CHARACTERIZATIONS OF CLASSES OF COUNTABLE
BOOLEAN INVERSE MONOIDS

MARK V. LAWSON AND PHILIP SCOTT

Abstract. A countably infinite Boolean inverse monoid that can be written
as an increasing union of finite Boolean inverse monoids (suitably embedded)
is said to be of finite type. Borrowing terminology from $C^*$-algebra theory,
we say that such a Boolean inverse monoid is AF (approximately finite) if the
finite Boolean inverse monoids above are isomorphic to finite direct products
of finite symmetric inverse monoids, and we say that it is UHF (uniformly
hyperfine) if the finite Boolean inverse monoids are in fact isomorphic to
finite symmetric inverse monoids. We characterize abstractly the Boolean
inverse monoids of finite type and those which are AF and, by using MV-
algebras, we also characterize the UHF monoids.

1. Introduction

In this paper, we shall be interested in countably infinite Boolean inverse monoids
$S$ which can be written in the form $S = \bigcup_{i=1}^{\infty} S_i$ where $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots$ and
where each $S_i$ is a subalgebra\footnote{This term will be defined later. It is much stronger than merely being an inverse submonoid.} of $S$. We shall characterize $S$ in three different
situations:

(1) The $S_i$ are finite Boolean inverse monoids. In this case, we say that $S$ is of
finite type. They are characterized in Theorem 4.3.

(2) The $S_i$ are finite fundamental Boolean inverse monoids. In this case, we
say that $S$ is an AF monoid. These were the subject of [19]. They are
characterized in Theorem 4.7. We characterize the Stone groupoids of AF
monoids as AF groupoids in Theorem 6.7.

(3) The $S_i$ are finite simple Boolean inverse monoids. In this case, we say that
$S$ is a UHF monoid. These are a special case of (2). They are characterized
in Section 5.

Cases (2) and (3) are clearly motivated by the theory of $C^*$-algebras.

We refer the reader to [12] for classical inverse semigroup theory, in particular
the theory of the natural partial order; to [33] for general results on Boolean inverse
semigroups; and to [2] for MV-algebras. In writing this paper, we have found the
thesis [30] very useful.

Remark 1.1. The following conventions are made throughout this paper:

(1) The only partial order considered in an inverse semigroup is the natural
partial order. Thus, in an inverse semigroup, meets and joins are always
taken with respect to this order.

(2) The meet operation in a Boolean algebra will always be denoted by con-
catenation. In the case of a unital Boolean algebra, the complement of the
element $e$ is denoted by $\bar{e}$.

(3) $\bigvee\emptyset = 0$ in any poset with a zero.
2 MARK V. LAWSON AND PHILIP SCOTT

(4) We make a distinction between a partial order and a strict partial order: thus we write \( \subseteq \) for containment and \( \subset \) for strict containment.

Acknowledgements The authors are very grateful to the anonymous referee for meticulously reading the original version of this paper and for the many constructive suggestions which led to improvements. Individual contributions by the referee are acknowledged where appropriate.

2. Background on inverse semigroups

In this section, we shall review the important definitions we shall need from inverse semigroup theory. See [11] for general semigroup theory (which we shall assume) and [12] for inverse semigroup theory.

2.1. General. Our inverse semigroups will have a zero and homomorphisms will be required to map zero to zero. In addition, if our semigroups are monoids then homomorphisms will be required to map identities to identities. If \( S \) is a monoid, we denote its group of \textit{units} (or \textit{invertible elements}) by \( U(S) \). If \( U(S) \) is trivial then the monoid is said to be \textit{conical}. If \( S \) is any semigroup and \( e \) is any idempotent in \( S \), then \( eSe \) is a monoid with identity \( e \) called a \textit{local monoid} of \( S \). If \( S \) is a semigroup then a subset \( I \) is an \textit{ideal} if \( SI, IS \subseteq I \); our ideals will always contain at least the zero and so are non-empty. If \( X \) is a non-empty set then \( I(X) \) denotes the \textit{symmetric inverse monoid} on \( X \); that is, the inverse monoid of all partial bijections of the set \( X \). If \( X \) is a finite set with \( n \) elements then we denote the corresponding symmetric inverse monoid by \( I_n \). If \( S \) is an inverse semigroup, we call the operation \( a \mapsto bab^{-1} \) \textit{conjugation}. We denote the set of idempotents of \( S \) by \( E(S) \). Observe that the conjugate of an idempotent is an idempotent. If \( X \subseteq S \) define \( E(X) = X \cap E(S) \). In the case where \( S \) is a monoid, the group of units of \( S \), \( U(S) \), acts on the semilattice of idempotents \( E(S) \) by \( e \mapsto geg^{-1} \). We call this the \textit{natural action} of \( U(S) \) on \( E(S) \).

The following is important. If \( a \) and \( b \) are elements of an inverse semigroup, we say that \( a \) and \( b \) are \textit{compatible}, written \( a \sim b \), if both \( a^{-1}b \) and \( ab^{-1} \) are idempotents. Being compatible is a necessary condition for \( a \) and \( b \) to have an upper bound.

We shall need a little groupoid theory. We refer the reader to [8] for the necessary background. For us, groupoids will be analogous to ‘groups with many identities’. If \( G \) is a groupoid, denote its set of identities by \( G_o \). We shall use the term \textit{principal groupoid} for the groupoid obtained from an equivalence relation. We shall also have to refer to \textit{connected groupoids}; a groupoid \( G \) is \textit{connected} if for any identities \( e \) and \( f \) there is an element \( g \) of the groupoid such that \( e = g^{-1}g \) and \( f = gg^{-1} \). This has nothing to do with topology. The connection between inverse semigroups and groupoids is this: underlying every inverse semigroup is a groupoid, which enables us to draw pictures of the elements of an inverse semigroup. If \( a \in S \), define \( d(a) = a^{-1}a \), called the \textit{domain} of \( a \), and \( r(a) = aa^{-1} \), called the \textit{range} of \( a \). Observe that if \( y \leq x \) and \( d(y) = d(x) \) (respectively, \( r(y) = r(x) \)) then, in fact, \( x = y \). We often write \( e \rightarrow f \) to mean that \( d(a) = e \) and \( r(a) = f \). In an inverse semigroup, this is equivalent to \( e \not D f \). Green’s relation \( D \) will play a very important role in this paper.

If \( d(a) = r(b) \) then define \( a \cdot b = ab \) is called the \textit{restricted product}. The inverse semigroup with respect to the restricted product is a groupoid. In this way, we can regard the elements of an inverse semigroup as being \textit{arrows}. Thus, we can also
Lemma 2.4. A version of the proof of part (2) below should include a proof because of its importance. We thank the referee for a simplified proof.

The key feature of factorizable inverse monoids is that they can be constructed from groups and semilattices in a simple way. The following is well-known but we shall prove it for the benefit of the reader.

Lemma 2.2. Let $S$ be an inverse semigroup. Let $e$ and $f$ be idempotents. Then $e \cog f$ if and only if $\mu(e) \cog \mu(f)$.

Proof. Only one direction needs proving. Suppose that $\mu(e) \cog \mu(f)$. Then there is an element $s \in S$ such that $\mu(e) = \mu(s^{-1}s)$ and $\mu(f) = \mu(ss^{-1})$. But $\mu$ is idempotent-separating. Thus $e = s^{-1}s$ and $f = ss^{-1}$. It follows that $e \cog f$. \hfill $\square$

In an inverse semigroup, $SeS \subseteq SfS$ is equivalent to $e \cog i \leq f$ for some idempotent $i$. We shall write $e \preceq f$ to mean $SeS \subseteq SfS$; this is not usual, but is convenient for us. This notation was first used in [19]. Observe that $\preceq$ is always a preorder on the set of idempotents of $S$. We write $e < f$ to mean $SeS \subset SfS$ where we wish to emphasize the fact that $e \neq f$.

We say that $S$ is Dedekind finite if $e \cog f \leq e$ implies that $e = f$ for any idempotents $e$ and $f$. An inverse monoid is said to be directly finite if for any idempotent $e$ we have that $e \cog 1$ implies that $e = 1$. Part (1) of the following is well-known or can be proved directly; see [19, Lemma 2.2]. The proof of Part (2) is immediate.

Lemma 2.3. Let $S$ be a Dedekind finite inverse semigroup.

(1) $\mathcal{D} = \mathcal{J}$.

(2) $S$ is directly finite.

2.2. Factorizable inverse monoids. An inverse monoid $S$ is said to be factorizable if every element is below a unit. It was proved in [4], that $\mathcal{I}(X)$ is factorizable if and only if $X$ is finite, and that factorizable inverse monoids are directly finite. The key feature of factorizable inverse monoids is that they can be constructed from groups and semilattices in a simple way. The following is well-known but we include a proof because of its importance. We thank the referee for a simplified version of the proof of part (2) below.

Lemma 2.4. Let $S$ be an inverse monoid.

(1) Let $G \leq U(S)$ be a subgroup and let $E \subseteq E(S)$ be a subsemilattice that contains the identity. Then $T = GE$ is an inverse submonoid of $S$ if and only if $E$ is closed under the natural action by $G$.

(2) If $G \leq U(S)$ is a subgroup and $E \subseteq E(S)$ is a subsemilattice that contains the identity such that $E$ is closed under the natural action by $G$ then $T = GE$ is a factorizable inverse monoid with group of units $G$ and semilattice of idempotents $E$. 

Observe that the above square commutes. Every product in an inverse semigroup can be written as a restricted product since $ab = (ae) \cdot (eb)$ where $e = d(a)r(b)$; observe that $ae \leq a$ and $eb \leq b$. The following can be proved directly or see [12, Section 3.1, Proposition 3].
(3) If \( T \) is a factorizable inverse submonoid of \( S \) then \( T = GE \) where \( G \) is the group of units of \( T \) and \( E \) is the semilattice of idempotents of \( T \) (and is closed under the natural action by \( G \)).

Proof. (1) Suppose first that \( E \) is closed under the natural action by \( G \); this means that if \( e \in E \) and \( g \in G \) then \( geg^{-1} \in E \). Let \( ge \) and \( hf \) be elements of \( T \). Then \((ge)(hf) = gh(h^{-1}eh)f \). But \( E \) is closed under the natural action by \( G \) and so \((ge)(hf) \in T \). Thus \( T \) is closed under multiplication. Let \( ge \in T \). Then \((ge)^{-1} = g^{-1}(geg^{-1}) \). But \( E \) is closed under the natural action by \( G \) and so \((ge)^{-1} \in T \). We have therefore proved that \( T \) is an inverse submonoid of \( S \). We now prove the converse: that if \( T \) is an inverse submonoid then \( E \) is closed under the natural action by \( G \). Let \( g \in G \) and \( e \in E \). We prove that \( geg^{-1} \in E \). Put \( s = ge \in T \). Then \( ss^{-1} \in T \), since \( T \) is an inverse submonoid of \( S \), from which it follows that \( geg^{-1} \in E \).

(2) By part (1), \( T = GE \) is an inverse submonoid of \( S \). Let \( ge \in T \) and suppose that \( d(ge) = 1 = r(ge) \). From \( d(ge) = 1 \) we deduce that \( e = 1 \). It follows that the group of units of \( T \) is \( G \). Let \( ge \) be an idempotent of \( T \). Then \( ge = (ge)^{-1} = eg^{-1} \). It follows that \( ge = (ge)(ge) = geg^{-1} = geg^{-1} \in E \). Thus the set of idempotents of \( T \) is \( E \). Observe that \( ge \leq g \) and so \( T \) is factorizable.

(3) Let \( G \) be the group of units of \( T \) and let \( E \) be its semilattice of idempotents. Clearly \( GE \subseteq T \). On the other hand we have equality since every element of \( T \) is beneath an element of \( G \). \( \square \)

2.3. Meet semigroups. An inverse semigroup \( S \) is called a meet-semigroup if each pair of elements has a meet. Meets have to be handled with care. There is a different way of thinking about meet-semigroups that will be important. Let \( S \) be an inverse semigroup. A function \( \phi : S \to E(S) \) is called a fixed-point operator if it satisfies the following two conditions:

(FPO1): \( s \geq \phi(s) \).
(FPO2): If \( s \geq e \) where \( e \) is any idempotent then \( \phi(s) \geq e \).

Thus, \( \phi(s) \) is the largest idempotent below \( s \). The following are proved in [20] or are immediate from the definition.

**Proposition 2.5.**

(1) An inverse semigroup \( S \) is an inverse meet-semigroup if and only if it possesses a fixed-point operator.

(2) In an inverse meet-semigroup, we have that \( \phi(s) = s \land s^{-1} \). In the monoid case, \( \phi(s) = s \lor 1 \).

(3) If \( \phi \) is the fixed-point operator then \( a \land b = \phi(ab^{-1})b \).

(4) \( \phi(s) = \phi(s^{-1}) \).

(5) \( \phi(s) = s \phi(s) = \phi(s) \).

(6) In an inverse meet-semigroup, we have that \( \phi(se) = \phi(s)e \) when \( e \) is an idempotent.

(7) If \( g \) and \( h \) are units then \( \phi(hgh^{-1}) = h \phi(g)h^{-1} \).

The following is proved as [12] Proposition 1.4.19 and shows how any binary meets in an inverse semigroup interact with the product.

**Lemma 2.6.** Let \( S \) be an inverse semigroup. Then for any \( a, b, c \in S \) such that \( a \land b \) exists, both \( ac \land bc \) and \( ca \land cb \) exist and \( (a \land b)c = ac \land bc \) and \( c(a \land b) = ca \land cb \).

We emphasize that in the following result, meets are to be formed in the larger Boolean inverse meet-monoid.

**Lemma 2.7.** Let \( T \) be an inverse submonoid of the inverse meet-monoid \( S \). Let \( \phi \) be the fixed-point operator for \( S \). Then \( T \) is closed under binary meets in \( S \) if and only if for each \( t \in T \) we have that \( \phi(t) \in T \).
Proof. Suppose first that \( T \) is closed under all binary meets in \( S \). Let \( t \in T \). We claim that \( \phi(t) \in T \). By Proposition 2.3, we have that \( \phi(t) = t \land 1 \). By assumption, \( T \) contains the identity of \( S \) (since it is a submonoid) and it is closed under binary meets. Thus \( \phi(t) \in T \). We now prove the converse. Suppose, now, that for each \( t \in T \) we have that \( \phi(t) \in T \). Let \( t_1, t_2 \in T \). Then, again by Proposition 2.3, we have that \( t_1 \land t_2 = \phi(t_1 t_2^{-1})t_2 \). Since \( T \) is, in particular, an inverse submonoid of \( S \), we have that \( t_1 t_2^{-1} \in T \). Thus, by assumption, \( \phi(t_1 t_2^{-1}) \in T \). Whence, \( t_1 \land t_2 \in T \)

The authors are grateful to the referee for simplifying the proof below.

**Lemma 2.8.** Let \( S \) be an inverse meet-monoid, with \( \phi \) as the fixed-point operator of \( S \), and let \( F \) be an inverse submonoid of \( S \) which is also factorizable. Then \( F \) is closed under binary meets in \( S \) if and only if \( \phi(g) \in F \) for every unit \( g \) in \( F \).

Proof. Suppose first that \( \phi(g) \in F \) for every unit \( g \) in \( F \). A typical element of \( F \) has the form \( \phi ge \) where \( g \) is a unit of \( F \) and \( e \) is an idempotent of \( F \). Then \( \phi(ge) = \phi(g)e \) by part (6) of Proposition 2.3. By assumption, \( \phi(g) \in F \), and \( e \in F \). It follows that \( \phi(ge) \in F \) and the result follows by Lemma 2.4. By Lemma 2.4, the proof of the converse is immediate.

### 2.4. Boolean algebras and posets

In this paper, generalized Boolean algebras are referred to as Boolean algebras and Boolean algebras will be called *unital* Boolean algebras. See [32]. Most of the paper deals exclusively with unital Boolean algebras. If \( e \) and \( f \) are elements of a Boolean algebra we write \( e \perp f \) if \( ef = fe = 0 \) and say that \( e \) and \( f \) are orthogonal. In the case where \( e \perp f \) the join \( e \lor f \) is often written \( e \uplus f \). In a Boolean algebra, if \( f \leq e \) then denote by \( e \setminus f \) the unique element such that \( e = (e \setminus f) \lor f \) and \( f \perp (e \setminus f) \). Observe that in a unital Boolean algebra, \( ef = 0 \) if and only if \( e \leq f \). The two-element Boolean algebra will be denoted by \( 2 \).

If \( P \) is a poset and \( a \in P \), we write \( a^\downarrow \) for the set \( \{ b \in P : b \leq a \} \) and \( a^\uparrow \) for the set \( \{ b \in P : a \leq b \} \). A subset \( Q \) of \( P \) is called an *order-ideal* if \( q \in Q \) and \( p \leq q \) implies that \( p \in Q \). An order-preserving map between posets will be called *isotone*. An *order-isomorphism* between two posets is an isotone bijection whose inverse is also isotone. Let \( P \) be a poset with zero. A non-zero element \( a \in P \) is called an *atom* if \( b \leq a \) implies that \( b = a \) or \( b = 0 \). The following definition will play a key role in this paper.

**Definition.** Let \( S \) be a group (respectively, a Boolean algebra, respectively an MV-algebra). Let \( X = \{ s_1, \ldots, s_n \} \) be a finite subset of \( S \). Then \( \langle s_1, \ldots, s_n \rangle \) denotes the subgroup (respectively, Boolean subalgebra, respectively MV-subalgebra) of \( S \) generated by \( X \). A structure is *locally finite* if every finitely generated substructure is finite.

We follow [5] in our definition of a locally finite MV-algebra. The following is proved in [5, Chapter 11].

**Lemma 2.9.** Unital Boolean algebras are locally finite

It will be useful to describe the elements of the Boolean subalgebra of the unital Boolean algebra \( B \) generated by the elements \( X = \{ p_1, \ldots, p_n \} \). A product \( x_1 \ldots x_n \) where each \( x_i \) is either \( p_i \) or \( \overline{p_i} \) is called a *minterm* [23]. The Boolean subalgebra generated by \( X \) consists of all finite joins of minterms. See [5, Pages 78–84]. To expand on this point, clearly the unital Boolean algebra \( \langle X \rangle \) contains all the minterms. On the other hand, the unital Boolean algebra generated by the minterms contains each element of \( X \). It follows that if we have a finite number of
elements of a unital Boolean algebra then we may assume that those elements are pairwise orthogonal.

3. **Background on Boolean inverse monoids**

In this section, we shall revise the important definitions around Boolean inverse monoids. For more on such monoids, see [33].

3.1. **General.** An inverse semigroup is said to be distributive if every compatible pair of elements has a join and multiplication distributes over such joins. A distributive inverse semigroup is said to be Boolean if its semilattice of idempotents is a Boolean algebra. The symmetric inverse monoids are always Boolean. We shall mostly be concerned with Boolean inverse monoids in this paper, and so their semilattices of idempotents form unital Boolean algebras.

**Example 3.1.** We can easily construct Boolean inverse monoids from groups. Let \( G \) be a group. Then \( G^0 \) is the group \( G \) with an adjoined zero. Inverse monoids of the form \( G^0 \) are examples of Boolean inverse monoids where the Boolean algebra of idempotent is simply the two-element unital Boolean algebra.

The following result will be used repeatedly. It can easily be directly proved.

**Lemma 3.2.** Let \( S \) be a Boolean inverse monoid.

1. If \( x = \bigvee_{i=1}^n x_i \) then \( x^{-1} = \bigvee_{i=1}^n x_i^{-1} \).
2. We have that 
   \[
   d \left( \bigvee_{i=1}^n x_i \right) = \bigvee_{i=1}^n d(x_i) \quad \text{and} \quad r \left( \bigvee_{i=1}^n x_i \right) = \bigvee_{i=1}^n r(x_i).
   \]

A morphism of Boolean inverse monoids is a homomorphism that preserves the 0 and 1 and maps finite joins to finite joins. A meet-morphism between Boolean inverse meet-monoids is a morphism of Boolean inverse monoids that also preserves binary meets. If \( a \) is an element of a Boolean inverse semigroup, define \( e(a) = d(a) \lor r(a) \), called the extent of \( a \). Observe that \( a \in e(a)Se(a) \) and so \( S = \bigcup_{e \in E(S)} eSe \).

We say that \( a \) and \( b \) are orthogonal, written \( a \perp b \), if \( a^{-1}b = 0 = ab^{-1} \). If an orthogonal pair of elements has a join then we speak of orthogonal joins. The join of \( a \) and \( b \) is written \( a \lor b \) and if they are orthogonal their orthogonal join is written \( a \oplus b \).

For the following, see [15, Lemma 2.8].

**Lemma 3.3.** Let \( S \) be a Boolean inverse semigroup. Suppose that \( a \sim b \). Then the following are equivalent:

1. \( d(a) \perp d(b) \).
2. \( r(a) \perp r(b) \).
3. \( a \perp b \).

Let \( S \) be a Boolean inverse monoid with group of units \( U(S) \) and Boolean algebra of idempotents \( E(S) \). The natural action of the group \( U(S) \) on the unital Boolean algebra \( E(S) \) is given by \( g \cdot e = ge^{-1} \). The following are all easy to prove:

1. \( g \cdot (e \lor f) = g \cdot e \lor g \cdot f \).
2. \( g \cdot (e \land f) = g \cdot e \land g \cdot f \).
3. \( g \cdot \bar{e} = \bar{(g \cdot e)} \).

---

2The context will determine what is meant by the symbol \( \oplus \) since we shall also use it to denote the binary operation in an MV-algebra.
It follows that the natural action of the group of units of a Boolean inverse monoid preserves the Boolean algebra structure of the Boolean algebra of idempotents.

The following result tells us about the structure of local monoids in Boolean inverse semigroups. See [15, Proposition 2.4].

**Lemma 3.4.** In a Boolean inverse semigroup S, the local monoids are always Boolean inverse monoids. If S is fundamental so, too, is any local monoid.

The following is what remains of one of the distributive laws in a distributive inverse semigroup. See [15, Lemma 2.5].

**Lemma 3.5.** Let S be a distributive inverse semigroup. Suppose that $\bigvee_{i=1}^{n} a_i$ and $c \land (\bigvee_{i=1}^{m} a_i)$ exist. Then all the meets $c \land a_i$ exist, the join $\bigvee_{i=1}^{m} c \land a_i$ exists and $c \land (\bigvee_{i=1}^{m} a_i) = \bigvee_{i=1}^{m} c \land a_i$.

The following result is basic to the theory of Boolean inverse monoids

**Lemma 3.6.** Let S be a Boolean inverse monoid. Suppose that $x \in S$ is such that $d(x) = \bigvee_{i=1}^{n} e_i$.

1. Then x is a join of the elements $xe_i$ and $r(x) = \bigvee_{i=1}^{n} r(xe_i)$.
2. If the idempotents $e_i$ are orthogonal, so too are the idempotents $r(xe_i)$.
3. If the idempotents $e_1, \ldots, e_n$ are $\mathcal{D}$-related, so too are the idempotents $r(xe_1), \ldots, r(xe_n)$.

**Proof.** (1) The elements $xe_i$ are pairwise compatible and so have a join $y \leq x$. But by Lemma 3.2, $d(y) = d(x)$ and so $x = \bigvee_{i=1}^{n} xe_i$. We now use Lemma 3.2 again to deduce that $r(x) = \bigvee_{i=1}^{n} r(xe_i)$.

(2) This follows by Lemma 3.3.

(3) If $e_i$ is $\mathcal{D}$-related to $e_{i+1}$ then $r(xe_i)$ is $\mathcal{D}$-related to $r(xe_{i+1})$. To see why, suppose that $e_i \xrightarrow{a} e_{i+1}$. Then we have the following diagram:

\[
\begin{array}{ccc}
  e_i & \xrightarrow{a} & e_{i+1} \\
  x e_i & \downarrow & x e_{i+1} \\
  r(xe_i) & \xrightarrow{\mathcal{D}} & r(xe_{i+1})
\end{array}
\]

This immediately implies that $r(xe_i) \mathcal{D} r(xe_{i+1})$. \qed

### 3.2. Rook matrices

We shall need a way of constructing Boolean inverse semigroups. The following is a matrix technique, the details of which can be found in [33], but we will outline it here now. Let S be a Boolean inverse semigroup. Let m and n be ordinals; we shall only need the cases where m and n are both finite or $m = n = \omega$. An $m \times n$ matrix over S is said to be a rook matrix if it satisfies the following two conditions:

1. In each column and each row only a finite number of elements are non-zero.
2. If $a$ and $b$ are in the same row but distinct columns then $a^{-1}b = 0$; if $a$ and $b$ are in the same column but distinct rows then $ab^{-1} = 0$.

We denote by $B_n(S)$ the set of all $n \times n$ rook matrices over S and by $R_n(S)$ the set of all $n \times n$ rook matrices over S in which each rook matrix contains only a finite number of non-zero entries. Multiplication of rook matrices is defined in the usual way except that addition is replaced by join (which makes sense because of the conditions we have imposed on the rows and columns). If A is a rook matrix then $A^*$ is defined to be the transpose of A with the inverse taken of all entries. It can be checked that $A^*$ is a rook matrix such that $A = AA^*A$ and $A^* = A^*AA^*$. If
Lemma 3.7. Let \( S \) be a Boolean inverse semigroup and let \( F \subseteq E(S) \) be a conjugate-closed additive ideal. Then \( F \) is closed under the \( \mathcal{D} \)-relation.

Proof. Let \( f \in F \) and suppose that \( e \not\mathcal{D} f \). Then \( e = aa^{-1} \) and \( f = a^{-1}a \) for some \( a \in S \). But \( e = aa^{-1} = a(a^{-1}a)a^{-1} = afa^{-1} \). By assumption, \( F \) is conjugate-closed. It follows that \( e \in F \). \( \square \)

If \( X \subseteq S \) define \( X \subseteq X^\vee \) to be the set of all joins (in \( S \)) of finite compatible subsets of \( X \).

Lemma 3.8. Let \( S \) be a distributive inverse semigroup. If \( I \) is a non-empty ideal of \( S \) then \( I^\vee \) is an additive ideal of \( S \).

Proof. A typical element of \( I^\vee \) is \( \bigvee_{i=1}^m a_i \) where \( a_i \in I \). If \( s \in S \) then \( s(\bigvee_{i=1}^m a_i) = \bigvee_{i=1}^m sa_i \), and dually. \( \square \)

Let \( S \) be a Boolean inverse semigroup. If \( X \subseteq S \) then \( SX \) is the smallest ideal of \( S \) containing \( X \). It follows by Lemma 3.8 that for any subset \( X \subseteq S \), we have that \( (SX)^\vee \) is an additive ideal of \( S \).

Lemma 3.9. Let \( S \) be a Boolean inverse semigroup.

1. Let \( I \) be an additive ideal of \( S \). Then \( E(I) \) is a conjugation-closed additive ideal.

2. If \( I \subseteq J \), where \( I \) and \( J \) are additive ideals, then \( E(I) \subseteq E(J) \).

3. If \( F \) is a conjugation-closed additive ideal of \( E(S) \) then \( (SF) \subseteq (SG) \).

4. If \( F \subseteq G \), where \( F \) and \( G \) are conjugation-closed additive ideals of \( E(S) \) then \( (SF)^\vee \subseteq (SG)^\vee \).

5. There is an order-isomorphism between the poset of additive ideals of \( S \) and the poset of conjugation-closed additive ideals of \( E(S) \).

Proof. (1) Let \( I \) be an additive ideal. Observe that if \( a \in I \) then \( d(a) \in I \), since \( I \) is an ideal. It follows that \( E(I) \) is non-empty. It is clear that \( E(I) \subseteq I \) and \( f \leq e \). Then \( e \in I \) and so \( f \in I \) from which we get that \( f \in E(I) \). We have proved that \( E(I) \) is an order-ideal. It is closed under binary joins since \( I \) is closed under binary joins and the join of idempotents is always an idempotent. Thus \( E(I) \) is an additive ideal of \( E(S) \). It remains to show that it is conjugation-closed. Let \( e \in E(I) \). Then \( e \in I \). Let \( s \in S \) be any element. Then \( ses^{-1} \in I \) since \( I \) is a semigroup ideal. But \( ses^{-1} \) is an idempotent. Thus \( ses^{-1} \in E(I) \). We have therefore proved that \( E(I) \) is a conjugation-closed additive ideal of \( E(S) \).

The proofs of (2), (3) and (4) are immediate.

(5) We have to prove that the constructions in (1) and (4) are mutually inverse. Let \( I \) be an additive ideal of \( S \). We shall prove that \( I = (SE(I)S)^\vee \). Clearly,
Let \(a \in I\). Then \(d(a) \in E(I)\) and so \(a \in SE(I)S\). The result now follows. Let \(F \subseteq E(S)\) be a conjugation-closed additive ideal. We prove that \(E((SFS)^{\vee}) = F\). It is clear that \(F \subseteq E((SFS)^{\vee})\). We prove the reverse inclusion. Let \(e \in E((SFS)^{\vee})\). Then \(e = f_1 \lor f_2\) where \(f_1, f_2 \in E(SFS)\). Thus \(f_1 = a_1 e_1 b_1\), where \(a_1, b_1 \in S\) and \(e_1 \in F\), and \(f_2 = a_2 e_2 b_2\), where \(a_2, b_2 \in S\) and \(e_2 \in F\). We can write each of these products as a restricted products. Thus \(f_1 = a_1' \cdot e_1' \cdot b_1'\) and \(f_2 = a_2' e_2' b_2'\) where the primes indicate potentially smaller elements using Lemma 2.1. Since \(F\) is an order-ideal, we have that \(e_1', e_2' \in F\). But \(f_1 \not\subseteq e_1'\) and \(f_2 \not\subseteq e_2'\). By Lemma 3.7 it follows that \(f_1, f_2 \in F\) since \(F\) is conjugation closed. We now have that \(f \in F\) since \(F\) is closed under binary joins. \(\square\)

Let \(e\) and \(f\) be idempotents in a distributive inverse semigroup. A finite set \(X\) is said to be a pencil from \(f\) to \(e\), denoted by \(f \circ e\), if \(f = \bigvee d(x)\) and \(\bigvee r(x) \leq e\).

**Lemma 3.10.** Let \(F\) be a conjugation-closed additive ideal of \(E(S)\), where \(S\) is a Boolean inverse semigroup. If \(e \in F\) and \(f \circ e\) then \(f \in F\).

**Proof.** By assumption, there is a finite set \(X = \{x_1, \ldots, x_m\}\) such that \(f = \bigvee_{i=1}^{m} d(x_i)\) and \(\bigvee_{i=1}^{m} r(x_i) \leq e\).

Because \(F\) is an order-ideal, we have that \(\bigvee_{i=1}^{m} r(x_i) \in F\). Thus, again using the fact that \(F\) is an order-ideal, we have that \(r(x_i) \in F\) for \(1 \leq i \leq m\). Because \(F\) is closed under conjugation, we may apply Lemma 3.7 and so \(d(x_i) \in F\) for \(1 \leq i \leq m\). We can now state the fact that \(F\) is closed under finite joins to get that \(f \in F\). \(\square\)

If \(S\) is a Boolean inverse monoid then both \(\{0\}\) and \(S\) are additive ideals and if these are the only additive ideals we say that \(S\) is 0-simplifying. A Boolean inverse semigroup which is both 0-simplifying and fundamental is called simple [14]. Theorem 4.18. Our use of the word ‘simple’ is justified by the following result.

**Lemma 3.11.** Let \(\theta: S \to T\) be a morphism between Boolean inverse monoids. If \(S\) is simple then \(\theta\) is injective.

**Proof.** The kernel of \(\theta\), that is the set \(K = \{a \in S: \theta(a) = 0\}\), is an additive ideal of \(S\). By assumption, \(S\) is 0-simplifying. Thus \(K = \{0\}\). Let \(e\) and \(f\) be idempotents of \(S\) such that \(\theta(e) = \theta(f)\). Then \(e \land f \leq e\) and \(\theta(e \land f) = \theta(e)\). It follows that \(\theta(e \land \overline{f}) = 0\). But we have already shown that the kernel of \(\theta\) consists only of zero. This means that \(e \land f = e\). By symmetry, \(e \land f = f\). This implies that \(e = f\). We have shown that \(\theta\) is idempotent-separating. But \(S\) is assumed to be fundamental. This implies that the congruence determined by \(\theta\) is the equality relation on \(S\) and so \(\theta\) is injective. \(\square\)

We now consider local monoids.

**Lemma 3.12.** Let \(S\) be a Boolean inverse monoid. If \(S\) is 0-simplifying so, too, is any local monoid.

**Proof.** Let \(I \subseteq eSe\) be any non-zero additive ideal. We shall prove that \(I = eSe\), from which we deduce that \(S\) is 0-simplifying. Put \(J = (SIS)^{\vee}\). This is an additive ideal of \(S\) and so, by assumption, \(J = S\). Let \(a \in eSe\) be arbitrary. Then \(a = a_1 \lor \ldots \lor a_n\) where \(a_i \in SIS\). Observe that since \(a \in eSe\) we can assume that \(a_i \in eSe\). In addition, \(eSe\) is closed under joins. Thus we may as well assume that \(a \in SIS\). It follows that \(a = xy\) where \(x, y \in S\) and \(b \in I\). Again, we may assume that \(x, y \in eSe\). It follows that \(a \in I\), as required. \(\square\)
3.4. Subalgebras. This section is important because we need to work with the right definition of ‘substructure’. Let $B$ be a unital Boolean algebra. We say that $B'$ is a subalgebra of $B$ if $B' \subseteq B$ contains the 0 and 1 and is closed under joins, meets and complements. Now, let $S$ and $T$ be Boolean inverse monoids. Each morphism $\theta: S \to T$ induces a homomorphism of Boolean algebras $\theta(\mathsf{E}(S)): \mathsf{E}(S) \to \mathsf{E}(T)$.

**Definition.** Let $S$ be a Boolean inverse monoid. An inverse submonoid $T$ of $S$ will be called a subalgebra if conditions (1) and (2) below hold:

1. If $a, b \in T$ and $a \sim b$ then $a \lor b \in T$, where $a \lor b$ means the join calculated in $S$.
2. $\mathsf{E}(T)$ is a subalgebra of $\mathsf{E}(S)$; because we are assuming that $T$ is closed under joins and products, in the presence of (1) this is equivalent to requiring that if $e \in \mathsf{E}(T)$ then $\bar{e} \in \mathsf{E}(T)$.
3. If $S$ is, in addition, a meet-monoid then we require that $T$ is closed under binary meets.

Wehrung [33, Definition 3.1.7] uses the terminology ‘additive inverse subsemigroup’. The justification for our use of the term ‘subalgebra’ is based on [33, Corollary 3.2.7].

**Remark 3.13.** Observe that if $S$ is a Boolean inverse monoid and $T$ is a subalgebra of $S$ then units of $T$ are units of $S$; this is immediate since $S$ has the same identity as any of its subalgebras.

**Lemma 3.14.**

1. Let $S$ and $T$ be Boolean inverse monoids. Let $\theta: S \to T$ be a morphism which is also injective. Then $\theta(S)$ is a subalgebra of $T$.
2. If $S$ is a subalgebra of $T$, then the natural embedding of $S$ into $T$ is a morphism.
3. Let $S$ and $T$ be Boolean inverse meet-monoids. Let $\theta: S \to T$ be a morphism which is also injective. Then $\theta(S)$ is a subalgebra of $T$.
4. If $S$ and $T$ are, in addition, inverse meet-monoids and $S$ is a subalgebra of $T$, then the natural embedding of $S$ into $T$ is a morphism.

**Proof.** (1) Let $\theta: S \to T$ be an injective morphism of Boolean inverse monoids. We prove that $\theta(S)$ is a subalgebra of $T$. By standard inverse semigroup theory, we have that $\theta(S)$ is an inverse submonoid of $T$. Let $\theta(a) \sim \theta(b)$ in $\theta(S)$. Then $a \sim b$ in $S$ since we are assuming that our map is injective. Thus $\theta(a \lor b) = \theta(a) \lor \theta(b)$ in $T$. It follows that joins in $\theta(S)$ are the same as joins in $T$. Now, let $\theta(e)$ be an idempotent in $\theta(S)$ where $e$ is an idempotent of $S$. Then $1 = e \lor \bar{e}$ and $0 = c\bar{e}$ in $S$. It follows that $\theta(e) = \theta(\bar{e})$.

The proofs of (2), (3) and (4) are now straightforward. □

**Lemma 3.15.** Let $S$ be a Boolean inverse monoid. Let $T$ be an inverse submonoid of $S$ such that $\mathsf{E}(T)$ is a Boolean subalgebra of $\mathsf{E}(S)$. Then $T' \lor$ is a subalgebra of $S$. In addition, if $T$ is finite then $T' \lor$ is finite.

**Proof.** Observe that since $T$ is an inverse submonoid of $S$ so too is $T' \lor$. The fact that $T' \lor$ is distributive is inherited from $S$. The idempotents of $T' \lor$ are the elements of the form $e = \bigvee_{i=1}^n e_i$ where the $e_i \in \mathsf{E}(T)$. It follows that the idempotents of $T' \lor$ are precisely the elements of $\mathsf{E}(T)$. But $\mathsf{E}(T)$ is a Boolean algebra. It follows that $T' \lor$ is a Boolean inverse monoid. It is therefore a subalgebra by construction. The fact that $T' \lor$ is finite if $T$ is finite follows from the fact that the number of finite compatible subsets of $T$ is finite. □
We shall now show how to construct examples of subalgebras. If $S$ is a Boolean inverse monoid and $T_1, \ldots, T_m \subseteq S$ are non-empty subsets then $T_1 \vee \cdots \vee T_m$ consists of all elements $t_1 \vee \cdots \vee t_m$ where $t_i \in T_i$ and the join is defined. If $T_i T_j = \{0\}$ for $i \neq j$ then we write $T_1 \oplus \cdots \oplus T_m$ instead of $T_1 \vee \cdots \vee T_m$.

**Lemma 3.16.** Let $S$ be a Boolean inverse monoid. Let each $T_i$ be a Boolean inverse submonoid of $e_i S e_i$ (so that the identity of $T_i$ is $e_i$) and put $e = e_1 \oplus \cdots \oplus e_n$. Then $T = T_1 \oplus \cdots \oplus T_m$ is a Boolean inverse submonoid of $e S e$. If each $T_i$ is a subalgebra of $e_i S e_i$ then $T$ is a subalgebra of $e S e$.

**Proof.** We begin by using the fact that each $T_i$ is an inverse subsemigroup of $S$. Let $t \in T$. Then $t = \bigvee_{i=1}^m t_i$, where $t_i \in T_i$. Thus $t^{-1} = \bigvee_{i=1}^m t_i^{-1}$ by Lemma 3.2. If $s = \bigvee_{i=1}^n s_i$, where $s_i \in T_i$, then $st = \bigvee_{i=1}^n s_i t_i$. Thus $T$ is an inverse submonoid of $e S e$. To prove the final claim, it is clear that joins are constructed point-wise. Let $f \in T$ be an idempotent. Then $f = \bigvee_{i=1}^m f_i$, where each $f_i$ is an idempotent of $T_i$. Define $f = \bigvee_{i=1}^m f_i$, where each $f_i$ is calculated in $T_i$. It is easy to check that $ff = 0$ and $f \vee f = e$. □

3.5. **Products of Boolean inverse monoids.** Let $S_1, \ldots, S_m$ be a finite number of Boolean inverse monoids. Then their product $S = S_1 \times \cdots \times S_m$ is a Boolean inverse monoid with the obvious componentwise operations. There are morphisms of Boolean inverse monoids $\pi_i : S \to S_i$ which simply project onto the $i$th coordinate.

However, the maps $\iota_i : S_i \to S_1 \times \cdots \times S_m$ given by $s \mapsto (0, \ldots, s, \ldots, 0)$, where the $s$ is in the $i$th position, are homomorphisms of semigroups, map zero to zero and map binary joins to binary joins but are not morphisms of Boolean inverse monoids since the identity of $S_i$ is not mapped to the identity of $S$. If we denote the identity of $S_i$ by $f_i$, then $S_i \cong \iota_i(f_i)$, a local monoid of $S$. Observe that the idempotents $\iota_i(e_i)$ are central in $S_i$, pairwise orthogonal and their join is the identity of $S$. This leads us to the following result which is the analogue of a result from ring theory.

**Lemma 3.17.** Let $S$ be a Boolean inverse monoid. Suppose that $e_1, \ldots, e_n$ is a finite set of orthogonal central idempotents such that $1 = e_1 \oplus \cdots \oplus e_n$. Then

$$S \cong e_1 S e_1 \times \cdots \times e_n S e_n.$$  

**Proof.** By assumption $1 = e_1 \oplus \cdots \oplus e_n$. Thus for each element $a \in S$ we have that $a = 1 a 1 = \prod_{1 \leq i, j \leq n} e_i a e_j$. By assumption, the idempotents $e_i$ are central and orthogonal. This means that $e_i a e_j = 0$ if $i \neq j$. Define the map $\theta$ from $S$ to $e_1 S e_1 \times \cdots \times e_n S e_n$ by $a \mapsto (e_1 a e_1, \ldots, e_n a e_n)$. Suppose that $a$ and $b$ map to the same element. Then $e_i a e_i = e_i b e_i$ for all $i$. By our result above, this implies that $a = b$. Now choose an arbitrary element $x$ of $e_1 S e_1 \times \cdots \times e_n S e_n$. This has components $e_i a e_i$. Define $b = e_1 a_1 e_1 \vee \cdots \vee e_n a_n e_n$, which makes sense because the idempotents are orthogonal. Then, using again the fact that the idempotents are orthogonal, we find that $b$ maps to $x$. We have shown that our function is bijective. It is now routine to check that this is a monoid homomorphism that preserves binary joins. It follows that $\theta$ is an isomorphism of Boolean inverse monoids. □

We shall need Lemma 3.17 in the following special case.

**Lemma 3.18.** Let $S$ be a Boolean inverse monoid. Let $T_i$ be a subalgebra of $e_i S e_i$. Let $1 = e_1 \oplus \cdots \oplus e_n$, an orthogonal join. Put $T = T_1 \oplus \cdots \oplus T_n$. Then $T \cong T_1 \times \cdots \times T_n$.

**Proof.** Let $a \in T_i$ and $b \in T_j$ where $i \neq j$. Then $ab = a e_i e_j b = 0$. We have therefore proved that $T_i T_j = \{0\}$ when $i \neq j$. It follows by Lemma 3.16 that $T$ is a subalgebra of $S$. Let $t \in T$. Then $t = t_1 \vee \cdots \vee t_n$ where $t_i \in T_i$. Observe that
It follows that \( e_i \) commutes with every element in \( T \). By Lemma 3.17, we have that \( T \cong e_1Te_1 \times \ldots \times e_nTe_n \). We prove that \( e_iTe_i = T_i \). We have that \( e_iTe_i = e_i(T_1 \vee \ldots \vee T_n)e_i = e_iT_ie_i = T_i \), using the fact that the idempotents \( e_1, \ldots, e_n \) are orthogonal, and \( e_i \) is the identity of \( T_i \). \( \square \)

### 3.6. Finite Boolean inverse monoids

The authors are grateful to the referee for spotting an error in the original statement of the lemma below.

**Lemma 3.19.** Every finite Boolean inverse semigroup is a monoid and has binary meets.

**Proof.** There are a finite number of idempotents and so their join is the identity. Let \( S \) be a finite Boolean inverse monoid containing elements \( a \) and \( b \). The set of elements beneath \( a \) and \( b \) is a finite compatible set and so has a join. This is precisely the meet of \( a \) and \( b \). \( \square \)

The following counterexample builds on one due to the referee.

**Example 3.20.** This example demonstrates that you have to be careful about just where meets are computed. Consider the finite Boolean inverse meet-monoid \( \mathcal{I}_3 \). It is convenient to regard \( \mathcal{I}_3 \) as the set of all partial bijections of the set \( \{1, 2, 3\} \). Denote the identity of this monoid by \( 1 \) and its zero by \( 0 \). Put \( g = (2, 3) \) a transposition. Put \( G = \{1, g\} \). Then \( G \) is a subgroup of the group of units of \( \mathcal{I}_3 \). If we adjoin the zero, we get that \( G^0 \) is a group with zero adjoined contained in \( \mathcal{I}_3 \). The meet \( g \wedge 1 \) in \( G^0 \) is 0, but \( g \wedge 1 \) calculated in \( \mathcal{I}_3 \) is the idempotent \( e = 1_{\{1\}} \).

The structure of finite Boolean inverse monoids can be described in two ways (but see below): in terms of finite groupoids or in terms of rook matrices. We can use the former to prove the latter. The theory of finite Boolean inverse monoids was described in [14, Theorem 4.18]. See, also, [23] and [11]. More generally, for the structure of Boolean inverse monoids with only a finite number of idempotents, see [34].

Let \( S \) be an inverse semigroup with zero. The set of atoms of an inverse semigroup \( S \) is denoted by \( \text{at}(S) \). If non-empty, \( \text{at}(S) \) is a groupoid with respect to the restricted product; in which case, we call this the minimal groupoid of \( S \). If \( G \) is a groupoid then a subset \( A \subseteq G \) is said to be a local bisection if \( g, h \in A \) and \( g^{-1}y = h^{-1}h \) (or \( g^{-1}y = hh^{-1} \)) then \( g = h \). The set of all local bisections of a groupoid \( G \) is denoted by \( \text{K}(G) \). This is always a Boolean inverse monoid.

For proofs of the following, see [14, Theorem 4.18].

**Theorem 3.21.**

1. The finite Boolean inverse monoids have the form \( \text{K}(G) \) where \( G \) is a finite groupoid. In fact, if \( S \) is a finite Boolean inverse monoid then \( S \cong \text{K}(\text{at}(S)) \).
2. \( \text{K}(G) \) is fundamental if and only if \( G \) is principal.
3. \( \text{K}(G) \) is 0-simplifying if and only if \( G \) is connected.

A Boolean inverse monoid is said to be matricial if it is isomorphic to a finite product of finite symmetric inverse monoids. By using the structure theory of (finite) groupoids, the above results can be rendered purely algebraic, as follows:

**Theorem 3.22.**

1. The finite Boolean inverse monoids are isomorphic to finite direct products of Rook matrices over groups with an adjoined zero.
2. The finite fundamental Boolean inverse monoids are isomorphic to the matricial inverse monoids.
3. The finite simple Boolean inverse monoids are isomorphic to the finite symmetric inverse monoids.
There is, in fact, a third way of describing the structure of finite Boolean inverse monoids which might be termed 'co-ordinate free'.

**Theorem 3.23.** Let $S$ be a finite Boolean inverse monoid. Then there are central idempotents $e_1, \ldots, e_n$ such that $1 = e_1 \oplus \ldots \oplus e_n$ and
\[ S \cong e_1S_{e_1} \times \ldots \times e_nS_{e_n}, \]
and each $e_iS_{e_i}$ is 0-simplifying. If $S$ is also fundamental then each $e_iS_{e_i}$ is simple.

**Proof.** Let $S$ be a finite Boolean inverse monoid. Let the $\mathcal{D}$-classes of the set of atoms be labelled $1, \ldots, n$. Define $e_i$ to be the join of the atomic idempotents in the $i$th $\mathcal{D}$-class. Observe, first, that $1 = e_1 \vee \ldots \vee e_n$ since every element of $S$ is a join of atoms. Let $a \in S$ be any non-zero element. Then $a = a_1 \vee \ldots \vee a_n$ where each $a_i$ is a join of atoms from the $i$th $\mathcal{D}$-class. Observe that $e_ja_i = a_je_i$. Now, let, $j \neq i$. We suppose first that $a_i$ is an atom in the $i$th $\mathcal{D}$-class. If $e_ja_i = a_i$ then $r(a_i) \leq e_j$. It follows that $r(a_i) = e_j$. But this contradicts the assumption that $a_i$ and $e_j$ are from different $\mathcal{D}$-classes of atoms. It follows that $e_ja_i = 0$. We may similarly show that $ae_j = 0$. If, now, $a$ is an arbitrary non-zero element of $S$, then $e_ja = ae_j$, which is equal to the join of the set of atoms in the $j$th $\mathcal{D}$-class below $a$. We have shown that the idempotents $e_1, \ldots, e_n$ are central. By Lemma 3.17 we have that
\[ S \cong e_1S_{e_1} \times \ldots \times e_nS_{e_n}. \]
It is clear by looking at the atoms that each $e_iS_{e_i}$ is 0-simplifying.

If $S$ is fundamental then each $e_iS_{e_i}$ is also fundamental by Lemma 3.17. It follows that in this case, each $e_iS_{e_i}$ is a finite simple Boolean inverse monoid. \(\square\)

Let $S$ be a finite fundamental Boolean inverse monoid. We call the simple Boolean inverse monoids $e_iS_{e_i}$ which appear in Theorem 3.23 the components of $S$. This concept will play a very important role in our work.

**Example 3.24.** Every finite matricial semigroup can be embedded as subalgebra of a finite simple Boolean inverse monoid. Recall that if $I_n$ is a finite symmetric inverse semigroup on the finite set with $n$ elements $X$ then the idempotents of $I_n$ are the elements of the form $1_A$ where $A \subseteq X$. We have that the local monoid $1_AI_n1_A \cong I_n(1_A)$. We can embed, for example, the matricial semigroup $I_3 \times I_4 \times I_5$ into the finite symmetric inverse monoid $I_{12}$ as follows. Regard $I_3$ as $I(\{1,2,3\})$, regard $I_4$ as $I(\{4,5,6,7\})$, and regard $I_5$ as $I(\{8,9,10,11,12\})$. With these identifications, each element of $I_3 \times I_4 \times I_5$ can be mapped to an element of $I_{12}$. This mapping preserves joins and is a monoid map that sends zero to zero. It embeds $I_3 \times I_4 \times I_5$ as a subalgebra of $I_{12}$.

### 3.7. Factorizable Boolean inverse monoids.

In the case of Boolean inverse monoids, the notion of factorizability assumes another form. Let $S$ be a Boolean inverse monoid. We say that a Boolean inverse monoid has $\mathcal{D}$-complementation if $e \mathcal{D} f$ implies that $\bar{e} \mathcal{D} \bar{f}$ for any idempotents $e, f$. The definition of factorizability presupposes the existence of an identity, but we can also make a definition that does not require the existence of such an identity. We say that a Boolean inverse semigroup $S$ is $\mathcal{D}$-cancellative if $e_1 \oplus f_1 \mathcal{D} e_2 \oplus f_2$ and $e_1 \mathcal{D} e_2$ implies that $f_1 \mathcal{D} f_2$. The proof of part (1) below can be found as [19 Proposition 2.7]. The proof of part (2) is essentially [19 Proposition 2.13].

**Proposition 3.25.** Let $S$ be a Boolean inverse monoid.

1. Then $S$ is factorizable if and only if $S$ has $\mathcal{D}$-complementation.
2. $S$ is $\mathcal{D}$-cancellative if and only if $S$ has $\mathcal{D}$-complementation.

The proof of the following is [19 Lemma 2.6] and the above proposition.
Lemma 3.26. Let \( S \) be a Boolean inverse monoid. If \( S \) is \( \mathcal{D} \)-cancellative then \( S \) is Dedekind finite.

Proposition 3.27. Let \( S \) be a Boolean inverse monoid. Then \( S \) is factorizable if and only if \( S/\mu \) is factorizable.

Proof. Only one direction needs proving. Let \( e \mathcal{D} f \) in \( S \) where \( e \) and \( f \) are idempotents. Then \( \mu(e) \mathcal{D} \mu(f) \). The inverse monoid \( S/\mu \) is Boolean by [33, Proposition 3.4.5]. By assumption, \( S/\mu \) is factorizable. Thus \( \mu(e) \mathcal{D} \mu(f) \) by Proposition 3.25. But \( \mu \) induces an isomorphism of the Boolean algebras of idempotents. Thus \( \mu(e) \mathcal{D} \mu(f) \). It follows that \( e \mathcal{D} f \) in \( S \) by Lemma 2.2. Thus \( S \) is factorizable. \( \square \)

Proposition 3.28. All finite Boolean inverse monoids are factorizable.

Proof. Let \( S \) be a finite Boolean inverse monoid. Then \( S/\mu \) is a finite fundamental Boolean inverse monoid. This is factorizable since it is a finite direct product of finite symmetric inverse monoids each of which is factorizable. The result now follows by Proposition 3.27. \( \square \)

3.8. Basic Boolean inverse monoids. A non-zero element \( a \) of an inverse semigroup is said to be an infinitesimal if \( a^2 = 0 \) [15]. We shall generalize infinitesimals in Section 5.4. Clearly, there are no idempotent infinitesimals and an infinitesimal cannot be above a non-zero idempotent. The following is proved as [15, Lemma 2.18(2)].

Lemma 3.29. In an inverse semigroup, the following are equivalent:

1. \( a \) is an infinitesimal.
2. \( \mathcal{d}(a) \perp \mathcal{r}(a) \).
3. \( a \perp a^{-1} \).

The proof of the following is straightforward.

Lemma 3.30. Let \( S \) be an inverse semigroup. Any element beneath an infinitesimal is either zero or an infinitesimal.

The proof of the following is also straightforward.

Lemma 3.31. Let \( S \) be an inverse semigroup and let \( x \) be any element. If \( a \) is an infinitesimal then \( xax^{-1} \) is either zero or an infinitesimal.

Recall that an involution in a group is a non-identity element \( g \) such that \( g^2 = 1 \). Let \( S \) be a Boolean inverse monoid. If \( a \) is any infinitesimal then \( g = a \lor a^{-1} \lor e(a) \) is an involution by [15, Lemma 2.18]. The following definition and an explanation of where it came from was first given in [15].

Definition. A Boolean inverse semigroup is said to be basic if each non-zero element can be written as a join of an idempotent (possibly 0) and a finite number of infinitesimals.

We shall only use the following in Section 6, but it does provide an explanation as to why being basic is a natural property. See [15, Proposition 4.31] [15].

Proposition 3.32. Let \( S \) be a Boolean inverse meet-monoid. Then \( S \) is basic if and only if the Stone groupoid \( G(S) \) is principal.

For the following, see [15, Remark 4.29].

---

3The result stated in that paper misses out the extra needed condition that \( S \) is a meet-monoid.
Lemma 3.33. Finite symmetric inverse monoids are basic.

The proof of the following is straightforward.

Lemma 3.34. A finite direct product of basic Boolean inverse semigroups is a basic Boolean inverse semigroup.

Let $S$ be a Boolean inverse monoid with group of units $U(S)$. Then $S$ is said to be piecewise factorizable if for each $a \in S$ we can write $a = \bigvee_{i=1}^{n} g_{i} e_{i}$ where the $g_{i}$ are units and the $e_{i}$ are idempotents. The following is proved as [15, Lemma 4.30].

Proposition 3.35. Let $S$ be a basic Boolean inverse monoid. Then $S$ is:

1. A meet-monoid.
2. Piecewise factorizable.
3. Fundamental.

Proposition 3.36. Let $S$ be a finite Boolean inverse monoid. Then $S$ is fundamental if and only if $S$ is basic.

Proof. By Proposition 3.35, every basic Boolean inverse semigroup is fundamental. To prove the converse, we use Lemma 3.33, Lemma 3.34 and Theorem 3.22. □

In general, we have to be careful with meets, but the following provides a situation where meets are preserved.

Lemma 3.37. Let $S$ be a Boolean inverse monoid in which $T$ is a subalgebra of $S$. Suppose that $T$ is basic. Then any meet computed in $T$ is still a meet in $S$.

Proof. By assumption, for each $a \in T$ we may write $a = a_{1} \lor \ldots \lor a_{n} \lor e$ where the $a_{i}$ are infinitesimals and $e$ is an idempotent. Because we are assuming that $T$ is a subalgebra of $S$, we also have that $a = a_{1} \lor \ldots \lor a_{n} \lor e$ in $S$. Let $f \leq a$ be any idempotent in $S$. Then by Lemma 3.35, $f = (f \land a_{1}) \lor \ldots \lor (f \land a_{n}) \lor (f \land e)$. Each of the elements $f \land a_{1}$ is an idempotent and an infinitesimal. It follows that $f \land a_{1} = 0$. We deduce that $f \leq x$. Thus inside $T$ we have that $\varphi(a) = e$. Now, let $a, b \in T$. Then $x = a \land b$ exists in $T$; we show that this is also a meet in $S$. Put $\varphi(ab^{-1}) = e$. Then by Proposition 2.26 we have that $x = ab$. We prove that $x$ is the meet of $a$ and $b$ in $S$. Clearly, $x \leq a, b$ in $S$. Let $c \leq a, b$. Then $cc^{-1} \leq ab^{-1}$. But $cc^{-1}$ is an idempotent and so $cc^{-1} \leq c$. Now, $c \leq b$ and so $c \leq eb = x$. It follows that $x$ is the meet of $a$ and $b$ in $S$. □

The following is an immediate corollary of Lemma 3.37 and Proposition 3.36.

Corollary 3.38. Let $S$ be a Boolean inverse monoid in which $T$ is a subalgebra of $S$. Suppose that $T$ is a finite fundamental Boolean inverse monoid. Then any meet computed in $T$ is still a meet in $S$.

4. Characterization theorems

In this section, we characterize the Boolean inverse monoids of finite type and those which are AF; recall that AF monoids were introduced in [13]. The characterization of UHF monoids is more complex and left until Section 5. The following lemma deals with what we might call ‘normalization’.

Lemma 4.1. Let $S$ be a Boolean inverse monoid.

1. Let $G \leq U(S)$ be a finite subgroup and let $E \subseteq E(S)$ be a finite Boolean subalgebra. Then there is a finite Boolean subalgebra $E' \subseteq E(S)$ such that $E \subseteq E'$ and $E'$ is closed under the natural action of $G$.
2. Let $G \leq U(S)$ be a finite subgroup and let $E \subseteq E(S)$ be a finite Boolean subalgebra. Then there is a finite subalgebra $T$ of $S$ such that $GE \subseteq T$. 

Proof. (1) Let $E'$ be the Boolean subalgebra of $E(S)$ generated by the finite set $F = \{eg^{-1}: g \in G, e \in E\}$. Observe that the set $F$ is closed under complements since $geg^{-1} = geg^{-1}$. Each element of $E'$ can be written as a join of terms each of which is a meet of elements of $F$. It is immediate that $E'$ is closed under the natural action by $G$.

(2) Construct the Boolean subalgebra $E'$ as in part (1). By Lemma 2.3, $GE'$ is a factorizable submonoid of $S$ with semilattice of idempotents $E'$ and group of units $G$. By Lemma 3.15 if we put $T = (GE')^\vee$ then $T$ is a finite subalgebra and Boolean inverse submonoid of $S$. \hfill \Box

We now come to our first main theorem.

Remark 4.2. We do not know if Boolean inverse monoids of finite type are automatically meet-monoids, though Example 3.20 suggests not.

Theorem 4.3 (Boolean inverse monoids of finite type). Let $S$ be a countably infinite Boolean inverse monoid. Then $S$ is of finite type if and only if $S$ is factorizable and its group of units is locally finite.

Proof. We prove the easy direction first. Suppose that $S$ is of finite type. Then $S = \bigcup_{i=1}^\infty S_i$ where $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots$ and where each $S_i$ is a finite subalgebra of $S$. Let $a \in S$. Then $a \in S_i$ for some $i$. But $S_i$ is factorizable by Proposition 3.28. Thus $a \leq g$ where $g$ is a unit of $S_i$. However, $g$ is a unit of $S$ since $S_i$ is a subalgebra of $S$. Thus $S$ is also factorizable. Now, let $X$ be any finite subset of the group of units of $S$. Then $X \subseteq S_i$ for some $i$ where $S_i$ is a subalgebra of $S$. It follows that $X$ are units of $S_i$, which is finite. Hence, the subgroup generated by $X$ is finite. We have therefore proved that the group of units of $S$ is locally finite.

We now prove the converse. Let $S$ be a countable Boolean inverse monoid which is factorizable and whose group of units is locally finite. Since $S$ is countable both the group $U(S)$ and the Boolean algebra $E(S)$ are countable. Let $U(S) = \{g_1 = 1, g_2, \ldots\}$ and $E(S) = \{e_1 = 1, e_2, \ldots\}$. Define $G_i = \langle g_1, \ldots, g_i \rangle$ and $E_i = \langle e_1, \ldots, e_i \rangle$. Since $S$ is locally finite, the groups $G_i$ are all finite. The Boolean algebras $E_i$ are finite since they are finitely generated using Lemma 2.9. By construction $G_1 \subseteq G_2 \subseteq \ldots$ and $E_1 \subseteq E_2 \subseteq \ldots$. By part (1) of Lemma 3.11 we may enlarge each Boolean algebra $E_i$ to a Boolean algebra $E'_i$ in such a way that $E'_i$ is closed under the natural action by $G_i$ and such that $E'_1 \subseteq E'_2 \subseteq \ldots$: to do this, first enlarge $E_i$ to $E'_i$ and then enlarge $\langle E'_1 \cup E'_2 \rangle$ to $E'_2$ and then enlarge $\langle E'_2 \cup E'_3 \rangle$ to get $E'_3$ and so on. We relabel $E'_i$ as $E_i$. We may therefore assume that each $E_i$ is closed under the natural action of $G_i$. We now apply part (2) of Lemma 4.1 to get an increasing sequence $F_i = (G_i E_i)^\vee$ of finite subalgebras where $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$. Now, let $a \in S$ be arbitrary. Then $a = g e$ for some unit $g$ and idempotent $e$ by factorizability. It follows that $a \in F_i$ for some $i$. We have therefore proved that $S$ is of finite type. \hfill \Box

We now turn to the characterization of AF inverse monoids, which requires the following result first. We have followed the referee’s suggestion and only assumed piecewise factorizability rather than that of being basic.

Lemma 4.4. Let $S$ be a countably infinite piecewise factorizable Boolean inverse monoid whose group of units is locally finite. Then $S$ is, in fact, factorizable.

Proof. Let $a \in S$. Using the fact that $S$ is piecewise factorizable, we have that $a = \bigvee_{i=1}^n g_i e_i$, where the $g_i$ are units and the $e_i$ are idempotents. Put $G = \langle g_1, \ldots, g_n \rangle$, which is finite since the group of units of $S$ is locally finite. Let $E = \{e_1, \ldots, e_n\}$. By part (1) of Lemma 3.11 there is a finite Boolean subalgebra $B$ of $E(S)$ containing $E$ invariant under the natural action by $G$. Put $T = (GB)^\vee$. By part (2) of
Lemma 4.1, it follows that $T$ is a finite subalgebra of $S$. However, $T$ is factorizable by Proposition 3.28. By construction, $a \in T$. The result now follows. 

Lemma 4.5. Let $S$ be a Boolean inverse monoid. Suppose that $S$ is basic and the group of units is locally finite. Let $T$ be a finite subalgebra of $S$. Then there is a finite fundamental subalgebra $T'$ of $S$ such that $T \subseteq T'$.

Proof. Since $T$ is a finite Boolean inverse monoid, we know that it is factorizable by Proposition 3.28. By Lemma 2.4 we can therefore write $T = U(T)E(T)$ where $U(T)$ is a finite subgroup of $U(S)$ and $E(T)$ is a finite Boolean subalgebra of the unital Boolean algebra $E(S)$. In addition, $E(T)$ is closed under the natural action by $U(T)$. By assumption, $S$ is basic. Thus for each $g \in U(T)$, we may write $g = a_1 \lor \ldots \lor a_m \lor e$ where the $a_1, \ldots, a_m$ are infinitesimals and $e$ is an idempotent. Since $d(g) = 1$, we have by Lemma 5.2 that $1 = d(a_1) \lor \ldots \lor d(a_m) \lor e$. Put $\iota(g) = \{d(a_1), \ldots, d(a_m), e\}$ — we shall use these later — and observe that $\phi(g) = e$. Put $E(T)'$ equal to the set $E(T)$ together with the sets $\iota(g)$ for each of the $g \in U(S)$. Then adjoining all the conjugates of these idempotents by elements of $U(T)$. Now, define $B$ to be the finite Boolean algebra generated by these idempotents. Thus

$$
E(T) \cup \left( \bigcup_{g \in U(T)} \iota(g) \right) \subseteq B.
$$

By part (1) of Lemma 2.4 $T_1 = U(T)B$ is a factorizable inverse submonoid of $S$ with $B$ as its Boolean algebra of idempotents and $U(T)$ as its group of units. It has all binary meets by Lemma 2.8, since for each $g \in U(T)$, we have that $\iota(g) \subseteq B$ and so $\phi(g) \in B$. By Lemma 4.1, we have that $T' = T_1'$ is a finite subalgebra of $S$ that contains $T$. It remains to show that $T'$ is fundamental. We shall do this by showing that $T'$ is basic and then use Proposition 3.36. Each element of $a \in T'$ is of the form $a = \bigvee_{i=1}^m g_i e_i$, where $g_i \in U(T)$ and $e_i \in B$. If we can show that each $g_i e_i$ is a join of infinitesimals and an idempotent then the element $a$ will also be a join of infinitesimals and an idempotent and we shall be done. Let $g \in U(T)$. By assumption $\iota(g) \subseteq B$. It follows that in $T'$, we can write $g = a_1 \lor \ldots \lor a_m \lor e$ where $a_1, \ldots, a_m \in U(S)B$ are infinitesimals and $e \in B$. If $f \in B$ then $gf = a_1 f \lor \ldots \lor a_m f \lor ef$. If we weed out the products which are zero, we get a representation of $gf$ as a sum of infinitesimals and an idempotent, using Lemma 3.30. We have therefore proved that $T'$ is basic and so it is fundamental by Proposition 3.35.

We now come to our second main theorem. Observe that AF monoids are automatically meet-monoids by Lemma 3.37.

Remark 4.6. It would be better if we could replace ‘basic’ by ‘fundamental’ in the statement of the result below, but we have not been able to do so. We think it unlikely. See Proposition 5.32 for an explanation of why being basic is a natural property. The Stone groupoids of AF inverse monoids are characterized in Section 6.5. There, it is proved that they correspond to AF groupoids; these are principal which is a much stronger condition than the corresponding monoid being fundamental.

Theorem 4.7 (AF monoids). Let $S$ be a countably infinite Boolean inverse monoid. Then $S$ is AF if and only if $S$ is basic and its group of units is locally finite.

Proof. We prove the easy direction first. Let $S$ be an AF inverse monoid. Then $S = \bigcup_{i=1}^\infty S_i$ where each $S_i$ is a matricial subalgebra of $S$; that is, each $S_i$ is a finite fundamental Boolean inverse monoid. By Proposition 3.30 each $S_i$ is basic and so, since $S_i$ is a subalgebra of $S$, it follows that $S$ is basic. The fact that the
group of units of \( S \) is locally finite follows by the same argument as that used in the proof of Theorem 4.3. We now prove the converse. By assumption, our semigroup is basic and so it is piecewise-factorizable by Proposition 3.35. By Lemma 4.4, we know that \( S \) is factorizable. Thus by Theorem 4.3, we can write \( S = \bigcup_{n=0}^{\infty} S_n \), where the \( S_n \) are finite Boolean inverse monoids each of which is a subalgebra of \( S \). By Lemma 5.1, every finite subalgebra of \( S \) is contained in a finite fundamental subalgebra of \( S \). Thus, we may find a finite fundamental subalgebra \( T_0 \) of \( S \) such that \( S_0 \subseteq T_0 \). By Theorem 3.22, \( T_0 \) is matricial. Because \( T_0 \) is finite, we may find an \( S_i \) such that \( T_0 \subseteq S_i \). By a similar argument to that above, we may find a finite matricial subalgebra \( T_1 \) of \( S \) such that \( S_1 \subseteq T_1 \). And so on. We have therefore shown that \( S = \bigcup_{n=0}^{\infty} T_i \), where each \( T_i \) is a finite matricial subalgebra of \( S \).  

5. Characterization of UHF inverse monoids

This turns out to be the most complex of the three characterizations presented in this paper. It is convenient to have a term for countably infinite Boolean inverse monoids which are basic, 0-simplifying, have a locally finite group of units and in which \( S/\mathcal{J} \) be linearly ordered. We shall refer to them as Mundici monoids. The following is immediate from Theorem 4.7.

**Lemma 5.1.** Every Mundici monoid is AF.

The following definition is taken from [11].

**Definition.** Let \( S \) be a Boolean inverse monoid. A function \( \nu: E(S) \to \mathbb{R}^{\geq 0} \) is said to be an invariant mean if it satisfies two conditions:

1. \( \nu(s^{-1}s) = \nu(ss^{-1}) \) for all \( s \in S \).
2. If \( e \) and \( f \) are orthogonal idempotents then \( \nu(e \lor f) = \nu(e) + \nu(f) \).

Observe that \( \nu(0) = 0 \) and if \( e \leq f \) then \( \nu(e) \leq \nu(f) \) by [11, Lemma 2.1]. We say that such an invariant mean is normalized if \( \nu(1) = 1 \). It follows from [11, Lemma 2.1], that if \( \nu \) is a normalized invariant mean then, in fact, \( \nu: E(S) \to [0, 1] \).

Our goal now is to prove the following theorem:

**Theorem 5.2 (UHF monoids).** Let \( S \) be a Boolean inverse monoid. Then \( S \) is a UHF monoid if and only if it is a Mundici monoid equipped with an invariant mean that assumes only rational values.

5.1. Proof of the easy direction. We shall first discuss the structure of the finite symmetric inverse monoids.

**Example 5.3.** Consider the finite symmetric inverse monoid \( I_n \) which we shall consider as acting on the set \( X = \{1, \ldots, n\} \). The idempotents in \( I_n \) are simply the identity functions defined on the subsets of \( X \). Define \( \nu: E(I_n) \to \mathbb{R}^{\geq 0} \) by \( \nu(1_A) = |A| \). Then \( \nu \) is an invariant mean from the usual properties of finite cardinalities. We get a normalized invariant mean if we define \( \nu(A) = \frac{|A|}{n} \) where we have used [11, Lemma 2.2].

The proof of the following can be found at [11, Lemma 2.17(1)], but we give a direct proof since we shall need the form taken by the normalized invariant mean in this case.

**Lemma 5.4.** Finite symmetric inverse monoids \( I_n \) have exactly one normalized invariant mean.

**Proof.** Each non-zero idempotent is the join of a finite number of orthogonal atomic idempotents and, since \( I_n \) is 0-simplifying, any two atomic idempotents are \( \mathcal{J} \)-related. Let the atomic idempotents be \( e_1, \ldots, e_n \). Then \( 1 = e_1 \oplus \ldots \oplus e_n \). It
follows that if \( \nu \) is a normalized invariant mean then \( \nu(e) = \frac{1}{n} \) where \( e \) is any atomic idempotent. Thus if a normalized invariant mean exists it is unique. We now prove existence. If \( A_n \) is any idempotent in \( I_n \), where \( A \) is some subset of \( \{1,2,3,\ldots,n\} \), define \( \nu(1_A) = \frac{|A|}{n} \). Then \( \nu \) is a normalized invariant mean. \( \square \)

We denote the unique normalized invariant mean on the finite simple Boolean inverse monoid \( S \) by \( \nu_S \).

**Lemma 5.5.** Let \( S \) be a finite simple Boolean inverse monoid. Then for \( e, f \in E(S) \) we have that \( e \unlhd f \) if and only if \( \nu_S(e) = \nu_S(f) \).

**Proof.** Only one direction needs proving. Suppose that \( \nu_S(e) = \nu_S(f) \). Observe that \( e = 1_A \) and \( f = 1_B \) and, by assumption, the cardinalities of \( A \) and \( B \) are the same. It follows that \( e \unlhd f \). \( \square \)

The proof of Lemma 5.6 below is immediate by the proof of Lemma 5.5.

**Lemma 5.6.** For a finite symmetric inverse monoid \( I_n \) the poset \( I_n/\mathcal{I} \) is linearly ordered.

**Lemma 5.7.** Let \( S \) and \( T \) be finite simple Boolean inverse monoids. Let \( \theta : S \to T \) be a morphism. Then for all \( e \in E(S) \) we have that \( \nu_T(\theta(e)) = \nu_T(\theta(\nu_S(e))) \).

**Proof.** Observe first that \( \nu_T \theta : E(S) \to [0,1] \). It is easy to check that this function satisfies the conditions to be a normalized invariant mean. By the uniqueness guaranteed by Lemma 5.4 it follows that \( \nu_T \theta \) restricted to the idempotents of \( S \) is just \( \nu_S \). \( \square \)

We now turn to the properties of UHF monoids. This is the easy direction of the proof of Theorem 5.2.

**Lemma 5.8.** Every UHF monoid is a Mundici monoid equipped with an invariant mean that assumes only rational values.

**Proof.** Every UHF monoid is an AF monoid and so it is basic and its group of units is locally finite by Theorem 4.7.

We prove that UHF monoids are 0-simplifying. Let \( I \) be a non-zero additive ideal of \( S \). We prove that \( I = S \). Let \( a \in I \) be any non-zero element of \( I \). Then \( a \in S_i \) for some \( i \). It follows that \( S_i \cap I \neq \{0\} \) and \( S_i \cap I \) is an additive ideal in \( S_i \). But \( S_i \) is 0-simplifying. It follows that \( S_i \subseteq I \). We can now see that \( S_j \cap I \neq \{0\} \) for all \( j \geq i \). In every case \( S_j \subseteq I \). We deduce that \( I = S \).

We prove that for a UHF monoid \( S \), we have that the poset \( S/\mathcal{I} \) is linearly ordered. Let \( a \) and \( b \) be non-zero elements of \( S \). Then \( a, b \in S_i \) for some \( i \). But \( S_i \) is a finite symmetric inverse monoid. On \( S_i \) the \( \mathcal{I} \) relation induces a linear order by Lemma 5.6. Without loss of generality, we may assume that \( a = xyb \) where \( x, y \in S \). It follows that \( SaS \subseteq SbS \) in \( S \), as required.

We shall define a function \( \nu : E(S) \to [0,1] \cap \mathbb{Q} \). Let \( e \) be any idempotent. Then \( e \in S_i \) for some \( i \). Denote the unique normalized invariant mean on \( S_i \) by \( \nu_i \). Define \( \nu(e) = \nu_i(e) \). Observe that this is a rational number. We now prove that our definition is independent of \( i \). Let \( e \in S_j \). There are two possibilities: either \( S_j \subseteq S_i \) or \( S_i \subseteq S_j \). Observe first that in either case the embedding is a morphism which we denote in both cases by \( \theta \). We deal with the first case first. We have that \( \theta : S_j \to S_i \). Now use Lemma 5.4 to get that \( \nu_j(e) = \nu_i(\theta(e)) \). The second case is the same as the first with the roles of \( i \) and \( j \) reversed. It follows that \( \nu \) is well-defined. It is now routine to check that \( \nu \) is a normalized invariant mean. \( \square \)

The rest of this section is dedicated to proving the converse of the above lemma.
5.2. MV-algebras. Our characterization of UHF monoids will use the theory of MV-algebras. We refer the reader to [2] for background information.

An MV-algebra \((A, \oplus, \neg, 0)\) is a commutative monoid \((A, \oplus, 0)\) together with a unary operation \(\neg\) that satisfies the following three axioms; in what follows, we define \(1 = \neg 0\):

1. \(\neg \neg x = x\).
2. \(x \oplus 1 = 1\).
3. \(\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x\).

Every Boolean algebra \(B\) becomes an MV-algebra when regarded as a structure \((B, \lor, x \mapsto \bar{x}, 0)\); both sides of (3) above reduce to \(x \lor y\). Of course, in a Boolean algebra \(x \lor x = x\). More generally, an idempotent in an MV-algebra is an element \(x\) such that \(x \oplus x = x\). By [2, Theorem 1.5.3], the idempotent elements of an MV-algebra form a Boolean algebra. It follows that the MV-algebras in which every element is an idempotent are precisely the Boolean algebras. If \(x\) is an element of an MV-algebra and \(n\) is a natural number we define \(nx\) as the orthogonal join of \(n\) elements of \(x\).

By [2, Lemma 1.1.2], we may define a partial order on an MV-algebra by \(x \leq y\) if \(y = z \oplus x\) for some \(z\). The unit interval \([0, 1]\) becomes an MV-algebra with respect to the following definitions:

\[x \oplus y = \min(x + y, 1)\] and \(\neg x = 1 - x\).

The partial order is the usual partial order. An ideal \(I\) of an MV-algebra \(A\) is a subset that contains 0, is an order-ideal and is closed under \(\oplus\). For \(n \geq 2\), define \(L_n\) to be the MV-algebra with cardinality \(n\) that contains the elements \(\frac{1}{n-1}\) where \(0 \leq r \leq n - 1\) and which is an MV-subalgebra of \([0, 1]\). These are precisely the finite simple MV-algebras.

We shall use the following property of finite simple MV-algebras in Subsection 5.5. First of all, the MV-order on \(L_n\) is the usual order on this subset of \([0, 1]\). The finite MV-algebra \(L_n\) has a smallest non-zero element \(\frac{1}{n-1}\) and every other element of \(L_n\) is a natural number multiple of \(\frac{1}{n-1}\). Observe that the ordering on the elements of \(L_n\) is determined by the numerators of the elements of \(L_n\).

More generally, we have the following by [2, Theorem 3.5.1]:

**Proposition 5.9.** The simple MV-algebras are precisely those which can be embedded in the MV-algebra \([0, 1]\).

By a rational MV-algebra we mean an MV-subalgebra of \(\mathbb{Q} \cap [0, 1]\). The following result is key to our work. It is a simple consequence of [3, Theorem 5.1].

**Proposition 5.10.** The rational MV-algebras are precisely the simple locally finite MV-algebras.

The finite simple MV-algebras are precisely those of the form \(L_n\) where \(n \geq 2\). by [2, Corollary 3.5.4].

5.3. Foulis monoids. We now need to connect Mundici monoids with MV-algebras. This is done using the concept of a Foulis monoid.

The following construction was first carried out in [19]. Let \(S\) be an arbitrary Boolean inverse monoid. Denote by \([e]\) denotes the set of all idempotents \(e_1\) such that...
Lemma 5.12. Define a partially defined addition on this set by

\[ [e] + [f] = [e'] \lor f' \]

if \( e \not\in e' \) and \( f \not\in f' \) and \( e' \) and \( f' \) are orthogonal. Put \( 0 = [0] \) and \( 1 = [1] \). Put \( \text{Int}(S) = E(S)/\mathcal{D} \) with the above partial addition. This structure is called the partial type monoid; Wehrung \[33\] Definition 4.1.3 calls this the type interval.

We shall now make some assumptions about \( S \) so that the partial type monoid has extra structure. We say that a Boolean inverse monoid \( S \) satisfies the lattice condition if \( S/\mathcal{J} \) is a lattice under subset-inclusion. A Foulis monoid is defined to be a factorizable Boolean inverse monoid satisfying the lattice condition.

Remark 5.11. By Proposition 5.25 and Lemma 5.20 Foulis monoids are Dedekind finite. Thus, by Lemma 2.3, we have that \( \mathcal{J} = \mathcal{F} \) in Foulis monoids. By Lemma 2.3 Foulis monoids are directly finite.

If \( S \) is a factorizable Boolean inverse monoid, then we may define \( \neg[e] = [\bar{e}] \) unambiguously by Proposition 3.25. In a Foulis monoid, the set \( [e] \) consists of all idempotents \( e' \) where \( SeS = Se' \).

Definition. Let \( S \) be a Foulis monoid. We define on the set \( \text{Int}(S) \) the structure of an MV-algebra which we denote by \( L(S) \). On the set \( \text{Int}(S) = E(S)/\mathcal{D} \) define

\[ [e] \land [f] = [i] \]

if \( SeS \cap SfS = SiS \). This makes sense by Remark 5.11. We now use the construction of [10]. If \( S \) is a Foulis monoid then \( (L(S), \oplus, \neg, 0, 1) \) is an MV-algebra where we define \( \neg[e] = \bar{e} \) and

\[ [e] \lor [f] = [e] + (\bar{e} \land [f]). \]

The following lemma simply reassures us that, in the case of Foulis monoids, we really are extending the partially defined addition.

Lemma 5.12. Let \( S \) be a Foulis monoid. Suppose that \( e \perp f \). Then \( [e] \lor [f] = [e \lor f] \).

Proof. By definition \( [e] \lor [f] = [e] + ([\bar{e}] \land [f]) \). Since \( e \perp f \), it follows that \( f \leq \bar{e} \).

We say that the MV-algebra \( A \) is coordinatized by the Foulis monoid \( S \) if \( A \cong L(S) \). It is known that every MV-algebra is coordinatized by some Boolean inverse monoid; the case of countable MV-algebras was proved in [19] and the general case was established in [33] Theorem 5.2.10. The key point is the following which is immediate from the definition of a Mundici monoid and the fact that \( S/\mathcal{F} \) being linearly ordered implies that \( S/\mathcal{J} \) is a lattice, and the fact that Mundici monoids are factorizable by Proposition 3.35 and Lemma 4.3.

Lemma 5.13. Every Mundici monoid is a Foulis monoid.

Example 5.14. With the finite symmetric inverse monoid \( I_\infty \), we may associate the finite MV-algebra \( L_{n+1} \) (since \( I_\infty \) contains \( n+1 \) \( \mathcal{P} \)-classes of idempotents). Thus the finite symmetric inverse monoids are classified by their associated MV-algebras.

\[ \text{The reader should note that this is an ordering on the principal ideals of the inverse semigroup and is therefore not the same as the natural partial order; for example, in the symmetric inverse monoid } \mathcal{J}_2 \text{ the idempotents } e = 1_{(1)} \text{ and } f = 1_{(2)} \text{ are such that } e \land f = 0 \text{ but they generate the same principal ideal since } e \mathcal{P} f. \]

\[ \text{This is different from the way the term was used in } [10] \text{ where the ‘lattice condition’ was omitted.} \]
If $S$ is a finite matricial Boolean inverse monoid, then its associated MV-algebra is a finite MV-algebra and every finite MV-algebra arises in this way. The finite matricial Boolean inverse monoids are classified by their associated MV-algebras. See [19] Theorem 2.14.

It can be shown that in an MV-algebra, $x \leq y$ if and only if $\neg x \oplus y = 1$ [2]. The following result is so important that we give a direct proof.

**Lemma 5.15.** Let $S$ be a Foulis monoid. Then $[e] \leq [f]$ in the MV-algebra $L(S)$ if and only if $e \leq f$ in $S$.

**Proof.** Suppose that $[e] \leq [f]$ in the MV-algebra. Then $[e] \oplus [f] = [1]$. By definition, $[e] \oplus [f] = [e] \hat{\land} ([e] \hat{\lor} [f])$. Suppose that $S \mathcal{S} \cap fS = S \mathcal{S}$. In particular, $S \mathcal{S} \subseteq S \mathcal{S}$ and so $i \leq f$. Then $[e] + ([e] \hat{\land} [f]) = [e] + [i] = [1]$. By definition of the partial addition, there are idempotents $e'$ such that $e \mathcal{D} e'$ and $i'$ such that $i \mathcal{R} i'$ where $e' \perp i'$. This implies that $[e] + [i] = [e' \lor i'] = [1]$. We therefore have that $e' \lor i' = 1$.

We now use the fact that Foulis monoids are directly finite to deduce that $e' \lor i' = 1$. But we also have that $e' \perp i'$ and we are working inside a unital Boolean algebra. It follows that $i' = e'$. But $e \mathcal{D} e'$. Now we use the fact that we are working in a factorizor Boolean inverse monoid and so use Proposition 3.25 to deduce that $e \mathcal{D} \mathcal{D} = i \mathcal{D} i$. Thus $e \mathcal{D} i$. But $i \leq f$ (see, above) and so $i \leq f$, as required. To prove the converse, suppose that $e \leq f$. Then $e \mathcal{D} e_1 \leq f$ for some idempotent $e_1$. Then $f = e_1 \oplus f_{\mathcal{D} e}$ using the theory of unital Boolean algebras. It follows that $[f] = [e] \oplus [f_{\mathcal{D} e}]$ using Lemma 5.12. Thus $[e] \oplus [f] = [1] \oplus [f_{\mathcal{D} e}] = [1]$, from which we deduce that $[e] \leq [f]$. \hfill $\Box$

The following result ties together the structure of a Foulis monoid $S$ with that of its MV-algebra $L(S)$.

**Proposition 5.16.** Let $S$ be a Foulis monoid. Then there is an order-isomorphism between the conjugate closed additive ideals of $\mathcal{E}(S)$ and the MV-algebra ideals of $L(S)$.

**Proof.** Let $F$ be a conjugate closed additive ideal of $\mathcal{E}(S)$. Put $[F] = \{[e] : e \in F\}$. Then $[F]$ is an ideal in $L(S)$. If $F_1 \subseteq F_2$ then $[F_1] \subseteq [F_2]$ with equality if and only if $F_1 = F_2$. Thus we have a map given by $F \mapsto [F]$.

Now let $I \subseteq L(S)$ be an ideal. Define $\mathcal{E}(I) = \{e \in \mathcal{E}(S) : [e] \in I\}$. Then $\mathcal{E}(I)$ is an additive ideal of $\mathcal{E}(S)$. There are three conditions to check each of which is easy: it is an order-ideal; it is closed under orthogonal binary joins and so it is closed under arbitrary binary joins; it is closed under the $\mathcal{D}$-relation. Suppose that $I_1 \subseteq I_2$. Then $\mathcal{E}(I_1) \subseteq \mathcal{E}(I_2)$ with equality if and only if $I_1 = I_2$. Thus we have a map given by $I \mapsto \mathcal{E}(I)$.

We now show that the above maps are mutually inverse. Let $F$ be an additive ideal in $\mathcal{E}(S)$. Then $[F]$ is an ideal in $L(S)$. Thus $\mathcal{E}([F])$ is an additive ideal in $\mathcal{E}(S)$. Observe that $F \subseteq \mathcal{E}([F])$. Let $f \in \mathcal{E}([F])$. Then $[f] \in [F]$. Thus $[f] = [e]$ where $e \in F$. But $E$ is closed under the $\mathcal{D}$-relation and so $f \in F$. We have proved that $F = \mathcal{E}([F])$.

Let $I$ be an ideal in $L(S)$. Then $\mathcal{E}(I)$ is an additive ideal in $\mathcal{E}(S)$. Thus $\mathcal{E}([I])$ is an ideal in $L(S)$. Let $[e] \in I$. Then $e \in \mathcal{E}(I)$. Thus $[e] \in \mathcal{E}([I])$. It follows that $I \subseteq \mathcal{E}([I])$. Let $[e] \in \mathcal{E}([I])$. Then $e \in \mathcal{E}(I)$. Thus $[e] \in I$. We have therefore proved that $I = \mathcal{E}([I])$. \hfill $\Box$

The following is immediate by Proposition 5.16 and part (5) of Lemma 5.9.

**Corollary 5.17.** Let $S$ be a Foulis monoid. Then there is an order-isomorphism between the additive ideals of $S$ and the MV-algebra ideals of $L(S)$. 
Thus by Lemma 5.15, it follows that the main goals of this paper but, we believe, interesting in its own right.

5.4. Characterizations of Classes of Countable Boolean Inverse Monoids

Corollary 5.18. Let $S$ be a Foulis monoid. Then $S$ is 0-simplifying if and only if $L(S)$ is simple.

It follows that if $S$ is a Mundici monoid then $L(S)$ is a simple MV-algebra. Let $S$ be a Mundici monoid equipped with a normalized invariant mean $\mu$. Define $\theta : L(S) \to [0, 1]$ by $\theta([e]) = \mu(e)$. This is a well-defined function since $[e] = [e_1]$ means that $e \not\leq e_1$ and so $\mu(e) = \mu(e_1)$.

Lemma 5.19. Let $S$ be a Mundici monoid equipped with a normalized invariant mean $\mu$. With the above definition, $\theta$ is an injective homomorphism of MV-algebras.

Proof. We show that this is a morphism of MV-algebras. We deal with negations first. By [11] Lemma 2.1, we have that $\mu([-e]) = 1 - \mu([e]) = \mu([-e])$.

We calculate $\theta([e] \oplus [f])$ and show it is equal to $\theta([e]) \oplus \theta([f])$. We are working inside a Foulis monoid where $S/\not\leq$ is linearly ordered. There are therefore two cases. Either $S\varepsilon S \subseteq SfS$ or $SfS \subseteq S\varepsilon S$. Suppose the former. Then $[e] \wedge [f] = [e]$. It follows that $[e] \oplus [f] = [1]$. Thus $\theta([e] \oplus [f]) = 1$. On the other hand, $\theta([e]) \oplus \theta([f]) = \mu(e) + (1 - \mu(e)) = 1$. Suppose now that $SfS \subseteq S\varepsilon S$. Then $[e] \oplus [f] = [e] + [f]$. Thus, there is an idempotent $f'$ such that $f' \not\leq e$. It follows that $[e] \oplus [f] = [e \vee f']$. Thus $\theta([e] \oplus [f]) = \theta([e]) \oplus \theta([f]) = \mu(e) \vee \mu(f)$. This is also equal to $\theta([e]) \oplus \theta([f])$. This map has to be an embedding since the kernel is an ideal of $L(S)$, which is simple by Corollary 5.18.

We have the following as an immediate consequence.

Corollary 5.20. Let $S$ be a Mundici monoid equipped with a normalized invariant mean $\mu$, such that $\mu(a)$ is rational for every $a \in S$. Then $L(S)$ is a simple, locally finite MV-algebra. In particular, $\mu$ induces an embedding of $L(S)$ in the MV-algebra $[0, 1] \cap \mathbb{Q}$.

It follows that MV-algebras are locally finite if they are associated with Mundici monoids equipped with a normalized invariant mean that assumes only rational values. We shall use this property in proving the hard direction of our theorem.

5.4. Co-ordinatizing simple MV-algebras. This section is tangential to the main goals of this paper but, we believe, interesting in its own right.

Proposition 5.21. Simple MV-algebras are co-ordinatized by 0-simplifying Foulis monoids. Such 0-simplifying Foulis monoids have the property that their lattice of principal ideals is linearly ordered.

Proof. Let $A$ be a simple MV-algebra. Then by [33] Theorem 5.2.10], there is a Foulis monoid $S$ which coordinatizes $A$. By Lemma 5.18, the fact that the MV-algebra $A$ is simple implies that $S$ is itself 0-simplifying. We now prove the last claim. By Proposition 5.19, $A$ is linearly ordered since it can be embedded in $[0, 1]$. Thus by Lemma 5.15, it follows that $S/\not\leq$ is linearly ordered.

Recall that if $S$ is a Boolean inverse monoid then by [33] Proposition 3.4.5], we have that $S/\mu$ is a Boolean inverse monoid. The following is not needed but is of independent interest.

Lemma 5.22. Let $S$ be a Boolean inverse monoid. If $S$ is 0-simplifying then $S/\mu$ is 0-simplifying.

Proof. Let $J$ be a non-zero additive ideal in $S/\mu$. Then $\mu^{-1}(J)$ is a semigroup ideal in $S$. Suppose that $a, b \in S$ are such that $a \sim b$ and $a, b \in \mu^{-1}(I)$. Then...
\( \mu(a), \mu(b) \in I \). Since \( a \sim b \), it follows that \( \mu(a) \sim \mu(b) \) in \( S/\mu \). Thus \( \mu(a) \lor \mu(b) \in I \). But \( \mu(a) \lor \mu(b) = \mu(a \lor b) \). It follows that \( a \lor b \in I \). By assumption, \( \mu^{-1}(I) = S \) and so \( I = S/\mu \). We have proved that \( S/\mu \) is \( 0 \)-simplifying.

The key result we need is the following.

**Lemma 5.23.** Let \( S \) be a Foulis monoid. Then \( S/\mu \) is a Foulis monoid and \( L(S) \cong L(S/\mu) \) as MV-algebras.

**Proof.** Let \( S \) be a Foulis monoid. We prove that \( S/\mu \) is also a Foulis monoid. Observe that \( S/\mu \) is a factorizable Boolean inverse monoid by Proposition 3.25 and Proposition 3.26. It therefore only remains to check that the poset of principal ideals of \( S/\mu \) forms a lattice. We shall in fact prove that it is isomorphic with the poset of principal ideals of \( S \). Put \( T = S/\mu \). Let \( SeS \) be a principal ideal in \( S \) where \( e \) is an idempotent. We shall map it to the principal ideal \( T\mu(e)T \). This is an isometric map. We shall now prove that it is an order-isomorphism. Observe that \( T\mu(e)T = T\mu(f)T \) if and only if \( \mu(e) \not\equiv \mu(f) \) if and only if \( e \not\equiv f \), where we have used Lemma 2.2 and Lemma 3.26. It follows that the map \( SeS \rightarrow T\mu(e)T \) is a bijection. (This map is well-defined. Observe that \( SeS = SfS \) if and only if \( e \not\equiv f \). But, by assumption, \( S \) is a Foulis monoid. It follows that \( e \not\equiv f \) since \( S \) is Dedekind finite. Thus \( e \not\equiv f \) and so \( T\mu(e)T = T\mu(f)T \).) We need to prove that it is an order-isomorphism. Suppose that \( T\mu(e)T \subseteq T\mu(f)T \). Then \( \mu(e) \not\equiv \mu(e') \leq \mu(f) \) for some idempotent \( e' \). We deduce that \( e \not\equiv e' \) and \( \mu(e') = \mu(e'f) \) by Lemma 2.2 where we have used the fact that \( \mu(e') \leq \mu(f) \) and so \( \mu(e') = \mu(e'f) \). But the fact that \( \mu \) is idempotent-separating together with Lemma 2.2 yields \( e \not\equiv e' \leq f \). Thus \( SeS \subseteq SfS \). It follows that \( S/\mu \) is a Foulis monoid if \( S \) is.

It remains to prove that \( L(S) \cong L(S/\mu) \) as MV-algebras. Define the map from \( L(S) \) to \( L(S/\mu) \) by \([e] \mapsto [\mu(e)]_\mu\) where \([e] \) is the idempotent \( \not\equiv \)-class in \( S \) and \([\mu(e)]_\mu \) is the idempotent \( \not\equiv \)-class in \( S/\mu \). Since \( e \not\equiv f \) implies \( \mu(e) \not\equiv \mu(f) \) this map is well-defined. In fact, by Lemma 2.2, it is a bijection that induces an order-isomorphism. Observe that the natural map associated with \( \mu \) induces an order-isomorphism between \( E(S) \) and \( E(S/\mu) \). Thus these Boolean algebras are isomorphic. It follows that, \([e] + [f] \) exists if and only if \([\mu(e)]_\mu + [\mu(f)]_\mu \) exists; and \([e] + [f] \) maps to \([\mu(e)]_\mu + [\mu(f)]_\mu \). The fact that \( L(S) \cong L(S/\mu) \) as MV-algebras now follows. \( \square \)

We can now strengthen Proposition 5.21 using the above lemma(s). We just need to remember that a Boolean inverse monoid is simple when it is \( 0 \)-simplifying and fundamental.

**Theorem 5.24.** Simple MV-algebras are co-ordinatized by simple Foulis monoids. Such simple Foulis monoids have the property that their lattice of principal ideals is linearly ordered.

The following is an open question: let \( S \) and \( T \) be simple Foulis monoids such that \( L(S) \cong L(T) \) as MV-algebras. Is it true that \( S \cong T \)?

We would also like an explicit description of a simple Foulis monoid that coordinatizes \([0,1] \).

5.5. Embedding results. This section is the key to proving the ‘hard direction’ of our theorem.

UHF monoids are defined using embeddings of finite symmetric inverse monoids as subalgebras. Let \( S \) be a Boolean inverse monoid containing the idempotent \( e \). Suppose that \( T \) is a subalgebra of \( eSe \), where \( T \) is isomorphic to the finite symmetric inverse monoid \( I_{n+1} \). Then \( T \) contains copies of the \( n \) partial bijections

\[
\begin{pmatrix}
  1 & 1 & \ldots & 1 \\
  i & i + 1 & \ldots & i + n
\end{pmatrix}
\]

where \( i = 1, \ldots, n \). Each of these is an infinitesimal; they form a restricted product; the join of the images of the idempotents that form the domains
and ranges of these elements is \( e \). This leads us to the following definition.

**Definition.** Let \( n \geq 2 \). By an \( n \)-infinitesimal \( a = (a_n, \ldots, a_1) \) in a Boolean inverse monoid \( S \) is meant a sequence of \( n \) elements of \( S \) such that

\[
e_{n+1} = a_n e_n a_{n-1} e_{n-1} = \cdots = e_3 a_2 e_2 a_1 e_1
\]

that is where \( d(a_{i+1}) = r(a_i) \), and where the \( n + 1 \) idempotents \( e_1, \ldots, e_{n+1} \) are orthogonal.

If \( a \) is an \( n \)-infinitesimal then the entries \( a_i \) are called its **components**. The components \( a_{i+1} \) and \( a_i \) are said to be **adjacent**. The idempotents \( e_1, \ldots, e_{n+1} \) are said to be in the \( n \)-infinitesimal \( a \). The **source** of \( a \) is the idempotent \( d(a_1) \), denoted by \( a_1(a) \), and the **target** of \( a \) is the idempotent \( r(a_n) \), denoted by \( \omega(a) \). The join of all of the idempotents in \( a \) is called the **extent** of the \( n \)-infinitesimal, denoted by \( e(a) \).

Observe that each component of an \( n \)-infinitesimal really is an infinitesimal. Thus \( 1 \)-infinitesimal is nothing other than an infinitesimal. We shall use the generic term **multi-infinitesimal** to mean any \( n \)-infinitesimal. Multi-infinitesimals are another way of encoding the multisectons of \([28]\). If \( a \) is a multi-infinitesimal then \( a^{-1} \) is the multi-infinitesimal with components \( a_i^{-1} \). Observe that \( a^{-1} \) interchanges the source and target of \( a \).

Let \( a \) be an \( n \)-infinitesimal. Put

\[
a_{n+1} = (a_n \ldots a_1)^{-1}.
\]

Then we can regard the \( n \)-infinitesimal as a circular directed graph with the idempotents labelling the vertices and the edges being labelled by \( a_1, \ldots, a_{n+1} \). The **rotation of** \( a \), denoted by \( \text{rot}(a) \), is the \( n \)-infinitesimal that arises from \( a \) by starting at any idempotent in this circle or ending at any idempotent in this circle. By using \( \text{rot}(a) \), we can choose as our source any idempotent in \( a \); similarly, we can choose as our target any idempotent in \( a \) (but not both in general).

If \( a = (a_1, \ldots, a_n) \) is an \( n \)-infinitesimal then we may obtain an \((n-1)\)-infinitesimal \( a' \), called the **truncation of** \( a \), by defining \( a' = (a_1, \ldots, a_{n-1}) \). The extent of a truncation is strictly less than the original extent. But we have that \( \omega(a') = \omega(a) \). It follows that we may apply any number of truncations to a multi-infinitesimal to obtain a multi-infinitesimal until we reach an individual infinitesimal when we stop. In such a sequence, the extents are strictly decreasing. We shall also call any multi-infinitesimal \( b \) a **truncation** if it arises by means of a sequence of truncations applied to the multi-infinitesimal \( a \). Observe that the target of a truncation of a multi-infinitesimal is the same as the target of the original multi-infinitesimal.

**Lemma 5.25.** Let \( S \) be a Boolean inverse monoid and let \( e \) be a nonzero idempotent. Then \( eSe \) contains a finite Boolean inverse monoid as a subalgebra isomorphic with \( I_{n+1} \) if and only if \( S \) contains the \( n \)-infinitesimal \( a = (a_n, \ldots, a_1) \) with extent \( e \).

**Proof.** We have established one direction above, we now prove the converse. Let \( a \) be an \( n \)-infinitesimal in the Boolean inverse monoid \( S \) with extent \( e \). For \( 1 \leq s, t \leq n + 1 \) we make the following definitions: if \( s > t \) define \( a_{st} = a_s \ldots a_t \); if \( s < t \) define \( a_{st} = (a_t \ldots a_s)^{-1} \); if \( s = t \), define \( a_{ss} = e_s \). Let \( \text{grpd}(a) \) be the set of all elements \( a_{st} \) as defined above. Then \( G = \text{grpd}(a) \) is a connected principal groupoid with \( n + 1 \) identities. We adjoin the zero of \( S \) to \( G \) to get the inverse subsemigroup \( G^0 \).

Put \( T = \text{inv}(a) = (G^0)^\vee \). Then \( T \) is a Boolean inverse subalgebra of \( eSe \) isomorphic to \( I_{n+1} \). \( \square \)
If \( a \) is an \( n \)-infinitesimal, we call \( \text{inv}(a) \), as defined in the above proof, the Boolean inverse monoid \( \text{generated by } a \). Observe that \( \text{inv}(a^{-1}) = \text{inv}(a) \).

We shall need to refer to the following property:

**Definition.** We say that a Boolean inverse monoid \( S \) satisfies the **Collapsing Property (CP)** if, whenever \( U \) is a finite simple subalgebra of \( eSe \), and \( V \) is a finite simple subalgebra of \( fSf \), where \( e \perp f \), then there is a finite simple subalgebra \( W \subseteq (e \oplus f)S(e \oplus f) \) such that \( U \oplus V \subseteq W \).

The significance of the above property is explained by the following lemma.

**Lemma 5.26.** Let \( S \) be a Boolean inverse monoid satisfying the (CP) and let \( T \) be a matricial subalgebra of \( S \). Then there is a subalgebra \( T' \) of \( S \) such that \( T \subseteq T' \) and \( T' \) is a finite simple Boolean inverse monoid.

**Proof.** By assumption, \( T = T_1 \oplus \ldots \oplus T_n \) where \( T_i \subseteq e_iSe_i \) is a subalgebra and finite simple Boolean inverse monoid. By definition, \( T \) has \( n \) components. Observe that \( T_1 \subseteq e_1Se_1 \) and \( T_2 \subseteq e_2Se_2 \). Thus, by the (CP), there exists a finite simple Boolean inverse monoid and subalgebra \( T_{12} \subseteq (e_1 \oplus e_2)S(e_1 \oplus e_2) \) that contains \( T_1 \oplus T_2 \). We may therefore write \( T = T_{12} \oplus T_3 \oplus \ldots \oplus T_n \). This has \( n-1 \) components. Iterating this process, we can find a \( T' \) with the required properties. \( \Box \)

If we can show that the (CP) is possessed by the monoids of interest to us, then we shall be done.

There are four operations we can apply to multi-infinitesimals: (co-)restriction, translation, splicing and (left and right) products. We deal with these operations in turn.

**Restriction/Corestriction**

**Lemma 5.27.** (Restriction). Let \( S \) be a Boolean inverse monoid. Let \( a \) be an \( n \)-infinitesimal and let \( f \leq \alpha(a) \). Then we may construct the \( n \)-infinitesimal \( (a|f) \), called the restriction of \( a \) to \( f \), which has components \( a_{i+1}r(a_i \ldots a_1f) \) and \( \alpha(a|f) = f \). Observe that \( e(a|f) \leq e(a) \).

**Proof.** In proving this result, we use repeatedly the fact that

\[
\text{r}(xy) = \text{r}(xr(y)).
\]

We are given that \( f \leq d(a_1) \). Each component of \( (a|f) \) has the form \( a_ik \leq a_i \) where \( k \) is an idempotent. It follows that \( \text{r}(a_ik) \leq \text{r}(a_i) \) and \( d(a_ik) \leq d(a_i) \). Thus the idempotents that arise are certainly orthogonal to each other, and the idempotents are \( \sqsubseteq \)-related to each other by construction. \( \Box \)

We shall need the following lemma below.

**Lemma 5.28.** Let \( S \) be a Boolean inverse monoid. Let \( a \) be an \( n \)-infinitesimal, let \( e \) and \( f \) be orthogonal idempotents, and suppose that \( e, f \leq \alpha(a) \). Then \( e(a|e) \perp e(a|f) \).

**Proof.** In proving this result, we use repeatedly the fact that

\[
\text{r}(xy) = \text{r}(xr(y)).
\]

By definition, \( e(a|e) \) is the join of \( e \) and all idempotents of the form \( r(a_i \ldots a_1e) \) where \( 1 \leq i \leq n \). By definition, \( e(a|f) \) is the join of \( f \) and all idempotents of the form \( r(a_i \ldots a_1f) \) where \( 1 \leq i \leq n \). Observe that \( e \) is orthogonal to \( f \) by assumption. We now prove that \( e \) is orthogonal to \( r(a_i \ldots a_1f) \). Observe that \( r(a_i \ldots a_1f) \leq r(a_i) \). So, it is enough to show that \( e \) is orthogonal to \( r(a_i) \). But \( r(a_i) \perp d(a_i) \geq e \). Thus \( e \) is orthogonal to \( r(a_i) \). We now prove that \( r(a_i \ldots a_1e) \) is orthogonal to
We therefore need to look at the product $ea_1^{-1}a_i^{-1}a_ja_1f$. If $i \neq j$ then the product is zero because $r(a_i)$ and $r(a_j)$ are orthogonal; if $i = j$ then the product is zero because $e$ and $f$ are orthogonal.

If $f \leq \omega(a)$ the we may define $(f|a) = (a^{-1}|f)^{-1}$. This is called the corestriction of $a$ to $f$.

**Translation**

**Lemma 5.29 (Translation).** Let $S$ be a Boolean inverse semigroup. Let $x \in S$ be an element and let $a$ be an $n$-infinitesimal such that $d(x) = e(a)$. Then we can define a new $n$-infinitesimal $x \cdot a$ such that $r(x) = e(x \cdot a)$, where $\alpha(x \cdot a) \not\subseteq \alpha(a)$ and $\omega(x \cdot a) \not\subseteq \omega(a)$.

**Proof.** Let $a$ have components labelled by $a_i$ and let the idempotents in $a$ be $e_1, \ldots, e_{n+1}$. By assumption, $d(x)$ is a join of the orthogonal idempotents $e_1, \ldots, e_{n+1}$. It follows that $x$ is a join of the orthogonal set $xe_1, \ldots, xe_{n+1}$. Thus $r(x)$ is a join of the orthogonal idempotents $r(xe_1), \ldots, r(xe_{n+1})$. Here, we have made free use of Lemma 3.30. Define a new $n$-infinitesimal $x \cdot a$ to have the components $(xe_{i+1})a_i(xe_i)^{-1}$.

**Splicing**

As we shall see, this is the key to our proof of the hard direction although at the moment it looks too special to be useful.

**Lemma 5.30 (Splicing).** Let $a$ be an $m$-infinitesimal such that $\alpha(a) = e$, and let $b$ be an $n$-infinitesimal such that $\omega(b) = f$ and let $x$ be any element $e \not\subseteq f$. and, suppose also, that $e(a) \not\subseteq e(b)$. Then there is a multi-infinitesimal $arb = (a_m, \ldots, a_1, x, b_n, \ldots, b_1)$ with extent $e(a) \cup e(b)$ such that $\alpha(arb) = \alpha(b)$ and $\omega(arb) = \omega(a)$. In addition, $\text{inv}(a) \cup \text{inv}(b) \subseteq \text{inv}(arb)$.

**Products**

To define the final operation needs a bit of preparation. As a first step to defining the product, we shall define a ‘grid’. Let $S$ be a Boolean inverse monoid which we regard as a groupoid with respect to the restricted product. By an $n \times m$ grid over $S$, we mean an $n \times m$ array of squares each side of which is labelled by an element of $S$, each vertex is labelled by an idempotent of $S$, the idempotents are pairwise orthogonal, and the squares commute (in the groupoid).

**Lemma 5.31.** Let $S$ be a Boolean inverse monoid. Let $a$ be an $n$-infinitesimal and let $b$ be an $m$-infinitesimal. Suppose that $e = \alpha(a) = e(b)$. Then we may form an $n \times m$ grid, the top of which is $b$. The join of the rows is $e_1 \subseteq \cdots \subseteq e_n$, respectively. The join of all the idempotents at the vertices is $e(a)$.

**Proof.** Let the idempotents contained in $b$ be $e_1, \ldots, e_{m+1}$. We may form the restrictions

$$(a|e_{m+1}), (a|e_n), \ldots, (a|e_1)$$

by Lemma 5.27. Using $b$ as the backbone, we may now attach each restriction $(a|e_i)$ to the idempotent $e_i$ in the backbone; in this way we obtain a ‘comb’. We now fill-in any horizontal lines starting at the top using commutativity in the groupoid. The fact that the idempotents are orthogonal to each other follows by Lemma 5.28.

We denote the grid we get by the above construction by $a \times b$. We call $b$ the spine of the grid. We shall now give an example of the above construction.

**Example 5.32.** Let $a$ be the 3-infinitesimal $(a_3, a_2, a_1)$. Let $b$ be the 2-infinitesimal $(b_2, b_1)$ containing the idempotents $e_3, e_2, e_1$. The grid $a \times b$ is the following (for
Lemma 5.36. Let $S$ be a Boolean inverse monoid. Let $a_1, \ldots, a_n$ be $n$ multi-infinitesimals. Suppose that the extent of $a_i$ is $e_i$, and that $1 = e_1 \oplus \cdots \oplus e_n$. Put $T_1 = \text{inv}(a_1)$ and put $T = T_1 \oplus \cdots \oplus T_n$. Then $T$ is a subalgebra of $S$ which is matricial. It has $n$ components, each of which is isomorphic to one of the $T_i$.

Proof. We use Lemma 3.16 and Lemma 5.25.

What follows is one situation in which we can ‘collapse’. We use splicing.

Lemma 5.37. Let $S$ be a Boolean inverse monoid. Let $T$ be a subalgebra of $S$ which is a matricial monoid where $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ be the decomposition of $T$ into finite simple Boolean inverse monoids. Let the identity of $T_i$ be $e_i$ and let $f_i$ be any atomic idempotent of $T_i$. Suppose that $f_1 \not\subseteq f_2$. Then $T \subseteq T'$ where $T'$ is matricial but has $n - 1$ components.

Proof. We first apply rot$(a)$ if needed. Let $a_1$ be a multi-infinitesimal which generates $T_1$ having source $f_1$. Let $a_2$ be a multi-infinitesimal which generates $T_2$ having target $f_2$. Since $f_1 \not\subseteq f_2$, there is an element $x$ of $S$ such that $f_1 \not\subseteq f_2$. We may
therefore form the multi-infinitesimal \(a_1 x a_2\) by Lemma \(5.30\). This has extent \(e_1 \oplus e_2\). We therefore have the following multi-infinitesimals: \(a_1 x a_2, a_3, \ldots, a_n\) with the extents \(e_1 \lor e_2, e_3, \ldots, e_n\). We now apply Lemma \(5.36\). It follows that these multi-infinitesimals give rise to a subalgebra \(T'\) such that \(T \subseteq T'\) and \(T'\) is isomorphic to the direct product of \((n-1)\) finite simple Boolean inverse monoids. 

We now introduce a condition that will play a key role in our proof of the hard direction.

**Definition.** We say that a Boolean inverse monoid \(S\) satisfies the Idempotent Condition \((IC)\) if whenever \(e \prec f\) there exists an orthogonal set of idempotents \(f_1, \ldots, f_m\) and an \(m < n\) such that \(f \not\sim f_1 \oplus \cdots \oplus f_n\) and \(e \not\sim f_1 \oplus \cdots \oplus f_m\) where \(f, f_j\).

The explanation for one direction of \((IC)\) follows from the lemma below.

**Lemma 5.38.** Let \(S\) be a Boolean inverse monoid. Let \(e\) and \(f\) be idempotents such that \(f \not\sim f_1 \oplus \cdots \oplus f_n\) and \(e \not\sim f_1 \oplus \cdots \oplus f_m\) where \(m < n\). Then \(e \prec f\).

**Proof.** Let \(f \not\sim f_1 \oplus \cdots \oplus f_n\). Then \(x = \bigvee^{n}_{i=1} x f_i\) and \(r(x) = \bigvee^{n}_{i=1} r(x f_i)\), an orthogonal join. The elements \(x f_1, \ldots, x f_m\) are pairwise orthogonal. Let \(f \not\sim f_1 \oplus \cdots \oplus f_m\). Then \(y = \bigvee^{m}_{i=1} y f_i\) and \(r(y) = \bigvee^{m}_{i=1} r(y f_i)\). Consider the elements \(x f_j y^{-1} = (x f_j)(y f_j)^{-1}\) where \(1 \leq j \leq m\). These are pairwise compatible. Put \(z = \bigvee^{m}_{j=1} x f_j y^{-1}\). We have that \(d(x f_j) = f_j = d(y f_j)\). It follows that \((x f_j)(y f_j)^{-1}\) is a restricted product. Thus \(d((x f_j)(y f_j)^{-1}) = r(y f_j)\) and \(r((x f_j)(y f_j)^{-1}) = r(x f_j)\). It follows that \(d(z) = e\) and \(r(z) \leq f\). We have therefore proved that \(e \leq f\). 

We now have the following result which is the key result of this subsection.

**Proposition 5.39.** Let \(S\) be a Boolean inverse monoid in which \(S/\mathcal{J}\) is linearly ordered and in which \(\mathcal{P} = \mathcal{J}\). If \(S\) satisfies \((IC)\) then \(S\) satisfies \((CP)\).

**Proof.** Suppose that \(U\) is a finite simple subalgebra of \(e S e\), and \(V\) is a finite simple subalgebra of \(f S f\), and \(e \perp f\). Let \(U = \text{inv}(a)\) and \(V = \text{inv}(b)\), where \(a\) and \(b\) are multi-infinitesimals. By assumption, \(e(a) = e\) and \(e(b) = f\) where \(e \perp f\). Put \(e_1 = \alpha(a)\) and \(f_1 = \omega(b)\). Because \(S/\mathcal{J}\) is linearly ordered there are three possibilities: \(Se_1 S \subseteq Sf_1 S, Se_1 S = Sf_1 S\) or \(Sf_1 S \subseteq Se_1 S\). The second possibility means that \(e_1 \not\sim f_1\) and so, by our assumption, \(e_1 \not\sim f_1\). Without loss of generality, we may therefore assume that either \(Se_1 S \subseteq Sf_1 S\) or \(e_1 \not\sim f_1\).

We deal with the second case first. For this case, we do not need the \((IC)\).

Let \(e_1 \not\sim f_1\). Then we can ‘splice’ the multi-infinitesimals \(a\) and \(b\) together, by Lemma \(5.30\) to obtain the multi-infinitesimal \(a x b\). Observe that \(U \lor V \subseteq \text{inv}(a x b) = W\) where \(e(a x b) = e \lor f\). By Lemma \(5.24\) we have that \(U \lor V \subseteq W \subseteq (e \lor f) S(e \lor f)\) where \(W\) is a subalgebra of \((e \lor f) S(e \lor f)\) and is a finite simple Boolean inverse monoid. Thus \((CP)\) holds in this case.

In what follows, therefore, we can assume that \(e_1 \prec f_1\). Now, we invoke the \((IC)\). There exist orthogonal \(\mathcal{P}\)-related idempotents \(f'_1, \ldots, f'_m\) and \(m < n\) such that \(f_1 \not\sim f'_1 \oplus \cdots \oplus f'_m\) and \(e_1 \not\sim f'_1 \oplus \cdots \oplus f'_m\). We deal with \(f_1\) first. By the above, there is a multi-infinitesimal \(e\) the idempotents in which are \(f'_1, \ldots, f'_m\); thus \(e(c) = f'_1 \oplus \cdots \oplus f'_m\). It follows that \(e(c) \not\sim f_1\). Let \(e(c) \not\sim f_1\). By Lemma \(5.29\) we may form the translated multi-infinitesimal \(x \cdot c\). By construction, \(e(x \cdot c) = f_1\).

By assumption,

\[\omega(b) = f_1 = e(x \cdot c).\]
By Lemma 5.35 we may therefore form the left product $b \triangleright (x \cdot c)$. We now have the following:

$$e(b \triangleright (x \cdot c)) = f, \quad \text{and} \quad V \subseteq \operatorname{inv}(b \triangleright (x \cdot c)), \quad \text{and} \quad \omega(b \triangleright (x \cdot c)) = \omega(x \cdot c).$$

This deals with $f_1$.

We now deal with $e_1$. By assumption, there is a truncation $c'$ of $c$ such that $e_1 \triangleright e(c')$. Let $e(e'(c')) \downarrow e_1$. By Lemma 5.29 we may form the translated multi-infinitesimal $y \cdot c'$. By construction, $e(y \cdot c') = e_1$. By Lemma 5.33 we may form the right product $a \triangleright (y \cdot c')$. We now have the following

$$e(a \triangleright (y \cdot c')) = e, \quad \text{and} \quad U \subseteq \operatorname{inv}(a \triangleright (y \cdot c')), \quad \text{and} \quad \alpha(a \triangleright (y \cdot c')) = \alpha(y \cdot c').$$

We can now conclude the proof. Observe that $\omega(x \cdot c) \triangleright \omega(c)$ and $\alpha(y \cdot c') \triangleright \alpha(c')$. But $\alpha(c') \triangleright \omega(c)$. Thus

$$\alpha(a \triangleright (y \cdot c')) = \omega(b \triangleright (x \cdot c)).$$

We may therefore splice $a \triangleright (y \cdot c')$ and $b \triangleright (x \cdot c)$ together. This gives rise to a single multi-infinitesimal which we shall denote by $x$ where

$$e(x) = e \oplus f \quad \text{and} \quad U \oplus V \subseteq \operatorname{inv}(x).$$

We have therefore proved that (CP) holds in this case. □

We now connect the (IC) with the properties of MV-algebras. This is a precursor to proving that the (IC) holds in the semigroups of interest to us.

**Lemma 5.40.** Let $S$ be a Foulis monoid in which $S/\mathcal{J}$ is linearly ordered. Then:

1. $[e] \oplus [f] < [1]$ if and only if $SfS \subset \bar{S}eS$, in which case, $[e] \oplus [f] = [e \vee f']$ where $f \triangleright f' \leq e$.
2. $[e] \oplus [e_1] = [h] < [1]$ if and only if $SeS \subset \bar{S}eS$ in which case $h \triangleright e_1 \oplus e_2$ where $e \triangleright e_1 \triangleright e_2$.
3. $n[e] = [f] < [1]$ if and only if $f \triangleright e_1 \oplus \cdots \oplus e_n$ and $e \triangleright e_i$ for each $i$.

**Proof.** (1) We shall prove that $[e] \oplus [f] < [1]$ if and only if $SfS \subset \bar{S}eS$. Suppose, first, that $SeS \subset SfS$. Then $[e] \leq [f]$ and so $[e] \wedge [f] = [e]$. It follows, from the definition, that $[e] \oplus [f] = [e] + [e] = [1]$. We now prove the converse. Suppose that $[e] \oplus [f] = [1]$. There are two possibilities: either $SfS \subset SeS$ or $SeS \subset SfS$. We shall rule out the former. Suppose that $SfS \subset \bar{S}eS$. Then $[e] \wedge [f] = [f]$. Thus $[e] \oplus [f] = [e] + [f'] = [e \vee f']$ where $f \triangleright f' \leq e$. By assumption, $e \vee f' \triangleright 1$. But Dedekind finite monoids are directly finite. Thus $e \vee f' = 1$. But $e \vee f' \leq e \leq e = 0$. In a unital Boolean algebra, we have that $f' = e$. Whence $SfS = SfS = \bar{S}eS$. It follows that $\bar{S}eS = SfS$ which is a contradiction.

(2) We use part (1). So that $e \triangleright e_2 \leq e$. Put $e_1 = e$. Then $e_1 \perp e_2$ and $h \triangleright e_1 \perp e_2$, as required.

(3) Assume first that $f \triangleright e_1 \oplus \cdots \oplus e_n$ and $e \triangleright e_i$ for each $i$. Then $[f] = [e_1 \oplus \cdots \oplus e_n] = [e_1] + \cdots + [e_n] = n[e]$ where we have used Lemma 5.12.

To prove the converse, we shall use induction since the case $n = 2$ was handled in part (2) above. Suppose that $n[e] = [f] < [1]$. Then $n[e] = [e] \oplus (n - 1)[e]$. Let $(n - 1)[e] = [g]$. By induction, we have that $g \triangleright g_2 \oplus \cdots \oplus g_n$ where $e \triangleright g$. We now have to calculate $[e] \oplus [g] = [f] < [1]$. By part (1), we have that $SgS \subset \bar{S}eS$. Thus $[e] \oplus [g] = [e \vee g']$ where $g \triangleright g' \leq e$. It follows that $f \triangleright e \triangleright g'$. We have that $g' \triangleright g_2 \oplus \cdots \oplus g_n$. Let $g' \triangleright g_2 \oplus \cdots \oplus g_n$. Put $e_i = r(yg_i)$. Then $g' = e_2 \oplus \cdots \oplus e_n$. Define $e_1 = e$. Then $f \triangleright e_1 \oplus e_2 \oplus \cdots \oplus e_n$ where $e \triangleright e_i$. □
5.6. Proof of the hard direction. Let $S$ be a Mundici monoid equipped with a normalized invariant mean which assumes rational values. We shall prove that such an $S$ is a UHF monoid. By Lemma 5.14, $S$ is a Foulis monoid. We prove that the (IC) holds. By Corollary 5.20, $L(S)$ is a locally finite MV-algebra. Now, let $e_1$ and $f_1$ be non-zero idempotents of $S$ such that $e_1 < f_1$. Then by Lemma 5.15 we have that $[e_1] < [f_1]$ in $L(S)$. But, $L(S)$ is locally finite. Thus $([e_1],[f_1])$ is a finite MV-subalgebra of $L(S)$. We can now apply our observations about finite MV-algebras made in Subsection 5.2. This means that there is an element $[e]$ of $L(S)$ such that $[e_1] = m[e]$ and $[f_1] = n[e]$ where $n > m$. Thus the (IC) holds. By Lemma 5.40 there is an $\mathbb{N}$-infinitesimal $a$ such that $f_1 \not\leq e(a)$ and there is a truncation $a'$ of $a$ such that $e_1 \not\leq e(a')$. The result now follows by Proposition 5.39. This completes the proof of Theorem 5.2.

Remark 5.41. As the referee has suggested, it is possible to simplify the statement of Theorem 5.2 at the price of using MV-algebras. Let $S$ be an AF Foulis monoid. Then $L(S)$ is an MV-algebra since $S$ is a Foulis monoid. Suppose that it is rational. Then $L(S)$ is a simple MV-algebra and so $S$ is 0-simplifying. There is an embedding $\theta$ of $L(S)$ in the MV-algebra $[0,1] \cap \mathbb{Q}$. Define $\nu: E(S) \to [0,1]$ by $\nu(e) = \theta([e])$. Then $\nu$ is a normalized invariant mean that assumes only rational values. Observe that $S/\not\leq$ is linearly ordered since $[0,1] \cap \mathbb{Q}$ is linearly ordered. It follows that UHF monoids may be characterized as those AF Foulis monoids $S$ in which $L(S)$ is a rational MV-algebra.

5.7. Complements. Our first result is a uniqueness result.

Lemma 5.42. Let $S$ be a UHF monoid. Then $S$ has exactly one normalized invariant mean.

Proof. This follows by Lemma 5.3 since any normalized invariant mean on $S$ must agree with the unique normalized invariant mean on a finite simple subalgebra. □

Our next result shows that every simple rational MV-algebra is co-ordinatized by some UHF monoid.

Theorem 5.43. Let $A$ be a simple rational MV-algebra. Then $A$ is co-ordinatized by some UHF monoid.

Proof. Clearly, $A$ is countable. By [19], there is therefore an AF Boolean inverse monoid which is also a Foulis monoid which co-ordinatizes $A$. This means that $L(S) \cong A$, for some isomorphism. By assumption, $A$ is simple and so $S$ is 0-simplifying by Corollary 5.18. $S/\not\leq$ is linearly ordered by our assumption on $A$ and Lemma 5.15. Let $\theta: L(S) \to A$ be the isomorphism. Define $\mu: E(S) \to [0,1] \cap \mathbb{Q}$ by $\mu(e) = \theta([e])$. This maps clearly satisfies condition (1) in the definition of a normalized mean. Now, let $e$ and $f$ be orthogonal. Then

$$\mu(e \vee f) = \theta([e \vee f]) = \theta([e] \oplus [f]) = \theta([e]) \oplus \theta([f]) = \mu(e) \oplus \mu(f),$$

where we have used Lemma 5.12. Observe that $\mu(1) = \theta([1]) = \theta(1) = 1$. We therefore have a normalized invariant mean. By Theorem 5.2 it follows that $S$ is, in fact, a UHF monoid. □

Before we can prove the next result, we need a little preparation.

Let $G$ be an abelian group with identity 0. The group $G$ is a lattice-ordered group or an $l$-group if it is equipped with a partial order which turns the group into a partially ordered monoid and a lattice. In such a group, $G$ has a subset $G^+$, called the positive cone of $G$, which consists of all elements greater than or equal to the identity of $G$. This submonoid completely determines the order (and the group). A homomorphism of $l$-groups must map positive elements to positive elements. It
follows that an isomorphism of \(l\)-groups induces an isomorphism of their positive cones. The positive cone of an \(l\)-groups can be precisely characterized: it is a conical cancellative abelian monoid such that for any \(a, b \in M\) there exists an element, denoted by \(a \lor b\), such that \(aM \cap bM = (a \lor b)M\); this latter condition is called Ore’s condition. See \([1]\) and \([29, \text{Section X.2}]\). Let \(G\) be an \(l\)-group. An order-unit \(u\) in \(G\) is an element \(u \in G\) such that for each \(g \in G\) there exists a natural number \(n\) such that \(g \leq nu\). We shall want to consider below the category of ordered pairs \((G, u)\) where \(G\) is an \(l\)-group and \(u \in G\) is a distinguished order-unit. A morphism from \((G, u)\) to \((H, v)\) is a homomorphism of \(l\)-groups that maps \(G\) to \(H\) and \(u\) to \(v\).

From \([7, \text{Chapter IV, Section 1}]\), each commutative monoid \(M\) has a universal group \(G(M)\). This is an abelian group equipped with a monomorphim \(\gamma: M \rightarrow G(M)\) which is universal. This is often known as the Grothendieck construction. The group \(G(M)\) can be constructed as a group of fractions of \(M\) so we can think of its elements as being equivalence classes of the form \(s - t\). The homomorphism \(\gamma\) is given by \(\gamma(s) = s - 0\) where the 0 is the identity of \(M\). The map \(\gamma\) is injective if and only if \(M\) is cancellative. If \(M\) is cancellative and conical then \(G(M)\) is a partially ordered abelian group. In the conical case, we can define a partial order \(\leq\) by \(x \leq y\) if and only if \(x + z = y\) for some \(z\). If \(M\) is a lattice with respect to this order then \(G(S)\) is an \(l\)-group. Every \(l\)-group is the Grothendieck group of its positive cone. Thus an isomorphism between positive cones of two \(l\)-groups induces an isomorphism of the corresponding \(l\)-groups.

With every Boolean inverse monoid, we can associate the partial type monoid \(\text{Int}(S)\). The partial type monoid can be embedded in a commutative monoid in two (equivalent) ways. One can either adopt the approach described in \([33, \text{Definition 4.1.3}]\) or the equivalent approach using \(R_\omega(S)\) in \([11]\). In either case, we denote the resulting monoid by \(\text{Typ}(S)\) and call it the \textit{type monoid} of \(S\); the binary operation in \(\text{Typ}(S)\) is also denoted by \(+\). There is a map \(\tau: E(S) \rightarrow \text{Typ}(S)\). The type monoid is always a refinement monoid. We now apply the above results to \(\text{Int}(S)\) and \(\text{Typ}(S)\) where \(S\) is a Boolean inverse monoid. The type monoid \(\text{Typ}(S)\) is always conical: by \([33, \text{Corollary 4.1.4}]\), \(\text{Typ}(S)\) is a conical refinement monoid and \(\text{Int}(S)\) is an order-ideal. By Proposition \([8,29,33, \text{Lemma 3.6}]\) and \([33, \text{Corollary 2.7.4}]\), if \(S\) is factorizable then \(\text{Typ}(S)\) is cancellative. From lemma \([5,15]\) we have that \(e \preceq f\) if and only if \([e] \leq [f]\) in \(\text{Int}(S)\). Thus, if \(S\) also satisfies the lattice condition, then \(\text{Int}(S)\) is a lattice; it follows by \([33, \text{Lemma 3.14}]\), that \(G(\text{Typ}(S))\) is an \(l\)-group with positive cone \(\text{Typ}(S)\). To ease notation, we shall regard \(\text{Typ}(S)\) as a subset of \(G(\text{Typ}(S))\): namely, its positive cone. In addition, we shall regard \(\text{Int}(S)\) as a subset of \(\text{Typ}(S)\). Thus \(\text{Int}(S)\), as a set, is the interval \([0, 1]\). The MV-algebra \(L(S)\) has maximum element \(1 = [1]\) and minimum element \(0 = [0]\). This gives rise to an order-unit \([1]\) in \(G(\text{Typ}(S))\). The following proposition summarizes the results so far.

**Proposition 5.44.** Let \(S\) be a Foulis monoid. Then \(G(\text{Typ}(S))\) is an \(l\)-group, \(1\) is an order-unit, the positive cone of this \(l\)-group is \(\text{Typ}(S)\), and the interval \([0, 1]\) = \(\text{Int}(S)\) is the partial type monoid.

It remains to connect MV-algebras with \(l\)-groups. This is done in \([26,27]\). See also \([2]\). Let such a \(G\) be an \(l\)-group and let \(u \in G\) be an order-unit. Denote by \([0, u]\) all the elements \(g \in G\) such that \(0 \leq g \leq u\). On the set \([0, u]\) make the following definitions: \(g \oplus h = u \land (g + h)\) and \(\neg g = u - x\). Then, with respect to these operations, \(\Gamma(G, u) = [0, u]\) is an MV-algebra. In fact, \(\Gamma\) defines a functor from the category of pairs \((G, u)\) to the category of MV-algebras. This is an equivalence of categories. It is useful to see how to go in the opposite direction. Let \(A\) be an MV-algebra with top element 1. Then we can construct a cancellative conical abelian
monoid \( M_A \) satisfying Ore’s condition. We now use the Grothendieck construction to obtain an abelian group \( G(M_A) \) which is an \( l \)-group with positive cone \( M_A \). In addition, from the \( 1 \in A \) we can construct an order-unit \( u_A \in G(M_A) \) such that \( A \cong \Gamma(G(M_A), u_A) \). The key result is the following. It is proved in [20] and also in [2] Lemma 7.2.1. It tells us, roughly, that the MV-algebra structure determines the \( l \)-group.

**Proposition 5.45.** Let \( A \) and \( B \) be MV-algebras such that \( \Gamma(G, u) \cong A \) and \( \Gamma(H, v) \cong B \). If \( A \) and \( B \) are isomorphic then \( (G, u) \) is isomorphic to \( (H, v) \).

We have the following from [19] Proposition 4.5.

**Proposition 5.46.** Let \( S \) be a Foulis monoid. Then \( L(S) = \Gamma(G(\text{Typ}(S)), 1) \).

Those MV-algebras which are rational correspond to those subgroups of \((\mathbb{Q}, +)\) that contain 1. The following is stated in [3] Section 2.

**Proposition 5.47.** Let \( A \) be a rational MV-algebra. Then we may construct a cancellative abelian monoid \( M_A \) such that its Grothendieck group \( G(M_A) \) is a subgroup of the rationals containing 1. In addition, \( A \cong \Gamma(G(M_A), 1) \).

We can now prove that that UHF monoids are completely determined by their MV-algebras.

**Theorem 5.48.** Let \( S \) and \( T \) be UHF monoids. If the MV-algebras \( L(S) \) and \( L(T) \) are isomorphic then \( S \) and \( T \) are isomorphic.

**Proof.** Let \( S \) and \( T \) be UHF monoids such that the MV-algebras \( L(S) \) and \( L(T) \) are isomorphic. Then \( \Gamma(G(\text{Typ}(S)), 1) \cong \Gamma(G(\text{Typ}(T)), 1) \). By Proposition 5.46 it follows that \( (G(\text{Typ}(S)), 1) \cong (G(\text{Typ}(T)), 1) \). Thus \( (\text{Typ}(S), 1) \) is isomorphic to \( (\text{Typ}(T), 1) \). But \( S \) and \( T \) are both AF by Lemma 5.1. Thus \( S \) and \( T \) are isomorphic by [20] Theorem 5.1.11.

The subgroups of \( \mathbb{Q} \) containing 1 are classified using supernatural numbers [31]. This means that the rational MV-algebra can be classified using supernatural numbers [3]. If \( n \) is any supernatural number, then \( S_n \) denotes a UHF monoid whose associated MV-algebra is the one associated with \( n \). Observe that when \( n \) is a finite supernatural number, that is a natural number, then \( S_n \cong \mathcal{I}_n-1 \). We have therefore generalized the finite symmetric inverse monoids into the supernatural realm.

**Example 5.49.** A UHF monoid which has \( [0, 1] \cap \mathbb{Q} \) as its MV-algebra is constructed in Section 4.3 of [21] and Section 3 of [22].

### 6. AF Monoids and AF Groupoids

The goal of this section is to characterize the Stone groupoids of AF monoids. This will require us to use non-commutative Stone duality. This theory is described from scratch in [17]. This theory was introduced in [13], in a special case, but in full generality in [18].

We quickly run through the definitions and notation we shall need. If \( G \) is a groupoid then its space of identities is denoted by \( G_o \). We say that a subgroupoid of a groupoid is wide if it contains all the identities. A topological space is said to be Boolean if it is Hausdorff, compact and 0-dimensional. An étale groupoid is said to be Boolean if its identity space is Boolean. Every Boolean space can be regarded as a Boolean groupoid. A local bisection of a groupoid \( G \) is any subset such that \( AA^{-1}, A^{-1}A \subseteq G_o \). We say that a local bisection is a bisection if \( AA^{-1} = A^{-1}A = G_o \). The set of all compact-open local bisections of a Boolean groupoid \( G \) is denoted by \( \text{KB}(B) \) and is a Boolean inverse monoid under subset multiplication. If \( S \) is a
Boolean inverse monoid then $G(S)$ denotes the set of all prime filters of $S$. It is a Boolean groupoid called the Stone groupoid of $S$. The essence of non-commutative Stone duality is that $S \cong KB(G(S))$ and $G \cong G(KB(G))$. It can be shown that $S$ is a meet-monoid if and only if $G(S)$ is Hausdorff.

Let $G$ be a Hausdorff Boolean groupoid. We say that a Boolean groupoid $G$ is $AF$ if $G = \bigcup_{i=1}^{\infty} G_i$ where each $G_i$ is a wide compact open principal subgroupoid of $G$.

**Proposition 6.1.** Let $S$ be a Boolean inverse meet-monoid. If $S$ is AF then $G(S)$ is AF.

**Proof.** Let $S$ be an AF monoid. Then $S = \bigcup_{i=1}^{\infty} S_i$ where each $S_i$ is a finite subalgebra of $S$ and isomorphic to a finite direct product of finite symmetric inverse monoids; thus, $S_i$ is a finite fundamental Boolean inverse monoid. For each $i$, define $G_i \subseteq G(S)$ to consist of all the prime filters of $S$ that contain at least one element of $S_i$. Since $S_i$ is a submonoid of $S$, it follows that $G_i$ contains all idempotent prime filters. Let $A \subseteq G_i$. Then $a \in A$ for some $a \in S_i$. Thus $a^{-1} \subseteq A^{-1}$. But $S_i$ is an inverse submonoid of $S$. It follows that $a^{-1} \subseteq S_i$. Thus $A^{-1} \subseteq G_i$. We have therefore proved that $G_i$ is closed under groupoid inverses. Let $A, B \subseteq G_i$ be such that $A \cdot B$ is defined. Let $a \in A \cap S_i$ and $b \in B \cap S_i$. Then $A = (aB(A))^{-1} \cdot B = (bB(A))^{-1}$. We have that $A \cdot B = (aB(A))^{-1}$. Since $S_i$ is an inverse submonoid of $S$, we have that $ab \in S_i$. It follows that $A \cdot B \subseteq G_i$. We have therefore proved that $G_i$ is a wide subgroupoid of $G(S)$. Observe that $G_i = \bigcup_{a \in S_i} U_a$. It follows that $G_i$ is open and compact. The Boolean groupoid associated with a basic Boolean inverse monoid is principal by [13] Proposition 4.31. Thus $G(S)$ is principal. It follows that each $G_i$ is principal. Observe that $G_1 \subseteq G_2 \subseteq \ldots$. We have therefore proved that the Boolean groupoid $G(S)$ is AF. □

We shall prove the converse of Proposition 6.1. To do this, we shall need a few lemmas first. If $X$ is a Boolean space, regarded as a groupoid, and $G$ is any Boolean groupoid, then $X \times G$ is a Boolean groupoid. Our first goal is to describe the compact-open bisections in $X \times G$.

**Lemma 6.2.** Let $X$ be a Boolean space and let $G$ be a Boolean groupoid. Let $A \subseteq X \times G$ be a non-empty compact-open local bisection

1. Then $A = \bigcup_{i=1}^{n} X_i \times A_i$, where the $X_i$ are clopen subsets of $X$, and the $A_i$ are compact-open local bisections of $G$.

2. If $A$ is actually a bisection, then $A = \bigcup_{i=1}^{n} X_i \times A_i$, where the $X_i$ are clopen subsets of $X$, the $A_i$ are compact-open local bisections of $G$, and $X$ is equal to the union of the $X_i$.

**Proof.** (1) From the definition of the product topology and the fact that $A$ is both open and compact we can write $X = \bigcup_{i=1}^{n} U_i \times V_i$ where the $U_i$ and $V_i$ are open. For each $i$, $U_i \times G$ is an open subset of $X$. But $X$ has a basis of clopen subsets. Thus we can write $U_i$ as a union of clopen subsets. Likewise, we can write $V_i$ as a union of compact-open local bisections. Thus we can write $X$ as a union of sets of the form $U \times V$, where both $U$ is a clopen subset of $X$, and $V$ is a compact-open local bisection. But $A$ is compact and so we can write $A$ as a finite union of such sets.

(2) From part (1), we have that $A = \bigcup_{i=1}^{n} X_i \times A_i$ where the $X_i$ are clopen subsets of $X$ and the $A_i$ are compact-open local bisections of $G$. Let $x \in X$ be arbitrary. Then for any $e \in G_e$, we have that $(x, e) \in (X \times G)$. By the assumption that $A$ is a bisection, there is $(y, g) \in A$ such that $d(y, g) = (x, e)$. It follows that $y = x$ and so $(x, g) \in A$ (remember that the first component lives in a Boolean space). Thus $(x, g) \in X_i \times A_i$ for some $i$. We have proved that $x \in X_i$. □
The following is immediate by part (2) Lemma 6.2 and properties of Boolean algebras as discussed at the end of Subsection 2.4 and the fact that
\[(\bigcup_{i=1}^{m} A_i) \times B = \bigcup_{i=1}^{m} A_i \times B.\]

**Lemma 6.3.** Let \(X\) be a Boolean space and let \(G\) be a Boolean groupoid. If \(A_1, \ldots, A_n\) is a finite set of compact-open bisections in \(X \times G\), then there is a clopen partition \(\{Y_j : 1 \leq j \leq p\}\) of \(X\) such that each \(A_k\), where \(1 \leq k \leq s\), can be written in the form \(A_k = \bigcup_{j=1}^{p} Y_j \times B_{jk}\) where \(B_{jk}\) is a compact-open bisection of \(G\).

We can now look at the product of two compact-open sections

**Lemma 6.4.** Let \(X\) be a Boolean space and let \(G\) be a Boolean groupoid. Let \(\{Y_j : 1 \leq j \leq p\}\) be a clopen partition of \(X\). Let \(A\) and \(B\) be compact-open bisections of \(X \times G\) such that \(A = \bigcup_{j=1}^{p} Y_j \times A_j\) and \(B = \bigcup_{j=1}^{p} Y_j \times B_j\) where \(A_j\) and \(B_j\) are compact-open local bisections of \(G\). Then \(AB = \bigcup_{j=1}^{p} X_j \times C_j\) where the \(C_j\) are compact-open local bisections of \(G\).

**Proof.** To calculate \(AB\) devolves down to calculating \((Y_i \times A_i)(Y_j \times B_j)\) where \(i\) and \(j\) could be different. For this set to be non-empty, we need \(X_i = X_j\). Since we are dealing with a partition, this implies that \(i = j\). It now follows that \((X_i \times A_i)(X_j \times B_j) = X_i \times A_i B_i\). Put \(C_i = A_i B_i\). \(\square\)

Denote by \(G_n\) the transitive principal groupoid on the set \(\{1, \ldots, n\}\); this is simply the groupoid \(\{1, \ldots, n\} \times \{1, \ldots, n\}\). This is a Boolean groupoid and \(\text{KB}(G_n) \cong \mathbb{Z}_n\) since the compact-open local bisections of \(G_n\) in this case are just the local bisections of \(G_n\), and each such local bisection gives rise to and is determined by a partial bijection of the set \(\{1, \ldots, n\}\). Let \(X\) be any Boolean space regarded as a groupoid. Then \(X \times G_n\) is a Boolean groupoid since it is the direct product of two Boolean groupoids.

**Lemma 6.5.** The group of compact-open sections of \(X \times G_n\) is locally finite.

**Proof.** Let \(A_1, \ldots, A_n\) be a finite set of compact-open bisections in \(X \times G\). By Lemma 6.3 there is a clopen partition \(\{Y_j : 1 \leq j \leq p\}\) of \(X\) such that each \(A_k\), where \(1 \leq k \leq s\), can be written in the form \(A_k = \bigcup_{j=1}^{p} Y_j \times B_{jk}\) where \(B_{jk}\) is a compact-open bisection of \(G\). By Lemma 6.4 every element of \(\langle A_1, \ldots, A_n\rangle\) can be written in the form \(\bigcup_{j=1}^{p} X_j \times C_j\) where each \(C_j\) is a local section of \(G_n\). But \(G_n\) has only a finite number of local sections. It follows that \(\langle A_1, \ldots, A_n\rangle\) is finite. \(\square\)

The following result is needed below and is of independent interest.

**Proposition 6.6.** Let \(G\) be a compact Hausdorff principal Boolean groupoid. Then its group of compact-open sections is locally finite.

**Proof.** Let \(G\) be a compact Hausdorff principal Boolean groupoid. By 3.4, we may write \(G\) as a finite disjoint union \(G = \bigcup_{i=1}^{n} X_i \times G_i\) where each \(X_i\) is a Boolean space. Each compact-open section of \(G\) is an \(n\)-tuple of compact-open sections of each \(X_i \times G_i\). It follows that the group of compact-open sections of \(G\) is a finite direct product of the groups of compact-open sections of each \(X_i \times G_i\). By Lemma 6.5 each of these groups is locally finite thus the group of compact-open sections of \(G\) is locally finite. \(\square\)

We can now prove the main result of our deliberations.

**Theorem 6.7.** Let \(S\) be a Boolean inverse meet-monoid. Then \(S\) is an AF monoid if and only if \(G(S)\) is an AF groupoid.
Proof. The proof that $G(S)$ is an AF groupoid is provided by Proposition \[15.1\]. So, we need only prove the converse. Suppose that $G(S)$ is AF where $G(S) = \bigcup_{i=1}^{\infty} G_i$. Then $S$ is basic by \[15. Proposition 4.31\] not forgetting to use the fact that we are dealing with a meet-monoid. It remains to show that the group of units of $S$ is locally finite. Let $A$ be a compact-open bisection of $G(S)$ (and so it is a typical element of the group of units of $S$.) Then $A \subseteq \bigcup_{i=1}^{\infty} G_i$. But $A$ is compact and the $G_i$ are open. In addition, we have that $G_1 \subseteq G_2 \subseteq \ldots$. It follows that there is an $i$ such that $A \subseteq G_i$. Thus $A$ is a compact-open bisection of $G_i$. If $A_1, \ldots, A_m$ is a finite set of compact-open bisections then, by the above reasoning, there is an $i$ such that $A_1, \ldots, A_m \subseteq G_i$. But we have proved above that the group of units of compact-open sections of $G_i$ is locally finite by Proposition \[6.6\]. Thus $\langle A_1, \ldots, A_m \rangle$ is finite.

\[\square\]

7. Selected results

The results of this section are not needed in the proof of our main theorems, but are of independent interest. The following result shows the place of Boolean algebras within the theory of Boolean inverse monoids.

**Lemma 7.1.** A commutative fundamental Boolean inverse monoid is a unital Boolean algebra.

**Proof.** We prove that every element is an idempotent. Let $a$ be an arbitrary element and let $e$ be any idempotent. Then, by commutativity, $ae = ea$. But we are assuming that the semigroup is fundamental. Thus $a$ is an idempotent. We have proved that every element is an idempotent.

We are next interested in when the MV-algebras associated with a Foulis monoid are, in fact, Boolean algebras.

**Lemma 7.2.** Let $S$ be a Foulis monoid. Then $\{e\} \in L(S)$ is an idempotent if and only if $SeS \cap S\bar{e}S = \{0\}$.

**Proof.** Suppose that $SeS \cap S\bar{e}S = \{0\}$. Then $\{e\} \wedge \{e\} = \{0\}$. Thus $\{e\} \oplus \{e\} = \{e\}$, and so $\{e\}$ is an idempotent. We now prove the converse. Suppose that $\{e\}$ is an idempotent and that $SeS \cap S\bar{e}S = SfS$. Then $\{e\} \oplus \{e\} = \{e\}$ where $f' \not\equiv f$ and $e' = 0$. Observe that $e \vee f' \equiv e \leq e \vee f$. But, by Dedekind finiteness, we have that $e = e \vee f'$ and so $f' \leq e$. But $f'f = 0$. It follows that $f' = 0$ and so $f = 0$, as required.

We now have the following result.

**Proposition 7.3.** Let $S$ be a Foulis monoid. Then $L(S)$ is a unital Boolean algebra if and only if $S$ has central idempotents.

**Proof.** Suppose that $S$ has central idempotents. Then $SeS \cap S\bar{e}S = Se \cap S\bar{e} = \{0\}$ and, so by Lemma \[7.2\] every element of $L(S)$ is an idempotent. Whence $L(S)$ is a Boolean algebra. Suppose, now, that $L(S)$ is a Boolean algebra. This means precisely that every element of $L(S)$ is an idempotent. By Lemma \[7.2\] this means that for every idempotent $e$ of $S$ we have that $SeS \cap S\bar{e}S = \{0\}$. Let $a$ be any element of $S$. Then $eae = 0 = eae$ since $SeS \cap S\bar{e}S = \{0\}$. We now use the fact that for every idempotent $e$ we have that $1 = e \vee \bar{e}$. Thus $ea = eae = ea(e \vee \bar{e}) = eae$ and $ae = 1ae = (e \vee \bar{e})ae = eae$. We have proved that $ae = ea$ and so every idempotent is central.

\[\square\]

Infinitesimals play an important role in our work. It is therefore important to know when they do, and do not, exist.
Proposition 7.4. Let $S$ be a Boolean inverse semigroup. Then $S$ has central idempotents if and only if $S$ contains no infinitesimals.

Proof. Suppose that $S$ has central idempotents. Let $a$ be an infinitesimal. Then $a^2 = 0$ thus $aa = 0$. It follows that $d(a) r(a) = 0$. But, in a semigroup with central idempotents, $d(a) = r(a)$. It follows that $d(a) = 0$ and so $a = 0$ which is a contradiction. Thus there are no infinitesimals.

To prove the converse, suppose that there are no infinitesimals. Let $a \in S$ be arbitrary and non-zero. We shall prove first that $d(a) = r(a)$. Suppose not. Put $e = d(a) r(a)$. If $e = 0$ then $a^{-1} a a a^{-1} = 0$ and so $a^2 = 0$. But this means that $a$ is an infinitesimal. It follows that $e \neq 0$. There are now two cases. Suppose, first, that $e = d(a)$. Then $d(a) < r(a)$. By assumption, there exists $0 < f \leq r(a)$ such that $f d(a) = 0$. Put $b = fa$. If $b = 0$ then it is easy to see that $f = 0$, which is a contradiction. It follows that $b \neq 0$. You can easily check that $b$ is an infinitesimal. This is a contradiction. We can now deal with the second case where $e < d(a)$.

By assumption, there is a non-zero idempotent $f$ such that $e f = 0$ and $f \leq d(a)$. Put $b = af$. If $b = 0$ then $(af)^{-1} af = 0$ which implies that $f = 0$. It follows that $b \neq 0$. However, $b^2 = af af = ad(a) f r(a) af = ac f af = 0$. But this contradicts the assumption that there are no infinitesimals. It follows that $d(a) = r(a)$ for all $a \in S$.

We can now prove that the idempotents are central. Let $e$ be any idempotent and let $a$ be any element. Then by the above, we have that $d(ae) = r(ae)$, so that $a^{-1} a e = e a a e^{-1}$. Multiply this equality on the left by $a$ to get $ea = a e a a^{-1} = a( aa a^{-1}) e$. We can then finish off by using the fact that $aa a^{-1} = a^{-1} a$.

We now turn to the existence of $n$-infinitesimals. The poset $a^\downarrow$ is order-isomorphic to the poset $d(a)^\downarrow$ and the poset $r(a)^\downarrow$. It is easy to check that $a$ is an atom if and only if $d(a)$ is an atom if and only if $r(a)$ is an atom. We say that a Boolean inverse monoid is atomless if it has no atoms. This mean that for every non-zero idempotent $e$, there is a non-zero idempotent $f < e$. The following was proved as [15 Lemma 5.15].

Lemma 7.5. Let $S$ be a 0-simplifying atomless Boolean inverse monoid. Let $F$ be any ultrafilter in the unital Boolean algebra $E(S)$ containing the idempotent $e$. Then the local monoid $eSe$ contains an infinitesimal $a$ such that $d(a) \in F$.

We can now prove the following.

Lemma 7.6. Let $S$ be an atomless 0-simplifying Boolean inverse monoid and let $e$ be any non-zero idempotent. Then the local monoid $eSe$ contains $n$-infinitesimals for any finite $n$.

Proof. Let $e$ be any non-zero idempotent. Then $eSe$ is a Boolean inverse monoid by Lemma 3.3, it is 0-simplifying by Lemma 3.12 and it is atomless since $eSe$ is an order-deal of $S$.

We will prove our result using induction. Let $e$ be any non-zero idempotent. Then from the theory of unital Boolean algebras, there is an ultrafilter $F$ of $E(S)$ such that $e \in F$. By Lemma 7.3, there is an infinitesimal $a \in eSe$ such that $d(a) \in F$. Thus $eSe$ contains the $1$-infinitesimal $a$. Suppose that $a$ is an $n$-infinitesimal contained in $eSe$. We will show that $eSe$ also contains an $(n + 1)$-infinitesimal. Let $f \neq 0$ be the target of $a$. This is orthogonal to every idempotent in $a$. Thus by Lemma 7.3 the local monoid $fSf$ contains an infinitesimal $b$. We now co-restrict the $n$-infinitesimal $a$ to obtain the $n$-infinitesimal $(d(b) a)$. Consider now the $(n + 1)$-sequence $(b, (d(b) a))$. Observe that since $b \in fSf$, we have that $r(b) \leq f$ and so is orthogonal to every idempotent in $(d(b) a)$. It follows that $(b, (d(b) a))$ is an $(n + 1)$-infinitesimal contained in $eSe$. \qed
The following is now immediate by Lemma \ref{lem:7.8}, Lemma \ref{lem:5.25} and the Wagner-Preston representation theorem \cite{Wagner1937}.

**Proposition 7.7.** Let $S$ be an atomless 0-simplifying Boolean inverse monoid. Then every non-zero local monoid of $S$ contains isomorphic copies of all finite inverse semigroups.

We can also say something about the groups of units of atomless 0-simplifying Boolean inverse monoids.

**Proposition 7.8.** Let $S$ be an atomless 0-simplifying Boolean inverse monoid. Then the group of units of $S$ contains isomorphic copies of all finite groups.

**Proof.** By Cayley’s theorem, every finite group can be embedded in a finite symmetric group. So, it is enough to prove that the group of units of $S$ contains an isomorphic copy of every finite symmetric group. By Proposition 7.7 we know that $S$ contains a symmetric copy $T$ of the finite symmetric monoid $I_n$. Let the identity of $T$ be $e$. Then $T \subseteq S e$. It follows that the symmetric group on $n$ letters is a subgroup of the group of units of $eS e$. Denote the group of units of $eS e$ by $G_e$. Define a function $\theta: G_e \to U(S)$ by $g \mapsto g \oplus e$. This is well-defined, since $d(\theta(g)) = r(\theta(g)) = 1$, and is clearly both injective and a homomorphism. It follows that the group of units of $S$ contains isomorphic copies of all finite symmetric groups. □

We refer the reader to \cite{Lawson2013} for the symmetric subgroup of $U(S)$ defined using infinitesimals, and the alternating subgroup defined using 2-infinitesimals. These two subgroups agree with the ones defined by \cite{Scott2003}.

We conclude this section with a result about being fundamental. Let $S$ be a Boolean inverse monoid. The group of units of $S$, denoted by $U(S)$ acts by conjugation on the unital Boolean algebra $E(S)$ by $e \mapsto geg^{-1}$. Observe that $geg^{-1} = e$ if and only if $ge = eg$. Denote by $N$ the normal subgroup of $U(S)$ that consists of all units that commute with every idempotent of $S$. We have the following result.

**Lemma 7.9.** Let $S$ be a Boolean inverse monoid. Then $Z(E(S)) = N^\perp$.

**Proof.** Let $a \in Z(E(S))$. Then $d(ea) = r(a)$. It follows that $g = a \vee (e(a))$ is a unit. It is easy to check that $g \in N$. We have therefore proved that $Z(E(S)) \subseteq N^\perp$. We now prove the reverse inclusion. Let $a \in N^\perp$. Then $a \leq g$ where $g \in N$. Thus $a = ga^{-1}a$. Let $e$ be an arbitrary idempotent. Then $ae = (ga^{-1}a)e = gea^{-1}a = ega^{-1}a = ea$. We have therefore proved that $a \in Z(E(S))$. □

If $N = \{1\}$ then $Z(E(S)) = E(S)$ and the semigroup is fundamental. On the other hand, if $Z(E(S)) = E(S)$ then the semigroup is fundamental. This means that $E(S) = N^\perp$. But every element of $N$ is beneath itself. We have therefore proved the following.

**Corollary 7.10.** Let $S$ be a Boolean inverse monoid. Then $S$ is fundamental if and only if $N = \{1\}$.

The above corollary was first proved, in a completely different way, as \cite[Lemma 5.5]{Lawson2013}.

8. Concluding remarks

The authors are grateful to Dr Ganna Kudryavtseva for her suggestion to include the following discussion.

Observe first that the groups studied in \cite{Lawson2013} are precisely the groups of units of our UHF monoids; this follows from \cite[Lemma 3.11]{Scott2003} and the fact that the ‘diagonal
embeddings’ of [10] are restrictions to the groups of units of standard maps [10]. We now apply non-commutative Stone duality [13] (and subsequent papers). By [15, Proposition 4.31], suppose that \( S \) is a Boolean inverse meet-monoid. Then \( S \) is basic precisely when its associated Boolean groupoid is principal.\(^7\) UHF monoids are simple: that is, they are 0-simplifying and fundamental (since they are AF and so basic); this follows by Proposition 3.35. These monoids are also meet-monoids by Proposition 3.35. Recall that we call the (unique) countable atomless Boolean algebra the Tarski algebra. A countable Boolean inverse meet-monoid which has a Tarski algebra of idempotents is called a Tarski monoid. By what we term the Dichotomy Theorem [15, Proposition 4.4], a simple countable Boolean inverse meet-monoid is either finite, and so isomorphic to a finite symmetric inverse monoid, or a Tarski monoid. It follows that an infinite UHF monoid is a Tarski monoid. By [10, Theorem 2.10] — which is an interpretation of work by Matui [24, Theorem 3.9] using non-commutative Stone duality — two infinite UHF monoids are isomorphic if and only if their groups of units are isomorphic. However, as we explained above, these are just the groups studied in [10]. This shows that, as we would expect, there is a close connection between our work on UHF monoids and the groups defined in [10]. However, we do not know of an abstract characterization of the groups in [10].

References

[1] G. Birkhoff, Lattice-ordered groups, *Ann. Math.* **43** (1942), 298–331.
[2] R. L. O. Cignoli, I. M. L. D’Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Springer, 2000.
[3] R. Cignoli, E. J. Duboc, D. Mundici, Extending Stone duality to multisets and locally finite MV-algebras, *J. Pure Appl. Math.* **189** (2004), 37–59.
[4] H. D’Alarcao, Factorizable as a finiteness condition, *Semigroup Forum* **20** (1980), 281–282.
[5] S. Givant, P. Halmos, *Introduction to Boolean algebras*, Springer, 2009.
[6] T. Giordano, I. Putnam, C. Skau, Affable equivalence relations and orbit structure of Cantor dynamical systems, *Ergod. The. & Dynam. Sys.* **23** (2004), 441–475.
[7] P. A. Grillet, *Semigroups*, Marcel Dekker, Inc., 1995.
[8] Ph. J. Higgins, *Categories and groupoids*, Van Nostrand Reinhold Company, London, 1971.
[9] J. M. Howie, *An introduction to semigroup theory*, Academic Press, 1976.
[10] N. K. Kroshko, V. I. Sushchansky, Direct limits of symmetric and alternating groups with strictly diagonal embeddings, *Archiv Math.* **71** (1998), 173–182.
[11] G. Kudryavtseva, M. V. Lawson, D. H. Lenz, P. Resende, Invariant means on Boolean inverse monoids, *Semigroup Forum* **92** (2016), 77–101.
[12] M. V. Lawson, *Inverse semigroups: the theory of partial symmetries*, World Scientific, 1998.
[13] M. V. Lawson, A non-commutative generalization of Stone duality, *J. Aust. Math. Soc.* **88** (2010), 385–404.
[14] M. V. Lawson, Non-commutative Stone duality: inverse semigroups, topological groupoids and \( C^\ast \)-algebras, *Inter. J. Algebra Comput.* **22**, 1250058 (2012) DOI:10.1142/S0218196712500580.
[15] M. V. Lawson, Subgroups of the group of homeomorphisms of the Cantor space and a duality between a class of inverse monoids and a class of Hausdorff étale groupoids, *Journal of Algebra* **462** (2016), 77–114.
[16] M. V. Lawson, Tarski monoids: Matui’s spatial realization theorem, *Semigroup Forum* **95** (2017), 379–404.
[17] M. V. Lawson, Non-commutative Stone duality, arXiv:2207.02096
[18] M. V. Lawson, D. Lenz, Pseudogroups and their étale groupoids, *Adv. Math.* **244** (2013), 117–170.
[19] M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, *J. Pure Appl. Alg.* **221** (2017), 45–74.
[20] J. Leech, Inverse monoids with a natural semilattice ordering, *Proc. London Math. Soc.* (3) **70** (1995), 146–182.

\(^7\)The condition that \( S \) be a meet-monoid is missing from the statement of the proposition but should be added.
[21] W. Lu, Topics in many-valued and quantum algebraic logic, MSc Thesis, University of Ottawa, 2016. [https://ruor.uottawa.ca/handle/10393/35173](https://ruor.uottawa.ca/handle/10393/35173).

[22] W. Lu, P. J. Scott, Coordinatizing some concrete MV algebras and a decomposition theorem, Semigroup Forum 98 (2019), 213–233.

[23] M. E. Malandro, Fast Fourier transforms for finite inverse semigroups, J. Alg. 324 (2010), 282–312.

[24] H. Matui, Topological full groups of one-sided shifts of finite type, J. Reine Angew. Math 705 (2015), 35–84.

[25] M. Morris Mano, M. D. Ciletti, Digital design, Pearson, 2013.

[26] D. Mundici, Mapping abelian l-groups with strong unit one-one into MV algebras, J. Algebra 98 (1986), 76–81.

[27] D. Mundici, Interpretation of AF C∗-algebras in Lukasiewicz sentential calculus, J. Functional Analysis 65 (1986), 15–63.

[28] V. Nekrashevych, Simple groups of dynamical origin, Ergod. Th. & Dynam. Sys. 39 (2019), 707–732.

[29] M. Petrich, Inverse semigroups, John Wilewy & Sons, 1984.

[30] K. Ravindran, On a structure theory of effect algebras, PhD thesis, Kansas State University, 1996. [https://www.proquest.com/pqdtglobal/results/8F1886F1ECA44544PQ/1?accountid=14701](https://www.proquest.com/pqdtglobal/results/8F1886F1ECA44544PQ/1?accountid=14701).

[31] M. Rørdam, F. Larsen, N. J. Laustsen, An introduction to K-theory for C∗-algebras, CUP, 2000.

[32] M. H. Stone, Postulates for Boolean algebras and generalized Boolean algebras, Am. J. Math. 57 (1935), 703-732.

[33] F. Wehrung, Refinement monoids, equidecomposability types, and Boolean inverse semigroups, LNM 2188, 2017.

[34] F. Wehrung, Varieties of Boolean inverse semigroups, J. Alg. 511 (2018), 114–147.