Schwinger Mechanism with Stochastic Quantization

Kenji Fukushima\textsuperscript{1} and Tomoya Hayata\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
\textsuperscript{2}Theoretical Research Division, Nishina Center, RIKEN, Wako 351-0198, Japan

We prescribe a formulation of the particle production with real-time Stochastic Quantization. To construct the retarded and the time-ordered propagators we decompose the stochastic variables into positive- and negative-energy parts. In this way we demonstrate how to derive the Schwinger mechanism under a time-dependent electric field. We also discuss a physical interpretation with help of numerical simulations and develop an analogue to the one-dimensional scattering with the non-relativistic Schrödinger equation. We can then reformulate the Schwinger mechanism as the high-energy quantum reflection problem rather than tunneling.

PACS numbers: 11.15.Ha, 11.15.Tk, 12.38.Mh

Introduction

Direct simulations of the quantum field theory formulated on discretized space-time, that is, lattice simulations, have proved to be a powerful numerical tool to reveal non-perturbative aspects of the theory. It is, however, not always guaranteed that one can dig meaningful information out from the lattice calculations. Because the numerical algorithm relies on the importance sampling, the method ceases to work as soon as the integrand becomes negative (or complex in general). In gauge theories the most notorious example to hinder the lattice numerical approach is the "sign problem" associated with finite density of fundamental fermions \cite{1, 2} (for reviews, see Ref. \cite{3}). The sign problem is activated also when the theory has a Chern-Simons term that is associated with finite density of fundamental fermions \cite{4–7}.

In addition to these Euclidean examples one cannot avoid encountering the complex nature if one attacks the real-time problem, which originates from the path-integral weight, $e^{iS}$, in Minkowskian space-time. The real-time simulation is one of the most challenging topics in modern quantum field theories; transport coefficients of a fluid, particle emission rate in strongly correlated systems, and so on, are needed in various physics circumstances. One can still utilize the conventional lattice technique as long as the analytical continuation from Euclidean space-time is a legitimate procedure \cite{8–11}. The applicability of such approach is, however, limited to static or steady phenomena or linear-response perturbation at best. Full quantum simulations would demand an alternative quantization machinery in a different direction from the importance sampling. For a promising candidate, in this work, we will advocate the Stochastic Quantization \cite{12, 13} (for reviews, see Ref. \cite{14}) by taking a concrete example of real-time physics.

One of the most important and most ubiquitous phenomena that call for real-time quantization is the problem of the particle production from the vacuum. In the quantum field theory, in fact, the vacuum is not empty but is full of quanta, and some of them could tunnel the potential barrier out from the vacuum. Celebrated examples of such tunneling phenomena include the Schwinger mechanism that refers to the vacuum-insulation breakdown under external electric fields \cite{15, 16} (for a review, see Ref. \cite{17}), and the Hawking radiation that refers to the spontaneous radiation process from black holes, namely, the particle production under external gravitational fields \cite{18, 19}.

In this work we shall focus specifically on a theoretical reformulation of the Schwinger mechanism on the basis of the Stochastic Quantization. (There are many reformulations; for recent examples, see Refs. \cite{20–22}.) Because the Stochastic Quantization is a functional description in terms of classical fields, we must first establish a prescription to derive various kinds of propagators without dealing with creation/annihilation operators. In Refs. \cite{23–25} it has been shown that the inclusive spectrum is to be expressed in the following manner:

$$\frac{dN}{d^3p} = \frac{1}{(2\pi)^3} \sum_{p} \frac{1}{2E_{\text{out}}(p)} \lim_{\tau \to \infty} \left[ \partial_{\tau} + iE_{\text{out}}(p) \right] \left( \partial_{\tau} - iE_{\text{out}}(p) \right) \left\langle \hat{\rho}_{in} \hat{\phi}(t', p) \hat{\phi}(t, p) \right\rangle .$$

The initial density matrix is assumed to be $\hat{\rho}_{in} = |0_{in}\rangle \langle 0_{in}|$ throughout this work. We note that this two-point function (called the Wightman function) is nothing but $D_F(t, p, t', -p) - D_R(t, p, t', -p)$ where $D_F(t, p, t', p')$ and $D_R(t, p, t', p')$, respectively, represent the time-ordered and the retarded propagators. In the present work we limit ourselves to the simplest case of complex scalar field theory (or scalar QED) under an external electric field, but an extension to spinor matter is just straightforward.

Stochastic quantization

The idea of the Stochastic Quantization is that one can quantize field theories using a classical equation of motion with one artificial axis (i.e., quantum or Suzuki-Trotter axis \cite{26}) denoted here by $\theta$ and with stochastic variables $n(x, \theta)$. We thus need to solve a complex Langevin equation, which turns out to be accompanied by $i$ in Minkowskian space-time. Let us take a quick flash at the way to retrieve free propag-
We inserted the two-point function to reach the following form:
\[ \langle \eta_p(t, \theta) \eta_p(t', \theta') \rangle = 2 \delta(t-t')(2\pi)^3 \delta^{(3)}(p + p') \delta(\theta - \theta') + \langle \eta_p(t, \theta) \eta_p(t, \theta') \rangle_{\eta} = 0. \] (3)

When we solve Eq. (2), the most useful boundary condition is \( \phi_p(t, 0) = 0 \). One could have taken a non-zero value, but then one should supplement a proper subtraction in the end. We can easily find a formal solution of the complex Langevin equation given explicitly as
\[ \phi_p(t, \theta) = \int_0^\theta d\theta' e^{i[-\partial_t^2 - E^2(p)](\theta - \theta')} \eta_p(t, \theta'). \] (4)

We inserted \( \imath \epsilon \) to guarantee the convergence in the \( \theta \to \infty \) limit, which corresponds to the \( \imath \epsilon \) prescription to derive the time-ordered propagator.

After taking the average we can simplify the expression of the two-point function to reach the following form:
\[ \langle \phi_p(t, \theta) \phi_{p'}(t', \theta') \rangle = \frac{1}{-\partial_t^2 - E^2(p) + \imath \epsilon} \times \left[ 1 - e^{2i(-\partial_t^2 - E^2(p) + \imath \epsilon)\theta} (2\pi)^3 \delta^{(3)}(p + p') \delta(t - t') \right]. \] (5)

When we take the \( \theta \to \infty \) limit, the exponential oscillatory term drops off, and the resultant expression is reduced to the standard form of the time-ordered propagator, i.e., \( D_T(t, p; t', p') \).

It is a non-trivial question how to construct other types of the propagators. Since the creation and annihilation operators correspond to the negative- and the positive-energy parts of the field operator, it is then quite natural to decompose the stochastic variable as \( \eta_p(t, \theta) = \eta_p^+(t, \theta) + \eta_p^-(t, \theta) \) where
\[ \eta_p^+(t, \theta) = \int_{-\infty}^0 \frac{d\omega}{2\pi} \eta_p(\pm \omega, \theta) e^{\pm \imath \omega t}. \] (6)

Here \( \tilde{\eta}_p(\omega, \theta) \) represents the Fourier transform of \( \eta_p(t, \theta) \). We also do the same for \( \tilde{\eta}_p(t, \theta) \) and then \( \delta(t - t') \) in Eq. (3) is replaced with \( 2\pi \delta(\omega' - \omega) \) in the two-point function of \( \tilde{\eta}_p(\omega, \theta) \) and \( \tilde{\eta}_p(\omega', \theta') \). Accordingly we can introduce variants of Eq. (4), namely, \( \phi_p^\pm(t, \theta) \) using \( \eta_p^\pm(t, \theta) \). Also we define,
\[ \psi_p^\pm(t, \theta) = \int_0^\theta d\theta' e^{i(-\partial_t^2 - E^2(p) + \imath \epsilon)(\theta - \theta')} \eta_p^\pm(t, \theta') = \frac{1}{\imath} \frac{\partial \phi_p^\pm(t, \theta)}{\partial \theta} - \frac{\partial \psi_p^\pm(t, \theta)}{\partial \theta}. \] (7)

which solves a slightly deformed equation of motion with the sign of \( i \) flipped in Eq. (2). The time-ordered propagator involves only the components with \( \phi_p^\pm(t, \theta) \) and our main proposition here is to utilize \( \psi_p^\pm(t, \theta) \) as an additional building block of other types of the propagators:
\[ D_R(t, p; t', p') = \lim_{\theta \to \infty} \langle \phi_p^+(t, \theta) \phi_{p'}^+(t', \theta') - \phi_p^-(t, \theta) \phi_{p'}^-(t', \theta') \rangle_{\eta}. \] (8)

We can also write the advanced propagator down in the same way by means of an appropriate combination of \( \phi_p^\pm(t, \theta) \) and \( \psi_p^\pm(t, \theta) \). In view of Eq. (1), therefore, we can identify an expression directly relevant to the particle production as
\[ D_F(t, p; t', p') - D_R(t, p; t', p') = \lim_{\theta \to \infty} \langle \phi_p^-(t, \theta) \phi_{p'}^+(t', \theta') + \psi_p^-(t, \theta) \psi_{p'}^+(t', \theta') \rangle_{\eta}. \] (9)

We emphasize that, though our prescription may look a bit artificial at first glance, this is almost a unique choice so that the convergence factor \( \imath \epsilon \) has a right sign in the propagator as \( p_0^2 - E^2(p) \pm \imath \epsilon \) \( \eta_p^0(t, \theta) \), after taking the Fourier transform from \( t \) to \( t_0 \).

Time-dependent background field From now on we shall turn the time-dependent potential on, denoted by \( V_p(t) \), which yields a complex Langevin equation,
\[ \frac{\partial \phi_p^\pm(t, \theta)}{\partial \theta} = i(\phi_p^\pm(t, \theta) + V_p(t)) + \eta_p^\pm(t, \theta) \] (10)
and a similar one for \( \psi_p^\pm(t, \theta) \) with \( i \) in the right-hand side changed to \( -i \). As long as \( V_p(t) \) does not involve momentum transfer, the spatial derivatives are diagonalized in this partially Fourier transformed representation. In the in- and the out-states the interaction falls off, so that \( \langle V_{p}(t \sim t_{i}) \rangle = -E_{\text{out}}(p) \) and \( \langle V_{p}(t \sim t_{f}) \rangle = -E_{\text{out}}(p) \) are asymptotically satisfied. Let us demonstrate how our formulas (1) and (9) work for the estimate of the produced particle number.

We can easily solve (10) for general \( V_p(t) \) to find the explicit form of the solution as
\[ \phi_p^\pm(t, \theta) = \int_0^\theta d\theta' e^{i[-\partial_t^2 + V_p(t) + \imath \epsilon(\theta - \theta')]} \eta_p^\pm(t, \theta') \] (11)
and we can solve for \( \psi_p^\pm(t, \theta) \) as well. We now get ready to compute \( D_R(t, p; t', p') \) according to our prescription.
The final answer should not depend on how we treat the \( \eta \)-average as long as \( \eta_p^\pm(t, \theta) \)'s are generated consistently as the Gaussian noise (3).

Instead of taking the Gaussian average straightforwardly, we can simplify the calculation by means of \( \eta_p^\pm(t, \theta) \) decomposed with a complete set of the solutions of the following equation of motion,

\[
[-\partial_t^2 + V_p(t)] \chi_\omega^\pm(t) = [\omega^2 - E_{\text{in}}^2(p)] \chi_\omega^\pm(t),
\]  

(12)

where in the right-hand side, \( \omega \) or \( \omega^2 - E_{\text{in}}^2(p) \) is an eigenvalue to label the complete set, and the superscript \( \pm \) corresponds to the boundary condition,

\[
\chi_\omega^\pm(t \to t_1) \to e^{\mp i \omega t},
\]  

(13)

which is chosen for convenience to meet the boundary condition of Eq. (6) at \( t = t_1 \).

We note that, as long as \( V_p(t) \) is real, \( \chi^\pm_{\omega, \omega}(t) = \chi^\pm_{\omega}(t) \) follows. Then, we can deform the definition of positive- and negative-energy parts at \( t = t_1 \) using this complete set:

\[
\eta_p^\pm(t, \theta) = \int_0^\infty d\omega \left\{ \frac{\eta_p(\pm \omega, \theta)}{\omega^2 - E_{\text{in}}^2(p) + \text{sgn}(\omega) \varepsilon}\right\} \chi_\omega^\pm(t),
\]  

which coincides with Eq. (6) in the in-state. We would emphasize again that this parametrization is just for practical convenience and we could have kept using the definition of Eq. (6) to come up to the same answer; the difference is whether we should cope with the complicated \( t \)-dependent evolution operator in the exponential as in Eq. (11) or make it \( t \)-independent with the complicated wave-function \( \chi^\pm_{\omega}(t) \) which is reminiscent of a transition between the Schrödinger and the Heisenberg pictures in quantum mechanics.

With help of eigenfunctions of Eq. (12) we can readily derive the following form of the retarded propagator,

\[
D_R(t, p; t', p') = (2\pi)^3 \delta^{(3)}(p + p') \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i \chi_{\omega}^+(t) \chi_{\omega}^-(t')}{\omega^2 - E_{\text{in}}^2(p) + \text{sgn}(\omega) \varepsilon}. \]  

(15)

For the particle production problem we need to calculate \( D_F - D_R \) which reads,

\[
D_F(t, p; t', p') - D_R(t, p; t', p') = (2\pi)^3 \delta^{(3)}(p + p') \times \int_{-\infty}^{0} \frac{d\omega}{2\pi} (-i) 2\pi \delta(\omega^2 - E_{\text{in}}^2(p)^2) \cdot i \chi_{\omega}^+(t) \chi_{\omega}^-(t') \\
= (2\pi)^3 \delta^{(3)}(p + p') \frac{\chi^+_{\text{in}}(p)(t) \chi^-_{\text{in}}(p)(t')}{2E_{\text{in}}(p)}. \]  

(16)

We note that the delta function picks up an eigenvalue of \( \omega = -E_{\text{in}}(p) \) only that makes the right-hand side of Eq. (12) vanishing! Therefore, \( \chi^\pm_{\text{in}}(p)(t) \) satisfies the ordinary classical equation of motion without any \( \theta \)-dependence augmented.

![FIG. 1. Evolution of the averaged field variable \( \varphi_p(t, \theta) \) with increasing fictitious time \( \theta \) with a fixed boundary condition at \( t = t_1 = 0 \).](image-url)

With the initial condition (13) the solution of the equation of motion should behave like \( \chi^-_{\text{in}}(p)(t) = e^{-iE_{\text{in}}(p)t} \) around \( t \sim t_1 \) in the in-state and we can parametrize,

\[
\chi^-_{\text{in}}(p)(t) = \sqrt{\frac{E_{\text{in}}(p)}{E_{\text{out}}(p)}} \left[ \alpha_p e^{-iE_{\text{out}}(p)t} + \beta_p e^{iE_{\text{out}}(p)t} \right],
\]  

(17)

around \( t \sim t_F \) in the out-state. From these asymptotic forms it is easy to find the following expression in the out-state as

\[
D_F(t, p; t', p') - D_R(t, p; t', p') = (2\pi)^3 \delta^{(3)}(p + p') \times \frac{1}{2E_{\text{out}}(p)} \left\{ |\alpha_p|^2 e^{iE_{\text{out}}(p)(t-t')} + |\beta_p|^2 e^{-iE_{\text{out}}(p)(t-t')} \right. \\
\left. + 2\text{Re}[\alpha_p \beta_p e^{-iE_{\text{out}}(p)(t-t')} \right\}, \]  

(18)

which recovers the results in Ref. [25], and leads to the well-known result of the produced particle spectrum [27]:

\[
\frac{dN}{dp^3} = \delta^{(3)}(0) |\beta_p|^2. \]  

(19)

**Interpretation** Now that we have reached a known expression of the particle production, let us deepen a physical insight from the point of view of more technical aspect.

In the numerical approach, we would stress, taking the \( \eta \)-average corresponds to solving a classical equation of motion, i.e., Eq. (12). Actually, let us first fix the initial values of \( \phi(t = t_1) \) and \( \phi(t = t_1) \) and then perform the \( \eta \)-average (except for one at \( t = t_1 \) and \( t_1 + \Delta t \)). Interestingly enough, we can explicitly confirm that the \( t \)-oscillation pattern gradually emerges from the \( \theta \)-average. More specifically, the \( \theta \)-averaged field as defined by

\[
\varphi_p(t, \theta) = \theta^{-1} \int_0^\theta d\theta' \phi_p(t, \theta'),
\]  

(20)
approaches the solution of the equation of motion as clearly observed in Fig. 1 for a non-interacting case, which can be taken as a reconfirmation of the simulation in Ref. [13]. Physical quantities are all made dimensionless by the time step $\Delta t$ and the site number along the $t$-axis is chosen as $N_t = 256$. The $\theta$-axis is discretized with $\Delta \theta = 5 \times 10^{-3}$ (which means that we update the $\theta$-evolution $2 \times 10^5$ times to get the results at $\theta = 1000$).

We chose $E_{in}(p) = 12 \times (2\pi/N_t)$, so that there are 12 periods included along the $t$-direction from $t = 0$ to $N_t \Delta t$. We input $\epsilon = 5 \times 10^{-3}$ to stabilize the $\theta$-evolution, and this is why the oscillation is slightly damped in Fig. 1. On the technical level, it is the most tough part to avoid unphysical “run-away” flows in $\theta$, which is overcome here by the Crank-Nicolson method (which will be reported in a separate publication [28]).

It is important to note that we imposed an “initial condition” as $\varphi_0(t_1, \theta) = (1/\sqrt{2E_{in}(p)}) e^{iE_{in}(p)t_1}$, so that the theoretical curve expected from the classical equation of motion should be $(1/\sqrt{2E_{in}(p)}) e^{iE_{in}(p)(t-\tau)/(2E_{in}(p))t_1}$, and indeed the results at $\theta = 1000$ are almost on the theoretical curve (and we have checked that the results at later time, e.g. $\theta = 10000$, sit precisely on the curve). Therefore, the decomposition to positive- and negative-energy parts with $\eta_{\pm}^2(t, \theta)$ is effectively taken into account with the proper initial condition imposed at $t = t_1$.

We could have changed our “initial condition” and such a choice should correspond to the fluctuations associated with the remaining $\eta$-average at $t = t_1$ and $t_1 + \Delta t$. Then, at this point, the theoretical description is to be reduced to the so-called classical statistical simulation, in which the real-time dynamics is approximated by a combination of the initial fluctuations and the solution of the classical equation of motion [29–31]. In such a way, one can, in principle, derive and/or improve the classical statistical approximation systematically (e.g. the renormalization issue could be resolved [32]).

We are now applying the above-formulated Stochastic Quantization to the numerical simulation with general profile of the electric field. Here, instead of going further into the numerical simulation, we shall point out a suggestive mapping to the one-dimensional scattering problem with the non-relativistic Schrödinger equation. In some literature it is often phrased that the particle production is a suggestive mapping to the one-dimensional scattering problem with the non-relativistic Schrödinger equation.

In such a way in Ref. [19]. We here emphasize that we can view the Schwinger mechanism as a scattering process rather than a quantum tunneling in Stochastic Quantization.

The key equation is Eq. (12), which can be interpreted as the (fictitious) time-independent Schrödinger equation with one-to-one correspondence of

$$\theta \rightarrow -t , \quad t \rightarrow x .$$

(21)

So, in this picture of the Schrödinger equation, $\theta$ plays a role as time, and $t$ is one of spatial coordinates, which forms a sort of “holographic” Quantum Mechanics. We can solve the time or $\theta$-independent Schrödinger equation by the separation of variables as the solution taking a form of $e^{iE(t)\theta} \tilde{\chi}^2(t)$ with the eigenenergy, $E(\omega) = \omega^2 - E^2_0$. In this picture, the condition on the Bogoliubov coefficients, $|\alpha|^2 = 1 + |\beta|^2$, can be given a plain interpretation as the probability current conservation.

The calculation of Eq. (16) implies that only the contribution at $\omega = -E_{in}(p)$ survives from the pole structure, and this fact corresponds to a high-energy scattering in our interpretation of the Schrödinger equation. If the particle mass $m$ is much larger than the typical scale of the electric field $\sqrt{e/|E|}$, the potential term $V_p(t)$ is negative large $\sim -m$. Because the scattering at $E = 0$ is relevant, the energy should be as large as $m$ which hardly feels a small effect of the potential barrier from the electric field if $\sqrt{e/|E|} \ll m$, as summarized schematically in Fig. 2. This picture offers a novel and profound explanation of why $\beta_0$ is suppressed by the balance between $m$ and $\sqrt{e/|E|}$. For a quick example, if we have a step-function potential (corresponding to an instantaneous electric field) with a height difference by $\Delta E$, the scattering problem at the energy $E$ is immediately solved, and the reflection coefficient is found to be $\beta \propto \Delta E/E$ for small $\Delta E$, leading to the particle production proportional to $(\Delta E/E)^2$. This estimate from only little calculation perfectly agrees with the result from complicated arithmetics [23].

On the practical level, also, our picture as a scattering problem may open a useful possibility to optimize the time profile of the electric field along the similar line as discussed in Ref. [33, 34]. To enhance the effect of the Schwinger mechanism, we should consider $E(t)$ that amplifies $N_p$ as much as possible. One way to do this, in our picture, is to introduce assisting $t$-oscillation to reduce the gap of $m$ [33]. We could attack this optimization program in an intuitive manner by virtue of the mapping to the scattering problem; we should consider the quantum reflection of electrons and positrons at high energy.
so that the reflection rate can be maximized, in other words, we should seek for the “giant quantum reflection” as done in condensed matter systems [35]. Our analysis along this line is now in progress.

In summary, we gave a derivation of the Schwinger mechanism with Stochastic Quantization, in which the most non-trivial part was how to prescribe the retarded propagator. We decomposed the stochastic variables into positive- and negative-energy parts, and this corresponds to imposing proper initial conditions in the numerical simulation. We pointed out that, if formulated with Stochastic Quantization with an extra coordinate, the Schwinger mechanism can be characterized as the quantum reflection of electrons and positrons at high energy. Such a mapping to our quite familiar setup has potential applications that prove the usefulness soon.

We thank Ryoji Anzaki, Yoshimasa Hidaka, Takashi Oka, Shoichi Sasaki for fruitful discussions. K. F. was supported by JSPS KAKENHI Grant Number 24740169. T. H. was supported by JSPS Research Fellowships for Young Scientists.

[1] I. M. Barbour, S. E. Morrison, E. G. Klepfish, J. B. Kogut and M. -P. Lombardo, Nucl. Phys. Proc. Suppl. 60A, 220 (1998) [hep-lat/9705042].
[2] M. G. Alford, A. Kapustin and F. Wilczek, Phys. Rev. D 59, 054502 (1999) [hep-lat/9807039].
[3] S. Muroya, A. Nakamura, C. Nonaka and T. Takaishi, Prog. Theor. Phys. 110, 615 (2003) [hep-lat/0306031]; G. Aarts, PoS LATTICE 2012, 017 (2012) [arXiv:1302.3028 [hep-lat]].
[4] V. Azcoiti, G. Di Carlo, A. Galante and V. Laliena, Phys. Rev. Lett. 89, 141601 (2002) [hep-lat/0203017].
[5] S. Aoki, R. Horsley, T. Izubuchi, Y. Nakamura, D. Pleiter, P. E. L. Rakow, G. Schierholz and J. Zanotti, arXiv:0808.1428 [hep-lat].
[6] E. Vicari and H. Panagopoulos, Phys. Rept. 470, 93 (2009) [arXiv:0803.1593 [hep-th]].
[7] M. D’Elia and F. Negro, Phys. Rev. Lett. 109, 072001 (2012) [arXiv:1205.0538 [hep-lat]]; M. D’Elia and F. Negro, Phys. Rev. D 88, 034503 (2013) [arXiv:1306.2919 [hep-lat]].
[8] M. Asakawa, T. Hatsuda and Y. Nakahara, Prog. Part. Nucl. Phys. 46, 459 (2001) [hep-lat/0011040].
[9] F. Karsch, E. Laermann, P. Petreczky, S. Stickan and I. Wetzorke, Phys. Lett. B 530, 147 (2002) [hep-lat/0110208].
[10] A. Nakamura and S. Sakai, Phys. Rev. Lett. 94, 072305 (2005) [hep-lat/0406009].
[11] H. B. Meyer, Phys. Rev. D 76, 101701 (2007) [arXiv:0704.1801 [hep-lat]].
[12] G. Parisi and Y.-S. Wu, Sci. Sinical 24, 483 (1981).
[13] J. Berges and I. -O. Stamatescu, Phys. Rev. Lett. 95, 202003 (2005) [hep-lat/0508030]; J. Berges, S. Borsanyi, D. Sexty and I. -O. Stamatescu, Phys. Rev. D 75, 045007 (2007) [hep-lat/0609058].