PHASING SPACES OF MATROIDS

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Abstract. We study the space of phasing classes of phased matroids over a fixed underlying matroid $M$. We generalize methods of Gel’fand, Rybnikov and Stone and we obtain three different characterizations: via phased projective orientations (in terms of the circuits and cocircuits of $M$), as a certain subspace of the Pontrjagin dual of the inner Tutte group of $M$, and as a subspace of the torus $(S^1)^S$.

As applications we provide an effectively computable formula for the rank of the inner Tutte group, we give an upper bound on the number of reorientation classes of oriented matroids on $M$ and we prove the existence of non-realizable non-chirotopal uniform phased matroids in any rank.

Introduction

Abstract theories of linear dependencies have generated a very fertile branch of combinatorics ever since Whitney’s foundational paper on matroids [Whi35]. Specific applications have given rise to different specializations of matroid theory such as oriented matroids, which stem from a combinatorial analysis of sign patterns of linear dependency in real vector spaces and quickly found applications in many fields of mathematics [BLVS+99].

Phased matroids are abstract generalizations of linear dependencies in complex vector spaces, introduced by Laura Anderson and the first author in [AD12] as a contribution to the quest for a “complex” counterpart to oriented matroid theory. During the last 20 years, this line of research led to different constructions, including Ziegler’s complex matroids [Zie93]. Below, Krummeck and Richter-Gebert’s phirotopes [BKRG03] and Dress and Wenzel’s matroids with coefficients [DW91].

Phased matroids subsume the phirotopes of [BKRG03] and exhibit many desirable aspects of a matroidal theory of complex dependencies: cryptomorphisms (i.e. equivalent reformulations of the theory), a duality theory, and a natural $S^1$-action. This last point hints at the fact that phased matroids are not completely discrete objects: a phased matroid can be seen as a decoration of a (“underlying”) matroid by means of elements of the set $S^1 \cup \{0\}$ of “complex signs” (or “phases”, see Section 1 for precise definitions).

By design, every finite set of vectors in a vector space defines a matroid (and an oriented or phased matroid, if the space is real or complex). Structures arising this way are called realizable (see e.g. Remark 1.3) and it is important to notice that there are (oriented, phased) matroids which cannot be realized by a set of vectors. In fact, characterizing realizability is a major research area in matroid theory. Matroids that underlie oriented, resp. phased matroids are called orientable resp. phasable. It is a natural question to study the various “moduli spaces” arising in this context. Mniév’s celebrated universality theorem [Mni88] addresses the space of all realizations of a given oriented matroid. The space of realizations of a phased matroids is studied in recent work by Ruiz [Rui13].

This paper tackles another fundamental question: what is the space of all phased matroids on a given underlying matroid? In the case of oriented matroids the corresponding space is a discrete and finite set and has been studied by Gel’fand,
Rybnikov and Stone \cite{GRS95}, who gave four different characterizations of it (up to a canonical operation on oriented matroids called “reorientation” which corresponds, in the realizable case, to rescaling the vectors by nonzero numbers), each fitting a particular interpretation of matroid theory.

Our task is made more delicate by the fact that since the set of phases is not discrete, also the set of phased matroids over a given underlying matroid has a non-trivial topology. The first step is then to verify that the different (“cryptomorphic”) definitions of phased matroids given in \cite{AD12} indeed give rise to homeomorphic spaces of phased matroids over a given matroid $M$, so that the definition of the space $\mathcal{R}_C(M)$ as ‘the’ topological space of phasing classes of phased matroids over $M$ is well-defined. This nice compatibility with the natural topologies (and the fact that the sets of oriented matroids defined in \cite{GRS95} embed as discrete subspaces by ‘complexification’, see Appendix A) can be seen as a further confirmation of the naturality of the concept of phased matroids. Extending the definitions of Gel’fand, Rybnikov and Stone, we then proceed to give four models for $\mathcal{R}_C(M)$, where the bijections of finite sets given in \cite{GRS95} generalize to homeomorphisms between $\mathcal{R}_C(M)$ and:

- the space $\mathcal{P}_C(M)$ of projective phasings of $M$, defined in terms of circuits and cocircuits of $M$;
- a subspace $\mathcal{H}_C$ of the Pontrjagin dual of $\tau^0_M$ (the inner Tutte group of $M$, introduced by Dress and Wenzel in \cite{DWS99});
- the space $\mathcal{G}_C(M)$ of generalized $S^1$-cross-ratios, described geometrically as a subset of the torus $(S^1)^N$ by equations encoded in a matrix we call GRS matrix of $M$.

Eliminating redundant information leads to a reduced GRS matrix which is effectively computable with modest computing power and describes

- a space $\mathcal{G}_R^C(M)$ homeomorphic to $\mathcal{G}_C(M)$, which affords easier geometric considerations.

These spaces are indeed all homeomorphic, but they have different geometric and algebraic properties, which can be exploited to treat specific problems. As a token of the advantages of this approach, we give three applications.

- Considering the dimension of $\mathcal{H}_C(M)$, we recover a result of Brändén and Gonzalez D’León \cite{BGD10} on the rank of Tutte groups and, via the homeomorphism with $\mathcal{G}_R^C(M)$, give an effectively computable formula for it.
- We obtain an upper bound in terms of circuits of $M$ for the number of reorientation classes of oriented matroids with underlying matroid $M$.
- We prove that there exist nonrealizable, nonchirotopal phased matroids over every uniform matroid $U_d(m)$ for $m \geq 5$ and $2 \leq d \leq m - 2$, extending the rank-2 result of \cite{BKRG03}. This is done by using topological and analytical arguments in order to show that the subspace of chirotopal classes cannot exhaust $\mathcal{R}_C(M)$.

Overview. Section 1 contains some basic definitions and results on matroids and phased matroids. In Section 2 we construct the topological spaces of phased matroids and phasing classes of phased matroids with a fixed underlying matroid $M$ and prove that the cryptomorphisms of \cite{AD12} translate to homeomorphisms.

Then we introduce the spaces $\mathcal{P}_C(M)$ (Section 3), $\mathcal{H}_C(M)$ (Section 4), and $\mathcal{G}_C(M)$ (Section 5). In Section 6 we perform the reduction of the GRS-matrix of $M$ which leads to the space $\mathcal{G}_R^C(M)$, described in Section 7.

Section 8 is devoted to the applications. Here, as in some previous sections, for readability’s sake we outsource some of the technical computations to dedicated appendices which conclude the paper.
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1. Matroids and phased matroids

In this section we recall some basic definitions and results about matroids and phased matroids, in order to set the stage (and some notation) for the remainder of the text. For a thorough treatment of matroid theory we point to the book of Oxley [Oxl92], for phased matroids to [AD12].

Remark 1.1 (On oriented matroids). We will often refer, for the sake of comparison, to oriented matroids. These references will be stated in separate remarks and can be safely skipped without harm for the comprehension of the core of our work. For this reason, we do not treat oriented matroids in this introduction and refer the interested reader to [BLVS+99] for a thorough introduction to the subject.

1.1. Matroids. A matroid $M$ can be defined as a pair $(E, \mathcal{I})$, where $E$ is a finite ground set and $\mathcal{I} \subseteq 2^E$ is a collection of subsets of $E$ satisfying the following three conditions:

(i1) $\emptyset \in \mathcal{I}$;

(i2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;

(i3) If $I$ and $J$ are in $\mathcal{I}$ and $|I| < |J|$, then there is an element $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

The members of $\mathcal{I}$ are the independent sets of $M$. Inclusion-maximal independent sets are called bases, and the set of bases of $M$ will be denoted by $\mathbf{B}$. A subset of $E$ that is not in $\mathcal{I}$ is called dependent. Minimal inclusion dependent sets are called circuits and the family of circuits of $M$ will be denoted by $\mathcal{C}$. We will write $\mathcal{I}(M)$, $\mathcal{C}(M)$, $\mathbf{B}(M)$ if specification of the matroid under consideration is called for.
The rank of a subset $S \subseteq E$ is defined by
$$\text{rk}(S) = \max \{|T \cap B| \mid B \in \mathcal{B}\}$$
and we define the rank of the matroid $M$ as $\text{rk}(M) := \text{rk}(E)$. A subset $S$ of $E$ is spanning if $\text{rk}S = \text{rk}M$.

**Remark 1.2 (Cryptomorphisms).** Our definition in terms of independent sets can be replaced by a set of requirements for any of the set systems described by an italicized word above. In fact, this availability of different reformulations is a distinctive feature of matroid theory, and the rules allowing to switch between these are called “cryptomorphisms” in the field’s jargon.

The family of complements of spanning sets of a matroid $M$ does satisfy conditions (I1), (I2) and (I3) above, thus is the set of independent sets of a matroid $M^*$ called dual to $M$. The rank function $\text{rk}^*$ of $M^*$ is linked to that of $M$ by
$$\text{rk}^*(A) = \text{rk}(E \setminus A) + |A| - \text{rk}(E).$$
Circuits and bases of $M^*$ are called cocircuits and cobases of $M$. We write $\mathcal{C}^*$ and $\mathcal{B}^*$ for the families of cocircuits and cobases of $M$.

**Remark 1.3.** A rank $d$ matroid $M$ with ground set $E = \{1, \ldots, m\}$ is called realizable over the field $\mathbb{F}$ if there exists a matrix $A \in M_{d,m}(\mathbb{F})$ of $d$ rows and $m$ columns with $\mathbb{F}$-coefficients and such that
$$\{S \subseteq E \mid \{A_i\}_{i \in S} \text{ is a linearly independent set}\}$$
is the collection of independent sets of $M$. Here, $A_i$ denotes the $i$-th column of $A$. We say that the matrix $A$ realizes the matroid $M$.

A matroid $M$ is binary if it is realizable over the field $\mathbb{F}_2$.

**Example 1 (The Fano matroid).** The Fano matroid is defined on the ground set $\{1, 2, \ldots, 7\}$ by the circuit set $\{\{1, 2, 3\}, \{4, 5, 6\}, \{1, 4, 7\}, \{1, 5, 6\}, \{3, 4, 5\}, \{3, 6, 7\}, \{2, 4, 6\}\}$. It is representable over $\mathbb{F}$ if and only if the characteristic of $\mathbb{F}$ is two [Oxl92, Proposition 6.4.8].

Given a subset $T$ of the ground set $E$ of the matroid $M$, the collection of all subsets of $T$ that are independent in $M$ is easily seen to satisfy (I1)-(I3), thus is the set of independent sets of a matroid called restriction of $M$ to $T$ and denoted by $M[T]$. The contraction of $T$ in $M$ is the matroid $(M^*[E \setminus T])^*$. A minor of $M$ is any matroid that can be obtained from $M$ through a sequence of restrictions and contractions.

**Remark 1.4.** We will often consider matroids “without minors of Fano or dual-Fano type”. By this we mean matroids for which neither the Fano matroid (see Example 4) nor its dual can arise as minors.

### 1.2. Phased matroids

The phase $\text{ph}(x)$ of a complex number $x \in \mathbb{C}$ is defined to be $0$ if $x = 0$ and $\frac{1}{\pi} \arg x$ otherwise. A phased vector is any $X \in (S^1 \cup \{0\})^E$ where, throughout this paper, $E := \{1, \ldots, m\}$ will be a finite ground set. To every complex vector $v \in \mathbb{C}^E$ we associate a phased vector $\text{ph}(v)$ defined componentwise by $\text{ph}(v)_e := \text{ph}(v_e)$ for each $e \in E$.

The support of a phased vector $X$ is the set
$$\text{supp}(X) := \{e \in E \mid X(e) \neq 0\}.$$ 
Given a finite set $S \subseteq (S^1 \cup \{0\})$ the phase convex hull of $S$ is the set $\text{pconv}(S)$ all phases of (real) strictly positive linear combinations of elements of $S$. We set
circuits of a phased matroid if and only if it satisfies

and let $A$ be a maximal phased vector. Then, $A$ is defined by such a phirotope is called realizable.

**Example 1.5** (Phirotopes). A function $\varphi : E^d \to (S^1 \cup \{0\})$ is called a rank $d$ phirotope if

1. $\varphi$ is non-zero;
2. $\varphi$ is alternating;
3. For any two subsets $\{x_1, \ldots, x_{d+1}\}$ and $\{y_1, \ldots, y_{d-1}\}$ of $E$,

$$0 \in \text{pconv}(\{(−1)^k \varphi(x_1, x_2, \ldots, \hat{x}_k, \ldots, x_{d+1}) \varphi(y_k, y_1, \ldots, y_{d-1})\}).$$

We say that $\varphi_1, \varphi_2$ are equivalent if $\varphi_1 = a\varphi_2$ for some $a \in S^1$.

Notice that these axioms ensure that the support of any phirotope $\varphi$ is the set of bases of a matroid on $E$ which we call $M_\varphi$. We then call a subset $I \subseteq E$ $\varphi$-independent if it is an independent set of the matroid $M_\varphi$. Accordingly, a $\varphi$-basis is a maximal $\varphi$-independent set.

**Example 2.** Let $E = \{1, \ldots, m\}$ be a finite ground set with $m \geq 1$. Let $1 \leq d \leq m$ and let $A \in M_{d,m}(\mathbb{C})$ be a rank $d$ matrix. The function $\varphi_A : E^d \to (S^1 \cup \{0\})$ defined by

$$\begin{equation}
(i_1, \ldots, i_d) \mapsto \text{ph}(\det(A^{i_1}, \ldots, A^{i_d})),
\end{equation}$$

where $A^j$ denotes the $j$-th column of $A$, is a rank $d$ phirotope. The phased matroid defined by such a phirotope is called realizable.

**Definition 1.6** (Phased circuits). A set $\mathcal{C} \subseteq (S^1 \cup \{0\})^E$ is the set of phased circuits of a phased matroid if and only if it satisfies

(C0) $\{0, \ldots, 0\} \notin \mathcal{C}$;
(C1) For all $X \in \mathcal{C}$ and all $\alpha \in S^1$, $\alpha X \in \mathcal{C}$;
(C2) For all $X, Y \in \mathcal{C}$ such that $\text{supp}(X) = \text{supp}(Y)$, $X = \alpha Y$ for some $\alpha \in S^1$;
(ME) For all $X, Y \in \mathcal{C}$ such that $\text{supp}(X), \text{supp}(Y)$ is a modular pair and all $e, f \in E$ with $X(e) = Y(e) \neq 0$ and $X(f) \neq Y(f)$, there is $Z \in \mathcal{C}$ with
(a) $f \in \text{supp}(Z) \subseteq (\text{supp}(X) \cup \text{supp}(Y)) \setminus \{e\};$
(b) $Z(g) \in \text{pconv}(\{X(g), Y(g)\})$ if $g \in \text{supp}(X) \cap \text{supp}(Y),$
$Z(g) \leq \max\{X(g), Y(g)\}$ else.
If $C$ is the set of phased circuits of a phased matroid, the set $\{\text{supp}(X) \mid X \in C\}$ is the set of circuits of a matroid $M_C$. The following theorem asserts that Definition 1.5 and Definition 1.6 encode equivalent data.

**Theorem 1.7** (Theorem A of [AD12]). There exists a bijection between the set of all equivalence classes of phirotopes on a set $E$ and the set of all sets of phased circuits of complex matroids on $E$, determined as follows. For a phirotope $\varphi$ and the corresponding set $C$ of phased circuits,

1. The set of all supports of elements of $C$ is the set of minimal nonempty $\varphi$-dependent sets;
2. The phases of $X \in C$ are determined by the rule
   \[ \frac{X(x_i)}{X(x_0)} = (-1)^i \frac{\varphi(x_0, x_1, \ldots, \hat{x}_k, \ldots, x_d)}{\varphi(x_1, \ldots, x_d)} \]
   for all $i = 0, \ldots, k$ where $x_0 \in \text{supp}(X)$ and $\{x_1, \ldots, x_d\}$ is any $\varphi$-basis containing $\text{supp}(X) \setminus \{x_0\}$.

Thus, we can refer to the phased matroid $M$ with phirotope $\varphi$ and phased circuit set $C$. In particular, in this case $M_\varphi = M_C$. We call this matroid the underlying matroid of $M$.

**Definition 1.8** (Circuit phasings, dual pairs; Definition 2.15 of [AD12]). Let $M$ be a matroid with ground set $E$. We say $C \subseteq (S^1 \cup \{0\})^E$ is a circuit phasing of $M$ if

1. (S1) for all $X \in C$ and all $\alpha \in S^1$, $\alpha X \in C$;
2. (S2) for all $X, Y \in C$ with $\text{supp}(X) = \text{supp}(Y)$, $X = \alpha Y$ for some $\alpha \in S^1$;
3. (S3) the set $\{\text{supp}(X) \mid X \in C\}$ is the set of circuits of $M$.

We say $D \subseteq (S^1 \cup \{0\})^E$ is a cocircuit phasing of $M$ if $D$ is a circuit phasing of $M^*$, the dual matroid to $M$. We call $C, D$ a dual pair of circuit phasings of $M$ if, moreover,

$C \perp D$.

**Theorem 1.9** (Theorem C, [AD12]). Let $C$ be a complex circuit phasing and $D$ be a complex cocircuit phasing of a matroid $M$. Then $C$ and $D$ are the set of phased circuits and cocircuits of a phased matroid with underlying matroid $M$ if and only if they are a dual pair.

**Remark 1.10.** It is an easy exercise to verify that the Fano matroid defined in Example 1 is not the underlying matroid of any phased matroid, e.g., by checking that it cannot support a dual pair of complex circuit/cocircuit phasings. Clearly, this implies that the dual of the Fano matroid is also not phasable.

## 2. Spaces of phased matroids

Let $M$ denote a matroid of rank $d$ on a ground set $E$. We want to study the set $\mathcal{M}_C(M)$ of phased matroids with a given underlying matroid $M$ and, paralleling [GRS95], also the space $\mathcal{R}_C(M)$ of “phasing classes” of phased matroids.

In comparison to the oriented matroid case, our task is made more delicate by the fact that in the phase case this sets have a nontrivial topology. Of the set of all given underlying matroid on the ground set $M$ has a natural topology as a subspace of $(S^1 \cup \{0\})^{E^d}$, the natural topology on the set of all sets of phased circuits of phased matroids on $M$ is less transparent.

In this section we will start with the more immediate case - phased matroids defined in terms of phirotopes -, and then offer a reformulation of the axiomatization in terms of phased circuits that is more convenient for topological considerations. The crux will be the proof that the cryptomorphism of Theorem 1.7 induces a
homeomorphism between the spaces of phased matroids on \( M \) defined in terms of phirotopes and, respectively, in terms of phased circuits, which in turn determines a homeomorphism between the respective spaces of phasing classes. Thus, in the following sections will be justified in studying “the” spaces \( \mathcal{M}_\mathcal{C}(M) \) and \( \mathcal{R}_\mathcal{C}(M) \).

### 2.1. Spaces of phirotopes.

The set of phirotopes with underlying matroid \( M \) will be denoted by \( \mathcal{N}_\mathcal{C}(M) \). This is naturally a topological subspace of

\[
\mathcal{N}_\mathcal{C}(M) \subseteq (S^1 \cup \{0\})^{E^2}.
\]

If we call \( \sim_p \) the equivalence relation among phirotopes introduced in Definition 2.1, we can state the following definition.

**Definition 2.1.** A (type \( p \)) phased matroid with underlying matroid \( M \) is an equivalence class of the relation \( \sim_p \) on \( \mathcal{N}_\mathcal{C}(M) \). The set of (type \( p \)) phased matroids with underlying matroid \( M \) is then \( \mathcal{M}_\mathcal{C}(M) := \mathcal{N}_\mathcal{C}(M)/ \sim_p \) with the quotient topology.

We now proceed to define the space of phasing classes of phased matroids in terms of phirotopes.

**Definition 2.2.** Two phirotopes \( \varphi_1, \varphi_2 \in \mathcal{N}_\mathcal{C}(M) \) are \( \sim_p \)-equivalent if there is a function \( h : E \to S^1 \) such that, for all \( (x_1, \ldots, x_d) \in E^d \),

\[
\varphi_1(x_1, \ldots, x_d) = \left( \prod_{j=1}^d h(x_j) \right) \varphi_2(x_1, \ldots, x_d).
\]

In this case we will write \( \varphi_1 = h \varphi_2 \).

A straightforward computation shows that \( \sim_p \) is an equivalence relation between phirotopes. Moreover, clearly \( \approx_p \) is coarser than \( \sim_p \), since whenever \( \varphi_1 = a \varphi_2 \) for some \( a \in S^1 \), then \( \varphi_1 = h \varphi_2 \) for \( h \) the constant function equal to \( a^{1/p} \).

**Definition 2.3.** Two (type \( p \)) phased matroids \( \Phi_1 \) and \( \Phi_2 \) with underlying matroid \( M \) are \( \cong_p \)-equivalent (denoted by \( \Phi_1 \cong_p \Phi_2 \)) if there exist phirotopes \( \varphi_1 \in \Phi_1 \) and \( \varphi_2 \in \Phi_2 \) such that \( \varphi_1 \cong_p \varphi_2 \).

**Remark 2.4.** In the previous notations, if \( \Phi_1 \cong_p \Phi_2 \), then \( \varphi'_1 \cong_p \varphi'_2 \) for all \( \varphi'_1 \in \Phi_1 \) and \( \varphi'_2 \in \Phi_2 \). Indeed, such phirotopes are of the form, say, \( \varphi'_1 = a_1 \varphi_1 \) and \( \varphi'_2 = a_2 \varphi_2 \) for some \( a_1, a_2 \in S^1 \), then \( \varphi'_1 \cong h \varphi'_2 \) if and only if \( \varphi'_1 = h \varphi'_2 \) for \( h = (a_1/a_2)^{1/d} \), where \( d \) is the rank of \( M \).

We can now define the quotient set of equivalence classes of \( \cong_p \).

**Definition 2.5.** Let \( M \) be a matroid. We define the space of phasing classes of (type \( p \)) phased matroids as the set

\[
\mathcal{R}_\mathcal{C}(M) := \mathcal{M}_\mathcal{C}(M)/\cong_p
\]

of \( \cong_p \) equivalence classes, endowed with the quotient topology.

### 2.2. Spaces of circuit signatures.

An \( S^1 \)-signature \( \gamma \) of the circuits of a matroid \( M \) is a set \( \{ \gamma C \}_{C \in \mathcal{C}(M)} \) of functions \( \gamma C : C \to S^1 \), one for each circuit of \( M \). Analogously, an \( S^1 \)-signature \( \delta \) of the cocircuits of \( M \) is a set of functions \( \delta D : D \to S^1 \), one for each cocircuit of \( M \). We say that an \( S^1 \) circuit signature \( \gamma \) and an \( S^1 \) cocircuit signature \( \delta \) are orthogonal (denoted by \( \gamma \perp \delta \)) if, for any circuit \( C \in \mathcal{C} \) and cocircuit \( D \in \mathcal{C}^* \) with non-empty intersection, i.e., with \( C \cap D \neq \emptyset \), we have

\[
0 \in \text{pconv} \left\{ \frac{\gamma C(x)}{\delta D(x)} \middle| x \in C \cap D \right\}.
\]
Definition 2.6. We denote by

\[ N_c^M(\mathcal{E}) \subseteq (S^1)^{\sum_{c \in \mathcal{E}}} \]

the set of \( S^1 \) circuit signatures of \( M \) such that there exists an \( S^1 \) cocircuit signature \( \delta \) of \( M \) satisfying \( \gamma \perp \delta \), with the subspace topology.

Remark 2.7. If the matroid \( M \) has no circuits, we have \( N_c^M(\mathcal{E}) = (S^1)^0 \), a single point, corresponding to the empty signature (which is orthogonal to every cocircuit signature).

Definition 2.8. Two \( S^1 \) circuit signatures \( \gamma_1 \) and \( \gamma_2 \) of a matroid \( M \) are called \( \sim_c \) equivalent (denoted by \( \gamma_1 \sim_c \gamma_2 \)) if there exists a function \( b : \mathcal{E}(M) \to S^1 \), \( C \mapsto b_C \), such that for every \( C \in \mathcal{E}(M) \) and every \( x \in C \)

\[ \gamma_1 C(x) = b_C \gamma_2 C(x). \]

One readily verifies that \( \sim_c \) is an equivalence relation. Moreover, whenever an \( S^1 \) cocircuit signature of \( M \) lies in \( N_c^M(\mathcal{E}) \), obviously so does its whole equivalence class.

A \( S^1 \) cocircuit signature \( \gamma \) of \( M \) gives rise to a circuit phasing

\[ \Psi(\gamma) := \{ a \gamma C \mid C \in \mathcal{E}, \ a \in S^1 \} \subseteq (S^1 \cup \{0\})^\mathcal{E}. \]

Therefore, in view of Theorem [L9], \( N_c^M(\mathcal{E}) \) is the set of all \( S^1 \) circuit signatures \( \gamma \) of \( M \) such that \( \Psi(\gamma) \) is the set of phased circuits of a phased matroid.

A straightforward - albeit laborious - check shows that indeed \( \Psi \) induces a bijection from \( N_c^M(\mathcal{E}) \) to the sets of phased circuits of phased matroids. We are thus led to the following definition.

Definition 2.9 (Compare Definition [L8]). A (type \( c \)) phased matroid with underlying matroid \( M \) is an equivalence class \( \Gamma \) of the relation \( \sim_c \) restricted to the set \( N_c^M(\mathcal{E}) \).

The set of (type \( c \)) phased matroids with underlying matroid \( M \) is

\[ \mathcal{M}_c^M := N_c^M(\mathcal{E})/\sim_c, \]

endowed with the quotient topology.

We now define an equivalence relation on the set of (type \( c \)) phased matroids with underlying matroid \( M \), in order to obtain the counterpart of the set of phasing classes of phirotopes on \( M \).

Definition 2.10. Two \( S^1 \) circuit signatures \( \gamma_1 \) and \( \gamma_2 \) of a matroid \( M \) are called \( \approx_c \) equivalent (denoted \( \gamma_1 \approx_c \gamma_2 \)) if there exists a function \( h : \mathcal{E} \to S^1 \) such that for \( C \in \mathcal{E} \) and \( x \in C \)

\[ \gamma_1 C(x) = h(x) \gamma_2 C(x). \]

We then write \( \gamma_1 = h \gamma_2 \).

It is easy to check that \( \approx_c \) is an equivalence relation which restricts to an equivalence relation on \( \mathcal{M}_c^M \).

Definition 2.11. Two phased matroids \( \Gamma_1, \Gamma_2 \in \mathcal{M}_c^M(M) \) are \( \approx_c \) equivalent (denoted by \( \Gamma_1 \approx_c \Gamma_2 \)) if there are \( \gamma_1 \in \Gamma_1 \) and \( \gamma_2 \in \Gamma_2 \) such that \( \gamma_1 \approx_c \gamma_2 \).

Again, \( \approx_c \) is obviously an equivalence relation.

Remark 2.12. We would like to point out a difference with the case of phirotopes. Suppose that \( \gamma_1 \approx_c \gamma_2 \), say \( \gamma_1 = h \gamma_2 \), and consider a representative \( \gamma'_i \) of the class of \( \gamma_i \), say \( \gamma'_i = b_i \gamma_i' \) where \( b_i \in (S^1)^\mathcal{E} \), for \( i = 1, 2 \). Then it might not be true that \( \gamma'_1 \approx_c \gamma'_2 \), however there will be \( \gamma'_2 := (b_1/b_2) \gamma'_2 \sim_c \gamma'_2 \) with \( \gamma'_1 \approx_c \gamma'_2 \).

As previously done, we conclude this section with the following:
Definition 2.13. Let $M = (E, \mathcal{X})$ be a matroid. The space of (type $c$) phasing classes of (type $c$) phased matroids over $M$ is

$$
\mathcal{R}_c^c(M) := \mathcal{M}_c^c(M)/\sim_c,
$$

which we endow with the quotient topology.

2.3. From cryptomorphism to homeomorphism. The aim of this section is to show that the cryptomorphisms of $[AD12]$ induce a homeomorphism between the topological spaces $\mathcal{M}_c^c(M)$ and $\mathcal{M}_p^p(M)$, as well as between $\mathcal{R}_c^c(M)$ and $\mathcal{R}_c^p(M)$. More precisely, we prove the following.

Proposition 2.14. Given a matroid $M$, there exists a homeomorphism

$$
\Upsilon : \mathcal{M}_c^c(M) \rightarrow \mathcal{M}_p^p(M),
$$

which induces a homeomorphism

$$
\Upsilon : \mathcal{R}_c^c(M) \rightarrow \mathcal{R}_c^p(M).
$$

such that

$$
\begin{array}{ccc}
\mathcal{M}_c^c(M) & \xrightarrow{\Upsilon} & \mathcal{M}_p^p(M) \\
\sim_c & & \sim_p \\
\mathcal{R}_c^c(M) & \xrightarrow{\Upsilon} & \mathcal{R}_c^p(M)
\end{array}
$$

(4)

is a commutative diagram of continuous functions.

The previous result enables us to freely switch between the point of view of phirotopes and that of circuits as long as the topology is concerned.

Definition 2.15. The space $\mathcal{M}_c(M)$ of phased matroids with underlying matroid $M$ is the topological space $\mathcal{M}_c(M) = \mathcal{M}_c^c(M) = \mathcal{M}_p^p(M)$ and is well defined up to homeomorphism.

The space $\mathcal{R}_c(M)$ of phasing classes of phased matroids with underlying matroid $M$ is the topological quotient $\mathcal{R}_c(M) = \mathcal{R}_c^c(M) = \mathcal{R}_c^p(M)$ and is well defined up to homeomorphism.

As a token of the fact that our construction naturally generalizes the case of oriented matroids as presented in [GRS95], let $\mathcal{M}_R(M)$ be the (finite) set of oriented matroids with underlying matroid $M$, let $\mathcal{R}_R(M)$ denote the (finite) set of reorientation classes of oriented matroids with underlying matroid $M$. There exist natural set injections

$$
i : \mathcal{M}_R(M) \hookrightarrow \mathcal{M}_c(M)
$$

and

$$
j : \mathcal{R}_R(M) \hookrightarrow \mathcal{R}_c(M)
$$

given by “complexification”, see Appendix A. Thus, identifying $\mathcal{M}_R(M)$ with $i(\mathcal{M}_R(M))$ and $\mathcal{R}_R(M)$ with $j(\mathcal{R}_R(M))$, we can endow $\mathcal{M}_R(M)$ and $\mathcal{R}_R(M)$ with the subspace topology of $\mathcal{M}_c(M)$ and $\mathcal{R}_c(M)$, respectively.

Proof of Proposition 2.14. Choose an ordered basis $B$ of $M$, and consider the function that assigns to any $\gamma \in \mathcal{N}_c^c(M)$ a phirotope $\Upsilon(\gamma)$ with value 1 on $B$ and satisfies equations

$$
\gamma C(x_d) = (-1)^i \frac{\Upsilon(\gamma)(x_0, \ldots, \hat{x}_i, \ldots, x_d)}{\Upsilon(\gamma)(x_1, \ldots, x_d)}
$$

(7)
for all circuits $C$ of $M$, all $x_0 \in C$ and all $i = 0, \ldots, d$, where $x_1, \ldots, x_d$ is any ordered basis containing $C \setminus x_0$. Theorem 2.7 shows that this gives a well-defined function $N^\mathbb{C}(M) \to N^\mathbb{R}(M)$ which induces a bijection

$$\Upsilon : \mathcal{M}_\mathbb{C}(M) \to \mathcal{M}_\mathbb{R}(M).$$

**Step 1: commutativity of diagram (4).** We have to check the equivalence of

$$\Gamma_1 \cong \Upsilon, \Gamma_2, \text{ and } \Upsilon(\Gamma_1) \cong \Upsilon(\Gamma_2).$$

For the left-to-right direction assume $\Gamma_1 \cong \Upsilon, \Gamma_2$. Then we may choose representatives $\gamma_1$ and $\gamma_2$ with $\gamma_1 \cong \gamma_2$ - say, $\gamma_1 = h_2 \gamma_2$ for some $h : E \to S^1$. Choose a representative $\varphi_2 \in \Upsilon(\Gamma_2)$. It is enough to prove that $h \varphi_2 \in \Upsilon(\Gamma_1)$, which amounts to checking that Equation (4) is satisfied with $\gamma_1$ on the left hand side and $h \varphi_2$ on the right hand side: a straightforward check of the definitions which we leave to the reader.

Now to the right-to-left implication. Here we can choose $\varphi_1 \in \Upsilon(\Gamma_1)$ and $\varphi_2 \in \Upsilon(\Gamma_2)$ with $\varphi_1 = h_2 \varphi_2$ for some $h : E \to S^1$ we must prove that $h \gamma_2 = b \gamma_1$ for some $b \in E^\mathbb{C}$. Choose, for every $C \in \mathcal{C}(M)$, an element $x_C \in C$. A check of the definitions shows that indeed for any $x \in C$

$$\frac{h \gamma_2 C(x)}{h \gamma_2 C(x_C)} = \frac{\gamma_1 C(x)}{\gamma_1 C(x_C)}$$

and thus $h \gamma_2 = b \gamma_1$ for $b : E \to S^1$ defined by $b_C = h \gamma_2(x_C)/\gamma_1(x_C)$.

**Step 2: continuity** We have left to show that (4) is indeed a commutative diagram of continuous functions.

Recall the definition of the elements $\{x_C\}_{C \in \mathcal{C}}$ from earlier in the proof. To every phirotope $\varphi \in N^\mathbb{R}_C(M)$ we assign a $S^1$ circuit signature $\xi(\varphi) \in N^\mathbb{R}_C(M)$ which satisfies

$$\xi(\varphi)C(x_i) \xi(\varphi)C(x_0) = (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_d)$$

(for all circuits $C$ of $M$, all $x_0 \in C$ and all $i = 0, \ldots, d$, $x_1, \ldots, x_d$ is any ordered basis containing $C \setminus x_0$), as well as $\xi(\varphi)C(x_C) = 1$ for every $C \in \mathcal{C}$. With [AD12, Section 4.1] and [AD12, Proposition 4.6] we have that this assignment induces the inverse map to $\Upsilon$. Now by the definition of their representatives (e.g., Equations (4) and (8)) we see that $\Upsilon$ and its inverse are continuous functions, and by commutativity of (4) they induce continuous maps in the quotient. \hfill \Box

3. Projective phasings

In this section we begin the study of the topological space $\mathcal{R}_C(M)$ of phasing classes of matroids with underlying matroid $M$ (see Definition 2.15). Our first characterization is combinatorial: we define projective phasings of $M$ in terms of its circuits and cocircuits and we prove that there exists a homeomorphism between $\mathcal{R}_C(M)$ and the topological space $\mathcal{P}_C(M)$ of projective phasings of $M$.

3.1. Definitions. We start with some basic constructions and definitions, generalizing [QRS95] to the phased context.

Given a matroid $M$, let us define

$$Q_M := \left\{ (C, D, x, y) \in \mathcal{C} \times \mathcal{C}^* \times E \times E \mid C \cap D \neq \emptyset, x, y \in C \cap D \right\}$$

and write $d_M := |Q_M|$. 


Definition 3.1. A projective phasing of a matroid $M$ is a function

$$P_C : Q_M \rightarrow S^1$$

whose values, denoted by $\begin{pmatrix} C \ D \ x \ y \end{pmatrix}$ as a shorthand for $P_C(C, D, x, y)$, satisfy conditions

(9) $\begin{pmatrix} C \ D \ x \ x \end{pmatrix} = 1,$

(10) $\begin{pmatrix} C \ D \ x \ y \end{pmatrix} \begin{pmatrix} C \ D \ y \ z \end{pmatrix} \begin{pmatrix} C \ D \ z \ x \end{pmatrix} = 1,$

(11) $\begin{pmatrix} C_1 \ D_1 \ x \ y \end{pmatrix} \begin{pmatrix} C_2 \ D_2 \ x \ y \end{pmatrix} = \begin{pmatrix} C_1 \ D_2 \ x \ y \end{pmatrix} \begin{pmatrix} C_2 \ D_1 \ x \ y \end{pmatrix}$

for all possible arguments, as well as

(12) $\forall y \in C \cap D, 0 \in pconv \left( \begin{pmatrix} C \ D \ x \ y \end{pmatrix} \left| x \in C \cap D \right. \right)$

for all $C \in \mathcal{C}, D \in \mathcal{C}^*.$

The set of projectivephasings of $M$ will be denoted by $\mathcal{P}_C(M)$, and we will regard this as a topological subspace $\mathcal{P}_C(M) \subseteq (S^1)^{dm}.$

Remark 3.2. When writing $\begin{pmatrix} C \ D \ x \ y \end{pmatrix}$, we will always assume that $(C, D, x, y) \in Q_M.$

Remark 3.3 (On the oriented case). This construction is an extension to the phased context of the well known oriented case. If $\mathcal{P}_R(M)$ denotes the set of projective orientations of $M$ as defined in [GRS95], we have $\mathcal{P}_R(M) = \mathcal{P}_C(M) \cap (S^1)^{dm}$, so that there exists a natural inclusion

(13) $\mathcal{P}_R(M) \subseteq \mathcal{P}_C(M).$

Since $\mathcal{P}_C(M)$ is a Hausdorff topological space, it induces the discrete topology on the finite subset $\mathcal{P}_R(M)$.

We now state for later reference some properties of projective phasings which follow easily from the definition.

Proposition 3.4. Let $P_C$ be a projective phasing of $M$. Then,

(14) $\begin{pmatrix} C \ D \ x \ y \end{pmatrix} \begin{pmatrix} C \ D \ x \ x \end{pmatrix} = 1.$

Moreover, for every circuit $C \in \mathcal{C}(M)$ and every cocircuit $D \in \mathcal{C}^*(M)$ such that $C \cap D = \{x, y\}$, we have

(15) $\begin{pmatrix} C \ D \ x \ y \end{pmatrix} = -1.$

Proof. Equation (14) is immediate from the definition. For (15) notice that from (9) and (12) we can derive

$$0 \in pconv \left( \begin{pmatrix} C \ D \ u \ y \end{pmatrix} \left| u \in \{x, y\} \right. \right) = pconv \left( \begin{pmatrix} 1, C \ D \ x \ y \end{pmatrix} \right),$$

and the claim follows. \hfill \Box

Proposition 3.5. If $P_C$ is a projective phasing of $M$, then condition (12) is equivalent to

(16) $\exists y_0 \in C \cap D \ such \ that \ 0 \in pconv \left( \begin{pmatrix} C \ D \ x \ y_0 \end{pmatrix} \left| x \in C \cap D \right. \right).$
Proof. Obviously (12) implies (16). For the reverse implication assume that property (16) holds and let \( z \in C \cap D \) be an arbitrary element. We want to prove that 
\[
0 \in \text{pconv} \left( \left\{ \begin{pmatrix} C \\ x \\ z \end{pmatrix} \middle| x \in C \cap D \right\} \right).
\]
From (16), using (10), we deduce 
\[
\left( \begin{pmatrix} C \\ x \\ z \end{pmatrix} \right) = \left( \begin{pmatrix} C \\ x \\ y_0 \end{pmatrix} \right) \left( \begin{pmatrix} C \\ D \\ y_0 \end{pmatrix} \right)
\]
for all \( x \in C \cap D \). Therefore, 
\[
0 \in \left( \begin{pmatrix} C \\ y_0 \\ z \end{pmatrix} \right) \text{pconv} \left( \left\{ \begin{pmatrix} C \\ x \\ y_0 \end{pmatrix} \middle| x \in C \cap D \right\} \right) = \text{pconv} \left( \left\{ \begin{pmatrix} C \\ y_0 \\ z \end{pmatrix} \middle| x \in C \cap D \right\} \right)
\]
as required. \( \square \)

3.2. Homeomorphism theorem. The goal of this section is to prove that \( \mathcal{R}_C(M) \) is homeomorphic to \( \mathcal{P}_C(M) \).

Theorem 3.6. Given a matroid \( M \) there exists a homeomorphism 
\[ \mathcal{R}_C(M) \leftrightarrow \mathcal{P}_C(M). \]

Remark 3.7 (On the oriented case). By (6) and (13), the bijection obtained in [GRS95, Theorem 1] can be now seen as the restriction of the homeomorphism of Theorem 3.6 to the subsets \( \mathcal{R}_E(M) \) and \( \mathcal{P}_E(M) \). In particular, we can conclude that the topology induced by \( \mathcal{R}_C(M) \) on \( \mathcal{R}_E(M) \) is discrete.

3.2.1. A case study: \( U_2(4) \). In order to prove Theorem 3.6 we need a preliminary result which characterizes the set of projective phasings of \( U_2(4) \), the uniform matroid of rank 2 over 4 elements. This is the matroid with ground set \( E = \{1, 2, 3, 4\} \), and bases set 
\[ \mathcal{B} = \{\{12\}, \{13\}, \{14\}, \{23\}, \{24\}, \{34\}\}. \]

Lemma 3.8. For any projective phasing \( \mathbb{P}_C \) of \( U_2(4) \) there exist orthogonal circuit and cocircuit \( S^1 \) signatures \( \gamma \) and \( \delta \) such that:

1. For every \( C \in \mathcal{E}, D \in \mathcal{E}^+ \) and \( x, y \in C \cap D \) identity 
\[
\begin{pmatrix} C \\ x \\ D \\ y \end{pmatrix} = \gamma C(x) \delta D(y) \]
holds;
2. The \( \Xi_C \) equivalence class of the phased matroid \((E, \Xi)\) is uniquely determined by the given phased projective orientation \( \mathbb{P}_C \).

Moreover, the space \( \mathcal{P}_C(U_2(4)) \) is homeomorphic to a topological subspace \( Y \) of the torus \( S^1 \times S^1 \), and \( Y \) has the homotopy type of \( S^1 \lor S^1 \).

Proof. Here (as well as in the general case of Theorem 3.6) the uniqueness claim follows from a straightforward adaptation of the arguments given in [GRS95]. We thus only prove existence of the desired pair \( \gamma, \delta \). Let \( \mathbb{P}_C \) be a projective phasing of \( U_2(4) \). We want to construct orthogonal circuit and cocircuit \( S^1 \) signatures such that (17) holds for all \( C \in \mathcal{E} \) and \( D \in \mathcal{E}^+ \).
Write \( \mathcal{E} = \{C_1, C_2, C_3, C_4\} \) and \( \mathcal{E}^* = \{D_1, D_2, D_3, D_4\} \), where \( C_i = D_i = E \setminus \{i\}, 1 \leq i \leq 4 \). If \( i \neq j \), \( |C_i \cap D_j| = 2 \) and then, from (15) we deduce that

\[
\begin{pmatrix}
C_\sigma(1) \\
\sigma(3)
\end{pmatrix}
\begin{pmatrix}
D_\sigma(2) \\
\sigma(4)
\end{pmatrix} = 1
\]

for each \( \sigma \in S_4 \). Hence, using (11) we find

\[
\begin{pmatrix}
C_i \\
j
\end{pmatrix}
\begin{pmatrix}
D_i \\
l
\end{pmatrix} =
\begin{pmatrix}
C_k \\
j
\end{pmatrix}
\begin{pmatrix}
D_k \\
l
\end{pmatrix}.
\]

From (14) and the previous observations, the only elements of interest are

\[
\begin{pmatrix}
C_1 \\
D_1
\end{pmatrix} = a,
\begin{pmatrix}
C_1 \\
D_2
\end{pmatrix} = b,
\begin{pmatrix}
C_2 \\
D_1
\end{pmatrix} = c.
\]

It is not hard to see that

\[
\begin{pmatrix}
C_1 \\
D_1
\end{pmatrix} = a,
\begin{pmatrix}
C_1 \\
D_2
\end{pmatrix} = b,
\begin{pmatrix}
C_2 \\
D_1
\end{pmatrix} = c.
\]

Now, set \( f = b \), \( g = ab \) and \( \mu = ac \). From (18) one obtains \( \mu^2 = 1 \). After some computations, applying (14) we find

\[
\begin{pmatrix}
C_1 \\
D_1
\end{pmatrix} = a,
\begin{pmatrix}
C_1 \\
D_2
\end{pmatrix} = b,
\begin{pmatrix}
C_2 \\
D_1
\end{pmatrix} = c.
\]

Using (12), from (19) we deduce that the following conditions are simultaneously satisfied:

\[
0 \in \text{pconv}(\{1, f, g\}) \quad 0 \in \text{pconv}(\{1, f, \mu g\})
\]

These force \( \mu = 1 \) and, therefore, (20) reduces to

\[
0 \in \text{pconv}(\{1, f, g\}).
\]

Now, we are able to construct orthogonal phased circuit and phased cocircuit signatures on \( U_2(4) \) satisfying (17). Assume \( \mu = 1 \) and let \( (f, g) \in S^1 \times S^3 \) such that \( 0 \in \text{pconv}(\{1, f, g\}) \). By direct computation, one can check that the following tables satisfy (17).

| \( E \) | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) |
|---|---|---|---|---|
| 1 | • | 1 | −1 | 1 |
| 2 | 1 | • | 1 | \( f \) |
| 3 | 1 | 1 | • | −g |
| 4 | 1 | −f | −g | • |

| \( E \) | \( D_1 \) | \( D_2 \) | \( D_3 \) | \( D_4 \) |
|---|---|---|---|---|
| 1 | • | \( \overline{g} \) | −\( \overline{f} \) | 1 |
| 2 | \( \overline{g} \) | • | 1 | 1 |
| 3 | \( \overline{f} \) | 1 | • | −1 |
| 4 | 1 | −1 | −1 | • |

For the topological part of the claim, let \( I = [0, 1] \) be the unit interval and let \( Y \subseteq S^1 \times S^1 \) be the set

\[
Y = \{(f, g) \in S^1 \times S^1 | 0 \in \text{pconv}(\{1, f, g\})\}.
\]
The previous arguments imply that there exists a one-to-one correspondence between $Y$ and $\mathcal{P}_C(U_2(4))$. By construction, such one-to-one correspondence is the restriction of continuous polynomial functions between $S^1 \times S^1$ and $(S^1)^d_{U_2(4)}$, so that $\mathcal{P}_C(U_2(4))$ and $Y$ are homeomorphic. It is not difficult to check that, as is suggested by Figure 2, the space $Y$ has the homotopy type of $S^1 \lor S^1$ and this concludes our proof. □

As noticed in [GRS95], there exist three (real) reorientation classes of $U_2(4)$. These classes correspond to the red, green and orange points in the previous figure.

3.2.2. The general case. We now turn to the general case and prove some preliminary results.

**Lemma 3.9.** Identity (17) determines a continuous map $\mathcal{R}_C(M) \longrightarrow \mathcal{P}_C(M)$.

**Proof.** For a matroid $M$, let $\Gamma \in \mathcal{M}_C(M)$ and choose $\gamma \in \Gamma$. It is not hard to see that, through equation (17), $\gamma$ defines a projective phasing of $M$ which depends only on the $\cong_c$ equivalence class of the phased matroid $\Gamma$. Thus, we can associate a well-defined projective phasing to any element of $\mathcal{R}_C(M)$. Moreover, this assignment is continuous, since it has a polynomial expression in terms of $S^1$ coordinate values, again through Equation (17). □

**Lemma 3.10.** For any projective phasing $\mathcal{P}_C$ of $M$ there exist orthogonal circuit and cocircuit $S^1$ signatures $\gamma$ and $\delta$ such that:

1. Identity (17) holds for every $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$ and $x, y \in C \cap D$;
2. The $\cong_c$ equivalence class of the phased matroid $\gamma$ is uniquely determined by the given phased projective orientation.

In particular, the map of Lemma 3.9 is bijective.

**Proof.** Uniqueness. Here, the arguments of [GRS95] can be generalized straightforwardly.

Existence. This is less obvious, and first we state some useful lemmas which will allow us to reduce to the case of $U_2(4)$. Both can be proved with an easy extension of the arguments of [GRS95, Lemma 2.5] and [GRS95, Lemma 2.6].

**Lemma 3.11.** Let $a \in C$ be an element of a circuit $C \in \mathfrak{C}$ of $M$. Let us assume $|C| \leq 2$. If the statement of Lemma 3.10 is true for $M' = M \setminus \{a\}$, then it is also true for $M$.

**Lemma 3.12.** If the statement of Lemma 3.11 is true for each proper minor of a connected matroid $M$ without parallel elements and such that $\text{rk} M^* > 2$, then it is also true for $M$.

**Figure 2.** the sets $Y$ (left) and $Z$ (right).
Now, we are ready to complete our proof of Lemma 3.10. By way of contradiction let $M$ be a matroid with the minimal number of elements, for which the lemma fails. Clearly if the statement of Lemma 3.10 is true for matroids $M_1$ and $M_2$, it holds for the direct sum $M_1 \oplus M_2$, too. We can thus assume $M$ to be connected (otherwise there exists a proper connected component of $M$ for which the claim also fails, contradicting the minimality of $M$). Moreover, by Lemma 3.11 $M$ has no parallel elements while by Lemma 3.12 we have $\text{rk}(M^*) \leq 2$. Hence, $\text{rk}(M^*) \leq 2$ and there are no coparallel elements in $M^*$. This implies that $M^*$ is isomorphic to $U_2(4)$. Since $U_2(4)$ is self-dual, we can conclude that $M$ is isomorphic to $U_2(4)$. However, by Lemma 3.8 the statement of Lemma 3.10 is true for $U_2(4)$, leading a contradiction. \hfill \Box

**Proof of Theorem 3.6.** Let us fix an arbitrary linear ordering of the ground set $E$ and order the circuits by the corresponding lexicographic order. In view of Lemma 3.9 and Lemma 3.10 it is enough to prove that the bijective map $\mathcal{P}_C(M) \rightarrow \mathcal{R}_C(M)$ constructed in Lemma 3.9 is indeed continuous. To see this, let $\pi_c : \mathcal{N}_C^i(M) \rightarrow \mathcal{R}_C^i(M)$ be the projection (obtained as composition of the quotient projections $\mathcal{N}_C(M) \rightarrow \mathcal{M}_C^i(M)$ and $\mathcal{M}_C^i(M) \rightarrow \mathcal{R}_C^i(M)$) and let $F^c : \mathcal{P}_C(M) \rightarrow \mathcal{N}_C^i(M)$ be the map which associates to a projective phasing $\mathcal{P}_C \in \mathcal{P}_C(M)$ the $S^1$ circuit signature $F^c(\mathcal{P}_C) \in \mathcal{N}_C^i(M)$ satisfying Lemma 3.10 and such that $F^c(\mathcal{P}_C)C(x) = 1$ whenever $x$ is the minimal element of $C \in \mathcal{E}$ or $C$ being the minimal circuit containing $x$. The proof of [GRS95, Lemma 2.5] generalized to the phase setting implies that each component of $F^c$ is a polynomial map in $S^1$-coordinates. So that, $F^c : \mathcal{P}_C(M) \rightarrow \mathcal{N}_C^i(M)$ is continuous. Therefore, $\pi_c \circ F^c : \mathcal{P}_C(M) \rightarrow \mathcal{R}_C^i(M)$ is continuous. By construction, this is exactly the inverse of the continuous map of Lemma 3.9. \hfill \Box

4. Pontrjagin duals of Tutte groups

The aim of this section is to characterize $\mathcal{R}_C(M)$ algebraically, using a certain abelian group associated to $M$, called the inner Tutte group of $M$. In fact, the reader familiar with this context will have noticed that Theorem 3.6 and especially its proof are naturally connected with the general theory of Tutte groups of matroids, as developed by Dress and Wenzel in [DW89] and subsequent papers.

We begin this section recalling some definitions and results from the theory of Tutte groups of matroids in a slightly modified form. The main difference with respect to [DW89] and [DW90] is that, following [GRS95], we use circuits instead of hyperplanes (complements of cocircuits).

#### 4.1. Tutte groups

We consider sets of the form $F = C_1 \cup \cdots \cup C_k$ with $C_i \in \mathcal{E}$, so $E \setminus F$ is a closed set for $M^*$. If $\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_d = F$ is a maximal chain of such sets, then $d$ depends only on $F$ and is called the dimension of $F$. We denote it by $\dim(F)$. Notice that $\dim(F) = |F| - \text{rk}F - 1$.

The extended Tutte group $\mathcal{T}_M^{\text{ex}}$ of a matroid $M$ is defined to be the multiplicative abelian group with formal generators given by the symbols $\epsilon$ and $C(x)$, one for each $C \in \mathcal{E}$ and each $x \in C$, with relations

$$\epsilon^2 = 1 \quad (21)$$

and

$$C_i(x_2)C_1(x_3)^{-1}C_2(x_3)C_2(x_1)^{-1}C_3(x_1)C_3(x_2)^{-1} = \epsilon$$

for $C_i \in \mathcal{E}$, $L = C_1 \cup C_2 \cup C_3 = C_i \cup C_j$ ($i \neq j$), $x_i \in L \setminus C_i$ and $\dim(L) = 1$.

The group $\mathcal{T}_M^{\text{ex}}$ is defined to be the multiplicative abelian group with formal generators $\epsilon$, $C(x)$ for $C \in \mathcal{E}$, $x \in C$ and $D(y)$ for $D \in \mathcal{E}^*$, $y \in D$, with relations
\[ C(x)D(x) = eC(y)D(y) \]

for \( C \in \mathcal{E}, D \in \mathcal{E}^* \) with \( \{x, y\} = C \cap D \).

The Tutte group \( T_M \) is then defined as the subgroup of \( \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \) generated by \( e \), \( C(x)C(y)^{-1} \) for \( C \in \mathcal{E}, x, y \in C \) and \( D(x)D(y)^{-1} \) for \( D \in \mathcal{E}^*, x, y \in D \).

From [DWS99, Theorem 1.3 and 1.4], we know that there exists an injective homomorphism \( \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \to \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \) with \( e \mapsto e \) and \( C(x) \mapsto C(x) \) for \( C \in \mathcal{E}, x \in C \). Therefore, we may view \( T_M^0 \) as a subgroup of \( \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \).

**Definition 4.1.** Let us consider the homomorphism \( T^{(0)}_M \to \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \) with \( e \mapsto e \) and \( C(x) \mapsto C(x) \) for \( C \in \mathcal{E}, x \in C \). The set \( T^{(0)}_M \cong \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \) is the kernel of this homomorphism.

It is not hard to see that \( T^{(0)}_M \subset T_M \subset \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \). Moreover, if \( c \) denotes the number of connected components of \( M \), with [DWS99, Theorem 1.5] we have

\[
\begin{align*}
T_M & \cong T^{(0)}_M \times \mathbb{Z}^{\dim(E) - c}, \\
T^{(0)}_M & \cong T_M \times \mathbb{Z}^c, \\
T^{\mathcal{E}, \mathcal{E}^*}_M & \cong T^{\mathcal{E}, \mathcal{E}^*}_M \times \mathbb{Z}^{c - 1}.
\end{align*}
\]

Thus, the isomorphism type of any of these groups is known as soon as we know the structure of \( T^{(0)}_M \).

According to [Wen89] Proposition 2.9, the inner Tutte group \( T^{(0)}_M \) is isomorphic to the subgroup of \( \mathbb{T}^{\mathcal{E}, \mathcal{E}^*}_M \) generated by \( e \) and all products of the form

\[
C_1(x)C_1(y)^{-1}C_2(y)C_2(x)^{-1}
\]

for \( C_1, C_2 \in \mathcal{E}, x, y \in C_1 \cap C_2 \) and \( \dim(C_1 \cup C_2) = 1 \). This suggest the following definition where, as in the following, for the sake of simplicity we write \( \mathcal{E}(x) \), instead of \( C(x)^{-1} \), in order to denote the inverse of the group element \( C(x) \).

**Definition 4.2.** Given a matroid \( M \) we denote by \( \mathcal{H}_C(M) \) the set of homomorphisms \( \Phi : T^{(0)}_M \to S^1 \) satisfying

\[
\begin{align*}
&\ (a) \quad e \mapsto -1; \\
&\ (b) \quad \text{For } x_1, x_2, x_3, x_4 \in L \subseteq E, \dim(L) = 1 \text{ and } C_1 = L \setminus \{x_1\} \text{ being pairwise distinct circuits,}
\end{align*}
\]

\[
\Phi \left( \frac{C_1(x_3)C_2(x_4)}{C_1(x_4)C_2(x_3)} \right), \Phi \left( \frac{C_4(x_3)C_2(x_1)}{C_4(x_1)C_2(x_3)} \right) \right) \right).
\]

Notice that \( \mathcal{H}_C(M) \) is naturally a topological subspace of the classical Pontrjagin dual hom\( (T^{(0)}_M, S^1) \). For more details on the classical Pontrjagin dual see [Lan02 Chapter 3, Section 6, Remark 2].

As in the case of phased projective orientations, this definition extends the oriented case to the phased setting. The set \( \mathcal{H}_R(M) \) is defined in [GRS95] as the set of group homomorphisms \( \chi : T^{(0)}_M \to S^0 \) satisfying

\[
\begin{align*}
&\ (a) \quad \epsilon \mapsto -1; \\
&\ (b) \quad \text{Condition (22) for } x_1, x_2, x_3, x_4 \in L \subseteq E, \dim(L) = 1 \text{ and } C_1 = L \setminus \{x_1\} \text{ being pairwise distinct circuits.}
\end{align*}
\]

By definition, \( \mathcal{H}_R(M) = \mathcal{H}_C(M) \cap \text{hom}(T^{(0)}_M, S^0) \). Thus, we have a natural inclusion

\[
\mathcal{H}_R(M) \subseteq \mathcal{H}_C(M).
\]
4. **Homeomorphic equivalence.** The purpose of this section is to relate $R(C(M))$ to $H(C(M))$. From Theorem 5.6 we already know that the spaces $R(C(M))$ and $P(C(M))$ are homeomorphic. Then, by composition of homeomorphisms it suffices to find a relation between $P(C(M))$ and $H(C(M))$.

**Theorem 4.3.** For a matroid $M$ there is a homeomorphism

$$P(C(M)) \leftrightarrow H(C(M)).$$

**Remark 4.4 (On the oriented case).** This will extend the results of [GRS95] to the phased case. In fact, from (6) and (23) the bijection given in [GRS95, Theorem 2] will correspond to the restriction of the homeomorphism between $R(C(M))$ and $H(C(M))$ obtained combining the results of Theorem 5.6 and Theorem 4.3.

To prove Theorem 4.3, a new presentation of the inner Tutte group with a bigger set of generators is needed (see [GRS95, Section 3]).

4.2.1. **The group $T_M^{(1)}$.** Let us denote by $T_M^{(1)}$ the multiplicative abelian group with formal generators given by the symbols $\epsilon$ and $\left| C \atop D \atop x \atop y \right|$ for $C \in C$, $D \in C^*$, $x, y \in C \cap D$, and relations $\epsilon^2 = 1$ as well as

(1) \[ \left| C \atop D \atop x \atop y \right| = 1, \]

(2) \[ \left| C \atop D \atop x \atop y \right| \left| C \atop D \atop y \atop z \right| = \left| C \atop D \atop z \atop x \right| = 1, \]

(3) \[ \left| C_1 \atop D_1 \atop x \atop y \right| \left| C_2 \atop D_2 \atop x \atop y \right| = \left| C_1 \atop D_2 \atop x \atop y \right| \left| C_2 \atop D_1 \atop x \atop y \right| \]

(4) \[ \left| C \atop D \atop x \atop y \right| = \epsilon, \]

if $C \cap D = \{x, y\}$,

(5) \[ \left| C_1 \atop D_1 \atop x_2 \atop x_3 \right| = \left| C_2 \atop D_2 \atop x_1 \atop x_4 \right|, \]

if dim$(C_1 \cup C_2) = 1$, $C_1 \cap D_1 = \{x_2, x_3, x_4\}$, $C_2 \cap D_2 = \{x_1, x_3, x_4\}$, and $C_1 \cap D_2 = C_2 \cap D_1 = \{x_3, x_4\}$.

Here we use $\left| C \atop D \atop x \atop y \right|$ not as $S^1$-valued functions, but as abstract generator of a group. Each time, when we write $\left| C \atop D \atop x \atop y \right|$, we mean that $C \in C$, $D \in C^*$ and $x, y \in C \cap D$.

**Proof of Theorem 4.3.** Let $T_M^{(0)}$ be the inner Tutte group of $M$ and let $T_M^{(1)}$ be the group previously introduced.

Thanks to [GRS95, Theorem 3], there exists a group homomorphism $\tau : T_M^{(1)} \longrightarrow T_M^{(0)}$ with $\tau(\epsilon) = \epsilon$ and

$$\tau \left( \left| C \atop D \atop x \atop y \right| \right) = \frac{C(x)D(x)}{C(y)D(y)},$$
which induces an isomorphism $T_M^{(1)} \cong T_M^{(0)}$ and, in turn, a homeomorphism of topological spaces

\[(24)\quad \text{hom}(T_M^{(1)}, S^1) \leftrightarrow \text{hom}(T_M^{(0)}, S^1).\]

By construction of $T_M^{(1)}$ we could homeomorphically identify $\mathcal{P}_C(M)$ with a topological subspace of $\text{hom}(T_M^{(1)}, S^1)$ such that \[(24)\] restricts to a homeomorphism $\mathcal{P}_C(M) \leftrightarrow H_C(M)$.

\[\square\]

**Corollary 4.5.** If $M$ is a binary matroid with no minors of Fano or dual-Fano type, then

\[|\mathcal{R}_C(M)| = |\mathcal{R}_\mathcal{R}(M)| = 1.\]

Moreover, in this case, the sets of phasing classes and reorientation classes (as defined in [GRS95]) coincide.

**Proof.** From [Wen89, Theorem 5.2] it follows that $T_M^{(0)} \cong (\epsilon) \cong \mathbb{Z}/2\mathbb{Z}$.

Hence, there exists only a homomorphism $\Phi : T_M^{(0)} \rightarrow S^1$ satisfying $\epsilon \mapsto -1$ and \[(22)\] for $x_1, x_2, x_3, x_4 \in L \subseteq E$, $\dim(L) = 1$ and $C_i = L \setminus \{x_i\}$ being pairwise distinct circuits. Moreover, such homomorphism is also an element of $H_C(M)$. This, together with Theorem 4.4 and [GRS95, Theorem 2], concludes our proof. $\square$

5. **Generalized cross-ratios**

Our goal for this section is to characterize the space of phasing classes of matroids in terms of some $S^1$-valued functions we call “generalized cross ratios” borrowing the terminology from the oriented case treated in [GRS95].

5.1. **Definitions.** Let $M$ be a matroid without minors of Fano or dual-Fano type and fix an arbitrary enumeration $\mathcal{E}(M) = \{C_i\}_{i \in I}$ of the set of circuits of $M$. An $S^1$ cross-ratio of $M$ is a family of values

\[\psi_{(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})} \in S^1\]

one for each quadruple $C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} \in \mathcal{E}(M)$ with $\dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1$ and $\{C_{i_1}, C_{i_2}\} \cap \{C_{i_3}, C_{i_4}\} = \emptyset$, satisfying

\[(25)\quad 0 \in \text{pconv}\{1, -\psi_{(C_{i_1} C_{i_2} C_{i_3} C_{i_4})}, -\psi_{(C_{i_4} C_{i_2} C_{i_3} C_{i_1})}\}\]

whenever $C_{i_1}, \ldots, C_{i_4}$ are pairwise distinct, as well as the following conditions \[(26), (27), (28), (29)\] and \[(30)\].

\[(26)\quad \psi_{(C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4})} = 1,\]

\[(27)\quad \psi_{(C_{i_1} C_{i_2} C_{i_3} C_{i_4})} = \psi_{(C_{i_3} C_{i_4} C_{i_1} C_{i_2})},\]

\[(28)\quad \psi_{(C_{i_1} C_{i_2} C_{i_3} C_{i_4})} \psi_{(C_{i_1} C_{i_4} C_{i_2} C_{i_3})} \psi_{(C_{i_1} C_{i_3} C_{i_2} C_{i_4})} \psi_{(C_{i_1} C_{i_4} C_{i_2} C_{i_3})} = -1\]

for all pairwise distinct $C_{i_1}, \ldots, C_{i_4}$;

\[(29)\quad \psi_{(C_{i_1} C_{i_2} C_{i_3} C_{i_4})} \psi_{(C_{i_1} C_{i_3} C_{i_4} C_{i_2})} \psi_{(C_{i_1} C_{i_2} C_{i_4} C_{i_3})} \psi_{(C_{i_1} C_{i_3} C_{i_2} C_{i_4})} = 1;\]

\[(30)\quad \psi_{(C_{i_1} C_{i_2} C_{i_4} C_{i_9})} \psi_{(C_{i_2} C_{i_3} C_{i_4} C_{i_7})} \psi_{(C_{i_3} C_{i_1} C_{i_5} C_{i_8})} \psi_{(C_{i_1} C_{i_3} C_{i_5} C_{i_8})} = 1\]

for any family of circuits $C_{i_1}, \ldots, C_{i_9} \in \mathcal{E}$ such that


(1) \(\dim(L_{pq}) = 1\) for \(L_{pq} = C_i \cup C_j\), where \(\{p, q, r\} = \{1, 2, 3\}\)

(2) \(\dim(P) = 2\) where \(P = C_i \cup C_j \cup C_k\)

(3) \(C_{i+s}, C_{i+s+2} \subseteq L_i\) for \(s = 1, 2, 3\)

(4) \(\dim(L_{ih}) = 1\) for \(L_{ih} = C_{i+2+h} \cup C_{i+4+h} \cup C_{i+6+h}, h \in \{1, 4\}\)

(5) \(\{C_{i1}, C_{i2}, C_{i3}\} \cap \{C_{i4}, \ldots, C_{i9}\} = \emptyset\)

We denote by \(\overline{\psi(C_i, C_{i2}, C_{i3}, C_{i4})}\) the complex conjugate of \(\psi(C_i, C_{i2}, C_{i3}, C_{i4})\).

**Definition 5.1.** Given a matroid \(M\) without minors of Fano or dual-Fano type we define the following integers.

- \(h_M\) : the number of 4-uples \((C_{i1}, C_{i2}, C_{i3}, C_{i4})\) defined for \(C_{i1}, C_{i2}, C_{i3}, C_{i4} \in \mathcal{C}\) with dim\((C_{i1} \cup C_{i2} \cup C_{i3} \cup C_{i4}) = 1\) and \(\{C_{i1}, C_{i3}\} \cap \{C_{i2}, C_{i4}\} = \emptyset\);
- \(p_M\) : the number of equations of type (26), (27), (28), (29) and (30).

The set of \(S^1\) cross-ratios of a matroid \(M\) without minors of Fano or dual-Fano type will be denoted by \(G_C(M)\). This is a topological subspace \(G_C(M) \subseteq (S^1)^{h_M}\).

**Remark 5.2 (On the oriented case).** The previous definitions generalize the one given in [GRS95] to the phased context. Let \(G_R(M)\) be the set of (oriented) cross-ratios as in the statement of [GRS95 Corollary 2]. By definition \(G_R(M) = G_C(M) \cap (S^0)^{h_M}\). Therefore, there exists a natural inclusion

\[(31) \quad G_R(M) \subseteq G_C(M).\]

### 5.2. Homeomorphic equivalence

The goal of this section is to establish a homeomorphism between \(\mathcal{R}_C(M)\) and \(\mathcal{G}_C(M)\). Again by composition of homeomorphisms, since the spaces \(\mathcal{R}_C(M)\) and \(\mathcal{H}_C(M)\) are homeomorphic, it is enough to construct a homeomorphism \(G_C(M)\) to \(\mathcal{H}_C(M)\).

**Theorem 5.3.** Given a matroid \(M\) without minors of Fano or dual-Fano type there exists a homeomorphism

\[\mathcal{H}_C(M) \leftrightarrow G_C(M).\]

**Remark 5.4 (On the oriented case).** As we noticed in Section 4 the one-to-one correspondence

\[\mathcal{R}_R(M) \leftrightarrow G_R(M).\]

studied in [GRS95 Corollary 2] can now be seen as the restriction of the homeomorphism

\[\mathcal{R}_C(M) \leftrightarrow G_C(M)\]

obtained combining the results of Theorem 4.2, Theorem 4.3 and Theorem 5.3.

#### 5.2.1. The group \(\mathcal{T}_M^{(2)}\)

In order to give a proof of Theorem 5.3 following once more [GRS95] we now define, for arbitrary \(M\), a multiplicative abelian group \(\mathcal{T}_M^{(2)}\) with formal generators given by the symbols \(\epsilon\) and \([C_1, C_{i2}, C_3, C_4]\), where \(C_{i1}, C_{i2}, C_{i3}, C_{i4} \in \mathcal{C}\), \(L = C_{i1} \cup C_{i2} \cup C_{i3} \cup C_{i4} = C_{i6} \cup C_{i7}\) for \(k = 1, 2 \leq 3, 4 \dim(L) = 1\) and relations \(\epsilon^2 = 1\) as well as

1. \[
\begin{bmatrix}
C_{i1} & C_{i2} \\
C_{i3} & C_{i4}
\end{bmatrix} = 1,
\]

2. \[
\begin{bmatrix}
C_{i1} & C_{i2} \\
C_{i3} & C_{i4}
\end{bmatrix} \begin{bmatrix}
C_{i1} & C_{i2} \\
C_{i3} & C_{i4}
\end{bmatrix} \begin{bmatrix}
C_{i6} & C_{i7} \\
C_{i8} & C_{i9}
\end{bmatrix} = 1,
\]

3. \[
\begin{bmatrix}
C_{i1} & C_{i2} \\
C_{i3} & C_{i4}
\end{bmatrix} = \begin{bmatrix}
C_{i1} & C_{i2} \\
C_{i3} & C_{i4}
\end{bmatrix}.
\]
indexing as transparent as possible by outsourcing some secondary technical considerations of the proofs in this section. We try nevertheless to make the argument

\[ \text{hom}(G) \]

defining the space \( G \) leads us to the definition of a new space \( T \)

induces an isomorphism

for any family of circuits \( C_{i_1}, \ldots, C_{i_6} \in C \) such as in (32).

**Proof of Theorem 5.3.** Let \( T^{(0)}_M \) be the inner Tutte group of \( M \) and let \( T^{(2)}_M \) be the group previously introduced. From [GRS95, Theorem 4] there is a group homomorphism \( \lambda: T^{(2)}_M \rightarrow T^{(0)}_M \), with \( \lambda(\varepsilon) = \varepsilon \) and

\[
\lambda \left( \begin{bmatrix} C_{i_1} & C_{i_2} \\ C_{i_3} & C_{i_4} \end{bmatrix} \right) = \begin{bmatrix} C_{i_1} & C_{i_2} \\ C_{i_3} & C_{i_4} \end{bmatrix} \frac{(x_{i_2})C_{i_2}(x_{i_1})}{(x_{i_4})C_{i_2}(x_{i_3})},
\]

where \( x_{i_3} \in L \setminus C_{i_3}, \ x_{i_4} \in L \setminus C_{i_4} \) with \( L = C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4} \). Moreover, \( \lambda \) induces an isomorphism \( T^{(2)}_M \cong T^{(0)}_M \). As a consequence of this, there exists a homeomorphism of topological spaces

\[ \text{hom}(T^{(2)}_M, S^1) \leftrightarrow \text{hom}(T^{(0)}_M, S^1). \]

By definition of \( T^{(2)}_M \) it is possible to homeomorphically identify \( G_C(M) \) with a topological subspace of \( \text{hom}(T^{(2)}_M, S^1) \) such that (32) restricts to a homeomorphism

\[ \mathcal{H}_C(M) \leftrightarrow G_C(M). \]

\[ \square \]

6. A reduction step

In view of our applications, we now embark on a closer analysis of the equations defining the space \( G_C(M) \) in order to find – and eliminate – redundancies. This leads us to the definition of a new space \( G^e_C(M) \), again homeomorphic to \( \mathcal{R}_C(M) \).

The usefulness of this construction in the applications below is paid for by the technicality of the proofs in this section. We try nevertheless to make the argument as transparent as possible by outsourcing some secondary technical considerations to the Appendix.

For a matroid \( M \) without minors of Fano or dual-Fano type we again choose an indexing \( \{C_i\}_I \) of the set of circuits of \( M \) and an arbitrary total order \( \prec \) on \( I \).

We start by noticing that, in (29), if \( C_{i_4} = C_{i_5} \) we easily find

\[ \psi(C_{i_4} C_{i_2} C_{i_3} C_{i_4}) = \psi(C_{i_4} C_{i_2} C_{i_3} C_{i_4}). \]

As a consequence of this, together with (27), it is possible to restrict to those \( S^1 \) cross-ratios such that \( \dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1 \) and \( i_1 < i_2, i_3 < i_4, i_1 < i_3 \).

A reduced \( S^1 \) cross-ratio of \( M \) is a family of values

\[ \phi(C_{i_1} C_{i_2} | C_{i_3} C_{i_4}) \]

defined for any family of circuits \( C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4} \in C \) with the properties that \( \dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4}) = 1, j_1 < j_2, j_3 < j_4 \) and \( j_1 < j_3 \), ranging in \( S^1 \), satisfying

\[ 0 \not\in \text{pcov}(\{1, -\phi(C_{q_1} C_{q_2} | C_{q_3} C_{q_4}), -\phi(C_{q_1} C_{q_3} | C_{q_2} C_{q_4})\}) \]

for any family of circuits \( C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4} \in C \) such that \( q_1 < q_2 < q_3 < q_4 \) and \( \dim(C_{q_1} \cup C_{q_2} \cup C_{q_3} \cup C_{q_4}) = 1 \), as well as conditions (35), (36) and (37) below.

\[ \phi(C_{q_1} C_{q_2} | C_{q_3} C_{q_4}) \phi(C_{q_1} C_{q_3} | C_{q_2} C_{q_4}) \phi(C_{q_1} C_{q_4} | C_{q_2} C_{q_3}) = -1 \]
for any family of circuits \( C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4} \in \mathcal{C} \) such that \( q_1 < q_2 < q_3 < q_4 \) and \( \dim(C_{q_1} \cup C_{q_2} \cup C_{q_3} \cup C_{q_4}) = 1 \).

\[
\begin{align*}
\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_4}C_{q_3})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_2}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_2}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4}) = 1 \\
\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_2}C_{q_3}|C_{q_3}C_{q_4}) = 1 \\
\end{align*}
\]

\( (36) \)

for each family of circuits \( C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4} \in \mathcal{C} \) with \( q_1 < q_2 < q_3 < q_4 < q_5 \) and \( \dim(C_{q_1} \cup C_{q_2} \cup C_{q_3} \cup C_{q_4} \cup C_{q_5}) = 1 \).

\[
\omega(C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4})\omega(C_{q_2}, C_{q_3}, C_{q_4})\omega(C_{q_3}, C_{q_4})\omega(C_{q_4}) = 1
\]

\( (37) \)

for each family \( \{C_{q_1}, \ldots, C_{q_6}\} \) of circuits of \( M \) as in \( (36) \) and with the additional properties that

1. \( q_1 < q_2 < q_3 \);
2. \( q_4 \geq q_7, q_5 \geq q_8 \) and \( q_6 \geq q_9 \) do not all hold at the same time, where the \( \omega(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) \) are the \( S^1 \)-valued functions defined by the formula

\[
\omega(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) = \begin{cases} 
1 & \text{if } d_1 = d_2 \text{ or } d_3 = d_4 \\
\phi(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2, d_3 < d_4, d_1 < d_3 \\
\phi(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2, d_3 < d_4, d_1 < d_3 \\
\phi(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2, d_3 < d_4, d_1 < d_3 \\
\phi(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2, d_3 < d_4, d_1 < d_3 \\
\phi(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2, d_3 < d_4, d_1 < d_3 \\
\phi(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2, d_3 < d_4, d_1 < d_3 \\
\phi(C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2, d_3 < d_4, d_1 < d_3 \\
\end{cases}
\]

\( (38) \)

for any family of circuits \( C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4} \in \mathcal{C} \) with the extra conditions \( \dim(C_{d_1} \cup C_{d_2} \cup C_{d_3} \cup C_{d_4}) = 1 \) and \( \{C_{d_1}, C_{d_2}\} \cap \{C_{d_3}, C_{d_4}\} = \emptyset \).

**Definition 6.1.** For a matroid \( M \) without minors of Fano or dual-Fano type we define the following integer numbers:

- \( k_M \): the number of 4-tuples \( (C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}) \) where \( C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4} \in \mathcal{C} \) are circuits of \( M \) such that \( \dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4}) = 1, j_1 < j_2, j_3 < j_4 \) and \( j_1 < j_3 \);
- \( q_M \): the number of equations of type \( w, w \) and \( w \),
- \( n_M \): the number of equations of type \( w \).

**Definition 6.2.** Given a matroid \( M \) with no minors of Fano or dual-Fano type, the set of reduced \( S^1 \)-cross-ratios of \( M \) will be denoted by \( G^R_C(M) \). This is a topological subspace \( G^R_C(M) \subseteq (S^1)^{k_M} \).

Now, we are ready to prove the main result of this section. As announced, we postpone in Appendix \( B \) some of the technical lemmas.
Theorem 6.3. For a matroid $M$ without minors of Fano or dual-Fano type there exists a homeomorphism

$$\mathcal{G}_c(M) \leftrightarrow \mathcal{G}_c^R(M).$$

Proof. Let $h_M$ and $k_M$ as in Definition 5.1 and Definition 6.1 respectively. First of all we identify the points $P$ of $(S^1)^{|M|}$ with the collections of $(S^1)$-valued functions $\{z_{C_1,C_2}^{P}|C_1,C_2 \in \mathcal{E}\}$ defined for any family of circuits $C_1$, $C_2$, $C_3$, $C_4 \in \mathcal{E}$ with the properties that $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$ and $\{C_1,C_2\} \cap \{C_3,C_4\} = \emptyset$.

In a similar way, let us identify the points $Q \in (S^1)^{|M|}$ with the collections of $(S^1)$-valued functions $\{w_{C_1,C_2,C_3}^{Q}|C_1,C_2,C_3 \in \mathcal{E}\}$ defined for any family of circuits $C_1$, $C_2$, $C_3 \in \mathcal{E}$ such that $\dim(C_1 \cup C_2 \cup C_3) = 1$ as well as $j_1 < j_2$, $j_3 < j_4$ and $j_1 < j_3$.

Let $\tilde{F} : (S^1)^{|M|} \rightarrow (S^1)^{|M|}$ be the function which maps a point $P \in (S^1)^{|M|}$ to the family $\tilde{F}(P) \in (S^1)^{|M|}$ of values

$$w_{(C_1,C_2)}^{\tilde{F}(P)} = z_{(C_1,C_2)}^P,$$

defined for any collection of pairwise circuits $C_1$, $C_2$, $C_3$, $C_4 \in \mathcal{E}$ with conditions $j_1 < j_2$, $j_3 < j_4$, $j_1 < j_3$ and $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$.

Let now $\tilde{G} : (S^1)^{|M|} \rightarrow (S^1)^{|M|}$ be the function which associates, to a point $Q \in (S^1)^{|M|}$, the family $\tilde{G}(Q)$ of $S^1$-valued functions

$$\tilde{G}(Q) = \tilde{z}_{(C_1,C_2)}^{Q},$$

built from the values $\{w_{(C_1,C_2)}^{Q}|C_1,C_2 \in \mathcal{E}\}$ using (35) and defined for any collection of circuits $C_1$, $C_2$, $C_3$, $C_4 \in \mathcal{E}$ with conditions $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$ and such that $\{C_1,C_2\} \cap \{C_3,C_4\} = \emptyset$.

By construction $\tilde{F}$ and $\tilde{G}$ are continuous. Hence, it is enough to show that $\tilde{F}$ and $\tilde{G}$ restrict to maps

$$F = \tilde{F} \big|_{\mathcal{G}_c(M)} : \mathcal{G}_c(M) \rightarrow \mathcal{G}_c^R(M)$$

and

$$G = \tilde{G} \big|_{\mathcal{G}_c^R(M)} : \mathcal{G}_c^R(M) \rightarrow \mathcal{G}_c(M)$$

which are one the inverse of the other.

(a) $F(\mathcal{G}_c(M)) \subseteq \mathcal{G}_c^R(M)$.

Let $P \in \mathcal{G}_c(M)$ be a $S^1$ cross-ratio expressed as a family $\{\psi^P\}$ and write $F(P)$ as a family $\{\phi^F(P)\}$. To check that $F(P) \in \mathcal{G}_c^R(M)$ it suffices to show that the collection $\{\phi^F(P)\}$ satisfies conditions (36), (37), (38) and (39).

(i) Proof of (35). Let $C_{j_1}$, $C_{j_2}$, $C_{j_3}$, $C_{j_4} \in \mathcal{E}$ be circuits such that $j_1 < j_2 < j_3 < j_4$ and $\dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4}) = 1$. From (28) and (32) we obtain

$$\psi^P_{(C_{j_1},C_{j_2})} \psi^P_{(C_{j_3},C_{j_4})} = -1.$$

From (39) we find then

$$\phi^F_{(C_{j_1},C_{j_2})} \phi^F_{(C_{j_3},C_{j_4})} = -1,$$

so that $\{\phi^F(P)\}$ satisfies (35).

(ii) Proof of (36). Let $C_{j_1}$, $C_{j_2}$, $C_{j_3}$, $C_{j_4} \in \mathcal{E}$ be circuits such that $j_1 < j_2 < j_3 < j_4$ and $\dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4}) = 1$.

Relation (33) allows us to check that $\{\phi^F(P)\}$ satisfies (36).
(iii) Proof of (37). Let $C_{j_1}, \ldots, C_{j_8} \in \mathcal{C}$ be a family of circuits as in (30) and such that $j_1 < j_2 < j_3$. Moreover, suppose that the conditions $j_6 \geq j_9$, $j_4 \geq j_2$ and $j_5 \geq j_8$ do not all hold. Using (33) from (30) it easily follows that the family \{\phi^F(P)\} satisfies condition (37).

(iv) Proof of (34). Let $C_{j_1}, \ldots, C_{j_4} \in \mathcal{C}$ be circuits such that $\dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4}) = 1$. Again from (33) and from (39) the condition

$$0 \in \text{pconv} \left( \{1, -\psi^{C_{j_1}C_{j_2}}[C_{j_3}C_{j_4}], -\psi^{C_{j_3}C_{j_4}}[C_{j_1}C_{j_2}] \} \right)$$

is equivalent to

$$0 \in \text{pconv} \left( \{1, -\phi^{C_{j_1}C_{j_2}}[C_{j_3}C_{j_4}], -\phi^{C_{j_3}C_{j_4}}[C_{j_1}C_{j_2}] \} \right).$$

(b) $G(\mathcal{G}_\mathcal{C}^R(M)) \subseteq \mathcal{G}_\mathcal{C}^R(M)$.

Let $Q \in \mathcal{G}_\mathcal{C}^R(M)$ be a reduced $S^1$ cross-ratio expressed as a family \{\phi^Q\} and write $G(Q) = \{\psi^{G(Q)}\}$ as a collection \{\psi^{G(Q)}\}. To prove that $G(Q)$ is a $S^1$ cross-ratio, we have to check conditions (20), (27), (28), (29) and (30) for the family \{\psi^{G(Q)}\}.

(i) Proof of (28). Clearly \{\psi^{G(Q)}\} satisfies (28) by definition \{(10)\}.

(ii) Proof of \{\psi^{G(Q)}\} fulfills condition (27). Similarly to the previous case, condition (27) holds by \{(10)\}.

(iii) Proof of \{28\}. In order to check \{28\}, let $C_{i_1}, \ldots, C_{i_4} \in \mathcal{C}$ be pairwise distinct circuits such that $\dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1$. Let $\sigma \in S_4$ be a permutation such that $i_{\sigma(1)} < i_{\sigma(2)} < i_{\sigma(3)} < i_{\sigma(4)}$. Since \{\psi^{G(Q)}\} fulfills \{28\}, \{27\} and \{33\} we can apply Lemma \{3.1\}. Hence, \{28\} is satisfied by \{\psi^{G(Q)}\} if and only if

$$\psi^{G(Q)}(C_{i_{\sigma(1)}}, C_{i_{\sigma(2)}}, C_{i_{\sigma(3)}}, C_{i_{\sigma(4)}}) \psi^{G(Q)}(C_{i_{\sigma(1)}}, C_{i_{\sigma(4)}}, C_{i_{\sigma(3)}}, C_{i_{\sigma(2)}}) \psi^{G(Q)}(C_{i_{\sigma(1)}}, C_{i_{\sigma(2)}}, C_{i_{\sigma(3)}}, C_{i_{\sigma(4)}}) = -1.$$

From \{33\} this equation is equivalent to

$$\phi^{G(Q)}(C_{i_{\sigma(1)}}, C_{i_{\sigma(2)}}, C_{i_{\sigma(4)}}, C_{i_{\sigma(3)}}) \phi^{G(Q)}(C_{i_{\sigma(1)}}, C_{i_{\sigma(3)}}, C_{i_{\sigma(2)}}, C_{i_{\sigma(4)}}) \phi^{G(Q)}(C_{i_{\sigma(1)}}, C_{i_{\sigma(4)}}, C_{i_{\sigma(3)}}, C_{i_{\sigma(2)}}) = -1.$$

which holds by hypothesis because $Q \in \mathcal{G}_\mathcal{C}^R(M)$ is a reduced $S^1$ cross-ratio.

(iv) Proof of \{29\}. Let $C_{i_1}, \ldots, C_{i_5} \in \mathcal{C}$ be circuits such that

$$\dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4} \cup C_{i_5}) = 1 \quad \text{and} \quad \{C_{i_1}, C_{i_2}\} \cap \{C_{i_3}, C_{i_4}\} = \emptyset$$

$$\dim(C_{i_1} \cup C_{i_2} \cup C_{i_4} \cup C_{i_5}) = 1 \quad \text{and} \quad \{C_{i_1}, C_{i_2}\} \cap \{C_{i_4}, C_{i_5}\} = \emptyset$$

$$\dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_5}) = 1 \quad \text{and} \quad \{C_{i_1}, C_{i_2}\} \cap \{C_{i_3}, C_{i_5}\} = \emptyset.$$

We want to prove that

$$\psi^{G(Q)}(C_{i_1}, C_{i_2}, C_{i_3}) \psi^{G(Q)}(C_{i_1}, C_{i_2}, C_{i_3}) \psi^{G(Q)}(C_{i_1}, C_{i_2}, C_{i_3}) = 1.$$

To see this, we have to distinguish between two cases:

(i) If $i_1 = i_2$ or $i_3 = i_4$ or $i_4 = i_5$ or $i_3 = i_5$, \{41\} follows from \{28\} and \{33\}.

(ii) Assume $i_1 \neq i_2$, $i_3 \neq i_4$, $i_4 \neq i_5$, $i_3 \neq i_5$. Hence, \{41\} follows from Lemma \{3.2\}.

(v) Proof of \{30\}. Let $C_{i_1}, \ldots, C_{i_9} \in \mathcal{C}$ be a family of circuits such as in \{28\}. We have to verify that

$$\psi^{G(Q)}(C_{i_1}, C_{i_2}, C_{i_3}) \psi^{G(Q)}(C_{i_1}, C_{i_2}, C_{i_3}) \psi^{G(Q)}(C_{i_1}, C_{i_2}, C_{i_3}) = 1.$$

For simplicity, we have to distinguish between two subcases. Case 1. If $i_6 = i_9$, $i_4 = i_7$ and $i_5 = i_8$ \{42\} is obviously verified since the family $G(Q)$ satisfies \{28\}.
Case 2. Conversely, there exists a permutation $\sigma \in S_9$ such that $i_\sigma(1) < i_\sigma(2) < i_\sigma(3)$ and conditions $i_\sigma(6) \geq i_\sigma(9)$, $i_\sigma(4) \geq i_\sigma(7)$ and $i_\sigma(5) \geq i_\sigma(8)$ do not all hold. From Lemma 6.5 equation (42) is equivalent to

$$\psi^G(Q) \left( C_{i_\sigma(1)} C_{i_\sigma(2)} | C_{i_\sigma(6)} C_{i_\sigma(9)} \right) \cdot \psi^G(Q) \left( C_{i_\sigma(2)} C_{i_\sigma(3)} | C_{i_\sigma(4)} C_{i_\sigma(7)} \right) \cdot \psi^G(Q) \left( C_{i_\sigma(3)} C_{i_\sigma(1)} | C_{i_\sigma(5)} C_{i_\sigma(8)} \right) = 1.$$ 

By construction of $G(Q)$ this is equivalent to (37).

(c) $F$ and $G$ are one the inverse of the other.

From the construction of $\tilde{F}$ and $\tilde{G}$ and from the fact that $F$ and $G$ are just the restrictions of such maps to the spaces $G_C(M)$ and $G_R^C(M)$, a straightforward check of definitions shows that $F$ and $G$ are exactly one the inverse of the other.

\[\square\]

7. Geometric description

This section is devoted to an explicit description of the sets $G_C(M)$ and $G_R^C(M)$ as solutions of systems of equations defined by means of explicitly computable matrices. These matrices are derived by the conditions on generalized cross-ratios and reduced cross ratios which are imposed in order to define $G_C(M)$ and $G_R^C(M)$. The matrices we obtain are implicit in [GRS95] and thus we call them GRS matrices. It is the explicit results in this section which allow us to use the geometric and algebraic needed in the applications (especially Section 8.3).

First, some setup. For $m, n \geq 1$ let $Q \subseteq (S^1)^n$ be the space of solutions of the system of $m$ equations in $n$ variables

$$\begin{align*}
z_1^q_1 \cdots z_n^q_n &= \beta_1 \\
&\vdots \\
z_1^q_m \cdots z_n^q_m &= \beta_m
\end{align*}$$

where $q_i^j \in \mathbb{Z}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ and $\beta_l \in S^1$ for $1 \leq l \leq m$. We call lift matrix of $Q$ the matrix $Q = (q^i_j) \in M_{m,n}(\mathbb{Z})$.

We will repeatedly use the following standard fact.

**Lemma 7.1.** The space $Q$ is either empty or homeomorphic to a finite disjoint union of subtori of $(S^1)^n$, each of dimension $\dim \ker Q$.

**Definition 7.2.** Consider a matroid $M$ without minors of Fano or dual-Fano type and let $h_M$ and $p_M$ as in Definition 5.1. Let

$$A \subseteq (S^1)^{h_M}$$

be the space of solutions of the system of the $p_M$ equations (26), (27), (28), (29) and (30). We define the GRS matrix of $M$ to be the lift matrix

$$A \in M_{p_M, h_M}(\mathbb{Z})$$

of $A$.

Now let $k_M$ and $q_M$ as in Definition 6.1. Let

$$B \subseteq (S^1)^{k_M}$$

be the space of solutions of the system of the $k_M$ equations (20), (22) and (24). We define the GRS matrix of $M$ to be the lift matrix

$$B \in M_{k_M, h_M}(\mathbb{Z})$$

of $B$. 
be the space of solutions of the system of the \( q_M \) equations \((35), (36), \) and \((37)\). We define the \( R \)-GRS matrix of \( M \) to be the lift matrix
\[
B \in M_{q_M, k_M}(\mathbb{Z})
\]
of \( B \). This is a matrix of \( q_M \) rows and \( k_M \) columns with integer coefficients.

**Remark 7.3.** Notice that the definition correctly handles the case \( h_M = 0 \), corresponding to the empty set of circuits. In this case, we have the empty matrix imposing a trivial condition on a one-point space, which is correct e.g. with Remark 7.4. The same considerations apply to the case \( k_M = 0 \).

With Lemma 7.1 we can describe some properties of \( \text{hom}(\mathbf{T}^{0}_{M}, S^1) \) using the GRS matrix of \( M \).

**Lemma 7.4.** Let \( M \) be a matroid without minors of Fano or dual-Fano type and let \( A \) be the GRS matrix of \( M \). The topological space \( \text{hom}(\mathbf{T}^{0}_{M}, S^1) \) is homeomorphic to a finite disjoint union of subtori of \((S^1)^{h_M} \), each of dimension \( \dim \ker A \).

**Proof.** Let \( \mathbf{T}^{(2)}_{M} \) be the group defined in [GRS95, Section 3] and let \( A \subseteq (S^1)^{h_M} \) be the space of solutions of equations \((26), (27), (28), (29) \) and \((30) \). Thanks to Lemma 7.1, this is either empty or homeomorphic to a finite disjoint union of subtori of \((S^1)^{h_M} \), each of dimension \( \dim \ker A \). Let now \( A' \subseteq (S^1)^{h_M} \) be the space of solutions of equations \((26), (27), (28), (29) \) and the additional equations
\[
(43) \quad \psi(C_{i_1}C_{i_2}|C_{i_3}C_{i_4})\psi(C_{i_1}C_{i_4}|C_{i_2}C_{i_3})\psi(C_{i_1}C_{i_3}|C_{i_2}C_{i_4}) = 1
\]
for all pairwise distinct \( C_{i_1}, C_{i_2}, C_{i_3}, \) and \( C_{i_4} \).

Again from Lemma 7.1, \( A' \) is either empty or homeomorphic to a finite disjoint union of subtori of \((S^1)^{h_M} \), each of dimension \( \dim \ker A \). Since \((28), (29) \) could not be satisfied at the same time, we must have \( A \cap A' = \emptyset \). Therefore, \( A \cup A' \) is either empty or a finite disjoint union of subtori of \((S^1)^{h_M} \), each of dimension \( \dim \ker A \). From \( \varepsilon^2 = 1 \) and from the definition of \( \mathbf{T}^{(2)}_{M} \) ([GRS95, p. 139-140] for more details) it is not hard to see that \( \text{hom}(\mathbf{T}^{(2)}_{M}, S^1) \) is homeomorphic to \( A \cup A' \). In particular, since there exists at least a homomorphism from \( \mathbf{T}^{(2)}_{M} \) to \( S^1 \), the space \( A \cup A' \) must be not empty. So that, \( A \cup A' \) is a finite disjoint union of subtori of \((S^1)^{h_M} \), each of dimension \( \dim \ker A \) and this completes our proof. \( \square \)

**Corollary 7.5.** Let \( M \) be a matroid without minors of Fano or dual-Fano type and let \( A \) and \( B \) be the GRS and the \( R \)-GRS matrix of \( M \), respectively. Then
\[
\dim \ker A = \dim \ker B.
\]

**Proof.** Let \( B \subseteq (S^1)^{h_M} \) be the space of solutions of equations \((35), (36), (37) \). Thanks to Lemma 7.1, this is either empty or homeomorphic to a finite disjoint union of subtori of \((S^1)^{k_M} \), each of dimension \( \dim \ker B \). Now, let \( B' \subseteq (S^1)^{k_M} \) be the space of solutions of equations \((36), (37) \) and
\[
(45) \quad \phi(C_{q_1}C_{q_2}|C_{q_3}C_{q_4})\phi(C_{q_1}C_{q_4}|C_{q_2}C_{q_3})\phi(C_{q_1}C_{q_3}|C_{q_2}C_{q_4}) = 1
\]
for any family of pairwise distinct circuits \( C_{q_1}, C_{q_2}, C_{q_3}, \) and \( C_{q_4} \) such that \( q_1 < q_2 < q_3 < q_4 \) and \( \dim(C_{q_1} \cup C_{q_2} \cup C_{q_3} \cup C_{q_4}) = 1 \). Again from Lemma 7.1, \( B' \) is either empty or homeomorphic to a finite disjoint union of subtori of \((S^1)^{k_M} \), each of dimension \( \dim \ker B \). Since \((37) \) and \((45) \) could not be satisfied at the same time, we must have \( B \cap B' = \emptyset \). Therefore, \( B \cup B' \) is either empty or a finite disjoint union of subtori of \((S^1)^{k_M} \), each of dimension \( \dim \ker B \). From the proof of Theorem 5.3 it follows that the maps \( \tilde{F} : (S^1)^{h_M} \rightarrow (S^1)^{k_M} \) and \( \tilde{G} : (S^1)^{k_M} \rightarrow (S^1)^{h_M} \) restrict to a homeomorphism
\[
A \cup A' \leftrightarrow B \cup B'.
\]
Thus, \( B \cup B' \) is non empty. As a consequence of [Hat02] Theorem 2.26 the manifolds \( A \cup A' \) and \( B \cup B' \) must have the same dimension (here \( A' \) is as in the proof of the lemma above). So that, (44) is verified. □

\[A \cup A\]

**Proposition 7.6.** Given a matroid \( M \) without minors of Fano or dual-Fano type, let \( q_M \) and \( k_M \) as in Definition 6.7. Let \( n_M \) be the number of equations of type (35) and let \( B \subseteq (S^1)^{k_M} \) be the space of solutions of (35), (36) and (37). The following properties hold:

(i) \( k_M = 3n_M \). Thus, \((S^1)^{k_M} = (S^1)^{3n_M}\);

(ii) \( B \) is either empty or homeomorphic to a finite disjoint union of subtori of \((S^1)^{3n_M}\), each of dimension \( \dim B \);

(iii) Up to permuting variables \( \mathcal{G}_C(M) \) equals

\[
\mathcal{B} \cap \prod_{j=1}^{n_M} (X \times S^1) \subseteq (S^1)^{3n_M}
\]

where \( X = \{(f, g) \in S^1 \times S^1 \mid 0 \in \text{pconv}\{1, -f, -g\}\}\).

**Proof.** Part (i) is immediate from Lemma 8.5. Part (ii) is a consequence of Lemma 7.1. For part (iii) notice first that, by definition and with part (i), the space \( \mathcal{G}_C(M) \) is the intersection \( B \cap W \subseteq (S^1)^{3n_M} \), where \( W \) is the space of points of \((S^1)^{3n_M}\) satisfying (35). Hence, it suffices to show that, up to permuting variables,

\[
W = \prod_{j=1}^{n_M} (X \times S^1).
\]

But this follows from the decomposition

\[
\{(\alpha_j, \beta_j, \gamma_j)_{1 \leq j \leq n_M} \in (S^1)^{3n_M} \mid 0 \in \text{pconv}\{1, -\alpha_j, -\beta_j\}\} =
\]

\[
\bigcap_{j=1}^{n_M} \left\{ (\alpha_j, \beta_j, \gamma_j)_{1 \leq j \leq n_M} \in (S^1)^{3n_M} \mid 0 \in \text{pconv}\{1, -\alpha_k, -\beta_k\} \right\} =
\]

\[
\bigcap_{j=1}^{n_M} \left[ \prod_{h=1}^{k-1} (S^1 \times S^1 \times S^1) \right] \times (X \times S^1) \times \left( \prod_{s=j+1}^{n_M} (S^1 \times S^1 \times S^1) \right) =
\]

\[
\prod_{j=1}^{n_M} (X \times S^1).
\]

□

**Corollary 7.7.** Let \( F : (S^1)^{3n_M} \rightarrow (S^1)^{2n_M} \) be the map \((\alpha_j, \beta_j, \gamma_j)_{1 \leq j \leq n_M} \mapsto (\alpha_j, \beta_j)_{1 \leq j \leq n_M}\). There exists a homeomorphism

\[
F|_{B \cap \prod_{j=1}^{n_M} (X \times S^1)} : B \cap \prod_{j=1}^{n_M} (X \times S^1) \rightarrow F(B) \cap \prod_{j=1}^{n_M} X.
\]

**Proof.** Let \( \tilde{B} \subseteq (S^1)^{3n_M} \) be the space of solutions of (35). \( \tilde{B} \) is non empty, since \((1, 1, -1, \ldots, 1, 1, -1) \in \tilde{B}\). Moreover, \( \tilde{B} \) is compact since it is a closed subspace of the compact space \((S^1)^{3n_M}\). The restriction map

\[
F|_B : \tilde{B} \rightarrow (S^1)^{2n_M}
\]

is a homeomorphism. In fact:

- \( F|_B : \tilde{B} \rightarrow (S^1)^{2n_M} \) is continuous, since is the restriction of the continuous map \( F : (S^1)^{3n_M} \rightarrow (S^1)^{2n_M} \);

- \( F|_B : \tilde{B} \rightarrow (S^1)^{2n_M} \) is bijective, with inverse \( G : (S^1)^{2n_M} \rightarrow \tilde{B} \) defined by \((u_j, v_j)_{1 \leq j \leq n_M} \mapsto (u_j, v_j, -\overline{u_j}v_j)_{1 \leq j \leq n_M}\).
– $F|_{\tilde{B}} : \tilde{B} \to (S^1)^{2nM}$ is closed, since it is a continuous map from a compact topological space to a Hausdorff space.

From the fact that restriction of a homeomorphism is also a homeomorphism and from

$$B \cap \prod_{j=1}^{nM} (X \times S^1) \subseteq \tilde{B}$$

we deduce that there exists a homeomorphism

$$F|_{B \cap \prod_{j=1}^{nM} (X \times S^1)} : B \cap \prod_{j=1}^{nM} (X \times S^1) \to F\left(B \cap \prod_{j=1}^{nM} (X \times S^1)\right).$$

To conclude our proof it suffices to show that

$$F(B) \cap \prod_{j=1}^{nM} X = F\left(B \cap \prod_{j=1}^{nM} (X \times S^1)\right)$$

holds. For the left-to-right inclusion let $P \in F(B) \cap \prod_{j=1}^{nM} X$ and write $P$ in the form $P = (u_j, v_j)_1 \leq j \leq m$. From $P \in F(B)$, there exists $Q \in B$ such that $P = F(Q)$. It remains to be checked that

$$Q \in \prod_{j=1}^{nM} (X \times S^1).$$

Since the points of $B$ satisfy (35), we must have $Q = (u_j, v_j, -\overline{w_j} v_j)_1 \leq j \leq m$. Thus, (46) is satisfied.

For the right-to-left inclusion, let $P \in F\left(B \cap \prod_{j=1}^{nM} (X \times S^1)\right)$. Hence, there is $Q \in B \cap \prod_{j=1}^{nM} (X \times S^1)$ such that $P = F(Q)$. Clearly $P \in F(B)$. Therefore, it suffices to show that

$$P \in \prod_{j=1}^{nM} X.$$

Write $P$ in the form $P = (u_j, v_j)_1 \leq j \leq m$. Since the points of $B$ satisfy (35), from $Q \in B$ and $P = F(Q)$ we deduce that $Q = (u_j, v_j, -\overline{w_j} v_j)_1 \leq j \leq m$. This, together with $Q \in \prod_{j=1}^{nM} (X \times S^1)$ and the definition of the map $F$, implies (47). □

8. Some applications

8.1. Rank of inner Tutte groups. The rank of the inner Tutte group was first computed by Brändén and González D’León in [BGD10]. This, together with [Wen89] Theorem 5.4, implies a full description of the inner Tutte group of any matroid with up to 7 elements. Here we prove that this rank equals the dimension of the kernel of the GRS matrix $A$ of $M$. In particular, this allows for a relatively efficient explicit computation of these groups. Some results obtained with SAGE on a standard laptop are listed in Table 1.

**Theorem 8.1.** Let $M$ be a matroid without minors of Fano or dual-Fano type. Let $A$ be the GRS matrix of $M$ and let $b_M$ be the rank of $\tau_M^{(0)}$. Then

$$b_M = \dim \ker A.$$  

Moreover, the same result holds for $B$, the R-GRS matrix of $M$. 
### Table 1

| Matroid | Inner Tutte Group |
|---------|-------------------|
| $F_7^-$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}$ |
| $(F_7^-)^*$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}$ |
| $O_7$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}^2$ |
| $P_6$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}^9$ |
| $P_7$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}^2$ |
| $Q_6$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}^5$ |
| $R_6$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}^4$ |
| $M(K_4)$ | $\mathbb{Z}_{2^2}$ |
| $W^3$ | $\mathbb{Z}_{2^2} \times \mathbb{Z}^2$ |

**Corollary 8.2.** Let $M$ be a matroid and assume $\sharp(E) \leq 7$. Let $A$ be the GRS matrix of $M$. Hence,

$$T_0^{(M)} = \begin{cases} \mathbb{Z}_{2^2} \times \mathbb{Z}^{\dim \ker A} & \text{if } M \text{ has no minors of Fano or dual-Fano type,} \\ 0 & \text{else.} \end{cases}$$

Moreover, the same result holds for $B$, the $R$-GRS matrix of $M$.

To prove Theorem 8.1 we need to recall a well known property of the classical Pontrjagin dual of finitely generated abelian groups.

**Lemma 8.3.** Let $G$ be a finitely generated abelian group and let $b_G$ be the rank of $G$. The classical Pontrjagin dual $\text{hom}(G, S^1)$ is homeomorphic to a finite disjoint union of tori, each of dimension $b_G$.

**Proof.** Thanks to the fundamental theorem of finitely generated abelian groups, there exists an isomorphism

$$\Phi : G \to \mathbb{Z}_{p_1^a_1} \times \cdots \times \mathbb{Z}_{p_l^a_l} \times \mathbb{Z}^{b_G},$$

where $p_1, \ldots, p_l \in \mathbb{N}$ are prime numbers and $a_1, \ldots, a_l \in \mathbb{N}_+$. Such isomorphism induces a homeomorphism between $\text{hom}(G, S^1)$ and the space

$$U_G = \{ (\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_{b_G}) \in (S^1)^{l+b_G} \mid p_i^{a_i} \equiv 0 \mod \text{ord}(\alpha_i) \} \subseteq (S^1)^{l+b_G}.$$

Clearly $U_G \subseteq (S^1)^{l+b_G}$ is a finite disjoint union of subtori of $(S^1)^{l+b_G}$, each of dimension $b_G$, proving our statement. \[\square\]

**Proof of Theorem 8.1.** With Corollary 7.5 it suffices to show that (18) holds. From Lemma 8.3, $\text{hom}(T_M^{(0)}, S^1)$ is homeomorphic to a finite disjoint union of tori, each of dimension $b_M$. According to Lemma 7.4, $\text{hom}(T_M^{(0)}, S^1)$ is homeomorphic to a finite disjoint union of tori, each of dimension $\dim \ker A$. The claim follows from [Hat02, Theorem 2.26]. \[\square\]

#### 8.2. An upper bound to reorientation classes

As a consequence of Theorem 8.1 and Corollary 7.7, we obtain an explicitly computable upper bound to the number of reorientation classes of oriented matroids over $M$. 
Proposition 8.4. For a matroid $M$ without minors of Fano or dual-Fano type let $n_M$ as in Definition 8.1 and let $R_R(M)$ be the set of reorientation classes of oriented matroids over $M$ as defined in [GRS95]. Hence,

$$|R_R(M)| \leq 3^{n_M}.$$  

Proof. From [0], [31], Theorem 5.3 and Corollary 7.7 it is not hard to see that the homeomorphism

$$\mathcal{R}_C(M) \leftrightarrow F(B) \cap \prod_{j=1}^{n_M} X$$

restricts to a one-to-one correspondence

$$\mathcal{R}_R(M) \leftrightarrow F(B) \cap \prod_{j=1}^{n_M} \{(1,-1),(-1,1),(-1,-1)\}. \quad (49)$$

Thus,

$$|R_R(M)| \leq \prod_{j=1}^{n_M} \{(1,-1),(-1,1),(-1,-1)\} = 3^{n_M}.$$

\hfill \Box

8.3. Non realizable uniform phased matroids. In this section we use some topological and geometric properties of the space introduced in Corollary 7.7 to prove the existence of non realizable, nonchirotopal uniform phased matroid in any rank.

First we recall the definitions of the terms of the statement. The uniform matroid $U_d(m)$ is the matroid with ground set $E = \{1, \ldots, m\}$ and set of bases

$$B = \{\{i_1 \cdots i_d\} \mid 1 \leq i_1 < \cdots < i_d \leq m\}.$$ 

Recall that a phased matroid $\Phi \in \mathcal{MC}(M)$ is called realizable if there exists a complex matrix $A$ such that $\varphi_A \in \Phi$ (see Example [2] for the definition of $\varphi_A$). Moreover, we call $\Phi \in \mathcal{MC}(M)$ chirotopal if there exists $\chi \in \Phi$ with $\chi(E) \subseteq \{0, +1, -1\}$ (this $\chi$ is then a chirotope in the sense of oriented matroid theory, see [BLVS99]).

Definition 8.5. Let $M$ be a matroid of rank $d$ over $m$ elements. We say that a phased class $P \in \mathcal{R}_C(M)$ is realizable if there exists a realizable phased matroid $\Phi \in \mathcal{MC}(M)$ such that $P = \pi(\Phi)$, where $\pi : \mathcal{MC}(M) \to \mathcal{R}_C(M)$ is the standard quotient map. We say that $P$ is non realizable otherwise.

Notice that, with [BKRG03, Lemma 1.6], the realizability of a phased class $P \in \mathcal{R}_C(M)$ is equivalent to requiring $\Phi$ being realizable for each phased matroid such that $P = \pi(\Phi)$.

We now state our theorem and outline the main steps of the proof, postponing the proofs of some technical lemmas in Appendix C.

Theorem 8.6. For $m \geq 5$ and $2 \leq d \leq m - 2$ there exists a non realizable non chirotopal phased matroid with underlying matroid $U_d(m)$.

Proof. Let us fix an arbitrary linear ordering of the ground set $E$ and, as above, an enumeration $C_1, C_2, \ldots$ of the circuits of $U_d(m)$ (e.g. according to the lexicographic order).

The idea of the proof is to exploit the topology of the subspace of realizable classes in order to prove that it cannot exhaust the space $\mathcal{R}_C(U_d(m))$. This is the content of Lemma C.7 which reduces our task to finding a certain type of realizable class, and more precisely to finding a matrix $A \in M_{d,m}(\mathbb{C})$ such that:
(A1) the phirotope $\varphi_A$ associated to $A$ has underlying matroid $U_d(m)$;

(A2) for any family $C_{d_1}, C_{d_2}, C_{d_3}, C_{d_4} \in \mathcal{C}$ of circuits of $U_d(m)$ with $d_1 < d_2 < d_3 < d_4$ and $\dim(C_{d_1} \cup C_{d_2} \cup C_{d_3} \cup C_{d_4}) = 1$, one has

$$
\begin{pmatrix}
\varphi_A(C_{d_1}(x_{d_1})) & \varphi_A(C_{d_2}(x_{d_2})) & \varphi_A(C_{d_3}(x_{d_3})) & \varphi_A(C_{d_4}(x_{d_4}))
\end{pmatrix}
\begin{pmatrix}
\varphi_A(C_{d_1}(x_{d_2})) & \varphi_A(C_{d_2}(x_{d_3})) & \varphi_A(C_{d_3}(x_{d_4})) & \varphi_A(C_{d_4}(x_{d_1}))
\end{pmatrix}^{-1} 
\notin \{(1, 1), (1, -1), (-1, 1)\}
$$

where $x_{d_i} = (C_{d_1} \cup C_{d_2} \cup C_{d_3} \cup C_{d_4}) \setminus C_{d_i}$. For $i \neq j \varphi_A(C_{d_i}(x_{d_j}))$ is the evaluation of the phirotope $\varphi_A$ on the ordered $d$-uple $C_{d_i}(x_{d_j}) = C_{d_i} \setminus \{x_{d_j}\}$. Notice that $U_d(m)$ is the uniform matroid, so that $C_{d_i}(x_{d_j})$ is a basis. Therefore, $\varphi_A(C_{d_i}(x_{d_j})) \neq 0$.

This can be done as follows. Let $U$ denote the space of all complex $d \times m$ matrices of the form

$$
A = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ddots & 1 & \ldots & 1
\end{pmatrix}
$$

and whose associated matroid is $U_d(m)$. Set $N = n_{U_d(m)}$ and consider the function $F : U \to \mathbb{C}^{2N}$ whose components are given by pairs

$$
\begin{pmatrix}
\det A(C_{d_1}(x_{d_1})) & \det A(C_{d_2}(x_{d_2})) & \det A(C_{d_3}(x_{d_3})) & \det A(C_{d_4}(x_{d_4})) \\
\det A(C_{d_1}(x_{d_2})) & \det A(C_{d_2}(x_{d_3})) & \det A(C_{d_3}(x_{d_4})) & \det A(C_{d_4}(x_{d_1}))
\end{pmatrix}
$$

one for each family of circuits as in (A2), were $A(C_{d_i}(x_{d_j}))$ denotes the $d \times d$ submatrix of $A$ with columns indexed by the ordered $d$-uple $C_{d_i}(x_{d_j})$. The same argument as in (A2) shows that the expressions in (51) are well defined. By Corollary [C.4] the components of the map $F$ are all nonconstant and thus, by Lemma [C.10] there exists $\tilde{A} \in U$ such that for each family of circuits appearing in (A2)

$$
\begin{pmatrix}
\det \tilde{A}(C_{d_1}(x_{d_1})) & \det \tilde{A}(C_{d_2}(x_{d_2})) & \det \tilde{A}(C_{d_3}(x_{d_3})) & \det \tilde{A}(C_{d_4}(x_{d_4})) \\
\det A(C_{d_1}(x_{d_2})) & \det A(C_{d_2}(x_{d_3})) & \det A(C_{d_3}(x_{d_4})) & \det A(C_{d_4}(x_{d_1}))
\end{pmatrix}
\in (\mathbb{C} \setminus \mathbb{R})^2.
$$

By definition, $\varphi_{\tilde{A}}$ satisfies (50), and the underlying matroid of $\varphi_{\tilde{A}}$ is $U_d(m)$ since $\tilde{A} \in U$.

**APPENDIX A. REORIENTATION CLASSES AND PHASING CLASSES**

We now provide detailed proof of the existence for set injections [5] and [6]. At first we need to recall some definitions from [GRS95].

Let $M$ be a matroid. A circuit signature $\sigma$ on $M$ is a set of functions, $\sigma C : C \to S^0$, one for each circuit $C \in \mathcal{C}$. Analogously, a cocircuit signature $\tau$ is a set of functions $\tau D : D \to S^0$, one for each cocircuit $D \in \mathcal{C}^*$. Thus, a circuit signature $\sigma$ is a $S^1$ circuit signature with $S^0$-values. The same for cocircuit signatures.

We say that a circuit signature $\sigma$ and a cocircuit signature $\tau$ are orthogonal (denoted by $\sigma \perp \tau$) if, for any circuit $C \in \mathcal{C}$ and cocircuit $D \in \mathcal{C}^*$ with non-empty intersection, i.e., with $C \cap D \neq \emptyset$, condition (3) is satisfied.

Since each element of $S^0$ coincide with its inverse, (3) is equivalent to

$$
\{\sigma C(x)\tau D(x) \mid x \in C \cap D\} = \{1, -1\}.
$$

Two circuit signatures $\sigma_1$ and $\sigma_2$ are ~-equivalent if for any $C \in \mathcal{C}$, either $\sigma_1 C(x) = \sigma_2 C(x)$ for all $x \in C$, or $\sigma_1 C(x) = -\sigma_2 C(x)$ for all $x \in C$. 




An oriented matroid with underlying matroid \( M \) is a \( \sim \)-equivalence class of circuit signatures on \( M \) such that there exists a cocircuit signature \( \tau \) on \( M \) satisfying \( \sigma \perp \tau \) for some representative \( \sigma \) of \( \Sigma \).

It is not hard to see that if \( \tau \) is an oriented cocircuit signature such that \( \sigma \perp \tau \) for some representative \( \sigma \) of \( \Sigma \), then \( \sigma' \perp \tau \) for each representative \( \sigma' \) of \( \Sigma \).

We will denote the set of oriented matroids with underlying matroid \( M \) by \( \mathcal{M}_\Sigma(M) \).

**Note A.1.** In contrast with Section 2, we do not stress the fact that we are working with oriented matroids defined in terms of circuit signatures. This because cryptomorphic equivalent definitions of oriented matroids are well known in the literature. For more details, see [BLVS+99] and [CRS95].

**Proposition A.2.** Let \( M \) be a matroid and let \( \sigma_1 \) and \( \sigma_2 \) be two circuit signatures. They are \( \sim \)-equivalent if and only if they are \( \sim_c \)-equivalent as \( S^1 \) circuit signatures. The same result holds for oriented cocircuit signatures \( \tau_1 \) and \( \tau_2 \).

**Proof.** It is obvious that if \( \sigma_1 \) and \( \sigma_2 \) are equivalent as oriented circuit signatures they are also equivalent as phased circuit signatures. Thus, we only have to check that if \( \sigma_1 \sim_c \sigma_2 \) as phased circuit signatures, then \( \sigma_1 \) and \( \sigma_2 \) are equivalent oriented circuit signatures. Let us assume \( \sigma_1 \sim_c \sigma_2 \). Hence, for every circuit \( C \in \mathcal{C} \) there exists \( b_C \in S^1 \) such that \( \sigma_1(C) = b_C \sigma_2(C) \) for all \( C \in C \). Since \( \sigma_1(C) \) and \( \sigma_2(C) \) are \( S^0 \)-valued functions for each circuit \( C \), we deduce that \( b_C \in S^0 \) for all \( C \in \mathcal{C} \). \( \square \)

**Corollary A.3.** Let \( M \) be a matroid. There exists a set injection

\[
i : \mathcal{M}_\Sigma(M) \hookrightarrow \mathcal{M}_{\Sigma}(M).
\]

**Proof.** It suffices to show that there exists a set injection

\[
i : \mathcal{M}_\Sigma(M) \hookrightarrow \mathcal{M}_{\Sigma}(M).
\]

Let \( \mathcal{N}_\Sigma(M) \) be the set of pairs \((E, \sigma)\), where \( \sigma \) is an oriented circuit signature on \( M \) such that there exists an oriented cocircuit signature \( \tau \) on \( M \) satisfying \( \sigma \perp \tau \). Let \( l : \mathcal{N}_\Sigma(M) \hookrightarrow \mathcal{N}_{\Sigma}^c(M) \) be the inclusion map and let \( \pi_{\sim_c} : \mathcal{N}_{\Sigma}^c(M) \rightarrow \mathcal{M}_{\Sigma}(M) \) be the quotient projection. Consider the map \( s : \mathcal{N}_\Sigma(M) \rightarrow \mathcal{M}_{\Sigma}^c(M) \) defined as composition \( s = \pi_{\sim_c} \circ l \). Let finally \((E, \sigma_1)\) and \((E, \sigma_2)\) be elements of \( \mathcal{N}_\Sigma(M) \).

With Proposition A.2 the following facts are equivalent:

1. The oriented circuit signatures \( \sigma_1 \) and \( \sigma_2 \) are in the same equivalence class;
2. \( s((E, \sigma_1)) = s((E, \sigma_2)) \).

Thus, there exists a set injection

\[
i : \mathcal{M}_\Sigma(M) \hookrightarrow \mathcal{M}_{\Sigma}^c(M)
\]

and this completes our proof. \( \square \)

Two oriented circuit signatures \( \sigma_1 \) and \( \sigma_2 \) are called \( \approx \) equivalent (denoted \( \sigma_1 \approx \sigma_2 \)) if there exists a subset \( A \subseteq E \) such that

\[
\sigma_1(C) = \begin{cases} 
\sigma_2(C) & \text{for } x \notin A \\
-\sigma_2(C) & \text{for } x \in A
\end{cases}
\]

We also consider the same relation for phased cocircuit signatures.

Finally, we say that two oriented matroids \((E, \Sigma_1)\) and \((E, \Sigma_2)\) are in the same reorientation class (denoted by \((E, \Sigma_1) \cong (E, \Sigma_2)\)) if there exist oriented circuit signatures \( \sigma_1 \in \Sigma_1 \) and \( \sigma_2 \in \Sigma_2 \) such that \( \sigma_1 \approx \sigma_2 \).

We will denote the set of reorientation classes of oriented matroids with underlying matroid \( M \) by \( \mathcal{R}_\Sigma(M) \). The same considerations of Note A.1 are valid here.
Proposition A.4. Let $M$ be a matroid. There exists a set injection
\[ j : R_S(M) \mapsto R_C(M). \]

Proof. It is enough to show that, given oriented matroids $(E, \Sigma_1)$ and $(E, \Sigma_2)$, the following facts are equivalent:

1. The oriented matroids $(E, \Sigma_1)$ and $(E, \Sigma_2)$ are in the same reorientation class;
2. The phased matroids $i((E, \Sigma_1))$ and $i((E, \Sigma_2))$ are in the same $\cong_e$ equivalence class.

If $(E, \Sigma_1)$ and $(E, \Sigma_2)$ are in the same reorientation class, connected.

With (52), we find $i((E, \Sigma_1))$ and $i((E, \Sigma_2))$ are in the same $\cong_e$ equivalence class, too. Hence, it remains to be proved that if $i((E, \Sigma_1)) \cong_e i((E, \Sigma_2))$, then the oriented matroids $(E, \Sigma_1)$ and $(E, \Sigma_2)$ are in the same reorientation class. Let us suppose $i((E, \Sigma_1)) \cong_e i((E, \Sigma_2))$. Let $\sigma_1$ and $\sigma_2$ be oriented circuit signatures such that $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. By definition of $\cong_e$-equivalence there exist families $\{b_C\} \subseteq S^1$ and $\{a(i)\} \subseteq S^1$ such that for $C \in C$ and $x \in C$

\[ \sigma_1 C(x) = a(x)b_C \sigma_2 C(x). \]

At first we consider the case when $M$ is connected. Under this assumption,
\[ a(i)b_C \in S^0 \]
for $C \in C$ and $i \in E$. To see this, let $C \in C$ and $i \in E$. We have to distinguish between two cases.

Case 1. If $i \in C$, (53) follows from (52) and the hypothesis that $\sigma_1 C(i)$ and $\sigma_2 C(i)$ are elements of $S^0$ for $C \in C$, $i \in C$.

Case 2. Let us assume $i \notin C$ and let $j \in E$ such that $j \notin C$. From the connection of $M$ and [Oxl92, Proposition 4.1.4], there exists a circuit $C' \in C$ such that $i, j \in C'$.

With (52), we find
\[ a(i)b_C = a(i)b_C \cdot a(j)b_{C'} = \frac{\sigma_1 C(i)}{\sigma_2 C(i)} \sigma_2 C'(j) \sigma_1 C(i) \sigma_2 C'(j) \sigma_1 C(i) \in S^0 \]
\[ \text{since } \sigma_1 \text{ and } \sigma_2 \text{ are oriented circuit signatures.} \]
\[ \text{Let } K \in C \text{ be a fixed circuit. For } C \in C \text{ we have} \]
\[ \frac{b_C}{b_K} \in S^0. \]

In order to prove this, pick $i \in C$ and $j \in K$ and consider a circuit $C' \in C$ with $i, j \in C'$. Such a circuit always exists, since $M$ is connected and therefore [Oxl92, Corollary 4.1.4] holds. Again with (52) and the fact that $\sigma_1$ and $\sigma_2$ are circuit signatures, we deduce that
\[ \frac{b_C}{b_K} = \frac{a(i)b_C}{a(j)b_{C'}} \cdot \frac{a(i)b_C}{a(j)b_{C'}} = \frac{\sigma_1 C(i)}{\sigma_2 C(i)} \sigma_2 K(j) \sigma_1 C'(j) \sigma_2 C'(j) \sigma_1 C'(j) \sigma_2 K(j) \sigma_1 C'(j) \sigma_2 C'(j) \sigma_1 C'(j) \in S^0. \]
\[ \text{For } C \in C \text{ and } x \in C \text{ set} \]
\[ \tilde{\sigma} C(x) = a(x)b_K \sigma_2 C(x). \]
\[ \text{With (53), this is indeed a circuit signature. By definition of } \tilde{\sigma} \text{ it is obvious that} \]
\[ \tilde{\sigma} \cong \sigma_1. \]
\[ \text{Moreover, from} \]
\[ \sigma_1 C(x) = \frac{b_C}{b_K} \tilde{\sigma} C(x) \]
and (54) we conclude that $\tilde{\sigma} \cong \sigma_1$. In particular, $\tilde{\sigma} \in \Sigma_1$. Thus, we have found circuit signatures $\tilde{\sigma} \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ such that $\tilde{\sigma} \cong \sigma_2$. So that, the oriented matroids $(E, \Sigma_1)$ and $(E, \Sigma_2)$ are in the same reorientation class, proving our statement for $M$ connected.
Now, let us assume $M$ not connected and let $T_1, \ldots, T_k$ ($k \geq 2$) be the connected components of $M$. For $1 \leq j \leq k$, let $M_j$ be the restriction $M|T_j$ and let $\mathcal{E}_j$ be the set of circuits of $M_j$. Let finally $K_j \in \mathcal{E}_j$ be a fixed circuit, one for each $j = 1, \ldots, k$. Thanks to [Oxl92, 4.2.16] we can define a function $\mu : \mathcal{E} \rightarrow \{1, \ldots, k\}$ which associates, to each circuit $C \in \mathcal{E}$, the only index $j$ such that $C \in \mathcal{E}_j$. Using the arguments of the connected case, we can deduce the following facts:

\begin{equation}
\tag{55}
a(i)b_C \in S^0
\end{equation}
for $1 \leq j \leq k$, $C \in \mathcal{E}_j$ and $i \in T_j$.

\begin{equation}
\frac{b_C}{b_{K_{\mu(C)}}} \in S^0
\end{equation}
for $C \in \mathcal{E}$.

For $C \in \mathcal{E}$, $x \in C$ set
\[\tilde{\sigma}C(x) = a(x)b_{K_{\mu(C)}}\sigma_2C(x)\] 
From (55) and the definition of the function $\mu$ it follows that $\tilde{\sigma}$ is a circuit signature satisfying $\tilde{\sigma} \approx \sigma_2$. Moreover, 
\[\sigma_1C(x) = \frac{b_C}{b_{K_{\mu(C)}}} \tilde{\sigma}C(x)\]
and (56) imply that $\tilde{\sigma} \sim \sigma_1$. Then, $\tilde{\sigma} \in \Sigma_1$. Hence, we have found circuit signatures $\tilde{\sigma} \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ such that $\tilde{\sigma} \approx \sigma_2$ proving that the oriented matroids $(E, \Sigma_1)$ and $(E, \Sigma_2)$ are in the same reorientation class.

\section*{Appendix B. Technical proofs of Section 6}

**Lemma B.1.** Let $c \in \mathbb{R}$ and let $\psi(C_{i_1}C_{i_2}|C_{i_3}C_{i_4})$ be values defined for circuits $C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} \in \mathcal{E}$ such that $\dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1$ and $\{C_{i_1}, C_{i_2}\} \cap \{C_{i_3}, C_{i_4}\} = \emptyset$, ranging in $S^1$, and satisfying (26), (27) and (33). Consider a permutation
\[\sigma = \left(\begin{array}{cccc}1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{array}\right) \in S_4.\]
Then,
\[\psi(C_{i_1}C_{i_2}|C_{i_3}C_{i_4})\psi(C_{i_1}C_{i_3}|C_{i_2}C_{i_4})\psi(C_{i_1}C_{i_2}|C_{i_3}C_{i_4}) = c.\]
is equivalent to
\[\psi(C_{\sigma(1)}C_{\sigma(2)}|C_{\sigma(3)}C_{\sigma(4)})\psi(C_{\sigma(1)}C_{\sigma(4)}|C_{\sigma(2)}C_{\sigma(3)})\psi(C_{\sigma(1)}C_{\sigma(3)}|C_{\sigma(4)}C_{\sigma(2)}) = c.\]

**Proof.** Since $S_4$ is generated by the transpositions (12), (23) and (34), it suffices to check our statement for $\sigma = (12)$, $\sigma = (23)$ and $\sigma = (34)$, which can be done straightforwardly using (26), (27) and (33). □

**Lemma B.2.** Let $\psi(C_{i_1}C_{i_2}|C_{i_3}C_{i_4})$ be values defined for circuits $C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} \in \mathcal{E}$ such that $\dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1$ and $\{C_{i_1}, C_{i_2}\} \cap \{C_{i_3}, C_{i_4}\} = \emptyset$, ranging in $S^1$ and satisfying (26), (27), (28) and (33). Consider a permutation
\[\sigma = \left(\begin{array}{cccc}1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{array}\right) \in S_4.\]
Then, relation (28) is equivalent to
\[0 \in \text{pconv}\left(\{1, -\psi(C_{\sigma(1)}C_{\sigma(2)}|C_{\sigma(3)}C_{\sigma(4)}), -\psi(C_{\sigma(4)}C_{\sigma(2)}|C_{\sigma(3)}C_{\sigma(1)})\}\right).\]
Proof. As in the proof of Lemma B.1, it is sufficient to check the thesis for transpositions (12), (23) and (34). Using (20), (27), (28) and (29) the claim can be verified by direct computation. □

Lemma B.3. Let \( \psi(C_{i_1}, C_{i_2}|C_{i_3}, C_{i_4}) \) be values defined for circuits \( C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} \in \mathcal{C} \) such that \( \dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1 \) and \( \{C_{i_1}, C_{i_2}\} \cap \{C_{i_3}, C_{i_4}\} = \emptyset \), ranging in \( S^1 \), and satisfying (20), (27) and (33). Let \( C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} \in \mathcal{C} \) be pairwise distinct circuits such that \( \dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1 \) and consider a permutation

\[
\sigma = \left( \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \right) \in S_6.
\]

The following families of equations are equivalent

\[
(1) \quad \psi(C_{i_1}, C_{i_2}|C_{i_3}, C_{i_4}) = 1,
\]

\[
(2) \quad \psi(C_{i_1}, C_{i_2}|C_{i_3}, C_{i_4}) = 1,
\]

Proof. As in the proof of the previous results, since \( S_6 \) is generated by transpositions (12), (23), (34), (45) it is sufficient to check the statement for \( \sigma \) being such transposition. Using (27) and (33) we can easily deduce the thesis. □

Lemma B.4. Let \( \psi(C_{i_1}, C_{i_2}|C_{i_3}, C_{i_4}) \) be values defined for circuits \( C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} \in \mathcal{C} \) such that \( \dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1 \) and \( \{C_{i_1}, C_{i_2}\} \cap \{C_{i_3}, C_{i_4}\} = \emptyset \), ranging in \( S^1 \) and satisfying (20), (27) and (33). Let \( C_{i_1}, \ldots, C_{i_9} \in \mathcal{C} \) such as in (30) and let \( \mu, \nu \in S_9 \) be the permutations

\[
\mu = \left( \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array} \right)
\]

\[
\nu = \left( \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array} \right)
\]
Proof. Step 1: inclusion and the set of triples \[ \nu = (47)(58)(69). \]

Let finally
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) & \sigma(6) & \sigma(7) & \sigma(8) & \sigma(9) \end{pmatrix} \in \langle \mu, \nu \rangle. \]

Then, relation (34) is equivalent to
\[ \psi(C_{\sigma(1)} C_{\sigma(2)} | C_{\sigma(1)-1(3)+3} C_{\sigma(1)-1(3)+6}), \]
\[ \cdot \psi(C_{\sigma(2)} C_{\sigma(3)} | C_{\sigma(1)-1(1)+3} C_{\sigma(1)-1(1)+6}) \]
\[ \cdot \psi(C_{\sigma(3)} C_{\sigma(1)+3} C_{\sigma(1)-1(2)+3} C_{\sigma(1)-1(2)+6}) = 1 \]

Proof. The subgroup \( \{ \mu, \nu \} < S_9 \) is generated by (12), (23) and \( \nu \). Hence, to conclude the thesis it is sufficient to verify the statement for such permutations. Thanks to (27) and (33) we can check our claim by direct computation. \( \square \)

Lemma B.5. There exists a one-to-one correspondence between
\[ X = \left\{ (q_1, q_2, q_3, q_4) \mid q_1 < q_2 < q_3 < q_4, q_1 < q_3 \right\} \]
and the set of triples
\[ Y = \left\{ \begin{array}{c|c} (d_1, d_2, d_3, d_4) & d_1 < d_2 < d_3 < d_4 \\ \hline (1, d_1, d_3, d_2) & C_{d_1, C_{d_2}}, C_{d_3} C_{d_4}, C_{q_1} C_{q_2}, C_{q_3} C_{q_4} \in C \text{ pairwise distinct} \\ \hline (1, d_1, d_2, d_3) & \text{dim}(C_{d_1} \cup C_{d_2} \cup C_{d_3} \cup C_{d_4}) = 1 \end{array} \right\}. \]

Proof. Step 1: inclusion \( \supset \) Let \( P \in X \). We have to distinguish between three cases:
(1) \( P = (d_1, d_2, d_3, d_4) \) and \( d_1 < d_2 < d_3 < d_4 \), then obviously \( P \in W \);
(2) \( P = (d_1, d_3, d_2, d_4) \) and \( d_1 < d_3 < d_2 < d_4 \). Hence, \( d_1 < d_3, d_2 < d_4 \) and \( d_1 < d_2 \) so that \( P \in W \);
(3) \( P = (d_1, d_4, d_2, d_3) \) and \( d_1 < d_4 < d_2 < d_3 \). In a similar way, \( d_1 < d_4, d_2 < d_3 \) and \( d_1 < d_2 \) and this proves that \( P \in X \).

Step 2: inclusion \( \subseteq \) Let \( Q \in X \). Then \( Q = (q_1, q_2, q_3, q_4) \) for \( q_1 < q_2 < q_3 < q_4, q_1 < q_3 \) and \( C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4} \in C \) being pairwise distinct circuits such that \( \text{dim}(C_{q_1} \cup C_{q_2} \cup C_{q_3} \cup C_{q_4}) = 1 \). We can distinguish between three cases:
(1) \( q_1 < q_2 < q_3 < q_4 \) so that clearly \( Q \in Y \);
(2) \( q_1 < q_3 < q_2 < q_4 \). By definition of \( Y \) we have \( Q \in Y \);
(3) \( q_1 < q_3 < q_4 < q_2 \). Again by definition of \( Y \) we find \( Q \in Y \) and this concludes our proof. \( \square \)

Appendix C. Technical Proofs of Section 8
Let us assume throughout \( 2 \leq d \leq m - 2 \).

Definition C.1. We denote by \( V \) the set of matrices \( A \in M_{d,m}(C) \) of the form
\[
\begin{pmatrix}
1 & 1 & \cdots & * \\
\vdots & 0 & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & * \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\]  (57)

Moreover, let \( U' \) be the subset defined by those matrices for which the expressions (54) are well-defined (i.e., the determinants in the denominators are nonzero) and recall that \( U \) denotes the subset of \( V \) of matrices representing the matroid \( U_d(m) \).
Obviously, $\mathcal{U} \subseteq \mathcal{U}' \subseteq V$, and $V$ can be diffeomorphically identified with $\mathbb{C}^{(d-1)(m-d-1)}$.

**Lemma C.2.** $\mathcal{U}$ is a nonempty open connected subset of $\mathcal{V}$.

**Proof.** First of all we prove that $\mathcal{U}$ is nonempty. Since $U_d(m)$ is realizable over $\mathbb{C}$ there exists a matrix $A \in M_{d,m}(\mathbb{C})$ with

$$\det(A^{i_1}, \ldots, A^{i_d}) \neq 0 \quad \text{for all} \quad 1 \leq i_1 < \cdots < i_d \leq m.$$ 

Here $A^j$ denotes the $j$-th column of $A$. By elementary linear algebra arguments, there exist matrices $B \in GL_d(\mathbb{C})$ and $D = (\delta_1, \ldots, \delta_m) \in GL_m(\mathbb{C})$ such that the new matrix $\tilde{A} = BAD$ is of the form \([57]\). Multiplying a matrix $A$ on the left or on the right side by a non singular matrix does not change the underlying matroid of $\varphi_A$. So that $A \in \mathcal{U}$, proving that $\mathcal{U} \neq \emptyset$.

To check that $\mathcal{U}$ is an open connected subset of $\mathcal{V}$ consider the family of curves in $\mathcal{V}$ defined by

$$\det(A^{i_1}, \ldots, A^{i_d}) \neq 0 \quad \text{for all} \quad 1 \leq i_1 < \cdots < i_d \leq m.$$ 

Notice that none of the polynomials $\det(A^{i_1}, \ldots, A^{i_d})$ is 0, since $\mathcal{U} \neq \emptyset$. Thus, the thesis comes from the following general result. \qed

**Lemma C.3.** Let $h, k \geq 1$ and let $F_1, \ldots, F_h \in \mathbb{C}[t_1, \ldots, t_k]$ such that $F_j \neq 0$ for $j = 1, \ldots, h$. Hence, the complement in $\mathbb{C}^h$ of the arrangements of curves $\{F_j = 0\}_{1 \leq j \leq h}$ is a nonempty open connected subset of $\mathbb{C}^h$.

**Proof.** Let $F \in \mathbb{C}[t_1, \ldots, t_k]$ be the polynomial defined by $F = \prod_{j=1}^h F_j$. Since none of the $F_j$ is zero, $F$ is non zero, too. Therefore, the complement is not empty. The regular part of the zero set

$$\{z \in \mathbb{C}^h \mid F(z) = 0, dF(z) \neq 0\}$$ 

has complex codimension 1, thus real codimension 2. So that the complement of this is connected. The singular part

$$\{z \in \mathbb{C}^h \mid F(z) = 0, dF(z) = 0\}$$ 

is of higher codimension. There are only finitely many such parts, thus the complement is connected. \qed

**Corollary C.4.** The set $\mathcal{U}'$ is a nonempty open connected subsets of $\mathcal{V}$.

**Proof.** The proof is analogue to that of Lemma C.2. \qed

**Lemma C.5.** Let $m, d \in \mathbb{N}$ and assume $2 \leq d \leq m - 2$. For $d + 1 \leq i \leq m$ and $1 \leq j \leq d - 1$ consider a set of variables $\{a_j^i\}$ and the matrix $A$ defined by

$$A = \begin{pmatrix}
1 & \cdots & 1 & a_1^{d+2} & \cdots & a_1^m \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \cdots & 1 & a_1^{d+2} & \cdots & a_{d-1}^{d-1} \\
1 & \cdots & 1 & 1 & \cdots & 1
\end{pmatrix}.$$

Let $\{i_1, i_2, i_3, i_4\}$ and $\{j_1, \ldots, j_{d-2}\}$ be two subsets of $\{1, \ldots, m\}$ such that

1. $|\{i_1, i_2, i_3, i_4\}| = 4$,
2. $|\{j_1, \ldots, j_{d-2}\}| = d - 2$,
3. $\{i_1, i_2, i_3, i_4\} \cap \{j_1, \ldots, j_{d-2}\} = \emptyset$.

Now, let $p$ and $q$ be the elements of the polynomial ring $\mathbb{C} \left[ a_j^i \right]$, $d + 1 \leq i \leq m$, $1 \leq j \leq d - 1$ defined by

$$p = \det(A^{i_1}, A^{i_2}, A^{i_3}, A^{i_4}) \det(A^{j_1}, A^{j_2}, \ldots, A^{j_{d-2}}),$$
$$q = \det(A^{i_1}, A^{i_2}, A^{i_3}, A^{i_4}) \det(A^{j_1}, A^{j_2}, \ldots, A^{j_{d-2}}),$$
where $A^l$ denotes the $l$-th column of $A$. The following properties hold:

(1) $p$ and $q$ are nonzero polynomials.
(2) There exist matrices $B_1, B_2 \in M_{d,m}(\mathbb{C})$ of the form (58) such that

$$p(B_1) \neq 0 \quad q(B_1) \neq 0 \quad p(B_2) \neq 0 \quad q(B_2) \neq 0$$

and

$$\frac{p(B_1)}{q(B_1)} \neq \frac{p(B_2)}{q(B_2)},$$

where, with a slight abuse of notation, we write $p(B)$ and $q(B)$ for the evaluation of the polynomials $p$ and $q$ on the entries of the matrix $B$.

**Proof.** In order to prove property (1), notice that we already know that $U \neq \emptyset$ so that there exists a matrix $B$ of the form (58) which belongs to $U$. By definition of the phirotope associated to a matrix, since $B \in U$ it follows that $p(B) \neq 0$ and $q(B) \neq 0$.

Property (2) then follows by a suitable choice of values for the variables $a'_i$ such that the determinants in the definition of $p$ and $q$ are Vandermonde-type minors of the matrix $A$ of the form (58). □

**Corollary C.6.** Every component of the function defined by the expressions (51) on $U'$ is nonconstant, and the same holds for its restriction to $U$.

**Proof.** The claim for $U'$ is the second part of the claim of Lemma [54], the claim on the restriction to $U$ follows from Corollary [55] and the open map theorem. □

**Lemma C.7.** For the uniform matroid $U_d(m)$ with $m \geq 5$ and $2 \leq d \leq m - 2$ assume, with the hypothesis and notations of Corollary [7], that the condition

$$F(B) \cap \prod_{j=1}^{n} [X \setminus \{ (1,1), (1,-1), (-1,1) \}] \neq \emptyset$$

is satisfied. Then, there exists a non realizable non chirotopal phased matroid with underlying matroid $U_d(m)$.

**Proof.** Consider the following subspace of $\mathcal{R}_C(U_d(m))$

$$\mathcal{Z}_C(U_d(m)) = \{ P \in \mathcal{R}_C(U_d(m)) \mid P \text{ is realizable} \} \setminus j(\mathcal{R}_E(U_d(m))).$$

We will use the following characterization of $\mathcal{Z}_C(U_d(m))$, whose proof, which relies on work of Ruiz [Rui13] Theorem 3.18, Theorem 5.1 and Section 3.3.6], we postpone.

**Lemma C.8.** Recall Definition [7] and let $\mathcal{T}$ be the subset of $\mathcal{V}$ consisting of matrices $A \in M_{d,m}(\mathbb{C})$ of the form (58) and such that

(C1) $\varphi_A$ has underlying matroid $U_d(m)$, and
(C2) $\varphi_A$ has at least a value in $(S^1 \cup \{0\}) \setminus \{0,1,-1\}$.

Let $\Lambda : \mathcal{T} \rightarrow \mathcal{R}_C(U_d(m))$ be the map which associates to a matrix $A \in \mathcal{T}$ the phasing class $\pi(\varphi_A)$ of the phased matroid $\varphi_A$. The following results hold:

(P1) $\mathcal{T}$ is either empty or diffeomorphic to a non empty open subset of $(\mathbb{C}^+)^{(d-1)(m-d-1)}$;
(P2) $\Lambda$ is continuous;
(P3) $\Lambda$ establishes a one-to-one correspondence between $\mathcal{T}$ and $\mathcal{Z}_C(U_d(m))$.

From Corollary [7] we can homeomorphically identify $\mathcal{R}_C(U_d(m))$ with the space $F(B) \cap \prod_{j=1}^{n} X$. Let $\Theta : \mathcal{R}_C(U_d(m)) \rightarrow F(B) \cap \prod_{j=1}^{n} X$ be such a homeomorphism. Clearly this restricts to a homeomorphism between $\mathcal{Z}_C(U_d(m))$ and a subspace of
$F(B) \cap \prod_{j=1}^{n_M} X$. Finally, set $\Sigma = \Theta \circ \Lambda$. Thus, by composition $\Sigma$ is a continuous map

$$\Sigma : \mathcal{T} \longrightarrow F(B) \cap \prod_{j=1}^{n_M} X.$$ 

Now, we have to distinguish between two cases.

Case 1. Assume that

$$\Theta(\mathcal{Z}(U_d(m))) \cap \left( F(B) \cap \prod_{j=1}^{n_M} \left[ X \setminus \{(1, 1), (1, -1), (-1, 1)\} \right] \right) = \emptyset.$$ 

By hypothesis (59), there exists $P \in F(B) \cap \prod_{j=1}^{n_M} X$ with $P = \Theta \circ \pi(\Phi)$ for some phased matroid $\Phi$ over $U_d(m)$. From (60) and (49) we conclude that $\Phi$ is non realizable and non chirotopal.

Case 2. Now, suppose that

$$\Theta(\mathcal{Z}(U_d(m))) \cap \left( F(B) \cap \prod_{j=1}^{n_M} \left[ X \setminus \{(1, 1), (1, -1), (-1, 1)\} \right] \right) \neq \emptyset.$$ 

Write from now

$$\hat{\mathcal{X}} := \prod_{j=1}^{n_M} \left[ X \setminus \{(1, 1), (1, -1), (-1, 1)\} \right]$$

and consider the subset $\mathcal{U} \subseteq \mathcal{T}$ defined by

$$\mathcal{U} = \Sigma^{-1} \left( F(B) \cap \hat{\mathcal{X}} \right).$$

$\mathcal{U}$ is non empty. Otherwise, $\Sigma(\mathcal{T}) \cap \left( F(B) \cap \hat{\mathcal{X}} \right) = \emptyset$ will contradict (61), since $\Sigma(\mathcal{T}) = \Theta(\mathcal{Z}(U_d(m)))$ by Lemma C.8. Moreover, $\mathcal{U}$ is an open subset of $\mathcal{T}$. This follows from the continuity of $\Sigma$ and from the fact that $F(B) \cap \hat{\mathcal{X}}$ is an open subset of $F(B) \cap \prod_{j=1}^{n_M} X$. Therefore, point (P1) in Lemma C.8 implies that $\mathcal{U}$ is diffeomorphic to a non empty open subset subset of $(\mathbb{C}^*)^{(d-1)(m-d-1)}$. In particular, $\mathcal{U}$ is a differentiable manifold of dimension $2(d-1)(m-d-1)$.

On the other hand, $F(B) \cap \hat{\mathcal{X}}$ is a non empty (by (61)) open subset of $F(B)$. From Proposition 7.6 and Corollary 7.7 we already know that $F(B)$ is a finite disjoint union of subtori of $(S^1)^{2n_M}$, each of dimension $\dim \ker B$, where $B$ is the R-GRS matrix of $U_d(m)$. Theorem 8.1 together with [DWS9] ensures us that all these rational expressions are well defined.
Thus, (62) is a smooth map of differentiable manifolds. Consider the inequality
\[
\binom{m}{d} - m > 2(d - 1)(m - d - 1)
\]
proved in Lemma C.11 below. From [Hir76, Chapter 3, Proposition 1.2] we finally deduce that there exists and element
\[
P \in (F(B) \cap \hat{X}) \setminus \Sigma(U).
\]
with \( P = \Theta \circ \pi(\Phi) \) for some phased matroid \( \Phi \) over \( U_d(m) \). In particular,
\[
(63) \quad P \in F(B) \cap \hat{X}.
\]
By definition of \( U \) we must have
\[
(64) \quad P \in \left( F(B) \cap \hat{X} \right) \setminus \Theta(Z_C(U_d(m))).
\]
To see this it suffices to check that \( P \not\in \Theta(Z_C(U_d(m))) \). Let us assume \( P \in \Theta(Z_C(U_d(m))) \). Hence, from point (P3) in Lemma C.8 we have
\[
P \in \Theta(\Sigma(U)) = \Sigma(T) \setminus \Sigma(U) = \Sigma(T \setminus U).
\]
Thus,
\[
P \in \left( F(B) \cap \hat{X} \right) \setminus \left( F(B) \cap \prod_{j=1}^{n_M} X \setminus \{(1,1),(1,-1),(-1,1)\} \right)
\]
contradicting (63). From (64), using the arguments of the previous case, the claim follows.

Proof of Lemma C.8.
Proof of property (P1). Clearly we can diffeomorphically identify the space of matrices \( A \in M_{d,m}(\mathbb{C}) \) of the form (58) with \( \mathbb{C}^{(d - 1)(m - d - 1)} \). If \( T \) is empty there is nothing to say. Let us assume \( T \) not empty. With (1) it is not hard to see that condition (C1) is equivalent to
\[
(65) \quad \det(A^{i_1}, \ldots, A^{i_d}) \neq 0 \quad \text{for all} \quad 1 \leq i_1 < \cdots < i_d \leq m.
\]
Similarly, condition (C2) is equivalent to
\[
(66) \quad \det(A^{i_1}, \ldots, A^{i_d}) \in \mathbb{C} \setminus \mathbb{R} \quad \text{for some} \quad 1 \leq i_1 < \cdots < i_d \leq m.
\]
Condition (65) and (66) define an open subset of \( (\mathbb{C}^*)^{(d - 1)(m - d - 1)} \). The first part of our statement is proved.

Proof of property (P2). Let \( N^p_C(M) \) as in (2) and let \( \Omega : T \rightarrow N^p_C(M) \) be the function \( A \mapsto \varphi_A \) which maps a matrix \( A \in T \) to the phirotope \( \varphi_A \) associated to \( A \). With (1) and (2), we can see that \( \Omega \) is continuous. Now, consider the projections
\[
\pi_{\neq_p} : N^p_C(U_d(m)) \rightarrow M^p_C(U_d(m))
\]
and
\[
\pi_{\sim_p} : M^p_C(U_d(m)) \rightarrow R^p_C(U_d(m)).
\]
From the definition of the topological spaces \( M^p_C(U_d(m)) \) and \( R^p_C(U_d(m)) \) (compare Definition 2.1 and Definition 2.3) these projections are continuous. Thus, \( \Lambda \) is continuous since it is composition of the continuous functions \( \pi_{\neq_p}, \pi_{\sim_p} \) and \( \Omega \).

Proof of property (P3). By definition of \( T \) and \( Z_C(U_d(m)) \) it follows that \( \Lambda(A) \) is an element of \( Z_C(U_d(m)) \). It needs check that \( \Lambda : T \rightarrow Z_C(U_d(m)) \) is bijective.
Lemma C.9. With the hypothesis and notations of Lemma C.7, let \( A, B \in T \) and assume there exist \( a \in S^1 \) and a family \( \{h(i)\}_{i \in E} \subseteq S^1 \) such that for \( (x_1, \ldots, x_d) \in E^d \)

\[
\varphi_A(x_1, \ldots, x_d) = a \left( \prod_{j=1}^{d} h(x_j) \right) \varphi_B(x_1, \ldots, x_d).
\]

Since \( A \) and \( B \) are of the form \((58)\), Lemma C.9 implies that

\[
\varphi_A = \varphi_B.
\]

Identity \((58)\), together with the proof of [Rui13, Theorem 5.1], implies \( A = B \). \(\square\)

Proof. For \( d \leq l \leq m \), consider the d-uple \((x_1^{(l)}, \ldots, x_d^{(l)})\) defined by setting

\[
x_j^{(l)} = \begin{cases} 
  j & \text{if } j \neq d, \\
  l & \text{if } j = d. 
\end{cases}
\]

Since the matrix \( A \) and \( B \) are of the form \((58)\), if we compute \( \varphi_A(x_1^{(l)}, \ldots, x_d^{(l)}) \) and \( \varphi_B(x_1^{(l)}, \ldots, x_d^{(l)}) \) keeping in mind \((1)\), \((67)\) and axiom \((\varphi2)\) in Definition 1.5, we deduce that

\[
\begin{align*}
1 &= ah(1) \cdots h(d-1)h(d) \\
1 &= ah(1) \cdots h(d-1)h(d+1) \\
1 &= ah(1) \cdots h(d-1)h(d+2) \\
\cdots \\
1 &= ah(1) \cdots h(d-1)h(m)
\end{align*}
\]

and then, \( h(d) = h(d+1) = h(d+2) = \cdots = h(m) \). In a similar way, for \( 1 \leq l \leq d \) consider the d-uple \((x_1^{(l)}, \ldots, x_d^{(l)})\) defined by setting

\[
x_j^{(l)} = \begin{cases} 
  j & \text{if } j \neq l, \\
  d+1 & \text{if } j = l. 
\end{cases}
\]

Using the same arguments of the previous case, we find

\[
1 = a \frac{h(1) \cdots h(d+1)}{h(l)}
\]

for \( 1 \leq l \leq d \). So that, \( h(1) = h(2) = \cdots = h(d) \). Therefore, we can define

\[
h = h(1) = \cdots = h(m).
\]

With \((69)\), this implies

\[
\varphi_A = ah^d \varphi_B.
\]
If we consider the d-uple \((1, \ldots, d)\), from (70) we have \(ad^d = 1\) and this concludes our proof. □

**Lemma C.10.** Let \( h \geq 1 \) and let \( \Omega \) be a nonempty open connected subset of \( \mathbb{C}^h \). Let \( k \geq 1 \) and let \( F : \Omega \rightarrow \mathbb{C}^k \) be a holomorphic map. Assume that none of the components \( F_1, \ldots, F_k \) of \( F \) are constant. Then, there exists \( z_0 \in \Omega \) such that \( F(z_0) \in (\mathbb{C} \setminus \mathbb{R})^k \).

**Proof.** Set \( \Omega^{(0)} = \Omega \) and let \( F_1 \) be the first component of \( F \). Since \( \Omega \) is a nonempty open connected subset of \( \mathbb{C}^h \) and \( F_1 : \Omega^{(0)} \rightarrow \mathbb{C} \) is a holomorphic non constant function, the open map theorem implies that there is \( z_0^{(1)} \in \Omega^{(0)} \) such that \( F_1(z_0^{(1)}) \in \mathbb{C} \setminus \mathbb{R} \). By continuity of \( F_1 \) it is possible to find an open connected neighborhood \( \Omega^{(1)} \) of \( z_0^{(1)} \) in \( \Omega^{(0)} \) with \( F_1(\Omega^{(1)}) \subseteq \mathbb{C} \setminus \mathbb{R} \).

Now, let \( F_2 \) be the second component of \( F \). Since \( \Omega^{(0)} \) is connected and \( F_2 \) is not constant on \( \Omega \), again from the open map theorem it follows that \( F_2 \) is not constant on \( \Omega^{(1)} \). So that, there exists \( z_0^{(2)} \in \Omega^{(1)} \) and an open connected neighborhood \( \Omega^{2} \) of \( z_0^{(2)} \) in \( \Omega^{(1)} \) such that \( F_2(\Omega^{(2)}) \subseteq \mathbb{C} \setminus \mathbb{R} \).

Hence, we can recursively find for each component \( F_j \) of \( F \) a point \( z_0^{(j)} \) and an open connected neighborhood \( \Omega^{(j)} \) of \( z_0^{(j)} \) in \( \Omega^{(j-1)} \) such that \( F_j(\Omega^{(j)}) \subseteq \mathbb{C} \setminus \mathbb{R} \). Set \( z_0 = z_0^{(k)} \). By contruction \( z_0 \in \Omega^l \) for each \( j = 1, \ldots, k \). Thus, \( F_j(z_0) \in \mathbb{C} \setminus \mathbb{R} \) for \( j = 1, \ldots, k \). □

**Lemma C.11.** If \( m \geq 5 \) and \( 2 \leq d \leq m-2 \), then
\[
\binom{m}{d} > m + 2(d-1)(m-d-1).
\]

**Proof.** Since \( 2 \leq d \leq m-2 \), we deduce that \( d-1 \geq 1 \) and \( m-d-1 \geq 1 \). For \( m \geq 5 \) we have
\[
\frac{m^2 - 2m + 4}{2} < \frac{m^2 - m}{2}.
\]
Moreover, for \( 2 \leq d \leq m-2 \), we always have
\[
\binom{m}{2} \leq \binom{m}{d}.
\]
From the inequality of arithmetic and geometric means, we conclude
\[
m + 2(d-1)(m-d-1) \leq m + 2 \left( \frac{(d-1) + (m-d-1)}{2} \right)^2 = m^2 - 2m + 4 \leq \frac{m^2 - m}{2} = \binom{m}{2} \leq \binom{m}{d}.
\]
□

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