DIFFERENTIAL OPERATORS ON QUANTIZED FLAG MANIFOLDS AT ROOTS OF UNITY, II

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To Etsuro Date on his 60th birthday

Abstract. We formulate a Beilinson–Bernstein-type derived equivalence for a quantized enveloping algebra at a root of 1 as a conjecture. It says that there exists a derived equivalence between the category of modules over a quantized enveloping algebra at a root of 1 with fixed regular Harish-Chandra central character and the category of certain twisted $D$-modules on the corresponding quantized flag manifold. We show that the proof is reduced to a statement about the (derived) global sections of the ring of differential operators on the quantized flag manifold. We also give a reformulation of the conjecture in terms of the (derived) induction functor.

§0. Introduction

0.1. Let $G$ be a connected, simply connected simple algebraic group over $\mathbb{C}$, and let $H$ be a maximal torus of $G$. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$, respectively. Let $Q$ and $\Lambda$ be the root lattice and the weight lattice, respectively. Let $h_G$ be the Coxeter number of $G$. We fix an odd integer $\ell > h_G$, which is prime to the order of $\Lambda/Q$ and prime to 3 if $\mathfrak{g}$ is of type $G_2$, $F_4$, $E_6$, $E_7$, $E_8$, and we consider the De Concini–Kac-type quantized enveloping algebra $U_\zeta$ at $q = \zeta = \exp(2\pi i\sqrt{-1}/\ell)$.

In [20], we started the investigation of the corresponding quantized flag manifold $B_\zeta$, which is a noncommutative scheme, and the category of $D$-modules on it. In view of a general philosophy saying that quantized objects at roots of 1 resemble ordinary objects in positive characteristics, it
is natural to pursue an analogue of the theory of $D$-modules on the ordinary flag manifolds in positive characteristics due to Bezrukavnikov, Mirković, and Rumynin [6]. Along this line, we have established in [20] certain Azumaya properties of the ring of differential operators on the quantized flag manifold. The aim of the present article is to investigate an analogue of another main point of [6] about the Beilinson–Bernstein-type derived equivalence.

0.2.

We denote by $\mathcal{D}_{\mathcal{B}_\zeta,1}$ the sheaf of rings of differential operators on the quantized flag manifold $\mathcal{B}_\zeta$. More generally, for each $t \in H$ we have its twisted analogue denoted by $\mathcal{D}_{\mathcal{B}_\zeta,t}$. It is obtained as the specialization $\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{C[H]} C$ of the universally twisted sheaf $\mathcal{D}_{\mathcal{B}_\zeta}$ with respect to the ring homomorphism $C[H] \to C$ corresponding to $t \in H$.

Let $\mathcal{B}$ be the ordinary flag manifold for $G$. Then we have a Frobenius morphism $Fr: \mathcal{B}_\zeta \to \mathcal{B}$, which is a finite morphism from a noncommutative scheme to an ordinary scheme. Taking the direct images, we obtain sheaves $Fr_* \mathcal{D}_{\mathcal{B}_\zeta}, Fr_* \mathcal{D}_{\mathcal{B}_\zeta,t} (t \in H)$ of rings on $\mathcal{B}$ (in the ordinary sense). Denote by $\text{Mod}_{\text{coh}}(Fr_* \mathcal{D}_{\mathcal{B}_\zeta,t})$ the category of coherent $Fr_* \mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. Let $Z_{\text{Har}}(U_\zeta)$ be the Harish-Chandra center of $U_\zeta$, and let $C_t$ be the corresponding 1-dimensional $Z_{\text{Har}}(U_\zeta)$-module. Denote by $\text{Mod}_f(U_\zeta \otimes Z_{\text{Har}}(U_\zeta) C_t)$ the category of finitely generated $U_\zeta \otimes Z_{\text{Har}}(U_\zeta) C_t$-modules. Then we have a functor

\[
(0.1) \quad R\Gamma(\mathcal{B}, \bullet): D^b(\text{Mod}_{\text{coh}}(Fr_* \mathcal{D}_{\mathcal{B}_\zeta,t})) \to D^b(\text{Mod}_f(U_\zeta \otimes Z_{\text{Har}}(U_\zeta) C_t))
\]

between derived categories. It is natural in view of [6] to conjecture that (0.1) gives an equivalence if $t$ is regular. By imitating the argument of [6], we can show that this is true if we have

\[
(0.2) \quad R\Gamma(\mathcal{B}, Fr_* \mathcal{D}_{\mathcal{B}_\zeta}) \cong U_\zeta \otimes Z_{\text{Har}}(U_\zeta) C[\Lambda].
\]

However, we do not know how to prove (0.2) at present; hence, we can only state it as a conjecture. We have also a stronger conjecture,

\[
(0.3) \quad R\Gamma(\mathcal{B}, Fr_* (\mathcal{D}_{\mathcal{B}_\zeta})_f) \cong U_{\zeta,f} \otimes Z_{\text{Har}}(U_\zeta) C[\Lambda],
\]

which is the analogue of (0.2) regarding the adjoint finite parts $(\mathcal{D}_{\mathcal{B}_\zeta})_f, U_{\zeta,f}$ of $\mathcal{D}_{\mathcal{B}_\zeta}, U_\zeta$, respectively. We will give a reformulation of (0.3) in terms of the induction functor (see Conjecture 5.2 below). It turns out that (0.3) is
equivalent to some assertions in Backelin and Kremnizer [2], [3] stated to be true under certain conditions on \( \ell \) (see Remark 5.4 below).

It is also an interesting problem to find a formulation which works even in the case when the parameter \( t \in H \) is singular. In the case of Lie algebras in positive characteristics, Bezrukavnikov, Mirković, and Rumynin in [5] have succeeded in giving a more general framework, which works even for singular parameters, using partial flag manifolds (quotients of \( G \) by parabolic subgroups). In their case, the parameter space is \( h^* \), and one can associate for each \( h \in h^* \) a parabolic subgroup whose Levi subgroup is the centralizer of \( h \); however, in our case the centralizer of \( t \in H \) is not necessarily a Levi subgroup of a parabolic subgroup, and hence the method in [5] cannot be directly applied to our case.

0.3.

This article has the following organization. In Section 1, we recall basic facts on quantized enveloping algebras at roots of 1 and the corresponding quantized flag manifolds. In Section 2, we investigate properties of the category of \( D \)-modules. In particular, we show that (0.2) implies (0.1) for regular \( t \) and that (0.3) implies (0.2). In Sections 3 and 4, we recall some known results on the representations of quantized enveloping algebras and the induction functor, respectively. Finally, in Section 5 we give a reformulation of (0.3) in terms of the induction functor.

§1. Quantized flag manifold

1.1. Quantized enveloping algebras

1.1.1. Let \( G \) be a connected simply connected simple algebraic group over the complex number field \( \mathbb{C} \). We fix Borel subgroups \( B^+ \) and \( B^- \) such that \( H = B^+ \cap B^- \) is a maximal torus of \( G \). Set \( N^+ = [B^+, B^+] \), and set \( N^- = [B^-, B^-] \). We denote the Lie algebras of \( G, B^+, B^-, H, N^+, N^- \) by \( \mathfrak{g}, \mathfrak{b}^+, \mathfrak{b}^-, \mathfrak{h}, \mathfrak{n}^+, \mathfrak{n}^- \), respectively. Let \( \Delta \subset \mathfrak{h}^* \) be the root system of \((\mathfrak{g}, \mathfrak{h})\). We denote by \( \Lambda \subset \mathfrak{h}^* \) and \( Q \subset \mathfrak{h}^* \) the weight lattice and the root lattice, respectively. For \( \lambda \in \Lambda \) we denote by \( \theta_\lambda \) the corresponding character of \( H \). The coordinate algebra \( \mathbb{C}[H] \) of \( H \) is naturally identified with the group algebra \( \mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e(\lambda) \) via the correspondence \( \theta_\lambda \leftrightarrow e(\lambda) \) for \( \lambda \in \Lambda \). We take a system of positive roots \( \Delta^+ \) such that \( \mathfrak{b}^+ \) is the sum of weight spaces with weights in \( \Delta^+ \cup \{0\} \). Let \( \{\alpha_i\}_{i \in I} \) be the set of simple roots, and let \( \{\varpi_i\}_{i \in I} \) be the corresponding set of fundamental weights. We denote by \( \Lambda^+ \) the set of dominant integral weights. We set \( Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \). Let \( W \subset}
GL(h*) be the Weyl group. For i ∈ I we denote by si ∈ W the corresponding simple reflection. We take a W-invariant symmetric bilinear form

( , ) : h* × h* → C

such that (α, α) = 2 for short roots α. For α ∈ ∆ we set α∨ = 2α/(α, α). For i ∈ I we fix ei ∈ gαi, fi ∈ g−αi such that [ei, fi] = α∨ i under the identification h = h* induced by ( , ).

1.1.2. For n ∈ Z≥0 we set

\[ [n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}], \]

\[ [n]_t! = [n]_t [n-1]_t \cdots [2]_t [1]_t \in \mathbb{Z}[t, t^{-1}]. \]

We denote by UF the quantized enveloping algebra over \( \mathbb{F} = \mathbb{Q}(q^{1/|Λ/Q|}) \) associated to g. Namely, UF is the associative algebra over \( \mathbb{F} \) generated by elements

\[ k_\lambda \quad (\lambda \in \Lambda), \quad e_i, f_i \quad (i \in I) \]

satisfying the relations

\[ k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda+\mu} \quad (\lambda, \mu \in \Lambda), \]

\[ k_\lambda e_i k_\lambda^{-1} = q^{(\lambda, \alpha_i)} e_i \quad (\lambda \in \Lambda, i \in I), \]

\[ k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda, \alpha_i)} f_i \quad (\lambda \in \Lambda, i \in I), \]

\[ e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I), \]

\[ \sum_{n=0}^{1-a_{ij}} (-1)^n e_i^{(1-a_{ij}-n)} e_j e_i^{(n)} = 0 \quad (i, j \in I, i \neq j), \]

\[ \sum_{n=0}^{1-a_{ij}} (-1)^n f_i^{(1-a_{ij}-n)} f_j f_i^{(n)} = 0 \quad (i, j \in I, i \neq j), \]

where \( q_i = q^{(\alpha_i, \alpha_i)/2}, k_i = k_{\alpha_i}, a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \) for i, j ∈ I, and

\[ e_i^{(n)} = e_i^n /[n]_{q_i}!, \quad f_i^{(n)} = f_i^n /[n]_{q_i}!. \]
for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$. We will use the Hopf algebra structure of $U_F$ given by

$$
\Delta(k_\lambda) = k_\lambda \otimes k_\lambda \quad (\lambda \in \Lambda),
$$

$$
\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \quad (i \in I),
$$

$$
\varepsilon(k_\lambda) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0 \quad (\lambda \in \Lambda, i \in I),
$$

$$
S(k_\lambda) = k_\lambda^{-1}, \quad S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_ik_i \quad (\lambda \in \Lambda, i \in I).
$$

Define subalgebras $U^0_F$, $U^+_F$, $U^-_F$, $U^\geq_0 F$, $U^\leq_0 F$ of $U_F$ by

$$
U^0_F = \langle k_\lambda \mid \lambda \in \Lambda \rangle, \quad U^+_F = \langle e_i \mid i \in I \rangle, \quad U^-_F = \langle f_i \mid i \in I \rangle,
$$

$$
U^\geq_0 F = \langle k_\lambda, e_i \mid \lambda \in \Lambda, i \in I \rangle, \quad U^\leq_0 F = \langle k_\lambda, f_i \mid \lambda \in \Lambda, i \in I \rangle.
$$

The multiplication of $U_F$ induces isomorphisms

$$(1.1) \quad U_F \cong U^-_F \otimes U^0_F \otimes U^+_F \cong U^+_F \otimes U^0_F \otimes U^-_F,$$

$$(1.2) \quad U^\geq_0 F \cong U^0_F \otimes U^+_F \cong U^+_F \otimes U^0_F,$$

$$(1.3) \quad U^\leq_0 F \cong U^0_F \otimes U^-_F \cong U^-_F \otimes U^0_F,$$

of $F$-modules. The fact $(1.1)$ is called the triangular decomposition of $U_F$.

For $\gamma \in Q$ we set

$$
U^\pm_{F, \gamma} = \{ u \in U^\pm_F \mid k_\mu uk_{-\mu} = q^{(\gamma, \mu)}u \quad (\mu \in \Lambda) \}.
$$

Then we have

$$
U^\pm_F = \bigoplus_{\gamma \in Q^\pm} U^\pm_{F, \pm \gamma}.
$$

For $i \in I$ we denote by $T_i$ the automorphism of the algebra $U_F$ given by

$$
T_i(k_\mu) = k_{s_i \mu}(\mu \in \Lambda),
$$

$$
T_i(e_j) = \begin{cases} 
\sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} e_j^{(-a_{ij} - k)} e_i^{(k)} & (j \in I, j \neq i), \\
-f_ik_i & (j = i), 
\end{cases}
$$

$$
T_i(f_j) = \begin{cases} 
\sum_{k=0}^{-a_{ij}} (-1)^k q_i^k f_i^{(k)} f_j f_i^{(-a_{ij} - k)} & (j \in I, j \neq i), \\
-k_i^{-1} e_i & (j = i)
\end{cases}
$$
Let $w_0$ be the longest element of $W$. We fix a reduced expression
\[ w_0 = s_{i_1} \cdots s_{i_N} \]
of $w_0$, and we set
\[ \beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad (1 \leq k \leq N). \]
Then we have $\Delta^+ = \{ \beta_k \mid 1 \leq k \leq N \}$. For $1 \leq k \leq N$ we set
\[ e_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(e_{i_k}), \quad f_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f_{i_k}). \]
Then \( \{ e_{\beta_N^m} \cdots e_{\beta_1^m} \mid m_1, \ldots, m_N \geq 0 \} \) (resp., \( \{ f_{\beta_N^m} \cdots f_{\beta_1^m} \mid m_1, \ldots, m_N \geq 0 \} \)) is an \( \mathbb{F} \)-basis of \( U^+_F \) (resp., \( U^-_F \)), called the PBW (Poincaré–Birkhoff–Witt) basis (see [14]). We have $e_{\alpha_i} = e_i$ and $f_{\alpha_i} = f_i$ for any $i \in I$.

Denote by
\begin{equation}
(1.4) \quad \tau : U^\geq_0 \times U^\leq_0 \to \mathbb{F}
\end{equation}
the Drinfeld pairing. It is characterized as the unique bilinear form satisfying
\[
\tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in U^\geq_0, y_1, y_2 \in U^\leq_0),
\]
\[
\tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in U^\geq_0, y \in U^\leq_0),
\]
\[
\tau(k_\lambda, k_\mu) = q^{-\langle \lambda, \mu \rangle} \quad (\lambda, \mu \in \Lambda),
\]
\[
\tau(k_\lambda, f_i) = \tau(e_i, k_\lambda) = 0 \quad (\lambda \in \Lambda, i \in I),
\]
\[
\tau(e_i, f_j) = \delta_{ij}/(q_i^{-1} - q_i) \quad (i, j \in I)
\]
(see [15], [18]). It also satisfies the following.

**Lemma 1.1** ([15, Section 1.2], [18, Proposition 2.1.1]). We have the following:

(i) $\tau(S(x), S(y)) = \tau(x, y)$ for $x \in U^\geq_0, y \in U^\leq_0$;

(ii) for $x \in U^\geq_0, y \in U^\leq_0$ we have
\[
yx = \sum_{(x)_2, (y)_2} \tau(x_0, S(y_0)) \tau(x_2, y_2) x(1)y(1),
\]
\[
xy = \sum_{(x)_2, (y)_2} \tau(x_0, y_0) \tau(x_2, S(y_2)) y(1)x(1);
\]
We define an algebra homomorphism
\[ \text{ad} : U_F \rightarrow \text{End}_F(U_F) \]
by
\[ \text{ad}(u)(v) = \sum_{(u)} u(0)v(Su(1)) \quad (u,v \in U_F). \]

1.1.3. We fix an integer \( \ell > 1 \) satisfying
(a) \( \ell \) is odd;
(b) \( \ell \) is prime to 3 if \( G \) is of type \( G_2, F_4, E_6, E_7, E_8 \);
(c) \( \ell \) is prime to \( |\Lambda/Q| \);
and a primitive \( \ell \)th root \( \zeta' \in \mathbb{C} \) of 1. Define a subring \( A \) of \( F \) by
\[ A = \{ f(q^{1/|\Lambda/Q|}) \mid f(x) \in \mathbb{Q}(x), f \text{ is regular at } x = \zeta' \}. \]
We set \( \zeta = (\zeta')^{|\Lambda/Q|} \). We note that \( \zeta \) is also a primitive \( \ell \)th root of 1 by condition (c).

We denote by \( U^L_A, U_A \) the \( A \)-forms of \( U_F \) called the Lusztig form and the De Concini–Kac form, respectively. Namely, we have
\[ U^L_A = \langle e^{(m)}_i, f^{(m)}_i, k_\lambda \mid i \in I, m \in \mathbb{Z}_{\geq 0}, \lambda \in \Lambda \rangle_{A\text{-alg}} \subset U_F, \]
\[ U_A = \langle e_i, f_i, k_\lambda \mid i \in I, \lambda \in \Lambda \rangle_{A\text{-alg}} \subset U_F. \]
We have obviously \( U_A \subset U^L_A \). The Hopf algebra structure of \( U_F \) induces Hopf algebra structures over \( A \) of \( U^L_A \) and \( U_A \). We set
\[ U^L_{A,b} = U^L_A \cap U^b_F, \quad U^b_A = U_A \cap U^b_F \quad (b = +, -, 0, \geq 0, \leq 0), \]
\[ U^L_{A,\pm \gamma} = U^L_A \cap U^\pm_{F, \pm \gamma}, \quad U^\pm_{A, \pm \gamma} = U_A \cap U^\pm_{F, \pm \gamma} \quad (\gamma \in Q^+). \]
Then we have triangular decompositions
\[ U_A \cong U^{L,-}_A \otimes_A U^{L,0}_A \otimes_A U^{L,+}_A, \]
\[ U_A \cong U^{-}_A \otimes_A U^{0}_A \otimes_A U^{+}_A. \]
Moreover, we have

\[ U_{\mathbb{A}}^{L,\pm} = \bigoplus_{\gamma \in Q^+} U_{\mathbb{A},\pm \gamma}^{L,\pm}, \quad U_{\mathbb{A}}^{\pm} = \bigoplus_{\gamma \in Q^+} U_{\mathbb{A},\pm \gamma}^{\pm}. \]

The Drinfeld pairing (1.4) induces

\[ L_{\pi}^A : U_{\mathbb{A}}^{L,\geq 0} \times U_{\mathbb{A}}^{\leq 0} \to \mathbb{A}, \quad L_{\pi}^A : U_{\mathbb{A}}^{\geq 0} \times U_{\mathbb{A}}^{L,\leq 0} \to \mathbb{A}. \]

**Lemma 1.2.** We have \( \text{ad}(U_{\mathbb{A}}^{L}) (U_{\mathbb{A}}) \subset U_{\mathbb{A}} \).

**Proof.** It is sufficient to show that

\[ \text{ad}(k\lambda)(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \quad (\lambda \in \Lambda), \]

\[ \text{ad}(e_i^{(n)})(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geq 0}), \]

\[ \text{ad}(f_i^{(n)})(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geq 0}). \]

The proof of (1.6) is easy and omitted. By the formulas

\[ \text{ad}(x)(uv) = \sum_{(x)} \text{ad}(x_{(0)})(u)\text{ad}(x_{(1)})(v) \quad (x \in U_{\mathbb{A}}^{L}, u, v \in U_{\mathbb{A}}), \]

\[ \Delta(e_i^{(n)}) = \sum_{r=0}^{n} q_i^{r(n-r)} e_i^{(n-r)} k_i^r \otimes e_i^{(r)} \quad (i \in I, n \geq 0), \]

\[ \Delta(f_i^{(n)}) = \sum_{r=0}^{n} q_i^{-r(n-r)} f_i^{(r)} \otimes k_i^{-r} f_i^{(n-r)} \quad (i \in I, n \geq 0), \]

we have only to show that

\[ \text{ad}(e_i^{(n)})(u) \in U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geq 0}, u = k\lambda, e_j, f_j), \]

\[ \text{ad}(f_i^{(n)})(u) \in U_{\mathbb{A}} \quad (i \in I, n \in \mathbb{Z}_{\geq 0}, u = k\lambda, e_j, f_j). \]

For \( \lambda \in \Lambda, i, j \in I \) with \( i \neq j \) and \( n \in \mathbb{Z}_{>0} \), we have

\[ \text{ad}(e_i^{(n)})(k\lambda) = \left( \frac{(-1)^n q_i^{n(n-1)} - q_i^{-2j}}{[n]_{q_i} !} \right) \prod_{j=0}^{n-1} (q_i^{\lambda,\alpha_j^\vee} - q_i^{-2j}) e_i^n k\lambda, \]

\[ \text{ad}(e_i^{(n)})(e_i) = q_i^{-n(n+1)/2} (q_i - q_i^{-1})^n e_i^{n+1}, \]
\[ \text{ad}(e_i^{(n)})(e_j) = \begin{cases} 
\sum_{r=0}^{n} (-1)^r q_i^{r(n-1+a_{ij})} e_i^{(n-r)} e_j e_i^{(r)} & (n < 1 - a_{ij}), \\
0 & (n \geq 1 - a_{ij}), 
\end{cases} \]
\[ \text{ad}(e_i^{(n)})(f_i k_i) = \begin{cases} 
(k_i^2 - 1)/(q_i - q_i^{-1}) & (n = 1), \\
(-1)^{n-1} q_i^{(n-1)(n+2)/2} (q_i - q_i^{-1})^{n-2} e_i^{n-1} k_i^2 & (n > 1), 
\end{cases} \]
\[ \text{ad}(e_i^{(n)})(f_j k_j) = 0, \]
and hence (1.9) holds. (Note that \([r]_q \) is invertible in \(A\) for \(r \leq -a_{ij}\).) The proof of (1.10) is similar and omitted.

1.1.4. Let us consider the specialization

\[ A \to \mathbb{C} \quad (q^{1/|\Lambda|/Q} \mapsto \zeta'). \]

Note that \( q \) is mapped to \( \zeta = (\zeta')^{|\Lambda|/Q} \in \mathbb{C} \), which is also a primitive \( \ell \)th root of 1. We set

\[ U_\zeta^{L} = \mathbb{C} \otimes_A U_\zeta^{L}, \quad U_\zeta = \mathbb{C} \otimes_A U_\zeta, \]
\[ U_\zeta^{L,0} = \mathbb{C} \otimes_A U_\zeta^{L,0}, \quad U_\zeta^0 = \mathbb{C} \otimes_A U_\zeta^0 \quad (b = +, -, 0, \geq 0, \leq 0), \]
\[ U_\zeta^{L,\pm} = \mathbb{C} \otimes_A U_\zeta^{L,\pm}, \quad U_\zeta^{\pm} = \mathbb{C} \otimes_A U_\zeta^{\pm,\gamma} \quad (\gamma \in Q^+). \]

Then \( U_\zeta^{L} \) and \( U_\zeta \) are Hopf algebras over \( \mathbb{C} \), and we have triangular decompositions

\[ U_\zeta^{L} \cong U_\zeta^{L,-} \otimes U_\zeta^{L,0} \otimes U_\zeta^{L,+}, \]
\[ U_\zeta \cong U_\zeta^{-} \otimes U_\zeta^{0} \otimes U_\zeta^{+}. \]

Moreover, we have

\[ U_\zeta^{L,\pm} = \bigoplus_{\gamma \in Q^+} U_{\zeta,\pm,\gamma}, \quad U_\zeta^{\pm} = \bigoplus_{\gamma \in Q^+} U_{\zeta,\pm,\gamma}. \]

By De Concini and Kac [8, Proposition 1.7], we have the following.

**Lemma 1.3.**

(i) The set \( \{ e_{\beta_N}^{m_N} \cdots e_{\beta_1}^{m_1} \mid m_1, \ldots, m_N \geq 0 \} \) (resp., \( \{ f_{\beta_N}^{m_N} \cdots f_{\beta_1}^{m_1} \mid m_1, \ldots, m_N \geq 0 \} \)) forms a \( \mathbb{C} \)-basis of \( U_\zeta^{+} \) (resp., \( U_\zeta^{-} \)).

(ii) The set \( \{ k_\lambda \mid \lambda \in \Lambda \} \) forms a \( \mathbb{C} \)-basis of \( U_\zeta^{0} \).
The Drinfeld pairings (1.5) induce
\[
L_{\tau} : U_{\zeta}^{L,\geq 0} \times U_{\zeta}^{\geq 0} \to \mathbb{C}, \quad \tau_{\zeta}^L : U_{\zeta}^{\geq 0} \times U_{\zeta}^{L,\leq 0} \to \mathbb{C}.
\]
Moreover, we have the following (see [20, Lemma 1.5]).

**Proposition 1.4.** For any \(\gamma \in Q^+\), the restrictions of \(L_{\tau}\) and \(\tau_{\zeta}^L\) to
\[
U_{\zeta,\gamma}^{L,\geq 0} \times U_{\zeta,-\gamma}^{\geq 0} \to \mathbb{C}, \quad U_{\zeta,\gamma}^{\geq 0} \times U_{\zeta,-\gamma}^{L,\leq 0} \to \mathbb{C},
\]
respectively, are nondegenerate.

By Lemma 1.2 we have an algebra homomorphism
\[
\text{ad} : U_{\zeta}^L \to \text{End}_\mathbb{C}(U_{\zeta}).
\]

In general, for a Lie algebra \(\mathfrak{s}\) we denote its enveloping algebra by \(U(\mathfrak{s})\). We denote by
\[
\pi : U_{\zeta}^L \to U(\mathfrak{g})
\]
Lusztig’s Frobenius homomorphism (see [14]). Namely, \(\pi\) is the \(\mathbb{C}\)-algebra homomorphism given by
\[
\pi(e_i^{(m)}) = e_i^{(m/\ell)}, \quad \pi(f_i^{(m)}) = f_i^{(m/\ell)} \quad \pi(k_\lambda) = 1
\]
for \(i \in I, \ m \in \mathbb{Z}_{\geq 0}, \ \lambda \in \Lambda\). Here, \(e_i^{(n)} = e_i^n / n!\), \(f_i^{(n)} = f_i^n / n!\) for \(i \in I\) and \(n \in \mathbb{Z}_{\geq 0}\). Then \(\pi\) is a homomorphism of Hopf algebras.

We recall the description of the center \(Z(U_{\zeta})\) of the algebra \(U_{\zeta}\) due to De Concini and Kac [8, Section 3] and De Concini and Procesi [9, Section 21]. Denote by \(Z(U_{\overline{F}})\) the center of \(U_{\overline{F}}\), and define a subalgebra \(Z_{\text{Har}}(U_{\zeta})\) of \(Z(U_{\zeta})\) by
\[
Z_{\text{Har}}(U_{\zeta}) = \text{Im}(Z(U_{\overline{F}}) \cap U_{\Lambda} \to U_{\zeta})\).
\]
We define a shifted action of \(W\) on the group algebra \(\mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C} e(\lambda)\) of \(\Lambda\) by
\[
w \circ e(\lambda) = \zeta^{(w,\lambda - \lambda, \rho)} e(w,\lambda) \quad (w \in W, \ \lambda \in \Lambda).
\]
Let
\[
\iota : Z_{\text{Har}}(U_{\zeta}) \to \mathbb{C}[\Lambda]
\]
be the composite of
\[ Z_{\text{Har}}(U_\zeta) \hookrightarrow U_\zeta \cong U_\zeta^- \otimes U_\zeta^0 \otimes U_\zeta^+ \xrightarrow{\varepsilon \otimes 1 \otimes \varepsilon} U_\zeta^0 \cong \mathbb{C}[\Lambda], \]
where \( U_\zeta^0 \cong \mathbb{C}[\Lambda] \) is given by \( k_\lambda \leftrightarrow e(\lambda) \). Then by [8, Lemma 3.9], \( \iota \) is an injective algebra homomorphism with image
\[ \mathbb{C}[2\Lambda]^{W_0} = \{ f \in \mathbb{C}[2\Lambda] \mid w \circ f = f \ (\forall w \in W) \}. \]
In particular, we have an isomorphism
\[ (1.15) \quad Z_{\text{Har}}(U_\zeta) \cong \mathbb{C}[2\Lambda]^{W_0} \]
of \( \mathbb{C} \)-algebras. By [8, Section 3.1] the elements
\[ e_\beta^\ell, \quad f_\beta^\ell, \quad k_\ell\lambda \quad (\beta \in \Delta^+, \lambda \in \Lambda) \]
are central in \( U_\zeta \). Let \( Z_{\text{Fr}}(U_\zeta) \) be the subalgebra of \( U_\zeta \) generated by them. It is a Hopf subalgebra of \( U_\zeta \). Define an algebraic subgroup \( K \) of \( B^+ \times B^- \) by
\[ K = \{(gh,g'h^{-1}) \mid h \in H, g \in N^+, g' \in N^-\}. \]
By [9, Section 19.1] we have an isomorphism
\[ (1.16) \quad Z_{\text{Fr}}(U_\zeta) \cong \mathbb{C}[K] \]
of Hopf algebras (see also [10, Theorem 7.4]). We refer the reader to [20, Section 1.5] for the explicit description of the isomorphism (1.16). By [9], \( Z(U_\zeta) \) is generated by \( Z_{\text{Fr}}(U_\zeta) \) and \( Z_{\text{Har}}(U_\zeta) \). Moreover, we have an isomorphism
\[ Z(U_\zeta) \cong Z_{\text{Har}}(U_\zeta) \otimes Z_{\text{Har}}(U_\zeta) \cap Z_{\text{Fr}}(U_\zeta) \]
of algebras.

1.2. Sheaves on quantized flag manifolds

1.2.1. We denote by \( C_\mathbb{F} \) the subspace of \( U_\mathbb{F}^+ = \text{Hom}_\mathbb{F}(U_\mathbb{F}, \mathbb{F}) \) spanned by the matrix coefficients of finite-dimensional \( U_\mathbb{F} \)-modules of type 1 in the sense of Lusztig, and we denote by
\[ (1.17) \quad \langle , \rangle : C_\mathbb{F} \times U_\mathbb{F} \rightarrow \mathbb{F} \]
the canonical pairing. Then \( C_\mathbb{F} \) is endowed with a Hopf algebra structure dual to \( U_\mathbb{F} \) via (1.17). We have a \( U_\mathbb{F} \)-bimodule structure of \( C_\mathbb{F} \) given by
\[ \langle u_1 \cdot \varphi \cdot u_2, u \rangle = \langle \varphi, u_2 uu_1 \rangle \quad (\varphi \in C_\mathbb{F}, u, u_1, u_2 \in U_\mathbb{F}). \]
Define a Λ-graded ring $A_F = \bigoplus_{\lambda \in \Lambda^+} A_F(\lambda)$ by

$$A_F = \{ \varphi \in C_F \mid \varphi \cdot f_i = 0 \ (i \in I) \},$$

$$A_F(\lambda) = \{ \varphi \in A_F \mid \varphi \cdot k_\mu = q^{(\mu, \lambda)} \varphi \ (\mu \in \Lambda) \}.$$

Note that $A_F$ is a left $U_F$-submodule of $C_F$. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$, we set

$$A_F(\lambda)_\xi = \{ \varphi \in A_F(\lambda) \mid k_\mu \cdot \varphi = q^{(\xi, \mu)} \varphi \}.$$

Then we have

$$A_F(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_F(\lambda)_\xi.$$

We define $A$-forms $C_A$, $A_{\hat{A}}$, $A_{\hat{A}}(\lambda) \ (\lambda \in \Lambda^+)$ of $C_F$, $A_F$, $A_F(\lambda)$, respectively, by

$$C_A = \{ \varphi \in C_F \mid \langle \varphi, U_A^L \rangle \subset A \}, \quad A_{\hat{A}} = A_F \cap C_A, \quad A_{\hat{A}}(\lambda) = A_F(\lambda) \cap C_A.$$

Then $C_A$ is a Hopf algebra over $A$, and $A_{\hat{A}}$ is its $A$-subalgebra. Moreover, $C_A$ is a $U_A^L$-bimodule, and $A_{\hat{A}}$ is its left $U_A^L$-submodule. We also set $A_{\hat{A}}(\lambda)_\xi = A_F(\lambda)_\xi \cap A_{\hat{A}}$ for $\lambda \in \Lambda^+$, $\xi \in \Lambda$.

We set

$$C_\zeta = C \otimes_A C_A, \quad A_\zeta = C \otimes_A A_A, \quad A_\zeta(\lambda) = C \otimes_A A_{\hat{A}}(\lambda) \ (\lambda \in \Lambda^+).$$

Then $C_\zeta$ is a Hopf algebra over $C$. Moreover, the $U_F$-bimodule structure of $C_F$ induces a $U_\zeta^L$-bimodule structure of $C_\zeta$. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$, we set $A_\zeta(\lambda)_\xi = C \otimes_A A_{\hat{A}}(\lambda)_\xi$. Then we have

$$A_\zeta(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_\zeta(\lambda)_\xi.$$

We have a natural pairing

(1.18) \qquad \langle \cdot, \cdot \rangle : C_\zeta \times U_\zeta^L \to C

induced by (1.17).
1.2.2. For a ring (resp., \( \Lambda \)-graded ring) \( \mathcal{R} \) we denote by \( \text{Mod}(\mathcal{R}) \) (resp., \( \text{Mod}_\Lambda(\mathcal{R}) \)) the category of \( \mathcal{R} \)-modules (resp., \( \Lambda \)-graded left \( \mathcal{R} \)-modules). Assume that we are given a homomorphism \( j : A \to B \) of \( \Lambda \)-graded rings satisfying

\[
(1.19) \quad j(A(\lambda))B(\mu) = B(\mu)j(A(\lambda)) \quad (\lambda, \mu \in \Lambda).
\]

For \( M \in \text{Mod}_\Lambda(B) \), let \( \text{Tor}(M) \) be the subset of \( M \) consisting of \( m \in M \) such that there exists \( \lambda \in \Lambda^+ \) satisfying \( j(A(\lambda + \mu))m = \{0\} \) for any \( \mu \in \Lambda^+ \). Then \( \text{Tor}(M) \) is a subobject of \( M \) in \( \text{Mod}_\Lambda(B) \) by (1.19). We denote by \( \text{Tor}(A,B) \) the full subcategory of \( \text{Mod}_\Lambda(B) \) consisting of \( M \in \text{Mod}_\Lambda(B) \) such that \( \text{Tor}(M) = M \). Note that \( \text{Tor}(A,B) \) is closed under taking subquotients and extensions in \( \text{Mod}_\Lambda(B) \). Let \( \Sigma(A,B) \) denote the collection of morphisms \( f \) of \( \text{Mod}_\Lambda(B) \) such that its kernel \( \text{Ker}(f) \) and its cokernel \( \text{Coker}(f) \) belong to \( \text{Tor}(A,B) \). Then we define an abelian category \( \mathcal{C}(A,B) = \text{Mod}_\Lambda(B)/\text{Tor}(A,B) \) as the localization

\[
\mathcal{C}(A,B) = \Sigma(A,B)^{-1}\text{Mod}_\Lambda(B)
\]

of \( \text{Mod}_\Lambda(B) \) with respect to the multiplicative system \( \Sigma(A,B) \) (see, e.g., [16] for the notion of localization of categories). We denote by

\[
(1.20) \quad \omega(A,B)^*: \text{Mod}_\Lambda(B) \to \mathcal{C}(A,B)
\]

the canonical exact functor. It admits a right adjoint

\[
(1.21) \quad \omega(A,B)_*: \mathcal{C}(A,B) \to \text{Mod}_\Lambda(B),
\]

which is left exact. It is known that \( \omega(A,B)^* \circ \omega(A,B)_* \cong \text{Id} \). By taking the degree 0 part of (1.21), we obtain a left exact functor

\[
(1.22) \quad \Gamma_{(A,B)}: \mathcal{C}(A,B) \to \text{Mod}(B(0)).
\]

The abelian category \( \mathcal{C}(A,B) \) has enough injectives, and we have the right derived functors

\[
(1.23) \quad R^i\Gamma_{(A,B)}: \mathcal{C}(A,B) \to \text{Mod}(B(0)) \quad (i \in \mathbb{Z})
\]

of (1.22).

We apply the above arguments to the case \( A = B = A_\zeta \). Then \( \text{Tor}(M) \) for \( M \in \text{Mod}_\Lambda(A_\zeta) \) consists of \( m \in M \) such that there exists \( \lambda \in \Lambda^+ \) satisfying \( A_\zeta(\lambda)m = \{0\} \) (see [20, Lemma 3.4]). We set

\[
(1.24) \quad \text{Mod}(O_{B_\zeta}) = \mathcal{C}(A_\zeta,A_\zeta).
\]
In this case, the natural functors (1.20), (1.21), (1.22) are simply denoted as

\[(1.25) \quad \omega^* : \text{Mod}_\Lambda(A_\zeta) \to \text{Mod}(O_{B_\zeta}),\]
\[(1.26) \quad \omega_* : \text{Mod}(O_{B_\zeta}) \to \text{Mod}_\Lambda(A_\zeta),\]
\[(1.27) \quad \Gamma : \text{Mod}(O_{B_\zeta}) \to \text{Mod}(C).\]

**Remark 1.5.** In the terminology of noncommutative algebraic geometry, \(\text{Mod}(O_{B_\zeta})\) is the category of quasicoherent sheaves on the quantized flag manifold \(B_\zeta\), which is a noncommutative projective scheme. The notations \(B_\zeta, O_{B_\zeta}\) have only symbolical meaning.

1.2.3. Using Lusztig’s Frobenius homomorphism (1.12), we will relate the quantized flag manifold \(B_\zeta\) with the ordinary flag manifold \(B = B^- \setminus G\). Taking the dual Hopf algebras in (1.12), we obtain an injective homomorphism \(C[G] \to C_\zeta\) of Hopf algebras. Moreover, its image is contained in the center of \(C_\zeta\) (see [14]). We will regard \(C[G]\) as a central Hopf subalgebra of \(C_\zeta\) in the following. Setting

\[A_1 = \{ \varphi \in C[G] \mid \varphi(ng) = \varphi(g) \ (n \in N^-, g \in G) \},\]
\[A_1(\lambda) = \{ \varphi \in A_1 \mid \varphi(tg) = \theta_\lambda(t)\varphi(g) \ (t \in H, g \in G) \} \quad (\lambda \in \Lambda^+),\]
we have a \(\Lambda\)-graded algebra

\[A_1 = \bigoplus_{\lambda \in \Lambda^+} A_1(\lambda).\]

We have a left \(G\)-module structure of \(A_1\) given by

\[(x\varphi)(g) = \varphi(gx) \quad (\varphi \in A_1, x, g \in G).\]

In particular, \(A_1\) is a \(U(g)\)-module. Moreover, for each \(\lambda \in \Lambda^+, A_1(\lambda)\) is a \(U(g)\)-submodule of \(A_1\) which is an irreducible highest-weight module with highest-weight \(\lambda\). Regarding \(C[G]\) as a subalgebra of \(C_\zeta\), we have

\[A_1 = A_\zeta \cap C[G], \quad A_1(\lambda) = A_\zeta(\ell\lambda) \cap C[G].\]

Since the \(\Lambda\)-graded algebra \(A_1\) is the homogeneous coordinate algebra of the projective variety \(B = B^- \setminus G\), we have an identification

\[(1.28) \quad \text{Mod}(O_B) = \mathcal{C}(A_1, A_1)\]
of abelian categories, where \( \text{Mod}(\mathcal{O}_B) \) denotes the category of quasicoherent \( \mathcal{O}_B \)-modules on the ordinary flag manifold \( B \). We set

\[
(1.29) \quad \omega_{B*} = \omega(A_1, A_1)_*: \text{Mod}(\mathcal{O}_B) \to \text{Mod}_\Lambda(A_1).
\]

For \( \lambda \in \Lambda \), we denote by \( \mathcal{O}_B(\lambda) \in \text{Mod}(\mathcal{O}_B) \) the invertible \( G \)-equivariant \( \mathcal{O}_B \)-module corresponding to \( \lambda \). Then under identification (1.28), we have

\[
\omega_{B*} M = \bigoplus_{\lambda \in \Lambda} \Gamma(B, M \otimes_{\mathcal{O}_B} \mathcal{O}_B(\lambda)) \quad (M \in \text{Mod}(\mathcal{O}_B)),
\]

where \( \Gamma(B, .): \text{Mod}(\mathcal{O}_B) \to \mathbb{C} \) is the global section functor for the algebraic variety \( B \). In particular, the functor \( \Gamma_{(A_1, A_1)}: \text{Mod}(\mathcal{O}_B) \to \text{Mod}(\mathbb{C}) \) is identified with \( \Gamma(B, .) \).

For a \( \Lambda \)-graded \( \mathbb{C} \)-algebra \( B \), we define a new \( \Lambda \)-graded \( \mathbb{C} \)-algebra \( B^{(\ell)} \) by

\[
B^{(\ell)}(\lambda) = B(\ell \lambda) \quad (\lambda \in \Lambda).
\]

Let

\[
(\cdot)^{(\ell)}: \text{Mod}_\Lambda(B) \to \text{Mod}_\Lambda(B^{(\ell)})
\]

be the exact functor given by

\[
M^{(\ell)}(\lambda) = M(\ell \lambda) \quad (\lambda \in \Lambda)
\]

for \( M \in \text{Mod}_\Lambda(B) \).

We have the following results (see [20, Lemma 3.9]).

**Lemma 1.6.** Let \( B \) be a \( \Lambda \)-graded \( \mathbb{C} \)-algebra. Assume that we are given a homomorphism \( j: A_\zeta \to B \) of \( \Lambda \)-graded \( \mathbb{C} \)-algebras. We denote by \( j': A_1 \to B^{(\ell)} \) the induced homomorphism of \( \Lambda \)-graded \( \mathbb{C} \)-algebras. Assume that

\[
j(A_\zeta(\lambda))B(\mu) = B(\mu)j(A_\zeta(\lambda)) \quad (\lambda, \mu \in \Lambda),
\]

\[
j'(A_1(\lambda))B^{(\ell)}(\mu) = B^{(\ell)}(\mu)j'(A_1(\lambda)) \quad (\lambda, \mu \in \Lambda).
\]

Then the exact functor

\[
(\cdot)^{(\ell)}: \text{Mod}_\Lambda(B) \to \text{Mod}_\Lambda(B^{(\ell)})
\]

induces an equivalence

\[
(1.31) \quad \text{Fr}_*: \mathcal{C}(A_\zeta, B) \to \mathcal{C}(A_1, B^{(\ell)})
\]

of abelian categories. Moreover, we have

\[
(1.32) \quad \omega(A_1, B^{(\ell)})_* \circ \text{Fr}_* = (\cdot)^{(\ell)} \circ \omega(A_\zeta, B)_*.
\]
Lemma 1.7. Let $F$ be a $\Lambda$-graded $\mathbb{C}$-algebra, and let $A_1 \to F$ be a homomorphism of $\Lambda$-graded $\mathbb{C}$-algebras. Assume that $\text{Im}(A_1 \to F)$ is central in $F$. Regard $F$ as an object of $\text{Mod}_\Lambda(A_1)$, and consider $\omega_B^* F \in \text{Mod}(O_B)$. Then the multiplication of $F$ induces an $O_B$-algebra structure of $\omega_B^* F$, and we have an identification

$$C(A_1, F) = \text{Mod}(\omega_B^* F) \tag{1.33}$$

of abelian categories, where $\text{Mod}(\omega_B^* F)$ denotes the category of quasicoherent $\omega_B^* F$-modules. Moreover, under identification (1.33) we have

$$\Gamma_{(A_1, F)}(M) = \Gamma(B, M) \in \text{Mod}(F(0)) \quad (M \in \text{Mod}(\omega_B^* F)).$$

We define an $O_B$-algebra $\text{Fr}_* O_{B_\zeta}$ by

$$\text{Fr}_* O_{B_\zeta} = \omega_B^*(A^{(f)}_\zeta).$$

We denote by $\text{Mod}(\text{Fr}_* O_{B_\zeta})$ the category of quasicoherent $\text{Fr}_* O_{B_\zeta}$-modules. By Lemmas 1.6 and 1.7, we have the following.

Lemma 1.8. We have an equivalence

$$\text{Fr}_* : \text{Mod}(O_{B_\zeta}) \to \text{Mod}(\text{Fr}_* O_{B_\zeta})$$

of abelian categories. Moreover, for $M \in \text{Mod}(O_{B_\zeta})$ we have

$$R^i \Gamma(M) \simeq R^i \Gamma(B, \text{Fr}_*(M)),$$

where $\Gamma(B,.) : \text{Mod}(O_B) \to \text{Mod}(C)$ on the right-hand side is the global section functor for $B$.

§2. The category of $D$-modules

2.1. Ring of differential operators

2.1.1. We define a subalgebra $D_F$ of $\text{End}_F(A_F)$ by

$$D_F = \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_F, u \in U_F, \lambda \in \Lambda \rangle,$$

where

$$\ell_\varphi(\psi) = \varphi \psi, \quad r_\varphi(\psi) = \psi \varphi, \quad \partial_u(\psi) = u \cdot \psi, \quad \sigma_\lambda(\psi) = q^{(\lambda, \mu)} \psi.$$
for $\psi \in A_F(\mu)$. In fact, we have

$$D_F = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_F, u \in U_F, \lambda \in \Lambda \rangle$$

by [20, Lemma 4.1].

We have a natural grading

$$D_F = \bigoplus_{\lambda \in \Lambda^+} D_F(\lambda),$$

$$D_F(\lambda) = \{ \Phi \in D_F \mid \Phi(A_F(\mu)) \subset A_F(\lambda + \mu) \ (\mu \in \Lambda) \} \quad (\lambda \in \Lambda)$$

of $D_F$. It is easily checked that

$$\partial_u \ell_\varphi = \sum_{(u)} \ell_{u(0)} \varphi \partial_u \ell_{u(1)} \quad (u \in U_F, \varphi \in A_F),$$

$$\partial_u \sigma_\lambda = \sigma_\lambda \partial_u \quad (u \in U_F, \lambda \in \Lambda),$$

$$\sigma_\lambda \ell_\varphi = q^{(\lambda, \mu)} \ell_\varphi \sigma_\lambda \quad (\lambda \in \Lambda, \varphi \in A_F(\mu)).$$

Set

$$E_F = A_F \otimes U_F \otimes F[\Lambda].$$

We have a natural $F$-algebra structure of $E_F$ such that $A_F \otimes 1 \otimes 1, 1 \otimes U_F \otimes 1, 1 \otimes 1 \otimes F[\Lambda]$ are subalgebras of $E_F$ naturally isomorphic to $A_F, U_F, F[\Lambda]$, respectively, and such that we have the relations

$$u \varphi = \sum_{(u)} (u(0) \cdot \varphi) u(1) \quad (u \in U_F, \varphi \in A_F),$$

$$ue(\lambda) = e(\lambda) u \quad (u \in U_F, \lambda \in \Lambda),$$

$$e(\lambda) \varphi = q^{(\lambda, \mu)} \varphi e(\lambda) \quad (\lambda \in \Lambda, \varphi \in A_F(\mu))$$

in $E_F$. Here, we identify $A_F \otimes 1 \otimes 1, 1 \otimes U_F \otimes 1, 1 \otimes 1 \otimes F[\Lambda]$ with $A_F, U_F, F[\Lambda]$, respectively. Then we have a surjective algebra homomorphism

$$E_F \to D_F$$

sending $\varphi \in A_F, u \in U_F, e(\lambda) \in F[\Lambda] \ (\lambda \in \Lambda)$ to $\ell_\varphi, \partial_u, \sigma_\lambda$, respectively. Moreover, $E_F$ has an obvious $\Lambda$-grading so that (2.1) preserves the $\Lambda$-grading.
2.1.2. Set

\[ D_A = \langle \ell \varphi, r \varphi, \partial u, \sigma \lambda \mid \varphi \in A_\Lambda, u \in U_\Lambda, \lambda \in \Lambda \rangle_{\Lambda\text{-alg}} \subset D_F, \]
\[ E_A = A_\Lambda \otimes U_\Lambda \otimes \Lambda[\Lambda] \subset E_F. \]

They are \( \Lambda \)-graded \( \Lambda \)-subalgebras of \( D_F \) and \( E_F \), respectively. Again, we have

\[ D_\Lambda = \langle \ell \varphi, \partial u, \sigma \lambda \mid \varphi \in A_\Lambda, u \in U_\Lambda, \lambda \in \Lambda \rangle_{\Lambda\text{-alg}} \]

by [20]. In particular, we have a surjective homomorphism

\[ E_\Lambda \rightarrow D_\Lambda \]

of \( \Lambda \)-graded algebras. Note that there is a canonical embedding

\[ D_\Lambda \rightarrow \text{End}_\Lambda(A_\Lambda). \]

2.1.3. We set

\[ D_\zeta = C \otimes A_D_A, \quad E_\zeta = C \otimes A_D_E_A = A_\zeta \otimes U_\zeta \otimes C[\Lambda]. \]

\( D_\zeta \) is a \( \Lambda \)-graded \( C \)-algebra generated by elements of the form

\[ \ell \varphi, \quad \partial u, \quad \sigma \lambda \quad (\varphi \in A_\zeta, u \in U_\zeta, \lambda \in \Lambda). \]

We have a surjective homomorphism

\[ E_\zeta \rightarrow D_\zeta \]

of \( \Lambda \)-graded \( C \)-algebras.

**Lemma 2.1.** Let \( z \in Z_{\text{Har}}(U_\zeta) \), and write \( \iota(z) = \sum_{\lambda \in \Lambda} c_\lambda k_{2\lambda} \) (\( c_\lambda \in C \)). Then we have

\[ \partial z = \sum_{\lambda \in \Lambda} c_\lambda \sigma_{2\lambda}. \]

**Proof.** This follows from the corresponding statement over \( F \), which is given in [19, Section 5.1]. \( \square \)

**Remark 2.2.** The natural algebra homomorphism \( D_\zeta \rightarrow \text{End}_C(A_\zeta) \) is not injective.
2.1.4. Define an $\mathcal{O}_B$-algebra $\text{Fr}_* \mathcal{D}_B$ by

$$\text{Fr}_* \mathcal{D}_B = \omega_B \mathcal{D}^{(\ell)}_B.$$  

We define $ZD^{(\ell)}_B$ to be the central subalgebra of $D^{(\ell)}_B$ generated by the elements of the form

$$\ell \varphi, \quad \partial_u, \quad \sigma_\lambda \quad (\varphi \in A_1, u \in Z_{\text{Fr}}(U), \lambda \in \Lambda),$$  

and we set

$$Z = \omega_B^* ZD^{(\ell)}_B.$$  

It is a central subalgebra of $\text{Fr}_* \mathcal{D}_B$. Define a subvariety $\mathcal{V}$ of $\mathcal{B} \times K \times H$ by

$$\mathcal{V} = \{(B^{-} g, k, t) \in \mathcal{B} \times K \times H \mid g \kappa(k) g^{-1} \in t^{2\ell} N^{-}\},$$  

where $\kappa : K \to G$ is given by $\kappa(k_1, k_2) = k_1 k_2^{-1}$. We denote by

$$p_\mathcal{V} : \mathcal{V} \to \mathcal{B}$$  

the projection. Now we can state the main results of [20].

**Theorem 2.3 ([20, Theorem 5.2]).** The $\mathcal{O}_B$-algebra $Z$ is naturally isomorphic to $p_\mathcal{V}^* \mathcal{O}_\mathcal{V}$.

Define an $\mathcal{O}_\mathcal{V}$-algebra $\mathcal{D}_B$ by

$$\mathcal{D}_B = p^{-1}_\mathcal{V} \text{Fr}_* \mathcal{D}_B \otimes_{p^{-1}_\mathcal{V} p_\mathcal{V}, \mathcal{O}_\mathcal{V}} \mathcal{O}_\mathcal{V}.$$  

**Theorem 2.4 ([20, Theorem 6.1]).** Here $\mathcal{D}_B$ is an Azumaya algebra of rank $\ell^2 (\mathcal{A}^+)$.

Define

$$\delta : \mathcal{V} \to K \times_{H/W} H$$  

by $\delta(B^{-} g, k, t) = (k, t)$, where $K \to H/W$ is given by $k \mapsto [h]$, where $h$ is an element of $H$ conjugate to the semisimple part of $\kappa(k)$, and $H \to H/W$ is given by $t \mapsto [t^{2\ell}]$.

**Theorem 2.5 ([20, Theorem 6.10]).** For any $(k, t) \in K \times_{H/W} H$, the restriction of $\mathcal{D}_B$ to $\delta^{-1}(k, t)$ is a split Azumaya algebra.
2.2. Category of $D$-modules

We define an abelian category $\text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ by

$$\text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) = \mathcal{C}(A_\zeta, D_\zeta).$$

By Lemmas 1.6 and 1.7, we have an equivalence

$$(2.2)\quad \text{Fr}_*: \text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) \to \text{Mod}(\text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})$$

of abelian categories, where $\text{Mod}(\text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta})$ denotes the category of quasicoherent $\text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta}$-modules. Moreover, for $M \in \text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ we have

$$(2.3)\quad R^i\Gamma_{(A_\zeta,D_\zeta)}(M) = R^i\Gamma(\mathcal{B},\text{Fr}_*(M)) \in \text{Mod}(D_\zeta(0)),$$

where $\Gamma(\mathcal{B},)$ on the right-hand side is the global section functor for the ordinary flag variety $\mathcal{B}$.

For $t \in H$ we define an abelian category $\text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ by

$$\text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) = \text{Mod}_{\Lambda,t}(D_\zeta)/(\text{Mod}_{\Lambda,t}(D_\zeta) \cap \text{Tor}_{\Lambda,+}(A_\zeta, D_\zeta)),$$

where $\text{Mod}_{\Lambda,t}(D_\zeta)$ is the full subcategory of $\text{Mod}_{\Lambda}(D_\zeta)$ consisting of $M \in \text{Mod}_{\Lambda}(D_\zeta)$ so that $\sigma_{\lambda}|_{M(\mu)} = \theta_{\lambda}(t)^{(\lambda,\mu)}\text{id}$ for any $\lambda, \mu \in \Lambda$. Then we can regard $\text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ as a full subcategory of $\text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ (see [19, Lemma 4.6]).

Set

$$\text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t} = \text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t,$$

where $\mathbb{C}_t$ denotes the 1-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda) \mapsto \theta_{\lambda}(t)$ for $\lambda \in \Lambda$. The equivalence (2.2) induces the equivalence

$$(2.4)\quad \text{Fr}_*: \text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) \to \text{Mod}(\text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}),$$

where $\text{Mod}(\text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ denotes the category of quasicoherent $\text{Fr}_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. In particular, for $M \in \text{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ we have

$$R^i\Gamma_{(A_\zeta,D_\zeta)}(M) = R^i\Gamma(\mathcal{B},\text{Fr}_*M) \in \text{Mod}(D_{\zeta,t}(0)),$$

where $D_{\zeta,t}(0) = D_{\zeta}(0)/\sum_{\lambda \in \Lambda} D_{\zeta}(0)(\sigma_{\lambda} - \theta_{\lambda}(t))$. 
2.3. Conjecture

By Lemma 2.1, the natural algebra homomorphism

$$U_\zeta \otimes \mathbb{C}[\Lambda] \to D_\zeta(0)$$

factors through

$$U_\zeta \otimes Z_{\text{Har}}(U_\zeta) \mathbb{C}[\Lambda] \to D_\zeta(0),$$

where $Z_{\text{Har}}(U_\zeta)$ is identified with $\mathbb{C}[2\Lambda]^W$ by (1.15). Hence, we have a natural algebra homomorphism

(2.5) $$U_\zeta \otimes Z_{\text{Har}}(U_\zeta) \mathbb{C}[\Lambda] \to \Gamma(B, Fr_* D_{B\zeta}).$$

For $t \in H$ we denote by $C_t$ the 1-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda)v = \theta_\lambda(t)v$ ($v \in C_t$). Then (2.5) induces an algebra homomorphism

(2.6) $$U_\zeta \otimes Z_{\text{Har}}(U_\zeta) C_t \to \Gamma(B, Fr_* D_{B\zeta,t}),$$

where $C_t$ is regarded as a $Z_{\text{Har}}(U_\zeta)$-module by $Z_{\text{Har}}(U_\zeta) \cong \mathbb{C}[2\Lambda] \subset \mathbb{C}[\Lambda]$.

Denote by $h_G$ the Coxeter number for $G$.

**Conjecture 2.6.** Assume that $\ell > h_G$. The algebra homomorphism (2.5) is an isomorphism, and we have

$$R^i \Gamma(B, Fr_* D_{B\zeta}) = 0$$

for $i \neq 0$.

**Proposition 2.7.** Let $\ell > h_G$, and assume that Conjecture 2.6 is valid. Then for $t \in H$ we have

$$\Gamma(B, Fr_* D_{B\zeta,t}) \cong U_\zeta \otimes Z_{\text{Har}}(U_\zeta) C_t$$

and

$$R^i \Gamma(B, Fr_* D_{B\zeta,t}) = 0 \quad (i \neq 0).$$

**Proof.** Define $f : V \to H$ to be the composite of the embedding $V \to B \times K \times H$ and the projection $B \times K \times H \to H$ onto the third factor. Since $p_V$ is an affine morphism, we have $R p_V_* D_{B\zeta} = p_V_* D_{B\zeta} = Fr_* D_{B\zeta}$. Hence, we have

$$U_\zeta \otimes Z_{\text{Har}}(U_\zeta) \mathbb{C}[\Lambda] = U_\zeta \otimes Z_{\text{Har}}(U_\zeta) \mathbb{C}[\Lambda] \cong R \Gamma(B, Fr_* D_{B\zeta}) = R \Gamma(V, D_{B\zeta}).$$
Here we use the fact that $C[\Lambda]$ is a free $Z_{\text{Har}}(U_{\zeta})$-module (see [17]). Denote by $O_t$ the $O_H$-module corresponding to the $C[\Lambda]$-module $C_t$. Similarly, we have

$$\text{Fr}_* D_{B_{\zeta},t} = pv_*(\tilde{D}_{B_{\zeta}} \otimes_{\mathcal{C}[\Lambda]} C_t) = Rp_{\nu_*}(\tilde{D}_{B_{\zeta}} \otimes_{\mathcal{C}[\Lambda]} C_t).$$

Since $f$ is flat, we have $Lf^* O_t = f^* O_t$. Hence, by Theorem 2.4 we have

$$\tilde{D}_{B_{\zeta}} \otimes_{\mathcal{O}_V} Lf^* O_t = \tilde{D}_{B_{\zeta}} \otimes_{\mathcal{O}_V} f^* O_t = \tilde{D}_{B_{\zeta}} \otimes_{\mathcal{O}_V} f^* O_t.$$

It follows that

$$\text{Fr}_* D_{B_{\zeta},t} = Rp_{\nu_*}(\tilde{D}_{B_{\zeta}} \otimes_{\mathcal{O}_V} Lf^* O_t) = Rp_{\nu_*}(\tilde{D}_{B_{\zeta}}) \otimes_{\mathcal{O}_H} O_t.$$

Hence we have

$$R\Gamma(B, \text{Fr}_* D_{B_{\zeta},t}) = R\Gamma(H, Rf_*(\tilde{D}_{B_{\zeta}} \otimes_{\mathcal{O}_V} Lf^* O_t))$$

$$= R\Gamma(H, Rf_* \tilde{D}_{B_{\zeta}} \otimes_{\mathcal{O}_H} O_t) = R\Gamma(H, Rf_* \tilde{D}_{B_{\zeta}} \otimes_{\mathcal{C}[\Lambda]} C_t)$$

$$= R\Gamma(V, \tilde{D}_{B_{\zeta}} \otimes_{\mathcal{C}[\Lambda]} C_t = U_{\zeta} \otimes_{Z_{\text{Har}}(U_{\zeta})} \mathcal{C}[\Lambda] \otimes_{\mathcal{C}[\Lambda]} C_t$$

$$= U_{\zeta} \otimes_{Z_{\text{Har}}(U_{\zeta})} \mathcal{C}_t. \qed$$

2.4. Derived Beilinson–Bernstein equivalence

We show that Conjecture 2.6 implies a variant of the Beilinson–Bernstein equivalence for derived categories.

Recall that we have an identification

$$Z_{\text{Har}}(U_{\zeta}) \cong \mathcal{C}[2\Lambda]^{\mathcal{W}_o} \subset \mathcal{C}[2\Lambda] \subset \mathcal{C}[\Lambda].$$

Recall also that we identify $\mathcal{C}[\Lambda]$ with the coordinate algebra $\mathcal{C}[H]$ of $H$. Set $H^{(2)} = H/\text{Ker}(H \ni t \mapsto t^2 \in H)$, and let $\pi : H \to H^{(2)}$ be the canonical homomorphism. Then we have a natural identification $\mathcal{C}[H^{(2)}] = \mathcal{C}[2\Lambda]$ so that $\pi^* : \mathcal{C}[H^{(2)}] \to \mathcal{C}[H]$ is identified with the inclusion $\mathcal{C}[2\Lambda] \subset \mathcal{C}[\Lambda]$. Denote the isomorphism $H \cong H^{(2)}$ corresponding to $\mathcal{C}[\Lambda] \ni e(\lambda) \leftrightarrow e(2\lambda) \in \mathcal{C}[2\Lambda]$ by $t \leftrightarrow t^{1/2}$. Then we have $\pi(t) = (t^2)^{1/2}$. The shifted action (1.13) of $W$ on $\mathcal{C}[2\Lambda]$ induces an action of $W$ on $H^{(2)}$ given by

$$w \circ t^{1/2} = \left(w(tt_{2\rho})t_{2\rho}^{-1}\right)^{1/2} \quad (w \in W, t \in H),$$
where \( t_{2\rho} \in H \) is given by \( \theta_\mu(t_{2\rho}) = \zeta^{2(\mu,\rho)} \) for any \( \mu \in \Lambda \) (note that \( 2(\mu,\rho) \in \mathbb{Z} \)), and \( \mathbb{Z}_{\text{Har}}(U_\zeta) \) is regarded as the coordinate algebra of the quotient variety \((W_0)\backslash H(2)\). For \( t \in H \) we denote by \( \chi_t : \mathbb{C}[\Lambda] \to \mathbb{C} \) the corresponding algebra homomorphism. By the above argument, we have
\[
\chi_{t_1}|_{\mathbb{Z}_{\text{Har}}(U_\zeta)} = \chi_{t_2}|_{\mathbb{Z}_{\text{Har}}(U_\zeta)} \iff (t_{2_1}^2)^{1/2} \in W \circ t_{2_2}^{1/2}.
\]
We say that \( t \in H \) is \textit{regular} if
\[
\{ w \in W \mid w \circ (t^2)^{1/2} = (t^2)^{1/2} \} = \{ 1 \}.
\]

We denote by \( \text{Mod}_{\text{coh}}(\text{Fr}_*\mathcal{D}_{B_\zeta},t) \) (resp., \( \text{Mod}_f(U_\zeta \otimes \mathbb{Z}_{\text{Har}}(U_\zeta) \mathbb{C}_t) \)) the category of coherent \( \text{Fr}_*\mathcal{D}_{B_\zeta},t \)-modules (resp., finitely generated \( U_\zeta \otimes \mathbb{Z}_{\text{Har}}(U_\zeta) \mathbb{C}_t \)-modules). We also denote by \( \text{Mod}_{\text{coh},t}(\text{Fr}_*\mathcal{D}_{B_\zeta}) \) (resp., \( \text{Mod}_f(U_\zeta) \)) the category of coherent \( \text{Fr}_*\mathcal{D}_{B_\zeta} \)-modules (resp., finitely generated \( U_\zeta \)-modules) killed by some power of the maximal ideal of \( \mathbb{C}[\Lambda] \) (resp., \( \mathbb{Z}_{\text{Har}}(U_\zeta) \)) corresponding to \( t \in H \).

**Theorem 2.8.** Let \( \ell > h_G \), and assume that Conjecture 2.6 is valid. If \( t \in H \) is regular, then the natural functors
\[
\begin{align*}
R\Gamma_\hat{t} : D^b(\text{Mod}_{\text{coh}}(\text{Fr}_*\mathcal{D}_{B_\zeta},t)) & \rightarrow D^b(\text{Mod}_f(U_\zeta)), \\
R\Gamma_t : D^b(\text{Mod}_{\text{coh}}(\text{Fr}_*\mathcal{D}_{B_\zeta},t)) & \rightarrow D^b(\text{Mod}_f(U_\zeta \otimes \mathbb{Z}_{\text{Har}}(U_\zeta) \mathbb{C}_t))
\end{align*}
\]
give equivalences of derived categories.

The proof of this result is completely similar to that of the corresponding fact for Lie algebras in positive characteristics given in [6, Theorem 5.3.1]. We give below only an outline of it. First note the following.

**Proposition 2.9 ([7, Theorem B]).** Here \( U_\zeta \) has finite homological dimension.

The functors
\[
R\Gamma_\hat{t} : D^b(\text{Mod}_{\text{coh}}(\text{Fr}_*\mathcal{D}_{B_\zeta})) \rightarrow D^b(\text{Mod}_f(U_\zeta)), \\
R\Gamma_t : D^{-}(\text{Mod}_{\text{coh}}(\text{Fr}_*\mathcal{D}_{B_\zeta},t)) \rightarrow D^{-}(\text{Mod}_f(U_\zeta \otimes \mathbb{Z}_{\text{Har}}(U_\zeta) \mathbb{C}_t))
\]
have left adjoints
\[
\begin{align*}
\mathcal{L}_\hat{t} : D^b(\text{Mod}_f(U_\zeta)) & \rightarrow D^b(\text{Mod}_{\text{coh}}(\text{Fr}_*\mathcal{D}_{B_\zeta})), \\
\mathcal{L}_t : D^{-}(\text{Mod}_f(U_\zeta \otimes \mathbb{Z}_{\text{Har}}(U_\zeta) \mathbb{C}_t)) & \rightarrow D^{-}(\text{Mod}_{\text{coh}}(\text{Fr}_*\mathcal{D}_{B_\zeta},t)).
\end{align*}
\]
Arguing exactly as in [6, Sections 3.3, 3.4] using Theorem 2.4 and Proposition 2.9, we obtain the following.
Proposition 2.10.

(i) If \( t \) is regular, the adjunction morphism \( \text{Id} \rightarrow R \Gamma_t \circ \mathcal{L}_t \) is an isomorphism on \( D^b(\text{Mod}_{f,t}(U_\zeta)) \).

(ii) For any \( t \), the adjunction morphism \( \text{Id} \rightarrow R \Gamma_t \circ \mathcal{L}_t \) is an isomorphism on \( D^{-}(\text{Mod}_{f}(U_\zeta \otimes \mathbb{Z}_{\text{Har}}(U_\zeta) \mathbb{C}_{t})) \).

Arguing exactly as in [6, Section 3.5] using Theorem 2.4, Proposition 2.10, and Lemma 2.11 below, we obtain Theorem 2.8. Details are omitted.

Lemma 2.11 ([21, Section 2.4]). The variety \( V \) is a symplectic manifold.

2.5. Finite part

2.5.1. In [20, Section 4], we also introduced a quotient algebra \( D'_\zeta \) of \( E_\zeta \), which is closely related to \( D_\zeta \). Let us recall its definition. Take bases \( \{x_p\}_{p} \), \( \{y_p\}_{p} \), \( \{x^L_p\}_{p} \), \( \{y^L_p\}_{p} \) of \( U_\zeta^+, U_\zeta^- \), \( U_\zeta^{L,+} \), \( U_\zeta^{L,-} \), respectively, such that

\[
\tau^L_\zeta(x_{1},y^L_{p}) = \delta_{p_1,p_2}, \quad L\tau_\zeta(x^L_{p_1},y_{p}) = \delta_{p_1,p_2}.
\]

We assume that

\[
x_p \in U_{\zeta,\beta_p}^+, \quad y_p \in U_{\zeta,-\beta_p}^-, \quad x^L_p \in U_{\zeta,\beta_p}^{L,+}, \quad y^L_p \in U_{\zeta,-\beta_p}^{L,-}
\]

for \( \beta_p \in Q^+ \).

For \( \varphi \in A_\zeta(\lambda,\xi) \) with \( \lambda \in \Lambda^+, \xi \in \Lambda \), we set

\[
\Omega'_1(\varphi) = \sum_p (y^L_p \cdot \varphi)x_p \in E_\zeta,\emptyset,
\]

\[
\Omega'_2(\varphi) = \sum_p ((Sx^L_p \cdot \varphi)y_p k_{\beta_p} k_{2\xi} e(-2\lambda)) \in E_\zeta,\emptyset,
\]

\[
\Omega'(\varphi) = \Omega'_1(\varphi) - \Omega'_2(\varphi) \in E_\zeta,\emptyset.
\]

We extend \( \Omega' \) to whole \( A_\zeta \) by linearity. Then \( D'_\zeta \) is defined by

\[
D'_\zeta = E_\zeta / \sum_{\varphi \in A_\zeta} A_\zeta \Omega'(\varphi) U_\zeta \mathbb{C}[\Lambda].
\]

We have a sequence

\[
E_\zeta \rightarrow D'_\zeta \rightarrow D_\zeta
\]

of surjective homomorphisms of \( \Lambda \)-graded algebras. Moreover, \( D'_\zeta \rightarrow D_\zeta \) induces an isomorphism

\[
(2.7) \quad \omega^* D'_\zeta \cong \omega^* D_\zeta
\]

in \( \text{Mod}(\mathcal{O}_{B_\zeta}) \) (see [20, Corollary 6.6]).
2.5.2. We set

\[ U^0_{F,\diamond} = \bigoplus_{\lambda \in \Lambda} \mathbb{F} k_{2\lambda} \subset U^0_F, \quad U_{F,\diamond} = S(U^0_F)U^0_{F,\diamond}U^+_F \subset U_F. \]

Then we see easily the following.

**Lemma 2.12.** The subspace \( U_{F,\diamond} \) is an \( \text{ad}(U_F) \)-stable subalgebra of \( U_F \).

Set

\( (2.8) \quad U_{F,f} = \{ u \in U_F \mid \dim \text{ad}(U_F)(u) < \infty \}. \)

Then \( U_{F,f} \) is a subalgebra of \( U_F \). Moreover, by [12] we have

\( (2.9) \quad U_{F,f} = \sum_{\lambda \in \Lambda^+} \text{ad}(U_F)(k_{-2\lambda}), \)

and hence \( U_{F,f} \) is a subalgebra of \( U_{F,\diamond} \). Note that \( U_{F,\diamond} \) and \( U_{F,f} \) are not Hopf subalgebras of \( U_F \); nevertheless, they satisfy the following.

**Lemma 2.13.** We have

\[ \Delta(U_{F,f}) \subset U_F \otimes U_{F,f}, \quad \Delta(U_{F,\diamond}) \subset U_F \otimes U_{F,\diamond}. \]

**Proof.** For \( u \in U_F \) and \( \lambda \in \Lambda^+ \), we have

\[
\Delta(\text{ad}(u)(k_{-2\lambda})) = \sum_{(u)} \Delta(u_{(0)}k_{-2\lambda}(Su_{(1)})) \\
= \sum_{(u)} u_{(0)}k_{-2\lambda}(Su_{(3)}) \otimes u_{(1)}k_{-2\lambda}(Su_{(2)}) \\
= \sum_{(u)} u_{(0)}k_{-2\lambda}(Su_{(2)}) \otimes \text{ad}(u_{(1)})(k_{-2\lambda}).
\]

Hence, the first formula follows from (2.9). Since \( U_{F,\diamond} \) is generated by \( e_i \), \( Sf_i \) for \( i \in I \) and \( k_{2\lambda} \) for \( \lambda \in \Lambda \), the second formula is a consequence of the fact that \( \Delta(e_i), \Delta(Sf_i), \Delta(k_{2\lambda}) \) belong to \( U_F \otimes U_{F,\diamond} \).

We set

\[ E_{F,\diamond} = A_F \otimes U_{F,\diamond} \otimes \mathbb{F}[\Lambda] \subset E_F, \]
\[ E_{F,f} = A_F \otimes U_{F,f} \otimes \mathbb{F}[\Lambda] \subset E_F. \]
By Lemma 2.13, they are subalgebras of $E_F$.

We set

$$U^0_{\Lambda,\Diamond} = U^0_{E,\Diamond} \cap U_{\Lambda} = \bigoplus_{\lambda \in \Lambda} \Lambda k_{2\lambda}, \quad U_{\Lambda,\Diamond} = U_{F,\Diamond} \cap U_{\Lambda} = S(U^-_{\Lambda})U^0_{\Lambda,\Diamond}U^+_\Lambda,$$

$$U_{\Lambda,f} = U_{\Lambda} \cap U_{F,f},$$

and

$$E_{\Lambda,\Diamond} = E_{\Lambda} \cap E_{F,\Diamond} = A_{\Lambda} \otimes U_{\Lambda,\Diamond} \otimes A[\Lambda] \subset E_{F,\Diamond},$$

$$E_{\Lambda,f} = E_{\Lambda} \cap E_{F,f} = A_{\Lambda} \otimes U_{\Lambda,f} \otimes A[\Lambda] \subset E_{F,f}.$$

We also set

$$E_{\zeta,\Diamond} = C \otimes A_{\zeta,\Diamond} = A_{\zeta} \otimes U_{\zeta,\Diamond} \otimes C[\Lambda] \subset E_{\zeta},$$

$$E_{\zeta,f} = C \otimes A_{\zeta,f} = A_{\zeta} \otimes U_{\zeta,f} \otimes C[\Lambda] \subset E_{\zeta},$$

and

$$D_{\zeta,\Diamond} = \text{Im}(E_{\zeta,\Diamond} \to D_{\zeta}), \quad D_{\zeta,f} = \text{Im}(E_{\zeta,f} \to D_{\zeta}),$$

$$D'_{\zeta,\Diamond} = \text{Im}(E_{\zeta,\Diamond} \to D'_{\zeta}), \quad D'_{\zeta,f} = \text{Im}(E_{\zeta,f} \to D'_{\zeta}).$$

By

$$E_{\zeta} \cong E_{\zeta,\Diamond} \otimes U_{\zeta,\Diamond} U_{\zeta}$$

we obtain

$$D'_{\zeta,\Diamond} = E_{\zeta,\Diamond} \bigg/ \sum_{\varphi \in A_{\zeta}} A_{\zeta} \Omega'_{\varphi} U_{\zeta,\Diamond} C[\Lambda],$$

(2.10)

$$D'_{\zeta} \cong D'_{\zeta,\Diamond} \otimes U_{\zeta,\Diamond} U_{\zeta}.$$

(2.11)

2.5.3. Since $U_{\zeta}$ is a free $U_{\zeta,\Diamond}$-module, we have

$$R^i \Gamma(\omega^* D'_{\zeta,\Diamond}) \cong R^i \Gamma(\omega^* D'_{\zeta,f}) \otimes U_{\zeta,\Diamond} U_{\zeta}$$

for any $i \in \mathbb{Z}$. Since $U_{\zeta,\Diamond}$ is a localization of $U_{\zeta,f}$ with respect to the Ore subset $\{k_{-2\lambda} \mid \Lambda \in \Lambda^+\}$, we have

$$R^i \Gamma(\omega^* D'_{\zeta,\Diamond}) \cong R^i \Gamma(\omega^* D'_{\zeta,f}) \otimes U_{\zeta,f} U_{\zeta,\Diamond}$$
for any \( i \in \mathbb{Z} \). It follows that
\[
R^i \Gamma (\omega^* D'_\zeta) \cong R^i \Gamma (\omega^* D'_{\zeta,f}) \otimes U_{\zeta,f} U_\zeta
\]
for any \( i \in \mathbb{Z} \). Note that
\[
R^i \Gamma (B, \text{Fr}_* D_{B_\zeta}) \cong R^i \Gamma (\omega^* D'_{\zeta})
\]
by Lemma 1.8 and (2.7). Hence Conjecture 2.6 is a consequence of the following stronger conjecture.

**Conjecture 2.14.** Assume that \( \ell > h_G \). We have
\[
\Gamma (\omega^* D'_{\zeta,f}) \cong U_{\zeta,f} \otimes \mathbb{Z}_{\text{Har}} (U_\zeta) \mathbb{C}[\Lambda],
\]
and
\[
R^i \Gamma (\omega^* D'_{\zeta,f}) = 0
\]
for \( i \neq 0 \).

In the rest of this article, we give a reformulation of Conjecture 2.14 in terms of the induction functor.

**§3. Representations**

**3.1.**

For simplicity, we introduce a new notation, \( \tilde{U}_F^- = S(U_F^-) \). Then we have \( \tilde{U}_F^- = \langle \tilde{f}_i \mid i \in I \rangle \), where \( \tilde{f}_i = f_i k_i \) for \( i \in I \). Moreover, setting
\[
\tilde{U}_{F,\gamma}^- = \{ u \in \tilde{U}_F^- \mid k_\mu u k_{-\mu} = q^{(\gamma,\mu)} u \ (\mu \in \Lambda) \}
\]
for \( \gamma \in Q \), we have
\[
\tilde{U}_F^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{F,\gamma}^-,
\]
\[
\tilde{U}_{F,\gamma}^- = U_{F,\gamma}^- k_\gamma \quad (\gamma \in Q^+).
\]
We also set
\[
\tilde{U}_A = U_A \cap \tilde{U}_F, \quad \tilde{U}_{A,\gamma} = U_A \cap \tilde{U}_{F,\gamma} \quad (\gamma \in Q^+),
\]
\[
\tilde{U}_\zeta = \mathbb{C} \otimes_A \tilde{U}_A, \quad \tilde{U}_{\zeta,\gamma} = \mathbb{C} \otimes_A \tilde{U}_{A,\gamma} \quad (\gamma \in Q^+).
\]
Then we have
\[
\tilde{U}_A^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{A,\gamma}^-, \quad \tilde{U}_\zeta^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{\zeta,\gamma}^-.
\]
3.2.

For \( \lambda \in \Lambda \), we define an algebra homomorphism \( \chi_{\lambda} : U_F^{0} \to F \) by \( \chi_{\lambda}(k_{\mu}) = q^{(\lambda,\mu)} \) (\( \mu \in \Lambda \)). For \( M \in \text{Mod}(U_F) \) and \( \lambda \in \Lambda \), we set

\[
M_{\lambda} = \{ m \in M \mid hm = \chi_{\lambda}(h)m \ (h \in U_F^{0}) \}.
\]

For \( \lambda \in \Lambda \), we define \( M_{+,F}(\lambda), M_{-,F}(\lambda) \in \text{Mod}(U_F) \) by

\[
M_{+,F}(\lambda) = U_F^{+} / \sum_{y \in U_F^{-}} U_F(y - \varepsilon(y)) + \sum_{h \in U_F^{0}} U_F(h - \chi_{\lambda}(h)),
\]

\[
M_{-,F}(\lambda) = U_F^{-} / \sum_{x \in U_F^{+}} U_F(x - \varepsilon(x)) + \sum_{h \in U_F^{0}} U_F(h - \chi_{\lambda}(h)),
\]

where \( M_{+,F}(\lambda) \) is a lowest-weight module with lowest-weight \( \lambda \), and \( M_{-,F}(\lambda) \) is a highest-weight module with highest-weight \( \lambda \). We have isomorphisms

\[
M_{+,F}(\lambda) \cong U_F^{+} \quad (\overline{u} \leftrightarrow u), \quad M_{-,F}(\lambda) \cong U_F^{-} \quad (\overline{u} \leftrightarrow u)
\]

of \( F \)-modules. Moreover, we have weight-space decompositions

\[
M_{+,F}(\lambda) = \bigoplus_{\mu \in \lambda + Q^{+}} M_{+,F}(\lambda)_{\mu}, \quad M_{-,F}(\lambda) = \bigoplus_{\mu \in \lambda - Q^{+}} M_{-,F}(\lambda)_{\mu}.
\]

For \( \lambda \in \Lambda^{+} \) we define \( L_{+,F}(-\lambda), L_{-,F}(\lambda) \in \text{Mod}_{f}(U_F) \) by

\[
L_{+,F}(-\lambda) = U_F^{+} / \sum_{y \in U_F^{-}} U_F(y - \varepsilon(y)) + \sum_{h \in U_F^{0}} U_F(h - \chi_{-\lambda}(h)) + \sum_{i \in I} U_F e_i((\lambda,\alpha_i^{\vee})+1),
\]

\[
L_{-,F}(\lambda) = U_F^{-} / \sum_{x \in U_F^{+}} U_F(x - \varepsilon(x)) + \sum_{h \in U_F^{0}} U_F(h - \chi_{\lambda}(h)) + \sum_{i \in I} U_F f_i((\lambda,\alpha_i^{\vee})+1).
\]

While \( L_{+,F}(-\lambda) \) is a finite-dimensional irreducible lowest-weight module with lowest-weight \(-\lambda\), here \( L_{-,F}(\lambda) \) is a finite-dimensional irreducible
highest-weight module with highest-weight $\lambda$. We have weight-space decompositions

$$L_{+,F}(-\lambda) = \bigoplus_{\mu \in -\lambda + Q^+} L_{+,F}(-\lambda)_\mu, \quad L_{-,F}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} L_{-,F}(\lambda)_\mu.$$ 

For $\lambda \in \Lambda^+$ we have isomorphisms

$$L_{+,F}(-\lambda) \cong U^L_{F^+} \big/ \sum_{i \in I} U^L_{F^+} e_i^{((\lambda, \alpha_i^\vee) + 1)} \ (\bar{u} \leftrightarrow \bar{u}),$$ 

$$L_{-,F}(\lambda) \cong \tilde{U}^L_{F^-} \big/ \sum_{i \in I} \tilde{U}^L_{F^-} f_i^{((\lambda, \alpha_i^\vee) + 1)} \ (\bar{u} \leftrightarrow \bar{u}).$$

of vector spaces (see [13]).

Let $M$ be a $U_F$-module with weight-space decomposition $M = \bigoplus_{\mu \in \Lambda} M_\mu$ such that $\dim M_\mu < \infty$ for any $\mu \in \Lambda$. We define a $U_F$-module $M^\star$ by

$$M^\star = \bigoplus_{\mu \in \Lambda} M^\star_\mu \subset M^\star = \text{Hom}_F(M, F),$$

where the action of $U_F$ is given by

$$\langle um^\star, m \rangle = \langle m^\star, (Su)m \rangle \ (u \in U_F, m^\star \in M^\star, m \in M).$$

Here $\langle , \rangle : M^\star \times M \to F$ is the natural pairing.

We set

$$M^\star_{\pm,F}(\lambda) = (M_{\mp,F}(-\lambda))^\star \ (\lambda \in \Lambda),$$ 

$$L^\star_{\pm,F}(\mp \lambda) = (L_{\mp,F}(\pm \lambda))^\star \ (\lambda \in \Lambda^+).$$

Since $L_{\mp,F}(\pm \lambda)$ is irreducible, we have

$$L^\star_{\pm,F}(\mp \lambda) \cong L_{\pm,F}(\mp \lambda) \ (\lambda \in \Lambda^+).$$

We define isomorphisms

$$\Phi_\lambda : U^+_F \to M^\star_{+,F}(\lambda), \quad \Psi_\lambda : \tilde{U}^-_F \to M^\star_{-,F}(\lambda)$$

of vector spaces by

$$\langle \Phi_\lambda(x), \bar{v} \rangle = \tau(x, v) \ (x \in U^+_F, v \in \tilde{U}^-_F),$$ 

$$\langle \Psi_\lambda(y), \overline{Su} \rangle = \tau(u, y) \ (y \in \tilde{U}^-_F, u \in U^+_F).$$
Lemma 3.1.

(i) The $U_\mathbb{F}$-module structure of $M^*_+\mathbb{F}(\lambda)$ is given by

\begin{align}
(3.2) \quad h \cdot \Phi_\lambda(x) &= \chi_{\lambda+\gamma}(h)\Phi_\lambda(x) \quad (x \in U^+_{\mathbb{F},\gamma}, h \in U^0_{\mathbb{F}}), \\
(3.3) \quad v \cdot \Phi_\lambda(x) &= \sum_{(x)} \tau(x(0), Sv)\Phi_\lambda(x(1)) \quad (x \in U^+_{\mathbb{F}}, v \in U^-_{\mathbb{F}}), \\
(3.4) \quad u \cdot \Phi_\lambda(x) &= \Phi_\lambda(k^-\lambda(\text{ad}(u)(k_\lambda x k_\lambda))k^-\lambda) \quad (x \in U^+_{\mathbb{F}}, u \in U^+_{\mathbb{F}}).
\end{align}

(ii) The $U_\mathbb{F}$-module structure of $M^*_-\mathbb{F}(\lambda)$ is given by

\begin{align}
(3.5) \quad h \cdot \Psi_\lambda(y) &= \chi_{\lambda-\gamma}(h)\Psi_\lambda(y) \quad (y \in \tilde{U}^-_{\mathbb{F},-\gamma}, h \in U^0_{\mathbb{F}}), \\
(3.6) \quad u \cdot \Psi_\lambda(y) &= \sum_{(y)} \tau(u, y(0))\Psi_\lambda(y(1)) \quad (y \in \tilde{U}^-_{\mathbb{F}}, u \in U^+_{\mathbb{F}}), \\
(3.7) \quad v \cdot \Psi_\lambda(y) &= \Psi_\lambda(k^-\lambda(\text{ad}(v)(k_\lambda y k_\lambda))k^-\lambda) \quad (y \in \tilde{U}^-_{\mathbb{F}}, v \in U^-_{\mathbb{F}}).
\end{align}

Proof. We will prove only (i). The proof of (ii) is similar and omitted.

Note that for $x \in U^+_{\mathbb{F}}$, $a \in U_{\mathbb{F}}$, and $v \in \tilde{U}^-_{\mathbb{F}}$, we have

\[ \langle a \cdot \Phi_\lambda(x), v \rangle = \langle \Phi_\lambda(x), (Sa)v \rangle. \]

Let us show (3.2). For $v \in \tilde{U}^-_{\mathbb{F},-\delta}$, we have

\[ \langle h \cdot \Phi_\lambda(x), v \rangle = \langle \Phi_\lambda(x), (Sh)v \rangle = \delta_{\gamma,\delta}\langle \Phi_\lambda(x), (Sh)v \rangle = \delta_{\gamma,\delta}\chi_{\lambda+\gamma}(h)\langle \Phi_\lambda(x), v \rangle. \]

Hence, (3.2) holds. Let us next show (3.3). For $v \in \tilde{U}^-_{\mathbb{F}}$, we have

\[ \langle y \cdot \Phi_\lambda(x), v \rangle = \langle \Phi_\lambda(x), (Sy)v \rangle = \tau(x, (Sy)v) = \sum_{(x)} \tau(x(0), Sy)\tau(x(1), v) \]

\[ = \langle \Phi_\lambda(\sum_{(x)} \tau(x(0), Sy)x(1)), v \rangle. \]

Hence, (3.3) also holds. Let us finally show (3.4). We may assume that $u \in U^+_{\mathbb{F},\beta}$ for some $\beta \in Q^+$. Then we can write

\[ \Delta u = \sum_{j} u_j k_{\beta'_j} \otimes u'_j \quad (\beta_j, \beta'_j \in Q^+, \beta_j + \beta'_j = \beta, u_j \in U^+_{\mathbb{F},\beta}, u_j \in U^+_{\mathbb{F},\beta'_j}). \]
For \( v \in \tilde{U}_F^- \), we have
\[
\langle u \cdot \Phi_\lambda(x), v \rangle = \langle \Phi_\lambda(x), (Su)v \rangle
= \sum_{(u)_{(2)}, (v)_2} \tau(Su_{(2)}, v_{(0)}) \tau(Su_{(0)}, Su_{(2)}) \langle \Phi_\lambda(x), v_{(1)}(Su_{(1)}) \rangle
= \sum_{j, (v)_2} \tau(Su'_j, v_{(0)}) \tau(u_j k^{\beta_j'}, v_{(2)}) \langle \Phi_\lambda(x), v_{(1)}(Sk^{\beta_j'}) \rangle
= \sum_{j, (v)_2} q^{(\lambda, \beta'_j - \beta_j)} \tau(Su'_j, v_{(0)}) \tau(u_j k^{\beta'_j}, v_{(2)}) \langle \Phi_\lambda(x), v_{(1)} k_{-\beta_j} \rangle
= \sum_{j, (v)_2} q^{(\lambda, \beta'_j - \beta_j)} \tau(Su'_j, v_{(0)}) \tau(x, v_{(1)}) \tau(u_j k^{\beta'_j}, v_{(2)})
= \sum_j q^{(\lambda, \beta'_j - \beta_j)} \tau(u_j k^{\beta'_j} x(Su'_j), v)
= \langle \Phi_\lambda(k_{-\lambda}(\text{ad}(u)(k_\lambda x k_\lambda))) k_{-\lambda}, v \rangle.
\]

Here, we have used Lemma 1.1. Note also that \( \Delta \tilde{U}_F^- \subset \sum_{\gamma \in Q^+} \tilde{U}_F^- k^{\gamma} \otimes \tilde{U}_F^- k_{-\gamma} \), and hence we have \( \Delta_2 \tilde{U}_F^- \subset \sum_{\gamma, \delta \in Q^+} \tilde{U}_F^- k^{\gamma + \delta} \otimes \tilde{U}_F^- k_{\delta} \otimes \tilde{U}_F^- k_{-\delta} \). Thus, (3.4) is proved.

For \( \lambda \in \Lambda \) we denote by \( \tilde{F}_\lambda^{\geq 0} = F_\lambda^{\geq 0} \) (resp., \( \tilde{F}_\lambda^{\leq 0} = F_\lambda^{\leq 0} \)) the 1-dimensional \( U_F^{\geq 0} \)-module (resp., \( U_F^{\leq 0} \)-module) such that \( h1^{\geq 0}_\lambda = \chi_\lambda(h)1^{\geq 0}_\lambda \), \( u1^{\geq 0}_\lambda = \varepsilon(u)1^{\geq 0}_\lambda \) for \( h \in U_F^- \) and \( u \in U_F^- \) (resp., \( h1^{\leq 0}_\lambda = \chi_\lambda(h)1^{\leq 0}_\lambda \), \( u1^{\leq 0}_\lambda = \varepsilon(u)1^{\leq 0}_\lambda \) for \( h \in U_F^- \) and \( u \in U_F^- \)).

Note that for any \( \lambda \in \Lambda \), \( k_{-2\lambda} U_F^- \) (resp., \( \tilde{U}_F^- k_{-2\lambda} \)) is \( \text{ad}(U_F^{\geq 0}) \)-stable (resp., \( \text{ad}(U_F^{\leq 0}) \)-stable). We see easily from Lemma 3.1 the following.

**Lemma 3.2.** Let \( \lambda \in \Lambda \).

(i) The linear map
\[
k_{-2\lambda} U_F^- \to M_{-\gamma,F}(-\lambda) \otimes \tilde{F}_\lambda^{\leq 0} \quad (k_{-\lambda} x k_{-\lambda} \mapsto \Phi_{-\lambda}(x) \otimes 1^{\leq 0}_\lambda)
\]
is an isomorphism of \( U_F^{\geq 0} \)-modules, where \( k_{-2\lambda} U_F^- \) is regarded as a \( U_F^{\leq 0} \)-module by the adjoint action.
(ii) The linear map
\[ \tilde{U}_\mathbb{F}^- k_{-2\lambda} \to \mathbb{F}^{-\leq 0}_- \otimes M^*_{\mathbb{F}^-}(\lambda) \quad (k_{-\lambda}y k_{-\lambda} \mapsto 1^-_{-\lambda} \otimes \Phi_\lambda(y)) \]

is an isomorphism of $U_{\mathbb{F}^-}^{-\leq 0}$-modules, where $\tilde{U}_\mathbb{F}^- k_{-2\lambda}$ is regarded as a $U_{\mathbb{F}^-}^{-\leq 0}$-module by the adjoint action.

We have an injective $U_{\mathbb{F}^-}$-homomorphism
\[
(3.8) \quad L_{\pm,\mathbb{F}}(\mp \lambda) \to M^*_{\pm,\mathbb{F}}(\mp \lambda) \quad (\lambda \in \Lambda^+) 
\]
induced by the natural homomorphism $M_{\pm,\mathbb{F}}(\mp \lambda) \to L_{\pm,\mathbb{F}}(\mp \lambda)$. For $\lambda \in \Lambda^+$ we define subspaces $U_{\mathbb{F}^+}^+(\lambda)$, $\tilde{U}_{\mathbb{F}^-}^-(\lambda)$ of $U_{\mathbb{F}^+}$, $\tilde{U}_{\mathbb{F}^-}$, respectively, by
\[
U_{\mathbb{F}^+}^+(\lambda) = \Phi^{-1}_\lambda(L_{+,\mathbb{F}}^-(\lambda)), \quad \tilde{U}_{\mathbb{F}^-}^-(\lambda) = \Psi^{-1}_\lambda(L_{-,\mathbb{F}}^-(\lambda)).
\]

**Lemma 3.3.**

(i) For $\lambda, \mu \in \Lambda^+$ we have
\[ U_{\mathbb{F}^+}^+(\lambda) \subset U_{\mathbb{F}^+}^+(\lambda + \mu), \quad \tilde{U}_{\mathbb{F}^-}^-(\lambda) \subset \tilde{U}_{\mathbb{F}^-}^-(\lambda + \mu). \]

(ii) We have
\[ U_{\mathbb{F}^+}^+ = \sum_{\lambda \in \Lambda^+} U_{\mathbb{F}^+}^+(\lambda), \quad \tilde{U}_{\mathbb{F}^-}^- = \sum_{\lambda \in \Lambda^+} \tilde{U}_{\mathbb{F}^-}^-(\lambda). \]

**Proof.** We will prove only the statements for $U_{\mathbb{F}^+}^+$. By definition, we have
\[ U_{\mathbb{F}^+}^+(\lambda) = \{ x \in U_{\mathbb{F}^+} \mid \tau(x, I_\lambda) = \{0\} \}, \]
where $I_\lambda = \sum_{i \in I} \tilde{U}_{\mathbb{F}^-}^- f_i^{(\lambda, \alpha_i^\vee) + 1}$. Hence, (i) is a consequence of $I_\lambda \supset I_{\lambda + \mu}$ for $\lambda, \mu \in \Lambda^+$. To show (ii) it is sufficient to show that for any $\beta \in Q^+$ there exists some $\lambda \in \Lambda^+$ such that $U_{\mathbb{F}^+}^+, \beta \subset U_{\mathbb{F}^+}^+(\lambda)$. Set $m = \text{ht}(\beta)$. If $\lambda \in \Lambda^+$ satisfies $(\lambda, \alpha_i^\vee) \geq m$ for any $i \in I$, then we have $I_\lambda \subset \bigoplus_{\gamma \in Q^+, \text{ht}(\gamma) > m} \tilde{U}_{\mathbb{F}^-}^- \gamma$. From this we obtain $\tau(U_{\mathbb{F}^+}^+, \beta, I_\lambda) = \{0\}$, and hence $U_{\mathbb{F}^+}^+, \beta \subset U_{\mathbb{F}^+}^+(\lambda)$. \Box

**Lemma 3.4.** For $\lambda \in \Lambda^+$, we have
\[ \tilde{U}_{\mathbb{F}^-}^- (\lambda) k_{-2\lambda} \subset U_{\mathbb{F}, f}, \quad k_{-2\lambda} U_{\mathbb{F}^-}^- (\lambda) \subset U_{\mathbb{F}, f}. \]

**Proof.** By Lemma 3.2, we have an isomorphism
\[ k_{-2\lambda} U_{\mathbb{F}^-}^- (\lambda) \to L_{+,-\mathbb{F}}^+(-\lambda) \otimes \mathbb{F}^{-\geq 0}_- \quad (k_{-\lambda} x k_{-\lambda} \mapsto \Phi_{-\lambda}(x) \otimes 1^{-\geq 0}_-) \]
of $U_{\mathbb{F}}^{\geq 0}$-modules. We have $L_{+,\mathbb{F}}^*(-\lambda) \cong L_{+,\mathbb{F}}(-\lambda)$, and hence $L_{+,\mathbb{F}}^*(-\lambda) \otimes U_{\mathbb{F}}^{\geq 0}$ is generated by $\Phi_{-\lambda}(1) \otimes 1_{\mathbb{F}}^{\geq 0}$ as a $U_{\mathbb{F}}^{\geq 0}$-module. It follows that

$$k_{-2\lambda}U_{\mathbb{F}}^+((\lambda) = \text{ad}(U_{\mathbb{F}}^{\geq 0})(k_{-2\lambda}) \subset U_{\mathbb{F},f}$$

by (2.9). The proof of $\tilde{U}_-((\lambda)k_{-2\lambda} \subset U_{\mathbb{F},f}$ is similar.

**3.3.**

It is well known that, for $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$, there exists $h \in U_{\mathbb{A}}^{L,0}$ such that $\chi_{\lambda}(h) = 1$ and $\chi_{\mu}(h) = 0$. In particular, we have $\chi_{\lambda} \neq \chi_{\mu}$ (see, e.g., [20, Lemma 2.3]).

For $M \in \text{Mod}(U_{\mathbb{A}}^{L})$ and $\lambda \in \Lambda$, we set

$$M_{\lambda} = \{ m \in M \mid hm = \chi_{\lambda}(h)m \ (h \in U_{\mathbb{A}}^{L,0}) \}. $$

For $\lambda \in \Lambda$, we define $M_{+\mathbb{A}}((\lambda), M_{-\mathbb{A}}((\lambda)) \in \text{Mod}(U_{\mathbb{A}}^{L})$ by

$$M_{+\mathbb{A}}((\lambda) = U_{\mathbb{A}}^{L} / \sum_{y \in U_{\mathbb{A}}^{L,-}} U_{\mathbb{A}}^{L}(y - \varepsilon(y)) + \sum_{h \in U_{\mathbb{A}}^{L,0}} U_{\mathbb{A}}^{L}(h - \chi_{\lambda}(h)),

$$

$$M_{-\mathbb{A}}((\lambda) = U_{\mathbb{A}}^{L} / \sum_{x \in U_{\mathbb{A}}^{L,+}} U_{\mathbb{A}}^{L}(x - \varepsilon(x)) + \sum_{h \in U_{\mathbb{A}}^{L,0}} U_{\mathbb{A}}^{L}(h - \chi_{\lambda}(h)).$$

By the triangular decomposition we have isomorphisms

$$M_{+\mathbb{A}}((\lambda) \cong U_{\mathbb{A}}^{L,0} \ (\overline{u} \leftrightarrow u), \quad M_{-\mathbb{A}}((\lambda) \cong U_{\mathbb{A}}^{L,0} \ (\overline{u} \leftrightarrow u)$$

of $\mathbb{A}$-modules. In particular, $M_{\pm\mathbb{A}}((\lambda)$ is a free $\mathbb{A}$-module, and we have $\mathbb{F} \otimes \mathbb{A} M_{\pm\mathbb{A}}((\lambda) \cong M_{\pm\mathbb{F}}((\lambda))$. Moreover, we have weight-space decompositions

$$M_{+\mathbb{A}}((\lambda) = \bigoplus_{\mu \in \Lambda \cap Q^{+}} M_{+\mathbb{A}}((\lambda)_{\mu}, \quad M_{-\mathbb{A}}((\lambda) = \bigoplus_{\mu \in \Lambda \cap Q^{+}} M_{-\mathbb{A}}((\lambda)_{\mu}. $$

For $\lambda \in \Lambda^{+}$, we define $L_{+\mathbb{A}}((\lambda) \in \text{Mod}(U_{\mathbb{A}}^{L})$ (resp., $L_{-\mathbb{A}}((\lambda) \in \text{Mod}(U_{\mathbb{A}}^{L})$) to be the $U_{\mathbb{A}}^{L}$-submodule of $L_{+,\mathbb{F}}((\lambda)$ (resp., $L_{-,\mathbb{F}}((\lambda)$) generated by $\overline{T} \in L_{+,\mathbb{F}}((\lambda)$ (resp., $\overline{T} \in L_{-,\mathbb{F}}((\lambda)$). By definition, $L_{+,\mathbb{A}}((\lambda)$ is a free $\mathbb{A}$-module, and we have $\mathbb{F} \otimes \mathbb{A} L_{+,\mathbb{A}}((\lambda) \cong L_{+,\mathbb{F}}((\lambda)$. Moreover, we have weight-space decompositions

$$L_{+\mathbb{A}}((\lambda) = \bigoplus_{\mu \in -\Lambda \cap Q^{+}} L_{+\mathbb{A}}((\lambda)_{\mu}, \quad L_{-\mathbb{A}}((\lambda) = \bigoplus_{\mu \in -\Lambda \cap Q^{+}} L_{-\mathbb{A}}((\lambda)_{\mu}. $$
The canonical surjective $U_F$-homomorphism $M_{\pm,F}(\mp\lambda) \to L_{\pm,F}(\mp\lambda)$ induces a surjective $U_L^\Lambda$-homomorphism

\[(3.9)\quad M_{\pm,\Lambda}(\mp\lambda) \to L_{\pm,\Lambda}(\mp\lambda) \quad (\lambda \in \Lambda^+).\]

Note that (3.9) is a split epimorphism of $\Lambda$-modules since $\Lambda$ is a PID (Principal Ideal Domain), and note that $M_{\pm,\Lambda}(\mp\lambda)_\mu$, $L_{\pm,\Lambda}(\mp\lambda)_\mu$ are torsion-free finitely generated $\Lambda$-modules for each $\mu \in \Lambda$.

Let $M$ be a $U_L^\Lambda$-module with weight-space decomposition $M = \bigoplus_{\mu \in \Lambda} M_\mu$ such that $M_\mu$ is a free $\Lambda$-module of finite rank for any $\mu \in \Lambda$. We define a $U_L^\Lambda$-module $M^*$ by

\[M^* = \bigoplus_{\mu \in \Lambda} \text{Hom}_\Lambda(M_\mu, \Lambda) \subset \text{Hom}_\Lambda(M, \Lambda),\]

where the action of $U_L^\Lambda$ is given by

\[\langle um^*, m \rangle = \langle m^*, (Su)m \rangle \quad (u \in U_L^\Lambda, m^* \in M^*, m \in M).\]

Here $\langle , \rangle : M^* \times M \to \Lambda$ is the natural pairing.

We set

\[M^*_{\pm,\Lambda}(\lambda) = (M_{\mp,\Lambda}(-\lambda))^\star \quad (\lambda \in \Lambda),\]

\[L^*_{\pm,\Lambda}(\mp\lambda) = (L_{\mp,\Lambda}(\pm\lambda))^\star \quad (\lambda \in \Lambda^+).\]

Then $M^*_{\pm,\Lambda}(\lambda)$ for $\lambda \in \Lambda$ and $L^*_{\pm,\Lambda}(\mp\lambda)$ for $\lambda \in \Lambda^+$ are free $\Lambda$-modules satisfying

\[F \otimes_\Lambda M^*_{\pm,\Lambda}(\lambda) \cong M^*_{\pm,F}(\lambda), \quad F \otimes_\Lambda L^*_{\pm,\Lambda}(\mp\lambda) \cong L^*_{\pm,F}(\mp\lambda).\]

Moreover, we can identify $M^*_{\pm,\Lambda}(\lambda)$ and $L^*_{\pm,\Lambda}(\mp\lambda)$ with $\Lambda$-submodules of $M^*_{\pm,F}(\lambda)$ and $L^*_{\pm,F}(\mp\lambda)$, respectively. Under this identification we have

\[(3.10)\quad L^*_{\pm,\Lambda}(\mp\lambda) = L^*_{\pm,F}(\mp\lambda) \cap M^*_{\pm,\Lambda}(\mp\lambda) \quad (\lambda \in \Lambda^+).\]

In particular, the $U_L^\Lambda$-homomorphism

\[(3.11)\quad L^*_{\pm,\Lambda}(\mp\lambda) \to M^*_{\pm,\Lambda}(\mp\lambda) \quad (\lambda \in \Lambda^+)\]

is a split monomorphism of $\Lambda$-modules.

By abuse of notation we write

\[(3.12) \quad \Phi_\lambda : U^+_{\Lambda} \to M^*_{\pm,\Lambda}(\lambda), \quad \Psi_\lambda : U^-_{\Lambda} \to M^*_{\pm,\Lambda}(\lambda)\]

for the isomorphisms of $\Lambda$-modules induced by (3.1). By Lemma 3.1 we have the following.
Lemma 3.5.
(i) The $U^L_A$-module structure of $M^*_{+,\mathbb{A}}(\lambda)$ is given by

(3.13) $h \cdot \Phi_\lambda(x) = \chi_{\lambda+\gamma}(h)\Phi_\lambda(x)$ \quad ($x \in U^+_{\mathbb{A},\gamma}, h \in U^L_A$),

(3.14) $v \cdot \Phi_\lambda(x) = \sum_{(x)} \tau^L_{\mathbb{A}}(x(0), Sv)\Phi_\lambda(x(1))$ \quad ($x \in U^+_A, v \in U^L_A$),

(3.15) $u \cdot \Phi_\lambda(x) = \Phi_\lambda(k_{\lambda} (\text{ad}(u)(k_{\lambda}xk_{\lambda})))k_{\lambda}^{-1}$ \quad ($x \in U^+_A, u \in U^L_A$).

(ii) The $U^L_A$-module structure of $M^*_{-,\mathbb{A}}(\lambda)$ is given by

(3.16) $h \cdot \Psi_\lambda(y) = \chi_{\lambda-\gamma}(h)\Psi_\lambda(y)$ \quad ($y \in \tilde{U}^-_{\mathbb{A},\gamma}, h \in U^L_A$),

(3.17) $u \cdot \Psi_\lambda(y) = \sum_{(y)} \tau^L_{\mathbb{A}}(u, y(0))\Psi_\lambda(y(1))$ \quad ($y \in \tilde{U}^+_A, u \in U^L_A$),

(3.18) $v \cdot \Psi_\lambda(y) = \Psi_\lambda(k_{\lambda} (\text{ad}(v)(k_{\lambda}yk_{\lambda})))k_\lambda^{-1}$ \quad ($y \in \tilde{U}^+_A, v \in U^L_A$).

For $\lambda \in \Lambda^+$ we define $\mathbb{A}$-submodules $U^+_A(\lambda), \tilde{U}^-_A(\lambda)$ of $U^+_A, \tilde{U}^-_A$, respectively, by

$$U^+_A(\lambda) = \Phi^{-1}_{-\lambda}(L^*_{+,\mathbb{A}}(-\lambda)), \quad \tilde{U}^-_A(\lambda) = \Psi^{-1}_{\lambda}(L^*_{-,\mathbb{A}}(\lambda)).$$

The embeddings

(3.19) $U^+_A(\lambda) \hookrightarrow U^+_A, \quad \tilde{U}^-_A(\lambda) \hookrightarrow \tilde{U}^-_A \quad (\lambda \in \Lambda^+)$

are split monomorphisms of $\mathbb{A}$-modules. By (3.10), we have

(3.20) $U^+_A(\lambda) = U^+_F(\lambda) \cap U^+_A, \quad \tilde{U}^-_A(\lambda) = \tilde{U}^-_F(\lambda) \cap \tilde{U}^-_A \quad (\lambda \in \Lambda^+)$.

In particular, we have

(3.21) $U^+_A(\lambda) \subset U^+_A(\lambda + \mu), \quad \tilde{U}^-_A(\lambda) \subset \tilde{U}^-_A(\lambda + \mu) \quad (\lambda, \mu \in \Lambda^+),

(3.22) U^+_A = \sum_{\lambda \in \Lambda^+} U^+_A(\lambda), \quad \tilde{U}^-_A = \sum_{\lambda \in \Lambda^+} \tilde{U}^-_A(\lambda),

(3.23) \tilde{U}^-_A(\lambda)k_{-2\lambda} \subset U^+_A, \quad k_{-2\lambda}U^+_A(\lambda) \subset U^+_A \quad (\lambda \in \Lambda^+)$

by Lemmas 3.3 and 3.4.
3.4. Let $\lambda \in \Lambda$. By abuse of notation we also denote by $\chi_\lambda : U^L_{\zeta,0} \to \mathbb{C}$ the $\mathbb{C}$-algebra homomorphism induced by $\chi_\lambda : U^L_{\bar{\kappa}} \to \mathbb{A}$. Then $\{\chi_\lambda\}_{\lambda \in \Lambda}$ is a linearly independent subset of the $\mathbb{C}$-module $\text{Hom}_{\mathbb{C}}(U^L_{\zeta,0}, \mathbb{C})$. For $M \in \text{Mod}(U^L_{\zeta})$ and $\lambda \in \Lambda$, we set 

$$M_\lambda = \{ m \in M \mid hm = \chi_\lambda(h)m \ (h \in U^L_{\zeta,0}) \}. \label{eq1}$$

For $\lambda \in \Lambda$ we set

$$M_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_\mathbb{A} M_{\pm,\zeta}(\lambda), \quad M^*_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_\mathbb{A} M^*_{\pm,\zeta}(\lambda). \label{eq2}$$

For $\lambda \in \Lambda^+$ we set

$$L_{\pm,\zeta}(\mp \lambda) = \mathbb{C} \otimes_\mathbb{A} L_{\pm,\zeta}(\mp \lambda), \quad L^*_{\pm,\zeta}(\mp \lambda) = \mathbb{C} \otimes_\mathbb{A} L^*_{\pm,\zeta}(\mp \lambda). \label{eq3}$$

We have canonical $U^L_{\zeta}$-homomorphisms

$$M_{\pm,\zeta}(\mp \lambda) \to L_{\pm,\zeta}(\mp \lambda) \quad (\lambda \in \Lambda^+), \label{eq4}$$

$$L^*_{\pm,\zeta}(\mp \lambda) \to M^*_{\pm,\zeta}(\mp \lambda) \quad (\lambda \in \Lambda^+). \label{eq5}$$

Note that (3.24) is surjective and that (3.25) is injective.

For any $\lambda \in \Lambda^+$ we have an isomorphism

$$A_{\zeta}(\lambda) \cong L^*_{-,-\zeta}(\lambda) \label{eq6}$$

of $U^L_{\zeta}$-modules (see, e.g., [11, Chapter 9], [20, Section 3.1]).

Let $\lambda \in \Lambda$. By abuse of notation we also denote by

$$\Phi_\lambda : U^+_{\zeta} \to M^*_{+,\zeta}(\lambda), \quad \Psi_\lambda : \tilde{U}^-_{\zeta} \to M^*_{-,\zeta}(\lambda)$$

the isomorphisms of $\mathbb{C}$-modules given by

$$\langle \Phi_\lambda(x), \overline{v} \rangle = \tau^L_{\zeta}(x, v) \quad (x \in U^+_{\zeta}, v \in \tilde{U}^-_{\zeta}), \quad \langle \Psi_\lambda(y), \overline{Su} \rangle = L \tau^L_{\zeta}(u, y) \quad (y \in \tilde{U}^-_{\zeta}, u \in U^+_{\zeta}).$$

By Lemma 3.5, we have the following.
LEMMA 3.6.

(i) The $U^L_\zeta$-module structure of $M^*_+\zeta(\lambda)$ is given by

\begin{align}
(3.27) & \quad h \cdot \Phi_\lambda(x) = \chi_{\lambda+\gamma}(h)\Phi_\lambda(x) \quad (x \in U^+_{\zeta,\gamma}, h \in U^{L,0}_\zeta), \\
(3.28) & \quad v \cdot \Phi_\lambda(x) = \sum_{(x)} \tau^L_\zeta(x(0), Sv)\Phi_\lambda(x(1)) \quad (x \in U^+_\zeta, v \in U^{L,-}_\zeta), \\
(3.29) & \quad u \cdot \Phi_\lambda(x) = \Phi_\lambda(k_{-\lambda}(\text{ad}(u)(k_\lambda x k_\lambda))k_{-\lambda}) \quad (x \in U^+_\zeta, u \in U^{L,+}_\zeta). 
\end{align}

(ii) The $U^L_\zeta$-module structure of $M^*_-\zeta(\lambda)$ is given by

\begin{align}
(3.30) & \quad h \cdot \Psi_\lambda(y) = \chi_{\lambda-\gamma}(h)\Psi_\lambda(y) \quad (y \in \tilde{U}^-_{\zeta,-\gamma}, h \in U^{L,0}_\zeta), \\
(3.31) & \quad u \cdot \Psi_\lambda(y) = \sum_{(y)} L^\tau_\zeta(u, y(0))\Psi_\lambda(y(1)) \quad (y \in \tilde{U}^-_{\zeta}, u \in U^{L,+}_\zeta), \\
(3.32) & \quad v \cdot \Psi_\lambda(y) = \Psi_\lambda(k_\lambda(\text{ad}(v)(k_{-\lambda} y k_{-\lambda}))k_\lambda) \quad (y \in \tilde{U}^-_{\zeta}, v \in U^{L,-}_\zeta). 
\end{align}

For $\lambda \in \Lambda^+$, we set

\begin{align*}
U^+_\zeta(\lambda) &= \mathbb{C} \otimes \mathbb{A} U^+_\mathbb{A}(\lambda), \\
\tilde{U}^-_\zeta(\lambda) &= \mathbb{C} \otimes \mathbb{A} \tilde{U}^-_\mathbb{A}(\lambda).
\end{align*}

Then $U^+_\zeta(\lambda)$ and $\tilde{U}^-_\zeta(\lambda)$ are the $\mathbb{C}$-submodules of $U^+_\zeta$ and $\tilde{U}^-_\zeta$, respectively, satisfying $\Phi_{-\lambda}(U^+_\zeta(\lambda)) = L^*_+\zeta(-\lambda)$ and $\Psi_\lambda(\tilde{U}^-_\zeta(\lambda)) = L^*_+\zeta(\lambda)$. We have linear isomorphisms

\begin{align}
(3.33) & \quad \Phi_{-\lambda} : U^+_\zeta(\lambda) \rightarrow L^*_+\zeta(-\lambda), \quad \Psi_\lambda : \tilde{U}^-_\zeta(\lambda) \rightarrow L^*_+\zeta(\lambda) \quad (\lambda \in \Lambda^+).
\end{align}

By (3.21), (3.22), and (3.23), we have

\begin{align}
(3.34) & \quad U^+_\zeta(\lambda) \subset U^+_\zeta(\lambda + \mu), \quad \tilde{U}^-_\zeta(\lambda) \subset \tilde{U}^-_\zeta(\lambda + \mu) \quad (\lambda, \mu \in \Lambda^+), \\
(3.35) & \quad U^+_\zeta = \sum_{\lambda \in \Lambda^+} U^+_\zeta(\lambda), \quad \tilde{U}^-_\zeta = \sum_{\lambda \in \Lambda^+} \tilde{U}^-_\zeta(\lambda), \\
(3.36) & \quad \tilde{U}^-_\zeta(\lambda)k_{-2\lambda} \subset U_{\zeta,f}, \quad k_{-2\lambda}U^+_\zeta(\lambda) \subset U_{\mathbb{A},f} \quad (\lambda \in \Lambda^+).
\end{align}

By (3.35) and (3.36), we can easily see the following.
Lemma 3.7. For any \( u \in U_\zeta \) there exists some \( \lambda \in \Lambda^+ \) such that \( uk_{-2\lambda} \in U_{\zeta,f} \).

§4. Induction functor

We set

\[
C_\zeta^{\leq 0} = C_\zeta/I, \quad I = \{ \varphi \in C_\zeta \mid \langle \varphi, U_\zeta^{L,\leq 0} \rangle = \{0\} \}.
\]

Then \( C_\zeta^{\leq 0} \) is a Hopf algebra, and we have a Hopf pairing

\[
\langle , \rangle : C_\zeta^{\leq 0} \times U_\zeta^{L,\leq 0} \to \mathbb{C}.
\]

We have a canonical Hopf algebra homomorphism

\[
\text{res} : C_\zeta \to C_\zeta^{\leq 0}.
\]

Following Backelin and Kremnizer [2, Section 3], we define abelian categories \( \mathcal{M}_\zeta \) and \( \mathcal{M}_\zeta^{\text{eq}} \) as follows.

An object of \( \mathcal{M}_\zeta \) is a triplet \((M, \alpha, \beta)\) with

1. \( M \) a vector space over \( \mathbb{C} \),
2. \( \alpha : C_\zeta \otimes M \to M \) a left \( C_\zeta \)-module structure of \( M \),
3. \( \beta : M \to C_\zeta^{\leq 0} \otimes M \) a left \( C_\zeta^{\leq 0} \)-comodule structure of \( M \)

such that \( \beta \) is a morphism of \( C_\zeta \)-modules. (Or, equivalently, \( \alpha \) is a morphism of \( C_\zeta^{\leq 0} \)-comodules.) A morphism from \((M, \alpha, \beta)\) to \((M', \alpha', \beta')\) is a linear map \( \varphi : M \to M' \) which is a morphism of \( C_\zeta \)-modules as well as that of \( C_\zeta^{\leq 0} \)-comodules.

An object of \( \mathcal{M}_\zeta^{\text{eq}} \) is a quadruple \((M, \alpha, \beta, \gamma)\) with

1. \( M \) a vector space over \( \mathbb{C} \),
2. \( \alpha : C_\zeta \otimes M \to M \) a left \( C_\zeta \)-module structure of \( M \),
3. \( \beta : M \to C_\zeta^{\leq 0} \otimes M \) a left \( C_\zeta^{\leq 0} \)-comodule structure of \( M \),
4. \( \gamma : M \to M \otimes C_\zeta \) a right \( C_\zeta \)-comodule structure of \( M \)

subject to the conditions that \((M, \alpha, \beta, \gamma) \in \mathcal{M}_\zeta \), that \( \beta \) and \( \gamma \) commute with each other, and that \( \gamma \) is a homomorphism of left \( C_\zeta \)-modules. A morphism from \((M, \alpha, \beta, \gamma)\) to \((M', \alpha', \beta', \gamma')\) is a linear map \( \varphi : M \to M' \) which is compatible with the left \( C_\zeta \)-module structure, the left \( C_\zeta^{\leq 0} \)-comodule structure, and the right \( C_\zeta \)-comodule structure.
For a coalgebra $C$ we denote by $\text{Comod}(C)$ (resp., $\text{Comod}^r(C)$) the category of left $C$-comodules (resp., right $C$-comodules). We define functors

$$\Xi: \mathcal{M}_\zeta^\text{eq} \to \text{Comod}(C_\zeta^{\leq 0}),$$

$$\Upsilon: \text{Comod}(C_\zeta^{\leq 0}) \to \mathcal{M}_\zeta^\text{eq}$$

by

$$\Xi(M) = \{ M \in \mathcal{M} \mid \gamma(m) = m \otimes 1 \},$$

$$\Upsilon(L) = C_\zeta \otimes L.$$

By Backelin and Kremnizer [2, Section 3.5], we have the following.

**Proposition 4.1.** The functor $\Xi: \mathcal{M}_\zeta^\text{eq} \to \text{Comod}(C_\zeta^{\leq 0})$ gives an equivalence of categories, and its quasi-inverse is given by $\Upsilon$.

**Remark 4.2.** For $M \in \mathcal{M}_\zeta^\text{eq}$ we have an isomorphism

$$\Xi(M) \cong \mathbb{C} \otimes_{C_\zeta} M$$

of vector spaces by Proposition 4.1. Here $C_\zeta \to \mathbb{C}$ is given by $\varepsilon$.

For $\lambda \in \Lambda$ we define $\chi^{\leq 0}_\lambda \in C_\zeta^{\leq 0} \subset \text{Hom}_\mathbb{C}(U^{L_\zeta,0}, \mathbb{C})$ by

$$\chi^{\leq 0}_\lambda(hu) = \chi_\lambda(h)\varepsilon(u) \quad (h \in U^{L_\zeta,0}, u \in U^{L_\zeta,-}).$$

We define left exact functors

(4.1) $\omega_{M^*}: \mathcal{M}_\zeta \to \text{Mod}_\Lambda(A_\zeta),$

(4.2) $\Gamma_M: \mathcal{M}_\zeta \to \text{Mod}(\mathbb{C})$

by

$$\omega_{M^*}(M) = \bigoplus_{\lambda \in \Lambda} (\omega_{M^*}(M))(\lambda) \subset M,$$

$$\big(\omega_{M^*}(M)\big)(\lambda) = \{ m \in M \mid \beta(m) = \chi^{\leq 0}_\lambda \otimes m \},$$

$$\Gamma_M(M) = \big(\omega_{M^*}(M)\big)(0).$$

We denote by $\text{Mod}^\text{eq}_\Lambda(A_\zeta)$ the category consisting of $N \in \text{Mod}_\Lambda(A_\zeta)$ equipped with a right $C_\zeta$-comodule structure $\gamma: N \to N \otimes C_\zeta$ such that
\[ \gamma(N(\lambda)) \subset N(\lambda) \otimes C_\zeta \] for any \( \lambda \in \Lambda \) and \( \gamma(\varphi n) = \Delta(\varphi)\gamma(n) \) for any \( \varphi \in A_\zeta \) and \( n \in N \). (Note that \( \Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta \).) By definition, (4.1) and (4.2) induce left exact functors

\[ \omega^{eq}_{\mathcal{M}*}: \mathcal{M}^{eq}_{\zeta} \to \text{Mod}^{eq}_{\Lambda}(A_\zeta), \]

\[ \Gamma^{eq}_{\mathcal{M}}: \mathcal{M}^{eq}_{\zeta} \to \text{Comod}^r(C_\zeta). \]

We also define a left exact functor

\[ \text{Ind}: \text{Comod}(C^{\leq 0}_\zeta) \to \text{Comod}^r(C_\zeta) \]

by \( \text{Ind} = \Gamma^{eq}_{\mathcal{M}} \circ \Upsilon \).

The abelian categories \( \mathcal{M}_{\zeta}, \mathcal{M}^{eq}_{\zeta}, \text{Comod}^r(C_\zeta) \) have enough injectives, and the forgetful functor \( \mathcal{M}^{eq}_{\zeta} \to \mathcal{M}_{\zeta} \) sends injective objects to \( \Gamma_{\mathcal{M}} \)-acyclic objects (see [2, Section 3.4]). Hence, we have the following.

**Lemma 4.3.** We have

\[ \text{For} \circ R^i\Gamma^{eq}_{\mathcal{M}} = R^i\Gamma_{\mathcal{M}} \circ \text{For} : \mathcal{M}^{eq}_{\zeta} \to \text{Mod}(C), \]

\[ R^i\text{Ind} \circ \Xi = R^i\Gamma^{eq}_{\mathcal{M}} : \mathcal{M}^{eq}_{\zeta} \to \text{Comod}^r(C_\zeta) \]

for any \( i \), where \( \text{For} : \text{Comod}^r(C_\zeta) \to \text{Mod}(C) \) and \( \text{For} : \mathcal{M}^{eq}_{\zeta} \to \mathcal{M}_{\zeta} \) are forgetful functors.

We define an exact functor

\[ \text{res}: \text{Comod}^r(C_\zeta) \to \text{Comod}(C^{\leq 0}_\zeta) \]

as follows. For \( V \in \text{Comod}^r(C_\zeta) \) with right \( C_\zeta \)-comodule structure \( \beta: V \to V \otimes C_\zeta \), we have \( \text{res}(V) = V \) as a \( C \)-module, and the left \( C^{\leq 0}_\zeta \)-comodule structure \( \text{res}(V) \to C^{\leq 0}_\zeta \otimes \text{res}(V) \) of \( \text{res}(V) \) is given by

\[ \beta(v) = \sum_k v_k \otimes \varphi_k \implies \gamma(v) = \sum_k \text{res}(S^{-1}\varphi_k) \otimes v_k. \]

The following fact is standard.

**Lemma 4.4.** For \( V \in \text{Comod}^r(C_\zeta), M \in \text{Comod}(C^{\leq 0}_\zeta) \), we have an isomorphism

\[ F: \text{Ind}(M) \otimes V \to \text{Ind}(\text{res}(V) \otimes M) \]
of right $C_\zeta$-comodules given by
\[ F\left(\left(\sum_i \varphi_i \otimes m_i\right) \otimes v\right) = \sum_{i,(v)} \varphi_i v_{(1)} \otimes v_{(0)} \otimes m_i, \]
where we write the right $C_\zeta$-comodule structure of $V$ by
\[ V \ni v \mapsto \sum_{(v)} v_{(0)} \otimes v_{(1)} \in V \otimes C_\zeta. \]

For $\lambda \in \Lambda$ we denote by $C_{\lambda}^{\leq 0} = C_1^{\leq 0}$ the object of $\text{Comod}(C_\zeta^{\leq 0})$ corresponding to the 1-dimensional right $U_L^{1,\leq 0}$-module given by $1_{\lambda}^{\leq 0} u = \chi_{\lambda}^{\leq 0}(u) 1_{\lambda}^{\leq 0}$ for $u \in U_L^{1,\leq 0}$. By definition, we have an isomorphism
\[ \text{Ind}(C_{\lambda}^{\leq 0}) \cong A_\zeta(\lambda) \quad (\lambda \in \Lambda^+) \]
of right $C_\zeta$-comodules.

Let $N \in \text{Mod}_\Lambda(A_\zeta)$. Then $C_\zeta \otimes A_\zeta N$ turns out to be an object of $\mathcal{M}_\zeta$ by
\[
\alpha(f \otimes (f' \otimes n)) = ff' \otimes n \quad (f, f' \in C_\zeta, n \in N), \\
\beta(f \otimes n) = \sum_{(f)} \text{res}(f_{(0)}) \chi_\lambda \otimes (f_{(1)} \otimes n) \quad (f \in C_\zeta, n \in N(\lambda)).
\]

Hence, we have a functor $\text{Mod}_\Lambda(A_\zeta) \to \mathcal{M}_\zeta$ sending $N$ to $C_\zeta \otimes A_\zeta N$.

**Lemma 4.5.** The functor $\text{Mod}_\Lambda(A_\zeta) \to \mathcal{M}_\zeta$ as above induces a functor
\[ \Phi : \text{Mod}(\mathcal{O}_{B_\zeta}) \to \mathcal{M}_\zeta. \]

**Proof.** It is sufficient to show that $C_\zeta \otimes A_\zeta A_\zeta/\Lambda^+ = \{0\}$ for any $\lambda \in \Lambda$. Hence, we have only to show that $C_\zeta A_\zeta(\lambda) = C_\zeta$ for any $\lambda \in \Lambda^+$. Take $\varphi \in A_\zeta(\lambda)$ such that $\varepsilon(\varphi) = 1$. We have $\Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta$, and hence we can write $\Delta(\varphi) = \sum_i \varphi_i \otimes \varphi_i'$ with $\varphi_i \in A_\zeta(\lambda), \varphi_i' \in C_\zeta$. Then we have $C_\zeta A_\zeta(\lambda) \ni \sum_i (S^{-1} \varphi_i') \varphi_i = 1$. 

We set
\[ \Psi = \omega^* \circ \omega_{M^*} : \mathcal{M}_\zeta \to \text{Mod}(\mathcal{O}_{B_\zeta}). \]

Backelin and Kremnizer [2, Section 3.3] obtained the following result using a result of Artin and Zhang [1, Theorem 4.5].
**Proposition 4.6.** The functor $\Phi: \text{Mod}(\mathcal{O}_B) \to \mathcal{M}_\zeta$ gives an equivalence of categories, and its quasi-inverse is given by $\Psi$. Moreover, we have an identification

$$\omega_{\mathcal{M}*} \circ \Phi = \omega_*: \text{Mod}(\mathcal{O}_B) \to \text{Mod}_A(A_\zeta)$$

of functors.

Hence we have the following.

**Lemma 4.7.** We have

$$R^i\Gamma = R^i\Gamma_M \circ \Phi: \text{Mod}(\mathcal{O}_B) \to \text{Mod}(\mathcal{C})$$

for any $i$.

We set

$$\text{Mod}^{eq}(\mathcal{O}_B) = \text{Mod}^{eq}_\Lambda(A_\zeta)/\text{Mod}^{eq}_\Lambda(A_\zeta) \cap \text{Tor}_A(A_\zeta).$$

Let $N \in \text{Mod}^{eq}_\Lambda(A_\zeta)$. We denote the right $C_\zeta$-comodule structure of $N$ by $\gamma': N \to N \otimes C_\zeta$. Then we have a right $C_\zeta$-comodule structure $\gamma: C_\zeta \otimes A_\zeta \to (C_\zeta \otimes A_\zeta) \otimes C_\zeta$ of $C_\zeta \otimes A_\zeta N$ given by

$$\gamma'(n) = \sum_k n_k \otimes \varphi_k \implies \gamma(f \otimes n) = \sum_{k,(f)} (f(0) \otimes n_k) \otimes f(1) \varphi_k.$$

This gives a functor $\text{Mod}^{eq}_\Lambda(A_\zeta) \to \mathcal{M}^{eq}_\zeta$. Hence, by Lemma 4.5 we have a functor

$$\Phi^{eq}: \text{Mod}^{eq}(\mathcal{O}_B) \to \mathcal{M}^{eq}_\zeta$$

induced by $\Phi$. Let $M \in \mathcal{M}^{eq}_\zeta$. The right $C_\zeta$-comodule structure of $M$ restricts to that of $\omega_{\mathcal{M}*}M$ so that $\omega_{\mathcal{M}*}M \in \text{Mod}^{eq}_\Lambda(A_\zeta)$. Hence, we have a functor

$$\Psi^{eq}: \mathcal{M}^{eq}_\zeta \to \text{Mod}^{eq}(\mathcal{O}_B)$$

induced by $\Psi$. By Proposition 4.6, we have the following.

**Proposition 4.8.** The functor $\Phi^{eq}: \text{Mod}^{eq}(\mathcal{O}_B) \to \mathcal{M}^{eq}_\zeta$ gives an equivalence of categories, and its quasi-inverse is given by $\Psi^{eq}$.

By Proposition 4.8 we see that (4.1) and (4.2) induce

$$\omega^{eq} = \omega^{eq}_{\mathcal{M}*} \circ \Phi^{eq}: \text{Mod}^{eq}(\mathcal{O}_B) \to \text{Mod}^{eq}_\Lambda(A_\zeta),$$

$$\Gamma^{eq} = \Gamma^{eq}_\mathcal{M} \circ \Phi^{eq}: \text{Mod}^{eq}(\mathcal{O}_B) \to \text{Comod}^{r}(C_\zeta).$$

By Lemma 4.3, we have the following.
**Lemma 4.9.** We have

\[ \text{For} \circ R^i\Gamma^\text{eq} = R^i\Gamma \circ \text{For} : \text{Mod}^\text{eq}(\mathcal{O}_{B_\xi}) \to \text{Mod}(\mathcal{C}) \]

for any \( i \), where \( \text{For} : \text{Comod}'(C_\xi) \to \text{Mod}(\mathcal{C}) \) and \( \text{For} : \text{Mod}^\text{eq}(\mathcal{O}_{B_\xi}) \to \text{Mod}(\mathcal{O}_{B_\xi}) \) are forgetful functors.

§5. Reformulation of Conjecture 2.14

5.1. Adjoint action of \( U^L_\xi \) on \( D'_\xi \)

Define a left \( U^F_\xi \)-module structure of \( E^F_\xi \) by

\[ \text{ad}(u)(P) = \sum_{(u)} u(0)P(Su(1)) \quad (u \in U_\xi, P \in E_\xi). \]

Then we have

\[ \text{ad}(u)(P_1P_2) = \sum_{(u)} \text{ad}(u(0))(P_1)\text{ad}(u(1))(P_2) \quad (P_1, P_2 \in E_\xi), \]

\[ \text{ad}(u)(\varphi) = u \cdot \varphi \quad (\varphi \in A_\xi \subset E_\xi), \]

\[ \text{ad}(u)(v) = \sum_{(u)} u(0)v(Su(1)) \quad (v \in U_\xi \subset E_\xi), \]

\[ \text{ad}(u)(e(\lambda)) = \varepsilon(u)e(\lambda) \quad (\lambda \in \Lambda, e(\lambda) \in \mathbb{F}[\Lambda] \subset E_\xi) \]

for \( u \in U_\xi \). We see from [20, Lemma 4.2] that this induces a left \( U_\xi \)-module structure of \( D'_\xi \). Moreover, the \( U_\xi \)-module structures of \( E_\xi \) and \( D'_\xi \) induce \( U^L_\Lambda \)-module structures of \( E_\Lambda, D'_\Lambda, E_{\Lambda,\diamond}, D'_{\Lambda,\diamond}, E_{\Lambda,\diamond,\xi}, D'_{\Lambda,\diamond,\xi} \), and \( D'_{\Lambda,\diamond,\xi,\varphi} \) by Lemmas 1.2 and 2.12. Hence, by specialization we obtain \( U^L_{\xi,\diamond} \)-module structures of \( E_{\xi,\diamond}, D'_{\xi,\diamond}, E_{\xi,\diamond,\xi}, D'_{\xi,\diamond,\xi,\varphi} \), and \( D'_{\xi,\diamond,\xi,\varphi} \) also denoted by \( \text{ad} \).

5.2.

We will regard \( E_{\xi,\diamond,\varphi}, D'_{\xi,\diamond,\varphi} \in \text{Mod}_\Lambda(A_{\xi}) \) as objects of \( \text{Mod}^\text{eq}_\Lambda(A_{\xi}) \) by the right \( C_{\xi,\diamond} \)-comodule structures induced from the left \( U^L_\xi \)-module structures

\[ (u, P) \mapsto \text{ad}(u)(P) \quad (u \in U^L_{\xi,\diamond}, P \in E_{\xi,\diamond,\varphi} \text{ or } D'_{\xi,\diamond,\varphi}). \]

Then for

\[ (\Xi \circ \Phi^\text{eq})(\omega^*D'_{\xi,\diamond,\varphi}) \in \text{Comod}(C_{\xi}^{\leq 0}) \]

we have

\[ R^i\Gamma(\omega^*D'_{\xi,\diamond,\varphi}) = R^i\text{Ind}((\Xi \circ \Phi^\text{eq})(\omega^*D'_{\xi,\diamond,\varphi})) \]

by Lemmas 4.3 and 4.9 and by (4.10).
Define a right $(U_{\zeta, \diamond} \otimes \mathbb{C}[\Lambda])$-module $V$ by

$$V = (U_{\zeta, \diamond} \otimes \mathbb{C}[\Lambda]) / \mathcal{I},$$

where

$$\mathcal{I} = (\bar{U}_{\zeta}^- \cap \text{Ker}(\varepsilon))U_{\zeta, \diamond} \mathbb{C}[\Lambda] + \sum_{\lambda \in \Lambda} (k_{2\lambda} - e(2\lambda))U_{\zeta, \diamond} \mathbb{C}[\Lambda].$$

By the triangular decomposition $\bar{U}_{\zeta}^- \otimes U_{\zeta, \diamond}^0 \otimes U_{\zeta}^+ \cong U_{\zeta, \diamond}$ we have

$$V \cong U_{\zeta}^+ \otimes \mathbb{C}[\Lambda]$$

as a vector space. Define a right action of $U_{L, \leq 0}^{\zeta}$ on $U_{\zeta, \diamond} \otimes \mathbb{C}[\Lambda]$ by

$$(u \otimes e(\lambda)) \ast v = \text{ad}(Sv)(u) \otimes e(\lambda) \quad (u \in U_{\zeta, \diamond}, \lambda \in \Lambda, v \in U_{\zeta}^{L, \leq 0}).$$

It induces a right action of $U_{L, \leq 0}^{\zeta}$ on $V$. Moreover, we see easily that this right $U_{\zeta}^{L, \leq 0}$-module structure gives a left $C_{\zeta}^{\leq 0}$-comodule structure of $V$.

**Proposition 5.1.** We have

$$(\Xi \circ \Phi^{eq})(\omega^* D'_{\zeta,f}) \cong V$$

as a left $C_{\zeta}^{\leq 0}$-comodule.

The proof is given in Section 5.3.

It follows from Proposition 5.1 that Conjecture 2.14 is equivalent to the following conjecture.

**Conjecture 5.2.** Assume that $\ell > h_G$. We have

$$\text{Ind}(V) \cong U_{\zeta,f} \otimes \text{Z_{Har}(U_{\zeta}) \mathbb{C}[\Lambda]},$$

and

$$R^i \text{Ind}(V) = 0$$

for $i \neq 0$.

**Remark 5.3.** We can show that

$$U_{\zeta,f} \cong (C_{\zeta})_{ad}, \quad V \cong \text{ad}(C_{\zeta}^{\leq 0}) \otimes \mathbb{C}[2\Lambda] \mathbb{C}[\Lambda],$$
where \((C_\xi)_{\text{ad}}\) (resp., \(\text{ad}(C_\xi^{\leq 0})\)) is given by the right (resp., left) adjoint coaction of \(C_\xi\) (resp., \(C_\xi^{\leq 0}\)) on itself. Hence, Conjecture 5.2 is equivalent to

\[
R \text{Ind}(\text{ad}(C_\xi^{\leq 0})) \cong (C_\xi)_{\text{ad}} \otimes_{\mathbb{C}[2\Lambda]} \mathbb{C}[2\Lambda].
\]

The corresponding statement for \(q = 1\) is

\[
R \text{Ind}(\text{ad}C[B^-]) \cong C[G]_{\text{ad}} \otimes_{\mathbb{C}[H/W]} C[H].
\]

We can prove this by a geometric method.

**Remark 5.4.**† A proof of Conjecture 5.2, when \(\ell\) is a prime greater than the Coxeter number, is given by Backelin and Kremnizer in [3, Proposition 3.25]; however, in a more recent article they admit that there are gaps in [3] (see [4, Version 3, Section 1.1.2]) and propose different proofs. But it is likely that problems still remain in the new proofs given in [4], as explained below.

The proof in [4, Versions 1 and 2] is wrong because all positive roots are assumed there to be dominant (see [4, Version 2, proof of Theorem 2.1]).

Another proof given in [4, Version 3] also has problems. In Step (b) of [4, Version 3, proof of Theorem 2.2.1], the authors compare certain weight multiplicities \(a_{q,\mu}\) and \(b_{q,\mu}\). But since those multiplicities are infinite, the argument there should be modified using multiplicities as \(U_q\)-modules. Let us assume for simplicity that \(q\) is generic and try to modify the original argument by replacing \(a_{q,\mu}, b_{q,\mu}, b'_{q,\mu}\) with their counterparts as multiplicities of \(U_q\)-modules. This even fails since \(a_{1,\mu}\) (resp., \(b'_{1,\mu}\)) is the dimension of the 0-weight space of the irreducible module (resp., Verma module) with highest-weight \(\mu\). We also point out that the reason that \(U_\lambda^q\) is an integral domain is not given in Step (a).

Note that the arguments in [4, Version 3, proof of Theorem 2.2.1] are partially similar to those in the earlier manuscripts (see [2, Proposition 4.8], [3, Proposition 3.25]). The main difference is that [4, Version 3] relies on a \(B_q\)-stable filtration with 1-dimensional subquotients instead of the Joseph–Letzter filtration used in [2] and [3]. For us, the original argument in [2] and [3] for generic \(q\) using the Joseph–Letzter filtration is not comprehensible either. In the notation of [2, proof of Proposition 4.8], the validity of the formula \(m_j(1) = \hat{n}_j(1)\) is not clear to us since the Joseph–Letzter filtration does not induce at \(q = 1\) the ordinary filtration for enveloping algebras and differential operators in general.

†This remark is added at the editor’s request.
5.3.
We will give a proof of Proposition 5.1 in the rest of this article. By Remark 4.2, we have

$$(\Xi \circ \Phi^\text{eq})(\omega^* D'_{\zeta,f}) \cong C \otimes_{A_{\zeta}} D'_{\zeta,f}$$

as a vector space, where $A_{\zeta} \to C$ is given by $\varepsilon$. Note that

$$C \otimes_{A_{\zeta}} E_{\zeta,\diamond} \cong U_{\zeta,\diamond} \otimes \mathbb{C}[\Lambda].$$

We first show the following.

**Lemma 5.5.** We have

$$C \otimes_{A_{\zeta}} D'_{\zeta,\diamond} \cong V.$$

**Proof.** By (2.10) we obtain

$$C \otimes_{A_{\zeta}} D'_{\zeta,\diamond} \cong \left( U_{\zeta,\diamond} \otimes \mathbb{C}[\Lambda] \right) / \sum_{\varphi \in A_{\zeta}} (1 \otimes \Omega'(\varphi))(U_{\zeta,\diamond} \otimes \mathbb{C}[\Lambda]),$$

where $1 \otimes \Omega'(\varphi)$ is the image of $\Omega'(\varphi)$ in $C \otimes_{A_{\zeta}} E_{\zeta,\diamond} = U_{\zeta,\diamond} \otimes \mathbb{C}[\Lambda]$. Note that $\varepsilon(A_{\zeta}(\lambda)\xi) = \{0\}$ for $\lambda \in \Lambda^+$, $\xi \in \Lambda$ with $\lambda \neq \xi$, and that $\varepsilon(A_{\zeta}(\lambda)\lambda) = C$ for $\lambda \in \Lambda^+$. Hence, for $\varphi \in A_{\zeta}(\lambda)\xi$ with $\lambda \in \Lambda^+$, $\xi \in \Lambda$ we have

$$1 \otimes \Omega_1'(\varphi) = \begin{cases} 0 & (\lambda \neq \xi), \\ \varepsilon(\varphi) & (\lambda = \xi). \end{cases}$$

Let us also compute $1 \otimes \Omega_2'(\varphi)$. Let

$$\tilde{\Psi}_\lambda : \tilde{U}_{\zeta}^-(\lambda) \to A_{\zeta}(\lambda)$$

be the composite of the linear isomorphism $\Psi_\lambda : \tilde{U}_{\zeta}^-(\lambda) \to L^*_{-\zeta}(\lambda)$ (see (3.33)) and an isomorphism $f : L^*_{-\zeta}(\lambda) \to A_{\zeta}(\lambda)$ of $U_\zeta^L$-modules. We have $\tilde{\Psi}_\lambda(\tilde{U}_{\zeta}^-(\lambda)_{-(\lambda-\xi)}) = A_{\zeta}(\lambda)\xi$ for any $\xi \in \Lambda$. Hence, we may assume that $\varepsilon = \varepsilon \circ \tilde{\Psi}_\lambda$ on $\tilde{U}_{\zeta}^-(\lambda)$. Let $\varphi \in A_{\zeta}(\lambda)\xi$, and take $v \in \tilde{U}_{\zeta}^-(\lambda)_{-(\lambda-\xi)}$ satisfying $\tilde{\Psi}_\lambda(v) = \varphi$. Then we have

$$\sum_p (S x_p^L) \cdot \varphi \otimes y_p k_{\beta_p} = \sum_p f((S x_p^L) \cdot \Psi_\lambda(v)) \otimes y_p k_{\beta_p}$$

$$= \sum_p \zeta^{-(\beta_p,\xi)} f((S x_p^L) k_{\beta_p} \cdot \Psi_\lambda(v)) \otimes y_p k_{\beta_p}.$$
Hence, we have

\[
\zeta^-(\beta_p, \xi) \frac{S \tau_c (((S x_p^L)_{k_{\beta_p}}, v(0)))}{\Psi_\lambda(v(1))} \otimes y_p k_{\beta_p}
\]

and hence

\[
1 \otimes \Omega^\prime_2(\varphi) = \sum_p \varepsilon((S x_p^L) \cdot \varphi) y_p k_{\beta_p} k_{2\xi} e(-2\lambda)
\]

\[
= \sum_p \zeta^-(\beta_p, \xi) L \tau_c (((S x_p^L)_{k_{\beta_p}}, v(0))) \varepsilon(v(1)) y_p k_{\beta_p} k_{2\xi} e(-2\lambda)
\]

\[
= \sum_p \zeta^-(\beta_p, \xi) L \tau_c (((S x_p^L)_{k_{\beta_p}}, v)) y_p k_{\beta_p} k_{2\xi} e(-2\lambda)
\]

\[
= \sum_p \zeta^-(\beta_p, \xi) L \tau_c (k_{-\beta_p, x_p^L}, S^{-1} v) y_p k_{\beta_p} k_{2\xi} e(-2\lambda)
\]

\[
= \sum_p \zeta^-(\beta_p, \xi) - (\beta_p, \beta_p) L \tau_c (x_p^L, S^{-1} v) y_p k_{\beta_p} k_{2\xi} e(-2\lambda)
\]

\[
= \sum_p \zeta^-(\beta_p, \xi) L \tau_c (x_p^L, S^{-1} v) y_p k_{\lambda - \xi} k_{2\xi} e(-2\lambda)
\]

\[
= \zeta^-(\beta_p, \xi) (S^{-1} v) k_{\lambda - \xi} k_{2\xi} e(-2\lambda).
\]

(Note that \( (S^{-1} v) k_{\lambda - \xi} \in \tilde{U}_\zeta^- (\lambda) (\lambda - \xi) \). It follows that

\[
1 \otimes \Omega^\prime(\varphi) = \begin{cases} 
-\zeta^-(\lambda - \xi, \lambda) (S^{-1} v) k_{\lambda - \xi} k_{2\xi} e(-2\lambda) & (\lambda \neq \xi), \\
\varepsilon(\varphi) (1 - k_{2\lambda} e(-2\lambda)) & (\lambda = \xi).
\end{cases}
\]

Hence, we have

\[
\sum_{\lambda \in \Lambda^+, \varphi \in A_\zeta(\lambda)_{\lambda - \gamma}} (1 \otimes \Omega^\prime(\varphi)) (U_{\xi, \varphi} \otimes \mathbb{C}[\Lambda])
\]

\[
= \sum_{\lambda \in \Lambda^+, \gamma \in Q^+ \{0\}} \tilde{U}_\zeta^- (\lambda)_{-\gamma} (U_{\xi, \varphi} \otimes \mathbb{C}[\Lambda]) + \sum_{\lambda \in \Lambda^+} (1 - k_{2\lambda} e(-2\lambda)) (U_{\xi, \varphi} \otimes \mathbb{C}[\Lambda])
\]

\[
= (\tilde{U}_\zeta^- \cap \text{Ker}(\varepsilon)) (U_{\xi, \varphi} \otimes \mathbb{C}[\Lambda]) + \sum_{\lambda \in \Lambda} (k_{2\lambda} - e(2\lambda)) (U_{\xi, \varphi} \otimes \mathbb{C}[\Lambda])
\]

by (3.35).
Lemma 5.6. We have
\[ C \otimes_{A_\zeta} D_{\zeta,f}' \cong V. \]

Proof. We need to show that the canonical homomorphism \( C \otimes_{A_\zeta} D_{\zeta,f}' \rightarrow C \otimes_{A_\zeta} D_{\zeta,\circ}' \) is bijective. The surjectivity is a consequence of (3.35) and (3.36). Let us give a proof of the injectivity. Set
\[ K = A_\zeta U_{\zeta,f} C[\Lambda] \cap \sum_{\varphi \in A_\zeta} A_\zeta \Omega'(\varphi) U_{\zeta,\circ} C[\Lambda] \subset A_\zeta \otimes U_{\zeta,f} \otimes C[\Lambda]. \]

Then it is sufficient to show that the natural map
\[ \mathbb{C} \otimes_{A_\zeta} ((A_\zeta \otimes U_{\zeta,f} \otimes C[\Lambda]) / K) \rightarrow (U_{\zeta,\circ} \otimes C[\Lambda]) / I \]
is injective. Let \( F : A_\zeta \otimes U_{\zeta,f} \otimes C[\Lambda] \rightarrow U_{\zeta,\circ} \otimes C[\Lambda] \) be the natural map. Then it is sufficient to show that
\[
(5.1) \quad I \cap (U_{\zeta,f} \otimes C[\Lambda]) \subset F(K).
\]

Indeed, assume that (5.1) holds. Denote by
\[
p : A_\zeta \otimes U_{\zeta,f} \otimes C[\Lambda] \rightarrow \mathbb{C} \otimes_{A_\zeta} ((A_\zeta \otimes U_{\zeta,f} \otimes C[\Lambda]) / K),
\]
\[
\pi : U_{\zeta,\circ} \otimes C[\Lambda] \rightarrow (U_{\zeta,\circ} \otimes C[\Lambda]) / I
\]
the natural maps. We have to show that \( \text{Ker}(\pi \circ F) \subset \text{Ker}(p) \). Take \( x \in \text{Ker}(\pi \circ F) \). Then \( F(x) \in I \cap (U_{\zeta,f} \otimes C[\Lambda]) \). Hence, by (5.1) there exists some \( v \in K \) such that \( F(x) = F(v) \). Then \( p(x) = p(x - v) + p(v) = p(x - v) \). Hence, we may assume that \( F(x) = 0 \) from the beginning. Note that \( p \) factors through
\[
p' : A_\zeta \otimes U_{\zeta,f} \otimes C[\Lambda] \rightarrow \mathbb{C} \otimes_{A_\zeta} (A_\zeta \otimes U_{\zeta,f} \otimes C[\Lambda]) (= U_{\zeta,f} \otimes C[\Lambda]).
\]
By \( F(x) = 0 \) we have \( p'(x) = 0 \), and hence \( p(x) = 0 \), as desired.

It remains to show (5.1). Let \( \lambda \in \Lambda^+ \), and let \( \varphi \in A_\zeta(\lambda) \). Then we have \[ \Omega'_1(\varphi) = \sum_p (y_p^L \cdot \varphi) x_p \in A_\zeta U_{\zeta}^+, \quad \Omega'_2(\varphi) = \varphi k_{2\lambda} e(-2\lambda). \]

Let us show that
\[
(5.2) \quad \Omega'_1(\varphi) = \sum_p (y_p^L \cdot \varphi) x_p \in A_\zeta U_{\zeta}^+(\lambda).
\]
This is equivalent to
\[
\sum_p (y_p^L \cdot \varphi) \otimes \Phi_{-\lambda}(x_p) \in A_\zeta \otimes L^*_+(-\lambda).
\]

This follows from
\[
\sum_p \langle \Phi_{-\lambda}(x_p), u f_i^{((\lambda, \alpha^+)_{\lambda}+1)} \rangle y_p^L \cdot \varphi = \sum_p \tau_{\zeta}^L(x_p, u f_i^{((\lambda, \alpha^+)_{\lambda}+1)} ) y_p^L \cdot \varphi = (u f_i^{((\lambda, \alpha^+)_{\lambda}+1)} ) \cdot \varphi = 0
\]
for \( u \in U_\zeta^{L_+}, i \in I \). Thus, (5.2) is verified. Hence, we have
\[
\Omega'(\varphi) k_{-2\lambda} \in \mathcal{K}.
\]

It follows that
\[
(5.3) \quad F(\mathcal{K}) \supset (k_{-2\lambda} - e(-2\lambda)) U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \quad (\lambda \in \Lambda^+).
\]

Now let \( u \in \mathcal{I} \cap (U_{\zeta,f} \otimes \mathbb{C}[\Lambda]) \). If we can show that \( k_{-2\mu} u \in F(\mathcal{K}) \) for some \( \mu \in \Lambda^+ \), then we obtain
\[
u = e(2\mu)(e(-2\mu) - k_{-2\mu}) u + e(2\mu)k_{-2\mu} u \in F(\mathcal{K})
\]
by (5.3). Hence, it is sufficient to show that for any \( u \in \mathcal{I} \) there exists some \( \mu \in \Lambda^+ \) such that \( k_{-2\mu} u \in F(\mathcal{K}) \). We may assume that there exists \( \nu \in Q \) such that \( k_{-2\mu} u = \zeta(\mu, \nu) u k_{-2\mu} \) for any \( \mu \in \Lambda \). Therefore, we have only to show that for any \( u \in \mathcal{I} \) there exists some \( \mu \in \Lambda^+ \) such that \( u k_{-2\mu} \in F(\mathcal{K}) \). By Lemma 5.5 we can take \( \varphi_i \in A_\zeta \), \( x_i \in U_{\zeta,\zeta} \otimes \mathbb{C}[\Lambda] \) \((i = 1, \ldots, N)\) such that
\[
u = 1 \otimes \sum_{i=1}^N \Omega'(\varphi_i) x_i.
\]

By Lemma 3.7 we can take \( \mu \in \Lambda^+ \) such that \( \Omega'(\varphi_i) x_i k_{-2\mu} \in A_\zeta \otimes U_{\zeta,\zeta} \otimes \mathbb{C}[\Lambda] \) for any \( i \). Then we have
\[
u u k_{-2\mu} = \sum_{i=1}^N F(\Omega'(\varphi_i) x_i k_{-2\mu}) \in F(\mathcal{K}).
\]
By Lemma 5.6 we obtain an isomorphism

$$(\Xi \circ \Phi^\text{eq})(\omega^* D_{\zeta, f}') \cong V$$

of vector spaces. We need to show that it is in fact an isomorphism of left $C^{\leq 0}_{\zeta}$-comodules. This is a consequence of the corresponding fact for $E_{\zeta, f}$. Note that we have

$$C \otimes A_{\zeta} E_{\zeta, f} \cong U_{\zeta, f} \otimes C[\Lambda],$$

and hence we have an isomorphism

$$(5.4) \quad (\Xi \circ \Phi^\text{eq})(\omega^* E_{\zeta, f}) \cong U_{\zeta, f} \otimes C[\Lambda]$$

of vector spaces. Hence, we have only to show the following.

**Lemma 5.7.** Under identification (5.4), the left $C^{\leq 0}_{\zeta}$-comodule structure of $U_{\zeta, f} \otimes C[\Lambda]$ is associated to the right $U_{\zeta}^{L, \leq 0}$-module structure given by

$$(u \otimes e(\lambda)) \cdot v = \text{ad}(Sv)(u) \otimes e(\lambda) \quad (u \in U_{\zeta, f}, \lambda \in \Lambda, v \in U_{\zeta}^{L, \leq 0}).$$

**Proof.** Note that the left $C^{\leq 0}_{\zeta}$-comodule structure of $U_{\zeta, f} \otimes C[\Lambda]$ is given by

$$U_{\zeta, f} \otimes C[\Lambda] \cong \Xi(C_{\zeta} \otimes (U_{\zeta, f} \otimes C[\Lambda])),$$

where $C_{\zeta} \otimes (U_{\zeta, f} \otimes C[\Lambda])$ is regarded as a left $C^{\leq 0}_{\zeta}$-comodule by the tensor product of $C_{\zeta}$ (with left $C^{\leq 0}_{\zeta}$-comodule structure $(\text{res} \otimes 1) \circ \Delta: C_{\zeta} \to C^{\leq 0}_{\zeta} \otimes C_{\zeta}$) and $U_{\zeta, f} \otimes C[\Lambda]$ with trivial left $C^{\leq 0}_{\zeta}$-comodule structure. Hence, it is sufficient to show that for a right $C_{\zeta}$-comodule $M$ the right $U_{\zeta}^{L, \leq 0}$-module structure of

$$M \cong \Xi(C_{\zeta} \otimes M) \in \text{Comod}(C^{\leq 0}_{\zeta})$$

is given by

$$m \cdot v = (Sv) \cdot m \quad (m \in M, v \in U_{\zeta}^{L, \leq 0}).$$

Denote by $M^{\text{triv}}$ the trivial right $C_{\zeta}$-comodule which coincides with $M$ as a vector space. We denote by $M \ni m \leftrightarrow m \in M^{\text{triv}}$ the canonical linear isomorphism. We have $C_{\zeta} \otimes M^{\text{triv}} \in \text{Comod}'(C_{\zeta})$ as the tensor product of $C_{\zeta} \in \text{Comod}'(C_{\zeta})$ and $M^{\text{triv}} \in \text{Comod}'(C_{\zeta})$. We can also define a left $C^{\leq 0}_{\zeta}$-comodule structure of $C_{\zeta} \otimes M^{\text{triv}}$ as the tensor product of the left
$C_{\zeta}^{\leq 0}$-comodules $C_{\zeta}$ and $M^{\text{triv}}$, where the left $C_{\zeta}^{\leq 0}$-comodule structure of $M^{\text{triv}}$ is given by the right $U_{\zeta}^{L,\leq 0}$-module structure

$$m \cdot v = (Sv) \cdot \overline{m} \quad (m \in M, v \in U_{\zeta}^{L,\leq 0}).$$

Then we have a linear isomorphism

$$C_{\zeta} \otimes M \ni \varphi \otimes m \mapsto \sum_{(m)} \varphi m(1) \otimes \overline{m(0)} \in C_{\zeta} \otimes M^{\text{triv}}$$

preserving the right $C_{\zeta}$-comodule structures and the left $C_{\zeta}^{\leq 0}$-comodule structures. It follows that

$$\Xi(C_{\zeta} \otimes M) \cong \Xi(C_{\zeta} \otimes M^{\text{triv}}) = M^{\text{triv}} \in \text{Comod}(C_{\zeta}^{\leq 0}).$$

The proof of Proposition 5.1 is complete.

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