A quantum secret ballot

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Abstract

The paper concerns the protection of the secrecy of ballots, so that the identity of the voters cannot be matched with their vote. To achieve this we use an entangled quantum state to represent the ballots. Each ballot includes the identity of the voter, explicitly marked on the "envelope" containing it. Measuring the content of the envelope yields a random number which reveals no information about the vote. However, the outcome of the elections can be unambiguously decided after adding the random numbers from all envelopes. We consider a few versions of the protocol and their complexity of implementation.
I. INTRODUCTION

Assume \( n \) parties participate in a vote to decide between a few alternatives. Each participant chooses his or her preference, designates it on the ballot, and puts it in the box. There are various ways in which the secrecy of the vote can be compromised, and we shall be particularly interested in the case of marked ballots. In this conspiracy the ballot is made to include the voter's identity, by secretly marking it prior to the vote or during it. To choose a rather paranoid scenario: Big Brother finds traces of the voter's DNA on the paper ballot. Or, in the case of an electronic ballot, a string containing the voter's identity, which has just been verified prior to voting, is stored together with the vote.

In this paper we use entangled qbits to prevent such schemes. Each ballot may very well include the identity of the voter, explicitly marked on the "envelope" containing it. However, this is inconsequential because reading the contents of the envelope reveals a random number, and no information about the vote. On the other hand, the outcome of the elections can be unambiguously decided after adding the random numbers from all envelopes.

A few quantum voting protocols have been proposed recently: Singh and Srikanth \[1\] suggested to use a quantum version of sealed envelopes; any attempt to read their content by unauthorised persons can be detected. Vaccaro et.al. \[2\] proposed a voting scheme in which the number of votes is coded into the phases of an entangled state and reading the result involves a complicated measurement. A protocol more similar to the present one has been proposed by Hillery et.al. \[3\]. In their protocol the election result is also encoded into the phases of a quantum state, and its reading involves a complicated measurement. Our mechanism is different from \[3\] in various respects which will be noted below. In particular, the voting result is coded and read directly from the computation basis states. The protocol can be implemented as soon as the implementation of the discrete Fourier transform becomes possible.

We begin with a protocol for a vote to decide between two alternatives. Although the protocol is valid for any number of voters \( n \geq 2 \), its implementation may be complicated when thousands of citizens participate in the elections. To ammend this situation we also propose an alternative version, whose complexity depends on the number of ballot boxes. The security of the protocol remains intact provided this number is \( \geq 2 \). Subsequently, the scheme is generalized to include a choice between more than two alternatives. Finally, the
II. THE PROTOCOL

Let $m$ be a natural number. Consider an $m$ dimensional space with basis vectors: $|0\rangle, |1\rangle, ..., |m-1\rangle$. The $m$-th order discrete Fourier transform is defined to be

$$F_m |j\rangle = \frac{1}{\sqrt{m}} \sum_{l=0}^{m-1} \exp\left(\frac{2\pi i j l}{m}\right) |l\rangle, \quad j = 0, 1, ..., m-1$$

(1)

Subsequently we shall suppress the subscript $m$, and denote the Fourier transform by $F$. Let $\Pi$ be the unitary operator which defines the following cyclic permutation on the basis elements:

$$\Pi |0\rangle = |1\rangle, \quad \Pi |1\rangle = |2\rangle, \quad ..., \quad \Pi |m-1\rangle = |0\rangle$$

(2)
or, in short $\Pi |j\rangle = |j \oplus 1\rangle$ where $\oplus$ represents addition mod $m$.

Suppose that we distribute among $n$ voters the entangled state $|W\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |j\rangle |j\rangle ... |j\rangle$

(3)

where each $|j\rangle$ is an $m$-dimensional basis state, and each product in the sum contains $n$ copies. The relation between $n$ and $m$ will be fixed later. Each voter has to vote either NO, in which case he applies $F$ to his bit; or YES, in which case she applies $\Pi F$ (that is, $F$ followed by $\Pi$). Suppose the votes were $a_1, a_2, ..., a_n$, with $a_k = 0$ in case of a NO vote by person $k$, and $a_k = 1$ in case of a YES vote. Put $\Pi^0 = I$ (identity) and $\Pi^1 = \Pi$, then after the vote the state is:

$$|V\rangle = (\Pi^{a_1} F) \otimes (\Pi^{a_2} F) \otimes ... \otimes (\Pi^{a_n} F) |W\rangle =$$

(4)

$$= \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} (\Pi^{a_1} F) |j\rangle \otimes (\Pi^{a_2} F) |j\rangle \otimes ... \otimes (\Pi^{a_n} F) |j\rangle =$$

$$= \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \left( \Pi^{a_1} \frac{1}{\sqrt{m}} \sum_{l_1=0}^{m-1} \exp\left(\frac{2\pi i j l_1}{m}\right) |l_1\rangle \right) \otimes ... \otimes \left( \Pi^{a_n} \frac{1}{\sqrt{m}} \sum_{l_n=0}^{m-1} \exp\left(\frac{2\pi i j l_n}{m}\right) |l_n\rangle \right)$$

Hillery et. al. [3] use the same initial state $|W\rangle$, apply $F$ for the YES vote and $I$ (identity) in the NO vote. The election outcome is then recorded in the phases of a complicated state.
Performing the tensor product we get:

$$|V\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \frac{1}{m^{n/2}} \sum_{l_1, \ldots, l_n} \exp\left(\frac{2\pi i j}{m} (l_1 + \ldots + l_n)\right) |l_1 \oplus a_1\rangle \otimes \ldots \otimes |l_n \oplus a_n\rangle$$  \hspace{1cm} (5)

exchanging the order of summation

$$|V\rangle = \frac{1}{m^{n+1}} \sum_{l_1, \ldots, l_n} \left(\sum_{j=0}^{m-1} \exp\left(\frac{2\pi i j}{m} (l_1 + \ldots + l_n)\right)\right) |l_1 \oplus a_1\rangle \otimes \ldots \otimes |l_n \oplus a_n\rangle$$  \hspace{1cm} (6)

Unless \( l_1 + \ldots + l_n \equiv 0 \pmod{m} \) we have \( \sum_{j=0}^{m-1} \exp\left(\frac{2\pi i j}{m} (l_1 + \ldots + l_n)\right) = 0 \). Hence the result of the vote is

$$|V\rangle = \frac{1}{m^{n+1}} \sum_{l_1, \ldots, l_n \equiv 0 \pmod{m}} \left| l_1 \oplus a_1\rangle \otimes \ldots \otimes |l_n \oplus a_n\rangle \right.$$  \hspace{1cm} (7)

Now, we measure the basis vectors and add the results mod \( m \). Since \( l_1 + \ldots + l_n \equiv 0 \pmod{m} \) for every component in the superposition in Eq.(7), we are left with the outcome \( a_1 + \ldots + a_n \pmod{m} \).

III. APPLICATIONS

1. In the simplest case we choose \( m > n \), preferably we let \( m \) be the smallest power of two greater than \( n \), so we can use qubits. Then, after adding the measurement results mod \( m \), we simply get \( a_1 + \ldots + a_n \), which is the number of YES votes. The secrecy of the vote is maintained because every individual “ballot” \( |l_r \oplus a_r\rangle \) contains the actual vote \( a_r \) added mod \( m \) to a random number \( l_r \) between 0 and \( m - 1 \). Note that the ballots are not mixed, and it may be public knowledge that the ballot \( |l_r \oplus a_r\rangle \) comes from voter \( r \) (we may even attach an extra probe carrying his or her name). However, this information is inconsequential, it only reveals the fact that person \( r \) participated in the poll.

Actually, we do not have to know in advance how many people will vote, just choose \( n \) to be sufficiently large. Since at the end of election day we know the exact number of people who participated, we push the NO button as many times as required to reach \( n \). After the measurement we subtract the number of fictional votes and announce the election results.

There is a classical protocol which is similar to the quantum ballot, but is nevertheless less secure: A sequence of \( n \) random numbers \( (l_1, \ldots, l_n) \) is generated and their sum \( y = l_1 + \ldots + l_n \) stored. When citizen \( r \) is voting \( a_r \), the electronic voting machine stores only the number
This way the privacy of the vote is protected. At the end of the day the stored numbers are added, and then $y$ subtracted. This protocol is secured only to the extent that the values of the random numbers are protected. In the classical world there is always an interval of time when the values of the $l_r$’s themselves are present in the system. In the quantum protocol, by contrast, the numbers $l_r$ are generated only upon measurement, and are present only in the compounds $l_r \oplus a_r$.

Note that an identical result obtains if we change the protocol slightly: Firstly, we distribute among the voters the state

$$|U\rangle = (\mathcal{F} \otimes \mathcal{F} \otimes ... \otimes \mathcal{F}) |W\rangle = \frac{1}{m^{N/2}} \sum_{l_1+...+l_n \equiv 0 \pmod{m}} |l_1\rangle \otimes ... \otimes |l_n\rangle,$$

and secondly, each voter applies $\Pi$ for a YES vote, or $I$ for NO. The choice between the two versions will depend on the technical detail of implementation.

It goes without saying that even a quantum protocol cannot be secured against all possible attacks by Big Brother, such as complete rewiring of the voting machine, or the installation of video cameras in the voting booths.

2. The single element that makes the protocol difficult to execute is the number of voters $n$. The difficulty is expressed in the structure of the initial state $|W\rangle$ in Eq.(3), where each component is a tensor product of $n$ states. If we consider a vote of a small committee then producing $|W\rangle$ seems feasible; but what if millions of people vote? Luckily we can simplify the protocol to include this case. To do this let $N$ stand for the number of ballot boxes, and assume that in the state $|W\rangle$ each component has $N$ copies, one for each box. However, we keep $m$ larger than the total number of voters $n$. Now, early in the morning on election day, an official performs $\mathcal{F}$ once for each box, and this is the last time the Fourier transform is applied to the box. Subsequently, any NO voter applies $I$ (identity) to the part of the state corresponding to his box, and any YES voter applies $\Pi$. By repeating the same calculation we get the post election state

$$|V\rangle = \frac{1}{m^{N/2}} \sum_{l_1+...+l_N \equiv 0 \pmod{m}} |l_1 + a'_1 + ... + a'_{k_1} \pmod{m}\rangle \otimes ... \otimes |l_N + a''_1 + ... + a''_{k_N} \pmod{m}\rangle$$

(9)

Where $a'_1, ..., a'_{k_1}$, are the votes cast in box 1, and so on, to $a''_1, ..., a''_{k_N}$, the votes in box $N$. Again, since $l_1 + ... + l_N \equiv 0 \pmod{m}$, then measuring the basis states and adding the results mod $m$ yields the sum of all YES votes from all boxes (recall that we kept $m$ larger than $n$).
So why not take \( N = 1 \), that is, only one box for all voters? In this case \( |W\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |j\rangle \), and \( \mathcal{F} |W\rangle = |0\rangle \), and thus \( |V\rangle = \Pi^{a_1}\Pi^{a_2}...\Pi^{a_n} |0\rangle = |a_1 + ... + a_n\rangle \) is the sum of all YES votes. In other words, using the protocol with a single ballot box brings us back to a classical voting system represented by an unentangled quantum state. But already with \( N = 2 \) there is a random element in the protocol, hiding the number of YES votes that each box contributes.

3. Suppose that there are more than two alternatives, not just YES and NO, but three candidates to choose from, call them I, II, and III. For \( n \) voters we choose \( m \) to be bigger than \( 2n \) and use two copies of \( |W\rangle \), call them \( |W\rangle_1 \) and \( |W\rangle_2 \). Now, each voter applies the following rule: For candidate I apply \( \mathcal{F} \) to \( |W\rangle_1 \) and \( \mathcal{F} \) to \( |W\rangle_2 \). For candidate II apply \( \Pi\mathcal{F} |W\rangle_1 \) and \( \Pi\mathcal{F} |W\rangle_2 \), and for III apply \( \Pi^2\mathcal{F} |W\rangle_1 \) and \( \Pi\mathcal{F} |W\rangle_2 \). Let \( n_I, n_{II} \) and \( n_{III} \) be the numbers of votes cast for the respective candidates. Applying a measurement to \( |V\rangle_1 \), the post elections state of \( |W\rangle_1 \), we obtain the outcome \( n_{II} + 2n_{III} \). Measuring \( |V\rangle_2 \) yields \( n_I + n_{II} \). Since we know \( n = n_I + n_{II} + n_{III} \), we can infer the election results. Generalizations to a larger number of alternatives is straightforward.

IV. COMPLEXITY OF IMPLEMENTATION.

The implementation of the protocol requires three steps:

1. **The creation of the state \( |W\rangle \).** Consider first the basis states copier defined on \( \mathbb{C}^m \otimes ... \otimes \mathbb{C}^m \) (\( n \) copies) and whose effect is, in particular

\[
|j\rangle \otimes |0\rangle \otimes ... \otimes |0\rangle \rightarrow |j\rangle \otimes |j\rangle \otimes ... \otimes |j\rangle \quad 0 \leq j \leq m
\]  

To implement this suppose \( m = 2^k \), then the operation \( |j\rangle \otimes |0\rangle \rightarrow |j\rangle \otimes |j\rangle \) can be achieved using bit by bit copying, each bit by the implementation of two CNOT gates, altogether \( 2^k \) gates. Generalizing to \( n \) copies we need \( O(kn) \) gates, \( k = \log_2 m \). To create \( |W\rangle \), therefore, we apply this copying mechanism to \( (m^{-\frac{1}{2}} \sum_{j=0}^{m-1} |j\rangle) \otimes |0\rangle \otimes ... \otimes |0\rangle \), where the state \( (m^{-\frac{1}{2}} \sum |j\rangle) \) is obtained from \( |0\rangle \) by the application of \( k \) Hadamard transforms, one for each bit.

2. **The application of the Fourier transform \( \mathcal{F} \).** By the central result of Shor [5], \( \mathcal{F} \) can be implemented using \( O((\log m)^2) \) gates, and we apply one Fourier transform per copy. Therefore, the fact that \( m \) has to be a large number, larger than the number of voters \( n \),
should not pose a big problem. In fact, with $k = 25$ binary digits we can accommodate elections in a mid size country. However, even smaller scale Fourier transforms suffice to implement an elections protocol using the following trick: Let $n$ be the number of voters and suppose first that we know $n$ in advance. Suppose, moreover, that $n = m_1 m_2 ... m_s$, where $m_1, m_2, ..., m_s$ are coprime. Now, perform the election in parallel on the $s$ states

$$|W\rangle_l = \frac{1}{\sqrt{m_l}} \sum_{j=0}^{m_l-1} |j\rangle |j\rangle ... |j\rangle, \quad 1 \leq l \leq s$$

(11)

Where each term in Eq (11) has $n$ copies. Assume that after the measurement on the $k$ post-election states we get the results $c_1, c_2, ..., c_k$. The number of YES votes $x$ is satisfying

$$x = c_1 (mod m_1), x = c_2 (mod m_2), ..., x = c_s (mod m_s),$$

(12)

and this set of congruences has a unique solution mod $n$.[6]

If we do not know $n$ in advance, or if there is no nice decomposition of $n$ to a product of coprimes, we can do as indicated previously: Choose a large enough $n$ to be on the safe side, and make sure it has a comfortable decomposition. After election day is over push the NO button as many times as needed to bring the number of votes to $n$, and subsequently subtract the fictional votes before the result is announced.

3. Application of Π: Is just an implementation of an algorithm that performs $j \rightarrow j+1 \mod m$, which takes $O(\log m)$ steps per copy.

Altogether, the complexity of the protocol is $O[n(\log_2 m)^2]$ where $n$ is the number of voters (or in another scheme, the number of ballot boxes) and $m$ is the least power of two greater than the number of voters.

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[6] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, Oxford (1988).