Effective sound speed in relativistic accretion discs around rotating black holes

Susovan Maity*, Md Arif Shaikh †, and Tapas Kumar Das‡1,2

1Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad 211 019, India
2Physics and applied mathematics unit, Indian Statistical Institute, Kolkata 700 108, India

November 14, 2018

Abstract

For axially symmetric accretion maintained in the hydrostatic equilibrium along the vertical direction in the Kerr metric, the radial Mach number does not become unity at the critical point. The sonic points are, thus, formed at a radial distance different from that where the critical points are formed. We propose that a modified dynamical sound speed can be defined through the linear perturbation of the full space-time dependent equations describing the aforementioned accretion flow structure. The linear stability analysis of such fluid equations leads to the formation of an wave equation which describes the propagation of linear acoustic perturbation. The speed of propagation of such perturbation can be used as the effective sound speed which makes the value of the Mach number to be unity when evaluated at the critical points. This allows the critical points to coalesce with the sonic points. We study how the spin angular momentum of the black hole (the Kerr parameter) influences the value of the effective sound speed.

1 Introduction

For low angular momentum axially symmetric accretion flow maintained in hydrostatic equilibrium along the vertical direction, the Mach number does not become unity at the critical point. For such flows, the critical points and the sonic points do not form at the same radial distance. Such non-isomorphism of the critical and the sonic points poses an intrinsic problem while making attempts to construct the phase portrait of stationary, transonic, integral accretion solutions. In a recent work, Shaikh et. al have shown that the introduction of certain effective sound speed, which can be conceived from linear stability analysis of the background steady state flow, may resolve the aforementioned issues and the critical points coincide with the sonic points. In a previous publication such works have been performed for disc accretion in the Schwarzschild metric. In the present work, we, however, are interested in introducing an effective dynamical sound speed for accretion discs around rotating black holes, and would like to investigate how the spin angular momentum of the black hole (the Kerr parameter) influences the value of the effective sound speed.

In what follows, we shall do the followings:
For a particular height function describing an accretion disc in hydrostatic equilibrium along the vertical direction, we first construct the phase portrait corresponding to the stationary transonic integral solutions. While doing so, we derive the so-called ‘critical point conditions’ to demonstrate that the value of the radial Mach number is not unity when evaluated at the critical point. We shall compute the value of the Mach number at the critical point and will discuss what will be the value of the modified sound speed to make the value of the Mach number to be unity at the critical point. We then linear perturb the complete space-time dependent general relativistic Euler and the continuity equations (tailored to describe the kind of accretion

* susovanmaity@hri.res.in
† arifshaikh@hri.res.in
‡ tapas@hri.res.in
flow we are interested to study in the present work) to demonstrate how a linear acoustic perturbation travels through the background fluid. We then find the speed of propagation of such linear acoustic perturbation and argue that instead of the expression of the standard thermodynamic speed of sound obtained through the equation

\[ c_s^2 = \left( \frac{\partial p}{\partial \epsilon} \right)_{\text{entropy}}, \]  

where \( p \) is pressure of the fluid and \( \epsilon \) is the total energy density, if one uses the speed of the propagation of the linear perturbation as the effective sound speed, then the value of the radial Mach number becomes unity at the critical point. Hence the critical points and the sonic points coincides.

2 Governing Equations and Choice of the Disc Height

We consider low angular momentum, inviscid, axially symmetric, irrotational accretion flow around a Kerr black hole.

2.1 Background metric of Kerr black hole:

The background spacetime metric could be written in the following form

\[ ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + 2g_{\phi t}d\phi dt + g_{\phi\phi}d\phi^2, \]  

where the metric elements are function of \( r, \theta \) and \( \phi \). The metric elements in the Boyer-Lindquist coordinates are given by

\[ g_{tt} = \left( 1 - \frac{2}{\mu r} \right), \quad g_{rr} = \frac{\mu r^2}{\Delta}, \quad g_{\theta\theta} = \mu r^2, \quad g_{\phi t} = g_{t\phi} = -\frac{2a \sin^2 \theta}{\mu r}, \quad g_{\phi\phi} = \frac{\Sigma}{\mu r^2} \sin^2 \theta, \]  

where

\[ \mu = 1 + \frac{a^2}{r^2} \cos^2 \theta, \quad \Delta = r^2 - 2r + a^2, \quad \Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta. \]  

Here \( a = \frac{J}{M} \) is the Kerr parameter. For a Kerr black hole, we define \( r_+ \) as

\[ r_+ = 1 + \sqrt{1 - a^2} \]  

in \( G = 1, c = 1, M = 1 \) unit.

2.2 The Euler and the continuity equations

The energy momentum tensor for a perfect fluid is given by

\[ T^{\mu\nu} = (p + \epsilon)v^\mu v^\nu + pg^{\mu\nu}, \]  

where \( \rho \) is the rest-mass energy density of the fluid, so that \( \epsilon = \rho + \epsilon_{\text{thermal}} \). \( v^\mu \) is the four-velocity with the normalization condition \( v^\mu v_\mu = -1 \).

At this point we would like to clarify the velocity and other quantities that will be used throughout this paper. \( v^\mu \) denotes the four-velocity in Boyer-Lindquist frame, \( u \) denotes advective velocity, which is the three component velocity in the co-rotating frame\(^1\). We will frequently denote \( v_0^\mu, u_0 \) of four velocity and advective velocity for the steady state flow. In general we will use the subscript zero to denote the value of physical variable corresponding to the stationary solutions of the steady flow, e.g, \( p_0, \rho_0 \) etc.

\(^1\)we refer Gamie Popham for the details description of expressions in various velocittties in different frames for rotating accretion flow in Kerr metric.
The equation of state for adiabatic flow is given by \( p = k \rho^\gamma \) where \( \gamma \) is the polytropic index and \( k \) is constant. Whereas for isothermal case \( p \propto \rho \). The sound speed for adiabatic flow (isoentropic flow) is given by

\[
v_s^2 = \frac{\partial p}{\partial \epsilon}_{\text{entropy}} = \frac{\rho}{h} \frac{\partial h}{\partial \rho},
\]

where \( h \) is the enthalpy given by

\[
h = \frac{p + \epsilon}{\rho}
\]
on the other hand the sound speed for isothermal flow can be defined as [1]

\[
v_s^2 = \frac{1}{h} \frac{dp}{d\rho}
\]

where \( h = \text{constant for isothermal case.} \)

The mass conservation equation and the energy-momentum conservation equations are given by, respectively,

\[
\nabla_\mu (\rho v^\mu) = 0
\]
and

\[
\nabla_\mu T^{\mu \nu} = 0
\]

using the expression for the sound speed the energy momentum conservation equation can be written in the following form

\[
v^\mu \nabla_\mu v^\nu + \frac{c_s^2}{\rho} (v^\mu v^\nu + g^{\mu \nu}) \partial_\mu \rho = 0
\]

where \( c_s \) for adiabatic case and isothermal case are given by equation (7) and equation (9), respectively.

2.3 Choice of disc height

For accretion flow maintained in the hydrostatic equilibrium along the vertical direction, the thickness of the flow becomes a complicated function of the local radial distance and the expression for the radial sound speed. As we will show in the subsequent sections that for the isothermal flow critical points and the sonic points overlap with each other for the disc heights considered in the present work. Thus we concentrate on the adiabatic accretion in the present work and hence such radial sound speed will be position dependent. For accreting fluid with reasonably low angular momentum (so that the inviscid flow can be studied), three different expressions for such flow thickness are available in the literature. The oldest, and most used expression for the flow thickness was provided by Novikov & Thorne [2] as -

\[
H_{NT}(r) = \left( \frac{\rho}{\rho_0} \right)^{\frac{1}{2}} \frac{r^3 + a^2r + 2a^2}{r^{\frac{3}{2}} + a} \sqrt{\frac{r^6 - 3r^5 + 2ar^2}{(r^2 - 2r + a^2)(r^4 + 4a^2r^2 - 4a^2r + 3a^4)}}
\]

It is to be noted that accretion flow described by the above disc thickness can not be extended upto \( r_+ \). The flow will be truncated at a distance \( r_T \), where

\[
(r_T)^{\frac{3}{2}} (r_T - 3) = 2a
\]

which is outside \( r_+ \). In reality of course the flow will exist upto \( r_+ \) but no stationary integral flow solutions can be constructed up to the close proximity of \( r_+ \) for accretion flow described by the Novikov-Thorne kind of disc heights.
Riffert and Herold [3] provided an expression of disc thickness by modifying the gravity-pressure balance condition of the treatment done in Novikov & Thorne as

\[ H_{RH}(r) = \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \sqrt{\frac{r^5 - 3r^4 + 2ar^\frac{5}{2}}{r^2 - 4ar^2 + 3ar^2}} \]  

(15)

Here the flow is also truncated at \( r_T \) as given by (14). Thus we see both the disc heights can be expressed in the form by \( H(r) = \left( \frac{p}{\rho} \right)^{\frac{1}{2}} f(r, a) \). The difference in these two models of disc thickness in vertical equilibrium is reflected by the difference in functional form of two different \( f(r, a) \). The essential difference arises because whereas Novikov-Thorne [2] balanced the vertical component of pressure with a particular Riemann tensor \( R_{0\alpha0}^z \), which was equivalent to the vertical component of gravitational acceleration, Riffert Herold [3] derived the gravity-pressure balance equation itself by imposing two particular orthonormality condition on vertical component of Euler equation. As we will observe that the Mach number evaluated at the critical points corresponding to the flow described by the thickness function proposed by Novikov and Thorne will be identical with that of the flow described by the thickness functions proposed by Riffert and Herold. Clearly, for the thickness expression proposed by Novikov and Thorne contains only a factor of the form \( f(r, a) \) apart from the \( \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \) factor because only Riemann tensor is invoked in the pressure-gravity balance equation and the Riemann tensor is entirely dependent on geometry. But for the thickness expression proposed by Riffert and Herold contains only a factor of the form \( f(r, a) \) apart from the \( \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \) factor because the choice of 4-velocity field in the accretion disc flow is chosen in such a way that the orthogonality relation of 4-velocity, \( v^\mu v_\mu = -1 \), and one component of the relativistic Euler equation reduces the velocity as a function of radial distance and Kerr parameter \( a \), which in turn ultimately introduces only the function of the form \( f(r, a) \) in the pressure-gravity balance equation.

Abramowicz, Lanza and Percival [4] introduced an expression for the disc thickness, given by

\[ H_{ALP}(r) = \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \sqrt{\frac{2r^4}{v_t^2 - a^2(v_t - 1)}}, \]  

(16)

for which the steady state accretion solutions can be obtained upto \( r_+ \). It is, however to be noted that linear perturbation of the flow equation described by the flow thickness as defined through (16) does not lead to the formation of any wave equation which can describe the propagation of linear perturbation embedded within the background accretion flow Kerr metric. In other words, it has not been possible to construct the acoustic metric corresponding to the flow with such disc height expression. One can not write down the wave equation (in the co-variant form) for a massless scalar field for perturbed flow described by such disc height expression. This seems to be a characteristic feature of the accretion in Kerr metric. The non-zero Kerr parameter introduces an additional term in the perturbation of height function which is dependent on linear perturbed radial velocity apart from the usual term which depends on the perturbation of density which arises in the height function as described in (13) and (15). This in turn causes the problem of inequality of the off-diagonal term when the index is interchanged. For accretion in the Schwarzschild metric, such issues do not arise as Kerr parameter becomes zero and the effective sound speed can be computed for accretion discs as described by height function in (16).

In our present work we thus present our calculation for accretion disc described by the height function due to Novikov & Thorne [2] only and denote it as \( H(r) \). Flow with the height function due to Riffert & Herold [3] will lead to the identical results, and flow with the thickness as provided by Abramowicz, Lanza & Percival [4] can not be handled in the Kerr metric for the present purpose. For axially symmetric accretion characterised by the disc height introduced by Novikov & Thorne [2], the critical points and the sonic points co-inside for the isothermal flow in Schwarzschild as well as in the Kerr metric. In the present work, we thus concentrate on the polytropic accretion only.
3 The Critical Point Analysis

To find the critical points of the accretion flow, we have to find the expression of the gradient of the advective velocity $u_0$, i.e., the expression of $du_0/dr$ for stationary accretion flow. In order to do that, we need two first integrals of motion for the stationary flow. The first one comes from the continuity equation and the second one comes from the momentum conservation equation. It is convenient to do a vertical averaging of the flow equations by integrating over $\theta$ and the resultant equation is described by the flow variables defined on the equatorial plane ($\theta = \pi/2$). In addition one also integrates over $\phi$ which gives a factor of $2\pi$ due to the axial symmetry of the flow. We do such vertical averaging as prescribed in to the continuity equation given by Eq. (10). Thus in case of stationary ($t$-independent) and axially symmetric ($\phi$-independent) flow with averaged $\nu^\theta \sim 0$, the continuity equation can be written as

$$\frac{\partial}{\partial r}(4\pi H_\theta \sqrt{-g} \rho_0 \nu^0_0) = 0 \quad (17)$$

$H_\theta$ arises due to the vertical averaging and is the local angular scale of flow. Thus one can relate the actual local flow thickness $H(r)$ to the angular scale of the flow $H_\theta$ as $H_\theta = H(r)/r$, where $r$ is the radial distance along the equatorial plane from the centre of the disc. $\tilde{g}$ is the value of the determinant of the metric $g_{\mu\nu}$ on the equatorial plane, $\tilde{g} = \det(g_{\mu\nu})|_{\theta = \pi/2} = -r^4$. The equation (17) gives the mass accretion rate $\dot{M}_0$ as

$$\dot{M}_0 = 4\pi \sqrt{-\tilde{g}} H_\theta \rho_0 \nu^0_0 = 4\pi H(r) r \rho_0 \nu^0_0. \quad (18)$$

The $r$ component of the four velocity, $\nu^r$, can be expressed in terms of $u_0$

$$\nu^r_0 = \frac{u_0}{\sqrt{g_{rr}(1 - u^2_0)}} = \frac{\sqrt{\Delta} u_0}{r \sqrt{1 - u^2_0}} \quad (19)$$

using $g_{rr} = r^2/\Delta$, with $\Delta = r(r - 2)$. Thus $\dot{M}_0$ can be written as

$$\dot{M}_0 = 4\pi H(r) \Delta^{1/2} \rho_0 \frac{u_0}{\sqrt{1 - u^2_0}} = \text{constant} \quad (20)$$

For adiabatic flow, one defines a new quantity $\dot{\Xi}$ from $\dot{M}_0$ by multiplying it with $(\gamma k)^{1/2}$. $\dot{\Xi}$ is a measure of entropy accretion rate and typically called as the entropy accretion rate. The concept of the entropy accretion rate is widely used in accretion astrophysics. The entropy accretion rate was first defined in the literature in [8, 9]. Expressing $\rho_0$ in terms of $\gamma, k$ and $c_{s0}$ gives

$$\dot{\Xi} = \left( \frac{(\gamma - 1)c_{s0}^2}{\gamma - 1 - c_{s0}^2} \right)^n \frac{4\pi H(r) \Delta^{1/2} u_0}{\sqrt{1 - u^2_0}} = \text{constant} \quad (21)$$

The second conserved quantity can be obtained from the time-component of the steady state relativistic Euler equation (12) which for stationary adiabatic case gives

$$\mathcal{E}_0 = h_0 \nu_{t0} = \text{constant}. \quad (22)$$

$\nu_{t0}$ can be further expressed in terms of $u_0$ as

$$\nu_{t0} = \sqrt{\frac{\Delta}{B(1 - u^2_0)}} \quad (23)$$

where $B = g_{\phi\phi} + 2\lambda_0 g_{tt\phi} - \lambda_0^2 g_{tt}$ and $\lambda_0 = -v_{\phi0}/v_{t0}$. Here $v_{\phi0}$ is the azimuthal component of covariant four-velocity. Thus

$$\mathcal{E}_0 = \frac{\gamma - 1}{\gamma - 1 - n c_{s0}^2} \sqrt{\frac{\Delta}{B(1 - u^2_0)}} \quad (24)$$

For adiabatic flow, the expression for $du_0/dr$ can be derived by using the expression of the two quantities, $\dot{\Xi}$ and $\mathcal{E}_0$ given by equation (21) and (24), respectively. Taking logarithmic derivative of both sides of equation (24) gives the gradient of sound speed as

$$\frac{dc_{s0}}{dr} = -\frac{\gamma - 1 - c_{s0}^2}{2c_{s0}} \left[ \frac{u_0}{1 - u_0^2} \frac{du_0}{dr} + \frac{1}{2} \left( \frac{\Delta'}{\Delta} - \frac{B'}{B} \right) \right] \quad (25)$$
In order to obtain the expression for the entropy density in terms of $u_0$ and $c_{s0}$ only, we need to write the expression of height in terms of $u_0$ and $c_{s0}$ also. For this we note that, for adiabatic equation of state, $\frac{p_0}{\rho_0}$ can be written as

$$\frac{p_0}{\rho_0} = \left(\frac{1}{\gamma}\right) \frac{(\gamma - 1)c_{s0}^2}{\gamma - 1 - nc_{s0}^2}$$

Thus we can write $H(r)$ from Eq. (13) as

$$H(r) = \left(\frac{1}{\gamma}\right)^{1/2} \frac{(\gamma - 1)c_{s0}^2}{\gamma - 1 - nc_{s0}^2} \left(\frac{1}{2}\right)^{1/2} f(r, a)$$

where

$$f(r, a) = \frac{r^3 + a^2 r + 2a^2}{r^2 + a} \sqrt{\frac{r^6 - 3r^5 + 2ar^2}{(r^2 - 2r + a^2)(r^4 + 4a^2r^2 - 4a^2r + 3a^4)}}.$$

Using the expression of $H(r)$, $\dot{\Xi}$ for this model can be written as

$$\dot{\Xi} = \sqrt{\frac{1}{\gamma}} \left(\frac{(\gamma - 1)c_{s0}^2}{\gamma - 1 - nc_{s0}^2}\right)^{\gamma+1\gamma} \frac{4\pi \Delta^{1/2}}{\sqrt{1 - u_0^2}} f(r, a)$$

Taking logarithmic derivative of both sides of the above equation and substituting $dc_{s0}/dr$ using Eq. (25) gives

$$\frac{du_0}{dr} = \frac{u_0(1 - u_0^2)}{u_0^2 - \frac{c_{s0}^2}{(\gamma + 1)^2}} = \frac{N}{D}$$

### 3.1 Obtaining Critical Points:

The critical points are obtained from the condition $D = 0$ which gives $u_0^2|_c = c_{s0}^2|_c/(\gamma + 1)$ or

$$u_0^2|_c = \frac{c_{s0}^2|_c}{1 + \beta}, \quad \text{where} \quad \beta = \frac{\gamma - 1}{2}.$$ (31)

Hereafter any quantity with a suffix $c$ will denote it’s value evaluated at the critical point. The other condition at critical point is $N = 0$, which gives

$$c_{s0}^2|_c = \left(\frac{1}{4}\right) \frac{B'}{B} - \frac{\Delta'}{\Delta}.$$ (32)

To obtain the critical points, the critical point condition (31) is used in (24), which gives

$$\mathcal{E}_0 = \frac{\gamma - 1}{\gamma - 1 - (c_{s0}^2)_c} \sqrt{\frac{(\gamma + 1)\Delta}{B(\gamma + 1 - (c_{s0}^2)_c)}}$$

where $(c_{s0}^2)_c$ is obtained as a function of $r$ and Kerr parameter $a$ from (32). The number of solution of this equation thus provide the number of critical points which can be more than one.

Now one obtains the value of $\frac{du_0}{dr}$ at critical point by using L’Hospital’s rule in (30) as both the numerator and denominator tends to zero at critical point. Then one obtains a quadratic equation of the form

$$\alpha_1 \left(\frac{du_0}{dr}\right)^2 - \alpha_2 \left(\frac{du_0}{dr}\right) - \alpha_3 = 0$$

(34)
$\alpha_1 = 2(u_0)c \left[ 1 - \frac{(c_{s0}^2)c - \gamma - 1}{(\gamma + 1)(1 - (u_0^2)c)} \right]$ \hfill (35)

$\alpha_2 = \frac{(c_{s0}^2)c - \gamma - 1}{(\gamma + 1)} \left[ \frac{\Delta'}{\Delta} - \frac{B'}{B} + \left( \frac{\Delta'}{\Delta} + \frac{2f_{NT}}{f_{NT}} \right) (u_0^2)c \right] + \left[ \frac{2}{\gamma + 1} (c_{s0}^2)c \left( \frac{\Delta'}{2\Delta} + \frac{f_{NT}'}{f_{NT}} \right) + \frac{1}{2} \left( \frac{B'}{B} - \frac{\Delta'}{\Delta} \right) \right] (1 - 3(u_0^2)c)$ \hfill (36)

$\alpha_3 = (u_0)c (1 - (u_0^2)c) \left[ \frac{1}{2} \alpha'^2 - \frac{\alpha''}{\gamma + 1} + \frac{2(c_{s0}^2)c}{\gamma + 1} \left( \frac{\Delta''}{2\Delta} + \frac{f_{NT}''}{f_{NT}} - \frac{1}{2} \frac{\Delta'}{\Delta} - \frac{f'^2}{f} \right) \right] + \left[ \frac{(c_{s0}^2)c - \gamma - 1}{(\gamma + 1)} \left( \frac{\Delta'}{2\Delta} + \frac{f_{NT}'}{f_{NT}} \right) \left( \frac{\Delta'}{\Delta} - \frac{B'}{B} \right) \right]$ \hfill (37)

by solving (34) we obtain two slopes at critical point as two roots of quadratic equation. Thus a particular set of values of $[\xi_0, \lambda, a, \gamma]$ the phase portrait of $u$ vs. $r$ can be plotted. Instead of $u$, we use Mach number, $M = \frac{u}{c_s}$. For a particular set of parameter $c_{s0}$ can be obtained by evaluating $u_0$. $u_0$ is obtained by integrating (30) and putting in (24). Below we demonstrate the phase portrait where multiple critical points are present and explain how multi-transonicity can be achieved if shock is present.

In this system three critical points are present for the parameter value of $a = 0.40, \xi_0 = 1.0045, \gamma = 1.35, \lambda = 3.0$, where the outermost ($C_{out}$) and innermost ($C_{in}$) critical points are saddle-type in nature, whereas, the middle one is center type in nature. Integrating (30) by using (32) and (31) at critical point (obtained by solving (33)) we draw the phase portrait of Mach number ($M$) vs radial distance ($r$). If we consider a fluid element in the disc accretion flow in hydrostatic equilibrium that starts far away from the gravitating body, in this case, the black hole, along the critical flow line then it will reach point $A$, move through the outer critical point $C_{out}$, where the Mach number is less than 1, and then passes through the sonic point $M_1$. Now, it will continue along the flow-line (the blue line in the online version of the figure) for all future instants, and will continue to be supersonic, if the flow variables and thermodynamic variables do not make a discontinuous jump to another branch (the green line in the online version of the figure) as a consequence of some mechanism. The mechanism by which this discontinuous jump happens is the presence of a infinitesimally thin shock, which we discuss next. For this moment we consider that the discontinuous jump occurs at $S_1$ in the blue flow line and reaches $S_2$ in the green flow line, for which the flow becomes subsonic again, then the fluid element eventually goes through $C_{in}$ and then $M_2$, where the flow again becomes supersonic. In this way the presence of shock makes the fluid element move through a flow line which is multi transonic and a subset of the whole critical flow lines.

### 4 Shock Formation in Critical Flow:

We have assumed a non-dissipative inviscid accretion flow. Therefore the flow has conserved specific energy and mass accretion rate. Thus the shock produced in such flow is assumed to be energy preserving Rankine Hugonoit type which satisfies the general relativistic Rankine Hugonoit conditions-

\[ [[\rho v^\mu n_\mu]] = [[\rho v^\tau]] = 0 \]
\[ [[T_{\mu\nu} n^\mu n^\nu]] = [[(p + \varepsilon) v^\tau v^\tau]] = 0 \]
\[ [[T_{\mu\nu} n^\mu n^\nu]] = [[(p + \varepsilon) (v^\rho v^\rho)]] = 0 \] \hfill (38)

Where $n_\mu = \delta^\mu_r$ is normal to the surface of shock formation. $[[f]]$ is defined as $[[f]] = f_+ - f_-$, where $f_+$ and $f_-$ are values of $f$ after and before the shock, respectively. The first condition comes from the conservation of mass accretion rate and the last two conditions come from the energy-momentum conservation. These
conditions are to be satisfied at the location of shock formation. In order to find out the location of shock formation, it is convenient to construct a shock invariant quantity, which depends only on \( u_0, c_{s0} \) and \( \gamma \), using the conditions above. The first and second conditions are trivially satisfied owing to the constancy of the mass accretion rate and the specific energy. The first condition is basically \((\dot{M}_0)_+ = (\dot{M}_0)_-\) and third condition is \((T^r)_+ = (T^r)_-\). Thus we can define a shock invariant quantity \( S_{sh} \) given by

\[
S_{sh} = \frac{T^r}{\dot{M}_0}
\]  

which also satisfies \([S_{sh}] = 0\). To calculate the shock invariant quantity we note that \( h_0 \) corresponds to the enthalpy of the stationary solutions of the steady state flow, given by equation Eq. (8). \( c_{s0} = (1/h_0)dp/d\rho = (1/h_0)k\gamma\rho^{-1} \), which gives \( \rho_0 \) (and hence also \( p \) and \( \epsilon \)) in terms of \( k, \gamma \) and \( c_{s0} \). Thus

\[
\rho = k^{-\frac{\gamma -1}{\gamma}} \left[ \frac{(\gamma -1) c_{s0}^2}{\gamma (\gamma -1 - c_{s0}^2)} \right]^{\frac{1}{\gamma -1}}
\]

\[
p = k^{-\frac{\gamma -1}{\gamma}} \left[ \frac{(\gamma -1) c_{s0}^2}{\gamma (\gamma -1 - c_{s0}^2)} \right]^{\frac{1}{\gamma -1}}
\]

\[
\epsilon = k^{-\frac{\gamma -1}{\gamma -1}} \left[ \frac{(\gamma -1) c_{s0}^2}{\gamma (\gamma -1 - c_{s0}^2)} \right]^{\frac{1}{\gamma -1}} \left( 1 + \frac{c_{s0}^2}{\gamma (\gamma -1 - c_{s0}^2)} \right)
\]  

Figure 1: Mach Number (\( M \)) vs radial distance (\( r \)) for \( \xi_0 = 1.0012, \lambda = 3.30, a = 0.30 \) and \( \gamma = 1.35\). \( S_1, S_2 \) are the shock positions. \( C_{out} \) is the outer critical point and \( C_{in} \) is the inner critical point. \( M_1, M_2 \) are the sonic points or the points where Mach number becomes 1.
Now \( \dot{M}_0 = \text{constant} \times r^2 \rho v_0^r \) and \( T^{rr} = (p + \varepsilon)(v_0^r)^2 + pg^{rr} \), where \( v_0^r = u_0/\sqrt{g_{rr}(1 - u_0^2)} \). Therefore the shock-invariant quantity \( S_{sh} = T^{rr}/\dot{M}_0 \) becomes

\[
S_{sh} = \frac{(u_0^2(\gamma - c_{s0}^2) + c_{s0}^2)}{u_0\sqrt{1 - u_0^2(\gamma - 1 - c_{s0}^2)}}
\]

(41)

where we have removed any overall factor of \( r \) as shock invariant quantity is to be evaluated at constant \( r = r_{sh} \).

5 Acoustic spacetime metric

In this section, we find out the acoustic spacetime metric by perturbing the equations describing the accretion flow and keeping only linear terms. Following standard linear perturbation analysis [5–7, 10], we write the time-dependent accretion variables, like the components of four velocity and pressure, as small time-dependent linear perturbations around their stationary values. Hence,

\[
\begin{align*}
  v^t(r, t) &= v_0^t(r) + v_1^t(r, t) \\
v^r(r, t) &= v_0^r(r) + v_1^r(r, t) \\
\rho(r, t) &= \rho_0(r) + \rho_1(r, t)
\end{align*}
\]

(42)

where the subscript ‘1’ denotes the small perturbations of some variable about the stationary value denoted by subscript ‘0’. Now we introduce \( \Psi = 4\pi \sqrt{-\tilde{g} \rho(r, t)v^r(r, t)H_{\theta}} \) which is the stationary mass accretion rate of the accretion flow. Thus

\[
\Psi(r, t) = \Psi_0 + \Psi_1(r, t)
\]

(43)

Where \( \Psi_0 \) is the stationary mass accretion rate defined in equation (18). We can redefine the mass accretion rate \( \Psi \) as \( \Psi = \sqrt{-\tilde{g} \rho(r, t)v^r(r, t)H_{\theta}} \) by absorbing the factor \( 4\pi \), without any loss of generality. Using the equations (42) we obtain

\[
\Psi_1 = \sqrt{-\tilde{g}}[\rho_1 v_0^r H_{\theta0} + \rho_0 v_1^r H_{\theta0} + \rho_0 v_0^r H_{\theta1}]
\]

(44)

We see that the perturbation \( \Psi_1 \) consists of a term which is the perturbation of angular height function \( H_{\theta} \). We recall \( H(r) \) as \( H_{\theta} = H(r)/r \).

For adiabatic flow, the pressure is given by the equation of state \( p = k \rho^\gamma \). Thus equation (8) can be rewritten as

\[
h = 1 + \frac{\gamma - 1}{\gamma - 1} \rho
\]

(45)

where the corresponding perturbation \( h_1 \) can be written as

\[
h_1 = \frac{h_0 c_{s0}^2}{\rho_0} \rho_1
\]

(46)

For adiabatic flow the irrotationality condition turns out to be [11]

\[
\partial_\mu (hv_\nu) - \partial_\nu (hv_\mu) = 0
\]

(47)

We yield some important relations next using (Eq. (47)), the normalization condition \( v^\mu v_\mu = -1 \) and the axis symmetry of the flow which are needed to derive the governing wave equation of the linear perturbation. From irrotationality condition Eq. (47) with \( \mu = t \) and \( \nu = \phi \) and with axis symmetric condition we have

\[
\partial_t (hv_\phi) = 0,
\]

(48)

further, for \( \mu = r \) and \( \nu = \phi \) and the axial symmetry, we have

\[
\partial_r (hv_\phi) = 0.
\]

(49)
Thus we yield that $h v_{\phi}$ is a constant of motion. Eq. (48) gives

$$\partial_{t} v_{\phi} = -\frac{v_{\phi} c_{s}^{2}}{\rho} \partial_{t} \rho.$$  

(50)

Substituting $v_{\phi} = g_{\phi \phi} v_{\phi} + g_{\phi t} v^{t}$ in the equation derived above gives

$$\partial_{t} v^{\phi} = -\frac{g_{\phi t}}{g_{\phi \phi}} \partial_{t} v^{t} - \frac{v_{\phi} c_{s}^{2}}{g_{\phi \phi}} \partial_{t} \rho.$$  

(51)

The previously mentioned normalization condition $v_{\mu} v_{\mu} = -1$ provides

$$g_{tt}(v^{t})^{2} = 1 + g_{rr}(v^{r})^{2} + g_{\phi \phi}(v^{\phi})^{2} + 2 g_{\phi t} v^{\phi} v^{t}$$  

(52)

which after taking derivative with respect to $t$ gives

$$\partial_{t} v^{t} = \alpha_{1} \partial_{t} v^{r} + \alpha_{2} \partial_{t} v^{\phi}$$  

(53)

where $\alpha_{1} = -\frac{g_{\phi t}}{v_{\phi}}$, $\alpha_{2} = -\frac{g_{\phi t}}{v_{\phi}}$, and $v_{t} = -g_{tt} v^{t} + g_{\phi t} v^{\phi}$. Substituting $\partial_{t} v^{\phi}$ in Eq. (53) using Eq. (51) gives

$$\partial_{t} v^{t} = \left(\frac{-\alpha_{2} v_{\phi} c_{s}^{2}/(g_{\phi \phi})}{1 + \alpha_{2} g_{\phi t}/g_{\phi \phi}}\right) \partial_{t} \rho + \left(\frac{\alpha_{1}}{1 + \alpha_{2} g_{\phi t}/g_{\phi \phi}}\right) \partial_{t} v^{r}$$  

(54)

Using Eq. (42) in Eq. (54) and keeping only the terms of first order in perturbed quantities we yield

$$\partial_{t} v_{1}^{t} = \eta_{1} \partial_{t} \rho_{1} + \eta_{2} \partial_{t} v_{1}^{r}$$  

(55)

where

$$\eta_{1} = -\frac{c_{s}^{2}}{A v_{0}^{2}} \left[\Lambda (v_{t}^{t})^{2} - 1 - g_{rr} (v_{t}^{r})^{2}\right], \quad \eta_{2} = \frac{g_{\phi t} v_{0}^{\phi}}{A v_{0}^{r}} \quad \text{and} \quad \Lambda = g_{tt} + \frac{g_{\phi t}^{2}}{g_{\phi \phi}}$$  

(56)

We now obtain $H_{\theta 1}$ in terms of perturbations previously introduced quantities as

$$
\frac{H_{\theta 1}}{H_{\theta 0}} = \left(\frac{\gamma - 1}{2}\right) \frac{\rho_{1}}{\rho_{0}} = \beta \frac{\rho_{1}}{\rho_{0}}
$$  

(57)

where $\beta = \frac{\gamma - 1}{2}$ has already been denoted in (31).

The continuity equation after taking vertical average takes the form

$$\partial_{t}(\sqrt{-g} p v^{t} H_{\theta}) + \partial_{r}(\sqrt{-g} p v^{r} H_{\theta}) = 0$$  

(58)

Using equation (42) and (43) in the previous equation and then using equation (55) and (57) and inserting them in (58) yields

$$-\frac{\partial_{r} \Psi_{1}}{\Psi_{0}} = \frac{\eta_{2}}{v_{0}^{2}} \partial_{t} v_{1}^{t} + v_{0}^{t} \left[1 + \beta + \frac{\eta_{1} \rho_{0}}{v_{0}^{t}}\right] \partial_{t} \rho_{1},$$  

(59)

and

$$\frac{\partial_{r} \Psi_{1}}{\Psi_{0}} = \frac{1}{v_{0}^{r}} \partial_{t} v_{1}^{r} + \frac{1 + \beta}{\rho_{0}} \partial_{t} \rho_{1}.$$  

(60)

With the two equations given by Eq. (59) and (60) we are able to write $\partial_{t} v_{1}^{t}$ and $\partial_{t} \rho_{1}$ only in terms of derivatives of $\Psi_{1}$ as

$$\partial_{t} v_{1}^{t} = \frac{1}{\sqrt{-g} H_{\theta} \rho_{0} \Lambda} \left[(v_{0}^{t}(1 + \beta) + \rho_{0} \eta_{1}) \partial_{t} \Psi_{1} + (1 + \beta) v_{0}^{t} \partial_{r} \Psi_{1}\right]$$  

(61)

$$\partial_{t} \rho_{1} = -\frac{1}{\sqrt{-g} H_{\theta} \rho_{0} \Lambda} \left[\rho_{0} \eta_{2} \partial_{t} \Psi_{1} + \rho_{0} \partial_{r} \Psi_{1}\right]$$

10
where $\hat{\Lambda}$ is given by
\[
\hat{\Lambda} = (1 + \beta) \left[ \frac{g_{rr}(v_0^r)^2}{\Lambda v_0^r} - v_0^t \right] + \frac{c_s^2}{\Lambda v_0^r} (\Lambda(v_0^t)^2 - 1 - g_{rr}(v_0^r)^2).
\] (62)

Now we go back to the irrotationality condition Eq. (47). Using $\mu = t$ and $\nu = r$ gives
\[
\partial_t (h g_{rr} v^r) - \partial_r (hv_t) = 0
\] (63)

For stationary flow this gives $\xi_0 = -h_0v_0 = \text{constant}$ which is the specific energy of the accreting system.

We now replace the density and velocities in Eq. (63) using Eq. (42), (46) and $\xi_0 = \text{constant}$. Finally replacing $\partial_t v^r_1$ and $\partial_t \rho_1$ in Eq. (68) using Eq. (61) we have
\[
\partial_t \left( \frac{g_{rr}}{v_0} \partial_t v^r_1 \right) + \partial_t \left( \frac{g_{rr} c_s^2 v_0^r}{\rho_0 v_0} \partial_t \rho_1 \right) - \partial_r \left( \frac{\hat{\eta}_1}{v_0} \partial_t v^r_1 \right) - \partial_r \left( \frac{\hat{\eta}_2}{v_0} + \frac{c_s^2}{\rho_0} \partial_t \rho_1 \right) = 0
\] (65)

Using Eq. (66) in the Eq. (65) and dividing that equation by $h_0 v_0$ yields
\[
\partial_t \left( \frac{g_{rr}}{v_0} \partial_t v^r_1 \right) + \partial_t \left( \frac{g_{rr} c_s^2 v_0^r}{\rho_0 v_0} \partial_t \rho_1 \right) - \partial_r \left( \frac{\hat{\eta}_1}{v_0} \partial_t v^r_1 \right) - \partial_r \left( \frac{\hat{\eta}_2}{v_0} + \frac{c_s^2}{\rho_0} \partial_t \rho_1 \right) = 0
\] (68)

where it has been used that $h_0 v_0 = \text{constant}$. Finally replacing $\partial_t v^r_1$ and $\partial_t \rho_1$ in Eq. (68) using Eq. (61) we have
\[
\partial_t \left[ k(r) \left( -g^{tt} + (v_0^t)^2 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \right) \right] + \partial_t \left[ k(r) \left( v_0^r v_0^t \left( 1 - \frac{1 + \beta}{c_s^2} \right) \right) \right] + \partial_r \left[ k(r) \left( g^{rr} + (v_0^r)^2 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \right) \right] = 0
\] (69)

where $k(r)$ is a conformal factor whose exact form is not required for the present analysis and
\[
g^{tt} = \frac{1}{\hat{\Lambda}} = \frac{1}{g_{tt} + g_{sr}^2 / g_{\phi \phi}}
\] (70)

Eq. (69) can be written as
\[
\partial_\mu (F^{\mu \nu} \partial_\nu \Psi_1) = 0
\] (71)

where $F^{\mu \nu}$ is obtained from the symmetric matrix
\[
f^{\mu \nu} = k(r) \left[ \begin{array}{cc} -g^{tt} + (v_0^t)^2 \left( 1 - \frac{1 + \beta}{c_s^2} \right) & v_0^r v_0^t \left( 1 - \frac{1 + \beta}{c_s^2} \right) \\ v_0^r v_0^t \left( 1 - \frac{1 + \beta}{c_s^2} \right) & g^{rr} + (v_0^r)^2 \left( 1 - \frac{1 + \beta}{c_s^2} \right) \end{array} \right]
\] (72)

The equation (71) how the perturbation $\Psi_1$ propagates in $1 + 1$ dimension effectively, where, $x^\mu = (t, r)$. Equation (71) represents the wave equation of a massless scalar field $\varphi$ in curved spacetime (with metric $g^{\mu \nu}$) given by
\[
\partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \varphi) = 0
\] (73)
where \( g \) is the determinant of the metric \( g_{\mu \nu} \). Comparing equation (71) and (73) one obtains the acoustic spacetime \( G^{\mu \nu} \) metric as
\[
\sqrt{-G} G^{\mu \nu} = f^{\mu \nu}
\]
where \( G \) is the determinant of \( G_{\mu \nu} \). Thus the acoustic metric \( G_{\mu \nu} \) would be given by
\[
G_{\mu \nu} = k_1(r) \left[ -g^{rr} - \frac{1 + \beta}{c_s^0} \left( v^r_0 \right)^2 \right. \\
\left. - \frac{1 - \frac{\beta}{c_s^0}}{1 + \frac{\beta}{c_s^0}} \left( v^r_0 \right)^2 \right. \\
\left. - g^{tt} \left( 1 - \frac{\beta}{c_s^0} \right) \left( v^t_0 \right)^2 \right]
\]
where \( k_1(r) \) is another conformal factor arising due to the process of inverting \( G^{\mu \nu} \) in order to yield \( G_{\mu \nu} \). For our present purpose we do not need the exact expression for \( k_1(r) \). As the effective metric is 1+1 dimensional, we only need the velocity components \( v_0 \), \( v_t \) and thus the metric components which are needed are \( g_{rr} \) and \( g_{tt} \).

### 6 Effective speed of acoustic perturbation

The acoustic metric for general relativistic axially symmetric disc in Kerr spacetime has been derived in the previous section for adiabatic flow which is given by equation (75). From the acoustic metric given by (75), the location of the acoustic horizon can be found. Just like the black hole event horizon in presence of matter-energy source, the acoustic horizon can be defined as a null surface with respect to the acoustic metric which is a one-way hypersurface for the acoustic perturbation. It means that the acoustic perturbations cannot propagate outside the acoustic horizon once it has reached in the closed hypersurface. In this particular case the coordinates have been chosen such that if we decrease \( r \) from infinity, then the \( r = \) constant hypersurfaces continue to be timelike once it reaches some fixed \( r = r_H \), which is null hypersurface. Thus the condition that the surface \( r = r_H \) constant is null with respect to the metric \( G_{\mu \nu} \) is given by
\[
G_{\mu \nu} n^\mu n^\nu = 0
\]
where \( n^\mu = (\partial_\alpha r)^\mu = \delta^\mu_\alpha \) is the normal to the surface \( r = r_H \) constant. Thus \( G^{rr} = 0 \) gives the location of the sonic horizon. Therefore, using the equation (19), the condition for the presence of the sonic or acoustic horizon is given by \( u_0^2 = c_s^2 / (1 + \beta) \). However the acoustic horizon is basically the transonic surface which thus requires that the effective speed of the acoustic perturbation is defined as \( c_{eff} = c_s / \sqrt{1 + \beta} \). Therefore, the critical point condition becomes \( u_0^2 = c_{eff}^2 \). Therefore, the fact that the critical points coincide with the acoustic horizon further implies that the critical points are the transonic points with effective sound speed given by
\[
c_{eff} = c_s / \sqrt{1 + \beta}.
\]
Hence, the apparent non homomorphism of the critical point and sonic point is resolved if the static sound speed \( c_s \) is abandoned from the analysis and effective speed of sound \( c_{eff} \) is used as the speed of propagation of acoustic perturbation and define the Mach number as the ratio of the dynamical bulk velocity \( u_0 \) and the effective sound speed \( c_{eff} \). Then we have the critical point and the sonic point (where the Mach number is 1) to be the same.

### 7 Spin dependence of effective sound speed at critical points

From (24), we can calculate \( \mathcal{E}_0 \) at the critical point using (31) and obtain-
\[
\mathcal{E}_0 = \frac{1}{1 - n \left( (c_s) \right)^2} \left( \frac{\Delta_0}{\Delta} \right) \sqrt{f(1 - [(u_0)_o]^2)}
\]
where
\[
[(c_s) \right)^2 = \left( 1 + \frac{1}{2 \eta} \right) \left( \frac{\Delta_0^\prime - \Delta_0^\prime}{\Delta_0^\prime + \frac{f_{NT}^2}{2 f_{NT}^2}} \right) = \left( 1 + \beta \right) \left( \frac{\Delta_0^\prime - \Delta_0^\prime}{\Delta_0 + \frac{f_{NT}^2}{2 f_{NT}^2}} \right).
\]
Now the effective sound speed at critical points is
\[
(c_{\text{eff}})_{c} = \left(\frac{c_{s0}}{\sqrt{1 + \beta}}\right) = \sqrt{\left(\frac{\beta'}{\beta} - \frac{\Delta'}{\Delta} \right) \left(\frac{\Delta'}{\Delta} + \frac{2f_{NT}}{f_{NT}}\right)}.
\]  

(80)

To obtain how the effective sound speed depends on spin at critical point, one have to obtain the multi critical solution by solving (78). Solving (78) numerically for \(E_{0} = 1.0045\), \(\lambda = 3.0\) and \(\gamma = 1.35\), and evaluating effective sound speed at the inner and outer critical point which are both saddle type critical points, we get the following dependence of critical point position \((r_{c})\) and effective sound speed at critical point \((c_{\text{eff}})\) with Kerr parameter\((a)\) as shown in the following figure.

![Graphs showing dependence of critical point position and effective sound speed on spin parameter](image)

(a) Inner critical point \((r_{in})\)  
(b) Outer critical point \((r_{out})\)

(c) Effective sound speed at inner critical point \((c_{\text{eff}}^{in})\)  
(d) Effective sound speed at outer critical point \((c_{\text{eff}}^{out})\)

Figure 2: Spin parameter \((a)\) dependence of location of critical points and effective sound speed at critical points for \(E_{0} = 1.0045\), \(\lambda = 3.0\) and \(\gamma = 1.35\)

From Fig. (2) it is seen that the location of critical point and also the effective sound speed depends on Kerr parameter strongly at the inner critical point, whereas the dependency is weak at the outer critical point. This happens for multi-transonic accretion because Kerr space-time is asymptotically flat and thus the effect of gravity on the matter in the flow is weaker as the radial distance from the gravitating body, in this case, the black hole, is increased. There is also another feature of these plots worth noting, which is
that the dependencies are not shown for the complete range of Kerr parameter, $a$, i.e., for the whole rage of $a \in [-1, +1]$. This happens as the multi transonicity occurs in a finite subspace of parameter space. But the dependence of radial distance of critical point and the effective sound speed at critical point can be studied for the complete range of Kerr parameter for mono transonic cases. We thus present below the dependence for a set of parameters where the dependence is present for complete range of Kerr parameter.

Figure 3: Spin parameter ($a$) dependence of (a)Radius of critical surface (b)effective sound speed at the critical point for $\mathcal{E}_0 = 1.0045$, $\lambda = 3.0$ and $\gamma = 1.35$ in monotransonic case

8 Causal Structure and Penrose Diagram of The Acoustic Metric

In section (6) we have used a simple argument in order to determine the acoustic horizon. One must formally analyze the causal structure in order to find the location of horizon and also to understand the nature of acoustic horizon as they may represent physically different hypersurfaces with particular characteristics. In this section we study the causal structure by using null-coordinates and also constructing the Penrose diagram of the acoustic metric.

8.1 Acoustic Metric and Other Preliminaries

From the obtained acoustic metric the line element is given by

$$ds^2 = G_{tt}dt^2 + 2G_{tr}dtdr + G_{rr}dr^2.$$  \hspace{1cm} (81)

where, the metric elements $G_{\mu\nu}$ are given by (75). The overall factor $k_1(r)$ is ignored as we want to finally focus on conformally transformed metric where this factor can be absorbed with the conformal factor.

To understand where coordinate singularity occurs we must rewrite all metric elements in terms of $u_0$ and $c_{\text{eff}}$, where $c_{\text{eff}}$ is defined in (77). In order to do that we have to write down $\nu^r, \nu^t$ in terms of $u_0$. Thus we have

$$\nu_0^r = \frac{u}{\sqrt{g_{rr}(1-u^2)}}$$  \hspace{1cm} (82)

and

$$\nu_0^t = \sqrt{\frac{(g_{\phi\phi} + \lambda g_{\phi t})^2}{(g_{\phi\phi} + 2\lambda g_{\phi t} - \lambda^2 g_{tt})(g_{\phi\phi} g_{tt} + g_{\phi t}^2)(1-u^2)}}$$  \hspace{1cm} (83)
where \( \lambda = -\frac{v_0}{u_0} \) is the specific angular momentum of the fluid. By using (82) and (83) in (75) we obtain the metric elements of acoustic metrics to be

\[
G_{tt} = \frac{u_0^2 - c_{\text{eff}}^2}{c_{\text{eff}}^2(1 - u_0^2)g_{rr}} \\
G_{tr} = G_{rt} = \frac{u_0(1 - c_{\text{eff}}^2)F_1(r, \lambda)}{c_{\text{eff}}^2(1 - u_0^2)} \\
G_{rr} = g_{rr} \left( \frac{r^2}{c_{\text{eff}}^2(1 - u_0^2)} - F_2(r) \right) \tag{84}
\]

where

\[
F_1(r, \lambda) = \frac{g_{\phi\phi} + \lambda g_{\theta\theta}}{\sqrt{(g_{\phi\phi} + 2\lambda g_{\theta\theta} - \lambda^2 g_{tt})(g_{\phi\phi} g_{tt} + g_{\phi\theta}^2)g_{rr}}} \\
F_2(r) = \frac{g_{\phi\phi}}{g_{\phi\phi} g_{tt} + g_{\theta\theta}^2} \tag{85}
\]

Thus we see that at critical point where \( u_0^2 = c_{\text{eff}}^2 \), the metric element \( G_{tt} \) becomes zero. So we have to transform coordinate in such a way that this coordinate singularity is removed. In the next section we go through a systematic procedure such that the singularity is removed from new metric elements. Furthermore we will conformally transform the final coordinates such that the infinite patch is mapped into a finite region of some coordinate space. This is helpful to study the causal structure of the acoustic metric by inspecting the Penrose-Carter Diagram.

### 8.2 Kruskal like Co-ordinate Transformation to Remove Singularity at Critical Point

First we choose null coordinates to write down the line element (81). We note that for null or lightlike curves \( ds^2 = 0 \), which implies

\[
(dt - A_+(r)dr)(dt - A_-(r)dr) = 0 \tag{86}
\]

where,

\[
A_\pm = -G_{tr} \pm \sqrt{G_{rr}^2 - G_{tt}G_{tr}} \tag{87}
\]

So instead of co-ordinates \( (t, r) \), we choose new co-ordinates to be null co-ordinates \( (\chi, \omega) \) such that

\[
d\omega = dt - A_+(r) \tag{88} \\
d\chi = dt - A_-(r) \tag{89}
\]

Thus the line element (81) becomes

\[
ds^2 = G_{tt}d\chi d\omega \tag{90}
\]

In order to remove the singularity of \( G_{tt} \) at the critical point by appropriate co-ordinate transformation one must study the behaviour of \( A_-(r) \) nd \( A_+(r) \) near critical points so that approximate analytical form of \( \chi \) and \( \omega \) can be examined in the vicinity of critical points. To study this behaviour we have to expand \( A_-(r) \) and \( A_+(r) \) upto first order of \( (r - r_c) \). Thus we expand \( u_0 \) near \( r_c \) as

\[
u_0(r) = -c_{\text{eff}}(r_c) + \left| \frac{du}{dr} \right|_{r_c} (r - r_c) + O \left( (r - r_c)^2 \right) \tag{91}
\]
where the negative sign of the effective sound speed implies the flow is towards the origin. We also note that from (91), near $r_c$ we can write

$$u_0^2 - c_{\text{eff}}^2 \approx -2 \left( c_{\text{eff}} \frac{du}{dr} \right)_{r_c} (r - r_c)$$

(92)

considering up to the first order term.

Now we expand $A_-(r)$ and $A_+(r)$ up to linear order of $(r - r_c)$. For that we first note that $G_{tt} \propto (u_0 - c_{\text{eff}})^2$ is very small near $r_c$ which implies $|\frac{G_u G_{\tau\tau}}{G_{tt}^2}| \ll 1$. Thus we obtain

$$A_+(r) = \frac{G_{\tau\tau} + G_{tr} \left(1 - \frac{G_u G_{\tau\tau}}{G_{tt}}\right)^{1/2}}{G_{tt}}$$

(93)

$$\approx -\frac{G_{tr}}{2G_{tt}}$$

(94)

and

$$A_-(r) = \frac{G_{\tau\tau} - G_{tr} \left(1 - \frac{G_u G_{\tau\tau}}{G_{tt}}\right)^{1/2}}{G_{tt}}$$

(95)

$$\approx -\frac{2G_{tr}}{G_{tt}}$$

(96)

$$= 2F(r, \lambda)\frac{u_0(c_{\text{eff}}^2 - 1)}{u_0^2 - c_{\text{eff}}^2}$$

(97)

$$\approx -\frac{F(r_c, \lambda)(u_{0c}^2 - 1)}{u_{0c}} \frac{1}{r - r_c}$$

(98)

$$= \frac{1}{\kappa} \frac{1}{r - r_c}$$

(99)

where

$$\kappa = \frac{u_{0c}^2}{F(r_c, \lambda)(u_{0c}^2 - 1)}.$$ (100)

So, we see that although

$$\chi \approx t - \frac{1}{\kappa} \ln |r - r_c|$$ (101)

shows a logarithmic divergence at $r \to r_c$, the form of $G_{rr}$ and $G_{tr}$ ensures that

$$\omega = t + \int \frac{G_{rr}}{2G_{tr}} dr$$ (102)

does not diverge at the critical point as the function inside integral is regular there. Thus we see that near critical point

$$e^{-\kappa \chi} \propto e^{-\kappa t} |r - r_c| \propto e^{-\kappa t}(u_0^2 - c_{\text{eff}}^2)$$ (103)

Now one can compare the null co-ordinates in this case with that of the Schwarzchild metric and guess a co-ordinate transformation such that the singularity of metric element at critical point is removed. The transformation equations can be given by

$$U(\chi) = -e^{-\kappa \chi}$$

$$W(\omega) = e^{\kappa \omega}$$ (104)
Using this new set of co-ordinates \((U, W)\), the line element can now be written as

\[
ds^2 = G_{tt} \frac{e^{\kappa (\chi - \omega)} (u_0^2 - c_{\text{eff}}^2)}{\kappa^2 c_{\text{eff}}^2 (1 - u_0^2)(1 - 2/r + a/r^2)^{-1}} dU dW. \tag{105}
\]

Examining the numerator of the new metric element, we see that the two factors multiplied together will cancel the divergence at critical points. Thus these new co-ordinates \((U, W)\) are similar to the Kruskal co-ordinates for the case of Schwarzchild metric which removes the co-ordinate singularity.

Now we have just one more transformation to compactify the infinite space into finite patch of some coordinates. For that the co-ordinates will be \((T, R)\) such that

\[
T = \frac{\tan^{-1}(W) + \tan^{-1}(U)}{2} \tag{106}
\]

\[
R = \frac{\tan^{-1}(W) - \tan^{-1}(U)}{2} \tag{107}
\]

Figure 4: \(\chi = \text{constant}\) lines are denoted by the blue lines and \(\omega = \text{constant}\) lines are denoted by the green lines in \((t, r)\) plane for \(\varepsilon_0 = 1.0045, \lambda = 3.0, a = 0.40\) and \(\gamma = 1.35\).

We have drawn \(\chi = \text{constant}\) and \(\omega = \text{constant}\) in \((t, r)\) plane in Figure(4). We can also draw \(r(\chi, \omega) = \text{constant}\) and \(t(\chi, \omega) = \text{constant}\) in \((\chi, \omega)\) plane. In order to draw the \(r = \text{constant}\) and \(t = \text{constant}\) in \((T, R)\)
plane from the previous data set we have to draw

\[ r \left( \frac{1}{\kappa} [\tan (-T - R)], \frac{1}{\kappa} [\tan (T - R)] \right) = \text{constant} \quad (108) \]

\[ t \left( \frac{1}{\kappa} [\tan (-T - R)], \frac{1}{\kappa} [\tan (T - R)] \right) = \text{constant} \quad (109) \]

\[ (110) \]

We must restrict our conformal diagram by \( r_T = \text{constant} \) because only up to that boundary The disc height prescribed by Novikov-Thorne is valid.

9 Concluding remarks

For accretion flow in hydrostatic equilibrium along the vertical direction, the Mach number does not become 1 at critical points and thus the critical points and the sonic points are non-isomorphic. This happens because, for such disc model, the flow thickness depends upon the sound speed. The deviation of Mach number from 1 is always observed for polytropic accretion because the sound speed depends upon polytropic flow.

The effective dynamical sound speed for the disc model described here is given by \( V_s = c_0 / \sqrt{1 + \beta} \). For NT and RH, we can write the disc height as \( H = \sqrt{p/\rho f(r)} \) where \( f(r) \) is function of the radial coordinate only.

The causal structure analysis also shows how the critical points become the horizon and thus necessitates redefinition of sound speed. One expects the Penrose diagram of the acoustic metric to clearly demonstrate the fact as the conformal diagram includes light cone which are similar at all points in the compact space time

10 Acknowledgement

MAS and SM acknowledge the visit at Sarojini Naidu College Kolkata to Sankha Subhra Naag and also at PAMU, ISI, Kolkata (supported by the Cosmology and High Energy Astrophysics Grant). TKD acknowledges the support from the Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata, India, in the form of a long-term visiting scientist (one-year sabbatical visitor).

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