Quantum-state comparison and discrimination

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We investigate the performance of discrimination strategy in the comparison task of known quantum states. In the discrimination strategy, one infers whether or not two quantum systems are in the same state on the basis of the outcomes of separate discrimination measurements on each system. In some cases with more than two possible states, the optimal strategy in minimum-error comparison is that one should infer the two systems are in different states without any measurement, implying that the discrimination strategy performs worse than the trivial “no-measurement” strategy. We present a sufficient condition for this phenomenon to happen. For two pure states with equal prior probabilities, we determine the optimal comparison success probability with an error margin, which interpolates the minimum-error and unambiguous comparison. We find that the discrimination strategy is not optimal except for the minimum-error case.

I. INTRODUCTION

The laws of quantum mechanics do not allow one to distinguish nonorthogonal quantum states perfectly [1–4]. First, this is because of the statistical nature of quantum measurement, which generally destroys the state of the system, and, further, one cannot clone an unknown quantum states [5].

Quantum state comparison is one of the problems which are directly related to this nature of quantum mechanics [6–12]. Suppose we are given two quantum systems, and the task is to optimally infer whether or not the two systems are in the same state. We can consider two different settings of the problem. One is the case in which the possible states are unknown; that is, we have no classical knowledge on the states. The other is the case in which the state is selected from a known set of states with some prior probabilities. We concentrate on the latter case.

Suppose two states are independently selected from a set of two known pure states. In Ref. [6], Barnett et al. showed that the optimal comparison in the minimum-error scheme is attained by the discrimination strategy, by which we mean we separately perform the optimal discrimination measurement on each system, and if the two outcomes are equal we infer that the two systems are in the same state, and otherwise they are in different states. On the other hand, they showed that the optimal comparison in the unambiguous scheme requires a collective measurement on the whole system as long as the error-margin condition is active.

As stated previously, for a two-state comparison, the discrimination strategy is not optimal in the unambiguous scheme, whereas it is optimal in the minimum-error setting. We determine the optimal comparison success probability with an error margin for two pure states with equal prior probabilities (Sec. II). This error-margin scheme interpolates the minimum-error and unambiguous settings [15–19]. We find that the discrimination strategy is optimal only in the minimum-error case; that is, the optimal comparison requires collective measurement on the whole system as long as the error-margin condition is active.

II. TWO-STATE COMPARISON AND DISCRIMINATION STRATEGY

Suppose we are given two quantum systems, which are independently prepared in one of two known pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ with equal prior probabilities. The state of the combined system is thus either one of $|\phi_1, \phi_1\rangle$, $|\phi_1, \phi_2\rangle$, $|\phi_2, \phi_1\rangle$, or $|\phi_2, \phi_2\rangle$ with probabilities $1/4$. The task is to infer whether the two systems are in the same state or not. This problem has been addressed and solved by Barnett et al. in Ref. [6]. In this section, we first reproduce their results, and then extend the results to a more general case.

Measurement is described by a positive operator-valued measure (POVM), $\{E=, E_\ne\}$, where measurement outcome “$=$” corresponds to the guess that two systems are in the same state whereas outcome “$\ne$” means their states are different. The probability of success in this
state comparison is given by
\[ P_o = \frac{1}{4} \sum_{k=1}^{2} \langle \phi_k, \phi_k | E = |\phi_k, \phi_k \rangle \]
\[ + \frac{1}{4} \sum_{k,j=1, k\neq j}^{2} \langle \phi_k, \phi_j | E \neq |\phi_j, \phi_k \rangle. \]  
(1)

In this section we adopt the minimum error scheme; we maximize the success probability \( P_o \) without any constraint on the probability of an erroneous guess. Using \( E = 1 - E_\neq \), we write the success probability \( P_o \) as
\[ P_o = \frac{1}{2} + \text{Tr} \, E = \Lambda, \]  
(2)
where
\[ \Lambda = \frac{1}{4} \sum_{k=1}^{2} |\phi_k \rangle \langle \phi_k | \otimes |\phi_k \rangle \langle \phi_k | \]
\[ - \frac{1}{4} \sum_{k,j=1, k\neq j}^{2} |\phi_k \rangle \langle \phi_j | \otimes |\phi_j \rangle \langle \phi_j |. \]  
(3)
Since \( 0 \leq E = \Lambda \leq 1 \), the maximal value of \( \text{Tr} \, E = \Lambda \) is given by the sum of all positive eigenvalues of the operator \( \Lambda \).

To obtain eigenvalues of \( \Lambda \), it suffices to work in the two-dimensional space spanned by the states \( |\phi_1 \rangle \) and \( |\phi_2 \rangle \). Introducing Bloch vectors \( n_k \) for the states \( |\phi_k \rangle \), we have
\[ |\phi_k \rangle \langle \phi_k | = \frac{1 + n_k \cdot \sigma}{2}, \]  
(4)
where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices. With this relation the operator \( \Lambda \) takes the form
\[ \Lambda = \frac{1}{16} (n_1 - n_2) \cdot \sigma \otimes (n_1 - n_2) \cdot \sigma. \]  
(5)
It is now easy to obtain the eigenvalues of \( \Lambda \), since the eigenvalues of \( (n_1 - n_2) \cdot \sigma \) are given by \( \pm |n_1 - n_2| \). Thus we find the optimal success probability is given by
\[ P_{o_{\text{opt}}} = \frac{1}{2} + \frac{1}{8} |n_1 - n_2|^2 = 1 - \frac{1}{2} |\langle \phi_1 | \phi_2 \rangle|^2, \]  
(6)
which is attained when \( E = \Lambda \) is the projecter onto the subspace spanned by the eigenstates of \( \Lambda \) with positive eigenvalues.

Let us examine the obtained results more closely. The optimal comparison is realized when \( E = \Lambda \) is given by
\[ E = e_1 \otimes e_1 + e_2 \otimes e_2, \]  
(7)
where \( e_1 \) and \( e_2 \) are the projectors onto the eigenspaces of \( (n_1 - n_2) \cdot \sigma \) with positive and negative eigenvalues, respectively. Now recall the minimal-error discrimination problem between the two pure states \( |\phi_1 \rangle \) and \( |\phi_2 \rangle \) with equal prior probabilities. The optimal measurement in this discrimination problem is given by the POVM \( \{ e_1, e_2 \} \), where \( e_k \) corresponds to the guess that the state is \( |\phi_k \rangle \). This implies that the optimal state comparison under consideration is reduced to the optimal discrimination; we separately perform the optimal discrimination measurement on each system of the two, and if the two outcomes are equal we infer that the two systems are in the same state, and otherwise they are in different states.

We can show that this conclusion holds in more general case: a state comparison of two mixed states \( \rho_1 \) and \( \rho_2 \) with arbitrary prior probabilities \( \eta_1 \) and \( \eta_2 \), respectively. In this case the success probability of comparison is written as
\[ P_o = \text{Tr} \, E = (\eta_1 \rho_1 \otimes \rho_2 + \eta_2 \rho_2 \otimes \rho_1)
\[ + \text{Tr} \, E \neq (\eta_1 \rho_1 \otimes \rho_2 + \eta_2 \rho_2 \otimes \rho_1) \]
\[ \geq 2\eta_1 \eta_2 + \text{Tr} \, E = (\eta_1 \rho_1 - \eta_2 \rho_2) \otimes (\eta_1 \rho_1 - \eta_2 \rho_2). \]  
(8)
The optimal POVM element \( E = \Lambda \) is clearly given in the form of Eq. (7) with \( e_1 \) (\( e_2 \)) being the projector onto the eigenspace of \( \eta_1 \rho_1 - \eta_2 \rho_2 \) with positive (negative) eigenvalues. This POVM \( \{ e_1, e_2 \} \) is the optimal POVM in the discrimination problem between \( \rho_1 \) and \( \rho_2 \) with prior probabilities \( \eta_1 \) and \( \eta_2 \), respectively. This is evident since the success probability \( Q_o \) of this discrimination problem is written as
\[ Q_o = \eta_1 \text{Tr} \, e_1 \rho_1 + \eta_2 \text{Tr} \, e_2 \rho_2 \]
\[ = \eta_2 + \text{Tr} \, e_1 (\eta_1 \rho_1 - \eta_2 \rho_2). \]  
(9)
In conclusion, the discrimination strategy is optimal in the general two-state minimum-error comparison.

### III. MANY-STATES COMPARISON

In the preceding section we have shown that the discrimination strategy is optimal in the general two-state minimum-error comparison. It is interesting whether this result holds in the minimum-error comparison of more than two states. To investigate this issue, we take an example of minimum-error comparison involving \( N \geq 2 \) states.

In a two dimensional space, we consider the following \( N \) pure states:
\[ |\phi_k \rangle = U_k |0 \rangle \, (k = 0, 1, \cdots, N - 1), \]  
(10)
where the initial state \( |0 \rangle \) is given by
\[ |0 \rangle = \frac{|0 \rangle + |1 \rangle}{\sqrt{2}}, \]  
(11)
and the phase shift operator \( U_k \) is defined as
\[ \left\{ \begin{array}{c}
U_k |0 \rangle = |0 \rangle \\
U_k |1 \rangle = e^{i \frac{2 \pi k}{N}} |1 \rangle.
\end{array} \right. \]  
(12)
Note that \( \{ U_k \}_{k=0}^{N-1} \) is a unitary representation of \( \mathbb{Z}_N \), consisting of two inequivalent one-dimensional irreducible representations. One irreducible representation
space is spanned by $|0\rangle$, and the other by $|1\rangle$. The discrimination problem of these $N$ states with equal prior probabilities has been analyzed in Ref. 20. The optimal minimum-error discrimination success probability $Q_\text{opt}$ is found to be

$$Q_\text{opt} = \frac{2}{N},$$  \hspace{1cm} (13)

with the optimal POVM $e_k^{\text{opt}}$, corresponding to the guess that the state is $|\phi_k\rangle$, given by

$$e_k^{\text{opt}} = \frac{2}{N} |\phi_k\rangle \langle \phi_k|.$$  \hspace{1cm} (14)

Now we consider the comparison problem of the $N$ states $|\phi_k\rangle$; the two systems are prepared in one of those states independently with equal prior probabilities, and the task is to guess whether the two systems are in the same state or not. As in the preceding section, we write the comparison success probability $P_\circ$ as

$$P_\circ = 1 - \frac{1}{N} + \text{tr} E_\circ \Lambda,$$  \hspace{1cm} (15)

where

$$\Lambda = \frac{1}{N^2} \sum_{k=0}^{N-1} |\phi_k\rangle \langle \phi_k| \otimes |\phi_k\rangle \langle \phi_k|$$

$$- \frac{1}{N^2} \sum_{k,j=0(k\neq j)}^{N-1} |\phi_k\rangle \langle \phi_k| \otimes |\phi_j\rangle \langle \phi_j|.$$  \hspace{1cm} (16)

The optimal POVM $E_\circ^{\text{opt}}$ is the projector onto the eigenspace of $\Lambda$ with positive eigenvalues. In order to calculate $\Lambda$, we again use the relation Eq. 4 where the Bloch vector for state $|\phi_k\rangle$ is now given by

$$n_k = \left( n_k^x, n_k^y, n_k^z \right)$$

$$= \left( \cos \frac{2\pi}{N} k, \sin \frac{2\pi}{N} k, 0 \right).$$  \hspace{1cm} (17)

The following formulas of Bloch vectors are useful for performing the summation over $k$ and $j$ in the expression of $\Lambda$:

$$\sum_{k=0}^{N-1} n_k^x = \sum_{k=0}^{N-1} n_k^y = 0,$$  \hspace{1cm} (18)

$$\sum_{k=0}^{N-1} n_k^x n_k^x = \sum_{k=0}^{N-1} n_k^y n_k^y = \begin{cases} 2, & (N = 2) \\ N/2, & (N \geq 3) \end{cases},$$  \hspace{1cm} (19)

$$\sum_{k=0}^{N-1} n_k^y n_k^y = \begin{cases} 0, & (N = 2) \\ N/2, & (N \geq 3) \end{cases},$$  \hspace{1cm} (20)

$$\sum_{k=0}^{N-1} n_k^x n_k^y = 0.$$  \hspace{1cm} (21)

We find that $\Lambda$ is expressed as follows:

$$\Lambda = \begin{cases} \frac{1}{4} \sigma_x \otimes \sigma_x, & (N = 2) \\ \frac{1}{N} (\sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x) + \frac{1}{N^2} - \frac{1}{4}, & (N \geq 3) \end{cases}.$$  \hspace{1cm} (22)

When $N = 2$, the problem is trivial since the states $|\phi_0\rangle$ and $|\phi_1\rangle$ are orthogonal and can be perfectly discriminated. In fact the sum of positive eigenvalues of $\Lambda$ is $1/2$, and we obtain $P_\circ^{\text{opt}} = 1$ with the optimal POVM given by the discrimination strategy

$$E^{\text{opt}}_\circ = E^{\text{disc}}_\circ \equiv \sum_{k=0}^{N-1} e_k^{\text{opt}} \otimes e_k^{\text{opt}}.$$  \hspace{1cm} (23)

Let us see the eigenvalues of $\Lambda$ when $N \geq 3$. The four eigenstates of $\Lambda$ are the Bell states:

$$\begin{cases} |\Psi_\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \\ |\Phi_\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) \end{cases}.$$  \hspace{1cm} (24)

We find that the eigenvalues associated with $|\Psi_\pm\rangle$ and $|\Phi_\pm\rangle$ are all negative, and the eigenvalue of $|\Phi_+\rangle$ is given by

$$\lambda_{\Phi_+} = \frac{1}{N} - \frac{1}{4} = \begin{cases} > 0, & (N = 3) \\ \leq 0, & (N \geq 4) \end{cases}.$$  \hspace{1cm} (25)

Therefore, in the case of $N = 3$, the optimal comparison success probability is given by $P_\circ^{\text{opt}} = 3/4$ with the optimal POVM $E^{\text{opt}}_\circ = |\Phi_+\rangle \langle \Phi_+|$. If we take the discrimination strategy, $E^{\text{disc}}_\circ = \sum_{k=0}^{N-1} e_k^{\text{opt}} \otimes e_k^{\text{opt}}$, we find $\text{tr} E^{\text{disc}}_\circ \Lambda$ vanishes. This implies that $P_\circ^{\text{disc}} = 2/3$, which is strictly less than $P_\circ^{\text{opt}} = 3/4$. Thus the state comparison is not reduced to the discrimination problem in this case. It should be noted that the success probability $2/3$ of the discrimination strategy can be obtained by a simpler strategy, where $E_\circ$ is set to zero and therefore $E_\circ = 1$; namely, we always infer the two systems are in different states without performing any measurement (no-measurement strategy).

When $N \geq 4$, no eigenvalue of $\Lambda$ is positive. Therefore the optimal comparison success probability is given by

$$P_\circ^{\text{opt}} = 1 - 1/N,$$  \hspace{1cm} (26)

with $E^{\text{opt}}_\circ = 0$. Thus, rather surprisingly, we find that the optimal strategy is just the no-measurement strategy. The discrimination strategy gives a worse result since $\text{tr} E^{\text{disc}}_\circ \Lambda$ is negative. The size relation of the success probabilities by the three strategies is summarized in Table I.

In general, we expect that the no-measurement strategy performs better when the number of possible states is large, since the probability of selecting two different states dominates. In what follows, we present a sufficient
TABLE I. The relation of the comparison success probabilities. $P_o^{\text{opt}}$ is the optimal success probability. $P_o^{\text{disc}}$ is the one based on the discrimination strategy, and the result of the no-measurement strategy ($E_w = 0$, $E_x = 1$) is denoted by $P_o^{\text{no}}$.

| $N$ | $N = 2$ | $N = 3$ | $N \geq 4$ |
|-----|---------|---------|------------|
| $P_o^{\text{no}} < P_o^{\text{disc}} = P_o^{\text{opt}}$ | $P_o^{\text{no}} = P_o^{\text{disc}} < P_o^{\text{opt}}$ | $P_o^{\text{disc}} < P_o^{\text{no}} = P_o^{\text{opt}}$ |

condition for the no-measurement strategy to be optimal. For general $N$ pure states $|\phi_k\rangle (k = 0, \ldots, N - 1)$ with equal prior probabilities, we write the $\Lambda$ operator corresponding to Eq. (10) in the following form:

$$\Lambda = \frac{2}{N} R - r \otimes r,$$

where we introduced two normalized density operators $R$ and $r$ defined as

$$R = \frac{1}{N} \sum_{k=0}^{N-1} |\phi_k\rangle \langle \phi_k|,$$

$$r = \frac{1}{N} \sum_{k=0}^{N-1} |\phi_k\rangle \langle \phi_k|.$$

Let $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ be the maximum and minimum of all nonzero eigenvalues of $r$, respectively. Note that $\lambda_{\text{min}} > 0$. We show that if

$$\lambda_{\text{min}} \geq \sqrt{\frac{2\lambda_{\text{max}}}{N}},$$

then $\Lambda$ is negative semidefinite, and consequently the no-measurement strategy is optimal. Let $V_r$ be the support of $r$. Clearly, the support of $r \otimes r$ is given by $V_r \otimes V_r$, and the support of $R$ is a subspace of $V_r \otimes V_r$. Now take an arbitrary vector $|\Phi\rangle$ in the total space and express it as $|\Phi\rangle = |\Phi||r\rangle$, where $|\Phi||r\rangle \in V_r \otimes V_r$ and $|\Phi||r\rangle$ is the component perpendicular to $V_r \otimes V_r$. Assume the condition (30) holds. Then we observe

$$\langle \Phi|\Lambda|\Phi\rangle = \frac{2}{N} \langle \Phi||R||\Phi\rangle - \langle \Phi||r \otimes r||\Phi\rangle \leq \left(\frac{2}{N} \lambda_{\text{max}} - \lambda_{\text{min}}^2\right) \langle \Phi||\Phi\rangle \leq 0,$$

where we used the fact that the maximum eigenvalue of $R$ never exceeds that of $r$, $\lambda_{\text{max}}$, which is proved in the Appendix. This completes the proof.

In the example considered in this section, we find that $\lambda_{\text{max}} = \lambda_{\text{min}} = 1/2$ for $N \geq 2$, implying that the sufficient condition (30) is fulfilled for $N \geq 4$. This perfectly agrees with the result obtained previously by detailed calculations.

IV. TWO-STATE COMPARISON WITH ERROR MARGIN

In this section we introduce an error margin in the problem of state comparison. If the error margin is sufficiently large, this scheme is reduced to the minimum-error comparison studied in the preceding sections. If the error margin is set 0, which implies that no error is allowed, the scheme is just unambiguous comparison.

We consider two pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ with equal prior probabilities. We first summarize the results of the discrimination problem between these states with an error margin $\mu$. The POVM now consists of $\{e_1, e_2, e_2\}$ with $e_2$ associated with the inconclusive result. The task is to maximize the discrimination success probability

$$Q_o = \frac{1}{2} \langle \phi_1|e_1|\phi_1\rangle + \frac{1}{2} \langle \phi_2|e_2|\phi_2\rangle,$$

subject to the condition that the error probability does not exceed an error margin $\mu$

$$Q_x = \frac{1}{2} \langle \phi_1|e_2|\phi_1\rangle + \frac{1}{2} \langle \phi_2|e_1|\phi_2\rangle \leq \mu,$$

and the POVM condition $e_1 + e_2 \leq 1$. This problem was solved in Ref. [18]. The results are

$$Q_o^{\text{opt}} = \left\{ \begin{array}{ll}
\frac{1}{2} \left( 1 + \sqrt{1 - |\langle \phi_1|\phi_2\rangle|^2} \right) & (\mu_e \leq \mu \leq 1), \\
\left( \frac{\sqrt{\mu} + \sqrt{1 - |\langle \phi_1|\phi_2\rangle|^2} \right) & (0 \leq \mu \leq \mu_e),
\end{array} \right.$$

$$Q_x^{\text{opt}} = \left\{ \begin{array}{ll}
\mu_e & (\mu_e \leq \mu \leq 1), \\
\mu & (0 \leq \mu \leq \mu_e),
\end{array} \right.$$

where

$$\mu_e = \frac{1}{2} \left( 1 - \sqrt{1 - |\langle \phi_1|\phi_2\rangle|^2} \right).$$

For the case of general prior probabilities, see Ref. [19].

In what follows we consider the comparison problem of $|\phi_1\rangle$ and $|\phi_2\rangle$ with equal prior probabilities. The POVM now includes an element $E_?$, for the inconclusive result in addition to $E_\pm$ and $E_x \neq$. In the discrimination strategy, the POVM takes the form

$$E_w^{\text{disc}} = e_1 \otimes e_1 + e_2 \otimes e_2,$$

$$E_x^{\text{disc}} = e_1 \otimes e_2 + e_2 \otimes e_1,$$

$$E_?^{\text{disc}} = 1 - E_w^{\text{disc}} - E_x^{\text{disc}}.$$

In this strategy, the comparison is successful if and only if the two discrimination inferences for subsystems are either both correct or both wrong. Therefore the comparison success probability $P_o$ is given by

$$P_o^{\text{disc}} = (Q_o^{\text{opt}})^2 + (Q_x^{\text{opt}})^2 = \left\{ \begin{array}{ll}
1 - \frac{1}{2} |\langle \phi_1|\phi_2\rangle|^2 & (\mu_e \leq \mu \leq 1), \\
\left( \frac{\sqrt{\mu} + \sqrt{1 - |\langle \phi_1|\phi_2\rangle|^2} \right)^4 + \mu^2 & (0 \leq \mu \leq \mu_e).
\end{array} \right.$$

The comparison in this scheme produces a wrong outcome if one discrimination is correct whereas the other is wrong. Thus we obtain

\[
P^\text{disc}_\times = 2Q_{\text{opt}}^1 Q_{\text{opt}}^2
\]

\[
= \left\{ \begin{array}{ll}
\frac{1}{2} |\langle \phi_1 | \phi_2 \rangle|^2 & (\mu_c \leq \mu \leq 1), \\
2\mu \left( \sqrt{\mu} + \sqrt{1 - |\langle \phi_1 | \phi_2 \rangle|^2} \right)^2 & (0 \leq \mu \leq \mu_c).
\end{array} \right.
\]  

(41)

We rewrite those results in terms of the margin \( m \) for the erroneous comparison probability; that is, under the error margin condition given by

\[
P_\times \leq m,
\]

(42)

the discrimination strategy gives the following results:

\[
P^\text{disc}_o
\]

\[
= \left\{ \begin{array}{ll}
1 - \frac{1}{2} |\langle \phi_1 | \phi_2 \rangle|^2 & (m_c \leq m \leq 1), \\
\left( \sqrt{\mu} + \sqrt{1 - |\langle \phi_1 | \phi_2 \rangle|^2} \right)^4 + \mu^2 & (0 \leq m \leq m_c),
\end{array} \right.
\]

(43)

where the critical error margin \( m_c \) is defined by

\[
m_c = \frac{1}{2} |\langle \phi_1 | \phi_2 \rangle|^2,
\]

(45)

and the margin \( \mu \) in the discrimination process is related to the margin \( m \) in the comparison in the following way:

\[
2\mu \left( \sqrt{\mu} + \sqrt{1 - |\langle \phi_1 | \phi_2 \rangle|^2} \right)^2 = m.
\]

(46)

Now we will determine the optimal comparison success probability with an error margin \( m \). The task is to maximize the success probability

\[
P_\times = \frac{1}{2} \text{tr} E_{\pi} \rho_\pi + \frac{1}{2} \text{tr} E_{\pi} \rho_\pi
\]

subject to the error margin condition

\[
P_\times = \frac{1}{2} \text{tr} E_{\pi} \rho_\pi + \frac{1}{2} \text{tr} E_{\pi} \rho_\pi \leq m,
\]

(48)

and the POVM conditions

\[
E_\pi, E_{\pi} \geq 0,
\]

(49)

\[
E_\pi + E_{\pi} \leq 1,
\]

(50)

where \( \rho_\pi \) and \( \rho_\pi \) are density operators defined to be

\[
\rho_\pi = \frac{1}{2} \sum_{k=1}^{2} |\phi_k, \phi_k \rangle \langle \phi_k, \phi_k |,
\]

(51)

\[
\rho_{\pi} = \frac{1}{2} \sum_{k,j=1}^{2} |\phi_k, \phi_j \rangle \langle \phi_j, \phi_k |.
\]

(52)

This is a discrimination problem with an error margin between two mixed states, which is generally hard to treat analytically. However, we can obtain analytical results by using two useful exchange-type symmetries in the problem.

The first symmetry we consider is concerned with the system swap operation \( \Pi \), whose action is \( \Pi |\phi \rangle \otimes |\psi \rangle = |\psi \rangle \otimes |\phi \rangle \) for any state \( |\phi \rangle \) and \( |\psi \rangle \). The second symmetry is a sort of state exchange symmetry. Suppose we choose phases of the states so that \( \langle \phi_1 | \phi_2 \rangle = \text{real} \). Then there exists a unitary \( U \) such that \( U |\phi_1 \rangle = |\phi_2 \rangle \) and \( U |\phi_2 \rangle = |\phi_1 \rangle \). The state exchange operator \( \Gamma \) is defined to be \( \Gamma = U \otimes U \). It is clear that \( \rho_\pi \) and \( \rho_{\pi} \) are invariant under these two operations.

\[
\Pi \rho_\pi \Pi^\dagger = \rho_\pi, \quad \Gamma \rho_{\pi} \Gamma^\dagger = \rho_{\pi},
\]

(53)

\[
\Pi \rho_{\pi} \Pi^\dagger = \rho_{\pi}, \quad \Gamma \rho_{\pi} \Gamma^\dagger = \rho_{\pi}.
\]

(54)

Therefore, the optimal success probability is achieved by a POVM which is invariant under the system swap and state exchange operations.

\[
\Pi E_\pi \Pi^\dagger = E_\pi, \quad \Gamma E_{\pi} \Gamma^\dagger = E_{\pi},
\]

(55)

\[
\Pi E_{\pi} \Pi^\dagger = E_{\pi}, \quad \Gamma E_{\pi} \Gamma^\dagger = E_{\pi}.
\]

(56)

The space \( V = \mathbb{C}^2 \otimes \mathbb{C}^2 \) can be decomposed into three orthogonal subspaces according to the symmetries with respect to \( \Pi \) and \( \Gamma \).

\[
V = V_{++} \oplus V_{+-} \oplus V_{-+},
\]

(57)

where \( V_{\pi, \gamma} \) is the eigenspace in which the eigenvalue of \( \Pi \) is \( \pi \) and the eigenvalue of \( \Gamma \) is \( \gamma \). Note that \( \pi = -1 \) immediately implies \( \gamma = -1 \). The space \( V_{++} \) is two-dimensional and spanned by

\[
|X_1 \rangle = |\phi_1, \phi_1 \rangle + |\phi_2, \phi_2 \rangle,
\]

(58)

\[
|X_2 \rangle = |\phi_1, \phi_2 \rangle + |\phi_2, \phi_1 \rangle,
\]

(59)

which are not orthogonal in general. The spaces \( V_{+-} \) and \( V_{-+} \) are both one-dimensional, and consist of scalar multiples of

\[
|Y_+ \rangle = |\phi_1, \phi_1 \rangle - |\phi_2, \phi_2 \rangle,
\]

(60)

\[
|Y_- \rangle = |\phi_1, \phi_2 \rangle - |\phi_2, \phi_1 \rangle,
\]

(61)

respectively.

By the invariance of POVM, Eqs. 100 and 101, we can write

\[
E_\pi = \tilde{E}_\pi + \alpha_+ |Y_+ \rangle \langle Y_+ | + \alpha_- |Y_- \rangle \langle Y_- |,
\]

(62)

\[
E_{\pi} = \tilde{E}_{\pi} + \beta_+ |Y_+ \rangle \langle Y_+ | + \beta_- |Y_- \rangle \langle Y_- |,
\]

(63)

where \( \alpha_\pm, \beta_\pm \geq 0 \) and \( \tilde{E}_\pi \) and \( \tilde{E}_{\pi} \) are semipositive definite operators on the subspace \( V_{++} \). Then \( P_\pi \) and
\[ P_x = \frac{1}{2} \left( \langle X_1 | \tilde{E}_\varphi | X_1 \rangle + \langle X_2 | \tilde{E}_\varphi | X_2 \rangle \right) 
+ \alpha_+ |\langle Y_+ | Y_+ \rangle|^2 + \beta_- |\langle Y_- | Y_- \rangle|^2 \right), \quad (64) \]

\[ P_x = \frac{1}{2} \left( \langle X_1 | \tilde{E}_\varphi | X_1 \rangle + \langle X_2 | \tilde{E}_\varphi | X_2 \rangle \right) 
+ \beta_+ |\langle Y_+ | Y_+ \rangle|^2 + \alpha_- |\langle Y_- | Y_- \rangle|^2 \right). \quad (65) \]

The POVM condition Eq. \[ (51) \] can also be decomposed into the conditions in subspaces. In \( V_{+-} \), it is given by

\[ (\alpha_+ + \beta_+) |Y_+ \rangle \langle Y_+ | \leq 1. \quad (66) \]

In order to maximize \( P_o \) subject to the condition \( P_x \leq m \), it is clear that we should have

\[ \alpha_+^{\text{opt}} = \frac{1}{(Y_+ | Y_+ \rangle}, \quad \beta_+^{\text{opt}} = 0. \quad (67) \]

Similarly, from the POVM condition in \( V_{-+} \), we obtain

\[ \alpha_-^{\text{opt}} = 0, \quad \beta_-^{\text{opt}} = \frac{1}{(Y_- | Y_- \rangle}. \quad (68) \]

The POVM operators \( \tilde{E}_\varphi \) and \( \tilde{E}_\varphi \) are yet to be determined. This should be carried out by maximizing

\[ P_x = \frac{1 + |\langle \phi_1 | \phi_2 \rangle|^2}{4} \left( \langle \tilde{X}_1 | \tilde{E}_\varphi | \tilde{X}_1 \rangle + \langle \tilde{X}_2 | \tilde{E}_\varphi | \tilde{X}_2 \rangle \right) 
+ \frac{1 - |\langle \phi_1 | \phi_2 \rangle|^2}{2}, \quad (69) \]

subject to the error-margin condition given by

\[ P_x = \frac{1 + |\langle \phi_1 | \phi_2 \rangle|^2}{4} \left( \langle \tilde{X}_1 | \tilde{E}_\varphi | \tilde{X}_1 \rangle + \langle \tilde{X}_2 | \tilde{E}_\varphi | \tilde{X}_2 \rangle \right) 
< m, \quad (70) \]

where \( |\tilde{X}_k \rangle \) is the normalized state of \( |X_k \rangle \) \((k = 1, 2)\). The POVM condition in \( V_{+-} \) is

\[ \tilde{E}_\varphi + \tilde{E}_\varphi \leq 1. \quad (71) \]

The problem in this form is, up to additive and multiplicative constants, equivalent to a discrimination problem with an error margin between two pure states \( |\tilde{X}_1 \rangle \) and \( |\tilde{X}_2 \rangle \) with equal prior probabilities. We can employ the optimal solution summarized at the beginning of this section. Note that the error margin in this equivalent discrimination problem should be taken as \( 2m/(1 + |\langle \phi_1 | \phi_2 \rangle|^2) \), and the inner product of the states to be discriminated is given by

\[ \langle \tilde{X}_1 | \tilde{X}_2 \rangle = \frac{2 \langle \phi_1 | \phi_2 \rangle}{1 + |\langle \phi_1 | \phi_2 \rangle|^2}. \quad (72) \]

Thus we finally obtain the optimal comparison success probability \( P_o^{\text{opt}} \) with an error margin \( m \).

\[
\begin{align*}
P_o^{\text{opt}} &= \begin{cases} 
1 - \frac{1}{2} |\langle \phi_1 | \phi_2 \rangle|^2 & (m_c \leq m \leq 1), \\
\frac{1}{2} + \frac{1}{2} (\sqrt{2m} + 1) (\sqrt{2m} + 1 - 2 |\langle \phi_1 | \phi_2 \rangle|) & (0 \leq m \leq m_c),
\end{cases} 
\end{align*}
\quad (73)
\]

where the critical error margin is given by \( m_c = |\langle \phi_1 | \phi_2 \rangle|^2/2 \), which is the same as the one in the discrimination strategy. When \( m \) is greater than \( m_c \), the optimal success probability is given by that of the minimum error scheme Eq. \[ (38) \]. The result of unambiguous comparison can be obtained by setting \( m = 0 \). We obtain

\[ P_o^{\text{opt}}(\text{unamb}) = 1 - |\langle \phi_1 | \phi_2 \rangle|, \quad (74) \]

which agrees with the result of Ref. \[ 6 \]. This probability happens to equal the unambiguous discrimination probability \( Q_o^{\text{opt}}(\text{unamb}) \). Note that the discrimination strategy gives \( P_o^{\text{disc}}(\text{unamb}) = (Q_o^{\text{opt}}(\text{unamb}))^2 \), which is strictly less than the optimal success probability \( P_o^{\text{opt}}(\text{unamb}) \). Thus, as emphasized in Ref. \[ 6 \], the unambiguous state comparison is not reduced to discrimination task for subsystems. What we have found is that this is also true for a general error margin as long as the error-margin condition is active; namely, we can show that \( P_o^{\text{opt}} > P_o^{\text{disc}} \) for \( 0 \leq m < m_c \). In Fig. \[ 1 \] we display \( P_o^{\text{opt}} \) and \( P_o^{\text{disc}} \) as functions of error margin \( m \) in the case of \( |\langle \phi_1 | \phi_2 \rangle| = 0.8 \).
V. SUMMARY AND CONCLUDING REMARKS

The aim of this paper was to investigate the performance of the discrimination strategy in the comparison task of known quantum states. In the case of minimum-error comparison of two states, the discrimination strategy always gives the optimal result. We have shown that this result can be generalized for mixed states with arbitrary prior probabilities. However, if the number of possible states is greater than two, this is no longer true. In some cases, the optimal comparison strategy is simply that we infer the two systems are in different states. Rather surprisingly the phenomenon to occur was presented.

We have also investigated how the constraint on the error probability (error margin) affects the performance of the discrimination strategy, and found that the discrimination strategy is optimal only in the minimum-error case. In the case where the error-margin condition is active (including the unambiguous scheme), the optimal comparison requires collective measurement on the whole system.

In this paper we assumed that the states are selected from a known set of states. We can consider a different scheme; that is, we have no classical knowledge of the possible states but some number of copies of the states are available instead [21, 22]. The corresponding discrimination task is sometimes called quantum state identification. It will be of interest in future studies to extend our investigation to the “identification” strategy in the state comparison in this setting.

Appendix: Maximum eigenvalues of $R$ and $r$

We prove the following theorem which is used in the end of Sec. III.

Theorem. Let \{\ket{\phi_0^A}, \ket{\phi_1^A}, \ldots, \ket{\phi_{N-1}^A}\} and \{\ket{\psi_0^B}, \ket{\psi_1^B}, \ldots, \ket{\psi_{N-1}^B}\} be sets of $N$ normalized pure states of system $A$ and $B$, respectively. Define three density operators $r^A_\phi$, $r^B_\psi$, and $R^{AB}$.

$$r^A_\phi = \frac{1}{N} \sum_{k=0}^{N-1} |\phi_k^A\rangle \langle \phi_k^A|,$$

$$r^B_\psi = \frac{1}{N} \sum_{k=0}^{N-1} |\psi_k^B\rangle \langle \psi_k^B|,$$

and

$$R^{AB} = \frac{1}{N} \sum_{k=0}^{N-1} |\phi_k^A, \psi_k^B\rangle \langle \phi_k^A, \psi_k^B|.$$

Then we have

$$\lambda_{\max}(R^{AB}) \leq \min\{\lambda_{\max}(r^A_\phi), \lambda_{\max}(r^B_\psi)\}, \quad (A.1)$$

where $\lambda_{\max}(\Omega)$ stands for the maximum eigenvalue of an operator $\Omega$.

Proof. For any normalized state $|\Phi^{AB}\rangle$ of system $AB$, we observe

$$\langle \Phi^{AB} | R^{AB} | \Phi^{AB} \rangle = \frac{1}{N} \sum_{k=0}^{N-1} \langle \Phi^{AB} | \psi_k^B \rangle \langle \phi_k^A | \psi_k^B \rangle \langle \phi_k^A | \psi_k^B \rangle | \Phi^{AB} \rangle \leq \frac{1}{N} \sum_{k=0}^{N-1} \langle \phi_k^A | \psi_k^B \rangle \langle \phi_k^A | \psi_k^B \rangle \langle \phi_k^A | \psi_k^B \rangle = \langle \Phi^{AB} | I | \Phi^{AB} \rangle,$$

where $\rho^A$ is the reduced density operator defined by

$$\rho^A = \text{tr}_B (|\Phi^{AB}\rangle \langle \Phi^{AB}|).$$

Since

$$\lambda_{\max}(R^{AB}) = \max_{|\Phi^{AB}|=1} \langle \Phi^{AB} | R^{AB} | \Phi^{AB} \rangle,$$

we obtain $\lambda_{\max}(R^{AB}) \leq \lambda_{\max}(r^A_\phi)$. Similarly we can show that $\lambda_{\max}(R^{AB}) \leq \lambda_{\max}(r^B_\psi)$. Combining these two results we obtain the desired inequality.

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