Stability of a stage-structure Rosenzweig-MacArthur model incorporating Holling type-II functional response

Lazarus Kalvein Beay1,2*, Agus Suryanto1, Isnani Darti1, and Trisilowati1

1Department of Mathematics, University of Brawijaya, Malang, East Java, Indonesia
2Department of Education and Culture, Provincial Government of Moluccas, Indonesia

*Corresponding Author: kalvein.beay@student.ub.ac.id.

Abstract. The local stability of the Rosenzweig-MacArthur predator-prey system with Holling type-II functional response and stage-structure for prey is studied in this paper. It is shown that the model has three equilibrium points. The trivial equilibrium point is always unstable while two other equilibrium points, i.e., the predator extinction point and the coexistence point, are conditionally stable. When the predation process on prey increases, the number of predator increases. If the predation rate is less than or equal to the reduction rate of the predator, then the predator will go to extinct. By using the Routh-Hurwitz criterion, the local stability of the interior equilibrium point is investigated. It is also shown that the model undergoes a Hopf bifurcation around the coexisting equilibrium point. The dynamics of the system are confirmed by some numerical simulations.

1. Introduction

Continuously, various studies on predator-prey interactions in the mathematical ecology are carried out [1]. There are several models known for studying the behavior of predator-prey systems, one of which is the Rosenzweig-MacArthur model [2]. The fundamental aspect about the Rosenzweig-MacArthur model is a well-known mechanism of the paradox of enrichment [3-6]. This is related to the extinction of species affected by the increasing of carrying capacity of prey in the ecosystem [4-5]. The effects of refuge on the Rosenzweig-MacArthur model have been studied by Kar [7] and Chen et al. [8]. Recently, studies of the dynamic behavior of fractional order Rosenzweig-MacArthur system with harvest and prey refuge have been studied by Javidi and Nyamoradi [9] and Moustofa et al. [5], respectively.

In studying predator-prey systems, functional responses are one of the important elements that affecting this dynamic, and one of them is the Holling-II type functional response [1,10]. Stage-structure is also considered to be one of the important factors in the predator-prey system. In some species, it is divisible into two stages, immature and mature [1,10-17].

In addition to the stage structure on the predator [11-15], various studies have been carried out with a consideration of the stage structure on the prey. Khajanchi and Banerjee [16] analysed the role of prey refuge of the predator-prey model incorporating ratio-dependent functional response and considered stage-structure for the prey. The model analysis has shown that the stage-structured predator-prey species is significantly influenced by the prey refuge. Falcioni et al. [17] considered a group defense mechanism and stage-structure for the prey. The analyzed model, assumed that immature and mature prey do not inhabit in the same location. It is found that the defense mechanism of the prey population...
has an impact the existence of the immature and mature prey and the predator. Next, stability analysis of a stage-structure model incorporating Holling type-IV functional response and considered competition on the mature prey was studied by Jia and Wei [1].

Motivated by these various studies, we consider the following Rosenzweig-MacArthu system incorporating stage-structure for prey and Holling type-II functional response. Here, we consider a mathematical model of the interaction between the densities of immature prey $x_1$, mature prey $x_2$, and predator $y$. By assuming that the predator just preys upon the immature prey, we get the following model

$$
\frac{dx_1}{dt} = rx_2 \left(1 - \frac{x_1}{K}\right) - ax_1 - \frac{v_0x_1y}{n_1 + x_1},
$$

$$
\frac{dx_2}{dt} = ax_1 - \delta_0 x_2,
$$

$$
\frac{dy}{dt} = \frac{av_0x_1y}{n_1 + x_1} - \delta_1 y.
$$

Here, $r$ and $K$ are respectively the intrinsic growth rate of immature prey and carrying capacity of immature prey, $\alpha$ and $v_0$ are the transition rate from the immature prey to the mature prey and the maximum predation rate. $n_1$ and $\alpha$ are the environment protection to immature prey and the predator conversion rate. The death rate of mature prey and the predator are respectively $\delta_0$ and $\delta_1$. All parameters $r, K, \alpha, v_0, n_1$, and $\alpha$ in the system (1) are positive.

We outline the format of this paper as follows. In section 2, we analyze our mathematical model which includes the determination of equilibrium points, local stability analysis and Hopf-bifurcation analysis of the coexistence point. Some numerical simulations are performed to corroborate our analytical finding in section 3. In the last section 4, we conclude the results of our analysis accompanied numerical simulations.

2. Model Analysis

To simplify the analysis of system (1), we reduce the number of parameters by writing system (2) in following non-dimensional form

$$
\frac{d{u}_1}{dt} = {u}_2(1 - {u}_1) - \alpha_1 {u}_1 - \frac{u_{1}{u}_3}{\omega + u_1},
$$

$$
\frac{d{u}_2}{dt} = \alpha_1 {u}_1 - \beta_1 {u}_2,
$$

$$
\frac{d{u}_3}{dt} = \frac{\alpha_2 {u}_1 {u}_3}{\omega + u_1} - \beta_2 {u}_3,
$$

where $u_1 = \frac{x_1}{K}, u_2 = \frac{x_2}{K}, u_3 = \frac{v_0y}{rK}, \omega = \frac{n_1}{K}, \alpha_2 = \frac{av_0}{r}, \alpha_4 = \frac{\alpha}{r}, \beta_1 = \frac{\delta_0}{r}, \beta_2 = \frac{\delta_1}{r}$, and $t = \frac{r}{t}$.

2.1. Equilibrium points analysis

By setting the right-hand sides of the system (2) equal to zero, we get the following equilibrium points:

(1) A trivial equilibrium $E_0 = (0,0,0)$ which is the extinction of all population in the ecosystem.

(2) The predator extinction equilibrium $E_1 = (\hat{u}_1, \hat{u}_2, 0)$, which exists if

$$
\beta_1 < 1,
$$

where

$$
\hat{u}_1 = 1 - \beta_1, \text{ and } \hat{u}_2 = \frac{\alpha_1 \hat{u}_1}{\beta_1}.
$$

Notice that the predator extinction equilibrium $E_1$ exists if the intrinsic growth rate of immature prey ($r$) greater than death rate of mature prey ($\delta_0$).

(3) A unique coexistence equilibrium $E^* = (u_1^*, u_2^*, u_3^*)$, which exists if

$$
\alpha_2 > \beta_2, \text{ and } \alpha_2 - \beta_2 > \beta_2 \omega,
$$

where

$$
u_1^* = \frac{\beta_2 \omega}{\alpha_2 - \beta_2}, u_2^* = \frac{\beta_1 u_1^*}{\alpha_1}, \text{ and } u_3^* = \frac{\alpha_2 u_2^* \alpha_4 (\alpha_2 - \beta_2) \hat{u}_1 - \beta_2 \omega}{\beta_1 (\alpha_2 - \beta_2)}.
$$
Condition (4) says that if the predation rate is larger than the death rate of the predator, then the coexistence equilibrium $E^*$ exists. It means that the predation process on prey determines the growing effects of the predator population.

2.2. Local stability of equilibrium point

The local stability of all equilibrium points can be studied from the linearization of system (2). The Jacobian matrix of the system (2) is given by

$$
J = \begin{pmatrix}
-\frac{u_2 + \frac{\beta_2 u_3}{(\omega + u_1)\alpha_2}}{u_1} & 1 - u_1 & -\frac{u_1}{\omega + u_1} \\
\frac{\omega \beta_2 u_3}{(\omega + u_1)^2} & -\beta_2 & 0 \\
\frac{\omega (\beta_2 u_3)}{(\omega + u_1)^2} & 0 & -\frac{\alpha_2 u_1}{\omega + u_1} - \beta_2
\end{pmatrix}.
$$

(6)

By observing the eigenvalues of the Jacobian matrix (6) at each equilibrium point, we have the following stability properties.

**Theorem 1.**

1. Equilibrium point $E_0 = (0,0,0)$ is always unstable.
2. If $(\alpha_2 - \beta_2)\bar{u}_1 < \beta_2\omega$ then $E_1 = (1 - \beta_1, \frac{\alpha_1\bar{u}_1}{\beta_1}, 0)$ is locally asymptotically stable, and otherwise then $E_1$ is unstable.

**Proof.**

(1) The Jacobian matrix (6) at equilibrium point $E_0$ is

$$
J_{E_0} = \begin{pmatrix}
\alpha_1 & -\beta_1 & 0 \\
0 & 1 & -\beta_1 \\
0 & 0 & -\beta_2
\end{pmatrix}.
$$

(7)

The eigenvalues of Jacobian matrix $J_{E_0}$ (7) are

$$
\lambda_1 = -\beta_2
$$

$$
\lambda_{2,3} = -\beta_1 \pm \frac{\beta_1^2 + 4\alpha_1}{2}
$$

It is clear that $\lambda_{1,2} < 0$ and $\lambda_3 > 0$, hence equilibrium point $E_0$ is always unstable.

(2) The Jacobian matrix (6) at equilibrium point $E_1$ is

$$
J_{E_1} = \begin{pmatrix}
-\frac{\alpha_1}{\beta_1} & \beta_1 & -\frac{\bar{u}_1}{\omega + \bar{u}_1} \\
\alpha_1 & -\beta_1 & 0 \\
0 & 0 & -\frac{\alpha_2 u_1}{\omega + u_1} - \beta_2
\end{pmatrix}.
$$

(9)

The eigenvalues of Jacobian matrix (9) are

$$
\lambda_1 = \frac{(\alpha_2 - \beta_2)\bar{u}_1 - \beta_2\omega}{\omega + \bar{u}_1}
$$

$$
\lambda_{2,3} = \frac{-(\alpha_1 + \beta_1)^2}{2} \pm \frac{\alpha_1 + \beta_1}{2} \frac{\alpha_2 u_1}{\omega + u_1} - \beta_2
$$

(10)

Since all parameters are positive, it is easily seen that $\lambda_{2,3} < 0$. Thus, the stability of equilibrium $E_1$ depends on $\lambda_1$. From equation (10), it can be concluded that $E_1$ is stable if $(\alpha_2 - \beta_2)\bar{u}_1 < \beta_2\omega$. Furthermore, if this condition is satisfied, then the coexistence equilibrium $E^* (u_1^*, u_2^*, u_3^*)$ does not exist.

**Theorem 2.**

The coexistence equilibrium point $E^* = (u_1^*, u_2^*, u_3^*) = \left(\frac{\beta_2\omega}{\alpha_2 - \beta_2}, \frac{\beta_1 u_1}{\alpha_1}, u_3^*\right)$ is locally asymptotically stable if it satisfies all following conditions:
(a) $\alpha_x^2\beta_1 u_1^* > \beta_x^2 u_3^*$
(b) $\omega \beta_2 > \beta_1 u_1^*$

Proof.
The Jacobian matrix (6) at equilibrium point $E^* = \left(\frac{\beta_2 \omega}{\alpha_x^2 \beta_2}, \frac{\beta_1 u_1^*}{\alpha_1}, u_3^*\right)$ is

$$J_{E^*} = \begin{pmatrix}
-\frac{\alpha_1}{\beta_1} + \frac{\beta_x^2 u_3^*}{\alpha_x^2 u_1^*} & 1 - u_1^* - \frac{\beta_2}{\alpha_2} \\
\frac{\alpha_1}{\beta_1} & -\beta_1 & 0 \\
\frac{\omega \beta_x^2 u_3^*}{\alpha_x^2 u_1^*} & 0 & 0
\end{pmatrix}. \quad (11)$$

The characteristic equation of the Jacobian matrix (11) is

$$\lambda^3 + \rho_1 \lambda^2 + \rho_2 \lambda + \rho_3 = 0. \quad (12)$$

where $\rho_1 = \frac{\alpha_x^2 u_1^* (\beta_x^2 + \alpha_1) - \beta_x^2 u_3^*}{\beta_x^2 u_1^*}$, $\rho_2 = \frac{\alpha_x^2 u_1^* + \beta_x^2 u_3^* (\omega \beta_2 - \beta_1 u_1^*)}{\alpha_x^2 u_1^*}$ and $\rho_3 = \frac{\omega \beta_x^2 u_3^*}{\alpha_x^2 u_1^*}$. The stability of $E^*$ is studied using the Routh-Hurwitz criterion [10]. Based on this criterion, it can be shown that the coexistence equilibrium point $E^*$ is locally asymptotically stable if $\alpha_x^2 \beta_1 u_1^* - \beta_x^2 u_3^* > 0$, $\omega \beta_2 - \beta_1 u_1^* > 0$, and $\rho_4 = \rho_1 \rho_2 - \rho_3 > 0$

$$= \left(\alpha_x^2 u_1^* (\beta_x^2 + \alpha_1) - \beta_x^2 u_3^*\right) \left(\alpha_x^2 u_1^* + \beta_x^2 (\omega \beta_2 - \beta_1 u_1^*)\right) - \omega \beta_x^2 u_3^* > 0.$$

**Theorem 3.**

System (2) undergoes a Hopf-bifurcation around coexistence equilibrium $E^* = (u_1^*, u_2^*, u_3^*)$ when parameter $\alpha_2$ passes through $\alpha_2^*$ where $\alpha_2^*$ satisfies $\rho_4(\alpha_2^*) = \rho_1(\alpha_2^*) \rho_2(\alpha_2^*) - \rho_3(\alpha_2^*) = 0$ provided that $\alpha_x^2 \beta_1 u_1^* > \beta_x^2 u_3^*$ and $\omega \beta_2 > \beta_1 u_1^*$.

Proof.

For $\alpha_2 = \alpha_2^*$, by the condition $\rho_4(\alpha_2^*) = 0$, the characteristic equation (12) from Theorem 2 can be written as

$$(\lambda^2 + \rho_2)(\lambda + \rho_1) = 0 \quad (13)$$

If $\alpha_x^2 \beta_1 u_1^* > \beta_x^2 u_3^*$ and $\omega \beta_2 > \beta_1 u_1^*$, then from the proof of Theorem 2, we have that $\rho_1 > 0$ and $\rho_2 > 0$. The roots of equation (14) are $\lambda_1 = -\rho_1$, and $\lambda_{2,3} = i \sqrt{\rho_2}$. For any $\alpha_2$, the characteristic roots are $\lambda_1(\alpha_2) = -\rho_1(\alpha_2)$ and $\lambda_{2,3}(\alpha_2) = \mu(\alpha_2) \pm i \theta(\alpha_2)$. Substituting $\lambda = \mu + i \theta$ into equation (13) and calculating the derivative, we have

$$A_1(\alpha_2)\mu'(\alpha_2) - A_2(\alpha_2)\theta'(\alpha_2) + A_3(\alpha_2) = 0 \quad (14a)$$
$$A_2(\alpha_2)\mu'(\alpha_2) + A_1(\alpha_2)\theta'(\alpha_2) + A_4(\alpha_2) = 0 \quad (14b)$$

where

$$A_1(\alpha_2) = 3(\mu^2 - \theta^2) + 2 \rho_1 \mu + \rho_2,$$
$$A_2(\alpha_2) = 6 \mu \theta + 2 \rho_1 \theta,$$
$$A_3(\alpha_2) = \rho_1(\mu^2 - \theta^2) + \rho_2 \mu + \rho_3,$$
$$A_4(\alpha_2) = 2 \rho_1 \mu \theta + \rho_3 \theta.$$

By solving system (15) and using $A_1(\alpha_2)A_3(\alpha_2) + A_2(\alpha_2)A_4(\alpha_2) \neq 0$, we get

$$\left(\frac{d\text{Re}(A)}{d\alpha_2}\right)_{\alpha_2 = \alpha_2^*} = \mu'(\alpha_2^*) = -\frac{(A_1(\alpha_2)A_3(\alpha_2) + A_2(\alpha_2)A_4(\alpha_2))}{A_1^2(\alpha_2) + A_2^2(\alpha_2)} \neq 0. \quad (15)$$

Thus, the transversality condition be in force, and Hopf-bifurcation come to pass at $\alpha_2 = \alpha_2^*$.

3. Numerical Simulations

From the previous analytical proceeds, we present some numerical results. In consequence of all parameters values of the model are not available, we use hypothetical parameters: $\alpha_1 = 0.91$, $\alpha_2 = \alpha_2^*$. 

4
0.09, \( \omega = 0.08, \beta_1 = 0.06, \beta_2 = 0.09 \). Using these values, we have \((\alpha_2 - \beta_2) \hat{u}_1 - \beta_2 \omega = -0.0072 < 0\), and hence the predator extinction point \( E_1 = (0.940, 1.426, 0) \) is asymptotically stable and \( E^* \) does not exist. This means that immature and mature prey will survive in the system, while predator will go extinct. This situation is clearly shown by our numerical result shown in Figure 1. If we increase the value of \( \alpha_2 \), i.e. using \( \alpha_2 = 0.12 \), then we have that \((\alpha_2 - \beta_2) \hat{u}_1 - \beta_2 \omega = 0.0228 > 0, \rho_1 = 0.780 > 0, \rho_2 = 0.002 < 0, \) and \( \rho_4 = -0.003 < 0 \). Observe that the stability conditions for both extinction of predator point and coexistence point are not satisfied and therefore \( E_1 \) and \( E^* \) are unstable. This situation is shown in Figure 2. Indeed, the numerical solution is not convergent to any equilibrium point, but there appears a periodic solution. If we further increase the parameter value of \( \alpha_2 \) such that \( \alpha_2 = 0.14 \) and consistently using the same parameter, we obtain \((\alpha_2 - \beta_2) \hat{u}_1 - \beta_2 \omega = 0.042 > 0, \rho_1 = 0.800 > 0, \rho_2 = 0.005 > 0, \) and \( \rho_4 = 0.001 > 0 \). Hence, the coexistence equilibrium point \( E^* = (1.440, 0.214, 0.270) \) is stable. This behavior is confirmed by our numerical simulation as depicted in see Figure 3. From Figure 2 and Figure 3, we see that there occurs a Hopf bifurcation which is driven by \( \alpha_2 \).

Figure 1. The phase portrait of model (2) with \( \alpha_2 = 0.09 \)

Figure 2. The phase portrait of model (2) with \( \alpha_2 = 0.12 \)
Figure 3. The phase portrait of model (2) with $\alpha_2 = 0.14$

4. Conclusions

From the analysis of the system (2), we obtain three equilibrium points namely the extinction populations point $E_0$, the extinction of predator point $E_1$, and the coexistence point $E^*$. $E_0$ is always unstable, whereas $E_1$ and $E^*$ are stable under certain conditions. If $(\alpha_2 - \beta_2)\hat{u}_1 < \beta_2\omega$ then $E_1$ stable and $E^*$ does not exist. Increasing the parameter value of the predation rate may stabilize equilibrium $E^*$. Here, Hopf-bifurcation around coexistence equilibrium point $E^*$ with respect to the predation rate. The analysis shows that if $\alpha_2$ is less than or equal to $\beta_2$ then it may lead to disappearance of coexistence equilibrium. Otherwise, if $\alpha_2$ is greater than $\beta_2$ then the coexistence equilibrium point exists and is stable.

References

[1] Jia J and Wei X 2016 On the stability and Hopf-bifurcation of a predator-prey model Advances in Difference Equations 86 1-11

[2] Rosenzweig M L and MacArthur R H 1963 Graphical representation and stability conditions of predator-prey interactions American Naturalist 97 209–223

[3] Kot M 2001 Elements of Mathematical Ecology (Cambridge University Press: United Kingdom)

[4] Turchin P 2003 Complex Population Dynamics: A Theoretical/Empirical Synthesis (Princeton University Press: Princeton and Oxford United Kingdom)

[5] Moustdofa M, Mohd M H, Ismail A I and Abdullah F A 2018 Dynamical analysis of a fractional order Rosenzweig-MacArthur model incorporating a prey refuge Chaos, Soliton and Fractals 109 1-13

[6] Ivanov T and Dimitrova N 2017 A predator–prey model with generic birth and death rates for the predator and Beddington–DeAngelis functional response Mathematics and Computers in Simulation 133 111–123

[7] Kar T K 2005 Stability analysis of a prey–predator model incorporating a prey refuge Communications in Nonlinear Science and Numerical 10 681–691

[8] Chen L, Chen F and Chen L 2010 Qualitative analysis of a predator–prey model with Holling type II functional response incorporating a constant prey refuge Nonlinear Analysis 11 246–252

[9] Javidi M and Nyamoradi N 2013 Dynamic analysis of a fractional order prey–predator interaction with harvesting Applied Mathematical Modelling 37 8946–8956

[10] Panja P and Mondal S K 2015 Stability analysis of coexistence of three species prey-predator model Nonlinear Dynamics 81 373-382

[11] Shi X, Zhou X and Song X 2011 Analysis of state-structured predator-prey model with Crowley-Martin function Journal of Applied Mathematics and Computing 36 459-472
[12] Kar T K and Jana S 2012 Stability and bifurcation analysis of a stage structured predator-prey model with time delay Applied Mathematics and Computation 219 3779-3792
[13] Khajanchi S 2014 Dynamic behavior of a Beddington-DeAngelis type stage structured predator-prey model Applied Mathematics and Computation 244 344-360
[14] Al-Omari J M F 2015 The effect of state dependent delay and harvesting on a stage-structured predator-prey model Applied Mathematics and Computation 271 142-153
[15] Khajanchi S 2017 Modelling the dynamics of stage-structure predator-prey system with Monod-Haldane type response function Applied Mathematics and Computation 302 122-143
[16] Khajanchi S and Banerjee S 2017 Role of constant prey refuge on stage structure predator-prey model with ratio dependent functional response Applied Mathematics and Computation 314 193-198
[17] Falconi M, Huenchucona M and Vidal C 2015 Stability and global dynamic of a stage-structured predator-prey model with group defense mechanism of the prey Applied Mathematics and Computation 270 47-61
[18] Murray J D 2002 Mathematical Biology: I. An Introduction Third Edition (Springer-Verlag: New York USA)