A note on the non-existence of prime models of theories of pseudo-finite fields

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Abstract

We show that if a field $A$ is not pseudo-finite, then there is no prime model of the theory of pseudo-finite fields over $A$. Assuming GCH, we extend this result to $\kappa$-prime models, for $\kappa$ an uncountable cardinal or $\aleph_\varepsilon$.

Introduction

In this short note, we show that prime models of the theory of pseudo-finite fields do not exist. More precisely, we consider the following theory $T(A)$: $F$ is a pseudo-finite field, $A$ a relatively algebraically closed subfield of $F$, and $T(A)$ is the theory of the field $F$ in the language of rings augmented by constant symbols for the elements of $A$. Our first result is:

Theorem 2.6 Let $T(A)$ be as above. If $A$ is not pseudo-finite, then $T(A)$ has no prime model.

Next we address the question of existence of $\kappa$-prime models of $T(A)$, where $\kappa$ is an uncountable cardinal or $\aleph_\varepsilon$. We assume GCH, and again show non-existence of $\kappa$-prime models of $T(A)$, unless $|A| < \kappa$ or $A$ is $\kappa$-saturated when $\kappa \geq \aleph_1$, and when the transcendence degree of $A$ is infinite if $\kappa = \aleph_\varepsilon$ (Theorem 3.5).

These results are not surprising, given that any complete theory of pseudo-finite fields has the independence property. However, the proofs do use some properties which are specific to pseudo-finite fields, so it is not clear that the results would hold in the general case of theories with IP. The question arose during the study of the existence (and uniqueness) of certain strengthenings of the notion of difference closure of difference fields of characteristic 0.

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In [2], we show that if \(K\) is an algebraically closed difference field of characteristic 0, and \(\kappa\) an uncountable cardinal or \(\aleph\), and if \(\text{Fix}(\sigma)(K)\) is a \(\kappa\)-saturated pseudo-finite field, then \(\kappa\)-prime models of ACFA (the theory of existentially closed difference fields) over \(K\) exist and are unique up to \(K\)-isomorphism. The question then arises of whether the hypothesis on the fixed field of \(K\) is necessary. This note shows that it is, under natural assumptions.

The paper is organised as follows. Section 1 recalls well-known facts about fields, Section 2 gives the results on the non-existence of prime models, and Section 3 those on the non-existence of \(\kappa\)-prime models.

1 Preliminaries

1.1. Convention and notation. Unless otherwise mentioned, all fields will be subfields of a large algebraically closed field. If \(K\) is a field, then \(K^s\) denotes the separable closure of \(K\), \(K^{\text{alg}}\) its algebraic closure, and \(G(K)\) its absolute Galois group \(\text{Gal}(K^s/K)\). If \(L\) is an extension of the field \(K\), and \(\sigma \in \text{Aut}(L/K)\), then \(\text{Fix}(\sigma)\) will denote the subfield of \(L\) consisting of elements fixed by \(\sigma\). If \(\sigma \in G(K)\), then \(\langle \sigma \rangle\) denotes the topological closure inside \(G(K)\) of the group generated by \(\sigma\).

1.2. Classical algebraic results on fields. (See chapter 3 of Lang’s book [6]) Let \(K \subset L\) be fields. Recall that \(L\) is regular over \(K\) if it is linearly disjoint from \(K^{\text{alg}}\) over \(K\). If \(K\) is perfect (i.e., of characteristic 0, or if of characteristic \(p > 0\), closed under \(p\)-th roots), then this is equivalent to \(L \cap K^s = K\). The perfect hull of \(K\) is \(K\) if \(\text{char}(K) = 0\), and the closure of \(K\) under \(p\)-th roots if \(\text{char}(K) = p > 0\). The field \(L\) is separable over \(K\) if it is linearly disjoint from the perfect hull of \(K\) over \(K\). Finally, if \(L\) is separable over \(K\), then \(L \cap K^s = K\) implies that \(L\) is regular over \(K\).

Recall also that a polynomial \(f \in K[\overline{X}]\) is called absolutely irreducible if it is irreducible in \(K^{\text{alg}}[\overline{X}]\). This corresponds to the field \(\text{Frac}(K[\overline{X}]/(f))\) being a regular extension of \(K\).

1.3. The Haar measure. Recall that if \(K\) is a field, then \(G(K)\) can be endowed uniquely with a measure \(\mu\) on the \(\sigma\)-algebra generated by open subsets of \(G(K)\), which satisfies \(\mu(G) = 1\), and is stable under translation. This measure is called the Haar measure. If \(L\) is a finite separable extension of \(K\), then \(\mu(G(L)) = \frac{1}{[L : K]}\). Furthermore, assume that \(L_i, i < \omega\), is a family of linearly disjoint algebraic extensions of \(K\) and \(A_i\) a non-empty set of left-cosets of \(G(L_i)\) in \(G(K)\). If \(\sum_i [L_i : K]^{-1} = \infty\), then \(\mu(\bigcup_i A_i) = 1\) (Lemma 18.5.2 in [4]).

1.4. Review on Hilbertian fields and their properties. All references are to the book of Fried and Jarden, [4].

(1) Recall that a field \(K\) is Hilbertian if whenever \(f \in K[T, X]\) (\((T, X)\) a tuple of indeterminates, \(|X| = 1\)) is separable in \(X\) and irreducible over \(K(T)[X]\), then there are infinitely many tuples \(a\) in \(K\) such that \(f(a, X)\) is irreducible over \(K\). There are many equivalent statements of this property, and in particular if it is satisfied for \(|T| = 1\), then it is satisfied for tuples \(T\) of arbitrary length (Proposition 13.2.2).
Facts 1.5. Some easy observations and reminders.

(2) Examples of Hilbertian fields include $\mathbb{Q}$ and any finitely generated infinite field. Function fields are Hilbertian, and if $K$ is Hilbertian, then so is any finite algebraic extension of $K$. An infinite separably algebraic extension $L$ of a Hilbertian field $K$ is not necessarily Hilbertian, but any finite proper separable extension of $L$ which is not contained in the Galois hull of $L$ (over $K$) is Hilbertian (Theorem 13.9.4).

(3) Let $M_1, M_2$ be Galois extensions of the Hilbertian field $K$, and $M$ a subfield of $M_1M_2$ containing $K$ and such that $M \nsubseteq M_i$ for $i = 1, 2$. Then $M$ is Hilbertian (Theorem 13.8.3).

(4) In order to state some properties of Hilbertian fields, it is convenient to define, for $K$ a field and irreducible polynomials $f_1, \ldots, f_m \in K[T][X]$ which are separable in $X$, and non-zero $g \in K[T]$, the separable Hilbert set $H_K(f_1, \ldots, f_m; g)$ as the set of $a \in K$ such that $g(a) \neq 0$ and $f_1(a, X), \ldots, f_m(a, X)$ are irreducible over $K$.

(5) Every separable Hilbert subset of $K^r$ contains one of the form $H_K(f)$, with $f$ monic irreducible and separable (Lemma 12.1.6). Hence if $K$ is Hilbertian then every separable Hilbert set is infinite.

(6) Let $L$ be a finite separable extension of $K$. Then every separable Hilbert subset of $L$ contains a separable Hilbert subset of $K$ of the forme $H_K(f)$ (Lemma 12.2.2).

(7) Let $K$ be a Hilbertian field, $f(T, X) \in K[T, X]$ irreducible and separable in $X$, and $G$ the Galois group of the Galois extension of $K(T)$ generated by the roots of $f(T, X) = 0$. Then there is a separable Hilbert set $H \subseteq K^r$ such that if $a \in H$, then the Galois group of the extension generated by the roots of $f(a, X) = 0$ is isomorphic to $G$. In particular, $f(a, X)$ is irreducible (Proposition 16.1.5).

Facts 1.5. Some easy observations and reminders.

(1) Let $B$ be a primary\footnote{i.e., $B \cap A^r = A$.} extension of the field $A$, and $\sigma \in G(A)$. Then $\sigma$ lifts to some $\sigma' \in G(B)$. Indeed, $\sigma$ has an obvious extension to $A^r \otimes_A B$ given by $\sigma' = \sigma \otimes id$; by primarity of $B/A$, $A^r \otimes_A B$ is a domain, and is isomorphic to $A^r B$. This automorphism $\sigma'$ of $A^r B$ extends to an automorphism of $B^r$ which is the identity on $B$. Recall that if $B$ is a regular extension of $A$ then it is primary.

(2) Let $K$ be a field, $\sigma \in G(K)$. Then $\langle \sigma \rangle \simeq \hat{\mathbb{Z}}$ if and only if $\langle \sigma \rangle$ has a quotient isomorphic to $\mathbb{Z}/4\mathbb{Z}$, and quotients isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for every odd prime $p$. The necessity is clear, the sufficiency follows from the fact that the only possible order of a torsion element of the absolute Galois group of a field is 2 (and then the field is of characteristic 0 and does not contain $\sqrt{-1}$) and that $\langle \sigma \rangle$ is the direct product of its Sylow subgroups. When $\text{char}(K)$ is positive, it suffices that $G(K)$ has a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for every prime $p$.

(3) Recall that by Theorem 11.2.3 of \cite{4}, if $L$ an algebraic extension of a field $K$, and every absolutely irreducible $f(X, Y) \in K[X, Y]$ has a zero in $L$, then $L$ is PAC.
(4) Let $K$ be a field, $f(X, Y) \in K[X, Y]$ an absolutely irreducible polynomial. If $f(X, Y)$ is not separable as a polynomial in $X$, then it is separable as a polynomial in $Y$. Indeed, otherwise it would not stay irreducible over the perfect hull of $K$.

(5) (Kummer theory). Let $K$ be a field of characteristic not 2, let $t$ be transcendental over $K$, and $a_1, \ldots, a_n$ distinct elements of $K$. Then the fields $K(t)(\sqrt{t+a_i})$ are linearly disjoint over $K(t)$, and they are proper Galois extensions of $K(t)$. Moreover, the field $K(t)(\sqrt{t+a_i} \mid 1 \leq i \leq n)$ is a regular extension of $K$. The general phenomenon is as follows: let $L$ be an extension of $K(t)$ generated by square roots of polynomials $f_i(t)$, $i = 1, \ldots, n$, and assume that the elements $f_i(t)$ are multiplicatively independent modulo the multiplicative subgroup $K^\times K(t)^{\times 2}$ of $K(t)^\times$; then $L$ is a regular extension of $K(t)$, and $\text{Gal}(L/K(t)) \simeq (\mathbb{Z}/2\mathbb{Z})^n$.

(6) (Artin-Schreier theory) Let $K$ be a field of characteristic 2, and $a_1, \ldots, a_n \in K$ be $\mathbb{F}_2$-linearly independent. Let $\alpha_i$ be a root of $X^2 + X + a_i t = 0$ for $i = 1, \ldots, n$. Then the fields $K(t)(\alpha_i)$ are linearly disjoint over $K(t)$, and are proper Galois extensions of $K(t)$. Moreover, the field $K(t, \alpha_1, \ldots, \alpha_n)$ is a regular extension of $K$. The general phenomenon is as follows: let $L$ be an extension of $K(t)$ generated by solutions of $X^2 + X + f_i(t) = 0$, $i = 1, \ldots, n$, where the $f_i(t)$ are elements of $K(t)$, which are $\mathbb{F}_2$-linearly independent modulo the additive subgroup $K + \{ f(t)^2 - f(t) \mid f(t) \in K(t) \}$ of $K(t)$; then $L$ is a regular extension of $K(t)$, and $\text{Gal}(L/K(t)) \simeq (\mathbb{Z}/2\mathbb{Z})^n$.

(7) (Linear disjointness). Recall that if if $M \subset N$ are fields, and $L$ is a Galois extension of $M$, then $N$ and $L$ are linearly disjoint over $N \cap L$. The same holds if $N$ is perfect and $L$ is the perfect hull of a Galois extension of $M$, because $L$ will then be a Galois extension of the perfect field $L \cap N$. This remark will be constantly used.

**Lemma 1.6.** Let $K \subset L_1, L_2$ be three algebraically closed fields, with $L_1$ and $L_2$ linearly disjoint over $K$, and consider the field composite $L_1 L_2$. Let $u \in L_1 \setminus K$, $v \in L_2 \setminus K$.

(1) If $\text{char}(K) \neq 2$, then $\sqrt{u+v} \notin L_1 L_2$.

(2) Assume $\text{char}(K) = 2$, and let $\alpha$ be a root of $X^2 + X + uv$. Then $\alpha \notin L_1 L_2$.

**Proof.** (I thank Olivier Benoist for this elegant proof.)

(1) If $\alpha = \sqrt{u+v} \in L_1 L_2$, then there are finite tuples $u_1 \in L_1$ and $v_1 \in L_2$ such that $\alpha \in K(u, u_1, v, v_1)$. As $L_1$ and $L_2$ are free over $K$, $t p_{\text{ACP}}(v, v_1/L_1)$ does not fork over $K$ (in the sense of the theory ACF of algebraically closed fields), and therefore is finitely satisfiable in $K$. In particular there are infinitely many elements $b \in K$ such that $u + b$ has a square root in $K(u, u_1)$. But this is impossible: as we saw above in [1.5(5)], the extensions $K(\sqrt{u+b}), b \in K$, are linearly disjoint over $K(u)$, and therefore $K(u, u_1)$ contains at most finitely many of them, since it is finitely generated over $K(u)$.

(2) Same proof: If $\alpha \in L_1 L_2$, then $\alpha \in K(u, u_1, v, v_1)$ for some $u_1 \in L_1$ and $v_1 \in L_2$. For all but finitely many $b \in K$, the equation $X^2 + X + bu$ would have a solution in $K(u, u_1)$, but this is impossible by [1.5(6)].
Corollary 1.7. Let $L_1$ and $L_2$ be regular extensions of the field $K$, which are linearly disjoint over $K$. Let $u \in L_1 \setminus K$, $v \in L_2 \setminus K$.

(1) If char($K$) $\neq 2$, then $L_1 L_2(\sqrt{u+v}) \cap L_1^{alg} L_2^{alg} = L_1 L_2$. Hence $L_1 L_2(\sqrt{u+v})$ is a regular extension of both $L_1$ and $L_2$.

(2) Assume char($K$) = 2, and let $\alpha$ be a root of $X^2 + X + uv$. Then $L_1 L_2(\alpha) \cap L_1^{alg} L_2^{alg} = L_1 L_2$, and $L_1 L_2(\alpha)$ is a regular extension of both $L_1$ and $L_2$.

Proof. (1) Our assumption implies that $L_1$ and $L_2$ are free over $K$, and therefore that their algebraic closures $L_1^{alg}$ and $L_2^{alg}$ are linearly disjoint over $K^{alg}$. By Lemma 1.6, $\sqrt{u+v} \notin L_1^{alg} L_2^{alg}$. I.e., $L_1 L_2(\sqrt{u+v}) \cap L_1^{alg} L_2^{alg} = L_1 L_2$, so that $L_1 L_2(\sqrt{u+v})$ is a regular extension of both $L_1$ and $L_2$. Same proof for (2).

Lemma 1.8. (Folklore) Let $G$ be a finite abelian group, $F$ a field, $t$ an indeterminate, and assume that $F$ has only finitely many Galois extensions with Galois group isomorphic to a quotient of $G$. Then there is a sequence $L_i$, $i \in \omega$, of linearly disjoint Galois extensions of $F(t)$ with Galois group isomorphic to $G$, and the field composite of which is a regular extension of $F$.

Proof. Let $M$ be the composite of the finitely many abelian Galois extensions of $F$ with Galois group isomorphic to a quotient of $G$. Let $u$ be a new indeterminate. By Proposition 16.3.5 of [H], letting $K = F(t)$, the field $K(u)$ has a Galois extension $L$ which is regular over $K$, and with Galois group $G$. Let $\alpha$ be a generator of $L$ over $K(u)$, and $f(u, X) \in K(u)[X]$ its minimal polynomial over $K(u)$. As $L$ is regular over $K$, $f(u, X)$ is irreducible over $M(t, u)$.

Observe that if $L'$ is a Galois extension of $K$ with Galois group $G$, and if $L' \cap M = F$, then $L'$ is regular over $F$. Indeed, $L' \cap F^s$ is a Galois extension of $F$, with Galois group isomorphic to a quotient of $G$, and therefore is contained in $M$. Our assumption therefore implies that $L' \cap F^s = F$. Furthermore, $L'$ is separable over $F$, hence regular over $F$.

As $K$ is Hilbertian, by (4) and (7) there is $a \in K = F(t)$ such that $f(a, X)$ is irreducible over $M(t)$, and such that the field $L_0$ generated over $F(t)$ by a root of $f(a, X)$ is Galois with Galois group isomorphic to $G$. Then $L_0 \cap M = F$, and by the discussion in the previous paragraph, $L_0$ is a Galois extension of $F(t)$ which is regular over $F$.

Replacing $M(t)$ by $ML_0$, we construct in the same fashion a Galois extension $L_1$ of $K$, with Galois group isomorphic to $G$, and which is linearly disjoint from $ML_0$ over $K$. We iterate the construction and build by induction a sequence $L_i$, $i \in \mathbb{N}$, of Galois extensions $L_i$ of $K$ with Galois group isomorphic to $G$, and such that for every $i$, $L_i$ is linearly disjoint from $ML_0 \cdots L_{i-1}$ over $K$. In particular, the field composite of all $L_i$’s is a regular extension of $F$.

1.9. Review on pseudo-finite fields and their properties. Recall that the theory of pseudo-finite fields is axiomatised by the following properties: the field is PAC (every absolutely irreducible variety defined over the field has a rational point); the absolute Galois group is isomorphic to $\hat{\mathbb{Z}}$ (= $\lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z}$); if the characteristic is $p > 0$, then the field is perfect (closed under $p$-th roots). We will mainly use the following five results:
(1) Let $F_1$ and $F_2$ be two pseudo-finite fields containing a common subfield $E$. Then

$$F_1 \equiv_E F_2$$

if and only if there is an $E$-isomorphism $F_1 \cap E^{alg} \to F_2 \cap E^{alg}$.

(2) Let $L$ be a relatively algebraically closed subfield of the perfect field $E$ and of the $|E|^+$-saturated pseudo-finite field $F$. Assume that $G(E)$ is procyclic. Then there is an $L$-embedding $\Phi$ of $E$ into $F$ such that $F/\Phi(E)$ is regular.

(3) If $E$ is a perfect field with pro-cyclic absolute Galois group, then it has a regular extension $F$ which is pseudo-finite.

(1) is a special case of 20.4.2 in [4].

(2) follows from the Embedding Lemma (20.2.2 and 20.2.4 in [4]) with $\Phi_0 = id$: the restrictions maps $\text{res}_{F/L} : G(F) \to G(L)$ and $\text{res}_{E/L} : G(E) \to G(L)$ are onto, and because $G(E)$ is procyclic and $G(F)$ is free, there is an onto map $\varphi : G(F) \to G(E)$ such that $\text{res}_{E/L} \varphi = \text{res}_{F/L}$. The lemma then gives the map $\Phi$, and because $\varphi$ is onto and $E$ is perfect, the extension $F/\Phi(E)$ is regular.

(3) is folklore, but I was not able to find an explicit statement of it: when $E$ is a subfield of the algebraic closure of the prime field $k$, this is given by Propositions 7 and 7’ of [4]. In the general case, $E$ is a regular extension of $L := \overline{k}^{alg} \cap E$, and $L$ has pro-cyclic Galois group and is perfect. By the above, there is some pseudo-finite field $F$ containing $L$, which is regular over $L$, and we may assume it is sufficiently saturated. Because $G(E)$ is procyclic, there is an onto map $\varphi : G(F) \to G(E)$ such that $\text{res}_{E/L} \varphi = \text{res}_{F/L}$, and we conclude as in (2).

These three results have several consequences. For instance, if $E \subset F_1$ is relatively algebraically closed in the pseudo-finite field $F_1$, then the theory Psf together with the quantifier-free diagramme of $E$ is complete (in the language $\mathcal{L}(E)$ of rings augmented by constant symbols for the elements of $E$). In particular, if $a \in F_1$ is transcendental over $E$, then $tp(a/E)$ is entirely axiomatised by the collection of $\mathcal{L}(E)$-formulas expressing that it is transcendental over $E$, as well as, for each finite Galois extension $L$ of $E(a)$, a formula which describes the isomorphism type over $E(a)$ of $L \cap F_1$. So this formula will say which polynomials $f(a, X) \in E[a, X]$ have a solution in $F_1$ and which do not. By (3) above, note that any subfield $K$ of $L$ which is a regular extension of $E$ and with $\text{Gal}(L/K)$ cyclic can appear as $L \cap F$ for some model $F$ of $T(E)$ which contains $a$.

(4) Hence, if $F_1$ is a pseudo-finite field containing $E$ and regular over $E$, and $a \in F_1$ is transcendental over $E$, then $tp(a/E)$ is not isolated. This follows easily from the description of types, and because $E(a)$ has infinitely many linearly disjoint extensions $L_i$ ($i \in \mathbb{N}$), the composite $L$ of which is regular over $E$ (see Lemma [5,8]). Indeed the type of $a$ is axiomatized by saying that $a$ is transcendental over $E$, and by saying which polynomials $f(a, X) \in E[a, X]$ have a root in $F_1$ and which have not. In particular, any $\mathcal{L}(E)$-formula $\varphi(x)$ will only give information about $F_1 \cap L_0 \cdots L_n$ for some $n$, and say nothing about $F_1 \cap L_{n+1}$, and whether it equals $L_{n+1}$ or not.

(5) If $F_1$ is pseudo-finite and $E \subset F_1$, then $acl(E) = E^{alg} \cap F_1$, see Proposition 4.5 in [4].
2 Non-existence of prime models

2.1 Setting. Let $T$ be a complete theory of pseudo-finite fields, $F$ a model of $T$, and $A \subset F$, $T(A)$ the $L(A)$-theory of $F$ ($L$ the language of rings $\{+, -, \cdot, 0, 1\}$). We want to show that unless $\operatorname{acl}(A)$ is a pseudo-finite field, then $T(A)$ has no prime model. As $T(A)$ describes the $A$-isomorphism type of $\operatorname{acl}(A) = A^\text{alg} \cap F$ over $A$, without loss of generality, we will assume that $A^\text{alg} \cap F = A$. Note that $A$ is perfect, $G(A)$ is procyclic, and we will fix a topological generator $\sigma$ of $G(A) = \operatorname{Gal}(A^s/A)$.

Notation 2.2. Let $A$ be a field, $F$ a regular field extension of $A$, and $t \in A$. We denote by $S(t, F)$ the set

$$S(t, F) = \begin{cases} \{a \in A \mid \sqrt{t + a} \in F\} & \text{if char}(A) \neq 2, \\ \{a \in A \setminus \{0\} \mid F \models \exists y y^2 + y = at\} & \text{if char}(A) = 2. \end{cases}$$

Remark 2.3. Observe that if $F \subseteq F'$, $F^\text{alg} \cap F' = F$ and $t \in F$, then $S(t, F) = S(t, F')$.

Proposition 2.4. Let $T$ and $A$ be as above, with $A$ not pseudo-finite. Then $T(A)$ has a model $F_0$ of transcendence degree 1 over $A$. Furthermore:

1. Assume that $A$ is countable, let $t$ be transcendental over $A$, and let $\tilde{\sigma}$ be a lifting of $\sigma$ to $G(A(t))$. Then for almost all $\tau \in G(A^s(t))$, the perfect closure of the subfield of $A(t)^s$ fixed by $\tilde{\sigma}\tau$ is a model of $T(A)$.

2. Assume that $|A| = \kappa \geq \aleph_0$. When char$(A) \neq 2$, we choose some $X \subset A \setminus \{0\}$. If char$(A) = 2$, we fix a basis $Z$ of the $\mathbb{F}_2$-vector space $A$ with $1 \in Z$, and take $X \subset Z$. Then there is a model $F_X$ of $T(A)$ which has transcendence degree 1 over $A$, and is such that for some $t \in F_X \setminus A$,

$$\begin{cases} S(t, F_X) = X & \text{when char}(A) \neq 2, \\ S(t, F_X) \cap Z = X & \text{when char}(A) = 2. \end{cases}$$

Proof. A model of $T(A)$ is a regular extension of $A$, with absolute Galois group isomorphic to $\hat{\mathbb{Z}}$, and which is PAC and perfect. For both items we will construct the model as an algebraic extension of $A(t)$: we will first work inside $A(t)^s$, then take the perfect closure. Recall that by Lemma 1.5.3, for the PAC condition, it suffices to build a regular extension of $A$ contained in $A(t)^s$, and in which every absolutely irreducible plane curve defined over $A(t)$ has a point. Then its perfect closure will be pseudo-finite. We first show (1). We will show the following:

1. if $f(X, Y) \in A(t)[X, Y]$ is absolutely irreducible, then for almost all $\tau \in G(A^s(t))$ (in the sense of the Haar measure $\mu$ on $G(A^s(t))$), $\operatorname{Fix}(\tilde{\sigma}\tau)$ contains a solution of $f(X, Y) = 0$.
2. for almost all $\tau \in G(A^s(t))$, for every $n \geq 2$, $\langle \tilde{\sigma}\tau \rangle$ has a quotient isomorphic to $\mathbb{Z}/n\mathbb{Z}$.
Towards (i), let \( f(X, Y) \in A(t)[X, Y] \) be absolutely irreducible; by Fact 1.3(4) we may assume that \( f \) is separable in \( Y \), and we let \( m \) be the degree of \( f \) in \( Y \). Let \( B \) be the subfield of \( A^s \) fixed by \( \sigma^m \). As \( A(t) \) is Hilbertian, as in the proof of Lemma 1.8 (using 1.4(6)), we build inductively a sequence \( L_i, i < \omega \), of finite separable extensions of \( A(t) \), and of elements \( a_i \in A(t) \), such that:

- the polynomial \( f(a_i, Y) \) is irreducible over \( BL_0 \cdots L_{i-1} \) for all \( i \) (over \( B(t) \) if \( i = 0 \));
- \( L_i = A(t, b_i) \) where \( f(a_i, b_i) = 0 \).

(For more details one may look at Theorem 18.6.1 in [4].) Note that because \([L_i : A(t)] \leq m\), it follows that \( L_i \) is linearly disjoint from \( A^sL_0 \cdots L_{i-1} \) over \( A(t) \) for every \( i \), and therefore that the field composite \( L \) of all \( L_i \)'s is a regular extension of \( A \). By Fact 1.1(1), \( A \) extends to some \( \sigma' \in G(L) \). Then, for every \( \tau \in \bigcup_i G(A^sL_i) \), \( \text{Fix}(\sigma' \tau) \) contains a solution of \( f(X, Y) = 0 \). Hence, for every \( \tau \in (\hat{\sigma}^{-1} \sigma')(\bigcup_i G(A^sL_i)) \), \( \text{Fix}(\hat{\sigma} \tau) \) contains a solution of \( f(X, Y) = 0 \). By 1.3.

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\mu(\bigcup_i G(A^sL_i)) = 1, \quad \text{so does its translate by } \hat{\sigma}^{-1} \sigma'. \quad \text{This shows (i).}
\]

(ii) is proved in the same fashion, using 1.3(2). Let \( n \) be a prime or 4, and use Lemma 1.8 to find a sequence \((L_i)_{i<\omega}\) of linearly disjoint Galois extensions of \( A(t) \), with \( \text{Gal}(L_i/A(t)) \simeq \mathbb{Z}/n\mathbb{Z} \), and such that the field composite \( L \) of all \( L_i \)'s is a regular extension of \( A \). As in (i), the set of \( \tau \in G(A^s(t)) \) such that for some \( i \), \( \tau |_{L_i} \) generates \( \text{Gal}(L_i/A(t)) \), has measure 1, and therefore so does its translate (on the left) by \( \hat{\sigma}^{-1} \sigma' \). This proves (ii).

A countable intersection of sets of Haar measure 1 has measure 1, and therefore the set of \( \tau \in G(A^s(t)) \) such that

\[
\text{every absolutely irreducible } f(X, Y) \text{ has a solution in } \text{Fix}(\hat{\sigma} \tau), \quad \langle \hat{\sigma} \tau \rangle \simeq \hat{\mathbb{Z}}
\]

has measure 1. For any such \( \tau \), the field \( \text{Fix}(\hat{\sigma} \tau) \) is therefore PAC, with absolute Galois group isomorphic to \( \hat{\mathbb{Z}} \), and its perfect closure is our desired pseudo-finite field.

(2) There are four cases to consider, depending on the characteristic, and whether \( A \) has an algebraic extension of degree 2 or not. Let \( t \) be an indeterminate over \( A \).

Case 1: \( \text{char}(A) \neq 2 \) and \( A^2 \neq A \):
Let \( c \in A \setminus A^2 \), and consider the Galois extension \( L_0 \) of \( A(t) \) defined as the field composite of all \( A(t, \sqrt{t + a}) \) for \( a \in X \), and \( A(t, \sqrt{ct + ca}) \) for \( a \in A \setminus X \). If \( B \supseteq L_0 \) is regular over \( A \), then \( S(t, B) = X \): indeed, \( c(t + a) \in B^2 \), \( c \notin B^2 \) imply \( (t + a) \notin B^2 \).

Case 2: \( \text{char}(A) \neq 2 \) and \( A^2 = A \):
We let \( L_0 \) be the field composite of all \( A(t, \sqrt{t + a}) \) for \( a \in X \), and all \( A(t, \sqrt{t^2 + at}) \) for \( 0 \neq a \in A \setminus X \). If \( B \supseteq L_0 \) is such that \( t \notin B^2 \), then \( S(t, B) = X \).

Case 3: \( \text{char}(A) = 2 \), and \( A \) has an extension of degree 2, say \( Y^2 + Y + c = 0 \) has no solution in \( A \):
Let \( L_0 \) be the field obtained by adjoining to \( A(t) \) a solution of \( Y^2 + Y + at = 0 \) if \( a \in X \), and a solution of \( Y^2 + Y + at + c = 0 \) if \( a \in Z \setminus X \). Then if \( B \supseteq L_0 \) is a regular extension of \( A \), we have \( S(t, B) \cap Z = X \).

Case 4: \( \text{char}(A) = 2 \), and \( A \) is closed under Artin-Schreier extensions:
Let $\alpha$ satisfy $Y^2 + Y + t^3 = 0$, and let $L_0$ be the Galois extension of $A(t)$ obtained by adjoining a solution of $Y^2 + Y + at = 0$ if $a \in X$, and $Y^2 + Y + at + t^3 = 0$ if $a \in Z \setminus X$. Again, if $B \supseteq L_0$ does not contain $\alpha$, then $S(t, B) \cap Z = X$.

Note that in all four cases, $L_0$ is regular over $A$ (by [1.5]5) and (6)), and is Hilbertian (by [1.6]3)). It therefore suffices to construct an algebraic extension of $L_0$ which is regular over $A$, does not contain the forbidden elements $t^{1/2}$ or $\alpha$ when $A$ has no proper algebraic extension of degree 2, and is pseudo-finite. To do the latter, we will construct inside $L_0^*$ a PAC field which contains $L_0$, and with Galois group isomorphic to $\hat{\mathbb{Z}}$. We first take care of the Galois group. To do that, to find some Galois extension $L$ of $A(t)$, which is linearly disjoint from $A^*L_0$ over $A(t)$, and such that $\text{Gal}(L/A(t)) \simeq \hat{\mathbb{Z}}$. Let $Q$ be the set of $n$ which are prime numbers or 4 and such that $G(A)$ does not have a quotient isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

For each odd $n \in Q$, using Lemma [1.16] we find a Galois extension $L_n$ of $A(t)$ with Galois group isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and which is regular over $A$. When $n$ is 2 or 4 we need to be a little more careful.

Case 3 is vacuous, as is Case 1 when $\text{char}(A) \neq 0$. In case 2, $A$ contains $\sqrt{-1}$, and we let $L_2 = L_4 = A(t^{1/4})$. Then $L_2$ is linearly disjoint from $A^*L_0$ over $A(t)$, with Galois group $\mathbb{Z}/4\mathbb{Z}$.

In case 4, we let $L_4 = L_4 = A(t)(a)$; it is linearly disjoint from $A^*L_0$ over $A(t)$. The case of $2 \not\in Q$ and $4 \in Q$, so that $\text{char}(A) = 0$, is more delicate, and we proceed as follows (it is a particular case of the construction given in Lemma 16.3.1 of [3]). We fix a square root $i$ of $-1$; then $\sigma(i) = -i$ (a generator of $G(A)$). Consider the element $1 + it$, and let $a \in A(t)^*$ satisfy $a^4 = 1 + it$. Such an element $a$ can be found in $A(i)[[t]]$ (by Hensel’s lemma), and we may therefore lift $\sigma$ to an element $\sigma_1 \in \text{Aut}(A(t, i, a)/A(t))$ with $\sigma_1^2 = \text{id}$. Let $b = a\sigma_1(a)^3$, and note that

$$b^4 = (1 + it)(1 - it)^3 = (1 + t^2)(1 - it^2),$$

and that $1 + t^2$ has no square root in $L_0(i)$. Hence $[A(t, i, b) : A(t, i)] = 4$, and $[A(t, i, b) : A(t)] = 8$. Define $\omega \in \text{Gal}(A(t, i, b)/A(t, i))$ by $\omega(b) = ib$. We now compute $\omega\sigma_1$ and $\sigma_1\omega$ on $i$ and on $b$. We have:

$$\omega\sigma_1(i) = \omega(-i) = -i, \quad \sigma_1\omega(i) = \sigma_1(i) = -i, \quad \sigma_1\omega(b) = \sigma_1(ib) = -i\sigma_1(b),$$

and one computes

$$\omega\sigma_1(b) = \omega(\sigma_1(a)a^3) = \omega(b^3\sigma_1(a)^{-8}) = -ib^3\sigma_1(a)^{-8} = -i\sigma_1(b).$$

(Here we use that $\sigma_1$ is an involution, that $\sigma_1(a)^8 \in A(t, i)$ is fixed by $\omega$). So $\sigma_1$ and $\omega$ commute, and $\text{Gal}(A(t, i, b)/A(t))$ is the direct product of the subgroups generated by $\sigma_1$ and by $\omega$. We take $L_4$ to be the subfield of $A(t, i, b)$ fixed by $\sigma_1$. It is regular over $A$, with Galois group over $A(t)$ isomorphic to $\mathbb{Z}/4\mathbb{Z}$. One computes that $\sqrt{1 + t^2} \in A(t, i, b)$ is fixed by $\sigma_1$ (since $1 + t^2 = (a^2\sigma_1(a)^2)^2$), and therefore $L_4$ is linearly disjoint from $A^*L_0$ over $A(t)$, since $\sqrt{1 + t^2} \notin L_0(i)$.

Let $L$ be the field composite of all $L_n$’s, $n \in Q$ (if $Q = \emptyset$, we let $L = A(t)$). Then the extensions $A^*(t), L_0$ and $L$ of $A(t)$ are all linearly disjoint over $A(t)$, and Galois, so that

$$\text{Gal}(A^*LL_0/A(t)) \simeq G(A) \times \text{Gal}(L/A(t)) \times \text{Gal}(L_0/A(t)) \simeq \hat{\mathbb{Z}} \times \text{Gal}(L_0/L) \simeq \hat{\mathbb{Z}} \times (\mathbb{Z}/2\mathbb{Z})^\kappa.$$
Let $f_\alpha(X, Y)$, $\alpha < \kappa$, be an enumeration of all absolutely irreducible polynomials of $L_0[X, Y]$ which are separable in $Y$. We will construct by induction on $\alpha < \kappa$ a chain $M_\alpha$ of algebraic extensions of $L_0$, which intersect $LA^s$ in $A(t)$ (and therefore are regular over $A$), such that each $M_{\alpha+1}$ is generated over $M_\alpha$ by a solution of $f_\alpha(X, Y) = 0$. We let $M_0 = L_0$, and when $\alpha$ is a limit ordinal, we let $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$. Assume $M_\alpha$ already constructed.

**Claim.** $M_\alpha$ is Hilbertian. Our assumption on $[M_{\beta+1} : M_\beta]$ being finite for $\beta < \alpha$ implies that $M_\alpha = M'_0 L_0$, where $M'_0$ is the union of $|\alpha|$ many finite algebraic extensions of $A(t)$; hence the Galois closure $\bar{M}_\alpha$ of $M'_0$ (over $A(t)$) is the union of $|\alpha|$ many finite Galois extensions of $A(t)$. Since $|\alpha| < \kappa$, $L_0$ is not contained in $\bar{M}_\alpha$, and Gal($L_0/A(t)$) $\cong$ $(\mathbb{Z}/2\mathbb{Z})^s$ and Property [1.3](2) give the result.

Construction of $M_{\alpha+1}$. We let $d$ be the degree of $f_\alpha$ in $Y$, and let $N_\alpha$ be the composite of all algebraic extensions of $L_0$ of degree $\leq d$ and contained in $A^s L_0 L$. Then $N_\alpha$ is a finite (Galois) extension of $L_0$. As $f_\alpha$ is absolutely irreducible, it remains irreducible over $M_\alpha N_\alpha$. Because $M_\alpha$ is Hilbertian, there is some $a_\alpha \in M_\alpha$ such that $f(a_\alpha, Y)$ is irreducible over $M_\alpha N_\alpha$ (by [1.3](6)). We let $M_{\alpha+1}$ be generated over $M_\alpha$ by a root $b_\alpha$ of $f_\alpha(a_\alpha, b_\alpha) = 0$. Since $[M_{\alpha+1} : M_\alpha] \leq d$, $M_\alpha \cap A^s L L_0 = L_0$, and $M_{\alpha+1}$ is linearly disjoint from $N_\alpha M_\alpha$ over $M_\alpha$, it follows that $M_{\alpha+1} \cap A^s L_0 L = L_0$.

We let $M_\kappa = \bigcup_{\alpha < \kappa} M_\alpha$. By construction, every absolutely irreducible polynomial $f(X, Y) \in L_0[X, Y]$ has a zero in $M_\kappa$, and therefore $M_\kappa$ is PAC (by Fact [1.3](3)).

Recall that $L$ is linearly disjoint from $A^s$ over $A$, and that Gal($L/A(t)$) is pro-cyclic. Hence the topological generator $\sigma$ of $G(A)$ lifts to an element $\sigma_2 \in$ Gal($A^s L/A(t)$) whose restriction to $L$ topologically generates Gal($L/A(t)$). As $M_\kappa$ is linearly disjoint from $A^s L$ over $L_0$, this $\sigma_2$ lifts to an element $\sigma' \in G(M_\kappa)$. Then $\langle \sigma' \rangle \cong \hat{\mathbb{Z}}$ by Remark [1.3](2). Fix($\sigma'$) is a regular extension of $A$ and is PAC. Hence the perfect hull of Fix($\sigma'$), $F_\kappa$, is pseudo-finite.

For the last assertion, note that $M_\kappa$ contains $L_0$, and that by construction of $L_0$ and $L$, we have that $S(t, F_\kappa) = S(t, M_\kappa) = X$ when char($A$) $\neq 2$, and $S(t, F_\kappa) \cap \hat{\mathbb{Z}} = X$ when char($A$) = 2.

**Remark 2.5.** Observe that if $A$ is infinite, and $F_X$ is as above, then the set of $Y$ such that $F_X \simeq_A F_Y$ has cardinality $\leq |A|$, since $|F_X| = |A|$. In particular, there are $2^{|A|}$-non-isomorphic models of $T(A)$ of the form $F_X$.

**Theorem 2.6.** Let $T$ and $A$ be as above, with $A$ not pseudo-finite. Then $T(A)$ has no prime model.

**Proof.** Let us first do the very easy case when $A$ is finite. Then $T(A)$ is countable, and the existence of a prime model would imply that isolated types are dense. If $|A| = q$, then every model of $T(A)$ must contain elements which are transcendental over $A$. In particular, by [1.9](4), the formula $x^q \neq x$ contains no isolated type over $A$.

Let us now assume that $A$ is infinite, and char($A$) $\neq 2$. By Proposition [2.4] a prime model $F$ of $T(A)$ has to (elementarily) $A$-embed in all $F_X$’s, and therefore have transcendence degree 1 over $A$. Then $F < F_X$ implies $F = F_X$. However, the set $S(F) = \{S(u, F) \mid u \in F \setminus A\}$ has size $|A|$, hence there is some subset $Y$ of $A$ which does not appear in $S(F)$.

I.e., $F$ cannot
A-embed elementarily in that $F_Y$. So, no prime model of $T(A)$ exists. A completely analogous discussion gives the result in characteristic 2.

Remark 2.7. The proof of Theorem 2.6 when $A$ is infinite only used item (2) of Proposition 2.4. The interest of the first item is its formulation and relation to the following result of Jarden (see Theorems 18.5.6 and 18.6.1 in [4]):

Let $K$ be a countable Hilbertian field. Then for almost all $\sigma$ in $G(K)$, the subfield of $K^{alg}$ fixed by $\sigma$ is pseudo-finite.

So, applying this to $K = A(t)$, we get that for almost all $\sigma$ in $G(A(t))$, the subfield of $K^{alg}$ fixed by $\sigma$ is pseudo-finite. But the set of $\sigma$ with fixed subfield a regular extension of $A$ has measure < 1 if $G(A) \neq 1$, and for instance if $A = \mathbb{F}_p$, it has measure 0. Item (1) of Proposition 2.4 is therefore the correct generalisation: once fixed a lifting of a generator of $G(A)$, its translates by almost all elements of $G(A^s(t))$ fix a regular extension of $A$ which is pseudo-finite. Note that by Theorem 18.8.8 of [4], this result is false when $A$ is uncountable.

3 Non-existence of prime saturated models

Definition 3.1. Let $T$ be a complete theory, $M$ a model of $T$.

(1) The model $M$ is $\aleph_\varepsilon$-saturated if whenever $A \subset M$ is finite, then every strong type over $A$ is realised in $M$. When $T = T^{eq}$, equivalently, for any finite $A \subset M$, any type over acl($A$) is realised in $M$.

(2) Let $\kappa$ be an infinite cardinal or $\aleph_\varepsilon$. We say that $M$ is $\kappa$-prime if $M$ is $\kappa$-saturated, and elementarily embeds into every $\kappa$-saturated model of $T$.

(3) Let $A \subset M$, and $T(A) := \text{Th}(M, a_{a \in A}$, $\kappa$ as in (2). We say that $N$ is $\kappa$-prime over $A$ if $N$ is a $\kappa$-saturated model of $T(A)$, and elementarily embeds into every $\kappa$-saturated model of $T(A)$.

In the remainder of this section we assume GCH.

Remark 3.2. It follows that if $\kappa$ is a regular cardinal larger than the cardinality of the language, and $A \subset M$ has cardinality $< \kappa$, then $T$ has $\kappa$-prime models over $A$, and furthermore they are all $A$-isomorphic: this follows easily observing that $T(A)$ has cardinality $< \kappa$, and the fact that our hypothesis on $\kappa$ guarantees that there are saturated models of $T(A)$ of cardinality $\kappa$. We will now show that when $T$ is the theory of a pseudo-finite field, this is essentially the only case when $\kappa$-prime models exist. The GCH hypothesis could be weakened to $2^{\aleph_\varepsilon} = \aleph_1$ when $\text{tr.deg}(A) = \aleph_0$, and to $\lambda^{<\kappa} \leq \lambda^+$, where $\lambda = |A|$.

3.3. Setting and strategy. We let $\kappa$ be an uncountable cardinal or $\aleph_\varepsilon$. Let $A$ be a perfect field of cardinality $\lambda \geq \kappa$, of infinite transcendence degree with absolute Galois group procyclic, and let $T(A)$ be the $L(A)$-theory whose models are the pseudo-finite fields which are regular
extensions of $A$. We also assume that $A$ is not $\kappa$-saturated, and fix a transcendence basis $Z$ of $A$. Given a model $F$ of $T(A)$ and $t \in F$, (almost) as before we define

$$S(t, F) = \begin{cases} 
\{a \in Z \mid \sqrt{t+a} \in F\} & \text{if } \text{char}(A) \neq 2, \\
\{a \in Z \mid F = \exists y \ y^2 + y = at\} & \text{if } \text{char}(A) = 2.
\end{cases}$$

We will show that given a subset $X$ of $Z$ with $|X| = \lambda$, there is a $\kappa$-saturated model $F_X$ of $T(A)$ of cardinality $\lambda^+$ and with the following property:

(*) For all $t \in F_X \setminus A$, there are some $b, c \in X$ such that $b \in S(t, F_X)$ and $c \notin S(t, F_X)$.

Note that this is equivalent to $S(t, F_X) \neq X$ and $S(t, F_X) \neq Z \setminus X$. Assume by way of contradiction that $F$ is a $\kappa$-prime model of $T(A)$. Then it embeds elementarily in all fields $F_X$ constructed above. Choose $t \in F \setminus A$, and let $Y = S(t, F)$. If $Y$ has size $\lambda$, then $F$ cannot elementarily $A$-embed into $F_Y$. If $|Y| < \lambda$, then $Z \setminus Y$ has size $\lambda$, and $F$ cannot elementarily $A$-embed into $F_{Z \setminus Y}$ either. This shows that there is no $\kappa$-prime model of $T(A)$.

The proof of the theorem needs a technical lemma, but let us first introduce a definition: Call a subset $C$ of $F^*$ small if it is relatively algebraically closed in $F^*$, and of transcendence degree $< \kappa$ if we are trying to build a $\kappa$-saturated model of $T(A)$, and of finite transcendence degree if we are trying to build an $\aleph_\gamma$-saturated model of $T(A)$.

**Lemma 3.4.** Let $A \subset M \subset F^*$, where $M$ is relatively algebraically closed in $F^*$, of cardinality $\lambda$, and has procyclic Galois group. Let $C$ be a small subfield of $F^*$, which is linearly disjoint from $M$ over $C \cap M$. Then there is a regular extension $N$ of $M$ and of $C$, with procyclic absolute Galois group, contained in $(CM)^{alg}$, and which satisfies $(\ast)$ over $M$: if $a \in N \setminus M$, then there are $b, c \in X$ such that $S(a, N)$ does not contain $b$ but contains $c$.

**Proof.** We fix an enumeration $b_\gamma$, $\gamma < \lambda$, of $(CM)^{alg} \setminus M$. Remark that $\lambda = |X| > \text{tr.deg}(C)$.

We will build $N$ as the union of an increasing chain $(C_\gamma)_{\gamma<\lambda}$ of subfields of $(CM)^*\setminus$, satisfying the following conditions:

- $C_0 = C$.
- If $\gamma$ is a limit ordinal, then $C_\gamma = \bigcup_{\delta<\gamma} C_\delta$.
- Each $MC_{\gamma+1}$ is a regular extension of $C_\gamma$ and of $M$, with $C_{\gamma+1}$ of finite transcendence degree over $C_\gamma$, and with $G(C_\gamma)$ procyclic.
- For each $\gamma$, $C_\gamma$ and $M$ are linearly disjoint over $C_\gamma \cap M$.
- For each $\gamma$, either $MC_\gamma(b_\gamma)$ is not a regular extension of $C_\gamma$ or of $M$, and then $b_\gamma \notin C_{\gamma+1}$; or $b_\gamma \in C_{\gamma+1}$, and there are some elements $b, c \in X$ such that $b \notin S(b_\gamma, C_{\gamma+1})$ and $c \in S(b_\gamma, C_{\gamma+1})$.

It will then follow that $N := \bigcup_{\gamma<\lambda} C_\gamma$ satisfies $(\ast)$, is a regular extension of $M$ and of $C$, and has procyclic Galois group. Indeed, recall that every element of $(CM)^{alg} \setminus M$ occurs as a $b_\gamma$, and therefore so does every element of $\bigcup_{\gamma<\lambda} C_\gamma \setminus M$, so that $\bigcup_{\gamma<\lambda} C_\gamma$ contains $CM$, is regular over $M$, with procyclic absolute Galois group, and satisfies $(\ast)$.

We now start with the construction of the $C_\gamma$'s. It is done by induction on $\gamma$, and if $\gamma$ is a
limit ordinal, then we set \( C_\gamma = \bigcup_{\beta < \gamma} C_\delta \). Assume that \( C_\gamma \) has been constructed, we will now construct \( C_{\gamma+1} \). If \( MC_\gamma(b_\gamma) \) is not regular over \( C_\gamma \) or over \( M \), then we let \( C_{\gamma+1} = C_\gamma \).

Assume now that \( MC_\gamma(b_\gamma) \) is regular over \( C_\gamma \) and over \( M \). Let \( X_0 \subset Z \) be finite and such that \( b_\gamma \in C_\gamma(X_0)^{alg} \), and let \( b, c \in X \) be transcendental and algebraically independent over \( C_\gamma(X_0) \). We will construct \( C_{\gamma+1} \) as a subfield of \( C_\gamma(X_0, b, c)^{alg} \). Let \( D = C_\gamma(X_0)^{alg} \cap M \).

**Claim 1.** \( C_\gamma D(b_\gamma) \) and \( M \) are linearly disjoint over \( D \), and \( C_\gamma D(b_\gamma) \) is a regular extension of \( D \).

**Proof of Claim 1.** As \( C_\gamma \) and \( M \) are linearly disjoint over \( C_\gamma \cap M \), and \( D \subset M \), it follows that \( C_\gamma D \) and \( M \) are linearly disjoint over \( D \); moreover, by Fact [1.57], as \( C_\gamma(X_0)^{alg}/C_\gamma \) is normal and \( M \) is perfect, \( C_\gamma(X_0)^{alg} \) is linearly disjoint from \( M \) over \( D \), and so is \( C_\gamma D(b_\gamma) \), which proves the first assertion. The second assertion follows from \( C_\gamma M(b_\gamma) \) being regular over \( M \), and \( M \) being regular over \( D \).

Let \( E = C_\gamma(X_0, b, c, \sqrt{b_\gamma + b}, \sqrt{b_\gamma + c}) \) if \( \text{char}(A) \neq 2 \), \( E = C_\gamma(X_0, b, c, d_1, d_2) \), where \( d_2^2 + d_1 = bb_\gamma \) and \( d_2^2 + d_2 = cb_\gamma \) if \( \text{char}(A) = 2 \), and let \( D_1 = C_\gamma(X_0, b, c)^{alg} \cap M \). Then, reasoning as in the proof of Claim 1, one shows that \( C_\gamma D_1(b_\gamma) \) and \( M \) are linearly disjoint over \( D_1 \), and are regular extensions of \( D_1 \). By Corollary [1.7] as \( D_1 \) and \( C_\gamma(b_\gamma) \) are free and linearly disjoint over \( D_1 \), we know that \( E \) is regular over \( D_1 \) and over \( C_\gamma(b_\gamma) \). Hence \( ME \) is regular over \( C_\gamma \) and \( M \), and \( M \) and \( D_1 E \) are linearly disjoint over \( D_1 \).

Therefore, if \( \sigma \) is a topological generator of \( G(M) \), and \( \sigma_\gamma \) is a topological generator of \( G(C_\gamma) \) which agrees with \( \sigma \) on \( M \cap C_\gamma \), then we may lift \( \sigma_\gamma \) to an element \( \sigma_{\gamma+1} \) of \( G(C_\gamma D(b_\gamma, b, c)) \) which extends \( \sigma \) on \( D_1(b_\gamma, b, c)^{alg} \), which is the identity on \( \sqrt{b_\gamma + b} \) and moves \( \sqrt{c + b_\gamma} \) if \( \text{char}(F) \neq 2 \), and is the identity on \( d_1 \) and moves \( d_2 \) if \( \text{char}(F) = 2 \). Then, lifting \( C_\gamma \) be the subfield of \( C_\gamma D(b_\gamma, b, c)^{alg} \) fixed by \( \sigma_{\gamma+1} \). Then \( MC_{\gamma+1} \) is a regular extension of \( C_\gamma \), and of \( M \). And by construction, \( S(b_\gamma, C_{\gamma+1}) \) does not contain \( c \) and contains \( b \). \( \square \)

**Theorem 3.5.** (GCH) Let \( A \) and \( T(A) \) be as in [2, 1]. Let \( \kappa \) be an uncountable cardinal, and suppose that \( A \) is not a \( \kappa \)-saturated model of \( T(A) \), and that \( \kappa \leq |A| \). Then \( T(A) \) has no \( \kappa \)-prime model.

If the transcendence degree of \( A \) is \( \geq \aleph_0 \) and \( A \) is not \( \aleph_0 \)-saturated, then \( T(A) \) has no \( \aleph_0 \)-prime model.

**Proof.** Let \( \lambda = |A| \), \( Z \) a transcendence basis of \( A \), and let \( X \subset Z \) of size \( \lambda \). We fix a \( \lambda^\ast \)-saturated model \( F^\ast \) of \( T(A) \). We will work both in \( F^\ast \) and in some large algebraically closed field containing \( F^\ast \). We will build a \( \kappa \)-saturated (resp. \( \aleph_0 \)-saturated) model \( F_X \) of \( T(A) \) satisfying the condition given in the strategy [3, 3]:

\[(*) \text{: For every } a \in F_X \setminus A, S(a, F_X) \text{ does not contain } X \text{ nor is it contained in } Z \setminus X.\]

Then as explained above, the existence of these models will imply the non-existence of \( \kappa \)-prime models and of \( \aleph_0 \)-prime models over \( A \). While realising types within \( F^\ast \) to obtain a \( \kappa \)-saturated submodel of size \( \lambda^\ast \) is easy, condition (\( *\)) requires some work.

Recall that by Corollary 3.3 of [3], if \( B \) is a relatively algebraically closed subfield of a pseudo-finite field \( F \) such that \( G(B) \simeq \mathbb{Z} \), then the \( L(B) \)-theory \( \text{Th}(F, a)_{a \in B} \) eliminates imaginaries.
In that case, it follows that strong types over \( B \) are simply types over \( B = B^{alg} \cap F \). We will therefore start our construction by defining \( F_0 = A \) if \( G(A) \simeq \hat{Z} \), and if \( G(A) \not\simeq \hat{Z} \), then using Proposition 2.4(1) we first choose some relatively algebraically closed subfield \( F_{-1} \) of \( F^* \) of transcendence degree 1 over the prime field and such that the map \( G(F^*) \to G(F_{-1}) \) is an isomorphism. We then let \( F_0 = (AF_{-1})^{alg} \cap F^* \). So, in both cases we have \( G(F_0) \simeq \hat{Z} \), and the same holds for all small subsets of \( F^* \) containing \( F_{-1} \). We will construct \( F_X \) as a chain of \( \lambda^+ \) subfields of \( F^* \). The reason for taking \( \lambda^+ \) instead of \( \lambda \) is two-fold: First of all, \( \lambda^+ \leq \lambda^+ = \lambda^+ \); and second, \( \lambda^+ \) is regular.

We use a diagonal argument, and build, by induction on \( \alpha < \lambda^+ \), an increasing sequence \( F_\alpha \) of subfields of \( F^* \), together with a collection of types \( (p_{\alpha, \beta})_{\beta < \lambda^+} \). We choose them satisfying the following conditions:

(a) \( F_0 \) is as above.
(b) \( F^* \) is a regular extension of \( F_\alpha \), and \( |F_\alpha| = \lambda \).
(c) If \( \alpha \) is a limit ordinal, \( F_\alpha = \bigcup_{\beta < \alpha} F_\beta \).
(d) \( (p_{\alpha, \beta})_{\beta < \lambda^+} \) enumerates all (finitary) types over small subsets of \( F_\alpha \).
(e) \( F_\alpha+1 \) contains realisations of \( p_{\beta, \delta} \) for all \( \delta, \beta \leq \alpha \).
(f) If \( a \in F_{\alpha+1} \setminus F_\alpha \), then there are some \( b, c \in X \) such that \( S(a, F_{\alpha+1}) \) contains \( b \) but not \( c \).

Items (a), (c) and (d) are straightforward. Items (e), (f) and (b) follow from Lemma 3.4.

Indeed, suppose \( F_\alpha \) constructed; we will build an increasing sequence of subfields \( (M_\beta)_{\beta \leq \alpha+1} \) of \( F^* \), satisfying

(a') \( M_0 = F_\alpha \).
(b') \( F^* \) is a regular extension of \( M_\beta \), and \( |M_\beta| = \lambda \).
(c') If \( \beta \) is a limit ordinal, \( M_\beta = \bigcup_{\gamma < \beta} M_\gamma \).
(e') If \( \beta \leq \alpha \), then \( M_{\beta+1} \) contains realisations of \( p_{\beta, \alpha} \) and of \( p_{\alpha, \beta} \).
(f') If \( a \in M_{\beta+1} \setminus M_\beta \), then there are some \( b, c \in X \) such that \( S(a, M_{\beta+1}) \) contains \( b \) but not \( c \).

Item (a') and (c') are straightforward. Assume \( M_\beta \) given, we will construct \( M_{\beta+1} \) as follows: let \( E \) be a small subset of \( M_\beta \) containing the bases of \( p_{\beta, \alpha} \) and \( p_{\alpha, \beta} \); choose \((a_1, a_2)\) realising \( (p_{\beta, \alpha}, p_{\alpha, \beta}) \) in \( F^* \), transcendental and algebraically independent over \( M_\beta \). Now apply Lemma 3.4 to \((C, M) = (E(a_1, a_2))^{alg} \cap F^*, M_\beta)\) to obtain \( N = M_{\beta+1} \) satisfying (b'), (e') and (f').

We then let \( F_{\alpha+1} = \bigcup_{\beta \leq \alpha+1} M_\beta \). Then item (b) and (f) hold (because at stage \( \alpha + 1 \) we are only realising \( \lambda \) many types); for (e), let \( \beta, \delta \leq \alpha \). If \( \beta, \delta < \alpha \), then \( p_{\beta, \delta} \) is realised in \( F_\alpha \). If \( \beta = \alpha \), then \( p_{\beta, \delta} \) is realised in \( M_{\delta+1} \), and if \( \delta = \alpha \), in \( M_{\beta+1} \), which shows (e).

By construction, \( F_X \) is \( \kappa \)-saturated: if \( q \) is a type over the small set \( B \subset F_X \), then \( B \subset F_\alpha \) for some \( \alpha < \lambda^+ \), and therefore \( q \) appears as a \( p_{\alpha, \beta} \) for some \( \beta < \lambda^+ \). Then \( q \) is realised in \( F_\gamma \), where \( \gamma = \sup\{\alpha, \beta\} + 1 \). In particular, \( F_X \) is PAC, perfect, with absolute Galois group isomorphic to \( \hat{Z} \), and therefore pseudo-finite.

Furthermore, if \( a \in F_X \setminus A \), then \( a \in F_\alpha \) for some \( \alpha < \lambda^+ \), and therefore \( S(a, F) = S(a, F_X) \) neither contains \( X \) nor is contained in \( X \setminus F \). This finishes the proof of the Theorem. \[\square\]
3.6. Concluding remarks. When $A$ has finite transcendence degree, then the above proof breaks down. It might be possible to fix it by choosing a suitable subset $Z$ of $A$.

In the absence of GCH, saturated models will in general not exist. The construction given above works when $\lambda^\kappa \leq \lambda^+$ when $\lambda = |A|$ is uncountable, and under CH when $\kappa = \aleph$, and $|A|$ has transcendence degree $\aleph_0$, and the same argument shows that $T(A)$ has no $\kappa$-prime model. I believe that Theorems 2.6 and 3.5 generalise to the case of arbitrary bounded PAC fields without assuming perfection (Recall that a field is bounded if it has finitely many Galois extensions of degree $n$ for every $n > 1$).

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