Steady heat transfer analysis of 2D plates using the numerical manifold method

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Abstract. In the numerical manifold method (NMM), the mathematical cover system can be inconsistent with the physical boundary, which can alleviate the meshing cost to some extent. In the present paper, the NMM is developed to solve two-dimensional steady heat conduction problems with the square mathematical elements. The corresponding formulations are derived and the proposed approach is validated through a typical example.

1. Introduction
The numerical manifold method (NMM) was proposed by Shi [1] in the beginning of 1990s for rock engineering problems. Attributing to the two cover systems, i.e., the mathematical cover system and the physical cover system, the NMM is capable of solving both continuous and discontinuous problems in the same framework. One of the major advantages of the NMM is that the mathematical cover system (similar to the concept of “mesh” in the finite element method (FEM)) can be independent of the physical boundaries.

Recent years, the NMM has been further developed and applied to solve various problems, e.g., see [2-16]. In the present paper, the NMM is further extended to study two-dimensional (2D) steady heat transfer problems. To this end, the rest of the paper is organized as follows. Firstly, the governing equations and associated boundary conditions are presented. Secondly, the NMM formulations for 2D heat conduction analysis are addressed; then, a typical numerical example is considered to verify the developed method. Finally, the corresponding conclusions are given.

2. Governing equations
The governing equations for steady heat conduction problems is [13]

\[ -\nabla \mathbf{q} + \mathbf{Q} = 0 \]  \hspace{1cm} (1)

where \( \nabla \) is the gradient operator, \( \mathbf{q} \) is the heat flux vector determined by the Fourier’s law as \( \mathbf{q} = -k\nabla T \) with \( k \) the thermal conductivity for materials and \( T \) the temperature, \( \mathbf{Q} \) is the heat source. The corresponding boundary conditions are

\[ T = \bar{T} \; (x \in \Gamma_T) \]  \hspace{1cm} (2)
\[ \mathbf{q} \cdot \mathbf{n} = \bar{q} \; (x \in \Gamma_q) \]  \hspace{1cm} (3)
where $\Gamma_T$ is the temperature boundary and $\Gamma_q$ is the flux boundary. $\mathbf{T}$ and $\mathbf{q}$ are, respectively, the prescribed temperature and flux on corresponding boundary. $\mathbf{n}$ is the outward unit normal to the domain.

3. The NMM for steady heat transfer problems

To solve a physical problem with the NMM, the mathematical cover (MC) system is firstly constructed. Theoretically, the MC formed by mathematical elements can be of any shape and the MC system may be non-conforming to all domain boundaries once it is large enough to cover the considered region. On each MC, a weight function is defined. Next, the physical cover (PC) system is formed by the intersection of MCs and physical domain. On each PC, the cover function is constructed to represent the local physical property. Then, the manifold elements (MEs) are obtained through the shared region of PCs. Accordingly, the NMM approximation on each ME is obtained by pasting the cover functions using the associated weight functions.

For the concerned problem, the temperature in any ME $e$ is expressed as [13]

$$T^e(x) = \sum_{i=1}^{n_p} w_i(x)T_i(x)$$

(4)

where $n_p$ is the number of total PCs shared by $e$. $w_i(x)$ is the weight function defined on the MC containing the $i$th PC, often borrowed from the shape functions in the FEM. $T_i(x)$ is the cover function for the $i$th PC, and is usually chosen as

$$T_i(x) = P(x)a_i$$

(5)

for 2D continuous problems. $P(x)$ is the polynomial basis being

$$P(x) = [1, x, y, ...]$$

(6)

and $a_i$ is the vector of unknowns defined on the $i$th PC.

The NMM discrete equations are derived using the weighted residual method as [13]

$$KT = F$$

(7)

where $T$ is the vector of unknowns; $K$ and $F$ are, respectively, the global thermal conductivity matrix and equivalent thermal load vector. The contributions of the ME $e$ to $K$ and $F$ are

$$K^e = \int_{\Omega^e} B^TBd\Omega + \lambda\int_{\Gamma_T} (w_iP)^T(w_iP)d\Gamma$$

(8)

$$F^e = \int_{\Omega^e} w_iP^TQd\Omega + \lambda\int_{\Gamma_T} w_iP^T\mathbf{T}d\Gamma - \int_{\Gamma_q} w_iP^T\mathbf{q}d\Gamma$$

(9)

where the superscript $T$ denotes the matrix transpose. $\Omega^e$ is the domain occupied by $e$. $\Gamma_T^e$ and $\Gamma_q^e$ are, respectively, the essential and natural boundary associated with $e$. $\lambda$ is the penalty factor for the enforcement of essential boundary condition. The elements of the matrix $B$ are
$$B_i = \begin{bmatrix} (w_i^T P)_x \\ (w_i^T P)_y \end{bmatrix}$$

(10)

To efficiently and accurately calculate the integration in equation (8) and (9), all the MEs are firstly divided into several sub-triangles, and then the 3-point Gaussian quadrature rules are used on each sub-triangle, the corresponding results on which finally adds up to the integration of the ME.

4. Numerical example

In this part, to verify the proposed method, a typical 2D heat conduction problem is tested under steady state.

Consider the square domain plotted in figure 1. The dimension of the plate, the Cartesian coordinates system and the boundary conditions are all provided in this figure. The analytical solutions to this problem can be derived as

$$T = 4x - \frac{x^2}{2}$$

(11)

When modeling, in view that the MC system can be independent of the physical domain, MCs composed of square mathematical elements are adopted to cover the plate. The discretized domain (when element size side length of square mathematical elements is $h=0.3$) is shown in figure 2, which includes 121 MEs and 144 PCs. Further, for simplicity, the polynomial basis in Eq. (6) is set to be constant. The penalty factor $\lambda$ in equation (8) and (9) is taken as $1.0 \times 10^{10} \, k$.

![Figure 1. Physical problem.](image1)

![Figure 2. Discretized plate when $h=0.3$.](image2)

The temperature contour is displayed in figure 3, which infers that the value of temperature only varies along $x$-direction, as revealed in equation (11). The computed temperatures at several sample points, i.e., (0.5, 1.5), (1.0, 1.5), (1.5, 1.5) … and (3.0, 1.5) by the NMM are listed in table 1, together with the exact solutions from equation (11). It is obvious that the NMM results agree very well with the analytical ones.
Figure 3. Distribution of temperature fields.

| Sample point | (0.5, 1.5) | (1.0, 1.5) | (1.5, 1.5) | (2.0, 1.5) | (2.5, 1.5) | (3.0, 1.5) |
|--------------|------------|------------|------------|------------|------------|------------|
| NMM results  | 1.869      | 3.494      | 4.864      | 5.994      | 6.869      | 7.499      |
| Exact solutions | 1.875      | 3.500      | 4.875      | 6.000      | 6.875      | 7.500      |

5. Concluding remarks
The numerical manifold method was developed to study 2D steady heat conduction problems. Details about the governing equations, the NMM discrete formulations and the numerical integration schemes are provided. A typical example is analyzed to validate the proposed method. Mathematical covers formed by square elements are adopted for numerical modeling due to the inconsistence of the mathematical cover system and physical boundaries. The great agreement between the NMM solutions and the reference results demonstrates the high accuracy of the present method.

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