Bulk Viscous Cosmological Model with G and Lambda Variables Through Dimensional Analysis

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Abstract—A model with flat FRW symmetries and G and \( \Lambda \), variable is considered in such a way that the momentum-energy tensor that describes the model is characterized by a bulk viscosity parameter. For this tensor the conservation principle is taken into account. In this paper it is showed how to apply the dimensional method in order to solve the outlined equations in a trivial way.

Keywords—Odes, AD, FRW Cosmologies, variable constant

I. Introduction.

Recently several models with FRW metric, where “constants” \( G \) and \( \Lambda \) are considered as dependent functions on time \( t \) have been studied. For these models, whose energy-momentum tensor describes a perfect fluid, it was demonstrated that \( G \propto t^\alpha \), where \( \alpha \) represents a certain positive constant that depends on the state equation imposed while \( \Lambda \propto t^{-2} \) is independent of the state equation (see [1], [2]).

More recently this type of model was generalized by Arbab (see [3]), who considers a fluid with bulk viscosity (or second viscosity in the nomenclature of Landau (see [1])). The role played by the viscosity and the consequent dissipative mechanism in cosmology has been studied by many authors (see [3]).

In the models studied by Arbab constants \( G \) and \( \Lambda \) are substituted by scalar functions that depend on time \( t \). The state equation that governs the bulk viscosity is: \( \xi \propto \xi_0 \rho^\gamma \) where \( \gamma \) is a certain indeterminate constant for the time being \( \gamma \in [0, 1] \).

As we shall see, this problem is already solved, but our aim is to solve it through Dimensional Analysis. We mean to point out how an adequate use of this technique let us find the solution of the outlined equations in a trivial way, even pointing out that it is not necessary to impose it. Several cases are also studied here by Arbab I. Arbab (see [3]), while in the other subsection a finer dimensional technique is showed. That is why we call it “not so simple method”. This section is based on dimensional techniques (groups and symmetries, see [8]), in order to reduce the number of variables intervening in the expounded ODEs. They are so simplified that its integration is immediate. We think that the technique showed here is so powerful that it shall be proved that imposing the condition \( \text{div}(T_{ij}) = 0 \) is not necessary to impose in order to solve the equations.

II. The model.

This problem was posed by Arbab (see [3]). The equations of the model are:

\[
R_{ij} - \frac{1}{2}g_{ij}R - \Lambda(t)g_{ij} = \frac{8\pi G(t)}{c^4}T_{ij} \tag{1}
\]

and it is imposed that:

\[
\text{div}(T_{ij}) = 0
\]

where \( \Lambda(t) \) represent (stand) the cosmological “constant”.

The basic ingredients of the model are:

1. The line element defined by:

\[
ds^2 = -c^2 dt^2 + f^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right] \tag{2}
\]

we only consider here the case \( k = 0 \).

2. The momentum-energy tensor defined by:

\[
T_{ij} = (\rho + p^*)u_i u_j - pg_{ij}
\]

where \( \rho \) is the energy density and \( p^* \) represents pressure \([\rho] = [p^*] \). The effect of viscosity is seen in:

\[
p^* = p - 3\xi H \tag{3}
\]

where: \( p \) is the thermostatic pressure, \( H = \left(f'/f\right) \) and \( \xi \) is the viscosity coefficient that follows the law:

\[
\xi = k_r \rho^\gamma \tag{4}
\]

1 we shall see that this condition it is not necessary to impose it.
where \( k, \) makes the equation be homogeneous i.e. it is a constant with dimensions and where the constant \( \gamma \in [0,1] \). And \( p \) also verifies the next state equation:

\[
   p = \omega \rho \quad \omega = \text{const.} \tag{5}
\]

where \( \omega \in [0,1] \) (i.e. it is a pure number) so that the momentum-energy tensor verifies the so-called energy conditions.

The field equations are:

\[
   2f'' + \left( \frac{f'}{f} \right)^2 - \frac{8\pi G(t)}{c^2} \rho^* + c^2 \Lambda(t) \tag{6}
\]

\[
   3\left( \frac{f'}{f} \right)^2 = \frac{8\pi G(t)}{c^2} \rho + c^2 \Lambda(t) \tag{7}
\]

deriving (7) and simplifying with (6) it yields

\[
   \rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^* H^2 = 0 \tag{8}
\]

and at the moment we consider this other equation.

\[
   \text{div}(T_{ij}) = 0 \Leftrightarrow \rho' + 3(\rho + \rho^*) \frac{f'}{f} = 0 \tag{9}
\]

if we develop the equation (9) we get:

\[
   \rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^* H^2 = 0 \tag{10}
\]

### III. Non Dimensional Method.

In this section we will mainly follow Singh et al work (see [10]). If we take the equation (8) regrouped, we get:

\[
   \rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^* H^2 = \left[ \frac{G'}{G} + \frac{\Lambda' e^4}{8\pi G} \right] \tag{11}
\]

if take into account the conservation principle

\[
   \rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^* H^2 = 0 \tag{12}
\]

then we solve this equation by solving the equation A2 in (11), in such a way that the equation to be solved is now:

\[
   \left[ \frac{G'}{G} + \frac{\Lambda' e^4}{8\pi G} \right] = 0 \tag{13}
\]

this equation is tried to be solved like this (see [10]). It is defined \( \Lambda = \frac{3\Lambda H^2}{8\pi G} \) where \( \beta \) is a numerical constant, (hypothesis by Arbab (see [3]) as well as by Singh et al (see [4]), condition that as we shall see, it is not necessary to impose in the solution through D.A.) and from the equation (7) the following relationship is obtained: \( 8\pi G\rho = 3(1 - \beta)H^2 \). Hence if all the equalities are replaced in the equation (13) it yields:

\[
   \frac{2}{(1 - \beta)} H' = \frac{\rho'}{\rho} \tag{14}
\]

which is easily integrated.

\[
   H = C_1 \rho^{1/d} \quad d = \frac{2}{(1 - \beta)} \tag{15}
\]

we get to the equation (12) with all these results

\[
   \rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^* H^2 = 0
\]

we arrive to the next equation:

\[
   \rho' + 3C_1(\omega + 1)\rho \frac{d\rho}{d\rho^*} - 9C_1^2 k_\gamma \rho \frac{d\rho^*}{d\rho} = 0 \tag{16}
\]

which has got a particular solution in the case \( \gamma = d^{-1} \) obtaining:

\[
   \rho(t) = \frac{1}{(a_0 t)^d} \quad / \quad a_0 = (3C_1(\omega + 1) - 9k_\gamma C_1^2) \frac{d^{-1}}
\]

and obtaining from it:

\[
   f(t) = C_2 t^{(3(\omega + 1) - 3k_\gamma C_1)(1 - \gamma)} \tag{18}
\]

This is the most developed solution reached by Singh et al (see [10]) which is slightly different from the one by Arbab (see [3]).

### IV. Dimensional Method.

We shall explore this section two dimensional methods. The first one, probably the simplest one, has the inconvenience of having to depend on Einstein criterion(see [4] and Barenblatt [5]), while the second one is more powerful and more elaborated. We shall finish showing an equation obtained without having to impose the condition \( \text{div}(T_{ij}) = 0 \).

#### A. Simple Method.

The dimensional way followed in this section is probably the most basic and simplest one. On one hand we integrate independently the equation

\[
   \text{div}(T_{ij}) = 0 \Leftrightarrow \rho' + 3(\omega + 1)\rho H = 0 \tag{17}
\]

not taking into account the term \( 9k_\gamma \rho^* H^2 \), since if we calculate its order of magnitude we verify that is very small. The other dimensional constant considered has been obtained from the state equation (4) i.e. \( \xi = k_\gamma \rho^* \), such constant \( k_\gamma \) will also have different dimensions depending on the value \( \gamma \), in such a way that the
problem is reduced to the following set of quantities and constants \( \mathcal{M} \):

\[
\mathcal{M} = (t, c, A_\omega, k_\gamma, a)
\]

where its respective dimensional equations in regard to a base \( B = \{L, M, T, \theta\} \) are (the base \( B \) of this type of models has been calculated in \( 8 \)):

\[
\begin{align*}
[f] &= T, \ [c] = LT^{-1}, \ [a] = L^{-1}MT^{-2}\theta^{-4} \ [k_\gamma] = L^{-1}M^{1-\gamma}T^{2n-1} \\
\end{align*}
\]

where \( a \) represents the radiation constant and it will be taken into account when we consider the thermodynamic quantities.

Having done these considerations our aim is, therefore to solve this model through D.A. The Pi-theorem will bring us to obtain two dimensionless monomials; one of them will be the obtained in the case of a perfect fluid ([2]) and the other monomial will contain information on viscosity, showing in this way that this type of models are very general, reproducing the results obtained in the case for perfect fluids. Since all solutions will depend on these two monomials we must take into account Barenblatt criterion if we mean to reach a satisfactory final solution coincident with the one obtained theoretically (see \( 3 \) and \( 10 \)).

**B. Solutions through D.A.**

We shall calculate through D.A. i.e. by applying Pi-Theorem variation of \( G(t) \) in function on \( t \), energy density \( \rho(t) \), the radius of Universe \( f(t) \), the entropy \( s(t) \), and the entropy density \( \rho(T) \) (see \( 6 \)) and \( 9 \)) bringing us to obtain two dimensionless monomials:

\[
\pi_1 = \frac{t^{1+3\omega}}{GA_\omega}, \quad \pi_2 = \frac{ct^{1+\beta}}{A_\omega^{(1+\beta)\gamma}k_\gamma^{\beta}}
\]

It is observed that the first monomial \( \pi_1 \) is identical to the one obtained in the paper \( 8 \) for perfect fluids, while the second monomial contains information on flow viscosity\(^2\). These two monomials lead us to the following solution:

\[
G \propto \frac{t^{1+3\omega}}{A_\omega} \cdot \varphi \left( \frac{ct^{1+\beta}}{A_\omega^{(1+\beta)\gamma}k_\gamma^{\beta}} \right) \quad (20)
\]

where \( \varphi \) represent an unknown function (i.e. at the moment we have obtained a “partial” solution, in order to reach a more satisfactory solution we must take into account the Barenblatt criterion) and \( \beta \) is:

\[
\beta = \frac{1}{3(\omega + 1)(\gamma - 1)}
\]

**B.2 Calculation of energy density \( \rho(t) \)**

\[
\rho = \rho(t, c, A_\omega, k_\gamma) \quad (21)
\]

**B.3 Calculation of radius of Universe \( f(t) \).**

\[
f = f(t, c, A_\omega, k_\gamma) \quad (22)
\]

**B.4 Calculation of temperature \( \theta(t) \).**

\[
\theta = \theta(t, c, A_\omega, a, k_\gamma) \quad (23)
\]

**B.5 Calculation of entropy \( S(t) \).**

\[
S = S(t, c, A_\omega, a, k_\gamma) \quad (24)
\]

**B.6 Entropy density \( s(t) \).**

\[
s = s(t, c, A_\omega, a, k_\gamma) \quad (25)
\]

**B.7 Calculation of cosmological “constant” \( \Lambda(t) \).**

\[
\Lambda = \Lambda(t, c, A_\omega, k_\gamma) \quad (26)
\]

**C. Different Cases.**

All the following cases that we go on to study now have been studied by Arbab (see \( 3 \)) confirming “it!” his solution \( 10 \).

In obtaining all solutions depending on two monomials we shall try to find a solution to the problem expounded by means of the Barenblatt criterion (for more details about the method used here see \( 3 \) and \( 10 \)).
C.1 $\gamma = 1/2$ and $\omega = 1/3$, Radiation predominance.

As we pointed out in the introduction the only models topologically equivalent to the ones of classic FRW are those that follow the law $\xi \propto R^{3/2}$ i.e $\gamma = 1/2$ for its viscous parameter. In this case we observe a Universe with radiation predominance $\omega = 1/3$. In order to obtain a complete solution we shall take into account Barenblatt criterion since, we have obtained the solutions depending on an unknown function $\varphi$. In this case the substitution of the values of $\omega$ and $\gamma$ leads us to:

$$G \propto \frac{t^2c^6}{A_\omega} \cdot \varphi \left( \frac{ct^{1/2}}{A_\omega^{1/4} k^{-1/2}_\gamma} \right)$$

To get rid of the unknown function $\varphi$ we apply Barenblatt criterion, for this purpose we need to know the order of magnitude of each monomial:\n
$$\pi_1 = \frac{GA_\omega}{t^2c^6} \approx 10^{-10.59} \quad \pi_2 = \frac{ct^{1/2}}{A_\omega^{1/4} k^{-1/2}_\gamma} \approx 10^{2.6} \quad \pi_3 = \frac{ct^{1/2}}{A_\omega^{1/4} k^{-1/2}_\gamma} \approx 10^{2.6}$$

$$G \propto \frac{ct^{1/2}}{A_\omega^{1/4} k^{-1/2}_\gamma}^m \quad m = \log \pi_1 \quad m = \log \pi_2$$

as we expected in having a model with $\gamma = 1/2$. We also obtain from this point that $k^2_\gamma = c^2/G$. With regard to the rest of quantities we operate identically finding without surprise that:

$$\rho \propto t^{-2} \quad f \propto t^{1/2} \quad \theta \propto t^{1/2} \quad S \propto t^0 \quad s \propto t^{2/3} \quad \Lambda \propto const.$$  

As we see the model shows the same behavior in the principal quantities as in the classic FRW model with radiation predominance.\\n
Let see, for example, how $f$ has been calculated: Following the same steps as we have seen in the case of calculations of $G$ it is observed that:

$$f \propto ct \cdot \varphi \left( \frac{ct^{1/2}k^{1/2}_\gamma}{A_\omega^{1/4}} \right)$$  

$$\pi_1 = \frac{f}{ct} \approx 10^{-2.6} \quad \pi_2 = \frac{ct^{1/2}k^{1/2}_\gamma}{A_\omega^{1/4}} \approx 10^{2.6}$$

$$f \propto \left( \frac{ct^{1/2}k^{1/2}_\gamma}{A_\omega^{1/4}} \right)^m \quad m = -1$$

$$f \propto \left( \frac{cA^{1/2}_\omega}{k^{1/2}_\gamma} \right)^{1/5} t^{1/5} \approx \left( \frac{A_\omega}{c^2} \right)^{1/5} t^{1/5}$$

C.2 $\gamma = 1/2$ and $\omega = 0$. Matter predominance

A model with matter predominance $\omega = 0$ is topologically equivalent to a classic FRW. In this case we find the following relationships:

Regarding to $G$ the solution obtained is (after replacing values $\gamma$ and $\omega$):

$$G \propto \frac{t^3c^5}{A_\omega} \cdot \varphi \left( \frac{ct^{1/3}}{A_\omega^{1/3} k^{-2/3}_\gamma} \right)$$

as we are working with a model described by matter instead of considering energy density we find more convenient to consider matter density which becomes a little dimensional readjustment in $A_\omega$ which becomes $[A_\omega] = M$ in such a way that the solution pointed out above for $G$ is still the following law:

$$G \propto \frac{t^3c^5}{A_\omega} \cdot \varphi \left( \frac{ct^2k^2_\gamma}{A_\omega^{1/3}} \right)^{1/3}$$

In regard to the rest of quantities if we operate as before, we get:

$$\rho \propto \frac{c^2}{Gt^2} \quad f \propto (MG)^{1/3} t^{2/3} \quad \Lambda \propto const.$$  

where we have used the equality $k^2_\gamma = c^2/G$ and we have identified $A_\omega$ with the total mass of Universe $M$ i.e. The same behavior has been obtained as in a FRW with matter predominance. Let see for instance how we calculate radius $f$:

For this quantity the obtained solution is:

$$f \propto \frac{ct^2k^2_\gamma}{A_\omega}^{1/3}$$

Barenblatt criterion brings us to:

$$\pi_1 = \frac{f}{ct} = 10^{0.5} \quad \pi_2 = \left( \frac{ctk^2_\gamma}{A_\omega} \right)^{1/3} = 10^{-0.47}$$

$$\pi_1 = (\pi_2)^m \quad m = -1 \quad f \propto (MG)^{1/3} t^{2/3}$$
C.3 \( \gamma = 3/4 \) and \( \omega = 1/3 \). An Universe with radiation predominance:

In this case, as \( \beta = -1 \) we find the following solutions:

\[
G \propto t^2 c^6 \frac{\varphi \left( \frac{ck}{A_{\omega}^{1/4}} \right)}{A_{\omega}}
\]
as the unknown function \( \varphi \) does not depend on \( t \) we can state fearlessly that

\[
\varphi \left( \frac{ck}{A_{\omega}^{1/4}} \right) = D = \text{const.}
\]
since \( c, k \) as well as \( A_{\omega} \) are constant through hypothesis, in such a way that

\[
G \propto D^t t^2
\]
where \( D' = Dc^6 / A_{\omega} \). In this case we do not need to resort to Barenblatt criterion in order to obtain a definitive solution. In regard to the other quantities we obtain the following behaviors:

\[
\rho \propto D \frac{A_{\omega}}{(ct)^2} \quad f \propto Dct \quad a^{1/4} \theta \propto D \frac{A_{\omega}^{1/4}}{ct}
\]

\[
S \propto D(A_{\omega}^3 a)^{1/4} \quad s \propto D \frac{(A_{\omega}^3 a)^{1/4}}{(ct)^3} \quad \Lambda \propto D(ct)^{-2}
\]

In short, the obtained behaviors are:

\[
\phi \propto t^{-4} \quad f \propto t \quad \theta \propto t^{-1}
\]

\[
S \propto \text{const.} \quad s \propto t^{-3} \quad \Lambda \propto t^{-2}
\]

this case follows an identical behavior to the one obtained in a model described by a perfect fluid with \( G \) and \( \Lambda \) variables (see [1] and [2]) showing in this way the generality that we have obtained when considering a bulk viscous fluid.

C.4 \( \gamma = 2/3 \) and \( \omega = 0 \) An Universe with matter predominance.

In this case also \( \beta = -1 \), finding the following relationships as in the previous case:

\[
G \propto \frac{t c^5}{A_{\omega}} \cdot \varphi \left( \frac{ck}{A_{\omega}^{1/4}} \right)
\]

that in the previous case leads us to:

\[
G \propto D \frac{c^5 t}{A_{\omega}}
\]

where \( D = \varphi \left( \frac{ck}{A_{\omega}^{1/4}} \right) \). Simplifying in the same way, without difficulty we reach:

\[
\rho \propto t^{-3} \quad f \propto t \quad \Lambda \propto t^{-2}
\]

These two last cases are identical to the ones studied in references (see [1] and [2]).

V. Not so simple method.

In this section we will combine dimensional techniques with standard techniques of ODEs integration. With the dimensional method, we go on to obtain dimensionless monomials, which will be replaced in the equations. Thus, the number of variables will be reduced in such a way that the resulting equation is integrable in a trivial way. We study two cases, the first in which we consider \( \text{div}(T_{ij}) = 0 \), while in the other, as we shall see, such hypothesis is not needed.

5 Considering the condition \( \text{div}(T_{ij}) = 0 \).

In this case we shall pay attention to the equation:

\[
\rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^2 H^2 + \rho \frac{G'}{G} + \frac{\Lambda c^4}{8\pi G} = 0
\]
taking into account the relationship \( \text{div}(T_{ij}) = 0 \) The following equality is brought up:

\[
\rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^2 H^2 = -\left[ \frac{G'}{A_1} + \frac{\Lambda c^4}{A_2} \right] = 0
\]

The idea is the following: By using D.A. we obtain two \( \pi \)-monomials, which are replaced in the equation, achieving a huge simplification of it. On the other hand we integrate (A1) and (A2), solving completely in this way the problem, this time without Barenblatt. Let see. The monomials obtained are: \( \pi_1 = \rho k_{\omega}^{1/4} t^{\sigma_1} \) and \( \pi_2 = \Lambda c^{2/3} \) i.e.

\[
\rho = ak_\gamma^{1/4} t^{\sigma_1 - 1} \quad \Lambda = \frac{d}{c^{2/3}}
\]

where \( a \) and \( d \) are numerical constants. In a generic way the solution is of the following form: \( \rho = ak_\gamma^{1/4} t^{\sigma_1 - 1} \) if we define \( b = \frac{1}{1-\gamma} \) then \( \rho = ak_\gamma^{1/4} t^{b-1} \) where \( a = \text{const.} \in \mathbb{R} \) then

\[
\rho' = -b k_\gamma^{1/4} t^{b-1} \quad \text{(paying attention only to the term (A1) of the equation) it yields:}
\]

\[
-b k_\gamma^{1/4} t^{b-1} + 3(\omega + 1)ak_\gamma^{1/4} t^{b-1} H - 9k_\gamma (ak_\gamma^{1/4} t^{b-1})^2 H^2 = 0
\]

(28)

that simplifying it is reduced to:

\[
9a(\gamma - 1) (f')^2 - 3w t^{-1} f' + b t^{-2} f^2 = 0
\]

(29)

\[
f' = \frac{1}{f} \left[ \frac{1}{6a^{\gamma - 1}} \left( w \pm (w^2 - 4a^{\gamma - 1})^{1/2} \right) \right]
\]

(30)

where \( w = (\omega + 1) \), if it is defined

\[
D = \left[ \frac{1}{6a^{\gamma - 1}} \left( w \pm (w^2 - 4a^{\gamma - 1})^{1/2} \right) \right]
\]

(31)

then, the solution has the following form:

\[
f = l B t^D
\]

(32)

where \( l \) is a certain numerical constant and \( B \) is an integration constant with dimensions, that can be identified with our result by making \( B = A_{\omega} k_\gamma \).
Now we shall solve the other term of the equation (the A2). The equation \[ \left( \rho \frac{G'}{G} + \frac{\rho \gamma}{c^2} \right) = 0 \] can be solved in a trivial way if we follow the next results. If we replace the monomials \( \pi_1 = \rho k_\gamma^{-1} t^\frac{1}{t} \) and \( \pi_2 = \Lambda e^t \) in such equation the integration of it becomes trivial:

\[
ak_{\gamma}^{-\frac{1}{t}} G' + \frac{dc^2}{4\pi G t^3} = 0
\]

\[
G' = \frac{dc^2}{4\pi k_{\gamma}^{-b}} G t^{b-3} = G(t) = \frac{dc^2}{4\pi k_{\gamma}^{-b}} G t^{b-2} \tag{33}
\]

where \( a, d \) and \( g \in \mathbb{R} \) (they are pure numbers). We can also observe that this integral needs not be solved since a more careful analysis about the number of \( \pi \)-monomials that we can obtain from the equation leads us to obtain a solution of the type:

\[
G = G(k_{\gamma}, c, t)
\]

which brings us to:

\[
G(t) = g k_{\gamma}^{-b} c^2 t^{b-2}
\]

This method, as we have seen, is more elaborated and the solution, therefore, finer though coincident with the previous one.

6 Case in which \( div(T_{ij}) = 0 \) is not considered.

Let see how we can tackle this problem from the D.A. point of view, without imposing the condition \( div(T_{ij}) = 0 \). The base B as before, is still \( B = \{ L, M, T \} \) while the fundamental set of quantities and constants this time is \( M = \{ t, c, k_{\gamma} \} \), with these data we can obtain the following monomials from the equation

\[
\rho' + 3(\omega + 1) \rho H - 9k_{\gamma}\rho^2 H^2 + \rho \frac{G'}{G} + \frac{\Lambda c^4}{8\pi G} = 0 \tag{34}
\]

considering that:

\[
\rho = a k_{\gamma}^{-\frac{1}{t}} t^\frac{1}{t} \quad \Lambda = \frac{d}{c^2 t^2} \tag{35}
\]

these two monomials are replaced into the equation, which is quite simplified:

\[
-ba k_{\gamma}^{-b} t^{-b-1} + 3(\omega + 1) ak_{\gamma}^{-b} t^{-b} H - 9k_{\gamma} \left( ak_{\gamma}^{-b} \right)^2 H^2 +
+ ak_{\gamma}^{-b} G' - \frac{dc^2}{4\pi G t^3} = 0 \tag{36}
\]

simplifying this equation, it yields:

\[
-9a(\gamma - 1) t H^2 + 3w H - bt^{-1} + \frac{G'}{G} - \frac{dc^2}{4\pi a k_{\gamma}^{-b}} t^{b-3} = 0 \tag{37}
\]

that along with the field equations (6) and (7) carry us to the next set of equations. For example we note that

\[
3H^2 = a \frac{8\pi G}{c^2} k_{\gamma}^{-b} t^{-b} + \frac{d}{t^2}
\]

that we replace into the equation that we are treating, resulting:

\[
-bt^{-1} + 3w \left( a \frac{8\pi k_{\gamma}^{-b} G t^{-b} + d}{3t^2} \right)^{\frac{1}{2}} -
-9a(\gamma - 1) \left( a \frac{8\pi k_{\gamma}^{-b} G t^{-b} + d}{3t^2} \right) t + \frac{G'}{G} - \frac{dc^2}{4\pi a k_{\gamma}^{-b}} t^{b-3} = 0
\]

that solving it results:

\[
G = g k_{\gamma}^{-b} c^2 t^{b-2} \tag{38}
\]

where \( g \in \mathbb{R} \) represents a numerical constant. We finally observe that as in the previous section we could have taken into account the three monomials obtained from the equation i.e.

\[
\rho = a k_{\gamma}^{-b} t^{-b} \quad \Lambda = \frac{d}{c^2 t^2} \quad G = g \frac{2t^{b-2}}{k_{\gamma}^{-b}}
\]

replacing them into the equation

\[
\rho' + 3(\omega + 1) \rho H - 9k_{\gamma}\rho^2 H^2 + \rho \frac{G'}{G} + \frac{\Lambda c^4}{8\pi G} = 0
\]

and calculate \( f \), arriving at the same solution obtained in the above section i.e.

\[
f = lB t^D
\]

We have proved that it is not necessary to impose the condition \( div(T_{ij}) = 0 \) since it is obtained, in this case, the same solution as the one obtain imposing it.

VI. Conclusions.

We have studied a cosmological model described by a momentum-energy tensor characterized by a fluid with bulk viscosity, in which, furthermore, we have considered the constants \( G \) and \( \Lambda \) as functions depending on time i.e. as variables and we have imposed the condition \( div(T_{ij}) = 0 \). We have proved how a suitable use of Dimensional Analysis enables us to find the solution of such model in a "trivial" way. With the "Pretty simple method", we have obtained two \( \pi \)-monomials, one of them is the one obtained in the case for a perfect fluid (6) and the other monomial contains the information about viscosity, showing, in this way, that this type of models is very general being able to reproduce the result obtained in the case of a perfect fluid. In order to solve the problem we have taken into account Barenblatt criterion being able to arrive to obtain a complete solution of the problem. Standing out that our results coincide with the solutions obtained by Arbab I. Arbab [3]. We have shown too that with the "not so simple method" we arrive to solve the problem without necessity of impose any condition. We believe, nevertheless, that the "simple method" can be more effective, since, we obtain more solutions with it or more complete solutions in the sense of finding in it solutions such as \( \Lambda \propto t^{-2} \) as well as \( \Lambda = \text{const} \), while the "not so simple method" the only solution that is obtained is \( \Lambda \propto t^{-2} \), but has the drawback of depending on Barenblatt criterion i.e. we depend on the always insecure numerical data.
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\omega = 1/3 & G & \epsilon & \rho & f & A_\omega & k_\gamma \\
-10.17 & 8.47 & -13.379 & 26 & 90.62 & 13.5 & 0.436 \\
\hline
\omega = 0 & -10.17 & 8.47 & -26.397 & 26 & 54 & 13.5 & 17 \\
\hline
\end{array}
\]

TABLE I

The values refer to a logarithmic scale i.e. \( G \approx 10^{-10.17} \)

etc.. measured in the International System \{m, kg, s\}. In

the case \( \omega = 0 \), \( \rho \) represents mass density while in the case

\( \omega = 1/3 \) represents energy density.

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