The Shifting Sands of Quantum Error Correction Thresholds

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The threshold probability for successful error correction of a quantum system is not a single, absolute value. Not only is it contingent on the noise model, but what the noise model is believed to be. This is exhibited for the 2D Toric code subject to local noise where, via a mapping to the 2D random bond Ising model, suitable thresholds can be established. The analytic calculation is founded on an assumption which is known to give a good approximate result. We also find the first order correction to this result based on renormalisation group calculations, and numerically evaluate lower bounds based on explicitly implementing an error correction strategy.

The possibility of error correcting quantum systems constitutes a cornerstone of quantum computation; without it, information processing in the quantum regime would be inherently limited to small systems, providing minimal computational advantage. Surface codes, specifically the Toric code in 2D [1], provided another massive step forward, since error correction can be performed fault tolerantly with a finite threshold [2], i.e. there is a non-zero per-qubit error rate below which error correction is successful even in the presence of faulty measurements, with a spatially local structure. This removes the need for the complicated hierarchy of concatenated codes present in the original threshold derivations which, while technically impressive, were highly impractical.

The simplicity, topology and locality of the Toric code have also engendered the idea of a quantum memory [1–4] wherein a quantum state is encoded in the ground state of a local Hamiltonian, and the structure of that Hamiltonian prevents any locally occurring errors from aggregating into topologically non-trivial errors which could corrupt the encoding, thereby minimising the rate at which error correction must be applied. Errors in the Hamiltonian, or the evolution of external errors driven by the Hamiltonian, typically do not manifest as independent local rotations. Nevertheless, they must be corrected even if their distribution is not explicitly known.

In this paper, we derive a series of new thresholds for the Toric code in 2D which elucidate the influence of the difference between the true noise model and our assumptions about its nature. For simplicity of exposition, error correction is assumed to be perfect, although imperfect measurements can be incorporated in principle.

The main technical tool is the previously established connection between syndrome measurements on a noisy Toric code and the random bond Ising model (RBIM) in 2D [2]. The phase transition of the RBIM locates the critical threshold of the Toric code. In [2], this connection was established for a noise model parametrised only by $p$, the error rate, and assumed that the noise model was known. We extend this to a noise model that contains two error rates $p_X$ and $p_Z$, and our assumptions about what these values are, $p_X$ and $p_Z$. The critical region of the RBIM can be determined by an ansatz [4] and improved upon by a renormalisation style expansion [7]. Numerical verification comes via the simulation of the performance of an explicit error correction procedure (thereby yielding a lower bound on the threshold), minimum weight perfect matching.

**Toric Code:** The Toric code [1] is the quintessential example of a planar code; one which can be defined for arbitrarily many qubits with a correspondingly increasing code distance, and which has a finite fault-tolerant threshold without requiring the additional complications of concatenation. We consider here, as in [3], a rotated version akin to the Wen code [8]. This only changes the interpretation of the types of noise that affect each site, making them more natural. Starting from an $N \times N$ square lattice with periodic boundary conditions, place a qubit in the middle of each edge. For each vertex $v$ and face $f$ of the lattice, there are four neighbouring qubits, two on horizontal edges $E_H$ and two on vertical edges $E_V$. The corresponding stabilizers are defined as

$$K_v = \prod_{e \in E_H} Z_e \prod_{e \in E_V} X_e, \quad K_f = \prod_{e \in E_H} X_e \prod_{e \in E_V} Z_e.$$ 

All terms mutually commute and each has eigenvalues $\pm 1$. The space of Toric code states $|\Psi_{ij}\rangle$ for $i, j \in \{0, 1\}$ are defined by the relations $K_v |\Psi_{ij}\rangle = |\Psi_{ij}\rangle$ and $K_f |\Psi_{ij}\rangle = |\Psi_{ij}\rangle$ for all $v, f$. There are $2N^2$ qubits and $2N^2 - 2$ independent stabilizers, leaving a four-fold degeneracy (the indices $i, j$) that represents two logical qubits, for which the logical Pauli operators correspond to long-range operations along a single row or column, looping around the entire torus. The logical $Z$ operators are two inequivalent loops of Paulis along a row; one made up of $X$ rotations along a set of horizontal edges and the other $Z$ rotations along a set of vertical edges. The logical $X$ operators similarly utilise rotations on two different columns. Starting from a logical state $|\Psi_{ij}\rangle$, and applying continuous segments of $X$ and $Z$ operators, it is possible to form closed loops (meaning that all stabilizers return $+1$ expectation). Provided those loops are topologically trivial (i.e. they don’t form loops around the torus), the state is the same as the original one while non-trivial
loops correspond to logical errors. This degeneracy of the code means that if a large set of errors has arisen, it is not necessary to establish from the locations of the anyons exactly which errors occurred in order to correct for them; one only has to form the closed loops which are most likely to be trivial.

Henceforth, we shall restrict the errors to being such that \( X \) and \( Z \) errors act independently on each site, with probabilities \( \hat{p}_X \) and \( \hat{p}_Z \) respectively. This assumption, which is relaxed in the Appendix, means that there are two independent sets of anyons that can never interact; those detected by the \( \{ K_f \} \) and \( \{ K_e \} \) respectively. It is sufficient to consider just one of these sets, as the other will be the same by symmetry. We concentrate on the \( \{ K_f \} \) centred on the vertices.

**Statistical Mechanics:** In \([2]\), a connection was proven between the ability to correct errors arising on the Toric code and the existence of a phase transition in the random bond Ising model. This was done for both perfect (the 2D RBIM) and imperfect stabilizer measurement (3D RBIM), assuming that \( X \) and \( Z \) errors each occur with probability \( p \) independently on each lattice site. We will now extend this argument for the case of perfect stabilizer measurement. To emphasise the difference with the actual error rates (\( \tilde{p}_X \) and \( \tilde{p}_Z \)), the assumed error rates are denoted by \( p_X \) and \( p_Z \) respectively. This is an important distinction because one should only be able to achieve the optimal recovery specified by \([2]\) if the nature of the noise is known exactly, but this cannot be extracted from the stabilizer measurements; only some net error rate \( p \) is apparent:

\[
(1 - 2p)^2 = (1 - 2\hat{p}_X)(1 - 2\hat{p}_Z).
\]

Having extracted the error syndrome by measuring the stabilizers, there are only four inequivalent consequences (\( 1, X_i^1, Z_i^1 \) and \( X_i^2 Z_i^2 \)) of the correction. Four possible corrections, error strings \( E_i, i = 0 \ldots 3 \), can be identified which would give topologically inequivalent results and need to be assigned a likelihood of having arisen according to the assumptions on the noise model

\[
p_i = \sum_{C \in S} p(E_i \cup C),
\]

where the set \( S \) corresponds to all trivial loops. If error correction is possible, then the expectation of the probability of getting the right answer over all actual error configurations should tend to 1 in the limit of large system size, \( N \), while the other probabilities should vanish. When error correction fails, all the \( p_i \) will be similar.

Let \( \tau_0 \) be a set of variables \( \pm 1 \) for each qubit corresponding to whether or not a rotation is applied in the correction \( E_0 \). Similarly, \( \tau_0^i \) is the set due to \( E_i \cup C \). The set of all closed loops is conveniently described by introducing variables \( \sigma_\ell \in \{ \pm 1 \} \) for each vertex of the dual lattice \([2,9]\). A qubit \( q \) on an edge of the primal lattice has two neighbouring vertices of the dual lattice, \( \nu_q^1 \) and \( \nu_q^2 \): \( \tau_{C,q}^0 = \tau_0^0 \sigma_{\nu_q^1} \sigma_{\nu_q^2} \), where it now suffices to sum over the variables \( \sigma_\ell \) without restriction. For probabilities

\[
p_q = \begin{cases} 
  p_Z & q \in V \\
  p_X & q \in H 
\end{cases}
\]

the probability of a given error string is

\[
p(\tau_0^0) = \prod_q (1 - p_q)^{(1+\tau_{C,q}^0)/2} p_q^{-(1-\tau_{C,q}^0)/2}.
\]

Removing a common factor and defining

\[
\frac{1 - p_X}{p_X} = e^{2J_H} \quad \frac{1 - p_Z}{p_Z} = e^{2J_V}
\]

then the probability \( p_0 \) is proportional to

\[
Z_0 = \sum_{\hat{\sigma}} e^{H(\hat{\sigma})}
\]

with

\[
H(\hat{\sigma}) = \sum_{q \in H} (\tau_{qH}^0 J_H) \sigma_q \sigma_{q_2} + \sum_{q \in V} (\tau_{qV}^0 J_V) \sigma_q \sigma_{q_2}.
\]

This is the Hamiltonian of the \( \pm J \) random bond Ising model on a square lattice, where the vector \( \tau_0^0 \) arises from the actual errors that occurred, and the coupling strengths constitute our assignment of the likelihood of different configurations. The transition in behaviour of the probabilities between successful correction (in asymptotically all cases of instances of the syndromes) and failure corresponds to a discontinuity of the free energy of this model \( F = \ln Z \) where \( Z = \sum_{\hat{\sigma}} Z_0 \). This is the well studied phase transition in the 2D RBIM. The salient details of this study are reproduced in the Appendix.

The most studied instance, \( \hat{p}_X = \hat{p}_Z = p_X = p_Z \), transforms to the Nishimori line of the RBIM, and reveals the critical probability quoted in \([2]\) via correspondence to the multicritical point of the RBIM.

**RBIM:** The analysis conveys an interest in the critical point of the RBIM, not only along the Nishimori
line, and for asymmetries between horizontal and vertical bond strengths. The discontinuity in the free energy is located by appealing to the replica method:

$$\bar{F} = \ln Z = \lim_{n \to 0} \frac{(Z^n - 1)}{n}.$$  

The angular brackets denote the average over possible configurations of the $\tau^0$ coefficients.

In the absence of randomness, the 2D Ising model is self dual, $Z(J_H, J_V) = Z^*(J_H, J_V) = Z(J_H^*, J_V^*)$ (the dual is denoted by *), where

$$e^{-2J_H} = \tanh(J_V) \quad e^{-2J_V} = \tanh(J_H).$$

These relations convey that under the duality transformation, horizontal edges are mapped to vertical ones and vice versa. In parameter regimes where the model exhibits only one phase transition, the duality relation reveals that point $J_H = J_H^*$ and $J_V = J_V^*$ (here, one relation implies the other). It is appealing to expect that the random model should exhibit similar properties and, indeed, the $n$ replicates are self dual but with a larger parameter set $\{x_k^H, x_k^V\}_{k=0}^n$ for the primal partition function and $\{x_k^{*H}, x_k^{*V}\}_{k=0}^n$ for the dual.

$$x_k^H = \tilde{p}_X e^{nJ_H} + (1 - \tilde{p}_X) e^{-nJ_H}$$  \hspace{1cm} (1)

$$x_k^0 = 2^{n/2} \cosh^n(J_H)$$  \hspace{1cm} (2)

The $V$ versions are equivalent, with the replacements $\tilde{p}_X \mapsto \tilde{p}_Z$ and $J_H \mapsto J_V$. Unfortunately, there is no satisfying assignment such that $x_k^H = x_k^V$ and $x_k^{*H} = x_k^{*V}$ for all $k$. Instead, it is traditional to proceed with the conjecture that the critical point is given by

$$x_0^H x_0^{*V} = x_0^{*H} x_0^V.$$  \hspace{1cm} (3)

Taking the limit $n \to 0$ yields

$$\tilde{p}_H \log_2 p_H + (1 - \tilde{p}_H) \log_2(1 - p_H)$$

$$+ \tilde{p}_V \log_2 p_V + (1 - \tilde{p}_V) \log_2(1 - p_V) = -1$$  \hspace{1cm} (4)

The above conjecture has been specifically postulated to work at the multicritical points of the RBIM ($\tilde{p}_X = p_X$ and $\tilde{p}_Z = p_Z$), being well justified by the existence of various symmetries, and well tested numerically, also in the asymmetric case $\tilde{p}_X > p_X$ and $\tilde{p}_Z > p_Z$. However, this quadrant can be resolved in two relevant cases.

Firstly, the absolute best that any error correction strategy could possibly perform is given precisely by the multicritical point, corresponding to perfect knowledge of the error model: $\tilde{p}_X = p_X$ and $\tilde{p}_Z = p_Z$. With the exception of $\tilde{p}_X = 0$, the measurement statistics only reveal the frequency $p$ of stabilizers being $\tilde{p}$. Without any further knowledge, one should assume that $\tilde{p}_X = p_Z$. How detrimental is this to the threshold? Under this condition, Eq. (4) reduces to an effective homogeneous system with $2\tilde{p} = \tilde{p}_X + \tilde{p}_Z$ and $2p = \tilde{p}_X + \tilde{p}_Z$.

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Although the conjecture of Eq. (5) compares favourably with numerical estimates on the square lattice, there is known to be a discrepancy with some exact renormalisation group calculations on hierarchical lattices. To account for this, a renormalisation inspired expansion has been introduced to account for corrections. While most natural for hierarchical lattices, it has been extended to square lattices and achieves an even tighter match with previous numerical results using only a first order correction. This technique proceeds by replacing the term $e^{J_H}$ in $x_0^H$ (and, similarly, the term $e^{-J_H}$ in $x_0^{*H}$) with an equivalent effective weight arising from an averaging effect over several neighbouring spins. This averaging involves horizontal and vertical edges equally, thereby simplifying the conjectured pos-
tion of the critical point to $x_0 = x^*_0$. The larger the block that is averaged over, the better the anticipated approximation of the position of the critical point. The first order approximation specified by Fig. 4 of [7] locates the critical point at

$$2 = \frac{1}{2} \sum_{\eta = \pm 1} (1 + \eta(1 - 2p)^4) \log_2(1 + \eta(1 - 2p)^4)$$

$$- \frac{1}{2} \sum_{n=0}^{2} \left( \frac{2}{m} \right) a_{nm}(\tilde{p}_X, \tilde{p}_Z) \log_2 a_{nm}(p_X, p_Z)$$

where

$$a_{nm}(r, s) = \frac{r^n s^m}{(1 - r)n^2(1 - s)m^2} + \frac{(1 - r)^n(1 - s)^m}{m^2 r^{n - 2}s^{m - 2}}.$$

The threshold values are barely impacted, although Fig. [2] indicates that they can surpass the Gilbert-Varshamov bound for stabilizer codes (which coincides with Eq. (4)). As the code is degenerate, there is no contradiction [12].

**Minimum Weight Perfect Matching:** Given the near-vertical phase boundary of the RBIM below the Nishimori line, any error correction strategy which assumes a lower ‘temperature’ has an almost identical critical probability. In particular, the 0 temperature case corresponds to correcting by minimum weight perfect matching [20]; an efficient algorithm which is readily implemented (however, it cannot deal with correlations between the probabilities of $X$ and $Z$ errors, and a polynomial scaling may be insufficient [13]). As such, this algorithm gives a lower bound on the threshold fidelities for comparison, and is anticipated to be quite tight. The distance measure is the log of the probability of a given error path, i.e. a sum over pair-wise matching of anyons separated by $l_H$ and $l_V$. Hence, each pair of vertices is assigned a weight

$$l_H \log \left( \frac{p_X}{1 - p_X} \right) + l_V \log \left( \frac{p_Z}{1 - p_Z} \right)$$

and the total weight is minimised [2]. The points in Fig. [2] represent a Monte Carlo simulation in a system with $N = 100$ based on the Blossom V algorithm [14]. For a fixed ratio $\tilde{p}_X/\tilde{p}_Z$, the fraction of 500 runs giving a logical error was computed for varying error rates, enabling determination of the failure probability (the threshold at which a transition in logical error rate from 0 to 50% occurs). Similar numerics, for a perfectly identified error model, are present in [15].

**Conclusions:** In this paper, we have demonstrated how any discrepancy between the actual and assumed noise models affects the error correcting threshold of the Toric code in 2D. For perfectly identified errors, the rotated Toric code that we examined (approximately) saturates, and even surpasses, the hashing bound, for all asymmetries, without resorting to random lattices [15]. In comparison, the non-rotated version has the much weaker threshold of $\max(p_X, p_Z) = p_C$. The present results are particularly important in the memory setting. In [3] it was demonstrated that a randomised Toric code could have a polynomial (in $N$) survival time between rounds of error correction. Any lack of knowledge obliterates that polynomial survival time, making it only a constant, due to the shifted threshold value. As detailed in the Appendix, correlations due to $Y$ errors can be incorporated into the analysis [10], with the equivalent of Eq. (4) being given in Eq. (A.1). In principle, the present results can be generalised to the situation of noisy measurements [2, 17], adding a third dimension to model.

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[19] Every column has an even number of $-1$ measurement results of the stabilizers. The chance of that happening if $\tilde{p}_X \neq 0$ is vanishingly small.
[20] Zero temperature corresponds to the limit $J \to \infty$, meaning the only term in $Z_\sigma$ that is worth considering is the $\sigma$ that gives the smallest value.
Appendix: Generalised Noise Model

As a generalisation to the main body of the paper, we want to derive the error correcting threshold of the Toric code in 2D when subject to a per-qubit noise model of

$$E(\rho) = (1 - \hat{q}_X - \hat{q}_Y - \hat{q}_Z)\rho + \hat{q}_X X \rho X + \hat{q}_Y Y \rho Y + \hat{q}_Z Z \rho Z$$

when we assume that the corresponding probabilities are $q_X$, $q_Y$ and $q_Z$. For comparison, the previous distribution had

$$q_X = p_X (1 - p_Z) \quad q_Z = p_Z (1 - p_X) \quad q_Y = p_X p_Z.$$

With $Y$ errors present, it will not be possible to divide the system into two independent systems. Let us denote by $L$ the lattice on which the toric code is defined (with the qubits in the middles of the edges). Let $L_1$ be a copy of $L$ and $L_2$ be the dual of $L_1$ (i.e. in the case of the periodic square lattice, the same lattice but shifted both horizontally and vertically by half a unit). With the edges $q$ (corresponding to a qubit on $L$) of each of the lattices $L_i$ we associate a variable $\tau^2_q \in \pm 1$ in the following way:

- If $q$ is a vertical bond of $L$, an $X$ error on $q$ is represented by $\tau^2_q = -1$.
- If $q$ is a horizontal bond of $L$, a $Z$ error on $q$ is represented by $\tau^2_q = -1$.
- If $q$ is a horizontal bond of $L$, an $X$ error on $q$ is represented by $\tau^4_q = -1$.
- If $q$ is a horizontal bond of $L$, a $Z$ error on $q$ is represented by $\tau^4_q = -1$.

Note that this means that $X$ errors are specified by horizontal edges of $L_1$ and $L_2$, while $Z$ errors are specified by vertical edges. For a given error configuration $\tau = (\tau^1, \tau^2)$, we’re interested in assessing our belief of it having arisen from a given sector, i.e.

$$p_0 = \sum_{\tau^1 \in S_1, \tau^2 \in S_2} p(\tau \cup \tau^1 \cup \tau^2).$$

where $S_1$ and $S_2$ are closed loops around $L_1$ and $L_2$ respectively. This $p$ is proportional to

$$Z_0 = \sum_{\tau^1, \tau^2} e^{\sum_q \tau^1_q J_H \tau^2_q + \tau^2_q J_V \tau^1_q + \tau^1_q \tau^2_q J_Y}$$

where the $\tau^1$ denotes the sum being restricted to closed loops and

$$e^{J_H} = \frac{1 - q_X - q_Y - q_Z}{q_Y q_X} \quad e^{J_V} = \frac{1 - q_X - q_Y - q_Z}{q_Y q_X} \quad e^{J_Y} = \frac{1 - q_X - q_Y - q_Z}{q_Y q_Z}.$$

A convenient way to parametrise the sum over closed loops is to instead let an edge $q$ specify the vertices $i$ and $j$ of the corresponding edge of $L_1$ and $i'$ and $j'$ of $L_2$ (see Fig. 3), and to write $\tau^1_q = \sigma_i \sigma_j$, $\tau^2_q = \sigma_i' \sigma_j'$. When one takes the product of values $\tau^1_q$ around a closed loop, every vertex appears an even number of times, and since $\sigma_i^2 = 1$, this gives the correct product. Hence, we have

$$Z_0 = \sum_{\sigma} e^H$$

with

$$H = \sum_{q \in H} \tau^1_q J_H \sigma_i \sigma_j + \tau^2_q J_V \sigma_i' \sigma_j' + \tau^1_q \tau^2_q J_Y \sigma_i \sigma_j \sigma_i' \sigma_j'.$$

The $\tau^1_q$ have a sign distribution specified by the error model. This is exactly the model derived in [10], and it corresponds to two planes of Ising models coupled by some 4-body terms. It is equivalent to the 8-vertex model. We are now interested in determining its phase transitions as this will convey the limit of when error correction is possible. In order to do this, the first thing to establish is duality of the non-random version. This can then be extended to the replica method in order to account for the configurational average. We note that if we take the previous parametrisation of the probabilities, $J_Y = 0$, we are returned to having two uncoupled 2D Ising lattices.
Duality

Duality is an important concept for us because if a model's partition function $Z(\vec{J})$ for a particular set of parameters can be mapped on to itself for a different set of parameters $Z(\vec{J}')$, then along a path in parameter space that contains its duals, if we know that there is a single discontinuity in $Z$, it can only happen at $\vec{J} = \vec{J}'$ (otherwise discontinuities would have to exist in symmetric pairs about this point). For simple noise models in which a single parameter describes the strength of the noise, as the strength increases the system must pass through a critical point demarking the failure of error correction, and there will only be one such point – adding extra noise should never bring us round full circle from correction being impossible to suddenly being able to correct again!

Let us approach the task of proving duality in the non-random model by expressing

$$Z = \sum_{\tau \in \mathbb{Z}^2} \prod_{q} u_q.$$

This is just writing exactly what we had before but with

$$u_q = e^{\tau_{\tau_{1}} J_H + \tau_{\tau_{2}} J_V + \tau_{\sigma_{1}} J_{\tau_{1}} J_{\sigma_{2}} + \tau_{\sigma_{1}} J_{\tau_{2}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{1}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{2}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{2}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{2}} J_{\sigma_{2}}},$$

Defining the dual functions as

$$2u_q^* = e^{\tau_{\tau_{1}} J_H + \tau_{\tau_{2}} J_V + \tau_{\sigma_{1}} J_{\tau_{1}} J_{\sigma_{2}} + \tau_{\sigma_{1}} J_{\tau_{2}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{1}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{2}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{2}} J_{\sigma_{2}} + \tau_{\sigma_{2}} J_{\tau_{2}} J_{\sigma_{2}}},$$

for $s_1^1, s_1^2 \in \{0, 1\}$, then $u_q$ can be expressed as

$$u_q = \frac{1}{2} \sum_{s_1^1, s_1^2 \in \{0, 1\}} u_q^* (-1)^{s_1^1 (\varepsilon_i + \varepsilon_1)} (-1)^{s_1^2 (\varepsilon_i + \varepsilon_2)},$$

having replaced $\sigma_i$ with $2\varepsilon_i - 1$ such that $\varepsilon_i \in \{0, 1\}$. Consider what happens when we examine the product of $u_q$, expressed as above, for all edges of $L$ leading out of a given vertex $i$, and performing the sum in $Z$ over that particular $\varepsilon_i$. The terms are 0 unless

$$\sum_{j} s_q^1 \mod 2 = 0,$$

which is entirely equivalent to the product of $s_q^1$ having to be +1 around vertices or, in other words, around closed loops of the dual lattice. Hence,

$$Z = \sum_{s_1^1, s_2^1} \prod_{q} u_q^*$$

meaning that the model is self dual. For instance, the values

$$u = (e^{J_H + J_V}, e^{J_H - J_V - J_Y}, e^{-J_H + J_V - J_Y}, e^{-J_H - J_V + J_Y})$$

$$u^* = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} u.$$
are equal if $J_H = J_V$ and $e^{-2J_V} = \sinh(2J_H)$. This is the only instance in which the model is self-dual. However, it also exhibits a pseudo mutual duality between parameter sets

$$(J_H, J_V, J_Y) \leftrightarrow (J_V, J_H, J_Y)$$

and all other permutations. This is the equivalent of the mutual duality applied to the anisotropic RBIM in the body of the paper and [6], and is succinctly expressed as the contour

$$q_X + q_Y + q_Z = \frac{1}{2}.$$  

This symmetry property helps to justify the applicability of the assumption we will make to determine the critical point of the random model [6].

**Replica Method**

The above duality was proven without any randomness present. In order to deal with the randomness of the bonds, we must take a configurational average, and use it to determine any discontinuity in the free energy, $\ln Z$. For $n$ copies, $Z_n = Z^\otimes n$ is dual to $Z_n^*$ with a corresponding set of parameters $\underline{u}^\otimes n$ and $\underline{u}^*\otimes n$. If, after averaging over the possible values of $\tau_q^i$, these two vectors are equal, there is a self-dual point. Unfortunately, this does not happen simultaneously for all elements of the vector. Instead, we make the zero order assumption that the critical point occurs when the first element of each vector (after averaging) is equal.

$$\lim_{n \to 0} x_n = x_n^*.$$

Note that changing the values of the $\tau_q^i$ permutes the elements of the vector $\underline{u}^\otimes n$, which means that the possible first elements are the $n^{th}$ power of the elements of $\underline{u}$. Hence,

$$x_n = (1 - \tilde{q}_X - \tilde{q}_Y - \tilde{q}_Z, \tilde{q}_Z, \tilde{q}_X, \tilde{q}_Y) \cdot \underline{u}^n.$$

On the other hand, the first element of the vector $\underline{u}^*\otimes n$ does not change as values $\tau_q^i$ are randomly selected. Hence,

$$x_n^* = \frac{1}{2^n} \left( e^{J_H+J_V+J_Y} + e^{J_H-J_V-J_Y} + e^{-J_H+J_V-J_Y} + e^{-J_H-J_V+J_Y} \right)^n.$$

Expanding this, and taking the limit, gives the equivalent of Eq. (4) for the generalised model:

$$(1 - \tilde{q}_X - \tilde{q}_Y - \tilde{q}_Z, \tilde{q}_Z, \tilde{q}_X, \tilde{q}_Y) \cdot \log_2(1 - q_X - q_Y - q_Z, q_Z, q_X, q_Y) = -1,$$

which coincides with the Hashing bound for stabilizer codes subject to Pauli error channels. This can be anticipated to be a good approximation whenever the error rates have been perfectly identified.

Now we can ask a variety of questions. For instance, if $q_X = q_Y = q_Z = \tilde{q}_X = \tilde{q}_Y = \tilde{q}_Z$, i.e. depolarising noise that we have perfectly identified, one has to solve the equation

$$(1 - 3q) \log_2(1 - 3q) + 3q \log_2(q) = -1$$

to find $3q = 0.18929$, which replicates the value given in [10], where this value was also numerically verified.

How effective is minimum weight perfect matching at correcting depolarising errors? Since it is not capable of taking the correlations introduced by $Y$ into account, it is not expected to be tight with the optimal correction. However, a good estimate is to set $q_Y = p^2$, $q_X = q_Z = p(1-p)$, $\tilde{q}_X = \tilde{q}_Y = \tilde{q}_Z = q$, and find the critical point. It’s described by

$$-\frac{1}{2} = (1 - 2q) \log_2(1 - p) + 2q \log_2(p),$$

and the largest value of $q$ is given by setting $p = 2q$, which reveals that $3q = 0.165$. This compares favourably with previous results [13].

One could improve the accuracy of these conclusions by utilising the higher order expansions [7]. However, at low orders (i.e. those which are computationally accessible), the differences are negligible, as evidenced in Fig. 2.