A Black Hole Farey Tail

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Abstract

We derive an exact expression for the Fourier coefficients of elliptic genera on Calabi-Yau manifolds which is well-suited to studying the AdS/CFT correspondence on $AdS_3 \times S^3$. The expression also elucidates an $SL(2, \mathbb{Z})$ invariant phase diagram for the D1/D5 system involving deconfining transitions in the $k \to \infty$ limit.
1. Introduction and Summary

One of the cornerstones of the AdS/CFT correspondence \cite{1} is the relation between the partition function $Z_X$ of a superstring theory on $AdS \times X$ and the partition function $Z_C$ of a holographically related conformal field theory $C$ on the boundary $\partial (AdS)$. Roughly speaking we have

$$Z_X \sim Z_C.$$  \hspace{1cm} (1.1)

While the physical basis for this relationship is now well-understood, the precise mathematical formulation and meaning of this equation has not been very deeply explored. This relationship is hard to test since it is difficult to calculate both sides in the same region of parameter space. In this paper we consider the calculation of a protected supersymmetric partition function and we will give an example of a precise and exact version of (1.1). In particular, we will focus on the example of the duality between the IIB string theory on $AdS_3 \times S^3 \times K3$ (arising, say, from the near horizon limit of $(Q_1, Q_5)$ (D1,D5) branes) and the dual conformal field theory with target space $\text{Hilb}^k(K3)$, the Hilbert scheme of $k = Q_1Q_5$ points on $K3$. We will analyze the so called “elliptic genus” which is a supersymmetry protected quantity and can therefore be calculated at weak coupling producing a result that is independent of the coupling. We will rewrite it in a form which reflects very strongly the sum over geometries involved in the supergravity side. We will then give an application of this formula to the study of phase transitions in the D1D5 system.

Since the main formulae below are technically rather heavy we will, in this introduction, explain the key mathematical results in a simplified setting, and then draw an analogy to the physics. The mathematical results are based on techniques from analytic number theory, and have their historical roots in the Hardy-Ramanujan formula for the partitions of an integer $n$ \cite{2}.

Let us consider a modular form for $\Gamma := SL(2, \mathbb{Z})$ of weight $w < 0$ with a $q$-expansion:

$$f(\tau) = \sum_{n \geq 0} F(n)q^{n+\Delta}$$  \hspace{1cm} (1.2)

where $F(0) \neq 0$. In the physical context $\Delta = -c/24$, where $c$ is the central charge of a conformal field theory, $w = -d/2$ if there are $d$ noncompact bosons in the conformal field theory, and $F(0)$ is a ground-state degeneracy. It is well-known to string-theorists and
number-theorists alike that the leading asymptotics of $F(\ell)$ for large $\ell$ can be obtained à la Hardy-Ramanujan from a saddle-point approximation, and are given by:

$$F(\ell) \sim \frac{1}{\sqrt{2}} F(0)|\Delta|^{1/4-w/2}(\ell + \Delta)^{w/2-3/4} \exp\left[4\pi \sqrt{|\Delta|/(\ell + \Delta)}\right] \left(1 + O(1/\ell^{1/2})\right). \quad (1.3)$$

This estimate is a key mathematical step when accounting for the entropy of extremal supersymmetric five-dimensional black holes in terms of D-brane microstates [3].

What is perhaps less well-known is that there is an exact version of the formula (1.3), which makes the asymptotics manifest. The Fourier coefficients are given by the expression:

$$F(\ell) = 2\pi \sum_{n+\Delta < 0} \left( \frac{\ell + \Delta}{|n + \Delta|}\right)^{(w-1)/2} F(n) \cdot \sum_{c=1}^{\infty} \frac{1}{c} Kl(\ell + \Delta, n + \Delta; c) I_{1-w} \left(\frac{4\pi}{c} \sqrt{|n + \Delta|/(\ell + \Delta)}\right). \quad (1.4)$$

The first sum is over the Fourier coefficients of the “polar part” $f^-$ of $f$, defined by

$$f^- := \sum_{n+\Delta < 0} F(n) q^{n+\Delta}. \quad (1.5)$$

In the second sum $I_{1-w}$ is the standard Bessel function and $Kl(\ell, m; c)$ is the “Kloosterman sum”

$$Kl(n, m; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} \exp\left[2\pi i (d \frac{n}{c} + d^{-1} \frac{m}{c})\right]. \quad (1.6)$$

The summation variable “$c$” in (1.4) is traditional. We hope it will not be confused with the central charge of a conformal field theory. Because $I_\nu(z) \sim z^\nu$ for $z \to 0$, the series (1.4) is absolutely convergent for $w < 1/2$ (see eq. (2.18) below). On the other hand, in view of the asymptotics $I_\nu(z) \sim \sqrt{\frac{1}{2\pi z}} e^z$ for $Re(z) \to +\infty$, (1.4) generalizes (1.3). Finally, to suppress some complications we will assume in this introduction (but not in subsequent sections) that $\Delta$ is a negative integer. [2]

The expression (1.4) may be usefully rewritten by introducing the map

$$f(\tau) \to Z_f(\tau) := \left(q \frac{\partial}{\partial q}\right)^{1-w} f \quad (1.7)$$

---

1. The formula (1.4) is due to Rademacher. The relatively trivial estimate (1.3) is usually referred to as the Hardy-Ramanujan formula, but in fact, their full formula is much closer to (1.4). See especially [4], eqs. 1.71-1.75. See [4],[5] for original papers. For some history see Littlewood’s Miscellany [6], p. 97, [7], ch. 5, the review article [8], and Selberg’s collected works.
We will call this the “fareytail transform.” Mathematically it is simply a special case of Serre duality, but the physical meaning should be clarified. (We comment on this below.)

In any case, it is in terms of this new modular form that formulas look simple. One readily verifies that, for \( w \) integral, the fareytail transform takes a form of modular weight \( w \) to a form of modular weight \( 2 - w \) with Fourier coefficients \( \tilde{F}(n) := (n + \Delta)^{1-w}F(n) \).

Moreover, the transform takes a polar expression to a polar expression:

\[
\mathcal{Z}_f^- = \mathcal{Z}_f^- = \sum_{n+\Delta < 0} \tilde{F}(n)q^{n+\Delta}.
\]

Notice that \( f \) and \( \mathcal{Z}_f \) contain the same information except for states with \( n + \Delta = 0 \). Now, using a straightforward application of the Poisson summation formula (see appendix C), one can cast (1.4) into the form of an average over modular transformations:

\[
\mathcal{Z}_f(\tau) = \sum_{\Gamma \in \mathbb{Q} \setminus \Gamma} (c\tau + d)^{w-2} \mathcal{Z}_f^-(\frac{a\tau + b}{c\tau + d}).
\]

Here \( \Gamma_\infty \) is the subgroup of the modular group \( \Gamma \) generated by \( \tau \to \tau + 1 \). It is necessary to average over the quotient \( \Gamma_\infty \setminus \Gamma \) rather than \( \Gamma \) to get a finite expression, since \( \exp[2\pi i (n + \Delta)] \) is invariant under \( \Gamma_\infty \). Note that \( \Gamma_\infty \setminus \Gamma \) can be identified with the set of relatively prime integers \((c, d)\) or, equivalently, with \( d/c \in \mathbb{Q} \cup \{\infty\} := \hat{\mathbb{Q}} \). In mathematics such averages over the modular group are called “Poincaré series.”

One final mathematical point is needed to complete the circle of mathematical formulae we will need. Using an integral representation of the Bessel function one can also write (1.4) in the form

\[
\tilde{F}(\ell) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{2\pi \beta(\ell + \Delta)} \tilde{Z}_f(\beta) d\beta
\]

where \( \epsilon \to 0^+ \) and we have introduced the “truncated sum”

\[
\hat{Z}_f(\beta) := \sum_{(\Gamma_\infty \setminus \Gamma)'} (c\tau + d)^{w-2} \mathcal{Z}_f^-(\frac{a\tau + b}{c\tau + d}).
\]

Here and throughout this paper \( \tau = i\beta \) in formulae of this type. When we want to emphasize modular aspects we use \( \tau \), when we want to stress the relation to statistical mechanics we use \( \beta \). Note that \( \beta \) is a complex variable with positive real part. The sum

\[2\] Actually, over two copies of \( \hat{\mathbb{Q}} \), if \( \Gamma = \text{SL}(2, \mathbb{Z}) \) and not \( \text{PSL}(2, \mathbb{Z}) \). This distinction becomes important if the weight \( w \) is odd.
in (1.11) is a truncated version of that in (1.9). The prime in the notation \((\Gamma_{\infty}\backslash \Gamma/\Gamma_{\infty})')\) means we omit the class of \(\gamma = 1\). Elements of the double-coset may be identified with the rational numbers \(-d/c\) between 0 and 1. For further details see equations 2.7 - 2.10 below.

We may now describe the physical interpretation of the formulae (1.2) to (1.11). \(f(\tau)\) will become a conformal field theory partition function. The fareytail transform \(Z_f(\tau)\) will be the dual supergravity “partition function.” The sum over the modular group in (1.9) will be a sum over solutions to supergravity. The fareytail transform is related to the truncated sum by

\[
Z_f(\tau) = Z_f^{-}(\tau) + \sum_{\ell \in \mathbb{Z}} \hat{Z}_f(\tau + \ell).
\]

In order to understand this relation, recall that in statistical mechanics a standard maneuver is to relate the canonical and microcanonical ensemble by an inverse Laplace transform:

\[
N(E) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} Z(\beta) e^{2\pi \beta E} d\beta.
\]

Regarding a modular form such as (1.2) as a partition function the corresponding microcanonical ensemble is given by

\[
N(E) = \sum_{n \in \mathbb{Z}} \delta(E - n) F(n)
\]

and thus we recognize the Rademacher expansion as the standard relation between the microcanonical and canonical ensembles, where the latter is a Poincaré series.

Let us pass now from the simplified version to the true situation. As we discuss in section two, the formulae (1.2) to (1.11) can be considerably generalized. In particular, they can be applied to Fourier coefficients of Jacobi forms of nonpositive weight, and thus can be applied to elliptic genera of Calabi-Yau manifolds. Recall that if \(X\) is a Calabi-Yau manifold then the elliptic genus may be defined in terms of the associated \((2,2)\) CFT trace:

\[
\chi(q, y; X) := \text{Tr}_{RR} e^{2\pi i \tau (L_0 - c/24)} e^{2\pi i \tilde{\tau} (\tilde{L}_0 - c/24)} e^{2\pi i z J_0} (-1)^F := \sum_{n \geq 0, r} c(n, r) q^n y^r.
\]

We will let \(q := e^{2\pi i \tau}, y := e^{2\pi i z}\), and \((-1)^F = \exp[i\pi (J_0 - \tilde{J}_0)]\). We collect some standard facts about the elliptic genus in section three.

As explained in section four, the analog of the fareytail transformation for a Jacobi form of weight \(w\) and index \(k\) will be the map

\[
\phi \to \text{FT} (\phi) := \left| q \partial_q - \frac{1}{4k} (y \partial_y)^2 \right|^{3/2-w} \phi.
\]
When $w$ is integral we interpret this as a pseudodifferential operator: $FT(\phi) = \sum \tilde{c}(n, \ell) q^n y^\ell$ with

$$\tilde{c}(n, \ell) := |n - \ell^2/4k|^{3/2-w} c(n, \ell).$$

(1.17)

Formal manipulations of pseudo-differential operators suggest that $FT(\phi)$ is a Jacobi form of weight $3-w$ and the same index $k$. However, these formal manipulations lead to a false result, as pointed out to us by D. Zagier. Nevertheless, as we show in section four, it turns out that for $n - \ell^2/4k > 0$ the coefficients $\tilde{c}(n, \ell)$ can be obtained as Fourier coefficients of a truncated Poincaré series $\hat{Z}_\phi$ defined in equation (4.6) below.

Our main result will be a formula for the fareytail transform of the elliptic genus $\chi$ for the Calabi-Yau manifold $X = \text{Hilb}^k(K3)$. The corresponding truncated Poincaré series takes the form:

$$Z_\chi(\beta, \omega) = 2\pi \sum_{(\Gamma_\infty \backslash \Gamma)_0} \frac{1}{(c\tau + d)^3} \sum_s D(s) \exp \left[ -2\pi i \Delta_s \frac{a\tau + b}{c\tau + d} \right] \Psi_s \left( \frac{\omega}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right)$$

(1.18)

The notation is explained in the following paragraph and a more precise version appears in equation (5.1) below. In section five we interpret this formula physically. The sum $(\Gamma_\infty \backslash \Gamma)_0$ is the sum over relatively prime pairs $(c,d)$ with $c > 0$ (together with the pair $(c,d) = (0,1)$.) We will interpret the average over $(\Gamma_\infty \backslash \Gamma)_0$ as a sum over an $SL(2,\mathbb{Z})$ family of black hole solutions of supergravity on $AdS_3 \times S^3$, related to the family discussed in [9]. The sum over $s$ is a finite sum over those particle states that do not cause black holes to form. They can be “added” to the black hole background, and the combined system has gravitational action $D(s) \exp(-2\pi i \Delta_s \tau)$ where $D(s)$ is a degeneracy of states. Finally $\Psi_s$ will be identified with a Chern-Simons wavefunction associated to $AdS_3$ supergravity.

In fact, if we introduce the “reduced mass”

$$L_0^\perp := L_0 - \frac{1}{4k} J_0^2 - \frac{k}{4}$$

(1.19)

then the polar part $Z_\chi^-$ of the supergravity partition function can be considered as a sum over states for which $L_0^\perp < 0$. The calculations of Cvetic and Larsen [10] show that the area of the horizon of the black hole and therefore its geometric entropy is precisely determined by a combination of the mass and (internal) angular momentum that is identical to $L_0^\perp$ (one has $S_{BH} = 2\pi \sqrt{kL_0^\perp} = \pi \sqrt{4kn - \ell^2}$.) Therefore the truncation of the partition function to states with $L_0^\perp < 0$ describes a thermal gas of supersymmetric particles in an AdS background, truncated to those (ensembles of) particles that do not yet form black
holes. It is with this truncated partition function that contact has been made through supergravity computations \[11\].

The sum over the quotient \((\Gamma_\infty \backslash \Gamma)_0\) as in \((1.18)\) adds in the black hole solutions. It is very suggestive that the full partition function can be obtained by taking the supergravity AdS thermal gas answer and making it modular invariant by explicitly averaging over the modular group. This sum has an interpretation as a sum over geometries and it seems to point to an application of a principle of spacetime modular invariance.

The relation to Chern-Simons theory makes it particularly clear that in the AdS/CFT correspondence the supergravity partition function is to be regarded as a vector in a Hilbert space, rather like a conformal block. Indeed, the Chern-Simons interpretation of RCFT \[12\] is a precursor to the AdS/CFT correspondence. This subtlety in the interpretation of supergravity partition functions has also been noted in a different context in \[13\]. The factor of \((c\tau + d)^{-3}\) in \((1.18)\) is perfect for the interpretation of \(Z_\chi\) as a half-density with respect to the measure \(dz \wedge d\tau\).

To be more precise, the wave-function is a section of a line-bundle \(\mathcal{L}^k \otimes \mathcal{K}\) where \(\mathcal{K}\) is the holomorphic canonical line bundle over the moduli space, and \(k\) is the level of the CS-supergravity theory. The invariant norm on sections on the line-bundle \(\mathcal{L}^k \otimes \mathcal{K}\) is given by

\[
\exp\left(-4\pi k \frac{(\text{Im} z)^2}{\text{Im} \tau}\right) |Z_\chi(\tau, z) d\tau dz|^2
\]

Assuming that this norm is invariant under modular transformations, one concludes that \(Z_\chi(\tau, z)\) is a Jacobi form of weight 3 and index \(k\).

One application of \((1.18)\) is to the study of phase transitions as a function of \(\tau\) in the \(k \to \infty\) limit. Since there are states with \(\Delta_\chi \sim k\) there will be sharp first order phase transitions as \(\tau\) crosses regions in an \(SL(2, \mathbb{Z})\) invariant tesselation of the upper half plane. We explain a proof of this phase structure in section six.

Finally, we may explain the title of this paper. In our proof of the result \((1.18)\) (see appendix B) the sum over rational numbers \(d/c\) is obtained by successive approximations by Farey sequences, a technique skillfully exploited by Rademacher, and going back to Hardy-Ramanujan. Only one term in the sum dominates the entropy of the \(D1D5\) system, the successive terms in longer Farey sequences constitute a tail of the distribution, but this tail is associated with a family of black holes. This, then, is our Black Hole Farey Tail.

\textit{Note added for V3, Dec. 8, 2007:} Don Zagier pointed out to us a serious error in versions 1 and 2 of this paper, namely, in those versions it was asserted under equation
that $O^{3/2-w}\phi$ is a Jacobi form of weight $3-w$ and index $k$. This turns out to be false. In addition to this, it turns out that when one attempts to convert the truncated Poincaré series $\hat{Z}_\phi$ of section 4 to a full Poincaré series one gets zero, and thus the series which one would expect to reproduce $O^{3/2}\chi$ in fact vanishes. As far as we are aware, the central formulae (4.5) and (4.6), which involve only the truncated Poincaré series are nevertheless correct. Fortunately, the physical interpretation we subsequently explain is based on this truncated series, so our main conclusions are unchanged.

A recent paper [14] has clarified somewhat the use of the Farey tail transform, and presented a regularized Poincaré series for $\chi$ rather than $Z\chi$.

2. The Generalized Rademacher Expansion

In this section we give a rather general result on the asymptotics of vector-valued modular forms. It is a slight generalization of results of Rademacher [15]. See also [4], ch.5, and [8].

Let us suppose we have a “vector-valued nearly holomorphic modular form,” i.e., a collection of functions $f_\mu(\tau)$ which form a finite-dimensional unitary representation of the modular group $PSL(2,\mathbb{Z})$ of weight $w$. Under the standard generators we have

$$f_\mu(\tau + 1) = e^{2\pi i \Delta_\mu} f_\mu(\tau)$$
$$f_\mu(-1/\tau) = (-i\tau)^w S_{\mu\nu} f_\nu(\tau)$$

and in general we define:

$$f_\mu(\gamma \cdot \tau) := (-i(c\tau + d))^w M(\gamma)_{\mu\nu} f_\nu(\tau)$$
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where, for $c > 0$ we choose the principal branch of the logarithm.

We assume the $f_\mu(\tau)$ have no singularities for $\tau$ in the upper half plane, except at the cusps $\mathcal{Q} \cup i\infty$. We may assume they have an absolutely convergent Fourier expansion

$$f_\mu(\tau) = q^{\Delta_\mu} \sum_{m \geq 0} F_\mu(m) q^m \quad \mu = 1, \ldots, r$$

with $F_\mu(0) \neq 0$ and that the $\Delta_\mu$ are real. We choose the branch $q^{\Delta_\mu} = e^{2\pi i \tau \Delta_\mu}$. We wish to give a formula for the Fourier coefficients $F_\mu(m)$. 
We will now state in several forms the convergent expansion that gives the Fourier coefficients of the modular forms in terms of data of the modular representation and the polar parts $f_{\mu}^-$. We assume that $w \leq 0$.\footnote{The arguments in appendices B, C are only valid for $w < 0$. The extension to $w = 0$ was already known in the 1930’s. See \cite{8}.}

The first way to state the result is

$$F_\nu(n) = \sum_{m + \Delta_\mu < 0} K_{n,\nu;m,\mu} F_\mu(m)$$

which holds for all $\nu, n$. The infinite × finite matrix $K_{n,\nu;m,\mu}$ is an infinite sum over the rational numbers in lowest terms $0 \leq -d/c < 1$:

$$K_{n,\nu;m,\mu} = \sum_{0 \leq -d/c < 1} K_{n,\nu;m,\mu}(d,c)$$

and for each $c, d$ we have:

$$K_{n,\nu;m,\mu}(d,c) := -i \tilde{M}(d,c)_{n,\nu;m,\mu}$$

$$\int_{1-i\infty}^{1+i\infty} d\beta (\beta c)^{w-2} \exp \left[ 2\pi c^2 \beta^{-1} - 2\pi (m + \Delta_\mu) \beta \right]$$

The matrix $\tilde{M}(d,c)_{n,\nu;m,\mu}$ is essentially a modular transformation matrix and is defined (in equation (2.10) below) as follows.

Let $\Gamma_\infty$ be the subgroup of modular transformations $\tau \rightarrow \tau + n$. We may identify the rational numbers $0 \leq -d/c < 1$ with the nontrivial elements in the double-coset $\gamma \in \Gamma_\infty \setminus PSL(2,\mathbb{Z})/\Gamma_\infty$ so the sum on $-d/c$ in (2.5) is more fundamentally the sum over nontrivial elements $[\gamma]$ in $\Gamma_\infty \setminus PSL(2\mathbb{Z})/\Gamma_\infty$. To be explicit, consider a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

where $c, d$ are relatively prime integers. Since

$$\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \ell c & b + \ell d \\ c & d \end{pmatrix}$$

the equivalence class in $\Gamma_\infty \setminus \Gamma$ only depends on $c, d$. When $c \neq 0$ we can take $0 \leq -d/c < 1$ because:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + a\ell \\ c & d + c\ell \end{pmatrix}$$

\footnote{The arguments in appendices B, C are only valid for $w < 0$. The extension to $w = 0$ was already known in the 1930’s. See \cite{8}.}
shifts $d$ by multiples of $c$. The term $c = 0$ corresponds to the class of $[\gamma = 1]$. It follows that

$$
\tilde{M}(d, c)_{n, \nu; m, \mu} := e^{2\pi i (n + \Delta_\nu) (d/c)} M(\gamma)_{\nu\mu}^{-1} e^{2\pi i (m + \Delta_\mu)(a/c)}
$$

(2.10)

only depends on the class of $[\gamma] \in \Gamma_\infty \backslash PSL(2, \mathbb{Z}) / \Gamma_\infty$ because of (2.8)-(2.9). In such expressions where only the equivalence class matters we will sometimes write $\gamma_{c, d}$.

Our second formulation is based on the observation that the integral in (2.6) is essentially the standard Bessel function $I_\rho(z)$ with integral representation:

$$
I_\rho(z) = \left(\frac{z}{2}\right)^\rho \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} t^{-\rho-1} e^{(t+z^2/(4t))} dt
$$

(2.11)

for $Re(\rho) > 0, \epsilon > 0$. This function has asymptotics:

$$
I_\rho(z) \sim \left(\frac{z}{2}\right)^\rho \frac{1}{\Gamma(\rho + 1)} \quad z \to 0
$$

(2.12)

$$
I_\rho(z) \sim \sqrt{\frac{1}{2\pi z}} e^z \quad Re(z) \to +\infty
$$

Thus, we can define

$$
\tilde{I}_\rho(z) := \left(\frac{z}{2}\right)^{-\rho} I_\rho(z)
$$

(2.13)

and re-express the formula (2.3) as:

$$
\mathcal{K}_{n, \nu; m, \mu} = +2\pi \sum_{0 \leq d/c < 1} c^{w-2} \tilde{M}(d, c)_{n, \nu; m, \mu} (2\pi|m + \Delta_\mu|)^{1-w}
$$

(2.14)

$$
\tilde{I}_{1-w} \left[ \frac{4\pi}{c} \sqrt{|m + \Delta_\mu|(n + \Delta_\nu)} \right]
$$

Finally, note that the integral in (2.6) does not depend on $d$, so one sometimes separates the summation over $d$ defining:

$$
K\ell(n, \nu, m, \mu; c) := \sum_{0 < -d < c; (d, c) = 1} e^{2\pi i (n + \Delta_\nu) (d/c)} M(\gamma_{c, d})_{\nu\mu}^{-1} e^{2\pi i (m + \Delta_\mu)(a/c)}
$$

(2.15)

for $c > 1$. We will call this a generalized Kloosterman sum. For $c = 1$ (the most important case!) we have:

$$
K\ell(n, \nu, m, \mu; c = 1) = S_{\nu\mu}^{-1}
$$

(2.16)
Putting these together we have the third formulation of the Rademacher expansion:

\[
F_\nu(n) = 2\pi \sum_{c=1}^\infty \sum_{\mu=1}^r c^{w-2} K\ell(n, \nu, m, \mu; c) \sum_{m + \Delta_\mu < 0} F_\mu(m) (2\pi|m + \Delta_\mu|)^{1-w} \tilde{I}_1 \left( \frac{4\pi}{c} \sqrt{|m + \Delta_\mu|(n + \Delta_\nu)} \right).
\] (2.17)

The function \(\tilde{I}_\nu(z) \to 1\) for \(z \to 0\). The Kloosterman sum is trivially bounded by \(c\), so we can immediately conclude that the sum converges for \(w < 0\). (In fact, by a deep result of A. Weil, the Kloosterman sum for the trivial representation of the modular group is bounded by \(c^{1/2}\).) From the proof in appendix B it follows that the series in fact converges to the value of the Fourier coefficient of the modular form.

We will give two proofs of the above results in appendices B and C. The first follows closely the method used by Rademacher [15][7]. This proof is useful because it illustrates the role played by various modular domains in obtaining the expression and is closely related to the phase transitions discussed in section six below. The second proof, which is also rather elementary, but only applies for \(w\) integral, establishes a connection with another well-known formula in analytic number theory, namely Petersson’s formula for Fourier coefficients of Poincaré series.

3. Elliptic genera and Jacobi Forms

3.1. Elliptic Genera and superconformal field theory

The elliptic genus for a \((2,2)\) CFT is defined to be:

\[
\chi(q, y) := \text{Tr}_{RR} e^{2\pi i \tau(L_0 - c/24)} e^{2\pi i \tilde{\tau}(\tilde{L}_0 - c/24)} e^{2\pi izJ_0} (-1)^F := \sum_{n \geq 0, r} c(n, r) q^n y^r
\] (3.1)

The Ramond sector spectrum of \(J_0, \tilde{J}_0\) is integral for \(\hat{c}\) even, and half-integral for \(\hat{c}\) odd so \((-1)^F = \exp[i\pi(J_0 - \tilde{J}_0)]\) is well-defined. In the path integral we are computing with worldsheet fermionic boundary conditions:

\[
e^{2\pi i z} \begin{array}{c} + \\ \otimes \\ + \end{array}.
\] (3.2)

We will encounter the elliptic genus for unitary \((4,4)\) theories. These necessarily have \(\hat{c} = 2k\) even integral and \(c = 3\hat{c} = 6k\). We choose the \(\mathcal{N} = 2\) subalgebra so that \(J_0 = 2J_0^3\) has integral spectrum.
General properties of CFT together with representation theory of $\mathcal{N} = 2$ superconformal theory show that $\chi(\tau, z)$ satisfies the following identities. First, modular invariance leads to the transformation laws for $\gamma \in SL(2, \mathbb{Z})$:

$$\chi \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = e^{2\pi ik \frac{cz^2}{c\tau+d}} \chi(\tau, z) \quad (3.3)$$

Second, the phenomenon of spectral flow is encoded in:

$$\chi(\tau, z + \ell \tau + m) = e^{-2\pi ik(\ell^2 \tau + 2\ell z)} \chi(\tau, z) \quad \ell, m \in \mathbb{Z} \quad (3.4)$$

### 3.2. Jacobi forms

It was pointed out in [16][17] that the fundamental identities $(3.3)(3.4)$ define what is known in the mathematical literature as a “weak Jacobi form of weight zero and index $c/2$.”

**Definition [18].** A Jacobi form $\phi(\tau, z)$ of (integral) weight $w$ and index $k$ satisfies the identities:

$$\phi \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = (c \tau + d)^w e^{2\pi ik \frac{cz^2}{c\tau+d}} \phi(\tau, z) \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) \quad (3.5)$$

$$\phi(\tau, z + \ell \tau + m) = e^{-2\pi ik(\ell^2 \tau + 2\ell z)} \phi(\tau, z) \quad \ell, m \in \mathbb{Z} \quad (3.6)$$

and has a Fourier expansion:

$$\phi(\tau, z) = \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell \quad (3.7)$$

where $c(n, \ell) = 0$ unless $4nk - \ell^2 \geq 0$.

**Definition [18].** A weak Jacobi form $\phi(\tau, z)$ of weight $w$ and index $k$ satisfies the identities $(3.5)(3.6)$ and has a Fourier expansion:

$$\phi(\tau, z) = \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell \quad (3.8)$$

where $c(n, \ell) = 0$ unless $n \geq 0$.

The notion of weak Jacobi form is defined in [18], p.104. In physics we must use weak Jacobi forms and not Jacobi forms since $L_0 - c/24 \geq 0$ in the Ramond sector of a
unitary theory. In a unitary theory the $U(1)$ charge $|\ell| \leq \frac{1}{2} \hat{c} = k$ for topological states so $4nk - \ell^2 = 2\hat{c}n - \ell^2 \geq -(\hat{c}/2)^2$.

Thanks to (3.6) the coefficients $c(n, \ell)$ satisfy

$$
c(n, \ell) = c(n + \ell s_1 + ks_1^2, \ell + 2ks_1)
$$

where $s_1$ is any integer. Therefore, if $\ell = \nu + 2ks_0$, with integral $s_0$,

$$
c(n, \ell) = c(n - \frac{\ell^2 - \nu^2}{4k}, \nu) = c(n - \nu s_0 - ks_0^2, \nu)
$$

Thus we obtain the key point, ([18], Theorem 2.2), that the expansion coefficients of the elliptic genus as an expansion in two variables $q, y$ are in fact given by:

$$
c(n, \ell) = c_{\ell}(2n\hat{c} - \ell^2)
$$

where $c_{\ell}(j)$ is extended to all values $\ell = \mu \mod \hat{c}$ by $c_{\ell}(N) = (-1)^{\ell-\mu}c_{\mu}(N)$.

It follows that we can give a theta function decomposition to the function $\phi(\tau, z)$ ([18] Theorem 5.1):

$$
\phi(\tau, z) = \sum_{-k+1 \leq \nu < k} \sum_{n \in \mathbb{Z}} c(n, \nu) q^{n-\nu^2/4k} \theta_{\nu,k}(z, \tau)
$$

where the sum is over integral $\mu$ for $2k$ even and over half-integral $\mu$ for $2k$ odd, and where $\theta_{\mu,k}(z, \tau)$, $\mu = -k + 1, \ldots, k$ are theta functions:

$$
\theta_{\mu,k}(z, \tau) := \sum_{\ell \in \mathbb{Z}, \ell = \mu \mod 2k} q^{\ell^2/(4k)} y^\ell
= \sum_{n \in \mathbb{Z}} q^{k(n+\mu/(2k))^2} y^{(\mu+2kn)}
$$

In the case of the elliptic genus we have:

$$
\chi(q, y; Z) = \sum_{\mu=-\hat{c}/2+1} h_\mu(\tau) \theta_{\mu,\hat{c}/2}(z, \tau)
$$

Physically, the decomposition (3.14) corresponds to separating out the $U(1)$ current $J$ and bosonizing it in the standard way $J = i\sqrt{\hat{c}}\partial \phi$. Then a basis of chiral conformal fields can be taken so that

$$
\mathcal{O} = \mathcal{O}_q e^{iq\phi/\sqrt{\hat{c}}}
$$
where $O_q$ is $U(1)$ neutral and has weight $h - q^2/(2\hat{c})$. This weight can be negative. The remaining “parafermion” contributions behave like:

$$h_\mu(\tau) = \sum_{j=-\mu^2 \text{mod} 2\hat{c}} c_\mu(j) q^{j/2\hat{c}} \quad 1 - \hat{c}/2 \leq \mu \leq \hat{c}/2$$

(3.16)

$$= c_\mu(-\mu^2) q^{-\mu^2/2\hat{c}} + \cdots$$

where the higher terms in the expansion have higher powers of $q$. Equation (3.6) is physically the statement of spectral flow invariance. Recall the spectral flow map [19]:

$$G_{n\pm a}^\pm \to G_{n\pm (a+\theta)}^\pm$$

$$L_0 \to L_0 + \theta J_0 + \theta^2 \frac{\hat{c}}{2} J_0$$

(3.17)

$$J_0 \to J_0 + \theta \hat{c}$$

leaves invariant the quantity $2\hat{c}L_0 - J_0^2$ for all $\theta$.

### 3.3. Digression: Elliptic Genera for arbitrary Calabi-Yau manifolds

We pause to note a corollary of the Rademacher expansion which might prove useful in other problems besides those discussed in this paper.

One of the primary sources of $(2,2)$ CFT’s are 2d susy sigma models with CY target. Let $X$ be a CY manifold. Let $\hat{c}$ be the complex dimension. The $\mathcal{N} = 2$ SUSY sigma model on $X$ has $c = 3\hat{c}$. The leading coefficient in the $q$-expansion of $h_\mu$ in (3.14) has a nice topological meaning [20][16]:

$$h_\mu = \chi_{\nu + \hat{c}/2}(X) q^{-\nu^2/2\hat{c}} + \cdots$$

(3.18)

where

$$\chi_{\mu}(X) := \sum_{q=0}^{\hat{c}} (-1)^{p+q} h^{p,q}(X)$$

(3.19)

is the holomorphic Euler character.

Applying the Rademacher expansion to the modular forms $h_\mu(\tau)$, we observe that the relevant Bessel function $I_{\nu}(z)$ is elementary, so we have:

$$\tilde{I}_{-1/2}(z) = \frac{2}{\sqrt{\pi z^2}} T(z)$$

(3.20)

$$T(z) = e^z (1 - 1/z) + e^{-z} (1 + 1/z)$$
Substituting the above into the general Rademacher series (2.14) we get the formula for the elliptic genus of an arbitrary Calabi-Yau manifold $X$ of complex dimension $\hat{c}$:

$$c(n, \ell; X) = \frac{\sqrt{\hat{c}}}{2\hat{c}n - \ell^2} \sum_{c=1}^{\infty} \sum_{4km - \mu^2 < 0 \mu = -\hat{c}/2 + 1} \sum_{\ell=1}^{\hat{c}/2} c^{-1/2} K\ell(n, \nu, m, \mu; c)c(2\hat{c}m - \mu^2; X)$$

$$|2\hat{c}m - \mu^2|^{1/2} T \left[ \frac{\pi}{c} \left( \frac{\hat{c}}{2} \right)^2 \sqrt{(\mu^2 - 2\hat{c}m)(2\hat{c}n - \ell^2)} \right]$$

where $\ell = \nu \mod \hat{c}$. (Since we are dealing with elliptic genera of different manifolds we will use the notation $c(n, \ell; X)$ when we wish to emphasize the dependence on the manifold $X$.)

Note that combining with (3.18)(3.19) one sees that almost all reference to the variety $X$ has disappeared except for a finite number of Chern classes. In fact, the elliptic genus carries no more topological information than the Hodge numbers as long as the only terms in the expansion of $h_\nu$ with negative powers of $q$ are the leading ones. That holds for

$$2\hat{c} - \left( \frac{\hat{c}}{2} \right)^2 \geq 0 \quad (3.22)$$

or $\hat{c} \leq 8$. However, a priori for CY manifolds of $\hat{c} \geq 9$ the elliptic genus will generally depend on other topological data besides the Hodge numbers. Using dimension formulas for the space of Jacobi forms, the above bound has been sharpened in [22] where it has been shown that Hodge numbers of the Calabi-Yau manifold determine the elliptic genus only if $\hat{c} < 12$ or $\hat{c} = 13$. This paper also contains many explicit computations of the elliptic genus of Calabi-Yau hypersurfaces in toric varieties.

### 3.4 Derivatives of Jacobi forms and fareytail transforms

We summarize here some formulae which are useful in discussing the fareytail transform of Jacobi forms.

Denote the space of (weak) Jacobi forms of weight $w$ and index $k$ by $J_{w,k}$. Introduce the operator

$$\mathcal{O} := \frac{\partial}{\partial \tau} - \frac{1}{8\pi i k} \left( \frac{\partial}{\partial z} \right)^2$$

An unfortunate clash of notation leads to three different meanings for “c” in this formula!

We thank V. Gritsenko for pointing out to us that the elliptic genus in general depends on more data than just the Hodge numbers. This statement becomes manifest in view of the Rademacher expansion. See also [21].
One then easily checks that if \( \phi \in J_{w,k} \) then
\[
\left( O + \frac{(w - 1/2)}{2i\Im \tau} \right) \phi
\]
transforms according the the Jacobi transformation laws of weight \( (w + 2) \) and index \( k \).

By composing operators of this type and “normal ordering” one can show that
\[
\sum_{j=0}^{\infty} \frac{\Gamma(n+1)\Gamma(n+w-1/2)}{j!\Gamma(n+1-j)\Gamma(n+w-1/2-j)} \left( \frac{1}{2i\Im \tau} \right)^j O^{n-j}
\]
maps an element of \( J_{w,k} \) to a function (in general, nonholomorphic) which transforms according to the Jacobi transformation rules with weight \( w+2n \) and index \( k \), at least for \( n \) integral. On the other hand, for \( w \) integral and \( n = 3/2 - w \) the expression above simplifies to a single term \( O^{3/2-w} \), where the latter should be interpreted as a pseudodifferential operator.

**Definition (The fareytail transform):** We define the fareytail transform \( FT(\phi) \) of \( \phi \in J_{w,k} \) to be \( FT(\phi) := |O^{3/2-w}|=\phi \). We also use the notation \( \tilde{\phi} = FT(\phi) \).

Note that if \( \phi \) is a weak Jacobi form and we define the polar part of \( \phi \) to be:
\[
\phi^- := \sum_{4kn-\ell^2 < 0} c(n, \ell)q^n y^\ell
\]
then \((FT(\phi))^-=FT(\phi^-)\).

As pointed out to us by Don Zagier, it turns out that \( FT(\phi) \) is not a (weak) Jacobi form. Nevertheless, it is related to a truncated Poincaré series as we explain in the next section.

4. The Rademacher expansion as a formula in statistical mechanics

As we discussed in the introduction, in statistical mechanics the canonical and microcanonical ensemble partition functions are related by an inverse Laplace transform:
\[
N(E) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} Z(\beta)e^{2\pi\beta E} d\beta.
\]

In this section we cast the Rademacher series into a form closely related to (4.1).
Consider the first formulation, (2.4) to (2.6) for \((n + \Delta_\nu) > 0\). We can make the change of variable in (2.6)

\[
\beta \rightarrow -\frac{n + \Delta_\nu}{m + \Delta_\mu} = \frac{n + \Delta_\nu}{|m + \Delta_\mu|}\beta
\]  

(4.2)

which is valid as long as it does not shift the contour through singularities of the integrand.

In the formula below we will find that \(Z(\beta)\) has a singularity at \(\beta = 0\), so we must have \(-\frac{n + \Delta_\nu}{m + \Delta_\mu}\) real and positive. Since \((m + \Delta_\mu) < 0\) this means we can only make the change of variable (4.2) for \((n + \Delta_\nu) > 0\). The contour deformations are valid in this case (for \(w < 2\)) so we can write:

\[
(n + \Delta_\nu)^{1-w} F_\nu(n) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \widehat{Z}_\nu(\beta) e^{2\pi \beta (n + \Delta_\nu)} d\beta
\]  

(4.3)

with

\[
\widehat{Z}_\nu(\beta) = 2\pi \sum_{0 \leq -d/c < 1} \sum_{m + \Delta_\mu < 0} (c\beta - id)^{w-2} M(\gamma_{c,d})^{-1} e^{2\pi i (m + \Delta_\nu)(a/c)} |m + \Delta_\mu|^{1-w} F_\mu(m) \exp[(2\pi |m + \Delta_\mu|)] c \theta_{\nu,k}(\omega, \beta, \omega)\]

(4.4)

In this equation we can take \(c > 0\) and since \(Re(\beta) > 0\) we can use the principal branch of the logarithm to define \((c\beta - id)^{w-2}\) when \(w\) is non-integral.

We now use this to derive the “statistical-mechanics” version of the Rademacher formula for weak Jacobi forms. These have an expansion of the form (3.8) so we aim to give a formula of the form:

\[
\tilde{c}(n, \ell) = \int_0^1 d\omega e^{-2\pi i \ell \omega} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\beta}{2\pi i} e^{2\pi \beta n} \widehat{Z}_\phi(\beta, \omega)
\]

(4.5)

where \(\tilde{c}\) are related to \(c\) by the fareytail transform (1.17). The basic idea of the derivation is to use the decomposition (3.14) and apply the generalized Rademacher series to the vector of modular forms given in (3.16). After some manipulation we find:

\[
\widehat{Z}_\phi(\beta, \omega) = 2\pi i (-1)^{w+1} \sum_{0 \leq -d/c < 1} \sum_{m: 4km - \mu^2 < 0} \tilde{c}_\mu(4km - \mu^2) (c\tau + d)^{-w-3} \exp\left[4\pi |m - \frac{\mu^2}{4k} | \frac{1}{2i} \frac{a\tau + b}{c\tau + d} \right] \exp[-2\pi i k \frac{c\omega^2}{c\tau + d}] \theta_{\mu,k}(\omega, \frac{a\tau + b}{c\tau + d})
\]

(4.6)
Recall that $\tau = i\beta$ in this formula. We will refer to our result (4.6) as the Jacobi-Rademacher formula. The most efficient way to proceed here is to work backwards by evaluating the right-hand-side of (4.5) and comparing to (4.4). The reader should note that the meaning of $w$ has changed in this equation, and it now refers to the weight of the Jacobi form: $w((4.6)) = w((4.4)) + \frac{1}{2}$. In particular, for the case of the elliptic genus $w((4.4)) = -\frac{1}{2}$ and $w((4.6)) = 0$. Finally, we must stress that the derivation of equation (4.5) is only valid for $n - \ell^2/4k > 0$.

It is useful to rewrite (4.6) in terms of the slash operator. In general the slash operator for Jacobi forms of weight $w$ and index $k$ is defined to be [18]:

$$
(p|w,k\gamma)(\tau,z):= (c\tau+d)^{-w} \exp\left[-2\pi i k \frac{cz^2}{c\tau+d}\right] p(\frac{a\tau+b}{c\tau+d} ; \frac{z}{c\tau+d}) \tag{4.7}
$$

Using this we can write

$$
\hat{Z}_\phi = \sum_{(\Gamma_{\infty}\backslash \Gamma/\Gamma_{\infty})_0'} \tilde{\phi}^-_{3-w,k\gamma} \tag{4.8}
$$

where we define the polar part to be

$$
\tilde{\phi}^- (\tau, z) := \sum_{4km-\ell^2<0} \tilde{c}_\ell (4km - \ell^2) e^{2\pi i m\tau+2\pi i \ell z}. \tag{4.9}
$$

Here $\Gamma = SL(2, \mathbb{Z})$, not $PSL(2, \mathbb{Z})$, the notation $(\Gamma_{\infty}\backslash \Gamma/\Gamma_{\infty})_0$ means that $c \geq 0$, and the prime indicates we omit the class of $c = 1$.

As with the full Jacobi form $\phi$ we can use spectral flow to decompose the polar part in terms of a finite sum of theta functions:

$$
\tilde{\phi}^- := \sum_{\mu=1}^k \tilde{h}_\mu \Theta_{\mu,k}^+ \tag{4.10}
$$

$$
\tilde{h}_\mu = \sum_{m:4km-\mu^2<0} \tilde{c}_\mu (4km - \mu^2) \exp\left[2\pi i (m - \frac{\mu^2}{4k})\tau\right]
$$

$$
\Theta_{\mu,k}^+(z, \tau) := \theta_{\mu,k}(z, \tau) + \theta_{-\mu,k}(z, \tau) \quad 1 \leq |\mu| < k
$$

$$
\Theta_{k,k}^+(z, \tau) := \theta_{k,k} \quad |\mu| = k
$$

As in equations (1.12) to (1.14) of the introduction the relation between the micro-canonical and canonical ensemble differs slightly from the relation between the Fourier coefficients and the truncated Poincaré series $\hat{Z}_\phi$. In order to write the full canonical partition function we extend the sum in (4.8), interpreted as a sum over reduced fractions
$0 \leq c/d < 1$ to a sum over all relatively prime pairs of integers $(c,d)$ with $c \geq 0$. (For $c = 0$ we only have $d = 1$.) Let us call the result $Z_\phi$. In order to produce a truly modular object we must also allow for $c < 0$, that is, we must sum over $\Gamma_\infty \backslash SL(2,\mathbb{Z})$. Extending the sum in this way produces zero, because the summand is odd under $(c,d) \rightarrow (-c,-d)$, and thus we fail to produce a Jacobi form whose Fourier coefficients match those of $\text{FT}(\phi)$, for $n - \ell^2/4k > 0$. This is just as well, since, as pointed out to us by Don Zagier, $\text{FT}(\phi)$ is not a Jacobi form.

5. Physical interpretation in terms of IIB string theory on $AdS_3 \times S^3 \times K3$

Let us apply the the Jacobi-Rademacher formula to the elliptic genus $\chi$ for $\text{Sym}^k(K3)$. In this case $w = 0$ and (4.8) becomes

$$Z_\chi(\beta,\omega) = -2\pi i \sum_{(c,d)=1,c\geq 0} \sum_{\mu=1}^k \sum_{4km-\mu^2<0} \bar{c}_\mu (4km - \mu^2; \text{Sym}^k(K3))$$

$$(c\tau + d)^{-3} \exp \left[ 2\pi i (m - \frac{\mu^2}{4k}) \frac{a\tau + b}{c\tau + d} \right] \exp \left[ -2\pi i k \frac{c\omega^2}{c\tau + d} \right] \Theta_{\mu,k}^+ \left( \frac{\omega}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right)$$

(5.1)

We claim that (5.1) has the interpretation as a sum over Euclidean geometries in the effective supergravity obtained by reducing IIB string on $AdS_3 \times S^3 \times K3$. The reasoning is the following: We begin with the trace for the $(4,4)$ CFT with target space $\text{Sym}^k(K3)$:

$$Z_{RR}(\tau, \tilde{\tau}, \omega, \tilde{\omega}) = \text{Tr}_{RR} e^{2\pi i \tau (L_0 - c/24)} e^{2\pi i \tilde{\tau} (\tilde{L}_0 - c/24)} e^{2\pi i \omega J_0} e^{-2\pi i \tilde{\omega} \tilde{J}_0} (-1)^F$$

(5.2)

As is standard, this trace can be represented as a partition function of the $(4,4)$ CFT on the torus. $\tau$ specifies the conformal structure of the torus relative to a choice of homology basis for space and time, while the fermion boundary conditions relative to this basis are

$$e^{2\pi i \omega} \quad \otimes \quad e^{2\pi i \tilde{\omega}}$$

(5.3)

By spectral flow $\omega \rightarrow \omega + \tau/2$ we may relate this to $Z_{NSNS}$ with boundary conditions

$$e^{2\pi i \omega} \quad - \quad e^{2\pi i \tilde{\omega}}$$

(5.4)

the precise relation being

$$Z_{RR}(\tau, \omega; \tilde{\tau}, \tilde{\omega}) = (q\bar{q})^{k/4} y^k \bar{y}^k Z_{NSNS}(\tau, \omega + \frac{1}{2}\tau; \tilde{\tau}, \tilde{\omega} + \frac{1}{2}\tilde{\tau})$$

(5.5)
where $Z_{NSNS}$ is defined as in (5.2) in the NSNS sector\textsuperscript{[1]}. Note that $\omega$ is inserted relative to $(-1)^F$.

Now we use the AdS/CFT correspondence \textsuperscript{[1]}. The conformal field theory partition function is identified with a IIB superstring partition function. The reduction on $AdS_3 \times S^3 \times K3$ leads to an infinite tower of massive propagating particles and a “topological multiplet” of extended $AdS_3$ supergravity \textsuperscript{[23],[11]}. The latter is described by a super-Chern-Simons theory \textsuperscript{[24]}, in the present case based on the supergroup $SU(2|1,1)_L \times SU(2|1,1)_R$ \textsuperscript{[11]}. The modular parameter $\tau$ and the twists in (5.3) are specified in the supergravity partition function through the boundary conditions on the fields in the super Chern-Simons theory. These involve the metric, the $SU(2)_L \times SU(2)_R$ gauge fields, and the gravitini. The boundary conditions are as follows:

1. The path integral over 3d metrics will involve a sum over asymptotically hyperbolic geometries which bound the torus of conformal structure $\tau \mod SL(2,\mathbb{Z})$. Thus, we sum over Euclidean 3-metrics with a conformal boundary at $r = \infty$ and

$$ds^2 \sim \frac{dr^2}{r^2} + r^2 g_{ij} dx^i dx^j,$$

(5.6)

where $g_{ij}$ is in the conformal class of a torus with modular parameter $\tau$, and $r$ is a radial coordinate near the conformal boundary.

2. Using the known relation between $SU(2)$ Chern-Simons theory and boundary current algebra \textsuperscript{[12]}, in particular, using the evaluation of the wavefunction on the torus given in \textsuperscript{[25]}, we see that the CFT trace with and insertion of $\exp[2\pi i \omega J_0]$ entails boundary conditions on $A \in su(2)_L$:

$$A_u du \rightarrow \frac{\pi}{23\tau} \omega^3 du$$

(5.7)

Note that in Chern-Simons theory we specify boundary conditions on $A_u$, but leave $A_\bar{u}$ undetermined. Similarly, we have:

$$\tilde{A}_u d\bar{u} \rightarrow \frac{\pi}{23\tau} \tilde{\omega}^3 d\bar{u}$$

(5.8)

3. Finally we must choose boundary conditions for the spinors, in particular for the gravitini in the fermionic part of the $SU(2|1,1)_L \times SU(2|1,1)_R$ connection. (These

---

Note that a complex $\omega$ in (5.3) is equivalent to inserting a phase $e^{2\pi i \omega_1}$ in the vertical direction and a phase $e^{2\pi i \omega_2}$ in the horizontal direction with $\omega = \omega_1 + \tau \omega_2$, where $\omega_1$, $\omega_2$ are real. In other words

$$e^{2\pi i \omega} = e^{2\pi i \omega_1} + e^{2\pi i \omega_2}$$

---
become the supercurrents in the boundary CFT.) Since these fermions are coupled to
the $SU(2)_L \times SU(2)_R$ gauge fields the fermion conditions can be shifted by turning
on a flat connection. If the boundary conditions on the gauge fields are given by
(5.7) and we wish to compute $Z_{RR}$, then the fermion boundary conditions should
be as in (5.3), but since we wish to specialize to the elliptic genus then we must put
\( \tilde{\omega} = 0 \) in (5.3), leaving us with
\[
\epsilon^{2\pi i \omega} \begin{array}{c}
+ \\
\otimes \\
+
\end{array}
\]  
(5.9)

There are many geometries that contribute to this partition function. Let us first
start by discussing the simplest ones. The simplest of these geometries are the ones which
can be obtained as solutions of the $SU(2|1,1)^2$ Chern Simons theory. This Chern Simons
theory is a consistent truncation of the six dimensional supergravity theory. So a solution
of the Chern Simons theory will also be a solution of the six dimensional theory. These
solutions correspond to choosing a way to fill in the torus. This corresponds to picking
a primitive one cycle $\gamma_r$ and filling in the torus so that $\gamma_r$ is contractible. The $U(1)_L$
gauge connection is flat; in a suitable gauge it just a given by constant $A_u$ and $\bar{A}_\bar{u}$. As
we said above the constant value of $A_u$ corresponds to the parameter $\omega$ in the partition
function through (5.7). In the classical solution $\bar{A}_\bar{u}$ is determined by demanding that the
full configuration is non-singular. This translates into the condition that the Wilson line
for a unit charge particle around the contractible cycle is minus one, in other words
\[
e^{i \int_{\gamma_r} A} = -1. \tag{5.10}
\]
This ensures that we will have a non-singular solution because the particles that carry odd
charge are fermions which in the absence of a Wilson line were periodic around $\gamma_r$. With
this particular value of the Wilson line they are anti-periodic, but this is precisely what we
need since the geometry near the region where $\gamma_r$ shrinks to zero size looks like the origin
of the plane. All that we have said for the $U(1)_L$ gauge field should be repeated for the
$U(1)_R$ gauge field. Since there are fermions that carry charges (1,0) or (0,1) we get the
condition (5.10) for both $U(1)_{L,R}$ connections. In this way we resolve the paradox that the
(++) spin structure in (5.9) cannot be filled in. For more details on these solutions in the
Lorentzian context, see [26]. Note that there is an infinite family of solutions that solves
(5.10) since we can always add a suitable integer to $\bar{A}_\bar{u}$. This corresponds to doing integral
units of spectral flow. For the purposes of this discussion we can take the $k \to \infty$ limit, and in this limit one can examine the classical equations, and hence specify the values of both $A_u$ and $A_{\bar{u}}$. Note that the final effective boundary conditions of the supergravity fields depends on both $A_u$ and $A_{\bar{u}}$. In particular, they are not purely given in terms of the field theory boundary conditions (which is the information contained in $A_u$). The final boundary conditions for the fermions in the supergravity theory depend also on the particular state that we are considering. The simple solutions that we have been discussing correspond to the $m = 0$ and $\mu = k$ term in (5.1). The sum over all possible contractible cycles corresponds to the sum over $c,d$ in (5.1). And the sum over integer values of spectral flow corresponds to the different terms in the sum over integers in the theta function in (5.1). Note that from the point of view of an observer in the interior all these solutions are equivalent, (up to a coordinate transformation), to Euclidean $(AdS_3 \times S^3)/\mathbb{Z}$. There is, however, nontrivial information in this sum since we saw that it is crucial for recovering the full partition function of the theory.

Now that we have discussed the simplest solutions we can ask about all the other terms, i.e. about the sum over $m, \mu$ in (5.1). These correspond to adding particles to the solutions described in the above paragraph. These particles are not contained in the Chern Simons theory. The six dimensional theory, reduced on $S^3$ gives a tower of KK fields that propagate on $AdS_3$. If we compute the elliptic genus for them only a very small subset contributes. From the point of view of the Chern Simons theory adding these particles is like adding Wilson lines for the $U(1)$ connection $\mathbb{Z}$ $\mathbb{Z}$. In this case we do not have the relation (5.11) near the boundary. This is not a problem since the connection is not flat any longer in the full spacetime. It is possible to find complete non-singular six (or ten) dimensional solutions which correspond to various combinations of $RR$ ground states $\mathbb{Z}$.

7 Note that this geometry is basically Euclidean $AdS_3 \times S^3$ with an identification in the time direction. This is sometimes loosely referred to as the “NS vacuum”. Nevertheless it also corresponds to a particular RR vacuum, the one with maximal angular momentum $\mathbb{Z}$. From the boundary field theory point of view we know that the NS sector and the RR sector are related by a simple spectral flow transformation that only changes the $U(1)$ charge of the state. The bosonic field corresponding to the bosonized $N = 2 \ U(1)$ current on the boundary is a singleton living on the boundary of $AdS$. So configurations in the boundary theory which only differ by the charge under this $U(1)$ are identical in the interior of $AdS$.

8 Note that the Chern Simons description only makes sense for distances much larger than the $AdS$ radius since some of the particles in question have compton wavelengths of the order of the AdS radius.
In the next several sections we discuss the physical interpretation of various aspects of the formula (5.1) to justify the claim that (5.1) takes exactly the form expected from an evaluation of the partition function of type IIB string theory on $\text{AdS}_3 \times S^3 \times K3$.

5.1. Interpreting the sum over $d/c$

We interpret the sum over relatively prime integers $(c,d)$ as a sum over the “$\text{SL}(2, \mathbb{Z})$ family of black holes” discussed by Maldacena and Strominger in [9]. Since this subject is apt to cause confusion we will be somewhat pedantic in this section, where we explain why we sum over $(c,d)$ and not all of $\text{SL}(2, \mathbb{Z})$.

Recall that we can identify Euclidean $\text{AdS}_3$, denoted by $\mathbb{I}H$, as the space of Hermitian matrices

$$X = \ell^{-1} \begin{pmatrix} T_1 + X_1 & X_2 + iT_2 \\ X_2 - iT_2 & T_1 - X_1 \end{pmatrix}$$

(5.11)

with $X_i, T_i$ real and $\det X = 1$. Here $\ell$ is the $\text{AdS}$ radius, for simplicity we usually choose units where $\ell = 1$.

We introduce global variables on $\mathbb{I}H$ via the Gauss decomposition

$$X = \begin{pmatrix} h + w\bar{w}/h & w/h \\ \bar{w}/h & 1/h \end{pmatrix} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{w} & 1 \end{pmatrix}$$

(5.12)

Here $h \neq 0$, and $\bar{w} \in \mathbb{C}$ the complex conjugate of $w$. Since $(T_1 \pm X_1) \neq 0$ we can always solve for $h, w, \bar{w}$ so these coordinates cover $\mathbb{I}H$ once. There are two connected components of $\mathbb{I}H$ and we restrict attention to the connected component defined by $h > 0$. The metric becomes:

$$ds^2 = \frac{1}{h^2} (dw\bar{w} + dh^2)$$

(5.13)

which is the standard model of hyperbolic space.

We now study the BTZ group action [29]. Abstractly, this is just an action of the additive group $\mathbb{Z}$ on $\mathbb{I}H$. A generator acts as

$$X \sim \begin{pmatrix} e^{-i\pi \tau} & 0 \\ 0 & e^{i\pi \tau} \end{pmatrix} X \begin{pmatrix} e^{i\pi \bar{\tau}} & 0 \\ 0 & e^{-i\pi \bar{\tau}} \end{pmatrix} := \rho(\tau)X\rho(\tau)^\dagger$$

(5.14)

where we must choose $\Im \tau > 0$ for a properly discontinuous action. The BTZ group acts as isometries in the hyperbolic metric. In the coordinates (5.12) the BTZ group action (5.14) is

$$w \sim qw \quad h \sim e^{-2\pi(\Im \tau)}h$$

(5.15)
From (5.15) it is now clear that the group action is not properly discontinuous at \( w = 0, h = 0 \). Therefore, we restrict to the domain \( \mathbb{H}^* \) defined by \( h > 0 \) and \( h = 0, w \neq 0 \) and take the quotient by \( \mathbb{Z} \). The quotient space \( \mathbb{H}^*/\mathbb{Z} \) is a solid torus with a boundary two-torus: The identification \( w \sim qw \) in \( \mathbb{C}^* \) defines a torus, then \( h > 0 \) fills it in.

We claim the modular parameter of the torus at the conformal boundary \( h = 0 \) is naturally given by \( \tau \), up to an ambiguity \( \tau \sim \tau + \mathbb{Z} \). Recall that the modular parameter of a torus is only defined up to a \( PSL(2, \mathbb{Z}) \) transformation until one chooses an oriented homology basis. Once we choose \( a \) and \( b \)-cycles (in that order!) we can define \( \tau := \int_b \omega / \int_a \omega \) where \( \omega \) is a globally nonvanishing holomorphic 1-form. When we are presented with a solid torus then there is a unique primitive contractible cycle. We take this to be the \( a \)-cycle. There is no unique noncontractible primitive cycle; any two differ by an integral multiple of the \( a \)-cycle. These different choices are related by diffeomorphisms of the solid torus which become Dehn twists about the \( a \)-cycle on the boundary. Thus, a choice of filling in the torus to a solid torus defines a modular parameter up to \( \tau \sim \tau + \mathbb{Z} \).

Now consider the quotient geometry \( \mathbb{H}^*/\mathbb{Z} \). From (5.13) the unique primitive contractible cycle is \( w(s) = w_0 e^{2\pi is}, h(s) = h_0 \). (Note that by making \( h_0 \) large we can make the length arbitrarily small.) One choice of noncontractible cycle is \( w(s) = \exp[-2\pi i \tau s]w_0 \), joining \( w_0 \) to \( q^{-1}w_0 \). With this choice of \( b \)-cycle the modular parameter is \( \tau \).

Now we would like to make a connection to the physics of black holes and thermal AdS. To begin we use the decomposition

\[
X = \left( \begin{array}{cc} e^u & 0 \\ 0 & e^{-u} \end{array} \right) \left( \begin{array}{cc} \cosh \rho & \sinh \rho \\ \sinh \rho & \cosh \rho \end{array} \right) \left( \begin{array}{cc} e^{-\bar{u}} & 0 \\ 0 & e^{\bar{u}} \end{array} \right)
\]

(5.16)

where \( \rho \geq 0 \) and \( u \in \mathbb{C} \). Comparing to (5.12) we have \( w = e^{2u} \tanh \rho \). In these coordinates the metric takes the form

\[
\text{ds}^2 = -\sinh^2 \rho (du - d\bar{u})^2 + \cosh^2 \rho (du + d\bar{u})^2 + d\rho^2
\]

(5.17)

These coordinates cover \( \mathbb{H}^* \) once, and therefore give global coordinates if we identify

\[
2u \sim 2u + 2\pi in \quad n \in \mathbb{Z}
\]

(5.18)

By the above remarks, \( u(s) = i\pi s, 0 \leq s \leq 1 \) is the unique contractible primitive cycle. On the other hand, the BTZ group action identifies

\[
2u \sim 2u + 2\pi in\tau \quad n \in \mathbb{Z}
\]

(5.19)

23
and defines one (of many) primitive noncontractible cycles. We will refer to (5.19) as the “BTZ cycle.” The metric at $\rho \rightarrow \infty$ is $ds^2 \sim e^{2\rho}|du|^2 + (d\rho)^2$ and, with respect to the homology basis (contractible cycle, BTZ cycle) the modular parameter is $\tau$.

So far we have only done geometry and no physics. The physics comes in when we introduce coordinates we want to call “space” and “time.” There is a unique complete, smooth, infinite volume hyperbolic 3-manifold with conformal boundary a single torus with Teichmüller parameter $\tau$ [30]. Nevertheless, for physics we must identify space and time, and physical quantities such as the gravitational action discussed in section 5.3 below are not invariant under global diffeomorphisms.

Suppose we define

$$2u = i(\phi + it_E)$$

(5.20)

with $\phi, t_E$ real. The notation suggests that $\phi$ is a spatial angular coordinate and $t_E$ is a Euclidean time. The identification (5.18) becomes

$$\phi \sim \phi + 2\pi n,$$

$$t_E \sim t_E, \quad n \in \mathbb{Z}$$

(5.21)

The BTZ group action (5.19) becomes the identification of Euclidean time with a spatial twist:

$$\phi \sim \phi + 2\pi n \tau_1,$$

$$t_E \sim t_E + 2\pi n \tau_2, \quad n \in \mathbb{Z}$$

(5.22)

The spatial ($\phi$) cycle (5.21) is the unique primitive contractible cycle. Substituting (5.20) into (5.17) we recognize the Euclidean thermal AdS.

On the other hand, we could instead define coordinates on $\mathbb{H}^*/\mathbb{Z}$ using (5.16):

$$2u := -(\tau_2 \phi + \tau_1 t_E) + i(\tau_1 \phi - \tau_2 t_E) = +i(\phi + it_E)$$

(5.23)

In these coordinates the identification (5.18) is equivalent to

$$\phi \sim \phi + 2\pi \Re(-1/\tau)n$$

$$t_E \sim t_E + 2\pi \Im(-1/\tau)n \quad n \in \mathbb{Z}$$

(5.24)

This defines the unique primitive contractible cycle in the solid torus. We call it the time cycle. The BTZ action (5.19) becomes

$$\phi \sim \phi + 2\pi n,$$

$$t_E \sim t_E, \quad n \in \mathbb{Z}$$

(5.25)
This defines a choice of noncontractible cycle within the handlebody. We call it the “space cycle.” Note that it is the spatial cycle \(5.23\) which is a noncontractible cycle: Thus we have a black hole since we have a hole in space. Indeed, identifying \(t_E\) as Euclidean time and \(\phi\) as an angular coordinate we can define the Schwarzschild coordinate \(r\) via

\[
\sinh^2 \rho = \frac{r^2 - \tau_2^2}{|\tau|^2}
\]

\((5.26)\)

with \(r \geq \tau_2\) to get the familiar Euclidean BTZ black hole in Schwarzschild coordinates:

\[
ds^2 = N^2(r) dt_E^2 + N^{-2}(r) dr^2 + r^2 (d\phi + N^\phi(r) dt_E)^2
\]

\[(5.27)\]

\[
N^2(r) = \frac{(r^2 - \tau_2^2)(r^2 + \tau_1^2)}{r^2}
\]

\[
N^\phi = + \frac{\tau_1\tau_2}{r^2}.
\]

Note that, in the second description, if we choose as oriented homology basis (space cycle, time cycle) then the modular parameter is \(-1/\tau\). In this way thermal AdS is related to the BTZ black hole by a modular transformation.

The general story is the following (we are switching here from a passive to an active viewpoint): We wish to find a complete hyperbolic 3-geometry with

1. \(ds^2 \rightarrow r^2 |d\phi + i dt|^2 + \frac{dr^2}{r^2}\) at \(r \rightarrow \infty\).

2. Periodicities \((\phi + it) \sim (\phi + it) + 2\pi(n + m\tau)\), \(n, m \in \mathbb{Z}\).

3. The unique primitive contractible cycle is defined by \(\Delta(\phi + it) = c\tau + d\), where \((c, d) = 1\).

The solution is to take the BTZ group action with \(\rho(\alpha \tau + b) \in (5.14)\), where

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),
\]

and to define coordinates

\[
2u = \frac{i}{c\tau + d} (\phi + it) \quad (5.28)
\]

Then the contractible cycle is \(\Delta(\phi + it) = 2\pi(c\tau + d)\) and the BTZ cycle is \(\Delta(\phi + it) = 2\pi(a\tau + b)\) so, with respect to the homology basis (contractible cycle, BTZ cycle) the

\[9\] One must include the special cases \((c = 0, d = 1), (c = 1, d = 0)\).
modular parameter is \( \frac{a \tau + b}{ct + d} \). The metric is (5.17). Substituting (5.28) into (5.17) we may bring the metric to the BTZ form (5.27) with \( \tau \rightarrow \tau'' \) where

\[
\rho^2 = \frac{(ct_1 + d)^2 \sinh^2 \rho + (ct_2)^2 \cosh^2 \rho}{|ct + d|^4}
\]

\[
\tau'' = \pm \frac{1}{ct + d}
\]  

(5.29)

where the sign in the second line is determined by \( \Im \tau'' > 0 \).

Finally, we may attempt to interpret the constraint \( c \geq 0 \) as follows: Modular transformations with \( (c, d) \) and \( (-c, -d) \) differ by the transformation \( \gamma = -1 \). While this is an orientation preserving transformation of the boundary, it can only be extended as a diffeomorphism of the handlebody if it is extended as an orientation reversing diffeomorphism. For some reason, which should be more clearly explained, we only sum over bounding geometries with fixed induced orientation.

The above construction of the “\( SL(2, \mathbb{Z}) \) family of black holes” of [9] shows that the family is perhaps more accurately described as a \( (\Gamma_\infty \backslash \Gamma)_0 \) family of choices of contractible cycle for the torus at infinity. (The subscript 0 indicates we only keep \( c \geq 0 \).) The geometries are really labelled by a pair of relatively prime integers \( (c, d) \) telling which cycle of the torus at infinity should be considered to be the contractible cycle.

5.2. Interpreting the sum on \( \mu, m \)

We now come to the physical interpretation of the sum over quantum numbers \( (m, \mu) \) with \( 4km - \mu^2 < 0 \). We will interpret the sum on \( \mu \) as arising from black holes which are spinning in the internal \( S^3 \) directions.

5.2.1. CFT description

First, let us clarify the meaning of the sum on \( (m, \mu) \) from the CFT viewpoint. It is useful to look at these quantum numbers in both the NS and R sectors. The two descriptions are related, of course, by spectral flow.
In the Ramond sector we plot \( m = L_0 - k/4 \) against \( \ell \) in fig. 1. The key region is: \( m \geq 0, |\ell| \leq k, -k^2 \leq 4km - \ell^2 < 0 \), and we will refer to this as the “Ramond sector particle region,” for reasons which will become clear in a moment. Physical states correspond to integral values of \( m, \ell \). All the topological states with \( m = 0 \) contribute, except for \((m = 0, \ell = 0)\). From the \( \mathcal{N} = 4 \) character formulae of \cite{[31]} one can check that other states besides just the \( \mathcal{N} = 4 \) descendents of topological states at \( m = 0 \) will contribute to the elliptic genus.

Now let us compare things in the NS sector. Recall \( 4kL_0 - J_0^2 \) is a spectral flow invariant. One should be careful to distinguish \( m \) from \( L_0 \) since they differ by \( m = L_0 - k/4 \) so that

\[
4km - \ell^2 = 4kL_0 - J_0^2 - k^2.
\]

The region in the NS sector is obtained by spectral flow and is illustrated in fig. 2. For flow by \( \theta = +\frac{1}{2} \) we get:

\[
0 \leq 4kL_0^{NS} - (J_0^{NS})^2 < k^2
\]

\[
0 \leq J_0^{NS} \leq 2k
\]

Physical states satisfy \((J_0, L_0) \in \mathbb{Z} \times \frac{1}{2}\mathbb{Z}_+\) with \( L_0 - \frac{1}{2}J_0 \in \mathbb{Z}_+ \).
Fig. 2: The shaded region is the NS sector particle region. It is obtained from fig. 1 by spectral flow by $\theta = +\frac{1}{2}$. The straight line describes the left chiral primary states, and all other states of the theory lie above this line. The parabola describes the cosmic censorship bound for black holes rotating in the $S^3$ directions. All black holes lie on or above the parabola. There is an extremal black hole state that is also a chiral primary with angular momentum $J_0 = k$.

It is interesting to see which chiral primary points $(J_0, L_0)$ contribute to the sum in the Jacobi-Rademacher series. For a chiral primary $L_0 = J_0/2$ so the spectral flow invariant (5.30) becomes $-(J_0 - k)^2$. Thus, all but the middle $\mathcal{N} = 2$ chiral primaries with $J_0 = k$ contribute. The point $(J_0, L_0) = (k, \frac{1}{2}k)$ is a distinguished point. These are the quantum numbers of a state which is both a chiral primary and - as we will discuss - a black hole. It does not contribute to the Rademacher sum because the restriction on the sum in (5.31) is given by a strict inequality. This is also the value of $J_0$ at which the number of chiral primaries is a maximum. For an antichiral primary the spectral flow invariant is $-(J_0 + k)^2$ and a similar discussion applies.

5.2.2. Supergravity interpretation

Now let us turn to the supergravity solutions which should contribute in the ADS/CFT correspondence. The Jacobi-Rademacher series involves a sum over quantum numbers of
Fig. 3: A more symmetric presentation of the NS particle region. By integer spectral flow \((3.17)\) with \(\theta = -1\) we can map the upper region in fig. 2 to the left, producing a domain symmetric about the \(L_0\) axis.

the \(SU(2)_L\) part of the \(SU(2|1,1)\) AdS algebra. In supergravity these quantum numbers are associated with the \(SU(2)_L\) Kaluza-Klein gauge theory from isometries of \(S^3 = SU(2)\) in the 6d geometry \(AdS_3 \times S^3\). Therefore, we should study supergravity solutions which are locally \(AdS_3 \times S^3\) and associated to spinning black holes. These have been discussed by Cvetic and Larsen \([10]\), and their solution may be summarized as follows. The Lorentz-signature geometry is given by a metric of the form

\[
ds^2 = ds^2_{BTZ} + A^a \otimes A^b (K^a, K^b) + A^a \otimes K_a + K_a \otimes A^a + ds^2_{S^3} \quad (5.32)
\]

where \(a\) is an adjoint \(so(4)\) index and \(K^a\) are \(so(4)\) Killing vectors for the round metric \(ds^2_{S^3}\) and \(K_a\) are the dual one-forms. Explicitly, we write the metric on \(S^3\) as

\[
ds^2_{S^3} = d\phi^2 + d\psi^2 + d\theta^2 + 2 \cos \theta d\phi d\psi \quad (5.33)
\]

in terms of the standard \([10]\) Euler angle parametrization of \(g \in SU(2)\):

\[
g(\phi, \theta, \psi) := \exp[i\frac{1}{2} \phi \sigma^3] \exp[i\frac{1}{2} \theta \sigma^1] \exp[i\frac{1}{2} \psi \sigma^3], \quad 0 \leq \psi \leq 4\pi, 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi. \quad (5.34)
\]

\[\text{Warning: } [10] \text{ uses the same notation for angles which are not Euler angles. We have } \psi^{C.L.} = \frac{1}{2}(\psi + \phi), \phi^{C.L.} = \frac{1}{2}(\psi - \phi), \theta^{C.L.} = \frac{1}{2} \theta.\]
In these coordinates, \( K_{3,L} = d\psi + \cos \theta d\phi, K_{3,R} = d\phi + \cos \theta d\psi \), are dual to \( K_{3,L} = \frac{\partial}{\partial \psi}, K_{3,R} = \frac{\partial}{\partial \phi} \), respectively. \( ds^2_{\text{BTZ}} \) is the standard Lorentz-signature BTZ black hole with coordinates \( t,r,\phi_b, r \geq r_+, \phi_b \sim \phi_b + 2\pi \). In these coordinates the Kaluza-Klein gauge fields are given by

\[
A_L^a = -4\delta^{a,3} j_L k (dt + d\phi_b) \hspace{1cm} (5.35) \\
A_R^a = -4\delta^{a,3} j_R k (dt - d\phi_b)
\]

Notice that \( A_L, A_R \) are left- and right-chiral flat gauge fields on the family of tori at fixed \( r \) foliating the BTZ black hole. They have nontrivial monodromy around the spatial cycle but, because there is a hole in space, there is no singularity in the geometry. Moreover, they can be removed by an improper gauge transformation on the \( S^3 = SU(2) \) coordinate \( g(\phi, \theta, \psi) \) by:

\[
g \rightarrow \tilde{g} := e^{-i\frac{j_L}{k} (t+\phi_b)\sigma^3} g e^{-i\frac{j_R}{k} (t-\phi_b)\sigma^3} \hspace{1cm} (5.36)
\]

This is not a good gauge transformation in general since it is not periodic in \( \phi_b \). The spins \( j_L \pm j_R \) are integrally quantized \( so(4) \) spins in the quantum theory. This transformation brings the metric to the standard \( AdS_3 \times S^3 \) form and allows us to complete the solution of the 6d \( (0,2) \) supergravity by writing the 3-form fieldstrength (for one of the 5 self-dual \( H \) fields of the \( (0,2) \) supergravity from IIB on K3) as:

\[
H = dt \wedge dr \wedge r d\phi_b + \frac{1}{12\pi} \text{Tr}(\tilde{g}^{-1} d\tilde{g})^3 \hspace{1cm} (5.37)
\]

Note that when expanded using (5.36) the \( H \) field will contain terms proportional to \( dt \pm d\phi_b \).

In the AdS/CFT correspondence the above geometries correspond to semiclassical states with \( J_0 = 2j_L, \bar{J}_0 = 2j_R \). In [10] it is shown that the cosmic censorship bound translates into the inequalities

\[
4kL_0 - J_0^2 \geq 0 \hspace{1cm} 4k\bar{L}_0 - \bar{J}_0^2 \geq 0 \hspace{1cm} (5.38)
\]

We thus interpret the lattice points \((\mu, m)\) in the particle region of fig. 2 as quantum numbers of particles which are not sufficiently massive to form black holes. The sum over the particle region in the Jacobi-Rademacher formula will thus involve a sum over these particles.
Now let us turn to the analytic continuation of the Cvetic-Larsen solutions. It is important to bear in mind that these represent saddle points in a thermal ensemble, and are not semiclassical descriptions of states with well-defined mass and spin in a quantum gravity Hilbert space. The Euclidean continuation of the BTZ geometry is \( t \to it_E, \phi_b \to \phi_b, r \to r, r_+ \to \tau_2, r_- \to i\tau_1, \) producing the geometry (5.27). We also continue \( j_L \to \omega_L, j_R \to \omega_R \) where \( \omega_L, \omega_R \) are complex spin fugacities. The resulting Euclidean solution has several interesting features.

First, the gauge fields are complex. Moreover, after Euclidean continuation the time coordinate \( t_E \) becomes periodic and the circle of time is contractible in the solid torus topology of (5.27). Thus the flat gauge fields have Wilson-line singularities at the “center” of the solid torus \( r = \tau_2 \), namely \( F^a_L \sim \delta^{a,3} \omega_L \delta^{(2)}(t_E, \phi) \). Similarly, \( H \) picks up a singularity, indicating the presence of a string at the Wilson line. The flat gauge fields

\[
\begin{align*}
A^3_L &= \omega_L(d\phi + idt_E) \\
A^3_R &= \omega_R(d\phi - idt_E)
\end{align*}
\]

are precisely of the right form to agree with the boundary conditions on the Euclidean path integral (5.7)(5.8) appropriate for evaluation of the \( SU(2)_L \times SU(2)_R \) Chern-Simons path integral.

Thus, we propose that the Cvetic-Larsen solutions are saddle point approximations to the geometries that contribute in the AdS path integral dual to the elliptic genus. We incorporate the sum over lattice points \((m, \ell)\) in the particle regions of fig. 1, fig. 2, fig. 3 as follows.

In the full theory there are (presumably smooth) 6d or (at shorter distances) 10d geometries which involve particles propagating along “worldlines” \( \gamma \). At long distances the metric is locally \( AdS_3 \) and best described by a Chern-Simons gauge field \( A \) of an \( SU(2|1,1)_L \times SU(2|1,1)_R \) Chern-Simons theory. The particles with worldline \( \gamma \) are described by the Wilson line \( \text{Tr}P \exp[\int_\gamma A] \). It would be very interesting to find explicit smooth solutions of the 6d \((0,2)\) supergravity equations corresponding to these particles. Consistency of this picture demands that we should only sum over particles which themselves do not form black holes. Accordingly, we should include Wilson lines for the representations corresponding to the quantum numbers \((m, \ell)\) which are lattice points in the particle region of fig. 2.

Remarks

31
1. Notice that when we perform the fareytail transform we omit the state with \((m, \ell) = (0, 0)\). This state is special because it is the unique state that is both a black hole and a chiral primary. From this point of view it seems natural that one should remove it from the partition function so that we get a simple expression such as \((1.12)(5.1)\).

2. As we have stressed above, the interpretation of the geometries as having an insertion of a Wilson line resolves a paradox regarding the continuation into three dimensions of the odd spin structure on the right movers required for computation of the elliptic genus.

3. The physical interpretation of the “worldline” \(\gamma\) depends on the physical interpretation of the solid torus geometry. If the geometry is that of Euclidean thermal AdS then \(\gamma\) is indeed a worldline. However, in a black hole geometry, \(\gamma\) is a spatial cycle at a fixed Euclidean time. Thus the Wilson line is associated with the virtual particles associated with Hawking radiation from the black hole. Indeed, in this interpretation of the spatial Wilson lines they represent a sequence of pair creation processes of the Hawking particles that surround the black hole like a virtual cloud. The contribution of these particles is included through a multi-particle generalization of the Schwinger calculation, and, just as for the original Schwinger calculation, give us information about the probability of pair creation in the gravitational field of the black hole. The fact that the Wilson lines are non-contractible makes clear that the processes involve particle-antiparticle pairs that, in the Euclidean world, go once or more times around the black hole before they again annihilate. The quantum mechanical probability of such a process is exponentially suppressed with an exponent that is proportional to the length of this euclidean path, which for the BTZ-black hole is \((\text{an integer multiple of}) \, \text{Im} (-1/\tau)\). In this interpretation the black hole Farey tail represents the trace of the density matrix of Hawking particles outside a black hole. Only, just as for the thermal AdS, the trace is again truncated to the subset of those ensembles of Hawking particles which themselves do not form a black hole.

5.2.3. Remark on the particle degeneracies \(c(m, \ell)\).

Notice that the only remnant of the fact that we have compactified the underlying microscopic superstring theory on a \(K3\) surface \(X\) is in the degeneracies \(c(4km - \ell^2; \text{Sym}^k X)\). The question arises as to whether these degeneracies themselves can be deduced purely from supergravity. In \([11]\) it is shown that, at least for the lattice points of fig. 3 with \(L_0^{NS} \leq \frac{k}{4}\), the degeneracies can indeed be obtained from Kaluza-Klein reduction of \(IIB\)
supergravity on \( AdS_3 \times S^3 \times X \). Extending this result to the full region of fig. 3 is an interesting open problem.

It is also worth stressing that the degeneracies \( c_\ell(4km - \ell^2; \text{Sym}^k X) \) can be quite large. They can be extracted from the formula

\[
\sum_{k=0}^{\infty} p^k \chi(\text{Sym}^k X; q, y) = \prod_{n>0,m\geq 0,r} \frac{1}{(1 - p^n q^m y^r)c(nm,r)}. \tag{5.40}
\]

In particular note that by taking \( q \to 0 \) we get the well-known result of

\[
\sum_{k=0}^{\infty} 2^k \sum_{\ell=0}^{2k} \chi_\ell(\text{Sym}^k X)p^k y^{\ell-k} = \prod_{n=1}^{\infty} \frac{1}{(1 - p^n y^{-1})^2(1 - p^n)^{20}(1 - p^n y^1)^2} \tag{5.41}
\]

where \( \chi_\ell(M) := \sum_s (-1)^{s+\ell} h_{s,\ell}(M) \) for a manifold \( M \). Using (5.41) we can obtain degeneracies at \( m = 0 \). As we scan \( \mu \) from \(-k\) to 0 at \( m = 0 \) the degeneracies increase from \( c(-k^2; \text{Sym}^k X) = k + 1 \) to \( \sim \exp[4\pi\sqrt{k}] \) near \( \mu \approx 0 \). Expressed in terms of the geometry of the D1D5 system, this degeneracy near \( \mu \approx 0 \) is of order \( \sim \exp[4\pi\sqrt{\tau_1\tau_5}/g_{\text{str}}] \), in the notation of [9]. In particular, there is a nonperturbatively large “ground state” degeneracy.

5.3. Interpreting the gravitational factor

We now propose that the factor

\[
(c\tau + d)^{-3} \exp \left[ 2\pi i \left( m - \frac{\mu^2}{4k} \frac{a\tau + b}{c\tau + d} \right) \right] \tag{5.42}
\]

is the contribution of \( SL(2,\mathbb{C}) \) Chern-Simons gravity.

Note first that the contribution of the \( SU(2|1,1) \) Chern-Simons path integral defines a wavefunction on the universal elliptic curve, parametrized by \((\tau, \omega)\). This should be a modular invariant half-density \( Z(\tau, \omega)d\omega \wedge d\tau \), so the modular transformation law given by \((c\tau + d)^{-3}\) is just right.

We now interpret the remaining exponential as the holomorphic part of the Euclidean gravitational action

\[
S = \frac{1}{16\pi G} \left[ \int_M d^3x \sqrt{-g}(\mathcal{R} - 2\Lambda) + 2 \int_{\partial M} K \right] \tag{5.43}
\]

for a BTZ black hole.

A straightforward computation shows the following. Paying due attention to the identification of space and time with \( a, b \)-cycles we conclude that if we define the gravitational
action of $BTZ(\tau)$ (where $\tau$ is defined with $a$-cycle = contractible cycle) relative to an AdS background\footnote{The action (5.43) is infinite, even including the boundary term. Thus one actually computes differences of actions for pairs of geometries with diffeomorphic asymptotics.} then in fact the gravitational action is:

$$\frac{\pi \ell}{4G} \Im(-1/\tau).$$  \hspace{1cm} (5.44)

Using $\ell/4G = k$, the generalization of this result to the $SL(2, \mathbb{Z})$ family gives the gravitational action

$$\exp[2\pi i \tau' h - 2\pi i \bar{\tau}' \bar{h}]$$  \hspace{1cm} (5.45)

where $\tau' = (a\tau + b)/(c\tau + d)$. In\footnote{The “parafermion” terms $c_{\mu, \mu'}(\tau)$ may themselves be written in terms of higher level theta functions.} Cvetić and Larsen show that extremal rotating black holes have $\bar{h} = 0$ and $h = m - \mu^2/4k$ so we obtain the expression in (5.1).

**5.4. Interpreting the $SU(2)$ factor**

Finally we interpret the factor

$$\exp[-2\pi ik \frac{c\omega^2}{c\tau + d}] \Theta^+_{\mu, k}(\frac{\omega}{c\tau + d}, \frac{a\tau + b}{c\tau + d})$$  \hspace{1cm} (5.46)

in (5.1) as the contribution of the $SU(2)_L$ Chern-Simons theory to the path integral.

The functions $\Theta^+_{\mu, k}(z, \tau)$ are closely related to the wavefunctions for $SU(2)$ Chern-Simons theory. It follows from the general reasoning of\footnote{The “parafermion” terms $c_{\mu, \mu'}(\tau)$ may themselves be written in terms of higher level theta functions.} that the $SU(2)$ level $k$ Chern-Simons path integral on a solid torus may be expanded (as a function of $z$) in terms of these functions. This was carried out more explicitly in\footnote{The “parafermion” terms $c_{\mu, \mu'}(\tau)$ may themselves be written in terms of higher level theta functions.} [25]. One basis for the level $k$ $SU(2)$ Chern-Simons wavefunctions is given by the characters of affine Lie algebras:

$$\Psi^{CS}_{\mu, k}(z, \tau) = \exp\left(\pi k \frac{z^2}{3\tau} \right) \frac{\Theta_{\mu+1,k+2}(z, \tau) - \Theta_{-\mu-1,k+2}(z, \tau)}{\Theta_{1,2}(z, \tau) - \Theta_{-1,2}(z, \tau)}$$  \hspace{1cm} (5.47)

This basis diagonalizes the Verlinde operators and is naturally associated with the path integral on the solid torus with a Wilson line in the spin $j = \mu/2$ representation labeling the $b$-cycle.

The space of wavefunctions (as functions of $z$) spanned by (5.47) is the same as the space spanned by even level $k$ theta functions because of the identity

$$\frac{\Theta_{\mu+1,k+2}(z, \tau) - \Theta_{-\mu-1,k+2}(z, \tau)}{\Theta_{1,2}(z, \tau) - \Theta_{-1,2}(z, \tau)} = \sum_{\mu' = 0}^{k} c_{\mu, \mu'}(\tau) \Theta^+_{\mu', k}(z, \tau)$$  \hspace{1cm} (5.48)
Now consider the exponential prefactor in (5.46). Take the Chern-Simons wavefunction
\[ \exp \left( \frac{\pi k}{3 \tau} \omega^2 \right) \Theta_{\mu,k}^+ (\omega, \tau) \] (5.49)

and substitute
\[ \omega \to \frac{\omega}{c \tau + d}, \quad \tau \to \frac{a \tau + b}{c \tau + d}. \] (5.50)

In the full elliptic genus only rightmoving BPS states contribute. Therefore we can take \( \tau, \bar{\tau} \) to be independent and only the term surviving the limit \( \bar{\tau} \to -i \infty \) can contribute to the elliptic genus. This limit gives the expression in (5.1):
\[ \exp \left[-2\pi i k \frac{c \omega^2}{c \tau + d} \right] \Theta_{\mu,k}^+ \left( \frac{\omega}{c \tau + d}, \frac{a \tau + b}{c \tau + d} \right). \] (5.51)

Note that the basis of \( SU(2) \) Chern-Simons wavefunctions \( \Theta^+ \) is preferred over the character basis (5.47). The character basis is usually thought of as the preferred basis because it diagonalizes the Verlinde operators. The function \( \Theta^+ \) sums states at definite values of \( J_{0,L}^3 \) in the current algebra, while (5.47) sums states at definite values of the Casimir, \( j(j+1) \). From the view of the sum over geometries, the \( \Theta^+ \) basis is preferred because the geometries are at definite values of \( J_{0,L}^3 \).

### 5.5. Comparison to previous approaches to the quantum gravity partition function and the Hartle-Hawking wavefunction

It is interesting to compare the above result for the supergravity partition function on the solid torus with the results of more traditional approaches to quantum gravity. The standard approaches, in the context of 3-dimensional gravity are described in [34].

The elliptic genus is a sum over states. By the AdS/CFT correspondence these states can be identified with states in the supergravity theory on \( AdS \), with the \( AdS \) time defining Hamiltonian evolution. In particular, the quantum gravity has a well-defined Hilbert space!

When time is made Euclidean and periodic it is traditional to replace the sum over states by a sum over Euclidean geometries. What has never been very clear is what class of geometries and topologies one should sum over. In the case of Euclidean geometries bounding a torus this question has been explicitly addressed by Carlip in [35]. In the case of a negative cosmological constant there are finite volume hyperbolic geometries which
bound the torus (see, e.g., [36]). These are hyperbolic three-manifolds with a cusp, i.e. a boundary at \( r = \infty \) where the metric behaves as

\[
ds^2 \sim \frac{1}{r^2} (dr^2 + g_{ij} dx^i dx^j),
\]

(5.52)

This is the opposite behaviour of the one considered in the AdS/CFT correspondence as induced by the D-branes. There are no uniqueness theorems for manifolds with this boundary behaviour. The entropy of these geometries overwhelms the action, making the Hartle-Hawking wavefunction ill-defined. In the AdS/CFT formulation of quantum gravity these geometries are eliminated by the boundary condition. The gravitational action of the infinite volume hyperbolic geometry must be regulated, but once this is done the sum over topologies (i.e. the sum over \((c, d)\)) is well-defined and convergent.

5.6. Puzzles

It will be clear, to the thoughtful reader, that the physical interpretation we have offered for the formula (5.1) is not complete. We record here several puzzles raised by the above discussion.

1. It would be nice to understand more clearly the hypothetical smooth six-dimensional geometries corresponding to “adding particles to black holes.” Related to this, it would be nice to go beyond the asymptotic topological field theory and understand more fully the \( SU(2)_L \times SU(2)_R \) gauge theory that arises in \((0, 2)\) supergravity on \( AdS_3 \times S^3 \). For some preliminary remarks see [23].

2. The procedure of taking \( \bar{\tau} \rightarrow -i\infty \) in (5.51) is ad hoc. Clearing up this point requires a careful discussion of the inner product of wavefunctions in the coherent state quantization of the full \( SU(2|1, 1) \) Chern-Simons theory.

3. It would also be interesting to understand more precisely the physical meaning of the fareytail transform. From the fact that the partition function becomes a wave function it seems reasonable to think that it corresponds to extracting some singleton degrees of freedom that live at the boundary of \( AdS \). Another hint is that the fareytail transform is just Serre duality, from the mathematical perspective. This is reminiscent of the fact that in the AdS/CFT correspondence supergravity modes and CFT operators are not equal, but rather in duality.

\[\text{13 It would be very interesting to find the appropriate interpretation (if any) of these cusps in the AdS/CFT correspondence.}\]
6. Large $k$ phase transitions

In this section we will derive the $SL(2, \mathbb{Z})$ invariant phase structure at large $k$ as a function of $\tau$. \[4\]

We begin with the form (4.8) (4.9) of the Jacobi-Rademacher expansion. We go to the NS sector by spectral flow: $Z_{NS,R}(\tau, 0) = (-1)^k e^{2\pi i\tau k/4} Z_{R,R}(\tau, \omega = \tau/2)$. The net result for $Z_{NS,R}(\tau, 0)$ is

$$e^{i\pi k/2} \sum_{(c,d)=1, c \geq 0} (c\tau + d)^{-3} e^{-2\pi ik c(\tau/2)^2/ (c\tau + d)} \sum_{4km - \ell^2 < 0} \tilde{c}_\ell(4km - \ell^2) e^{2\pi im a\tau + b + 2\pi i\ell \tau/2} (6.1)$$

We will estimate the magnitude of the various terms in the sum (6.1). In order to do this we need the following identities (valid for $c \neq 0$):

$$\frac{a\tau + b}{c\tau + d} = \frac{a}{c} + \frac{-1}{c(c\tau + d)}$$

$$\frac{\tau/2}{c\tau + d} = \frac{1}{2c} + \frac{d}{2c(c\tau + d)}$$

$$\frac{c(\tau/2)^2}{(c\tau + d)} = \frac{\tau}{4} - \frac{d}{4c} - \frac{(d/2)^2}{c(c\tau + d)}$$

(6.2)

Using these identities one can evaluate the absolute norm of the terms in the sum (6.1). We find the norm:

$$\frac{1}{|c\tau + d|^3} |\tilde{c}(4km - \ell^2)| \exp \left[ -2\pi \Im \left( \frac{a\tau + b}{c\tau + d} \right) \left( k\frac{d^2}{2} + m + \ell d/2 \right) \right]$$

(6.3)

We will show that the sum (on $m, \ell$) is bounded by a constant for large values of $c, d$. Then the sum over $(c, d)$ is absolutely convergent, because of the factor $(c\tau + d)^{-3}$. Therefore, to study the large $k$ limit we can study the terms for fixed $(c, d)$ separately. For fixed $(c, d)$ we now analyze which terms in the sum over $(m, \ell)$ dominate in the large $k$ limit. The sum over $(m, \ell)$ is over lattice points between the parabolas $4km - \ell^2 = 0$ and $4km - \ell^2 = -k^2$. Since $\Im \left( \frac{a\tau + b}{c\tau + d} \right) > 0$, we minimize as a function of $m$ for $m = \ell^2/4k - k/4$. Next we minimize with respect to $\ell$. The minimum is taken at $\ell_* = -kd, m_* = k(d^2 - 1)/4$. (In order to guarantee that this is an integer we take the limit $k \to \infty$ with $k$ divisible by 4.) Noting that $|\tilde{c}(4km - \ell^2)|$ is bounded by a constant depending only on $k$, we conclude that the dominant term for fixed $(c, d)$ has magnitude:

$$\exp \left[ 2\pi k \frac{\Im \tau}{4 |c\tau + d|^2} \right]$$

(6.4)

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14 The phase structure of the D1D5 system has been discussed in a different way in [37].
and all other terms in the sum on $m,\ell$ are exponentially smaller. This statement also holds for the case $(c = 1, d = 0)$. If we now consider the sum over $(c,d)$ we see that in each region of an $SL(2,\mathbb{Z})$ invariant tessellation of the upper half plane (corresponding to the keyhole region and its modular images) there is a unique $(c,d)$ which dominates. The reason is that the keyhole region has the property that the modular image of any point $\tau \in \mathcal{F}$ has an imaginary part $\text{Im}\tau' = \text{Im}\tau/|c\tau + d|^2 \leq \text{Im}\tau$. Thus the phase transitions are located at the boundary of the region $\Gamma_\infty \cdot \mathcal{F}$ and its images under $SL(2,\mathbb{Z})$.

Thus we conclude that $\log Z_{NS}$ is a piecewise continuous function with discontinuous derivatives across the boundaries of the standard $SL(2,\mathbb{Z})$ invariant tessellation of the upper half plane: There are first order phase transitions across the boundaries of the $SL(2,\mathbb{Z})$ invariant tessellation.

In order to understand the physics of the phase transitions more clearly let us focus on the phase transition for along the imaginary $\tau$ axis, $\Re(\tau) = 0$. There is a phase transition as $\Im\tau$ crosses from $1-\epsilon$ to $1+\epsilon$, and the derivative of the free energy exhibits a discontinuity as shown in fig. 4. The free energy is defined to be $F := -\frac{1}{\beta} \log Z_{NSNS}(\tau = \frac{i\beta}{2\pi})$. At low temperatures the contribution of the leftmovers to the free energy is:

$$F = -\frac{c}{24} = -\frac{k}{4}$$

while at high temperatures the contribution of the leftmovers to the free energy is

$$F = -\frac{k \pi^2}{4 \beta^2}.$$
We have done the calculation for the elliptic genus with \((NS,R)\) boundary conditions. If we want both left and rightmovers to have NS boundary conditions the formulae (6.5)(6.6) get multiplied by 2.

We now give an heuristic argument which explains the physical nature of this phase transition. Consider IIB theory on \(AdS_3 \times S^3 \times X\) where \(X\) is a K3 surface (or a torus \(T^4\)). At the orbifold point, a CFT with target space \(\text{Sym}^k(X)\) is equivalent to a gas of strings with total winding number \(k\) moving on \(X\) \([32]\). We are interested in putting this conformal field theory on a circle with antiperiodic (NS) boundary conditions for the fermions. There is a unique ground state where all strings are singly wound and in their ground states on \(X\). The lowest energy state is when they are singly wound because the twist field that multiply winds them has positive conformal weight, of order \(\Delta \sim w/4\). The energy of this NS vacuum state is (6.5). We can excite the system by putting oscillations on the strings or by multiply winding them. Since we have to symmetrize the state we could think of these excitations as creating second quantized string states in a multiple string hilbert space \([32]\). The number of such states is independent of \(k\) at low energies \(E \ll k\). The energies of these states are of order one. They will not contribute very much to the free energy at temperatures of order one. Another set of states can be obtained by multiply winding the strings. If we multiply wind a string \(w\) times we have to supply an energy of the order of \(w/4\) but we decrease the energy gap of the system which is now of the order of \(1/w\) \([38]\). Then we can have an entropy of the order of \(S = 2\pi^2 w/\beta\) coming from the oscillations of the string. Notice that we can apply the large temperature approximation for calculating the entropy for large \(w\), when the gap is very small. Taking into account the energy of the oscillations and the energy necessary to multiply wind it we see that the contribution of these states to the free energy is

\[
F = E - TS \sim w/4(1 - 4\pi^2/\beta^2)
\]

so that it becomes convenient to produce them above a critical temperature, \(\beta < 2\pi\). The maximum winding number is \(k\) and we see that then the free energy becomes (6.4) (accounting for the factor of two as noted above.) On the other hand, approaching the phase transition from the low temperature side we see a Hagedorn density of string states, with a Hagedorn temperature \(T = 1/(2\pi)\) (see below), so as long as the temperature is smaller than this, the contribution of these states to the free energy is finite and independent of \(k\) and therefore subleading compared to (6.3). This Hagedorn density of states appears only at the orbifold point and

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\(^{15}\) The analysis in \([38]\) considered the system in the Ramond sector where strings can be multiply wound at no energy cost.
it is not seen in supergravity. It appears from the supergravity analysis that most of these states get large masses.

From the supergravity point of view this is the usual Hawking-Page transition between thermal AdS and a black hole background as described in [39]. Actually we need to be more precise since we are considering only the left movers and we are considering the R ground states on the right. If we start in the (NS,NS) vacuum, then the R ground states on the right correspond to chiral primaries. So we need to understand which black holes are chiral primaries on the right. Here again we use the cosmic censorship bound of [10] [11] [10]. As we have discussed, this bound implies \( \tilde{L}_0 \geq \frac{\tilde{J}_0^2}{4k} \) with a similar bound for \( L_0 \) and \( J_0 \). This bound is in general stronger than the chiral primary bound, except for \( \tilde{J}_0 = k \) when they coincide (see fig. 2.). So the black hole will have these right-moving quantum numbers.

The left moving part of this black hole has essentially the same free energy as the BTZ solution (we take \( J_L = 0 \) in the absence of chemical potentials for \( J_L \)). So that the free energy is \( F_{sugra} = -\frac{1}{4} \frac{4\pi^2}{\beta^2} \).

The smallest value of \( L_0 \) for a black hole consistent with cosmic censorship is \( L_0 = k/4 \). This naturally “explains” why de Boer [11] found agreement with supergravity up to this stage; at this point black holes start contributing to the elliptic genus. Beyond this point we only seem to find agreement in the asymptotic form of the sugra and CFT elliptic genus. In fact, beyond this point the CFT form of the elliptic genus and the gravity form for it continue to agree to leading order in \( k \), they both show an exponentially large (Hagedorn) number of states \( e^{2\pi \sqrt{kh}} \) where \( h = L_0 \) is the energy in the NS sector relative to the NS ground state. Notice that our results indicate that if we include all black hole contributions and all particles around them that do not form black holes then we also find agreement.

Similarly notice that the extremal rotating black hole with \( \frac{1}{2} J_0 = \frac{1}{2} \tilde{J}_0 = L_0 = \tilde{L}_0 = k/2 \) is a (chiral, chiral) primary. This gives a reason for expecting that disagreement between supergravity and CFT spectra of (chiral, chiral) appears when \( J_0 = \tilde{J}_0 = k \). Indeed this is the point where the “exclusion principle” becomes operational [11] [11].

Remarks
1. There is an interesting analogy between the phase transitions discussed here and those in the four-dimensional \( U(N) \) case discussed by Witten [39]. The analogy is that the permutation symmetry is analogous to the gauge group, i.e. \( S_k \) is analogous to \( U(N) \). In the low temperature phase all oscillations of different strings have to be symmetrized (i.e., made gauge invariant) and this reduces the number of independent excitations, which becomes independent of \( k \). On the other hand in the high temperature regime,
it is as if all strings are distinct and one can put independent excitations on each of the strings. So in this high temperature phase the permutation symmetry is “deconfined.” Of course what is making this possible is the multiple winding since the whole configuration should be gauge invariant.

2. $SL(2, \mathbb{Z})$ invariant phase diagrams have appeared before in different contexts. In four-dimensional abelian lattice gauge theory one finds such transitions as a function of $\tau = \theta + i/e^2$ [12]. Also, in the dissipative Hofstadter model one finds an $SL(2, \mathbb{Z})$ invariant phase diagram as a function of $\tau = B + i\eta$ where $B$ is a magnetic field and $\eta$ is the dissipative parameter of the Caldeira-Leggett model. See [43]. The phase boundaries in these models are Ford circles (see appendix B), which are modular equivalent to $3\tau = 1$. In our example, as we have explained, the phase boundaries are at the boundary of the region $\Gamma_\infty \cdot F$ and its images under $SL(2, \mathbb{Z})$.

7. Conclusions and Future Directions

In this paper we have applied some techniques and results of analytic number theory to the study of supersymmetric black holes. This has lead to a formula (5.1) for the elliptic genus. Using this formula in (6.1) we were able to derive an $SL(2, \mathbb{Z})$ invariant phase diagram. We have also offered some physical interpretations of these formulae. It should be stressed that if it turns out that there are flaws in these interpretations this would not invalidate the basic formula (5.1), nor the derivation of the phase diagram in section six. There are several interesting avenues for further research and possible applications of these results. Among them are:

1. The Jacobi-Rademacher series gives a useful way of controlling subleading terms in the exact entropy formula for black hole degeneracy provided by the elliptic genus. For this reason the result (5.1) might help in proving or disproving some of the conjectures of [44] relating black hole entropy to some issues in analytic number theory.

2. It is interesting to compare with Witten’s discussion of the partition function of $d = 4$ $U(N)$ gauge theory via the AdS/CFT correspondence in [39]. There are only two obvious ways to fill in $S^{n-1} \times S^1$ topologically for $n > 2$, but in our case, with $n = 2$, there are infinitely many ways. Note that the right-moving odd spin structure of the elliptic genus at first sight suggests that we cannot fill in the torus with a solid torus at all. This did not in fact kill the sum over instantons because of the the Wilson line defects. Thus, this example raises the question of whether similar things might also happen in
higher dimensions, i.e., whether there might be other terms in the Euclidean partition sum besides the geometries $X_1, X_2$ considered in the calculation of Witten.

Another lesson for higher dimensional calculations is that other boundary geometries with nontrivial diffeomorphism groups (such as $T^4$) will probably lead to interesting infinite sums over instanton contributions. It would be quite interesting to reproduce, for example, the formulae of Vafa and Witten from the ADS/CFT correspondence [45].

3. Notice that the exact result for the elliptic genus turns out to depend very little on most of the data one needs to define the full partition function on $AdS_3 \times S^3$. For example, we did not need to specify the boundary conditions for the massive fields. In general the full partition function will depend on these boundary conditions, it is only for a semi-topological quantity like the elliptic genus that these do not enter.

4. There are several variations on the above results which would be interesting to investigate. It should be straightforward to extend the above discussion for $K3$ to the case of $T^4$. This will involve a $J_0^2$ insertion in the path integral, as in [10]. One can also ask about higher genus partition functions, as well as about extensions to nonholomorphic quantities such as the NS-NS partition function at $\omega = \bar{\omega} = 0$.

5. It would be interesting to see if similar formulae apply to the other AdS $(p, q)$ supergravities of [24].

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Version 3 of this paper was stimulated by an important observation of Don Zagier, as explained in the introduction.
Appendix A. Notation & Conventions

A, \tilde{A} \quad SU(2) \times SU(2) \text{ gauge fields from KK reduction of 6d gravity on } AdS_3 \times S^3.

\beta \quad \text{Inverse temperature. Also, the analytic continuation of } \beta. \text{ It might be complex.}

c \quad \text{A central charge. We use } c = 6k \text{ instead to avoid confusion with integers in } SL(2, \mathbb{Z}) \text{ matrices.}

c, d \quad \text{Relatively prime integers. Often entries in an } SL(2, \mathbb{Z}) \text{ matrix } \gamma.

\hat{c} \quad \text{The superVirasoro level of an } \mathcal{N} = 2 \text{ superconformal algebra.}

e(x) \quad \exp(2\pi ix).

G \quad 3\text{D Newton constant. Has dimensions of length.}

\Gamma \quad SL(2, \mathbb{Z}).

\Gamma_\infty \quad \text{The subgroup of } \Gamma \text{ stabilizing } \tau = i\infty.

j \quad \text{An } SU(2) \text{ spin. } j \in \frac{1}{2}\mathbb{Z}_+ \text{. Associated with harmonics of } S^3.

J^{2+1} \quad \text{Black hole spin on } AdS_3.

J(z) \quad \text{Leftmoving } U(1) \text{ current in a } d = 2, \mathcal{N} = 2 \text{ superconformal algebra.}

\tilde{J}(z) \quad \text{Rightmoving } U(1) \text{ current in a } d = 2, \mathcal{N} = 2 \text{ superconformal algebra.}

J_0 \quad \text{The zeromode of the leftmoving current } J(z). \text{ For } \mathcal{N} = 4 \text{ reps } J_0 = 2J_3^0 \text{ has integral eigenvalues.}

\tilde{J}_0 \quad \text{The zeromode of the rightmoving current } \tilde{J}(z).

J_3^0 \quad \frac{1}{2}-\text{integer moded. From the } d = 2, \mathcal{N} = 4 \text{ superconformal algebra.}

k \quad \text{The level } k = Q_1Q_5 \text{ in the D1 D5 system. Positive integral.}

\ell \quad \text{The radius of curvature of } AdS_3, \text{ and its quotients. } \Lambda = -1/\ell^2. \text{ In the D1D5 system } \ell^2 = g_6k.

\ell \quad \text{Also, a nonnegative integral eigenvalue of } J_0 \text{ in the CFT with target Sym}^k X.

L_0, \tilde{L}_0 \quad \text{Left, right Virasoro generators. These are dimensionless.}

q \quad q = \exp[2\pi i \tau].

Q_1, Q_5 \quad \text{Positive integer numbers of D1, D5 branes.}
\(\omega, \tilde{\omega}\) Fugacities for \(SU(2)_L \times SU(2)_R\) spins. These are complex.

\(X\) A Calabi-Yau manifold.

\(y\) \(y = e^{2\pi iz}\) in the context of Jacobi forms and \(y = e^{2\pi i\omega}\) in the context of black hole statistical mechanics.

\(Z_f, Z_\phi\) Fareytail transform of a modular form \(f\) or Jacobi form \(\phi\).

\(Z_{NS}, Z_R\) Partition functions.

Appendix B. First proof of the Rademacher expansions

B.1. Preliminaries: Ford circles, Farey fractions, and Rademacher paths

Before proving the above result we need to give a few preliminary definitions and results.

Fig. 5: The Rademacher path for \(N = 1\) goes from \(A\) to \(B\) to \(C\).

Fig. 6: Ford circles used to construct the Rademacher path for \(N = 3\)
Fig. 7: Ford circles used in the Rademacher path for $N = 5$

**Definition:** The Ford circle $C(d, c)$ is defined for $d/c$ in lowest terms to be the circle of radius $1/(2c^2)$ tangent to the $x$-axis at $d/c$. It is given analytically by:

$$\tau(\theta) := \frac{d}{c} + \frac{i}{c^2} z(\theta) = \frac{d}{c} + \frac{i}{c^2} \left( \frac{1 + e^{i\theta}}{2} \right)$$

(B.1)

Note that $z$ runs over a circle of radius $1/2$ centered on $1/2$ shown in fig. 8:

$$z(\theta) := \frac{1 + e^{i\theta}}{2} = \cos(\theta/2)e^{i\theta/2}$$

(B.2)

**Definition:** The Farey numbers $\mathcal{F}_N$ are the fractions in lowest terms between 0 and 1 (inclusive) with denominator $\leq N$.

We will now need some facts about Ford circles and Farey numbers. These can all be found in [7], ch. 5, and in [13]. First, two Ford circles are always disjoint or tangent at exactly one point. Moreover, two Ford circles $C(d_1, c_1)$ and $C(d, c)$ are tangent iff $d_1/c_1 < d/c$ are consecutive numbers in some Farey series ([7], Theorem 5.6). If $d/c \in \mathcal{F}_N$ then denote the neighboring entries in the Farey series $\mathcal{F}_N$ by $d_1/c_1 < d/c < d_2/c_2$.

Call the intersection points of the Ford circle $C(d, c)$ with the neighboring Ford circles $\alpha_-(d, c; N), \alpha_+(d, c; N)$. Explicit formulae for $\alpha_{\pm}(d, c; N)$, are ([7], Thm. 5.7):

$$\alpha_-(d, c; N) = \frac{d}{c} - \frac{c_1}{c(c^2 + c_1^2)} + \frac{i}{c^2 + c_1^2}$$

$$\alpha_+(d, c; N) = \frac{d}{c} + \frac{c_2}{c(c^2 + c_2^2)} + \frac{i}{c^2 + c_2^2}$$

(B.3)
Fig. 8: Path of integration for $z$. Let $\beta = 1/z$, then the path is vertical, parallel to the imaginary $z$-axis.

In mapping to the $z$-plane by (B.1) we get

$$z_-(d, c; N) = \frac{c^2}{(c^2 + c_1^2)} + \frac{icc_1}{c^2 + c_1^2}$$

$$z_+(d, c; N) = \frac{c^2}{(c^2 + c_2^2)} - \frac{icc_2}{c^2 + c_2^2}$$

A key property used in the estimates below is that if $z$ is on the chord joining $z_-(d, c; N)$ to $z_+(d, c; N)$ then ([7], Thm. 5.9):

$$|z| < \frac{\sqrt{2}c}{N}$$

**Definition:** The Rademacher path $\mathcal{P}(N)$ is

$$\mathcal{P}(N) = \cup_{d/c \in \mathcal{F}_N} \gamma_{d,c}^{(N)}$$

where $\gamma_{d,c}^{(N)}$ is the arc of $C(d, c)$ above $d/c$ which lies between the intersection with the Ford circles of the adjacent Farey fractions in $\mathcal{F}_N$. Call the intersection points $\alpha_-(d, c; N), \alpha_+(d, c; N)$ as above. We orient the path from $\alpha_-$ to $\alpha_+$. 

46
Qualitatively, as $N \to \infty$ the Rademacher path approaches more and more closely an integration around the complete Ford circles associated with all the rational numbers in $[0, 1)$. The arcs associated with $C(0, 1), C(1, 1)$ require special treatment, since the integration path is only over a half-arc. However, by translating $C(1, 1)$ under $\tau \to \tau - 1$ these two arcs become a good approximation to the integral over the full Ford circle $C(0, 1)$.

**B.1.1. Modular transformations of Ford circles**

We will make modular transformations on the Ford circles to what we call the “standard circle.” This is the circle:

$$\tau(\theta) = i z(\theta) = i \frac{1 + e^{i \theta}}{2} = i \cos(\theta/2) e^{i \theta/2}. \quad (B.7)$$

Under the modular transformation $\tau \to -1/\tau$ the standard circle maps to a line parallel to the $x$-axis:

$$-1/\tau(\theta) = \tan(\theta/2) + i \quad (B.8)$$

It is also very useful to introduce the parameter

$$\beta(\theta) := 1/z(\theta) = 1 - i \tan(\theta/2) \quad (B.9)$$

We see from the above that the modular transformation $\tau \to -1/\tau$ takes the Ford circle $C(0, 1)$ to the line $\Im \tau = 1$. In fact, any Ford circle $C(-d, c)$ can be mapped to the line $\Im \tau = 1$ by a transformation of the form

$$\gamma_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad - bc = 1 \quad (B.10)$$

$\gamma_{c,d}$ is well-defined up to left-multiplication by $\Gamma_\infty$. See (2.8).

**B.2. Proof of the theorem**

We now give the proof of (2.4)(2.5)(2.6). Using (2.3) we have

$$F_\nu(n) = \int_{\gamma} d\tau e^{-2\pi i (n+\Delta_\nu) \tau} f_\nu(\tau), \quad (B.11)$$

which holds for any path $\gamma$ in the upper half plane with $\gamma(1) = \gamma(0) + 1$. In particular, we may we take the Rademacher path $\mathcal{P}(N)$:

$$F_\nu(n) = \sum_{d/c \in \mathcal{F}_N} \int_{\gamma^{(N)}(d,c)} d\tau e^{-2\pi i (n+\Delta_\nu) \tau} f_\nu(\tau) \quad (B.12)$$
Written out explicitly this is:

\[ F_\nu(n) = \int_{\alpha_-(1,1;N)}^{\alpha_+(0,1;N)-1} d\tau e^{-2\pi i(n+\Delta_\nu)\tau} f_\nu(\tau) + \sum_{d/c \in \mathcal{F}_N, 0 < d/c < 1} \int_{\alpha_-(d,c;N)}^{\alpha_+(d,c;N)} d\tau e^{-2\pi i(n+\Delta_\nu)\tau} f_\nu(\tau) \]  \hspace{1cm} (B.13)

In the case \( d/c = 0/1, 1/1 \) we should translate the arc above \( \mathcal{C}(1,1) \) by \( \tau \rightarrow \tau - 1 \) to get a single arc above \( \mathcal{C}(0,1) \). Denote the integrals in \( (B.13) \) over the arc \( \gamma^{(N)}(d,c) \) by \( \mathcal{I}_\nu(d,c;N) \).

Now for each of the integrals in the Ford circles we make a modular transformation of the form \( (B.10) \) which maximizes the imaginary part of the top of the Ford circle. This brings the Ford circle to a standard circle, which we can take to be the \( z \)-circle, centered on \( z = 1/2 \), or the “circle” given by \( i/z \) which is the line \( \Im \tau = 1 \):

\[ \tau = d/c + iz/c^2 \]
\[ \tau' = \gamma_{c,-d} \cdot \tau = a/c + i/z = a/c + \tan(\theta/2) + i \]  \hspace{1cm} (B.14)

So, using the modular transformation law for \( f_\mu \) the resulting integral for this arc is

\[ \mathcal{I}_\nu(d,c;N) = \mathcal{I}_\nu^{-}(d,c;N) + \mathcal{I}_\nu^{+}(d,c;N) \]

with the integral on \( z \) over a circle of radius 1/2 shown in fig. 8, with the orientation given by integrating from \( \theta \cong +\pi - \epsilon \) to \( \theta \cong -\pi + \epsilon \). For \( c > 0 \) and \( w \) half-integral we use the principal branch of the logarithm.

We now split the Fourier sum for \( f_\mu \) into its polar and nonpolar pieces

\[ f_\mu(\tau) = f_\mu^{-}(\tau) + f_\mu^{+}(\tau) \]
\[ f_\mu^{-}(\tau) := \sum_{m+\Delta_\mu < 0} F_\mu(m) e^{2\pi i(m+\Delta_\mu)\tau} \]
\[ f_\mu^{+}(\tau) := \sum_{m+\Delta_\mu \geq 0} F_\mu(m) e^{2\pi i(m+\Delta_\mu)\tau} \]  \hspace{1cm} (B.16)

and similarly define \( \mathcal{I}_\nu^\pm \) such that \( \mathcal{I}_\nu(d,c;N) = \mathcal{I}_\nu^{-}(d,c;N) + \mathcal{I}_\nu^{+}(d,c;N) \).

The integral \( \mathcal{I}_\nu^{-}(d,c;N) \) will become the sum of I-Bessel functions for \( N \rightarrow \infty \). We will show that \( \sum_{d/c \in \mathcal{F}_N} \mathcal{I}_\nu^{-}(d,c;N) \) goes to zero for \( N \rightarrow +\infty \).

\[ \text{Under modular transformations the maximal imaginary part of the image of the top of a Ford circle is achieved by the transformations } (B.10): \text{ We wish to maximize } \frac{\Im \tau}{|c'/\tau + d'|} \text{ for } \tau = \frac{d}{c} + \frac{i}{c'}z \text{ for } z \cong 1. \text{ Clearly we can minimize the denominator by taking } \gamma_{c',-d'} \text{ with } c' = c, d' = d. \]
The basic integral we need to estimate is:

\[ \int_{z_-(d,c;N)}^{z_+(d,c;N)} dz z^{-w} e^{2\pi(n+\Delta \nu)z/c^2} e^{-2\pi(m+\Delta \mu)/z} \]  

The integral is over the circular contour in fig. 9, oriented from \( z_- \) to \( z_+ \). Already saddle point techniques show that the behavior of the integral is very different depending on the sign of \( (m + \Delta \mu) \).

If \( (m + \Delta \mu) \geq 0 \) then we deform in integral along the arc \( z(\theta) \) to an integral along the chord joining \( z_-(d,c;N) \) to \( z_+(d,c;N) \) shown in fig. 9. More fundamentally, in terms of \( \beta(\theta) \), we can deform the contour into the right half-plane. As \( N \to \infty \), \( z_\pm(d,c;N) \to 0 \). Therefore, the chord is near zero. Along the chord if we write \( z = \epsilon e^{i\phi} \) then \( \Re(1/z) = (1/\epsilon) \cos(\phi) \) can get very large. Therefore this contour deformation will only be useful for \( (m + \Delta \mu) \geq 0 \). In this case, the minimal value of \( \Re(1/z) \) is taken by the points on the Ford circle, so for \( z \) on the chord:

\[ \Re(1/z) \geq \min[\Re(1/z_+), \Re(1/z_-)] = 1 \]  

(B.18)
Therefore, a crude estimate of (B.17) is
\[ \left| \int_{z_-(d,c;N)}^{z_+(d,c;N)} dz z^{-w} e^{2\pi (n+\Delta \nu)/z} \right| \leq |z_+(d, c; N) - z_-(d, c; N)|. \]  
(B.19)

\( \cdot (\text{Max}_{\text{chord}}|z|^{-w}) \cdot (\text{Max}_{\text{chord}}|e^{2\pi/c^2(n+\Delta \nu)}|) e^{-2\pi(m+\Delta \mu)} \quad (m + \Delta \mu) \geq 0 \)

Using (B.5) ( [7], Thm 5.9 ) we have \( |z| \leq \sqrt{2c}/N \) along the chord. Therefore
\[ \text{Max}_{\text{chord}}|e^{2\pi/c^2(n+\Delta \nu)}| = \text{Max}_{\text{chord}}e^{2\pi/c^2(n+\Delta \nu)}\Re(z) \leq \text{Max}[1, e^{2\pi(n+\Delta \nu)\sqrt{2}/(Ne)}] \]  
(B.20)

To estimate the other factors in (B.17) we now take \( w \leq 0 \) and use (B.5) so:
\[ \left| \int_{z_-(d,c;N)}^{z_+(d,c;N)} dz z^{-w} e^{2\pi/c^2(n+\Delta \nu)}z e^{-2\pi(m+\Delta \mu)/z} \right| \leq 2 \left( \frac{\sqrt{2c}}{N} \right)^{1-w} \text{Max}[1, e^{2\pi(n+\Delta \nu)\sqrt{2}/(Ne)}] e^{-2\pi(m+\Delta \mu)} \quad (m + \Delta \mu) \geq 0 \]  
(B.21)

Using the absolute convergence of \( F_{\nu}(\tau = i) \) we see that the sum on \( m \) in \( f^+ \) gives an \( N \)-independent constant and we need to estimate:
\[ \left| \sum_{d/c \in \mathcal{F}_N} \mathcal{I}_\nu(d, c; N)^+ \right| \leq \sum_{\mu} \sum_{d/c \in \mathcal{F}_N} |M_{\nu\mu}^{-1}(\gamma_{c,-d})|c^{-w-2} \]  
(B.22)

\[ 2 \left( \frac{\sqrt{2c}}{N} \right)^{1-w} e^{2\pi|n+\Delta \nu|\sqrt{2}/(Ne)} |f_\mu^+(\tau = i)| \]

We can now estimate the error from (B.22) as follows:

1. The sum on \( d \) has \( \phi(c) \leq c \) terms.
2. \( |M_{\nu\mu}^{-1}(\gamma_{c,d})| \leq 1 \) because \( M_{\nu\mu} \) are unitary matrices.
3. The sum on \( c \) has \( N \) terms.

Thus we get an upper bound of
\[ \left| \sum_{d/c \in \mathcal{F}_N} \mathcal{I}_\nu(d, c; N)^+ \right| \leq N^w r 2^{(3-w)/2} \max_{\mu} |f_\mu^+(\tau = i)| e^{2\pi|n+\Delta \nu|\sqrt{2}/N} \]  
(B.23)

For \( w < 0 \) this goes to zero for \( N \to \infty \).

Now we come to the finite number of terms with \( (m + \Delta \mu) < 0 \). In (B.17) we write
\[ \int_{z_-(d,c;N)}^{z_+(d,c;N)} dz z^{-w} e^{2\pi/c^2(n+\Delta \nu)}z e^{-2\pi(m+\Delta \mu)/z} = \]
\[ \int_0^{z_+(d,c;N)} dz z^{-w} e^{2\pi/c^2(n+\Delta \nu)}z e^{-2\pi(m+\Delta \mu)/z} + \int_0^{z_-(d,c;N)} dz z^{-w} e^{2\pi/c^2(n+\Delta \nu)}z e^{-2\pi(m+\Delta \mu)/z} \]  
(B.24)
The last integral is essentially the Bessel function of the theorem. All integrals are along
the Ford circle. However, along the circle \( \Re(1/z) = \Re(1 - i \tan(\theta/2)) = 1 \), while
\[
\Re(z) \leq \Re(z^+) \leq |z^+| = \frac{c}{\sqrt{c^2 + c^2}}
\]
so once again we use (B.5) to get the estimate on the error
\[
\left| \int_0^{z^+(d,c;N)} dz z^{-w} e^{2\pi/c^2 (n+\Delta)z} e^{-2\pi(m+\Delta)/z} \right| \leq \pi |z^+(d,c;N)| \cdot \left( \max_{\text{arc}} |z|^{-w} \right) \cdot \left( \max[1, e^{2\pi(n+\Delta)\sqrt{2}/(Nc)}] \right) e^{-2\pi(m+\Delta)}/z
\]
(B.25)

So again, as in (B.23), the error in dropping these terms is \( \sim N^w \). Similar remarks apply
to the integral from 0 to \( z^- \). This leaves the integral
\[
\int_1^{\infty} d\beta \beta^{w-2} \exp \left[ \frac{2\pi n + \Delta}{c^2} \frac{1}{\beta} + 2\pi|m + \Delta|/|\beta| \right]
\]
which can be expressed in terms of the \( I \)-Bessel function. Note
\[
\beta = \frac{1}{z} = 1 - i \tan(\theta/2)
\]
(B.27)

The above argument can be extended to \( w = 0 \). See [8].

Appendix C. An elementary proof of the Rademacher expansion

In this appendix we give a much simpler proof of the Rademacher expansion, making
use of the mathematical transformation that appears when relating the conformal field
theory and supergravity partition functions.

For simplicity we take the case of a one-dimensional representation of \( SL(2, \mathbb{Z}) \) without
multiplier system and with integral weight \( w < 0 \). The proof is simply the following:

1. Observe that \( (q \frac{\partial}{\partial q})^{1-w} f \) transforms with modular weight 2 - \( w \) > 2.

2. Note that \( (q \frac{\partial}{\partial q})^{1-w} f \) has no constant term and is orthogonal to all the cusp forms
   in \( M_{2-w} \). (We will prove this below.)

3. Therefore \( F_{\mu} := (q \frac{\partial}{\partial q})^{1-w} f_{\mu} \) is fully determined by applying the Poincaré series
   operation to the polar part (negative powers of \( q \)) \( F_{\mu^-} \), since the difference of two such
would be a Poincare series defining an ordinary modular form in $M_{2-w}$, that is, a cusp form. The Poincaré series will converge absolutely for $w < 0$. Note that

$$F_\mu^- = (q \frac{\partial}{\partial q})^{1-w} f_\mu^-$$  \hfill (C.1)

Now that we have weight $2 - w > 2$ and can represent $F_\mu$ as an Poincaré series we can apply the Petersson formula for Fourier coefficients of Poincaré series. This gives exactly the Rademacher formula.

Proof of step 2: We will show that the Petersson inner product is

$$\int_{\mathcal{F}} \frac{dxdy}{y^2} y^{2-w} (q \frac{\partial}{\partial q})^{1-w} f(\tau) \tilde{g}(\bar{\tau}) = 0$$ \hfill (C.2)

for cusp forms $g \in M_{2-w}$. Since $f$ has a polar part the integral is understood in the usual sense of cutting off $\Im \tau < \Lambda$ and then taking $\Lambda \to \infty$. We can justify this using integration by parts with an operator similar to (3.25), namely, $\nabla_W = (\frac{\partial}{\partial \tau} + \frac{W-2}{2iy})$ which takes modular forms of weight $W - 2$ to forms of weight $W$. We have:

$$\int_{\mathcal{F}} \frac{dxdy}{y^2} y^W (\nabla_{W-2} f(\tau)) \tilde{g}(\bar{\tau}) = 0$$ \hfill (C.3)

simply by integration by parts. Note we need $f(\tau) \tilde{g}(\bar{\tau})$ to have no constant term. Now, for $w < 0$ we have

$$\left(2\pi i q \frac{\partial}{\partial q}\right)^1 f(\tau) = \nabla_{2-w} \nabla_{-w} \cdots \nabla_{w+2} \nabla_w f(\tau)$$ \hfill (C.4)

and now step 2 follows. ♠

The point of this derivation is that the proof of Petersson’s formula (we recall it below) is more elementary and straightforward than Rademacher’s method based on Farey series and Ford circles. Note also that it is consistent with Lemma 9.1 of [47].

C.1. Petersson’s formula

Here we recall a standard formula from analytic number theory. See, for examples, texts by Iwaniec [48] or Sarnak [49] for further discussion. We let $w > 0$ be a positive weight. Consider the Poincaré series:

$$f_p(\tau) := \sum_{\Gamma \in \Gamma} (c\tau + d)^{-w} p(\gamma \cdot \tau)$$ \hfill (C.5)
This is well-defined if \( p(\tau + 1) = p(\tau) \). Unless \( w \) is even integral we must fix \( c > 0 \). We will specialize to \( p(\tau) = e^{2\pi im\tau}, m \in \mathbb{Z} \). If \( w > 2 \) the series is absolutely convergent. Then \( f_p(\tau) \) is a modular form, although for \( m < 0 \) we allow poles at the cusps. (This is usually excluded, e.g. in [48] one takes \( m \geq 0 \) but we want \( m < 0 \) for our application. The sign of \( m \) does not change the convergence properties.)

Petersson’s formula gives an expression for the Fourier coefficients in

\[
f_p(\tau) = \sum_{\ell \in \mathbb{Z}} F(\ell) q^\ell
\]

The derivation is elementary. We write

\[
f_p(\tau) = p(\tau) + \sum_{(\Gamma_\infty \setminus \Gamma) / \Gamma_\infty} \sum_{\ell \in \mathbb{Z}} [c(\tau + \ell) + d]^{-w} p\left( \frac{a(\tau + \ell) + b}{c(\tau + \ell) + d} \right)
\]

where we used the Poisson summation formula. Now we specialize again to \( p(\tau) = e(m\tau) \). By a simple change of variables the integral in (C.7) becomes

\[
e(\hat{\ell}\tau + \hat{\ell}d/c + ma/c) c^{-1} \int_{-\infty + icy}^{+\infty + icy} e(-\hat{\ell}v/c - m/(cv))v^{-w}dv
\]

where we use the standard notation \( e(x) := \exp[2\pi ix] \). The contour integral does not depend on \( y \) by Cauchy’s theorem, and for \( \hat{\ell} \leq 0 \) we can close the contour in the upper halfplane and get zero. (For \( \hat{\ell} = 0 \) we must have \( w > 1 \) for this.) For \( \hat{\ell} > 0 \) we close in the lower half-plane and the countour becomes a Hankel countour surrounding the lower imaginary axis. This gives the standard result in terms of the Bessel function \( J_{w-1} \) for \( m > 0, \hat{\ell} > 0 \):

\[
F(\ell) = -2\pi i^{-w} \sum_{c=1}^{\infty} \frac{1}{c} Kl(\ell, m; c)(\frac{\ell}{m})^{(w-1)/2} J_{w-1}(\frac{4\pi}{c} \sqrt{m\ell})
\]

where \( Kl(\ell, m; c) \) is the Kloosterman sum:

\[
Kl(\ell, m; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} e(\ell d/c)e(md^{-1}/c)
\]
We can also do the contour integral if \( m < 0, \ell > 0 \). In this case we get an \( I \)-Bessel function. One quick way to see this is to use the relation between \( J \)- and \( I \)-Bessel functions. (See, e.g., Gradshteyn and Ryzhik, GR 8.406):

\[
J_\nu(e^{i\pi /2} z) = e^{i \frac{\nu}{2} \pi} I_\nu(z).
\]

Doing the integral we get (for \( p(\tau) = e(m\tau), m < 0 \)):

\[
F(\ell) = \sum_{c=1}^{\infty} \frac{2\pi}{c} Kl(\ell, m; c)(\frac{\ell}{|m|})^{(w-1)/2} I_{w-1}\left(\frac{4\pi}{c} \sqrt{|m|} \ell\right). \tag{C.11}
\]
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