Comparative testing of MOST and Mac-Cormack numerical schemes to calculate tsunami wave propagation

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Abstract. The problem of fast, reliable, and accurate calculation of tsunami wave propagation is addressed. Based on the shallow water approximation, two numerical approaches to calculate tsunami wave propagation are compared at the same computational mesh. To evaluate precision of numerical solutions the available exact solutions (for the cases of sloping and parabolic bottom) are used. Results of several numerical tests are presented. Time required to obtain tsunami wave height maxima distribution is briefly discussed, too.

1. Introduction

It is agreed among the tsunami researchers that linear or nonlinear shallow water system pretty well describes tsunami wave propagation at the deep-water area. Dispersive effects could be neglected provided that the wave amplitude is concerned. Applying the modern achievements in numerical methods it is not a problem to calculate tsunami wave propagation even over the globe.

Currently, several software packages for modeling tsunami propagation have been created. This, for example, is the TUNAMI N2, developed at the University of Tohoku (Japan) [1]. The COMCOT package, also popular is the Asian-Pacific region [2], should be mentioned, too. Among the most popular software instruments the MOST (Method of Splitting Tsunamis) software package should be highlighted. Numerical code for MOST was proposed and originally developed in the Computing Center of Siberian Branch the Russian Academy of Sciences by V.V. Titov at the end of the 80s of the last century [3]. The software package was created in Pacific Marine Environmental Laboratory (National Ocean and Atmosphere Administration of the USA) [4], and now it is used as official instrument of the US NOAA tsunami warning service.

In the first decade of this century, the MOST calculation scheme was tuned at the Novosibirsk State University, to use the Graphic Processing Units (GPUs) for better performance. Calculation become 100 times faster compared to the original sequential computer code provided that Nvidia K40 platform is used. At the same time, the accuracy of the calculations remains similarly good [5].

The MOST calculation algorithm is based on the so-called splitting method with respect to space variables. One should solve consequently the 1D shallow water system twice, with respect to $x$ and $y$ variables.

GPU is a processor with SIMT (Single Instruction Multiple Thread) architecture. The algorithm of solving shallow water differential equations is well-suited for this architecture: we can process each grid point independently in parallel. We also decided to keep all data on GPU in invariant variables.
and convert them back into the physical values only if we need to save data on CPU side. To implement the code on GPU we used CUDA C historically. Later, we have marked the original code with OpenACC directives.

The Mac-Cormack finite difference scheme (of the second order approximation) has been applied in [6] for numerical solution of the shallow water system. It is currently in use in the Institute of Computational Technologies SB RAS. The authors adopt this scheme for parallel implementation at the Field-Programmable Gate Array (FPGA) based specialized Calculator [5].

FPGA is an integrated circuit designed to be configured after manufacturing. FPGAs contain an array of programmable logic blocks, and a reconfigurable interconnects between them. This allows to implement application-specific calculators with finest-grain parallelism. To enjoy advantages of the FPGA features, the stream processor architecture was proposed for the Mac-Cormack algorithm implementation. The proposed Calculator contains several processor elements (PEs) connected into a chain and running parallel. Each of these elements is included into pipeline which performs a one timestep on sequential data stream with one point per cycle performance. Calculation speed-up achieved by storing whole modeling data is stored on “On board” DRAM and using FPGA inner memory (BRAM) for implementing cache buffer optimized for computation scheme.

2. Model and numerical approach

2.1. Mathematical model

Following [3], we use the following equivalent form of a shallow water system (which does not take into account such external forces as sea bed friction, Coriolis force and others):

\[ H_t + (uH)_x + (vH)_y = 0, \]
\[ u_t + uu_x + vv_y + gH_x = gD_x, \]
\[ v_t + uv_x + vv_y + gH_y = gD_y, \]

(1)

where \( H(x, y, t) = \eta(x, y, t) + D(x, y) \) is the entire height of water column, \( \eta \) being the sea surface disturbance (wave height), \( D(x, y) \) – depth (which is supposed to be known at all grid points), \( u \) and \( v \) components of velocity vector, \( g \) – acceleration of gravity.

2.2. Mac-Cormack numerical scheme

To achieve better performance results by hardware code acceleration, the explicit two steps Mac-Cormack finite difference scheme of the second order approximation has been used [5]. This scheme looks similar to the splitting method (with respect to space variables), which is used in the MOST software package. Indeed, in order to calculate the values of the sought functions at point \((i, j, n + 1)\) at \(n + 1\) time step, the values at 3 points of the previous time step are used, first time at the points \((i, j, n), (i - 1, j, n), (i, j - 1, n)\), and the second time – at the points \((i, j, n), (i + 1, j, n), (i, j + 1, n)\). However, the proposed version to realize the three-points calculation stencil seems to be preferable compared to the one from the MOST package.

3. Comparison of MOST and Mac-Cormack numerical schemes

To verify the reliability of the results of numerical calculations obtained using the above algorithms, comparison with some of the well-known exact solutions [7–9] has been arranged. The first numerical test was carried out in order to verify the correctness of modeling the reflection of a wave from a completely reflecting boundary located at an angle of 45 degrees to the direction of motion of the flat wave front. In a rectangular computational domain of 1000x2000 computational nodes, a long wave about 1 m height generated by a one-dimensional source parallel to the inner boundary of the region propagates over the region and is reflected from the reflecting boundary extending from the right boundary (upper right corner) to the left one at an angle of 45 degrees to ordinate axis.
Figure 1 shows the distribution of the vertical displacement of the water surface in the entire region calculated using the Mac-Cormack difference scheme (a) and the MOST algorithm (b). The dark line shows the isoline of the tsunami height corresponding to the value of 0.4 m. The grey line outlines the water area with the surface displacement \( \pm 0.4 \). Both figures confirm the correctness of numerical modeling of the process of wave reflection from a completely reflecting boundary. It is clear from the figures that the direction of motion of the reflected wave in both cases is orthogonal to the direction of the incident wave.

The second test consists of calculation of the tsunami wave propagation from a round source in an area with a sloping bottom profile. An area of 1000x1000 km with a computational grid having spatial steps \( \Delta x = \Delta y = 1000 \) m was considered. The center of the circular tsunami source with the radius of 50 km was located in the middle of the region at a distance of 300 km from the lower boundary, where the depth was vanishing. In the rest of the computational domain, the depth linearly increased according to the formula \( D(x,y) = 0.01y \), where \( y \) is the distance to the lower boundary of the region. The initial vertical displacement \( h(r) \) in the source is determined by the formula...
Here, the parameter \( r \) presents the distance to the center of the source. Thus, in the center of the source area, the initial displacement of the water surface was \(+2 \text{ m}\). This source generates a circular wave having a height of \(0.95 \text{ m}\) at a distance of \(50 \text{ km}\) from the center. It was such a wave height that was used at the initial circular wave front with a radius of \(50 \text{ km}\) in the course of estimating the wave amplitude at all points of the region according to the ray approximation [9]. This distribution of tsunami wave amplitude (obtained in an analytic form) was compared with the same distributions obtained by numerically modeling of the tsunami propagation using the MOST and Mac-Cormack algorithms (figure 2).

\[
h(r) = 1 + \cos \left( \frac{\pi r}{r_0} \right), \quad 0 \leq r \leq r_0,
\]

(2)

Figure 2 shows that at sufficiently large depths (exceeding \(500 \text{ m}\)), the contours of all three distributions of tsunami height maxima are quite close to each other. The proximity of the results of numerical calculations for the two algorithms under consideration is also preserved near the coast. The increasing difference in comparison of numerical and wave-ray approaches wave heights distribution in the coastal zone is caused by the neglection of the effect of increasing wave height due to reflection from the coast.

Based on the results of calculations of the wave propagation over a bottom slope, the correctness of the wave front kinematics modeling was also estimated by the two considered numerical methods. Figure 3 shows the comparison of the wave front position with an interval of 5 minutes calculated according to the Mac-Cormack scheme with the exact solution of the kinematic problem at the same time points. In order to prevent the points from merging (as it happens at the initial moment), the moments of output of the points of the calculated wave front are taken 3 seconds later than the moment of the corresponding exact solution of the kinematic problem. We do not present the wave
front positions as a result of numerical simulations of the tsunami according to the MOST software, since the positions of the front points exactly coincide with the positions of the corresponding points as a result of the numerical calculation of the wave dynamics using the Mac-Cormack scheme.

The third test was similar to the second one in case of a parabolic bottom relief. Consider the same computational area of 1000×1000 km with a computational grid having spatial steps \( \Delta x = \Delta y = 1000 \text{ m} \). The center of the circular source with a radius of 50 km was also located in the middle of the region at a distance of 300 km from the lower boundary, where the depth is equal to zero. In the rest of the computational domain, the depth increased according to the formula \( D(x, y) = 10^{-8} y^2 \), where \( y \) is the distance to the lower boundary of the region. The initial vertical displacement inside the circular focus is determined by the formula (2). Figure 4 presents, similarly to figure 2, the isolines of the distributions of tsunami height maxima calculated by the MOST and Mac-Cormack algorithms, as well as the estimates of these maxima in the framework of the ray model.

![Figure 3](image-url)

**Figure 3.** Comparison of the tsunami wave isochrones over the sloping bottom: obtained by numerical experiments with MOST and Mac-Cormack algorithms (grey lines) and exact solution (black points).

From figure 4, one can output the same conclusions as from figure 2. Namely, at sufficiently large depths (more than 200 m), the contours of all three distributions (obtained numerically by MOST and Mac-Cormack algorithms and distribution of the wave-ray solution) of tsunami height maxima are quite close to each other. Here, the similarity of the results of numerical calculations by the two algorithms under consideration is observed up to the coastline itself (the lower boundary of the region), while there is an increase in the difference between these results in the coastal zone with estimates by the wave-ray approximation.
4. Discussion
The first numerical experiment testing the reflection of a wave from an oblique wall showed the correctness of the implementation of the boundary conditions for wave reflection, which are set, in particular, along the coastline. At the same time, free boundary conditions (set at the left and right boundaries of the computational domain, see figure 1) result in decrease of the wave energy (it is “pumping off” the simulation domain). According to the theory this should not take place in case of quasi-one-dimensional wave propagation. It should be noted here that when calculating according to the MOST algorithm, the wave amplitude decrease is greater as compared to calculation using the Mac-Cormack scheme (see figure 1).

In case of the so-called near field tsunami event, factor of time is very important to avoid casualties and to safe the infrastructure. After the tsunamigenic earthquake offshore Japan, the wave approaches the nearest coast in about 20 min. Fast and reliable evaluation of the expected tsunami wave height maxima is on demand. Our experience shows that such a highly reputed tool as the MOST software package does not allow effective parallelization at PC basis. On the contrary, the mathematical operations of Mac-Cormack algorithm are implemented as a calculation pipeline, with performance of 1 node treatment at one clock cycle.

Figure 4. Isolines of maximal wave heights distribution from [4], obtained by: the wave-ray solution to shallow water system (grey lines), numerical solution at the FPGA based Calculator (dashed lines), and using the MOST package (black lines). The offshore distance is measuring along vertical axis relative to the figure’s bottom boundary.

5. Conclusion
The test calculations of the tsunami wave propagation using both the Mac-Cormack difference scheme implemented on the FPGA and the MOST algorithm demonstrated good agreement between the results of numerical calculations among themselves and with the solutions obtained by the wave-ray model. Moreover, the Mac-Cormack difference scheme yielded results closer to the exact solutions.
obtained in the framework of the ray approximation. This was especially evident during the first computational experiment when testing tsunami wave reflection from an oblique wall.

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