Abstract

Practical applications that use treewidth algorithms have graphs with treewidth \( k = O(\sqrt[3]{n}) \). Given such \( n \)-vertex graphs we present a word-RAM algorithm to compute vertex separators using only \( O(n) \) bits of working memory. As an application of our algorithm, we show an \( O(1) \)-approximation algorithm for tree decomposition. Our algorithm computes a tree decomposition in \( c' n (\log^* n) \log \log n \) time using \( O(n) \) bits for some constant \( c' \).

We finally show that our tree-decomposition algorithm can be used to solve several monadic second-order problems using \( O(n) \) bits as long as the treewidth of the graph is smaller than \( c' \log n \) for some constant \( 0 < c' < 1 \).

1 Introduction

For solving problems in the context of the ever-growing field of big data we require algorithms and data structures that do not only focus on runtime efficiency, but consider space as an expensive and valuable resource. Some reasons for saving memory are that less slower memory in the memory hierarchy has to be used, less cache faults arise and the available memory allows us to run more parallel tasks on a given problem.

As a solution researchers began to provide space-efficient algorithms and data-structures to solve basic problems like connectivity problems \([2, 4, 11, 14]\), memory initialization \([10, 20]\), dictionaries with constant-time operations \([5, 8, 13]\) or graph interfaces \([3, 17]\) space efficiently, i.e., they designed practicable algorithms and data-structures that run (almost) as fast as standard solutions for the problem under consideration while using asymptotically less space. Our model of computation is the word RAM, where we assume to have the standard operations to read, write as well as arithmetic operations (addition, subtraction, multiplication and bit shift) take constant time on a word of size \( \Theta(\log n) \) bits, where \( n \in \mathbb{N} \) is the size of the input. To measure the total amount of memory that an algorithm requires we distinguish between the input memory, i.e., the read-only memory that stores the input, and the working memory, i.e., the read-write memory an algorithm additionally occupies during the computation.

To the authors knowledge, we are the first to present a space-efficient algorithm for NP-hard problems, which are often solved by a so-called FPT algorithm.
An usual approach to find an FPT algorithm for graph problems is to decompose the
given graph into a tree decomposition consisting of a tree where each node of the tree has a
bag containing a small fraction of the original vertices of the graph. The quality of a tree
decomposition is measured by its width, i.e., the number of vertices of the largest bag minus
1. The treewidth of a graph is the smallest width over all tree decompositions for the graph.
Having a tree decomposition of a graph, the problem is solved by first determining the solution
size of the problem in a bottom-up traversal and second in a top-down process computing a
solution for the whole graph. Treewidth is also a topic in current interdisciplinary research,
such as smart contracts using cryptocurrency [6] or computational quantum physics [10],
which are fields that often work with big data sets. So it is important to have space-efficient
algorithms.

Several algorithms are known for computing a tree decomposition. For the following, we
assume that the given graphs have $n$ vertices and treewidth $k$. Reed [22] showed an algorithm
for computing a tree decomposition of width $O(k)$ in $O(c^kn \log n)$ for some constant $c$. His
algorithm repeatedly uses a so-called balanced separator that splits the input graph into
roughly equally sized subgraphs, each used as an input for a recursive call of the algorithm.
Further tree-decomposition algorithms can be found in [2, 7, 11, 14]. The basic strategy of
repeated separator searches is the foundation of all treewidth approximation algorithms, as
mentioned by Bodlaender et al. [4]. Using the same strategy, Bodlaender et al. also presented
an algorithm that runs in $2^O(k)n$ time and finds a tree decomposition having width $5k + 4$.

To obtain a space-efficient approximation algorithm for treewidth we modify Reed’s
algorithm. We finally use a hybrid approach, which combines our new algorithm and
Bodlaender et al.’s algorithm [4] to find a tree decomposition in $b^k n (\log^* n) \log n$ time for
some constant $b$. The general idea for the runtime improvement is to use our space-efficient
algorithm for treewidth only for constructing the nodes of height at most $b^k \log \log n$. For
the subgraph induces by the bags of the vertices below a node of height $b^k \log \log n$, we use
Bodlaender et al.’s algorithm. The most computationally difficult task of this paper is the
computation of the separators.

Finding separators requires finding vertex-disjoint paths for which running DFS as a
subroutine is needed. All recent space-efficient DFS require $\Omega(n)$ bits [2, 7, 11, 14]. Moreover,
Tompa [24] showed that certain natural algorithmic approaches for the s-t-connectivity
problem require super-polynomial time if $o(n)$ bits of working memory are available. Thus,
our challenge was to compute a separator and subsequently a tree decomposition with $O(n)$
bits.

To compute a separator of size at most $k$ with $O(n \log k)$ bits, the idea is to store up to $k$
vertex disjoint paths by assigning a color $c \in \{0, \ldots, k\}$ to each vertex $v$ such that we know to
which path $v$ belongs. We also number the vertices along a path with 1, 2, 3, 1, 2, 3, etc. so that
we know the direction of the path. Since we want to find separators with only $O(n)$ bits, we
further show that it suffices to store the color information only at every $\Theta(k)$th vertex.
We so manage to find separators of size at most $k$ with $O(n + k^2 (\log k)^2 \log n)$ bits. If
$k = O(\sqrt{n})$, we thus use $O(n)$ bits.

Our solution to find a separator is in particular interesting because previous space-efficient
graph-traversal algorithms either reduce the space from $O(n \log n)$ bits to $O(n)$, e.g., depth
first search (DFS) or breath first search (BFS) [7, 11, 14], or reduce the space from $O(m \log n)$
to $O(m)$, e.g., Euler partition [17] and cut-vertices [18]. In contrast, we reduce the space for
the separator search from $O((n + m) \log n)$ bits to $O(n)$ bits for small treewidth $k$.

Besides the separator search, algorithms for treewidth store large subgraphs of the $n$-
vertex, $m$-edge input graph during recursive calls, i.e., they require $\Omega((n + m) \log n)$ bits. We
modify and extend the algorithm presented by Reed with space-efficiency techniques (e.g., store recursive graph instances with the so-called subgraph stack) to present an iterator that allows us to output the bags of a tree decomposition of width \(O(k)\) in an Euler-traversal order using \(O(kn)\) bits in \(c^kn \log n \log^* n\) time for some constant \(c\). To lower the space bound further, we use the subgraph stack only to store the vertices of the recursive graph instances. For the edges we present a new problem specific solution. This allows us to lower the space for storing recursive graph instances to \(O(n + k^2 \log n)\) bits.

In Section 2 we summarize known data structures and algorithms that we use afterwards. Our main result, the computation of \(k\)-vertex disjoint paths is shown in Section 3. We sketch Reed’s algorithm in Section 4 where we also show a space-efficient computation of a balanced vertex separator using \(O(n)\) bits. In Section 5 we present an iterator that outputs the bags of a tree decomposition using \(O(kn)\) bits. In the following section we lower the space bound to \(O(n)\) bits for small treewidth, and show our hybrid approach. We conclude the paper by showing that our tree decomposition iterator can be used to solve several monadic second order problems with \(O(n)\) bits on graphs with small treewidth. The following table summarizes the space bound of the algorithms described in this paper.

|                  | standard             | intermediate goal                  | final goal       |
|------------------|----------------------|------------------------------------|------------------|
| \(k\) vertex-disjoint paths | \(\Omega(kn \log n)\) | \(\Theta(n)\) (Sect. 3) |                  |
| balanced vertex separator | \(\Omega(n \log(n/k))\) | \(\Theta(n)\) (Sect. 4) |                  |
| subgraph stack   | \(\Omega(kn \log n)\) | \(\Theta(kn)\) (Sect. 5) | \(\Theta(n)\) (Sect. 6) |
| iterator for a t.d. | \(\Omega(kn \log n)\) | \(\Theta(kn)\) (Sect. 5) | \(\Theta(n)\) (Sect. 6) |

Table 1 This table shows the different parts of the algorithm to compute a tree decomposition (t.d.) for an \(n\)-vertex graph with treewidth \(k = O({\sqrt{n}})\), and their space requirements in bits.

## 2 Preliminaries

Let \(G = (V, E)\) be an undirected \(n\)-vertex \(m\)-edge graph. If it is helpful, we consider an edge \(\{u, v\}\) as two directed edges (called arcs) \((u, v)\) and \((v, u)\). As usual for graph algorithms we define \(V = \{1, \ldots, n\}\). For every vertex \(v \in V\), we access the degree of \(v\) through a function \(\text{deg}_G : V \to \mathbb{N}\) that returns the number of edges with an endpoint in \(v\). For \(A = \{(v, k) \in V \times \mathbb{N} | 1 \leq k \leq \text{deg}_G(v)\}\), let \(\text{head}_G : A \to V\) be a function such that \(\text{head}_G(v, k)\) returns the \(k\)th neighbor of \(v\). If space is not a concern, it is custom to store each graph that results from a transformation separately. To save space, we always use the given graph \(G\) and store only some auxiliary information that helps us to implement the following graph interface for a graph transformed.

**Definition 2.1. (graph interface)** A data structure for a graph \(G = (V, E)\) implements the graph interface (with adjacency arrays) exactly if it provides the functions \(\text{deg}_G : V \times \mathbb{N}\), \(\text{head}_G : A \to V\), where \(A = \{(v, k) \in V \times \mathbb{N} | 1 \leq k \leq \text{deg}_G(v)\}\), and gives access to the number \(n\) of vertices and the number \(m\) of edges.

Let \(G = (V, E)\) be a graph and let \(V' \subseteq V\). Unless stated otherwise, we assume that our input graphs always provide a graph interface with a mate function. During our computation, some of our graph interfaces can support \(\text{head}_G(v, k)\) and \(\text{deg}_G(v)\) only for vertices \(v \notin V'\). For vertices in \(V'\), we can access their neighbors via adjacency lists, i.e., we can use the functions \(\text{adjfirst} : V' \to P \cup \{\text{null}\}\), \(\text{adjhead} : P \to V\) and \(\text{adjnext} : P \to P \cup \{\text{null}\}\) for a set of pointers \(P\) to output the neighbors of a vertex \(v\) as follows: \(p := \text{adjfirst}(v)\);
while (p != null) { print adjhead(p); p := adjnext(p); }. We then say that we have a graph interface with \(|V|\) access-restricted vertices.

In an undirected graph it is common to store an edge at both endpoints, hence, every undirected edge \(\{u, v\}\) is stored as an arc (directed edge) \((u, v)\) at the endpoint \(u\) and as an arc \((v, u)\) at the endpoint \(v\). Call those two arcs mates of each other. For many algorithms and graph transformations it is useful to access the mate of an arc.

**Definition 2.2.** (mate function) A graph interface of a graph \(G\) supports the mate function if it provides the function \(\text{mate}_G : A \to A\) that, for a given adjacency entry \((v, k)\) \(\in A\), returns its mate \((u, j) \in A\) such that \(\text{head}_G(v, k) = u \land \text{head}_G(u, j) = v\). Note that \(\text{mate}_G(\text{mate}_G(v, k)) = (v, k)\).

We also use the two space-efficient data structures below.

**Definition 2.3.** (subgraph stack \([17]\)) The subgraph stack is a data structure initialized for an \(n\)-vertex \(m\)-edge undirected graph \(G_0 = (V, E)\) that manages a finite list \((G_0, \ldots, G_{\ell})\), called the client list, of ordered graphs such that \(G_i\) is a proper subgraph of \(G_{i-1}\) for \(0 < i \leq \ell\). It allows us to append (push) or remove (pop) graphs at the end of the client list. For each graph in the client list, the subgraph stack implements the graph access interface with the mate function. More exactly, a graph \(G_i = (V_i, E_i)\) on the client list has vertices \(V_i = 1, \ldots, n_i\) with \(n_i = |V_i|\) such that \(G_i\) is isomorphic to the subgraph it represents. This isomorphism is accessible via a translation function on the vertices.

Pushing a new subgraph \(G_{\ell+1}\) is done by calling the push operation with two parameters consisting of the bit vectors \(B_V\) and \(B_A\) which represent the vertices and arcs included in \(G_{\ell+1}\). \(B_V\) is of length \(|V_{\ell}|\) and \(B_A\) is of length \(2|E_{\ell}|\). Bits are set to 1 in the bit vector if the respective vertex or arc is included in \(G_{\ell+1}\). We extend the functionality of push to optionally only take \(B_V\) as a parameter, which means that the vertex induced subgraph is pushed.

The graph-access and transformation operations are evaluated in \(O(\log^* \ell)\) time. Translation operations are evaluated in \(O(\log^* \ell - \log^* i + 1)\) time when translating from \(G_\ell\) to \(G_i\) and vice versa, i.e., the running time is dependent on the distance between the graphs on the client list. To speedup the graph-access operations for \(G_\ell\) (and the translation operations from \(G_\ell\) to \(G_0\)) to \(O(1)\) time, an operation \(\text{toptime}\) can be called. It runs in \(O((n_i + m_i) \log^* \ell)\) time and uses \(O(n + m)\) bits. The entire subgraph stack occupies \(O(n + m)\) bits when the size of subsequent subgraphs on the client list shrinks by a constant factor \(0 < c < 1\).

**Definition 2.4.** (rank-select \([3]\)) Given access to an \(\ell\)-long-bit sequence \(B = (b_1, \ldots, b_\ell) = \{0, 1\}^\ell (\ell \in \mathbb{N})\) a rank-select structure supports an operation \(\text{rank}_B(j) = 1 \sum_{i=1}^{j} b_i (j \in \{1, \ldots, \ell\})\) that counts the number of bits set to 1 in \((b_1, \ldots, b_\ell)\), as well as an operation \(\text{select}_B(k) = \min\{j \in \{1, \ldots, \ell\} : \text{rank}_B(j) = k\}\) that returns the \(k\)th position of the \(k\)th bit set to 1 in \(B\). Both operations can be implemented to have constant time evaluation after \(O(\ell)\) time for an initialization of an auxiliary structure of \(o(\ell)\) bits.

We now formally define a tree decomposition.

**Definition 2.5.** (tree decomposition \([23]\), bag) A tree decomposition of a graph \(G = (V, E)\) is a pair \((T, B)\) where \(T = (W, F)\) is a tree and \(B\) is a mapping \(W \to \{V' \mid V' \subseteq V\}\) such that the following properties hold: (TD1) \(\bigcup_{w \in W} G[B(w)] = G\), and (TD2) for each vertex \(v \in V\), the nodes \(w\) with \(v \in B(w)\) induce a subtree in \(T\). For each node \(w \in W\), \(B(w)\) is called the bag of \(w\).

We recall from the introduction that the width of a tree decomposition is defined as the number of vertices in a largest bag minus 1 and the treewidth of a graph \(G\) as the minimum
width among all possible tree decompositions for \( G \). We subsequently also use the well known fact that an \( n \)-vertex graph \( G \) with treewidth \( k \) has \( O(kn) \) edges \cite{23}.

Our algorithms use space-efficient BFS and DFS. On \( n \)-vertex \( m \)-edge graphs there exists a BFS \cite{11} that runs in \( O(n + m) \) time using \( O(n) \) bits and a DFS \cite{7} that runs in \( O(m + n \log n) \) time using \( O(n) \) bits. The later result is only obtained by replacing a randomized dictionary \cite{7} Lemma 3 \cite{13} Corollary 5.3. The DFS assumes that we provide a graph interface with adjacency arrays. Subsequently, when we run a DFS on an \( n \)-vertex graph \( G \) with treewidth \( k \), we can only provide a graph interface with \( O(k) \) access-restricted vertices. Let \( V' \) be the set of access-restricted vertices. The reason why the DFS wants to use adjacency arrays is that it stores on its stack tuples of a vertex \( v \in V \) and an approximation of an index pointer into the adjacency array of \( v \), which is used to store the status of the current iteration over the neighbors of \( v \). We can modify the DFS to iterate over the neighbors of \( v' \in V' \) by storing a pointer for \( v' \) such that no adjacency-array access is needed for \( v' \). This results in a DFS using \( O(n + k \log n) \) bits instead of \( O(n) \), but does not affect the asymptotic runtime.

3. \( k \) Vertex-Disjoint Paths using \( O(n + k^2(\log k)^2 \log n) \) Bits

Let \( G = (V,E) \) be an \( n \)-vertex graph with treewidth \( k \leq n \) and let \( s,t \in V \). In this section we develop a so-called storage scheme that allows us to store a set \( P \) of \( \ell \leq k \) vertex-disjoint \( s-t \)-paths with \( O(n + k^2(\log k)^2 \log n) \) bits such that we can use a network-flow algorithm to increase the size of \( P \) by one. As we see later, the storage scheme might slightly change the given vertex disjoint \( s-t \)-path before storing them.

To store the paths with \( o(n \log n) \) bits, we start to number the vertices along each path with \( 1,2,3,1,2,3 \), etc. To avoid ambiguities when following a path we require that each vertex-disjoint path is chordless, i.e., only subsequent vertices of the path are connected by an edge. Using the standard approach for computing \( k \) vertex disjoint paths in a so-called residual network, together with making the paths chordless, we get easily the following lemma.

Lemma 3.1. \( \text{(Network-Flow Technique \cite{7})} \) Given an \( n \)-vertex \( m \)-edge graph \( G = (V,E) \), an integer \( k \), and two vertices \( s \in V \) and \( t \in V \) there is an algorithm that can compute up to \( k \) chordless vertex-disjoint paths from \( s \) to \( t \) using \( O((n + m) \log n) \) bits by executing \( k \) times a depth-first search, i.e., in \( O(k(m + n)) \) time.

Proof. The paths can be easily constructed by the standard network-flow technique. It remains to show to make the paths chordless. We call the subpath \( P' \subseteq P \) from \( u \) to \( v \) skippable exactly if there exists a chord from \( u \) to \( v \). The idea is to first construct an \( s-t \) path \( P \) with a DFS and directly after finding \( t \), to backtrack the DFS stack and remove skippable parts until we arrive at \( s \).

In detail, construct first an \( s-t \) path with a DFS. Then, start to backtrack from \( s \) to \( t \). We store the direction of the path from \( s \) to \( t \). During the backtracking of the DFS we mark the vertices in reverse order. We refer to this as adding a vertex to the path \( P \). To store \( P \) we use a bit vector with a bit at index \( v \) set to 1 exactly if \( v \) is on the path. The idea is to only add vertices of the original path to \( P \) if they are not skippable. For this we distinguish between two cases, initially starting in Case 1. In Case 1 we add vertices to \( P \), and in Case 2 we skip vertices. We stop when we backtracked through the entire original path. A chord can be identified as a back edge of the DFS.

Case 1 Add the vertex \( v \) the DFS currently visits to \( P \). Then, if \( v \) has no back edges, we continue to the next vertex. Otherwise add the endpoints of all back edges to an initially
empty set \( B \), switch to Case 2 and backtrack to the next vertex.

**Case 2** If the vertex \( v \) the DFS currently visits is contained in \( B \), remove it from \( B \). Now, if \( B = \emptyset \), add \( v \) to \( P \), backtrack to the next vertex and switch to Case 1. If \( B \neq \emptyset \), just backtrack to the next vertex.

Observe that we end Case 2 always after having found the chord spanning the largest distance. During Case 2 we ignore further back edges because at least one endpoint of that back edge will not be contained in our final path, i.e., skipping is ok.

To store \( B \) we use a bitset of size \( n \) that can be initialized in constant time and to check if \( B = \emptyset \) we count the number of elements added, decrementing it when we remove a vertex.

Numbering the vertices along each of the \( k \) vertex-disjoint paths does not uniquely define the chordless vertex-disjoint paths since there can be edges between the paths called **cross points** where the choice of the next vertex of a path is ambiguous (see Fig. 1). In detail, a cross point is a gadget between two paths consisting of either (1) a clique of four vertices where each path has two of these vertices or (2) an edge between the two paths. To avoid these ambiguous choices a straightforward idea is to color each vertex along a path with a color representing a path uniquely and so get a solution using \( O(n \log k) \) bits. However, we focus on a better space bound. Let \( V' \) be the set consisting of all vertices that belong to the chordless vertex-disjoint paths. Our idea is to color a fraction of vertices in \( V' \) and use this coloring to determine the paths in between colored vertices whenever we are interested in the exact routing of the paths. In detail, we color \( O(n/\log k) \) vertices of \( V' \) such that the following condition holds: by removing all colored vertices in \( G[V'] \) we get \( \Theta(n/(k \log k)) \) connected components called **regions** of size \( \Theta(k \log k) \) and all uncolored vertices part of a cross point are within the same region.

![Figure 1](image.png)

**Figure 1** This figure shows some vertex-disjoint paths from a vertex \( s \) to a vertex \( t \) stored in our storage scheme where \( O(n/\log k) \) vertices are colored. The edges between the black vertices define the cross points.

A region is separated by two **borders**, where each border consists of \( k \) colored vertices. Due to performance reasons, we additionally color all vertices that belong to a path and have large degree, i.e., a degree greater than \( k \log k \). Furthermore, we also color the neighbors of a large-degree vertex on its path. Intuitively speaking, we so introduce additional small borders at vertices of large degree. Since a graph with treewidth \( k \) has at most \( kn \) edges, at most \( \lceil n/\log k \rceil \) vertices can have degree greater than \( k \log k \) such that the whole storage scheme can be realized with \( O(n) \) bits. To sum up, our **storage scheme** consists of 1. numbering the vertices along a path using numbers out of \{1, 2, 3\}, 2. storing a color for \( \Theta(n/\log k) \) border vertices, and 3. storing a color for \( O(n/\log k) \) vertices of degree greater than \( k \log k \) and their two neighbors of the same path.
We next show that our storage scheme allows us to construct distinguishable paths within a region whenever we want to know the exact routing of the paths through the region. Whenever we want to know the paths in a region, the idea is to reconstruct first the region and then run a network-flow algorithm to construct the paths between the color borders.

Lemma 3.2. In an \( n \)-vertex graph \( G \) with treewidth \( k \), the storage scheme stores a set of \( \ell \leq k \) vertex-disjoint \( s \)-t-paths such that, for each vertex \( v \), we can answer the following question in \( O(\text{deg}(v) + \ell(k^2 \log^2 k)) \) time using \( O(k^2 (\log k)^2 \log n) \) bits: is \( v \) part of one of our vertex-disjoint paths and, if so, which of \( v \)'s incident edges are used to visit \( v \) and to leave \( v \) while moving from \( s \) to \( t \) on the path.

Proof. In the case that \( v \) is a large-degree vertex we answer the question by iterating over \( v \)'s adjacency list and searching for the two neighbors having the same color as \( v \). By the numbering, we know the incoming and outgoing edge of the path through \( v \).

For the remaining vertices, we need to determine the paths in the region to which \( v \) belongs. Thus, we need to explore the connected component containing \( v \) in that region including the border vertices and create the graph \( G' \) consisting of that connected component induced by these vertices. To explore the connected component we run a space-efficient BFS from \( v \). However, while the BFS iterates over the neighbors of a vertex, we ignore every vertex that is not part of any path. Since we want to ensure that \( G' \) looks always the same (no matter which vertex of the connected component we hit), we sort the vertices of \( G' \) and its adjacency arrays using an in-place linear time radix sort [12].

By the numbers on the paths, we know which vertices belong to a left border \( S' \) and a right border \( T' \). To fully construct \( G' \) we create two additional vertices \( s' \) and \( t' \) and connect them with the vertices of \( S' \) and \( T' \), respectively. By applying the deterministic algorithm from Lemma 3.1 on \( G' \) we always obtain the same paths for the connected component. Now we can answer the question of the incoming and outgoing neighbor by iterating over the neighborhood of \( v \) in \( G' \) and thus in \( G \).

By our storage scheme, \( G' \) consists of \( n' = \Theta(k \log k) \) vertices and \( m' = O((k \log k)^2) \) edges—recall that large-degree vertices belong to the border. Hence, constructing, storing \( G' \) and computing the vertex-disjoint paths can be done with \( O((n' + m') \log n) = O((k^2 (\log k)^2) \log n) \) bits.

We finally determine the running time to explore a connected component of a region and the construction of the paths. The construction is based on running a linear-time BFS that considers only the neighborhood of at most \( n' \) uncolored vertices. Thus, we have to consider only \( m' \) edges for computing the connected component and the construction of \( G' \), including radix sort, can be done in \( O(n' + m') \) time. The application of Lemma 3.1 costs us \( O(\ell(m' + n' \log^* n')) = O(\ell(k^2 \log^2 k))) \) time. If \( v \) is a large-degree vertex, we only have to iterate over its neighbors to find its two colored neighbors belonging to the same path. Summarized, the algorithm runs in \( O(\text{deg}(v) + \ell(k^2 \log^2 k)) \) time.

The next lemma shows that we can compute a storage scheme if the given paths have monotone cross points, i.e., there is a numbering of the cross points of all pairs of paths such that we can move over each path and the numbers of the seen cross points strictly increase. To compute the storage scheme, we basically have to compute the regions with the colored border vertices, which is done by counting vertices during a parallel run over the paths and coloring vertices whenever \( \Theta(k \log k) \) vertices have been seen. Our idea is sketched in Fig. 2.

Lemma 3.3. Let \( G = (V, E) \) be an \( n \)-vertex graph with treewidth \( k \) and \( s, t \in V \). Assume that \( \ell \leq k \) chordless vertex-disjoint \( s \)-t-paths with monotone cross points are given. The paths
can be given explicitly by a bit vector storing the vertices of the paths or can be stored in our storage scheme. Then, we can construct the storage scheme for $\ell$ chordless vertex-disjoint $s$-$t$-paths in $O(n\ell k^2 \log^2 k)$ time using $O(n + k^2 (\log k)^2 \log n)$ bits.

Assume that, for some $\ell \leq k$, we have already computed $\ell$ vertex-disjoint $s$-$t$-paths with monotone cross points and that the paths are stored in our storage scheme. Intuitively, the next lemma is a space-efficient version of the network-flow technique \cite{1} to increase the size of a set of vertex-disjoint $s$-$t$-paths by one.

\begin{lemma}
Given our storage scheme for a set of $\ell$ vertex-disjoint $s$-$t$-paths $P$ in $G$, $O(n(\log^* n)k^3 \log^2 k)$ time and $O(n + k^2 (\log k)^2 \log n)$ bits suffice to either compute an $(\ell+1)$th $s$-$t$-path $P^*$ such that $P^*$ has common vertices (called conflict points) with the $\ell$ paths only if $P^*$ runs at all conflict points in reverse over old paths (i.e., $P^*$ uses the edges in the residual network of $G$ and $P$), or output that no such path exists.

If $P^*$ is returned, then it is chordless and it is represented by a numbering 1, 2, 3, 1, 2, 3, etc. along the path in a bit vector $P$ of $2n$ bits.
\end{lemma}

\begin{proof}
We construct a kind of a residual network of $G$ and $P$ on the fly. In literature, a residual networks for edge-disjoint paths is well-known. In contrast, we want to have a residual network for vertex-disjoint paths. This means for the construction of a next path $P^*$ that, whenever we reach a vertex $v$ of $G$ that belongs to a path $P$ in $P$, we have to run to the predecessor of $v$ on $P$. Vertices $u$ and $v$ are conflict points. After finishing our construction of $P^*$, we can resolve the conflict points as shown in Fig. 3 and described in the next lemma.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{An illustration of the algorithm to remove conflict points.}
\end{figure}

\end{proof}

To avoid costly access to vertices of large degree over our storage scheme we change the graph presented to the DFS. We add a virtual vertex $w = n + 1$ to the graph, which the DFS always views as black, i.e., as already visited by the DFS.
When the DFS arrives at a vertex $v$ of large degree it wants to know the next possible vertex $u$ to consider, or backtrack if no such vertex exists. In the case that the vertex $v$ is part of a path $P \in \mathcal{P}$ and the vertex $v'$ on the DFS stack right below $v$ is not part of $P$, we want $u$ to be the successor of $v$ on $P$. For this we want that the DFS assumes that all neighbors of $u' \neq u$ are black, i.e., we virtually present vertex $w = n + 1$ for each vertex $u'$. We so have avoided that our storage scheme has to return directly the successor of $v$ on $P$. In all other cases, we use the storage scheme to present the neighbors in the residual network.

Then we construct a chordless path in the residual network as described in the proof of Lemma 3.1. If the construction of $P^*$ fails, we output that no such path exists.

Concerning the running time we can observe that whenever the DFS reaches a colored vertex $v$ of large degree we can present the neighbors in constant time without applying Lemma 3.2. For the remaining vertices we use Lemma 3.2 and pay with a factor of $O(k^3 \log^2 k)$ in the running time. Hence, we get a total running time of $O(n(\log^* n)k^3 \log^2 k)$. The total space used for this algorithm is $O(n + k^2(\log k)^2 \log n)$ bits since we have to apply Lemma 3.2 and store the path $P^*$ in an $O(n)$ bit vector.

We next want to reroute the paths in $\mathcal{P}$ and a path $P^*$ to remove non-monotone cross points as well as conflict points. Concerning the non-monotone cross points, our idea is to move over $P^*$ starting from $s$ to find the first vertex $u$ that is incident to a cross point between $P^*$ and some paths in $\mathcal{P}$. Starting from $u$, move over the paths in $\mathcal{P}$ backwards and search for further cross points, which are then not monotone with $u$. All the cross points found can be removed by a simple rerouting of the paths. For a sketch, see Fig. 4. In a similar way, conflict points can be cleared.

![Figure 4](image-url) Steps and data structures of the algorithm to remove non-monotone cross points.
Lemma 3.5. There is an algorithm to include a new chordless s-t-path $P^*$ into our storage scheme storing $\ell$ old s-t-paths $P$ such that, afterwards, the storage scheme stores $\ell + 1$ paths that have only monotone cross points and no conflict points. The algorithm runs in $O(n(\log^* n)k^3\log^2 k)$ time and uses $O(n + k^2 \log^2 \log n)$ bits.

Proof. We start to describe how to determine a pair of non-monotone cross points and compute a rerouting. Follow the path $P^*$ from the vertex $p = s$ and search for a cross point with any of the old $\ell$ paths. If we are unable to find one, then we are done with removing non-monotone cross points. Otherwise, assume that we have a cross point at some vertex $v$ with an old path $P^+$ as seen in Fig. 4b.

Since we want monotone cross points, $P^*$ is not allowed to have a cross point with any vertex that can be reached from $v$ as follows. Run $P^+$ backwards. Whenever we reach a cross point during the backward run, we continue running backwards at all paths touched by that cross point. For an illustration see at the orange vertices in Fig. 4b. All vertices that we reached are stored in an $n$-bit vector $C$, and are called orange vertices. Each cross point between an orange vertex and $P^*$ is a non-monotone cross point.

We are now interested in the last vertex $v$ of $P^*$ that is connected with an orange vertex and that does not belong to the same cross point. Note that $u$ and $v$ are then non-monotone cross points. Since we want to find $v$ in a time linear to the length of the subpath of $P^*$ from $p$ to $v$, we proceed as follows: during the exploration of the orange vertices for the vector $C$, whenever we hit a vertex that has a neighbor in $P^*$, i.e., whenever we hit a vertex that belongs to a non-monotone cross point, we add the neighbor into an initially empty choice dictionary [15, 19].

Afterwards, we start from vertex $u$ and follow the path $P^*$. Whenever we reach a vertex stored in the choice dictionary, remove it from the choice dictionary and check if it was the last vertex in it. If it was, we stop following $P^*$. The vertex of $P^*$ where we stopped is vertex $v$.

To fix the non-monotone cross-point with respect to $u$ and $v$, we want to find a red path $Q^*$ from $u$ to $v$ hitting all cross points of the $\ell$ paths where we can reroute the paths using only vertices stored in $C$. For an illustration see Fig. 4c: To find the path we run a slightly modified standard DFS from $u$. One idea is to force the DFS to first use edges that are part of a path in $P$. It is not easy to see that even if we forbid to switch to a path a second time, by the idea, we still find $v$. To avoid switching to a path for a second time, we use a bit vector of size $k$ to mark each path $P^+ \in P$ whenever we visit a first vertex of $P^+$. Furthermore, we can store the DFS stack with $O(k \log n)$ bits by storing the vertices and their edges that are used to switch from one path to another. After $v$ is found, the stack of the DFS contains all necessary information to describe a path from $u$ to $v$.

Note that, during the DFS, we can avoid to query our storage scheme on vertices $v$ of large degree. Instead, we iterate once over all neighbors to find the predecessor on the same path—recall that we run backwards over the paths—which the DFS visits first. Then, as usual, the DFS iterates over the remaining vertices. If we are done, we iterate again once over all neighbors to find the successor on the same path, which is the vertex to which we return (unless the stack tells us that we should switch back to another path). For the iteration over the neighbors of the remaining vertices $v'$, we need to know the colors of $v'$ and of the neighbors. We use Lemma 3.2 and pay the usage with an extra factor of $O(k^3 \log^2 k)$ in the runtime.

We next want to remove all non-monotone cross points on the orange path by rerouting the paths as sketched in Fig. 4d. Run backwards over the vertices on the stack and mark every visited vertex in a bit vector $Q$. Whenever reaching a vertex that is on the DFS stack,
pop the vertex from $S'$ with the edge $(w, w')$ that was taken to switch a path. Switch at a vertex $w$ from one path in $P$ to a vertex $w'$ of another path in $P$. We store in a switch vector $S$, realized by an ragged dictionary [14, Corolarry 5.3], the color of other paths at vertex $w$ and $w'$. Similar information is stored for $v$ and $u$. Since we forced our DFS to prioritize edges part of a path in $P$, we have to store for each red path at most $k$ times the colors for two vertices due to a switch. Thus, with each red path we extend $S$ by $O(k)$ entries.

Now the internal $u$-$v$-subpath of $P^*$ can be removed from $P^*$ and the numbering of each vertex $w \neq u, v$ of $P$, with $w$ being on the red path, can be cleared. Since the path can have additional non-monotone cross points, we repeat the steps above with $p = v$ as the new start vertex of the path $P^*$. Note that we do not need to search for a non-monotone cross point in the previously orange colored vertices. We therefore add all orange colored vertices into an initial zero bit vector $C^*$ and do not consider vertices in $C^*$ whenever we explore the orange vertices above.

The switch vector consists of entries of $O(\log k)$ bits each. Hence, after adding $\Theta(n / \log k)$ entries to $S$, we clean up the paths as follows. We apply Lemma 3.3 to store the rerouted path including the visited vertices of $P^*$ in a new storage scheme (Since Lemma 3.3 requires a set of $s$-$t$-paths, we connect the last seen vertex of $P^*$ with $t$ by possibly a new edge). After constructing the new storage scheme we continue with $p = v$ and repeat the whole process until we have visited all vertices of $P^*$.

Concerning the running time, we travel along $P^*$ as well as explore the orange vertices of $P$ only once. This time sums up to $O(kn)$ accesses to a vertex or an edge. To write a switch color in $S$ we need to determine the color of the paths, i.e., we need to determine the color for $O(n)$ vertices. Both can be done by Lemma 3.2 in $O(n(\log^* n)k^3 \log^2 k)$ total time. Computing a new storage scheme can be done in $O(n(\log^* n)k^2 \log^2 k)$ time. We do it every time we have collected up to $O(n / \log k)$ switch colors, i.e., at most $O(\log k)$ times for a total of $O(n(\log^* n)k^2 \log^3 k)$ time.

It is easy to see that all of our data structures including the modified standard DFS work with $O(n)$ bits. Applying Lemma 3.3 and 3.2 uses $O(n + k^2 \log^2 \log n)$ bits. Therefore, our algorithm for removing non-monotone cross points runs in $O(n(\log^* n)k^3 \log^2 k) + O(n(\log^* n)k^2 \log^3 k) = O(n(\log^* n)k^3 \log^2 k)$ time and uses $O(n + k^2 \log^2 \log n)$ bits.

We also have to resolve the conflict points. Similar as for non-monotone cross points, we look for conflict points and store switch vertices in a switch vector $S'$ of $O(n)$ bits to resolve the conflict points by rerouting our paths. A sketch of the following algorithm is also shown in Fig. 2. Follow path $P^*$ until a conflict point is found, i.e., until the path uses a vertex $u$ of some other path $P^+$. Store the color of $P^+$ with vertex $u'$ in $S'$ and proceed following $P^*$ until a vertex $v$ where the path $P^*$ is about to leave $P^+$. Store the color of $P^+$ with vertex $v'$ in $S'$. Proceed on $P^*$ and repeat this process until $S'$ contains $\Theta(n / \log k)$ entries. No we compute a new storage scheme by considering the switch vector $S'$ and repeat this process until the whole path $P^*$ is processed. The time and space consumption is the same as for removing non-monotone cross points. Hence, our asymptotic bounds do not change. ▶

Repeating Lemma 3.4 and 3.5 several times we can conclude our final theorem for computing vertex-disjoint paths.

\textbf{Theorem 3.6.} Given an $n$-vertex graph $G = (V, E)$ with treewidth $k$ and two vertices $s \in V$ and $t \in V$ there is an algorithm that can compute up to $k$ chordless vertex-disjoint paths from $s$ to $t$ in $O(n(\log^* n)k^2 \log^2 k)$ time using $O(n + k^2 (\log k)^2 \log n)$ bits.
Knowing a maximum set of vertex-disjoint paths between two vertices $s$ and $t$, we can easily construct a vertex separator for $s$ and $t$.

**Corollary 3.7.** Given an $n$-vertex graph $G = (V, E)$ with treewidth $k$, and two vertices $s \in V$ and $t \in V$, $O(n(\log^* n)k^4 \log^2 k)$ time and $O(n + k^2(\log k)^2 \log n)$ bits suffice to construct a bit vector $S$ marking all vertices of a vertex separator for $s$ and $t$.

**Proof.** We use a standard approach, where we explore the graph and mark every explored vertex, but never cross any vertex part of a vertex-disjoint path. The vertices part of a vertex-disjoint path that have an unmarked neighbor are selected for the vertex separator.

In detail, construct at most $k$ vertex-disjoint paths. Run a space-efficient BFS from $s$ and, whenever the DFS visits a vertex $u$, mark $u$ in a bit vector $B$. If $u$ is a vertex that belongs to any of the vertex-disjoint paths, store $u$ in a choice dictionary $C$, backtrack from $u$, and continue as described. Now, iterate over the vertices $v$ in $C$ and over the neighbors of $v$. If $v$ has a neighbor $w$ that is not marked in $B$, then mark $v$ in a bit vector $S$, that marks all vertices part of the vertex separator. Finally, return $S$.

A BFS and an iteration over the members of $C$ run in $O(n + m)$ time. Moreover, the choice dictionary and the bit use $O(n)$ bits. The time and space bound is negligible to the time and space bound of the computation of at most $k$-vertex disjoint paths, such runs in $O(n(\log^* n)k^4 \log^2 k)$ time using $O(n + k^2(\log k)^2 \log n)$ bits.

Practical applications that use treewidth algorithms have graphs with treewidth $k = O(\sqrt[4]{n})$, and then our space consumption is $O(n)$ bits.

## 4 Reed’s Algorithm

In this section we first sketch Reed’s algorithm to compute a tree decomposition and then the computation of a balanced X-separator, i.e., a set of vertices whose removal splits the graph in at least two connected components. In the following sections, we modify his algorithm to make it space efficient.

Reed’s algorithm [22] takes an undirected $n$-vertex $m$-edge graph $G = (V, E)$ with treewidth $k$ and an initially empty vertex set $X$ as input and outputs a balanced tree decomposition of width $8k + 6$. If $n \leq 8k + 6$, we return a tree decomposition $(T, B)$ consisting of a tree with one node $r$ (the root node) and a mapping $B$ with $B(r) = V$. Otherwise, we search for a so-called balanced $X$-separator $S$ of size $2k + 2$ that divides $G$ such that $G[V \setminus S]$ consists of $x \geq 2$ vertex-disjoint connected components $\Gamma = \{G_1, \ldots, G_x\}$. Then, we create a new tree $T$ with a root node $r$, a mapping $B$, and set $B(r)$ to $X \cup S$. For each graph $G_i \in \Gamma$ with $1 \leq i \leq x$, we proceed recursively with $G' = G_i[V(G_i) \cup S]$ and $X' = (X \cap V(G')) \cup S$. Every recursive call returns a tree decomposition $(T_i, B_i)$ ($i = 1, \ldots, x$). We connect the root of $T_i$ to $r$, we then set $B(w) = B_i(w)$ for all nodes $w \in T_i$. After processing all elements of $\Gamma$, return the tree decomposition $(T, B)$.

Since a balanced $X$-separator is used, the tree has a depth of $O(\log n)$, and thus there are at most $O(\log n)$ stack frames on the call stack—each stack frame is associated with a node $w$ of $T$. A standard implementation of the algorithm needs a new graph structure for each recursive call. In the worst-case, each of these graphs contains $2/3$ of the vertices of the previous graph. Thus, the graphs on the stack frame use $\Theta((n + m) \log n) = \Theta(kn \log n)$ bits. Storing the tree decomposition $(T, B)$ requires $\Theta(kn \log n)$ bits as well. The various other structures needed can be realized within the same space bound. In conclusion, a standard implementation of Reed’s algorithm requires $\Theta(kn \log n)$ bits.
Lemma 4.1. Given an \( n \)-vertex graph \( G = (V, E) \) with treewidth \( k \) and \( X \subseteq V \), there is an algorithm for finding a balanced \( X \)-separator of size \( 2k + 2 \) in \( G \) that runs in \( O(c^k \log^* n) \) time and uses \( O(n + k^2 \log k)^2 \log n \) bits. For some constant \( c \), the algorithm searches a set consisting of \( k \) vertex-disjoint paths \( c \) times and executes \( O(1) \) extra DFS.

Proof. We now sketch Reed’s ideas to compute a balanced \( X \)-separator. For a graph \( G = (V, E) \), an \( X \)-separator is a set \( S \subseteq V \) such that \( S \) separates \( X \) among the connected components of \( G[V \setminus S] \) such that no component contains more than \( 2/3|X| \) vertices of \( X \). A balanced \( X \)-separator \( S \) is an \( X \)-separator with the additional property that no component of \( G[V \setminus S] \) contains more than \( 2/3|V| \) vertices. To compute a balanced \( X \)-separator we compute first an \( X \)-separator \( S_1 \). To make it balanced, we compute an additional \( R \)-separator \( S_2 \) where \( R \) is a set of vertices that is in some sense equally distributed in \( G \). Then \( S = S_1 \cup S_2 \) is a balanced \( X \)-separator.

Graph transformation and edge reversal are typically realized using \( \Theta(m) \) or even \( \Theta(m \log n) \) bits.

Reed computes an \( X \)-separator by iterating over all \( 3^{|X|} \) possibilities to split \( X \) into three vertex disjoints sets \( X_1, X_2 \subseteq V \) and \( X_S \) with \( |X_S| \leq k \) and \( X_1, X_2 \leq \max\{k, 2/3|X|\} \). For each iteration compute vertex disjoint paths to find a separator \( S \) and check if \( X_S \subseteq S \) holds.

We now shortly describe Reed’s computation of the set \( R \). Run a DFS on the graph \( G \) and compute in a bottom up process for each vertex \( v \in G \) of the resulting DFS tree the number of descendants of \( v \). Whenever this number exceeds \( n/(8k + 6) \), add \( v \) to the initially empty set \( R \) and reset the number of descendants of \( v \) to zero. At the end of the DFS, the set \( R \) consist of at most \( 8k + 6 \) vertices, which can be used to compute a balanced \( X \)-separator as described above. By doing the same to compute an \( R \)-separator we so get a running time of \( O((3^{|X|} + 2^{15.7k})k(m + n \log^* n)) = O((3^{|X|} + 2^{15.7k})(k^2n + kn \log^* n)) \) by using \( O(n \log k) \) bits.

It remains to show how the set \( R \) is computed in \( O(m + n \log^* n) \) time using \( O(n) \) bits.

The idea is to use a balanced parenthesis representation for the DFS-tree used during the computation. The representation allows us to compute for each node \( w \) of the tree the position within an \( n \)-bit vector where we can store the number of descendants of \( w \) as a self-delimiting number.

For the next lemma, we describe its runtime additionally by a number of DFS runs because we later use the lemma on a graph interface that changes the runtime of the DFS.

To prove the lemma, we use the following observation. Whenever the number of descendants for a node \( u \) is computed, the numbers of \( u \)’s children are not required anymore. The idea is to use a so-called balanced parentheses representation of the DFS tree to manage the memory and to store all needed information. Moreover, we use self-delimiting numbers and reuse the space of the children of a node \( u \) to store its number. This works since a balanced parentheses representation of a tree consists of an open parenthesis for every node, followed by the balanced parentheses representation of the subtrees of every child, and a matching closed parenthesis.

In other words, if the open parenthesis for a vertex \( v \) with \( x \) descendants is at position \( i \) and its closed parenthesis is at position \( j \), then the difference between \( i \) and \( j \) is \( 2x \). To store \( x \) as a self-delimiting number requires \( 2[\log \log x] + 1 + [\log x] < 2x \) bits.

To construct \( R \) we run a space-efficient DFS of \( O(n) \) bits in \( O(m + n \log^* n) \) time twice, first to construct a balanced parentheses representation of the DFS tree, which is used to compute the descendants of each vertex in the DFS tree and so choose vertices for the set \( R \), and a second time to translate the labels of the chosen vertices since the balanced parentheses representation is an ordinal tree, i.e., we lose our original vertex labels and the vertices get a numbering in the order the DFS visited the vertices. However, after choosing the vertices
that belong to the set \( R \) and marking them in a bit vector \( R' \) we run the DFS again and create a bit vector \( R^* \) that marks every vertex \( v \) that the DFS visits as the \( i \)th vertex if and only if \( i \) is marked in \( R' \).

It remains to show how to compute the bit vector \( R' \). Let \( P \) be a bit vector of \( 2n \) bits storing the balanced parentheses representation, and let \( A \) be a bit vector of \( 2n \) bits that we use to store the numbers of descendants for some vertices. Note that a leaf is identified by an immediately closed parenthesis. Moreover, since the balanced representation is computed via a DFS in pre-order, we will visit the vertices by running through \( P \) in the same order. Note that Munro and Raman [21] showed a succinct data structure for balanced representation that initializes in \( O(n) \) time and allows to compute the position of a matching parenthesis, i.e., given an index \( i \) of an open (closed) parenthesis there is an operation \( \text{findclose}(i) \) (\( \text{findopen}(i) \)) that returns the position \( j \) of the matching closed (open) parenthesis.

The algorithm starts in Case 1 with \( i = 1 \) \((i \in \{1, \ldots, 2n\})\).

**Case 1** Iterate over \( P \) until a leaf is found at position \( i \), i.e., a find an \( i \) with \( P[i] = 0 \wedge P[i + 1] = 1 \). Since we found a leaf we write a 0 as a self-delimiting number in \( A[i, i + 1] \). Set \( i := i + 2 \) and check if \( P[i] = 1 \). If so, move to Case 2, otherwise repeat Case 1.

**Case 2** At position \( i \) is a closing parenthesis, i.e., \( P[i] = 1 \). In this case we reached the end of a subtree with \( j = \text{findopen}(i) \) being the position of a corresponding open parenthesis. That means we have already computed all numbers for the whole subtree.

By the parenthesis representation we know that the number of children in this subtree is \( c = (i - j)/2 \). Using an integer variable \( x \), sum up all the self-delimiting numbers in \( A[j + 1, i - 1] \). Check if the sum \( x + c \) exceeds \( \ell \). If it does write 0 as a self-delimiting number in \( A[j, i] \) and set \( R'[j] = 1 \), otherwise write \( x + c \) in \( A[j, i] \). Note that we store only one self-delimiting number between an open the matching closed parenthesis and this number does not necessary occupy the whole space available. Hence, using \( \text{findclose} \) operation we jump to the end of the space that is reserved for a number and start reading the second.

After writing the number we set \( i := i + 1 \). We end the algorithm if \( i \) is out of \( P \), otherwise we check in which case we fall next and proceed with it.

This completes the proof of the lemma.

\section{5 Iterator for Tree Decompositions using \( O(kn) \) bits}

We now introduce our iterator by showing a data structure, which we call tree-decomposition iterator. We think of it as an agent moving through a tree decomposition \((T, B)\), one node at a time in a specific order. We implement such an agent to traverse \( T \) in the order of an Euler-traversal and, when visiting some node \( w \in T \), being able to return the tuple \((B(w), d_w)\) with \( d_w \) being the depth of the node \( w \).

The tree-decomposition iterator provides the following operations:

- \( \text{init}(G, k) \): Initializes the structure for an undirected \( n \)-vertex graph \( G \) with treewidth \( k \).
- \( \text{next} \): Moves the agent to the next node according to an Euler-traversal and returns true unless the traversal of \( T \) has already finished. In that case, it returns false.
- \( \text{show} \): Returns the tuple \((B(w), d_w)\) of the node \( w \) where the agent is currently positioned.

We refer to initializing such an iterator and using it to iterate (call \( \text{show}() \) after every call of \( \text{next()} \)) over the entire tree decomposition \((T, B)\) of a graph \( G \) as iterating over a
tree-decomposition \((T, B)\) of \(G\). Our goal in this section is to use \(O(kn)\) bits to iterate over the bags of a tree decomposition in time \(c^k n \log n \log^* n\) for some constant \(c\).

To save space, we are often working with bit vectors and therefore assume that the (vertex) sets used in the following lemma and subsequent proof are bit vectors with a bit at index \(i\) set to 1 exactly if \(i\) is contained in the set. We refer to a data structure that implements the functionality of this lemma as a connected-component finder.

\[\textbf{Lemma 5.1.}\] Given an undirected \(n\)-vertex graph \(G = (V, E)\) and a vertex set \(S \subseteq V\), there is an algorithm that finds all connected components of \(G[V \setminus S]\) and the number of vertices contained in each component. The algorithm runs in \(O(n + m)\) time and uses \(O(n)\) bits and outputs a component as a set \(C\) of vertices, i.e., an \(n\)-bit vector.

\textbf{Proof.}\ As auxiliary structures we store in a vertex set \(A\) all vertices of the connected components which have been found and a pin, which is a pointer to some vertex contained in a not-yet-found connected component of \(G\). The vertex set \(A\) together with the pin stores the state of the algorithm, i.e., they allow us to collect the next connected component and find the size of the next connected component. Initially \(A\) contains no vertices, i.e., \(A\) is a bit vector of size \(n\) with all bits set to 0. We initialize the pin to be the first vertex in \(V\) that is not contained in \(S\) or \(A\). To realize this we iterate over \(V\) until such a vertex is found. We refer to this as updating the pin. If the pin can not be updated, there are no more connected components to be found, and we set the pin to null. To find the next connected component start a BFS at the pin. The BFS is not allowed to traverse a vertex contained in \(S\) and each time a new vertex \(v\) is visited, add \(v\) to \(A\). Once finished, update the pin to a not-yet-found connected component if possible. To determine the size of the next connected component skip the updating process and simply count and return the number of vertices visited. Updating the pin until it can not be updated anymore runs in \(O(n)\) time. Running a BFS over all vertices runs in \(O(n + m)\) total time and uses \(O(n)\) bits. Storing the vertex set \(A\) uses \(O(n)\) bits and the pin \(O(\log n)\) bits. Thus, we can find all connected in \(O(n + m)\) time and \(O(n)\) bits.

To turn Reed’s recursive algorithm into an iterative version, we use a stack structure called record-stack that manages a set of data structures to determine the current state of the algorithm. Informally, the record-stack allows us to pause Reed’s algorithm at specific time-points and continue from the last paused point. With each recursive call of Reed’s algorithm we need the following information: an undirected \(n_i\)-vertex graph \(G_i = (V_i, E_i)\) \((i = 0, 1, 2, \ldots)\) of treewidth \(k\), a vertex set \(X_i\), a separator \(S_i\), an instance \(F_i\) of the connected-component-finder data structure that iterates over the connected components of \(G_i \setminus S_i\) and outputs the vertices of each component in a bit vector. We call the combination of these elements a record. Although we use a single record-stack structure, often we think of the record-stack to be a combination of specialized stack structures: a subgraph-stack, which manages to store the recursive graphs used as a parameter for the call of Reed’s algorithm, a stack for iterating over the connected components of \(G[V \setminus S]\), called component-finder stack, a stack containing the separators as bit vectors, called \(S\)-stack, a stack containing the vertex sets \(X\) as bit vectors, called \(X\)-stack. The bit vectors \(S_i\), \(X_i\) and \(F_i\) contain information referring to \(G_i\) and are thus of size \(O(n_i)\). On top of \(S_i\) and \(X_i\) we create rank-select data structures. We require these structures to calculate the bag associated with the current record in \(O(k)\) time. Pushing a record \(r_{\ell+1} = (G_{\ell+1}, S_{\ell+1}, X_{\ell+1}, F_{\ell+1})\) to the record-stack is equivalent to pushing each element in \(r_{\ell+1}\) to the corresponding stack (and analogous for popping).
Lemma 5.2. When a record-stack $R$ is initialized for an undirected $n$-vertex graph $G$ with treewidth $k$ such that each subgraph $G_i$ of $G_0 = G$ on the subgraph-stack of $R$ contains $2/3$ of the vertices of $G_{i-1}$ for $0 < i < \ell$ and $\ell = O(\log n)$, then the record-stack occupies $O(kn + k \log n) = O(kn)$ bits.

Proof. We know that the size of the subgraph-stack structure is $O(n + m)$ bits when the size of the subgraphs shrink with every push by a factor $0 < c < 1$. Since each subgraph of $G_0$ has also a treewidth $k$, the number of edges of each subgraph is bound by $k$ times the number of vertices. Thus, the subgraph stack uses $O(n + m) = O(kn)$ bits. The size of the bit vectors $X_i$, $S_i$ (including the respective rank-select structures) and the component-finder $F_i$ is $O(n_i)$ for $0 \leq i \leq \ell$. This means the total size of the stacks containing these elements is $O(n)$ bits since they shrink in the same way as the vertex sets of the subgraphs. Storing the bag that is currently being output uses $O(k \log n)$ bits. Thus, the size of the record-stack is $O(kn + k \log n) = O(kn)$ bits.

We call a tree decomposition $(T, B)$ balanced if $T$ has logarithmic height, and binary if $T$ is binary. Using our structures and Lemma 4.1 for finding balanced $X$-separators we are now able to show the following theorem.

Theorem 5.3. Given an undirected $n$-vertex graph $G$ with treewidth $k$, there exists an iterator that outputs a balanced and binary tree decomposition $(T, B)$ of width $8k + 6$ in Euler-traversal order using $O(kn)$ bits and $c^2 n \log n \log^* n$ time for some constant $c$.

Proof. We use the tree-decomposition iterator structure to realize this and show the implementation of $\text{init}$, $\text{next}$ and $\text{show}$. When $\text{init}(G = (V, E))$ is called for a graph $G$ with $n > 8k + 6$ vertices, we initialize a flag $f = 0$, which indicates that the agent’s traversal is not yet finished, and a record-stack. The record stack is initialized by first initializing its subgraph stack with a reference to $G$ as the first graph $G_0$. Next, we push the empty vertex set $X_0$ on the $X$-stack in form of an initial-zero bit vector $X_0$ of length $n$. Now, using the techniques described in Lemma 4.1, we find a balanced $X_0$-separator $S_0$ of $G_0$ and push it on the $S$-stack. Then we create a new connected-component-finder instance $F_0$ (Lemma 5.1) and push $F_0$ on the component-finder stack. Since the tree $T$ of our tree decomposition does not exist as a real structure, we only virtually move the agent to the next node by advancing the state of Reed’s algorithm. For this we also store a boolean value $f = 0$ to indicate that the agent has not yet finished its traversal of $T$.

We now view our implementation of $\text{next}()$, which has the task to calculate the next bag on the fly. If $f = 1$, we return false (the agent can not be moved) and do not change the state of the record-stack. If $n_\ell \leq 8k + 6$, we pop the record stack (the agent is moving backwards from a leaf). Otherwise, we check if the connected-component-finder instance $F_\ell$ has finished its search. If this is the case and the record-stack contains more than one record, we pop it (the agent is moving backwards from a processed node). If the record-stack contains only one element, we set $f = 1$ (the agent’s traversal is finished). Then, we return true (the agent has moved).

If $F_\ell$ has not finished its search (the agent is moving to a previously untraversed node) we proceed as follows: first, we initialize a bit vector $C = [0_1, \ldots, 0_n]$, use $F_\ell$ to collect the next connected-component of $G_\ell[V_\ell \setminus S_\ell]$ in $C$, and push the vertex-induced subgraph of $C \cup S_\ell$ on the subgraph stack as $G_{\ell+1} = (V_{\ell+1}, E_{\ell+1})$. Now, if $n_\ell \leq 8k + 6$, we are calculating the bag of a leaf of $T$ by setting $B(w) = V_{\ell+1}$. We do this by pushing a bit vector with all bits set to 1 on the $S$-stack and $X$-stack and an empty component-finder on the component-finder stack.
If \( nt > 8k + 6 \), the agent is moving to an internal node and we are thus calculating the bag of an internal node. In this case, we proceed as follows: we push a new bit vector \( X_{t+1} = (X_t \cap V_{t+1}) \cup S_t \) on the \( X \)-stack. We then find the balanced \( X_{t+1} \)-separator \( S_{t+1} \) of \( G_{t+1} \) and push it on the \( S \)-stack. Then, create a new connected component-finder \( F_{t+1} \) for \( G_{t+1} \) and \( S_{t+1} \) and push it on the component-finder stack and return true. Anytime we pop or push a new record, we call the \text{toptune} function of the subgraph stack to speed up the graph-access operations.

We can now implement \text{show()} to simply return the tuple \((B(w), d_w)\) with \( B(w) \) being the current bag, and \( d_w \) being the number of records of the record-stack. The current bag is defined as \( S \cup X \). Thus, we iterate over all elements of \( S \cup X \) via their rank-select data structures. Note that, since the subgraph \( G_t \) on top of the record stack is toptuned, we can return the bag as containing elements of \( G_0 \) or \( G_t \) in \( O(k) \) time.

A quick analysis of the structure shows that the tree-decomposition iterator uses \( O(kn + k \log n) = \Theta(n) \) bits. The entire iterator only needs a record-stack structure which occupies \( O(kn) \) bits. For finding separators, we use the balanced \( X \)-separator search of Lemma 4.1 with its internal path construction algorithm. The DFS has a runtime of \( O(kn \log^* n) \) on an \( n \)-vertex graph \( G \) with treewidth \( k \). It uses \( O(n) \) bits since all input graphs of the DFS provide a graph interface. The total runtime over all separator searches is \( c^kn \log n \log^* n \), which makes the overhead for topline calls of the subgraph stack negligible. All other operations, such as finding all connected components, have runtime \( O(n \log n) \). We thus arrive at a runtime of \( c^kn \log n \log^* n \) for some constant \( c \).

Now, we only have to show how \( T \) can be made binary. The balanced \( X \)-separator \( S \) partitions \( V \setminus S \) into any number of vertex disjoint sets between 2 and \( n \) such that no set contains more than \( 2/3 \) of the vertices of \( V \) (and \( X \)). The idea is to combine these vertex sets into exactly two sets such that neither contains more than \( 2/3|V| \) vertices. For this we change our usage of the connected component finder slightly. Previously, before retrieving the next connected component with some connected component finder \( F_t \) (initialized for the graph \( G_t \) on top of the subgraph stack) we have allocated a new bit vector \( C \) of size \( nt \), with all bits set to 0, and then collected the next connected component in \( C \). Now, after we first initialize \( F_t \) we also initialize two bit vectors \( C_1 \) and \( C_2 \) of size \( nt \) each with all bits set to 0. We also store the number of bits set to 1 for each of the bit vectors as \( s_1 \) and \( s_2 \), i.e., the number of vertices contained in them (initially 0). We now want to collect all connected components of \( G_t[V_t \setminus S_t] \) in \( C_1 \) and \( C_2 \). While there are still connected components to be returned by \( F_t \), this is done by obtaining the size of the next connected component via \( F_t \) as \( s \). If \( s_1 + s \leq 2/3|V_t| \), we collect the next connected component in \( C_1 \) and increment \( s_1 = s_1 + s \). Otherwise, we do the same but for \( C_2 \) and \( s_2 \). Doing this until all connected components are found results in \( C_1 \) and \( C_2 \) to contain all connected components of \( G_t[V_t \setminus S_t] \). For \((C_1, C_2)\) we implement a function that returns \( C_1 \) if it was not yet returned, or \( C_2 \) if it was not yet returned, or null otherwise. We store \((C_1, C_2)\) with the respective functions on the connected component finder stack (instead of \( F_t \)). Anytime we do this during our iterator, the graph \( G_t \) is toptuned, resulting in constant time graph access operations. The previous runtime and space requirements still hold.

Often it is needed to access the subgraph \( G[B(w)] \) induced by a bag \( B(w) \) of a tree decomposition \((T, B)\) for further computations. We call such a subgraph \text{bag-induced}. For this we show the following:

\begin{itemize}
\item \textbf{Lemma 5.4.} Given an undirected \( n \)-vertex graph \( G \) with treewidth \( k \) and an iterator \( \mathcal{A} : G \rightarrow (T, B) \) that outputs a balanced tree decomposition of width \( O(k) \) we can additionally
\end{itemize}
output the bag-induced subgraphs using $O(k^2 \log n)$ bits additional space and $O(kn \log n)$ additional time.

Proof. To obtain the edges we create a bit matrix $M_l$ of size $O(k^2)$ with bits at index $[v][u]$ set to 1 exactly if there exists and edge $(v, u)$ in $G[B(w)]$. We create $M_l$ anytime a new record $r_l$ is pushed on the record stack, and anytime $r_l$ is popped, we throw away $M_l$. We can see that we store at most $O(\log n)$ matrices this way, since the record stack contains $O(\log n)$ records (i.e., the height of the tree decomposition). When we push a bit matrix $M_l$ we first use $M_{l-1}$ to initialize edges that were contained in $B(w)_{l-1}$ and are still contained in $B(w)_l$. Reason being, that to obtain the edges of vertices $v$ in $B(w)$, we have to iterate over all the edges that $v$ has in $G_l$ and check if both endpoints are in $B(w)$. Because of the definition (TD2) of a tree-decomposition we only have to iterate over the edges of such a vertex once, i.e., the first time they are contained in a bag, instead of every time we create $M_l$. To quickly find vertices of the respective bags we use rank-select data structures on $S$ and $X$. Storing all bit matrices uses $O(k^2 \log n)$ bits and initializing all bit matrices takes $O(kn \log n)$ time, including initializing and storing the rank-select structures, if not already present.

We conclude the section with a quick remark on the output scheme of our iterator. The specific order of an Euler-traversal encompasses many other orders of tree traversal such as pre-order, in-order or post-order. To achieve these orders we simply filter the output of our iterator, i.e., skip some output values.

6 Modifying the Record Stack to Work with $O(n + k^2 \log n)$ Bits

The space requirements of the record stack used by the iterator shown in Theorem 5.3 is $O(kn)$ bits. Assume that, for $\ell = O(\log n)$, a graph $G_\ell = (V_\ell, E_\ell)$ with $n_\ell$ vertices is on top of the record stack. When considering the record $r_\ell$ on top of the record stack we see that most structures use $O(n_\ell)$ bits: a separator $S_\ell$, a vertex set $X_\ell$ and a connected component finder $F_\ell$. The only structure that uses more space is the subgraph stack, which uses $O(kn_\ell)$ bits. This is due to the storage of the edge set $E_\ell$ using $O(kn_\ell)$ bits. The strategy we want to pursue is to store only the vertices of the subgraphs such that the space requirement of the subgraph stack is $O(n)$ bits. We call such a subgraph stack a minimal subgraph stack.

In the following we always assume that the number of subgraphs on the minimal subgraph stack is $O(\log n)$ and that the subgraphs shrink by a constant factor. This is in particular the case for the subgraphs generated by Reed’s algorithm.

In the following we make a distinction between complete and incomplete vertices. Complete vertices have all their original edges, i.e., they have the same degree in the original graph as they do in the subgraph. The number of incomplete vertices in each subgraph is $O(k)$, which follows directly from Reed’s algorithm. To clarify, a vertex in the subgraph $G_\ell$ on top of the subgraph stack is incomplete exactly if it is contained in a separator of the parent graphs $G_i$ with $0 \leq i < \ell$ and if it is still contained in $G_\ell$.

Lemma 6.1. Given an undirected $n$-vertex graph $G = (V, E)$ with treewidth $k$ and a toptuned minimal subgraph stack $(G_0 = G, \ldots, G_\ell)$ with $G_\ell$ containing $O(k)$ incomplete vertices, we can iterate over the arcs of all vertices in $G_\ell$ in $O(k^2 n_\ell)$ time. The modified subgraph stack can be realized with $O(n + k^2 \log n)$ bits and allows us to push an $n_{\ell+1}$-vertex graph $G_{\ell+1}$ on top of a minimal subgraph stack in $O(k^2 n_{\ell+1} \log^* \ell)$ time. The resulting graph interface allows us to access the adjacency array of the non-restricted vertices in constant time.
time whereas an iteration over the adjacency list of all restricted vertices runs in $O(k^2n)$ time.

**Proof.** Recall that we consider every edge as a pair of directed arcs. Let $\phi$ be the vertex translation between $G_\ell$ and $G_0$. Each complete vertex of $G_\ell = (V_\ell, E_\ell)$ has the same degree in $G_\ell$ and in $G_0$. Thus, to iterate over all arcs of a complete vertex $v \in V_\ell$, iterate over every arc $(\phi(v), u)$ of $\phi(v)$ and return the arc $(v, \phi^{-1}(u))$. Since $G_\ell$ contains $n_\ell - O(k)$ complete vertices and $O(kn_\ell)$ arcs with complete vertices at both endpoints, this can be done in $O(kn_\ell)$ time. To iterate over the arcs of an incomplete vertex $v$, we consider two cases: (1) the arcs to a complete vertex and (2) the arcs to another incomplete vertex. To iterate over all arcs of (1) we iterate over all complete vertices $u$ of $G_\ell$ and check in $G_0$ if $\phi(u)$ has an edge to $\phi(v)$. If it does, $(v, u)$ is an arc of $v$. Since there are at most $O(k)$ incomplete vertices, and iterating over all complete vertices runs in $O(kn_\ell)$ time, the iteration over all arcs of (1) runs in $O(k^2n_\ell)$ total time. For the arcs according to (2), proceed as follows.

Every time after a new graph $G_{\ell+1}$ is pushed on the subgraph stack, we create a bit matrix $M_{\ell+1}$ of size $k^2$ and a rank-select data structure $I$ of size $n_{\ell+1}$ with $I[v] = 1$ exactly if $v$ is incomplete. $M_{\ell+1}$ is used to store the information if $G_{\ell+1}$ contains an edge $(u_{\ell+1}, v_{\ell+1})$ between two incomplete vertices $u_{\ell+1}$ and $v_{\ell+1}$, which is the case exactly if $M[\text{rank}(u_{\ell+1})][\text{rank}(v_{\ell+1})] = 1$ and $M[\text{rank}(v_{\ell+1})][\text{rank}(u_{\ell+1})] = 1$. First, we initialize $M_{\ell+1}$ to contain only 0 for all bits. Then we use $M_\ell$ to find edges between incomplete vertices of $G_\ell$ and set the respective bits in $M_{\ell+1}$ to 1 if those incomplete vertices are still contained in $G_{\ell+1}$ (if $\ell = 0$, we set all bits to 0). Afterwards we are able to find edges between incomplete vertices that are incomplete in the previous graph as well. We still need to update $M_{\ell+1}$ to contain the information of the edges between the vertices that are complete in $G_\ell$, but are not complete in $G_{\ell+1}$. Since they are complete in $G_\ell$, we can simply iterate over all complete edges $e$ of $G_\ell$ in $O(kn_\ell)$ time and check if both endpoints of $e$ are incomplete in $G_{\ell+1}$ via $I$. If so, we set the respective bits in $M$ to 1. Queries on $M$ allow us to iterate over all arcs of (2) in $O(k^2)$ time. This results in a combined runtime of $O(kn_\ell k^2 + k^2 n_\ell + k^2) = O(k^2 n_\ell)$.

Storing all bit matrices $M$ uses $O(k^2 \log \ell)$ bits and the space used by the rank-select structures is negligible.

We can now support a graph interface with $O(k)$ restricted vertices. The adjacency arrays of the non-restricted vertices can be accessed in constant time whereas over the adjacency list of all restricted vertices runs in $O(k^2n)$ time. The list interface is realized by storing a pointer for each vertex. For constant time evaluation of the degree of the restricted vertices it is need to iterate over all restricted vertices once (in $O(k^2 n_\ell)$ time) to find the respective degree and store it. This uses a negligible additional $\Theta(k \log n)$ bits for implementing the interface. □

The last lemma allows us to store all recursive instances of Reed’s algorithm with $O(n)$ bits. We use the result in the next section to show our first $O(n)$-bit iterator to output a tree decomposition on graphs of small treewidth.

### 7 Iterator for Tree Decomposition using $O(n)$ Bits for $k = O(\sqrt[4]{n})$

In this section we show an iterator to output a tree decomposition of an $n$-vertex graph $G$ with treewidth $k = O(\sqrt[4]{n})$ using $O(n)$ bits. The space reduction is realized with small compromises in the runtime. To obtain the new result, we use well-known techniques for manipulating tree decompositions. For an introduction of these techniques, see [9].
By combining the iterator of Theorem 5.3 with our modified subgraph stack (Lemma 6.1) we are able to show the following theorem:

**Theorem 7.1.** There exists an iterator to output a balanced and binary tree decomposition \((T, B)\) of width \(8k + 6\) for an \(n\)-vertex graph \(G\) with treewidth \(k\) in Euler-traversal order using \(O(n + (k \log k)^2 \log n)\) bits and \(c^k n \log n \log^* n\) time for some constant \(c\). For \(k = O(\sqrt{n})\), our space consumption is \(O(n)\) bits.

**Proof.** Recall that the algorithm of Theorem 3.6 wants to find \(k\) vertex-disjoint paths. As described in the proof of Lemma 3.4, each path is constructed in such a way that colored vertices of large degree do not query the storage scheme for their neighbors on the paths. The same is true for the construction of the red path in the proof of Lemma 3.5. Therefore, the term \(\deg(v)\) in the runtime of Lemma 3.3 can be ignored.

Since the DFS has to access the graph through the graph interface of the minimal subgraph stack and our storage scheme (Lemma 3.3), we get an additional factor of \(O(k)\) and \(O(k^3 \log^k)\), respectively, in the runtime. Therefore, we can construct the \(\ell\)-disjoint paths in \(O(k(k^3 \log^2 k)(kn \log^* n))\) time. Note that it suffices to compute the set \(R\) on the graph induced by the complete vertices and the runtime for the computation of \(R\) is \(O(kn)\).

For constants \(c\) and \(d\), this results in a runtime of \(d^k (k^6 (\log k)^2 n \log^* n) = c^k n \log^* n\) when searching for a balanced \(X\)-separator for \(G\) (Lemma 4.1). We know that the runtime of calculating the entire balanced tree decomposition for \(G\) is based on instances consisting of \(O(n)\) total vertices in each recursion level and that the recursion depth is \(O(\log n)\). Therefore, our total runtime is \(c^k n \log n \log^* n\).

A recent algorithm by Bodlaender et al. finds a tree decomposition for a given \(n\)-vertex graph \(G\) of treewidth \(k\) in \(b^k n\) time for some constant \(b\) [4]. The resulting tree decomposition has a width of \(5k + 4\). The general strategy pursued by them is to first compute a tree decomposition of large width and then use dynamic programming on that tree decomposition to obtain the final tree decomposition of width \(5k + 4\). For an overview of the construction, we refer to [4] p. 3. The final tree decomposition is balanced due to the fact that its construction uses balanced \(X\)-separators at every second level, alternating between an \(8/9\)-balanced and an unbalanced \(X\)-separator [4] p. 26]. Further details of the construction of different kinds of the final tree decomposition can be found in [4] p. 20, and p. 39]. Since its runtime is \(b^k n\), it can write at most \(b^k n\) words and thus has a space requirement of \(b^k n \log n\) bits.

Our following idea is to use a hybrid approach to improve the runtime of our iterator. We first run our iterator (Theorem 7.1). Once the height of the record-stack of our tree-decomposition iterator is equal to \(z = b^k \log \log n\), the call of \(\text{next()}\) uses an unbalanced \(X\)-separator \(S^*\). This ensures that the size of the bag is at most \(4k + 2\) instead of \(8k + 6\). (We later add all vertices in the bag to all following bags.) Note that using a single unbalanced \(X\)-separator \(S^*\) on all root-to-leaf paths of our computed tree decomposition increases the height of the tree decomposition only by one. A following call of \(\text{next()}\) toptunes the graph \(G_t\) and then uses Bodlaender et al.’s linear-time tree-decomposition algorithm [4] to calculate a tree decomposition \((T', B')\) of an \(n_t\) vertex subgraph \(G_t\), which we then turn by folklore techniques into a binary tree decomposition \((T'', B'')\) by neither increasing the asymptotic size nor the width of the tree decomposition. In detail, this is done by repeatedly replacing all nodes \(w\) with more than two children by a node \(w_0\) with two children \(w_1\) and \(w_2\), with \(B'(w_0) = B'(w_1) = B'(w_2) = B(w)\), followed by adding the original children of \(w\) to \(w_1\) and \(w_2\), alternating between them both. To ensure property \((TD2)\) of a tree decomposition, we add the vertices in \(S^*\) to all bags of \((T'', B'')\). We so get a tree decomposition of the width \((5k + 4) + (4k + 2) = 9k + 6\).
Since $G_{\ell}$ contains $n_{\ell} = O(n/2^\ell) = O(n/(b^k \log n))$ vertices, the space usage of the linear-time tree-decomposition algorithm is $b^k n_{\ell} \log n = O(n)$ bits. The runtime of the algorithm is $b^k n_{\ell}$. Once we obtain $(T', B')$, we also need to transform each bag $b'$ of $B'$ since $B'$ contains mappings in relation to $G'$, but we want them to contain mappings in relation to $G$. This can be done in negligible time since $G'$ was touched before. We then initialize a tree-decomposition iterator $I$ for $(T', B')$ as described in the beginning of Subsection 5.

Now, as long as $I'$ has not finished its traversal of $(T', B')$, a call to $\text{next}$ on $I$ is equal to a call to $\text{next}$ on $I'$. Similarly, a call to $\text{show}$ on $I$ now returns the tuple $(B'(w), d_w)$ with $d_w$ being the depth of $w$ in $T'$ plus the size of the record stack of $I$. Once iterator $I'$ is finished, we throw away $(T', B')$. Then, the operations of $\text{next}$ and $\text{show}$ work normally on $I$ until the size of the record stack again is $O(b^k \log \log n)$ or until the iteration is finished. Since we use our iterator only to recursion depth $z$, our algorithm runs in $a^k n (\log^* n) z$ time for some constant $a$. The total runtime is $a^k n (\log^* n) (b^k \log \log n) + b^k n = c^k n \log \log n \log^* n$ for some constant $c$.

**Theorem 7.2.** There is an iterator to output a balanced binary tree-decomposition $(T, B)$ of width $9k + 6$ for an $n$-vertex $G = (V, E)$ with treewidth $k$ in Euler-traversal order in $c^k n \log \log n \log^* n$ time for some constant $c$ using $O(n + (k \log k)^2 \log n)$ bits. For $k = O(\sqrt[3]{n})$, the space consumption is $O(n)$ bits.

If we try to run our iterator from the last theorem on a graph that has a treewidth greater than $k$, then either the computation of a vertex separator or the computation of Bodleander et al.'s tree decomposition fails [4]. In both cases, our iterator stops and outputs that the treewidth of $G$ is larger than $k$.

## 8 Applications

As an example we first give an algorithm for vertex cover. Afterwards we conclude this section by giving a list of problems that can be solved with the same asymptotic time and space bound.

A vertex cover of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that, for each edge $(u, v) \in E$, $u \in C \lor v \in C$. For graphs with a small treewidth, one can find a minimum vertex cover by first computing a tree decomposition of the graph and then, using dynamic programming, calculate a minimal vertex cover. We start to sketch this standard approach.

Let $(T, B)$ be a tree decomposition of width $O(k)$ of an undirected graph $G$ with treewidth $k$. Now, iterate over $T$ in Euler-traversal order and, if a node $w$ is visited for the first time, calculate and store in a table $T_w$ all possible solutions of the vertex cover problem for $G[B(w)]$. Also store the value of each solution, which is equal to the number of vertices used for the cover. If the solution is not valid, store $\infty$ instead.

When visiting a node $w$ and $T_w$ already exists, we update $T_w$ by using $T_w$ with $w'$ being the node visited during the Euler-traversal right before $w$ ($w'$ is a child of $w$). The update process is done by comparing each solution $s$ in $T_w$ with each overlapping solution in $T_w$. A solution $s' \in T_w$ is chosen if it has the smallest value among overlapping solution. The value of $s'$ is added to the value of $s$, and the two solutions are linked with a pointer structure. Two solutions $s$ and $s'$ are overlapping exactly if, for each $v \in B(w) \cap B(w')$, $(v \in s \land v \not\in s') \lor (v \not\in s \land v \not\in s')$. Once the Euler-traversal is finished, the table $T_r$, with $r$ being the root of $T$, contains the size of the minimum vertex cover $C$ of $G$ as the smallest value of all solutions. This is the first step of the algorithm.

The second step is obtaining $C$, which is done by traversing top-down through all tables with the help of the pointer structures, starting at the solution with the smallest value in $T_r$. 
and adding the vertices used by the solutions to the initially empty set \(C\) if they are not yet contained in \(C\). The set of all tables connected via the pointer structure form a tree.

For an \(n\)-vertex graph \(G\) with treewidth \(k\) and a given tree decomposition \((T, B)\) of width \(O(k)\) the runtime of the algorithm is \(O(2^k n)\). A table \(T_w\) constructed for a bag \(B(w)\) consists of a bit vector of size \(O(k)\) for each of the \(O(2^k)\) possible solutions, and their respective values and pointer structures. This uses \(O(2^k (k + \log n)) = a^k \log n\) bits per table, for some constant \(a\). Thus, storing the tables for the entire tree decomposition uses \(O(a^k n \log n)\) bits. Our goal is to obtain the optimal vertex cover using only \(O(n)\) bits for both the tree decomposition \((T, B)\) and the storage of the tables. For obtaining only the size of \(C\), i.e., the first step of the algorithm, we only need to store the tables for the current root-node path of the tree decomposition iterator. The reason is that once a table has been used to update its parent table it is only needed for later obtaining the final cover via the pointer structures. We can iterate over a balanced binary tree decomposition of width \(O(k)\) in \(c^k n \log \log n \log^* n\) using \(O(n)\) bits (Theorem 7.2). To obtain the bag-induced subgraphs we use Lemma 5.4. We have to store \(O(\log n)\) tables, which results in \(O(a^k \log^2 n)\) bits used, which for \(k \leq \log_a n - 2 \log_a \log n\) equals \(O(n)\) bits \((a^k \log^2 n \leq n \Rightarrow a^k \leq n \log^2 n \Rightarrow k \leq \log_a n - 2 \log_a \log n)\). Initializing and updating all tables can be done in \(O(2^k n)\) time. From this we can conclude the following lemma:

**Lemma 8.1.** Given an \(n\)-vertex graph \(G\) with treewidth \(k \leq \log n - 2 \log \log n\) we can calculate the size of the optimal vertex cover \(C\) of \(G\) in \(O(c^k n \log \log n \log^* n)\) time using \(O(n)\) bits.

To obtain the final vertex cover we need access to all tables and bags they have been initially created for. Since we want to only use \(O(n)\) bits, we are not able to store all of them. We now use the previous lemma with some modifications to define an operation. In addition to storing the tables of the current root-node path we store all tables \(T_w\) with \(w\) having a depth \(d_w < \ell\) (\(\ell\) is specified later), which we call the upper tree, and all subtrees with a root at depths \(\ell\), a lower tree. We refer to this as the operation \(\text{vc}_1(G, \ell)\), which calculates and stores all tables of the upper tree of \(G\).

The first step is to call \(\text{vc}_1(G, \ell)\) followed by iterating over \((T, B)\) a second time. We then start to follow the pointer structures starting at \(T_r\) with \(r\) being the root of \((T, B)\). Anytime we output a new bag \(B(w')\) we move to \(T_{w'}\) via the pointer structure and process the vertices of the best solution in that table, together with the graph \(G[B[w']]\). When we arrive at a table and the pointer structure is invalid (because the next table does not exist) we call \(\text{vc}_1(G', \ell)\), with \(G'\) being the graph on top of the record stack used by our tree decomposition iterator (Section 5). Once the call is finished, we can continue to follow the pointer structures since the next tables now exist. We repeat this anytime we try to follow a pointer that is invalid until we arrive at a leaf (at which point we backtrack). When we have processed all tables of some upper tree, we can (recursively) throw away all tables in the subtrees below it since those tables have been processed by that point.

We want each upper tree to have a depth of \(\ell = \log \log n\), and thus contain \(O(\log n)\) tables. The maximum number of concurrent upper trees for which we store tables is \(\log n / \log \log n\). We thus need to store \(O(\log^2 n / \log \log n)\) tables. For \(k \leq \log_a n - 3 \log_a \log n\) this uses \(O(n)\) bits \((a^{k \log n / \log \log n} \leq n \Rightarrow a^k \leq n^{\log \log n / \log n} \Rightarrow k \leq \log_a n - 3 \log_a \log n + \log_a \log \log n)\). It remains to show the impact on runtime. Anytime we want to obtain the tables of a lower tree, we need to obtain the tables of all its lower tree, recursively. Thus the tables of the lower trees at the bottom need to be calculated \((\log n / \log \log n)-times\), the tables above that \((\log n / \log \log n - 1)-times\) and so forth. This can be thought of as iterating over the tree.
decomposition \((T, B)\) of \(G\) for \((\log n / \log \log n)\) times. Combined with the previous lemma we can conclude:

\begin{itemize}
  \item \textbf{Lemma 8.2.} Given an \(n\)-vertex graph \(G\) with treewidth \(k \leq \log_a n - 3 \log_a \log n = c' \log n\) for some constant \(0 < c' < 1\) we can calculate the optimal vertex cover \(C\) of \(G\) in \(c^k n \log^a n \log^* n\) time using \(O(n)\) bits for some constant \(c\).
\end{itemize}

From [9, Theorem 7.9] we know that there is an algorithm that solves all problems mentioned in Theorem 8.3 and many other monadic second-order problems on \(n\)-vertex graph with treewidth \(k \leq c' \log n\) for some constant \(0 < c' < 1\). The general strategy used for solving these problems is almost identical. First, traverse the tree decomposition bottom-up and compute a table for each node \(w\). The table stores the size of all best possible solutions in the graph induced by all bags belonging to nodes below \(w\) under certain conditions for the vertices in bag \(B(w)\). E.g., for Vertex Cover the table contains \(2^{k+1}\) solutions \((v \in B(w)\) does belong or does not belong to the solution) and for Dominating Set it contains \(4^{k+1}\) solutions (one additionally differs, if a vertex is already dominated or not). In general, the table has \(c^k\) solutions. For each possible solution, the table stores the size of the solution and thus uses \(O(\log n)\) bits. After the bottom-up traversal, the minimal/maximal solution size in the table at the root is the solution for the minimization/maximization problem, respectively. An optimal solution set can be obtained in a top-down traversal by using the tables.

It is clear that, for large \(k\), we cannot store all tables when trying to use \(O(n)\) bits. Our strategy is to store the tables only for the nodes on a single root-leaf path of the tree decomposition. For a balanced tree decomposition this results in \(O(c^k \log^2 n)\) bits used for these tables. Using this strategy we have all information to use the standard bottom-up traversal to compute the size of the solution for the given problem for \(G\).

To obtain an optimal solution set we need a balanced and binary tree decomposition that has an \(O(1)\)-approximation ratio.

\begin{itemize}
  \item \textbf{Theorem 8.3.} Let \(G\) be an \(n\)-vertex graph with treewidth \(k \leq c' \log n\) for some constant \(0 < c' < 1\). Using \(O(n)\) bits and \(c^k n \log n \log^* n\) time for some constant \(c\) we can solve the following problems: Vertex Cover, Independent Set, Dominating Set, Odd Cycle Transversal, MaxCut and \(q\)-Coloring.
\end{itemize}

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