A TYPE $F_\infty$ GROUP OF PIECEWISE PROJECTIVE HOMEOMORPHISMS.

YASH LODHA

Abstract. The purpose of this article is to prove that the group of homeomorphisms of the circle introduced in [15] is of type $F_\infty$. This provides the first example of a type $F_\infty$ group which is nonamenable and does not contain nonabelian free subgroups. We also obtain a normal form, or a canonical unique choice of words for the elements of the group.

1. Introduction

Recall that the von Neumann-Day problem asks whether every nonamenable group contains nonabelian free subgroups (see [20] and [11]). In 1980, Ol’shanskii solved the von Neumann-Day problem by producing a counterexample [18]. Soon after, Adyan showed that certain Burnside groups are also counterexamples [2, 3]. Ol’shanskii and Sapir constructed the first finitely presented counterexample in 2003 [19].

In [17] Monod gave a remarkable family of counterexamples to the von Neumann-Day problem. In particular he showed that the group of piecewise $\text{PSL}_2(\mathbb{R})$ homeomorphisms of $\mathbb{R}P^1$ that fix infinity is nonamenable and does not contain nonabelian free subgroups. However, Monod’s examples are not finitely presentable.

In [15] we constructed a finitely presented nonamenable subgroup of Monod’s group with 3 generators and 9 relations. The group is generated by $a(t) = t + 1$ together with the following two homeomorphisms of $\mathbb{R}$:

$$b(t) = \begin{cases} 
  t & \text{if } t \leq 0 \\
  \frac{1}{t} & \text{if } 0 \leq t \leq \frac{1}{2} \\
  3 - \frac{1}{t} & \text{if } \frac{1}{2} \leq t \leq 1 \\
  t + 1 & \text{if } 1 \leq t 
\end{cases}$$

$$c(t) = \begin{cases} 
  \frac{2t}{t+1} & \text{if } 0 \leq t \leq 1 \\
  \frac{1}{t} & \text{otherwise}
\end{cases}$$

It is worth noting that $a, b$ generate an isomorphic copy of Thompson’s group $F$.

In 1979 Geoghegan made the following conjectures about Thompson’s group $F$.

1. $F$ has type $F_\infty$.
2. $F$ has no nonabelian free subgroups.
3. $F$ is nonamenable.
4. All homotopy groups of $F$ at infinity are trivial.

Conjectures (1) and (4) were proved by Brown and Geoghegan [7]. Conjecture (2) was proved by Brin and Squier [8]. The status of (3) still remains open. There

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is considerable interest in Conjecture (3) especially because if $F$ is nonamenable it would be an elegant counterexample to the von Neumann-Day problem.

It is natural to inquire about the higher finiteness conditions satisfied by $\langle a, b, c \rangle$, especially in light of Geoghegan’s conjectures about Thompson’s group $F$. For a topologist, a finite group presentation describes a finite, connected, 2-dimensional CW complex. An Eilenberg-Maclane complex for a group $G$ is a connected CW complex $X$ which is aspherical and satisfies that $\pi_1(X) = G$. A group is said to be of type $F_\infty$ if it admits an Eilenberg-Maclane complex with finitely many cells in each dimension.

In this article we establish that this group is of type $F_\infty$. More particularly, we will show the following.

**Theorem 1.1.** The group $G = \langle a, b, c \rangle$ acts on a connected cell complex $X$ by cell permuting homeomorphisms such that the following holds.

1. $X$ is contractible.
2. The quotient $X/G$ has finitely many cells in each dimension.
3. The stabilizers of each cell are of type $F_\infty$.

It follows that the group $G$ is of type $F_\infty$.

The article is structured as follows.

- In Section 2 we will give background information on the following topics:
  1. Finiteness properties of groups.
  2. Nonpositively curved cube complexes.
  3. Vertex transitive graphs.
  4. An infinite presentation $\langle S \mid R \rangle$ of $\langle a, b, c \rangle$ and standard forms for the elements of the group which were introduced in [15]. The infinite presentation will be used throughout the article.

An effort has been made to make the article self contained, however the reader will benefit from reading [15] as preparation.

- In Section 3 we will describe a normal form for the presentation $\langle S \mid R \rangle$, i.e. a canonical unique choice of word for each element in the group. We will describe an algorithm to convert each word into a word in normal form using the relations. The normal forms will be used in the subsequent sections, but are also of independent interest.

- In Section 4 we will introduce the notion of a *special form*, which will serve as a basic building block for our complex. We will prove some basic properties about special forms which will be used throughout the article.

- In Section 5 we will build a CW-complex $X$ and an action of $G = \langle a, b, c \rangle$ on $X$. We will prove that the stabilizer of each cell is of type $F_\infty$, and that the quotient $X/G$ has finitely many cells in each dimension.

- In Section 6 we will prove that $X$ is simply connected.

The goal of Sections 7, 8, 9 will be to show that $\pi_i(X)$ is trivial for each $i > 1$.

- In Section 7 we will define an equivalence relation on the 1-cells of $X$ and also describe criteria to detect when two 1-cells are equivalent.

- In Section 8 we will describe various technical procedures that take as an input a finite set of 1-cells in $X$ and a set of 0-cells incident to them, and produce a new set of 1-cells and 0-cells in $X$. These procedures will be crucial in the proofs in Section 9.
In Section 9 we will prove that for every finite subcomplex $Y$ of $X$, there is a subcomplex $Y'$ of $X$ such that $Y \subseteq Y'$ and $Y'$ is homeomorphic to a nonpositively curved cube complex. We will conclude that the higher homotopy groups of $X$ are trivial.

2. Preliminaries

In this section will review some terminology needed later in the paper. Readers may wish to skim or skip the material and refer back to it only as necessary.

2.1. Finiteness properties of groups. The classical finiteness properties of groups are that of being finitely generated and finitely presented. These notions were generalized by C.T.C. Wall [21]. In this paper we are concerned with the properties type $F_n$. These properties are quasi-isometry invariants of groups [1]. In order to discuss these properties first we need to define Eilenberg-Maclane complexes.

An Eilenberg-Maclane complex for a group $G$, or a $K(G, 1)$, is a connected CW-complex $X$ such that $\pi_1(X) = G$ and $\tilde{X}$ is contractible. It is a fact that for any group $G$, there is an Eilenberg-Maclane complex $X$ which is unique up to homotopy type. A group is said to be of type $F_n$ if it has an Eilenberg-Maclane complex with a finite $n$-skeleton. Clearly, a group is finitely generated if and only if it is of type $F_1$, and finitely presented if and only if it is of type $F_2$. (For more details see [12].)

For a group $G$, consider the group ring $\mathbb{Z}G$. We view $\mathbb{Z}$ as a $\mathbb{Z}G$-module where the action of $G$ is trivial, i.e. $g \cdot 1 = 1$ for every $g \in G$. A module is called projective if it is the direct summand of a free module. The group $G$ is said to be of type $FP_n$ if there is a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$ (an exact sequence):

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z}$$

of the trivial $\mathbb{Z}G$ module $\mathbb{Z}$ such that for each $1 \leq i \leq n P_i$ is finitely generated as a $\mathbb{Z}G$ module.

A group is of type $FP_n$ if and only if it is finitely generated. If a group is of type $F_n$ then it is of type $FP_n$. In general for $n > 1$ type $FP_n$ does not imply type $F_n$. (There are examples due to Bestvina-Brady [5].) However, whenever a group is finitely presented and of type $FP_n$, it is also of type $F_n$. For a detailed exposition about homological finiteness properties, we refer the reader to [12].

The following is a special case of a well known result (Proposition 1.1 in [6]).

**Proposition 2.1.** Let $\Gamma$ be a group that acts on a cell complex $X$ by cell permuting homeomorphisms such that the following holds:

1. $X$ is acyclic, i.e. $\tilde{H}_n(X) = 0$ for each $n \geq 0$.
2. The quotient $X/\Gamma$ has finitely many cells in each dimension.
3. The stabilizers of each cell are of type $FP_\infty$.

Then $\Gamma$ is of type $FP_\infty$.

We remark that if a group is finitely presented and of type $FP_\infty$, then it is of type $F_\infty$. So for a finitely presented group $\Gamma$ we can replace type $FP_\infty$ with type $F_\infty$ in the above proposition. In our article we shall be concerned with the property $F_\infty$. It will be our goal to construct a CW-complex $X$ and an action of $G$ on $X$ which satisfies the hypothesis of the above proposition.
2.2. **CAT(0) Cube complexes.** Although the complex we construct in this article is not a cube complex, cube complexes will play an important role in the proof of contractibility of our complex.

By a *regular n-cube* $\square^n$ we mean a cube in $\mathbb{R}^n$ which is isometric to the cube $[0,1]^n$ in $\mathbb{R}^n$. Informally a cube complex is a cell complex of regular Euclidean cubes glued along their faces by isometries. More formally, a cube complex is a cell complex $X$ that satisfies the following conditions. The metric on such cube complexes is the piecewise Euclidean metric. (see [9] for details).

**Definition 2.2.** Given a face $f$ of a regular cube $\square^n$, let $x$ be the center of this face. The link $Lk(f, \square^n)$ is the set of unit tangent vectors at $x$ that are orthogonal to $f$ and point in $\square^n$. This is a subset of the unit sphere $S^{n-1}$ which is homeomorphic to a simplex of dimension $n - \dim(f) - 1$. This admits a natural spherical metric, in which the dihedral angles are right angles.

**Definition 2.3.** Let $f$ be a cell in $X$ and let $S = \{ C | C$ is a cube in $X$ that contains $f$ as a face $\}$

The link $Lk(f, X) = \bigcup_{C \in S} Lk(f, C)$. This is a complex of spherical “all right” simplices glued along their faces by isometries. This admits a natural piecewise spherical metric.

Given a geodesic metric space $X$, a triple $x, y, z \in X$ forms a geodesic triangle $\triangle(x, y, z)$ obtained by joining $x, y, z$ pairwise by geodesics. Let $\triangle_{\mathbb{E}^2}(x', y', z')$ be a Euclidean triangle such that the Euclidean distance between each pair $(x', y'), (x', z')$, $(y', z')$ is equal to the distance between the corresponding pair $(x, y), (x, z), (y, z)$. The triangle $\triangle_{\mathbb{E}^2}(x', y', z')$ is called a comparison triangle for $\triangle(x, y, z)$. If $p$ is a point on the geodesic joining $x, y$, there is a point $p'$ in the geodesic joining $x', y'$ such that $d_X(x, p) = d_{\mathbb{E}^2}(x', p')$. This is called a comparison point.

$X$ is said to be CAT(0) if for any geodesic triangle $\triangle(x, y, z)$ and a pair $p, q \in \triangle(x, y, z)$, the corresponding comparison points $p', q' \in \triangle_{\mathbb{E}^2}(x', y', z')$ have the property that $d_X(p, q) \leq d_{\mathbb{E}^2}(p', q')$. It is a well known fact that CAT(0) spaces are contractible.

Gromov gave the following characterization of CAT(0) cube complexes by combinatorial conditions on the links of vertices in [13]. A nice survey of these results can be found in [9].

**Definition 2.4.** A simplicial complex $Z$ is called a “flag” complex if any set $v_1, ..., v_n$ of vertices of $Z$ that are pairwise connected by an edge span a simplex. This is also known as the “no empty triangles” condition.

**Definition 2.5.** A cube complex $X$ is said to be nonpositively curved if the link of each vertex is a flag complex.

**Theorem 2.6.** (Gromov) A cube complex $X$ is CAT(0) if and only if it is nonpositively curved and simply connected.

It follows that any nonpositively curved cube complex $C$ has the property that the universal cover $\tilde{C}$ is contractible, and so in particular the group $\pi_i(C)$ is trivial for each $i > 1$. 
2.3. **Vertex transitive graphs.** The following is a natural generalization of Cayley graphs. Let $G$ be a group and $H$ be a subgroup of $G$. Let $S$ be a finite set of elements of $G$ such that:

1. $S \subseteq G \setminus H$.
2. For each $s \in S$ we have $s^{-1} \in S$.
3. $S \cup H$ generates $G$.

Then we can form the so called *coset graph* $\text{Cos}(G, H, S)$ as follows. The vertices of $\text{Cos}(G, H, S)$ are the right cosets of $H$ in $G$, and two cosets $Hg_1, Hg_2$ are connected by an edge if $g_1g_2^{-1} \in H(S)H = \cup_{s \in S}HsH$. Here $HsH = \{h_1sh_2 \mid h_1, h_2 \in H\}$ is a double coset.

The group $G$ acts on the graph $\text{Cos}(G, H, S)$ on the right, and this action is vertex transitive. In particular, the group of graph automorphisms of $\text{Cos}(G, H, S)$ is vertex transitive. By the following theorem (see Theorem 3.8 in [16]), in fact every vertex transitive graph can be described in this way.

**Theorem 2.7.** Let $\Omega$ be some vertex transitive graph. Then $\Omega$ is isomorphic to some coset graph $\text{Cos}(G, H, S)$.

The 1-skeleton of our complex will emerge as a vertex transitive graph in this way.

2.4. **Binary sequences.** We will take $\mathbb{N}$ to include 0; in particular all counting will start at 0. Let $2^\mathbb{N}$ denote the collection of all infinite binary sequences and let $2^{<\mathbb{N}}$ denote the collection of all finite binary sequences. If $i \in \mathbb{N}$ and $u$ is a binary sequence of length at least $i$, we will let $u \upharpoonright i$ denote the initial part of $u$ of length $i$. If $s$ and $t$ are finite binary sequences, then we will write $s \subseteq t$ if $s$ is an initial segment of $t$ and $s \subset t$ if $s$ is a proper initial segment of $t$. If neither $s \subseteq t$ nor $t \subseteq s$, then we will say that $s$ and $t$ are *incompatible*. The set $2^{<\mathbb{N}}$ is equipped with a lexicographic order defined by $s <_\text{lex} t$ if $t \subset s$ or $s$ and $t$ are incompatible and $s(i) < t(i)$ where $i$ is minimal such that $s(i) \neq t(i)$.

If $\xi$ and $\eta$ are infinite binary sequences, then we will say that $\xi$ and $\eta$ are *tail equivalent* if there are $s, t,$ and $\zeta$ such that $\xi = s \cdot \zeta$ and $\eta = t \cdot \zeta$.

2.5. **The $(a, b, c)$ group.** We direct the reader to the standard reference [4] for the definition and properties of Thompson’s group $F$; additional information can be found in [10]. We shall mostly follow the notation and conventions of [4] [8].

Now we recall the infinite presentation of group $(a, b, c)$ described in [15]. We start with the following two primitive functions:

$$
\xi \cdot x = \begin{cases} 
0\eta & \text{if } \xi = 00\eta \\
10\eta & \text{if } \xi = 01\eta \\
11\eta & \text{if } \xi = 1\eta
\end{cases}
\quad \xi \cdot y = \begin{cases} 
0(\eta.y) & \text{if } \xi = 00\eta \\
10(\eta.y^{-1}) & \text{if } \xi = 01\eta \\
11(\eta.y) & \text{if } \xi = 1\eta
\end{cases}
$$

From these functions, we define families of functions $x_s$ ($s \in 2^{<\mathbb{N}}$) and $y_s$ ($s \in 2^{<\mathbb{N}}$) which act just as $x$ and $y$, but localized to those binary sequences which extend $s$.

$$
\xi \cdot x_s = \begin{cases} 
s^{-1}(\eta.x) & \text{if } \xi = s^{-1}\eta \\
\xi & \text{otherwise}
\end{cases}
\quad \xi \cdot y_s = \begin{cases} 
s^{-1}(\eta.y) & \text{if } \xi = s^{-1}\eta \\
\xi & \text{otherwise}
\end{cases}
$$

If $s$ is the empty-string, it will be omitted as a subscript.
Let $S = \{y_s \mid s \in 2^{<\mathbb{N}}, s \neq 0^k, s \neq 1^k\}$. Our group is generated by functions in the set $S$.

We now list an infinite set of relations $R$ satisfied by the generators in $S$ such that $G = \langle S, R \rangle$.

(Here $s$ and $t$ are finite binary sequences.)

1. if $t.x_s$ is defined, then $x_t x_s = x_s x_t x_s$;
2. $x_s^2 = x_{s0} x_s x_s$;
3. if $t.x_s$ is defined, then $y_t x_s = x_s y_t x_s$;
4. if $s$ and $t$ are incompatible, then $y_s y_t = y_t y_s$;
5. $y_s = x_s y_{s0} y_{s10}^{-1} y_{s11}$.

Note that there is no occurrence of the functions $y_s^\pm$ where $s = 1^k$ or $0^k$.

We remark that the generators

$$\{x_t^i \mid s \in 2^{<\mathbb{N}}, t \in \mathbb{Z}\}$$

together with the relations

1. if $t.x_s$ is defined, then $x_t x_s = x_s x_t x_s$;
2. $x_s^2 = x_{s0} x_s x_s$;

describe an infinite presentation for Thompson’s group $F$.

We proved in [15] that:

**Theorem 2.8.** The group $G = \langle S \mid R \rangle$ is finitely presented, nonamenable, and does not contain nonabelian free subgroups.

Recall from [15] the following definitions.

**Definition 2.9.** An $X$-word is a word in the letters $\{x_t^i \mid s \in 2^{<\mathbb{N}}, t \in \mathbb{Z}\}$ of $F$. A $Y$-word is a word in the letters $\{y_s^i \mid s \in 2^{<\mathbb{N}}, s \text{ is nonconstant}, t \in \mathbb{Z}\}$. An $S$-word is a word in the generators $\{y_s^i, x_u^v \mid s, u \in 2^{<\mathbb{N}}, s \text{ is nonconstant}, t, v \in \mathbb{Z}\}$. The elements of the set $\{y_s^i \mid s \in 2^{<\mathbb{N}}, s \text{ is nonconstant}, t \in \mathbb{Z}\}$ are called percolating elements.

**Definition 2.10.** An $S$-word $\omega$ is in **standard form** if it is the concatenation of an $X$-word followed by a $Y$-word and whenever $\omega(i) = y_s^m$, $\omega(j) = y_t^n$, and $s \subseteq t$, then $j \leq i$. We will write **standard form** to mean an $S$-word in standard form. It is convenient to denote a standard form as $f\lambda$, where $f$ is an $X$-word and $\lambda$ is a $Y$-word in standard form. The **depth** of $f\lambda$ is the least $l$ such that there is binary sequence $s$ of length $l$ such that $y_s$ occurs in $f\lambda$ (if $\lambda$ is the empty word, i.e. $f\lambda$ is an $X$-word, then we say that $f\lambda$ has infinite depth).

Associated with a standard form $f\lambda$ and a sequence $\sigma \in 2^\omega$ is a **calculation**, which is a string in letters $y, y^{-1}, 0, 1$. The evaluation of $f\lambda$ on $\sigma$ comprises of a prefix replacement that is determined by the transformation $f \in F$ followed by an infinite sequence of applications of the transformations described by the percolating elements. The latter is encoded as an infinite string with letters $y, y^{-1}, 0, 1$. The calculation is equipped with the following substitutions:

- $y00 \rightarrow 0y$
- $y01 \rightarrow 10y^{-1}$
- $y1 \rightarrow 11y$
- $y^{-1}0 \rightarrow 00y^{-1}$
- $y^{-1}10 \rightarrow 01y$
- $y^{-1}11 \rightarrow 1y^{-1}$

For example, for the word $y_0^{-1}0y_1$ and the binary sequence 1001111... the calculation string is 10y0y^{-1}1111111.... The output of the evaluation of the word on the
binary string is the limit of the strings obtained from performing these substitutions. A calculation has a potential cancellation if upon performing a finite set of substitutions we encounter a substring of the form $yy^{-1}$ or $y^{-1}y$. When a calculation has no potential cancellations, we say that it has exponent $n$ if $n$ is the number of occurrences of the symbols $y^\pm$. Note that there is no potential cancellation in the example above, and the exponent is 2.

In what follows, we will say that an $S$-word $\Omega_1$ is derived from an $S$-word $\Omega_0$ if it is the result of applying permissible substitutions of the following forms:

1. (Rearranging) $y_i^j x_s^{\pm 1} \rightarrow x_s^{\pm 1} y_i^{j+1}$ where $s, t \in 2^{<\mathbb{N}}$ are such that $t \prec s$ is defined.
2. (Expansion) $y_i \rightarrow x_s y_s y_t^{-1} y_s^{-1}$ where $s \in 2^{<\mathbb{N}}$.
3. (Commuting) $y_u y_v \leftrightarrow y_v y_u$ where $u, v \in 2^\mathbb{N}$ and $u, v$ are incompatible.
4. $x^{i+j} \leftrightarrow x^i x^j$.
5. $y^{i+j} \leftrightarrow y^i y^j$.
6. (Cancellation) Delete an occurrence of $y^i y^{-i}$.

We will write this symbolically as $\Omega_0 \rightarrow \Omega_1$. Notice that each of these substitutions corresponds either to a relation in $R$ or to a group-theoretic identity.

The following lemma is extracted from the proof of finite presentability in [15] (see Lemmas 4.10 and 4.11 in [15]). We mention it here without proof.

**Lemma 2.11.** Let $f \lambda$ be a standard form, and let $\sigma \in 2^\mathbb{N}$ be a sequence such that the calculation of $f \lambda$ at $\sigma$ is non trivial with no potential cancellations, and exponent $n$. Then there is a rational interval $[\tau_0, \tau_1]$ on which the action of $f \lambda$ is an action of the form $h y_\sigma^n$ for some $h \in F, u \in 2^\mathbb{N}$. In particular, $f \lambda \notin F$.

### 3. Normal Forms

Recall that for a given group presentation, a normal form is a canonical, unique choice of a word representing each group element. In this section we shall describe a normal form for $\langle S \mid R \rangle$. This normal form will be useful in the sections to follow, but is also of independent interest since it can be used as a tool to study the group. We will also define a notion of percolating support of a normal form and prove some of its basic properties. This will be crucial in the proof that the higher homotopy groups of $X$ are trivial. First, recall the notion of a potential cancellation in a calculation from Section 2. We extend this notion to standard forms in the following definition.

**Definition 3.1.** (Potential cancellation) A calculation is said to have a potential cancellation if an application of a finite sequence of permissible substitutions produces an occurrence of a subword $yy^{-1}$ or $y^{-1}y$. A standard form $f y^{t_1}_s \ldots y^{t_n}_s$ is said to have a potential cancellation if there is an infinite binary sequence $\tau$ such that the calculation of $y^{t_1}_s \ldots y^{t_n}_s$ on $\tau$ contains a potential cancellation.

**Definition 3.2.** (Neighboring pairs) Let $f y^{t_1}_s \ldots y^{t_n}_s$ be a standard form. A pair $y^{t_i}_s, y^{t_j}_s$ is said to be a neighboring pair if:

1. $s_i \neq s_j$, and $s_i$ is an initial segment of $s_j$, i.e. $s_i \subset s_j$.
2. for any finite binary sequence $u$ such that $s_i \subset u \subset s_j$ we have that $u \neq s_k$ for all $1 \leq k \leq n$. 

Such a neighboring pair is said to contain a potential cancellation if the associated calculation of the subword $y_t^i y_s^j$ on some (or any) binary sequence $v$ that extends $s_j$ contains a potential cancellation.

**Lemma 3.3.** Any standard form can be converted to a standard form with no potential cancellations using the relations.

**Proof.** We prove this by induction on the number of percolating elements in the standard form. The base case is trivial. Let this be true for $t_i \in \{1, -1\}$. We remark that the condition for $t_i$’s is simply a matter of notation, i.e. for example we write $y_s^i$ as $y_i s_i$.

By the inductive hypothesis it follows that we can convert $f y_{s_1}^1 ... y_{s_{n-1}}^n$ into a standard form with no potential cancellations using the relations. Moreover, from Lemmas 4.4 and 4.9 in [15] it follows that this can further be converted into a standard form $g y_{p_1}^1 ... y_{p_m}^m$ with no potential cancellations and depth $l > |s_n|$. It follows that $g y_{p_1}^1 ... y_{p_m}^m y_{s_n}^n$ is a standard form.

Let $\tau$ be an infinite binary sequence for which the associated calculation is a potential cancellation. It follows that the potential cancellation must occur between the first two occurrences of $y^{\pm}$. in the associated calculation. In particular, the potential cancellation is produced by a neighboring pair $y_{s_n}^t y_{r_i}^t$ for some $1 \leq i \leq m$. It follows that we can convert $g y_{p_1}^1 ... y_{p_m}^m y_{s_n}^n$ into a standard form $h(y_{r_1}^l ... y_{r_l}^l)(y_{u_1}^l ... y_{u_o}^l) y_{s_n}^n$ such that

1. $|r_i| > |u_j|$ for each $1 \leq i \leq l, 1 \leq j \leq o$.
2. Each pair $y_{s_n}^t, y_{r_i}^t$ is a neighboring pair that has a potential cancellation.
3. No neighboring pair of the form $y_{s_n}^t, y_{r_i}^t$ contains a potential cancellation.

These conditions allow us to perform a sequence of expansion and rearranging substitutions on $y_{s_n}^t$, followed by cancellations to obtain a standard form with no potential cancellations.

**Definition 3.4.** A standard form is said to contain a potential contraction if either of the following holds.

1. It contains an occurrence of a subword of the form $y_{s_0} y_{s_{10}}^{-1} y_{s_{11}}$ but no occurrences of $y_{s1}^\pm$.
2. It contains an occurrence of a subword of the form $y_{s_0}^{-1} y_{s_01} y_{s_{11}}^{-1}$ but no occurrences of $y_{s10}^\pm$.

If a standard form $f y_{t_1}^1 ... y_{t_n}^n$ contains a potential contraction of the form $y_{s_0} y_{s_10}^{-1} y_{s_{11}}$ (but no occurrences of $y_{s_{11}}^\pm$), we can replace the subword with the word $x_s^{-1}$ and then apply a rearranging substitution to move the $x_s^{-1}$ to the left of the percolating elements. Note that since there are no occurrences of $y_{s_0}^\pm$ it follows that the resulting word is a standard form. We call this a contraction substitution. The contraction substitution for the other case is defined in a similar way.

**Lemma 3.5.** Any standard form can be converted into a standard form with no potential contractions and no potential cancellations.

**Proof.** By Lemma 3.3 we can convert our standard form into a standard form with no potential cancellations. On the resulting standard form, we perform contraction substitutions, one by one, until no contraction substitutions can be performed. This process terminates, since upon performing a contraction substitution we obtain a standard form which is smaller in the well founded ordering on standard forms.
that was defined in the proof of Lemma 5.6 in \cite{15} (see the second paragraph.)
Moreover, it is clear that we do not introduce any potential cancellations in this process. (This is a consequence of Lemma 5.9 in \cite{15}.)

\textbf{Lemma 3.6.} Let $y_{s_1}^{u_1}y_{s_2}^{u_2}$ be a standard form such that $v, t \in \{1, -1\}$. Now let $f y_{s_1}^{u_1} \cdots y_{s_n}^{u_n}$ be a standard form such that:

1. $f y_{s_1}^{u_1} \cdots y_{s_n}^{u_n} = (y_{u_1}^{v_1}y_{s_1}^{v_1})^{-1}$.
2. $f y_{s_1}^{u_1} \cdots y_{s_n}^{u_n}$ is obtained by performing only expansion substitutions on $y_{s_1}^{-1}$, and its offsprings, in the word $y_{s_1}^{-1}y_{u_1}^{-1}$.

If $y_{u_1}^{v_1}y_{s_1}^{v_1}$ has no potential cancellation, then $f y_{s_1}^{u_1} \cdots y_{s_n}^{u_n}$ has no potential cancellations.

\textbf{Proof.} By Lemma 5.9 in \cite{15} it suffices to show this for the following cases.

1. $t = 1, u = s_0, v = 1$.
2. $t = 1, u = s_0, v = -1$.
3. $t = -1, u = s_1, v = 1$.
4. $t = -1, u = s_1, v = -1$.

Further, by considering the involution $y \mapsto y^{-1}$ 0 $\mapsto 1$ described in \cite{15} (see the paragraph before Lemma 5.10 in \cite{15}), it suffices to show this for the first two cases.

Case (1): $(y_{s_0}y_{s_1})^{-1} = y_{s_1}^{-1}y_{s_0} = x_s^{-1}s^{-1}y_0s_0y_0s_1y_0s_1^{-1}y_0s_1^{-1}$ This word has no potential cancellations.

Case (2): $(y_{s_0}^{-1}y_{s_1})^{-1} = y_{s_1}^{-1}y_{s_0} = x_s^{-1}s^{-1}y_0s_0y_0s_1y_0s_1^{-1}y_0s_1^{-1}$ This word has no potential cancellations.

\textbf{Lemma 3.7.} Let $y_{s_1}^{u_1} \cdots y_{s_n}^{u_n}$ be a standard form with no potential cancellations. The process for converting the word $(y_{s_1}^{u_1} \cdots y_{s_n}^{u_n})^{-1} = y_{s_n}^{-t_n} \cdots y_{s_1}^{-t_1}$ into a standard form does not involve any cancellation substitution and produces a standard form with no potential cancellations.

\textbf{Proof.} We proceed by induction on the length of the word. Assume as before that $t_i \in \{1, -1\}$, and that the inductive hypothesis is satisfied by the subword \((y_{s_1}^{u_1} \cdots y_{s_{n-1}}^{u_{n-1}})^{-1}\). Let the resulting standard form for this word be $f y_{s_1}^{u_1} \cdots y_{s_n}^{u_n}$. It follows that

\[(y_{s_1}^{u_1} \cdots y_{s_n}^{u_n})^{-1} = y_{s_n}^{-t_n} (f y_{s_1}^{u_1} \cdots y_{s_n}^{u_n})\]

Let $y_{s_n}^{-t_n} = gy_{p_i}^{q_i} \cdots y_{p_l}^{q_l}$ be a standard form such that $f$ acts on $p_i$ for each $1 \leq i \leq l$ and $p_i \cdot f = p'_i$ has the property that for each $1 \leq j \leq m$ either $u_j \subset p'_i$ or $p'_i, u_j$ are incompatible.

It follows that

\[(y_{s_1}^{u_1} \cdots y_{s_n}^{u_n})^{-1} = gf(y_{p_i}^{q_i} \cdots y_{p_l}^{q_l})(y_{u_1}^{u_1} \cdots y_{u_m}^{u_m})\]

and that $gf(y_{p_i}^{q_i} \cdots y_{p_l}^{q_l})(y_{u_1}^{u_1} \cdots y_{u_m}^{u_m})$ is a standard form.

Since any potential cancellation is witnessed by a pair of neighboring elements, it suffices to show that no neighboring pair $y_{u_i}^{u_i}, y_{p_i}^{p_i}$ contains a potential cancellation. This follows from the previous lemma.

Recall from \cite{10} the normal form for Thompson’s group $F$. Note that this normal form uses the infinite generating set $X' = \{x_0, x_1, x_{11}, x_{111}, \ldots\} \subseteq X$.

\textbf{Theorem 3.8.} For an element of $G$, let $f y_{s_1}^{u_1} \cdots y_{s_n}^{u_n}$ be a word representing the group element such that
(1) It is a standard form with no potential contractions or potential cancellations.

(2) If \( i < j \) then \( s_i \prec \text{lex} s_j \).

(3) \( f \in F \) is an \( \mathcal{X} \)'-word in a normal form for \( F \).

Such a word exists and is unique. Moreover, for any finite binary sequence \( s \) such that \( s_i \subset s \) for some \( 1 \leq i \leq n \) it follows that there is an element of \( [s\bar{0}, s\bar{1}] \) on which the action of \( y_{s_1} \cdots y_{s_n} \) does not preserve tail equivalence.

Proof. From Lemma \[ \ref{lem:1} \] and \[ \ref{lem:2} \] it follows that for any given group element we can find a word satisfying the hypothesis of the theorem. It remains to show that this is unique and that it satisfies the property concerning tail equivalence.

By way of contradiction assume that \( f_{y_{s_1} \cdots y_{s_n}} \cdot g_{y_{t_1} \cdots y_{t_m}} \) are two standard forms that satisfy conditions (1) and (2) and represent the same element of \( G \). We can assume without loss of generality that \( s_n \prec \text{lex} t_m \), or else we can perform a cancellation on the right until we achieve this condition or else contradict our assumption that the words are distinct or represent the same group element. We also assume that \( q_m > 0 \). (The proof for \( q_m < 0 \) is similar.) By our assumption it follows that the word \( (y_{s_1} \cdots y_{s_n} y_{p_m}^{-1})(y_{p_m}^{-q_m-1} \cdots y_{p_m}^{-q_1}) \in F \).

First observe that since \( s_n < \text{lex} t_m \), the word \( (y_{s_1} \cdots y_{s_n} y_{p_m}^{-1}) \) is in standard form. Moreover, it is true that there is an infinite binary sequence \( \psi \) such that the calculation of \( y_{s_1} \cdots y_{s_n} y_{p_m}^{-1} \) on the binary sequence \( p_m \psi \) does not contain any potential cancellations. If this were not the case, then this would contradict the assumption that the word \( y_{s_1} \cdots y_{s_n} \) does not have any potential contractions.

Now let \( g_1 y_{w_1} \cdots y_{w_l} \) be a standard form for the word \( y_{p_m}^{-q_m-1} y_{p_m-1} \cdots y_{p_1} \). Let \( l_1, l_2 \) be sufficiently large numbers such that \( l_2 > |w_j| \) for every \( 1 \leq j \leq m \) and \( g_1 \) acts on every finite binary sequence of length at least \( l_1 \) to produce a finite binary sequence of length at least \( l_2 \).

Now by Lemma 5.4 in [15] we can convert the word \( (y_{s_1} \cdots y_{s_n} y_{p_m}^{-1}) \) into a standard form \( g_2 y_{w_1} \cdots y_{w_j} \) with depth greater than \( l_1 \) and no potential cancellations. By our observation above it follows that this word is not empty and there is a sequence \( w_j \cdot \nu \) such that the associated calculation of \( y_{w_1} \cdots y_{w_j} \) on this sequence has no potential cancellation.

Now by our assumptions about depth it follows that \( g_1 \) acts properly on \( w_1, \ldots, w_j \), and that

\[
(g_2 y_{w_1} \cdots y_{w_j}^z)(g_1 y_{w_1} \cdots y_{w_k}^y)
\]

is a standard form.

By Lemma \[ \ref{lem:3} \] it follows that the calculation of the standard form

\[
(y_{w_1}^z \cdots y_{w_j}^z)(y_{w_1}^y \cdots y_{w_k}^y)
\]
on the sequence \( (w_j \cdot g_1) \nu \) has at least an occurrence of \( y, y^{-1} \) and no potential cancellations. By Lemmas 4.10 and 4.11 from [15] this contradicts the assumption that this word is an element of \( F \). So the original standard forms must have been the same to begin with.

The statement about tail equivalence follows from the no potential cancellation assumption together with Lemmas 4.10, 4.11 from [15]. \( \square \)

The following is immediate.
Corollary 3.9. Let $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a $Y$-word in normal form. Then if $y_u^{v_1}...y_u^{v_m}$ is a $Y$-word in normal form such that $y_u^{v_1}...y_u^{v_m}$ is an element of the right coset $F(fy_{s_1}^{t_1}...y_{s_n}^{t_n})$ then it follows that $y_u^{v_1}...y_u^{v_m} = y_{s_1}^{t_1}...y_{s_n}^{t_n}$.

This provides a normal form representative for right cosets of $F$ in $G$. This normal form will be useful in the rest of the article.

4. Right cosets of $F$ in $G$ and Special forms

In this section we will define special forms, which are standard forms that will play a crucial role in both the definition our complex as well the group action.

Definition 4.1. Let $\Omega$ be the set of right cosets of $F$ in $G$. The group $G$ acts naturally on $\Omega$ by right multiplication. A $Y$-standard form $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is called a special form if the following holds:

1. There is a finite rooted binary tree with leaves $l_1, ..., l_m$, such that $l_i \leq l_j$ if $i < j$ for which there is an $1 \leq i < m$ such that $s_1 = l_{i+1}, s_2 = l_{i+2}, ..., s_n = l_{i+n}$.
2. For each $1 \leq j \leq n$, $t_j \in \{1, -1\}$ and $t_{j+1} = (-1)t_j$.

Some basic examples of special forms are $y_s$, $y_s^{-1}$, $y_0y_{s_1}^{-1}$, and $y_0y_{s_1}^{-1}y_{s_2}^{-1}$. A special form $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is of type 1 if $t_1 = -1$ and type 2 if $t_1 = 1$. The parity of the special form is the parity of $n$. In particular, the special form is said to have odd or even length if $n$ is odd or even.

If $\lambda = y_{s_1}^{t_1}...y_{s_n}^{t_n}$ is a special form, then $\lambda^{-1} = y_{s_n}^{-t_n}...y_{s_1}^{-t_1}$ is not a special form. However since $y_{s_1}^{-t_1}, y_{s_2}^{-t_2}$ commute in $G$, we have that the group element described by $\lambda^{-1}$ can be represented by the special form $y_{s_1}^{-t_1}...y_{s_n}^{-t_n}$. Note that if $\lambda$ is of type 1 then this special form representation of $\lambda^{-1}$ is of type 2 and vice versa.

Definition 4.2. Let $\Gamma \subset \Omega$ be the cosets that contain special forms. If $\tau_1, \tau_2$ are special forms such that $F\tau_1 = F\tau_2$ then we say that $\tau_1 \sim \tau_2$, and that $\tau_1, \tau_2$ are equivalent special forms.

We describe some basic manipulations of special forms.

Definition 4.3. Given a special form $\lambda = y_{s_1}^{t_1}...y_{s_n}^{t_n}$, an expansion substitution at $y_{s_i}^{t_i}$ entails replacing $y_{s_i}^{t_i}$ in $\lambda$ by $y_{s_0}y_{s_1}^{-1}y_{s_i}^{-1}$ if $t_i = 1$ or replacing $y_{s_i}^{-1}$ by $y_{s_0}^{-1}y_{s_0}^{-1}y_{s_i}^{-1}$ if $t_i = -1$. Replacing $y_{s_0}y_{s_1}^{-1}y_{s_1}^{-1}$ by $y_{s_0}$ or replacing $y_{s_0}^{-1}y_{s_0}^{-1}y_{s_1}^{-1}$ by $y_{s_1}^{-1}$ is called a contraction substitution.

The following is a basic property of special forms.

Lemma 4.4. The following conditions hold for special forms.

1. Performing contraction and expansion substitutions on a special form produces special forms that belong to the same coset.
2. Given any pair of special forms that belong to the same coset, one can be obtained from the other by performing a sequence of expansion and contraction substitutions.

Proof. Let $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ be a special form. If a substitution $y_s \to x_s y_{s_0}y_{s_1}^{-1}y_{s_1}^{-1}$ (or $y_s^{-1} \to x_s^{-1} y_{s_0}^{-1}y_{s_0}^{-1}y_{s_1}^{-1}$) is performed, then $x_s$ (or $x_s^{-1}$) commutes with every percolating element of $y_{s_1}^{t_1}...y_{s_n}^{t_n}$ that occurs to its left in the word. This means that $x_s$ (or $x_s^{-1}$) can be deleted from the word to obtain a special form that lies in the same coset.
Let \( y_{s_1}^1 \cdots y_{s_n}^m \) and \( y_{p_1}^0 \cdots y_{p_m}^0 \) be special forms that lie in the same coset. By applying expansion rules we can assume that whenever \( s_i, p_j \) are compatible, \( s_i \) extends \( p_j \). By our assumption \((y_{s_1}^1 \cdots y_{s_n}^m)(y_{p_1}^0 \cdots y_{p_m}^m)^{-1} = (y_{s_1}^1 \cdots y_{s_n}^m)(y_{p_1}^0 \cdots y_{p_m}^0)^{-1}\) can be reduced to an \( X \)-word by performing substitutions. Using the algorithm in [15], we perform expansion substitutions on the letters \( y_{p_i}^0 \) and their offsprings one by one, until all the offsprings are cancelled. This means that the special form \( y_{s_1}^1 \cdots y_{s_n}^m \) can be obtained from \( y_{p_1}^0 \cdots y_{p_m}^0 \) by performing expansion substitutions.

**Corollary 4.5.** The parity and type of special forms that lie in the same coset is the same.

The following lemma produces a unique canonical choice of special form for each equivalence class of \( \Gamma \). This is an immediate corollary of Theorem 3.8.

**Lemma 4.6.** If \( \lambda \) is a special form then there is a unique special form \( \tau \) in \( F\lambda \) such that any special form \( \chi \) in \( F\lambda \) can be derived from \( \tau \) by expansion substitutions. Moreover, given a special form \( \psi_1 \in F\lambda \) there is a special form \( \psi_2 \in F\lambda \) that can be obtained from both \( \psi_1, \lambda \) by expansion substitutions.

**Proof.** Given \( \lambda \), we perform a sequence of contraction substitutions, one by one, whenever possible until no contraction substitution can be performed. This produces a special form \( \tau \in F\lambda \) that is unique by Theorem 3.8. For the second statement of the Lemma, observe that we can perform a finite sequence of expansion substitutions on \( \tau \) to obtain the required special form \( \psi_2 \).

We now discuss the action of \( F \) on special forms.

**Definition 4.7.** Let \( \nu = y_{s_1}^1 \cdots y_{s_n}^m \) be a special form. We say that an element \( f \in F \) acts on \( \nu \) if it acts on each sequence in the set \( \{s_1, \ldots, s_n\} \). In particular, we obtain that

\[
(F\nu) \cdot f = F(y_{s_1}^1 \cdots y_{s_n}^m.f) = F(y_{f(s_1)}^1 \cdots y_{f(s_n)}^m)
\]

We remark that if in the above \( f \) does not act on \( \nu \) then we can use the relations \( y_s \rightarrow x_s y_{s0} y_{s1}^{-1} y_{s01} y_{s1}^{-1} \) and \( y_s^{-1} \rightarrow x_s^{-1} y_{s0}^{-1} y_{s01}^{-1} y_{s1}^{-1} \) to obtain a word \( g\nu' = \nu \) such that:

(1) \( g \in F \) and \( \nu' \) is a special form.

(2) \( f \) acts on \( \nu' \) and \( \nu'f = hv'' \) for a special form \( \nu'' \).

It follows that

\[
\nu f = g\nu' f = gh\nu''
\]

and so in particular \( F\nu \cdot f = F\nu'' \).

**Lemma 4.8.** Consider the action of \( F \) on \( \Omega \). The following conditions hold.

(1) \( \Gamma \) is invariant under the action of \( F \) on \( \Omega \). Also, the type and parity of the special forms representing the equivalence class is preserved under this action.

(2) The action of \( F \) on elements \( \Gamma \) of the same type and parity is transitive. The action on \( \Gamma \) has precisely four orbits, one for each combination of type and parity.

**Proof.** If \( \tau \) is a special form and \( f \in F \) then by performing expansion substitutions on \( \tau \) we obtain special form \( \tau' \) such that \( \tau' \in F\tau, f \) acts on \( s_1, \ldots, s_n \) and \( F\tau \cdot f = F(y_{f(s_1)}^1 \cdots y_{f(s_n)}^m) \). Now \( y_{f(s_1)}^1 \cdots y_{f(s_n)}^m \) is a special form since the conditions on the \( t_i \)'s hold and \( f(s_1), \ldots, f(s_n) \) are consecutive leaves of a finite binary tree if
Lemma 4.10. If $\nu$ are special forms $n$ by performing expansion substitutions we can assume that

$$\{ \text{observation. For each } s \text{ satisfying the property that if } \lambda \text{ is a special form then } \nu \text{ can be obtained from } \lambda \text{ by performing expansion substitutions. Let } f \in \text{Stab}_F(\lambda) \text{. Now } F \lambda \cdot f = F \nu \text{ where } \lambda f = g \nu \text{ for } g \in F \text{ for a special form } \nu. \text{ By our assumption } \nu \sim \lambda, \text{ and hence } \nu \text{ can be obtained from } f \text{ by performing expansion substitutions.}

The set of special forms obtained by performing expansion substitutions on $\tau = y_{s_1} \ldots y_{s_n}$ is in a natural bijection correspondence with finite 3-branching forests with $n$ roots.

Let $F_{\tau}$ be the set of elements of $F$ that fix the binary sequences $s_1, \ldots, s_n$. We note that $F_{\tau}$ forms a group under composition which is isomorphic to $F \times F$.

Let $F'_{\tau}$ be the set of elements $f \in F$ such that $F \tau \cdot f = F \tau$ and $f$ fixes each binary sequence that does not extend a sequence in the set $\{s_1, \ldots, s_n\}$. There are special forms $\nu_1 = y_{p_1}^{s_1} \ldots y_{p_m}^{s_m}, \nu_2 = y_{h_1}^{s_1} \ldots y_{h_m}^{s_m}$ which can be obtained from $\nu_1$ by performing expansion substitutions such that $f(p_i) = h_i$. A tree diagram for an element $f \in F'_{\tau}$ can be obtained by considering a tree pair $(P, Q)$ where $p_1, \ldots, p_m$ occur as leaves of $P$, $h_1, \ldots, h_m$ occur as leaves of $Q$, $p_i$ is mapped to $h_i$ and every leaf of $P$ that is not in the set $\{p_1, \ldots, p_m\}$ is mapped to a leaf with the same address in $Q$. Let $P', Q'$ be the forest of subtrees of $P, Q$ with roots $s_1, \ldots, s_n$ that are obtained by deleting every vertex that has an address that does not extend $s_1, \ldots, s_n$. The tree pair $P', Q'$ describes an element of $F_{2, n}$ which naturally corresponds to an element of $F_{3, n}$ in the following way: The expansion substitution $y_s \rightarrow y_s^{y_{s_{i_1}} \ldots y_{s_{i_k}}} (\text{or } y_s^{-1} = y_s^{y_{s_{i_1}} \ldots y_{s_{i_k}}^{-1}})$ corresponds naturally to a “3-branching” that is uniquely determined by the exponent of $y_s$. So the trees $P', Q'$ correspond naturally to a tree pair $U, V$ in $F_{3, n}$. This correspondence is well defined under composition and inverses and hence produces a group homomorphism between $F'_{\tau}$ and $F_{3, n}$ which is easily seen to be a group isomorphism. Recall that for each $n \in \mathbb{N}$, $F_{3, n} \cong F_{3, 1} = F_3$ and $F_{2, n} \cong F_{2, 1} = F_2 = F$. The stabilizer $\text{Stab}_F(F \tau)$ has a natural product decomposition as

$$F_{\tau} \times F'_{\tau} \cong F_3 \times F \times F$$
there is a finite binary tree with leaves $s$ has the property that

$$\tau = \tau_1 \ldots \tau_n$$

is a sorted list of special forms. Such a list is said to be **alternating sign**.

**Definition 4.11.** Two special forms $\tau_1 = y_{s_1} \ldots y_{s_k}, \tau_2 = y_{p_1} \ldots y_{p_m}$ are said to be **incompatible** if each pair $s_i, p_j$ is incompatible.

The definition of incompatibility extends naturally to cosets of special forms. We make this precise in the next lemma.

**Lemma 4.12.** If $\nu_1, \nu_2$ are independent special forms and $\lambda_1, \lambda_2$ are special forms such that $F\lambda_1 = F\nu_1, F\lambda_2 = F\nu_2$, then $\lambda_1, \lambda_2$ are independent special forms.

**Proof.** This follows immediately from Lemma 4.4 and the observation that $y^\pm_s, y^\pm_i$ are incompatible if and only if the pairs

$$\{s, t0\}, \{s, t00\}, \{s, t10\}, \{s, t01\}, \{s, t1\}, \{s, t11\}$$

are all incompatible.

The following Lemma is a straightforward generalization of Lemma 4.10.

**Lemma 4.13.** Let $\tau_1, \ldots, \tau_n$ be special forms that are pairwise independent. Then $\cap_{1 \leq i \leq n} \text{Stab}_F(F\tau_i)$ is a finite product of copies of $F$ and $F_3$ and in particular is of type $F_\infty$.

Now we discuss some more basic properties of products of special forms. These properties will be crucial in the construction of the cell complex $X$ in the next section.

**Definition 4.14.** Let $\tau_1, \ldots, \tau_n$ be a list of special forms that are pairwise independent. Such a list is said to be **sorted** if for each $1 \leq i < n$

$$\tau_i = y_{s_i} \ldots y_{s_{k_i}}, \tau_{i+1} = y_{p_i} \ldots y_{p_m}$$

has the property that $s_i <_{\text{lex}} p_j$ for each $1 \leq k \leq l, 1 \leq j \leq m$.

We say that our sorted list of special forms is **consecutive** if for each $1 \leq i < n$, there is a finite binary tree with leaves $r_1, \ldots, r_m$ (in $<_{\text{lex}}$ order) and $1 \leq k \leq m$ such that $\tau_i = y_{s_i} \ldots y_{s_k}, \tau_{i+1} = y_{p_i} \ldots y_{p_m}$ and $r_{k+1} = s_1, r_{k+2} = s_2, \ldots, r_{k+l} = s_l, r_{k+l+1} = p_1, \ldots, r_{k+l+m} = p_m$.

We say that a sorted list of special forms is of **alternating sign** if for each $1 \leq i < n$, $\tau_i = y_{s_i} \ldots y_{s_k}, \tau_{i+1} = y_{p_i} \ldots y_{p_m}$ satisfy that $t_i = -q_i$.

The following lemmas follow immediately from the definitions.

**Lemma 4.15.** If $\tau_1, \ldots, \tau_n$ is a sorted list of pairwise independent special forms then $\tau = \tau_1 \ldots \tau_n$ is a special form if and only if our list is consecutive and of alternating sign.

**Lemma 4.16.** Let $\lambda_1, \ldots, \lambda_n$ be special forms such that $\lambda_1 \ldots \lambda_n$ is a special form. Let $\tau$ be a special form and let $1 \leq i \leq n$ be such that $\lambda_i, \tau$ are consecutive and of alternating type. Then the $Y$-word $\tau(\lambda_1 \ldots \lambda_n)$ can be converted into a standard form $fv$ with the property that there is an infinite binary string $u$ such that the calculation of $fvu$ on $u$ contains no potential cancellations and two occurrences of $y$ or $y^{-1}$.
\[ \Psi = \{ \text{right action of } G \}\text{ is a standard form, with } f = y^{t_1} \ldots y^{t_n} \text{ and } \tau. \] By performing expansion substitutions we obtain \( \tau = fy^{u_1} \ldots y^{u_m} \) where \( f \in F \) and \( u_1 = s_{t+1}v \) where \( v \) is a finite binary string consisting only of 0's. By our assumptions it follows that \( t_i = -t_{i+1} = -v_1 \) and that the calculation of the associated standard form \( f\nu \) on the sequence \( s_{t+1}0 \) has no potential cancellation and two occurrences of \( y^{n+1} \).

**Lemma 4.17.** Let \( v_1, \ldots, v_n \) and \( \lambda_1, \ldots, \lambda_n \) be sorted lists of pairwise independent special forms. Then there is an \( f \in F \) such that \( F\nu \cdot f = F\lambda_1 \cdot f \) if and only if for each \( 1 \leq i \leq n \) the following holds:

1. The forms \( \nu_i \) have the same type and parity as \( \lambda_i \).
2. For \( i < n \), the pair \( \nu_i, \nu_{i+1} \) is consecutive if and only if \( \lambda_i, \lambda_{i+1} \) is consecutive.

**Proof.** We proceed by induction on \( n \). The case \( n = 1 \) follows from Lemma 4.8. The action of \( F \) preserves the type and parity. First we observe that the conditions are necessary. Let \( \psi_1, \psi_2 \) be special forms and \( f \in F \). Let \( \psi_1' \sim \psi_1 \) be special forms such that \( f \) acts on \( \psi_1' \) and \( \psi_1'' = f\psi_1' \) for a special form \( \psi_1'' \). It follows that \( \psi_1', \psi_2' \) are independent and sorted and that \( F\psi_i \cdot f = F\psi_i'' \). By Lemma 4.8, we know that \( \psi_i'' \) has the same type and parity as \( \psi_i \).

Now \( \psi_1', \psi_2' \) are consecutive if and only if \( \psi_1, \psi_2 \) are consecutive. Let

\[ \psi_1' = y^{u_{11}} \ldots y^{u_{1m}}, \quad \psi_2' = y^{u_{21}} \ldots y^{u_{2m}} \]

Then \( s_1, \ldots, s_i, u_1, \ldots, u_m \) are consecutive leaves in a finite rooted binary tree if and only if \( f(s_1), \ldots, f(s_i), f(u_1), \ldots, f(u_m) \) have this property. It follows that \( \psi_1'', \psi_2'' \) are consecutive if and only if \( \psi_1, \psi_2 \) are consecutive.

To see that the conditions are sufficient, observe that by applying expansion substitutions we can find special forms \( \lambda_i', \lambda_j' \) such that \( \lambda_i', \lambda_j' \) have the same number of percolating elements. The conditions allow us to construct an element of \( F \) which maps the indices of the percolating elements of \( \lambda_i' \) (which are finite binary sequences) to the indices of the percolating elements \( \nu_i' \) in the \( <_{\text{lex}} \) order simultaneously for each \( i \). \( \square \)

### 5. The Complex \( X \)

In this section we will build a CW complex \( X \) and an action of \( G \) on \( X \) by cell permuting homeomorphisms. First we will describe the 1-skeleton of \( X \). Then we will introduce a class of finite subgraphs of \( X^{(1)} \) called **clusters**.

**Definition 5.1.** We define \( X^{(0)} = \Omega \). For \( \lambda_1, \lambda_2 \in G \) the 0-cells \( F\lambda_1, F\lambda_2 \) of \( X^{(0)} \) are connected by a 1-cell in \( X^{(1)} \) if and only if \( F(\lambda_1\lambda_2^{-1}) \in \Gamma \). By Lemma 4.9, this condition is well defined for any choice of coset representatives of \( F\lambda_1, F\lambda_2 \). The right action of \( G \) on \( X^{(0)} \) extends to a right action of \( G \) on \( X^{(1)} \).

Recall the discussion about vertex transitive graphs from subsection 2.3. Let \( \Psi = \{ y_100y_101, y_{100}y_{101}, y_{100}y_{101} \} \). For \( \lambda_1, \lambda_2 \in G \) the condition \( F(\lambda_1\lambda_2^{-1}) \in \Gamma \) is equivalent to the condition that \( \lambda_1\lambda_2^{-1} \in F(\Psi)F \). This means that \( X^{(1)} = \text{Cos}(G, F, \Psi) \) by Theorem 2.7. \( X^{(1)} \) is vertex transitive.

It will often be convenient to choose a coset representative \( f\lambda \) of a 0-cell which is a standard form, with \( f \in F \) and \( \lambda \) a \( Y \)-standard form.

Now we will show that the action of \( G \) on \( X^{(1)} \) is cocompact and the stabilizers of cells are of type \( F_{\infty} \).
Lemma 5.2. The action of $G$ on the set of 0-cells of $X^{(1)}$ is transitive. The action of $G$ on the set of 1-cells of $X^{(1)}$ has precisely two orbits.

Proof. The action of $G$ on the 0-cells is easily seen to be transitive. Given a 1-cell $\{F\nu_1, F\nu_2\}$,

$$\{F\nu_1, F\nu_2\} \cdot \nu_1^{-1} = \{F, F\nu_2\nu_1^{-1}\}$$

Further, if $\{F, F\lambda\}$ is a 1-cell and $\lambda$ is a type 2 special form, then $\{F, F\lambda\} \cdot \lambda^{-1} = \{F\lambda^{-1}, F\}$ is an 1-cell incident to $F$ and $\lambda^{-1}$ is a type 1 special form. This means that any 1-cell contains in its $G$-orbit a 1-cell $\{F, F\tau\}$ where $\tau$ is of type 1.

So it suffices to show that given $\{F, F\lambda_1\}$, $\{F, F\lambda_2\}$, where $\lambda_1, \lambda_2$ are special forms of type 1 there is an $f \in F$ such that $\{F \cdot f, F\lambda_1 \cdot f\} = \{F, F\lambda_2\}$ if and only if $\lambda_1, \lambda_2$ have the same parity. This is true by Lemma 4.10.

Lemma 5.3. For each $\nu \in X^{(0)}, e \in X^{(1)}$ the following is true:

1. $\text{Stab}_G(e)$ is isomorphic to $F$.
2. $\text{Stab}_G(v)$ is isomorphic to $F^3 \times F \times F$.

Proof. The fact (1) is true by construction. For (2), first observe that $\text{Stab}_G(\{F\lambda_1, F\lambda_2\})$ is conjugate in $G$ to $\text{Stab}_G(\{F\lambda_1\lambda_2^{-1}, F\})$.

So it suffices to understand the stabilizer of a 1-cell $\{F\lambda, F\}$, where $\lambda$ is a special form. If $F\lambda \cdot \nu = F$ then $\lambda \nu = f$ for some $f \in F$ and $\nu = \lambda^{-1} f$. We can perform expansion substitutions on percolating elements of $\lambda^{-1}$ to obtain a special form $\psi$ such that:

1. $\lambda^{-1} = f_1 \psi$ for $f_1 \in F$.
2. $f$ acts on $\psi$.
3. $\psi f = f \psi'$ for a special form $\psi'$. Note that $\psi'$ must have the same parity and type as both $\psi$ and $\lambda^{-1}$

So $F \cdot \nu = F \psi' \neq F\lambda$, since $\lambda, \psi'$ have different types. This means that if $\nu \in G$ has the property that $\{F\lambda, F\} \cdot \nu = \{F\lambda, F\}$ then in fact $F\lambda \cdot \nu = F\lambda, F\psi \cdot \nu = F$.

So it follows that

$$\text{Stab}_G(\{F\lambda, F\}) = \text{Stab}_G(F\lambda) \cap F = \text{Stab}_F(F\lambda) \cong F^3 \times F \times F$$

by Lemma 4.10.

Now we define a class of finite subgraphs of $X^{(1)}$ called clusters. Recall the definition of a sorted list of special forms from the previous section. (Special forms in a sorted list are in particular pairwise independent.) Also recall that special forms $\tau_1, \tau_2$ are said to be equivalent, or $\tau_1 \sim \tau_2$ if $F\tau_1 = F\tau_2$.

Definition 5.4. Let $\lambda_1, ..., \lambda_n$ be a sorted list of pairwise independent special forms and $\tau \in G$. The subgraph of $X^{(1)}$ consisting of 0-cells in the set

$$I = \{F(\prod_{i \in A} \lambda_i)\tau \mid A \subseteq \{1, ..., n\}\}$$

and 1-cells

$$J = \{e \mid e = \{u, v\} \text{ is a 1-cell of } X^{(1)} \text{ such that } u, v \in I\}$$

is called an $n$-cluster. For this $n$-cluster the group element $\tau$ is said to be the basepoint, $\lambda_1, ..., \lambda_n$ are said to be parameters, and the 0-cell $F\tau$ the base-vertex. Informally, this $n$-cluster is said to be based at $\tau$ and parametrized by $\lambda_1, ..., \lambda_n$. 
Observe that each 1-cell in $X^{(1)}$ is a 1-cluster. In particular any 1-cell $e$ in $X^{(1)}$ can be described by a pair $\lambda, \tau$ where $\lambda$ is a special form and $\tau \in G$ and $e = \{ F\lambda\tau, F\tau \}$. We remark that if $f \in F$ the pairs $\lambda, \tau$ and $\lambda, f\tau$ do not necessarily describe the same 1-cell since $F\lambda(f\tau), F\lambda\tau$ can be distinct 0-cells even though $F(f\tau) = F\tau$.

**Definition 5.5.** Consider the cluster in the previous definition. If $\tau = f\tau'$ for $f \in F$ and $\lambda_i f = f_i \lambda'_i$ for $f_i \in F$ and special forms $\lambda'_i$, then this cluster has the following equivalent parametrization: It is based at $\tau'$ and parametrized by $\lambda'_1, \ldots, \lambda'_n$. These parametrizations are related by the fact that $\tau \sim \tau'$.

Another equivalent parametrization is of the following form: Let $\tau' = (\prod_{i \in A} \lambda_i)\tau$ for some $A \subseteq \{1, \ldots, n\}$. Then the cluster based at $\tau'$ is parametrized by $\nu_1, \ldots, \nu_n$ where $\nu_i = \lambda_i$ if $i \notin A$ and $\nu_i = \lambda_i^{-1}$ if $i \in A$ is the same as above.

Moreover, a third kind of equivalent parametrization can be obtained by combining the first with the second, i.e. when the basepoint $\tau'$ satisfies that $\tau' \sim (\prod_{i \in A} \lambda_i)\tau$ for some $A \subseteq \{1, \ldots, n\}$.

**Definition 5.6.** Let $\Delta$ be a cluster at $\tau$ with parameters $\lambda_1, \ldots, \lambda_n$. We say that the parametrization is balanced if $\lambda_1, \ldots, \lambda_n$ are of alternating sign. If this is a balanced parametrization and $\lambda_1$ is of type 1 (or type 2), then $\tau$ is said to be a type 1 (or a type 2) basepoint for $\Delta$. We say that this parametrization is consecutive if $\lambda_1, \ldots, \lambda_n$ are consecutive.

In the next Lemma we show that these notions are invariant under equivalent parametrizations of clusters provided the new basepoint is equivalent to the old basepoint.

**Lemma 5.7.** Let $\Delta$ be a cluster at $\tau$ with parameters $\lambda_1, \ldots, \lambda_n$. Let $\tau = f\tau'$ and $\nu_i$ be special forms such that $\lambda_i f = g_i \nu_i$ for $g_i \in F$. Then the following holds.

1. $\nu_1, \ldots, \nu_n$ is an sorted list of pairwise independent special forms.
2. The equivalent parametrization based at $f\tau'$ and parametrized by $\nu_1, \ldots, \nu_n$ is either consecutive or of alternating sign if and only if the original parametrization is respectively consecutive or of alternating sign.

**Proof.** For any $A \subseteq \{1, \ldots, n\}$ observe that

$$\left(\prod_{i \in A} \lambda_i\right)\tau = \left(\prod_{i \in A} \lambda_i\right)f\tau' = \left(\prod_{i \in A} g_i f\left(\prod_{i \in A} \nu_i\right)\right)\tau'$$

such that $g_i$ is an element of $F$ that is a product of words $x_s, x_s^{-1}$ obtained by replacing $y_s$ by $x_s y_{s0} y_{s01}^{-1} y_{s11}$ or $y_s^{-1}$ by $x_s^{-1} y_{s00} y_{s01} y_{s11}^{-1}$ where $y_s$ or $y_s^{-1}$ is a percolating element of $\lambda_i$ or an offspring of such a percolating element. Clearly each $g_i$ commutes with $\lambda_j$ for each $j \neq i$. In particular,

$$F\left(\prod_{i \in A} \lambda_i\right)\tau = F\left(\prod_{i \in A} \nu_i\right)\tau'$$

Recall that the action of $F$ on sorted lists of pairwise independent special forms preserves the properties of consecutive and alternating sign. So $\nu_1, \ldots, \nu_n$ are consecutive or of alternating sign if and only if $\lambda_1, \ldots, \lambda_n$ have the respective property. □
We may occasionally drop the reference to \( n \) above and call such a subgraph a *cluster*. Clusters will be denoted by the symbol \( \Delta \), often with subscripts. The set of clusters are ordered by inclusion.

**Lemma 5.8.** For any cluster there are precisely two balanced parametrizations. A cluster base-vertex \( F_\tau \) with a balanced parametrization by basepoint \( \tau \) and parameters \( \lambda_1, \ldots, \lambda_n \) also admits the base-vertex \( F(\prod_{i \in \{1, \ldots, n\}} \lambda_i) \tau \) and a balanced parameterization with basepoint \( (\prod_{i \in \{1, \ldots, n\}} \lambda_i) \tau \) and parameters \( \lambda_1^{-1}, \ldots, \lambda_n^{-1} \). Moreover, this cluster does not admit any other base-vertex for which there is a balanced parametrization. 

**Proof.** Recall that a special form \( \tau \) is of type 1 if and only if \( \tau^{-1} \) is of type 2. Let \( \tau \) be a basepoint of a cluster parametrized by \( \lambda_1, \ldots, \lambda_n \). There is a unique set of numbers \( l_1, \ldots, l_n \in \{1, -1\} \) such that \( \prod_{1 \leq i \leq n} \lambda_i^{l_i} \) is of type 1 and \( \prod_{1 \leq i \leq n} \lambda_i^{-l_i} \) is of type 2. Our assertion follows immediately. \( \Box \)

### 5.1. Subclusters

The set of clusters is ordered by inclusion as subgraphs of \( X^{(1)} \). In this subsection we study subclusters of clusters.

**Definition 5.9.** A cluster \( \Delta_1 \) is said to be a subcluster of a cluster \( \Delta_2 \) if there is an inclusion \( \Delta_1 \subseteq \Delta_2 \) as graphs in \( X^{(1)} \).

First we show that if we have clusters \( \Delta_1, \Delta_2 \) such that \( \Delta_1 \subseteq \Delta_2 \) as graphs, then there is a natural inclusion of these clusters in the sense of parameters. We then proceed to define some types of subclusters and show that any subcluster of a cluster must be of one of these types.

**Lemma 5.10.** Let \( \Delta_1, \Delta_2 \) be clusters such that \( \Delta_1 \subseteq \Delta_2 \). By reparametrizing if necessary, we can assume that they have a common basepoint \( \tau \) and parameters \( \lambda_1, \ldots, \lambda_n \) and \( \nu_1, \ldots, \nu_m \) respectively (not necessarily balanced). Then there are \( C_1, \ldots, C_n \subseteq \{1, \ldots, m\} \) that have the following properties:

1. The sets \( C_1, \ldots, C_n \) are pairwise disjoint.
2. Each \( C_i = [a_i, b_i] \cap \{1, \ldots, m\} \) such that \( b_i < a_j \) for \( i < j \).
3. \( \prod_{j \in C_i} \nu_j \) are special forms.
4. \( \lambda_i \sim \prod_{j \in C_i} \nu_j \).

**Proof.** Let \( e_1, \ldots, e_n \) be the 1-cells incident to \( F_\tau \) in \( \Delta_1 \) such that \( e_i = \{F(\lambda_i \tau), F_\tau\} \). Since these 1-cells belong in \( \Delta_2 \) as well, for each edge \( e_i \) there is a set \( C_i \subseteq \{1, \ldots, m\} \) such that \( \prod_{j \in C_i} \nu_j \sim \lambda_i \) and \( e_i = \{F(\prod_{j \in C_i} \nu_j), F_\tau\} \). By Lemma 5.12 it follows that \( C_i \cap C_j = \emptyset \). Now \( \prod_{j \in C_i} \nu_j \sim \lambda_i \) is a special form and so the elements of each set \( \{\nu_j \mid j \in C_i\} \) listed in increasing order of index must be consecutive and of alternating sign. In particular the conditions (2) and (3) must hold. The order \( b_i < a_j \) for \( i < j \) in condition (2) is satisfied since both \( \lambda_1, \ldots, \lambda_n \) and \( \nu_1, \ldots, \nu_m \) are sorted lists. \( \Box \)

Now we introduce the *types* of subclusters.

**Definition 5.11.** Let \( \Delta \) be a cluster at \( \tau \) parametrized by \( \lambda_1, \ldots, \lambda_n \). A subcluster \( \Delta' \) of \( \Delta \) is a face of \( \Delta \) if there are sets \( A, B \subseteq \{1, 2, \ldots, n\} \) such that:

1. \( A \cap B = \emptyset \).
2. \( \Delta' \) has a basepoint \( \tau' = (\prod_{i \in B} \lambda_i) \tau \).
3. The parameters of \( \Delta' \) are elements of the set \( \{\lambda_i \mid i \in A\} \).
If \( \{1, ..., n\} \setminus A \) is a singleton then \( \Delta' \) is a maximal face.

**Definition 5.12.** Let \( \Delta \) be a cluster at \( \tau \) parametrized by \( \lambda_1, ..., \lambda_n \). A subcluster \( \Delta' \) of \( \Delta \) is a diagonal subcluster if:

1. \( \Delta' \) has a basepoint at \( \tau \).
2. There are numbers \( 1 = a_0 < a_1 < a_2 ... < a_m-1 < a_m = n + 1 \) such that \( \Delta' \) is parametrized by \( \nu_1, ..., \nu_m \) where \( \nu_i = \prod_{a_{i-1} \leq t < a_i} \lambda_t \).

We remark that each \( \nu \) must be a special form and so the terms in the product \( \prod_{a_{i-1} \leq t < a_i} \lambda_t \) must be consecutive special forms of alternating type.

To motivate the above definition, imagine a unit Euclidean square that is subdivided to include a diagonal at a specified pair of opposite points. If we take a product of three such squares, the product contains as a subcomplex the product of the three diagonals. So this cube is sitting “diagonally” inside the product which is a subdivided 6-cube.

Let \( \Delta \) be a cluster at \( \tau \) that has a balanced parametrization by \( \lambda_1, ..., \lambda_n \). Let \( A, B \subseteq \{1, ..., n\} \) such that the vertices \( F \prod_{i \in A} \lambda_i \tau, F \prod_{i \in B} \lambda_i \tau \) are connected by an edge. This means that \( (\prod_{i \in A} \lambda_i)(\prod_{j \in B} \lambda_j)^{-1} \) is a special form. Since \( \lambda_1, ..., \lambda_n \) is a balanced parametrization, it must be the case that either \( A \setminus B \) or \( B \setminus A \) is empty. Furthermore, in the former case, from the definition of balanced and alternating type it must be true that \( A \setminus B = [i, j] \cap \{1, ..., n\} \) (and in the latter case that \( B \setminus A = [i, j] \cap \{1, ..., n\} \)) for some \( 1 \leq i < j \leq n \).

Now we will show that the set of types for subclusters that we have defined is exhaustive.

**Lemma 5.13.** Let \( \Delta \) be a cluster at \( \tau \) parametrized by \( \lambda_1, ..., \lambda_n \). A subcluster \( \Delta' \) of \( \Delta \) is either a diagonal subcluster of \( \Delta \), a face of \( \Delta \) or a diagonal subcluster of a face of \( \Delta \).

**Proof.** We assume without loss of generality that the subcluster is also based at \( \tau \).

By Lemma 5.10 the subcluster must have a parametrization as follows: There are sets \( C_1, ..., C_k \subseteq \{1, ..., n\} \) such that:

1. The sets \( C_1, ..., C_k \) are pairwise disjoint.
2. \( C_i = [a_i, b_i] \cap \{1, ..., m\} \) such that \( b_i < a_j \) for \( i < j \).
3. \( \nu_i = (\prod_{j \in C_i} \lambda_j) \) are special forms.

Then our subcluster is parametrized by \( \nu_1, ..., \nu_k \). The subcluster is a diagonal subcluster of \( \Delta \) if and only if \( C_1 \cup C_2 \cup ... \cup C_k = \{1, ..., n\} \). Otherwise, we have the following two possibilities. It is a face of \( \Delta \) if and only if \( C_1, ..., C_k \) are single element sets, and otherwise it is a diagonal subcluster of a face of \( \Delta \). \( \square \)

Now we shall show that the intersection of two clusters is a cluster. First we need to prove a technical lemma.

**Lemma 5.14.** Let \( \lambda_1, ..., \lambda_n \) and \( \nu_1, ..., \nu_m \) be sorted lists of pairwise independent special forms such that

\[
F( \prod_{1 \leq i \leq n} \lambda_i) = F( \prod_{1 \leq j \leq m} \nu_j)
\]

Let

\[
I_1 = \{ J \subseteq \{1, ..., n\} \mid \prod_{i \in J} \lambda_i \text{ is a special form.} \}
\]
and
\[ I_2 = \{ J \subseteq \{1, \ldots, m\} \mid \prod_{i \in J} \nu_i \text{ is a special form.} \} \]

Let \( \{K_1, \ldots, K_s\} \) and \( \{L_1, \ldots, L_t\} \) be the maximal elements of \( I_1, I_2 \) respectively (with respect to inclusion). Then \( s = t \) and after re-ordering the indices if necessary we have that
\[ F( \prod_{i \in K_j} \lambda_i ) = F( \prod_{i \in L_j} \nu_i ) \]

**Proof.** It suffices to show that for any \( 1 \leq o \leq s \) there is a \( 1 \leq r \leq t \) such that
\[ F( \prod_{i \in K_o} \lambda_i ) = F( \prod_{i \in L_r} \nu_i ) \]

Since
\[ F( \prod_{1 \leq i \leq n} \lambda_i ) = F( \prod_{1 \leq j \leq m} \nu_i ) \]
it follows that we can perform expansion and contraction substitutions on the special form \( \prod_{i \in K_o} \lambda_i \) to obtain a subword of \( F( \prod_{1 \leq j \leq m} \nu_j ) \) that is a special form. It follows that there is an element \( A \in I_2 \) such that
\[ \prod_{i \in K_o} \lambda_i \sim \prod_{i \in A} \nu_i \]

Let \( A = \{ j_1, j_1 + 1, \ldots, j_1 + w_1 \} \) and \( K_o = \{ j_2, j_2 + 1, \ldots, j_2 + w_2 \} \). We claim that \( A \in \{ L_1, \ldots, L_t \} \). If this is not the case, then either the special forms \( \nu_{j-1}, \prod_{i \in A} \nu_i \) are consecutive and of alternating type, or \( \prod_{i \in A} \nu_i, \nu_{j+w+1} \) are consecutive and of alternating type. Let us assume the latter (the former case is similar).

There are two possibilities.

1. We can perform expansion and contraction moves on \( \prod_{j_1 \leq i \leq j_1 + w_1 + 1} \nu_i \) to obtain a subword of \( \prod_{j_2 \leq i \leq j_2 + w_2 + 1} \lambda_i \) that properly contains \( \prod_{j_2 \leq i \leq j_2 + w_2} \lambda_i \).

2. We can perform expansion and contraction moves on \( \prod_{j_2 \leq i \leq j_2 + w_2 + 1} \lambda_i \) to obtain a subword of \( \prod_{j_1 \leq i \leq j_1 + w_1 + 1} \nu_i \) that properly contains \( \prod_{j_1 \leq i \leq j_1 + w_1} \nu_i \).

In either case we obtain that in fact \( \prod_{j_2 \leq i \leq j_2 + w_2 + 1} \lambda_i \) is a special form. This contradicts the assumption that \( K_o \) is a maximal element of \( I_2 \).

This means that our assumption that \( A \notin \{ L_1, \ldots, L_t \} \) must be false. \( \square \)

**Lemma 5.15.** Let \( \Delta_1, \Delta_2 \) be clusters with basepoints \( \tau_1, \tau_2 \) and parameters \( \lambda_1, \ldots, \lambda_n \) and \( \nu_1, \ldots, \nu_m \) respectively. If \( \Delta_3 = \Delta_1 \cap \Delta_2 \) is nonempty, then \( \Delta_3 \) is a cluster.

**Proof.** By reparametrization of the clusters if necessary we can assume that \( \tau_1 = \tau_2 = \tau \). Let \( e_1, \ldots, e_k \) be the set of 1-cells incident to \( F\tau \) that are contained in both \( \Delta_1 \) and \( \Delta_2 \). Let \( C_1, \ldots, C_k \subseteq \{1, \ldots, n\} \) and \( D_1, \ldots, D_k \subseteq \{1, \ldots, m\} \) be such that \( \sigma_i = \prod_{j \in C_i} \lambda_j \) and \( \psi_i = \prod_{j \in D_i} \nu_j \), and \( e_i = \{ F\sigma_i \tau, F\tau \} = \{ F\psi_i \tau, F\tau \} \). By our assumption we have \( \psi_i \sim \sigma_i \). Our proof consists of four steps.

Step (1): We claim that \( \sigma_i, \sigma_j \) are independent if and only if \( \psi_i, \psi_j \) are independent.

First note that \( \psi_i \sim \sigma_i \) are equivalent to a special form. Since they are also expressed as products of independent special forms they must be special forms. Since \( \psi_i \sim \sigma_i \) for each \( 1 \leq i \leq k \) our claim follows from Lemma 4.12.
Step (2): Let $C_i, C_j$ be such that $C_i \cap C_j \neq \emptyset$. Then by step (1) it also holds that $D_i \cap D_j \neq \emptyset$. Let

$$\sigma_{i,j} = \prod_{k \in C_i \cap C_j} \lambda_k$$

and

$$\psi_{i,j} = \prod_{k \in D_i \cap D_j} \nu_k$$

In this step we will show that $\sigma_{i,j} \sim \psi_{i,j}$.

Define

$$\sigma_{i,j} = \prod_{k \in C_i \cap C_j} \lambda_k, \sigma_{j\setminus i} = \prod_{k \in C_j \setminus C_i} \lambda_k$$

Similarly define

$$\psi_{i,j} = \prod_{k \in D_i \cap D_j} \nu_k, \psi_{j\setminus i} = \prod_{k \in D_j \setminus D_i} \nu_k$$

We have $\sigma_i \sigma_j = \sigma_{i,j} \sigma_{i,j}^{2, \sigma_j}$ and $\psi_i \psi_j = \psi_{i,j} \psi_{i,j}^{2, \psi_j} \psi_{j\setminus i}$.

Recall that $\psi_{i,j} = \psi_{i,j} \psi_{j\setminus i} \sigma_j = \sigma_{i,j} \sigma_{j\setminus i} \psi_j$ by the second statement in Lemma 4.6 it follows that we can apply expansion substitutions to $\psi_{i,j} \psi_{j\setminus i}$ to obtain special forms $\rho_{i,j}, \rho_{j\setminus i}$ such that $\rho_{i,j} \sim \psi_{i,j}, \rho_{j\setminus i} \sim \psi_{j\setminus i}$ and $\rho_{i,j} \rho_{j\setminus i}$ can be obtained from the special form $\sigma_{i,j} \sigma_{j\setminus i}$ by expansion substitutions.

Let $\rho_{i,j} = y_{s_1} \cdots y_{s_m}$. In particular we can apply expansion substitutions to $\sigma_{i,j}$ to obtain a special form that contains a subword of the form $y_{s_1}^{l_1} \cdots y_{s_m}^{l_m}$ for some $1 \leq m \leq l$.

Since $\sigma_{i,j} \sim \psi_{i,j}$ we can apply expansion substitutions $\sigma_{i,j}$ to obtain a standard form that contains a subword $y_{s_1}^{l_1} \cdots y_{s_m}^{l_m}$ for some $1 \leq m \leq l$. These observations together imply that we can apply expansion substitutions to $\sigma_{i,j}$ to obtain a word that contains as a subword $y_{s_1}^{l_1} \cdots y_{s_m}^{l_m}$. This means that a subword of $\sigma_{i,j}$ is equivalent to $\psi_{i,j}$.

But doing the same analysis above with $\sigma$ and $\psi$ interchanged we obtain that $\psi_{i,j}$ contains a subword that is equivalent to $\sigma_{i,j}$. So in fact $\psi_{i,j} \sim \sigma_{i,j}$. This also implies that $\psi_{i\setminus j} \sim \sigma_{i\setminus j}$ and $\psi_{j\setminus i} \sim \sigma_{j\setminus i}$.

In particular, it follows that these special forms satisfy $\psi_{i\setminus j}, \psi_{i,j}, \psi_{j\setminus i} \in \{\psi_1, \ldots, \psi_k\}$ and $\sigma_{i\setminus j}, \sigma_{i,j}, \sigma_{j\setminus i} \in \{\sigma_1, \ldots, \sigma_k\}$.

Step (3): The elements $\{C_1, \ldots, C_k\}$ (respectively $\{D_1, \ldots, D_k\}$) are partially ordered by inclusion. We claim that the set of minimal elements $P_1 \subseteq \{C_1, \ldots, C_k\}, P_2 \subseteq \{D_1, \ldots, D_k\}$ in this ordering satisfy the following.

Let $P_1 = \{C_1, \ldots, C_{i_s}\}$ and $P_2 = \{D_{j_1}, \ldots, D_{j_t}\}$. Then the following holds:

1. For each $1 \leq i \leq k$, $C_i$ (and respectively $D_i$) can be expressed as a union of sets in $P_1$ (and respectively $P_2$).
2. The sets $C_{i_1}, \ldots, C_{i_s}$ (respectively $D_{j_1}, \ldots, D_{j_t}$) are pairwise disjoint.
3. $s = t$.

By step (2) it follows that if $C_i \cap C_j \neq \emptyset$ and $D_i \cap D_j \neq \emptyset$ then there are $1 \leq s_1, s_2, s_3 \leq k$ such that:

1. $\prod_{i \in C_{i_1}} \lambda_i \sim \prod_{i \in D_{i_2}} \nu_i$ and $C_{s_1}, C_{s_3}$ (respectively $D_{s_1}, D_{s_3}$) are disjoint for $l \neq m, 1 \leq l, m \leq 3$.
2. $C_i = C_{s_1} \cup C_{s_2}$ and $D_i = D_{s_1} \cup D_{s_2}$.
3. $C_j = C_{s_2} \cup C_{s_3}$ and $D_j = D_{s_2} \cup D_{s_3}$.
4. $\sigma_{i,j} = \prod_{i \in C_{i,j}} \lambda_i \sim \psi_{i,j} = \prod_{i \in D_{i,j}} \nu_i$. 

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This proves our assertion. Moreover, our analysis produces a natural bijective correspondence between the sets \( \{C_{a_1}, ..., C_{a_s}\} \) and \( \{D_{b_1}, ..., D_{b_t}\} \), so in particular \( s = t \).

For the rest of the proof we let \( C_j' = C_{a_j}, D_j' = D_{b_j} \), and \( \sigma'_i = \sigma_{a_i}, \psi'_i = \psi_{b_i} \). Since we know that \( s = t \), so we shall use \( s \) for the remainder of the argument.

Step (4): Now we claim that the subgraph \( \Delta_1 \cap \Delta_2 \) is precisely the cluster \( \Delta_3 \) with basepoint \( \tau \) and parameters \( \sigma'_i = \prod_{j \in C_j'} \lambda_j \sim \psi'_i = \prod_{j \in D_j'} \nu_j \).

Let \( F_\zeta \) be a 0-cell in \( \Delta_1 \cap \Delta_2 \), and let \( I_1 \subseteq \{1, ..., n\}, I_2 \subseteq \{1, ..., m\} \) such that

\[
\zeta \sim (\prod_{i \in I_1} \lambda_i) \tau \sim (\prod_{j \in I_2} \nu_j) \tau
\]

It suffices to show that there are \( A, B \subseteq \{1, ..., s\} \) such that \( I_1 = \cup_{i \in A} C_{i}' \) and \( I_2 = \cup_{j \in B} D_{j}' \). It will then follow that

\[
\zeta \sim (\prod_{i \in (\cup_{j \in A} C_{i}')} \lambda_i) \tau \sim (\prod_{j \in (\cup_{i \in B} D_{j}')} \nu_j) \tau
\]

and so \( F_\zeta \in \Delta_3 \).

Let \( \{J_1, ..., J_l\}, \{K_1, ..., K_o\} \) be the maximal elements of the sets

\[
\{I \subseteq I_1 | \prod_{i \in I} \lambda_i \text{ is a special form}\}
\]

\[
\{I \subseteq I_2 | \prod_{i \in I} \nu_i \text{ is a special form}\}
\]

respectively.

Note that \( \cup_{1 \leq j \leq l} J_j = I_1, \cup_{1 \leq j \leq o} K_j = I_2 \). By Lemma 5.14 it follows that \( l = o \) and by reordering the indices if necessary,

\[
\prod_{w \in J_j} \lambda_w \sim \prod_{w \in K_j} \nu_w
\]

for each \( 1 \leq j \leq l = o \). This in particular implies that \( J_1, ..., J_l \in \{C_{1}, ..., C_{k}\} \) and \( K_1, ..., K_o \in \{D_{1}, ..., D_{k}\} \). Recall that for each \( 1 \leq i \leq k \), \( C_i \) (and respectively \( D_i \)) can be expressed as a union of sets in \( P_1 = \{C_{i}', ..., C_{i}'\} \) (and respectively \( P_2 = \{D_{i}', ..., D_{i}'\} \)). This means that \( I_1 \) (and respectively \( I_2 \)) can be expressed as a union of sets in \( P_1 = \{C_{1}', ..., C_{k}'\} \) (and respectively \( P_2 = \{D_{1}', ..., D_{k}'\} \)).

This proves our assertion and \( F_\zeta \) is a 0-cell of the cluster \( \Delta_3 \). \( \square \)

**Lemma 5.16.** Let \( \lambda_1, ..., \lambda_n \) be special forms and

\[
\{F_{\tau}, F_{\lambda_1 \tau}, \{F_{\tau}, F_{\lambda_2 \tau}, ..., \{F_{\tau}, F_{\lambda_n \tau}\}
\]

be 1-cells incident to \( F_\tau \) such that \( \{F_{\lambda_i \tau}, F_{\tau}\}, \{F_{\lambda_j \tau}, F_{\tau}\} \) are facial 1-cells of a 2-cluster at \( \tau \) for each \( i, j \in \{1, ..., n\}, i \neq j \). Then

\[
\{F_{\lambda_1 \tau}, F_{\tau}, \{F_{\lambda_2 \tau}, F_{\tau}, ..., \{F_{\lambda_n \tau}, F_{\tau}\}
\]

are facial 1-cells of the n-cluster parametrized with basepoint at \( \tau \) and parameters \( \lambda_1, ..., \lambda_n \).

**Proof.** We show this using induction on \( n \). Assume this holds for the first \( n - 1 \) edges in our list. The pairs of edges \( \{F_{\lambda_i \tau}, F_{\tau}\}, \{F_{\lambda_n \tau}, F_{\tau}\} \) belong to a 2-cluster for each \( 1 \leq i < n \). This 2-cluster can be parametrized by \( \nu_i, \psi_i \) based at \( \tau \) where \( \nu_i \sim \lambda_i \) and \( \psi_i \sim \lambda_n \) are special forms such that \( \nu_i, \psi_i \) are independent. By Lemma
it follows that \( \lambda_n \) is independent with all the forms in the list \( \lambda_1, \ldots, \lambda_{n-1} \) and so our \( n \)-cluster has the parameters \( \lambda_1, \ldots, \lambda_n \) with basepoint \( \tau \).

For each \( n \) the group \( G \) acts on the set of \( n \)-clusters. We now study the orbits of this action.

**Lemma 5.17.** The action of \( G \) on the set of \( n \)-clusters has precisely \( 2^{(2n-1)} \) orbits.

**Proof.** Observe that given an \( n \)-cluster with a type 1 basepoint \( \tau \), the action of \( \tau^{-1} \) moves this basepoint to \( \emptyset \). Moreover the subgroup of \( G \) that stabilizes the trivial coset \( F \) is \( F \). So it suffices to show that the action of \( F \) on the set of \( n \)-clusters parametrized with type 1 basepoint \( \emptyset \) has precisely \( 2^{(2n-1)} \) orbits.

By Lemma 4.17 together with the assumption about types there is an \( f \in F \) such that \( \Delta_1 \cdot f = \Delta_2 \) (or \( F\lambda_i \cdot f = F\nu_i \)) if and only if all of the following conditions hold:

1. \( \nu_i \) has the same type and parity as \( \lambda_i \).
2. The pair \( \lambda_i, \lambda_{i+1} \) is consecutive if and only if \( \nu_i, \nu_{i+1} \) is consecutive.

Observe that the type of \( \lambda_1 \) together with the parity of \( \lambda_1, \ldots, \lambda_n \) determines the type of \( \lambda_2, \ldots, \lambda_n \). Since the types of \( \lambda_1, \nu_1 \) are assumed to be the same the first condition reduces to requiring that the parity of \( \nu_i \) be the same as the parity of \( \lambda_i \). Our claim follows immediately by counting the possibilities. □

**5.2. Higher dimensional cells.** In this section we shall describe a method to add higher cells to each \( n \)-cluster to obtain a CW complex homeomorphic to an \( n \)-cube.

**Definition 5.18.** Let \( \Delta \) be an \( n \)-cluster based at \( \tau \) and parametrized by \( \nu_1, \ldots, \nu_n \) such that the parametrization has the property that if \( \nu_i, \nu_{i+1} \) are consecutive special forms, then they have alternating sign. Such a parametrization is called a proper parametrization. Note that in particular a balanced parametrization is proper, but not vice versa. Let the above be a proper parametrization, which we call \( \rho \). Let \( A_{\Delta, \rho} \) be the following set:

\[
\{ B \subseteq \{1, \ldots, n\} \mid |B| = 1 \text{ or } B = \{i, i+1, \ldots, i+k\}, \text{ and } \nu_i, \nu_{i+k} \text{ are consecutive} \}
\]

We will show that the set \( A_{\Delta, \rho} \) is independent of the choice of a proper parametrization. As a consequence we will drop the \( \rho \) in the notation and simply denote the set by \( A_\Delta \).

**Lemma 5.19.** Let \( \Delta \) be a cluster with two different proper parametrizations:

1. \( \rho_1 \): Based at \( \tau_1 \) and with parameters \( \lambda_1, \ldots, \lambda_n \).
2. \( \rho_2 \): Based at \( \tau_2 \) and with parameters \( \nu_1, \ldots, \nu_n \).

Then \( A_{\Delta, \rho_1} = A_{\Delta, \rho_2} \).

**Proof.** The proof comprises of three steps:

1. First we consider the case where \( \tau_1 \sim \tau_2 \).
2. Next, we consider the case where \( \prod_{i \in I} \lambda_i \tau_1 = \tau_2 \) for some \( I \subseteq \{1, \ldots, n\} \).
3. Finally, we consider the general case where \( \prod_{i \in I} \lambda_i \tau_1 \sim \tau_2 \) for some \( I \subseteq \{1, \ldots, n\} \).
Case 1: Let $\tau_1 = f \tau_2$. There are special forms $\nu'_i$ such that $\nu_i \sim \nu'_i \sim \lambda_i f$. By Lemma 4.4, $\nu_i, \nu_{i+1}$ are consecutive and of alternating sign if and only if $\nu'_i, \nu'_{i+1}$ have this property. In particular by Lemma 4.8 this is the case if and only if $\lambda_i, \lambda_{i+1}$ have this property. So it follows that in this case $A_{\Delta, \rho_1} = A_{\Delta, \rho_2}$.

Case 2: A list of parameters for $\Delta$ with respect to the basepoint $(\prod_{i \in I} \lambda_i)\tau$ is $\lambda'_1, ..., \lambda'_n$ where $\lambda'_i = \lambda_i$ if $i \notin I$ and $\lambda'_i = \lambda_i^{-1}$ if $i \in I$. It follows from our assumption that $\nu_i \sim \lambda'_i$. Now since this is a proper parametrization, it must be the case that if $\lambda_i, \lambda_{i+1}$ are consecutive, then $i \in I$ if and only if $i+1 \in I$. So $\lambda_i, \lambda_{i+1}$ is consecutive and of alternating sign if and only if $\lambda'_i, \lambda'_{i+1}$ has this property. It follows that $\nu_i, \nu_{i+1}$ are consecutive and of alternating sign if and only if $\lambda_i, \lambda_{i+1}$ has this property. Therefore in this case $A_{\Delta, \rho_1} = A_{\Delta, \rho_2}$.

Case 3: This follows from combining the first two cases. □

Consider a unit $n$-cube $\square^n = [0, 1] \times \ldots \times [0, 1]$ in $\mathbb{R}^n$. We shall use $(z_1, ..., z_n)$ as notation for the set of coordinates of vectors in $\mathbb{R}^n$ with respect to the standard choice of orthonormal basis for $\mathbb{R}^n$. We now produce a cellular decomposition of $\square^n$ whose 1-skeleton is $\Delta$.

**Definition 5.20.** Let $\Delta$ be an $n$-cluster. A $\Delta$-system of equations and inequalities on $z_1, ..., z_n$ comprises of a subcollection of equalities and inequalities from the following collection:

1. The equalities
   
   \[ z_j = z_{j+1} = \ldots = z_{j+k} \]
   
   where $\{j, ..., j+k\} \in A_{\Delta}$.

2. The equalities
   
   \[ z_j = \ldots = z_{j+k} = b_i \]
   
   for $b_i \in \{0, 1\}$ and $\{j, ..., j+k\} \in A_{\Delta}$.

3. The inequalities
   
   \[ z_j = \ldots = z_{j+k} < z_{j+k+1} = \ldots z_{j+k+t} \]
   
   and
   
   \[ z_{j+k+1} = \ldots z_{j+k+t} < z_j = \ldots = z_{j+k} \]
   
   for $\{j, ..., j+k\}, \{j+k+1, ..., j+k+t\} \in A_{\Delta}$

4. For each $z_i$, the inequalities
   
   \[ z_i < 1, z_i > 0, 0 < z_i < 1 \]
   
   and equations
   
   \[ z_i = 1, z_i = 0 \]

**Definition 5.21.** Given an $\Delta$-system $\Xi$ of inequalities and equalities, there is a convex set $\text{Conv}_{\square^n}(\Xi) = \mathbb{R}^n \cap \square^n$ (possibly empty) that is defined by the system.

The convex sets $\{\text{Conv}_{\square^n}(\Xi) \mid \Xi \text{ is a } \Delta\text{-system}\}$ are partially ordered by inclusion. The minimal elements of this partial order form a cellular decomposition of $\square^n$ which we call $\Delta(\square^n)$.

In the next lemma we will show that the graphs $(\Delta(\square^n))^{(1)}$ are $\Delta$ are naturally isomorphic.
Lemma 5.22. Let $\Delta$ be parametrized by $\lambda_1, \ldots, \lambda_n$ at $\tau$, such that this is a proper parametrization. The map $f : \Delta^{(0)} \to \rho(\square^n)^{(0)}$ that maps

$$F((\prod_{i \in I} \lambda_i)\tau) \to (\chi_I(1), \ldots, \chi_I(n))$$

extends to a graph isomorphism.

Proof. First we show that if for sets $I, J \subseteq \{1, \ldots, n\}$ the 0-cells

$$F((\prod_{i \in I} \lambda_i)\tau), F((\prod_{i \in J} \lambda_i)\tau)$$

form an edge in $\Delta$, then the 0-cells $(\chi_I(1), \ldots, \chi_I(n)), (\chi_J(1), \ldots, \chi_J(n))$ of $\Delta(\square^n)^{(1)}$ form an edge. We know that $(\prod_{i \in I} \lambda_i)(\prod_{j \in J} \lambda_j)^{-1}$ is a special form. Now consider

$$(\prod_{i \in I} \lambda_i)(\prod_{j \in J} \lambda_j)^{-1} = (\prod_{i \in I \setminus J} \lambda_i)(\prod_{j \in J \setminus I} \lambda_j)^{-1}$$

Since this is a special form and the parametrization is proper, it must be the case that either $I \subseteq J$ or $J \subseteq I$. Let us assume that $J \subseteq I$. (The other case is similar.)

Now consider the partition of $\{1, \ldots, n\}$ consisting of singletons $L_j = \{j\}$ for $j \notin I$ and the sets $I \setminus J, J$. The system of equations and inequalities,

1. $x_j = 0$ if $j \notin I$.
2. $x_j = 1$ if $j \in J$.
3. $x_{i+1} = x_{i+2} = \ldots = x_{i+k}$ where $I \setminus J = \{i+1, i+2, \ldots, i+k\}$.

is a $\Delta$-system which defines a 1-dimensional subspace of $\mathbb{R}^n$ whose intersection with $\square^n$ is precisely the 1-cell connecting the vertices $(\chi_I(1), \ldots, \chi_I(n)), (\chi_J(1), \ldots, \chi_J(n))$.

Now we show the other direction. Let $e$ be a 1-cell in $\rho(\square^n)$. By definition there is a $\rho$-system $\Xi$ that describes a 1-dimensional affine subspace of $\mathbb{R}^n$ whose intersection with $\square^n$ is the 1-cell $e$. Such a system must have the following form.

There are disjoint sets $I', J' \subseteq \{1, \ldots, n\}$ and a partition consisting of the set $I'$ and sets $\{x_i\}$ for each $i \notin I'$ such that the system is described by the following list of equations and inequalities.

1. $x_i = \chi_{J'}(i)$ for each $i \notin I'$.
2. $x_{j+1} = x_{j+2} = \ldots = x_{j+l}$ where $I' = \{j+1, j+2, \ldots, j+l\} \in A_\Delta$.

Then it follows that $e$ is the image of the 1-cell in $\Delta$ connecting the 0-cells

$$F((\prod_{i \in J'} \lambda_i)\tau), F((\prod_{i \notin J' \cup I'} \lambda_i)\tau)$$

This proves our assertion. \qed

We shall show that each $n$-cluster can be filled in this way such that the following holds.

1. The filling of a cluster is independent of choice of proper parametrization.
2. If $\Delta_1$ is a subcluster of $\Delta_2$ then the subcomplex of the filling of $\Delta_2$ that comprises of the cells whose boundary 1-skeleton is a subset of $\Delta_1$ is isomorphic with the filling of $\Delta_1$ in a natural way. Since by Lemma 5.15 the intersection of a set of clusters is a cluster, the filling of any two clusters in $X^{(1)}$ is compatible.
Observe that in Lemma 5.22 the map depends upon the choice of base-vertex $F\tau$, i.e. the 0-cell $F\tau$ is mapped to the vector $(0,...,0)$ in $\mathbb{R}^n$. We will show that the prescribed CW-filling of $\Delta$ is independent of choice of basepoint, or the proper parametrization. This will imply (1) above.

**Lemma 5.23.** Let $\Delta$ be a cluster with two different proper parametrizations.

1. $\rho_1$: Based at $\tau_1$ and with parameters $\lambda_1,...,\lambda_n$.
2. $\rho_2$: Based at $\tau_2$ and with parameters $\nu_1,...,\nu_n$

Then the identity map $i : \Delta \rightarrow \Delta$ extends to an isomorphism of the CW complexes that are obtained from the fillings with respect to the two parameterizations.

**Proof.** The proof comprises of two steps:

1. We consider the case where $(\prod_{i\in I} \lambda_i)\tau_1 = \tau_2$ for some $I \subseteq \{1,...,n\}$.
2. We consider the case where $F((\prod_{i\in I} \lambda_i)\tau_1) = F\tau_2$ or equivalently $(\prod_{i\in I} \lambda_i)\tau_1 \sim \tau_2$ for some $I \subseteq \{1,...,n\}$.

Case (1): In this case, $\tau_2 = (\prod_{i\in I} \lambda_i)\tau_1$ for some $I \subseteq \{1,...,n\}$ and $\nu_i = \lambda_i$ if $i \notin I$ and $\nu_i = \lambda_i^{-1}$ if $i \in I$. It follows from the definition of proper parametrization that if $\lambda_i, \lambda_{i+1}$ are consecutive, then $i \in I$ if and only if $i+1 \in I$.

In this case we shall consider two different coordinate systems for $\mathbb{R}^n$. These coordinate system have the same underlying $\mathbb{R}^n$, but different origin and orthonormal bases vectors. However the two coordinate systems are related by the composition of an affine transformation with orientation reversing transformations.

The first coordinate system consists of an origin $o_1 = (0,...,0)$ and unit basis vectors $e_1 = (1,0,...,0), e_2 = (0,1,0,...,0),...,e_n = (0,0,...,1)$. We consider the unit square $\square^n = [0,1] \times ... \times [0,1]$ in this coordinate system. We denote this system by $(\mathbb{R}^n, C_1)$.

We now consider a different, but related coordinate system denoted by $(\mathbb{R}^n, C_2)$. The origin for this system is the point $o_2 = (\chi_I(1), \chi_I(2),...,\chi_I(n))$ in our original system. The orthonormal basis vectors $(u_1,...,u_n)$ in $(\mathbb{R}^n, C_2)$ correspond to the 1-cells incident to $(\chi_I(1),...,\chi_I(n))$ in $\square^n$. More formally, in $(\mathbb{R}^n, C_1)$, the vectors $u_i$ are $u_i = e_i$ if $i \notin I$ and $u_i = -e_i$ if $i \in I$.

We claim that the set of minimal elements of the partial order of convex sets of $\mathbb{R}^n$ obtained by $\Delta$ systems in the coordinate system $(\mathbb{R}^n, C_1)$ is the same as those obtained from $\Delta$ systems with the coordinate system $(\mathbb{R}^n, C_2)$.

Given an inequality $z_i < z_j$ in a $\Delta$ system, by definition of proper parametrization and 5.20 we know that $\lambda_i, \lambda_j$ must be consecutive elements so in particular $i \in I$ if and only if $j \in I$. So the inequality $z_i < z_j$ in the $\Delta$ system represents the same region of $\mathbb{R}^n$ in the system $(\mathbb{R}^n, C_2)$ as $z_j < z_i$ in the $\Delta$ system in the coordinate system $(\mathbb{R}^n, C_2)$. An equality $x_i = x_j$ produces the same hyperplane of $\mathbb{R}^n$ in each system. If $i \notin I$ the equality $x_i = a$ for $a \in (0,1)$ in the coordinate system $(\mathbb{R}^n, C_1)$ corresponds to the same hyperplane in $\mathbb{R}^n$ as $x_i = b$ in the coordinate system $(\mathbb{R}^n, C_2)$ for $b \in (0,1) \setminus \{a\}$. If $i \in I$, they correspond to the same hyperplane of $\mathbb{R}^n$ in each system. This proves our claim.

Case (2): This case follows from combining the first case together with Lemma 5.19 and Definitions 5.20 and 5.21.

This justifies introducing the following notation.

**Definition 5.24.** We denote the (unique) filling of a cluster $\Delta$ with respect to some (any) proper parametrization as $\tilde{\Delta}$.
Lemma 5.25. Let $\Delta_1$ be a subcluster of $\Delta_2$. The inclusion $\phi : \Delta_1 \hookrightarrow \Delta_2$ induces an isomorphism of the CW complex $\bar{\Delta}_1$ with the subcomplex of $\bar{\Delta}_2$ which consists of the union of cells in $\bar{\Delta}_1$ whose boundary 1-skeleton is a subgraph of $\bar{\Delta}_1$.

Proof. The proof has two cases.

(1) $\Delta_1$ is a facial subcluster of $\Delta_2$.

(2) $\Delta_1$ is a diagonal subcluster of $\Delta_2$.

In the first case, we have proper parametrizations of $\Delta_1, \Delta_2$ as follows: We start with a proper parametrization $\rho_2$ of $\Delta_2$ as based at $\tau$ and parametrized by $\lambda_1, ..., \lambda_n$. Then there are disjoint sets $I, J \subseteq \{1, ..., n\}$ such that $\Delta_1$ is based at $(\prod_{j \in I} \lambda_j)\tau$ and parametrized by $\lambda_1, ..., \lambda_i$ for the ordered set $\{i_1, ..., i_l\} = J$, for which $i_j < i_k$ if $j < k$. It follows that this is a proper parametrization for $\Delta_1$.

Let $\Xi$ be a $\Delta_2$-system describing a cell of $\Delta_2$ whose boundary 1-skeleton is a subgraph of $\Delta_1$. It follows that the following equations must hold in $\Xi$.

(1) $z_j = 0$ for each $j \in \{1, ..., n\} \setminus (I \cup J)$.

(2) $z_j = 1$ for each $j \in I$.

Observe that any such system naturally corresponds to the cell of $\Delta_1$ that is represented by the $\Delta_1$-system $\Xi'$ which consists of the same equations and inequalities in $\Xi$ except that the equalities or inequalities that contain $z_j$ are removed for each $j \notin J$.

Now consider a cell $e$ in the filling of $\Delta_1$ given by an $\Delta_1$ system $\Xi'$. This naturally corresponds to an $\Delta_2$ system $\Xi$ obtained by replacing $z_j$ with $z_{i,j}$ in $\Xi'$ together with the equations $z_j = 0$ if $j \notin I \cup J$ and $z_j = 1$ if $j \in I$.

This system represents the cell $\phi(e)$ in $\Delta_2$. Therefore $\phi$ is an isomorphism of $\bar{\Delta}_1$ with the image $\phi(\Delta_1)$ in $\bar{\Delta}_2$.

Now we consider the second case. Let $\Delta_2$ be parametrized by $\rho_2$ as above. Assume that $\Delta_1$ is not a diagonal subcluster of a proper facial subcluster of $\Delta_2$, since otherwise from the first case of our proof we can replace $\Delta_2$ with that facial subcluster in the statement of the Lemma. By Lemma 5.10, we can parametrize $\Delta_1$ in the following way: It is based at $\tau$ and the parameters are $\nu_1, ..., \nu_k$ such that:

(1) There is a partition $I_1, ..., I_k$ of $\{1, ..., n\}$ such that each $\prod_{i \in I_j} \lambda_i$ is a special form, and if $i < j$ and $p \in I_i, q \in I_j$ then $p < q$.

(2) $\nu_j = \prod_{i \in I_j} \lambda_i$.

We call this parametrization $\rho_1$.

Let $\Xi$ be an $\Delta_1$ system that represents a cell of $\bar{\Delta}_1$. We produce a $\Delta_2$ system $\Xi'$ that respresents the cell $\phi(e)$. This system comprises of the following set of inequalities and equalities:

(1) For each $1 \leq l \leq k$ and each $i, j \in I_l$ an equation $z_i = z_j$.

(2) We choose representatives $j_i \in I_l$ for each $1 \leq i \leq k$. For each inequality/equation $\mu$ in $\Xi$, we include the inequality/equation $\mu'$ obtained by replacing the symbol $z_i$ (that represents a coordinate in $\mathbb{R}^k$) with the symbol $z_{i,j}$ (that represents a coordinate in $\mathbb{R}^l$) and include it in $\Xi'$.

The system $\Xi'$ defines the cell $\phi(e)$ in $\bar{\Delta}_2$.

Next we show that every $\Delta_2$ system that defines a cell whose boundary 1-skeleton is in $\Delta_1 \subset \Delta_2$ is the $\phi$-image of a cell represented by a corresponding $\Delta_1$ system.
Given such a cell \( e \), by convexity it follows that \( e \) is contained in the convex hull of the boundary vertices of \( e \) which are all in \( \Delta_1([n]) \subseteq \Delta_2([n]) \). In fact, the subcomplex of \( \Delta_2 \) with 1-skeleton \( \Delta_1 \) is convex in \( \mathbb{R}^n \).

So in particular it follows that if \( \Xi \) is a \( \Delta_2 \)-system defining \( e \) then the following equations must hold in \( \Xi \). (possibly in addition to other equations and inequalities.) The equations \( z_i = z_j \) for each \( 1 \leq i, j \leq n \) such that \( i, j \in I_1 \) for some \( 1 \leq l \leq k \).

Now we construct an \( \Delta_1 \) admissible system \( \Xi' \) as follows. For any occurrence of \( z_i \) in an equation or inequality of \( \Xi \), we replace the symbol \( z_i \) by \( z_j \) where \( i \in I_1 \). The resulting \( \Delta_2 \) system is \( \Xi' \), and it defines a cell \( e' \) in \( \Delta_1 \) such that \( \phi(e') = e \). □

By Lemma 5.26 that since the intersection of a set of clusters is a cluster. So the filling is naturally well defined under intersection. Consider clusters \( \Delta_1, ..., \Delta_n \) and \( \Delta = \bigcap_{1 \leq i \leq n} \Delta_i \). Since \( \Delta \) is a subcluster of every cluster \( \Delta_1, ..., \Delta_n \), the filling of \( \Delta \) agrees with the filling inherited by \( \Delta \) from each cluster \( \Delta_i \). Now we can finally define our complex \( X \).

**Definition 5.26.** The complex \( X \) is the union of filled clusters in the set

\[ \{\tilde{\Delta} \mid \Delta \text{ is a cluster of } X^{(1)}\} \]

The following lemma follows from the definitions.

**Lemma 5.27.** Let \( \Delta \) be a cluster in \( X^{(1)} \). Then \( \tilde{\Delta} \) is a topological n-cube. If \( \Delta' \) is a k-facial subcluster of \( \Delta \), then \( \tilde{\Delta}' \) is an k-cube which is a face of \( \tilde{\Delta} \). Moreover, if \( F \) is an k-subcube of \( \tilde{\Delta} \) then \( F^{(1)} \) is a k-facial subcluster of \( \Delta \).

Now we show that every \( n \)-cell \( e \) of \( X \) is contained in the filling of an \( n \)-cluster.

**Lemma 5.28.** Let \( e \) be an \( n \)-cell of \( X \). Then \( e \) is contained in \( \tilde{\Delta} \) where \( \Delta \) is an \( n \)-cluster of \( X \).

**Proof.** Let \( e \) be the \( n \)-cell of a \( k \)-cluster \( \Delta \) that is based at \( \tau \) and parametrized by \( \lambda_1, ..., \lambda_k \). Assume that \( k > n \). We will show that \( e \) is contained in a subcluster of \( \Delta \) of dimension less than \( k \). Let \( \rho \) be a proper parametrization of \( \Delta \).

Recall that by definition 5.20 we know that \( e \) corresponds to a convex subset of \( \mathbb{R}^n \), in particular a cell of \( \Delta([n]) \). This is represented by an \( \Delta \)-system of equations and inequalities, which we denote as \( \Xi \). We claim that there is an equation of the form \( z_i = z_j \), where \( i = 0 \) or \( i = 1 \) holds for the system \( \Xi \).

If there are no equations of the above type, then the convex region is described by \( H \cap [n] \), where \( H \) is an intersection of open half spaces of \( \mathbb{R}^n \). Such a region must be either empty or \( k \)-dimensional. Since \( e \) is \( n \)-dimensional and we assumed that \( n < k \), it follows that there must be an equation of the above type. There are three cases to consider.

1. If \( z_i = z_j \) for \( i < j \), by definition 5.23 the equation \( z_i = z_{i+1} = ...z_{i+l} \) (where \( j = i + l \)) is in \( \Xi \). So in fact \( \Xi \) describes a cell of a subcluster properly parametrized with basepoint \( \tau \), and parameters \( \lambda_1, ..., \lambda_i, \lambda_{i+1}, ..., \lambda_k \).

2. If \( z_i = 1 \) (and \( z_i = 0 \)), then the cell \( e \) lies in the subcluster properly parametrized with basepoint \( \lambda_1\tau \) (or respectively \( \tau \)) and parameters \( \lambda_1, ..., \lambda_i-1, \lambda_{i+1}, ..., \lambda_k \).

This proves our assertion. □
We end this section by showing that an $n$-cell of an $n$-cluster is incident to the “extreme points” of a proper parametrization.

**Lemma 5.29.** Let $e$ be an $n$-cell in $\Delta$, where $\Delta$ is an $n$-cluster with a proper parametrization based at $\tau$ and with parameters $\lambda_1, \ldots, \lambda_n$. Then the 0-cells

$$F\tau, F\left(\prod_{i \in \{1,n\}} \lambda_i\right)$$

are incident to $e$.

**Proof.** By definition 5.20 the $n$-cell $e$ is precisely the intersection of $n$-halfspaces of the form $z_i > 0$, $z_i < 1$ or $z_i < z_j$. The 0-cells $F\tau, F\left(\prod_{i \in \{1,n\}} \lambda_i\right)$ lie in the boundary of every such half space, and we claim that as a consequence they must lie in the boundary of the intersection if the intersection is non-trivial. The intersection of two closed convex sets is closed and convex. Also, the intersection of open half spaces in $\mathbb{R}^n$, if nonempty, is $n$-dimensional. So assuming this intersection is nonempty, the intersection of the corresponding closed half spaces must contain our 0-cells as a limit point of a sequence of points in the intersection of the open half spaces. \qed

**Remark 5.30.** For the rest of the article, we shall denote a filled cluster $\bar{\Delta}$ as simply $\Delta$.

### 5.3. The action of $G$ on $X$.

In this section it will be our goal to show that the stabilizer of each cell of $X$ in $G$ is of type $F_\infty$. The following lemma is a crucial first step in understanding the stabilizer of cells.

**Lemma 5.31.** Let $e$ be an $n$-cell in $X$ contained in $\Delta$ where $\Delta$ is an $n$-cluster with a proper parametrization $\rho$ with basepoint $\emptyset$ and parameters $\lambda_1, \ldots, \lambda_n$. If $\nu \in \text{Stab}_G(e)$ then $\nu \in F$.

**Proof.** First note that by Lemma 5.29 we know that the 0-cells $F, F\left(\prod_{i \in \{1,n\}} \lambda_i\right)$ are incident to $e$. Define

$$P = \{I \subseteq \{1,n\} \mid F\left(\prod_{i \in I} \lambda_i\right) \text{ is a 0-cell incident to } e\}$$

By our observation above we know that $\{\emptyset\}, \{1,...,n\} \in P$.

Let $\nu \in G$ be an element such that $e \cdot \nu = e$. Since $e \cdot \nu = e$ it follows that the action of $\nu$ induces a permutation of the set of 0-cells $V(e) = \{F\left(\prod_{i \in I} \lambda_i\right) \mid I \in P\}$. So in particular, it induces a permutation of the set $P$.

Let $I_1 \in P$ be the set with the property that $F \cdot \nu = F\left(\prod_{i \in I_1} \lambda_i\right)$. This means that $\nu = f\left(\prod_{i \in I_1} \lambda_i\right)$ for some $f \in F$. We claim that $I_1 = \emptyset$. Assume that this is not the case and let $I_2 \in P$ be the set with the property that

$$F\left(\prod_{i \in \{1,n\}} \lambda_i\right) \cdot f\left(\prod_{i \in I_1} \lambda_i\right) = F\left(\prod_{i \in I_2} \lambda_i\right)$$

Let $\lambda'_i$ be special forms such that $\lambda'_i \sim \lambda_i \cdot f$. It follows that

$$F\left(\prod_{i \in \{1,n\}} \lambda_i\right) \cdot f\left(\prod_{i \in I_1} \lambda_i\right) = F\left(\prod_{i \in I_1} \lambda'_i\right)\left(\prod_{i \in I_1} \lambda_i\right)$$

Let $I_2 \in P$ be the set with the property that

$$F\left(\prod_{i \in I_2} \lambda_i\right) \cdot F\left(\prod_{i \in I_1} \lambda_i\right) = F\left(\prod_{i \in I_2} \lambda'_i\right)\left(\prod_{i \in I_1} \lambda_i\right)$$
It follows that
\[ F(\prod_{i \in I_2} \lambda'_i) = F(\prod_{i \in I_2} \lambda_i) \]

We claim that \( I_2 = \{1, \ldots, n\} \). Assume that this is not the case. Consider the smallest \( k \in \{1, \ldots, n\} \) such that either \( k \in I_2, k + 1 \notin I_2 \) or \( k \notin I_2, k + 1 \in I_2 \). Let us assume the former case. (The latter case is similar.) We claim that this implies that
\[ F(\prod_{i \in \{1, \ldots, n\}} \lambda_i) \cdot (f \prod_{j \in I_1} \lambda_j) \]
is not a 0-cell in \( e \).

Now we have
\[ F(\prod_{i \in \{1, \ldots, n\}} \lambda_i) \cdot (f \prod_{j \in I_1} \lambda_j) = F(\prod_{i \in I_1} \lambda_i)(\prod_{j \in I_2} \lambda'_j)(\prod_{i \in I_1} \lambda_i) \]

There is a \( j \in I_1 \) such that
1. \( \lambda_j, \lambda'_k \) are consecutive special forms of alternating type for some \( k \in I_2 \).
2. \( \lambda_j, \lambda'_{k-1} \) have subwords \( \tau_1, \tau_2 \) and \( \nu_1, \nu_2 \) such that \( \lambda_j = \tau_1 \tau_2, \lambda'_k = \nu_1 \nu_2 \) and \( \tau_2, \nu_2 \) are equivalent special forms.

This means that the subword \( \lambda'_{k-1} \lambda_j \) of
\[ F(\prod_{i \in I_2} \lambda'_i)(\prod_{i \in I_2} \lambda'_i)(\prod_{i \in I_1} \lambda_i) \]
contains a calculation of exponent 2. So by Lemma 4.16 it follows that the word
\[ (\prod_{i \in I_1} \lambda_i)(\prod_{i \in I_2} \lambda'_i)(\prod_{i \in I_1} \lambda_i) \]
contains a calculation with no potential cancellations and two occurrences of \( y, y^{-1} \).

This contradicts the fact that
\[ F(\prod_{i \in I_1} \lambda'_i)(\prod_{i \in I_1} \lambda'_i)(\prod_{i \in I_1} \lambda_i) \]
is a 0-cell of \( e \).

So it must be the case that \( I_2 = \{1, \ldots, n\} \). So we have
\[ F(\prod_{i \in \{1, \ldots, n\}} \lambda_i)(f \prod_{j \in I_1} \lambda_j) = F(\prod_{i \in \{1, \ldots, n\}} \lambda_i) \]
This implies that
\[ F(\prod_{i \in \{1, \ldots, n\}} \lambda'_i) = F(\prod_{i \in I_2} \lambda_i) \]

But this means that for any \( I \in P \)
\[ F(\prod_{i \in I} \lambda_i) \cdot (f \prod_{i \in I} \lambda_i) = F(\prod_{i \in I} \lambda'_i)(\prod_{i \in I} \lambda_i) \neq F \]

This means that there is no 0-cell \( u \) that gets mapped to \( F \) under this action, which is a contradiction. Therefore, we must have that \( I_1 \) is empty and hence \( \nu \in F \).

\[ \square \]

Lemma 5.32. Let \( e \) be an \( n \)-cell of \( X \). Every element of \( G \) that stabilizes \( e \) must stabilize every 0-cell of \( e \).
Proof. Let \( \Delta \) be an \( n \)-cluster in \( X \) containing \( e \). We can assume without loss of generality that \( \Delta \) is properly parametrized as follows. It is based at \( \emptyset \), and has parameters \( \lambda_1, \ldots, \lambda_n \). This is because if \( \Delta \) is properly parametrized and based at \( \tau \), then the stabilizer of cells of \( \Delta \) in \( G \) is conjugate to the stabilizer of corresponding cells of the cluster \( \Delta \cdot \tau^{-1} \) in \( G \). By the previous Lemma, we know that \( \text{Stab}_G(e) = \text{Stab}_F(e) \).

Just as in the proof of the previous Lemma, we define

\[
P = \{ I \subseteq \{1, \ldots, n\} \mid F(\prod_{i \in I} \lambda_i) \text{ is a 0-cell incident to } e \}\]

By our assumption above we know that \( \{\emptyset\}, \{1, \ldots, n\} \in P \).

We now define a linear order on the 0-cells incident to \( e \), as follows. Let \( I_1, I_2 \in P \) and \( F(\prod_{i \in I_1} \lambda_i), F(\prod_{i \in I_2} \lambda_i) \) be 0-cells incident to \( e \). Then

\[F(\prod_{i \in I_1} \lambda_i) < F(\prod_{i \in I_2} \lambda_i)\]

if either \( I_1 \subseteq I_2 \) or \( \inf(I_2 \setminus I_1) < \inf(I_1 \setminus I_2) \). This is easily seen to be transitive and it follows immediately that this is a linear order.

We claim that the action of \( \text{Stab}_F(e) \) preserves this linear order. Let \( I_1, I_2 \in P \) such that \( F(\prod_{i \in I_1} \lambda_i) < F(\prod_{i \in I_2} \lambda_i) \) and let \( F(\prod_{i \in I_1} \lambda_i) \cdot f = F(\prod_{i \in I_2} \lambda_i) \) and \( F(\prod_{i \in I_2} \lambda_i) \cdot f = F(\prod_{i \in I_1} \lambda_i) \). We claim that \( F(\prod_{i \in I_1} \lambda_i) < F(\prod_{i \in I_2} \lambda_i) \). Observe that \( F(\prod_{i \in I_1} \lambda_i) < F(\prod_{i \in I_2} \lambda_i) \) if and only if the word

\[(\prod_{i \in I_1} \lambda_i)(\prod_{i \in I_2} \lambda_i)^{-1} = (\prod_{i \in I_2 \setminus I_1} \lambda_i^{-1})(\prod_{i \in I_1 \setminus I_2} \lambda_i)\]

expressed in standard form as \( \lambda_{t_1}^{s_1} \ldots \lambda_{t_n}^{s_n} \) with \( t_i \in \{1, -1\} \), \( s_i \in \{1, \ldots, n\} \) and \( s_i \leq_{\text{lex}} s_j \) if \( i < j \) has the property that \( t_1 = -1 \). It is easily checked by considering the action of \( F \) that this property also holds for

\[(\prod_{i \in I_1} \lambda_i)(\prod_{i \in I_2} \lambda_i)^{-1} = (\prod_{i \in I_2 \setminus I_1} \lambda_i^{-1})(\prod_{i \in I_1 \setminus I_2} \lambda_i)\]

expressed in standard form as \( \lambda_{q_1}^{p_1} \ldots \lambda_{q_m}^{p_m} \) just as above. That is, \( q_1 = -1 \).

Since an order preserving bijection of a finite totally ordered set is in fact the identity map, our claim follows.

\[\square\]

**Proposition 5.33.** The action of \( G \) on the complex \( X \) satisfies the following properties:

1. \( X/G \) has finitely many cells in each dimension.
2. The stabilizer of each cell is of type \( F_{\infty} \).

**Proof.** We know by Lemma 5.17 that the action of \( G \) on the set of \( n \)-clusters has precisely \( 2^{(2n-1)} \) orbits. Moreover, if \( e \) is an \( n \)-dimensional cell, then by Lemma 5.28 it is contained in the filling of an \( n \)-cluster. So the first statement holds.

We now prove the second statement. Let \( e \) be an \( n \)-cell. Let \( \Delta \) be the \( n \)-cluster containing \( e \), which without loss of generality is assumed to be properly parametrized as based at \( \emptyset \) and with parameters \( \lambda_1, \ldots, \lambda_n \). We know from the previous two Lemmas that \( \text{Stab}_G(e) = \text{Stab}_F(e) \) and elements of \( \text{Stab}_F(e) \) stabilize each vertex of \( e \).
As in the previous two Lemmas, let

\[ P = \{ I \subseteq \{ 1, \ldots, n \} \mid F(\prod_{i \in I} \lambda_i) \text{ is a 0-cell incident to } e \} \]

We first observe a general fact before proceeding to prove the Lemma. Let \( I_1, I_2 \subseteq \{ 1, \ldots, n \} \) be sets such that \( I_1 = \{ i, i+1, \ldots, i+l \} \), \( I_2 = \{ j, j+1, \ldots, j+m \} \). If \( f \in F \) stabilizes the cosets \( F(\prod_{i \in I_1} \lambda_i), F(\prod_{i \in I_2} \lambda_i) \) then in fact \( f \) stabilizes the cosets \( F(\prod_{i \in I_1 \cap I_2} \lambda_i), F(\prod_{i \in I_1 \setminus I_2} \lambda_i), F(\prod_{i \in I_2 \setminus I_1} \lambda_i) \). We remark that \( \prod_{i \in I_1 \cap I_2} \lambda_i, \prod_{i \in I_1 \setminus I_2} \lambda_i, \prod_{i \in I_2 \setminus I_1} \lambda_i \) are special forms by definition unless they are empty.

Let \( P' \) be the smallest collection of subsets of \( \{ 1, \ldots, n \} \) such that

1. \( P \subseteq P' \).
2. \( P' \) is closed under the operations of taking intersection and set difference.

The elements of \( P' \) are ordered by inclusion. Let \( I_1, \ldots, I_k \) be the minimal elements of \( P' \) in this ordering. Observe that,

1. \( I_1, \ldots, I_k \) forms a partition of \( \{ 1, \ldots, n \} \).
2. \( \nu_i = \prod_{j \in I_i} \lambda_j \) is a special form for each \( 1 \leq i \leq k \).
3. \( \nu_i, \nu_j \) are independent if \( i \neq j \).

It follows that \( \text{Stab}_F(e) = \cap_{1 \leq i \leq k} \text{Stab}_F(F(\prod_{j \in I_i} \lambda_j)) \). By Lemma 4.13 this is a group of type \( F_{\infty} \).

\[ \square \]

6. \( X \) is simply connected.

In this section we will show that \( X \) is simply connected. This is shown by providing an explicit homotopy for every loop with a trivial loop. Given a loop \( l \) in \( X \), we can assume that \( l \) is a path along the 1-skeleton that is parametrized by a sequence of 0-cells

\[ F \tau, F(\nu_k \tau), F(\nu_{k-1} \nu_k \tau), \ldots, F(\nu_1 \ldots \nu_k \tau), F \tau \]

where \( \tau \) is a \( Y \)-word, \( \nu_1, \ldots, \nu_k \) are special forms, and the \( Y \)-word \( \prod_{1 \leq i \leq k} \nu_i \) is equivalent to a special form. (Note that it may not be the case that \( \nu_1 \ldots \nu_k \) is itself a special form, or that pairs \( \nu_i, \nu_j \) are independent. Since the last two vertices in the sequence are connected by an edge we deduce that \( F(\nu_1 \ldots \nu_k) \in \Gamma \) and in particular \( \nu_1 \ldots \nu_k = f \psi \) where \( f \in F \) and \( \psi \) is a special form.)

There are some basic homotopies that correspond to moves in the analysis of standard forms. We list them below:

1. The expansion substitution \( y_\sigma \to x_\sigma y_\sigma^0 y_\sigma^{-1} y_{\sigma 11} \) corresponds to homotopic paths \( F \tau, F(y_\sigma \tau) \) and

\[ F \tau, F(y_{\sigma 11} \tau), F(y_{\sigma 10} y_{\sigma 11} \tau), F(y_\sigma y_{\sigma 10} y_{\sigma 11} \tau) \]

2. Similarly the expansion substitution \( y_\sigma^{-1} \to x_\sigma^{-1} y_\sigma^0 y_\sigma 0 y_\sigma^{-1} \) corresponds to a homotopy between paths \( F \tau, F(y_\sigma \tau) \) and

\[ F \tau, F(y_{\sigma 1} \tau), F(y_{\sigma 10} y_{\sigma 1} \tau), F(y_{\sigma 00} y_{\sigma 10} y_{\sigma 1} \tau) \]

(These paths are homotopic in \( X \) because they are homotopic in a 3-cluster of \( X \).)
(2) Consider the path $F\tau, F(y_s^t f \tau)$. Performing expansion substitutions on $y_s^t$ we obtain special form $y_s^{t_1} \ldots y_s^{t_n} \sim y_s^t$ such that $f$ acts on $s_1, \ldots, s_n$. Our path is homotopic to the path described by the sequence

\[ F\tau, F(y_s^{t_1} f \tau), F(y_s^{t_1-1} y_s^{t_2} f \tau), \ldots, F(y_s^{t_1} \ldots y_s^{t_n} f \tau) \]

Applying the rearranging substitution

\[ y_s^{t_1} \ldots y_s^{t_n} f = f y_s^{t_1} \ldots y_s^{t_n} = y_s^{t_1} \ldots y_s^{t_n} \]

produces a reparametrization of this path as

\[ F, F(y_s^{t_1} f \tau), F(y_s^{t_1-1} y_s^{t_2} f \tau), \ldots, F(y_s^{t_1} \ldots y_s^{t_n} f \tau) \]

(3) A commuting substitution $y_s^i y_p^q = y_p^q y_s^i$ for $t, q \in \{-1, 1\}$ and $s, p$ incompatible, corresponds to a homotopy between paths of the form $F\tau, F(y_s^t f \tau), F(y_p^q y_s^t f \tau)$ and $F\tau, F(y_p^q f \tau), F(y_p^q y_s^t f \tau)$, for any $\tau \in G$. This can be performed in the complex since these paths are homotopic in a 2-cluster.

(4) A cancellation substitution $y_s^i y_p^{-t} \rightarrow \emptyset$ corresponds to shrinking a path of the form $F\tau, F(y_s^{t-\tau} f \tau), F(y_s^{t-\tau} f \tau)\rightarrow \text{trivial path}$.

**Lemma 6.1.** Given a loop

\[ L = \{ F\tau, F(\lambda_n \tau), F(\lambda_{n-1} \lambda_n \tau), \ldots, F(\lambda_1 \ldots \lambda_n \tau) \} \]

where $\lambda_1, \ldots, \lambda_n$ are special forms, it is homotopic to a loop of the form

\[ F\tau, F(y_s^{t_m} \tau), F(y_s^{t_{m-1}} y_s^{t_m} \tau), \ldots, F(y_s^{t_1} \ldots y_s^{t_m} \tau) \]

**Proof.** We show this by induction on $n$. For $n = 1$, let $\lambda_1 = y_p^{q_1} \ldots y_p^{q_k}$. Now the 1-cell $\{F\tau, F(\lambda_1 \tau)\}$ is the cross diagonal 1-cell of the $k$-cluster at $\tau$ parametrized by the special forms $y_p^{q_1}, \ldots, y_p^{q_k}$. It follows that this 1-cell is homotopic to the path

\[ F\tau, F(y_p^{q_1} \tau), F(y_p^{q_1-1} y_p^{q_2} \tau), \ldots, F(y_p^{q_1} \ldots y_p^{q_k} \tau) \]

The inductive step is essentially the same as the base case, since by the inductive hypothesis we replace the path

\[ \{F\tau, F(\lambda_1 \tau), F(\lambda_{n-1} \lambda_n \tau), \ldots, F(\lambda_2 \ldots \lambda_n \tau), F(\lambda_1 \ldots \lambda_n \tau) \} \]

by a path

\[ F\tau, F(y_s^{t_1} \tau), F(y_s^{t_1-1} y_s^{t_2} \tau), \ldots, F(\lambda_1 y_s^{t_1} \ldots y_s^{t_n} \tau) \]

and then argue the last edge

\[ F(y_s^{t_1} \ldots y_s^{t_n} \tau), F(\lambda_1 y_s^{t_1} \ldots y_s^{t_n} \tau) \]

traversed in the path is homotopic to path of a suitable sequence of edges as in the base case.

\[ \square \]

**Proposition 6.2.** The complex $X$ is simply connected.

**Proof.** Let $L$ be a loop described as a path in $X^{(1)}$. By considering the group action we can assume that the loop $L$ begins and ends at $F$. By the previous Lemma this loop is homotopic to a loop of the form

\[ F, F(y_s^{t_n} \tau), F(y_s^{t_{n-1}} y_s^{t_n} \tau), \ldots, F(y_s^{t_1} \ldots y_s^{t_n} \tau) \]

Since the 0-cells $F = F(y_s^{t_1} \ldots y_s^{t_n} \tau)$ it follows that the word $y_s^{t_1} \ldots y_s^{t_n}$ represents an element of $F$. By the proof that $\langle S \mid R \rangle \cong \langle a, b, c \rangle$ in [22], we know that this word can be reduced to an $X$-word by applying the expansion, commuting and
rearranging substitutions. Since an application of each substitution produces a loop homotopic to $L$, we observe that this process provides an explicit homotopy between $L$ and the trivial loop.

7. A RELATION ON THE 1-CELLS OF $X$.

The rest of the paper will be devoted to proving that the higher homotopy groups of $X$ are trivial. In particular we will show that for any finite subcomplex $Y$ of $X$, there is a subcomplex $Y'$ of $X$ such that $Y \subseteq Y'$ and $Y'$ is homeomorphic to a nonpositively curved cube complex.

In this section we will define an equivalence relation on the 1-cells of $X$. This will play a crucial role in the proof.

**Definition 7.1.** Let $e_1, e_2$ be 1-cells in $X$ that admit parametrizations of the form $e_1 = \{F(\lambda \tau_1), F(\tau_1)\}$, $e_2 = \{F(\lambda \tau_2), F(\tau_2)\}$. Such parametrizations are called common parametrizations for $e_1, e_2$.

Given a common parametrization as above, if $\lambda'$ is a special form of the same type and parity as $\lambda$ then we can find a common parametrization

$$e_1 = \{F(\lambda' \tau_1'), F(\tau_1')\}, e_2 = \{F(\lambda' \tau_2'), F(\tau_2')\}$$

as follows. Let $f \in F$ such that $\lambda \cdot f \sim \lambda'$. Then $F(\lambda \tau_i) = F(\lambda' f^{-1} \tau_i)$, and so we get $\tau_i' = f^{-1} \tau_i$.

Now we define the notion of percolating support of a normal form, which will be useful in this section.

**Definition 7.2.** Given a word $\lambda$ in the infinite generating set, recall that the support of $\lambda$ in $\mathbb{R}$, or $\text{supp}(\lambda)$, is the closure of the set of points in $\mathbb{R}$ that are not fixed by $\lambda$. Given a normal form $f y_{s_1}^{t_1} ... y_{s_n}^{t_n}$ we denote the percolating support of $f y_{s_1}^{t_1} ... y_{s_n}^{t_n}$ as $\text{supp}_p(f y_{s_1}^{t_1} ... y_{s_n}^{t_n})$, and this is defined as $\text{supp}_p(f y_{s_1}^{t_1} ... y_{s_n}^{t_n}) = \text{supp}(y_{s_1}^{t_1} ... y_{s_n}^{t_n})$. So if an element $\lambda \in G$ has a normal form $f \nu$ for $f \in F$ and a $Y$-normal form $\nu$, then we define $\text{supp}_p(\lambda) = \text{supp}_p(f \nu) = \text{supp}(\nu)$. By our definition it follows that if $\psi \in G$ and $f \in F$, then $\text{supp}_p(\psi) = \text{supp}_p(f \psi)$.

The following lemmas are important technical properties of the notion of a percolating support. The first three are corollaries of Theorem 7.1 and we leave proofs of these to the reader.

**Lemma 7.3.** Let $g y_{s_1}^{t_1} ... y_{s_n}^{t_n}$ be a normal form where $g \in F$ and $y_{s_1}^{t_1} ... y_{s_n}^{t_n}$ is a $Y$-normal form. Let $I \subseteq \mathbb{R}$ be the set of reals for which the action of $y_{s_1}^{t_1} ... y_{s_n}^{t_n}$ does not preserve tail equivalence. Then

$$\text{supp}_p(g y_{s_1}^{t_1} ... y_{s_n}^{t_n}) = \text{supp}(y_{s_1}^{t_1} ... y_{s_n}^{t_n}) = I$$

**Lemma 7.4.** Let $f \in F$ and let $g y_{s_1}^{t_1} ... y_{s_n}^{t_n}$ be a normal form. Let $h y_{v_1}^{u_1} ... y_{v_m}^{u_m}$ be a normal form for the group element described by $f^{-1}(g y_{s_1}^{t_1} ... y_{s_n}^{t_n})$. Let $I = \text{supp}(g y_{s_1}^{t_1} ... y_{s_n}^{t_n})$ Then $\text{supp}(h y_{v_1}^{u_1} ... y_{v_m}^{u_m}) = f(I)$.

**Lemma 7.5.** Let $f \in F$ and let $\nu \in G$ be such that $\text{supp}(f) \subseteq \text{supp}((\nu)$. Then it follows that $\text{supp}_p(\nu) = \text{supp}_p(f \nu f^{-1})$.

**Lemma 7.6.** Let $\nu_1, \nu_2 \in G$ and let $I \subseteq \mathbb{R}$. If $I \subseteq \text{supp}_p(\nu_1)$ and $I \cap \text{supp}_p(\nu_2)$ is a nullset, then it follows that $I \subseteq \text{supp}_p(\nu_1 \nu_2)$. 
Proof. Let \( f y_{s_1}^{t_1} ... y_{n}^{t_n}, gy_{u_1}^{v_1} ... y_{u_m}^{v_m} \) be normal forms for the elements \( \nu_1, \nu_2 \) respectively such that \( f, g \in F \).

First observe that by our assumption it follows that in fact \( I \cap \text{supp}(g), I \cap \text{supp}(y_{u_1}^{v_1} ... y_{u_m}^{v_m}) \) are nullsets. The set \( I \cap \text{supp}(g) \) a nullset because otherwise there is an interval \( I_1 \subseteq (I \cap \text{supp}(g)) \) with the property that \( I_1 \cap \text{supp}(gy_{u_1}^{v_1} ... y_{u_m}^{v_m}) \) is a nullset. It follows that for any \( r \in I_1, (gy_{u_1}^{v_1} ... y_{u_m}^{v_m})(r) = r \). This is impossible by Theorem 13. It also follows that \( I \cap \text{supp}(y_{u_1}^{v_1} ... y_{u_m}^{v_m}) \) is a nullset.

Now let \( hy_{p_1}^{q_1} ... y_{p_l}^{q_l} \) be a standard form such that

1. \( hy_{p_1}^{q_1} ... y_{p_l}^{q_l} = y_{s_1}^{t_1} ... y_{s_n}^{t_n} \).
2. \( hy_{p_1}^{q_1} ... y_{p_l}^{q_l} \) is obtained by performing expansion and rearranging substitutions on percolating elements of \( y_{s_1}^{t_1} ... y_{s_n}^{t_n} \).
3. \( g \) acts on \( p_1, ..., p_n \), i.e. it acts on \( y_{p_1}^{q_1} ... y_{p_l}^{q_l} \).
4. For each \( 1 \leq i \leq l, 1 \leq j \leq m \) we have \( |g(p_i)| > |u_j| \).

Note that it follows that \( y_{p_1}^{q_1} ... y_{p_l}^{q_l} \) has no potential cancellations and that \( \text{supp}(y_{p_1}^{q_1} ... y_{p_l}^{q_l}) = \text{supp}(y_{s_1}^{t_1} ... y_{s_n}^{t_n}) \).

Now we have

\[
\nu_1 \nu_2 = (fh y_{p_1}^{q_1} ... y_{p_l}^{q_l})(gy_{u_1}^{v_1} ... y_{u_m}^{v_m})
\]

Now converting the word

\[
(fhg)(y_{g(p_1)}^{q_1} ... y_{g(p_l)}^{q_l})y_{u_1}^{v_1} ... y_{u_m}^{v_m}
\]

into a normal form does not involve any cancellations along infinite binary sequences \( u \in I \) since \( I \cap \text{supp}(g) \) and \( I \cap \text{supp}(y_{u_1}^{v_1} ... y_{u_m}^{v_m}) \) are null sets, it follows that the resulting normal form \( f' \lambda \) (with \( f' \in F \) and \( \lambda \) a \( Y \)-normal form), has the property that \( I \subseteq \text{supp}(\lambda) \). This proves our assertion. \( \square \)

Now we shall describe the equivalence relation.

Definition 7.7. Let \( e_1, e_2 \) be two 1-cells of \( X \). We say that \( e_1 \simeq e_2 \) if there is a common parametrization

\[
e_1 = \{ F \tau_1, F(\lambda_1) \}, e_2 = \{ F \tau_2, F(\lambda_2) \}
\]

such that \( \text{supp}(\lambda) \cap \text{supp}(\tau_2 \tau_1^{-1}) \) is a nullset.

If this holds, consider the parametrizations

\[
e_1 = \{ F(\lambda_1), F(\lambda^{-1}(\lambda_1)) \}, e_2 = \{ F(\lambda_2), F(\lambda^{-1}(\lambda_2)) \}
\]

(with basepoints \( \lambda_1, \lambda_2 \) and parameter \( \lambda^{-1} \)). Since \( \text{supp}(\lambda) \cap \text{supp}(\tau_2 \tau_1^{-1}) \) is a nullset, we have that

\[
\lambda_1(\lambda_2)^{-1} = \lambda_1 \tau_1^{-1} \lambda^{-1} = \tau_2 \tau_1^{-1}
\]

It follows that

\[
\text{supp}(\lambda^{-1}) \cap \text{supp}(\lambda_2(\lambda_1^{-1})) = \text{supp}(\lambda) \cap \text{supp}(\tau_2 \tau_1^{-1})
\]

is a nullset. So this parametrization also satisfies the definition of equivalence. This motivates defining the following notion. In the above the pairs of 0-cells \( F \tau_1, F \tau_2 \) and \( F(\lambda_1), F(\lambda_2) \) are defined to be associated pairs for the equivalence \( e_1 \simeq e_2 \).

We remark that in the above, the orientations \( e_1 = \{ F(\lambda_1), F(\lambda^{-1}(\lambda_1)) \} \) (with parameter \( \lambda^{-1} \)) and \( e_2 = \{ F(\tau_2), F(\lambda_2) \} \) (with parameter \( \lambda \)) do not satisfy the condition of the definition. This is made precise in the Lemma below.
Lemma 7.8. Let \( e_1 = \{ u_1 = F(\lambda_1), u_2 = F(\tau_1) \}, e_2 = \{ v_1 = F(\lambda_2), v_2 = F\tau_2 \} \) be 1-cells such that \( e_1 \not\equiv e_2 \) and the given parametrization satisfies the definition of equivalence. Then \( u_1, v_1 \) are associated pairs for this equivalence but there is no parametrization for which \( u_1, v_2 \) or \( u_2, v_1 \) are associated pairs for this equivalence.

Proof. Let \( e_1 = \{ F(\psi_1), F\eta_1 \}, e_2 = \{ F(\psi_2), F\eta_2 \} \) be a parametrization satisfying \( e_1 \not\equiv e_2 \) and so that \( F(\psi_1) = u_2, F\eta_2 = v_2 \). Note that since \( F(\psi\eta) = F(\lambda\tau), F\eta_2 = F\tau_2 \), it follows that the type of \( \psi, \lambda \) must be the same. But since \( F(\psi\eta_1) = F\tau_1, F\eta_2 = F(\tau_1) \), it follows that the type of \( \psi, \lambda \) is different. This is a contradiction. \( \square \)

Lemma 7.9. The relation \( \equiv \) is an equivalence relation.

Proof. We will show that if \( e_1 \equiv e_2, e_2 \equiv e_3 \) then \( e_1 \equiv e_3 \). Let \( e_1 = \{ F\tau_1, F(\lambda\tau_1) \}, e_2 = \{ F\tau_2, F(\lambda\tau_2) \} \) be parametrizations such that \( \text{supp}(\tau_2\tau_1^{-1}) \cap \text{supp}(\lambda) \) is a nullset. Also, let \( e_2 = \{ F\psi_1, F(\nu\psi_1) \}, e_3 = \{ F\psi_2, F(\nu\psi_2) \} \) be parametrizations such that \( \text{supp}(\nu) \cap \text{supp}(\psi_1\psi_2^{-1}) \) is a nullset and \( F\psi_1 = F\tau_2 \).

Now there is a \( g \in F \) such that \( \psi_1 = g\tau_2 \). By replacing \( \nu \) with an equivalent special form if necessary, we can assume that \( g \) acts on \( \nu \). Moreover, since \( F(\nu g\tau_2) = F(\lambda\tau_2) \) it follows that \( F(\nu g) = F\lambda \) and \( e_2 = \{ F(\lambda g^{-1}\tau_2), F(g^{-1}\tau_2) \} \). We also obtain that \( e_3 = \{ F(\lambda g^{-1}\psi_1), F(g^{-1}\psi_2) \} \).

Now since \( \text{supp}(g^{-1}\nu g) = \text{supp}(\lambda) \) and \( \text{supp}(g^{-1}\nu g) \cap \text{supp}(g^{-1}\psi_1\psi_2^{-1}) \) is a nullset, it follows that \( \text{supp}(\lambda) \cap \text{supp}(g^{-1}\psi_1\psi_2^{-1}) \) is a nullset.

Therefore

\[
\text{supp}(\lambda) \cap \text{supp}((g^{-1}\psi_2)^{-1}(g^{-1}\psi_1^{-1}))
\]

and

\[
\text{supp}(\lambda) \cap \text{supp}(g^{-1}\psi_2\psi_1^{-1})
\]

are nullsets. This proves our claim. \( \square \)

From the proof of the previous lemma it follows that for 1-cells \( e_1, \ldots, e_n \) in \( X \), such that \( e_i \equiv e_j \) for each \( 1 \leq i, j \leq n \), there are elements \( \tau_1, \ldots, \tau_n \in G \) and a special form \( \lambda \) such that the following holds.

(1) \( e_i = \{ F\tau_i, F(\lambda\tau_i) \} \).

(2) For every \( 1 \leq i, j \leq n, i \neq j \), \( \text{supp}(\lambda) \cap \text{supp}(\tau_i\tau_j^{-1}) \) are nullsets.

Such a description will be useful in the proofs to follow. Now we show that any two distinct 1-cells \( e_1, e_2 \) that share a 0-cell are inequivalent.

Lemma 7.10. Let \( e_1, e_2 \) be two distinct 1-cells of \( X \) that are incident to the same 0-cell. Then \( e_1, e_2 \) are inequivalent.

Proof. Let \( e_1 = \{ F\tau, F(\lambda_1\tau) \}, e_2 = \{ F\tau, F(\lambda_2\tau) \} \). Note that if \( \lambda_1, \lambda_2 \) are of a different parity, then it is impossible to have a common parametrization for \( e_1, e_2 \) and hence they must be inequivalent. So we assume that \( \lambda_1, \lambda_2 \) have the same parity.

There are two cases to consider.

(1) \( \lambda_1, \lambda_2 \) have the same type.

(2) \( \lambda_1, \lambda_2 \) have different types.
Case (1): There is a common parametrization
\[ e_1 = \{ F(\psi(f\tau)), F(f\tau) \}, e_2 = \{ F(\psi(g\tau)), F(g\tau) \} \]
such that \( f, g \in F \) and \( \text{supp}(\psi) \cap \text{supp}((f\tau)(g\tau)^{-1}) \) is a nullset. By replacing \( \psi \) by an equivalent special form if necessary we can assume that \( f, g \) act on \( \psi \).

Let \( \psi_1, \psi_2 \) be special forms such that \( \psi \cdot f = \psi_1, \psi \cdot g = \psi_2 \). Then \( F(\psi f) = F(\psi_1) = F(\lambda_1), F(\psi g) = F(\psi_2) = F(\lambda_2) \). Since \( \text{supp}(\psi) \cap \text{supp}(fg^{-1}) \) is a nullset, it follows that \( (fg^{-1})\psi(fg^{-1}) = \psi \), and so \( f^{-1}\psi f = g^{-1}\psi g \). Therefore \( F\psi_1 = F\psi_2 \) and hence \( F\lambda_1 = F\lambda_2 \). This contradicts or assumption that \( e_1, e_2 \) are distinct.

Case (2): There is a common parametrization
\[ e_1 = \{ F(\psi'(\psi^{-1}f\tau)), F(\psi^{-1}f\tau) \}, e_2 = \{ F(\psi(g\tau)), F(g\tau) \} \]
with parameter \( \psi' \) such that \( f, g \in F \) and \( \text{supp}(\psi') \cap \text{supp}(\psi^{-1}f\tau(g\tau)^{-1}) \) is a nullset. A consequence of fact that \( fg^{-1} \in F \), together with the statement about tail equivalence in Theorem 3.8 is that \( \text{supp}(\psi') \subseteq \text{supp}(\psi^{-1}fg^{-1}) \). This is a contradiction. \( \square \)

We obtain the following Corollary immediately from the previous lemma together with the definition of equivalence.

**Corollary 7.11.** Let \( \Delta \) be a cluster based at \( \tau \) and parametrized by \( \lambda_1, \ldots, \lambda_n \). Let
\[ e_1 = \{ F((\prod_{i \in I} \lambda_i)\tau), F((\lambda_j \prod_{i \in I} \lambda_i)\tau) \} \]
and
\[ e_2 = \{ F((\prod_{i \in J} \lambda_i)\tau), F(\lambda_k \prod_{i \in J} \lambda_i)\tau) \} \]
for \( I, J \subseteq \{1, ..., n\} \) and \( j \notin I, k \notin J \) be two facial 1-cells. If \( e_1 \equiv e_2 \) then it follows that \( e_1, e_2 \) are “parallel” facial 1-cells of the cluster, i.e. \( j = k \).

We now define two (related) notions called disparate 1 and disparate 2, which will be crucial in determining when two 1-cells are inequivalent.

**Definition 7.12.** (Disparate 1) Consider a pair of a parametrizations
\[ e = \{ F(\lambda\tau_1), F\tau_1 \}, u = F\tau_2 \]

(1) We say that the pair of parametrizations is **disparate with associated pair**
\( F\tau_1, F\tau_2 \) if \( \text{supp}(\lambda) \subseteq \text{supp}(\tau_2\tau_1^{-1}) \).

(2) We say that the above pair of parametrizations is **disparate with associated pair**
\( F(\lambda\tau_1), F\tau_2 \) if \( \text{supp}(\lambda^{-1}) \subseteq \text{supp}(\tau_2\tau_1^{-1}\lambda^{-1}) \).

If both the conditions hold for the given parametrizations, then we simply say that the pair of parametrizations is **disparate**.

Now we show that the above definition is independent of parametrization. Therefore the property of being disparate (with associated pairs or otherwise) is a property of the pair \( e, u \), rather than just a property of the parametrizations. This is done in the next few lemmas. We will also show that if \( e, u \) is disparate, then there is no \( e' \) incident to \( u \) such that \( e \sim e' \).

**Lemma 7.13.** Let \( e = \{ F(\lambda\tau_1), F\tau_1 \} \) and \( u = F\tau_2 \) be such that the parametrizations satisfy the definition of disparate with associated pair \( F\tau_1, F\tau_2 \). Then if \( \lambda' \sim \lambda \) is a special form, the parametrization \( e = \{ F(\lambda'\tau_1), F\tau_1 \} \) and \( u = F\tau_2 \) also satisfy the definition of disparate with the associated pair \( F\tau_1, F\tau_2 \). Moreover, we can
use any special form $\tau'_2 \sim \tau_2$ as the representative for the coset $F\tau_2$ to satisfy the definition of disparate.

**Proof.** It is clear that $\text{supp}(\lambda') = \text{supp}(\lambda)$, whenever $\lambda', \lambda$ are equivalent special forms. So it follows that $\text{supp}(\lambda') \subseteq \text{supp}_Y(\tau_2\tau_1^{-1})$. The last claim of the Lemma follows from the fact that $\text{supp}_Y(f\tau_2\tau_1^{-1}) = \text{supp}_Y(\tau_2\tau_1^{-1})$ for any $f \in F$. □

**Lemma 7.14.** Let $e = \{F(\lambda_1\tau_1), F\tau_1\} = \{F(\lambda_2\tau_2), F\tau_2\}$ and $u = F\psi$ be a pair such that $F\tau_1 = F\tau_2$ and $e$, $u$ is disparate with respect to the parametrization $e = \{F(\lambda_1\tau_1), F\tau_1\}$ with associated pair $F\tau_1, F\psi$. Then it is disparate with respect to the parametrization $\{F(\lambda_2\tau_2), F\tau_2\}$ with associated pair $F\tau_2, F\psi$.

Moreover, if $e = \{F(\lambda_1\tau_1), F\tau_1\}$ and $u = F\psi$ are disparate, the parametrization $e = \{F(\lambda_2\tau_2), F\tau_2\}$ and $u = F\psi$ also satisfies the definition of disparate.

**Proof.** First we shall prove the first claim of our lemma. We know that $\text{supp}(\lambda_1) \subseteq \text{supp}_Y(\psi\tau_1^{-1})$. Now $\tau_1 = f\tau_2$ for some $f \in F$. We can assume that $f$ acts on $\lambda_1$ by replacing $\lambda_1$ with an equivalent special form if necessary. Let $\lambda'_1$ be a special form such that $\lambda'_1 = f\lambda_1 f^{-1}$. It follows that $\lambda_2 \sim \lambda'_1$. So for the rest of the proof we can assume that $\lambda_2 = f^{-1}\lambda_1 f$. (By Lemma 7.13 this does not change the fact that the definition of disparate is satisfied.) We also note that $\text{supp}(\lambda_2) = \text{supp}(f\lambda_1 f^{-1})$.

We know that $\psi\tau_2^{-1} = \psi\tau_1^{-1} f$ and so

$$\text{supp}_Y(\psi\tau_2^{-1}) = \text{supp}_Y(\psi\tau_1^{-1} f) = \text{supp}_Y(f^{-1}\psi\tau_1^{-1} f)$$

Since $\text{supp}(f^{-1}\lambda_1 f) \subseteq \text{supp}_Y(f^{-1}\psi\tau_1^{-1} f)$ we conclude that $\text{supp}(\lambda_2) \subseteq \text{supp}_Y(\psi\tau_2^{-1})$.

So the pair is disparate (for the given associated pair) with the new parametrization.

For the second claim, our assumption says that in addition to the above, it is also true that $\text{supp}(\lambda_1) \subseteq \text{supp}_Y(\psi\tau_1^{-1}\lambda_1^{-1})$. It suffices to show that $\text{supp}(\lambda_2) \subseteq \text{supp}_Y(\psi\tau_2^{-1}\lambda_2^{-1})$.

Now

$$\text{supp}_Y(\psi\tau_2^{-1}\lambda_2^{-1}) = \text{supp}_Y(f^{-1}\psi\tau_2^{-1}\lambda_2^{-1})$$

$$= \text{supp}_Y(f^{-1}\psi\tau_1^{-1} f(f^{-1}\lambda_1^{-1} f)) = \text{supp}_Y(f^{-1}\psi\tau_1^{-1}\lambda_1^{-1} f)$$

Now we know that since $\text{supp}(\lambda_1) \subseteq \text{supp}_Y(\psi\tau_1^{-1}\lambda_1^{-1})$, it follows that

$$\text{supp}(f^{-1}\lambda_1^{-1} f) \subseteq \text{supp}_Y(f^{-1}\psi\tau_1^{-1}\lambda_1^{-1} f)$$

This implies that

$$\text{supp}(\lambda_2) \subseteq \text{supp}_Y(\psi\tau_1^{-1}\lambda_1^{-1} f) = \text{supp}_Y(\psi\tau_2^{-1}\lambda_2^{-1})$$

proving our assertion. □

We now show that the notion of disparate with respect to a fixed 0-cell is actually a property of equivalence classes of 1-cells. We make this precise in the next lemma.

**Lemma 7.15.** Let $e_1 = \{F(\lambda_1\tau_1), F\tau_1\}, e_2 = \{F(\lambda_2\tau_2), F\tau_2\}$ be 1-cells such that $e_1 \sim e_2$ and the given parametrizations satisfy the definition of equivalence with associated pairs $F(\lambda_1\tau_1), F(\lambda_2\tau_2)$ and $F\tau_1, F\tau_2$. Let $u = F\tau_3$ be a 0-cell such that $e_1, u$ are disparate with associated pair $F\tau_1, F\tau_3$. Then it follows that $e_2, u$ are disparate with associated pair $F\tau_2, F\tau_3$.

**Proof.** From the definitions of disparate and the equivalence relation, it follows that:

1. $\text{supp}(\lambda) \cap \text{supp}(\tau_1\tau_2^{-1})$ are nullsets.
2. $\text{supp}(\lambda) \subseteq \text{supp}(\tau_3\tau_1^{-1})$. 


It follows from Lemma 7.6 that

\[ \text{supp}(\lambda) \subseteq \text{supp}(\tau_3 \tau_1^{-1}) \cap \text{supp}(\tau_3 \tau_2^{-1}) = \text{supp}(\tau_3 \tau_2^{-1}) \]

\[ \square \]

**Lemma 7.16.** If the pair \( e, u \) is disparate, and \( e = e' \), then \( e', u \) is disparate. If \( e, u \) is disparate, then there is no 1-cell \( e' \) incident to \( u \) such that \( e = e' \).

**Proof.** The first statement follows from the Lemmas 7.15 and 7.14. For the second statement, assume by way of contradiction that there are parametrizations

\[ e = \{ F(\lambda t_2), F t_2 \}, e' = \{ F(\lambda t_2), F t_2 \}, u = F t_2 \]

such that \( e = e' \) with associated pair \( F t_1, F t_2 \). Then it follows that \( \text{supp}(\lambda) \cap \text{supp}(\tau_2 \tau_1^{-1}) \) is a nullset. This contradicts the fact that \( e, u \) are disparate. \[ \square \]

The following criterion allows us to detect equivalent 1-cells.

**Lemma 7.17.** Let \( e_1 = \{ F(\lambda t_3), F t_3 \} \) be a 1-cell and \( u = F t_2 \) be a 0-cell such that \( \text{supp}(\lambda) \cap \text{supp}(\tau_2 \tau_1^{-1}) \) is a nullset. Then there is a 1-cell \( e_1 = \{ F(\lambda t_3), F t_3 \} \) such that \( e_1 = e_2 \) with associated pairs \( F t_1, F t_3, F(\lambda t_1), F(\lambda t_3) \), and so that \( F t_3 = F t_2 \).

**Proof.** Let \( f \nu \) be a normal form such that \( f \in F, \nu \) is a \( Y \)-normal form and \( f \nu = \tau_2 \tau_1^{-1} \). By our assumption it follows that

\[ \text{supp}(\lambda) \cap \text{supp}(\nu) = \text{supp}(\lambda) \cap \text{supp}(\nu) \cap \text{supp}(f^{-1} \tau_2 \tau_1^{-1}) \]

is a nullset.

It follows that the 1-cell \( e_1 = \{ F(\lambda t_3), F t_3 \} \) where \( t_3 = f^{-1} \tau_2 \) has the required property. \[ \square \]

Now we will define a second notion of disparate. First we need to define a relation on pairs of elements of \( \Omega \).

**Definition 7.18.** (Overlay for elements of \( \Omega \)) Let \( F \lambda_1, F \lambda_2 \in \Omega \). We say that the pair \( F \lambda_1, F \lambda_2 \) contains an overlay if there are special forms \( y_1^{f_1} \cdots y_n^{f_n} \) and \( y_1^{v_1} \cdots y_m^{v_m} \) such that \( y_1^{f_1} \cdots y_n^{f_n} = F \lambda_1, y_1^{v_1} \cdots y_m^{v_m} = F \lambda_2 \), and

\[ \{ y_1^{f_1}, \ldots, y_n^{f_n} \} \cap \{ y_1^{v_1}, \ldots, y_m^{v_m} \} \neq \emptyset \]

(Cancellation free.) Let \( F \lambda_1, F \lambda_2 \in \Omega \). The pair \( \lambda_1, \lambda_2 \) is said to be cancellation free if \( F \lambda_1, F \lambda_2 \) does not contain an overlay.

The following is an immediate consequence of the above definition and Lemma 5.9 in [15].

**Lemma 7.19.** Let \( \lambda_1, \lambda_2 \) be special forms such that \( F \lambda_1, F \lambda_2 \) are cancellation free. Let \( f \in F \) be such that \( f \) acts on \( \lambda_1, \lambda_2 \), and let \( v_1, v_2 \) be special forms such that \( v_i = f^{-1} \lambda_i f \). Then the pair \( F v_1, F v_2 \) is cancellation free.

We can now introduce the second notion of disparate which is defined for pairs of 1-cells that share an incident 0-cell.

**Definition 7.20.** (Disparate 2) Let \( e_1 = \{ F(\lambda_1 t), F t \}, e_2 = \{ F(\lambda_2 t), F t \} \) be parametrizations of 1-cells in \( X \). Then the parametrizations are said to be disparate if \( F \lambda_1, F \lambda_2 \) is cancellation free. Otherwise the parametrizations are said to be coupled. (Orthogonal 1-cells) Let \( e_1 = \{ F(\lambda_1 t), F t \}, e_2 = \{ F(\lambda_2 t), F t \} \) be
parametrizations of 1-cells such that $supp(\lambda_1) \cap supp(\lambda_2)$ is a nullset. Then the parametrizations are said to be orthogonal. We remark that if the parametrizations are orthogonal then they are also disparate.

We remark that we shall use the word disparate for both definitions and the specific definition that is being used will be clear from the context. We will now show that these notions of disparate and orthogonal are both properties of 1-cells that are independent of parametrizations. We leave the proof of the following lemmas to the reader. (The proofs are a routine application of Lemma 7.19)

**Lemma 7.21.** Let $e_1 = \{F(\lambda_1 \tau), F\tau\}, e_2 = \{F(\lambda_2 \tau), F\tau\}$ be two 1-cells in $X$ such that $e_1, e_2$ are disparate in the given parametrization. Let $e_1 = \{F(\nu_1 \psi), F\psi\}, e_2 = \{F(\nu_2 \psi), F\psi\}$ be another parametrization such that $F\psi = F\tau$. Then $e_1, e_2$ are disparate with respect to this parametrization as well.

**Lemma 7.22.** Let $e_1 = \{F(\lambda_1 \tau), F\tau\}, e_2 = \{F(\lambda_2 \tau), F\tau\}$ be two 1-cells in $X$ such that $e_1, e_2$ are orthogonal in the given parametrization. Let $e_1 = \{F(\nu_1 \psi), F\psi\}, e_2 = \{F(\nu_2 \psi), F\psi\}$ be another parametrization such that $F\psi = F\tau$. Then $e_1, e_2$ are orthogonal with respect to this parametrization as well.

8. Expansions and systems of cells.

In this section we will describe two procedures that take as input a finite set of 1-cells of $X$ and a finite set of 0-cells incident to them, and produce a new set of 1-cells and 0-cells. These procedures will be used in the proof that the higher homotopy groups of $X$ are trivial, which will be shown in Section 9. In particular we will start with the 1-skeleton of a finite subcomplex $Y$ of $X$, and perform a sequence of these procedures to obtain a set of 0-cells and 1-cells in $X$, and we shall use these as a building block to build a subcomplex $Y'$ of $X$ that is homeomorphic to a nonpositively curved cube complex and contains $Y$.

We will now define an operation that takes as input a parametrized 1-cell $e$ of $X$ incident to a 0-cell $u$ of $X$, and produces a set of 1-cells $e_1, ..., e_n$ incident to $u$.

**Definition 8.1.** (Decomposition) Let $\lambda$ be a special form and $\nu_1, ..., \nu_n$ be a sorted list of pairwise independent special forms such that $\lambda \sim \nu_1 ... \nu_n$. Then the formal expression $\nu_1 ... \nu_n$ is said to be a decomposition of $\lambda$ as a product of special forms, or simply a decomposition.

**Definition 8.2.** (Expansion) Input: A 1-cell $e = \{F(\lambda \tau), F\tau\}$, a decomposition $\lambda \sim \nu_1 ... \nu_n$, and a 0-cell $F\tau$ incident to $e$.

Output: The expansion operation at $F\tau$ for this decomposition produces the following 1-cells as the output:

$$e_1 = \{F(\nu_1 \tau), F\tau\}, e_2 = \{F(\nu_2 \tau), F\tau\}, ..., e_n = \{F(\nu_n \tau), F\tau\}$$

**Definition 8.3.** (op-expansion) An expansion of $e = \{F(\lambda \tau), F\tau\}$ with respect to the decomposition $\lambda \sim \nu_1 ... \nu_n$ at $F\lambda \tau$ is the defined to be the expansion at $F((\nu_1 ... \nu_n) \tau) = F\lambda \tau$ with respect to the parametrization

$$\{F((\nu_1 ... \nu_n)^{-1}(\nu_1 ... \nu_n) \tau), F(\nu_1 ... \nu_n) \tau)\}$$

and decomposition $(\nu_1 ... \nu_n)^{-1} = \nu_1^{-1} ... \nu_n^{-1}$. (Note that $\nu_i, \nu_j$ commute, so $\nu_1^{-1} ... \nu_n^{-1} = \nu_n^{-1} ... \nu_1^{-1}$.) This produces the 1-cells

$$e_1' = \{F(\nu_1^{-1}(\nu_1 ... \nu_n) \tau), F(\nu_1 ... \nu_n) \tau)\}, ..., e_n' = \{F(\nu_n^{-1}(\nu_1 ... \nu_n) \tau), F(\nu_1 ... \nu_n) \tau)\}$$
We will call this second type of expansion an \textit{op-expansion}, since this is an analogous expansion for the opposite orientation of the parameters.

Finally, we can combine the two notions of expansion above to obtain a \textit{two sided expansion}.

\textbf{Definition 8.4.} (Two sided expansion) A two sided expansion of $e = \{F(\lambda \tau), F\tau\}$ with the above decomposition produces both families of 1-cells

\begin{align*}
e_1 &= \{F(\nu_1 \tau), F\tau\}, ..., e_n = \{F(\nu_n \tau), F\tau\} \\
e'_1 &= \{F(\nu_1^{-1}(\nu_1 ... \nu_n \tau)), F((\nu_1 ... \nu_n \tau))\}, ..., e'_n &= \{F(\nu_n^{-1}(\nu_1 ... \nu_n \tau)), F((\nu_1 ... \nu_n \tau))\}
\end{align*}

We remark that this can be viewed as performing expansions at both incident 0-cells of $e$ using analogous decompositions.

For the above expansions consider the $n$-cluster $\Delta$ based at $\tau$ and parametrized by $\nu_1, ..., \nu_n$. It follows that the 1-cells $e_1, ..., e_n$ of the above expansion of $e$ at $F\tau$ are precisely the facial 1-cells of $\Delta$ incident to $F\tau$. And the 1-cells $e'_1, ..., e'_n$ of the op-expansion are precisely the facial 1-cells of $\Delta$ incident to $F(\nu_1 ... \nu_n \tau)$.

In particular it follows that $e_i \cong e'_i$ for each $1 \leq i \leq n$ and the associated pairs for each equivalence are $F\tau, F(\nu_1^{-1}(\nu_1 ... \nu_n \tau))$ and $F(\nu_i \tau), F((\nu_1 ... \nu_n \tau))$.

\textbf{Definition 8.5.} (Offspring and parent) For the above expansions we say that the cells $e_1, ..., e_n$ (or $e'_1, ..., e'_n$) are offspring of $e$, and $e$ is the parent. If we perform a sequence of expansions on $e$ and its offspring (at the same 0-cell each time), then an offspring of an offspring is also considered an offspring of $e$.

The proof of the following lemma is left to the reader.

\textbf{Lemma 8.6.} (Independence of choice of parameters) Let $e = \{F(\lambda \tau), F\tau\}$ be a 1-cell and $\lambda = \nu_1 ... \nu_n$ a decomposition. Let $e = \{F(\lambda' (f \tau)), F(f \tau)\}$ be another parametrization, with the decomposition $\lambda' \sim \nu'_1 ... \nu'_n$ where $\nu'_1 \sim \nu_1 \cdot f^{-1}$. Both associated expansions of $e$ at $F\tau$ produce the same 1-cells. It also follows that both associated expansions of $e$ at $F(\lambda \tau)$ produce the same 1-cells.

(Sequence of expansions) If $e$ is a 1-cell, and if $e_1, ..., e_n$ are 1-cells that are produced after performing a sequence of expansions on $e$ and its offspring, all at the same 0-cell incident to $e$, then we can in fact obtain $e_1, ..., e_n$ from a single expansion of $e$ at that 0-cell.

\textbf{Definition 8.7.} Let $e$ be a 1-cell incident to a 0-cell $u$. We define $E(e, u)$ to be the set of expansions of $e$ at $u$. The set of expansions $E(e, u)$ is endowed with a natural partial order defined in the following manner. Let $e = \{F(\lambda \tau), F\tau\}, u = F\tau$. (By Lemma 8.6 we can choose any set of parameters for the sake of the definition.) Let $\alpha, \beta \in E(e, u)$ be expansions and let $\nu_1 ... \nu_n \sim \lambda$ and $\psi_1 ... \psi_m \sim \lambda$ be the respective decompositions of $\lambda$ associated with $\alpha, \beta$. Then $\alpha < \beta$ if there is a partition $\{i_0 = 1, 2, ..., i_1 - 1\}, \{i_1, ..., i_2 - 1\}, ..., \{i_{m-1}, i_{m-1} + 1, ..., i_m - 1 = n\}$ of $\{1, ..., n\}$ such that $\nu_{i_k-1} ... \nu_{i_k-1} + 1 ... \nu_{i_k-1} \sim \psi_k$ for each $1 \leq k \leq m$. In particular it holds that the expansion $\alpha$ is a composition of the expansion $\beta$ followed by expansions on the offsprings of $e$ in $\beta$.

The proofs of the following lemmas are straightforward and are left to the reader.

\textbf{Lemma 8.8.} Let $e$ be a 1-cell incident to a 0-cell $u$. For each $\alpha, \beta \in E(e, u)$ there is a $\gamma \in E(e, u)$ such that $\gamma < \alpha$ and $\gamma < \beta$. 

Lemma 8.9. Let \( e_1 = \{ F(\lambda_1 \tau), F\tau \}, e_2 = \{ F(\lambda_2 \tau), F\tau \} \) be 1-cells such that \( e_1, e_2 \) are disparate. Let \( \lambda_1 = \psi_1 ..., \psi_n, \lambda_2 = \psi_1 ..., \psi_m \) be decompositions. Let \( e_1, ..., e_1, n \) and \( e_2, ..., e_2, m \) be the 1-cells obtained from the respective expansions of \( e_1, e_2 \) at \( F\tau \). Then each pair of 1-cells in the set \( \{ e_1, ..., e_1, n, e_2, ..., e_2, m \} \) is disparate.

Lemma 8.10. Let \( e = \{ v_1, v_2 \} \) be a 1-cell and \( u \) be a 0-cell such that \( e, u \) is disparate with associated pair \( v_1, u \). If \( e_1, ..., e_n \) are offsprings of an expansion of \( e \) at \( v_1 \), then for each \( 1 \leq j \leq n \) \( e_j, u \) is disparate with associated pair \( v_1, u \). Moreover from this and Lemma 7.11, it follows that if \( e, u \) is disparate, then \( e_1, u \) is disparate for each \( 1 \leq i \leq n \).

Lemma 8.11. Let \( e_1, e_2 \) be 1-cells incident to 0-cells \( u, v \) respectively such that \( e_1 \overset{=}{} e_2 \) and \( u, v \) is an associated pair for this equivalence. Let \( e_1', e_2' \) be offsprings of expansions of \( e_1, e_2 \) respectively at \( u, v \) such that \( e_1' \overset{=}{} e_2' \). Then it follows that \( u, v \) is an associated pair for \( e_1', e_2' \).

Recall that if \( e_1, e_2 \) are 1-cells incident to a 0-cell \( u \) have the property that they are not disparate, then they are said to be coupled. We shall define a similar notion for a collection of 1-cells that are incident to a given 0-cell.

Definition 8.12. Let \( I = \{ e_1, ..., e_n \} \) be a set of 1-cells that are incident to a 0-cell \( u \). This set of 1-cells is said to be coupled if for any pair \( e_i, e_j \), there are 1-cells \( e_i = e_{i_1}, ..., e_{i_k} = e_{j_1} \in I \) such that the pair \( e_{i_1}, e_{j_1} \) is coupled for each \( 1 \leq i \leq k \).

This motivates us to define the notion of a coupling graph.

Definition 8.13. (Coupling graph) Let \( I = \{ e_1, ..., e_n \} \) be 1-cells in \( X \) and let \( u \) be a 0-cell. The coupling graph \( \Gamma_u(I) \) is defined as follows. The vertices are elements \( \{ e_{i_1}, ..., e_{i_k} \} \) of \( I \) that are incident to \( u \). Two vertices are connected by an edge if the corresponding pair of 1-cells is coupled at \( u \). For each cell \( e \in \{ e_{i_1}, ..., e_{i_k} \} \) we define \( \Gamma_u,e(i) \) as the connected component containing \( e \) in \( \Gamma_u(I) \).

Definition 8.14. (Equivariant isomorphism for coupling graphs.) Let \( I \) be a finite set of 1-cells in \( X \) and let \( u, v \) be 0-cells in \( X \). Let \( \{ e_1, ..., e_k \}, \{ e'_1, ..., e'_k \} \) be the vertices of connected components \( \Gamma_u,e_1(I), \Gamma_v,e'_1(I) \) respectively. We say that \( \Gamma_u,e_1(I), \Gamma_v,e'_1(I) \) are equivariantly isomorphic if there is a graph isomorphism \( \phi : \Gamma_u,e_1(I) \to \Gamma_v,e'_1(I) \) such that \( e_l \overset{=}{} \phi(e_l) \) for each \( 1 \leq l \leq k \) with associated pair \( u, v \).

We say that \( \Gamma_u,e_1(I), \Gamma_v,e'_1(I) \) are op-equivariantly isomorphic if there is a graph isomorphism \( \phi : \Gamma_u,e_1(I) \to \Gamma_v,e'_1(I) \) such that \( e_l \overset{=}{} \phi(e_l) \) for each \( 1 \leq l \leq k \) such that \( u, v \) is not an associated pair for the equivalence.

Definition 8.15. (The decoupling problem for a 0-cell.) Given a 0-cell \( u \) and 1-cells \( e_1, ..., e_n \) incident to \( u \), the decoupling problem requires one to perform expansions on \( e_1, ..., e_n \) at \( u \) to obtain 1-cells \( e'_1, ..., e'_m \) incident to \( u \) such that any pair \( e'_i, e'_j \) of distinct 1-cells is disparate.

Now we will describe a decoupling procedure that solves the decoupling problem at a 0-cell. First this is described for the case of two 1-cells.

Lemma 8.16. (Decoupling lemma 1) Let \( e_1, e_2 \) be 1-cells incident to a 0-cell \( u \). Then we can perform expansions on \( e_1, e_2 \) at \( u \) such that the union of the set of outputs of the expansions has the property that any distinct pair of 1-cells in the set is disparate.
Proof. Let $e_1 = \{ F(\lambda_1 \tau), F\tau \}, e_2 = \{ F(\lambda_2 \tau), F\tau \}$. By considering the right action of the group, we can assume without loss of generality that $\tau = 0$. There is a special form $\lambda_1 \sim \lambda'_1$ such that $\lambda_1 \lambda_2^{-1}$ is a standard form and so $\lambda_1 \lambda_2^{-1} = f\lambda'_1 \lambda_2^{-1}$ for some $f \in F$.

We know from Lemma 8.15 that $f\lambda'_1 \lambda_2^{-1}$ can be further refined into a standard form with no potential cancellations. But since our word $\lambda_1 \lambda_2^{-1}$ is a standard form that is a product of two special forms, this can be done in the following way. For every percolating element $y_1$ of $\lambda_1^{-1}$, there is a percolating element $y_2^n$ in $\lambda_2^{-1}$ such that the pair $y_1', y_2^n$ is a neighboring pair with a potential cancellation, or there is no such percolating element of $\lambda_2^{-1}$ with this property. It follows that can find a special form $y_1'^{i_1}...y_n'^{i_n} \sim \lambda'_1$ such that there is a partition $I_1 \cup I_2$ of $\{ 1, ..., n \}$ such that if $i \in I_1$ then $y_1'^i$ faces potential cancellation from a neighboring percolating element $y_2^n$ of $\lambda_2^{-1}$ and if $i \in I_2$ then $y_1'^i$ does not face such a potential cancellation. Moreover, we can find such a word that satisfies that for any $i \in I_1, j \in I_2 |s_i| < |s_j|$.

We apply expansion and rearranging substitutions on percolating elements $y_1'^{i_1}...y_n'^{i_n} \lambda_2^{-1}$ until all the potential cancellations for neighboring pairs have been performed. Therefore there is a word $y_1'^{i_1}...y_n'^{i_n} \sim \lambda_2$ such that for each $1 \leq i \leq n, 1 \leq j \leq m$ either $y_1'^{i_j} = y_2^{i_j}$ or $y_1'^{i_j}, y_2^{i_j}$ are cancellation free. It follows that the expansions associated with the decompositions $\lambda_1 \sim y_1'^{i_1}...y_n'^{i_n}, \lambda_2 \sim y_1'^{i_1}...y_n'^{i_n}$ satisfy the required property.

In the proof of the previous Lemma, we have actually proved something stronger.

**Corollary 8.17.** Let $\nu_1, ..., \nu_n$ be special forms that are pairwise independent, or equivalently that the supports $\text{supp}(\nu_1), ..., \text{supp}(\nu_n)$ are pairwise disjoint. Let

$$e_1 = \{ F(\nu_1 \tau), F\tau \}, ..., e_n = \{ F(\nu_n \tau), F\tau \}$$

and $e = \{ F(\lambda \tau), F\tau \}$ be 1-cells for some 0-cell $F\tau$. Then we can perform expansions on the 1-cells $e_1, ..., e_n, e$ to obtain 1-cells $e'_1, ..., e'_n$ that are pairwise disparate.

**Proof.** In the proof of the previous lemma replace $\lambda_1$ with $\nu_1...\nu_n$. (This is not a special form, but the method of reducing $\nu_1...\nu_n \lambda^{-1}$ to a standard form with no potential cancellations can be applied here.)

**Lemma 8.18.** Let $e_1, ..., e_n$ be 1-cells incident to a 0-cell $u$ that are pairwise disparate. Let $e$ be a 1-cell incident to $u$. Then we can perform expansions on $e_1, ..., e_n, e$ to obtain 1-cells $e'_1, ..., e'_n$ that are pairwise disparate.

**Proof.** We prove this by induction on $n$. The case $n = 1$ is simply the statement of Lemma 8.16. For the general case, assume that we have performed expansions on $e_1, ..., e_{n-1}, e$ to obtain 1-cells $e'_1, ..., e'_n$ that are pairwise disparate. Let $e'_i$ be an offspring of $e$ at the end of the sequence of expansions that were performed on $e$ in the process. We know from Lemma 8.9 that if $e_i$ is also an offspring of $e_l$ for some $1 \leq l \leq n - 1$, then $e_i, e_n$ are disparate. Let

$$\{ e'_{i_1}, ..., e'_{i_k} \} := \{ e'_j \mid 1 \leq j \leq m, e'_j \text{ is an offspring of } e \text{ but not of } e_1, ..., e_{n-1} \}$$

Let

$$e'_{i_1} = \{ F(\nu_1 \tau), F\tau \}, e'_{i_2} = \{ F(\nu_2 \tau), F\tau \}, ..., e'_{i_k} = \{ F(\nu_k \tau), F\tau \}$$

and $e_n = \{ F(\lambda \tau), F\tau \}$. Since $e'_{i_1}, ..., e'_{i_k}$ are offsprings of $e$ it follows that the supports $\text{supp}(\nu_1), ..., \text{supp}(\nu_k)$ are pairwise disjoint. Now from Corollary 8.17 it
follows that we can perform expansions on \( e_1', ..., e_n' \) to obtain the required set of 1-cells.

**Lemma 8.19.** (Decoupling lemma 2) Let \( e_1, ..., e_n \) be 1-cells in \( X \) that are incident to the same 0-cell. Then we can perform expansions on \( e_1, ..., e_n \) at the given 0-cell to produce 1-cells \( e_1', ..., e_n' \) such that any pair \( e_i', e_j' \) of distinct 1-cells is disparate.

**Proof.** We proceed by induction on \( n \). The case \( n = 2 \) was shown in Lemma 8.16.

By the inductive hypothesis we assume that this has been shown for \( n - 1 \). Consider a set of \( n \) 1-cells \( e_1, ..., e_n \) that are incident to the same 0-cell. By our inductive hypothesis we perform expansions on \( e_1, ..., e_{n-1} \) at the 0-cell to obtain 1-cells \( e_1', ..., e_{n-1}' \) that are pairwise disparate. Now by Lemma 8.16 we can perform expansions on \( e_1', ..., e_{n-1}', e_n \) at the given 0-cell to obtain the required set of 1-cells.

Now we shall define a notion of simultaneous expansion for equivalent 1-cells.

**Definition 8.20.** (Equivariant expansion) Let

\[ e_1 = \{F(\lambda_1), F\tau_1\}, ..., e_n = \{F(\lambda_n), F\tau_n\} \]

be 1-cells such that

\[ e_1 \doteq e_2 \doteq ... \doteq e_n \]

and that the given set of parameters satisfies the definition of the equivalence, i.e.

\[ \text{supp}(\lambda) \cap \text{supp}(\tau_i^{-1}) = \emptyset \]

for each \( 1 \leq i, j \leq n \). Then a decomposition \( \lambda = \nu_1...\nu_m \) can be used to perform an equivariant expansion to produce 1-cells

\[ e_{i,1} = \{F(\nu_1 \tau_i), F\tau_1\}, ..., e_{i,m} = \{F(\nu_m \tau_i), F\tau_1\} \]

for each \( 1 \leq i \leq n \).

(Mixed Equivariant expansion) Let \( e_1, ..., e_n \) be as above. Let \( I, J \subseteq \{1, ..., n\} \) such that \( I \cap J = \emptyset \), \( I \cup J = \{1, ..., n\} \). We can use the decomposition as above to perform the mixed equivariant expansion that produces 1-cells

\[ e_{i,1} = \{F(\nu_1 \tau_i), F\tau_1\}, ..., e_{i,m} = \{F(\nu_m \tau_i), F\tau_i\} \]

for each \( i \in I \) and

\[ e_{j,1} = \{F(\nu_1^{-1}(\nu_1...\nu_n) \tau_j), F(\nu_1...\nu_n \tau_j)\}, ..., e_{j,m} = \{F(\nu_1^{-1}(\nu_1...\nu_m \tau_j)), F(\nu_1...\nu_m \tau_j)\} \]

for each \( j \in J \).

This notion of equivariant expansion is useful in making the following important observation.

**Lemma 8.21.** Let \( e_1, e_2 \) be 1-cells incident to \( u \) and \( e_1', e_2' \) be 1-cells incident to \( v \) such that \( e_1 \doteq e_1' \) and \( e_2 \doteq e_2' \). Then if \( e_1, e_2 \) are disparate, it also follows that \( e_1', e_2' \) are disparate.

**Proof.** Assume that \( e_1', e_2' \) are not disparate. Then there are expansions of \( e_1', e_2' \) at \( v \) with a common offspring \( e_3' \). Following the expansion of \( e_i' \), we perform the analogous equivariant expansions (or mixed equivariant expansions) of \( e_1, e_2 \) at \( u \) and we obtain a common offspring \( e_3 \doteq e_3' \) incident to \( u \). This contradicts the assumption that \( e_1, e_2 \) are disparate.

Now we are ready to define the notion of a system of cells.
Definition 8.22. (Systems of cells in $X^{(1)}$) Let $I$ be a set of 1-cells in $X^{(1)}$, and let $J$ be a set of the 0-cells in $X^{(1)}$ incident to 1-cells in $I$ (Not necessarily all the 0-cells incident to 1-cells in $I$) such that the following holds. For each $e = \{u,v\} \in I$, there are 1-cells $e_1 = \{u_1,u_2\}$, $e_2 = \{v_1,v_2\}$ such that $u_1,v_1 \in J$ and $e = e_1$ with associated pair $u,u_1$ and $e_2 = e_2$ with associated pair $v,v_1$. We call such a pair $(I,J)$ a system of cells in $X^{(1)}$.

(Balanced system) We say that a system $(I,J)$ is balanced if for each $u_i,u_j \in J, e_i \in I$ such that $e_i$ is incident to $u_i$, either $e_i,u_j$ is disparate, or there is a $e_k \in I$ incident to $u_j$ such that $e_i \equiv e_k$, or equivalently (By Lemmas 8.21 and 8.11) $\Gamma_{u_i,e_i}$ is either equivariantly isomorphic to $\Gamma_{u_j,e_k}$ or op-equivariantly isomorphic to $\Gamma_{u_j,e_k}$.

(Free system) We say that a system $(I,J)$ is free if it is balanced and for each $u \in J$ the graph $\Gamma_u(I)$ has no edges, i.e. for any pair of 1-cells $e_1,e_2 \in I$ incident to $u$, $e_1,e_2$ is disparate.

We define a partial order on systems as follows.

Definition 8.23. Let $(I_1,J_1),(I_2,J_2)$ be two systems. We say that $(I_1,J_1) \leq (I_2,J_2)$ if for each $e \in I_1$ either $e \in I_2$, or for some $n \geq 2$ there are 1-cells $e_1,...,e_n \in I_2$ and a 0-cell $u \in J$ incident to $e$, such that $e_1,...,e_n$ is the set of 1-cells produced by performing an expansion of $e$ at $u$.

The remainder of the section is structured as follows.

- Let $Y$ be a finite subcomplex of $X^{(1)}$, $I$ be the set of 1-cells of $Y$ and $J$ be the set of 0-cells of $Y$. First, we shall describe a procedure called the Separation procedure that takes the system $(I,J)$ as input, and by performing expansions on elements of $I$, produces a balanced system $(I',J')$ such that $(I,J) \leq (I',J')$.

- Then we describe Equivant decoupling procedure that takes the balanced system $(I',J')$ as input and produces a free system $(I'',J')$ such that $(I',J') \leq (I'',J')$ and hence $(I,J) \leq (I'',J')$.

8.1. Separation procedure. We shall first describe the case for a single pair, and then we shall describe a general separation procedure. This procedure takes as input a 1-cell $e$ and a 0-cell $u$, and produces a two-sided expansion of $e$ producing 1-cells $e_1,...,e_n$ such that for each $1 \leq i \leq n$ either there is a 1-cell $e'_i$ incident to $u$ such that $e'_i \equiv e_i$ or $e_i,u$ are disparate. To do this we first need to prove the following lemma.

Lemma 8.24. Let $\lambda$ be a special form and let $f\nu$ be a normal form. There is a special form $y_{11}^i,...,y_{nn}^i \sim \lambda$ such that for each $1 \leq i \leq n$ either $[s_i0,s_i1] \cap \text{supp}(f\nu)$ is a nullset or $[s_i0,s_i1] \subseteq \text{supp}(f\nu)$ where $\text{supp}(y_{si}^i) = [s_i0,s_i1] \subseteq 2^N$.

Proof. Given any special form $f\nu$, by Theorem 3.8 it follows that $\text{supp}(f\nu)$ is always of the form $\bigcup_{1 \leq i \leq m} [u_i0,v_i1]$ for finite binary sequences $u_1,...,u_m,v_1,...,v_m$ such that $u_i \leq v_i$ and $v_i \leq u_{i+1}$. Given any such set and a special form $\lambda$, any special form $y_{11}^i,...,y_{nn}^i \sim \lambda$ of depth greater than $\sup\{|v_1|,...,|v_m|,|u_1|,...,|u_m|\}$ satisfies the property that if $1 \leq i \leq n$ either $\text{supp}(y_{si}^i) \cap \bigcup_{1 \leq j \leq m} [u_j0,v_j1]$ is a nullset or $\text{supp}(y_{si}^i) \subseteq \bigcup_{1 \leq j \leq m} [u_j0,v_j1]$. □

Lemma 8.25. Let $e = \{F(\lambda_1),F(\tau_1)\}$ be a 1-cell and $u = F(\tau_2)$ be a 0-cell such that $e,u$ is not disparate. Then we can perform a two-sided equivariant expansion of $e$ to obtain 1-cells $e_1,...,e_n$ incident to $F(\tau_1)$ and $e'_1,...,e'_n$ incident to $F(\lambda_1)$ such
that for each $1 \leq i \leq n$ either the pairs $e_i, u$ and $e_i', u$ are both disparate, or there is a 1-cell $e_i''$ incident to $u$ such that $e_i \sim e_i' \sim e_i''$.

Proof. Let $F\tau_1 = v_1, F(\lambda \tau_1) = v_2$. By Lemma 8.24 there is an expansion $\alpha$ with an associated decomposition $\lambda \sim y_{i_1}^1 \ldots y_{i_n}^n$ such that for each $1 \leq i \leq n$ either $\text{supp}(y_{i_1}^1) \cap \text{supp}(\tau_2\tau_1^{-1})$ is a nullset or $\text{supp}(y_{i_1}^1) \subseteq \text{supp}(\tau_2\tau_1^{-1})$

In particular there is an expansion $\alpha \in E(e, v_1)$ such that the offsprings $e_1, \ldots, e_n$ of $e$ satisfy the property that for each $1 \leq i \leq n$ either $e_i, u$ is disparate with associated pair $v_1$ or there is a 1-cell $e'$ incident to $u$ such that $e_i \sim e'$ with associated pair $v_1, u$.

Also by Lemma 8.24 and an argument similar to the first paragraph of our proof, it follows that there is an expansion $\beta \in E(e, v_2)$ for which the offsprings $e_1', \ldots, e_n'$ of $e$ satisfy the property that for each $1 \leq i \leq m$ either $e_i, u$ is disparate with associated pair $v_2, u$ or there is a 1-cell $e'$ incident to $u$ such that $e_i' \sim e'$ with associated pair $v_2, u$.

Let the op-expansion of $\beta \in E(e, v_2)$ be the expansion $\beta' \in E(e, v_1)$. By Lemma 8.8 it follows that there is an expansion $\gamma \in E(e, v_1)$ such that $\gamma < \beta'$ and $\gamma < \alpha$. The corresponding op-expansion $\gamma' \in E(e, v_1)$ has the property that $\gamma' < \beta$.

By Lemma 8.10 the two sided expansion of $e$ given by $\gamma, \gamma'$ produces the desired outcome. \hfill $\Box$

Now we outline the next step towards defining the separation procedure.

Lemma 8.26. (Procedure A) Let $e_1, \ldots, e_n$ be 1-cells and $u_1, \ldots, u_k$ be the set of 0-cells incident to them. Then we can perform two sided equivariant expansions on $e_1, \ldots, e_n$ to produce 1-cells $e_1', \ldots, e_m'$ such that for each pair $1 \leq i \leq m, 1 \leq j \leq k$ either $e_i', u_j$ are disparate or there is a 1-cell $e_{i,j}$ incident to $u_j$ such that $e_i' \sim e_{i,j}$.

Proof. For a fixed $1 \leq i \leq n$, let $e = \{u_s, u_t\}$. For each $1 \leq l \leq k$, let $\alpha_l \in E(e_i, u_s), \alpha'_l \in E(e_i, u_t)$ be two sided expansions that perform the separation procedure for the pair $e_i, u_l$.

By Lemma 8.3 there is are dual expansions $\alpha \in E(e_i, u_s), \alpha' \in E(e_i, u_t)$ such that for each $1 \leq l \leq k$ we have that $\alpha < \alpha_l, \alpha' < \alpha'_l$.

It follows that the associated two sided expansion of $e_i$ has the property that if $e$ is an offspring of this expansion and $1 \leq j \leq k$ then the pair $e, u_j$ satisfies the conclusion of the lemma. We perform these two sided expansions for each $e_1, \ldots, e_n$ to finish the procedure. (We remark that the expansions produce new 0-cells incident to the offsprings, but in the statement of the lemma the set of 0-cells $\{u_1, \ldots, u_k\}$ is fixed.) \hfill $\Box$

Definition 8.27. Let $(U, V)$ be a system of cells in $X^{(1)}$. Let $I$ be the set of 1-cells $e$ in $X$ for which there are cells $e' \in U, v \in V$ such that $e$ is incident to $v$ and $e \sim e'$.

(In particular, $U \subseteq I$.) The system $(I, V)$ is called the equivariant closure of the system $(U, V)$.

We are now ready to define the separation procedure.

Definition 8.28. (Separation Procedure) Input: A system $(U, V)$ of cells in $X^{(1)}$.

Let $(U_1, V)$ be the system obtained by performing Procedure A from Lemma 8.20

Output: The system $(U_2, V)$, where $(U_2, V)$ is the equivariant closure of $(U_1, V)$.

Now our work in this section culminates in this important proposition.
Proposition 8.29. Let $Y$ be a finite subcomplex of $X^{(1)}$. Let $U$ be the sets of 1-cells of $Y$ and let $V$ be the set of 0-cells of $Y$. Let $(U_3, V)$ be the system obtained after performing Procedure $A$ on the system $(U, V)$ and let $(U_2, V)$ be the equivariant closure of $(U_1, V)$. In particular, $(U_2, V)$ is the system obtained after performing the separation procedure on $(U, V)$. Then $(U, V) \leq (U_2, V)$ and $(U_2, V)$ is balanced.

Proof. The claim that $(U, V) \leq (U_2, V)$ is obvious from the definitions of procedure $A$ and the equivariant closure. We will show that $(U_2, V)$ is balanced. It suffices to check the following.

1. For each $e \in U_3$, $u \in V$ either $e$, $u$ is disparate, or there is a $e' \in U_3$ incident to $u$ such that $e \cong e'$.
2. Let $e_1, e_2, e_1', e_2' \in U_3, u_1, u_2 \in V$ such that:
   a) $e_1, e_2$ are incident to $u_1$.
   b) $e_1, e_2$ are not disparate.
   c) $e_1'$ is incident to $u_2$ such that $e_1 \cong e_1'$.
   Then there is an $e_2'$ in $U_3$ incident to $u_2$ such that $e_2 \cong e_2'$. Moreover, $u_1, u_2$ is an associated pair for the equivalence $e_1 \cong e_1'$ if and only if it is the associated pair for the equivalence $e_2 \cong e_2'$.

Proof of (1): By the definition of equivariant closure there is a 1-cell $e_1 \in U_3$ such that $e \cong e_1$. By Procedure $A$ either $e_1, u$ are disparate (and hence by Lemma 7.15 $e, u$ are disparate), or else there is a 1-cell $e' \in U_2$ incident to $u$ such that $e_1 \cong e'$ (so $e \cong e'$ and $e' \in U_3$). The 1-cell $e'$ is in $U_3$ by definition of the equivariant closure.

Proof of (2): By the definitions of Procedure $A$ and equivariant closure there is a 1-cell $e_3 \in U_2$ incident to some 0-cell $u_3 \in V$ such that $e_3 \cong e_2$ and $u_1, u_3$ are associated pairs for the equivalence. Now since $e_1, e_2$ are not disparate it follows that there is a common offspring $e_4$ of expansions of $e_1, e_2$ at $u_1$. It follows that the equivariant expansion of $e_1'$ at $u_2$ has an offspring $e_4'$ such that $e_4 \cong e_4'$. Similarly there is an equivariant expansion of $e_3$ at $u_3$ with an offspring $e_5$ such that $e_5 \cong e_5'$. But then it follows that $e_5, u_2$ are not disparate, and in particular by Lemma 8.20 this means that $e_5, u_2$ are not disparate and there is a $e_2' \in U_3$ such that $e_2 \cong e_3 \cong e_2'$. The last part of (2) is a consequence of Lemmas 8.11 and 7.8. □

8.2. The Equivariant decoupling procedure. Now we define a procedure that takes an input a balanced system $(U, V)$ and a free system $(U', V)$ with the property that $(U, V) \leq (U', V)$. We shall adapt the proof from Lemma 8.19 to show the following.

Lemma 8.30. (Equivariant decoupling procedure) Let $(U, V)$ be a balanced system and let $V = \{v_1, ..., v_n\}$. We can perform mixed equivariant expansions of 1-cells in $U$ at $\{v_1, ..., v_n\}$ to produce a set of 1-cells $U_1$ such that the system $(U_1, V)$ is free. We remark that since we are only performing expansions at elements of $V$ we have that $(U, V) \leq (U_1, V)$.

Proof. Let $v_i, v_j \in V$ let $\{e_1, ..., e_n\}, \{e_1', ..., e_m\}$ be 1-cells incident to $v_i, v_j$ respectively such that the following holds.

1. $\{e_1, ..., e_n\}$ is the set of vertices of a connected component of $\Gamma_{v_i}(U)$ and $\{e_1', ..., e_m\}$ is the set of vertices of a connected component of $\Gamma_{v_j}(U)$.
2. $e_1 \cong e_1'$.

□
By the definition of balanced it follows that in fact \( m = n \) and by reordering the indices if necessary, \( e_i \cong e_i' \) and the map sending \( e_i \) to \( e_i' \) is a graph isomorphism between these components. Moreover either \( v_1, v_2 \) is an associated pair for each equivalence, or \( v_1, v_2 \) is not an associated pair for each equivalence.

For \( v_i \) by Lemma 8.19 we can perform expansions on the 1-cells \( e_1, \ldots, e_k \) incident to to solve the decoupling problem for this component at \( u_i \). Now given the expansion for \( e_i \) we perform the corresponding equivariant (or op-equivariant) expansion on each \( e_i' \) at \( u_j \), and by Lemma 8.21 it follows that these expansions solve the decoupling problem for the respective components of \( \Gamma_{v_1}(I), \Gamma_{v_2}(I) \). By an application of the above observation for all pairs of elements of \( V \), and by Lemma 8.18 we can easily generalize this to all vertices in \( V \).

\[ \square \]

9. Cube complexes and higher homotopy groups of \( X \)

In this section it will be our goal to show that \( \pi_i(X) \) is trivial for each \( i > 1 \). We shall do so by showing that for each finite subcomplex \( Y \) of \( X \), there is a subcomplex \( Y' \) of \( X \) such that \( Y \subseteq Y' \) and \( Y' \) is homeomorphic to a nonpositively curved cube complex.

First we will introduce a certain class of subcomplexes of \( X \), which we call cluster-cube complexes. These are subcomplexes that are naturally homeomorphic to cube complexes.

\[ \text{Definition 9.1.} \text{ Let } \{ \Delta_i \mid i \in I \} \text{ be a set of clusters in } X. \text{ Let } Y \text{ be a subcomplex of } X \text{ such that } Y = \bigcup_{i \in I} \Delta_i. \text{ We say that } Y \text{ is a cluster-cube complex if for each } i, j \in I \text{ the cluster } \Delta_i \cap \Delta_j \text{ is a facial subcluster of both } \Delta_i \text{ and } \Delta_j. \]

We observe that upon replacing each cluster in the set \( \{ \Delta_i \mid i \in I \} \) of \( Y \) by a Euclidean cube of the same dimension, we obtain a cube complex \( C(Y) \) which is homeomorphic to the complex \( Y \). The clusters \( \Delta_i \) naturally correspond to cubes in \( C(Y) \). Given any facial subcluster \( \Delta_i' \) of \( \Delta_i \), \( \Delta_i' \) naturally corresponds to a subcube of the cube \( \Delta_i \) in \( C(Y) \). Any cluster \( \Delta \) in \( Y \) that corresponds to a cube in \( C(Y) \) is called a cubical cluster of \( Y \). We call the complex \( Y \) a cluster-cube complex, but formally it is a complex of clusters in \( X \).

We now describe a procedure that takes as an input a free system \( (U, V) \), and produces a cluster-cube complex \( Y \) such that \( C(Y) \) is nonpositively curved.

\[ \text{Lemma 9.2.} \text{ (The cubulation procedure)} \text{ Let } (U, V) \text{ be a free system. Let } \Upsilon(U, V) \text{ be the set of 1-cells } e \text{ in } X \text{ for which there is an } e_i \in U \text{ such that } e \cong e_i. \text{ Let } Y \text{ be the subcomplex of } X \text{ described in the following way. } Y \text{ is a union of clusters } \Delta \text{ in } X \text{ such that all the facial 1-cells of } \Delta \text{ are in } \Upsilon(U, V). \text{ Then } Y \text{ is homeomorphic to a nonpositively curved cube complex.} \]

\[ \text{Proof.} \text{ First we claim that any pair of 1-cells } e_1, e_2 \in \Upsilon(U, V) \text{ incident to a 0-cell } u \text{ of } X \text{ satisfy that } e_1, e_2 \text{ are disparate. Assume that this is not the case. Let } e_1', e_2' \text{ be 1-cells in } U \text{ incident to } v_1, v_2 \in V \text{ respectively such that } e_i' \cong e_i. \text{ It follows that we can perform an expansion of } e_1', e_2' \text{ at } v_1, v_2 \text{ respectively such that there are equivalent offsprings } e_1'', e_2''. \text{ This means that the pairs } e_1', v_2 \text{ and } e_2', v_1 \text{ are not disparate. Since a free system is balanced it follows that there is a 1-cell } e_j'' \in U \text{ such that } e_j'' \text{ is incident to } v_1 \text{ and } e_j' \cong e_j'' \text{. But this means that } e_1', e_j'' \in U \text{ are not disparate, contradicting the definition of a free system. Our claim follows.} \]

Let \( T \) be the set of clusters satisfying the hypothesis of the lemma. From our characterization of subclusters in section 4 and the fact that the intersection of two
clusters is a cluster, it follows that if two clusters $\Delta_1, \Delta_2$ in $T$ have the property that $\Delta_1 \cap \Delta_2$ is a diagonal subcluster of $\Delta_2$, then we can find facial 1-cells $e_1, e_2$ in $\Delta_1, \Delta_2$ respectively such that $e_2$ is a diagonal 1-cell of $\Delta_1$ and $e_1$ is an offspring of $e_2$. In particular $e_1, e_2$ are incident to a common 0-cell and are not disparate. Since they are facial 1-cells of $\Delta_1, \Delta_2$ in $T$, $e_1, e_2 \in \Upsilon(U, V)$. This contradicts our observations above.

It follows that any pair of clusters $\Delta_1, \Delta_2$ in $T$ with a nonempty intersection must intersect in a facial subcluster. It follows that $Y$ is homeomorphic to a cube complex.

Now we will show that this is nonpositively curved. Let $u$ be a 0-cell in $Y$ such that $Lk(u, C(Y))$ has an empty $n$-simplex. This empty $n$-simplex corresponds to 1-cells $e_1, \ldots, e_n$ in $Y$ incident to $u$ such that each pair $e_i, e_j$ satisfy the property that there is a 2-cluster $\Delta_{i,j}$ which is a cluster in $T$ and has $e_i, e_j$ as its facial 1-cells. In particular it follows that $e_1, \ldots, e_n \in \Upsilon(U, V)$. By Lemma 5.16 it follows that in fact $e_1, \ldots, e_n$ are facial 1-cells of an $n$-cluster $\Delta$ which must be in $T$, since there are $n$ equivalence classes of the facial 1-cells of $\Delta$ with representatives $e_1, \ldots, e_n$, and so all the facial 1-cells of $\Delta$ are in fact in $\Upsilon(U, V)$. This means that this simplex in the link is not empty. By Gromov's characterization of nonpositively curved cube complexes, we conclude that $C(Y)$ is nonpositively curved. \(\square\)

We are now ready to prove the main result of this section.

**Proposition 9.3.** Let $Y$ be a finite subcomplex of $X$. Then there is a subcomplex $Y'$ of $X$ such that $Y \subseteq Y'$ and $Y'$ is homeomorphic to a nonpositively curved cube complex. In particular, it follows that $\pi_1(X)$ is trivial for $i > 1$.

**Proof.** Let $U_1$ be the set of 1-cells of $Y$ and $V$ be the set of 0-cells of $Y$. For the system $(U_1, V)$, following Proposition 8.29 we first perform the separation procedure to obtain a system $(U_2, V)$ which is balanced. Now we perform the equivariant decoupling procedure for the balanced system $(U_2, V)$ to obtain a free system $(U_3, V)$. Let $Y'$ be the nonpositively curved cube complex obtained from performing the cubulation procedure on the free system $(U_3, V)$. We know by Lemma 9.2 that $Y'$ is homeomorphic to a nonpositively curved cube complex. It remains to check that $Y \subseteq Y'$.

Let $\Delta$ be a cluster in $Y$ that is based at $\tau$ and parametrized by $\lambda_1, \ldots, \lambda_n$. Let $e_1 = \{F(\lambda_1 \tau), F\tau\}, \ldots, e_n = \{F(\lambda_n \tau), F\tau\}$. We know that $F\tau \in V$, and since $(U, V) \subseteq (U_3, V)$, we know that there are 1-cells $e_1', \ldots, e_m' \in U_3$ such that the set $\{e_1', \ldots, e_m'\}$ is precisely the set of offsprings obtained after performing some expansions on $e_1, \ldots, e_n$ at $\tau$.

Now $e_1', \ldots, e_m' \in \Upsilon(U_3, V)$ and since they are offsprings of 1-cells $e_1, \ldots, e_n$ it follows that there is an $m$-cluster $\Delta'$ whose facial 1-cells are $e_1', \ldots, e_m'$, which are all in $\Upsilon(U_3, V)$ Clearly, $\Delta$ is a diagonal subcluster of $\Delta'$ and $\Delta' \subseteq Y'$. Therefore $Y \subseteq Y'$. \(\square\)

Now we are ready to prove the main theorem.

**Theorem 9.4.** The group $G$ acts on a cell complex $X$ by cell permuting homeomorphisms such that the following holds.

1. $X$ is contractible.
2. The quotient $X/G$ has finitely many cells in each dimension.
3. The stabilizers of each cell are of type $F_\infty$. 

Since $G$ is finitely presented, it follows from the above that $G$ is of type $F_{\infty}$.

Proof. Claim 1 follows from propositions [9.3] and [6.2]. Claims 2 and 3 follow from proposition [5.33].

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Department of Mathematics, Cornell University, Ithaca, NY 14853–4201, USA
E-mail address: y1763@cornell.edu