Almost sharp Sobolev trace inequalities in the unit ball under constraints

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Abstract

We establish three families of Sobolev trace inequalities of orders two and four in the unit ball under higher order moments constraint, and are able to construct smooth test functions to show all such inequalities are almost optimal. Some distinct feature in almost sharpness examples between the fourth order and second order Sobolev trace inequalities is discovered. This has been neglected in higher order Sobolev inequality case in [21]. As a byproduct, the method of our construction can be used to show the sharpness of the generalized Lebedev-Milin inequality under constraints.

Keywords: Conformally covariant operators, Sobolev trace inequality, almost optimal, Lebedev-Milin inequality, Ache-Chang inequality.

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1 Introduction

The study of optimal constants in the Sobolev trace inequality has a long history. The purpose of this paper is to study three new families of Sobolev trace inequalities in the unit ball, under constraint of higher order moments with respect to the standard volume element on its boundary, and give examples to show these inequalities are almost optimal. Such optimal constants are in connection with the cubature formulas on spheres. We expect that such almost sharp inequalities will bring us more interesting applications to geometric problems in the future.

We would like to give a brief survey on the history of Sobolev trace inequalities in the unit ball. For \(n \geq 2\) and \(m \in \mathbb{N}\), we define

\[ P_m = \{ \text{all polynomials on } \mathbb{R}^n \text{ with degree at most } m \}, \]
\[ \hat{P}_m = \left\{ p \in P_m; \int_{S^{n-1}} pd\mu_{S^{n-1}} = 0 \right\}. \]

For the second order Sobolev trace inequality in the unit disk, Lebedev-Milin [23] established the following inequality in 1951.

**Theorem 1** (Lebedev-Milin). For \(f \in C^\infty(S^1)\), let \(u\) be a smooth extension of \(f\) to the unit disk \(\mathbb{B}^2\), then

\[
\log \left( \frac{1}{2\pi} \int_{S^1} e^f d\mu_{S^1} \right) \leq \frac{1}{4\pi} \int_{\mathbb{B}^2} |\nabla u|^2 dx + \frac{1}{2\pi} \int_{S^1} f d\mu_{S^1}. \tag{1}
\]

The generalization of Lebedev-Milin inequality was first proved in 1988 by Osgood-Phillips-Sarnak [29] for the first order moment and later extended by Widom [33] to all higher order moments.

**Theorem 2** (Osgood-Phillips-Sarnak & Widom). Let \(f \in C^\infty(S^1)\) satisfy \(\int_{S^1} e^f p d\mu_{S^1} = 0\) for all \(p \in \hat{P}_m\), and \(u\) be a smooth extension of \(f\) to the unit disk \(\mathbb{B}^2\), then

\[
\log \left( \frac{1}{2\pi} \int_{S^1} e^f d\mu_{S^1} \right) \leq \frac{1}{4\pi(m+1)} \int_{\mathbb{B}^2} |\nabla u|^2 dx + \frac{1}{2\pi} \int_{S^1} f d\mu_{S^1}. \tag{2}
\]

In higher dimension \(n \geq 3\), Beckner [4] proved a sharp Sobolev inequality in the unit ball \(\mathbb{B}^n\) with boundary \(S^{n-1}\).
Theorem 3 (Beckner). Let $n \geq 3$ and $1 < q \leq n/(n-2)$, then for all $u \in H^1(\mathbb{B}^n)$, there holds
\[
|S^{n-1}|^{\frac{q-1}{q+1}} \left( \int_{S^{n-1}} |u|^{q+1} d\mu_{S^{n-1}} \right)^{\frac{2}{q+1}} \leq (q-1) \int_{\mathbb{B}^n} |\nabla u|^2 dx + \int_{S^{n-1}} u^2 d\mu_{S^{n-1}},
\]
where $|S^{n-1}|$ denotes the volume of the standard unit sphere $S^{n-1}$.

For $n \geq 3$ and $1 < q \leq n/(n-2)$, we define
\[
c(n, q) = \frac{q-1}{|S^{n-1}|^{\frac{1}{q+1}}}
\]
and for all $u \in H^1(\mathbb{B}^n)$ with $u \neq 0$ on $S^{n-1}$,
\[
E_q[u] = \frac{(q-1) \int_{\mathbb{B}^n} |\nabla u|^2 dx + \int_{S^{n-1}} u^2 d\mu_{S^{n-1}}}{(\int_{S^{n-1}} |u|^{q+1} d\mu_{S^{n-1}})^{\frac{2}{q+1}}}.
\]

In the critical case $q = n/(n-2)$, Escobar [15] employed the Obata type argument to characterize all positive minimizers of the above functional $E_q$; Y. Y. Li -Zhu [25] applied the method of moving spheres to classify all positive critical points of the Euler-Lagrange equation of $E_q$; see also Y. Y. Li-Zhang [24]. In the subcritical case $q \in (1, n/(n-2))$, Guo-Wang [20] recently proved a Liouville type theorem for the Euler-Lagrange equation of $E_q$.

Following an argument in Aubin’s book [3, pp.61-63], we can prove the following Sobolev trace inequality under the vanishing first order moment of the boundary volume element (see Chang-Xu-Yang [11, Inequality (2.5) or Lemma 2.3]): Let $n \geq 3$ and $1 < q \leq n/(n-2)$, for any $0 < \varepsilon < 1$ and any $u \in H^1(\mathbb{B}^n)$ with
\[
\int_{S^{n-1}} x_i |u| \frac{2(n-1)}{n-2} d\mu_{S^{n-1}} = 0, \quad 1 \leq i \leq n,
\]
then there exists a positive constant $C_\varepsilon$ such that
\[
\left( \int_{S^{n-1}} |u|^{q+1} d\mu_{S^{n-1}} \right)^{\frac{2}{q+1}} \leq c(n, q) \left( 2^{\frac{1-q}{2}} + \varepsilon \right) \int_{\mathbb{B}^n} |\nabla u|^2 dx + C_\varepsilon \int_{S^{n-1}} u^2 d\mu_{S^{n-1}}.
\]

The above inequality has applications to the prescribed boundary mean curvature problem in $\mathbb{B}^n$. For instance, given $0 < h \in C^\infty(S^{n-1})$ with $\max_{S^{n-1}} f / \min_{S^{n-1}} f < 2^{1/(2(n-2))}$ and $2 \leq q < n/(n-2)$, let $u_q$ be a positive smooth minimizer of
\[
\inf_{u \in \mathcal{S}_h^q} \frac{\int_{\mathbb{B}^n} |\nabla u|^2 dx + \frac{n-2}{2} \int_{S^{n-1}} u^2 d\mu_{S^{n-1}}}{(\int_{S^{n-1}} h |u|^{q+1} d\mu_{S^{n-1}})^{\frac{2}{q+1}}},
\]
where
\[
\mathcal{S}_h^q = \left\{ u \in H^1(\mathbb{B}^n); \int_{S^{n-1}} x_i |u|^{q+1} d\mu_{S^{n-1}} = 0 \quad \text{and} \quad \int_{S^{n-1}} h |u|^{q+1} d\mu_{S^{n-1}} = 1 \right\}.
\]

The above inequality (4) plays a central role in deriving a key estimate, the lower bound of $\|u_q\|_{L^2(S^{n-1})}$, in the subcritical approximations method; see [11].
In a celebrated paper of Ache-Chang [1], authors proved the fourth order sharp Sobolev trace
inequalities in $\mathbb{B}^4$ and $\mathbb{B}^n$ for $n \geq 5$, which are natural counterparts of Lebedev-Milin and Beckner
inequalities, respectively. For readers’ convenience, we restate Ache-Chang sharp Sobolev trace
inequalities as follows.

**Theorem 4 (Ache-Chang).** Let $f \in C^\infty(S^{n-1})$ and $n \geq 5$. Suppose $u$ is a smooth extension of $f$
to $\mathbb{B}^n$ which satisfies Neumann boundary condition
\[
\frac{\partial u}{\partial r} = -\frac{n-4}{2} f \quad \text{on } S^{n-1},
\]
where $\partial_r$ is the outward unit normal derivative on $S^{n-1}$. Then
\[
c_n |S^{n-1}| \frac{3}{n-1} \left( \int_{S^{n-1}} |f|^{\frac{2(n-1)}{n-4}} d\mu_{S^{n-1}} \right)^\frac{n-4}{n} 
\leq \int_{\mathbb{B}^n} (\Delta u)^2 dx + 2 \int_{S^{n-1}} |\nabla f|^2 d\mu_{S^{n-1}} + b_n \int_{S^{n-1}} f^2 d\mu_{S^{n-1}}, \tag{5}
\]
where $c_n = n(n-2)(n-4)/4$ and $b_n = n(n-4)/2$. Moreover, equality holds if and only
if $u$ is the biharmonic extension of $f_{z_0}(x) = c |1 - z_0 \cdot x|^{(4-n)/2}$ on $S^{n-1}$, also satisfying
the above Neumann boundary condition, where $c \in \mathbb{R}, z_0 \in \mathbb{B}^n$. In particular, if $f = 1$, then
$u(x) = 1 + (n-4)(1-|x|^2)/4, x \in \mathbb{B}^n$.

**Theorem 5 (Ache-Chang).** Given $f \in C^\infty(S^3)$, let $u$ be a smooth extension of $f$ to $\mathbb{B}^4$
coupled with zero Neumann boundary condition, that is
\[
\frac{\partial u}{\partial r} = 0 \quad \text{on } S^3.
\]
Then with $\tilde{f} := \int_{S^3} f d\mu_{S^3}/(2\pi^2)$, there holds
\[
\log \left( \frac{1}{2\pi^2} \int_{S^3} e^{3(\tilde{f} - f)} d\mu_{S^3} \right) \leq \frac{3}{16\pi^2} \left[ \int_{\mathbb{B}^4} (\Delta u)^2 dx + 2 \int_{S^3} |\nabla f|^2 d\mu_{S^3} \right]. \tag{6}
\]
Moreover, equality holds if and only if $u$ is a biharmonic extension of some function $f_{z_0}(x) =
-\log |1-z_0 \cdot x|+C$ on $S^3$, and satisfies zero Neumann boundary condition, where $z_0 \in \mathbb{B}^4, C \in \mathbb{R}$.

In 2019, Chang-Hang [10] initiated a study on Moser-Trudinger-Onofri inequalities under
higher order moments constraint, which are similar improvements of (1) or (2). Subsequently,
Hang-Wang [22] made an extension of Sobolev inequality for functions in $W^{1,p}(\mathbb{S}^n)$ with $1 < p < n$
under the same constraint.

In this paper, we continue with an effort for Sobolev trace inequalities in the unit ball under
higher order moments constraint and establish three families of *almost sharp* Sobolev trace
inequalities.

To continue, we need to set up some notations. For $0 < \theta < 1$, as in [22] we define
\[
\mathcal{M}_m^\infty(S^{n-1}) = \{ \nu; \ \nu \text{ is a probability measure on } S^{n-1} \text{ supported on }
\]

\{\xi_i; i \in \mathbb{N}\} \text{ s.t. } \int_{S^{n-1}} p d\nu = 0, \ \forall \ p \in \hat{\mathcal{P}}_m \}

and

\[ \Theta(m, \theta, n - 1) = \inf \left\{ \sum_i \nu_i^\theta; \nu \in \mathcal{M}_m^c (S^{n-1}) \text{ is supported on } \{\xi_i\} \subset S^{n-1}, \right\} \]

Indeed, it has been proved by Putterman [30, Proposition 3.1] that \(\Theta(m, \theta, n - 1)\) can only be achieved by a Dirac probability measure supported on finitely many points. This directly implies that the infimum for \(\Theta(m, \theta, n - 1)\) is a minimum, which can follow from [30, Corollary 3.2]) or the proof of Proposition 4 below implicitly. Moreover, some known exact values of \(\Theta(m, \theta, n - 1)\) are \(\Theta(1, \theta, n - 1) = 2^{1-\theta}\) and \(\Theta(2, \theta, n - 1) = (n+1)^{1-\theta}\) by Hang-Wang [22] and \(\Theta(3, \theta, n - 1) = (2n)^{1-\theta}\) by Putterman [30, Theorem 5.1], whose method is related to the idea of deriving cubature formulas, such as the technique of reproducing kernels on spheres, etc.

We first state the second order Sobolev trace inequality in higher order moments case, which is a natural generalization of Beckner inequality.

**Theorem 6.** Let \(n \geq 3, m \in \mathbb{N} \text{ and } 1 < q \leq n/(n - 2)\), then for any \(u \in H^1 (\mathbb{B}^n)\) with

\[ \int_{S^{n-1}} p |u|^{q+1} d\mu_{S^{n-1}} = 0 \]  

for all \(p \in \hat{\mathcal{P}}_m\), and for any \(\varepsilon > 0\) we distinguish it into two cases:

(i) when \(1 < q < n/(n - 2)\), there exists a positive constant \(C_\varepsilon\) such that

\[ \left( \int_{S^{n-1}} |u|^{q+1} d\mu_{S^{n-1}} \right)^{\frac{2}{q+1}} \leq \varepsilon \int_{\mathbb{B}^n} |\nabla u|^2 dx + C_\varepsilon \int_{S^{n-1}} u^2 d\mu_{S^{n-1}}; \]

(ii) when \(q = n/(n - 2)\), there exists a positive constant \(C_\varepsilon\) such that

\[ \left( \int_{S^{n-1}} |u|^{\frac{2(n-1)}{n-2}} d\mu_{S^{n-1}} \right)^{\frac{n-2}{n-1}} \leq \left( \frac{c(n, \frac{n}{n-2})}{\Theta(m, \frac{n-2}{n-1}, n - 1)} + \varepsilon \right) \int_{\mathbb{B}^n} |\nabla u|^2 dx + C_\varepsilon \int_{S^{n-1}} u^2 d\mu_{S^{n-1}}, \]  

where

\[ c(n, \frac{n}{n-2}) = \frac{2}{n-1} \cdot \frac{\theta}{|S^{n-1}|^{\frac{1}{n-1}}}. \]

The proof of Theorem 6, as well as Theorems 7 and 8 below, relies on a modified compactness and concentration argument of Lions. One of our main advances is the use of conic way to connect a Borel measure in the ball with a Borel measure on the sphere for the deduction of concentration compactness principle in this setting; see the proof of Lemma 1 for example.

Moreover, we can show that the number \(c(n, \frac{n}{n-2})/\Theta(m, \frac{n-2}{n-1}, n - 1)\) in (8) is almost optimal.
Proposition 1. If \( n \geq 3, m \in \mathbb{N} \) and there exist \( a, b \in \mathbb{R} \) such that
\[
\left( \int_{S^{n-1}} |u|^{\frac{2(n-1)}{n-2}} \, d\mu_{S^{n-1}} \right)^{\frac{n-2}{n-4}} \leq a \int_{\mathbb{B}^n} |\nabla u|^2 \, dx + b \int_{S^{n-1}} u^2 \, d\mu_{S^{n-1}}
\]  
(9)
for any \( u \in H^1(\mathbb{B}^n) \) with
\[
\int_{S^{n-1}} p |u|^{\frac{2(n-1)}{n-2}} \, d\mu_{S^{n-1}} = 0, \quad \forall \, p \in \mathcal{P}_m,
\]  
(10)
then
\[
a \geq \frac{c(n, \frac{n}{n-2})}{\Theta(m, \frac{n-4}{n-1}, n-1)}.
\]

Next we transfer to the other two families of fourth order Sobolev trace inequalities under constraints.

Theorem 7. Suppose \( n \geq 5, m \in \mathbb{N} \) and for any \( \varepsilon > 0 \) and any \( f \in C^\infty(S^{n-1}) \) with
\[
\int_{S^{n-1}} p |f|^{\frac{2(n-1)}{n-4}} \, d\mu_{S^{n-1}} = 0
\]  
(11)
for all \( p \in \mathcal{P}_m \), then there exists a positive constant \( C_\varepsilon \) such that
\[
\left( \int_{S^{n-1}} |f|^{\frac{2(n-1)}{n-4}} \, d\mu_{S^{n-1}} \right)^{\frac{n-4}{n-1}} \leq \left( \frac{\alpha(n)}{\Theta(m, \frac{n-4}{n-1}, n-1) + \varepsilon} \right) \int_{\mathbb{B}^n} (\Delta u)^2 \, dx + C_\varepsilon \int_{S^{n-1}} (|\nabla f|^2_{S^{n-1}} + b_n f^2) \, d\mu_{S^{n-1}},
\]
(11)
where \( u \) is an \( H^2(\mathbb{B}^n) \) norm extension of \( f \) and satisfies the Neumann boundary condition
\[
\frac{\partial u}{\partial r} = -\frac{n-4}{2} f \quad \text{on } S^{n-1}
\]  
(12)
and
\[
b_n = \frac{n(n-4)}{2}, \quad \alpha(n) = \frac{4}{n(n-2)(n-4)|S^{n-1}|^{\frac{n-4}{n-1}}}.
\]

For non-vanishing Neumann (precisely, Robin) boundary condition (12), our conic proof of Theorem 7 has some advantage and thus sounds interesting to readers.

As in [10] we define
\[
\mathcal{N}_m(S^{n-1}) = \left\{ N \in \mathbb{N}; \exists \, x_1, \ldots, x_N \in S^{n-1} \text{ and } \nu_1, \ldots, \nu_N \in [0, \infty) \quad \text{with} \quad \sum_{i=1}^N \nu_i = 1 \text{ and } \sum_{i=1}^N \nu_i p(x_i) = 0, \quad \forall \, p \in \mathcal{P}_m \right\}.
\]
The smallest number in $N_m(S^{n-1})$ is denoted as $N_m(S^{n-1})$, i.e. $N_m(S^{n-1}) = \min N_m(S^{n-1})$.
Moreover, Chang-Hang gave an elementary proof in [10] to show that $N_m(S^1) = m + 1$ for all $m \in \mathbb{N}$ and $N_1(S^2) = 2$, $N_2(S^2) = 4$. Later Hang [21] extended to prove $N_1(S^{n-1}) = 2$ and $N_2(S^{n-1}) = n + 1$ for all $n \geq 2$. Lower bounds of $N_m(S^{n-1})$ can be found in Dai-Xu’s book [13, Theorems 6.1.2 and 6.1.4] (see also Delsarte-Goethals-Seidel [14] in 1977):

$$N_m(S^{n-1}) \geq C_{n-1+\frac{m}{2}}^n + C_{n+\frac{m}{2}-2}^n$$
if $m$ is even;
$$N_m(S^{n-1}) \geq 2C_{n-1+\frac{m-1}{2}}^n$$
if $m$ is odd.

(13)

Besides [21], some examples meeting the above lower bounds, called “tight $m$-spherical designs” in [14], can help us to derive other exact values of $N_m(S^{n-1})$; see also [5, 6, 7]. In particular, $N_3(S^{n-1}) = 2n$ by virtue of Lemma 5. The exact value of $N_m(S^{n-1})$ is intimately related to cubature formulas on spheres; see [13, Chapter 6], [28] etc.

**Theorem 8.** Given $f \in C^\infty(S^3)$, suppose $u$ is a smooth extension of $f$ to $\mathbb{B}^4$ which also satisfies zero Neumann boundary condition. Assume that $f$ satisfies $\int_{S^3} pe^{3f} d\mu_{S^3} = 0$ for all $p \in \mathcal{P}_m$ with some $m \in \mathbb{N}$, then for any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ such that

$$\log \left( \frac{1}{2\pi^2} \int_{S^3} e^{3f-\bar{f}} d\mu_{S^3} \right) \leq \left( \frac{3}{16\pi^2 N_3(S^3)} + \varepsilon \right) \left[ \int_{\mathbb{B}^4} \langle \nabla u \rangle^2 dx + 2 \int_{S^3} \langle \nabla f \rangle^2_{S^3} d\mu_{S^3} \right] + C_\varepsilon,$$

(14)

where $\bar{f} = \int_{S^3} f d\mu_{S^3}/(2\pi^2)$.

Since there exist several more challenging obstructions in addition to the second order case, almost sharpness examples for the fourth order Sobolev trace inequalities become more subtle and interesting. At the same time, without the second order example, the ones for fourth order case are impossible to appear.

**Proposition 2.** Suppose $n \geq 5$, $m \in \mathbb{N}$ and there exist $a, b \in \mathbb{R}$ such that

$$\left( \int_{S^{n-1}} |f|^{\frac{2(n-1)}{n-4}} d\mu_{S^{n-1}} \right)^{\frac{n-4}{n-1}} \leq a \int_{\mathbb{B}^n} \langle \nabla u \rangle^2 dx + b \int_{S^{n-1}} (\langle \nabla f \rangle^2_{S^{n-1}} d\mu_{S^{n-1}} + b_n f^2) d\mu_{S^{n-1}}$$

for any $u \in H^2(\mathbb{B}^n)$ satisfying

$$\int_{S^{n-1}} p|f|^{\frac{2(n-1)}{n-4}} d\mu_{S^{n-1}} = 0 \quad \forall \ p \in \mathcal{P}_m,$$

coupled with boundary conditions:

$$u = f, \quad \frac{\partial u}{\partial r} = -\frac{n-4}{2} f \quad \text{on} \ S^{n-1}.$$
Then
\[ a \geq \frac{\alpha(n)}{\Theta(m, \frac{n-1}{n-4}, n-1)} \]
with
\[ \alpha(n) = \frac{4}{n(n-2)(n-4)|\mathbb{S}^{n-1}|^{\frac{1}{n-4}}}. \]

For the construction of examples in Propositions 1 and 2, the following fact is enough to our use: The number \( \Theta(m, \theta, n-1) \) is achieved by some \( \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \in \mathcal{M}_m(S^{n-1}) \) for some \( N \geq N_m(S^{n-1}) \).

**Proposition 3.** Suppose there exist \( a, b \in \mathbb{R} \) such that for all \( u \in H^2(B^4) \) satisfying boundary conditions:
\[ u = f, \quad \frac{\partial u}{\partial r} = 0 \text{ on } S^3, \]
and \( \int_{S^3} p e^{2f} \, d\mu_{S^3} = 0 \) for all \( p \in \mathcal{P}_m \) with \( m \in \mathbb{N} \), we have
\[ \log \left( \frac{1}{2\pi^2} \int_{S^3} e^{3(f-f)} \, d\mu_{S^3} \right) \leq a \left[ \int_{B^4} (\Delta u)^2 \, dx + 2 \int_{S^3} |\nabla f|^2 \, d\mu_{S^3} \right] + b, \]
where \( \tilde{f} = (2\pi^2)^{-1} \int_{S^3} f \, d\mu_{S^3} \). Then
\[ a \geq \frac{3}{16\pi^2 N_m(S^3)}. \]

As have shown before, optimal constants are related to two numbers \( \Theta(m, \theta, n-1) \) and \( N_m(S^{n-1}) \). Furthermore, it is of independent interest that a natural relationship between these two numbers has been discovered.

**Proposition 4.** For \( n \geq 2 \) and all \( m \in \mathbb{N} \), there holds
\[ \lim_{\theta \searrow 0} \Theta(m, \theta, n-1) = N_m(S^{n-1}). \]

Compared with Chang-Hang [10] and Hang-Wang [22], some additional difficulties arise from compact manifolds with boundary and high order of conformally covariant operators. The following are the novelties of our constructions. First, we introduce a union of finitely many conic annuli \( \bigcup_{i=1}^{N} A_{\delta}(x_i) \) (see (24) for the definition and Figure 3 (a) for example) to isolate the dominated terms, which contribute to the sharp constant, and its complement in \( \mathbb{R}^n \) controls higher order terms with delicate computations and observations. Second, the Neumann boundary conditions are of geometric favor, arising from conformally covariant boundary operator \( P^3_3 \), GJMS operator of order three. As the involved operator is fourth order, precisely, the Paneitz operator, we need to make an appropriate correction to the test function like the second order one such that the new test function satisfies Neumann boundary condition. Third, since the extremal metric in Ache-Chang’s sharp Sobolev trace inequalities of order four in [1] is not the flat metric, but “the adapted metric”, first introduced by Case-Chang [9], this forces us to know the exact geometric local bubbles, which are resolved in Section 2. In dimension \( n \geq 5 \), a further correction is still needed to the geometric local bubble to ensure that the new local bubble satisfies the Neumann
boundary condition and controls higher order terms. By the way, the local bubble for the second order case is well-known in the study of boundary Yamabe problem. Geometric intuitions play an important role in all constructions. Finally, in contrast to the example for \( n \geq 5 \), we soon realize that the Chang-Hang type estimate [10, (3.12)]\(^1\) is not enough in dimension four. Whenever we struggled in this optimal constant, a geometric intuition/Branson’s intuitive proof always indicates that such an example should be there. That is exactly our motivation for improving the Chang-Hang type estimate. Such an essential improvement rescues us from above dilemma. We eventually achieve this goal in Section 5.2.1. This also demonstrates one of main differences of higher order Sobolev trace inequality from the second order one.\(^2\)

To the best knowledge of authors, *almost sharp* examples in this paper seem to be the first ones of fourth order Sobolev trace inequalities.

To demonstrate the relationships among these Sobolev trace inequalities in the unit ball, we draw a diagram for readers’ convenience.

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\(^1\)The estimate (3.12) is also crucial in the Chang-Hang’s example. At first glance, it sounds very tough to be improved. Fortunately, a clever observation on \( \mathcal{P}_1 \) in Proposition 6 motivates us to achieve this goal for all \( m \in \mathbb{N} \) in Proposition 7.

\(^2\)After the completion of our examples, we look back to the example for higher order Sobolev inequality in [21, p.21]. We do not know how to construct the example as the author advised in [21] without our new type estimate as in Proposition 7.
and the Widom (called Osgood-Phillips-Sarnak for \( m = 1 \)) inequality is also sharp.

We outline a unified approach to our constructions for fourth order Sobolev trace inequalities in three steps. **Step 1**, a localized analysis in the conic annulus \( \mathcal{A}_\delta(x_i) \) together with "a local bubble" \( \phi_{\varepsilon,i} \) is used to find the restriction of the test function to the boundary, say, \( v \), satisfying higher order moments constraint. **Step 2**, find a natural way to extend \( v \) to a global function \( u \) in \( \mathbb{B}^n \) for \( n \geq 4 \), satisfying the Neumann boundary condition in the following ways.

- For \( n \geq 5 \), we denote the test function by
  \[
  u^{2/(n-4)} = v^{2/(n-4)} + (1 - r)\tilde{g}(x), \quad r = |x|, x \in \mathbb{B}^n,
  \]
  where \( 1 - r \) is exactly the boundary defining function. Again the Neumann boundary condition (12) enables us to obtain the exact expression of \( \tilde{g} \).

- For \( n = 4 \), with delicate selections of suitable cut-off functions and the ‘nice’ properties of the “local bubble” \( \phi_{\varepsilon,i} \), we find the scheme used in the second order case still valid in this case except for an improved Chang-Hang type estimate.

**Step 3**, delicate calculations and deep insights are employed to capture optimal constants and to control higher order terms. We shall convince geometric intuitions through an analytic way.

The following is the organization of this paper. In Section 2, we revisit Ache-Chang’s elegant proof to understand “local bubbles” and their ‘nice’ geometric properties in the viewpoint of conformal geometry. In Section 3, we prove the second order Sobolev trace inequality and give an example of precise test functions to show the almost sharpness. A proof of Proposition 4 is also presented there. In Section 4.1, we adapt our conic proof to the fourth order Sobolev trace inequality for \( n \geq 5 \). In Section 4.2, we complete the construction of our example for \( n \geq 5 \) and thus finish the proof of Proposition 2. Section 5 is devoted to the fourth order Sobolev trace inequality established in Section 5.1 and the four dimensional example. In Section 5.2.1, we give an elementary proof of the exact value of \( N_3(S^{n-1}) \) and establish an improvement of the Chang-Hang type estimate, which enables us to construct an example to complete the proof of Proposition 3 in Section 5.2.2. In Appendix A, we employ an example to show the sharpness of Widom inequality as a warm-up of the four dimensional example, which is a subtle case as we know.

### 2 Geometric interpretations of local bubbles

Before presenting our concrete constructions of test functions, we think it important to describe geometric ideas behind them, as it will be a long journey.

As we have pointed out before, the local bubble in the second order is originated from the study of boundary Yamabe problem. Precisely, the bubble function is the conformal factor of a conformal metric in the class of the flat metric in upper half-space, which is scalar-flat with positive constant boundary mean curvature. So, we only focus on the ones associated to the fourth order Sobolev trace inequalities.

The **Paneitz operator** on a smooth Riemannian manifold \((M, g)\) of dimension \( n \geq 3 \) is defined by

\[
P^g_4 = \Delta^2_g + \delta_g(4A_g - (n - 2)J_g g)(\nabla \cdot, \cdot) + \frac{n - 4}{2}Q^g_4,
\]
where $\delta_g$ is the divergent operator, $A_g = \frac{1}{n-2} [\text{Ric}_g - \frac{R_g}{2(n-1)} g]$, $R_g$ is the scalar curvature, $J_g = \text{tr}_g(A_g)$ and $Q^g_4$ is the $Q$-curvature

$$Q^g_4 = -\Delta_g J_g + \frac{n}{4} J_g^2 - 2 |A_g|^2_g.$$  

It is well-known that $P^g_4$ is conformally covariant in the sense that if we let $\tilde{g} = u^{4/(n-4)} g \in [g]$ for $n \geq 5$, then for all $\psi \in C^\infty(\mathcal{M})$,

$$u^{\frac{n+4}{n-4}} P^\tilde{g}_4 \psi = P^g_4 (u\psi) = \Delta (u\psi)$$

and if we let $\tilde{g} = e^{2u} g$ for $n = 4$, then $P^\tilde{g}_4 = e^{-4u} P^g_4$ and

$$P^g_4 u + Q^g_4 = Q^\tilde{g}_4 e^{4u}.$$  

Since our construction is of not only analytic but also geometric favors, it sounds important to grasp some key ideas in Ache-Chang’s elegant proof. It will enable us to discover “a local bubble” naturally associated to “the adapted metric” in $\mathbb{B}^n$:

$$g^* := e^{2\tau(x)} |dx|^2 \quad \text{with} \quad \tau(x) = \frac{1 - |x|^2}{2} \quad \text{and} \quad n = 4$$

and

$$g^* := \psi(x) \frac{n-4}{n-4} |dx|^2 \quad \text{with} \quad \psi(x) = 1 + \frac{n-4}{2} \tau(x) \quad \text{and} \quad n \geq 5.$$  

These correspond to the extremal metrics in Ache-Chang’s Sobolev trace inequality of order four in [1, Theorem B] and [1, Theorem A], respectively. Moreover, the adapted metric $g^*$ satisfies the following properties (cf. a particular case of [1, Proposition 2.1], and first shown by Case-Chang [9] on a Poincaré-Einstein manifold):

- $Q^g_4 = 0$;
- $\mathbb{S}^{n-1} = \partial \mathbb{B}^n$ is totally geodesic with respect to $g^*$ and $g^* = g_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$;
- Let $P_3 = P^g_3 \mathbb{S}^{n-1} = (B - 1) B (B + 1)$ be the fractional GJMS operator defined in [1, (4.1)] with

$$B = \sqrt{-\Delta_{\mathbb{S}^{n-1}} + \frac{(n-2)^2}{4}}$$

and its associated $Q$-curvature

$$Q^g_{3 \mathbb{S}^{n-1}} := \frac{2}{n-4} P_3(1) = \frac{n(n-2)}{4}.$$  

Then

$$\frac{1}{2} E_4(g^*)[U_f] = \int_{\mathbb{S}^{n-1}} f P_3 f \, d\sigma_g^* - \frac{n-4}{2} \int_{\mathbb{S}^{n-1}} Q^g_{3 \mathbb{S}^{n-1}} f^2 \, d\sigma_g^*$$

$$= \int_{\mathbb{S}^{n-1}} f P_3 f \, d\mu_{\mathbb{S}^{n-1}} - \frac{n(n-2)(n-4)}{8} \int_{\mathbb{S}^{n-1}} f^2 \, d\mu_{\mathbb{S}^{n-1}}.$$  

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We give a brief summary on Ache-Chang’s proof: Authors started with the following energy identity

\[ 0 = \int_{\mathbb{B}^n} U_f P_4^{g^*} U_f d\mu_{g^*} = E_4(g^*)[U_f] + J(g^*)[U_f] \]

for all \( U_f \) satisfying

\[
\begin{align*}
P_4^{g^*} U_f &= 0 \quad \text{in } \mathbb{B}^n; \\
U_f &= f \quad \text{on } S^{n-1}; \\
\frac{\partial U_f}{\partial \nu_{g^*}} &= 0 \quad \text{on } S^{n-1},
\end{align*}
\]

(15)

where \( \nu_{g^*} \) is the outward unit normal with respect to the metric \( g^* \) on \( S^{n-1} \). Here

\[ E_4(g^*)[U_f] = \int_{\mathbb{B}^n} [(\Delta g^* U_f)^2 - (4A_{g^*} - (n-2)J_{g^*} g^*) (\nabla U_f, \nabla U_f)] d\mu_{g^*} \]

and \( J(g^*)[U_f] \) is the boundary terms arising from the integration by parts.

It follows from conformal invariance property of \( P_4^{g^*} \) that if \( n \geq 5 \), then for all \( u \in C^\infty(\mathbb{B}^n) \),

\[
\psi \frac{n+4}{n-4} P_4^{g^*} u = P_4^{g^*} |d\psi|^2 (u \psi) = \Delta^2 (u \psi)
\]

and if \( n = 4 \), then \( P_4^{g^*} = e^{-4\tau} P_4^{g_0} |d\psi|^2 = e^{-4\tau} \Delta^2 \). This together with (15) indicates

- If \( n \geq 5 \), then

\[
\begin{align*}
\Delta^2 (\psi U_f) &= 0 \quad \text{in } \mathbb{B}^n; \\
\psi U_f &= f \quad \text{on } S^{n-1}; \\
\frac{\partial (\psi U_f)}{\partial r} &= -\frac{n-4}{2} f \quad \text{on } S^{n-1},
\end{align*}
\]

(16)

- If \( n = 4 \), then

\[
\begin{align*}
\Delta^2 U_f &= 0 \quad \text{in } \mathbb{B}^n; \\
U_f &= f \quad \text{on } S^{n-1}; \\
\frac{\partial U_f}{\partial r} &= 0 \quad \text{on } S^{n-1}.
\end{align*}
\]

(17)

Direct consequences of (16) and (17) are

\[
0 = \int_{\mathbb{B}^n} |\Delta (\psi U_f)|^2 dx + \int_{S^{n-1}} f \frac{\partial}{\partial r} (\psi U_f) d\mu_{S^{n-1}} - \int_{S^{n-1}} \frac{\partial (\psi U_f)}{\partial r} (\Delta (\psi U_f)) d\mu_{S^{n-1}}
\]

\[
:= \int_{\mathbb{B}^n} |\Delta (\psi U_f)|^2 dx + J(g_0)[\psi U_f]
\]

for \( n \geq 5 \); and

\[
0 = \int_{\mathbb{B}^n} (\Delta U_f)^2 dx + \int_{S^{n-1}} f \frac{\partial}{\partial r} U_f d\mu_{S^{n-1}} := \int_{\mathbb{B}^n} (\Delta U_f)^2 dx + J(g_0)[U_f]
\]
for $n = 4$.

One of key ingredients in Ache-Chang’s proof is to calculate the exact expression of boundary term $J(g^*)[U_f]$ at the cost of lengthy computations. Finally, a bridge to Ache-Chang’s sharp Sobolev trace inequalities is the following Beckner’s sharp Sobolev inequalities (see [4] or [1, Theorem 4.2]):

- If $n \geq 5$, then for all $f \in C^\infty(S^{n-1})$,
  $$\frac{n(n-2)(n-4)}{8} \left( \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |f|^{\frac{2(n-1)}{n-4}} \, d\mu_{S^{n-1}} \right)^{\frac{n-4}{n-1}} \leq \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f P_3 f \, d\mu_{S^{n-1}}.$$

- If $n = 4$, then for all $f \in C^\infty(S^3)$,
  $$\log \left( \frac{1}{2\pi^2} \int_{S^3} e^{3(f-\bar{f})} \, d\mu_{S^3} \right) \leq \frac{3}{8\pi^2} \int_{S^3} f P_3 f \, d\mu_{S^3},$$
  where $\bar{f} = (2\pi^2)^{-1} \int_{S^3} f \, d\mu_{S^3}$.

With the help of differences of $J(g^*)[U_f] - J(g_0)[\psi U_f]$ for $n \geq 5$ and $J(g^*)[U_f] - J(g_0)[U_f]$ for $n = 4$, respectively, authors obtained the desired assertions, see [1, Theorems A and B].

From the viewpoint of conformal geometry, we collect some elementary facts together, which stimulate us to find aforementioned “local bubbles”.

- For $n \geq 2$, let $F : (B^n, |dx|^2) \to (\mathbb{R}_+^n, |dz|^2)$ be an inversion with respect to the sphere $\partial B_{\sqrt{2}}(-e_n)$ with radius $\sqrt{2}$ and center at $-e_n$, that is,
  $$z = F(x) = -e_n + \frac{2(x + e_n)}{|x + e_n|^2},$$
  then $F$ is a conformal map with property that
  $$F_* |dx|^2 = \left( \frac{2}{(1+z_n)^2 + |z'|^2} \right)^2 |dz|^2 := U(z) \frac{n+4}{2} |dz|^2, \quad z = (z', z_n) \in \mathbb{R}_+^n.$$

- Notice that
  $$F_* (\psi^{\frac{n+4}{n-4}} |dz|^2) = \hat{\psi}(z) \frac{n+4}{n-4} |dz|^2$$
  with
  $$\hat{\psi}(z) = \left[ 1 + \frac{n-4}{2} \frac{2z_n}{(1+z_n)^2 + |z'|^2} \right] U(z).$$

For $\varepsilon > 0$, by scalings we define
  $$\hat{\psi}_\varepsilon(z) = \varepsilon^{\frac{4-n}{2}} \hat{\psi}(\frac{z}{\varepsilon}).$$
• Combining Case [8, Corollary 5.2] and Ache-Chang [1, Theorem 4.2 (b)], we can immediately obtain the following sharp Sobolev trace inequality $H^2(R^n) \hookrightarrow H^{3/2}(R^{n-1}) \hookrightarrow L^{2(n-1)/(n-4)}(R^{n-1})$: For all $u \in C^\infty_c(R^n_+)$ with $\partial_z u = 0$ on $R^{n-1}$, there holds

$$
\frac{n(n-2)(n-4)}{4} |S^{n-1}|^{-\frac{3}{n-1}} \left( \int_{R^{n-1}} |u|^{\frac{2(n-4)}{n-4}} \, dz \right)^{\frac{n-4}{n-1}} \leq \int_{R^n_+} (\Delta u)^2 \, dz.
$$

Moreover, when equality holds, the extremal metric is exactly $\hat{\psi}^{4/(n-2)}|dz|^2$. In particular, $\hat{\psi}$ satisfies

$$
\begin{cases}
\Delta^2 \hat{\psi} = 0 & \text{in } R^n_+,
\hat{\psi} = f & \text{on } R^{n-1},
\partial_z \hat{\psi} = 0 & \text{on } R^{n-1},
\end{cases}
$$

where

$$
f(z') = \left( \frac{2}{1 + |z'|^2} \right)^{\frac{n-4}{2}}.
$$

As a byproduct, we obtain

$$
\int_{R^n_+} (\Delta \hat{\psi})^2 \, dz = \frac{n(n-2)(n-4)}{4} |S^{n-1}|.
$$

In this specific case, we give an explicit extremal functions in [8, Corollary 5.2].

• This arrives at the construction of the aforementioned “local bubble” in $\mathbb{B}^n$. Based on successful experience of the second order example, we replace $z_n$ by $1 - r$ and $|z'|$ by $\rho$ in $\psi_\epsilon = \hat{\psi}_\epsilon(z_n, |z'|)$, which is called “a local geometric bubble”. However, in order to satisfy the Neumann boundary condition, with a further modification on $\psi_\epsilon(r, \rho) := \hat{\psi}_\epsilon(1 - r, \rho)$ we define

$$
\phi_\epsilon(r, \rho) := \psi(r) \psi_\epsilon(r, \rho) = \psi(r) \left[ 1 + \frac{n-4}{2} \frac{2\epsilon(1-r)}{(\epsilon + 1 - r)^2 + \rho^2} \right] \left( \frac{2}{(\epsilon + 1 - r)^2 + \rho^2} \right)^{\frac{n-4}{2}}
$$

for the flat metric

$$
|dx|^2 = dr^2 + r^2 (d\rho^2 + \sin^2 \rho g_{S^{n-2}})
$$

for $x = r\xi \in \mathbb{B}^n$ and $\rho(\xi) = d_{S^{n-1}}(\xi, x_i) = \overline{\xi x_i}$. For convenience, we call $\phi_\epsilon$ “a local bubble”, which will save us a lot of energy on calculations.

Then it is not hard to verify that

$$
\partial_r \phi_\epsilon = \hat{\psi}_\epsilon \partial_r \psi + \psi \partial_r \hat{\psi}_\epsilon = -\frac{n-4}{2} \hat{\psi}_\epsilon = -\frac{n-4}{2} \phi_\epsilon \quad \text{on } S^{n-1}. \quad (18)
$$

The above property is fundamental in the positivity of our test function $u$ in $\mathbb{B}^n$, as well as the control of higher order terms.
As above, we can apply the same scheme to find the “local geometric bubble” in dimension \( n = 4 \) as follows.

- Notice that
  \[
  F_*\left(\frac{2}{(1 + z_n)^2 + |z'|^2}\right)^2 |dz|^2 = e^{2U_1(z)}|dz|^2
  \]
  and then
  \[
  F_* g^* = F_* \left(e^{2\tau(x)}|dz|^2\right) = e^{2\tau \circ F^{-1}(z)} e^{2U_1(z)}|dz|^2 = e^{2\hat{\tau}(z)}|dz|^2. \tag{19}
  \]
  This implies
  \[
  U_1(z) = \log \frac{2}{(1 + z_n)^2 + |z'|^2}
  \]
  and
  \[
  \hat{\tau}(z) = \tau \circ F^{-1}(z) + U_1(z) = \frac{2z_n}{(1 + z_n)^2 + |z'|^2} + \log \frac{2}{(1 + z_n)^2 + |z'|^2}
  \]
  for \( z = (z', z_n) \in \mathbb{R}^n_+ \).

Since
  \[
  Q_4^{g^*} = 0 \implies Q_4^{F_* (g^*)} = Q_4^{g^*} \circ F^{-1} = 0,
  \]
the \( Q_4 \)-curvature equation together with (19) yields
  \[
  P_4 |dz|^2 \hat{\tau} = Q_4^{F_* (g^*)} e^{n\hat{\tau}} = 0 \implies \Delta^2 \hat{\tau} = 0 \text{ in } \mathbb{R}^n_+.
  \]
Thus, we obtain
  \[
  \begin{cases}
  \Delta^2 \hat{\tau} = 0 & \text{in } \mathbb{R}^n_+,
  \\
  \hat{\tau} = f_1 & \text{on } \mathbb{R}^{n-1},
  \\
  \partial_{z_n} \hat{\tau} = 0 & \text{on } \mathbb{R}^{n-1},
  \end{cases}
  \]
where
  \[
  f_1(z') = \log \frac{2}{1 + |z'|^2}.
  \]
For \( \varepsilon > 0 \), by scalings we define
  \[
  \hat{\tau}_\varepsilon(z) = \tau\left(\frac{z}{\varepsilon}\right) - \log \varepsilon
  \]
  \[
  = \frac{2\varepsilon z_n}{(\varepsilon + z_n)^2 + |z'|^2} + \log \frac{2\varepsilon}{(\varepsilon + z_n)^2 + |z'|^2}.
  \]

- For each \( A_\delta(x_i) \subset \mathbb{B}^n \), under the local coordinates near \( x_i \), the flat metric can be expressed as
  \[
  |d\xi|^2 = dr^2 + r^2 (d\rho^2 + \sin^2 \rho g_{S^2})
  \]
  for \( x = r\xi \in \mathbb{B}^n \) and \( \rho(\xi) = d_{S^{n-1}}(\xi, x_i) = \xi x_i \).
As before, replacing \( z_n \) and \( |z'| \) in \( \hat{\rho}(z) = \hat{\rho}(z_n, |z'|) \) by \( 1 - r \) and \( \rho \), respectively, up to a constant \( \log(2\varepsilon) \), we define "a local geometric bubble" in dimension four by

\[
\phi_\varepsilon(r, \rho) = -\log \left( \left( \varepsilon + 1 - r \right)^2 + \rho^2 \right) + \frac{2\varepsilon(1-r)}{(\varepsilon + 1 - r)^2 + \rho^2}.
\]

A direct computation yields

\[
\partial_r \phi_\varepsilon = 0 \quad \text{on} \quad \mathbb{S}^{n-1}.
\]

(20)

This implies that \( \phi_\varepsilon \) satisfies zero Neumann boundary condition, which is crucial to control higher order terms.

With the above geometric intuitions at hand, we are confident that our strategy is feasible. A confirmation unavoidably involves lengthy calculations and deep insights.

3 Almost sharp Sobolev trace inequality of order two under constraints in dimension three and higher

The second order example is a basis for the ones of the fourth order case. The successful experience on the second order example opens the doors to the construction of the fourth order examples. It is our first time to introduce the conic proof and the conic annulus \( A_\delta(x_i) \) in the process of our example.

3.1 Second order Sobolev trace inequality

We first generalize the concentration and compactness principle in \( \mathbb{R}^n \) or closed manifolds (e.g., Lions \[26, 27\], Struwe \[32, \text{Section 4.8} \] and Chang-Hang \[10\] etc.) to the unit ball.

Lemma 1. Let \( n \geq 3 \) and \( 1 < q < n/(n - 2) \). Suppose that as \( k \to \infty \), \( u_k \rightharpoonup u \) weakly in \( H^1(\mathbb{B}^n) \) and

\[
|\nabla u_k|^2 dx \to |\nabla u|^2 dx + \lambda \quad \text{as measure in} \quad \mathbb{B}^n,
\]

\[
|u_k|^{q+1} d\nu_{\mathbb{S}^{n-1}} \to |u|^{q+1} d\mu_{\mathbb{S}^{n-1}} + \nu \quad \text{as measure in} \quad \mathbb{S}^{n-1},
\]

where \( \lambda \) and \( \nu \) are bounded nonnegative Borel measures on \( \mathbb{B}^n \) and \( \mathbb{S}^{n-1} \), respectively. For any Borel set \( \Omega \subset \mathbb{S}^{n-1} \), we define a nonnegative Borel measure on \( \mathbb{S}^{n-1} \) associated to \( \lambda \) through

\[
\hat{\lambda}(\Omega) := \lambda(C(\Omega)),
\]

where \( C(\Omega) \) is a cone with vertex at the origin and base \( \Omega \), i.e.,

\[
C(\Omega) = \{(r, \xi) \in \mathbb{B}^n; 0 \leq r \leq 1, \xi \in \Omega\}.
\]

(21)

Then there exist countably many points \( x_i \in \mathbb{S}^{n-1} \) with \( \nu_i = \nu(\{x_i\}) > 0 \) such that

\[
\nu = \sum_{i} \nu_i \delta_{x_i}, \quad \hat{\lambda} \geq \frac{1}{c(n, q)} \sum_{i} \nu_i^{ \frac{q+2}{q+1}} \delta_{x_i} \quad \text{for} \quad q = \frac{n}{n-2};
\]

and

\[
\nu = 0, \quad \text{for} \quad 1 < q < \frac{n}{n-2}.
\]
Proof. Let \( v_k := u_k - u \), then up to a subsequence, as \( k \to \infty \), \( v_k \to 0 \) weakly in \( H^1(\mathbb{B}^n) \) and \( v_k \to 0 \) in \( L^2(\mathbb{B}^n) \), and \( v_k \to 0 \) in \( L^{q+1}(S^{n-1}) \) for any \( 0 < s < n/(n-2) \); also \( \nu_k := |u_k|^{q+1}d\mu_{S^{n-1}} - |u|^{q+1}d\mu_{S^{n-1}} \to \nu \) and \( \lambda_k := |\nabla v_k|^2 \, dx \to \lambda \) in the weak sense of measures.

If \( 1 < q < n/(n-2) \), then the compact embedding of Sobolev trace inequality from \( H^1(\mathbb{B}^n) \) to \( L^{q+1}(S^{n-1}) \) forces \( \nu = 0 \).

In the following, it suffices to consider \( q = n/(n-2) \).

For any \( \varphi \in C^1(\mathbb{B}^n) \), by the sharp Sobolev trace inequality in (3) we have

\[
\int_{S^{n-1}} |\varphi|^{q+1} \, d\nu = \lim_{k \to \infty} \int_{S^{n-1}} |\varphi|^{q+1} \, d\nu_k = \lim_{k \to \infty} \int_{S^{n-1}} |v_k \varphi|^{q+1} \, dx
\]

\[
\leq c(n, q)^{q+1} \lim_{k \to \infty} \left[ \int_{\mathbb{B}^n} |\nabla (v_k \varphi)|^2 + (v_k \varphi)^2 \, dx \right]^{\frac{q+1}{2}}
\]

\[
=c(n, q)^{\frac{q+1}{2}} \lim_{k \to \infty} \left( \int_{\mathbb{B}^n} \varphi^2 |\nabla v_k|^2 \, dx \right)^{\frac{q+1}{2}}
\]

\[
=c(n, q)^{\frac{q+1}{2}} \left( \int_{\mathbb{B}^n} \varphi^2 \, d\lambda \right)^{\frac{q+1}{2}}
\]

that is,

\[
\int_{S^{n-1}} |\varphi|^{q+1} \, d\nu \leq c(n, q)^{\frac{q+1}{2}} \left( \int_{\mathbb{B}^n} \varphi^2 \, d\lambda \right)^{\frac{q+1}{2}}
\]

(22)

for all \( \varphi \in C^1(\mathbb{B}^n) \).

Now let \( J \subset S^{n-1} \) be the set of the atoms of the measure \( \nu \). From the assumption and the Sobolev trace inequality of \( H^1(\mathbb{B}^n) \hookrightarrow L^{q+1}(S^{n-1}) \) that

\[
\int_{S^{n-1}} d\nu \leq \lim_{k \to \infty} \int_{S^{n-1}} |u_k|^{q+1} \, d\mu_{S^{n-1}} \leq C \lim_{k \to \infty} \|u_k\|_{H^1(\mathbb{B}^n)}^{q+1} < \infty,
\]

we know that \( J \) is an at most countable set denoted by \( J = \{ x_i \mid i \in \mathbb{N} \} \). We decompose

\[
\nu = \nu_0 + \sum_{i} \nu_i \delta_{x_i},
\]

where \( \nu_0 \) is the singular continuous part of the measure \( \nu \) and also a nonnegative Borel measure.

We first claim that \( \nu_0 = 0 \).

To this end, for any open set \( \mathcal{O} \subset S^{n-1} \), choosing a sequence \( \{ \varphi_k \} \subset C^1(\mathbb{B}^n) \) in (22) such that \( \varphi_k \) converges to the characteristic function of \( \mathcal{C}(\mathcal{O}) \) and letting \( k \to \infty \), we obtain

\[
\int_{\mathcal{O}} d\nu \leq c(n, q)^{\frac{q+1}{2}} \left( \int_{\mathcal{C}(\mathcal{O})} d\lambda \right)^{\frac{q+1}{2}} = c(n, q)^{\frac{q+1}{2}} \left( \int_{\mathcal{O}} d\lambda \right)^{\frac{q+1}{2}} < \infty.
\]
This obviously implies that $\nu$ is absolutely continuous with respect to $\hat{\lambda}$ and thus from the Radon-Nikodym theorem that there exists a nonnegative $f \in L^1(S^{n-1}, \hat{\lambda})$ such that $d\nu = f \, d\hat{\lambda}$. Moreover, for $\hat{\lambda}$-almost every $x \in S^{n-1}$ and $B_\rho(x) \subset S^{n-1}$, we have

$$f(x) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} d\nu}{\int_{B_\rho(x)} d\hat{\lambda}}.$$

In particular, if we choose $x \in S^{n-1}$ such that the segment $\overline{ax}$ does not carry any atom of $\lambda$, then

$$f(x) = \lim_{\rho \to 0} \left( \frac{\int_{B_\rho(x)} d\nu}{\int_{B_\rho(x)} d\hat{\lambda}} \right)^{\frac{q}{q+1}} \leq c(n, q) \lim_{\rho \to 0} \left( \int_{\mathcal{C}(B_\rho(x))} d\hat{\lambda} \right)^{\frac{q-1}{q+1}} = 0.$$

Furthermore, $\lambda$ has only countably many atoms, so does $\hat{\lambda}$ by definition. Hence, we conclude that for $\hat{\lambda}$-almost everywhere on $S^{n-1}$, $f(x) = 0$. This together with the fact that $\nu_0$ contains no atoms implies that $\nu_0 = 0$.

Next we go to establish the desired inequality.

For each $x_i \in J$, we choose $\varphi \in C^1(B^n)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on the segment $\overline{ax_i}$. Hence, with $\hat{\lambda}_i = \hat{\lambda}(\{x_i\})$, we can apply (22) with the above $\varphi$ to conclude that

$$c(n, q) \hat{\lambda}_i \geq \nu_i \frac{2}{q+1}$$

and thus

$$\hat{\lambda} \geq \frac{1}{c(n, q)} \sum_i \nu_i \frac{2}{q+1} \delta_{x_i}.$$ 

This completes the proof.

We are now in a position to prove Theorem 6.

**Proof of Theorem 6.** We shall prove these results by contradiction. If either (7) or (8) is not true, then we can find a sequence $\{u_k\} \subset H^1(B^n)$ and some $\alpha > 0$ for $1 < q < n/(n-2)$ or

$$\alpha = \frac{c(n, \frac{n}{n-2})}{\Theta(m, \frac{n-2}{n-1}, n-1)} + \varepsilon \quad \text{for} \quad q = \frac{n}{n-2},$$

respectively, such that

$$\left( \int_{S^{n-1}} |u_k|^{q+1} d\mu_{S^{n-1}} \right)^{\frac{2}{q+1}} > \alpha \int_{B^n} |\nabla u_k|^2 dx + k \int_{S^{n-1}} |u_k|^2 d\mu_{S^{n-1}}$$

and

$$\int_{S^{n-1}} p |u_k|^{q+1} d\mu_{S^{n-1}} = 0, \quad \forall p \in \mathcal{P}_m.$$
We may normalize \( \int_{S^{n-1}} |u_k|^{q+1} d\mu_{S^{n-1}} = 1 \) such that
\[
\int_{\mathbb{B}^n} |\nabla u_k|^2 dx \leq \frac{1}{\alpha}
\]
and
\[
\int_{S^{n-1}} |u_k|^2 d\mu_{S^{n-1}} \leq \frac{1}{k}.
\]

The Sobolev inequality
\[
Y(\mathbb{B}^n, \partial \mathbb{B}^n) \|u_k\|_{L^{2n/(n-2)}(\mathbb{B}^n)}^2 \leq \int_{\mathbb{B}^n} |\nabla u_k|^2 dx + 2(n-1) \int_{S^{n-1}} u_k^2 d\mu_{S^{n-1}}
\]
indicates that \( \|u_k\|_{L^{2n/(n-2)}(\mathbb{B}^n)} \) is uniformly bounded, so is \( \|u_k\|_{L^2(\mathbb{B}^n)} \). Readers are referred to [16] for the definition of the Yamabe constant \( Y(\mathbb{B}^n, \partial \mathbb{B}^n) \), which is positive. Thus, \( u_k \) is uniformly bounded in \( H^1(\mathbb{B}^n) \) and then up to a subsequence, \( u_k \to 0 \) weakly in \( H^1(\mathbb{B}^n) \) as \( k \to \infty \).

(i) If \( 1 < q < n/(n-2) \), then \( u_k \to 0 \) in \( L^{q+1}(S^{n-1}) \) as \( k \to \infty \). A contradiction!

(ii) For \( q = n/(n-2) \), there exist nonnegative Borel measures \( \lambda \) and \( \nu \) on the \( \sigma \)-algebras of \( \mathbb{B}^n \) and \( S^{n-1} \), respectively, such that, up to a subsequence, as \( k \to \infty \)
\[
|\nabla u_k(x)|^2 dx \rightharpoonup \lambda \quad \text{as measures in } \mathbb{B}^n
\]
and
\[
|u_k|^{q+1} d\mu_{S^{n-1}} \rightharpoonup \nu \quad \text{as measures in } S^{n-1}.
\]

Let \( \hat{\lambda} \) be the Borel measure on the \( \sigma \)-algebra of \( S^{n-1} \) associated to \( \lambda \) as in Lemma 1. Again by Lemma 1, we can find countably many points \( x_j \in S^{n-1} \) such that 
\[
\nu = \sum_j \nu_j \delta_{x_j}
\]
with \( \nu_j = \nu(\{x_j\}) \) and
\[
\hat{\lambda} \geq \frac{1}{e(n, \frac{n}{n-2})} \sum_j \nu_j^{\frac{n-2}{n-1}} \delta_{x_j}.
\]

Notice that
\[
\nu(S^{n-1}) = 1 \quad \text{and} \quad \hat{\lambda}(S^{n-1}) \leq \frac{1}{\alpha}.
\]

By definition of the weak convergence for measures, we know that 
\[
\int_{S^{n-1}} pd\nu = 0 \quad \text{for } p \in \mathcal{P}_m.
\]
By definition of $\Theta(m, (n-2)/(n-1), n)$, with $\hat{\lambda}_j = \hat{\lambda} \{{x_j}\}$ we have

$$\Theta(m, \frac{n-2}{n-1}, n-1) \leq \sum_j \nu_j^{n-2} \leq \sum_j c(n, \frac{n}{n-2}) \hat{\lambda}_j$$

$$\leq c(n, \frac{n}{n-2}) \hat{\lambda} \left(\mathbb{S}^{n-1}\right) \leq \frac{c(n, \frac{n}{n-2})}{\alpha}.$$ 

Hence,

$$\alpha \leq \frac{c(n, \frac{n}{n-2})}{\Theta(m, \frac{n-2}{n-1}, n-1)}.$$ 

However, this contradicts the choice of $\alpha$.  \qed

### 3.2 Almost sharpness

We start with a brief discussion on the number $\Theta(m, \theta, n-1)$. Using the idea of proof in Putterman [30, Proposition 3.1], one can prove that if $\nu = \sum_{i=1}^N \nu_i \delta_{x_i} \in \mathcal{M}_m^\circ \left(\mathbb{S}^{n-1}\right)$ for any $N > \bar{N} := \dim(\mathcal{P}_m)$, then $\nu$ can not be an extremal element of $\Theta(m, \theta, n-1)$. A direct consequence is that the infimum in $\Theta(m, \theta, n-1)$ is a minimum by virtue of [30, Corollary 3.2].

Some exact values have been known.

- For $m = 1$, the [22, Proposition 3.1] states that
  $$\Theta(1, \theta, n-1) = 2^{1-\theta}$$
  is achieved by $\nu_1 \in \mathcal{M}_1^\circ \left(\mathbb{S}^{n-1}\right)$ if and only if $\nu_1 = \frac{1}{2}(\delta_\xi + \delta_{-\xi})$ for any $\xi \in \mathbb{S}^{n-1}$.

- For $m = 2$, the [22, Proposition 3.2] states that
  $$\Theta(2, \theta, n-1) = (n+1)^{1-\theta}$$
  is achieved by $\nu_2 \in \mathcal{M}_2^\circ \left(\mathbb{S}^{n-1}\right)$ if and only if $\nu_2 = \left(\sum_{i=1}^{n+1} \delta_{x_i}\right)/(n+1) \in \mathcal{M}_2^\circ \left(\mathbb{S}^{n-1}\right)$, where $x_1, \ldots, x_{n+1} \in \mathbb{S}^{n-1}$ are the vertices of a regular $(n+1)$-simplex embedded in $\mathbb{R}^n$.

- For $m = 3$, the [30, Theorem 1.2] states that
  $$\Theta(3, \theta, n-1) = (2n)^{1-\theta}$$
  is achieved by $\nu_3 \in \mathcal{M}_3^\circ \left(\mathbb{S}^{n-1}\right)$ if and only if
  $$\nu_3 = \frac{1}{2n} \sum_{i=1}^n \left(\delta_{e_i} + \delta_{-e_i}\right),$$
  up to an isometry on $\mathbb{S}^{n-1}$, where $\{e_i; 1 \leq i \leq n\}$ is the standard basis in $\mathbb{R}^n$.

We would like to point out that a combination of the cubature formulas and examples meeting the lower bounds (13) in [14] is helpful to know more exact values of $\Theta(m, \theta, n-1)$.

Based on the above known results, it is expected that as $\theta \rightarrow 0$, the limit of $\Theta(m, \theta, n-1)$ should be $N_m(\mathbb{S}^{n-1})$.  

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**Proof of Proposition 4.** First, we show that
\[
\limsup_{\theta \searrow 0} \Theta(m, \theta, n-1) \leq N_m(S^{n-1}) := N_m.
\]

To this end, it follows from the definition of \(N_m(S^{n-1})\) that there exist \(\{\xi_i; 1 \leq i \leq N_m\} \subset S^{n-1}\) and \(\nu_i > 0, \sum_{i=1}^{N_m} \nu_i = 1\), such that \(\nu = \sum_{i=1}^{N_m} \nu_i \delta_{\xi_i} \in M_3^c(S^{n-1})\). By definition of \(\Theta(m, \theta, n-1)\), we have
\[
\Theta(m, \theta, n-1) \leq \sum_{i=1}^{N_m} \nu_i^\theta.
\]

Letting \(\theta \searrow 0\) in the above inequality, the desired assertion follows.

Next, it suffices to show
\[
\Theta := \liminf_{\theta \searrow 0} \Theta(m, \theta, n-1) \geq N_m.
\]

To that end, it follows from [30, Proposition 3.1 and Corollary 3.2] that \(\forall \theta \in (0, 1)\), there exist \(\xi_i \in S^n\) and \(\nu_i \geq 0\) for \(1 \leq i \leq \bar{N}\), such that \(\sum_{i=1}^{\bar{N}} \nu_i = 1, \sum_{i=1}^{\bar{N}} \nu_i p_j(\xi_i) = 0\) for a basis \(\{p_j\} \subset \hat{P}_m\) and
\[
\Theta(m, \theta, n-1) = \sum_{i=1}^{\bar{N}} \nu_i^\theta.
\]

We can find a sequence of real numbers \(\{\theta_k\}\) such that \(\theta_k \to 0\) and
\[
\Theta(m, \theta_k, n-1) \to \Theta \quad \text{as} \quad k \to \infty.
\]

For any fixed \(\theta_k\), there exist \(\xi_i^{(k)} \in S^n\) and \(\nu_i^{(k)} \geq 0\) for \(1 \leq i \leq \bar{N}\), such that
\[
\sum_{i=1}^{\bar{N}} \nu_i^{(k)} = 1, \quad \sum_{i=1}^{\bar{N}} \nu_i^{(k)} p_j(\xi_i^{(k)}) = 0 \quad \text{for all} \quad p_j \in \hat{P}_m
\]
and
\[
\Theta(m, \theta_k, n-1) = \sum_{i=1}^{\bar{N}} \left( \nu_i^{(k)} \right)^{\theta_k}.
\]

Then up to a subsequence, there exist \(\nu_i^{(\infty)} \geq 0\) and \(\xi_i^{(\infty)} \in S^{n-1}, 1 \leq i \leq \bar{N}\), such that for each \(i\),
\[
\nu_i^{(k)} \to \nu_i^{(\infty)}, \quad \xi_i^{(k)} \to \xi_i^{(\infty)} \quad \text{as} \quad k \to \infty.
\]

Letting \(k \to \infty\) in (23) we obtain
\[
\sum_{i=1}^{\bar{N}} \nu_i^{(\infty)} = 1, \quad \sum_{i=1}^{\bar{N}} \nu_i^{(\infty)} p_j(\xi_i^{(\infty)}) = 0 \quad \text{for all} \quad p_j \in \hat{P}_m,
\]
whence
\[ \nu^{(\infty)} := \sum_{i=1}^{N} \nu_i^{(\infty)} \delta_{x_i^{(\infty)}} \in \mathcal{M}_m^c(\mathbb{S}^{n-1}). \]

It follows from the definition of \( N_m(\mathbb{S}^{n-1}) \) that
\[ N_m \leq \sharp \left\{ \nu_i^{(\infty)} > 0, \ 1 \leq i \leq N \right\}. \]

On the other hand, we notice that if \( \nu_i^{(\infty)} > 0 \), then \( \left( \nu_i^{(k)} \right)^{\theta_k} \to 1 \) as \( k \to \infty \); if \( \nu_i^{(\infty)} > 0 \), then \( \liminf_{k \to \infty} \left( \nu_i^{(k)} \right)^{\theta_k} \geq 0 \). Hence, putting these facts together we obtain
\[
\Theta = \lim_{k \to \infty} \Theta(m, \theta_k, n-1) \\
= \lim_{k \to \infty} \sum_{i=1}^{N} \left( \nu_i^{(k)} \right)^{\theta_k} \\
\geq \lim_{k \to \infty} \sum_{1 \leq i \leq N, \ \nu_i^{(\infty)} > 0} \left( \nu_i^{(k)} \right)^{\theta_k} = \sharp \left\{ \nu_i^{(\infty)} > 0, \ 1 \leq i \leq N \right\} \geq N_m
\]
as desired. \( \square \)

We now transfer to the construction of precise test functions in order to show that the constant \( c(n, \frac{n}{n-2})/\Theta(m, \frac{n-2}{n-1}, n-1) \) in the inequality (8) is almost optimal.

**Proof of Proposition 1.** For each \( m \), there exist some natural number \( N \geq N_m(\mathbb{S}^{n-1}) \) and \( \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \in \mathcal{M}_m^c \) for \( 1 \leq i \leq N \), such that
\[
\Theta(m, \frac{n-2}{n-1}, n-1) = \sum_{i=1}^{N} \nu_i^{\frac{n-2}{n-1}}.
\]

We denote by \( \tilde{xy} \) the geodesic distance between \( x \) and \( y \) in \( \mathbb{S}^{n-1} \). Fix \( \delta > 0 \) small enough such that \( \mathcal{A}_{2\delta}(x_i) \cap \mathcal{A}_{2\delta}(x_j) = \emptyset \) for \( 1 \leq i < j \leq N \), where
\[
\mathcal{A}_\delta(x_i) := \left\{ x = r\xi \in \mathbb{B}^n, \xi \in \mathbb{S}^{n-1}, \ 1-r < \delta, \ \tilde{x_i}\xi < \delta \right\}
\]
for each \( x_i \in \mathbb{S}^{n-1} \). In other words, \( \mathcal{A}_\delta(x_i) \) is a conic annulus, where the cone has the origin as its vertex and a geodesic ball \( B_\delta(x_i) \subset \mathbb{S}^{n-1} \) as its base.

For \( 0 < \varepsilon < \delta \) and each \( 1 \leq i \leq N \), under the above coordinates we define
\[
\phi_{\varepsilon, i}(x) = \chi_i(x) \left( (\varepsilon + 1 - r)^2 + \tilde{x_i}^2 \right)^{\frac{2-n}{2}},
\]
where \( \chi_i(x) \) is a smooth cut-off function, \( \chi_i(x) = 1 \) in \( \mathcal{A}_\delta(x_i) \) and \( \chi_i(x) = 0 \) outside \( \mathcal{A}_{2\delta}(x_i) \).
Define
\[ v(x) = \sum_{i=1}^{N} \nu_i^{n-2} \phi_{\varepsilon,i}(x), \]
then a direct computation yields
\[
\int_{S^{n-1}} v^{2(n-1)} \frac{\mu_{S^{n-1}}}{n-2} \\
= \sum_{i=1}^{N} \nu_i \int_{B_{2\varepsilon}(x_i)} \phi_{n-2}^{2(n-1)} \frac{\mu_{S^{n-1}}}{n-2} \\
= (2\varepsilon)^{1-n} |S^{n-1}| + \begin{cases} 
O(\log \varepsilon^{-1}) & \text{if } n = 3; \\
O(\varepsilon^{3-n}) & \text{if } n \geq 4;
\end{cases} \\
= (2\varepsilon)^{1-n} |S^{n-1}| + O(\varepsilon^{3-n} \log \varepsilon^{-1}) \quad \text{as } \varepsilon \to 0. 
\]

(25)

For any \( p \in \mathcal{P}_m \), we have
\[
\int_{S^{n-1}} v^{2(n-1)} p \frac{\mu_{S^{n-1}}}{n-2} \\
= \sum_{i=1}^{N} \nu_i \int_{B_{2\varepsilon}(x_i)} \phi_{n-2}^{2(n-1)} p(x) \frac{\mu_{S^{n-1}}}{n-2} \\
= \sum_{i=1}^{N} \int_{B_{2\varepsilon}(x_i)} \left[ \phi_{n-2}^{2(n-1)} (x) p(x_i) + \phi_{n-2}^{2(n-1)} (\widehat{x} x_i) \frac{2(n-1)}{2} \right] \frac{\mu_{S^{n-1}}}{n-2} \\
= O(\varepsilon^{3-n} \log \varepsilon^{-1}),
\]

(26)

where the last equality follows from
\[ \nu \in \mathcal{M}_m^c(S^{n-1}) \implies \sum_{i=1}^{N} \nu_i p(x_i) = 0. \]

Obviously, we shall make a further correction for the above \( v \) to fulfill the condition (10). It is shown in [31, Theorem 2.1 in Chapter IV] that there exists a basis \( \{ P_1, \cdots, P_L \} \) of \( \mathcal{P}_m \), such that
\[ p_1 = P_1|_{S^{n-1}}, \cdots, p_L = P_L|_{S^{n-1}} \]
are spherical harmonics, where \( L = n + \sum_{i=2}^{m} (C_{n+1-i} - C_{n-1-i}) \). Then for each \( 1 \leq i \leq N \), we claim that there exist \( \psi_1, \cdots, \psi_L \in C_\infty^\infty(\mathbb{R}^n \setminus \bigcup_{i=1}^{m} \mathcal{A}_{2\varepsilon}(x_i)) \) such that the determinant
\[
\det \left[ \int_{S^{n-1}} \psi_j p_k \frac{\mu_{S^{n-1}}}{n-2} \right]_{1 \leq j, k \leq L} \neq 0.
\]

(27)
To this end, we can choose a nonzero smooth function $\eta \in C^\infty_\infty \left( B^n \setminus \bigcup_{i=1}^N A_{2\delta} (x_i) \right)$ such that $\eta P_1, \cdots, \eta P_L$ are linearly independent. It follows that the Gram matrix

$$\left[ \int_{S^{n-1}} \eta^2 p_j p_k d\mu_{S^{n-1}} \right]_{1 \leq j, k \leq L}$$

is positive definite, then $\psi_j = \eta^2 P_j$ satisfies (27).

The fact (27) enables us to find $\beta_1, \cdots, \beta_L \in \mathbb{R}$ such that

$$\int_{S^{n-1}} \left( v^{\frac{2(n-1)}{n-2}} + \sum_{j=1}^L \beta_j \psi_j \right) p_k d\mu_{S^{n-1}} = 0 \quad \forall 1 \leq k \leq L. \tag{28}$$

Moreover, it follows from (26) that for all $1 \leq j \leq L$, $\beta_j = O(\varepsilon^3 - n \log \varepsilon - 1)$ as $\varepsilon \to 0$. As a consequence we can find a constant $c_1 > 0$ such that

$$\sum_{j=1}^L \beta_j \psi_j + c_1 \varepsilon^3 - n \log \varepsilon - 1 \geq \varepsilon^3 - n \log \varepsilon - 1.$$

We define the test function by

$$u^{\frac{2(n-1)}{n-2}} = v^{\frac{2(n-1)}{n-2}} + \sum_{j=1}^L \beta_j \psi_j + c_1 \varepsilon^3 - n \log \varepsilon - 1. \tag{29}$$

Clearly, it follows from (28) and (29) that

$$\int_{S^{n-1}} p u^{\frac{2(n-1)}{n-2}} d\mu_{S^{n-1}} = 0, \quad \forall p \in \hat{P}_m.$$

Hence, as $\varepsilon \to 0$, by (25) and (29) we have

$$\| u \|_{L^2(S^{n-1})}^2 \geq \left[ \int_{S^{n-1}} \left( v^{\frac{2(n-1)}{n-2}} + \sum_{j=1}^L \beta_j \psi_j + c_1 \varepsilon^3 - n \log \varepsilon - 1 \right) d\mu_{S^{n-1}} \right]^{\frac{n-2}{n-1}} = (2\varepsilon)^{2-n} |S^{n-1}|^{\frac{n-2}{n-1}} (1 + O(\varepsilon^2 \log \varepsilon - 1)). \tag{30}$$

Next, we estimate the term $\| u \|_{L^2(S^{n-1})}$. To this end, we can apply (29) to show

$$u^{\frac{2(n-1)}{n-2}} \leq v^{\frac{2(n-1)}{n-2}} + C \varepsilon^3 - n \log \varepsilon - 1,$$

which directly yields

$$u^2 \leq \left( v^{\frac{2(n-1)}{n-2}} + C \varepsilon^3 - n \log \varepsilon - 1 \right)^{\frac{n-2}{n-1}} \leq v^2 + C(\varepsilon^3 - n \log \varepsilon - 1)^{\frac{n-2}{n-1}}.$$
Through a direct computation showing that in $A$ where the last identity follows from the estimate that in $A$ for $\alpha, \beta$

$$\int_{S^{n-1}} u^2 \, d\mu_{S^{n-1}} \leq \int_{S^{n-1}} \left( v^2 + C(\varepsilon^{3-n} \log \varepsilon^{-1})^{\frac{n-2}{n-1}} \right) \, d\mu_{S^{n-1}}$$

$$= \sum_{i=1}^{N} \nu_i^{\frac{n-2}{n-1}} \int_{B_{2\delta}(x_i)} \phi_{\epsilon,i}^2 \, d\mu_{S^{n-1}} + C(\varepsilon^{3-n} \log \varepsilon^{-1})^{\frac{n-2}{n-1}}$$

$$= O(\varepsilon^{3-n} \log \varepsilon^{-1})^{\frac{n-2}{n-1}} = O(\varepsilon^{2-n})(\varepsilon^2 \log \varepsilon^{-1})^{\frac{n-2}{n-1}}. \quad (31)$$

To estimate $||\nabla u||^2_{L^2(B^n)}$. Near each $x_i, 1 \leq i \leq N$ the flat metric $|dx|^2$ in $\mathbb{B}^n$ is expressed as

$$|dx|^2 = dr^2 + r^2 \left( d\rho^2 + \sin^2 \rho g_{S^{n-2}} \right)$$

for $x = r \xi \in \mathbb{B}^n$ and $\rho = d_{S^{n-1}}(\xi, x_i) \hat{=} \xi x_i$. Under this coordinate system, we rewrite

$$\phi_{\epsilon,i}(x) = \chi_i((\varepsilon + 1 - r)^2 + \rho^2)^{\frac{2-n}{2}}.$$

Recall that the Beta function is defined by

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} \, dx = B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

for $\alpha, \beta \in \mathbb{C}$ with $\text{Re}(\alpha), \text{Re}(\beta) > 0$.

We are ready to calculate

$$||\nabla u||^2_{L^2(B^n)}$$

$$= \sum_{i=1}^{N} \int_{A_{2\delta}(x_i)} |\nabla u|^2 \, dx + O \left( \varepsilon^{2(3-n)} \log \varepsilon^{-2} \right)$$

$$= \sum_{i=1}^{N} \nu_i^{\frac{n-2}{n-1}} \int_{A_{\delta}(x_i)} \left( 1 + c_1 \nu_i^{-1} \varepsilon^{3-n} \log \varepsilon^{-1} \phi_{\epsilon,i}^{\frac{2(1-n)}{n-2}} \right)^{-\frac{n}{n-1}} |\nabla \phi_{\epsilon,i}|^2 \, dx + O \left( \varepsilon^{2(3-n)} \log \varepsilon^{-2} \right),$$

where the last identity follows from the estimate that in $A_{2\delta}(x_i)$,

$$u^{\frac{2(\alpha-1)}{n-2}} = \nu_i^{\frac{2(n-1)}{n-2}} + c_1 \varepsilon^{3-n} \log \varepsilon^{-1} \implies |\nabla u|^2 = \nu_i^2 \left( \frac{\phi_{\epsilon,i}}{u} \right)^{\frac{2n}{n-2}} |\nabla \phi_{\epsilon,i}|^2.$$

Through a direct computation showing that in $A_{\delta}(x_i)$,

$$|\nabla \phi_{\epsilon,i}|^2 = (n-2)^2 \frac{(\varepsilon + 1 - r)^2 + r^2 \rho^2}{((\varepsilon + 1 - r)^2 + \rho^2)^n}.$$

Hence, the above integral on the right hand side can be estimated by

$$\int_{A_{\delta}(x_i)} \left( 1 + c_1 \nu_i^{-1} \varepsilon^{3-n} \log \varepsilon^{-1} \phi_{\epsilon,i}^{\frac{2(1-n)}{n-2}} \right)^{-\frac{n}{n-1}} |\nabla \phi_{\epsilon,i}|^2 \, dx$$

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\[ \int_{\frac{1}{2}}^{1-r} (n-2)^2 |S|^{n-2} \left| \int_0^\delta \int_0^t \frac{((\varepsilon + t)^2 + \rho^2)^{1-n} (1-t)^{n-1} \sin^{n-2} \rho}{(1 + c_1 \nu_1^2 \varepsilon^{3-n} \log \varepsilon^{-1} ((\varepsilon + t)^2 + \rho^2)^{n-1})^{\frac{n}{n-1}}} \, d\rho \, dt + I_1 \right. \]
\[ \leq (n-2)^2 |S|^{n-2} \left| \int_0^\delta \int_0^t \frac{((\varepsilon + t)^2 + \rho^2)^{1-n} \rho^{n-2}}{(1 + c_1 \nu_1^2 \varepsilon^{3-n} \log \varepsilon^{-1} ((\varepsilon + t)^2 + \rho^2)^{n-1})^{\frac{n}{n-1}}} \, d\rho \, dt + O(\varepsilon^{3-n}) \right. \]
\[ \leq (n-2)^2 |S|^{n-2} \left| \int_0^\delta \int_0^t ((\varepsilon + t)^2 + \rho^2)^{1-n} \rho^{n-2} \, d\rho \, dt + O(\varepsilon^{3-n}) \right. \]
\[ = (n-2)^2 |S|^{n-2} |e^{2-n} \int_0^\delta \int_0^t ((1 + t)^2 + \rho^2)^{1-n} \rho^{n-2} \, d\rho \, dt + O(\varepsilon^{3-n}) \]
\[ = (n-2)^2 |S|^{n-2} |e^{2-n} \int_0^{\infty} \int_0^t ((1 + t)^2 + \rho^2)^{1-n} \rho^{n-2} \, d\rho \, dt + O(\varepsilon^{3-n}) \]
\[ = \frac{n-2}{2} |S|^{n-2} |B(\frac{n-1}{2}, \frac{n-1}{2})| \varepsilon^{2-n} + O(\varepsilon^{3-n}) , \]

where the first inequality follows from

\[ I_1 = (n-2)^2 |S|^{n-2} \left| \int_0^\delta \int_0^t \frac{((\varepsilon + t)^2 + \rho^2)^{1-n} t(2-t)(1-t)^{n-3}}{(1 + c_1 \nu_1^2 \varepsilon^{3-n} \log \varepsilon^{-1} ((\varepsilon + t)^2 + \rho^2)^{n-1})^{\frac{n}{n-1}}} \, d\rho \, dt \right. \]
\[ \leq 2(n-2)^2 |S|^{n-2} \left| \int_0^\delta \int_0^t ((\varepsilon + t)^2 + \rho^2)^{1-n} t \rho^{n-2} \, d\rho \, dt = O(\varepsilon^{3-n}) \right. \]

On the other hand, we can further require that

\[ (1-r)^2 \leq \varepsilon^{1-\frac{2(1-\rho_0)}{n-1}} < \delta^2 \quad \text{and} \quad \rho^2 \leq \varepsilon^{1-\frac{2(1-\rho_0)}{n-1}} < \delta^2 \]

for some \( 0 < \epsilon_0 < 1/2 \), then

\[ 1 + c_1 \nu_1^2 \varepsilon^{3-n} \log \varepsilon^{-1} ((\varepsilon + 1-r)^2 + \rho^2)^{n-1} = 1 + O(\varepsilon^{2\epsilon_0} \log \varepsilon^{-1}) . \]

With this estimate at hand, it is not hard to show that \( I_1 = O(\varepsilon^{3-n}) \).

Hence, we obtain

\[ \int_{A_d(x_i)} \left( 1 + c_1 \nu_1^2 \varepsilon^{3-n} \log \varepsilon^{-1} \frac{\delta}{\varepsilon^{1-\rho_0}} \frac{\phi_{\varepsilon,i}}{\phi_{\varepsilon,i}^{2-n-2}} \right) \, |\nabla \phi_{\varepsilon,i}|^2 \, dx \]
\[ = (n-2)^2 |S|^{n-2} \left| \int_0^\delta \int_0^t \frac{((\varepsilon + t)^2 + \rho^2)^{1-n} \rho^{n-2}}{(1 + c_1 \nu_1^2 \varepsilon^{3-n} \log \varepsilon^{-1} ((\varepsilon + t)^2 + \rho^2)^{n-1})^{\frac{n}{n-1}}} \, d\rho \, dt \right. \]
\[ + I_2 + O(\varepsilon^{3-n}) \]
\[ \geq (n-2)^2 |S|^{n-2} (1 + O(\varepsilon^{2\epsilon_0} \log \varepsilon^{-1})) \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} ((\varepsilon + t)^2 + \rho^2)^{1-n} \rho^{n-2} \, d\rho \, dt \right) \]
\[ + O(\varepsilon^{3-n}) \]
\[ (n - 2)^2 |S^{n-2}| \varepsilon^{2-n} \int_0^\infty \int_0^\infty (1 + t)^2 + \rho^2)^{1-n} \rho^{n-2} \, d\rho \, dt + O \left( \varepsilon^{2-n+2\alpha} \log \varepsilon^{-1} \right) \]

where the above inequality follows from

\[ \| \nabla \phi \|_{L^2(B^n)}^2 = n \sum_{i=1}^N \int_{A_3(x_i)} \left( 1 + c_1 \nu_i \right)^{1-n} \varepsilon^{3-n} \log \varepsilon^{-1} |\phi_{\varepsilon,i}| \, dx \]

In summary, for each \( 1 \leq i \leq N \) we have

\[ \int_{A_3(x_i)} \left( 1 + c_1 \nu_i \right)^{1-n} \varepsilon^{3-n} \log \varepsilon^{-1} |\phi_{\varepsilon,i}| \, dx = \varepsilon^{2-n} n - 2 |S^{n-2}| B \left( \frac{n-1}{2}, \frac{n-1}{2} \right) + O \left( \varepsilon^{2-n+2\alpha} \log \varepsilon^{-1} \right). \]

Consequently, we conclude that

\[ \| \nabla u \|_{L^2(B^n)}^2 = \sum_{i=1}^N \int_{A_3(x_i)} \left( 1 + c_1 \nu_i \right)^{1-n} \varepsilon^{3-n} \log \varepsilon^{-1} |\phi_{\varepsilon,i}| \, dx + O \left( \varepsilon^{2-n+2\alpha} \log \varepsilon^{-1} \right) \]

Therefore, inserting (30), (31), (32) into (9) and dividing both sides by \( \varepsilon^{2-n} \), next letting \( \varepsilon \to 0 \), we obtain

\[ 2^{2-n} |S^{n-1}|^{\frac{n-2}{n-1}} \leq a 2^{2-n} \Theta(m, \frac{n-2}{n-1}, n-1)(n-2) |S^{n-1}|, \]

i.e.,

\[ a \geq \frac{c(n, \frac{n-2}{n-1})}{\Theta(m, \frac{n-2}{n-1}, n-1)}. \]

This completes our construction.
4 Almost sharp Sobolev trace inequality of order four under constraints in dimension five and higher

We utilize the conic proof to derive Sobolev trace inequality of fourth order for \( n \geq 5 \). With some deep insight on “local bubble”, we are able to complete the construction of almost sharp example.

4.1 Fourth order Sobolev trace inequality

Inspired by the proof of the second order Sobolev trace inequality, we know that Ache-Chang’s sharp Sobolev trace inequalities of order four in Theorem 4 is a prerequisite for the following compactness and concentration lemma. For the almost sharp example, we do care about the equality in (5) that if \( f = 1 \), then the extremal metric \( u^{4/(n-4)} dx^2 \) is the adapted metric in the Poincaré model \((\mathbb{B}^{n}, \mathcal{S}^{n-1}, g_{\mathbb{H}})\) of hyperbolic space for \( n \geq 5 \). first introduced by Case-Chang [9]; see also [1, Proposition 2.2].

Based on Theorem 4, as before we begin with the refinement of the concentration and compactness principle, whose proof presented below is similar in spirit to the one of Lemma 1.

Lemma 2. For \((\mathbb{B}^{n}, |dx|^2)\) and \( n \geq 5 \), let \( u_k \in H^2(\mathbb{B}^{n}) \) be a sequence of \( H^2(\mathbb{B}^{n}) \) extensions of \( f_k \in C^\infty(\mathcal{S}^{n-1}) \) satisfying the Neumann boundary condition

\[
\frac{\partial u_k}{\partial r} = -\frac{n-4}{2} f_k \quad \text{on } \mathcal{S}^{n-1}.
\]

Assume that as \( k \to \infty \), \( u_k \rightharpoonup u \) weakly in \( H^2(\mathbb{B}^{n}) \) and

\[
\frac{2(n-1)}{n-4} |f_k|^\frac{2(n-1)}{n-4} d\mu_{\mathcal{S}^{n-1}} \to |f|^\frac{2(n-1)}{n-4} d\mu_{\mathcal{S}^{n-1}} + \nu \quad \text{as measure in } \mathbb{B}^{n},
\]

\[
|f_k|^{\frac{2(n-1)}{n-4}} d\mu_{\mathcal{S}^{n-1}} \to |f|^\frac{2(n-1)}{n-4} d\mu_{\mathcal{S}^{n-1}} + \nu \quad \text{as measure in } \mathcal{S}^{n-1},
\]

where \( \lambda \) and \( \nu \) are bounded nonnegative measures on \( \mathbb{B}^{n} \) and \( \mathcal{S}^{n-1} \), respectively. Then there exist countably many points \( x_i \in \mathcal{S}^{n-1} \) with \( \nu_i = \nu(\{x_i\}) > 0 \) such that

\[
\nu = \sum_i \nu_i \delta_{x_i} \quad \text{and} \quad \hat{\lambda} \geq \frac{1}{\alpha(n)} \sum_i \nu_i^{\frac{n-4}{n-1}} \delta_{x_i}.
\]

Proof. Let \( v_k := u_k - u \), then it follows from the assumptions that up to a subsequence, as \( k \to \infty \), \( v_k \rightharpoonup 0 \) weakly in \( H^2(\mathbb{B}^{n}) \) and \( v_k \to 0 \) in \( H^1(\mathbb{B}^{n}) \) and \( f_k \to 0 \) in \( H^1(\mathcal{S}^{n-1}) \); also \( \nu_k := |v_k|^{\frac{2(n-1)}{n-4}} d\mu_{\mathcal{S}^{n-1}} - |u|^{\frac{2(n-1)}{n-4}} d\mu_{\mathcal{S}^{n-1}} \to \nu \) and \( \lambda_k := (\Delta u_k)^2 dx \to \lambda \).

For any \( \varphi \in C^2(\mathbb{B}^{n}) \) with \( \partial \varphi / \partial r = 0 \) on \( \mathcal{S}^{n-1} \), by the Sobolev trace inequality of order four in Theorem 4 we have

\[
\left( \int_{\mathcal{S}^{n-1}} |\varphi|^{\frac{2(n-1)}{n-4}} d\nu \right)^{\frac{n-4}{n-1}} = \left( \lim_{k \to \infty} \int_{\mathcal{S}^{n-1}} |\varphi|^{\frac{2(n-1)}{n-4}} d\nu_k \right)^{\frac{n-4}{n-1}} = \left( \lim_{k \to \infty} \int_{\mathcal{S}^{n-1}} |v_k \varphi|^{\frac{2(n-1)}{n-4}} d\mu_{\mathcal{S}^{n-1}} \right)^{\frac{n-4}{n-1}}
\]
\[ \leq \alpha(n) \lim_{k \to \infty} \left[ \int_{\mathbb{B}^n} |\Delta (u_k \varphi)|^2 \, dx \right] \]
\[ = \alpha(n) \lim_{k \to \infty} \left[ \int_{\mathbb{B}^n} \varphi^2 (\Delta u_k)^2 \, dx \right] \]
\[ = \alpha(n) \int_{\mathbb{B}^n} \varphi^2 \, d\lambda, \]
that is,
\[ \left( \int_{\mathbb{S}^{n-1}} |\varphi|^{2(n-1)} \, d\nu \right)^{\frac{n-4}{n-1}} \leq \alpha(n) \int_{\mathbb{B}^n} \varphi^2 \, d\lambda \quad (33) \]
for all \( \varphi \in C^2(\mathbb{B}^n) \) with \( \partial \varphi / \partial r = 0 \) on \( \mathbb{S}^{n-1} \).

Now let \( J \subset \mathbb{S}^{n-1} \) be the set of the atoms of the measure \( \nu \). From the assumption and the Sobolev trace inequality that
\[ \int_{\mathbb{S}^{n-1}} d\nu \leq \lim_{k \to \infty} \int_{\mathbb{S}^{n-1}} |u_k|^{2(n-1)} \, d\mu_{\mathbb{S}^{n-1}} \leq C \liminf_{k \to \infty} \| u_k \|_{H^2(\mathbb{B}^n)}^2 < \infty, \]
we know that \( J \) is an at most countable set denoted by \( J = \{ x_i ; i \in \mathbb{N} \} \). We decompose
\[ \nu = \nu_0 + \sum_i \nu_i \delta_{x_i} \]
where \( \nu_0 \) is the singular continuous part of the measure \( \nu \) and also a nonnegative measure. For any Borel set \( \Omega \subset \mathbb{S}^{n-1} \), same as in Lemma 1, we define
\[ \hat{\lambda}(\Omega) := \lambda(C(\Omega)), \]
where \( C(\Omega) \) is the cone defined in (21). Since both \( \nu \) and \( \lambda \) are bounded Borel measures, so does \( \hat{\lambda} \) by definition. Therefore, given any Borel set \( \Omega \) in \( \mathbb{S}^{n-1} \) and any \( \varepsilon > 0 \), there exist an open set \( \mathcal{O} \) and a closed set \( K \) such that \( K \subset \Omega \subset \mathcal{O} \) with
\[ \nu(\mathcal{O} \setminus \Omega) < \varepsilon, \quad \hat{\lambda}(\mathcal{O} \setminus \Omega) < \varepsilon, \]
and
\[ \nu(\Omega \setminus K) < \varepsilon, \quad \hat{\lambda}(\Omega \setminus K) < \varepsilon. \]
For all \( \varphi \in C^\infty(\mathbb{B}^n) \) with \( \text{supp} \varphi \subset C(\mathcal{O}) \) such that \( \varphi|_{C(\mathcal{K})} = 1, 0 \leq \varphi \leq 1, \partial \varphi / \partial r = 0 \) on \( \mathbb{S}^{n-1} \), we conclude from (33) that
\[ \left( \int_K d\nu \right)^{\frac{n-4}{n-1}} \leq \alpha(n) \left( \int_{C(\mathcal{O})} d\lambda \right). \]
Hence,
\[ (\nu(\Omega) - \varepsilon)^{\frac{n-4}{n-1}} \leq \alpha(n) \left( \hat{\lambda}(\Omega) + \varepsilon \right). \]
Letting \( \varepsilon \to 0 \), we have
\[
\nu(\Omega)^{\frac{n-4}{n-1}} \leq \alpha(n) \tilde{\lambda}(\Omega) < \infty.
\]

This indicates that \( \nu \) is absolutely continuous with respect to \( \tilde{\lambda} \) and thus from the Radon-Nikodym theorem that there exists \( f \in L^1(\mathbb{B}^n; \tilde{\lambda}) \) such that \( d\nu = f d\tilde{\lambda} \). Moreover, for \( \tilde{\lambda} \)-almost every \( x \in \mathbb{S}^{n-1} \) and \( B_\rho(x) \subset \mathbb{S}^{n-1} \), we have
\[
f(x) = \lim_{\rho \searrow 0} \frac{\int_{B_\rho(x)} d\nu}{\int_{B_\rho(x)} d\tilde{\lambda}}.
\]

In particular, if we choose \( x \in \mathbb{S}^{n-1} \) and the segment \( \overline{ox} \) does not carry any atom of \( \lambda \), then
\[
f(x) = \lim_{\rho \searrow 0} \left( \frac{\int_{B_\rho(x)} d\nu}{\int_{B_\rho(x)} d\tilde{\lambda}} \right)^{\frac{n-4}{n-1}} \leq \alpha(n) \lim_{\rho \searrow 0} \left( \int_{C(B_\rho(x))} \frac{\beta_{\varepsilon}^3}{3} d\mu \right) = 0.
\]

From this and (33), a similar argument in Lemma 1 yields that \( \nu_0 = 0 \) and
\[
\hat{\lambda} \geq \frac{1}{\alpha(n)} \sum_i \nu_i^{\frac{n-4}{n-1}} \delta_{x_i}.
\]

This completes the proof. \( \square \)

We are now ready to prove Theorem 7.

**Proof of Theorem 7.** Following the same lines in the proof of [1, Theorem A], we only need to consider the case when \( u \) is a biharmonic extension of \( f \) to \( \mathbb{B}^n \). For brevity, we define
\[
\beta_{\varepsilon} = \frac{\alpha(n)}{\Theta(m, \frac{n-4}{n-1}, n-1)} + \varepsilon.
\]

By contradiction, if (11) is not true, then we can find some \( \beta_{\varepsilon} \) and a sequence of functions \( \{u_k\} \subset H^2(\mathbb{B}^n) \), which are the biharmonic extensions of \( f_k \) to \( \mathbb{B}^n \) satisfying the Neumann boundary condition
\[
\frac{\partial u_k}{\partial r} = -\frac{n-4}{2} f_k \quad \text{on} \quad \mathbb{S}^{n-1},
\]

such that
\[
\left( \int_{\mathbb{S}^{n-1}} |f_k|^2 \frac{2(n-1)}{n-4} d\mu_{\mathbb{S}^{n-1}} \right)^{\frac{n-4}{n-1}} > \beta_{\varepsilon} \int_{\mathbb{B}^n} (\Delta u_k)^2 \, dx + k \int_{\mathbb{S}^{n-1}} (|\nabla f_k|^2 \frac{2(n-1)}{n-4} + b_n f_k^2) \, d\mu_{\mathbb{S}^{n-1}}
\]

and
\[
\int_{\mathbb{S}^{n-1}} p |f_k|^2 \frac{2(n-1)}{n-4} d\mu_{\mathbb{S}^{n-1}} = 0, \quad \forall \, p \in \tilde{\mathcal{P}}_m.
\]

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We may normalize $\int_{S^{n-1}} |f_k|^2(n-1)/(n-4) \, d\mu_{S^{n-1}} = 1$ such that
\[
\int_{\mathbb{B}^n} |\Delta u_k|^2 \, dx \leq \frac{1}{\beta \varepsilon}, \quad \int_{S^{n-1}} f_k^2 \, d\mu_{S^{n-1}} \leq \frac{1}{b_n k},
\]
and
\[
\int_{S^{n-1}} |\nabla f_k|^2 \, d\mu_{S^{n-1}} \leq \frac{1}{k}.
\]
It follows from the standard elliptic theory (e.g., [18, Theorem 2.16]) that $u_k$ is uniformly bounded in $H^2(\mathbb{B}^n)$. Hence, up to a subsequence, as $k \to \infty$, $u_k \to u$ weakly in $H^1(\mathbb{B}^n)$, $f_k \to 0$ in $H^1(S^{n-1})$; and there exist two nonnegative Borel measures $\lambda$ and $\nu$ such that
\[
|\Delta u_k|^2 \, dx \rightharpoonup |\Delta u|^2 \, dx + \lambda \quad \text{as measures in } \mathbb{B}^n,
\]
\[
|f_k|^{2(n-1)/(n-4)} \, d\mu_{S^{n-1}} \rightharpoonup \nu \quad \text{as measures in } S^{n-1}.
\]
It follows from Lemma 2 that there exist countably many points $x_i \in S^{n-1}$ with $\nu_i = \nu(\{x_i\}) > 0$ such that
\[
\nu = \sum_i \nu_i \delta_{x_i} \quad \text{and} \quad \hat{\lambda} \geq \frac{1}{\alpha(n)} \sum_i \nu_i^{n-4} \delta_{x_i}.
\]
Notice that
\[
\nu(S^{n-1}) = 1 \quad \text{and} \quad \hat{\lambda}(S^{n-1}) = \lambda(\mathbb{B}^n) \leq \frac{1}{\beta \varepsilon}.
\]
By definition of weak convergence for measures, we know that
\[
\int_{S^n} p \, d\nu = 0 \quad \text{for all } p \in \mathcal{P}_m.
\]
Let $\theta = (n-4)/(n-1)$ for simplicity. By definition of $\Theta(m, \theta, n-1)$ and Lemma 2, we have
\[
\Theta(m, \theta, n-1) \leq \sum_j \nu_j^\theta \leq \alpha(n) \hat{\lambda}(S^{n-1}) \leq \frac{\alpha(n)}{\beta \varepsilon}.
\]
Hence,
\[
\beta \varepsilon \leq \frac{\alpha(n)}{\Theta(m, \theta, n-1)}.
\]
Obviously, this contradicts the choice of $\beta \varepsilon$. \qed
4.2 Almost sharpness

It is worth pointing out that \(\alpha(n)/\Theta(m, n-1, n-1)\) in (11) is also almost optimal. Such an example appears as the first one to the fourth order Sobolev trace inequality.

**Proof of Proposition 2.** For each \(m\), there exist some natural number \(N \geq N_m(S^{n-1})\) and \(\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i} \in M^n_m\) for \(1 \leq i \leq N\), such that

\[
\Theta(m, n-4, n-1) = \sum_{i=1}^{N} \nu_i^{n-4}.
\]

Let \(\bar{d}(x, y)\) denote the geodesic distance between \(x\) and \(y\) in \(S^{n-1}\), and \(A_\delta(x_i)\) be the conic annulus as in (24) for each \(x_i \in S^{n-1}\). Fix \(\delta > 0\) small enough such that \(A_{2\delta}(x_i) \cap A_{2\delta}(x_j) = \emptyset\) for \(1 \leq i < j \leq N\).

For each \(1 \leq i \leq N\), let \(\chi_i(x)\) be a smooth cut-off function such that \(\chi_i(x) = 1\) in \(A_{\delta}(x_i)\) and \(\chi_i(x) = 0\) outside \(A_{2\delta}(x_i)\). Moreover, \(\partial_r \chi_i \geq 0\) in \(B^n\) and \(\partial_r \chi_i = 0\) on \(S^{n-1}\). Indeed, one possible way to choose the cut-off function as \(\chi_i(x) = \eta_i(r)\bar{\chi}_i(x, \rho, \theta)\). Here we can take \(\bar{\chi}_i(x, \rho, \theta)\) to be a nonnegative smooth cut-off function supported on \(B_{2\delta}(x_i) \subset S^{n-1}\) such that \(\bar{\chi}_i = 1\) on \(B_{\delta}(x_i)\) and \(\eta_i\) to be a non-decreasing smooth cut-off function on \([1 - 2\delta, 1]\) such that \(0 \leq \eta_i \leq 1\) and \(\eta_i = 1\) on \([1 - \delta, 1]\), \(\eta_i = 0\) on \([0, 1 - 2\delta]\).

For each \(x_i\), we use the polar coordinates \(x = (r, \rho, \theta)\) to express the flat metric \(|dx|^2\) in \(B^n\) near \(x_i\) as

\[
|dx|^2 = dr^2 + r^2 (d\rho^2 + \sin^2 \rho g_{S^{n-2}})
\]

for \(x = r\xi \in B^n, \rho = d_{S^{n-1}}(\xi, x_i) = \xi x_i\) and \(\theta \in S^{n-2}\).

Under the above coordinates, we define

\[
\psi_{\varepsilon,i}(r, \rho) = \left[1 + \frac{n-4}{2} \left(\frac{2\varepsilon(1 - r)}{\varepsilon + 1 - r} + \rho^2\right)\right]^{\frac{n-4}{2}}
\]

for any \(0 < \varepsilon < \delta\) with some sufficiently small \(\delta\), and

\[
\phi_{\varepsilon,i}(x) = \chi_i(x) \psi(r) \psi_{\varepsilon,i}(r, \rho).
\]

Define

\[
v(x) = \sum_{i=1}^{N} \nu_i^{\frac{n-4}{2(n-1)}} \phi_{\varepsilon,i}(x),
\]

then

\[
\int_{S^{n-1}} v^{\frac{2(n-1)}{n-4}} d\mu_{S^{n-1}} = \sum_{i=1}^{N} \nu_i \int_{B_{2\delta}(x_i)} \phi_{\varepsilon,i}(x)^{\frac{2(n-1)}{n-4}} d\mu_{S^{n-1}}
\]

\[
= \left( \sum_{i=1}^{N} \nu_i \right) |S^{n-2}| \int_{0}^{\delta} (\varepsilon^2 + \rho^2)^{1-n} \sin^{n-2} \rho d\rho + O(\varepsilon^{1-n}) \int_{\frac{\delta}{\varepsilon}}^{2\delta} (1 + t^2)^{1-n} t^{n-2} dt
\]

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For any $p \in \mathcal{P}_m$, we have

$$
\int \frac{2(n-1)}{v^{-(n-4)}} p d\mu_{S^{n-1}}
\begin{align*}
= & \sum_{i=1}^{N} \nu_i \int_{B_{\delta}(x_i)} \frac{2(n-1)}{v^{-(n-4)}} (\xi)p(\xi) d\mu_{S^{n-1}} \\
= & \sum_{i=1}^{N} \int_{B_{\delta}(x_i)} \left[ \frac{2(n-1)}{v^{-(n-4)}} (\xi)\nu_i p(x_i) + \frac{2(n-1)}{v^{-(n-4)}} (\xi)O \left( \frac{1}{v^{2}} \right) \right] d\mu_{S^{n-1}} \\
= & O \left( \varepsilon^{3-n} \right),
\end{align*}
$$

where the last equality follows from

$$
\nu \in \mathcal{M}_m \left( S^{n-1} \right) \implies \sum_{i=1}^{N} \nu_i p(x_i) = 0.
$$

It is shown in [31, Theorem 2.1 in Chapter IV] that there exists a basis \( \{ P_1, \cdots, P_L \} \) of \( \mathcal{P}_m \), such that

$$
p_1 = P_1 |_{S^{n-1}}, \cdots, p_L = P_L |_{S^{n-1}}
$$

are spherical harmonics, where \( L = n + \sum_{i=2}^{m} (C_{n+i-1}^{n-1} - C_{n+i-3}^{n-1}) \). Then for each \( 1 \leq i \leq N \), we claim that there exist \( \psi_1, \cdots, \psi_L \in C_\infty \left( \mathbb{R}^n \setminus \bigcup_{i=1}^{N} A_{\delta_i} (x_i) \right) \) such that the determinant

$$
det \left[ \int_{S^{n-1}} \psi_j p_k d\mu_{S^{n-1}} \right]_{1 \leq j, k \leq L} \neq 0. \quad (35)
$$

To this end, we can choose a nonzero smooth function \( \eta \in C_\infty \left( \mathbb{R}^n \setminus \bigcup_{i=1}^{N} A_{\delta_i} (x_i) \right) \) such that \( \eta P_1, \cdots, \eta P_L \) are linearly independent. It follows that the Gram matrix

$$
\begin{bmatrix}
\int_{S^{n-1}} \eta^2 p_j p_k d\mu_{S^{n-1}} \\
\end{bmatrix} \neq 0
$$

is positive definite, then \( \psi_j = \eta^2 P_j \) satisfies (35).

The fact (35) enables us to find \( \beta_1, \cdots, \beta_L \in \mathbb{R} \) such that

$$
\int_{S^{n-1}} \left( \frac{2(n-1)}{v^{-(n-4)}} + \sum_{j=1}^{L} \beta_j \psi_j \right) p_k d\mu_{S^{n-1}} = 0 \quad \forall \ 1 \leq k \leq L. \quad (36)
$$

Moreover, it follows from (34) that for all \( 1 \leq j \leq L \), \( \beta_j = O \left( \varepsilon^{3-n} \right) \) as \( \varepsilon \to 0 \). In the following, we shall use \( \partial_r = \sum_{i=1}^{n} x_i \partial_{x_i} \). As a consequence we can find a constant \( c_1 > 0 \) such that

$$
\sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} \geq \varepsilon^{3-n} \quad (37)
$$
\[ n - 1 \left( \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} \right) + \frac{1}{2} \sum_{j=1}^{L} \beta_j \frac{\partial \psi_j}{\partial r} > 0. \] (38)

We need further corrections to satisfy higher order moments constraint and Neumann boundary condition. To this end, we define a test function in the form of

\[ u^{2(n-1)}_{n-4} = \varepsilon^{2(n-1)} + \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} + g(x)(1 - r^2) \]

\[ := u_1(x) + g(x)(1 - r^2). \] (39)

Here \( c_1 \in \mathbb{R}_+ \) and \( g(x) = g(r, \rho, \theta) \) is a smooth function in \( \mathbb{R}^n \), which are to be determined later. Our goal is to capture the optimal constant and have a good control of higher order terms at the same time.

**Step 1.** We need to find some good candidates of \( c_1 \) and \( g \) such that \( u \) satisfies the following conditions:

i) Neumann boundary condition:

\[ \frac{\partial u}{\partial r} = -\frac{n-4}{2} u \quad \text{on} \quad S^{n-1}. \]

ii) \( u > 0 \) in \( \mathbb{R}^n \).

iii) Vanishing higher order moments constraint:

\[ \int_{S^{n-1}} p_j u^{2(n-1)} \frac{1}{n-4} d\mu_{S^{n-1}} = 0, \quad 1 \leq j \leq L. \]

To that end, we shall handle them term by term. Keep in mind that Neumann boundary condition is an additional difficulty to Sobolev inequality on closed manifolds, as well as the second order example. As we shall see that the property (18) of \( \phi_\varepsilon \) plays an important role.

i) We may choose the restriction of \( g \) on \( S^{n-1} \) as

\[ g(1, \rho, \theta) = \frac{1}{2} \left( \frac{\partial u_1}{\partial r} + (n-1)u_1 \right) \]

\[ = \frac{n-1}{n-4} \frac{\partial v}{\partial r} + \sum_{j=1}^{L} \beta_j \frac{\partial \psi_j}{\partial r} + \frac{n-1}{2} \left( \frac{\partial v}{\partial r} + \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} \right) \]

\[ = \frac{n-1}{n-4} \frac{\partial v}{\partial r} + \frac{n-4}{2} v + \sum_{j=1}^{L} \beta_j \frac{\partial \psi_j}{\partial r} + \frac{n-1}{2} \left( \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} \right) \]

to satisfy the Neumann boundary condition

\[ \frac{\partial u}{\partial r} = -\frac{n-4}{2} u \quad \text{on} \quad S^{n-1}. \]
We claim that \( u > 0 \) on \( S^{n-1} \) if \( c_1 \) is sufficiently large.

Thanks to the properties of the “local bubble”, we divide our discussion into three distinct domains of \( S^{n-1} \) to show

1. On \( S^{n-1} \setminus \bigcup_{i=1}^{N} B_{2\delta}(x_i) \),
   \[
   \frac{\partial v}{\partial r} + \frac{n-4}{2} v = 0,
   \]
   then
   \[
   g(1, \rho, \theta) = \frac{n-1}{2} \left( \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} \right) + \frac{1}{2} \sum_{j=1}^{L} \beta_j \frac{\partial \psi_j}{\partial r} > 0
   \]
   by virtue of (38).
2. On \( \bigcup_{i=1}^{N} B_{2\delta}(x_i) \),
   \[
   \frac{\partial v}{\partial r} + \frac{n-4}{2} v = \sum_{i=1}^{N} \frac{\partial \chi_i}{\partial r} \psi_{\varepsilon,i},
   \]
   then we have
   \[
   g(1, \rho, \theta) = \frac{n-1}{2} \left( \sum_{i=1}^{N} \frac{\partial \chi_i}{\partial r} \psi_{\varepsilon,i} \right) + \frac{n-1}{2} c_1 \varepsilon^{3-n}
   \]
   since \( \frac{\partial}{\partial r} \chi_i = 0 \) on \( S^{n-1} \).
3. On \( \bigcup_{i=1}^{N} B_{\delta}(x_i) \),
   \[
   \frac{\partial v}{\partial r} + \frac{n-4}{2} v = 0,
   \]
   then we have
   \[
   g(1, \rho, \theta) = \frac{n-1}{2} c_1 \varepsilon^{3-n} > 0.
   \]

In summary, we obtain \( u > 0 \) on \( S^{n-1} \) and \( g(1, \rho, \theta) = O(\varepsilon^{3-n}) \). This finishes the proof of the claim.

Next we plan to extend \( g(1, \rho, \theta) \) to \( g(x) \) in the unit ball \( \mathbb{B}^n \).

To simplify calculations, we may choose \( g(x) \) in the form of
\[
   g(x) = g(r, \rho, \theta) = \eta_2(r) g(1, \rho, \theta) \geq 0 \quad \text{in} \quad \mathbb{B}^n,
\]
where \( \eta_2(r) \in C^\infty([0, 1]) \) such that \( \eta_2(r) = 1 \) for \( 1 - 2\delta \leq r \leq 1 \), \( \eta_2(r) = 0 \) for \( 0 \leq r \leq 1 - 4\delta \).

For future reference, we define
\[
   U := u \frac{2(n-1)}{n-4} = v \frac{2(n-1)}{n-4} + \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} + g(r, \rho, \theta) \left( 1 - r^2 \right)
\]
\[ \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} \geq \varepsilon^{3-n} > 0 \]

by virtue of (37). This directly implies that \( U > 0 \) in \( B_n \), so does \( u \).

iii) Clearly, it follows from (36) and (39) that
\[
\int_{S^{n-1}} p_j u^{2(n-1)} \mu_{S^{n-1}} = 0, \quad 1 \leq j \leq L.
\]

**Step 2.** Sharp constant and a good control of higher order terms.

We are now in a position to estimate the involved terms.

For the first main term, it follows from (34) and definition of \( u \) in (39) that
\[
\| u \|_{L^2(S^{n-1})}^2 \leq \varepsilon^{3-n},
\]
which directly yields
\[
u^2 \leq \left( v^{2(n-1)} + C \varepsilon^{3-n} \right)^2 \leq v^2 + C \varepsilon^{(3-n)(n-4)}.
\]

Hence, we obtain
\[
\int_{S^{n-1}} u^2 \, d\mu_{S^{n-1}} \leq \int_{S^{n-1}} \left( v^2 + C \varepsilon^{(3-n)(n-4)} \right) \, d\mu_{S^{n-1}}
\]
\[
\leq \sum_{i=1}^{N} \nu_i^{n-4} \int_{B_{2\delta}(x_i)} \phi_{\varepsilon,i}^2 \, d\mu_{S^{n-1}} + C \varepsilon \frac{(3-n)(n-4)}{n-1}
\]
\[
\leq \varepsilon^{(3-n)(n-4)} \frac{n-4}{n-1} + \sum_{i=1}^{N} \int_{0}^{2\delta} \left( \varepsilon^2 + \rho^2 \right)^{4-n} \rho^{n-2} \, d\rho
\]
\[
\leq \varepsilon^{(3-n)(n-4)} \frac{n-4}{n-1}.
\]

To estimate \( \| \nabla u \|_{L^2(S^{n-1})}^2 \), it follows from (39) that
\[
\partial_{\rho} u |_{S^{n-1}} = U \frac{n-4}{2(n-1)} v \frac{2(n-1)}{n-4} \partial_{\rho} v + \frac{n-4}{2(n-1)} U \frac{n-4}{2(n-1)} \sum_{j=1}^{L} \beta_j \partial_{\rho} \psi_j
\]
and
\[ \nabla u|_{\mathbb{S}^{n-1}} = U \left( \frac{n-4}{2(n-1)} \right) v \left( \frac{n-4}{2(n-1)} \right)^{-1} \nabla v + \frac{n-4}{2(n-1)} U \left( \frac{n-4}{2(n-1)} \right)^{-1} \sum_{j=1}^{L} \beta_j \nabla \psi_j. \]

Notice that
\[ |\nabla u|_{\mathbb{S}^{n-1}}^2 = |\partial u|^2 + \frac{1}{\sin^2 \rho} |\nabla u|_{\mathbb{S}^{n-2}}^2. \]

Since
\[ \int_{B_{2\delta}(x_i)} |\partial u|^2 d\mu_{\mathbb{S}^{n-1}} = \int_{B_{2\delta}(x_i)} \left( U \left( \frac{n-4}{2(n-1)} \right) v \left( \frac{n-4}{2(n-1)} \right)^{-1} \partial v \right)^2 d\mu_{\mathbb{S}^{n-1}} \]
\[ \lesssim \int_{B_{2\delta}(x_i)} |\partial \chi_i \psi_{\varepsilon,i} + \chi_i \partial u \psi_{\varepsilon,i}|^2 d\mu_{\mathbb{S}^{n-1}} \]
\[ \lesssim \int_{B_{2\delta}(x_i)} \left[ (\varepsilon^2 + \rho^2)^{4-n} + \rho^2 (\varepsilon^2 + \rho^2)^{2(4-n-1)} \right] \rho^{n-2} d\rho \]
\[ \lesssim \varepsilon^{2(2-n)+n+1} \int_0^{\frac{\pi}{\varepsilon}} (1 + t^2)^{2-n} t^n dt \]
\[ \lesssim \varepsilon^{5-n} \begin{cases} O(1), & n \geq 6 \\ O(\log \frac{1}{\varepsilon}), & n = 5 \end{cases} \lesssim \varepsilon^{5-n} \log \frac{1}{\varepsilon}, \]

and
\[ \int_{\mathbb{S}^{n-1}} \frac{1}{\sin^2 \rho} |\nabla u|_{\mathbb{S}^{n-2}}^2 d\mu_{\mathbb{S}^{n-1}} \]
\[ \lesssim \sum_{i=1}^{n+1} \int_{B_{2\delta}(x_i)} \frac{|\nabla \chi_i \psi_{\varepsilon,i}|^2}{\sin^2 \rho} d\mu_{\mathbb{S}^{n-1}} + \int_{\mathbb{S}^{n-1}} \frac{1}{\sin^2 \rho} U \left( \frac{n-4}{2(n-1)} \right)^{-2} \sum_{j=1}^{L} |\beta_j|^2 |\nabla \psi_j|_{\mathbb{S}^{n-2}}^2 d\mu_{\mathbb{S}^{n-1}} \]
\[ \lesssim \varepsilon^{(3-n)(n-4)} + \varepsilon^{5-n} \log \frac{1}{\varepsilon}, \]

we obtain
\[ \int_{\mathbb{S}^{n-1}} |\nabla u|_{\mathbb{S}^{n-1}}^2 d\mu_{\mathbb{S}^{n-1}} = O(\varepsilon^{(3-n)(n-4)}) + O(\varepsilon^{5-n} \log \frac{1}{\varepsilon}) = o(\varepsilon^{4-n}). \]  \( (42) \)

It is left to calculate \( \| \Delta u \|^2_{L^2(\mathbb{B}^n)} \) containing the second main term.

Notice that
\[ \Delta u = \text{div} \left( \frac{n-4}{2(n-1)} U \pi_{\mathbb{n}^4}^{n-4} \nabla U \right) \]
\[ = \frac{n-4}{2(n-1)} U \pi_{\mathbb{n}^4}^{n-4} \Delta U + \frac{n-4}{2(n-1)} \left( \frac{n-4}{2(n-1)} - 1 \right) U \pi_{\mathbb{n}^4}^{n-4} |\nabla U|^2. \]

We divide \( \mathbb{B}^n \) into three distinct domains accordingly.
On $\mathbb{R}^n \setminus A_{2\delta}(x_i)$,

$$U = \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} + \eta(r) \left[ \frac{n-1}{2} \left( \sum_{j=1}^{L} \beta_j \psi_j + c_1 \varepsilon^{3-n} \right) + \frac{1}{2} \sum_{j=1}^{L} \beta_j \frac{\partial \psi_j}{\partial r} \right] (1 - r^2).$$

A direct computation shows $|\nabla U| \lesssim \varepsilon^{3-n}$ and $|\Delta U| \lesssim \varepsilon^{3-n}$. Thus, we obtain

$$|\Delta u| \lesssim \varepsilon^{(3-n)\frac{(n-4)}{2(n-1)}}$$

and

$$\int_{\mathbb{R}^n \setminus A_{2\delta}(x_i)} |\Delta u|^2 dx \lesssim \varepsilon^{(3-n)\frac{(n-4)}{2(n-1)}}.$$

On $A_{2\delta} \setminus A_{\delta}(x_i)$, according to the selection of cut-off functions $\chi_i$ we have

$$U = \nu_i \phi_{\varepsilon,i}^{\frac{2(n-1)}{(n-4)}} + c_1 \varepsilon^{3-n} + \frac{n-1}{2} c_1 \varepsilon^{3-n} (1 - r^2).$$

A direct computation yields

$$\partial_r \psi_{\varepsilon,i} = \frac{(n-4)(1-r)}{((\varepsilon+1-r)^2 + \rho^2)^{\frac{n-1}{2}}} \left[ (\varepsilon + 1 - r)^2 + \rho^2 + (n-2)\varepsilon \varepsilon(1-r) \right]$$

and

$$\partial_\rho \psi_{\varepsilon,i} = -\frac{(n-4)\rho}{((\varepsilon+1-r)^2 + \rho^2)^{\frac{n-1}{2}}} \left[ (\varepsilon + 1 - r)^2 + \rho^2 + (n-2)\varepsilon (1-r) \right],$$

$$\partial_\rho^2 \psi_{\varepsilon,i} = -\frac{n-4}{((\varepsilon+1-r)^2 + \rho^2)^{\frac{n-1}{2}+1}} \left[ (n-1)(\varepsilon + 1 - r)^2 + \rho^2)^2 - n(n-2)\rho^2 \varepsilon (1-r) \right. \left. -2(n-2)\rho^2 ((\varepsilon + 1 - r)^2 + \rho^2) \right].$$

Then

$$|\nabla \psi_{\varepsilon,i}|^2 = (\partial_r \psi_{\varepsilon,i})^2 + r^{-2} (\partial_\rho \psi_{\varepsilon,i})^2$$

$$= \left\{ (1-r)^2 \left[ (n-1)\varepsilon^2 + \rho^2 + (1-r)^2 + n\varepsilon (1-r) \right]^2 
+ \frac{\rho^2}{r^2} \left[ \varepsilon^2 + (1-r)^2 + n\varepsilon (1-r) + \rho^2 \right]^2 \right\} \frac{(n-4)^2}{(\varepsilon + 1 - r)^2 + \rho^2)^{2n}}$$

$$= O(1),$$

and

$$\Delta \psi_{\varepsilon,i} = r^{1-n} \partial_r \left( r^{n-1} \partial_r \psi_{\varepsilon,i} \right) + r^{-2} \sin^{2-n} \rho \partial_\rho (\sin^n \rho \partial_\rho \psi_{\varepsilon,i})$$

$$= \partial_r^2 \psi_{\varepsilon,i} + \frac{n-1}{r} \partial_r \psi_{\varepsilon,i} + r^{-2} (\partial_\rho^2 \psi_{\varepsilon,i} + (n-2) \cot \rho \partial_\rho \psi_{\varepsilon,i}).$$
Thus, we have
\[ |\nabla U| \lesssim \varepsilon^{3-n} \quad \text{and} \quad |\Delta U| \lesssim \varepsilon^{3-n}. \]
Recall that \( \phi_{\varepsilon,i} = \chi_i \psi_{\varepsilon,i} \). Hence, putting these facts together we obtain
\[
\int_{\mathcal{A}_\delta(x_i)} |\Delta u|^2 \, dx \lesssim \varepsilon^{(3-n)(n-4)/(n-1)}. 
\]

- On \( \mathcal{A}_\delta(x_i) \),
\[
U = \nu_i (\psi_{\varepsilon,i})^{2(n-1)/n-4} + c_1 \varepsilon^{3-n} + \frac{n-1}{2} c_1 \varepsilon^{3-n} (1 - r^2). 
\]

Note that
\[
\nabla (\psi_{\varepsilon,i})^{2(n-1)/n-4} = \frac{2(n-1)}{n-4} (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 1 \nabla (\psi_{\varepsilon,i}). 
\]

All terms \(|\nabla \psi_{\varepsilon,i}|^2 \) involved in the expression of \( \Delta u \) are
\[
\left( \frac{2(n-1)}{n-4} - 1 \right) U^{2(n-1)/n-4} (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 2 |\nabla \psi_{\varepsilon,i}|^2 \psi^2 \left( U - \nu_i (\psi_{\varepsilon,i})^{2(n-1)/n-4} \right) 
= \left( \frac{2(n-1)}{n-4} - 1 \right) U^{2(n-1)/n-4} (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 2 |\nabla \psi_{\varepsilon,i}|^2 \psi^2 c_1 \varepsilon^{3-n} \left( 1 + \frac{n-1}{2} (1 - r^2) \right). 
\]

Thus, we obtain
\[
|\Delta u| \lesssim \frac{n-4}{2(n-1)} U^{2(n-1)/n-4} \left| \Delta \left( \frac{n-1}{2} c_1 \varepsilon^{3-n} (1 - r^2) \right) \right| 
+ \frac{n-4}{2(n-1)} \nu_i c_1 \varepsilon^{3-n} \left( 1 - \frac{n-4}{2(n-1)} \right) U^{2(n-1)/n-4} - 2 |\nabla (\psi_{\varepsilon,i})|^2 (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 1 \langle \nabla (\psi_{\varepsilon,i}), \nabla r \rangle 
+ \frac{n-4}{2(n-1)} \left( 1 - \frac{n-4}{2(n-1)} \right) \nu_i U^{2(n-1)/n-4} - 1 (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 2 |\nabla \psi_{\varepsilon,i}|^2 c_1 \varepsilon^{3-n} \left( 1 + \frac{n-1}{2} (1 - r^2) \right) 
+ \frac{n-4}{2(n-1)} \nu_i U^{2(n-1)/n-4} - 1 \psi_{\varepsilon,i} (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 1 \langle \nabla \psi_{\varepsilon,i}, \nabla \psi_{\varepsilon,i} \rangle 
+ \frac{n-4}{2(n-1)} \nu_i U^{2(n-1)/n-4} - 1 \psi_{\varepsilon,i} (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 2 |\nabla \psi_{\varepsilon,i}|^2 \psi^2 c_1 \varepsilon^{3-n} \left( 1 + \frac{n-1}{2} (1 - r^2) \right) 
+ \frac{n-4}{2(n-1)} \nu_i U^{2(n-1)/n-4} - 1 \psi_{\varepsilon,i} (\psi_{\varepsilon,i})^{2(n-1)/n-4} - 1 \langle \nabla (\psi_{\varepsilon,i}), \nabla (\psi_{\varepsilon,i}) \rangle.
\]

We first deal with two easier terms:
\[
\int_{\mathcal{A}_\delta(x_i)} \left[ U^{2(n-1)/n-4} - 1 \Delta \left( \frac{n-1}{2} c_1 \varepsilon^{3-n} (1 - r^2) \right) \right]^2 \, dx
\]
\[ \lesssim \varepsilon^{2(3-n)} \int_{A_\delta(x_i)} U \frac{n-4}{n-1} - 2 \, dx \]
\[ \lesssim \varepsilon^{2(3-n)} \varepsilon^{(3-n)(\frac{n-4}{n-1}) - 2} |A_\delta(x_i)| \]
\[ \lesssim \varepsilon^{(3-n) \frac{n-4}{n-1}} \]

and
\[ \lesssim \varepsilon^{4(3-n)} \int_{A_\delta(x_i)} U \frac{n-4}{(n-1)^2} - 4 \, dx \]
\[ \lesssim \varepsilon^{3-n} \varepsilon^{n-4} \varepsilon^{(n-4)(n-1)^2} \left| \mathbf{A}_\delta \right| \]

In the following, we shall use the change of variables: \( s = (1 - r)/\varepsilon, t = \rho/\varepsilon \) and \( \tau = t/(1 + s) \).

By (43) we have
\[ \lesssim \int_{A_\delta(x_i)} \left[ c_1 \varepsilon^{3-n} U \left( \frac{n-4}{n-1} \right) - 2 \left\{ \nabla \left( \frac{n-1}{2} c_1 \varepsilon^{3-n} \left( 1 - r^2 \right) \right) \right\}^2 \right] \, dx \]
\[ \lesssim \int_{A_\delta(x_i)} \left[ \left\{ \nabla (\psi_\varepsilon, i), \nabla r^2 \right\} \right]^2 \, dx \]
\[ \lesssim \int_{A_\delta(x_i)} \left| \partial_r \psi_\varepsilon, i \right|^2 \, dx \]
\[ \lesssim \int_{1-\delta}^1 \int_0^\delta \left[ (\varepsilon + 1 - r)^2 + \rho^2 \right]^{-n} (1 - r)^2 \]
\[ \cdot \left[ (\varepsilon + 1 - r)^2 + \rho^2 + (n - 2)\varepsilon (\varepsilon + 1 - r) \right]^2 \rho^{n-2} d\rho dr \]
\[ \lesssim \varepsilon^{\delta - n} \int_0^{\delta/e} \int_0^{\delta/e} \left( (1 + s)^2 + t^2 \right)^{-n} s^2 \left[ (1 + s)^2 + t^2 + (n - 2)(1 + s) \right]^2 t^{n-2} d\rho dt \]
\[ \lesssim \varepsilon^{6-n}. \]

For a real number \( \alpha \leq 0 \), we have
\[ \int_{A_\delta(x_i)} \psi_\varepsilon, i \, dx \]
\[ \lesssim \int_{A_\delta(x_i)} \left( 1 + \frac{n-4}{2} \frac{2\varepsilon (1 - r)}{(\varepsilon + 1 - r)^2 + \rho^2} \right)^2 \left( (\varepsilon + 1 - r)^2 + \rho^2 \right)^{4-n} \, dx \]
\[ \lesssim \delta^{-2\alpha} \int_{1-\delta}^\delta \int_0^\delta \left( (\varepsilon + 1 - r)^2 + \rho^2 \right)^{4-n+\alpha} \rho^{-2} d\rho dr \]
\[ \lesssim \varepsilon^{8-n+2\alpha} \int_0^{\delta/e} \int_0^{\delta/e} \left( (1 + s)^2 + t^2 \right)^{4-n+\alpha} t^{n-2} d\rho dt \]
Similarly, for a real number \( \alpha \leq 0 \), by (46) we have

\[
\int_{A_\delta(x_i)} \left[ U^{\frac{n-4}{n-1}} \left( \psi \frac{\partial^2 \psi}{\partial x_i^2} \right)^2 + \frac{4(n-1)}{n-4} \frac{\nabla \psi}{\psi} \right]^2 \ dx \leq \int_{A_\delta(x_i)} \psi^2 \ dx = o(\varepsilon^{4-n}).
\]
Thus, we obtain \( \hat{y} \) and \( \hat{z} \) such that

\[
\hat{y} \approx \hat{z} \approx \hat{A} \varepsilon \hat{d} \delta \varepsilon i \alpha (2 + \alpha) n + 6 \int_{0}^{\delta/\varepsilon} (1 + s)^{2(n + \alpha(n - 1)) + n - 1} ds \int_{0}^{\delta/\varepsilon} (1 + \tau^{2})^{n + \alpha(n - 1)} \tau^{n - 2} d\tau.
\]

We emphasize that the condition \( n \geq 5 \) has been used to choose

\[-2 < \alpha < -\frac{3n}{2(n - 1)} \]

\[\implies \alpha(n + 1) + n + 6 > 4 - n \quad \text{and} \quad 3n + 2\alpha(n - 1) - 1 < -1\]

such that

\[\int_{0}^{\delta/\varepsilon} (1 + s)^{2(n + \alpha(n - 1)) + n - 1} ds < \infty\]

and

\[\int_{0}^{\delta/\varepsilon} (1 + \tau^{2})^{n + \alpha(n - 1)} \tau^{n - 2} d\tau < \infty.\]

Thus, we obtain

\[\varepsilon^{2(3-n)} \int_{A_{0}(x_{i})} U^{\frac{n-4}{2(n-1)-2}} (\psi_{i}, \frac{4(n-1)-2}{n-4}) |\nabla \psi_{i}|^{2} dx = o(\varepsilon^{4-n}).\]

By (43) we have

\[\int_{A_{0}(x_{i})} \left| \frac{U^{n-4}}{2(n-1)-2} (\psi_{i}, \frac{4(n-1)-2}{n-4}) (\nabla \psi_{i}, \nabla \psi_{i}) \right|^{2} dx \]

\[\lesssim \int_{A_{0}(x_{i})} U^{n-4} (\psi_{i}, \frac{4(n-1)-2}{n-4}) (r \partial_{r} \psi_{i})^{2} r^{n-1} \sin^{n-2} \rho^{2} d\rho dr \]

\[\lesssim \int_{A_{0}(x_{i})} U^{n-4} (\psi_{i}, \frac{4(n-1)-2}{n-4}) ((\varepsilon + 1 - r)^{2} + \rho^{2})^{-n} \]

\[\cdot (1 - r)^{2} ((\varepsilon + 1 - r)^{2} + \rho^{2} + (n - 2)\varepsilon(\varepsilon + 1 - r))^{2} \rho^{n-2} d\rho dr \]

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\begin{align*}
&\lesssim \left. \int_{1-\delta}^{1} \int_{0}^{\delta} ((\varepsilon + 1 - r)^2 + \rho^2)^{-n} (1 - r)^2 \\
&\quad \cdot \left[ ((\varepsilon + 1 - r)^2 + \rho^2 + (n - 2)\varepsilon(\varepsilon + 1 - r)]^2 \rho^{-n-2}d\rho dr \right.
&\lesssim \varepsilon^{6-n} \int_{0}^{\delta} \int_{0}^{\delta/\varepsilon} \left[ ((1 + s)^2 + t^2)^{-n-2} s^2 \left( (1 + s)^2 + t^2 + (n - 2)(1 + s)^2 \right)^{-n} \right] \rho^{-n-2}dsdt
&\lesssim \varepsilon^{5-n}.
\end{align*}

Similarly,
\begin{align*}
&\lesssim \int_{A_{\delta}(x_i)} \left[ \int_{A_{\delta}(x_i)} \left[ U\frac{n-4}{2(n-1)} \left( \psi_{\psi,\varepsilon,i} \cdot \frac{2(n-1)}{n-4} - 2 \psi_{\psi,\varepsilon,i} \right) \right] ^2 \right. \\
&\quad \cdot (\varepsilon + 1 - r)^2 \left[ (\varepsilon + 1 - r)^2 + \rho^2 + (n - 2)\varepsilon(\varepsilon + 1 - r)]^2 \rho^{-n-2}d\rho dr \right.
&\lesssim \int_{1-\delta}^{1} \int_{0}^{\delta} \left[ (\varepsilon + 1 - r)^2 + \rho^2 \right]^{-n} (1 - r)^2 \\
&\quad \cdot \left[ (\varepsilon + 1 - r)^2 + \rho^2 + (n - 2)\varepsilon(\varepsilon + 1 - r)]^2 \rho^{-n-2}d\rho dr
&\lesssim \varepsilon^{5-n}.
\end{align*}

The remaining term is exactly the second main term
\[ \nu_i^2 \int_{A_{\delta}(x_i)} \left[ U\frac{n-4}{2(n-1)} \left( \psi_{\psi,\varepsilon,i} \cdot \frac{2(n-1)}{n-4} - 1 \psi_{\Delta \psi,\varepsilon,i} \right) \right] ^2 dx. \]

Keep in mind that
\[ U = \nu_i \left( \psi_{\psi,\varepsilon,i} \right)^{2/\left( n-4 \right)} + c_1 \varepsilon^{3-n} + \frac{n-1}{2} c_1 \varepsilon^{3-n} \cdot (1 - r^2). \]

Observe that
\begin{align*}
&\nu_i^2 \int_{A_{\delta}(x_i)} \left[ U\frac{n-4}{2(n-1)} \left( \psi_{\psi,\varepsilon,i} \cdot \frac{2(n-1)}{n-4} - 1 \psi_{\Delta \psi,\varepsilon,i} \right) \right] ^2 dx \\
&\lesssim \nu_i^2 \int_{A_{\delta}(x_i)} \left( \psi_{\psi,\varepsilon,i} \cdot \frac{2(n-1)}{n-4} - 1 \psi_{\Delta \psi,\varepsilon,i} + \frac{2(n-1)}{n-4} - 1 \psi_{\Delta \psi,\varepsilon,i} \right) ^2 dx \\
&\lesssim \nu_i^2 \int_{A_{\delta}(x_i)} \left( \psi_{\Delta \psi,\varepsilon,i} \right) ^2 dx \\
&= \nu_i^2 \int_{A_{\delta}(x_i)} \left( \Delta \psi_{\varepsilon,i} \right) ^2 dx + O(1) \int_{A_{\delta}(x_i)} (1 - r^2)(\Delta \psi_{\varepsilon,i}) ^2 dx.
\end{align*}

Notice that
\[ \psi_{\varepsilon,i}(r, \rho) = \varepsilon^{4-n} \left[ 1 + \frac{n-4}{2} \frac{2(1-r)^3}{(1 + \frac{1-r}{\varepsilon})^2 + (\frac{\rho}{\varepsilon})^2} \right] \left( \frac{1 + \frac{1-r}{\varepsilon}}{\varepsilon} \right)^2 \left( \frac{\rho}{\varepsilon} \right)^2. \]
Let us deal with

\[ \psi \left( \frac{1 - r}{\varepsilon}, \frac{\rho}{\varepsilon} \right) \]

by recalling that \( s = (1 - r)/\varepsilon, t = \rho/\varepsilon \) and

\[ \hat{\psi}(s, t) = \left[ 1 + \frac{n - 4}{2} \frac{2s}{(1 + s)^2 + t^2} \right] \left( 1 + s + t^2 \right)^{\frac{n-6}{2}}. \]

Let us deal with

\[ \Delta \psi_{\varepsilon, i} = r^{1-n} \partial_r \left( r^{n-1} \partial_r \psi_{\varepsilon, i} \right) + r^{-2} \sin^{2-n} \rho \partial_\rho (\sin^{n-2} \rho \partial_\rho \psi_{\varepsilon, i}) \]

\[ = \partial_r^2 \psi_{\varepsilon, i} + \frac{n - 1}{r} \partial_r \psi_{\varepsilon, i} + r^{-2} \left( \partial_\rho^2 \psi_{\varepsilon, i} + (n - 2) \cot \rho \partial_\rho \psi_{\varepsilon, i} \right) \]

\[ - \partial_r \psi_{\varepsilon, i} + \partial_\rho^2 \psi_{\varepsilon, i} + \frac{n - 2}{\rho} \partial_\rho \psi_{\varepsilon, i} + \frac{n - 1}{r} \partial_r \psi_{\varepsilon, i} + (r^{-2} - 1) \partial_\rho^2 \psi_{\varepsilon, i} \]

\[ + (n - 2)(r^{-2} \cot \rho - \rho^{-1}) \partial_\rho \psi_{\varepsilon, i} \]

\[ = e^{2-n} \left( \partial_r^2 \hat{\psi} + \partial_t^2 \hat{\psi} + \frac{n - 2}{t} \partial_t \hat{\psi} \right) \]

\[ + \frac{n - 1}{r} \partial_r \psi_{\varepsilon, i} + (r^{-2} - 1) \partial_\rho^2 \psi_{\varepsilon, i} + (n - 2)(r^{-2} \cot \rho - \rho^{-1}) \partial_\rho \psi_{\varepsilon, i} \]

Here we regard \( \Delta \hat{\psi} \) as the Laplacian of a function \( \hat{\psi}(s, t) = \hat{\psi}(z) \) with \( s = z_n, t = |z'| \) for \( z = (z', z_n) \in \mathbb{R}^n_+ \).

By (43) we have

\[ \int_{A_s(x_1)} \left( \frac{n - 1}{r} \partial_r \psi_{\varepsilon, i} \right)^2 \, dx \]

\[ \leq \int_{1-\delta}^1 \int_0^{\delta \left( \varepsilon + 1 - \varepsilon^2 - \rho^2 \right)^2} \left[ (\varepsilon + 1 - r)^2 + \rho^2 + (n - 2)\varepsilon(\varepsilon + 1 - r) \right]^2 \, dr \, d\rho \]

\[ \leq e^{6-n} \int_0^{\delta \varepsilon} \int_0^{(1 + s)^2 + t^2} \left[ (1 + s)^2 + t^2 + (n - 2)(1 + s) \right]^2 \, ds \, dt \]

\[ \leq e^{6-n} \left[ \int_0^{\delta \varepsilon} \frac{(1 + s)^3}{s^2} \, ds \int_0^{\delta \varepsilon} (1 + \tau^2)^{2-n} \tau^{-2} \, d\tau \right. \]

\[ + (n - 2) \int_0^{\delta \varepsilon} \frac{(1 + s)^2}{s^2} \, ds \int_0^{\delta \varepsilon} (1 + \tau^2)^{1-n} \tau^{-2} \, d\tau \]

\[ \left. + (n - 2) \int_0^{\delta \varepsilon} \frac{(1 + s)^1}{s^2} \, ds \int_0^{\delta \varepsilon} (1 + \tau^2)^{-n} \tau^{-2} \, d\tau \right] \]

\[ \leq e^{5-n}. \]

By (45) and (44) we obtain

\[ \int_{A_s(x_1)} (r^{-2} - 1)^2 \left( \partial_\rho^2 \psi_{\varepsilon, i} \right)^2 \, dx \]
\[ \lambda \int_{1-\delta}^{1} \int_{0}^{\delta} \frac{(1-r)^2}{((\varepsilon + 1-r)^2 + \rho^2)n-2} r^{n-1} \rho^{n-2} dr d\rho \]

\[ \lesssim \varepsilon^{6-n} \int_{0}^{\delta/\varepsilon} (1+s)^{5-n} ds \int_{0}^{\delta/\varepsilon} (1+\tau^2)^{2-n} \tau^{n-2} d\tau \]

and

\[ \int_{\mathcal{A}_\delta(x_i)} (n-2)^2 (r^{-2} \cot \rho - \rho^{-1})^2 (\partial_\rho \psi_{\varepsilon,i})^2 dx \]

\[ \lesssim \int_{\mathcal{A}_\delta(x_i)} \frac{(1-r)^2}{((\varepsilon + 1-r)^2 + \rho^2)n-2} r^{n-1} \rho^{n-2} dr d\rho \]

\[ \lesssim \varepsilon^{5-n} \]

A direct computation yields

\[ \Delta \hat{\psi}(s,t) = -2(n-4) ((s+1)^2 + t^2 + (n-2)(1+s)) ((1+s)^2 + t^2)^{-\frac{3}{2}}. \]

Then we have

\[ \int_{\mathcal{A}_\delta(x_i)} (\Delta \psi_{\varepsilon,i})^2 dx \]

\[ = |S^{n-2}| \int_{\mathcal{A}_\delta(x_i)} (\Delta \psi_{\varepsilon,i})^2 r^{n-1} (\sin \rho)^{n-2} dr d\rho \]

\[ = |S^{n-2}| \int_{\mathcal{A}_\delta(x_i)} (\Delta \psi_{\varepsilon,i})^2 \rho^{-2} dr d\rho + |S^{n-2}| \int_{\mathcal{A}_\delta(x_i)} (\Delta \psi_{\varepsilon,i})^2 [r^{n-1} (\sin \rho)^{n-2} - \rho^{-2}] dr d\rho \]

\[ = 4(n-4)^2 \varepsilon^{4-n} |S^{n-2}| \int_{0}^{\delta/\varepsilon} \int_{0}^{\delta/\varepsilon} ((s+1)^2 + t^2 + (n-2)(1+s))^2 \left( \frac{t^{n-2} ds dt}{((1+s)^2 + t^2)^n} \right) \]

\[ + O(\varepsilon^n) \]

Now we deal with the opposite direction of the above main term. Fix \( \delta > \varepsilon^{1/2} \), notice that on \( \mathcal{A}_{\varepsilon^{1/2}} \),

\[ \psi_{\varepsilon,i} \gtrsim \varepsilon^{\frac{4-n}{2}}, \]

whence

\[ U = (\psi \psi_{\varepsilon,i})^{\frac{2(n-1)}{n-4}} + c_1 \varepsilon^{3-n} + \frac{n-1}{2} c_1 \varepsilon^{3-n} \cdot (1-r^2) \]

\[ \lesssim (\psi \psi_{\varepsilon,i})^{\frac{2(n-1)}{n-4}} (1 + O(\varepsilon^{3-n+n-1})) \]

\[ = (\psi \psi_{\varepsilon,i})^{\frac{2(n-1)}{n-4}} (1 + O(\varepsilon^2)). \]

Then we obtain

\[ \int_{\mathcal{A}_\delta(x_i)} \left[ U^{\frac{n-4}{2(n-1)-1}} (\psi \psi_{\varepsilon,i})^{\frac{2(n-1)}{n-4}-1} \psi \Delta \psi_{\varepsilon,i} \right]^2 dx \]
This completes our construction.

which implies by virtue of

Finally, we combine (40), (41), (42) and (47), as well as other higher order terms, to show

Therefore, putting these facts together we conclude that

Based on the above estimates, it is not hard to see

Hence, with the above choice of \( \delta \), the main term is

A direct calculation shows

Therefore, putting these facts together we conclude that

by virtue of \( 2^{n-2} B \left( \frac{n-1}{2}, \frac{n-1}{2} \right) \mid S^{n-2} \mid = \mid S^{n-1} \mid \).

Finally, we combine (40), (41), (42) and (47), as well as other higher order terms, to show

which implies

This completes our construction.

\[ \square \]
5 Almost sharp Sobolev trace inequality of order four under constraints in dimension four

For the Sobolev trace inequality, our conic proof in dimension four can be compared to the one in the pioneering work of Chang-Hang [10]. For the four dimensional example, as mentioned before, we need to settle the existing obstructions and make an improvement of Chang-Hang type estimate. These enable us to complete the construction.

5.1 Fourth order Sobolev trace inequality

Due to the same reason as in Section 4.1, it is important to understand Ache-Chang’s sharp Sobolev trace inequality in Theorem 5 well first. Moreover, notice in the equality case in (6) with \( f = 0 \) that \( u(x) = (1 - |x|^2)/2 \) and thus the extremal metric \( e^{2u} |dx|^2 \) is the Fefferman-Graham metric defined in [17].

We employ the regularity theory for bi-Laplace boundary value problem in Gazzola-Grunau-Sweers [18] to prove a preliminary result.

Lemma 3. Given \( f_i \in C^\infty(S^3) \), let \( u_i \) be the biharmonic extension of \( f_i \) to the unit ball \( B^4 \) satisfying zero Neumann boundary condition. Assume that as \( i \to \infty \), \( u_i \to 0 \) in \( H^2(B^4) \) and \( f_i \to 0 \) in \( H^{3/2}(S^3) \). Denote by \( U_i \) the biharmonic extension of \( \varphi f_i \) to \( B^4 \), where \( \varphi \in C^\infty(\mathbb{B}^3) \) satisfies \( \partial \varphi / \partial r = 0 \) on \( S^3 \). Then there holds

\[
\lim_{i \to \infty} \| \varphi u_i - U_i \|_{H^2(B^4)} = 0.
\]

Proof. By definition of \( U_i \), we know that \( U_i \) satisfies

\[
\begin{align*}
\Delta^2 U_i &= 0 \quad \text{in} \quad \mathbb{B}^4, \\
U_i &= \varphi f_i \quad \text{on} \quad S^3, \\
\frac{\partial U_i}{\partial r} &= 0 \quad \text{on} \quad S^3.
\end{align*}
\]

By assumption, a direct computation yields that \( \varphi u_i \) satisfies

\[
\begin{align*}
\Delta^2 (\varphi u_i) &= \Delta^2 \varphi u_i + 4 \langle \nabla u_i, \nabla \Delta \varphi \rangle + 2 \Delta \varphi \Delta u_i + 4 \langle \nabla^2 \varphi, \nabla^2 u_i \rangle + 4 \langle \nabla \varphi, \nabla \Delta u_i \rangle \quad \text{in} \quad \mathbb{B}^4, \\
\varphi u_i &= \varphi f_i \quad \text{on} \quad S^3, \\
\frac{\partial}{\partial r} (\varphi u_i) &= 0 \quad \text{on} \quad S^3.
\end{align*}
\]

Thus, if we let \( v_i \in H^2(\mathbb{B}^n) \cap H^1_0(\mathbb{B}^n) \) be a weak solution\(^3\) of

\[
\begin{align*}
\Delta^2 v_i &= \Delta^2 \varphi u_i + 4 \langle \nabla u_i, \nabla \Delta \varphi \rangle \quad \text{in} \quad \mathbb{B}^4, \\
v_i &= 0 \quad \text{on} \quad S^3, \\
\frac{\partial v_i}{\partial r} &= 0 \quad \text{on} \quad S^3,
\end{align*}
\]

\(^3\)See [18, (2.42) on p.41 and Theorem 2.31 on p.52] for the precise definition of weak solution.
then \( \varphi u_i - U_i - v_i \) weakly satisfies
\[
\begin{aligned}
\Delta^2 (\varphi u_i - U_i - v_i) &= 2\Delta \varphi \Delta u_i + 4\langle \nabla^2 \varphi, \nabla^2 u_i \rangle + 4\langle \nabla \varphi, \nabla \Delta u_i \rangle \\
\varphi u_i - U_i - v_i &= 0 \\
\frac{\partial}{\partial r} (\varphi u_i - U_i - v_i) &= 0
\end{aligned}
\] in \( \mathbb{B}^4 \), on \( \mathbb{S}^3 \), on \( \mathbb{S}^3 \).

Hence, it follows from [18, Theorem 2.22] that
\[
\| \varphi u_i - U_i - v_i \|_{H^2(\mathbb{B}^4)} \leq C\| u_i \|_{H^1(\mathbb{B}^4)}.
\]

Moreover, by [18, Theorem 2.16] we have
\[
\| v_i \|_{H^2(\mathbb{B}^4)} \leq C\| u_i \|_{H^1(\mathbb{B}^4)}.
\]

Consequently, we conclude that
\[
\| \varphi u_i - U_i \|_{H^2(\mathbb{B}^4)} \leq \| v_i \|_{H^2(\mathbb{B}^4)} + \| \varphi u_i - U_i - v_i \|_{H^2(\mathbb{B}^4)} \leq C\| u_i \|_{H^1(\mathbb{B}^4)} \to 0 \quad \text{as } i \to \infty.
\]

This completes the proof. \(\square\)

**Lemma 4.** For \( f_i \in C^\infty(\mathbb{S}^3) \) with \( \overline{f_i} = 0 \), let \( u_i \) be the biharmonic extension of \( f_i \) to the unit ball \( \mathbb{B}^4 \) satisfying zero Neumann boundary condition. We also assume that as \( i \to \infty \), \( u_i \to u \) weakly in \( H^2(\mathbb{B}^4) \), \( u_i \to u \) a.e. in \( \mathbb{B}^4 \) and
\[
(\Delta u_i) dx \to (\Delta u) dx + \sigma \quad \text{as measures.} \quad (48)
\]

If \( K \subset \mathbb{S}^3 \) is a compact subset with \( \sigma(C(K)) < 1 \), then for any \( 1 < p < \frac{1}{\sigma(C(K))} \), \( e^{12\pi^2 f_i^2} \) is bounded in \( L^p(K, g_{\mathbb{S}^3}) \), i.e.,
\[
\sup_i \int_K e^{12\pi^2 f_i^2} d\mu_{\mathbb{S}^3} < \infty. \quad (49)
\]

**Proof.** Let \( v_i = u_i - u \) and \( g_i = f_i - f \), then as \( i \to \infty \), \( v_i \to 0 \) weakly in \( H^2(\mathbb{B}^4) \), \( v_i \to 0 \) in \( H^1(\mathbb{B}^4) \), \( g_i \to 0 \) in \( H^1(\mathbb{S}^3) \). Thus, for any \( \varphi \in C^\infty(\overline{\mathbb{B}^4}) \), as \( i \to \infty \) we obtain
\[
\| \Delta (\varphi v_i) \|_{L^2(\mathbb{B}^4)}^2 = \int_{\mathbb{B}^4} (\Delta^2 \varphi v_i + 2\langle \nabla \varphi, \nabla v_i \rangle + \varphi \Delta v_i)^2 dx
\]
\[
= \int_{\mathbb{B}^4} |\Delta \varphi|^2 v_i^2 dx + 4\int_{\mathbb{B}^4} (\nabla \varphi, \nabla v_i)^2 dx + \int_{\mathbb{B}^4} (\varphi \Delta v_i)^2 dx
\]
\[
+ \int_{\mathbb{B}^4} (4v_i \Delta \varphi \nabla \varphi \cdot \nabla v_i) - 2\varphi \Delta v_i \langle \nabla u, \nabla v_i \rangle + 2\varphi \Delta \varphi v_i \Delta v_i) dx
\]
\[
\to \int_{\mathbb{B}^4} \varphi^2 d\sigma.
\]

If \( 1 < p_1 < \frac{1}{\sigma(C(K))} \), then \( \sigma(C(K)) < \frac{1}{p_1} \). Hence, there exists \( \varphi \in C^\infty(\overline{\mathbb{B}^4}) \) such that \( \partial \varphi / \partial r = 0 \) on \( \mathbb{S}^3 \), \( \varphi |_{C(K)} = 1 \) and \( \int_{\mathbb{B}^4} \varphi^2 d\sigma < \frac{1}{p_1} \).
On the other hand, a direct consequence of Lemma 3 is

\[ P_3^{S^3} = -\Delta_{S^3} (-\Delta_{S^3} + 1)^{1/2}, \]

which coincides with the one in a Poincaré-Einstein manifold introduced by C. Graham and M. Zworski [19] via scattering theory, and enjoys the conformal covariance property that

\[ P_3^{S^3} e^{-2\phi} = e^{-2\phi} P_3^{S^3} \quad \text{for } \phi \in C^\infty(S^3). \]

Denote by \( U_i \) the biharmonic function in \( \mathbb{B}^4 \) of \( \varphi_i \) with zero Neumann boundary condition, then it follows from [1, (5.5)] that

\[ 2 \int_{S^3} \varphi_i P_3^{S^3}(\varphi_i) \, d\mu_{S^3} = \int_{\mathbb{B}^4} |\Delta U_i|^2 \, dx + 2 \int_{S^3} |\nabla g_i|_{S^3}^2 \, d\mu_{S^3}. \]

On the other hand, a direct consequence of Lemma 3 is

\[ \lim_{i \to \infty} \int_{\mathbb{B}^4} |\Delta (U_i - \varphi v_i)|^2 \, dx = 0. \]

Therefore, putting these facts together, we obtain that for \( i \) sufficiently large,

\[ 2 \int_{S^3} \varphi_i P_3^{S^3}(\varphi_i) \, d\mu_{S^3} < \frac{1}{p_1}. \]

We are now ready to estimate

\[
\int_K e^{12\pi^2 p_1 (g_i - \varphi g_i)^2} \, d\mu_{S^3} \leq \int_{S^3} e^{12\pi^2 p_1 (\varphi g_i - \varphi g_i)^2} \, d\mu_{S^3} \\
\leq \int_{S^3} e^{6\pi^2 \frac{(\varphi g_i - \varphi g_i)^2}{\mu_{S^3}}} \, d\mu_{S^3} \\
\leq C,
\]

where the second inequality follows from the Moser-Trudinger type inequality for \( P_3^{S^3} \); see Chang-Yang [12, Proposition 4.4].

With the above estimates at hand, we obtain that for any \( \varepsilon > 0 \),

\[
f_i^2 = (g_i - \varphi g_i + f + \varphi g_i)^2 \\
= (g_i - \varphi g_i)^2 + 2 (g_i - \varphi g_i) (f + \varphi g_i) + (f + \varphi g_i)^2 \\
\leq (1 + \varepsilon) (g_i - \varphi g_i)^2 + (1 + \varepsilon^{-1}) (f + \varphi g_i)^2 \\
\leq (1 + \varepsilon) (g_i - \varphi g_i)^2 + 2 (1 + \varepsilon^{-1}) f^2 + 2 (1 + \varepsilon^{-1}) \varphi g_i^2.
\]

Hence,

\[ e^{12\pi^2 f_i^2} \leq e^{12\pi^2 (1+\varepsilon) (g_i - \varphi g_i)^2} e^{24\pi^2 (1+\varepsilon^{-1}) f^2} e^{24\pi^2 (1+\varepsilon^{-1}) \varphi g_i^2}. \]

Given \( 1 \leq p < \frac{1}{\sigma(C(K))} \), we can choose some \( p_1 \in \left( p, \frac{1}{\sigma(C(K))} \right) \) and small enough \( \varepsilon > 0 \) such that \( \frac{p_1}{1+\varepsilon} > p \). Notice that \( e^{12\pi^2 (1+\varepsilon) (g_i - \varphi g_i)^2} \) is bounded in \( L^p \left( \frac{p_1}{1+\varepsilon} (K) \right) \), \( e^{24\pi^2 (1+\varepsilon^{-1}) f^2} \in L^q (K, g_{S^3}) \) for any \( q > 0 \) (e.g., see [10, Lemma 2.1]) and \( e^{24\pi^2 (1+\varepsilon^{-1}) \varphi g_i^2} \to 1 \) as \( i \to \infty \). Therefore, by Hölder’s inequality we conclude that \( e^{12\pi^2 f_i^2} \) is bounded in \( L^p (K, g_{S^3}) \). ∎
Corollary 1. With the same assumption as in Lemma 4, let
\[ \kappa = \max_{x \in S^3} \sigma (\{x\}) \leq 1, \]
then
\[ \begin{align*}
&i) \text{ if } \kappa < 1, \text{ then for any } 1 \leq p < \frac{1}{\kappa}, \quad e^{12\pi^2 f_i^2} \text{ is bounded in } L^p \left( S^3 \right). \text{ In particular, as } i \to \infty, \quad e^{12\pi^2 f_i^2} \to e^{12\pi^2 f^2} \text{ in } L^1 \left( S^3 \right); \\
&ii) \text{ if } \kappa = 1, \text{ then } \sigma = \delta_{x_0} \text{ for some } x_0 \in S^3, u = f = 0 \text{ and after passing to a subsequence, as } i \to \infty, \quad e^{12\pi^2 f_i^2} \to 1 + c_0 \delta_{x_0} \text{ as measures,}
\end{align*} \]
for some constant \( c_0 \geq 0 \).

Proof. Since the proof is similar in spirit to the one of [10, Corollary 2.1], we omit it here. \( \square \)

Proposition 5. Assume \( \alpha > 0, m_i \to 0, m_i \to \infty \). For \( f_i \in C^\infty \left( S^3 \right) \) with \( \bar{f}_i \equiv 0 \) and \( \int_{S^3} f_i P_{S^3} f_i d\mu_{S^3} = \left( \int_{B^4} (\Delta u_i)^2 dx + 2 \int_{S^3} |\nabla f_i|_{S^3}^2 d\mu_{S^3} \right) / 2 = 1 \), where \( u_i \) be the biharmonic extension of \( f_i \) to the unit ball \( B^4 \) satisfying zero Neumann boundary condition, and
\[ \log \int_{S^3} e^{3m_i f_i} d\mu_{S^3} \geq \alpha m_i^2. \]
We also assume as \( i \to \infty, u_i \rightharpoonup u \) weakly in \( H^2 \left( B^4 \right) \), \( \int_{S^3} f_i P_{S^3} f_i d\mu_{S^3} \to \int_{S^3} f P_{S^3} f d\mu_{S^3} + \sigma \) as measures and
\[ \int_{S^3} e^{3m_i f_i} d\mu_{S^3} \rightharpoonup \nu \text{ as measures.} \]
Let
\[ \left\{ x \in S^3; \sigma (\{x\}) \geq \frac{16}{3} \pi^2 \alpha \right\} = \{x_1, \ldots, x_N\}, \]
then
\[ \nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}, \]
where \( \nu_i \geq 0 \text{ and } \sum_{i=1}^{N} \nu_i = 1. \)

Proof. First we claim that if \( K \) is a compact subset of \( S^3 \) with \( \sigma (\mathcal{C}(K)) < \frac{16}{3} \pi^2 \alpha \), then \( \nu (K) = 0. \) To this end, we can find another compact set \( K_1 \) such that \( K \subset K_1 \), the interior of \( K_1 \), and \( \sigma (\mathcal{C}(K_1)) < \frac{16}{3} \pi^2 \alpha. \) Fix a number \( p \) such that
\[ \frac{3}{16\pi^2 \alpha} < p < \frac{1}{\sigma (\mathcal{C}(K_1))}, \]
then it follows from Lemma 4 that with a positive constant \( C \) independent of \( i \), there holds
\[ \int_{K_1} e^{12\pi^2 pu_i^2} d\mu_{S^3} \leq C. \]
Notice that
\[ 3m_iu_i \leq 12\pi^2 pu_i^2 + \frac{3m_i^2}{16\pi^2 p}, \]
this together with Lemma 4 yields
\[ \int_{K_1} e^{3m_iu_i}d\mu_{g_3} \leq Ce^{\frac{3m_i}{16\pi^2 p}}. \]
It follows that
\[ \frac{\int_{K_1} e^{4m_iu_i}d\mu_{g_3}}{\int_{\mathbb{S}^3} e^{3m_iu_i}d\mu_{g_3}} \leq Ce^{\left(\frac{3}{16\pi^2 p} - \alpha\right)m_i^2}. \]
Hence
\[ \nu(K) \leq \nu(K_1) \leq \liminf_{i \to \infty} \frac{\int_{K_1} e^{4m_iu_i}d\mu_{g_3}}{\int_M e^{4m_iu_i}d\mu_{g_3}} = 0, \]
whence \( \nu(K) = 0. \)

If \( \sigma(\{x\}) < \frac{16}{3} \pi^2 \alpha, \) then we choose \( r_x > 0 \) small enough such that \( \sigma(C(B_{r_x}(x))) < \frac{16}{3} \pi^2 \alpha. \) It follows from the above claim that \( \nu(B_{r_x}(x)) = 0. \) Hence,
\[ \nu(S^3 \setminus \{x_1, \cdots, x_N\}) = 0. \]

In other words, \( \nu = \sum_{i=1}^N \nu_i \delta_{x_i} \) with \( \nu_i \geq 0 \) and \( \sum_{i=1}^N \nu_i = 1. \)

\textbf{Proof of Theorem 8.} Let \( \alpha_\varepsilon = \frac{3}{16\pi^2 N_m(S^3)} + \varepsilon. \) By the proof of [1, Theorem A], we only need to prove the case when \( u \) is a biharmonic extension of \( f. \)

By contradiction, if (14) is not true, then there exist some \( \varepsilon > 0 \) and \( v_i \in H^2(B^4) \) to be the biharmonic extension of \( f_i \in C^\infty(S^3), \) where \( f_i = 0, \int_{S^3} P_{g_3} f_i d\mu_{g_3} = 0 \) for all \( i \in \mathbb{N}, \) such that
\[ \log \int_{S^3} e^{3f_i}d\mu_{g_3} - 2\alpha_\varepsilon \int_{S^3} f_i P_{g_3}^3 f_i d\mu_{g_3} \to \infty \quad \text{as } i \to \infty. \]
Then \( \log \int_{S^3} e^{3f_i}d\mu_{g_3} \to \infty. \) It follows from [4, Theorem 1] that
\[ \log \int_{S^3} e^{3f_i}d\mu_{g_3} \leq \frac{3}{8\pi^3} \int_{S^3} f_i P_{g_3}^3 f_i d\mu_{g_3}, \]
whence \( \int_{S^3} f_i P_{g_3}^3 f_i d\mu_{g_3} \to \infty. \) Let
\[ m_i = \left( \int_{S^3} f_i P_{g_3}^3 f_i d\mu_{g_3} \right)^{\frac{1}{2}}, \quad u_i = \frac{v_i}{m_i} \quad \text{and} \quad g_i = \frac{f_i}{m_i}, \]
then \( m_i \to \infty \) and \( u_i \) is the biharmonic extension of \( g_i \) such that \( \int_{S^3} g_i P_{g_3}^3 g_i d\mu_{g_3} = 1, \overline{g_i} = 0. \)

Up to a subsequence, as \( i \to \infty \) we have
\[ u_i \to u \quad \text{weakly in } H^2(B^4), \]
\[ g_i \to g \quad \text{in } H^1(S^3), \]
51.
\[
\log \int_{S^3} e^{3m g_i} d\mu_{S^3} - \alpha_\varepsilon m_1^2 \to \infty,
\]
\[
\int_{S^3} (\Delta u_i)^2 \, dx \to \int_{S^3} (\Delta u)^2 + \sigma \text{ as measures,}
\]
\[
\int_{S^3} e^{3m g_i} \, d\mu_{S^3} \to \nu \text{ as measures.}
\]

Let
\[
\left\{ x \in S^3; \sigma (\{x\}) \geq \frac{16}{3} \pi^2 \alpha_\varepsilon \right\} = \{ x_1, \ldots, x_N \},
\]
then it follows from Proposition 5 that
\[
\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i},
\]
here \( \nu_i \geq 0 \) and \( \sum_{i=1}^{N} \nu_i = 1 \). On the other hand, we have
\[
\int_{S^4} p d\nu = 0
\]
for all \( p \in \hat{\mathcal{P}}_m \). In other words, we conclude that
\[
\frac{16}{3} \pi^2 \alpha_\varepsilon N \leq 1 \quad \text{and} \quad \sum_{i=1}^{N} \nu_i p(x_i) = 0 \quad \text{for all } p \in \hat{\mathcal{P}}_m.
\]
This indicates that \( N \in \mathcal{N}_m(S^3) \) and thus \( N \geq N_m \). Moreover,
\[
\alpha_\varepsilon \leq \frac{3}{16\pi^2 N} \leq \frac{3}{16\pi^2 N_m(S^3)}.
\]
However, this contradicts the choice of \( \alpha_\varepsilon \).

5.2 Almost sharpness

Provided that we adopt Chang-Hang type test function in [10] as the restriction of our test function to the boundary, we shall face a dilemma: How to extend it to the interior of the unit ball, since Chang-Hang’s test function is only piecewise Lipschitz? To demonstrate our idea, as a good warm-up we construct an example to show the sharpness of Widom inequality in Theorem 2. For the completeness, it is left to Appendix A. Thanks to this stimulating example together ‘nice’ properties of “geometric local bubble”, it extricates us from the above dilemma.
5.2.1 An improvement of the Chang-Hang type estimate

With the help of the lower bounds (13) of \( N_m(S^{n-1}) \), we give an elementary proof of the exact value of \( N_m(S^{n-1}) \), which is originally due to Mysovskih [28].

**Lemma 5.** For \( n \geq 2 \), there holds \( N_3(S^{n-1}) = 2n \).

**Proof.** It suffices to consider \( n \geq 3 \). On one hand, we have \( N_3(S^{n-1}) \geq 2n \) by virtue of (13). On the other hand, a basis of \( \hat{P}_3 \) can be chosen as

\[
\left\{ x_i, \ 1 \leq i \leq n; \ x_i^2 - \frac{|x|^2}{n}, \ 1 \leq i \neq j \leq n - 1; \ x_i x_j, \ 1 \leq i \neq j \leq n; \ x_i (|x|^2 - (n + 2)x_j^2), \ 1 \leq i \neq j \leq n; \ x_i x_j x_k, \ 1 \leq i \neq j \neq k \leq n \right\}.
\]

We can choose \( \nu_i = 1/(2n) \) for \( 1 \leq i \leq n \) and vertices as \( \{ \pm e_i; 1 \leq i \leq n \} \). Then it is straightforward to check that for any element \( p_j \) belonging to the above basis,

\[
\sum_{i=1}^{n} \nu_i (p_j(e_i) + p_j(-e_i)) = 0.
\]

Putting these facts together, the desired assertion follows. \( \square \)

A key observation on \( \hat{P}_1 \) is very important to our improved estimate.

**Proposition 6.** Let

\[
\phi_{\varepsilon,1}(x) = -\log(\varepsilon^2 + \text{dist}_{S^n}(x, N)^2) \quad \text{and} \quad \phi_{\varepsilon,2}(x) = -\log(\varepsilon^2 + \text{dist}_{S^n}(x, S)^2),
\]

where \( N \) and \( S \) are the north and south poles on \( S^n \), respectively, then for small \( \delta > 0 \),

\[
\int_{B_{\delta}(N)} e^{3\phi_{\varepsilon,1}(x)} x_i d\mu_{S^n} + \int_{B_{\delta}(S)} e^{3\phi_{\varepsilon,2}(x)} x_i d\mu_{S^n} = 0
\]

for all \( 1 \leq i \leq 4 \).

**Proof.** For brevity, we set \( \rho = \text{dist}_{S^n}(x, N) \) and \( x = (\sin \rho \xi, \cos \rho), \xi \in S^2 \). Then we distinguish our discussion into two cases:

- For \( 1 \leq i \leq 3 \), it follows from symmetry that

\[
\int_{B_{\delta}(N)} e^{3\phi_{\varepsilon,1}(x)} x_i d\mu_{S^n} + \int_{B_{\delta}(S)} e^{3\phi_{\varepsilon,2}(x)} x_i d\mu_{S^n}
\]

\[
= \left[ \int_0^{\delta} \frac{1}{(\varepsilon^2 + \rho^2)^3} \sin^3 \rho d\rho + \int_{\pi - \delta}^{\pi} \frac{1}{(\varepsilon^2 + (\pi - \rho)^2)^3} \sin^3 \rho d\rho \right] \cdot \left( \int_{S^2} \xi_i d\mu_{S^2} \right)
\]

\[
= 0.
\]
• For \( i = 4 \),
\[
\int_{B_\delta(N)} e^{3\phi_{\varepsilon,1}(x)} x_i \, d\mu_{\mathbb{S}^3} + \int_{B_\delta(S)} e^{3\phi_{\varepsilon,2}(x)} x_i \, d\mu_{\mathbb{S}^3} = |S^2| \left[ \int_0^{\delta} \frac{1}{(\varepsilon^2 + \rho^2)^3} \cos \rho \sin^2 \rho \, d\rho + \int_{\pi-\delta}^{\pi} \frac{1}{(\varepsilon^2 + (\pi - \rho)^2)^3} \cos \rho \sin^2 \rho \, d\rho \right]
= 0.
\]

Putting these facts together, the desired assertion follows.

Though \( n = 4 \) is enough to our later use, Proposition 6 above motives us to prove a generic result.

**Proposition 7.** For every \( p \in \tilde{\mathcal{P}}_m \) with \( m \geq 1 \), there exist \( \nu_1, \ldots, \nu_{n+1} \in (0, 1) \) and pairwise distinct points \( \{\bar{x}_k; 1 \leq k \leq n+1\} \subset S^{n-1} \) with \( n \geq 3 \) such that
\[
\sum_{k=1}^{n+1} \nu_k = 1 \quad \text{and} \quad \sum_{k=1}^{n+1} \nu_k p(\bar{x}_k) = 0.
\]

Choose \( \delta > 0 \) sufficiently small such that geodesic balls \( \{B_\delta(\bar{x}_k); 1 \leq k \leq n+1\} \subset S^{n-1} \) satisfy
\[
B_\delta(\bar{x}_i) \cap B_\delta(\bar{x}_j) = \emptyset, \quad \forall \ 1 \leq i \neq j \leq n + 1.
\]

Then
\[
\sum_{k=1}^{n+1} \nu_k \int_{B_\delta(\bar{x}_k)} e^{(n-1)\phi_{\varepsilon,k}(x)} p(x) \, d\mu_{S^{n-1}} = \sum_{k=1}^{n+1} \nu_k \int_{B_\delta(\bar{x}_k)} e^{(n-1)\phi_{\varepsilon,k} O(\rho^3)} \, d\mu_{S^{n-1}},
\]
where
\[
\phi_{\varepsilon,k}(x) = -\log \left( \varepsilon^2 + \text{dist}_{S^{n-1}}(x, \bar{x}_k)^2 \right), \quad 1 \leq k \leq n+1.
\]

**Proof.** Since
\[
\tilde{\mathcal{P}}_m = \bigoplus_{l=1}^{m} \mathcal{H}_l \quad \text{on } S^{n-1}
\]
by virtue of [31, Theorem 2.1], where \( \mathcal{H}_l \) is the set of all homogeneous harmonic polynomials of degree \( l \) in \( \mathbb{R}^n \), it suffices to consider each \( \mathcal{H}_l \) instead of \( \tilde{\mathcal{P}}_m \). Without loss of generality, we assume \( p \in \mathcal{H}_{l_0} \) for some \( 1 \leq l_0 \leq m \).

Fix each \( \bar{x}_k \), rotate \( \bar{x}_k \) properly to the north pole \( N \). In other words, we can find some \( Q \in SO(n) \) such that \( Q \bar{x}_k = N \). Under the change of variables: \( y = Qx \), we let \( \bar{p}(y) = p(Q^T y) \) and now choose local coordinates around \( N \) such that
\[
y = (\sin \rho \, \xi, \cos \rho) \in S^{n-1}, \quad \xi \in S^{n-2}.
\]

Notice that
\[
\frac{\partial \bar{p}}{\partial \rho} = \sum_{i=1}^{n-1} \frac{\partial \bar{p}}{\partial y_i} \frac{\partial y_i}{\partial \rho} + \frac{\partial \bar{p}}{\partial y_n} \frac{\partial y_n}{\partial \rho},
\]
\[
\frac{\partial^2 \tilde{p}}{\partial \rho^2} = \sum_{i,j=1}^{n-1} \frac{\partial^2 \tilde{p}}{\partial y_i \partial y_j} \frac{\partial y_i}{\partial \rho} \frac{\partial y_j}{\partial \rho} + \sum_{i=1}^{n-1} \frac{\partial \tilde{p}}{\partial y_i} \frac{\partial^2 y_i}{\partial \rho^2} + 2 \sum_{i=1}^{n-1} \frac{\partial \tilde{p}}{\partial y_i} \frac{\partial y_i}{\partial \rho} \frac{\partial y_n}{\partial \rho} + \frac{\partial \tilde{p}}{\partial y_n} \frac{\partial^2 y_n}{\partial \rho^2}.
\]

In particular, at \( \rho = 0 \) we have

\[
\frac{\partial \tilde{p}}{\partial \rho}(N) = \sum_{i=1}^{n-1} \frac{\partial \tilde{p}}{\partial y_i}(N) \xi_i,
\]

\[
\frac{\partial^2 \tilde{p}}{\partial \rho^2}(N) = \sum_{i,j=1}^{n-1} \frac{\partial^2 \tilde{p}}{\partial y_i \partial y_j}(N) \xi_i \xi_j - \left( y_n \frac{\partial \tilde{p}}{\partial y_n}(N) \right).
\]

For brevity, we define

\[
a_{ij} = \frac{1}{2} \frac{\partial^2 \tilde{p}}{\partial y_i \partial y_j}(N), \quad b_i = \frac{\partial \tilde{p}}{\partial y_i}(N), \quad c_n = \frac{1}{2} \left( y_n \frac{\partial \tilde{p}}{\partial y_n}(N) \right)
\]

for \( 1 \leq i, j \leq n - 1 \). Then the Taylor’s expansion of \( \tilde{p}(y) \) in a neighborhood around \( N \) is

\[
\tilde{p}(y) = \tilde{p}(N) + \left( \sum_{i=1}^{n-1} b_i \xi_i \right) \rho + \left( \sum_{i,j=1}^{n-1} a_{ij} \xi_i \xi_j - c_n \right) \rho^2 + O(\rho^3).
\]

Observe that

\[
\int_{B_{\delta}(\bar{x}_k)} e^{(n-1)\phi_{c,k} p(x)} d\mu_{S^{n-1}} = \int_{B_{\delta}(N)} e^{(n-1)\phi_{c,k}(Q^T y) \tilde{p}(y)} d\mu_{S^{n-1}}.
\]

Thus by symmetry we obtain

\[
\int_{B_{\delta}(N)} e^{(n-1)\phi_{c,k}(Q^T y) \tilde{p}(y)} d\mu_{S^{n-1}} + c_n \int_{B_{\delta}(N)} e^{(n-1)\phi_{c,k}(Q^T y) \rho^2} d\mu_{S^{n-1}}
\]

\[
= \tilde{p}(N) |S^{n-2}| \int_{0}^{\delta} \frac{1}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho
\]

\[
+ \sum_{i=1}^{n-1} b_i \int_{0}^{\delta} \frac{\rho}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho \int_{S^{n-2}} \xi_i d\mu_{S^{n-2}}
\]

\[
+ \sum_{i,j=1}^{n-1} \left( a_{ij} \int_{S^{n-2}} \xi_i \xi_j d\mu_{S^{n-2}} \right) \int_{0}^{\delta} \frac{\rho^2}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho
\]

\[
+ \int_{B_{\delta}(N)} e^{(n-1)\phi_{c,k}(Q^T y) O(\rho^3)} d\mu_{S^{n-1}}
\]

\[
= \tilde{p}(N) |S^{n-2}| \int_{0}^{\delta} \frac{1}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho
\]

\[
+ \frac{|S^{n-2}|}{n-1} \left( \sum_{i=1}^{n-1} a_{ii} \right) \int_{0}^{\delta} \frac{\rho^2}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho
\]

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+ \int_{B_\delta(N)} e^{(n-1)\phi_{\epsilon,k}(Q^T y)} O(\rho^3) d\mu_{S^{n-1}}.

Next, we notice that $p(x) \in H_{t_0}$ implies $\tilde{p}(y) \in H_{t_0}$. Observe that
\[
\sum_{i=1}^{n-1} a_{ii} = \sum_{i=1}^{n-1} \frac{\partial^2 \tilde{p}}{\partial y_i^2} = \Delta_{S^{n-1}} \tilde{p} = -l_0(n + l_0 - 2) \tilde{p} \quad \text{at } N.
\]
Moreover, we claim that
\[
2c_n = \left[ y_n \partial_{y_n} \tilde{p}(y) \right] \bigg|_{y=N} = l_0 p(\bar{x}_k).
\]
To that end, denote by $Q = (q_{ij})_{n \times n}$ and then $x_i = q_{ij}y_j$ by definition, hereafter we shall use Einstein summation notation. Notice that
\[
Q \bar{x}_k = N \iff \bar{x}_k = (q_{1n}, \ldots, q_{nn})^\top.
\]
Using
\[
\partial_{y_n} \tilde{p} = \frac{\partial x_l}{\partial y_n} \partial_{x_l} \tilde{p} = q_{ln} \partial_{x_l} p,
\]
we have
\[
\left. (y_n \partial_{y_n} \tilde{p}) \right|_{y=N} = \left. (q_{ln} x_l q_{sn} \partial_{x_s} p) \right|_{x=\bar{x}_k} = (q_{ln} q_{ln}) (x_s \partial_{x_s} p) \bigg|_{x=\bar{x}_k} = l_0 p(\bar{x}_k).
\]
Hence, the desired claim follows.

Going back to the original variable $x$, we combine all facts together to conclude that
\[
\int_{B_\delta(\bar{x}_k)} e^{(n-1)\phi_{\epsilon,k}(x)} p(x) d\mu_{S^{n-1}} = \mathcal{C}_{\epsilon,\delta,n} p(\bar{x}_k) + \int_{B_\delta(\bar{x}_k)} e^{(n-1)\phi_{\epsilon,k} O(\rho^3)} d\mu_{S^{n-1}},
\]
where
\[
\mathcal{C}_{\epsilon,\delta,n} = |S^{n-2}| \int_0^\delta \frac{1}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho - \frac{|S^{n-2}|}{n - 1} l_0(n + l_0 - 2) \int_0^\delta \frac{\rho^2}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho - \frac{l_0}{2} |S^{n-2}| \int_0^\delta \frac{\rho^2}{(\varepsilon^2 + \rho^2)^{n-1}} \sin^{n-2} \rho d\rho
\]
and $\mathcal{C}_{\epsilon,\delta,n}$ is a constant independent of $k$. Consequently, we arrive at
\[
\sum_{k=1}^{n+1} \nu_k \int_{B_\delta(\bar{x}_k)} e^{(n-1)\phi_{\epsilon,k}(x)} p(x) d\mu_{S^{n-1}}
\]
\[ = \epsilon^n \delta \sum_{k=1}^{n+1} \nu_k \rho_k(\bar{x}_k) + \sum_{k=1}^{n+1} \nu_k \int_{B_\delta(\bar{x}_k)} e^{(n-1)\phi_{r,k}} O(\rho^3) d\mu_{S^n-1} \]
\[ = \sum_{k=1}^{n+1} \nu_k \int_{B_\delta(\bar{x}_k)} e^{(n-1)\phi_{r,k}} O(\rho^3) d\mu_{S^n-1} . \]

This finishes the proof. \hfill \Box

5.2.2 Construction of test functions

With preparations above, we are now in a position to construct our example in dimension four.

Proof of Proposition 3. For each \( m \), we can find a \( \nu = \sum_{i=1}^{N_m(S^3)} \nu_i \delta_{x_i} \in M_{n_0}^\circ(S^3) \). Due to the precise estimate of \( m = 1 \) in Proposition 6, we shall point out modifications if necessary.

![Figure 2: A regular 4-simplex \( R \)](image)

To illustrate the number \( N_m(S^3) \), we take \( m = 2 \) for example. It follows from [21, Lemma 3.4] that \( N_2(S^3) = 5 \) can be attained by some \( \nu = \sum_{i=1}^5 \nu_i \delta_{x_i} \in M_2^\circ(S^3) \), where \( \nu_i = 1/5 \) and

![Figure 3: On \( \{ x_4 = -\frac{1}{4} \} \cap \mathcal{R} \): (a) Conic Annuli (b) A regular 3-simplex](image)
distinct points $x_i \in S^3$ for $1 \leq i \leq 5$ are vertices of a regular 4-simplex embedded in $\mathbb{B}^4$. One possible choice of vertices could be

$$
x_1 = (0, 0, 0, 1),
$$
$$
x_2 = (0, 0, \sqrt{15}/4, -1/4),
$$
$$
x_3 = (0, \sqrt{30}/6, -\sqrt{15}/12, -1/4),
$$
$$
x_4 = (\sqrt{10}/4, -\sqrt{30}/12, -\sqrt{15}/12, -1/4),
$$
$$
x_5 = (-\sqrt{10}/4, -\sqrt{30}/12, -\sqrt{15}/12, -1/4).
$$

In the following, we set $N_m = N_m(S^3)$ for brevity.

For each $1 \leq i \leq N_m$, choose conic annuli $A_\delta(x_i)$ and local coordinates around $x_i$ as before, and let $\chi_i(x)$ be a smooth cut-off function such that $\chi_i(x) = 1$ in $A_\delta(x_i)$ and $\chi_i(x) = 0$ outside $A_{2\delta}(x_i)$. Under the above coordinates, for any $0 < \varepsilon < \delta$ we define

$$
\phi_\varepsilon(r, \rho) = -\log \left( (\varepsilon + 1 - r)^2 + \rho^2 \right) + \frac{2\varepsilon(1-r)}{(\varepsilon + 1 - r)^2 + \rho^2}
$$

and

$$
\phi_{\varepsilon,i}(r, \rho) = \phi_\varepsilon(r, \rho) + \frac{1}{3} \log \nu_i,
$$

where $x = r\xi$ with $r = |x|$ and $\rho = \xi \cdot x_i$.

We estimate

$$
\sum_{i=1}^{N_m} \int_{S^3} \chi_i(x) e^{3\phi_{\varepsilon,i}(x)} d\mu_{S^3} = \sum_{i=1}^{N_m} \int_{B_{2\delta}(x_i)} \chi_i e^{3\phi_{\varepsilon,i}} d\mu_{S^3}
$$

$$
= \sum_{i=1}^{N_m} \int_{B_{2\delta}(x_i)} \chi_i(x) \nu_i \frac{1}{(\varepsilon^2 + \rho^2)^3} d\mu_{S^3}
$$

$$
= |S^2| \int_0^\delta \frac{1}{(\varepsilon^2 + \rho^2)^3} \sin^2 \rho d\rho + O(\varepsilon^{-3}) \int_\delta^2 \frac{2\varepsilon^2}{\rho} (1 + \varepsilon^2)^{-3} \rho^2 d\rho
$$

$$
= \frac{\pi^2}{4} \varepsilon^{-3} + O\left( \frac{1}{\varepsilon} \right).
$$

and then

$$
\int_{S^3} e^{3u} d\mu_{S^3} = \sum_{i=1}^{N_m} \int_{B_{2\delta}(x_i)} \chi_i e^{3\phi_{\varepsilon,i}} d\mu_{S^3} + \int_{S^3} \left( c_1 \log \frac{1}{\varepsilon} + \sum_{j=1}^{L} \beta_j \psi_j \right) d\mu_{S^3}
$$

$$
= \frac{\pi^2}{4} \varepsilon^{-3} + O(\log \frac{1}{\varepsilon}).
$$

This implies

$$
\log \left( \frac{1}{2\pi^2} \int_{S^3} e^{3u} d\mu_{S^3} \right) = 3 \log \frac{1}{\varepsilon} + O(\log \log \frac{1}{\varepsilon}).
$$

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For every \( p \in \mathcal{P}_m \), it follows from Proposition 7 that

\[
\int_{S^3} \left( \sum_{i=1}^{N_m} \chi_i(x)e^{3\phi_{e,i}(x)} \right) p d\mu_{S^3} = \sum_{i=1}^{N_m} \int_{B_\delta(x_i)} e^{3\phi_{e}} O(\rho^3) d\mu_{S^3} \\
= |S^2| \sum_{i=1}^{N_m} \int_0^\delta O(\rho^3) \frac{\sin^2 \rho d\rho}{(\varepsilon^2 + \rho^2)^3} + O(1) \\
= O(\log \frac{1}{\varepsilon}). \tag{52}
\]

Similar to the proof of Proposition 2, we can find a basis of spherical harmonics \( \{p_1, \cdots, p_L\} \) of \( \mathcal{P}_m \) with \( L = m(2m^2 + 9m + 13)/6 \), and functions \( \psi_1, \cdots, \psi_L \in C_c^\infty \left( \mathbb{R}^4 \setminus \bigcup_{i=1}^{N_m} A_{2\delta}(x_i) \right) \) such that the determinant

\[
\det \left[ \int_{S^3} \psi_j p_k d\mu_{S^3} \right]_{1 \leq j, k \leq L} \neq 0. \tag{53}
\]

To this end, we can choose a nonzero smooth function \( \eta \in C_c^\infty \left( \mathbb{R}^4 \setminus \bigcup_{i=1}^{N_m} A_{2\delta}(x_i) \right) \) such that \( \eta p_1, \cdots, \eta p_L \) are linearly independent. Actually, one possible way of \( \eta \) is to choose \( \eta(x) = \eta_1(r) \tilde{\chi}(\rho, \theta) \), where \( \tilde{\chi}(\rho, \theta) \) is a smooth cut-off function supported in \( S^3 \setminus \bigcup_{i=1}^{N_m} B_{2\delta}(x_i) \) and \( \eta_1 \) is the smooth cut-off function used in Section 4.2. Then it follows that the Gram matrix

\[
\left[ \int_{S^3} \eta_j^2 p_j p_k d\mu_{S^3} \right]_{1 \leq j, k \leq L}
\]

is positive definite, then \( \psi_j = \eta^2 p_j \) satisfies (53) and \( \partial_r \psi_j = 0 \) on \( S^3 \).

The fact (53) enables us to find \( \beta_1, \cdots, \beta_L \in \mathbb{R} \) such that

\[
\int_{S^3} \left( \sum_{i=1}^{N_m} \chi_i(x)e^{3\phi_{e,i}(x)} + \sum_{j=1}^{L} \beta_j \psi_j \right) p_k d\mu_{S^3} = 0 \quad \forall \ 1 \leq k \leq L. \tag{54}
\]

Moreover, it follows from (52) that for all \( 1 \leq j \leq L \), \( \beta_j = O \left( \log \frac{1}{\varepsilon} \right) \). As a consequence we can find a constant \( c_1 > 0 \) such that

\[
\sum_{j=1}^{L} \beta_j \psi_j + c_1 \log \frac{1}{\varepsilon} \geq \log \frac{1}{\varepsilon}.
\]

In dimension four, we may choose our test function as

\[
e^{3u} = \sum_{i=1}^{N_m} \chi_i e^{3\phi_{e,i}} + \sum_{j=1}^{L} \beta_j \psi_j + c_1 \log \frac{1}{\varepsilon}. \tag{55}\]

Here we would like to take \( \chi_i \) to be a variable separation cut-off function in order to ensure zero Neumann boundary condition without additional correction. To that end, as in Section 4.2 we may
choose \( \chi_i(x) = \eta_1(r)\tilde{\chi}_i(\rho, \theta) \) such that \( \partial_r \chi_i = 0 \) on \( S^3 \). Since \( \partial_r \phi_{\varepsilon,i} = 0 \) on \( S^3 \) by virtue of (20), it follows from (55) that

\[
3e^{3u} \partial_r u = \sum_{i=1}^{N_m} \partial_r \chi_i e^{3\phi_{\varepsilon,i}} + 3 \sum_{i=1}^{N_m} \chi_i e^{3\phi_{\varepsilon,i}} \partial_r \phi_{\varepsilon,i} + \sum_{j=1}^{L} \beta_j \partial_r \psi_j = 0 \quad \text{on} \quad S^3
\]

\[\implies \partial_r u = 0 \quad \text{on} \quad S^3.\]

By (54) and (55) we have

\[
\int_{S^3} e^{3u} p_k d\mu_{S^3} = 0 \quad \text{for} \quad 1 \leq k \leq L.
\]

On one hand, we have

\[
\hat{u} |_{S^3} = \int_{S^3} u d\mu_{S^3} = \int_{S^3} \frac{1}{3} \log(e^{3u}) d\mu_{S^3}
\]

\[= \frac{1}{3} \int_{S^3} \log \left( \sum_{i=1}^{N_m} \chi_i e^{3\phi_{\varepsilon,i}} + c_1 \log \frac{1}{\varepsilon} + \sum_{j=1}^{L} \beta_j \psi_j \right) d\mu_{S^3}
\]

\[\lesssim \frac{1}{3} \sum_{i=1}^{N_m} \int_{B_{\delta}(x_i)} \left[ \log \left( 2 e^{3\phi_{\varepsilon,i}} \right) + \log \left( 2c_1 \log \frac{1}{\varepsilon} \right) \right] d\mu_{S^3}
\]

\[+ \frac{1}{3} \sum_{i=1}^{N_m} \int_{S^3 \setminus B_{\delta}(x_i)} \log \left( O(\log \frac{1}{\varepsilon}) \right) d\mu_{S^3}
\]

\[\lesssim \frac{1}{3} \int_{S^3} \delta^3 \log \left( \frac{\rho^2}{\varepsilon} \right) d\mu_{S^3} + \int_{S^3 \setminus B_{\delta}(x_i)} \frac{2\rho_i \beta_{\varepsilon,i}}{\varepsilon^2(1 + (\frac{\rho_i}{\varepsilon})^2)} d\mu_{S^3} + O(\log \log \frac{1}{\varepsilon})
\]

\[\lesssim \frac{1}{3} \int_{S^3} \delta^3 \log \left( 1 + t^2 \right) dt + O(\log \log \frac{1}{\varepsilon})
\]

\[\lesssim \frac{1}{3} \int_{S^3} \left[ t^2 \log \frac{1}{\varepsilon^2} dt - \left( \frac{2 \delta}{3 \varepsilon} - \frac{2}{9} (\frac{\delta}{\varepsilon})^3 + \frac{1}{3} (\frac{\delta}{\varepsilon})^3 \log[1 + (\frac{\delta}{\varepsilon})^2] \right) \right] + O(\log \log \frac{1}{\varepsilon})
\]

\[\lesssim O(\log \log \frac{1}{\varepsilon}),
\]

where

\[
\int_{0}^{\delta/\varepsilon} t^2 \log \left( 1 + t^2 \right) dt
\]

\[= \frac{1}{9} \left[ 6t^2 - 2(\delta^2 + 6 \arctan t + 3t^3 \log(1 + t^2) \right] \bigg|_{t = \frac{\delta}{\varepsilon}}
\]

\[= \frac{2}{3} \varepsilon - \frac{2}{9} (\frac{\delta}{\varepsilon})^3 + \frac{1}{3} (\frac{\delta}{\varepsilon})^3 \log[1 + (\frac{\delta}{\varepsilon})^2] + O(1).
\]

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On the other hand,

$$\bar{u}|_{S^3} = \int_{S^3} \frac{1}{3} \log \left( \sum_{i=1}^{N_m} \chi_i e^{3\phi_{\varepsilon,i}} + c_1 \log \frac{1}{\varepsilon} + \sum_{j=1}^{L} \beta_j \psi_j \right) d\mu_{S^3}$$

$$\geq \int_{S^3} \frac{1}{3} \log \left( c_1 \log \frac{1}{\varepsilon} \right) d\mu_{S^3}$$

$$\geq O(\log \log \frac{1}{\varepsilon}).$$

Thus, we put these together to show

$$\bar{u} = O(\log \log \frac{1}{\varepsilon}). \quad (56)$$

Under local coordinates of the flat metric near each $x_i$

$$|dx|^2 = dr^2 + r^2(d\rho^2 + \sin^2 \rho g_{S^2}),$$

there holds

$$\Delta u = \partial_r^2 u + \frac{3}{r} \partial_r u + \frac{1}{r^2} \left( \partial_{\rho}^2 u + 2 \cot \rho \partial_{\rho} u + \sin^{-2} \rho \Delta_{S^2} u \right)$$

$$= \partial_r^2 u + \partial_{\rho}^2 u + \frac{2}{\rho} \partial_{\rho} u + \left( \frac{2 \cot \rho}{r^2} - \frac{2}{\rho} \right) \partial_{\rho} u + \left( \frac{1}{r^2} - 1 \right) \partial_{\rho}^2 u + \frac{1}{r^2} \frac{1}{\sin^2 \rho} \Delta_{S^2} u.$$

In $\mathbb{B}^4 \setminus A_\delta(x_i)$, by (55) we have

$$u = \frac{1}{3} \log \left( \chi_i e^{3\phi_{\varepsilon,i}} + c_1 \log \frac{1}{\varepsilon} \right) \quad \text{in} \ A_\delta \setminus A_\delta(x_i)$$

and

$$u = \frac{1}{3} \log \left( c_1 \log \frac{1}{\varepsilon} + \sum_{j=1}^{L} \beta_j \psi_j \right) \quad \text{in} \ \mathbb{B}^4 \setminus A_\delta(x_i).$$

Then it is not hard to verify that

$$|\nabla u|_{S^3} + |\Delta u| = O(1) \quad \mathbb{B}^4 \setminus A_\delta(x_i)$$

In each $A_\delta(x_i)$, again by (55) we have

$$u = \frac{1}{3} \log \left( e^{3\phi_{\varepsilon,i}} + c_1 \log \frac{1}{\varepsilon} \right).$$

Recall that $\phi_{\varepsilon,i} = \phi_{\varepsilon} + \frac{1}{3} \log \nu_i$. For convenience, we decompose $\phi_{\varepsilon} = \phi_1 + \phi_2$, where

$$\phi_1(r, \rho) = -\log \left( (\varepsilon + 1 - r)^2 + \rho^2 \right)$$

and

$$\phi_2(r, \rho) = \frac{2\varepsilon(1-r)}{(\varepsilon + 1 - r)^2 + \rho^2}. $$
A direct computation yields

\[ \partial_r \phi_1 = \frac{2(\varepsilon + 1 - r)}{(\varepsilon + 1 - r)^2 + \rho^2}, \]
\[ \partial_\rho \phi_1 = \frac{-2\rho}{(\varepsilon + 1 - r)^2 + \rho^2} \]

and

\[ \partial_t^2 \phi_1 = -\frac{2}{(\varepsilon + 1 - r)^2 + \rho^2} + \frac{4(\varepsilon + 1 - r)^2}{((\varepsilon + 1 - r)^2 + \rho^2)^2}, \]
\[ \partial_\rho^2 \phi_1 = -\frac{2}{(\varepsilon + 1 - r)^2 + \rho^2} + \frac{4\rho^2}{((\varepsilon + 1 - r)^2 + \rho^2)^2}. \]

Also,

\[ \partial_r \phi_2 = \frac{-2\varepsilon}{(\varepsilon + 1 - r)^2 + \rho^2} + \frac{4\varepsilon(1 - r)(\varepsilon + 1 - r)}{((\varepsilon + 1 - r)^2 + \rho^2)^2}, \]
\[ \partial_\rho \phi_2 = -\frac{4\varepsilon(1 - r)\rho}{((\varepsilon + 1 - r)^2 + \rho^2)^2} \]

and

\[ \partial_t^2 \phi_2 = -\frac{8\varepsilon^2 - 12\varepsilon(1 - r)}{((\varepsilon + 1 - r)^2 + \rho^2)^2} + \frac{16\varepsilon(1 - r)(\varepsilon + 1 - r)^2}{((\varepsilon + 1 - r)^2 + \rho^2)^3}, \]
\[ \partial_\rho^2 \phi_2 = -\frac{4\varepsilon(1 - r)}{((\varepsilon + 1 - r)^2 + \rho^2)^2} + \frac{16\varepsilon(1 - r)\rho^2}{((\varepsilon + 1 - r)^2 + \rho^2)^3}. \]

Hence, it is straightforward to show

\[ \partial_\rho u = \frac{e^{3\phi_\varepsilon}}{e^{3\phi_\varepsilon} + c_1 \log \frac{1}{\varepsilon}} \partial_\rho \phi_\varepsilon = \frac{\partial_\rho \phi_\varepsilon}{1 + c_1 \nu_\varepsilon^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3} \]

and

\[ \partial_\rho^2 u = \frac{\partial_\rho^2 \phi_\varepsilon e^{3\phi_\varepsilon}}{e^{3\phi_\varepsilon} + c_1 \log \frac{1}{\varepsilon}} + \frac{3c_1 \nu_\varepsilon^{-1} \log \frac{1}{\varepsilon} (\partial_\rho \phi_\varepsilon)^2 e^{3\phi_\varepsilon}}{((e^{3\phi_\varepsilon} + c_1 \log \frac{1}{\varepsilon})^2)} \]
\[ = \frac{\partial_\rho^2 \phi_\varepsilon}{1 + c_1 \nu_\varepsilon^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3} + \frac{3c_1 \nu_\varepsilon^{-1} \log \frac{1}{\varepsilon} e^{-3\phi_\varepsilon} (\partial_\rho \phi_\varepsilon)^2}{(1 + c_1 \nu_\varepsilon^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^2} \]

as well as similar formulae hold for \( \partial_r u \) and \( \partial_r^2 u \).

In the following estimates, as before we shall use the change of variables: \( s = (1 - r)/\varepsilon, t = \rho/\varepsilon \) and \( \tau = t/(1 + s) \).

We first handle higher order terms. By (55) we estimate

\[ \int_{\Sigma^3} |\nabla u|^2_{\Sigma^3} d\mu_{\Sigma^3} \]

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\[ = \int_{\mathbb{S}^3} \left[ (\partial_\rho u)^2 + \sin^{-2} \rho |\nabla u|_\mathbb{S}^2 \right] d\mu_{\mathbb{S}^3} \]
\[ = O(1) + O(1) \int_{0}^{\delta} \frac{\rho^2}{(\varepsilon^2 + \rho^2)^2} \sin^2 \rho \rho d\rho \]
\[ = O(1). \quad (57) \]

Based on the above calculations, we have

\[ \int_{A_\delta(x_i)} \left| \left( \frac{2 \cot \rho}{r^2} - \frac{2}{\rho} \right) \partial_\rho u \right|^2 dx \]
\[ \lesssim \int_{A_\delta(x_i)} (O(\rho^2) + O((1 - r)^2))(\partial_\rho u)^2 dx \]
\[ \lesssim \int_{A_\delta(x_i)} (O(\rho^2) + O((1 - r)^2))(\partial_\rho \phi \varepsilon)^2 dx \]
\[ \lesssim \int_{A_\delta(x_i)} (O(\rho^2) + O((1 - r)^2)) \left( \frac{\rho^2}{(\varepsilon + 1 - r)^2 + \rho^2} \right)^2 \rho^2 d\rho dr \]
\[ \lesssim \varepsilon^4 \int_{0}^{\frac{\delta}{\varepsilon}} \int_{0}^{\delta} (O(t^2) + O(s^2)) \frac{t^4}{((1 + s)^2 + t^2)^2} dt ds = O(1) \quad (58) \]

and

\[ \int_{A_\delta(x_i)} \left( \frac{1}{r^2} - 1 \right) \partial_\rho^2 u \right)^2 dx \]
\[ \lesssim \int_{A_\delta(x_i)} O((1 - r)^2)((\partial_\rho \phi \varepsilon)^4 + |\partial_\rho^2 \phi \varepsilon|^2) \rho^2 d\rho dr \]
\[ \lesssim \int_{A_\delta(x_i)} O((1 - r)^2) \left( \frac{1}{(\varepsilon + 1 - r)^2 + \rho^2} \right)^2 \rho^2 d\rho dr \]
\[ \lesssim \int_{A_\delta(x_i)} O((1 - r)^2) \left( \frac{1}{(\varepsilon + 1 - r)^2 + \rho^2} \right)^2 \rho^2 d\rho dr \]
\[ \lesssim \varepsilon^2 \int_{0}^{\frac{\delta}{\varepsilon}} \int_{0}^{\delta} s^2 t^2 dt ds \frac{1}{((1 + s)^2 + t^2)^2} = O(1). \quad (59) \]

Next, we focus on the remaining main term. Collecting the above calculations together we obtain

\[ \Delta_1 u := \partial_\rho^2 u + \partial_\rho^2 u + \frac{2}{\rho} \partial_\rho u \]
\[ = \frac{\partial_\rho^2 \phi \varepsilon + \partial_\rho^2 \phi \varepsilon + \frac{2}{\rho} \partial_\rho \phi \varepsilon}{1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3} \]
\[ + 3(\partial_\rho \phi \varepsilon)^2 \frac{c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3}{1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3} \]

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On the other hand, we have

\[
B_0 = - \frac{4}{(\varepsilon + 1 - r)^2 + \rho^2} 1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3,
\]

\[
B_1 = 3(\partial_r \phi_r) \frac{c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3}{1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3},
\]

\[
B_2 = 3(\partial_r \phi_r) \frac{c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3}{1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3},
\]

\[
B_3 = - \frac{8\varepsilon(\varepsilon + 1 - r)}{((\varepsilon + 1 - r)^2 + \rho^2)^2} 1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3.
\]

On one hand, we obtain an upper bound of

\[
\int_{A_\delta(x_1)} B_0^2 \, dx
\]

\[
= 16 \int_{A_\delta(x_1)} \frac{1}{((\varepsilon + 1 - r)^2 + \rho^2)^2} \left( 1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3 \right)^2 \, dx
\]

\[
\leq 16 |S^2| \int_{1-\delta}^{1} \int_{0}^{\delta} \frac{\rho^2}{((\varepsilon + 1 - r)^2 + \rho^2)^2} \, d\rho \, dr
\]

\[
= 16 |S^2| \int_{0}^{\delta} \frac{t^2}{((1+s)^2 + t^2)^2} \, dt
\]

\[
= 16 |S^2| \frac{\pi}{4} \log \frac{1}{\varepsilon} + O(1).
\]

On the other hand, we have

\[
\int_{A_\delta(x_1)} \left( \frac{1}{((\varepsilon + 1 - r)^2 + \rho^2)^2} \right) \left( 1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3 \right) \, dx
\]

\[
= |S^2| \int_{0}^{\delta} ds \int_{0}^{\delta} \frac{t^2}{((1+s)^2 + t^2)^2} \left( 1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((1+s)^2 + t^2)^3 \right)^2 \, dt + O(1)
\]

\[
\geq |S^2| \int_{0}^{-1} \frac{1}{(\log \frac{1}{\varepsilon})^2} \frac{1}{1+s} \int_{0}^{-1} \frac{1}{(\log \frac{1}{\varepsilon})^2} \frac{\tau^2}{(1+\tau^2)^2} \left( 1 + O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right) \right) \, d\tau + O(1)
\]

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Hence, we know
\[ \hat{\tau} = \frac{1}{1 + s} \left( \int_0^\infty \frac{\tau^2}{1 + \tau^2} d\tau + O(\varepsilon \log \frac{1}{\varepsilon}) \right) ds + O(1) \]
\[ = \frac{\pi}{4} \left| S^2 \right| \log \frac{1}{\varepsilon} + O(\log \log \frac{1}{\varepsilon}). \]

Hence, we know
\[ \int_{A_1(x_1)} B_0^2 dx = 4\pi \left| S^2 \right| \log \frac{1}{\varepsilon} + O(\log \log \frac{1}{\varepsilon}). \]

Notice that
\[ B_1 + B_2 \]
\[ = \frac{3c_1 \nu_i^{-1} \log \frac{1}{\varepsilon}}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} (\varepsilon^2 + \gamma (1 - r)^2 + \rho^2)^3)^2} \]
\[ \cdot (\varepsilon + 1 - r)^2 + \rho^2 (4(1 - r)^2 + 4\rho^2) + 16\varepsilon^2 (1 - r)^2 \]
\[ + 16\varepsilon (1 - r)^2 (\varepsilon + 1 - r) + 16\varepsilon \rho^2 (1 - r) \].

Now let us consider
\[ \int_{A_3(x_1)} (B_1 + B_2)^2 dx \]
\[ \leq O((\log \frac{1}{\varepsilon})^2) \left[ \int_{A_3(x_1)} \frac{\varepsilon^4 (1 - r)^4}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^4} dx \right] \]
\[ + \int_{A_3(x_1)} \frac{\varepsilon^4 (1 - r)^4}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^4} dx \]
\[ + \int_{A_3(x_1)} \frac{\varepsilon^2 (1 - r)^4 (\varepsilon + 1 - r)^2}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^4} dx \]
\[ + \int_{A_3(x_1)} \frac{\varepsilon^2 \rho^4 (1 - r)^2}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^4} dx \]
\[ := O((\log \frac{1}{\varepsilon})^2 |S^2| (I_1 + I_2 + I_3 + I_4), \]

where
\[ I_1 = \int_{1 - \delta}^1 \int_0^\delta \frac{((\varepsilon + 1 - r)^2 + \rho^2)^2 (4(1 - r)^2 + 4\rho^2)^2}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^4} \rho^2 d\rho dr, \]
\[ I_2 = \int_{1 - \delta}^1 \int_0^\delta \frac{\varepsilon^4 (1 - r)^4}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^4} \rho^2 d\rho dr, \]

\[ \text{and \quad } \int_0^\delta \frac{((\varepsilon + 1 - r)^2 + \rho^2)^2 (4(1 - r)^2 + 4\rho^2)^2}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)^4} \rho^2 d\rho dr, \]
where we have used the following estimates:

\[ I_3 = \int_{1-\delta}^1 \int_0^\delta \frac{\varepsilon^2 (1-r)^4 (\varepsilon + 1-r)^2}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1-r)^2 + \rho^2)^3)} 4\rho^2 d\rho dr, \]

\[ I_4 = \int_{1-\delta}^1 \int_0^\delta \frac{\varepsilon^2 \rho^4 (1-r)^2}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1-r)^2 + \rho^2)^3)} 4\rho^2 d\rho dr. \]

We shall estimate \( I_i \) term by term. Let

\[ \gamma = \frac{1}{\varepsilon (\log \frac{1}{\varepsilon})^{b}}, \]

where \( b \in \mathbb{R}_+ \) is to be determined later, then

\[ I_1 = \int_{1-\delta}^1 \int_0^\delta \frac{((\varepsilon + 1-r)^2 + \rho^2)(4(1-r)^2 + 4\rho^2)^2}{(1 + c_1 \nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1-r)^2 + \rho^2)^3)} 4\rho^2 d\rho dr \]

\[ \lesssim \varepsilon^{12} \left[ \int_0^\gamma ds \int_0^s \frac{((1+s)^2 + t^2)^2(s^2 + t^2)^2 t^2}{(1 + \frac{1}{\varepsilon})^4 \varepsilon 24 ((1+s)^2 + t^2)^{12}} dt \right. \]

\[ + \int_0^\gamma ds \int_0^s \frac{((1+s)^2 + t^2)^2(s^2 + t^2)^2 t^2}{(1 + \frac{1}{\varepsilon})^4 \varepsilon 24 ((1+s)^2 + t^2)^{12}} dt \]

\[ + \int_\gamma^\tilde{s} ds \int_0^s \frac{((1+s)^2 + t^2)^2(s^2 + t^2)^2 t^2}{(1 + \frac{1}{\varepsilon})^4 \varepsilon 24 ((1+s)^2 + t^2)^{12}} dt \right] \]

\[ \lesssim \varepsilon^{12} \left( \gamma^{12} + \frac{(\log \frac{1}{\varepsilon})^{12b-4}}{\varepsilon^{12}} \right) \]

\[ \lesssim (\log \frac{1}{\varepsilon})^{12b-4}, \]

where we have used the following estimates:

\[ \int_0^\gamma ds \int_0^s \frac{((1+s)^2 + t^2)^2(s^2 + t^2)^2 t^2}{(1 + \frac{1}{\varepsilon})^4 \varepsilon 24 ((1+s)^2 + t^2)^{12}} dt \]

\[ \lesssim (\log \frac{1}{\varepsilon})^4 \varepsilon 24 \int_0^\gamma \frac{ds}{(1+s)^{13}} \int_\gamma^{(1+s)} \frac{\tau^2}{(1 + \tau^2)^8} d\tau \]

\[ \lesssim \gamma^{-12} (\log \frac{1}{\varepsilon})^{12b-4} \lesssim (\log \frac{1}{\varepsilon})^{12b-4} \]

and

\[ \int_0^\gamma ds \int_0^s \frac{((1+s)^2 + t^2)^2(s^2 + t^2)^2 t^2}{(1 + \frac{1}{\varepsilon})^4 \varepsilon 24 ((1+s)^2 + t^2)^{12}} dt \]

\[ \lesssim (\log \frac{1}{\varepsilon})^4 \varepsilon 24 \int_\gamma^s \frac{ds}{(1+s)^{13}} \int_0^{(1+s)} \frac{\tau^2}{(1 + \tau^2)^8} d\tau \]

\[ \lesssim (\log \frac{1}{\varepsilon})^{12b-4} \]
\[
\beta \frac{1}{(\log \frac{1}{\varepsilon})^{42}} \int_{\gamma} (1 + s)^{-13} ds \\
\lesssim (\log \frac{1}{\varepsilon})^{-12} \frac{1}{(\log \frac{1}{\varepsilon})^{42}} \lesssim \frac{(\log \frac{1}{\varepsilon})^{12b-4}}{\varepsilon^{12}}.
\]

Hence, we may take \(12b - 2 < 1\), i.e., \(b \in (0, 1/4)\) such that

\[
(\log \frac{1}{\varepsilon})^2 I_1 = o(\log \frac{1}{\varepsilon}).
\]

Let us deal with the other three terms:

\[
I_2 = \int_{1-\delta}^{1} \int_{0}^{\delta} \frac{\varepsilon^4 (1 - r)^4}{(1 + c_1 \nu_1^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)} \rho^2 d\rho dr
\]

\[
= 16\varepsilon^{12} \int_{0}^{\frac{\delta}{2}} \int_{0}^{\frac{\delta}{2}} \frac{s^4 t^2 dt ds}{(1 + c_1 \nu_1^{-1} \varepsilon^6 \log \frac{1}{\varepsilon} ((1 + s)^2 + t^2)^3)^4}
\]

\[
\lesssim 16\varepsilon^{12} \int_{0}^{\frac{\delta}{2}} \int_{0}^{\frac{\delta}{2}} s^4 t^2 dt ds \lesssim \varepsilon^4
\]

\[
I_3 = \int_{1-\delta}^{1} \int_{0}^{\delta} \frac{\varepsilon^2 (1 - r)^4 (\varepsilon + 1 - r)^2}{(1 + c_1 \nu_1^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)} \rho^2 d\rho dr
\]

\[
\lesssim \varepsilon^{12} \int_{0}^{\frac{\delta}{2}} \int_{0}^{\frac{\delta}{2}} (1 + s)^2 s^4 t^2 dt ds \lesssim \varepsilon^2
\]

and

\[
I_4 = \int_{1-\delta}^{1} \int_{0}^{\delta} \frac{\varepsilon^2 \rho^4 (1 - r)^2}{(1 + c_1 \nu_1^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3)} \rho^2 d\rho dr
\]

\[
\lesssim \varepsilon^{12} \int_{0}^{\frac{\delta}{2}} \int_{0}^{\frac{\delta}{2}} t^6 s^2 dt ds \lesssim \varepsilon^2.
\]

Hence, putting these facts together we obtain

\[
\int_{A_b(x_i)} (B_1 + B_2)^2 dx \\
\lesssim (\log \frac{1}{\varepsilon})^2 \left( I_1 + I_2 + I_3 + I_4 \right) \\
\lesssim (\log \frac{1}{\varepsilon})^{12b-2} = o(\log \frac{1}{\varepsilon}),
\]

where \(b \in (0, 1/4)\).
We turn to show
\[
\int_{A_\delta(x_i)} B_3^2 \, dx \\
= \int_{A_\delta(x_i)} \left( \frac{8\varepsilon(\varepsilon + 1 - r)}{((\varepsilon + 1 - r)^2 + \rho^2)^2} \frac{1}{1 + c_1\nu_i^{-1} \log \frac{1}{\varepsilon} ((\varepsilon + 1 - r)^2 + \rho^2)^3} \right)^2 \, dx \\
\lesssim |S^2| \int_{1-\delta}^1 \int_0^\delta \left( \frac{8\varepsilon(\varepsilon + 1 - r)}{((\varepsilon + 1 - r)^2 + \rho^2)^2} \rho^2 \, dp \, dr \\
\lesssim \int_0^{\frac{\delta}{3}} \int_0^{\frac{\delta}{3}} \frac{(1 + s)^2 t^2}{((1 + s)^2 + t^2)^4} \, dt \, ds \\
\lesssim \int_0^{\frac{\delta}{3}} \frac{ds}{(1 + s)^3} \int_0^{\frac{\delta}{3}} \frac{r^2}{(1 + r^2)^4} \, dr \\
\lesssim \int_0^{\frac{\delta}{3}} \frac{1}{(1 + s)^3} \, ds = O(1). 
\]

Consequently, it follows from Cauchy inequality that
\[
\int_{A_\delta(x_i)} (\Delta u)^2 \, dx = 4\pi |S^2| \log \frac{1}{\varepsilon} + o(\log \frac{1}{\varepsilon}). 
\]

By (58), (59), (60) and summing i from 1 to N_m, we obtain
\[
\int_{B^4} (\Delta u)^2 \, dx = 4\pi |S^2| N_m \log \frac{1}{\varepsilon} + o(\log \frac{1}{\varepsilon}). 
\]

Therefore, from the assumption that
\[
\frac{3}{4\pi |S^2| N_m} - 3\bar{u},
\]
putting the above estimates (51), (56), (57) and (61) together, we conclude that
\[
a \geq \frac{3}{4\pi |S^2| N_m} = \frac{3}{16\pi^2 N_m}. 
\]
This completes our construction.

A An example on the sharpness of Lebedev-Milin inequality under constraints

The purpose of this appendix is to demonstrate our idea in dimension four through an example for Lebedev-Milin inequality in higher order moments case, which will be presented in a concise manner, as the same idea prevails throughout the paper.

For clarity, we restate Widom [33] inequality, a generalization of Lebedev-Milin inequality: If \( u \in H^1(B^2) \) satisfies \( \int_{S^1} e^{u} d\mu_{S^1} = 0 \) for all \( p \in P_m \), then
\[
\log \left( \frac{1}{2\pi} \int_{S^1} e^{u-a} d\mu_{S^1} \right) \leq \frac{1}{4\pi N_m(S^1)} \int_{S^2} |\nabla u|^2 \, dx,
\]
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where \( \bar{u} = (2\pi)^{-1} \int_{\mathbb{S}^1} e^{u} d\mu_{\mathbb{S}^1} \) and \( N_m(\mathbb{S}^1) = m + 1 \) as shown in [10]. For \( m = 1 \), the above inequality was first proved by Osgood-Phillips-Sarnak [29]. Furthermore, if we relax \( m \) to define \( \mathcal{P}_0 = \emptyset \) and \( N_0(\mathbb{S}^1) = 1 \), then the above inequality with \( m = 0 \) reduces to Lebedev-Milin inequality. An alternative proof of the above Widom’s inequality is also available in [10, Section 6].

The sharpness of the above inequality can be shown in the following way: If there exists \( a \in \mathbb{R}_+ \) such that for all \( u \in H^1(\mathbb{B}^2) \) satisfying \( \int_{\mathbb{S}^1} e^{u} p d\mu_{\mathbb{S}^1} = 0 \) for all \( p \in \mathcal{P}_m \), we have

\[
 a \int_{\mathbb{B}^2} |\nabla u|^2 dx \geq \log \left( \frac{1}{2\pi} \int_{\mathbb{S}^1} e^{u - \bar{u}} dx \right),
\]

then

\[
 a \geq \frac{1}{4\pi N_m}.
\]

For brevity, we let \( N_m = N_m(\mathbb{S}^1) \).

In dimension two, for each \( x_i := (\cos \theta_i, \sin \theta_i) \in \mathbb{S}^1 \) we define

\[
 \mathcal{A}_\delta(x_i) := \{ x = r\xi \in \mathbb{B}^n; \xi \in \mathbb{S}^{n-1}, 1 - r < \delta, |\theta - \theta_i| < \delta \}.
\]

Near \( x_i \), we use the polar coordinates

\[
 |dx|^2 = dr^2 + r^2 d\theta^2
\]

for \( x = (r, \theta) \) with \( \xi = (\cos \theta, \sin \theta) \).

Define

\[
 \phi_{\epsilon,i}(r, \theta) = -\log \left( \epsilon^2 + (1 - r)^2 + (\theta - \theta_i)^2 \right).
\]

Let \( \rho = \theta - \theta_i \), then

\[
 \sum_{i=1}^{N_m} \int_{\mathbb{S}^1} \chi_i e^{\phi_{\epsilon,i} + \log \nu_i} d\theta
\]

\[
 = \int_{-\delta}^{\delta} \frac{d\rho}{\epsilon^2 + \rho^2} + O(1) \int_{-\delta}^{\delta} \frac{d\rho}{\epsilon^2 + \rho^2}
\]

\[
 = 2\epsilon^{-1} \int_{0}^{\delta/\epsilon} \frac{dt}{1 + t^2} + O(1)
\]

\[
 = \pi \epsilon^{-1} + O(1)
\]

and for all \( p \in \mathcal{P}_m \),

\[
 \sum_{i=1}^{N_m} \int_{\mathbb{S}^1} \chi_i e^{\phi_{\epsilon,i} + \log \nu_i} p(x) d\theta
\]

\[
 = \sum_{i=1}^{N_m} \nu_i \int_{-\delta}^{\delta} \chi_i e^{\phi_{\epsilon,i}} O(|\theta - \theta_i|^2) d\theta + O(1)
\]

\[
 = O(1).
\]
We can choose a test function as
\[ e^u = \sum_{i=1}^{N_m} \chi_i e^{\phi_{\varepsilon,i}} + \log \nu_i + c_1 + \sum_{j=1}^{2m} \beta_j \psi_j \]

As before, we can find \( \psi_j \in C_\infty^\infty ([0, 2\pi] \setminus \cup_{i=1}^{m+1} (\theta_i - 2\delta, \theta_i + 2\delta)) \) and \( c_1 \in \mathbb{R}_+ \) such that
\[ \int_{S^1} e^u p_k d\mu_{S^1} = 0 \quad 1 \leq k \leq 2m \]

and
\[ c_1 + \sum_{j=1}^{2m} \beta_j \psi_j > 1, \]

then it in turn implies \( \beta_j = O(1) \).

Now we check that \( \bar{u} = \frac{1}{2\pi} \int_{S^1} u d\mu_{S^1} \)
\[ = O(1) + \frac{1}{2\pi} \int_{\delta}^{0} \log \left( \frac{1}{\varepsilon^2 + (\theta - \theta_i)^2} + O(1) \right) d\theta \]
\[ = O(1) + \frac{1}{2\pi} \int_{\delta}^{0} \log \left( \frac{1}{\varepsilon^2 + \rho^2} \right) d\rho \]
\[ = \frac{1}{\pi} \int_{0}^{\delta} \left( 2 \log \varepsilon^{-1} + \log \frac{1}{1 + (\frac{\rho}{\varepsilon})^2} \right) d\rho + O(1) \]
\[ = \frac{1}{\pi} \left[ 2\delta \log \varepsilon^{-1} - \int_{0}^{\delta/\varepsilon} \log(1 + t^2) dt \right] + O(1) \]
\[ = \frac{1}{\pi} \left[ 2\delta \log \varepsilon^{-1} - \delta \log(1 + \delta^2 \varepsilon^{-2}) \right] + O(1) \]
\[ = O(1). \]

Observe that
\[ \partial_r \phi_{\varepsilon,i} = \frac{2(1 - r)}{\varepsilon^2 + (1 - r)^2 + (\theta - \theta_i)^2} \]
and
\[ \partial_\theta \phi_{\varepsilon,i} = \frac{-2(\theta - \theta_i)}{\varepsilon^2 + (1 - r)^2 + (\theta - \theta_i)^2}, \]
then
\[ |\nabla \phi_{\varepsilon,i}|^2 = \frac{4 ((1 - r)^2 + (\theta - \theta_i)^2)}{\varepsilon^2 + (1 - r)^2 + (\theta - \theta_i)^2} + \frac{4(\theta - \theta_i)^2}{\varepsilon^2 + (1 - r)^2 + (\theta - \theta_i)^2} \left( \frac{1}{r^2} - 1 \right). \]

Under the change of variables: \( s = (1 - r)/\varepsilon \) and \( t = \rho/\varepsilon \), we obtain
\[ \int_{A_{\delta}(x_i)} \frac{4 ((1 - r)^2 + (\theta - \theta_i)^2)}{\varepsilon^2 + (1 - r)^2 + (\theta - \theta_i)^2} r dr d\theta \]

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\[= \int_{0}^{\delta} \int_{-\delta}^{\delta} \frac{4 ((1-r)^2 + \rho^2)^2}{(\varepsilon^2 + (1-r)^2 + \rho^2)^2} \, d\rho \, dr + \int_{0}^{\delta} \int_{-\delta}^{\delta} \frac{4 ((1-r)^2 + \rho^2)^2}{(\varepsilon^2 + (1-r)^2 + \rho^2)^2} (r-1) \, d\rho \, dr \]
\[= 2 \int_{0}^{\delta/\varepsilon} \int_{0}^{\delta/\varepsilon} \frac{4 (s^2 + t^2)^2}{(1+s^2 + t^2)^2} \, ds \, dt + O(1) \]
\[= 8 \int_{0}^{\delta/\varepsilon} \int_{0}^{\delta/\varepsilon} \frac{4 s^2 + \tau^2 (1+s^2)}{(1+s^2)^{3/2} (1+\tau^2)^2} \, d\tau \, ds + O(1) \]
\[= 8 \int_{0}^{\delta/\varepsilon} \frac{ds}{\sqrt{1+s^2}} \int_{0}^{\delta/\varepsilon} \frac{d\tau}{\sqrt{1+\tau^2}} \]
\[= 4\pi \log \frac{1}{\varepsilon} + O(1) \]

and
\[\int_{A_t(x_i)} \frac{4(\theta - \theta_i)^2}{(\varepsilon^2 + (1-r)^2 + (\theta - \theta_i)^2)^2} \left( \frac{1}{r^2} - 1 \right) \, dx \]
\[\lesssim \varepsilon \int_{\delta}^{\varepsilon/\delta} \int_{\delta}^{\varepsilon/\delta} \frac{t^2 s}{(1+s^2 + t^2)^2} \, ds \, dt = O(1). \]

Thus, we obtain
\[\int_{\mathbb{B}^2} |\nabla u|^2 \, dx = 4\pi N_m \log \frac{1}{\varepsilon} + O(1). \]

On the other hand, we have
\[\int_{\mathbb{S}^1} e^u \, d\mu_{\mathbb{S}^1} = \int_{-\delta}^{\delta} \frac{1}{\varepsilon^2 + \rho^2} \, d\rho + O(1) = \pi \varepsilon^{-1} + O(1). \]

This gives
\[\log \left( \frac{1}{2\pi} \int_{\mathbb{S}^1} e^u \, d\mu_{\mathbb{S}^1} \right) = \log \frac{1}{\varepsilon} + O(1). \]

Hence, it follows from the assumption that
\[a \int_{\mathbb{B}^2} |\nabla u|^2 \, dx \geq \log \left( \frac{1}{2\pi} \int_{\mathbb{S}^1} e^u \, d\mu_{\mathbb{S}^1} \right) - \bar{u}. \]

Putting these facts together, we conclude that
\[a \geq \frac{1}{4\pi N_m}. \]

**Remark 1.** The above strategy can also provide an alternative of Chang-Hang’s test function in [10] as follows: we can replace the piecewise Lipschitz function \( \phi_{\varepsilon}(t) \) in Chang-Hang [10, p.10] by a global function
\[\phi_{\varepsilon}(t) = -\log(\varepsilon^2 + t^2), \]
indeed, Chang-Hang’s $\phi_\varepsilon$ also occurred in Lions’ example for Moser-Trudinger inequality in [26, pp.195-199]. By choosing constants $\beta_j, c_1$ and smooth cut-off functions $\chi_i, \psi_j$ as Chang-Hang’s, we define a smooth test function

$$e^{2u} = \sum_{i=1}^{N} \chi_i e^{2\phi_\varepsilon(x_i)} + \log \nu_i + \sum_{j=1}^{m^2+2m} \beta_j \psi_j + c_1 \log \frac{1}{\varepsilon}.$$ 

We follow the same lines of Chang-Hang to give an example to show almost sharpness of Moser-Trudinger-Onofri inequality under constraints.

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