Long time asymptotics of large data in the Kadomtsev–Petviashvili models

Argenis J Mendez¹,5, Claudio Muñoz²,6,7, Felipe Poblete³,7,8 and Juan C Pozo⁴,8

¹ Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Santiago, Chile
² CNRS and Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (AFB170001 and UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile
³ Instituto de Ciencias Físicas y Matemáticas, Facultad de Ciencias, Universidad Austral de Chile, Valdés, Chile
⁴ Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Santiago, Chile

E-mail: felipe.poblete@uach.cl, amendez@dim.uchile.cl, cmunoz@dim.uchile.cl and jpozo@uchile.cl

Received 30 May 2023; revised 28 February 2024
Accepted for publication 19 March 2024
Published 2 April 2024

Recommended by Dr Karima Khusnutdinova

Abstract

We consider the Kadomtsev–Petviashvili (KP) equations posed on \( \mathbb{R}^2 \). For both models, we provide sequential in time asymptotic descriptions of solutions obtained from arbitrarily large initial data, inside regions of the plane not containing lumps or line solitons, and under minimal regularity assumptions. The proof involves the introduction of two new virial identities adapted to the KP dynamics. This new approach is particularly important in the KP-I case, where no monotonicity property was previously known. The core of our results

---

This work was partially supported by CMM Conicyt Proyecto Basal AFB170001 and by Postdoc FONDECYT Project 3210727.

C M’s work was funded in part by Chilean research Grants Exploration Project 13220060, FONDECYT 1191412, 1231250 and Centro de Modelamiento Matemático (CMM), ACE210010 and FB210005, BASAL funds for centers of excellence from ANID-Chile, Project France–Chile ECOS-Sud C18E06, MathAmSud EQUADDS II, MathAmSud WAFFLE 23-MATH-18 and CMM ANID PIA AFB170001.

F P’s work is partially supported by ANID Exploration Project 13220060, ANID Project FONDECYT 1221076 and MathAmSud WAFFLE 23-MATH-18

J C P’s work was partially supported by Chilean research Grants Exploration Project 13220060 and FONDECYT 1221271.

Author to whom any correspondence should be addressed.

© 2024 IOP Publishing Ltd & London Mathematical Society
do not require the use of the integrability of KP and are adaptable to well-posed perturbations.

Keywords: KP equations, KPI, KPII, asymptotics, decay, virial estimate, large data
Mathematics Subject Classification numbers: 35Q53, 35Q05

1. Introduction and main results

1.1. Setting

Consider the Kadomtsev–Petviashvili (KP) equations posed in $\mathbb{R}^2$,

$$
\partial_t u + \partial_x^3 u + \kappa \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0,
$$

(1.1)

where $u = u(x,t) \in \mathbb{R}$, for $t \in \mathbb{R}$ and $x = (x,y) \in \mathbb{R}^2$. The nonlocal operator $\partial_x^{-1}f$ is formally defined in the literature as

$$
(\partial_x^{-1}f)(x,y) := \int_{-\infty}^{x} f(s,y) \, ds,
$$

and $\kappa \in \{-1,1\}$. The KP equations were first introduced by Kadomtsev and Petviashvili in 1970 [20] for modelling long and weakly nonlinear waves propagating essentially along the $x$ direction, with a small dependence in the $y$ variable. It is a universal integrable two-dimensional generalisation of the well-known Korteweg–de Vries (KdV) equation, since many other integrable systems can be obtained as reductions [24]. The nonlocal term $\partial_x^{-1} \partial_y^2$ arises from weak transverse effects, with the sign of $\kappa$ allowing to take into account the surface tension. More precisely, when $\kappa = -1$ KP models capillary gravity waves, in the presence of strong surface tension. In such a case KP is known as the KP-I equation. When $\kappa = 1$, the equation (1.1) is known as the KP-II model and it allows to study capillary gravity waves in the presence of small surface tension. A rigorous derivation of both models from the symmetric $abcd$ Boussinesq system was obtained in [31–33]. We refer the reader to the monographs by Klein and Saut [23], [24], chapter 5 for a complete mathematical and physical description of the KP model.

The nonlocal term makes KP models hard from the mathematical point of view, and the understanding of the dynamics is far from satisfactory. Despite their apparent similarity, KP-I and KP-II differ significantly with respect to their underlying mathematical structure and the behaviour of their solutions. For instance, from the point of view of well-posedness theory KP-II is much better understood than KP-I. Indeed, since the foundational work of Bourgain [2] in 1993, one knows that KP-II is globally well-posed on $L^2(\mathbb{R}^2)$ (see also Ukai [61] and Iorio-Nunes [14] for early results, and [11, 18, 60] for results improving Bourgain’s results). The global well-posedness is proved via the contraction principle in $X^{s,b}$ spaces and the conservation of the $L^2$-norm

$$
M[u](t) := \frac{1}{2} \int_{\mathbb{R}^2} u^2(x,y,t) \, dx \, dy = \text{const.}
$$

(1.2)

Note that this conservation holds for both KP-I and KP-II. Meanwhile, the KP-I global theory in the energy space took years to be solved. Without being exhaustive, we emphasise the foundational global well-posedness work by Molinet, Saut and Tzvetkov [48, 49], and the improved global well-posedness by Kenig [21], which will be used in this paper several times.
(see also [8]). A key breakthrough was obtained in 2008 by Ionescu, Kenig and Tataru [17] (see also [16]), who proved that KP-I is globally well-posed in the natural energy space of the equation

\[ E^1(\mathbb{R}^2) := \{ u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} + \|\partial_x u\|_{L^2} + \|\partial_x^{-1} \partial_y u\|_{L^2} < +\infty \}, \]

in the sense that the flow map extends continuously from suitable smooth data into \( E^1(\mathbb{R}^2) \), see [17] for further details (it was known from [49] that KP-I behaves badly with respect to perturbative methods). The space \( E^1(\mathbb{R}^2) \) arises naturally from the conservation of energy \((\kappa = 1 \text{ KP-II}, \kappa = -1 \text{ KP-I})\)

\[ E[u](t) := \int_{\mathbb{R}^2} \left( \frac{1}{2} (\partial_t u)^2 - \frac{1}{2} \kappa (\partial_x^{-1} \partial_y u)^2 - \frac{1}{3} u^3 \right) (x, y, t) \, dx \, dy = \text{const.} \tag{1.3} \]

It is worth to mention that the well-posedness of KP-I in the space \( E^1(\mathbb{R}^2) \) will be essential for the validity of the functional tools that we shall use below. Additionally, very important for us will be the momentum conservation law, valid for solutions in \( E^1(\mathbb{R}^2) \):

\[ P[u](t) := \frac{1}{2} \int_{\mathbb{R}^2} (u \partial_x^{-1} \partial_y u)(x, y, t) \, dx \, dy = \text{const.} \tag{1.4} \]

Also, for the purposes of this paper, we shall also need the global well-posedness result for KP-I established by Kenig [21], in the so-called second energy space

\[ E^2(\mathbb{R}^2) := \{ u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} + \|\partial_x^{-1} \partial_y u\|_{L^2} + \|\partial_y^2 u\|_{L^2} + \|\partial_x^{-2} \partial_y^2 u\|_{L^2} < +\infty \}. \tag{1.5} \]

This space is motivated by the conservation of the second energy [63]

\[ F[u](t) := \int_{\mathbb{R}^2} \left( \frac{3}{2} (\partial_x^2 u)^2 + 5 (\partial_t u)^2 + \frac{5}{6} (\partial_x^{-2} \partial_y^2 u)^2 - \frac{5}{6} u^2 \partial_x^{-2} \partial_y^2 u ight) - \frac{5}{6} u (\partial_x^{-1} \partial_y u)^2 + \frac{5}{4} u^2 \partial_y^2 u + \frac{5}{24} u^4 \right) (x, y, t) \, dx \, dy = \text{const.}, \]

a property that has been rigorously proved in [48]. From [21], we also know that \( \|\partial_t u\|_{L^2} \lesssim \|u\|_{L^2} \). Finally, the Cauchy problem for the fractional version of KP was addressed in [34], see also references therein for more details on the low dispersion problem for KP.

Once a suitable well-posedness theory is available, one may wonder about the long time behaviour of globally defined solutions. This is a difficult open question not yet solved either by inverse scattering technique (IST) or partial differential equations (PDE) methods. Moreover, the answer may be strongly dependent on the choice of model, KP-I or KP-II. Except by some particular cases (see below the case of lumps and line solitons, their orbital and asymptotic stability, and the case of suitable small data scattering solutions), no results about large data behaviour in KP models, starting from Cauchy data, are available, as explained in [24, chapter 2].

1.2. New results

In this paper we provide the first arbitrarily large data, long time asymptotic description of global finite energy solutions to the KP equation (1.1), with the minimal data assumptions required to run the global well-posedness theory with uniform in time bounds.

Basically, we only assume data in \( L^2(\mathbb{R}^2) \) in the KP-II case, and data in \( E^1(\mathbb{R}^2) \) and \( E^2(\mathbb{R}^2) \) in the KP-I case. In particular, for both KP-I and KP-II we describe via a sequence of times the
Figure 1. Left. Schematic figure depicting the set \( \Omega_1(t) \), in the centred case \( l_1 = l_2 = 0 \), as defined in (1.6). Right. Schematic figure depicting the sets \( \Omega_{1,1}(t) \) and \( \Omega_{1,2}(t) \), as defined in (1.8) and (1.9), respectively.

The final dynamical behaviour of any global solution in regions of the plane growing unbounded in time (namely, containing any compact region in \( \mathbb{R}^2 \)), not containing the zone where the lumps are present, and confirming (for a sequence of times) the KP soliton resolution conjecture in this region and under physical regularity assumptions. Recall that the lump, the soliton-like solution to KP-I with algebraic decay in space, does not decay in time. It was very recently proved to be orbitally stable in the space \( E^1(\mathbb{R}^2) \) in another breakthrough by Liu and Wei [38]. Consequently, and in view of the results that we will describe below, the asymptotic stability of this object seems a fundamental open problem in the field.

Let us state our main results. Let \( t \gg 1 \). Let \( \Omega_1(t) \) denote the following rectangular box

\[
\Omega_1(t) = \{ (x,y) \in \mathbb{R}^2 : |x - \ell_1^{m_1}| \leqslant \ell^b, |y - \ell_2^{m_2}| \leqslant \ell^{br} \},
\]  

with \( \ell_1, \ell_2 \in \mathbb{R} \),

\[
\frac{5}{3} < r < 3, \quad 0 < b < \frac{2}{3 + r},
\]

\[
0 \leq m_1 < 1 - \frac{b}{2} (r + 1), \quad \text{and} \quad 0 \leq m_2 < 1 - \frac{b}{2} (3 - r).
\]

Let also \( \sigma_1, \sigma_2 \geq 0 \), not both equal zero. We define \( \Omega_{2,1}(t) \) as

\[
\Omega_{2,1}(t) := \{ (x,y) \in \mathbb{R}^2 : \sigma_1 |x| + \sigma_2 |y| \geq t \log^{1+\gamma} t \},
\]

for any fixed \( \gamma > 0 \). We also define, for \( \beta > 0 \) and \( \sigma_3 \in \mathbb{R} \),

\[
\Omega_{2,2}(t) := \{ (x,y) \in \mathbb{R}^2 : x + \sigma_3 y \geq \beta t \}.
\]

See figure 1 for details. Also, figure 2 shows a comparison between the exponents \( b \) and \( br \) of \( \Omega_1 \) in (1.6). Note that inside the square \( |x| \sim t, |y| \sim t \), the set \( \Omega_1(t) \) represents a central region, while \( \Omega_{2,1}(t) \) and \( \Omega_{2,2}(t) \) far regions. In terms of \( \Omega_1(t) \), \( \Omega_{2,1}(t) \) and \( \Omega_{2,2}(t) \) the following two theorems are the main results of this work. First, we consider the central region far from the lumps.
Figure 2. Recall that $\frac{5}{2} < r < 3$, $0 < b < \frac{2}{3 + r}$, and $0 < br < \frac{2}{3 + r}$. The largest value of $b$ is $\sim \frac{4}{3}$, and $br \sim \frac{5}{6}$, obtained by $r \sim \frac{5}{3}$. Given $r$, $f_1(r)$ and $f_2(r)$ represents the supremum value of $b$ and $br$, respectively.

Theorem 1.1 (central decay). Every solution $u = u(x, y, t)$ of KP obtained from arbitrary initial data $u_0$ in $L^2(\mathbb{R}^2)$ in the KP-II case, and $u_0$ in the energy space $E^1(\mathbb{R}^2)$ for KP-I, satisfies

$$\liminf_{t \to \infty} \int_{\Omega_1(t)} u^2(x, y, t) \, dx \, dy = 0. \quad (1.10)$$

Note that property (1.10) in theorem 1.1 is satisfied by both KP-I and KP-II. This is due to the fact that both models contain a quadratic nonlinearity, and the proof does not depend on the sign of $\kappa$ in (1.1), namely it does not depend on the linear behaviour of each KP model. The asymptotic description is strictly triggered by the nonlinear behaviour of KP. This new point of view of the dynamics has a key advantage: each particular zone should be treated via a different virial method, leading to different outcomes. In the particular setting $\ell_1 = \ell_2 = 0$ (the centred case), (1.10) reveals that the zone of strong influence of the linear dynamics is outside $\Omega_1(t)$, although under additional regularity assumptions, one may show $L^\infty$ decay in the non centred case, see [7] for results in that direction in the one dimensional KdV model.

In the KP setting, data in $E^1(\mathbb{R}^2)$ does not assure boundedness $L^\infty(\mathbb{R}^2)$. In the non centred case $\ell_1, \ell_2 \neq 0$, theorem 1.1 also provides a description of the dynamics except in the region dominated by nonlinear objects, which corresponds to the region $|x| \sim t$. Our second result establishes full decay in far regions in $\mathbb{R}^2$:

Theorem 1.2 (far regions decay). Let $u = u(x, y, t)$ a solution of KP obtained from arbitrary initial data $u_0$ in $L^2(\mathbb{R}^2)$ in the KP-II case, and $u_0$ in the energy space $E^1(\mathbb{R}^2)$ for KP-I. The following statements are satisfied:

If $u \in L^\infty([0, \infty); E^1(\mathbb{R}^2))$ is a solution to KP

$$\lim_{t \to \infty} \int_{\Omega_2(t)} u^2(x, y, t) \, dx \, dy = 0. \quad (1.11)$$
If $u$ is a solution to KP-II and $\beta$ is sufficiently large (only depending on $\sigma_3$ and the $E^1(\mathbb{R}^2)$ norm of the initial datum), then

$$\lim_{t \to \infty} \int_{\Omega_{1/2}(t)} u^2(x,y,t) \, dx \, dy = 0. \tag{1.12}$$

Indeed, the region possibly containing lumps or line solitons $|x| \sim t$ is of different nature, characterised by shifts of order $\sim t$, and must be treated in a separated way, depending on each KP model one considers. However, in the KP-II setting, more can be said: Kenig and Martel [22] showed strong decay on the right half-plane, in the following form: for any $\beta > 0$,

$$\lim_{t \to \infty} \int_{x > \beta} u^2(x,y,t) \, dx \, dy = 0, \tag{1.13}$$

provided the initial data is small enough in $L^1 \cap L^2$. Thus, in some sense, (1.11) and (1.12) of theorem 1.2 extend (1.13) to large data in the energy space $E^1(\mathbb{R}^2)$ and different regions, and provide new results for both KP models. The only condition needed is boundedness in time of the energy norm, valid for KP-I, but not clear for KP-II (see [3] for a similar assumption).

The remaining limsup property in (1.10) holds in a particular region if the solution is in $L^\infty_t (L^1_y \cap L^2_y)$, see [51]; but the required hypothesis is more demanding than data the energy space only. There is ground to believe that the results in theorem 1.1 are sharp. Indeed, note that our previous results in the one dimensional KdV setting [43] are sharp, in the sense that, assuming asymptotic linear decay of the nonlinear dynamics, space-time pointwise estimates on the linear profile lead to an agreement with the decay property established by our techniques.

From the proofs, it will become clear that theorem 1.1 is stable under perturbations of the nonlinearity in the form $f(u) = u^2 + o_{u \to 0}(u^2)$, provided the well-posedness theory of the corresponding KP equation is available. This is a delicate subject; for that reason, the treatment of the term $o_{u \to 0}(u^2)$ requires some care. For details on generalised KP models, see Saut [58] and Liu [37].

Given the result of Bourgain [2], property (1.10) provides a sequential $L^2$ description of the dynamics in the KP-II case on any compact set of $\mathbb{R}^2$. This can be complemented with the previous works by de Bouard and Saut [5] and de Bouard and Martel [3], where nonexistence of KP-II compact solutions uniformly bounded in $E^1(\mathbb{R}^2)$ was proved. Although applied to a different problem and setting, some of the elements in the proofs of [3] will be adapted and extended to general data in order to prove theorem 1.3 below, concerning the particular case of KP-I.

In order to state our third result, we introduce the notion of compact solutions [3] in the KP-I setting:

**Definition 1.1.** Let $u = u(x,y,t)$ be an $E^0(\mathbb{R}^2)$ solution to KP-I. We say that $u$ is compact as $t \to \infty$, if there are $C^1$ functions $x(t), y(t) : [0, \infty) \to \mathbb{R}$, such that for all $\varepsilon > 0$ there exists $R(\varepsilon) > 0$, such that

$$\sup_{t \geq 0} \int_{\|x-x(t), y-y(t)\| \geq R(\varepsilon)} u^2(x,y,t) \, dx \, dy < \varepsilon.$$

A similar definition can be introduced for $t \to -\infty$.

Note that we only ask for information about the solution for times $t \geq 0$ (or $t \leq 0$). Clearly the lump (1.21) stated below is a compact solution to KP-I satisfying $x(t) \sim t$ and $y(t) \sim t$. Definition 1.1 requires data in $E^0(\mathbb{R}^2)$, a natural space in KP-I. In this paper, as a consequence of theorem 1.1, we extend [3] to the KP-I setting and show nonexistence of periodic compact zero-speed solutions to KP-I (a.k.a. breathers):
Corollary 1.1. Assume that \( u \) is a compact solution to \( \text{KP-I} \) as in definition 1.1 satisfying, for some \( C_0 > 0 \) and \( t \geq 0 \),

\[
|x(t)| \leq C_0 |t|^{m_1}, \quad |y(t)| \leq C_0 |t|^{m_2},
\]

with \( m_1, m_2 \) as in (1.17). Then \( u \equiv 0 \).

This result rules out any low speed solution in \( \text{KP-I} \).

We now comment on previous asymptotic results in \( \text{KP} \) models. Standard \( L^1 \to L^\infty \) estimates of the linear \( \text{KP} \) flow show decay of order \( 1/t \), see Saut [58]. However, as explained by Hayashi and Naumkin in [13], the quadratic nonlinearity in \( \text{KP} \) is of critical order for scattering methods. From [13, 15], one has explicit space/time decay estimates for both \text{linear} \( \text{KP} \) models, assuming sufficient decay, smallness and smoothness on the initial datum. In the nonlinear \( \text{KP} \) setting, scattering and decay estimates have been extensively studied during past years, essentially under small data assumptions. The large data case poses serious obstructions to either inverse scattering methods, or linear dispersive techniques. Concerning to PDE methods, Hadac, Herr and Koch [11] showed scattering of small solutions for \( \text{KP-II} \) in the critical space \( \dot{H}^{-1/2,0}([\mathbb{R}^2]) \) (below the \( L^2 \) regularity). Hayashi, Naumkin and Saut [15] proved scattering for small data in the case of (1.1) with power nonlinearity bigger than 3. This was done in suitable weighted spaces for both \( \text{KP-I} \) and \( \text{KP-II} \) models (see also [54]). In that work (see also [58] for previous remarks), the parabolic regions

\[
x + \frac{y^2}{4 \kappa t} = \text{const.,}
\]

became important as essentially the worst possible decay regions in the small data \( \text{KP} \) regime. The somehow strange constraint \( r > \frac{2}{5} \) in (1.7) is easily explained by this parabolic heuristic: under the limiting values \( x \sim r^t \), \( y \sim r^t \) and \( x \sim \frac{r^t}{2} \), the largest value of \( b \) (\( \sim \frac{2}{27} \), namely, the largest window in the \( x \) variable) is attained when \( r \sim \frac{2}{17} \). See more about this fact in figure 3. In a similar direction, (1.9) in the case of the \( \text{KP-II} \) model, the superposition of the half planes \( \bigcup_{\sigma \in \mathbb{R}} \Omega_{2,2}(t) \) (see section 3) describes the exterior of a parabolic region with border

\[
x + \frac{y^2}{4t} = \text{const.} \cdot t.
\]

The limiting value in (1.7) for \( r = 3 \), \( b = \frac{1}{4} \) (maximal rectangular \( y \) side of (1.6)) shows that the \( y \) coordinate is asymptotically limited by the parabola (1.15), since \( \Omega(t) = \{ (x,y) : |x| \leq r^t, |y| \leq t^t \} \) and if \( x = r^t \) in (1.15) yields \( r^t + \frac{y^2}{4t} = \text{const.} \cdot t \) implying that \( |y| \sim t \). From this perspective \( \text{KP} \) models seem to be geometrically restricted, see figure 3. Probably this fact is related to the family of line solitons of \( \text{KP-II} \)

\[
u(x,y,t) = \frac{c^2}{2} \text{sech} \left( \frac{c}{2} (x + dy) - (c^2 + d^2) t \right),
\]

which have its crests on the straight lines \( x + dy = (c^2 + d^2)t \). These crests describe the same family of parabolas when \( t \) is fixed and \( d \) slides. Recall that line solitons do not fit in theorem 1.1 since they have infinite mass. However, they satisfy its conclusion, as (1.12) of theorem 1.2.

Later, Hayashi and Naumkin [13] improved their previous results and consider the actual KP model. They describe for the first time the asymptotics of small KP solutions in high regularity.
Figure 3. Schematic figure depicting the rectangle $\Omega_1(t)$ from theorem 1.1 obtained when $r \sim 3$ (shadow region on the left), and $r \sim \frac{5}{3}$ (shadow region on the right), in the KP-II case. Here, $\tilde{x} = \frac{x}{t^{1/3}}$ and $\tilde{y} := \frac{y}{t^{1/3}}$ are the standard self-similar variables for KP, see e.g. [58, p 1015]. Additionally, the parabolic regions $\tilde{x} \sim -\tilde{y}^2 + t^{2/3}$ (right) and $\tilde{x} \sim -\tilde{y}^2$ (right) from [15, 58] and (1.15) are depicted, showing agreement with the limiting values $r \sim \frac{5}{3}$ and $r = 3$, which correspond to the cases where (in terms of theorem 1.1), the largest and widest possible parabolic regions are contained, respectively. Finally, the first quadrant limit points of the sets $\Omega_1(t)$, $r$ variable (see the bullets), which are of the form $(\tilde{x}, \tilde{y}) \sim (t^{\frac{1}{3(1+\epsilon)}}, t^{\frac{1}{3(1+\epsilon)}})$, with $\frac{5}{3} < r < 3$, follow the curve $\tilde{x}^3 \tilde{y} \sim t^{1/3}$ above depicted. The upper limit $|y| \sim t^{1/3}$ in the case $r \sim 3$, which corresponds to $|y| \sim t$, is in agreement with the existence of lumps moving in the $y$ direction with speeds $\sim 1$, see (1.21).

weighted Sobolev spaces. This description is done in terms of $\| \partial_t u(t) \|_{L^\infty}$, which decays as $1/t$ if the initial data $\partial_x^{-1} u_0$ is sufficiently small in $H^7 \cap H^{5.4}$. In 2017, Harrop-Griffits, Ifrim and Tataru [12] showed scattering of small data for KP-I in Galilean invariant spaces, with precise asymptotics, by using testing with wave packets. These spaces are also larger than the ones considered by Hayashi and Naumkin. Additionally, an improved description of the dynamics in terms of the parabolic and its complementary region was obtained in this work. Note also that our space $L^2$ in theorem 1.1 is a Galilean invariant space. Klein and Saut [25] developed precise numerical simulations describing the KP dynamics, including the long time behaviour of lumps and instability properties. In another relevant topic in KP models, Isaza, Linares and Ponce [19] proved propagation of regularity in KP-II and recently in KP-I by Levandosky [64]. On the other hand, many authors have considered the KP equations via ISTs. See e.g. the extensive description given in the monograph by Konopelchenko [28, chapter 2]. In [62], precise asymptotics are given for KP-II in the case of small data with eight derivatives in $L^1 \cap L^2$. In the KP-I case, a similar description is established for the case of small data in the Schwartz class [59]. See [23, 24] for a complete description of the state of art via the use of rigorous ISTs.

Now we introduce our third result. One may wonder if there is additional decay in the case of KP-I, where the data is in the energy space $E^1(\mathbb{R}^2)$ and not only in $L^2(\mathbb{R}^2)$. It turns out that this question seems quite hard, and it is deeply related to the fact that small KP-I lumps can travel arbitrarily fast, see remark 1.2 for details. In this paper, we show decay for $\partial_x u$ and
Figure 4. Schematic figure depicting the sets $\tilde{\Omega}_1(t)$ and $\tilde{\Omega}_2(t)$ defined in (1.16) and (1.17) respectively, in the centred case $l_1 = l_2 = 0$ taking as reference $\Omega_1(t)$.

The principal result of this paper, in terms of the complexity of its proof, is the following

**Theorem 1.3.** Assume now any initial data $u_0$ in the second energy space $E^2(\mathbb{R}^2)$ for KP-I. Then the corresponding solution $u$ satisfies

$$\liminf_{t \to \infty} \int_{\tilde{\Omega}_1(t)} (\partial_x^{-1} \partial_y u)^2 (x, y, t) \, dx \, dy = 0,$$

and

$$\liminf_{t \to \infty} \int_{\tilde{\Omega}_2(t)} (\partial_y u)^2 (x, y, t) \, dx \, dy = 0.$$  

**Remark 1.1.** Theorem 1.3, together with theorem 1.1, states that all the members of the energy space norm decay locally in space along sequences of time, provided the initial data is in the subclass $E^2(\mathbb{R}^2)$. The family of KP-I lumps belong to this class, see remark 1.2. It is not clear to us whether or not the condition $u \in E^2(\mathbb{R}^2)$ is also necessary. We conjecture that theorem 1.3 is valid for data in $E^1(\mathbb{R}^2)$ only. Such extension would require the introduction of a set of unknown virial identities for KP-I.
We finish the presentation of our results with a corollary, obtained from theorems 1.1 and 1.3 in conjunction with [3, lemma 4]:

**Corollary 1.2.** Any solution \( u \in E^2(\mathbb{R}^2) \) to KP-I satisfies

\[
\liminf_{t \to +\infty} \|u(t)\|_{L^p(K)} = 0, \quad 2 \leq p \leq 6,
\]

for any compact set \( K \subseteq \mathbb{R}^2 \).

1.3. **Lumps and line solitons**

KP-I has lump solutions, namely solutions of the form

\[
u(x, y, t) = Q_c(x - ct, y), \quad c > 0.
\]

The function \( Q_c \) is given as

\[
Q_c(x, y) := cQ(\sqrt{cx}, cy), \quad Q \text{ is the fixed profile}
\]

\[
Q(x, y) = 12\partial_x^2 \log (x^2 + y^2 + 3) = \frac{24(3 - x^2 + y^2)}{(x^2 + y^2 + 3)^2}.
\]

This profile satisfies the elliptic nonlocal PDE in \( \mathbb{R}^2 \)

\[
\partial_x^2 Q - Q + \frac{1}{2} Q^2 - \partial_x^{-2} \partial_y^2 Q = 0, \quad Q \in E^1(\mathbb{R}^2).
\]

(1.20)

From this formula one clearly sees that lumps are in \( E^2(\mathbb{R}^2) \) and have zero mean. Lumps were first found by Satsuma and Ablowitz [57] via an intricate limiting process of complex-valued algebraic solutions to KP-I. Also, multi-lump solutions were found in the same reference. de Bouard and Saut [4–6] described, via PDE techniques, qualitative properties of KP-I lumps solutions which are also ground states (not necessarily equal to \( Q \)). In particular, these ground states decay as \( 1/(x^2 + y^2) \) as \( \| (x, y) \| \) tends to infinity. However, whether or not lump solutions \( Q \) are ground states is still an unknown open problem in the field.

Important advances in the understanding of this problem were obtained by Liu and Wei in the aforementioned work [38]. By using linear Bäcklund transformation techniques, they proved the orbital stability of the lump \( Q \) in the space \( E^1(\mathbb{R}^2) \), hinting that \( Q \) it is probably the unique (modulo translations) ground state of (1.20).

**Remark 1.2.** Lumps can travel quite fast. Indeed, recall the invariances of the KP equations (1.1) [3]:

1. Shifts:

\[
u(x, y, t) \mapsto u(x + x_0, y + y_0, t + t_0).
\]

2. Scaling: if \( c > 0, \)

\[
u(x, y, t) \mapsto cu(c^{1/2}x, cy, c^{-1/2}t).
\]

3. Galilean invariance: for any \( \beta \in \mathbb{R} \), if \( u(x, y, t) \) is solution to KP, then

\[
\begin{align*}
u(x, y, t) &\mapsto u(x - \beta^2 t - \beta(y - 2\beta t), y - 2\beta t, t), \quad \text{KP-I} \\
\nu(x, y, t) &\mapsto u(x + \beta^2 t + \beta(y - 2\beta t), y - 2\beta t, t), \quad \text{KP-II},
\end{align*}
\]

define new solutions to KP.
Using this, one can construct a moving lump solution, of arbitrary size and speed: for any \( \alpha, \beta \in \mathbb{R} \),

\[
Q_{c, \alpha}(x, y, t) := Q_c \left(x - \alpha t - \beta^2 t - \beta (y - 2\beta t), y - 2\beta t \right) \tag{1.21}
\]
is a moving lump solution of KP-I with speed \( \beta \in \mathbb{R} \). Moreover, a simple computation reveals that (see (1.2)–(1.4))

\[
M[Q_{c, \beta}] = e^{1/2}M[Q],
E[Q_{c, \beta}] = E[Q_c] + \beta^2 M[Q_c] = e^{3/2} E[Q] + \beta^2 e^{1/2} M[Q],
P[Q_{c, \beta}] = P[Q_c] - \beta M[Q_c] = cP[Q] - \beta e^{1/2} M[Q] = -\beta e^{1/2} M[Q].
\]

The previous computations imply that small lumps (in the energy space) may have arbitrarily large speeds \( \beta \gg 1 \) and \( c \ll 1 \) such that \( \beta^2 e^{1/2} \ll 1 \). Precisely, this property of lumps ensures that no monotonicity on the right of the solution may hold in the KP-I case, in the sense of Martel and Merle (note that this is not the case in KP-II, see (1.13) and de Bouard–Martel [3]). A general solution may contain a small fast lump solution that will appear after some time on the right of the main part of the solution. This simply implies that an arbitrary portion of mass on the right of the plane cannot be almost preserved in time. In that sense, the proof of theorem 1.3 overcomes this difficulty by adding one more degree of regularity, and using the momentum as a trigger of decay, instead of considering only the mass as in standard monotonicity properties.

In view of the results by de Bouard–Saut [5], and de Bouard–Martel [3], it is known that KP-II has no lump structures. However, any KdV soliton becomes an (infinite energy) line-soliton solution of KP. Such structure is stable in the KP-II case, as proved by Mizumachi and Tzvetkov [47], and asymptotically stable in a series of deep works by Mizumachi [45, 46]. Moreover, multi-line-soliton structures are known to exist via IST methods [24], and nothing is known about their stability in rigorous terms. In the KP-I case, it is transversally unstable, as proved by Rousset and Tzvetkov [55, 56]. Finally, in order to study the stability of the KdV soliton under the flow of the KP-II equation, Molinet et al [50] proved global well-posedness in \( L^2(\mathbb{R} \times \mathbb{T}) \) and \( L^2(\mathbb{R}^2) \) (see also Koch and Tzvetkov [27]).

1.4. Idea of proofs

As mentioned before, the proof of theorems 1.1 and 1.3 is based in the introduction and manipulation of two new virial identities. The first of these identities is an extension to KP of an early but fundamental estimate worked out by Gustavo Ponce and the second author in [51, 52], where the KdV and Benjamin–Ono models were considered. This first estimate has been improved step by step to the point of being able to consider data only in the ‘energy space’; this last achievement allowed us to describe the local dynamics in the Zakharov–Kuznetsov (ZK) model, see [43]. Some virial identities are independent of the integrability of the equation, consequently, are valid for plenty of dispersive models, see e.g. [9, 29, 35, 36, 40–42, 44] and references therein.

The idea goes as follows. For technical reasons, we will need the introduction of six scaling factors \( \lambda_i(t) \), three attenuation functions \( \eta_k(t) \), and two shift parameters \( \rho_l(t) \), plus one completely unrelated shift/scaling factor \( \theta(t) \). The key point is to have in mind the following array of inequalities:

\[
\lambda_1(t) \ll \lambda_2(t) \ll \lambda_4(t) \ll \lambda_3(t) \ll \lambda_5(t) \ll \lambda_6(t), \tag{1.22}
\]
and

$$\eta_1(t) \gg \eta_2(t) \gg \eta_3(t) .$$  \hspace{1cm} (1.23)

(See section 2.4 for further details.) Here, $a \gg b$ for $a, b > 0$ means that for all possible constant $C > 0$, one has $a \geq Cb$. We will describe here the proof in the KP-I case, since the KP-II one is in some sense simpler. The starting point is the classical Kato smoothing functional related to the mass of the solution. For a two-variable function $\Phi$ which is essentially of the form $\tanh x \tanh y$, we introduce the Kato type functional

$$K(t) = \frac{1}{\eta_1(t)} \int_{\mathbb{R}^2} u^2(x, y, t) \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy,$$

with $\tilde{x} = x - \rho_1(t)$ and $\tilde{y} = y - \rho_2(t)$ as translated variables. For simplicity, one can assume $\tilde{x} = x$ and $\tilde{y} = y$, having in mind that the results in theorems 1.1 and 1.3 do hold even in the shifted case, with the necessary modifications.

This functional (with different weights associated to the soliton dynamics) has been widely used in the KdV setting by Martel and Merle [40–42], and in the ZK setting in [9, 43], because it presents nice Kato smoothing properties (sometimes also called monotonicity properties, because suitable nonnegative quantities are almost preserved in time). However, in the KP setting these are not available anymore. The influence of the transversal term $\partial_x^{-1} \partial_y^2 u$ in the equation becomes evident and poses strong difficulties to the obtention of monotonicity properties a la Martel–Merle. Up to now, no such property was known in the KP-I case, and in the KP-II case it holds under suitable assumptions (see [3]). One of the main objectives of this paper is, via modified virial identities, recover a new monotonicity property for KP-I.

The first virial estimate, related to $K(t)$, reads as follows: it is possible to bound $\partial_t u L^2$ locally in space, provided one has local $L^2$ control on $\partial_x^{-1} \partial_t u$ and local $L^2$ control on $u$. Indeed, for constants $\sigma_0, C_0 > 0$,

$$\frac{\sigma_0}{r \log r} \int_{\mathbb{R}^2} \left( \partial_t u \right)^2 \sech^2 \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \sech^2 \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy \leq - \frac{dK(t)}{dr} + \frac{C_0}{r \log r} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_y u \right)^2 \sech^2 \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \sech^2 \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + C_0 \eta_1(t) \int_{\mathbb{R}^2} \left| u \right|^3 \sech^2 \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \sech^2 \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + K_{\text{int}}(t) , \hspace{1cm} (1.24)$$

with $K_{\text{int}}(t) \in L^1(\{ t \gg 1 \})$, see proposition 4.1 for further details. This is done with a correct choice of $\lambda_3, \lambda_4$ and $\eta_3$ chosen in order to maximise the area of decay as much as possible. Note that the terms on the right hand side (r.h.s) are still uncontrolled. Indeed, the term with $\partial_x^{-1} \partial_y u$ appears precisely with the wrong sign because of the lack of monotonicity in KP-I, and it is hard to control even in the small data case.

In this paper we overcome this problem by adding one more degree of regularity to the initial data. This is done by working in the second energy space $E^2(\mathbb{R}^2)$ introduced by Kenig [21]. We define the functional

$$\mathcal{F}(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} (u \partial_x^{-1} \partial_y u)(x, y, t) \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy,$$

valid for solutions in $E^1(\mathbb{R}^2)$. Contrary to the previous case ($K(t)$ and $\Phi$), here $\Psi$ is similar to $\sech^3 \tanh y$. This functional is inspired in the one introduced by de Bouard–Martel [3] for the KP-II setting, which was $\int y u \partial_x^{-1} \partial_y u \, dx \, dy$. For this functional we prove the following. Let $u \in E^2(\mathbb{R}^2)$ be a globally defined solution to KP-I. Under appropriate choices of $\lambda_3, \lambda_4$ and $\eta_2$
above, and having in mind the chain of inequalities in (1.22) and (1.23), there exist constants \( \sigma_1, C_1 > 0 \) such that
\[
\frac{\sigma_1}{\log t} \int_{\mathbb{R}^2} (\partial_{x}^{-1} \partial_y u)^2 \sech^2 \left( \frac{x}{\lambda_3(t)} \right) \sech^2 \left( \frac{y}{\lambda_4(t)} \right) \, dx \, dy
\leq -\frac{d}{dt} J(t) + \frac{C_1}{\log t} \int_{\mathbb{R}^2} |u|^3 \sech^2 \left( \frac{x}{\lambda_3(t)} \right) \sech^2 \left( \frac{y}{\lambda_4(t)} \right) \, dx \, dy + J_{\text{int}}(t),
\]
(1.25)
where \( J_{\text{int}}(t) \in L^1(\{t \gg 1\}) \); see proposition 5.1 for more details. Proving (1.25) is not direct, mainly because the equation formally satisfied by \( \partial_{x}^{-1} \partial_y u \) contains several new high order nonlocal terms which enter to the virial estimate. These terms are hard to control unless one has better regularity estimates. One of those terms is \( \partial_{x}^{-2} \partial_y^2 u \), which is only well-defined in a proper subspace of the energy space \( E^1(\mathbb{R}^2) \). Here is when the space \( E^2(\mathbb{R}^2) \) in (1.5) enters to the scene: bounded in time solutions in this space make (1.25) possible.

The previous virial estimate allows us to bound the term \( \partial_{x}^{-1} \partial_y u \) in (1.24) just in terms of local \( L^2 \) and \( L^3 \) terms on \( u \). Note that in principle, these last terms do not integrate in time. Therefore, to conclude we need suitable integrability bounds on those terms, independent of the size of the data.

Here is when a third virial estimate is needed. Generalising our previous results for the ZK equation [43], we introduce the \( L^1 \) functional
\[
\mathcal{I}(t) = \frac{1}{\eta_5(t)} \int_{\mathbb{R}^2} u(x,y,t) \psi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{y}{\lambda_6(t)} \right) \, dx \, dy,
\]
with \( q > 1 \), and \( \lambda_5, \lambda_6 \) and \( \eta_5 \) well-chosen, always following (1.22) and (1.23). The weight \( \psi \) behaves as tanh and \( \phi \) as sech. At this moment the quadratic character of KP enters and \( \psi \) is introduced by technical reasons; the \( \phi \) contains several new high order nonlocal terms which enter to the virial estimate. These terms are hard to control unless one has better regularity estimates. One of those terms is \( \partial_{x}^{-2} \partial_y^2 u \), which is only well-defined in a proper subspace of the energy space \( E^1(\mathbb{R}^2) \). Here is when the space \( E^2(\mathbb{R}^2) \) in (1.5) enters to the scene: bounded in time solutions in this space make (1.25) possible.

The previous virial estimate allows us to bound the term \( \partial_{x}^{-1} \partial_y u \) in (1.24) just in terms of local \( L^2 \) and \( L^3 \) terms on \( u \). Note that in principle, these last terms do not integrate in time. Therefore, to conclude we need suitable integrability bounds on those terms, independent of the size of the data.

In principle, \( \mathcal{I}(t) \) is not well-defined in \( L^2 \); this makes very important to consider the weights above introduced. The weight with power \( q \) is introduced by technical reasons; the most important contribution is always related to the weight with scaling \( \lambda_5(t) \). For this functional, we prove, regardless one considers KP-I or KPI-II, the following: assume data only in \( L^2(\mathbb{R}^2) \) for KP-II, or data in \( E^1(\mathbb{R}^2) \) in the KP-I case. Then there exists \( \sigma_2 > 0 \) and \( \varepsilon_0 > 0 \) small enough such that, for any \( t \gg 1 \), one has the bound
\[
\frac{\sigma_2}{t} \int_{\mathbb{R}^2} u^q \sech^2 \left( \frac{x}{\lambda_5(t)} \right) \sech^2 \left( \frac{y}{\lambda_6(t)} \right) \, dx \, dy \leq \frac{d\mathcal{I}}{dt}(t) + \mathcal{I}_{\text{int}}(t),
\]
(1.26)
provided \( q = 1 + \varepsilon_0 \) in \( \mathcal{I}(t) \) (6.1), and where \( \mathcal{I}_{\text{int}}(t) \) are terms that belong to \( L^1(\{t \gg 1\}) \) (see proposition 6.1). Corollary 1.1 follows directly from this estimate. A similar estimate was proved in the ZK case in [43]. However, there are some key differences in (1.26) in the KP and ZK cases. At first sight, a standard estimate as in [43] reveals that (1.26) should not hold in
the KP case because the nonlocal operator $\partial_{x}^{-1}$ does not send $L^{2}$ into $L^{2}$, leading to a critical loss of time decay. Fortunately, we will benefit of a new decay relation between the nonlocal operator $\partial_{x}^{-1}$ and well-chosen virial weights $\psi$ and $\phi$. After some work, it will be proven that an estimate as in (1.26) is also valid in the KP setting. The price to pay is the more restrictive condition $r > \frac{5}{4}$ in $\Omega_{1}(t)$. Recall that $r$ is a parameter related to the $y$ variable, but also reduces the $x$ variable, and $r$ larger means that the size of $\Omega_{1}(t)$ is smaller in $x$ than in the ZK case. This difference is an intriguing feature obtained from the proof, and we believe that it is not caused by technical reasons, but because of weaker spatial decay properties in KP.

Estimates (1.25) and (1.26) are, as far as we know, new in the KP setting. We emphasise that no smallness condition is needed for proving both estimates, only the validity of data in the corresponding energy spaces. Moreover, combining cumulatively (1.26), (1.25) and (1.24) (in that order) allows us to conclude that (see lemma 7.1 and section 7.2) every solution to KP-I in $E^{2}(\mathbb{R}^{2})$ satisfy the integrability properties

\[
\frac{1}{t \log t} \int_{\mathbb{R}^{2}} |u|^{3} (x, y, t) \text{sech}^{2} \left( \frac{\tilde{x}}{\lambda_{3}(t)} \right) \text{sech}^{2} \left( \frac{\tilde{y}}{\lambda_{4}(t)} \right) \, dx \, dy \in L^{1} \left( \{ t > 1 \} \right),
\]

\[
\frac{1}{t \log t} \int_{\mathbb{R}^{2}} \left( \partial_{x}^{-1} \partial_{x} \right)^{2} (x, y, t) \text{sech}^{2} \left( \frac{\tilde{x}}{\lambda_{3}(t)} \right) \text{sech}^{2} \left( \frac{\tilde{y}}{\lambda_{4}(t)} \right) \, dx \, dy \in L^{1} \left( \{ t > 1 \} \right),
\]

\[
\frac{1}{t \log t} \int_{\mathbb{R}^{2}} \left( \partial_{x} \partial_{y} \right)^{2} (x, y, t) \text{sech}^{2} \left( \frac{\tilde{x}}{\lambda_{1}(t)} \right) \text{sech}^{2} \left( \frac{\tilde{y}}{\lambda_{2}(t)} \right) \, dx \, dy \in L^{1} \left( \{ t > 1 \} \right).
\]

In view of (1.24), the last estimate on the gradient can be recast as the first monotonicity property obtained for solutions to KP-I. Unlike the gKdV and ZK setting, it requires a combination of the functionals $K(t)$, $J(t)$ and $I(t)$. This combination has no apparent sign, in concordance with the existence of fast small lumps. Following the ideas in [52], (1.26) leads to (1.10) in theorem 1.1 (valid for data only in $L^{2}$ or $E^{1}(\mathbb{R}^{2})$), and the three estimates in (1.27), combined with Propositions 4.1 and 5.1 ((1.24) and (1.25)), lead to theorem 1.3.

Finally, (1.11) and (1.12) of theorem 1.2 are proved using a modification of the technique introduced in [53], which follows a different procedure, starting with another virial functional in the spirit of $K(t)$, but with a completely different choice of parameters.

We finish this long introduction by mentioning that we believe that the techniques introduced in the proofs above mentioned will be very useful for the proof of asymptotic stability of the KP-I lump (1.21) as well.

1.5. Organisation of this paper

This work is organised as follows. In section 2 we introduce the necessary preliminary results needed for the proof of main theorems. Section 3 is devoted to the proof of (1.11) and (1.12) of theorem 1.2. Section 4 deals with a virial estimate for the derivative $\partial_{x}u$ of the solution in the KP-I case. Section 5 is devoted to a new virial estimate for the variable $\partial_{x}^{-1} \partial_{y}u$ in the KP-I case. Section 6 is concerned with the proof of theorem 1.1 of $L^{2}$-decay in both KP models. Finally, section 7 ends with the proof of theorem 1.3 and corollary 1.1.

2. Preliminaries

This section is devoted to establish or recall a series of relevant results needed for the proofs of theorems 1.1, 1.3 and 1.11.
2.1. Interpolation inequalities

By $H^s_1(\mathbb{R}^2)$, $s \in \mathbb{R}$, we denote the classical Sobolev spaces equipped with the norm

$$
\|f\|_{H^s_1} := \left\| (1 + |\xi|^2)^{-\frac{s}{2}} \mathcal{F}(f)(\xi) \right\|_{L^{2}_{\xi}},
$$

(2.1)

where $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $\mathcal{F}$ denotes the Fourier transform. The space $H^\infty_1(\mathbb{R}^2)$, is defined as the intersection of all the $H^s_1(\mathbb{R}^2)$. For the proof of the following interpolation inequality, see L. Molinet, J-C Saut, and N Tzvetkov [48].

**Lemma 2.1.** For $2 \leq p \leq 6$ there exist $c > 0$ such that for every $f \in H^\infty_1(\mathbb{R}^2)$

$$
\|f\|_{L^p} \leq c \|f\|_{L^2}^{\frac{4}{6-p}} \|\partial_x f\|_{L^2}^{\frac{4-p}{6-p}} \|\partial_x^{-1} \partial_y f\|_{L^2}^{\frac{p-6}{6-p}}.
$$

(2.2)

This estimate pass to the limit and by density holds true in the energy space $E^1(\mathbb{R}^2)$.

2.2. Local and global well-posedness

First we recall the standard $L^2$ global well-posedness proved by Bourgain [2] in his seminal paper on KP models; consider KP-II. If the initial datum $u_0 \in L^2(\mathbb{R}^2)$, the the solution $u(t)$ exists globally in time in the $L^2$ space, with continuity and boundedness in time thanks to the mass conservation. This result is applied several times along this work, the most important being lemma 6.1.

Let us first mention some of the consequences of the global well-posedness of Ionescu et al [17]. Assume that $u_0 \in E^1(\mathbb{R}^2)$; then the corresponding solution to KP-I satisfies

$$
\sup_{t \geq 0} \left( \|u(t)\|_{L^2_x}^2 + \|\partial_x u(t)\|_{L^2_x}^2 + \|\partial_x^{-1} \partial_y u(t)\|_{L^2_x} \right) \leq C,
$$

(2.3)

for a fixed constant $C = C(\|u_0\|_{E^1})$. Estimate (2.3) is a standard consequence of the conservation of mass (1.2) and energy (1.3), together with (2.2) for $p = 3$.

More involved is the following boundedness property obtained from Kenig [21]. Recall the space $E^2(\mathbb{R}^2)$ introduced in (1.5). We also need the auxiliary space, defined for $s \in \mathbb{R}$:

$$
Y_s := \left\{ u \in L^2(\mathbb{R}^2) : \|u\|_{Y_s} := \|u\|_{L^2_x}^2 + \|\langle D_x \rangle^s u\|_{L^2_x}^2 + \|\partial_x^{-1} \partial_y u\|_{L^2_x} < \infty \right\}.
$$

Recall that, in terms of Fourier variables, $\mathcal{F}(\langle D_x \rangle^s u)(\xi, \mu) = (1 + |\xi|^2)^{s/2} \mathcal{F}(u)(\xi, \mu)$.

**Lemma 2.2.** Let $u_0 \in E^2(\mathbb{R}^2)$ and let $u$ be the global in time solution of KP-I with initial data $u(t = 0) = u_0$. Then one has

$$
\sup_{t \geq 0} \left( \|\partial_x u(t)\|_{L^2_x}^2 + \|\partial_x^2 u(t)\|_{L^2_x}^2 + \|\partial_x^{-1} \partial_y^2 u(t)\|_{L^2_x} \right) \leq C,
$$

(2.4)

with constant only dependent on $\|u_0\|_{E^2}$.

Indeed, in [21], for any $s \in (\frac{1}{2}, 2]$ a unique global solution $u \in C(\mathbb{R}_+, Y_s) \cap L^\infty(\mathbb{R}_+, E^2(\mathbb{R}^2))$ was constructed from initial data $u_0 \in E^2(\mathbb{R}^2)$. ($E^2(\mathbb{R}^2)$ is continuously embedded in $Y_s$ for $s \in (\frac{1}{2}, 2)$). Moreover, one has $\sup_{t \geq 0} \|u(t)\|_{E^2} \leq C(\|u_0\|_{E^2})$. Finally, from [21], we also know that $\|\partial_x u(t)\|_{L^2_x} \leq \|u(t)\|_{E^2}$. These arguments provide (2.4).
2.3. Weighted functions adapted to KP

Let \( \phi \) be a smooth even and positive function such that

(i) \( \phi' \leq 0 \) on \( \mathbb{R}^+ \),
(ii) \( \phi|_{[0,1]} = 1 \), \( \phi(x) = e^{-x} \) on \( [2, \infty) \), \( e^{-x} \leq \phi(x) \leq 3e^{-x} \) on \( \mathbb{R}^+ \).
(iii) The derivatives of \( \phi \) satisfy:

\[
|\phi'(x)| \leq c \phi(x) \quad \text{and} \quad |\phi''(x)| \leq c \phi(x),
\]

for some positive constant \( c \).

Let

\[
\psi(x) := \int_0^x \phi(s) \, ds, \quad \psi'(x) = \phi(x).
\] (2.5)

Then \( \psi \) is an odd function such that \( \psi(x) = x \) on \( [-1,1] \) and \( |\psi(x)| \leq 3 \). Notice that

\[
\psi(x) = x \quad \text{on} \quad [-1,1],
\]

\[
|\psi(x)| \leq 3, \quad e^{-|x|} \leq \phi(x) \leq 3e^{-|x|} \quad \text{on} \quad \mathbb{R}.
\] (2.6)

Finally, recall that \( \psi \) is odd and \( \phi \) is even. The following result is key for the proof of theorem 1.1.

**Lemma 2.3.** Let \( a, b > 0 \) and \( x_0 \in \mathbb{R} \). Then

\[
\partial_{x}^{-1} \left( \psi \left( \frac{x - x_0}{a} \right) \phi \left( \frac{x - x_0}{b} \right) \right)(x)
\]

is Schwartz in \( x \) and the following inequality holds

\[
\left| \partial_{x}^{-1} \left( \psi \left( \frac{x - x_0}{a} \right) \phi \left( \frac{x - x_0}{b} \right) \right)(x) \right| \leq 9b \phi \left( \frac{x - x_0}{b} \right), \quad \text{for all} \quad x \in \mathbb{R}.
\] (2.7)

**Proof.** First, note that \( \psi \left( \frac{x - x_0}{a} \right) \phi \left( \frac{x - x_0}{b} \right) \) is odd with respect to \( x_0 \) and Schwartz in \( x \). By virtue of (2.6), the term \( \partial_{x}^{-1} \left( \psi \left( \frac{x - x_0}{a} \right) \phi \left( \frac{x - x_0}{b} \right) \right)(x) \) is then well-defined because

\[
\partial_{x}^{-1} \left( \psi \left( \frac{x - x_0}{a} \right) \phi \left( \frac{x - x_0}{b} \right) \right)(x) = \int_{-\infty}^{x} \psi \left( \frac{s - x_0}{a} \right) \phi \left( \frac{s - x_0}{b} \right) \, ds
\]

\[
= \int_{-\infty}^{x-x_0} \psi \left( \frac{s}{a} \right) \phi \left( \frac{s}{b} \right) \, ds.
\]

With no loss of generality, we can assume \( x_0 = 0 \) below. To obtain the bound in (2.7) we shall examine the following cases:

(I) Case \( x \leq 0 \) : by definition we have

\[
\partial_{x}^{-1} \left( \psi \left( \frac{x}{a} \right) \phi \left( \frac{x}{b} \right) \right)(x) = \int_{-\infty}^{x} \psi \left( \frac{s}{a} \right) \phi \left( \frac{s}{b} \right) \, ds = b \int_{-\infty}^{x/b} \psi \left( \frac{bs}{a} \right) \phi(s) \, ds.
\]
Thus,
\[
\left| \partial_{x}^{-1} \left( \psi \left( \frac{x}{a} \right) \phi \left( \frac{x}{b} \right) \right) (x) \right| \leq b \int_{-\infty}^{x/b} \left| \psi \left( \frac{bx}{a} \right) \right| \phi (s) \, ds \\
\leq 9b \int_{-\infty}^{x/b} e^{s} \, ds = 9be^{-|x|/x} \leq 9b \phi \left( \frac{x}{b} \right).
\]

(II) Case $x \geq 0$: since the function $\psi \left( \frac{x}{a} \right) \phi \left( \frac{x}{b} \right)$ is odd, we have from the definition of $\partial_{x}^{-1}$ that
\[
\left| \partial_{x}^{-1} \left( \psi \left( \frac{x}{a} \right) \phi \left( \frac{x}{b} \right) \right) (x) \right| = -\int_{x}^{\infty} \psi \left( \frac{s}{a} \right) \phi \left( \frac{s}{b} \right) \, ds \\
\leq \int_{x}^{\infty} \left| \psi \left( \frac{s}{a} \right) \right| \phi \left( \frac{s}{b} \right) \, ds \\
\leq 9b \int_{x/b}^{\infty} e^{-s} \, ds \leq 9b \phi \left( \frac{x}{b} \right).
\]

Thus, combining both cases we obtain the bound
\[
\left| \partial_{x}^{-1} \left( \psi \left( \frac{x}{a} \right) \phi \left( \frac{x}{b} \right) \right) (x) \right| \leq 9b \phi \left( \frac{x}{b} \right), \quad x \in \mathbb{R}.
\]

This finishes the proof.

2.4. Scaling and attenuation parameters

We introduce now a set of time-dependent parameters that allow us to capture decay properties in the long time regime. We classify them in three classes: scaling parameters $\lambda_j(t)$, and $j = 1, \ldots, 6$, attenuation parameters $\eta_1(t), \eta_2(t)$, and $\eta_3(t)$, and a sort of shift/scaling parameter $\theta(t)$.

But first, we recall the exponents introduced in the definition of the set $\Omega_1(t)$ in (1.6).

Let $b, r > 0$ and $q \in (1, 2)$ numbers satisfying the conditions
\[
b \leq \frac{2}{2 + q + r} < \frac{2}{3 + r}, \quad \frac{5}{3} < r < 3.
\]

Note that for $\frac{5}{3} < r < 3$ we easily have
\[
br \leq \frac{2r}{3 + r} < 1.
\]

Always consider $t \gg 1$. The key idea to have in mind is that
\[
\lambda_1 (t) \ll \lambda_2 (t) \ll \lambda_4 (t) \ll \lambda_5 (t) \ll \lambda_6 (t), \\
\eta_1 (t) \ll \eta_2 (t) \ll \eta_3 (t).
\]

First, we shall introduce $\lambda_5 (t)$, $\lambda_6 (t)$ and $\eta_3 (t)$, as follows:
\[
\lambda_5 (t) := \frac{p}{\log t} \quad \text{and} \quad \eta_3 (t) := \ell \log t,
\]

where $p$ is a positive constant satisfying
\[
p + b = 1.
\]
For some technical reasons, we introduce
\[ \tilde{\lambda}_5(t) := \frac{\lambda_5'(t)}{\lambda_5(t) + \lambda_5(t)} \lambda_5(t) < \lambda_5(t). \] (2.12)
We also consider
\[ \lambda_6(t) := \lambda_6'(t) \quad \text{where} \quad \frac{5}{3} < r < 3. \] (2.13)
Then,
\[ \frac{\lambda_5'(t)}{\lambda_5(t)} \sim \frac{\eta_3'(t)}{\eta_3(t)} \sim \frac{1}{t} \quad \text{for} \quad t \gg 1. \]
Also,
\[ \lambda_5'(t) = \frac{1}{t^{1-k} \log t} \left( b \log t - 1 \right) \quad \text{and} \quad \lambda_5(t) \eta_3(t) = t. \]
These parameters will be important to understand the \( L^2 \)-decay of KP solutions, mainly in section 6.

Based on the previous parameters \( \lambda_5(t), \lambda_6(t) \) and \( \eta_3(t) \), we construct \( \eta_2(t), \lambda_3(t) \) and \( \lambda_4(t) \).
Let \( \eta_2(t), \lambda_3(t), \lambda_4(t) \) locally bounded functions defined as follows: for \( q = 1 + \varepsilon_0 > 1, \varepsilon_0 > 0 \) small enough,
\[ \eta_2(t) := t^{1-p} \log^4 t, \quad \lambda_3(t) := \frac{\lambda_5(t)}{\log t} = \frac{\rho}{\log^2 t}, \]
\[ \lambda_4(t) := \frac{\rho}{\log^3 t} \ll \lambda_6(t) = \frac{\rho}{\log^3 t}, \quad \left( r > \frac{5}{3} \right); \] (2.14)
such that
\[ \lambda_3(t) \gg \lambda_4(t), \quad \eta_2(t) \lambda_4(t) = t \log t, \quad \eta_2(t) \lambda_3(t) = t \log^2 t. \] (2.15)
Note that
\[ \frac{1}{\eta_2(t) \lambda_4(t)} \notin L^1 \left( \{ t \gg 1 \} \right), \quad \frac{1}{\eta_2(t) \lambda_3(t)} \in L^1 \left( \{ t \gg 1 \} \right). \] (2.16)
The previously defined parameters will be important in section 5.
Finally, let \( \lambda_1(t), \lambda_2(t), \) and \( \eta_1(t) \) be time-dependent functions by
\[ \eta_1(t) := t^{1-p} \log^6 t, \quad \lambda_1(t) := \frac{\lambda_3(t)}{\log^2 t} = \frac{\rho}{\log^2 t}, \]
\[ \lambda_2(t) := \frac{\rho}{\log^3 t} \ll \lambda_4(t) = \frac{\rho}{\log^3 t}; \] (2.17)
such that
\[ \lambda_2(t) \gg \lambda_1(t), \quad \eta_1(t) \lambda_1(t) = t \log t, \quad \eta_1(t) \lambda_2(t) = t \log^2 t. \] (2.18)
Finally, note that
\[ \frac{1}{\eta_1(t) \lambda_1(t)} \notin L^1 \left( \{ t \gg 1 \} \right), \quad \frac{1}{\eta_1(t) \lambda_2(t)} \in L^1 \left( \{ t \gg 1 \} \right). \]
Next, we introduce the shift parameters needed for \( \Omega_1(t) \) in (1.6). For constants \( \ell_1, \ell_2 \in \mathbb{R} \) and \( m_1, m_2 \geq 0 \), let
\[ \rho_1(t) := \ell_1 t^{m_1}, \quad \rho_2(t) := \ell_2 t^{m_2}. \] (2.19)
For further purposes, we need the following additional restrictions on $m_1$ and $m_2$:

$$0 \leq m_1 < 1 - \frac{b}{2} (r+1), \quad 0 \leq m_2 < 1 - \frac{1}{2} b(q + 2 - r). \tag{2.20}$$

Note that $r > 1$, therefore $\frac{1}{2}(r+1) < r$. Consequently, $\frac{b}{2}(r+1) < b < 1$. On the other hand, since $b \leq \frac{2}{2 + q + r}$,

$$\frac{1}{2} b(q + 2 - r) \leq \frac{2 + q - r}{2 + q + r} < \frac{1}{2} + q < 1.$$

Finally, fixed $m_2$ such that $0 \leq m_2 < 1 - \frac{1}{2} b(3 - r)$, as in (1.7), for $q = 1 + \varepsilon_0$ and $\varepsilon_0 > 0$ small enough, one has $0 \leq m_2 < 1 - \frac{1}{2} b(q + 2 - r)$. These last facts corroborate that the conditions (2.20) are well-defined.

### 3. Proof of decay properties (1.11) and (1.12)

#### 3.1. Proof of (1.12)

In this section we present a result that extends the ones obtained by Kenig and Martel [22] for bounded in time finite energy solutions of the KP-II equation. More precisely, in [22] it was proved strong decay on the right half-plane, in the following form: for any $\beta > 0$,

$$\lim_{t \to \infty} \int_{\Omega > \beta t} u^2(x, y, t) \, dx \, dy = 0,$$

provided the initial data is small enough in $L^1 \cap L^2$. In this section we complement this result for solutions of the KP-II equation that are uniformly bounded in the energy space $E^1(\mathbb{R}^2)$ without size assumptions on the initial data and in different regions of the plane, as we describe below.

**Proof of (1.12).** The proof is standard, see e.g. [39]. We consider the standard function $\varphi_L$,

$$\varphi_L(s) = \frac{2}{\pi} \arctan(\varepsilon t^L), \quad s \in \mathbb{R}, \quad L > 0.$$  

Recall the notation for $\Omega_{2,2}(t)$ in (1.9). Consider an increasing function $\beta(t)$, with $\beta(t) \gtrsim t$. Next, for $t, t_0 > 0$ fixed we define the functional

$$L_{t_0}(t) = \int_{\mathbb{R}^2} u^2(x, y, t) \varphi_L(x + \sigma_3 y - \beta(t) - \beta(t_0)) \, dx \, dy, \quad \sigma_3 \in \mathbb{R}.$$

The functional above satisfies the identity

$$\frac{d}{dt} L_{t_0}(t) = -3 \int_{\mathbb{R}^2} (\partial_x u)^2 \varphi'_L \, dx \, dy + \int_{\mathbb{R}^2} u^2 \varphi''_L \, dx \, dy - \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_x u)^2 \varphi'_L \, dx \, dy$$

$$+ 2 \sigma_3 \int_{\mathbb{R}^2} u \partial_x^{-1} \partial_x u \varphi'_L \, dx \, dy + \frac{2}{3} \int_{\mathbb{R}^2} u^3 \varphi'_L \, dx \, dy - \beta'(t) \int_{\mathbb{R}^2} u^2 \varphi'_L \, dx \, dy.$$

Since $|\varphi'_L(s)| \leq \frac{1}{L} \varphi'_L(s), \forall s \in \mathbb{R}$,

$$\left| \int_{\mathbb{R}^2} u^2 \varphi''_L \, dx \, dy \right| \leq \frac{1}{L^2} \int_{\mathbb{R}^2} u^2 \varphi'_L \, dx \, dy.$$

Without loss of generality, we assume $\sigma_3 \neq 0$, since the case $\sigma_3 = 0$ was addressed by Kenig–Martel [22].
Hence,
\[
\left| \int_{\mathbb{R}^2} u \partial_t^{-1} \partial_x u \varphi_L' \, dx \, dy \right| \leq \delta \int_{\mathbb{R}^2} u^2 \varphi_L' \, dx \, dy + \frac{1}{4\delta} \int_{\mathbb{R}^2} (\partial_t^{-1} \partial_x u)^2 \varphi_L' \, dx \, dy, \quad \text{for all } \delta > 0.
\]

Thus, by combining the estimates above we get,
\[
\frac{d}{dt} \mathcal{L}_n(t) \leq -3 \int_{\mathbb{R}^2} (\partial_t u)^2 \varphi_L' \, dx \, dy - \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_t u)^2 \varphi_L' \, dx \, dy + \frac{1}{L^2} \int_{\mathbb{R}^2} u^2 \varphi_L' \, dx \\
+ 2|\sigma_3| \delta \int_{\mathbb{R}^2} u^2 \varphi_L' \, dx \, dy + \frac{|\sigma_3|}{2\delta} \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_t u)^2 \varphi_L' \, dx \, dy - \beta'(t) \int_{\mathbb{R}^2} u^2 \varphi_L' \, dx \, dy \\
+ \frac{2AC}{3} \left( \tau \int_{\mathbb{R}^2} u^2 \varphi_L' \, dx \, dy + \frac{1}{4\tau^2} \int_{\mathbb{R}^2} (\partial_t u)^2 \varphi_L' \, dx \, dy \right),
\]

where we have used the lemma A.1 in appendix to handle the term with the cubic power, and \( A \) is a constant only depending on the energy of the initial data. Now, we choose \( \beta, \delta, \tau, \) and \( \beta'(t) \) satisfying
\[
\tau = \left( \frac{AC}{18} \right)^{1/2}, \quad \delta = \frac{|\sigma_3|}{2}, \quad \beta'(t) \geq \frac{1}{L^2} + \frac{2AC}{3} \left( \frac{AC}{18} \right)^{1/3} > 0, \quad (3.1)
\]
to obtain \( \mathcal{L}_n'(t) \leq 0 \). In particular, this implies that \( \mathcal{L}_n(t) \) is a decreasing function, which allow us to guarantee that \( \mathcal{L}_n(t_0) \leq \mathcal{L}_n(2) \), for any \( t_0 > 0 \). This condition reads
\[
\int_{\mathbb{R}^2} u^2 (x,y,t_0) \varphi_L (x + \sigma_3 y - \beta(t_0)) \, dx \, dy \leq \int_{\mathbb{R}^2} u^2 (x,y,2) \varphi_L (x + \sigma_3 y - \beta(2) - \beta(t_0)) \, dx \, dy.
\]

The condition (3.1) show us that for any \( L > 0 \), the function \( \beta(t) \gtrsim_L t \). This condition implies that pointwise in \( (x,y) \),
\[
\lim_{t_0 \to +\infty} \varphi_L (x + \sigma_3 y - \beta(t_0) - \beta(2)) = 0.
\]

Thus, by dominated convergence theorem,
\[
\limsup_{t_0 \to +\infty} \int_{\mathbb{R}^2} u^2(x,y,t_0) \varphi_L (x + \sigma_3 y - \beta(t_0)) \, dx \, dy \leq \lim_{t_0 \to +\infty} \int_{\mathbb{R}^2} u^2(x,y,2) \varphi_L (x + \sigma_3 y - \beta(2) - \beta(t_0)) \, dx \, dy = 0.
\]

Therefore, for all \( \sigma_3 \neq 0 \) and \( \beta(\cdot) \) as in (3.1), one concludes (1.12). \( \square \)

As can be evidenced in the result proved above, the technique is quite restricted to the structure of the KP-II equation. Also, the geometry seems to be quite restricted by the technique, since we only are able to show decay in half-spaces moving to the r.h.s of the plane. However, the same technique does not provide any information for subsets in the l.h.s of the plane, even for far regions. Also, the internal structure of the mass in the KP-I equation does not offer any information for decay of solutions as in the KP-II case.

As mentioned above KP-II seems to be geometrically restricted. In this sense we can see that, for a time \( t \), the half planes described in (1.9) combined with (3.1) (\( \sigma_3 \) as a parameter) are such that, for some \( N_0 > 0 \), they circumscribe the parabola
\[
x - N_0t = -\frac{1}{4r^2}.
\]
Indeed, if $N_0 = \frac{1}{4} t^2 + \frac{24c}{7} \left( \frac{4c}{7} t \right)^{1/3}$, the line $x + \sigma_3 y = (N_0 + \sigma_3^2) t$ associated to (3.1) is tangential to the parabola $x - N_0 t = -\frac{1}{2} y^2$ in the point $(N_0 - \sigma_3^2 t, 2\sigma_3 t)$. We believe that (1.12) still valid in the right exterior parabolic region of (3.2) but with a virial modification that includes the fact that the half parabolic regions are self-similar in time. The proof of this fact is an interesting open problem.

In another direction, one can see the centered rectangular region (1.6) in the $y$ coordinate is asymptotically limited by the parabola (3.2). In fact in (1.6) for $r = 3$, $b = \frac{1}{2}$ (maximal rectangular $y$ side) we have $\Omega(t) = \{(x, y) : |x| \leq t^3, |y| \leq t\}$. Now if $x = t^3$ in (3.2) we have $t^3 - N_0 t = -\frac{1}{2} y^2$ that implies $|y| = \sqrt{N_0 t^2 - 4 t^3} \sim t$.

3.2. Proof of (1.11)

In this section we present new results that extend those considered in [43] for solutions of the ZK equation. In this sense we introduce a variation of the functional used in [43] to show the decay to zero in regions that are far away from the solitonic region.

**Proof.** We will provide the proof for solutions of the KP-II equation. Nevertheless, the same arguments apply for solutions of the KP-I equation as well. Without loss of generality, in what follows we can assume $x \leq 0, y \in \mathbb{R}$. The remaining cases are treated in a similar fashion by changing the respective cut-off function.

Let us assume that $u$ is a solution to the KP-II equation with $u_0 \in E^1(\mathbb{R}^2)$, and for some $K_0 > 0$,

$$\sup_{t \in \mathbb{R}} \left( \|u(t)\|_{L_{\infty}^0}^2 + \|\partial_x^{-1} \partial_x u(t)\|_{L_{\infty}^0}^2 + \|\partial_x u(t)\|_{L_{\infty}^0}^2 \right) \leq K_0. \quad (3.3)$$

(Without loss of generality we assume $u \neq 0$.) Let us consider $\epsilon, \gamma > 0$ and $t, t_0, t_1 \in \mathbb{R}$, such that $t_1 > t_0 > 2$, $t \geq 2$ and

$$2\theta K_0 \int_{t_0}^{\infty} \frac{ds}{s \log^{1+\gamma} s} < \epsilon, \quad (3.4)$$

where $\theta$ is a positive constant to be fixed.

As part of our analysis we will require a suitable weight function $\chi$ satisfying the following properties:

(i) $\chi \in C^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$.

(ii) For $s \in \mathbb{R}$,

$$\chi(s) = \begin{cases} 1, & s \leq -1 \\ 0, & s \geq 0. \end{cases}$$

(iii) $\chi'(s) < 0$, for $s \in (-1, 0)$.

(iv) $|\chi^{(k)}(s)| \leq 2^k$ for $k = 1, 2, 3$, for all $s \in \mathbb{R}$.

Let $\mu(t) = \frac{c_1}{2} t \log^{1+\gamma} t$, where $c_1, \gamma > 0$. For $\chi$ satisfying the properties described above, we set

$$\varphi(x, y, t) := \chi \left( \frac{\sigma_1 x + \sigma_2 y + \frac{1}{2} (\mu(t) + \mu(t_1))}{\mu(t)} \right),$$

where $\sigma_1, \sigma_2 \in \mathbb{R}$ are not both zero.
As a next step, we define the functional
\[ M(t) := \frac{1}{2} \int_{\mathbb{R}^2} u^2(x,y,t) \varphi(x,y,t) \, dx \, dy, \quad \text{for} \quad t > 0. \]

The functional \( M \) is clearly well defined for solutions of the KP-II equations [2].

The corresponding virial-type identity associated to \( M \) is given by
\[
\frac{d}{dt} M(t) = \int_{\mathbb{R}^2} u \partial_t u \varphi \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} u^2 \partial_x^2 \varphi \, dx \, dy
\]
\[ = M_1(t) + M_2(t). \]

Since \( u \) is a solution to the KP-II equation, we obtain after integrating by parts
\[
M_1(t) = -\frac{3}{2} \int_{\mathbb{R}^2} (\partial_t u)^2 \partial_x \varphi \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} u^2 \partial_x^2 \varphi \, dx \, dy - \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_t u)^2 \partial_x \varphi \, dx \, dy
\]
\[ + \int_{\mathbb{R}^2} u \partial_x^{-1} \partial_t u \partial_x \varphi \, dx \, dy + \frac{1}{3} \int_{\mathbb{R}^2} u^3 \partial_x \varphi \, dx \, dy
\]
\[ = -\frac{3\sigma_1}{2\mu(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 \chi \, dx \, dy + \frac{\sigma_1}{2\mu(t)} \int_{\mathbb{R}^2} u^2 \chi'' \, dx \, dy - \frac{\sigma_1}{2\mu(t)} \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_t u)^2 \chi \, dx \, dy
\]
\[ + \frac{\sigma_2}{\mu(t)} \int_{\mathbb{R}^2} u \partial_x^{-1} \partial_t u \chi \, dx \, dy + \frac{\sigma_1}{3\mu(t)} \int_{\mathbb{R}^2} u^3 \chi' \, dx \, dy
\]
\[ = M_{1,1}(t) + M_{1,2}(t) + M_{1,3}(t) + M_{1,4}(t) + M_{1,5}(t). \]

Since (3.3) holds we obtain
\[
|M_{1,1}(t)| \leq \frac{6|\sigma_1|K_0}{c_1 \log^{1+\gamma} t} \in L^1 \{ t \gg 1 \}.
\]

A similar argument produces
\[
|M_{1,2}(t)| \leq \frac{32K_0|\sigma_1|^3}{c_1^3 \log^{3(1+\gamma)} t} \in L^1 \{ t \gg 1 \},
\]
\[
|M_{1,3}(t)| \leq \frac{2K_0|\sigma_1|}{c_1 \log^{1+\gamma} t} \in L^1 \{ t \gg 1 \},
\]

and
\[
|M_{1,4}(t)| \leq \frac{4K_0|\sigma_2|}{c_1 \log^{1+\gamma} t} \in L^1 \{ t \gg 1 \}.
\]

To handle \( M_{1,5} \) we apply lemma 2.1 to obtain
\[
M_{1,5}(t) = \frac{\sigma_1}{3\mu(t)} \int_{\mathbb{R}^2} |u|^3 |\chi'| \, dx \, dy
\]
\[ \leq \frac{4c|\sigma_1|K_0}{3c_1 \log^{1+\gamma} t} \in L^1 \{ t \gg 1 \},
\]
where \( c \) is the positive constant provided by the embedding (2.2).

Finally, the term \( M_2 \) satisfies
\[
M_2(t) = \frac{\mu'(t)}{2\mu(t)} \int_{\mathbb{R}^2} u^2 \left( \frac{1}{2} - \frac{\sigma_1 \chi + \sigma_2 \gamma + \frac{1}{2} (\mu(t) + \mu(t_1))}{\mu(t)} \right) \chi' \, dx \, dy.
\]
Nevertheless, by construction of $\chi$, we have

$$\frac{3}{2} \chi' \leq \chi' \left( \frac{1}{2} \left( \frac{\sigma_1 x + \sigma_2 y + \frac{1}{2} (\mu(t) + \mu(t_1))}{\mu(t)} \right) \right) \leq \frac{1}{2} \chi' \leq 0,$$

which implies $M_2(t) \leq 0$ for all $t > 2$.

Thus, we obtain

$$\frac{d}{dt} M(t) \leq \frac{\theta K_0}{t \log^{1+\gamma} t},$$

where

$$\theta := \max \left\{ \frac{6|\sigma_1|}{c_1}, \frac{32|\sigma_1|^3}{c_1^3}, \frac{2|\sigma_2|}{c_1}, \frac{4|\sigma_1|}{3c_1}, \frac{4c_1|\sigma_1|}{3c_1} \right\} > 0,$$

which after integrating in the time interval $[t_0, t_1]$, and using (3.4), allows us to obtain

$$M(t_1) \leq M(t_0) + \int_{t_0}^{t_1} \frac{K_0}{t \log^{1+\gamma} t} dt \leq M(t_0) + \frac{1}{2} \varepsilon.$$  

From the last inequality above we see that

$$\int_{\Omega_{2,1}(t_1)} u^2(x, y, t_1) \, dx \, dy \leq \int_{\mathbb{R}^2} u^2(x, y, t_0) \chi \left( \frac{\sigma_1 x + \sigma_2 y + \frac{1}{2} (\mu(t_0) + \mu(t_1))}{\mu(t_0)} \right) \, dx \, dy + \varepsilon,$$

where we shall remind that

$$\Omega_{2,1}(t) := \left\{ (x, y) \in \mathbb{R}^2 : \sigma_1 |x| + \sigma_2 |y| \geq t \log^{1+\gamma} t \right\}.$$

Next, we make $t_1 \rightarrow +\infty$ to obtain by the dominated convergence theorem,

$$\lim_{t_1 \rightarrow +\infty} \int_{\Omega_{2,1}(t_1)} u^2(x, y, t_1) \, dx \, dy \leq \lim_{t_1 \rightarrow +\infty} \int_{\mathbb{R}^2} u^2(x, y, t_0) \chi \left( \frac{\sigma_1 x + \sigma_2 y + \frac{1}{2} (\mu(t_0) + \mu(t_1))}{\mu(t_0)} \right) \, dx \, dy + \varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{t \rightarrow +\infty} \int_{\Omega_{2,1}(t)} u^2(x, y, t) \, dx \, dy = 0.$$

\[ \square \]

**Remark 3.1.** A proof of strong decay for $\partial_x u$ and $\partial_x^{-1} \partial_y u$ as in (1.11) is an interesting open problem. In the case of regions around the origin, the answers are provided in (1.18) and (1.19). This is possible thanks to the chain of virial identities that we will prove in forthcoming sections. In the case of the region $\Omega_2(t)$, these arguments need important improvements to hold true.
4. **Kato virial estimates in the KP-I case**

4.1. Preliminaries

Let $\Phi(x,y)$ be the smooth function defined as (see (2.5))

$$\Phi(x,y) := \psi(x) \phi(y).$$  (4.1)

Let $\lambda_1(t), \lambda_2(t)$, and $\eta(t)$ be the time-dependent functions introduced in section 2.4. We consider now the classical mass virial functional [3]

$$K(t) := \frac{1}{\eta_1(t)} \int_{\mathbb{R}^2} u^2 \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) d\tilde{x}d\tilde{y},$$  (4.2)

with

$$\tilde{x} := x - \rho_1(t), \quad \tilde{y} := y - \rho_2(t),$$  (4.3)

and $\rho_1(t), \rho_2(t)$ defined in (2.19).

In order to give a first insight on how to show (1.19), we will prove the following estimate.

**Proposition 4.1.** Let $u_0 \in E^1(\mathbb{R}^2)$. Let $u$ be the corresponding global solution to (1.1) with initial data $u(t=0) = u_0$ in the KP-I case. Let $K(t)$ be the functional defined in (4.2). Then, under the setting of section 2.4, one has that $K(t)$ is well-defined and bounded in time, and for a fixed $\sigma_0, C_0 > 0$,

$$\frac{\sigma_0}{t \log t} \int_{\mathbb{R}^2} (\partial_2 u)^2 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) d\tilde{x}d\tilde{y} \leq - \frac{dK(t)}{dt} + \frac{C_0}{t \log t} \int_{\mathbb{R}^2} (\partial_2^{-1} \partial_1 u)^2 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) d\tilde{x}d\tilde{y} + K_{\text{int}}(t) \quad (4.4)$$

with $K_{\text{int}}(t) \in L^1(\{t \gg 1\})$.

**Remark 4.1.** We believe that in the KP-II setting, estimate (4.4) should be better, in the sense that one should have

$$\frac{\sigma_0}{t \log t} \int_{\mathbb{R}^2} (\partial_1 u)^2 + (\partial_2^{-1} \partial_1 u)^2 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) d\tilde{x}d\tilde{y} \leq - \frac{dK(t)}{dt} + \frac{C_0}{t \log t} \int_{\mathbb{R}^2} |u|^3 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) d\tilde{x}d\tilde{y} + K_{\text{int}}(t) \quad (4.5)$$

Indications leading to this conjecture are given in de Bouard–Martel [3], where similar estimates are proved in the case of bounded solutions in $E^1(\mathbb{R}^2)$, and where the KP-II Cauchy problem in $E^1(\mathbb{R}^2)$ was showed globally well-posed. However, in our case some key uniform in time bounds in $E^1(\mathbb{R}^2)$ are missing. The proof of (4.5) is an interesting open problem.

Estimate (4.4) shows that we need two additional estimates to control the remaining terms on the right of (4.4), since the classical Kato smoothing estimate fails in the KP case. This will be done in next sections. For the moment, we prove proposition 4.1.
4.2. Virial computations

The following computations are somehow classical, see [3, 4] for instance. We have,

\[
\frac{d}{dt} K(t) = \frac{2}{\eta_1(t)} \int_{\mathbb{R}^2} u \partial_t u \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dxdy - \frac{\eta_1(t)}{\eta_1(t)} \int_{\mathbb{R}^2} u^2 \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dxdy
\]

\[
+ \frac{1}{\eta_1(t)} \int_{\mathbb{R}^2} u^2 \partial_t \left( \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \right) \, dxdy
\]

\[
= K_1(t) + K_2(t) + K_3(t).
\]

We easily bound \( K_2(t) \) as follows:

\[
|K_2(t)| \lesssim \frac{1}{\eta_1(t)} \|u_0\|_{L^2_{\eta}}^2 \in L^1 \{ t \gg 1 \},
\]

since \( \eta_1(t) \gg \log^2 t \). On the one hand, \( K_3(t) \) is bounded as follows. First of all,

\[
K_3(t) = \frac{1}{\eta_1(t)} \int_{\mathbb{R}^2} u^2 \partial_t \left( \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \right) \, dxdy
\]

\[
= - \frac{\lambda'_1(t)}{\lambda_1(t) \eta_1(t)} \int_{\mathbb{R}^2} u^2 \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \partial_t \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dxdy
\]

\[
- \frac{\lambda'_2(t)}{\lambda_2(t) \eta_1(t)} \int_{\mathbb{R}^2} u^2 \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \partial_t \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dxdy
\]

\[
- \frac{\rho'_1(t)}{\lambda_1(t) \eta_1(t)} \int_{\mathbb{R}^2} u^2 \partial_t \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dxdy
\]

\[
= : K_{3,1}(t) + K_{3,2}(t) + K_{3,3}(t) + K_{3,4}(t).
\]

The terms \( K_{3,1}(t) \) and \( K_{3,2}(t) \) are easily bounded: we have

\[
|K_{3,1}(t)| \lesssim \|u_0\|^2_{L^2_{\eta}} \in L^1 \{ t \gg 1 \},
\]

and the same holds for \( K_{3,2}(t) \). On the other hand, the term \( K_{3,3}(t) \) is bounded as follows: from (2.17),

\[
|K_{3,3}(t)| = \left| \frac{\rho'_1(t)}{\lambda_1(t) \eta_1(t)} \int_{\mathbb{R}^2} u^2 \partial_t \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dxdy \right| \lesssim \frac{|\ell_1 m_1| \|u_0\|^2_{L^2_{\eta}}}{t^{2-m_1 \log t}},
\]

which integrates in time since \( m_1 < 1 \) in (2.20). Bounding the term \( K_{3,4}(t) \) follows the same idea. Consequently,

\[
K_3(t) \in L^1 \{ t \gg 1 \}.
\]
On the other hand, $K_1(t)$ is treated as follows. Using (1.1),

$$K_1(t) = \frac{2}{\eta_1(t)} \int_{\mathbb{R}^2} u \partial_t u \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy$$

$$= \frac{2}{\eta_1(t)} \int_{\mathbb{R}^2} u \left( -\partial^2_{\tilde{x}} u - \kappa \partial^{-1}_{\tilde{x}} \partial^2_{\tilde{y}} u - u \partial_t u \right) \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy$$

$$= \frac{2}{\eta_1(t)} \int_{\mathbb{R}^2} \partial^2_{\tilde{x}} u \partial_t u \left( \tilde{u} \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \right) \, dx \, dy$$

$$+ \frac{2\kappa}{\eta_1(t)} \int_{\mathbb{R}^2} \partial^{-1}_{\tilde{x}} \partial_t u \partial_t \left( \tilde{u} \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \right) \, dx \, dy$$

$$+ \frac{2}{3\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} \partial^3_{\tilde{x}} \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy \quad (4.9)$$

As for the term $K_{1,1}(t)$, we have

$$K_{1,1}(t) = -\frac{1}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 \partial_x \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy$$

$$+ \frac{2}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} \partial^2_{\tilde{x}} u \partial_t \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy$$

$$= -\frac{3}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 \partial_x \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy$$

$$+ \frac{1}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} \partial^3_x \Phi \left( \frac{\tilde{x}}{\lambda_1(t)} - \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy.$$
We have from (2.9)
\[
\left| \frac{1}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_y u \right)^2 \partial_x \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \right| \leq \frac{1}{t \log t} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_y u \right)^2 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy.
\]

This last term cannot be estimated properly, unless we have independent estimates. Later, we will prove that (see (5.2)),
\[
\left| \frac{1}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_y u \right)^2 \partial_x \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \right| \in L^1(\{t \gg 1\}),
\]
but for the moment we will save this term for later purposes. Collecting (4.11), we conclude
\[
|K_{1,2}(t)| \lesssim \frac{1}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_y u \right)^2 \partial_x \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + K_{1,2,\text{int}}(t),
\]
with $K_{1,2,\text{int}}(t) \in L^1(\{t \gg 1\})$. Finally, $K_{1,3}$ is simply bounded as follows:
\[
|K_{1,3}(t)| \lesssim \frac{2}{3 \eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} u^3 \partial_y \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy \leq \frac{1}{t \log t} \int_{\mathbb{R}^2} |u|^3 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy.
\]
Gathering this last estimate, (4.12) and (4.10), we conclude that for some fixed constant $c > 0$,
\[
K_1(t) \leq -\frac{3}{\eta_1(t) \lambda_1(t)} \int_{\mathbb{R}^2} \left( \partial_x u \right)^2 \partial_x \Phi \left( \frac{\tilde{x}}{\lambda_1(t)}, \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + \frac{1}{t \log t} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_y u \right)^2 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + \frac{c}{t \log t} \int_{\mathbb{R}^2} |u|^3 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + K_{1,\text{int}}(t),
\]
with $K_{1,\text{int}}(t) \in L^1(\{t \gg 1\})$. Coming back to (4.6), and collecting (4.7), (4.8) and (4.13), we conclude that for some fixed constant $\sigma_0, C_0 > 0$,
\[
\frac{\sigma_0}{t \log t} \int_{\mathbb{R}^2} \left( \partial_x u \right)^2 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy \leq -\frac{dK(t)}{dt} + \frac{C_0}{t \log t} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_y u \right)^2 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + \frac{C_0}{t \log t} \int_{\mathbb{R}^2} |u|^3 \phi \left( \frac{\tilde{x}}{\lambda_1(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy + K_{\text{int}}(t),
\]
with $K_{\text{int}}(t) \in L^1(\{t \gg 1\})$. In the last identity we have also used (2.18) and (4.1). This last identity is nothing but (4.4). The proof is complete.
5. Virial estimate for $\partial_x^{-1}\partial_t u$ in the KP-I case

Proposition 4.1 gives us a control on the derivative $\partial_t u$, but depending on $\partial_x^{-1}\partial_t u$ and $u$ locally in $L^2$ and $L^3$, respectively. Obtaining additional control on these variables for arbitrary large data is key to get a decay property for solutions to KP-I.

In order to prove this result, and following a similar argument as in the previous section, we will introduce a new virial identity, in the spirit of [3]. In this reference, the authors consider compact solutions bounded in time in $E^1(\mathbb{R}^2)$ in the KP-II case. Some particular complications arise when dealing with solutions in the energy space, leading us to assume more regularity in the data than expected.

5.1. Preliminaries

Recall the functions $\lambda_3(t)$, $\lambda_4(t)$ and $\eta_2(t)$ introduced in (2.14), and for $\phi$ and $\psi$ in (2.5), let

$$\Psi(x,y) := \phi(x) \psi(y).$$

(5.1)

Note the difference with the choice in (4.1).

For a solution of the KP-I equation in $E^1(\mathbb{R}^2)$, we set (as in [4]) $v := \partial_x^{-1}\partial_t u$, for simplicity of notation. Consider the functional

$$\mathcal{J}(t) := \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} uv \Psi \left( \frac{x}{\lambda_3(t)}, \frac{y}{\lambda_4(t)} \right) \, dx \, dy.$$ 

Clearly $\mathcal{J}$ is well-defined and bounded uniformly in time for data in $E^1(\mathbb{R}^2)$, since

$$\sup_{t \in \mathbb{R}} \left( ||u(t)||_{L^2} + ||\partial_t u(t)||_{L^2} + ||v(t)||_{L^2} \right) \leq C,$$

for data in $E^1(\mathbb{R}^2)$. However, since we are assuming data in $E^2(\mathbb{R}^2)$, we also have from lemma 2.2

$$\sup_{t \in \mathbb{R}} \left( ||\partial_t u(t)||_{L^2} + ||\partial^2_t u(t)||_{L^2} + ||\partial_x^{-1}\partial_t v(t)||_{L^2} \right) \leq C.$$ 

These bounds are key to the proof of decay. Indeed, we will prove that

**Proposition 5.1.** Let $u \in E^2(\mathbb{R}^2)$ be a globally defined solution to KP-I. Then there exist constants $\sigma_1, C_1 > 0$ such that

$$\frac{\sigma_1}{t \log t} \int_{\mathbb{R}^2} u^2 (\partial_j \Psi) \left( \frac{x}{\lambda_3(t)}, \frac{y}{\lambda_4(t)} \right) \, dx \, dy \leq -\frac{d}{dt} \mathcal{J}(t) + \frac{C_1}{t \log t} \int_{\mathbb{R}^2} |u|^3 \phi \left( \frac{x}{\lambda_3(t)} \right) \phi \left( \frac{y}{\lambda_4(t)} \right) \, dx \, dy + \mathcal{J}_{\text{int}}(t),$$

(5.2)

where $\mathcal{J}_{\text{int}}(t) \in L^1(\{t \gg 1\}).$

Proving (5.2) for data only in $E^1(\mathbb{R}^2)$ remains an important open question. The proof of proposition 5.1 requires several key estimates, and it will be carried out in next subsection.
5.2. Virial computations

We estimate the variation in time of the functional $J$. In this sense, we have that

$$
\frac{d}{dt} J(t) = \frac{d}{dt} \left( \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u v \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \right)
= -\frac{\eta'_2(t)}{\eta_2^2(t)} \int_{\mathbb{R}^2} u v \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
+ \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_1(uv) \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
+ \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} uv \partial_1 \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
=: J_1(t) + J_2(t) + J_3(t).
$$

(5.3)

In the first place, we have for $t \gg 1$, the following:

$$
|J_1(t)| = \left| \frac{\eta'_2(t)}{\eta_2^2(t)} \int_{\mathbb{R}^2} u v \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \right|
\lesssim \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} |uv| \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
\lesssim \frac{1}{\eta_2(t)} \|u_0\|_{L^2_x} \|v(t)\|_{L^2_y} \lesssim \frac{1}{\eta_2(t)} \in L^1(\{t \gg 1\}),
$$

(5.4)

because of (2.3) and (2.14).

Next, for $J_2(t)$ we have that

$$
J_2(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_1(uv) \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
= \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_1uv \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
+ \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_1uv \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
=: J_{2,1}(t) + J_{2,2}(t).
$$

(5.5)

Recall that if $u$ satisfies (1.1)-I then $v = \partial_x^{-1} \partial_x u$ formally satisfies the equation

$$
\partial_t v + \partial_x^3 v - \partial_x^{-1} \partial_x^2 v + u \partial_x u = 0.
$$

(5.6)
It is clear that (after a density argument), using that \( u \) solves (1.1)-I,

\[
\mathcal{J}_{2,1}(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_x \partial_y \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
= \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \left( \partial_x^{-1} \partial_x^2 u - \partial_x^2 u - \partial_x \nu \right) \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
= \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_x^{-1} \partial_x^2 \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
- \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_x \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
- \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u \partial_x \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
= \mathcal{J}_{2,1,1}(t) + \mathcal{J}_{2,1,2}(t) + \mathcal{J}_{2,1,3}(t).
\]

For the term \( \mathcal{J}_{2,1,1}(t) \), we have that

\[
\mathcal{J}_{2,1,1}(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_x \partial_y \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
- \frac{2}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} v^2 \left( \partial_y \nu \right) \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy.
\]

This term has a good sign, and it will be preserved until the end.

Instead, for \( \mathcal{J}_{2,1,2} \), and integrating by parts once, we have

\[
\mathcal{J}_{2,1,2}(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_x^2 u \partial_x \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
= \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_x^2 u \partial_x \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
+ \frac{1}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} \partial_x^2 \nu \partial_x \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy.
\]

After another integration,

\[
\mathcal{J}_{2,1,2}(t) = -\frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_x \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
- \frac{2}{\eta_2(t) \lambda_3(t) \lambda_4(t)} \int_{\mathbb{R}^2} \partial_x u \partial_x \nu \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy
\]

\[
- \frac{1}{\eta_2(t) \lambda_3^2(t)} \int_{\mathbb{R}^2} \partial_x \nu \left( \partial_x^2 \nu \right) \left( \frac{\hat{x}}{\lambda_3(t)}, \frac{\hat{y}}{\lambda_4(t)} \right) \, dxdy.
\]
Rearranging terms,

\[
\mathcal{J}_{2,1,2}(t) = \frac{1}{2\eta_2(t)\lambda_4(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 (\partial_3 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
- \frac{1}{\eta_2(t)\lambda_3(t)} \int_{\mathbb{R}^2} \partial_z u \partial_3 u (\partial_3 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
- \frac{1}{2\eta_2(t)\lambda_3^2(t)\lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_3^2 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{\eta_2(t)\lambda_3^3(t)\lambda_4(t)} \int_{\mathbb{R}^2} uv (\partial_3^2 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
= : \mathcal{J}_{2,1,2,1}(t) + \mathcal{J}_{2,1,2,2}(t) + \mathcal{J}_{2,1,2,3}(t) + \mathcal{J}_{2,1,2,4}(t)
\] (5.9)

In what follows, we estimate \( \mathcal{J}_{2,1,2,2}(t) \), \( \mathcal{J}_{2,1,2,3}(t) \) and \( \mathcal{J}_{2,1,2,4}(t) \), which are bad terms here. On the other hand, \( \mathcal{J}_{2,1,2,1}(t) \) is a good term, to be saved for later.

First of all, using (2.16) and (2.4), and Cauchy–Schwarz,

\[
|\mathcal{J}_{2,1,2,2}(t)| \lesssim \frac{1}{\eta_2(t)\lambda_3(t)} \|\partial_3 u(t)\|_{L^2} \|\partial_3 u(t)\|_{L^2} \lesssim \frac{1}{\eta_2(t)\lambda_3(t)} \in L^1 \left( \{ t \gg 1 \} \right).
\] (5.10)

Similarly, thanks to the previous estimate and (2.3),

\[
|\mathcal{J}_{2,1,2,3}(t)| \lesssim \frac{1}{\eta_2(t)\lambda_3^2(t)\lambda_4(t)} \in L^1 \left( \{ t \gg 1 \} \right),
\] (5.11)

and

\[
|\mathcal{J}_{2,1,2,4}(t)| \lesssim \frac{1}{\eta_2(t)\lambda_3^3(t)} \in L^1 \left( \{ t \gg 1 \} \right).
\]

We conclude from (5.9) that

\[
\mathcal{J}_{2,1,2}(t) = \frac{1}{2\eta_2(t)\lambda_4(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 (\partial_3 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + \mathcal{J}_{2,1,2,1}(t)
\] (5.12)

with \( \mathcal{J}_{2,1,2,1}(t) \in L^1 \left( \{ t \gg 1 \} \right) \).

Coming back to (5.7), one has

\[
\mathcal{J}_{2,1,3}(t) = - \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u \partial_3 u \Psi \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
= \frac{1}{2\eta_2(t)} \int_{\mathbb{R}^2} u^2 \partial_3 \Psi \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
= - \frac{1}{6\eta_2(t)\lambda_4(t)} \int_{\mathbb{R}^2} u^3 (\partial_3 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{2\eta_2(t)\lambda_3(t)} \int_{\mathbb{R}^2} u^2 v (\partial_3 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy.
\]
The last term above is actually integrable in time. Indeed, by Cauchy’s inequality and (2.16),

\[
\left| \frac{1}{2\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} u^2 v (\partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \right| \\
\leq \frac{1}{4\eta_2(t) \lambda_3(t)} \left\{ \int_{\mathbb{R}^2} u^4 (\partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + \int_{\mathbb{R}^2} v^2 (\partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \right\} \\
\lesssim \frac{C}{\eta_2(t) \lambda_3(t)} \in L^1 \left\{ \{ t > 1 \} \right\},
\]

thanks to lemma 2.1 and (2.3).

Hence, after gathering all the terms we get

\[
J_{2,1}(t) = - \frac{1}{2\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} v^2 (\partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{2\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 (\partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
- \frac{1}{6\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^3 (\partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + J_{2,1,\text{int}}(t),
\]

where \( J_{2,1,\text{int}}(t) \in L^1 \left\{ \{ t > 1 \} \right\} \).

Now, we consider the term \( J_{2,2}(t) \) in (5.5). By virtue of (5.6) we have that

\[
J_{2,2}(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u \partial_\lambda v \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
= \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u (-\partial_\lambda^2 v + \partial_\lambda^{-1} \partial_\lambda^2 v - u \partial_\lambda u) \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
= - \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u \partial_\lambda^2 v \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u \partial_\lambda^{-1} \partial_\lambda^2 v \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
- \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u^2 \partial_\lambda u \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
= J_{2,2,1}(t) + J_{2,2,2}(t) + J_{2,2,3}(t).
\]

First of all, by integration by parts and using \( \partial_\lambda v = \partial_\lambda u \) we obtain

\[
J_{2,2,1}(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_\lambda u \partial_\lambda u u \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} u \partial_\lambda u \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy,
\]

so that,

\[
J_{2,2,1}(t) = - \frac{1}{2\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} (\partial_t u)^2 (\partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
- \frac{1}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} \partial_t u \partial_\lambda u \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{2\eta_2(t) \lambda_3(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_\lambda^2 \partial_\lambda \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy.
\]
Performing similar estimates as in (5.10) and (5.11), one can conclude that \( J_{2,2,1}(t) \) in (5.15) follows the decomposition

\[
J_{2,2,1}(t) = -\frac{1}{2n(2) \lambda_4(t)} \int_{\mathbb{R}^2} (\partial_1 u)^2 (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy + J_{2,2,1,int}(t),
\]

with \( J_{2,2,1,int}(t) \in L^1(\{t \gg 1\}) \). Note that the first term above and the one in (5.12) cancels out. Additionally,

\[
\begin{align*}
J_{2,2,2}(t) &= \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u \partial_x^{-1} \partial_y^2 v \nu \psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy \\
&= \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} \partial_1 u \partial_x^{-1} \partial_1 \nu \psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy \\
&\quad - \frac{1}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u \partial_x^{-1} \partial_y v (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy \\
&=: J_{2,2,2,1}(t) + J_{2,2,2,2}(t).
\end{align*}
\]

Using that \( \partial_1 u = \partial_y \nu \), and integrating by parts, the term \( J_{2,2,2,1}(t) \) is bounded as follows:

\[
J_{2,2,2,1}(t) = \frac{1}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} v \partial_x^{-1} \partial_y \nu (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy \\
+ \frac{1}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} \partial_1 v \nu \psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy.
\]

The first term on the r.h.s above satisfies

\[
\left\| \frac{1}{\eta_2(t) \lambda_3(t)} \left\| v(t) \right\|_{L^2} \left\| \partial_x^{-1} \partial_y \nu (\partial_1 \Psi) \right\|_{L^1} \right\|_{L^\infty} \lesssim \frac{1}{\eta_2(t) \lambda_3(t)} \in L^1(\{t \gg 1\}),
\]

thanks to (2.4) and (2.16). We conclude

\[
J_{2,2,2,1}(t) = -\frac{1}{2n(2) \lambda_4(t)} \int_{\mathbb{R}^2} v^2 (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy + J_{2,2,2,1,int}(t),
\]

with \( J_{2,2,2,1,int}(t) \in L^1(\{t \gg 1\}) \). This term adds up to the one in (5.8).

On the other hand, \( J_{2,2,2,2}(t) \) in (5.17) is treated as follows. Using Young’s inequality,

\[
|J_{2,2,2,2}(t)| \leq \frac{\log t}{2n(2) \lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy \\
+ \frac{1}{2 \log t \eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} (\partial_1 \partial_x^{-1} v)^2 (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy.
\]

The second term above integrates in time. Indeed, from (2.4) and (2.15),

\[
\begin{align*}
&\frac{1}{2 \log t \eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} (\partial_1 \partial_x^{-1} v)^2 (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy \\
&\lesssim \frac{1}{t \log t} \sup_{t \gg 1} \left\| \partial_x^{-1} \partial_y v (\partial_1 \Psi) \right\|_{L^2}^2 \in L^1(\{t \gg 1\}).
\end{align*}
\]

Therefore,

\[
J_{2,2,2,2}(t) \lesssim \frac{1}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_1 \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dxdy + J_{2,2,2,2,int}(t),
\]

with \( J_{2,2,2,2,int}(t) \in L^1(\{t \gg 1\}) \).
with $J_{2,2,2,2,\text{int}}(t) \in L^1(\{t \gg 1\})$. We conclude from (5.18) and (5.19)

$$
J_{2,2,2}(t) \leq -\frac{1}{2\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} v^2 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{c \log t}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + J_{2,2,2,\text{int}}(t),
$$

(5.20)

with $J_{2,2,2,\text{int}}(t) \in L^1(\{t \gg 1\})$ and some constant $c > 0$.

Now, we deal with $J_{2,2,3}(t)$ in (5.14). Integrating by parts,

$$
J_{2,2,3}(t) = -\frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} u^2 \partial_t u \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
= \frac{1}{3\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^3 (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy.
$$

(5.21)

We conclude from (5.16), (5.20) and (5.21) that

$$
J_{2,2}(t) \leq -\frac{1}{2\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} (\partial_x u)^2 (\partial_y \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
- \frac{1}{2\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} v^2 (\partial_y \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{c \log t}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{2\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^3 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + J_{2,2,\text{int}}(t),
$$

with $J_{2,2,2,\text{int}}(t) \in L^1(\{t \gg 1\})$. Adding this result to the one in (5.13) for $J_{2,1}(t)$, we conclude

$$
J_2(t) \leq -\frac{1}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} v^2 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{c \log t}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{1}{6\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^3 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + J_{\text{int}}(t),
$$

(5.22)

where $J_{\text{int}}(t) \in L^1(\{t \gg 1\})$.

Finally, we have that $J_3(t)$ in (5.3) satisfies

$$
J_3(t) = \frac{1}{\eta_2(t)} \int_{\mathbb{R}^2} uv \partial_t \left( \Psi \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \right) \, dx \, dy \\
- \frac{1}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} uv \nabla \Psi \cdot \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \lambda_3'(t) \lambda_4(t) \right) \, dx \, dy \\
- \frac{\rho_1'(t)}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} uv (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
- \frac{\rho_2'(t)}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} uv (\partial_y \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy.
$$
Expanding terms,

\[
\mathcal{J}_3(t) = -\frac{\lambda_1'(t)}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} u v (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_3(t)} \right) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_3(t)} \right) d\tilde{x} d\tilde{y} \\
- \frac{\lambda_2'(t)}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u v (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) d\tilde{x} d\tilde{y} \\
- \frac{\rho_1'(t)}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} u v (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_3(t)} \right) d\tilde{x} d\tilde{y} \\
- \frac{\rho_2'(t)}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u v (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) d\tilde{x} d\tilde{y} \\
=: \mathcal{J}_{3,1}(t) + \mathcal{J}_{3,2}(t) + \mathcal{J}_{3,3}(t) + \mathcal{J}_{3,4}(t).
\]

The terms \(\mathcal{J}_{3,1}(t)\) and \(\mathcal{J}_{3,2}(t)\) are easy to control. Indeed, we have then from (2.3) and (2.14) that combined with the Cauchy–Schwarz inequality yield

\[
|\mathcal{J}_{3,1}(t)| \lesssim \frac{1}{\eta_2(t)} \in L^1(\{t \gg 1\}),
\]

and similar with \(\mathcal{J}_{3,2}(t)\) since \(b < 1\). On the other hand, \(\mathcal{J}_{3,3}(t)\) and \(\mathcal{J}_{3,4}(t)\) require some care. Using (2.14) and (2.20),

\[
|\mathcal{J}_{3,3}(t)| = \left| \frac{\rho_1'(t)}{\eta_2(t) \lambda_3(t)} \int_{\mathbb{R}^2} u v (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_3(t)} \right) d\tilde{x} d\tilde{y} \right| \lesssim \frac{|t_1 m_1|^{p_{m_1}-1}}{t \log^2 t} \in L^1(\{t \gg 1\}),
\]

since \(m_1 < 1\). The estimate for \(\mathcal{J}_{3,4}(t)\) is completely analogous and we skip it. We conclude

\[
\mathcal{J}_3(t) \in L^1(\{t \gg 1\}). \quad (5.23)
\]

Gathering estimates (5.4), (5.22) and (5.23), we conclude in (5.3) that

\[
\frac{d}{dt} \mathcal{J}(t) \leq -\frac{1}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} v^2 (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) d\tilde{x} d\tilde{y} \\
+ \frac{C}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} |u|^3 (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) d\tilde{x} d\tilde{y} \\
+ \frac{C \log t}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} u^2 (\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) d\tilde{x} d\tilde{y} + \mathcal{J}_{int}(t),
\]

where \(\mathcal{J}_{int}(t) \in L^1(\{t \gg 1\})\). Using the identity (see (5.1))

\[
(\partial_t \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) = \tilde{\Phi} \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \tilde{\Phi} \left( \frac{\tilde{y}}{\lambda_4(t)} \right),
\]

and the fact that \(\eta_2(t) \lambda_4(t) = t \log t\), we conclude (5.2).
6. $L^2$ decay in KP-I and KP-II

This section is devoted to the proof of theorem 1.1 and by extension the proof of theorem 1.3, which will be proved in next section. We consider both cases KP-I and KP-II, since the proof does not depend on the sign in the equation (1.1), only on the well-posedness theory available for each model.

Theorem 1.1 will be a consequence of the following integrability result. Recall the set $\Omega_1(t)$ already introduced in (1.6):

$$\Omega_1(t) = \{(x,y) \in \mathbb{R}^2 : |x - \ell_1 r_m| \leq t^p, |y - \ell_2 r_n| \leq \tilde{t}^p\},$$

with $\frac{5}{3} < r < 3$, $0 < b < \frac{2}{3+r}$, $0 \leq m_1 < 1 - \frac{5}{3} (r + 1)$, and $0 \leq m_2 < 1 - \frac{1}{3} b (q+2-r)$.

**Theorem 6.1.** Assume that $u_0 \in L^2(\mathbb{R}^2)$ in the KP-II case, and $u_0 \in E^1(\mathbb{R}^2)$ in the KP-I case. Let $u = u(x,y,t)$ be the corresponding unique solution of the IVP (1.1) with $\kappa = \pm 1$. Then, there exists a constant $c > 0$, such that

$$\int_{(r \geq 1)} \frac{1}{t^2} \left( \int_{\Omega_1(t)} u^2(x,y,t) \, dx \, dy \right) \, dt \leq c.$$

This bound, and a very similar argument to the one performed in [52], allows to conclude theorem 1.1 (1.10) easily. We skip the details.

6.1. Setting

Recall the weighted functions $\psi$ and $\phi$ defined in (2.5).

For $u$ a solution of the KP equation (1.1), we set the functional

$$\mathcal{I}(t) = \mathcal{I}_u(t) := \frac{1}{\eta_3(t)} \int_{\mathbb{R}^2} u(x,y,t) \psi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{y}{\lambda_6(t)} \right) \, dx \, dy,$$

where $x, y$ were defined in (4.3), and for $t \gg 1$, $\lambda_5(t)$, $\lambda_6(t)$ and $\eta_3(t)$ were introduced in section 2.4.

**Lemma 6.1.** For $u \in L^2(\mathbb{R}^2)$, the functional $\mathcal{I}$ is well defined and bounded in time.

**Proof.** By Cauchy–Schwarz inequality we obtain

$$|\mathcal{I}(t)| \leq \frac{1}{\eta_3(t)} \|u(t)\|_{L^2_{\lambda_5}} \|\psi\|_{L^\infty} \left\| \phi \left( \frac{\cdot}{\lambda_5(t)} \right) \phi \left( \frac{\cdot}{\lambda_6(t)} \right) \right\|_{L^1_{\lambda_5}}$$

$$= \frac{\lambda_5^2(t) \lambda_6(t)}{\eta_3(t)} \|u_0\|_{L^2_{\lambda_5}} \|\psi\|_{L^\infty} \|\phi\|_{L^2} \|\phi\|_{L^2}$$

$$\leq \left( \frac{1}{\log^{2+q+r/2} t} \right) \|u_0\|_{L^2_{\lambda_5}} \|\psi\|_{L^\infty} \|\phi\|_{L^2} \|\phi\|_{L^2}.$$

Since (2.8) is satisfied we have

$$\sup_{t \geq 1} |\mathcal{I}(t)| < \infty,$$

which finishes the proof. 

---
6.2. Dynamics for $I(t)$

In what follows, we compute and estimate the dynamics of $I(t)$ in the long time regime. Recall from (2.12),

$$\tilde{\lambda}_5(t) = \frac{\lambda_5^q(t)}{\lambda_5^q(t) + \lambda_5(t)} \lambda_5(t) < \lambda_5(t).$$

Notice that as time tends to infinity, $\tilde{\lambda}_5(t)/\lambda_5(t)$ tends to 1.

**Proposition 6.1.** There exist $\sigma_2 > 0$ and $\varepsilon_0 > 0$ small enough such that, for any $t \geq 2$, one has the bound

$$\frac{\sigma_2}{t} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_0(t)} \right) \, dx \, dy \leq \frac{dI}{dt}(t) + I_{\text{int}}(t),$$

(6.3)

provided $q = 1 + \varepsilon_0$ in $I(t)$ (6.1), and where $I_{\text{int}}(t)$ are terms that belong to $L^1(\{t \gg 1\})$.

Assuming this estimate and lemma 6.1, one concludes theorem 6.1 as in [52]. The rest of the section will be devoted to the proof of proposition 6.1. We have

$$\frac{dI}{dt}(t) = \frac{1}{\eta_5(t)} \int_{\mathbb{R}^2} \partial_t \left( u \phi \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_0(t)} \right) \right) \, dx \, dy$$

$$- \frac{\eta_5(t)}{\eta_5^2(t)} \int_{\mathbb{R}^2} u \psi \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_0(t)} \right) \, dx \, dy$$

$$=: I_1(t) + I_2(t).$$

First, we bound $I_2$, that in virtue of (6.2) the same analysis yields

$$|I_2(t)| \leq \frac{\eta_5(t)}{\eta_5^2(t)} \int_{\mathbb{R}^2} u \psi \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_0(t)} \right) \, dx \, dy$$

$$\lesssim \|u_0\|_{L^2_{x,y}} \left( \lambda_5^q(t) \lambda_0(t) \right)^{1/2} \eta_5^2(t)$$

$$\lesssim \frac{1}{t} \frac{1}{\eta_5(t)} \left( \lambda_5(t) \right)^{-\left(q+r\right)/2} = \frac{1}{t^{2-q-r} \log^{1/2}(q+r)}.$$

(6.5)

From (2.10) and (2.8) we have $b \leq \frac{2}{q+r}$, then $I_2 \in L^1(\{t \gg 1\})$. 

37
Unlike $I_2$, to bound $I_1$ it is required to take into consideration the dispersive part associated to the KP equation. More precisely,

$$
I_1 (t) = \frac{1}{\eta_3 (t)} \int_{\mathbb{R}^2} \partial_t u \psi \left( \frac{x}{\lambda_5 (t)} \right) \phi \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
- \frac{\lambda_5^2 (t)}{\lambda_3 (t) \eta_3 (t)} \int_{\mathbb{R}^2} u \psi' \left( \frac{x}{\lambda_5 (t)} \right) \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{x}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
- q \lambda_5 (t) \eta_3 (t) \int_{\mathbb{R}^2} u \psi \left( \frac{x}{\lambda_5 (t)} \right) \phi \left( \frac{x}{\lambda_3^2 (t)} \right) \phi' \left( \frac{y}{\lambda_6 (t)} \right) \phi' \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
- \frac{\lambda_3^2 (t)}{\lambda_6 (t) \eta_3 (t)} \int_{\mathbb{R}^2} u \psi \left( \frac{x}{\lambda_5 (t)} \right) \left( \frac{x}{\lambda_3^2 (t)} \right) \Phi' \left( \frac{y}{\lambda_6 (t)} \right) \phi' \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
- \frac{r_1 (t)}{\lambda_5 (t) \eta_3 (t)} \int_{\mathbb{R}^2} u \psi' \left( \frac{x}{\lambda_5 (t)} \right) \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
- \frac{r_3 (t)}{\lambda_5 (t) \eta_3 (t)} \int_{\mathbb{R}^2} u \psi \left( \frac{x}{\lambda_5 (t)} \right) \phi' \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
- \frac{r_5 (t)}{\lambda_3 (t) \eta_3 (t)} \int_{\mathbb{R}^2} u \psi \left( \frac{x}{\lambda_5 (t)} \right) \phi' \left( \frac{x}{\lambda_3^2 (t)} \right) \phi' \left( \frac{y}{\lambda_6 (t)} \right) \phi' \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
=: I_{1,1} (t) + I_{1,2} (t) + I_{1,3} (t) + I_{1,4} (t) + I_{1,5} (t) + I_{1,6} (t) + I_{1,7} (t).
$$

Concerning $I_{1,1}$, we have by (1.1) and integration by parts

$$
I_{1,1} (t) = - \frac{1}{\eta_3 (t)} \int_{\mathbb{R}^2} \left( \partial_t^2 u + \kappa \partial_x^{-1} \partial_y^2 u + u \lambda_3 \psi \right) \left( \frac{x}{\lambda_5 (t)} \right) \phi \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
- \frac{1}{\eta_3 (t)} \int_{\mathbb{R}^2} \psi \partial_x^2 u \left( \frac{x}{\lambda_5 (t)} \right) \phi \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
+ \frac{1}{2 \eta_3 (t) \lambda_5 (t)} \int_{\mathbb{R}^2} \partial_x^2 \psi \left( \frac{x}{\lambda_5 (t)} \right) \phi \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
=: I_{1,1,1} (t) + I_{1,1,2} (t) + I_{1,1,3} (t) + I_{1,1,4} (t).
$$

For $I_{1,1,1}$ we have, after integration by parts,

$$
I_{1,1,1} (t) = \frac{1}{\eta_3 (t) \lambda_3^2 (t)} \int_{\mathbb{R}^2} u \psi''' \left( \frac{x}{\lambda_5 (t)} \right) \phi \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
+ \frac{3}{\eta_3 (t) \lambda_3^{2+q} (t)} \int_{\mathbb{R}^2} u \psi'' \left( \frac{x}{\lambda_5 (t)} \right) \phi \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
+ \frac{3}{\eta_3 (t) \lambda_3^{2+q} (t)} \int_{\mathbb{R}^2} u \psi' \left( \frac{x}{\lambda_5 (t)} \right) \phi' \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy
$$

$$
+ \frac{1}{\eta_3 (t) \lambda_3^{2+q} (t)} \int_{\mathbb{R}^2} u \psi \left( \frac{x}{\lambda_5 (t)} \right) \phi''' \left( \frac{x}{\lambda_3^2 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \phi \left( \frac{y}{\lambda_6 (t)} \right) \, dx \, dy.
$$
First, we bound each term using the Cauchy–Schwarz inequality, as follows:

\[
|\mathcal{I}_{1,1,1}(t)| \leq \frac{(\lambda_s(t) \lambda_6(t))^{1/2}}{\eta_3(t) \lambda_5^2(t)} \|u_0\|_{L_2^2} \|\psi'''\|_{L_2^2} \|\phi\|_{L_\infty^2} \|\phi\|_{L_2^2} + \frac{(\lambda_s(t) \lambda_6(t))^{1/2}}{\eta_3(t) \lambda_5^{2+q}(t)} \|u_0\|_{L_2^2} \|\psi''\|_{L_2^2} \|\phi\|_{L_\infty^2} \|\phi\|_{L_2^2} + \frac{(\lambda_s(t) \lambda_6(t))^{1/2}}{\eta_3(t) \lambda_5^2(t)} \|u_0\|_{L_2^2} \|\psi\|_{L_2^2} \|\phi\|_{L_\infty^2} \|\phi\|_{L_2^2} + \frac{(\lambda_s(t) \lambda_6(t))^{1/2}}{\eta_3(t) \lambda_5^{2q}(t)} \|u_0\|_{L_2^2} \|\psi\|_{L_2^2} \|\phi\|_{L_\infty^2} \|\phi\|_{L_2^2}.
\]

Consequently, replacing (2.13), (2.11) and (2.10), and since \(q > 1\),

\[
|\mathcal{I}_{1,1,1}(t)| \lesssim \frac{1}{\eta_3(t) (\lambda_5(t))^{3/2 - r/2}} + \frac{1}{\eta_3(t) (\lambda_5(t))^{3/2 + q - r/2}} + \frac{1}{\eta_3(t) (\lambda_5(t))^{1/2 + 2r - r/2}} \lesssim \frac{1}{\eta_3(t) (\lambda_5(t))^{3/2 - r/2}} = \frac{1}{t^{1 + \frac{2}{r}(3 - r) - \frac{5}{2r} \log t}}.
\]

From (2.8) we easily see that \(\mathcal{I}_{1,1,1} \in L^1\{t \gg 1\}\).

To bound the term \(\mathcal{I}_{1,1,2}(t)\) we will require lemma 2.3 as key step. Combining integration by parts, (2.7) and Cauchy–Schwarz inequality,

\[
\mathcal{I}_{1,1,2}(t) = -\frac{\kappa}{\eta_3(t) \lambda_5(t)} \int_{\mathbb{R}^2} \partial_x^{-1} \partial_y^{-1} \psi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{x}{\lambda_3(t)} \right) \phi' \left( \frac{x}{\lambda_6(t)} \right) \ dx dy
\]

so that,

\[
|\mathcal{I}_{1,1,2}(t)| \leq \frac{|\kappa|}{\eta_3(t) \lambda_5(t)} \int_{\mathbb{R}^2} |u| \left| \partial_x^{-1} \psi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{x}{\lambda_3(t)} \right) \phi'' \left( \frac{x}{\lambda_6(t)} \right) \right| \ dx dy
\]

\[
\lesssim \frac{\lambda_5^2(t)}{\eta_3(t) \lambda_5^2(t)} \int_{\mathbb{R}^2} |u| \phi \left( \frac{x}{\lambda_3(t)} \right) \phi'' \left( \frac{x}{\lambda_6(t)} \right) \ dx dy
\]

\[
\lesssim \frac{\lambda_5^{3/2}(t)}{\eta_3(t) \lambda_6^{3/2}(t)} \|u_0\|_{L_2^2} \|\phi\|_{L_2^2} \|\phi\|_{L_2^2}
\]

\[
\sim \frac{1}{\eta_3(t) \lambda_5^{2(r-q)}(t)} \int_{\mathbb{R}^2} \frac{1}{t^{1 + \frac{2}{r}(3 - r) - \frac{5}{2r} \log t}}.
\]

Since \(r > 5/3\), taking \(q = 1 + \varepsilon_0\), with \(\varepsilon_0 > 0\) sufficiently small, we obtain that \(\mathcal{I}_{1,1,2} \in L^1\{t \gg 1\}\).
We emphasise that the term $I_{1,1,3}$ in (6.7)

$$I_{1,1,3}(t) = \frac{1}{2\eta_3(t)\lambda_5(t)} \int_{\mathbb{R}^2} u^2 \psi' \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{x}}{\lambda_5^3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) dxdy,$$

(6.8)

is the term to be estimated after integrating in time, leading to the left hand side in (6.3). Therefore, it will be taken until the end of the proof.

The term $I_{1,1,4}$ in (6.7) satisfies the following estimate

$$|I_{1,1,4}(t)| \leq \frac{1}{2\eta_3(t)\lambda_5^2(t)} \int_{\mathbb{R}^2} u^2 \psi \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi' \left( \frac{\tilde{x}}{\lambda_5^2(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) dxdy \leq \frac{1}{2\eta_3(t)\lambda_5^2(t)} \|u_0\|_{L^2_x}^2 \|\psi\|_{L^\infty_x} \|\phi\|_{L^\infty_x} \|\phi\|_{L^\infty_x} \sim \frac{1}{t^{1+(q-1)b} \log^{1-q} t}.$$ 

Since (2.8) are satisfied, and $q = 1 + \epsilon_0 > 1$, $I_{1,1,4} \in L^1\left( \{ t \gg 1 \} \right)$.

This last estimate ends the study of the term $I_{1,1}$ in (6.6), concluding that

$$I_{1,1}(t) = \frac{1}{2\eta_3(t)\lambda_5(t)} \int_{\mathbb{R}^2} u^2 \psi' \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{x}}{\lambda_5^3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) dxdy + I_{1,1,int}(t),$$

(6.9)

with $I_{1,1,int}(t) \in L^1\left( \{ t \gg 1 \} \right)$.

Now, we focus our attention in the remaining terms in (6.6). First, by means of Young’s inequality, we have for $\epsilon > 0$,

$$|I_{1,2}(t)| = \frac{\lambda_5^2(t)}{\lambda_5(t)\eta_3(t)} \int_{\mathbb{R}^2} |u| \psi' \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi' \left( \frac{\tilde{x}}{\lambda_5^3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) dxdy$$

$$\leq \frac{1}{4\epsilon} \frac{\lambda_5^2(t)}{\lambda_5(t)\eta_3(t)} \int_{\mathbb{R}^2} u^2 \psi' \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{x}}{\lambda_5^3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) dxdy$$

$$+ \epsilon \frac{\lambda_5^2(t)}{\lambda_5(t)\eta_3(t)} \int_{\mathbb{R}^2} \psi' \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \left( \frac{\tilde{x}}{\lambda_5(t)} \right)^2 \phi \left( \frac{\tilde{x}}{\lambda_5^3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) dxdy$$

$$= \frac{1}{4\epsilon} \frac{\lambda_5^2(t)}{\lambda_5(t)\eta_3(t)} \int_{\mathbb{R}^2} u^2 \psi' \left( \frac{\tilde{x}}{\lambda_5(t)} \right) \phi \left( \frac{\tilde{x}}{\lambda_5^3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) dxdy$$

$$+ \epsilon \frac{\lambda_5^2(t)\lambda_6(t)}{\eta_3(t)} \|\cdot\|_{L^1_x} \|\phi'\|_{L^\infty_x} \|\phi\|_{L^\infty_x} \|\phi\|_{L^1_x},$$

where $\epsilon > 0$ is a small constant.
so that, taking $\epsilon = \lambda_2^t(\lambda(t)) > 0$ for $t \gg 1$; it is clear that

$$|I_{1,2}(t)| \leq \frac{1}{4\lambda_5(t)\eta_3(t)} \int_{\mathbb{R}^2} u^2 \phi' \left( \frac{\tilde{x}}{\lambda_5^t(t)} \right) \phi \left( \frac{\tilde{x}}{\lambda_3^t(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) d\tilde{x} d\tilde{y}
$$

$$+ c \left( \frac{\lambda_2^t(\lambda(t))^2}{\lambda_3^t(t)} \right) \frac{\lambda_6(t)}{\eta_3(t)(\lambda_5^t(t))^{-2}}
$$

$$= \frac{1}{2} I_{1,1,3}(t) + I_{1,2}^t(t).$$

Note that the first term in the r.h.s is the quantity to be estimated (see (6.8)), unlike the remaining term $I_{1,2}^t$ which satisfies

$$0 \leq I_{1,2}^t(t) \leq \frac{\lambda_3^t(\lambda(t))^2}{\lambda_5^t(t)} \frac{\lambda_6(t)}{\eta_3(t)(\lambda_5^t(t))^{-2}} \leq \frac{1}{t^2} \frac{1}{\eta_3(t)(\lambda_5^t(t))^{-2\gamma}}$$

$$= \frac{1}{t^{3-3\gamma+r} \log^{3-r+t}}.$$

The term $I_{1,2}^t$ belongs in $L^1(\{t \gg 1\})$, since (2.8) implies that $b \leq \frac{2}{2+q+r} < \frac{2}{2+r}$. This ends the estimate of $I_{1,2}(t)$, concluding that

$$|I_{1,2}(t)| \leq \frac{1}{2} I_{1,1,3}(t) + I_{1,2,\text{int}}(t),$$

with $I_{1,2,\text{int}}(t) \in L^1(\{t \gg 1\})$.

Now, we consider the term $I_{1,3}(t)$. Combining the properties attribute to $\phi$ and Young’s inequality we get for $\theta(t) = \psi/\log t$,

$$|I_{1,3}(t)| \leq q \left\| \frac{\lambda_3^t(\lambda(t))}{\lambda_5^t(\lambda(t))} \right\|_1 \int_{\mathbb{R}^2} u^2 \phi' \left( \frac{\tilde{x}}{\lambda_5^t(\lambda(t))} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) d\tilde{x} d\tilde{y}
$$

$$+ q \left| \frac{\lambda_2^t(\lambda(t))^2}{\lambda_3^t(\lambda(t))} \right| \int_{\mathbb{R}^2} u^2 \phi' \left( \frac{\tilde{x}}{\lambda_5^t(\lambda(t))} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) d\tilde{x} d\tilde{y}
$$

$$+ q \left| \frac{\lambda_2^t(\lambda(t))^2}{\lambda_3^t(\lambda(t))} \right| \int_{\mathbb{R}^2} u^2 \phi' \left( \frac{\tilde{x}}{\lambda_5^t(\lambda(t))} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) d\tilde{x} d\tilde{y}
$$

$$\leq q \left\| \frac{\lambda_3^t(\lambda(t))}{\lambda_5^t(\lambda(t))} \right\|_1 \left\| \psi \right\|_{L^\infty} \left\| \phi \right\|_{L^2} + q \left| \frac{\lambda_2^t(\lambda(t))^2}{\lambda_3^t(\lambda(t))} \right| \left\| \phi \right\|_{L^2} \left\| \phi \right\|_{L^2}.$$

Hence,

$$|I_{1,3}(t)| \leq \frac{1}{t^{\log^2(t)} + 1} + \frac{1}{t^{\log^2(t)} + 1} \frac{1}{\eta_3(t)(\lambda_5^t(\lambda(t)))^{-2\gamma}}$$

$$= \frac{1}{t^{\log^2(t)} + 1} + \frac{1}{t^{\log^2(t)} + 1} \frac{1}{\eta_3(t)(\lambda_5^t(\lambda(t)))^{-2\gamma}}$$

$$= \frac{1}{t^{\log^2(t)} + 1} + \frac{1}{t^{\log^2(t)} + 1} \frac{1}{(2+q+r) \log^{2+r+t}}.$$
As before, we get for $\theta(t) = \varphi' / \log t$,
\[
|I_{1,4}(t)| \leq \frac{\lambda'_6(t)}{\lambda_6(t) \eta_3(t)} \int_{\mathbb{R}^2} |u|^2 |\psi'(\frac{x}{\lambda_5(t)})| \left| \phi \left( \frac{x}{\lambda_5(t)} \right) \right| \left| \frac{\tilde{y}}{\lambda_6(t)} \right| |\phi'(\frac{\tilde{y}}{\lambda_6(t)})| \, dx \, dy
\]
\[
\leq \frac{\lambda'_6(t) \theta(t)}{2\lambda_6(t) \eta_3(t)} \int_{\mathbb{R}^2} u^2 |\psi'(\frac{x}{\lambda_5(t)})| \left| \frac{\tilde{y}}{\lambda_6(t)} \right| |\phi'(\frac{\tilde{y}}{\lambda_6(t)})| \, dx \, dy
\]
\[
+ \left| \frac{\lambda'_6(t) \theta(t)}{2\lambda_6(t) \eta_3(t)} \right| \int_{\mathbb{R}^2} \phi^2 \left( \frac{x}{\lambda_5(t)} \right) \left| \frac{\tilde{y}}{\lambda_6(t)} \right| |\phi'(\frac{\tilde{y}}{\lambda_6(t)})|^2 \, dx \, dy
\]
\[
\leq \frac{\lambda'_6(t) \theta(t) ||u_0||^2_{L^2} ||\psi||^2_{\infty}}{2\lambda_6(t) \eta_3(t)} + \frac{\lambda'_6(t) \lambda_5^2(t) ||u_0||^2_{L^2} ||\psi||^2_{\infty}}{2\eta_3(t) \theta(t)} |\phi||^2_{L^2} |\phi'(\cdot)|^2_{L^2}
\]
\[
\leq \frac{1}{t \log^2 t} + \frac{1}{r^{(2+\varphi+q)/(2+\varphi+q)} \log^{q\varphi+q} t},
\]
(6.12)

Just like in (6.11) we obtain that $I_{1,4} \in L^1(\{t \gg 1\})$.

Now we deal with the term $I_{1,5}$ in (6.6). We have
\[
\left| \frac{\rho'_1(t)}{\lambda_5(t) \eta_3(t)} \int_{\mathbb{R}^2} u \psi' \left( \frac{x}{\lambda_5(t)} \right) \phi' \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{x}{\lambda_5(t)} \right) \, dx \, dy \right|
\]
\[
\leq \frac{\rho'_1(t) ||u_0||^2_{L^2}}{\lambda_5^2(t) \eta_3(t)} \leq \frac{|\epsilon_1 m_1|}{t^{2-\frac{1}{2}b(r+1)-m_1 \log^{1+\frac{1}{2}b(r+1)} t}},
\]
which integrates in time because of (2.20). The term $I_{1,6}$ is treated very similarly: one has for $t \gg 1$,
\[
\left| \frac{\rho'_1(t)}{\lambda_5(t) \eta_3(t)} \int_{\mathbb{R}^2} u \psi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{x}{\lambda_5(t)} \right) \phi' \left( \frac{x}{\lambda_5(t)} \right) \, dx \, dy \right|
\]
\[
\leq \frac{\rho'_1(t) ||u_0||^2_{L^2}}{\lambda_5^2(t) \eta_3(t)} \leq \frac{|\epsilon_1 m_1| ||u_0||^2_{L^2}}{t^{2-\frac{1}{2}b(r+1)-m_1 \log^{1+\frac{1}{2}b(r+1)} t}},
\]
(6.13)

since $q > 1$. Finally, the term $I_{1,7}$ can be treated as follows:
\[
\left| \frac{\rho'_2(t)}{\lambda_6(t) \eta_3(t)} \int_{\mathbb{R}^2} u \psi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{x}{\lambda_5(t)} \right) \phi' \left( \frac{x}{\lambda_5(t)} \right) \, dx \, dy \right|
\]
\[
\leq \frac{\rho'_2(t) ||u_0||^2_{L^2}}{\lambda_5^2(t) \eta_3(t)} \leq \frac{|\epsilon_2 m_3| ||u_0||^2_{L^2}}{t^{2-\frac{1}{2}b(r+1)-m_3 \log^{1+\frac{1}{2}b(r+1)} t}},
\]
which integrates in time because of (2.20).

We conclude from (6.6)–(6.12),
\[
I_1(t) \geq \frac{1}{4\eta_3(t) \lambda_5(t)} \int_{\mathbb{R}^2} u^2 \phi' \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{y}{\lambda_6(t)} \right) \, dx \, dy - I_{1,\text{int}}(t),
\]
(6.14)

with $I_{1,\text{int}}(t) \in L^1(\{t \gg 1\})$. Finally, combining (6.4), (6.5) and (6.14), we have
\[
\frac{1}{4\eta_3(t) \lambda_5(t)} \int_{\mathbb{R}^2} u^2 \phi' \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{x}{\lambda_5(t)} \right) \phi \left( \frac{y}{\lambda_6(t)} \right) \, dx \, dy \leq \frac{dI}{dt}(t) + I_{\text{int}}(t),
\]
Since proving the inequality in this region. Finally, in the region $\phi \phi$ the inequality fixed, independent of time, we have $\gamma (t) = t$, $q > 1$ and for $t \gg 1$ :

$$\frac{\sigma_2}{t} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) \phi \left( \frac{\bar{y}}{\lambda_5 (t)} \right) \frac{\bar{y}}{\lambda_5 (t)} \frac{\bar{y}}{\lambda_5 (t)} dxdy$$

$$\leq \frac{1}{4 \gamma (t) \lambda_5 (t)} \int_{\mathbb{R}^2} u^2 \psi' \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) \phi \left( \frac{\bar{y}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{y}}{\lambda_5 (t)} \right) dxdy.$$

Now we consider the term $\phi \phi$. Without loss of generality, we assume $\bar{x} \geq 0$. Since $t$ is arbitrarily large, $2 \lambda_5 (t) \ll \lambda_5^2 (t)$ and $\frac{1}{\lambda_5^2 (t)} \ll \frac{1}{\lambda_5 (t)}$. Therefore,

$$\phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) = \begin{cases} 1, & \bar{x} \leq \lambda_5 (t), \\ \phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right), & \lambda_5 (t) \leq \bar{x} \leq 2 \lambda_5 (t), \\ \exp \left( - \frac{\bar{x}}{\lambda_5 (t)} \right), & 2 \lambda_5 (t) \leq \bar{x} \leq \lambda_5^2 (t), \\ \exp \left( - \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right), & \lambda_5^2 (t) \leq \bar{x} \leq \lambda_5^3 (t), \\ \exp \left( - \frac{1}{\lambda_5 (t)} + \frac{1}{\lambda_5^2 (t)} \right) \bar{x}, & \bar{x} \geq 2 \lambda_5^3 (t). \end{cases}$$

Consequently,

$$\phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) \geq e^{-2} \phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right), \quad \bar{x} \leq \lambda_5^3 (t) = \frac{\lambda_5^3 (t)}{\lambda_5 (t) + \lambda_5 (t) \lambda_5 (t) < \lambda_5 (t)}.$$

Indeed, in the region $\bar{x} \leq \lambda_5^2 (t)$, one has

$$\phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) = \phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right).$$

Since $\bar{x} \leq \lambda_5 (t)$, and since $\phi$ is nonincreasing, one easily has in the region $\bar{x} \leq \lambda_5^2 (t)$ the inequality $\phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) \geq \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right)$. In the region $\lambda_5^2 (t) \leq \bar{x} \leq 2 \lambda_5^2 (t)$, one has $\phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) \geq e^{-2}$, so that

$$\phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \phi \left( \frac{\bar{x}}{\lambda_5^2 (t)} \right) \geq e^{-2} \phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) \geq e^{-2} \phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right),$$

proving the inequality in this region. Finally, in the region $\bar{x} \geq 2 \lambda_5^2 (t)$,

$$\phi \left( \frac{\bar{x}}{\lambda_5 (t)} \right) = \exp \left( - \left( \frac{1}{\lambda_5 (t)} + \frac{1}{\lambda_5^2 (t)} \right) \bar{x} \right).$$
proving the last part of the inequality. We conclude that for \( \sigma_2 = e^{-2} \),
\[
\frac{\sigma_2}{T} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_5(t)} \right) \, dx \, dy \\
\leq \frac{1}{4\eta_3(t)\lambda_5(t)} \int_{\mathbb{R}^2} u^2 \phi' \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_5(t)} \right) \, dx \, dy.
\]
This ends the proof of lemma 6.1.

7. Proof of theorem 1.3

This last section is devoted to connect propositions 4.1 and 5.1 and theorem 1.1 to obtain the proof of theorem 1.3.

7.1. End of proof of theorem 1.3, part (1.18)

Recall propositions 5.1 and 6.1. We have
\[
\frac{\sigma_1}{T} \int_{\mathbb{R}^2} \nu^2 (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
\leq - \frac{d}{dt} J(t) + \frac{C_1}{T \log T} \int_{\mathbb{R}^2} |u|^3 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
+ \frac{C_1}{T} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + J_{int}(t),
\]
where \( J_{int}(t) \in L^1([t \gg 1]) \).

Note that the last term above is integrable in time, thanks to lemma 6.1. Indeed, using the fact that from (2.14) one has
\[
\lambda_3(t) = \frac{\lambda_5(t)}{\log t}, \quad \tilde{\lambda}_5(t) = \frac{\lambda_5^2(t)}{\lambda_3(t)} + \lambda_5(t),
\]
and then
\[
\lambda_3(t) \ll \tilde{\lambda}_5(t), \quad \lambda_4(t) \ll \lambda_6(t),
\]
we obtain using (6.3),
\[
\frac{1}{T} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \\
\lesssim \frac{1}{T} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_5(t)} \right) \, dx \, dy \in L^1([t \gg 1]).
\]

The final conclusion in (5.2) follows from the following result. Recall (5.24).

**Lemma 7.1 (control of cubic terms).** Let \( u \in E^1(\mathbb{R}^2) \) be a solution of the KP-I equation verifying (2.3), then
\[
\frac{1}{\eta_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} |u|^3 (x, y, t) (\partial_x \Psi) \left( \frac{\tilde{x}}{\lambda_3(t)} , \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \in L^1([t \gg 1]).
\]
First, we apply Young’s inequality properly, from where we obtain
\[
\frac{1}{\eta_2(t)\lambda_4(t)} \int_{\mathbb{R}^2} |u|^3 \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
\]
\[
\lesssim \frac{1}{t} \int_{\mathbb{R}^2} u^2 \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
\]
\[
+ \frac{1}{t^{1/3} \log^{4/3} t} \int_{\mathbb{R}^2} u^\theta \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy.
\]

Thus, in virtue of \((7.3)\), we obtain
\[
\frac{1}{t} \int_{\mathbb{R}^2} u^2 \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \sim \frac{1}{t} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
\]
\[
\lesssim \frac{1}{t} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_6(t)} \right) \, dx \, dy \in L^1(\{ t \gg 1 \}).
\]

Thus, coming back to \((7.2)\) we obtain \((7.1)\). \(\Box\)

Next, combining H"older’s inequality and \((2.3)\), we find that
\[
\frac{1}{\eta_2(t)\lambda_4(t)} \int_{\mathbb{R}^2} |u|^3 \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy
\]
\[
\lesssim \frac{1}{t} \int_{\mathbb{R}^2} u^2 \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy + \frac{1}{t \log^4 t} \left\| u \right\|_{L^6_\infty}^6 \left\| \left( \frac{\tilde{x}}{\lambda_3(t)}, \frac{\tilde{y}}{\lambda_4(t)} \right) \right\|_{L^\infty_{t}}^6.
\]

We use now \((2.5), (2.15), (5.1)\), and the fact that from \((2.14)\) we have
\[
\lambda_3(t) \ll \tilde{\lambda}_3(t), \quad \lambda_4(t) \ll \lambda_6(t).
\]

Now we conclude. By taking \( b, r \) slightly smaller but fixed if necessary, and using \((2.14)\), we obtain
\[
\int_{\{ t \gg 1 \}} \frac{1}{t \log t} \int_{\mathbb{R}^2} v^2 \, dx \, dy \, dt \lesssim \int_{\{ t \gg 1 \}} \frac{1}{t \log t} \int_{\mathbb{R}^2} v^2 \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \left( \frac{\tilde{y}}{\lambda_4(t)} \right) \, dx \, dy \, dt < \infty.
\]

Here, \(\tilde{\Omega}_1(t)\) is the subset of the plane \(\mathbb{R}^2\) introduced in \((1.16)\). This estimate and \([52]\) readily implies \((1.18)\).

72. End of proof of theorem 1.3, part (1.19)

Recall proposition 4.1. From \((7.3)\) and \(v = \partial_{t}^{-1} \partial_{2}u\),
\[
\frac{1}{t \log t} \int_{\mathbb{R}^2} \left( \partial_{t}^{-1} \partial_{2}u \right)^2 \phi \left( \frac{\tilde{x}}{\lambda_3(t)} \right) \phi \left( \frac{\tilde{y}}{\lambda_2(t)} \right) \, dx \, dy \in L^1(\{ t \gg 1 \}).
\]
Here we have used that $\lambda_1(t) \ll \lambda_3(t)$ and $\lambda_2(t) \ll \lambda_4(t)$, see (2.9). On the other hand, from (5.24), (2.9), (2.15) and (7.1),

$$\frac{1}{t} \log t \int_{\mathbb{R}^2} |u|^3 \phi \left( \frac{\bar{x}}{\lambda_1(t)} \right) \phi \left( \frac{\bar{y}}{\lambda_2(t)} \right) \, dx \, dy \leq \frac{1}{n_2(t) \lambda_4(t)} \int_{\mathbb{R}^2} |u|^3 (x, y, t) \left( \partial_x \Psi \right) \left( \frac{\bar{x}}{\lambda_3(t)}, \frac{\bar{y}}{\lambda_4(t)} \right) \, dx \, dy \in L^1 \left( \{ t > 1 \} \right).$$

We conclude

$$\int_{t > 1} \frac{1}{t} \log t \int_{\mathbb{R}^2} \left( \partial_x u \right)^2 \phi \left( \frac{\bar{x}}{\lambda_1(t)} \right) \phi \left( \frac{\bar{y}}{\lambda_2(t)} \right) \, dx \, dy \, dr < +\infty.$$ 

Using (1.17),

$$\liminf_{t \to \infty} \int_{\Omega(t)} \left( \partial_x u \right)^2 \, dx \, dy \leq \liminf_{t \to \infty} \int_{\mathbb{R}^2} \left( \partial_x u \right)^2 \phi \left( \frac{\bar{x}}{\lambda_1(t)} \right) \phi \left( \frac{\bar{y}}{\lambda_2(t)} \right) \, dx \, dy = 0.$$

This ends the proof.

7.3. Proof of corollary 1.1

This corollary is a direct consequence of proposition 6.1. Fix $\varepsilon > 0$ and consider $R(\varepsilon) > 0$ from definition 1.1. Assume (1.14). Notice that theorem 1.1 remains valid if $\Omega_1(t)$ in (1.6) is changed by

$$\Omega_1(t) = \{ (x, y) \in \mathbb{R}^2 : |x - x(t)| \leq \tilde{\ell}, |y - y(t)| \leq \tilde{r} \},$$

with $x(t)$ and $y(t)$ $C^1$ curves satisfying $|x(t)| \leq |\tilde{e}|$ and $|y(t)| \leq |\tilde{g}|$, with $m_1, m_2$ satisfying (1.7), and $b$ is taken accordingly to this choice. This fact is justified because $\tilde{\ell}$ and $\tilde{g}$ are the worst possible scenarios for the shifts in $x$ and $y$, respectively, and $b$ decreases if $m_1$ or $m_2$ increases. Having this in mind, and being $m_1, m_2$ fixed, let us choose $b > 0$ accordingly to the definition of this new $\Omega_1(t)$, and set $\tilde{x} = x - x(t)$, $\tilde{y} = y - y(t)$. Consider $\tilde{\lambda}_5(t)$ and $\tilde{\lambda}_6(t)$ bigger than $R(\varepsilon)$ for $t$ large (possible since we can always choose $b$ small but positive), as defined in (2.14). One has

$$\frac{\sigma_2}{t} \int_{\mathbb{R}^2} u^2 (x, y, t) \, dx \, dy \leq \frac{\sigma_2}{t} \int_{|x - x(t)| \leq \tilde{\lambda}_5(t), |y - y(t)| \leq \tilde{\lambda}_6(t)} u^2 (x, y, t) \, dx \, dy \leq \frac{\sigma_2}{t} \int_{\mathbb{R}^2} u^2 \phi \left( \frac{\tilde{x}}{\tilde{\lambda}_5(t)} \right) \phi \left( \frac{\tilde{y}}{\tilde{\lambda}_6(t)} \right) \, dx \, dy \leq \frac{d\Omega}{dt}(t) + \mathcal{I}_{\text{in}}(t).$$
with \( q = 1 + \varepsilon_0 \) in \( I(t) \) (6.1), and where \( I_{\text{int}}(t) \) are terms that belong to \( L^1 \{ t \gg 1 \} \). The last inequalities imply

\[
\int_{\{ t \gg 1 \}} \frac{\sigma_t}{t} \int_{\| (x-x(y(t)), y-y(t)) \| \leq R(\varepsilon)} u^2(x,y,t) \, dx \, dy < +\infty. \tag{7.4}
\]

Decompose

\[
\int_{\mathbb{R}^2} u^2 \, dx \, dy = \int_{\| (x-x(y(t)), y-y(t)) \| \leq R(\varepsilon)} u^2 \, dx \, dy + \int_{\| (x-x(y(t)), y-y(t)) \| > R(\varepsilon)} u^2 \, dx \, dy
\]

By assumption, the second contribution is less than \( \varepsilon \). By (7.4) (following the proof of theorem 1.1), there exists \( t_0 > 0 \) such that the first contribution is less than \( \varepsilon \). Hence there exists \( t_0 > 0 \) such that \( \int_{\mathbb{R}^2} u^2(0, \cdot) \, dx \leq 2\varepsilon \). Using the conservation of the \( L^2 \) norm, and since the above is valid for any \( \varepsilon > 0 \), we find that \( u|_{t=0} = 0 \). We thank the anonymous referee for pointing out this simplification of the proof of corollary 1.1.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

We thank Yvan Martel and Jean-Claude Saut for many useful discussions regarding the KP models along the years, Jean-Claude Saut for making available to us the monograph by Klein and Saut [24], and Felipe Linares and Jean-Claude Saut for useful comments on a first version of this paper.

Appendix. Statement and proof of lemma A.1

In this part we describe some interpolation inequalities used in section 3 to handle the nonlinear part of the equations (KP-I and KP-II).

For \( L > 0 \), we set the function

\[
\varphi_L(x) = \frac{2}{\pi} \arctan(e^x), \quad x \in \mathbb{R}.
\]

Lemma A.1. Let \( u \) be a function such that \( u \in E^1(\mathbb{R}^2) \). Suppose that \( u(t) \) is uniformly bounded in \( E^1 \), that is

\[
\sup_{t \in \mathbb{R}} \left( \| u(t) \|_{L^2_{\omega_0}}^2 + \| \partial_x u(t) \|_{L^2_{\omega_0}}^2 + \| \partial_x^{-1} \partial_y u(t) \|_{L^2_{\omega_0}}^2\right) < \infty.
\]
Let $\sigma_1, \sigma_2 \in \mathbb{R}$ not both null, and $\beta : \mathbb{R} \to \mathbb{R}$ a positive, smooth, and nondecreasing function. Then there exist a positive constant $C$, such that for any $\tau > 0$,

$$\int_{\mathbb{R}^2} |u(x,y,\tau)|^3 \varphi_L^\tau (\sigma_1 x + \sigma_2 y + \beta(t)) \, dx \, dy \leq C A \left( \tau \int_{\mathbb{R}^2} (u(x,y,\tau))^2 \varphi_L^\tau (\sigma_1 x + \sigma_2 y + \beta(t)) \, dx \, dy \right. $$

$$\left. + \frac{1}{4\tau^3} \int_{\mathbb{R}^2} (\partial_u u(x,y,\tau))^2 \varphi_L^\tau (\sigma_1 x + \sigma_2 y + \beta(t)) \, dx \, dy \right),$$

where

$$A := \sup_{t, y \in \mathbb{R}} \|u(\cdot,y,t)\|_{L^4} \leq \sup_{t} \|\partial_u^{-1} \partial_t u(t)\|_{L^4}^{\frac{1}{2}} \|\partial_t u(t)\|_{L^4}^{\frac{3}{2}} < +\infty.$$ 

**Proof.** First, we decompose the term in left hand side above as follows: let $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\chi \equiv 1 \quad \text{on } [0,1], \quad 0 \leq \chi \leq 1 \quad \text{and} \quad \text{supp}(\chi) \subseteq (-1,2).$$

For $m_1 \in \mathbb{Z}$, we set

$$\chi_{m_1}(x,y) := \chi(\sigma_1 x + \sigma_2 y - \beta(t) - m_1),$$

where $\beta(t)$ is a function depending on the variable $t$.

Next, with an abuse of notation we denote $\varphi_L^\tau = \varphi_L^\tau (x,y,\tau) = \varphi_L^\tau (\sigma_1 x + \sigma_2 y - \beta(t))$. We get

$$\int_{\mathbb{R}^2} |u|^3 (x,y,t) \varphi_L^\tau \, dx \, dy = \sum_{m_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} |u(x,y,t)|^3 \varphi_L^{\tau,m_1} \, dx \, dy \leq \sum_{m_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} |u(x,y,t)\chi_{m_1}(x,y)|^3 \varphi_L^\tau \, dx \, dy$$

$$\leq \sum_{m_1 \in \mathbb{Z}} \left( \sup_{(x,y) \in \Delta_{m_1}} \varphi_L^\tau (x,y) \right) \int_{\mathbb{R}^2} |u(x,y,t)\chi_{m_1}(x,y)|^3 \, dx \, dy$$

$$= \sum_{m_1 \in \mathbb{Z}} \left( \sup_{(x,y) \in \Delta_{m_1}} \varphi_L^\tau (x,y) \right) \int_{\mathbb{R}^2} \|u(\cdot,y,t)\chi_{m_1}(\cdot,y)\|_{L^4}^3 \, dy,$$

where

$$\Delta_{m_1} := \{(x,y) \in \mathbb{R}^2 | m_1 \leq \sigma_1 x + \sigma_2 y - \beta(t) \leq m_1 + 1\}, \quad \text{for } m_1 \in \mathbb{Z}.$$ 

Next, by Gagliardo–Nirenberg–Sobolev inequality

$$\|u\chi_{m_1}\|_{L^4} \leq c \|D_s^\tau (u\chi_{m_1})\|_{L^4}^{\frac{1}{s}} \|u\chi_{m_1}\|_{L^4}^{\frac{3-s}{s}}, \quad \text{whenever } s > \frac{1}{6},$$

and the constant $c$ is independent of $m_1$. 

48
Thus, for $m_1 \in \mathbb{Z}$

\[ \int_{\mathbb{R}} \Vert u (\cdot, y, t) \chi_{m_1} (\cdot, y) \Vert_{L_t^4}^3 \, dy \leq c \int_{\mathbb{R}} \Vert D_x^4 (u \chi_{m_1}) \Vert_{L_y^2}^{\frac{1}{2}} \Vert u \chi_{m_1} \Vert_{L_y^2}^{\frac{6-1}{2}} \, dy \]

\[ \leq c \Vert D_x^4 (u \chi_{m_1}) \Vert_{L_y^2}^{\frac{1}{2}} \Vert u \chi_{m_1} \Vert_{L_y^2}^{\frac{6-1}{2}} \sup_{y \in \mathbb{R}} \Vert u (\cdot, y, t) \chi_{m_1} (\cdot, y) \Vert_{L_t^4} \]

\[ \leq c \Vert D_x^4 (u \chi_{m_1}) \Vert_{L_y^2}^{\frac{1}{2}} \Vert u \chi_{m_1} \Vert_{L_y^2}^{\frac{6-1}{2}} \sup_{y \in \mathbb{R}} \Vert u (\cdot, y, t) \Vert_{L_t^4}, \]

whenever $s > \frac{1}{4}$.

Henceforth, we will denote $A := \sup_{y \in \mathbb{R}} \sup_{t \in \mathbb{R}} \Vert u (\cdot, y, t) \Vert_{L_t^4} < \infty$. Later on, we will provide an upper bound for this particular term.

Gathering the estimates above, we obtain that for any $\tau > 0$, the following inequality holds true

\[ \sum_{m_1 \in \mathbb{Z}} \left( \sup_{(x, y) \in \Delta_{m_1}} \varphi'_L (x, y, t) \right) \int_{\mathbb{R}} \Vert u (\cdot, y, t) \chi_{m_1} (\cdot, y) \Vert_{L_t^4}^3 \, dy \]

\[ \leq AC \sum_{m_1 \in \mathbb{Z}} \left( \sup_{(x, y) \in \Delta_{m_1}} \varphi'_L (x, y, t) \right) \left( \tau \int_{\mathbb{R}^2} (u \chi_{m_1})^2 \, dx dy + \frac{3}{2 \pi^3} \int_{\mathbb{R}^2} (\partial_t (u \chi_{m_1}))^2 \, dx dy \right) \]

\[ \leq AC \left( \tau \int_{\mathbb{R}^2} u^2 \varphi'_L \, dx dy + \frac{1}{4 \tau^3} \int_{\mathbb{R}^2} (\partial_t u)^2 \varphi'_L \, dx dy \right), \]

since

\[ \sup_{(x, y) \in \Delta_{m_1}} \varphi'_L (x, y, t) \leq c' \inf_{(x, y) \in \Delta_{m_1}} \varphi'_L (x, y, t), \]

where $c'$ is a positive constant that does not depend on $m_1$ nor $t$. For more details on how to decouple the term $\int_{\mathbb{R}^2} (\partial_t (u \chi_{m_1}))^2 \, dx dy$ above, see our previous work on the solutions of the ZK equation in the case 3d in [43]. **Claim 2:** For $u \in L_t^\infty E^1 (\mathbb{R}^2)$,

\[ \sup_{t, y \in \mathbb{R}} \| u (\cdot, y, t) \|_{L_t^4} < \infty. \]

Notice that for all $y \in \mathbb{R}$, and $u \in E^1$

\[ \| u (\cdot, y, t) \|_{L_t^4}^2 = \int_{\mathbb{R}} u^2 (x, y, t) \, dx \]

\[ = 2 \int_{\mathbb{R}} \int_{-\infty}^t \partial_t u (x, s, t) u (x, s, t) \, ds dx \]

\[ \leq 2 \| \partial_x^{-1} \partial_t u (t) \|_{L_x^2} \| \partial_t u (t) \|_{L_t^2}. \]

Therefore,

\[ \sup_{t, y \in \mathbb{R}} \| u (\cdot, y, t) \|_{L_t^4} \leq \sup_t \| \partial_x^{-1} \partial_t u (t) \|_{L_x^2} \| \partial_t u (t) \|_{L_t^2}^{\frac{1}{2}} < \infty. \]

Finally, we gather the estimates above to obtain (A.1). \[ \square \]
References

[1] Bona J L, Souganidis P E and Strauss W A 1987 Stability and instability of solitary waves of Korteweg-de Vries type Proc. R. Soc. A 411 395–412
[2] Bourgain J 1993 On the Cauchy problem for the Kadomtsev–Petviashvili equation Geom. Funct. Anal. 3 315–41
[3] de Bouard A and Martel Y 2004 Non existence of $L^2$-compact solutions of the Kadomtsev–Petviashvili II equation Math. Ann. 328 525–44
[4] de Bouard A and Saut J-C 1997 Symmetry and decay of the generalized Kadomtsev–Petviashvili solitary waves SIAM J. Math. Anal. 28 104–1085
[5] de Bouard A and Saut J-C 1997 Solitary waves of the generalized KP equations Ann. Inst. Henri Poincare C 14 211–36
[6] de Bouard A and Saut J-C 1996 Remarks on the stability of the generalized Kadomtsev–Petviashvili solitary waves Mathematical Problems in The Theory of Water Waves (Contemporary Mathematics vol 200) ed F Dias J-M Ghidaglia and J-C Saut (AMS) pp 75–84
[7] Cavalcante M and Muñoz C 2021 Asymptotic stability of KdV solitons in the half line (https://doi.org/10.4171/rmi/1102)
[8] Colliander M, Kenig C E and Staffillani G 2008 Weighted low-regularity solutions of the KP-I initial-value problem Discrete Contin. Dyn. Syst. 20 219–58
[9] Cote R, Muñoz C, Pilod D and Simpson G 2016 Asymptotic stability of high-dimensional Zakharov–Kuznetsov solitons Arch. Ration. Mech. Anal. 220 639–710
[10] Hadac M 2008 Well-posedness for the Kadomtsev–Petviashvili II equation and generalisations Trans. Am. Math. Soc. 360 6555–72
[11] Hadac M, Herr S and Koch H 2009 Well-posedness and scattering for the KP-II equation in a critical space Ann. Inst. Henri Poincare C 26 917–41
[12] Harrop-Griffits B, Ifrim M and Tataru D 2017 The lifespan of small solutions to the KP-I Int. Math. Res. Notices 1 1–28
[13] Hayashi N and Naumkin P 2014 Large time asymptotics for the Kadomtsev–Petviashvili equation Commun. Math. Phys. 332 505–33
[14] Iório R and Nunes W 1998 On equations of KP type Proc. R. Soc. Edinburgh 128A 725–43
[15] Hayashi N, Naumkin P and Saut J-C 1999 Asymptotics for large time of global solutions to the generalized Kadomtsev–Petviashvili equation Commun. Math. Phys. 201 577–90
[16] Ionescu A D and Kenig C E 2007 Local and global wellposedness of periodic KP-I equations Mathematical Aspects of Nonlinear Dispersive Equations (Ann. of Math. Stud. vol 163) (Princeton University Press) pp 181–211
[17] Ionescu A D, Kenig C E and Tataru D 2008 Global well-posedness of the KP-I initial-value problem in the energy space Invent. Math. 173 265–304
[18] Isaza P and Mejía J 2001 Local and global Cauchy problems for the Kadomtsev–Petviashvili (KP-II) equation in Sobolev spaces of negative indices Commun. PDE 26 1027–54
[19] Isaza P, Linares F and Ponce G 2016 On the propagation of regularity of solutions of the Kadomtsev–Petviashvili equation SIAM J. Math. Anal. 48 1006–24
[20] Kadomtsev B B and Petviashvili V I 1970 On the stability of solitary waves in weakly dispersing media Dokl. Akad. Nauk SSSR 192 753–6
[21] Kenig C E 2004 On the local and global well-posedness theory for the KP-I equation Ann. Inst. Henri Poincare C 21 827–38
[22] Kenig C E and Martel Y 2006 Global well-posedness in the energy space for a modified KP II equation via the Miura transform Trans. AMS 358 2447–88
[23] Klein C and Saut J-C 2015 IST versus PDE: a comparative study Hamiltonian Partial Differential Equations and Applications (Fields Institute Communications) vol 75 (Fields Inst. Res. Math. Sci) pp 383–449
Klein C and Saut J-C 2021 Nonlinear dispersive equations. Inverse scattering and PDE methods (in preparation) (https://doi.org/10.1007/978-3-030-91427-1)

Klein C and Saut J-C 2012 Numerical study of blow-up and stability of solutions to generalized Kadomtsev–Petviashvili equations J. Nonlinear Sci. 22 763–811

Koch H and Tzvetkov N 2003 On the local well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$ Int. Math. Res. Not. 2003 1449–64

Koch H and Tzvetkov N 2008 On finite energy solutions of the KP-I equation Math. Z. 258 55–68

Konopelchenko B G 1992 Introduction to Multidimensional Integrable Equations: The Inverse Spectral Transform in 2 + 1 Dimensions (Plenum Press) p 292

Kowalczyk M, Martel Y and Muñoz C 2017 Kink dynamics in the $\phi^4$ model: asymptotic stability for odd perturbations in the energy space J. Am. Math. Soc. 30 769–98

Kwak C Muñoz C, Pozo J C and Pozo J C 2019 The scattering problem for Hamiltonian ABCD Boussinesq systems in the energy space J. Math. Pures Appl. 127 121–59

Lannes D 2013 Water Waves: Mathematical Theory and Asymptotics (Math. Surveys and Monographs) vol 188 (AMS)

Lannes D 2002 Consistency of the KP Approximation Proc. 4th. Int. Conf. on Dynamical Systems and Differential Equations (Wilmington, NC, 24–27 May) pp 517–25

Lannes D and Saut J-C 2006 Weakly transverse Boussinesq systems and the KP approximation Nonlinearity 19 2853–75

Linares F, Pilod D and Saut J-C 2018 The Cauchy problem for the fractional Kadomtsev–Petviashvili equations SIAM J. Math. Anal. 50 3172–209

Linares F and Mendez A J 2020 On long time behavior of solutions of the Schrödinger-Korteweg-de Vries system (https://doi.org/10.1137/20M137553X)

Linares F, Mendez A J and Ponce G 2019 Asymptotic behavior of solutions of the dispersive generalized Benjamin-Ono equation J. Dyn. Diff. Equ. (submitted) (https://doi.org/10.1007/s10884-020-09843-6)

Liu Y 2000 Blow-up and instability of solitary wave solutions to a generalized Kadomtsev–Petviashvili equation Trans. AMS 353 191–208

Liu Y and Wei J-C 2019 Nondegeneracy. Morse index and orbital stability of the lump solution to the KP-I equation Arch. Mech. Ration. Anal. 234 1335–89

Martel Y and Merle F 2005 Asymptotic stability of solitons of the subcritical gKdV equations revisited Nonlinearity 18 55–80

Martel Y and Merle F 2000 A Liouville theorem for the critical generalized Korteweg-de Vries equation J. Math. Pures Appl. 79 339–425

Martel Y and Merle F 2001 Asymptotic stability of solitons for subcritical generalized KdV equations Arch. Ration. Mech. Anal. 157 219–54

Martel Y and Merle F 2002 Blow up in finite time and dynamics of blow up solutions for the $L^2$–critical generalized KdV equation J. Am. Math. Soc. 15 617–64

Mendez A J, Muñoz C, Poblete F and Pozo J C 2021 On local energy decay for large solutions of the Zakharov–Kuznetsov equation Commun. PDE. 46 1440–87

Merle F 2001 Existence of blow-up solutions in the energy space for the critical generalized Kdv equation J. Am. Math. Soc. 14 555–78

Mizumachi T 2015 Stability of Line Solitons for the KP-II Equation inMemoires of the AMS vol 238 (https://doi.org/10.1090/memo/1125)

Mizumachi T 2018 Stability of line solitons for the KP-II equation in $\mathbb{R}^2$. II Proc. R. Soc. A 143 149–98

Mizumachi T and Tzvetkov N 2012 Stability of the line soliton of the KP-II equation under periodic transverse perturbations Math. Ann. 352 659–90

Molinet L, Saut J-C and Tzvetkov N 2002 Global well-posedness for the KP-I equation Math. Ann. 324 255–75

Molinet L, Saut J-C and Tzvetkov N 2002 Well-posedness and ill-posedness results for the Kadomtsev–Petviashvili-I equation Duke Math. J. 115 353–84

Molinet L, Saut J-C and Tzvetkov N 2011 Global well-posedness for the KP-II equation on the background of a non-localized solution Ann. Inst. Henri Poincare C 28 653–76

Muñoz C and Ponce G 2019 Breakers and the dynamics of solutions in KdV type equations Commun. Math. Phys. 367 581–98

Muñoz C and Ponce G 2019 On the asymptotic behavior of solutions to the Benjamin-Ono equation Proc. Am. Math. Soc. 147 5303–12
[53] Muñoz C, Ponce G and Saut J-C 2019 On the long time behavior of solutions to the Intermediate Long Wave equation SIAM J. Math. Anal (submitted) (https://doi.org/10.1137/19M1293181)
[54] Niizato T 2011 Large time behavior for the generalized Kadomtsev–Petviashvili equations Differ. Equ. Appl. 3 299–308
[55] Rousset F and Tzvetkov N 2011 Transverse instability of the line solitary water waves Invent. Math. 184 257–388
[56] Rousset F and Tzvetkov N 2009 Transverse nonlinear instability for two-dimensional dispersive models Ann. Inst. Henri Poincare 26 477–96
[57] Satsuma J and Ablowitz M J 1979 Two dimensional lumps in nonlinear dispersive systems J. Math. Phys. 20 1496
[58] Saut J-C 1993 Remarks on the generalized Kadomtsev–Petviashvili equations Indiana Univ. Math. J. 42 1011–26
[59] Sung L Y 1999 Square integrability and uniqueness of the solutions of the Kadomtsev–Petviashvili-I equation Math. Phys. Anal. Geometry 2 1–24
[60] Takaoka H and Tzvetkov N 2001 On the local regularity of the Kadomtsev–Petviashvili-II equation Int. Math. Res. Not. 2001 77–114
[61] Ukai S 1989 Local solutions of the Kadomtsev–Petviashvili equation J. Fac. Sci. Univ. Tokyo A 36 193–209
[62] Wickerhauser M V 1987 Inverse scattering for the heat equation and evolutions in (2+1) variables Commun. Math. Phys. 108 67–89
[63] Zakharov V E and Schulman E I 1980 Degenerate dispersion laws, motion invariants and kinetic equations Physica D 1 192–202
[64] Levandovsky J 2023 Propagation of regularity for solutions to the KP-I equation Nonlinear Anal. 234 113315