Abstract

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\[ \text{BKT PHASE TRANSITIONS} \]
\[ \text{IN TWO-DIMENSIONAL SYSTEMS} \]
\[ \text{WITH INTERNAL SYMMETRIES} \]

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\[ \text{Abstract} \]

The Berezinsky-Kosterlitz-Thouless (BKT) type phase transitions in two-dimensional systems with internal abelian continuous symmetries are investigated. The necessary conditions for they can take place are: 1) conformal invariance of kinetic part of the model action; 2) vacuum manifold must be degenerated with abelian and discrete homotopy group \( \pi_1 \). Then topological excitations have logarithmically divergent energy and they can be described by effective field theories generalizing the two-dimensional euclidean sine-Gordon theory, which is an effective theory of the initial \( \text{XY} \)-model. In particular, the effective actions for the two-dimensional chiral models on maximal abelian tori \( T_G \) of simple compact groups \( G \) are found. Critical properties of possible effective theories are determined and it is shown that they are characterized by the Coxeter numbers \( h_G \) of lattices from the series \( A, D, E, Z \) and can be interpreted as those of conformal field theories with integer central charge \( C = n \), where \( n \) is a rank of the groups \( \pi_1 \) and \( G \). A possibility of restoration of full symmetry group \( G \) in massive phase is also discussed.

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1. Introduction

A discovery of possibility of the phase transition (PT) in two-dimensional $XY$-model \cite{1} from very beginning has attracted a great attention of theoreticians due to its unusual properties. First of all it seems that such PT contradicts to the well-known theorems by Peierls - Landau \cite{2,3} and Bogolyubov - Goldstone \cite{4,5} telling us that spontaneous magnetization and spontaneous breaking of continuous symmetry cannot exist in low-dimensional ($d \leq 2$) systems \cite{6,7}. Secondly, due to the absence of spontaneous magnetization, correlation functions in low-temperature phase must fall off algebraically \cite{8,9}, what means that the whole low-temperature phase has to be massless.

All these controversies were brilliantly resolved in series of papers by Berezinsky \cite{10}, Popov \cite{11} and by Kosterlitz and Thouless \cite{12,13}, who have proven for the first time an important role of topological excitations - vortices - with logarithmically divergent energies in this PT. An existence of vortices is connected with the fact that the manifold of values of $XY$-model $\mathcal{M} = S^1$ has nontrivial topology with the discrete abelian homotopy group $\pi_1(\mathcal{M}) = \mathbb{Z}$, while a logarithmic divergence of the energy is connected with conformal symmetry of the model. An account of vortices transforms continuous compact symmetry $U(1)$ into dual discrete symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is an automorphism group of $S^1$ corresponding to the reflection symmetry of $U(1) = S^1$. Analogous PT take place in other systems with the same symmetry: two-dimensional SOS and 6-vertex lattice models, $XXZ$ quantum spin chains \cite{14} and euclidean sine-Gordon (SG) model with non-compact field \cite{15-18}. It appears that all these systems belong to the same critical universality class. The sine-Gordon model can be considered as an effective theory of the BKT PT like the Ginzburg - Landau - Wilson theories are the effective theories of the PT of II order (see, for example \cite{19}).

The BKT PT can be also connected with some conformal theory, but now there is a peculiarity. In contrast with usual situation of two-dimensional II order phase transitions, when an infinite-dimensional conformal symmetry with a rational central charge $C$ takes place only at phase transition point \cite{20}, in systems with BKT PT an infinite-dimensional conformal symmetry with integer central charge $C = 1$ exists not only at PT point (with logarithmic corrections), but in a whole low-temperature phase. Thus we see that the BKT PT is intimately related with two fundamental properties of the systems: 1) a nontrivial topology described by discrete abelian homotopy
group $\pi_1, 2$) a conformal symmetry.

It is interesting to consider the BKT PT properties of such systems with internal symmetries. These systems are related with tori, which are the natural generalization of circle $S^1$ with necessary properties. It is clear that the same properties take place in two-dimensional chiral models on tori $T^n$ with $\pi_1(T^n) = \mathbb{Z}^n$. This case effectively reduces to the previous one, since only excitations with minimal topological charges $e_i = \pm 1, i = 1, ..., n$ are important and such charges with different $i$ do not interact between themselves. Similar properties have $\sigma$-models on general tori, connected with arbitrary nondegenerate lattices $L$ [21]. But, as has been shown in [21], besides $T^n$, there are the maximal abelian tori $T_G$ of the simple compact Lie groups $G$, which have (in case of simply connected $G$) $\pi_1(T_G) = L_v \neq \mathbb{Z}^n$ (here $L_v$ is a dual root lattice of the corresponding Lie algebra $\mathcal{G}$) and where excitations with different vector topological charges interact with each other.

The next question arises naturally: how the properties of the above-mentioned topological phase transition will depend on $G$? This question is important, for example, for string theory, where a compactification on $T_G$ (more rigorously, on the simplified tori $T^n = T_U(n)$ or $T_L = \mathbb{R}^n/L$, where $L$ is some nondegenerate lattice of rank $n$) is considered in different aspects [22, 23], or for chiral models on $G$ with reduced (or partially broken) symmetry $G \rightarrow T_G$ [24].

In this paper it will be shown that:

1) all critical properties of nonlinear $\sigma$-models on compact $T_G$ can be described in terms of effective field theories with discrete symmetries, generalizing SG theory;

2) they depend only on the Coxeter number $h_L$ of the corresponding lattice of topological charges $L_l$;

3) different classes of universality of the BKT PT are defined by $A, D, E, Z$ series of the integer-valued lattices;

4) all critical and low-temperature properties of these $\sigma$-models (except logarithmic corrections at criticality) can be described by corresponding conformal theories with integer central charge $C = n$, where $n$ is a rank of groups $\pi_1(T_G)$ and $G$.

A possible restoration of the full symmetry group $G$ in massive phase will be also discussed.
2. Nonlinear $\sigma$-models on $T_G$ and vortices.

Now we pass to consideration of the euclidean two-dimensional chiral field theories on $T_G$, generalizing nonlinear $\sigma$-model on a circle $S^1$ or continuous $XY$-model. Their action has the following form

$$S = \frac{1}{2\alpha} \int d^2 x Tr_\tau (t^{-1}_\nu t_\nu) = \frac{(2\pi)^2}{2\alpha} \int d^2 x Tr_\tau (H \phi_\nu)^2 = \frac{(2\pi)^2}{2\alpha} N_\tau \int d^2 x (\phi_\nu)^2, \quad (1)$$

where $t = e^{2\pi i(H \phi)} \in T_G$, $H = (H_1, ..., H_n) \in C$, which is the maximal Cartan subalgebra of the corresponding Lie algebra $G$, $[H_i, H_j] = 0$, $n$ is a rank of $G$, $\phi_\nu = \partial_\nu \phi$, $\nu = 1, 2$, and an isotropness property of the weight system $\{w_\tau\}$ of any $\tau(G)$-representation, which is a consequence of invariance of the weight systems under discrete Weyl group $W_G$, is used

$$\sum_a w^a_i w^a_k = N_\tau \delta_{ik}, \quad a = 1, ..., \text{dim} \tau(G). \quad (1a)$$

It will be convenient below to include a factor $N_\tau$ as a normalization factor into trace $Tr_\tau$. This gives us a canonical euclidean metric in space of topological charges.

Theories (1), like other two-dimensional chiral models, are invariant under direct product of right (R) and left (L) groups $N_{G}^{R(L)}$, which are a semi-direct product of $T_G$ and $W_G$

$$N_G = T_G \times W_G. \quad (2)$$

The group $N_G$ is called a normalizator of $T_G$ and is a symmetry group of torus $T_G$.

These theories are the multicomponent generalization of $XY$-model, having properties analogous to those of $XY$-model:

1) a zero beta-functions $\beta(\alpha)$ due to flatness of $T_G$;
2) non-trivial homotopy group $\pi_1$ and corresponding vortex solutions.

The classical equations of motion

$$(\partial_\nu)^2 (H \phi) = 0, \quad (3)$$

have vortex solutions in a region $R > r > a$, where $R$ is a radius of a system and $a$ is a short-wave cut-off parameter (for example, size of a vortex core).

$$t(\vartheta) = e^{2\pi i(H \phi(\vartheta))}, \quad \phi = \frac{1}{2\pi} q(\vartheta) \quad (4)$$
where $\vartheta$ is an angular and $r$ is a radial coordinates on a plane $\mathbb{R}^2$, $\mathbf{q} \in L^t_r = L^{-1}_r$, $\mathbf{q}$ is a vector topological charge of a vortex, and $L^t_r$ is a lattice of all possible topological charges of $\tau$ - representation, $L^{-1}_r$ is a lattice of vectors, inverse to all weights of $\tau$-representation

$$
\mathbf{q} \in L^t_r, \quad \mathbf{w}_a \in \{\mathbf{w}_r\}, \quad (\mathbf{q}\mathbf{w}_a) \in \mathbb{Z}.
$$

(5)

For minimal fundamental representations of the simply connected groups $\tau(G) = \text{min}$ a lattice $L^t_{\text{min}} = L_v$ and for adjoint representations $\tau = \text{ad}$ a lattice $L^t_{\text{ad}} = L^{-1}_r = L_{w^*}$, where $L_r$ is a root lattice of group $G$ and $L_{w^*}$ is a lattice of dual weights or a weight lattice of dual group $G^*$. Just these solutions for groups $G$, such, that $L^t_{\tau} \supset L_{w^*}$, can give the topological interpretation of all their quantum numbers [21]. The energy of these vortices is logarithmically divergent

$$
E = \frac{(2\pi)^2}{2\alpha} \int (\partial_\mu \Phi)^2 d^2 x = \frac{2\pi}{2\alpha} (\mathbf{q})^2 \ln (R/a).
$$

(6)

Due to (2), which defines an effective metric in space of topological charges [21], there is a logarithmic interaction between vortices with different vector topological charges

$$
E = (\mathbf{q}_1, \mathbf{q}_2) \frac{2\pi}{2\alpha} \ln \frac{|x_1 - x_2|}{a}.
$$

(7)

The general $N$-vortex solutions have the next form [20]

$$
\Phi(x) = \sum_{i=1}^{N} \mathbf{q}_i \frac{1}{\pi} \arctan \left( \frac{y - y_i}{x - x_i} \right),
$$

(8)

$$
\mathbf{q}_i \in L^t_r, \quad (\mathbf{q}_i \mathbf{w}_a) \in \mathbb{Z}, \quad (x, y) \in \mathbb{R}^2.
$$

The energy of $N$-vortex solution, $E_N$, with a whole topological charge $\sum_{i=1}^{N} \mathbf{q}_i = 0$ is

$$
E_N = \sum_i E_{\mathbf{q}_i}^0 + E_{\text{int}}, \quad E_{\mathbf{q}_i}^0 = \frac{1}{2\alpha} C(a)(\mathbf{q}_i \mathbf{q}_i),
$$

$$
E_{\text{int}} = \frac{2\pi}{2\alpha} \sum_{i \neq k} (\mathbf{q}_i \mathbf{q}_k) \ln \frac{|x_i - x_k|}{a},
$$

(9)
where $E_{q_i}^0$ is a "self-energy" (or an energy of the core) of vortex with topological charge $q_i$ and $C(a)$ is some nonuniversal constant, depending on type of the core regularization. Only such solutions give finite contribution to partition function of the theory $Z$. Since $E_q \sim q^2$ and $q \in L_t$, the maximal contribution in each $N$-vortex sector of solutions will give vortices with minimal $|q_i|$. Thus, in quasi-classical approximation (or in low $T$ expansion) one can represent a partition function of theory (5) in the form of grand partition function of classical neutral Coulomb gas of vortex solutions with minimal isovectorial topological charges $q_i \in \{q\}_\tau$, where $\{q\}_\tau$ is a set of minimal vectors of lattice $L^\tau$

$$Z = Z_0 Z_{CG}, \quad Z_{CG} = \sum_{N=0}^{\infty} \frac{\mu^{2N}}{N!} \sum' \{q\}_N (\{q\}|\beta). \quad (11)$$

Here $\sum'$ goes over all neutral configurations of minimal charges $q_i \in \{q\}_\tau$ with $\sum_1^N q_i = 0$, $Z_0$ is a partition function of free massless isovectorial boson field, which corresponds to "spin waves" of $XY$-model

$$Z_0 = \int D\phi \exp(-S_0[\phi]), \quad (12)$$

$$Z_N(\{q\}|\beta) = \prod_{i=1}^N \int d^2x_i \exp(-\beta H_N(\{q\})) \quad (13)$$

$$H_N(\{q\}) = \sum_{i<j} (q_i q_j) D(x_i - x_j), \quad (14)$$

$$D(x) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx} - 1}{f(ka)/k^2} \sim \frac{1}{2\pi} \ln \frac{|x|}{a} \quad (15)$$

where

$$\mu^2 = a^{-2} y^2 det, \quad y^2 = e^{-E_{q_i}^0} \quad (15a)$$

is a chemical activity of Coulomb gas, $det$ is a determinant of the fluctuations over one vortex solution (further we will suppose that it is equal to some constant of order $O(1)$ and assume that $det = 1$),

$$\beta = 2(\pi)^2/\alpha, \quad (15b)$$

6
\[ f(ka) \text{ is a regularisator such that} \]
\[ \lim_{k \to 0} f(ka) = 1, \quad \lim_{k \to \infty} f(ka) = 0. \]

How the account of vortices reduces the initial symmetry group \( N_G \) of the \( \sigma \)-model becomes obvious in the next section.

3. Duality of the compact and noncompact theories.

In case of \( XY \)-model in long-wave quasi-classical approximation there is an important connection between partition function of compact chiral theory (1) on \( S^1 \) and a partition function of the noncompact SG theory [15-17] with action (modulo \( Z_0 \))

\[ S_{SG} = \int d^2x \left( \frac{1}{2\beta} (\partial_\mu \phi)^2 - \mu^2 \cos(\phi) \right). \quad (16) \]

This action has explicit invariance under dual discrete group \( Z_2 \times Z \). Analogous connection exist between compact chiral models on \( T_G \) and noncompact generalized SG theories.

To see this we note that the grand partition function \( Z_{CG} \) from (11) is in its turn equivalent to partition function of noncompact scalar isovectorial field theory

\[ Z_{CG} = \int D\phi e^{-S_{eff}}, \quad S_{eff} = \int \frac{1}{2\beta} (\partial \phi)^2 - \mu^2 V(\phi), \quad (17) \]

\[ V(\phi) = \sum_{\{q\}} \exp i(q\phi). \quad (18) \]

where \( \sum_{\{q\}} \) goes over the set of minimal topological charges, and \( \phi \in \mathbb{R}^n \). Strictly speaking the theories (17) with arbitrary parameters \( \mu \) and \( \beta \) are more general than initial \( \sigma \)-models (1). The last have only one parameter, a coupling constant \( \alpha \). The representation of \( \sigma \)-models in the form (11,17) gives their embedding into general theories (17), since there are the restrictions (15a,b), relating parameters \( \mu \) and \( \beta \). This fact will be important later when a possible increasing of the symmetry of \( \sigma \)-models will be discussed (section 6).

Since the set of minimal charges \( \{q\}_\tau \) is invariant under dual Weyl group \( W_{G^*} \), we see that the account of vortices reduces the initial symmetry group
\( N_G \) into discrete dual group \( W_G \times L_q^{-1} \). Here a lattice \( L_q^{-1} \) is a periodicity lattice of potential \( V \) and, consequently, is inverse to all \( q \in \{q\} \). It follows from their definitions that \( L_q^{-1} = L_\tau \). This dual group generalizes the dual group \( Z_2 \times \mathbb{Z} \) of \( XY \)-model.

Thus, in this semiclassical and long wavelength approximation compact theory on a torus \( T_G \) with continuous symmetry \( N_G \) appears to be equivalent (again modulo \( \mathbb{Z}_0 \)) to noncompact theory with periodic potential and an infinite discrete symmetry. These potentials contain the sum over all minimal vectors \( \{q\} \) and can coincide with characters of some representations of group \( G \). For example, in case of \( L_\tau^{-1} = L_v \) the sum in (18) goes over all dual minimal roots. Thus the corresponding potentials \( V \) for simply laced groups from series \( A, D, E \) coincide with characters of adjoint representations of these groups (modulo some constant, corresponding to zero weight). In this case the general theories (17) can describe systems with symmetry \( G \) broken to \( N_G \) [25].

The noncompact theories (17) can be considered also as corresponding linear \( \sigma \)-models. Thus we have shown that compact nonlinear \( \sigma \)-models on \( T_G \) are equivalent in this approximation to noncompact linear \( \sigma \)-models on Cartan tori of dual group \( T_G^* \).

For further consideration we need to classify all possible effective theories of this type. It follows from (17,18) that they are determined by the set of minimal vectors \( \{q\} \) of lattice \( L_\tau^t \), which satisfies the next restriction

\[
L_w^* \supseteq L_\tau^t \supseteq L_v.
\]

For \( \tau = \text{min} \) a lattice \( L_\tau^t = L_v \), for \( \tau = \text{ad} \) a lattice \( L_\tau^t = L_w^* \). The lattices \( L_v \) and \( L_w^* \) differ by a factor, which is isomorphic to the centre \( Z_G \) of group \( G \)

\[
L_w^*/L_v = Z_G.
\]

Thus the set \( \{q\} \) can vary from the set of minimal vectors (it forms the so called Voronoi region or Wigner - Seitz cell of the corresponding lattice) of the weight lattice till that of the root lattice. All possible cases are determined by subgroups of the centre \( Z_G \). For groups \( G \) with \( Z_G = 1 \) the lattices \( L_v \) and \( L_w^* \) coincide.

4. Phase transition in chiral models on \( T_G \).

In this section we consider topological phase transitions in chiral models on \( T_G \) using above obtained approximate equivalence of these compact theo-
ries with noncompact generalized SG field theories (17). These field theories can be considered as effective theories for topological phase transitions in chiral models defined on $T_G$ like the sine-Gordon theory for $XY$-model [15-17] and the Ginzburg-Landau-Wilson theories for the II order phase transitions [13].

The investigation of the BKT type phase transition for all effective field theories of the form (17) was done by the renorm-group method in [25]. It was shown there that the new critical properties can have only theories connected with even integer-valued lattices of $A, D, E$ type. They have a structure of root lattices of the corresponding simple groups $G$ from simply laced series $A, D, E$. All theories connected with other lattices have the same critical properties as SG theory or its superpositions connected with lattice $Z^n$. Here we give brief description and write out obtained results, paying the main attention to the symmetry and universality properties.

Under renormalization transformations both parameters $\mu$ and $\beta$ are renormalized. It is convenient to introduce two dimensionless parameters

$$(\mu a)^2 = g, \quad \delta = \frac{\beta q^2 - 8\pi}{8\pi}$$

(20)

where $q^2$ is a square of the norm of the minimal vector topological charges from $\{q\}$. The theories (17) are renormalizable only if the vectors from $\{q\}$ belong to some lattice (here $L^l_T$). A new critical properties can appear only if a geometry of $\{q\}$ is such that each vector $q \in \{q\}$ can be represented as a sum of two other vectors from $\{q\}$ [25]. The last property is very restrictive and coincides with a definition of the root systems $\{r\}$ of simple groups from series $A, D, E$ [25] or with a definition of the root set of the even integer-valued (in some scale) lattices of $A, D, E$ types [26]. The sets of minimal roots (or minimal dual roots) of all simple groups belong to four series of integer-valued lattices $A, D, E, Z$. For theories with sets $\{q\} \notin A, D, E$, all critical properties will be the same as for theories with $\{q\} \in Z^n$ [25].

The RG equations for theories (17) with $\{q\}$ from these lattices have the next form [25] (here $G = A, D, E$)

$$\frac{dg}{dl} = -2\delta g + B_G g^2, \quad \frac{d\delta}{dl} = -C_G g^2.$$  

(21)

Here $B_G = \pi \theta_G$, $\theta_G$ is the multiplicity of the reproduction of $V(\phi)$ under renormalization of (17) or the number of times by which each root can be
represented as a sum of two other roots, and \( C_G = 2\pi K_G \), where \( K_G \) is the value of the second Casimir operator in adjoint representation (where \( \mathbf{w}_a = r_a \))

\[
\sum_a r^a_i r^a_j = K_G \delta_{ij}.
\] (22)

The RG equations of type (21) with coefficients corresponding to the case \( G = A_2 \) have been obtained firstly in [27] under investigation of the melting of the two-dimensional triangle lattice.

The eigenvalue of the second Casimir operator \( K_G \) for groups from series \( A, D, E \) can be expressed in terms of the corresponding Coxeter number \( h_G \)

\[
K_G = 2h_G, \quad h_G = \frac{\text{(number of all roots)}}{\text{(rank of group)}}.
\] (23)

This definition of the Coxeter number coincides with that of the Coxeter number of the corresponding lattices from series \( A, D, E \). The coefficient \( B_G \) can be calculated by different methods, and can be expressed also through the Coxeter number

\[
\theta_G = K_G - 4 = 2(h_G - 2)
\] (23)

Thus we see that all coefficients of RG equations depend only on the Coxeter number \( h_G \) or on the second Casimir value \( K_G \). The RG equations (21) have two separatrices [25, 27]:

\[
u_1,2 \equiv \left(\frac{g}{\delta}\right)_{1,2} = \frac{1}{2C_G} \left[ \pm (B^2_G + 8C_G)^{1/2} - B_G \right],
\] (24)

where \( u_1 \) corresponds to the phase separation line. The critical exponent \( \nu_G \), determining divergence of the correlation length \( \xi \)

\[\xi \sim a \exp(A\tau^{-\nu_G}), \quad \tau = \frac{T - T_c}{T_c},\]

is given by the next expression

\[
\nu_G = 1/\kappa_G = u_1 \left[(B_G/C_G)^2 + 8/C_G\right]^{-1/2},
\] (25)

where \( 1/\kappa_G \) is the Lyapunov index of the separatrix 1 [23]. Substituting corresponding values one finds the declinations of two separatrices

\[
u_1,2 = \left\{ \begin{array}{ll}
1/\pi h_G, & \\
-1/2\pi.
\end{array} \right.
\] (26)

Note that $u_2 = -1/2\pi$ does not depend on $G$ and is universal constant. This fact is very important for a possibility of restoration of full symmetry group $G$ (see below section 6). A schematic phase diagram is depicted on Fig.1.

![Schematic phase diagram](image_url)

Fig.1. A schematic phase diagram

The dashed line of the initial values corresponds to the initial $\sigma$-model. This line is defined by the dependencies of the parameters $\beta$ and $\mu$ on coupling constant $\alpha$ (equations (15a,b)). To low-temperature phase (decompactified, massless) corresponds region I, other regions answer to a high-temperature (compact, massive) phase. In the region I the correlation length $\xi = \infty$, and in the region II, near separatrix 1,

$$\xi \sim ae^{\lambda r^{-\nu_G}}, \; \nu_G = 2/(2 + h_G) = 4/(4 + K_G).$$  \hspace{1cm} (27)

Using known values for the Coxeter numbers $h_G$ and a geometry of the minimal dual root sets, we obtain the following expressions for critical exponents $\nu_G$ (see Table 1).

| $G$ | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\nu_G$ | $\frac{2}{n+3}$ | $\frac{1}{n}$ | $\frac{1}{2}$ | $\frac{1}{n}$ | $\frac{2}{5}$ | $\frac{1}{4}$ | $\frac{1}{7}$ | $\frac{1}{10}$ | $\frac{1}{16}$ |
It is interesting to note, that groups $D_{16} = O(32)$ and $E_8$, used in a construction of the anomaly-free theories of strings [22], have identical $\nu_G$ (together with $A_{29}$). The greatest number of possible values $\nu_G$ is given by $A_n$: $1/k$ and $2/(2k+1)$, where $k$ is an integer. For the theories with $V$, containing a set of the minimal roots, all indices remain the same, except $\nu_{B_n}$ and $\nu_{C_n}$ which interchange themselves due to mutual duality of their groups.

5. Low temperature phase and conformal symmetry

The equality $\xi \to \infty$ everywhere in low-temperature phase means an existence of conformal symmetry in it at large distancies. It can be seen also from renormalized effective action $S_{eff}$ of the theory, which has in IR limit the next form

$$S_{eff} = \int d^2x \frac{1}{2\bar{\beta}} (\partial \phi)^2,$$

(28)

where $\bar{\beta}$ is the IR limit value of the renormalized $\beta(l)$

$$\bar{\beta} = \lim_{l \to \infty} \beta(l)$$

(29)

At the PT point $\bar{\beta} = \beta^* = 8\pi/q_{\text{min}}^2$. In other points of low T phase $\bar{\beta}$ depends on initial values of the system. It is well known that action (28) describes free conformal theory with central charge $C = n$, here $n$ is a rank of group $G$. It means that long-wave low T properties of $\sigma$-models, defined on different torus $T_G$, are the same for all groups with equal rank $n$. Only logarithmic corrections at PT point will depend on group $G$ through the Coxeter number $h_G$. In this relation it becomes clear why the critical indexes depend only on $h_G$ or $C_{2dG}$. It agrees with the fact that on compact groups all quantum conformal anomalies depend also only on $h_G$ (or dual $\tilde{h}_G$) [28]. In this connection it is interesting, that $\nu_G$ coincides with a "screening" factor in the formula for the central charge $C_k$ of affine algebra $\hat{G}$ [28]

$$C_k = \frac{k}{k + h_G} \text{dim} G$$

(30)

at a level $k = 2$ or in the formula for $C_k$ in "coset" realization $\hat{G}_k \otimes \hat{G}_1/\hat{G}_{k+1}$ of minimal unitary conformal models [29]

$$C_k = n \left( 1 - \frac{h_G(h_G+1)}{(h_G+k)(h_G+k+1)} \right)$$

(31)
at a level $k = 1$.

The fact that low $T$ phase is effectively free field phase permits the calculation of correlation functions. For example, for correlation functions of the exponentials one obtains the next expression

$$
\left\langle \prod_{s=1}^{t} \exp(i(r_s \phi(x_s))) \right\rangle = \prod_{i \neq j} \left| \frac{x_i - x_j}{a} \right|^{\tilde{\beta}(r_i, r_j)/2\pi}, \quad \sum_{i=1}^{t} r = 0.
$$

At the PT point (where $\tilde{\beta} = \beta^* = 8\pi/q^2 = 4\pi$) an additional logarithmic factor, related with the ”null charge” behaviour of $g$ and $\delta$ on the critical separatrix, the phase separation line, appears in them:

$$
\prod_{i \neq j}^{t} \left( \ln \left| \frac{x_i - x_j}{a} \right| \right)^{\beta^*(r_i, r_j)/2\pi A_G} = \prod_{i \neq j}^{t} \left( \ln \left| \frac{x_i - x_j}{a} \right| \right)^{h_G \cos(r_i, r_j)},
$$

where $A_G = 4/h_G$ is a coefficient in RG equations for $\delta$ on the critical separatrix.

6. Massive phase, asymptotic freedom and global symmetry

The regions II and III answer in IR-limit to the high-temperature (in statistical mechanics language), massive (in field theory language), phase. In UV-limit the region III will be asymptotically free. A separatrix 2 with decline $u_2 = -1/2\pi$ plays also an important role. In UV-limit it is a boundary of the asymptotically free region III. There is also another possibility of the enhancement of the symmetry of the initial nonlinear $\sigma$-model on this separatrix. $\sigma$-model (1) has at classical level two symmetries: 1) scale (or conformal) symmetry, 2) isotopic global symmetry $N_G = T_G \times W_G$. At quantum level the first symmetry is, in general, spontaneously broken in IR region by vortices (see (11)). For this reason $\sigma$-model has in massive phase a finite correlation length $\xi \sim m^{-1}$, where $m$ is a characteristic mass scale of the theory. This mass must depend on the coupling constant $\alpha$ or $\beta$. The behaviour of $m$ near PT point is described by formula (27), where

$$
\tau \sim \frac{\alpha - \alpha_c}{\alpha_c}.
$$

There is another region in massive phase, the separatrix 2, where $m(\alpha)$ can be found. Since in IR-limit this separatrix attracts all trajectories in massive
(or high-temperature) phase, it is very important to know an effective mass scale on it. In main approximation on $g$ it is given by pole in RG equation or by the formula

$$m \sim \Lambda \exp(-\int g \, dx/\beta(x))$$

where $\Lambda \sim a^{-1}$ is UV cut-off parameter in momentum space, $\beta(x)$ is a $\beta$-function on the separatrix 2

$$\beta(g) = 2\pi g^2 K_G^2/2 = 2\pi g^2 h_G.$$ 

Therefore one obtains

$$m \sim a^{-1} \exp(-1/2\pi gh_G).$$

The numerical factor in $\beta$-function can vary, in dependence on a normalization of the coupling constants, but the fact that on a separatrix 2 $\beta \sim K_G^2 \sim h_G$ is a corollary of the property that the declination of this separatrix $u_2$ does not depend on $G$.

This expression for a mass scale on separatrix 2, depending only on $K_G$, coincides with those for chiral models on groups and with those, obtained from an exact solution of the appropriate fermion theories in main approximation on $g$

$$m \sim \Lambda \exp(-2\pi/(gK_G/2)), $$

Thus, it appears, that a mass scale on separatrix 2 coincides (at least for $G = A, D, E$) with that for $G$-invariant theories (chiral and fermionic), connected with simple Lie groups $G$, and can be expressed only through the Casimir operator $K_G$ by the universal formula (ours $g \rightarrow g/(2\pi)^2$)

$$m \sim \Lambda e^{-4\pi/K_G}.$$ 

It means that on this separatrix the theory (17) can become $G$-invariant. This can be seen also from the equivalence of the effective field theories (17) in cases $G = A_{n-1} = SU(n), G = D_n = O(2n), G = E_{6,7}$ to the fermion theories with the same glodal symmetry groups $G$ [25a].

From here it follows, that in massive phase of chiral theory on $T_G$ ($G = A, D, E$) in minimal representation (when $L_{\text{min}}^i = L_v$) there is strong dependence of the mass scale on coupling constant, which interpolates between
formula (27) near the PT point and formula (34) near the separatrix 2. The first region corresponds to symmetry of $T_G$, which is a torus normaliser $N_G = T_G \times W_G$, while the second one corresponds to more symmetric, $G$-invariant, situation. Analogous crossover in $m(\alpha)$ takes place in $\sigma$-models on other groups and in other representations, but the symmetry properties in two limiting regions remain not so clear.

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