CURVATURE OF HYPERKÄHLER QUOTIENTS

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Abstract. We prove estimates for the sectional curvature of hyperkähler quotients and give applications to moduli spaces of solutions to Nahm’s equations and Hitchin’s equations.

This note was motivated by the following observation: the sectional curvature of the moduli space of charge $k$ SU(2)-monopoles is bounded (by an explicit constant depending on normalisations). Unlike most statements about monopole metrics, this one has a remarkably easy proof which led us to investigate estimates on sectional curvature of general (finite or infinite-dimensional) Kähler and hyperkähler quotients.

We recall that there is an explicit formula, due to J. Jost and X.-W. Peng [7], for the sectional curvature of a large class of quotients, which include hyperkähler quotients. The quotients in [7] are formed by taking a Riemannian Banach manifold $(M,g)$ with a smooth and isometric action of Banach Lie group $G$ which is free on an invariant the level set $\phi^{-1}(c)$ of a suitable smooth map $\phi$. Jost and Peng compute the curvature of $\phi^{-1}(c)/G$ by giving variational formulae for the second fundamental form of the embedding $\phi^{-1}(c) \hookrightarrow M$ and for the O’Neill tensor of the submersion $\phi^{-1}(c) \to \phi^{-1}(c)/G$.

Our aim is to give only pointwise estimates on the sectional curvature of hyperkähler quotients and so our proofs are much simpler than in [7]. We conclude that for 1- and 2-dimensional gauge theories, i.e. moduli spaces of solutions to Nahm’s equations and to Hitchin’s equations, one gets bounds on the curvature for free, i.e. without seeking any apriori bounds on solutions of relevant differential equations. In the 1-dimensional case, this is a consequence of the Sobolev embedding $W^{1,2}(a,b) \to L^\infty(a,b)$, while in dimension 2 this follows from an analogous embedding of $W^{1,2}(Z)$ into the Orlicz space $L_{e^{2^{-1}}}(Z)$.

We also give a simple criterion for a hyperkähler quotient of a finite-dimensional vector space to have asymptotically null curvature.

1. Infinite-dimensional hyperkähler quotients

1.1. Riemannian Banach manifolds. Let $M$ be a smooth Banach manifold modelled on a Banach space $E$ (see [8] for basics on Banach manifolds). We have a well-defined tangent bundle $TM$ and the cotangent bundle $T^*M$ (bundle of continuous linear functionals). Both of these are Banach manifolds. Since $E^* \otimes E^*$ is not necessarily complete (with the norm $\|\alpha\| = \sup\{\alpha(x,y); \|x\|_E = \|y\|_E = 1\}$), we consider its completion $E^* \widehat{\otimes} E^*$ and the corresponding bundle $T^*M \widehat{\otimes} T^*M$.

Definition 1.1. A weak Riemannian metric on $M$ is a smooth section $g$ of $T^*M \widehat{\otimes} T^*M$ which induces a (continuous) positive definite symmetric bilinear form on each tangent space $T_mM$. The metric $g$ is called strong if the topology induced by $g$ on each fibre is equivalent to the topology of the model Banach space $E$. 
If $(M, g)$ is a weak Riemannian Banach manifold, then the usual proof of existence and uniqueness of the Levi-Civita connection tells us what $g(\nabla_X Y, Z)$ should be for any $Z$ (and so proves the uniqueness), but it does not guarantee existence. However, if we assume that a smooth Levi-Civita connection exists, then other Riemannian notions such as parallel transport, geodesics, exponential map, curvature make sense and have usual properties.

In what follows we shall assume that $(M, g)$ is a weak Riemannian Banach manifold such that the (smooth) Levi-Civita connection exists.

**Definition 1.2.** A weak hyperkähler Banach manifold is a weak Riemannian manifold $(M, g)$ with Levi-Civita connection $\nabla$ and three smooth anti-commuting almost complex structures $I_1, I_2, I_3$, which are fibre-wise isometries satisfying $I_1I_2I_3 = -1$ and which commute with $\nabla$.

The definition of a weak Kähler Banach manifold is analogous.

1.2. **Group actions and quotients.** Let $G$ be a Banach Lie group with Lie algebra $\mathfrak{g}$. If $G$ acts smoothly on a Banach manifold $M$, then for any $\rho \in \mathfrak{g}$ we denote by $\mathbf{\rho}$ the corresponding fundamental vector field on $M$. We write $\mathbf{\mathfrak{g}}$ for the “subbundle” of the tangent bundle generated by the vector fields $\mathbf{\rho}$.

If $M$ has a weak Riemannian metric $g$, then we define a $\mathfrak{g}^*$-valued 1-form $\Lambda$ by

$$\Lambda(v)(\rho) = g(v, \mathbf{\rho}),$$

i.e. the pointwise adjoint of the mapping $l_m : \rho \mapsto \mathbf{\rho}m$ with respect to $g$.

**Definition 1.3.** A smooth action of a Banach Lie group $G$ on a weak Riemannian manifold $(M, g)$ is called **elliptic** at a point $m$ if $\Lambda(T_m M) = \Lambda(\mathfrak{g}_m)$.

In many infinite-dimensional applications, the map $\rho \mapsto \Lambda(\mathbf{\rho}_m)$ is a second order linear differential operator whose ellipticity guarantees that $\Lambda(T_m M) = \Lambda(\mathfrak{g}_m)$.

We also observe that the condition of ellipticity at $m$ is equivalent to the addition map $\mathfrak{g}_m \times \mathfrak{g}_m^\perp \to T_m M$ being an isomorphism (here $\mathfrak{g}_m^\perp$ is the subspace $g$-orthogonal to $\mathfrak{g}_m$). In particular, it implies that $\mathfrak{g}_m$ is a closed subspace.

Recall that an action is called **proper** if the map $G \times M \to M \times M$, $(g, m) \to (gm, m)$ is proper.

**Proposition 1.1.** Let there be given a free, proper, isometric and elliptic action of a Banach Lie group $G$ on a weak Riemannian Banach manifold $(M, g)$ for which a Levi-Civita connection exists. Then the space of orbits $M/G$ is canonically a weak Riemannian Banach manifold with a Levi-Civita connection.

**Proof.** Ellipticity and properness imply that the orbits are closed submanifolds of $M$. One then constructs a slice $S_m$ using the exponential mapping for the Levi-Civita connection at a particular point $m$ of an orbit in the directions of the subbundle $\mathfrak{g}_m^\perp$. The properness of the action guarantees that $S_m$ can be chosen small enough to be a slice to the action. The properness of the action also implies that $M/G$ is Hausdorff and hence a Banach manifold. The tangent space to $M/G$ at $Gm$ is canonically identified with $\mathfrak{g}_m^\perp$ and this gives us a metric and the Levi-Civita connection.

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1 We resist the temptation to call such manifolds medium Riemannian Banach manifolds.

2 In general, $\mathfrak{g}$ is not locally trivial.
1.3. Hyperkähler quotients.

Definition 1.4. A smooth action of $G$ on a weak hyperkähler Banach manifold $(M, g, I_1, I_2, I_3)$ is called tri-Hamiltonian if there exist (moment) maps $\mu_1, \mu_2, \mu_3 : M \to \mathfrak{g}^*$ which are smooth, equivariant and satisfy

\begin{equation}
<d\mu_i(v), \rho> = g(v, I_i\bar{\rho}), \quad i = 1, 2, 3,
\end{equation}

for any tangent vector $v$ and any $\rho \in \mathfrak{g}$.

In infinite dimensions, if the metric is only weak, the image of $d\mu_i$ (which is the same as image of $\Lambda$) will only be a dense subspace of $\mathfrak{g}^*$ and so there is no hope that the moment map $\mu = (\mu_1, \mu_2, \mu_3)$ will be a submersion. We can give a simple criterion for a level set of the moment map to be a manifold. Observe first, that if $M$ is connected and $c_i \in \text{Im}\mu_i$, then $\text{Im}\mu_i \subset c_i + \text{Im}\Lambda$.

Proposition 1.2. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a moment map for a tri-Hamiltonian action of a Banach Lie group $G$ on a connected weak hyperkähler Banach manifold $M$. Let $c$ be an element of $\mathfrak{g}^* \otimes \mathbb{R}^3$ fixed by the coadjoint action of $G$ and such that the action of $G$ is locally free and elliptic at points of $\mu^{-1}(c)$. Suppose that the point-wise image $V = \Lambda(T_m M)$ of $\Lambda$ does not depend on $m$ and that $V$ can be made into a Banach space with respect to a norm $\|\cdot\|_V$, which is stronger than the one defining the topology of $\mathfrak{g}^*$ and for which $\mu : M \to c + V \otimes \mathbb{R}^3$ remains smooth. Then $\mu^{-1}(c)$ is a submanifold.

Proof. Let $M_c = \mu^{-1}(c)$. Acting by $I_i$ on the splitting $T_m M = \mathfrak{g}_m \oplus \mathfrak{g}^*_m$, we have $T_m M = I_1\mathfrak{g}_m \oplus \text{Ker}(d\mu_1)_m$. At a point $m$ of $M_c$, the spaces $I_i\mathfrak{g}$ are mutually orthogonal, and, hence, $d\mu_m |_{m}$ is an isomorphism between $(\text{Ker} d\mu)^\perp = I_1\mathfrak{g} \oplus I_2\mathfrak{g} \oplus I_3\mathfrak{g}$ and $V \otimes \mathbb{R}^3$. Therefore $\mu : M \to c + V \otimes \mathbb{R}^3$ is a submersion with respect to a stronger topology on $V$ and, so, $M_c$ is a submanifold. □

Finally, we have:

Proposition 1.3. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a moment map for a tri-Hamiltonian action of a Banach Lie group $G$ on a weak hyperkähler Banach manifold. Let $c$ be an element of $\mathfrak{g}^* \otimes \mathbb{R}^3$ fixed by the coadjoint action of $G$ such that $\mu^{-1}(c)$ is a submanifold and the action of $G$ on $\mu^{-1}(c)$ is free, isometric, elliptic and proper. Then $Q = \mu^{-1}(c)/G$ is a weak hyperkähler Banach manifold.

Proof. Proposition 1.1 shows that $Q = M_c/G$ is a Banach manifold. Its tangent space at any $Gm$ is identified with the subspace $H$ of $T_m M$ orthogonal to $\mathfrak{g} \oplus I_1\mathfrak{g} \oplus I_2\mathfrak{g} \oplus I_3\mathfrak{g}$. It is clear that $H$ inherits the Riemanian metric and it is invariant under $I_1, I_2, I_3$. Moreover, the Levi-Civita connection $\nabla$ of $Q$ is simply defined by $\nabla_{XY} = \pi_H(\nabla_X Y)$, where $\pi_H$ is the orthogonal projection onto $H$ and $X, Y$ are sections of $H$. The induced complex structures commute with $\nabla$. □

2. Curvature estimates

We assume that we are in the situation of Proposition 1.3, i.e. we have a weak hyperkähler manifold $(M, g, I_1, I_2, I_3)$ with a tri-Hamiltonian action of a Banach Lie group $G$ which is free, proper, isometric and elliptic on the $c$-level set of the hyperkähler moment map $\mu = (\mu_1, \mu_2, \mu_3) (c \in (\mathfrak{g}^* \otimes \mathbb{R}^3)^G)$. Thus, at every point $m \in \mu^{-1}(c)$, there is a splitting $T_m M = H \oplus \mathfrak{g} \oplus I_1\mathfrak{g} \oplus I_2\mathfrak{g} \oplus I_3\mathfrak{g}$ as an orthogonal sum of closed subspaces. Moreover, $\mu^{-1}(c)$ is a submanifold and hence the hyperkähler
Proposition 2.1. Under the above assumptions the sectional curvature $K_Q$ of the hyperkähler quotient $Q$ of $M$ by $G$ satisfies the pointwise estimate

$$|K_Q(p)(\pi) - K_M(m)(\tilde{\pi})| \leq 9V(m)^2$$

where $m$ is any point in $M$ projecting to $p$ and $\tilde{\pi}$ is the horizontal lift of a plane $\pi \subset T_pQ$ to $T_mM$.

**Proof.** Recall that $M_c$ denotes the c-level set of the hyperkähler moment map. The space $T_pQ$ is identified with the horizontal subspace of $T_mM$, which in turn is the subspace of $T_mM$ orthogonal to $\hat{g} \oplus I_1\hat{g} \oplus I_2\hat{g} \oplus I_3\hat{g}$. At points of $M_c$, this last decomposition is orthogonal. Let $X, Y$ be horizontal vector fields on $M_c$ and let us decompose $\nabla_X Y = Z + Z^\perp$ where $Z$ is horizontal and $Z^\perp = \hat{\rho}_0 + I_1\hat{\rho}_1 + I_2\hat{\rho}_2 + I_3\hat{\rho}$. Thus

$$\rho_i = A(X, -I_i Y) \quad i = 0, 1, 2, 3,$$

where $I_0 = -1$. Hence

\begin{equation}
|\hat{\rho}_i| \leq V(m)||Y||X|.
\end{equation}

We compute the sectional curvature $K_{M_c}(X, Y)$ of plane in $T_mM_c$ spanned by orthonormal and horizontal vectors $X, Y$. From the Gauss equation [2] (which remains true in infinite-dimensional setting), we know that

$$K_{M_c}(X, Y) = K_M(X, Y) + g(\alpha(X, X), \alpha(Y, Y)) - g(\alpha(X, Y), \alpha(X, Y))$$

where $\alpha$ is the second fundamental form of the embedding $M_c \to M$, i.e. $\alpha(X, Y) = (\nabla_X Y)^\perp$ for any extension of $Y$ to a vector field near $m$. From the above discussion $\alpha(X, Y) = I_1\hat{\rho}_1 + I_2\hat{\rho}_2 + I_3\hat{\rho}$ (and similarly for $\alpha(X, X), \alpha(Y, Y)$) and so, using the mutual orthogonality of $I_i\hat{g}_m$, the estimate (2.1) and the fact $|X| = |Y| = 1$, we obtain

\begin{equation}
|K_{M_c}(X, Y) - K_M(X, Y)| \leq 6V(m)^2.
\end{equation}
We now compare $K_M(X,Y)$ to the sectional curvature $K_Q(X,Y)$ in the quotient $Q = M_c/G$. The O’Neill formula \[ (2.2) \] shows that
\[
K_Q(X,Y) = K_M(X,Y) + 3 \left( \nabla_X Y \right)^v,
\]
where the superscript $v$ denotes the vertical part. In the above notation $(\nabla_X Y)^v = \tilde{\rho}_0$, which together with \[ (2.1) \] shows that
\[
|K_Q(X,Y) - K_M(X,Y)| \leq 3V(m)^2.
\]

We observe that the proof obviously works as well for Kähler quotients and gives a similar estimate. In the Kähler case, moreover, the curvature must increase by at least $V(m)$ on some planes:

**Proposition 2.2.** Let $M$ be a weak Kähler Banach manifold with a Hamiltonian action of a Banach Lie group $G$, which is free, proper and strongly isometric on the $c$-level set of the Kähler moment map $\mu$ ($c \in (\mathfrak{g}^* \otimes \mathbb{R})^G$). The sectional curvature $K_Q$ of the Kähler quotient $Q = \mu^{-1}(c)/G$ of $M$ by $G$ satisfies the pointwise estimate
\[
|K_Q(p)(\pi) - K_M(m)(\tilde{\pi})| \leq 5V(m)^2
\]
where $m$ is any point in $M$ projecting to $p$ and $\tilde{\pi}$ is the horizontal lift of a plane $\pi \subset T_pQ$ to $T_mM$. Moreover
\[
\sup_{\pi} (K_Q(p)(\pi) - K_M(m)(\tilde{\pi})) \geq V(m).
\]

**Proof.** The proof is essentially the same. The estimate \[ (2.2) \] can be replaced by
\[
|K_M(X,Y) - K_M(X,Y)| \leq 2V(m)^2.
\]
This, together with \[ (2.3) \] and the definition of $V(m)$, proves the estimates. \[ \square \]

We now wish to give estimates on $V(m)$, which can actually be computed in applications. Let $(M,g,I_1,I_2,I_3)$, $G$ and $c \in (\mathfrak{g}^* \otimes \mathbb{R})^G$ be as above. Fix an $m \in M_c = \mu^{-1}(c)$. We choose any norm $\| \cdot \|_m$ on $\mathfrak{g}$. This norm can be completely different from the norm used to define the Banach Lie algebra on $\mathfrak{g}$. Moreover, this norm may depend on $m \in M_c$.

**Definition 2.3.** Let $B : (\mathfrak{g}^* \otimes \mathbb{R})^G$ denote the bilinear map given by $B(X,\rho) \mapsto Z$, where $Z$ is the $(\mathfrak{g}^* \otimes \mathbb{R})^G$-part of $\nabla_X \tilde{\rho}$.

We now define $F(m)$ as the norm of $B_m$ with respect to the norm $\| \cdot \|_m$ on $\mathfrak{g}$ and $| \cdot | = \sqrt{g(\cdot,\cdot)}$ on $T_mM$:
\[
F(m) = \sup \left\{ \frac{B(X,\rho)}{|X||\rho|_m} \mid (X,\rho) \in (\mathfrak{g}^* \otimes \mathbb{R})^G_m \times \mathfrak{g}, \ X,\rho \neq 0 \right\}.
\]
Consider again the $\mathfrak{g}^*$-valued 1-form $\Lambda$ given by $\Lambda(v)(\rho) = g(v,\tilde{\rho})$. In other words
\[
\Lambda = -I_1d\mu_1 = -I_2d\mu_2 = -I_3d\mu_3.
\]
We denote by $\| \cdot \|_*^m$ the “norm” induced by $\| \cdot \|_m$ on $\text{Im} \Lambda \subset \mathfrak{g}^*$, i.e. $\|L\|_*^m = \sup\{ L(\rho) ; \| \rho \| = 1 \}$. Let $l(m)$ be the norm of $(\Lambda_m)^{-1} : \text{Im} \Lambda \rightarrow \hat{\mathfrak{g}}$:

$$l(m) = \sup_{\rho \neq 0} \frac{| \rho_m |}{\| \Lambda(\rho_m) \|_*^m}.$$  

(2.6)

Without further assumptions both $l(m)$ and $F(m)$ can be infinite. We have:

**Proposition 2.3.** For any norm $\| \cdot \|_m$ on $\mathfrak{g}$ the following inequality holds:

$$V(m) \leq l(m)F(m).$$

**Proof.** Let $X,Y \in (\hat{\mathfrak{g}}^\mathbb{H})_m^\perp$ and let $\hat{X},\hat{Y}$ be local sections of $(\hat{\mathfrak{g}}^\mathbb{H})_m^\perp$ extending $X$ and $Y$. Let $\hat{\rho}$ be the fundamental vector field whose value at $m$ is the $\hat{\mathfrak{g}}$-part of $\nabla_X Y$ (as the action is free at $m$, this is well defined). We need to estimate $| \hat{\rho}_m |$. Let $\nu$ be any element of $\mathfrak{g}$. Since $\hat{Y}$ is horizontal, $g(\hat{Y}, \nu) \equiv 0$, and since $\hat{X}$ is also horizontal

$$g(\nabla_{\hat{X}} \hat{Y}, \nu) = -g(\hat{Y}, \nabla_{\hat{X}} \nu).$$

At the point $m$ we can rewrite this as

$$\langle \Lambda(\hat{\rho}), \nu \rangle = -g(\hat{Y}, \nabla_{\hat{X}} \nu).$$

Therefore $\| \Lambda(\hat{\rho}_m) \|_*^m \leq F(m)\| X \| Y \|$, and using the definition of $l(m)$ we get

$$| \rho_m | \leq l(m)F(m)\| Y \| X \|, \quad i = 0, 1, 2, 3,$$

which proves the estimate. \qed

Let us discuss this. We start with $F(m)$. From its definition, $F(m)$ is finite if the bilinear operator $B_m$ is continuous for the norms $\| \cdot \|_m$ and $| \cdot |$. This tells us which norms $\| \cdot \|_m$ are allowed on $\mathfrak{g}$. If the hyperkähler quotient is finite-dimensional, it is easier to decide on $\| \cdot \|_m$:

**Lemma 2.4.** If $\dim Q < +\infty$, then $F(m)$ is bounded providing the linear operator $\rho \rightarrow (\nabla X \hat{\rho})_m$ is bounded for every $X \in (\hat{\mathfrak{g}}^\mathbb{H})_m^\perp$ with respect to the norms $\| \cdot \|_m$ on $\mathfrak{g}$ and $| \cdot |$ on $T_m M$.

**Proof.** As $(\hat{\mathfrak{g}}^\mathbb{H})_m^\perp$ is finite-dimensional, the bilinear operator $B_m$ is separately continuous in both variables. The Mazur-Orlicz theorem (see e.g. Corollary 8 in [10]) implies that $B_m$ is continuous. \qed

We observe next that it is easy estimate $F(m)$ for a flat $M$:

**Lemma 2.5.** Let $(M,g)$ be a flat weak hyperkähler manifold, i.e. $(M,g,I_1, I_2, I_3)$ is isomorphic to an open subset of a quaternionic Banach space $E$ with a continuous bilinear form $g$ on $E$. Suppose that, under this isometry, $\hat{\rho}_m = L(\rho, m) + P(\rho)$, where $L : \mathfrak{g} \times E$ is a bilinear operator and $P$ is independent of $m \in E$. Let $\| \cdot \|$ be a norm on $\mathfrak{g}$ such that $L$ is continuous with respect to $\| \cdot \|$ on $\mathfrak{g}$ and $g$ on $E$. Then $F(m)$ is uniformly bounded for $\| \cdot \|_m = \| \cdot \|$. \qed

In particular $F$ is bounded for linear actions of Hilbert Lie groups on Hilbert spaces (with strong metrics).

On the other hand, we have the following estimates on $l(m)$:
Lemma 2.6.

\[ l(m) \leq \sup_{\rho \neq 0} \frac{\|\rho\|_m}{|\hat{\rho}_m|}, \]

\[ l(m) \leq \sup_{\rho \neq 0} \left( \frac{\|\rho\|_m}{\|\Lambda(\hat{\rho}_m)\|_m} \right)^{1/2}. \]

Proof. From the definition of \( \Lambda \), we have, for any \( \rho \in g \), \( \langle \Lambda(\hat{\rho}), \rho \rangle = g(\hat{\rho}, \hat{\rho}) \) and hence

\[ |\hat{\rho}_m|^2 \leq \|\Lambda(\hat{\rho}_m)\|_m \cdot \|\rho\|_m. \]

The estimates follow thanks to (2.6).

Thus \( l \) is uniformly bounded (resp. is asymptotically null on \( Q \)) if, for every \( \rho \in g \) with \( \|\rho\|_m = 1 \), the length of \( \hat{\rho}_m \) is bound away from zero (resp. is asymptotically infinite) on the \( c \)-level set of the hyperkähler moment map. In quotients of infinite-dimensional manifolds, it is the second inequality that is useful: the map \( \rho \mapsto \Lambda(\hat{\rho}_m) \) is often a positive-definite self-adjoint elliptic operator and one easily gets an estimate on \( \|\rho\| \) in terms of the norm of \( \Lambda(\hat{\rho}_m) \).

If we set \( \|\rho\|_m = |\hat{\rho}_m| \) in the first inequality of the above lemma, we get that \( l(m) \leq 1 \), and hence we obtain

Corollary 2.7. \( V(m) \) is bounded by the norm of the second O’Neill tensor at \( m \), i.e. the norm of the bilinear operator \( C : (\mathfrak{g}^\perp)_m^\perp \times \mathfrak{g}_m \to (\mathfrak{g}^\perp)_m^\perp, C(X, U) \) is the \( (\mathfrak{g}^\perp)^\perp \)-part of \( (\nabla_X \hat{\rho})_m \), where \( \rho \) is the unique element of \( \mathfrak{g} \) such that \( \hat{\rho}_m = U \).

3. Finite-dimensional quotients

We give a simple application to hyperkähler quotients of finite-dimensional vector spaces.

Let \( M = \mathbb{H}^d \) with its Euclidean hyperkähler structure and \( G \) be a closed subgroup of \( Sp(d) \) acting linearly on \( M \). If we identify \( sp(d) \) and \( sp(d)^* \) with quaternionic matrices \( A \) satisfying \( A^\dagger = -A \), where \( \dagger \) is transposition followed by quaternionic conjugation, then the hyperkähler moment map for the action of \( Sp(d) \) is

\[ q \mapsto (qiq^\dagger, qq^\dagger, qkq^\dagger) \]

and we denote by \( \mu = (\mu_1, \mu_2, \mu_3) \) the projection of (3.1) onto three copies of the Lie algebra \( \mathfrak{g} \) of \( G \). \( \mu \) is a particular hyperkähler moment map for \( G \).

We have

Theorem 3.1. With the above notation, suppose that \( G \) acts locally freely on the set \( \mu^{-1}(0) - \{0\} \). Let \( c = (c_1, c_2, c_3) \) with each \( c_i \) a central element of \( \mathfrak{g} \) and \( G \) acting freely on \( \mu^{-1}(c) \). Then the curvature of the hyperkähler quotient \( Q = \mu^{-1}(c)/G \) is asymptotically null.

Proof. We use Propositions 2.1 and 2.3. In the finite-dimensional case any norm \( \| \cdot \| \) on \( \mathfrak{g} \) will do. As observed in Lemma 2.3, \( F \) is uniformly bounded. We are going to estimate \( l \) from the first statement of Lemma 2.6. Let \( S \) denote the unit sphere in \( \mathbb{H}^d \). Since \( G \) is compact and acts locally freely on \( \mu^{-1}(0) - \{0\}, G \) acts locally freely on \( V = S \cap \mu^{-1}(D), \) where \( D \) is some small closed neighbourhood of 0 in the center of \( \mathfrak{g} \). Therefore, for any \( \rho \in \mathfrak{g} \), the minimum of \( |\hat{\rho}| \) over \( V \) is a non-zero number, say \( \lambda(\rho) \). Consider now the asymptotic behaviour of \( |\hat{\rho}| \) on \( \mu^{-1}(c) \).
Let \( q \in \mu^{-1}(c) \) and \( |q| = R \). Then \( q/R \in \mu^{-1}(c/R^2) \) and so, for large enough \( R \), \( |\tilde{\rho}_q| \geq \lambda(\rho) \). Therefore \( |\tilde{\rho}_q| \geq R\lambda(\rho) \) and the result is proven. \( \square \)

This fact is of course to be expected as \( \mu^{-1}(c)/G \) is asymptotically isometric to \( \mu^{-1}(0)/G \) and the latter is a cone over a compact orbifold.

The example of \( T^*\mathbb{P}^1 \times T^*\mathbb{P}^1 \), which is the hyperkähler quotient of \( \mathbb{H}^2 \times \mathbb{H}^2 \) by \( S^1 \times S^1 \) (acting separately on each \( \mathbb{H}^2 \)), shows that, without the assumption, the quotient does not have to have an asymptotically null curvature.

### 4. Moduli spaces of solutions to Nahm’s equations

We wish to estimate the sectional curvature of moduli spaces of solutions to Nahm’s equations on an interval with prescribed poles at the end of the interval. We first describe the hyperkähler quotient construction of this moduli space in order to check that the conditions of Proposition 1.3 hold.

#### 4.1. Construction.

Let \( G \) be a compact Lie group with Lie algebra \( \mathfrak{g} \) and let \( \langle \ , \ \rangle \) denote a positive-definite \( \text{Ad}G \)-invariant inner product on \( \mathfrak{g} \) with respect to which

\[
\| [A, B] \| \leq 2 |A| |B|, \quad \text{for all } A, B \in \mathfrak{g}.
\]

Let \( \alpha_i, \beta_i \in \mathfrak{g}, \ i = 1, 2, 3 \) satisfy \( [\alpha_i, \alpha_j] = \epsilon_{ijk} \alpha_k \) and similarly for \( \beta_i \), so that these define homomorphisms \( \text{su}(2) \to \mathfrak{g} \).

We are going to construct the moduli space of solutions to Nahm’s equations on an interval \((a, b)\) with simple poles at \( a, b \) and residues \( \alpha_i, \beta_i \).

Let \( e_1, e_2, e_3 \) denote the right multiplication by \( i, j, k \) on \( \mathbb{H} \) and consider the linear operators \( L_\alpha = \sum_{i=1}^3 (\text{ad} \alpha_i) \otimes e_i \) and \( L_\beta = \sum_{i=1}^3 (\text{ad} \beta_i) \otimes e_i \) on \( \mathfrak{g} \otimes \mathbb{H} \).

We define a space \( E \) as the space of \( \mathfrak{g} \otimes \mathbb{H} \)-valued continuously differentiable functions \( u \) on \((a, b)\) such that

\[
L(u)(s) = \frac{du}{ds} - \frac{L_\alpha(u)(s)}{s-a} - \frac{L_\beta(u)(s)}{s-b}
\]

is continuous on \([a, b]\). The right multiplication by quaternions preserves this space. We put a norm on \( E \) by

\[
\|u\| = \|u\|_{C^0} + \|L(u)\|_{C^0},
\]

where the \( C^0 \)-norms are the sup-norms defined on \( \mathfrak{g} \otimes \mathbb{H} \)-valued functions using the usual norm on the quaternions and the chosen invariant inner product on \( \mathfrak{g} \). With this norm \( E \) is a Banach space (and a closed subspace of \( C^1(a, b) \)).

We consider the following metric \( g \) on \( E \): if, after identifying \( \mathbb{H} \simeq \mathbb{R}^4 \), \( v = (t_0, t_1, t_2, t_3) \), \( v' = (t'_0, t'_1, t'_2, t'_3) \) are in \( E \), then

\[
g(v, v') = \sum_{i=0}^3 \int_a^b \langle t_i, t'_i \rangle ds.
\]

We now consider an affine space \( \mathcal{M} \) defined as \( iS_1(t) + jS_2(t) + kS_3(t) + E \), where \( S_i(s) = \frac{\partial S_i}{\partial a} + \frac{\partial S_i}{\partial b}, \ i = 0, 1, 2, 3 \). We view \((\mathcal{M}, g)\) as a flat weak hyperkähler Banach manifold (modelled on \( E \)) consisting of \( \mathfrak{g} \)-valued quadruples \((T_0, T_1, T_2, T_3)\) with prescribed boundary behaviour.
We also define $G$ as the group of gauge transformations $g : [a,b] \rightarrow G$ whose Lie algebra are maps $\rho : [a,b] \rightarrow \mathfrak{g}$ of class $C^2$ on $(a,b)$ satisfying $\rho(a) = \rho(b) = 0$ and

\begin{equation}
(4.4) \quad P(\rho)(s) = \frac{d^2 \rho}{ds^2} + \sum_{i=1}^{3} \frac{(ad^2 \alpha_i)\rho(s)}{(s-a)^2} + \sum_{i=1}^{3} \frac{(ad^2 \beta_i)\rho(s)}{(s-b)^2}
\end{equation}

continuous on $[a,b]$. We equip this Lie algebra with the Banach norm $\|\rho\|_{C^1} + \|P(\rho)\|_{C^0}$ (all norms are sup-norms on $(a,b)$).

The group $G$ acts smoothly on $\mathcal{M}$ by

\begin{align*}
T_0 &\mapsto Ad(g)T_0 - \dot{g}^{-1} \\
T_i &\mapsto Ad(g)T_i, \quad i = 1, 2, 3.
\end{align*}

Differentiating, we obtain that for any $\rho \in \text{Lie } G$

\begin{equation}
(4.6) \quad \dot{\rho}_T = (-\dot{\rho} + [\rho, T_0], [\rho, T_1], [\rho, T_2], [\rho, T_3]).
\end{equation}

This action is Hamiltonian and the moment map equations are:

\begin{equation}
(4.7) \quad \mu_i(T_0, T_1, T_2, T_3)(\rho) = \int_a^b \langle \dot{T}_i + [T_0, T_i] - \frac{1}{2} \sum_{j,k=1,2,3} \epsilon_{ijk}[T_j, T_k], \rho \rangle ds.
\end{equation}

The 0-level set of $\mu = (\mu_1, \mu_2, \mu_3)$ is given by the Nahm’s equations: $\dot{T}_1 + [T_0, T_1] - [T_2, T_3] = 0$ etc. We need to check that the conditions of Proposition 1.3 are satisfied. The action of $G$ is isometric, free and proper everywhere. The form $\Lambda$ is given by:

\begin{equation}
(4.8) \quad \Lambda(t_0, t_1, t_2, t_3)(\rho) = \int_a^b \langle \dot{t}_0 + [T_0, t_0] + [T_1, t_1] + [T_2, t_2] + [T_3, t_3], \rho \rangle ds.
\end{equation}

Thus the image of $\Lambda$ is contained in the subspace $V$ of $(\text{Lie } G)^*$ given by the pairing $\int_a^b \langle f, \rho \rangle ds$ for $f \in C^0([a,b]) \otimes \mathfrak{g}$. According to Propositions 1.2 and 1.3, we need to show that any element of $V$ can be obtained as $\Lambda(\dot{\rho})$. Substituting (4.6) into (4.8) expresses $\Lambda(\dot{\rho}) = \langle f, \cdot \rangle$ as the equation

\begin{equation}
(4.9) \quad -\dot{\rho} + 2[\rho, T_0] + [\rho, \dot{T}_0] - \sum_{i=0}^{3} [T_i, [T_i, \rho]] = f.
\end{equation}

It is easy to obtain a solution in $C^2(a,b)$ of this equation (with bounded $C^1(a,b)$-norm), e.g. by finding a solution in $W^{1,1}(a,b)$ using variational methods or by approximation method as in Lemmas 2.18-2.20 in [3]. It is then automatic that $\rho(t) \in \text{Lie } G$.

Thus, the quotient of the space of solutions to (4.7) by $G$ is a (finite-dimensional) hyperkähler manifold $Q$, and the tangent space at a solution $(T_0, T_1, T_2, T_3)$ can be identified with the space of solutions to the following system of linear equations:

\begin{align*}
(4.10) \quad \dot{t}_0 + [T_0, t_0] + [T_1, t_1] + [T_2, t_2] + [T_3, t_3] &= 0, \\
\dot{t}_1 + [T_0, t_1] - [T_1, t_0] - [T_2, t_3] + [T_3, t_2] &= 0, \\
\dot{t}_2 + [T_0, t_2] + [T_1, t_3] - [T_2, t_0] - [T_3, t_1] &= 0, \\
\dot{t}_3 + [T_0, t_3] - [T_1, t_2] + [T_2, t_1] - [T_3, t_0] &= 0.
\end{align*}

The first equation is the condition that $(t_0, t_1, t_2, t_3)$ is orthogonal to the infinitesimal gauge transformations and the remaining three are linearisations of Nahm’s equations.
4.2. Curvature of the Nahm moduli spaces. We shall estimate the sectional curvature of the above moduli space $Q$ of solutions to Nahm’s equations from Theorem 2.1. We need first some definitions.

**Definition 4.1.** Let $\lambda : [a, b] \to \mathbb{R}$ be a continuous function. Let $E = \{ u \in W^{2,1}(a, b); u(a) = u(b) = 0 \}$ and let $L_\lambda : E \to L^1(a, b)$ be the linear operator

$$L_\lambda(u)(s) = \ddot{u}(s) - \lambda(s)^2 u(s).$$

$L$ has a continuous inverse and we define $N(\lambda)$ as the norm of $j \circ L_\lambda^{-1}$ where $j : W^{2,1}(a, b) \to C^0([a, b])$ is the embedding (and $C^0([a, b])$ is equipped with usual max norm).

**Theorem 4.1.** Let $T = (T_0, T_1, T_2, T_3)$ be a solution to Nahm’s equations on $[a, b]$ which is an element of a hyperkähler quotient $Q$ constructed above. Let $\lambda(s)$ be a continuous real function such that, for every $s \in [a, b]$, $\lambda^2(s)$ is not greater than the smallest eigenvalue of the operator $H(s) = -(\text{ad} T_1(s))^2 - (\text{ad} T_2(s))^2 - (\text{ad} T_3(s))^2$. Then the sectional curvature of $Q$ at $T$ is bounded by $18N(\lambda)^{1/2}$.

**Proof.** We shall use Propositions 2.1, 2.3 together with Lemmata 2.5 and 2.6. For every solution $T$ to Nahm’s equations the norm $\| \cdot \|_T$ on $\text{Lie} \mathbb{G}$ will be the $L^\infty$-norm, so the dual norm $\| \cdot \|_T^*$ is the $L^1$-norm. Since the Levi-Civita connection on $\mathcal{A}$ is simply the directional derivative, we have, using (4.10), for a horizontal tangent vector $X = (t_0, t_1, t_2, t_3)$,

$$\nabla_X \dot{\rho} = ([\rho, t_0], [\rho, t_1], [\rho, t_2], [\rho, t_3]).$$

Hence, because of the normalisation (4.11),

$$g(\nabla_X \dot{\rho}, \nabla_X \dot{\rho}) \leq 4g(X, X)\|\rho\|_{L^\infty}^2,$$

and consequently

$$F(T) \leq 2.$$

We now estimate $l(T)$ using the second statement in Lemma 2.6. We first observe that under the action of a gauge transformation $g(s)$, $\rho, \dot{\rho}$ and $\Lambda(\dot{\rho})$ are all pointwise conjugated by $g(s)$ and therefore the values of $l$ at $T$ and at $g.T$ are the same. Similarly, the eigenvalues of $H$ do not change under the action of $g$. Therefore we can assume that $T_0 = 0$.

The form $\Lambda$ is given by the equation (4.8). Substituting (4.6) into (4.8) we get (as $T_0 = 0$)

$$\dot{\rho} = H(s)\rho - \Lambda(\dot{\rho}),$$

where $H(s)$ is the positive-definite Hermitian operator defined in the statement of the theorem. Therefore

$$\frac{d^2}{ds^2} \langle \rho, \rho \rangle = 2\langle \dot{\rho}, \dot{\rho} \rangle + 2\langle \dot{\rho}, \rho \rangle \geq 2\langle \dot{\rho}(s), \rho(s) \rangle + 2\lambda(s)^2|\rho(s)|^2 - |\Lambda(\dot{\rho})(s)||\rho(s)|.$$

Combining this with

$$\frac{d^2}{ds^2}|\rho(s)|^2 = 2|\rho(s)||d^2|\rho(s)||ds^2 + 2\left(\frac{d|\rho(s)|}{ds}\right)^2 \leq 2|\rho(s)|^2\frac{d^2|\rho(s)|}{ds^2} + 2\left|\frac{d\rho(s)}{ds}\right|^2,$$

we get

$$\frac{d^2}{ds^2}|\rho(s)| \geq \lambda(s)^2|\rho(s)| - |\Lambda(\dot{\rho})(s)|.$$
From this one easily concludes that $|\rho(s)|$ is point-wise bounded by the solution $u(s)$ to $L_\lambda(u) = -|\Lambda(\hat{\rho})(s)|$ with $u(a) = u(b) = 0$. The definition of $N(\lambda)$ gives now $\|\rho\|_{L^\infty} \leq N(\lambda)\|\Lambda(\hat{\rho})\|_{L^1}$, and Lemma 2.6 implies that $l(T) \leq N(\lambda)^{1/2}$. \hfill \Box

**Corollary 4.2.** The sectional curvature of a moduli space of solutions to Nahm’s equations on $(a, b)$ with the boundary conditions described above and the metric \( g \) satisfying \((4.3)\) is bounded by \( 9\sqrt{b-a} \).

**Proof.** We use the last theorem with \( \lambda = 0 \). Thus we need to estimate the $C^0$-norm of the unique solution $u$ of the $\bar{\partial}z$-equations with $u(a) = u(b) = 0$ in terms of the $L^1$-norm of $h$. We have an explicit solution to this boundary problem:

$$u(s) = -\frac{1}{b-a} (b-s) \int_a^s (\tau-a)h(\tau)d\tau - \frac{1}{b-a} (s-a) \int_s^b (b-\tau)h(\tau)d\tau.$$  

From this

$$|u(s)| \leq \frac{(b-s)(s-a)}{b-a} \|h\|_{L^1},$$

and so $N(0) = (b-a)/4$. \hfill \Box

**Remark 4.1.** The moduli space of charge $k$ $SU(2)$-monopoles arises when $G = U(k)$ and the residues $\alpha_i, \beta_i$ of the solutions to Nahm’s equations at $a, b$ define irreducible representations of $su(2)$. Therefore the curvature of this moduli space is bounded. One can extend the above proof to more complicated moduli spaces of solutions to Nahm’s equations, such as those in \([9]\), to show that the moduli spaces of $G$-monopoles with maximal symmetry breaking have bounded curvature ($G$ - a classical compact group).

**Remark 4.2.** If we are only interested in a global bound on the curvature, and not in the finer estimates of Theorem 4.1 then the proof is much simpler. We can use Corollary 2.7 if $|\hat{\rho}| \leq 1$, then $\|\hat{\rho}\|_{L^1} \leq 1$ and now the Poincaré inequality together with the Sobolev embedding $W^{1,2}(a, b) \to C^0([a, b])$ implies that $\|\rho\|_{L^\infty} \leq K$ for some constant $K$ depending only on $b-a$. The estimate \((4.11)\) and Corollary 2.7 give us a bound on the sectional curvature.

5. **Moduli spaces of solutions to Hitchin’s equations**

We briefly recall the hyperkähler quotient construction of the moduli space of solutions to Hitchin’s equations on a Riemann surface with boundary \([5, 4]\). Since the fields do not have singularities, this is actually simpler than for Nahm’s equations considered in the previous section.

Let $Z$ be a compact connected 2-dimensional Kähler manifold $Z$ with boundary $\partial Z$ which may be empty. Let $P$ be a principal bundle over $Z$ with a compact structure group $G$ and a $G$-trivialisation over $\partial Z$. For a positive integer $k$, we denote by $\Omega^1_k(ad\, P)$ the space of $ad\, P$-valued 1-forms of Sobolev class $W^{k,2}$, i.e. those whose first $k$ derivatives are square-integrable. Similarly for other forms. We now consider the affine manifold $M = (D_0 + \Omega^0_4(ad\, P \otimes \mathbb{C})) \oplus \Omega^1_4(ad\, P \otimes \mathbb{C})$, where $D_0$ is a fixed $G$-connection. We identify $\Omega^1_4(ad\, P \otimes \mathbb{C})$ with $\Omega^0_4(ad\, P)$ and we view elements of $D_0 + \Omega^0_4(ad\, P \otimes \mathbb{C})$ as $G$-connections. The metric on $Z$ and an invariant inner product on $\mathfrak{g}$ induce an $L^2$-metric on $\Omega^2_4(Ad\, P \otimes \mathbb{C})$ and on $\Omega^0_4(Ad\, P \otimes \mathbb{C})$. With this metric, $M$ is a flat weak hyperkähler Hilbert manifold.
The three anti-commuting complex structures are given, on each tangent space \( \Omega^{0,1}(\text{Ad} \, P \otimes \mathbb{C}) \oplus \Omega^{1,0}(\text{Ad} \, P \otimes \mathbb{C}) \), by:

\[
I_1(a, \phi) = (ia, i\phi), \quad I_2(a, \phi) = (-\phi^*, a^*), \quad I_3(a, \phi) = (-i\phi^*, ia^*).
\]

The Levi-Civita connection is again provided by the directional derivative.

Proposition 5.2. Let \( \mu \) be a solution to Hitchin’s equations on \( Z \) and \( \mu \) is bounded by \( C \cdot \text{sup}_{\rho \in \text{Lie} \, G - \{0\}} \| \rho \|_{W^{1,2}} \) for any \( X, \dot{\rho} \) whose \( L^2 \)-norms are 1. Since \( \nabla \) is simply the directional derivative, it follows that for \( X = (a, \phi) \)

\[
\nabla_X \dot{\rho} = ([\rho, a], [\rho, \phi]).
\]

The desired bound on \( \nabla_X \dot{\rho} \) follows, by using local trivialisations and partitions of unity, from the following extension of the Sobolev multiplication theorem to the critical case:

**Proposition 5.2.** Let \( Y \) be a compact 2-dimensional Riemannian manifold \( Y \) with or without boundary. Then the multiplication of functions is a well-defined continuous bilinear operator

\[
L^2(Y) \times W^{1,2}(Y) \to L^2(Y).
\]
Proof. Although we do not have an embedding of $W^{1,2}(Y)$ into $L^\infty(Y)$, we do have a continuous embedding into certain Orlicz space, namely, if $\psi \in W^{1,2}(Y)$, then $e^{\|\psi\|} \in L^1(Y)$ and

\begin{equation}
  \int_Y e^{\|\psi\|} d\mu \leq C \exp(\alpha \|\psi\|^2_{W^{1,2}})
\end{equation}

for some constants $C, \alpha$ depending only on $Y$ (see, e.g., Theorem 2.46 in [1] for a proof). Let now $\phi \in L^2(Y)$ and $\psi \in W^{1,2}(Y)$ both have norm 1 in the respective spaces. Define, for a nonnegative integer $K$,

$$Y_K = \{ z \in Y; \ K \leq |\psi(z)| \leq K + 1 \}.$$

It follows from (5.2) that $\mu(Y_K) \leq Ce^{\alpha-K}$. We now compute

$$\int_Z |\phi\psi|^2 d\mu = \sum_{K \geq 0} \int_{Y_K} |\phi\psi|^2 d\mu \leq \sum_{K \geq 0} (K+1)^2 Ce^{\alpha-K},$$

and, hence, the $L^2$-norm of $\phi\psi$ is bounded by some constant depending only on $Y$. \hfill \Box

**Corollary 5.3.** Let $Z$ be a compact connected Riemann surface with a non-empty boundary and $P$ a principal $G$-bundle trivialised on $\partial Z$. Then the sectional curvature of the moduli space $Q$ of solutions to Hitchin’s equations on $(Z, Ad P \otimes \mathbb{C})$ framed on $\partial Z$ is bounded.

Proof. Let $(A, \Phi) \in Q$ and $\rho$ be a fundamental vector field whose $L^2$-norm is 1 at $(A, \Phi)$. Then $\|\nabla_A \rho\|_{L^2} \leq 1$ and, since $\rho$ vanishes on the boundary, the Kato and Poincaré inequalities imply that the $W^{1,2}$-norm of $\rho$ is bounded by some constant, which does not depend on $(A, \Phi)$. Therefore the expression in the statement of the last theorem is finite and does not depend on $(A, \Phi)$. \hfill \Box

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