A CONJECTURE ON CRITICAL GRAPHS AND CONNECTIONS TO THE PERSISTENCE OF ASSOCIATED PRIMES

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Abstract. We introduce a conjecture about constructing critically \((s + 1)\)-chromatic graphs from critically \(s\)-chromatic graphs. We then show how this conjecture implies that any unmixed height two square-free monomial ideal \(I\) in a polynomial ring \(R\), i.e., the cover ideal of a finite simple graph, has the persistence property, that is, \(\text{Ass}(R/I^s) \subseteq \text{Ass}(R/I^{s+1})\) for all \(s \geq 1\). To support our conjecture, we prove that the statement is true if we also assume that \(\chi_f(G)\), fractional chromatic number of the graph \(G\), satisfies \(\chi(G) - 1 < \chi_f(G) \leq \chi(G)\). We give an algebraic proof of this result.

1. Introduction

Let \(G = (V_G, E_G)\) be a finite simple graph with vertex set \(V_G = \{x_1, \ldots, x_n\}\) and edge set \(E_G\). We say that \(G\) has a \(s\)-{f c}oloring if there exists a partition \(V_G = C_1 \cup \cdots \cup C_s\) such that for every \(e \in E_G\), \(e \not\subseteq C_i\) for \(i = 1, \ldots, s\). The minimal integer \(s\) such that \(G\) has a \(s\)-coloring is called the \textbf{chromatic number} of \(G\), and is denoted \(\chi(G)\). The coloring of graphs is one of the main branches in the field of graph theory and has many applications to other fields. In this short note, we propose a conjecture about the coloring of critical graphs that arose out of our study of an algebraic question about the associated primes of powers of square-free monomial ideals.

To state our conjecture, we recall the following definitions and construction. A graph \(G\) is said to be \textbf{critically \(s\)-chromatic} if \(\chi(G) = s\) but \(\chi(G \setminus x) = s - 1\) for every \(x \in V_G\), where \(G \setminus x\) denotes the graph obtained from \(G\) by removing the vertex \(x\) and all edges incident to \(x\). A graph that is critically \(s\)-chromatic for some \(s\) is called \textbf{critical}. For any vertex \(x_i \in V_G\), the \textbf{expansion} of \(G\) at the vertex \(x_i\) is the graph \(G' = G[\{x_i\}]\) whose vertex set is given by \(V_{G'} = (V_G \setminus \{x_i\}) \cup \{x_i, 1, x_i, 2\}\) and whose edge set has form \(E_{G'} = \{\{u, v\} \in E_G \mid u \neq x_i\ and v \neq x_i\} \cup \{\{u, x_{i,1}\}, \{u, x_{i,2}\} \mid \{u, x_i\} \in E_G\} \cup \{x_{i,1}, x_{i,2}\}\).

Equivalently, \(G[\{x_i\}]\) is formed by replacing the vertex \(x_i\) with the clique \(K_2\) on the vertex set \(\{x_{i,1}, x_{i,2}\}\). For any \(W \subseteq V_G\), the \textbf{expansion} of \(G\) at \(W\), denoted \(G[W]\), is formed by successively expanding all the vertices of \(W\) (in any order). We propose the following conjecture:

\textbf{Conjecture 1.1.} Let \(s\) be a positive integer, and let \(G\) be a finite simple graph that is critically \(s\)-chromatic. Then there exists a subset \(W \subseteq V_G\) such that \(G[W]\) is a critically \((s + 1)\)-chromatic graph.
If Conjecture 1.1 holds, then given any critically $s$-chromatic graph, one can then construct a critically $(s+d)$-chromatic graph for any integer $d \geq 1$ by repeatedly applying the conjecture. Note that Conjecture 1.1 is true if we also assume that $G$ is a perfect graph since the only perfect critically $s$-chromatic graph is $K_s$, the clique of size $s$. If we expand $K_s$ at any vertex, we obtain the critically $(s+1)$-chromatic graph $K_{s+1}$.

Conjecture 1.1 arose out of our investigations of the associated primes of powers of the cover ideal of a graph [6]. Let $k$ be a field, and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over $k$. For any ideal $I \subseteq R$, a prime ideal $P$ is an associated prime of $R/I$ if there exists an element $T \in R$ such that $I : (T) = \{F \in R \mid FT \in I\} = P$. The set of associated primes of $R/I$ is denoted by $\text{Ass}(R/I)$. Consider the sets $\text{Ass}(R/I^s)$ as $s$ varies. Then Brodmann (see [1]) proved that there exists an integer $a$ such that

$$\bigcup_{s=1}^{\infty} \text{Ass}(R/I^s) = \bigcup_{s=1}^{a} \text{Ass}(R/I^s).$$

However, it is not true in general that $\text{Ass}(R/I^s) \subseteq \text{Ass}(R/I^{s+1})$ for all $s \geq 1$ (see the paper of Herzog and Hibi [8] for examples involving monomial ideals). We say that $I$ has the persistence property if $\text{Ass}(R/I^s) \subseteq \text{Ass}(R/I^{s+1})$ for all $s \geq 1$.

Conjecture 1.1 implies the persistence property for the following class of ideals:

**Theorem 1.2.** Suppose that $I$ is any unmixed square-free monomial ideal of height 2. If Conjecture 1.1 holds for $(s + 1)$, then $\text{Ass}(R/I^s) \subseteq \text{Ass}(R/I^{s+1})$.

In particular, if Conjecture 1.1 holds for all $s$ then $I$ has the persistence property.

Any unmixed square-free monomial ideal of height two is the Alexander dual of an edge ideal of a graph (for an introduction, see Chapter 6 of [16]). We prove Theorem 1.2 in Section 2 by using our results in [6] which link the irredundant irreducible decomposition of a power of a square-free monomial ideal to critical subgraphs.

In Section 3, we prove Conjecture 1.1 if $\chi_f(G)$, the fractional chromatic number of $G$, is “close enough” to $\chi(G)$.

**Theorem 1.3.** Conjecture 1.1 holds if we also assume $\chi(G) - 1 < \chi_f(G) \leq \chi(G)$.

We give an algebraic proof of Theorem 1.3. As a corollary, all odd holes and odd antiholes, both examples of critical graphs, satisfy Conjecture 1.1.

We view this paper as part of the ongoing dialogue between graph coloring problems and commutative algebra. As examples of this discussion, we point the reader to the following papers. In [15], Sturmfels and Sullivant identify the generators of secant ideals of the edge ideal of a graph $G$ in terms of the colorability of induced subgraphs of $G$ in order to give an algebraic interpretation of the Strong Perfect Graph Theorem. A previous paper of the authors [6], which extends results of [15], examines how the associated primes of the powers of the cover ideal encode coloring information of the graph. Moreover, this work inspired the conjecture of this paper. In a different direction, Steingrímsson constructed a square-free monomial ideal $K_G$ in a polynomial ring $R$ where the monomials of $K_G$ are in one-to-correspondence with the colorings of the graph $G$ [14]. Additionally, there is a
nice CoCoA tutorial that investigates a Gröbner basis method for determining whether a graph can be colored with three colors [9, Tutorial 26]. Connections between the chromatic number and ideals that are not necessarily monomial can be found in [2, 4]. As well, other variations on graph coloring can be tackled using commutative algebra, as in the paper of Miller [10] where orthogonal colorings of planar graphs and connections to the minimal free resolutions of monomial ideals of \(k[x, y, z]\) are studied.

**Remark 1.4.** As a final comment, one can also formulate a hypergraph version of Conjecture 1.1, and by adapting the proof of Theorem 1.2 one can remove the unmixed and height hypotheses from Theorem 1.2. Thus a proof of the hypergraph version of Conjecture 1.1 would prove the persistence property for *any* square-free monomial ideal. We have decided to emphasize the graph theory question, at the expense of the more general discussion, in the hope of attracting graph theorists to this problem. As well, we have less evidence that the hypergraph analog of Conjecture 1.1 is true.

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2. Persistence of cover ideals

The goal of this section is to prove Theorem 1.2. We begin by encoding a finite simple graph into a monomial ideal.

**Definition 2.1.** Let \(G\) be a finite simple graph on the vertex set \(V_G = \{x_1, \ldots, x_n\}\). The **cover ideal** of \(G\) is the monomial ideal

\[
J = J(G) = \bigcap_{\{x_i, x_j\} \in E_G} (x_i, x_j) \subseteq R = k[x_1, \ldots, x_n].
\]

From the definition, it follows that there is a one-to-one correspondence between cover ideals of finite simple graphs (with at least one edge and no isolated vertices) and unmixed square-free monomial ideals of height two.

Recall that a subset \(W \subseteq V_G\) is called a **vertex cover** of \(G\) if \(e \cap W \neq \emptyset\) for every \(e \in E_G\). A vertex cover is a **minimal vertex cover** if no proper subset is a vertex cover. An subset \(W\) is an **independent set** if \(V_G \setminus W\) is a vertex cover, and is a **maximal independent set** if \(V_G \setminus W\) is a minimal vertex cover. The name cover ideal makes sense in light of the following lemma.

**Lemma 2.2.** Let \(G\) be a finite simple graph. Then

\[
J(G) = (x_{i_1} \cdots x_{i_r} \mid W = \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a minimal vertex cover of } G).
\]

Irreducible monomial ideals will play a large role in the following discussion:
Definition 2.3. A monomial ideal in $R = k[x_1, \ldots, x_n]$ of the form $m^b = (x_i^{b_i} \mid b_i \geq 1)$ with $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$ is called an **irreducible ideal**. An irreducible decomposition of a monomial ideal $I$ is an expression of the form

$$I = m^{b_1} \cap \cdots \cap m^{b_r}$$

for some vectors $b_1, \ldots, b_r \in \mathbb{N}^n$. The decomposition is **irredundant** if none of the $m^{b_i}$ can be omitted.

Every monomial ideal has a unique irredundant irreducible decomposition [11, Theorem 5.27]. In our previous paper [6], we explored what information about the graph $G$ is encoded into the irredundant irreducible decompositions of the ideals $J(G)^s$ as $s$ varies, that is, the powers of the cover ideals. To state our results, we need the following terminology.

Definition 2.4. Let $G = (V_G, E_G)$ be a graph with vertices $V_G = \{x_1, \ldots, x_n\}$. For each $s$, we define the $s$-th expansion of $G$ to be the graph obtained by replacing each vertex $x_i \in V_G$ by a collection $\{x_{ij} \mid j = 1, \ldots, s\}$, and replacing $E_G$ by the edge set that consists of edges $\{x_{il}, x_{ij}\}$ whenever $\{x_i, x_l\} \in E_G$ and edges $\{x_{il}, x_{ik}\}$ for $l \neq k$. We denote this graph by $G^s$. We call the new variables $x_{ij}$ the **shadows** of $x_i$.

Observe that in the $s$-th expansion of $G$, we are replacing each vertex of $G$ with a clique of size $s$. The connection between the irredundant irreducible decomposition of $J(G)^s$ and the graph $G^s$ is summarized in the following theorem.

**Theorem 2.5** ([6, Corollary 4.14]). Let $G$ be a finite simple graph with cover ideal $J(G)$. The following are equivalent:

(i) $(x_1^{a_{i1}}, \ldots, x_r^{a_{ir}})$ appears in the irredundant irreducible decomposition of $J(G)^s$

(ii) The induced subgraph of $G^s$ on the vertex set

$$Y = \{x_{i1,1}, \ldots, x_{i1,s-a_{i1}+1}, \ldots, x_{ir,1}, \ldots, x_{ir,s-a_{ir}+1}\},$$

is a critically $(s+1)$-chromatic graph.

For any monomial ideal $I$, $(x_1, \ldots, x_r) \in \text{Ass}(R/I)$ if and only if there exists an irreducible ideal of the form $(x_1^{a_{i1}}, \ldots, x_r^{a_{ir}})$ in the irredundant irreducible decomposition of $I$. We come to the main result of this section, which shows that if Conjecture [1.1] is true, unmixed height two square-free monomial ideals have the persistence property.

**Theorem 2.6.** Let $G$ be a finite simple graph with cover ideal $J = J(G)$. Let $s \geq 1$ and assume that Conjecture [1.1] holds for $(s+1)$. Then

$$\text{Ass}(R/J^s) \subseteq \text{Ass}(R/J^{s+1}).$$

In particular, if Conjecture [1.1] holds for all $s$ then $J$ has the persistence property.

**Proof.** Let $P = (x_1, \ldots, x_r) \in \text{Ass}(R/J^s)$. To simplify our notation, we can assume, after relabeling, that $P = (x_1, \ldots, x_r)$. Thus, there exist positive integers $a_1, \ldots, a_r$ such that $(x_1^{a_1}, \ldots, x_r^{a_r})$ appears in the irredundant irreducible decomposition of $J^s$. By Theorem 2.5, this means that the induced graph of $G^s$ on the vertex set

$$Y = \{x_{1,1}, \ldots, x_{1,s-a_1+1}, \ldots, x_{r,1}, \ldots, x_{r,s-a_r+1}\}$$
is a critically \((s + 1)\)-chromatic graph. Let \(H = (G^s)_Y\) denote this induced subgraph.

Since we are assuming that Conjecture \(\mathcal{C}\) holds for \((s + 1)\), we can find a subset \(W \subseteq Y\) such that the expansion of \(H\) at \(W\), that is, the graph \(H[W]\), is a critically \((s + 2)\)-chromatic graph. For each \(i = 1, \ldots, r\), let \(b_i\) denote the number of shadows of \(x_i\) that are contained in \(W\).

We claim that \(0 \leq b_i \leq a_i\) for each \(i\). Indeed, suppose that \(W\) contains \(b_i \geq a_i + 1\) shadows of \(x_i\). The induced graph of \(H\) on \(\{x_{i,1}, \ldots, x_{i,s-a_i+1}\}\) is a clique of size \(s - a_i + 1\). If we expand \(H\) at the \(b_i\) shadows of \(x_i\) in \(W\), we end up with a clique of size \(b_i + s - a_i + 1 \geq a_i + 1 + s - a_i + 1 = s + 2\), i.e., \(H[W]\) will contain a clique of size at least \(s + 2\) as an induced subgraph. On the other hand, because \(H\) is a critically \((s + 1)\)-chromatic graph, and because the induced graph on \(\{x_{i,1}, \ldots, x_{i,s-a_i+1}\}\) has size at most \(s - a_i + 1 \leq s < s + 1\), the graph \(H\) has at least one other edge. But then this edge (or a shadow of this edge) also belongs to \(H[W]\) and does not belong to the clique of size of \(s + 2\) in \(H[W]\). We now have a contradiction, since \(H[W]\) is a critically \((s + 2)\)-chromatic graph that contains a \(\mathcal{K}_{s+2}\) as a proper induced subgraph.

Now consider the induced subgraph of \(G^{s+1}\) on the vertex set

\[Y' = \{x_{1,1}, \ldots, x_{1,s-a_1+b_1+1}, \ldots, x_{r,1}, \ldots, x_{r,s-a_r+b_r+1}\}.\]

By comparing the constructions of \((G^{s+1})_Y\) and \(H[W]\), one can verify that these two graphs are isomorphic. Thus \((G^{s+1})_Y\) is a critically \((s + 2)\)-subgraph of \(G^{s+1}\). Theorem 2.5 then implies that the irreducible ideal

\[(x_1^{a_1-b_1+1}, \ldots, x_r^{a_r-b_r+1})\]

appears in the irredundant irreducible decomposition of \(J(G)^{s+1}\). Since \(0 \leq b_i \leq a_i\) for each \(i\), we have \(a_i - b_i + 1 \geq 1\) for each \(i\). Hence \((x_1, \ldots, x_r) \in \text{Ass}(R/J^{s+1})\), as desired. \(\square\)

**Remark 2.7.** In \([6]\), we proved that \(J = J(G)\) has the persistence property if \(G\) is perfect. Our proof relies heavily on the fact that Conjecture \(\mathcal{C}\) is true for perfect critical graphs.

### 3. Special cases of the conjecture

In this section, we prove that Conjecture \(\mathcal{C}\) holds for all \(s\) if the fractional chromatic number of \(G\) is “close enough” to \(\chi(G)\). We begin with a lemma that shows that we can prove Conjecture \(\mathcal{C}\) if there exists a maximal independent set \(W \subseteq V_G\) such that \(\chi(G[W]) = \chi(G) + 1\).

**Lemma 3.1.** Suppose that \(G\) is a critically \(s\)-chromatic graph and \(W\) is a maximal independent set such that \(\chi(G[W]) = s + 1\). Then \(G[W]\) is critically \((s + 1)\)-chromatic.

**Proof.** It suffices to prove that \(G[W]\) is critically \((s + 1)\)-chromatic since we are given \(\chi(G[W]) = s + 1\). After relabeling the vertices, we can assume that \(W = \{x_1, \ldots, x_r\}\) and \(V_G \setminus W = \{x_{r+1}, \ldots, x_n\}\). Thus, the vertices of \(G[W]\) are

\[V_G[W] = \{x_{11}, x_{12}, \ldots, x_{r1}, x_{r2}, x_{r+1,1}, \ldots, x_n\}.\]

Fix an \(x_j \in \{x_{r+1}, \ldots, x_n\}\), and set \(H = G \setminus x_j\). The \(H[W] = G[W] \setminus x_j\). We can color \(G \setminus x_j\) with \(s - 1\) colors, and in particular, we use the \(s - 1\) colors to color the induced subgraph of \(H[W]\) on \(\{x_{11}, \ldots, x_{r1}, x_{r+1,1}, \ldots, x_n\}\) \(\setminus \{x_j\}\). We then color the
decomposition is the ideal of vertices \( \{x_{12}, \ldots, x_r\} \) with the \( s \)-th color. Since \( W \) is an independent set, we can color all of these vertices the same color. But this means that \( H[W] = G[W] \setminus x_j \) is \( s \)-colorable.

So, now fix an \( x_j \in \{x_{11}, x_{21}, \ldots, x_r\} \). We will show that we can color \( G[W] \setminus x_j \) with \( s \) colors. If we remove \( x_j \) from \( G \), we can color \( G \setminus x_j \) with \( s - 1 \) colors. This gives us a \((s - 1)\)-coloring of the vertices of \( \{x_{11}, \ldots, x_r, x_{r+1}, \ldots, x_n\} \setminus \{x_j\} \). The independent set \( \{x_{12}, \ldots, x_r\} \) in \( G[W] \setminus x_j \) can now be colored with the \( s \)-th color. A similar argument works if \( x_j \in \{x_{12}, x_{22}, \ldots, x_r\} \) is removed.

\[\square\]

**Remark 3.2.** Suppose \( G \) is critically \( s \)-chromatic, and let \( W \) be a non-maximal independent set in \( G \); that is, there exists a vertex \( x \) so that \( W \cup \{x\} \) is an independent set. It can be seen that \( \chi(G \setminus x) = s - 1 \), so by assigning the \( s \)-th color to \( x \) and shadows of the vertices in \( W \), we have \( \chi(G[W]) = s \). This says that for \( G[W] \) to be critically \((s + 1)\)-chromatic, \( W \) cannot be a non-maximal independent set. On the other hand, the proof of Lemma \( 3.1 \) suggests that the easiest way to prove Conjecture \( 1.1 \) is to show the existence of a maximal independent set \( W \subseteq V_G \) with the property that \( \chi(G[W]) = \chi(G) + 1 \). Our initial hope was that any maximal independent set, or at the very least, any maximum independent set (a maximal independent set with largest possible cardinality), would have the desired property. The following two examples, due to Bjarne Toft and Anders Sune Pedersen, show that this is not true.

**Example 3.3.** Let \( G = C_9 \), the cycle on \( V_G = \{x_1, \ldots, x_9\} \). Consider the maximal independent set \( W = \{x_2, x_5, x_8\} \). When we construct the graph \( G[W] \), the resulting graph has \( \chi(G[W]) = \chi(G) = 3 \).

**Example 3.4.** As in the previous example, we begin with the cycle \( C_9 \) on the vertex set \( V_G = \{x_1, \ldots, x_9\} \) which is a critically 3-chromatic graph. We now apply the construction of Mycielski \( [12] \) to make a new graph \( G' \) as follows: let \( V_{G'} = \{x_1, \ldots, x_9, y_1, \ldots, y_9, z\} \) where \( z \) is adjacent to \( y_1, \ldots, y_9 \), \( y_i \) is adjacent to the neighbors of \( x_i \) for each \( i = 1, \ldots, 9 \), and the induced graph on \( \{x_1, \ldots, x_9\} \) is the original graph \( G = C_9 \). The new graph \( G' \) has the property that it is critically 4-chromatic. Furthermore, in this new graph \( G' \), the set \( W = \{y_1, \ldots, y_9\} \) is a maximum independent set.

We claim that the graph \( G'[W] \) has the property that \( \chi(G'[W]) = \chi(G') = 4 \). To see this, color the vertices \( x_1, \ldots, x_9 \) in the following order: red, blue, green, red, green, orange, red, orange, blue. One can color vertices \( \{y_{11}, y_{12}, \ldots, y_{91}, y_{92}\} \) using only the colors green, blue and orange. Finally, one colors the vertex \( z \) red.

Using Macaulay 2 \( [7] \), we calculated the irredundant irreducible decomposition of the \( J^4 \), where \( J = J(G') \) is cover ideal of \( G' \). (Computing this decomposition will take about an hour on a standard laptop.) One of the irredundant irreducible ideals appearing in the decomposition is the ideal

\[
(x_1^3, x_2^4, x_3^3, x_4^3, x_5^2, x_6^3, x_7^4, x_8^4, x_9^4, y_1^3, y_2^3, y_3^3, y_4^4, y_5^4, y_6^4, y_7^4, y_8^4, y_9^4, z^4).
\]

By Theorem \( 2.5 \) the induced graph on the vertex set

\[
\{x_{11}, x_{12}, x_{21}, x_{31}, x_{32}, x_{41}, x_{51}, x_{52}, x_{61}, x_{71}, x_{72}, x_{81}, x_{91}, y_{11}, y_{12}, y_{21}, y_{31}, y_{32}, y_{41}, y_{51}, y_{52}, y_{61}, y_{71}, y_{72}, y_{81}, y_{91}, z_{11}\}
\]
in the graph $(G')^2$ is a critically 5-chromatic graph. This induced graph is isomorphic to
the graph one obtains by expanding $G'$ at the vertex set $Z = \{x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7\}$,
which is a maximal independent set. So, although expanding $G'$ at one maximum independent set does not result in graph whose chromatic number increases, at another maximal independent set we will have this property, i.e., $\chi(G'[Z]) = \chi(G') + 1$.

Remark 3.5. The above example suggests that a stronger version of Conjecture 1.1 may hold, namely, one can pick the set $W$ so that $W$ is also a maximal independent set.

We now give an algebraic proof that Conjecture 1.1 is true provided that the fractional chromatic number of $G$ is “close enough” to the chromatic number of $G$. We begin by recalling some needed definitions and results.

Definition 3.6. A $b$-fold coloring of a graph $G$ is an assignment to each vertex a set of $b$ distinct colors such that adjacent vertices receive disjoint sets of colors. The minimal number of colors needed to give $G$ a $b$-fold coloring is called the $b$-fold chromatic number of $G$ and is denoted $\chi_b(G)$.

When $b = 1$, then $\chi_b(G) = \chi(G)$. The following result, which appeared in [6], relates the value of $\chi_b(G)$ to an ideal membership question.

Theorem 3.7. Let $G$ be a finite simple graph on $V = \{x_1, \ldots, x_n\}$ with cover ideal $J$. Then
$$\chi_b(G) = \min \{d \mid (x_1 \cdots x_n)^{d-b} \in J^d\}.$$ If follows directly from the definition that the $b$-fold chromatic number has the subadditivity property, that is, $\chi_{a+b}(G) \leq \chi_a(G) + \chi_b(G)$ for all $a$ and $b$. Thus, we can define the following invariant of $G$ (cf. [13, Section 3.1 and Appendix 4]):

Definition 3.8. The fractional chromatic number of a graph $G$, denoted $\chi_f(G)$, is defined by
$$\chi_f(G) := \lim_{b \to \infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}.$$ Note that $\chi_f(G) \leq \frac{\chi_a(G)}{a}$ for all $a$. Moreover, by [13, Corollary 1.3.2], there always exists a number $b$ such that $\chi_f(G) = \frac{\chi_b(G)}{b}$. This enables us to prove the following case of Conjecture 1.1:

Theorem 3.9. Let $G$ be a finite simple graph that is critically $s$-chromatic. Suppose that $\chi(G) - 1 < \chi_f(G) \leq \chi(G)$. Then there exists a set $W \subseteq V_G$ such that $\chi(G[W])$ is critically $(s + 1)$-chromatic.

We will, in fact, prove a stronger result, by showing that we can pick $W$ to be a maximal independent set, thus suggesting a stronger version of Conjecture 1.1 holds (see Remark 3.5). We need a technical algebraic result.

Lemma 3.10. Let $W \subseteq V_G$ be any subset of vertices and set $G' = G[W]$. Then
$$\frac{(x_1 \cdots x_n)^{\chi_b(G')-b}}{m_W^b} \in J(G)^{\chi_b(G')}$$ for any integer $b \geq 1$. 
where \( m_W = \prod_{x_i \in W} x_i \).

**Proof.** For simplicity, let \( W = \{x_1, \ldots, x_t\} \), and we assume that
\[
V_{G'} = \{x_1, y_1, \ldots, x_t, y_t, x_{t+1}, \ldots, x_n\},
\]
i.e., we use \( x_i \) and \( y_i \) to represent the shadows of \( x_i \). By Theorem 5.7, we have
\[
(x_1 \cdot \cdot \cdot x_n y_1 \cdot \cdot \cdot y_t)^{\chi_b(G') - b} \in J(G')^{\chi_b(G')},
\]
This means that
\[
(x_1 \cdot \cdot \cdot x_n)^{\chi_b(G') - b} = m_1 \cdot \cdot \cdot m_{\chi_b(G')} M
\]
for some monomial \( M \), where the \( m_i \) are minimal generators of \( J(G') \). We can write each \( m_i \) as \( m_i = m_{i,x}m_{i,y} \) where \( m_{i,x} \) is a monomial in \( \{x_1, \ldots, x_n\} \) and \( m_{i,y} \) is a monomial in the \( y_i \)'s.

We note that each \( m_{i,x} \in J(G) \); to see this, observe that the support of \( m_i \) that is contained in \( \{x_1, \ldots, x_n\} \) would cover the vertices of \( G \). Thus
\[
(x_1 \cdot \cdot \cdot x_n)^{\chi_b(G') - b} = m_{1,x} \cdot \cdot \cdot m_{\chi_b(G'),x} M_x \in J(G)^{\chi_b(G')},
\]
where \( M = M_x M_y \). Note that if \( y_i \not| m_j \), then \( x_i \) and \( N(x_i) \), the neighbors of \( x_i \), must divide \( m_j \), since the neighbors of \( y_i \) are \( x_i \) and \( N(x_i) \). In other words, \( x_i \) and \( N(x_i) \) both divide \( m_{j,x} \). But then \( m_{j,x} \) is not a minimal vertex cover since we can remove \( x_i \) and still have a vertex cover. That is, \( m_{j,x}/x_i \in J(G) \).

Now, for \( i = 1, \ldots, t \), the variable \( y_i \) is missing from at least \( b \) of \( \{m_1, \ldots, m_{\chi_b(G')}\} \) because each of these monomials is square-free. This means that for \( i = 1, \ldots, t \), the variable \( x_i \) divides (at least) \( b \) of the elements of \( \{m_{1,x}, \ldots, m_{\chi_b(G'),x}\} \), and the resulting monomials still correspond to vertex covers. Hence
\[
\frac{(x_1 \cdot \cdot \cdot x_n)^{\chi_b(G') - b}}{(x_1 \cdot \cdot \cdot x_t)^b} = \frac{m_{1,x} \cdot \cdot \cdot m_{\chi_b(G'),x} M_x}{(x_1 \cdot \cdot \cdot x_t)^b} \in J(G)^{\chi_b(G')}. \]

**Proof of Theorem 3.9.** By Lemma 3.1 it suffices to find a maximal independent set \( W \) such that \( \chi(G[W]) = s + 1 \). Suppose, for a contradiction, that for every maximal independent set \( W \), \( \chi(G[W]) = \chi(G) \).

Fix any maximal independent set \( W \), and let \( G' = G[W] \). There exists an integer \( b \) such that \( \chi_f(G') = \frac{\chi_f(G)}{b} \). Furthermore, by Lemma 3.10
\[
\frac{(x_1 \cdot \cdot \cdot x_n)^{\chi_b(G') - b}}{m_W^b} \in J(G)^{\chi_b(G')},
\]
(Note that the value of \( b \) will depend upon the choice of \( W \); the fact remains that for every maximal independent set \( W \), there exists an integer \( b \) so that the above monomial belongs to some power of \( J(G) \).)

Now we consider \( \chi_f(G) \). Again, there exists an integer \( c \) such that \( \chi_f(G) = \frac{\chi_c(G)}{c} \). We thus have \((x_1 \cdot \cdot \cdot x_n)^{\chi_c(G) - c} \in J(G)^{\chi_c(G)} \) by Theorem 3.7. This means that there exist minimal generators of \( J(G) \) such that
\[
(x_1 \cdot \cdot \cdot x_n)^{\chi_c(G) - c} = M_1 W_1 \cdot \cdot \cdot M_W W_2 \cdot \cdot \cdot M_{W_{\chi_c(G)}} M.
\]
Since $m_{W_1}$ corresponds to a minimal vertex cover $W_1$, the complement $V_G \setminus W_1$ is a maximal independent set of $G$. Hence, by the discussion in the previous paragraph, we also have
\[
(x_1 \cdots x_n)^{x_h(G') - 2b} m_{W_1}^b \in J(G)^{x_h(G')},
\]
where $\frac{x_h(G')}{b} = x_f(G')$ and $G'$ is the expansion of the independent set $V \setminus W_1$.

When we combine together the above results, we get:
\[
(x_1 \cdots x_n)^{x_h(G') - 2b} m_{W_1}^b (m_{W_2} \cdots m_{W_{\chi_c(G)}} M)^b \leq J(G)^{x_h(G') + b(x_c(G) - 1)}
\]
\[
(x_1 \cdots x_n)^{x_h(G') - 2b + b\chi_c(G) - \chi_c(G) - b} \leq J(G)^{x_h(G') + b\chi_c(G) - b}
\]
\[
(x_1 \cdots x_n)^{x_h(G') + b\chi_c(G) - \chi_c(G) - (c+1)b} \leq J(G)^{x_h(G') + b\chi_c(G) - b}
\]
The last expression, coupled with Theorem 3.7, implies that $(c+1)b$-fold chromatic number of $G$ is bounded above by
\[
\chi_{(c+1)b}(G) \leq \chi_b(G') + b\chi_c(G) - b.
\]
If we divide both sides of the equation by $(c+1)b$ we then get
\[
\frac{\chi_{(c+1)b}(G)}{(c+1)b} \leq \frac{\chi_b(G')}{(c+1)b} + \frac{b\chi_c(G)}{(c+1)b} - \frac{b}{(c+1)b} = \frac{\chi_f(G') + \chi_c(G) - 1}{c + 1},
\]
where we use the fact that $\chi_f(G') = \frac{x_h(G')}{b}$.

We have the following inequalities and equalities, where the first inequality is our hypothesis, and the last equality comes from our assumption that $\chi(G | W) = \chi(G)$ for every maximal independent set:
\[
\chi(G) - 1 < \chi_f(G) \leq \chi_f(G') \leq \chi(G) = \chi_f(G).
\]
So, $\chi_f(G') < \chi_f(G) + 1$, whence
\[
\frac{\chi_f(G') + \chi_c(G) - 1}{c + 1} < \frac{\chi_f(G) + 1 + \chi_c(G) - 1}{c + 1} = \frac{\chi_f(G) + \chi_c(G)}{c + 1}
\]
\[
= \frac{\chi_c(G)/c + \chi_c(G)}{c + 1} = \frac{\chi_c(G) + \chi_c(G)}{c + 1} = \frac{\chi_c(G)}{c + 1} = \chi_f(G).
\]
For any integer $a$, we have $\chi_f(G) \leq \frac{x_h(G)}{a}$. So, if we put our inequalities together, we get
\[
\chi_f(G) \leq \frac{\chi_{(c+1)b}(G)}{(c+1)b} < \chi_f(G)
\]
which gives our desired contradiction. Hence there exists a maximal independent set such that when we expand it, we get a graph whose chromatic number goes up. \hfill \Box

**Corollary 3.11.** Conjecture 1.1 holds for the following critical graphs: cliques, odd holes, and odd antiholes.
Proof. It suffices to show that for each type of graph, $\chi(G) - 1 < \chi_f(G) \leq \chi(G)$. For cliques, this is immediate since $\chi_f(G) = \chi(G)$. When $G = C_{2n+1}$ is an odd hole, $\chi_f(C_{2n+1}) = 2 + \frac{1}{n}$ and $\chi(G) = 3$, so the result holds. Finally, when $G = C_{2n+1}$, the chromatic number of $G$ is $n + 1$, while $\chi_f(G) = n + \frac{1}{2}$, so again the result holds.

\[\square\]

Remark 3.12. Critically 3-chromatic graphs are induced odd cycles. Let $C$ be an induced odd cycle on the vertices $\{x_1, \ldots, x_{2r+1}\}$ for some $r \geq 1$. Let $W = \{x_2, x_4, \ldots, x_{2r}\}$. Then it is not hard to see that $\chi(C[W]) = 4 = \chi(C) + 1$. This says that Conjecture [1] holds for $s = 3$. As a consequence, for any cover ideal $J = J(G)$ of a graph $G$, we always have

$$\text{Ass}(R/J^2) \subseteq \text{Ass}(R/J^3).$$

In [5], we characterize elements of $\text{Ass}(R/J^2)$. Thus, all primes of the form $(x_i, x_j)$, where $\{x_i, x_j\}$ is an edge in $G$, and $P = (x_{i_1}, \ldots, x_{i_r})$, where $r$ is odd and the induced subgraph of $G$ on $\{x_{i_1}, \ldots, x_{i_r}\}$ is an odd cycle, belong to $\text{Ass}(R/J^3)$.

References

[1] M. Brodmann, Asymptotic stability of $\text{Ass}(M/I^nM)$. Proc. Amer. Math. Soc. 74 (1979), 16–18.
[2] J.I. Brown, Chromatic polynomials and order ideals of monomials. Discrete Math. 189 (1998), 43–68.
[3] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it
[4] J. A. De Loera, C. Hillar, P. N. Malkin, M. Omar, Recognizing Graph Theoretic Properties with Polynomial Ideals. (2010) Preprint. arXiv:1002.4435
[5] C.A. Francisco, H.T. Ha, and A. Van Tuyl, Associated primes of monomial ideals and odd holes in graphs. To appear, J. Algebraic Combin.. arXiv:0806.1159
[6] C.A. Francisco, H.T. Ha, and A. Van Tuyl, Colorings of hypergraphs, perfect graphs, and associated primes of powers of monomial ideals. (2009) Preprint. arXiv:0908.1505v1
[7] D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry. http://www.math.uiuc.edu/Macaulay2/
[8] J. Herzog, T. Hibi, The depth of powers of an ideal. J. Algebra 291 (2005), 534–550.
[9] M. Kreuzer, L. Robbiano, Computational commutative algebra. 1. Springer-Verlag, Berlin, 2000. x+321 pp.
[10] Miller, Planar graphs as minimal resolutions of trivariate monomial ideals. Doc. Math. 7 (2002), 43–90
[11] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra. GTM 227, Springer-Verlag, New York, 2004.
[12] J. Mycielski, Sur le coloriage des graphes. Colloq. Math. 3 (1955), 161–162.
[13] E.R. Scheinerman, D.H. Ullman, Fractional Graph Theory. John Wiley & Sons, Inc., New York, 1997.
[14] E. Steingrímsson, The coloring ideal and coloring complex of a graph. J. Algebraic Combin. 14 (2001), no. 1, 73–84.
[15] B. Sturmfels, S. Sullivant, Combinatorial secant varieties. Pure Appl. Math. Q. 2 (2006), no. 3, part 1, 867–891.
[16] R.H. Villarreal, Monomial algebras. Monographs and Textbooks in Pure and Applied Mathematics, 238. Marcel Dekker, Inc., New York, 2001.
