Chaos in the Music of the Spheres

J. Robert Buchler∗, Zoltan Kolláth† and Robert Cadmus∗∗

∗University of Florida, USA
†Konkoly Observatory, HUNGARY
∗∗Grinnell College, USA

Abstract. The light curves (time series of the radiated energy) of most large amplitude, pulsating stars such as the well known Cepheid stars are regular. However, a smaller group of variable stars that are located next to them in the Hertzsprung - Russell diagram undergoes irregular light variations and exhibits irregular radial velocities as well. The mechanism behind this irregular behavior was a long standing mystery. A flow reconstruction technique based on the observed lightcurves of six separate stars shows that their underlying dynamics is chaotic and low dimensional (d = 4). Furthermore, we present evidence that the physical mechanism behind the behavior is the nonlinear interaction of just two pulsation eigenmodes. In a generalized Shil’nikov scenario, the pulsation energy alternates continuously, but irregularly between a lower frequency mode that is linearly unstable and thus growing, and a stable overtone that gets entrained through a low order resonance (2:1), but that wants to decay. The flow reconstruction from the stellar light curve thus yields interesting physical insight into the pulsation mechanism.

INTRODUCTION

Almost every type of star becomes unstable to self-excited vibrations at least once during its lifetime. Typically, the stars that exhibit large amplitude variations in light are undergoing radial pulsations, i.e. pulsations in which the star remains spherically symmetric at all times. Typically, only the fundamental pulsation mode or the first two overtones are involved in the pulsation. In contrast, most small amplitude variable stars such as the Sun or white dwarfs undergo nonradial pulsations, but with many modes participating in the motion.

The classical Cepheid variables and RR Lyrae variables are the best known and best studied variable stars of the radial pulsator category. They owe this interest at least in part to their role as primary cosmological distance indicators; they are almost perfectly periodic and obey a period-luminosity relation that can be used for measuring their distances.

This paper concerns itself with variable stars that lie next to and slightly below the classical Cepheids in a long and broad curved strip in the Hertzsprung - Russell diagram (luminosity vs. temperature diagram) [1]. For historical reasons the stars at the lower luminosity end of the strip are called W Virginis type stars, further up RV Tauri type stars, then Semiregular stars and finally Mira variables, even though the transition between the different classes appears to be gradual. For convenience we shall refer to all of them here as semiregular stars, largo sensu. The W Vir stars that lie at the low luminosity end of the strip with concomitantly low period (P < 20d) have regular light curves, but for the slightly more luminous stars stars the observations show alternations
FIGURE 1. Smoothed observed light curves in the pulsation cycles that appear to be due to period doubling [2]. In fact, numerical hydrodynamical models display cascades of period doublings [3], [4], [5] as well as tangent bifurcations [6], [7]. Observations also show that stars with higher luminosity have increasingly irregular pulsations [8].

Traditional astronomers like to think of stars being multi-periodic (meaning that
they consider the pulsation to be a superposition of a number of pulsation frequencies with steady amplitudes). It is true that a phenomenologically motivated multiperiodic fit will always be 'successful' (as an interpolation), but on physical grounds it is possible to rule out such a description [9]. In a nutshell:

(1) The frequency peaks in the Fourier spectrum of the light curve are numerous, but in this type of star, and in the frequency range of interest, there do not exist enough radial pulsation eigenmodes, nor even low order nonradial pulsation modes that could be observable and excitable to such large amplitudes. No multi-periodic fits are possible that can reproduce the light curves within the observational accuracy while using modal frequencies that are acceptable on a physical basis.

(2) The frequencies in the Fourier spectra appear randomly variable from one section of the light curve data to another (for R Sct cf. [10]). If a multi-periodic star were evolving slowly, the structure of the peaks should evolve slowly as well. The absence of such correlations eliminates any explanation of the light curve as that of slowly evolving multi-periodic stars.

(3) One recent paper ([11]) ‘explains’ the irregular behavior of R Sct as the superposition of stochastically excited linear oscillators. While such a stochastic ‘explanation’ is perhaps mathematically valid, it is not meaningful on physical grounds because no mechanism is proposed (nor can be found) that could excite damped modes to such large pulsation amplitudes (up to factors of 40 in the light curve of R Sct).

Unfortunately the cycling times of these variable stars vary from as low as 20 days for the lower luminosity W Vir stars to as much as a year for the Semiregular variables. It is almost needless to say that very few stars have been observed with a coverage that lends itself to modern nonlinear analyses such as a flow reconstruction. Furthermore, for obvious reasons, the observations are generally not spaced with equal time intervals, so that an interpolation with concomitant introduction of noise is unavoidable.

A few years ago we were fortunate to obtain access to and analyze the data sets of two irregular variable stars, called R Scuti [12] [13] and AC Herculis [14], that have a sufficiently good coverage and exhibit a large enough number of pulsation cycles (cf. Fig. 1) to be representative of their long term behavior.

Our analysis was based on a flow reconstruction with a multivariate polynomial map $\mathcal{M}$ (or o.d.e.) in an embedding space of dimension $d_e$

$$X_{n+1} = \mathcal{M} \cdot X_n$$

where the state vector $X \in \mathbb{R}^{d_e}$ is constructed from the scalar time-series $\{x_i\}$,

$$X_n = (x_i, x_{i-\tau}, x_{i-2\tau}, \ldots x_{i-(d_e-1)\tau})$$

where $\tau$ is a delay parameter that should be long enough so that noise does not kill the reconstruction, but small enough so that the map does not become too nonlinear [15]. We briefly summarize here the results that were obtained.
The light curve data of R Sct were shown to be generated by a 4-dimensional dynamics \((d=4)\) \[12\] \[13\]. This conclusion was based on several facts: (a) the minimum embedding dimension is \(d_e = 4\): first, synthetic light curves generated
in 3D bear no resemblance to the observations; second, the error of the fit levels off at \( d_e = 4 \); third, the nearest neighbor method indicates 4 as the minimum dimension. However, the strongest argument comes from a comparison of the Fourier spectra of the synthetic light curves (generated through an iteration of the map) with those of that observational data, and of a comparison of the respective Broomhead-King projections (onto the eigenvectors of the correlation matrix).

(b) the fractal (Lyapunov) dimensions \([16]\) derived from the Lyapunov exponents of the synthetic signals fell in the range \(3.1 - 3.2\) for R Sct, independent of the embedding dimension. This is therefore a lucky situation where the bounds \(d_L \sim 3.15 < d \leq d_e = 4\) uniquely determine the physical dimension to be \(d = 4\).

(c) There is no a priori guarantee that a map or flow should capture the underlying dynamics. The fact that one of the Lyapunov exponents is always close to zero very strongly suggests that we have been successful in the flow reconstruction. Indeed, a map with a short time-step should be close to a flow for which we know that the corresponding Lyapunov exponent is exactly zero. Actually we have also reconstructed a true flow (system of o.d.e.'s) for R Sct, although we find that the reconstruction of the flow is a little less robust than that of the map.

(d) Perhaps the most interesting physical result comes from the linearization of the successful maps around their fixed points: Two spiral roots \( \pm i\omega + \xi \) are found with the following properties: \( \omega_2 \approx 2\omega_1 \), with \( \xi_2 > 0 \), and \( \xi_1 < 0 \), and \( |\xi_2| > \xi_1 \). This is of course reminiscent of the Shil'nikov criterion \([17]\).

These results allow us to give a physical interpretation of the motion. We already mentioned that the regular, classical Cepheid variables undergo pulsations in one of the two lowest modes of pulsation. It is seen that the semiregular stars to the same: the complex amplitudes of vibrational modes are the 'natural' coordinates for the phase space, so that \(d = 4\) implicates the involvement of two vibrational modes. Furthermore, the low frequency mode, of frequency \( \omega_1 \) is linearly unstable (self-excited) and grows in amplitude. In the nonlinear regime it interacts with a second mode because of a low order resonance condition \((\omega_2 = 2\omega_1)\). However, this entrained mode is linearly stable and wants to decay. A chaotic motion of alternating growth and decay ensues.

Our analysis of AC Herculis \([14]\) was a little less conclusive. The constructed maps and flows were less robust. By that we mean that for many seed values the iteration blew up much more rapidly. We believe that this lack of robustness may be due to the lower signal to noise ratio (Fig. 1) for AC Her. Despite these short-comings of the reconstruction, however, everything points at the minimal embedding dimension again being 4. The fractal dimension \(d_L\) turned out to be lower \(d_L \sim 2.3\) than for R Sct.

As Fig. 1 shows the light curve of AC Her has a lot smaller amplitude swings, and hardly any phases of low amplitude oscillations. The linear part of the map which describes the vicinity of the fixed point is therefore less well determined, and so are the stability roots. We were therefore unable to verify whether the same resonant entrainment as in R Sct is operative in AC Her.
NEW RESULTS

In this paper we present an analysis of observational data of four additional stars, taken by one of us (RC) at Grinnell College over a 15 year period (cf. Fig. 1). These data are much higher quality than the amateur astronomer data previously analyzed, but they suffer from the drawback of lesser coverage. The stars go under the names of SX Herculis, R Ursae Minoris, RS Cygni and V Canum Venaticorum, and astronomers classify them as Semiregular variables (or Mira type for V CVn). The light curves of these stars are shown together with those of R Sct and AC Her which we analyzed earlier. The light curves of Fig. 1 represent cubic spline fits to the observational data.

Our cubic spline fits depend on the amount of smoothing which is controlled by the parameter $\sigma$ ([18]), and by the size of the gaps that we allow. Interpolation is necessary because our analysis requires a time series with equal time intervals. Typically we use a value of $\sigma = 0.02$.

In Fig. 2 we also display the amplitude Fourier spectra of the six stars. Of the four stars RS Cyg has the most harmonic power whereas SX Her has almost none. But, as is well known, we do not learn too much about the dynamics from linear analyses, such as Fourier, MEM, ARMA or time-frequency [20].

The star SX Herculis

The smoothed 5500 day long observational light curve (with 1 day sampling) of SX Her is shown on top of Fig. 3 for reference. In the flow reconstruction we have
used astronomical magnitudes, $m \equiv -2.5 \log L$, rather than the more physical luminosity ($L$, energy radiated per unit time). Underneath, we display two of the best results of our global flow reconstructions in the form of segments of synthetic light curves that were generated with the help of 4D maps and two different values of $\tau$, indicated in the upper right corner of the graphs. The synthetic light curves have been obtained through a 10500 fold iteration of the maps with different initial seeds, thus generating 10000 d long signals. (We have discarded the first 500 iterations to avoid transients).

We note however that for a given map not all seeds lead to this type of signal, some go to fixed points, some blow up. We attribute these failures at least partially to the shortness of the observational data set that was used to train the map. After all, the observational signal samples only a small part of phase space, and not very densely for that matter. As the map is iterated the trajectory eventually ends up in a part of phase space where the map is not well determined because of the training signal had no or few points there, and it runs off to infinity. For other parameter values the opposite can also occur, namely that the map is too stable leading to limit cycles because of a paucity of phase space points around this limit cycle. Stable synthetic signals resembling the observations could only be found for some values of $\tau$ (7 and 9), indicating that this reconstruction lacks robustness. Despite these shortcomings, however, the synthetic light curves display many of the characteristic features of the observed light curves, such as asymmetric bursts, but they are somewhat more regular.

In the upper left corners we show the Lyapunov dimensions of the synthetic signal. They have been computed with relatively short time-series (10000 points) and are thus not very precise. Nevertheless the maps that successfully produce synthetic signals with the characteristics of the observations all have a fractal dimension $d_L$ in the range from $3.1 - 3.7$. 

**FIGURE 4.** Flow reconstructions with the luminosity of SX Her.
Reconstructions in 3D are not successful at all. In 5D they are successful, but not very robust, most likely because of the short number of modulation cycles. However, the few synthetic signals that we were able to reconstruct had a fractal dimension less than 4, despite the fact that the embedding space was 5. This is of importance because it suggests again, as for R Sct that the physical dimension is $d = 4$.

When we linearize our 4D maps around their fixed point, we find that for the good reconstructions the fixed points of the maps are of a doubly spiral nature, as in the case of R Sct. (There can be more than one fixed point, but generally only one is located at the 'center' of the motion). The ratio of the two frequencies is found to be $2.1 \sim 2.2$, and the stable growth rate of the higher frequency root (real part of the linear spiral eigenvalues) is always larger than the unstable one, again a generalized Shil’nikov scenario for chaos [17]. The linearization of the map is of course is closely related to the study of the linear vibrational properties of the equilibrium star. Thus for SX Her, as for R Sct earlier on, we arrive at a physical picture of the irregular pulsation mechanism, namely one of continual and alternating exchange of energy between a growing pulsation mode of frequency $f_0$ and a decaying resonant mode of frequency $\sim 2f_0$.

We have also made flow reconstructions with the luminosities (obtained from transforming the smoothed magnitude data) which are shown in Fig. 4. This reconstruction is seen to lead to better synthetic signals, although again only for two values of $\tau = 7$ and 8. However, physically, the results are not that different. One finds fractal Lyapunov dimensions $3.2 \sim 3.4$, i.e. again larger than 3, but less than 4, and period ratios again close to and slightly greater than 2. (Here it has not been possible to construct robust 5D maps).

Although the quality of the reconstructions for SX Her are hampered by the shortness
of the data set (not many modulation cycles), which shows up among other things in a relatively poor stability of the maps and in a narrow range of values of \( \tau \) over which the reconstruction is possible, one can reasonably conclude that that the embedding dimension is 4 and that the fractal dimension is between 3.1 and 3.8 < 4.

**The star R Ursae Minoris**

In Figure 5 on top we display the smoothed 5500 d long R UMi observational signal that we have used to train our maps. Underneath are two segments of synthetic signals that were generated from the iteration of four different reconstructed 4D maps, for values of \( \tau \) going from 4 to 10 (days). Good synthetic signals are seen to be obtainable from a range of \( \tau \) values, even though there are some intermediate values for which no good reconstruction has been obtained. The best 3D maps that we can construct are incapable of producing synthetic signals that bear a resemblance to the observational light curve, suggesting an embedding dimension \( d_e = 4 \).

The maps for R UMi are not as robust as those for R Sct, even though the observational accuracy is much higher. One of the probable reasons is that there are fewer cycles in the data. Indeed, for a successful flow reconstruction to be possible the data has to sample sufficiently well the dominant features of the dynamics in phase space. If some regions are only covered very lightly, then the maps can have ‘leaks’ in such regions, and the lack of robustness manifests itself in trajectories that blow up quickly. The synthetic signals that are constructed from the maps are therefore relatively short. It is also hard to obtain accurate Lyapunov exponents and fractal dimensions.

As for AC Her, the R UMi signal does not explore very well the linear neighborhood of the fixed point. Consequently it is not possible to derive any information about a possible resonance.

We find that the fractal dimensions for good synthetic signals hover around \( d_L = 3.25 \pm 0.15 \). Reassuringly this dimension is found to be stable in the sense that reconstructions in 5D also yield \( d_L \) in the same range, and importantly that they are less than 4, i.e. independent of the dimension of the embedding space. The conclusion that imposes itself, at least tentatively, is that the physical dimension which is sandwiched between \( d_L \sim 3.25 \) and \( d_e = 4 \), is \( d = 4 \) for this star as well.

**The star V Canum Venaticorum**

We have not been able to make a reconstruction with the smoothed magnitude data themselves, even with various values of smoothing, although the light curve data span some 27 cycles. The reason why our reconstruction is not successful could be that over the 15 years of observations the light curve does not explore its whole potential, i.e. it does not explore enough of the physical phase space to allow us to make a successful reconstruction. We cannot exclude the possibility that a higher order dynamics might be at work, but then again we would need more observational information to lay it bare. Perhaps the star undergoes intermittent mass loss which causes obscuration and leads
to variations in the overall luminosity. We think the first of these former may be the problem, because, after a prior conversion of the same magnitude data to luminosities we arrive at a reasonably good reconstruction (with two values of $\tau$, 8 and 9). The smoothed observational luminosity data that are used in the reconstruction are shown in Fig. 6 together with two typical synthetic luminosity curves.

In conclusion, the data for V CVn are just barely sufficient for a reconstruction which therefore remains fragile.

The star RS Cygni

The smoothed observed light curve of RS Cyg is displayed in Fig. 7, followed by a couple of 4D synthetic light curves. The flow reconstructions are reasonably robust (with a range of $\tau$ from 11 to 17). The synthetic light curves appear good, but have some difficulties capturing the moving small RV Tau-like feature, and they have added wiggles that are nonexistent in the data. These difficulties are not astonishing because of the shortness of the data set, viz. only about 13 cycles – we should not expect to get more out the data than is in them. A comparison of the amplitude Fourier spectrum of the synthetic light curves with that of the data also shows that the reconstruction is somewhat lacking. The Lyapunov dimensions have a wide range of values 2.1 – 3.1. A 2:1 resonance condition is approximately satisfied for $\tau = 11, 12$ and 15.

We find that flow reconstructions in 3D are not satisfactory. As to 5D there are too few data points to attempt a reconstruction, but our results, in particular Fig. 7, suggest that for RS Cyg an embedding dimension $d_e = 4$ is sufficient. The physical phase space
dimension $d$ is probably equal to 4 in order to accommodate two complex vibrational modes (strictly speaking, if $d_L < 3$ a value of 3 cannot be ruled out from $d_L < d \leq d_e$). Possibly there is again a 2:1 resonance condition between the two modes in this star. The analysis of RS Cyg is thus promising, but for a reliable flow reconstruction it will be necessary to gather more information about the light curve of this star.

**DISCUSSION AND CONCLUSIONS**

The reader may have been left wondering why the classical Cepheids undergo periodic pulsations while their neighbors in the Hertzsprung-Russell diagram pulsate irregularly. We want to address the difference between these two classes of stars briefly. They have the same luminosity range, but the irregular stars have lower masses, by about a factor of 10. As a result the coupling between the heat flow and the acoustic oscillation is strongly enhanced. This can best be seen with a linear stability analysis of the equilibrium stellar models. Let us call $\omega$ the eigenvalues of the pulsation modes (for an assumed $\exp(i\omega t)$ dependence). Then for the modes that are relevant here, i.e. the lowest frequency modes, we find that the relative growth rates of these modes $|\text{Im} \omega / \text{Re} \omega|$ are of the order of a few percent for the classical Cepheids, but they are of order unity for the semiregular stars. The mathematical consequence is that there exists a center manifold for the classical Cepheids, and their behavior can be captured with amplitude equations (normal forms) [21]; in particular, the periodic pulsations of the classical Cepheids owe their existence to the proximity of a broad Hopf bifurcation.

Clearly, a large value of the relative growth rates is a prerequisite for the occurrence
of chaos (appreciable amplitude changes must be possible over a period). For the W Virginis type stars at the low luminosity end the growth rates are still small, and periodic pulsations (limit cycles) prevail, just as for the classical Cepheids. For the more luminous ones, such as the stars analyzed in this paper, the growth rates are of order unity, thus fulfilling the necessary condition for chaotic behavior.

We have applied the global flow reconstruction technique to the high quality observational data of SX Her, R UMi, V CVn and RS Cyg. In view of the relatively small number of observed cycles and the complexity of the light curves our conclusions must be taken with some caution. The flow reconstructions lack the robustness of those obtained earlier from the large amplitude variable star R Sct. Consequently, it is not possible to produce very long synthetic light curves by iterating the map, and the Lyapunov exponents and the fractal dimension are therefore beset with large fluctuations from one synthetic signal to another. Our flow reconstructions are best for SX Her and R UMi. For RS Cyg the time span of the observed light curve is clearly too short. Our poor success with V CVn is perhaps related to the complexity of its light curve.

We find that the light curves of six different large amplitude, irregularly pulsating stars all indicate a minimum embedding dimension of 4, although the only reconstructions that have some robustness are those for R Sct and SX Her. For both of these the fractal Lyapunov dimensions fall in the range 3.1 – 3.5, and one can conclude that the dimension of the physical phase space of the dynamics is 4. This in turn suggests that the 'natural' generalized coordinates in this phase space are the complex amplitudes of two vibrational modes.

Our analysis of both R Sct and SX Her suggests that the irregular behavior has the same physical mechanism. The irregular behavior arises through the nonlinear interaction between two vibrational modes, one of lower frequency that is unstable and one with higher frequency that is stable. Furthermore, the ratio of the two frequencies is close to 2:1 in both stars. We thus arrive at the same physical picture of the underlying mechanism for the irregular light curve. A lower frequency mode is self-excited and entrains an otherwise linearly stable overtone through a low order (2:1) resonance. The irregular pulsation occurs as a result of continual exchange of energy between the two resonant modes. The minimum embedding dimensions of the other four stars corroborate this finding although no information about a resonance could be gotten from the linearization of the map.

The fact that the underlying dynamics of these types of stars are low dimensional and chaotic is not surprising, and it was in fact predicted by numerical hydrodynamical simulations of W Vir type stars \[15\] \[22\]. Furthermore a topological analysis of the attractor \[23\] corroborated these conclusions.

The flow reconstruction technique thus has shed new light on the old mystery of the nature of the irregular pulsations of a large class of large amplitude pulsating variable stars.

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