Type I Hilltop Inflation and the Refined Swampland Criteria

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In this paper, I show that type I hilltop inflation under the current observational constraints could have $M_P^2 \frac{V''}{V} \sim -O(10^{-2})$ in the parameter space which may be marginally compatible with the refined swampland criteria. On the other hand, $M_P^2 \frac{V''}{V} \sim -O(1)$ can be achieved if type I hilltop inflation on the brane is considered.

I. INTRODUCTION

There are many inflation models and string theory hopefully might be able to put some (if not all) of them into the Swampland. It was proposed that if an effective field theory can be embedded consistently in quantum gravity, it has to satisfy two Swampland criteria [1, 2].

- **The distance conjecture:**
  \[ \frac{\Delta \phi}{M_P} < O(1), \]  

- **The de Sitter conjecture:**
  \[ M_P \frac{|V'|}{V} > c \sim O(1), \]  

The distance conjecture states that scalar field excursion in reduced Planck units in field space are bounded from above [3] and the de Sitter conjecture states that the slope of the scalar field potential satisfies a lower bound whenever $V > 0$ [4]. The cosmological consequences of these Swampland criteria is investigated in [5–37].

These Swampland criteria strongly constrain particle physics and cosmology models and some modifications have been suggested in [5, 7, 37, 38]. Recently the refined version of the de Sitter conjecture is proposed in [39]. For a single field, it can be stated as the following:

- **Refined de Sitter conjecture:**
  \[ M_P \frac{|V'|}{V} > c \sim O(1) \text{ or } M_P^2 \frac{V''}{V} < -c' \sim -O(1). \]  

The refined version is weaker and allows a scalar field with a potential maxima, namely a hilltop to exist. Therefore it not only evades the problems with the Higgs potential and the QCD axion potential [38, 40, 42] but also hilltop inflation [43, 44]. Some cosmological consequences of the refined Swampland conjecture have been discussed in [45–48].

Slow-roll inflation is usually characterized by the slow-roll parameters:

\[ \epsilon \equiv \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2 = \frac{1}{2} c^2, \]  

\[ \eta \equiv M_P^2 \frac{V''}{V} = -c'. \]
Hilltop inflation is a small field inflation model, therefore the distance conjecture is automatically satisfied. Interestingly, the alternative condition for the de Sitter conjecture even prefers a convex potential which is the character of a hilltop inflation. However, if $c \sim c' \sim \mathcal{O}(1)$, (conventional) slow-roll inflation does not happen. I will show that if we can tune the parameter to $c' \sim \mathcal{O}(10^{-2})$, type I hilltop inflation would be a viable model.

Another way to satisfy the refined Swampland criteria is to consider inflation on the brane [19, 34], because the Hubble parameter is enhanced during inflation due to the modification of the Friedmann equation. Since the Hubble parameter effectively provides an ”friction term” for the equation of motion of the inflaton field, slow-rolling is enhanced. This mechanism can also be applied to hilltop inflation. This case will be considered in Section V.

II. TYPE I HILLTOP INFLATION

It is widely known that the inflaton field $\phi$ of a hilltop quartic model has a potential of the form

$$V = V_0 - \lambda \phi^4.$$  \hspace{1cm} (6)

When compared with the latest Planck satellite result [49], this model can fit the data very well. However, since the slow roll parameter $\eta$ of the model vanishes at the top of the hill when $\phi = 0$, this model is incompatible with the refined Swampland criteria. In this paper, I consider the hilltop inflation with the potential

$$V(\phi) = V_0 - \frac{1}{2} m^2 \phi^2 - \lambda \frac{\phi^p}{M_{P}^{p-4}} \equiv V_0 \left( 1 + \frac{1}{2} \eta_0 \frac{\phi^2}{M_{P}^2} \right) - \lambda \frac{\phi^p}{M_{P}^{p-4}},$$  \hspace{1cm} (7)

where

$$\eta_0 \equiv \frac{m^2 M_{P}^2}{V_0}. \hspace{1cm} (8)$$

This model was originally considered as the first type of hilltop inflation in [44] where three types of hilltop inflation models are proposed and analyzed by the author and collaborators. Therefore in this paper, I call it a type I hilltop inflation.

In order to achieve slow-roll inflation, we need $V \simeq V_0$ giving

$$\frac{V'}{V} = \frac{\eta_0 \phi}{M_{P}^2} - p \lambda \frac{\phi^{p-1}}{V_0 M_{P}^{p-4}}$$ \hspace{1cm} (9)

$$\frac{V''}{V} = \frac{\eta_0}{M_{P}^2} - p(1-p \lambda \frac{\phi^{p-2}}{V_0 M_{P}^{p-4}}).$$ \hspace{1cm} (10)

From Eq. (10), we can see that $\eta_0$ corresponds to the value of the slow-roll parameter $\eta$ at the top of the potential hill, namely at $\phi = 0$. The field value increases during inflation and $\eta$ becomes more negative than $\eta_0$. Therefore when we compare $M_{P}^2 V''/V$ with $c'$ for the refined Swampland criteria,
we should focus on $\eta_0$ instead of $\eta$. The hilltop quartic model corresponds to $\eta_0 = 0$. The number of e-folds is given by

$$N = \frac{1}{M_P^2} \int_{\phi_e}^{\phi_i} \frac{V}{V'} d\phi,$$  

(11)

which can be integrated analytically in this model to obtain

$$\left( \frac{\phi}{M_P} \right)^{p-2} = \left( \frac{V_0}{M_P^2} \right) \eta_0 e^{(p-2)\eta_0 N} \frac{\eta_0 x + p\lambda(e^{(p-2)\eta_0 N} - 1)}{\eta_0^{2-p} (\eta_0 x - p\lambda)^2},$$

(12)

leading to the predictions for the spectrum of the primordial density perturbation $P_\zeta$, the spectral index $n_s$, and the running spectral index $\alpha$ as

$$P_\zeta = \frac{1}{12\pi^2} \left( \frac{V_0}{M_P^4} \right)^{\frac{p-2}{2}} e^{-2\eta_0 N} \frac{[p\lambda(e^{(p-2)\eta_0 N} - 1) + \eta_0 x]^{\frac{2p-2}{2}}}{\eta_0^{2-p} (\eta_0 x - p\lambda)^2}$$

(14)

$$n_s - 1 = 2\eta = 2\eta_0 \left[ 1 - \frac{\lambda p(p - 1)e^{(p-2)\eta_0 N}}{\eta_0 x + p\lambda(e^{(p-2)\eta_0 N} - 1)} \right]$$

(15)

$$\alpha = 2\eta_0^2 \lambda p(p - 1)(p - 2) \frac{e^{(p-2)\eta_0 N}(\eta_0 x - p\lambda)}{[\eta_0 x + p\lambda(e^{(p-2)\eta_0 N} - 1)]^2}.$$  

(16)

From Eq. (10), we can see that $\eta$ becomes more and more negative during inflation when the field value of $\phi$ increases and inflation ends when $\eta = -1$ at $N = 0$. From Eq. (15), this implies

$$x = \frac{\lambda p(p - 1)}{\eta_0 + 1}.$$  

(17)

By using the definition of $x$ from Eq. (13), we can obtain

$$\frac{V_0}{M_P^4} = \frac{\lambda p(p - 1)}{\eta_0 + 1} \left( \frac{\phi_e}{M_P} \right)^{p-2}.$$  

(18)

This is in accordance with the well known fact that a low scale slow-roll inflation is a small field inflation. In the following, I consider two cases where $p = 4$ and $p = 6$ as examples.

III. CASE $p = 4$

For the case $p = 4$, from Eq. (13), by fixing $N = 60$ for the horizon exit of the CMB scale I obtain

$$n_s = 1 + 2\eta_0 \left[ 1 - \frac{3e^{12\eta_0}(\eta_0 + 1)}{3\eta_0 + (e^{12\eta_0} - 1)(\eta_0 + 1)} \right].$$  

(19)

It is interesting to note that this prediction only depends on $\eta_0$. The spectral index $n_s$ as a function of $|\eta_0|$ is given in Fig. This can be compared with the Planck result $n_s \sim 0.96$ [49]. We can see from the figure that $|\eta_0|$ can be as large as $|\eta_0| \sim 0.01$ and the spectral index is still within the experimental bounds.
From Eqs. (15) and (17), by imposing CMB normalization $P_\zeta \sim (5 \times 10^{-5})^2$, we obtain

$$\lambda = (2.96 \times 10^{-7}) \times \frac{e^{120\eta_0} \times \eta_0^3 \left(\frac{12\eta_0}{\eta_0+1} - 4\right)^2}{\left[4(e^{120\eta_0} - 1) + \frac{12\eta_0}{\eta_0+1}\right]^3}. \tag{20}$$

This result also only depends on $\eta_0$. I plot $\lambda$ as a function of $|\eta_0|$ in Fig. 2.

The running spectral index $\alpha$ can be obtained from Eqs. (16) and (17) as

$$\alpha = 48\eta_0^2 \frac{e^{120\eta_0} \left(\frac{12\eta_0}{\eta_0+1} - 4\right)}{\left[\frac{12\eta_0}{\eta_0+1} + 4(e^{120\eta_0} - 1)\right]^2}. \tag{21}$$

The running spectral index for $p = 4$ as a function of $\eta_0$ is plotted in Fig. 3. This is safely within the experimental bound $|\alpha| < 0.01$.

**IV. CASE $p = 6$**

For the case $p = 6$, from Eq. (15), by fixing $N = 60$ for the horizon exit we obtain

$$n_s = 1 + 2\eta_0 \left[1 - \frac{5e^{240\eta_0}(\eta_0 + 1)}{5\eta_0 + (e^{240\eta_0} - 1)(\eta_0 + 1)}\right]. \tag{22}$$

The spectral index $n_s$ as a function of $|\eta_0|$ is given in Fig. 4. This fits the experimental result $n_s \sim 0.96$ perfectly well for $|\eta_0| \lesssim 0.01$. 
From Eqs. (15), (17) and (18), by imposing CMB normalization $P_\zeta \sim (5 \times 10^{-5})^2$, we can obtain

$$\lambda = (2.96 \times 10^{-7}) \times e^{120\eta_0} \times \frac{\eta_0^{\frac{3}{2}}}{\eta_0^{\frac{3}{2}+1}} \times \frac{30\eta_0}{(30^{\eta_0+1})} \times \frac{\phi_e^2}{M_P^2} \left( \frac{\eta_0 + 1}{30} \right)^{\frac{1}{2}}. \quad (23)$$

This result depends not only on $\eta_0$ but also on the inflaton field value at the end of inflation $\phi_e$. I plot
\[ \lambda \text{ as a function of } |\eta_0| \text{ in Fig. 5 for } \phi_e = M_P \text{ and } \phi_e = 10^{-7} M_P. \text{ Here } \phi_e = M_P \text{ should be regarded as the upper bound for the field value. As can be seen in the plot, by decreasing } \phi_e, \lambda \text{ increases. It is even possible to obtain } \lambda \sim \mathcal{O}(1) \text{ if } \phi_e \text{ is small enough. Because of the relation between } \phi_e \text{ and } V_0 \text{ from Eq. (17), one may wonder whether such a small field value could result in an unacceptable small inflation scale } V_0. \text{ Therefore } V_0^{1/4} \text{ is plotted in Fig. 6. As can be seen in the plot, } V_0^{1/4} \text{ in this range of } \phi_e \text{ is still much larger than the requirement } V_0^{1/4} \gtrsim T_R \gtrsim \text{MeV} \sim 10^{-21} M_P \text{ from successful Big Bang Nucleosynthesis (BBN) [50].}

The running spectral index } \alpha \text{ can be obtained from Eqs. (16) \text{ and (17) as}

\[
\alpha = 240 \eta_0^2 e^{240 \eta_0} \frac{\left( \frac{30 \eta_0}{\eta_0 + 1} - 6 \right)}{\left[ \frac{30 \eta_0}{\eta_0 + 1} + 6(e^{240 \eta_0} - 1) \right]^2}.
\]\n
(24)

The running spectral index for } p = 6 \text{ as a function of } \eta_0 \text{ is plotted in Fig. 7. This is again safely within the experimental bound } |\alpha| < 0.01.

V. TYPE I HILLTOP INFLATION ON THE BRANE

For inflation on the brane, the slow-roll parameters are modified into [51]

\[
\epsilon \equiv \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2 \frac{1}{\left( 1 + \frac{V}{2 \Lambda} \right)^2} \left( 1 + \frac{V}{\Lambda} \right),
\]\n
(25)

\[
\eta \equiv M_P^2 \left( \frac{V''}{V} \right) \left( \frac{1}{1 + \frac{V}{2 \Lambda}} \right),
\]\n
(26)
where $\Lambda$ provides a relation between the four-dimensional Planck scale $M_4$ and five-dimensional Planck scale $M_5$ through

$$M_4 = \sqrt{\frac{3}{4\pi}} \left( \frac{M_5^2}{\sqrt{\Lambda}} \right) M_5,$$  \hspace{1cm} (27)$$

where $M_P = M_4/\sqrt{8\pi}$. 

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**FIG. 5:** $\lambda$ as a function of $|\eta_0|$ for $p = 6$. 

**FIG. 6:** $V_0^{1/4}$ as a function of $|\eta_0|$ for $p = 6$. 

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If $V \gg \Lambda$, the slow-roll parameters are given by
\begin{align}
\epsilon &= \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2 \frac{1}{(4\Lambda)}, \\
\eta &= \frac{M_P^2}{2} \left( \frac{V''}{V} \right) \frac{1}{(2\Lambda)},
\end{align}
(28) (29)

The number of e-folds is
\[ N = \int_{\phi_i}^{\phi_e} \left( \frac{V}{V'} \right) \left( \frac{V}{2\Lambda} \right)^{-1} d\phi. \]
(30)

The spectrum is
\[ P_R = \frac{1}{12\pi^2} \frac{V^3}{V'^2} \left( \frac{V}{2\Lambda} \right)^3. \]
(31)

For hilltop inflation, we have $V \simeq V_0$. Therefore from the above equations, we can see that the result of the previous sections can apply simply by replacing $V_0$ with $V_0^2/\Lambda$. The only exception is the expression for $\epsilon$, but it does not matter because $\epsilon$ is very small and negligible in hilltop inflation due to the fact that it is a small field inflation model. This also implies the predicted tensor-to-scalar ratio in general is very small. We can see from Eq. (26) that we can achieve $\eta_0 = -0.01$ by having $M_P^2 V''/V = -1$ and $V/\Lambda = 200$. Therefore evade the refined Swampland criteria completely.

VI. CONCLUSION AND DISCUSSION

The refined Swampland criteria maybe have rescued hilltop inflation with its topological eternal inflation \[52, 54\] from the swampland. In this paper, I have shown that type I hilltop inflation
is compatible with the Planck data even when $|\eta_0|$ as large as $|\eta_0| = 0.01$. This means we have $M_P^2V''/V < -0.01$ when the field value is non-zero during inflation. Although this is in tension with the proposal $c' \sim 1$, the precise value of $c'$ may depend on the detailed string construction and it may be slightly deviate from unity along the line of argument given in [10]. Thus I conclude that although hilltop quartic inflation model is incompatible with the refined swampland criteria, type I hilltop inflation model is marginally compatible with it. On the other hand, type I hilltop inflation on the brane can completely satisfy the swampland criteria.

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