PICARD GROUP AND FUNDAMENTAL GROUP OF THE MODULI OF HIGGS BUNDLES ON CURVES

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ABSTRACT. Let $X$ be an irreducible smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Let $G$ be a connected reductive affine algebraic group over $\mathbb{C}$. Let $M^\delta_{G,Higgs}$ be the moduli space of semistable principal $G$–Higgs bundles on $X$ of topological type $\delta \in \pi_1(G)$. In this article, we compute the fundamental group and Picard group of $M^\delta_{G,Higgs}$.

1. INTRODUCTION

Let $X$ be an irreducible smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Let $G$ be a connected reductive affine algebraic group over $\mathbb{C}$. The topological types of holomorphic principal $G$–bundles on $X$ are parametrized by $\pi_1(G)$. Let $M^\delta_G$ be the moduli space of semistable holomorphic principal $G$–bundles on $X$ of topological type $\delta \in \pi_1(G)$. This is an irreducible normal complex projective variety (generally non-smooth) of dimension $(g-1)\cdot \dim(G) + \dim(Z(G))$, where $Z(G)$ is the center of the group $G$. Geometry of moduli spaces of bundles over projective curves is an important topic to study in algebraic geometry. Picard group of $M^\delta_G$ was studied in [KN] for simply connected semisimple complex affine algebraic groups. Later A. Beauville, Y. Laszlo and C. Sorger studied the Picard group of $M^\delta_G$ case by case for almost all classical semisimple complex affine algebraic groups (see [BLS]). The case of reductive groups was studied in [BH] for moduli stacks. The fundamental group of moduli space (and stack) of $G$–bundles was studied in [BMP]. The case of moduli of $G$–Higgs bundles was open. In this article we study the fundamental group and Picard group of the moduli spaces of semistable $G$–Higgs bundles over $X$.

Topological type of a holomorphic principal $G$–Higgs bundle on $X$ is defined by the topological type of the underlying principal $G$–bundle on $X$. Let $M^\delta_{G,Higgs}$ be the moduli space of semistable holomorphic principal $G$–Higgs bundles on $X$ of topological type $\delta \in \pi_1(G)$. The space $M^\delta_{G,Higgs}$ is nonempty and connected, for all $\delta \in \pi_1(G)$, [GO, Theorem 1.1, p. 791]. Let $M^\delta_G$ be the moduli space of semistable principal $G$–bundles on $X$ of topological type $\delta \in \pi_1(G)$. For any $\mathbb{C}$–scheme $Z$, we denote by $\text{Pic}(Z)$ (resp., $\pi_1(Z)$) the Picard group (resp., fundamental group) of $Z$.

Assume that either $g \geq 3$, or there is no homomorphism of $G$ onto $\text{PSL}_2(\mathbb{C})$ whenever $g = 2$. Then we have the following

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Theorem 1.1. There are isomorphisms $\text{Pic}(M_{G,\text{Higgs}}^\delta) \cong \text{Pic}(M_G^\delta)$ and $\pi_1(M_{G,\text{Higgs}}^\delta) \cong \pi_1(M_G^\delta)$.

Therefore, it follows from [BMP, Theorem 1.1] that $\pi_1(M_{G,\text{Higgs}}^\delta) \cong \mathbb{Z}^{2gd}$, where $d = \dim Z(G)$ is the dimension of the center of $G$. In particular, $M_{G,\text{Higgs}}^\delta$ is simply connected, whenever $G$ is connected and semisimple. Therefore, together with the results of [KN, BLS], the above theorem determines Picard group of $M_{G,\text{Higgs}}^\delta$ for essentially all classical semisimple complex affine algebraic groups.

We further remark that, our method of determining the fundamental group and Picard group generalize to the case of moduli stack of $G$–Higgs bundles over $X$.

2. PRELIMINARIES

2.1. Some results on locally trivial fibrations. Let $Z$ be a smooth quasi-projective variety over $\mathbb{C}$. We first recall some well-known results related to fundamental groups and Picard groups of vector bundles over $Z$. Let $p : E \longrightarrow Z$ be a vector bundle of finite rank over $Z$.

**Lemma 2.1.** The natural homomorphism $p^* : \text{Pic}(Z) \longrightarrow \text{Pic}(E)$ is an isomorphism.

**Proof.** Let $n$ be the rank of the vector bundle $E$ over $Z$. Note that the fibers of $p$ are affine $n$–spaces $\mathbb{A}^n_C$. We have a short exact sequence of groups

$$\text{Pic}(Z) \xrightarrow{p^*} \text{Pic}(E) \longrightarrow \text{Pic}(\mathbb{A}^n_C) ,$$

where the second map is given by the restriction of a line bundle to the fiber $\mathbb{A}^n_C$. Since $\text{Pic}(\mathbb{A}^n_C)$ is trivial, $p^*$ is surjective. To show $p^*$ injective, let $L \in \text{Pic}(Z)$ be such that $p^*L$ is a trivial bundle on $E$. Since $E$ is Zariski locally trivial over $Z$, we must have $L$ is trivial on $Z$. \hfill \Box

**Lemma 2.2.** The natural homomorphism $q_* : \pi_1(E) \longrightarrow \pi_1(Z)$ is an isomorphism.

**Proof.** Since the morphism $p$ is locally trivial, we have an exact sequence of homotopy groups

$$\pi_1(\mathbb{A}^n_C) \longrightarrow \pi_1(E) \xrightarrow{p_*} \pi_1(Z) \longrightarrow \pi_0(\mathbb{A}^n_C) .$$

Since the fibers $\mathbb{A}^n_C$ are connected and contractible, we conclude that $p_*$ is an isomorphism. \hfill \Box

**Lemma 2.3.** Let $\iota : U \hookrightarrow Z$ be a Zariski open subset of $Z$, whose complement has codimension at least 2 in $Z$. Then we have isomorphisms $\text{Pic}(Z) \overset{\iota^*}{\longrightarrow} \text{Pic}(U)$ and $\pi_1(U) \overset{\iota_*}{\cong} \pi_1(Z)$.

**Proof.** The first isomorphism follows from [Ha, Proposition 1.6, p. 126], and for the second isomorphism, see [Mi, p. 42]. \hfill \Box
2.2. Principal $G$–Higgs bundles. Let $Y$ be an irreducible smooth projective variety over $\mathbb{C}$. Let $\Omega^1_Y$ be the sheaf of differential 1-forms on $Y$. For any $i \geq 1$, let $\Omega^i_Y = \wedge^i \Omega^1_Y$.

Let $G$ be a connected reductive affine algebraic group over $k$. Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of $G$. The adjoint action of $G$ on its Lie algebra $\mathfrak{g}$ is denoted by

$$\text{ad} : G \rightarrow \text{End}(\mathfrak{g}).$$

Let $p : E_G \rightarrow Y$ be a principal $G$–bundle on $Y$. The right $G$–action on $E_G$ and the adjoint action of $G$ on $\mathfrak{g}$ produces a $G$–action on $E_G \times \mathfrak{g}$ defined by

$$(z, \xi) \cdot g = (z \cdot g, \text{ad}(g^{-1})(\xi)),$$ \quad \forall (z, \xi) \in G \times \mathfrak{g}, g \in G.

Then the associated quotient $E_G \times^G \mathfrak{g} := (E_G \times \mathfrak{g}) / G$ is a vector bundle $\text{ad}(E_G)$ on $Y$, called the adjoint vector bundle of $E_G$. A Higgs field on $E_G$ is a section $\theta \in H^0(Y, \text{ad}(E_G) \otimes \Omega^1_Y)$ such that $\theta \wedge \theta = 0$ in $H^0(Y, \text{ad}(E_G) \otimes \Omega^2_Y)$. A principal $G$–Higgs bundle on $Y$ is a pair $(E_G, \theta)$ consisting of a principal $G$–bundle $E_G$ on $Y$ and a Higgs field $\theta$ on $E_G$.

**Definition 2.4.** A principal $G$–Higgs bundle $(E_G, \theta)$ on $Y$ is said to be semistable (respectively, stable) if for any reduction $\sigma : U \rightarrow E_G / P$ of the structure group of $E_G$ to a proper parabolic subgroup $P \subset G$ over an open subset $U \subset Y$ whose complement in $Y$ has codimension at least 2, such that $\theta \in H^0(X, \text{ad}(E_P) \otimes \Omega^1_Y)$, we have

$$\deg (\sigma^*(T_{\text{rel}})) \geq (\text{respectively, } >) 0,$$

where $T_{\text{rel}}$ is the relative tangent bundle of the projection $E_G / P \rightarrow Y$.

Note that, any principal $G$–bundle $E_G$ on $Y$ is a principal $G$–Higgs bundle on $Y$ with Higgs field $\theta = 0$. Taking $\theta = 0$ in the above definition, we get the definition of semistability and stability of principal $G$–bundles $E_G$ on $Y$.

3. Fundamental Group and Picard Group of $M^\delta_{G, \text{Higgs}}$

Fix $\delta \in \pi_1(G)$. Let $M_G^\delta$ be the moduli space of semistable holomorphic principal $G$–bundles on $X$ of topological type $\delta$. This is a normal complex projective variety (generally non-smooth) of dimension $(g - 1) \cdot \dim(G) + \dim(Z(G))$. This contains a smooth open subvariety $M^{s, \delta}_G$ parametrizing stable principal $G$–bundles on $X$ of topological type $\delta$. Let $M^\delta_{G, \text{Higgs}}$ be the moduli space of semistable holomorphic principal $G$–Higgs bundles on $X$ of topological type $\delta$, and let $M^{s, \delta}_{G, \text{Higgs}}$ be the subvariety parametrizing the stable principal $G$–Higgs bundles on $X$ of topological type $\delta$. Note that $M^{s, \delta}_{G, \text{Higgs}}$ is a smooth quasi-projective variety over $\mathbb{C}$.

(*) Assume that either $g = \text{genus}(X) \geq 3$ or there is no nontrivial homomorphism of $G$ onto $\text{PGL}(2, \mathbb{C})$.

**Proposition 3.1.** There is an open embedding

$$\phi : T^*M_G^\delta \hookrightarrow M^{s, \delta}_{G, \text{Higgs}},$$

The complement of the image of $\phi$ in $M^{s, \delta}_{G, \text{Higgs}}$ has codimension at least 2.
Theorem 3.2. There are isomorphisms

\[ \pi_1(M_G^{s, \delta}) \cong \pi_1(M_G^{s, \delta, \text{Higgs}}) \text{ and } \text{Pic}(M_G^{s, \delta}) \cong \text{Pic}(M_G^{s, \delta, \text{Higgs}}). \]

Proof. Consider the open embedding \( \phi : T^*(M_{G}^{s, \delta}) \hookrightarrow M_{G}^{s, \delta, \text{Higgs}} \),

where \( T^*(M_G^{s, \delta}) \) is the cotangent bundle of \( M_G^{s, \delta} \). It follows from [F, Theorem II.6 (iii), p. 534] that \( M_{G}^{s, \delta, \text{Higgs}} \setminus \phi(T^*(M_{G}^{s, \delta})) \) has codimension \( \geq 2 \) in \( M_{G}^{s, \delta, \text{Higgs}} \) (here we are using the condition (*)). \( \square \)

Theorem 3.2. There are isomorphisms

\[ \pi_1(M_G^{s, \delta}) \cong \pi_1(M_G^{s, \delta, \text{Higgs}}) \text{ and } \text{Pic}(M_G^{s, \delta}) \cong \text{Pic}(M_G^{s, \delta, \text{Higgs}}). \]

Proof. Consider the open embedding \( \phi \) in the Proposition 3.1. Then by Lemma 2.3, we have isomorphisms

\[ \pi_1(T^*(M_G^{s, \delta})) \cong \pi_1(M_G^{s, \delta, \text{Higgs}}), \]

and

\[ \text{Pic}(T^*M_G^{s, \delta}) \cong \text{Pic}(M_G^{s, \delta, \text{Higgs}}). \]

Since \( p : T^*M_G^{s, \delta} \rightarrow M_G^{s, \delta} \) is a vector bundle over a smooth base \( M_G^{s, \delta} \), from Lemma 2.2 we have an isomorphism

\[ \pi_1(M_G^{s, \delta}) \cong \pi_1(T^*M_G^{s, \delta}). \]

Similarly, from Lemma 2.2, we have an isomorphism of Picard groups

\[ p^* : \text{Pic}(M_G^{s, \delta}) \rightarrow \text{Pic}(T^*M_G^{s, \delta}). \]

Then from (3.2) and (3.4), we have an isomorphism \( \pi_1(M_G^{s, \delta}) \cong \pi_1(M_G^{s, \delta, \text{Higgs}}) \). Similarly, combining (3.3) and (3.5), we have \( \text{Pic}(M_G^{s, \delta}) \cong \text{Pic}(M_G^{s, \delta, \text{Higgs}}) \). \( \square \)

Corollary 3.3. There are isomorphisms

\[ \pi_1(M_G^{\delta}) \cong \pi_1(M_G^{\delta, \text{Higgs}}) \text{ and } \text{Pic}(M_G^{\delta}) \cong \text{Pic}(M_G^{\delta, \text{Higgs}}). \]

Proof. It follows from [KN, Proposition 3.4] and openness of stability that \( M_G^{s, \delta} \) is a smooth open subscheme of \( M_G^{\delta} \), and the complement \( M_G^{\delta} \setminus M_G^{s, \delta} \) has codimension \( \geq 2 \). Then from Lemma 2.3, we have isomorphisms

\[ \pi_1(M_G^{s, \delta}) \cong \pi_1(M_G^{\delta}) \text{ and } \text{Pic}(M_G^{s, \delta}) \cong \text{Pic}(M_G^{\delta}). \]
Similarly, $M^s_{G,Higgs,\delta}$ contains $M^{s,\delta}_{G,Higgs}$ as a smooth open subscheme such that the complement $M^s_{G,Higgs,\delta} \setminus M^{s,\delta}_{G,Higgs}$ has codimension $\geq 2$ (see [F, Theorem II.6]). Then from Lemma 2.3, we have isomorphisms
\[
\pi_1(M^{s,\delta}_{G,Higgs}) \cong \pi_1(M^s_{G,Higgs,\delta}) \quad \text{and} \quad \text{Pic}(M^{s,\delta}_{G,Higgs}) \cong \text{Pic}(M^s_{G,Higgs,\delta}).
\]
(3.7)
Then the corollary follows from Theorem 3.2 and the above isomorphisms in (3.6) and (3.7).

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