SYMMETRIES OF THE HYDROGEN ATOM AND ALGEBRAIC FAMILIES

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To the memory of Joseph L. Birman

Abstract. We show how the Schrödinger equation of the hydrogen atom in two dimensions gives rise to an algebraic family of Harish-Chandra pairs that codifies the hidden symmetries. The hidden symmetries vary continuously between $\text{SO}(3)$, $\text{SO}(2,1)$ and the Euclidean group $\text{O}(2) \ltimes \mathbb{R}^2$. We show that solutions of the Schrödinger equation naturally come from an algebraic family of Harish-Chandra modules for the family of Harish-Chandra pairs. Furthermore, Jantzen filtration techniques are used to algebraically recover the spectrum of the Schrödinger operator. This is a first application to physics of algebraic families of Harish-Chandra pairs and modules in the sense of [BHS16, BHS17b].

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1. INTRODUCTION AND MAIN RESULTS

The Schrödinger equation of the hydrogen atom has an obvious \( O(3) \) symmetry. Almost a century ago, in the case of bound states (negative energy), Pauli discovered new invariants, independent of the angular momentum (the generators of the \( O(3) \) symmetry), and used them to derive the spectrum of the hydrogen atom \[Pau26\]. A few years later Fock showed that an \( SO(4) \) symmetry governs the degeneracy of the energy eigenspaces \[Foc35\]. Shortly after, Bargmann explained how the works of Pauli and Fock are related \[Bar35\]. The larger symmetry group \( SO(4) \) is known as the hidden symmetry. Later on it was shown that for scattering states (positive energy) there is an \( SO_0(3,1) \) hidden symmetry and for zero energy there is an \( SO(3) \times \mathbb{R}^3 \) (The Euclidean group) hidden symmetry. Moreover, this was generalized to any dimension \( n \geq 2 \) of the configuration space, e.g., see \[Bl66a, Bl66b\]. It is important to note that the hidden symmetry can only be detected upon restriction of the system to an energy eigenspace.

The purpose of this paper is to show that an algebraic family of Harish-Chandra pairs can be associated with the Schrödinger equation and the hidden symmetries (for all possible energy values) can be obtained from the algebraic family. We will show that the parameter space for the family naturally contains the spectrum of the Schrödinger operator and can be thought of as the space of all “pseudo-energies”. We further show that the collection of all physical solutions of the Schrödinger equation (for all possible energy values) arise from an algebraic family of Harish-Chandra modules. Here we shall only consider the case of the two dimensional system. The general case will be considered elsewhere. The two dimensional system was studied before e.g., see \[CM69, TdC98, PP02\] and reference therein. We shall now describe the setup and results more carefully.

The symmetries and group theoretical aspects of the hydrogen atom system, as well as its classical analogue, the Kepler-Coulomb system, was studied extensively throughout the years e.g., see \[GS90, Sin05, Wul11\]. The Schrödinger equation of the hydrogen atom in \( n \) dimensions (\( n \)-dimensional configuration space) with \( n \geq 2 \) is given
by

\[ H \psi(x) = E \psi(x) \]

\[ H = -\frac{\hbar^2}{2\mu} \Delta - \frac{k}{r} \]

where \( x = (x_1, x_2, ..., x_n) \) is the coordinate vector, \( r = \sqrt{\sum_{i=1}^{n} x_i^2} \), \( \mu \) the reduced mass, \( k \) a positive constant, \( \hbar \) the reduced Planck’s constant, and \( \Delta \) is the laplacian operator \( \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \). It has an obvious \( O(n) \) symmetry but also a not that obvious larger symmetry group, its hidden symmetry which we shall now describe. It is known for quite some time that on every eigenspace of \( H \) with eigenvalue \( E \) components of the angular momentum vector operator and components of the quantum Laplace-Runge-Lenz vector operator generate a Lie algebra \( G_E \) such that

\[ G_E \simeq \begin{cases} 
  \mathfrak{so}(n,1), & E > 0 \\
  \mathfrak{so}(n) \ltimes \mathbb{R}^n, & E = 0 \\
  \mathfrak{so}(n+1), & E < 0 
\end{cases} \]

e.g., see [Bl66a, Bl66b]. These Lie algebras are the *infinitesimal hidden symmetries*. Moreover, the spectrum of \( H \) composed of three disjoint pieces

\[ \text{Spec}(H) = \mathcal{E} = \mathcal{E}_b \sqcup \mathcal{E}_0 \sqcup \mathcal{E}_s \]

where \( \mathcal{E}_b \) corresponds to bound states and given explicitly by \( \mathcal{E}_b = \{ E_n = -\frac{C}{|n_0+n|} | n = 0, 1, 2,... \} \) for some positive \( C \) and \( n_0 \) (see [Ada94, Eq. 4.115] for exact formula), \( \mathcal{E}_s \) corresponds to scattering states and given explicitly by \( \mathcal{E}_s = (0, \infty) \), and \( \mathcal{E}_0 = \{ 0 \} \). For any \( E \) in the spectrum, the solution space of the Schrödinger equation, \( \text{Sol}(E) \), is invariant under its hidden symmetry group. For \( E \in \mathcal{E}_b \) the space \( \text{Sol}(E) \), carries a unitary irreducible representation of \( \text{SO}(n+1) \), for \( E \in \mathcal{E}_s \) it carries a unitary irreducible (principal series) representation of \( \text{SO}_0(n,1) \) and for \( E = 0 \) a unitary representation of the Euclidean group \( \text{SO}(n) \ltimes \mathbb{R}^n \). In general, it is not known if the representation of \( \text{SO}(n) \ltimes \mathbb{R}^n \) is irreducible, for \( n = 2 \) it does, (this follows from [TdCJ98]). In fact, the picture with \( E = 0 \) is not completely clear even for \( n = 3 \), see [Bri11].

So far we reviewed some known facts, to explain the novelty of this work we shall first discuss algebraic families of Harish-Chandra pairs and modules. The idea of *contraction* of Lie groups and their representation as a formalism for deformation of Lie groups and their representations is well known in the in mathematical-physics
literature, see e.g., [IW53, Sal61, DR85, Gil94, SBBM12]. Recently, it was demonstrated how algebraic families gives a good mathematical framework for contractions [BHS16, BHS17b]. We will show that the energy is a natural deformation parameter (or contraction parameter) for the hydrogen atom system and for that we use the language of algebraic families introduced in [BHS16, BHS17b].

Roughly speaking, an algebraic family of complex Lie algebras over a complex algebraic variety $X$ is a collection of complex Lie algebras $\mathfrak{g} = \{\mathfrak{g}_x\}_{x \in X}$ that vary algebraically in $x$. Similarly, there is a natural notion for a family of complex algebraic groups over $X$. For our purpose it is enough to consider constant families of groups, that is, families of the form $K = X \times K$ where $K$ is a fixed complex algebraic group. An algebraic family of Harish-Chandra pairs over $X$ is a pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is an algebraic family of complex Lie algebras over $X$ and $K$ is an algebraic family of complex algebraic groups over $X$ satisfying some compatibility conditions. In Section 3 we shall see that in the case of the two dimensional hydrogen atom, the Schrödinger operator $H$ gives rise to an algebraic family of Harish-Chandra pairs $(\mathfrak{g}, K)$ over $X := \mathbb{C}$, where $K = \mathbb{C} \times O(2, \mathbb{C})$ and for any $x \in X$ the fibers of $\mathfrak{g}$ satisfy

$$\mathfrak{g}_x \simeq \begin{cases} 
\mathfrak{so}(3, \mathbb{C}) & x \neq 0 \\
\mathfrak{so}(2, \mathbb{C}) \times \mathbb{C}^2 & x = 0 
\end{cases}$$

The physical interpretation of the family gives rise to a canonical real structure $\sigma$ on $(\mathfrak{g}, K)$ (see Section 3.3) which leads to a family of real Harish-Chandra pairs $(\mathfrak{g}^\sigma, K^\sigma)$ over $X^\sigma = \mathbb{R}$ with $K^\sigma = \mathbb{R} \times O(2)$ and for $x \in \mathbb{R}$

$$\mathfrak{g}^\sigma_x \simeq \begin{cases} 
\mathfrak{so}(2, 1), & x > 0 \\
\mathfrak{so}(2) \times \mathbb{R}^2, & x = 0 \\
\mathfrak{so}(3), & x < 0 
\end{cases}$$

Moreover, by construction, points of $X$ correspond to “generalized” eigenvalues of $H$, this allow us to regard $\text{Spec}(H)$ as a subset of $X^\sigma$. We will prove the following theorem.

**Theorem 1.** For any $E \in \text{Spec}(H) \subset X^\sigma$ the obvious symmetry of the Schrödinger equation is given by $K^\sigma_E$ and the (infinitesimal) hidden symmetry is given by $\mathfrak{g}^\sigma_E$. Furthermore, $\mathfrak{g}^\sigma$ can be lifted to a family of Lie
groups that correspond to the hidden symmetries. That is, there is an algebraic family of complex algebraic group $G$ over $X$ with a real structure $\sigma_G$ such that for every $E \in X^\sigma$

$$G|_E^{\sigma_G} \simeq \begin{cases} 
\text{SO}(2,1), & E > 0 \\
\text{O}(2) \ltimes \mathbb{R}^2, & E = 0 \\
\text{SO}(3), & E > 0
\end{cases}$$

There is an obvious notion for an algebraic family of Harish-Chandra modules for $(\mathfrak{g}, K)$ (also known as an algebraic family of $(\mathfrak{g}, K)$-modules) see [BHS16, Sec. 2.4]. In simple cases (like the examples considered here) an algebraic family of Harish-Chandra modules for $(\mathfrak{g}, K)$ is an algebraic vector bundle $\mathcal{F}$ over $X$ which carries compatible actions of $\mathfrak{g}$ and $K$. In particular, for $x \in X$ the fiber of $\mathcal{F}$ at $x$ is a Harish-Chandra module for $(\mathfrak{g}|_x, K|_x)$. The family $\mathfrak{g}$ has a natural analogue for a Casimir element of a semisimple Lie algebra, we call it the regularized Casimir $\Omega$ (see Section 4.1). On generically irreducible families of $(\mathfrak{g}, K)$-modules $\Omega$ must act via multiplication by some polynomial function. In Section 4.2 we show that the physical realization of $\mathfrak{g}$ forces $\Omega$ to acts on $\text{Sol}(E)$ via multiplication by $\omega(E) = -\frac{E^4}{4} - \frac{k^2}{2}$.

**Theorem 2.** Let $\mathcal{F}$ be a generically irreducible and quasi-admissible family of $(\mathfrak{g}, K)$-modules on which $\Omega$ acts by multiplication by $\omega(E) = -\frac{E^4}{4} - \frac{k^2}{2}$. Then the collection of all the reducibility points of $\mathcal{F}$ coincides with $E_b$.

Using the real structure of $(\mathfrak{g}, K)$, to any family $\mathcal{F}$ of Harish-Chandra modules we can define a dual family $\mathcal{F}^{(\sigma)}$ (the $\sigma$-twisted dual see Section 4.4.1 and [BHS17b, Sec. 2.4]). A nonzero morphism of $(\mathfrak{g}, K)$-modules from $\mathcal{F}$ to $\mathcal{F}^{(\sigma)}$ (an intertwining operator) induces a canonical filtration on every fiber of $\mathcal{F}$. This is the Jantzen filtration, see Section 4.4 and [BHS17b, Sec. 4]. The Jantzen filtration gives a new algebraic way to calculate the spectrum of $H$ and also the part of the spectrum corresponding to bound states. In Section 4.3 we show that there are exactly two families of generically irreducible and quasi-admissible family of $(\mathfrak{g}, K)$-modules on which $\Omega$ acts by multiplication by $\omega(E) = -\frac{E^4}{4} - \frac{k^2}{2}$ and that are generated by their trivial $\text{SO}(2, \mathbb{C})$-type, $\mathcal{F}_0$. We prove the following.

**Theorem 3.** Let $\mathcal{F}$ be any one of the two generically irreducible and quasi-admissible families of $(\mathfrak{g}, K)$-modules on which $\Omega$ acts via multiplication...
by \( \omega(E) = -\frac{E}{2} - \frac{k^2}{2E} \) and that are generated by \( \mathcal{F}_0 \). Then the spectrum of \( \mathcal{H} \) coincides with the set of all \( E \in \mathbb{X} \) for which \( \mathcal{F}_E \), has a nonzero infinitesimally unitary Jantzen quotient. Moreover, \( \mathcal{E}_b \) coincides with the set of all \( E \in \mathbb{X} \) for which \( \mathcal{F}_E \) has a nontrivial Jantzen filtration.

For any one of the two families \( \mathcal{F} \) we show that for each \( E \in \text{Spec}(\mathcal{H}) \) the fiber \( \mathcal{F}_E \) has exactly one infinitesimally unitary Jantzen quotient \( J(\mathcal{F}_E) \). Furthermore, we prove the following result.

**Theorem 4.** Let \( \mathcal{F} \) be any one of the two families from Theorem 3. Then for any \( E \in \text{Spec}(\mathcal{H}) \) the Jantzen quotient \( J(\mathcal{F}_E) \) can be integrated to the unitary irreducible representation of the connected component of \( G^{\sigma_G}_E \) which is isomorphic to \( \text{Sol}(E) \).

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## 2. Algebraic Families

In this section we shall review the needed formalism for algebraic families of Harish-Chandra pairs and their modules. We shall avoid technicalities and convey some ideas by examples. More details can be found in [BHS16, BHS17]. Algebraic families can be defined over any complex algebraic variety, for our purpose it is enough to consider families over the simplest non-trivial affine variety, \( \mathbb{A}^1_{\mathbb{C}} = \mathbb{C} \). So, throughout this note we let \( \mathbb{X} \) be the complex affine variety line i.e., \( \mathbb{X} = \mathbb{C} \). As usual, we shall denote the structure sheaf of regular functions of \( \mathbb{X} \) by \( \mathcal{O}_\mathbb{X} \). Moreover, we shall freely identify various families (various sheaves) over \( \mathbb{X} \) with their space of global sections.

### 2.1. Families of complex Lie algebras

An algebraic family of complex Lie algebras \( \mathfrak{g} \) over \( \mathbb{X} \) is a locally free sheaf of \( \mathcal{O}_\mathbb{X} \)-modules equipped with an \( \mathcal{O}_\mathbb{X} \)-linear Lie bracket. Since \( \mathbb{X} \) is affine, such a family is nothing else but a Lie algebra over the ring of \( C[\mathbb{X}] = \mathcal{O}_\mathbb{X}(\mathbb{X}) \). Intuitively, \( \mathfrak{g} \) should be thought of as a collection of complex Lie algebras (the fibers of \( \mathfrak{g} \)) parameterized by \( \mathbb{X} \) that vary algebraically. Recall that the fiber of \( \mathfrak{g} \) at \( x_0 \in \mathbb{X} \) is \( \mathfrak{g}|_{x_0} := \mathfrak{g}/I_{x_0}\mathfrak{g} \) where \( I_{x_0} \) is the maximal ideal of \( C[\mathbb{x}] \) consisting of all functions the vanish at \( x_0 \). Of course, \( \mathfrak{g}|_{x_0} \) is a Lie algebra over \( \mathbb{C} \cong C[x]/I_{x_0} \).
Example 1. Consider the constant family of Lie algebras over $X$ with fiber $\mathfrak{gl}_3(C)$, that is, the sheaf of regular (algebraic) sections of the bundle $X \times \mathfrak{gl}_3(C)$ over $X$. This is an algebraic family of Lie algebras over $X$, we shall denote it by $\mathfrak{gl}_3(C)$. Each fiber of $\mathfrak{gl}_3(C)$ is canonically identified with $\mathfrak{gl}_3(C)$. The family $\mathfrak{gl}_3(C)$ contains interesting non-constant subfamilies. We shall now describe one such subfamily that will play a role in what follows. Let $\widetilde{\mathfrak{so}}_3$ be the subfamily that is characterized by the following property. For every $x \in X$ the fiber $\widetilde{\mathfrak{so}}_3|_x$ is given (under the above mentioned identification) by

$$\begin{pmatrix}
0 & \alpha & \beta \\
-\alpha & 0 & \gamma \\
-x\beta & -x\gamma & 0
\end{pmatrix} \quad | \quad \alpha, \beta, \gamma \in C.$$

Note that $\widetilde{\mathfrak{so}}_3|_x \simeq \begin{cases} 
\mathfrak{so}(3, C) & x \neq 0 \\
\mathfrak{so}(2, C) \rtimes \mathbb{C}^2 & x = 0.
\end{cases}$

Let $e_{ij}$ with $i, j \in \{1, 2, 3\}$ be the standard basis of $\mathfrak{gl}_3(C)$. The maps (from $\mathbb{C}$ into $\mathfrak{gl}_3(C)$) given by

(2.1) $j_1(x) = e_{23} - xe_{32}$

(2.2) $j_2(x) = e_{13} - xe_{31}$

(2.3) $j_3(x) = e_{12} - e_{21}$

define a basis for $\widetilde{\mathfrak{so}}_3$ as a Lie algebra over $\mathbb{C}[x]$. In particular

$$\widetilde{\mathfrak{so}}_3 = \mathbb{C}[x]j_1 \oplus \mathbb{C}[x]j_2 \oplus \mathbb{C}[x]j_3.$$

The commutation relations are determined by

(2.4) $[j_1(x), j_2(x)] = xj_3(x), [j_2(x), j_3(x)] = j_1(x), [j_3(x), j_1(x)] = j_2(x)$

2.2. Families of complex algebraic groups. Formally an algebraic family of complex algebraic groups is a smooth group scheme $G$ over $X$ with $G$ being a smooth complex algebraic variety, see [BHS16, Sec. 2.2]. As in the case of families of Lie algebras, we can think about it as a collection of complex algebraic groups that vary algebraically. For us, the most important example is the constant family of groups over $X$ with fiber $\text{GL}_3(C)$, we shall denote it by $\text{GL}_3(C)$. Any fiber of $\text{GL}_3(C)$ is canonically identified with $\text{GL}_3(C)$. We shall now define a subfamily $\widetilde{\text{SO}}_3$ of $\text{GL}_3(C)$, whose associated family of Lie algebras ([BHS16, Sec. 2.2.1]) is $\widetilde{\mathfrak{so}}_3$ from the previous section.

Example 2. The family $\widetilde{\text{SO}}_3$ is uniquely determined by its fibers which we shall now describe. For a nonzero $x \in X$, $\widetilde{\text{SO}}_3|_x$ is given
by all \( A \in \text{SL}_3(\mathbb{C}) \) such that \( A^t D x A = D_x \) where \( D_x \) is the diagonal matrix in \( \text{GL}_3(\mathbb{C}) \) with diagonal entries \((x, x, 1)\). In particular \( \widetilde{SO}_3|_x \simeq O(3, \mathbb{C}) \). The remaining fiber is

\[
\widetilde{SO}_3|_0 = \left\{ \begin{pmatrix} A & \nu \\ 0 & |A| \end{pmatrix} \bigg| A \in O(2, \mathbb{C}), \nu \in \mathbb{C}^2 \right\} \simeq O(2, \mathbb{C}) \ltimes \mathbb{C}^2
\]

To show that \( \widetilde{SO}_3 \) is indeed an algebraic family of complex algebraic groups one should follows the same calculations as in [BHS17].

### 2.3. Families of Harish-Chandra pairs

Before we discuss families of Harish-Chandra pairs we recall the definition of a (classical) Harish-Chandra pair. A Harish-Chandra pair consists of a pair \((g, K)\) where \( g \) is a complex Lie algebra, \( K \) a complex algebraic group acting on \( g \), and an embedding of Lie algebras \( \iota : \text{Lie}(K) \rightarrow g \) such that the following two conditions hold.

1. \( \iota \) is equivariant where \( K \) acts on \( \text{Lie}(K) \) by the adjoint action.
2. The action of \( \text{Lie}(K) \) on \( g \) coming from the derivative of the action of \( K \) coincides with the action that is obtained by composing \( \iota \) with the adjoint action of \( g \) on itself.

For more details see e.g., [BBG97, KV95]. An algebraic family of a Harish-Chandra pairs over \( X \) is defined in analogous way. The main difference is the replacement of \( g \) by a family of Lie algebras \( g \) over \( X \) and replacement of \( K \) by an algebraic family of complex algebraic groups \( K \) over \( X \). We shall only consider the case in which \( K \) is a constant family over \( X \). For precise definition see [BHS16, Sec. 2.3]. Below is an example that will be of main interest for us.

#### Example 3

The family of \( \widetilde{SO}_3 \) contains a constant subfamily \( O_2 \) with fiber isomorphic to \( O(2, \mathbb{C}) \). Explicitly for any \( x \in X \)

\[
O_2|_x = \left\{ \begin{pmatrix} A & 0 \\ 0 & |A| \end{pmatrix} \bigg| A \in O(2, \mathbb{C}) \right\}
\]

Note that any matrix in \( O_2|_x \) is generated by matrices of the form

\[
\tilde{R}(\theta) := \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{s} := \begin{pmatrix} s & 0 \\ 0 & -1 \end{pmatrix}
\]

where

\[
R(\theta) := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
with $\theta \in \mathbb{C}$. The family $O_2$ naturally acts on $\tilde{\mathfrak{so}}_3$ via conjugation. This action can be summarized by the formulas
\[
\begin{align*}
\tilde{R}(\theta) \cdot j_1 &= \cos(\theta)j_1 - \sin(\theta)j_2, \\
\tilde{R}(\theta) \cdot j_2 &= \sin(\theta)j_1 + \cos(\theta)j_2, \\
\tilde{R}(\theta) \cdot j_3 &= j_3, \\
\tilde{s} \cdot j_1 &= -j_2, \\
\tilde{s} \cdot j_2 &= -j_1, \\
\tilde{s} \cdot j_3 &= -j_3.
\end{align*}
\]

The family of Lie algebras $\mathfrak{o}_2$ associated with $O_2$ coincides with the subfamily of $\tilde{\mathfrak{so}}_3$ that is generated by $j_3$. All in all, the pair $(\tilde{\mathfrak{so}}_3, O_2)$ is an algebraic family of Harish-Chandra pairs over $\mathbb{X}$.

2.4. **Real structure for families.** In this section we discuss real structures of a family of complex groups and show how it gives rise to a family of Lie groups.

In general, for any complex algebraic variety $X$ we denote its complex conjugate variety by $\overline{X}$. The underlying set of $\overline{X}$ coincides with that of $X$ and a complex valued function on $\overline{X}$ is regular if (by definition) its complex conjugate is a regular function of $X$. In addition $\overline{X}$ is canonically isomorphic to $X$. An antiholomorphic morphism from a complex variety $X$ to another complex variety $Y$ is an algebraic morphism from $X$ to $\overline{Y}$. Given any morphism $\psi : X \longrightarrow Y$ there is a canonical morphism $\overline{\psi} : \overline{X} \longrightarrow Y$ such that $\psi = \overline{\psi}$ as set theoretic maps. A real structure, or antiholomorphic involution, on $X$ is a morphism $\sigma_X : X \longrightarrow \overline{X}$ such that the composition
\[
X \xrightarrow{\sigma_X} \overline{X} \xrightarrow{\sigma_X} X
\]
is the identity. See [BHS16, Sec. 2.5]. The fixed points of $\sigma$ is denoted by $X^\sigma$. We treat $X^\sigma$ as a topological space (equipped with the subspace topology as a subset of $X$ with its analytic topology). We shall mainly work with $X = \mathbb{C}$ and $\sigma$ being the usual complex conjugation. In that case $X^\sigma = \mathbb{R}$.

If $G$ is an algebraic family of complex algebraic groups over $X$ then $\overline{G}$ is an algebraic family of complex algebraic groups over $\overline{X}$. A real structure, or antiholomorphic involution, on $G$ is a pair of involutions $\sigma_X : X \longrightarrow \overline{X}$ and $\sigma_G : G \longrightarrow \overline{G}$ such that
\[
\begin{array}{ccc}
G & \xrightarrow{\sigma_G} & \overline{G} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma_X} & \overline{X}
\end{array}
\]
is commutative and induces a morphism $G \longrightarrow \sigma_X^* \overline{G}$ of algebraic families of complex algebraic groups over $X$. See [BHS16, Sec. 2.5].
The morphism \( \sigma_G \) induces an antiholomorphic involution of the complex algebraic group \( G \) for each \( x \in X^\sigma \). Letting \( G |_x^{\sigma} \) stand for the fixed points of the mentioned involution, then \( G |_x^{\sigma} \) is a Lie group. In this way we get a collection of Lie groups parameterized by \( X^\sigma \). We shall refer to this collection as a family of Lie groups over \( X^\sigma \) and denote it by \( G^{\sigma} \).

For our purpose it is enough to understand the simplest case in which \( X \) is the variety \( X = \mathbb{C} \), \( \sigma_X \) is complex conjugation of complex numbers, \( G \) is the constant family \( \mathbb{C} \times \text{GL}_n(\mathbb{C}) \) and \( \sigma_G(x, g) = (\overline{x}, \overline{g}) \) where \( \overline{x} \) is the complex conjugate of \( x \in \mathbb{C} \) and \( \overline{g} \) is the complex conjugate of the matrix \( g \in \text{GL}_n(\mathbb{C}) \). In that case the family \( G^{\sigma} \) is the constant family \( \mathbb{R} \times \text{GL}_n(\mathbb{R}) \). We will obtain interesting non-constant families of Lie groups as subfamilies of \( \mathbb{R} \times \text{GL}_n(\mathbb{R}) \) as in the following example.

**Example 4.** Consider the family \( \widetilde{SO}_3 \) from example 2 with the above mentioned real structure. Then for any \( x \in \mathbb{R} \) the group \( \widetilde{SO}_3 |_x^{\sigma} \) is given by all real matrices in \( \widetilde{SO}_3 |_x \). In particular \( \widetilde{SO}_3 |_x^{\sigma} = SO(3) \), \( \widetilde{SO}_3 |_x^\sigma = SO(2,1) \), and

\[
\widetilde{SO}_3 |_x^{\sigma} = \left\{ \begin{pmatrix} A & \nu \\ 0 & |A| \end{pmatrix} \right| A \in O(2), \nu \in \mathbb{R}^2 \right\}
\]

There are no other isomorphism classes and

\[
\widetilde{SO}_3 |_x^{\sigma} \simeq \begin{cases} 
SO(2,1) & x < 0 \\
O(2) \ltimes \mathbb{R}^2 & x = 0 \\
SO(3) & x > 0 
\end{cases}
\]

There are similar definitions for a real structure, or antiholomorphic involution of algebraic families of complex Lie algebras and algebraic families of Harish-Chandra pairs see [BHS16, Sec. 2.5]. In the above example the real structure on \( \widetilde{SO}_3 \) induces one on \( (\widetilde{so}_3, O_2) \) from example 3. The corresponding family of real Harish-Chandra pairs is given by looking on real matrices in each of the fibers \( (\widetilde{so}_3 |_x, O_2 |_x) \). Explicitly, for any \( x \in \mathbb{R} \)

\[
\widetilde{so}_3 |_x^{\sigma} = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -x \beta & -x \gamma & 0 \end{pmatrix} \right| \alpha, \beta, \gamma \in \mathbb{R} \right\}
\]

and

\[
O_2 |_x^{\sigma} = \left\{ \begin{pmatrix} A & 0 \\ 0 & |A| \end{pmatrix} \right| A \in O(2) \right\}
\]
In addition
\[ \widehat{so}_3^\sigma = \mathbb{R}[x]_j_1 \oplus \mathbb{R}[x]_j_2 \oplus \mathbb{R}[x]_j_3 \]

The homogenous spaces \( \widehat{SO}_3^\sigma / SO_2^\sigma \) for \( x \in \mathbb{R} \) carries a geometric meaning that is easy to describe. For \( x > 0 \) \( \widehat{SO}_3^\sigma / SO_2^\sigma \) it is a two sheeted hyperboloid, for \( x < 0 \) it is ellipsoid and as \( x \) approaches zero the space “flattened” into two parallel planes. See figure 1.

3. THE FAMILY OF HARISH-CHANDRA PAIRS AND HIDDEN SYMMETRIES

In this section we show how the Schrödinger equation of the hydrogen atom in two dimensions gives rise to an algebraic family of Harish-Chandra pairs equipped with a real structure. We then show that the various hidden symmetries can be recovered from the algebraic family.

3.1. The infinitesimal hidden symmetry. The Schrödinger equation of the hydrogen atom in two dimensions is given by

\[ H\psi = E\psi \]

\[ H = -\frac{1}{2}(\partial_{xx} + \partial_{yy}) - \frac{k}{\sqrt{x^2 + y^2}} \]

Here \( k \) is a positive constant and we use atomic units, that is \( \mu = \hbar = 1 \). We regard \( H \) as a differential operator on smooth complex valued functions on \( S = \mathbb{R}^2 \setminus \{0\} \). We let \( \mathcal{D} = \mathcal{D}(S) \) be the algebra of all such differential operators. The algebra \( \mathcal{D} \) is naturally filtered.
by the degree of a differential operator. We denote this filtration by \( D = \bigcup_{n \geq 0} D_n \). We denote the centralizer of \( H \) in \( D \) by \( C_D(H) \) and the space \( C_D(H) \cap \bigcup_{n=0}^2 D_n \) by \( C_D^2(H) \).

**Lemma 1.** The complex vector space \( C_D^2(H) \) is spanned by:

\[
L := y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad L^2, \quad I \quad \text{(the identity operator)}
\]

\[
H = -\frac{1}{2}(\partial_{xx} + \partial_{yy}) - \frac{k}{\sqrt{x^2 + y^2}}
\]

\[
A_y := \frac{i}{\sqrt{2}} \left(y \partial_{xx} - x \partial_{xy} - \frac{1}{2} \partial_y + k \frac{y}{r}\right),
\]

\[
A_x := \frac{i}{\sqrt{2}} \left(x \partial_{yy} - y \partial_{xy} - \frac{1}{2} \partial_x + k \frac{x}{r}\right)
\]

The proof is by direct calculation.

We shall consider the complex associative sub-algebra of \( C_D(H) \) that is generated (as an algebra) by \( C_D^2(H) \). We denote it by \( B \). Observe that \( B \) is in fact an associative algebra over the free polynomial ring \( C[H] \) and it is generated as an algebra over \( C[H] \) by \( \{A_x, A_y, L, I\} \).

The generating set \( \{A_x, A_y, L, I\} \) is minimal in two ways. There are no smaller sets of generators and the sum of degrees (with respect to the natural filtration of \( D \)) of the generators is minimal among all other sets of generators\(^2\). In fact, all such minimal sets of four generators with their sum of their degrees equal to 5 span the same four dimensional vector subspace \( V_4 := \text{span}_C \{A_x, A_y, L, I\} \). Let \( V_4 \) be the \( C[H] \)-module generated by \( V_4 \).

**Lemma 2.** \( V_4 \) is a Lie algebra over \( C[H] \), saying differently, an algebraic family of complex Lie algebras over \( C \).

**Proof.** By definition and the linear independence of \( \{A_x, A_y, L, I\} \)

\[
V_4 = C[H]L \oplus C[H]A_x \oplus C[H]A_y \oplus C[H]I
\]

By direct calculation

\[
[A_y, L] = A_x, \quad [L, A_x] = A_y, \quad [A_x, A_y] = -HL
\]

and, \( I \) commutes with everything else. \( \square \)

For purposes that will be made clear below, from now on we shall use the notation \( E \) for the indeterminate \( H \). For any nonzero \( E \in C \)

\(^2\)That is, \( B \) can not be generated by less than four generators and there are no other sets of four generators \( \{p_1, p_2, p_3, p_4\} \) of \( B \) such that the sum of their degrees is larger then \( 2 + 2 + 1 + 0 = 5 \) (the sum of the degrees of \( \{A_x, A_y, L, I\} \)).
the fiber $V_4|_E$ is a complex reductive Lie algebra over $\mathbb{C}$ (it is isomorphic to $gl_2(\mathbb{C})$). Let $\mathfrak{g}$ be the largest Lie subalgebra (over $\mathbb{C}[E]$) of $V_4$ such that for any nonzero $E \in \mathbb{C}$ the fiber $\mathfrak{g}|_E$ is semisimple. The commutation relations imply that $\{A_x, A_y, L\}$ form a basis for $\mathfrak{g}$ over $\mathbb{C}[E]$.

**Proposition 1.** $\mathfrak{g}$ is isomorphic to $\tilde{\mathfrak{so}}_3$.

**Proof.** As mentioned above

\[ \mathfrak{g} = \mathbb{C}[E]L \oplus \mathbb{C}[E]A_x \oplus \mathbb{C}[E]A_y \]

and

\[ [A_y, L] = A_x, [L, A_x] = A_y, [A_x, A_y] = -EL \]  

The correspondence

\[ x \leftrightarrow -E, \quad A_x \leftrightarrow j_1, \quad A_y \leftrightarrow j_2, \quad L \leftrightarrow j_3 \]

defines an isomorphism between $\mathfrak{g}$ and $\tilde{\mathfrak{so}}_3$. \qed

**Remark 1.** The commutation relations of span$_\mathbb{C}\{A_x, A_y, L\}$ shows that it is not a Lie algebra over $\mathbb{C}$ and suggest to consider it as a Lie algebra over $\mathbb{C}[E]$.

**Remark 2.** We shall see in Sections 3.2 and 3.3 that the isomorphism extends to an isomorphism of algebraic families of Harish-Chandra pairs equipped with a real structure. This isomorphism is essentially unique; all such isomorphisms are given by

\[ x \leftrightarrow -rE, \quad A_x \leftrightarrow cj_1, \quad A_y \leftrightarrow c^{-1}j_2, \quad L \leftrightarrow j_3 \]

for some positive $r$ and $c$.

3.2. **The non-hidden symmetries.** The group $O(2)$ naturally acts on $S = \mathbb{R}^2 \setminus \{0\}$. This action induces an action of $O(2, \mathbb{C})$ on $\mathbb{D}$ which descends to $\mathfrak{g}$ and defines an action of $\mathcal{K}$, the constant family over $X$ with fiber $O(2, \mathbb{C})$. Explicitly

\[
R(\theta) \cdot A_x = \cos(\theta)A_x - \sin(\theta)A_y, \quad s \cdot A_x = -A_y \\
R(\theta) \cdot A_y = \sin(\theta)A_x + \cos(\theta)A_y, \quad s \cdot A_y = -A_x \\
R(\theta) \cdot L = L, \quad s \cdot L = -L
\]

In particular we see that under the canonical isomorphism $O_2 \simeq \mathcal{K}$ the following hold.

**Lemma 3.** The isomorphism is canonically extended to an isomorphism of algebraic families of Harish-Chandra pairs over $X$ between $(\tilde{\mathfrak{so}}_3, O_2)$ and $(\mathfrak{g}, \mathcal{K})$. \qed

---

3It is the stabilizer of $S$ inside the Euclidean group of $\mathbb{R}^2$. 
3.3. **Real structure on** \((\mathfrak{g}, \mathcal{K})\). We shall now describe a natural real structure on \((\mathfrak{g}, \mathcal{K})\), under the isomorphism \((\mathfrak{g}, \mathcal{K}) \simeq (\widetilde{\mathfrak{so}}_3, \mathcal{O}_2)\) the real structure coincides with the one that was given in Example 4.

The natural inner product on \(L^2(\mathbb{R}^2 \setminus 0)\) allows us to define a real structure on \((\mathfrak{g}, \mathcal{K})\) in the following way. For \(T \in \mathbb{D} \subset \text{End}(L^2(\mathbb{R}^2 \setminus 0))\) the formula

\[
\sigma(T) = -T^*
\]

where \(T^*\) is the adjoint of \(T\), defines a conjugate linear involution \(\mathbb{D}\) which descends to an antiholomorphic involution of \(\mathfrak{g}\). Similarly the action of \(\mathcal{O}(2, \mathbb{C})\) on \(L^2(\mathbb{R}^2 \setminus 0)\) embeds it into \(\text{Aut}(L^2(\mathbb{R}^2 \setminus 0))\) we can define an antiholomorphic involution of \(\mathcal{O}(2, \mathbb{C})\) by

\[
\sigma_{\mathcal{O}(2,\mathbb{C})}(g) = (g^*)^{-1}
\]

where here we identify \(g \in \mathcal{O}(2, \mathbb{C})\) with its image in \(\text{Aut}(L^2(\mathbb{R}^2 \setminus 0))\). By direct calculation \(\sigma_{\mathcal{O}(2,\mathbb{C})}\) in terms of the matrix group \(\mathcal{O}(2, \mathbb{C})\) turns out to be the usual complex conjugation of matrices. As was described in Section 2.4 this defines a real structure \(\sigma_{\mathcal{K}}\) on the constant family \(\mathcal{K}\).

**Remark 3.** In quantum mechanics it is often the case that a Lie group acts by unitary operators. Our definition for real structure is consistent with the quantum mechanical description of the hydrogen atom system.

**Lemma 4.** The isomorphism \((\mathfrak{g}, \mathcal{K}) \simeq (\widetilde{\mathfrak{so}}_3, \mathcal{O}_2)\) is compatible with the real structure on \((\mathfrak{g}, \mathcal{K})\), introduced above, and the real structure on \((\widetilde{\mathfrak{so}}_3, \mathcal{O}_2)\), introduced in example 4. □

By Example 4 we know that \(\widetilde{\mathfrak{so}}_3\) with its real structure can be lifted to the family \(\widetilde{\mathcal{SO}}_3\) with a compatible real structure. We summarize the results of this section in the following theorem.

**Theorem 1.** For any \(E \in \text{Spec}(\mathcal{H}) \subset \mathcal{X}^{\sigma}\) the obvious symmetry of the Schrödinger equation is given by \(K_{\mathbf{e}}^{\sigma}\) and the (infinitesimal) hidden symmetry is given by \(g_{\mathbf{e}}^{\sigma}\). Furthermore, \(g_{\mathbf{e}}^{\sigma}\) can be lifted to a family of Lie groups that correspond to the hidden symmetries. That is, there is an algebraic family of complex algebraic group \(\mathbf{G}\) over \(\mathcal{X}\) with a real structure \((\sigma_{\mathcal{K}}, \sigma_{\mathbf{G}})\) such that for every \(E \in \mathcal{X}^{\sigma}\)

\[
G_{\mathbf{e}}^{\sigma} \simeq \begin{cases} 
\text{SO}(2, 1), & E > 0 \\
\mathcal{O}(2) \ltimes \mathbb{R}^2, & E = 0 \\
\text{SO}(3), & E > 0 
\end{cases}
\]

\(^4\text{Here } \sigma(ST) = \sigma(T)\sigma(S).\)
4. THE FAMILY OF HARISH-CHANDRA MODULES AND HIDDEN SYMMETRIES

In this section we show how the physical realization of the family \((g, K)\) induces an essentially unique family of \((g, K)\)-modules from which one can recover the solutions for the Schrödinger equation.

4.1. Algebraic families of Harish-Chandra modules. Let \((g, K)\) be the algebraic family of Harish-Chandra pairs introduced in Section 3. Roughly speaking, an algebraic family of Harish-Chandra modules for \((g, K)\) (or algebraic family of \((g, K)\)-modules) is a family of complex vector spaces parameterized by \(X\) that carries compatible actions of \(g\) and \(K\). More precisely, an algebraic family of Harish-Chandra modules is a flat quasicoherent \(O_X\)-module, \(F\), that is equipped with compatible actions of \(g\) and \(K\), see [BHS16, Sec. 2.4]. Since \(X\) is the affine variety \(\mathbb{C}\), \(F\) can be identified with its space of global sections and can be considered as a \(\mathbb{C}[E]\)-module that carries a representation of \(g\) as a Lie algebra over \(\mathbb{C}[E]\) and also carries a compatible action of \(K = O(2, \mathbb{C})\), the fiber of the constant family \(K\). We shall freely change our perspective between sheaves of \(O_X\)-modules and \(\mathbb{C}[E]\)-modules with no further warning. Since \(K\) is a constant family there is a canonical decomposition of \(F\) into \(K\)-isotopic subsheaves (or submodules)

\[F = \bigoplus_{\tau \in \hat{K}} F_{\tau}\]

We call \(\tau \in \hat{K}\) with nonzero \(F_{\tau}\) a \(K\)-type of \(F\). We shall only consider families that are quasi-admissible and generically irreducible (see [BHS16, Sec. 2.2 & 4.1]). For such families over \(X\) each \(F_{\tau}\) is a free \(K\)-equivariant \(O_X\)-module of finite rank. In simpler terms, the space of global sections of each \(F_{\tau}\) is isomorphic to \(\mathbb{C}[E] \otimes \mathbb{C} V_\tau\), with \(V_\tau\) isomorphic to a finite direct sum of the representation \(\tau\). The number of summands in this direct sum is called the multiplicity of the \(K\)-type \(\tau\) in \(F\). In addition, for almost any \(x \in X\) the fiber \(F|_x\) is an irreducible admissible \((g|_x, K|_x)\)-module. Now, for almost any \(x \in X\) the Harish-Chandra pair \((g|_x, K|_x)\) is isomorphic to the pair arising from \(SO(2, 1)\), that is the pair \((\mathfrak{so}(2, 1)_C, O(2, \mathbb{C}))\) where \(\mathfrak{so}(2, 1)_C\) is the complexification of \(\mathfrak{so}(2, 1)\) and \(O(2, \mathbb{C})\) sits, as before, block diagonally inside \(SL(3, \mathbb{C})\). The classification of irreducible admissible \((\mathfrak{so}(2, 1)_C, O(2, \mathbb{C}))\) is well known\(^5\) and determines the possible lists

\(^5\)For example it can be deduce from the classification of the irreducible admissible \((\mathfrak{sl}_2(\mathbb{C}), SO(2, \mathbb{C}))\)-modules given at e.g., at [Vog81, ch.1], (or [HT92, sec. II. 1] or [Tay86, ch. 8]) together with Clifford theory [Cli37]. Or it can be directly deduced from [Nai64].
of K-types for generically irreducible and quasi-admissible families of \((g, K)\)-modules. Such a list of K-types is an invariant of an isomorphism class of such families of \((g, K)\)-modules. Using this invariant and a few more, the classification of generically irreducible and quasi-admissible families of Harish-Chandra modules for a closely related family of Harish-Chandra pairs was given in [BHS16, Sec. 4]\(^6\). By the same methods one can classify generically irreducible and quasi-admissible families of \((g, K)\)-modules. The relation between the family of Harish-Chandra pairs in [BHS16] and the family \((g, K)\) introduced in Section 3 is analogous to the relation between \(SU(1,1)\) and \(SO(2,1)\). More precisely, the families of Lie algebras are isomorphic while the fibers of the two constant families of groups are related by quotient by a two element subgroup and extension by a two element group. We note that similar families of representations were studied in [Ada17].

Before we state the needed parts from the mentioned classification we shall describe the center of the enveloping algebra of \(g\). Let \(\mathcal{U}(g)\) the enveloping algebra of \(g\), by Poincaré-Birkhoff-Witt theorem as a module over \(\mathbb{C}[E]\) we can write

\[
\mathcal{U}(g) = \bigoplus_{i,j,k \in \mathbb{N}_0} \mathbb{C}[E]\Lambda_x^i \Lambda_y^j \Lambda_E^k.
\]

By direct calculation \(Z(g)\), the center of \(\mathcal{U}(g)\), is a free polynomial algebra over \(\mathbb{C}[E]\) with one generator

\[
\Omega := A_x^2 + A_y^2 - E \Lambda_E^2
\]

which we call the regularized Casimir. On a generically irreducible and quasi-admissible family of \((g, K)\)-modules the regularized Casimir \(\Omega\) must act by multiplication by a function \(\omega(E) \in \mathbb{C}[E]\), see [BHS16 Sec. 4.4]. The function \(\omega(E)\) is another invariant of generically irreducible and quasi-admissible family of \((g, K)\)-modules. The following proposition describes a class of families of \((g, K)\)-modules that are completely determined by the two invariants mentioned above.

**Proposition 2.** Let \(\tau\) be an irreducible algebraic representation of \(O(2, \mathbb{C})\). Up to an isomorphism, a generically irreducible and quasi-admissible family of \((g, K)\)-modules \(\mathcal{F}\) that is generated by its \(\tau\) isotopic piece \(F_\tau\) is determined by \(\omega(E)\).

\(^6\)In fact in [BHS16] families over \(\mathbb{C}P^1\) were classified, the case with \(X = \mathbb{C}\) is simpler.
The proposition follows from [BHS16, Thm. 4.9.3] and the classification of admissible irreducible \((\mathfrak{sl}_2(\mathbb{C}), \text{SO}(2, \mathbb{C}))\)-modules. In section 4.3 we shall be interested in a specific case of such families and in that case we explicitly show how \(\omega(E)\) determines \(F\).

4.2. The families of Harish-Chandra modules imposed by the physical realization. The family \(g\) is more than just an abstract algebraic family of Lie algebras, it is a family that is given in a concrete realization. The realization is induced from the realization of the Schrödinger operator \(H\) as a differential operator on smooth complex valued functions on \(S = \mathbb{R}^2 \setminus \{0\}\). As such, there are (algebraic) relations between the elements of \(C_D(H)\). For our purposes the relevant relation is given by

\[
A_x^2 + A_y^2 + \frac{1}{2} k^2 = H(L^2 - \frac{1}{4})
\]

This can be verified directly. In fact the mentioned relation is nothing but the Casimir relation for the quantum superintegrable system that is determined by \(H\), (this superintegrable system is known as \(E_{18}\)) see [KMW76, KKPM01] and specifically [MPW13, Sec. 3.1]. Now on the solution space for the Schrödinger equation with eigenvalue \(E\) realized as a space of functions on \(\mathbb{R}^2 \setminus \{0\}\) the Lie algebra \(\mathfrak{g}|_E\) naturally acts via its realization as differential operators on functions on \(\mathbb{R}^2 \setminus \{0\}\\):

\[
L = y \partial_x - x \partial_y,
\]

\[
A_y = \frac{1}{\sqrt{2}} \left( y \partial_{xx} - x \partial_{xy} - \frac{1}{2} \partial_y + \frac{k}{r} y \right),
\]

\[
A_x = \frac{1}{\sqrt{2}} \left( x \partial_{yy} - y \partial_{xy} - \frac{1}{2} \partial_x + \frac{k}{r} x \right)
\]

This induces a morphism of algebras \(\mathfrak{u}(\mathfrak{g}) \rightarrow C_D(H)\). Hence equation 4.1 implies that \(\Omega = A_x^2 + A_y^2 - EL^2\) acts via multiplication by the function \(\omega(E) = -\frac{E}{4} - \frac{k^2}{2}\). As was mentioned above, one can classify all generically irreducible and quasi-admissible families of \((\mathfrak{g}, K)\)-modules. These calculations follow from those in [BHS16, Sec. 4]. We shall only need the following fact that can be derived from the mentioned classification.

**Fact 1.** Let \(\mathcal{F}\) be a generically irreducible and quasi-admissible family of \((\mathfrak{g}, K)\)-modules on which \(\Omega\) acts by multiplication by \(\omega(E) = -\frac{E}{4} - \frac{k^2}{2}\). Then any irreducible algebraic representation of \(\text{SO}(2, \mathbb{C})\) is an \(\text{SO}(2, \mathbb{C})\)-types of \(\mathcal{F}\), each appears with multiplicity one.
A point $E \in X$ such that $F|_E$ is reducible is called a reducibility point of $F$. We are now ready to calculate the bounded spectrum of the Schrödinger operator.

**Theorem 2.** Let $F$ be a generically irreducible and quasi-admissible family of $(g, K)$-modules on which $\Omega$ acts by multiplication by $\omega(E) = -\frac{E}{4} - \frac{k^2}{2}$. Then the collection of all the reducibility points of $F$ coincides with $E_b$.

Before proving Theorem 2, we shall recall the natural parameterization of $\widehat{SO}(2, \mathbb{C})$, the set of equivalence classes of irreducible algebraic representations of $SO(2, \mathbb{C})$, and then do the same for $O(2, \mathbb{C})$.

For every $n \in \mathbb{Z}$ the formula $\chi_n(R(\theta)) = e^{in\theta}$ define a one dimensional irreducible algebraic representation of $O(2, \mathbb{C})$, and up to equivalence these are all such representations. This identify $\mathbb{Z}$ with $\widehat{SO}(2, \mathbb{C})$. For $n \in \mathbb{N}$ the $SO(2)$ representation $\chi_n \oplus \chi_{-n}$ can be turned into an irreducible representation of $O(2, \mathbb{C})$ by defining the action of $s$ on the underlying vector space $\mathbb{C}^2$ to be given by $s \cdot (z, w) = (w, z)$. In addition the trivial representation of $SO(2, \mathbb{C})$ can be extended in exactly two ways to an irreducible algebraic representation of $O(2, \mathbb{C})$. One is the trivial representation of $O(2, \mathbb{C})$, denoted by $\chi^+_0$ and the other the natural determinant representation of $O(2, \mathbb{C})$, denoted by $\chi^-_0$. The collection $\{\chi^+_0, \chi^-_0, \chi_n \oplus \chi_{-n} | n \in \mathbb{N}\}$ contains exactly one representative for each class in $\widehat{O}(2, \mathbb{C})$.

**Proof.** Let $F$ be a family satisfying the above hypotheses of Theorem 2. By Fact 1 there is a sequence $\{f_n | n \in \mathbb{Z}\} \subset F$ with

$$R(\theta)f_n = e^{in\theta}f_n$$

and such the decomposition of $F$ with respect to the action of $SO(2, \mathbb{C})$ is given by

$$F = \bigoplus_{n \in \mathbb{Z}} F_n$$

with $F_n = \mathbb{C}[E]f_n$. Define a new basis for $g$ by $J = iL, A_+ = \frac{1}{\sqrt{2}}(A_x + iA_y), A_- = \frac{1}{\sqrt{2}}(A_x - iA_y)$. In particular for every $n \in \mathbb{Z}$, $Jf_n = nf_n$. The commutation relations of these basis elements are given by

$$[J, A_+] = A_+, [J, A_-] = -A_-, [A_+, A_-] = EJ$$

The regularized Casimir in terms of the new basis is given by

$$\Omega = EJ^2 + EJ + 2A_-A_+ = EJ^2 - EJ + 2A_+A_-$$
Hence for every $n \in \mathbb{Z}$ we have

\[
\mathcal{A}_- \mathcal{A}_+ f_n = \frac{1}{2} \left( \omega(E) - E(n^2 + n) \right) f_n = -\frac{1}{2} \left( \frac{k^2}{2} + E \left( n + \frac{1}{2} \right)^2 \right) f_n
\]

\[
\mathcal{A}_+ \mathcal{A}_- f_n = \frac{1}{2} \left( \omega(E) - E(n^2 - n) \right) f_n = -\frac{1}{2} \left( \frac{k^2}{2} + E \left( n - \frac{1}{2} \right)^2 \right) f_n
\]

and the reducibility points are given by $\left\{ E_n := -\frac{k^2}{2(n+1/2)^2} \middle| n \in \{0, 1, 2, \ldots\} \right\}$ which is nothing but $\mathcal{E}_b$. $\square$

4.3. Concrete families of Harish-Chandra modules. There are many generically irreducible and quasi-admissible families of $(\mathfrak{g}, K)$-modules on which $\Omega$ acts by multiplication by $\omega(E) = -\frac{k^2}{4} - \frac{k^2}{2}$. The analysis that follows can be apply to any one of these. For concreteness we shall focus on such families that are generated by their isotypic sub-sheaf corresponding to the trivial $SO(2, \mathbb{C})$-type. Combining Proposition 2 and Fact 1 we see that there are at most two such families. We shall see below that indeed there are two such families. These two families must have the same $SO(2, \mathbb{C})$-types but they differ by one $O(2, \mathbb{C})$-type. One of them contains the trivial $O(2, \mathbb{C})$-type and the other family contains the determinant $O(2, \mathbb{C})$-type.

From now on we let $\mathcal{F}$ be a generically irreducible and quasi-admissible family of $(\mathfrak{g}, K)$-modules on which $\Omega$ acts by multiplication by $\omega(E) = -\frac{k^2}{4} - \frac{k^2}{2}$ and that is generated by $\mathcal{F}_0$, the isotypic piece corresponding to the trivial $SO(2, \mathbb{C})$ representation. In this case all the $SO(2, \mathbb{C})$-types appear, each has multiplicity one, and each $\mathcal{F}_n$ is a free rank one $\mathbb{C}[E]$-module.

We choose a basis $f_0$ of $\mathcal{F}_0$ and define

\[ f_n = \begin{cases} (\mathcal{A}_+)^nf_0 & n > 0 \\ (\mathcal{A}_-)^{-n}f_0 & n < 0 \end{cases} \]

The commutation relations 4.2 imply that $f_n \in \mathcal{F}_n$. Since $\mathcal{F}$ is generated by $\mathcal{F}_0$, each of the $f_n$ vanish nowhere. That is, at any $E \in \mathbb{C}$ each $f_n$ defines a non zero vector in the fiber of $\mathcal{F}$ at $E$. In particular, $\{f_n|n \in \mathbb{Z}\}$ is a basis for $\mathcal{F}$ and $\mathcal{F}_n = \mathbb{C}[E]f_n$. This, together with the commutation relations 4.2, completely determines the action of $\mathfrak{g}$ to
be given by

\begin{align}
\mathcal{J}f_n &= nf_n \\
\mathcal{A}_+ f_n &= \begin{cases} 
  f_{n+1} & n \geq 0 \\
  -\frac{1}{2} \left( \frac{k^2}{2} + E \left( n + \frac{1}{2} \right)^2 \right) f_{n+1} & n < 0 
\end{cases} \\
\mathcal{A}_- f_n &= \begin{cases} 
  -\frac{1}{2} \left( \frac{k^2}{2} + E \left( n - \frac{1}{2} \right)^2 \right) f_{n-1} & n > 0 \\
  f_{n-1} & n \leq 0 
\end{cases}
\end{align}

Of course the action of $SO(2, \mathbb{C})$ is given by

$$R(\theta)f_n = e^{in\theta}f_n.$$ 

In addition, one can show that the either one of the two formulas (that only differ by a sign)

$$s \cdot f_n = \pm (-i)^n f_n$$

extends $\mathcal{F}$ to a generically irreducible and quasi-admissible algebraic family of $(\mathfrak{g}, \mathbb{K})$-modules over $\mathbb{X}$. The two families obtained in this way are not isomorphic.

4.4. The Jantzen Filtration. In this section we recall the $\sigma$-twisted dual, which is a certain dual family to $\mathcal{F}$. We calculate the space of intertwining operators from $\mathcal{F}$ to its $\sigma$-twisted dual and we recall how such an intertwining operator gives rise to the Jantzen filtration of the fibers of $\mathcal{F}$.

4.4.1. The $\sigma$-twisted dual. Using the real structure of $(\mathfrak{g}, \mathbb{K})$, with any algebraic family $\mathcal{F}$ of $(\mathfrak{g}, \mathbb{K})$-modules we can associate another family of $(\mathfrak{g}, \mathbb{K})$-modules, $\mathcal{F}^{(\sigma)}$, the $\sigma$-twisted dual of $\mathcal{F}$. See [BHS17b, Sec. 2.4] for precise definition. For the two families $\mathcal{F}$ that were defined in Section 4.3 the $\sigma$-twisted dual has a basis $\{q_n | n \in \mathbb{Z}\}$ such that

$$\mathcal{F}^{(\sigma)} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^{(\sigma)}_n$$

with $\mathcal{F}^{(\sigma)}_n = \mathbb{C}[E]q_n$ and is given explicitly by

\begin{align}
\mathcal{J}q_n &= nq_n \\
\mathcal{A}_+ q_n &= \begin{cases} 
  \frac{1}{2} \left( \frac{k^2}{2} + E \left( n + \frac{1}{2} \right)^2 \right) q_{n+1} & n \geq 0 \\
  -q_{n+1} & n < 0 
\end{cases} \\
\mathcal{A}_- q_n &= \begin{cases} 
  -q_{n-1} & n > 0 \\
  \frac{1}{2} \left( \frac{k^2}{2} + E \left( n - \frac{1}{2} \right)^2 \right) q_{n-1} & n \leq 0 
\end{cases} \\
R(\theta)q_n &= e^{in\theta}q_n, \quad s \cdot q_n = \pm (-i)^n q_{-n}.
\end{align}
4.4.2. Intertwining operators. In this section we describe the space of intertwining operators from $\mathcal{F}$ to $\mathcal{F}^{(\sigma)}$.

**Proposition 3.** $\text{Hom}_{(\mathfrak{g},K)}(\mathcal{F}, \mathcal{F}^{(\sigma)}) \simeq \mathbb{C}[E]$.

**Proof.** By definition $\text{Hom}_{(\mathfrak{g},S(K))}(\mathcal{F}, \mathcal{F}^{(\sigma)})$ is the space of algebraic intertwining operators from $\mathcal{F}$ to $\mathcal{F}^{(\sigma)}$. Such a map is given by a map of $\mathbb{C}[E]$-modules $\psi : \mathcal{F} \rightarrow \mathcal{F}^{(\sigma)}$ that is equivariant with respect to the action of $\mathfrak{g}$ and the action of $K$. The equivariance with respect to $K$ implies that for every $n \in \mathbb{Z}$

$$\psi(f_n) = \psi_n(E) q_n$$

for some $\psi_n \in \mathbb{C}[E]$ with

$$\psi_{-n}(E) = \psi_n(E)$$

The equivariance with respect to $\mathfrak{g}$ further implies that

$$\psi(A_{\pm} f_n) = A_{\pm} \psi_n(E) q_n, \quad \forall n \in \mathbb{Z}$$

This leads to recursion relation between the $\psi_n(E)$ and the explicit form of $\psi_n(E)$ is given by

$$\psi_n(E) = \psi_{-n}(E) = \frac{1}{2^{n+1}} \psi_0(E) \prod_{m=1}^{|n|} \left( \frac{k^2}{2} + E \left( \frac{m - 1}{2} \right)^2 \right)$$

(4.11)

Hence the map that assigns to $\psi \in \text{Hom}_{(\mathfrak{g},K)}(\mathcal{F}, \mathcal{F}^{(\sigma)})$ the function $\psi_0(E) \in \mathbb{C}[E]$ is an isomorphism (of $\mathbb{C}[E]$-modules).

4.4.3. The Jantzen filtration. In this section we describe the Jantzen filtration. For more information see [BHS17b, Sec. 4.1].

Let $\psi : \mathcal{F} \rightarrow \mathcal{F}^{(\sigma)}$ be a nonzero intertwining operator. For a fixed $e \in \mathbb{C}$ we can localize $\mathbb{C}[E]$ at $e$, obtaining the ring $\mathbb{C}[E]_e$ of all rational functions that are defined near $e$, that is

$$\mathbb{C}[E]_e = \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[E], g(e) \neq 0 \right\}$$

Similarly, we can form the localizations of $\mathcal{F}$ and $\mathcal{F}^{(\sigma)}$ at $e$:

$$\mathcal{F}_e := \mathbb{C}[E]_e \otimes_{\mathbb{C}[E]} \mathcal{F}, \quad \mathcal{F}^{(\sigma)}_e := \mathbb{C}[E]_e \otimes_{\mathbb{C}[E]} \mathcal{F}^{(\sigma)}.$$  

The morphism $\psi$ induces a morphism of $\mathbb{C}[E]_e$-modules (that we shall also denote by $\psi$)

$$\psi : \mathcal{F}_e \rightarrow \mathcal{F}^{(\sigma)}_e$$

Using $\psi$ we obtain decreasing filtration of $\mathcal{F}_e$ defined by

$$\mathcal{F}^n_e = \left\{ f \in \mathcal{F}_e \mid \psi(f) \in (E - e)^n \mathcal{F}_e \right\}$$
for every \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). The fiber of \( \mathcal{F} \) at \( e \) is defined by:

\[
\mathcal{F}|_e := \mathbb{C} \otimes_{\mathbb{C}[E]} \mathcal{F} = \mathbb{C} \otimes_{\mathbb{C}[E]} \mathcal{F}|_e.
\]

The natural surjections from the localization at \( e \) to the fiber at \( e \) give rise to a decreasing filtration \( \{\mathcal{F}|_e^n\} \) of \( \mathcal{F}|_e \). These are the Jantzen filtrations. The Jantzen quotients (of \( \mathcal{F}|_e \)) are \( \mathcal{F}|_e^n/\mathcal{F}|_e^{n-1} \). Up to a shift of the filtration parameter \( n \), the Jantzen quotients are independent of (a non-zero) \( \psi \).

**Proposition 4.** Let \( \mathcal{F} \) and \( \psi \) be as above and let \( e \in \mathbb{C} \). The Jantzen filtration of \( \mathcal{F}|_e \) is trivial iff \( e \in \mathbb{C} \setminus \mathcal{E}_b \). If \( e \in \mathcal{E}_b \), there are exactly two nonzero quotients. Furthermore, for \( e = e_m = -\frac{k^2}{2(m+1/2)} \) one of the quotients is of dimension \( 2m + 1 \) with \( \text{SO}(2, \mathbb{C}) \)-types \( \{-m, -m+1, \ldots, m\} \) and the other is infinite dimensional with \( \text{SO}(2, \mathbb{C}) \)-types \( \{l \in \mathbb{Z} : |l| > m\} \).

**Proof.** As mentioned above the quotients are independent of \( \psi \), so we can and will assume that \( \psi_0(E) \) is the constant function 1. Note that by equation 4.11, \( \psi(f_m) = \psi_m(E)q_n \) where \( \psi_m(E) \) is a polynomial of degree \( m \) and its \( m \) different roots are

\[
\left\{ e_n = -\frac{k^2}{2(n+1/2)^2} : 0 \leq n < m \right\} \subset \mathcal{E}_b.
\]

Of course each root has multiplicity one. Hence for any \( e \in \mathbb{C} \setminus \mathcal{E}_b \) none of the \( \psi_n(E) \) vanish at \( e \) so \( \mathcal{F}|_e^0 = \mathcal{F}|_e, \mathcal{F}|_e^1 = \{0\} \) and the unique nonzero quotient is \( \mathcal{F}|_e^1/\mathcal{F}|_e^0 \approx \mathcal{F}|_e \). For \( e = e_m = -\frac{k^2}{2(m+1/2)^2} \) all the \( \psi_n(E) \) with \( |n| \leq m \) do not vanish at \( e_m \) and all the other \( \psi_n(E) \) do. Hence the non zero quotients as \( \text{SO}(2, \mathbb{C}) \)-modules are

\[
\mathcal{F}|_e^0/\mathcal{F}|_e^1 \approx \chi_{-m} \oplus \chi_{-m+1} \oplus \cdots \oplus \chi_m \quad \text{and} \quad \mathcal{F}|_e^1/\mathcal{F}|_e^2 \approx \bigoplus_{|n| \geq m} \chi_n.
\]

**Remark 4.** The \( \text{O}(2, \mathbb{C}) \)-types of the Jantzen quotients come in two flavors, corresponding to the two families \( \mathcal{F} \) with \( \omega(E) = -\frac{E}{4} - \frac{k^2}{2} \).

In one of the families \( \mathcal{F} \), all the trivial \( \text{SO}(2, \mathbb{C}) \) representations in the various quotients extend to a trivial \( \text{O}(2, \mathbb{C}) \) representation. In the other family all the trivial \( \text{SO}(2, \mathbb{C}) \) representations in the various quotients extend to the nontrivial one-dimensional \( \text{O}(2, \mathbb{C}) \) representation.

---

\(^7\text{There is a canonical isomorphism between the two descriptions of the fibers.}\)
4.5. The Hermitian form on the Jantzen quotients. In this section we describe the invariant Hermitian form on the Jantzen quotients. For more information see [BHS17b, Sec. 4.2].

The module $\mathcal{F}^{(a)}$ can be naturally identified with the space of functions from $\mathcal{F}$ to $\mathbb{C}[E]$ that are conjugate $\mathbb{C}[E]$-linear and vanish on all but finitely many of the $K$ isotypic summands of $\mathcal{F}$. That is, for $h \in \mathcal{F}^{(a)}$ and $f \in \mathcal{F}$ and $p \in \mathbb{C}[E]$

$$h(pf) = \sigma(p)h(f)$$

where $\sigma(p)(E) = \overline{p(E)}$. The bases $\{f_n\}$ and $\{q_n\}$ can be chosen such that under the above mentioned identification

$$q_n(f_m) = \delta_{mn}.$$ 

This allows us to equip each of the Jantzen quotients, $\mathcal{F}|_e^n/\mathcal{F}|_e^{n-1}$ with a non-degenerate sesquilinear form $\langle \cdot, \cdot \rangle_{e,n}$ defined by

$$\langle [f], [f'] \rangle_{e,n} = ((E - e)^{-n}\psi(f)(f'))_{E=e}$$

That is, for every $[f], [f'] \in \mathcal{F}|_e^n$ with representatives $f, f' \in \mathcal{F}$ we apply the linear functional $\psi(f_1)$ to $f_2$, then we divide by $(E - e)^n$ (this makes sense since $[f], [f'] \in \mathcal{F}|_e^n$) and finally we evaluate this complex valued function of $E$ at $e$. The forms $\langle \cdot, \cdot \rangle_{e,n}$ are invariant under the actions of $K|_e$ and $g|_e$ in the sense

$$\langle X \cdot [f], [f'] \rangle_{e,n} + \langle [f], \sigma(X) \cdot [f'] \rangle_{e,n} = 0$$

$$\langle g \cdot [f], \sigma(g) \cdot [f'] \rangle_{e,n} = \langle [f], [f'] \rangle_{e,n}$$

for all $[f], [f'] \in \mathcal{F}|_e^n$, all $X \in g|_e$, and all $g \in K|_e$. Up to a shift of the filtration parameter $n$, the Jantzen quotients are independent of (a nonzero) $\psi$. The intertwining operator $\psi$ can be chosen such that for every $e \in \mathbb{R} \subset \mathbb{C}$ the forms $\langle \cdot, \cdot \rangle_{e,n}$ will be Hermitian. If the invariant Hermitian form is of definite sign we say that the Jantzen quotient is infinitesimally unitary.

**Proposition 5.** Let $\mathcal{F}$ and $\psi$ be as above with $\psi_0(E) \equiv 1$. The form is of definite sign (positive definite or negative definite) exactly on the following Jantzen quotients:

1. $\mathcal{F}|_e^0/\mathcal{F}|_e^1 \simeq \mathcal{F}|_e$ with $e > 0$.
2. $\mathcal{F}|_e^0/\mathcal{F}|_0 \simeq \mathcal{F}|_0$.
3. $\mathcal{F}|_e^{0/m}/\mathcal{F}|_e^{1/m}$ with $e_m = -\frac{k^2}{2(m+1/2)^2}$, for some $m \in \mathbb{N}_0$.

**Proof.** By direct calculation on $\mathcal{F}|_e^n/\mathcal{F}|_e^{n+1}$ the form satisfies

$$\langle [f_s], [f_t] \rangle_{e,n} = (E - e)^{-n} \prod_{i=1}^{|s|} \left( \frac{k^2}{2} + E \left( 1 - \frac{1}{2} \right)^2 \right) \delta_{st}$$
If $e$ is not real then the form can not be of definite sign. If $e \in \mathbb{R} \setminus E_b$ then by Proposition 4, the unique nonzero Jantzen quotient is $\mathcal{F}|^0_e/\mathcal{F}|^1_e$ on which the form is given by

$$\langle [f_s], [f_t] \rangle_{e,n} = \left( \frac{1}{2^{|s|}} \prod_{i=1}^{|s|} \left( \frac{k^2}{2} + e \left(1 - \frac{1}{2}\right) \right) \right) \delta_{st}$$

which is obviously positive definite for $e \geq 0$ and not of definite sign for $e < 0$. The case with $e = e_m$ is proven similarly.

For each $e \in E$ we shall denote by $J(\mathcal{F}|_e)$ the unique infinitesimally unitary Jantzen quotient of $\mathcal{F}|_e$. We combine Proposition 4 and Proposition 5 to the following theorem.

**Theorem 3.** Let $\mathcal{F}$ be any one of the two generically irreducible and quasi-admissible families of $(\mathfrak{g}, K)$-modules on which $\Omega$ acts via multiplication by $\omega(E) = -\frac{k^2}{2} - \frac{e^2}{2}$ and that are generated by $\mathcal{F}_0$, their isotypical piece corresponding to the trivial $SO(2, \mathbb{C})$ representation. Then the spectrum of $\mathcal{H}$ coincides with the set of all $E \in X$ for which $\mathcal{F}|_E$, has a nonzero infinitesimally unitary Jantzen quotient. Moreover, $E_b$ coincides with the set of all $E \in X$ for which $\mathcal{F}|_E$ has a nontrivial Jantzen filtration.

### 4.6. From Jantzen quotients to group representations and solutions of the Schrödinger equation.

In this section we describe how the infinitesimally unitary Jantzen quotients can be integrated to unitary group representations. We show that these group representations are isomorphic to the solution spaces of the Schrödinger equation.

In Proposition 5 we saw that for each $e \in E \subset \mathbb{R}$ there is exactly one infinitesimally unitary Jantzen quotient, $J(\mathcal{F}|_e)$. Recall that $J(\mathcal{F}|_e)$ is a $(\mathfrak{g}|_e, K|_e)$-modules. In fact each quotient is an irreducible $(\mathfrak{g}|_e, K|_e)$-modules. The discussion in [BHS17b, Sec. 4.3] explains how a theorem of Nelson [Nel59] implies that each of the $J(\mathcal{F}|_e)$ with $e \in E$, as a representation of $\mathfrak{g}|_e$, can be integrated to a unitary representation of the simply connected Lie group with Lie algebra $\mathfrak{g}|_e$. Here we shall proceed using other approach. We shall separately deal with the cases of $e > 0$, $e = 0$, and $e = e_m$ for some $m \in \mathbb{N}_0$. Explaining why the representations of the Lie algebras on the the various $J(\mathcal{F}|_e)$ can be integrated to unitary irreducible representations of $SO_0(2,1)$, $SO(2) \ltimes \mathbb{R}^2$ and $SO(3)$, depending on the value of $e$.

#### 4.6.1. The case of $SO_0(2,1)$.

For $e > 0$, $SO(2,1) \simeq \tilde{SO}_3|_e$ is reductive with a maximal compact subgroup $O(2) \simeq K|_e$. By a theorem of Harish-Chandra and Lepowsky (see e.g., [Vog81] Sec. 0.3)) the Jantzen quotient $J(\mathcal{F}|_e)$ can be integrated to a unitary irreducible
representation of \( SO(2, 1) \). The \( O(2) \)-types and infinitesimal character are enough to parametrize the unitary dual of \( SO(2, 1) \). Using this we can determine that for each of the two families \( \mathcal{F} \), the corresponding unitary irreducible representation of \( SO(2, 1) \cong \tilde{SO}_3|_e \) obtained by integrating \( J(\mathcal{F}|_e) \) is a unitary principal series on which the Casimir acts by multiplication by \( \Omega(e) \). Both options restrict to the same unitary irreducible representation of \( SO_0(2, 1) \), the connected component of \( SO(2, 1) \). This representation is isomorphic to the solution space of the Schrödinger equation of the hydrogen atom in two dimensions with energy eigenvalue \( e \), compare to [Bl66b].

4.6.2. The case of \( SO(2) \rtimes \mathbb{R}^2 \). The unitary duals of \( O(2) \rtimes \mathbb{R}^2 \) and \( SO(2) \rtimes \mathbb{R}^2 \) can be calculated using the Mackey machine. Again, the \( O(2) \)-types in the case of \( O(2) \rtimes \mathbb{R}^2 \) and the \( SO(2) \)-types in the case of \( SO(2) \rtimes \mathbb{R}^2 \) together with the action of the center of the enveloping algebra provide enough invariants to parameterize the unitary duals. For each of the two families \( \mathcal{F} \) it is straightforward, in this low-dimensional case, to find the unitary irreducible representation of \( O(2) \rtimes \mathbb{R}^2 \cong \tilde{SO}_3|_e \) that have their underlying Harish-Chandra module given by \( J(\mathcal{F}|_0) \). As in the case of \( e > 0 \) the two non-isomorphic irreducible unitary representations of \( O(2) \rtimes \mathbb{R}^2 \) have the same unitary irreducible restriction to \( SO(2) \rtimes \mathbb{R}^2 \). This representation is isomorphic to the solution space of the Schrödinger equation of the hydrogen atom in two dimensions with energy eigenvalue \( e = 0 \), compare to [TdCJ98].

4.6.3. The case of \( SO(3) \). For \( e = e_m \) for some \( m \in \mathbb{N}_0 \), \( J(\mathcal{F}|_{e_m}) \) is a \((2m + 1)\)-dimensional irreducible representation of \( so(3) \cong \mathfrak{g}|_{e_m} \) and hence can integrated to the unique unitary irreducible \((2m + 1)\)-dimensional representation of \( SO(3) \cong \tilde{SO}_3|_{e_m} \). This representation is isomorphic to the solution space of the Schrödinger equation of the hydrogen atom in two dimensions with energy eigenvalue \( e = e_m = -\frac{k^2}{2(m+1/2)^2} \), compare to [Bl66a].

Summarizing, we have shown the following result.

**Theorem 4.** Let \( \mathcal{F} \) be any one of the two families from Theorem 3. Then for any \( e \in \text{Spec}(\mathcal{H}) \) the Jantzen quotient \( J(\mathcal{F}|_e) \) can be integrated to the unitary irreducible representation of the connected component of \( SO_3|_e \) which is isomorphic to \( \text{Sol}(e) \).
5. DISCUSSION

The energy spectrum of the bound states of the hydrogen atom is a fundamental example for quantization. This work shows how quantization of the spectrum is related to discretely appearing infinitesimally unitary Jantzen quotients. We believe that this connection should be studied further. For example, we believe that similar results hold in the case of the $n$-dimensional hydrogen atom. Another important feature is the algebraic/analytic structure of the symmetries and solutions of the hydrogen atom. In principal knowing $\mathcal{F}$ over some open subset of $X = \mathbb{C}$ is enough to completely determine $\mathcal{F}$ everywhere on $\mathbb{C}$. For example, this implies that if one knows the solutions spaces for the Schrödinger equation with energy eigenvalues confined to some open sub-interval of $(0, \infty)$ one can recover all solutions for all possible energy values. One can wonder what should be the physical meaning of this; does the solution spaces for scattering states know about the bound state solution spaces and vice versa?

From physics perspective there are a couple of natural questions that need to be asked.

- Is there a physical meaning that can be attached to the fibers $\mathcal{F}|_{e}$ with $e$ not in to the spectrum of $H$?
- Is there a physical explanation for the existence of the two (rather than just one) families of $(\mathfrak{g}, K)$-modules that leads to the solutions of the Schrödinger equation?

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8This can even be seen with our human eyes letting light emitted from an excited hydrogen atoms to pass through a prism.
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