Uniformly Convergence Of The Spectral Expansions Of The Schrödinger Operator On A Closed Domain

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Abstract. In this work uniformly convergent problems of the eigenfunction expansions of the Schrödinger operator $-\Delta + q(y_1, y_2)$ with singular potential from $W^{1,2}_2(\Omega)$ are investigated. Using the estimation of the spectral function of the Schrödinger operator on closed domain and mean value formula for the eigenfunction the uniformly convergent of the eigenfunction expansions of the functions continuous on the closed domain is proved.

1. Introduction

Solving the boundary value problems of equations of mathematical physics in bounded domains leads to the investigation of convergence and summability problems, related to the eigenfunction expansion on a closed domain. This particular problem requires estimating eigenfunctions near boundary of the domain. Methodology of estimation eigenfunctions in compact subsets of the domain is well developed and known (see in [1]) . But in estimation of eigenfunctions in closed domains some difficulties occurs near the boundary. These difficulties can be avoided if we consider boundary conditions that help to estimate eigenfunctions near the boundary with more accurate values. E.I. Moiseev [2] proved an estimation of eigenfunctions of the first boundary value problem for the Laplace operator. Some latest results on the uniformly convergent of the spectral expansions of the distributions are obtained by Rakhimov (see [6]).

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial \Omega$ and $q(y_1, y_2) \geq 0$ be a potential functions from Sobolev’s space $W^{1,2}_2(\Omega)$.

We investigate the solutions of the following Schrödinger equation:

$$-\Delta \psi(y_1, y_2) + q(y_1, y_2) \psi(y_1, y_2) + \lambda \psi(y_1, y_2) = 0, (y_1, y_2) \in \Omega, \quad (1)$$

with Dirichlet boundary condition:

$$\psi(y_1, y_2) = 0, (y_1, y_2) \in \partial \Omega. \quad (2)$$

It is well known that the (1) and (2) have the countable set of solutions: $\{\psi_n(x_1, x_2)\}_{n=1}^{\infty}$ ; $\{\lambda_n\}$, which are called eigenfunctions and eigenvalues of Schrödinger operator, respectively.
The main aim of the paper is to investigate conditions for uniformly convergence of eigenfunctions expansions

\[ \sum_{n=1}^{\infty} f_n \psi_n(x_1, x_2) \]

by Riesz means, which can be defined as follows:

\[ E_\alpha^\lambda f(x_1, x_2) = \sum_{\lambda_n < \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^\alpha f_n \psi_n(x_1, x_2), \alpha > 0, \]

where \( f_n \) denotes the Fourier coefficients of the function \( f(x_1, x_2) \):

\[ f_n = \int \int_\Omega f(x_1, x_2) \psi_n(x_1, x_2) dx_1 dx_2, \quad n = 1, 2, \ldots. \]

2. Main results and formulation

**Theorem 2.1.** For the eigenfunctions of the Schrödinger operator on \( \Omega \subset R^2 \) with Dirichlet boundary conditions one has

\[ \sum_{|\sqrt{\lambda_n} - \lambda| \leq 1} \psi_n^2(x_1, x_2) \leq C \lambda \ln^2 \lambda, \quad \lambda \to +\infty \]

uniformly on \((x_1, x_2) \in \overline{\Omega}\).

Theorem 2.1 is proved in [5] using the techniques of the work [2].

Using the estimation in Theorem 2.1 we prove uniformly convergence of spectral expansions on closed domain as follows:

**Theorem 2.2.** Let \( f \in C(\overline{\Omega}) \). If \( \alpha > \frac{1}{2} \), then for the Riesz mean \( E_\alpha^\lambda f(x_1, x_2) \) of spectral expansions of Schrödinger operator corresponding to Dirichlet boundary conditions we have

\[ \lim_{\lambda \to +\infty} E_\alpha^\lambda f(x_1, x_2) = f(x_1, x_2), \] uniformly on \((x_1, x_2) \in \overline{\Omega}\).

In the proof of the estimation for eigenfunctions and in establishing the facts on convergent the following Lemma plays main role.

**Lemma 2.3** (Mean Value Formula): For the eigenfunctions of the Schrödinger operator one has

\[ \int_0^{2\pi} \psi_n(x_1 + r \cos \theta, x_2 + r \sin \theta) d\theta \\
= 2\pi \psi_n(x_1, x_2) J_\nu(r \sqrt{\lambda_n}) + \frac{\pi}{2} \int_0^r \{ J_\nu(t \sqrt{\lambda_n}) Y_\nu(r \sqrt{\lambda_n}) - Y_\nu(t \sqrt{\lambda_n}) J_\nu(r \sqrt{\lambda_n})\} \rho_n(t) dt, r \geq 0, \]

where \( J_\nu(x) \) and \( Y_\nu(x) \) are Bessel Functions of the first and second kind, respectively and we use notation:

\[ \rho_n(t) = \int_0^{2\pi} q(x_1 + t \cos \theta, x_2 + t \sin \theta) \psi_n(x_1 + t \cos \theta, x_2 + t \sin \theta) d\theta. \]
We fix \((x_1, x_2) \in \Omega\) and consider

\[
g(r) = \begin{cases} r^{-1-\alpha}J_{1+\alpha}(r\sqrt{\lambda}), & r \leq R \\ 0, & r > R \end{cases}
\]

where

\[
r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.
\]

For the Fourier coefficients of \(g(r)\) we obtain

\[
g_n = \int_{\Omega} g(y_1, y_2)\psi_n(y_1, y_2)dy_1dy_2 = \int_{r \leq R} r^{-1-\alpha}J_{1+\alpha}(r\sqrt{\lambda})\psi_n(y_1, y_2)dy_1dy_2.
\]

After transforming to polar coordinates we obtain by using Mean Value Formula

\[
g_n = \int_0^R r^{-\alpha}J_{1+\alpha}(r\sqrt{\lambda})\int_0^{2\pi} \psi_n(x_1 + r\cos\theta, x_2 + r\sin\theta)d\theta dr
\]

\[
= \int_0^R r^{-\alpha}J_{1+\alpha}(r\sqrt{\lambda})[2\pi\psi_n(x_1, x_2)J_o(r\sqrt{\lambda_n})
\]

\[
+ \frac{\pi}{2} \int_0^R \{J_o(t\sqrt{\lambda_n})Y_o(r\sqrt{\lambda_n}) - Y_o(t\sqrt{\lambda_n})J_o(r\sqrt{\lambda_n})\}\rho_n(t)tdt|dr
\]

\[
= \int_0^R r^{-\alpha}J_{1+\alpha}(r\sqrt{\lambda})[2\pi\psi_n(x_1, x_2)J_o(r\sqrt{\lambda_n}) + S^n_q(r)]dr,
\]

where it is denoted

\[
S^n_q(r) = \frac{\pi}{2} \int_0^r [J_o(t\sqrt{\lambda_n})Y_o(r\sqrt{\lambda_n}) - Y_o(t\sqrt{\lambda_n})J_o(r\sqrt{\lambda_n})]\rho_n(t)tdt.
\]

The integral on the right side of the latter \((6)\) can be simplified as follows:

\[
g_n = 2\pi\psi_n(x_1, x_2)\left\{\frac{2^{-\alpha}}{\lambda^{\frac{3}{2}}\Gamma(\alpha + 1)}(1 - \frac{\lambda_n}{\lambda})^\alpha - 2\pi\psi_n(x_1, x_2)I_n(R) + T_n(R)\right\}.
\]

Here we used the notations:

\[
I_n(R) = \int_R^{\infty} r^{-\alpha}J_{1+\alpha}(r\sqrt{\lambda})J_o(r\sqrt{\lambda_n})dr
\]

\[
T_n(R) = \int_0^R r^{-\alpha}J_{1+\alpha}(r\sqrt{\lambda})S^n_q(r)dr.
\]

From well known Integral formula for Bessel functions (see [10]):

\[
\int_0^{\infty} r^{-\alpha}J_{1+\alpha}(r\sqrt{\lambda})J_o(r\sqrt{\lambda_n})dr = \begin{cases} \frac{2^{-\alpha}}{\lambda^{\frac{3}{2}}\Gamma(\alpha + 1)}(1 - \frac{\lambda_n}{\lambda})^\alpha, & \lambda_n \leq \lambda \\
0, & \lambda_n > \lambda,
\end{cases}
\]

we derive

\[
g_n = 2\pi\psi_n(x_1, x_2)\left\{\frac{2^{-\alpha}}{\lambda^{\frac{3}{2}}\Gamma(\alpha + 1)}(1 - \frac{\lambda_n}{\lambda})^\alpha \delta_{\lambda_n} - 2\pi\psi_n(x_1, x_2)I_n(R) + T_n(R)\right\}.
\]
where $\delta_{\lambda n}^{\lambda}$ is a Kronecker number:

$$\delta_{\lambda n}^{\lambda} = \begin{cases} 1, & \lambda_n \leq \lambda \\ 0, & \lambda_n > \lambda. \end{cases}$$

Substitution of (8) into Parseval’s formula

$$\int \int \int \Omega f(x_1, x_2)g(x_1, x_2)dx_1dx_2 = \sum_{n=1}^{\infty} f_ng_n$$

gives

$$\int \int \int_{r \leq R} f(y_1, y_2)r^{-1-\alpha} J_{1+\alpha}(r\sqrt{\lambda})dy_1dy_2$$

$$= \sum_{n=1}^{\infty} f_n(2\pi \psi_n(x_1, x_2)[\frac{2-\alpha}{\lambda^{1+\alpha}} (1 - \frac{\lambda_n}{\lambda})^{\alpha} \delta_{\lambda n}^{\lambda}] - 2\pi \psi_n(x_1, x_2)I_n(R) + T_n(R))$$

$$= 2\pi \sum_{n=1}^{\infty} f_n \psi_n(x_1, x_2)[\frac{2-\alpha}{\lambda^{1+\alpha}} (1 - \frac{\lambda_n}{\lambda})^{\alpha} \delta_{\lambda n}^{\lambda}] - 2\pi \sum_{n=1}^{\infty} f_n \psi_n(x_1, x_2)I_n(R) + \sum_{n=1}^{\infty} f_n T_n(R).$$

From the definition of Riesz Means we have a representation:

$$E_{\lambda}^{\alpha} f(x_1, x_2) = \sum_{\lambda_n < \lambda} (1 - \frac{\lambda_n}{\lambda})^{\alpha} f_n \psi_n(x_1, x_2),$$

(9)

where

$$E_{\lambda}^{\alpha} f(x_1, x_2) = \frac{1}{2\pi}(2^{\alpha} \lambda^{\frac{1+\alpha}{2}} \Gamma(\alpha+1)) \int \int \int \int_{r \leq R} f(y_1, y_2)r^{-1-\alpha} J_{1+\alpha}(r\sqrt{\lambda})dy_1dy_2$$

$$+ (2^{\alpha} \lambda^{\frac{1+\alpha}{2}} \Gamma(\alpha+1)) \sum_{n=1}^{\infty} f_n \psi_n(x_1, x_2)I_n(R) - \frac{1}{2\pi}(2^{\alpha} \lambda^{\frac{1+\alpha}{2}} \Gamma(\alpha+1)) \sum_{n=1}^{\infty} f_n T_n(R)$$

$$= \frac{1}{2\pi}(2^{\alpha} \lambda^{\frac{1+\alpha}{2}} \Gamma(\alpha+1)) \int \int \int \int_{r \leq R} f(y_1, y_2)r^{-1-\alpha} J_{1+\alpha}(r\sqrt{\lambda})dy_1dy_2$$

$$+ (2^{\alpha} \lambda^{\frac{1+\alpha}{2}} \Gamma(\alpha+1)) \sum_{n=1}^{\infty} f_n \psi_n(x_1, x_2) \int_{R}^{\infty} r^{-\alpha} J_{1+\alpha}(r\sqrt{\lambda})J_{o}(r\sqrt{\lambda})dr$$

$$- \frac{1}{2\pi}(2^{\alpha} \lambda^{\frac{1+\alpha}{2}} \Gamma(\alpha+1)) \sum_{n=1}^{\infty} f_n \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r\sqrt{\lambda})S_{q}^{n}(r)dr.$$
To prove the Theorem 2.2 it is sufficient to establish the following results:

**Lemma 2.4**: Let \( f(x_1, x_2) \) be any continuous function with compact support in \( \Omega \). For any \( \alpha > 0 \), one has uniformly for all \((x_1, x_2) \in \overline{\Omega}\):

\[
| \frac{1}{2\pi} (2^\alpha \lambda^{\frac{1}{2}} \Gamma(\alpha + 1)) \int \int_{r \leq R} f(x_1, x_2) r^{-1-\alpha} J_{1+\alpha}(r \sqrt{\lambda}) dy_1 dy_2 | \leq C \| f \|_{\infty}, R > 0, \lambda \rightarrow +\infty
\]

For the proof see we refer to [6].

To estimate \( \sigma_\alpha^q f(x_1, x_2) \) we have

**Lemma 2.5**: Let \( f(x_1, x_2) \) be any continuous function with compact support in \( \Omega \). If \( \alpha > \frac{1}{2} \), then we have

\[
| \sigma_\alpha^q f(x_1, x_2) | \leq C \| f \|_{\infty}, \lambda \rightarrow +\infty
\]

uniformly for all \((x_1, x_2) \in \overline{\Omega}\).

For the proof see [6].

**Lemma 2.6**: Let \( f(x_1, x_2) \) be any continuous function in \( \Omega \) with compact support. If \( \alpha > \frac{1}{2} \), we have

\[
| \tau_\alpha^q f(x_1, x_2) | \leq C(\sqrt{\lambda})^{\frac{1}{2}-\alpha} \| f \|_{L_2(\Omega)} \| q \|_{W_1^1(\Omega)}
\]

uniformly on \((x_1, x_2) \in \overline{\Omega}\).

Proof of the Lemma 2.6:

\[
\tau_\alpha^q f(x_1, x_2) = -\frac{1}{2\pi} (2^\alpha \lambda^{\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_n \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r \sqrt{\lambda}) S_n(r) dr.
\]

Substitute equation (8) into \( \tau_\alpha^q f(x_1, x_2) \) then we have

\[
\tau_\alpha^q f(x_1, x_2) = -\frac{1}{2\pi} (2^\alpha \lambda^{\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_n \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r \sqrt{\lambda})
\]

\[
\frac{\pi}{2} \int_{0}^{r} |J_\alpha(t \sqrt{\lambda}_n) Y_\alpha(t \sqrt{\lambda}_n) - Y_\alpha(t \sqrt{\lambda}_n) J_\alpha(t \sqrt{\lambda}_n)| \rho_n(t) t dt dr.
\]

We expand \( \tau_\alpha^q f(x_1, x_2) \) into two main parts as follows:

\[
\tau_\alpha^q f(x_1, x_2) = \frac{1}{4} (2^\alpha \lambda^{\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_n \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r \sqrt{\lambda}) \int_{0}^{r} Y_\alpha(t \sqrt{\lambda}_n) J_\alpha(t \sqrt{\lambda}_n) \rho_n(t) t dt dr
\]

\[
- \frac{1}{4} (2^\alpha \lambda^{\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_n \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r \sqrt{\lambda}) \int_{0}^{r} J_\alpha(t \sqrt{\lambda}_n) Y_\alpha(t \sqrt{\lambda}_n) \rho_n(t) t dt dr.
\]

And denoted

\[
\tau_\alpha^q f(x_1, x_2) = (\tau_\alpha^q f(x_1, x_2))_1 - (\tau_\alpha^q f(x_1, x_2))_2 = (\tau_\alpha^q f)_1 - (\tau_\alpha^q f)_2,
\]

where we defined

\[
(\tau_\alpha^q f)_1 = \frac{1}{4} (2^\alpha \lambda^{\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_n \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r \sqrt{\lambda}) \int_{0}^{r} Y_\alpha(t \sqrt{\lambda}_n) J_\alpha(t \sqrt{\lambda}_n) \rho_n(t) t dt dr
\]

\[
(\tau_\alpha^q f)_2 = - \frac{1}{4} (2^\alpha \lambda^{\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_n \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r \sqrt{\lambda}) \int_{0}^{r} J_\alpha(t \sqrt{\lambda}_n) Y_\alpha(t \sqrt{\lambda}_n) \rho_n(t) t dt dr.
\]
Finally we have
\[ \tau \] and using integration by parts and we obtain
\[ \int r^{-\nu} J_{\nu+1}(r\sqrt{\lambda}) dr = -r^{-\nu}(\sqrt{\lambda})^{-1} J_{\nu}(r\sqrt{\lambda}) 
- r^{-\nu} J_{\nu+1}(r\sqrt{\lambda}) = \frac{d}{dr} \{ r^{-\nu}(\sqrt{\lambda})^{-1} J_{\nu}(r\sqrt{\lambda}) \} 
- r^{-\nu}\sqrt{\lambda} J_{\nu+1}(r\sqrt{\lambda}) = \frac{d}{dr} \{ r^{-\nu} J_{\nu}(r\sqrt{\lambda}) \} \]
and for \((\tau_{n}^\alpha f)_{1}\) and \((\tau_{n}^\alpha f)_{2}\) we obtain
\[ (\tau_{n}^\alpha f)_{1} = \frac{1}{4}(2^\alpha \lambda^{-\frac{1}{2}} \Gamma(\alpha + 1)) \]
\[ \sum_{n=1}^{\infty} f_{n} \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r\sqrt{\lambda}) J_{\alpha}(r\sqrt{\lambda}) \int_{0}^{r} Y_{\alpha}(t\sqrt{\lambda}) Y_{\alpha}(r\sqrt{\lambda}) \int_{\gamma \leq r} \psi_{n}(y_{1}, y_{2}) q(y_{1}, y_{2}) d\theta dt dr \]
\[ (\tau_{n}^\alpha f)_{2} = \frac{1}{4}(2^\alpha \lambda^{-\frac{1}{2}} \Gamma(\alpha + 1)) \]
\[ \sum_{n=1}^{\infty} f_{n} \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r\sqrt{\lambda}) J_{\alpha}(r\sqrt{\lambda}) \int_{0}^{r} Y_{\alpha}(t\sqrt{\lambda}) Y_{\alpha}(r\sqrt{\lambda}) \int_{\gamma \leq r} \psi_{n}(y_{1}, y_{2}) q(y_{1}, y_{2}) d\theta dt dr. \]

Estimations for \((\tau_{n}^\alpha f)_{1}\) and \((\tau_{n}^\alpha f)_{2}\) are similar.
We estimate \((\tau_{n}^\alpha f)_{2}\) by using the recurrence relation of Bessel functions as follows:
\[ \int r^{-\nu} J_{\nu+1}(r\sqrt{\lambda}) dr = -r^{-\nu}(\sqrt{\lambda})^{-1} J_{\nu}(r\sqrt{\lambda}) \]
\[ -r^{-\nu} J_{\nu+1}(r\sqrt{\lambda}) = \frac{d}{dr} \{ r^{-\nu}(\sqrt{\lambda})^{-1} J_{\nu}(r\sqrt{\lambda}) \} \]
\[ -r^{-\nu}\sqrt{\lambda} J_{\nu+1}(r\sqrt{\lambda}) = \frac{d}{dr} \{ r^{-\nu} J_{\nu}(r\sqrt{\lambda}) \} \]
and for \((\tau_{n}^\alpha f)_{1}\) and \((\tau_{n}^\alpha f)_{2}\) we obtain
\[ (\tau_{n}^\alpha f)_{2} = \frac{1}{4}(2^\alpha \lambda^{-\frac{1}{2}} \Gamma(\alpha + 1)) \]
\[ \sum_{n=1}^{\infty} f_{n} Y_{\alpha}(r\sqrt{\lambda}) Y_{\alpha}(t\sqrt{\lambda}) \int_{0}^{\beta \leq t} \psi_{n}(y_{1}, y_{2}) q(y_{1}, y_{2}) d\theta dt \]
\[ = \frac{1}{4}(2^\alpha \lambda^{-\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_{n} Y_{\alpha}(r\sqrt{\lambda}) \]
\[ [J_{\alpha}(r\sqrt{\lambda}) \int_{\beta \leq r} \psi_{n}(y_{1}, y_{2}) q(y_{1}, y_{2}) d\theta + \int_{0}^{\sqrt{\lambda} \alpha}(J_{\alpha}(t\sqrt{\lambda}) \int_{\beta \leq t} \psi_{n}(y_{1}, y_{2}) q(y_{1}, y_{2}) d\theta dt] \]
\[ = \frac{1}{4}(2^\alpha \lambda^{-\frac{1}{2}} \Gamma(\alpha + 1)) \sum_{n=1}^{\infty} f_{n} \int_{0}^{R} r^{-\alpha} J_{1+\alpha}(r\sqrt{\lambda}) Y_{\alpha}(r\sqrt{\lambda}) Y_{\alpha}(r\sqrt{\lambda}) J_{\alpha}(r\sqrt{\lambda}) dr \int_{r \leq R} \psi_{n}^{2} d\theta. \]
Finally we have
\[ |(\tau_{n}^\alpha f)_{2}|^{2} \leq C \lambda^{-\frac{1}{2}} \sum_{n=1}^{\infty} f_{n}^{2} r^{-2\alpha} J_{1+\alpha}(r\sqrt{\lambda}) Y_{\alpha}^{2}(r\sqrt{\lambda}) J_{\alpha}(r\sqrt{\lambda}) dr \int_{r \leq R} \psi_{n}^{2} d\theta||q||_{2} \]
The following estimations can be obtained from Theorem 2.1. Applying them we have
\[ \sum_{\lambda_n > \lambda} \psi_n^2 \lambda^{(\varepsilon - 1)} = O(\lambda^{-\varepsilon}) \ln^2 \lambda, \lambda > 0 \]
\[ \sum_{\lambda_n < \lambda} \psi_n^2 \lambda^{(\varepsilon - 1)} = O(\lambda^\varepsilon) \ln^2 \lambda. \]

After we have proved this estimation, the Theorem 2.2 can be proved as follows,
\[ |(\tau_\alpha^\lambda f)_2| \leq C\lambda^{(\frac{1}{4} - \frac{\alpha}{2})} \|f\|_{L^2(\Omega)} \|q\|_{W^{1,2}_2(\Omega)}. \]

For any function from the space \( W^{1,2}_2(\Omega) \) convergence of Riesz Means \( E^\alpha_\lambda f(x) \), \( \alpha > 0 \) will be uniformly convergent on closed domain \( \Omega \). From compactness of \( W^{1,2}_2(\Omega) \) in the space of continuous function on \( \Omega \), we conclude that the function \( f \) which satisfies conditions of the theorem can be approximated by functions from \( W^{1,2}_2(\Omega) \), which has supports in \( \Omega \).

The proof of Theorem 2.2 is completed.

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