Ground States for Logarithmic Schrödinger Equations on Locally Finite Graphs

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Abstract
In this paper, we study the following logarithmic Schrödinger equation:

\[-\Delta u + a(x)u = u \log u^2 \quad \text{in } V,\]

where \(\Delta\) is the graph Laplacian, \(G = (V, E)\) is a connected locally finite graph, the potential \(a : V \to \mathbb{R}\) is bounded from below and may change sign. We first establish two Sobolev compact embedding theorems in the case when different assumptions are imposed on \(a(x)\). This leads to two kinds of associated energy functionals, one of which is not well defined under the logarithmic nonlinearity, while the other is \(C^1\). The existence of ground state solutions are then obtained by using the Nehari manifold method and the mountain pass theorem respectively.

Keywords  Ground state solutions · Logarithmic Schrödinger equations · Locally finite graphs · Variational methods

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1 Introduction and Main Results

In this paper, we are interested in the following logarithmic Schrödinger equation:

\[- \Delta u + a(x)u = u \log u^2 \text{ in } V\]  \hspace{1cm} (1)

on a connected locally finite graph \( G = (E, V) \), where \( V \) denotes the vertex set and \( E \) denotes the edge set. A graph is said to be \textit{locally finite} if for any \( x \in V \), there are only finitely many \( y \in V \) such that edge \( xy \in E \). A graph is \textit{connected} if any two vertices \( x \) and \( y \) can be connected via finite edges. For any edge \( xy \in E \), we assume the weight \( \omega_{xy} \) satisfies \( \omega_{xy} = \omega_{yx} > 0 \). The \textit{degree} of \( x \in V \) is defined as

\[ \deg(x) = \sum_{y \sim x} \omega_{xy} \]  \hspace{1cm} (2)

In recent years, the study of partial differential equations on graphs has drawn much attention, see [1–11] and references therein. Specially in [11], Grigor’yan, Lin, and Yang studied the following nonlinear Schrödinger equation:

\[- \Delta u + b(x)u = f(x, u) \text{ in } V \]  \hspace{1cm} (3)

on a connected locally finite graph \( G \), where the potential \( b : V \to \mathbb{R}^+ \) satisfies \( b(x) \geq b_0 \) for some constant \( b_0 > 0 \), and one of the following hypotheses holds:

\((B_1)\) \( b(x) \to +\infty \) as \( d(x, x_0) \to +\infty \) for some fixed \( x_0 \in V \);

\((B_2)\) \( 1/b(x) \in L^1(V) \).

When \( f \) satisfies the so-called Ambrosetti-Rabinowitz ((AR) for short) condition:

\((f_1)\) there exists a constant \( \theta > 2 \) such that for all \( x \in V \) and \( s > 0 \),

\[ 0 < \theta F(x, s) = \theta \int_0^s f(x, t)dt \leq sf(x, s), \]  \hspace{1cm} (4)

and some additional assumptions, they applied the mountain pass theorem and showed that equation (3) admits strictly positive solutions. In [12], Zhang and Zhao established the existence and convergence (as \( \lambda \to +\infty \)) of ground state solutions for equation (3), when \( b(x) = \lambda a(x) + 1 \) and \( f(x, u) = |u|^{p-1}u \), where \( a(x) \geq 0 \) and the potential well \( \Omega = \{ x \in V : a(x) = 0 \} \) is a non-empty connected and bounded domain in \( V \). One may find more related works in [13–15] and their references.

On the other hand, this type of problem has been studied extensively in the Euclidean space, see for example, [16–24]. In recent years, the logarithmic Schrödinger equation

\[- \Delta u + b(x)u = u \log u^2 \text{ in } \mathbb{R}^N \]  \hspace{1cm} (5)
has attracted much interest. It is closely related to the time-dependent logarithmic Schrödinger equation:

\[ i \frac{\partial u}{\partial t} - \Delta u + b(x)u - u \log u^2 = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \]

which has wide applications in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity, and Bose–Einstein condensation, see for example, [25–27].

In fact, there are several challenges when studying the logarithmic Schrödinger equation. One of the main challenges is that the associated energy functional of equation (5) is not well defined in \( H^1(\mathbb{R}^N) \) because there exists \( u \in H^1(\mathbb{R}^N) \) such that \( \int_{\mathbb{R}^N} u^2 \log u^2 \, dx = -\infty \). Different approaches have been developed to overcome this technical difficulty. Cazenave [28] worked in an Orlicz space endowed with a Luxemburg type norm in order to make the functional to be \( C^1 \). Squassina and Szulkin [29] studied the existence of multiple solutions by using non-smooth critical point theory (see also [30–32]). Tanaka and Zhang [33] applied the penalization technique to study multi-bump solutions of equation (5). For the idea of penalization, see also Guerrero et al. [34]. In [35], Wang and Zhang proved that the ground state solutions of the power-law scalar field equations

\[ -\Delta u + \lambda |u|^{p-2} u = 0, \quad \text{as} \quad p \downarrow 2, \]

converge to the ground state solution of the logarithmic-law equation

\[ -\Delta u = \lambda u \log u^2. \]

By using the direction derivative and constrained minimization method, Shuai [36] obtained the existence of positive and sign-changing solutions of equation (5) under different types of potentials. For more studies on logarithmic Schrödinger equations, one may refer to [37–47], and their references.

The goal of this paper is to investigate the logarithmic Schrödinger equation on connected locally finite graphs. The problem (1) discussed in this paper can be viewed as the discrete model of equation (5). By developing variational methods on discrete graphs and establishing new embedding results, we shall obtain the existence of ground state solutions of equation (1).

Throughout this paper, we always assume that there exists some constant \( \mu_{\text{min}} > 0 \) such that the measure \( \mu(x) \geq \mu_{\text{min}} > 0 \) for all \( x \in V \). For the potential \( a(x) \), we assume

\[(A_1) \quad a : V \to \mathbb{R} \ \text{satisfies} \ \inf_{x \in V} a(x) \geq a_0 \ \text{for some constant} \ a_0 \in (-1, 0); \]

\[(A_2) \ \text{for every} \ M > 0 \ \text{such that the volume of the set} \ D_M \ \text{is finite, namely}, \]

\[ Vol(D_M) = \sum_{x \in D_M} \mu(x) < \infty, \]

where \( D_M = \{x \in V : a(x) \leq M\} \).

Our first result is as follows.

**Theorem 1** Let \( G = (V, E) \) be a connected locally finite graph. Assume the potential \( a : V \to \mathbb{R} \) satisfies \((A_1)\) and \((A_2)\). Then the problem (1) admits a ground state solution.
We note that the assumption \((A_2)\) is weaker than the coercivity condition \((B_1)\) in [11] even when \(D_M\) is empty. This assumption was originally introduced by Bartsch and Wang in [20] to study the existence and multiplicity of solutions of nonlinear Schrödinger equations in \(\mathbb{R}^N\). The potential \(a(x)\) in [20] is positive and bounded away from 0, and later was relaxed to the critical frequency case, namely, \(\inf_{x \in V} a(x) = 0\), by Byeon and Wang [48, 49] and Sirakov [50]. In [51], in a similar spirit, Ding and Szulkin dealt with the case when \(a(x)\) is sign changing. Inspired by these studies on nonlinear Schrödinger equations in the Euclidean space, we consider problem (1) when \(a(x)\) may change sign. To the best of our knowledge, there is no result about nonlinear Schrödinger equations on discrete graphs with potentials of sign changing. Furthermore, if we generalize \((B_2)\) in [11] to the sign-changing potential case, we shall obtain the following result.

**Theorem 2.** Let \(G = (V, E)\) be a connected locally finite graph. Assume the potential \(a : V \to \mathbb{R}\) satisfies \((A_1)\) and

\[(A_2') \text{ there exists } M_0 > 0 \text{ such that } 1/a(x) \in L^1(V \setminus D_{M_0}), \text{ where } D_{M_0} = \{x \in V : a(x) \leq M_0\} \text{ and the volume of } D_{M_0} \text{ is finite.}\]

Then the problem (1) admits a ground state solution.

To prove Theorem 1 and Theorem 2, the difficulties are mainly twofold.

Firstly, in most previous works of nonlinear Schrödinger equations on locally finite graphs, the potentials have positive lower bound. Introducing the corresponding workspace

\[\mathcal{E} = \left\{ u \in D^{1,2}(V) : \int_V a(x)u^2 \, d\mu < +\infty \right\},\]

endowed with \(\|u\|_\mathcal{E}^2 = \int_V (|\nabla u|^2 + a(x)u^2) \, d\mu\), we know that \(\mathcal{E}\) is continuously embedded into \(H^1(V)\). However, due to the assumption that \(a(x)\) may change sign, it is not clear if the quantity \(\| \cdot \|_\mathcal{E}\) is a norm or not and hence the embedding is not available.

Second, it is known that the logarithmic Sobolev inequality plays an important role in the study of logarithmic Schrödinger equations (see [29, 35, 36, 41, 46] etc.). A logarithmic Sobolev inequality on discrete graphs is proved under a positive curvature condition, which requires the measure \(\mu\) to be finite (see [52] for details). Unfortunately, this inequality cannot be applied here because, in our case, the measure \(\mu\) has a uniform positive lower bound.

Moreover, note that in previous results on connected locally finite graphs, the \(\text{(AR)}\) condition plays a crucial role in obtaining bounded Palais-Smale sequences for the associated energy functional. However, because of the logarithmic term, the \(\text{(AR)}\) condition is not satisfied and it makes our proof more difficult.

Indeed, to deal with the logarithmic term, two Sobolev compact embedding results are established under \((A_1) - (A_2)\) and \((A_1) - (A_2')\), respectively. In each case, we encounter different obstacles in seeking the solutions of (1). Specifically, when \(a(x)\) satisfies \((A_1)\) and \((A_2)\), the associated energy functional is not well defined and the
previous arguments on locally finite graphs do not work. We borrow the ideas in \[35, 36\] to overcome this difficulty. By restricting \(u^2 \log u^2 \in L^1(V)\), we develop an approach based on the direction derivative and the Nehari manifold method for locally finite graphs to obtain the existence of ground state solutions of (1). For the other case, i.e., when \(a(x)\) satisfies (A1) and (A2), the associated energy functional is \(C^1\), and we can apply a variant of mountain pass theorem to obtain the existence of ground state solutions.

The paper is organized as follows. In Sect. 2, we present some notations, definitions, and lemmas that will be used throughout the paper, and then two Sobolev compact embedding results will be given. In Sect. 3, we develop the Nehari manifold method for locally finite graphs to prove Theorem 1. In Sect. 4, we apply the mountain pass theorem with the Cerami condition to give the proof of Theorem 2. For simplicity, we denote by \(C, C_1, C_2, \cdots\) the positive constants which may be different in different places.

### 2 Some Preliminary Results

Let us start with some notations. For any function \(u : V \to \mathbb{R}\), the integral of \(u\) over \(V\) is defined by

\[
\int_V u d\mu = \sum_{x \in V} \mu(x)u(x).
\]

The gradient form of two functions \(u, v\) on \(V\) is defined as follows:

\[
\Gamma(u, v)(x) = \frac{1}{2|\mu(x)|} \sum_{y \sim x} \omega_{xy} (u(y) - u(x))(v(y) - v(x)).
\]

Write \(\Gamma(u) = \Gamma(u, u)\), and sometimes we use \(\nabla u \nabla v\) to replace \(\Gamma(u, v)\). The length of the gradient for \(u\) is defined by

\[
|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2|\mu(x)|} \sum_{y \sim x} \omega_{xy} (u(y) - u(x))^2\right)^{1/2}.
\]

Denote by \(C_c(V)\) the set of all functions with compact support, and let \(H^1(V)\) be the completion of \(C_c(V)\) under the norm

\[
\|u\|_{H^1(V)} = \left(\int_V \left(|\nabla u|^2 + u^2\right) d\mu\right)^{1/2}.
\]

Then, \(H^1(V)\) is a Hilbert space with the inner product

\[
\langle u, v \rangle = \int_V (\Gamma(u, v) + uv) d\mu, \ \forall u, v \in H^1(V).
\]
We write $\|u\|_p = (\int_V |u|^p d\mu)^{1/p}$ for $p \in [1, +\infty)$ and $\|u\|_{L_\infty} = \sup_{x \in V} |u(x)|$.

We introduce the space

$$
\mathcal{H} = \left\{ u \in H^1(V) : \int_V a(x)u^2 d\mu < +\infty \right\}
$$

with the norm

$$
\|u\|_{\mathcal{H}} = \left( \int_V \left( |\nabla u|^2 + (a(x) + 1) u^2 \right) d\mu \right)^{1/2}
$$

induced by the inner product

$$
\langle u, v \rangle_{\mathcal{H}} = \int_V (\Gamma(u, v) + (a(x) + 1) uv) d\mu, \ \forall u, v \in \mathcal{H}.
$$

Clearly, $\mathcal{H}$ is also a Hilbert space.

Next, we establish two Sobolev embedding results when $a(x)$ satisfies $(A_1) - (A_2)$ and $(A_1) - (A'_2)$, respectively.

**Lemma 3** Assume $\mu(x) \geq \mu_{\text{min}} > 0$ and $a(x)$ satisfies $(A_1), (A_2)$. Then $\mathcal{H}$ is weakly pre-compact and $\mathcal{H}$ is compactly embedded into $L^p(V)$ for all $2 \leq p \leq +\infty$.

**Proof** At any vertex $x_0 \in V$, by $(A_1)$, we have

$$
\|u\|_{\mathcal{H}}^2 = \int_V \left( |\nabla u|^2 + (1 + a(x)) u^2 \right) d\mu \\
\geq (1 + a_0) \int_V u^2 d\mu \\
= (1 + a_0) \sum_{x \in V} \mu(x)u^2(x) \\
\geq (1 + a_0)\mu_{\text{min}}u^2(x_0),
$$

which implies $|u(x_0)| \leq \sqrt{\frac{1}{(1+\mu_{\text{min}})u^2}} \|u\|_{\mathcal{H}}$. Thus $\mathcal{H} \hookrightarrow L^\infty(V)$ continuously. Hence, by interpolation, we deduce $\mathcal{H} \hookrightarrow L^p(V)$ continuously for all $2 \leq p \leq +\infty$. Since $\mathcal{H}$ is a Hilbert space and then reflexive, if $\{u_k\}$ is bounded in $\mathcal{H}$, then we have that, up to a subsequence, $u_k \rightharpoonup u$ in $\mathcal{H}$ for some $u \in \mathcal{H}$. In particular, $\{u_k\} \subset \mathcal{H}$ is also bounded in $L^2(V)$ and we have the weak convergence in $L^2(V)$, i.e., for any $\varphi \in L^2(V)$,

$$
\lim_{k \to \infty} \int_V (u_k - u)\varphi d\mu = \lim_{k \to \infty} \sum_{x \in V} \mu(x) (u_k(x) - u(x)) \varphi(x) = 0. \quad (9)
$$

Take any $x_0 \in V$ and let

$$
\varphi_0(x) = \begin{cases} 
1, & x = x_0, \\
0, & x \neq x_0.
\end{cases}
$$
Obviously, $\varphi_0(x) \in L^2(V)$. By substituting $\varphi_0$ into (9), we can get that $\lim_{k \to \infty} u_k(x) = u(x)$ for any fixed $x \in V$.

We now prove $u_k \to u$ in $L^p(V)$ for all $2 \leq p \leq \infty$. We first show that $u_k \to u$ in $L^2(V)$. It suffices to prove that $\alpha_k := \|u_k\|_2^2 \to \|u\|_2^2$. We assume that, up to a subsequence, $\alpha_k \to \alpha$ such that $\alpha \geq \|u\|_2^2$. Let $x_0 \in V$ be fixed. We claim that for every $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{d(x, x_0) > R} u_k^2(x) d\mu < \epsilon \text{ uniformly in } k. \quad (10)$$

Since $\{x \in V : d(x, x_0) \leq R\}$ is a finite set and $u_k \to u$ for any $x \in V$ as $k \to \infty$, it follows that $\lim_{k \to \infty} \int_{d(x, x_0) \leq R} |u_k - u|^2 d\mu = 0$. If (10) holds, then

$$\int_V u^2 d\mu = \int_{d(x, x_0) \leq R} u^2 d\mu + \int_{d(x, x_0) > R} u^2 d\mu \geq \lim_{k \to \infty} \int_{d(x, x_0) \leq R} u_k^2 d\mu \geq \lim_{k \to \infty} \int_{d(x, x_0) \leq R} u_k^2 d\mu - \lim_{k \to \infty} \int_{d(x, x_0) > R} u_k^2 d\mu \geq \alpha - \epsilon.$$ 

Since $\epsilon$ is arbitrary, we get $\|u\|_2^2 \geq \alpha$ and hence $u_k \to u$ in $L^2(V)$.

It remains to prove (10). For fixed $\epsilon > 0$, we choose constants $M > \frac{2}{\epsilon} \sup_k \|u_k\|_H^2$, $p \in (1, \infty)$ and

$$C > \sup_{u \in H \setminus \{0\}} \frac{\|u\|_{\frac{2p}{p}}^2}{\|u\|_H^2}.$$ 

Denote $V_1 := \{x \in V : d(x, x_0) > R\}$. By $(A_2)$ it follows that, for $R > 0$ large enough,

$$Vol(V_1 \cap D_M) := \sum_{x \in V_1 \cap D_M} \mu(x) \leq \left(\frac{\epsilon}{2C \sup_k \|u_k\|_H^2}\right)^q,$$

where $q$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\int_{V_1} u_k^2 d\mu = \int_{V_1 \setminus D_M} u_k^2 d\mu + \int_{V_1 \cap D_M} u_k^2 d\mu.$$
For \( \int_{V_1 \setminus D_M} u_k^2 d\mu \), we have

\[
\int_{V_1 \setminus D_M} u_k^2 d\mu \leq \int_{V_1 \setminus D_M} \frac{a(x)}{M} u_k^2 d\mu \leq \frac{\|u_k\|_{L^2}^2}{M} \leq \frac{\epsilon}{2}.
\]

For \( \int_{V_1 \cap D_M} u_k^2 d\mu \), by Hölder’s inequality

\[
\int_{V_1 \cap D_M} u_k^2 d\mu \leq \left( \int_{V_1 \cap D_M} |u_k|^{2p} d\mu \right)^{\frac{1}{p}} \left( \int_{V_1 \cap D_M} 1 d\mu \right)^{\frac{1}{q}} \leq \|u_k\|_{L^{2p}}^2 \cdot \left( \sum_{x \in V_1 \cap D_M} \mu(x) \right)^{\frac{1}{q}} \leq C \|u_k\|_{L^{2p}}^2 \cdot \frac{\epsilon}{2C \sup_k \|u_k\|_{L^{2p}}} \leq \frac{\epsilon}{2}.
\]

Hence (10) holds.

Since

\[
\|u_k - u\|_\infty^2 \leq \frac{1}{\mu_{\min}} \int_V |u_k - u|^2 d\mu,
\]

we have for any \( 2 < p < \infty \),

\[
\int_V |u_k - u|^p d\mu \leq \left( \frac{1}{\mu_{\min}} \right)^{\frac{p-2}{2}} \left( \int_V |u_k - u|^2 d\mu \right)^{\frac{p}{2}}.
\]

This completes the proof. \( \square \)

**Lemma 4** Assume \( \mu(x) \geq \mu_{\min} > 0 \) and \( a(x) \) satisfies (A1), (A2'). Then \( \mathcal{H} \) is weakly pre-compact and \( \mathcal{H} \) is compactly embedded into \( L^p(V) \) for all \( 1 \leq p \leq +\infty \).

**Proof** As in the proof of Lemma 3, we have \( \mathcal{H} \hookrightarrow L^p(V) \) continuously for all \( 2 \leq p \leq \infty \). Next, we prove that \( \mathcal{H} \hookrightarrow L^p(V) \) continuously for any \( 1 \leq p < 2 \). Indeed, since \( D_{M_0} \) is a bounded set, it follows that \( \mathcal{H} \hookrightarrow L^p(D_{M_0}) \) continuously for any
$1 \leq p < 2$. By $(A_2')$, for any $u \in \mathcal{H}$,

$$
\int_{V \setminus D_{M_0}} |u| d\mu = \int_{V \setminus D_{M_0}} \left( \frac{1}{a(x)} \right)^{\frac{1}{2}} (a(x))^{\frac{1}{2}} |u| d\mu \\
\leq \left( \int_{V \setminus D_{M_0}} \frac{1}{a(x)} d\mu \right)^{\frac{1}{2}} \left( \int_{V \setminus D_{M_0}} a(x) u^2 d\mu \right)^{\frac{1}{2}} \\
\leq C_1 \left( \int_{V \setminus D_{M_0}} \frac{1}{a(x)} d\mu \right)^{\frac{1}{2}} \|u\|_{\mathcal{H}},
$$

which implies that $\mathcal{H} \hookrightarrow L^p(V \setminus D_{M_0})$ continuously for any $1 \leq p < 2$.

Let $\{u_k\}$ be a bounded sequence in $\mathcal{H}$. By similar arguments as in Lemma 3, we have $u_k \to u$ point-wisely in $V$ as $k \to \infty$. In what follows, we prove $u_k \to u$ in $L^p(V)$ for all $1 \leq p \leq \infty$. Since $\{u_k\}$ is bounded in $\mathcal{H}$ and $u \in \mathcal{H}$, there exists some constant $C_2$ such that

$$
\int_{V \setminus D_{M_0}} a(x) |u_k - u|^2 d\mu \leq C_2.
$$

Fix $x_0 \in V$. For any $\varepsilon > 0$, using $(A_2')$, there exists $R > 0$ such that

$$
\int_{V \setminus D_{M_0}} a(x) d\mu < \varepsilon^2.
$$

Then, together with the Hölder’s inequality, it holds that

$$
\int_{V \setminus D_{M_0}} |u_k - u| d\mu = \int_{V \setminus D_{M_0}} \left( \frac{1}{a(x)} \right)^{\frac{1}{2}} (a(x))^{\frac{1}{2}} |u_k - u| d\mu \\
\leq \left( \int_{V \setminus D_{M_0}} \frac{1}{a(x)} d\mu \right)^{\frac{1}{2}} \left( \int_{V \setminus D_{M_0}} a(x) |u_k - u|^2 d\mu \right)^{\frac{1}{2}} \\
\leq \sqrt{C_2 \varepsilon}. \quad (11)
$$

Since

$$
\lim_{k \to \infty} \left( \int_{V \setminus D_{M_0}} |u_k - u| d\mu + \int_{d(x, x_0) \leq R} |u_k - u| d\mu \right) = 0,
$$

by (11), we get

$$
\liminf_{k \to \infty} \int_{V} |u_k - u| d\mu = 0.
$$
which implies that $u_k \to u$ in $L^1(V)$. In view of

$$\|u_k - u\|_{L^\infty(V)} \leq \frac{1}{\mu_{\text{min}}} \int_V |u_k - u|d\mu,$$

it follows that

$$\int_V |u_k - u|^p d\mu \leq \frac{1}{\mu_{\text{min}}} \left( \int_V |u_k - u|d\mu \right)^p, \forall p \in (1, +\infty).$$

Hence, $u_k \to u$ in $L^p(V)$ for all $1 \leq p \leq +\infty$. $\square$

### 3 Proof of Theorem 1

In this section, under the assumptions $(A_1)$ and $(A_2)$, we prove that equation (1) admits a ground state solution by using the Nehari manifold method.

We note that equation (1) is formally associated with the energy functional $J : H^1(V) \to \mathbb{R} \cup \{ +\infty \}$ given by

$$J(u) = \frac{1}{2} \int_V \left( |\nabla u|^2 + (a(x) + 1) |u|^2 \right) d\mu - \frac{1}{2} \int_V u^2 \log u^2 d\mu.$$  

But $J$ may fail to be $C^1$ in $H^1(V)$. In fact, for some $G = (V, E)$ with suitable measure $\mu$, there exists $u \in H^1(V)$ but $\int_V u^2 \log u^2 d\mu = -\infty$ (see the 'Appendix' for the details).

When $a(x)$ satisfies $(A_1)$ and $(A_2)$, we consider the functional $J$ on the set

$$\mathcal{D} = \left\{ u \in H : \int_V u^2 |\log u^2|d\mu < \infty \right\},$$

that is,

$$J(u) = \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{2} \int_V u^2 \log u^2 d\mu, \forall u \in \mathcal{D}.$$  

**Definition 1**

1. Define

$$J'(u) \cdot v := \int_V (\Gamma(u, v) + a(x)uv) d\mu - \int_V uv \log u^2 d\mu, \quad \forall u, v \in \mathcal{D}.$$  

Clearly, $\int_V uv \log u^2 d\mu$ is well defined for $u, v \in \mathcal{D}$.

2. We say that $u \in H$ is a critical point of $J$ if $u \in \mathcal{D}$ and $J'(u) \cdot v = 0$ for all $v \in \mathcal{D}$, and we say that $c \in \mathbb{R}$ is a critical value for $J$ if there exists a critical point $u \in H$ such that $J(u) = c$. 

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Clearly, \( u \) is a weak solution to equation (1) if and only if \( u \) is a critical point of \( J \). Furthermore,

**Proposition 5** If \( u \in \mathcal{D} \) is a weak solution of equation (1), then \( u \) is a point-wise solution of equation (1).

**Proof** If \( u \in \mathcal{D} \) is a weak solution of (1), then for any \( \varphi \in \mathcal{D} \), there holds

\[
\int_\mathcal{V} (\Gamma(u, \varphi) + a(x)u \varphi) \, d\mu = \int_\mathcal{V} u \varphi \log u^2 \, d\mu.
\]

Let \( C_c(\mathcal{V}) \) be the set of all functions on \( \mathcal{V} \) with compact support. It is easily seen that it is dense in \( \mathcal{D} \). By integration by parts, we get

\[
\int_\mathcal{V} (-\Delta u + a(x)u) \varphi \, d\mu = \int_\mathcal{V} u \varphi \log u^2 \, d\mu, \quad \forall \varphi \in C_c(\mathcal{V}).
\]  

(12)

For any fixed \( x_0 \in \mathcal{V} \), define

\[
\varphi(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}
\]

Clearly, \( \varphi \in C_c(\mathcal{V}) \). Taking \( \varphi \) as a test function in (12), we deduce

\[-\Delta u(x_0) + a(x_0)u(x_0) - u(x_0) \log (u(x_0))^2 = 0.\]

Since \( x_0 \) is arbitrary, we conclude that \( u \) is a point-wise solution of (1). \( \square \)

We define

\[ \mathcal{N} := \{ u \in \mathcal{D} \setminus \{0\} : J'(u) \cdot u = 0 \} \]

and

\[ d = \inf_{u \in \mathcal{N}} J(u). \]

First we have

**Lemma 6** For any \( u \in \mathcal{D} \setminus \{0\} \), there exists a unique \( t_u > 0 \) such that \( t_u u \in \mathcal{N} \). Furthermore, \( J(t_u u) > J(tu) \) for all \( t \geq 0 \) but \( t \neq t_u \). Specially, if \( u \in \mathcal{N} \), then \( t_u = 1 \).

**Proof** Fixed \( u \in \mathcal{D} \setminus \{0\} \), define \( j(t) = J(tu) \), \( \forall t \geq 0 \), i.e.,

\[ j(t) = \frac{t^2}{2} \|u\|_H^2 - \frac{t^2}{2} \int_\mathcal{V} u^2 \log u^2 \, d\mu - \frac{t^2 \log t^2}{2} \int_\mathcal{V} u^2 \, d\mu. \]
Clearly, $j(0) = 0$, and there exist $\delta_2 > \delta_1 > 0$ such that $j(t) > 0$ for $t \in (0, \delta_1)$ and $j(t) < 0$ for $t \in (\delta_2, +\infty)$. Hence, $j$ has a maximum at some $t_u \in [\delta_1, \delta_2]$, which implies that $j'(t_u) = 0$. Moreover, we note that
\[
\frac{j'(t)}{t} = \frac{J'(tu) \cdot (tu)}{t^2} = \|u\|^2_H - \|u\|^2_2 - \int_V u^2 \log u^2 d\mu - \log t^2 \|u\|^2_2,
\]
which implies that $\frac{j'(t)}{t}$ is strictly decreasing with respect to $t > 0$. Hence $t_u$ is unique. □

Next we prove the following result.

**Lemma 7** Supposed $(A_1)$ and $(A_2)$ hold. Then $d > 0$ is achieved.

**Proof** By Lemma 6, $\mathcal{N} \neq \emptyset$. Taking a minimizing sequence $\{u_k\} \subset \mathcal{N}$ of $J$ yields
\[
\lim_{k \to +\infty} J(u_k) = \lim_{k \to +\infty} \left[ J(u_k) - J'(u_k) \cdot u_k \right] = \lim_{k \to +\infty} \frac{1}{2} \|u_k\|^2_2 = d. \tag{13}
\]
By Lemma 3, the Hölder’s inequality and Young inequality, for any $\varepsilon \in (0, 1)$, there exist $C_\varepsilon, C_\varepsilon', C_\varepsilon'' > 0$ such that
\[
\int_V u_k^2 \log u_k^2 d\mu \leq \int_V (u_k^2 \log u_k^2)^+ d\mu \leq C_\varepsilon \int_V |u_k|^{2+\varepsilon} d\mu \\
\leq C_\varepsilon \left( \int_V |u_k|^2 d\mu \right)^{\frac{1}{2}} \left( \int_V |u_k|^{2(1+\varepsilon)} d\mu \right)^{\frac{1}{2}} \\
\leq C_\varepsilon' \|u_k\|_2 \|u_k\|_H^{1+\varepsilon} \\
\leq \frac{1}{2} \|u_k\|^2_H + C_\varepsilon'' \|u_k\|_2^{\frac{2}{1+\varepsilon}}.
\]
Since $\{u_k\} \subset \mathcal{N}$, we deduce that
\[
\|u_k\|^2_H = \int_V u_k^2 \log u_k^2 d\mu + \|u_k\|^2_2 \leq \frac{1}{2} \|u_k\|^2_H + C_\varepsilon'' \|u_k\|_2^{\frac{2}{1+\varepsilon}} + \|u_k\|^2_2. \tag{14}
\]
This together with (13) implies that $\{u_k\}$ is bounded in $\mathcal{H}$. Thus, by Lemma 3, up to a subsequence, there exists $u_0 \in \mathcal{H}$ such that
\[
\begin{cases}
  u_k \rightharpoonup u_0 \text{ weakly in } \mathcal{H}, \\
  u_k \to u_0 \text{ point-wisely in } V, \\
  u_k \to u_0 \text{ strongly in } L^p(V) \text{ for } p \in [2, +\infty].
\end{cases}
\]
Then, by the weak-lower semi-continuity of norm and Fatou’s lemma, together with
the Lebesgue dominated convergence theorem, we get

\[
\int_V \left( |\nabla u_0|^2 + (a(x) + 1) u_0^2 \right) d\mu - \int_V \left( u_0^2 \log u_0^2 \right) d\mu \\
\leq \liminf_{k \to \infty} \left[ \int_V \left( |\nabla u_k|^2 + (a(x) + 1) u_k^2 \right) dx - \int_V (u_k^2 \log u_k^2) d\mu \right] \\
= \liminf_{k \to \infty} \int_V \left[ u_k^2 + (u_k^2 \log u_k^2)^+ \right] d\mu \\
= \int_V u_0^2 d\mu + \int_V (u_0^2 \log u_0^2)^+ d\mu.
\]

It follows that

\[
\int_V \left( |\nabla u_0|^2 + a(x) u_0^2 \right) d\mu - \int_V u_0^2 \log u_0^2 d\mu \leq 0. \tag{15}
\]

In view of Lemma 6, there exists a constant \( t_0 > 0 \) such that \( t_0 u_0 \in \mathcal{N} \). By (15), we have

\[
0 = J'_0(t_0 u_0) \cdot (t_0 u_0) \\
= t_0^2 \int_V \left( |\nabla u_0|^2 + a(x) u_0^2 \right) d\mu - t_0^2 \int_V u_0^2 \log u_0^2 d\mu - t_0^2 \log t_0 \| u_0 \|^2_2 \\
\leq -t_0^2 \log t_0 \| u_0 \|^2_2,
\]

which implies that \( 0 < t_0 \leq 1 \). Then

\[
d \leq J(t_0 u_0) = J(t_0 u_0) - \frac{1}{2} J'(t_0 u_0) \cdot (t_0 u_0) = \frac{1}{2} \| t_0 u_0 \|^2_2 \leq \liminf_{k \to \infty} \frac{t_0^2}{2} \| u_k \|^2_2 \\
= t_0^2 \liminf_{k \to \infty} \left[ J(u_k) - \frac{1}{2} J'(u_k) \cdot u_k \right] = t_0^2 \liminf_{k \to \infty} J(u_k) = t_0^2 d \leq d.
\]

Hence \( t_0 = 1 \) and thus \( J(u_0) = d \).

We claim that \( d > 0 \). In fact, if \( d = 0 \), then

\[
0 = J(u_0) - J'(u_0) \cdot u_0 = \frac{1}{2} \| u_0 \|^2_2.
\]

By (14), we have \( \| u_0 \|_{\mathcal{H}} = 0 \).

On the other hand, by Lemma 3, for any \( q > 2 \), there exists \( C_q > 0 \) such that

\[
\| u_0 \|^2_\mathcal{H} = \int_V u_0^2 \log u_0^2 d\mu \leq \int_V (u_0^2 \log u_0^2)^+ d\mu \leq C_q \int_V |u_0|^q d\mu \leq C \| u_0 \|^q_\mathcal{H},
\]
which implies
\[ \|u_0\|_{H^1} \geq \left( \frac{1}{C} \right)^{\frac{1}{q-2}} > 0. \]

Contradiction! Hence the claim holds. \( \square \)

The following lemma is crucial, which will completes the proof of Theorem 1.

**Lemma 8** If \( J(u) = c \) for some \( u \in \mathcal{N} \), then \( u \) is a weak solution of equation (1).

**Proof** By contradiction, we can find a function \( \phi \in C_c(V) \) such that
\[
\int_V (\nabla u \nabla \phi + a(x)u \phi) \, d\mu - \int_V u \phi \log u^2 \, d\mu \leq -1,
\]
which implies that, for some \( \varepsilon > 0 \) small enough,
\[
J'(su + \sigma \phi) \cdot \phi \leq -\frac{1}{2}, \text{ for all } |s - 1| + |\sigma| \leq \varepsilon. \tag{16}
\]

In what follows, we estimate \( \sup_s J(su + \varepsilon \eta(s) \phi) \), where \( \eta \) is a cut-off function such that
\[
\eta(s) = \begin{cases} 
1 & \text{if } |s - 1| \leq \frac{\varepsilon}{2}, \\
0 & \text{if } |s - 1| \geq \varepsilon.
\end{cases}
\]

Since \( \eta(s) = 0 \) for \( |s - 1| \geq \varepsilon \), it suffices to consider the case \( |s - 1| \leq \varepsilon \). Indeed, by (16) we obtain
\[
J(su + \varepsilon \eta(s) \phi) = J(su + \varepsilon \eta(s) \phi) - J(su) + J(su)
\]
\[
= J(su) + \int_0^1 J'(su + \tau \varepsilon \eta(s) \phi) \cdot (\varepsilon \eta(s) \phi) \, d\tau
\]
\[
= J(su) + \varepsilon \eta(s) \int_0^1 J'(su + \tau \varepsilon \eta(s) \phi) \cdot \phi \, d\tau
\]
\[
\leq J(su) - \frac{1}{2} \varepsilon \eta(s),
\]
which implies that
\[
J(u + \varepsilon \eta(1) \phi) \leq J(u) - \frac{1}{2} \varepsilon \eta(1) = J(u) - \frac{1}{2} \varepsilon.
\]

In addition, by Lemma 6 we have \( J(su) < J(u) \) for all \( s \neq 1 \). Then
\[
J(su + \varepsilon \eta(s) \phi) \leq J(su) < J(u) \text{ for all } s \neq 1.
\]
Hence, we deduce $J(s_0u + \varepsilon \eta(s)\phi) < J(u) = c$. Therefore, for $0 < \varepsilon < 1 - \varepsilon$, there exists $s_0 \in [\varepsilon, 2 - \varepsilon]$ such that

$$J(s_0u + \varepsilon \eta(s)\phi) = \sup_{\varepsilon \leq s \leq 2 - \varepsilon} J(su + \varepsilon \eta(s)\phi) < c.$$ 

Denote $v = su + \varepsilon \eta(s)\phi$ and define $F(s) = J'(v) \cdot v$. It is easily seen that $F(\varepsilon) > 0$, $F(2 - \varepsilon) < 0$, which implies that there exists $s_1 \in [\varepsilon, 2 - \varepsilon]$ such that $F(s_1) = J'(v) \cdot v|_{s=s_1} = 0$. Let $\tilde{u} = s_1u + \varepsilon \eta(s_1)\phi$. Clearly, $\tilde{u} \in \mathcal{N}$ and $J(\tilde{u}) < c$. This gives a contradiction to the definition of $c$. 

$\square$

### 4 Proof of Theorem 2

This section is devoted to proving Theorem 2. We recall that $J(u) = \frac{1}{2}\|u\|^2_{\mathcal{H}} - \frac{1}{2} \int_{\mathcal{V}} u^2 \log u^2 d\mu$. Note that, for any $0 < \varepsilon < 1$, there exists $C_\varepsilon > 0$ such that

$$|u^2 \log u^2| \leq C_\varepsilon \left(|u|^{2-\varepsilon} + |u|^{2+\varepsilon}\right). \quad (17)$$

When $a(x)$ satisfies $(A_1)$ and $(A'_2)$, from Lemma 4, the embedding $\mathcal{H} \hookrightarrow L^p(V)$ is compact for $p \in [1, +\infty]$. Then, in view of [24, Lemma 2.16], we can easily conclude the following result.

**Proposition 9** Assume that $a(x)$ satisfies $(A_1)$ and $(A'_2)$. Then the functional $J \in C^1(\mathcal{H}, \mathbb{R})$. Moreover, if $u \in \mathcal{H}$ is a critical point of $J$, then it is a weak solution of equation (1), i.e.,

$$\int_{\mathcal{V}} (\Gamma(u, v) + a(x)uv) d\mu = \int_{\mathcal{V}} uv \log u^2 d\mu, \ \forall v \in \mathcal{H}.$$ 

Similar as Proposition 5, we have following result.

**Proposition 10** If $u \in \mathcal{H}$ is a weak solution of (1), then $u$ is a point-wise solution of equation (1).

We recall that, for $c \in \mathbb{R}$, $J$ satisfies the Cerami condition at level $c$, if for any sequence $\{u_k\} \subset \mathcal{H}$ with

$$J(u_k) \rightarrow c, \ (1 + \|u_k\|_{\mathcal{H}})\|J'(u_k)\|_{\mathcal{H}'} \rightarrow 0 \quad (18)$$

there is a subsequence $\{u_k\}$ such that $\{u_k\}$ converges strongly in $\mathcal{H}$. Any sequence $\{u_k\} \subset \mathcal{H}$ for which (18) holds true is called a Cerami sequence of $J$ at level $c$.

To prove Theorem 2, we shall apply the following version of mountain pass theorem, which provides the existence of a Cerami sequence at the mountain pass level.

**Lemma 11** [53] Let $X$ be a real Banach space, $\Phi \in C^1(X, \mathbb{R})$ satisfies

$$\max\{\Phi(0), \Phi(e)\} \leq \alpha < \beta \leq \inf_{\|u\|_{X} = \rho} \Phi(u)$$

\(\square\) Springer
for some $\alpha, \beta, \rho > 0$ and $e \in X$ with $\|e\|_X > \rho$. Set

$$c = \inf_{\gamma} \max_{0 \leq \tau \leq 1} \Phi(\gamma(\tau)),$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Then there exists a Cerami sequence $\{u_k\} \subset \mathcal{H}$ of $\Phi$ at level $c$.

Firstly, we show $J$ admits a mountain pass geometry.

**Lemma 12** (i) There exists positive constants $\rho$ and $\delta$ such that $J(u) \geq \delta$ for all $u \in \mathcal{H}$ with $\|u\|_\mathcal{H} = \rho$.

(ii) There exists $\varphi \in \mathcal{H} \setminus \{0\}$ such that $J(t\varphi) \to -\infty$ as $t \to +\infty$.

**Proof** For (i), by the definition of $J$ and Lemma 4, for any $q > 2$, there exist constants $C_q, C > 0$ such that

$$J(u) = \frac{1}{2} \|u\|_\mathcal{H}^2 - \frac{1}{2} \int_V u^2 \log u^2 \, d\mu$$

$$\geq \frac{1}{2} \|u\|_\mathcal{H}^2 - \frac{1}{2} \int_V (u^2 \log u^2)^+ \, d\mu$$

$$\geq \frac{1}{2} \|u\|_\mathcal{H}^2 - \frac{C_q}{2} \|u\|_q^q$$

$$\geq \frac{1}{2} \|u\|_\mathcal{H}^2 - \frac{C}{2} \|u\|_\mathcal{H}^q.$$

Taking $\rho > 0$ small enough, there exists a constant $\delta > 0$ such that

$$J(u) \geq \delta$$

for all $u \in \mathcal{H}$ with $\|u\|_\mathcal{H} = \rho$.

For (ii), for any $\varphi \in \mathcal{H} \setminus \{0\}$, we have

$$J(t\varphi) = \frac{t^2}{2} \|\varphi\|_\mathcal{H}^2 - \frac{t^2}{2} \int_V \varphi^2 \log \varphi^2 \, d\mu - \frac{t^2}{2} \log t^2 \|\varphi\|_2^2.$$

This shows that $J(t\varphi) \to -\infty$ as $t \to +\infty$. $\square$

Next, we prove that $J$ satisfies the Cerami condition.

**Lemma 13** The functional $J$ satisfies the Cerami condition at any level $c > 0$.

**Proof** Assume that $\{u_k\} \subset \mathcal{H}$ is a Cerami sequence of $J$. We claim that $\{u_k\}$ is bounded in $\mathcal{H}$. If not, we may assume that $\{u_k\}$ is unbounded in $\mathcal{H}$. Set $w_k = \frac{u_k}{\|u_k\|_\mathcal{H}}$. Then,
passing to a subsequence if necessary, we may assume that there exists $w \in \mathcal{H}$ such that

$$
\begin{align*}
\begin{cases}
  w_k &\to w \text{ in } \mathcal{H}, \\
  w_k &\to w \text{ point-wise in } V, \\
  w_k &\to w \text{ in } L^p(V), \; p \in [1, +\infty].
\end{cases}
\end{align*}
$$

We will then show the claim by discussing the following two cases:

**Case 1: $w = 0$.** Let $t_k \in [0, 1]$ such that

$$
J(t_k u_k) = \max_{t \in [0,1]} J(t u_k).
$$

By the unboundedness of $\{u_k\}$, for any given $\tau > 0$, there exists $N > 0$ such that

$$
\frac{\tau}{\|u_k\|_{\mathcal{H}}} \in (0, 1), \; k \geq N.
$$

Denote $\overline{w}_k = (4\tau)^{\frac{1}{2}} w_k$. By (17) and the Lebesgue dominate convergence theorem, it follows that

$$
\begin{align*}
\lim_{k \to \infty} \int_V (\overline{w}_k^2 \log \overline{w}_k^2) d\mu &= \lim_{k \to \infty} \left(4\tau \int_V w_k^2 \log w_k^2 d\mu + 4\tau \log(4\tau) \int_V w_k^2 d\mu\right) \\
&= 4\tau \left(\int_V w^2 \log w^2 d\mu + \log(4\tau) \int_V w^2 d\mu\right) \\
&= 0,
\end{align*}
$$

which implies that, for $k$ large enough,

$$
J(t_k u_k) \geq J\left(\frac{(4\tau)^{\frac{1}{2}}}{\|u\|_{\mathcal{H}}} u_k\right) = J(\overline{w}_k)
$$

$$
= \frac{1}{2} \|\overline{w}_k\|_{\mathcal{H}}^2 - \frac{1}{2} \int_V \overline{w}_k^2 \log \overline{w}_k^2 d\mu
$$

$$
\geq \tau.
$$

Since $\tau > 0$ is arbitrary, we deduce

$$
\lim_{k \to \infty} J(t_k u_k) = +\infty.
$$

(19)
However, by \( J(0) = 0 \) we have \( t_k \in (0, 1) \). Then \( \frac{d}{dt} J(t_k u_k) |_{t=t_k} = 0 \). Hence we obtain

\[
J(t_k u_k) = J(t_k u_k) - \frac{1}{2} J'(t_k u_k) \cdot (t_k u_k)
\]

\[
= \frac{1}{2} \int_V |t_k u_k|^2 d\mu
\]

\[
\leq \frac{1}{2} \int_V u_k^2 d\mu
\]

\[
= J(u_k) - \frac{1}{2} J'(u_k) \cdot u_k
\]

\[
\leq C,
\]

which is contrary to (19).

Case 2: \( w \neq 0 \). Set \( V' = \{ x \in V : w \neq 0 \} \). Then \( |u_k(x)| \rightarrow +\infty \) point-wisely in \( V' \). Since \( \{\|u_k\|_{\mathcal{H}}\} \) is unbounded and \( J(u_k) \leq c \), we have

\[
\frac{J(u_k)}{\|u_k\|_{\mathcal{H}}} \rightarrow 0, \text{ i.e.,}
\]

\[
\frac{1}{2} - \frac{1}{2} \int_{V'} \frac{u_k^2 \log u_k^2}{\|u_k\|_{\mathcal{H}}^2} d\mu - \frac{1}{2} \int_{V \setminus V'} \frac{u_k^2 \log u_k^2}{\|u_k\|_{\mathcal{H}}^2} d\mu = o_k(1),
\]

which is equivalent to the following

\[
C - \int_{\{x \in V \setminus V' : |u_k(x)| \leq 1\}} \frac{u_k^2 \log u_k^2}{\|u_k\|_{\mathcal{H}}^2} d\mu
\]

\[
= \int_{\{x \in V \setminus V' : |u_k(x)| > 1\}} \frac{u_k^2 \log u_k^2}{\|u_k\|_{\mathcal{H}}^2} d\mu + \int_{V'} \frac{u_k^2 \log u_k^2}{\|u_k\|_{\mathcal{H}}^2} d\mu.
\]  (20)

On the left side of the (20), from the definition of \( w_k \), we know \( |u_k(x)| < +\infty \) point-wisely in \( V \setminus V' \). By (17) and Lemma 4, we deduce

\[
0 \leq \lim_{k \to \infty} \int_{\{x \in V \setminus V' : |u_k(x)| \leq 1\}} \frac{-u_k^2 \log u_k^2}{\|u_k\|_{\mathcal{H}}^2} d\mu
\]

\[
\leq \lim_{k \to \infty} \int_{\{x \in V \setminus V' : |u_k(x)| \leq 1\}} \frac{2C \varepsilon |u_k|^2 - \varepsilon}{\|u_k\|_{\mathcal{H}}^2} d\mu
\]

\[
\leq \lim_{k \to \infty} \int_{\{x \in V \setminus V' : |u_k(x)| \leq 1\}} \frac{2C \varepsilon |u_k|}{\|u_k\|_{\mathcal{H}}^2} d\mu
\]

\[
\leq \lim_{k \to \infty} \frac{2C \varepsilon C}{\|u_k\|_{\mathcal{H}}^2}
\]

\[
= 0.
\]
On the right side of the (20):

\[
\lim_{k \to \infty} \left[ \int_{\{x \in V \setminus V' : |u_k(x)| > 1\}} \frac{u_k^2 \log u_k^2}{\|u_k\|_{H}^2} d\mu + \int_{V'} \frac{u_k^2 \log u_k^2}{\|u_k\|_{H}^2} d\mu \right] \\
\geq \lim_{k \to \infty} \int_{V'} w_k^2 \log u_k^2 d\mu = +\infty,
\]

which provides a contradiction.

Thus \(\{u_k\}\) is bounded in \(H\). Hence, by Lemma 4, there exists \(u \in H\) such that, up to a subsequence,

\[
\begin{align*}
&\{u_k \to u\} \quad \text{in } H, \\
&u_k \to u \quad \text{point-wisely in } V, \\
&u_k \to u \quad \text{in } L^p(V), \quad p \in [1, +\infty].
\end{align*}
\]

Since \(\{u_k\}\) is a Cerami sequence, we have

\[
\lim_{k \to \infty} J'(u_k) \cdot (u_k - u) = 0.
\] (21)

On the other hand,

\[
\begin{align*}
&\lim_{k \to \infty} J'(u) \cdot (u_k - u) \\
&= \lim_{k \to \infty} \left[ \int_V (\nabla u \nabla (u_k - u) + a(x)u(u_k - u)) d\mu - \int_V u(u_k - u) \log u_k^2 d\mu \right].
\end{align*}
\]

By using the Hölder's inequality and Lemma 4, we obtain

\[
\lim_{k \to \infty} \int_V (u_k - u) u \log u_k^2 d\mu \\
\leq \lim_{k \to \infty} \left( \int_V |u_k - u|^2 d\mu \right)^{\frac{1}{2}} \left( \int_V u^2 (\log u^2)^2 d\mu \right)^{\frac{1}{2}} \\
= 0.
\]

Hence, in view of \(u_k \to u\), it follows that

\[
\lim_{k \to \infty} J'(u) \cdot (u_k - u) = 0.
\] (22)

Note that

\[
J'(u_k) \cdot (u_k - u) - J'(u) \cdot (u_k - u) \\
= \left[ \int_V (\nabla u_k \nabla (u_k - u) + a(x)u_k(u_k - u)) d\mu - \int_V u_k(u_k - u) \log u_k^2 d\mu \right]
\]
\[
- \left[ \int_V (\nabla u \nabla (u_k - u) + a(x)u(u_k - u)) \, d\mu - \int_V u(u_k - u) \log u_k^2 \, d\mu \right] \\
= ||u_k - u||^2_{H^1} - \int_V (u_k - u)u_k \log u_k^2 \, d\mu + \int_V (u_k - u)u \log u^2 \, d\mu - ||u_k - u||^2.
\]

By Lemma 4 again,

\[
\lim_{k \to \infty} \int_V (u_k - u)u_k \log u_k^2 \, d\mu \\
\leq \lim_{k \to \infty} \left( \int_V |u_k - u|^2 \, d\mu \right)^{\frac{1}{2}} \left( \int_V u_k^2 (\log u_k^2)^2 \, d\mu \right)^{\frac{1}{2}} \\
\leq \lim_{k \to \infty} \left( \int_V |u_k - u|^2 \, d\mu \right)^{\frac{1}{2}} \cdot \left( C_\varepsilon \int_V (|u_k|^{2-2\varepsilon} + |u_k|^{2+2\varepsilon}) \, d\mu \right)^{\frac{1}{2}} \\
= 0.
\]

Then, combing with (21–22) and letting \( k \to \infty \) in (23), we deduce

\[
\lim_{k \to \infty} ||u_k - u||^2_{H^1} = 0.
\]

The proof is complete. \( \square \)

**Completion of the proof of Theorem 1.** By Lemma 12, taking \( e = t_1 \varphi \) for some \( t_1 > 0 \) large enough, we can apply Lemma 11 to conclude that there exists a Cerami sequence of \( J \) at level \( c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J(u) \), where \( \Gamma \) is defined as in Lemma 11. Using Lemma 13 it follows that \( c \) is a critical value of \( J \), i.e., there exists some \( u \in H \setminus \{0\} \) such that \( J(u) = c \) and \( J'(u) = 0 \).

To get ground state solution, we denote by \( K \) the critical set of \( J \). Set

\[
m = \inf \{ J(u) : u \in K \setminus \{0\} \}.
\]

For any \( u \in K \), we have

\[
J(u) = J(u) - \frac{1}{2} J'(u) \cdot u = \frac{1}{2} ||u||^2 \geq 0,
\]
which implies $m \geq 0$. On the other hand, using the Fatou’s lemma, we have

\[
J(u) = J(u) - \frac{1}{2} J'(u) \cdot u \\
= \frac{1}{2} \| u \|_2^2 \\
\leq \frac{1}{2} \liminf_{k \to \infty} \| u_k \|_2^2 \\
= \liminf_{k \to \infty} \left[ J(u_k) - \frac{1}{2} J'(u_k) \cdot u_k \right] \\
= c.
\]

Therefore, $0 \leq m \leq c$.

Next, let $\{v_k\}$ be a sequence of non-trivial critical points of $J$ satisfying

\[ J(v_k) \to m. \]

By Lemma 13 we can see that $v_k$ converges to some $v \neq 0$. Using the Fatou’s lemma again, we obtain

\[
m \leq J(v) \leq \liminf_{k \to \infty} J(v_k) = m.
\]

Hence $J(v) = m$ and $J'(v) = 0$. $\square$

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**Data Availability** Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**Declarations**

**Conflict of interest** The authors declare that they have no competing interests.

**Appendix**

In this appendix, we present two examples to show that there exists $u \in H^1(V)$ but

\[
\int_V u^2 \log u^2 \, dx = -\infty.
\]
Example 1 We consider a connected locally finite graph $G = (V, E)$ such that $V := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of natural numbers. Fixed $x_0 = 0 \in V$. Let

$$u(x) = \begin{cases} (|x| \log |x|)^{-1}, & |x| \geq 3, \\ 0, & |x| \leq 2, \end{cases}$$

where $|x| := d(x, x_0)$, the measure

$$\mu(x) = \begin{cases} x, & x \geq 1, \\ 1, & x = 0. \end{cases}$$

For simplicity, we assume that $\omega_{xy} = 1$.

Let’s recall the following fact

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \begin{cases} < \infty, & \text{if } p > 1, \\ = \infty, & \text{if } 0 \leq p \leq 1. \end{cases}$$

First of all, we prove that $u \in H^1(V)$. By the definition of $u$ and $\mu$, we have

$$\int_V u^2 d\mu = \sum_{x \in \mathbb{N}} \mu(x) u^2(x) = \sum_{x \geq 3} x \frac{x}{x^2 (\log x)^2} = \sum_{x \geq 3} \frac{1}{x (\log x)^2} < \infty$$

and

$$\int_V |\nabla u|^2 d\mu = \frac{1}{2} \sum_{x \in \mathbb{N}} \sum_{y \sim x} (u(y) - u(x))^2$$

$$= \frac{1}{2} (u(3) - u(2))^2 + \frac{1}{2} (u(2) - u(3))^2 + \frac{1}{2} (u(4) - u(3))^2$$

$$+ \frac{1}{2} \sum_{x \geq 4} \sum_{y \sim x} (u(y) - u(x))^2$$

$$= \frac{1}{3^2 (\log 3)^2} + \sum_{x \geq 3} \left( \frac{1}{(x + 1) \log (x + 1)} - \frac{1}{x \log x} \right)^2$$

$$\leq \frac{1}{3^2 (\log 3)^2} + \sum_{x \geq 3} \left( \frac{1}{(x + 1)^2 (\log (x + 1))^2} + \frac{2}{x^2 (\log x)^2} \right)$$

$$\leq \frac{1}{3^2 (\log 3)^2} + \sum_{x \geq 3} \frac{2}{x^2 (\log x)^2}$$

$$< \infty.$$
Next, we prove that \( \int_{V} u^2 \log u^2 d\mu = -\infty \). Note that
\[
   u^2(x) \log u^2(x) = -\left( \frac{2}{x^2 \log x} + \frac{2 \log(\log x)}{x^2 (\log x)^2} \right),
\]
we have
\[
   \int_{V} u^2 \log u^2 d\mu = \sum_{x \in \mathbb{N}} \mu(x) u^2(x) \log u^2(x)
   = \sum_{x \geq 3} x \cdot u^2(x) \log u^2(x)
   = -\left( \sum_{x \geq 3} \frac{2}{x \log x} + \sum_{x \geq 3} \frac{2 \log(\log x)}{x (\log x)^2} \right)
   = -(I + II).
\]
Since \( II > 0 \), it suffices to prove the following fact
\[
   \sum_{x \geq 3} \frac{2}{x \log x} = \infty.
\]
But this is obvious, and thus we completes the proof.

**Example 2** Assume that \( G = (V, E) \) is a connected locally finite graph, and there exist some constants \( \mu_{\min}, \mu_{\max} > 0 \) such that \( 0 < \mu_{\min} \leq \mu(x) \leq \mu_{\max} \) for all \( x \in V \).

Fixed \( x_0 \in V \). Let
\[
   u(x) = \begin{cases} 
   \left( |x| \frac{1}{2} \log |x| \right)^{-1}, & |x| \geq 3, \\
   0, & |x| \leq 2,
   \end{cases}
\]
where \( |x| := d(x, x_0) \). Then \( u(x) \in H^1(V) \) and \( \int_{V} u^2 \log u^2 d\mu = -\infty \). In fact, from the definition of \( u \) and \( \mu \), we have
\[
   \int_{V} u^2 d\mu = \sum_{x \in V} \mu(x) u^2(x) = \sum_{|x| \geq 3} \frac{\mu(x)}{|x| (\log |x|)^2} \leq \mu_{\max} \sum_{|x| \geq 3} \frac{1}{|x| (\log |x|)^2} < \infty
\]
\[
\int_V |\nabla u|^2 d\mu = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (u(y) - u(x))^2 \\
= \frac{1}{2} \sum_{|x|=2} \sum_{|y|=3} \frac{\omega_{xy}}{3(\log 3)^2} + \frac{1}{2} \sum_{|x|=3} \sum_{|y|=2} \frac{\omega_{xy}}{3(\log 3)^2} \\
+ \frac{1}{2} \sum_{|x|=3} \sum_{y \sim x} \omega_{xy} \left( \frac{1}{\sqrt{4 \log 4} - \frac{1}{\sqrt{3 \log 3}}} \right)^2 + \frac{1}{2} \sum_{|x| \geq 4} \sum_{y \sim x} \omega_{xy} (u(y) - u(x))^2 \\
= \sum_{|x|=3} \sum_{|y|=2} \frac{\omega_{xy}}{3(\log 3)^2} + \sum_{|x| \geq 3} \sum_{|y|=|x|+1} \omega_{xy} \left( \frac{1}{(|x| + 1)^2 \log (|x| + 1) - |x|^2 \log |x|} \right)^2 \\
\leq \sum_{|x|=3} \sum_{|y|=2} \frac{\omega_{xy}}{3(\log 3)^2} + \omega_{\text{max}} \sum_{|x| \geq 3} \sum_{|y|=|x|+1} \frac{1}{(|x| + 1)(\log (|x| + 1))^2 + |x|^2 \log |x|} \\
\leq \sum_{|x|=3} \sum_{|y|=2} \frac{\omega_{xy}}{3(\log 3)^2} + \omega_{\text{max}} \sum_{|x| \geq 3} \sum_{|y|=|x|+1} \frac{2}{|x|^2 \log |x|^2} \\
< \infty.
\]

In what follows, we prove that \( \int_V u^2 \log u^2 d\mu = -\infty \). By direct calculations, we have

\[
\int_V u^2 \log u^2 d\mu = \sum_{x \in V} \mu(x) u^2(x) \log u^2(x) \\
= \sum_{|x| \geq 3} \mu(x) u^2(x) \log u^2(x) \\
= -\left( \sum_{|x| \geq 3} \frac{\mu(x)}{|x| \log |x|} + \sum_{|x| \geq 3} \frac{2\mu(x) \log (\log |x|)}{|x|^2 (\log |x|)^2} \right) \\
= -\infty.
\]

References

1. Bianchi, D., Setti, A.G., Wojciechowski, R.K.: The generalized porous medium equation on graphs: existence and uniqueness of solutions with \( L^1 \) data. Calc. Var. Partial Differ. Equ. 61(5), 42 (2022)
2. Huang, X.P.: On uniqueness class for a heat equation on graphs. J. Math. Anal. Appl. 393, 377–388 (2012)
3. Lin, Y., Wu, Y.T.: The existence and nonexistence of global solutions for a semilinear heat equation on graphs. Calc. Var. Partial Differ. Equ. 56(4), 22 (2017)
4. Huang, A., Lin, Y., Yau, S.-T.: Existence of solutions to mean field equations on graphs. Comm. Math. Phys. 377(1), 613–621 (2020)
5. Hou, S.B., Sun, J.M.: Existence of solutions to Chern–Simons–Higgs equations on graphs. Calc. Var. Partial Differ. Equ. 61(4), 13 (2022)

6. Ge, H.B., Jiang, W.F.: Kazdan–Warner equation on infinite graphs. J. Korean Math. Soc. 55(5), 1091–1101 (2018)

7. Grigor’yan, A., Lin, Y., Yang, Y.Y.: Kazdan–Warner equation on graph. Calc. Var. Partial Differ. Equ. 55(4), 13 (2016)

8. Ge, H.B.: A p-th Yamabe equation on graph. Proc. Amer. Math. Soc. 146(5), 2219–2224 (2018)

9. Ge, H.B., Jiang, W.F.: Yamabe equations on infinite graphs. J. Math. Anal. Appl. 460(2), 885–890 (2018)

10. Grigor’yan, A., Lin, Y., Yang, Y.Y.: Yamabe type equations on graphs. J. Differ. Equ. 261(9), 4924–4943 (2016)

11. Grigor’yan, A., Lin, Y., Yang, Y.Y.: Existence of positive solutions for nonlinear equations on graphs. Sci. China Math. 60(7), 1311–1324 (2017)

12. Zhang, N., Zhao, L.: Convergence of ground state solutions for nonlinear Schrödinger equations on graphs. Sci. China Math. 61(8), 1481–1494 (2018)

13. Han, X.L., Shao, M.Q., Zhao, L.: Existence and convergence of solutions for nonlinear biharmonic equations on graphs. J. Differ. Equ. 268, 3936–3961 (2020)

14. Lin, Y., Yang, Y.: Calculus of variations on locally finite graphs. Rev. Mat. Complut. 35, 791–813 (2022)

15. Xu, J.Y., Zhao, L.: Existence and convergence of solutions for nonlinear elliptic systems on graphs. Commun. Math. Stat. (2023). https://doi.org/10.1007/s40304-022-00318-2

16. Ambrosetti, A., Badiale, M., Cingolani, S.: Semiclassical states of nonlinear Schrödinger equations. Arch. Ration. Mech. Anal. 140(3), 285–300 (1997)

17. Ambrosetti, A., Malchiodi, A., Perturbation methods and semilinear elliptic problems on \( \mathbb{R}^N \). Progress in Mathematics. Birkhäuser Verlag, Basel (2006)

18. Ambrosetti, A., Malchiodi, A., Felli, V.: Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity. J. Eur. Math. Soc. 7(1), 117–144 (2005)

19. Bartsch, T., Pankov, A., Wang, Z.-Q.: Nonlinear Schrödinger equations with steep potential well. Commun. Contemp. Math. 3(4), 549–569 (2001)

20. Bartsch, T., Wang, Z.-Q.: Existence and multiplicity results for some superlinear elliptic problems on \( \mathbb{R}^N \). Comm. Partial Differ. Equ. 20(9–10), 1725–1741 (1995)

21. Cerami, G., Passaseo, D., Solimini, S.: Infinitely many positive solutions to some scalar field equations with nonsymmetric coefficients. Comm. Pure Appl. Math. 66(3), 372–413 (2013)

22. Li, Y., Wang, Z.Q., Zeng, J.: Ground states of nonlinear Schrödinger equations with potentials. Ann. Inst. Poincaré Anal. Non Lineaire. 23(6), 829–837 (2006)

23. Rabinowitz, P.H.: On a class of nonlinear Schrödinger equations. Z. Angew. Math. Phys. 43(2), 270–291 (1992)

24. Willem, M.: Minimax Theorems. Birkhäuser Verlag, Boston (1996)

25. Carles, R., Gallagher, I.: Universal dynamics for the defocusing logarithmic Schrödinger equation. Duke Math. J. 167(9), 1761–1801 (2018)

26. Cazenave, T.: Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics Vol. 10 (New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society) (2003)

27. Zloshchastiev, K.G.: Logarithmic nonlinearity in the theories of quantum gravity: origin of time and observational vanishing at infinity. J. Phys. A 43, 185210 (2010)

28. Cazenave, T.: Stable solutions of the logarithmic Schrödinger equation. Nonlinear Anal. 7(10), 1127–1140 (1983)

29. Squassina, M., Szulkin, A.: Multiple solutions to logarithmic Schrödinger equations with periodic potential. Calc. Var. Partial Differ. Equ. 54(1), 585–597 (2015)

30. d'Avenia, P., Montefusco, E., Squassina, M.: On the logarithmic Schrödinger equation. Commun. Math. Phys. 316(2), 15 (2014)

31. d'Avenia, P., Squassina, M., Zenari, M.: Fractional logarithmic Schrödinger equations. Math. Methods Appl. Sci. 38(18), 5207–5216 (2015)

32. Ji, C., Szulkin, A.: A logarithmic Schrödinger equation with asymptotic conditions on the potential. J. Math. Anal. Appl. 437(1), 241–254 (2016)

33. Tanaka, K., Zhang, C.X.: Multi-bump solutions for logarithmic Schrödinger equations. Calc. Var. Partial Differ. Equ. 56(2), 33–35 (2017)
34. Guerrero, P., López, J.L., Nieto, J.: Global $H^1$ solvability of the 3D logarithmic Schrödinger equation. Nonlinear Anal. Real World Appl. 11(1), 79–87 (2010)
35. Wang, Z.-Q., Zhang, C.X.: Convergence from power-law to logarithmic-law in nonlinear scalar field equations. Arch. Ration. Mech. Anal. 231(1), 45–61 (2019)
36. Shuai, W.: Multiple solutions for logarithmic Schrödinger equations. Nonlinearity 32(6), 2201–2225 (2019)
37. Alves, C.O., Ji, C.: Multiple positive solutions for a Schrödinger logarithmic equation. Discrete Contin. Dyn. Syst. 40, 2671–2685 (2020)
38. Alves, C.O., Ji, C.: Existence of a positive solution for a logarithmic Schrödinger equation with saddle-like potential. Manuscripta Math. 164, 555–575 (2021)
39. Alves, C.O., Ji, C.: Multi-peak positive solutions for a logarithmic Schrödinger equation via variational methods, Israel J. Math. to appear (2023)
40. Alves, C.O., Ji, C.: Multi-bump positive solutions for a logarithmic Schrödinger equation with deepening potential well. Sci. China Math. 65(8), 1577–1598 (2022)
41. Alves, C.O., Ji, C.: Existence and concentration of positive solutions for a logarithmic Schrödinger equation via penalization method. Calc. Var. Partial Differ. Equ. 59(1), 21–27 (2020)
42. Alves, C.O., Moussaoui, A., Tavares, L.: An elliptic system with logarithmic nonlinearity. Adv. Nonlinear Anal. 8(1), 928–945 (2019)
43. Carles, R.: Logarithmic Schrödinger equation and isothermal fluids. EMS Surv. Math. Sci. 9(1), 99–134 (2022)
44. Cazenave, T., Lions, P.L.: Orbital stability of standing waves for some nonlinear Schrödinger equations. Commun. Math. Phys. 85(4), 549–561 (1982)
45. Ikoma, N., Tanaka, K., Wang, Z.-Q., Zhang, C.X.: Semi-classical states for logarithmic Schrödinger equations. Nonlinearity 34(4), 1900–1942 (2021)
46. Shuai, W.: Existence and multiplicity of solutions for logarithmic Schrödinger equations with potential. J. Math. Phys. 62(5), 22 (2021)
47. Zhang, C.X., Zhang, X.: Bound states for logarithmic Schrödinger equations with potentials unbounded below. Calc. Var. Partial Differ. Equ. 59(1), 31 (2020)
48. Byeon, J., Wang, Z.-Q.: Standing waves with a critical frequency for nonlinear Schrödinger equations. Arch. Ration. Mech. Anal. 165(4), 295–316 (2002)
49. Byeon, J., Wang, Z.-Q.: Standing waves with a critical frequency for nonlinear Schrödinger equations. II. Calc. Var. Partial Differ. Equ. 18(2), 207–219 (2003)
50. Sirakov, B.: Standing wave solutions of the nonlinear Schrödinger equation in $\mathbb{R}^n$. Ann. Mat. Pura Appl. 181(1), 73–83 (2002)
51. Ding, Y., Szulkin, A.: Bound states for semilinear Schrödinger equations with sign-changing potential. Calc. Var. Partial Differ. Equ. 29(3), 397–419 (2007)
52. Lin, Y., Liu, S., Song, H.Y.: Log-Sobolev inequalities on graphs with positive curvature. Mat. Fiz. Komp’yut. Model. 3, 99–110 (2017)
53. Schechter, M.: A variation of the mountain pass lemma and applications. J. London Math. Soc. 44(3), 491–502 (1991)

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