A Modified Wigner's Inequality for Secure Quantum Key Distribution

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(Dated: March 26, 2018)

In this report we discuss the insecurity with present implementations of the Ekert protocol for quantum-key distribution based on the Wigner Inequality. We propose a modified version of this inequality which guarantees safe quantum-key distribution.

PACS numbers: 03.67.Dd, 03.67.-a, 03.65.Ud

Following the first proposal by Bennett and Brassard [1] and the later introduction of the Ekert protocol invoking entangled states [2], various systems of quantum key distribution (QKD) have been implemented and tested by groups around the world, turning QKD into the most advanced application in quantum information science.

QKD offers the possibility that two remote parties, conventionally called Alice and Bob, exchange a secret random key to implement a secure encryption-decryption algorithm, without meeting [1, 2, 3]. QKD provides a significant advantage over the public-key cryptography because the security of the distributed key relies on the laws of quantum physics [1, 2, 3], i.e. the wave packet collapse prohibits gaining information from a quantum channel without disturbing it. Indeed, any attempt by a third party (Eve) to obtain information about the key is detected.

Two main goals underly the implementation of QKD schemes. One is to create and preserve authentic quantum channels against decoherence effects induced by any interaction with the environment [4], eventually reducing or even destroying invulnerability of quantum channels against Eve’s attack. The other is to provide a true guarantee of absolute security against any possible eavesdropping attack i.e. the security is not simply based on technological feasibility.

Scientists are currently using one of two photon sources for practical QKD, faint lasers [5, 6, 7, 8, 9, 10] or spontaneous parametric down-conversion to generate entangled photon states (SPDC) [11, 12, 13, 14, 15].

Both schemes have disadvantages, although Brassard et al. [16] proved theoretically that QKD schemes based on entangled photons offer enhanced performance and security relative to schemes based on weak coherent pulses.

In fact, entangled states allow a further test of security based on the completeness of quantum mechanics; in other words, the Ekert protocol exploits either the Clauser-Horne-Shimony-Holt (CHSH) [2] or the Wigner inequality [13] as an ultimate test of eavesdropping attack.

Wigner’s inequality was originally intended [13] to provide an easier but as equally reliable eavesdropping check as the CHSH test in implementing the Ekert protocol. In this report we disclose the weakness of the Wigner inequality as a security test when Eve has the total control of quantum channels.

In Fig. 1 we depict a typical scheme for QKD by means of entangled photons. QKD performed by pure entangled states relies on the realization of two quantum-correlated optical channels yielding quasi single-photon polarization states. Alice’s measurement of the polarization of a photon of the pair, automatically projects Bob’s photon into a defined polarization state, and reversibility. Alice’s and Bob’s detection apparatures are built using polar-
ization analyzer systems (PAS), polarizing beam splitters (PBS), detectors ($+A, -A, +B, -B$), data storage systems (computers), and synchronization systems.

Consider, as an example, type II parametric down-conversion entangled states $|\psi^-angle$, where the output two-photon states are a quantum superposition of orthogonally polarized photons, i.e. in the singlet state

$$|\psi^-angle = \frac{1}{\sqrt{2}} (|H_A\rangle |V_B\rangle - |V_A\rangle |H_B\rangle).$$

Without any loss of generality, a particular QKD procedure can be described by the transformations, $\hat{T}_z$, with $z = A, B$, on the polarization state of a photon, e.g. a rotation of angles $\alpha_A$ and $\alpha_B$ respectively according to

$$\hat{T}_z |H_z\rangle = \cos \alpha_z |H_z\rangle + \sin \alpha_z |V_z\rangle,$$
$$\hat{T}_z |V_z\rangle = \sin \alpha_z |H_z\rangle - \cos \alpha_z |V_z\rangle.$$

We assume ideal PBS in channel $z$ transmitting photons in the state $|H_z\rangle$ towards detectors $+z$ and reflecting photons in the state $|V_z\rangle$ towards detector $-z$.

The joint probabilities of each possible pair of detectors firing due to the arrival of a photon pair with PAS angles of $\alpha_A$ and $\alpha_B$ are given by

$$p_{\alpha_A}\alpha_B (+A, +B) = \left| \langle H_B | \langle H_A | \hat{T}_A \hat{T}_B | \psi^-angle \right|^2$$
$$= \frac{1}{2} \sin^2 (\alpha_A - \alpha_B),$$
$$p_{\alpha_A}\alpha_B (+A, -B) = \left| \langle V_B | \langle H_A | \hat{T}_A \hat{T}_B | \psi^-angle \right|^2$$
$$= \frac{1}{2} \cos^2 (\alpha_A - \alpha_B),$$
$$p_{\alpha_A}\alpha_B (-A, +B) = \left| \langle H_B | \langle V_A | \hat{T}_A \hat{T}_B | \psi^-angle \right|^2$$
$$= \frac{1}{2} \cos^2 (\alpha_A - \alpha_B),$$
$$p_{\alpha_A}\alpha_B (-A, -B) = \left| \langle V_B | \langle V_A | \hat{T}_A \hat{T}_B | \psi^-angle \right|^2$$
$$= \frac{1}{2} \sin^2 (\alpha_A - \alpha_B).$$

Here, we consider the variant of the Ekert protocol based on the Wigner inequality originally proposed in ref. [13]. For the analyzer settings the possible combined choices by Alice and Bob can be split into two groups: the first for key distribution and the second testing security. We assume Alice and Bob measure randomly among four combined analyzer settings. Alice’s possible choices are $\alpha_A = (\alpha_{A,1} = -\pi/6, \alpha_{A,2} = 0)$ and Bob’s possible choices are $\alpha_B = (\alpha_{B,2} = 0, \alpha_{B,3} = \pi/6)$. The key is obtained from the subset of measurements corresponding to parallel PAS settings (i.e. $\alpha_{A,i} = \alpha_{B,i} = 0$) and the firing detectors $+A$ and $-A$.

The Wigner inequality is based on the Wigner parameter

$$W = p_{\alpha_{A,1}\alpha_{B,2} (+A, +B)} + p_{\alpha_{A,2}\alpha_{B,3} (+A, +B)}$$
$$- p_{\alpha_{A,1}\alpha_{B,3} (+A, +B)}. \tag{1}$$

Note that for the maximally entangled states $W = -1/8$ while for any local realistic theory $W \geq 0$.

To obtain the Wigner inequality $W \geq 0$ it is necessary to summarize the Wigner argument [13]. Two assumptions are stipulated in the proofs of the Wigner inequality: locality and realism. Locality means that Alice’s measurement results do not influence Bob’s results, and vice versa. Realism means that, given any physical property its value exists independently of its observation or measurement.

In the present case this is translated in terms of a classical probability distribution, $P(x_1, x_2; y_2, y_3)$, where $x_1$ and $x_2$ are the hidden variables associated with the physical property inducing Alice’s measurement outcome in the presence of the PAS rotation angle $\alpha_{A,i}$. Similarly $y_2$ and $y_3$ correspond to the physical property inducing Bob’s outcomes when PAS angle is set as $\alpha_{B,2}$ and $\alpha_{B,3}$. Thus, we can identify the possible values of $x_1, x_2$ and $y_2, y_3$ with Alice and Bob’s measurement outcomes, in other words $x_{1,2} = +A, -A$ and $y_{2,3} = +B, -B$. Following the Wigner approach we write,

$$p_{\alpha_{A,1}\alpha_{B,2} (+A, +B)} = \sum_{x_2, y_2} P(+A, x_2; y_2, +B)$$
$$= P(+A, +A; +B, +B) + P(+A, -A; -B, +B)$$
$$+ P(+A, +A; -B, +B) + P(+A, -A; +B, +B). \tag{2}$$

In Wigner’s original paper [13], $\alpha_{A,2} = \alpha_{B,2}$ and $P(x_1, +A; +B, y_3) = P(x_1, -A; -B, y_3) = 0$. The assumption of perfect anticorrelation is obviously reasonable in the test of realism and locality of a physical theory, because it reflects the classical counterpart of a quantum system prepared in the singlet state, i.e. $p_{\alpha_{A,2}\alpha_{B,2} (+A, +B)} = p_{\alpha_{A,2}\alpha_{B,2} (-A, -B)} = 0$. Thus, the inequality $W \geq 0$ is obtained from Eq. 2 by simply observing that $P(+A, -A; +B, +B) < \sum_{x_2, y_2} P(+A, x_2; +B, y_2)$ and $P(+A, -A; -B, +B) < \sum_{x_1, y_2} P(x_1, +A; y_2, +B) = p_{\alpha_{A,2}\alpha_{B,2} (+A, +B)}$.

Refs. [13, 19] suggest to be cautious in applying Wigner’s inequality to test the security of cryptography schemes, since authors are aware that the Wigner’s inequality is derived assuming perfect anticorrelations, which are only approximately realized in practical situations. Nevertheless a punctual quantification of the insecurity induced by this assumption in the Ekert protocol based on Wigner’s inequality has never been pointed out.

In particular, after stating that the violation of the Wigner’s inequality ascertains the security of the quantum channels, Ref. [13] suggests to replace the Wigner’s inequality by the generalization of the Bell’s inequality presented in Ref. [20] for a substantial deviation from perfect anticorrelations. This last point has been overlooked by most readers probably because it is not clear what a ”substantial deviation from the perfect anticorrelations” is.

When the eavesdropper, Eve, measures photons on either one or both of Alice’s and Bob’s channels, her pres-
ence might be expected to be revealed by a higher value of $W$ than the local realistic theory limit, as it happens for the CHSH inequality \[2\]. Unfortunately this is not the case. Even if it can be proved that, for Eve adopting intercept-resend strategy and detecting only one photon of the pair, the limit becomes $W_{\text{eve}} \geq 1/16$, this is not for eavesdropping on both channels, because in this case there is no bound.

A proof of this last assertion, according to ref. \[2\], can be obtained by considering Eve preparing each particle of the pairs separately, so that each individual particle has a well defined polarization direction, possibly varying from pair to pair. We write the probability $P(\Phi_A, \Phi_B)$ of having Alice’s particle in state $|\Phi_A\rangle$, and Bob’s particle in state $|\Phi_B\rangle$, where $\Phi_A$ and $\Phi_B$ are two angles of Eve’s polarization preparation i.e. $|\Phi_2\rangle = \cos \Phi_2 |H_2\rangle + \sin \Phi_2 |V_2\rangle$. If the density operator associated with Eve’s pairs of photons is $\rho_{\text{eve}} = \int P(\Phi_A, \Phi_B) |\Phi_A\rangle \langle \Phi_B| \langle \Phi_A|d\Phi_Ad\Phi_B$, then

$$W_{\text{eve}} = \int P(\Phi_A, \Phi_B) \left[ \frac{\cos^2(\Phi_A + \frac{\pi}{6}) \cos^2 \Phi_B + \cos^2 \Phi_A \cos^2(\Phi_B - \frac{\pi}{6}) - \cos^2(\Phi_A + \frac{\pi}{6}) \cos^2(\Phi_B - \frac{\pi}{6})}{\cos^2(\Phi_A + \frac{\pi}{6}) \cos^2 \Phi_B} \right] d\Phi_A d\Phi_B. \quad (3)$$

In the case of single-channel eavesdropping with intercept-resend strategy, the anticorrelation between the two photons in the Eve measurement base is preserved because $\Phi_B = \Phi_A - \pi/2$, and thus $W_{\text{ eve}} \geq 1/16$. In the most general case, when Eve has the total control over the state of individual particles, there is no physical bound, as $W_{\text{ eve}}$ results in some cases below the limit of local realistic theory, i.e. $W_{\text{ eve}} < 0$ and surprisingly also below the quantum limit $W_{\text{ eve}} < -1/8$. For instance, if Eve gains total control of the source of photons, she can send photons in proper polarization states to avoid disclosure by a security test by Alice and Bob, and for $P(\Phi_A, \Phi_B) = \delta(\Phi_A - 0.6\pi)\delta(\Phi_B - 0.4\pi)$ the value $W_{\text{ eve}} = -0.1995$ is below the quantum limit.

In other words, the guarantee of Wigner’s security test against eavesdropping strategies is limited only to the detection of one photon of the pairs while the CHSH security is independent from the adopted strategy \[2\].

Therefore, we end up with a modified Wigner’s parameter $\tilde{W}$ suitable for QKD security test by removing from Wigner’s original argument the anticorrelation assumption when $\alpha_{A,2} = \alpha_{B,2}$. Starting again from Eq. \[2\] but rejecting the assumption $p_{\alpha_{A,2},\alpha_{B,2}}(+A, +B) = p_{\alpha_{A,2},\alpha_{B,2}}(-A, -B) = 0$ and observing that $P(+A, +A; +B, +B) + P(+A, -A; +B, +B) \leq p_{\alpha_{A,1},\alpha_{B,2}}(+A, +B)$, $P(+A, +A; -B, +B) \leq p_{\alpha_{A,2},\alpha_{B,3}}(+A, +B)$ and $P(+A, -A; -B, +B) \leq p_{\alpha_{A,2},\alpha_{B,2}}(-A, -B)$, we obtain

$$\tilde{W} = p_{\alpha_{A,1},\alpha_{B,2}}(+A, +B) + p_{\alpha_{A,2},\alpha_{B,3}}(+A, +B) + p_{\alpha_{A,2},\alpha_{B,2}}(-A, -B) - p_{\alpha_{A,1},\alpha_{B,2}}(+A, +B) \geq 0. \quad (4)$$

for any local realistic theory.

We finally observe that for the singlet state we still obtain $\tilde{W} = -1/8$ also in this case. This result happens because the modified parameter $\tilde{W}$ equals the original $W$ except for the additional term $p_{\alpha_{A,2},\alpha_{B,2}}(-A, -B)$, which is zero in the case of singlet state.

To demonstrate the robustness of this security test against an eavesdropping attack we perform a calculation analogous to Eq. \[2\].

$$\tilde{W}_{\text{ eve}} = \int P(\Phi_A, \Phi_B) \left[ \frac{\cos^2(\Phi_A + \frac{\pi}{6}) \cos^2 \Phi_B + \cos^2 \Phi_A \cos^2(\Phi_B - \frac{\pi}{6}) + \sin^2(\Phi_A) \sin^2(\Phi_B) - \cos^2(\Phi_A + \frac{\pi}{6}) \cos^2(\Phi_B - \frac{\pi}{6})}{\cos^2(\Phi_A + \frac{\pi}{6}) \cos^2 \Phi_B} \right] d\Phi_A d\Phi_B. \quad (5)$$

Even in the general case when Eve has total control of the polarization states of photons in the two channels, we obtain that the minimum of $\tilde{W}_{\text{ eve}} = 0.04428$, well above the limit for local realistic theory. This result is completely equivalent to the one obtained with the security test based on the CHSH inequality \[2\].

Some further analysis of $\tilde{W}$ must be considered for the practical implementation of the Ekert protocol based on Wigner’s inequality. According to \[8\], we highlight that the Ekert’s protocol based on modified Wigner’s inequality still guarantees a simplification with respect to the one based on the CHSH inequality, because Alice and Bob randomly choose between two bases rather than three. Though the necessity of an experimental evaluation of the term $p_{\alpha_{A,2},\alpha_{B,2}}(-A, -B)$ forces Alice and Bob to sacrifice part of the key for the sake of security. We
note that in any practical implementation of QKD protocols, Alice and Bob distill from the noisy sifted key a nearly noise-free corrected key by means of error correction procedures subjected to the constraint of knowing the quantum bit error rate (QBER). Also, the QBER is estimated at the cost of losing part of the key. Thus, we suggest using the same sacrificed part of the key to estimate both $W$ and QBER.

To compare the performances of the Ekert protocol based on Wigner’s inequality versus the one CHSH-based we consider a protocol (Table I) where Alice and Bob measure randomly using three analyzer settings (as in the case of CHSH), where $\alpha_{A,1} = \alpha_{B,1} = -\pi/6$, $\alpha_{A,2} = \alpha_{B,2} = 0$ and $\alpha_{A,3} = \alpha_{B,3} = \pi/6$. $\tilde{W}$ and $\tilde{W}'$ correspond to two distinct test parameters and $K$ to the key distribution. This protocol is more efficient than the protocol based on CHSH. In terms of key generation we observe that in the case of CHSH only 2/9 of the qubits exchanged are devoted to the key generation while in the case of our protocol this quantity is a number between 2/9 and 1/3, depending on the security needs.

Even if, for some strict security request, all the qubits exchanged by the two parties with analyzer settings $\alpha_{A,2}$, $\alpha_{B,2}$ are devoted to the evaluation of $\tilde{W}$ ($\tilde{W}'$), and also for this protocol only 2/9 of the data still contribute to the key, in this protocol none of the qubits exchanged are discarded (Table I) while in the case of CHSH 1/3 of the qubits are discarded.

The results so far obtained allow us to quantify the so-called “substantial deviation from perfect anticorrelations” as that value of the QBER such that the quantum key distribution fails, because the key is insecure even if the Wigner’s inequality is violated.

For the original protocol introduced in the QBER of the key is the probability of correlated results when analyzer settings are $\alpha_{A,2}, \alpha_{B,2}$, i.e.

$$\text{QBER} = p_{\alpha_{A,2}, \alpha_{B,2}}(-A, -B) + p_{\alpha_{A,2}, \alpha_{B,2}}(+A, +B).$$

Thus, this leads to

$$0 \leq \tilde{W} \leq W + p_{\alpha_{A,2}, \alpha_{B,2}}(-A, -B) + p_{\alpha_{A,2}, \alpha_{B,2}}(-A, -B),$$

which can be easily rewritten in terms of QBER as

$$\text{QBER} + W \geq 0.$$  \hspace{1cm} (5)

Thanks to the modified Wigner’s inequality a limit is found for both $W$ and the anticorrelation check. Thus, for the Ekert’s protocol based on the original Wigner’s inequality a secure key distribution is provided when, instead of the violation of the Wigner’s inequality ($W \geq 0$), the Wigner’s parameter $W$ satisfies $W \leq -\text{QBER}$. Eq. 5 and the related considerations obviously still hold for the enlarged protocol presented in Table I. Here one has to consider only the QBER of that part of the key obtained with analyzers settings $\alpha_{A,2}, \alpha_{B,2}$, and not the QBER of the whole key.

As a final remark we highlight that our approach not only solves the problem of estimating the “substantial deviation from the perfect anticorrelations” but completely overcomes it. Following our approach the experimentalist has simply to check the modified Wigner’s inequality from his experimental data to guarantee non-locality, i.e. security of the quantum channels as in the original Ekert’s idea without any further anticorrelation check.

Furthermore, the experimentalist in a noisy environment can discard a secure key when Eq. 5 is used instead of the modified Wigner’s inequality, since $\tilde{W} \leq W + \text{QBER}$.

In conclusion, this paper discusses the security of Ekert’s protocol based on the Wigner inequality. We showed that the QKD Ekert protocol based on Wigner’s inequality presents a serious lack of security against some eavesdropping strategies other than intercept-resend. We emphasized the motivation beneath the missing security and propose a modified test ultimately guaranteeing secure QKD.

We would like to thank M. Rasetti and I. Ru Berchera, G. Di Giuseppe, A. M Colla, F. Bovino, P. Varisco for useful discussion and helpful suggestions. This work was developed in collaboration with Elsag S.p.A., Genova (Italy), within a project entitled ”Quantum Cryptographic Key Distribution” co-funded by the Italian Ministry of Education, University and Research (MIUR) - grant n. 67679/ L. 488. In addition S. C. acknowledges the partial support of the DARPA QuIS program and M. L. R. acknowledges the partial support by INFM.

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\[\begin{array}{|c|c|c|c|}
\hline
\alpha_{A,1} & \alpha_{A,2} & \alpha_{A,3} \\
\hline \hline
\alpha_{B,1} & K & \tilde{W}' & \tilde{W}' \\
\hline \hline
\alpha_{B,2} & \tilde{W} & K & \tilde{W}' \\
\hline \hline
\alpha_{B,3} & \tilde{W} & \tilde{W} & K \\
\hline
\end{array}\]

Table I: Distribution of data for Alice and Bob’s analyzer settings $\alpha_{A,1}, \alpha_{A,2}$ and $\alpha_{B,2}, \alpha_{B,3}$ respectively. The dotted square is the original protocol of T. Jennewein, C. Simon, G. Weihs, H. Weinfurter and A. Zeilinger, Phys. Rev Lett. 84, 4729 (2000).
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