Accepted for publication in the Annals of Mathematics

AX-LINDEMANN FOR $A_g$

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Abstract. We prove the Ax-Lindemann theorem for the coarse moduli space $A_g$ of principally polarized abelian varieties of dimension $g \geq 1$. We affirm the André-Oort conjecture unconditionally for $A_g$ for $g \leq 6$, and under GRH for all $g$.

1. INTRODUCTION

In this paper we prove the “Ax-Lindemann” theorem for $A_g = A_{g,1}, g \geq 1$, the moduli space of principally polarized Abelian varieties of dimension $g$. The statement of the theorem is as follows (for definitions and conventions see §2). Let $\mathbb{H}_g$ be the Siegel upper-half space and $\pi_g : \mathbb{H}_g \to A_g$ the $\text{Sp}_{2g}(\mathbb{Z})$-invariant uniformisation.

Theorem 1.1. Let $V \subset A_g$ be a subvariety and $W \subset \pi_g^{-1}(V)$ a maximal algebraic subvariety. Then $W$ is weakly special.

As explained in [31], this theorem may be viewed as an analogue for the map $\pi_g$ of part of Ax’s theorem [4] establishing the differential field version of Schanuel’s conjecture for the exponential function (see [26], p30), namely the part which corresponds to the classical Lindemann (or Lindemann-Weierstrass) theorem (see [26], p77). Hence the neologism “Ax-Lindemann”. The proof of Theorem 1.1 combines various arithmetic estimates with the Counting Theorem of Pila-Wilkie [33] and with the idea of Ullmo-Yafaev [46] to use hyperbolic volume at the boundary.

The André-Oort conjecture (AO) is a compositum of conjectures made by André [1] and Oort [29]. A full proof of AO under the assumption of GRH for CM fields has been announced by Klingler, Ullmo, and Yafaev [22, 43]. An appropriate Ax-Lindemann theorem is a key ingredient in proving cases of AO unconditionally using o-minimality and point-counting [9, 31, 32, 42], following the basic strategy originally proposed by Zannier for re-proving the Manin-Mumford conjecture [34]. It provides a geometric characterisation of the exceptional set in the Counting Theorem, and in this role is analogous to functional transcendence statements in the context of other Zilber-Pink type problems (e.g. [27, 20]). The following theorem affirms AO unconditionally for $A_g$ for $g \leq 6$, and for all $g$ under the assumption of GRH.

Theorem 1.2. Let $V \subset A_g, g \leq 6$ be a subvariety. Then $V$ contains only finitely many maximal special subvarieties. Under the assumption of GRH (for CM fields) the same conclusion holds for all $g$. 

1
Both theorems rely on the definability in the o-minimal structure $\mathbb{R}_{\text{an, exp}}$ of the map $\pi_g : \mathbb{H}_g \to \mathcal{A}_g$ when restricted to a standard fundamental domain for the $\text{Sp}_{2g}(\mathbb{Z})$ action. This result, stated in §2.7, is due to Peterzil-Starchenko [30]. The o-minimality of $\mathbb{R}_{\text{an, exp}}$ is due to van den Dries and Miller [14], building on the fundamental work of Wilkie [47]; for further references on o-minimality see §2.

The restriction to $g \leq 6$ for the unconditional statement in 1.2 is due to another ingredient which is crucial to the strategy: a suitable lower bound for the size of the Galois orbit of a special point. These have been established by the second author [41] unconditionally for $g \leq 6$, and for all $g$ on GRH (the former for $g \leq 3$ and the latter were shown independently by Ullmo-Yafaev [45]). However, we show that such bounds are the only remaining obstacle to proving AO for $\mathcal{A}_g$ in general. This gives a new proof of AO for $\mathcal{A}_g$ for all $g$ assuming GRH, by different methods to the ones employed by Klingler-Ullmo-Yafaev [22, 43].

In the course of preparation of this manuscript, the preprint [42] by Ullmo appeared showing how to deduce AO from these ingredients (Ax-Lindemann, height upper bound for preimages of special points, Galois lower bounds, and uniformisation with suitable definability) for any Shimura variety. Ullmo thereby proves (in [42]) AO unconditionally for all projective Shimura subvarieties of $\mathcal{A}_g^n$, $n \geq 1$, using the Ax-Lindemann theorem for projective Shimura varieties established by Ullmo-Yafaev [46], the height upper bounds in [32], and the Galois lower bounds of [41], as the definability requirements are easily seen to be satisfied (in $\mathbb{R}_{\text{an}}$) in the case of a projective Shimura variety. The deductions of AO in [42] and here differ in detail but both depend on the structure of weakly special subvarieties of $\mathcal{A}_g$ and make further crucial use of o-minimality (as in [31, 32]). We have retained our treatment in order to keep our paper self-contained.

We begin in §2 by reviewing $\mathbb{H}_g$ and our basic definitions and conventions. Then we prove some norm estimates in §3, and estimates about volumes of curves in fundamental domains (§4) and near the boundary of $\mathbb{H}_g$ (§5). With these preparations we prove 1.1 in §6. The deduction of AO from 1.1 and the various other “ingredients” is carried out in §7.

Acknowledgements. We thank Martin Orr, Kobi Peterzil, and Sergei Starchenko for their comments. We are extremely grateful to the referees for their thorough reading of earlier versions of this paper and their detailed comments and corrections.

2. Basic conventions and notation

2.1. $\text{Sp}_{2g}(\mathbb{R})$ and $\mathbb{H}_g$. The symplectic group $\text{Sp}_{2g}(R)$ with entries in a ring $R$ is the group of matrices $T \in M_{2g}(R)$ satisfying

$$T J T^t = J$$
where \( J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \) is the standard alternating matrix of degree \( 2g \), and \( T^t \) is the transpose of \( T \); see e.g. [21]. If we write

\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in M_g(R)
\]

then the condition \( T J T^t = J \) is equivalent (see [19], p183) to

\[
AB^t = B A^t, \quad CD^t = D C^t, \quad AD^t - BC^t = I_g.
\]

We know (see [19], p184) that

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(R) \Rightarrow A^t C = C^t A, B^t D = D^t B,
\]

so that \( \text{Sp}_{2g}(R) \) is closed under transposition.

The Siegel upper half-space \( \mathbb{H}_g \) is defined to be

\[
\mathbb{H}_g = \{ Z \in M_g(\mathbb{C}) : Z = Z^t, \text{Im}(Z) > 0 \}.
\]

Thus \( \mathbb{H}_g \) is an open domain in the space \( M_g(\mathbb{C})^{\text{sym}} \) of symmetric complex \( g \times g \) matrices, which we may identify with \( \mathbb{C}^{g(g+1)/2} \). There is an action of \( \text{Sp}_{2g}(\mathbb{R}) \) on \( \mathbb{H}_g \) given by

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.
\]

Denote by \( A_g \) the coarse moduli space of complex principally polarized abelian varieties of dimension \( g \). The quotient \( \text{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g \) is isomorphic to \( A_g \). We write \( \pi_g : \mathbb{H}_g \to A_g \) for the projection map, which may be given explicitly by scalar Siegel modular forms of suitable weight; see [19], §10 and §11.

### 2.2. Varieties and subvarieties

We identify varieties with their sets of complex-valued points. By a subvariety \( V \) of a quasiprojective variety \( W \) we mean a Zariski closed subset. An irreducible algebraic subvariety of \( \mathbb{H}_g \), considered as a subset of \( \mathbb{C}^{g(g+1)/2} \), means an irreducible component (as a complex analytic variety) of \( \mathbb{H}_g \cap Y \) where \( Y \subset \mathbb{C}^{g(g+1)/2} \) is a subvariety. Likewise, if \( \mathbb{H} \) is any hermitian domain, then there is a natural algebraic variety \( \mathbb{H}_0 \) of which \( \mathbb{H} \) is an open subset (see [44]), and we define an irreducible algebraic subvariety of \( \mathbb{H} \) to be an irreducible component, as a complex analytic variety, of \( \mathbb{H} \cap Y \), where \( Y \) is an algebraic subvariety of \( \mathbb{H}_0 \). For \( A \subset \mathbb{H}_g \), a maximal algebraic subvariety of \( A \) is an irreducible algebraic subvariety \( W \subset \mathbb{H}_g \) with \( W \subset A \) such that if \( W' \subset \mathbb{H}_g \) is an irreducible algebraic subvariety with \( W \subset W' \subset A \) then \( W = W' \).

### 2.3. The metric

There is an invariant metric on \( \mathbb{H}_g \) for the action of \( \text{Sp}_{2g}(\mathbb{R}) \), and this is given (see e.g. [40], p17) by

\[
d\mu(Z) = \text{Tr}(Y^{-1}dZY^{-1}dZ).
\]
2.4. Shimura data and special subvarieties. We gather here some basic facts about Shimura varieties; for a more thorough account with proofs see [10] or [11]. We work exclusively with connected Shimura varieties, of which an excellent account can be found in [35].

Define the Deligne torus $\mathbb{S}$ be the real torus given by Weil restricting $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$. Thus, the real points $\mathbb{S}(\mathbb{R})$ can be identified with $\mathbb{C}^\times$. A connected Shimura datum is defined to be a pair $(G, X^+)$ where $G$ is a reductive group over $\mathbb{Q}$ and $X^+$ is a connected component of a $G(\mathbb{R})$-conjugacy class $X$ of homomorphisms $h : \mathbb{S} \to G(\mathbb{R})$ satisfying the following conditions:

- For all $h \in X$, only the weights $(0,0), (1,-1), (-1,1)$ may occur in the adjoint action of $\mathbb{S}$ on the complexified Lie algebra of $G$.
- $\text{ad} \circ h(i)$ is a Cartan involution on the adjoint group $G^{\text{ad}}$ and $G^{\text{ad}}$ has no simple $\mathbb{Q}$-factor $H$ such that $H(\mathbb{R})$ is compact.

These axioms ensure that if we set $K_\infty$ to be the stabilizer of some $h \in X$, then $X = G(\mathbb{R})/K_\infty$ is a finite union of hermitian symmetric domains on which $G(\mathbb{R})$ acts transitively and biholomorphically. We say that $(H, X_H)$ is a Shimura subdatum of $(G, X_G)$ if $H \subset G$ and $X_H \subset X_G$. A connected Shimura variety is defined to be $\Gamma \backslash X^+$, where $\Gamma \subset G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{R})$ that stabilises $X^+$ ([35]). Thus, to give a connected Shimura variety is the same as to give a triple $(G, X^+, \Gamma)$. A connected Shimura variety can be given the structure of an algebraic variety defined over a number field.

We denote by $G^{\text{ad}}$ the adjoint form of a reductive group $G$. Given a Shimura datum $(G, X_G)$ we get an associated Shimura datum $(G^{\text{ad}}, X_G^{\text{ad}})$ such that we can identify $X_G^+$ with $X_G^{\text{ad}}$.

A morphism of connected Shimura varieties

$$\phi : (H, X_H^+, \Gamma_H) \to (G, X_G^+, \Gamma_G)$$

is a morphism of $\mathbb{Q}$-groups $\phi : H \to G$ which carries $X_H^+$ to $X_G^+$ and $\Gamma_H$ to $\Gamma_G$. The image $\phi(\Gamma_H \backslash X_H^+)$ is an algebraic subvariety of $\Gamma_G \backslash X_G^+$.

**Definition.** A subvariety $V \subset \Gamma_G \backslash X_G^+$ is called a special subvariety if there exists a morphism of connected Shimura varieties $\phi : (H, X_H^+, \Gamma_H) \to (G, X_G^+, \Gamma_G)$ such that $\phi(\Gamma_H \backslash X_H^+) = V$. By abuse of notation, we also refer to $\phi(X_H^+)$ as a special subvariety of $X_G^+$. Keep in mind that $\phi(X_H^+)$ is an irreducible algebraic subvariety of $X_G^+$ under its semi-algebraic structure as a hermitian symmetric space. A special point is a special subvariety of dimension zero (in $\Gamma_G \backslash X_G^+$ or in $X_G^+$).

For our purposes, we shall need a slightly more general notion (see [41]).

**Definition.** We say that $V \subset \Gamma_G \backslash X_G^+$ is a weakly special subvariety if there exist Shimura varieties $(H_i, X_i^+, \Gamma_i)$ for $i = 1, 2$ and a Shimura subdatum $(H, X_H^+) \subset (G, X_G^+)$ such that

$$(H^{\text{ad}}, X_H^{\text{ad}}) = (H_1, X_1^+) \times (H_2, X_2^+)$$
and \( y \in \Gamma_2 \backslash X^+_2 \) such that \( V \) is the image of \( X^+_1 \times \{ y \} \) in \( \Gamma_G \backslash X^+_G \). By the same abuse of notation as above, we refer to the image of \( X^+_1 \times \{ y \} \) in \( X^+_G \) as a weakly special subvariety of \( X^+_G \).

A weakly special subvariety is special if and only if it contains a special point; see [28].

The following result of Ullmo-Yafaev is very useful in working with weakly special subvarieties:

**Lemma 2.1.** [41] An irreducible algebraic variety \( S \subset \mathbb{H}_g \) is weakly special iff \( \pi_g(S) \) is algebraic.

**Corollary 2.2.** An irreducible component of the intersection of weakly special subvarieties is weakly special.

**Lemma 2.3.** Let \((G, X^+_G)\) be a connected Shimura datum. Let \( V \) be a weakly special subvariety of \( X^+_G \). Then there exists a semi-simple subgroup \( G_0 \subset G \) defined over \( \mathbb{Q} \), with no compact \( \mathbb{Q} \)-factors, and a point \( Z_0 \in X^+_G \) such that the image of \( G_0, R \) in \( G_0^{ad} \) is fixed by the Cartan involution corresponding to \( Z_0 \) and \( V = G_0(\mathbb{R})^+ \cdot Z_0 \).

**Proof.** Since \( V \) is weakly special, there are connected Shimura data

\[
(H^{ad}, X^{ad}_{H^{ad}}) = (H_1, X^+_1) \times (H_2, X^+_2)
\]

where \((H, X_H)\) is a Shimura subdatum of \((G, X_G)\), and a point \( y \in X^+_2 \) such that \( V \) is the image of \( \phi(X^+_1 \times \{ y \}) \) in \( \Gamma_G \backslash X^+_G \).

Pick \( h \in X^+_1 \) and set \( Z_0 = (h, y) \), where as always we identify \( X^{ad}_{H^{ad}} \) with \( X^+_H \). Finally, take \( G_1 \subset H \) to be \( \phi \) composed with the pullback under the map \( H \to H^{ad} \) of \( H_1 \times \{ 1_{H_2} \} \). Then the Cartan involution corresponding to \( Z_0 \) is given by conjugation by a lift to \( H(\mathbb{R}) \) of \((h, y)(i)\). Thus, \( G_{1, R} \) is fixed under this involution. By definition, we have

\[
V = X^+_1 \times \{ y \} = G_1(\mathbb{R})^+ \cdot Z_0,
\]

as desired. Finally, we let \( G_0 \) be the derived subgroup of \( G_1 \). The fact that \( G_0^{ad} \) has no compact \( \mathbb{Q} \)-factors follows from the fact that \( H_1^{ad} \) has none. \( \square \)

Now, given an irreducible variety \( V \subset A_g \), let \( S(V) \) be the minimal weakly special subvariety containing \( V \). Let \( V^{sm} \) be the smooth locus of \( V \), and let \( U_0 \) be a connected component of \( \pi_g^{-1}(V^{sm}) \). Define the monodromy group \( \Gamma_V \subset \text{Sp}_{2g}(\mathbb{Z}) \) to be the stabilizer of \( U_0 \). Let \( G_0 \) be the identity component of the Zariski closure of \( \Gamma_V \).

**Lemma 2.4.** The group \( G_0 \) is semi-simple and \( \pi_g(G_0(\mathbb{R})^+ \cdot v) = S(V) \) for all \( v \in U_0 \).

**Proof.** The semi-simplicity would follow from Theorem 1 of [2], except for the fact that \( \text{Sp}_{2g}(\mathbb{Z}) \) does not act freely on \( \mathbb{H}_g \) and thus \( A_g \) does not possess a family of polarized Hodge structures. However, consider the full congruence group \( \Gamma(3) \subset \text{Sp}_{2g}(\mathbb{Z}) \). This subgroup does act freely on \( \mathbb{H}_g \), so that \( A_{g,3} := \)}
Γ(3)\H_g does have a family of polarized Hodge structures. Now, we have the finite map \(\phi_3 : A_{g,3} \to A_g\). Let \(V_3 \subset A_{g,3}\) be an irreducible component of \(\phi_3^* V\). We can apply Theorem 1 of \([2]\) to \(V_3\) and get that the identity component of the Zariski closure of \(\Gamma V_3\) is semi-simple. However, \(\Gamma V_3\) is a finite index normal subgroup of \(\Gamma V\) and thus their Zariski closures have the same identity component, proving the semi-simplicity of \(G_0\). The rest of the lemma follows from \([28]\), Lemmas 3.7 and 3.9.

2.5. Fundamental domains. We set \(S_g\) to be the usual Siegel fundamental domain for the action of \(\text{Sp}_{2g}(\mathbb{Z})\) on \(H_g\) (see \([30]\), or \([21]\), §3.3).

**Lemma 2.5.** Suppose \(X + iY \in S_g\). Then there exists a constant \(c_g > 0\) such that
(a) All the coefficients of \(X\) are bounded in absolute value by \(\frac{1}{2}\).
(b) \(|Y| \leq \prod_{i=1}^{g} y_{ii} \leq c_g |Y|\), where \(|Y|\) is the determinant of \(Y^t\).
(c) \(\sqrt{2} \leq y_{11} \leq y_{22} \leq y_{33} \cdots \leq y_{gg}\)
(d) \(\forall 1 \leq i \neq j \leq g, |y_{ij}| \leq \frac{1}{2} \min(y_{ii}, y_{jj})\).

**Proof.** Parts (a) and (d) are true by definition, while (c) follows from \([21]\), Lemma 15, p.195. Part (b) follows from Minkowski’s 2nd result on successive minima, together with \([23]\), Theorem 2. \(\square\)

2.6. Norms. We define the norm of a matrix \(Z = (z_{ij}) = X + iY \in \mathbb{H}_g\), where \(X, Y \in M_g(\mathbb{R})\), to be
\[h(Z) = \max(1, |z_{ij}|, |Y|^{-1}),\]
and we define the norm of a matrix \(M = (m_{ij}) \in M_n(\mathbb{R})\) to be
\[h(M) = \max(1, |m_{ij}|)\]

2.7. O-minimality and definability. In this paper definable will mean definable in the o-minimal structure \(\mathbb{R}_{\text{an exp}}\). See \([31]\) for an introduction o-minimality and \([14, 13]\) for the properties of \(\mathbb{R}_{\text{an exp}}\). An essential input to enable the o-minimal machinery to be applied is the following result due to Peterzil-Starchenko \([30]\):

**Theorem 2.6.** The projection map \(\pi_g : S_g \to A_g\) is definable.

3. Some norm bounds

In this section we prove some basic lemmas concerning the norms introduced in 2.6 above. We shall only care about asymptotic growth, and moreover we only wish to establish that certain quantities do not exhibit super-polynomial growth. Thus, we introduce some notation.

**Definition.** Let \(M\) be a set and let \(F, G\) be functions mapping \(M\) to \(\mathbb{R}_{>0}\). We say that \(F\) is polynomially bounded in \(G\) if there exist constants \(a, b > 0\) with \(F(m) \leq a \cdot G(m)^b\), and write \(F \prec G\). If \(F \ll G\) and \(G \ll F\) we write \(F \asymp G\).
Clearly, $\prec$ is transitive, whereas $\sim$ is an equivalence relation. It is also clear that $F < G \iff G^{-1} < F^{-1}$. We record some basic facts.

**Lemma 3.1.** If $Z \in \mathbb{H}_g$ and $M_1, M_2 \in \text{Sp}_{2g}(\mathbb{R})$ we have:

1. $h(M_1M_2) < h(M_1)h(M_2)$
2. $h(M_1) \sim h(M_1^{-1})$
3. $h(M_1Z) < h(M_1)h(Z)$

**Proof.** (1) Clear.

(2) By symmetry, it is enough to show $h(M_1^{-1}) < h(M_1)$. This is obvious because the minors of $M_1$ are polynomial in the entries of $M_1$, and $|M_1| = 1$.

(3) Write $M_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $M_1Z = (AZ + B)(CZ + D)^{-1}$. We first bound $|CZ + D|$ from below. Note the identity ([19], p184)

$$ (^*) \quad (CZ + D)^t(AZ + B) - (AZ + B)^t(CZ + D) = 2iY. $$

Let $f = |CZ + D|$. Then there exists a vector $\xi \in \mathbb{C}^g$ with $||\xi|| = 1$ and $||(CZ + D)\xi|| \sim f$, where $||\xi||$ is defined to be $\bar{\xi} \cdot \xi$.

Complete $\xi$ to a unitary basis $\xi = \xi_1, \xi_2, \ldots, \xi_g$. Then we have

$$ |Y| \leq \prod_{i=1}^g |\xi_i^tY\xi_i| < |\xi^tY\xi|h(Y) $$

so that

$$ |\xi^tY\xi|^{-1} < h(Y) \leq h(Z). $$

Finally, (*) gives

$$ |\xi^tY\xi| < f h(M_1)h(Z), $$

so that $f^{-1} < h(M_1)h(Z)$. It is now clear that all the entries of $M_1Z$ are polynomially bounded by $h(M_1)h(Z)$. It remains to show that if $M_1Z = X' + iY'$ then $|Y'|^{-1} < h(M_1)h(Z)$.

Again using (*) and the fact that $M_1Z$ is symmetric, we derive $(Y')^{-1} = (CZ + D)Y^{-1}(CZ + D)^t$. Thus all the coefficients of $(Y')^{-1}$ are polynomially bounded by $h(M_1)h(Z)$, and thus so is $|Y'|^{-1}$. This completes the proof.

We restate here Lemma 3.2 from [32]. For a point $Z \in \mathbb{H}_g$ there is a unique $\gamma Z \in \text{Sp}_{2g}(Z)$ such that $\gamma Z \cdot Z \in S_g$.

**Lemma 3.2.** $h(\gamma Z) \prec h(Z)$. 

4. Volumes of algebraic curves in $\mathbb{H}_g$.

4.1. Volumes of algebraic curves in $\mathbb{H}_g$. Recall that there is a complex structure on $\mathbb{H}_g$ via the imbedding $\mathbb{H}_g \hookrightarrow M_g(\mathbb{C})^{\text{sym}} = \mathbb{C}^{g(g+1)/2}$. Take $C \subset \mathbb{H}_g$ to be a curve in $\mathbb{H}_g$ (i.e. an irreducible algebraic subvariety of dimension 1). We define the degree of $C$ to be the degree of the Zariski closure of $C$ in $\mathbb{C}^{g(g+1)/2}$. The restriction of the metric $d\mu(Z)$ gives a Riemannian metric on $C$, and thus an induced Volume form $dC$. Our goal in this section is to prove the following theorem.

**Theorem 4.1.** For a curve $C \subset \mathbb{H}_g$ of degree $k$ we have the bound

$$\int_{C \cap S_g} dC \ll k$$

where the implied constant depends on $g$.

**Proof.** The main ingredient in the proof of the theorem is the following lemma.

**Lemma 4.2.** Within $S_g$, we have

$$d\mu(Z) = \text{Tr}(Y^{-1}dZY^{-1}d\overline{Z}) \leq O_g(1) \cdot \sum_{i,j} |dz_{ij}|^2 \cdot \frac{1}{y_{ii}y_{jj}}$$

**Proof.** By Lemma 2.5 we have that $|Y| \gg \prod_{i=1}^g y_{ii}$, and

(*) $\forall i \neq j, |y_{ij}| \leq \min(y_{ii}, y_{jj})/2$.

Now, let $y'_{ij}$ denote the entries of $Y^{-1}$, and let $M_{ij}$ be the $(i,j)$'th minor of $Y$. Then by (*) and the expansion of $M_{ij}$ along columns we see that $|M_{ij}| \ll \prod_{l \neq j} y_{il}$, and thus $|y'_{ij}| = |Y|^{-1}|M_{ij}| \ll y_{jj}^{-1}$. Likewise $|y'_{ij}| \ll y_{ii}^{-1}$. Thus $y'_{ij} \leq |y_{ii} \cdot y_{jj}|^{-\frac{1}{2}}$. Thus we have

$$\text{Tr}(Y^{-1}dZY^{-1}d\overline{Z}) = \sum_{i,j,m,l} y'_{ij}dz_{jm}d\overline{y'_{ml}} \overline{dz_{li}}$$

$$\ll \sum_{i,j,m,l} |dz_{jm}| |(y'_{ij}y'_{jl}y'_{ml}y_{li})^{-\frac{1}{2}}|$$

$$\leq \sum_{i,j,m,l} \frac{|dz_{jm}|^2}{y_{jj}y_{mm}} + \frac{|dz_{li}|^2}{y_{ii}y_{ll}}$$

which is what we wanted to show. \qed

Since the volume form for the conformal metric $|dz_{ij}|^2$ is $dx_{ij}dy_{ij}$, by the lemma above, it is enough to show that for all $(i,j)$ we have

$$\int_{C \cap S_g} \frac{dx_{ij}dy_{ij}}{y_{ii}y_{jj}} \ll k.$$
We first consider the case $i = j$. In this case, consider the projection map onto the $z_{ii}$ coordinate, $\pi_{ii} : \mathbb{H}_g \to \mathbb{H}_1$. By Lemma 2.5, the image of $S_g$ under $\pi_{ii}$ is contained in the Siegel set

$$y_{ii} > \sqrt{\frac{3}{2}}, |x_{ii}| \leq \frac{1}{2}.$$ 

Moreover, when the map $\pi_{ii}$ is restricted to $C$ it is either constant, in which case the differential $dz_{ii}$ vanishes along $C$, or it has finite fibers. In fact, since $C$ has degree $k$, the map $\pi_{ii} : C \to \mathbb{H}_1$ is at most $k$ to 1. Thus we have

$$\int_{C \cap S_g} \frac{dx_{ij}dy_{ij}}{y_{ii}^2} \leq k \int_{\substack{y_{ii} = \sqrt{3/2} \\ x_{ii} = -\frac{1}{2}}}^{\infty} \frac{dx_{ij}dy_{ij}}{y_{ii}^2} = \frac{2k}{\sqrt{3}}$$

We now consider the case of $i \neq j$. Here we use that since $y_{ii}, y_{jj} > \sqrt{\frac{3}{2}}$, we have $\frac{1}{y_{ii}y_{jj}} \leq \min(\frac{1}{y_{ij}}, \frac{1}{y_{jj}})$. Thus projecting to the $z_{ij}$ coordinate and reasoning as before, we get

$$\int_{C \cap S_g} \frac{dx_{ij}dy_{ij}}{y_{ii}y_{jj}} \leq k \int_{\substack{y_{ij} = -\infty \\ x_{ij} = -\frac{1}{2}}}^{\infty} \min(\frac{4}{3}, \frac{1}{y_{ij}^2})dx_{ij}dy_{ij} \ll k$$
as desired. \hfill \Box

4.2. Volumes of algebraic curves in weakly special subvarieties. Let $G$ be a connected semi-simple algebraic subgroup of $\text{Sp}_{2g}$ defined over $\mathbb{Q}$, with no simple compact $\mathbb{Q}$-factors, and $Z_0 \in \mathbb{H}_g$ such that $\mathbb{H}_G := G(\mathbb{R})^+ \cdot Z_0$ is a weakly special subvariety with $G^\text{ad}(\mathbb{R})$ fixed by the Cartan involution corresponding to $Z_0$. Our goal is to prove a similar statement to Theorem 4.1 with $\mathbb{H}_G$ replacing $\mathbb{H}_g$.

Let $V_G := \pi_g(\mathbb{H}_G)$ be the weakly special subvariety of $A_g$, and let

$$\Gamma_G \subset \text{Sp}_{2g}(\mathbb{Z})$$

be the lattice which stabilizes $\mathbb{H}_G$. Necessarily, $\Gamma_G$ contains $G(\mathbb{Z})^+$, the intersection of $G(\mathbb{R})^+$ with $\text{Sp}_{2g}(\mathbb{Z})$, as a subgroup of finite index. Note that $\pi_g^{-1}(V_G)$ is the (not disjoint) union over $\gamma \in \text{Sp}_{2g}(\mathbb{Z})$ of $\gamma \mathbb{H}_G$. Now, consider $\pi_g^{-1}(V_G) \cap S_g$. By Theorem 2.6, this intersection is definable and thus has only finitely many connected components. Thus, there are finitely many elements $\gamma_i \in \text{Sp}_{2g}(\mathbb{Z}), 1 \leq i \leq m$ such that

$$\pi_g \left( \bigcup_{i=1}^m \gamma_i^{-1} \mathbb{H}_G \cap S_g \right) = V_G$$

and thus

$$\pi_g \left( \bigcup_{i=1}^m \mathbb{H}_G \cap \gamma_i S_g \right) = V_G.$$  

Lemma 4.3. With notation as above, $S^0_G := \bigcup_{i=1}^m \gamma_i S_g \cap \mathbb{H}_G$ contains a fundamental domain for the action of $\Gamma_G$. 

Proof. Indeed, let \( Z \in \mathbb{H}_G \). Then as \( S_g \) is a fundamental domain for \( \text{Sp}_{2g}(\mathbb{Z}) \), there exists \( \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \) such that \( \gamma \cdot Z \in S_g \). Now, as above this means that there is an \( i \) with \( 1 \leq i \leq m \) such that \( \gamma \cdot Z \in \gamma_i^{-1} \cdot \mathbb{H}_G \) and \( \gamma \cdot \mathbb{H}_G = \gamma_i^{-1} \cdot \mathbb{H}_G \). Thus there exists a \( g \in \Gamma_G \) with \( \gamma = \gamma_i^{-1} \cdot g \). Thus \( \gamma_i^{-1} g Z \in S_g \cap \gamma_i^{-1} \mathbb{H}_G \) and so \( gZ \in \gamma_i S_g \cap \mathbb{H}_G \subset S^0_G \) as desired. \( \square \)

Now, by picking coset representatives for \( G(\mathbb{Z})^+ \) in \( \Gamma_G \) we can find a finite union of elements \( \beta_j \in \text{Sp}_{2g}(\mathbb{Z}) \) such that \( S_G := \bigcup_j \beta_j S_g \cap \mathbb{H}_G \) contains a fundamental domain for the action of \( G(\mathbb{Z})^+ \) on \( \mathbb{H}_G \).

Summarizing the above discussion, we arrive at the following theorem:

**Theorem 4.4.** There is a semi-algebraic set \( S_G \subset \mathbb{H}_G \) such that

1. \( G(\mathbb{Z})^+ \cdot S_G = \mathbb{H}_G \)
2. For an irreducible algebraic curve \( C \subset \mathbb{H}_G \) of degree \( k \), we have the bound
   \[
   \int_{C \cap S_G} dC \ll k
   \]
   where the implied constant depends on \( G \)
3. The projection map \( \pi_g : S_G \to A_g \) is definable.

**Proof.** Assertion (1) was already proven in the discussion above. For (2) we need only observe that if \( C \) is algebraic of degree \( k \), then so is \( \gamma \cdot C \). Hence
   \[
   \int_{C \cap S_G} dC \leq \sum_{i=1}^m \sum_{j=1}^n \int_{C \cap \beta_j \gamma_i S_g} dC = \sum_{i=1}^m \sum_{j=1}^n \int_{C \cap (\beta_j \gamma_i)^{-1} C \cap S_g} d(\beta_j \gamma_i)^{-1} C \ll k
   \]
   where the last inequality follows from Theorem 4.1. Finally, (3) follows easily from the fact that \( \pi_g : S_g \to A_g \) is definable, as per Theorem 2.6. \( \square \)

5. Volumes of algebraic curves near the boundary

Take \( C \subset M_g(C)_{\text{sym}} \) to be an irreducible algebraic curve of degree \( k \) in \( \mathbb{H}_g \). Then its Zariski closure \( C_{\text{zar}} \) must intersect the boundary \( \partial \mathbb{H}_g \) in some real algebraic curve \( C_0 \); to see that \( C_{\text{zar}} \) cannot be contained in \( \mathbb{H}_g \) recall that there is a birational algebraic map taking \( \mathbb{H}_g \) to a bounded set. Our goal in this section is to show that the volume of \( C \) near \( C_0 \) is large. This idea and its execution are due to Ullmo and Yafaev [46].

**Theorem 5.1.** For \( M > 1 \) set \( C_M := \{ Z \in C \mid h(Z) \leq M \} \). Then
   \[
   M \ll \int_{C_M} dC.
   \]

**Proof.** We proceed as in [46]. Pick a smooth compact piece \( I \subset C_0 \), and a point \( p \in I \). For \( 0 < \alpha < \beta < 2\pi \) set
   \[
   \Delta_{\alpha, \beta} := \{ z = re^{i\theta} \mid 0 \leq r < 1, \alpha \leq \theta \leq \beta \}
   \]
and
\[ C_{\alpha, \beta} := \{ z = e^{i\theta} \mid \alpha \leq \theta \leq \beta \}. \]

We may find \( \alpha, \beta \) and a real analytic map
\[ \psi : \Delta_{\alpha, \beta} \to C \]
which extends to a real analytic function from a neighbourhood of \( \Delta_{\alpha, \beta} \cup C_{\alpha, \beta} \) to \( \mathbb{C}^{\text{zar}} \) such that \( \psi(C_{\alpha, \beta}) \subset \partial \mathbb{H}_g \). Composing with \( |Y| \) gives a real analytic function on a neighbourhood of \( \Delta_{\alpha, \beta} \cup C_{\alpha, \beta} \) which is positive on \( \Delta_{\alpha, \beta} \) and vanishes exactly when \( 1 - z \bar{z} \) vanishes. Thus, there exists \( \lambda > 0 \) such that
\[ \log |Y| = \frac{\lambda}{2} \log(1 - z \bar{z}) + O(1). \]

Now, as in Ullmo-Yafaev we can also ensure that if \( \omega \) denotes the Kähler form on \( \mathbb{H}_g \) and \( \omega_{\Delta} = idz \wedge d\bar{z} / (1 - |z|^2)^2 \) then
\[ \psi^*(\omega) = s\omega_{\Delta} + \eta \]
where \( \eta \) is smooth in some neighbourhood of \( C_{\alpha, \beta} \), and \( s \) is some positive integer. Finally, for \( \delta < 1 \) we set \( I_\delta = \Delta_{\alpha, \beta} \cap \{|z| < 1 - \delta\} \).

A computation gives we have
\[ \int_{I_\delta} \omega_{\Delta} \gg \frac{1}{\delta}. \]

Combining this with equations (1) and (2) gives the result. \( \Box \)

6. PROOF OF AX-LINDEMANN

We can now prove the Ax-Lindemann (or Ax-Lindemann-Weierstrass) theorem for \( A_g \). We give a more general formulation applicable to semi-algebraic subsets \( W \subset \pi^{-1}(V) \). Such a set \( W \) will be called irreducible if it is not the union of two non-empty relatively closed proper subsets in the topology induced on it by the Zariski topology of algebraic sets defined over \( \mathbb{R} \) (see [18]).

**Theorem 6.1.** Let \( V \subset A_g \) be an irreducible algebraic variety, and suppose that \( W \subset \pi_g^{-1}(V) \) is a connected irreducible semi-algebraic subset of \( \mathbb{H}_g \).
Then there exists a weakly special subvariety \( S \subset V \) such that \( W \subset \pi_g^{-1}(S) \).

**Proof.** We proceed by induction on the dimension \( V \), the case of \( \dim(V) = 0 \) being obvious, as all points are weakly special; for the same reason we may assume \( W \) has positive dimension. We may assume without loss of generality that \( W \) is maximal (i.e. if \( W' \) is semi-algebraic with \( W \subset W' \subset \pi_g^{-1}(V) \) then \( W' \) has \( W \) as a component). Likewise, we may assume by our induction hypothesis that \( V \) is minimal in the sense that there does not exist \( V' \subset V \) with \( \dim V' < \dim V \) and \( W \subset \pi_g^{-1}(V') \). Note that since \( W \) is maximal, by
Lemma 4.1 of [32], \( W \) must be a complex algebraic subvariety. To prove the theorem we must show that \( V \) is weakly special, so that \( W \) is an irreducible component of \( \pi^{-1}_g(V) \), by Lemma 2.4.

Take now \( S_0 \) to be the minimal weakly special subvariety of \( A_g \) containing \( V \) and write \( U_0 \) for the irreducible component of \( \pi^{-1}_g(S_0) \) containing \( W \) (so we have to show that \( S_0 = V \) and \( W = U_0 \)).

Let \( \Gamma_V \subset \text{Sp}_{2g}(\mathbb{Z}) \) be the monodromy group of \( V \), and let \( G_0 \) be the connected component of the Zariski closure of \( \Gamma_V \). Then by Lemma 2.4, \( G_0 \) is a semi-simple group over \( \mathbb{Q} \), and for any point \( u_0 \in U_0 \) we have \( U_0 = G_0(\mathbb{R})^+ \cdot u_0 \). Assume without loss of generality that \( W \cap S_{G_0} \neq \emptyset \).

Write \( Y \) for a connected component of \( \pi^{-1}_g(V) \) which intersects the open part of \( S_{G_0} \), and set \( Y^0 = Y \cap S_{G_0} \). Note that \( Y^0 \) is definable (in \( \mathbb{R}_{\text{an,exp}} \); [30]). Write

\[
X = \{ \gamma \in G_0(\mathbb{R}) \mid \dim (\gamma \cdot W \cap Y^0) = \dim W \},
\]

which likewise is definable. Observe that \( \gamma \cdot W \subset Y \) for all \( \gamma \in X \) by the dimension assumptions and analytic continuation.

For a set \( A \subset \text{Sp}_{2g}(\mathbb{R}) \) and a real number \( T \geq 1 \) define the counting function

\[
N(A, T) = \# \{ M \in A \cap \text{Sp}_{2g}(\mathbb{Z}) : H(M) \leq T \}.
\]

Here \( H(M) \) is the multiplicative Weil height of \( M \), i.e. the maximum size of its integer entries, so that \( H(M) = h(M) \) as defined in 2.6 (as not all its entries can vanish).

**Lemma 6.2.** We have \( T \ll N(X, T) \), implied constants depending on \( X \).

**Proof.** Let \( C \subset W \) be an algebraic curve. For \( T > 0 \), define

\[
C_T = \{ Z \in C \mid h(Z) \leq T \}, \quad X_T = \{ \gamma \in X \mid h(\gamma) \leq T \}.
\]

Since \( S_{G_0} \) contains a fundamental domain, for each point \( Z \in C \) there exists a \( \gamma \in \Gamma_V \) such that \( \gamma \cdot Z \in Y^0 \). By Lemma 3.2 there exists \( M > 0 \) such that \( H(\gamma) \leq h(Z)^M \) for sufficiently large \( h(Z) \). Hence for \( T \gg 1 \), we must have

\[
C_T \subset \bigcap_{\gamma \in X_T \cap \Gamma_V} \gamma^{-1} S_{G_0}.
\]

In particular, we have that

\[
\text{Vol}(C_T) \leq \sum_{\gamma \in X_T \cap \Gamma_V} \text{Vol}(\gamma \cdot C \cap S_{G_0}).
\]

Combining Theorems 4.4 and 5.1 now gives the result. \( \square \)

Since \( N(X, T) \) grows faster than some positive power of \( T \), the Counting Theorem (in the form [31], Theorem 3.6) implies that there are semi-algebraic varieties \( W_1 \subset X \) of positive dimension containing arbitrarily many points \( \gamma \in \Gamma_V \) such that \( W_1 \cdot W \subset Y \). To see this, supposing \( N(X, T) \gg T^\eta \) with \( \eta > 0 \), apply [31], Theorem 3.6 with \( \mu = 0, k = 1 \), and \( 0 < \epsilon < \eta \). As the \( T^\eta \) rational points up to height \( T \) are contained in \( \ll T^\epsilon \) blocks (as
defined there) provided by the theorem, there must be a block containing $
exists \geq T^{\varepsilon}$ rational points. Note that the simpler version \cite{33}, Theorem 3.2 (which is \cite{33}, Theorem 1.9 in the case $k=1$) does not give this conclusion. For such $W_1$ and $\gamma \in W_1 \cap \Gamma_V$ we have $\gamma^{-1}W_1 \cdot W$ contains $W$, and so by our maximality assumption we have

$$\gamma^{-1}W_1 \cdot W = W.$$  

Now let $\Theta$ be the algebraic group over $\mathbb{R}$ which is the Zariski closure in $\Gamma_0,\mathbb{R}$ of the subset

$$\{ \gamma \in \Gamma_0(\mathbb{R}) \mid \gamma : W = W \}.$$  

Let $\Theta^0$ be its connected component. Since $\gamma^{-1}W_1 \subset \Theta$, it follows that $\Theta^0$ has positive dimension. Let $H$ be the maximal connected algebraic subgroup of $\Gamma_0$ defined over $\mathbb{Q}$ such that $H(\mathbb{R}) \subset \Theta^0(\mathbb{R})$. We know that $\Theta^0$ has infinitely many rational (in fact integral) points, hence so does $H(\mathbb{R})$, and so $\dim(H(\mathbb{R})) > 0$.

Suppose that $Y$ is NOT invariant under $H(\mathbb{R})^+$. Since $H(\mathbb{Q})$ is dense in $H(\mathbb{R})$ (see e.g. \cite{37}, Cor. 3.5) we can find an element $h \in H(\mathbb{Q})^+$ such that $Y$ is not invariant under $h$. Let $Y' = Y \cap hY$. Then $\pi_g(Y')$ is a closed algebraic proper subvariety $V' \subset V$ with and $W \subset Y'$ (in fact $V'$ is a component of the intersection of $V$ with one of its Hecke translates). This contradicts our minimality assumption on $V$, hence $Y$ is invariant under $H(\mathbb{R})^+$.

Let $H'$ be the smallest algebraic subgroup of $\Gamma_0$ containing the conjugates of $H$ by all $\gamma \in \Gamma_V$. Since $Y$ is invariant under $H(\mathbb{R})^+$ and under $\Gamma_V$ it is invariant under $H'(\mathbb{R})^+$. Now $H'$ is an algebraic group that is invariant by conjugation under $\Gamma_V$, and hence also under $\Gamma_0$, the Zariski closure of $\Gamma_V$. Hence $H'$ is normal in $\Gamma_0$. Note that $H'(\mathbb{R})^+ \cdot W \subset Y'$ so by maximality of $W$ we conclude that $H'(\mathbb{R})^+ \cdot W = W$, so that $H' = H$.

Next consider the map $\phi : \Gamma_0 \to \Gamma_0^{ad}$, where $\Gamma_0^{ad}$ is the adjoint form of $\Gamma_0$. We can therefore write $\Gamma_0^{ad} = \prod_{i=1}^{r} G_i$ where the $G_i$ are $\mathbb{Q}$-simple algebraic groups. Therefore there is some non-empty subset $I \subset \{1, \ldots, r\}$ such that

$$\phi(H) = \prod_{i \in I} G_i.$$  

We write $U_0 \cong \prod_i U_i$ where the $U_i$ are hermitian symmetric spaces associated to $G_i$. Thus $W$ can be written as $\prod_{i \in I} U_i \times W'$, where $W'$ is an irreducible algebraic subvariety of $\prod_{i \in I'} U_i$. If $W'$ is a point then $\pi_g(W)$ is weakly special, and so we must have $W = U_0$ as desired. Hence, we assume from now on that $\dim(W') > 0$.

Now pick any point $u_I \in \prod_{i \in I} U_i$ and consider the variety

$$W'_{new} = u_I \times W'.$$  

Moreover, let

$$V'_{new} = V \cap \pi_g(u_I \times \prod_{j \in I^c} U_j),$$  

where $I^c$ is the complement of $I$. We know that $V'_{new}$ is a proper subvariety of $V'$. If $V'_{new} = \emptyset$ then $\pi_g(W')$ is not weakly special, and so we must have $\pi_g(W') = \emptyset$ as desired. Hence, we assume from now on that $\dim(V'_{new}) > 0$.

Now pick any point $u_{I'} \in \prod_{i \in I'} U_i$ and consider the variety

$$W'_{new} = u_{I'} \times W'.$$  

Moreover, let

$$V'_{new} = V \cap \pi_g(u_{I'} \times \prod_{j \in I^c} U_j),$$  

where $I^c$ is the complement of $I$. We know that $V'_{new}$ is a proper subvariety of $V'$. If $V'_{new} = \emptyset$ then $\pi_g(W')$ is not weakly special, and so we must have $\pi_g(W') = \emptyset$ as desired. Hence, we assume from now on that $\dim(V'_{new}) > 0$.
which is evidently an algebraic variety as \(\prod_{j \in I} U_j\) is weakly special. Then
\[
\pi_g(W^\text{new}) \subset V^\text{new}
\]
and so by our induction hypothesis there is a weakly special variety
\[
S^\text{new} \subset u_I \times \prod_{j \in I} U_j
\]
such that \(\pi_g(S^\text{new}) \subset V^\text{new}\) and \(W^\text{new} \subset S^\text{new}\). By projection, we can identify \(S^\text{new}\) with its image in \(\prod_{j \in I} U_j\). Since weakly special varieties are closed under intersection (Corollary 2.2) we can consider \(S_{\text{min}} \subset \prod_{j \in I} U_j\) to be the minimal weakly special variety containing \(W'\). By the discussion above, for each point \(u_I \in \prod_{i \in I} U_i\) we know that \(\pi_g(u_I \times S_{\text{min}}) \subset V\). In other words, the variety
\[
S = \prod_{i \in I} U_i \times S_{\text{min}}
\]
is weakly special and satisfies both \(W \subset S\) and \(\pi_g(S) \subset V\). By our maximality assumption, we conclude that \(W = S = U_0\) and \(V = S_0\) as desired. \(\square\)

7. Application to Andre-Oort

In this section we give the application to the Andre-Oort conjecture. As already mentioned, the size of the Galois orbit of a special point plays a crucial role. For a point \(x \in A_g\) let \(A_x\) denote the corresponding \(g\)-dimensional principally polarized abelian variety, \(R_x = \mathbb{Z}(\text{End}(A_x))\) the centre of the endomorphism ring of \(A_x\), and \(\text{Disc}(R_x)\) its discriminant (for further details see \[41\]). In general one expects to have the following lower bound suggested by Edixhoven in \[15\].

**Conjecture 7.1.** Let \(g \geq 1\). Then, for a special point \(x \in A_g\),
\[
|\text{Disc}(R_x)| \ll |\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|
\]
(with the implied constants depending on \(g\)).

For \(g = 1\) this is known by the theory of complex multiplication of elliptic curves (see e.g. \[6\]) and the Landau-Siegel lower bound for the class number of an imaginary quadratic field \[25, 39\]. As already mentioned, the second author \[41\] has affirmed this conjecture
1. for \(g \leq 6\), and
2. for all \(g\) under GRH for CM fields (see also Ullmo-Yafaev \[45\]).

All the unconditional lower bounds are ineffective; hence Theorem 1.2, which we now establish by proving Theorem 7.1 below, is ineffective (other aspects of the proof are also ineffective, including the Counting Theorem, though this ineffectivity seems less serious). The only non-trivial case of the Andre-Oort conjecture known unconditionally and effectively is that of the product of two modular curves, due independently to Kuhne \[24\] and Bilu-Masser-Zannier \[5\] (André’s original proof \[3\] is ineffective).
Theorem 7.1. Suppose Conjecture 7.1 holds for \( \mathcal{I} \). Let \( V \) be an irreducible closed algebraic subvariety of \( A_g \). Then \( V \) contains only finitely many maximal special subvarieties.

Before proving this theorem we need some lemmas, in which we take \( V \subset A_g \) to be an irreducible closed algebraic subvariety. We do not for the moment assume Conjecture 7.1.

We set \( G \subset \text{GSp}_{2g,\mathbb{Q}} \) to be the generic Mumford-Tate group of an irreducible component \( Z \) of \( \pi_g^{-1}(V) \). Then, for any \( z \in Z \), \( \pi_g(G(\mathbb{R})^+ \cdot z) \) is the smallest special subvariety containing \( V \). We may assume that \( V \) contains special points, so that this is also the smallest such weakly special subvariety. Let \( S_G \) be as defined above Theorem 4.4, and recall it contains a fundamental domain for \( G(\mathbb{Z})^+ \). Let \( Y^0 = \pi_g^{-1}(V) \cap S_G \), which, as we have already observed in Theorem 4.4, is a definable set.

We consider orbits \( H(\mathbb{R})^+ \cdot z, z \in \mathbb{H}_g \) that lie inside \( \pi_g^{-1}(V) \), where \( H \subset \text{Sp}_{2g,\mathbb{R}} \) is a semi-simple group. As such orbits are real algebraic, we observe that if
\[
\dim H(\mathbb{R})^+ \cdot z = \dim \left( H(\mathbb{R})^+ \cdot z \cap (\pi_g^{-1}(V) \cap S_G) \right)
\]
then, by analytic continuation, we have \( H(\mathbb{R})^+ \cdot z \subset \pi_g^{-1}(V) \). We call such an orbit maximal if it is not contained in an orbit \( K(\mathbb{R})^+ \cdot z \), where \( K \subset \text{Sp}_{2g,\mathbb{R}} \) is semi-simple, \( K(\mathbb{R})^+ \cdot z \) is contained in \( \pi_g^{-1}(V) \) and has larger dimension than \( H(\mathbb{R})^+ \cdot z \). By Ax-Lindemann [6,11], every maximal orbit is weakly-special, and Lemma 2.3 implies that there is a connected, semi-simple \( \mathbb{Q} \)-subgroup \( F \subset G \) with no compact \( \mathbb{Q} \)-factors, such that \( F(\mathbb{R})^+ \cdot z = H(\mathbb{R})^+ \cdot z \).

Definition. We set \( \mathcal{C} \) to be the set of all connected semi-simple \( \mathbb{Q} \)-subgroups \( F \subset G \) which have no compact \( \mathbb{Q} \)-factors, and such that there exists \( y \in \mathbb{H}_g \) for which \( F(\mathbb{R})^+ \cdot y \) is weakly special.

For \( F \in \mathcal{C} \) and for every weakly special subvariety \( F(\mathbb{R})^+ \cdot y \subset \mathbb{H}_g \) there is a Shimura subdatum \( (H, X_H) \subset (G, X_G) \) and a splitting
\[
(H_{\text{ad}}, X_{H_{\text{ad}}}, \Gamma) = (H_1, X_1^+, \Gamma_1) \times (H_2, X_2^+, \Gamma_2),
\]
and a point \( h_2 \in X_2^+ \) such that \( X_1^+ \times \{ h_2 \} = F(\mathbb{R})^+ \cdot y \).

Consider the group
\[
\Gamma' := \{ \gamma \in G(\mathbb{Z})^+ \mid \gamma \cdot F(\mathbb{R})^+ \cdot y = F(\mathbb{R})^+ \cdot y \}.
\]
This group has the property that \( \Gamma' \backslash F(\mathbb{R})^+ \cdot y \) is finite volume. However, letting \( \tilde{H}_1 \) denote the pullback of \( H_1 \) to \( H \), the subgroups \( F(\mathbb{Z})^+ \) and \( \tilde{H}_1(\mathbb{Z})^+ \) also have this property, and hence their images in \( G_{\text{ad}} \) are commensurable. Since neither \( F \) nor \( H_1 \) have compact \( \mathbb{Q} \)-factors and both are connected, \( F \cdot Z(\tilde{H}_1) = \tilde{H}_1 \) by [36], Theorem 4.10. Letting \( N(F) \) denote the connected component of the normalizer of \( F \) in \( G \), it follows that \( H \subset N(F) \).

Lemma 7.2. Let \( F \in \mathcal{C} \). Then \( N(F) \) is a reductive \( \mathbb{Q} \)-group.
Proof. Let \( y \in \mathbb{H}_g \) be such that \( F(\mathbb{R})^+ \cdot y \) is a weakly special subvariety. Recall that \( y \) gives a homomorphism \( y : \mathbb{S} \rightarrow G_\mathbb{R} \), and conjugation by \( y(i) \) gives a Cartan involution of \( G_\mathbb{R}^{ad} \) which preserves the image of \( F_\mathbb{R} \) in \( G_\mathbb{R}^{ad} \). Therefore, it must also preserve \( N(F)_\mathbb{R} \). By \[35\] §I, 4.3, \( N(F) \) is reductive. The fact that it is defined over \( \mathbb{Q} \) follows easily from the fact that \( F \) is. \( \square \)

Write \( \text{Com}(N(F)) \) for the product of all compact \( \mathbb{Q} \)-factors of \( N(F)_\mathbb{R}^{ad} \) and let \( N(F)_{sh} \) denote the identity component of the kernel of the projection \( N(F) \rightarrow \text{Com}(N(F)) \). Then \( y : \mathbb{S} \rightarrow G_\mathbb{R} \) factors through \( \phi(H_1 \times H_2) \) and hence also through \( N(F)_{sh} \). Thus \( N(F)_{sh}(\mathbb{R})^+ \cdot y \) defines a special subvariety of \( G(\mathbb{R})^+ \cdot y \).

Lemma 7.3. Let \( F \in \mathcal{C} \). Then the set of special subvarieties of \( \mathbb{H}_g \) of the form \( N(F)_{sh}(\mathbb{R})^+ \cdot y \) such that \( y(\mathbb{S}) \subset N(F)_{sh,\mathbb{R}} \) is finite as \( y \) varies over \( \mathbb{H}_g \).

Proof. The requirement is that \( y : \mathbb{S} \rightarrow G_\mathbb{R} \) has image in \( N(F)_{sh,\mathbb{R}} \), and we would like to show that the number of such images that arise is finite up to \( N(F)_{sh}(\mathbb{R})^+ \) conjugacy. Since the homomorphisms we are interested in correspond to Shimura varieties, it means each such homomorphism also defines a hermitian symmetric domain, which has as its group of biholomorphisms the corresponding adjoint real group \( N(F)_{sh,\mathbb{R}}^{ad} \). It is well known that the number of such domains of a bounded dimension is finite, hence there are finitely many types \( (N(F)_{sh,\mathbb{R}}^{ad}, y : \mathbb{S} \rightarrow N(F)_{sh,\mathbb{R}}^{ad}) \) corresponding to a hermitian symmetric space up to an automorphism of \( N(F)_{sh,\mathbb{R}}^{ad} \). Since \( N(F)_{sh}^{ad} \) is semi-simple, its outer automorphism group is finite. This means that, up to \( N(F)_{sh}(\mathbb{R})^+ \) conjugacy, there are only finitely many homomorphisms \( y : \mathbb{S} \rightarrow N(F)_{sh,\mathbb{R}}^{ad} \) that arise from points \( y \in \mathbb{H}_g \). But recall that we know that the weights of the action \( \text{ad} \circ y \) of \( \mathbb{S} \) on the complexified Lie algebra of \( \text{Sp}_{2g} \) are restricted to a finite set. This implies that for each homomorphism to \( N(F)_{sh,\mathbb{R}}^{ad} \) only finitely many possible lifts to \( N(F)_{sh,\mathbb{R}} \) actually occur in the \( G(\mathbb{R})^+ \)-conjugacy class. This completes the proof of the assertion. \( \square \)

For the proof of Theorem 7.1, it will be useful to make the following definition:

Definition. For each \( F \in \mathcal{C} \), define \( S(F) \) to be the union of \( \pi_g(F(\mathbb{R})^+ \cdot z) \) over the maximal \( F \)-orbits \( F(\mathbb{R})^+ \cdot z \), which are also special subvarieties and satisfy

\[
\dim F(\mathbb{R})^+ \cdot z = \dim \left( F(\mathbb{R})^+ \cdot z \cap \left( \pi_g^{-1}(V) \cap S_G \right) \right).
\]

We let \( W(F) \) be the corresponding union over maximal \( F \)-orbits which are weakly special.

Finally, we need the following result, which is well known to the experts but for which we know of no easy reference in the literature:

Lemma 7.4. For any \( B > 0 \), there are only finitely many special points \( x \in \mathcal{A}_g \) such that \( |\text{Disc}(R_x)| < B \).
Proof. Let $x$ be such a point and consider its lift $Z$ to the fundamental domain $S_g$. Then by Theorem 3.1 in [32] we know that $H(Z) < |\text{Disc}(R_x)|$, where $H(Z)$ is the height of the point $Z$. Since all the co-ordinates of $Z$ are algebraic of degree at most $2g$, there are only finitely many such $Z$ by Northcott’s Theorem. This completes the proof. □

We can now prove Theorem 7.1.

Proof. We now assume that Conjecture 7.1 holds for $g$. The proof is by induction on $\dim V$, therefore we may assume that $V$ contains a Zariski-dense set of special points, and we must then show that $V$ is a special subvariety. Since special points are defined over $\mathbb{Q}$, we must have that $V$ is defined over a field $K$ of finite degree over $\mathbb{Q}$.

Let $x$ be a special point in $V$, and let $Z$ be a preimage of $x$ contained in the standard fundamental domain $S_g$ of $\text{Sp}_{2g}(\mathbb{Z})$ as in [32]. By Theorem 3.1 in [32], we know that the height $H(Z)$ of $Z$ satisfies $H(Z) \prec |\text{Disc}(R_x)|$. The conjugates $x'$ of $x$ over $K$ again lie on $V$, are special points of the same discriminant, and have pre-images $Z'$ in $S_g$. Thus by Conjecture 7.1, if $|\text{Disc}(R_x)|$ is sufficiently large, the Counting Theorem (in the form [31], Theorem 3.2) implies that “many” of the $x'$ (i.e. a positive power of $|\text{Disc}(R_x)|$ of them) are contained in the algebraic part (as defined there) of $Y^0$, which thus contains a semi-algebraic set of positive dimension containing at least one (in fact many, if we use [33], 3.6, but one will suffice) preimage $Z'$ of a Galois conjugate of $x$ over $K$. By Ax-Lindemann (the semi-algebraic version: Theorem 6.1) $Z'$ is contained in a weakly special subvariety $S \subset \pi^{-1}(V)$ which we may choose to be maximal, and is in fact special as $x$ is special. This implies that $\pi_g(S)$ is algebraic, and hence the union of all Galois conjugates of $\pi_g(S)$ over $K$ contains $x$ and lies inside $V$. Hence, by Lemma [7.4] all but finitely many special points $x \in V$ are contained in a positive dimensional special subvariety of $V$.

It therefore suffices to establish that there are only finitely many positive dimensional maximal special subvarieties of $V$, for then the Zariski-density of special points will imply that $V$ itself is special.

By Lemma [2.34] any such special subvariety is the image under $\pi_g$ of an orbit of $F(\mathbb{R})^+$ for some $F \in C$. We first show that, for each $F \in C$, the set $S(F)$ is a finite union of special subvarieties.

We first consider the case where $N(F)_{\text{sh}}$ is not equal to all of $G$. Then by Lemma [7.3] special subvarieties of $V$ which are images under $\pi_g$ of orbits of $F(\mathbb{R})^+$ lie in the union of finitely many special subvarieties whose generic Mumford-Tate group $M$ satisfies $M_{\text{def}} = N(F)_{\text{sh}}$. But the smallest special subvariety containing $V$ has generic Mumford-Tate group $G$, and so the intersections of these special subvarieties with $V$ must all be proper. Hence we are done in this case by induction on the dimension of $V$.

Now consider the case that $N(F)_{\text{sh}}$ is equal to $G$. This means that $F$ is normal in $G$, and so $G$ splits as an almost direct product $G = F \cdot Z_G(F)$. Let $y$ be a point in $\pi_g^{-1}(V) \cap S_G$, so that $G(\mathbb{R})^+ \cdot y$ is a hermitian symmetric...
space with \( \pi_g(G(\mathbb{R})^+ \cdot y) \) the minimal special subvariety containing \( V \). Then one has a splitting of the hermitian symmetric space \( G(\mathbb{R})^+ \cdot y \) as \( F(\mathbb{R})^+ \cdot y \times Z_G(F)(\mathbb{R})^+ \cdot y \).

Let \( z_{CM} \in S_G \) be a special point. Then \( F(\mathbb{R})^+ \cdot y \cap Z_G(F)(\mathbb{R})^+ \cdot z_{CM} \) consists of a single point, and moreover this point is special if and only if \( F(\mathbb{R})^+ \cdot y \) is a special subvariety, if and only if \( F(\mathbb{R})^+ \cdot y \) contains special points (see [28], Theorem 4.3). Thus special subvarieties of the form \( F(\mathbb{R})^+ \cdot y \) are in bijection with special points on \( Z_G(F)(\mathbb{R})^+ \cdot z_{CM} \). Moreover, if we let \( X^+_F, X^+_G \), and \( X^+_ZG(F) \) denote the hermitian symmetric spaces corresponding respectively to \( F, G \) and \( Z_G(F) \), then we get a finite algebraic map of Shimura varieties

\[
\psi : F(\mathbb{Z})^+ \times Z_G(F)^+ \backslash X^+_F \times X^+_ZG(F) \rightarrow G(\mathbb{Z})^+ \backslash X^+_G.
\]

Put \( W = \pi_g(Z_G(F)(\mathbb{R})^+ \cdot z_{CM}) \) and let

\[
V = \{ x \in W \mid x \in V \land F(\mathbb{Z})^+ \backslash X^+_F \times \text{pr}_2 \psi^{-1}\{x\} \subset \psi^{-1}(V) \}.
\]

Then \( V \) is a closed algebraic subvariety of \( A_g \) of dimension lower than \( V \) (as the orbits of \( F(\mathbb{R})^+ \) have positive dimension), and special points in \( V \) correspond to special subvarieties of the form \( \pi_g(F(\mathbb{R})^+ \cdot y) \) in \( V \).

By induction on dimension the conclusion of 7.1 is true for \( V' \) and thus the set of special points in \( V' \) is contained in a finite union of special subvarieties of \( V' \). Hence, the images of maximal orbits \( \pi_g(F(\mathbb{R})^+ \cdot y) \) correspond to the finite set of isolated special points in \( V' \). Thus we have finiteness also in this case.

**Lemma 7.5.** The union of all positive dimensional weakly special subvarieties of \( V \) is a countable union of algebraic varieties.

**Proof.** Since there are only countably many \( \mathbb{Q} \)-subgroups of \( G \), we need only consider those weakly special subvarieties arising from a single Shimura subdatum \( (H, X_H) \subset (G, X_G) \) and a splitting

\[
(H^\text{ad}, X_H^\text{ad, +}) = (H_1, X_1^+) \times (H_2, X_2^+).
\]

We then get an algebraic morphism

\[
\Gamma_1 \backslash X_1^+ \times \Gamma_2 \backslash X_2^+ \rightarrow G(\mathbb{Z})^+ \backslash X_G^+,
\]

where \( \Gamma_1 \subset H_1 \) and \( \Gamma_2 \subset H_2 \) are appropriate arithmetic subgroups. Then the set of points \( y \in \Gamma_2 \backslash X_2^+ \) such that \( X_1^+ \times \{y\} \) maps inside of \( V \) is clearly an algebraic variety. This completes the proof. \( \square \)

From Lemma 7.5 it follows that \( \bigcup_{F \in \mathcal{C}} W(F) \) is a countable union of algebraic varieties. We state two further lemmas (which do not require the assumption of Conjecture 7.1).

**Lemma 7.6.** The union of all weakly special subvarieties of \( V \) of positive dimension is a definable subset of \( A_g(\mathbb{C}) \).
Proof. Let $T \subset V$ be weakly special. Since $S_G$ contains a fundamental domain for the action of $G(\mathbb{Z})$, every point of $T$ is contained in an image of the form $\pi_g(H(\mathbb{R})^+ \cdot z \cap S_G)$ for $H$ a semi-simple subgroup of $G$ and $z \in S_G$. There are only finitely many semi-simple real groups that embed into $\text{Sp}_{2g,\mathbb{R}}$, and the embeddings come in finitely many families up to conjugacy by (16, A.1). We restrict for the moment to a particular semi-simple subgroup $H \subset G_{\mathbb{R}}$. For each $y \in G(\mathbb{R})$ let $H_y := yHy^{-1}$. Since the dimension of orbits is definable, it follows that the set

$$T_H := \{(y, z) \in \text{Sp}_{2g}(\mathbb{R}) \times S_G : H_y(\mathbb{R})^+ \cdot z \text{ is maximal}\}$$

is definable (the maximality is definable as there are only finitely many conjugacy classes of groups to compare with), and the union of the

$$\pi_g(H_y(\mathbb{R})^+ \cdot z \cap S_G)$$

over $(y, z) \in T_H$ is thus also definable. Taking the union over the finitely many conjugacy classes of semi-simple groups $H$ proves the result. □

Lemma 7.7. Let $A$ be a quasiprojective complex algebraic variety. Suppose $V_i$ is a countable sequence of closed complex algebraic subvarieties of $A$ whose union is definable. Then there is a natural number $m$ such that

$$\bigcup_{i \in \mathbb{N}} V_i = V_1 \cup V_2 \cup \cdots \cup V_m.$$ 

Proof. Let $X = \bigcup_{i \in \mathbb{N}} V_i$. Since $X$ is definable (in $\mathbb{R}_{\text{an \ exp}}$) it has an analytic cell decomposition ([14]), $X = \cup_j U_j$. For each $j$, there is an $i$ such that $U_j \cap V_i$ has the same dimension as $U_j$. But since $U_j$ is a connected real analytic variety, if a polynomial vanishes on an open set of $U_j$ it must vanish on $U_j$ identically. Hence $U_j \subset V_i$. □

By applying the preceding lemmas we see that there is a finite set $V \subset C$ such that the union of all maximal weakly special subvarieties of $V$ of positive dimension is $\bigcup_{F \in \mathcal{V}} W(F)$. Consider a positive dimensional special subvariety $T \subset V$. We have $T \subset \bigcup_{F \in \mathcal{V}} W(F)$. Now special points are Zariski-dense in $T$, but the special points in each $W(F)$ are contained in $S(F)$. Thus $T \subset \bigcup_{F \in \mathcal{V}} S(F)$. We conclude that all maximal special subvarieties of $V$ of positive dimension are contained in the finite union $\bigcup_{F \in \mathcal{V}} S(F)$. Hence the theorem is true for $V$. □

References

[1] Y. André, G-functions and Geometry, Aspects of Mathematics E13, Vieweg, Braunschweig, 1989.
[2] Y. André, Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. Compositio Math. 82 1992, no. 1, 1-24.
[3] Y. André, Finitude des couples d’invariants modulaires singuliers sur une courbe algébrique plane non modulaire. J. Reine Angew. Math. 505 (1998), 203-208.
[4] J. Ax, On Schanuel's conjectures, *Annals of Mathematics* **93** (1971), 252–268.
[5] Y. Bilu, D. Masser, and U. Zannier, An effective “theorem of André” for CM-points on a plane curve *Math. Proc. Cambridge Philos. Soc.* **154** (2013), 145–152.
[6] A. Borel, S. Chowla, C. S. Herz, K. Iwasawa and J.-P. Serre, *Seminar on complex multiplication*, (1957-8), Lecture Notes in Mathematics **21**, Springer, 1966.
[7] A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups, *Annals of Mathematics* **75** (1962), 485-535.
[8] C.-L. Chai, *Compactification of Siegel Moduli Schemes*, LMS Lecture Note Series **107**, CUP, 1985.
[9] C. Daw and A. Yafaev, An unconditional proof of the André-Oort conjecture for Hilbert modular surfaces, *Manuscripta Math.* **135** (2011), 263-271.
[10] P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki, Exposé 389, Février 1971, *Lecture notes in Maths.* 244, Springer-Verlag, Berlin 1971, p.123-165.
[11] P. Deligne, *Variétés de Shimura: interprétation modulaire et techniques de construction de modèles canoniques, Automorphic Forms, Representations, and L-functions Part. 2; Proc. of Symp. in Pure Math. 33*, American Mathematical Society 1979, p.274-290.
[12] L. van den Dries, *Tame Topology and O-minimal Structures*, LMS Lecture Note Series **248**, CUP, 1998.
[13] L. van den Dries, A. Macintyre, D. Marker, The elementary theory of restricted analytic fields with exponentiation, *Annals of Mathematics* **140** (1994), 183–205.
[14] L. van den Dries and C. Miller, On the real exponential field with restricted analytic functions, *Israel J. Math.* **85** (1994), 19–56.
[15] S.J. Edixhoven, B.J.J. Moonen, and F. Oort, editors, Open problems in algebraic geometry, *Bull. Sci. Math.* **125** (2001), 1–22.
[16] M. Einseidler, G. Margulis, and A. Venkatesh, Effective equidistribution for closed orbits of semi-simple groups on homogeneous spaces, *Inventiones* **177** (2009), 137–212.
[17] P. Erdős, P.M. Gruber, J. Hammer, *Lattice Points*, Longman Scientific & Technical, New York, 1989.
[18] A. Galligo and N. Vorobjov, Complexity of finding irreducible components of a semi-algebraic set, *J. Complexity* **11** (1995), 174–193.
[19] G. van der Geer, Siegel modular forms and their applications, in *The 1-2-3 of Modular Forms*, Brunier, van der Geer, Harder, and Zagier, editors, Springer-Verlag, Berlin, 2008.
[20] P. Habegger and J. Pila, Some unlikely intersections beyond André-Oort, *Compositio* **148** (2012), 1–27.
[21] J.-I. Igusa, *Theta Functions*, Grund. math. Wissen. **194**, Springer-Verlag, Berlin, 1972.
[22] B. Klingler and A. Yafaev, The André-Oort conjecture, arXiv preprint.
[23] A. Krieg, Primitive minima of positive definite quadratic forms, *Acta Arithmetica LXIII.1*, 1993, 91–96
[24] L. Kühne, An effective result of André-Oort type, *Annals of Mathematics* **176** (2012), 651–671.
[25] E. Landau, Bemerkung zum Heilbronnischen Satz, *Acta Arithmetica* **1** (1935), 2–18.
[26] S. Lang, *Introduction to Transcendental Numbers*, Addison-Wesley, Reading, MA, 1966.
[27] D. Masser and U. Zannier, Torsion anomalous points and families of elliptic curves, *Amer. J. Math.* **132** (2010), 1677–1691.
[28] B. J. J. Moonen Linearity properties of Shimura varieties, I, *J. Algebraic Geom.* **7** (1998), 539–567.
[29] F. Oort, Canonical lifts and dense sets of CM points, *Arithmetic Geometry, Cortona, 1994*, 228–234, F. Catanese, editor, Symposia. Math., XXXVII, CUP, 1997.
[30] Y. Peterzil and S. Starchenko, Definability of restricted theta functions and families of abelian varieties, Duke. Math. J. 162 (2013), 731–765.
[31] J. Pila, O-minimality and the André-Oort conjecture for \( \mathbb{C}^n \), Annals of Mathematics 173 (2011), 1779–1840.
[32] J. Pila and J. Tsimerman, The André-Oort conjecture for the moduli space of abelian surfaces, Compositio 149 (2013), 204–216.
[33] J. Pila and A. J. Wilkie, The rational points of a definable set, Duke Math. J. 133 (2006), 591–616.
[34] J. Pila and U. Zannier, Rational points in periodic analytic sets and the Manin-Mumford conjecture, Rend. Lincei Mat. Appl. 19 (2008), 149–162.
[35] R. Pink, A combination of the conjectures of Mordell-Lang and André-Oort, Geometric Methods in Algebra and Number Theory, Progress in Mathematics 235, 2005, 251–282.
[36] V. Platonov and A. Rapinchuk, Algebraic Groups and Number Theory, Pure and Applied Mathematics 139, Academic Press, Boston, 1994.
[37] J.J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, Crelle 327 (1981), 12–80.
[38] I. Satake, Algebraic Structures of Symmetric Domains, Vol. 4 of Kano Memorial Lectures, Iwanami Shoten, Tokyo, 1980.
[39] C.-L. Siegel, Uber die Classenzahl quadratischer Zahlkorper, Acta Arithmetica 1 (1935), 83–86.
[40] C.-L. Siegel, Symplectic Geometry, Academic Press, New York - London, 1964.
[41] J. Tsimerman, Brauer-Siegel for arithmetic tori and lower bounds for Galois orbits of special points, J. Amer. Math. Soc. 25 (2012), 1091–1117.
[42] E. Ullmo, Quelques applications du théorème de Ax-Lindemann hyperbolique, preprint, 2012, Compositio, to appear.
[43] E. Ullmo and A. Yafaev, Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture, arXiv preprint.
[44] E. Ullmo and A. Yafaev, A characterization of special subvarieties, Mathematika 57 (2011), 263–273.
[45] E. Ullmo and A. Yafaev, Nombres de classes des tores de multiplication complexe et bornes inférieures pour orbites Galoisiannes de points spéciaux, arXiv preprint, 2011.
[46] E. Ullmo and A. Yafaev, Hyperbolic Ax-Lindemann theorem in the cocompact case, arXiv preprint, 2011, Duke Math. J., to appear.
[47] A. J. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, J. Amer. M. Soc. 9 (1996), 1051–1094.