Dispersionless transport in washboard potentials revisited

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Abstract

We reassess the ‘dispersionless transport regime’ of Brownian particles in tilted periodic potentials. We show that the particles exhibit normal diffusive motion right after transitioning into the running state dragged by the constant bias force. No special transient dynamics appears, contrary to conjectures in the previous studies. The observed flat segment in the dispersion evolution curve is solely due to the broad spatial distribution of particles formed in the early superdiffusion stage. We quantitatively describe the whole evolution of the distribution function during superdiffusion and the transition to the normal diffusion that follows, in the framework of the two-well potential in the velocity space model. We show that the superdiffusion exponent is $\alpha = 3$. The duration of the ostensible ‘dispersionless regime’ is derived analytically. It is shown to diverge exponentially as the temperature decreases to zero.

Keywords: Brownian motion, transport processes, Langevin equation, driven diffusive systems, nonequilibrium and irreversible thermodynamics, stochastic analysis methods

(Some figures may appear in colour only in the online journal)

1. Introduction

Phenomena of Brownian particle transport and diffusion in tilted periodic potentials are realized in many diverse systems. Superionic conductors [1], magnetic ratchets [2], optical lattices [3], charge-density waves [4], granular gases, Josephson junctions, automatic phase-lock

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frequency control systems are some prominent examples [5]. These phenomena have been studied meticulously in recent decades [6–8].

These investigations produced a host of intriguing discoveries, including giant diffusion [9], negative mobility [10, 11], temperature abnormal diffusivity (TAD) [12, 13], noise-assisted transport [14], stochastic resonance [15]. Dispersionless motion of the packet of particles formed after them leaving the initial potential well under the action of the constant bias force was reported in [16].

The regime of transport with strongly suppressed thermal dispersion would be of great interest in nanophysics. Various tools of manipulating nanoscale objects are being perfected, such as the scanning tunnelling microscopy [17], atomic force microscopy [18], optical [19, 20] and plasmonic [21, 22] tweezers. Creation of nanostructures on the surface of crystals is of particular technological importance. The fundamental issue in nanotechnology is the precision with which one can manipulate ensembles of nano-objects. Classical thermal noise leads to the dispersion of the object velocities and coordinates. Special coherent transport mechanisms of particles or excitations are known to arise when particles interact with space-periodic structures. Examples are channeling of high-energy particles in single-crystals [23], emergence of focusons along the close-packed directions in crystals [24]. Channeling is realized under special entrance conditions of the high-energy particles into the crystal; the particles then move coherently along the specific ‘channel’ in the crystal. In the case of focusons, the momenta of the particles get focused towards the closed-packed direction of atoms along which the excitation propagates.

The results found in [16] are thus promising as they could suggest new methods of manipulating nano-objects in periodic structures. The authors showed that the time evolution of dispersion $\sigma^2(t)$ of the Brownian particles moving in tilted one-dimensional (1D) periodic potential appeared to have a long plateau for a certain range of the system parameters. The distribution of particle coordinates had exponential shape and stayed nearly unchanged during that plateau interval. The same regime was observed in two-dimensional systems [25]. The authors of [16] called the regime of the particle motion ‘dispersionless’ during the plateau phase. The duration of that phase could be seen to progressively increase at low temperatures, thus for a certain range of parameters it was impossible to determine the diffusion coefficient, cf figure 7 in [25]. In [16] the authors suggested that they observed coherent transport and proposed an intuitive model based on the specifics of the motion of the particles in the space-periodic field.

Diffusion is quantified by the particle mean square displacement (dispersion) evolution with time. In many situations of interest this follows a power-law $\sigma^2(t) \equiv \langle (x(t) - \langle x(t) \rangle)^2 \rangle \propto t^\alpha$. Here angle brackets $\langle \ldots \rangle$ stand for averaging over the ensemble. $x$ is the particle radius-vector (we only consider 1D setup below) and $t$ is the time. Normal diffusion is characterized by $\alpha = 1$ (Fick’s law; $\sigma^2(t) = 2Dt$). Anomalous diffusion corresponds to $\alpha > 1$ (superdiffusion) or $\alpha < 1$ (subdiffusion) [26].

Anomalous diffusion oftentimes occurs as a transient regime; asymptotically at late times the particle ensemble spreading evolves towards normal diffusion. In such transient situations we define exponent $\alpha$ differentially as

$$\alpha = d \ln(\sigma^2)/d \ln t$$

in what follows. Likewise, we define the normal diffusion coefficient differentially as $D = d\sigma^2(t)/2dt$.

We consider Langevin stochastic differential equation for $x(t)$ in dimensionless units

$$\dot{x} = -\sin x - \gamma \dot{x} + f + \sqrt{2\gamma} \zeta(t).$$
Figure 1. The dispersion evolution for different values of the external force. $Q = 0.5$. Power-law scaling $\sigma^2 \propto t^3$ in early superdiffusion stage, $\sigma^2 \approx \text{const}$ in the putative ‘dispersionless regime’ and asymptotic $\sigma^2 \propto t^1$ are shown at corresponding time intervals.

The parameters of the system are $f$ ([constant] bias force), $\gamma$ (viscous friction coefficient) and $Q$ (temperature). $\zeta(t)$ is the Gaussian white noise with intensity 1,

$$\langle \zeta(t_i)\zeta(t_j) \rangle = \delta(t_i - t_j). \quad (3)$$

Overdot stands for differentiation over $t$. Periodic potential is chosen cosinusoidal $U(x) = -\cos x$ in this work. The frequency of small oscillations at the potential minima at small $\gamma$ is $\omega_0 = 1$.

In underdamped systems, characterized by $\gamma < 1$, two types of solutions exist for a range of $f \in [f_1; f_3]$ (in Risken’s [5] notation) in the deterministic limit $Q = 0$: running solutions (which drift on average in the direction of the bias force $f$) and locked ones (with $\langle dx/dt \rangle = 0$).

When studying this (underdamped) problem in [16], the authors observed a horizontal plateau (cf figure 1 below, curve 3) in $\sigma^2(t)$ curve, spanning $\sim 2$ decades for the specified force range. The authors provided analytical arguments on the nature of that plateau, which they interpreted as a manifestation of special novel ‘dispersionless transport regime’. In particular, by studying the motion of an individual Brownian particle after it leaving its initial locked state, the authors suggested that the particle spends long time on a trajectory that does not deviate further than the lattice period $2\pi$ from deterministic $x = vt + x_0$ before eventually switching to normal diffusive behavior.

The findings of [16] got differing interpretations in later works. ‘Coherent motion’ was discussed in [27] in the absence of net time-averaged particle motion—in the setup with the constant bias force $f$ replaced by a time-periodic force $\tilde{f}(t)$ with zero mean. In a series of works [28–30] the authors observed the same horizontal plateau in $\sigma^2(t)$ evolution in setups with either constant $f$ [28] or time-periodic $\tilde{f}(t)$ forcing. They called the corresponding stage of system evolution ‘subdiffusion’, without checking if the known mechanisms behind subdiffusion [26] were realized. The reason for identification of the plateau stage in $\sigma^2(t)$ evolution with ‘subdiffusion’ stemmed from the observation that the ‘diffusion coefficient’ defined (in that stage, when the system did not show normal diffusive behavior) as $D(t) = \sigma^2/(2t)$, decreased with time.

In this paper we show that in a proper sense no special ‘dispersionless’ physics happens in the tilted periodic potential with Brownian particles subject to white Gaussian noise. Nor
does subdiffusion occur following the initial superdiffusion stage. Instead, Brownian particles demonstrate normal Brownian motion (normal diffusion) shortly after transitioning into the running state. The (nearly) invariance of $\sigma^2(t)$ and spatial distribution function $n(x, t)$ in the comoving frame are solely due to the initial broad spatial distribution formed during the superdiffusion stage, with no novel physics on top. We also demonstrate how erroneous conclusions may be (and often are) drawn by visual inspection of dispersion $\sigma^2(t)$ curves and by improper definition of the ‘diffusion coefficient’ $D$ in transient regime, in which $\sigma^2(t)$ deviates substantially from the normal diffusion Fick’s law $\sigma^2 = 2Dt$ form.

2. Method

We analyze the ‘dispersionless’ transport in the system governed by (2) and (3). The same dimensionless form is used in [27, 28], whereas extra factors of 2 and $2\pi$ appear in [16].

For the simulations we used the friction coefficient value of $\gamma = 0.03$ that is close to $\gamma_0 = 0.035$ used in [27], and somewhat larger than the equivalent $\gamma = \gamma_{LSLS}/\pi \sqrt{2} \approx 0.0090$ friction coefficient to that used in [16] (if brought to the same units as used here in (2)). The temperature parameter $Q = 0.5$ is mostly used in what follows, close to equivalent $Q = 2T_{LSLS} = 0.4$ of the temperature used in [16]. $f = 0.25$ for the in-depth studied setup. At this $f$ nearly all the particles are in the running state: at $Q = 0.5$ we find the fraction of locked particles $1.25(8) \times 10^{-4}$ by fitting the velocity distribution function by the sum of two Maxwellians.

Equation (2) was integrated numerically using a Verlet-type algorithm [31, 32], essentially equations (24) and (25) of [32] used to update the particle coordinate and velocity. The same procedure was used in [33] with further tests against analytical results presented. The time step was selected in such a way that the maximal distance covered by the particle in one step was within $10^{-3}$ of the lattice spacing. The statistical averaging was performed over the ensemble of at least $5 \times 10^4$ particles. The ensemble of $10^6$ particles was used to derive particle velocity and coordinate distribution functions.

The simulations were initiated with all the particles at $x = 0$, their velocities chosen at random, with the Maxwellian distribution for a given $Q$. Thermalization over 100 oscillation periods, in the potential $U(x)$ (with no bias), was performed to achieve equilibrium particle distribution over both coordinate and velocity. All the particles were returned to the initial elementary cell $x \in [-\pi; \pi]$ by translation over an integer multiple of the lattice constant $2\pi$. After that the bias $F$ was switched on, corresponding moment of time declared $t = 0$ in the results presented below.

3. Results

Figure 1 shows the time evolution of the dispersion of Brownian particles for different values of the external force $f$. At early times after the initial transient we see superdiffusion phase with $\sigma^2 \propto t^\alpha$, $\alpha > 1$. At late times normal diffusion sets in, $\sigma^2 \propto t$. Curves 3 and 4 show the nearly horizontal section between the early superdiffusion and the late normal diffusion phase, $\sigma^2(t) \approx \text{const}$. Such sections were interpreted earlier as coherent motion phases [16].

As the first step towards interpreting this dispersionless section, in top figure 2 we plot Péclet number, which characterizes the degree of transport coherence [34–36]. In general Pe quantifies the relative importance of convective vs diffusive flux of heat or species concentration. The higher Pe corresponds to more coherent transport. Applied to the flux of Brownian particles in the present 1D problem, $\text{Pe} = \langle \nu \rangle / l$. Here $l$ is the characteristic spatial scale; taken the lattice spacing here, $l = 2\pi$. 


Figure 2. Time dependence of the Péclet number $\text{Pe}(t) = 2\pi \langle v(t) \rangle / D(t)$ (top), average velocity and the dispersion derivative (bottom; right and left scale, resp.). The horizontal dashed line in the top plot shows the asymptotic late time value of $\text{Pe}$. Curve 1 shows the $\text{Pe}$ computed with differential definition of diffusivity, $D_{\text{diff}} = \sigma^2 / 2t$, while curve 2 depicts $\text{Pe}_{\text{int}}$ computed with $D_{\text{int}} = \sigma^2(t) / (2t)$. The horizontal dashed line in the bottom plot is at the expected asymptotic value $\sigma^2 / d \sigma^2 / dt \mid_{t \to \infty} = 2D$. $f = 0.25$, $Q = 0.5$.

In the regime when the asymptotic normal diffusion has not set in, the proper way to define $D$ is via $D(t) = \sigma^2 / 2t$. With so defined $D(t)$ the time-dependent Péclet number $\text{Pe}(t) = \langle v(t) \rangle / D(t)$ characterizes the ratio of the particle packet average displacement over unit time to the packet diffusive broadening over the same time. For comparison of definitions of $D$ we also show the Péclet number defined with integrally defined diffusion coefficient $D_{\text{int}}(t) = \sigma^2(t) / (2t)$ (as used in particular in [28–30]): curve 2 in the top figure 2.

Figure 2 demonstrates that the Péclet number shows no spectacular features in the segment of the ‘dispersionless transport’

$$t \in [t_1; t_2]:$$

$$t_1 \approx 1.5 \times 10^3,$$  \hspace{1cm} (4)

$$t_2 \approx 3.5 \times 10^4 \quad \text{(at } f = 0.25).$$  \hspace{1cm} (5)

$\text{Pe}$ is nowhere infinite, as it must have been for the truly coherent transport. Moreover, the $\text{Pe}_{\text{int}}(t)$ computed with the $D_{\text{int}}(t)$ stays below 1 throughout the ‘dispersionless regime’. Whereas the $\text{Pe}$ defined with the differential definition of $D$ (line 1) reaches the constant value of somewhat over 3 at the very beginning of the ‘dispersionless’ regime, and statistically oscillates around that value into the asymptotic normal diffusion regime.

These conclusions based on the $\text{Pe}(t)$ behaviour (contradicting the reported ‘dispersionless transport’ features) are corroborated by the dispersion growth rate $d\sigma^2/dt$ in the $[t_1; t_2]$ interval shown in the lower part of figure 2. We see that not only $d\sigma^2/dt$ never becomes zero, but in fact it never drops (in statistically significant manner) below the asymptotic value of $2D$. 

[5]
Figure 3. The dispersion growth with time. $f = 0.25$, $Q = 0.5$. (a) log–log scale, (b) linear scale. The solid line is the simulation data. The dash-dotted line is $\sigma^2(t) = 2Dt$ asymptote, the dashed line shows the approximation $\sigma^2(t) = 2Dt + \sigma_1^2$, and the dotted line shows constant $\sigma_1^2$. The arrows show the starting and the ending time of the ‘dispersionless’ phase, $t_1$ and $t_2$.

do\sigma^2/dt simply approaches that value from above at $t \approx 4 \times 10^3$, within the ‘dispersionless’ interval, and statistically fluctuates around $2D$ at all later times.

To elucidate the somewhat paradoxical look of $\sigma^2(t)$ curves in figure 1 that has led different authors to searching for novel physics in time interval $[t_1; t_2]$ we compare in figure 3 the appearance of the dispersion evolution in the log–log scale (customarily used in the references) and in the linear scale.

Superficial impression arising from figure 3(a) is that $\sigma^2(t)$ undergoes rapid (superdiffusive) growth between $t \approx 10$ and $t = t_1$; the dispersion stays unchanged at $t \in [t_1; t_2]$; and eventually after $t = t_2$ normal diffusion with $\alpha = 1$ ensues. However, the flatness of $\sigma^2(t)$ curve on $t \in [t_1; t_2]$ is in a sense illusory; it is an artifact of the log–log scale and the preceding fast-growing segment of the dispersion curve. The red dashed line in upper figure 3 shows the linear fit $\sigma^2(t) = 2Dt + \sigma_1^2$, seen to be virtually indistinguishable from the simulated $\sigma^2(t)$ through the whole ‘dispersionless’ interval $[t_1; t_2]$ and the late-time normal diffusion stage.

Figure 3(b) makes the fact of the linear $\sigma^2(t)$ growth on $[t_1; t_2]$ more overt, with the linear scale used at both axes. The fast superdiffusion with $\sigma^2(t) \propto t^3$ is observed at $t < t_1$. It results in the broad distribution of particles at $t_1$ that would have required 100 times longer time to be formed if diffused according to the linear diffusion law $\sigma^2 = 2Dt$. With that, it takes much time, till $t_2$, for the linear growth of $\sigma^2(t)$ after $t_1$ to become apparent in the logarithmic scale used in figure 3(a). Modulo this artifact of the initial broad distribution at $t_1$ — spreading of the particle ensemble at $t \in [t_1; t_2]$ follows usual linear Fick’s law.
Figure 4. Time evolution of the velocity distribution function. $f = 0.25$, $Q = 0.5$, $t = 1560$ roughly corresponds to $t_1$, the start of the ‘dispersionless stage’.

The shown behavior of $d\sigma^2/dt$ provides further support for our advice to define the diffusion coefficient differentially as $D_{\text{diff}}(t) = d\sigma^2/2dt$, instead of the integral definition $D_{\text{int}}(t) = \sigma^2(t)/(2t)$ customarily used. In situations when the diffusion regime changes with time, e.g. from initial superdiffusion to late-time normal diffusion, $\sigma^2(t)$ stays dominated for long time by the spatial width of the particle distribution formed during the early superdiffusion phase. This necessitates using very long times to get the asymptotic diffusivity $D = D_{\text{diff}}(\infty) = D_{\text{int}}(\infty)$.

Following the usual integral definition, $D$ could not be obtained in [25] (cf figure 7) for a wide range of bias forces, at which ‘dispersionless regime’ was realized. From the appearance of curve 2 in top figure 2 above (that curve 2 uses $D_{\text{int}}(t)$ for definition of $\text{Pe}(t)$) it is clear that to get $D_{\text{int}}(t)$ approximating asymptotic $D$, $t \gtrsim 10^7$ must be used. On the other hand, bottom figure 2 shows that $D_{\text{diff}}(t) \approx D$ already at any $t \gtrsim 4 \times 10^3$. This allows to immensely save computation time by using $D_{\text{diff}}(t)$ as an approximation for asymptotic $D$.

Let us explain the evolution of the spatial distribution function by considering the process of transitioning of the particles into the running state in the two-well velocity potential theory [33]. The effective potential $W(v)$ has two minima that correspond to locked and running particles, located at $v_l = 0$ and $v_r = f/\gamma$, and a maximum between them at $v = v_{cr}(f)$ [33, 37]. In this potential, the particles transition from the locked state to the running state with the rate of $w_{lr}$ and in the opposite direction with rate $w_{rl}$.

Figure 4 shows the simulated according to (2) particle velocity distribution $n(v)$ at different times. At $t = 0$ all the particles are in the locked state. With time, the particles gradually transition into the running state. This transition is virtually finished by $t = t_1$, when $n(v; t)$ assumes its final asymptotic shape, corresponding to all the particles having transitioned into the running state. This behavior (with which the ‘dispersionless regime’ is observed) occurs at the parameter values at which $w_{lr} \gg w_{rl}$, so the reverse transitions from the running to the locked state can be ignored [33, 37].

The problem is thus transformed to computation of the exit of the particles in contact with the thermal reservoir from the potential well. This is described by the Poisson process. The number of particles transitioning in $[t; t + dt]$ into the running state is $w_{lr} \exp(-w_{lr}t)dt$. The average velocity of such particles after the transition is $v_r = f/\gamma$. This yields exponential distribution of the running particles over $x$,

$$n(x; t) = \frac{w_{lr}}{v_r} \exp\left[-w_{lr} \left(\frac{x}{v_r}\right)\right] \theta(t)\theta(v_r t - |x|),$$

(7)
Figure 5. Evolution of the spatial distribution function. Main figure: curve 1 and symbols (squares) 2 are simulated \( n(x; t) \) in the frame moving with the velocity of the running population \( v_r \) at times \( t_1 \) and \( t_2 \) respectively. Circles 3 are the simulated \( n(x; t) \) at \( t = 10^6 \); by that time the profile has diffused to a close to the Gaussian shape on scales above the potential lattice constant. Dotted curves 2 and 3 were obtained by integrating the diffusion equation with constant diffusivity \( D \) starting from the initial profile 1 at \( t_1 \) till time moments \( t_2 \) and \( 10^6 \). Inset: rest frame, formation of the exponential spatial \( n(x) \) at early times in superdiffusion stage, when the particles gradually transition from locked to running state. Curve 4’ corresponds to time \( t = 1560 \approx t_1 \) (cf figure 4) when the transition is largely finished.

where \( \theta(y) = \begin{cases} 1 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases} \) is the Heaviside function.

In deriving (7) the two-state approximation was used, i.e. we assumed all the running particles having velocity \( v = f/\gamma \), all the locked ones having \( v = 0 \); thermal scatter of the velocities around these values neglected. This approximation is valid at sufficiently low temperatures, \( Q \ll v_r^2 \).

The inset in figure 5 shows the \( n(x, t) \) we found in simulations. The formation of a profile of the form (7) with the steep front and exponential tail is observed. That profile mainly moves in the direction of \( f \) with velocity \( v_r \); its thermal broadening only becomes noticeable at significantly later times.

Given the distribution (7) its momenta may be calculated as

\[
\langle x^k \rangle(t) = \int_0^{v_r t} x^k n(x; t) \, dx,
\]

yielding

\[
\sigma^2(t) \equiv \langle x^2 \rangle - \langle x \rangle^2 = \frac{v_r^2}{w_{lr}^2} \left( 1 - e^{-2w_{lr}t} - w_{lr} t e^{-w_{lr}t} \right).
\]

The \( t \to \infty \) limit of (9) yields the dispersion at the start of the ‘dispersionless regime’ (when the particles have transitioned into the running state, but the \( n(x, t) \) has not yet undergone a noticeable further thermal broadening)

\[
\sigma^2_{\text{max}} = \frac{v_r^2}{w_{lr}^2} = \frac{f^2}{\gamma^2 w_{lr}^2}. \tag{10}
\]

At early times \( t \ll w_{lr}^{-1} \), Taylor expansion of (9) yields the leading power series terms
Figure 6. Dispersion $\sigma^2(t)$ in superdiffusion phase. Dot-dashed horizontal line is its asymptotic value (10), achieved as all particles transition into the running state. Short dashed line is the $\sigma^2(t)$ obtained in the two-state approximation in the velocity space (9). Dash-dot-dot line shows the leading cubic term in (11), explaining the superdiffusion exponent $\alpha = 3$ observed in figure 1, while the dotted line is the two term truncation (11) that shows that noticeable deviation from cubic $\sigma^2(t)$ growth starts at $t \approx 2 \times 10^4$.

$$\sigma^2(t) = (1/3)w_i v_r^2 t \left[ 1 - w_i t + O(w_i t^2) \right].$$  \hspace{1cm} (11)

Predictions (9)–(11) are demonstrated to accurately hold in figure 6. $Q = 0.15$ was used, that yields longer superdiffusion stage than that at $Q = 0.5$ used in figure 1. We thus conclude that the superdiffusion phase must universally precede the ‘dispersionless regime’, it must have $\alpha = 3$, the whole time evolution is quantitatively understood within the two-state model in the velocity space.

As follows from the above, the formation of the exponential coordinate distribution of particles is largely completed by $t \approx 3/w_i$. $\sigma^2(3/w_i) \approx 0.848\sigma_{\text{max}}^2$ according to approximation (9), thus indeed $t \approx 3/w_i$ on the log–log graph of $\sigma^2(t)$ would look close to the left starting point of the plateau. After that time ordinary diffusion becomes the main process slowly reshaping $n(x; t)$. The main plot in figure 5 compares the result of the normal diffusion starting at $t_1$ with the $\sigma^2(t)$ found from simulations of (2). Solid line 1 shows the nonequilibrium distribution formed at $t = t_1$. Square and circle symbols show the numerically obtained $n(x; t)$ at $t_2 = 3.5 \times 10^4$ corresponding to visual completion of the ‘dispersionless’ phase, and at $t = 10^6$ in the late-time normal diffusion regime. The dotted lines (overlapped with the corresponding sets of symbols 2 and 3) show the solution of the diffusion equation with constant diffusivity $D$ (from asymptotic late-time $\sigma^2(t)$), evolved from the initial condition given by curve 1 at $t = t_1$ to time moments $t_2$ and $10^6$. We see excellent agreement between the simulated $n(x; t)$ and the $n(x; t)$ obtained by normal diffusion of the initial profile of $t = t_1$.

In the log–log graph of $\sigma^2(t)$ the ostensible ‘horizontal’ segment ends when the normal diffusive broadening of the $n(x; t)$ becomes of the order of the broad $n(x; t_1)$ width formed by the end of the superdiffusion stage. Therefore it is natural to define $t_2$ by $2Dt_2 = \sigma^2(t_1)$. Accordingly, the timespan of the ‘horizontal’ section is

$$\Delta t_D = t_2 - t_1 \approx t_2 = \sigma^2(t_1)/(2D) = f^2/(2D\gamma^2 w_i^2).$$  \hspace{1cm} (12)
Since most of the particles are in the running state, $D$ is close to the particle diffusion coefficient in viscous medium, $D_{\text{visc}} = Q/\gamma$. Hence the duration of the ‘dispersionless stage’ should grow $\propto 1/(Qw_{lr}^2)$ at temperature decreasing $Q \to 0$. According to \cite{33,37}

$$w_{lr}^2 = \frac{\gamma^2 v_{cr}^2}{8\pi Q} \exp \left( -\frac{v_{cr}^2}{Q} \right),$$

yielding exponential divergence at low temperature $Q$

$$\Delta t_{\text{DL}} \simeq \frac{4\pi f^2}{\gamma [v_{cr}(f)]^2} \exp \left( \frac{[v_{cr}(f)]^2}{Q} \right).$$

The data obtained in earlier simulations \cite{28,38–40} agree with the conclusion of $\Delta t_{\text{DL}}$ growing at the temperature decreasing.

As discussed above, $w_h \gg w_{lr}$ must be satisfied for the appearance of the ‘horizontal’ section in the log–log graph of $\sigma^2(t)$. This imposes constraints on the form of the stationary particle velocity distribution. TAD is observed when the stationary velocity distribution has two maxima \cite{33,37}, both locked and running states occupied with a significant fraction of all particles. For the perceivable ‘dispersionless regime’ the tilt $f$ must be above the upper limit $f_r$ of TAD-yielding tilts, so that the fraction of the particles in the locked state is negligible. As found in \cite{37} at low temperatures $f_r = \gamma v_{cr}(1 + \sqrt{2})$. For the parameters used in this paper $f_r \approx 0.1$. The impression of ‘coherent phase’ in the $\sigma^2(t)$ graph should appear at $f \in [f_r; 1]$. This agrees with the behavior observed in figure 1.

4. Conclusions

In conclusion, we have analyzed the ‘dispersionless transport’ \cite{16} of Brownian particles in the washboard potential, which received conflicting interpretations in recent works \cite{27–30}. We have shown that the whole phenomenon is due to the broad spatial distribution of the particles formed by the end $t_1$ of the superdiffusion phase, which requires significant time $t_2 - t_1 \gg t_1$ to be noticeably broadened further by slow normal diffusion. $t_1$ is the time needed for most of the particles to transition from the initial locked state to the running state, with $\langle v \rangle = f/\gamma$ for the running particles. The superdiffusion phase has been proven to have exponent $\alpha = 3$. The evolution of the spatial ensemble dispersion $\sigma^2(t)$ during the superdiffusion phase, its transition into normal diffusion after $t_1$ have been quantitatively described in the two-well potential in the velocity space theory \cite{33}. We have shown that the evolution of the distribution function $n(x; t)$ follows normal diffusion equation with constant diffusion coefficient $D$ right after $t_1$; no novel transient physical behavior of particles right after transitioning into the running state (as theorized in \cite{16}) was observed.

The phenomenon of ‘dispersionless transport’ occurs at bias force values $f > f_r$, $f_r$ being the upper boundary of the bias at which TAD is observed. At these values in the stationary distribution in the velocity space formed after $t_1$ the vast majority of the particles are in the running state. The timespan of this ‘dispersionless’ regime is given by (12); it increases at the temperature $Q$ decreasing, mainly exponentially, $\propto \exp(Q f_r/\gamma Q)$.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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