Limiting absorption principle and virtual levels of operators in Banach spaces

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Abstract We review the concept of the limiting absorption principle and its connection to virtual levels of operators in Banach spaces.

RÉSUMÉ Nous passons en revue le principe d’absorption limite et sa relation avec les niveaux virtuels pour des opérateurs dans les espaces de Banach.

Keywords limiting absorption principle · nonselfadjoint operators · threshold resonances · virtual levels · virtual states

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To Alexander Shnirelman on the occasion of his 75th birthday

1 Limiting absorption principle

The idea of introducing a small absorption into the wave equation for selecting particular solutions goes back to Ignatowsky [Ign05] and is closely related to the Sommerfeld radiation condition [Som12]. We start with the Helmholtz equation

\[-\Delta u - zu = f(x) \in L^2(\mathbb{R}^3), \quad u = u(x), \quad x \in \mathbb{R}^3.\]  

(1.1)

For \( z \in \mathbb{C} \setminus \mathbb{R}_+ \), equation (1.1) has a unique \( L^2 \)-solution \( (-\Delta - zI)^{-1}f \), with \( (-\Delta - zI)^{-1} \) represented by the convolution with \( e^{-|x|\sqrt{-z}}/(4\pi |x|) \), \( \text{Re} \sqrt{-z} > 0 \).
For \( z \geq 0 \), there may be no \( L^2 \)-solution; moreover, when \( z > 0 \), there are different solutions of similar norm, and one faces the question of choosing an appropriate one. The way to select a unique solution is known as the radiation principle. V.I. Smirnov, in his widely renowned “Course of higher mathematics” [Smi41], credits the radiation principle to V.S. Ignatowsky [Ign05] and to A. Sommerfeld [Som12]; the work of Ignatowsky was also publicized by A.N. Tikhonov, both in his lectures at mechmat at Moscow State University and in the textbook written jointly with A.A. Samarskii [TS51] (and even in their ’1950 Russian translation of A. Sommerfeld’s textbook [Som48]). In [Ign05], Ignatowsky considered the electromagnetic field scattered by a wire using the expression \( Z(t, x) = Ae^{i(\omega t - \kappa x)} \) for the electric field, with \( \omega \) and \( \kappa \) certain parameters. The absorption in the medium corresponded to \( \kappa \) having nonzero imaginary part; its sign was taken into account when choosing an appropriate solution to the Helmholtz equation. Following this idea, A.G. Sveshnikov, a student of Tikhonov, specifies in [Sve50] a solution to (1.1) by

\[
  u(x) = \lim_{\varepsilon \to 0^+} \left( -\Delta - (z + i\varepsilon)I \right)^{-1} f, \quad k > 0,
\]

(1.2)
calling this approach the limiting absorption principle (LAP) and attributing it to Ignatowsky.\(^1\) We note that (1.2) leads to

\[
  u(x) \sim \lim_{r \to \infty} \lim_{\varepsilon \to +0} e^{ir\sqrt{z+i\varepsilon}} = e^{ikr}, \quad k = z^{1/2} > 0, \quad r = |x|,
\]

(1.3)

\(^1\) We suppose that in the twenties and thirties, between [Ign05] and [Smi41], the idea of the limiting absorption principle was being refined when V.S. Ignatowsky worked at St. Petersburg University, where in particular he taught mathematical theory of diffraction and likely was in contact with V.I. Smirnov. Let us mention that, besides his work on diffraction, Ignatowsky is known for his contributions to the theory of relativity (see [VG87]) and for developing optical devices while heading the theoretical division at GOMZ, the State Association for Optics and Mechanics (which later became known as “LOMO”). On 6 November 1941, during the blockade of St. Petersburg, Ignatowsky was arrested by NKVD (an earlier name of KGB), as a part of the “process of scientists”, and shot on 30 January 1942. (During this process, V.I. Smirnov was “credited” by NKVD the role of a Prime Minister in the government after the purportedly planned coup; Smirnov avoided the arrest because he was evacuated from St. Petersburg in August 1941, shortly before the blockade began.) As a result, Ignatowsky’s name has been unknown to many: the reference to his article disappeared from Smirnov’s “Course of higher mathematics” until post-1953 editions (see e.g. the English translation [Smi64, §230]).

Russians are used to such rewrites of the history, joking about the “History of the history of the Communist Party”, a reference to a mandatory and ever-changing Soviet-era ideological course in the first year of college. As the matter of fact, the very “Course of higher mathematics” mentioned above was started by V.I. Smirnov together with J.D. Tamarkin, with the first two volumes (published in 1924 and 1926) bearing both names, but after Tamarkin’s persecution by GPU (another earlier name of KGB) and his escape from the Soviet Union with smugglers over frozen lake Chudskoe in December 1924 [Hil47], Tamarkin’s authorship eventually had to disappear. His coauthor Smirnov spent the next year pleading with the authorities (and succeeding!) for Tamarkin’s wife Helene Weichardt – who tried to follow her husband’s route with the smugglers over the icy lake but was intercepted at the border and jailed – to be released from prison and allowed to leave the Soviet Union to join her husband [AN18].
where the choice of the branch of the square root in the exponent is dictated by the
need to avoid the exponential growth at infinity. Sveshnikov points out that Ignatowsky’s approach does not depend on the geometry of the domain and hence is of
more universal nature than that of A. Sommerfeld [Som12], which is the selection of
the solution to (1.1) satisfying the Sommerfeld radiation condition

$$
\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - iku \right) = 0,
$$

(1.4)
in agreement with (1.3).

Let us also mention the limiting amplitude principle [TS48] (the terminology also
introduced in [Sve50]) which specifies a solution to (1.1) by

$$
u(x, t) = \lim_{t \to +\infty} \psi(x, t) e^{ikt},
$$

where \(\psi(x, t)\) is a solution to the wave equation

$$
\partial^2_t \psi - \Delta \psi = f(x) e^{-ikt}, \quad t > 0; \quad \psi|_{t=0} = 0, \quad \partial_t \psi|_{t=0} = 0.
$$

(1.5)

Thus, \(u\) is the limiting amplitude of the periodic vibration building up under the action
of a periodic force for a long time. This corresponds to using the retarded Green func-
tion, represented by the convolution with \(G_{\text{ret}}(x, t) = \frac{\delta(t-|x|)}{4\pi|x|}\), yielding the solution
to (1.5) in the form

$$
\psi(x, t) = G_{\text{ret}} \ast (f(x) e^{-ikt}) = \iint_{|x-y|<t} \frac{f(y) e^{-ik(t-|x-y|)}}{4\pi|x-y|} dy \sim e^{ik|x|-ikt},
$$
in agreement both with the limiting absorption principle (1.2) (cf. (1.3)) and with the
Sommerfeld radiation condition (1.4).

Presently, a common meaning of the LAP is the existence of a limit of the resolvent
at a given point of the essential spectrum. While the resolvent of \(A : X \to X\) cannot
have a limit at the essential spectrum as an operator in \(X\), it can have a limit as a mapping

\[(A - zI)^{-1} : E \to F,\]

where \(E\) and \(F\) are some Banach spaces such that the embeddings \(E \hookrightarrow X \hookrightarrow F\) are
dense and continuous. Historically, this idea could be traced back to eigenfunction ex-
pansions [Wey10, Car34, Tit46] and Krein’s method of directing functionals [Kre46, 
Kre48] (see [AG81, Appendix II.7]). This was developed in [Pov50, Pov53, GK55, 
Ber57, Bir61] (see also rigged spaces in [GV61, I§4], also known as equipped spaces
and related to Gelfand’s triples from [GK55]). Gradually the theory takes the form
of estimates on the limit of the resolvent at the essential spectrum in certain spaces;
this further development becomes clearer in [Eid62, Vai66, Eid69] (the convergence
of the resolvent is in the sense of distributions), then in [Rej69, Lemma 6.1] (where
" certain spaces are introduced), and finally in [Agm70, Theorem 2.2], [Yam73, Theo-
rem 4.1] (for Dirac operators), and [Agm75, Appendix A], where the convergence of
the resolvent is specified with respect to weighted \(L^2\) spaces. See also [Kur78] and
[BAD87]. Let us also mention that in [Agm98] this same approach – to consider
the resolvent as a mapping from $\mathbf{E}$ to $\mathbf{F}$, with the embeddings $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}$ being dense
and continuous – is used to define resonances of an operator as poles of the analytic
continuation of its resolvent.

**Remark 1.1** Such an approach is not universal since such a definition of resonances
depends on the choice of regularizing spaces $\mathbf{E}$, $\mathbf{F}$. By [Agm98, Proposition 4.1],
the set of resonances is the same if $\mathbf{E}_i$ and $\mathbf{F}_j$, $i = 1, 2$, satisfy the following additional
assumptions:

(I) The set $\mathbf{E}_1 \cap \mathbf{E}_2$ (identified with a subset of $\mathbf{X}$) is dense in both $\mathbf{E}_1$ and $\mathbf{E}_2$;

(II) There exists a Banach space $\mathbf{F}$ containing both $\mathbf{F}_1$ and $\mathbf{F}_2$ as linear subsets with
embeddings which are continuous.

See Example 1.2 and Theorem 3.22 below.

Perhaps the simplest example of LAP is covered by S. Agmon in [Agm75, Lemma
A.1]: by that lemma, the operator $(\partial_x - z I)^{-1}$, $z \in \mathbb{C}$, $\text{Re} z \neq 0$, is uniformly bounded
as an operator from $L^2_s(\mathbb{R})$ to $L^2_{-s}(\mathbb{R})$, $s > 1/2$, and has a limit (in the uniform
operator topology) as $\text{Re} z \to \pm 0$. For example, for $\text{Re} z < 0$, the solution to the equation $(\partial_x - z)u = f$ is given by the operator $f \mapsto u(x) = \int_{-\infty}^{\infty} e^{z(x-y)} f(y) \, dy,$
which is bounded from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ and hence from $L^2_s(\mathbb{R})$ to $L^2_{-s}(\mathbb{R})$, $s > 1/2,$
uniformly in $z \in \mathbb{C}$, $\text{Re} z < 0$. Here we use the standard notation

\[ L^p_s(\mathbb{R}^d) = \{ u \in L^p_{\text{loc}}(\mathbb{R}^d); \langle \cdot \rangle^s u \in L^p(\mathbb{R}^d), \| u \|_{L^p_s} = \| \langle \cdot \rangle^s u \|_{L^p} \}, \tag{1.6} \]

for any $p \in [1, +\infty]$, $s \in \mathbb{R}$, $d \in \mathbb{N}$, with $\langle x \rangle = (1 + x^2)^{1/2}$. Agmon then shows
that the LAP is available for the Laplacian when the spectral parameter approaches
the bulk of the essential spectrum: by [Agm75, Theorem 4.1], for $s, s' > 1/2$, the
resolvent

\[ R_0^{(d)}(z) = (-\Delta - z I)^{-1} : L^2_s(\mathbb{R}^d) \to L^2_{-s'}(\mathbb{R}^d), \quad z \in \mathbb{C} \setminus \mathbb{R}^+, \quad d \geq 1, \tag{1.7} \]

is bounded uniformly for $z \in \Omega \setminus \overline{\mathbb{R}^+}$, for any open neighborhood $\Omega \subset \mathbb{C}$ such that $\mathbb{R} \not\ni \{0\}$, and has limits as $z \to z_0 \pm i0$, $z_0 > 0$. For the sharp version (the $\mathbf{B} \to \mathbf{B}^*$ continuity of the resolvent in the Agmon–Hörmander spaces), see [Yaf10,
Proposition 6.3.6].

While the mapping (1.7) has a limit as $z \to z_0 \pm i0$ with $z_0 > 0$, for any $d \geq 1$, the behaviour at $z_0 = 0$ depends on $d$. For example, in three dimensions, as long as $s, s' > 1/2$ and $s + s' \geq 2$, the mapping (1.7), represented by the convolution with

\[ (4\pi |x|)^{-1} e^{-|x| \sqrt{-z}}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{R}^+}, \quad \text{Re} \sqrt{-z} > 0, \]

remains uniformly bounded and has a limit as $z \to z_0 = 0$. A similar boundedness of the resolvent in an open neighborhood of the threshold $z_0 = 0$ persists in higher dimensions, but breaks down in dimensions $d \leq 2$. In particular, for $d = 1$, the resolvent is represented by the convolution with

\[ e^{-|x| \sqrt{-z}}/(2\sqrt{-z}), \quad z \in \mathbb{C} \setminus \overline{\mathbb{R}^+}, \quad \text{Re} \sqrt{-z} > 0, \]

and cannot have a limit as $z \to 0$ as a mapping $\mathbf{E} \to \mathbf{F}$ as long as $\mathbf{E}, \mathbf{F}$ are weighted Lebesgue spaces at the same time,
see Example 1.2 below). There is a similar situation in two dimensions. We say that the threshold $z_0 = 0$ is a regular point of the essential spectrum for $d \geq 3$ and that it is a virtual level if $d \leq 2$.

**Example 1.2** While the limit of the resolvent $(-\partial_x^2 - zI)^{-1}$, $z \to z_0 = 0$, does not exist in the weak operator topology of mappings $L^2_s(\mathbb{R}) \to L^2_{-s}(\mathbb{R})$ with arbitrarily large $s > 1$, this limit exists in the weak operator topology of mappings $E \to F$ if one takes

$$E = \{ u \in L^2_s(\mathbb{R}) : \hat{u}(\xi) \text{ vanishes of order } \tau > 1 \text{ at } \xi = 0 \}, \quad F = L^2_{-s}(\mathbb{R}), \quad s > 1,$$

with $\|u\|_E := \|u\|_{L^2_s} + \limsup_{\xi \to 0} |\xi|^{-\tau} |\hat{u}(\xi)|$. Both $L^2_s(\mathbb{R})$ and $E$ are densely and continuously embedded into $X = L^2(\mathbb{R})$, while $E \cap L^2_s(\mathbb{R}) = E$ is not dense in $L^2_s(\mathbb{R})$ (cf. Remark 1.1): for a fixed $v \in L^2_s(\mathbb{R}^d)$ with $\hat{v}(0) \neq 0$ and for any $u \in E$, one has

$$\| u - v \|_{L^2_s} = \| \hat{u} - \hat{v} \|_{H^s(\mathbb{R}^d)} \geq c_s |\hat{u}(0) - \hat{v}(0)| = c_s |\hat{v}(0)|,$$

where $c_s > 0$ depends only on $s > d/2$; thus the left-hand side cannot approach zero.

### 2 Virtual levels

**History of virtual levels.** Virtual levels appeared first in the nuclear physics, in the study of neutron scattering on protons by E. Wigner [Wig33]. While a proton and a neutron with parallel spins form a spin-one deuteron (Deuterium’s nucleus), which is stable, with the binding energy around 2.2 MeV, when the spins of the particles are antiparallel, their binding energy is near zero. It was not clear for some time whether the corresponding spin-zero state is real or virtual, that is, whether the binding energy was positive or negative; see, for instance, [Fer35], where the word “virtual” appears first. It turned out that this state was virtual indeed [AF36], with a small negative binding energy, around $-67$ KeV. The resulting increase in the total cross-section of the neutron scattering on protons is interpreted as a resonance of the incoming wave with this “virtual state” corresponding to the energy $E \approx 0$.

Mathematically, virtual levels correspond to particular singularities of the resolvent at the essential spectrum. This idea goes back to J. Schwinger [Sch60b] and was further addressed by M. Birman [Bir61], L. Faddeev [Fad63], B. Simon [Sim73, Sim76], B. Vainberg [Vai68, Vai75], D. Yafaev [Yaf74, Yaf75], J. Rauch [Rau78], and A. Jensen and T. Kato [JK79], with the focus on Schrödinger operators in three dimensions. Higher dimensions were considered in [Jen80, Yaf83, Jen84]. An approach to more general symmetric differential operators was developed in [Wei99]. The virtual levels of nonselfadjoint Schrödinger operators in three dimensions appeared in [CP05]. Dimensions $d \leq 2$ require special attention since the free Laplace operator has a virtual level at zero (see [Sim76]). The one-dimensional case is covered in [BGW85, BGK87]. The approach from the latter article was further developed in
[BGD88] to two dimensions (if $\int_{\mathbb{R}^2} V(x) \, dx \neq 0$) and then in [JN01] (with this condition dropped) who give a general approach in all dimensions, with the regularity of the resolvent formulated via the weights which are square roots of the potential (and consequently not optimal). There is an interest in the subject due to dependence of dispersive estimates on the presence of virtual levels at the threshold point, see e.g. [JK79, Yaf83, ES04, Yaj05] in the context of Schrödinger operators; the Dirac operators are treated in [Bou06, Bou08, EG17, EGT19]. Let us mention the dichotomy between a virtual level and an eigenvalue manifested in the large-time behavior of the heat kernel and the behavior of the Green function near criticality; see [Pin92, Pin04]. We also mention recent articles [BBV20] on properties of virtual states of selfadjoint Schrödinger operators and [GN20] proving the absence of genuine (non-$L^2$) virtual states of selfadjoint Schrödinger operators and massive and massless Dirac operators, as well as giving classification of virtual levels and deriving properties of eigenstates and virtual states.

Equivalent characterizations of virtual levels. The definition of virtual levels has been somewhat empirical; one would say that there were a virtual level at the threshold of the Schrödinger operator if a certain arbitrarily small perturbation could produce a (negative) eigenvalue. To develop a general approach for nonselfadjoint operators, we notice that the following properties of the threshold $z_0 = 0$ of the Schrödinger operator $H = -\Delta + V(x), x \in \mathbb{R}^d, d \geq 1, V \in C_{\text{comp}}(\mathbb{R}^d, \mathbb{C})$, are related:

(P1) There is a nonzero solution to $H \psi = z_0 \psi$ from $L^2$ or a certain larger space;

(P2) $R(z) = (H - zI)^{-1}$ has no limit in weighted spaces as $z \to z_0$;

(P3) Under an arbitrarily small perturbation, an eigenvalue can bifurcate from $z_0$.

For example, properties (P1) – (P3) are satisfied for $H = -\partial_x^2$ in $L^2(\mathbb{R})$ considered with domain $\mathcal{D}(H) = H^2(\mathbb{R})$. Indeed, the equation $-\partial_x^2 \psi = 0$ has a bounded solution $\psi(x) = 1$; while non-$L^2$, it is “not as bad as a generic solution” to $(-\partial_x^2 + V(x))\psi = 0$ with $V \in C_{\text{comp}}(\mathbb{R})$, which may grow linearly at infinity. The integral kernel of the resolvent $R^{(1)}_0(z) = (-\partial_x^2 - zI)^{-1}, z \in \mathbb{C} \setminus \mathbb{R}_+$, contains a singularity at $z = 0$:

$$R^{(1)}_0(x, y; z) = \frac{e^{-|x-y|\sqrt{-z}}}{2\sqrt{-z}}, \quad x, y \in \mathbb{R}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \quad \text{Re} \sqrt{-z} > 0, \quad (2.1)$$

and has no limit as $z \to 0$ even in weighted spaces. Under a small perturbation, an eigenvalue may bifurcate from the threshold (see e.g. [Sim76]). Indeed, for the perturbed operator $H_g = -\partial_x^2 - g\mathbb{1}_{[-1,1]}, 0 < g \ll 1$, there is a relation

$$(-\partial_x^2 - g\mathbb{1}_{[-1,1]})\psi(x) = -\kappa^2 \psi(x), \quad \psi(x) = \begin{cases} c_1 e^{-\kappa|x|}, & |x| > 1, \\ c_2 \cos \left( x \sqrt{g - \kappa^2} \right), & |x| \leq 1, \end{cases}$$

where we assume that $\kappa \in (0, g^{1/2})$. The eigenvalue $E_g := -\kappa^2$ is obtained from the continuity of $-\partial_x \psi / \psi$ at $x = 1 \pm 0$:

$$\kappa = \sqrt{g - \kappa^2} \tan \sqrt{g - \kappa^2} = g - \kappa^2 + O((g - \kappa^2)^2),$$
hence $\kappa = g + O(g^2)$, leading to $E_g = -\kappa^2 = -g^2 + O(g^3)$. In this case, when properties \((P1) - (P3)\) are satisfied, one says that $z_0 = 0$ is a \textit{virtual level}; the corresponding nontrivial bounded solution $\psi(x) = 1$ of $-\partial_x^2 \psi = 0$ is a \textit{virtual state}.

On the contrary, properties \((P1) - (P3)\) are not satisfied for $H = -\Delta$ in $L^2(\mathbb{R}^3)$, with $\mathcal{D}(H) = H^2(\mathbb{R}^3)$. Regarding \((P1)\), we notice that nonzero solutions to $(-\Delta + V)\psi = 0$ (with certain compactly supported potentials) can behave like the Green function, $\sim |x|^{-1}$ as $|x| \to \infty$, and one expects that this is what virtual states should look like, while nonzero solutions to $\Delta \psi = 0$ cannot have uniform decay as $|x| \to \infty$, so should not qualify as virtual states; the integral kernel of $R_0^{(3)}(z) = (-\Delta - zI)^{-1}$,

$$R_0^{(3)}(x, y; z) = \frac{e^{-\sqrt{-z}|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \quad \text{Re} \sqrt{-z} > 0, \quad (2.2)$$

remains pointwise bounded as $z \to 0$ and has a limit in the space of mappings $L^2(\mathbb{R}^3) \to L^2, s' > 1/2, s + s' > 2$ (see e.g. [JK79]), failing \((P2)\); finally, small perturbations cannot produce negative eigenvalues (this follows from the Hardy inequality), so \((P3)\) also fails. In this case, we say that $z_0 = 0$ is a \textit{regular} point of the essential spectrum.

We claim that the properties \((P1) - (P3)\) are essentially equivalent, even in the context of the general theory [BC21]. These properties are satisfied when $z_0$ is either an eigenvalue of $H$ or, more generally, a \textit{virtual level}. To motivate the general theory, we can start from the Laplace operator in one dimension, considering the problem

$$(-\partial_x^2 - z)u(x) = f(x), \quad u(x) \in \mathbb{C}, \quad x \in \mathbb{R}. \quad (2.3)$$

For any $f \in C_{\text{comp}}(\mathbb{R})$, there is a $C^2$-solution to \((2.3)\). If we consider $z \in \mathbb{C} \setminus \mathbb{R}_+$, then the natural choice of a solution is

$$u(x) = (R_0^{(1)}(z)f)(x) := \int_\mathbb{R} R_0^{(1)}(x, y; z)f(y) \, dy,$$

where the resolvent $R_0^{(1)}(z) = (-\Delta - zI)^{-1}$ has the integral kernel $R_0^{(1)}(x, y; z)$ from \((2.1)\). This integral kernel is built of solutions $e^{\pm x\sqrt{-z}}$; the choice of such a combination is dictated by the desire to avoid solutions exponentially growing at infinity. For $z \neq 0$, since $R_0^{(1)}(x, y; z)$ is bounded, the mapping $f \mapsto R_0^{(1)}f$ defines a bounded mapping $L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$. This breaks down at $z = 0$, since $e^{\pm x\sqrt{-z}}$ are linearly dependent when $z = 0$. To solve \((2.3)\) at $z = 0$, one can use the convolution with the fundamental solution $G(x) = |x|/2 + xC$, with any $C \in \mathbb{C}$. While such fundamental solutions provide a solution $u = G \ast f$ to \((2.3)\), this solution may no longer be from $L^\infty$; any of the above choices of $G$ would no longer be bounded as a mapping $L^1 \to L^\infty$. This problem is resolved if a potential $V \in C_{\text{comp}}(\mathbb{R}, \mathbb{C})$ is introduced into \((2.3)\),

$$(-\partial_x^2 + V - z_0)u = f, \quad x \in \mathbb{R}, \quad (2.4)$$
so that the Jost solution $\theta_-(x)$ to $(-\partial_x^2 + V)u = 0$ with $\lim_{x \to -\infty} \theta_-(x) = 1$, tends to infinity as $x \to +\infty$ and is linearly independent with the Jost solution $\theta_+(x)$, $\lim_{x \to +\infty} \theta_+(x) = 1$. To construct a fundamental solution to (2.4) at $z_0 = 0$, we set

$$G(x, y; z_0) = \frac{1}{W[\theta_+, \theta_-](y)} \begin{cases}
\theta_-(y)\theta_+(x), & x > y, \\
\theta_-(x)\theta_+(y), & x < y,
\end{cases}$$

with $W[\theta_+, \theta_-](y) = \theta_+(y)\theta_-'(y) - \theta_+'(y)\theta_-'(y)$, the Wronskian. This will work if $|\theta_-'(x)|$ grows as $x \to +\infty$ (and similarly if $|\theta_+'(x)|$ grows as $x \to -\infty$); if, on the other hand, $\theta_\pm$ remain bounded, then, as the matter of fact, these functions are linearly dependent, their Wronskian is zero, and (2.5) is not defined. In this construction the space $L^\infty$ appears twice: it contains the range of $G(z_0)|_{L^\infty}$, $s > 3/2$, when $\theta_\pm$ are linearly independent (see [BC21]), and it is the space where $\theta_\pm$ live when they are linearly dependent. This is not a coincidence: from $-u'' = f \in C_{\text{comp}}^\infty(\mathbb{R})$, we can write $-u'' + Vu = f + Vu$, and then $u = (-\partial^2_x + V - z_0I)^{-1}(f + Vu)$ is in the range of $(-\partial^2_x + V - z_0I)^{-1}(C_{\text{comp}}^\infty(\mathbb{R})) \subset L^\infty(\mathbb{R})$.

We point out that in the case of general exterior elliptic problems the above dichotomy – either boundedness of the truncated resolvent or existence of a nontrivial solution to a homogeneous problem with appropriate radiation conditions – was studied by B. Vainberg [Vai75].

**Example 2.1** Here is an example of virtual levels at $z_0 = 0$ of a Schrödinger operator in $\mathbb{R}^3$ from [Yaf75]. Let $u$ be a solution to $-\Delta u + Vu = 0$ in $\mathbb{R}^3$. Taking the Fourier transform, we arrive at $\hat{u}(\xi) = -\hat{V}u(\xi)/\xi^2$. The right-hand side is not in $L^2_{\text{loc}}(\mathbb{R}^3)$ if $\hat{V}u(\xi)$ does not vanish at $\xi = 0$; this situation corresponds to zero being a virtual level, with the corresponding virtual state $u(x) \sim |x|^{-1}$, $|x| \gg 1$. One can see that in the case of the Schrödinger operator in $\mathbb{R}^3$ the space of virtual levels is at most one-dimensional. A similar approach in two dimensions gives

$$\hat{u}(\xi) = -\frac{\hat{V}u(\xi)}{\xi^2} = -\frac{c_0 + c_1\xi_1 + c_2\xi_2 + O(\xi^2)}{\xi^2}, \quad \xi \in \mathbb{R}^2,$$

indicating that the space of virtual states at $z_0 = 0$ of the Schrödinger operator in $\mathbb{R}^2$ could consist of up to one “s-state” approaching a constant value as $|x| \to \infty$ and up to two “p-states” behaving like $\sim (c_1x_1 + c_2x_2)/|x|^2$ for $|x| \gg 1$.

**Relation to critical Schrödinger operators.** In the context of positive-definite symmetric operators, a dichotomy similar to having or not properties (P1) – (P3) – namely, either having a particular Hardy-type inequality or existence of a null state – is obtained by T. Weidl [Wei99], at that time a PhD. student of M. Birman and E. Laptev, as a generalization of Birman’s approach [Bir61, §1.7] which was based on closures of the space with respect to quadratic forms corresponding to symmetric positive-definite operators (in the spirit of the Krein–Vishik–Birman extension theory [Kre47, Vis52, Bir56]). This approach is directly related to the research on subcritical and critical Schrödinger operators [Sim81, Mur86, Pin88, Pin90, GZ91, PT06,
PT07, TT08, Dev14, LP18, LP20]. Let us present the following result from [PT06], which we write in the particular case of $\Omega = \mathbb{R}^d$ and $V \in C_{\text{comp}}(\mathbb{R}^d, \mathbb{R})$:

Let $H = -\Delta + V$ with $V \in C_{\text{comp}}(\mathbb{R}^d, \mathbb{R})$ be a Schrödinger operator in $L^2(\mathbb{R}^d)$, and assume that the associated quadratic form

$$a[u] := \int_{\mathbb{R}^d} (|\nabla u|^2 + V|u|^2) \, dx$$

is nonnegative on $C^\infty_{\text{comp}}(\mathbb{R}^d)$. Then either there is a continuous function $w(x) > 0$ such that $\int_{\mathbb{R}^d} w|u|^2 \, dx \leq a[u]$ for any $u \in C^\infty_{\text{comp}}(\mathbb{R}^d)$ (one says that $a[\cdot]$ has a weighted spectral gap), or there is a sequence $\varphi_j \in C^\infty_{\text{comp}}(\mathbb{R}^d)$ such that $a[\varphi_j] \to 0$, $\varphi_j \to \varphi > 0$ locally uniformly on $\mathbb{R}^d$ (then one says that $a[\cdot]$ has a null state $\varphi$).

Let us mention that in the former case, when $a[\cdot]$ has a weighted spectral gap, the operator $H$ is subcritical (that is, it admits a positive Green’s function), and that in the latter case, when $a[\cdot]$ has a null state, $H$ is critical. This null state coincides with Agmon’s ground state, which can be characterized as a state with minimal growth at infinity from [Agm82, Definitions 4.1, 5.1]. See [Pin88, Pin90, PT06] for more details.

A null state, or Agmon’s ground state, corresponds to a virtual level at the bottom of the spectrum, in the following sense:

**Lemma 2.2** A nonnegative Schrödinger operator $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$, with $V \in C_{\text{comp}}(\mathbb{R}^d, \mathbb{R})$, has a null state $\varphi$ if any compactly supported negative perturbation $H - W$ of $H$, with $W \in C_{\text{comp}}(\mathbb{R}^d)$, $W \geq 0$, $W \neq 0$, produces a negative eigenvalue.

For the converse, we impose a stronger requirement that $V \in C^m_{\text{comp}}(\mathbb{R}^d, \mathbb{R})$, $m \geq \max(0, \lfloor n/2 \rfloor - 1)$, supp $V \subset K \subset \mathbb{R}^d$. If an arbitrary negative perturbation $H - W$ of $H$, with $W \in C^m_{\text{comp}}(\mathbb{R}^d, \mathbb{R})$, supp $W \subset K$, $W \geq 0$, $W \neq 0$, produces a negative eigenvalue, then $H$ has a null state.

**Proof.** Let $\varphi > 0$ be a null state of $H$ and let $\varphi_j$ be a sequence such that $\varphi_j \to \varphi$ locally uniformly on $\mathbb{R}^d$ and such that $a[\varphi_j] \to 0$ as $j \to \infty$. Let $W \in C_{\text{comp}}(\mathbb{R}^d)$, $W \geq 0$, $W \neq 0$. Then

$$\lim_{j \to \infty} \langle \varphi_j, (H - W) \varphi_j \rangle = \lim_{j \to \infty} (a[\varphi_j] - \langle \varphi_j, W \varphi_j \rangle) = -\langle \varphi, W \varphi \rangle < 0$$

(we took into account the convergence $\varphi_j \to \varphi$, locally uniformly on $\mathbb{R}^d$), hence $\langle \varphi_j, (H - W) \varphi_j \rangle < 0$ for some $j \in \mathbb{N}$, and so the Rayleigh quotient for $H - W$ is strictly negative, leading to $\sigma(H - W) \cap \mathbb{R}_+ \neq \emptyset$.

Let us prove the converse statement. Let supp $V \subset K \subset \mathbb{R}^d$ and let there be perturbations $W_j \in C^m_{\text{comp}}(\mathbb{R}^d, \mathbb{R})$, $j \in \mathbb{N}$, with supp $W_j \subset K$, $W_j \geq 0$, $W_j \neq 0$ for all $j$, and with $\sup_{x \in \mathbb{R}^d} |\partial^\beta_j W_j(x)| \to 0$ as $j \to \infty$ for all multiindices $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m$. By the assumption of the Lemma, $\lambda_j := \inf \sigma(H + W_j) < 0$ (thus $\lambda_j \to 0-$ as $j \to \infty$). Let $\psi_j \in L^2(\mathbb{R}^d, \mathbb{C})$ be the corresponding eigenfunctions, which can be shown to be from $H^{m+2}(\mathbb{R}^d) \subset C^\alpha(\mathbb{R}^d)$, $\forall \alpha < 1/2$ (having the uniform bound in
due to Harnack’s inequality for Schrödinger operators \( \| \psi \| \) of distributions. Since converges, uniformly on compacts. The limit function where \( \langle \psi, (H + W_j)\psi \rangle = \lambda_j \langle \psi_j, \psi_j \rangle < 0 \quad \forall j \in \mathbb{N}, \)

where \( \langle \psi_j, W_j\psi_j \rangle \to 0 \) (due to the convergence \( \psi_j \to \varphi \), \( \text{supp } W_j \subset K \), and due to \( \| W_j \|_{L^\infty} \to 0 \) as \( j \to \infty \)) while \( a[\psi_j] \geq 0 \), one can see that \( a[\psi_j] \to 0 \). Moreover, due to Harnack’s inequality for Schrödinger operators [CFG86, Theorem 2.5], since \( \varphi(x_0) = 1 \), one has \( \varphi(x) \neq 0 \) for all \( x \in \mathbb{R}^d \). (In [CFG86], the proof is given for \( d \geq 3 \) but is shown to apply to \( d = 2 \) as well; the statement for \( d = 1 \) is trivial by the ODE uniqueness theory.) Thus the limit function \( \varphi \) is a null state. \( \square \)

3 General theory of virtual levels in Banach spaces

We now sketch our approach to virtual levels from [BC21]. Let \( X \) be an infinite-dimensional complex Banach space and let \( A \in \mathcal{C}(X) \) be a closed operator with dense domain \( \mathcal{D}(A) \subset X \). We assume that there are some complex Banach spaces \( E, F \) with embeddings \( E \hookrightarrow X \hookrightarrow F \). We will assume that the operator \( A \) and the “regularizing” spaces \( E \) and \( F \) satisfy the following assumption.

**Assumption 3.1** 1. The embeddings

\[
E \hookrightarrow X \hookrightarrow F
\]

are dense and continuous.

2. The operator \( A : X \to X \), considered as a mapping \( F \to F \),

\[
A_{E \hookrightarrow F} : F \to F, \quad \mathcal{D}(A_{E \hookrightarrow F}) = j(\mathcal{D}(A)), \quad A_{E \hookrightarrow F} : y \mapsto j(Ax) \quad \text{if } y = j(x),
\]

is closable in \( F \), with closure \( \hat{A} \in \mathcal{C}(F) \) and domain \( \mathcal{D}(\hat{A}) \supset j(\mathcal{D}(A)) \).

3. Denote

\[
\mathcal{D}(A_{E \hookrightarrow F}) = \{ \phi \in E : j(\mathcal{D}(A)), A_{E \hookrightarrow F} \}
\]

and

\[
\mathcal{D}(\hat{A}_{E \hookrightarrow F}) = \{ \phi \in E : j \circ i(\mathcal{D}(\hat{A})), \hat{A}_{E \hookrightarrow F} \}
\]

The space \( j \circ i(\mathcal{D}(A_{E \hookrightarrow E})) \) is dense in \( j \circ i(\mathcal{D}(\hat{A}_{E \hookrightarrow E})) \) in the topology induced by the graph norm of \( \hat{A} \), defined by

\[
\| \psi \|_{\hat{A}} = \| \psi \|_F + \| \hat{A}\psi \|_F, \quad \psi \in \mathcal{D}(\hat{A}) \subset F.
\]
We note that Assumption 3.1 is readily satisfied in the usual examples of differential operators. For convenience, from now on, we will assume that $E \subset X \subset F$ (as vector spaces) and will omit $i$ and $j$ in numerous relations.

**Definition 3.2 (Virtual levels)** Let $A \in \mathcal{C}(X)$ and $E \hookrightarrow X \hookrightarrow F$ satisfy Assumption 3.1. Let

$$
\Omega \subset \mathbb{C} \setminus \sigma(A)
$$

be a connected open set such that $\sigma_{\text{ess}}(A) \cap \partial \Omega$ is nonempty. We say that a point $z_0 \in \sigma_{\text{ess}}(A) \cap \partial \Omega$ is a point of the essential spectrum of $A$ of rank $r \in \mathbb{N}_0$ relative to $(\Omega, E, F)$ if it is the smallest value for which there is $B \in \mathcal{B}_{00}(F, E)$ (with $\mathcal{B}_{00}$ denoting bounded operators of finite rank) of rank $r$ such that

$$
\Omega \cap \sigma(A + B) \cap \mathbb{D}_{\delta}(z_0) = \emptyset
$$

for some $\delta > 0$, and there exists the following limit in the weak operator topology of mappings $E \to F$:

$$
(A + B - z_0I)^{-1}_{\Omega, E, F} := \text{w-lim}_{z \to z_0, z \in \Omega} (A + B - zI)^{-1} : E \to F.
$$

Points of rank $r = 0$ relative to $(\Omega, E, F)$ (so that there is a limit (3.2) with $B = 0$) are called regular points of the essential spectrum relative to $(\Omega, E, F)$.

If $z_0$ is of rank $r \geq 1$ relative to $(\Omega, E, F)$, we call it an exceptional point of rank $r$ relative to $(\Omega, E, F)$, or a virtual level of rank $r$ relative to $(\Omega, E, F)$. The corresponding virtual states are defined as elements of the space

$$
\mathcal{M}_{\Omega, E, F}(A - z_0I) := \{ \Psi \in \mathcal{H}(A - z_0I)^{-1}_{\Omega, E, F}) : (A_{F \to F} - z_0I)\Psi = 0 \};
$$

with any $B \in \mathcal{B}_{00}(F, E)$ such that the limit (3.2) is defined (this space is of dimension $r$ and does not depend on the choice of $B$; see Theorem 3.16 below).

Above, $\sigma_{\text{ess}}(A)$ is F. Browder’s essential spectrum [Bro61, Definition 11]. It can be characterized as $\sigma(A) \setminus \sigma_0(A)$, with the discrete spectrum $\sigma_0(A)$ being the set of isolated points of $\sigma(A)$ with corresponding Riesz projectors having finite rank (see e.g. [BC19, Lemma III.125]). Let us emphasize that the existence of the limit (3.2) implicitly implies that there is $\delta > 0$ such that $\Omega \cap \sigma(A + B) \cap \mathbb{D}_{\delta}(z_0) = \emptyset$.

**Remark 3.3** Definition 3.2 allows one to treat generalized eigenfunctions corresponding to “threshold resonances” of a Schrödinger operator $A$ (not necessarily selfadjoint) and solutions to $(A - z_0I)u = 0$ with $z_0$ from the bulk of $\sigma_{\text{ess}}(A)$ which satisfy the Sommerfeld radiation condition as the same concept of virtual states $\Psi \in \mathcal{M}_{\Omega, E, F}(A - z_0I)$ (with appropriate choice of $\Omega$).

**Remark 3.4** In case when $z_0$ is a virtual level but not an eigenvalue, it seems reasonable to call it an (embedded) resonance. Note that the name threshold resonance seems misleading, since in the nonselfadjoint case a virtual level could be located at
any point of contact of the essential spectrum with the resolvent set, not necessarily at a threshold. (According to [How74], thresholds could be defined as (i) a branch point of an appropriate function, (ii) a point where the absolutely continuous part changes multiplicity, or (sometimes) (iii) an end point of the spectrum.)

**Remark 3.5** The dimension of the null space of a square matrix $M$ can be similarly characterized as $\dim \ker(M) = \min \{ \text{rank } N : \det(M + N) \neq 0 \}$. For example, for $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we can take $N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, in agreement with $\dim \ker(M) = 1$.

**Example 3.6** Let $A = -\Delta$ in $L^2(\mathbb{R}^3)$, $\mathcal{D}(A) = H^2(\mathbb{R}^3)$. By [Agm75, Appendix A], for any $s$, $s' > 1/2$ and $z_0 > 0$, the resolvent $(-\Delta - zI)^{-1}$ converges as $z \to z_0 \pm 0i$ in the uniform operator topology of continuous mappings $L^2_s(\mathbb{R}^3) \to L^2_{-s'}(\mathbb{R}^3)$. The two limits differ; the integral kernels of the limiting operators $(-\Delta - zI)^{-1}_{C_{\pm}}$ are given by $e^{\pm i|x-y|/z_0}/(4\pi|x-y|), x, y \in \mathbb{R}^3$. It follows that $z_0 > 0$ is a regular point of the essential spectrum of $-\Delta$ relative to $\Omega = \mathbb{C}_\pm$. Moreover, according to [JK79], there is a limit of the resolvent as $z \to z_0 = 0, z \in \mathbb{C} \setminus \mathbb{R}_+$, in the uniform operator topology of continuous mappings $L^2_s(\mathbb{R}^3) \to L^2_{-s'}(\mathbb{R}^3)$, $s, s' > 1/2$, $s + s' > 2$, hence $z_0 = 0$ is also a regular point of the essential spectrum (relative to $\Omega = \mathbb{C} \setminus \mathbb{R}_+$).

**Example 3.7** Consider the differential operator $A = -i\partial_x + V : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\mathcal{D}(A) = H^1(\mathbb{R})$, with $V$ the operator of multiplication by $V \in L^1(\mathbb{R})$. The solution to $(-i\partial_x + V - zI)u = f \in L^1(\mathbb{R}), z \in \mathbb{C}_+$, is given by

$$u(x) = i \int_{-\infty}^{x} e^{iz(x-y)-iW(x)+iW(y)} f(y) \, dy, \quad W(x) := \int_{-\infty}^{x} V(y) \, dy, \quad W \in L^\infty(\mathbb{R}).$$

For each $z \in \mathbb{C}_+$, the mapping $(A - zI)^{-1} : f \mapsto u$ is continuous from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$, with the bound uniform in $z \in \mathbb{C}_+$. Moreover, one can see that for each $z_0 \in \mathbb{R}$ there exists a limit $(A - z_0I)^{-1}_{\mathbb{C}_+, L^1, L^\infty} = \lim_{z \to z_0, z \in \mathbb{C}_+} (A - zI)^{-1}$ in the strong operator topology of mappings $L^1 \to L^\infty$; thus, any $z_0 \in \mathbb{R}$ is a regular point of the essential spectrum relative to $(\mathbb{C}_+, L^2_s(\mathbb{R}), L^2_{-s'}(\mathbb{R}))$ (and similarly relative to $(\mathbb{C}_-, L^2_s(\mathbb{R}), L^2_{-s'}(\mathbb{R}))$).

**Example 3.8** Consider the left shift $L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$, with $\sigma(L) = \sigma_{\text{ess}}(L) = \overline{\Delta_1}$. The matrix representations of $L - zI$ and $(L - zI)^{-1}$, $|z| > 1$, are given by

$$L - zI = \begin{bmatrix} z^{-1} & z^{-2} & \cdots \\ 0 & z^{-1} & \cdots \\ 0 & 0 & \ddots \end{bmatrix}, \quad z \in \mathbb{C}; \quad (L - zI)^{-1} = -\begin{bmatrix} z^{-1} & z^{-2} & \cdots \\ 0 & z^{-1} & \cdots \\ 0 & 0 & \ddots \end{bmatrix}, \quad z \in \mathbb{C} \setminus \overline{\Delta_1}.$$ 

From the above representation, one has $|((L - zI)^{-1})_i| \leq |z^{-1}x_i| + |z^{-2}x_{i+1}| + \cdots \leq \|x\|_{\ell^1}$, and moreover $\lim_{z \to 0} ((L - zI)^{-1})_i = 0$, for any $x \in \ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$ and any $z \in \mathbb{C}, |z| > 1$, hence $(L - zI)^{-1}$ defines a continuous linear mapping $\ell^1(\mathbb{N}) \to \ell^2(\mathbb{N})$. 

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c_0(\mathbb{N}), with the norm bounded (by one) uniformly in \( z \in \mathbb{C}, |z| > 1 \). For any \( |z_0| = 1 \), the mappings \((L - zI)^{-1} : \ell^1(\mathbb{N}) \to c_0(\mathbb{N})\) have a limit as \( z \to z_0, |z| > 1 \), in the weak operator topology (also in the strong operator topology). It follows that any of the boundary points of the spectrum of \( L \) (i.e., any \( z_0 \in \mathbb{C} \) with \( |z_0| = 1 \)) is a regular point of the essential spectrum relative to \((\mathbb{C} \setminus \overline{D}_1, \ell^1(\mathbb{N}), c_0(\mathbb{N}))\).

Let us construct an operator with a virtual level at \( z_0 \in \mathbb{C}, |z_0| = 1 \). Assume that \( K \in \mathcal{B}_{00}(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N})) \) has eigenvalue \( 1 \in \sigma(K|_\mu) \), with the corresponding eigenfunction \( \phi \in \ell^1(\mathbb{N}) \). Then the operator \( A = L - K(L - z_0I), \mathcal{D}(A) = \ell^2(\mathbb{N}) \), has a virtual level at \( z_0 \) since \( z_0 \) is a regular point of \( A + B \), with \( B = K(L - z_0I) : c_0(\mathbb{N}) \to \ell^1(\mathbb{N}) \) of finite rank (we note that \( L \) has a bounded extension onto \( c_0(\mathbb{N}) \)). The function \( \Psi = (L - z_0I)^{-1}_{\Omega, \ell^1, c_0} \phi \in c_0(\mathbb{N}) \) is a virtual state of \( \hat{A} = L - K(L - z_0I) \) corresponding to \( z_0 \), relative to \((\mathbb{C} \setminus \overline{D}_1, \ell^1(\mathbb{N}), c_0(\mathbb{N}))\), satisfying \((\hat{A} - z_0I)\Psi = 0\), with \( \hat{A} \) a closed extension of \( A \) onto \( c_0(\mathbb{N}) \).

**Example 3.9** Let \( X \) be an infinite-dimensional Banach space and let \( Z : X \to X, \psi \mapsto 0, \forall \psi \in X \), be the zero operator with \( \sigma(Z) = \sigma_{\text{ess}}(Z) = \{0\} \). Assume that \( E, F \) are Banach spaces with dense continuous embeddings \( E \hookrightarrow X \hookrightarrow F \). Let \( B \in \mathcal{B}_{00}(F, E) \). Let \( \epsilon > 0 \) be smaller than the absolute value of the smallest nonzero eigenvalue of \( B \) (there are finitely many nonzero eigenvalues since \( B \) is of finite rank), and define

\[
P_0 = -\frac{1}{2\pi i} \oint_{|\zeta| = \epsilon} (B - \zeta I)^{-1} d\zeta : X \to X
\]

to be a projection onto the eigenspace of \( B \) corresponding to eigenvalue \( \lambda = 0 \). Then, for \( z \in \mathbb{C} \setminus \{0\} \),

\[
(Z + B - zI)^{-1} P_0 = (B - zI)^{-1} P_0 = -z^{-1} P_0 : X \to X,
\]

hence

\[
(Z + B - zI)^{-1} P_0 = -z^{-1} P_0 : E \to F, \quad z \in \mathbb{C} \setminus \{0\},
\]

with the norm unbounded as \( z \to 0, z \neq 0 \). Thus, \( z_0 = 0 \) is an exceptional point of the essential spectrum of \( Z \) of infinite rank relative to \( \mathbb{C} \setminus \{0\} \) and arbitrary \( E, F \).

**Remark 3.10** Let us contrast virtual levels to spectral singularities [Nai54, Sch60a, Pav66, Lja67, Gus09, KLV19] (for a more general setting, see [Nag86]). We note that selfadjoint operators have no spectral singularities, although they could have virtual levels at threshold points; this shows that these two concepts differ.

**Remark 3.11** There is no direct relation of virtual levels to pseudospectrum [Lan75]. For \( A \in \mathcal{C}(X) \), one defines the \( \epsilon \)-pseudospectrum by

\[
\sigma_\epsilon(A) = \sigma(A) \cup \{ z \in \mathbb{C} \setminus \sigma(A) : \|(A - zI)^{-1}\| \geq \epsilon^{-1}\}.
\]

Since \( \sigma_\epsilon(-\Delta|_{L^2(\mathbb{R}^d)}) = \{ z \in \mathbb{C} : \text{dist } (z, \mathbb{R}_+) \leq \epsilon \} \) does not depend on the dimension \( d \geq 1 \), the behaviour of pseudospectrum near the threshold \( z_0 = 0 \) does not distinguish the presence of a virtual level at \( z_0 \) for \( d \leq 2 \) and its absence for \( d \geq 3 \).
The following key lemma is essentially an abstract version of [JK79, Lemma 2.4].

**Lemma 3.12 (Limit of the resolvent as the left and right inverse)** Let \( A \in \mathcal{C}(X) \) and \( E \hookrightarrow X \hookrightarrow F \) satisfy Assumption 3.1. Let \( \Omega \subset \mathbb{C} \setminus \sigma(A) \). Assume that \( z_0 \in \sigma_{\text{ess}}(A) \cap \partial \Omega \) is a regular point of the essential spectrum relative to \((\Omega, E, F)\), so that there exists a limit
\[
(A - z_0 I)^{-1}_{\Omega, E, F} := \lim_{z \to z_0, z \in \Omega} (A - z I)^{-1} : E \to F.
\]
This limit is both the left and the right inverse of \( \hat{A} - z_0 I : \mathcal{R}((A - z_0 I)^{-1}_{\Omega, E, F}) \to E \).

In applications one needs to consider not only finite rank perturbations but also relatively compact perturbations, allowing in place of \( B \) in (3.2) operators which are \( \hat{A} \)-compact, in the following sense.

**Definition 3.13** Let \( \hat{A} : F \to F \) and \( B : F \to E \) be linear, with \( \mathcal{D}(B) \supset \mathcal{D}(\hat{A}) \). We say that \( B \) is \( \hat{A} \)-compact if \( \mathcal{R}(B\{x \in \mathcal{D}(\hat{A}) : \|x\|_{\hat{A}} + \|Ax\|_E \leq 1\}) \subset E \) is precompact.

We denote the set of \( \hat{A} \)-compact operators for which the limit (3.2) exists by
\[
\mathcal{Q}_{\Omega, E, F}(A - z_0 I) = \{ B \text{ is } \hat{A} \text{-compact; } \exists \delta > 0, \Omega \cap \sigma(A + B) \cap \mathbb{D}_\delta(z_0) = \emptyset,
\exists \text{ w-lim}_{z \to z_0, z \in \Omega} (A + B - z I)^{-1} : E \to F \}.
\]

**Theorem 3.14 (Independence from the regularizing operator)** Let \( A \in \mathcal{C}(X) \) and \( E \hookrightarrow X \hookrightarrow F \) satisfy Assumption 3.1. Let \( \Omega \subset \mathbb{C} \setminus \sigma(A) \). Assume that \( z_0 \in \sigma_{\text{ess}}(A) \cap \partial \Omega \) is a regular point of the essential spectrum relative to \((\Omega, E, F)\), so that there is a limit 
\[
(A - z_0 I)^{-1}_{\Omega, E, F} := \lim_{z \to z_0, z \in \Omega} (A - z I)^{-1} : E \to F.
\]
Assume that \( B \in \mathcal{C}(F, E) \) is \( \hat{A} \)-compact. Then:

1. For each \( B \in \mathcal{C}(F, E) \) which is \( \hat{A} \)-compact and such that there exists \( \delta > 0 \) which satisfies \( \Omega \cap \sigma(A + B) \cap \mathbb{D}_\delta(z_0) = \emptyset \), the following statements are equivalent:
   (a) There is no nonzero solution to \((\hat{A} + B - z_0 I)\Psi = 0, \Psi \in \mathcal{R}((A - z_0 I)^{-1}_{\Omega, E, F})\);
   (b) There exists a limit
   \[
   (A + B - z_0 I)^{-1}_{\Omega, E, F} := \lim_{z \to z_0, z \in \Omega} (A + B - z I)^{-1} : E \to F.
   \]
   (That is, there is the inclusion \( B \in \mathcal{Q}_{\Omega, E, F}(A - z_0 I) \)).

2. If any (and hence both) of the statements from Part 1 is satisfied, then:
   (a) \( \mathcal{R}((A - z_0 I)^{-1}_{\Omega, E, F}) = \mathcal{R}((A + B - z_0 I)^{-1}_{\Omega, E, F}) \);
   (b) If the operators \((A - z I)^{-1}\) converge as \( z \to z_0, z \in \Omega \), in the strong or uniform operator topology of mappings \( E \to F \), then \((A + B - z I)^{-1}\) converge as \( z \to z_0, z \in \Omega \), in the same topology;
   (c) If there are Banach spaces \( E_0 \) and \( F_0 \) with dense continuous embeddings \( E \hookrightarrow E_0 \hookrightarrow X \hookrightarrow F_0 \hookrightarrow F \), such that the operator \((A - z_0 I)^{-1}_{\Omega, E, F} \) extends to a bounded mapping \((A - z_0 I)^{-1}_{\Omega, E, F} : E_0 \to F_0 \), then \((A + B - z_0 I)^{-1}_{\Omega, E, F} \) also extends to a bounded mapping \( E_0 \to F_0 \).
Remark 3.15  Regarding Theorem 3.14 (2c), it is possible that \((A + B - z_0 I)^{-1}_{\Omega,E,F}\) extends to a bounded map \(E_0 \to F_0\), yet there is no convergence \((A + B - z I)^{-1} \to (A + B - z_0 I)^{-1}_{\Omega,E,F}\) in the weak operator topology of mappings \(E_0 \to F_0\). For example, the resolvent of the Laplacian in \(\mathbb{R}^d\), \(d \geq 5\), converges in the weak operator topology of continuous linear mappings \(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), s + s' > 2\), as \(z \to z_0 = 0\), \(z \in \mathbb{C} \setminus \mathbb{R}_+\), only as long as \(s, s' > 1/2\), while the limit \((-\Delta - z_0 I)^{-1}_{\Omega,E,F}\) extends to continuous mappings \(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), L^2(\mathbb{R}^d) \to L^2_{-2}(\mathbb{R}^d)\).

Now we introduce the space of virtual states \(\mathcal{M}\). This space appears in [JK79] in the context of Schrödinger operators in \(\mathbb{R}^3\) (see also [Bir61, §1.7]).

Theorem 3.16 (LAP vs. existence of virtual states)  Let \(A \in \mathcal{C}(X)\) and \(E \hookrightarrow X \hookrightarrow F\) satisfy Assumption 3.1. Let \(\Omega \subset \mathbb{C} \setminus \sigma(A)\). Let \(z_0 \in \sigma_{ess}(A) \cap \partial \Omega\) be of rank \(r \in \mathbb{N}_0\) relative to \((\Omega, E, F)\). For \(B \in \mathcal{D}_{\Omega,E,F}(A - z_0 I)\) (which is nonempty), define the space of virtual states by

\[
\mathcal{M}_{\Omega,E,F}(A - z_0 I) := \{\psi \in \mathcal{M}((A + B - z_0 I)^{-1}_{\Omega,E,F}) : (A - z_0 I)\psi = 0\} \subset F,
\]

where \((A + B - z_0 I)^{-1}_{\Omega,E,F} : E \to F\). Then:

1. \(\mathcal{M}_{\Omega,E,F}(A - z_0 I)\) does not depend on the choice of \(B \in \mathcal{D}_{\Omega,E,F}(A - z_0 I)\);
2. There is the inclusion \(E \cap \ker (A - z_0 I) \subset \mathcal{M}_{\Omega,E,F}(A - z_0 I)\);
3. \(\dim \mathcal{M}_{\Omega,E,F}(A - z_0 I) = r\).

Example 3.17  Let \(A = -\partial_x^2\) in \(L^2(\mathbb{R})\), with \(\mathcal{D}(A) = H^2(\mathbb{R})\). We note that its resolvent \(R^{(1)}_0(z) = (A - z I)^{-1}, z \in \mathbb{C} \setminus \mathbb{R}_+\), with the integral kernel \(R^{(1)}_0(x, y; z) = e^{-\sqrt{-z}|x-y|}/(2\sqrt{-z})\), Re \(-z > 0\), does not extend to a linear mapping \(L^2(\mathbb{R}) \to L^2(\mathbb{R})\), for some particular \(s, s' \geq 0\), which would be bounded uniformly for \(z \in \mathbb{D}_\delta \setminus \mathbb{R}_+\) with some \(\delta > 0\). At the same time, if \(V \in C_{\text{comp}}([-a, a], \mathbb{C})\) is any potential such that the solution \(\theta_+(x)\) to \((-\partial_x^2 + V)\theta = 0, \theta|_{x = 0} = 1\), remains unbounded for \(x \leq 0\) (one can take \(V \geq 0\) not identically zero), so that it is linearly independent with \(\theta_-(x)\) (solution which equals one for \(x < -a\)), then for any \(s, s' > 1/2, s + s' \geq 2\), the resolvent \(R_V(z) = (A + V - z I)^{-1}\) extends to a bounded linear mapping \(L^2(\mathbb{R}) \to L^2_{-s}(\mathbb{R})\) for all \(z \in \mathbb{D}_\delta \setminus \mathbb{R}_+\) with some \(\delta > 0\) and has a limit in the strong operator topology as \(z \to z_0 = 0, z \not\in \mathbb{R}_+\); thus, \(z_0 = 0\) is a regular point of \(A + V\) relative to \(\mathbb{C} \setminus \mathbb{R}_+\). Since the operator of multiplication by \(V(x)\) is \(A\)-compact, \(z_0 = 0\) is a virtual level of \(A = -\partial_x^2\) in \(L^2(\mathbb{R})\) (relative to \(\mathbb{C} \setminus \mathbb{R}_+\)).

Definition 3.18 (Genuine virtual levels)  If \(\mathcal{M}_{\Omega,E,F}(A - z_0 I) \not\subset X\), then we say that \(z_0\) is a genuine virtual level of \(A\) relative to \(\Omega\), and call any \(\psi \in \mathcal{M}_{\Omega,E,F}(A - z_0 I) \setminus X\) a virtual state of \(A\) corresponding to \(z_0\) relative to \(\Omega\). A virtual level can be both an eigenvalue and a genuine virtual level, with a corresponding eigenfunction \(\psi \in X\) and a virtual state \(\psi \in \mathcal{M}_{\Omega,E,F}(A - z_0 I) \setminus X\).

Theorem 3.19 (LAP vs. bifurcations)  Let \(A \in \mathcal{C}(X)\) and \(E \hookrightarrow X \hookrightarrow F\) satisfy Assumption 3.1. Let \(\Omega \subset \mathbb{C} \setminus \sigma(A)\). Assume that \(z_0 \in \sigma_{ess}(A) \cap \partial \Omega\).
1. If there is a sequence of perturbations \( V_j \in \mathcal{B}(\mathbf{F}, \mathbf{E}) \), \( \lim_{j \to \infty} \| V_j \|_{\mathbf{F} \to \mathbf{E}} = 0 \), and a sequence of eigenvalues \( z_j \in \sigma_d(\Lambda + V_j) \cap \Omega \), \( z_j \to z_0 \), then there is no limit \( \text{w-lim}_{z \to z_0, z \in \Omega} (\Lambda - zI)^{-1} \) in the weak operator topology of mappings \( \mathbf{E} \to \mathbf{F} \).

2. Assume that \( z_0 \) is a virtual level of \( A \) of finite rank relative to \( (\Omega, \mathbf{E}, \mathbf{F}) \), and moreover assume that there is \( \delta > 0 \) and \( B \in \mathcal{B}_{00}(\mathbf{F}, \mathbf{E}) \) such that there is a limit

\[
(A + B - zI)^{-1}_{\Omega, \mathbf{E}, \mathbf{F}} := \text{s-lim}_{z \to z_0, z \in \Omega \cap \Omega_{\delta}(z_0)} (A + B - zI)^{-1}
\]

in the strong operator topology of mappings \( \mathbf{E} \to \mathbf{F} \). There is \( \delta_1 \in (0, \delta) \) such that for any sequence \( z_j \in \Omega \cap \Omega_{\delta_1}(z_0), z_j \to z_0 \), there is a sequence \( V_j \in \mathcal{B}_{00}(\mathbf{F}, \mathbf{E}), \quad \| V_j \|_{\mathbf{F} \to \mathbf{E}} \to 0, \quad z_j \in \sigma_d(\Lambda + V_j), \quad j \in \mathbb{N} \).

Example 3.20 (Virtual levels of \(-\Delta + V\) at \( z_0 \geq 0 \)) For \( x \in \mathbb{R}^3 \) and \( \zeta \in \overline{\mathbb{C}_+} \), define

\[
\psi(x, \zeta) = \begin{cases} 
\frac{e^{i\zeta|x|}}{|x|}, & |x| \geq 1, \\
\frac{(3 - |x|^2)/2)e^{i\zeta(1+|x|^2)/2}}, & 0 \leq |x| < 1,
\end{cases}
\]

so that \(-\Delta \psi = \zeta^2 \psi\) for \( x \in \mathbb{R}^3 \setminus \mathbb{B}_1^3 \). Since \( \psi(x, \zeta) \neq 0 \) for all \( x \in \mathbb{R}^3 \) and \( \zeta \in \overline{\mathbb{C}_+} \), we may define the potential \( V(x, \zeta) \) by the relation \(-\Delta \psi + V \psi = \zeta^2 \psi\). Then, for each \( \zeta \in \overline{\mathbb{C}_+} \), \( V(\cdot, \zeta) \in L^\infty(\mathbb{R}^3) \) is piecewise smooth, with \( \text{supp} V \subset \mathbb{B}_1^3 \). For \( \zeta \in \mathbb{C}_+ \), one has \( z = \zeta^2 \in \sigma_p(-\Delta + V(\zeta)) \), so for each \( \zeta_0 \geq 0 \) there is an eigenvalue family bifurcating from \( z_0 = \zeta_0^2 \in \sigma_{\text{ess}}(-\Delta + V(\cdot, \zeta_0)) = \mathbb{R}_+ \) into \( \mathbb{C}_+ \). By Theorem 3.19 (1), \( z_0 = \zeta_0^2 \) is a virtual level of \(-\Delta + V(x, \zeta)\).

Lemma 3.21 (The Fredholm alternative) Let \( A \in \mathcal{C}(\mathbf{X}) \) and \( \mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F} \) satisfy Assumption 3.1. Let \( \Omega \subset \mathbb{C} \setminus \sigma(A) \). Assume that \( z_0 \in \sigma_{\text{ess}}(A) \cap \partial \Omega \) is of rank \( r \in \mathbb{N}_0 \) relative to \( (\Omega, \mathbf{E}, \mathbf{F}) \). Then there is a projector \( P \in \text{End} (\mathbf{E}) \), with rank \( P = r \), such that for any \( B \in \mathcal{B}_{\Omega, \mathbf{E}, \mathbf{F}}(A - z_0I) \) the problem

\[
(\Lambda - z_0I)u = f, \quad f \in \mathbf{E}, \quad u \in \mathcal{N}(\Lambda - z_0I)^{-1}_{\Omega, \mathbf{E}, \mathbf{F}} \subset \mathbf{F},
\]

has a solution if and only if \( Pf = 0 \). This solution is unique under an additional requirement \( Qu = 0 \), with \( Q \in \text{End} (\mathbf{F}) \) a projection onto \( \mathcal{N}(\Lambda - z_0I)^{-1}_{\Omega, \mathbf{E}, \mathbf{F}} \subset \mathbf{F} \).

The next result is related to [Agm98, Proposition 4.1] (see Remark 1.1).

Theorem 3.22 (Independence from regularizing spaces) Let \( A \in \mathcal{C}(\mathbf{X}) \). Let \( \mathbf{E}_i \) and \( \mathbf{F}_i, \quad i = 1, 2 \), be Banach spaces with dense continuous embeddings

\[
\mathbf{E}_i \xleftarrow{\imath_i} \mathbf{X} \xrightarrow{\jmath_i} \mathbf{F}_i, \quad i = 1, 2.
\]

Assume that \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) are mutually dense, in the sense that \( \imath_i^{-1}(\imath_1(\mathbf{E}_1) \cap \imath_2(\mathbf{E}_2)) \) are dense in \( \mathbf{E}_i, \quad i = 1, 2 \), that \( \mathbf{F}_1, \mathbf{F}_2 \) are continuously embedded into a Hausdorff
vector space $G$, with $j_1(x) = j_2(x)$ (as an element of $G$) for each $x \in X$, and that there is an extension $\hat{A} \in \mathcal{C}(F_1 + F_2)$ of $A$ with dense domain $\mathcal{D}(\hat{A}) \subset F_1 + F_2$. Let
\[
\mathcal{D}(A_{E_i} \star E_i) = \{ \phi \in E_i : \iota_i(\phi) \in \mathcal{D}(A), \ A\iota_i(\phi) \in \iota_i(E_i) \}, \quad i = 1, 2
\]
and
\[
\mathcal{D}(\hat{A}_{E_i} \star E_i) = \{ \phi \in E_i : \iota_i(\phi) \in \mathcal{D}(\hat{A}), \ \hat{A}\iota_i(\phi) \in \iota_i(E_i) \}, \quad i = 1, 2,
\]
and assume that for $i = 1, 2$ the space $\iota_i(\mathcal{D}(A_{E_i} \star E_i))$ is dense in the space $\iota_i(\mathcal{D}(\hat{A}_{E_i} \star E_i))$ in the topology induced by the graph norm of $\hat{A}$ (it follows that both triples $A, E_1, F_1$ and $A, E_2, F_2$ satisfy Assumption 3.1 (3)).

Let $\Omega \subset \mathbb{C} \setminus \sigma(A)$ and assume that $z_0 \in \sigma_{ess}(A) \cap \partial \Omega$ and that $\mathcal{D}_{\Omega, E_i, F_i}(A - z_0 I) \neq \emptyset, i = 1, 2$. Then $z_0$ is of the same rank relative to $(\Omega, E_1, F_1)$ and relative to $(\Omega, E_2, F_2)$, and moreover
\[
\mathcal{M}_{\Omega, E_1, F_1}(A - z_0 I) = \mathcal{M}_{\Omega, E_2, F_2}(A - z_0 I).
\]

Above, $E_1 \cap E_2 = \{ \phi \in X : \exists (\phi_1, \phi_2) \in E_1 \times E_2, \ i_1(\phi_1) = i_2(\phi_2) = \phi \}$, with
\[
\|\phi\|_{E_1 \cap E_2} = \|\phi\|_{E_1} + \|\phi\|_{E_2},
\]
and $F_1 + F_2 = \{ \psi_1 + \psi_2 \in G : (\psi_1, \psi_2) \in F_1 \times F_2 \}$, with the norm
\[
\|\psi\|_{F_1 + F_2} = \inf_{\psi = \psi_1 + \psi_2, (\psi_1, \psi_2) \in F_1 \times F_2} (\|\psi_1\|_{F_1} + \|\psi_2\|_{F_2}).
\]

We point out that if $E_1$ and $E_2$ are not mutually dense (or similarly if $F_1$ and $F_2$ are not inside a common vector space $G$), then there is a nontrivial dependence of the rank of $z_0 \in \sigma_{ess}(A)$ from the choice of regularizing subspaces; see Example 1.2.

**Lemma 3.23 (Virtual levels of the adjoint)** Let $A \in \mathcal{C}(X)$ and $E \rightarrow X \inj X^n \rightarrow F$ satisfy Assumption 3.1. Moreover, assume that $E$ be reflexive and let $A^* : X^* \rightarrow X^*$ have a closable extension to a mapping $A^* : E^* \rightarrow E^*$ with domain $\mathcal{D}(A^*) \subset E^*$.

Let
\[
\mathcal{D}(\hat{A}^*_{F^*} : E^*) \rightarrow \mathcal{D}(A^* : E^*) = \{ \phi \in \mathcal{D}(A^*) \cap j^*(F^*), A^* \phi \in j^*(F^*) \}
\]

and
\[
\mathcal{D}(\hat{A}^*_{F^*} : E^*) = \{ \phi \in \mathcal{D}(A^*) \cap j^*(F^*), \hat{A}^* \phi \in j^*(F^*) \}
\]
and assume that $\mathcal{D}(\hat{A}^*_{F^*} : E^*)$ is dense in $\mathcal{D}(\hat{A}^*_{F^*} : E^*)$ in the topology induced by the graph norm of $\hat{A}^*$ (that is, $A^*$ and $F^* \rightarrow X^* \leftarrow E^*$ satisfy Assumption 3.1).
Let $\Omega \subset \mathbb{C} \setminus \sigma(A)$. Assume that $z_0 \in \sigma_{\text{ess}}(A) \cap \partial \Omega$ is an exceptional point of the essential spectrum of $A$ of rank $r \in \mathbb{N}_0 \cup \{\infty\}$ relative to $(\Omega, E, F)$. Then $\bar{z}_0 \in \sigma_{\text{ess}}(A^*)$ is an exceptional point of the essential spectrum of $A^*$ of rank $r$ relative to $(\Omega^*, F^*, E^*)$, with $\Omega^* := \{\zeta \in \mathbb{C} : \bar{\zeta} \in \Omega\}$.

Above, the assumption that $E$ is reflexive is needed so that the existence of a limit of the resolvent $(A - zI)^{-1}$, $z \to z_0$, $z \in \Omega$, in the weak operator topology of mappings $E \to F$ also provides the existence of a limit of $(A^* - \zeta I)^{-1}$, $\zeta \to \bar{z}_0$, $\bar{\zeta} \in \Omega$, in the weak operator topology of mappings $F^* \to E^*$.

4 Application to the Schrödinger operators

Let us illustrate how our approach from Section 3 can be applied to the study of properties of virtual states and LAP estimates of Schrödinger operators with nonselfadjoint potentials. According to the developed theory, it suffices to derive the estimates for model operators. If $d \geq 3$, one derives optimal LAP estimates for the Laplacian (see [GM74, Proposition 2.4] and [BC21, §3.3]); for $d = 2$, one considers the regularization $-\Delta + g1_{\mathbb{R}^2}, 0 < g \ll 1$, destroying the virtual level at $z_0 = 0$ [BC21, §3.2]. For $d = 1$, one could proceed in the same way as for $d = 2$, although the estimates can be derived directly for any $V \in L^1_{\rho}(\mathbb{R}, \mathbb{C})$ [BC21, §3.1]. The resulting estimates will be valid for all complex-valued potentials when there are no virtual levels. When there are virtual levels, then the corresponding virtual states can be characterized as functions in the range of the regularized resolvent (Theorem 3.16).

**Theorem 4.1** Assume that $V \in L^\infty_{\rho}(\mathbb{R}^d, \mathbb{C}), \rho > 2$; if $d = 1$, it suffices to have $V \in L^1_{\rho}(\mathbb{R}^1, \mathbb{C})$ (see (1.6)). Let $A = -\Delta + V$ in $L^2(\mathbb{R}^d)$, $\mathfrak{D}(A) = H^2(\mathbb{R}^d), d \geq 1$.

- If $z_0 = 0$ is not a virtual level of $A$ relative to $\Omega = \mathbb{C} \setminus \mathbb{R}_+$, $E = L^2_{s'}(\mathbb{R}^d), F = L^2_{s'}(\mathbb{R}^d)$, with $s, s' > 0$ sufficiently large, then the following mappings are continuous:

$$(A - z_0 I)^{-1}_{\Omega} : L^2_s(\mathbb{R}^d) \to L^2_{s'}(\mathbb{R}^d), \quad \begin{cases} s + s' \geq 2, & s, s' > 1/2, \quad d = 1; \\ s, s' > 2 - d/2, & s, s' \geq 0, \quad d \geq 2. \end{cases}$$

Moreover, for $1 \leq d \leq 3$,

$$(A - z_0 I)^{-1}_{\Omega} : L^1(\mathbb{R}^d) \to L^2_{s}(\mathbb{R}^d), \quad L^2_{s}(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d), \quad \forall s > 2 - d/2.$$  

- If $z_0 = 0$ is a virtual level of $A$, then there is a nonzero solution to the following problem:

$$(A - z_0 I)\Psi = 0, \quad \Psi \in \begin{cases} L^\infty(\mathbb{R}^d), & d \leq 2; \\ L^\infty(\mathbb{R}^d) \cap L^2_{2 - 1/2 - 0}(\mathbb{R}^d), & d = 3; \\ L^2_{-1}(\mathbb{R}^d), & d = 4; \\ L^2(\mathbb{R}^d), & d \geq 5. \end{cases}$$
For more details and references, see [BC21]. Related results on properties of eigenstates and virtual states are in [GN20] (Schrodinger and massive Dirac operators in dimension $d \geq 3$ and massless Dirac operators in $d \geq 2$) and in [BBV20, Theorem 2.3] (Schrodinger operators in $d \leq 2$). Let us note that, prior to [BC21], the nonselfadjoint case has not been considered (although some results appeared in [CP05]). Moreover, as far as we know, even in the selfadjoint case, the LAP in dimension $d = 2$ at the threshold when it is a regular point of the essential spectrum was not available. Although the $L^1 \to L^2_{-\delta}$ and $L^2 \to L^\infty$ estimates stated above are straightforward in dimension $d = 1$ and $d = 3$, we also do not have a reference.

**Remark 4.2** According to Theorem 3.22, the absence of uniform estimates of the form $(-\Delta - zI)^{-1} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ for $z \in \mathbb{C} \setminus \mathbb{R}^+$ for $d \leq 2$ [KL20] is directly related to the fact that there is a virtual level of $\Delta$ at $z_0 = 0$ in dimensions $d \leq 2$ relative to $(\mathbb{C} \setminus \mathbb{R}^+, L^2_s(\mathbb{R}), L^2_{s'}(\mathbb{R}))$, with arbitrarily large $s, s' \geq 0$.

**Example 4.3** Since $\Psi \equiv 1$ is an $L^\infty$-solution to $\partial_x^2 u = 0$, by Theorem 4.1, $z_0 = 0$ is not a regular point of the essential spectrum of the Laplacian in $\mathbb{R}$ relative to $(\mathbb{C} \setminus \mathbb{R}^+, L^2_s(\mathbb{R}), L^2_{s'}(\mathbb{R}))$, with $s, s' > 1/2, s + s' \geq 2$.

Now let us show that $z_0 = 0$ is a virtual level of rank $r = 1$ (relative to the same triple $(\mathbb{C} \setminus \mathbb{R}^+, L^2_s(\mathbb{R}), L^2_{s'}(\mathbb{R}))$). Consider a rank one perturbation of the Laplacian,

$$A = -\partial_x^2 + \mathbbm{1}_{[-1,1]} \otimes \langle \mathbbm{1}_{[-1,1]}, \cdot \rangle, \quad A \in \mathcal{C}(L^2(\mathbb{R})), \quad \mathcal{D}(A) = H^2(\mathbb{R}),$$

with $\mathbbm{1}_{[-1,1]}$ the characteristic function of the interval $[-1, 1]$. We claim that $z_0 = 0$ is a regular point of $\sigma_{\text{ess}}(A)$. Indeed, the relation $Au = 0$ takes the form

$$u''(x) = c \mathbbm{1}_{[-1,1]}(x), \quad x \in \mathbb{R}, \quad c := \int_{-1}^1 u(y) \, dy. \tag{4.1}$$

The requirement $u \in L^\infty(\mathbb{R})$ implies that $u(x) = a_-$ for $x < -1$ and $u(x) = a_+$ for $x > -1$, with some $a_{\pm} \in \mathbb{C}$; for $-1 < x < 1$, one has $u = a + bx + cx^2/2$, with some $a, b \in \mathbb{C}$. The continuity of the first derivative at $x = \pm 1$ leads to $b - c = 0$ and $b + c = 0$, hence $b = c = 0$; at the same time, the relation $0 = c = \int_{-1}^1 a \, dx$ implies that $a = 0$ and thus $u(x)$ is identically zero. Hence, there is no nontrivial $L^\infty$-solution to (4.1). By Theorem 4.1, $z_0 = 0$ is a regular point of $\sigma_{\text{ess}}(A)$, hence it is a virtual level of rank one of $-\partial_x^2$.

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