Resolving Singularities in (0,2) Models

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In contrast to the familiar (2, 2) case, the singularities which arise in the (0, 2) setting can be associated with degeneration of the base Calabi–Yau manifold and/or with degenerations of the gauge bundle. We study a variety of such singularities and give a procedure for resolving those which can be cured perturbatively. Among the novel features which emerge are models in which smoothing singularities in the base yields a gauge sheaf as opposed to a gauge bundle as the structure to which left moving fermions couple. Supersymmetric σ-models with target data being an appropriate sheaf on a Calabi–Yau space therefore appear to be the natural arena for $N = 1$ string models in four dimensions. We also indicate a variety of singularities which would require a nonperturbative treatment for their resolution and briefly discuss applications to heterotic models on K3.

May 1996

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1. Introduction

After many years of intensive investigation, string compactifications with \((2,2)\) world sheet supersymmetry continue to yield new and remarkable physical consequences. It is important to realize, though, that \((2,2)\) models are likely to be but a small slice through the more general class of \((0,2)\) compactifications. Historically, \((0,2)\) Calabi–Yau compactifications have received less attention because they are technically more difficult to construct and to analyze than their \((2,2)\) counterparts. The work of [1] went a long way towards ameliorating this unpleasant aspect by providing a new tool—the linear \(\sigma\)-model—for dealing with both \((2,2)\) and \((0,2)\) models. The linear \(\sigma\)-model provides a non-conformal member of the universality class of a superconformal theory which captures many features of the latter while avoiding much of its complexity. Furthermore, the linear \(\sigma\)-model provides a bridge between (non-conformal) Calabi–Yau \(\sigma\)-models, with either \((2,2)\) or \((0,2)\) world sheet supersymmetry, and Landau–Ginsburg mean field theory models. The latter are well understood, relatively easy to analyze and share a number of important physical characteristic with the Calabi–Yau’s to which they are connected. Hence, they provide another important tool for detailed study. A number of papers [2–7] have used these new tools to initiate a comprehensive investigation of \((0,2)\) models. It is important to note that in [8] and in greater detail in [9,10] it was shown that methods of toric geometry are equivalent to those of the linear \(\sigma\)-model but in certain circumstances provide a more powerful analytic tool. We shall avail ourselves of this approach in the sequel.

Ultimately we hope to have as complete an understanding of \((0,2)\) models as we presently have for \((2,2)\) models. Although this goal is still rather far off, the work of [2–7] and, hopefully, the present paper, are steps in this direction. More specifically, the mathematics and physics of apparent singularities in a variety of contexts has played a key role in numerous recent developments in string theory and in field theory. This is true, in particular, for \((2,2)\) string compactifications. The mirror symmetry construction of [11], for example, relies on Calabi–Yau orbifolds which generally have singularities. The phase structure of \((2,2)\) moduli space found in [11,12] shows that parameter spaces for numerous conformal theories adjoin along common walls, which are geometrically interpretable as singular configurations. This phenomenon was dramatically augmented through the work of [12,13] in which, at the level of nonperturbative type II string theory, many and possibly all vacuum configurations were shown to join together through mathematically singular but physically smooth transitions. And much of the exciting work on string dualities focuses
on various singularities as key points of physical interest [14–17]. It therefore seems quite important to understand both the mathematics and the physics of singularities in (0, 2) Calabi–Yau conformal field theory. In this paper we begin such a study.

In section II we review the linear $\sigma$-model/toric geometry approach to (2, 2) models and show how it naturally fits singular configurations into a phase diagram which contains appropriate desingularizations. We then review the linear $\sigma$-model approach to (0, 2) models, as presented in [1, 2, 18], and indicate the singularities which arise. In section III we give a procedure for extending the phases analysis to the (0, 2) case and thereby resolving the singularities encountered. Although adequate for resolving singularities we still seek a more unified toric treatment. In the course of resolving (0, 2) singularities, we shall find a number of interesting physical differences from the (2, 2) case. After pointing out these differences, we illustrate them in section IV with a number of explicit examples. In section V we give some brief conclusions and indicate directions for future work.

2. (2, 2) and (0, 2) Models: A Linear $\sigma$-Model Approach

2.1. Bosonic Fields

We begin with a brief discussion of the linear $\sigma$-model introduced in [1], and its extension discussed in [9] to which the reader should refer for more detail. Rather than being completely general, we review the case which corresponds to a Calabi–Yau hypersurface in a weighted projective four space. Later in this paper we will consider some generalizations.

Witten found that an interesting class of two dimensional models with (2, 2) world sheet supersymmetry could be constructed by starting with an $N = 2$ supersymmetric gauge theory with gauge group $U(1)$ and action

$$S = S_{\text{kinetic}} + S_W + S_{\text{gauge}} + S_{\text{FI-D term}}.$$  \hfill (2.1)

The term $S_W$ takes the form

$$S_W = \int d^2 z d^2 \theta W(P, S_1, ..., S_5)$$  \hfill (2.2)

where $W$ is the superpotential of the theory, $P, S_1, ..., S_5$ are chiral superfields whose $U(1)$ charges are denoted $q_0, q_1, ..., q_5$ and $W$ is chosen to be a $U(1)$ invariant holomorphic function of the form

$$W = PG(S_1, ..., S_5).$$  \hfill (2.3)
In this expression \( G \) is a transverse quasihomogeneous function of \( S_1, \ldots, S_5 \) whose overall \( U(1) \) charge is \(-q_0\). For future reference we note that we can explicitly integrate out one of the superspace coordinates in (2.2), say \( \theta^- \) and write this contribution to the action as

\[
\int d^2 z d\theta^+ \Gamma G + P \Lambda_i F_i \tag{2.4}
\]

In this expression we have expressed a general \((2, 2)\) chiral superfield in terms of its \((0, 2)\) chiral field content, namely,

\[
\Phi^{(2, 2)} = \Phi^{(0, 2)} + \theta^- \Psi^{(0, 2)} + i \theta^- \overline{\theta}^-( -\partial \Phi^{(0, 2)}) \tag{2.5}
\]

where \( \Phi^{(0, 2)} \) and \( \Psi^{(0, 2)} \) are \((0, 2)\) bosonic and fermionic multiplets, respectively. Our notation is that, in the \((2, 2)\) context, \( P^{(0, 2)} \) and \( \Gamma^{(0, 2)} \) constitute \( P^{(2, 2)} \), while \( S_i^{(0, 2)} \) and \( \Lambda_i^{(0, 2)} \) constitute \( \Phi_i^{(2, 2)} \). The quantity \( F_i \) denotes \( \partial G / \partial S_i \). Typically we will drop the \((2, 2)\) and \((0, 2)\) superscripts, as we have done in all previous equations. The Fayet–Illiopoulos \( D \)-term takes the form

\[
S_{\text{FI–D term}} = t \int d\theta^+ d\overline{\theta}^- \Sigma + \text{c.c.} \tag{2.6}
\]

where \( \Sigma \) is a twisted chiral superfield and \( t = r + i \theta \) is a complex parameter. Witten showed that this model has a nontrivial phase structure in the sense that for \( r \) large and positive it reduces in the infrared to a Calabi–Yau \( \sigma \)-model on the Calabi–Yau space \( G = 0 \) in \( \mathbb{P}^4 \) with homogeneous coordinates \( (s_1, \ldots, s_5) \) \((s_i \text{ is the scalar part of the superfield } S_i)\) while for \( r \) large and negative it reduces to a Landau–Ginsburg model with superpotential given by \( G \). Seeing this is a straightforward exercise in studying the bosonic potential

\[
U = |G(s_i)|^2 + |p|^2 \sum_i |\partial G / \partial s_i|^2 + \frac{1}{2e^2} D^2 + 2|\sigma|^2 \left( \sum_i q_i^2 |s_i|^2 + q_0^2 |p|^2 \right) \tag{2.7}
\]

with

\[
D = -e^2 \left( \sum_i q_i |s_i|^2 - q_0 |p|^2 - r \right) \tag{2.8}
\]

for the two cases distinguished by the sign of \( r \). For \( r > 0 \), vanishing of the potential requires \( p = 0, \sigma = 0, G = 0, \sum_i q_i |s_i|^2 = r \). Together with the \( U(1) \) gauge symmetry

\footnote{We use \( \theta^+ \) and \( \overline{\theta}^+ \) as our right moving (i.e. \((0, 2)\)) fermionic coordinates and \( \theta^- \) and \( \overline{\theta}^- \) as our left moving \((2, 0)\) fermionic coordinates. Hence \(+,-\) refer to right moving and left moving, respectively, and an overline denotes opposite \( U(1) \) eigenvalue.}
identifications, this yields the stated Calabi–Yau \( \sigma \)-model. For \( r < 0 \), we find \( s_i = 0, \sigma = 0, p = \sqrt{(-r/q_0)} \) and the resulting model describing fluctuations around this vacuum configuration is a Landau–Ginsburg model with potential proportional to \( G \). These two descriptions of the physical model can be thought of as different phases of the overarching Lagrangian \((2.1)\). The fact that \( r \) is actually part of a complex parameter which includes a theta angle \( t = r + i\theta \) plays a key role in establishing that the transition from one phase to the other is smooth \([1]\).

An important point for the present study is that the typical Calabi–Yau obtained in this manner is singular as it is embedded in the weighted projective space \( W\mathbb{P}^4_{q_1,q_2,...,q_5} \), which itself has singularities unless all of the charges are relatively prime. The physical model is well behaved even with these singularities, but their presence signals that the phase analysis indicated above, for such a model, is incomplete. Namely, there are marginal operators associated with resolving the singularities. These marginal operators can be used to deform the original model to a desingularized form and thereby probe regions of the moduli space not encountered by our previous discussion. In particular, in the above discussion we assumed that we had a single \( U(1) \) gauge symmetry which gave rise to one Kähler moduli space parameter \( r \). As indicated, this is but a one-dimensional slice through the complete moduli space and hence we are led to embed this linear \( \sigma \)-model in one which has a \( U(1)^s \) gauge symmetry with corresponding parameters \( r_1,...,r_s \) where \( s \) equals \( h^{11} \) of the resolved space. To do so, we need to know the charges of our fields under this full gauge symmetry group. Furthermore, each \( U(1) \) factor gives rise to its own Fayet–Iliopoulos \( D \)-term whose vanishing cuts down the vacuum configuration by one complex dimension. Since this dimension is fixed, each of the \( s - 1 \) additional \( U(1) \) factors must be accompanied by an additional chiral superfield and hence we also need to know the \( U(1)^s \) charges of these additional \( s - 1 \) chiral superfields \( \chi_1,...,\chi_{s-1} \). Knowledge of this data provides us with a moduli space containing that of the original singular Calabi–Yau, but enlarged to include regions corresponding to its desingularization. How, therefore, do we find this data? Methods of toric geometry prove to be the most efficient and systematic way of doing so.

The link between the linear \( \sigma \)-model and toric geometry arises because mathematically, setting the \( D \)-term to zero and taking proper account of the \( U(1) \) phase symmetry corresponds to taking a \textit{symplectic} quotient. As is well known, this can be rephrased as a holomorphic quotient and toric geometry is a formalism for studying the latter. Roughly
speaking, toric geometry provides a systematic method for studying spaces that can be realized as holomorphic quotients of the form \((\mathbb{C}^n - F_\Delta)/(\mathbb{C}^*)^m\) with \(F_\Delta \subset \mathbb{C}^n\). A weighted projective space such as \(W\mathbb{P}^4_{q_1,q_2,...,q_5}\), for example, can be realized in this manner via \((\mathbb{C}^5 - (0,0,0,0,0))/\mathbb{C}^*\) with \(\mathbb{C}^*\) action being \((z_1, ..., z_5) \to (\lambda^{q_1} z_1, ..., \lambda^{q_5} z_5), \lambda \in \mathbb{C}^*\). Toric geometry gives us a simple procedure for desingularizing such holomorphic quotients which we now briefly review. The reader should consult [8,9,10,19] for more details.

As in [20,8] the toric varieties of interest to us are associated to reflexive Gorenstein cone in \(\mathbb{C}^5\) with apex at the origin and edges given by five vectors \(v_1, ..., v_5\). Phases of the physical model correspond to distinct triangulations of a transverse hyperplane section of the cone lying at a unit distance from the origin—that is, triangulations of the polytope with vertices given by the above edges. To be concrete, consider the example of the quintic hypersurface with \((q_0, ..., q_5) = (-5, 1, 1, 1, 1, 1)\) and edges

\[
v_1 = (1, 0, 0, 0, 1), v_2 = (0, 1, 0, 0, 1), v_3 = (0, 0, 1, 0, 1),
\]
\[
v_4 = (0, 0, 0, 1, 1), v_5 = (-1, -1, -1, -1, 1).
\]

We see that there are two possible triangulations of this polytope: the triangulation consisting of the polytope itself and the triangulation of the polytope into the five sections with vertices \(\{v_0, ..., \hat{v}_i, ..., v_5\}\) where \(v_0\) is the interior point \((0,0,0,0,1)\) and \(\hat{v}_i\) denotes omitting the \(i^{th}\) vertex. Toric geometry associates each of these triangulations to holomorphic quotients of the the form \((\mathbb{C}^6 - F_\Delta)/\mathbb{C}^*\) where the form of \(F_\Delta\) is determined by the triangulation and the action of \(\mathbb{C}^*\) is determined by the point set \(v_0, ..., v_5\), as explained in [8,9]. In this example with \(\mathbb{C}^6\) variables labeled \((s_1, ..., s_5, p)\), the first triangulation yields

\[
F^{(1)}_\Delta = (s_1, ..., s_5, 0)
\]

while the second gives

\[
F^{(2)}_\Delta = (0, ..., 0, p)
\]

both with \(\mathbb{C}^*\) action

\[
(s_1, ..., s_5, p) \to (\lambda s_1, ..., \lambda s_5, \lambda^{-5} p).
\]

Examination of these holomorphic quotients reveals that the first corresponds to the \(\mathbb{C}^5/\mathbb{Z}_5\) configuration space of the \(r < 0\) phase while the second corresponds to the \(\mathcal{O}(-5)\) over \(\mathbb{CP}^4\) \(r > 0\) phase of the linear \(\sigma\)-model. The quintic itself is recovered in the \(r > 0\) phase by requiring that the full bosonic potential vanishes thereby yielding the locus of a quintic
in $\mathbb{CP}^4$. In this way we see that the linear $\sigma$-model provides the physical counterpart of such constructions of toric varieties.

We can make powerful use of this link since in more complicated examples than the quintic, for which the linear $\sigma$-model analysis becomes increasingly difficult, the toric methods remain tractable. In particular, how do we resolve the singularities which such toric constructions may yield? For a detailed discussion see [8,9]; here we will only briefly summarize the procedure. Intuitively speaking, we want to excise a neighborhood of the singular loci and glue in a smooth space which has the same boundary. The size and shape of the space we glue in are extra degrees of freedom that arise on the smooth model. As discussed earlier, this translates into the linear $\sigma$-model language as the existence of more $U(1)$ gauge symmetries and more chiral superfields giving a $U(1)^s$ gauge symmetry and chiral superfields $P, S_1, ..., S_5, \chi_1, ..., \chi_{s-1}$. As mentioned, specifying the model requires that we give the $U(1)^s$ charges of all of these chiral superfields. These charges are most systematically determined by the kernel of the toric $(s+5) \times 5$ point set matrix $\mathcal{A}$, describing the polytopic base $\mathcal{P}$ of the associated reflexive Gorenstein cone as discussed in [9]. This, therefore, is how toric geometry gives us the linear $\sigma$-model data for the resolved model.

In particular, the linear $\sigma$-model in this augmented form has $s$ Fayet–Illiopoulos $D$-terms with coefficients $r_1, ..., r_s$. The space of all possible values of these parameters naturally divides up into phase regions in a manner similar to the simple case of the quintic described above. One can determine these phase regions by varying the $r_1, ..., r_s$ in the bosonic potential of the linear $\sigma$-model and studying the result minima or, alternatively, by studying the possible triangulations of $\mathcal{P}$. Often the latter approach is much easier. Among the different phases are those which correspond to Calabi–Yau $\sigma$-models on the possible (crepant) desingularizations of the initial Calabi–Yau space. These correspond to maximal triangulations of $\mathcal{P}$. Thus, this linear $\sigma$-model/toric geometrical approach provides a systemic procedure for resolving local quotient singularities and clearly delineates the relationship between the various smooth and singular geometric configurations.

All of the above discussion is in the context of $(2,2)$ models. Our interest in this paper is in the larger class of $(0,2)$ models. The essential difference between the two cases is the treatment of the left moving world sheet fermions $\lambda^a$ which in $(2,2)$ models

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2 An important fact [3] is that there need not be a unique maximal triangulation and therefore there can be distinct ways of resolving the Calabi–Yau singularities (as the triangulation determines the set $F_\Delta$, their difference arises in the identity of the latter).
lie in supermultiplets with the world sheet scalars $\phi_i$ but do not in the $(0,2)$ case. That is, from (2.3) we recall that a $(2,2)$ chiral superfield decomposes into a $(0,2)$ bosonic chiral superfield and a $(0,2)$ fermionic chiral superfield, the latter containing left moving chiral fermions, $\lambda$. In the $(2,2)$ setting the properties of these fermions are determined via the left moving supersymmetry from the properties of their bosonic partners. In the $(0,2)$ case, though, the absence of the left moving supersymmetry yields newfound independence for the left moving fermions both in terms of their number and their interactions. To understand the range of possibilities associated with these degrees of freedom, let us first recall their properties in $(2,2)$ models and then pass to the more general $(0,2)$ setting.

2.2. Fermionic Fields

We carry out our discussion in a fully resolved Calabi–Yau region of the moduli space. The $(2,2)$ world sheet supersymmetry implies that a complex scalar field $\phi_i$ lies in a supermultiplet with both a left-moving $\lambda^i$ and a right moving $\psi^i$ world sheet fermion. Our goal is to study these theories in the far infrared and therefore we need to determine which fermions are massless and hence survive into the long distance limit. There are two types of terms in the linear $\sigma$-model action which can give mass to these fermions. The first comes from the gauge field part of the action and yields couplings of the form

$$\sum_i q_i^{(k)} \alpha_k \psi^i \phi_i + q_i^{(k)} \beta_k \lambda^i \phi_i$$

where $\alpha_k$ and $\beta_k$ are world sheet fermions from the $k^{th}$ vector $U(1)$ vector multiplet, $k = 1, \ldots, s$. The second comes from the superpotential and yields couplings of the form

$$\sum_i (\gamma \psi^i + \pi \lambda^i) \frac{\partial G}{\partial \phi_i}$$

where $\gamma$ and $\pi$ are the left and right moving fermionic partners to $p$. These couplings imply that the fermions which remain massless are those which are in the kernel of the map

$$g : (\psi_1, \ldots, \psi_5) \rightarrow \sum_i \psi_i \frac{\partial G}{\partial \phi_i} \quad (2.13)$$

(so that the second type of couplings do not contribute to their mass) but not in the image of

$$f : (y_1, \ldots, y_s) \rightarrow \left( \sum_k q_1^{(k)} y_k \phi_1, \sum_k q_2^{(k)} y_k \phi_2, \ldots, \sum_k q_5^{(k)} y_k \phi_5 \right) \quad (2.14)$$

(so that the first type of couplings do not contribute to their mass). Geometrically, we can interpret $\phi_i$ as a section of the bundle $\mathcal{O}(q_i^{(1)}, q_i^{(2)}, \ldots, q_i^{(k)})$ over the resolved Calabi–Yau. This is the line bundle whose first Chern class is $\sum_k q_i^{(k)} J_k$ where the $J_k$ are $(1,1)$ forms generating the integral 2-forms on the desingularized Calabi–Yau manifold. Using
this structure, we can describe the massless fermions as the cohomology of the (non-exact) sequence

\[\begin{align*}
0 \to \oplus^k \mathcal{O} \to \oplus_{i=1}^6 \mathcal{O}(q_i^{(1)}, q_i^{(2)}, \ldots, q_i^{(k)}) \to \mathcal{O}(q_0^{(1)}, q_0^{(2)}, \ldots, q_0^{(k)}) \to 0. 
\end{align*}\]  

(2.15)

This same discussion, due to the left/right symmetry holds identically for the fermions \(\lambda_i\).

As previously emphasized, we carried out this discussion in a smooth Calabi–Yau phase. It is easily repeated in any other phase and hence we can follow the smooth tangent bundle on the resolved Calabi–Yau to any other phase, for instance, to the phase associated with the original singular weighted projective space. It is not hard to show that in this phase the massless fermions arise from the cohomology of the sequence

\[\begin{align*}
0 \to \mathcal{O} \to \oplus_{i=1}^5 \mathcal{O}(q_i) \to \mathcal{O}(\sum_{i=1}^5 w_i) \to 0
\end{align*}\]  

(2.16)

over \(W_{\mathbb{P}^4_{q_1,\ldots,q_5}}\) as one would expect from a naïve application of the linear \(\sigma\)-model ignoring all issues associated with singularities.

We can now see how the \((0,2)\) case differs from the \((2,2)\) case. In the \((0,2)\) case the above discussion applies to the \(\psi^i\)'s as they are still the superpartners to the \(\phi_i\)'s and hence have the same gauge charges. However, the \(\lambda\)'s now have no relation to the \(\phi_i\)'s: both the number of \(\lambda\)'s and their charges are data which are ingredients in the definition of a \((0,2)\) model. This data must be chosen in a consistent manner, i.e. to ensure anomaly freedom, but is otherwise unconstrained. As we will discuss in more detail in the next section, after such a choice has been made, we can re-run through the above discussion, in any phase and analyze the structure defining the massless fermions. In the phase associated to the original singular weighted projective space, the gauge bundle of the left moving fermions will take the form studied in previous works such as [1-4, 8]. Now, though, the structure of toric geometry allows us to move from this phase to others such as smooth Calabi–Yau phases. A consistent choice of gauge charges is geometrically translated, in such a phase, into the data defining the resolution of the gauge bundle. In the next section we describe this in greater detail.
3. (0, 2) Singularities and Their Resolution

As follows from the above discussion, our (0, 2) models differ from (2, 2) models in two essential ways:

1. The bosonic and fermionic chiral multiplets, which in a (2, 2) model occur in pairs that join into a chiral multiplet, are independent both in terms of their number and gauge charges.

2. The supersymmetric gauge transformations come in two varieties—bosonic and fermionic gauge symmetries. Again, in a (2, 2) model these occur in pairs which join into a chiral multiplet parameterizing gauge transformations. In a (0, 2) model, though, they are independent both in number and in their action on the fields in the theory.

It is the above data—the list of bosonic and fermionic (0, 2) chiral superfields along with the action of the bosonic and fermionic gauge symmetries upon them, together with a gauge invariant superpotential—which constitutes a (0, 2) model. From the above discussion, it is clear that there are two complementary ways of framing our discussion. We can describe this data, which is necessary to construct a full phase diagram for a (0, 2) model, and show how in various phases it contains the singular models of [1, 2, 18] while in others these singularities are resolved. Or, we can start with the kind of models studied in [1, 2, 18] and show how to embed them in a larger phase diagram that includes fully resolved regions. Let’s take the latter approach.

For concreteness, we again work with base spaces that arise from hypersurfaces in a weighted projective four space or desingularizations thereof. From [1, 2, 18] and our discussion above, the data we begin with are the $U(1)$ gauge charges $q_1, \ldots, q_5$ of the (0, 2) chiral superfields $S_1, \ldots, S_5$ (which contain $(s_1, \psi_1), \ldots, (s_1, \psi_1)$ as components) and the charge $\tilde{q}_0$ of a (0, 2) Fermi multiplet $\Gamma$ (which contains $\gamma$ as a component). We then seek a consistent choice of $U(1)$ gauge charges $(\tilde{q}_1, \ldots, \tilde{q}_n)$ of the (0, 2) Fermi multiplets $\Lambda_1, \ldots, \Lambda_n$ (which contain $\lambda_1, \ldots, \lambda_n$ as components) and also the gauge charge $q_0$ of the (0, 2) chiral multiplet $P$ (which contains $(p, \pi)$ as components). The (0, 2) superpotential describing this model takes the form

$$\int d^2z \theta (\Gamma G + \Lambda^a P F_a) \tag{3.1}$$

where $F_a$ are homogeneous polynomials in the chiral bosonic superfields with $U(1)$ charges $- q_0 - \tilde{q}_a$. The reader may find it instructive to compare this with [2.7] where one sees that the $F_a$ are no longer determined by $G$ but rather are independent degrees of freedom. We shall sometimes refer to the first term in (3.1) as $\Lambda^0 F_0$ for uniformity of notation. We
see from this expression that we must choose \( \tilde{q}_0 = -(\text{degree of homogeneity, } d, \text{ of } G) \) and \( q_0 = -(\text{degree of homogeneity of } F_a + \tilde{q}_a) \) which must be independent of \( a \). Consideration of anomalies constrains these choices in the following way:

\[
\tilde{q}_0 = d = \sum_{i=1}^{5} q_i \quad \text{and} \quad q_0 = -\sum_{a=1}^{n} \tilde{q}_a \tag{3.2}
\]

\[
\sum_{i=0}^{5} q_i^2 = \sum_{a=1}^{n} \tilde{q}_a^2 \tag{3.3}
\]

There is one immediate solution to these equations: \( n = 5 \) and \( \tilde{q}_i = q_i \). This, of course, takes us back to a \((2,2)\) model. More generally, though, solutions of these equations (modulo some other more subtle anomalies discussed a bit later) gives us data defining a consistent \((0,2)\) model.

By studying the structure of the bosonic potential, one again finds that this linear \( \sigma \)-model has two phases depending upon the sign of the Fayet–Illiopoulos parameter \( r \). For \( r \) positive, the theory reduces, in the infrared, to a \((0,2)\) Calabi–Yau \( \sigma \)-model with base manifold given by the vanishing of \( G \) in \( WP^4 q_1, ..., q_5 \) (with right moving fermions coupling to the tangent bundle of this Calabi–Yau\(^3\)) and left moving fermions coupling to the bundle defined by the cohomology of the sequence

\[
0 \to \mathcal{O} \to \oplus_{i=1}^{n} \mathcal{O}(\tilde{q}_i) \to \mathcal{O}(\sum_{i=1}^{n} \tilde{q}_i) \to 0. \tag{3.4}
\]

As in the \((2,2)\) case, singularities in the Calabi–Yau indicate that we have not probed the full moduli space to which this model belongs. In the \((2,2)\) setting, though, resolving the Calabi–Yau space automatically resolved the tangent bundle and hence determined how the left and right moving fermions behave in the desingularized model. Concretely, the charges of the bosonic fields under the full \( U(1)^n \) gauge symmetry determines the charges of the fermions as they are joined together into \((2,2)\) supermultiplets. In the \((0,2)\) setting this is still true for the right moving fermions \( \psi^i \) but it is not true for the left moving fermions \( \lambda^a \). Rather, we have to resolve the Calabi–Yau space and then consider all possible ways of consistently pulling back the gauge bundle to this smooth space. Consistency here,

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\(^3\) More precisely, we should refer to the tangent bundle as a tangent \( V \)-bundle, since the Calabi-Yau may have orbifold singularities and hence be a \( V \)-manifold. We will typically not make this linguistic distinction.
from the geometrical point of view, is that the first Chern class of the resolved bundle must vanish and its second Chern class must equal that of the resolved tangent bundle. From a physical point of view, this translates into the full $U(1)^s$ gauge symmetry being anomaly free\footnote{In fact, demanding anomaly freedom is a somewhat stronger condition that the second Chern class constraint as the former amounts to requiring equality of the corresponding differential four forms viewed as elements of a free module, i.e. not taking into account cohomology relations. It would be interesting to see if sense can be made of linear sigma models in which only the latter, weaker condition is satisfied.}. In particular, if we denote the $U(1)^s$ gauge charges of the right moving fermions $(\pi, \psi_1, ..., \psi_5)$ by $(q_0^{(k)}, q_1^{(k)}, ..., q_5^{(k)})$ and the left moving fermions $(\gamma, \lambda_1, ..., \lambda_n)$ by $(\tilde{q}_0^{(k)}, \tilde{q}_1^{(k)}, ..., \tilde{q}_n^{(k)})$ with $k = 1, ..., s$, then the conditions generalizing (3.2) and (3.3) are

\begin{align}
q_0^{(k)} &= \sum_{i=1}^{5} q_i^{(k)} \text{ and } q_0^{(k)} = -\sum_{a=1}^{n} \tilde{q}_a^{(k)} \tag{3.5}
\end{align}

\begin{align}
\sum_{i=0}^{5} q_i^{(j)} q_i^{(k)} &= \sum_{a=0}^{n} \tilde{q}_i^{(j)} \tilde{q}_i^{(k)} \tag{3.6}
\end{align}

With one of the $U(1)$ charges, say for $k = 1$, being those of the singular model used in (3.4), the other $U(1)^{s-1}$ charges give us the data required to define the desingularization of the model. Going back to (3.1) we see that after choosing the left moving gauge charges, we must also modify our set of $n$ polynomials $F_a, a = 1, ..., n$ of the bosonic fields to have multicharges $(-q_0 - \tilde{q}_1^{(a)}, -q_0 - \tilde{q}_2^{(a)}, ..., -q_0 - \tilde{q}_s^{(a)})$. Finally, we note that in addition to these bosonic $U(1)^s$ gauge symmetries, we also have the choice of imposing some number of independent fermionic gauge symmetries.

If we impose $m$ such symmetries, we must introduce $m(n + 1)$ homogeneous polynomials $E_{(j)}^a, a = 0, ..., n; j = 1, ..., m$ where $E_{(j)}^a$ has $U(1)^s$ charges $(\tilde{q}_1^{(a)}, ..., \tilde{q}_s^{(a)})$, for all $j$. These symmetries act on the fermionic chiral multiplets as

\begin{align}
\Gamma &\rightarrow \Gamma + 2E_{(j)}^a(\Phi)\Omega_j^a, \quad \Lambda^a \rightarrow \Lambda^a + 2E_{(j)}^a(\Phi)\Omega_j^a \tag{3.7}
\end{align}

where the $\Omega^j$ are fermionic chiral superfields which are neutral under the $U(1)^n$ gauge symmetry. Invariance of (3.1) requires

\begin{align}
E_{(j)}^0 F_0 + P \sum_{a=1}^{n} E_{(j)}^a F_a = 0. \tag{3.8}
\end{align}
To make the kinetic terms invariant, we need to introduce, for every fermionic gauge symmetry, an unconstrained complex fermionic superfield $\Sigma^j$, which transforms as $\Sigma^j \to \Sigma^j + \Omega^j$. The detailed form of the action for the $\Sigma^j$ is explained in [18]. For our purposes, it suffices to note that it leads to a term in the scalar potential of the form

$$V_\sigma = \sum_j |\sigma_j|^2 (|E^a_{(j)}(\phi)|^2 + |p|^2 |E^0_{(j)}(\phi)|^2)$$

So long as, for each $j$, the quantity in parentheses is everywhere nonzero, then the $\sigma_j$ are massive, and drop out of the infrared theory.

Now, if the $E^a_{(j)}$ for fixed but arbitrary $j$ with $a = 1, \ldots, n$ do not simultaneously vanish on the Calabi–Yau, as well as the same being true for the $F_a$ with $a = 1, \ldots, n$, then the same reasoning which led to (2.15) implies that the massless left moving fermions couple to the vector bundle $V$ which is the cohomology of the sequence

$$0 \to \oplus^m \mathcal{O} \overset{\oplus E^a_{(j)}}{\to} \oplus^n_{a=1} \mathcal{O}(\tilde{q}^{(1)}_i, \tilde{q}^{(2)}_i, \ldots, \tilde{q}^{(k)}_i) \overset{\oplus F}{\to} \mathcal{O}(\tilde{q}^{(1)}_0, \tilde{q}^{(2)}_0, \ldots, \tilde{q}^{(k)}_0) \to 0 \quad (3.9)$$

over the resolved space (where in this expression $a = 1, \ldots, n$).

If this condition can not be met on the $E$’s and $F$’s, $V$ may not be a bundle on the resolved space. Depending on how severely the vanishing conditions on these maps is violated, the model may still make perturbative sense, as we shall see.

Notice that in general there isn’t a unique set of charges $\tilde{q}_i$ satisfying (3.5) and (3.6). Thus, a given singular model may admit many desingularizations. We emphasize that the well known and physically important statement in (2, 2) theories [8]—that there can be distinct ways of resolving the singularities of a given Calabi–Yau space refers to the distinct maximal triangulations of the point set $P$, for a fixed and identifiable set of gauge charges. Geometrically, this ambiguity corresponds to the freedom of flopping certain rational curves in a given Calabi–Yau yielding topological distinct cousins. In the (0, 2) setting we see even additional freedom in resolving the model as there are desingularizations whose gauge charges differ as well.

We can summarize the algorithm for resolving a (0, 2) model on a Calabi–Yau hypersurface $M$ in $\mathbb{P}^4 q_1, \ldots, q_5$ as follows:

- Using the machinery of toric geometry, find the charges of the $h^{11} + 5$ chiral superfields under the full $U(1)^{h^{11}}$ gauge symmetry required to desingularize the base space $M$. 
• For a chosen rank of the gauge bundle, find a set of charges of the left moving fermions under this $U(1)^{h_{11}}$ gauge symmetry meeting the anomaly cancelation conditions (3.3) (3.6).

• Choose $m$ and a set of $F_a$ and $E^a_{(j)}$ $a = 1, ..., n, j = 1, ..., m$ meeting the conditions given above.

Our discussion makes it apparent that there are a number of new features that arise in the more general $(0, 2)$ setting. From our perspective the two most prominent are: (1) it is often the case that upon resolution of singularities in the base Calabi–Yau the $F_a$ ’s can not be chosen to avoid them simultaneously vanishing. Nonetheless, it can be arranged in many cases for the vacuum field configuration space to remain compact thus giving rise to an apparently well defined theory in which the left moving fermions couple to a sheaf rather than a bundle. Hence, this appears to greatly broaden the geometrical setting for $(0, 2)$ models. (2) A given $(0, 2)$ model defined on a singular Calabi–Yau space in the manner of [1,4,18] can admit distinct desingularizations. These, of course, are in addition to the known distinct desingularizations of the base Calabi–Yau which played a prominent role in [1,8] and are of a rather different character as the charges of the vacuum bundle differ and hence appear to lead to a multi critical phenomenon. We give an example of this below and will explore this aspect more fully elsewhere.

4. Examples

4.1. Two Distinct $(0, 2)$ Resolutions

For our first example, we consider $M$, a $12^{th}$ order hypersurface in $W\mathbb{P}^4_{1,1,2,6}$. The $(2,2)$ compactification on this manifold was studied using the techniques of mirror symmetry in [21,22].

$M$ has a $\mathbb{Z}_2$ orbifold singularity where $\phi_1 = \phi_2 = 0$. To resolve the singularity, we introduce another scalar, $\chi$, neutral under the original $U(1)$, and a second $U(1)$ under which $\chi$ is charged.

One $(0, 2)$ model that we could build on $M$ is, of course, a holomorphic deformation of the tangent bundle. In that case, the $\lambda$s have exactly the same charges as the $\phi$s, and the bundle $V_1$ is the cohomology of the monad

$$0 \rightarrow \mathcal{O} \oplus \mathcal{O} \xrightarrow{\otimes E^a_{(\phi)}} \mathcal{O}(1, -2) \oplus \mathcal{O}(0, 1) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(3, 0) \xrightarrow{\otimes F_a_{(\phi)}} \mathcal{O}(6, 0) \rightarrow 0 \quad (4.1)$$
Table 1: $U(1)$ charges of the (bosonic and fermionic) fields for the resolved model. The original charge is $Q = 2Q_1 + Q_2$.

| Field | $Q_1$ | $Q_2$ |
|-------|-------|-------|
| $\phi_{1,2}$ | 0 | 1 |
| $\phi_{3,4}$ | 1 | 0 |
| $\phi_5$ | 3 | 0 |
| $\chi$ | 1 | -2 |
| $p$ | -6 | 0 |
| $\lambda_{1,2}$ | 1 | -1 |
| $\lambda^3$ | 0 | 2 |
| $\lambda^4$ | 1 | 0 |
| $\lambda^5$ | 3 | 0 |
| $\gamma$ | -6 | 0 |

For the tangent bundle, $E_{1,2}^a = q_{1,2}^a \phi^a$ (no sum on $a$), and $F_a(\phi) = \frac{\partial G}{\partial \phi^a}$. The quasihomogeneity of $G(\phi)$ implies

$$E_1^a(\phi)F_a(\phi) = 6G(\phi), \quad E_2^a(\phi)F_a(\phi) = 0 \quad (4.2)$$

The general bundle $V_1$ is defined by any set of $E$s and $F$s in (4.1) which satisfy (4.2).

What is more interesting is that we can build another rank three bundle, $V_2$, which is the cohomology of the monad

$$0 \to O \to O(1, -1)^{\oplus 2} \oplus O(0, 2)^{\oplus 2} \oplus O(1, 0) \to O(3, 0) \to O(6, 0) \to 0 \quad (4.3)$$

This is, on the resolved manifold, a completely different bundle than $V_1$, and the (0,2) linear $\sigma$-model is very different. Rather than two fermionic gauge symmetries (two $\Sigma$ multiplets) there is now only one. Correspondingly, there are only five $\lambda$s instead of six. And, of course, the weights of the $E^a$ and the $F_a$ are different. Nevertheless, $V_1$ and $V_2$ become isomorphic over the singular locus where we blow down the orbifold singularity.

This means, of course, that the net number of generation in the two models must be the same. But, in fact, one can study the number of generations and antigenerations separately for the two models. First, we consider built with $V_1$, the deformation of the tangent bundle. We write the monad (4.1) as a pair of exact sequences

$$0 \to O \oplus O \to O(1, -2) \oplus O(0, 1)^{\oplus 2} \oplus O(1, 0)^{\oplus 2} \oplus O(3, 0) \to E \to 0$$

$$0 \to V_1 \to E \to O(6, 0) \to 0 \quad (4.4)$$

Almost all of the line bundles in (4.4) satisfy $h^i(O(n, m)) = 0$, $i > 0$. The exceptions are $O$, which has $h^0(O) = h^3(O) = 1$, and $O(1, -2)$, which has $h^0(O(1, -2)) = 1$ and $h^2(O(1, -2)) = 1$.\footnote{We are implicitly assuming the existence of suitable $E$’s and $F$’s for this model, which in practice, can be tedious to find explicitly.}
\(h^1(\mathcal{O}(1, -2)) = 2\). The number of holomorphic sections of the other line bundles can be determined by counting monomials or by computing the index:

\[
\text{Ind}_{\mathcal{O}(n,m)} = \frac{n(13 + 2n^2)}{3} + m(2 + n^2)
\] (4.5)

Tracing through the long exact sequences in cohomology associated to (4.4), one finds that \(h^1(V_1) = 128\), and \(h^2(V_1) = 2\). These give, respectively, the number of 27s and \(\overline{27}\)s. One notes that this is exactly the same number as in the (2,2) model. It might have happened, that as one deformed away from (2,2), the extra 27s and \(\overline{27}\)s paired up and became massive. This does not happen.

We can perform the same cohomological calculation for the bundle \(V_2\), defined by (4.3). Few of the details change. Again, only one of the line bundles involved has nonvanishing higher cohomology groups. In this case, it is \(\mathcal{O}(0, 2)\) which has \(h^0(\mathcal{O}(0, 2)) = 3\) and \(h^2(\mathcal{O}(0, 2)) = 1\). Again, tracing through the long exact sequences in cohomology, we find \(h^1(V_2) = 128\), and \(h^2(V_2) = 2\).

For a little more insight, we can compute the spectrum of these two theories at Landau–Ginzburg. In fact, of course, at the Landau–Ginzburg point, they are isomorphic, so there is really only one calculation to do. \(\gamma\), \(\lambda_6\) and \(\chi\) are massive, , and after some rescaling, the Landau–Ginzburg superpotential is

\[
\mathcal{W} = \int d\theta \sum_{a=1}^{5} \Lambda^a F_a(\Phi)
\]

One finds 126 generations in the untwisted sector, realized as twelfth order polynomials modulo the ideal generated by the \(F_a\). There are two more generations whose 16\(_{1/2}\) components comes from the \(k = 12\) twisted sector, \(\phi^{3,4}_0|k = 12\). The (16\(_{1/2}\) components of the) two antigenerations are the ground states of the \(k = 6\) and \(k = 14\) sectors.

In this example, then we have seen two distinct resolutions of the vector bundle when one resolves the manifold. In our next example, we will see that what one obtains on the resolved manifold need not even be a vector bundle.

4.2. Protuberances and Reflexive Sheaves

Consider the manifold \(M\), an octic hypersurface in \(W\mathbb{P}^{4,1,2,2,2}\). The (2,2) compactification on this manifold was studied using the techniques of mirror symmetry in [21,22].
Instead of the tangent bundle, we consider a rank three vector bundle, $V$, defined as the kernel in the exact sequence

$$0 \to V \to \mathcal{O}^{\oplus 2}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(5) \otimes F_a \to \mathcal{O}(9) \to 0$$

That is, in addition to the scalars $\phi_i$, with charges $(1,1,2,2,2)$, and the fermion $\gamma$, with charge $-8$, we have fermions $\lambda^a$, with charges $(1,1,2,5)$, and a scalar, $p$, with charge $-9$.

$M$ has a $\mathbb{Z}_2$ orbifold singularity where $\phi_1 = \phi_2 = 0$. To resolve the singularity, we introduce another scalar, $\chi$, neutral under the original $U(1)$, and a second $U(1)$ under which $\chi$ is charged. The net effect is to blow up the singularity, replacing each point on the orbifold locus by a $\mathbb{C}P^2$.

| Field   | $Q_1$ | $Q_2$ |
|---------|-------|-------|
| $\phi_{1,2}$ | 0     | 1     |
| $\phi_{3,4,5}$ | 1     | 0     |
| $\chi$      | 1     | -2    |
| $p$         | -5    | 1     |
| $\lambda^{1,2}$ | 0     | 1     |
| $\lambda^3$  | 2     | -2    |
| $\lambda^4$  | 3     | -1    |
| $\gamma$    | -4    | 0     |

**Table 2:** $U(1)$ charges of the (bosonic and fermionic) fields for the resolved model. The original charge is $Q = 2Q_1 + Q_2$.

The charge assignments of the fields under the two $U(1)$s are given in table 2. We have chosen a basis for the $U(1)$s such that the original $U(1)$ is given by

$$Q = 2Q_1 + Q_2$$

The D-terms which follow from these charge assignments are

$$|\phi_3|^2 + |\phi_4|^2 + |\phi_5|^2 + |\chi|^2 + 5|p|^2 = r_1$$

$$|\phi_1|^2 + |\phi_2|^2 - 2|\chi|^2 + |p|^2 = r_2$$

In the Calabi–Yau phase, for $r_1, r_2 \gg 0$, we expect that these D-terms, together with the superpotential terms, force $\langle p \rangle = 0$ (this turns out to be not quite correct here, as we shall see shortly).

Each point on the erstwhile singularity $\phi_1 = \phi_2 = 0$ is replaced by a $\mathbb{C}P^1$, given by $\chi = 0$, $|\phi_1|^2 + |\phi_2|^2 = r_2$.

In similar fashion, we expect that (4.6) is replaced by

$$0 \to \tilde{V} \to \mathcal{O}^{\oplus 2}(0, 1) \oplus \mathcal{O}(2, -2) \oplus \mathcal{O}(3, -1) \mathcal{O}(5, -1) \to 0$$

(4.8)
The new feature about this sequence (4.8) is that it is not exact! The problem is that there are points on the resolved manifold $\tilde{M}$ where all of the $F_a$'s vanish simultaneously. To see this, let us write the general form of the $F_a$, denoting their charges by superscripts.

\[
F^{(5,-2)}_1 = \chi \tilde{F}_1^{(4,0)} \\
F^{(5,-2)}_2 = \chi \tilde{F}_2^{(4,0)} \\
F^{(3,1)}_3 = \phi_1 f^{(3,0)} + \phi_2 g^{(3,0)} \\
F^{(2,0)}_4
\]

Note that setting $\chi = 0$ automatically guarantees $F_1 = F_2 = 0$. To satisfy the other two equations, we can set to zero all terms in $F_{3,4}$ which contain $\chi$. Then $F_4$ is a quadric in $\phi_3, \phi_4, \phi_5$, and

\[
F_3 = \phi_1 f + \phi_2 g
\]

with $f, g$ being cubics in $\phi_3, \phi_4, \phi_5$. For each solution to $F_4 = 0$, we get a linear equation for $\phi_1, \phi_2$ from setting $F_3 = 0$.

Since, on setting $\chi = 0$, the equation for $\tilde{M}$ is a quartic in $\phi_3, \phi_4, \phi_5$, there are, all in all, 8 = 4 × 2 points on $\tilde{M}$ where all of the $F$s vanish, all of them located on the locus $\chi = 0$. At these points, the sequence (4.8) fails to be exact, because the last map isn’t onto. To get an exact sequence, we need include the cokernel of this map:

\[
0 \to \tilde{V} \to \mathcal{O} \oplus 2(0,1) \oplus \mathcal{O}(2,-2) \oplus \mathcal{O}(3,-1) \oplus F_3 \mathcal{O}(5,-1) \to \mathcal{O}_S(5,-1) \to 0 \quad (4.9)
\]

where the skyscraper sheaf $\mathcal{O}_S(5,-1)$ is the restriction of the line bundle $\mathcal{O}(5,-1)$ to the set $S = \{ \chi = F_3 = F_4 = 0 \} \subset \tilde{M}$.

The $\tilde{V}$ we obtain in this way, however, is not a vector bundle! It fails to be locally free precisely at the points of $S$. If we remove those points, $\tilde{V}$ is a vector bundle over $\tilde{M} \setminus S$.

Even over those “bad” points, $\tilde{V}$ has the property of being isomorphic to its double dual $(\tilde{V})^{**}$. Any sheaf which is isomorphic to its double dual is called reflexive; such sheaves are locally free, except for a “singularity set” of complex codimension 3. So in our low-energy nonlinear $\sigma$-model, the left-moving fermions couple to $\tilde{V}$, a rank three reflexive sheaf over $\tilde{M}$.

This is not all that surprising. We know that strings are perfectly happy propagating on spaces which are not quite manifolds but, rather, have certain suitably-mild singularities (e.g. orbifolds). Here we see that the left-movers can happily couple to objects which are
not quite vector bundles, but rather have certain suitably-mild singularities (e.g. reflexive sheaves).

Before we proceed to calculate the effect of this modification, let us see how exactly this reflexive sheaf is realized in the linear $\sigma$-model? The answer is very simple. Away from the points in $S$, the linear $\sigma$-model just gives the same solution for $\tilde{M}$ that one expects from geometry. However, at those points, where all the $F$s vanish, $p$ is no longer forced to be zero. Instead, from the D-terms (4.7), one finds that there is a $\mathbb{CP}^1$ sitting over each point in $S$. The linear $\sigma$-model solution looks like the manifold $\tilde{M}$, with 8 $\mathbb{CP}^1$s glued on. These protuberances are similar to the exoflops found in [8] in certain phases of (2,2) models. Here we would like to interpret them as the linear $\sigma$-model’s way of coping with the skyscraper sheaf which makes (4.8) fail to be exact.

Let us compute the Chern classes of $\tilde{V}$, defined by (4.9). Since, $h^{1,1} = 2$, we have two fundamental cohomology classes, $\eta_1, \eta_2$, which are, respectively, the first Chern classes of the hyperplane bundles $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. The standard calculation of the Stanley–Reisner ideal [23] for this manifold yields

$$
\eta_1^3 = 8, \quad \eta_1^2 \eta_2 = 4, \quad \eta_2^2 = 0 \quad (4.10)
$$

To compute the Chern classes of $\mathcal{O}_S(5, -1)$, we use the Koszul complex, which give a locally free resolution of $L_S$, for any line bundle $L$. Since $S$ is the complete intersection $\chi = F_3 = F_4 = 0$, we have

$$
0 \rightarrow L(-6, 1) \rightarrow L(-5, -1) \oplus L(-4, 1) \oplus L(-3, 2) \rightarrow L(-3, -1) \oplus L(-2, 0) \oplus L(-1, 2) \rightarrow L \rightarrow L_S \rightarrow 0 \quad (4.11)
$$

The total Chern class of $L_S$ is

$$
c(L_S) = \frac{c(L)c(L(-5, -1))c(L(-4, 1))c(L(-3, 2))}{c(L(-3, -1))c(L(-2, 0))c(-1, 2)c(L(-6, 1))} = 1 + 12\eta_1^3 - 20\eta_1^2 \eta_2 \quad (4.12)
$$

From (4.10), we find $c_3(L_S) = 16$.  

---

6 For a complete intersection of divisors $D_i$, the Koszul complex is

$$
0 \rightarrow \bigoplus_{i < j < k} \mathcal{O}(-D_i - D_j - D_k) \rightarrow \bigoplus_{i < j} \mathcal{O}(-D_i - D_j) \rightarrow \bigoplus_i \mathcal{O}(-D_i) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_S \rightarrow 0
$$
Now we can compute the Chern classes of $\tilde{V}$.

$$c(\tilde{V}) = 1 + 6\eta_1^2 + 2\eta_1\eta_2 - 18\eta_1^3 - 12\eta_1^2\eta_2$$

(4.13)

So

$$c_3(\tilde{V}) = -192$$

(4.14)

which predicts that the net number of generations is 96. Note that, without the correction due to $c_3(\mathcal{O}_S(-5,1))$, we would have gotten $c_3(\tilde{V}) = -208$. Actually, with a little more work, we can do better than this and compute the number of generations and anti-generations separately. From (4.9), we can derive a long exact sequence in cohomology:

$$0 \to H^0(\mathcal{O}(0,1))^\oplus 2 \oplus H^0(\mathcal{O}(2, -2)) \oplus H^0(\mathcal{O}(3, -1)) \to H^0(\mathcal{O}(5, -1)) \to$$

$$\to H^1(\tilde{V}) \to H^1(\mathcal{O}(2, -2)) \to H^0(\mathcal{O}_S(5, -1)) \to H^2(\tilde{V}) \to 0$$

(4.15)

The facts needed to derive this sequence are: i) The higher cohomology groups for all of the line bundles (and the skyscraper sheaf) which appear in (4.9) vanish except for $H^1(\mathcal{O}(2, -2))$, which is 6-dimensional and ii) the restriction of $H^0(\mathcal{O}(5, -1))$ to the set $S$ vanishes.

The key point now is to study the map $H^1(\mathcal{O}(2, -2)) \xrightarrow{\alpha} H^0(\mathcal{O}_S(5, -1))$. On the overlap of patches $U_{\{\phi_1 \neq 0\}} \cap U_{\{\phi_2 \neq 0\}}$, $H^1(\mathcal{O}(2, -2))$ has representatives of the form

$$\frac{P_2(\phi_3, \phi_4, \phi_5)}{\phi_1\phi_2}$$

for a quadric polynomial $P_2$. (As announced, $h^1(\mathcal{O}(2, -2)) = 6$.) The restriction to the set $S$ annihilates one of these quadric polynomial (which we called $F_4$ above), so the cokernel of the map $\alpha$ has dimension $8 - (6 - 1) = 3$. So we conclude that the number of antigenerations is $h^2(\tilde{V}) = 3$ and, either using the index (4.14), or the explicit counts of holomorphic sections of the line bundles appearing in (4.15) ($h^0(\mathcal{O}(0,1)) = 2$, $h^0(\mathcal{O}(2, -2)) = 6$, $h^0(\mathcal{O}(3, -1)) = 30$, $h^0(\mathcal{O}(5, -1)) = 138$), we arrive a the number of generations, $h^1(\tilde{V}) = 99$.

Now let us turn to the Landau–Ginzburg phase, and compute the massless spectrum at the Landau–Ginzburg point. The generations, $27$s of $E_6$, are obtained as follows. (We state the results for the right-handed $16_{-1/2}$s of $SO(10) \times U(1)$, that is, for the states with $(q, \eta) = (-1/2, -1/2)$ arising from $k$-even sectors. The $10_1$ and $1_{-2}$ are obtained from the neighboring $k$-odd sectors.)
There are 98 generations obtained as nonic polynomials acting on the untwisted vacuum, \( P_9(\phi)|k = 0 \). These nonic polynomials are taken modulo the ideal generated by \( W(\phi), F_a(\phi) \). There is also one more generation, coming from the \( k = 10 \) twisted sector, which has the form \( \lambda_{-1/9}^3|k = 9 \).

The 3 antigenerations \( ((q, \bar{q}) = (1/2, -1/2)) \) also come from the \( k = 10 \) sector, and have the form \( \phi^{3,4,5}_{-1/9}|k = 10 \). All in all, there are 99 generations and 3 antigenerations, exactly as predicted by the geometrical analysis above.

One readily check that the \( \mathbb{Z}_{18} \) discrete R-symmetry which stems from the quantum symmetry of the LG theory is nonanomalous. Recall that there are possible anomalies due to \( E_6, E_8 \) and gravitational instantons. The corresponding anomaly coefficients, \( A_{1,2,3} \), must satisfy

\[
A_1 = A_2 \mod 2mr
\]

\[
24A_1 = 24A_2 = A_3 \mod 2mr
\]

where, here, \( m = 9 \) and \( r = 3 \).

We compute \( A_1 \) by taking a trace in the right-moving Ramond sector (spacetime fermions) of tensor product of the “internal” \((0,2)\) SCFT and the CFT consisting of 10 free left-moving Majorana–Weyl fermions, which carry the \( E_6 \) gauge degrees of freedom:

\[
A_1 = Tr_R \left( (kr - 2q)\frac{q^2}{2r}(-1)^{F_R} \right) \mod 2mr
\]

The charge \( (kr - 2q) \) act homogeneously on \( E_6 \) representations. The factor of \( \frac{q^2}{2r} \), being the square of an \( E_6 \) generator, when traced gives the index of the corresponding \( E_6 \) representation. Also, we get equal contributions from each right-handed fermion, and the left-handed charge-conjugate state, so, up to a factor of 2, we can count the contribution only of right-handed fermions, weighted by the index of the representation:

\[
A_1 = 2((-3)c(78) + 98(1)c(27) + (31)c(27) + 3(29)c(27))
\]

\[
= -18 \mod 54
\]

where \( c(78) = 12, c(27) = c(\overline{27}) = 3 \) \( A_2 \) receives contributions only from the gluinos of the second \( E_8 \), and so is

\[
A_2 = -60r = -18 \mod 54
\]

The calculation of the gravitational contribution to the anomaly is the most involved, because it receives contributions from all fermions, including the gauge singlets.
The right-handed gauge singlets arise as follows: there are 318 which arise from the “untwisted” $k = 1$ sector. These have the general form of an oscillator mode of a fermion ($\lambda^a$ or $\gamma$) times a polynomial of the appropriate degree in the $\phi$s. 19 singlets come from the $k = 3$ sector, and have the form $\lambda^4_{-1/9} \phi^{1/2}_{-5/18} |k = 5\rangle$, and 6 come from the $k = 11$ sector and have the form $\lambda^{1/2}_{-1/9} \phi^{1/2}_{-7/18} |k = 11\rangle$ and $\gamma_{-1/9} \phi^{1/2}_{-7/18} |k = 11\rangle$. There are also singlets from the $k = 9$ sector, but since they don’t contribute to the anomaly, we won’t write them down.

Including by hand the contributions of the gluino, the dilatino and the gluinos from the second $E_8$, the gravitational anomaly is

$$A_3 = -452 r + Tr_R \left( (kr - 2q)(-1)^F_R \right) \mod 2mr$$
$$= -452 r + 2[(-3)78 + (98 + 31 + 3 \cdot 29)27 + 19(9) + 2(15) + 6(33)]$$
$$= 0 \mod 2mr$$

So the anomalies satisfy (4.16), and can be canceled by assigning a suitable inhomogeneous transformation law to the axion.

| Field | $Q_1$ | $Q_2$ |
|-------|-------|-------|
| $\phi_{1,2}$ | 0 | 1 |
| $\phi_{3,4,5}$ | 1 | 0 |
| $\chi$ | 1 | -2 |
| $p$ | -5 | 1 |
| $\lambda^{1,2}$ | 1 | -1 |
| $\lambda^3$ | 1 | 0 |
| $\lambda^4$ | 3 | -1 |
| $\lambda^4$ | -1 | 2 |
| $\gamma$ | -4 | 0 |

Table 3: $U(1)$ charges of the (bosonic and fermionic) fields for an unstable rank four bundle on the same manifold.

The resolved model that we have constructed has three antigenerations. It is conceivable that, at Landau–Ginzburg, a flat direction exists where we turn on a $27$-$\overline{27}$ pair and break the gauge symmetry to $SO(10)$. At first sight, there is an obvious candidate linear $\sigma$-model realization of this idea.

We start again with the same singular model as in the previous subsection. But now, in the process of resolving the singularities, we “borrow” a pair of left-moving Majorana–Weyl fermions from the 10 free Majorana–Weyl fermions which represent the gauge degrees of freedom. When we blow up, this pair becomes an interacting Weyl fermion, an integral part of the “internal” $(0,2)$ model. The charge assignments are given in Table 3.

The fifth fermion $\lambda^5$ is uncharged under the original $U(1)$, $Q = 2Q_1 + Q_2$, and so is free along the singular locus. That is, along the
singular locus, the gauge group is $E_6$, as before. But, away from the singular locus, the
gauge group is broken to $SO(10)$.

The vacuum gauge bundle, $V'$, on the resolved space $\tilde{M}$, is the kernel in the exact
sequence

$$0 \to V' \to O(1, -1)^\oplus 2 \oplus O(1, 0) \oplus O(3, -1) \oplus O(-1, 2)^\oplus 3 \to O(5, -1) \to 0 \quad (4.17)$$

This time, there is no funny business; nowhere on $\tilde{M}$ do all of the $F_a$ simultaneously vanish,
so the sequence (4.17) really is exact. $V'$ has rank four; along the singular locus, it is the
direct sum of a rank three bundle and a trivial line bundle. This is exactly what we want
to represent the breaking of $E_6$ to $SO(10)$ as we move away from the singular locus.

Unfortunately, $V'$, as defined by (4.17), is not stable. The higher cohomology groups of
all of the line bundles in (4.17) vanish except for $O(-1, 2)$, which has $h^3(O(-1, 2)) = 1$ and
$h^2(O(-1, 2)) = 3$. Tracing through the long exact sequence in cohomology associated to
(4.17), one finds $h^1(V') = 98$, $h^2(V') = h^2(O(-1, 2)) = 3$, and $h^3(V') = h^3(O(-1, 2)) = 1$.
The last is a disaster. It means that $V'$ is not stable, as its dual bundle, $(V')^*$ has a global
section.

Physically, the postulated deformation, in which we turn on a $27-\overline{27}$ pair is not a flat
direction.

In our next example, we will look at simple monad on a complete-intersection Calabi-
Yau manifold.
4.3. A Monad Example

Consider the complete intersection of two sextics in $\mathbb{P}^5_{1,1,1,3,3,3}$. This has a $\mathbb{Z}_3$ orbifold singularity at $\phi_1 = \phi_2 = \phi_3 = 0$. We torically blow up the singularity in the usual way, which adds a second $U(1)$ and a field $\chi$. For the vector bundle, we choose a bundle of rank $3$. On the original, unresolved manifold, this is a kernel,

$$0 \rightarrow V \rightarrow \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(6) \rightarrow \mathcal{O}(9) \rightarrow 0$$

On the resolved space, it is the cohomology of the monad

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1,-3) \oplus \mathcal{O}(0,1)^{\oplus 3} \oplus \mathcal{O}(2,0) \rightarrow \mathcal{O}(3,0) \rightarrow 0$$

(4.18)

The fields and their charges under the two $U(1)$s are listed in Table 4.

| Field  | $Q_1$ | $Q_2$ |
|--------|------|------|
| $\phi_{1,2,3}$ | 0    | 1    |
| $\phi_{4,5,6}$ | 1    | 0    |
| $\chi$     | 1    | $-3$ |
| $p$        | $-3$ | 0    |
| $\lambda^0$ | 1    | $-3$ |
| $\lambda^{1,2,3}$ | 0    | 1    |
| $\lambda^4$  | 2    | 0    |
| $\gamma^{1,2}$ | $-2$ | 0    |

Table 4: $U(1)$ charges of the (bosonic and fermionic) fields for the resolved model. The original charge is $Q = 3Q_1 + Q_2$.

Working out the Stanley-Reisner ideal for this manifold, we find

$$\eta_1^3 = 36, \quad \eta_1^2 \eta_2 = 12, \quad \eta_1 \eta_2^2 = 4, \quad \eta_2^3 = 0$$

(4.19)

so $c_3(T) = -4\eta_1^3 - 3\eta_1^2 \eta_2 + 9\eta_1 \eta_2^2 = -144$ and $c_3(V) = -6\eta_1^3 - 3\eta_1^2 \eta_2 + 9\eta_1 \eta_2^2 = -216$. Thus the net number of generations is 108.

We can, as in the previous examples, calculate $h^1(V)$ and $h^2(V)$ separately. Split the monad (4.18) into two short exact sheaf sequences,

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1,-3) \oplus \mathcal{O}(0,1)^{\oplus 3} \oplus \mathcal{O}(2,0) \rightarrow \mathcal{E} \rightarrow 0$$

$$0 \rightarrow V \rightarrow \mathcal{E} \rightarrow \mathcal{O}(3,0) \rightarrow 0$$

(4.20)

Of the line bundles in (4.20), only $\mathcal{O}$ and $\mathcal{O}(1,-3)$ have nonzero higher cohomology groups:

$$h^0(\mathcal{O}) = h^0(\mathcal{O}(1,-3)) = h^3(\mathcal{O}) = 1, \quad h^2(\mathcal{O}(1,-3)) = 3$$

---

7 This example also arose in conversations with T. M. Chiang.
$H^2(O(1, -3))$ is generated by the cocycles $\frac{\phi_{1,5,6}}{\phi_1\phi_2\phi_3}$ on the triple overlap $U_1 \cap U_2 \cap U_3$. The number of global sections of the other line bundles is given by the index:

$$h^0(O(n, m)) = \frac{1}{12} (c_1(O(n, m))c_2(T) + 2c_1(O(n, m))^3)
= 7n + 3m + 6n^3 + 6n^2m + 2nm^2$$  \hspace{1cm} (4.21)

Putting all this together, we find

$$H^1(V) = \frac{H^0(O(3, 0))}{H^0(O(1, -3)) \oplus H^0(O(0, 1))^6 \oplus H^0(O(2, 0))}$$

which yields $h^1(V) = 112$ and we find the exact sequence

$$0 \rightarrow H^2(O(1, -3)) \rightarrow H^2(V) \rightarrow H^3(O) \rightarrow 0$$

which yields $h^2(V) = 4$.

This is readily compared with the spectrum at Landau-Ginzburg. There are 112 27s of $E_6$ of the form $P_9(\phi) | k = 0$, from the untwisted sector. There are a total of four 27s from the twisted sectors. Two are the ground states of the $k = 6$ and $k = 8$ twisted sectors; the remaining two have the form $\phi_0^{4,5,6} | k = 6$, modulo the $Q^+\lambda_0^4 | k = 6$. So, indeed, the number of generations is unchanged from the prediction at large radius.

For our last example, we construct a (0,2) model on K3

4.4. A K3 example

We consider a rank-3 bundle on K3, which we take to be the intersection of two quartics in $\mathbb{P}^{4,1,1,2,2}$. Upon resolving the singularities, this toric construction gives a two-dimensional slice through the twenty-dimensional Kähler moduli space of K3. The bundle is constructed as a kernel (no fermionic gauge symmetries) and, unlike the previous example, remains a kernel when resolved. The charges of the fields for the resolved manifold are listed in Table 5. The bundle $V$ is given by

$$0 \rightarrow V \rightarrow O(0, 1)^{\oplus 2} \oplus O(1, -1) \oplus O(2, -2) \rightarrow O(3, -1) \rightarrow 0$$  \hspace{1cm} (4.22)

| Field | $Q_1$ | $Q_2$ |
|-------|-------|-------|
| $\phi_{1,2}$ | 0 | 1 |
| $\phi_{3,4,5}$ | 1 | 0 |
| $\chi$ | 1 | $-2$ |
| $p$ | $-3$ | 1 |
| $\lambda^{1,2}$ | 0 | 1 |
| $\lambda^3$ | 1 | $-1$ |
| $\lambda^4$ | 2 | $-2$ |
| $\gamma^{1,2}$ | $-2$ | 0 |

**Table 5:** $U(1)$ charges of the (bosonic and fermionic) fields for the resolved model. The original charge is $Q = 2Q_1 + Q_2$. 

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The number of hypermultiplets is given by the index

\[ h^1(V) = -\text{Ind}(\overline{\partial} V) = c_2(V) - 2r = 18 \]

since we have set \( c_2(V) = c_2(T) = 24 \) for K3, and \( r(V) = 3 \).

We can compare this with the Landau-Ginzburg calculation. How this goes in six dimensions may be a little unfamiliar, so let us pause to make a few comments. As far as the left-moving degrees of freedom (which assemble the gauge representations), the structure is the same as what we have seen in four dimensions. However, the right-movers work a little differently. The fermions in six-dimensional vector multiplets have \( \bar{q} = \pm 1 \).

As in the four-dimensional case, half of them (those with \( \bar{q} = -1 \)) come in a completely canonical way from the \( k = 0, 1, 2 \) sectors. The others come from the conjugate sectors (in this case, \( 10 - k \)). The fermions in hypermultiplets have \( \bar{q} = 0 \), but again come in pairs, respectively in representations \( R \) and \( \overline{R} \) of the gauge group, from conjugate twisted sectors of the LG orbifold. In our case, we will simply list which sectors the \( 27 \)s (actually, just the \( 16_{-1/2} \) components of the \( 27 \)) arise in; the \( \overline{27} \)s arise in the conjugate sectors.

One finds 14 \( 27 \)s in the untwisted sector, realized as \( P_5(\phi)|k = 0 \). There are 3 \( 27 \)s in the \( k = 4 \) sector, \( \overline{\phi}_{-1/5}^{3,4,5}|k = 4 \), and one \( 27 \) in the \( k = 6 \) sector, \( \lambda_{-1/5}^4|k = 6 \). All in all, this is \( 14 + 3 + 1 = 18 \) hypermultiplets, as expected.

5. Prospects

As we have seen, the linear \( \sigma \)-model allows one to define a wider class of (0,2) theories than previously considered. Rather than having the left-moving fermions couple to a stable holomorphic vector bundle, the linear \( \sigma \)-model is perfectly happy having the left-moving fermions coupling to a stable torsion-free (or reflexive) sheaf. Such a sheaf is a vector bundle outside of a “singularity set” of complex codimension 2 (in the case of a torsion-free sheaf) or 3 (in the case of a reflexive sheaf).

The linear \( \sigma \)-model realizes the singularity set as a protuberance, that is, as an additional branch of zeroes of the scalar potential, glued in at the location of the singularity set on \( M \). In the first example in section 4.2, the singularity set consisted of 8 points, and over each point, the protuberance was a \( \mathbb{C}P^1 \). The key point is that the protuberance
is compact, so there is no divergence of the linear $\sigma$-model, and hence, presumably, no divergence of the superconformal field theory to which it flows in the infrared.

Since there was no divergence, we can presume that no interesting nonperturbative physics is associated with this type of protuberance. In particular, there is no signal of new massless states appearing for those values of the moduli.

There are, however, situations where new nonperturbative physics is dictated. First, in some cases, it can happen that, when all of the $F_a(\phi)$ vanish at some point in $M$, the new branch of the space of zeroes of the scalar potential is noncompact. This is the (0,2) analogue of the (complex structure side of) the conifold, in which the polynomials defining the manifold $M$ fail to be transverse.

Another possibility, when $V$ is defined as the cohomology of a monad, is that at some point in the Kähler moduli space, the $E_a$s might all simultaneously vanish. In this case, we again get a noncompactness of the space of zeroes of the scalar potential, this time associated to the complex scalar $\sigma$ in the $\Sigma$ multiplet. This is the (0,2) analogue of the Kähler side of the conifold (the singularity in the Kähler moduli space at $r = \theta = 0$).

In both of these cases, the linear $\sigma$-model, and hence the superconformal field theory, is singular. We hope to discuss the physics of these singularities in a future work.\footnote{As this manuscript was nearing completion, we learned of work of Kachru, Seiberg and Silverstein\cite{Kachru:2002he}, which addresses this issue, in the case of (0,2) models obtained as deformations of (2,2) models.}

Some examples of the physics which can ensue are provided by the work of Witten and collaborators on “small instantons”\cite{34, 25, 26}. This is a particular example of the first type of singularity, where the $F$s simultaneously vanish on some set of complex codimension 2 on $M$. In the case of $M = \text{K3}$ (itself of complex dimension 2), this singularity set is a collection of points, the locations of instantons shrunk to zero size. The findings of\cite{34, 25, 26} can be translated into the language of torsion-free sheaves as follows: for each point in the singularity set which contributes $n$ to $c_2(V)$ (we persist in calling the torsion-free sheaf which couples to the left-movers “$V$”), there arises a nonperturbative $\text{Sp}(n)$ gauge symmetry in 6 dimensions. When $M$ has complex dimension 3 (\textit{i.e.} a Calabi–Yau manifold), the singularity set could be complex codimension 2 or 3.

An exciting prospect is that at some of these points where nonperturbative physics is required, our experience with (2,2) models leads us to suspect that there may well be additional branches of the (0,2) moduli space that can be reached in a physically smooth
manner. Some such transitions would presumably involve topology changing transitions of the base Calabi-Yau manifolds together with nontrivial changes in the bundle structure as well. We hope to report on such processes shortly.

Acknowledgments

The research of J.D. is supported by NSF grant PHY-9511632, the Robert A. Welch Foundation and an Alfred P. Sloan Foundation Fellowship. The research of B.R.G. is supported by the Alfred P. Sloan Foundation, a National Young Investigator Award, and by the National Science Foundation. The research of D.R.M. is supported by NSF grant DMS-9401447.
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