THE ORDER COMPLEX OF $PGL_2(p^n)$ IS CONTRACTIBLE WHEN $p$ IS ODD

EMILIO PIERRO

Department of Mathematics,
London School of Economics and Political Science,
Houghton Street,
London, WC2A 2AE

Abstract. Given a group $G$, its lattice of subgroups $L(G)$ can be viewed as a simplicial complex in a natural way. The inclusion of $1_G, G \in L(G)$ implies that $L(G)$ is contractible, and so we study the topology of the order complex $\hat{L}(G) := L(G) \setminus \{1_G, G\}$. In this short note we consider the homotopy type of $\hat{L}(G)$ where $G \cong PGL_2(p^{2n}), p \geq 3, n \geq 1$ and show that $\hat{L}(G)$ is contractible. This is consistent with a conjecture of Shareshian on the homotopy type of order complexes of finite groups.

1. Introduction

Given a poset $P$, the order complex $\Delta(P)$ is the simplicial complex whose $k$ chains are the chains of length $k$ in $P$. In the case where $P$ is the subgroup lattice $L(G)$ of a finite group $G$, this yields a contractible space and so its topology is rather uninteresting. For a finite group $G$, then, we consider the order complex of $\hat{L}(G) := L(G) \setminus \{G, 1_G\}$. The following example serves to illustrate that if $G$ and $H$ are two groups with identical composition factors, then their order complexes can still wildly.

Example 1. Let $G$ be a group with $|G| = p^2$ where $p$ is prime. If $G$ is cyclic, then $\hat{L}(G)$ is homotopy equivalent to a point. If $G$ is elementary abelian, then $\hat{L}(G)$ is a wedge of $p$ spheres (the disjoint union of $p + 1$ disconnected points).

In the case where $G$ is a solvable group, it was shown by Shareshian [10] that the order complex of $G$ is shellable if and only if $G$ is solvable. Recall that a shellable topological space is homotopy equivalent to a wedge of equidimensional spheres. This led to the following conjecture of Shareshian [11, Conjecture 1.1 (A)]

Conjecture 2 (Shareshian). Let $G$ be a finite group and let $H < G$. Then, the order complex of $\hat{L}(H, G) = \{K \mid H < K < G\}$ has the homotopy type of a wedge of spheres.

If $G$ has a non-trivial Frattini subgroup then a result of Quillen, which we state in the following section, shows that the order complex of $G$ is contractible. Hence it seems natural to consider the order complex of $G$ in the case where $G$ is an almost simple group.

The homotopy type of minimal simple groups (i.e. those which do not contain simple subgroups) were considered by Shareshian [10, Lemma 3.8]. To the best of the author’s knowledge, the homotopy type of $\hat{L}(G)$ is known in only a small number of other cases, appearing in the work of Kramarev and Lokutsievskiy [6]. In unpublished work of Shareshian and the author, they determined that when $G \cong PSL_2(8):3 \cong R(3), \hat{L}(G)$ is homotopy equivalent to a wedge of $504 = |G'|$ 2-spheres.

Our main result is the following.

Proposition 3. Let $G \cong PGL_2(p^{2n})$ where $p \geq 3$ is prime. Then $\hat{L}(G)$ is contractible.

We do not consider the groups $PGL_2(2^n)$ since these are simple. In the case $G \cong PSL_2(4)$, the homotopy type of $G$ was determined by Shareshian [11] and it is homotopy equivalent to a wedge of 60 spheres. The groups $PSL_2(q^2)$, where $q$ is odd, also admit field automorphisms and the product...
of a field and a diagonal automorphism yields a non-split extension. We mention that the non-split extension \( PSL_2(9).2 \) is isomorphic to a point-stabiliser of \( M_{11} \) in its natural permutation representations on 11 points. It is immediate \(^8\) Proposition 1.8) that when \( G \) is isomorphic to the non-split extension \( PSL_2(q^2).2 \), \( \hat{L}(G) \) is contractible. On the contrary, the extensions \( P\Sigma L_2(q^4) \) seem too difficult to approach in general.

We follow the notation of \(^3\) and explicitly mention the following conventions. The notation \( nX \) is used to denote a conjugacy class of elements of order \( n \) and where \( |C_G(nA)| \geq |C_G(nB)| \), etc. The notation \( 2A, B_2 \) denotes an elementary abelian group of exponent 2 containing one element from the conjugacy class \( 2A \) and two elements from the conjugacy class \( 2B \).

2. PROOF OF PROPOSITION \(^3\)

In general, to consider the full poset \( \hat{L}(G) \) for a finite group \( G \) would be quite challenging. The first reduction we can make follows from Quillen’s Fiber Lemma \(^8\) which we state in the following specific form.

**Lemma 4** (Quillen). Let \( f : X \to X \) be the inclusion map of the poset \( X \) such that \( f(x) \leq x \) for all \( x \in X \). Let \( f/x = \{ x \in X | f(x) \leq x \} \), the “fiber” of \( x \). If \( f/x \) is contractible for all \( x \in X \), then \( f \) is a homotopy equivalence.

For a finite group \( G \) this allows us to restrict \( \hat{L}(G) \) to the subposet consisting of subgroups which occur as the intersection of maximal subgroups. Dually, we can then consider the subposet consisting of all subgroups generated by their “minimal”, i.e., prime order, elements. By abuse of notation we shall denote this subposet by \( \hat{L}(G) \). We note that in the case of \( PGL_2(q) \), where \( q \equiv 1 \mod 4 \), this allows us to omit the maximal parabolic subgroups, since all of their prime order elements are contained in \( PSL_2(q) \).

The connection between the order complex of \( \hat{L}(G) \) and the Möbius function of \( G \) is well known \(^9\). Although we shall not need the following result, the most well known connection is that \( \mu_G(1_G) + 1 = \chi(\hat{L}(G)) \). In particular, for the determination of the Möbius function of \( G \) it is necessary to know which subgroups of \( G \) occur as intersections of maximal subgroups. This was determined for the groups \( PGL_2(q) \) by Downs in his thesis \(^4\), which, unfortunately the author has not been able to obtain.

Nevertheless, the maximal subgroups of \( G \cong PGL_2(q) \) can be found in \(^2\) Table 8.7\) and since we restrict ourselves to the case where \( G \) does not contain maximal subfield subgroups, we are able to easily determine the subgroups necessary. It is worth mentioning that the case of determining \( \hat{L}(G) \) when \( G \) is an almost simple group is in general much easier than the case where \( G \) is simple since we are able to use the following result of Björner and Walker \(^3\) Theorem 1.1\).

**Theorem 5** (Björner–Walker). If \( L \) is a bounded lattice, \( s \in \hat{L} \) and the complements of \( s \) form an antichain, then

\[
\hat{L} \simeq \bigvee_{s \perp s} \left( \sum_{x \perp s} (\langle \hat{0}, x \rangle \ast (x, 1)) \right).
\]

We use the above result in conjunction with the following specific case of a result of Kratzer and Thévenaz \(^5\) Proposition 1.6\).

**Lemma 6** (Kratzer–Thévenaz). Let \( G \) be a finite group with order complex \( \hat{L}(G) \) and let \( H \leq G \). If the subposet \( (H, G) = \{ K \mid H < K < G \} \) (or dually \( (1_G, H) = \{ K \leq G \mid 1 < K < H \} \)) is contractible, then \( \hat{L}(G) \) is homotopy equivalent to \( \hat{L}(G) \setminus \{ H \} \).

We now consider the case where \( G \cong PGL_2(q) \), where \( q = p^{2n}, p \geq 3 \) is an odd prime and \( n \geq 1 \). There is a unique conjugacy class of involutions in \( G \setminus \text{soc}(G) \)^5 Table 4.5.1\) which we denote \( 2B \).

The centraliser in \( G \) of an element \( x \in 2B \) is isomorphic to \( D_{2(2q+1)} \). To apply Lemma \(^5\) we use the fact that the set \( \{ \langle x \rangle \mid x \in 2B \} \) is the set of complements of \( G' \). By showing that the subcomplex \( \langle \{ x \}, G' \rangle = \{ H \leq G \mid \langle x \rangle < H < G \} \) is contractible, Lemma \(^5\) shows that removing the subgroups \( \langle x \rangle \) where \( x \in 2B \) does not change the homotopy type of \( \hat{L}(G) \). Combining the two results yields that \( \hat{L}(G) \) has the homotopy type of an empty wedge of spheres, in other words, a single point, and so \( \hat{L}(G) \) is contractible.

It remains, then, to prove the following.
Lemma 7. Let $G \cong PGL_2(q)$ where $q = p^a \geq 9$. Let $2B$ denote the unique conjugacy class of involutions in $G \setminus \text{soc}(G)$ and let $x \in 2B$. The ascending link of $x$ is contractible.

Proof. The maximal subgroups of $G$ are known [2, Table 8.7] and those intersection non-trivially with an element from $2B$ are isomorphic to $D_{2(q-1)}$ or $D_{2(q+1)}$. The intersection of any pair of maximal subgroups of these isomorphism types, both containing $x$, is isomorphic to a four-group. Such a four-group must have shape $2A_1B_2$. Hence, the ascending link of $x$ consists of subgroups of these three isomorphism types.

An easy calculation shows these are:

1. $1 + (q + 1)/2$ subgroups isomorphic to $D_{2(q+1)}$, one of which is $C_G(x)$;
2. $(q + 1)/2$ subgroups isomorphic to $D_{2(q-1)}$; and,
3. $(q + 1)/2$ subgroups of shape $2A_1B_2$.

Since $C_G(2A) \cong D_{2(q-1)}$, it is clear that in the ascending link of $x$, each maximal subgroup isomorphic to $D_{2(q-1)}$ contains a unique subgroup of shape $2A_1B_2$, and so these can be removed without changing the homotopy type. Similarly, each of the subgroups isomorphic to $D_{2(q+1)}$ except for $C_G(x)$ can be removed without changing the homotopy type. Finally, we are left with the $(q + 1)/2$ subgroups of shape $2A_1B_2$, contained in the unique maximal subgroup $C_G(x)$. It is clear that this contractible, completing the proof. 

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