Regime Transition in the Energy Cascade of Rotating Turbulence

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Transition from a split to a forward kinetic energy cascade system is explored in the context of rotating turbulence using direct numerical simulations with a three-dimensional isotropic random force uncorrelated with the velocity field. Our parametric study covers confinement effects in large aspect ratio domains and a broad range of rotation rates. The data here presented add substantially to previous works, which, in contrast, focused on smaller and shallower domains. Results indicate that for fixed geometrical dimensions the Rossby number acts as a control parameter, whereas for a fixed Rossby number the product of the domain size along the rotation axis and forcing wavenumber governs the amount of energy that cascades inversely. The regime transition criterion hence depends on both control parameters.

INTRODUCTION

The energy cascade is the fundamental mechanism in turbulent flows that describes the energy exchange between the various scales of motion [1]. A forward cascade from large to small scales is commonly observed in three-dimensional (3D) flows, whereas an inverse energy cascade from small towards large scales is the hallmark of two-dimensional (2D) flows [2, 3]. Predicting the energy cascade direction, therefore, requires anticipating if, for a given set of control parameters, the resulting flow field resembles best 3D or 2D flow dynamics. In lack of analytical predictions, a typical approach consists of carefully designing numerical experiments, where the system’s parameters are individually varied to produce a phase transition diagram. Throughout this study we consider a large number of forced direct numerical simulations (DNS) and analyze the influence of geometric confinement and system rotation on the cascade direction in homogeneous rotating turbulence.

Inertial waves, i.e. plane wave solutions to the linearized Navier-Stokes equations, can modulate the energy transfer in rotating turbulence [4, 5]. By considering high rotation rates and exploiting the fact that rotating turbulence is a multi-timescale problem, Waleffe [6] suggested that the nonlinear dynamics are modified by wave interactions. Resonant wave interactions can explain the favored energy transfer towards horizontal modes, whereas non-resonant wave interactions are considered to damp and inhibit the triadic interactions typical of homogeneous turbulence [7, 8]. This mechanism also persists at lower rotation rates due to homochiral interactions that transfer energy into the plane orthogonal to the rotation axis [9]. As a consequence, when rotating homogeneous flows are forced at wavenumber $\kappa_f$, the injected energy can cascade both to larger ($\kappa < \kappa_f$) and smaller scales ($\kappa > \kappa_f$); this is hereafter referred to as split energy cascade. These findings help to explain the preferential upscale of energy typically found in numerical and experimental investigations of rotating turbulent flows [8, 10–14]. Nevertheless, we must bear in mind that a large network of triadic interactions as in the Navier-Stokes equations can evolve differently than a set of isolated triads, as previously pointed out in Refs. [15, 16].

Among different theories that elucidate the phenomenon of rotating turbulence, the work of Galtier [17] is regarded as an important contribution. Based on wave turbulence theory, which deals with systems where interactions are governed by waves, he derived scaling laws for the energy spectrum. These laws were also shown to follow from phenomenological arguments for the spectral transfer time — a typical energy transfer timescale. For infinitely large domains, as required by wave turbulence theory [18], the weak inertial-wave theory of Galtier [17] predicts that energy cascades forward and to small scales. However, a passage from a split to a forward energy cascade system by approaching the large-box limit has not yet been confirmed by DNS.

In the absence of rotation, however, the geometrical dimensions of the system itself influences the energy cascade direction. Using a two-dimensional two-component (2D2C) horizontal force, Smith et al. [19] and Celani et al. [20] found that the ratio $L_\alpha/\ell_f$, where $L_\alpha$ is the vertical domain extension and $\ell_f$ is the forcing lengthscale, is a governing control parameter. They showed that large $L_\alpha/\ell_f$ results in a forward energy cascade, whereas inverse energy transfer was triggered and split the energy cascade for $L_\alpha/\ell_f \leq 1/2$. More recently, numerical simulations by Benavides and Alexakis [21] explored transitions in a thin layer of fluid subjected to free-slip boundary conditions. Transition from a forward to a split energy cascade was shown to be critical and depend on the ratio of forcing lengthscale to wall separation.

Regime transitions in rotating homogeneous turbulence are therefore affected by geometrical dimensions and rotation rate. Deusebio et al. [22] studied hyper-viscous fluids in rotating small aspect ratio domains subjected to 2D2C forcing and found that large rotation rates as well as small $L_\alpha/\ell_f$ suppress enstrophy production and
induce an inverse energy cascade. Their data proves, at least for weak rotation rates, that transition from a split to a forward cascade is possible by controlling either rotation rate or domain size. For strong rotation, however, almost the entire injected energy cascaded inversely. Although transition was not observed, they hypothesized that it could still take place for sufficiently large \( L_3/\ell_f \). This conjecture, however, remains to be verified by either forcing smaller flow scales or by increasing the domain size [23].

The present work sheds light on the question whether a transition from a split to a forward cascade system always exists in forced homogeneous rotating turbulence. We conduct a systematic parametric study that covers several rotation rates and an unprecedented range of geometric confinements by considering strongly elongated domains and large forcing wavenumbers \( \kappa_f \). This new database is complementary to previous studies, which focused on the confinement induced transition in smaller and shallower domains. Through large-scale forcing, we construct isotropic flow fields that are posteriorly subject to rotation. Differently from previous studies, we employ a three-dimensional three-component (3D3C) forcing scheme that by design provides a constant energy input independent of the velocity field. We believe this results in a neater and more general framework where anisotropy originates solely from rotation.

The external force \( f \) injects energy to the system at rate \( \varepsilon_I \), see Ref. [28]. The force's spectrum \( F(\kappa) \), from which \( f \) in Eq. (2) is assembled, is Gaussian distributed, centered around a wavenumber \( \kappa_f \) and has standard deviation \( \varepsilon = 0.5: F(\kappa) = A \exp(- (\kappa - \kappa_f)^2 / \varepsilon) \). For given \( \kappa_f \) and \( \varepsilon \), the prefactor \( A \) is uniquely determined from the desired energy input rate \( \varepsilon_I \). In the absence of rotation, we obtain isotropic velocity fields and a balance between energy input rate and viscous dissipation, i.e. \( \varepsilon_I = \varepsilon_\nu \). This forcing scheme ensures through projection that the force and velocity field are uncorrelated at every instant of time [28]. As a consequence, \( \varepsilon_I \) is solely determined by the force-force correlation and is independent of the velocity field. Thus, we can define a priori true control parameters from which the governing non-dimensional numbers are derived.

The domain size, \( L_1 \) and \( L_\perp \), the forcing wavenumber \( \kappa_f \), the viscosity \( \nu \), the rotation rate \( \Omega \) and the energy input rate \( \varepsilon_I \) can all be freely chosen. Regarding \( \varepsilon_I \), it could be additionally decomposed in three contributions stemming from the power injected in each direction. However, because the forcing is isotropic, it is sufficient to consider the total power input \( \varepsilon_I \) only. These six parameters \( \{\kappa_f, \nu, \varepsilon_I, \Omega, L_\perp, L_1\} \) form the set of true control parameters and are the basis for the non-dimensional similarity numbers. The characteristic length, velocity and time-

### METHODOLOGY AND GOVERNING PARAMETERS

We solve the incompressible Navier-Stokes equations in a frame rotating at rate \( \Omega \):

\[
\nabla \cdot \mathbf{u} = 0, \quad (1)
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + (2 \mathbf{\Omega} + \mathbf{\omega}) \times \mathbf{u} = -\nabla q + \nu \nabla^2 \mathbf{u} + \mathbf{f}. \quad (2)
\]

Here, \( \mathbf{u}, \mathbf{\omega} \) and \( \mathbf{f} \) are velocity, vorticity and an external force, respectively. The reduced pressure into which the centrifugal force is incorporated is given by \( q \), and \( \nu \) denotes the kinematic viscosity.

Equations (1) and (2) are discretized in space by a dealiased Fourier pseudo-spectral method (2/3-rule) in a triply-periodic domain of size \( 2\pi L_1 \times 2\pi L_2 \times 2\pi L_3 \) [24, 25]. The rotation axis is assumed aligned with the vertical direction, i.e. \( \mathbf{\Omega} = \Omega \hat{e}_3 \), and we restrict ourselves to cases where the domain size in the direction perpendicular to the axis of rotation are equal: \( L_1 = L_2 = L_\perp = 1 \). Accordingly, \( L_1 \) replaces \( L_3 \) to denote the domain size in the direction parallel to the rotation axis, and can be arbitrarily chosen. We use Rogallo’s integrating factor technique for exact time integration of the viscous and Coriolis terms and a third-order Runge-Kutta scheme for the nonlinear terms [26, 27].

### TABLE I: List of direct numerical simulations at \( \text{Re}_f \approx 55 \). The \( \text{Ro}_f \) numbers are given in the footnote.

| Case  | \( \kappa_f L_\perp \) | \( \kappa_f L_1 \) | \( A_r \) | \( N_p \) |
|-------|----------------------|--------------------|----------|--------|
| kf02-a01a | 2 | 2 | 1 | 192³ |
| kf04-a01a | 4 | 4 | 1 | 384³ |
| kf04-a02b | 4 | 8 | 2 | 384² × 768 |
| kf04-a04b | 4 | 16 | 4 | 384² × 1536 |
| kf04-a08b | 4 | 32 | 8 | 384² × 3072 |
| kf04-a16b | 4 | 64 | 16 | 384² × 6144 |
| kf04-a32b | 4 | 128 | 32 | 384² × 12288 |
| kf08-a01a | 8 | 8 | 1 | 768³ |
| kf08-a02b | 8 | 16 | 2 | 768³ × 1536 |
| kf08-a04b | 8 | 32 | 4 | 768³ × 3072 |
| kf08-a08b | 8 | 64 | 8 | 768³ × 6144 |
| kf08-a16b | 8 | 128 | 16 | 768² × 12288 |
| kf16-a01a | 16 | 16 | 1 | 1536³ |
| kf16-a02b | 16 | 32 | 2 | 1536² × 3072 |
| kf16-a04b | 16 | 64 | 4 | 1536² × 6144 |
| kf32-a01b | 32 | 32 | 1 | 3072³ |

a \( \text{Ro}_f \approx 0.31, 0.06 \)
b \( \text{Ro}_f \approx 0.06 \)
c \( \text{Ro}_f \approx 1.25, 0.63, 0.31, 0.27, 0.24, 0.22, 0.19, 0.16, 0.14, 0.11, 0.09, 0.08, 0.06 \)
scale follow naturally as \( \ell_f = \kappa_f^{-1} \), \( u_f = \xi_f^{1/3} \kappa_f^{-1/3} \), and \( \tau_f = \kappa_f^{2/3} \xi_f^{-1/3} \), respectively. In addition, a timescale based on the rotation rate is taken as \( \tau_\Omega = 1/(2\Omega) \).

The Reynolds and Rossby numbers are now unambiguously defined as

\[
\text{Re}_\varepsilon = \frac{\xi_f^{1/3} \kappa_f^{-4/3}}{\nu} \quad \text{and} \quad \text{Ro}_\varepsilon = \frac{\kappa_f^{2/3} \xi_f^{1/3}}{2\Omega}.
\]

From the problem’s geometry and the forcing wavenumber, we define two other non-dimensional numbers, i.e., \( \kappa_f \mathcal{L}_\perp \) and \( \kappa_f \mathcal{L}_\parallel \). Hence, we obtain a set of four independent governing non-dimensional numbers that fully describes our numerical experiments: \( \text{Re}_\varepsilon \), \( \text{Ro}_\varepsilon \), \( \kappa_f \mathcal{L}_\perp \) and \( \kappa_f \mathcal{L}_\parallel \). As the final goal is to investigate dimensional and rotational effects on forced homogeneous rotating turbulence, we fix \( \text{Re}_\varepsilon \) and allow \( \text{Ro}_\varepsilon \), \( \kappa_f \mathcal{L}_\perp \) and \( \kappa_f \mathcal{L}_\parallel \) to vary. We remark that this set is not unique and other non-dimensional groups exist. For instance, \( \text{Re}_\varepsilon \) and \( \text{Ro}_\varepsilon \) could be combined to form the micro-scale Rossby number \( \text{Ro}_\varepsilon = \text{Re}_\varepsilon^{1/2} \text{Ro}_\varepsilon \) (ratio of rotation and Kolmogorov timescale \([7]\)) or \( \kappa_f \mathcal{L}_\parallel \) and \( \kappa_f \mathcal{L}_\perp \) could be related to obtain the domain’s aspect ratio \( \mathcal{A} = \mathcal{L}_\parallel/\mathcal{L}_\perp \). Initial conditions were generated by performing DNS of non-rotating forced isotropic turbulence. We started from a zero-velocity field and marched in time until a fully developed steady-state was achieved. After the initial transient statistics, were sampled over at least 24 \( \tau_f \), corresponding to approximately ten large-eddy turnover times. Following this procedure, a reference isotropic solution was computed for every entry in Tab. I.

The initially imposed \( \text{Re}_\varepsilon \approx 55 \) ultimately led to homogeneous non-rotating turbulent fields with a characteristic Taylor micro-scale Reynolds number \( \text{Re}_\lambda \approx 68 \). The spatial resolution in terms of the Kolmogorov lengthscale \( \eta \) was kept constant throughout this study, i.e., \( \kappa_{\text{max}} \eta \approx 1.5 \), where \( \kappa_{\text{max}} \) is the largest represented wavenumber. For the case with largest \( \kappa_f \mathcal{L}_\parallel \), the integral lengthscale in the direction of rotation is about 600 times smaller than the respective domain size.

Figure 1 compares the 3D spherically averaged energy spectrum \( E(\kappa) \) for cases with aspect ratio \( \mathcal{A}_r = 1 \), which contain “a01” in its name description, and two additional simulations with \( \mathcal{A}_r = 16 \) and \( \mathcal{A}_r = 32 \) (cases kf04-a32 and kf08-a16 in Tab. I). This data proves the equivalence between initial conditions for DNS forced at different wavenumbers and those computed with distinct \( \kappa_f \mathcal{L}_\parallel \) and \( \kappa_f \mathcal{L}_\perp \). We find that the energy spectra perfectly coincide and that \( E(\kappa) \) scales best with \( \kappa^2 \) at wavenumbers \( \kappa < \kappa_f \), in agreement with Ref. [29]. The obtained isotropic velocity fields were used as initial condition for the simulations with different rotation rates. The statistical variability of the results for small domains was reduced by ensemble averaging. For the smallest domain kf02-a01 we ensemble averaged 10 independent realizations and cases kf04 with \( \mathcal{A}_r > 1 \) are averages of 3 realizations. For all other cases, the data represents a single numerical experiment.

**RESULTS**

First we assess the effects of geometrical dimension and rotation on the time evolution of box-averaged kinetic energy \( K \) and viscous dissipation \( \varepsilon_\nu \). The non-dimensional geometric parameters \( \kappa_f \mathcal{L}_\perp \) and \( \kappa_f \mathcal{L}_\parallel \) are varied for two fixed rotation rates: weak \( (\text{Ro}_\varepsilon = 0.31; \text{Fig. 2}) \) and strong \( (\text{Ro}_\varepsilon = 0.06; \text{Fig. 3}) \). Additionally, for a fixed and large domain, \( \kappa_f \mathcal{L}_\perp = 8 \) and \( \kappa_f \mathcal{L}_\parallel = 64 \) (case kf08-a08; Fig. 4), we investigate the Rossby number range \( 0.06 < \text{Ro}_\varepsilon < 0.25 \). For more details about the simulation parameters, please refer to Tab. I.

All cases undergo a transient of roughly 10 \( \tau_f \) from the onset of rotation (Figs. 2 to 4), which converges towards a unique solution for sufficiently large \( \kappa_f \mathcal{L}_\parallel \). We find that the results are independent of the transversal domain size for \( \kappa_f \mathcal{L}_\perp \geq 4 \); see Fig. 3, where the lines for different \( \kappa_f \mathcal{L}_\perp \) and identical \( \kappa_f \mathcal{L}_\parallel \) coincide. Departing from an isotropic state, where the energy cascade is strictly forward (\( \varepsilon_\nu/\varepsilon_f = 1 \)), \( \varepsilon_\nu \) decreases monotonically until it is lowest at approximately 3 \( \tau_f \) (Figs. 2b, 3b and 4b). For fixed \( \text{Ro}_\varepsilon \), Figs. 2b and 3b show that both \( \kappa_f \mathcal{L}_\perp \) and \( \kappa_f \mathcal{L}_\parallel \) have no influence on the minimum of \( \varepsilon_\nu \). On the other hand, Fig. 4b suggests a direct proportionality between the minimum value of \( \varepsilon_\nu \) and \( \text{Ro}_\varepsilon \).

After \( t \approx 3 \tau_f \), \( \varepsilon_\nu \) increases towards \( \varepsilon_f \). Nevertheless, the strong and weak rotation cases lead to a different final state for \( \varepsilon_\nu \). While increasing \( \kappa_f \mathcal{L}_\parallel \) restores \( \varepsilon_\nu = \varepsilon_f \) for the weak rotating case (Fig. 2b), the imbalance \( \varepsilon_\nu < \varepsilon_f \), although lower than 0.075 \( \varepsilon_f \) for \( \kappa_f \mathcal{L}_\parallel = 128 \), persists up to the final time for the strong rotating case (Fig. 3b).
Similarly to Fig. 2b, increasing $\text{Ro}_z$ reestablishes a forward energy cascade for a fixed domain size (Fig. 4b). After the initial transient ($t > 10\,\tau_f$), $\varepsilon_{\nu}$ follows mostly a slow linear decay (Fig. 3b) or remains nearly constant (Figs. 2b and 4b). Consequently, $K$, which evolves in time as $dK/dt = \varepsilon_I - \varepsilon_{\nu}$, grows quasi-linearly (Figs. 2a, 3a and 4a). Based on this idea we define the inverse energy flux $\varepsilon_{\text{inv}} = \varepsilon_I - \varepsilon_{\nu}$ from the imbalance between energy injection rate and viscous dissipation. To estimate $\varepsilon_{\text{inv}}$, which is equal to the local slope of $K(t)$, a linear least-square fit is applied to $15\,\tau_f < t < 30\,\tau_f$ in the time evolution of $K$ (Figs. 2a, 3a and 4a). The r.m.s. residual between the actual and fitted data indicates that the linear regression model is appropriate. For the worst case, $\kappa_f L_\perp = 8$, the r.m.s. residual is 0.65\% of the mean value. Assuming that the linear law is exact and the noise is essentially Gaussian, one obtains 0.0004 for the standard error of the slope coefficient. Results for the inverse energy flux are thus shown in Figs. 5 and 6 in form of a phase transition diagram.

From Fig. 5a, we see that the inverse energy flux $\varepsilon_{\text{inv}}$ decreases monotonically with $\kappa_f L_\perp$ for both $\text{Ro}_z \approx 0.31$ and $\text{Ro}_z \approx 0.06$. Moreover, results for the strong rotating case suggest that increasing $\kappa_f L_\perp$ while retaining $\kappa_f L_\parallel$ leads to negligible differences in $\varepsilon_{\text{inv}}$ — see the overlapping circles with different colors for $\text{Ro}_z \approx 0.06$. 

FIG. 2: Time evolution of box-averaged kinetic energy (a) and energy dissipation rate (b) for $\text{Ro}_z \approx 0.31$ (weak rotation). Lines corresponding to same $\kappa_f L_\parallel$ are grouped by color: $\kappa_f L_\parallel = 2$ (■), $\kappa_f L_\parallel = 4$ (■), $\kappa_f L_\parallel = 8$ (■), $\kappa_f L_\parallel = 16$ (■). Lines corresponding to the same $A_r$ are grouped by line types: $A_r = 1$ (——), $A_r = 8$ (——), cf. Tab. I.
FIG. 4: Time evolution of box-averaged kinetic energy (a) and energy dissipation rate (b) for $\kappa_f L_\perp = 8$ and $\kappa_f L_\parallel = 64$. Different line colors correspond to the range $0.06 < \text{Ro}_\varepsilon < 1.25$, see Tab. I.

FIG. 5: Phase transition diagram for weak and strong rotation and varying geometrical dimensions (a) and for constant geometrical dimension and varying $\text{Ro}_\varepsilon$ (b). Color scheme of (a) is the same as in Fig. 3. In (a), the data point for $\kappa_f L_\perp = \kappa_f L_\parallel = 32$ (case $\text{kf}32$–$\text{a}01$) is almost identical to case $\text{kf}04$–$\text{a}08$ ($\kappa_f L_\perp = 4$; $\kappa_f L_\parallel = 32$), and is therefore not visible.

Transition from a split to a forward cascade system occurs gradually. For $\text{Ro}_\varepsilon \approx 0.31$ and $\kappa_f L_\parallel = 64$ less than 0.004 $\varepsilon_I$ is transferred in the inverse direction, whereas for $\text{Ro}_\varepsilon \approx 0.06$ a split cascade is still present at $\kappa_f L_\parallel = 128$. For a fixed domain size with $\kappa_f L_\perp = 8$ and $\kappa_f L_\parallel = 64$ (case $\text{kf}08$–$\text{a}08$; Fig. 5b), $\varepsilon_{\text{inv}}$ is continuously suppressed for increasing $\text{Ro}_\varepsilon$ and transition to a forward cascade system occurs in the vicinity of $\text{Ro}_\varepsilon = 1$.

A question that follows from these results is for which combination of governing non-dimensional parameters regime transition occurs. From literature, a possible criteria is $\text{Ro}_\varepsilon \kappa_f L_\parallel = C$, where $C$ is a constant [2, 23]. To test this hypothesis, Fig. 6 presents the data from Fig. 5, but juxtaposed in a single diagram and scaled accordingly with $\text{Ro}_\varepsilon$. The curves for different $\text{Ro}_\varepsilon$ do not line up; hence, this criteria disagrees with our data. A discussion on a possible reason is given in the next section.

Now we turn our attention to the influence of $\kappa_f L_\parallel$ and $\kappa_f L_\perp$ on the spectral energy flux and energy spectra. Hereafter we present results for the strong rotating case with $\text{Ro}_\varepsilon \approx 0.06$ only, as differences are more pronounced than in the weak rotating case. Although we show instantaneous data at $t = 30 \tau_f$, the trend described in what follows also holds for other instants of time. Conservation of energy requires the portion of the injected energy that is not dissipated to be accumulated. By analyzing the spectral energy flux $\Pi(\kappa)$, we find that the net energy transfer $T(\kappa) = -d\Pi/d\kappa$ is positive for $\kappa < \kappa_f$. In other words, wavenumbers in this range gain energy and we ob-
serve an upscale energy transfer. Evidence is presented in Fig. 7, which also highlights how sensitive $\Pi(\kappa)$ is with respect to changes in $\kappa_f \mathbf{L}_\parallel$ and $\kappa_f \mathbf{L}_\perp$. In this regard, Fig. 7a, where $\kappa_f \mathbf{L}_\parallel$ is constant and $\kappa_f \mathbf{L}_\perp = \{8, 16, 32\}$, shows that the shape of $\Pi(\kappa)$ remains unaltered for different $\kappa_f \mathbf{L}_\perp$. On the other hand, varying $\kappa_f \mathbf{L}_\perp$ from 16 to 64 while $\kappa_f \mathbf{L}_\parallel$ is constant, reduces the magnitude of the inverse energy flux and the range of wavenumbers for which an upscale energy transfer takes place, see Fig. 7b. Therein, greater values of $\kappa_f \mathbf{L}_\parallel$ are also associated with an enhanced spectral energy flux for $\kappa > \kappa_f$. This is a consequence of the fixed energy input rate $\varepsilon_I$, which causes the step in $\Pi(\kappa)$ at $\kappa = \kappa_f$ to be the same for all cases.

The three-dimensional energy spectra $E(\kappa)$ for the same cases are shown in Fig. 8. Additionally, the energy spectrum of case $\kappa = 32$ with $\kappa_f \mathbf{L}_\parallel = \kappa_f \mathbf{L}_\perp$ at the onset of rotation is included as reference. Figure 8a reinforces that $\kappa_f \mathbf{L}_\parallel$ dictates the degree of energy accumulation, as the curves for different $\kappa_f \mathbf{L}_\perp$ and constant $\kappa_f \mathbf{L}_\parallel$ overlap. In agreement with results in Fig. 7 for $\Pi(\kappa)$, we observe significantly higher levels of energy for $\kappa < \kappa_f$ with respect to the isotropic reference spectrum. These are reduced for increasing $\kappa_f \mathbf{L}_\parallel$, see Fig. 8b.

As for the distribution of energy in terms of $\kappa_\parallel$ and $\kappa_\perp$, Fig. 9 presents the two-dimensional energy spectrum $E(\kappa_\parallel, \kappa_\perp)$. Results are shown exclusively for case $\kappa = 32$ with $\kappa_f \mathbf{L}_\perp = \kappa_f \mathbf{L}_\parallel = 32$, as it contains most large scale resolution. The energy spectrum is non-dimensionalized with $2\pi \kappa_\perp$, in such a way that contour levels of isotropic spectra appear as circles centered at the origin. In agreement with previous works, Fig. 9 confirms that the kinetic energy has the tendency to accumulate at lower $\kappa_\parallel/\kappa_f$. Hence, $E(\kappa_\parallel, \kappa_\perp)$ is anisotropic and contour levels display an elliptical shape with major axis aligned with the $\kappa_\perp$-direction. This is observed even for high wavenumbers and suggests that all scales of motion are influenced by rotation; indeed, for this case, $\kappa_f \eta = 1.1$, where $\kappa_f \eta = (\Omega^2/\varepsilon_I)^{1/2}$ is the Zeman wavenumber [14]. At the same time, the energy input remains isotropic. See the inset for the imprint of the isotropic forcing scheme, which delineates the bright area located at $\kappa_\parallel^2 + \kappa_\perp^2 = \kappa_f^2$. In addition, we see higher energy levels in the vicinity of $\kappa_\parallel/\kappa_f = 0$.

An anisotropic distribution of energy is predicted by the weak inertial-wave theory, which suggests that the energy spectrum has the form $E(\kappa_\parallel, \kappa_\perp) \sim \kappa_\perp^{-5/2} \kappa_\parallel^{-1/2}$ [17]. To test if our data presents any sign of this scaling law, we show in Fig. 10 instantaneous one-dimensional energy spectra along the perpendicular and parallel directions, i.e. $E(\kappa_\perp)$ and $E(\kappa_\parallel)$ for $t = 0, 10, 20$ and $30 \tau_f$. Figure 10a shows that energy levels increase progressively for $\kappa_\perp < \kappa_f$, whereas for $\kappa_\perp > \kappa_f$, the distribution of energy is nearly unaltered. Also for $\kappa_\perp > \kappa_f$, we observe

![Image of Figure 6](image6.png)

**FIG. 6:** Phase transition diagram in terms of combined control parameter $\mathrm{Ro}_x \kappa_f \mathbf{L}_\parallel$ for all data points of Fig. 5. Colored circles represent data from Fig. 5a, and squares data from Fig. 5b.

![Image of Figure 7](image7.png)

**FIG. 7:** Spectral energy flux for $\mathrm{Ro}_x \approx 0.06$ and cases with $\kappa_f \mathbf{L}_\parallel = 32$ (a) and $\kappa_f \mathbf{L}_\perp = 16$ (b). In (a), $\kappa_f \mathbf{L}_\perp = 8$ ( ), $\kappa_f \mathbf{L}_\perp = 16$ ( ) and $\kappa_f \mathbf{L}_\perp = 32$ ( ). In (b), $\kappa_f \mathbf{L}_\parallel = 16, 32$ and 64 ( ). Arrow denotes the direction of increase.
FIG. 8: Three-dimensional spherically averaged energy spectrum for \( \kappa_fL_\parallel = 32 \) (a) and \( \kappa_fL_\perp = 16 \) (b) with \( \text{Ro}_t \approx 0.06 \). Line styles are the same as in Fig. 7, apart from the reference energy spectrum of Fig. 1 with \( \kappa_fL_\perp = \kappa_fL_\parallel = 32 \) (----).

that a narrow wavenumber range develops from the initial state and approaches best a \( \kappa_\perp^{-5/2} \) scaling law. Regarding \( E(\kappa) \), Fig. 10b, the energy content for \( \kappa_\parallel > \kappa_f \) is significantly lower than at the onset of rotation. This corroborates the idea that rotation lessen the flow field dependency on the direction parallel to the rotation axis. As time evolves, the range \( \kappa_\parallel < \kappa_f \) resembles best a \( \kappa_\parallel^{-1/2} \) scaling law for all time instants. We emphasize that this result is essentially different from predictions of the weak inertial-wave theory, as the latter estimates \( E(\kappa) \sim \kappa_\parallel^{-1/2} \) for \( \kappa_\parallel \) larger than the forcing wavenumber.

DISCUSSION

This work investigated through direct numerical simulations the effects of domain size and rotation rate on the energy cascade direction of rotating turbulence. The

data here presented add substantially to previous works, which, in contrast, focused on smaller and shallower domains (\( \kappa_fL_\parallel \) and \( \kappa_fL_\perp < 8 \) [19, 22]). The presented results, therefore, contribute towards a complete picture of the phase diagram, which unveils transition from inverse to forward through a split energy cascade in rotating turbulence.

Our results support \( \kappa_fL_\parallel \) as the primary control parameter provided that \( \text{Ro}_t \) is constant and \( \kappa_fL_\perp > 4 \). In this scenario, transversal finite-size effects of \( \kappa_fL_\perp \) on the inverse energy transfer \( \varepsilon_{inv} \) are negligible for our cases with aspect ratio \( A_r \geq 1 \). For weak rotation with \( \text{Ro}_t \approx 0.31 \), transition from a split to a forward cascade was observed at \( \kappa_fL_\parallel \approx 64 \). For the strong rotating case, however, although strongly suppressed, a portion of the injected energy (\( \varepsilon_{inv} \approx 0.075 \varepsilon_f \)) still cascaded inversely and accumulated at the large scales for \( \kappa_fL_\parallel = 128 \).

We attribute the fact that \( \varepsilon_{inv} \) does not become exactly zero for \( \text{Ro}_t \approx 0.31 \) to two effects. First, the simulations considered in this study are limited to \( \text{Re}_t \approx 68 \). A higher Reynolds number could contribute to a stronger forward cascade, possibly reducing \( \varepsilon_{inv} \) to zero. Second, although effects of the geometric non-dimensional parameter \( \kappa_fL_\perp \) are minor, results hint that larger values of \( \kappa_fL_\perp \) could also contribute to a reduction of \( \varepsilon_{inv} \). In this manner, indefinite increase of \( \kappa_fL_\perp \) could potentially change the phase diagram in the vicinity of \( \varepsilon_{inv}/\varepsilon_f = 0 \), and could cause regime transition to be sharp rather than smooth. The recent study of Benavides and Alexakis [21] has shown that a continuous increase of horizontal domain dimensions shifts the transition behavior for thin layer turbulence from smooth to critical. We hope that
FIG. 10: One-dimensional energy spectra for Roₐ ≈ 0.06 and κ_fLₚ = κ_fL_f = 32 (case k/32-a01) along directions κ⊥ (a) and κ∥ (b). Lines represent the time evolution of the energy spectrum: t = 0 (---), and t = 10, 20 and 30τ_f (---). A reference line for the scaling laws that best agrees with the presented data is also shown (---).

further studies will help to fill the parameter space for higher Reynolds numbers and even longer domain sizes.

For Roₐ ≈ 0.06, we agree with Deusebio et al. [22] and believe that a continuous increase of κ_fL_f would result in transition to a forward energy cascade. Nevertheless, results for the weak case suggest a slow-paced transition and significantly larger values for κ_fL_f might be required. Interestingly, the transition of ε_inv in terms of κ_fL_f resembles a logistic function, similar to what has been found for regime transitions in thin layer turbulence [21].

In search of a criteria for transition between a forward and a split cascade system, we made an attempt to express ε_inv/ε_f for all parameter points as a function of Roₐκ_fL_f. As the different curves do not overlap, we believe that a criteria for transition should stem from a more general match of timescales. A criteria such as Roₐκ_fL_f = C, can be obtained by requiring the slowest inertial wave frequency 1/τ_w = 2Ω/κ_fL_f and the eddy turnover frequency u_fκ_f at the forcing scale to be of same order [2, 23]. Alternatively, we can frame the problem within the idea that rotation alters the spectral transfer time τ_s at which energy is transferred to smaller scales. Thus, it follows that ε_inv ∼ u_f²/τ_s, with u_f a velocity scale characteristic of eddies of size ℓ, and τ_s ∼ τ_f²/ℓ. Here, τ_{nl} ≈ ℓ/u_f is the nonlinear timescale and τ_s is the relaxation time of triple velocity correlations. The relaxation time in isotropic turbulence simplifies to τ_{nl} to recover the dissipation law, i.e. ε_inv ∼ u_f²/ℓ.

Now the condition Roₐκ_fL_f = C can be obtained by requiring ε_inv = ε_f, and assuming u ∼ u_f, τ_s ≈ τ_f and τ_s ≈ τ_w. So, Roₐκ_fL_f = C is equivalent to state that in the presence of rotation the nonlinear timescale remains of the order of τ_f, and that the relaxation timescale τ_s is given by the inverse of the slowest inertial wave frequency, i.e. τ_s ∼ τ_w. A generalization of the previous reasoning would be to consider a τ_{nl} obtained from a measured velocity property, like the r.m.s. velocity, and the lengthscale ℓ possibly as ℓ⊥, as the triadic interactions are expected to be depleted in the direction parallel to the rotation axis [32]. The relaxation time τ_s could be sought as a function of both τ_f and τ_w. In this manner, more general criteria like Roₐκ_fL_f = C arise, where a and b are yet undetermined exponents.

Results for scaling laws of the energy spectrum are here not conclusive, and there is no clear sign of an inertial range over several decades. This is plausible since our initial and isotropic field with Roₐ ≈ 68 does not contain a clear inertial range. In spite of that, the narrow wavenumber region after κ⊥ = κ_f develops and approaches best a κ⊥⁻⁵/² scaling law. Our results also show that the κ⊥⁻⁵/² and κ∥⁻¹/² scalings appear at different wavenumber ranges, and that the κ⊥⁻⁵/² scaling prevails in the 3D energy spectrum, see Fig. 8.

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