The BIC of a conical fibration.

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Abstract

In the paper we introduce the notions of a singular fibration and a singular Seifert fibration. These notions are natural generalizations of the notion of a locally trivial fibration to the category of stratified pseudomanifolds. For singular foliations defined by such fibrations we prove a de Rham type theorem for the basic intersection cohomology introduced the authors in a recent paper. One of important examples of such a structure is the natural projection onto the leaf space for the singular Riemannian foliation defined by an action of a compact Lie group on a compact smooth manifold.

The failure of the Poincaré duality for the homology and cohomology of some singular spaces led Goresky and MacPherson to introduce a new homology theory called intersection homology which took into account the properties of the singularities of the considered space (cf. [10]). This homology is defined for stratified pseudomanifolds. The initial idea was generalized in several ways. The theories of simplicial and singular homologies were developed as well as weaker notions of the perversity were proposed (cf. [11, 12]). Several versions of the Poincaré duality were proved taking into account the notion of dual perversities. Finally, the deRham intersection cohomology was defined by Goresky and MacPherson for Thom-Mather stratified spaces. The first written reference is the paper by J.L. Brylinski, cf. [3]. This led to the search for the ”de Rham - type” theorem for the intersection cohomology.

The first author in his thesis and several subsequent publications, (cf. [14]) presents the de Rham intersection cohomology of stratified spaces using essentially the Verona resolution of singularities (cf. [19]). The main idea is the introduction of the complex (sheaf) of liftable intersection forms (cf. [14]).

In [15] the authors introduced the basic (de Rham) intersection cohomology (BIC) for singular Riemannian foliations using the fact that such foliations define a stratification of the manifold (cf. [13, 21]) respect to the leaves of the foliation. In fact, the BIC can be defined for a larger class of foliations which we call conical foliations.

In the present paper we study BIC for a particular class of conical foliations which are defined by the so called singular fibration. In this case the space of leaves is a stratified pseudomanifold and the BIC of this foliation is the intersection cohomology of the space of leaves. This result is
a version of the well-known de Rham theorem for the basic de Rham intersection cohomology of a foliation defined by a generalized fibration.

Such singular fibrations can be met in several natural situations, for example the foliation defined by an action of a compact Lie group is defined by a singular fibration (cf. [7]).

1 Conical foliations.

1.1 Singular foliations.

A regular foliation on a manifold $M$ is a partition $F$ of $M$ by connected immersed submanifolds, called leaves, with the following local model:

$$(\mathbb{R}^m, \mathcal{H})$$

where leaves of $\mathcal{H}$ are defined by $\{dx_1 = \cdots = dx_p = 0\}$. We shall say that $(\mathbb{R}^m, \mathcal{H})$ is a simple foliation. Notice that the leaves have the same dimension.

A singular foliation on a manifold $M$ is a partition $F$ of $M$ by connected immersed submanifolds, called leaves, with the following local model:

$$(\mathbb{R}^{m-n-1} \times \mathbb{R}^{n+1}, \mathcal{H} \times \mathcal{K})$$

where $(\mathbb{R}^{m-n-1}, \mathcal{H})$ is a simple foliation and $(\mathbb{R}^{n+1}, \mathcal{K})$ is a singular foliation having the origin as a leaf. When $n = -1$ we just have a regular foliation. Notice that the leaves may have different dimensions. This local model is exactly the local model of a foliation of Sussmann [17] and Stefan [16]; so, they are singular foliations in our sense.

Classifying the points of $M$ following the dimension of the leaves one gets a stratification $S_F$ of $M$ whose elements are called strata. The foliation is regular when this stratification has just one stratum $\{M\}$.

A smooth map $f: (M, F) \rightarrow (M', F')$ between singular foliated manifolds is foliated if it sends the leaves of $F$ into the leaves of $F'$. When $f$ is an embedding it preserves the dimension of the leaves and therefore it sends the strata of $M$ into the strata of $M'$. The group of foliated diffeomorphisms preserving the foliation will be denoted by Diff $(M, F)$. Examples, more properties and the singular version of the Frobenius theorem the reader can find in [13, 11, 10, 17, 18].

1.1.1 Examples.

(a) In any open subset $U \subset M$ we have the singular foliation $F_U = \{\text{connected components of } L \cap U / L \in F \}$. The associated stratification is $S_{F_U} = \{\text{connected components of } S \cap U / S \in S_F \}$.

(b) We shall say that a foliated embedding of the form

$$\varphi: (\mathbb{R}^{m-n-1} \times \mathbb{R}^{n+1}, \mathcal{H} \times \mathcal{K}) \longrightarrow (U, F_U),$$

is a foliated chart modelled on $(\mathbb{R}^{m-n-1} \times \mathbb{R}^{n+1}, \mathcal{H} \times \mathcal{K})$.

(c) Consider $(N, \mathcal{K})$ a connected regular foliated manifold. In the product $N \times M$ we have the singular foliation $\mathcal{K} \times F = \{L_1 \times L_2 / L_1 \in \mathcal{K}, L_2 \in F \}$. The associated stratification is $S_{\mathcal{K} \times F} = \{N \times S / S \in S_F \}$.

(d) Consider $\mathbb{S}^n$ the sphere of dimension $n$ endowed with a singular foliation $\mathcal{G}$ without 0-dimensional leaves. Identify the disk $D^{n+1}$ with the cone $c\mathbb{S}^n = \mathbb{S}^n \times [0, 1] / \mathbb{S}^n \times \{0\}$ by the map
x \mapsto [x/\|x\|, \|x\|] where [u, t] is a generic element of the cone. We shall consider on $D^{n+1}$ the singular foliation
\[ cG = \{ F \times \{ t \} / F \in G, t \in [0, 1] \cup \{ \vartheta \}, \]
where $\vartheta$ is the vertex $[u, 0]$ of the cone. The induced stratification is
\[ S_{cG} = \{ S \times [0, 1] / S \in S_G \} \cup \{ \vartheta \}, \]
since $G$ does not possess 0-dimensional leaves. For technical reasons we allow $n = -1$, in this case $S^n = \emptyset$ and $cS^n = \{ \vartheta \}$. Unless otherwise stated, if $(S^n, G)$ is a singular foliation, we shall consider on $D^{n+1} = cS^n$ the foliation $cG$.

1.2 Conical foliations.

The strata of this kind of foliations are not necessarily manifolds and their relative position can be very wild. Consider $(\mathbb{R}, F)$ where $F$ is given by a vector field $f \partial / \partial t$; there are two kinds of strata. The connected components of $f^{-1}(\mathbb{R} - \{0\})$ and these of $f^{-1}(0)$. In other words, any connected closed subset of $S$ can be a stratum. In order to support a intersection cohomology structure, the stratification $S_F$ asks for a certain amount of regularity and conicalicity (see [10] for the case of stratified pseudomanifolds). This leads us to introduce the following definition.

A singular foliation $(M, F)$ is said to be a conical foliation if any point $x \in M$ possesses a foliated chart $(U, \varphi)$ modelled on
\[ (\mathbb{R}^{m-1} \times cS^n, \mathcal{H} \times cG), \]
where $(S^n, G)$ is a conical foliation without 0-dimensional leaves. We shall say that $(U, \varphi)$ is a conical chart of $x$ and that $(S^n, G)$ is the link of $x$. Notice that, if $S$ is the stratum containing $x$ then $\varphi^{-1}(S \cap U) = \mathbb{R}^{m-1} \times \{ \vartheta \}$. This definition is made by induction on the dimension of $M$.

Notice that each stratum is an embedded submanifold of $M$. The leaf (resp. stratum) of $F$ (resp. $S_F$) containing $x$ is sent by $\varphi$ to the leaf of $F$ containing 0 (resp. $\mathbb{R}^{m-1}$). Although a point $x$ may have several charts the integer $n + 1$ is an invariant: it is the codimension of the stratum containing $x$. This integer cannot to be 1 since the conical foliation $(S^n, G)$ has not 0-dimensional leaves.

We also use the notion of conical foliation in a manifold with boundary. In this case, the points of the boundary have conical charts modelled on $(\mathbb{R}^{m-n-2} \times [0, 1] \times cS^n, H \times I \times cG)$ where $I$ is the foliation by points of $[0, 1]$. The boundary $(\partial M, F_{\partial M})$ is also a conical foliated manifold.

The above local description implies some important facts about the stratification $S_F$. Notice for example that the family of strata is finite in the compact case and locally finite in the general case. The closure of a stratum $S \in S_F$ is a union of strata. Put $S_1 \preceq S_2$ if $S_1, S_2 \in S_F$ and $S_1 \subset S_2$. This relation is an order relation and therefore $(S_F, \preceq)$ is a poset.

The depth of $S_F$, written depth $S_F$, is defined to be the largest $i$ for which there exists a chain of strata $S_0 \prec S_1 \prec \cdots \prec S_i$. So, depth $S_F = 0$ if and only if the foliation $F$ is regular. We also have depth $S_{F_U} \leq$ depth $S_F$ for any open subset $U \subset M$ and depth $S_\mathcal{G} =$ depth $S_{\mathcal{H} \times \mathcal{G}} <$ depth $S_{\mathcal{H} \times \mathcal{G}}$ (cf. 1.1.1).

The minimal strata are exactly the closed strata. An inductive argument shows that the maximal strata are the strata of dimension $m$. They are called regular strata and the others singular strata. The union of singular strata is written $\Sigma_F$. Since the codimension of singular strata is at least 2, then only one regular $R$ strata appears, which is an open dense subset.
For the definition of perverse forms we exploit the compatibility between the foliated charts given in the following Lemma whose proof can be found in [15].

**Lemma 1.2.1** Let \((U_1, \varphi_1), (U_2, \varphi_2)\) be two foliated charts of a point \(x\) of \(M\) with \(U_1 \subset U_2\). There exists an unique foliated embedding
\[
\varphi_{1,2}: (\mathbb{R}^{m-n-1} \times S^n \times [0, 1], \mathcal{H}_1 \times \mathcal{G}_1 \times \mathcal{I}) \to (\mathbb{R}^{m-n-1} \times S^n \times [0, 1], \mathcal{H}_2 \times \mathcal{G}_2 \times \mathcal{I})
\]
making the following diagram commutative
\[
\begin{array}{ccc}
\mathbb{R}^{m-n-1} \times S^n \times [0, 1] & \xrightarrow{\varphi_{1,2}} & \mathbb{R}^{m-n-1} \times S^n \times [0, 1] \\
\otimes & \downarrow{P} & \downarrow{P} \\
\mathbb{R}^{m-n-1} \times cS^n & \xrightarrow{\varphi_2^{-1} \circ \varphi_1} & \mathbb{R}^{m-n-1} \times cS^n \\
\end{array}
\]
where the smooth map \(P\) is defined by \(P(u, \theta, t) = (u, [\theta, t])\).

This result also implies that the link \((S^n, \mathcal{G})\) of a point \(x\) is the same for any point of the stratum \(S\) containing \(x\). For this reason we shall say that \((S^n, \mathcal{G})\) is the *link of the stratum \(S*.

### 1.3 Partial blow up.

A useful tool for inductive arguments is the desingularisation. Fix \((M, \mathcal{F})\) a conical foliated manifold with strictly positive depth. A *partial blow up* is a conical foliated manifold with boundary \((\hat{M}, \hat{\mathcal{F}})\) and a foliated smooth map \(L_M: (\hat{M}, \hat{\mathcal{F}}) \to (M, \mathcal{F})\) such that:

1) The restriction \(L: \hat{M} - \partial \hat{M} \to M - S_{\text{min}}\) is a diffeomorphism, where \(S_{\text{min}}\) is the union of closed (minimal) strata.

2) For any point \(x \in S_{\text{min}}\) there exists a commutative diagram
\[
\begin{array}{ccc}
\mathbb{R}^{m-n-1} \times S^n \times [0, 1] & \xrightarrow{\varphi} & \hat{M} \\
\otimes & \downarrow{L} & \\
\mathbb{R}^{m-n-1} \times cS^n & \xrightarrow{\varphi} & M \\
\end{array}
\]
where \((U, \varphi)\) is a conical chart of \(x\) and \(\varphi: (\mathbb{R}^{m-n-1} \times S^n \times [0, 1], \mathcal{H} \times \mathcal{G}) \to (\hat{M}, \hat{\mathcal{F}})\) is a foliated embedding.

Notice that each restriction \(L: L^{-1}(S) \to S\), where \(S\) is a closed stratum, is a fiber bundle possessing a foliated atlas whose structural group is \(\text{Diff}(S^n, \mathcal{G})\) relatively to a conical foliation \((S^n, \mathcal{G})\).

**Proposition 1.3.1** The partial blow up always exists and is unique. It also verifies

3) \(\text{depth } S_{\hat{\mathcal{F}}} = \text{depth } S_{\mathcal{F}} - 1\).

4) \((\partial \hat{M}, \hat{\mathcal{F}}_{\partial \hat{M}})\) is a conical foliated manifold.

The proof of Proposition 1.3.1 is presented in [15].
2 Basic Intersection cohomology.

There are two ways to define perverse forms, the first one uses a system of tubular neighborhoods (cf. [6]) and the second one uses a global blow up ([14]). In the study of conical foliations we have decided to introduce intersection forms in an intermediate way: iterating the local blow up \( M \) we obtain a manifold \( \widetilde{M} \) with borders where the foliation become regular. A similar procedure has been developed in [14]. Some of the differential forms on the regular part of \( M \) can be extended to \( \widetilde{M} \), these are the perverse forms. We present this notion in an intrinsic way without constructing \( \widetilde{M} \).

We are going to deal with differential forms on a product \( N \times [0,1]^p \), where \( N \) is a manifold: they are restrictions of differential forms defined on \( N \times [-1,1] \).

2.1 Perverse forms.

The differential complex \( \Pi^*(M \times [0,1]^p) \) of perverse forms is introduced by induction on the depth \( S_F \). When this depth is 0 then \( \Pi^*(M \times [0,1]^p) = \Omega^*(M \times [0,1]^p) \). In the general case we shall put \( \omega \in \Pi^*(M \times [0,1]^p) \) if \( \omega \) is a differential form on \( \Omega^*((M - \Sigma_F \times [0,1]^p)) \) such that any \( x \in M \) possesses a conical chart \((U,\varphi)\) such that

\[
(\varphi \times \text{id}_{[0,1]^p})^* \omega \quad \text{extends to} \quad \omega \varphi \in \Pi^*(\mathbb{S}^{m-n-1} \times \mathbb{S}^n \times [0,1]^{p+1}).
\]

In fact, \( \Pi^*(M \times [0,1]^p) \) is a differential graduated commutative algebra (dgca in short) since \((\omega + \eta)\varphi = \omega \varphi + \eta \varphi, (\omega \wedge \eta)\varphi = \omega \varphi \wedge \eta \varphi \) and \((d\omega)\varphi = d\omega \varphi \).

Notice that the notion of perverse form depends on the foliation \( F \) through the stratification \( S_F \).

2.1.1 Properties (see [15]).

(a) Let \((N,\mathcal{H})\) be a regular foliated manifold. The partial blow up of \( N \times \epsilon \mathbb{S}^n \) is \( N \times \mathbb{S}^n \times [0,1] \), with \((x,\theta,t) \mapsto (x,[\theta,t])\). Here the factor \([0,1] \) appears. Further desingularisation produces extra \([0,1] \) factors.

(b) Consider \( L_M : \widetilde{M} \rightarrow M \) the partial blow up of a conical foliated manifold. Any perverse form \( \omega \) on \( M \) defines a perverse form \( \tilde{\omega} \) on \( \widetilde{M} \) extending \( L_M^*\omega \). This map \( \omega \mapsto \tilde{\omega} \) defines an isomorphism between \( \Pi^*(M) \) and \( \Pi^*(\widetilde{M}) \).

(c) Any perverse form and any conical chart verify the equation (1).

(d) There are smooth functions on \( M - \Sigma_F \) which are not perverse. Any differential form \( \omega \) of \( M \) is perverse.

(e) Consider \( \{U,V\} \) an open covering of \( M \). There exist a subordinated partition of the unity made up with smooth functions defined on \( M \). The Mayer-Vietoris short sequence

\[
0 \rightarrow \Pi^*(M) \rightarrow \Pi^*(U) \oplus \Pi^*(V) \rightarrow \Pi^*(U \cap V) \rightarrow 0,
\]

where the map are defined by \( \omega \mapsto (\omega,\omega) \) and \((\alpha,\beta) \mapsto \alpha - \beta \), is exact.
2.2 Perverse degree.

The amount of transversality of a perverse form $\omega \in \Pi^*(M)$ with respect to a singular stratum $S$ is measured by the perverse degree. Given a point $x$ of a singular stratum $S$, we define the \textit{perverse degree} $||\omega||_x$ as the smallest integer $k$ verifying:

$$i_{\zeta_0} \cdots i_{\zeta_k} \omega \varphi \equiv 0$$

for each conical chart $(U, \varphi)$ of $x$, with $\varphi^{-1}(x) = (a, \emptyset)$, and each family of vector fields $\{\zeta_i\}_{i=0}^k$ on $\{a\} \times (\mathbb{S}^n - \Sigma_{\emptyset}) \times \{0\}$. Here $i.$ denotes the interior product. When the form $\omega \varphi$ vanishes on $[a - \epsilon, a + \epsilon \times (\mathbb{S}^n - \Sigma_{\emptyset}) \times \{0\}$ then we shall write $||\omega||_x = -\infty$. We define the \textit{perverse degree} $||\omega||_S$ by

$$||\omega||_S = \sup\{||\omega||_x / x \in S\}.$$ 

2.2.1 Properties.

We recall some basic properties of the perverse form, cf. [15].

(a) A perverse form $\omega$ can be extended to $\overline{M}$. Its perverse degree is in fact the degree of $\omega$ on the added part when passing from $M$ to $\overline{M}$.

(b) The perversity conditions (2) and (3) do not depend on the choice of the conical chart.

(c) For two perverse forms $\omega$ and $\eta$ and a singular stratum $S$ we have:

$$||\omega + \eta||_S \leq \max\{||\omega||_S, ||\eta||_S\}, ||\omega \wedge \eta||_S \leq ||\omega||_S + ||\eta||_S.$$ 

(d) The perverse degree of a perverse function is 0 or $-\infty$. The perverse degree of a differential form $\omega$ of $M$ is 0 or $-\infty$.

(e) Let $(N, \mathcal{H})$ be a regular foliation and $a_0 \in N$ a basis point. Put $pr: N \times M \to M$ the canonical projection and $i: M \to N \times M$ the inclusion defined by $i(x) = (a_0, x)$. For any $\omega \in \Pi^*(M)$ and $\eta \in \Pi^*(N \times M)$ we have

$$||pr^*\omega||_{N \times S} \leq ||\omega||_S \quad \text{and} \quad ||i^*\omega||_S \leq ||\omega||_{N \times S}$$

for each singular stratum of $M$.

(f) Consider $\omega \in \Pi^*(\mathbb{S}^n)$ a perverse form. Its perverse degree relatively to the vertex is:

$$||\omega||_{(\emptyset)} = \begin{cases} -\infty & \text{if } \omega \varphi \equiv 0 \text{ on } \mathbb{S}^n \times \{0\} \\ \deg \omega & \text{if not,} \end{cases}$$

where $(U, \varphi)$ is any conical chart of $\emptyset$.

(g) A perverse form with $||\omega||_S \leq 0$ and $||d\omega||_S \leq 0$ induces a differential form $\omega_S$ on $S$. When this happens for each stratum we conclude that $\omega \equiv \{\omega_S\}$ is a Verona’s \textit{controlled form} (cf. [19]).

2.3 Basic cohomology.

Consider $(M, \mathcal{F})$ a foliated manifold. A differential form $\omega \in \Omega^r(M)$ is basic if

$$i_X \omega = i_X d\omega = 0,$$

for each vector field $X$ on $M$ tangent to the foliation. By $\Omega^r(M/\mathcal{F})$ we denote the complex of basic forms. Since the sum and the product of basic forms are still basic forms then the complex of basic forms is a dga. Its cohomology $H^*(M/\mathcal{F})$ is the \textit{basic cohomology} of $(M, \mathcal{F})$, which is a dga. If $F \subset M$ is a saturated closed subset we shall write $\Omega^r((M, F)/\mathcal{F})$ the subcomplex of basic forms of $M$ vanishing on $F$. Its cohomology will be denoted by $H^*((M, F)/\mathcal{F})$. November 14, 2018
2.3.1 Properties.

(a) A smooth foliated map \( f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2) \) induces the dgca operator \( f^* : \Omega^*(M_2/\mathcal{F}_2) \to \Omega^*(M_1/\mathcal{F}_1) \).

(b) An open covering \( \{U, V\} \) of \( M \) by saturated open subsets is a basic covering when there exists a subordinated partition of the unity made up of basic functions. They may or may not exist. Then the Mayer-Vietoris short sequence

\[
0 \to \Omega^*(M/F) \to \Omega^*(U/F_U) \oplus \Omega^*(V/F_V) \to \Omega^*((U \cap V)/F_{U \cap V}) \to 0,
\]

where the map are defined by \( \omega \mapsto (\omega, \omega) \) and \( (\alpha, \beta) \mapsto \alpha - \beta \), is exact. This result is not longer true for more general coverings.

2.4 Basic intersection cohomology.

Consider \((M, \mathcal{F})\) a conical foliated manifold. A perversity is a map \( \overline{p} : S_F \to \mathbb{Z} \). There are two particular perversities: the zero perversity \( \overline{0} \) and the top perversity \( \overline{1} \) defined by \( \overline{0}(S) = 0 \) and \( \overline{1}(S) = \text{codim} S - 2 \). Associated to a smooth foliated map \( f : (M', \mathcal{F}') \to (M, \mathcal{F}) \) (resp. \( f : (M, \mathcal{F}) \to (M', \mathcal{F}') \)) there exists a perversity on \((M', \mathcal{F}')\), still written \( \overline{p} \), defined by \( \overline{p}(S') = \overline{p}(S) \) where \( S' \in S_F \) and \( S \in S_F \) with \( f(S') \subset S \) (resp. \( S = f^{-1}(S') \)).

The basic intersection cohomology appears when one considers basic forms whose perverse degree is controlled by a perversity. We shall write

\[
\Omega^*_\overline{p}(M/\mathcal{F}) = \{ \omega \in \Omega^*((M - \Sigma)/F_{M - \Sigma}) / \max(||\omega||_S, ||d\omega||_S) \leq \overline{p}(S) \ \forall S \in S_F \}
\]

the complex of basic forms whose perverse degree (and that of the their derivative) is bounded by the perversity \( \overline{p} \). It is a differential complex, but it is not an algebra, in fact the wedge product acts in this way:

\[
\wedge : \Omega^i\overline{p}(M/\mathcal{F}) \times \Omega^j\overline{p}(M/\mathcal{F}) \to \Omega^{i+j}\overline{p}(M/\mathcal{F})
\]

The cohomology \( H^*_{\overline{p}}(M/\mathcal{F}) \) of this complex is the basic intersection cohomology of \( M \), or BIC for short, relatively to the perversity \( \overline{p} \).

2.4.1 Properties.

(a) A foliated embedding \( f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2) \) induces the dgca operator \( f^* : \Omega^*_\overline{p}(M_2/\mathcal{F}_2) \to \Omega^*_\overline{p}(M_1/\mathcal{F}_1) \).

(b) Consider \( \{U, V\} \) a basic covering of \( M \). Then the Mayer-Vietoris short sequence

\[
0 \to \Omega^*_\overline{p}(M/\mathcal{F}) \to \Omega^*_\overline{p}(U/\mathcal{F}_U) \oplus \Omega^*_\overline{p}(V/\mathcal{F}_V) \to \Omega^*_\overline{p}((U \cap V)/\mathcal{F}_{U \cap V}) \to 0,
\]

where the map are defined by \( \omega \mapsto (\omega, \omega) \) and \( (\alpha, \beta) \mapsto \alpha - \beta \), is exact. This result is not longer true for more general coverings.

The basic intersection cohomology coincides with the basic cohomology when the foliation \( \mathcal{F} \) is regular. But it also generalizes the intersection cohomology of Goresky-MacPherson (cf. \( \Box \)) when the leaf space \( B \) lies in the right category, that of stratified pseudomanifolds (cf. Theorem \( \Box \)).

The intersection basic cohomology, as the basic cohomology, are not easily computed. Nevertheless, the typical calculations for intersection basic cohomology are the following.
**Proposition 2.4.2** Let $(\mathbb{R}^k, \mathcal{H})$ be a simple foliation and let $(M, \mathcal{F})$ be a conical foliation. For any perversity $\upplambdabar$ the projection $\text{pr}: \mathbb{R}^k \times (M - \Sigma) \to (M - \Sigma)$ induces an isomorphism

$$\mathbb{I}^* \mathcal{H}_\upplambdabar (\mathbb{R}^k \times M/\mathcal{H} \times \mathcal{F}) \cong \mathbb{I}^* \mathcal{H}_\upplambdabar (M/\mathcal{F}).$$

**Proposition 2.4.3** Let $\mathcal{G}$ be a conical foliation on the sphere $\mathbb{S}^n$. For any perversity $\upplambdabar$ the projection $\text{pr}: (\mathbb{S}^n - \Sigma_\mathcal{G}) \times ]0,1[ \to (\mathbb{S}^n - \Sigma_\mathcal{G})$ induces the isomorphism

$$\mathbb{H}^i \upplambdabar (c\mathbb{S}^k/c\mathcal{G}) = \begin{cases} 
\mathbb{H}^i \upplambdabar (S^k/G) & \text{if } i \leq \upplambdabar(\{\vartheta\}) \\
0 & \text{if } i > \upplambdabar(\{\vartheta\}).
\end{cases}$$

This result shows that the basic intersection cohomology is not completely determined by the cohomology of $M$.

### 3 Stratified pseudomanifolds.

In some cases, mainly when the leaves are compact, the orbit space $B$ of a conical foliated manifold $(M, \mathcal{F})$ has a nice topological structure, that of stratified pseudomanifold. This notion has been introduced in [10]. In the smooth context, a *stratified pseudomanifold* $X$ is given by

- a paracompact space $X$
- a locally finite partition $S_X$, called *stratification*, made up of connected smooth manifolds, called *strata*.

The local structure must be conical, that is, the local model is $\mathbb{R}^p \times cL$, where $L$ is a smaller compact stratified pseudomanifold. More exactly, each point of $X$ has a neighborhood which is homeomorphic to the local model, and this homeomorphism sends diffeomorphically a stratum into a stratum. Moreover, there exists a dense stratum $R$, the *regular stratum*.

The work initiated by Goresky and MacPherson have proved that the right homology to study this kind of singular spaces is the intersection homology (see [10] and also [12], [11]). The main result of this works establishes an isomorphism between the intersection homology of $B$ and the basic intersection cohomology of $\mathcal{F}$.

#### 3.1 Without holonomy

In this section, we are interested in the case where the conical foliation $\mathcal{F}$ is proper leaves without holonomy. We write $B$ the leaf space and $\pi: M \to B$ the canonical projection. We shall say that a conical foliated manifold $(M, \mathcal{F})$ is a *singular fibration* when, for each singular singular stratum $S$ of $S_\mathcal{F}$, we have

a) the restriction $\pi: S \to \pi(S)$ is a smooth fiber bundle (*holonomy condition for $\mathcal{F}_S$*),

b) the link $(\mathbb{S}^n, \mathcal{G})$ of $S$ is a singular fibration (*inductive condition*) and

c) the trace of a leaf of $\mathcal{F}$ on $\mathbb{S}^n$ is connected (*holonomy condition for $S$ on $M$*).

For this kind of foliations we have the following result.

**Proposition 3.1.1** The leaf space of a singular fibration is a stratified pseudomanifold.
Let \((M, \mathcal{F})\) be the singular fibration and let \(B\) be the leaf space. We prove first that \(B\) is a paracompact space. This comes from the fact that, being \(M\) second countable, we can find a basis \(\{(\varphi, U)\}\) of the topology made up with good conical charts. The induced family \(\{(\psi, \pi(U))\}\) is a countable basis of \(B\); the orbit space \(B\) is therefore second countable. It is also clearly regular. This implies that \(B\) is paracompact (cf. [20]).

From a) and from the fact that the family \(S_\mathcal{F}\) is a locally finite then we get that the family \(S_B = \{\pi(S) / S \in S_\mathcal{F}\}\) is a stratification.

Let us study the local structure of this stratification. We proceed by induction on the depth of the stratification. In the first step of the induction we have that the projection \(\pi\) is a stratified pseudomanifold whose depth is 0. The Proposition becomes the usual deRham Theorem.

We consider, in the generic case, a conical chart \((\varphi, U)\) of a point of a stratum \(S\). Since \((S, \mathcal{F}_S)\) is a fibration, then any point of \(S\) admits a chart modelled on \((\mathbb{R}^{m-n-1}, \mathcal{H})\) \(\equiv (\mathbb{R}^r, \mathcal{K}) \times (\mathbb{R}^s, \mathcal{I})\), where \(\mathcal{K}\) is the foliation with one leaf and \(\mathcal{I}\) is the foliation by points, with the property that each point of \(\mathbb{R}^s\) belongs to a different leaf of \(\mathcal{F}\). We shall say that \((\varphi, U)\) is a good conical chart if it is modelled on

\[
(\mathbb{R}^r \times \mathbb{R}^s \times cS^n, \mathcal{K} \times \mathcal{I} \times c\mathcal{G}).
\]

A good conical chart \(\varphi\) induces the stratified embedding (homeomorphism sending diffeomorphically the strata into the strata) \(\psi: \mathbb{R}^s \times c\pi_{s\mathcal{G}}(S^n) \to B\) (cf. c)), where \(\pi_{s\mathcal{G}}\) is the canonical projection on \(S^n\). Here, since \((S^n, \mathcal{G})\) is a singular fibration (cf. b)), we can apply the induction hypothesis.

With the same technics, one shows directly that there exists a regular stratum \(R\).

We present now some examples.

### 3.1.2 Examples.

a) We begin with a simple example. Fix \((p, q) \in \mathbb{R}^2\) and put \(R: \mathbb{C}^2 \to \mathbb{C}^2\) the map defined by \(R(z_1, z_2) = (e^{2\pi ip} \cdot z_1, e^{2\pi i q} \cdot z_2)\). The suspension of \(R\) defines on the manifold \(M = S^1 \times cS^3\) a conical flow \(\mathcal{F}\) without singularities. We have a singular fibration just in the case \(p/q = 1\). When \(p/q\) is irrational then the leaf space \(B\) is not a stratified pseudomanifold (even paracompact!). In the other cases the leaf space \(B\) is the cone over a lens space and therefore an orbifold.

b) Consider a smooth action of a compact Lie \(G\) on a smooth compact manifold \(M\). Such an action defines a singular (Riemannian) foliation \(\mathcal{F}\) on \(M\). Subsets of points of orbits of the same dimension are submanifolds of \(M\) and define a stratification (cf. [4]). On each stratum \(S\), the induced foliation \(\mathcal{F}_S\) is a regular Riemannian foliation whose leaves are compact with finite holonomy. The space of leaves \(S/\mathcal{F}_S\) is an orbifold. If we assume that our action has just one type of orbit in each stratum and that the isotropy subgroups are connected then \((M, \mathcal{F})\) is a singular fibration.

c) The considerations in [3] permits us to formulate the following generalization for compact singular riemannian foliated manifold \((M, \mathcal{F})\). The subsets consisting of points of leaves of the same dimension are smooth submanifolds and define a stratification of the manifold. On each stratum \(S\) the foliation \(\mathcal{F}\) defines a regular Riemannian foliation \(\mathcal{F}_S\). The leaf space \(S/\mathcal{F}_S\) is an orbifold. Each leaf has finite holonomy. When all these holonomies are trivial, then \((M, \mathcal{F})\) is a singular fibration.

The intersection homology \(\mathcal{H}^\bullet_{\mathcal{P}}(B)\) has been introduced by Goresky-MacPherson in order to study the stratified pseudomanifolds (see [10] and also [2], [1]). The main result of this works
relates this intersection homology with the basic intersection cohomology of $F$. Recall that two perversities $\overline{p}$ and $\overline{q}$ on $B$ are dual if $\overline{q}(\pi(S)) + \overline{p}(\pi(S)) = \text{codim}_{\mathcal{B}}(\pi(S)) - 2$ for each singular stratum $\pi(S)$. Notice that a perversity $\overline{p}$ defines a perversity on $B$ by putting $\overline{p}(\pi(S)) = \overline{p}(S)$.

**Theorem 3.1.3** Let $(M, F)$ be a singular fibration. Put $B$ the orbit space. Then we have

$$\overline{IH}^*(M/F) \cong \overline{IH}^*(B),$$

where $\overline{p}$ and $\overline{q}$ are dual perversities.

**Proof.** See Section 4.

\[\blacklozenge\]

### 3.2 With holonomy.

We consider now the case where the holonomy is not trivial, in fact finite. We get the same relationship between the BIC of $F$ and the intersection homology of $B$, but some technical difficulties appear. First of all we present an representative example.

#### 3.2.1 Example. Consider $S^3$ endowed with the riemannian foliation $\mathcal{H}$ defined by the Hopf action, that is, $z \cdot (u_1, u_2) = (z \cdot u_1, z \cdot u_2)$. On the other hand, we consider $\mathbb{R}^6 = cS^5$ endowed with the singular riemannian foliation $c\mathcal{Q}$ defined by the torus action $\mathbb{T}^2 \times cS^5 \to cS^5$ given by

$$(z_1, z_2) \cdot [(v_1, v_2, v_3), t] = [(z_1 \cdot v_1, z_1 z_2 \cdot v_2, z_1 z_2^3 \cdot v_3), t].$$

There are three different types of leaves: points, circles and tori. There are three different types of strata: a point (whose link is $(S^5, Q)$), three cylinders (whose link is $(S^3, \mathcal{H})$) and the regular stratum. In fact, the singular part is the wedge $\mathbb{D}^2 \vee \mathbb{D}^2 \vee \mathbb{D}^2$ endowed with the foliation induced by the natural action of $S^1$ on each disk. The finite group $\mathbb{Z}_3$ acts freely on the product $S^3 \times cS^5$ by

$$e^{2\pi i/3} \cdot ((u_1, u_2), [(u_1, u_2, u_3), t]) = ((u_1, e^{2\pi i/3} \cdot u_2), [(u_3, u_1, u_2), t]).$$

There are three different types of leaves: circles, tori and three-dimensional tori. There are three different types of strata: $S^3$, three copies of $S^1 \times ]0, 1[$ and the regular stratum. Notice that this action permutes the three four-dimensional strata. Since this actions preserves the foliation $\mathcal{H} \times c\mathcal{Q}$, then we get on the quotient manifold

$$M = S^3 \times_{\mathbb{Z}_3} cS^5$$

a singular riemannian foliation $F$. There are three different types of leaves: circles, tori and three-dimensional tori. There are three different types of strata. The minimal stratum $S$ is the lens space $L(1, 3) = S^3/\mathbb{Z}^3$, its link is $(L_S, \mathcal{G}) = (S^5, Q)$. There is one four-dimensional stratum, the product $S^3 \times S^1 \times ]0, 1[$ whose link is $(L_S, \mathcal{G}) = (S^3, \mathcal{H})$. The last stratum is the regular stratum.

Put $B$ the leaf space $M/F$ and $\pi: M \to B$ the canonical projection. The image $\pi(S) = L(3, 1)/F = S^2/\mathbb{Z}^2$ is not a manifold but an orbifold. We already see a difference with the singular fibrations. Here, the strata are not necessarily manifolds but orbifolds (see for example [4]). But there is another important difference related to the "links". The natural projection $\tau: M \to S$ is a foliated tubular neighborhood (the charts preserve the foliation and the conical structure) of $S$ with fiber $cS^5$. The induced map $\sigma: \pi(T) \to \pi(S)$ is a tubular neighborhood (the charts preserve the conical structure) of $\pi(S)$ with fiber the quotient $(S^5/\mathcal{G})/\mathbb{Z}_3$. The "link" $L_{\pi(S)}$ of $\pi(S)$ is therefore $(S^5/\mathcal{Q})/\mathbb{Z}_3$. In other words,

we don’t have $L_{\pi(S)} = L_S/\mathcal{G}$, but $L_{\pi(S)} = (L_S/\mathcal{G})/\mathbb{Z}_3$. We can see that the trace of $F$ on the link of $S$ are exactly the obits of $\mathbb{Z}_3 \cdot Q$. 

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3.2.2 Orbifolds category. These considerations lead us to introduce the following notions. A stratified pseudorbifold is defined as the stratified pseudomanifold changing "manifold" by orbifold and "diffeomorphism" by "isomorphism between orbifolds". In other words, a stratified pseudorbifold is given by

- a paracompact space $X$
- a locally finite partition $S_X$, called stratification, made up of connected smooth orbifolds, called strata.

The local structure must be conical, that is, the local model is $U \times cL$, where $U$ is an orbifold, $L$ is a smaller compact stratified pseudorbifold. More exactly, each point of $X$ has a neighborhood which is homeomorphic to the local model and this homeomorphism sends a stratum into a stratum by an isomorphism. Moreover, there exists a dense stratum $R$, the regular stratum.

We also extend the notion of singular fibration. We shall say that a conical foliated manifold $(M, F)$ is a singular Seifert fibration when, for each singular singular stratum $S$ of $S_F$, we have

a) the restriction $\pi: S \to \pi(S)$ is a Seifert bundle,

b) the link $(S^n, G)$ of $S$ is a singular Seifert fibration and

c) the trace of a $F$ on $S^n$ is given by the orbits of $H \cdot G$, where $H$ is a finite subgroup of $O(n + 1, G) = O(n + 1) \cap \text{Diff}(S^n, G)$.

Proposition 3.2.3 The leaf space of a singular Seifert fibration is a stratified pseudorbifold.

Proof. Let $(M, F)$ be the singular fibration and let $B$ be the leaf space. We know from [13] that the elements of the family $S_B = \{ \pi(S) \} / S \in S_F$ are orbifolds. Proceeding as in Proposition 3.1.1 we get that $B$ is a paracompact space and that the family $S_B$ is locally finite.

Let us study the local structure of this stratification. We proceed by induction on the depth of the stratification. In the first step of the induction we have that the projection $\pi$ is a Seifert fibration and $B$ is a stratified pseudorbifold whose depth is 0.

We consider, in the generic case, a conical chart $(\varphi, U)$ of a point of a stratum $S$. The local model of a point of $\pi(S)$ is a product $(\mathbb{R}^p / \Gamma, c(S^n / G) / H)$ where $\Gamma \subset O(p)$ is a finite subgroup, $(S^n, G)$ is the link of $S$ and $H \subset O(n + 1, G)$ is a finite subgroup. An inductive argument gives that $S^n / G$ is a stratified pseudorbifold. Thus $(S^n / G) / H$ is also a stratified pseudorbifold. ♣

3.2.4 Singular riemannian foliations. From [2] (see also [13]) we know that each stratum $S$ of a singular riemannian foliation $(M, F)$ possesses a tubular neighborhood, called foliated tubular neighborhood, $\tau: T \to S$ with a with a foliated atlas

$$\{ \varphi: (\tau^{-1}(U), F) \to (U \times cS^n, F \times cG) \}$$

whose structural group is $O(n + 1, F) = \{ A \in O(n + 1) / A \text{ preserves } G \}$. Here $(S^n, G)$ denotes the link of $S$.

A singular riemannian foliation gives the main example of a stratified pseudomanifold.

Proposition 3.2.5 The leaf space of a singular riemannian foliation with compact leaves is a stratified pseudorbifold.
Proof. Put \((M, \mathcal{F})\) the singular riemannian foliation, \(B\) the leaf space and \(\pi: M \to B\) the natural projection. Since the the restriction of \(\mathcal{F}\) to \(S\) is a riemannian foliation with compact leaves then the restriction \(\pi: S \to \pi(S)\) is a Seifert bundle (see [13]). This gives a).

For b) we notice that \((S^n, \mathcal{G})\) is a singular riemannian foliated manifold with depth \(S_G < \) depth \(S_F\). An inductive argument on the depth of the stratification gives b). In the first step of the induction we have that the projection \(\pi\) is just a Seifert fibration. The Proposition becomes the deRham Theorem for orbifolds.

The foliation \(\mathcal{F}\) does not always induces on \(S^n\) the foliation \(\mathcal{G}\) (see the Example 3.2.1). Consider \(\tau: T \to S\) a foliated tubular neighborhood whose structural group is \(O(n+1, \mathcal{G})\). The trace of \(\mathcal{F}\) on the generic fiber \(S^n\) is given by the orbits \(H \cdot \mathcal{G}\) where \(H = \{A \in O(n+1, \mathcal{G})\text{ preserving the leaves}\}\). Since \(H\) preserves the regular stratum then, for dimensional reasons, we have \(\dim H = 0\). This gives c).

As in the without-holonomy case we get that the basic intersection cohomology generalizes the intersection cohomology of the leaf space.

**Theorem 3.2.6** Let \((M, \mathcal{F})\) be a singular Seifert fibration. Put \(B\) the leaf space. Then we have

\[
IH^*_\pi(M/\mathcal{F}) \cong IH^*_\pi(B),
\]

where \(\overline{p}\) and \(\overline{q}\) are dual perversities.

Proof. The same procedure followed in the proof of the Theorem [3.1.3] reduces the problem to a local question on \(B\). Recall that the local model of \(B\) is

\[
(\mathbb{R}^p/\Gamma, c\pi(S^n)/H)
\]

where \(\Gamma \subset O(p)\) is a finite subgroup, \((S^n, \mathcal{G})\) is the link of \(S\), \(\pi: S^n \to \pi(S^n)\) denotes the canonical projection associated to \(\mathcal{G}\) and \(H \subset O(n+1, \mathcal{G})\) is a finite subgroup. By retracting the first factor, we transform the problem to the proof of the following statement:

\[
H^*\left(\Omega_\pi(\pi(S^n)/H)\right) \cong H^*\left(\text{Hom} (SC_{\pi}(\pi(S^n))/H, \mathbb{R})\right)
\]

Since \(S^n/\mathcal{G}\) is a stratified pseudorbifold, an inductive argument gives

\[
H^*\left(\Omega_\pi(\pi(S^n))\right) \cong H^*\left(\text{Hom} (SC_{\pi}(\pi(S^n)), \mathbb{R})\right).
\]

A classic argument using the finiteness of \(H\) (see for example [4]) gives \(H^*\left(\Omega_\pi(\pi(S^n)/H)\right) \cong \left(H^*\left(\Omega_\pi(\pi(S^n))\right)\right)^H\) and \(H^*\left(\text{Hom} (SC_{\pi}(\pi(S^n))/H, \mathbb{R})\right) \cong \left(H^*\left(\text{Hom} (SC_{\pi}(\pi(S^n)), \mathbb{R})\right)\right)^H\), the subspaces of fixed points. The proof ends by noticing that the isomorphism (1) is natural. ♠

Let \(\mathcal{F}\) be a conical foliation on \(M\) with compact leaves, which we will call a compact conical foliation. Then on each stratum of \(M\) the foliation induces a regular compact foliation. The results of [3, 8] permit us to formulate the following Corollary.

**Corollary 3.2.7** Let \((M, \mathcal{F})\) be a compact conical foliation. Let \(S_F\) be the partition of \(M\) by subsets consisting of points of leaves of \(\mathcal{F}\) of the same dimension. If one of the following is satisfied:
(i) on each stratum the volume function of leaves is locally bounded;
(ii) in each stratum leaves of the induced foliation have finite holonomy;
(iii) the codimension of the foliation in each stratum is 2;
then the leaf space $B$, is a stratified pseudomanifold and, for any perversity $\overline{p}$,
$$\mathbb{H}^{*}_{\overline{p}}(M/F) \cong \mathbb{H}^{*}_{\overline{p}}(B),$$
where $\overline{q}$ is the dual perversity of $p$.

Proof. The conditions (i),(ii) and (iii) assure that in each stratum the induced foliation is Riemannian. Therefore the leaf space $B$ endowed with the stratification $\{\pi(S) / S \in S_F\}$ is a stratified pseudomanifold. On each stratum the natural projection is a Seifert fibration. The rest follows from Theorem 3.2.6.

4 Proof of the Theorem 3.1.3.

We proceed in five steps.

- We reduce the problem to a question on the leaf space $B$ by giving a presentation of the BIC of $(M,F)$ using a complex of differential forms living on $\pi(R)$: the complex $\Omega^{*}_{\overline{p}}(B)$.

- We present the intersection homology of $B$ by using the complex $SC^{\overline{p}}_{*}(B)$ of $\overline{p}$-intersection chains.

- Since these two complexes are not comparable by integration, we introduce the subcomplex $LC^{\overline{p}}_{*}(B)$ of smooth $\overline{p}$-intersection chains.

- These complexes are related by the restriction $\Lambda: \text{Hom}(LC^{\overline{p}}_{*}(B),\mathbb{R}) \rightarrow \text{Hom}(SC^{\overline{p}}_{*}(B),\mathbb{R})$ and the integration $\Theta: \Omega^{*}_{\overline{p}}(B) \rightarrow \text{Hom}(LC^{\overline{p}}_{*}(B),\mathbb{R})$.

- We prove that the operators $\Theta$ and $\Lambda$ are quasi-isomorphisms.

4.1 The complex $\Omega^{*}_{\overline{q}}(B)$.

The result we are going to prove is a comparison between a cohomology defined on $M$ with a cohomology defined on $B$. In order to simplify the proof we are going to present the basic intersection cohomology of $(M,F)$ using differential forms on $B$. We define
$$\Omega^{*}_{\overline{p}}(B) = \{ \eta \in \Omega^{*}(\pi(R)) / \pi^{*}\eta \in \Omega^{*}_{\overline{p}}(M/F) \}.$$ Clearly, the differential operator
$$\pi^{*}: \Omega^{*}_{\overline{p}}(B) \rightarrow \Omega^{*}_{\overline{p}}(M/F)$$
is an isomorphism.
Consider \{V,W\} an open covering of \(B\). Recall that there exists a subordinated partition of the unity made up of controlled functions, elements of \(\Omega^0_\pi(B)\) (cf. [14]). So, the covering \{\pi^{-1}(U), \pi^{-1}(V)\} is a basic covering and we get from 2.4.1 (b) the Mayer-Vietoris short sequence

\[0 \to \Omega^*_\pi(B) \to \Omega^*_\pi(V) \oplus \Omega^*_\pi(W) \to \Omega^*_\pi(V \cap W) \to 0.\]

The fact that a differential form of \(\pi(R)\) lives on \(\Omega^*_\pi(B)\) is a local question. But we have more than that. Notice that, any two points of \(M\) can be related by a local foliated diffeomorphism (use good conical charts). So, in order to verify that a differential for \(m\) on \(M\) than that. Notice that, any two points of \(M\) (i.e. a finite product of simplices).

For each integer \(i\) we shall write \(\Sigma \) the liftable prism \(\Delta \times \Sigma \) of \(\Delta\). We shall write \(\Sigma \) the union of strata with dimension less or equal to \(i\). The union of singular strata of \(\Sigma \) is \(\Sigma\). The liftable prism \(\Delta \times \Sigma \) is \(\pi\)-allowable.

A liftable prism \(\pi : P \times \Delta \to B \times [0,1]^a\) is a continuous map verifying the two following properties:

i) The restriction \(\pi : P \times \text{int}(\Delta) \to \pi(R) \times [0,1]^a\) is smooth.

ii) Each \(\pi^{-1}(B_i \times [0,1]^a)\) is of the form \(P \times F_i\), where \(F_i\) is a face of \(\Delta\).

The liftable prism \(\mathbf{c}\) is \(\pi\)-allowable if it verifies

iii) \(\text{codim}_B \pi(S) \leq \text{codim}_\Delta F_{\dim \pi(S)} + \mathbf{p}(\pi(S))\) for each singular stratum \(\pi(S)\) of \(B\).

A singular chain \(\xi\) is an \(\pi\)-intersection chain when the chains \(\xi\) and \(\partial\xi\) are made up with \(\pi\)-allowable simplices. We shall write \(SC^*_\pi(B)\) the complex of \(\pi\)-intersection chains. The intersection homology \(H^*_\pi(B)\) of \(B\) can be computed using the complex of intersection chains \(SC^*_\pi(B)\) (cf. [11], [14]).

### 4.3 The complex \(LC^*_\pi(B)\).

First of all we need establish some results about the blow up of a standard simplex.

#### 4.3.1 Linear blow-up

Consider \(\Delta = \Delta_0 \ast \cdots \ast \Delta_k\) a decomposition of \(\Delta\). We can think \(P \times \Delta\) as a stratified prism with singular strata \(\{P \times \Delta_0, P \times ((\Delta_0 \ast \Delta_1) - \Delta_0), \ldots, P \times ((\Delta_0 \ast \cdots \ast \Delta_{k-1}) - (\Delta_0 \ast \cdots \ast \Delta_{k-2}))\}\) and with a regular stratum \(R_{P \times \Delta} = P \times (\Delta - (\Delta_0 \ast \cdots \ast \Delta_{k-1}))\). The depth of the stratified prism is depth \(P \times \Delta = k\).

When this depth is strictly positive we can desingularise \(P \times \Delta\) in the following way. The linear blow-up of \(P \times \Delta\) is the smooth map

\[\mathcal{L}_{P \times \Delta} : (P \times \pi \Delta_0) \times (\Delta_1 \ast \cdots \ast \Delta_k) \to P \times \Delta\]
defined by \( L_{P \times \Delta}(x, [x_0, t_0], y) = (x, t_0x_0 + (1 - t_0)y) \). Here \( \overline{\Delta}_0 \) denotes the closed cone \( \frac{\Delta_0 \times [0,1]}{\Delta_0 \times \{0\}} \).

We shall write \( (P \times \Delta)^{blu} = (P \times \overline{\Delta}_0) \times (\Delta_1 \times \cdots \times \Delta_k) \), which is a stratified prism with depth \( (P \times \Delta)^{blu} \) \( \leq \) depth \( (P \times \Delta) \). The inverse image \( L_{P \times \Delta}^{-1}(R_{P \times \Delta}) = R_{(P \times \Delta)^{blu}} - ((P \times \Delta_0 \times \{1\}) \times \{\Delta_1 \times \cdots \times \Delta_k\}) \) is a dense subset of \( (P \times \Delta)^{blu} \) and the restriction

\[
L_{P \times \Delta} : L_{P \times \Delta}^{-1}(R_{P \times \Delta}) \to R_{P \times \Delta}
\]

is a diffeomorphism. The same properties hold for \( int(P \times \Delta) \subset R_{P \times \Delta} \), here

\[
L_{P \times \Delta}^{-1}(int(P \times \Delta)) = int((P \times \Delta)^{blu}).
\]

Notice that, when \( k = 1 \) and \( \dim \Delta_1 = 0 \), then \( L_{P \times \Delta} \) itself is in fact a diffeomorphism.

### 4.3.2 Smooth intersection homology.

A liftable simplex \( c : \Delta \to B \) induces a natural decomposition on \( \Delta \). Consider \( \{i_0, \ldots, i_k\} \) the family of indices verifying \( F_i \neq F_{i-1} \) and put \( \Delta_j \) the face of \( \Delta \) with \( F_{i_j} = F_{i_{j-1}} \). This defines on \( \Delta \) the \( c \)-decomposition \( \Delta = \Delta_0 \times \cdots \times \Delta_k \). We have \( c^{-1}(\Sigma_B \times [0,1^n]) = P \times (\Delta_0 \times \cdots \times \Delta_{k-1}) \).

The prism is smooth when it also verifies the condition:

iv) There exists a smooth map \( c : P \times \Delta \to M \times [0,1^n] \) with \( \pi \circ c = c \).

Since \( c^{-1}(\pi(R) \times [0,1^n]) = R_{P \times \Delta} \) then the restriction \( c : R_{P \times \Delta} \to R \times [0,1^n] \) is smooth and therefore it verifies a stronger condition than i), namely

i') The restriction \( c : R_{P \times \Delta} \to \pi(R) \times [0,1^n] \) is smooth.

A singular chain \( \xi \) is a smooth \( \overline{\mathcal{P}} \)-intersection chain when the chains \( \xi \) and \( \partial \xi \) are made up with smooth \( \overline{\mathcal{P}} \)-allowable simplices. We shall write \( LC^*_{\overline{\mathcal{P}}}(B) \) the complex of smooth \( \overline{\mathcal{P}} \)-intersection chains. It will be shown in the next section that this complex also computes the intersection homology.

### 4.4 The operators \( \Theta \) and \( \Lambda \).

The natural inclusion \( LC^*_{\overline{\mathcal{P}}}(B) \hookrightarrow SC^*_{\overline{\mathcal{P}}}(B) \) induces the differential operator

\[
\Lambda : \text{Hom}(SC^*_{\overline{\mathcal{P}}}(B), \mathbb{R}) \to \text{Hom}(SC^*_{\overline{\mathcal{P}}}(B), \mathbb{R}).
\]

The difficulty to integrate a differential form \( \omega \in \Omega^*(B) \) on an intersection chain \( \xi \) lies on the fact that \( \omega \) is defined only on the regular stratum of \( B \) while \( \xi \) is defined on \( B \). For this reason we need some preparatory results.

The linear blow up is compatible with the barycentric subdivision in the following way. Let \( \nabla \) an element of the barycentric subdivision of \( \Delta \), endowed with the induced decomposition. That is, \( \nabla = \nabla_1 \times \cdots \times \nabla_l \) where \( \{i_1, \ldots, i_l\} = \{i \in \{1, \ldots, k\} / \Delta_0 \times \cdots \times \Delta_i \neq \Delta_0 \times \cdots \times \Delta_{i-1}\} \) and \( \Delta_0 \times \cdots \times \Delta_i = \Delta_0 \times \cdots \times \Delta_{i-1} \times \nabla_j \). Notice that \( R_{\nabla} = R_{\Delta} \cap \nabla \). It has been proved in [3, page 220] that

**Proposition 4.4.1** Given an element \( \nabla \) of the barycentric subdivision of \( \Delta \) and \( I : (P \times \nabla)^{blu} \to (P \times \Delta)^{blu} \) the natural inclusion, then there exists a smooth map \( I : P \times \nabla \to P \times \Delta \) verifying \( I \circ L_{P \times \nabla} = L_{P \times \Delta} \circ I \).
4.4.2 Differential forms on $P \times \Delta$.

We shall write $\Pi^*(P \times \Delta)$ the complex of liftable forms. When depth $(P \times \Delta) = 0$ then we put $\Pi^*(P \times \Delta) = \Omega^*(P \times \Delta)$. In the generic case, we shall say that a differential form $\omega \in \Omega^*(\pi(R) \times [0,1]^a)$ is liftable if there exists a liftable form $\hat{\omega} \in \Pi^*(P \times \Delta^{blu})$ with $\mathcal{L}_{P \times \Delta}^\omega = \hat{\omega}$ on $\mathcal{L}_{P \times \Delta}^{-1}(R_{P \times \Delta})$. Notice that, when $k = 1$ and dim $\Delta = 0$, the form $\omega$ is defined in fact in $P \times \Delta$. By density, the lifting $\hat{\omega}$ is unique. Then we have $\hat{d}\omega = d\hat{\omega}$. The complex $\Pi^*(P \times \Delta)$ is thus differential. As usual, we shall write

$$\int_{P \times \Delta} \omega = \int_{\text{int}(P \times \Delta)} \omega,$$

which is not always well defined. But in our context

**Lemma A** Let $\omega$ be a liftable form, then the integral $\int_{P \times \Delta} \omega$ is finite.

**Proof.** We proceed by induction on the depth. When this depth is 0 then the result is clear. For the generic step we have

$$\int_{P \times \Delta} \omega = \int_{\text{int}(P \times \Delta)} \mathcal{L}_{P \times \Delta}^\omega = \int_{\text{int}(P \times \Delta)} \hat{\omega} = \int_{P \times \Delta} \hat{\omega}$$

since (3) and int $(P \times \Delta) \subset R_{P \times \Delta}$. By induction hypothesis this number is finite. ♣

**Lemma B** Let $c : P \times \Delta \to B \times [0,1]^a$ be a smooth prism. If $\eta \in \Omega^*(\pi(R) \times [0,1]^a)$ with $\pi^*\eta \in \Pi^*(M \times [0,1]^a)$ then $c^*\eta$ is liftable.

**Proof.** Since $c^*\eta \in \Omega^*(R_{P \times \Delta})$ (cf. i')) then it suffices to construct the lifting $c^*\eta \in \Pi^*(P \times \Delta)$.

We proceed in several steps.

I - Localizing $M$ and $B$. Remark that for any element $\nabla$ of the barycentric subdivision of $\Delta$, the restriction $c^* \nabla : P \times \Delta \to B \times [0,1]^a$ is a smooth prism. The statement becomes a local one. So, we can identify

$$(M, \mathcal{F}) \equiv (\mathbb{R}^r \times \mathbb{R}^e \times c\mathbb{S}^n, \mathcal{I} \times \mathcal{J} \times c\mathcal{G}), \quad B \equiv \mathbb{R}^e \times c\pi_{gn}(\mathbb{S}^n)$$

and suppose that $\mathfrak{X}c$ meets $\mathbb{R}^{blu} \times \{\varnothing\} \times [0,1]^a$. Then $c^{-1}(\mathbb{R}^e \times \{\varnothing\} \times [0,1]^a) = P \times \Delta_0$.

Notice that a neighborhood of $\Delta_0$ on $\Delta$ is a product of $\Delta_0 \times c\nabla$, where $\nabla$ is a simplex. From Lemma 1.2.1 we get a commutative diagram

$$\begin{array}{ccc}
P \times \Delta & \xrightarrow{c} & \mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0,1][0,1]^a \\
\mathcal{L}_{P \times \Delta} & \downarrow & \\
P \times \Delta & \xrightarrow{c} & \mathbb{R}^{m-n-1} \times c\mathbb{S}^n \times [0,1]^a
\end{array}$$

where $\hat{c}$ is smooth.

II - Blowing up $c$. Consider now the continuous map

$$\hat{c} = \hat{\pi} \circ c : P \times \Delta \to \mathbb{R}^e \times \pi_{gn}(\mathbb{S}^n) \times [0,1][0,1]^a,$$

where $\hat{\pi}: \mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0,1][0,1]^a \to \mathbb{R}^e \times \pi_{gn}(\mathbb{S}^n) \times [0,1][0,1]^a$ is the projection defined by
Let us see that $\widehat{c}$ is a smooth prism.

i) Since $\widehat{\pi}$ and $\widehat{c}$ are smooth it suffices to prove that

$$\widehat{c}^{-1}(\mathbb{R}^s \times \pi_{gn}(\Sigma_G) \times [0,1)^{[n+1]} \subset ((P \times \pi_{\Delta_0}) \times ((\Delta_1 \ast \cdots \ast \Delta_k) - \text{int}(\Delta_1 \ast \cdots \ast \Delta_k))$$

This comes from

$$\widehat{c}^{-1}(\mathbb{R}^s \times \pi_{gn}(\Sigma_G) \times [0,1)^{[n+1]} = \widehat{c}^{-1}(\mathbb{R}^{m-n} \times \Sigma_{\tau} \times [0,1)^{[n+1]} =$$

$$\widehat{c}^{-1}(P \times \pi_{\Delta_0})^{-1}(\mathbb{R}^{m-n} \times \pi_{\Delta_0} \times [0,1)^{[n]} = \mathcal{L}_{P \times \Delta}^{-1}(\mathbb{R}^{m-n} \times \pi_{\Delta_0} \times [0,1)^{[n]} =$$

$$\mathcal{L}_{P \times \Delta}^{-1}(P \times (\Delta_0 \ast \cdots \ast \Delta_k)) \subset ((P \times \pi_{\Delta_0}) \times ((\Delta_1 \ast \cdots \ast \Delta_k) - \text{int}(\Delta_1 \ast \cdots \ast \Delta_k))$$

ii) Proceeding as before we get

$$\widehat{c}^{-1}(\mathbb{R}^s \times ((\pi_{gn}(\mathbb{S}^n))_j \times [0,1)^{[n+1]} = \mathcal{L}_{P \times \Delta}^{-1}(\mathbb{R}^s \times (c\pi_{gn}(\mathbb{S}^n))_{j+1} \times [0,1)^{[n]} =$$

$$\mathcal{L}_{P \times \Delta}^{-1}(P \times (\Delta_0 \ast \cdots \ast \Delta_h)) = P \times (\Delta_0 \ast \cdots \ast \Delta_h),$$

for some $h \in \{1, \ldots, k\}$.

iii) By construction.

III - Lifting $\eta$. The differential forms $\pi^*\eta \in \Pi^*(\mathbb{R}^{m-n} \times c\mathbb{S}^n \times [0,1)^{[n]}$ lifts into the differential form $\pi^*\eta \in \Pi^*(\mathbb{R}^{m-n} \times \mathbb{S}^n \times [0,1)^{[n+1]}$. Since $\pi^*\eta$ is basic and the restriction of $P \times \pi_{\Delta_0} \times [0,1)^{[n]}$, where $R_0$ is the regular stratum of $\mathbb{S}^{m-n}$, is the identity then $\pi^*\eta$ is also basic. There exists $\hat{\eta} \in \Omega^2(\mathbb{R}^s \times \pi_{\Delta_0}(R_0) \times [0,1)^{[n+1]}$ with $\hat{\pi}^*\hat{\eta} = \pi^*\eta$. The differential form $\hat{\eta}$ is in the conditions of the Lemma.

IV - Final step. We proceed by induction on depth $\mathbb{S}_F$. The result is clear when this depth is 0, that is, when $B = \pi(R)$. For the generic case, notice that depth $\mathbb{S}_{H \times G} < \text{depth } \mathbb{S}_{H \times G}$ then the induction argument gives that $\hat{c}^*\hat{\eta}$ is liftable. It remains to prove that $\hat{c}^*\hat{\eta} = \hat{c}^*\hat{\eta}$, which is

$$\mathcal{L}_{P \times \Delta}^{-1}(R_{P \times \Delta}).$$

Since $c \mathcal{L}_{P \times \Delta} = (P \times \pi_{\Delta_0})^* \pi^*\eta$ we have

$$\mathcal{L}_{P \times \Delta}^{-1}(R_{P \times \Delta}).$$

Since $\pi^*\eta$ is liftable then

$$\mathcal{L}_{P \times \Delta}^{-1}(R_{P \times \Delta}).$$

Given a form $\eta \in \Omega^r_\pi(B)$ and a smooth $\overline{p}$-allowable simplex $c: \Delta \to B$ we can define the integral

$$\Theta(\omega)(c) = \int_{-\Delta} c^*\eta$$

(cf. Lemma 4 and Lemma 5). It remains to prove that the operator

$$\Theta: \Omega^r_\pi(B) \to \text{Hom}(LC^r_\pi(B), \mathbb{R}).$$

is a differential one. We also need some preparatory results.
4.4.3 Boundary.

There are two types of (one codimensional) faces on \((P \times \Delta)^{blu}\).

T1) The faces \((Q \times \nabla)^{blu}\), where \(Q\) is a face of \(P\) and \(\nabla = \Delta\) or \(Q = P\) and \(\nabla\) is a face of \(\Delta\).

The restriction of \(L_{P \times \Delta}\) is the linear blow up \(L_{Q \times \nabla}\).

T2) The face \(F_{P \times \Delta} = (P \times \Delta_0 \times \{1\}) \times (\Delta_1 * \cdots * \Delta_k)\). The restriction of \(L_{P \times \Delta}\) is just the canonical projection over \(P \times \Delta_0\).

The faces of type T1) run over the boundary of \(P \times \Delta\). The face \(F_{P \times \Delta}\) is the extra face produced by the linear blow up. We have the formula

\[
\partial((P \times \Delta)^{blu}) = (\partial(P \times \Delta))^{blu} + F_{P \times \Delta}.
\]

4.4.4 More differential forms on \(P \times \Delta\). We shall write \(\Gamma^*(P \times \Delta)\) the complex of regular forms. When depth \((P \times \Delta) = 0\) then we put \(\Gamma^*(P \times \Delta) = \Omega^*(P \times \Delta)\). In the generic case, we shall say that a liftable form \(\omega \in \Pi^*(P \times \Delta)\) is regular if \(\hat{\omega} \in \Gamma^*(P \times \Delta)\) and if its restriction to \(\text{int} F_{P \times \Delta}\) vanishes. Notice that \(\Gamma^*(P \times \Delta)\) is a differential complex. From Proposition [4.4.1] we know that the restriction of a regular form to a element of the barycentric subdivision is again regular. For these kind of forms we have the following Stoke’s Theorem.

**Lemma C** Let \(\omega\) a regular form, then

\[
\int_{P \times \Delta} d\omega = \int_{\partial(P \times \Delta)} \omega.
\]

**Proof.** We proceed by induction on the depth. When this depth is 0 then the result is clear. For the generic step we notice that \(\omega\) is defined in the interior of any face \(F\) of \(P \times \Delta\) except in the case where \(\dim \Delta_k = 0\) and \(F = \Delta_0 * \cdots * \Delta_{k-1}\). But here \(\omega\) extends to a form on \(\text{int} F\). To see that, we apply the Proposition [4.4.1] to reduce the problem to \(\Delta = \Delta_0 * \Delta_1\) and \(\dim \Delta_1 = 0\). Here we know that \(L_{P \times \Delta}\) is a diffeomorphism and that \(\hat{\omega}\) is defined everywhere, then \(\omega\) is defined everywhere. The two terms of the equality to show make sense.

The induction hypothesis gives

\[
\int_{P \times \Delta} \hat{\omega} = \int_{\partial(P \times \Delta)} \hat{\omega}
\]

and therefore (cf. [3] and [1])

\[
\int_{P \times \Delta} d\omega = \int_{P \times \Delta} \hat{d}\omega = \int_{\partial(P \times \Delta)} \hat{\omega} = \int_{\partial(P \times \Delta)} \hat{\omega} + \int_{F_{P \times \Delta}} \hat{\omega} = \int_{\partial(P \times \Delta)} \omega + \int_{F_{P \times \Delta}} \hat{\omega}.
\]

Now it suffices to notice that \(\hat{\omega}\) vanishes on \(F_{P \times \Delta}\).

**Lemma D** Let \(c: P \times \Delta \to B \times [0,1]^a\) be a smooth \(\overline{p}\)-allowable prism. If \(\eta \in \Omega^*(B)\) then \(c^*\eta\) is regular.

**Proof.** Since \(c^*\eta\) is liftable, it remains to prove that \(\hat{c^*}\eta\) vanishes on \(\text{int} F_{P \times \Delta}\). In fact, it suffices to prove that that, we have

\[
\hat{c^*}\eta(u_1, \ldots, u_a, v_1, \ldots, v_b, 0, w_1, \ldots w_c) = 0,
\]

where \(\{u_1, \ldots, u_a = \dim P\}\) are tangent vectors to \(\text{int} P\), \(\{v_1, \ldots, v_b = \dim \Delta_0\}\) are tangent vectors to \(\text{int} \Delta_0\) and \(\{w_1, \ldots, w_c = \dim \Delta_1 * \cdots * \Delta_k\}\) are tangent vectors to \(\text{int} (\Delta_1 * \cdots * \Delta_k)\).
Since the question is finally local, we can proceed as before and identify \((M, F)\) with \((\mathbb{R}^r \times \mathbb{R}^s \times cS^n, K \times I \times cG)\), identify \(B\) with \(\mathbb{R}^s \times c\pi_{gn}(S^n)\), identify \(S\) with \(\mathbb{R}^{m-n-1} \times \{\vartheta\}\) and suppose that \(\mathfrak{c}\) meets \(\mathbb{R}^{blu} \times \{\vartheta\} \times [0, 1]^a\). Then \(c^{-1}(\mathbb{R}^s \times \{\vartheta\} \times [0, 1]^a) = P \times \Delta_0\). The diagram [(4.4)] becomes

\[
\begin{array}{ccc}
F_{P \times \Delta} & \xrightarrow{\widehat{c}} & \mathbb{R}^{m-n-1} \times \mathbb{S}^n \times \{0\} \times [0, 1]^a \\
\mathcal{L}_{P \times \Delta} & \xrightarrow{c} & \mathbb{R}^{m-n-1} \times \{\vartheta\} \times [0, 1]^a \\
P \times \Delta_0 & \xrightarrow{\pi} & P \times \text{Id}_{[0, 1]^a}
\end{array}
\]

Here \(\mathcal{L}_{P \times \Delta}(x_1, \ldots, x_a, y_1, \ldots, y_b, 0, z_1, \ldots, z_c) = (x_1, \ldots, x_a, y_1, \ldots, y_b)\). The prism \(\widehat{c}\) verifies i)’ and therefore \(\widehat{c}\) sends int \(F_{P \times \Delta}\) into \(\mathbb{R}^{m-n-1} \times (\mathbb{S}^n - \Sigma G) \times \{0\} \times [0, 1]^a\). Since \(\widehat{c} = \widehat{\pi}_0 \circ \widehat{c}\) and \(\widehat{\pi}_0^* \eta = \widehat{\pi}_c^* \eta\) then the condition [(3)] is equivalent to

\[
\widehat{\pi}_0^* \eta(c_s u_1, \ldots, c_s u_{a}, c_s v_1, \ldots, c_s v_b, 0, c_s w_1, \ldots, c_s w_c) = 0.
\]

Since \(\text{codim}_\Delta F_{\dim \pi(S)} = \text{codim}_\Delta F_s = \text{codim}_\Delta \Delta_0 = c + 1\) then the condition iv) implies that

\[
c \geq \text{codim}_B \pi(S) - \pi(S) - 1 = \pi(S) + 1 > \pi(S).
\]

Finally, since \((\mathcal{L}_{P \times \Delta}) w_j = 0\) then \(\widehat{c}_s w_j\) is a vector of \(\mathbb{R}^{m-n-1} \times (\mathbb{S}^n - \Sigma G) \times \{0\} \times [0, 1]^a\) tangent to the fibers of \((P \times \text{Identity}_{[0, 1]^a})\) and therefore we get (10).

\[\clubsuit\]

The Lemma [C] and the Lemma [D] show that the operator \(\Theta: \Omega^*_\Sigma(B) \rightarrow \text{Hom}(LC_{\Sigma}(B), \mathbb{R})\) is a differential operator.

### 4.5 The quasi-isomorphisms \(\Theta\) and \(\Lambda\).

Consider the statement \(Q(B)\) about the leaf space of singular fibrations:

\[
Q(B) = "\Omega^*_\Sigma(B) \xrightarrow{\Theta} \text{Hom}(LC_{\Sigma}(B), \mathbb{R}) \xleftarrow{\Lambda} \text{Hom}(SC_{\Sigma}(B), \mathbb{R})\) are quasi-isomorphisms",
\]

and we shall prove it by using the Bredon’s Trick [3] pag. 289].

**Bredon’s Trick.** Let \(X\) be a paracompact topological espace and let \(\{U_\alpha\}\) be an open covering, closed for finite intersection. Suppose that \(Q(U)\) is a statement about open subsets of \(M\), satisifying the following three properties:

1. **(BT1)** \(Q(U_\alpha)\) is true for each \(\alpha\);
2. **(BT2)** \(Q(U), Q(V)\) and \(Q(U \cap V) \implies Q(U \cup V)\), where \(U\) and \(V\) are open subsets of \(M\);
3. **(BT3)** \(Q(U_i) \implies Q(\bigcup U_i)\), where \(\{U_i\}\) is a disjoint family of open subsets of \(M\).

Then \(Q(X)\) is true.

We proceed by induction on depth \(S_F\), then this depth is 0 then \(Q(B)\) is the usual de Rham Theorem. Let us suppose that \(Q(B)\) is proved when depth \(S_F < \ell\). We proceed in several steps.
(i) The foliated manifold \((M, \mathcal{F})\) is \((\mathbb{R}^u \times \mathbb{R}^v \times cS^w, \mathcal{K} \times \mathcal{I} \times c\mathcal{G})\) where \(\mathcal{K}\) is the foliation with one leaf, \(\mathcal{I}\) is the foliation by points and \((S^w, \mathcal{G})\) is a singular fibration with depth \(\mathcal{S}_G < \ell\).

Using the canonical projection \(\Pr : \mathbb{R}^v \times c\pi_{gw}(S^w) \to c\pi_{gw}(S^w)\), restricted to the regular part, we already know that
\[
\Pr^* : \Omega^*_\mathcal{F}(c\pi_{gw}(S^w)) \to \Omega^*_\mathcal{F}(\mathbb{R}^v \times c\pi_{gw}(S^w))
\]
is a quasi-isomorphism (cf. Proposition 2.4.2 and (8)). Following the same procedure as in [3] we prove that the operators
\[
\Pr^* : \text{Hom}(SC^*_\mathcal{F}(c\pi_{gw}(S^w)), \mathbb{R}) \to \text{Hom}(SC^*_\mathcal{F}(\mathbb{R}^v \times c\pi_{gw}(S^w)), \mathbb{R})
\]
and
\[
\Pr^* : \text{Hom}(LC^*_\mathcal{F}(c\pi_{gw}(S^w)), \mathbb{R}) \to \text{Hom}(LC^*_\mathcal{F}(\mathbb{R}^v \times c\pi_{gw}(S^w)), \mathbb{R})
\]
are quasi-isomorphisms. Now, since \(\Pr^*\) commutes with \(\Theta\) and \(\Lambda\), then \(Q(c\pi_{gw}(S^w)) \implies Q(\mathbb{R}^v \times c\pi_{gw}(S^w))\). It remains to prove \(Q(c\pi_{gw}(S^w))\).

From Proposition 2.4.3 and (8) we know that the canonical projection \(\Pr : (S^w - \Sigma_G) \times ]0, 1[ \to (S^w - \Sigma_G)\) induces the quasi-isomorphism
\[
\mathcal{H}^i(\pi_{gw}(S^w)) = \left\{ \begin{array}{ll}
\mathcal{H}^i(\pi_{gw}(S^w)) & \text{if } i \leq \mathcal{p}(\varnothing) \\
0 & \text{if } i > \mathcal{p}(\varnothing) 
\end{array} \right.
\]
Following the same procedure as in [3] we prove that \(\Pr^*\) induces the isomorphisms
\[
\mathcal{H}^i(LC^*_\mathcal{F}(c\pi_{gw}(S^w))) = \left\{ \begin{array}{ll}
\mathcal{H}^i(LC^*_\mathcal{F}(c\pi_{gw}(S^w))) & \text{if } i \leq k - 1 - \mathcal{p}(\varnothing) \\
0 & \text{if } i > k - 1 - \mathcal{p}(\varnothing) 
\end{array} \right.
\]
and
\[
\mathcal{H}^i(SC^*_\mathcal{F}(c\pi_{gw}(S^w))) = \left\{ \begin{array}{ll}
\mathcal{H}^i(SC^*_\mathcal{F}(c\pi_{gw}(S^w))) & \text{if } i \leq k - 1 - \mathcal{p}(\varnothing) \\
0 & \text{if } i > k - 1 - \mathcal{p}(\varnothing) 
\end{array} \right.
\]
where \(k = \dim \pi_{gw}(S^w)\). Now, since \(\Theta\) and \(\Lambda\) commute with \(\Pr^*\) it suffices to apply \(Q(\pi_{gw}(S^w))\) and to take into account that the perversities \(\mathcal{p}\) and \(\mathcal{q}\) on \(B\) are dual.

(ii) The foliated manifold \((M, \mathcal{F})\) is an open subset of \((\mathbb{R}^u \times \mathbb{R}^v \times cS^w, \mathcal{K} \times \mathcal{I} \times c\mathcal{G})\) where \(\mathcal{K}\) is the foliation with one leaf, \(\mathcal{I}\) is the foliation by points and \((S^w, \mathcal{G})\) is a singular fibration with depth \(\mathcal{S}_G < \ell\).

Since \(B\) is paracompact we can apply the Bredon’s Trick to the following covering
\[
\{ V \times c_\epsilon \pi_{gw}(S^w) \} / V = ]a_1, \epsilon[ \times \cdots \times ]a_v, b_v[ \subset \mathbb{R}^u, \epsilon \in [0, 1] \cup \{ \pi(U) / U \subset \mathbb{R}^u \times S^w \times ]0, 1[ \text{ open }, \}
\]
where \(c_\epsilon \pi_{gw}(S^w) = \pi_{gw}(S^w) \times ]0, \epsilon[ / \pi_{gw}(S^w) \times \{ 0 \}\). This family is closed for finite intersections. Let us verify the three conditions (BT1-3).

(BT1) Apply (i) to \(V \times c_\epsilon \pi_{gw}(S^w)\) and apply the induction hypothesis to \(\pi(U)\) noticing that depth \(\mathcal{S}_{\pi(U)} < \ell\).

(BT2) This is Mayer-Vietoris.

(BT3) By construction.

(iii) The depth of the foliated manifold \((M, \mathcal{F})\) is \(\ell\).
Since $B$ is paracompact we can apply the Bredon’s Trick relatively to the following covering:

$$\{ V \subset B \mid V \text{ is an open subset of } \pi(U) \text{ where } (\varphi, U) \text{ is a good conical chart} \}. $$

This family is closed for finite intersections. Let us verify the three conditions (BT1-3).

(BT1) Apply(2) and (ii) using the fact that $\ell = \text{depth } S_{K \times I \times G} = \text{depth } S_G + 1$.

(BT2) This is Mayer-Vietoris.

(BT3) By construction.

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