Derivations for the even part of the Hamiltonian superalgebra in positive characteristic

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Abstract: In this paper we consider the derivations for even part of the finite-dimensional Hamiltonian superalgebra $H$ over a field of prime characteristic. We first introduce an ideal $\mathfrak{N}$ of $H_\mathbb{C}$ and show that the derivation space from $H_\mathbb{C}$ into $W_\mathbb{C}$ can be obtained by the derivation space from $\mathfrak{N}$ into $W_\mathbb{C}$, the even part of the generalized Witt superalgebra $W$. For further application we also give the generating set of the ideal $\mathfrak{N}$. Then we describe three series of exceptional derivations from $H_\mathbb{C}$ into $W_\mathbb{C}$. Finally, we determine all the derivations vanishing on the non-positive $\mathbb{Z}$-graded part of $H_\mathbb{C}$, the odd $\mathbb{Z}$-homogeneous derivations, and negative $\mathbb{Z}$-homogeneous derivations from $H_\mathbb{C}$ into $W_\mathbb{C}$.

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0. Introduction

During last few decades the theory of Lie superalgebras has undergone a remarkable evolution both in mathematics and in physics (see 11). For example, the classifications by V.G. Kac of finite-dimensional simple Lie superalgebras and infinite-dimensional simple linearly compact Lie superalgebras over algebraically closed fields of characteristic zero have been completed (see 2, 3). For modular Lie superalgebras, as far as we know, 4 and 10 may be the earliest papers.

In this paper we consider derivations for the even parts of modular Lie superalgebras of Cartan type $H$. Our work is originally motivated by the work on modular Lie algebras of Cartan type (see 11,12,13). Note that the superderivation algebras have been determined for the finite-dimensional modular Lie superalgebras of Cartan type $W$, $S$, $H$, and $K$. (see 2, 14, 17). The superderivation algebra was determined for the finite-dimensional odd Hamiltonian superalgebra $HO$ in 8. We should mention that the derivations of the even parts have also been studied sufficiently for the Lie superalgebras of Cartan type $W$, $S$.

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and \( HO (\text{see } \cite{6}, \cite{7}) \); in particular, the derivations from the even parts into the odd parts have been determined for \( W \) and \( S \) (see \cite{5}). However, the present work differs greatly from the ones mentioned above; in particular, we find “more” outer derivations for the even part of \( H \) than \( HO, \) \( W, \) or \( S \) (see also Remark \cite{3}, \cite{13}).

This paper is organized as follows. In Section 1 we give the necessary notation and concepts. In Section 2, we first introduce an ideal \( \mathfrak{M} \) of the even part of \( H \), which is crucial for our aim. Then we give the generating set of the ideal \( \mathfrak{N} \) for future application. In Section 3 we mainly construct three series of outer derivations from the even part of \( H \) into the even part of \( W \), the generalized Witt superalgebra. In Section 4, we determine all the derivations vanishing on the non-positive \( \mathbb{Z} \)-graded part of \( H_0^\mathfrak{p} \), the odd \( \mathbb{Z} \)-homogeneous derivations, and negative \( \mathbb{Z} \)-homogeneous derivations from \( H_0^\mathfrak{p} \) into \( W_0^\mathfrak{p} \).

1. Preliminaries

Let \( \mathbb{Z}_2 = \{0, 1\} \) be the field of two elements. For a vector superspace \( V = V_0 \oplus V_1 \), we denote by \( p(a) = \theta \) the \textit{parity of a homogeneous element} \( a \in V_0, \theta \in \mathbb{Z}_2 \). We assume throughout that the notation \( p(x) \) implies that \( x \) is a \( \mathbb{Z}_2 \)-homogeneous element.

Let \( \mathfrak{g} \) be a Lie algebra and \( V \) a \( \mathfrak{g} \)-module. A linear mapping \( D: \mathfrak{g} \to V \) is called a \textit{derivation} from \( \mathfrak{g} \) into \( V \) if \( D(xy) = x \cdot D(y) - y \cdot D(x) \) for all \( x, y \in \mathfrak{g} \). A derivation \( D: \mathfrak{g} \to V \) is called \textit{inner} if there is \( v \in V \) such that \( D(x) = x \cdot v \) for all \( x \in \mathfrak{g} \). Following \cite{13}, p. 13, denote by \( \text{Der}(\mathfrak{g}, V) \) the \textit{derivation space} from \( \mathfrak{g} \) into \( V \). Then \( \text{Der}(\mathfrak{g}, V) \) is a \( \mathfrak{g} \)-submodule of \( \text{Hom}_F(\mathfrak{g}, V) \). Assume in addition that \( \mathfrak{g} \) and \( V \) are finite-dimensional and that \( \mathfrak{g} = \oplus_{r \in \mathbb{Z}} \mathfrak{g}[r] \) is \( \mathbb{Z} \)-graded and \( V = \oplus_{r \in \mathbb{Z}} V[r] \) is a \( \mathbb{Z} \)-graded \( \mathfrak{g} \)-module. Then \( \text{Der}(\mathfrak{g}, V) = \oplus_{r \in \mathbb{Z}} \text{Der}[r](\mathfrak{g}, V) \) is a \( \mathbb{Z} \)-graded \( \mathfrak{g} \)-module by setting

\[
\text{Der}[r](\mathfrak{g}, V) := \{ D \in \text{Der}(\mathfrak{g}, V) \mid D(\mathfrak{g}[r]) \subset V[r+\ell] \text{ for all } \ell \in \mathbb{Z} \}.
\]

In the case \( V = \mathfrak{g} \), the \textit{derivation algebra} \( \text{Der}(\mathfrak{g}) \) coincides with \( \text{Der}(\mathfrak{g}, \mathfrak{g}) \) and \( \text{Der}(\mathfrak{g}) = \oplus_{r \in \mathbb{Z}} \text{Der}[r](\mathfrak{g}) \) is a \( \mathbb{Z} \)-graded Lie algebra. If \( \mathfrak{g} = \oplus_{-r \leq i \leq s} \mathfrak{g}[i] \) is a \( \mathbb{Z} \)-graded Lie algebra, then \( \oplus_{-r \leq i \leq s} \mathfrak{g}[i] \) is called the \textit{top of} \( \mathfrak{g} \) (with respect to the gradation). Let \( V = \oplus_{r \in \mathbb{Z}} V[r] \) be a \( \mathbb{Z} \)-graded vector space and \( x \in V \) is a \( \mathbb{Z} \)-homogeneous element. Then we let \( zd(x) \) denote the \( \mathbb{Z} \)-degree of \( x \).

In the following we recall the notions of the generalized Witt modular superalgebra and the Hamiltonian modular superalgebra and their natural gradation structures. We also introduce the notation, terminology and convention which will be used throughout.

In the sequel \( F \) denotes a field of characteristic \( p > 3 \). In addition to the standard notation \( \mathbb{Z} \), we use \( \mathbb{N} \) for the set of positive integers and \( \mathbb{N}_0 \) for the set of nonnegative integers. Henceforth, we will let \( m \) and \( n \) denote fixed positive integers without notice. Given \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \), we put \( |\alpha| := \sum_{i=1}^{m} \alpha_i \). Following \cite{12}, denote by \( O(m) \) the \textit{divided power algebra} over \( F \) with an \( F \)-basis \( \{ x^{(\alpha)} \mid \alpha \in \mathbb{N}_0^m \} \). For \( \varepsilon_i = (\delta_{i1}, \ldots, \delta_{im}) \), we abbreviate \( x^{(\varepsilon_i)} \) to \( x_i \), \( i = 1, \ldots, m \). Let \( \Lambda(n) \) be the \textit{exterior superalgebra} over \( F \) in \( n \) variables \( x_{m+1}, \ldots, x_{m+n} \). Denote the tensor product by \( O(m, n) = O(m) \otimes_F \Lambda(n) \). Obviously, \( O(m, n) \) is an associative superalgebra with a \( \mathbb{Z}_2 \)-gradation induced by the trivial \( \mathbb{Z}_2 \)-gradation of \( O(m) \) and the natural \( \mathbb{Z}_2 \)-gradation of \( \Lambda(n) \). Evidently, \( O(m, n) \) is super-commutative.

For \( g \in O(m), f \in \Lambda(n) \), we abbreviate \( g \otimes f \to gf \). The following formulas hold in \( O(m, n) \):

\[
x^{(\alpha)} x^{(\beta)} = \left( \frac{\alpha + \beta}{\alpha} \right) x^{(\alpha + \beta)} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^m;
\]
$x_ix_j = -x_jx_i$ for $i, j = m + 1, \ldots, m + n$;

$x^{(a)}x_j = x_jx^{(a)}$ for $\alpha \in \mathbb{N}_0^m$, $j = m + 1, \ldots, m + n$,

where $(\alpha + \beta) := \prod_{i=1}^{m} (\alpha_i + \beta_i)$.

For convenience, put $Y_0 := \{1, 2, \ldots, m\}$, $Y_1 := \{m + 1, m + 2, \ldots, m + n\}$ and $Y := Y_0 \cup Y_1$.

Let

$$\mathbb{B}_k := \{(i_1, i_2, \ldots, i_k) \mid m + 1 \leq i_1 < i_2 < \cdots < i_k \leq m + n\}$$

be the set of $k$-tuples of strictly increasing integers between $m + 1$ and $m + n$, and $\mathbb{B} := \bigcup_{k=0}^{n} \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. For $u := (i_1, i_2, \ldots, i_k) \in \mathbb{B}_k$, set $|u| := k$, $|\emptyset| := 0$, $x^0 := 1$, and $x^u := x_{i_1}x_{i_2} \cdots x_{i_k}$; we also let $u$ stand for the set $\{i_1, i_2, \ldots, i_k\}$ if no confusion occurs.

Put $\mathbb{B}^0 := \{u \in \mathbb{B} \mid |u| \text{ even}\}$. Clearly, $\{x^{(a)}x^u \mid \alpha \in \mathbb{N}_0^m, u \in \mathbb{B}\}$ is an $\mathbb{F}$-basis of $\mathcal{O}(m, n)$.

Let $\partial_1, \partial_2, \ldots, \partial_{m+n}$ be the linear transformations of $\mathcal{O}(m, n)$ such that

$$\partial_i(x^{(a)}x^u) = \begin{cases} x^{(\alpha - e_i)}x^u, & i \in Y_0 \\ x^{(a)}x^u/\partial x_i, & i \in Y_1. \end{cases}$$

Then $\partial_1, \partial_2, \ldots, \partial_{m+n}$ are superderivations of the superalgebra $\mathcal{O}(m, n)$.

Let

$$W(m, n) := \left\{ \sum_{i \in Y} a_i \partial_i \mid a_i \in \mathcal{O}(m, n), i \in Y \right\}.$$

Then $W(m, n)$ is a Lie superalgebra contained in $\text{Der}\mathcal{O}(m, n)$.

One may verify that

$$[aD, bE] = aD(bE) - (-1)^{p(aD)p(bE)}bE(a)D + (-1)^{p(D)p(b)}ab[D, E] \quad (1.1)$$

for $a, b \in \mathcal{O}(m, n)$, $D, E \in \text{Der}\mathcal{O}(m, n)$. Consequently, the following formula holds in $W(m, n)$:

$$[a\partial_i, b\partial_j] = a\partial_i(b\partial_j) - (-1)^{p(a\partial_i)p(b\partial_j)}b\partial_j(a)\partial_i$$

for $a, b \in \mathcal{O}(m, n), i, j \in Y$. We note that $p(\partial_i) = \mu(i)$, where

$$\mu(i) := \begin{cases} 0, & i \in Y_0 \\ 1, & i \in Y_1. \end{cases}$$

In the sequel suppose $m = 2r$ is even. Define the linear mapping $D_H : \mathcal{O}(m, n) \to W(m, n)$ by means of

$$D_H(a) := \sum_{i \in Y} \tau(i)(-1)^{\mu(i)p(a)}\partial_i(a)\partial_{i'} \quad \text{for all } a \in \mathcal{O}(m, n),$$

where

$$i' := \begin{cases} i + r, & 1 \leq i \leq r \\ i - r, & r < i \leq 2r \\ i, & i \in Y_1; \end{cases} \quad \tau(i) := \begin{cases} 1, & 1 \leq i \leq r \\ -1, & r < i \leq 2r \\ 1, & i \in Y_1. \end{cases}$$

Then the following identity holds

$$[D_H(a), D_H(b)] = D_H(D_H(a)(b)) \quad \text{for all } a, b \in \mathcal{O}(2r, n). \quad (1.2)$$
Let 
\[ \mathfrak{L} := (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m, \quad \pi := (\pi_1, \pi_2, \ldots, \pi_m) \]
where \( \pi_i := p^i - 1, i \in \mathbb{Y}_0 \). Put \( \mathcal{A} := \mathcal{A}(m; \mathfrak{L}) := \{ \alpha \in \mathbb{N}_{\leq 0}^m | \alpha_i \leq \pi_i, i = 1, 2, \ldots, m \} \). Then
\[ \mathcal{O}(m, n; \mathfrak{L}) := \text{span}_F \{ x^{(\alpha)} x^u | \alpha \in \mathcal{A}, u \in \mathbb{B} \} \]
is a finite-dimensional subalgebra of \( \mathcal{O}(m, n) \), with a natural \( \mathbb{Z} \)-gradation \( \mathcal{O}(m, n; \mathfrak{L}) = \bigoplus_{i=0}^{\xi} \mathcal{O}(m, n; \mathfrak{L})_i \), where \( \mathcal{O}(m, n; \mathfrak{L})_i = \text{span}_F \{ x^{(\alpha)} x^u | |\alpha| + |u| = i \} \), \( \xi := |\pi| + n \). Set
\[ W(m, n; \mathfrak{L}) := \left\{ \sum_{i \in Y} a_i \partial_i | a_i \in \mathcal{O}(m, n; \mathfrak{L}), i \in Y \right\}. \]
Then \( W(m, n; \mathfrak{L}) \) is a subalgebra of \( W(m, n) \). In particular, it is a finite-dimensional simple Lie superalgebra (see \[16\]). As in the case of Lie algebras, \( W(m, n; \mathfrak{L}) \) is called the generalized Witt superalgebras. Obviously, \( W(m, n; \mathfrak{L}) \) is a free \( \mathcal{O}(m, n; \mathfrak{L}) \)-basis \( \{ \partial_1, \partial_2, \ldots, \partial_{m+n} \} \).

Set
\[ H(m, n; \mathfrak{L}) := \{ \mathcal{D}_H(a) | a \in \bigoplus_{i=0}^{\xi-1} \mathcal{O}(m, n; \mathfrak{L})_i \} \],
where \( \xi = |\pi| + n \). Then \( H(m, n; \mathfrak{L}) \) is a finite-dimensional simple Lie superalgebra, which is called the Hamiltonian superalgebra (see \[15\] \[17\]).

The \( \mathbb{Z} \)-gradation of \( \mathcal{O}(m, n; \mathfrak{L}) \) induces naturally a \( \mathbb{Z} \)-gradation structure of the generalized Witt superalgebra \( W(m, n; \mathfrak{L}) = \bigoplus_{i=0}^{\xi-1} W(m, n; \mathfrak{L})_i \), where
\[ W(m, n; \mathfrak{L})_i := \text{span}_F \{ f \partial_s | s \in Y, f \in \mathcal{O}(m, n; \mathfrak{L})_{i+1} \} \].
Note that \( H(m, n; \mathfrak{L}) \) is a \( \mathbb{Z} \)-graded subalgebras of \( W(m, n; \mathfrak{L}) \).

In the following sections, since the positive integer \( m \) is even, we write \( 2m \) instead of \( m \), and we usually write \( \mathcal{O}, W, \) and \( H \) for \( \mathcal{O}(2m, n; \mathfrak{L}), W(2m, n; \mathfrak{L}), \) and \( H(2m, n; \mathfrak{L}) \), respectively. For convenience, the even parts of \( W \) and \( H \) will be denoted by \( W \) and \( H \), respectively.

In this paper we suppose \( m > 1, n > 3 \) and \( p > 3 \) for the sake of simplicity although sometimes a weak hypothesis is sufficient.

2. The ideal \( \mathfrak{N} \) and its generating set

As mentioned in the introduction, our main object is to discuss the derivations for the even part \( \mathcal{H} \) of the finite-dimensional Hamiltonian superalgebra \( H \). Precisely speaking, we want to formulate certain homogeneous derivations from \( \mathcal{H} \) into \( W \). However, in contrast to the setting of \( W \) and \( S \) (see \[5\] \[7\]), we shall first deal with the derivations from an ideal \( \mathfrak{N} \) of \( \mathcal{H} \) into \( W \) rather than the derivations from \( \mathcal{H} \) into \( W \). Of course, we can guarantee that this transformation behaves without any influence on our aim (see Remark \[2.5\]).

Recall
\[ \mathcal{H} := H_0(2m, n; \mathfrak{L}) = \text{span}_F \{ \mathcal{D}_H(x^{(\alpha)} x^u) | \alpha \in \mathcal{A}, u \in \mathbb{B}^0, (\alpha, u) \neq (\pi, \omega) \} \]
and put
\[ \mathfrak{N} := \text{span}_F \{ \mathcal{D}_H(x^{(\alpha)} x^u) | \alpha \in \mathcal{A}, u \in \mathbb{B}^0, (\alpha, u) \neq (\pi, \omega), (\alpha, u) \neq (\pi, \emptyset) \}, \]
where $B^0 = \{ u \in B \mid |u| \text{ even} \}$ and $\omega := \langle m + 1, \ldots, m + n \rangle \in B_n$. Evidently, $\mathfrak{N}$ is a subspace of $\mathcal{H}$ of codimension 1:

$$\mathcal{H} = \mathfrak{N} \oplus F_{\mathcal{H}}(x^{(\pi)}).$$

(2.1)

In the following we shall demonstrate that in order to determine the derivations from $\mathcal{H}$ into $\mathcal{W}$ it suffices to determine the derivations from $\mathfrak{N}$ into $\mathcal{W}$. It is clear that this will simplify our consideration. To that end, we need the following two propositions.

**Proposition 2.1.** $\mathfrak{N}$ is an ideal of $\mathcal{H}$.

**Proof.** We first show that $\mathfrak{N}$ is a subalgebra of $\mathcal{H}$. Given any two linear generators $D_H(x^{(\alpha)}x^u), D_H(x^{(\beta)}x^v) \in \mathfrak{N}$, we assert that

$$[D_H(x^{(\alpha)}x^u), D_H(x^{(\beta)}x^v)] \in \mathfrak{N}.$$ (2.2)

In fact, if $u = v = 0$, in view of the theory of Hamiltonian algebras [13], we get (2.2) immediately. If $u \neq 0$ or $v \neq 0$, noticing that $|u|, |v|$ are all even, we obtain (2.2) by using the formula (1.2). This proves that $\mathfrak{N}$ is a subalgebra of $\mathcal{H}$. We next show that $\mathfrak{N}$ is an ideal of $\mathcal{H}$. By (2.1), it suffices to show that

$$D_H(x^{(\pi)}), \mathfrak{N} \subset \mathfrak{N}. $$

(2.3)

Clearly,

$$[D_H(x^{(\pi)}), \mathfrak{N}] \subset \mathcal{H}. $$

(2.4)

For every linear generator $D_H(x^{(\alpha)}x^u)$ of $\mathfrak{N}$, if $zd([D_H(x^{(\pi)}), D_H(x^{(\alpha)}x^u)]) \neq |\pi| - 2$, then we see from (2.4) that $[D_H(x^{(\pi)}), D_H(x^{(\alpha)}x^u)] \in \mathfrak{N}$ and thereby (2.3) holds. It remains only to consider the case $|\alpha| + |u| = 2$. If $|u| = 2$, then $[D_H(x^{(\pi)}), D_H(x^u)] = 0 \in \mathfrak{N}$; if $\alpha = 2$, by the theory of Hamiltonian Lie algebras, we also have $[D_H(x^{(\pi)}), D_H(x^{(\alpha)})] \in \mathfrak{N}$ and therefore, (2.3) holds.

Following [7], put $\mathcal{G} := \text{span}_F \{ x^u \partial_r \mid r \in Y, u \in B, p(x^u\partial_r) = 0 \}$, then $C_{\mathcal{W}}(\mathcal{W}_{-1}) = \mathcal{G}$; in particular, $C_{\mathcal{H}}(\mathcal{H}_{-1}) \subset \mathcal{G}$. Note that $\mathcal{G}$ is a $\mathbb{Z}$-graded subalgebra of $\mathcal{W}$.

**Proposition 2.2.** Let $|\omega| = n$ be even. Then $C_{\mathcal{W}}(\mathfrak{N}) = F D_{\mathcal{H}}(x^{\omega})$.

**Proof.** For any arbitrary basis element $D_H(x^{(\alpha)}x^u)$ of $\mathfrak{N}$, $u \in B^0$, it is clear that

$$[D_H(x^{\omega}), D_H(x^{(\alpha)}x^u)] = 0.$$ (2.5)

Therefore, $F D_{\mathcal{H}}(x^{\omega}) \subset C_{\mathcal{W}}(\mathfrak{N})$. We propose to prove the converse inclusion. Since $C_{\mathcal{W}}(\mathfrak{N})$ is a $\mathbb{Z}$-subalgebra of $\mathcal{H}$, it suffices to show that if $D \in C_{\mathcal{W}}(\mathfrak{N})$ is homogeneous then $D \in F D_{\mathcal{H}}(x^{\omega})$. Noting that $\mathcal{W}_{[1]} = \mathcal{H}_{[1]} = \mathfrak{N}_{[1]}$, we see that $D \in \mathcal{G}$. Thus one may assume that

$$D = \sum_{r \in Y} f_r \partial_r \quad \text{where } f_r \in \Lambda(n).$$

(2.5)

For any $i \in Y_0$, since $D_H(x^{(2\epsilon_i)}) \in \mathfrak{N}$, we have $[D, D_H(x^{(2\epsilon_i)})] = 0$ and therefore, $f_i \partial_{\epsilon_r} = 0$. This proves that $f_i = 0$ for all $i \in Y_0$. Thus, by (2.5), we have

$$D = \sum_{r \in Y_1} f_r \partial_r \quad \text{where } f_r \in \Lambda(n).$$

(2.6)
Case (i): $1 < zd(f_r) \leq n - 3$ for $r \in Y_1$. Assume that $f_{r_0} \neq 0$ for some $r_0 \in Y_1$. By our assumption one may choose $k, l \in Y_1$ with $k \neq l$ such that $x_l \partial_k(f_{r_0}) \neq 0$ and $k \neq r_0$, $l \neq r_0$. Note that $D_H(x_k x_l) = x_l \partial_k - x_k \partial_l \in \mathfrak{N}$. We have
\[ 0 = [D_H(x_k x_l), \sum_{r \in Y_1} f_r \partial_r] = \sum_{r \in Y_1} (x_l \partial_k(f_r) - x_k \partial_l(f_r)) \partial_r - f_l \partial_k + f_k \partial_l. \tag{2.7} \]

Then a comparison of the coefficients of $\partial_r$ in (2.7) yields
\[ x_l \partial_k(f_{r_0}) - x_k \partial_l(f_{r_0}) = 0. \]
However, this equation implies $x_l \partial_k(f_{r_0}) = 0$, a contradiction. This proves that $D = 0$.

Case (ii): $zd(f_r) = n - 1$ for $r \in Y_1$. Then one may assume that
\[ f_r = \sum_{s \in Y_1} c_{rs} x^{(s)} \text{ where } c_{rs} \in \mathbb{F}. \tag{2.8} \]
Here we put $\langle s \rangle := \omega - \langle s \rangle$. Fixing $r \in Y_1$ and replacing $f_r$ in (2.7) by (2.8), for $k, l \in Y_1 \setminus r$ with $k \neq l$, we obtain that $x_l \partial_k(\sum_{s \in Y_1} c_{rs} x^{(s)}) = 0$ and therefore, $c_{rt} x^{(k)} = 0$. This implies that $c_{rt} = 0$ whenever $r \neq l$. Consequently, $f_r = c_{rr} x^{(r)}$ for $r \in Y_1$. Observing the coefficient of $\partial_k$ in (2.7), we obtain that $-x_l \partial_l(c_{kk} x^{(k)}) = -c_{ll} x^{(l)} = 0$. It follows that $c_{kk} = (-1)^{k+1} c_{ll}$ for $k, l \in Y_1$. Let $\lambda := c_{m+1,m+1}$. So far, we have proved that
\[ D = \lambda \sum_{r \in Y_1} (-1)^{r+1} x^{(r)} = \lambda D_H(x^\omega). \]

Case (iii): $zd(f_r) = 1$. As in Case (i) one may easily get the desired result.

\[ \square \]

**Proposition 2.3.** Let $|\omega| = n$ be odd. Then $C_W(\mathfrak{N}) = 0$.

**Proof.** Arguing just as in the proof of Proposition 2.2 we see that every element $D \in C_W(\mathfrak{N})$ may be written as (2.6). Given $r_0 \in Y_1$, we obtain from (2.7) that
\[ x_l \partial_k(f_{r_0}) = 0 \text{ whenever } k, l \in Y_1 \setminus r_0 \text{ with } k \neq l. \tag{2.9} \]
Without loss of generality, one may assume that $D$ is homogeneous and so is $f_{r_0} = \sum u c_{r_0 u} x^u$. If $1 < zd(f_{r_0}) < n - 1$, then by (2.9), $f_{r_0} = 0$. In the case $zd(f_{r_0}) = 1$, it is easily shown that $D = 0$. Assume that $f_{r_0} \neq 0$. Then $zd(f_{r_0}) \geq n - 1$ as we have shown that $x_l \partial_k(f_{r_0}) = 0$ whenever $k, l \in Y_1$ with $k \neq l$. Because $n$ is odd, this forces $zd(f_{r_0}) = n$ and hence, $f_{r_0} = c_{r_0 \omega} x^\omega$. Now, using (2.7) one may easily deduce that $c_{r_0 \omega} = 0$, contradicting the assumption that $f_{r_0} \neq 0$. Summarizing, we have shown that $D = 0$. The proof is complete.

\[ \square \]

**Theorem 2.4.** Suppose $\varphi \in \text{Der}(\mathcal{H}, \mathcal{W})$ and $\varphi(\mathfrak{N}) = 0$. Then the following statements hold.

(i) If $|\omega| = n$ is odd then $\varphi = 0$.

(ii) If $|\omega| = n$ is even then $\varphi(D_H(x^{(\pi)})) = \lambda D_H(x^\omega)$ for some $\lambda \in \mathbb{F}$. Conversely, any mapping $\varphi : \mathcal{H} \rightarrow \mathcal{W}$ vanishing on $\mathfrak{N}$ and satisfying $\varphi(D_H(x^{(\pi)})) = \lambda D_H(x^\omega)$ for any fixed $\lambda \in \mathbb{F}$ is necessarily a derivation from $\mathcal{H}$ into $\mathcal{W}$.

**Proof.** By Proposition 2.1 $[\varphi(D_H(x^{(\pi)})), \mathfrak{N}] = 0$; that is, $\varphi(D_H(x^{(\pi)})) \in C_W(\mathfrak{N})$. Now (i) follows from Proposition 2.3 and the decomposition (2.1). The first part of (ii) is an immediate consequence of Propositions 2.1, 2.2 and the decomposition (2.1). The second part follows immediately from Propositions 2.1 and 2.2.

\[ \square \]
Remark 2.5. Suppose that the structure of $\text{Der}(\mathfrak{R}, \mathcal{W})$ has been determined and all the derivations from $\mathfrak{R}$ into $\mathcal{W}$ may extend to $\mathcal{H}$. Then, in the light of Theorem 2.4, one may easily determine the derivation space $\text{Der}(\mathcal{H}, \mathcal{W})$. We thereby pay our attention to the ideal $\mathfrak{R}$ in future.

Remark 2.6. Define the linear mapping $\Gamma_\lambda : \mathcal{H} \to \mathcal{W}$ by means of $\Gamma_\lambda(\mathfrak{R}) := 0$ and $\Gamma_\lambda(D_H(x^{(\eta)})) := \lambda D_H(x^{(\omega)})$ for any fixed $\lambda \in \mathbb{F}$. By Theorem 2.4, $\Gamma_\lambda$ is a derivation from $\mathcal{H}$ into $\mathcal{W}$. Note that $zd(\Gamma_\lambda) = |\omega| - |\pi|$. In general, $\Gamma_\lambda$ is outer. In the next section, we shall give in addition three series of the so-called exceptional derivations from $\mathcal{H}$ into $\mathcal{W}$ in the setting that $|\omega| = n$ is even. In general, they are outer derivations.

In the following we study the generating set of $\mathfrak{R}$. Put
\[
\mathcal{M} := \{D_H(x^{(q_i,\varepsilon_i)}) \mid 1 \leq q_i \leq \pi_i, \ i \in Y_0\},
\]
and
\[
\mathcal{N} := \{D_H(x_i x^u) \mid i \in Y_0, u \in \mathbb{B}_2\}.
\]
We conclude this section with the following generating theorem.

**Theorem 2.7.** $\mathfrak{R}$ is generated by $\mathcal{M} \cup \mathcal{N} \cup \mathfrak{R}_{[0]}$.

**Proof.** Let $\mathcal{L}$ be the subalgebra of $\mathfrak{R}$ generated by $\mathcal{M} \cup \mathcal{N} \cup \mathfrak{R}_{[0]}$. For $i, j \in Y_0$, direct computation shows that
\[
D_H(x^{(2\varepsilon_i)} x_j) = \tau(i)(1 + \delta_{ij})^{-1}[D_H(x^{(3\varepsilon_i)}), D_H(x_i x_j)] \in \mathcal{L}. \tag{2.10}
\]
Furthermore,
\[
D_H(x_i x_j x_j) = \tau(i)(1 - \frac{1}{3} \delta_{ij})[D_H(x^{(2\varepsilon_i)}), D_H(x^{(2\varepsilon_j)} x_j)] \in \mathcal{L}. \tag{2.11}
\]
Using (2.11) we get for $i \neq j'$,
\[
D_H(x^{(\pi_i \varepsilon_i)} x_j) = -\tau(i)(1 + \delta_{ij})^{-1}[D_H(x^{(\pi_i \varepsilon_i)}), D_H(x_i x_j)] \in \mathcal{L}. \tag{2.12}
\]
It follows from (2.10) and (2.12) that for $i \neq j, j'$,
\[
D_H(x^{(\pi_i \varepsilon_i)} x_j x_{j'}) = \tau(j)[D_H(x^{(2\varepsilon_j)} x_{j'}), D_H(x^{(\pi_i \varepsilon_i)} x_{j'})] \in \mathcal{L}. \tag{2.13}
\]
An application of (2.13) yields
\[
D_H(x^{(\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2)}) = -[D_H(x^{(\pi_1 \varepsilon_1)}), D_H(x_1 x_1 x^{(\pi_2 \varepsilon_2)})] \in \mathcal{L};
\]
and therefore,
\[
D_H(x^{(\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2 + \pi_3 \varepsilon_3)}) = -[D_H(x^{(\pi_1 \varepsilon_1 + \pi_2 \varepsilon_2)}), D_H(x_1 x_1 x^{(\pi_3 \varepsilon_3)})] \in \mathcal{L}.
\]
By induction we may easily obtain that
\[
D_H(x^{(\pi_1 \varepsilon_1 + \cdots + \pi_m \varepsilon_m)}) \in \mathcal{L}. \tag{2.14}
\]
Using (2.12) and (2.13) we obtain that
\[
D_H(x^{(\pi_1 \varepsilon_1 + \cdots + \pi_m \varepsilon_m + \pi_i \varepsilon_i)}) = -[D_H(x^{(\pi_1 \varepsilon_1 + \cdots + \pi_m \varepsilon_m)}), D_H(x_m x_m x^{(\pi_i \varepsilon_i)})] \in \mathcal{L}.
\]
By induction one may easily show that
\[ D_H(x^{(\pi - 2m \varepsilon_m)}) \in \mathcal{L}. \] (2.15)

By (2.12), \( D_H(x_m x^{(\pi \varepsilon_m)}) \in \mathcal{L} \). It follows from (2.15) that
\[ D_H(x^{(\pi - \varepsilon_m)}) = -[D_H(x^{(\pi - \varepsilon_m)}), D_H(x_m x^{(\pi \varepsilon_m)})] \in \mathcal{L}. \] (2.16)

In general, we have
\[ D_H(x^{(\pi - r)}) \in \mathcal{L} \quad \text{for all } r \in Y_0. \]

Therefore,
\[ D_H(x^{(\alpha)}) \in \mathcal{L} \quad \text{for all } \alpha \neq \pi. \] (2.17)

We want to prove that
\[ D_H(x^{(\alpha)} x^u) \in \mathcal{L} \quad \text{for all } \alpha \neq \pi, \emptyset \neq u \in \mathbb{B}^0. \] (2.18)

We proceed by induction on \(|\alpha| + |u|\). When \(|\alpha| + |u| = 2\), we have \( u \in \mathbb{B}_2 \) and \( \alpha = 0 \) and therefore, \( D_H(x^u) \in \mathcal{L} \), since \( \mathcal{H}_{[-1]} \cup \mathcal{N} \subset \mathcal{L} \); that is, (2.18) holds. When \(|\alpha| + |u| = 3\), necessarily \( |u| = 2 \) and \( |\alpha| = 1 \) and therefore, (2.18) holds. Assume that \(|\alpha| + |u| > 3\). Find \( w \in \mathbb{B}_2 \) and \( v \in \mathbb{B} \) such that \( x^v x^w = x^u \). If \( \alpha_i < \pi_i \) for some \( i \in Y_0 \), then \( |\alpha + \varepsilon_i| + |v| < |\alpha| + |u|, 1 + |w| < |\alpha| + |u| \). Thus,
\[ D_H(x^{(\alpha)} x^u) = \tau(i)[D_H(x^{(\alpha + \varepsilon_i)} x^v), D_H(x_v x^u)] \in \mathcal{L}; \]
that is, (2.18) holds.

It remains to show that
\[ D_H(x^{(\pi)} x^u) \in \mathcal{L} \quad \text{for } u \in \mathbb{B} \quad \text{with } u \neq \omega, u \neq \emptyset. \] (2.19)

To do that, we first show that
\[ D_H(x^u) \in \mathcal{L} \quad \text{for all } u \in \mathbb{B}^0. \] (2.20)

Clearly, \( D_H(x^u) \in \mathcal{L} \) for all \( u \in \mathbb{B}_2 \). Assume that \( u \in \mathbb{B}^0 \) and \( |u| > 2 \). Find \( v, w \in \mathbb{B}^0 \) satisfying \( |v| < |u| \) and \( |w| < |u| \), such that \( x^v x^w = x^u \). Then
\[ D_H(x^u) = [D_H(x_1 x^u), D_H(x_1 x^u)]. \]

Since \( \mathcal{N} \subset \mathcal{L} \), using induction one may easily prove (2.20). When \( |u| = 2 \), let \( u = (k, l) \). Take \( r \in Y_1 \setminus \{k, l\} \). By (2.18),
\[ D_H(x^{(\pi)} x_k x_l) = [D_H(x^{(\pi - \pi_1 \varepsilon_1)} x_k), D_H(x^{(\pi_1 \varepsilon_1)} x_l)] \in \mathcal{L}; \] (2.21)
that is, (2.19) holds in this case. Now suppose \( |u| > 2 \) and \( u \neq \omega \). Find \( v, w \in \mathbb{B} \) with \( |v| = 1 \) such that \( x^v x^w = x^u \). Take \( r \in Y_1 \setminus \{u\} \). Then (2.20) and (2.21) ensure that
\[ D_H(x^{(\pi)} x^u) = [D_H(x^{(\pi)} x_v x^u), D_H(x_v x^u)] \in \mathcal{L}, \]
proving (2.19). The proof is complete. \( \square \)
3. Exceptional derivations

In this section we shall give three series of the so-called *exceptional derivations from* $\mathcal{H}$ into $\mathcal{W}$. As we shall see, in general these exceptional derivations are all outer. We note that these exceptional derivations have no analogs in the setting for the even part of the odd Hamiltonian superalgebra or the special superalgebra (see [3, 7]). Thus, roughly speaking, the even part of Hamiltonian modular Lie superalgebra possesses “more” outer derivations than the even part of the special Lie superalgebra. The other reason we are interested in this phenomenon is that it does not occur in the “super” setting (see [13, Theorem 2.13]). Throughout this section assume that $|\omega| = n$ is even.

Let us define the first series of exceptional derivations from $\mathcal{H}$ into $\mathcal{W}$. Given $q \in \mathbb{N}$ and $i \in Y_0$, define

$$\Phi^{(q)}_i : \mathcal{H} \rightarrow \mathcal{W}, \quad D_H(f) \mapsto \partial^q_i(f)D_H(x^\omega).$$

As $\text{Ker}(D_H) = F 1$, $\Phi^{(q)}_i$ is well defined. Clearly, $\text{zd}(\Phi^{(q)}_i) = n - p^q$. Moreover, we have

**Proposition 3.1.** Let $|\omega| = n$ be even and $i \in Y_0$. Then $\Phi^{(q)}_i \in \text{Der}(\mathcal{H}, \mathcal{W})$.

**Proof.** It is sufficient to verify the following equation for $D_H(x^{(\alpha)}x^u), D_H(x^{(\beta)}x^v) \in \mathcal{H}$:

$$\Phi^{(q)}_i([D_H(x^{(\alpha)}x^u), D_H(x^{(\beta)}x^v)]) = [\Phi^{(q)}_i(D_H(x^{(\alpha)}x^u)), D_H(x^{(\beta)}x^v)] + [D_H(x^{(\alpha)}x^v), \Phi^{(q)}_i(D_H(x^{(\beta)}x^v))]. \quad (3.1)$$

The verification is divided into three parts.

**Case (i):** $u = v = \emptyset$. The left-hand side of (3.1) is as follows:

$$\Phi^{(q)}_i([D_H(x^{(\alpha)}), D_H(x^{(\beta)})]) = \Phi^{(q)}_i(D_H(D_H(x^{(\alpha)})(x^{(\beta)}))) = \partial^q_i(D_H(x^{(\alpha)})(x^{(\beta)}))D_H(x^\omega). \quad (3.2)$$

By (1.1), the right-hand side equals:

$$[x^{(\alpha - p^q\epsilon_i)}D_H(x^\omega), D_H(x^{(\beta)})] + [D_H(x^{(\alpha)}), x^{(\beta - p^q\epsilon_i)}D_H(x^\omega)] = (-D_H(x^{(\beta)})(x^{(\alpha - p^q\epsilon_i)})) + D_H(x^{(\alpha)})(x^{(\beta - p^q\epsilon_i)}))D_H(x^\omega). \quad (3.3)$$

Noticing that $\partial^q_i$ is a derivation, one can compute the coefficient of $D_H(x^\omega)$ in (3.2):

$$\partial^q_i(D_H(x^{(\alpha)})(x^{(\beta)})) = \partial^q_i\left(\sum_{r \in Y_0} \tau(r)x^{(\alpha - \epsilon_r)}\partial_r(x^{(\beta)})\right) = \sum_{r \in Y_0} \left(\tau(r)x^{(\alpha - \epsilon_r - p^q\epsilon_i)}x^{(\beta - \epsilon_r)} + \tau(r)x^{(\alpha - \epsilon_r)}x^{(\beta - \epsilon_r - p^q\epsilon_i)}\right)$$

The coefficient of $D_H(x^\omega)$ in (3.3) is as follows:

$$-D_H(x^{(\beta)})(x^{(\alpha - p^q\epsilon_i)}) + D_H(x^{(\alpha)})(x^{(\beta - p^q\epsilon_i)}) = \sum_{r \in Y_0} \tau(r)x^{(\alpha - \epsilon_r - p^q\epsilon_i)}x^{(\beta - \epsilon_r)} + \sum_{r \in Y_0} \tau(r)x^{(\alpha - \epsilon_r)}x^{(\beta - \epsilon_r - p^q\epsilon_i)} = \sum_{r \in Y_0} \left(\tau(r)x^{(\alpha - \epsilon_r - p^q\epsilon_i)}x^{(\beta - \epsilon_r)} + \tau(r)x^{(\alpha - \epsilon_r)}x^{(\beta - \epsilon_r - p^q\epsilon_i)}\right).$$
Hence (3.1) holds in this case.

Case (ii): \( u \neq \emptyset \) and \( v \neq \emptyset \). Then \(|u| \geq 2, |v| \geq 2\). By the definition of \( \Phi_i^{(q)} \), it is easily seen that the two sides of (3.1) are all zero.

Case (iii): \( u = \emptyset \) \( v \neq \emptyset \). By the definition of \( \Phi_i^{(q)} \), the left-hand side is as follows:

\[
\Phi_i^{(q)}([D_H(x^{(\alpha)}), D_H(x^{(\beta)}x^v)]) = \Phi_i^{(q)}(D_H(D_H(x^{(\alpha)})(x^{(\beta)}x^v))) = \partial_i^{(q)}(D_H(x^{(\alpha)})(x^{(\beta)}x^v))D_H(x^\omega) = 0. \quad (\text{Note } |v| \geq 2)
\]

The right-hand side is as follows:

\[
\begin{align*}
[\Phi_i^{(q)}(D_H(x^{(\alpha)})), D_H(x^{(\beta)}x^v)] + [D_H(x^{(\alpha)}), \Phi_i^{(q)}(D_H(x^{(\beta)}x^v))] & = [\partial_i^{(q)}(x^{(\alpha)}D_H(x^{(\beta)}x^v)), D_H(x^{(\beta)}x^v)] + [D_H(x^{(\alpha)}), \partial_i^{(q)}(x^{(\beta)}x^v)(D_H(x^\omega))] \\
 & = [\partial_i^{(q)}(x^{(\alpha)}D_H(x^{(\beta)}x^v)), D_H(x^{(\beta)}x^v)] \\
 & = -D_H(x^{(\beta)}x^v)(\partial_i^{(q)}(x^{(\alpha)}D_H(x^\omega))) + \partial_i^{(q)}(x^{(\alpha)})([D_H(x^\omega), D_H(x^{(\beta)}x^v)]) \\
 & = -D_H(x^{(\beta)}x^v)(\partial_i^{(q)}(x^{(\alpha)}))D_H(x^\omega) \\
 & = 0.
\end{align*}
\]

The proof is complete.

Now we define the second series of exceptional derivations. We have known that \((\text{ad} \partial_r)^p\) is a derivation of \( \mathcal{H} \). Define for \( i \in Y_0 \) and \( q \in \mathbb{N}, \)

\[
\Theta_i^{(q)} : \mathcal{H} \to \mathcal{W}, \quad D_H(f) \mapsto x^\omega(\text{ad} \partial_r)^p(D_H(f)).
\]

By the definition, \( zd(\Theta_i^{(q)}) = n - p^q \). Clearly, \( \Theta_i^{(q)} \) may be naturally extended to a linear mapping of \( \mathcal{W} \). Note that \( \Theta_i^{(q)}(D_H(f)) = x^\omega D_H(\partial_i^{(q)}(f)) \). Moreover, we have the following

**Proposition 3.2.** Let \( |\omega| = n \) be even and \( i \in Y_0 \). Then \( \Theta_i^{(q)} \in \text{Der}(\mathcal{H}, \mathcal{W}) \).

**Proof.** View \( \Theta_i^{(q)} \) as the linear mapping of \( \mathcal{W} \) and consider the standard basis elements \( x^{(\alpha)}x^u\partial_r \) of \( \mathcal{W} \) satisfying that \( r \neq u \) and that if \( u = \emptyset \) then \( r \in Y_0 \). Since every element of \( D_H(f) \in \mathcal{H} \) must be a linear combination of such standard basis elements, it suffices to show that for such standard basis elements \( x^{(\alpha)}x^u\partial_r \) and \( x^{(\beta)}x^v\partial_s \), the following holds:

\[
\Theta_i^{(q)}([x^{(\alpha)}x^u\partial_r, x^{(\beta)}x^v\partial_s]) = [\Theta_i^{(q)}(x^{(\alpha)}x^u\partial_r), x^{(\beta)}x^v\partial_s] + [x^{(\alpha)}x^u\partial_r, \Theta_i^{(q)}(x^{(\beta)}x^v\partial_s)]. \quad (3.4)
\]

When \( u = v = \phi \), it is easy to see that (3.4) holds. When \( u \neq \emptyset \) and \( v \neq \emptyset \), the two sides of (3.4) vanish. It remains only the case that \( u = \emptyset \) but \( v \neq \emptyset \). Note that \( r \in Y_0 \) in this case. Consequently, the left-hand side of (3.4) vanishes. As \( s \notin v \), the first summand in the right-hand side of (3.4) is zero. Clearly, the second is also zero.

**Remark 3.3.** From the proof of Proposition 3.2, it is easily seen that in general, \( \Theta_i^{(q)} \notin \text{Der}\mathcal{W} \) and \( \Theta_i^{(q)} \notin \text{Der}(\mathcal{S}, \mathcal{W}) \) where \( \mathcal{S} \) denotes the even part of the special superalgebra and \( \Theta_i^{(q)} \) is naturally extended.
Let us consider the third series of exceptional derivations. Define for $i \in Y_0$, 
\[ \Psi^{(i)} : \mathcal{H} \rightarrow \mathcal{W}, \quad \mathcal{D}_H(f) \mapsto \partial_i \partial_v(f) \mathcal{D}_H(x^\omega) \quad \text{for} \quad f \in \mathcal{O}(m, n; \mathbb{L}). \]
As $\ker(\mathcal{D}_H) = \mathbb{F} 1$, the linear mapping $\Psi^{(i)}$ is well defined and $zd(\Psi^{(i)}) = n - 2$. Moreover, we have the following

**Proposition 3.4.** Let $|\omega| = n$ be even and $i \in Y_0$. Then $\Psi^{(i)} \in \text{Der}(\mathcal{H}, \mathcal{W})$.

**Proof.** We want to verify that for $\alpha, \beta \in \Lambda$, $u, v \in \mathbb{B}^0$, the following holds:
\begin{align*}
\Psi^{(i)}([\mathcal{D}_H(x^{(\alpha)} x^u), \mathcal{D}_H(x^{(\beta)} x^v)]) &= \left[ \Psi^{(i)}(\mathcal{D}_H(x^{(\alpha)} x^u)), \mathcal{D}_H(x^{(\beta)} x^v) \right] + [\mathcal{D}_H(x^{(\alpha)} x^u), \Psi^{(i)}(\mathcal{D}_H(x^{(\beta)} x^v))]. \tag{3.5} \end{align*}

**Case (i):** $u \neq 0, v \neq 0$. Since $|u| \geq 2, |v| \geq 2$ in this case, two sides of (3.5) vanish.

**Case (ii):** $u \neq 0, v = 0$. Then the right-hand side of (3.5) is as follows:
\begin{align*}
[\mathcal{D}_H(x^{(\alpha)} x^u), x^{(\beta - \epsilon_i - \epsilon_{\rho'})} \mathcal{D}_H(x^\omega)] &= \mathcal{D}_H(x^{(\alpha)} x^u)(x^{(\beta - \epsilon_i - \epsilon_{\rho'})} \mathcal{D}_H(x^\omega)) + x^{(\beta - \epsilon_i - \epsilon_{\rho'})}[\mathcal{D}_H(x^{(\alpha)} x^u), \mathcal{D}_H(x^\omega)] \\
&= 0.
\end{align*}
The left-hand side is as follows
\begin{align*}
\Psi^{(i)}(\mathcal{D}_H(\mathcal{D}_H(x^{(\alpha)} x^u)(x^{(\beta)}))) &= \Psi^{(i)} \left( \mathcal{D}_H \left( \sum_{r \in Y_0} \tau(r) x^{(\alpha - \epsilon_r)} x^u x^{(\beta - \rho')} \right) \right) = 0.
\end{align*}
This proves (3.5) in this case.

**Case (iii):** $u = v = \phi$. The right-hand side of (3.5) is as follows:
\begin{align*}
x^{(\alpha - \epsilon_i - \epsilon_{\rho'})} \mathcal{D}_H(x^\omega), \quad [\mathcal{D}_H(x^{(\alpha)}), \quad x^{(\beta - \epsilon_i - \epsilon_{\rho'})} \mathcal{D}_H(x^\omega)] &= -\mathcal{D}_H(x^{(\beta)})(x^{(\alpha - \epsilon_i - \epsilon_{\rho'})}) \mathcal{D}_H(x^\omega) + \mathcal{D}_H(x^{(\alpha)})(x^{(\alpha - \epsilon_i - \epsilon_{\rho'})}) \mathcal{D}_H(x^\omega).
\end{align*}
The left-hand side is as follows:
\begin{align*}
\Psi^{(i)}(\mathcal{D}_H(\mathcal{D}_H(x^{(\alpha)})(x^{(\beta)}))) &= \partial_i \partial_v(\mathcal{D}_H(x^{(\alpha)})(x^{(\beta)})) \mathcal{D}_H(x^\omega) \\
&= \partial_i \partial_v \left( \sum_{r \in Y_0} \tau(r) x^{(\alpha - \epsilon_r)} x^{(\beta - \epsilon_{\rho'})} \right) \mathcal{D}_H(x^\omega) \\
&= \partial_i \left( \sum_{r \in Y_0} \tau(r) x^{(\alpha - \epsilon_r - \epsilon_{\rho'})} x^{(\beta - \epsilon_{\rho'})} + x^{(\alpha - \epsilon_r)} x^{(\beta - \epsilon_{\rho'})} \right) \mathcal{D}_H(x^\omega) \\
&= \sum_{r \in Y_0} \tau(r) x^{(\alpha - \epsilon_r - \epsilon_{\rho'})} x^{(\beta - \epsilon_{\rho'})} \mathcal{D}_H(x^\omega) + \sum_{r \in Y_0} \tau(r) x^{(\alpha - \epsilon_r)} x^{(\beta - \epsilon_{\rho'})} \mathcal{D}_H(x^\omega) \\
&= - \left( \sum_{r \in Y_0} \tau(r') x^{(\alpha - \epsilon_r - \epsilon_{\rho'})} x^{(\beta - \epsilon_{\rho'})} \right) \mathcal{D}_H(x^\omega) + \left( \sum_{r \in Y_0} \tau(r) x^{(\alpha - \epsilon_r)} x^{(\beta - \epsilon_{\rho'})} \right) \mathcal{D}_H(x^\omega) \\
&= ( - \mathcal{D}_H(x^{(\beta)})(x^{(\alpha - \epsilon_{\rho'})}) + \mathcal{D}_H(x^{(\alpha)})(x^{(\beta - \epsilon_{\rho'})}) ) \mathcal{D}_H(x^\omega).
\end{align*}
Summarizing, (3.5) holds, completing the proof. \qed
4. $\mathbb{Z}$-homogeneous derivations

In this section, we first determine the derivations from $\mathfrak{H}$ into $\mathcal{W}$ which vanish on the top of $\mathfrak{H}$. To that aim, one needs to investigate the action on the generators of $\mathfrak{H}$ for such a derivation. Recall the generating theorem established in Section 2. We shall consider the set $\mathcal{M}$ and $\mathcal{N}$ separately. Recall

$$G = C_{\mathcal{M}}(H_{-1}) = \text{span}_F \{ x^u \partial_r | r \in Y, u \in \mathbb{R}, p(x^u \partial_r) = 0 \}.$$ 

For simplicity, put

$$E(G) := \oplus_{r \in \mathbb{Z}} G_{2r}, \quad O(G) := \oplus_{r \in \mathbb{Z}} G_{2r+1}.$$ 

We shall frequently use the following simple fact.

**Lemma 4.1.** Suppose $\phi \in \text{Der}(\mathfrak{N}, \mathcal{W})$ satisfies $\phi(\mathfrak{N}_{-1}) = 0$. Then for $D \in \mathfrak{N}$, $\phi([D, \mathfrak{N}_{-1}]) = 0$ if and only if $\phi(D) \in G$.

Let us first consider the elements in $\mathcal{M}$.

**Lemma 4.2.** Let $\phi \in \text{Der}(\mathfrak{N}, \mathcal{W})$ be homogeneous such that $\phi(\mathfrak{N}_{-1} \oplus \mathfrak{N}_{0}) = 0$. Suppose $\text{zd}(\phi) + a$ is odd and $\phi(D_H(\epsilon_i)) = 0$ for all $b < a$, where $a \leq \pi_i$ is a fixed positive integer and $i \in Y_0$. Then the following statements hold.

(i) If $a \equiv 1 \mod p$ then $\phi(D_H(x^{(a\epsilon_i)})) = 0$.

(ii) If $a \equiv 1 \mod p$ and $a - 1$ is not any $p$-power, then $\phi(D_H(x^{(a\epsilon_i)})) = 0$.

(iii) If $a - 1 = p^q$ for some $q \in \mathbb{N}$, then there are $\mu_i^{(q)}, \eta_i^{(q)} \in \mathbb{F}$ such that $\mu_i^{(q)} \eta_i^{(q)} = 0$ and

$$\phi(D_H(x^{(a\epsilon_i)})) = 0.$$ 

**Proof.** Since $\phi(\mathfrak{N}_{-1}) = 0$ and $\text{zd}(\phi) + a$ is odd, by Lemma 4.1, $\phi(D_H(x^{(a\epsilon_i)})) \in O(G)$. Thus one may assume that

$$\phi(D_H(x^{(a\epsilon_i)})) = \sum_{r \in Y_0} f_r \partial_r \quad \text{where} \quad f_r \in \Lambda(n).$$

For arbitrary $j \in Y_0 \setminus \{i, i'\}$, we have $[D_H(x_j x_{j'}), D_H(x^{(a\epsilon_i)})] = 0$. Applying $\phi$ to this equation, one gets

$$[\tau(j)x_{j'} \partial_{j'} + \tau(j')x_j \partial_j, \sum_{r \in Y_0} f_r \partial_r] = 0.$$ 

It follows that $f_j = 0$ for all $j \in Y_0 \setminus \{i, i'\}$ and therefore,

$$\phi(D_H(x^{(a\epsilon_i)})) = f_i \partial_i + f_i' \partial_i'.$$

Similarly, applying $\phi$ to the equation $[D_H(x^{(2\epsilon_i)}), D_H(x^{(a\epsilon_i)})] = 0$, one gets $f_i = 0$. Thus

$$\phi(D_H(x^{(a\epsilon_i)})) = f_i' \partial_{i'}.$$ 

For arbitrary $k, l \in Y_1$ with $k \neq l$, we have

$$0 = [D_H(x_k x_l), \phi(D_H(x^{(a\epsilon_i)}))] = (x_l \partial_k - x_k \partial_l)(f_i') \partial_{i'}.$$
Hence \( x_i \partial_k (f_{i'} - x_k \partial_l (f_{i'}) = 0 \) and therefore, \( x_k \partial_l (f_{i'}) = 0 \). This implies that \( f_{i'} \in \mathbb{F} x^\omega \) or \( f_{i'} \in \mathbb{F} \). Therefore, there is \( \mu_i \in \mathbb{F} \), such that

\[
\phi(D_H(x^{(a_\varepsilon i)})) = \mu_i x^\omega \partial_{i'} \quad \text{or} \quad \mu_i \partial_{i'}.
\]

(4.1)

Suppose \( a - 1 = p^q \) for some \( q \in \mathbb{N} \). Then it is easily seen that (iii) holds. Note that \( [D_H(x_i x_{i'}), D_H(x^{(a_\varepsilon i)})] = \tau(i') a D_H(x^{(a_\varepsilon i)}). \) Applying \( \phi \), we have

\[
\left[ \tau(i) x_{i'} \partial_i + \tau(i') x_i \partial_{i'}, \phi(D_H(x^{(a_\varepsilon i)})) \right] = \tau(i') a \phi(D_H(x^{(a_\varepsilon i)})).
\]

It follows from (4.1) that \( (a - 1) \mu_i = 0 \). Consequently, (i) holds.

To prove (ii), suppose \( a \equiv 1 \pmod{p} \) and \( a - 1 \) is not any \( p \)-power. Write \( a - 1 \) to be the \( p \)-adic form \( a - 1 = \sum_{r=1}^t c_r p^r \) where \( c_t \neq 0 \). Then \( \binom{a_i}{p^r} \neq 0 \pmod{p} \). Note that

\[
[D_H(x^{(p^r \varepsilon_i)} x_{i'}), D_H(x^{(a-p^r+1) \varepsilon_i})] = \tau(i') \binom{a}{p^r} D_H(x^{(a_\varepsilon i)}).
\]

(4.2)

It is clear that \( a - p^t + 1 < a \). Then \( \phi(D_H(x^{(a-p^r+1) \varepsilon_i})) = 0 \). We want to show that \( \phi(D_H(x^{(p^r \varepsilon_i)} x_{i'})) = 0 \). Since \( p^t < a - 1 \), it follows that

\[
\phi(D_H(x^{(p^r \varepsilon_i)} x_{i'})) = \tau(i') [D_H(x^{(2 \varepsilon_i)}), \phi(D_H(x^{(p^r+1) \varepsilon_i}))] = 0.
\]

Thus we obtain from (4.2) that \( \phi(D_H(x^{(a_\varepsilon i)})) = 0 \) that is, (ii) holds.

\[ \Box \]

**Lemma 4.3.** Let \( \phi \in \text{Der}(\mathcal{R}, W) \) be homogeneous such that \( \phi(\mathcal{R}_{-1} \oplus \mathcal{R}_{q}) = 0 \). Suppose \( a \in \mathbb{N} \) and \( i \in Y_0 \) such that \( \phi(D_H(x^{(b_\varepsilon i)})) = 0 \) for all \( b < a \). If \( zd(\phi) + a \) is even, then the following statements holds.

(i) If \( a \not\equiv 0 \pmod{p} \) then \( \phi(D_H(x^{(a_\varepsilon i)})) = 0 \).

(ii) If \( a \equiv 0 \pmod{p} \) and \( p \) is not any \( p \)-power, then \( \phi(D_H(x^{(a_\varepsilon i)})) = 0 \).

(iii) If \( a = p^t \) for some \( q \in \mathbb{N} \), then there is \( \lambda_i(q) \in \mathbb{F} \) such that

\[
(\phi - \lambda_i(q) \Phi_i(q))(D_H(x^{(a_\varepsilon i)})) = 0.
\]

**Proof.** Since \( \phi(\mathcal{R}_{-1}) = 0 \) and \( zd(\phi) + a \) is even, by Lemma 4.1 \( \phi(D_H(x^{(a_\varepsilon i)})) \in E(G) \). Thus we may assume that

\[
\phi(D_H(x^{(a_\varepsilon i)})) = \sum_{r \in Y_1} f_r \partial_{r} \quad \text{where} \quad f_r \in \Lambda(u).
\]

(4.3)

(i) Applying \( \phi \) to the equation

\[
[D_H(x_i x_{i'}), D_H(x^{(a_\varepsilon i)})] = \tau(i') a D_H(x^{(a_\varepsilon i)})
\]

one may obtain \( \phi(D_H(x^{(a_\varepsilon i)})) = 0 \), since \( a \not\equiv 0 \pmod{p} \).

(ii) Write \( a \) to be the \( p \)-adic form \( a = \sum_{r=1}^t c_r p^r \), \( c_t \neq 0 \). Then \( \binom{a}{p^r} \neq 0 \pmod{p} \). Just as in the proof of Lemma 4.2, we have

\[
[D_H(x^{(p^r \varepsilon_i)} x_{i'}), D_H(x^{(a-p^r+1) \varepsilon_i})] = \tau(i') \binom{a}{p^r} D_H(x^{(a_\varepsilon i)}).
\]

(4.4)

Clearly, \( \phi(D_H(x^{(a-p^r+1) \varepsilon_i})) = 0 \), since \( a - p^r + 1 < a \). On the other hand, we have also

\[
[D_H(x^{(2 \varepsilon_i)}), D_H(x^{(a_\varepsilon i)})] = \tau(i') D_H(x^{(a-1) \varepsilon_i} x_{i'}).
\]
It follows from (4.3) and the equation above that \( \phi(D_H(x^{(a-1)\varepsilon_i}x_i')) = 0 \). Then 
\[
\phi(D_H(x^{(p')\varepsilon_i}x_i')) = 0,
\]
since \( p' < a-1 \). Now (ii) follows from (4.4).

(iii) For \( k, l \in Y_1 \) with \( k \neq l \), we have
\[
0 = [D_H(x_kx_l), \phi(D_H(x^{(a\varepsilon_i)}))] = \sum_{r \in Y_1} (x_l \partial_k - x_k \partial_l)(f_r)\partial_r - f_l \partial_k + f_k \partial_l.
\]
(4.5)

It follows that \( (x_l \partial_k - x_k \partial_l)(f_r) = 0 \) for \( k, l \in Y_1 \setminus \{r\} \). This implies that \( x_k \partial_l(f_r) = 0 \) whenever \( k, l \in Y_1 \setminus \{r\}, k \neq l \). From this one may deduce that 
\[
f_r \in \Lambda(n)_{[n-1]} \cup \Lambda(n)_{[n]} \cup \mathbb{F} \quad \text{for all } r \in Y_1.
\]
Clearly, \( f_r \) cannot be nonzero in \( \mathbb{F} \), since \( zd(\phi) + a \) is even. On the other hand, if \( f_r \in \Lambda(n)_{[n]} \), one may obtain \( f_r = 0 \) from (4.3). Summarizing, \( f_r \in \Lambda(n)_{[n-1]} \). This implies that if \( n \) is odd then \( f_r = 0 \) for all \( r \in Y_1 \). Thus we assume that \( n \) is even in the following. From the fact that \( x_k \partial_l(f_r) = 0 \) whenever \( k, l \in Y_1 \setminus \{r\}, k \neq l \), one may deduce that \( f_r \in \mathbb{F}x^{\omega<r>} \). Hence we may assume that 
\[
\phi(D_H(x^{(a\varepsilon_i)})) = \sum_{r \in Y_1} c_r x^{\omega<r>} \partial_r \quad \text{where } c_r \in \mathbb{F}.
\]
(4.6)

From (4.5) we have
\[
-x_k \partial_l(c_k x^{\omega<k>}) \partial_k - c_l x^{\omega<l>} \partial_k + x_l \partial_k(c_l x^{\omega<l>}) \partial_l + c_k x^{\omega<k>} \partial_l = 0.
\]
Furthermore,
\[
x_k \partial_l(c_k x^{\omega<k>}) + c_l x^{\omega<l>} = 0.
\]
Without loss of generality, we may assume that \( k < l \). Then the equation yields that 
\[
(-1)^{l-2}(-1)^{k-1}c_k + c_l = 0;
\]
that is, \( c_k = (-1)^{k+l}c_l \). Recall that \( a = p^q \). Put \( \lambda_i^{(q)} := c_{2m+1} \). Then we obtain from (4.6) that 
\[
\phi(D_H(x^{(a\varepsilon_i)})) = \lambda_i^{(q)} D_H(x^{\omega}).
\]
Therefore,
\[
(\phi - \lambda_i^{(q)} \Phi_i^{(q)})(D_H(x^{(a\varepsilon_i)})) = 0.
\]

The proof is complete. \( \square \)

Now we can reduce the derivations vanishing on the top of \( \mathcal{N} \) to be vanishing on \( \mathcal{M} \).

**Lemma 4.4.** Let \( \phi \in \text{Der}(\mathcal{N}, \mathcal{W}) \). Suppose \( zd(\phi) \) is even and \( \phi(\mathcal{N}_{[-1]} \oplus \mathcal{N}_{[0]}) = 0 \). Then \( \phi(D_H(x^{(a_i\varepsilon_i)})) = 0 \) for all \( i \in Y_0 \) and \( a_i \leq \pi_i \). That is, \( \phi(M) = 0 \).

**Proof.** We proceed by induction on \( a_i \). Assume that the assertion holds for \( a_i - 1 \). Let us consider the case \( a_i \). If \( a_i = p^q \) for some \( q \in \mathbb{N} \), then \( zd(\phi) + a_i \) is odd. By Lemma 4.2(i), \( \phi(D_H(x^{(a_i\varepsilon_i)})) = 0 \). Assume that \( a_i \) is not any \( p \)-power. The discuss is divided into the following two cases (i) \( a_i \equiv 0 \) (mod \( p \)) and (ii) \( a_i \not\equiv 0 \) (mod \( p \)). In the case (i), if \( zd(\phi) + a_i \) is odd, then Lemma 4.2(ii) ensures that \( \phi(D_H(x^{(a_i\varepsilon_i)})) = 0 \); if \( zd(\phi) + a_i \) is even, then Lemma 4.3(ii) ensures that \( \phi(D_H(x^{(a_i\varepsilon_i)})) = 0 \), since \( a_i \) is not any \( p \)-power. In the case (ii), if \( zd(\phi) + a_i \) is even, then Lemma 4.3(i) ensures that \( \phi(D_H(x^{(a_i\varepsilon_i)})) = 0 \); if \( zd(\phi) + a_i \) is odd, then \( a_i \) is odd and there is no \( q \) such that \( a_i - 1 = p^q \) and therefore, \( \phi(D_H(x^{(a_i\varepsilon_i)})) = 0 \) by Lemma 4.2. The proof is complete. \( \square \)
Lemma 4.5. Let $\phi \in \text{Der}(\mathcal{N}, \mathcal{W})$ and zd($\phi$) be odd. Suppose $\phi(\mathcal{N}_{[-1]} \oplus \mathcal{N}_{[q]}) = 0$. Then there are $\lambda_r^{(s_r)}, \mu_r^{(s_r)}, \eta_r^{(s_r)} \in \mathbb{F}$, where $r \in Y_0$ and $1 \leq s_r < t_r$, such that $\mu_r^{(s_r)} \eta_r^{(s_r)} = 0$ and

$$
\left( \phi - \sum_{r \in Y_0} \sum_{s_r = 1}^{t_r - 1} \left( \lambda_r^{(s_r)} \Phi_r^{(s_r)} + \mu_r^{(s_r)} \Theta_r^{(s_r)} + \eta_r^{(s_r)} (\text{ad}_{\partial_r})^{p^{s_r}} \right) \right) (D_H(x^{(a_1, \varepsilon_1)})) = 0
$$

for all $i \in Y_0$, $1 \leq a_i \leq \pi_i$.

Proof. For $1 < a_i < p$, by virtue of Lemmas 4.2(i) and 4.3(i), one may show by induction on $a_i$ that

$$
\phi(D_H(x^{(a_i, \varepsilon_i)})) = 0, \quad 1 < a_i < p, \ i \in Y_0.
$$

Put $\phi^{(0)} := \phi$. Assume inductively that we have constructed a derivation $\phi^{(q)}$ from $\mathcal{N}$ into $\mathcal{W}$ for $q \geq 0$ such that

$$
\phi^{(q)}(D_H(x^{(a_i, \varepsilon_i)})) = 0 \quad \text{for all } i \in Y_0 \text{ and } 1 < a_i < p^{q+1}.
$$

We first consider $\phi^{(q)}(D_H(x^{(p^{q+1}, \varepsilon_i)}))$. Note that zd($\phi$) + $p^{q+1}$ is even. By Lemma 4.3(iii), there is $\lambda^{(q+1)} \in \mathbb{F}$ such that

$$
(\phi^{(q)} - \lambda^{(q+1)} \Phi_r^{(q+1)})(D_H(x^{(p^{q+1}, \varepsilon_i)})) = 0 \quad \text{for } r \in Y_0.
$$

Put $\phi^{(q+1)} := \phi^{(q)} - \sum_{r \in Y_0} \lambda^{(q+1)} \Phi_r^{(q+1)}$. Then

$$
\phi^{(q+1)}(D_H(x^{(p^{q+1}, \varepsilon_i)})) = 0 \quad \text{for all } r \in Y_0.
$$

By Lemma 4.2(iii), there are $\mu_r^{(q+1)}, \eta_r^{(q+1)} \in \mathbb{F}$ such that $\mu_r^{(q+1)} \eta_r^{(q+1)} = 0$ and

$$
(\phi^{(q+1)} - \tau(r) \mu_r^{(q+1)} \Theta_r^{(q+1)} - \tau(r) \eta_r^{(q+1)} (\text{ad}_{\partial_r})^{p^{q+1}})(D_H(x^{((p^{q+1}+1), \varepsilon_i)})) = 0, \ r \in Y_0.
$$

Put $\phi^{(q+1)} := \phi^{(q+1)} - \sum_{r \in Y_0} \tau(r) \mu_r^{(q+1)} \Theta_r^{(q+1)} - \sum_{r \in Y_0} \tau(r) \eta_r^{(q+1)} (\text{ad}_{\partial_r})^{p^{q+1}}$. Then

$$
\phi^{(q+1)}(D_H(x^{((p^{q+1}+1), \varepsilon_i)})) = 0 \quad \text{for all } i \in Y_0.
$$

Suppose $p^{q+1} + 1 < a_i < p^{q+2}$. Then $a_i$ and $a_i - 1$ are not any $p$-power. Thus, by Lemmas 4.2 and 4.3 using induction on $a_i$, one may show that

$$
\phi^{(q+1)}(D_H(x^{(a_i, \varepsilon_i)})) = 0 \quad \text{for all } i \in Y_0 \text{ and } p^{q+1} + 1 < a_i < p^{q+2}.
$$

Summarizing, $\phi^{(q+1)}(D_H(x^{(a_i, \varepsilon_i)})) = 0$ for all $i \in Y_0$ and $1 \leq a_i < p^{q+2}$. Thus we construct inductively a series of derivations

$$
\phi = \phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(k)}, \ldots
$$

Let $k = \max\{t_1, \ldots, t_{2m}\}$. Then $\phi^{(k)}(D_H(x^{(a_i, \varepsilon_i)})) = 0$ for all $i \in Y_0$ and $a_i \leq \pi_i$. Moreover, by the process of construction, $\phi^{(k)}$ coincides with

$$
\phi - \sum_{r \in Y_0} \sum_{s_r = 1}^{t_r - 1} (\lambda_r^{(s_r)} \Phi_r^{(s_r)} + \mu_r^{(s_r)} \Theta_r^{s_r} + \eta_r^{(s_r)} (\text{ad}_{\partial_r})^{p^{s_r}})
$$

on the elements $D_H(x^{(a_i, \varepsilon_i)}$ for all $i \in Y_0$ and $a_i \leq \pi_i$. The proof is complete. □
For convenience, put $\Gamma' := \sum_{r \in Y_1} x_r \partial_r$. For the elements in $\mathcal{N}$, we have:

**Lemma 4.6.** Let $\phi \in \text{Der}(\mathcal{N}, \mathcal{W})$ be homogeneous such that $\phi(\mathcal{M}_{\{-1\}} \oplus \mathcal{N}_{\{0\}}) = 0$. Then there is $\lambda \in \mathbb{F}$ such that $(\phi - \lambda \text{ad}\Gamma')(\mathcal{N}) = 0$.

**Proof.** Recall that $\mathcal{N} = \{D_H(x_i x^u) \mid i \in Y_0, u \in \mathbb{B}_2\}$. Since $\phi(\mathcal{M}_{\{0\}}) = 0$, we may assume that

$$\phi(D_H(x_i x^u)) = \sum_{r \in Y} f_{i,u,r} \partial_r \quad \text{where } f_{i,u,r} \in \Lambda(n), \ i \in Y_0, \ u \in \mathbb{B}_2. \quad (4.7)$$

If $zd(\phi)$ is odd then $zd(\phi(D_H(x_i x^u)))$ is even and therefore, $f_{i,u,r} = 0$ for all $r \in Y_0$. Thus we obtain from (4.7) that

$$\phi(D_H(x_i x^u)) = \sum_{r \in Y_1} f_{i,u,r} \partial_r. \quad (4.8)$$

Note that $[D_H(x_i x^i), D_H(x_i x^u)] = \tau(i')D_H(x_i x^u)$. Applying $\phi$ to this equation, we obtain from (1.11) that $\phi(D_H(x_i x^u)) = 0$.

Now suppose $zd(\phi)$ is even. Then $f_{i,u,r} = 0$ for all $r \in Y_1$ and therefore,

$$\phi(D_H(x_i x^u)) = \sum_{r \in Y_0} f_{i,u,r} \partial_r. \quad (4.9)$$

Obviously,

$$[D_H(x_i^{(2i)}), \phi(D_H(x_i x^u))] = \tau(i)\phi(D_H(x_i x^u)). \quad (4.10)$$

From (4.9) and (4.10) we have

$$\phi(D_H(x_i x^u)) = f_{i,u,i'} \partial_{i'}. \quad (4.11)$$

By our hypothesis that $n \geq 4$, we can find $k, l \in Y_1 \setminus u$ with $k \neq l$. Then

$$[D_H(x_k x_l), D_H(x_i x^u)] = 0. \quad (4.12)$$

Applying $\phi$ to this equation and using (1.11) we obtain that $(x_l \partial_k - x_k \partial_l)(f_{i,u,i'}) = 0$ and therefore, $x_k \partial_l(f_{i,u,i'}) = 0$. This forces $f_{i,u,i'} \in \mathbb{F} \cup \mathbb{F} x^\omega \cup \mathbb{F} x^u$. Let $u = \{r, s\}$. Take $k \in Y_1 \setminus \{r, s\}$. Then $[D_H(x_k x_r), D_H(x_i x_k x_s)] = D_H(x_i x^u)$ and therefore,

$$[D_H(x_k x_r), \phi(D_H(x_i x_k x_s))] = \phi(D_H(x_i x^u)). \quad (4.12)$$

If $f_{i,u,i'} \in \mathbb{F} \cup \mathbb{F} x^\omega$, then it is easily seen from (4.12) that $\phi(D_H(x_i x^u)) = 0$. If $f_{i,u,i'} \in \mathbb{F} x^u$, let $f_{i,u,i'} = \lambda_{i,u,i'} x^u$. Denote $\lambda_{i,u} := \lambda_{i,u,i'}$ in the following. Then we obtain from (4.12) that $\lambda_{i,<k,s>} = \lambda_{i,<r,s>}$. This proves that $\lambda_{i,u}$ is independent of the choice of $u$. We denote $\lambda_i := \lambda_{i,u}$ for $i \in Y_0$. Then $\phi(D_H(x_i x^u)) = \lambda_i x^u \partial_{i'}$ for all $u \in \mathbb{B}_2$. On the other hand, for $i, j \in Y_0$, we have

$$[D_H(x_i x_j), D_H(x_j x^u)] = \tau(j')(1 + \delta_{i,j'})D_H(x_i x^u).$$

Applying $\phi$ to the equation above, one gets

$$\tau(i)\lambda_i = \tau(j)\lambda_j \quad i, j \in Y_0.$$

Let $\lambda := \frac{1}{2}\tau(i)\lambda_i$. Then

$$(\phi - \lambda \text{ad}\Gamma')(D_H(x_i x^u)) = \lambda_i x^u \partial_{i'} - 2\tau(i)\lambda x^u \partial_{i'} = 0.$$
Now we can give an explicit description for the derivations from $\mathcal{R}$ into $\mathcal{W}$ vanishing on the top of $\mathcal{R}$.

**Theorem 4.7.** Let $\phi \in \text{Der}(\mathcal{R}, \mathcal{W})$ be homogeneous and $\phi(\mathcal{R}_{[-1]} \oplus \mathcal{R}_{[0]}) = 0$. Then there are $\lambda$, $\lambda_r^{(s_r)}$, $\mu_r^{(s_r)}$, $\eta_r^{(s_r)} \in \mathbb{F}$, where $r \in Y_0$ and $1 \leq s_r < t_r$, such that $\mu_r^{(s_r)}\eta_r^{(s_r)} = 0$ and

$$\phi = \lambda \text{ad} \Gamma' + \sum_{r=1}^{2m} \sum_{s_r=1}^{t_r-1} (\lambda_r^{(s_r)} \Phi_r^{(s_r)} + \mu_r^{(s_r)} \Theta_r^{(s_r)} + \eta_r^{(s_r)}(\text{ad} \partial_r)^{p_{s_r}}).$$

In particular, if $n$ is odd then $\phi = \lambda \text{ad} \Gamma' + \sum_{r=1}^{2m} \sum_{s_r=1}^{t_r-1} \eta_r^{(s_r)}(\text{ad} \partial_r)^{p_{s_r}}$.

**Proof.** By Lemmas 4.4 and 4.5 there are $\lambda_r^{(s_r)}$, $\mu_r^{(s_r)}$, $\eta_r^{(s_r)} \in \mathbb{F}$ such that

$$\tilde{\phi} := \phi - \sum_{r=1}^{2m} \sum_{s_r=1}^{t_r-1} (\lambda_r^{(s_r)} \Phi_r^{(s_r)} + \mu_r^{(s_r)} \Theta_r^{(s_r)} + \eta_r^{(s_r)}(\text{ad} \partial_r)^{p_{s_r}})$$

vanishes on $\mathcal{M}$. By Lemma 4.6 there is $\lambda \in \mathbb{F}$ such that $\varphi := \tilde{\phi} - \lambda \text{ad} \Gamma'$ vanishes on $\mathcal{N}$. It is easy to see that $\varphi$ vanishes on $\mathcal{M}$ and Theorem 2.7 ensures that $\varphi = 0$. The remaining is clear.

It is our present aim to reduce certain derivations to be vanishing the top of $\mathcal{R}$.

**Lemma 4.8.** (see [7, Proposition 2.1.6]) Let $\mathcal{L}$ be a $\mathbb{Z}$-graded subalgebra of $\mathcal{W}$ such that $\mathcal{L}_{-1} = \mathcal{W}_{-1}$. If $\phi \in \text{Der}_1(\mathcal{L}, \mathcal{W})$ where $t := zd(\phi) \geq 0$, then there exists $E \in \mathcal{W}_t$ such that

$$(\phi - \text{ad}E)(\mathcal{L}_{-1}) = 0.$$

In view of Lemma 4.8 and Theorem 4.7, it suffices to reduce the homogeneous derivations vanishing on $\mathcal{R}_{[-1]}$ to be vanishing on the top of $\mathcal{R}$.

**Lemma 4.9.** Let $\phi \in \text{Der}(\mathcal{R}, \mathcal{W})$ be $\mathbb{Z}$-homogeneous with odd degree such that $\phi(\mathcal{R}_{[-1]}) = 0$. Then there is $D \in \mathcal{W}$ such that $(\phi - \text{ad}D)(\mathcal{R}_{[-1]} \oplus \mathcal{R}_{[0]}) = 0$.

**Proof.** Since $\phi(\mathcal{R}_{[-1]}) = 0$ and $zd(\phi)$ is odd, by Lemma 4.1 $\phi(\mathcal{R}_{[0]}) \in O(\mathcal{G})$. For $1 \leq i \leq m$, one may assume that

$$\phi(D_H(x_i x_i')) = \sum_{r \in Y_0} f_{i,r} \partial_r \quad \text{where } f_{i,r} \in \Lambda(n). \quad (4.13)$$

Let $1 \leq i \neq j \leq m$. Then $[D_H(x_i x_i'), D_H(x_j x_j')] = 0$ and therefore,

$$[\phi(D_H(x_i x_i')), D_H(x_j x_j')] = [\phi(D_H(x_j x_j')), D_H(x_i x_i')]. \quad (4.14)$$

One may easily deduce from (4.13) and (4.14) that $f_{ij} = f_{ji} = 0$. Thus,

$$\phi(D_H(x_i x_i')) = f_{ii} \partial_i + f_{i'i'} \partial_{i'} \quad \text{for } 1 \leq i \leq m.$$

Put $\psi := \phi - \text{ad}(\sum_{r=1}^{m} (f_{r'r'} \partial_r - f_{rr} \partial_r))$. Then $\psi(D_H(x_i x_i')) = 0$ for $i \in Y_0$. We propose to show that

$$\psi(D_H(x_i x_j)) = 0 \quad \text{for all } i, j \in Y_0.$$
First, we consider the case \( j \neq i, i' \). Clearly, \([D_H(x_{i'i'}), D_H(x_{ij})] = \tau(i')D_H(x_{ij})\). Applying \( \psi \) to this equation, we have
\[
[D_H(x_{i'i'}), \psi(D_H(x_{ij}))] = \tau(i')\psi(D_H(x_{ij})).
\]
Similarly,
\[
[D_H(x_{j'j'}), \psi(D_H(x_{ij}))] = \tau(j')\psi(D_H(x_{ij})).
\]
Noticing \( \psi(D_H(x_{ij})) \in O(\mathcal{G}) \) and \( j \neq i, i' \), one gets from the above two equations that \( \psi(D_H(x_{ij})) = 0 \).

Next, we consider \( \psi(D_H(x_{(2e_i)})) \). Clearly,
\[
[D_H(x_{i'i'}), D_H(x_{(2e_i)})] = 2\tau(i')D_H(x_{(2e_i)}).
\]
Assume that \( \psi(D_H(x_{(2e_i)})) = \sum_{r \in Y_0} g_{ir}\partial_r, \ g_{ir} \in \mathbb{F} \). Applying \( \psi \) to the equation above, we have \( g_{ir} = 0 \) for \( r \neq i, i' \) and \( 3\tau(i')g_{ii} = 0, \tau(i)g_{ii} = 0 \). Thus \( \psi(D_H(x_{(2e_i)})) = 0 \).

For \( k, l \in Y_1 \), applying \( \psi \) to the equation \([D_H(x_{i'i'}), D_H(x_{k'j})] = 0 \) for \( i \in Y_0 \), one may easily show that \( \psi(D_H(x_{k'j})) = 0 \). Summarizing, we have shown that \( \psi(\mathfrak{M}_{[0]}) = 0 \) and then \( \psi(\mathfrak{M}_{[-1]} \oplus \mathfrak{M}_{[0]}) = 0 \). \( \square \)

Put \( \text{Der}^-(\mathfrak{M}, \mathcal{W}) := \oplus_{r \leq -1} \text{Der}^r(\mathfrak{M}, \mathcal{W}) \).

**Theorem 4.10.** If \( n \) is even, then
\[
\text{Der}^-(\mathfrak{M}, \mathcal{W}) = \text{ad} \mathcal{W}_{[1]} + \sum_{1 \leq r \leq 2m} \sum_{1 \leq s_r \leq t_r - 1} \sum_{n - p^r < 0} (\mathbb{F} \Phi_r^{(s_r)} + \mathbb{F} \Theta_r^{s_r}) + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\text{ad} \partial_r)^{p^{sr}}.
\]
If \( n \) is odd, then \( \text{Der}^-(\mathfrak{M}, \mathcal{W}) = \text{ad} \mathcal{W}_{[1]} + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\text{ad} \partial_r)^{p^{sr}}. \)

**Proof.** Suppose \( n \) is even. Let \( \phi \in \text{Der}(\mathfrak{M}, \mathcal{W}) \) be homogeneous. If \( zd(\phi) \leq -2 \), then Theorem 4.7 shows that
\[
\phi \in \sum_{1 \leq r \leq 2m} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\Phi_r^{(s_r)} + \Phi_r^{s_r}) + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\text{ad} \partial_r)^{p^{sr}}.
\]
Now assume that \( zd(\phi) = -1 \). By Lemma 4.9, there is \( D \in \mathcal{W} \) such that \((\phi - \text{ad}D)(\mathfrak{M}_{[-1]} \oplus \mathfrak{M}_{[0]}) = 0 \). Again Theorem 4.7 entails that
\[
\phi \in \text{ad} \mathcal{W} + \mathbb{F} \text{ad} \mathcal{W}' + \sum_{1 \leq r \leq 2m} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\Phi_r^{(s_r)} + \Phi_r^{s_r}) + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\text{ad} \partial_r)^{p^{sr}}.
\]
Since \( \phi \) is \( Z \)-degree \(-1 \), one may easily obtain the desired result. In the case that \( n \) is odd, the argument is similar. \( \square \)

As an direct application of Theorem 4.7 and Lemma 4.9 we can determine all the derivations of odd \( Z \)-degree:

**Theorem 4.11.** If \( n \) is even, then
\[
\sum_{k \text{ odd}} \text{Der}^k(\mathfrak{M}, \mathcal{W}) = \sum_{i \text{ odd}} \text{ad} \mathcal{W}_{[i]} + \sum_{1 \leq r \leq 2m} \sum_{1 \leq s_r \leq t_r - 1} (\mathbb{F} \Phi_r^{(s_r)} + \mathbb{F} \Theta_r^{s_r}) + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\text{ad} \partial_r)^{p^{sr}}.
\]
If \( n \) is odd, then
\[
\sum_{k \text{ odd}} \text{Der}^k(\mathfrak{M}, \mathcal{W}) = \sum_{i \text{ odd}} \text{ad} \mathcal{W}_{[i]} + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r - 1} \mathbb{F} (\text{ad} \partial_r)^{p^{sr}}.
\]
Let us consider the derivations of even $\mathbb{Z}$-degree.

**Proposition 4.12.** Let $\phi \in \text{Der}(\mathfrak{H}, \mathcal{W})$ be of even $\mathbb{Z}$-degree such that $\phi(\mathfrak{H}_{-1}) = 0$. Then there are $\lambda_r \in \mathbb{F} \ (1 \leq r \leq m)$ such that

$$\left( \phi - \sum_{1 \leq r \leq m} \lambda_r \Psi(r) \right)(D_H(x_i x_j)) = 0 \quad \text{for all } i, j \in Y_0.$$  

*Proof.* Since $zd(\phi)$ is even and $\phi(\mathfrak{H}_{-1}) = 0$, Lemma 4.1 ensures that $\phi(\mathfrak{H}_0) \subset E(\mathcal{G})$. Thus one may assume that for $1 \leq i \leq m$,

$$\phi(D_H(x_i x_{ir})) = \sum_{r \in Y_1} f_{ir} \partial_r \quad \text{where } f_{ir} \in \Lambda(n).$$

Applying $\phi$ to $[D_H(x_k x_l), D_H(x_i x_{ir})] = 0$ for $k, l \in Y_1$, we have

$$[D_H(x_k x_l), \phi(D_H(x_i x_{ir}))] = 0$$

since $\phi(D_H(x_k x_l)) \in E(\mathcal{G})$. Thus, $[x_l \partial_k - x_k \partial_l; \sum_{i \in Y_1} f_{ir} \partial_r] = 0$ and therefore,

$$\sum_{r \in Y_1} (x_l \partial_k - x_k \partial_l)(f_{ir}) \partial_r - (f_{il} \partial_k + f_{ik} \partial_l) = 0. \quad (4.15)$$

This implies that

$$\sum (x_l \partial_k - x_k \partial_l)(f_{ir}) = 0 \quad \text{whenever } k, l \neq r; \quad (4.16)$$

$$\quad (x_l \partial_k - x_k \partial_l)(f_{ik}) - f_{il} = 0. \quad (4.17)$$

By (4.16), one may deduce that $f_{ir} \in \mathbb{F} x_r \cup \mathbb{F} x^u \cup \mathbb{F} x^{\omega_--<r>}$. Assume that $|\omega| = n$ is odd and $0 \neq f_{ir} \in \mathbb{F} x^u$. Then one may reach a contradiction from (4.15). It remains to consider the following two cases, since $\phi$ is homogeneous.

**Case (i):** $f_{ir} \in \mathbb{F} x_r$ for every $r \in Y_1$. Assume that

$$\phi(D_H(x_i x_{ir})) = \sum_{r \in Y_1} \mu_{ir} x_r \partial_r, \quad \mu_{ir} \in \mathbb{F}.$$  

From (4.17) we have $\mu_{ir} = \mu_{ts}$ for all $r, s \in Y_1$. Denote $\mu_i := \mu_{ir}$ for $r \in Y_1$. Thus,

$$\phi(D_H(x_i x_{ir})) = \mu_i \Gamma', \quad r \in Y_0.$$  

Recall our assumption that $n \geq 4$. Let $u \in \mathbb{B}_4$. Clearly, $\phi(D_H(x^u)) \in E(\mathcal{G})$. Applying $\phi$ to the equation $[D_H(x_i x_{ir}), D_H(x^u)] = 0$, we have

$$2\mu_i D_H(x^u) = [\mu_i \Gamma', D_H(x^u)] = [\phi(D_H(x_i x_{ir})), D_H(x^u)] = 0.$$  

Consequently, $\mu_i = 0$. Hence $\phi(D_H(x_i x_{ir})) = 0$ for $i \in Y_0$. For $i, j \in Y_0$ with $j \neq i'$, we have $[D_H(x_i x_{i'}), D_H(x_j)] = \tau(i')(1 + \delta_{ij})D_H(x_i x_j)$. It follows that $\phi(D_H(x_i x_j)) = 0$ for $i, j \in Y_0$ with $j \neq i'$. Hence

$$\phi(D_H(x_i x_j)) = 0 \quad \text{for } i, j \in Y_0.$$
Corollary 4.16. If \( \Gamma \) is odd, then \( \phi(D_H(x, x')) = \sum_{r \in Y_1} \lambda_{ir} x^{(r-i)} \partial_r \), \( \lambda_{ir} \in \mathbb{F} \).

For \( k, l \in Y_1 \) with \( k \neq l \), we have \( [\phi(D_H(x, x')), D_H(x_k x_l)] = 0 \). A direct computation shows that \((-1)^k \lambda_{ik} = (-1)^l \lambda_{il} \). Denote \( \lambda_i := (-1)^k \lambda_{ik} \) for \( k \in Y_1 \). Then

\[
\phi(D_H(x, x')) = \lambda_i D_H(x^r).
\]

Put \( \psi := \phi - \sum_{1 \leq r \leq m} \lambda_r \psi^{(r)} \). Then \( \psi(D_H(x, x')) = 0 \). Just as in Case (i), we have \( \psi(D_H(x, x')) = 0 \) for all \( i, j \in Y_0 \). The proof is complete. \( \square \)

 Remark 4.13. In view of Lemma 4.8, Proposition 4.12, Theorems 4.7, 4.10 and 4.11, in order to determine the derivation space of \( \mathcal{H} \) to \( \mathcal{W} \), it suffices to reduce every even \( \mathbb{Z} \)-degree derivation in \( \text{Der}(\mathcal{H}, \mathcal{W}) \) vanishing on \( \mathcal{H}[1] \) to be the one vanishing on \( \{D_H(x_k x_l) \mid k, l \in Y_1 \} \) modulo a suitable known derivation (such as a linear combination of an inner derivation and some exceptional derivations). However, in contrast to the situation of \( \mathcal{H} \) or \( \mathcal{W} \) (see [7, 17]), it seems that we must find a new approach other than the method of canonical torus used there. We believe that this problem will be settled in future.

Using Theorems 4.4, 4.7, 4.9, 4.10 and 4.11, we obtain the corresponding results on the even part \( \mathcal{H} \) of the finite-dimensional Hamiltonian superalgebra:

Corollary 4.14. Let \( \phi \in \text{Der}(\mathcal{H}, \mathcal{W}) \) and \( \phi(\mathcal{H}[-1] \oplus \mathcal{H}[0]) = 0 \). Then there are \( \lambda, \mu, \lambda_r(\psi_r), \mu_r(\psi_r), \eta_r(\psi_r) \in \mathbb{F} \), where \( r \in Y_0 \) and \( 1 \leq s_r < t_r \), such that \( \mu_r(\psi_r), \eta_r(\psi_r) = 0 \) and

\[
\phi = \Gamma + \mu(\text{ad} \Gamma) + \sum_{r=1}^{2m} \sum_{s_r=1}^{t_r-1} (\lambda_r(\psi_r) \Phi_1(\psi_r) + \mu_r(\psi_r) \Theta_1 + \eta_r(\psi_r) (\text{ad} \partial_r)^{p^{(s_r)}}).
\]

Corollary 4.15. If \( n \) is even, then

\[
\text{Der}^{-}(\mathcal{H}, \mathcal{W}) = \text{ad} \mathcal{W}[1] + \mathbb{F} \delta_{[\pi]} | \omega | \Gamma_1 + \sum_{1 \leq r \leq 2m} \sum_{1 \leq s_r \leq t_r-1} \sum_{n=p^{(s_r)}<0} (\mathbb{F} \Phi_1(\psi_r) + \mathbb{F} \Theta_1) + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r-1} \sum_{n=p^{(s_r)}<0} \mathbb{F} (\text{ad} \partial_r)^{p^{(s_r)}},
\]

where \( \Gamma_1 \) is defined as in Remark 2.6. If \( n \) is odd, then

\[
\text{Der}^{-}(\mathcal{H}, \mathcal{W}) = \text{ad} \mathcal{W}[1] + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r-1} \mathbb{F} (\text{ad} \partial_r)^{p^{(s_r)}}.
\]

Corollary 4.16. If \( n \) is even, then

\[
\sum_{k \text{ odd}} \text{Der}_{[k]}(\mathcal{H}, \mathcal{W}) = \sum_{i \text{ odd}} \text{ad} \mathcal{W}[i] + \sum_{1 \leq r \leq 2m} \sum_{1 \leq s_r \leq t_r-1} (\mathbb{F} \Phi_1(\psi_r) + \mathbb{F} \Theta_1) + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r-1} \sum_{n=p^{(s_r)}<0} \mathbb{F} (\text{ad} \partial_r)^{p^{(s_r)}}.
\]

If \( n \) is odd, then

\[
\sum_{k \text{ odd}} \text{Der}_{[k]}(\mathcal{H}, \mathcal{W}) = \sum_{i \text{ odd}} \text{ad} \mathcal{W}[i] + \sum_{r \in Y_0} \sum_{1 \leq s_r \leq t_r-1} \mathbb{F} (\text{ad} \partial_r)^{p^{(s_r)}}.
\]
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