QCD Pomeron as a soliton wave

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Abstract
I review a recent progress in understanding of QCD Pomeron and its relation to exactly solvable models.

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QCD POMERON AS A SOLITON WAVE

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I review a recent progress in understanding of QCD Pomeron and its relation to exactly solvable models.

1. Understanding of the mechanism responsible for the rise of the structure function of deeply inelastic scattering (DIS) at small $x$ still remains a challenge for QCD. The structure function measures the distribution density of partons inside the proton carrying the fraction $x$ of the proton momentum and having the transverse size $\sim 1/Q$. At intermediate $x$ and large $Q^2$, the density of partons is small, their interaction is weak being proportional to $\alpha_s(Q^2)$ and the proton can be thought of as a dilute system of quasifree partons whose distribution density is governed by the DGLAP evolution equation. The situation is changed however at small $x$. The rise of the structure function as $x \to 0$ indicates that the density of partons increases and although the interaction between partons is still weak at large $Q^2$ we are not allowed to neglect multiparton correlations anymore.

Thus, at small $x$ we enter into a new regime of QCD, well-known since a long time as a Regge asymptotic limit, in which one has to deal with the dynamics of strongly correlated system of partons. It is widely believed that in the Regge limit quarks and gluons should form a new collective excitations, Reggeons and Pomerons, and QCD has to be replaced by an effective Reggeon field theory (with dual models, QCD string etc. among potential candidates). It remains unclear however what are the critical values of $Q^2$ and small $x$ at which the Regge dynamics will take over under the DGLAP evolution and what is the origin of QCD Pomeron in DIS.

In perturbative QCD approach to the Pomeron for the sake of simplicity, we replace the nonperturbative hadronic states by perturbative onium states built from two heavy quarks with mass $M$. In this case, the hadron-hadron scattering amplitude $A(s,t)$ (and the structure function of DIS, $\sim \text{Im } A(s = Q^2/x, t = 0)/s$, in particular) can be calculated in the Regge limit, $s \gg -t, M^2$, as a sum of Feynman diagrams describing the multi-gluon exchanges in the $t$–channel. The result of calculation of $A(s, t)$ in the leading logarithmic approximation (LLA) was interpreted as an emergence of the perturbative Reggeon. Although the Reggeon is built from an infinite number of “bare” gluons it behaves as a point-like particle with gluon quantum numbers. In perturbative Regge limit, the hadrons scatter each other by exchanging Reggeons and their interaction is described by an effective $S$–matrix. The Reggeons propagate in the $t$–channel between two hadrons and interacting with each other they change their 2-dim transverse momenta, $k_\perp$, but preserve the strong ordering of the longitudinal momenta, $k_\parallel$. As a result the Reggeon rapidity $y = \ln \frac{k_\parallel}{k_\perp}$ can be interpreted as a “time” in the $t$–channel and evolution of the system of interacting Reggeons in the $t$–channel is governed by the effective $(2 + 1)$–dim Reggeon $S$–matrix whose exact expression is unknown yet. In what follows we will evaluate the scattering amplitudes in the generalized LLA. In this approximation one preserves unitarity of the $S$–matrix in the direct channels but not in the subchannels.

2. In the generalized LLA the hadron-hadron scattering amplitude is given by the sum of effective Reggeon ladder diagrams, in which an arbitrary number of Reggeons propagate in the $t$–channel between two hadrons. The interaction between Reggeons is elastical and pair-wise. The number of Reggeons in the $t$–channel, $N$, is conserved and for given $N$ the scattering amplitude satisfies the BKP equation. The solutions of this Bethe-Salpeter like equation define the color-singlet compound states built from $N$ interacting Reggeons, perturbative QCD Pomerons and Odderon. Their contribution to the scattering amplitude takes the standard Regge form

$$A(s, t) = i s \sum_{N=2}^{\infty} \alpha_s^{N-2} \sum_{\{a\}} \sum_{\{b\}} \sum_{\{c\}} \langle \{a\} \rangle \langle \{b\} \rangle \langle \{c\} \rangle \pi_{A \rightarrow A}^{(a)} \beta_{N \rightarrow B}^{(b)}(t) s^{E_{N}(s)}$$
where indices $A$ and $B$ refer to the scattered hadrons. Here, the energy of the $N$ Reggeon compound state, $E_{N,\{q\}}$, is defined as an eigenvalue of the $N$ Reggeon Hamiltonian, $\mathcal{H}_N$, acting on the 2-dim transverse Reggeon momenta

$$\mathcal{H}_N|\chi_{N,\{q\}}\rangle = E_{N,\{q\}}|\chi_{N,\{q\}}\rangle \quad (1)$$

with $\{q\}$ being some set of quantum numbers parameterizing all possible solutions. The residue functions $\beta_{N\to A(B)} = \langle A(B)|\chi_{N,\{q\}}\rangle$ measure the coupling of the $N$ Reggeon state to the hadronic states. For given $N$ the scattering amplitude $A(s,t)$ gets a leading contribution from the Reggeon states with the maximal energy

$$\alpha_N - 1 = \max_{\{q\}} E_{N,\{q\}} \quad (2)$$

and $\alpha_N$ can be interpreted as an intercept of a Regge trajectory. Its character (Regge cut or pole) depends on the distribution density of the energy levels $E_{N,\{q\}}$ close to $\alpha_N - 1$.

Being rewritten in the configuration space, the Schrödinger equation (1) describes the dynamics of $N$ pair-wise interacting Reggeons on the 2-dim plane of impact parameters $b = (x,y)$ with the effective QCD Hamiltonian $\mathcal{H}_N$ having the following remarkable properties in the multi-color limit, $N_c \to \infty$ and $\alpha_N N_c = $ fixed. Firstly, the dynamics of Reggeons in holomorphic, $z = x + iy$, and antiholomorphic, $\bar{z} = x - iy$, directions turns out to be independent on each other and $\mathcal{H}_N$ splits into the sum of mutually commuting holomorphic and antiholomorphic 1-dim hamiltonians

$$\mathcal{H}_N = \frac{\alpha_N N_c}{4\pi} (H_N + \overline{H_N}) ,$$

where $H_N$ and $\overline{H_N}$ describe the nearest-neighbour interaction between $N$ Reggeons on the line with (anti)holomorphic coordinates $z_k$ and $\bar{z}_k$ ($k = 1,\ldots,N$) and periodic boundary conditions. Secondly, the operator $H_N$ (and $\overline{H_N}$) was identified as a hamiltonian of completely integrable 1-dim XXX Heisenberg magnet for a noncompact $SL(2,\mathbb{C})$ spin $s = 0$ and with the number of sites equal to the number of Reggeons $N$. As a result, the system of $N$ Reggeons contains the family of $N - 1$ mutually commuting holomorphic conserved charges

$$q_k = \sum_{N \geq j_1 \ldots j_k \geq 1} i^k \xi_{j_1 j_2 \ldots j_k j_1} \partial_{j_1} \partial_{j_2} \ldots \partial_{j_k}$$

with $z_{jk} = z_j - z_k$ and their eigenvalues together with the corresponding antiholomorphic eigenvalues form the set of quantum numbers of $N$ Reggeon compound state. Finally, the spectrum of the $N$ Reggeon states is defined as

$$E_{N,\{q\}} = \frac{\alpha_N N_c}{4\pi} (\varepsilon_{N,\{q\}} + \bar{\varepsilon}_{N,\{q\}}) , \quad \chi_{N,\{q\}}(z,\bar{z}) = \varphi_{N,\{q\}}(z) \overline{\varphi_{N,\{q\}}(\bar{z})} ,$$

where $\varepsilon_{N,\{q\}}$ and $\varphi_{N,\{q\}}(z)$ are the (holomorphic) energy and the wave function of the Heisenberg magnet. The intercept (2) can be identified as a ground state energy of the XXX Heisenberg magnet with $N$ sites.

3. To find the explicit expression for $E_{N,\{q\}}$ from (3) one has to derive the quantization conditions for $q_2, \ldots, q_N$ and establish the dependence of $\varepsilon_{N,\{q\}}$ on their eigenvalues. This can be done by using the generalized Bethe Ansatz developed in [5] and based on the separation of variables [4]. The operators $q_k$ act on holomorphic coordinates of $N$ Reggeons and their diagonalization is reduced to solving of a complicated system of $N$ coupled Schrödinger equations for eigenvalues of $q_k$. Instead of dealing with this system we perform a unitary transformation, $z_k \to x_k = U^\dagger z_k U$, in order to replace the original set of Reggeon coordinates $z_k$ by a new set of separated variables $x_k$ in terms of which the same system of equations decouples into $N$ independent Schrödinger equations and the Reggeon wave function takes the following factorized form in new coordinates [6]

$$\varphi_{N,\{q\}}(x_1,\ldots,x_N) = Q(x_1)\ldots Q(x_{N-1}) e^{iPx_N} ,$$

where $P$ is the total (holomorphic) momentum of the Reggeon state, $x_N = \frac{i}{P} \sum_k z_k$ is the center-of-mass coordinate and the function $Q(x)$ satisfies the Baxter equation [6]

$$x^{-N} \Lambda(x) Q(x) = Q(x+i) + Q(x-i) \quad (5)$$

where $N$ is the number of Reggeons inside the compound state, $q_k$ are the corresponding quantum numbers and $\Lambda(x) = 2x^N - q_2 x^{N-2} + \ldots + q_N$. Having solved the Baxter equation one can obtain the wave function of the $N$ Reggeon compound state (4) and calculate its holomorphic energy using the relation

$$\varepsilon_{N,\{q\}} = \left. \frac{d}{dx} \ln Q(x-i) \right|_{x=0} .$$
This expression determines the dependence of the energy on the quantum numbers \( q_k \). The Reggeon wave function belongs to the principal series representation of the \( SL(2,\mathbb{C}) \) group and the conserved charges \( q_k \) can be interpreted as higher Casimir operators. In particular, \( q_2 = -h(h-1) \) is the quadratic Casimir. Its eigenvalue \( h \) takes the following quantized values

\[
h = \frac{1 + m}{2} + i\nu, \quad m = \mathbb{Z}, \quad \nu = \mathbb{R},
\]

which define the conformal weight of the \( N \) Reggeon state. Here, integer \( m \) is the Lorentz spin of the Reggeon state \( \chi_{N,q} \), corresponding to the rotations in the 2-dimensional impact parameter space. The quantization conditions for the remaining charges \( q_3, ..., q_N \) are much more involved.

To find the solution to the Baxter equation (3) one has to specify the appropriate boundary conditions on the function \( Q(x) \). Their general form was not found yet (for recent progress see Ref. 9), except of the subclass of polynomial solutions of the Baxter equation corresponding to the special values of quantized conformal weight (3), \( h = \mathbb{Z}_+ \) and \( h \geq N \), and leading to the following expression

\[
Q(x) = x^N \prod_{j=1}^{h-N} (x - \lambda_j),
\]

where roots \( \{\lambda_j\} \) satisfy the Bethe equations for the XXX magnet of spin \( s = 0 \). Using polynomial solutions one can calculate the spectrum of \( N \) Reggeon states and then analytically continue the resulting expressions to arbitrary quantized values of the conformal weight.

4. The study of the polynomial solutions reveals the following interesting properties of the Baxter equation (3). For given integer \( h \geq N \), the space of polynomial solutions is finite-dimensional. The possible values of roots \( \lambda_k \), as well as the values of quantized \( q_k \) and the energy \( \varepsilon_N \), turn out to be real and simple

\[
\text{Im} \lambda_j = \text{Im} q_k = \text{Im} \varepsilon_N = 0
\]

and they can be parameterized by the set of integers \( \{n\} = n_1, ..., n_{N-2} \)

\[
q_k = q_k(h; \{n\}), \quad \varepsilon_N = \varepsilon_N(h; \{n\})
\]

such that \( n_1, ..., n_{N-2} \geq 0 \) and \( \sum_{k=1}^{N-2} n_k \leq h - N \). As an example, the quantized values of \( q_3 \) and holomorphic energy \( \varepsilon_3 \) for \( N = 3 \) Reggeon compound states obtained from numerical solutions of the Baxter equation for integer \( h \) are shown by dots in Figs.1 and 2, respectively. The Baxter equation has the form of a discretized 1-dim Schrödinger equation and one can apply the WKB expansion to find its asymptotic solution as

\[
Q(x) = \exp(iS_{\text{WKB}}(x)), \quad S_{\text{WKB}} = S_0 + S_1 + ...
\]

The leading term \( S_0(x) \) defines the semiclassical Reggeon dynamics in the collective coordinates \( x_k \). To describe the classical trajectories of \( N \) Reggeons it is convenient to introduce the complex curve \( w \)

\[
\Gamma_N : \quad \omega + \frac{1}{\omega} = x^N \Lambda(x)
\]

with \( \Lambda(x) \) defined in (3), which is a single-valued function of complex \( x \) on the hyperelliptic Riemann surface obtained by gluing together two...
sheets of the complex $x-$plane along the cuts running between the branch points $\sigma_j$ defined as

$$\sigma_j^{-N} \Lambda(\sigma_j) = \pm 2.$$ 

The genus of the Riemann surface $\Gamma_N$, $g = N - 2$, depends on the number of Reggeons inside the compound state. Then, the Reggeon momentum in the separated coordinates is given by $p = \ln|\omega(x)|$ and the action $S_0$ can be obtained as an integral of a meromorphic differential on $\Gamma_N$

$$S_0 = \int^Q dx \ln \omega \cong - \int^Q x \frac{d\omega}{\omega}$$

with $Q = (x, \pm)$ being the point on the Riemann surface belonging to either upper or lower sheet of $\Gamma_N$. In the center-of-mass frame, $x_N = 0$, the classical motion of Reggeons corresponds to the points on $\Gamma_N$ with real coordinates and momenta, $(x, p = \ln|\omega|)$. These points belong to the $N - 1$ cycles $\alpha_j$ on $\Gamma_N$ which surround the cuts running between the branch points $[\sigma_{2j-1}, \sigma_{2j}]$ and forming $N-1$ compact intervals on the real axis. The branch points become the turning points of the classical trajectories.

The phase space of $N$ Reggeons is given by the direct product of the cycles $\alpha_j$ on the Riemann surface $\Gamma_N$ times the center-of-mass motion. The set of points $Q_1, ..., Q_{N-1}$ situated one each on the $\alpha-$cycles corresponds to the real values of the Reggeon coordinates $(x_j, p_j)$ and provides the coordinates on the level surface $q_k = \text{const}$. The conserved charges $q_k$ of the $N$ Reggeon state play the role of hamiltonians generating the hamiltonian flows of Reggeons on $\Gamma_N$ in “times” $\tau_k$. The corresponding evolution equations for the Reggeon coordinates have the form

$$d\tau_k = \sum_{j=1}^{N-1} dx_j \frac{x_j^{N-k}}{\Lambda^2(x_j) - 4x_j^{2N}}.$$ 

They are similar to the soliton equations of the KP/Toda hierarchy and their solution defines the Reggeon soliton wave in 2-dim plane of the impact parameters $(z, \bar{z})$. The quantum numbers $q_k$ enter as parameters into the $N$ Reggeon soliton waves and their possible values are constrained by the Bohr-Sommerfeld quantization conditions

$$\oint_{\alpha_k} dS_{\text{WKB}} = 2\pi n_k$$

with integers $n_k$ defined in (7). Their solutions define the $N-2$ parametric families of curves (8), which can be interpreted as corresponding to the Whitham deformation of the Reggeon soliton waves in a “slow” time $h$, the conformal weight. For large $h$ one can develop the asymptotic expansion of $q_k$ and $\varepsilon_N$ in inverse powers of the conformal weight

$$q_k = \frac{1}{h^k} \sum_{l=0}^{\infty} q^{(i)}_k (\{n\}) h^{-l},$$

$$\varepsilon_N = -2N\ln h + \sum_{l=0}^{\infty} \varepsilon^{(i)}_N (\{n\}) h^{-l},$$

where $k = 3, ..., N$ and the coefficients $q^{(i)}_k$ and $\varepsilon^{(i)}_N$ depend on the integers $n_k$. For $N = 2$ Reggeon state, the BFKL Pomeron, all coefficients are known exactly. For $N = 3$ Reggeon states $q_3$ and $\varepsilon_3$ were calculated up to $O(h^{-2})$ order \cite{8} and the results are shown by dotted lines in Figs.1 and 2. The asymptotic approximation to the intercept of the $N = 3$ Reggeon state, perturbative Odderon, can be obtained from (8) as

$$\alpha^{\text{app}}_3 = 1 + \frac{\alpha N_c}{\pi} 2.4131$$

and it is smaller than the intercept of the BFKL Pomeron \cite{9}, $\alpha_2 = 1 + \frac{\alpha N_c}{\pi} 4\ln2$.

References

1. V.N. Gribov, Sov. Phys. JETP 26, 414 (1968); Nucl. Phys. B 106, 189 (1976).
2. E.A. Kuraev, L.N. Lipatov and V.S. Fadin, Sov. Phys. JETP 44, 443 (1976); 45, 199 (1977). Ya.Ya. Balitsky and L.N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978).
3. J. Bartels, Nucl. Phys. B 175, 365 (1980); J. Kwiecinski and M. Praszalowicz, Phys. Lett. B 94, 413 (1980).
4. L.N. Lipatov, Phys. Lett. B 251, 284 (1990); 309, 394 (1993).
5. L.D. Faddeev, G.P. Korchemsky and L.N. Lipatov, in Proceedings of the 27th ICHEP, 20-27 July 1994, Glasgow, Scotland.
6. G.P. Korchemsky, Nucl. Phys. B 443, 255 (1995); 462, 333 (1996).
7. E.K. Sklyanin, Prog. Theor. Phys. Suppl. 118, 35 (1995).
8. G.P. Korchemsky, preprint LPTHE-Orsay-96/76 [hep-th/9609123].
9. R. Janik and J. Wosiek, these proceedings.