POSITIVE DEHN TWIST EXPRESSION FOR A $\mathbb{Z}_3$ ACTION ON $\Sigma_g$

HISAAKI ENDO AND YUSUF Z. GUR T

Abstract. A positive Dehn twist product for a $\mathbb{Z}_3$ action on the 2-dimensional closed, compact, oriented surface $\Sigma_g$ is presented. The homeomorphism invariants of the resulting symplectic 4-manifolds are computed.

Introduction

This article attempts to answer a question raised by Feng Luo in [6] which asks for a Dehn twist expression for the generator of a $\mathbb{Z}_3$ action with $g + 2$ fixed points on the 2-dimensional closed, compact, oriented surface $\Sigma_g$. By the work of Nielsen, there is only one such action on $\Sigma_g$, [6].

In Section 2 we build a closed genus $g$ surface $\Sigma_g$ using $g$ copies of tori with boundary as building blocks in order to realize that action on $\Sigma_g$. We simply take an order three element from the mapping class group $\mathcal{M}_1$ of torus and juxtapose its Dehn twist expression in $\mathcal{M}_g$, considering torus with boundary as a subsurface of $\Sigma_g$ and taking the orientation into consideration in the gluing process. We start with a torus with one boundary component oriented positively. Then glue a torus with two boundary components oriented negatively to it. Then keep adding more tori with boundary with alternating orientations and finally cap it off with a torus with one boundary component oppositely oriented as the previous copy. We aim at a Dehn twist product for the generator of the $\mathbb{Z}_3$ action on $\Sigma_g$ that uses only positive exponents in order to make sure that the 4-manifold it defines as Lefschetz fibration carries symplectic structure. This becomes a challenge because the negatively oriented bounded tori introduce into the expression many elements with negative exponents and there are still some negative powers to be eliminated from the expression for genus $g > 6$. Therefore this work is still in progress.

In Section 3 we show explicitly how to obtain a positive Dehn twist product for the generator of the $\mathbb{Z}_3$ action on $\Sigma_g$, $g \leq 6$. What seems to be working for low genus doesn’t generalize to higher genus easily and the construction evolves rather ad hoc, at least partially.

In Section 4 we compute the Euler characteristic and signatures of the 4-manifolds given by the words that are obtained in Section 3. The method introduced by the first author and S.Nagami is used for signature computations, [1].

1991 Mathematics Subject Classification. Primary 57M07; Secondary 57R17, 20F38.

Key words and phrases. low dimensional topology, symplectic topology, mapping class group, Lefschetz fibration.
1. Review of Relations in $\mathcal{M}_1, \mathcal{M}_1^1$, and $\mathcal{M}_1^3$

The mapping class group $\mathcal{M}_1$ of torus is generated by Dehn twists about the cycles $\alpha$ and $\beta$, Figure 1, subject to the relations

\begin{align*}
\alpha \beta \alpha &= \beta \alpha \beta \\
(\alpha \beta)^6 &= 1.
\end{align*}

(1.1)

Here, by abuse of notation, we use $\alpha$ and $\beta$ to mean Dehn twists about them for simplicity. The first relation is called braid relation and it exists between every pair of curves that intersect transversely.

Torus with one, two, and three boundary components are subject to the relations

\begin{align*}
(\alpha \beta)^6 &= \delta \\
(\beta \alpha \beta \gamma)^3 &= (\alpha \beta \gamma)^4 = \delta_1 \delta_2, \quad \text{and} \quad (\alpha_1 \alpha_2 \alpha_3 \beta)^3 = \delta_1 \delta_2 \delta_3
\end{align*}

(1.2)

respectively. The last one is also called star relation, \( \text{[3]} \).

The basic idea that is used in this paper is to glue several copies of torus with two boundary components together and cap the resulting bounded surface off with two copies of torus with one boundary component, one on each end, to get a closed surface of genus $g$. We take the word

\begin{align*}
(\alpha \beta)^2
\end{align*}

(1.3)

on the two end copies and the word

\begin{align*}
\beta \alpha \beta \gamma
\end{align*}

(1.4)

on each of the remaining copies in between and juxtapose them with alternating signs to come up with an order three element in the mapping class group of the resulting closed genus $g$ surface.

2. Construction of the Order Three Element on $\Sigma_g$

In this section we will construct an order three element in the mapping class group of closed genus $g$ surface using the words (1.3) and (1.4) according to their position in the gluing process. First case is when genus $g$ is even.
2.1. Genus \(g\)-even. We juxtapose the words of type (1.3) and (1.4) on each of the bounded surfaces above by paying careful attention to the orientation:

\[
\begin{align*}
(c_1 c_2)^2 \\
(c_4 e_1 c_4 f_1)^{-1} \\
c_6 e_2 c_6 f_2 \\
(c_8 e_3 c_8 f_3)^{-1} \\
& \quad \vdots \\
(c_{2g-4} e_{g-3} c_{2g-4} f_{g-3})^{-1} \\
c_{2g-2} e_{g-2} c_{2g-2} f_{g-2} \\
(c_{2g+1} c_{2g})^{-2}
\end{align*}
\]

(2.1)

Every other surface will be negatively oriented so that we can glue the boundaries together. Using the chain relation

\[
(c_{2i+2} e_i c_{2i+2} f_i)^3 = d_i \bar{d}_i
\]

on torus with two boundary components and

\[
(c_{2g+1} c_{2g})^6 = d_{g-1}
\]

on torus with one boundary component, the expressions containing negative exponents in (2.1) can be written as

\[
\begin{align*}
(c_4 e_1 c_4 f_1)^{-1} &= d_2^{-1} (c_4 e_1 c_4 f_1)^2 \bar{d}_2^{-1} \\
(c_8 e_3 c_8 f_3)^{-1} &= d_4^{-1} (c_8 e_3 c_8 f_3)^2 \bar{d}_4^{-1} \\
& \quad \vdots \\
(c_{2g-4} e_{g-3} c_{2g-4} f_{g-3})^{-1} &= d_{g-3}^{-1} (c_{2g-4} e_{g-3} c_{2g-4} f_{g-3})^2 \bar{d}_{g-3}^{-1}
\end{align*}
\]

and the last one as

\[
(\mathbf{2.3})
\]

\[
\begin{align*}
(c_{2g+1} c_{2g})^{-2} &= d_{g-1}^{-1} (c_{2g+1} c_{2g})^4 \\
\end{align*}
\]
We glue the bounded surfaces together and use the lantern relations

\[
\begin{align*}
\delta_1x_3 &= c_1c_1f_1 \\
\delta_2x_5 &= e_1f_1e_2f_2 \\
&\vdots \\
\delta_{g-2}x_{2g-3} &= e_{g-3}f_{g-3}e_{g-2}f_{g-2} \\
\delta_{g-1}x_{2g-1} &= e_{g-2}f_{g-2}e_{2g+1}f_{2g+1} 
\end{align*}
\]

(2.4)

\[
(2.5)
\]

to eliminate the negative exponents of \( \overline{d}_i \) and \( d_i \) in (2.2) and (2.3) using the fact that \( \overline{d}_i = d_{i+1} = \delta_i \) after gluing. Solving (2.4) for the \( \delta_i^{-1} \) we get

\[
\begin{align*}
\delta_1^{-1} &= x_1c_3c_1c_1^{-1}e_1^{-1}f_1^{-1} \\
\delta_2^{-1} &= x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1} \\
&\vdots \\
\delta_{g-2}^{-1} &= x_{g-2}c_{2g-3}e_{g-3}^{-1}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1} \\
\delta_{g-1}^{-1} &= x_{g-1}c_{2g-1}e_{2g+1}^{-1}f_{2g+1}^{-1}e_{2g+1}^{-1} 
\end{align*}
\]

Therefore equations (2.2) and (2.3) become

\[
(2.6)
\]

\[
(2.7)
\]

and (2.1) becomes

\[
(2.8)
\]
Juxtaposing these words, we obtain

\[ (c_1e_2c_1e_f)^2 x_1 c_1c_1^{-1} e_1^{-1} f_1^{-1} (c_4e_1c_4f_1)^2 x_2e_5c_1^{-1} e_1^{-1} f_1^{-1} c_6e_2c_6f_2 \]
\[ x_3c_7e_2^{-1} f_2^{-1} e_3^{-1} f_3^{-1} (c_8e_3c_8f_3)^2 x_4c_9e_3^{-1} e_4^{-1} f_4^{-1} c_10e_4c_{10}f_4 \]
\[ x_5c_{11}e_4^{-1} f_4^{-1} e_5^{-1} f_5^{-1} (c_{12}e_5c_{12}f_5)^2 x_6c_{13}e_5^{-1} f_5^{-1} e_6^{-1} f_6^{-1} c_{14}e_{16}c_{14}f_6 \]

\[ (2.9) \]

Next, we will eliminate the negative exponents using braid and commutativity relations only. Let’s expand the parenthesis in the top three lines in (2.8):

\[ c_2g^{-6} e_g^{-4} c_2g^{-6} f_g^{-4} x_g^{-3} c_2g^{-5} e_g^{-4} f_g^{-1} x_g^{-3} c_2g^{-4} e_g^{-4} f_g^{-3} x_g^{-2} c_2g^{-3} e_g^{-4} c_2g^{-3} f_g^{-1} x_g^{-1} c_2g^{-2} e_g^{-3} c_2g^{-2} f_g^{-1} x_g^{-1} c_2g^{-1} e_g^{-2} c_2g^{-1} f_g^{-1} (c_2g+1c_2g)^4 \]

and use braid relation for the underlined triples:

\[ (2.10) \]

Next, cancel the underlined pairs above using commutativity:

\[ c_1c_2c_1x_1c_3c_1^{-1} f_1^{-1} e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} c_6e_2c_6f_2 \]
\[ x_3c_7e_2^{-1} f_2^{-1} e_3^{-1} f_3^{-1} (c_8e_3c_8f_3)^2 x_4c_9e_3^{-1} e_4^{-1} f_4^{-1} c_10e_4c_{10}f_4 \]
\[ x_5c_{11}e_4^{-1} f_4^{-1} e_5^{-1} f_5^{-1} (c_{12}e_5c_{12}f_5)^2 x_6c_{13}e_5^{-1} f_5^{-1} e_6^{-1} f_6^{-1} c_{14}e_{16}c_{14}f_6 \]

Now, using commutativity and the fact that \( t_{f(\alpha)} = ft_{\alpha}f^{-1} \), for any simple closed curve \( \alpha \) in \( \Sigma_g \) and any diffeomorphism \( f : \Sigma_g \to \Sigma_g \), where \( t_{\alpha} \) and \( t_{f(\alpha)} \) are Dehn twists about the curves \( \alpha \) and \( f(\alpha) \) respectively, we can write
\[ c_1 \prod_{i=1}^{14} f_i^{c_1 c_2 c_1^{-1}} x_1 c_3 f_1^{-1} c_4 f_1 c_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 f_3^{-1} c_8 f_3 c_3 c_8 x_4 c_9 f_4^{-1} c_{10} x_5 c_{11} f_5^{-1} c_{12} f_5 c_5 c_{12} x_6 c_{13} f_6^{-1} c_{14} \]

as

\[(2.11)\]
\[ c_1 d x_1 c_3 r_1 c_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 r_3 c_3 c_8 x_4 c_9 f_4^{-1} c_{10} x_5 c_{11} r_5 c_3 c_{12} x_6 c_{13} f_6^{-1} c_{14} x_7 c_{15}, \]

where \( d = c_1 c_2 c_1^{-1} \) and \( r_i = f_1^{-1} c_{2i+2} f_i, i = 1, 3, 5. \)

The last portion of the word in \( (2.9) \) will be simplified using the same procedure:

\[ c_{2g-6} c_{2g-4} c_{2g-6} f_{g-4} x_{g-3} c_{2g-5} e_{g-4} f_{g-4} e_{g-4} f_{g-4} c_{2g} c_{2g} f_{g-3} c_{2g-4} c_{2g} f_{g-3} c_{2g-4} c_{2g} f_{g-3} \]
\[ x_{g-2} c_{2g-3} e_{g-3} c_{2g-3} f_{g-3} c_{2g-3} f_{g-3} c_{2g-3} f_{g-3} e_{g-3} c_{2g} f_{g-3} \]
\[ e_{g-4} c_{2g-6} e_{g-4} f_{g-4} x_{g-3} c_{2g-5} e_{g-4} f_{g-4} e_{g-4} f_{g-4} c_{2g} c_{2g} f_{g-3} c_{2g-4} c_{2g} f_{g-3} c_{2g-4} c_{2g} f_{g-3} \]
\[ x_{g-2} c_{2g-3} e_{g-3} c_{2g-3} f_{g-3} c_{2g-3} f_{g-3} c_{2g-3} f_{g-3} e_{g-3} c_{2g} f_{g-3} \]
\[ e_{g-2} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_{2g} c_
2.2. **Genus g-odd**. Most of the argument will be similar to the even case; we just need to make some changes on the indices.

The following are the words from each component listed with alternating signs

\[(c_1c_2)^2\]
\[(c_4e_1c_4f_1)^{-1}\]
\[c_6e_2c_6f_2\]

(2.14)

\[\vdots\]
\[(c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^{-1}\]
\[(c_{2g+1}c_{2g})^2\]

Now, (2.14) becomes

\[(c_4e_1c_4f_1)^{-1} = d_2^{-1} (c_4e_1c_4f_1)^2 \overline{d_2}^{-1}\]

(2.15)

\[(c_8e_3c_8f_3)^{-1} = d_3^{-1} (c_8e_3c_8f_3)^2 \overline{d_3}^{-1}\]

\[\vdots\]

\[(c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^{-1} = d_{g-2}^{-1} (c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^2 \overline{d_{g-2}}^{-1}\]

(2.16)

and (2.15) are still the same. Therefore (2.6) becomes

\[(c_4e_1c_4f_1)^{-1} = x_1c_3c_1^{-1}c_1e_1^{-1}f_1^{-1} (c_4e_1c_4f_1)^2 x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1}\]
\[(c_8e_3c_8f_3)^{-1} = x_3c_7c_2^{-1}f_2^{-1}e_3^{-1}f_3^{-1} (c_8e_3c_8f_3)^2 x_4c_9e_3^{-1}f_3^{-1}e_4^{-1}f_4^{-1}\]

(2.17)

\[\vdots\]

\[(c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^{-1} = x_{g-2}c_{2g-3}e_{g-3}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1} (c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^2 x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}c_{2g+1}\]

using lantern relations in (2.15) to replace \(d_i^{-1}\) and \(\overline{d_i}^{-1}\).

Then (2.3) becomes

\[(c_1c_2)^2\]
\[x_1c_3c_1^{-1}c_1e_1^{-1}f_1^{-1} (c_4e_1c_4f_1)^2 x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1}\]
\[c_6e_2c_6f_2\]
\[x_3c_7c_2^{-1}f_2^{-1}e_3^{-1}f_3^{-1} (c_8e_3c_8f_3)^2 x_4c_9e_3^{-1}f_3^{-1}e_4^{-1}f_4^{-1}\]
\[c_{10}e_4c_{10}f_4\]

(2.18)

\[\vdots\]

\[x_{g-2}c_{2g-3}e_{g-3}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1} (c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^2 x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}c_{2g+1}\]

and juxtaposing them results in

\[(c_1c_2)^2 x_1c_3c_1^{-1}c_1e_1^{-1}f_1^{-1} (c_4e_1c_4f_1)^2 x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1}c_6e_2c_6f_2\]
\[x_3c_7c_2^{-1}f_2^{-1}e_3^{-1}f_3^{-1} (c_8e_3c_8f_3)^2 x_4c_9e_3^{-1}f_3^{-1}e_4^{-1}f_4^{-1}c_{10}e_4c_{10}f_4\]
\[ x_5 c_{11} e_4^{-1} f_4^{-1} e_0^{-1} f_0^{-1} (c_8 e_3 c_8 f_3)^2 x_6 c_{13} e_4^{-1} f_4^{-1} e_0^{-1} f_0^{-1} c_{14} c_6 c_{14} f_6 \]

\[ (2.18) \]

\[ c_{2g-4} e_{g-3} c_{2g-4} f_{g-3} x_g-2 c_{2g-3} e_{g-3}^{-1} f_{g-3} e_{g-2} f_{g-2} c_{2g-2} e_{g-2} f_{g-2} \]

\[ c_{2g-2} e_{g-4} c_{2g-2} f_{g-2} x_g-1 c_{2g-1} e_{g-2} f_{g-2} c_{2g+1} c_{2g} c_{2g+1} c_{2g} \]

Eliminating the negative exponents using braid and commutativity relations in the first half of (2.18) results in

\[ c_1 d x_1 c_3 r_1 e_1 e_1 c_4 x_2 f_2 c_6 x_3 c_7 r_3 e_3 e_8 c_4 f_4^{-1} c_{10} x_5 c_{11} r_5 e_5 c_5 c_{12} x_6 c_{13} f_6^{-1} c_{14} x_7 c_{15}, \]

\[ (2.19) \]

where \( d = c_1 c_2 c_1^{-1} \) and \( r_i = f_i^{-1} c_{2i+2} f_i, i = 1, 3, 5, \) just like in the even case (see (2.9) - (2.11)).

The last part is also simplified using braid relation first

\[ c_{2g-4} e_{g-3} c_{2g-4} f_{g-3} x_g-2 c_{2g-3} e_{g-3}^{-1} f_{g-3} e_{g-2} f_{g-2} c_{2g-2} e_{g-2} f_{g-2} \]

\[ c_{2g-2} e_{g-4} c_{2g-2} f_{g-2} x_g-1 c_{2g-1} e_{g-2} f_{g-2} c_{2g+1} c_{2g} c_{2g+1} c_{2g} \]

and commutativity relation along with cancelation next

\[ e_{g-3} c_{2g-4} e_{g-3} f_{g-3} x_g-2 c_{2g-3} e_{g-3}^{-1} f_{g-3} e_{g-2} f_{g-2} c_{2g-2} e_{g-2} f_{g-2} \]

\[ e_{g-2} c_{2g-4} e_{2g-2} f_{g-2} x_g-1 c_{2g-1} e_{g-2} f_{g-2} c_{2g+1} c_{2g} c_{2g+1} c_{2g} \]

Finally, defining \( r_{g-2} = f_{g-2}^{-1} c_{2g-2} f_{g-2} \) and using commutativity one more time we obtain:

\[ e_{g-3} c_{2g-4} e_{g-4} c_{2g-3} f_{g-4} x_g-2 c_{2g-2} e_{g-2} f_{g-2} c_{2g-2} x_g-1 c_{2g-1} c_{2g} c_{2g+1} \]

Putting the two ends together, the word now has the form

\[ c_1 d x_1 c_3 r_1 e_1 e_1 c_4 x_2 f_2 c_6 x_3 c_7 r_3 e_3 e_8 c_4 f_4^{-1} c_{10} x_5 c_{11} r_5 e_5 c_5 c_{12} x_6 c_{13} f_6^{-1} c_{14} x_7 c_{15}, \]

\[ (2.20) \]

\[ c_{2g-4} e_{g-2} c_{2g-3} f_{g-2} x_g-2 c_{2g-2} x_g-1 c_{2g-1} c_{2g} c_{2g+1} \]

In a more compact form we have

\[ c_1 d x_1 c_3 r_1 e_1 e_1 c_4 x_2 f_2^{-1} W_6 W_8 \cdots W_{g-1} W_g, \]

where \( W_i = c_{2i-6} c_{2i-5} c_{2i-4} c_{2i-3} c_{2i-2} f_{2i-3}^{-1} \) and

\[ W_g = c_{2g-6} x_g-3 c_{2g-5} x_g-3 c_{2g-4} x_g-3 c_{2g-3} x_g-3 c_{2g-2} x_g-3 c_{2g-1} x_g-1 c_{2g} c_{2g+1}. \]

The next section deals with lower genus.
3. Low genus

For genus 2 and 3 there isn’t much difficulty with eliminating the terms with negative exponents. For genus 4, 5, and 6 however, we use additional lantern relations to eliminate them.

3.1. genus 2. We glue two tori with one one boundary component together and juxtapose the words \((c_1c_2)^2\) and \((c_5c_4)^{-2}\) on the resulting closed surface.

Next, we use the first relation in \((1.2)\) to replace \((c_5c_4)^{-2}\) by \(\delta^{-1}(c_5c_4)^4\).

Now using the lantern relation

\[
\delta xc_3 = c_1^2 c_5^2
\]

we substitute \(\delta^{-1} = x c_4 c_1^{-2} c_5^{-2}\) and obtain

\[
(c_1c_2)^2 (c_5c_4)^{-2} = (c_1c_2)^2 c_1^{-2} c_2^{-1} c_5 (c_5c_4)^{4} = (c_1c_2)^2 x c_3 c_5 c_4^{-2} c_5 (c_5c_4)^{4}.
\]

Expanding the expression and using commutativity gives

\[
c_1c_2c_1^2 c_2c_1^{-2} x c_3 c_5 c_4^{-2} c_5 c_4 c_3 c_4 c_5 c_4 = c_1c_2c_1^{-2} x c_3 c_5 c_4 c_3 c_4 c_5 c_4 c_3 c_4 = c_1 c_2 c_1^{-1} x c_3 c_4 c_3 c_4 c_3 c_4.
\]

Using braid relation on the underlined terms and doing the obvious cancelations afterward we arrive at

\[
c_1 c_2 c_1^{-1} x c_3 c_4 c_3 c_4 c_5 c_4 c_5 c_4 c_5 c_4 = c_1 c_2 c_1^{-1} x c_3 c_4 c_3 c_4 c_5 c_4 c_5 c_4 c_5 c_4.
\]

Even though \(c_1 c_2 c_1^{-1}\) represents a positive twist, we can do away with this conjugation with little effort: Just bring the left most \(c_1\) in the third power of the word to the right end and see the cancelations that occur between the underlined terms as you go from right to left.

\[
(c_1 c_2 c_1^{-1} x c_3 c_4 c_3 c_4 c_5 c_4 c_5 c_4)^3
\]

\[
\equiv (c_1 c_2 c_1^{-1} x c_3 c_4 c_3 c_4 c_5 c_4 c_3 c_4 c_5 c_4)^3
\]

\[
\equiv (c_1 c_2 c_1^{-1} x c_3 c_4 c_3 c_4 c_5 c_4 c_3 c_4 c_5 c_4)^3
\]

\[
\equiv (c_1 c_2 c_1^{-1} x c_3 c_4 c_3 c_4 c_5 c_4 c_3 c_4 c_5 c_4)^3
\]

\[
(3.1) = (c_1 c_2 x c_3 c_4 c_5 c_4 c_5 c_4 c_5 c_4)^3 = 1
\]
3.1.1. *An Alternate Expression.* We can obtain an alternate expression out of (3.1) by inserting into it lantern relations as follows:

\[
\begin{align*}
&(c_1 c_2 c_3 c_4 c_5 c_4 c_5 c_4)^3 \\
&\equiv (c_1 c_2 c_3 c_4 c_3^{-2} c_3^{-2} c_4 c_5 c_4)^3 \\
&\equiv (c_1 c_2 c_3 c_4 c_3^{-2} k_1 h_1 c_4 c_5 c_4)^3 \\
&\equiv (c_2 x c_3 c_4 c_3^{-2} k_1 h_1 c_5^2 c_5^{-2} c_4 c_5 c_4)^3 \\
&\equiv (c_2 x c_3 c_4 c_3^{-2} k_1 h_1 c_3 \delta c_5^{-2} c_4 c_5 c_4)^3 \\
&\equiv (c_2 x c_3 c_4 c_3^{-1} k_1 h_1 \delta x c_5^{-2} c_5 c_4 c_5)^3 \\
&\equiv (c_2 x (c_3 c_4 c_3^{-1}) k_1 h_1 \delta x (c_5^{-1} c_4 c_5))^3 \\
&\equiv (c_2 x t_2 k_1 h_1 \delta x s_2)^3 = 1
\end{align*}
\]

where \( t_2 = c_3 c_4 c_3^{-1} \) and \( s_2 = c_5^{-1} c_4 c_5 \). The cycles that are used in the first lantern relation

\[ c_1 k_1 h_1 = c_3^2 c_5^2 \]

used in (3.2) are shown in Figure 4.

![Figure 4](image_url)

3.2. *genus 3.* To get the word for genus 3 we will use (2.17) with \( g = 3 \):

\[
\begin{align*}
&(c_1 c_2)^2 \\
x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} (c_4 e_1 c_4 f_1)^2 x_2 c_5 e_1^{-1} f_1^{-1} e_7^{-1} c_7^{-1} \\
&(c_7 c_6)^2,
\end{align*}
\]
In the end, using the identification
\[ c_1c_2c_1x_1c_3c_1^{-1}c_1^{-1}f_1^{-1}c_4e_1c_4f_1c_4e_1c_4f_1x_2c_5c_1^{-1}f_1^{-1}c_7^{-1}c_7^{-1}c_1c_6c_7c_6. \]

After this initial cancelation we use braid relation on the underlined terms
\[ c_1c_2c_1c_2 x_1c_3c_1^{-1}c_1^{-1}f_1^{-1}c_4e_1c_4f_1c_4e_1c_4f_1x_2c_5c_1^{-1}c_7^{-1}c_6c_7c_6, \]
and get
\[ c_1c_1c_2c_1 x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}c_1c_4c_1f_1c_4e_1c_4f_1x_2c_5c_1^{-1}c_7^{-1}c_7c_6c_7. \]

Cancelation of the underlined terms gives
\[ (3.3) \quad c_1c_2c_1x_1c_3c_1^{-1}f_1^{-1}c_4e_1c_4f_1c_4x_2c_5c_6c_7. \]

Now, rearranging the terms using commutativity and letting \( r = f_1^{-1}c_4f_1 \) we obtain
\[ c_1c_2c_1c_1^{-1}x_1c_3r e_1c_4x_2c_5c_6c_7. \]

Using the same kind of rotation as in (3.1) will allow us to eliminate the conjugation \( c_1c_2c_1^{-1} \) and we will get
\[ (3.4) \quad (c_1c_2x_1c_3rc_8c_8c_4x_2c_5c_6c_7)^3 = 1 \]

in the end, using the identification \( e_1 = c_8 \) as shown in Figure 3.

3.2.1. An Alternate Expression. An alternate expression is obtained when \( f_1^{-1} \) is eliminated from (3.3) using the lantern relation
\[ f_1tv = c_1c_3c_5c_7. \]

Substituting \( f_1^{-1} = tvc_1^{-1}c_5^{-1}c_7^{-1}c_7 \) into (3.3) we get

\[ c_1c_1c_2x_1c_3c_1^{-1}tv^{-1}c_1^{-1}c_5^{-1}c_7^{-1}c_4e_1f_1c_4x_2c_5c_6c_7. \]

We can cancel the underlined terms and rewrite the rest of the word as
\[ c_1c_1c_2c_1^{-1}c_1^{-1}x_1tv^{-1}c_4e_1f_1c_4x_2c_5c_7^{-1}c_6c_7 \]

using commutativity. Now, because \( c_5(\bar{x}_2) = x_2 \) (i.e., Dehn twist about \( c_5 \) maps \( \bar{x}_2 \) to \( x_2 \)) we have
\[ c_5\bar{x}_2c_5^{-1} = x_2, \text{ i.e., } c_5\bar{x}_2 = x_2c_5. \]

Substituting \( c_5\bar{x}_2 \) in place of \( x_2c_5 \) and inserting a \( c_5c_5^{-1} \) using commutativity results in
\[ c_1c_1c_2c_1^{-1}c_1^{-1}x_1tv^{-1}c_5c_5e_1f_1e_1c_5^{-1}c_4c_3\bar{x}_2c_2^{-1}c_6c_7. \]

Now, all we have to do is rename the conjugations. If we let

\[ \text{Figure 5.} \]
then the final form of the word becomes
\[
(\bar{y}_1 x_1 t v s_2 c_8 f_1 c_8 s_2 x_2 r_3)^3 = 1,
\]
using the identification \( e_1 = c_8 \) in Figure 3.

3.2.2. Another Alternate Expression. One other expression is obtained using the relation
\[
(c_1 c_2 c_3)^4 = e_1 f_1
\]
in order to substitute \( (c_1 c_2 c_3)^4 \) in place of \( c_8 f_1 = e_1 f_1 \). The alternate expression is
\[
(\bar{y}_1 x_1 t v s_2 (c_1 c_2 c_3)^4 c_8 s_2 x_2 r_3)^3 = 1,
\]

3.3. genus 4. Using (2.8) with \( g = 4 \) we have
\[
(c_1 c_2)^2 \quad x_1 c_3 e_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} (c_4 e_1 c_4 f_1)^2 x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} c_6 c_6 f_2
\]
\[
x_3 c_7 e_2^{-1} f_2^{-1} c_9^{-1} e_9^{-1} (c_9 c_8)^4,
\]
which will be the same as the top line in (2.10) for the most part after juxtaposing:
\[
(c_1 c_2 c_3 x_1 c_1 c_1^{-1} c_1^{-1} c_1^{-1} e_1^{-1} e_1^{-1} c_4 e_1 c_1 c_4 e_1 f_1 c_4 e_1 c_4 e_1 f_1 c_2 c_5 e_1^{-1} f_1^{-1} c_6 c_6 f_2 x_3 c_7 e_2^{-1} f_2^{-1} c_9^{-1} e_9^{-1} (c_9 c_8)^4.
\]

Cancelation of the underlined terms gives
\[
c_1 c_1 c_2 x_1 c_1 c_1^{-1} f_1^{-1} c_4 e_1 f_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 c_9^{-1} e_9^{-1} c_9 c_9 c_9 c_9 c_9 c_9 c_9 c_9.
\]

Rearranging some commuting terms along with braid relation on the underlined triple we get
\[
c_1 c_1 c_2 c_1^{-1} x_1 c_3 f_1^{-1} c_4 f_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 c_9^{-1} c_9 c_9 c_9 c_9 c_9 c_9 c_9 c_9,
\]
and by setting \( d = c_1 c_2 c_1^{-1} , r_1 = f_1^{-1} c_4 f_1 \) and canceling the underlined pair we arrive at
\[
(c_1 d x_1 c_3 r_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 c_9 c_9 c_9 c_9 c_9 c_9).
\]
We will have to use another lantern relation to eliminate \( f_2^{-1} \) and that will be
\[
f_2 t v = f_1 c_5 c_7 c_9
\]
as shown in Figure 4. We solve it for \( f_2^{-1} \)
\[
f_2^{-1} = t v f_1^{-1} c_5^{-1} c_7^{-1} e_9^{-1}
\]
and substituting that in (3.7) using commutativity gives:

\[ c_1dx_1c_3r_1e_1e_1c_4f_1^{-1}x_2c_2tv_c5^{-1}c_7^{-1}c_6x_3c_7c_9^{-1}c_8c_9c_9c_8c_8. \]

Because \( c_7(\bar{x}_3) = x_3 \) (i.e., Dehn twist about \( c_7 \) maps \( \bar{x}_3 \) to \( x_3 \)) we have

\[ c_7\bar{x}_3c_7^{-1} = x_3, \text{ i.e., } c_7\bar{x}_3 = x_3c_7. \]

Substituting that in and using braid relation and cancelation on the underlined parts we get

\[ c_1dx_1c_3r_1e_1e_1c_4f_1^{-1}x_2tv_c7^{-1}c_6c_7c_3^{-1}c_8c_9c_9c_8c_8. \]

Renaming \( c_7^{-1}c_6c_7 = s_3 \) and using braid relation on the underlined part again gives

\[ c_1dx_1c_3r_1e_1e_1c_4f_1^{-1}x_2tv_sx_3^{-1}c_9c_9c_9c_8c_8. \]

The following is how we eliminate \( f_1^{-1} \) from the underlined portion:

\[ r_1e_1e_1c_4f_1^{-1} = f_1^{-1}c_4f_1e_1e_1c_4f_1^{-1} = \]

\[ f_1^{-1}c_4e_1e_1f_1c_4f_1^{-1} = \]

\[ f_1^{-1}c_4e_1c_4^{-1}f_1^{-1}c_4e_1c_4^{-1}f_1^{-1}c_4e_1c_4^{-1}f_1c_4 = \]

\[ f_1^{-1}c_4e_1c_4^{-1}f_1f_1^{-1}c_4e_1c_4^{-1}f_1c_4 = y_2y_2c_4, \]

where \( y_2 = f_1^{-1}c_4e_1c_4^{-1}f_1 \). Now (3.9) becomes

\[ c_1dx_1c_3y_2y_2c_4x_2tv_sx_3c_9c_9c_9c_8c_8. \]

Using the same rotation operation as in (3.11) allows us to eliminate the conjugation \( d = c_1c_2c_1^{-1} \) and we get

\[ (c_1c_2x_1c_3y_2y_2c_4x_2tv_sx_3c_9c_9c_9c_8c_8)^3 = 1. \]

3.3.1. An Alternate Expression. An alternate expression is obtained when \( f_1^{-1} \) is eliminated from (3.9) using the lantern relation

\[ f_1t_{1,4}v_{1,4} = c_1c_3v_c9. \]

Substituting \( f_1^{-1} = t_{1,4}v_{1,4}c_1^{-1}c_3^{-1}v_1c_9^{-1} \) into (3.9) yields

\[ c_1dx_1c_3r_1e_1e_1c_4t_{1,4}v_{1,4}c_1^{-1}c_3^{-1}v_1c_9^{-1}x_2tv_sx_3c_9c_9c_9c_8c_8. \]

Using commutativity and inserting identity where necessary we can rewrite the last expression as

\[ c_1c_1c_2c_1^{-1}c_1^{-1}x_1c_3r_1c_3^{-1}c_1e_1c_3c_4c_3^{-1}t_{1,4}v_{1,4}v_1c_9^{-1}x_2v_1c_9^{-1}tv_sx_3c_9^{-1}c_8c_9c_9c_8c_8, \]
remembering $d = c_1c_2c_1^{-1}$. Now, all we have to do is rename the conjugations:

$$\tilde{y}_1 = c_1c_1c_2c_1^{-1}c_1^{-1}, u_1 = c_3r_1c_3^{-1}, \tilde{s}_2 = c_3c_4c_3^{-1}, w = v^{-1}x_2v, z = v^{-1}tv \text{ and } r_4 = c_9^{-1}c_8c_9.$$  

Then (3.13) becomes $\tilde{y}_1x_1u_1e_1e_1\tilde{s}_2t_1u_1x_1wzv_t^3x_3v_3r_4c_9c_8c_3$. Therefore the final form of the alternate word for genus 4 is

$$\begin{align*}
(\tilde{y}_1x_1u_1e_1e_1\tilde{s}_2t_1u_1x_1wzv_t^3x_3v_3r_4c_9c_8c_3)^3 &= 1.
\end{align*}$$

3.3.2. An alternate Construction. An alternate gluing operation for genus 4 can be performed as shown in Figure 9.
We use the words
\[ b_1a_1b_1a_1 \]
\[ b_2a_2b_2a_2 \]
\[ b_3a_3b_3a_3 \]
\[ (\alpha_1\alpha_2\alpha_3d)^{-1} \]
on the four bounded surfaces taking the one in the center with the opposite orientation. Using the star relation (1.2)
\[ (\alpha_1\alpha_2\alpha_3d)^3 = \bar{\delta}_1\bar{\delta}_2\bar{\delta}_3, \]
we write
\[ (\alpha_1\alpha_2\alpha_3d)^{-1} = \bar{\delta}_1^{-1}\bar{\delta}_2^{-1}\bar{\delta}_3^{-1} (\alpha_1\alpha_2\alpha_3d)^2 \]
and using the lantern relations
\[ \delta_1 x_1 c_1 = a_1 a_1 a_1 \alpha_2 \]
\[ \delta_2 x_2 c_2 = a_2 a_2 a_3 \alpha_3 \]
\[ \delta_3 x_3 c_3 = a_3 a_3 a_1 \alpha_1 \]
and the fact that \( \delta_1 = \bar{\delta}_1, \delta_2 = \bar{\delta}_2, \delta_3 = \bar{\delta}_3 \) we write

**Figure 10.**

\[ \delta_1^{-1} = x_1 c_1 a_1^{-1} a_1^{-1} a_1^{-1} a_2^{-1} \]
\[ \delta_2^{-1} = x_2 c_2 a_2^{-1} a_2^{-1} a_2^{-1} a_3^{-1} \]
\[ \delta_3^{-1} = x_3 c_3 a_3^{-1} a_3^{-1} a_3^{-1} a_1^{-1} \]

Substituting all these in (3.15) and juxtaposing we obtain
\[ b_1 a_1 b_1 a_1^{-1} b_2 a_2 b_2 a_2^{-1} b_3 a_3 b_3 a_3^{-1} x_1 c_1 x_2 c_2 x_3 c_3 a_1^{-1} a_1^{-1} a_2^{-1} a_2^{-1} a_3^{-1} a_3^{-1} a_3^{-1} a_1^{-1} (\alpha_1 \alpha_2 \alpha_3 d)^2 \]

We can cancel the underlined terms right away using commutativity and rearrange rest of the word as
\[ b_1 a_1 b_1 a_1^{-1} b_2 a_2 b_2 a_2^{-1} b_3 a_3 b_3 a_3^{-1} x_1 c_1 x_2 c_2 x_3 c_3 a_1^{-1} a_2^{-1} a_3^{-1} a_1 a_2 a_3 da_1 a_2 a_3 d \]
using commutativity again. Now, using braid relation and cancelation on the underlined portion, the word reduces to

\[ a_1b_1a_1^{-1}a_2b_2a_2^{-1}a_3b_3x_1x_2a_1a_2c_1c_2x_3c_3r \]

Further cancelation and renaming \( r = (\alpha_1\alpha_2\alpha_3)^{-1}d\alpha_1\alpha_2\alpha_3d \) gives the positive relation

\[ (3.16) \quad (a_1b_1a_2b_2a_3x_1x_2c_1x_3c_3r)^3 = 1 \]

3.3.3. Another alternate expression. We can modify (3.16) in order to insert the lantern relation

\[ \alpha_2t = a_1c_1a_2c_2 \]

into it and obtain a new expression.

![Figure 11](image-url)
We will use the same additional lantern relation as in genus 4 to eliminate $f_2^{-1}$: Substitute
\[ f_2^{-1} = tvf_1^{-1}c_5^{-1}c_7^{-1}f_3^{-1} \]
in (3.18) using commutativity
\[ c_1dx_1c_3r_1e_1e_4c_2x_5f_2^{-1}c_6x_3c_7e_3c_8x_4c_9c_{10}c_{11} \]
and eliminate $f_1^{-1}, c_5^{-1},$ and $c_7^{-1}$ in the exact same way as in genus 4 following Figure 12. Therefore we can borrow the portion of (3.11) up to $c_8$ and write
\[
\begin{align*}
(3.19) & \quad c_1dx_1c_3y_2x_2e_4x_2tv_3x_3f_2^{-1}c_8e_3c_8x_4c_9c_{10}c_{11}.
\end{align*}
\]
The following is how we deal with $f_3^{-1}$:
\[
\begin{align*}
f_3^{-1}c_8e_3c_8 & = f_3^{-1}f_3^{-1}c_8f_3e_3c_8 = \\
f_3^{-1}c_8f_3c_8^{-1}e_3c_8 & = f_3^{-1}c_8f_3c_8^{-1}e_3c_8^{-1}c_8 = r_3^-c_3^-f_3,
\end{align*}
\]
where $r_3 = f_3^{-1}c_8f_3, r_3 = c_8^{-1}c_3$. Therefore the final form of the genus 5 word is
\[
\begin{align*}
(3.20) & \quad (c_1c_2x_1c_3y_2x_2e_4x_2tv_3x_3c_3^-x_3c_4c_9c_{10}c_{11})^3 = 1
\end{align*}
\]
after a rotation similar to (3.11) applied.

3.5. genus 6. Setting $g = 6$ in (3.8) we obtain the components
\[
\begin{align*}
(c_1c_2)^2 & \\
x_1c_3c_1^{-1}c_1^{-1}f_1^{-1}(c_4e_1c_4f_1)^2 & x_2c_5c_5^{-1}f_1^{-1}c_5^{-1}f_2^{-1} \\
c_6e_2c_6f_2 & \\
x_3c_7c_2^{-1}f_2^{-1}c_3^{-1}f_3^{-1}(c_8e_3c_8f_3)^2 & x_4c_9c_9^{-1}f_3^{-1}c_9^{-1}f_4^{-1}
\end{align*}
\]
(3.21)
\[
\begin{align*}
c_{10}e_4c_{10}f_4 & \\
x_5c_{11}e_4^{-1}f_4^{-1}c_3^{-1}c_3^{-1}(c_3c_12)^4
\end{align*}
\]
of the word on bounded subsurfaces before juxtaposition. After juxtaposing them we arrive at
\[
\begin{align*}
c_1dx_1c_3r_1e_1e_4c_2x_5f_2^{-1}c_6x_3c_7r_3e_3c_8x_4c_9f_4^{-1}c_{10}x_5c_{11}c_{12}c_{13}c_{14}c_{15}c_2c_3c_12 & \\
as in (2.13) with $g = 6$. If we substitute
\[ f_2^{-1} = t_{2.4}v_{2.4}f_1^{-1}c_5^{-1}c_7^{-1}f_3^{-1} \]
and $f_4^{-1} = t_{4.6}v_{4.6}f_3^{-1}c_9^{-1}c_{13}^{-1}c_{13}$
then we get
\[
\begin{align*}
c_1dx_1c_3r_1e_1e_4c_2x_5t_{2.4}v_{2.4}f_1^{-1}c_5^{-1}c_7^{-1}f_3^{-1}c_6x_3c_7r_3e_3c_8x_4c_9t_{4.6}v_{4.6}f_3^{-1}c_9^{-1}c_{13}^{-1}c_{10}x_5c_{11}c_{12}c_{13}c_{14}c_{15}c_2c_3c_12.
\end{align*}
\]
We do the obvious cancelations and use (3.8) again to write
\[ x_3c_7 = c_7\bar{x}_3 \] and likewise \[ x_5c_{11} = c_{11}\bar{x}_3. \]

Using commutativity as well yields
\[ c_4dx_0c_1c_1c_4f_1^{-1}x_2t_{2.4}v_{2.4}c_7^{-1}c_6c_7\bar{x}_3f_3^{-1}r_3c_3c_6c_8f_3^{-1}x_4t_{4.6}v_{4.6}c_{11}^{-1}c_{10}c_{11}\bar{x}_5c_{13}^{-1}c_{12}c_{13}c_{12}c_{13}c_{12}c_{13}c_{12}. \]

Following the same argument given in (3.10) for the underlined portions and renaming \[ c_7^{-1}c_6c_7 = s_3, \] \[ c_{11}^{-1}c_{10}c_{11} = s_5 \] and \[ c_{13}^{-1}c_{12}c_{13} = s_6 \] we get
\[ c_4dx_0c_1c_2c_4x_2t_{2.4}v_{2.4}s_3\bar{x}_3f_3^{-1}s_4y_4c_8x_4t_{4.6}v_{4.6}s_5\bar{x}_5s_{12}c_{12}c_{13}c_{12}. \]

One more lantern substitution is needed to eliminate \( f_3 \) and that is
\[ f_3^{-1} = t_{2.6}v_{2.6}f_1^{-1}v_{2.4}c_{13}^{-1}. \]

Result from substituting that is
\[ c_4dx_0c_1y_2c_4x_2t_{2.4}v_{2.4}s_3\bar{x}_3t_{2.6}v_{2.6}f_1^{-1}v_{2.4}v_{4.6}c_{13}^{-1}y_4y_4c_8x_4t_{4.6}v_{4.6}s_5\bar{x}_5s_{12}c_{12}c_{13}c_{12}. \]

The idea in (3.8) can be used for \( t_{4.6} \) and \( v_{4.6} \) to write
\[ t_{4.6}v_{4.6} = v_{4.6} \bar{t}_{4.6}. \]

Now, conjugating \( y_4, c_8, x_4 \) by \( v_{4.6} \) and renaming them as \( y_{4,6}c_{8,12}x_{4,5} \), respectively, and likewise renaming \( v_{2.4}\bar{x}_3v_{2.4} = \bar{x}_3 \) and \( y_6 = c_{13}^{-1}s_6c_{13} \) gives
\[ c_4dx_0c_1c_2c_4x_2t_{2.4}v_{2.4}s_3\bar{x}_3t_{2.6}v_{2.6}f_1^{-1}y_4y_4c_8x_{12}x_{4,5}t_{4.6}v_{4.6}s_5\bar{x}_5y_{6}c_{12}c_{13}c_{12}. \]

The final lantern substitution is to replace \( f_1^{-1} \) and it is
\[ f_1^{-1} = t_{1.6}v_{1.6}c_1^{-1}c_3^{-1}v_{2.6}c_{13}^{-1}, \]
where \( t_{1.6} \) and \( v_{1.6} \) are defined in the same fashion. Substituting that results in
\[ c_4dx_0c_1y_2c_4x_2t_{2.4}s_3\bar{x}_3t_{2.6}v_{2.6}t_{1.6}v_{1.6}c_1^{-1}c_3^{-1}v_{2.6}c_{13}^{-1}y_4y_4c_8x_{12}x_{4,5}t_{4.6}v_{4.6}s_5\bar{x}_5y_{6}c_{12}c_{13}c_{12}. \]
After renaming two conjugations $t_{1,2,6} = e_{2,6} t_{1,6} e_{2,6}^{-1}$ and $w_{6} = c_{13}^{-1} y_{6} c_{13}$ following the braid relation $c_{1} c_{12} = c_{13} c_{12}$, we will push $c_{1}^{-1}$ to the right end in order to cancel it with the $c_{1}$ at the left end of the next copy. Following the same idea for $c_{3}^{-1}$ gives $u_{1} = c_{3}^{-1} d c_{3}$ after we invoke one more time in order to write $x_{1} c_{3} = c_{3} x_{1}$. All of these changes are realized in the final form of the genus 6 word that follows:

\[(u_{1} x_{1} y_{2} c_{4} x_{2} t_{2,4} s_{3} x_{3,2} t_{2,6} t_{1,2,6} v_{1,6} y_{4,6} y_{4,6} c_{8,12} x_{4,5} 7_{4,6} s_{5} w_{6} c_{12} c_{13})^{3} = 1.\]

### 4. Applications

In this section we will compute the homeomorphism invariants of the 4-manifolds defined by the words in the previous section. We will denote by $X_{g}$ the manifolds that are given by the words (3.1), (3.4), and (3.16) and those that are obtained from them by inserting $k$ lantern relations will be denoted by $X_{g,k}$.

**Proposition 4.1.** The signature and Euler characteristic of the Lefschetz fibration $X_{g}, g = 2, 3, 4$, is given by $\sigma(X_{g}) = -2 (g + 7)$ and $\chi(X_{g}) = 2g + 22$, respectively.

**Proof:** By checking the respective equations we see that the number of cycles in those that define $X_{2}$, $X_{3}$, and $X_{4}$ is $3 (2g + 6)$; therefore their Euler characteristics are given by the formula

\[\chi(X_{g}) = 2 (2 - 2g) + 3 (2g + 6) = 2g + 22.\]

Here we used the well known fact from the theory of Lefschetz fibrations that the Euler characteristic of a Lefschetz fibration $X^{4} \rightarrow S^{2}$ is given by the formula

\[\chi(X) = 4 - 4g + s,\]

where $g$ is the genus of the fiber and $s$ is the number of singular fibers, i.e., the number of vanishing cycles [4].

For signature computations that follow the reader is referred to article II. First we compute $\sigma(X_{2})$.

Let $C_{2}$ denote a chain of length 2 in $M_{2}$, such as $(c_{1} c_{2})^{6} \delta^{-1}$ and $(c_{5} c_{4})^{6} \delta^{-1}$. Following the construction of the word in [3.1] we have

\[
\begin{align*}
C_{2} & \cdot C_{2}^{-1} \\
& = (c_{1} c_{2})^{6} \delta^{-1} \delta (c_{5} c_{4})^{-6} = (c_{1} c_{2})^{6} (c_{5} c_{4})^{-6} \\
& = \left((c_{1} c_{2})^{2} (c_{5} c_{4})^{-2}\right)^{3} \quad \text{(commutativity)} \\
& = \left((c_{1} c_{2})^{2} \delta^{-1} (c_{5} c_{4})^{6} (c_{5} c_{4})^{-2}\right)^{3} \quad \text{(chain relation $C_{2}$)} \\
& = \left((c_{1} c_{2})^{2} \delta^{-1} (c_{5} c_{4})^{4}\right)^{3} \quad \text{(cancelation )} \\
& = \left((c_{1} c_{2})^{2} x c_{3} c_{4} x c_{3}^{-2} c_{5}^{-2} (c_{5} c_{4})^{4}\right)^{3} \quad \text{(lantern relation)} \\
& = \quad \ldots \\
& = (c_{1} c_{2} x c_{3} c_{4} c_{5} c_{4} c_{5} c_{4} c_{4} c_{4})^{3} \quad \text{(commutativity, braid relations)}
\end{align*}
\]
Cancelations do not change the signature and commutativity and braid relations have zero signature (\(\text{II}\), Proposition 3.6); therefore we have

\[
\sigma(X_2) = I(C_2) - I(C_2) + 3I(C_2) + 3I(L) \\
= -7 - (-7) + 3(-7) + 3(+1) \\
= -18
\]

Next, we compute \(\sigma(X_3)\). Let \(C_2\) denote either of the two chains \((c_1c_2)^6\delta_1^{-1}\) or \((c_7c_6)^6\delta_2^{-1}\) of length 2 and \(C_3\) denote the chain \((e_1c_4f_1)^4\delta_1^{-1}\delta_2^{-1}\) of length 3 in \(M_3\). Then construction of the word in [3,4] gives

\[
\begin{align*}
C_2 \cdot C_3^{-1} \cdot C_2 &= (c_1c_2)^6\delta_1^{-1}(e_1c_4f_1)^{-4}\delta_2\delta_1^{-1}(c_7c_6)^6 \\
&= ((c_1c_2)^2(e_1c_4f_1c_4)^{-1}(c_7c_6)^2)^3 \quad \text{(commutativity and cancelations)} \\
&= ((c_1c_2)^2(e_1c_4f_1c_4)^{-1}(e_1c_4f_1c_4)^3\delta_1^{-1}\delta_2^{-1}(c_7c_6)^2)^3 \quad \text{(chain relation } C_3) \\
&= ((c_1c_2)^2\delta_1^{-1}(e_1c_4f_1c_4)^2\delta_2^{-1}(c_7c_6)^2)^3 \quad \text{(commutativity and cancelations)} \\
&= ((c_1c_2)^2x_1c_2c_1^{-2}e_1^{-1}f_1^{-1}(e_1c_4f_1c_4)^2x_2c_5e_1^{-1}f_1^{-1}c_7^{-2}(c_7c_6)^2)^3 \quad \text{(2 lantern relations } L) \\
&= \ldots. \\
&= (c_1c_2x_1c_3c_8c_8c_4x_2c_5c_6c_7)^3 = 1 \quad \text{(commutativity, braid relations)}
\end{align*}
\]

Keeping track of the relations that are used in the process we obtain

\[
\sigma(X_3) = I(C_2) - I(C_3) + I(C_2) + 3I(C_3) + 3I(L) + 3I(L) \\
= -7 - (-6) + (-7) + 3(-6) + 3(+1) + 3(+1) \\
= -20
\]

Here we also used the fact that \((e_1c_4f_1)^4 = (e_1c_4f_1c_4)^3\).

We compute \(\sigma(X_4)\) last. Following its construction in [3,3] we obtain

\[
\begin{align*}
C_2 \cdot C_2 \cdot C_2 \cdot E^{-1} &= (b_1a_1)^6\delta_1^{-1}(b_2a_2)^6\delta_2^{-1}(b_3a_3)^6\delta_3^{-1}(\alpha_1\alpha_2\alpha_3d)^{-3}\delta_1\delta_2\delta_3 \\
&= ((b_1a_1)^2(b_2a_2)^2(b_3a_3)^2(\alpha_1\alpha_2\alpha_3d)^{-1})^3 \quad \text{(commutativity and cancelations)} \\
&= ((b_1a_1)^2(b_2a_2)^2(b_3a_3)^2\delta_1^{-1}\delta_2^{-1}\delta_3^{-1}(\alpha_1\alpha_2\alpha_3d)^3(\alpha_1\alpha_2\alpha_3d)^{-1})^3 \quad \text{(star relation } E) \\
&= ((b_1a_1)^2(b_2a_2)^2(b_3a_3)^2\delta_1^{-1}\delta_2^{-1}\delta_3^{-1}(\alpha_1\alpha_2\alpha_3d)^2)^3 \quad \text{(commutativity and cancelations)} \\
&= ((b_1a_1)^2(b_2a_2)^2(b_3a_3)^2x_1c_1a_1^{-1}a_1^{-1}a_1^{-1}a_2^{-1}x_2c_2a_2^{-1}a_2^{-1}a_3^{-1}
\quad x_3c_3a_3^{-1}\alpha_3^{-1}\alpha_1^{-1}(\alpha_1\alpha_2\alpha_3d)^2)^3 \quad \text{(3 lantern relations } L) \\
&= \ldots. \\
&= (a_1b_1a_2b_2a_3b_3x_1c_1x_2c_2x_3c_3r)^3 = 1 \quad \text{(commutativity, braid relations)}
\end{align*}
\]
From this we obtain
\[ \sigma(X_4) = 3I(C_2) - I(E) + 3I(E) + 3I(L) + 3I(L) \]
\[ = 3(-7) - (-5) + 3(-5) + 3(+1) + 3(+1) + 3(+1) \]
\[ = -22 \]

Consider now the fibrations \( X_{g,k} \) given by the words \((3.2), (3.5), \) and \((3.17)\) which are obtained from \((3.1), (3.4), \) and \((3.16)\) by substituting \( k \) lantern relations.

**Proposition 4.2.** The Euler characteristic and the signature of the manifold \( X_{g,k} \) are given by \( \sigma(X_{g,k}) = \sigma(X_g) + k \) and \( \chi(X_{g,k}) = \chi(X_g) - k, g = 2, 3, 4. \)

**Proof:** The only substitutions used in \((3.2), (3.5), \) and \((3.17)\) that have nonzero signature are lantern relations. The rest of the modifications which result from commutativity and braid relations do not have nonzero contributions \((3.6)\). Cancellations also do not effect the signature. Since the signature of each lantern relation is +1 half of the proof follows. The other half follows from \((3.1)\) and the fact that each time we substitute a lantern relation the length of the word reduces by one.

**Remark 1.** To be more specific about \( k \) we need to point out that \( 1 \leq k \leq 6 \) for genus 2 and \( 1 \leq k \leq 3 \) for genus 3, 4. Therefore
\[ -18 \leq \sigma(X_{2,k}) \leq -12, \quad -20 \leq \sigma(X_{3,k}) \leq -17, \quad -22 \leq \sigma(X_{4,k}) \leq -19 \]

**Remark 2.** In order to see that we have a positive relator for each \( k \) we will show what the word for \( X_{2,1} \) becomes, for example. \( c_1c_2x_3c_4c_5c_6c_7^2khc_1c_4c_5c_4 \) in the third line of \((3.2)\) can be rewritten as \( c_1c_2c_4c_1c_4c_5c_4c_4c_4c_4c_1c_5c_4c_5c_4c_5c_4c_5c_4c_5c_4 \) and this becomes the positive relator \( c_1c_2x_{mnp}c_1c_5c_4, \) where \( m = c_4^{-1}c_3c_4c_3^{-1}c_4, n = c_4^{-1}k, \) and \( p = c_4^{-1}h. \) Therefore the monodromy of \( X_{2,1} \) is
\[ c_1c_2x_{mnp}c_1c_5c_4(c_2c_3c_4c_5c_4c_5c_4c_5c_4)^2 = 1. \]
This is a fibration with \( \sigma(X_{2,1}) = \sigma(X_2)+1 = -17 \) and \( \chi(X_{2,1}) = \chi(X_2)+1 = 25. \)

**Remark 3.** An interesting thing to observe here is the effect of substituting a lantern relation into the monodromy of \( X_g \) on its homeomorphism invariants. Proposition \((4.2)\) shows that it has the same effect on \( X_g \) as that of a rational blow-down operation on it. Therefore it’s an interesting question to investigate whether or not \( X_g \) and \( X_{g,k} \) are diffeomorphic. See \([2]\) for examples that answer this question in the negative.

Next in our list is the word \((5.6)\) obtained from \((5.5)\) by substituting \( m \) chain relations of length 3 into \( X_{3,k}, \) which will be denoted by \( X_{3,k,m}, 1 \leq m \leq k \leq 3. \) This notation does not reflect the length of the chain for the sake of simplicity. Note that chain substitution must follow a lantern substitution; therefore \( m \leq k. \)

**Proposition 4.3.** \( \sigma(X_{3,k,m}) = -20 + k - 6m \) and \( \chi(X_{g,k,m}) = 28 - k + 10m \) for \( 1 \leq m \leq k \leq 3. \)

**Proof:** The signature of \( X_3 \) is \(-20 \) by Proposition \((4.1)\) and the signature of \( X_{3,k} \) was found to be \(-20 + k \) in Proposition \((4.2). \) Since \( X_{3,k,m} \) is obtained from \( X_{3,k} \)
by substituting $m$ chain relations of length 3 and $C_3$ has signature $-6$ (Proposition 3.10, [1]), we have

$$\sigma (X_{3,k,m}) = \sigma (X_{3,k}) + m I (C_3) = -20 + k + m(-6), 1 \leq m \leq k \leq 3.$$ 

Proposition 4.11 gives $\chi (X_t) = 28$ and according to Proposition 4.22 $\chi (X_{g,k}) = 28 - k$. Since substitution of each $C_3$ results in increasing overall number of cycles by 10, its contribution to the Euler characteristic will be 10 according to (4.1). Therefore we have $\chi (X_{g,k,m}) = 28 - k + 10m, 1 \leq m \leq k \leq 3$.

**Remark 4.** Possible values for $\sigma (X_{3,k,m})$ are $-23, -24, -25, -29, -30, -35$ and possible values for $\chi (X_{g,k,m})$ are 35, 36, 37, 45, 46, 55.

Next, we will compute the signatures of the achiral Lefschetz fibrations (2.13) and (2.21), denote them by $Z_g$. Assume, first, $g$ is even and greater than 7. $Z_g$ has monodromy

$$c_1 dx_1 c_2 r_1 c_1 c_2 c_3 c_4 c_5 f_1^{-1} W_6 W_8 \cdots W_g c_2 g - 2 x_2 g - 1 c_2 g - 1 c_2 g c_2 g c_2 g c_2 g _1 c_2 g c_2 g c_2 g ,$$

where $W_i = c_2 i - 6 x_i - 3 c_2 i - 5 r_i - 3 c_2 i - 3 c_2 i - 4 x_i - 2 c_2 i - 3 f_i^{-1}, i = 6, 8, \ldots, g$.

From its construction in 2.1 we can see that this word originally contains two chains of length 2, one on each end, and $g - 2$ chains of length 3 half of which are negatively oriented. Then we substituted $3(g - 2)/2$ additional chains of length 3 in order to replace the negatively oriented ones by positive exponents. We also substituted 3 chains of length 2 for the same reason. These substitutions resulted in $3(g - 1)$ separating negatively oriented boundary curves. Finally we introduced $3(g - 1)$ lantern relations to eliminate them. The rest of the operations until we obtained 2.13 are cancellations, commutativity and braid relations, which have zero contribution to the signature. Combining all of that we can compute the signature of $Z_g$ as

$$\sigma (Z_g) = I (C_2) - \frac{g}{2} I (C_3) + \frac{g}{2} I (C_3) - I (C_2) + \frac{3}{2} (g - 2) I (C_3) + 3 I (C_2) + 3 (g - 1) I (L)$$

$$= 0 + \frac{3 (g - 2)}{2} (-6) + 3(-7) + 3(g - 1)(+1) = -6g - 6$$

Suppose now that $g$ is odd and greater than 6. This time $Z_g$ is given by the monodromy

$$c_1 dx_1 c_2 r_1 c_2 g + 2 c_2 g + 2 c_4 x_2 c_5 f_2^{-1} W_6 W_8 \cdots W_g - 1 W_g,$$

where $W_i = c_{2 i - 6 x_i - 3 c_2 i - 3 c_2 i - 2 c_2 g _1 i - 2 c_2 i - 4 x_i - 2 c_2 i - 3 f_i^{-1}, i = 6, 8, \ldots, g - 1$ and

$$W_g = c_{2 g - 6 x_g - 3 c_2 g - 5 r_g - 3 c_2 g - 2 c_2 g - 2 c_2 g - 2 c_2 g - 2 c_2 g - 2 c_2 g - 2 c_2 g - 2 c_2 g - 2 c_2 g - 2 c_2 g - 1 c_2 g - 1 c_2 g - 1 c_2 g - 1 c_2 g - 1 c_2 g + 1.$$

Using a similar argument we calculate the signature of $Z_g$ as

$$\sigma (Z_g) = I (C_2) - \frac{g - 1}{2} I (C_3) + \frac{g - 3}{2} I (C_3) + I (C_2) + \frac{3 (g - 1)}{2} I (C_3) + 3 (g - 1) I (L)$$

$$= -7 - \frac{g - 1}{2} (-6) + \frac{g - 3}{2} (-6) + (-7) + \frac{3 (g - 1)}{2} (-6) + 3 (g - 1)(+1) = -6g - 2$$

Note that $\sigma (Z_g) = \sigma (X_g)$ for $g = 2, 3$. This is because the simplified form of the general construction leads to a positive relator. The existence of negative powers in the expression for higher genus, however, requires further substitution of lantern relations. We’ll denote by $Y_g$ a genus $g$ Lefschetz fibration that is obtained from
either of the achiral Lefschetz fibrations (2.13) or (2.20) by substituting into them a number of lantern relations until a positive relator is obtained. In that regard the fibration given by (3.11) that is obtained from (3.7) via 3 lantern substitutions will be denoted by $Y_4$. If $k$ additional lantern substitutions are made into these positive words then the resulting manifold will be denoted by $Y_{g,k}$. For example the positive relator (3.14) is denoted by $Y_4$, $3$ because it is obtained via 3 lantern substitutions into (3.11), which is equivalent to (3.9). We will now compute the signatures of $Y_{g}$, $Y_{g,k}$.

**Proposition 4.4.** $\sigma(Y_4) = -27$, $\sigma(Y_{4,k}) = -27 + k$, $\sigma(Y_5) = -29$, $\sigma(Y_6) = -30$.

**Proof:** $Y_4$ is obtained from $Z_4$ by substituting lantern relation 3 times; therefore

$$\sigma(Y_4) = \sigma(Z_4) + 3(+1) = -6 \cdot 4 - 6 + 3 = -27.$$

Similarly $Y_{4,k}$ is obtained from $Y_4$ by substituting $k$ lantern relations, $1 \leq k \leq 3$; therefore

$$\sigma(Y_{4,k}) = \sigma(Y_4) + k(+1) = -27 + k.$$

$Y_5$ is obtained from $Z_5$ by substituting lantern relation 3 times; therefore

$$\sigma(Y_5) = \sigma(Z_5) + 3(+1) = -6 \cdot 5 - 2 + 3 = -29.$$

A careful analysis shows that $Y_6$ is obtained from $Z_6$ by substituting lantern relation 12 times (3 for each of the negative powers $f_2^{-1}$, $f_4^{-1}$, $f_5^{-1}$, $f_1^{-1}$); therefore

$$\sigma(Y_6) = \sigma(Z_6) + 12(+1) = -6 \cdot 6 - 6 + 12 = -30.$$

\[\square\]

We summarize what we found in the following table, which includes the Euler characteristic $\chi$, signature $\sigma$, the holomorphic Euler characteristic $\chi_h = \frac{1}{4}(\sigma + \chi)$, and the self-intersection of the first Chern class $c_1^2 = 3\sigma + 2\chi$. The latter two are defined for manifolds having almost complex structure and symplectic Lefschetz fibrations are known to possess that.

| $X_g$      | $\chi$ | $\sigma$ | $\chi_h$ | $c_1^2$ | $\chi_1$ |
|------------|--------|----------|----------|---------|----------|
| $X_2$      | 26     | -18      | 2        | -2      | 1        |
| $X_{2,k}$  | $26 - k$ | -18 + k | 2        | -2 + k  | 1        |
| $X_{2,6}$  | 20     | -12      | 2        | 4       | $\mathbb{Z}_3$ |
| $X_f$      | 28     | -20      | 2        | -4      | 1        |
| $X_{3,k}$  | $28 - k$ | -20 + k | 2        | -4 + k  | 1        |
| $X_{3,k,m}$| $28 - k + 10m$ | $-20 + k - 6m$ | 2 + m | $-4 + k + 2m$ | 1 |
| $X_4$      | 30     | -22      | 2        | -6      | 1        |
| $X_{4,k}$  | $30 - k$ | -22 + k | 2        | -6 + k  | 1        |
| $Y_4$      | 39     | -27      | 3        | -3      | 1        |
| $Y_{4,k}$  | $39 - k$ | -27 + k | 3        | -3 + k  | 1        |
| $Y_5$      | 41     | -29      | 3        | -5      | 1        |
| $Y_6$      | 46     | -30      | 4        | 2       | 1        |

**Acknowledgment**

The authors thank Ronald J. Stern for helpful comments. They also thank Yusuke Kuno for the correction of the signature of the star relation $E$ in Proposition 3.13 of [1] that was calculated as +5.
References

1. H. Endo and S. Nagami, *Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations*, Trans. Amer. Math. Soc. 357 (2005), 3179-3199.

2. H. Endo and Y. Gurtas, *Lantern Relations and Rational Blowdowns*, arXiv:0808.0386.

3. S. Gervais, *Presentation and Central Extensions of Mapping Class Groups*, Transactions of the American Mathematical Society, Vol. 348 8 (1996), 3097-3132.

4. R. Gompf and A. Stipsicz, *An Introduction to 4-manifolds and Kirby Calculus*, AMS Graduate Studies in Mathematics, 20 (1999).

5. B. Ozbagci, *Signatures of Lefschetz fibrations*, Pacific Journal of Mathematics, Vol. 202 1 (2002), 99-118.

6. F. Luo, *Torsion elements in the mapping class group of a surfaces*, preprint, arXiv:math.GT/0004048.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: endo@math.sci.osaka-u.ac.jp

Department of Mathematics and Computer Science, St. Louis University, MO, USA

E-mail address: ygurtas@slu.edu