Abstract. In this note, we prove that a 3-dimensional steady Ricci soliton is rotationally symmetric if its scalar curvature $R(x)$ satisfies
\[
\frac{C_0}{\rho(x)} \leq R(x) \leq \frac{C_0}{\rho(x)}
\]
for some constant $C_0 > 0$, where $\rho(x)$ denotes the distance from a fixed point $x_0$. Our result doesn’t assume that the soliton is $\kappa$-noncollapsed.

1. Introduction

In his celebrated paper [13], Perelman conjectured that all 3-dimensional $\kappa$-noncollapsed steady (gradient) Ricci solitons must be rotationally symmetric. The conjecture is solved by Brendle in 2012 [1]. For a general dimension $n \geq 3$, under an extra condition that the soliton is asymptotically cylindrical, Brendle also proves that any $\kappa$-noncollapsed steady Ricci soliton with positive sectional curvature must be rotationally symmetric [2]. In general, it is still open whether an $n$-dimensional $\kappa$-noncollapsed steady Ricci soliton with positive curvature operator is rotationally symmetric for $n \geq 4$. For $\kappa$-noncollapsed steady Kähler-Ricci solitons with nonnegative bisectional curvature, the authors have recently proved that they must be flat [10], [11].

Recall from [2],

**Definition 1.1.** An $n$-dimensional steady Ricci soliton $(M,g,f)$ is called asymptotically cylindrical if the following holds:
(i) Scalar curvature $R(x)$ of $g$ satisfies
\[
\frac{C_0^{-1}}{\rho(x)} \leq R(x) \leq \frac{C_0}{\rho(x)}, \quad \forall \rho(x) \geq r_0,
\]
where \( C_0 > 0 \) is a constant and \( \rho(x) \) denotes the distance of \( x \) from a fixed point \( x_0 \).

(ii) Let \( p_m \) be an arbitrary sequence of marked points going to infinity. Consider rescaled metrics \( g_m(t) = r_m^{-1}\phi_m^*g \), where \( r_m R(p_m) = \frac{n-1}{2} + o(1) \) and \( \phi_t \) is a one-parameter subgroup generated by \( X = -\nabla f \). As \( m \to \infty \), flows \((M, g_m(t), p_m)\) converge in the Cheeger-Gromov sense to a family of shrinking cylinders \((\mathbb{R} \times S^{n-1}(1), \tilde{g}(t)), t \in (0, 1)\). The metric \( \tilde{g}(t) \) is given by

\[
\tilde{g}(t) = dr^2 + (n-2)(2-2t)g_{S^{n-1}(1)},
\]

where \( S^{n-1}(1) \) is the unit sphere of euclidean space.

In this note, we discuss 3-dimensional steady (gradient) Ricci solitons without assuming the \( \kappa \)-noncollapsed condition. \( \footnote{It is proved by Chen that any 3-dimensional ancient solution has nonnegative sectional curvature \[7\].} \)

We prove

**Theorem 1.2.** Let \((M, g, f)\) be a 3-dimensional steady Ricci soliton. Then, it is rotationally symmetric if the scalar curvature \( R(x) \) of \((M, g, f)\) satisfies

\[
\frac{C_0^{-1}}{\rho(x)} \leq R(x) \leq \frac{C_0}{\rho(x)},
\]

for some constant \( C > 0 \), where \( \rho(x) \) denotes the distance from a fixed point \( x_0 \).

Under the condition \ref{2}, we need to check the property (ii) in Definition \ref{1} to prove Theorem \ref{2}. Actually, we show that for any sequence \( p_i \to \infty \), there exists a subsequence \( p_{i_k} \to \infty \) such that

\[
(M, g_{p_{i_k}}(t), p_{i_k}) \to (\mathbb{R} \times S^2, \tilde{g}(t), p_{\infty}), \text{ for } t \in (-\infty, 1),
\]

where \( g_{p_{i_k}}(t) = R(p_{i_k})g(R^{-1}(p_{i_k})t) \) and \((\mathbb{R} \times S^2, \tilde{g}(t))\) is a shrinking cylinders flow, i.e.

\[
\tilde{g}(t) = dr^2 + (2-2t)g_{S^2}.
\]

As in \cite{10}, we study the geometry of neighborhood \( M_{p, k} = \{ x \in M \mid f(p) - \frac{k}{\sqrt{R(p)}} \leq f(x) \leq f(p) + \frac{k}{\sqrt{R(p)}} \} \) around level set \( \Sigma_r = \{ f(x) = f(p) = r \} \) for any \( p \in M \). We are able to give a uniform injective radius estimate for \((M, R_{p, k}g)\) at each sequence of \( p_i \). Then we can still get a limit flow for rescaled flows \((M, g_{p_i}(t))\), which will split off a line. By using a classification result of Daskalopoulos-Hamilton-Sesum for ancient flows on a compact surface \cite{8}, we finish the proof of Theorem \ref{2}.

We remark that the curvature condition in Theorem \ref{2} cannot be removed, since there does exist a 3-dimensional non-flat steady Ricci soliton with exponential curvature decay. For example, \((\mathbb{R}^2 \times S^1, g_{cigar} + ds^2)\), where
(\mathbb{R}^2, g_{cigar}) is a cigar soliton. Also, Theorem \ref{thm:main} is not true for dimension \( n \geq 4 \) by Cao’s examples of steady Kähler-Ricci solitons with positive sectional curvature \cite{Cao1994}.

At last, we remark that it is still open whether there exists a 3-dimensional collapsed steady Ricci soliton with positive curvature. Hamilton has conjectured that there should exist a family of collapsed 3-dimensional complete gradient steady Ricci solitons with positive curvature and \( S^1 \)-symmetry (cf. \cite{Hamilton1982}). Our result shows that the curvature of Hamilton’s examples could not have a linear decay.

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2. Positivity of Ricci curvature

\((M, g, f)\) is called a gradient steady Ricci soliton if a Riemannian metric \( g \) on \( M \) satisfies

\[ \text{Ric}(g) = \nabla^2 f, \]

for some smooth function \( f \). We first show the positivity of Ricci curvature of \((M, g, f)\) under \eqref{eq:condition} of Theorem \ref{thm:main}.

Lemma 2.1. Under \eqref{eq:condition}, \((M, g, f)\) has positive sectional curvature.

Proof. We need to show that \((M, g)\) has positive Ricci curvature. On the contrary, \((M, g)\) locally splits off a flat piece of line by Shi’s splitting theorem \cite{Shi1997}. Then, the universal cover \((\tilde{M}, \tilde{g})\) of \((M, g)\) is isometric to a product Riemannian manifold of a real line and the cigar soliton. Namely, \((\tilde{M}, \tilde{g}) = (\mathbb{R}^2 \times \mathbb{R}, g_{cigar} + ds^2)\). Let \( \pi : \tilde{M} \rightarrow M \) be a universal covering. We fix \( x_0 \in M \) and \( \tilde{x}_0 \in \tilde{M} \) such that \( \pi(\tilde{x}_0) = x_0 \). For any \( x \in M \) and \( \tilde{x} \in \tilde{M} \) such that \( \pi(\tilde{x}) = x \), one sees

\[ \rho(x, x_0) \leq \tilde{\rho}(\tilde{x}, \tilde{x}_0), \]

where \( \rho \) and \( \tilde{\rho} \) are the distance functions w.r.t \( g \) and \( \tilde{g} \) respectively. Let \( \{\tilde{x}_i\}_{i \geq 1} \) be a sequence of points so that \( \tilde{x}_i = (p_i, 0) \in \mathbb{R}^2 \times \mathbb{R} \) tend to infinity. Then, one may check that

\[ \tilde{R}(\tilde{x}_i) \rho(\tilde{x}_i, \tilde{x}_0) \rightarrow 0, \text{ as } i \rightarrow \infty. \]

Since \( R(x_i) = \tilde{R}(\tilde{x}_i) \rightarrow 0 \), where \( x_i = \pi(\tilde{x}_i) \), we see \( d(x_i, x_0) \rightarrow \infty \) by \eqref{eq:condition}. Again by \eqref{eq:condition} and \eqref{eq:dis}, we get

\[ C_1 \leq R(x_i) d(x_i, x_0) \leq \tilde{R}(\tilde{x}_i) d(\tilde{x}_i, \tilde{x}_0). \]
This is a contradiction to (2.3). Hence, the lemma is proved.

\[ \square \]

**Corollary 2.2.** \((M, g, f)\) in Theorem 1.2 has a unique equilibrium point \(o\), i.e., \(\nabla f(o) = 0\). As a consequence, \(\Sigma_r = \{ f(x) = r \}\) is diffeomorphic to \(S^2\), for any \(r > f(o)\).

**Proof.** Note that

\[ |\nabla f|^2 + R = A. \tag{2.5} \]

By taking covariant derivatives on both sides of (2.5), it follows

\[ 2\text{Ric}(\nabla f, \nabla f) = -\langle \nabla R, \nabla f \rangle. \tag{2.6} \]

On the other hand, by (1.2), there exists a point \(o\) such that

\[ \sup_M R(x) = R(o) = R_{\text{max}}. \]

In particular, \(\nabla R(o) = 0\). Thus

\[ \text{Ric}(\nabla f, \nabla f) = 0. \]

By Lemma 2.1, \(\nabla f(o) = 0\). The uniqueness also follows from the positivity of Ricci curvature.

By the Morse theorem, \(\Sigma_r = \{ f(x) = r > f(o) \}\) is diffeomorphic to \(S^2\) (cf. [10], Lemma 2.1).

\[ \square \]

3. Geometry of \(M_{p,k}\)

For any \(p \in M\) and number \(k > 0\), we set

\[ M_{p,k} = \{ x \in M \mid f(p) - \frac{k}{\sqrt{R(p)}} \leq f(x) \leq f(p) + \frac{k}{\sqrt{R(p)}} \}. \]

Let \(g_p = R(p)g\) be a rescaled metric and denote \(B(p, r; g_p)\) a \(r\)-geodesic ball centered at \(p\) with respect to \(g_p\). Then by Corollary 2.2, we have (cf. [10], Lemma 3.3)

**Lemma 3.1.** Under (1.2), for any \(p \in M\) and number \(k > 0\) with \(f(p) - \frac{k}{\sqrt{R(p)}} > f(o)\), it holds

\[ B(p, \frac{k}{\sqrt{R_{\text{max}}}}; g_p) \subset M_{p,k}. \tag{3.1} \]

By Lemma 3.1, we prove

**Lemma 3.2.** Under (1.2), there exists a constant \(C\) such that

\[ \frac{|\Delta R(p)|}{R^2(p)} \leq C, \forall p \in M. \tag{3.2} \]
Proof. Fix any \( p \in M \) with \( f(p) \geq r_0 >> 1 \). Then
\[
|f(x) - f(p)| \leq \frac{1}{\sqrt{R(p)}}, \quad \forall \ x \in M_{p,1}.
\]
(3.3)

It is known by \cite{4},
\[
c_1 \rho(x) \leq f(x) \leq c_2 \rho(x), \quad \forall \ \rho(x) \geq r_0.
\]
(3.4)

Thus by (1.2), (3.3) and (3.4), we get
\[
c_2 \rho(x) \geq f(p) - \frac{1}{\sqrt{R(p)}} \geq c_1 \rho(p) - \sqrt{C_0 \rho(p)}.
\]
It follows
\[
\frac{R(x)}{R(p)} \leq \frac{C_0^2 \rho(p)}{\rho(x)} \leq \frac{2 c_2 C_0^2}{c_1}, \quad \forall \ x \in M_{p,1}.
\]
(3.5)

On the other hand, by (3.1), we have
\[
B(p, \frac{1}{\sqrt{R_{\max}}}; g_p) \subseteq M_{p,1}.
\]
Hence
\[
R(x) \leq C' R(p), \quad \forall \ x \in B(p, \frac{1}{\sqrt{R_{\max}}}; g_p).
\]
(3.6)

Let \( \phi_t \) be generated by \( -\nabla f \). Then \( g(t) = \phi_t^* g \) satisfies the Ricci flow,
\[
\frac{\partial g(t)}{\partial t} = -2 \text{Ric}(g(t)).
\]
(3.7)

Also rescaled flow \( g_p(t) = R(p)g(R^{-1}(p)t) \) satisfies (3.7). Since the Ricci curvature is positive,
\[
B(p, \frac{1}{\sqrt{R_{\max}}}; g_p(-t)) \subseteq B(p, \frac{1}{\sqrt{R_{\max}}}; g_p(0)), \quad t \in [-1, 0].
\]
Combining with (3.6), we get
\[
R_{g_p(t)}(x) \leq C', \quad \forall \ x \in B(p, \frac{1}{\sqrt{R_{\max}}}; g_p(0)), \quad t \in [-1, 0].
\]
(3.8)

Thus, by Shi’s higher order estimates, we obtain
\[
|\Delta_{g_p(t)} R_{g_p(t)}|(x) \leq C'_1, \quad \forall \ x \in B(p, \frac{1}{2\sqrt{R_{\max}}}; g_p(-1)), \quad t \in [-\frac{1}{2}, 0].
\]
It follows
\[
|\Delta R|(x) \leq C'_1 R^2(p), \quad \forall \ x \in B(p, \frac{1}{2\sqrt{R_{\max}}}; g_p(-1)).
\]
In particular, we have
\[
|\Delta R|(p) \leq C'_1 R^2(p), \quad \text{as} \ \rho(p) \geq r_0.
\]
The lemma is proved. \( \square \)
Remark 3.3. Under (1.2), by the same argument as in the proof of Lemma 3.2, for each \( k \in \mathbb{N} \), there exists a constant \( C(k) \) such that
\[
\frac{|\nabla^k R(p)|}{R^{\frac{k+2}{2}}(p)} \leq C(k), \quad \forall \, p \in M.
\]

Next, we want to show that \( M_{p,k} \) is bounded by a finite ball \( B(p, 2\pi \sqrt{B} + \frac{2k}{\sqrt{R_{\text{max}}}}; g_p) \), where \( C \) is a uniform constant. We need to use the Gauss formula,
\[
R(X,Y,Z,W) = R(X,Y,Z,W) + \langle B(X,Z), B(Y,W) \rangle - \langle B(X,W), B(Y,Z) \rangle,
\]
where \( X,Y,Z,W \in T\Sigma_r \) and \( B(X,Y) = \langle \nabla_X Y, \nabla f \rangle \cdot |\nabla f| \).

We choose a normal basis \( \{e_1, e_2\} \) on \((\Sigma_r, \bar{g})\) with the induced metric \( \bar{g} \). Then \( \{e_1, e_2, \nabla f / |\nabla f|\} \) spans a normal basis of \((M, g)\). Thus
\[
R_{11} = \bar{R}_{11} + R(\frac{\nabla f}{|\nabla f|}, e_1, e_1, \frac{\nabla f}{|\nabla f|}) - \frac{R_{11} R_{22} - R_{12} R_{21}}{|\nabla f|^2},
\]
\[
R_{22} = \bar{R}_{22} + R(\frac{\nabla f}{|\nabla f|}, e_2, e_2, \frac{\nabla f}{|\nabla f|}) - \frac{R_{11} R_{22} - R_{12} R_{21}}{|\nabla f|^2}.
\]

Since \((\Sigma_r, \bar{g})\) is a surface, \( K = \bar{R}_{11} = \bar{R}_{22} \). Hence, we get

**Lemma 3.4.** The Gauss curvature of \((\Sigma_r, \bar{g})\) is given by
\[
K = \frac{R}{2} - \frac{\text{Ric}(\nabla f, \nabla f)}{|\nabla f|^2} + \frac{R_{11} R_{22} - R_{12} R_{21}}{|\nabla f|^2}.
\]

**Lemma 3.5.** Under (1.2), there exists a uniform \( B > 0 \) such that the following is true: for any \( k \in \mathbb{N} \), there exists \( r_0 = r_0(k) \) such that
\[
M_{p,k} \subset B(p, 2\pi \sqrt{B} + \frac{2k}{\sqrt{R_{\text{max}}}}; g_p), \quad \forall \, p(\rho) \geq r_0.
\]

**Proof.** By (1.2) and (3.4), we have
\[
\frac{R_{\text{max}}}{2} \leq |\nabla f|^2(x) \leq R_{\text{max}}, \quad \forall \, x \in M_{p,k},
\]
as long as $\rho(p) \geq r_0 >> 1$. Then by Lemma 3.3 and Lemma 3.2, we get

$$|K - \frac{R}{2}| = |\frac{\text{Ric}(\nabla f, \nabla f)}{\nabla f^2} + \frac{R_{11}R_{22} - R_{12}R_{21}}{\nabla f^2}|$$

$$\leq \frac{|\text{div} R \times \nabla f|}{2\nabla f^2} + \frac{R^2}{\nabla f^2}$$

$$\leq \frac{|\Delta R + 2|\text{Ric}|^2}{2|\nabla f|^2} + \frac{R^2}{\nabla f^2}$$

$$\leq \frac{(C + 4)R^2}{R_{\text{max}}}.$$ 

It follows

$$(3.13) \quad \frac{R(x)}{4} \leq K(x) \leq \frac{3R(x)}{4}, \quad \forall \ x \in M_{p,k}, \ \rho(p) \geq r_0.$$ 

On the other hand, by (1.2), (3.3) and (3.4), we see

$$c^{-1}_2 \left( c_1 \rho(p) - k \sqrt{\rho(p)} C_0 \right) \leq \rho(x)$$

$$\leq c^{-1}_1 (c_2 \rho(p) + k \sqrt{\rho(p)} C_0), \quad \forall \ x \in M_{p,k},$$

as long as $\rho(p) \geq r_0$. Then similar to (3.5), there exists a $\bar{r}_0 \geq r_0$ such that

$$(3.15) \quad R(x) \geq \frac{c_1}{2c_2 C_0^2} R(p), \quad \forall \ x \in M_{p,k}.$$ 

Thus by (3.13), we get

$$\overline{R}_{ij} \geq B^{-1} R(p) \overline{g}_{ij}, \quad \forall \ x \in \Sigma_{f(p)}, \ \rho(p) \geq \bar{r}_0,$$

where $B > 0$ is a uniform constant. By the Myer’s theorem, the diameter of $\Sigma_{f(p)}$ is bounded by

$$\text{diam}(\Sigma_{f(p)}, g) \leq \text{diam}(\Sigma_{f(p)}, \overline{g}_{f(p)}) \leq 2\pi \sqrt{\frac{B}{R(p)}}.$$ 

As a consequence,

$$(3.16) \quad \Sigma_{f(p)} \subset B(p, 2\pi \sqrt{B}; R(p)g).$$ 

For any $q \in M_{p,k}$, there exists $q' \in \Sigma_{f(p)}$ such that $\phi_s(q) = q'$ for some $s \in \mathbb{R}$. Then by (3.16) and (3.12), we have
\[ d(q, p) \leq d(q', p) + d(q, q') \]
\[ \leq \text{diam}(\Sigma_{f(p)}, g) + \mathcal{L}(\phi_{\tau}|_{[0,s]}) \]
\[ \leq 2\pi \sqrt{\frac{B}{R(p)}} + \int_0^s \left| \frac{d\phi_{\tau}(q)}{d\tau} \right| d\tau \]
\[ = 2\pi \sqrt{\frac{B}{R(p)}} + \int_0^s |\nabla f(\phi_{\tau}(q))| d\tau \]
\[ \leq 2\pi \sqrt{\frac{B}{R(p)}} + \int_0^s \left| \nabla f(\phi_{\tau}(q)) \right|^2 \cdot \frac{2}{\sqrt{R_{\max}}} d\tau \]
\[ = 2\pi \sqrt{\frac{B}{R(p)}} + \int_0^s \left| \nabla f(\phi_{\tau}(q)) \right|^2 \cdot \frac{2}{\sqrt{R_{\max}}} d\tau \]
\[ \leq 2\pi \sqrt{\frac{B}{R(p)}} + |f(q) - f(p)| \cdot \frac{2}{\sqrt{R_{\max}}} \]
\[ \leq \left( 2\pi \sqrt{B} + \frac{2k}{\sqrt{R_{\max}}} \right) \cdot \frac{1}{\sqrt{R(p)}}. \]

Thus

\[ M_{p,k} \subset B(p, 2\pi \sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; R(p)g). \]

The lemma is proved.

\[ \square \]

By Lemma 3.5, we get the following volume estimate of \( B(p, s; g_p) \).

**Proposition 3.6.** Under (1.2) of Theorem 1.2, there exists \( s_0 \) and \( c > 0 \) such that

\[ \text{Vol} B(p, s; g_p) \geq cs^3, \quad \forall s \leq s_0 \text{ and } \rho(p) \geq r_0 \gg 1. \]

Moreover, the injective radius of \((M, g_p)\) at \( p \) has a uniform lower bound \( \delta > 0 \), i.e.,

\[ \text{inj}(p, g_p) \geq \delta, \quad \forall \rho(p) \geq r_0. \]

**Proof.** By Lemma 3.5 we have

\[ M_{p,1} \subset B(p, 2\pi \sqrt{B} + \frac{2}{\sqrt{R_{\max}}}; g_p). \]

In the following, we give an estimate of \( \text{Vol}(\Sigma_{l, \bar{g}}) \) for any \( l \) with \( f(p) - \frac{1}{\sqrt{R(p)}} \leq l \leq f(p) + \frac{1}{\sqrt{R(p)}} \).
By (3.5) and (3.15), we see
\[ C_1^{-1} \leq \frac{R(x)}{R(p)} \leq C_1, \quad \forall \rho(p) \geq r_0 \text{ and } x \in M_{p,1}. \]

By (3.13), it follows that the Gauss curvature \( K_i \) of \((\Sigma_i, g_p|_{\Sigma_i})\) satisfies
\[ \frac{1}{4C_1} \leq K_i \leq \frac{3C_1}{4}. \]

Thus
\[ \text{Vol}(\Sigma_i, \bar{g}) = \frac{1}{R(p)} \text{Vol}(\Sigma_i, g_p|_{\Sigma_i}) \geq \frac{64\pi C_1}{R(p)}. \]

By the Co-Area formula, we get
\[
\text{Vol}(M_{p,1}, g) = \int_{f(p)-\frac{1}{\sqrt{R(p)}}}^{f(p)+\frac{1}{\sqrt{R(p)}}} \frac{\text{Vol}(\Sigma_i, \bar{g})}{|\nabla f|} dl \geq 128\pi C_1 R_{\text{max}}^{-\frac{1}{2}} R^{-\frac{3}{2}}(p).
\]

Hence
\[ (3.19) \quad \text{Vol}(B(p, 2\pi \sqrt{B} + \frac{2}{\sqrt{R_{\text{max}}}}; g_p)) \geq \text{Vol}(M_{p,1}, g_p) \geq 128\pi C_1 R_{\text{max}}^{-\frac{1}{2}}. \]

By the volume comparison theorem, we derive from (3.19),
\[
\frac{\text{Vol}(B(p, s; g_p))}{s^3} \geq \frac{\text{Vol}(B(p, 2\pi \sqrt{B} + \frac{2}{\sqrt{R_{\text{max}}}}; g_p))}{(2\pi \sqrt{B} + \frac{2}{\sqrt{R_{\text{max}}}})^3} \geq \frac{128\pi C_1 R_{\text{max}}^{-\frac{1}{2}}}{(2\pi \sqrt{B} + \frac{2}{\sqrt{R_{\text{max}}}})^3},
\]
for any \( s \leq 2\pi \sqrt{B} + \frac{2}{\sqrt{R_{\text{max}}}} \). This proves (3.17). By (3.17), we can apply a result of Cheeger-Gromov-Taylor for Riemannian manifolds with bounded curvature to get the injective radius estimate (3.18) immediately \[6\].

\[ \square \]

4. Proof of Theorem 1.2

First we prove the following convergence of rescaled flows.

**Lemma 4.1.** Under (1.2), let \( p_i \to \infty \). Then by taking a subsequence of \( p_i \) if necessary, we have
\[
(M, g_{p_i}(t), p_i) \to (\mathbb{R} \times N, \bar{g}(t); p_\infty), \quad \text{for } t \in (-\infty, 0],
\]
where \( g_{p_i}(t) = R(p_i)g(R^{-1}(p_i)t) \), \( \bar{g}(t) = ds \otimes ds + g_N(t) \) and \((N, g_N(t))\) is an ancient solution of Ricci flow on \( N \).
Proof. For a fixed $\tilde{\tau}$, as in (3.5), it is easy to see that there exists a uniform $C_1$ independent of $\tilde{\tau}$ such that
\begin{equation}
R(x) \leq C_1 R(p_i), \quad \forall \ x \in M_{p_i, \tilde{\tau}\sqrt{R_{\max}}}
\end{equation}
as long as $i$ is large enough. By Lemma 3.3 it follows
\begin{equation}
R_{g_{p_i}}(x) \leq C_1, \quad \forall \ x \in B(p_i, \bar{\tau}; g_{p_i}),
\end{equation}
where $g_{p_i} = g_{p_i}(0)$. Since the scalar curvature is increasing along the flow, we get
\begin{equation}
|Rm_{g_{p_i}(t)}(x)|_{g_{p_i}(t)} \leq 3R_{g_{p_i}(t)}(x)
\end{equation}
\begin{equation}
\leq 3R_{g_{p_i}}(x) \leq 3C_1, \quad \forall \ x \in B(p_i, \bar{\tau}; g_{p_i}), \ t \in (-\infty, 0].
\end{equation}
Thus together with the injective radius estimate in Proposition 3.6 we can apply the Hamilton compactness theorem to see that $g_{p_i}(t)$ converges subsequently to a limit flow $(\tilde{M}, \tilde{g}(t); p_{\infty})$ on $t \in (-\infty, 0]$. Moreover, the limit flow has uniformly bounded curvature. It remains to prove the splitting property.

By Remark 3.3 we have
\begin{equation}
|Ric|(x) \leq CR(x), \quad \forall \ x \in B(p_i, \bar{\tau}; g_{p_i}).
\end{equation}
It follows from (4.1),
\begin{equation}
|Ric|(x) \leq CR(p_i), \quad \forall \ x \in B(p_i, \bar{\tau}; g_{p_i}).
\end{equation}
Let $X_{(i)} = R(p_i)^{-\frac{1}{2}} \nabla f$. Then
\begin{equation}
\sup_{B(p_i, r_0; g_{p_i})} |\nabla_{(g_{p_i})} X_{(i)}|_{g_{p_i}} = \sup_{B(p_i, r_0; g_{p_i})} \frac{|Ric|}{\sqrt{R(p_i)}}
\end{equation}
\begin{equation}
\leq C \sqrt{R(p_i)} \to 0.
\end{equation}
On the other hand, by Remark 3.3 we also have
\begin{equation}
\sup_{B(p_i, r_0; g_{p_i})} |\nabla_{(g_{p_i})}^m X_{(i)}|_{g_{p_i}} \leq C_n \sup_{B(p_i, r_0; g_{p_i})} |\nabla_{(g_{p_i})}^{m-1} Ric(g_{p_i})|_{g_{p_i}} \leq C_1.
\end{equation}
Thus $X_{(i)}$ converges subsequently to a parallel vector field $X_{(\infty)}$ on $(\tilde{M}, \tilde{g}(0))$. Moreover,
\begin{equation}
|X_{(i)}|_{g_{p_i}}(x) = |\nabla f|(p_i) = \sqrt{R_{\max}} + o(1) > 0, \quad \forall \ x \in B(p_i, r_0; g_i),
\end{equation}
as long as $f(p_i)$ is large enough. This implies that $X_{(\infty)}$ is non-trivial. Hence, $(\tilde{M}, \tilde{g}(t))$ locally splits off a piece of line along $X_{(\infty)}$. It remains to show that $X_{(\infty)}$ generates a line through $p_{\infty}$.

By Lemma 3.5
\begin{equation}
M_{p_i, k} \subset B(p_i, 2\pi \sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; g_{p_i}(0)), \quad \forall \ p_i \to \infty.
\end{equation}
Let \( \gamma_{i,k}(s), s \in (-D_{i,k}, E_{i,k}) \) be an integral curve generated by \( X_{(i)} \) through \( p_{i,k} \), which restricted in \( M_{p_{i,k}} \). Then \( \gamma_{i,k}(s) \) converges to a geodesic \( \gamma_{\infty}(s) \) generated by \( X_{(\infty)} \) through \( p_{\infty} \), which restricted in \( B(p_{\infty}, 2\pi \sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; \bar{g}(0)) \). Let \( L_{i,k} \) be lengths of \( \gamma_{i,k}(s) \) and \( L_{\infty,k} \) length of \( \gamma_{\infty}(s) \),
\[
L_{i,k} = \int_{-D_{i,k}}^{E_{i,k}} |\nabla f|_{g_{p_{i,k}}(0)} ds = \int_{f(p_{i})}^{f(p_{i})+\frac{k}{\sqrt{R(p_{i})}}} \sqrt{R(p_{i})} \|\nabla f\|_{\bar{g}} ds \geq R_{\max}^{-\frac{1}{2}} \int_{f(p_{i})}^{f(p_{i})+\frac{k}{\sqrt{R(p_{i})}}} \sqrt{R(p_{i})} \|\nabla f\|_{\bar{g}} ds \geq R_{\max}^{-\frac{1}{2}} \int_{f(p_{i})}^{f(p_{i})+\frac{k}{\sqrt{R(p_{i})}}} \sqrt{R(p_{i})} ds = 2R_{\max}^{-\frac{1}{2}} k,
\]
and so,
\[
L_{\infty,k} \geq \frac{1}{2} L_{i,k} \geq R_{\max}^{-\frac{1}{2}} k.
\]
Thus \( X_{(\infty)} \) generates a line \( \gamma_{\infty}(s) \) through \( p_{\infty} \) as \( k \to \infty \). As a consequence, \( (\bar{M}, \bar{g}(0)) \) splits off a line and so does the flow \( (\bar{M}, \bar{g}(t); p_{\infty}) \). The lemma is proved.

\( \square \)

Next we estimate the curvature of \( (N, g_N(t)) \).

**Lemma 4.2.** Under \( (\ref{4.2}) \), there exists a constant \( C \) independent of \( t \) such that the scalar curvature \( R_N(t) \) of \( (N, g_N(t)) \) satisfies
\[
\frac{R_N(x, t)}{R_N(y, t)} \leq C, \quad \forall \ x, y \in N, \ t \in (-\infty, 0].
\]

**Proof.** Let \( \bar{R}(x, t) \) be the scalar curvature of \( (\mathbb{R} \times N, \bar{g}(t)) \). It suffices to prove the following is true:
\[
\frac{\bar{R}(x, t)}{\bar{R}(y, t)} \leq C, \quad \forall \ x, y \in \mathbb{R} \times N, \ t \in (0, \infty],
\]
for some constant \( C \). For any \( x, y \in \mathbb{R} \times N \), we choose \( \bar{\tau} > 0 \) such that \( x, y \in B(p_{\infty}, \bar{\tau}; \bar{g}(0)) \). By the convergence of \( g_{p_{i,t}}(t) \), there are sequences \( \{x_i\} \) and \( \{y_i\} \) in \( B(p_{i,t}, \bar{\tau}; g_{p_{i,t}}(0)) \) such that \( x_i \) and \( y_i \) converge to \( x \) and \( y \) in the Cheeger-Gromov sense, respectively. By Lemma \( \ref{3.1} \) we have
\[
x_i, y_i \subseteq B(p_{i,t}, \bar{\tau}; g_{p_{i,t}}(0)) \subseteq M_{p_{i,t}, \bar{\tau} \sqrt{R_{\max}}},
\]
Thus
\[
f(x_i) = (1 + o(1)) f(p_{i,t}) \quad \text{and} \quad f(y_i) = (1 + o(1)) f(p_{i,t}), \quad \text{as} \ p_{i,t} \to \infty.
\]
On the other hand, for a fixed \( t < 0 \),
\[
\frac{f(\phi_{R^{-1}(p_i) t}(x_i)) - f(x_i)}{|R^{-1}(p_i) t|} = \frac{\int_{R^{-1}(p_i) t}^{0} \nabla f^2 ds}{|R^{-1}(p_i) t|} \to R_{\max}, \text{ as } p_i \to \infty
\]
and
\[
\frac{f(\phi_{R^{-1}(p_i) t}(y_i)) - f(y_i)}{|R^{-1}(p_i) t|} = \frac{\int_{R^{-1}(p_i) t}^{0} \nabla f^2 ds}{|R^{-1}(p_i) t|} \to R_{\max}, \text{ as } p_i \to \infty.
\]
By (4.4) and the fact
\[
C_1 \leq R(x) f(x) \leq C_2, \quad \forall \text{ } f(x) >> 1,
\]
we get
\[
\frac{f(\phi_{R^{-1}(p_i) t}(x_i))}{f(\phi_{R^{-1}(p_i) t}(y_i))} \to 1, \text{ as } p_i \to \infty.
\]
It follows
\[
\frac{R(\phi_{R^{-1}(p_i) t}(x_i))}{R(\phi_{R^{-1}(p_i) t}(y_i))} \leq \frac{C_2}{C_1}.
\]
Hence we obtain
\[
\frac{R_N(x, t)}{R_N(y, t)} = \lim_{i \to \infty} \frac{R^{-1}(p_i) R(x_i, R^{-1}(p_i) t)}{R^{-1}(p_i) R(y_i, R^{-1}(p_i) t)} = \frac{\lim_{i \to \infty} \frac{R^{-1}(p_i) R(\phi_{R^{-1}(p_i) t}(x_i))}{R^{-1}(p_i) R(\phi_{R^{-1}(p_i) t}(y_i))}}{C_2} \leq \frac{C_2}{C_1}.
\]
This proves (4.3).

The proof of Theorem 1.2 is completed by the following lemma.

**Lemma 4.3.** \((N, g_N(t))\) in Lemma 4.1 is a shrinking spheres flow. Namely,

\[
(N, g_N(t)) = (S^2, (2 - 2t)g_{S^2}).
\]

**Proof.** By Lemma 4.2, the Gauss curvature of \((N, g_N(0))\) has a uniform positive lower bound. Then \(N\) is compact by Myer’s Theorem. On the other hand, by a classification theorem of Daskalopoulos-Hamilton-Sesum [8], an ancient solution on a compact surface \(N\) is either a shrinking spheres flow or a Rosenau solution. The Rosenau solution is obtained by compactifying \((\mathbb{R} \times S^1(2), h(x, \theta, t) = u(x, t)(dx^2 + d\theta^2))\) by adding two points, where \(u(x, t) = \frac{\sinh(-t)}{\cosh(x) + \cosh(t)}\) and \(t \in (-\infty, 0)\). By a direct computation,

\[
R_{h(t)} = \frac{\cosh(t) \cosh(x) + 1}{\sinh(-t)(\cosh(x) + \cosh(t))}.
\]
It is easy to check that $R_{h(t)}$ doesn’t satisfy (4.2) in Lemma 4.2 as $t \to -\infty$. Hence, $(N, g_{N(t)})$ must be a shrinking spheres flow on $S^2$. Note that $\bar{R}(p_\infty, 0) = 1$. Then it is easy to see that $g_{S^2(t)} = (2 - 2t)g_{S^2}$.

□

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