Generalized Linear Model for Gamma Distributed Variables via Elastic Net Regularization

Xin Chen, Aleksandr Y. Aravkin, and R. Douglas Martin

April 24, 2018

Abstract

The Generalized Linear Model (GLM) for the Gamma distribution (glmGamma) is widely used in modeling continuous, non-negative and positive-skewed data, such as insurance claims and survival data. However, model selection for GLM depends on AIC/BIC criteria, which is computationally impractical for even a moderate number of variables. In this paper, we develop variable selection for glmGamma using elastic net regularization (glmGammaNet), for which we provide an algorithm and implementation. The glmGammaNet model is more challenging than other more common GLMs as the likelihood function has no global quadratic upper bound, and we develop an efficient accelerated proximal gradient algorithm using a local model. We report simulation study results and discuss the choice of regularization parameter. The method is implemented in the R package glmGammaNet.

1 Introduction

Generalized Linear Models (GLMs) (McCullagh and Nelder 1989) are used for inference when outcomes are binary, multinomial, count, or non-negative. Regularization plays a key role for many GLM formulations; in particular the $\ell_1$ norm (Tibshirani 1996) and elastic net (Zou and Hastie 2005), a linear combination of the $\ell_1$ and quadratic loss, are frequently used to select the most important predictors and predictor groups from a large set of candidate variables.

We focus on GLM models with Gamma-distributed response variables (i.e. the responses are non-negative). This work is motivated by a recent effort to estimate the standard errors of nonparametric sample estimators for risk and performance measures (Chen and Martin 2017). Estimation
and cross-validation requires many evaluations of glmGammaNet, so the approach must be parallelizable. There is currently no R package that implements a parallelizable GLM for Gamma, so the current work fills this gap. Table 1 is a summary of existing R packages for GLM, to the authors’ best knowledge. In particular, we provide an efficient, parallelizable package that can fit a GLM model with EN regularization for the Gamma family.

| Package      | Function     | Support Gamma Dist | Model Selection          | Multicore Parallel |
|--------------|--------------|--------------------|--------------------------|--------------------|
| glmnet       | glmnet()     | No                 | EN Regularization        | Yes                |
| h2o          | h2o.glm()    | No                 | EN Regularization        | No                 |
| stats        | glm()        | Yes                | AIC/BIC                  | Yes                |
| bestglm      | bestglm()    | No                 | Subset AIC/BIC           | Yes                |
| glmGammaNet  | glmGammaNet()| Yes                | EN Regularization        | Yes                |

Table 1: Comparison of R implementations for GLM

The optimization problem for the Gamma family is more challenging than that for linear or logistic regression. The objective function required to perform the inference does not have a global quadratic upper bound. Such bounds are very useful for designing simple and efficient first-order methods for penalized log-likelihood estimation. Without the bound, a line search is needed in theory to ensure descent. Instead, we estimate a quadratic bound locally using the functional form of the Gamma to get a fast and robust method for the problem. We implement a safeguard line search, but it is never activated.

The paper proceeds as follows. In Section 1.1, we give a brief survey of GLM use cases and algorithms. We also discuss the role of regularization and its impact on choice of algorithm. In Section 2, we formulate the Gamma inference problem, discuss first-order methods for elastic-net (EN) regularization, and detail the algorithm we implemented. Section 4 presents simulation results. We end with a discussion in Section 5.

1.1 Related work

First introduced by Nelder and Wedderburn (1972), GLM has been used for a variety of applications, including Binary logistic regression, Multinomial logistic regression, ordinal logistic regression and Poisson regression, see for example McCullagh and Nelder (1989) and Dobson and Barnett (2008). Gamma GLMs are used to model right-skewed non-negative data, such as insurance claims (Jong 2008), Semiconductor Wafer sensitivity (Myers and Montgomery 1997), clotting times of blood (McCullagh and Nelder 1989) and Survival Function of Diabetic Nephropathy Pa-
tients (Grover, Sabharwal, and Mittal 2013). In R, GLMs are often fit using the \texttt{glm()} function in the \texttt{stats} package.

Model selection is essential for GLMs. While classic approaches use AIC/BIC criteria (see e.g. Bozdogan 1987, Burnham and Anderson 2003, Burnham and Anderson 2004), sparsity-based regularization is very useful (Tibshirani 1996; Zou and Hastie 2005).

## 2 First-Order Methods for Regularized GLM

The GLM inference problem is formulated as follows. Suppose we wish to predict an output $b$ of a certain system on an input $a \in \mathbb{R}^n$. Let us also make the following two assumptions: (i) the relationship between the input $a$ and the output $b$ is fairly simple and (ii) we have available examples $a_i \in \mathbb{R}^n$ together with inexact observed responses $b_i$ for $i = 1, \ldots, m$. The tuples $\{(b_i, a_i)\}_{i=1}^m$ comprise the training data. The responses $b_i$ can have special restrictions; for example they may be counts, indicate class membership, or be non-negative, such as concentration of sugar in the blood. To build the GLM, suppose the distribution of $b_i$ is parametrized by $(\mu_i, \sigma^2)$:

$$L(b_i | \mu_i, \sigma^2) = g_1(b_i, \sigma^2) \exp \left( \frac{b_i \mu_i - g_2(\mu_i)}{g_3(\sigma^2)} \right).$$

To obtain the GLM objective, set $\mu_i := \langle a_i, x \rangle$, and take the negative log-likelihood (ignore $g_1$ and $g_3$ as they do not depend on $x$):

$$\min_x \sum_{i=1}^m g_2(\langle a_i, x \rangle) - b_i \langle a_i, x \rangle \quad (1)$$

Common examples are shown in Table 2.

| Model   | Restriction on $b_i$ | $g_2(z)$ |
|---------|----------------------|----------|
| Regression | None | $\frac{1}{2}||z||^2$ |
| Classification | $b_i \in \{0, 1\}$ | $\log(1 + \exp(z))$ |
| Counts       | $b_i \in \mathbb{Z}_+$ | $\exp(z)$ |
| Non-negative | $b_i \geq 0$ | $-\ln(z)$ |

Table 2: Common Generalized Linear Models
We are interested in an extension of (1) that includes nonsmooth regularization terms $R(x)$ (including 1-norm, elastic net, or constraints):

$$
\min_x \sum_{i=1}^m g_2(\langle a_i, x \rangle) - b_i \langle a_i, x \rangle + R(x)
$$  \hspace{1cm} (2)

A simple strategy for optimizing models of form (2) is to develop simple upper bounds and minimize them. If $g$ is smooth, its gradient is said to satisfy the Lipschitz property with constant $L$ if

$$
\|\nabla g(x) - \nabla g(y)\| \leq L \|x - y\| \quad \forall x, y.
$$  \hspace{1cm} (3)

When $g$ is twice continuously differentiable, any bound on the operator norm of $\nabla^2 g$ is a Lipschitz constant for $\nabla g$. For example, if $g(x) = \frac{1}{2} \|Ax - b\|^2$, the Lipschitz constant for $\nabla g(x) = A^T(Ax - b)$ is the largest eigenvalue of $A^T A$. Any Lipschitz constant for $\nabla g$ gives a simple tight global upper bound for $g$:

$$
g(x) \leq g(x_0) + (x - x_0)^T \nabla g(x_0) + \frac{L}{2} \|x - x_0\|^2.
$$

In the context of model (2), suppose that $\nabla g_2$ has Lipschitz constant $l$. Then if we define

$$
g(x) = \sum_{i=1}^m g_2(\langle a_i, x \rangle) - b_i \langle a_i, x \rangle,
$$

the reader can immediately see that the Lipschitz constant of $\nabla g$ is bounded above by $l \|A\|^2$. A simple iterative strategy is to minimize the upper bound for $g$ at each iteration, without modifying $R(x)$, which may be non-smooth (1-norm) or infinite valued (box-constraint). Given iterate $x^k$, the next iterate $x^+$ is found as follows:

$$
x^+ = \arg \min_x (x - x^k)^T \nabla g(x^k) + \frac{L}{2} \|x - x^k\|^2 + R(x)
$$

$$
= \arg \min_x \frac{1}{2L} \|x - (x^k - \frac{1}{L} \nabla g(x^k))\|^2 + R(x)
$$

$$
:= \text{prox}_{\frac{1}{L} R}(x^k - \frac{1}{L} \nabla g(x^k)).
$$  \hspace{1cm} (4)
The proximity operator $\operatorname{prox}_{\frac{1}{L}R}(z)$ defined in (4) should be thought of as a simple subroutine. It is the minimizer of the problem

$$\min_x \frac{1}{2L} \|x - z\|^2 + R(x),$$

and is available in closed form for a wide variety of regularizers $R(x)$, including 1-norm, the elastic net, and simple constraints (Combettes and Pesquet 2011).

**Algorithm 1** FISTA for Regularized GLM

1. Initialize $x^1 = 0$, $\omega = 0$, $\kappa = 0$, $s_1 = 1$, compute $d^1 = \nabla g(x^1)$. Let $L$ be a Lipschitz constant for $g$.

2. While $\|\operatorname{prox}_{R}(\omega^\kappa - d^\kappa)\| > \epsilon$
   - Set $\kappa = \kappa + 1$.
   - update $x^\kappa = \operatorname{prox}_{L^{-1}R}(\omega^{\kappa-1} - \alpha d^{\kappa-1})$.
   - set $s_\kappa = \frac{1 + \sqrt{1 + 4s_{\kappa-1}^2}}{2}$
   - set $\omega^\kappa = x^\kappa + \frac{s_{\kappa-1}}{s_\kappa}(x_\kappa - x_{\kappa-1})$.
   - Compute $d^\kappa = \nabla g(x^\kappa)$.

3. Output $x^\kappa$.

The iteration (4) is known as the proximal gradient iteration, and converges with the same rates as gradient descent on the smooth function $g$ (Nesterov 2013). The iteration can be accelerated to achieve a better rate of convergence by using an auxiliary iterative sequence; the most famous example of such an algorithm is FISTA (Beck and Teboulle 2009). The FISTA algorithm is only slightly more complicated than (4), and is detailed in Algorithm 1.

Linear regression and logistic regression have easily computable Lipschitz constants of $\|A\|^2$ and $\frac{1}{4}\|A^2\|$ respectively. Unfortunately, the models for the Gamma and Poisson GLMs are constructed using logarithmic and exponential $g_2$, and these functions do not have a global quadratic upper bound. A safeguard linesearch is required to ensure descent, i.e., that the objective at $x^{+}$ is smaller than the objective at $x^{\kappa}$ unless $x^{\kappa}$ is a stationary point. A FISTA with line search replaces $L^{-1}$ in Algorithm 1 with an iterative step-size selection procedure to ensure

$$g(x^{k+1}) + R(x^{k+1}) < g(x^k) + R(x^k).$$
However, in practice Algorithm 1 is a descent method when $L$ is locally estimated at each iteration using a simple heuristic. We discuss the heuristic and other specifics of the Gamma GLM in the next section.

3 Adapting FISTA to the Gamma Family with EN Regularization

In this section, we adapt the general scheme of fitting GLM model with regularization term to the special case of GLM for gamma distributed response variables with the elastic net regularization (glmGammaNet). The optimization problem is as follows

$$\min_x NLL(x) + R_{EN}(x; \lambda, \alpha)$$

(5)

where $NLL(x)$ is the negative log-likelihood, $\lambda$ is the regularization parameter and $R_{EN}(x; \lambda, \alpha)$ is the Elastic Net regularization term. The section proceeds as follows. Section 3.1 derives the negative log-likelihood (NLL), the gradient of the NLL and the proximal of the EN regularization term. In Section 3.2 we develop a customized method of obtaining a local approximation of the Lipschitz constant to avoid a line search (which requires additional function evaluations). Section 3.3 shows how cross-validation can be used to select the optimal regularization parameter and presents the complete algorithm to fit GLM for gamma-distributed responses with EN regularization.

3.1 NLL, $\nabla NLL$ and Proximal of EN

The probability density function of gamma distribution is given by

$$f(b; k, \theta) = \frac{1}{\Gamma(k)\theta^k} b^{k-1} e^{-\frac{b}{\theta}}, \ k > 0, \ \theta > 0,$$

where $k$ is the shape parameter and $\theta$ is the scale parameter. The expectation of a gamma random variable $B$ is given by

$$E(B) = k\theta.$$
Using the logarithm link function, the relationship between the expectation of $B_i$ and the linear component of GLM is given by

$$E(B_i) = \log(k \theta_i) = A_i \cdot x,$$

where $k$ is the shape parameter, which is assumed to be the same for all examples, $\theta_i$ is the scale parameter for the $i$th example, $A_i$ is the $i$th row of the data matrix $A$, $x$ is the coefficients for the GLM model. Therefore, the scale parameter can be written as

$$\theta_i = e^{A_i \cdot x} / k.$$

The objective function of problem 5 over all examples is given by

$$H(x; b, k, \lambda, \alpha) = NLL(b; k, x) + R_{EN}(x; \lambda, \alpha)$$

$$= -\sum_{i=1}^{N} \log f(b_i; k, \theta_i) + R_{EN}(x; \lambda, \alpha)$$

$$= \sum_{i=1}^{N} \log \Gamma(k) + k \log \theta_i - (k - 1) \log b_i + \frac{b_i}{\theta_i} + R_{EN}(x; \lambda, \alpha)$$

$$= \sum_{i=1}^{N} \log \Gamma(k) + k \cdot A_i \cdot x - k \cdot \log k - (k - 1) \log b_i + k \cdot b_i e^{-A_i \cdot x} + \lambda \left( \alpha \|x\|_1 + \frac{1 - \alpha}{2} \|x\|_2^2 \right).$$

(6)

The partial derivative of $NLL$ with respect to $x_j$ is

$$\frac{\partial NLL}{\partial x_j} = \sum_{i=1}^{N} k \cdot \left( 1 - b_i e^{-A_i \cdot x} \right) A_{ij}.$$  

(7)

In order to use FISTA, we also need the proximity operator (prox) of the elastic regularization term

$$R_{EN}(x; \lambda, \alpha) = \lambda \left( \alpha \|x\|_1 + \frac{1 - \alpha}{2} \|x\|_2^2 \right).$$

From (Parikh and Boyd 2014), the prox of $R_{EN}(x; \lambda, \alpha)$ is given by
\[ \text{prox}_{tR \kappa}(v) = \frac{1}{1 + t \lambda (1 - \alpha)} \text{sgn}(v) \max(|v| - t \lambda \alpha, 0), \]  

(8)

where \( v \) is a vector, while \( \text{sgn}(v) \) and \( \max(v) \) act on \( v \) element-wise.

### 3.2 Computing a Local Upper Bound

Algorithm 1 requires a global Lipschitz constant for the gradient of the objective. However, the gradient does not have a global Lipschitz constant, because the exponential function cannot have a global quadratic upper bound. A local quadratic approximation can be computed efficiently and used in lieu of a global Lipschitz constant. The local quadratic upper bound for the Gamma model is given by

\[ L(x) = \|A\|_F^2 \left( \sum_{i=1}^{N} k^2 \cdot \left(1 - b_i e^{-A_i \cdot x}\right)^2 \right). \]  

(9)

The idea behind (9) is to get a data-dependent local quadratic upper bound, analogous to those we have for linear and logistic regression. The bound we use is conservative in practice, since we never need to activate the safe-guard line search. However, it cannot be too conservative, as we see fast performance across the testbed of problems, as discussed in the numerical experiments.

### 3.3 Optimal \( \lambda \) via Cross Validation

With the results from section (3.1) and (3.2), we are able to solve the optimization problem (5) if \( \lambda \) and \( \alpha \) are given. The choice of \( \lambda \) and \( \alpha \) has a strong impact on the problem. Following the suggestions by (Friedman, Hastie, and Tibshirani 2010), we assume that \( \alpha \) is determined by the user and focus on using cross-validation to find the optimal value for \( \lambda \). The cross-validation procedure is as follows:

1. Compute the smallest \( \lambda \) that gives an all-zero solution for the regularized problem, call this \( \lambda_{\text{max}} \).

2. The lower bound of the grid is given by \( \lambda_{\text{min}} = \epsilon \cdot \lambda_{\text{max}} \), where \( \epsilon \) is a user-defined constant with a default value of 0.001.
3. The grid of $\lambda$ consists of $n_{\lambda}$ values between $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ equally spaced in the log scale, where $n_{\lambda}$ is a user-defined constant with a default value of 100.

4. For each $\lambda$, perform n-fold cross validation and compute the mean NLL, where n is a user-defined constant with a default value of 10.

5. Choose the $\lambda$ with the smallest NLL for use in the final model.

**Computing $\lambda_{\text{max}}$**

The value $\lambda_{\text{max}}$ should be such that $x_k = 0$ satisfies the optimality conditions for the problem. Denoting the $j$th element of $x_k$ as $x_j^j$, optimality is equivalent to a fixed point condition across all $j$:

$$
 x_j^j = \text{prox}_{\frac{1}{L} \nabla \text{NLL}}(x_j^j - \frac{1}{L} \nabla \text{NLL})^j = \frac{1}{1 + \frac{1}{L} \lambda (1 - \alpha)} \text{sgn}\left(x_j^j - \frac{1}{L} \nabla \text{NLL}\right) \max\left(\left|x_j^j - \frac{1}{L} \nabla \text{NLL}\right| - \frac{1}{L} \lambda \alpha, 0\right)
$$

The definition of $\lambda_{\text{max}}$ requires the fixed point condition above to hold when $x_j^j = 0$ for all $j$, so we have

$$
 0 = \text{prox}_{\frac{1}{L} \nabla \text{NLL}}(-\frac{1}{L} \nabla \text{NLL})^j = \frac{1}{1 + \frac{1}{L} \lambda (1 - \alpha)} \text{sgn}\left(-\frac{1}{L} \nabla \text{NLL}\right) \max\left(\left|\frac{1}{L} \nabla \text{NLL}\right| - \frac{1}{L} \lambda \alpha, 0\right) \quad (10)
$$

A sufficient condition to ensure above is that

$$
 \lambda \alpha \geq \left|\frac{1}{L} \nabla \text{NLL}\right| = \left|\sum_{i=1}^{N} \left(k - \frac{b_i}{\theta_i}\right) A_{ij}\right| = \left|\sum_{i=1}^{N} \left(k - \frac{b_i}{1/k}\right) A_{ij}\right| = \left|\sum_{i=1}^{N} k (1 - b_i) A_{ij}\right|
$$

To make sure that $x_k = 0$ satisfies the optimality condition across all $j$, we take $\lambda_{\text{max}}$ to be the largest of all such $\lambda$:

$$
 \lambda_{\text{max}} = \max_j \left|\sum_{i=1}^{N} k (1 - b_i) A_{ij}\right|.
$$
3.4 GLM for Gamma Response Variables with Elastic Net (glmGammaNet)

Algorithm 2 gives the complete pseudo code for glmGammaNet.

**Algorithm 2 glmGammaNet**

1. Set $A$, $b$, $\lambda_{max}$, $\epsilon$ and $n_\lambda$
2. Compute $\lambda_{max} = \max_j \left| \sum_{i=1}^{N} k (1 - b_i) A_{ij} \right|$
3. Compute $\lambda_{min} = \epsilon \ast \lambda_{max}$
4. Compute vector of candidate $\lambda$’s, $\lambda_{vec} = \exp(\text{seq}(\log(\lambda_{min}), \log(\lambda_{max}), \text{length} = n_\lambda))$
5. For $j = 1 : n_\lambda$
   - $\lambda = \lambda_{vec}(j)$
   - for $i = 1 : k$
     - randomly divide $A$ into $k$ partitions. Let $A_{test}$ be one of the partitions, $A_{train}$ be the union of the rest of the partitions.
     - randomly divide $b$ into $k$ partitions. Let $b_{test}$ be one of the partitions, $b_{train}$ be the union of the rest of the partitions.
     - use the FISTA algorithm on the model defined $A_{train}$, $b_{train}$ and $\lambda$ to find the solution $x_{train}$
     - $\text{NLL}_{ij} = \text{NLL}(x_{train}, A_{test}, b_{test})$
   - $\text{NLL}_{j} = \sum_i \text{NLL}_{ij}$
6. $\lambda_{best}$ is the $\lambda$ that results in the smallest value among all $\text{NLL}_{j}$
7. use the FISTA algorithm on the model defined $A$, $b$ and $\lambda_{best}$ to find the solution $x_{best}$

Two alternative ways of choosing the best $\lambda$

In Algorithm 2, we choose the $\lambda$ that minimizes the $\text{NLL}$, which seems to be the optimal choice. However, this is not always the case. It is worth noting that the $\text{NLL}$ computed in Algorithm 2 are just estimates of the true prediction errors. Therefore, there are uncertainties associated with these estimates. To account for these uncertainties, we propose two alternative ways of choosing the “best” $\lambda$. The first alternative is to choose the maximum $\lambda$ with the corresponding $\text{NLL} \leq \text{NLL}_{min} + SD_{\text{NLL}_{min}}$. The second alternative is to choose the maximum $\lambda$ with the corresponding $\text{NLL}$ smaller than the $\alpha$th percentile of all the $\text{NLLs}$ from cross-validation. We discuss the performance of both alternatives along with Algorithm 2 in the numerical experiment section.
4 Numerical Experiment

We use Monte Carlo (MC) simulations to demonstrate the superior performance of the three variants of glmGammaNet compared with the standard GLM method. glmGamma is the standard GLM for gamma responses. glmGammaNet is our method described in Algorithm 2. glmGammaNet.percentile is the percentile variant described in section 3.4 with 10th percentile threshold, and glmGammaNet.percentile.nonzero is the result of fitting the glmGamma without the zero coefficients identified by glmGammaNet.percentile. glmGammaNet.1sd is the one-standard-deviation variant described in section 3.4 and glmGammaNet.1sd.nonzero is the result of fitting the glmGamma without the zero coefficients identified by glmGammaNet.1sd. We run 1000 MC simulations. In the $i$th MC run, the following steps are performed:

1. Set $n = 100$ and $p = 15$, where $n$ is the number of examples and $p$ is the dimension of the coefficient vector $x$.
2. Generate the $n \times p$ predictor matrix $A$ from i.i.d. normal distribution $N \sim (0, 1)$.
3. Generate vector of length $p$ from i.i.d. normal distribution $N \sim (0, 1)$ and randomly set 10 of the elements to zero. Denote the resulting vector as $x_{true}$.
4. Compute vector of the true responses $b_{true} = \exp(Ax_{true})$.
5. Compute the vector of rates $\lambda_{true} = k/b_{true}$.
6. Generate $n \times 1$ vector of response variables $b$ by getting one sample from the gamma distribution specified by each element in the vector $\lambda_{true}$.
7. Use different methods to compute the solution $x_{METHOD}^{(i)}$ using $A$, $b$ as the input and save $x_{METHOD}^{(i)}$.

4.1 Error of Fitted Coefficients

We compute the following performance metrics for the error of the fitted coefficients

1. L1 Norm of the difference between $x_{METHOD}$ and $x_{true}$

$$\text{error.L1}_{METHOD} = \|x_{METHOD} - x_{true}\|$$
2. \( \text{error.L1}_{METHOD} \) as a percentage of \( \|x_{true}\| \)

\[
\% \text{ error.L1}_{METHOD} = \frac{\|x_{METHOD} - x_{true}\|}{\|x_{true}\|} \cdot 100
\]

Table 3 summarizes the error of fitted coefficients for different GLM methods.

| Method                     | error.L1 | % error.L1 |
|----------------------------|----------|------------|
| glmGamma                   | 0.41     | 10.4       |
| glmGammaNet                | 0.37     | 9.3        |
| glmGammaNet.percentile    | 0.36     | 9.2        |
| glmGammaNet.1sd           | 0.55     | 14.0       |
| glmGammaNet.percentile.nonzero | 0.33     | 8.3        |
| glmGammaNet.1sd.nonzero   | 0.23     | 5.8        |

Table 3: Performance Summary of Different GLM methods

As shown in Table 3, the glmGamma method does a reasonably good job, with a percentage error of 10.4%. By adding Elastic Net regularization and choosing the \( \lambda \) with the smallest NLL in cross-validation, we get a slight improvement in percentage error, down to 9.3%. The glmGammaNet.percentile method has a similar percentage error of 9.2%. The glmGammaNet.1sd method has the highest percentage error of 14%. However, if we drop the zero coefficients identified by glmGammaNet.percentile and glmGammaNet.1sd, and then perform a regular glmGamma, the percentage errors drop significantly. The glmGammaNet.percentile.nonzero method has a percentage error of 8.3%, down from 9.2% and the glmGammaNet.1sd.nonzero method has a percentage error of 5.8%, down from 14%. We conjecture that the reduction in percentage error is due to the variable selection power of our new methods. In the next subsection we explore the variable selection performance of different methods.

### 4.2 Variable Selection Performance Analysis

We compare the variable selection performance of different GLM methods by examining the following two statistics:

1. Number of correctly identified zero coefficients, denoted as \( \text{zeros.correct}_{METHOD} \)

2. Number of correctly identified nonzero coefficients denoted as \( \text{nonzeros.correct}_{METHOD} \)
Table 4 summarizes the variable selection performance for different GLM methods.

| Method                  | zeros.correct | nonzeros.correct |
|-------------------------|---------------|------------------|
| glmGamma                | 0             | 5                |
| glmGammaNet             | 1.976         | 5                |
| glmGammaNet.percentile | 4.771         | 5                |
| glmGammaNet.1sd         | 7.815         | 5                |

Table 4: Performance Summary of Different GLM methods

All the methods studied have successfully included the nonzero coefficients in their solution, yet the number of identified zero coefficients varies a lot. Out of the 10 zero coefficients in the true solution, the glmGamma method fails to identify any zero coefficients. On average, the glmGammaNet method manages to find roughly 2 zero coefficients. The glmGammaNet.percentile method identifies approximately 5 zero coefficients, which presents a great improvement. Most notably, the glmGammaNet.1sd method finds an incredible 8 zero coefficients on average and 80% chance of correctly identifying 7 or more zero coefficients. By dropping the variables with zero coefficients, we are essentially removing noise in the dataset, therefore, the percentage error of these methods are much better, as shown in the previous section.

Figure 1 further visualizes the distribution of the number of correctly identified zero coefficients. The blue bars represent the results for glmGammaNet. Notice that there is a 35% chance that glmGammaNet does not identify any zero coefficients and the probability decreases as the number of correctly identified zero coefficients increases. This shows that glmGammaNet is very conservative in terms of variable selection. The red bars show the results for glmGammaNet.percentile. The distribution is roughly bell-shaped and the peak occurs at 6 zero coefficients with a probability of 15%. This indicates that glmGammaNet.percentile is considerably more aggressive than glmGammaNet, but still not satisfactory. The black bars show the results for glmGammaNet.1sd. The distribution is concentrated around 8 and 9 zero coefficients, which account for more than 0.5 probability. This shows that the glmGammaNet.1sd method performs extremely well in variable selection.
5 Discussion

In this paper, we developed a parallel implementation for GLM fitting with Gamma distributed data and elastic net regularization. One reason the Gamma may not be available in standard software is that the objective function is a composition of an exponential model with a linear map, and so does not have a global quadratic upper bound. We developed a customized accelerated proximal gradient method by using local quadratic estimates; although a safeguard line search is implemented, it was never activated across our entire suite of experiments. We also provide a straightforward cross-validation scheme to determine the optimal value of the regularization parameter. Numerical experiments are show the advantage of these methods over standard GLM without regularization. The new methods have both smaller error in fitted coefficients and superior variable selection performance. The choice of regularization parameter is very important, and we recommend two simple strategies: (1) conservative: using the parameter that corresponds to the smallest negative loglikelihood in the cross validation, and (2) aggressive: using the one-standard-deviation rule.
References

Beck, A. and M. Teboulle (2009). “A fast iterative shrinkage-thresholding algorithm for linear inverse problems”. *SIAM journal on imaging sciences* 2.1, pp. 183–202 (cit. on p. 5).

Bozdogan, H. (1987). “Model selection and Akaike’s information criterion (AIC): The general theory and its analytical extensions”. *Psychometrika* 52.3, pp. 345–370 (cit. on p. 3).

Burnham, K. P. and D. R. Anderson (2003). *Model selection and multimodel inference: a practical information-theoretic approach*. Springer Science & Business Media (cit. on p. 3).

– (2004). “Multimodel inference: understanding AIC and BIC in model selection”. *Sociological methods & research* 33.2, pp. 261–304 (cit. on p. 3).

Chen, X. and R. D. Martin (2017). “Standard Errors of Risk and Performance Estimators with Serially Correlated Returns”. *Working Paper* (cit. on p. 1).

Combettes, P. L. and J.-C. Pesquet (2011). “Proximal splitting methods in signal processing”. *Fixed-point algorithms for inverse problems in science and engineering*. Springer, pp. 185–212 (cit. on p. 5).

Dobson, A. J. and A. Barnett (2008). *An introduction to generalized linear models*. CRC press (cit. on p. 2).

Friedman, J., T. Hastie, and R. Tibshirani (2010). “Regularization paths for generalized linear models via coordinate descent”. *Journal of statistical software* 33.1, p. 1 (cit. on p. 3).

Grover, G., A. S. A. Sabharwal, and J. Mittal (2013). “An application of gamma generalized linear model for estimation of survival function of diabetic nephropathy patients”. *International Journal of Statistics in Medical Research* 2.3, pp. 209–219 (cit. on p. 3).

Jong, P. d. (2008). *Generalized linear models for insurance data*. eng. Cambridge ; New York (cit. on p. 2).

McCullagh, P. and J. A. Nelder (1989). *Generalized linear models*. Vol. 37. CRC press (cit. on pp. 1, 2).

Myers, R. H. and D. C. Montgomery (1997). “A tutorial on generalized linear models”. *Journal of Quality Technology* 29.3, p. 274 (cit. on p. 2).

Nelder, J. A. and R. W. M. Wedderburn (1972). “Generalized Linear Models”. *Journal of the Royal Statistical Society. Series A (General)* 135.3, pp. 370–384. issn: 00359238. url: [http://www.jstor.org/stable/2344614](http://www.jstor.org/stable/2344614) (cit. on p. 2).

Nesterov, Y. (2013). *Introductory lectures on convex optimization: A basic course*. Vol. 87. Springer Science & Business Media (cit. on p. 5).

Parikh, N., S. Boyd, et al. (2014). “Proximal algorithms”. *Foundations and Trends® in Optimization* 1.3, pp. 127–239 (cit. on p. 7).
Tibshirani, R. (1996). “Regression shrinkage and selection via the lasso”. *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 267–288 (cit. on pp. 1, 3).

Zou, H. and T. Hastie (2005). “Regularization and variable selection via the elastic net”. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 67.2, pp. 301–320 (cit. on pp. 1, 3).