Equilibrium stability of a cylindrical body subject to the internal structure of the material and inelastic behaviour of the completely compressed matrix

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Abstract. The mathematical model describing the stress-strain state of a cylindrical body under the uniform radial compression effect is constructed. The model of the material is the porous medium model. The compressed skeleton of the porous medium possesses hardening elastic-plastic properties. Deforming of the porous medium under the specified compressive loads is divided into two stages: elastic deforming of the porous medium and further elastic-plastic deforming of the material with completely compressed matrix. The analytical relations that define the fields of stress and displacement at each stage of the deforming are obtained. The influence of the porosity and other physical, mechanical and geometric parameters of the construction on the size of the plastic zone is evaluated. The question of the ground state equilibrium instability is investigated within the framework of the three-dimensional linearized relationships of the stability theory of deformed bodies.

1. Introduction
At present, the construction and protection of underground structures of various kinds, including underground cylindrical cavities, remain the topical issue. As well, there is a number of related problems that need to be considered, such as rock burst control, underground rock blasting, pollution prevention, seismic problems, etc. The underground structural analysis includes (a) the definition of the fields of stress and displacement that arise in the structural components; (b) the definition of the conditions of structural strength and stability. According to the results of the structural analysis, the rational designs of the support constructions and the optimum dimensions of their sections are selected to ensure reliable operation of structures at the minimal cost. Ensuring the stability of shallow excavations with sufficiently strong housing rocks is generally not very difficult. At present, however, there is a steady increase in volume of excavations at great depths and at complex geological conditions (permafrost, high seismicity, neotectonic phenomena, etc.), partly due to the active development of Arctic space. When the condition of a favorable combination of depth and strength of materials is violated, the stability of the excavations and their support constructions becomes a complex engineering and scientific problem [1-11]. To solve all these problems, it is necessary to view the destruction of rock massifs and the stability of the support constructions. The destruction of the underground structure support construction can happen as a result of following two situations: (1) The stress-strain state reaches
the strength limits; (2) The stress-strain state reaches the critical values corresponding to the loss of stability (failure) of the support construction. The solution of the first problem is based on the comparison of the stress-strain state, which can be found in analytical or numerical form, and the material ultimate strength [1]. In the second case, the initial phase of the stability problem solution is to find the ground stress-strain state of the structure in the analytical view [2-8]. As a result, it is a matter of urgency to obtain analytical relations describing the subcritical stress-strain state in the analytical form. Cylindrical constructions are widely used in the study of rock mechanics, building mechanics and engineering. Therefore, the question of the definition and analysis of stress-strain states and the study of its stability, taking into account the different physical, mechanical and geometric parameters for cylindrical structures, is the subject of considerable recent and ongoing research.

2. Material model

In the paper under consideration, the deforming of the porous medium in the inelastic behavior of the completely compressed skeleton is described within the framework of the model analysis proposed in the paper [4].

The deforming of the porous material with a characteristic value $\varepsilon_0$, determined by the specific volume of pores, is divided into two stages: deforming of the material before and after the pores are compressed at a given point in space. The approach used to determine the stress-strain state in this paper differs from the one used in work [5]. (In [5] some analytical relations that use the values of applied loads and allow to determine the existence of partly compressed pores at a given point in space were obtained. To study the process of pore compression for the part of the medium with completely compressed pores it was necessary to introduce an additional term in the rheological relationships).

In this paper, as well as in [5], we obtain the conditions of complete pore compression at a given point in space. As opposed to [5], to determine the stress-strain state in the completely compressed area, we have to consider two independent problems: (1) obtaining the history of deforming and deformations before the pores are compressed (the first stage); (2) investigation of the further deforming and obtaining the deformations of the compressed skeleton (the second stage). In the first stage, we consider elastic deforming of the porous medium under the pressure of the loads at which complete pore compression occurs (these loads will be further defined). In the second stage, we consider inelastic deforming of the compressed skeleton under the effects of the initial loads without the portion of loads that caused the complete pore compression.

Thus, the stress-strain state of the first stage is assumed to be the initial state of the body for the second stage of the deforming. The resulting stress-strain state is obtained by composition of solutions from the first and the second stages.

The obtained results have made it possible to move forward with the stability of the ground stress-strain state in the framework of three-dimensional linearized stability theory of the deformed bodies.

3. Main relationships used for model-based analysis of the stress-strain state of the porous body in the inelastic behaviour of the completely compressed skeleton

The resulting stress-strain state is obtained by composition of solutions from each stage. We use the following formulas for displacement, deformation and stress, respectively

$$u_i = u^{(1)}_i + u^{(2)}_i; \quad \varepsilon_{ij} = \varepsilon^{(1)}_{ij} + \varepsilon^{(2)}_{ij} + \varepsilon^{(1)}_{ij} \varepsilon^{(2)}_{ij}; \quad \sigma_{ij} = \sigma^{(1)}_{ij} + \sigma^{(2)}_{ij},$$

where the values with (1)-superscript refer to the first stage, the ones with (2)-superscript refer to the second stage.

The relation between stresses and deformations at the first stage of deforming is taken from Hooke’s law for the compressible body. In the second stage, the elastic deformation of the compressed skeleton is subjected to the Hooke’s law for the incompressible body. In the plastic deforming zone of the compressed skeleton, we use the incompressible hardening elastic-plastic body model [12] with the load surface
where \( S^\beta_j \) are the components of the deviator stress tensor; \( \varepsilon^\beta_j \) are the components of plastic strain tensor; \( c \) is the strengthening factor of the material with the completely compressed matrix; \( k \) is the fluidity limit of the material with the completely compressed matrix.

In the inelastic deforming zone that occurs in a region of completely compressed pores, the total deformation is composed of the elastic and plastic parts

\[
\varepsilon^\beta_j = \varepsilon^\beta_e + \varepsilon^\beta_p .
\]  

(3)

The plastic and elastic components of the volumetric deformation respectively satisfy the conditions of incompressibility

\[
\varepsilon^\beta_n = 0 , 
\varepsilon_e^\beta_n = -\varepsilon_0 ,
\]  

(4)

where \( \varepsilon^\beta_j \) and \( \varepsilon^\beta_e \) are the components of the total deformation tensor and elastic deformation tensor, respectively, \( \varepsilon_0 \) is the value determined by the specific volume of pores.

4. Mathematical modelling of the stress-strain state of the porous cylindrical body under uniform radial compression

Consider the stress-strain state problem for a cylindrical heavy-wall body with outer and inner radii \( b \) and \( a \), respectively (figure 1). The outer and inner surfaces are under uniformly distributed compressive loads with the intensities \( q_b \) and \( q_a \), respectively.

![Figure 1. The cylindrical body under uniform radial compression](image)

The axisymmetric stress-strain state is evaluated at the first stage of deforming within the framework of plane deformation, in the cylindrical coordinate system \((r, \theta, z)\). We will use the following relations.
of geometric linear theory:
the equilibrium equation
\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0;
\] (5)
the Cauchy relations
\[
e_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = 0;
\] (6)
the Hooke’s law for the elastic compressible body
\[
\sigma_r = (\lambda_1 + 2\mu_1)\epsilon_r + \lambda_1\epsilon_\theta, \quad \sigma_\theta = \lambda_1\epsilon_r + (\lambda_1 + 2\mu_1)\epsilon_\theta, \quad \sigma_z = \lambda_1(\epsilon_r + \epsilon_\theta).
\] (7)
Hereinafter, \(u\) is the radial component of the displacement vector, \(\lambda_1\) and \(\mu_1\) are Lamé parameters for the compressible body.
The boundary conditions
\[
\sigma_r \bigg|_{r=b} = -q_s, \quad \sigma_\theta \bigg|_{r=a} = -q_s.
\] (8)
From (5)–(7), the volumetric deformation can be written as \(\epsilon_r + \epsilon_\theta = 2C\), where \(C\) is an integration constant.
Hence, the volumetric deformation is independent on a coordinate, being the same at each point of the body under consideration. Therefore, complete pore compression occurs in the entire body when the volumetric deformation reaches some given value.
Let the limit value of the volumetric deformation be equal to \(-\epsilon_0\) (\(\epsilon_0 > 0\) is the value defining by the ratio of the total specific volume of pores to the volume of the medium). Then the condition of the existence of uncompressed pores in the cylindrical body is
\[-2C < \epsilon_0.\] (9)
Subject to the boundary conditions (8), inequality (9) takes the form
\[
\frac{q_s b^2 - q_a a^2}{(\lambda_1 + \mu_1)(b^2 - a^2)} < \epsilon_0.
\] (10)

4.1. Displacement, deformation and stress at the elastic deforming stage of the porous medium
If inequality (10) takes place, the body has uncompressed pores and the stress-strain state is described by the following relations
\[
u = -\frac{\epsilon_0}{2} + \frac{q_a - \epsilon_0(\lambda_1 + 1)}{2} \cdot \frac{a^2}{r^3};
\]
\[
ev = -\frac{\epsilon_0}{2} - \frac{q_a - \epsilon_0(\lambda_1 + 1)}{2} \cdot \frac{a^2}{r^2}, \quad \epsilon_\theta = \frac{\epsilon_0}{2} + \frac{q_a - \epsilon_0(\lambda_1 + 1)}{2} \cdot \frac{a^2}{r};
\]
\[
\sigma_r = -\epsilon_0(\lambda_1 + 1) - \frac{(q_a - \epsilon_0(\lambda_1 + 1))a^2}{r^2}, \quad \sigma_\theta = -\epsilon_0(\lambda_1 + 1) + \frac{(q_a - \epsilon_0(\lambda_1 + 1))a^2}{r^2}, \quad \sigma_z = -\lambda_1\epsilon_0.
\] (11)
In (11) all values are used without the superscripts since the second stage of the deforming doesn’t occur due to the inequality (10). Hence, equalities (11) give the definitive result for the stress-strain state.
Hereinafter, all relations are written in the non-dimensional form. All values of the stress dimension are referred to the parameter \(\mu_1\). All values of the length dimension are referred to the parameter \(b\).
If intensities \(q_s = q_s^*\) and \(q_a = q_a^*\) satisfy the condition
\[
q_s^* = \epsilon_0(\lambda_1 + 1)(1 - a^2) + q_a^* a^2,
\] (12)
then all pores are completely compressed in the cylindrical body.
Subject to (12), the stress-strain state at the time of complete pore compression has the form
\[ u^{(1)} = \frac{\varepsilon_0}{2} r + \frac{q^* \cdot f(\varepsilon_0) - \varepsilon_0 (\lambda_1 + 1)}{2 a^2} r^2, \]
\[ \varepsilon_r^{(1)} = -\frac{\varepsilon_0}{2} - \frac{q^* \cdot f(\varepsilon_0) - \varepsilon_0 (\lambda_1 + 1)}{2 a^2} r^2, \]
\[ \varepsilon_0^{(1)} = -\frac{\varepsilon_0}{2} + \frac{q^* \cdot f(\varepsilon_0) - \varepsilon_0 (\lambda_1 + 1)}{2 a^2} r^2; \]
\[ \sigma_r^{(1)} = -\varepsilon_0 (\lambda_1 + 1) - \frac{q^* \cdot f(\varepsilon_0) - \varepsilon_0 (\lambda_1 + 1)}{r^2} a^2, \]
\[ \sigma_0^{(1)} = -\varepsilon_0 (\lambda_1 + 1) + \frac{q^* \cdot f(\varepsilon_0) - \varepsilon_0 (\lambda_1 + 1)}{r^2} a^2, \quad \sigma_z^{(1)} = -\lambda_1 \varepsilon_0, \]
where \( f(\varepsilon_0) = \begin{cases} 1, & \text{if } \varepsilon_0 \neq 0 \\ 0, & \text{if } \varepsilon_0 = 0 \end{cases} \)
ensures the possibility of limiting process when \( \varepsilon_0 \to 0 \).

4.2. The stress-strain state at the stage of inelastic deforming of the material with the completely compressed matrix

The stage of deforming a material with the completely compressed matrix for the problem under consideration occurs when the condition \( q_0 > \varepsilon_0 (\lambda_1 + 1)(1 - a^2) + q_o a^2 \) holds. If at the same time the plasticity condition (2) holds, an inelastic deforming zone forms and starts growing near the inner surface of the cylindrical body. Material hardening holds back this expansion, i.e. compressed skeleton deforms like a hardening incompressible elastic-plastic material with parameters \( \mu = 1 + \mu_0, \ k, \ c, \) where \( \mu_0 \) is the displacement modulus of the element of the incompressible material. In this case, the continuous medium divides into two zones – elastic and plastic.

At the second stage of deforming, equations (3)–(6) and the following rheological relations describe the stress-strain state:

in the elastic zone
\[ s_r = 2 \mu \varepsilon_r, \ s_\theta = 2 \mu \varepsilon_\theta, \ s_z = 2 \mu \varepsilon_z; \]
in the plastic zone
\[ s_r = 2 \mu \varepsilon_r^p, \ s_\theta = 2 \mu \varepsilon_\theta^p, \ s_z = 2 \mu \varepsilon_z^p; \]
\[ (s_r - c \varepsilon_r)^2 + (s_\theta - c \varepsilon_\theta)^2 + (s_z - c \varepsilon_z)^2 = 2k^2. \]
The boundary conditions at current stage have the form
\[ \sigma_r|_{r=a} = -(q_b - q_o), \quad \sigma_r|_{r=b} = -(q_b - q_o). \]
The consistency conditions at the edge of the elastic and plastic zones have the form
\[ \sigma_r^p|_{r=a} - \sigma_r^e|_{r=a} = 0, \quad \varepsilon_r^p|_{r=a} = 0. \]
The stress-strain state of the cylindrical body at the stage of inelastic deforming of the material with the completely compressed matrix is found from the solution of the boundary value problem (5), (6), (13)–(18) in the following way:

in the elastic area \( (\gamma < r < 1) \)
\[ \sigma_r^{(2)} = -(q_b - q_o) + 2 \mu D \left( 1 - \frac{1}{r^2} \right), \quad \sigma_0^{(2)} = -(q_b - q_o) + 2 \mu D \left( 1 + \frac{1}{r^2} \right); \]
in the plastic area \( (a < r < \gamma) \)
\[ \varepsilon_r^{p(2)} = \frac{1}{2 \mu + c} \left( \frac{2 D \mu}{r^2} + \kappa \sqrt{r^2 - \mu^2 \varepsilon_z^2} \right); \]
The displacements and total deformations in the elastic and plastic regions are determined by the relations

\[ \sigma_{p}^{(2)} = -\left(q_a - q_a^{*}\right) - 4\mu \left( \frac{Dc}{2(\mu + c)} \left( \frac{1}{r^2} - \frac{1}{a^2} \right) + \left( \frac{k^2 - \mu^2}{2\mu + c} - \epsilon_0 \right) \ln \frac{r}{a} \right), \]

\[ \sigma_{\theta}^{(2)} = -\left(q_a - q_a^{*}\right) + 4\mu \left( \frac{Dc}{2(\mu + c)} \left( \frac{1}{a^2} + \frac{1}{r^2} \right) - \left( \frac{k^2 - \mu^2}{2\mu + c} + \epsilon_0 \right) \ln \frac{r}{a} \right) \ln - 1 \right). \]

(21)

The equation for defining the radius \( \gamma \), which separates the elastic and plastic deforming zones of the cylindrical body with the completely compressed skeleton, has the form

\[ \left(q_b - q_b^{*}\right) - \left(q_a - q_a^{*}\right) + 4\mu \left( \frac{O}{2} \left( \frac{1}{a^2} - 1 \right) + \frac{D\mu}{2(\mu + c)} \left( \frac{1}{\gamma^2} - \frac{1}{a^2} \right) - \left( \frac{k^2 - \mu^2}{2\mu + c} + \epsilon_0 \right) \ln \frac{\gamma}{a} \right) = 0. \]

(23)

Here in (19)–(23)

\[ \chi = \text{sign} \left( \left(q_b - q_b^{*}\right) - \left(q_a - q_a^{*}\right) \right), \quad D = \frac{-\chi \gamma^2 \sqrt{k^2 - \mu^2 \epsilon_0^2}}{2\mu}. \]

Thus, the resulting stress-strain state of the cylindrical body under consideration within the framework of the assumed material model is determined by the relations (1). Besides, the components with (1)-superscript are defined by relations (13) and the components with (2)-superscript are defined by relations (19)–(23).

We note, that for the obtained solutions (13), (19)–(23) it is possible to pass to the limit when \( \epsilon_0 \to 0 \). In this case, we obtain the results presented in the paper [7].

5. The results of a numerical simulation of the ground stress-strain state of the cylindrical body subject to the initial internal material structure

A numerical experiment is the form of the obtained solutions describing the stress-strain state of the cylindrical body during the stage of the inelastic material deforming. The results of the numerical experiment are presented in figures 2, 3.

![Graph of radius of the elastic-plastic interface \( \gamma \) against value of the initial pore size \( \epsilon_0 \)](a)

![Graph of radius of the elastic-plastic interface \( \gamma \) against value of the initial pore size \( \epsilon_0 \)](b)

Figure 2. Graph of radius of the elastic-plastic interface \( \gamma \) against value of the initial pore size \( \epsilon_0 \) (a) at varying values of the material fluidity limit \( k \) (curve 1: \( k = 0.007 \); curve 2: \( k = 0.008 \); curve 3: \( k = 0.009 \)); (b) at varying values of the parameter \( \mu_0 \) (curve 1: \( \mu_0 = 2 \); curve 2: \( \mu_0 = 3 \); curve 3: \( \mu_0 = 5 \)).
Figure 3. Graph of principal stresses $\sigma_r, \sigma_\theta$ against the radius $r$

(a) at varying values of the parameter $\lambda$ (curve 1: $\lambda = 1.5$; curve 2: $\lambda = 3$; curve 3: $\lambda = 5$);
(b) at varying values of the material fluidity limit $k$
(curve 1: $k = 0.0035$; curve 2: $k = 0.0045$; curve 3: $k = 0.0055$).

The values of other physical, mechanical and geometric parameters are as follows: $q_{\lambda} = 0.001$, $q_{\theta} = 0.005$, $q_\phi = 0$, $a = 0.7$, $b = 1$, $\varepsilon_0 = 0.0004$, $k = 0.005$, $c = 0.005$, $\mu_0 = 2$, $\lambda = 2$, $\mu_\lambda = 1$.

The obtained analytical solutions describing the subcritical stress-strain state of the cylindrical construction in question can be used for studying stability of this state.

6. Basic relationships that model the equilibrium stability of the bodies in case of non-homogeneous subcritical states

Investigation of stability of the systems under elastic-plastic deformations is conducted within the framework of the perturbation method assuming the concept of continuing loading [8]. According to this method, the stability of the ground state of such systems is specified by the behaviour of small perturbations of the corresponding linearized problem in the stable unloading areas forming in subcritical state.
The perturbed state of the nonconservative systems in question is described by the relationships of three-dimensional linearized stability theory of deformed bodies [7, 8].

The equilibrium equations for each area of elastic and plastic deforming of the material with the completely compressed matrix have the form

$$\nabla_i \left( \sigma^i_0 + \sigma_{a0} \nabla^a u_j \right) = 0, \quad (24)$$

where $\nabla$ is the symbol of covariant differentiation; values without “0”-superscript correspond to the components of perturbed state; values with “0”-superscript correspond to the components of the ground unperturbed state defined by the relationships (19)–(23).

The boundary conditions on the part of body surface with given stresses have the form

$$N_i \left( \sigma^i + \sigma_{a0} \nabla^a u_j \right) = 0, \quad (25)$$

where $N_i$ are the components of the surface unit normal vector.

The matching conditions of stresses and displacements on the elastic-plastic interface $\gamma$ have the form

$$\left[ N_i \left( \sigma^i + \sigma_{a0} \nabla^a u_j \right) \right] = 0, \quad \left[ u_j \right] = 0, \quad (26)$$

where the brackets indicate the difference of the corresponding values belonging to the elastic and plastic areas.

The equations of state for the material with the completely complex elastic-plastic matrix have the form

$$\sigma^\beta_j = \left( x_{\beta\alpha} g^{\alpha\alpha} \nabla^\alpha u_\alpha + p \right) g^\beta_j + \left( 1 - g^\beta_j \right) g^{\alpha\beta} \mu \left( \nabla_\beta u^\alpha_\beta + \nabla^\beta u^\alpha_\beta \right), \quad (27)$$

where

$$x_{\beta\alpha} = 2\mu g_{\beta\alpha} - \nu f_{\alpha\beta} f_{\beta\gamma}, \quad f^\alpha_j = S^\alpha_j - c \epsilon^\alpha_j, \quad \nu = \begin{cases} \frac{4\mu^2}{k^2(2\mu + c)}, & \text{in the plastic area;} \\ 0, & \text{in the elastic area.} \end{cases} \quad (28)$$

The incompressibility condition in perturbations due to linearity can be rewritten as

$$\nabla^a u_\alpha = 0 \quad (29)$$

(for each area of elastic and plastic deforming).

7. Spatial equilibrium stability of the cylindrical body under uniform radial compression

Stability analysis of the ground state (1), (13), (19)–(23) is based on investigating the closed system of equations (24)–(29). It is a partial differential equations system in terms of the perturbations of the components of the displacement vector $(u, v, w)$ and the hydrostatic pressure $p$. The nontrivial solution of this problem corresponds to the instability of the ground state.

We will seek a double trigonometric series solution in elastic and plastic deforming areas

$$u = \sum_{n}^{\infty} A_{nm}(r) \cos(m\theta) \cos(nz), \quad v = \sum_{n}^{\infty} B_{nm}(r) \sin(m\theta) \cos(nz), \quad w = \sum_{n}^{\infty} C_{nm}(r) \cos(m\theta) \sin(nz), \quad p = \sum_{n}^{\infty} D_{nm}(r) \cos(m\theta) \cos(nz), \quad (30)$$

where $n$ and $m$ are the wave formation parameters.

The system (24)–(29) is a linear homogeneous system, hence, it can be written with the same values of the wave formation parameters for each member. Later on we don’t write $m$ and $n$ indices for notational convenience.

Taking into account (27)–(30), the boundary value problem (24)–(26) in terms of the perturbations...
of displacements and stresses after a number of transformations is rearranged to the following boundary
value problem in terms of the functions \( A(r), B(r), D(r) \).

Equations of equilibrium
\[
\begin{align*}
\xi_0^0(r) A(r) + \xi_0^0(r) A'(r) + \xi_0^0(r) A''(r) + \xi_0^0(r) B(r) + \xi_0^0(r) B'(r) + r \cdot D'(r) &= 0, \\
\xi_0^0(r) A(r) + \xi_0^0(r) A'(r) + \xi_0^0(r) B(r) + \xi_0^0(r) B'(r) + m \cdot D(r) &= 0, \\
\xi_0^0(r) A(r) + \xi_0^0(r) A'(r) + \xi_0^0(r) A''(r) + \xi_0^0(r) B'(r) + m \cdot B''(r) + r \cdot m \cdot B''(r) + r \cdot m \cdot D(r) &= 0, \\
\xi_0^1(r) A(r) + \xi_0^1(r) A'(r) + \xi_0^1(r) A''(r) + \xi_0^1(r) B'(r) + m \cdot B''(r) + m \cdot B''(r) + m \cdot D(r) &= 0,
\end{align*}
\]
(31)
where
\[
\begin{align*}
\xi_0^0(r) &= \frac{1}{r} \left[ r^2 \rho \omega^2 - \left( \sigma_0 + \mu \right) \left( 1 + m^2 \right) + \nu \cdot \psi_1 - r^2 \left( \sigma_0 + \mu \right) \right] n^2, \\
\xi_0^1(r) &= \frac{m}{r} \left(-2 \sigma_0 - 2 \mu + \nu \cdot \psi_1 \right), \\
\xi_0^2(r) &= m \left( \nu \cdot \psi_1 - 2 \mu \right), \\
\xi_0^3(r) &= \frac{1}{r} \left[ r^2 \rho \omega^2 - \sigma_0 - r^2 \left( \mu + \sigma_0 \right) \right] n^2 - 3 \mu \cdot \psi_1 + \sigma_0 \right) m^2 - \mu, \\
\xi_0^4(r) &= \sigma_r + r \sigma_r + \mu, \\
\xi_0^5(r) &= \frac{1}{r} \left[ r^2 \rho \omega^2 - n^2 \left( 2 \mu + \sigma_0 \right) \right] m^2 + \mu \cdot \psi_1 + \mu n^2 r^2 + \mu + \sigma_r - r \sigma_r, \\
\xi_0^6(r) &= \frac{1}{r} \left[ r^2 \rho \omega^2 - n^2 \left( 2 \mu + \sigma_0 \right) \right] m^2 + \mu \cdot \psi_1 + \mu n^2 r^2 - \mu - \sigma_r + r \sigma_r, \\
\xi_0^7(r) &= \frac{1}{r} \left[ r^2 \rho \omega^2 - n^2 \left( 2 \mu + \sigma_0 \right) \right] m^2 + \mu \cdot \psi_1 + \mu n^2 r^2 - \mu - \sigma_r - r \sigma_r, \\
\psi_1 &= \frac{\mu \cdot \psi_0}{3} + \sqrt{\frac{k^2}{1 - 3 \mu^2 \cdot \psi_0}}, \\
\psi_2 &= \frac{\mu \cdot \psi_0}{3} - \sqrt{\frac{k^2}{1 - 3 \mu^2 \cdot \psi_0}}, \\
\psi_3 &= \sqrt{\frac{2 \mu^2 \cdot \psi_0}{3}} - k^2.
\end{align*}
\]

The equilibrium equations (31) are valid for the plastic zone if we add “p”-superscript to all variables in (31). If we put in (31) \( \nu = 0 \) and place “e”- superscript to all variables, the equilibrium equations (31) are valid for the elastic zone.

The boundary conditions on the inner \( (r = a) \) and outer \( (r = b) \) surfaces of the body, respectively, are written as
\[
\begin{align*}
\left( A^p(a) + B^p(a) m \right) \phi_1 + \phi_2 A^p(a) + D^p(a) &= 0, \\
m \mu A^p(a) + \mu B^p(a) + \phi_2 B^p(a) &= 0, \\
\phi_1 A^p(a) + \left( A^p(a) a + A^p(a) a^2 - B^p(a) m + B^p(a) am \right) \phi_3 &= 0; \\
A^c(1) \phi_3 + D^c(1) &= 0, \\
\left( A^c(1) + A^c(1) a + A^c(1) a^2 - B^c(1) m + B^c(1) m \right) \phi_3 &= 0, \\
m \mu A^c(1) + \mu B^c(1) - \phi_2 B^c(1) &= 0, \\
A^c(1) \phi_3 + \left( A^c(1) + A^c(1) a - B^c(1) m + B^c(1) m \right) \phi_3 &= 0,
\end{align*}
\]
(32)
\[
\phi_1 = -\nu \cdot \psi_1, \quad \phi_2 = -q_a - \nu \cdot \psi_1 + 2\mu, \quad \phi_3 = -a (\mu - q_a), \quad \phi_4 = n^2 a^2 \mu - \mu - q_a, \quad \phi_5 = \mu - q_a,
\]
\[
\phi_6 = 2\mu - q_a, \quad \phi_7 = n^2 \mu - \mu - q_a.
\]

The matching conditions on the elastic-plastic interface \( \gamma \) have the form
\[
\phi_8 [A'] + [D] + \phi_9 (A'' + mB'') - A''\nu \cdot \psi_1 = 0,
\]
\[
m\mu [A] + \mu [B] + \phi_{10} [B'] = 0,
\]
\[
\phi_{11} [A] - \gamma [A'] - \gamma^2 [A''] + m[B] - \gamma m[B'] = 0,
\]
where
\[
\phi_8 = \sigma_0^\gamma (\gamma) + 2\mu, \quad \phi_9 = -\nu \cdot \psi_1, \quad \phi_{10} = -\gamma \left( \sigma_0^\gamma (\gamma) + \mu \right), \quad \phi_{11} = 1 - \frac{\mu n^2 \gamma^2}{\sigma_0^\gamma (\gamma) + \mu}.
\]

It is not possible to obtain an exact analytical solution of the boundary value problem (31)–(34). The approximate solution can be obtained by the use of the finite-difference method [13]. As a result we have the infinite system of homogeneous algebraic equations linear with respect to parameters \( A_{nm}, B_{nm}, D_{nm} \). Hence, sizing the critical internal load \( q_a \) corresponding to instability of the cylindrical body under uniform radial compression for the material with a completely compressed skeleton resolves into a matrix equation solvability. In calculating the algebraic system determinant we must take into account the ground stress-strain state for each of the areas of elastic and plastic deforming (1), (13), (19)–(22) and the equation (23). The latter determines the position of the elastic-plastic interface \( \gamma \) entirely covering the internal boundary of the cylindrical body under consideration. The minimization is for the following parameters: the difference interval, the wave formation parameters \( n \) and \( m \), the material and structure parameters \( \lambda, \). Thus, we have a multidimensional \( q_a \)-optimization problem depending on parameters \( n \) and \( m \) subject to the requirement that the resulting algebraic system determinant equal to zero.

8. Conclusions

The results are summarized as follows.

1. Mathematical models describing the stress-strain states of the cylindrical body under radial compression have been constructed within the framework of the assumed material model at the stages of the elastic deforming of the compressible porous medium and the elastic-plastic deforming of the body with the completely compressed matrix.

2. The analytical model-based calculation of the non-homogeneous subcritical stress-strain state of the cylindrical body in question has been performed within the framework of plane deformation of the body under static uniformly distributed radial loads.

3. The influence of the physical and mechanical parameters of the material (characteristic value determined by the specific volume of pores, the material fluidity limit, the strengthening factor, etc.), the external loads and the geometric parameters of the cylindrical body on its stress-strain state has been evaluated.

In particular, it has been found that
- the value of elastic-plastic interface \( \gamma \) decreases when the Lamé parameter \( \mu_0 \) of the compressible porous material becomes large;
- increase of the characteristic value \( \varepsilon_0 \) determined by the specific volume of pores leads to enlarging the region of the compressed matrix inelastic deformations;
- the value of elastic-plastic interface \( \gamma \) decreases when the values of material fluidity limit \( k \) and
strengthening factor $c$ of the material with the completely compressed matrix increase.

4. The question of the nonhomogeneous ground state equilibrium instability of the cylindrical body under uniform radial compression has been investigated within the framework of three-dimensional linearized stability theory of the deformed bodies. The question has been considered within the bounds of model that takes into account the porous structure of the material, which compressed skeleton has elastic and plastic properties simultaneously.

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