Comparing optimized renormalization schemes for QCD observables

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(Dated: February 15, 2022)

In this article based on the McKeon et al. approach, it can be shown that the renormalization group equation while it is related to the radiatively mass scale \( \mu \), operates to do a summation over QCD perturbative terms. Employing the full QCD \( \beta \)-function within this summation, all logarithmic corrections can be presented as the log-independent contributions. In another step of this approach, the renormalization scheme dependence for QCD observable which is characterised by Stevenson, can be examined by specifying the renormalization scheme in which the \( \beta \)-function beyond two-loop orders is utilized. In this regard two choices of renormalization schemes would be exposed. In one of them the QCD observable is expressed while involves two powers of running coupling constant such that the perturbative series contains just two terms. In another choice the perturbative series expansion is written as an infinite series in terms of two-loop running coupling which can be presented by Lambert \( W \)-function. In both cases the QCD observable involves parameters which are renormalization scheme invariant and the coupling constant which is independent of renormalization scale. We then consider the other approach which is called the complete renormalization group improvement (CORGI). In this approach, using the self consistency principle it is possible to reconstruct the conventional perturbative series in terms of scheme invariant quantities and the coupling constant as a function of Lambert-\( W \) function. It should be noted while in the renormalization group summation method of McKeon et al., scheme dependence of observables is investigated separately from their scale dependence, in CORGI approach, through the principle of self consistency, both scale and scheme parameters are utilized together. In continuation we examine numerically these two approaches, considering two QCD observable. The first one is \( R_{e^+e^-} \) ratio, investigated at three different colliding energies and the second observable is Higgs decay width to gluon-gluon. Comparison for \( R_{e^+e^-} \) with the available experimental is done. The results based on the McKeon et al. approach are in better agreement with the experimental data.

Keywords: Scheme and scale dependence, renormalization scheme summation, complete renormalization group improvement.

I. INTRODUCTION

There are different approaches to optimize the QCD observable. One of them is called principle of maximum conformality (PMC) which was intimated by S.J.Brodsky and collaborators [1-5]. In this approach the perturbative series is divided into two conformal and non-conformal parts. The conformal part does involve the terms which are independent on the employed scheme. The non-conformal parts are absorbed into the coupling constant by choosing proper scales. During the absorption, the renormalization scales would be fixed and determined and final result is such that the conformal series to be independent of the used scheme and scales. The division of the series can be done, using an auxiliary scheme that is called \( R_3 \) scheme [2]. In connection to the PMC approach some other research activities have been done which can be found in [6, 7].

Another approach to optimize QCD observable is called complete renormalization group improvement (CORGI) [8]. In this approach, using self consistency principle, it is possible to write each expansion coefficient at high order, in terms of coefficients of lower orders and an invariant scheme independent terms which are unknown at that high order. If one resumes the contribution of each expansion terms at lower orders then it can be shown that the result would be scale and scheme independent. This provides an opportunity to reconstruct the initial perturbative series in terms of scale and scheme independent terms and also coupling constant which are determined at the physical energy scales and consequently the scale ambiguity is removed.

There might be some connections between the PMC and CORGI approaches which have been discussed in [9]. The most important one is that the predictable terms in CORGI approach can be assigned to the non-conformal part since the predictable terms are scale and scheme dependence while unpredictable terms can be related to the conformal part of PMC which is scheme independent. For more details, see [9].

As a third approach which has been initiated by D. C. McKeon and his collaborators and we call it McKeon et al. approach, it can be shown that upon resummation of all perturbative series terms to all orders, using the full QCD-\( \beta \)-function, the result would be scale independent and just depends on the physical energy scale, \( Q \), without any ambiguity. In continuation using the scheme...
dependence of coupling constant, it is shown how the perturbative coefficients up to specified high order, can be written in terms of coefficients at lower orders toward the lowest order coefficient [10–14]. This stage is like the one in CORGI approach but with one little difference. In CORGI approach the perturbative coefficients at specified high order are obtained in terms of coefficients at lower orders based on self consistency principle. In this regard, it is required to obtain the derivative of perturbative coefficients not only with respect to the scheme parameters, but also with respect to renormalization scale which is casted in terms of $\tau$ variable, see [15]. Therefore, in comparison with McKeon et al. approach, the result of CORGI approach for perturbative coefficient at a specified high order, not only contains the coefficients at lower orders, but also involves more extra terms. This occurs, as we refereed above, because the separation between scale and scheme dependence does not happen during the optimizing procedure in CORGI approach while it has been supplied by the utilized approach of Mackeon et al.

We are know planning to discuss in more details the differences and similarities between CORGI and Mackecon et al. approach and to investigate which one has more preference. We specially examine this procedure, considering two QCD observable: one is the ratio of electron positron annihilation to hadrons and the other is the Higgs decay width to gluon-gluon. But at first we need to review the basic concepts of these two approaches which is done in the further sections.

The organization of this paper consists of the following section. In Sec.II basic concepts of McKeon et al. approach is reviewed. This section contains two parts. At first part the renormalization group summation is described which is done in Subsec.IIA. The scheme characterizing of QCD observable is done in Subsect.IIB. Further on we are going to examine the Mackecon et al. approach and the CORGI one as well for some QCD observable which are done in Sec.III. At first we inspect $R_{e^+e^-}$ ratio in Subsect.IIIA and Subsect.IIIIB, considering McKeon et al. approach and CORGI one respectively. We then employ the McKeon et al. approach but for Higgs decay width to gluon-gluon in Sec.IV where we first there indicates how to do the RG summation. The scheme dependence for this observable is done in Subsect.IVA. Numerical investigation for this observable in McKeon et al. approach is presented in Subsect.IV B. This observable is considered numerically as well in CORGI approach in Sec.V. Finally we give our conclusion in Sec.VI.

II. AN OVERVIEW ON McKEON et al. APPROACH

Here we give a brief description to the method, initiated by McKeon et al. on renormalization group summation. Most of the relations in this section and the further ones are rephrased from [10, 11]. We are inevitable to resort them in order the readers can follow easily and precisely the concerned subject. As an example how the renormalization group summation has been done, the QCD observable for $e^+e^-$ annihilation is examined. In following, it is shown how by achieving a recurrence relation between perturbative terms, the result of summation on all perturbative terms would be independent of renormalization scale as an unphysical parameter.

The cross section of $e^+e^-$ annihilation into hadrons, after normalizing it to $\mu^+\mu^-$ pair production, can be written as:

$$R_{e^+e^-} = (3 \sum_i q_i^2)[1 + R].$$  \hspace{1cm} (1)

The expansion of $R$ in terms of coupling constant $a$ has the following form:

$$R = \sum_{n=0}^{\infty} r_n a^{n+1}, \hspace{1cm} r_0 = 1.$$  \hspace{1cm} (2)

The coefficient of $r_n$, arising from contribution of Feynman diagrams to the observable $R$ can be represented by [10]:

$$r_n = \sum_{m=0}^{n} T_{nm} L^m.$$ \hspace{1cm} (3)

In this equation $T_{00} = 1$ and $L = b \ln(\frac{M^2}{\mu^2})$ where $\mu$ is renormalization scale while $Q$ is the physical center of mass energy in $e^+e^-$ collision.

Since $R$ is independent of renormalization scale $\mu$, so the renormalization group equation (RGE) implies that:

$$\frac{d}{d\mu} R = (\frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a}) R = 0.$$ \hspace{1cm} (4)

The $\mu$-dependence of the coupling is governed by the QCD $\beta$-function equation [16]:

$$\beta(a) = \mu \frac{\partial a}{\partial \mu} = -ba^2(1 + ca + c_2a^2 + c_3a^3 + \ldots).$$ \hspace{1cm} (5)

Here $b = (33 - 2N_f)/6$ and $c = (153 - 19N_f)/12b$ are renormalization scheme (RS) invariant where $N_f$ is denoting to the number of active quark flavor. The higher coefficients $c_2,c_3,\ldots$ serve to label the RS dependence [17].

To indicate the coefficients $b$ and $c$ in Eq.(5) are scheme independent, one can consider two different coupling $a$ and $a^*$ at two different scheme, such that [11]:

$$a^* = a + x_1a^2 + x_3a^3 + \ldots,$$ \hspace{1cm} (6)

therefor

$$\beta^*(a^*) = \mu \frac{\partial a^*}{\partial \mu} = \beta(a) \frac{\partial a^*}{\partial a}.$$ \hspace{1cm} (7)
Then one can show:

\[
\beta^*(a^*) = -b^* a^{2*} [1 + c^* a^* + c_2^* a^{2*} + \ldots] = [1 + 2 x_2 a + 3 x_3 a^2 + \ldots] [(-b a^2)][1 + c a + c_2 a^2 + \ldots].
\]

(8)

If we substitute \( a^* \) from Eq. (6) on the first row of Eq. (8) and compare the result with the second row of this equation, then one will arrive at:

\[
\begin{align*}
\beta &= b^* \\
\gamma &= c^* \\
c_2 &= c^*_2 + c x_2 + x_2^2 - x_3 \\
\cdots & \\
& \\
&
\end{align*}
\]

(9)

After this introductory subject we now deal with in details the McKeon et al. approach.

A. Renormalization group summation

Here based on the McKeon et al. approach, it can be shown that by a full resummation on the QCD perturbative series the unphysical parameter, \( \mu \), would be disappeared and the final result for any QCD observable, as it is expected, is independent of the unphysical parameter. One of the feature of McKeon et al. approach with respect to the CORGI approach is based on this fact that to achieve this approach, it does not need to use the self consistency principle and all required considerations to construct the McKeon et al. approach totally back to the RGE for QCD \( \beta \)-function. In CORGI approach one needs to employ the self consistency principle not only for the scheme parameters but also for the scale parameter \( \mu \). Therefore the scale and scheme parameters are used simultaneously during the optimization process in CORGI approach while in the McKeon et al. approach, a separation between the renormalization scale parameter in one hand side and scheme parameters on the other hand side are realized perfectly.

Now we do a brief review on the McKeon et al. approach to show how the unphysical parameters would be removed in perturbative series for any QCD observable, just using RGE of QCD-\( \beta \) function. For this purpose, by substituting Eq. (3) in Eq. (2) the following result would be arrived [10]:

\[
R = R_{\text{pert}} = \sum_{n=0}^{\infty} r_n a^{n+1} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{n,m} L^m a^{n+1}.
\]

(10)

To satisfy the condition \( r_0 = 1 \) in Eq. (2), it is required that \( T_{0,0} = 1 \). From Eq. (10), the following expression should be assigned to \( r_n \):

\[
r_n = \sum_{m=0}^{n} T_{n,m} L^m.
\]

(11)

If one set \( m = 0 \) then

\[
r_n = T_n \Rightarrow R = \sum_{n=0}^{\infty} T_n a^{n+1},
\]

(12)

which is similar to Eq. (2).

Now a new grouping, denoted by \( A_n \) is introduced [10] :

\[
A_n = \sum_{m=0}^{\infty} T_{n+m,n} a^{n+m+1}.
\]

(13)

This makes a possibility to sum the contribution to \( R \), considering the RGE. Therefore using Eq. (13) in above, the \( R \) in Eq. (10) would have the following presentation:

\[
R = R_A = \sum_{n=0}^{\infty} A_n(a) L^n.
\]

(14)

Substituting Eq. (14) in Renormalization Group Equation, Eq. (4), it can be shown:

\[
\sum_{n=0}^{\infty} (b n A_n(a)L^{n-1} + \beta(a) A_n(a)L^n) = 0.
\]

(15)

Using this equation and rearranging the order of sums, \( A_n \) can be written as:

\[
A_n(a) = -\frac{\beta(a)}{n b} \frac{d}{da} A_{n-1}(a).
\]

(16)

Considering QCD \( \beta \)-function, Eq. (5), one will arrive at:

\[
A_n(a(\ln \frac{\mu}{\Lambda})) = -\frac{1}{n} \frac{d}{d(\ln \frac{\mu}{\Lambda})} A_{n-1}(a(\ln \frac{\mu}{\Lambda})),
\]

(17)

where \( \Lambda \) is related to be boundary condition on Eq. (5) such that:

\[
\ln \frac{\mu}{\Lambda} = \int_0^{\mu} \frac{dx}{\beta(x)} + \int_0^{\infty} \frac{dx}{b x^2 (1 + c x)}.
\]

(18)

Defining \( \eta = \ln \frac{\mu}{\Lambda} \) and using the recurrence relation, Eq. (17), the following result would be obtained:

\[
A_n(a(\eta)) = -\frac{1}{b n} \frac{d}{d\eta} A_{n-1}(a(\eta)) = \frac{1}{n!} \left( -\frac{1}{b} \frac{d}{d\eta} \right)^n A_0(a(\eta)).
\]

(19)

Substituting Eq. (19) into Eq. (14) will lead to

\[
R_A = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{L}{b} \right)^n \frac{d^n}{d\eta^n} A_0(a(\eta)) = A_0(a(\eta - \frac{L}{b})).
\]

(20)

To prove the above relation, it is easily to show:

\[
R(A) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{L}{b} \right)^n \frac{d^n}{d\eta^n} A_0(a(\eta)) = \exp((-\frac{L}{b}) \frac{d}{d\eta}) A_0(a(\eta)).
\]

(21)
On the other hand, using the Taylor expansion for the right hand side of Eq.\( \text{(20)} \), one can write
\[
A_0(a(\ln \frac{\mu}{\Lambda} - L)) = A_0(a(\ln \frac{\mu}{\Lambda})) - LA_0'(a(\ln \frac{\mu}{\Lambda})) \nonumber \\
+ \frac{L^2}{2!} A_0''(a(\ln \frac{\mu}{\Lambda})) + \ldots , \tag{22}
\]
which is equivalent to the right hand side of Eq.\( \text{(21)} \), considering the expansion of exponential term. Consequently, based on the definitions of \( L \) and \( \eta \) parameters, one can see that \[ 11] :
\[
R_A = A_0(a(\ln \frac{Q}{\Lambda})) . \tag{23}
\]
This equation shows that all dependence of \( R \) on \( \mu \) scale has been cancelled. This is a pleasant result since \( \mu \) is an unphysical parameter and it is expected to be removed by doing a full resumation on the QCD perturbative series.

B. Renormalization scheme dependence

In previous section, it has been shown that the final result for perturbative series of any QCD observable is independent of renormalization scale, \( \mu \), as unphysical parameter. Here a brief review on the McKeon et al. approach is done to indicate how the perturbative series of QCD observable would be scheme independent and therefore the final result for the QCD perturbative series is reliable and independent of the scale and scheme parameters.

Let us to start the subject by reminding that the first two \( b \) and \( c \) parameters in QCD \( \beta \)-function are independent of renormalization scheme while the expansion parameters \( c_i (i \geq 2) \) are renormalization scheme dependent. It is now shown explicitly how \( R_A \) in Eq.\( \text{(14)} \) depends on \( c_i \) parameters. Since \( R_A \) is arising out from the sum over all perturbative terms, it should be independent of any choice for renormalization scheme. Therefore one can write \[ 17] :
\[
(\frac{\partial}{\partial c_i} + \beta_i(a) \frac{\partial}{\partial a})R_A = 0 \ . \tag{24}
\]
Here \( \beta_i(a) \) is defined by
\[
\frac{\partial a}{\partial c_i} = \beta_i(a) = -\beta(a) \int_0^a \frac{x^{i+2}}{(\beta(x))} \, dx \ , \tag{25}
\]
which indicates how the coupling constant depends on the \( c_2, c_3, \ldots \) scheme parameters where \( \hat{\beta}(a) = \beta(a)/b \). The solution of this equation is as following \[ 17] :
\[
\frac{\partial a}{\partial c_i} = \frac{1}{i-1} a^{i+1} \left[ 1 - \frac{(i-2)}{i} c_0 a + \left( \frac{(i-1)(i-2)}{i(i+1)} c^2 - \frac{(i-3)}{(i+1)} c_2 \right) a^2 + \ldots \right] . \tag{26}
\]
By setting \( T_{n,0} \equiv T_n \) in Eq.\( \text{(10)} \) and based on Eq.\( \text{(24)} \) one can derive the following result:
\[
\sum_{n=0}^{\infty} n^{n+1} \frac{\partial T_n}{\partial c_i} + (n + 1) \beta_i(a) T_n a^n = 0 \ . \tag{27}
\]
This leads to a set of nested equations for \( T_n \):
\[
\frac{\partial T_0}{\partial c_i} = 0 \Rightarrow T_0 = \tau_0 = 1 , \\
\frac{\partial T_1}{\partial c_i} = 0 \Rightarrow T_1 = \tau_1 = \text{Const.} \tag{28}
\]
Reminding that \( T_{0,0} = T_0 = 1 \).

For the \( T_2 \) coefficients one can derive
\[
\frac{\partial T_2}{\partial c_2} + 1 = 0 \Rightarrow T_2 = -c_2 + \tau_2 , \tag{29}
\]
where \( \tau_2 \) is again constant and an scheme invariant.

For the \( T_3 \) coefficient one leads to the two following set of equations:
\[
\frac{\partial T_3}{\partial c_2} + 2\tau_1 = 0 , \\
\frac{\partial T_3}{\partial c_3} + \frac{1}{2} = 0 . \tag{30}
\]
Simultaneous solution of these two differential equations will end to
\[
T_3 = -2c_2\tau_1 - \frac{1}{2} c_3 + \tau_3 \ , \tag{31}
\]
where again \( \tau_3 \) is constant of integration and scheme invariant.

For the \( T_4 \) coefficient, one will get:
\[
\frac{\partial T_4}{\partial c_2} + \frac{1}{3} c_2 + 3T_2 = 0 , \\
\frac{\partial T_4}{\partial c_3} + T_1 - \frac{c}{6} = 0 , \\
\frac{\partial T_4}{\partial c_4} + \frac{1}{3} = 0 . \tag{32}
\]
Solving the differential equations in Eq.\( \text{(32)} \) compatibly with each other, give us the following result:
\[
T_4 = -\frac{1}{3} c_4 - c_3(\tau_1 - \frac{c}{6}) + \frac{4}{3} c_2^2 - 3c_2\tau_2 + \tau_4 , \tag{33}
\]
where once again \( \tau_4 \) is constant of integration and \( RS \) invariant.
Following the procedure, introduced in [10] two specific choices of renormalization scheme are considered. In the first scheme, the \( c_i \) are selected so that \( T_n = 0 \ (n \geq 2) \). By this choice one then find from Eqs. Eqs. (28,29,31,33) the following results:

\[
\begin{align*}
&c_2 = \tau_2 , \\
&c_3 = 2(-2c_2\tau_1 + \tau_3) , \\
&c_4 = \frac{3}{2}c_3(-\frac{c}{3} + 2\tau_1) + 4e_2^2 - 9c_2\tau_2 + 3\tau_4 .
\end{align*}
\]

(34)

In this case, the expansion series in Eq. (10) or equivalently Eq. (23) contains just two terms [10]:

\[
R_1 = a(1) + \tau_1a_1^2(\ln \frac{Q}{\Lambda}) ,
\]

(35)

where the coupling \( a(1) \) in addition to universal parameters \( b \) and \( c \), depends also on the \( c_2, c_3 \) and other scheme parameters. It should be noted that since Eq. (23) is independent of renormalization scale \( \mu \), the finite expansion in Eq. (35) has been written in terms of physical energy scale \( Q \). As can be seen the result for the observable \( R \) is scheme independent since \( \tau_1 \) is RS invariant and coupling constant \( a(1) \) is independent of renormalization scale.

In the second case, the scheme parameters are used such that \( c_i = 0 \ (i \geq 2) \) which is corresponding to the ’t Hooft scheme [18]. With this choice, considering again Eqs. (28,29,31,33) will lead to \( T_n = \tau_n \). In this case Eq. (10) contains the infinite series:

\[
R_2 = \sum_{n=0}^{\infty} \tau_n a_{(2)}^n(\ln \frac{Q}{\Lambda}) = a(2)(\ln \frac{Q}{\Lambda}) + \tau_1a_1^2(\ln \frac{Q}{\Lambda})
\]

\[+ \tau_2a_{(2)}^2(\ln \frac{Q}{\Lambda}) + ... .
\]

(36)

Like the expansion in Eq. (35) and with the similar reason, just the physical energy scale \( Q \) is appearing in series expansion of Eq. (35) while \( \tau_1, \tau_2 \) and etc. are RS invariant.

The \( a(2) \) coupling in Eq. (36) is obtained from the solution of QCD-\( \beta \) function while just two universal parameters \( b \) and \( c \) are kept there. Therefore it can be expressed in terms of the Lambert-W function. For details, it is required to back to Eq. (5) while the general solution reduces to:

\[
\begin{align*}
&1 + c \ln(\frac{ca}{1+ca}) = \tau - \int \left( \frac{1}{-a^2(1+ca+c_2a^2+...)} + \frac{1}{a^2(1+ca)} \right) da , \\
&+ c \ln(\frac{ca_0}{1+ca_0}) = b \ln(\frac{Q}{\Lambda_R}) .
\end{align*}
\]

(37)

where \( \tau \) is RS parameter, defined by \( \tau = b \ln(\frac{Q}{\Lambda}) \). By setting \( c_n = 0 \) in Eq. (37), one will get:

\[
\frac{1}{a_0} + c \ln\left(\frac{ca_0}{1+ca_0}\right) = b \ln\left(\frac{Q}{\Lambda_R}\right) .
\]

(38)

To solve this equation the \( W(Q) \) function is defined such that:

\[
1 + W(Q) = -\frac{1}{ca_0} .
\]

(39)

Here \( a_0 = a_0(Q) \). Finally one can write:

\[
-\frac{1}{ca_0} - \ln\left(\frac{ca_0}{1+ca_0}\right) = \ln\left(\frac{Q}{\Lambda_R}\right) - b/c \Rightarrow W e^W = -\frac{1}{e} \left(\frac{Q}{\Lambda_R}\right) - b/c .
\]

(40)

If \( z(Q) = -\frac{1}{e} \left(\frac{Q}{\Lambda_R}\right) - b/c \) is introduced then Eq. (40) will lead to:

\[
W(z)e^W(z) = z .
\]

(41)

The solution of this equation is called Lambert W-function [19, 20] then from Eq. (39) the following result can be obtained:

\[
a(2) = a_0(Q^2) = -\frac{1}{c[1+\Lambda_W^{-1}(z(Q))]} .
\]

(42)

However two different cases for the McKeon et al. approach have been introduced but it can be shown that the expansions for \( R_1 \) and \( R_2 \) given by Eq. (35) and Eq. (36) are equivalent to each other. For this purpose two different couplings \( a_c \) and \( a_d \) are considered which evaluated by different renormalization schemes associated with parameters \( c_i \) and \( d_i \) respectively [10]:

\[
a_c = a_d + \lambda_2, d_1) a_2^2 + \lambda_3, d_1) a_3^2 + ... .
\]

(43)

The couplings \( a_c \) and \( a_d \) are being satisfied, each one, by their related QCD-\( \beta \)-functions such that:

\[
\begin{align*}
\beta_c(a_c) &= -ba_c^2(1 + ca_c + c_2a_c^2 +...) , \\
\beta_d(a_d) &= -ba_d^2(1 + ca_d + d_2a_d^2 +...) .
\end{align*}
\]

(44)

Since \( \frac{d\ln a}{da} = 0 \) then it can be written

\[
\left(\frac{\partial}{\partial d_i} + \beta_j(a_d) \frac{\partial}{\partial a_d}\right) \sum_{n=1}^{\infty} \lambda_n(c_i, d_i) a_n^2 = 0 .
\]

(45)

Using the boundary condition \( \lambda_n(c_i, c_i) = 0 \), a set of differential equations can be obtained for \( \lambda_n \) whose solutions lead to the following relation for couplings:

\[
\begin{align*}
a_c &= a_d - (d_2 - c_2)a_2^3 - \frac{1}{2}(d_3 - c_3)a_3^4 \\
&+ \frac{1}{6}(d_2^2 - c_2^2) + \frac{3}{2}(d_2 - c_2)^2 \\
&+ \frac{c}{6}(d_3 - c_3) - \frac{1}{3}(d_4 - c_4) a_5^2 + ...
\end{align*}
\]

(46)

Substituting this equation in Eq. (35) and using ‘t Hooft scheme in which \( d_i = 0 \ (i \geq 2) \) then the expansion for \( R(1) \) reads to:

\[
R_1 = a_d + c_2a_d^3 + (1/2)c_3a_d^4 + \tau_1(a_d + c_2a_d^3 + (1/2)c_3a_d^4)^2 .
\]

(47)
Evaluating $c_i$ from Eq. (34) in terms of $\tau_i$s, and substituting them in above equation, after proper rearranging the desired series expansion, one will arrive at [10]:

$$R_{(1)} = a_{(2)} + \tau_1 a_{(2)}^2 + \tau_2 a_{(2)}^3 + \tau_3 a_{(2)}^4 + \ldots$$

(48)
which is completely corresponding to series expansion for $R_{(2)}$ in Eq. (36). Remembering that the $a_d$ coupling in Eq. (47) is evaluated while the scheme parameters $d_i = 0 \ (i \geq 2)$. Therefore it can be shown by $a_{(2)}$ as the Lambert-W function and this gives us a full agreement between Eq. (35) and Eq. (36).

III. CONSIDERING SOME OBSERVABLE IN MCKEON AND AT AL AND CORGI APPROACHES

After achieving the required mathematical framework for the McKeon et al. approach, we now examine it, considering two QCD observable. We do their numerical calculations and compare them in further section with the results which will be obtained from CORGI approach. At the first we employ the McKeon et al. approach to obtain the ratio of cross sections for the electron-positron annihilation into hadrons at the center of mass energy $\sqrt{s} = 31.6 \text{ GeV}$.

A. Electron-positron annihilation in Mceoon and at al approach

The expansion up to fourth order for $R_{e+e^-}$ as the ratio for electron-positron annihilation into hadrons with $N_f = 5$ can be written as [21, 22]:

$$R_{e+e^-}(s) = \frac{11}{3} [1 + a_s + 1.40902a^2_s - 12.80a^3_s - 80.434a^4_s + \ldots]$$

(49)
According to the notation of McKeon at al approach, one can write

$$T_0 = 1 , T_1 = 1.409 , T_2 = -12.80 , T_3 = -80.434$$

(50)
Using Eqs. (28, 29, 31) the numerical values for the required RS invariants are given by:

$$\tau_1 = T_1 = 1.409 \ , \ \tau_2 = T_2 + c_2 = -11.3252$$
$$\tau_3 = T_3 + \frac{1}{2} c_3 + 2c_2\tau_1 = -71.3600$$

(51)
We take the number of active quark flavour $N_f = 5$ at colliding energy $\sqrt{s} = 31.6 \text{ GeV}$ and consider the QCD cut off value in the $\overline{MS}$ scheme as $\Lambda_{\overline{MS}} = 419^{+222}_{-168} \text{ MeV}$ which is raised from empirical value for $R_{e+e^-}$ [2]. Final numerical result for the ratio of cross section in the McKeon at al approach up to fourth order would be

$$\frac{3}{\Pi} R_{e+e^-} = 1 + a + \tau_1 a^2 + \tau_2 a^3 + \tau_3 a^4 = 1.0556^{+0.006}_{-0.006}.$$  

(52)

which is in good agreement with its available experimental value $R_{e^+e^-} = 1.0527^{+0.005}_{-0.005}$ [23]. Reminding that the coupling constant $a$ in Eq. (52) is given in terms of Lambert-W function (see Eq. (42)).

There is a possibility to get more precise numerical value for $R_{e^+e^-}$ if we do the calculations at first in Euclidean space and then by contour improved back the result to Minkowski space. This is because the perturbative coefficients for the concerned observable have been computed more precisely in Euclidean space. In this regard we need to a relation between the ratio for $e^+e^-$ annihilation and Adler D-function in Euclidean space which can be found in [30]:

$$D(Q^2) = \frac{Q^2}{2\pi} \int_0^\infty R_{e^+e^-} \left( \frac{Q^2 + m^2}{Q^2} \right) dQ$$

(53)
The Adler D-function would have the following expansion:

$$D(Q^2) = a(1 + \sum_{n>0} d_n a^n)$$

(54)
where $d_i$ coefficients are known in [21, 22]. Nevertheless if one is interesting, can do some computations which finally yields $d_i$s coefficients in terms of $\tau_i$s ones. Similar computations but for Higgs decay width to gluon-gluon in details is done in Subsec. IV B. Doing the same calculations but for $R_{e^+e^-}$ will lead to:

$$d_1 = r_1 \ , \ \ d_2 = r_2 + \frac{1}{3} \beta_0^2 \pi^2 \ , \ \ d_3 = r_3 + \pi^2 r_1 \beta_0^2 + \frac{5}{6} \beta_0^4 \beta_1 \ .$$

(55)
From Eqs. (28, 29, 31) one can obtain $\tau_i$ as it follows where in these equations $T_i$ are replaced by $d_i$. Therefore one can write:

$$\tau_1 = d_1 \ , \ \ \tau_2 = d_2 + c_2 \ , \ \ \tau_3 = d_3 + \frac{1}{2} c_4 + 2c_2 d_1 \ .$$

(56)
To back the result to Minkowski space it is required to use the analytic continuation then the observable $R$ can be written as [31]:

$$R(s) = \frac{1}{2\pi} \int_0^\infty W(\theta) D(se^{i\theta}) d\theta \ .$$

(57)
Here $W(\theta)$ is the weight function which is taken to be 1 for the $R_{e^+e^-}$. Contour improved numerical result, taking $\Lambda_{\overline{MS}} = 419^{+222}_{-168} \text{ MeV}$ at the center of mass energy $\sqrt{s} = 31.6 \text{ GeV}$ would be: $1.0547^{+0.005}_{-0.005}$ which is, as it expected, in better agreement with the available experimental data $3^{+1}_{-1} R_{e^+e^-} \left( \sqrt{s} = 31.6 \text{ GeV} \right) = 1.0527^{+0.005}_{-0.005}$ [23].

If we take $\Lambda_{\overline{MS}} = 210^{+14}_{-14} \text{ MeV}$ which is corresponding to world average value $\alpha_s(M_Z) = 0.1181$ [24] then we get
$\frac{3}{2} R_{e^+e^-} = 1.0471^{+0.0006}_{-0.0007}$. The related conventional value for this observable is $\frac{3}{2} R_{e^+e^-} = 1.0461^{+0.0015}_{-0.0008}$ [5].

Since there are experimental data for $R_{e^+e^-}$ up to center of mass energy 208 GeV [25] or even more, we do as well the required calculations in the McKeon et al. approach, taking $\Lambda_{\overline{MS}} = 210^{+14}_{-222}$ MeV at the energy scales 42.5 and 56.5 GeV. What we obtain for $\frac{3}{2} R_{e^+e^-}$ at these energy scales are 1.0463 and 1.0441 respectively which are comparable with their experimental values, i.e. 1.0554 [26] and 1.0745 [27]. It seems that by increasing energy scales the experimental data becomes little by little far away from the theoretical prediction as can also be seen from [28] (see Fig.8.4 of this Ref.). This is the reason we suspend to consider the $R_{e^+e^-}$ at the other high energy scales. A summery of numerical results is listed in Table I.

**B. Electron-positron annihilation in CORGI approach**

Here we are going to do the same calculation for electron-positron annihilation into hadrons but in CORGI approach. The required information for this approach can be found in [8, 9, 15]. We just remind that in this approach, using the self consistency principle while the scheme parameters as well as renormalization scale are taken into account, it is possible to obtain an expression for perturbative coefficients of QCD observable at high order in terms of coefficients at lower order. Then doing the resumation over the perturbative expansion but at a specified order, the resummed result for instant at next-to-leading order (NLO), or the resummed result at next to NLO (NNLO) and etc. is such that to be RS invariant while the renormalization scale is disappeared as well. Hence there is a possibility to reconstruct the perturbative series in terms of RS invariants while this new constructed series does not depend as well on renormalization scale as an unphysical parameter. By this brief introductory we are now intending to present the numerical result for the $R_{e^+e^-}$ ratio in the CORGI approach.

The full theoretical expression for the electron-positron annihilation into hadrons can be written as [32]:

$$R(s) = N \sum \frac{Q_f^2}{f} \left(1 + \frac{3}{4} C_f \tilde{R}(s)\right) + \left(\sum \frac{Q_f}{f} \right)^2 \tilde{R}(s),$$

where $\tilde{R}$ is the perturbative corrections to the parton model result and has an expansion as it follows:

$$\tilde{R}(s) = a(1 + \sum n r_n a^n).$$

In Eq.(58) $\tilde{R}$ refers to “light-by ligh” contribution and due to factor $\sum Q_f^2$ which is zero for light quark flavors, it would be eliminate from the calculations.

In the Minkowski space, based on the CORGI approach, the perturbative part of intended observable has the following presentation [9]:

$$\bar{R}(s) = a_0 + X_2 a_0^3 + X_3 a_0^4 + ... .$$

The $X_2$ and $X_3$ are scheme invariant quantities and according to notation of McKeon et al. approach can be written in terms of $T_i$ coefficients (see Eq.(12)). The explicit expressions for these quantities have been presented in Ref.[33] (see Eq. (26) of this reference). Then we will arrive at these numerical results:

$$X_2 = -15.0870 \ , \ X_3 = -16.6423 .$$

By substituting the above numerical values in Eq.(60) while $a_0$ is coupling constant at two loop levels which is written in terms of Lambert-W function and then inserting Eq.(60) in Eq.(58), the following result for $R_{e^+e^-}$ ratio in Minkowski space would be obtained: $R_{e^+e^-}(\sqrt{s}) = 31.6 GeV = 1.05440 \pm 0.006 . As before we take into account $\Lambda_{\overline{MS}} = 419^{+222}_{-168}$ MeV and the energy scale $\sqrt{s} = 31.6 GeV$.

If one decides to do the calculations at first in preferable Euclidean space, then the connection between Adler D-function in this space with the $\tilde{R}(s)$ observable in Minkowski space which is like Eq.(53), is again needed. The perturbative expansion of Adler D-function in Eq.(54) has the following form in CORGI approach

$$\bar{D}(s) = a_0 + X_2 a_0^3 + X_3 a_0^4 + ... + X_n a_0^{n+1} .$$

Here $X_i$s are RS invariants and can be written in term of $d_i$ coefficients [9]. The $X_2$ and $X_3$ numerical results for the Adler D-function are as following :

$$X_2 = -7.2775 \ , \ X_3 = 39.9935 .$$

Using the analytic continuation, given by Eq.(57) and substituting Eq.(62) into it, the numerical value for the full expression $R(s)$ in Minkowski space can be obtained. Numerical result for the concerned observable at the NNLO approximation, based on the CORGI approach, arouse out from analytical continuation and taking $\Lambda_{\overline{MS}} = 419^{+222}_{-168}$ MeV would be $R_{e^+e^-}(\sqrt{s}) = 31.6 GeV = 1.0523 \pm 0.005$. This result is in better agreement with reported experimental data: $1.0527 \pm 0.005$ [23] than the result, attained before, from direct calculations in Minkowski space.

Like to what we did in previous approach, we do as well the calculations in CORGI approach for $\frac{3}{2} R_{e^+e^-}$ at the energy scales 42.5 and 56.5 GeV, taking $\Lambda_{\overline{MS}} = 210 \pm 14$. What we obtain are 1.0436 and 1.0425 respectively while their experimental values are 1.0554 [26] and 1.0745 [27]. Similar calculations in CORGI approach at $\sqrt{s} = 31.6 GeV$ but with $\Lambda_{\overline{MS}} = 210 \pm 14$ have also been done. The related numerical values are listed in Table I.
IV. HIGGS DECAY TO GLUON IN MCKEON ET AL. APPROACH

As a second observable in McKeon et al. approach we take into account the Higgs decay width to a gluon-gluon pair \((H \to gg)\). This observable will be of fundamental phenomenological importance and its dominant contribution is given by [34]:

\[
\Gamma(H \to gg) = \frac{4G_F M_H^3}{9\sqrt{2\pi}} R(M_H) , \tag{64}
\]

where \(G_F\) is the fermi constant and \(M_H\) is higgs boson mass. The numerical and perturbative series expansion of \(R(M_H)\) is given later on in Subsec.IV.B and Sec.V respectively.

As before we first review how the scale renormalization will be removed when we do a resummation over all perturbative terms of this observable. The procedure to do RG summation for \(H \to gg\) in many senses is like the one for \(R_{e+e^-}\) ratio. Nonetheless it involves its own worth to do it again for this observable. According to the notation of McKeon et al. approach, the expansion series of \(R(M_H)\) has the following representation:

\[
R(M_H) = a_s^2 \sum_{n=1}^{\infty} \sum_{m=0}^{n} T_{n,m} a_s^n L^m , \tag{65}
\]

where \(T_{0,0} = 1\) and \(L = \ln \frac{\mu}{\Lambda}\). Eq.(65) depends on renormalization scale \(\mu\) as unphysical quantity and it is possible to show, as for the perturbative expansion in Eq.(10), that all dependence on \(\mu\) in \(\Gamma(H \to gg)\) can be removed by doing a summation over all logarithmic terms. We should note that the series expansion in Eq.(65) has an extra factor \(a_s^2\) with respect to Eq.(10). In this regard, the other grouping is introduced:

\[
A_n(a) = \sum_{m=0}^{\infty} T_{n+m,n} a_s^{n+m+2} \quad (n = 0, 1, 2, ...). \tag{66}
\]

Now Eq.(64) can be represented by:

\[
\Gamma(H \to gg) = \frac{4G_F M_H^3}{9\sqrt{2\pi}} \sum_{n=0}^{\infty} A_n(a) L^n . \tag{67}
\]

The RG equation implies that

\[
(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a}) \Gamma(H \to gg) = 0 , \tag{68}
\]

which like before leads to

\[
A_n(a) = -\frac{\beta(a)}{n} \frac{d}{da} A_{n-1}(a) . \tag{69}
\]

By introducing \(\eta\) scale as it follows [11]

\[
\eta = \int_{a_1}^{a(\eta)} \frac{dx}{\beta(x)}, \quad a_1 = \text{const.} \tag{70}
\]

Then

\[
\beta(a) \frac{d}{da} = \frac{d}{d\eta} . \tag{71}
\]

By substituting Eq.(71) into Eq.(69) one will arrive at

\[
A_n(a) = -\frac{1}{n} \frac{d}{d\eta} A_{n-1}(a(\eta)) , \tag{72}
\]

where \(\Lambda\) is defined by Eq.(18). Now one can show that \(A_0, A_1, \ldots\) and finally \(A_n\) can be represented in terms of \(A_0\). Therefore we can write

\[
A_1 = -\frac{1}{2} \frac{d}{d\ln \frac{\mu}{\Lambda}} A_0 ,
\]

\[
A_2 = -\frac{1}{2} \frac{d^2}{d\ln^2 \frac{\mu}{\Lambda}} A_1 = -\frac{1}{2} \frac{d}{d\ln \frac{\mu}{\Lambda}} \left( \frac{1}{2} \frac{d}{d\ln \frac{\mu}{\Lambda}} A_0 \right) = \frac{1}{2} \frac{d^2}{d\ln^2 \frac{\mu}{\Lambda}} A_0 ,
\]

\[
A_3 = -\frac{1}{3} \frac{d^3}{d\ln^3 \frac{\mu}{\Lambda}} A_2 = -\frac{1}{3} \frac{d}{d\ln \frac{\mu}{\Lambda}} \left( \frac{1}{2} \frac{d}{d\ln \frac{\mu}{\Lambda}} A_0 \right) = \frac{1}{3} \frac{d^3}{d\ln^3 \frac{\mu}{\Lambda}} A_0 ,
\]

\[
\ldots
\]

\[
A_n = (-1)^n \frac{d^n}{d^n \ln \frac{\mu}{\Lambda}} A_0 . \tag{73}
\]

If one defines \(\eta = \ln \frac{\mu}{\Lambda}\) then

\[
A_n(a) = \frac{1}{n!} \left( \frac{d}{d\eta} \right)^n A_0(a) , \tag{74}
\]

and finally

\[
R(M_H) = \sum_{n=0}^{\infty} \frac{1}{n!} (-L)^n \left( \frac{d}{d\eta} \right)^n A_0(a(\eta)) = \exp(-L \frac{d}{d\eta}) A_0(a(\eta)) . \tag{75}
\]

The series expansion of exponential term in Eq.(75) makes the right hand side of this equation like the Taylor expansion of \(A_0(a(\ln \frac{\mu}{\Lambda} - L))\). For this purpose, as before one can write

\[
A_0(a(\ln \frac{\mu}{\Lambda} - L)) = A_0(a(\ln \frac{\mu}{\Lambda})) - L A'_0(a(\ln \frac{\mu}{\Lambda})) + \frac{L^2}{2!} A''_0(a(\ln \frac{\mu}{\Lambda})) + \ldots.
\]

Consequently the sum in Eq.(75) results in

\[
R(M_H) = A_0(a(\eta - L)) = A_0(a(\ln \frac{\mu}{\Lambda} - \ln \frac{\mu}{M_H})) = A_0(a(\frac{M_H}{\Lambda})) . \tag{77}
\]
As shown by Eq.(77), the renormalization scale \( \mu \) for Higgs decay width to gluon-gluon has been removed by doing a summation over all its perturbative series terms.

Finally using Eq.(66) with \( m = 0 \), Eq.(77) would be appeared as:

\[
\Gamma(H \to gg) = \frac{4G_F M_H^3}{9\sqrt{2}\pi} a \left( \frac{M_H}{\Lambda} \right)^2 \left( 1 + \sum_{n=1}^{\infty} T_n a \left( \frac{M_H}{\Lambda} \right)^n \right) .
\] (78)

A. Renormalization Scheme dependence for \( H \to gg \)

Here we specifically investigate how the series expansion for Higgs decay width to gluon-gluon can be appeared in which it becomes RS invariant.

As before the RS dependence of the expansion coefficients \( T_n \) in Eq.(78) can be founded by using the RG equation

\[
\frac{d}{dc_1} + \beta_i(a) \frac{d}{da} \Gamma(H \to gg) = 0 ,
\] (79)

where \( \beta_i(a) \) has been given by Eq.(25). Considering perturbative part of Higgs decay in Eq.(78) as

\[
R(M_H) = a^2 + \sum_{n=1}^{\infty} T_n a^{n+2} ,
\] (80)

and employing Eq.(79) we then arrive at

\[
2a\beta_i(a) + \sum_{n=1}^{\infty} a^{n+2} \frac{d T_n}{dc_i} + (n+2)a^{n+1} T_n = 0 .
\] (81)

This equation leads to a sequence of equations for \( T_n \) order by order in terms of \( a \) that their solutions are

\[
\frac{d T_1}{dc_1} = 0 \Rightarrow T_1 = \lambda_1 ,
\]
\[
\frac{d T_2}{dc_2} + 2 = 0 \Rightarrow T_2 = -2c_2 + \lambda_2 ,
\] (82)

where \( \lambda_1 \) and \( \lambda_2 \) are constants of integrations and RS invariant. For \( T_3 \) coefficient, one can write

\[
\frac{d T_3}{dc_2} + 3T_1 = 0 ,
\]
\[
\frac{d T_3}{dc_3} + 1 = 0 ,
\] (83)

which leads to

\[
T_3 = -3\lambda_1 c_2 - c_3 + \lambda_3 .
\] (84)

One can derive for \( T_4 \) coefficient in similar manner the following differential equations:

\[
\frac{d T_4}{dc_2} + \frac{2c_2}{3} + 4T_2 = 0 ,
\]
\[
\frac{d T_4}{dc_3} - \frac{c}{3} + \frac{3}{2} T_1 = 0 ,
\]
\[
\frac{d T_4}{dc_4} + \frac{2}{3} = 0 ,
\] (85)

which leads to the final result:

\[
T_4 = \frac{11}{3} c_2^2 - 4\lambda_2 c_2 + \frac{c}{3} c_3 - \frac{3}{2} \lambda_1 c_3 - \frac{2}{3} c_4 + \lambda_4 .
\] (86)

This process can be followed to obtain the other expressions for higher order of \( T_i \) coefficients.

Like for \( T_1 \) and \( T_2 \) the \( \lambda_3 \) and \( \lambda_4 \) in coefficient of \( T_3 \) and \( T_4 \) are constant of integration and RS invariant that can be determined once \( T_i \) and \( c_i \) have been evaluated in some mass independent RS.

Since the summation of concerned perturbative series are scheme invariant, again two particular RS are of special interest. In the first scheme one can set \( c_i = 0 \) (\( i \geq 2 \)) and consequently \( T_n = \lambda_n \). In this case Eq.(80) is appearing as

\[
R_{(2)}(M_H) = a_{(2)}^2 + \sum_{n=1}^{\infty} \lambda_n a_{(2)}^{n+2} .
\] (87)

In the other case with setting \( T_i = 0 \) (\( i \geq 2 \)), Eq.(80) would involve just two terms

\[
R_{(1)}(M_H) = a^2 + \lambda_1 a^3 .
\] (88)

Reminding that the coupling \( a \) in Eq.(88) depends on the scheme parameters \( c_i \) while the coupling Eq.(87) depends just on two \( b \) and \( c \) universal scheme parameters that can be expressed in terms of the Lambert function \( W(x) \) i.e \( x = W(x)e^{W(x)} \).

Again, considering the relation between the couplings at two different schemes, one can show that two presentation of Eq.(88) and Eq.(87) are equivalent such that

\[
R_{(1)}(M_H) = R_{(2)}(M_H) .
\] (89)

B. Numerical value for Higgs decay to gluon

In this section we do numerical investigation for the Higgs decay width to gluon-gluon in McKeon et al. approach. The Higgs boson decay to gluon-gluon is given by Eq.(64) where \( R(M_H) \) in this equation has a numerical expansion in conventional perturbative QCD as it follows [34]

\[
R(M_H) = a^2(1+17.9167a+153.15787a^2+393.822a^3+...). \] (90)
According to the notation of McKeon et al. approach, one can obtain from Eq.(90) the following numerical values for the $T_i$ coefficients:

\[ T_1=17.9167 \ , \ T_2=153.1578788 \ , \ T_3=393.822046 \ . \]  

(91)

From Eqs.(82,84) the following numerical values for the $\lambda_i$ as RS invariants would be obtained:

\[ \lambda_1=T_1=17.9167 \]
\[ \lambda_2=T_2+2c_2=156.1074561 \]
\[ \lambda_3=T_3+c_3+3c_2\lambda_1 = 482.927999 \]

(92)

Now Eq.(87) which has been obtained, using 't Hooft scheme, becomes

\[ R(M_H) = a^2(1+\sum_{n=1}^{\infty} d_n a^n) = a^2 + d_1 a^3 + d_2 a^4 + d_3 a^5 + \ldots \ . \]  

(93)

In this equation, as illustrated before, coupling constant $a_2$ can be written in terms of Lambert-W function which is depends on physical energy scale $Q$ and $\lambda_i$ are specified scheme invariants. To do numerical calculation for the decay width $H \to gg$ in the McKeon et al. approach we consider again $G_F = 1.16638 \times 10^{-5}$ $GeV^{-2}$ and the Higgs mass $M_H = 126 \pm 4 GeV$ while we take the number of active quark flavor $N_f = 5$ and QCD cut off $\Lambda_{QCD} = 210 \pm 14$ MeV. What we achieve for the concerned decay width is: $\Gamma(H \to gg) = 0.383 \pm 0.01$ $MeV$ which can be compared to the result of the conventional value: $0.349 \pm 0.05$ $MeV$ [36, 37].

As for $R_{e^+e^-}$ ratio it is possible to follow the numerical calculation in Euclidian space and then by contour improved back to Minkowski space to get more precise result.

Back to Eq.(93) the related Adler-D function should have the following expansion

\[ D(M_H) = a^2(1+\sum_{n=1}^{\infty} d_n a^n) = a^2 + d_1 a^3 + d_2 a^4 + d_3 a^5 + \ldots \ . \]  

(94)

There are the following relations between that $d_i$ coefficients and the $T_i$ ones in Eq.(80) where we deal with about them in next section:

\[ d_1 = T_1 \ , \ d_2 = T_2 + \beta_0^2 \pi^2 \ , \]
\[ d_3 = T_3 + 2\pi^2 T_1 \beta_0^2 + \frac{2}{3} \beta_0^2 \beta_1 \ . \]  

(95)

(96)

(97)

Now using Eqs. (82,84) one can obtain $\lambda_i$ invariants as it follows:

\[ \lambda_1=T_1 \ , \ \lambda_2=d_2+2c_2 \ , \ \lambda_3=d_3+c_3+3c_2\lambda_1 \ . \]  

(98)

Finally the perturbative part in Minkowski space converts to

\[ D(M_H) = a^2(1+\sum_{n=1}^{\infty} d_n a^n) = a^2 + d_1 a^3 + d_2 a^4 + d_3 a^5 + \ldots \ . \]  

(99)

where in addition to $\lambda_i$ quantities as RS invariants, $a_2$ coupling that is written in terms of Lambert-W function, is independent of renormalization scheme and depends just on physical energy scale $Q = M_H$. Now the results for Higgs decay width to gluon-gluon in the McKeon et al. approach but in the Euclidian space is accessible. Using the contour improved integration as in below we could get the decay width in the Minkowski space:

\[ R(M_H) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(M_H e^{i\theta}) d\theta \ . \]  

(100)

Finally by taking $M(H) = 126 \pm 4$ $GeV$ and $\Lambda_{QCD} = 210 \pm 14$ $MeV$ the result would be $\Gamma(H \to gg) = 0.370 \pm 0.01$ $MeV$ which is compatible with the convention one: $0.349 \pm 0.05$ $MeV$. At the moment there is not any experimental data for Higgs decay width to gluon-gluon. But it is expected the result, arising out from counter improved, shall be closer to experimental value. Similar calculations but with updated value for Higgs mass, $M_H = 125.18 \pm 0.16$ $GeV$ [24] have also been done. The concerned numerical results have been brought in Table II.

V. HIGGS DECAY TO GLUON IN CORGI APPROACH

Here we review how to obtain the result for Higgs decay to gluon-gluon in CORGI approach. More details can be found in [9]. Nevertheless in order the readers can follow the subject clearly, some main steps to arrive at CORGI approach for $\Gamma(H \to gg)$ are given here. Reminding that in conventional perturbative series, the Higgs decay width to gluon-gluon is given by

\[ \Gamma(H \to gg) = \frac{4G_F M_H^3}{9\pi^3} \left[ a_s^2 + r_1 a_s^3 \right. \]
\[ + r_2 a_s^4 + 3c_2 a_s^5 + \mathcal{O}(a_s^6) \right] \]  

(101)

where $a = \frac{\alpha_s}{\pi}$ is coupling constant and coefficients $r_i$ are known in [34]. As it is obvious the perturbative part of Eq.(101) is presented by:

\[ R = a^2 + r_1 a^3 + r_2 a^4 + r_3 a^5 + \ldots \ . \]  

(102)

Self-consistency principle for this observable up to $n$-th order, yields us [17]

\[ \frac{\partial R^{(n)}}{\partial(RS)} = \mathcal{O}(a^{n+1}) \ , \]  

(103)

where $n = 1, 2, \ldots, n$ and RS stands for the scale parameter $\tau = b \ln \frac{\mu^2}{\Lambda}$ as well as the scheme parameters such as $c_2, c_3, c_4, \ldots$ [9, 15]. Considering two terms in Eq.(102) that is

\[ R = a^2 + r_1 a^3 \ , \]  

(104)
and using self-consistency principle with respect to $\tau$ variable, one then concludes
\[
\frac{\partial r_1}{\partial \tau} = 2 \Rightarrow r_1 - 2\tau = \rho_1 .
\] (105)
Here $\rho_1$ is RS invariant and independent of un-physical scale parameter $\tau$. Adding the third term to Eq.(104) which contains the $r_2$ coefficient then it appears as:
\[
R(Q) = a^2 + r_1a^3 + r_2a^4 .
\] (106)
Then by derivation of $R(Q)$ with respect to the related RS parameters, one arrives at
\[
\begin{align*}
\frac{\partial r_2}{\partial c_2} &= -2, \\
\frac{\partial r_2}{\partial r_1} &= c + 3 \frac{r_1}{2} ,
\end{align*}
\] (107)
where in deriving the first above differential equation, the Eq.(26) has been used. Simultaneous solution of differential equations (107) will lead to:
\[
r_2 = cr_1 + \frac{3}{4}r_1^2 - 2c_2 + X_2 .
\] (108)
Here $X_2$ is the constant of integration and RS invariant. If the forth term of the series expansion in Eq.(102) is considered such that
\[
R(Q) = a^2 + r_1a^3 + r_2a^4 + r_3a^5 ,
\] (109)
then self consistency principle implies that
\[
\begin{align*}
\frac{\partial r_3}{\partial c_2} &= -3r_1 , \\
\frac{\partial r_3}{\partial c_3} &= -1 , \\
\frac{\partial r_3}{\partial r_1} &= \frac{3}{2}cr_1 + 2r_2 + c_2 .
\end{align*}
\] (110)
Considering the above differential equations, the result for $r_3(r_1, c_2, c_3)$ is
\[
r_3(r_1, c_2, c_3) = \frac{r_1^3}{2} + \frac{7}{4}cr_1^2 - 3c_2r_1 + 2X_2r_1 - c_3 + X_3 ,
\] (111)
where $X_3$ is the constant of integration and RS invariant. Substituting Eqs.(108,111) in Eq.(102) the result would be
\[
R(Q) = a^2 + r_1a^3 + (cr_1 + \frac{3}{4}r_1^2 - 2c_2 + X_2)a^4 + (\frac{r_1^3}{2} + \frac{7}{4}cr_1^2 - 3c_2r_1 + 2X_2r_1 - c_3 + X_3)a^5 + ... .
\] (112)
Doing resummation for the individual NLO, NNLO and etc. contributions in above equation and employing the ’t Hooft scheme at any specified order such that $c_2 = c_3 = ... = c_n = 0$ and for convenience to set $r_1 = 0$ then one will arrive at the following result [9]
\[
R(Q) = a_0^2 + X_2a_0^3 + X_3a_0^5 + ... .
\] (113)
In this equation $a_0$ and $X_i$ are scale and scheme invariant and $a_0$ can be expressed in term of Lambert-W function. At first we present the numerical result for Higgs decay width to gluon-gluon directly in Minkowski space. For this propose we require initially to calculate the numerical values for the $X_2$ and $X_3$ in Eq.(113) which can be done using Eqs.(108,111) where $r_i$ coefficients have been calculated in [25]. The results for these RS invariant quantities are as following:
\[
X_2 = -107.2392 , \quad X_3 = 3269.9755
\] (114)
By substituting the above results in Eq.(113) and taking $M(H) = 126 \pm 4$ GeV and $\Lambda_{QCD}^{57} = 210 \pm 10$ MeV the numerical result for Higgs decay width would be: $\Gamma(H \to gg) = 0.393 \pm 0.025$ MeV.
Since the calculation can be done in Euclidean space as in the $R_{\tau+e^-,e^-}$, it is needed to convert the calculations from the Minkowski to Euclidean space. For this propose at first one needs to a relation between coupling constants at different scales:
\[
a(Q) = a(\mu)(1+h_1a(\mu)+h_2a(\mu)^2+h_3a(\mu)^3+h_4a(\mu)^4+... ) .
\] (115)
The following relations exist for the $h_i$ s coefficients [35]:
\[
\begin{align*}
h_1 &= -\beta_0L , \\
h_2 &= \beta_0^2L^2 - \beta_1L , \\
h_3 &= -\beta_0^3L^3 + \frac{5}{2}\beta_0\beta_1L^2 - \beta_2L , \\
h_4 &= \beta_0^4L^4 - \frac{13}{3}\beta_0^2\beta_1L^3 + 3\left(\frac{1}{2}\beta_0^2 + \beta_0\beta_2\right)L^2 - \beta_3L .
\end{align*}
\] (116)
Here $L = \ln(\frac{\mu}{\mu_0})$ where $\mu$ is new scale and $\beta_i$s are the coefficients of QCD $\beta$ function . Substituting Eq.(115) in Eq.(102) will yield us the observable $R$ in terms of the $s$ variable. Using the dispersion relation, Eq.(53), but for Higgs decay width to gluon-gluon, will yield us this observable in Euclidean space. To back the result form the $\mu$ scale to the $Q$ one, it is needed to use again the displacement relation, Eq.(115), in a proper state. Finally we arrive at
\[
D(Q) = a(Q)^2 + r_1a(Q)^3 + (r_2 + \beta_0^2a(Q)^2)a(Q)^4 + (\frac{7}{3}\beta_0\beta_1s^2 + r_3 + 2\beta_0^2\beta_2s^2)\alpha(Q)^5 + O(aQ)^6
\] (117)
By comparing Eq.(102) and Eq.(117) the following relations between $d_i$, s and $r_i$ s coefficients would be obtained:
\[
\begin{align*}
d_1 &= r_1 , \\
d_2 &= r_2 + \beta_0^2s^2 , \\
d_3 &= r_3 + 2\pi^2r_1\beta_0^2 + \frac{7}{3}\beta_0\pi^2\beta_1 ,
\end{align*}
\] (118)
Using the counter improved integration as in below we could get back the decay width to the Minkowski space:

$$\Gamma(H \rightarrow gg) = \frac{4G_F M_H^3}{9\sqrt{2}\pi} \frac{1}{2\pi} \int_0^{2\pi} \left[ a_0^2 + X_2a_0^4 + X_3a_0^5 + \ldots \right] d\theta.$$  \hspace{1cm} (119)

where

$$X_2 = 472.8741 \quad , \quad X_3 = 10096.8174$$  \hspace{1cm} (120)

To compute the above numerical results, we need again to Eqs.(108,111) while \( r_1 \) coefficients are replaced by \( d_i \) ones, given by Eq.(118). As before \( a_0(s_0e^{i\theta}) \) is coupling constant in CORGI approach. To do numerical calculation for the decay width \( H \rightarrow gg \) in the CORGI approach, we take as before \( G_F = 1.16638 \times 10^{-5} \text{ GeV}^{-2} \) and the Higgs mass \( M_H = 126 \pm 4 \text{ GeV} \) while we take \( N_f = 5 \) and \( \Lambda_{\overline{MS}} = 210 \pm 14 \text{ MeV} \). What we obtain for the concerned decay width would be:\( \Gamma(H \rightarrow gg) = 0.379 \pm 0.03 \text{ MeV} \) which can be compared with the conventional result : \( 0.349 \pm 0.05 \text{ MeV} \) \cite{36,37}. The numerical results, considering updated value for Higgs mass, \( M_H = 125.18 \pm 0.16 \text{GeV} \) \cite{24}, have been added in Table II.

VI. CONCLUSION

In this article, we reviewed first the principles that the McKeon et al. approach has been based on them \cite{10–14}. In this approach two essential steps have been done. At first by defining a new summed group, denoted by \( A(n) \), it was tried by taking into account the RGE, to find a pattern which gives a recurrence relation between the \( n^\text{th} \) and the \((n-1)^\text{th} \) order of \( A(n) \). Finally the concerned quantity at \( n^\text{th} \) order can be written in terms of the first order. In deriving this pattern the role of QCD \( \beta \)-function is very essential. Then by constructing the primary series of QCD observable in terms of \( A(n) \) and doing a resummation over all perturbative terms, the renormalization scale as an unphysical quantity was disappeared and the final result for the QCD observable depended just on physical energy scale \( Q \) (see Eq.(23)).

In the second step of this approach, RS dependence of perturbative series coefficients were being investigated. Considering the differential equation for the coupling constant with respect to scheme parameters \( c_i, (i \geq 2) \), based on the Eq.(25) one could rewrite the perturbative series of QCD observable in terms of scheme invariant quantities while the coupling constant depends just on the physical energy scale, written in terms of Lambert-W function (see Eq.(36)). It should be reminded that in this approach there is a possibility to render the RS invariant and scale independence of series expansion such that the series contains just two terms. In this case the related coupling constant, however is scale independence, but depends on the scheme parameters \( c_2, c_3 \) and etc. But the series is overall independent of renormalization scale and scheme (see Eq.(35)). At the end of subsection II B it has been shown that these two series expansion are equivalent.

In the second approach which was called CORGI, using the self consistency principle there would be a possibility to express the \( r_1 \) coefficient at \( n^\text{th} \) order in terms of the coefficients at lower orders which finally include just the coefficient at first order, i.e. \( r_1 \). During to employ this principle, not only the derivative of observable with respect to scheme parameters \( c_i, (i \geq 2) \) is considered but also the derivative with respect to \( r_1 \) coefficient is required to obtain such that in \( \frac{\partial R}{\partial \overline{R}} \) derivative, the RS's can be labeled by \( (r_1, c_2, c_3, \ldots) \) parameters \cite{17}. The parameter \( r_1 \) can be casted latter on in terms of \( \tau = \ln \frac{\Lambda}{\mu} \) as scale parameter \cite{9}. By simultaneous solution of differential equations which have been obtained for \( r_1 \), using self-consistency principle, with respect to \( (r_1, c_2, c_3, \ldots) \) parameters, the \( r_i \) coefficients can be presented as a function of these parameters. Rewriting the initial perturbative series of QCD observable, Eq.(2) in terms of relations which have been attained for \( r_i \) coefficients, a possibility would be provided to do a resummation over the perturbative series at individual NLO approximation, later on at NNLO approximation and so on. Then the desired perturbative series can be rewritten in terms of scheme invariants and coupling constant which just depends on physical energy scale.

From this point of views the CORGI approach and the the McKeon et al. approach have been achieved the same goal. Both of them were able to reconstruct the conventional perturbative series in terms of the scheme invariants and the coupling constant which does not depend on renormalization scheme. But which one has more advantage with respect to the other one?. At first consideration, one may assign the priority to the McKeon and et al. approach. In this approach the renormalization scale parameter and scheme parameters are investigating separately, as we explained it in introduction section while in CORGI approach this separation, utilizing the self-consistency principle, has not occurred.

We reexamined these two approaches and employed them to do numerical computations for \( R_{e+e^-} \) ratio and the Higgs decay width to gluon-gluon. We listed the numerical results, based on the contour improved integration, in Tables I and II. Inspection the Table I indicates that the results of McKeon et al. approach are in better agreement with the available data.

In addition to these two approaches, there is another approach which is called principle of maximum comfortability (PMC). We explained about this approach in introduction section. Based on this approach there is a possibility to absorb the non-conformal parts of the perturbative series coefficients into the renormalized coupling constant and finally to attain a perturbation series in terms of just conformal parts while the renormalization scales are also fixed at any specified order. It is possible to convert the McKeon et al. approach to PMC one which has been done in \cite{13}. Some other works which are related to PMC approach can be found in \cite{6,7}.
Converting the CORGI approach to the PMC one, is the other research task which is not so easy to do it and we hope to be able to report about it in future. The McKeon et al. approach can be considered at infrared fixed point where QCD $\beta$-function is considered to be zero [38–40]. This subject is also interesting to be investigated from the other literature points of view like the infrared safe mode of QCD [41] or the Ads/CFT [42] and would be valuable to follow it as our further research activity.

ACKNOWLEDGMENT

Authors acknowledge Yazd university for provided facilities to do this project. Authors are indebted M. R. Khellat for his reading of the manuscript and constructive comments.

Table I: Numerical results for $R_{e^+e^-}$ at $\sqrt{s} = 31.6, 42.5$ and $56.5$ GeV, using two different approaches.

| $R_{e^+e^-}$ | CORGI approach | Conventional pQCD | Experimental value | $\Lambda_{QCD}$ |
|-------------|----------------|------------------|-------------------|---------------|
| $\sqrt{s} = 31.6$ GeV | 1.04711$^{+0.0097}_{-0.0097}$ | 1.04615$^{+0.0006}_{-0.0006}$ | 210 $^{+14}$ MeV |
| $\sqrt{s} = 42.5$ GeV | 1.0463$^{+0.0004}_{-0.0004}$ | 1.0436$^{+0.0003}_{-0.0003}$ | 210 $^{+14}$ MeV |
| $\sqrt{s} = 52.5$ GeV | 1.0441$^{+0.0004}_{-0.0004}$ | 1.0425$^{+0.0004}_{-0.0004}$ | 210 $^{+14}$ MeV |

Table II: Numerical results for Higgs decay to gluon-gluon at two different approaches.

| $H \rightarrow gg(M_H = 126 \pm 4)$ | CORGI approach | Conventional pQCD | Experimental value |
|-----------------|----------------|------------------|--------------------|
| $H \rightarrow gg(M_H = 125.18 \pm 0.16)$ | 0.379$^{+0.042}_{-0.042}$ | 0.349$^{+0.007}_{-0.007}$ | $210 \pm 14$ MeV |

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