A Min-plus Model of Age-of-Information with Worst-case and Statistical Bounds

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Abstract—We consider networked sources that generate update messages with a defined rate and we investigate the age of that information at the receiver. Typical applications are in cyber-physical systems that depend on timely sensor updates. We phrase the age of information in the min-plus algebra of the network calculus. This facilitates a variety of models including wireless channels and schedulers with random cross-traffic, as well as sources with periodic and random updates, respectively. We show how the age of information depends on the network service where, e.g., outages of a wireless channel cause delays. Further, our analytical expressions show two regimes depending on the update rate, where the age of information is either dominated by congestive delays or by idle waiting. We find that the optimal update rate strikes a balance between these two effects.

I. INTRODUCTION

Networked cyber-physical systems rely on timely status information provided by all kinds of remote sensors. At a data sink, the freshness of a status information depends on network delays but also on the update rate of the sensor. Age of information quantifies this freshness, measuring the time that has elapsed between the generation of a sensor reading and its use. A common illustration [1] of the progression of the age of information $\Delta(t)$ over time $t \geq 0$ is shown in Fig. 1, where $T_A(n)$ denotes the arrival time of status update $n \geq 1$ from the sensor to the network and $T_D(n)$ its departure time from the network to the sink. For an example, select a time $t^* \geq T_D(1)$, determine the latest status update at the sink $n^* = \max\{n \geq 1 : T_D(n) \leq t^*\}$ that was generated at time $T_A(n^*)$ to find the age of information $\Delta(t^*) = t^* - T_A(n^*)$.

The notion of age of information has been introduced in vehicular networks [1] where it has been referred to also as status age [2], information freshness [3], or message lifetime [4]. It emerged as a very active area of research, being of general importance for a variety of applications in the areas of cyber-physical systems and the Internet of Things. There, particular challenges arise in networked feedback control systems [5], [6]. Further applications extend to cache updating and microblogging [7]. For a recent, comprehensive survey see [8].

A focus of age of information research are wireless channels, such as multiple access channels [1], [9], memoryless on-off channels [6], fading channels [9], and wireless networks with interference constraints [10]. With the general aim of minimizing the age of information, a frequent subject of investigation is the optimal update rate [2], [6], [11], [12]. Typical update processes are periodic [1], [6], [12] or random [2], [5], [9]–[11], [13]. Predominantly, these works use analytical models, such as queueing theory [2], [11], [14], to derive time averages of mean and peak age of information.

Most closely related to this work are two papers that apply techniques common to the network calculus: [9] derives statistical delay bounds using a $(\min, \times)$-algebra with Mellin transform and moment bound; and [6] employs a max-plus algebra to derive statistical age of information bounds by Chernoff’s bound. The max-plus formulation works with time stamps of packet arrivals and departures. It can express delay most easily and extends naturally to age of information.

Differently from [6], we choose a min-plus algebra in this work. The min-plus approach uses cumulative arrivals and departures, i.e., bits as functions of time, that are pseudo-inverse functions of the max-plus representation [15]. Unlike max-plus, min-plus can conveniently model multiplexing of traffic flows and time-varying services that arise, e.g., due to wireless communications or scheduling algorithms. We will take advantage of this and derive statistical age of information bounds for these systems. Our results show how service outages, congestive delays, and idle waiting affect the age of information. They enable finding the optimal update rate.

The remainder of this paper is structured as follows. We will derive our min-plus model of the age of information in Sec. II, where we also show worst-case bounds for periodic updates. In Sec. III we derive a solution for random service and show statistical age of information bounds for a Markov channel. Random arrivals are considered in Sec. IV and scheduling of multiple sources with different priorities in Sec. V. We present brief conclusions in Sec. VI.

II. MIN-PLUS AOI MODEL

We consider a queueing system, such as a buffered link, a scheduler, or a network thereof. We denote arrivals $A(t)$ the cumulative amount of data arriving at the system in $[0, t)$. By definition, the function $A(t)$ is non-negative, non-decreasing and passes through the origin. For convenience we extend the definition $A(t) = 0$ for $t < 0$. Hence, $A(t) \in F_0$ where
\( \mathcal{F}_0 = \{ f(t) : f(t) \geq f(\tau) \geq 0 \ \forall t \geq \tau \geq 0, f(t) = 0 \ \forall t \leq 0 \} \).

We use shorthand notation \( A(\tau, t) = A(t) - A(\tau) \) for \( t \geq \tau \).

We employ a continuous time model where by convention \( A(t) \) is left-continuous. Similarly, the cumulative departures of a system are denoted \( D(t) \in \mathcal{F}_0 \), where in addition \( D(t) \leq A(t) \) for all \( t \) for causality.

We use the concept of dynamic server \([16]\) to model systems. A system has service process \( S(\tau, t) \) for \( t \geq \tau \geq 0 \) if

\[
D(t) \geq \inf_{\tau \in [0, t]} \{ A(\tau) + S(\tau, t) \} =: A \otimes S(t),
\]

for all \( t \geq 0 \), where \( S(\tau, t) \) is non-negative, non-decreasing with \( t \), and non-increasing with \( \tau \). For a basic example, a buffered, lossless, and work-conserving link with constant service rate \( c > 0 \) has service process \( S(\tau, t) = c(t - \tau) \). Another example is a link with a time-varying service rate \([17]\).

In a continuous time model, it is convenient to assume that data behaves like fluid, i.e., a system may serve any amount of data regardless of packet or message boundaries. The effects that are due to packet boundaries are modeled by a packetizer that converts fluid input \( x \in \mathbb{R}_+ \) to packetized output \( P_L(x) \) \([17, 18]\). Given packets of length \( l(n) > 0 \) with packet index \( n \in \mathbb{N} \), we denote the cumulative packet length \( L(n) = \sum_{\nu=1}^{n} l(n) \) and \( L(0) = 0 \). The output of the packetizer is

\[
P_L(x) = \max_{n \in \mathbb{N}} \{ L(n) \mathbb{1}_{\{ L(n) \leq x \}} \},
\]

where \( \mathbb{1}_{\{ \}} \) is the indicator function that is one if the argument is true and zero otherwise. It follows that \( x \geq P_L(x) \geq x - l_{\text{max}} \), where \( l_{\text{max}} = \max_{n \in \mathbb{N}} \{ l(n) \} \) is the maximal packet length. A function \( A(t) \) is packetized if \( A(t) = P_L(A(t)) \).

Fig. 2 shows an example of a packetized arrival function \( A(t) \) (left-continuous, marked by empty and full circles) and departure function \( D(t) \), fluid (dashed line) and packetized (solid line), respectively.

Now, consider the series of a fluid system and a packetizer. The system has service process \( S(\tau, t) \), packetized arrivals \( A(t) = P_L(A(t)) \), and fluid departures \( D(t) \geq A \otimes S(t) \). For the packetized departures \( P_L(D(t)) \) it is known that \([17, 18]\)

\[
P_L(D(t)) \geq \inf_{\tau \in [0, t]} \{ P_L(A(\tau) + S(\tau, t)) \}
\]

\[
\geq \inf_{\tau \in [0, t]} \{ A(\tau) + [S(\tau, t) - l_{\text{max}}]_+ \},
\]

where \( [x]_+ = \max\{0, x\} \). The first line follows since \( P_L(x) \) is non-decreasing and right-continuous. In the second line, \( P_L(x) \geq x - l_{\text{max}} \) and \( P_L(A(\tau) + S(\tau, t)) \geq P_L(A(\tau)) = A(\tau) \) since \( S(\tau, t) \) non-negative are used. As a result, the effects that are due to packetization are integrated into the service process, i.e., for \( t \geq 0 \) the combination of the fluid system and the packetizer offers service process

\[
S_{P_L}(\tau, t) = [S(\tau, t) - l_{\text{max}}]_+.
\]

A. Definition of Age of Information

Equipped with basics of the network calculus, we now derive the age of information \( \Delta(t) \) at time \( t \geq 0 \). For a system with first-come first-served (fcfs) policy, we define

\[
\Delta(t) = \sup\{ \delta \in [0, t] : D(t) - A(t - \delta) \leq 0 \},
\]

i.e., the last bit that departed before or at \( t \) arrived no earlier than \( t - \Delta(t) \). Hence, \( \Delta(t) \) is the age of that bit in the system at time \( t \). For an example, Fig. 2 shows \( \Delta(t) \) for \( t = 8 \). For comparison, Fig. 2 also includes the virtual delay \( V(t) \) for \( t = 3 \). The virtual delay is the delay metric that is commonly used in network calculus. It is virtual in the sense that it is not conditioned on an actual arrival at \( t \).

Most closely related works \([6, 14]\) use a max-plus model where \( T_A(n) \) and \( T_D(n) \) denote the arrival and departure time stamps of packet \( n \in \mathbb{N} \). There is a duality of max-plus and min-plus models, see \([15]\). The max-plus definition of age of information is \( \Delta(t) = t - \max_{n \geq 1} \{ T_A(n) : T_D(n) \leq t \} \) \([6, 14]\). The equivalence with \( (5) \) is seen in Fig. 2, exemplified for \( t = 8 \) and \( n = 2 \).

Next, we derive the age of information for a system with service process \( S(\tau, t) \). By insertion of \( (1) \) into \( (5) \) it follows for \( t \geq 0 \) that

\[
\Delta(t) = \sup \{ \delta \in [0, t] : \inf_{\tau \in [0, t]} \{ S(\tau, t) + A(\tau - A(t - \delta)) \} \leq 0 \},
\]

which is equivalent to

\[
\Delta(t) = \sup \{ \delta \in [0, t] : \inf_{\tau \in [0, t]} \{ S(\tau, t) + A(t - \delta, \tau) \} \leq 0 \},
\]

Here, we have to pay attention to the second line of \( (6) \), where \( \inf_{\tau \in [t - \delta]} \{ S(\tau, t) + A(t - \delta, \tau) \} \geq 0 \) trivially, but may nevertheless attain the outer infimum if the functions have plateaus, i.e., if they are not strictly increasing.

B. Worst-Case Analysis

The network calculus uses deterministic envelope functions for worst-case analysis. An upper envelope \( \mathcal{E}(t) \in \mathcal{F}_0 \) and a lower envelope \( \mathcal{E}(t) \in \mathcal{F}_0 \) of the arrivals satisfy for all \( t \geq 0 \) that

\[
\mathcal{E}(t - \tau) \geq A(\tau, t) \geq \mathcal{E}(t - \tau).
\]
Also, a deterministic lower envelope \( S(t) \in \mathcal{F}_0 \) of the service satisfies for all \( t \geq \tau \geq 0 \) that
\[
S(\tau, t) \geq S(t - \tau),
\]
(8)
By insertion into (1) it follows that \( D(t) \geq A \otimes S(t) = \inf_{\tau \in [0, t]} \{ A(\tau) + S(t - \tau) \} \), where \( S(t) \) is known as deterministic lower service curve [17], [18]. By insertion of (7) and (8) into (6), a variable substitution, and letting \( t \to \infty \), we find the worst-case age of information bound
\[
\Delta_{\text{max}} \leq \sup \left\{ \delta \geq 0 : \inf_{\tau \in [0, \delta]} \{ S(\tau) - E(\tau - \delta) \}, \inf_{\tau \in [0, \delta]} \{ S(\tau) + E(\delta - \tau) \} \right\} \leq 0.
\]
(9)

C. Periodic Updates
We consider a source that periodically sends update messages resulting in packets of length \( l > 0 \). The width of the update interval \( w \geq 0 \), hence \( A(t) = l \lfloor t/w \rfloor \), and \( A(t) \) has the upper and lower envelope functions
\[
l [t/w] \geq A(t, t + \tau) \geq l [t/w],
\]
for all \( \tau, t \geq 0 \). For a first example, we use a fluid service curve \( S(t) = ct \) that models a minimal capacity guarantee \( c \). Including the packetizer into the service curve model (4), we obtain with \( l_{\text{max}} = l \) that \( S_p(t) = [ct - l]_+ = c[t - l/c]_+ \). Here, packetization is expressed as a latency \( t_0 = l/c \), so that the resulting service curve is of the latency-rate type \( S_p(t) = c[t - t_0]_+ \). By insertion into (9) we have
\[
\Delta_{\text{max}} \leq \sup \left\{ \delta \geq 0 : \inf_{\tau \in [0, \delta]} \{ c[t - t_0]_+ - l \lfloor (\tau - \delta)/w \rfloor \}, \inf_{\tau \in [0, \delta]} \{ c[t - t_0]_+ + l \lfloor (\delta - \tau)/w \rfloor \} \right\} \leq 0.
\]
With the stability condition \( c \geq l/w \), we find that the first infimum is smaller or equal zero only if \( \delta < l/c + t_0 \), consider \( \tau = l/c + t_0 \), and larger than zero otherwise. The second infimum is zero only if \( \delta < w + t_0 \), consider \( \tau = t_0 \), so that
\[
\Delta_{\text{max}} \leq \max \left\{ \frac{l}{c}, w \right\} + t_0,
\]
and since \( w \geq l/c \) from the stability condition, we have
\[
\Delta_{\text{max}} \leq w + t_0.
\]
(10)
We can easily see how the bound is attained if the system transmits messages with rate \( c \). Clearly, for a system with deterministic service curve, the maximal age of information can be reduced by decreasing the width of the update interval up to \( w = l/c \), i.e., full utilization, so that \( \Delta_{\text{max}} \leq 2l/c \).

We consider systems with a random service, such as a wireless channel, where the state of the channel determines the success of a transmission, or a scheduler with cross-traffic.

III. RANDOM SERVICE

We define a statistical service curve \( S_\varepsilon(t) \in \mathcal{F}_0 \) that models a minimal capacity guarantee that has probability of underflow \( \varepsilon \in [0, 1] \), where \( \varepsilon \) is typically small, e.g., \( 10^{-6} \). It is important to note that (11) considers the probability of underflow along an entire sample path \( \tau \in [0, t] \). Thus, it can be directly inserted into (1). It follows that \( P[D(t) < A \otimes S_\varepsilon(t)] \leq \varepsilon \), where \( S_\varepsilon(t) \) (with a certain parameterization) is known as statistical service curve [21] or weak stochastic service curve [20]. With (6) we obtain a statistical age of information bound \( P[\Delta(t) > \Delta_\varepsilon] \leq \varepsilon \) for \( t \geq 0 \) where \( \Delta_\varepsilon \) is given by substitution of \( S_\varepsilon(t) \) into (9).

To derive \( \varepsilon \) for a certain function \( S_\varepsilon(t) \), we use a bound of the moment generating function of \( S(\tau, t) \) defined as
\[
E[e^{-\theta S(\tau, t)}] \leq e^{-\theta (\rho(\theta)(t - \tau - \sigma(\theta))},
\]
(12)
for \( \theta \geq 0 \) and envelope parameters \( \sigma(\theta) \geq 0 \) and \( \rho(\theta) > 0 \). The bound is a variant of the \( (\sigma(\theta), \rho(\theta)) \) traffic characterization of [17] applied to service processes, see [19] for details.

We define a sampling interval \( \tau_0 > 0 \) and number the intervals by \( \kappa \in \mathbb{N}_0 \), i.e., the interval \( [0, t] \) is extended to the left and divided into subintervals \( [t - (\kappa + 1)\tau_0, t - \kappa\tau_0] \) for \( \kappa \in [0, t/\tau_0] - 1 \). Since \( S_\varepsilon(t) \in \mathcal{F}_0 \) is non-decreasing and \( S(\tau, t) \) is non-negative and non-increasing with increasing \( \tau \), we have for \( \kappa \in [0, t/\tau_0] - 1 \) that if
\[
S(t - \kappa\tau_0, t) \geq S_\varepsilon((\kappa + 1)\tau_0) \Rightarrow S(\tau, t) \geq S_\varepsilon(t - \tau), \forall \tau \in [t - (\kappa + 1)\tau_0, t - \kappa\tau_0].
\]
Hence, with application of the union bound
\[
P[\exists \tau \in [0, t] : S(\tau, t) < S_\varepsilon(t - \tau)] \leq \sum_{\kappa=0}^{t/\tau_0 - 1} P[S(t - \kappa\tau_0, t) < S_\varepsilon((\kappa + 1)\tau_0)].
\]
(13)
We let \( S_\varepsilon(t) = [rt - b]_+ \) with \( 0 < r < \rho(\theta) \) and \( b \geq r\tau_0 + \sigma(\theta) \). It follows that \( S_\varepsilon(\tau_0) = 0 \) and the summand for \( \kappa = 0 \) in (13) is zero trivially since \( S(\tau, t) \) is non-negative. For the same reason, we drop the \( [\cdot]_+ \) condition of \( S_\varepsilon(t) \) in the following step. To estimate (13), we apply Chernoff’s lower bound \( P[X \leq x] \leq e^{\delta^2/2}E[e^{-\delta X}] \) for a random variable \( X \) and
\( \theta > 0 \). We insert \( S(t - \kappa \tau_0, t) \) for \( X, S_x((\kappa + 1)\tau_0) \) for \( x, \) and estimate \( E[\epsilon^{-\theta^2(t-\kappa \tau_0)}] \) by (12) to obtain

\[
\begin{align*}
& P[\exists \tau \in [0, t] : S(\tau, t) < S_x(t - \tau)] \\
& \leq \sum_{k=1}^{\infty} e^{-\theta(\rho(\theta) - \kappa \tau_0 - \sigma(\theta))} e^{\theta((\kappa + 1)\tau_0 - b)} \\
& \leq e^{-\theta(b - \kappa \tau_0 - \sigma(\theta))} \sum_{k=1}^{\infty} e^{-\theta(\rho(\theta) - \kappa \tau_0)} \\
& \leq e^{-\theta(b - \kappa \tau_0 - \sigma(\theta))} \int_0^\infty e^{-\theta(\rho(\theta) - r)\Delta x} dx \\
& = e^{-\theta(b - \kappa \tau_0 - \sigma(\theta))} \theta(\rho(\theta) - r)\tau_0. 
\end{align*}
\]

(14)

In the third line, we let \( t \to \infty \), and in the fourth line we made use of the fact that the summands are decreasing with \( \kappa \) since \( \rho(\theta) > r > 0 \), so that each summand is bounded from above by an integral of unit size left of that summand. Equating (14) with \( \epsilon \) we can solve for

\[
b = -\frac{1}{\theta} \ln(\theta(\rho(\theta) - r)\tau_0) + r\tau_0 + \sigma(\theta). \tag{15}
\]

Below, we insert the envelope parameters \( (\sigma(\theta), \rho(\theta)) \) of the service process. A variety of service models are known, including wireless channels [22] and scheduling algorithms [23], which we will employ in Sec. II-C, to find the statistical age of information bound

\[
\Delta_\epsilon = w + \frac{b + l}{r} \tag{16}
\]

under the stability condition \( l/w \leq r < \rho(\theta) \) and \( \theta > 0 \).

B. Markov Channel

An established model of a wireless channel with memory is the Markov on-off model with transition rates \( \mu > 0 \) from on to off state and \( \lambda > 0 \) from off to on state. The probability of the on state is \( p_{on} = \lambda/(\lambda + \mu) \) and \( p_{off} = 1 - p_{on} \). The memory of the channel is characterized by the burstiness parameter \( \beta = 1/\lambda + 1/\mu \) that is the mean time to change state twice. The transmission rate in on state is \( \epsilon \), and in off state \( 0 \). The channel has mean rate \( \gamma = \epsilon p_{on} \), and for \( \theta > 0 \) envelope parameters \( \sigma(\theta) = 0 \) and [22], [24]

\[
\rho(\theta) = -\frac{1}{\theta} \sqrt{\lambda - \mu \theta} + 2\theta. \tag{17}
\]

For a numerical evaluation we insert (17) into (15) and then use (16) for periodic updates. We optimize the free parameters \( r \in [l/w, \rho(\theta)] \), \( \theta > 0 \), and \( \tau_0 > 0 \) numerically to find the smallest statistical age of information bound \( \Delta_\epsilon \).

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Statistical age of information bound \( \Delta_\epsilon \) and virtual delay bound \( V_\epsilon \) with probability \( \epsilon \) for a Markov channel and message generation interval \( w \).}
\end{figure}\]

In Fig. 3, we show \( \Delta_\epsilon \) (16) for a periodic source with packet size \( l = 1 \) kb and different update intervals \( w \) in ms. The source is transmitted via a Markov on-off channel. The probability of the on state is \( p_{on} = 0.9 \), the burstiness of the channel is \( \beta = 8 \) ms, and the mean rate of the channel is \( \gamma = 1 \) Mbs. We use \( \epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\} \). For comparison, we also include the statistical virtual delay bound \( V_\epsilon \).

Regarding (16), \( \Delta_\epsilon \) depends in two ways on \( w \). Firstly, \( \Delta_\epsilon \) grows linearly with \( w \). This is clearly visible in Fig. 3 where \( \Delta_\epsilon \) but not \( V_\epsilon \) increases if \( w \) becomes large. Secondly, \( \Delta_\epsilon \) and \( V_\epsilon \) are affected by delays that occur during any periods of the channel. This is expressed by parameter \( b \). The effect that \( w \) has on \( b \) and thus on \( \Delta_\epsilon \) and \( V_\epsilon \) is via the stability condition \( l/w \leq r < \rho(\theta) \), where small \( w \) implies small \( \theta \) resulting in large \( b \) (15). This corresponds to queueing that arises during off periods of the channel if the utilization is high. This effect is visible in Fig. 3 for \( \Delta_\epsilon \) and \( V_\epsilon \) if \( w \) becomes small.

IV. RANDOM ARRIVALS

So far, we considered periodic updates that are common in time-triggered systems. Event-triggered systems, on the other hand, generate update messages in case a defined event occurs, e.g., if a sensor reading exceeds a certain threshold. The occurrence of events can be modeled as a random process.

The analysis of random updates is dual to that of random service in Sec. III. We reuse some of the notation, where \( \varepsilon_A, \rho_A, \sigma_A, r_A, \) and \( b_A \) refer to the arrivals \( A(t) \). For readability, we skip the subscript \( A \) where it is not ambiguous.

A. Statistical Upper Arrival Envelope

A statistical version of the upper arrival envelope (7) is

\[
P[\exists \tau \in [0, t] : A(\tau, t) > \overline{E}_\epsilon(t - \tau)] \leq \overline{\tau},
\]

for all \( t \geq 0 \) and \( \overline{E}_\epsilon(t) \in F_0 \).

We use the \( (\sigma(\theta), \rho(\theta)) \) traffic characterization of [17], i.e.,

\[
E[e^{\rho(\theta)(t-\tau)} + \sigma(\theta)] \leq e^{\rho(\theta)(t-\tau)} + \sigma(\theta),
\]

for \( \theta \geq 0, \sigma(\theta) \geq 0, \) and \( \rho(\theta) > 0 \).
We choose $\mathcal{E}_v(t) = rt + b$, where $r > \rho(\theta)$ and $b \geq r\tau_0 + \sigma(\theta)$, and use the same steps as in Sec. III to derive for $\theta > 0$ that
\[
b = -\frac{1}{\theta} \ln(\theta(r - \rho(\theta))\tau_0\varepsilon) + r\tau_0 + \sigma(\theta) .
\] (18)

B. Statistical Lower Arrival Envelope

Further, (9) uses a lower arrival envelope. A statistical lower envelope $\mathcal{E}_v(t) \in \mathcal{F}_0$ that fits in with (6) has the form
\[
P[\exists t \geq \tau : A(\tau, t) < \mathcal{E}_v(t - \tau)] \leq \varepsilon,
\] (19)
for $\tau \geq 0$. We observe that the age of information (9) depends only on the first non-zero value of $\mathcal{E}_v(t)$. Hence, we define
\[
\mathcal{E}_v(t) = 1_{\{t > u\}}l_{\text{min}}
\]
where $l_{\text{min}} > 0$ is the minimal packet size and parameter $u > 0$ is a statistical measure of the time until the next update occurs. By insertion into (19) it follows for $A(t)$ non-decreasing that
\[
P[\exists t \geq \tau : A(\tau, t) < 1_{\{t > u\}}l_{\text{min}}] \leq P[A(\tau, \tau + u) < l_{\text{min}}].
\]

We equate the right hand side with $\varepsilon$ and apply the desired probability distribution, e.g., Poisson, in Sec. IV-C.

To obtain the age of information, we insert $\mathcal{E}_v(t) = rt + b$, and $\mathcal{E}_v(t) = 1_{\{t > u\}}l_{\text{min}}$, as well as the latency rate server model $S(t) = c[t - t_0]_+$ into (9)
\[
\Delta_v \leq \sup \left\{ \delta \geq 0 : \inf_{\tau > 0} \left\{ c[\tau - t_0]_+ - r(\tau - \delta) - b \right\},
\right. \\
\left. \inf_{\tau \in [0, \delta]} \left\{ c[\tau - t_0]_+ + 1_{\{\delta \geq t > u\}}l_{\text{min}} \right\} \leq 0 \right\},
\]
where $\varepsilon = \bar{\tau} + \underline{\varepsilon}$ by the union bound. With $c \geq r$ for stability, it follows that
\[
\Delta_v \leq \max \left\{ \frac{b}{c}, u \right\} + \tau_0 .
\] (20)

C. Poisson Arrivals

To evaluate (20) we use arrivals with constant packet length $l > 0$ and exponential inter-arrival times with mean value $w > 0$. Hence, the arrivals form a Poisson process with arrival rate $1/w$. The envelope parameters of the Poisson process are $\sigma(\theta) = 0$ and
\[
\rho(\theta) = \frac{e^{\theta l} - 1}{\theta w} ,
\] (21)
for $\theta > 0$ [19]. By insertion of $\rho(\theta) < r$ into (18) and choice of $\bar{\tau}$ parameter $b$ is determined and can be inserted into (20). To fix the remaining parameter $u$, we use the Poisson distribution
\[
P[A(\tau, \tau + u) = \eta l] = \frac{e^{-\frac{\eta l}{\bar{\tau}}}(\frac{\eta l}{\bar{\tau}})^\eta}{\eta!} ,
\]
for $\eta \geq 0$. With $\underline{\varepsilon} = P[A(\tau, \tau + u) < l] = 1_{\{\tau \geq 0\}} = \bar{\tau}^{-1}$ we can solve for $u = -w \ln \underline{\varepsilon}$.

In Fig. 4 we depict the age of information (20) for a system with capacity $c = 1$ Mb/s, packet length $l = 1$ kb, $t_0 = l/c = 1$ ms and $\varepsilon \in \{10^{-3}, 10^{-6}, 10^{-9}\}$. We optimize parameters $r \in (\rho(\theta), c)$, $\theta > 0$, and $\tau_0 > 0$ numerically and select $\bar{\tau} = \underline{\varepsilon}$ so that $\bar{\tau} + \underline{\varepsilon} = \varepsilon$ and $b/c = u$.

Fig. 4 clearly shows two regimes that govern the age of information. For small $w$, congestive queuing delays are dominant that are derived from the upper arrival envelope in the first line of (9) corresponding to the term $b/c$ in (20). For large $w$, the age of information is mostly due to idle waiting represented by the lower arrival envelope in the second line of (9) and parameter $u$ in (20), respectively. The optimal update interval $w$ strikes a balance between these two effects.

V. MULTIPLEXING AND SCHEDULING

The min-plus representation of the network calculus naturally extends to tandem systems, including multiplexing and scheduling of traffic flows. Here, we consider several sources with different priorities that send periodic updates via a wireless channel.

A. Priority Scheduler

We consider a work-conserving system with service process $S(\tau, t)$ and service parameters $(\sigma_A(\theta), \rho_A(\theta))$. The system serves a number of independent and identically distributed (iid) flows $A_i(t)$ for $i \in \mathbb{N}$ that are numbered in descending order of priority. We analyze the age of information of flow $m + 1$, i.e., there are $m$ flows of higher priority. Given each of the flows has the envelope parameters $(\sigma_A(\theta), \rho_A(\theta))$, it is known that the aggregate of $m$ flows has parameters $(m\sigma_A(\theta), m\rho_A(\theta))$. Further, the service that is left over for flow $m + 1$ satisfies the definition of service process (1) with envelope parameters $(\sigma_A(\theta) + m\sigma_A(\theta), \rho_A(\theta) - m\rho_A(\theta))$ [19]. We will use the Markov channel model from Sec. III-B with $\sigma_A(\theta) = 0$ and $\rho_A(\theta)$ given by (17) and $(\sigma_A(\theta), \rho_A(\theta))$ for periodic updates.

B. Periodic Updates

For the moment generating function of a periodic source with packet length $l > 0$ and update interval width $w > 0$ it
enable finding the update interval that achieves the minimal age of information. Owing to the properties of the network calculus, our analysis can be easily extended to include further traffic and service models as well as multi-hop networks.

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**VI. Conclusions**

We phrased age of information in the min-plus network calculus and derived worst-case and statistical age of information bounds. We obtained solutions for sources with periodic updates and random updates, respectively, and systems with a random service, including wireless channels and schedulers with random cross-traffic. We showed different effects how the update interval affects the age of information. Our results

![Fig. 5. Statistical age of information bound $\Delta_s$ for priority $m + 1$.](image-url)