Remarks on the Vanishing Viscosity Process of State-Constraint Hamilton–Jacobi Equations

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Abstract
We investigate the convergence rate in the vanishing viscosity process of the solutions to the subquadratic state-constraint Hamilton–Jacobi equations. We give two different proofs of the fact that, for non-negative Lipschitz data that vanish on the boundary, the rate of convergence is $O(\sqrt{\varepsilon})$ in the interior. Moreover, the one-sided rate can be improved to $O(\varepsilon)$ for non-negative compactly supported data and $O(\varepsilon^{1/p})$ (where $1 < p < 2$ is the exponent of the gradient term) for non-negative data $f \in C^2(\Omega)$ such that $f = 0$ and $Df = 0$ on the boundary. Our approach relies on deep understanding of the blow-up behavior near the boundary and semiconcavity of the solutions.

Keywords First-order Hamilton–Jacobi equations · Second-order Hamilton–Jacobi equations · State-constraint problems · Optimal control theory · Rate of convergence · Viscosity solutions · Semiconcavity · Boundary layer

Mathematics Subject Classification 35B40 · 35D40 · 49J20 · 49L25 · 70H20

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1 Introduction

1.1 Settings

Let $\Omega$ be an open, bounded and connected domain in $\mathbb{R}^n$ with $C^2$ boundary, $f \in C(\Omega) \cap W^{1,\infty}(\Omega)$. For $\varepsilon > 0$, let $u^\varepsilon \in C^2(\Omega)$ (see [21] for the existence and the uniqueness) be the solution to

$$
\begin{cases}
  u^\varepsilon(x) + H(Du^\varepsilon(x)) - f(x) - \varepsilon \Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\
  \lim_{\text{dist}(x, \partial\Omega) \to 0} u^\varepsilon(x) = +\infty,
\end{cases}
$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$ is a given continuous Hamiltonian. The solution that blows up uniformly on the boundary is also called a \textit{large solution}. A typical Hamiltonian that has been considered in the literature is $H(\xi) = |\xi|^p$ for $\xi \in \mathbb{R}^n$ where $1 < p \leq 2$, and Eq. (1) becomes

$$
\begin{cases}
  u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon \Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\
  \lim_{\text{dist}(x, \partial\Omega) \to 0} u^\varepsilon(x) = +\infty.
\end{cases}
$$

(PDE$_\varepsilon$)

It turns out that for this specific subquadratic Hamiltonian, $u^\varepsilon$ is also the unique solution to the second-order state-constraint problem (see [21])

$$
\begin{cases}
  u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon \Delta u^\varepsilon(x) \leq 0 & \text{in } \Omega, \\
  u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon \Delta u^\varepsilon(x) \geq 0 & \text{on } \overline{\Omega}.
\end{cases}
$$

We focus on this Hamiltonian in our paper, which follows the setting of [21], where the specific structure of the Hamiltonian enables more explicit estimates for the solution of (PDE$_\varepsilon$). In fact, for $1 < p \leq 2$, the solution to equation (PDE$_\varepsilon$) is the value function associated with a minimization problem in stochastic optimal control theory with state constraints [13, 21]. We briefly recall the setting and all the domains and target spaces are omitted for simplicity. For a given stochastic control $\alpha(\cdot)$, we can solve for a solution (a state process) of the feedback control system

$$
\begin{cases}
  dX_t = \alpha(X_t) \, dt + \sqrt{2\varepsilon} \, dB_t & \text{for } t > 0, \\
  X_0 = x.
\end{cases}
$$

(3)

Here, $B_t \sim N(0, t)$ is the Brownian motion with mean zero and variance $t$. To constrain the state $X_t$ inside $\overline{\Omega}$, we define

$$
\hat{A}_x = \left\{ \alpha(\cdot) \in C(\Omega) : \mathbb{P}(X_t \in \Omega) = 1 \text{ for all } t \geq 0 \right\}
$$
and hope to minimize a cost function in expectation to get the value function

\[ u^\varepsilon(x) = \inf_{\alpha \in \hat{\mathcal{A}}} \mathbb{E} \left[ \int_0^\infty e^{-t} L(X_t, -\alpha(X_t)) \, dt \right], \quad (4) \]

where \( L(x, v) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R} \) is the running cost. More specifically, \( L(x, v) = c|v|^q + f(x) \) is the Legendre transform of \( H(x, \xi) := |\xi|^p - f(x) \) with \( q > 1 \), \( f \in C(\overline{\Omega}) \) non-negative, and some constant \( c \). Using the Dynamic Programming Principle (see [21]), we expect the value function (4) to solve (2), which means that \( u^\varepsilon \) is a subsolution in \( \Omega \) and a supersolution on \( \overline{\Omega} \).

We are interested in studying the asymptotic behavior of \( \{u^\varepsilon\}_{\varepsilon > 0} \) as \( \varepsilon \to 0^+ \). Heuristically, the solution of the second-order state-constraint equation converges to that of a first-order state-constraint equation associated with the deterministic optimal control problem, namely,

\[
\begin{cases}
  u(x) + |Du(x)|^p - f(x) \leq 0 & \text{in } \Omega, \\
  u(x) + |Du(x)|^p - f(x) \geq 0 & \text{on } \overline{\Omega}.
\end{cases} \quad \text{(PDE0)}
\]

and indeed equation (PDE0) admits a unique viscosity solution \( u \in C(\overline{\Omega}) \) (see [10, 30]). From the viewpoint of optimal control theory, as \( \varepsilon \to 0^+ \), the stochastic control system (3) becomes a deterministic control system. In particular, let \( \mathcal{A}_x = \{\xi \in AC([0, \infty); \overline{\Omega}) : \xi(0) = x\} \) and we have

\[ u(x) = \inf_{\xi \in \mathcal{A}_x} \int_0^\infty e^{-t} L(\xi(t), -\dot{\xi}(t)) \, dt \]

where \( L(x, v) \) is again the Legendre transform of \( H(x, \xi) := |\xi|^p - f(x) \).

The problem is interesting since in the limit there is no blowing up behavior near the boundary, as \( u \in C(\overline{\Omega}) \). In this paper, we investigate the rate of convergence of \( u^\varepsilon \to u \) as \( \varepsilon \to 0^+ \). What is intriguing and delicate here is the blow-up behavior of \( u^\varepsilon \) in a narrow strip near \( \partial \Omega \) as \( \varepsilon \to 0^+ \). This is often called the boundary layer theory in the literature.

Note that a comparison principle holds for (PDE0) since we always assume \( \Omega \) is an open, bounded and connected domain in \( \mathbb{R}^n \) with \( C^2 \) boundary [10, 30].

### 1.2 Relevant Literature

There is a vast amount of work in the literature on viscosity solutions with state constraints and large solutions. We would like to first mention that the problem (PDE0) with general Hamiltonian is a huge subject of research interest, started with the pioneer work [30] (see also [17, 18]). Some of the recent work related to the asymptotic behavior of solutions of (PDE0) can be found in [19, 20, 25, 33]. The problem (PDE\(_\varepsilon\)) was first studied in [21] and subsequently many works have been done in understanding deeper the properties of solutions (see [2, 24, 27, 28] and the references therein).
time-dependent version of (1) was also studied by many works, for instance, [5, 6, 22, 26] and the references therein.

In terms of rate of convergence, that is, the convergence rate of $u^\varepsilon \to u$ as $\varepsilon \to 0^+$, to the best of our knowledge, such a question has not been studied in the literature. For the case where (PDE$_\varepsilon$) is equipped with the Dirichlet boundary condition, a rate $O(\sqrt{\varepsilon})$ is well known with multiple proofs (see [3, 11, 32]).

1.3 Main Results

For $1 < p \leq 2$, define

$$\alpha = \frac{2-p}{p-1} \in [0, \infty).$$

Let $\Omega$ be an open, bounded and connected subset of $\mathbb{R}^n$ with boundary $\partial \Omega$ of class $C^2$. For small $\delta > 0$, denote $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \delta\}$ and $\Omega^\delta = \{x \in \mathbb{R}^n : \text{dist}(x, \overline{\Omega}) < \delta\}$.

**Definition 1** Define

$$\delta_{0, \Omega} = \frac{1}{2} \sup \{\delta > 0 : x \mapsto \text{dist}(x, \partial \Omega) \text{ is } C^2 \text{ in } \Omega^\delta \setminus \overline{\Omega_\delta}\}. \tag{5}$$

We will write $\delta_0$ instead of $\delta_{0, \Omega}$ when the underlying domain is understood.

The reader is referred to [16] for the regularity of the distance function defined in $\Omega^\delta \setminus \Omega_\delta$. We then extend $\text{dist}(x, \partial \Omega)$ to a function $d(x) \in C^2(\mathbb{R}^n)$ such that

$$d(x) \geq 0 \text{ for } x \in \Omega \text{ with } d(x) = +\text{dist}(x, \partial \Omega) \text{ for } x \in \Omega \setminus \Omega_\delta,$$

$$d(x) \leq 0 \text{ for } x \notin \Omega \text{ with } d(x) = -\text{dist}(x, \partial \Omega) \text{ for } x \in \Omega^\delta \setminus \Omega. \tag{6}$$

**Assumption on $f$.** We assume that $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ with $f = \min_{\overline{\Omega}}$ on $\partial \Omega$.

By replacing $f$ by $f - \min_{\overline{\Omega}}$, without loss of generality, we can assume $\min_{\overline{\Omega}} f = 0$ and $f = 0$ on $\partial \Omega$. The reason why this assumption is needed is elaborated in Remark 1. The main results of the paper are the following theorems.

**Theorem 1** Let $\Omega$ be an open, bounded and connected subset of $\mathbb{R}^n$ with $C^2$ boundary. Assume that $1 < p \leq 2$ and $f$ is non-negative and Lipschitz with $f = 0$ on $\partial \Omega$. Let $u^\varepsilon$ be the unique solution to (PDE$_\varepsilon$) and $u$ be the unique solution to (PDE$_0$). Then there exists a constant $C$ independent of $\varepsilon \in (0, 1)$ such that for $x \in \Omega$,

$$-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C \left(\sqrt{\varepsilon} + \frac{\varepsilon^{\alpha+1}}{d(x)^\alpha}\right), \quad 1 < p < 2,$$

$$-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C \left(\sqrt{\varepsilon} + \varepsilon |\log(d(x))|\right), \quad p = 2.$$
Remark 1 To the best of our knowledge, this theorem is new in the literature. The precise boundary behavior is very delicate and deserves further investigation. The condition $f = 0$ on $\partial \Omega$ is a little bit restrictive but is naturally needed in the proof. As is illustrated in the proof of Theorem 1, we will first show the result for $f$ that is compactly supported and non-negative. Then, to further generalize the main result, if we make the assumption that $f = 0$ on $\partial \Omega$ and $f$ is non-negative, we can approximate $f$ uniformly in $L^\infty(\Omega)$ by a sequence of compactly supported Lipschitz functions with uniformly bounded Lipschitz constants. Using the previous result obtained for the case where $f$ is compactly supported and non-negative, we can pass to the limit and prove Theorem 1 for non-negative $f$ with $f = 0$ on $\partial \Omega$, which is more general than the compactly supported case. At the current moment, we do not yet know how to extend the result to general $f$ where $f$ does not vanish or is not equal to its minimum on the boundary.

To prove the result for the case where $f$ is compactly supported and non-negative, it is natural to consider the doubling variable method. Indeed, for instance, if $1 < p < 2$, one would consider constructing an auxiliary function with

$$
\psi^\varepsilon(x) := u^\varepsilon(x) - \frac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}
$$

and $u(x)$, where $C_\alpha \varepsilon^{\alpha+1} d(x)^{-\alpha}$ is the leading order term in the asymptotic expansion of $u^\varepsilon(x)$ as $d(x) \to 0^+$ with $C_\alpha := \alpha^{-1}(\alpha + 1)^{\alpha+1}$. If we take the derivative of (7) formally, it becomes

$$
D \psi^\varepsilon(x) = Du^\varepsilon(x) + C_\alpha \alpha \left( \frac{\varepsilon}{d(x)} \right)^{\alpha+1} Dd(x). \tag{8}
$$

We will see that $D \psi^\varepsilon(x)$ is uniformly bounded if $d(x) \geq \varepsilon$ (Lemma 17). Indeed,

$$
-C_\alpha \alpha \left( \frac{\varepsilon}{d(x)} \right)^{\alpha+1} Dd(x)
$$

is more or less the leading order term in the asymptotic expansion of $Du^\varepsilon$ near $\partial \Omega$. Heuristically, this means that the boundary layer is $O(\varepsilon)$ from the boundary.

However, to get a useful estimate by the doubling variable method, at the maximum point $x_0$ of $\psi^\varepsilon(x) - u(x)$, we need to have $d(x_0) \geq \varepsilon^\gamma$ for $\gamma < 1$ so that the latter term in (8) vanishes as $\varepsilon \to 0^+$. Otherwise, we cannot obtain a convergence rate via the doubling variable method as there are still nonvanishing constant terms. In the other case where $d(x_0) < \varepsilon^\gamma$, we introduce a new localization idea, that is, we construct a blow-up solution in the ball of radius $\varepsilon^\gamma$ from the boundary. Finally, a technical (and common for the doubling variable method) computation leads to $\gamma = 1/2$.

As a different approach, the convexity of $|\xi|^p$ and the semiconcavity of the solution to (PDE0) give us a better one-sided $O(\varepsilon)$ estimate for non-negative compactly supported $f$ which is semiconcave in its support (see Theorem 2). Such an one-sided $O(\varepsilon)$ rate is well known for the Dirichlet boundary problem (see [3, 32]). Moreover,
the result in Theorem 2 further provides us with a better one-sided estimate $O(\varepsilon^{1/p})$ than that in Theorem 1, as in Corollary 3. We recall that $f$ is (uniformly) semiconcave in $\Omega$ with linear modulus (or semiconcavity constant) $c > 0$ if

$$f(x + h) - 2f(x) + f(x - h) \leq c |h|^2,$$

$\forall x, h \in \mathbb{R}^n$ such that $x + h, x, \text{ and } x - h \in \overline{\Omega}$.

Note that any $f \in C^2_c(\Omega)$ is semiconcave on its support with the constant

$$c = \max \left\{ D^2 f(x) \xi \cdot \xi : |\xi| = 1, x \in \Omega \right\}$$

in the above definition. It is well known that the solution $u$ to (PDE0) is locally semiconcave given $f$ is uniformly semiconcave in $\Omega$. Using tools from the optimal control theory, we provide the explicit blow-up rate of the semiconcavity modulus of $u(x)$ when $x$ approaches $\partial \Omega$. As an application, we can improve the rate of convergence as follows.

**Theorem 2** (One-sided $O(\varepsilon)$ rate for non-negative compactly supported data)

*Under the conditions of Theorem 1, suppose $f$ also satisfies the following conditions:

- $f$ is semiconcave in its support;
- $f$ has a compact support in $\Omega_\kappa := \{ x \in \Omega : \text{dist}(x, \Omega) > \kappa \}$ for some $\kappa \in (0, \delta_0)$, $0 < \delta_0 < 1$ defined in (5).

Then there exist two constants $\nu > 1$ and $C$ independent of $\varepsilon$ and $\kappa$ such that $\forall x \in \Omega$,

$$u^\varepsilon(x) - u(x) \leq \frac{\nu C \varepsilon^{\alpha+1}}{d(x)\alpha} + C \left( \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+2} \right) + \frac{Cn \varepsilon}{\kappa}, \quad \text{if } p < 2,$$

$$u^\varepsilon(x) - u(x) \leq \nu \varepsilon \log \left( \frac{1}{d(x)} \right) + C \left( \left( \frac{\varepsilon}{\kappa} \right)^{\alpha} + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+2} \right) + \frac{Cn \varepsilon}{\kappa}, \quad \text{if } p = 2.$$ 

**Remark 2** If $f \in C^2_c(\Omega)$, then the last term $(Cn \varepsilon) \kappa^{-1}$ in the equations above can be improved to $nc\varepsilon$, where $c$ is the semiconcavity constant of $f$. This improvement is due to the fact that we can prove $u$ is uniformly semiconcave with a semiconcavity constant that only depends on the semiconcavity constant $c$ of $f$ (see Theorem 19 in the Appendix). Hence, in the proof of Theorem 2, in Eq. (50), instead of $C\kappa^{-1}$, we can bound $c(x_0)$ by the semiconcavity constant $c$ of $f$, independent of $\kappa$. Similarly, see Remark 8 for this improvement on the last term. It turns out that in general, if $f$ can be extended to a semiconcave function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $f = 0$ on $\Omega^c$, then $u$ is uniformly semiconcave, and hence this improvement happens. See Fig. 1 for two examples where $f$ can and cannot be extended to a semiconcave function in the whole space by setting $f = 0$ outside $\Omega$. 

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Fig. 1 The one on the right corresponds to a general \( f \) in Theorem 2, while the one on the left corresponds to the situation in Remark 2 where the improvement happens.

**Corollary 3** (One-sided \( O(\varepsilon^{1/p}) \) rate) Let \( 1 < p < 2 \). If \( f \in C^2(\Omega) \) is non-negative, \( f = 0 \) and \( Df = 0 \) on \( \partial\Omega \), then there exists a constant \( C \) independent of \( \varepsilon \in (0, 1) \) such that

\[
-C\varepsilon^{1/2} \leq u^\varepsilon(x) - u(x) \leq C \left( \varepsilon^{1/p} + \frac{\varepsilon^{\alpha+1}}{d(x)^\alpha} \right)
\]

for all \( x \in \Omega \).

**Remark 3** While the second approach looks more powerful, we need the gradient bound of \( u^\varepsilon \) (Lemma 17), the blow-up rate of the semiconcavity constant of \( u \) (Theorem 16), and higher regularity on \( f \). On the other hand, the first approach by doubling variable is relatively simple and does not require any explicit asymptotic behavior of \( Du^\varepsilon \), except the fact that it is locally bounded.

**Organization of the Paper**

Section 2 contains some preliminary results. The proof of Theorem 1 is given in Sect. 3. Then in Sect. 4, we give the proof of Theorem 2 and Corollary 3. Finally, the proofs of some useful lemmas are presented in Appendix.

**2 Preliminaries**

Let \( K_0 := \max_{x \in \Omega} |d(x)| \), \( K_1 := \max_{x \in \Omega} |Dd(x)| \), and \( K_2 := \max_{x \in \Omega} |\Delta d(x)| \). Note that \( d(x) = \text{dist}(x, \partial\Omega) \) for \( x \in \Omega \) and \( |Dd(x)| = 1 \) in the classical sense in \( \Omega^{\delta_0} \setminus \Omega_{\delta_0}. \) Denote by \( L^\varepsilon : C^2(\Omega) \rightarrow C(\Omega) \) the operator

\[
L^\varepsilon[u](x) := u(x) + |Du(x)|^p - f(x) - \varepsilon \Delta u(x), \quad x \in \Omega.
\]

**2.1 Local Gradient Estimate**

For \( \varepsilon \in (0, 1) \) and \( p > 1 \), we state an a priori estimate for \( C^2 \) solutions to (PDE\( _\varepsilon \)) ([21, Appendix]). Since we are working with smooth solutions, the proof is relatively simple by the classical Bernstein method [7], which is provided in Appendix for the reader’s convenience.
Theorem 4 Let \( f \in C(\Omega) \cap W^{1,\infty}(\Omega) \) and \( u^\varepsilon \in C^2(\Omega) \) be a solution to \( L^\varepsilon [u^\varepsilon] = 0 \) in \( \Omega \) with \( 1 < p \leq 2 \). Let \( m := \max_{\Omega} f(x) \). Then for \( \delta > 0 \), there exists \( C_\delta = C(m, p, \delta, \|DF\|_{L^\infty(\Omega)}) \) such that
\[
\sup_{x \in \Omega_\delta} \left( |u^\varepsilon(x)| + |Du^\varepsilon(x)| \right) \leq C_\delta
\]
for \( \varepsilon \) small enough.

2.2 Well-Posedness of (PDE\( _\varepsilon \))

In this section, we recall the existence and the uniqueness of solutions to (PDE\( _\varepsilon \)) for \( 1 < p \leq 2 \) and \( f \in C(\Omega) \cap W^{1,\infty}(\Omega) \). In fact, the assumption of \( f \) can be relaxed to \( f \in L^\infty(\Omega) \) [21].

Theorem 5 Let \( f \in C(\Omega) \cap W^{1,\infty}(\Omega) \). There exists a unique solution \( u^\varepsilon \in C^2(\Omega) \) of (PDE\( _\varepsilon \)) such that:

(i) If \( 1 < p < 2 \), then
\[
\lim_{d(x) \to 0} \left( u^\varepsilon(x) d(x)^\alpha \right) = C_\alpha \varepsilon^{\alpha+1},
\]
where \( \alpha = (p - 1)^{-1} (2 - p) \) and \( C_\alpha = \alpha^{-1} (\alpha + 1)^{\alpha+1} \).

(ii) If \( p = 2 \), then
\[
\lim_{d(x) \to 0} \left( -\frac{u^\varepsilon(x)}{\log(d(x))} \right) = \varepsilon.
\]

Furthermore, \( u^\varepsilon \) is the maximal subsolution among all the subsolutions \( v \in W^{2,r}_{loc}(\Omega) \) for all \( r \in [1, \infty) \) of (PDE\( _\varepsilon \)).

This is Theorem I.1 in [21] with an explicit dependence on \( \varepsilon \). The proof of this theorem is carried out explicitly in Appendix for later use. Also, it is useful to note that \( \alpha + 1 = (p - 1)^{-1} \). More results on the behavior of the gradient of \( u^\varepsilon \) can be found in [28] and Lemma 17, where we show \( |Du^\varepsilon| \leq C + C \left( \frac{\varepsilon}{d(x)} \right)^{\alpha+1} \). We believe Lemma 17 is new in the literature.

2.3 Convergence Results

We first state the following Lemma [10], which characterizes the solution to the first-order state-constraint equation (PDE\( _0 \)).

Lemma 6 Let \( u \in C(\overline{\Omega}) \) be a viscosity subsolution of (PDE\( _0 \)) such that, for any viscosity subsolution \( v \in C(\overline{\Omega}) \) of (PDE\( _0 \)), one has \( v \leq u \) on \( \overline{\Omega} \). Then \( u \) is a viscosity supersolution of (PDE\( _0 \)) on \( \overline{\Omega} \).
Again, the proof of Lemma 6 is given in Appendix for the reader’s convenience.

**Lemma 7** Assume $1 < p \leq 2$. Let $u^\varepsilon \in C^2(\Omega)$ be the solution to (PDE$_\varepsilon$) and $u \in C(\overline{\Omega})$ be the solution to (PDE$_0$). We have $\{u^\varepsilon\}_{\varepsilon > 0}$ is uniformly bounded from below by a constant independent of $\varepsilon$. More precisely, $u^\varepsilon \geq \min_\Omega f$ and $u \geq \min_\Omega f$.

**Proof** For $m \in \mathbb{N}$, let $u^\varepsilon_m \in C^2(\Omega) \cap C(\Omega)$ solve the Dirichlet problem

\[
\begin{cases}
    u^\varepsilon_m(x) + |Du^\varepsilon_m(x)|^p - f(x) - \varepsilon \Delta u^\varepsilon_m(x) = 0 & \text{in } \Omega, \\
    u^\varepsilon_m(x) = m & \text{on } \partial \Omega.
\end{cases}
\]

(PDE$_{\varepsilon,m}$)

We have $u^\varepsilon_m(x) \to u^\varepsilon(x)$ in $\Omega$ as $m \to \infty$. Let $\varphi(x) = \inf_\Omega f$ for $x \in \Omega$. Then $\varphi(x)$ is a classical subsolution of (PDE$_{\varepsilon,m}$) in $\Omega$ with

\[
\varphi(x) = \inf_\Omega f \leq m = u^\varepsilon_m(x) \quad \text{on } \partial \Omega
\]

for $m$ large enough. By the comparison principle of the uniformly elliptic equation (PDE$_{\varepsilon,m}$),

\[
\inf_\Omega f \leq u^\varepsilon_m(x) \quad \text{for all } x \in \Omega.
\]

As $m \to \infty$, we obtain $u^\varepsilon \geq \min_\Omega f$. The inequality $u \geq \min_\Omega f$ follows from the comparison principle of (PDE$_0$) applied to the supersolution $u$ on $\overline{\Omega}$ and the subsolution $\varphi$ in $\Omega$. \hfill \Box

We present here a simple proof of the convergence $u^\varepsilon \to u$ using Lemma 6. See also [10, Theorem VII.3].

**Theorem 8** (Vanishing viscosity) Let $u^\varepsilon$ be the solution to (PDE$_\varepsilon$). Then there exists $u \in C(\overline{\Omega})$ such that $u^\varepsilon \to u$ locally uniformly in $\Omega$ as $\varepsilon \to 0$ and $u$ solves (PDE$_0$).

**Proof** By the a priori estimate (Theorem 4),

\[
|u^\varepsilon(x)| + |Du^\varepsilon(x)| \leq C_\delta \quad \text{for } x \in \overline{\Omega}_\delta.
\]

(11)

By the Arzelà–Ascoli theorem, there exists a subsequence $\varepsilon_j \to 0$ and a function $u \in C(\Omega)$ such that $u^\varepsilon_j \to u$ locally uniformly in $\Omega$. From the stability of viscosity solutions, we easily deduce that

\[
u(x) + |Du(x)|^p - f(x) = 0 \quad \text{in } \Omega.
\]

(12)

From Lemma 7, $u^\varepsilon(x) \geq \min_\Omega f$ and $u(x) \geq \min_\Omega f$ for all $x \in \Omega$. Together with (12), we obtain $|\xi|^p \leq \max_\Omega f - \min_\Omega f$ for all $\xi \in D^+ u(x)$ and $x \in \Omega$. This implies there exists a constant $C_0$ such that

\[
|u(x) - u(y)| \leq C_0 |x - y| \quad \text{for all } x, y \in \Omega.
\]

(13)
Thus, we can extend $u$ uniquely to $u \in C(\overline{\Omega})$. We use Lemma 6 to show that $u$ is a supersolution of (PDE$_0$) on $\overline{\Omega}$.

It suffices to show that $u \geq w$ on $\Omega$, where $w \in C(\overline{\Omega})$ is the unique solution to (PDE$_0$). For $\delta > 0$, let $u_\delta \in C(\overline{\Omega}_\delta)$ be the unique viscosity solution to

$$
\begin{cases}
  u_\delta(x) + |Du_\delta(x)|^p - f(x) \leq 0 & \text{in } \Omega_\delta, \\
  u_\delta(x) + |Du_\delta(x)|^p - f(x) \geq 0 & \text{on } \overline{\Omega}_\delta.
\end{cases}
$$

(14)

Since $u_\delta \to w$ locally uniformly as $\delta \to 0^+$ (see [20]) and $w$ is bounded, $\{u_\delta\}_{\delta > 0}$ is uniformly bounded. Let $v^\varepsilon_\delta \in C^2(\Omega_\delta) \cap C(\overline{\Omega}_\delta)$ be the unique solution to the Dirichlet problem

$$
\begin{cases}
  v^\varepsilon_\delta(x) + |Dv^\varepsilon_\delta(x)|^p - f(x) = \varepsilon \Delta v^\varepsilon_\delta(x) & \text{in } \Omega_\delta, \\
  v^\varepsilon_\delta(x) = u_\delta & \text{on } \partial \Omega_\delta.
\end{cases}
$$

(15)

It is well known that $v^\varepsilon_\delta \to u_\delta$ uniformly on $\overline{\Omega}_\delta$ as $\varepsilon \to 0$ [11, 15, 31].

For $\delta$ small enough, $u_\delta \leq u^\varepsilon$ on $\partial \Omega_\delta$. Hence, by the maximum principle, $v^\varepsilon_\delta \leq u^\varepsilon$ on $\overline{\Omega}_\delta$. Now we first let $\varepsilon \to 0$ to obtain $u_\delta \leq u$ on $\overline{\Omega}_\delta$. Then let $\delta \to 0$ to get $w \leq u$ in $\Omega$, which implies $w \leq u$ on $\overline{\Omega}$ since both $w, u$ belong to $C(\overline{\Omega})$. \qed

### 3 Rate of Convergence

In this section, we focus on the rate of convergence for the case where $f \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ is non-negative. As a consequence, $u^\varepsilon(x), u(x) \geq 0$ for $x \in \Omega$ by Lemma 7. In our main results, we have an additional assumption that $f = 0$ on $\partial \Omega$.

Before we show any result about the rate of convergence, we would like to mention a lower bound of $u^\varepsilon - u$ and some properties of $u$ from its optimal control formulation.

**Theorem 9** Let $u^\varepsilon$ be the unique solution to (PDE$_\varepsilon$) and $u$ be the unique solution to (PDE$_0$). Then there exists a constant $C$ independent of $\varepsilon$ such that

$$- C \sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \quad \text{for all } x \in \Omega.$$  

(16)

**Proof** The proof relies on a well-known rate of convergence for vanishing viscosity of the viscous Hamilton–Jacobi equation with the Dirichlet boundary condition (see [11, 12, 14, 31]). Let $g(x) = u(x)$ for $x \in \partial \Omega$. Let $v^\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$ be the unique viscosity solution to

$$
\begin{cases}
  v^\varepsilon(x) + |Dv^\varepsilon(x)|^p - f(x) - \varepsilon \Delta v^\varepsilon(x) = 0 & \text{in } \Omega, \\
  v^\varepsilon(x) = g(x) & \text{on } \partial \Omega.
\end{cases}
$$

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It is well known that $v^\varepsilon \to u$ [11, 15, 31]. Furthermore, there exists a positive constant $C$ independent of $\varepsilon \in (0, 1)$ such that

$$
|v^\varepsilon(x) - u(x)| \leq C\sqrt{\varepsilon} \quad \text{for } x \in \overline{\Omega}.
$$

(17)

By the comparison principle for $(PDE_\varepsilon)$, we have

$$
v^\varepsilon(x) \leq u^\varepsilon(x) \quad \text{for } x \in \Omega.
$$

(18)

From (17) and (18), we obtain the lower bound (16).

$\square$

**Lemma 10** Assume $f \geq 0$ in $\Omega$. Then $u(x) = 0$ if and only if $f(x) = 0$. In particular, $f \equiv 0$ implies $u \equiv 0$.

**Proof** It is clear to see that $f \equiv 0$ implies $u \equiv 0$ by the uniqueness of $(PDE_0)$.

It is not hard to prove the converse by contradiction. Suppose $u \equiv 0$ and $f(x_0) > 0$. Then there exists $\varepsilon, \delta > 0$ such that $f(x) > \varepsilon$ for all $x \in B_\delta(x_0)$. Let $\eta \in AC([0, \infty); \overline{\Omega})$ such that $\eta(0) = x_0$ and $t$ be the time that $\eta$ first hits $\partial B_\delta(x_0)$. Note that $t$ could be $+\infty$. Then

$$
\int_0^\infty e^{-s} (|\dot{\eta}(s)|^q + f(\eta(s))) \, ds \geq \int_0^t e^{-s} (|\dot{\eta}(s)|^q + f(\eta(s))) \, ds
$$

$$
\geq \frac{1}{e^{tq-1}} \left( \int_0^t |\dot{\eta}(s)|^q \, ds \right) + \varepsilon \left( 1 - e^{-t} \right)
$$

$$
\geq \frac{\delta^q}{e^{tq-1}} + \varepsilon \left( 1 - e^{-t} \right),
$$

where we used Jensen’s inequality in the second line. This implies $u(x_0) > 0$ since $q \geq 2$, which is a contradiction.

$\square$

The following lemma is about a crucial estimate that will be used. It is a refined construction of a supersolution for $(PDE_\varepsilon)$.

**Lemma 11** Let $\delta_0$ be defined as in (5). There exist positive constants $\nu = \nu(\delta_0) > 1$ and $C_{\delta_0} = O\left(\delta_0^{-(\alpha+2)}\right)$ such that

$$
\begin{cases}
\nu C_\alpha e^{\alpha + 1} d(x) + \max f + C_{\delta_0} e^{\alpha + 2}, & p < 2, \\
\nu \varepsilon \log \left( \frac{1}{d(x)} \right) + \max f + C_{\delta_0} e^2, & p = 2,
\end{cases}
$$

(19)

is a supersolution of $(PDE_\varepsilon)$ in $\Omega$.

**Proof** Let us first consider $1 < p < 2$. Recall from Theorem 5 that $C_\alpha^p \alpha^p = C_\alpha \alpha (\alpha + 1)$ and $p(\alpha + 1) = \alpha + 2$. Compute

$$
|Dw(x)|^p = v^p \frac{(C_\alpha^p \alpha^p)^{p(\alpha+1)}}{d(x)^{p(\alpha+1)}} |Dd(x)|^p = v^p \frac{C_\alpha \alpha (\alpha + 1) e^{\alpha + 2}}{d(x)^{\alpha + 2}} |Dd(x)|^p
$$
and
\[
\varepsilon \Delta w(x) = \nu \frac{C_{\alpha} \alpha (1 + 1) \varepsilon^{q + 2} + C_{\alpha} \alpha \varepsilon^{q + 2} \Delta d(x)}{d(x)^{q + 2}} |Dd(x)|^2 - \nu \frac{\Delta d(x)}{d(x)^{q + 1}}.
\]

We have
\[
L^\varepsilon [w] = \frac{\nu C_{\alpha} \varepsilon^{q + 1}}{d(x)^{q}} + \max_{\Omega} f - f(x) + C_{\delta_0} \varepsilon^{q + 2} + C_{\alpha} \varepsilon^{q + 2} |Dd(x)|^2 + \nu \frac{d(x) \Delta d(x)}{q + 1}.
\]

**Case 1** If \( 0 < d(x) \leq \delta_0 \), we have \( |Dd(x)| = 1 \). Recall that \( K_2 = \| \Delta d \|_{L^\infty} \) and observe
\[
\frac{|d(x) \Delta d(x)|}{\alpha + 1} \leq \frac{\delta_0 \| \Delta d \|_{L^\infty}}{\alpha + 1} \leq \frac{K_2 \delta_0}{\alpha + 1} \leq K_2 \delta_0.
\]

Therefore,
\[
\nu^p - \nu + \nu \frac{d(x) \Delta d(x)}{\alpha + 1} \geq \nu^{p - 1} - (1 + K_2 \delta_0) \nu^{p - 1} - (1 + K_2 \delta_0) \nu^p.
\]

We will choose \( \nu \) as follows. For \( \gamma > 1 \), we have the inequality
\[
|\gamma | + |y| \leq \gamma (|x| + |y|)^{\gamma - 1} |y|
\]
for \( x, y \in \mathbb{R} \), which implies that
\[
0 \leq (1 + K_2 \delta_0)^{q + 1} - 1 \leq \frac{(1 + K_2 \delta_0)^q}{K_2} \delta_0.
\]

Hence, \( (1 + K_2 \delta_0)^{q + 1} \leq 1 + C_2 \delta_0 \). Since \( \alpha + 1 = \frac{1}{p - 1} \),
\[
(1 + K_2 \delta_0) \leq (1 + C_2 \delta_0)^{\frac{1}{p - 1}} = (1 + C_2 \delta_0)^{p - 1}.
\]

Choose \( \nu = 1 + C_2 \delta_0 \) in (20) and we obtain \( \mathcal{L}[w] \geq 0 \) in \( \{ x \in \Omega_5 : \delta < d(x) \leq \delta_0 \} \).

**Case 2** If \( d(x) \geq \delta_0 \), recall that \( K_0 = \| d \|_{L^\infty} \) and \( K_1 = \| Dd \|_{L^\infty} \). And we have
\[
\mathcal{L}[w] = \frac{\nu C_{\alpha} \varepsilon^{q + 1}}{d(x)^{q}} + \max_{\Omega} f - f(x)
\]
\[
+ \nu^p \frac{C_{\alpha} \varepsilon^{q + 2} + C_{\alpha} \varepsilon^{q + 2} \Delta d(x)}{d(x)^{q + 2}} |Dd(x)|^2 - \nu \frac{\Delta d(x)}{d(x)^{q + 1}} + C_{\delta_0} \varepsilon^{q + 2}
\]
\[ \geq C_\alpha \alpha (\alpha + 1) \varepsilon \alpha + 2 \frac{\nu \alpha}{d(x)\alpha + 2} \left( v^p |Dd(x)|^p - \nu |Dd(x)|^2 + v \frac{d(x) \Delta d(x)}{\alpha + 1} \right) + C_{\delta_0} \epsilon \alpha + 2 \]

\[ \geq \left[ C_{\delta_0} - C_3 \left( \frac{1}{\delta_0} \right) \alpha + 2 \right] \varepsilon \alpha + 2, \]

where

\[ C_3 = C_\alpha \alpha (\alpha + 1) \left( \nu^p K_1^p + \nu K_1^2 + \frac{K_0 K_2}{\alpha + 1} \right). \]

We can choose \( C_{\delta_0} = C_3 \delta_0^{-\alpha - 2} \) to obtain \( L[w] \geq 0 \) in \( \{ x \in \Omega : d(x) \geq \delta_0 \} \).

If \( p = 2 \), then \( \alpha = 0 \). We can easily see that the similar calculation holds true with \( \nu := 1 + K_2 \delta_0 \) and \( C_{\delta_0} := \delta_0^{-2} \nu (\nu K_1^2 + K_1^2 + K_0 K_2) \).

Now we begin to present the rate of convergence for the special case where \( f = C_f \) in \( \Omega \) for some constant \( C_f \).

**Theorem 12** (Constant data) Assume \( f \equiv C_f \) in \( \Omega \). Let \( u^\varepsilon \) be the unique solution to \( (\text{PDE}_\varepsilon) \) and \( u \equiv C_f \) be the unique solution to \( (\text{PDE}_0) \). Then there exists a constant \( C \) independent of \( \varepsilon \in (0, 1) \) such that

\[ 0 \leq u^\varepsilon(x) - u(x) \leq C \left( \varepsilon^{\alpha + 1} \frac{\nu^p}{d(x)\alpha} + \frac{\varepsilon^{\alpha + 2}}{\delta_{0,\Omega}^{\alpha + 2}} \right), \quad \text{if } 1 < p < 2, \]

\[ 0 \leq u^\varepsilon(x) - u(x) \leq C \left( \varepsilon \log \left( \frac{1}{d(x)} \right) + \frac{\varepsilon^2}{\delta_{0,\Omega}^2} \right), \quad \text{if } p = 2, \]

for \( x \in \Omega \), where \( \delta_{0,\Omega} \) is defined as in (5). In particular,

(i) if \( 1 < p < 2 \), we have \( C_f \leq u^\varepsilon(x) \leq C_f + C \varepsilon \) for \( x \in \Omega_\varepsilon \), and

(ii) for any \( K \subset \subset \Omega \), there holds \( \| u^\varepsilon - u \|_{L^\infty(K)} \leq C \epsilon^{\alpha + 1} \).

**Proof** Lemma 10 implies \( u \equiv C_f \) in \( \Omega \). And Lemma 7 tells us \( u^\varepsilon - u = u^\varepsilon - C_f \geq 0 \). By the comparison principle of \( (\text{PDE}_\varepsilon) \) and Lemma 11, the conclusion follows.

**Remark 4** The conclusion of Theorem 12 also holds if \( f = C_f + O(\varepsilon^\beta) \) for \( \beta \geq \alpha + 1 \).

Even this special case (Theorem 12) is new in the literature. As an immediate consequence, we obtain the rate of convergence on any compact subset that is disjoint from the support of \( f \).

**Corollary 13** Assume \( f \) is Lipschitz with compact support and \( K \) is a connected compact subset of \( \Omega \) that is disjoint from \( \text{supp}(f) \). Then there exists a constant \( C = C(K) \) independent of \( \varepsilon \in (0, 1) \) such that

\[ \| u^\varepsilon - u \|_{L^\infty(K)} \leq C \epsilon^{\alpha + 1}. \]
Proof We choose an open, bounded and connected set $U$ such that $\partial U$ is $C^2$ and $K \subset U \subset \Omega$. Let $w^\varepsilon$ be the solution to \((\text{PDE}_\varepsilon)\) with $\Omega$ replaced by $U$. Then by Theorem 12, we have

$$0 \leq w^\varepsilon(x) \leq C \left( \varepsilon^{\alpha+1} + \varepsilon^{\alpha+2} \right), \quad x \in K,$$

where $C$ depends on $\text{dist}(K, \partial U)$ and $U$. Recall that $u = 0$ outside the support of $f$. By the comparison principle in $U$, we see that $u^\varepsilon \leq w^\varepsilon$ and thus the conclusion follows. \hfill \Box

For the general result of non-negative compactly supported data, we have the following theorem.

Theorem 14 (Non-negative compactly supported data) Assume that $f$ is non-negative and Lipschitz with compact support in $\Omega_\kappa$ for some $\kappa > 0$. Let $u^\kappa$ be the unique solution to \((\text{PDE}_\kappa)\) and $u$ be the unique solution to \((\text{PDE}_0)\). Then there exists a constant $C$ independent of $\varepsilon \in (0, 1)$ and $\kappa$ such that

$$-C \sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C \left( \sqrt{\varepsilon} + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+2} \right) + \frac{\nu C_{\alpha} \varepsilon^{\alpha+1}}{d(x)^\alpha}, \quad p < 2, \quad (23)$$

$$-C \sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C \left( \sqrt{\varepsilon} + \left( \frac{\varepsilon}{\kappa} \right)^2 \right) + \nu \varepsilon \log \left( \frac{1}{d(x)} \right), \quad p = 2, \quad (24)$$

for any $x \in \Omega$. As a consequence, $|u^\varepsilon(x) - u(x)| \leq C \sqrt{\varepsilon}$ for all $x \in \Omega_\varepsilon$.

We state the following lemma as a preparation.

Lemma 15 Let $0 < \kappa < \delta_0$ and $U_\kappa = \{ x \in \Omega : 0 < \text{dist}(x, \partial \Omega) < \kappa \} = \Omega \setminus \overline{\Omega}_\kappa$. There holds

$$\text{dist}(x, \partial U_\kappa) = \kappa - \text{dist}(x, \partial \Omega) \quad \text{for all } x \in U_\kappa.$$

As a consequence, $x \mapsto \text{dist}(x, \partial U_\kappa) = \min \{ \text{dist}(x, \partial \Omega_\kappa), \text{dist}(x, \partial \Omega) \}$ is twice continuously differentiable for $x \in \Omega \setminus \overline{\Omega}_{\kappa/2}$. Hence, we can choose

$$\delta_{0, U_\kappa} \geq \frac{\kappa}{4} \quad (25)$$

where $\delta_{0, \Omega}$ is defined as in (5).

Proof By the definition of $\delta_0 = \delta_{0, \Omega}$, we have $d(x) = \text{dist}(x, \partial \Omega)$ is twice continuously differentiable in the region $U_{\delta_0} = \Omega \setminus \overline{\Omega}_{\delta_0}$. The proof follows from [16, p. 355]. \hfill \Box

Proof of Theorem 14 Without loss of generality, assume that $f$ is supported in $\Omega_\kappa$ where $0 < \kappa < \delta_0$. Let $g_\kappa = u^\kappa$ on $\partial \Omega_\kappa$. Then the solution $u^\varepsilon$ of \((\text{PDE}_\varepsilon)\) also solves

$$\begin{cases}
  u^\varepsilon(x) + |Du^\varepsilon(x)|^p - \varepsilon \Delta u^\varepsilon(x) = 0 \quad \text{in } U_\kappa, \\
  u^\varepsilon(x) = +\infty \quad \text{on } \partial \Omega, \\
  u^\varepsilon(x) = g_\kappa \quad \text{on } \partial \Omega_\kappa,
\end{cases}$$
in $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_\kappa = \{ x \in \Omega : 0 < d(x) < \kappa \}$. Let $\tilde{u}^\varepsilon \in C^2(U_\kappa)$ be the solution to the following problem

$$\begin{cases}
\tilde{u}^\varepsilon(x) + |D\tilde{u}^\varepsilon(x)|^p - \varepsilon \Delta \tilde{u}^\varepsilon(x) = 0 & \text{in } U_\kappa, \\
\tilde{u}^\varepsilon(x) = +\infty & \text{on } \partial U_\kappa = \partial \Omega \cup \partial \Omega_\kappa,
\end{cases}$$

whose existence is guaranteed by Theorem 5. Here the boundary condition is understood in the sense that $\tilde{u}^\varepsilon(x) \to \infty$ as $d_\kappa(x) \to 0$, where $d_\kappa(\cdot)$ is the distance function from the boundary of $U_\kappa$, i.e.,

$$d_\kappa(x) = \min \{ \text{dist}(x, \partial \Omega_\kappa), \text{dist}(x, \partial \Omega) \} \leq d(x) \text{ for } x \in U_\kappa.$$ 

Since $f = 0$ in $U_\kappa$, by Lemma 10, $u = 0$ in $U_\kappa$. Hence, $u$ is also the unique state-constraint solution to

$$\begin{cases}
u u(x) + |Du(x)|^p \geq 0 & \text{on } \partial U_\kappa = \partial \Omega \cup \partial \Omega_\kappa.
\end{cases}$$

The vanishing viscosity of $\tilde{u}^\varepsilon \to 0$ in $U_\kappa$ can be quantified by Theorem 12, which gives us

$$0 \leq \tilde{u}^\varepsilon(x) \leq \nu C_0 \varepsilon^{\frac{\alpha+1}{\alpha}} \sigma + C_3 \left( \frac{\varepsilon}{\delta_{0, U_\kappa}} \right)^{\frac{\alpha+2}{\alpha}} d_\kappa(x) $$

for $p < 2$,

$$0 \leq \tilde{u}^\varepsilon(x) \leq \nu \varepsilon \log \left( \frac{1}{d_\kappa(x)} \right) + C \left( \frac{\varepsilon}{\delta_{0, U_\kappa}} \right)^2$$

for $p = 2$,

for $x \in U_\kappa$. From (25) and the comparison principle in $U_\kappa$, we have

$$0 \leq u^\varepsilon(x) \leq \tilde{u}^\varepsilon(x) \leq \nu C_0 \varepsilon^{\frac{\alpha+1}{\alpha}} \sigma + C_3 \left( \frac{4\varepsilon}{\kappa} \right)^{\frac{\alpha+2}{\alpha}} d_\kappa(x) $$

for $p < 2$, (26)

$$0 \leq u^\varepsilon(x) \leq \tilde{u}^\varepsilon(x) \leq \nu \varepsilon \log \left( \frac{1}{d_\kappa(x)} \right) + C \left( \frac{4\varepsilon}{\kappa} \right)^2$$

for $p = 2$, (27)

for $x \in U_\kappa$.

We proceed with the doubling variable method. For $p < 2$, consider the auxiliary functional

$$\Phi(x, y) = u^\varepsilon(x) - u(y) - \frac{C_0 |x - y|^2}{\sigma} - \nu C_0 \varepsilon^{\frac{\alpha+1}{\alpha}} \frac{d(x)^\alpha}{d_\kappa(x)\sigma}, \quad (x, y) \in \overline{\Omega} \times \overline{\Omega},$$

where $C_0$ is the Lipschitz constant of $u$ from (13), $\sigma \in (0, 1)$. The fact that $d(x)^\alpha u^\varepsilon(x) \to C_\alpha \varepsilon^{\alpha+1}$ as $d(x) \to 0^+$ implies

$$\max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Phi(x, y) = \Phi(x_\sigma, y_\sigma) \quad \text{for some } (x_\sigma, y_\sigma) \in \Omega \times \overline{\Omega}.$$
From $\Phi(x_\sigma, y_\sigma) \geq \Phi(x_\sigma, x_\sigma)$, we can deduce that

$$|x_\sigma - y_\sigma| \leq \sigma. \quad (28)$$

If $d(x_\sigma) \geq \frac{1}{2} \kappa$, since $x \mapsto \Phi(x, y_\sigma)$ has a maximum over $\Omega$ at $x = x_\sigma$, the subsolution test for $u^\varepsilon(x)$ gives us

$$u^\varepsilon(x_\sigma) + \left| \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} - \frac{\nu C_0 \alpha \varepsilon^{\alpha+1} Dd(x_\sigma)}{d(x_\sigma)^{\alpha+1}} \right|^p - f(x_\sigma)$$

$$- \varepsilon \left( \frac{2nC_0}{\sigma} + \frac{\nu C_0 \alpha (\alpha + 1) \varepsilon^{\alpha+1} |Dd(x_\sigma)|^2}{d(x_\sigma)^{\alpha+2}} - \frac{\nu C_0 \alpha \varepsilon^{\alpha+1} \Delta d(x_\sigma)}{d(x_\sigma)^{\alpha+1}} \right) \leq 0. \quad (29)$$

Since $y \mapsto \Phi(x_\sigma, y)$ has a maximum over $\overline{\Omega}$ at $y = y_\sigma$, the supersolution test for $u(y)$ gives us

$$u(y_\sigma) + \left| \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} \right|^p - f(y_\sigma) \geq 0. \quad (30)$$

For simplicity, define

$$\xi_\sigma := \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} \quad \text{and} \quad \zeta_\sigma := -\frac{\nu C_0 \alpha \varepsilon^{\alpha+1} Dd(x_\sigma)}{d(x_\sigma)^{\alpha+1}}.$$

From (28) and $d(x_\sigma) \geq \frac{1}{2} \kappa$,

$$|\xi_\sigma| \leq 2C_0, \quad \text{and} \quad |\xi_\sigma| \leq \nu K_1 C_0 \alpha \left( \frac{\varepsilon}{d(x_\sigma)} \right)^{\alpha+1} \leq \nu K_1 C_0 \alpha \left( \frac{2\varepsilon}{\kappa} \right)^{\alpha+1}.$$

Using the inequality (21) with $\gamma = p > 1$, we deduce that

$$|\xi_\sigma + \zeta_\sigma|^p - |\xi_\sigma|^p \leq p (|\xi_\sigma| + |\zeta_\sigma|)^{p-1} |\zeta_\sigma|$$

$$\leq p \left[ 2C_0 + \nu K_1 C_0 \alpha \left( \frac{2\varepsilon}{\kappa} \right)^{\alpha+1} \right] \nu K_1 C_0 \alpha \left( \frac{2\varepsilon}{\kappa} \right)^{\alpha+1}.$$

(31)

Combine (31) together with (29), (30) and $|f(x_\sigma) - f(y_\sigma)| \leq C |x_\sigma - y_\sigma| \leq C \sigma$ to obtain

$$u^\varepsilon(x_\sigma) - u(y_\sigma) \leq p \left[ 2C_0 + \nu K_1 C_0 \alpha \left( \frac{2\varepsilon}{\kappa} \right)^{\alpha+1} \right] \nu K_1 C_0 \alpha \left( \frac{2\varepsilon}{\kappa} \right)^{\alpha+1} + C \sigma$$

$$+ 2nC_0 \left( \frac{\varepsilon}{\sigma} \right) + \nu K_1^2 C_0 \alpha (\alpha + 1) \left( \frac{2\varepsilon}{\kappa} \right)^{\alpha+2} + \nu K_2 C_0 \alpha \left( \frac{2\varepsilon}{\kappa} \right)^{\alpha+1}.$$
\[
\leq C \left[ \sigma + \frac{\varepsilon}{\sigma} \left( 1 + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} \right)^{p-1} \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+2} \right].
\]

By the fact that \((1 + x)^\gamma \leq 1 + x^\gamma\) for \(x \in [0, 1]\) and \(\gamma \in [0, 1]\), we know
\[
\left( 1 + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} \right)^{p-1} \leq 1 + \left( \frac{\varepsilon}{\kappa} \right),
\]
as \(0 \leq p - 1 \leq 1\). Therefore,
\[
u \varepsilon(x_\sigma) - u(y_\sigma) \leq C \left[ \sigma + \frac{\varepsilon}{\sigma} + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+2} \right],
\]
where \(C\) is independent of \(\kappa\) and \(\varepsilon\). Now choose \(\sigma = \sqrt{\varepsilon}\) to get (with \(\kappa\) fixed)
\[
\Phi(x_\sigma, y_\sigma) \leq u^\varepsilon(x_\sigma) - u(y_\sigma) \leq C \sqrt{\varepsilon}. \tag{32}
\]

If \(d(x_\sigma) < \frac{1}{2}\kappa\), then \(x_\sigma \in U_\kappa\) and furthermore \(\text{dist}(x_\sigma, \partial \Omega_\kappa) > \frac{1}{2}\kappa\). Indeed, for any \(y \in \partial \Omega\) and \(z \in \partial \Omega_\kappa\), we have \(|x_\sigma - z| + |x_\sigma - y| \geq |y - z|\). Taking the infimum over all \(y \in \partial \Omega\), we deduce that
\[
|x_\sigma - z| + d(x_\sigma) \geq \inf_{y \in \partial \Omega} |y - z| = d(z) = \kappa
\]
since \(z \in \partial \Omega_\kappa = \{x \in \Omega : d(x) = \kappa\}\). Thus, \(|x_\sigma - z| \geq \kappa - d(x_\sigma) > \frac{1}{2}\kappa\) for all \(z \in \partial \Omega_\kappa\), which implies that \(\text{dist}(x_\sigma, \partial \Omega_\kappa) > \frac{1}{2}\kappa\) and hence \(d_\kappa(x_\sigma) = d(x_\sigma)\). By (26) and the fact that \(u \geq 0\), we have
\[
\Phi(x_\sigma, y_\sigma) \leq u^\varepsilon(x_\sigma) - \frac{\nu C_\alpha e^{\alpha+1}}{d(x_\sigma)^\alpha} \leq C_3 \left( \frac{4\varepsilon}{\kappa} \right)^{\alpha+2} \tag{33}
\]
Since \(\Phi(x, x) \leq \Phi(x_\sigma, y_\sigma)\) for all \(x \in \Omega\), we obtain from (32) and (33) that
\[
u u^\varepsilon(x) - u(x) - \frac{\nu C_\alpha e^{\alpha+1}}{d(x)^\alpha} \leq C \sqrt{\varepsilon} + C_3 \left( \frac{4\varepsilon}{\kappa} \right)^{\alpha+2}
\]
and thus (23) follows.

For \(p = 2\), we consider instead the functional
\[
\Phi(x, y) = u^\varepsilon(x) - u(y) - \frac{C_0 |x - y|^2}{\sigma} - \nu \varepsilon \log \left( \frac{1}{d(x)} \right), \quad (x, y) \in \overline{\Omega} \times \overline{\Omega}.
\]
Similar to the previous case where \(1 < p < 2\), the maximum of \(\Phi\) occurs at some point \((x_\sigma, y_\sigma) \in \Omega \times \overline{\Omega}\) and \(|x_\sigma - y_\sigma| \leq \sigma\). If \(d(x_\sigma) \geq \frac{1}{2}\kappa\), by the subsolution test.
for $u^\varepsilon(x)$, we have
\[
\begin{align*}
&\quad u^\varepsilon(x_\sigma) + \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} - \nu \frac{Dd(x_\sigma)}{d(x_\sigma)} - f(x_\sigma) \\
&\quad - 2nC_0 \left( \frac{\varepsilon}{\sigma} \right) - \nu |Dd(x_\sigma)|^2 \left( \frac{\varepsilon}{d(x_\sigma)} \right)^2 + \nu \Delta d(x_\sigma) \left( \frac{\varepsilon^2}{d(x_\sigma)} \right) \\ 
&\quad \leq 0. 
\end{align*}
\]
(34)

By the supersolution test for $u(y)$, we have
\[
\begin{align*}
&\quad u(y_\sigma) + \frac{2C_0(x_\sigma - y_\sigma)}{\sigma} - f(y_\sigma) \\ 
&\quad \geq 0. 
\end{align*}
\]
(35)

Subtract (35) from (34) to get
\[
\begin{align*}
&\quad u^\varepsilon(x_\sigma) - u(y_\sigma) \leq \left( 4C_0 + \nu \frac{Dd(x_\sigma)}{d(x_\sigma)} \right) \left( \frac{\varepsilon}{d(x_\sigma)} \right) \\
&\quad + C\sigma + 2nC_0 \left( \frac{\varepsilon}{\sigma} \right) + \nu |Dd(x_\sigma)|^2 \left( \frac{\varepsilon}{d(x_\sigma)} \right)^2 + \nu |\Delta d(x_\sigma)| \frac{\varepsilon^2}{d(x_\sigma)}. 
\end{align*}
\]
Using $d(x_\sigma) \geq \frac{1}{2} \kappa$ and bounds on $d(x)$, we see that
\[
\begin{align*}
&\quad \Phi(x_\sigma, y_\sigma) \leq u^\varepsilon(x_\sigma) - u(y_\sigma) \\
&\quad \leq 4K_1^2 \nu (1 + \nu) \left( \frac{\varepsilon}{\kappa} \right)^2 + C\sigma + 2nC_0 \left( \frac{\varepsilon}{\sigma} \right) + 2\nu(K_2\varepsilon + 4C_0K_1) \left( \frac{\varepsilon}{\kappa} \right) \\
&\quad \leq C \left( \sigma + \frac{\varepsilon}{\sigma} + \frac{\varepsilon}{\kappa} + \left( \frac{\varepsilon}{\kappa} \right)^2 \right) \leq C \sqrt{\varepsilon} 
\end{align*}
\]
(36)

if we choose $\sigma = \sqrt{\varepsilon}$.

If $d(x_\sigma) < \frac{1}{2} \kappa$, then $x_\sigma \in U_\kappa$. Again, we have $d_\kappa(x_\sigma) = d(x_\sigma)$ and from (27)
\[
\begin{align*}
&\quad \Phi(x_\sigma, y_\sigma) \leq u^\varepsilon(x_\sigma) - \nu \varepsilon \log \left( \frac{1}{d(x_\sigma)} \right) \leq C \left( \frac{4\varepsilon}{\kappa} \right)^2. 
\end{align*}
\]
(37)

Since $\Phi(x, x) \leq \Phi(x_\sigma, y_\sigma)$ for $x \in \Omega$, we obtain from (36) and (37) that
\[
\begin{align*}
&\quad u^\varepsilon(x) - u(x) - \nu \varepsilon \log \left( \frac{1}{d(x)} \right) \leq C \sqrt{\varepsilon} + C \left( \frac{4\varepsilon}{\kappa} \right)^2 
\end{align*}
\]
and thus (24) follows.

\[\square\]

Remark 5 For general non-negative Lipschitz data $f \in C(\bar{\Omega})$, it is natural to try a cutoff function argument. Let $\chi_\kappa \in C^\infty_c(\Omega)$ such that $0 \leq \chi_\kappa \leq 1$, $\chi_\kappa = 1$ in $\Omega_{2\kappa}$ and supp $\chi_\kappa \subset \Omega_\kappa$. Let $u^\varepsilon_\kappa \in C^2(\Omega) \cap C(\bar{\Omega})$ solve (PDE\varepsilon) with data $f \chi_\kappa$. Then

\[\square\] Springer
\( u^\varepsilon \rightarrow u^\varepsilon \) as \( \kappa \rightarrow 0 \) (since \( f \chi_\kappa \rightarrow f \) in the weak* topology of \( L^\infty(\Omega) \) and we have the continuity of the solution to (PDE_\varepsilon) with respect to data in this topology [21, Remark II.1]). However, it is not clear at the moment how to quantify this rate of convergence, since \( f \chi_\kappa \) does not converge to \( f \) in the uniform norm, unless \( f = 0 \) on \( \partial\Omega \).

### 3.1 A Rate for Non-negative Zero Boundary Data

We prove the rate of convergence for the case where \( f \) is non-negative with \( f = 0 \) on \( \partial\Omega \).

**Proof of Theorem 1** Let \( L = \| Df \|_{L^\infty(\Omega)} \) be the Lipschitz constant of \( f \). For \( \kappa > 0 \) small such that \( 0 < \kappa < \delta_0 \) and \( x \in \Omega \setminus \Omega_\kappa \), let \( x_0 \) be the projection of \( x \) onto \( \partial\Omega \). We observe that

\[
    f(x) = f(x) - f(x_0) \leq L \| x - x_0 \| = L\kappa. \quad (38)
\]

Define

\[
    g_\kappa(x) = \begin{cases} 
      0 & \text{if } 0 \leq d(x) \leq \kappa/2, \\
      2L \left( d(x) - \kappa/2 \right) & \text{if } \kappa/2 \leq d(x) \leq \kappa.
    \end{cases}
\]

It is clear that for \( x \in \partial\Omega_\kappa \), \( g_\kappa(x) = L\kappa \geq f(x) \) since (38). Therefore, we can define the following continuous function

\[
    f_\kappa(x) = \begin{cases} 
      0 & \text{if } 0 \leq d(x) \leq \kappa/2, \\
      \min \{ g_\kappa(x), f(x) \} & \text{if } \kappa/2 \leq d(x) \leq \kappa, \\
      f(x) & \text{if } \kappa \leq d(x).
    \end{cases} \quad (39)
\]

A graph of \( f_\kappa \) is given in Fig. 2.

The continuity at \( x \in \partial\Omega_\kappa \) comes from the fact that when \( d(x) = \kappa \), we have \( g_\kappa(x) = L\kappa \geq f(x) \) by (38). It is clear that \( f_\kappa \) is Lipschitz with \( \| f_\kappa \|_{L^\infty(\Omega)} \leq L \) as well and \( f_\kappa \rightarrow f \) uniformly as \( \kappa \rightarrow 0 \). Indeed, we have \( 0 \leq f_\kappa \leq f \) and

\[
    0 \leq \max_{x \in \Omega} (f(x) - f_\kappa(x)) \leq \max_{x \in \Omega \setminus \Omega_\kappa} \max_{x \in \Omega \setminus \Omega_\kappa} (f(x) - f_\kappa(x)) = \max_{x \in \Omega \setminus \Omega_\kappa} f(x) \leq L\kappa.
\]

Let \( u_\kappa^\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega}) \) be the solution to (PDE_\varepsilon) with data \( f \chi_\kappa \) and \( u_\kappa \in C(\overline{\Omega}) \) be the corresponding solution to (PDE_0) with data \( f \chi_\kappa \). By the comparison principle ([21, Corollary II.1]), we have

\[
    0 \leq u^\varepsilon(x) - u_\kappa^\varepsilon(x) \leq L\kappa \quad \text{for } x \in \Omega. \quad (40)
\]

By the comparison principle for (PDE_0), we also have

\[
    0 \leq u(x) - u_\kappa(x) \leq L\kappa \quad \text{for } x \in \Omega. \quad (41)
\]
If $1 < p < 2$, by Theorem 14, there exists a constant $C$ independent of $\kappa$ such that

$$-C\sqrt{\varepsilon} \leq u^\varepsilon_\kappa(x) - u_\kappa(x) \leq C \left[\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha + 2} + \frac{\varepsilon^{\alpha + 1}}{d(x)^\alpha}\right], \quad x \in \Omega. \quad (42)$$

Combining (40), (41) and (42), we obtain

$$-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) = \left(u^\varepsilon(x) - u^\varepsilon_\kappa(x)\right) + \left(u^\varepsilon_\kappa(x) - u_\kappa(x)\right) + \left(u_\kappa(x) - u(x)\right)
\leq L_\kappa + C \left[\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^{\alpha + 2} + \frac{\varepsilon^{\alpha + 1}}{d(x)^\alpha}\right], \quad x \in \Omega.
$$

Choose $\kappa = \sqrt{\varepsilon}$ and we deduce that

$$-C\sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C\sqrt{\varepsilon} + \frac{C\varepsilon^{\alpha + 1}}{d(x)^\alpha}$$

for $x \in \Omega$. Thus, the conclusion follows.

If $p = 2$, by Theorem 14, there exists a constant $C$ independent of $\kappa$ such that

$$-C\sqrt{\varepsilon} \leq u^\varepsilon_\kappa(x) - u_\kappa(x) \leq C \left[\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^2 + \varepsilon \log \left(\frac{1}{d(x)}\right)\right], \quad x \in \Omega. \quad (43)$$
Combining (40), (41) and (43), we obtain

\[-C \sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) = \left( u^\varepsilon(x) - u^\varepsilon_\kappa(x) \right) + \left( u^\varepsilon_\kappa(x) - u_\kappa(x) \right) + \left( u_\kappa(x) - u(x) \right) \]
\[\leq L\kappa + C \left[ \sqrt{\varepsilon} + \left( \frac{\varepsilon}{\kappa} \right)^2 + \varepsilon \log \left( \frac{1}{d(x)} \right) \right], \quad x \in \Omega.\]

Choose \( \kappa = \varepsilon \) and we deduce that

\[-C \sqrt{\varepsilon} \leq u^\varepsilon(x) - u(x) \leq C \sqrt{\varepsilon} + \varepsilon \log \left( \frac{1}{d(x)} \right)\]

for \( x \in \Omega \). Thus, the conclusion follows. \( \square \)

### 4 Improved One-Sided Rate of Convergence

In this section, we assume \( f \in C^2(\overline{\Omega}) \) (or uniformly semiconcave in \( \overline{\Omega} \)) such that \( f = 0 \) on \( \partial \Omega \) and \( f \geq 0 \). It is known that for the problem on \( \mathbb{R}^n \), namely,

\[ u(x) + |D u(x)|^p - f(x) = 0 \quad \text{in } \mathbb{R}^n, \]

if \( f \) is semiconcave in the whole space \( \mathbb{R}^n \), then the solution \( u \) is also semiconcave (Theorem 19, see also [8]).

**Remark 6** The heuristic idea that we will use in this section is the following. Assume that \( u^\varepsilon(x) - u(x) \) has a maximum over \( \overline{\Omega} \) at some interior point \( x_0 \in \Omega \). Then by the equation (PDE\( \varepsilon \)) at \( x_0 \) and the supersolution test for (PDE\( 0 \)) at \( x_0 \), we obtain

\[ \max_{x \in \Omega} \left( u^\varepsilon(x) - u(x) \right) \leq u^\varepsilon(x_0) - u(x_0) \leq \varepsilon \Delta u^\varepsilon(x_0). \]

If \( u \) is uniformly semiconcave in \( \overline{\Omega} \), then \( \Delta u^\varepsilon(x_0) \leq \Delta u(x_0) \leq C \). Thus, we obtain a better one-sided rate \( O(\varepsilon) \) for \( u^\varepsilon - u \). However, there are a couple of problems with this argument. Firstly, as \( u^\varepsilon = +\infty \) on \( \partial \Omega \), we need to subtract an appropriate term from \( u^\varepsilon \) to make a maximum over \( \overline{\Omega} \) happen in the interior. Secondly, unless \( f \in C^2_{\text{loc}}(\Omega) \), in general, \( u \) is not uniformly semiconcave but only locally semiconcave. In this section, we provide estimates on the local semiconcavity constant of \( u \) and rigorously show how the upper bound of \( u^\varepsilon - u \) can be obtained.

From Lemma 10, we have \( u = 0 \) on \( \partial \Omega \). It is clear that the solution \( u \) to (PDE\( 0 \)) is also the unique solution to the following Dirichlet boundary problem

\[
\begin{aligned}
\begin{cases}
u(x) + |D u(x)|^p &= f(x) \quad \text{in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]

\( \square \ Springer
Since $H(x, \xi) = |\xi|^p - f(x)$, the corresponding Legendre transform is

$$L(x, v) = C_p |v|^q + f(x)$$

where $p^{-1} + q^{-1} = 1$ and $C_p$ is defined in Lemma 10. Let us extend $f$ to a function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $\tilde{f}(x) = 0$ for $x \notin \Omega$.

**Definition 2** Define

$$C^k_0(\Omega) = \left\{ \varphi \in C^k(\Omega) : D^\beta \varphi(x) = 0 \text{ on } \partial \Omega \text{ with } |\beta| \in [0, k] \right\},$$

where $\beta$ is a multiindex and $|\beta|$ is its order.

We summarize the results about the semiconcavity of $u$ as follows.

**Theorem 16** (Semiconcavity) Assume $f \geq 0$, $f = 0$ on $\partial \Omega$ and $f$ is uniformly semiconcave in $\Omega$ with semiconcavity constant $c$. Let $u$ be the solution to (PDE0).

(i) If $\tilde{f}$ is uniformly semiconcave in $\mathbb{R}^n$, then $u$ is uniformly semiconcave in $\Omega$.

(ii) In general, $u$ is locally semiconcave. More specifically, there exists a constant $C > 0$ independent of $x \in \Omega$ such that $\forall x \in \Omega$,

$$u(x + h) - 2u(x) + u(x - h) \leq \frac{C}{d(x)} |h|^2, \quad (45)$$

$\forall h \in \mathbb{R}^n$ with $|h| \leq M_x$ for some constant $M_x$ that depends on $x$.

The proof of Theorem 16 is given at the end of this section.

**Remark 7** If $f \in C^2_c(\mathbb{R}^n)$ (or $C^2_0(\Omega)$), then $f$ is uniformly semiconcave with semiconcavity constant

$$c = \max \left\{ D^2 f(x) \xi \cdot \xi : |\xi| = 1, x \in \mathbb{R}^n \right\} \geq 0. \quad (46)$$

Also, the condition that $\tilde{f}$ is semiconcave in $\mathbb{R}^n$ holds for $C^2_c(\Omega)$ and $C^2_0(\Omega)$.

The following lemma is a refined version of the local gradient bound in Theorem 4. We follow [1, Theorem 3.1] where the authors use Bernstein’s method inside a doubling variable argument and explicitly keep track of all the dependencies. We refer the reader to [4, 9] and the references therein for related versions of the gradient bound. We believe this result is new in the literature since it is uniform in $\varepsilon$, namely, we give the explicit dependence of the gradient bound on $d(x)$. It also indicates that the boundary layer is a strip of size $O(\varepsilon)$ from the boundary.

**Lemma 17** For all $\varepsilon$ small enough, there exists a constant $C$ independent of $\varepsilon$ such that

$$|Du^\varepsilon(x)| \leq C \left( 1 + \left( \frac{\varepsilon}{d(x)} \right)^{\alpha + 1} \right) \text{ for } x \in \Omega. \quad (47)$$
Proof of Lemma 17  Fix $x_0 \in \Omega \setminus \Omega_{\delta_0}$. Let $\delta := \frac{1}{4}d(x_0)$ and

$$v(x) := \frac{1}{\delta}u^\varepsilon(x_0 + \delta x), \quad x \in B(0, 2).$$

Then $v$ solves

$$\delta v(x) + |Dv(x)|^p - \tilde{f}(x) - \frac{\varepsilon}{\delta} \Delta v(x) = 0 \quad \text{in} \ B(0, 2), \quad (48)$$

where $\tilde{f}(x) := f(x_0 + \delta x)$ on $\overline{B(0, 2)}$. Note that $\|\tilde{f}\|_{L^\infty} \leq \|f\|_{L^\infty}$ and $B(x_0, 2\delta) \subset \Omega_{2\delta} \subset \subset \Omega$.

By Lemma 11, there is a constant $C$ independent of $\delta, \varepsilon$ such that

$$\delta \|v\|_{L^\infty(B(0, \frac{3}{2}))} \leq \|u^\varepsilon\|_{L^\infty(\Omega_{2\delta})} \leq C \left( 1 + \frac{e^{\alpha+1}}{\delta^\alpha} \right).$$

Apply Theorem 3.1 in [1] to obtain

$$\sup_{x \in B(0, 1)} |Dv(x)| \leq C \left[ \left( \frac{\varepsilon}{\delta} \right)^{\frac{1}{p-1}} + \left( \|f\|_{L^\infty} + \delta \|v\|_{L^\infty(B(0, \frac{3}{2}))} \right)^{\frac{1}{p}} \right]$$

$$\leq C \left[ \left( \frac{\varepsilon}{\delta} \right)^{\alpha+1} + \left( 1 + \frac{e^{\alpha+1}}{\delta^\alpha} \right)^{\frac{\alpha+1}{\alpha+2}} \right] \leq C \left( 1 + \left( \frac{\varepsilon}{\delta} \right)^{\alpha+1} \right),$$

where $p = \frac{\alpha+2}{\alpha+1}$ and $\alpha + 1 = \frac{1}{p-1}$. Plugging in $\delta = \frac{1}{4}d(x_0)$, we obtain

$$|Du^\varepsilon(x_0)| = |Dv(0)| \leq C \left( 1 + \left( \frac{\varepsilon}{d(x_0)} \right)^{\alpha+1} \right).$$

In other words, we have (47) for all $x \in \Omega \setminus \Omega_{\delta_0}$. On the other hand, from Theorem 4, there exists a constant $C$ independent of $\varepsilon$ such that $|Du^\varepsilon(x)| \leq C$ for all $x \in \Omega_{\delta_0}$. Thus, the proof is complete.

Proof of Theorem 2  For $1 < p < 2$, we proceed as in the proof of Theorem 14 to obtain

$$0 \leq u^\varepsilon(x) \leq \bar{u}^\varepsilon(x) \leq \frac{\nu C_{\alpha} e^{\alpha+1}}{d(x)^\alpha} + C_3 \left( \frac{4\varepsilon}{\kappa} \right)^{\alpha+2}$$

for $x \in U_\kappa$. Let

$$\psi^\varepsilon(x) := u^\varepsilon(x) - \frac{\nu C_{\alpha} e^{\alpha+1}}{d(x)^\alpha}, \quad x \in \Omega,$$
where \( \nu > 1 \) is chosen as in Lemma 11. It is clear that \( u - \psi^\varepsilon \) has a local minimum at some point \( x_0 \in \Omega \) since \( \psi^\varepsilon(x) \to -\infty \) as \( x \to \partial \Omega \). The normal derivative test gives us

\[
D\psi^\varepsilon(x_0) = Du^\varepsilon(x_0) + \nu C_{\alpha \alpha} \left( \frac{\varepsilon}{d(x_0)} \right)^{\alpha + 1} Dd(x_0) \in D^-u(x_0).
\]

There are two cases to consider:

- If \( d(x_0) < \frac{1}{2} \kappa \), then as in the proof of Theorem 14, \( x_0 \in U_\kappa \) and \( d_\kappa (x_0) = d(x_0) \).

  By the definition of \( x_0 \), for any \( x \in \Omega \), there holds

  \[
  u(x) - \left( u^\varepsilon(x) - \frac{\nu C_{\alpha}}{\varepsilon} \frac{d(x_0)^{\alpha + 1}}{d(x_0)^{\alpha}} \right) \geq u(x_0) - \left( u^\varepsilon(x_0) - \frac{\nu C_{\alpha}}{\varepsilon} \frac{d(x_0)^{\alpha + 1}}{d(x_0)^{\alpha}} \right).
  \]

  Therefore,

  \[
  u^\varepsilon(x) - u(x) - \frac{\nu C_{\alpha}}{\varepsilon} \frac{d(x_0)^{\alpha + 1}}{d(x_0)^{\alpha}} \leq \left( u^\varepsilon(x_0) - \frac{\nu C_{\alpha}}{\varepsilon} \frac{d(x_0)^{\alpha + 1}}{d(x_0)^{\alpha}} \right) - u(x_0) \leq C_3 \left( \frac{4\varepsilon}{\kappa} \right)^{\alpha + 2}
  \]

  thanks to (49). Thus, in this case

  \[
  u^\varepsilon(x) - u(x) \leq \frac{\nu C_{\alpha}}{\varepsilon} \frac{d(x_0)^{\alpha + 1}}{d(x_0)^{\alpha}} + C_3 \left( \frac{4\varepsilon}{\kappa} \right)^{\alpha + 2}, \quad x \in \Omega.
  \]

- If \( d(x_0) \geq \frac{1}{2} \kappa \), from the fact that \( u \) is semiconcave in \( \Omega \) with a linear modulus \( c(x) \) as in Theorem 16, we have

  \[
  D^2 \psi^\varepsilon(x_0) \leq c(x_0) \mathbb{I}_n,
  \]

  where \( \mathbb{I}_n \) denotes the identity matrix of size \( n \). This implies that

  \[
  \Delta \psi^\varepsilon(x_0) \leq nc(x_0) \leq \frac{Cn}{d(x_0)} \leq \frac{Cn}{\kappa}.
  \]

In other words, we have

\[
\varepsilon \Delta u^\varepsilon(x_0) - \frac{\nu C_{\alpha}}{d(x_0)^{\alpha + 2}} \left| Dd(x_0) \right|^2 + \frac{\nu C_{\alpha}}{d(x_0)^{\alpha + 1}} \cdot \Delta d(x_0) \leq \frac{Cn \varepsilon}{\kappa}.
\]

Since \( d(x_0) \geq \frac{1}{2} \kappa \), we can further deduce that

\[
\varepsilon \Delta u^\varepsilon(x_0) \leq \frac{Cn \varepsilon}{\kappa} + \frac{C \varepsilon^{\alpha + 2}}{d(x_0)^{\alpha + 2}} \leq \frac{Cn \varepsilon}{\kappa} + C \left( \frac{\varepsilon}{\kappa} \right)^{\alpha + 2}, \quad (51)
\]
where $C$ is independent of $\varepsilon$. Since $\psi^\varepsilon \in C^2(\Omega)$, the viscosity supersolution test for $u$ gives us
\begin{equation}
    u(x_0) + \left| Du^\varepsilon(x_0) + \frac{\nu C_a \varepsilon \alpha^{\alpha+1}}{d(x_0)^{\alpha+1}} Dd(x_0) \right|^p - f(x_0) \geq 0.
\end{equation}

On the other hand, since $u^\varepsilon$ solves $(\text{PDE}_\varepsilon)$, we have
\begin{equation}
    u^\varepsilon(x_0) + \left| Du^\varepsilon(x_0) \right|^p - f(x_0) - \varepsilon \Delta u^\varepsilon(x_0) = 0.
\end{equation}

Combine (52) and (53) to obtain that
\begin{equation}
    u^\varepsilon(x_0) - u(x_0) \leq \left| Du^\varepsilon(x_0) + \frac{\nu C_a \varepsilon \alpha^{\alpha+1}}{d(x_0)^{\alpha+1}} Dd(x_0) \right|^p + \varepsilon \Delta u^\varepsilon(x_0).
\end{equation}

By Lemma 17, we can bound $Du^\varepsilon(x_0)$ as
\begin{equation}
    \left| Du^\varepsilon(x_0) \right| \leq C + C \left( \frac{\varepsilon}{d(x_0)} \right)^{\alpha+1} \leq C + C \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1}
\end{equation}

since $d(x_0) \geq \frac{1}{\kappa}$. We estimate the gradient terms on the right hand side of (54) using (55) as follows.
\begin{align*}
    &\left| Du^\varepsilon(x_0) + \frac{\nu C_a \varepsilon \alpha^{\alpha+1}}{d(x_0)^{\alpha+1}} Dd(x_0) \right|^p - \left| Du^\varepsilon(x_0) \right|^p \\
    \leq &\ p \left( \left| Du^\varepsilon(x_0) \right| + \frac{\nu C_a \varepsilon \alpha^{\alpha+1}}{d(x_0)^{\alpha+1}} |Dd(x_0)| \right)^{p-1} \frac{\nu C_a \varepsilon \alpha^{\alpha+1}}{d(x_0)^{\alpha+1}} |Dd(x_0)| \\
    \leq &\ p \left( C + C \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} \right)^{p-1} C \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} \left( 1 + \left( \frac{\varepsilon}{\kappa} \right) \right),
\end{align*}

where $C$ is a constant depending only on $\nu, \alpha$, and $d$. Plugging (51) and (56) in the right hand side of (54), we get
\begin{equation}
    u^\varepsilon(x_0) - u(x_0) \leq \frac{C n \varepsilon}{\kappa} + C \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} \left( 1 + \left( \frac{\varepsilon}{\kappa} \right) \right).
\end{equation}

Therefore,
\begin{equation}
    u^\varepsilon(x) - u(x) \leq \frac{\nu C_a \varepsilon^{\alpha+1}}{d(x)^{\alpha}} + C \left( \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+1} + \left( \frac{\varepsilon}{\kappa} \right)^{\alpha+2} \right) + \frac{C n \varepsilon}{\kappa}, \quad x \in \Omega.
\end{equation}

For $p = 2$, the argument is similar. We take $\psi^\varepsilon(x) := u^\varepsilon(x) - \nu \varepsilon \log \left( \frac{1}{d(x)} \right)$ instead and still $u - \psi^\varepsilon$ attains a local minimum at some point $x_0 \in \Omega$. Carrying out the similar computations as in the case of $1 < p < 2$, we have:
If \( d(x_0) < \frac{1}{2} \kappa \), then
\[
    u^\varepsilon(x) - u(x) \leq \nu \varepsilon \log \left( \frac{1}{d(x)} \right) + C \left( \frac{4\varepsilon}{\kappa} \right)^2, \quad x \in \Omega.
\]

If \( d(x_0) \geq \frac{1}{2} \kappa \), then
\[
    u^\varepsilon(x) - u(x) \leq \nu \varepsilon \log \left( \frac{1}{d(x)} \right) + C \left( \frac{\varepsilon}{\kappa} + \frac{\varepsilon}{\kappa} \right)^2 + \frac{Cn\varepsilon}{\kappa}, \quad x \in \Omega.
\]

From these two cases, the conclusion for \( p = 2 \) follows.

\( \Box \)

**Remark 8** If \( f \in C^2(\overline{\Omega}) \) with \( f = 0 \), \( Df = 0 \) and \( D^2 f = 0 \) on \( \partial \Omega \), then (50) can be improved to \( \Delta \psi^\varepsilon(x_0) \leq nc \) where \( c \) is the semiconcavity constant of \( f \), and thus the final estimate becomes
\[
    u^\varepsilon(x) - u(x) \leq \nu C \alpha \varepsilon^\alpha + C \left( \frac{\varepsilon}{d(x_0)^\alpha} + \frac{\varepsilon}{d(x_0)^{\alpha+2}} \right) + nc\varepsilon, \quad x \in \Omega.
\]

**Remark 9**

- We only need the local gradient bound in Theorem 4 to obtain the local rate of convergence \( O(\varepsilon) \) in (56). However, to make the dependence on \( \kappa \) explicit, we need to bound \( Du^\varepsilon(x_0) \) as in (56).

- Another way to get (55) without using Lemma 17 (which is true for all \( x \in \Omega \)) is using the fact that \( D\psi^\varepsilon(x_0) \in D^-u(x_0) \), which implies
\[
    \left| D\psi^\varepsilon(x_0) \right| = \left| Du^\varepsilon(x_0) + \nu C_0 \left( \frac{\varepsilon}{d(x_0)^\alpha} \right)^{\alpha+1} Dd(x_0) \right| \leq C_0
\]

since \( u \) is Lipschitz with constant \( C_0 \).

Before giving the proof of Corollary 3, we need to modify the construction of the cutoff function in the proof of Theorem 1.

**Lemma 18** Assume \( f \in C^2(\overline{\Omega}) \) such that \( f = 0 \), \( Df = 0 \) and \( D^2 f = 0 \) on \( \partial \Omega \). For all \( \kappa > 0 \) small enough, there exists \( f_\kappa \in C^2(\Omega) \) such that
\[
    \| f_\kappa - f \|_{L^\infty(\Omega)} \leq C\kappa \quad \text{and} \quad \| D^2 f_\kappa \|_{L^\infty(\Omega)} \leq C
\]

where \( C \) is independent of \( \kappa \).

**Proof** Choose a smooth function \( \chi \in C^\infty(\mathbb{R}) \) such that \( \chi \geq 0, \chi = 0 \) if \( x \leq 1, \chi = 1 \) if \( x \geq 2 \) and \( 0 \leq \chi' \leq 2 \) in \( \mathbb{R} \).

For \( \kappa > 0 \) such that \( 0 < 2\kappa < \delta_0 \) and \( x \in \Omega \setminus \Omega_{2\kappa} \), let \( x_0 \) be the projection of \( x \) onto \( \partial \Omega \) and denote by \( \nu(x_0) \) the outward unit normal vector at \( x_0 \). Write \( x = x_0 - d(x)\nu(x_0) \)
where \( d(x) \leq 2 \kappa \). We have

\[
 f(x) = f(x_0) - Df(x_0) \cdot v(x_0) d(x) + \int_0^{d(x)} (d(x) - s) v(x_0) \cdot D^2 f(x_0 - sv(x_0)) \cdot v(x_0) ds.
\]

Since \( f = 0 \) and \( Df = 0 \) on \( \partial \Omega \), we deduce that

\[
|f(x)| \leq \left\| \frac{1}{2} D^2 f \right\|_{L^\infty(\Omega)} d(x)^2 \leq C \kappa^2
\]

for all \( d(x) \leq 2 \kappa \). Define

\[
 f_k(x) = f(x) \chi \left( \frac{d(x)}{\kappa} \right)
\]

for \( x \in \Omega \). It is clear that \( 0 \leq f_k(x) \leq f(x) \) for all \( x \in \Omega \) and \( f_k(x) = f(x) \) if \( d(x) \geq 2 \kappa \). Furthermore, we observe that

\[
0 \leq \max_{x \in \Omega} \left( f(x) - f_k(x) \right) \leq \max_{0 \leq d(x) \leq 2 \kappa} (f(x) - f_k(x)) \leq \max_{0 \leq d(x) \leq 2 \kappa} f(x) \leq C \kappa^2.
\]

We have

\[
 Df_k(x) = Df(x) \chi \left( \frac{d(x)}{\kappa} \right) + f(x) \chi' \left( \frac{d(x)}{\kappa} \right) \frac{Dd(x)}{\kappa}
\]

and

\[
 D^2 f_k(x) = D^2 f(x) \chi \left( \frac{d(x)}{\kappa} \right) + 2 \chi' \left( \frac{d(x)}{\kappa} \right) \frac{Df(x) \otimes Dd(x)}{\kappa} + f(x) \left( \chi'' \left( \frac{d(x)}{\kappa} \right) \frac{Dd(x) \otimes Dd(x)}{\kappa^2} + \chi' \left( \frac{d(x)}{\kappa} \right) \frac{D^2 d(x)}{\kappa} \right)
\]

is uniformly bounded thanks to (56).

**Proof of Corollary 3** Let \( u_k^\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega}) \) be the solution to (PDE\(_\varepsilon\)) and \( u_k \) be the solution to (PDE\(_0\)) with \( f \) replaced by \( f_k \), respectively. It is clear that

\[
0 \leq u^\varepsilon(x) - u_k^\varepsilon(x) \leq C \kappa \quad \text{for } x \in \Omega
\]

and

\[
0 \leq u(x) - u_k(x) \leq C \kappa \quad \text{for } x \in \Omega.
\]
Therefore,

\[
    u^\varepsilon(x) - u(x) \leq 2C\kappa + \left(u_k^\varepsilon(x) - u_k(x)\right). \tag{57}
\]

By Theorem 2 and Remark 8, as \( f_\kappa \in C^2_c(\Omega) \) with a uniform bound on \( D^2 f_\kappa \), we have

\[
    u_k^\varepsilon(x) - u_k(x) \leq \nu C_{\kappa} \epsilon^{\alpha + 1} + C \left( \frac{\epsilon}{\kappa} \right) + 4nC\epsilon, \quad p < 2,
\]

\[
    u_k^\varepsilon(x) - u_k(x) \leq \nu\epsilon \log \left( \frac{1}{d(x)} \right) + C \left( \frac{\epsilon}{\kappa} \right)^2 + 4nC\epsilon, \quad p = 2
\]

for some constant \( C \) independent of \( \kappa \). Choose \( \kappa = \epsilon^\gamma \) with \( \gamma \in (0, 1) \). Then (57) becomes

\[
    u^\varepsilon(x) - u(x) \leq C\epsilon^\gamma + C\epsilon + \frac{C\epsilon^{\alpha + 1}}{d(x)^\alpha} + C\epsilon^{(1-\gamma)(\alpha + 1)}, \quad p < 2,
\]

\[
    u^\varepsilon(x) - u(x) \leq C\epsilon^\gamma + C\epsilon + C\epsilon |\log d(x)| + C\epsilon^{1-\gamma}, \quad p = 2.
\]

If \( p = 2 \), then \( \gamma = 1/2 \) is the best value to choose, which implies the \( O(\sqrt{\epsilon}) \) estimate in Theorem 1. If \( p < 2 \), by setting \( \gamma = (1 - \gamma)(\alpha + 1) \), we can get the best value of \( \gamma \), that is,

\[
    \gamma = \frac{\alpha + 1}{\alpha + 2} = \frac{1}{p} > \frac{1}{2},
\]

and we obtain a better estimate \( O(\epsilon^{1/p}) \).

\[ \square \]

**Remark 10** If we do not assume \( Df = 0 \) on \( \partial \Omega \), then the best we can get from the above argument is

\[
    u^\varepsilon(x) - u(x) \leq C\epsilon^\gamma + C\epsilon^{1-\gamma} + \frac{C\epsilon^{\alpha + 1}}{d(x)^\alpha} + C\epsilon^{(1-\gamma)(\alpha + 1)}, \quad p < 2
\]

and we obtain the rate \( O(\epsilon^{1/2}) \) again.

**Proof of Theorem 16** (i) It is clear that

\[
    \tilde{u}(x) = \begin{cases} 
    u(x) & \text{if } x \in \Omega, \\
    0 & \text{if } x \notin \Omega,
    \end{cases}
\]

solves the equation \( \tilde{u}(x) + |D\tilde{u}(x)|^p - \tilde{f}(x) = 0 \) in \( \mathbb{R}^n \). Now we can use a classical doubling variable argument to show that \( -D^2u \geq -c I_n \) in \( \mathbb{R}^n \) where

\[
    c = \max \left\{ D^2 f(x)\xi \cdot \xi : |\xi| = 1, x \in \mathbb{R}^n \right\} \geq 0.
\]
We give the proof of this fact in Appendix for the reader’s convenience (see also [8]).

(ii) Fix \( x \in \Omega \) and let \( \eta \) be a minimizing curve for \( u(x) \). Then

\[
\eta(x) = \int_0^\infty e^{-s} \left( C_q |\dot{\eta}(s)|^q + f(\eta(s)) \right) ds.
\]

Since \( \eta(0) = x \in \Omega \), then there exists \( T > 0 \) such that \( \eta(s) \in \Omega \), \( \forall 0 \leq s \leq T \). In fact, we can choose \( T \geq \frac{d(x)}{C_0} \) for some constant \( C_0 \) independent of \( x \), since \( \|\dot{\eta}\|_\infty \leq C \) where \( C \) is independent of \( x \). Note that

\[
u(x) = \int_0^T e^{-s} \left( C_q |\dot{\eta}(s)|^q + f(\eta(s)) \right) ds + e^{-T} u(\eta(T)).
\]  

Define \( \tilde{\eta} : [0, +\infty) \to \mathbb{R}^n \) by

\[
\tilde{\eta}(s) := \begin{cases} 
\eta(s) + \left( 1 - \frac{s}{T} \right) h, & \text{if } 0 \leq s \leq T, \\
\eta(s), & \text{if } s \geq T.
\end{cases}
\]

Choose \( h \) small enough so that \( \tilde{\eta}(s) \in \Omega \), \( \forall s \geq 0 \). (This can be done because there exists \( r > 0 \) such that \( B(\eta(s), r) \subset \Omega \), for all \( 0 \leq s \leq T \).) By the optimal control formula of \( u(x + h) \) and \( u(x - h) \), we have

\[
\begin{align*}
u(x + h) & \leq \int_0^T e^{-s} \left( C_q \left| \dot{\eta}(s) - \frac{h}{T} \right|^q + f \left( \eta(s) + \left( 1 - \frac{s}{T} \right) h \right) \right) ds + e^{-T} u(\eta(T)), \\
\nu(x - h) & \leq \int_0^T e^{-s} \left( C_q \left| \dot{\eta}(s) + \frac{h}{T} \right|^q + f \left( \eta(s) - \left( 1 - \frac{s}{T} \right) h \right) \right) ds + e^{-T} u(\eta(T)).
\end{align*}
\]

Hence, from (58), (59), and (60), for \( h \) small enough,

\[
\begin{align*}
u(x + h) + \nu(x - h) - 2\nu(x) & \leq \int_0^T e^{-s} C_q \left( \left| \dot{\eta}(s) - \frac{h}{T} \right|^q + \left| \dot{\eta}(s) + \frac{h}{T} \right|^q - 2 |\dot{\eta}(s)|^q \right) ds \\
& \quad + \int_0^T e^{-s} \left( f \left( \eta(s) + \left( 1 - \frac{s}{T} \right) h \right) + f \left( \eta(s) - \left( 1 - \frac{s}{T} \right) h \right) - 2 f(\eta(s)) \right) ds \\
& \leq \int_0^T e^{-s} C_q \left( \left| \dot{\eta}(s) - \frac{h}{T} \right|^q + \left| \dot{\eta}(s) + \frac{h}{T} \right|^q - 2 |\dot{\eta}(s)|^q \right) ds \\
& \quad + C |h|^2 \int_0^T e^{-s} \left( 1 - \frac{s}{T} \right)^2 ds.
\end{align*}
\]
where the second inequality follows from the semiconcavity of $f$. By Taylor’s theorem, for any $y \in \mathbb{R}^n$,

$$
\begin{aligned}
|y + \frac{1}{T}h|^q &= |y|^q + q |y|^{q-2} y \cdot \frac{1}{T}h \\
&+ \int_0^1 q(q-2) |y + \frac{t}{T}h|^{q-4} \left( \left( y + \frac{t}{T}h \right) \cdot \frac{h}{T} \right)^2 (1-t)dt \\
&+ \int_0^1 q |y + \frac{t}{T}h|^{q-2} \left| \frac{h}{T} \right|^2 (1-t)dt \\
&\leq |y|^q + q |y|^{q-2} y \cdot \frac{1}{T}h + C \int_0^1 |y + \frac{t}{T}h|^{q-2} \left| \frac{h}{T} \right|^2 dt \\
&\leq |y|^q + q |y|^{q-2} y \cdot \frac{1}{T}h + C \left( |y|^{q-2} \left| \frac{h}{T} \right|^2 + \left| \frac{h}{T} \right|^q \right)
\end{aligned}
$$

(62)

and similarly

$$
\begin{aligned}
|y - \frac{1}{T}h|^q &\leq |y|^q - q |y|^{q-2} y \cdot \frac{1}{T}h + C \left( |y|^{q-2} \left| \frac{h}{T} \right|^2 + \left| \frac{h}{T} \right|^q \right),
\end{aligned}
$$

(63)

which implies

$$
\begin{aligned}
&|\dot{\eta}(s) - \frac{h}{T}|^q + |\dot{\eta}(s) + \frac{h}{T}|^q - 2 |\dot{\eta}|^q \\
&\leq C \left( \left| \frac{h}{T} \right|^2 + \left| \frac{h}{T} \right|^q \right) \\
&\leq C \left| \frac{h}{T} \right|^2
\end{aligned}
$$

(64)

where $q \geq 2$, $C = C(q, \|\dot{\eta}\|_{\infty})$, and $h$ is chosen to be small enough so that $\left| \frac{h}{T} \right| \leq 1$. Plugging (64) into (61), we get

$$
\begin{aligned}
u(x+h) + u(x-h) - 2u(x) &\leq C |h|^2 \int_0^T e^{-s} \frac{T}{T^2} ds + C |h|^2 \int_0^T e^{-s} \left( 1 - \frac{s}{T} \right)^2 ds \\
&\leq C |h|^2 \int_0^T e^{-s} ds + C |h|^2 \int_0^T e^{-s} ds \\
&\leq C \left( 1 + \frac{1}{T} \right) |h|^2 \leq C \left( 1 + \frac{1}{d(x)} \right) |h|^2 \leq \frac{C}{d(x)} |h|^2
\end{aligned}
$$

(65)

since $T \geq \frac{d(x)}{C_0}$.

\hfill \square
4.1 Future Work

In the end, we would like to mention some questions that are worth investigating in the future.

– **General $f$.** As is mentioned earlier, one interesting question is to figure out the rate of convergence for the case of general $f$ where $f$ is not equal to its minimum on the boundary.

– **General $H$.** In our proof, an explicit estimate of the asymptotic behavior of the solution $u^\varepsilon$ near the boundary is obtained due to the specific form of Hamiltonian $H(\xi) = |\xi|^p$. We believe that a similar but more technical computation can be done to establish such an estimate of the asymptotic behavior of the solution for Hamiltonian that satisfies

$$
\delta^\frac{p}{p-1} H \left( \delta^{-\frac{1}{p-1}} \xi \right) = |\xi|^p
$$

locally uniformly in $\xi$ as $\delta \to 0$. This condition is mentioned in [29]. For more general Hamiltonian, the question is still open.

– **The case $p > 2$.** In this case, the solution to the second order state-constraint equation is no longer blowing up near the boundary and we do not know any explicit boundary information, which becomes a main difficulty. In fact, loss of boundary data can happen in this case, that is, the Dirichlet boundary problem may not be solvable for any boundary condition in the classical sense [5].

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### Declarations

#### Conflict of interest

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### Appendix

#### A.1 Estimates on Solutions

We present here a proof for the gradient bound of the solution to (PDE$^\varepsilon$) using Bernstein’s method (see also [21, 23]). Another proof using Berstein’s method inside a doubling variable argument is given in [1].

**Proof of Theorem 4** Let $\theta \in (0, 1)$ be chosen later, $\varphi \in C^\infty_c(\Omega), 0 \leq \varphi \leq 1$, supp $\varphi \subset \Omega$ and $\varphi = 1$ on $\Omega_\delta$ such that

$$
|\Delta \varphi(x)| \leq C \varphi^\theta \quad \text{and} \quad |D \varphi(x)|^2 \leq C \varphi^{1+\theta}, \quad \forall x \in \Omega, \quad (66)
$$

where $C = C(\delta, \theta)$ is a constant depending on $\delta, \theta$. 

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Define \( w(x) := |Du^\varepsilon(x)|^2 \) for \( x \in \Omega \). The equation for \( w \) is given by
\[
-\varepsilon \Delta w + 2p |Du^\varepsilon|^{p-2} Du^\varepsilon \cdot Dw + 2w - 2Df \cdot Du^\varepsilon + 2\varepsilon |D^2u^\varepsilon|^2 = 0 \quad \text{in} \quad \Omega.
\]

Then an equation for \((\varphi w)\) can be derived as follows.
\[
-\varepsilon \Delta (\varphi w) + 2p |Du^\varepsilon|^{p-2} Du^\varepsilon \cdot D(\varphi w) + 2(\varphi w) + 2\varepsilon |D^2u^\varepsilon|^2 + 2\varepsilon \frac{D\varphi}{\varphi} \cdot D(\varphi w)
= \varphi (Df \cdot Du^\varepsilon) + 2p |Du^\varepsilon|^{p-2} (Du^\varepsilon \cdot D\varphi)w - \varepsilon w \Delta \varphi + 2\varepsilon \frac{|D\varphi|^2}{\varphi} w \quad \text{in} \quad \text{supp} \ \varphi.
\]

Assume that \( \varphi w \) achieves its maximum over \( \overline{\Omega} \) at \( x_0 \in \Omega \). And we can further assume that \( x_0 \in \text{supp} \ \varphi \), since otherwise the maximum of \( \varphi w \) over \( \Omega \) is zero. By the classical maximum principle,
\[
-\varepsilon \Delta (\varphi w)(x_0) \geq 0 \quad \text{and} \quad |D(\varphi w)(x_0)| = 0.
\]

Use this in the equation of \( \varphi w \) above to obtain
\[
\varepsilon \varphi |D^2u^\varepsilon|^2 \leq \varphi |Df| w^{\frac{1}{2}} + 2Cp w^{\frac{p-1}{2} + \frac{1+\theta}{2}} + C\varepsilon w^{\frac{1}{2} + \theta} + 2C \varepsilon w^\theta.
\]

By Cauchy–Schwartz inequality, \( n \ |D^2u^\varepsilon|^2 \geq (\Delta u^\varepsilon)^2 \). Thus, if \( n\varepsilon < 1 \), then
\[
\varepsilon |D^2u^\varepsilon|^2 \geq \frac{(\varepsilon \Delta u^\varepsilon)^2}{n\varepsilon} \geq (\varepsilon \Delta u^\varepsilon)^2 = (u^\varepsilon + |Du^\varepsilon|^p - f)^2
\]
\[
\geq |Du^\varepsilon|^{2p} - 2C |Du^\varepsilon|^p \geq \frac{|Du^\varepsilon|^{2p}}{2} - 2C,
\]
where \( C \) depends on \( \max_{\overline{\Omega}} f \) only. Using (68) in (67), we obtain that
\[
\varphi \left( \frac{1}{2} w^p - 2C \right) \leq \varphi |Df| w^{\frac{1}{2}} + 2C p w^{\frac{p-1}{2} + \frac{1+\theta}{2}} + 3C \varepsilon w^\theta.
\]

Multiply both sides by \( \varphi^{p-1} \) to deduce that
\[
(\varphi w)^p \leq 4C \varphi^{p-1} + 2 \|Df\|_{L^\infty} \varphi^p w^{\frac{1}{2}} + 4C p \varphi^{\frac{2p-1}{2} + \frac{1+\theta}{2}} w^{\frac{p+1}{2}} + 6C \varepsilon \varphi^p \varphi^{p+\theta-1} w.
\]
Choose $2p + \theta - 1 \geq p + 1$, i.e., $p + \theta \geq 2$. This is always possible with the requirement $\theta \in (0, 1)$, as $1 < p < \infty$. Then we get

$$\tag{69} (\varphi w)^p \leq C \left( 1 + (\varphi w)^{\frac{1}{2}} + (\varphi w)^{\frac{p+1}{2}} + (\varphi w) \right).$$

As a polynomial in $z = (\varphi w)\Phi(x_0)$, this implies that $(\varphi w)\Phi(x_0) \leq C$ where $C$ depends on coefficients of the right hand side of (69), which gives our desired gradient bound since $w(x) = (\varphi w)(x) \leq (\varphi w)(x_0)$ for $x \in \overline{\Omega}_\delta \subset \text{supp } \varphi$. □

A.2 Well-Posedness of (PDE$_\varepsilon$)

Proof of Theorem 5 If $p \in (1, 2)$, we use the ansatz $u(x) = C_{\varepsilon} d(x)^{-\alpha}$ to find a solution to (PDE$_\varepsilon$). Plug the ansatz into (PDE$_\varepsilon$) and compute

$$|Du(x)|^p = \frac{(\alpha C_{\varepsilon})^p}{d(x)^{p(\alpha + 1)}} |Dd(x)|^p,$$

$$\varepsilon \Delta u(x) = \frac{\varepsilon C_{\varepsilon} \alpha (\alpha + 1)}{d(x)^{\alpha + 2}} |Dd(x)|^2 - \frac{\varepsilon C_{\varepsilon} \alpha}{d(x)^{\alpha + 1}} \Delta d(x).$$

Since $|Dd(x)| = 1$ for $x$ near $\partial \Omega$, as $x \to \partial \Omega$, the explosive terms of the highest order are

$$C_{\varepsilon}^p \alpha^p d^{-(\alpha + 1)p} - \varepsilon C_{\varepsilon} \alpha (\alpha + 1) d^{-(\alpha + 2)}.$$

Set the above to be zero to obtain that

$$\alpha = \frac{2 - p}{p - 1} \quad \text{and} \quad C_{\varepsilon} = \left( \frac{1}{\alpha} (\alpha + 1) \right)^{\frac{1}{p-1}} \varepsilon^{\frac{1}{p-1}} = \frac{1}{\alpha} (\alpha + 1)^{\alpha + 1} \varepsilon^{\alpha + 1}. \quad (70)$$

For $0 < \delta < \frac{1}{2} \delta_0$ and $\eta$ small, define

$$\overline{w}_{\eta, \delta}(x) := \frac{(C_{\alpha} + \eta) \varepsilon^{\alpha + 1}}{(d(x) - \delta)^{\alpha}} + M_{\eta}, \quad x \in \Omega_\delta,$$

$$w_{\eta, \delta}(x) := \frac{(C_{\alpha} - \eta) \varepsilon^{\alpha + 1}}{(d(x) + \delta)^{\alpha}} - M_{\eta}, \quad x \in \Omega_\delta,$$

where $C_{\alpha} := \frac{1}{\alpha} (\alpha + 1)^{\alpha + 1}, M_{\eta}$ to be chosen. Next, we show that $\overline{w}_{\eta, \delta}$ is a supersolution of (PDE$_\varepsilon$) in $\Omega_\delta$, while $w_{\eta, \delta}$ is a subsolution of (PDE$_\varepsilon$) in $\Omega_\delta$. Compute

$$L^\varepsilon \left[ \overline{w}_{\eta, \delta} \right] (x) = \frac{(C_{\alpha} + \eta) \varepsilon^{\alpha + 1}}{(d(x) - \delta)^{\alpha}} + M_{\eta} + \frac{(C_{\alpha} - \eta) \alpha^p \varepsilon^{\alpha + 2}}{(d(x) + \delta)^{\alpha + 2}} |Dd(x)|^p - f(x)$$

$$- \frac{(C_{\alpha} + \eta) \alpha (\alpha + 1) \varepsilon^{\alpha + 2}}{(d(x) - \delta)^{\alpha + 2}} |Dd(x)|^2 + \frac{(C_{\alpha} + \eta) \alpha \varepsilon^{\alpha + 2}}{(d(x) + \delta)^{\alpha + 1}} \Delta d(x) \geq M_{\eta} - f(x)$$
where we use \((C_\alpha \alpha)^p = C_\alpha \alpha (\alpha + 1)\) and \(\nu = \frac{C_\alpha - \eta}{C_\alpha} \in (1, 2)\) for small \(\eta\). Let

\[
d_\eta := \frac{\alpha + 1}{K_2} \left[ \nu^{p-1} - 1 \right]
\]

and \(d_\eta \to 0\) as \(\eta \to 0\). To get \(\mathcal{L}_e \left[ \overline{w}_{\eta, \delta} \right] \geq 0\), there are two cases to consider, depending on how large \(d(x) - \delta\) is.

- If \(0 < d(x) - \delta < d_\eta < \delta_0\) for \(\eta\) small and fixed, then \(|Dd(x)| = 1\), and thus \(I \geq 0\). Hence, \(\mathcal{L}_e \left[ \overline{w}_{\eta, \delta} \right] \geq 0\) if we choose \(M_\eta \geq \max_{\Omega} f\).
- If \(d(x) - \delta \geq d_\eta\), then

\[
I \leq \left( \frac{1}{d_\eta} \right)^{\alpha+1} \nu C_\alpha \alpha (\alpha + 1) \left[ \nu^{p-1} \left( K_1^p + K_2^2 + K_2 K_0 \right) e^{\alpha+2} \right].
\]

Thus, we can choose \(M_\eta = \max_{\Omega} f + C e^{\alpha+2}\) for \(C\) large enough (depending on \(\eta\)) so that \(\mathcal{L}_e \left[ \overline{w}_{\eta, \delta} \right] \geq 0\).

Therefore, \(\overline{w}_{\eta, \delta}\) is a supersolution in \(\Omega_\delta\).

Similarly, we have

\[
\mathcal{L}_e \left[ w_{\eta, \delta} \right] (x) = \frac{\nu C_\alpha \alpha (\alpha + 1) e^{\alpha+2}}{(d(x) + \delta)^{\alpha+2}} - \left( \frac{C_\alpha - \eta}{C_\alpha} \right)^p - |Dd(x)|^p - f(x)
\]

where \(\nu = \frac{C_\alpha - \eta}{C_\alpha} \in (0, 1)\) for small \(\eta\). Let

\[
d_\eta := \left( 1 - \nu^{p-1} \right) \left( \frac{\alpha (\alpha + 1) \ve}{1 + K_2 \alpha \ve} \right)
\]

and \(d_\eta \to 0\) as \(\eta \to 0\). To obtain \(\mathcal{L}_e \left[ w_{\eta, \delta} \right] \leq 0\), there are two cases to consider depending on how large \(d(x) + \delta\) is.

- If \(0 < d(x) + \delta < d_\eta < \delta_0\) for \(\eta\) small and fixed, then \(|Dd(x)| = 1\), and thus \(J \leq 0\). Hence, \(\mathcal{L}_e \left[ w_{\eta, \delta} \right] \leq 0\) if we choose \(M_\eta \geq - \max_{\Omega} f\).
If \( d(x) + \delta \geq \delta \eta \), then

\[
|J| \leq \left( \frac{1}{\delta \eta} \right)^{\alpha + 2} \nu C_\alpha \alpha (\alpha + 1) \left[ \nu^{\alpha - 1} K_1^p + K_1^2 + \frac{(K_0 + 1)K_2}{\alpha + 1} + \frac{(K_0 + 1)^2}{\alpha(\alpha + 1)\epsilon} \right] \epsilon^{\alpha + 2}
\]

Thus, we can choose \( M_\eta = -\max_\Omega f - C \epsilon^{\alpha + 2} \) for \( C \) large enough (depending on \( \eta \)) so that \( L^\epsilon [w_{\eta, \delta}] \leq 0 \).

Therefore, \( w_{\eta, \delta} \) is a subsolution in \( \Omega^\delta \).

For \( p = 2 \), we use the ansatz \( u(x) = -C \epsilon \log(d(x)) \) instead. Similar to the previous case, one can find \( u(x) = -\epsilon \log(d(x)) \). For \( 0 < \delta < \frac{1}{2} \delta_0 \), define

\[
\bar{w}_{\eta, \delta}(x) = -(1 + \eta)\epsilon \log(d(x) - \delta) + M_\eta, \quad x \in \Omega_\delta,
\]

\[
w_{\eta, \delta}(x) = -(1 - \eta)\epsilon \log(d(x) + \delta) - M_\eta, \quad x \in \Omega_\delta,
\]

where \( M_\eta \) is to be chosen so that \( \bar{w}_{\eta, \delta}(x) \) is a supersolution in \( \Omega_\delta \) and \( w_{\eta, \delta} \) is a subsolution in \( \Omega_\delta \). The computations are omitted here, as they are similar to the previous case.

We divide the rest of the proof into 3 steps. We first construct a minimal solution, then a maximal solution to \( \text{PDE}_\epsilon \), and finally show that they are equal to conclude the existence and the uniqueness of the solution to \( \text{PDE}_\epsilon \).

\textbf{Step 1} There exists a minimal solution \( u \in C^2(\Omega) \) of \( \text{PDE}_\epsilon \) such that \( v \geq u \) for any other solution \( v \in C^2(\Omega) \) solving \( \text{PDE}_\epsilon \).

\textbf{Proof} Let \( w_{\eta, \delta} \in C^2(\Omega) \) solve

\[
\begin{align*}
L^\epsilon [w_{\eta, \delta}] &= 0 \quad \text{in } \Omega, \\
w_{\eta, \delta} &= w_{\eta, \delta} \quad \text{on } \partial \Omega.
\end{align*}
\]

– Fix \( \eta > 0 \). As \( \delta \to 0^+ \), the value of \( w_{\eta, \delta} \) blows up on the boundary. Therefore, by the standard comparison principle for the second-order elliptic equation with the Dirichlet boundary, \( \delta_1 \leq \delta_2 \) implies \( w_{\eta, \delta_1} \geq w_{\eta, \delta_2} \) on \( \overline{\Omega} \).

– For \( \delta' > 0 \), since \( w_{\eta, \delta'} \) is a subsolution in \( \overline{\Omega} \) with finite boundary,

\[
0 < \delta \leq \delta' \implies w_{\eta, \delta'} \leq w_{\eta, \delta} \quad \text{on } \overline{\Omega}.
\]

– Similarly, since \( \bar{w}_{\eta, \delta'} \) is a supersolution on \( \Omega_{\delta'} \) with infinity value on the boundary \( \partial \Omega_{\delta'} \), by the comparison principle,

\[
w_{\eta, \delta} \leq \bar{w}_{\eta, \delta'} \quad \text{in } \Omega_{\delta'} \implies w_{\eta, \delta} \leq \bar{w}_{\eta, 0} \quad \text{in } \Omega.
\]

From (72) and (73), we have

\[
0 < \delta \leq \delta' \implies w_{\eta, \delta'} \leq w_{\eta, \delta} \leq w_{\eta, \delta} \leq \bar{w}_{\eta, 0} \quad \text{in } \Omega.
\]
Thus, $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $L^\infty_{\text{loc}}(\Omega)$ ($\{w_{\eta,\delta}\}_{\delta>0}$ is uniformly bounded from below). Using the local gradient estimate for $w_{\eta,\delta}$ solving (71), we deduce that $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $W^{1,\infty}_{\text{loc}}(\Omega)$. Since $w_{\eta,\delta}$ solves (71), we further have that $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $W^{2,r}_{\text{loc}}(\Omega)$ for all $r < \infty$ by Calderon–Zygmund estimates.

Local boundedness of $\{w_{\eta,\delta}\}_{\delta>0}$ in $W^{2,r}_{\text{loc}}(\Omega)$ implies weak* compactness, that is, there exists a function $u \in W^{2,r}_{\text{loc}}(\Omega)$ such that (via subsequence and monotonicity)

$$w_{\eta,\delta} \rightharpoonup u$$ weakly in $W^{2,r}_{\text{loc}}(\Omega)$, and

$$w_{\eta,\delta} \to u$$ strongly in $W^{1,r}_{\text{loc}}(\Omega)$.

In particular, $w_{\eta,\delta} \to u$ in $C^1(\Omega)$ as $\delta \to 0$, we have $F[w_{\eta,\delta}](x) \to F(x)$ uniformly in $U$ as $\delta \to 0$, where

$$F[w_{\eta,\delta}](x) = w_{\eta,\delta}(x) + H(x, Dw_{\eta,\delta}(x)).$$

Since $w_{\eta,\delta} \to u$ in $C^1(U)$ as $\delta \to 0$, we have $F[w_{\eta,\delta}](x) \to F(x)$ uniformly in $U$ as $\delta \to 0$, where

$$F(x) = u(x) + H(x, Du(x)).$$

In the limit, we obtain that $u \in L^2(U)$ is a weak solution of $\varepsilon \Delta u = F$ in $U$ where $F$ is continuous. Thus, $u \in C^2(\Omega)$ and by stability, $u$ solves $\mathcal{L}^\varepsilon[u] = 0$ in $\Omega$. From (74), we also have

$$w_{\eta,0} \leq u \leq \bar{w}_{\eta,0} \quad \text{in} \ \Omega.$$ 

Moreover, $u(x) \to \infty$ as $\text{dist}(x, \partial \Omega) \to 0$ with the precise rate like (9) or (10). Note that by construction, $u$ may depend on $\eta$. But next, we will show that $u$ is independent of $\eta$, by proving $u$ is the unique minimal solution of $\mathcal{L}^\varepsilon[u] = 0$ in $\Omega$ with $u = +\infty$ on $\partial \Omega$.

Let $v \in C^2(\Omega)$ be a solution to (PDE). Fix $\delta > 0$. Since $v(x) \to \infty$ as $x \to \partial \Omega$ while $w_{\eta,\delta}$ remains bounded on $\partial \Omega$, the comparison principle yields

$$v \geq w_{\eta,\delta} \quad \text{in} \ \Omega.$$ 

Let $\delta \to 0$ and we deduce that $v \geq u$ in $\Omega$. This concludes that $u$ is the minimal solution in $C^2(\Omega)(\forall r < \infty)$ and thus $u$ is independent of $\eta$. □

**Step 2** There exists a maximal solution $\overline{u} \in C^2(\Omega)$ of (PDE) such that $v \leq \overline{u}$ for any other solution $v \in C^2(\Omega)$ solving (PDE).

**Proof** For each $\delta > 0$, let $u_\delta \in C^2(\Omega_\delta)$ be the minimal solution to $\mathcal{L}^\varepsilon[u_\delta] = 0$ in $\Omega_\delta$ with $u_\delta = +\infty$ on $\partial \Omega_\delta$. By the comparison principle, for every $\eta > 0$, there holds

$$w_{\eta,\delta} \leq u_\delta \leq \bar{w}_{\eta,\delta} \quad \text{in} \ \Omega_\delta,$$
and
\[ 0 < \delta < \delta' \quad \implies \quad u_\delta \leq u_{\delta'} \quad \text{in} \quad \Omega_{\delta'}. \]

The monotonicity, together with the local boundedness of \( \{u_\delta\}_{\delta > 0} \) in \( W^{2,r}_{loc}(\Omega) \), implies that there exists \( u \in W^{2,r}_{loc}(\Omega) \) for all \( r < \infty \) such that \( u_\delta \to u \) strongly in \( C^1_{loc}(\Omega) \).

Using the equation \( L_\varepsilon[u_\delta] = 0 \) in \( \Omega_{\delta'} \) and the regularity of Laplace’s equation, we can further deduce that \( u \in C^2(\Omega) \) solves (PDE\( _\varepsilon \)) and
\[ w_{\eta,0} \leq u \leq w_{\eta,0} \quad \text{in} \quad \Omega \]
for all \( \eta > 0 \). As \( u_\delta \) is independent of \( \eta \) by the previous argument in Step 1, it is clear that \( u \) is also independent of \( \eta \). Now we show that \( u \) is the maximal solution of (PDE\( _\varepsilon \)). Let \( v \in C^2(\Omega) \) solve (PDE\( _\varepsilon \)). Clearly \( v \leq u_\delta \) on \( \Omega_\delta \). Therefore, as \( \delta \to 0 \), we have \( v \leq u \).

In conclusion, we have found a minimal solution \( \underline{u} \) and a maximal solution \( \overline{u} \) in \( C^2(\Omega) \) such that
\[ w_{\eta,0} \leq u \leq \overline{u} \leq w_{\eta,0} \quad \text{in} \quad \Omega \quad (75) \]
for any \( \eta > 0 \). This extra parameter \( \eta \) now enables us to show that \( \overline{u} = u \) in \( \Omega \). The key ingredient here is the convexity in the gradient slot of the operator.

**Step 3** We have \( \overline{u} \equiv u \) in \( \Omega \). Therefore, the solution to (PDE\( _\varepsilon \)) in \( C^2(\Omega) \) is unique.

**Proof** Let \( \theta \in (0, 1) \). Define \( w_\theta = \theta \overline{u} + (1 - \theta) \inf_{\Omega} f \). It can be verified that \( w_\theta \) is a subsolution to (PDE\( _\varepsilon \)). Then one may argue that by the comparison principle,
\[ w_\theta = \theta \overline{u} + (1 - \theta) \inf_{\Omega} f \leq u \quad \text{in} \quad \Omega, \]
and conclude that \( \overline{u} \leq u \) by letting \( \theta \to 1 \). But we have to be careful here. As they are both explosive solutions, to use the comparison principle, we need to show that \( w_\theta \leq u \) in a neighborhood of \( \partial \Omega \). From (75), we see that
\[
1 \leq \frac{\overline{u}(x)}{u(x)} \leq \frac{w_{\eta,0}(x)}{w_{\eta,0}(x)} = \frac{(C_\alpha + \eta) + M_\eta d(x)^\alpha}{(C_\alpha - \eta) - M_\eta d(x)^\alpha}, \quad 1 < p < 2,
\]
\[
1 \leq \frac{\overline{u}(x)}{u(x)} \leq \frac{w_{\eta,0}(x)}{w_{\eta,0}(x)} = \frac{-(1 + \eta) \log(d(x)) + M_\eta}{-(1 - \eta) \log(d(x)) - M_\eta}, \quad p = 2,
\]
for \( x \in \Omega \). Hence,
\[
1 \leq \lim_{d(x) \to 0} \frac{\overline{u}(x)}{u(x)} \leq \frac{C_\alpha + \eta}{C_\alpha - \eta}, \quad 1 < p < 2,
\]
\[
1 \leq \lim_{d(x) \to 0} \frac{\overline{u}(x)}{u(x)} \leq \frac{-(1 + \eta)}{-(1 - \eta)}, \quad p = 2.
\]
Since $\eta > 0$ is chosen arbitrary, we obtain

$$
\lim_{d(x) \to 0} \left( \frac{\overline{u}(x)}{u(x)} \right) = 1.
$$

This means for any $\zeta \in (0, 1)$, there exists $\delta_1(\zeta) > 0$ small such that

$$
\frac{\overline{u}(x)}{u(x)} \leq (1 + \zeta) \implies \left( \frac{1}{1 + \zeta} \right) \overline{u}(x) \leq u(x) \quad \text{in } \Omega \setminus \Omega_{\delta_1}.
$$

For a fixed $\theta \in (0, 1)$, one can always choose $\zeta$ small enough so that $(1 + \zeta)^{-1} \geq \frac{1 + \theta}{2}$. Since $\overline{u}(x) \to +\infty$ as $d(x) \to 0$, there exists $\delta_2 > 0$ such that $\overline{u}(x) \geq 2 \inf_{\Omega} f$ for all $x \in \Omega \setminus \Omega_{\delta_2}$. Now we have

$$
u(x) \geq \left( \frac{1}{1 + \zeta} \right) \overline{u}(x) \geq \theta \overline{u}(x) + \left( \frac{1 - \theta}{2} \right) \overline{u}(x) \geq \theta \overline{u}(x) + (1 - \theta) \left( \inf_{\Omega} f \right)
$$

for all $x \in \Omega \setminus \Omega_{\delta}$ where $\delta := \min[\delta_1, \delta_2]$. This implies for any fixed $\theta \in (0, 1)$, $w_{\theta} \leq u$ in a neighborhood of $\partial \Omega$. Hence, by the comparison principle,

$$w_{\theta} = \theta \overline{u} + (1 - \theta) \inf_{\Omega} f \leq u \quad \text{in } \Omega,$$

for any $\theta \in (0, 1)$. Then let $\theta \to 1$ to get the conclusion.

This finishes the proof of the well-posedness of (PDE)$_p$ for $1 < p \leq 2$. \hfill \qed

**Proof of Lemma 6** The proof is a variation of Perron’s method (see [10]) and we proceed by contradiction. Let $\varphi \in C(\overline{\Omega})$ and $x_0 \in \overline{\Omega}$ such that $u(x_0) = \varphi(x_0)$ and $u - \varphi$ has a global strict minimum over $\overline{\Omega}$ at $x_0$ with

$$\varphi(x_0) + H(x_0, D\varphi(x_0)) < 0. \quad (76)$$

Let $\varphi^\varepsilon(x) = \varphi(x) - |x - x_0|^2 + \varepsilon$ for $x \in \overline{\Omega}$. Let $\delta > 0$. We see that for $x \in \partial B(x_0, \delta) \cap \overline{\Omega}$,

$$\varphi^\varepsilon(x) = \varphi(x) - \delta^2 + \varepsilon \leq \varphi(x) - \varepsilon$$

if $2\varepsilon \leq \delta^2$. We observe that

$$\varphi^\varepsilon(x) - \varphi(x_0) = \varphi(x) - \varphi(x_0) + \varepsilon - |x - x_0|^2$$

$$D\varphi^\varepsilon(x) - D\varphi(x_0) = D\varphi(x) - D\varphi(x_0) - 2(x - x_0)$$

for $x \in B(x_0, \delta) \cap \overline{\Omega}$. By the continuity of $H(x, p)$ near $(x_0, D\varphi(x_0))$ and the fact that $\varphi \in C^1(\overline{\Omega})$, we can deduce from (76) that if $\delta$ is small enough and $0 < 2\varepsilon < \delta^2$,
then
\[ \varphi^e(x) + H(x, D\varphi^e(x)) < 0 \quad \text{for } x \in B(x_0, \delta) \cap \overline{\Omega}. \quad (77) \]

We have found \( \varphi^e \in \mathcal{C}^1(\overline{\Omega}) \) such that \( \varphi^e(x_0) > u(x_0) \), \( \varphi^e < u \) on \( \partial B(x_0, \delta) \cap \overline{\Omega} \) and (77). Let

\[ \tilde{u}(x) = \begin{cases} 
\max \{ u(x), \varphi^e(x) \} & x \in B(x_0, \delta) \cap \overline{\Omega}, \\
\max \{ u(x) \} & x \notin B(x_0, \delta) \cap \overline{\Omega}.
\end{cases} \]

We see that \( \tilde{u} \in \mathcal{C}(\overline{\Omega}) \) is a subsolution of (PDE) in \( \Omega \) with \( \tilde{u}(x_0) > u(x_0) \), which is a contradiction. Thus, \( u \) is a supersolution of (PDE) on \( \overline{\Omega} \).

\[ \square \]

A.3 Semiconcavity

We present a proof for the semiconcavity of solution to first-order Hamilton–Jacobi equation using the doubling variable method (see also [8]).

**Theorem 19** Let \( H(x, p) = G(p) - f(x) \) where \( G \geq 0 \) with \( G(0) = 0 \) is a convex function from \( \mathbb{R}^n \to \mathbb{R}^n \) and \( f \in \mathcal{C}^2_c(\mathbb{R}^n) \). Let \( u \in \mathcal{C}_c(\mathbb{R}^n) \) be a viscosity solution to \( u + H(x, Du) = 0 \) in \( \mathbb{R}^n \). Then \( u \) is semiconcave, i.e., \( u \) is a viscosity solution of

\[ -D^2u \geq -c \mathbb{I}_n \]

in \( \mathbb{R}^n \) where

\[ c = \max \{ D_{\xi\xi} f(x) : |\xi| = 1, x \in \mathbb{R}^n \} \geq 0. \]

**Proof** Consider the auxiliary functional

\[ \Phi(x, y, z) = u(x) - 2u(y) + u(z) - \frac{\alpha}{2} |x - 2y + z|^2 - \frac{c}{2} |y - x|^2 - \frac{c}{2} |y - z|^2 \]

for \((x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\). By the a priori estimate, \( u \) is bounded and Lipschitz. Thus, we can assume \( \Phi \) achieves its maximum over \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) at \((x_\alpha, y_\alpha, z_\alpha)\). The viscosity solution tests give us

\[ u(x_\alpha) + G\left(p_\alpha + c(x_\alpha - y_\alpha)\right) \leq f(x_\alpha) \]
\[ u(z_\alpha) + G\left(p_\alpha + c(z_\alpha - y_\alpha)\right) \leq f(z_\alpha) \]
\[ u(y_\alpha) + G\left(p_\alpha + \frac{c}{2}(x_\alpha - y_\alpha) + \frac{c}{2}(z_\alpha - y_\alpha)\right) \geq f(y_\alpha), \]

where \( p_\alpha = \alpha(x_\alpha - 2y_\alpha + z_\alpha) \). By the convexity of \( G \), we have

\[ 2G\left(p_\alpha + \frac{c}{2}(x_\alpha - y_\alpha) + \frac{c}{2}(z_\alpha - y_\alpha)\right) \leq G(p_\alpha + c(x_\alpha - y_\alpha)) + G(p_\alpha + c(z_\alpha - y_\alpha)) \]

Therefore,

\[ u(x_\alpha) - 2u(y_\alpha) + u(z_\alpha) \leq f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha). \]
\( \Phi(x_\alpha, y_\alpha, z_\alpha) \geq \Phi(0, 0, 0) \) gives us
\[
\frac{\alpha}{2} |x_\alpha - 2y_\alpha + z_\alpha|^2 + \frac{c}{2} |y_\alpha - x_\alpha|^2 + \frac{c}{2} |y_\alpha - z_\alpha|^2 \leq C.
\]

Thus, \((x_\alpha - y_\alpha) \to h_0 \) and \((y_\alpha - z_\alpha) \to h_0 \) as \( \alpha \to \infty \) for some \( h_0 \in \mathbb{R}^n \).

\(- \Phi(x_\alpha, y_\alpha, z_\alpha) \geq \Phi(y_\alpha + h_0, y_\alpha, y_\alpha - h_0) \) gives us
\[
u(x_\alpha) - 2u(y_\alpha) + u(z_\alpha) - \frac{\alpha}{2} |x_\alpha - 2y_\alpha + z_\alpha|^2 - \frac{c}{2} |x_\alpha - y_\alpha|^2 - \frac{c}{2} |y_\alpha - z_\alpha|^2 \\
\geq u(y_\alpha + h_0) - 2u(y_\alpha) + u(y_\alpha - h_0) - c|h_0|^2.
\]

Therefore, by the fact that \( u \) is Lipschitz, we have
\[
\frac{\alpha}{2} |x_\alpha - 2y_\alpha + z_\alpha|^2 \leq c \left( \frac{2|h_0|^2 - |x_\alpha - y_\alpha|^2 - |y_\alpha - z_\alpha|^2}{2} \right) \\
+ C \left( |(x_\alpha - y_\alpha) + h_0| + |(z_\alpha - y_\alpha) + h_0| \right) \to 0
\]
as \( \alpha \to \infty \).

For any \( x \in \mathbb{R}^n \), we have \( \Phi(x_\alpha, y_\alpha, z_\alpha) \geq \Phi(x + h, x, x - h) \), i.e.,
\[
u(x + h) - 2u(x) + u(x - h) - c|h|^2 \\
\leq f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha) \\
- \frac{\alpha}{2} |x_\alpha - 2y_\alpha + z_\alpha|^2 - \frac{c}{2} |y_\alpha - x_\alpha|^2 - \frac{c}{2} |y_\alpha - z_\alpha|^2.
\]

If \( \{y_\alpha\} \) is unbounded, then since \( f \in C^2(\mathbb{R}^n) \), we have \( f(y_\alpha) \to 0 \) as \( \alpha \to \infty \). As a consequence, \( x_\alpha, z_\alpha \to \infty \) as well and thus \( f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha) \to 0 \) as \( \alpha \to \infty \). Therefore,
\[
u(x + h) - 2u(x) + u(x - h) - c|h|^2 \leq 0.
\]

If \( \{y_\alpha\} \) is bounded, then \( y_\alpha \to y_0 \) for some \( y_0 \in \mathbb{R}^n \) as \( \alpha \to \infty \). Thus,
\[
u(x + h) - 2u(x) + u(x - h) - c|h|^2 \\
\leq f(y_0 + h_0) - 2f(y_0) + f(y_0 - h_0) - c|h_0|^2.
\]

Let \( \xi = h_0 \) and we have
\[
\begin{align*}
\left\{ f(y_0 + h_0) - f(y_0) = \int_0^1 D_x f(y_0 + t\xi) \cdot \xi dt, \\
f(y_0) - f(y_0 - h_0) = \int_0^1 D_x f(y_0 - \xi + t\xi) \cdot \xi dt.
\end{align*}
\]
Therefore,
\[
f(y_0 + h_0) - 2f(y_0) + f(y_0 - h_0) = \int_0^1 \left( D_x f(y_0 + t\xi) - D_x f(y_0 - \xi + t\xi) \right) \cdot \xi \, dt
\]
\[
= \int_0^1 \int_0^1 \xi^T D^2 f(y_0 - \xi + t\xi + s\xi) \xi \, dsdt.
\]
which implies
\[
|f(y_0 + h_0) - 2f(y_0) + f(y_0 - h_0)| \leq \left( \max_{|\xi| = 1} D_{\xi\xi} f \right) |\xi|^2.
\]
Hence,
\[
u(x + h) - 2\nu(x) + \nu(x - h) - c|h|^2 \leq 0
\]
and thus \( \nu \) is semiconcave. It is easy to see that if \( \varphi \) is smooth and \( \nu - \varphi \) has a local min at \( x \), then \( D^2 \varphi(x) \leq c I \), i.e., \( -D^2 \varphi(x) \geq -c I \).

\[\square\]

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