TRANSCENDENCE MEASURES FOR SOME $U_m$-NUMBERS RELATED TO LIOUVILLE CONSTANT

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Abstract. In this note, we shall prove that the sum and the product of an algebraic number $\alpha$ by the Liouville constant $L = \sum_{j=1}^{\infty} 10^{-j^2}$ is a $U$-number with type equals to the degree of $\alpha$ (with respect to $\mathbb{Q}$). Moreover, we shall have that

$$\max\{w_m^*(\alpha L), w_m^*(\alpha + L)\} \leq 2m^2n + m - 1, \text{ for } n = 1, \ldots, m - 1.$$

1. Introduction

A real number $\xi$ is called a Liouville number, if for any positive real number $w$ there exist infinitely many rational numbers $p/q$, with $q \geq 1$, such that

$$0 < |\xi - \frac{p}{q}| < \frac{1}{q^w}.$$ Transcendental number theory began in 1844 when Liouville [6] showed that all Liouville numbers are transcendental establishing thus the first examples of such numbers. For instance, the number

$$L = \sum_{j=1}^{\infty} 10^{-j^2} = 0.11000100000000000000001000\ldots,$$

which is known as Liouville’s constant, is a Liouville number and therefore transcendental. In 1962, Erdős [3] proved that every nonzero real number can be written as the sum and the product of two Liouville numbers.

In 1932, Mahler [7] split the set of the transcendental numbers in three disjoint sets named $S$-, $T$- and $U$-numbers. Particularly, the $U$-numbers generalizes the concept of Liouville numbers. We denote by $w_m^*(\xi)$ as the supremum of the real numbers $w^*$ for which there exist infinitely many real algebraic numbers $\alpha$ of degree $n$ satisfying

$$0 < |\xi - \alpha| < H(\alpha)^{-w^*-1},$$

where $H(\alpha)$ (so-called the height of $\alpha$) is the maximum of absolute value of coefficients of the minimal polynomial of $\alpha$ (over $\mathbb{Z}$). The number $\xi$ is said to be a $U_m^*$-number (according to LeVeque [4]) if $w_m^*(\xi) = \infty$ and $w_m^*(\xi) < \infty$ for $1 \leq n < m$ ($m$ is called the type of the $U$-number). We point out that we actually have defined a Koksma $U_m^*$-number instead of a Mahler $U_m$-number. However, it is well-known that they are the same [2] cf. Theorem 3.6] and [1]. We observe that the set of $U_1$-numbers is precisely the set of Liouville numbers.

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The existence of $U_m$-numbers, for all $m \geq 1$, was proved by LeVeque \cite{LeVeque}. In 1993, Pollington \cite{Pollington} showed that for any positive integer $m$, every real number can be expressed as a sum of two $U_m$-numbers.

Since two algebraically dependent numbers must belong to the same Mahler’s class \cite[Theorem 3.2]{Gutting}, then $\alpha L$ and $\alpha + L$ are $U$-numbers, for any nonzero algebraic number $\alpha$. But what are their types?

In this note, we use Gutting’s method \cite{Gutting} for proving that the sum and the product of every $m$-degree algebraic number $\alpha$ by $L$ is a $U_m$-number. Moreover, we obtain an upper bound for $w^*_n$.

**Theorem 1.** Let $\alpha$ be an algebraic number of degree $m$ and let $L$ be the Liouville’s constant. Then $\alpha L$ and $\alpha + L$ are $U_m$-numbers, with

\[
\max\{w^*_n(\alpha L), w^*_n(\alpha + L)\} \leq 2m^2n + m - 1, \text{ for } n = 1, \ldots, m - 1.
\]

2. Auxiliary Results

Before the proof of the main result, we need two technical results. The first one follows as an easy consequence of the triangular inequality and binomial identities.

**Lemma 1.** Given $P(x) \in \mathbb{Z}[x]$ with degree $m$ and $a/b \in \mathbb{Q}\{0\}$. If $Q_1(x) = a^m P(\frac{b}{a} x)$ and $Q_2(x) = b^m P(x - \frac{b}{a})$, then

(i) $H(Q_1) \leq \max\{|a|, |b|\}^m H(P)$;

(ii) $H(Q_2) \leq 2^{m+1}\max\{|a|, |b|\}^m H(P)$.

**Proof.** (i) If $P(x) = \sum_{j=0}^{m} a_j x^j$, then $Q_1(x) = \sum_{j=0}^{m} b_j a^{m-j} x^j$. Supposing, without loss of generality, that $|a| \geq |b|$, we have $|a|^m |a_j| \geq |a|^{m-j} |a_j| |b|^j$ for $0 \leq j \leq m$. Hence, we are done. For (ii), write $Q_2(x) = \sum_{i=0}^{m} c_i x^i$, where

\[
c_i = b^m \sum_{j=i}^{m} a_j \left(\begin{array}{c} j \\ i \end{array}\right) (-1)^{j-i} \left(\frac{a}{b}\right)^{j-i}
\]

Therefore

\[
|c_i| \leq H(P) \sum_{k=0}^{m-i} \left(\begin{array}{c} k+i \\ k \end{array}\right) |a|^k |b|^{m-k} \leq \max\{|a|, |b|\}^m H(P) \sum_{k=0}^{m-i} \left(\begin{array}{c} k+i \\ k \end{array}\right).
\]

Since $\sum_{k=0}^{m-i} \left(\begin{array}{c} k+i \\ k \end{array}\right) = \left(\begin{array}{c} m+1 \\ m-i \end{array}\right) \leq 2^{m+1}$, we finally have

\[
|c_i| \leq 2^{m+1}\max\{|a|, |b|\}^m H(P),
\]

which completes our proof. \hfill \Box

In addition to Lemma 1, we use the fact that algebraic numbers are not well approximable by algebraic numbers.

**Lemma 2.** (Cf. Corollary A.2 of \cite{Gutting}). Let $\alpha$ and $\beta$ be two distinct nonzero algebraic numbers of degree $n$ and $m$, respectively. Then we have

\[
|\alpha - \beta| \geq (n+1)^{-m/2} (m+1)^{-n/2} \max\{2^{-n}(n+1)^{-(m-1)/2}, 2^{-m}(m+1)^{-(n-1)/2}\} \times H(\alpha)^{-m} H(\beta)^{-n}.
\]

**Proof.** A sketch of the proof can be found in the Appendix A of \cite{Gutting}. \hfill \Box
3. Proof of the Theorem

For \( k \geq 1 \), set
\[
p_k = 10^k \sum_{j=1}^{k} 10^{-j}, \quad q_k = 10^k \quad \text{and} \quad \alpha_k = \frac{p_k}{q_k}.
\]
We observe that \( H(\alpha_{k+1}) < H(\alpha_k) = 10^k = H(\alpha_{k-1})^k \) and
(3.1) \[ |L - \alpha_k| > \frac{10}{9} H(\alpha_k)^{k-1}. \]
Thus, setting \( \gamma_k = \alpha \alpha_k \), we obtain of (3.1)
(3.2) \[ |\alpha L - \alpha \alpha_k| \leq c H(\alpha_k)^{-k-1}, \]
where \( c = 10^{10} |\alpha| / 9 \). It follows by the Lemma 1 (i) that \( H(\alpha_k)^m \geq H(\alpha)^{-1} H(\gamma_k) \)
and thus we conclude that
(3.3) \[ |\alpha L - \alpha \alpha_k| \leq c H(\alpha)^{k+1} H(\gamma_k)^{-k-1}. \]
Consequently, \( \alpha \beta \) is a \( U \)-number with type at most \( m \) (since \( \gamma_k \) has degree \( m \)).

Again, we use Lemma 1 (i) for obtaining
(3.4) \[ H(\gamma_{k+1}) \leq H(\alpha) H(\alpha_{k+1})^m = H(\alpha) H(\alpha_k)^{(k+1)m} \leq H(\alpha) H(\gamma_k)^{(k+1)m}. \]

Now, let \( \gamma \) be an \( n \)-degree real algebraic number, with \( n < m \) and \( H(\gamma) > H(\gamma_1) \). Thus, one may ensure the existence of a sufficient large \( k \) such that
(3.5) \[ H(\gamma_k) < H(\gamma)^{2m^2} < H(\gamma_{k+1}) \leq H(\alpha) H(\gamma_k)^{(k+1)m}. \]
So, by Lemma 2 it follows that
(3.6) \[ |\gamma_k - \gamma| \geq f(m, n) H(\gamma)^{-m} H(\gamma_k)^{-n}, \]
where \( f(m, n) \) is a positive number which does not depend on \( k \) and \( \gamma \). Therefore by (3.5)
(3.7) \[ |\gamma_k - \gamma| \geq f(m, n) H(\alpha)^{-1/2m} H(\gamma_k)^{-(k+1)/2-n}. \]

By taking \( H(\gamma) \) large enough, the index \( k \) satisfies
(3.8) \[ H(\gamma_k)^{(k+1)/2-n} \geq 2 c f(m, n)^{-1} H(\alpha)^{(k+1)/2m}. \]
Thus, it follows from (3.6), (3.7) and (3.8) that \( |\gamma_k - \gamma| \geq 2|\alpha L - \gamma_k| \). Therefore, except for finitely many algebraic numbers \( \gamma \) of degree \( n \) strictly less than \( m \), we have
\[
|\alpha L - \gamma| \geq |\gamma_k - \gamma| - |\alpha L - \gamma_k| \\
\geq \frac{1}{2} |\gamma_k - \gamma| \\
\geq \frac{f(m, n)}{2} H(\gamma)^{-m} H(\gamma_k)^{-n} > \frac{f(m, n)}{2} H(\gamma)^{-2m^2 n - m},
\]
where we use the left-hand side of (3.5). It follows that \( w^*_n(\alpha L) \leq 2m^2 n + m - 1 \) which finishes our proof.

The case \( \alpha + L \) follows the same outline, where we use Lemma 1 (ii) rather than (i). □
4. THE GENERAL CASE AND FURTHER COMMENTS

Let $\beta$ be a Liouville number. Since that a $U$-number keeps its type when multiplied by any nonzero rational number, we can consider $0 < \beta < 1$. Set

$$S_\beta = \{ (p_k/q_k)_{k \geq 1} \in \mathbb{Q}^\infty : |\beta - p_k/q_k| < \frac{1}{q_k}, \ k = 1, 2, \ldots \}.$$ 

By the assumption on $\beta$, we may suppose $1 \leq p_k \leq q_k$ and then $H(p_k/q_k) = q_k$, for all $k$. Note that $S_\beta$ is an infinite set.

As is customary, the symbols $\ll, \gg$ mean that there is an implied constant in the inequalities $\leq, \geq$, respectively. In our process for proving the Theorem 1, the key step happens when holds an inequality like in (3.5). Thus it follows that

**Theorem 2.** Let $\alpha$ be an $m$-degree algebraic number and let $\beta$ be a Liouville number. If there exists a sequence $(p_k/q_k)_{k \geq 1} \in S_\beta$ such that $q_k \ll q_{k+1} \ll q_k^{k+1}$ for all $k \gg 1$, then the numbers $\alpha \beta$ and $\alpha + \beta$ are $U_m$-numbers and

$$\max\{w_n^*(\alpha \beta), w_n^*(\alpha + \beta)\} \leq 2m^2n + m - 1,$$

for $n = 1, \ldots, m - 1$.

**Example 1.** For any integer number $m \geq 2$ and any $a_j \in \{1, \ldots, 9\}$, the number $\sum_{j=1}^{\infty} a_j 10^{-j!}$ is a Liouville number satisfying the hypothesis of the previous theorem.

**Corollary 1.** For any $m \geq 1$, there exists an uncountable collection of Liouville numbers that are expressible as sum of two algebraically dependent $U_m$-numbers.

**Proof.** Set $\beta = \sum_{j=1}^{\infty} a_j 10^{-j!}$, where $a_j \in \{1, 2\}$. The result follows immediately of Theorem 2 and of writing $\beta = (\frac{\beta + \sqrt{2}}{2}) + (\frac{\beta - \sqrt{2}}{2})$. \hfill $\Box$

There exist several lower estimates for the distance between two distinct algebraic numbers, e.g., Liouville’s inequality and Lemma 2. A too-good-to-be-true Conjecture due to Schmidt [9] states that

**Conjecture 1.** For any number field $K$ and any positive real number $\epsilon$, we have

$$|\alpha - \beta| > c(K, \epsilon)(\max\{H(\alpha), H(\beta)\})^{-2-\epsilon},$$

for any distinct $\alpha, \beta \in K$, where $c(K, \epsilon)$ is some constant depending only on $K$ and on $\epsilon$.

We conclude by pointing that if the Schmidt’s conjecture is true, then the sum and the product of any $m$-degree algebraic number $\alpha$ by any Liouville number $\beta$ is a $U_m$-number and the inequality (4.1) can be considerable improved for

$$\max\{w_n^*(\alpha \beta), w_n^*(\alpha + \beta)\} \leq 1.$$

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