GENERALIZED PARTON DISTRIBUTION FUNCTIONS VIA QUANTUM SIMULATION OF QUANTUM FIELD THEORY IN LIGHT-FRONT COORDINATES.

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ABSTRACT
Quantum simulation of quantum field theories offers a new way to investigate properties of the fundamental constituents of matter. We develop quantum simulation algorithms based on the light-front formulation of relativistic field theories. The process of quantizing the system in light-cone coordinates will be explained for Hamiltonian formulation, which becomes block diagonal, each block approximating the Fock space with a certain harmonic resolution $K$. We analyze Yukawa theories in $1+1D$ and $2+1D$. We compute the analogues of parton distribution functions and generalized parton distribution functions for mesonic composite particles – like hadrons - in these theories. The dependence of such analyses on the scaling of the number of qubits is compared with other schemes and conventional computations. There is a notable advantage to the light-front formulation.

1 Formalism

1.1 The Hadronic Model in $2 + 1D$

A fermionic field $\psi$, interacting with a bosonic field, $\phi$, can be described by the Yukawa model Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m_B^2 \phi^2 + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu \psi - (m_F + \lambda \phi) \bar{\psi} \psi$$

(1.1)

where $m_B, m_F$ are the bare bosonic and fermionic masses (for single flavor particles), respectively. $\lambda$ describes the coupling between the fields. Light-front (LF) coordinates describe the motion of a (massless) particle moving at the speed of light on the light-cone and will be utilized throughout this paper. Given equal time coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, the coordinate transformation to LF coordinates is:

$$x^\pm = x^0 \pm x^3.$$  

(1.2)

The fields are given as

$$\phi(x) = \sum_{n=1}^{\Lambda} \frac{1}{(4\pi n)^{\frac{3}{2}}} \left( a_n e^{-ik_n^+ x^+} + a_n^\dagger e^{ik_n^+ x^+} \right)$$

(1.3)

and

$$\psi(x) = \sum_{n=1}^{\Lambda} \frac{u}{(2L)^{\frac{3}{2}}} \left( b_n e^{-ik_n^+ x^+} + d_n^\dagger e^{ik_n^+ x^+} \right)$$

(1.4)

with discretized single-particle light-cone momenta

$$k_n^+ = \frac{2\pi}{L} n; \quad n = 1, 2, ... \Lambda$$

(1.5)

where $\Lambda$ is the momentum cut-off which must be chosen by hand, and $L$ is the size of the box which the fields are quantized onto. Luckily, in LF coordinates, there are three constants of motion (which constitute a complete set of commuting observables): $Q$ the charge, $P^+$ the LF momentum, and $P^-$ the LF energy, which isolate the dependence of the box size, $L$. In LF coordinates, $Q = \sum_n (b_n^\dagger b_n - d_n^\dagger d_n)$, $P^+ = \frac{2\pi}{L} K$, $P^- = \frac{L}{2\pi} H$, where a new operator, $K$ the harmonic resolution

$$K = \sum_n n (a_n^\dagger a_n + b_n^\dagger b_n + d_n^\dagger d_n)$$

(1.6)
gives the total momentum of the hadron. Physically, one can think of each parton inside the hadron having light-cone momentum \( k^+ \) and the harmonic resolution \( K \) (the total LF hadronic momentum) is the sum of all of the individual parton momenta.

### 1.2 Fock States

We look at Fock states in second-quantized momentum space for fermions, antifermions and bosons:

\[
|\bar{\psi}\rangle = |n_1, \ldots n_N; \bar{n}_1, \ldots, \bar{n}_N; \bar{\bar{n}}_1\bar{\bar{n}}_1, \ldots, \bar{\bar{n}}_1\bar{\bar{n}}_N\rangle (1.7)
\]

which can be built up from the vacuum state \(|0\rangle\) via creation operators \(a^+_i, b^+_i, d^+_i\) subject to the constraint\(\sum_n n_n + \bar{n}_i = K\) and \(\sum_n k_{n\perp} = \bar{P}_\perp\), where \(\bar{P}_\perp\) is the total transverse momentum. The first \(N\) terms refer to fermions with \(n_i \in \{0, 1\}\), the second \(\bar{N}\) terms refer to antifermions again with \(\bar{n}_i \in \{0, 1\}\), while the final \(\bar{N}\) terms refer to bosons with \(\bar{\bar{n}}_i\) bosons in a given mode. Note that only occupied modes are shown in a given Fock state. The set of quantum numbers that each fermion can take on are \(\{n^+, n_\perp, \lambda, c, f\}\) where \(\lambda\) refers to the helicity of the fermion, \(c\) is the color, and \(f\) is the flavor. In a given Fock state, instead of each fermionic mode representing a single quantum number, each fermionic mode represents a set of numbers.

This can obviously increase the number of qubits quite rapidly as there are 2 possible helicities, 3 possible colors and 6 possible flavors for each \(n^+, n_\perp\) so this scales exponentially large as \(n^+, n_\perp\) increase. There are a few ways to decrease the number of qubits that must be used to represent this system. First, we use the Tamm-Dancoff cutoff which looks only at the \(|q\bar{q}\rangle\) or \(|q\bar{q}\bar{q}\rangle\) sectors (for the purposes of this paper, we only look at the \(|q\bar{q}\rangle\) sector). With this cutoff, we are essentially looking at mesonic bound states such as the pion. Additionally, we disregard the helicity, color and flavor and only keep the quantum numbers \(n^+\) and \(n_\perp\).

### 1.3 The Bound State problem

In this section, the bound state problem will be discussed (note that much of this information comes from the Pauli, Brodsky, Pinsky paper [3] and the Wöhl paper [2]). The eigenvalue problem we must solve is

\[
H|\Psi\rangle = \frac{M^2 + P^2}{2P^+} |\Psi\rangle . (1.8)
\]

\(|\Psi\rangle\) is the hadronic statefunction defined in the set of basis Fock states \(|\mu_n\rangle\) for the partons. A given Fock state has the form:

\[
|q\bar{q}: k^+_i, \bar{K}_{i,\perp}\rangle = b^\dagger(q_1) d^\dagger(q_2) |0\rangle (1.9)
\]

where \(b^\dagger(q_1)\) creates a quark with the quantum numbers \(q_1\) and \(d^\dagger(q_2)\) creates an antiquark with the quantum numbers \(q_2\). There are a finite number of Fock states of this form satisfying \(\sum_i k^+_i = P^+\) and \(\sum_i k_{i,\perp} = \bar{P}_\perp\). This set of Fock states forms a basis to which the hadronic state \(\Psi\) can be written:

\[
|\Psi\rangle = \sum_n (\langle \mu_n | \Psi \rangle |\mu_n\rangle .
\]

Solving the bound state problem using VQE will allow us to determine the ground state energy as well as the coefficients \(\langle \mu_n | \Psi \rangle\) in the expansion of \(|\Psi\rangle\) which allows us to write down the ground state wavefunction.

### 1.4 2 + 1D Yukawa Hamiltonian

The 2 + 1D Yukawa Hamiltonian is given by

\[
H_{LC} = P^\mu P_\mu + P^+ \bar{P}_\perp - P_\perp^2 (1.10)
\]

where \(P^+\) and \(P_\perp\) are parameters that are chosen. \(P^+ = K\) and it is permissible to take \(P_\perp = 0\) as there exists a reference frame such that the total hadronic transverse momentum is zero, but the individual partonic transverse momenta are non-zero but sum to zero. Thus the Hamiltonian is \(H_{LC} = K P^-\) where \(P^-\) is more complicated. Given explicitly in Pauli, Brodsky, Pinsky[3], the Hamiltonian is the sum of a kinetic term and an interaction (potential) term \(H = T + U\). The kinetic piece is diagonal in number operators:

\[
T = \sum_q \frac{m_q^2 + k_q^2}{x} (a^\dagger_q a_q + b^\dagger_q b_q + d^\dagger_q d_q) (1.11)
\]

where \(x = n^+ / K\) and the sum goes over the set of all possible quantum numbers:

\[
\sum_q = \sum_{n^+} \sum_{n^\perp} \ldots
\]

where the dots include any other quantum numbers not used in our problem (such as helicity, color, flavor). The interaction term is a sum of four pieces: \(U = S + C + V + F\). See the Pauli, Brodsky, Pinsky [3] paper for the full interaction matrix for an SU(N)-Meson. In the \(|q\bar{q}\rangle\) sector we are interested in, the only part of the interaction that comes into play is the seagull part \((S)\) given as:
\[ S_{q\bar{q}} = \sum_{q_1, q_2} \left( \frac{2\tilde{g}^2}{(x_1 + x_2)^2} - \frac{1}{(x_1 - x_2)^2} \right) \delta^2(k_1^+ + k_2^+ | k_1^+ + k_2^+; \bar{q}_1, \bar{q}_2) = \frac{1}{(m_B/\Omega)^2} b_1^d, d_2^b \] (1.12)

where

\[ \Delta = \delta(k_1^+ + k_2^+ | k_1^+ + k_2^+; \bar{q}_1, \bar{q}_2) \]

\[ \tilde{g}^2 = g^2 P^+/(2\Omega) \]

\[ b_1^d \] annihilates a quark and an antiquark with the quantum numbers described by \( q_1, q_2 \) while \( b_1^d, d_2^b \) creates a quark and an antiquark with the quantum numbers described by \( q_1', q_2' \). The additional \( (m_B/\Omega)^2 \) term is added to the exchange boson propagator to remove the singularity. With the explicit form of the Hamiltonian in the \(|q\bar{q}\rangle\) sector, we can write the Hamiltonian in the basis of Fock states:\[4\]:

\[ \langle \mu_n | H | \mu_n \rangle. \]

2 The Generalized Parton Distribution Function (GPD)

2.1 Overview of the GPD

The GPD [6] is a description of the probability amplitude to find a parton with a fraction of the total light-front momentum after an interaction, such as that with a photon, as well as an increase in transverse momentum. The handbag diagram for this process is given in figure 1.

The handbag diagram shows a hadron (shown as the ovular region) which has initial momentum \(|P\rangle\), which is struck by a virtual photon \( \gamma^* \), that has been emitted by an electron with momentum \(|k\rangle\). From here, the virtual photon strikes a parton inside the hadron which changes its momentum and thus the total momentum of the hadron is changed to \(|P'\rangle\). After the photon’s interaction with the parton, a real photon is emitted \( \gamma \) to satisfy momentum conservation. The GPD \( F^q(x, \xi, t) \), describes the off-diagonal matrix elements between \(|P\rangle\) and \(|P'\rangle\). We are now working in 2 + 1D (the formalism for the added dimension will be discussed in the following section) so the GPD is a function of two new variables besides the standard momentum fraction \( x = n/K \). \( \xi \) is called the skewness, the transfer of longitudinal momentum, and \( t = (P - P')^2 \) is the invariant momentum transfer.

\[ F^q(x, 0, 0) = f_q(x). \]
2.3 GPD in terms of Fock operators

\[ F^q(x, \xi) = \frac{1}{2^{P^+}} \int \frac{d^2k_T}{2\sqrt{x^2 - \xi^2}|(2\pi)^3}} \sum_{\lambda} \langle \langle P' | b_\lambda((x-\xi)P^+, k_T - \Delta_T)b_\lambda((x+\xi)P^+, k_T) | P \rangle \theta(x \geq \xi) \]

\[ + \langle P' | d_\lambda((x+\xi)P^+, -k_T + \Delta_T)b_{-\lambda}(x+\xi)P^+, k_T) | P \rangle \theta(-\xi < x < \xi) \]

\[ - \langle P' | d_\lambda((x-\xi)P^+, k_T - \Delta_T)d_{-\lambda}((-x+\xi)P^+, k_T) | P \rangle \theta(x \leq \xi) \]

(2.2)

Golec-Biernat and Martin (as well as Ji)\(^7\)\(^8\) give the explicit form of the GPD (off-diagonal distribution) in terms of creation/annihilation operators by writing the quark field operators \(q(\frac{x}{2})\) in terms of these operators (eq. 2.2). This is called the off-diagonal distribution because it connects different Fock states \(|P\rangle\) and \(|P'\rangle\) where as the PDF (forward distributions) are diagonal and connect the same states. \(F^q\) simplifies when we look at the zero skewness case (\(\xi = 0\)) since we are interested in hadrons with the same resolutions before and after an interaction.

\(|P\rangle\) is the hadronic bound state that is obtained by running VQE with the 2 + 1D Hamiltonian above. This state will be a superposition of Fock states defined for a particular \(|q\bar{q}\rangle\) sector depending on \((P^+, P_\perp)\). \(|P\rangle\) can be any chosen output state (not all output states will give non-zero GPDs as not all states can be "reached" by the interaction from a photon). The four-vector \(P^\mu = (P^+, P_\perp, P^\perp)\) allows us to choose \(P^\perp\) and \(P_\perp\) and subsequently create a new set of Fock states \(\{|\mu_\perp\rangle\}\) that satisfy the parameters given by \(P^\mu\). Each of these primed Fock states can be used as \(|P'\rangle\).

We can prepare the state \(|P\rangle\) on a quantum computer and treat \(F^q(x, \xi)\) as an observable which can be measured on a quantum computer. An observable \(Q\) on a quantum computer is measured by measuring its expectation value in some state \(|\psi\rangle\): \(\langle Q \rangle = \langle \psi | Q | \psi \rangle\). In order to treat \(F^q\) as an observable, we must transform the state \(|P\rangle\) to \(\langle P\rangle\) so that we can calculate the expectation value and thus measure the observable. There exists some operator, \(O\) such that \(|P'\rangle = O|P\rangle\) (or \(|P'\rangle = |P\rangle O\)). We need \(O\) to be analogous to the Lorentz boost in the transverse direction (i.e. leaves \(P^\perp\) unchanged while changing \(P_\perp\) to \(P'_\perp\)).

In equal time coordinates, \(P^\mu = (p^0, p^1, p^2, p^3)\) and a Lorentz boost in the transverse direction is given by:

\[
\begin{align*}
P'^i &= \gamma(p_\perp - \beta p^0); i = 1, 2 \\
p^0 &= \gamma(p^0 - \beta P_\perp) \\
p^3 &= p^3.
\end{align*}
\]

In LF coordinates, \(P^\mu = (P^+, P_\perp, P_\perp)\) \(P^\pm = P^0 \pm P^3,\)

\(P_\perp = (P^1, P^2).\) \(P^0 = \frac{1}{2}(P^+ + P^-),\) \(P^3 = \frac{1}{2}(P^+ - P^-).\)

Thus,

\[
\begin{align*}
P'^\perp &= \gamma(P_\perp - \frac{1}{2}\beta \cdot (P^+ + P^-)) \\
P'^\pm &= e^{\mp \varphi} P^\pm
\end{align*}
\]

with \(\varphi = \frac{1}{2} \ln \left(\frac{1 + \beta}{1 - \beta}\right)\). This gives

\[ \Lambda_{\mu_F}(\varphi) = \begin{pmatrix} e^{-\varphi} & 0 & 0 \\ 0 & e^{\varphi} & 0 \\ -\beta/2 & -\beta/2 & \gamma \end{pmatrix}_{LF} \]

(2.3)

Then \(\Omega\) the Lorentz-transformed creation an annihilation operators are

\[ U(\Lambda)a^\dagger(p)U(\Lambda) = a^\dagger(\Lambda p) \]

(2.4) and

\[ U(\Lambda)a(p)U(\Lambda) = a(\Lambda p). \]

(2.5)

Given \(|P\rangle\) = \(|q\bar{q} : q_1, q_2\rangle\), \(|P'\rangle\) = \(|q\bar{q} : q'_1, q'_2\rangle\), where \(q_i = \{k_i^+, \bar{k}_i, \bar{k}_i^\perp\}\), we need \(O\) to first annihilate the quantum numbers \(q_i\) in \(|P\rangle\) and then create the quantum numbers \(q'_j\) depending on what we choose for \(|P'\rangle\) i.e.

\[ O = \sum_{q_i, q_j} c^\dagger(q_j)c(q_i) \]

(2.6)

where the sum goes over all Fock states in \(|P\rangle\) (c = b for fermions and c = d for antifermions).

Since we are disregarding helicity, we can drop the summation over \(\lambda\) and rewrite \(F^q\) in the discrete case (with \(\xi = 0\)):

\[ F^q(x, 0) = \frac{1}{4(2\pi)^{3/2}xP^+} \sum_{k_T} \langle P|F^q|P\rangle \]

(2.7)

where

\[ F^q = \sum_{q_i, q_j} c^\dagger(q_i)c(q_j)b^\dagger(n, k_T - \Delta_T)b(n, k_T)\theta(n) \]

\[ - d^\dagger(-n, k_T - \Delta_T)d(-n, k_T)\theta(-n)\].

(Here \(n = xP^+\)). The first operator in \(F^q\) is \(O^\dagger\) such that when it acts on \(|P\rangle\), it returns \(|P'\rangle\). \(F^q\) is not Hermitian, so in order to measure this observable on a quantum computer, we must decompose this operator into its real and imaginary components:

\[ \text{Re}\{F^q\} = \frac{1}{2}(F + F^\dagger) \]

\[ \text{Im}\{F^q\} = \frac{1}{2i}(F - F^\dagger) \].
We can measure the real and imaginary components separately on a quantum computer which will give us a value for the observable $F_q$ and thus $F^{\gamma}$. In order to find $|P\rangle$, we search for the ground bound state in the K-block of the $|q\bar{q}\rangle$ sector of the lightfront Hamiltonian.

### 2.4 GPDs

In what follows, the parameters chosen are $P^+ = P^{+\prime} = 3$, $P_\perp = 0$, $P_\perp^{\prime} = 1$, $\Lambda_\perp = 1$, $\Lambda_{nc} = 5832$ (the total momentum cutoff), $g = 0.001$, and $m_F = 36$, $m_B = 0.5$ which gives the bound state mass for the Pion as $m_\pi = 139.7$ MeV/c$^2$. Essentially, we are interested in the GPD of a pion gaining another unit of transverse momentum only after an interaction.

When running VQE with the Hamiltonian above, as well as a trial state from the Fock space given the parameters for $|P\rangle$ above, the ground state is found via Qiskit’s Statevector class to be:

$$|P\rangle = 0.7078|q\bar{q} : (1, 0); (2, 0)\rangle - 0.7063|q\bar{q} : (2, 0); (1, 0)\rangle.$$  \hspace{1cm} (2.8)

It was found that the initial parameters for the circuit must be negative so that we achieve an anti-symmetric statefunction, necessary to describe fermions.

From here, we can calculate the GPD on a quantum computer as an observable via equation (2.7). We can alter the value of $n$ and then graph our GPD values for various $x$ given in figure 2.

### 3 Summary

The GPD is a transition amplitude (not probability) of transitioning from the state $|P\rangle$ to the state $|P^{\prime}\rangle$ defined above. In the zero-skewness case $\xi = 0$, the only states that the hadron can transition to are states with the same light-front momentum but with different transverse momentum. We look at $|P^{\prime}\rangle$ such that the fermion gains an increased unit of transverse momentum.

By concentrating on the $q\bar{q}$ sector, we have simulated a $\pi^0$ type hadron, whose wave function is predominantly valence quarks. The Yukawa bosons are stand-ins for gluons, but have mass and spin 0. While we have not solved the full QCD Hamiltonian, using light front quantization, we have shown how a simpler quantum field theory gives rise to hadronic states composed of Fock states. Those states are mapped onto qubit states - the state preparation being a crucial part of the connection to quantum computers. From the VQE optimization, states are constructed whose resulting pdf’s and GPD’s can be extracted, ultimately by quantum computers. The resulting distributions show very reasonable resemblances to known measured distribution functions.

In future work, by extending the Hamiltonian to include flavors and colors, as well as a third space dimension, a more realistic representation of hadrons via quantum computers will be sought. Additionally, we hope to study baryonic and diquark-quark bound states.
4 Appendix

To define the operator that maps $|P\rangle$ to $|P'\rangle$, we consider an example. We start with $|P\rangle$ defined in equation 2.5. We take $|P'\rangle$ to be the state such that the fermion in the first state of the superposition gains an increased unit of transverse momentum, i.e.:

$$|P'\rangle = 0.7078|\bar{q}q : (1, 1); (2, 0)\rangle - 0.7063|\bar{q}q : (2, 1); (1, 0)\rangle.$$  \hspace{1cm} (4.1)

To construct $O$, we clearly must include a $b^\dagger b$ term that annihilates the fermions with quantum numbers $(1, 0)$ and creates a fermion with quantum numbers $(1, 1)$. Additionally, we will want a term that annihilates the fermion with quantum numbers $(2, 0)$ and creates a fermion with an increased unit of transverse momentum: $(2, 1)$. Thus, in this case,

$$O = b^\dagger(1, 1)b(1, 0) + b^\dagger(2, 1)b(2, 0)$$

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