On the Solution of Locally Lipschitz BSDE Associated to Jump Markov Process.

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Abstract

In this study, we consider a class of backward SDE driven by jump Markov process. An existence and uniqueness result to this kind of equations is obtained in a locally Lipschitz case. We essentially approximate the initial problem by constructing a convenient sequence of globally Lipschitz BSDEs having the existence and the uniqueness propriety. Then, we show, by passing to the limits, the existence and uniqueness of a solution to the initial problem. After that, a stability theorem is also proved in the local Lipschitz setting. Applying the aforementioned result, we give an application to European option pricing with constraint.

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1 Introduction

The history of backward stochastic differential equations driven by continuous Brownian motion goes back to the work of J.M. Bismut [8], in 1973. However, the theory that deals with this type of equations was developed in 1990 by E. Pardoux and S. Peng, in their original paper [19]. In [19],

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the authors, announced the first result of existence and uniqueness concerning the nonlinear case. From this work, many authors attempt to relax the regularity of the generator with respect to the state variables, this consists in trading of the Lipschitz condition by imposing more weaker assumptions. Let us mention that there are a list of large literature in this respect, see for example [1 2 3 4 14 18 19], for the continuous Brownian case; we also refer the reader to [5 7 9 10 13 16 17] and the references therein for BSDEs driven by random jumps processes.

Since the aim of this paper is to study locally Lipschitz setting for one kind of BSDE with jump Markov process, we present some existing results that go in this direction and are investigated for BSDEs driven by Brownian motion (without Markov jump part). The first paper treats the locally Lipschitz case, is given by S. Hamadene [14], in 1996; in that paper, the author studied a kind of one dimensional BSDEs with locally Lipschitz generator satisfying a suitable growing condition assuming that the terminal condition is bounded. Then, in Bahlali [2], the previous result has been generalized in the multidimensional case with both local assumption on the coefficient and only square integrable terminal data. Subsequently, the same results have been addressed in Bahlali [3] for BSDEs driven by Teugel’s martingales and an independent Brownian motion. After that, an existence and uniqueness result to backward stochastic nonlinear Volterra integral equations under local Lipschitz continuity condition on the drift, has been investigated in A. Auguste and N. Modeste [1].

More recently, BSDEs driven by a random measure related to a pure jump Markov process have been studied by F. Confortola, M. Fuhrman in [9, 10], where they provided existence and uniqueness results for such equations. Furthermore, they applied these results to study nonlinear variants of the Kolmogorov equation of the Markov process and to solve optimal control problems related to this topic. Herein, we are going to extend the contribution of the important papers [2, 9] by involving random measure associated to jump Markov process in the state BSDE.

Motivated by the above results, we study a class of backward stochastic differential equations driven by a random measure associated to jump Markov process. We tackle an existence and uniqueness result on top of a stability propriety for the solution of the following type of backward stochas-
tic differential equation.

\[ Y_s = h(X_T) + \int_{s}^{T} f(r, X_r, Y_r, Z_r(\cdot)) \, dr - \int_{s}^{T} \int_{\Gamma} Z_r(y) q(dr, dy), \quad s \in [t, T]. \] (1.1)

Where \( q(dt, dy) \) is a random measure, \( X \) is a normal jump Markov process and \( h(X_T) \) is the terminal condition. Noting that \( Z \), under some appropriate measurability conditions, is a random field on \( \Gamma \). The generator \( f \) depends on \( Y \) and \( Z \) in a general functional way. An adapted solution to BSDE (1.1) (if there exists) is a couple \((Y, Z)\) which belongs to \( B \) and satisfies equation (1.1).

In our setting, two main results are established. The first result consists in proving the existence and uniqueness of the solution to the BSDE (1.1) under the locally Lipschitz condition satisfied by the generator \( f \). In this case, we prove our main result assuming that the terminal condition is square integrable, the generator \( f \) satisfies the polynomial growth condition and the Lipschitz condition in the ball \( B(0, M) \) with a Lipschitz constant inferior or equal to \( L + \sqrt{\log(M)} \), where \( L \) is a universal positive constant. As a second result, we prove a stability of solutions under the same conditions.

The remainder of the current paper is organized as follows. In Section 2, we give a brief introduction into jump Markov process theory and we recall an existence and uniqueness Theorem for BSDEs with globally lipschitz coefficients. In Section 3, we study BSDEs with locally Lipschitz coefficients. To illustrate our theoretical results, an example in finance is treated in Section 4.

2 Preliminaries.

2.1 Overview of Jump Markov Process

Throughout this paper, the real positive number \( T \) stands for horizon, and \((\Omega, \mathcal{F}, \mathbb{P})\) stands for a complete probability space. We define \( (\Gamma, \mathcal{E}) \) as a measurable space such that \( \mathcal{E} \) contains all one-point sets; we also define \( \Delta \) as a point not included in \( \Gamma \). For a given a normal jump Markov process \( X \), we denote by \( \mathbb{F}^t := \sigma(X_r, r \leq s) \vee \mathcal{N} \) the filtration \((\mathcal{F}_{[t,s]})_{s \in [t,\infty]}\) generated by the process \( X \) and completed by \( \mathcal{N} \), where \( \mathcal{N} \) is the totality of \( \mathbb{P} \)-null sets.
We assume further that the jump Markov process $X$ satisfies the following conditions:

1. $\mathbb{P}^{t,x}(X_t = x) = 1$ for every $t \in [0, \infty[$, $x \in \Gamma$.

2. For every $0 \leq t \leq s$ and $A \in \mathcal{E}$ the function $x \rightarrow \mathbb{P}^{t,x}(X_s \in A)$ is $\mathcal{E}$-measurable.

3. For every $0 \leq r \leq t \leq s$, $A \in \mathcal{E}$ we have $\mathbb{P}^{r,x}(X_s \in A \mid \mathcal{F}_{r,t}) = \mathbb{P}^{t,X}(X_s \in A)$, $\mathbb{P}^{t,x}$-a.s.

4. All the trajectories of the pure jump process $X$ have right limits when $\Gamma$ is endowed with its discrete topology (the one where all subsets are open). In other words, for every $\omega \in \Omega$ and $t \geq 0$ there exists $\delta > 0$ such that $X_s(\omega) = X_t(\omega)$ for $s \in [t, t + \delta]$.

5. For every $\omega \in \Omega$ the number of jumps of the trajectory $t \rightarrow X_t(\omega)$ is finite on every bounded interval, which implies that $X$ is non explosive process.

Let $\mathcal{P}$ be the predictable $\sigma$-algebra, and $Prog$ be the progressive $\sigma$-algebra on $\Omega \times [0, \infty[$, the same symbols will also denote the restriction to $\Omega \times [t, T]$. We define a transition measure (also called rate measure) $v(t, x, A)$, $t \in [0, T]$, $x \in \Gamma$, $A \in \Gamma$ from $[0, \infty) \times \Gamma$ to $\Gamma$, such that $\sup_{t \in [0, T], x \in \Gamma} v(t, x, \Gamma) < \infty$ and $v(t, x, \{x\}) = 0$. Let $T_n$ be the jump times of $X$, we consider the marked point process $(T_n, X_{T_n})$, and the associated random measure $p(dt \ dy) := \sum_n \delta_{(T_n, X_{T_n})}$ on $(0, \infty) \times \Gamma$, where $\delta$ stands for the Dirac measure. The compensator (also called the dual predictable projection) $\tilde{p}$ of $p$ is $\tilde{p} = v(t, X_t, dy)dt$, so that $q(dr \ dy) := p(dr \ dy) - v(r, X_{r-}, dy)dr$ is an $\mathbb{F}^t$-martingale.

We recall the representation theorem of marked point process martingales; it is one of the important tools to prove the existence and uniqueness of solution to BSDE. This theorem states that every integrable martingale adapted to the natural filtration generated by a jump Markov process, can be written in terms of stochastic integral with respect to this jump Markov process (for further information in this subject see for example [10, 12]).

In the remainder of this study, we will work on the following spaces
• $\mathcal{L}^m(p)$, $m \in [1, \infty[$ the space of $\mathcal{P} \otimes \mathcal{E}$-measurable real functions $W_s(\omega, y)$ defined on $\Omega \times [t, \infty[ \times \Gamma$, such that

$$
\mathbb{E} \int_t^T \int_{\Gamma} |W_s(y)|^m p'(ds \, dy) = \mathbb{E} \int_t^T \int_{\Gamma} |W_s(y)|^m v(s, X_s, dy)ds < \infty.
$$

• $\mathcal{L}_{loc}^1(p)$, the space of the real functions $W$ such that $W \mathbb{I}_{[t, \tau_n]} \in \mathcal{L}^1(p)$ for some increasing sequence of $\mathbb{F}^t$-stopping times $\tau_n$ diverging to $+\infty$.

• $L^2(\Gamma, \mathcal{E}, v(s, x, dy))$ the space of processes $z : \Gamma \rightarrow \mathbb{R}$ such that

$$
\|z(\cdot)\| = \left( \int_{\Gamma} |z(y)|^2 v(s, X_s, dy) \right)^{1/2} < \infty.
$$

• $B^{t,x}$ the space of processes $(Y, Z)$ on $[t, T]$ such that

$$
\|(Y, Z)\|^2_B = \mathbb{E} \int_s^T |Y_t|^2 dr + \mathbb{E} \int_s^T \|Z_r(\cdot)\|^2 dr < \infty.
$$

The space $B$, endowed with this norm, is a Banach space.

Remark 1 A stochastic integral $\int_t^s \int_{\Gamma} W_r(y)q^i(dr \, dy)$ is a finite variation martingale if $W \in \mathcal{L}^1(p')$.

2.2 BSDE with Globally Lipschitz coefficients

In this subsection we recall a result of existence and uniqueness of solution to BSDE (1.1) in the globally Lipschitz framework. Let us first introduce the following Hypothesis:

**Hypothesis 1**

$H_1$ The final condition $h : \Gamma \rightarrow \mathbb{R}$ is $\mathcal{F}_{[t,T]}$-measurable function satisfies

$$
\mathbb{E} |h(X_T)|^2 < \infty.
$$

$H_2$ For every $s \in [0, T], x \in \Gamma, r \in \mathbb{R}$, $f(s, x, r, \cdot)$ is a mapping $L^2(\Gamma, \mathcal{E}, v(s, x, dy)) \rightarrow \mathbb{R}$.

$H_3$ For every bounded and $\mathcal{E}$-measurable function $z : \Gamma \rightarrow \mathbb{R}$, the mapping $(s, x, r) \rightarrow f(s, x, r, z(\cdot))$ is $B([0, T]) \otimes \mathcal{E} \otimes B(\mathbb{R})$-measurable.
There exist $L \geq 0$, $\dot{L} \geq 0$ such that for every $s \in [0,T]$, $x \in \Gamma$, $r$, $r' \in \mathbb{R}$, $z$, $z' \in L^2(\Gamma, \mathcal{E}, v(s, x, dy))$,

$$|f(s, x, r, z(.)) - f(s, x, r', z(.))| \leq \dot{L}|r - r'| + L \|z(.) - \dot{z}(.)\|.$$ 

Noting that, under the assumptions ($H_1$), ($H_2$) and ($H_3$), Lemma(3.2) in [10], shows that the mapping $(\omega, s, y) \rightarrow f(s, X_{s-}(\omega), y, Z_s(\omega, .))$ is $\mathcal{P} \otimes B(\mathbb{R})$-measurable, if $Z \in L^2(p)$. Furthermore, if $Y$ is Prog-measurable process then, $(\omega, s) \rightarrow f(s, X_{s-}(\omega), Y_s(\omega), Z_s(\omega, .))$ is Prog-measurable.

The following Theorem has been proved in [10]. The authors showed, in the case where the generator does not depend on the state variables $Y$ and $Z$, that BSDE(11) admits a unique solution. This done thanks to the representation Theorem for the $\mathcal{F}$–martingales by means the stochastic integrals with respect to $q$. Using this last result and a fixed point argument, they proved the existence and uniqueness of global solution to BSDE(11) in the case where the generator depends on state variables $Y$ and $Z$.

**Theorem 1** Suppose that Hypothesis1 holds true. Then BSDE (11) has a unique solution $(Y, Z)$ which belongs to $\mathcal{B}$.

### 3 BSDE with locally Lipschitz coefficients

The first purpose of this section is to prove existence and uniqueness of solutions to BSDE (11) in the case where the generator $f$ is merely locally Lipschitz. The second purpose is to prove a stability propriety for the solutions if they exist. For this end, we need to impose some regularity assumptions on the coefficients. These conditions are gathered and listed in following basic assumptions:

**H2.1.** $f$ is continuous in $(y, z)$ for almost all $(t, \omega)$.

**H2.2.** There exists $\lambda > 0$ and $\alpha \in ]0, 1[$ such that

$$|f(s, x, y, z(.))| \leq \lambda [1 + |y|^{\alpha} + \|z(.)\|^\alpha].$$
H$_{2.3}$. For every $M \in \mathbb{N}$, there exists a constant $L_M > 0$ such that
\[
|f(s, x, y, z(.)) - f(s, x, \dot{y}, \dot{z}(.))| \leq L_M \left( |y - \dot{y}| + \|z(.) - \dot{z}(.)\| \right),
\]
\[\mathbb{P}\text{-a.s. } \forall t \in [0, T],\]
and $\forall y, \dot{y}, z, \dot{z}$ such that $|y| < M$, $|\dot{y}| < M$, $\|z(.)\| < M$, $\|\dot{z}(.)\| < M$.

H$_{2.4}$ The final condition $h : \Gamma \rightarrow \mathbb{R}$ is $\mathcal{F}_{[t,T]}$ measurable function satisfies $\mathbb{E} |h(X_T)|^2 < \infty$.

We assume that $f$ satisfies (H$_{2.1}$) and (H$_{2.2}$), then we define the family of semi–norms $(\Phi_n(f))_{n \in \mathbb{N}}$
\[
\Phi_n(f) = \left( \mathbb{E} \int_0^T \sup_{|y|,\|z(.)\| \leq n} |f(s, X_s, y, z(.))|^2 \, ds \right)^{1/2}.
\] (3.1)

3.1 A priori estimations and results

In order to prove the existence and uniqueness result, we start by giving the following very useful three Lemmas. They involve some priori estimates of solutions for BSDE (1.1) on top of some estimates between two solutions. These lemmas will be a key tool in proving the essential results of the next section. For later use, we denote respectively BSDE (1.1), $h(X_T)$ by BSDE $(f, h)$ and $\xi$.

**Lemma 1** Let $(Y, Z)$ be a solution of BSDE $(f, \xi)$.

(i) If $f$ satisfies (H$_{2.2}$) then there exists a positive constant $C = C(\lambda, \xi, T)$, which depends on $\lambda^2$, $\mathbb{E} |\xi|^2$ and $T$, such that for every $s \in [t, T]$, $\mathbb{E} |Y_s|^2 + \mathbb{E} \int_s^T \|Z_r(.)\|^2 \, dr \leq C$.

(ii) Moreover, if the terminal condition $\xi := h(X_T)$ is bounded, there exists a positive constant $K$, such that $\sup_{t \in [0, T]} |Y_t|^2 \leq K$.

**Proof of Lemma (i)**

First, we prove (i). Using Itô’s formula for semimartingales (cf. Theorem 32 in [20]) to $|Y_s|^2$ and integrating on the time interval $[s, T]$ we get
\[
|Y_s|^2 = |\xi|^2 + 2 \int_s^T Y_r f_r (r, X_r, Y_r, Z_r(.)) \, dr - 2 \int_s^T \int Y_r Z_r (y) q(dr \, dy) - \sum_{s \leq r \leq T} |\Delta Y_r|^2.
\] (3.2)
Noting that the process \( \left( \int_s^T Y_{r-} Z_r(y) \, q(dr, dy) \right)_{s \in [0,T]} \) is an \( \mathbb{F}^t \)-martingale. Due the fact that \( Y_{r-} Z_r(y) \in \mathcal{L}^1(p) \), one can easily check that the process \( \left( \int_s^T Y_{r-} Z_r(y) \, q(dr, dy) \right)_{s \in [0,T]} \) is an \( \mathbb{F}^t \)-martingale. Indeed, from Young’s inequality and the fact that \( \sup_{t \in [0,T], x \in \Gamma} v(t, x, \Gamma) < \infty \), we get

\[
\int_{s \in [0,T]} \int \frac{|Y_{r-}| \, |Z_r(y)| \, v(r, X_r, dy) \, dr \leq \frac{1}{2} \sup_{t, x} \int \int |Y_r|^2 \, dr.
\]

\[
+ \frac{1}{2} \int_{0 \in \Gamma} \int |Z_r(y)|^2 \, v(r, X_r, dy) \, dr < \infty.
\]

In addition, we can rewrite the last term in the equality (3.2) as the following,

\[
\sum_{s \leq r \leq T} |\Delta Y_r|^2 = \int_{s \in \Gamma} \int |Z_r(y)|^2 \, p(dr, dy) \tag{3.3}
\]

\[
= \int_{s \in \Gamma} \int |Z_r(y)|^2 \, q(dr, dy) + \int_{s \in \Gamma} \int |Z_r(y)|^2 \, v(r, X_r, dy) \, dr.
\]

Keeping in mind that the stochastic processes \( q \) is an \( \mathbb{F}^t \)-martingale, plugging the equality (3.3) into (3.2) and taking the expectation, one can get

\[
\mathbb{E}(|Y_s|^2) + \int_{s \in [t, T]} |Z_r(y)|^2 \, v(r, X_r, dy) \, dr = \mathbb{E} |\xi|^2 + 2\mathbb{E} \int_{s \in \Gamma} Y_{r-} f_r(r, X_r, Y_r, Z_r(.) \, dr
\]

\[
s \in [t, T].
\]

By invoking (H2.2) and using the inequality \(|y|^\alpha \leq 1 + |y|\) for each \( \alpha \in [0, 1] \), we get

\[
\mathbb{E}(|Y_s|^2) + \int_{s \in [t, T]} \|Z_r(.)\|^2 \, dr \leq \mathbb{E} |\xi|^2 + 6\lambda \mathbb{E} \int_{s \in [t, T]} Y_{r} \, dr + 2\lambda \mathbb{E} \int_{s \in [t, T]} Y_r^2 \, dr + 2\lambda \mathbb{E} \int_{s \in [t, T]} Y_{r-} \|Z_r(.)\| \, dr. \tag{3.4}
\]

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we find, thanks to Young’s inequality $2xy \leq \varepsilon x^2 + \frac{y^2}{\varepsilon}$ with $\varepsilon = 1$

$$\mathbb{E}(|Y_s|^2) \leq \mathbb{E} |\xi|^2 + 9T + (1 + 3\lambda^2)\mathbb{E} \int_s^T |Y_r|^2 \, dr.$$  

From Gronwall’s Lemma, we get

$$\mathbb{E}(|Y_s|^2) \leq (\mathbb{E} |\xi|^2 + 9T) \exp ((1 + 3\lambda^2)T) = C_1. \quad (3.5)$$

We turn back to inequality (3.4), it follows that, using Young’s inequality once again with $\varepsilon = 2$ and the inequality (3.5)

$$\mathbb{E} \int_s^T \|Z_r(\cdot)\|^2 \, dr \leq 2 (\mathbb{E} |\xi|^2 + 9T) + 2T(1 + 4\lambda^2)C_1 = C_2.$$  

This proves (i). We proceed now to prove (ii). Firstly, by invoking and replacing the inequality (3.3) into (3.2), we get

$$|Y_s|^2 = \mathbb{E} |\xi|^2 + 2 \int_s^T Y_r f_r (r, X_r, Y_r, Z_r (\cdot)) \, dr - 2 \int_s^T \int_{\Gamma} Y_r Z_r (y) q(dr \, dy)$$

$$- \int_s^T \int_{\Gamma} |Z_r(y)|^2 q(dr \, dy) - \int_s^T \int_{\Gamma} |Z_r (y)|^2 v(r, X_r, dy)dr.$$  

Taking the conditional expectation with respect to $\mathcal{F}_{[0,s]}$, using Assumption (H_{2.2}), the inequality $|y|^\alpha \leq 1 + |y|$ for $\alpha \in [0, 1]$ together with Young’s inequality, gives

$$|Y_s|^2 \leq C + 9T + (2\lambda^2 + 2\lambda) \int_s^T \mathbb{E} (|Y_r|^2 \mid \mathcal{F}_{[0,s]}) \, dr.$$  

For any time $t \leq s$, using once again $\mathbb{E} (\cdot \mid \mathcal{F}_{[0,t]})$ in both sides of the previous inequality, we get

$$\mathbb{E} (|Y_s|^2 \mid \mathcal{F}_{[0,t]}) \leq C + 9T + (2\lambda^2 + 2\lambda) \int_s^T \mathbb{E} (|Y_r|^2 \mathcal{F}_{[0,t]}) \, dr.$$
Then, Gronwall’s Lemma, yields
\[ \mathbb{E} \left( |Y_s^2| \mid \mathcal{F}_{[0,t]} \right) \leq \left[ C + 9T \right] \exp \left[ \left( 9T + \lambda^2 + 2\lambda \right) (T - s) \right]. \]
In particular, if \( t = s \), we immediately find
\[ |Y_s^2| \leq \left( C + 9T \right) \exp \left[ \left( 9T + \lambda^2 + 2\lambda \right) (T - s) \right]. \]
Hence
\[ \sup_{s \in [0,T]} |Y_s^2| \leq K_1. \]

**Lemma 2** Let \( f_1 \) and \( f_2 \) be two functions satisfy (H2.1) and (H2.2). Let \((Y^1, Z_1)\) (resp. \((Y^2, Z_2)\)) be a solution of the BSDE \((f_1, \xi_1)\) (resp. BSDE \((f_2, \xi_2)\)), where \( \xi_1 \) and \( \xi_2 \) are two final conditions satisfy \((H2.4)\). Then for every locally Lipschitz function \( f \) and every \( M > 1 \), the following estimates hold:
\[ \mathbb{E} \left( \left| Y_r^1 - Y_r^2 \right|^2 \right) \leq \left[ \mathbb{E} |\xi_1 - \xi_2|^2 + \Phi_M^2 (f - f_2) + \Phi_M^2 (f_1 - f) \right. \\
+ \left. \frac{C(\xi_1, \xi_2, \lambda)}{(1 + 2L_M^2) M^{2(1-\alpha)}} \right] \exp \left[ (4 + 4L_M^2)(T - s) \right]. \]
\[ \mathbb{E} \int_s^T \left\| Z_r^1(.) - Z_r^2(.) \right\|^2 dr \leq C(\xi_1, \xi_2, \lambda) \left[ \mathbb{E} \left| \xi_1 \right|^2 + \left( \mathbb{E} \int_s^T \left| Y_r^1 - Y_r^2 \right|^2 dr \right)^{\frac{1}{2}} \right]. \]
Where \( C(\xi_1, \xi_2, \lambda) \) is a constant depends only on \( \lambda^2 \), \( \mathbb{E} |\xi_1|^2 \) and \( \mathbb{E} |\xi_2|^2 \).

**Proof of Lemma (2)**
We set \( \check{Y} = Y^1 - Y^2 \), \( \check{Z} = Z^1 - Z^2 \), \( \check{f}_s = f_1(s, X_s, Y^1_s, Z^1_r(.)) - f_2(s, X_s, Y^2_s, Z^2_r(.)) \), \( \check{\xi} = \xi_1 - \xi_2 \). By Itô’s formula we have
\[ \mathbb{E} \left( \left| \check{Y}_s \right|^2 \right) + \mathbb{E} \int_s^T \left\| \check{Z}_r(.) \right\|^2 dr = \mathbb{E} \left| \check{\xi} \right|^2 + 2\mathbb{E} \int_s^T \check{Y}_r \check{f}_r dr \quad s \in [t, T]. \tag{3.6} \]
For a given \( M > 1 \), let \( L_M \) be the Lipschitz constant of \( f \) in the ball \( B(0, M) \), we define
\[ D_M := \left\{ (s, \omega) : \left| Y^1_s \right|^2 + \left\| Z^1_r(.) \right\|^2 + \left| Y^2_s \right|^2 + \left\| Z^2_r(.) \right\|^2 \geq M^2 \right\}, \quad \bar{D}_M := \Omega \setminus D_M, \]
This make it possible to rewrite (3.6) as the following

$$
\mathbb{E}(\left|\tilde{Y}_s\right|^2) + \mathbb{E} \int_s^T \left\| \tilde{Z}_r(.) \right\|^2 dr = \mathbb{E} |\xi|^2 + 2\mathbb{E} \int_s^T \tilde{Y}_r \tilde{f}_r \mathbb{I}_{D_M} dr
$$

$$
+ 2\mathbb{E} \int_s^T \tilde{Y}_r \tilde{f}_r \mathbb{I}_{D_M} dr \quad s \in [t, T].
$$

(3.7)

where $\mathbb{I}_{D_M}$ stands for the indicator function of the set $D$. We proceed to estimate the last term in the previous equality

$$
2\mathbb{E} \int_s^T \tilde{Y}_r \tilde{f}_r \mathbb{I}_{D_M} dr = 2\mathbb{E} \int_s^T \tilde{Y}_r [(f_1 - f)(r, X_r, Y_r^1, Z_r^1(.))] \mathbb{I}_{D_M} dr
$$

$$
+ 2\mathbb{E} \int_s^T \tilde{Y}_r [(f - f_2)(r, X_r, Y_r^2, Z_r^2(.))] \mathbb{I}_{D_M} dr
$$

(3.8)

$$
+ 2\mathbb{E} \int_s^T \tilde{Y}_r [f(r, X_r, Y_r^1, Z_r^1(.)) - f(r, X_r, Y_r^2, Z_r^2(.))] \mathbb{I}_{D_M} dr
$$

$$
= I_1 + I_2 + I_3.
$$

Then, the inequality $2xy \leq x^2 + y^2$ together with the definition of the semi-norm (3.11), leads to

$$
I_1 \leq \mathbb{E} \int_s^T \left| \tilde{Y}_r \right|^2 dr + \Phi_M^2(f_1 - f),
$$

(3.9)

and

$$
I_2 \leq \mathbb{E} \int_s^T \left| \tilde{Y}_r \right|^2 dr + \Phi_M^2(f - f_2).
$$

(3.10)

Since $f$ is $L_M$-Lipschitz in the ball $B(0, M)$, we obtain

$$
I_3 \leq 2L_M \left[ \mathbb{E} \int_s^T \left| \tilde{Y}_r \right|^2 dr + \mathbb{E} \int_s^T \left| \tilde{Y}_r \right| \left\| \tilde{Z}_r(.) \right\| dr \right].
$$

From Young’s inequality $2xy \leq \frac{x^2}{2} + \frac{2}{y^2}y^2$, we find

$$
I_3 \leq (L_M^2 + 1) \mathbb{E} \int_s^T \left| \tilde{Y}_r \right|^2 dr + \frac{2}{T} \mathbb{E} \int_s^T \left| \tilde{Y}_r \right|^2 dr + \frac{2L_M^2}{\gamma^2} \mathbb{E} \int_s^T \left\| \tilde{Z}_r(.) \right\|^2 dr,
$$

$$
\leq \left( L_M^2 + 1 + \frac{\gamma^2}{2} \right) \mathbb{E} \int_s^T \left| \tilde{Y}_r \right|^2 dr + \frac{2L_M^2}{\gamma^2} \mathbb{E} \int_s^T \left\| \tilde{Z}_r(.) \right\|^2 dr.
$$

(3.11)
From the inequalities (3.9), (3.10) and (3.11), one can get
\[
2\mathbb{E}\int_{s}^{T} \tilde{Y}_{r} \tilde{f}_{r} \mathbb{I}_{D_{M}} dr \leq \Phi_{M}^{2}(f - f_{2}) + \Phi_{M}^{2}(f_{1} - f) + \left(3 + L_{M}^{2} + \frac{\lambda^{2}}{2}\right) \mathbb{E}\int_{s}^{T} |\tilde{Y}_{r}|^{2} dr + \frac{2L_{M}^{2}}{\rho^{2}} \mathbb{E}\int_{s}^{T} \|\tilde{Z}_{r}(\cdot)\|^{2} dr. \tag{3.12}
\]
Now, we turn out to estimate the second term in the inequality (3.7). Using Young’s inequality \(2xy \leq \rho^{2}x^{2} + \frac{L^{2}}{\rho^{2}}y^{2}\), we get
\[
2\mathbb{E}\int_{s}^{T} \tilde{Y}_{r} \tilde{f}_{r} \mathbb{I}_{D_{M}} dr \leq \rho^{2}\mathbb{E}\int_{s}^{T} |\tilde{Y}_{r}|^{2} \mathbb{I}_{D_{M}} dr + \frac{2L_{M}^{2}}{\rho^{2}} \mathbb{E}\int_{s}^{T} |f_{1}(s, X_{s}, Y_{s}^{1}, Z_{s}^{1}(\cdot))|^{2} \mathbb{I}_{D_{M}} dr
+ \frac{4}{\rho^{2}} \mathbb{E}\int_{s}^{T} |f_{2}(s, X_{s}, Y_{s}^{1}, Z_{s}^{1}(\cdot))|^{2} \mathbb{I}_{D_{M}} dr,
\]
by using the assumption (H_{2.2}) and the inequality \((a + b + c)^{2} \leq 3(a^{2} + b^{2} + c^{2})\), a simple calculation shows that
\[
2\mathbb{E}\int_{s}^{T} \tilde{Y}_{r} \tilde{f}_{r} \mathbb{I}_{D_{M}} dr \leq \rho^{2}\mathbb{E}\int_{s}^{T} |\tilde{Y}_{r}|^{2} \mathbb{I}_{D_{M}} dr + \frac{6L_{M}^{2}}{\rho^{2}} \mathbb{E}\int_{s}^{T} \left[2 + |Y_{s}^{1}|^{2\alpha} + \|Z_{s}^{1}(\cdot)\|^{\alpha}\right] \mathbb{I}_{D_{M}} dr.
\]
From Lemma (1), Holder’s inequality and the fact that
\[
\mathbb{I}_{D_{M}} \leq M^{-2} \left[|Y_{s}^{1}|^{2} + \|Z_{s}^{1}(\cdot)\|^{2} + |Y_{s}^{2}|^{2} + \|Z_{s}^{2}(\cdot)\|^{2}\right],
\]
we arrive at
\[
\mathbb{E}\int_{s}^{T} |Y_{s}^{1}|^{2\alpha} \mathbb{I}_{D_{M}} dr \leq \left(\mathbb{E}\int_{s}^{T} |Y_{s}^{1}|^{2} dr\right)^{\alpha} \left(\mathbb{E}\int_{s}^{T} \mathbb{I}_{D_{M}} dr\right)^{1-\alpha},
\leq \left(\mathbb{E}\int_{s}^{T} |Y_{s}^{1}|^{2} dr\right)^{\alpha} \mathbb{E}\int_{s}^{T} \left[\left(|Y_{s}^{1}|^{2} + \|Z_{s}^{1}(\cdot)\|^{2} + |Y_{s}^{2}|^{2} + \|Z_{s}^{2}(\cdot)\|^{2}\right) dr\right]^{1-\alpha},
\leq (CT)^{\alpha} \left[\left(C + \hat{C}\right)(T + 1)\right]^{1-\alpha} \frac{1}{M^{2(1-\alpha)}}.
\]
Applying the same method to each one of the terms \(\mathbb{E}\int_{s}^{T} |Y_{s}^{1}|^{2\alpha} \mathbb{I}_{D_{M}} dr,\)
\(\mathbb{E}\int_{s}^{T} \|Z_{s}^{1}(\cdot)\|^{\alpha} \mathbb{I}_{D_{M}} dr\) and \(\mathbb{E}\int_{s}^{T} \|Z_{s}^{2}(\cdot)\|^{\alpha} \mathbb{I}_{D_{M}} dr,\) to get
\[
2\mathbb{E}\int_{s}^{T} \tilde{Y}_{r} \tilde{f}_{r} \mathbb{I}_{D_{M}} dr \leq \rho^{2}\mathbb{E}\int_{s}^{T} |\tilde{Y}_{r}|^{2} dr + \frac{C(\xi_{1}, \xi_{2}, \lambda)}{\rho^{2}M^{2(1-\alpha)}}. \tag{3.13}
\]
If we choose \( \gamma^2 = 2L_M^2 \) and \( \rho^2 = (2L_M^2 + 1) \), and plugging the inequalities (3.12) and (3.13) into (3.7), we find
\[
\mathbb{E}\left( Y_s^2 \right) \leq \mathbb{E}\left( \xi^2 \right) + \Phi_M^2(f - f_2) + \Phi_M^2(f_1 - f) + \frac{C(\xi_1, \xi_2, \lambda)}{(1 + 2L_M^2)M^{2(1-\alpha)}} + (4 + 4L_M^2)\mathbb{E} \int_s^T \left| \bar{Y}_r \right|^2 dr.
\]

We deduce, using Gronwall’s Lemma
\[
\mathbb{E}\left( \left| \bar{Y}_s \right|^2 \right) \leq \left[ \mathbb{E}\left( \xi^2 \right) + \Phi_M^2(f - f_2) + \Phi_M^2(f_1 - f) + \frac{C(\xi_1, \xi_2, \lambda)}{(1 + 2L_M^2)M^{2(1-\alpha)}} \right] \exp \left[ (4 + 4L_M^2)(T - s) \right].
\]

To prove the second inequality we turn back to the equality (3.6), and we use Schwartz inequality, to obtain
\[
\mathbb{E}\left( \left| \bar{Y}_s \right|^2 \right) \leq \left[ \mathbb{E}\left( \xi^2 \right) + \Phi_M^2(f - f_2) + \Phi_M^2(f_1 - f) + \frac{C(\xi_1, \xi_2, \lambda)}{(1 + 2L_M^2)M^{2(1-\alpha)}} \right] \left( \mathbb{E} \int_s^T \left| \bar{Y}_r \right|^2 dr \right)^{\frac{1}{2}}.
\]

this achieve the proof of Lemma 2.

The following Lemma is the main tool in proving our main results; it allows us to approximate the initial locally Lipschitz problem by constructing a sequence of globally Lipschitz BSDEs. By taking advantage of the fact that each BSDE from this sequence has a unique solution, we can prove, by passing to the limits, that our initial locally Lipschitz BSDE has also a unique solution under some suitable conditions. Since the proof of this Lemma uses similar arguments to those goes as that of Lemma 4.4 in [3], we omit it in here.

**Lemma 3** Let \( f \) be a function satisfies (H2.1), (H2.2) and (H2.3). Then, there exists a sequence of functions \( f_n \) such that,

(i) For each \( n \), \( f_n \) is globally Lipschitz and satisfying (H2.2).

(ii) \( \sup_n |f_n(s, x, y, z(.))| \leq |f(s, x, y, z(.))| \leq \lambda(1 + |y|^a + \|z(.)\|^q). \) \( \mathbb{P} \)-a.s, \( \forall s \in [t, T] \).

(iii) For every \( M \), \( \Phi_M(f_n - f) \to 0 \) as \( n \to \infty \).
Now we are able to state and prove the main results of this paper. It is important to point out that the usual localization techniques by means of stopping time do not work when the generator of BSDE (1.1) are merely local.

3.2 The main Theorems

3.2.1 Existence and Uniqueness

Theorem 2 Suppose that (H_{2.1}), (H_{2.2}), (H_{2.3}) and (H_{2.4}) hold true. If there exists a positive constant $L$ such that $L_M \leq L + \sqrt{\log(M)}$, the BSDE (1.1) has a unique solution $(Y, Z)$ which belongs to $\mathcal{B}$.

Proof Suppose that there exists two solutions of the BSDE $(f, \xi)$: $(Y^1, Z^1)$ and $(Y^2, Z^2)$. The proof of the uniqueness is straightforward of Lemma (2) applied with $f_1 = f_2 = f$, $\xi_1 = \xi_2 = h(X_T)$, and Lebesgue’s dominated convergence theorem.

To prove the existence, with the sequence of generators $(f_n)_{n \in \mathbb{N}}$, we define a family of approximating BSDEs obtained by replacing the generator $f$ in BSDE(1.1) by $f_n$ defined in Lemma (3)

$$Y^n_s = h(X_T) + \int_s^T f^n(r, X_r, Y^n_r, Z^n_r(\cdot))dr - \int_s^T \int_{\Gamma} Z^n_r(y)q(dr, dy), \quad s \in [t, T].$$

In view of Theorem (1), the above BSDE has a unique solution, for each integer $n$, which will be denoted by $(Y^n, Z^n)$. Using similar arguments in the proof of Lemma (1), one can easily find

$$\sup_n \mathbb{E} \left( |Y^n_r|^2 + \int_s^T \|Z^n_r(\cdot)\|^2 dr \right) \leq C. \quad (3.14)$$

Noting that for each $n \geq N + 1$, $|f_n(s, x, y, z(\cdot)) - f_n(s, x, \hat{y}, \hat{z}(\cdot))| \leq L_M (|y - \hat{y}| + \|z(\cdot) - \hat{z}(\cdot)\|)$. We split the remain of the proof into the following three steps

Step 1: In this step, let us first assume that $T$ is small enough such that $T < \frac{(1 - \alpha)}{4}$. Then, we prove that $(Y^n, Z^n)$ is a Cauchy sequence in the Banach
space \((\mathcal{B}, ||\cdot||)\). Let us also assume (without loss the generality) that \(L = 0\), so that \(L \leq \sqrt{\log M}\). We apply Lemma (2) to \((Y^n, Z^n, f_n, \xi), (Y^m, Z^m, f_m, \xi)\), to obtain
\[
\mathbb{E}|Y^n_r - Y^m_r|^2 \leq \left[ \Phi^2_M(f_n - f) + \Phi^2_M(f - f_m) + \frac{C(\xi_1, \xi_2, \lambda)}{(1 + 2L^2_M)M^{2(1-\alpha)}} \right] \exp \left[(4 + 4L^2_M)T\right].
\]
Since \(L \leq \sqrt{\log M}\), we get
\[
\mathbb{E}|Y^n_r - Y^m_r|^2 \leq K(M, \alpha) \left[ \Phi^2_M(f_n - f) + \Phi^2_M(f - f_m) + C(\xi_1, \xi_2, \lambda)(2 \log M + 1)M^{2(1-\alpha)} \right],
\]
such that \(K(M, \alpha) = \exp (1 - \alpha) M^{(1-\alpha)}\). Passing to the limits successively on \(n, m, M\), we obtain
\[
\mathbb{E}|Y^n_r - Y^m_r|^2 \to_{n, m, M \to \infty} 0.
\]
We use (3.14) and the Lebesgue’s dominated convergence Theorem, to get
\[
\mathbb{E} \int_s^T |Y^n_r - Y^m_r|^2 \, dr \to_{n, m \to \infty} 0.
\]
And then
\[
\mathbb{E} \int_s^T \|Z^n_r(.) - Z^m_r(.)\|^2 \, dr \leq C(\xi_1, \xi_2, \lambda) \left[ \mathbb{E} \left( \int_s^T |Y^n_r - Y^m_r|^2 \, dr \right)^{\frac{1}{2}} \right] \to_{n, m \to \infty} 0.
\]
The two previous limits imply that \((Y^n, Z^n)\) is a Cauchy sequence in the Banach space \((\mathcal{B}, ||\cdot||)\). That is,
\[
\exists (Y, Z) \in \mathcal{B} \text{ such that } \lim_{n \to \infty} ||(Y^n, Z^n) - (Y, Z)||_\mathcal{B} = 0. \quad (3.15)
\]

**Step2** In this step, we assume at first that \(T\) is an arbitrary large time duration. Then, we will prove \((Y^n, Z^n)\) is a Cauchy sequence in the Banach space \((\mathcal{B}, ||\cdot||)\) on the time interval \([0, T]\). Firstly, let \([[T_i, T_{i+1}]]_{i=0}^{\infty}\) be
a subdivision of \([0, T]\), such that for any \(0 \leq i \leq k\), \(|T_{i+1} - T_i| \leq \delta\), where 
\(\delta\) is a strictly positive number satisfy \(\delta < \frac{(1-\alpha)}{4}\). Now, for \(t \in [T_{k-1}, T_k]\), we consider the following BSDE

\[
Y_s^n = h(X_T) + \int_s^{T_k} f(r, X_r, Y_r^n, Z_r^n(.))dr - \int_s^{T_k} \int_{\Gamma} Z_r^n(y)q(dr,dy). \tag{3.16}
\]

It is obvious from step 1 that, the relation (3.15) remain valid on the small interval time \([T_{k-1}, T_k]\). Next, for \(t \in [T_{k-2}, T_{k-1}]\), we consider the following BSDE

\[
Y_s^n = Y_{T_{k-1}}^n + \int_s^{T_{k-1}} f(r, X_r, Y_r, Z_r(.))dr - \int_s^{T_{k-1}} \int_{\Gamma} Z_r(y)q(dr,dy). \tag{3.17}
\]

Due to fact that, \(T_{k-1}\) is an element from the previous subinterval \([T_{k-1}, T_k]\), \(Y_{T_{k-1}}^n\) converges to \(Y_{T_{k-1}}\), and thus, \((Y^n, Z^n)\) is a Cauchy sequence and converges to an element in \(B\), on the small interval \([T_{k-2}, T_{k-1}]\). Repeating this procedure backwardly for \(i = k, ..., 1\), we obtain the desired result on the whole time interval \([0, T]\).

**Step 3:** In this step, we prove a global existence and uniqueness solution to BSDE(1.1). We have, from step 2, the following result

\[\exists (Y, Z) \in B \text{ such that } \lim_{n \to \infty} \|(Y^n, Z^n) - (Y, Z)\|_B = 0.\]

It remain to prove that \(\int_s^T f^n(r, X_r, Y_r^n, Z_r^n(.))dr\) converges to \(\int_s^T f(r, X_r, Y_r, Z_r(.))dr\), in probability. Denoting \(\tilde{Y}^n = Y^n - Y, \tilde{Z}^n = Z^n - Z\) and we set for \(M > 1\)

\[A^n_M := \{(s, \omega) : |Y_s| + \|Z_s(.)\| + |Y_s^n| + \|Z_s^n(.)\| \geq M\}, \quad \tilde{A}^n_M := \Omega \setminus A^n_M.\]

We further, denote by \(L_M\) the Lipschitz constant of \(f\) in the ball \(B(0, M)\)
and

\[ \mathbb{E} \int_s^T |f_n(r, X_r, Y^n_r, Z^n_r(.)) - f(r, X_r, Y_r, Z_r(.))| \, dr \]

\[ = \mathbb{E} \int_s^T (|f_n - f| (r, X_r, Y^n_r, Z^n_r(.))) \, \mathbb{I}_{A^m_\lambda} \, dr \]

\[ + \mathbb{E} \int_s^T (|f_n - f| (r, X_r, Y^n_r, Z^n_r(.))) \, \mathbb{I}_{A^m} \, dr \]

\[ + \mathbb{E} \int_s^T |f(r, X_r, Y^n_r, Z^n_r(.)) - f(r, X_r, Y_r, Z_r(.))| \, \mathbb{I}_{A^m} \, dr \]

\[ + \mathbb{E} \int_s^T |f(r, X_r, Y^n_r, Z^n_r(.)) - f(r, X_r, Y_r, Z_r(.))| \, \mathbb{I}_{A^m} \, dr. \]

Then, since \( f \) is \( L_M \)–locally Lipschitz, \((H_2,2)\) and \(|y|^\alpha \leq 1 + |y|\) for each \( \alpha \in [0, 1] \), we obtain

\[ \mathbb{E} \int_s^T |f_n(r, X_r, Y^n_r, Z^n_r(.)) - f(r, X_r, Y_r, Z_r(.))| \, dr \]

\[ \leq \mathbb{E} \int_s^T \sup_{|y|, ||z(.)|| \leq M} |(f_n - f)(r, X_r, y, z(.))| \, dr \]

\[ + 2\lambda \mathbb{E} \int_s^T [4 + |Y^n_r| + ||Z^n_r(.)||] \, \mathbb{I}_{A^m} \, dr + L_M \left( \mathbb{E} \int_s^T \left[ \|\bar{Y}^n_r\| + \|\bar{Z}^n_r(.\)\| \right] \, dr \right) \]

\[ + \lambda \mathbb{E} \int_s^T [6 + |Y_r| + ||Z_r(.)|| + |Y^n_r| + ||Z^n_r(.)||] \, \mathbb{I}_{A^m} \, dr. \]

Keeping in mind that

\[ \mathbb{I}_{A^m_\lambda} \leq M^{-1} \left[ |Y_s| + ||Z_s(.)|| + |Y^n_s| + ||Z^n_s(.)|| \right] , \]

then, Schwartz inequality and Lemma (I) show that there exists a constant depends on \( \xi_1, \xi_2, \lambda \) and \( T \) such that the second and the last term in the previous inequality are bounded by \( \frac{C(\xi_1, \xi_2, \lambda)}{M^{\frac{1}{2}}} \) and thus,

\[ \mathbb{E} \int_s^T |f_n(r, X_r, Y^n_r, Z^n_r(.)) - f(r, X_r, Y_r, Z_r(.))| \, dr \]

\[ \leq \mathbb{E} \int_s^T \sup_{|y|, ||z(.)|| \leq M} |(f_n - f)(r, X_r, y, z(.))| \, dr \]

\[ + L_M \left( \mathbb{E} \int_s^T \left[ \|\bar{Y}^n_r\| + \|\bar{Z}^n_r(.\)\| \right] \, dr \right) + \frac{C(\xi_1, \xi_2, \lambda)}{M^{\frac{1}{2}}}. \]
Noting that thanks to Lemma (3) (iii) and Schwartz inequality, the first term in the previous inequality tends to 0 when \( n \) goes to infinity. Then, by using Schwartz inequality and step 1, one can easily check that

\[
E \int_s^T \left( |Y_t^n| + \|Z_t^n(\cdot)\| \right) dr \to 0.
\]

Finally, due the fact that the constant \( C(\xi_1, \xi_2, \lambda) \) is independent of \( M \), the last term goes to 0 by sending \( M \) to infinity. And therefore, \( \int f_n(r, X_r, Y_r^n, Z_r^n(\cdot)) dr \) converges to \( \int f(r, X_r, Y_r, Z_r(\cdot)) dr \). Theorem is proved.

### 3.2.2 Stability of the solutions

Next we will give a stability Theorem for the solution to BDSE \((f, \xi)\) as a second main result in this paper. Our starting point is to define a sequence \((f_n)_n \in \mathbb{N}\) of Prog–progressively measurable functions, \((\xi_n)_n \in \mathbb{N}\) a sequence of \(\mathcal{F}_{[t,T]}\)-measurable and square integrable random variables. For each integer \(n\), we suppose that BSDE \((f_n, \xi_n)\) has a (not necessarily unique) solution \((Y^n, Z^n)\). Furthermore, We assume also that \((f_n, \xi_n)\) satisfies the following assumptions:

**H3.1.** For every \(M\), \( \Phi_M(f_n - f) \to 0 \) as \( n \to \infty \).

**H3.2.** \( E|\xi_n - \xi|^2 \to 0 \) as \( n \to \infty \).

**H3.3.** There exist \( \lambda > 0 \) such that:

\[
\sup_n |f_n(s, x, y, z(\cdot))| \leq \lambda \left[ 1 + |y|^{\alpha} + \|z(\cdot)\|^\alpha \right], \text{P-a.s.} \forall s \in [t, T].
\]

**Theorem 3 (Stability Theorem)** Suppose that \((f, \xi)\) satisfies Assumptions \((H2.1), (H2.2), (H2.3)\) and \((H2.4)\), \((f_n, \xi_n)\) satisfies \((H3.1), (H3.2)\) and \((H3.3)\). Then we have

\[
\lim_{n \to \infty} E \int_s^T (|Y_t^n - Y_t|^2) dr + E \int_s^T \|Z_t^n(\cdot) - Z_t(\cdot)\|^2 dr = 0.
\]

**Proof** We apply Lemma (2) to \((Y, Z, f, \xi)\) and \((Y^n, Z^n, f^n, \xi^n)\), we get

\[
E(|Y_t^n - Y_t|^2) \leq \left[ E(|\xi_n - \xi|^2) + \Phi_M^2(f_n - f) + \frac{C(\xi_1, \xi_2, \lambda)}{(1 + 2L^2_M) M^{2(1-\alpha)}} \right] \exp \left[ (4 + 4L^2_M)(T - s) \right].
\]
\[
\mathbb{E} \int_s^T \|Z^n_r(.) - Z_r(.)\|^2 \, dr \leq C(\xi_1, \xi_2, \lambda) \left[ \mathbb{E} (|\xi_n - \xi|^2) + \mathbb{E} \left( \int_s^T |Y^n_r - Y_r|^2 \, dr \right)^{\frac{1}{2}} \right].
\]

Passing to the limits, first on \(n\) and next on \(M\), and using Lebesgue’s dominated convergence Theorem, we arrive at

\[
\mathbb{E} \int_s^T |Y^n_r - Y_r|^2 \, dr \xrightarrow{n,M \to 0} 0,
\]

\[
\mathbb{E} \int_s^T \|Z^n_r(.) - Z_r(.)\|^2 \, dr \xrightarrow{n \to 0} 0.
\]

The Theorem is proved.

4 Example in finance

In this section, we adopt the notations of section 2 and we shall give an example to illustrate our theoretical results. We consider an application to European option pricing in the constraint case. Noting that the continuos Brownian case of this model have been treated in [11], under the globally Lipschitz setting. We impose here to work with a quite general semimartingale framework assuming that the wealth process is driven by pure jump Markov process. Roughly speaking, this kind of models comes naturally as consequence of the lack of continuity in many real world of applications. Indeed, the empirical distribution of wealth process tend to deviate from normal distributions. For instance, due to inspected dusters or huge profits, or even many successive incidents. We start again with a measurable space \((\Gamma, \mathcal{E})\), tarnation measure \(v\) on \(\Gamma\), satisfying \(\sup_{t \in [0,T], x \in \Gamma} v(t, x, \Gamma) < \infty\) and \(v(t, x, \{x\}) = 0\). The jump Markov process \(X\) is also constructed as described in section 2. Our main object is to prove the existence of feasible strategy to the following general wealth equation

\[
\begin{cases}
-dY_s &= g(s, X_s, Y_s, \pi_s) \, ds - \int_\Gamma \pi_s \sigma_s q(ds, dy) \\
Y_T &= h(X_T) \quad s \in [0, T].
\end{cases}
\]

such that \(g\) is a \(\mathbb{R}\) valued function defined on \([0, T] \times \Gamma \times \mathbb{R} \times \mathbb{R}\), \(\pi\) is the portfolio process, \(\sigma\) is the volatility traded security, \(\xi := h(X_T) \geq 0\) is
the contingent claim, which is supposed to be $\mathcal{F}_{[t,T]}$-measurable and square integrable random variable. Here $Y_s$ is the price of the contingent claim $\xi$ at time $s$. Noting that this model can be considered as generalization of Merton’s model in two directions: (i) By letting the generator takes a general form (not necessarily linear). (ii) By involving the Markov jump part in the wealth process, so that the dynamic contain a discontinuous part.

In the other hand, if we set $Z(t):=\sigma \pi$, and we suppose that $\sigma^{-1}$ is uniformly bounded, the instantaneous wealth process becomes

\[
\begin{align*}
-dY_s &= f(s, X_s, Y_s, Z(.) ) ds - \int_{\Gamma} Z(y) q(ds, dy) \\
Y_T &= h(X_T), \quad s \in [0,T].
\end{align*}
\] (4.1)

where $g(s, X_s, Y_s, (\sigma^{-1} Z(.) )=f(s, X_s, Y_s, Z(.) )$.

Furthermore, if the generator $g$ satisfies the hypotheses ($H_{2.1}$), ($H_{2.2}$) and ($H_{2.3}$), Theorem 2 confirmed to us that BSDE (4.1) has a unique solution $(Y, Z)$. Therefore, there exists a unique wealth-portfolio strategy $(Y, \pi)$ which belongs to $\mathcal{B}$, such that $\pi = \sigma^{-1} Z$.

It remain to prove that the unique existing strategy is feasible (that is, the constraint of non-negative holds true: $Y_s \geq 0$ a.s. $t \in [0,T]$). To fulfill this, we need to impose the following sufficient condition, $f(s, X_s, 0, 0) \geq 0$. Indeed, to benefit from the comparison Theorem for globally Lipschitz BSDE, we first construct a sequence of globally Lipschitz generators $(f^n)_{n \in \mathbb{N}}$, by replacing $f$ in BSDE (4.1) by $f^n$ defined in Lemma 3, to obtain the following family of approximating BSDEs

\[
Y^n_s = h(X_T) + \int_{s}^{T} f^n(r, X_r, Y^n_r, Z^n_r(\cdot))dr - \int_{s}^{T} \int_{\Gamma} Z^n_r(y) q(dr, dy), \quad s \in [t,T],
\] (4.2)

In view of Theorem 1, BSDE (4.2) has a unique solution $(Y^n, Z^n)$, for each integer $n$. Since $h(X_T) \geq 0$, and $f^n \geq 0$, the comparison Theorem (cf. Theorem 3.9 in [6]), leads to $Y^n \geq 0$. Using similar arguments in the proof of Theorem 2, one can easily show that $(Y^n, Z^n)$ is a Cauchy sequence then it converges to $(Y, Z)$. This implies that $Y \geq 0$. Moreover if the the contingent claim is bounded, lemma 1(ii) shows that the wealth process is also bounded in the sense that there exists a positive real constant $K$ such that for any $0 \leq t \leq T, 0 < Y_t \leq K$. 

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5 Conclusion

In this work, we have studied a class of backward stochastic differential equations driven by a random measure associated to jump Markov process. Motivated by the work of Confortola F, Fuhrman M [10], we have proved an existence and uniqueness result to this kind of equations by assuming weaker assumptions on the coefficients. More precisely, we have treated the locally Lipschitz case by using similar techniques developed in Bahlali [2] with some suitable changes due to the difference between the processes and the spaces. We note that pretty much of the technical difficulties coming from the pure jump Markov process part are due to the fact that its quadratic variation, which can be represented as an integral with respect to the random measure \( p(dt \, dy) \), is not absolutely continuous with respect to the Lebesgue measure. To overcome these difficulties, we use the fact that \( q(dr \, dy) := p(dr \, dy) - v(r, X_r, dy)dr \) is an \( \mathbb{F}^t \)-martingale, where \( v(r, X_r, dy)dr \) represents the dual predictable projection of \( p(dr \, dy) \).

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