EMBEDDING PROPER ULTRAMETRIC SPACES INTO $\ell_p$ AND ITS APPLICATION TO NONLINEAR DVORETZKY’S THEOREM

KEI FUNANO

Abstract. We prove that every proper ultrametric space isometrically embeds into $\ell_p$ for any $p \geq 1$. As an application we discuss an $\ell_p$-version of nonlinear Dvoretzky’s theorem.

1. Introduction

Recall that a metric space $(X, \rho)$ is called an ultrametric space if for every $x, y, z \in X$ we have $\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}$. Such spaces naturally appear and have applications in various areas such as number theory, $p$-adic analysis, and computer science (see [9], [10], [16, 17]).

Let us briefly review several results with respect to isometric embedding of ultrametric spaces. Timan and Vestfrid [21, 22] proved that any separable ultrametric space embeds isometrically into $\ell_2$. Vestfrid [24] later proved that the result is also true if one replace $\ell_2$ by $\ell_1$ and $c_0$ by constructing a universal ultrametric space for the class of separable ultrametric space and using its property. Vestfrid [23] also proved that a certain class of countable ultrametric spaces embed isometrically into $\ell_p$ for $p \geq 1$. Lemin [10] proved that any separable ultrametric space embeds isometrically into the Lebesgue space. He also raised a problem whether any separable ultrametric space embeds isometrically into any infinite dimensional Banach space. Motivated by Lemin’s problem, Shkarin [20] proved that every finite ultrametric space embeds into every infinite dimensional Banach space. From these results ultrametric spaces have attracted much attention in embedding theory.

In this paper we tackle Lemin’s problem in the case where the target Banach space is $\ell_p$. It is already well-known that every separable ultrametric space embeds isometrically into the function space $L_p$ for any $p \geq 1$. In fact, it follows from Timan and Vestfrid’s result mentioned above and the fact that $\ell_2$ embeds isometrically into $L_p$. Since $\ell_2$ does not embed bi-Lipschitzly into $\ell_p$ for any $p \neq 2$ ([11 Corollary 2.1.6]), embedding separable ultrametric spaces into $\ell_p$ is left as a problem. Our main theorem is the following: Recall that a metric space is proper if every closed ball in $X$ is compact.

Theorem 1.1. Every proper ultrametric space isometrically embeds into $\ell_p$ for any $p \geq 1$.
The case of general separable ultrametric spaces remains open. A similar method of the proof of Theorem 1.1 also implies an isometric embedding into $c_0$ (see Remark 23). Our construction of isometric embeddings into $\ell_1, \ell_2,$ and $c_0$ is different from the one by 21, 22, 23, 24 in the case of proper ultrametric spaces.

As an application of Theorem 1.1 we obtain an $\ell_p$-version of nonlinear Dvoretzky’s theorem, see Section 3.

2. Proof of the main theorem

We use some basic facts of compact ultrametric spaces (see 8, 12, Section 2). Let $(X, \rho)$ be a compact ultrametric space and put $r_0 := \text{diam } X$. Consider the relation $\sim_0$ on $X$ given by $x \sim_0 y \iff \rho(x, y) < r_0$. Since $\rho$ is ultrametric $\sim_0$ is an equivalence relation on $X$. The compactness of $X$ implies that each equivalence class is a closed ball of radius strictly less than $r_0$ (see 12, Section 2). Since the distance between two distinct equivalence classes is exactly $r_0$ and $X$ is totally bounded, there are only finitely many equivalence classes, say, $\{B_1, \ldots, B_k\}$, where each $B_i$ is a closed ball of radius $r_i = \text{diam } B_i < r_0$. Note that for any $x \in B_i$ and $y \in B_j$ ($i \neq j$) we have $\rho(x, y) = r_j$. For each $i$ we choose $x_i \in B_i$ and fix it. As above for each $i_1 = 1, \ldots, k_1$ we consider the equivalence relation $\sim_{i_1}$ on $B_{i_1}$ given by $x \sim_{i_1} y \iff \rho(x, y) < r_{i_1}$. Then we can divide $X_{i_1}$ into finitely many equivalence classes, i.e., $B_{i_1} = \bigcup_{i_2=1}^{k_{i_1}} B_{i_1 i_2}$, where $B_{i_1 i_2}$ is a closed ball of radius $r_{i_1 i_2} = \text{diam } B_{i_1 i_2} < r_{i_1}$. We may assume that $x_{i_1} \in B_{i_1 1}$. For each $i_1, i_2$, we choose a point $x_{i_1 i_2} \in B_{i_1 i_2}$ so that $x_{i_1 1} = x_{i_1}$ and we fix $x_{i_1 i_2}$. Repeatedly we get a sequence $P_k = \{B_{i_1 \ldots i_k} \mid i_1, \ldots, i_k\}$ of partitions of $X$ satisfying the following:

1. Each $B_{i_1 \ldots i_k}$ is a closed ball of radius $r_{i_1 \ldots i_k} = \text{diam } B_{i_1 \ldots i_k}$.
2. If $r_{i_1 \ldots i_k} \neq 0$, then $r_{i_1 \ldots i_k} > r_{i_1 \ldots i_{k-1}}$.
3. $B_{i_1 \ldots i_{k-1}} = \bigcup_{i_k=1}^{k_{i_{k-1}}} B_{i_1 \ldots i_{k-1} i_k}$.

For each $i_1, \ldots, i_k$ we choose $x_{i_1 \ldots i_k} \in B_{i_1 \ldots i_k}$ so that $x_{i_1 \ldots i_k_{1-1}} = x_{i_1 \ldots i_k}$. The compactness of $X$ yields the following:

Lemma 2.1 (cf. 12, Section 2). $\lim_{k \to \infty} \max_{i_1, \ldots, i_k} r_{i_1 \ldots i_k} = 0$.

In particular, $\bigcup_{k=1}^{\infty} \{x_{i_1 \ldots i_k} \mid i_1, \ldots, i_k\}$ is a countable dense subset of $X$.

Lemma 2.2 (cf. 12, Section 2). For every closed ball $B$ in $X$, there exist $k$ and $B_{i_1 \ldots i_k} \in P_k$ such that $B = B_{i_1 \ldots i_k}$.

Proof of Theorem 1.1. We first prove the theorem for compact ultrametric spaces. Let $(X, \rho)$ be a compact ultrametric space and let $P_k = \{B_{i_1 \ldots i_k} \mid i_1, \ldots, i_k\}$, and $x_{i_1 \ldots i_k}$ as above. Put $N_k := \# P_k$. We consider each coordinate of an element of $\ell^N_p$ is indexed by $(i_1, \ldots, i_k)$. We define a map $f_k : \{x_{i_1 \ldots i_k} \mid i_1, \ldots, i_k\} \to \ell^N_p$ as follows: $(f_k(x_{i_1 \ldots i_k})(j_1, \ldots, j_k) := 0$ if $(j_1, \ldots, j_k) \neq (i_1, \ldots, i_k)$ and

\[
(f_k(x_{i_1 i_k}))(j_1, \ldots, j_k) := \frac{(r_{i_1 \ldots i_k})^k}{2^k} \quad \text{and} \quad (f_k(x_{i_1 \ldots i_k}))(j_1, \ldots, j_k) := \frac{(r_{i_1 \ldots i_k})^k - r_{i_1 \ldots i_k})^k}{2^k} \quad \text{if } k \geq 2.
\]

Note that $f_k(x_{i_1 \ldots i_k}) \perp f_k(x_{j_1 \ldots j_k})$ for two distinct $(i_1, \ldots, i_k)$, $(j_1, \ldots, j_k)$.

We define a map $f : \bigcup_{k=1}^{\infty} \{x_{i_1 \ldots i_k} \mid i_1, \ldots, i_k\} \to \ell_p$ as follows. For each $x_{i_1 \ldots i_k}$, putting $i_m := 1$ for $m > k$, we define

\[
f(x_{i_1 i_k \ldots i_m}) := (f_1(x_{i_1}), f_2(x_{i_1 i_2}), \ldots, f_m(x_{i_1 \ldots i_m}), \ldots).
\]
The right-hand side in the above definition is actually the element of $\ell_p$ since
\[
\sum_{m=1}^{\infty} \|f_m(x_{i_1\ldots i_m})\|_p^p = \sum_{m=1}^{\infty} \frac{r_{i_1\ldots i_{m-1}}^p - r_{i_1\ldots i_m}^p}{2} = \frac{r_0^p}{2} < +\infty
\]
by Lemma 2.1. Note that $f$ is well-defined in the sense that $f(x_{i_1\ldots i_k1\ldots 1}) = f(x_{i_1\ldots i_k})$.

We shall prove that $f$ is an isometric embedding. Since $\bigcup_{k=1}^{\infty} \{x_{i_1\ldots i_k}\}_{i_1\ldots i_k}$ is dense in $X$ this implies the theorem. Taking two distinct elements $x_{i_1\ldots i_k}$ and $x_{j_1\ldots j_l}$ we may assume that $k \leq l$. Put $i_m := 1$ for $m > k$. Then we have $(i_1, \ldots, i_l) \neq (j_1, \ldots, j_l)$. Letting
\[
n := \min\{m \leq l \mid i_m \neq j_m\}
\]
we get $\rho(x_{i_1\ldots i_k}, x_{j_1\ldots j_l}) = \text{diam } B_{i_1\ldots i_{n-1}} = r_{i_1\ldots i_{n-1}}$ if $n \geq 2$ and $\rho(x_{i_1\ldots i_k}, x_{j_1\ldots j_l}) = r_0$ if $n = 1$. Since $f_m(x_{i_1\ldots i_m}) = f_m(x_{j_1\ldots j_m})$ for $m < n$ and $f_m(x_{i_1\ldots i_m}) \mid_{x_{j_1\ldots j_m}}$ for $m \geq n$,
\[
\|f(x_{i_1\ldots i_k}) - f(x_{j_1\ldots j_l})\|_p^p = \sum_{m=n}^{\infty} \|f(x_{i_1\ldots i_m})\|_p^p + \sum_{m=n}^{\infty} \|f(x_{j_1\ldots j_m})\|_p^p
\]
\[
= r_{i_1\ldots i_{n-1}}^p
= \rho(x_{i_1\ldots i_k}, x_{j_1\ldots j_l})^p.
\]
This completes the proof of the theorem for compact ultrametric spaces.

Let $(X, \rho)$ be a proper ultrametric space and fix a point $x_0 \in X$. For any $r > 0$ we denote by $B(x_0, r)$ the closed ball of radius $r$ centered at $x_0$. For any $R > 0$ let $f_1 : B(x_0, R) \to \ell_p$ be an isometric embedding constructed as in the above way. It suffices to prove that for any $R' > R$ we can construct an isometric embedding $f_2 : B(x_0, R') \to \ell_p$ as in the above way, which extends $f_1$ in the following sense: There exists an isometry $T : \ell_p \to \ell_p$ such that $T \circ f_2|_{B(x_0, R)} = f_1$. This is possible by the above construction. In fact, keep dividing $B(x_0, R')$ as in the above way. Then at finite steps we reach at $B(x_0, R)$ by Lemma 2.2 since $B(x_0, R')$ is compact. From the above construction we easily see the existence of $f_2$ and $T$. This completes the proof of the theorem.

**Remark 2.3.** A similar method of the above proof implies new isometric embeddings of proper ultrametric spaces into $c_0$. In fact, let us consider first the case of compact ultrametric spaces. Using the same notation as above, for each $k$ we define $g_k : \{x_{i_1\ldots i_k}\}_{i_1\ldots i_k} \to \ell_p$ as follows: $(g_k(x_{i_1\ldots i_k}))(j_1, \ldots, j_k) := 0$ if $(j_1, \ldots, j_k) \neq (i_1, \ldots, i_k)$ and
\[
(g_1(x_{i_1}))(i) := r_0 \quad \text{and} \quad (g_k(x_{i_1\ldots i_k}))(i_{1\ldots k-1}i) := r_{i_1\ldots i_{k-1}} \quad \text{if} \quad k \geq 2.
\]
Then we define a map $g : \bigcup_{k=1}^{\infty} \{x_{i_1\ldots i_k}\}_{i_1\ldots i_k} \to c_0$ by
\[
g(x_{i_1i_2\ldots i_k}) := (g_1(x_{i_1}), g_2(x_{i_1i_2}), \ldots, g_m(x_{i_1\ldots i_m}), \ldots),
\]
where as in the above proof we put $i_m := 1$ for $m > k$. Note that the right-hand side of the above definition is in $c_0$ by Lemma 2.1. We can easily check that the map $g : \bigcup_{k=1}^{\infty} \{x_{i_1\ldots i_k}\}_{i_1\ldots i_k} \to c_0$ is an isometric embedding. As in the proof of Theorem 1.1, this construction also implies an isometric embedding from every proper ultrametric space into $c_0$. 
3. $\ell_p$-version of nonlinear Dvoretzky’s theorem

In this section we apply Theorem 1.1 to obtain an $\ell_p$-version of nonlinear Dvoretzky’s theorem. Refer to [3], [5] for the case of finite metric spaces.

We say that a metric space $X$ is embedded with distortion $D \geq 1$ in a metric space $Y$ if there exist a map $f : X \to Y$ and a constant $r > 0$ such that

$$rd_X(x, y) \leq d_Y(f(x), f(y)) \leq Dr d_X(x, y)$$

for all $x, y \in X$.

Dvoretzky’s theorem states that for every $\varepsilon > 0$, every $n$-dimensional normed space contains a $k(n, \varepsilon)$-dimensional subspace that embeds into a Hilbert space with distortion $1 + \varepsilon$ ([6]). This theorem was conjectured by Grothendieck ([7]). See [14] and [15], [19] for the estimate of $k(n, \varepsilon)$. Bourgain, Figiel, and Milman [4] first studied Dvoretzky’s theorem in the nonlinear setting. They obtained that for every $\varepsilon > 0$, every finite metric space $X$ contains a subset $S$ of sufficiently large size which embeds into a Hilbert space with distortion $1 + \varepsilon$. See [2], [11], [18] for further investigation. Recently Mendel and Naor [12, 13] studied another variant of nonlinear Dvoretzky’s theorem, answering a question by T. Tao. For example they obtained the following: For a metric space $X$ we denote by $\dim_H(X)$ the Hausdorff dimension of $X$.

**Theorem 3.1 (cf. [13 Theorem 1.7]).** There exists a universal constant $c \in (0, \infty)$ such that for every $\varepsilon \in (0, \infty)$, every compact metric space $X$ contains a closed subset $S \subseteq X$ that embeds with distortion $2 + \varepsilon$ in an ultrametric space, and

$$\dim_H(S) \geq \frac{c \varepsilon}{\log(1/\varepsilon)} \dim_H(X).$$

Note that since every separable ultrametric space isometrically embed into $\ell_1$, $\ell_2$, and $c_0$ ([24]), the above $S$ embeds into these spaces.

Applying Theorem 1.1 to Theorem 3.1 we obtain the following $\ell_p$-version of nonlinear Dvoretzky’s theorem:

**Corollary 3.2.** There exists a universal constant $c \in (0, \infty)$ such that for every $\varepsilon \in (0, \infty)$, every compact metric space $X$ contains a closed subset $S \subseteq X$ that embeds with distortion $2 + \varepsilon$ in $\ell_p$, and

$$\dim_H(S) \geq \frac{c \varepsilon}{\log(1/\varepsilon)} \dim_H(X).$$

Mendel and Naor also obtained the following impossibility result for distortion less than 2:

**Theorem 3.3 (cf. [13 Theorem 1.8]).** For every $\alpha > 0$ there exists a compact metric space $(X, d)$ of Hausdorff dimension $\alpha$, such that if $S \subseteq X$ embeds into a Hilbert space with distortion strictly smaller than 2 then $\dim_H(S) = 0$.

We shall consider an impossibility problem for the $\ell_p$-version of nonlinear Dvoretzky’s theorem.

In the proof of Theorem 3.3 Mendel and Naor used the following result: Let $G$ be the random graph on $n$-vertices of the Erdős-Reyni model $G(n, 1/2)$, i.e., every edge is present independently with probability $1/2$. From $G$ we construct a metric space $W_n$ by assigning the distance between each two vertices of $G$ by 1 if they are joined by an edge, and 2 if they are not joined by an edge. Then the obtained metric space $W_n$ satisfies the following property ([2]). There exists $K \in (0, \infty)$ such
that for any $n \in \mathbb{N}$ there exists an $n$-point metric space $W_n$ such that for every $\delta \in (0, 1)$ any subset of $W_n$ of size larger than $2 \log n + K(\delta^{-2} \log(2/\delta))^2$ must incur distortion at least $2 - \delta$ when embedded into $\ell_2$.

Bartal, Linial, Mendel, and Naor obtained a similar result for the same $W_n$ when considering $\ell_p$ instead of $\ell_2$ [3]. Then Charikar and Karagiozova [5, Theorem 1.3] improved the result in [3]: For any $\delta \in (0, 1)$ and $p \geq 1$, there is a constant $c(p, \delta)$ depending only on $p$ and $\delta$ such that any subset of $W_n$ of size larger than $c(p, \delta) \log n$ must incur distortion at least $2 - \delta$ when embedded into $\ell_p$.

Then applying this result to the proof in [13, Section 7.3] implies the following:

**Proposition 3.4.** For every $p \geq 1$ and $\alpha > 0$, there exists a compact metric space $(X, d)$ with $\dim H(X, d) = \alpha$, such that if $S \subseteq X$ embeds into $\ell_p$ with distortion strictly smaller than 2 then $\dim H(S) = 0$.

The case of the distortion 2 remains open for any $p \geq 1$.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502
E-mail address: kfunano@kurims.kyoto-u.ac.jp