OLSON’S THEOREM FOR CYCLIC GROUPS

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Abstract. Let \( n \) be a large number. A subset \( A \) of \( \mathbb{Z}_n \) is complete if \( S_A = \mathbb{Z}_n \), where \( S_A \) is the collection of the subset sums of \( A \). Olson proved that if \( n \) is prime and \( |A| > 2n^{1/2} \), then \( S_A \) is complete. We show that a similar result for the case when \( n \) is a composite number, using a different approach.

1. Introduction

Let \( G \) be an additive group. For a subset \( A \subset G \), we denote by \( S_A \) the collection of the subset sums of \( A \)

\[
S_A = \left\{ \sum_{x \in B} x \mid B \subset A, |B| < \infty \right\}.
\]

Following [1], we say that \( A \) is complete (with respect to \( G \)) if \( S_A = G \); in other words, every element of \( G \) can be represented as a sum of different elements of \( A \).

In this short note, we investigate the case when \( G = \mathbb{Z}_n \), the cyclic group of order \( n \), where \( n \) is a large positive integer.

A well-known result of Olson [5], answering a question of Erdős and Heilbronn, shows that if \( n \) is a prime and \( |A| > 2n^{1/2} \), then \( A \) is complete.

Theorem 1.1. If \( n \) is a prime and \( A \) is a subset of \( \mathbb{Z}_n \) with cardinality larger than \( 2n^{1/2} \), then \( A \) is complete.

The bound is sharp. To see this observe that if the sum of the elements in \( A \) (viewed as integers between 1 and \( n-1 \)) is less than \( n \), then \( A \) is not complete.

We extend this result for the case when \( n \) is a composite number. Our result is

Theorem 1.2. There is a constant \( C \) such that the following holds. Let \( n \) be a sufficiently large positive integer and \( A \) be a subset of \( \mathbb{Z}_n \), where \( |A| \geq Cn^{1/2} \) and the elements of \( A \) are co-primes with \( n \). Then \( A \) is complete.

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Remark 1.3. The assumption that the elements of $A$ are co-primes with $n$ is necessary. For instance, if $n$ is divisible by 3 then it is possible to have an incomplete set of size $n/3$. Without the co-prime assumption, the problem of bounding $|A|$ is known as Diderrich's problem. It has been proved that the sharp bound for $|A|$ is $p + n/p - 2$, where $p$ is the smallest prime divisor of $n$ (see [4] for the case of cyclic groups and [3] for the general case of arbitrary abelian groups).

In the current proof, the constant $C$ in Theorem 1.2 is fairly large. However, we believe that the constant $C$ in Theorem 1.2 can be set to (the asymptotically optimal value) $2 + o(1)$.

It seems to be of interest to investigate the general case. Given a finite abelian group $G$, one would like to find a parameter $f(G)$ so that if $A$ is a set of at least $f(G)$ primitive elements, then $A$ is complete. This problem can be seen as a variant of Diderrich’s problem. On the other hand, by comparing the bounds in Theorem 1.2 and Remark 1.3, it is plausible that the answer would be quite different. In fact, we think that the nature of this problem is closer to that of Olson’s than to Diderrich’s.

Notation. In the whole paper, we understand that the elements of a set are different. If there are possible repetitions we use the phrase multi-set instead.

2. Lemmas

For a set $A$ of integers and a positive integer $l \leq |A|$, let $l^*A$ denote the set of sums of $l$ different elements of $A$

$$l^*A = \{a_1 + \ldots + a_l | a_i \in A, a_i \neq a_j \}.$$ 

Denote by $[n]$ the set $\{1, 2, \ldots, n\}$. In [7], Szemerédi and the author proved the following theorem.

**Theorem 2.1.** There are positive constants $C$ and $c$ such that the following holds. Let $l$, $n$ be positive integers and $A$ be a subset of $[n]$ such that $|A|/2 \geq l$ and $l|A| \geq Cn$. Then $l^*A$ contains an arithmetic progression of length $cl|A|$.

Since $l^*A \subset S_A$, this theorem implies the following corollary.

**Corollary 2.2.** There is a positive constant $C$ such that the following holds. For every sufficiently large integer $n$ and a subset $A$ of $[n]$ of cardinality at least $Cn^{1/2}$, $S_A$ contains an arithmetic progression of length $n$.

**Remark 2.3.** The bounds on both $|A|$ and the length of the arithmetic progression is sharp, up to constant factors. Freiman [2] and Sárközy [6], independently, showed that the same statement holds under the stronger assumption that $|A| \geq C\sqrt{n\log n}$.
We also need the following simple lemma.

**Lemma 2.4.** Let \( n \) be a positive integer and \( A \) be a multi-set of \( n \) integers co-prime to \( n \). Then \( S_A \) contains every residue modulo \( n \).

**Proof of Lemma 2.4.** Assume that \( a_1, a_2, \ldots, a_n \) are the elements of \( A \). We are going to prove, by induction, that \( |S_{A_i}| \geq i \), where \( A_i = \{a_1, \ldots, a_i\} \). The case \( i = 1 \) is trivial. Assume that the statement holds for \( i - 1 \). Let \( b_1, \ldots, b_{i-1} \) be \( i - 1 \) different elements (modulo \( n \)) of \( S_{A_{i-1}} \). Since the statement is invariant under dilation, we can assume that \( a_i = 1 \). Consider the elements

\[
b_1, \ldots, b_{i-1}, 1, 1 + b_1, \ldots, 1 + b_{i-1}.
\]

At least \( i \) of the above must be different (modulo \( n \)) and this concludes the proof. \( \square \)

3. **Proof of Theorem 1.2.**

Assume that \( A \) has at least \( 2\lceil Cn^{1/2} \rceil \) elements, where \( C \) is the constant in Corollary 2.2. For convenience, we think of the elements of \( A \) as positive integers between one and \( n - 1 \). We are going to prove that \( S_A \) contains every residue modulo \( n \).

Let \( A' \) be a subset of \( A \) of \( \lceil Cn^{1/2} \rceil \) elements. Apply Corollary 2.2 to \( A' \) to get an arithmetic progression \( P' \) of length \( n \). If the difference \( d' \) of \( P' \) is co-prime to \( n \), then \( P' \) contains every residue modulo \( n \) and we are done. If \( d' \) is not co-prime to \( n \), set \( d = \gcd(d', n) \).

Since the largest element in \( S_{A'} \) is less than \( Cn^{3/2} \), \( d \) is less than \( Cn^{1/2} \). The set \( B = A \setminus A' \) has at least \( \lceil Cn^{1/2} \rceil > d \) elements, each of which is co-prime to \( n \) (and thus co-prime to \( d \)). By Lemma 2.4, we conclude that \( S_{B'} \) contains every residue modulo \( d \).

The set \( S_{A'} + S_B \) thus contains every residue modulo \( n \). But this set is clearly a subset of \( S_A \), completing the proof. \( \square \)

Notice that the proof requires that the elements of \( B \) are co-prime to \( n \); but for \( A' \), it is enough to assume that its elements are non-zero modulo \( n \).

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