Brane-Waves and Strings

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Abstract

Recently, solutions of the Born-Infeld theory representing strings emanating from a Dirichlet p-brane have been constructed. We discuss the embedding of these Born-Infeld solutions into the non-abelian theory appropriate to multiple overlapping p-branes. We also prove supersymmetry of the solutions explicitly in the full nonlinear theory. We then study transverse fluctuations, both from the worldbrane point of view and by analyzing a test-string in the supergravity background of a Dp-brane. We find agreement between the two approaches for the cases p=3,4.

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1 Introduction

The defining property of Dirichlet $p$-branes in type II superstring theory is that fundamental strings can have endpoints inside a $p$-brane worldvolume, whereas in the bulk of spacetime, away from any $p$-branes, the theory only contains closed strings. D-brane dynamics is governed by worldvolume terms in the spacetime effective action of type II string theory which arise from the open-string sector of the worldsheet theory. In a flat supergravity background the worldvolume action is given by the dimensional reduction of a ten-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory. In general this is some complicated non-abelian theory involving high-derivative terms whose detailed structure is not known at present, but for many applications it may be approximated by supersymmetric Yang-Mills theory, which is then dimensionally reduced in a straightforward manner. For the special case of a single $p$-brane the worldvolume gauge theory is abelian and is approximated, for slowly varying fields, by the dimensional reduction of ten-dimensional supersymmetric Born-Infeld non-linear electrodynamics. We also expect the Born-Infeld theory to govern any dynamics that only involves an abelian subsector of the more complicated non-abelian theory.

The non-linear Born-Infeld electrodynamics can itself be approximated by a linear theory, which is just ordinary Maxwell electrodynamics, but in doing so one loses some interesting features. It was for example recently shown, by Callan and Maldacena [1], by Gibbons [2], and by Howe et al. [3], that the full non-linear Born-Infeld theory has simple classical solutions with the interpretation of macroscopic strings extending from the $p$-brane worldvolume. It is interesting to see the open fundamental string arise as a classical solution in a gauge theory which was obtained from open string dynamics in the first place.

In the present paper we study these strings further in a number of ways. We first show how the Born-Infeld solution, describing some number of parallel fundamental strings emanating from a single $p$-brane, is generalized to the case of fundamental strings extending from a collection of parallel $p$-branes. This configuration lives in an abelian subsector of the gauge theory and does not involve non-abelian directions, and so does not require knowledge of the full non-abelian theory.

We then turn our attention to the supersymmetry of these configurations. We explicitly show that the string solutions of the Born-Infeld theory satisfy the supersymmetry conditions of the full non-linear theory, with the fraction of unbroken supersymmetry being $\nu = 1/4$.

Finally, we address some dynamical issues. Callan and Maldacena [1] considered fluctuations in the background of a string attached to a three-brane. They obtained a wave equation for the propagation of modes that are transverse to both the three-brane worldvolume and the string. In the bulk of the three-brane worldvolume the equation describes free propagation in $3 + 1$ dimensions but along the string it reduces to the $1 + 1$-dimensional wave equation. In the throat region where the string attaches to the three-brane the transverse modes encounter a potential which leads to non-trivial reflection and transmission amplitudes. In the low-frequency limit, Callan and Maldacena found that incident modes on the string undergo perfect reflection from the three-brane in a manner consistent with a Dirichlet boundary condition.

We obtain the corresponding equations for transverse fluctuations in the case of a general
a macroscopic string extending from an extremal black $p$-brane. Unfortunately, we do not have the appropriate solution of the full supergravity equations at our disposal, so we instead work within a test-string approximation. In this approximation we consider a macroscopic fundamental string in the curved background geometry of an extremal $p$-brane and obtain a wave-equation for transverse fluctuations traveling along the string. Remarkably, for $p = 3, 4$, the two approaches give identical results upon identification of certain parameters. We finish with a discussion of the range of validity of the two calculations and some suggestions for further work.

2 Fundamental strings emanating from D-branes

The classical dynamics of a Dirichlet $p$-brane are governed by a non-linear effective action. Kappa-symmetric gauge-invariant actions for Dirichlet $p$-branes have been constructed for arbitrary supergravity backgrounds in [4] and [5]. In a remarkable way, the requirement of kappa-symmetry puts the supergravity background on-shell. In this section, however, we will be studying D$p$-branes in the weak-coupling regime where the supergravity background may be taken to be flat. Kappa- and super-symmetry were studied for flat backgrounds, and in detail in the static gauge, by Aganagic et al. [6]. For our purposes the static gauge will be convenient, and so we will use the results of [6].

Our conventions are essentially similar to those of that work, but not identical, so we list them here. The spacetime signature is $(-, +, \ldots, +)$. Greek indices are curved, and roman indices are flat. Indices $\alpha, \beta, a, b = 0, \ldots, p$ are worldbrane; $\rho, \sigma, r, s = (p + 1) \ldots 9$ are transverse to the brane; and $\mu, \nu, m, n = 0 \ldots 9$ are ten dimensional. Where we need to enumerate indices and confusion could arise, we will use hats to denote flat indices. Worldbrane coordinates are $\sigma^\alpha$, target space coordinates are $x^\mu$. Lastly, we take $\alpha' = 1$.

The action for the $Dp$-brane then has two terms, the first being the Born-Infeld term and the second the Wess-Zumino term which allows branes to end on other branes. The relative coefficients are fixed by kappa-symmetry.

The static gauge is an embedding choice which sets the $p + 1$ longitudinal target, and worldbrane, coordinates equal:

$$x^\alpha = \sigma^\alpha .$$

(1)

Kappa symmetry may be used to eliminate half of the fermionic degrees of freedom; in static gauge one of the two spinors of the type II theory may be set to zero. This has the consequence that the WZ term vanishes in static gauge. The remaining spinor is renamed $\lambda$, and is the superpartner of the gauge and scalar fields.
2.1 Strings emanating from a lone D-brane

The equation of motion for a purely bosonic background in flat spacetime, using ten-dimensional notation, is

\[ \left( \frac{1}{\eta - F^2} \right)_\nu \partial_\nu F^\lambda = 0 , \]

where \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \) is the field strength of an abelian gauge field living in the \( p \)-brane worldvolume. The remaining components of \( F_{\mu\nu} \) are \( F_{\alpha\rho} = -\partial_\alpha \phi_\rho \), where \( \phi_\rho(\sigma^\alpha) \) are scalar fields describing the transverse displacement of the \( p \)-brane worldvolume, and finally \( F_{\rho\sigma} = 0 \) reflects the fact that all fields in the action depend only on worldvolume coordinates.

The Born-Infeld equation (2) has many interesting solutions. We focus on a class of static solutions, described recently by Callan and Maldacena [1] and by Gibbons [2], which have the interpretation of macroscopic fundamental strings extending away from a \( p \)-brane in a perpendicular direction (closely related solutions were also obtained by Howe et al. [3]). Such a solution may be obtained by a simple ansatz, describing the electric field \( E_\alpha \) due to a point charge in the \( p \)-brane worldvolume with only one of the scalar fields excited, say \( \phi_9 \), such that \( \partial_\alpha \phi_9 = \pm E_\alpha \). The choice of sign determines whether the string extends in the positive or negative \( x^9 \) direction. This solution is easily extended to include strings protruding from different locations on the \( p \)-brane [1, 2].

With this ansatz one needs to keep track of only the \( \mu, \nu = 0, \ldots, p, 9 \) components of the field strength tensor, which we combine into a \( p + 2 \) dimensional matrix

\[ F_{\mu\nu} = \begin{bmatrix} 0 & \pm \tilde{\nabla} \phi_9 & 0 \\ \mp \tilde{\nabla} \phi_9 & 0 & -\tilde{\nabla} \phi_9 \\ 0 & +\tilde{\nabla} \phi_9 & 0 \end{bmatrix} , \]

where \( \tilde{\nabla} \) denotes the spatial gradient on the worldvolume. The Born-Infeld equation (2) then reduces to

\[ \tilde{\nabla}^2 \phi_9 = 0 , \]

which is solved by the \( p \)-dimensional Coulomb potential

\[ \phi_9(r) = \pm \frac{b_p}{p - 2} \frac{1}{r^{p-2}} , \]

where \( r = \sqrt{x^2} \) is a radial worldvolume coordinate. For \( p = 2 \) the potential is logarithmic so we will only consider \( p \geq 3 \).

Since \( \phi_9 \) has the interpretation of the location of the brane in the \( \hat{9} \) direction, we see that the \( Dp \)-brane has developed a one-dimensional spike, emanating from the origin of worldbrane coordinates \( r = 0 \), directed along \( x^9 \). The spike is interpreted as a string [1, 2]. This identification was solidified in [1] where it was shown that \( b_p \) is quantized such that the energy per unit length of the spike is the fundamental string tension. The solution (5) is generalized to the case of multiple strings extending from \( r = 0 \) by multiplying the unit of charge \( b_p \) by the number of fundamental strings \( N_1 \).
2.2 Multiple D-branes

The discussion so far has been limited to the case where the number of \( p \)-branes was \( N_p = 1 \). It is natural to ask whether or not this construction can be carried over to the case where there are multiple \( p \)-branes. Later on we will compare gauge theory results to classical supergravity calculations which are only reliable if the \( p \)-brane carries a macroscopic R-R charge, \textit{i.e.} it is a collection of many coincident parallel branes.

For \( N_p > 1 \), the worldvolume theory is described by a non-abelian \( U(N_p) \) theory, in which the transverse coordinates \( \phi^i \) become matrix-valued. In abelian subsectors this action is expected to be just the Born-Infeld action again. Here we will look for the ground state of the multi-string-brane system for given \( N_1 \) and \( N_p \).

In our solutions, \( \partial_r \phi^9 = \pm F_{0r} \), so the \( U(N_p) \) matrices \( F_{0r} \) and \( \phi^9 \) may be simultaneously diagonalized, and since we do not excite any other directions we do not need to worry about subtleties of the non-abelian generalization of the Born-Infeld action. Each diagonal entry of \( \phi^9 \) then represents the \( x^9 \) coordinate of each \( p \)-brane. For now, let \( N_1 \) be larger than, and divisible by, \( N_p \), and imagine distributing the fundamental strings among the \( p \)-branes. Put \( n_i \) strings on the \( i \)-th \( p \)-brane such that

\[
N_1 = \sum_{i=1}^{N_p} n_i .
\]

(6)

Then the diagonal entries of \( \phi^9 \) are

\[
\phi^9_i(r) = \frac{n_i}{p - 2} \left( \frac{b_p}{N_p} \right) .
\]

(7)

Now let us introduce a cutoff \( \delta \) on the worldbrane radial coordinate \( r \) to keep the string length finite, and calculate the total energy of the system,

\[
E(\delta) = \left\{ N_1 + \frac{N_p^2}{N_1} (\Delta n)^2 \right\} T_f \phi^9_{cm}(\delta) .
\]

(8)

Here \( T_f \) is the fundamental string tension, \( \phi^9_{cm}(r) = \frac{1}{N_p} \text{tr} \phi^9(r) \) is the center of mass coordinate of the strings, and \( \Delta n \) is the standard deviation of the distribution \( \{n_i\} \). It follows that the energy is minimized when the strings are evenly distributed among the branes. The minimum energy is proportional to the center of mass coordinate of the strings.

In this naive picture, the exact ground state energy can be obtained only when \( N_1 \) is divisible by \( N_p \). It is, however, possible for the branes to “share” the strings for arbitrary \( N_1 \) in the following way. In the gauge theory, separate the center-of-mass \( U(1) \times SU(N_p) \) from \( U(N_p) = [U(1) \times SU(N_p)]/Z_n \). The energy density due to the center-of-mass electric field is \( (N_p/2g_p) F^2 \), where the factor of \( N_p \) can be understood as the total mass of the mechanical analogue. To obtain the minimum value of the energy, we recall that flux quantization is modified in the presence of multiple coincident \( p \)-branes. Fractionation occurs and the unit electric flux on each brane is reduced by \( 1/N_p \) by comparison with the case of the lone brane.

Then, since the unit charge for the single brane is \( b_p \), the center of mass transverse position and electric field become

\[
\phi^9_{cm}(r) = \pm \frac{1}{p - 2} \left( \frac{N_1}{N_p} b_p \right) , \quad F^0_{0r} = \frac{N_1}{N_p} \left( \frac{b_p}{r^{p-2}} \right) .
\]

(9)
The ground state energy of the system is
\[ E(\delta) = N_1 T f \phi_{cm}^9(\delta) \]
which exactly equals the ground state energy in (8), but is no longer subject to the same restrictions on \( N_1 \) and \( N_p \).

The upshot of all this is that whenever we consider a multiple-brane spike, we can work entirely within the center-of-mass \( U(1) \) and the equations are identical to those for the lone-brane spike. We only have to remember to replace the unit charge \( b_p \) by \( b_p(N_1/N_p) \).

3 Supersymmetry

In [1, 2] it was conjectured on the basis of calculations in the Maxwell limit that the brane-spike configuration is supersymmetric. We now proceed to show how this works in the full Born-Infeld theory. We will work explicitly with the case \( N_p = 1 \) but the calculation would be identical for the multiple-brane case by the reasoning at the end of the last section.

For the purpose of investigating supersymmetry, it is convenient to use the dimensionally reduced form of the Born-Infeld action. Again we follow [6]. The action is
\[ S = -T_p \int d^{p+1}x \sqrt{-\det \left( G^{(p)}_{\alpha\beta} + F^{(p)}_{\alpha\beta} \right)} \]
where
\[ G^{(p)}_{\alpha\beta} = \eta_{\alpha\beta} + \partial_\alpha \phi_\rho \partial_\beta \phi^\rho - \bar{\lambda}(\Gamma_\alpha + \Gamma_\rho \partial_\alpha \phi^\rho) \partial_\beta \lambda \\
- \bar{\lambda}(\Gamma_\beta + \Gamma_\rho \partial_\beta \phi^\rho) \partial_\alpha \lambda + \bar{\lambda} \Gamma^\mu \partial_\alpha \lambda \bar{\lambda} \Gamma_\mu \partial_\beta \lambda , \]
\[ F^{(p)}_{\alpha\beta} = F_{\alpha\beta} - \bar{\lambda}(\Gamma_\alpha + \Gamma_\rho \partial_\alpha \phi^\rho) \partial_\beta \lambda + \bar{\lambda}(\Gamma_\alpha + \Gamma_\rho \partial_\alpha \phi^\rho) \partial_\beta \lambda . \]
In these expressions, \( \lambda \) are the the fermionic superpartners of the gauge and scalar fields.

The choice of static gauge is consistent with kappa-symmetry and supersymmetry. The resulting supersymmetry variations are [3]
\[ \delta \bar{\lambda} = \bar{\epsilon} \left[ \Delta^{(p)} + \zeta^{(p)} \right] + \xi^\alpha \partial_\alpha \bar{\lambda} \]
\[ \delta \phi^\rho = \bar{\epsilon} \left[ \Delta^{(p)} - \zeta^{(p)} \right] \Gamma^\rho \lambda + \xi^\alpha \partial_\alpha \phi^\rho , \]
\[ \delta A_\alpha = \bar{\epsilon} \left[ \zeta^{(p)} - \Delta^{(p)} \right] (\Gamma_\alpha + \Gamma_\rho \partial_\alpha \phi^\rho) \lambda + \bar{\epsilon} \left[ \frac{1}{3} \Delta^{(p)} - \zeta^{(p)} \right] \Gamma_\mu \lambda \bar{\lambda} \Gamma^\mu \partial_\alpha \lambda \]
\[ + \xi^\rho \partial_\rho A_\alpha + \partial_\alpha \xi^\rho A_\rho \]
where the compensating general coordinate transformation parameter is [3]
\[ \xi^\alpha = \bar{\epsilon} (\zeta^{(p)} - \Delta^{(p)}) \Gamma^\alpha \lambda . \]
In the above we have used the abbreviation
\[ \Delta^{(p)} \equiv \pm \Gamma_9 \Gamma_8 \ldots \Gamma_{p+1}, \] (19)
where the ± sign is determined by reducing from ten dimensions. The crucial quantity \( \zeta^{(p)} \) depends on the gauge field via \( F \).

For bosonic backgrounds, the only nontrivial supersymmetry transformation is
\[ \delta \bar{\lambda} = \bar{\epsilon} \left[ \Delta^{(p)} + \zeta^{(p)} \right]. \] (20)

In order to proceed further, we need the form of \( \zeta^{(p)} \). We can extract it from the formulæ in section 3.2 of [6]. The equations are slightly different for the IIA and IIB cases, in that \( \Gamma_{11} \) appears in IIA equations while \( SO(2) \) matrices \( \{\tau_1, \tau_3, i\tau_2\} \) appear in those of IIB.

For definiteness, let us exhibit the relevant equations for the IIB theory with \( p \) odd:
\[ \gamma^{(p)} = \begin{pmatrix} 0 & \zeta^{(p)} \\ \tilde{\zeta}^{(p)} & 0 \end{pmatrix}, \] (21)
\[ \rho^{(p)} = \sqrt{-\det (G + F)} \gamma^{(p)}, \] (22)
\[ \rho^{(p)} = \frac{1}{(p+1)!} \epsilon^{\alpha_1 \ldots \alpha_{p+1}} \rho_{\alpha_1 \ldots \alpha_{p+1}}, \] (23)
\[ \rho_{p+1} = \frac{1}{(p+1)!} \rho_{\alpha_1 \ldots \alpha_{p+1}} d\sigma^\alpha \ldots d\sigma^{p+1}, \] (24)
\[ \rho_B = \sum_{p \text{ odd}} \rho_{p+1} = e^F C_B(\psi) \tau_1, \] (25)
\[ C_B(\psi) = (\tau_3) + \frac{1}{2!} \psi^2 + \frac{1}{4!} (\tau_3) \psi^4 + \frac{1}{6!} \psi^6 + \ldots, \] (26)
\[ \psi = \gamma_\alpha d\sigma^\alpha = (\partial_\alpha X^\mu - \bar{\lambda} \Gamma^\mu \partial_\alpha \lambda) \Gamma_\mu d\sigma^\alpha. \] (27)

In order to untangle this, let us inspect our background configuration. We have that the gauge field is purely electric and that the scalar \( \phi^9 \) is related to it by
\[ \partial_r \phi^9 = \chi F_{0r}, \] (28)
where \( \chi = \pm 1 \) and \( r \) is the radial variable in the worldbrane. For a configuration of this form, the Born-Infeld determinant conveniently collapses to unity.

In addition, since \( F \) is purely electric and our background is bosonic, the matrices \( \rho \) take the form
\[ \rho_{p+1} = \frac{1}{(p+1)!} (\tau_3^{(p-1)/2} \tau_1) \psi^{p+1} + \frac{1}{(p-1)!} (\tau_3^{(p+1)/2} \tau_1) F \psi^{p+1}. \] (29)

From now on, we will concentrate on the threebrane for definiteness, but the supersymmetry equations for other \( p = 4, \ldots, 9 \) follow in an exactly analogous fashion. Of course, this is to be expected from T-duality.

For the threebrane,
\[ \rho_4 = \frac{1}{4!} (\tau_3 \tau_1) \psi^4 + \frac{1}{2!} (\tau_1) F \psi^2, \] (30)
and so $\zeta^{(3)}$ becomes

$$
\zeta^{(3)} = \epsilon^{a_1 a_2 a_3 a_4} \left[ \frac{1}{4!} \gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} + \frac{1}{2!} F_{a_1 a_2} \gamma_{a_3} \gamma_{a_4} \right],
$$

where

$$
\gamma_{a} = \left( \partial_{\alpha} X^\mu - \bar{\lambda} \Gamma^\mu \partial_{\alpha} \lambda \right) \Gamma_{\mu},
$$

$$
= \Gamma_{\alpha} + \Gamma_{\rho} \partial_{\alpha} \phi^\rho,
$$

because we are in a bosonic background and in static gauge. Recall also that in a bosonic background $F = F$.

Our supersymmetry equation is now

$$
\delta \bar{\lambda} = \bar{\epsilon} \left[ F_{0 \bar{r}} \left( 1 + \chi (\Gamma_9 \Gamma_0) \right) \Gamma_{\bar{r}} \Gamma_{\phi} \right].
$$

To get this, we used the fact that the IIB spinor $\epsilon$ obeys the chirality condition $\Gamma_{11} \epsilon = \epsilon$.

Let the spinor $\epsilon$ also obey the condition

$$
(\Gamma_0 \Gamma_9) \epsilon = \chi_1 \epsilon
$$

appropriate to a string, with $\chi_1 = \pm 1$.

Then we have unbroken supersymmetry for

$$
\chi = -\chi_1.
$$

We see from (35) that the fraction of supersymmetry unbroken by the brane with spike is $\nu = 1/4$, as expected.

### 4 Brane-waves from Born-Infeld

In this section we will consider perturbations in the background of a $p$-brane with a string attached, from the worldbrane point of view. Let us expand the ten dimensional gauge field as

$$
F_{\mu \nu} = \tilde{F}_{\mu \nu} + \delta F_{\mu \nu},
$$

where $\tilde{F}_{\mu \nu}$ is the background value (3) and the fluctuations can in general have components both parallel and transverse to the $p$-brane,

$$
\delta F_{\alpha \beta} = \partial_{\alpha} (\delta A_{\beta}) - \partial_{\beta} (\delta A_{\alpha}),
$$

$$
\delta F_{\alpha \rho} = -\partial_{\alpha} (\delta \phi_{\rho}).
$$
For the multiple brane case we work within the center of mass $U(1)$. Let $B_{\nu}^\mu$ be the matrix prefactor to the derivative of $F$ in the Born-Infeld equation of motion (II). Then to first order in fluctuations,

$$0 = \bar{B}^\nu_\lambda \partial_\nu (\delta F)_\mu^\lambda + (\delta B)^\nu_\lambda \partial_\nu \bar{F}^\lambda_\mu$$

$$= \bar{B}^\nu_\lambda \partial_\nu (\delta F)_\mu^\lambda + (\bar{B}[F \delta F + \delta F \bar{F}] \bar{B})^\nu_\lambda \partial_\nu \bar{F}^\lambda_\mu .$$

(40)

The simplest case to consider is perturbations $\delta \phi^\perp(x^\alpha)$, that are transverse to both the string and the $p$-brane. The fluctuation equation then collapses to

$$\left[(1 + (\bar{D}\phi^9)^2) \partial_t^2 - \partial_u^2 \right] \delta \phi^\perp = 0 .$$

(41)

This equation has several interesting aspects. The first, as noted for three-branes in [1], is that ripples in transverse position satisfy the appropriate one-dimensional wave equation along the string. To see this, let us switch the radial coordinate to

$$u = \frac{b_p(N_1/N_p)}{(p-2)r^{p-2}} ,$$

(42)

for which the background scalar takes the simple form $\phi^9_{\text{cm}} = u$. The region near $u \to \infty$ (i.e. $r \to 0$) describes the string-like spike and there the fluctuation equation reduces to

$$\left(\partial_t^2 - \partial_u^2 \right) \delta \phi^\perp = 0 .$$

(43)

The worldvolume angular information is irrelevant in the $u \to \infty$ region, and the equation becomes two-dimensional. This further supports the identification of the the brane spike as a string in this region.

In the region $u \to 0$ (i.e. $r \to \infty$), on the other hand, the fluctuation equation (II) becomes the free wave equation in $p$ spatial dimensions and describes wave propagation in the bulk of the $p$-brane worldvolume. The most interesting region is in between where the transition between bulk $p$-brane behavior and one-dimensional string behavior occurs.

In the following section, we will compare these transverse fluctuations in Born-Infeld theory to a supergravity calculation of waves propagating along a string attached to a $p$-brane. The level of approximation in the gravity calculation is to treat the string as a test string in the curved background geometry of a $p$-brane worldvolume. The test string can carry no information about the angular variables of the worldvolume and accordingly we will restrict to $s$-waves in the remainder of this section.

Anticipating the supergravity result, we will also restrict the value of $p$. The three-brane is known to have a core region which is nonsingular and so we expect that the worldbrane and supergravity perturbation equations have a chance of agreeing. Similarly, in the case of the four-brane we expect that agreement is possible]. The reasoning is that the fourbrane-string system may be thought of as the dimensional reduction of the M-theory five-brane with an M-theory two-brane attached and the core region of the five-brane geometry is also known to be nonsingular. Thus we will concentrate on the cases $p = 3, 4$ from now on.

\[5\]We thank Igor Klebanov for a discussion on this point.
A mode of frequency $\omega$ satisfies
\[
\left[ \frac{d^2}{dy^2} + 1 + a_p y^{\frac{2p-2}{p-2}} \right] \delta \phi^\perp = 0 ,
\]
where
\[a_p = \left( \frac{\omega}{p - 2} \right)^{\frac{2p-2}{p-2}} b_{p-2}^{\frac{2}{p-2}} ,
\]
and we have defined a rescaled variable $y = \omega u$. Note that the power of $y$ that appears in the differential equation is integer valued for $p = 3, 4$ only, which is another indication that these values are special.

At this point it is useful to define a new coordinate, $\xi \in (-\infty, +\infty)$, which blows up the $y \sim 0$ region. Our choice (for $p = 3, 4$) is
\[\frac{d\xi}{dy} = \sqrt{h_p(y)} ,\]
where
\[h_p(y) = \left( 1 + \frac{a_p}{y^{p-1}} \right) .\]

Due to the change of variables from $y$ to $\xi$, our differential equation develops a term with a linear $\xi$ derivative. In order to get rid of it, we rescale $\delta \phi^\perp$ as follows,
\[\delta \phi^\perp_{(p)} = h_p(y)^{-1/4} \delta \tilde{\phi}^\perp_{(p)} .\]

The equation for the transverse brane fluctuations takes the form of a one-dimensional Schrödinger equation,
\[
\left[ -\frac{d^2}{d\xi^2} + V_{(p)} \right] \delta \tilde{\phi}^\perp_{(p)} = \delta \tilde{\phi}^\perp_{(p)} ,
\]
with the potential given by
\[V_{(3)}(\xi) = \frac{5a_3}{y^6} h_3(y)^{-3} ,\]
\[V_{(4)}(\xi) = \frac{3a_4}{y^8} \left[ 1 + \frac{a_4}{16y^2} \right] h_4(y)^{-3} ,\]
where $a_3 = \omega^4(b_3 N_1/N_3)^2$ and $a_4 = \omega^3(b_4 N_1/N_4)/8$.

Callan and Maldacena [1] analyzed the scattering problem for the three-brane by approximating the potential by a delta function in the long wavelength limit. They found that the reflection amplitude $R = -1$ in that limit and interpreted it as a Dirichlet boundary condition on string modes incident on the three-brane. The same argument goes through for the four-brane case, although the details of the calculation are somewhat different.
5 Test string in a \( p \)-brane supergravity background

The full supergravity solution for a string extending from a \( p \)-brane at right angles is not known. We and others \[8\] have attempted to construct such solutions, but the closest we have got is to U-dualize a solution of \[9\] in which one or other of the branes is partially delocalized. This solution reduces to a pure \( p \)-brane geometry or a pure string geometry in appropriate limits but does not capture the physics of the two together in a suitable fashion for our present discussion. In the absence of a fully localized supergravity solution we will make do with a test-string approximation. We will treat the \( p \)-brane as a supergravity background for a calculation of wave propagation along a macroscopic fundamental string, which is embedded in the \( p \)-brane geometry. An analogous treatment of a string extending out from a Schwarzschild black hole has been carried out by Lawrence and Martinec \[10\].

Let the string lie along a radial direction \( u \) in the space transverse to the \( p \)-brane, and use static gauge:

\[
\begin{align*}
X^0_{\text{cl}}(\tau, \sigma) &= \tau, \\
X^u_{\text{cl}}(\tau, \sigma) &= \sigma, \\
X^\parallel_{\text{cl}}(\tau, \sigma) &= 0, \\
X^\perp_{\text{cl}}(\tau, \sigma) &= 0.
\end{align*}
\]

Here \( X^\parallel \) denotes the coordinates longitudinal to the threebrane but perpendicular to the string, and \( X^\perp \) denotes the purely transverse coordinates.

The motion of a string in a supergravity background is governed by the usual two-dimensional sigma-model Lagrangian. In studying fluctuations of a string around a classical position as given above, it is most convenient to use the normal coordinate expansion and work in tangent space rather than curved space. The relevant part of the sigma-model action to second order in perturbations is \[11\]

\[
S = \frac{1}{2} \int d^2 \sigma \sqrt{-h} \left[ \eta_{mn} h^{\alpha\beta} (D_\alpha \eta)^m (D_\beta \eta)^n + R_{\mu mn\nu} \eta^m \eta^n h^{\alpha\beta} \partial_\alpha X^\mu_{\text{cl}} \partial_\beta X^\nu_{\text{cl}} \right].
\]

In this action, \( h_{\alpha\beta} \) is the intrinsic worldsheet metric, which in static gauge becomes the pullback of the background metric to the worldsheet. The variables \( \eta^m \) are the tangent space Riemann normal coordinates,

\[
\eta^m = e^m_\mu (X_{\text{cl}}) \eta^\mu,
\]

whose covariant derivatives in tangent space are defined via the spin-connection \( \omega^{[mn]}_\mu \) as

\[
(D_\alpha \eta)^m = \partial_\alpha \eta^m + \omega^{mn}_\mu \partial_\alpha X^\mu \eta_n.
\]

We ignore the ghost sector because we are only interested in transverse fluctuations \( \eta^\perp \). We also ignore the dilaton term in the worldsheet action because it turns out not to contribute to transverse fluctuation equations. There is no antisymmetric tensor term present in the action because we are working in a \( p \)-brane background with only Ramond-Ramond charge.
The next step toward finding the equations for transverse fluctuations is to calculate the various components of the spin-connection and the Riemann tensor in the $p$-brane supergravity background. For this, we need the metric. In string frame, it is

$$ds^2 = H_p(u)^{-1/2}[-dt^2 + (d\bar{x})^2] + H_p(u)^{1/2}(d\bar{x})^2,$$

where

$$H_p(u) = 1 + \frac{R_p^{7-p}}{u^{7-p}},$$

$R_p$ is essentially the gravitational radius of the $p$-brane, and $u = |\bar{x}|$ is the radial coordinate of the transverse space. There is also a dilaton present, $e^{-2\Phi} = H_p(u)^{(p-3)/2}$, which is constant in the case of a three-brane.

Before we calculate the spin-connections and curvatures, we change to tortoise-type coordinates via

$$\frac{d\xi}{du} = \sqrt{H_p(u)},$$

so that the coordinate along the string runs over the whole real axis, $\xi \in (-\infty, \infty)$, and the metric takes the form,

$$ds^2 = H_p(u)^{-1/2}[-dt^2 + (d\bar{x})^2 + d\xi^2] + H_p(u)^{1/2}u^2d\Omega^2_{8-p},$$

Note the similarity between (61) and the analogous redefinition (46) in the gauge theory.

We then find that the relevant components of the spin-connection are

$$\omega^\xi_\xi = \frac{(7 - p) R_p^{7-p}}{4u^{8-p}}H_p(u)^{-5/4},$$

$$\omega^{\hat{r}\hat{s}}_\xi = \delta^{\hat{r}\hat{s}}\frac{1}{u}\left[1 + \frac{(p - 3) R_p^{7-p}}{4u^{7-p}}\right]H_p(u)^{-5/4},$$

where $\hat{r}, \hat{s}$ denote directions transverse to both the string and $p$-brane. The components of the Riemann tensor needed for our calculation are

$$R^{(p)}_{t\hat{r}\hat{s}t} = -\delta_{\hat{r}\hat{s}}\frac{(7 - p) R_p^{7-p}}{4u^{9-p}}\left[1 + \frac{(p - 3) R_p^{7-p}}{4u^{7-p}}\right]H_p(u)^{-3},$$

$$R^{(p)}_{\xi\hat{r}\hat{s}\xi} = +\delta_{\hat{r}\hat{s}}\frac{(7 - p)^2 R_p^{7-p}}{4u^{9-p}}H_p(u)^{-3}.$$

The resulting equation for a transverse fluctuation mode of frequency $\omega$, using rescaled variables $\xi = \omega\xi$ and $y = \omega u$, takes the form

$$\left[-\frac{d^2}{d\xi^2} + V(p)\right]\eta^{\hat{s}}_p = \eta^{\hat{s}}_p,$$

where

$$V(p)(\xi) = \frac{1}{4(7 - p)(8 - p)}\frac{\omega R_p^{7-p}}{y^{9-p}}\left[1 + \frac{(p - 3)}{4(8 - p)}\frac{(\omega R_p)^{7-p}}{y^{7-p}}\right]H_p(y)^{-3}.$$
For the cases \( p = 3, 4 \), in particular, the potentials are
\[
V(3) (\tilde{\xi}) = \frac{5(\omega R_3)^4}{y^6} H_3(y)^{-3}, \tag{69}
\]
\[
V(4) (\tilde{\xi}) = \frac{3(\omega R_4)^3}{y^5} \left[ 1 + \frac{(\omega R_4)^3}{16y^3} \right] H_4(y)^{-3}, \tag{70}
\]
which is identical to the gauge theory results (50) and (51), provided we make the frequency-independent identifications
\[
R_3^4 \leftrightarrow \left[ b_3(N_1/N_3) \right]^2, \quad R_4^3 \leftrightarrow \left[ b_4(N_1/N_4) \right]/8. \tag{71}
\]
In addition to giving the same end result, the gauge theory parallels the supergravity calculation in a number of ways. For example, the scale factor multiplying the gauge theory fluctuation in (48) corresponds precisely to the appropriate component of the zehnbein that relates tangent space and curved space fluctuations of the test string.

It is remarkable that the Born-Infeld theory, a gauge theory formulated in flat spacetime, is able to correctly describe dynamics that arise from the curved geometry of a \( p \)-brane on the supergravity side. It is equally remarkable that fluctuations on a test string deep in the throat of the \( p \)-brane geometry correspond, in the gauge theory, to bulk wave propagation in the \( p \)-brane worldvolume.

6 Discussion

An obvious question is whether the identification of parameters (71) is physically sensible. In spite of the exact agreement of the form of the fluctuation equations, the actual values of the parameters have markedly different physical origins in the two approaches.

The quantity \( b_p \) controls the strength of the electric field on the brane, and it is directly related to the tension of a fundamental string [1]. In our conventions,
\[
b_p = \frac{(2\pi)^{p-1}}{\Omega_{p-1}} g. \tag{72}
\]
On the other hand, the gravitational quantities \( R_3^4, R_4^3 \) in the test string approach are related to the gravitational size of the supergravity \( p \)-brane. In order to get the precise coefficient, it is simplest to compare the known ADM mass of the supergravity \( p \)-brane to the known tension of a Dp-brane. For the first quantity, we have [12]
\[
M_{ADM} = \frac{\Omega_{8-p}}{2\kappa_2^{10}} (7 - p) R_p^{7-p} \text{Vol}_p, \tag{73}
\]
where \( \text{Vol}_p \) is the volume of the brane, whereas for the second we have [13]
\[
\frac{M}{\text{Vol}_p} = \frac{\sqrt{\pi}}{\kappa_10 (2\pi)^{p-3}} N_p. \tag{74}
\]
Therefore, the supergravity parameter $R_p$ is related to the number of $p$-branes $N_p$ by:

$$R_p^{7-p} = \frac{2\sqrt{\pi}}{\Omega_{8-p}(2\pi)^{p-3}(7-p)}\kappa_{10}N_p.$$  \hfill (75)

Note also the relationship between the gravitational radius of the fundamental string, $R_{F1}$, and the number of strings:

$$R_{F1}^6 = \frac{1}{2\pi^5}\kappa_{10}^2N_1.$$  \hfill (76)

Putting all of this together, and using the fact that $\alpha' = 1$ implies that $2\kappa_{10}^2 = (2\pi)^7g^2$, we find

$$N_3^3 = g\pi^3N_1^2,$$  \hfill (77)

$$N_4^2 = \pi N_1.$$  \hfill (78)

The physical meaning of this identification is not entirely transparent. We may get a handle on it by rewriting it, up to pure numbers of order one, as

$$(g^2N_1)^2 \sim (gN_3)^3,$$  \hfill (79)

$$(g^2N_1) \sim (gN_4)^2.$$  \hfill (80)

In our worldvolume approach we were assuming that the supergravity background was flat. In order for this to be a good approximation, we need for the gravitational radii of the D-branes to be less than the string scale. In addition, for our identification of the brane spike to be consistent, we need that the gravitational radius of the fundamental string be less than the string scale. In our unit conventions, these conditions read

$$gN_3 \ll 1, \quad gN_4 \ll 1, \quad g^2N_1 \ll 1.$$  \hfill (81)

On the supergravity side, we treated the string as a test string in the background of the much more massive three- and four-branes. This is a good approximation if the gravitational influence of the string is much smaller than that of the supergravity $p$-brane:

$$g^2N_1 \ll gN_3, \quad g^2N_1 \ll gN_4.$$  \hfill (82)

From (81) and (82), we see that the agreement conditions (79) and (80) are satisfied if the gravitational fields of both the D-brane and the fundamental string are small in string units. This is actually a regime in which we cannot trust the supergravity solution, but it is intriguing that naive extrapolation of the test-string-supergravity result to substringy scales gives agreement with the Born-Infeld calculation. Perhaps the fact that the threebrane, and the M-theory analog of the fourbrane, are nonsingular in their cores is playing an important role here.

Another reason why the agreement is perhaps unexpected is that the brane-spike solutions of Born-Infeld theory are suspect near $r = 0$, where derivatives of the field strength are inevitably large. It is conceivable that the supersymmetry of these configurations leads them to also satisfy the higher-derivative equations of motion, and this would explain why the gauge theory captures the correct physics far out along the string.
In any case, the agreement that we have found suggests further avenues of investigation. It would, for example, be interesting to investigate fermionic fluctuations, for which supersymmetry may play a useful role. Another issue to tackle is Hawking radiation along the string subsequent to adding energy to the BPS system. We would also like to find the exact supergravity solution corresponding to the fundamental string emanating from a Dp-brane, at least for the BPS case and with luck for the nonextremal case as well. Once that is known, we can allow the string to flex its gravitational muscles. This supergravity solution would also be U-dual to the supergravity solution for a “T”-shaped M-theory fivebrane, which is of interest in its own right.

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Note Added:
After completion of this paper, we were informed by one of the authors of [3] that supersymmetry of a related class of Born-Infeld solutions was proven there.

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