Characterizing Block Graphs
in Terms of One-vertex Extensions

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ABSTRACT. Block graphs has been extensively studied for many decades. In this paper we present a new characterization of the class in terms of one-vertex extensions. To this purpose, a specific representation based on the concept of boundary cliques is presented, bringing about some properties of the graph.

Keywords: block graph, one-vertex-extension.

1 INTRODUCTION

Block graphs have been extensively studied for many decades with characterizations based on different approaches since the first one in 1963 until today. In this paper we present a new characterization of the class in terms of one-vertex extensions. As block graphs are a subclass of chordal graphs, properties of this class can be successfully particularized: a specific representation of block graphs based on the concept of boundary cliques is presented, bringing about some properties of the graph.

Harary [7] introduced the definition of a block graph based on structural properties and presented a classical characterization: the block graph $B(G)$ of a given graph $G$ is that graph whose vertices are the blocks (maximal 2-connected components) $B_1, \ldots, B_k$ of $G$ and whose edges are determined by taking two vertices $B_i$ and $B_j$ as adjacent if and only if they contain a cut-vertex (its removal disconnects the graph) of $G$ in common. A graph is called a block graph if it is the block graph of some graph.

Characterization 1. [7] A graph is a block graph if and only if all its blocks are complete.

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Howorka [8], Bandelt and Mulder [2] and Behtoei et al. [3] presented characterizations based on metric conditions.

**Characterization 2.** [8] A graph is a block graph if and only if for every four vertices $u, v, w, x$, the larger two of the distance sums $d(u, v) + d(w, x), d(u, w) + d(v, x), d(u, x) + d(v, w)$ are equal.

**Characterization 3.** [2] A graph $G = (V, E)$ is a block graph if and only if for every vertices $u, v, w \in V$, $G$ satisfies

1. $I(u, v) \subseteq I(u, w) \cup I(v, w)$
2. $I(u, w) \subseteq I(u, v) \cup I(v, w)$
3. $I(v, w) \subseteq I(u, v) \cup I(u, w)$

where $I(x, y) = \{z \in V; d(x, y) = d(x, z) + d(z, y)\}$, $x, y \in V$.

**Characterization 4.** [3] A graph $G$ is a block graph if and only if it satisfies:

1. the shortest path between any two vertices of $G$ is unique and
2. for each edge $e = uv \in E$, if $x \in N_e(u)$ and $y \in N_e(v)$, then, and only then, the shortest path between $x$ and $y$ contains the edge $e$, where $N_e=uv(v) = \{w \in V; d(v, w) < d(u, w)\}$.

Bandelt and Mulder [1] presented a characterization based on forbidden subgraphs.

**Characterization 5.** [1] A graph is a block graph if and only if it is $C_{n \geq 4}$-free and diamond-free.

Mulder and Nebeský [11] characterized block graphs using an algebraic approach, a binary operation + (leap operation) on a finite nonempty set $V$ such that for $u, v, w \in V$,

1. $(u + v) + u = u$.
2. if $u \neq v$ then $(u + v) + v \neq u$.
3. if $u + v \neq v$ then $((u + v) + v) + u \neq u$.
4. if $u \neq v = u + v, u + w \neq v$ and $v + w \neq u$ then $u + w = v + w$.

The underlying graph $G_+ = (V, E_+)$ of + is such that $uv \in E_+$ if and only if $u \neq v, u + v = v$ and $v + u = u$.

Intuitively, for any two vertices $u$ and $w$ in different blocks, the leap operation produces the cut-vertex $z$ in the block of $u$ on the way to $w$, i.e., $u + w = z$. If $u$ and $w$ are in the same block, then $u + w = w$.

**Characterization 6.** [11] $G$ is a block graph if and only if it is the underlying graph of a leap operation on $V$. 

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Recently, the subject was resumed. Dress et al. [5] characterized block graphs in terms of their vertex-induced partitions: any partition of a given finite set \( V \) is a \( V \)-partition, and \( P_V = \{p_v\}_{v \in V} \) is a \( V \)-indexed family of \( V \)-partitions. A family \( P_V \) is a compatible family of \( V \)-partitions if, for any two distinct elements \( u, v \in V \), the union of the set in \( p_u \) that contains \( u \) and the set in \( p_v \) that contains \( v \) coincides with the set \( V \), and \( \{v\} \in p_v \) holds for all \( v \in V \). Let \( P(V) \) denote the set of all compatible families of \( V \)-partitions.

**Characterization 7.** [5] There is a bijective function between the block graphs with vertex set \( V \) and \( P(V) \).

Mulder [10] presented a surprisingly simple characterization.

**Characterization 8.** [10] The graph \( G \) is a block graph if and only if there exists a unique induced path between any two vertices in \( G \).

## 2 BACKGROUND

Basic concepts about chordal graphs are assumed to be known and can be found in Blair and Peyton [4] and Golumbic [6]. In this section, the most pertinent concepts are reviewed.

Let \( G = (V(G), E(G)) \), or simply \( G = (V, E) \), be a connected graph, where \( |E| = m \) and \( |V| = n \). The set of neighbors of a vertex \( v \in V \) is denoted by \( N_G(v) = \{w \in V; vw \in E\} \) and its closed neighborhood by \( N_G[v] = N_G(v) \cup \{v\} \). Two vertices \( u \) and \( v \) are true twins in \( G \) if \( N_G[u] = N_G[v] \) and false twins in \( G \) if \( N_G(u) = N_G(v) \). For any \( S \subseteq V \), the subgraph of \( G \) induced by \( S \) is denoted \( G[S] \). The set \( S \) is a clique if \( G[S] \) is complete. A vertex \( v \in V \) is said to be simplicial in \( G \) when \( N_G(v) \) is a clique in \( G \).

It is worth mentioning two kinds of cliques in a chordal graph \( G \). A simplicial clique is a maximal clique containing at least one simplicial vertex. A simplicial clique \( Q \) is called a boundary clique if there exists a maximal clique \( Q', Q \neq Q' \), such that \( Q \setminus Q' \) is a set of simplicial vertices of \( G \).

A perfect elimination ordering (peo) of a graph \( G = (V, E) \) is a bijective function \( \sigma : \{1, \ldots, n\} \rightarrow V \) such that \( \sigma(i) \) is a simplicial vertex in the induced subgraph \( G_i = G[\{\sigma(i), \ldots, \sigma(n)\}] \), for \( 1 \leq i < n \). A peo is ultimately an arrangement of \( V \) in a sequence \( \sigma(V) = [\sigma(1), \ldots, \sigma(n)] \). It is well known that a graph \( G \) is chordal if and only if \( G \) admits a perfect elimination ordering.

## 3 BOUNDARY REPRESENTATION

In this section we present a representation of block graphs based on the concept of a perfect elimination ordering of the graph. As in a peo, where a vertex is eliminated when it is simplicial in the remaining graph, in this proposed representation, a maximal clique is eliminated when it is a boundary clique in the remaining graph. As all elements of the maximal clique are stored, the graph can be easily recovered. The representation is defined as follows; its structure is similar to the one presented in [9].
Let \( G = (V, E) \) be a block graph with \( \ell \) maximal cliques. A \textit{boundary representation} of \( G \) is the sequence of pairs

\[
BR(G) = [(P_1, s_1), \ldots, (P_\ell, s_\ell)]
\]
such that

- \( P_i \cup \{s_i\} = Q_i \), \( i = 1, \ldots, \ell - 1 \), is a boundary clique of graph \( G[V \setminus (Q_1 \cup \ldots \cup Q_{i-1})] \) where
  - \( P_i \subset V \) is the set of simplicial vertices of \( Q_i \) and
  - \( s_i \in V \) is the cut-vertex of the clique \( Q_i \);
- \( P_\ell = Q_\ell \) is a maximal clique of \( G \) and \( s_\ell = \emptyset \) (the symbol \( \emptyset \) denotes the absence of the parameter).

This representation makes possible to deduce some structural properties of the graph.

**Property 1.** \( P_1 \cup \{s_1\}, P_2 \cup \{s_2\}, \ldots, P_\ell \) are the maximal cliques of \( G \).

**Property 2.** The set \( \{P_1, \ldots, P_\ell\} \) is a partition of \( V \).

![Block graph](image-url)
Property 3. The sequence provided by all vertices of $P_1$, followed by all vertices of $P_2$, and so on, up to $P_t$ is a perfect elimination ordering of $G$. Observe that, since there is no order in the set $P_i$, $i = 1, \ldots, \ell$, several sequences can be built.

Employing the algorithm for the graph in Figure 1 we have:

$$BR(G) = [(\{a\},b), (\{e\},f), (\{k\},g), (\{j\},i), (\{o,p\},m), (\{b\},d), (\{m,n,\ell\},h), (\{d,h,g,i\},\emptyset)]$$

4 ONE-VERTEX EXTENSIONS

The concept of one-vertex extension was introduced by Bandelt and Mulder [1].

Let $G = (V,E)$ be a graph, $v \in V$ and $u \notin V$. An extension of $G$ to a graph $G' = (V',E')$ is a one-vertex extension if it obeys one of the following three rules:

(α) $V' = V \cup \{u\}$ and $E' = E \cup \{vu\}$, i.e., $N_{G'}(u) = \{v\} \ (u \text{ is a pendant vertex}).$

(β) $V' = V \cup \{u\}$ and $E' = E \cup \{xu; x \in N_G[v]\}$, i.e., $N_{G'}[u] = N_{G'}[v] \ (u \text{ is a true twin of } v)$.

(γ) $V' = V \cup \{u\}$ and $E' = E \cup \{xu; x \in N_G(v)\}$, i.e., $N_{G'}(u) = N_{G'}(v) \ (u \text{ is a false twin of } v)$.

The special cases of (α), (β) and (γ) restricted to a simplicial vertex $v \in V$ are denoted by (α*), (β*) and (γ*), respectively.

In order to generate a graph $G = (V,E)$, it is possible to establish a building sequence. A one-extension sequence (oes) of $G$ is a sequence of triples

$$\Pi(G) = [\pi(1), \ldots, \pi(n)]$$

being $\pi(i) = (e_i, v_i, u_i), i = 2, \ldots, n$, such that

1. $e_i \in \{ (\alpha), (\beta), (\gamma), (\alpha^*), (\beta^*), (\gamma^*) \}$;
2. $v_i = u_j$, for some $j < i$;
3. $u_i \neq u_k, 1 \leq k \leq i - 1$;

and $\pi(1)$ is the special initial triple $(\emptyset, \emptyset, u_1)$.

Bandelt and Mulder [1] presented characterizations of distance hereditary graphs and ptolemaic graphs; the first one using the extensions (α), (β) and (γ), and the second one using (α), (β) and (γ*). Theorem 4.1, presented below, shows a characterization of block graphs using one-vertex extensions.

Consider a graph $G$, $CV(G)$ the set of cut-vertices, $Simp(G)$ the set of simplicial vertices and $Q(G)$ the set of maximal cliques of the graph.

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Theorem 4.1. A graph $G = (V,E)$ is a block graph if and only if there is a sequence $\Pi(G)$ of $G$ composed by type $(\alpha)$ and type $(\beta^*)$ extensions.

Proof. Consider a block graph $G$ with $\ell$ maximal cliques and its boundary representation $BR(G) = [(P_1,s_1),\ldots,(P_\ell,\varnothing)]$. It is possible to construct a sequence $\Pi(G)$ by transversing the boundary representation in reverse order.

Let $(P_\ell,\varnothing)$, $v \in P_\ell$ and $\pi(1) = (\varnothing,\varnothing,v)$. Consider $w \in P_\ell \setminus \{v\}$ and $\pi(2) = (\alpha, v, w)$. For $x \in P_\ell \setminus \{v,w\}$, let be the triple $(\beta^*,w,x)$. So, there are the following elements of the sequence $\Pi(G)$: $\pi(i) = (\beta^*,w,x)$, $i = 3,\ldots,|P_\ell|$. Thus, a first maximal clique of $G$ is obtained.

Let $(P_{\ell-j},s_{\ell-j})$, $j = 1,\ldots,\ell - 1$. By the definition of boundary representation, $s_{\ell-j} \in Q_k$, $\ell - j + 1 \leq k \leq \ell$. Consider $v \in P_{\ell-j}$. The graph obtained by the extension $\pi(1 + \sum_{k=\ell-j+1}^\ell |P_k|) = (\alpha, s_{\ell-j}, v)$ has $v$ as a pendant vertex and $s_{\ell-j}$ as a cut-vertex. The vertices $v$ and $s_{\ell-j}$ belong to a new maximal clique $Q$. For $x \in P_{\ell-j} \setminus \{v\}$, let be the triple $(\beta^*, v, x)$. Thus, there are the following elements of the sequence $\Pi(G)$: $\pi(i) = (\beta^*, v, x)$, $i = 2 + \sum_{k=\ell-j+1}^\ell |P_k|, \ldots, \sum_{k=\ell-j}^\ell |P_k|$. These extensions increase the clique $Q$ to which vertex $v$ belongs in $G$. Then, we obtain the one-extension sequence of $G$, $\Pi(G)$, composed by type $(\alpha)$ and type $(\beta^*)$ extensions.

Conversely, consider $\Pi(G) = [(\varnothing, \varnothing, v)]$. The resulting graph $G$ is a trivial graph $(\{v\}, \emptyset)$ and it is a block graph.

Consider $H = (V(H),E(H))$ a block graph with $n - 1$ vertices obtained by $\Pi(H) = [\pi(1),\ldots,\pi(n-1)]$ a sequence of $(\alpha)$ and $(\beta^*)$ extensions. Let $v \in V(H)$, $Q$ the maximal clique to which it belongs in $H$ and $u \not\in V(H)$.

Let $\Pi(G) = \Pi(H) \|(\pi(n) = [\pi(1),\ldots,\pi(n-1),\pi(n)]$.

If $\pi(n) = (\alpha, v, u)$, two cases must be analyzed.

1. $v$ is a simplicial vertex of $H$. Then, $CV(G) = CV(H) \cup \{v\}$ and $Simp(G) = (Simp(H) \setminus \{v\}) \cup \{u\}$.

2. $v$ is a cut-vertex of $H$. Then, $CV(G) = CV(H)$ and $Simp(G) = Simp(H) \cup \{u\}$.

In both cases, the set of maximal cliques $Q(G) = Q(H) \cup \{vu\}$.

If $\pi(n) = (\beta^*, v, u)$, $v$ must be a simplicial vertex in $H$. So, $CV(G) = CV(H)$, $Simp(G) = Simp(H) \cup \{u\}$ and $Q(G) = (Q(H) \setminus Q) \cup \{Q'\}$ where $Q$ is a maximal clique such that $v \in Q$ and $Q' = Q \cup \{u\}$.

In any case, $G$ is a block graph. □
The proof of Theorem 4.1 provides a possible one-extension sequence of a block graph. As an example, consider the block graph $G$ in Figure 1 and the boundary representation of the same graph presented in Section 3. The one-extension sequence obtained from $BR(G)$ is

$$\Pi(G) = [((\emptyset, \emptyset, d), (\alpha, d, h), (\beta^*, h, g), (\beta^*, h, i), (\alpha, h, m), (\beta^*, m, n), (\beta^*, m, \ell), (\alpha, d, b), (\alpha, m, o), (\beta^*, o, p), (\alpha, i, j), (\alpha, g, k), (\alpha, d, e), (\beta^*, e, f), (\alpha, d, c), (\alpha, b, a)].$$

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