SPECTRAL PROPERTIES OF KILLING VECTOR FIELDS OF CONSTANT LENGTH

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Abstract. This paper is devoted to the study of properties of Killing vector fields of constant length on Riemannian manifolds. If \( g \) is a Lie algebra of Killing vector fields on a given Riemannian manifold \((M, g)\), and \( X \in g \) has constant length on \((M, g)\), then we prove that the linear operator \( \text{ad}(X) : g \to g \) has a pure imaginary spectrum. More detailed structure results on the corresponding operator \( \text{ad}(X) \) are obtained. Some special examples of vector fields of constant length are constructed.

2010 Mathematical Subject Classification: 53C20, 53C25, 53C30.

Key words and phrases: geodesic orbit space, homogeneous Riemannian space, Killing vector field of constant length.

1. Introduction

We study general properties of a given Killing vector field of constant length \( X \) on an arbitrary Riemannian manifold \((M, g)\). A comprehensive survey on classical results in this direction could be found in [5, 6]. Important properties of Killing vector fields of constant length (abbreviated as KVFCL) on compact homogeneous Riemannian spaces are studied in [20]. Some recent results about Killing vector field of constant length on some special Riemannian manifolds are obtained in [25, 26]. All manifolds in this paper are supposed to be connected.

Let us consider a Riemannian manifold \((M, g)\) and any Lie group \( G \) acting effectively on \((M, g)\) by isometries. We will identify the Lie algebra \( g \) of \( G \) with the corresponded Lie algebra of Killing vector field on \((M, g)\) as follows. For any \( U \in g \) we consider a one-parameter group \( \exp(tU) \subset G \) of isometries of \((M, g)\) and define a Killing vector field \( \vec{U} \) by a usual formula

\[
\vec{U}(x) = \frac{d}{dt}\exp(tU)(x)\bigg|_{t=0}.
\]

It is clear that the map \( U \to \vec{U} \) is linear and injective, but \([\vec{U}, \vec{V}] = -\overline{[U, V]}_g\), where \([\cdot, \cdot]_g\) is the Lie bracket in \( g \) and \([\cdot, \cdot]\) is the Lie bracket of vector fields on \( M \). We will use this identification repeatedly in this paper.

Any \( X \in g \) determines a linear operator \( \text{ad}(X) : g \to g \) acting by \( Y \mapsto [X, Y] \). If we consider \( X \) as a Killing vector field on \((M, g)\), then some geometric type assumptions on \( X \) imply special properties (in particular, spectral properties) of the corresponding operator \( \text{ad}(X) \). In this paper, we study the property of \( X \) to be of constant length.

For a Lie algebra \( g \), we denote by \( n(g) \) and \( \tau(g) \) the nilradical (the maximal nilpotent ideal) and the radical of \( g \) respectively. A maximal semi-simple subalgebra of \( g \) is called a Levi factor or a Levi subalgebra. There is a semidirect decomposition \( g = \tau(g) \ltimes s \), where \( s \) is an arbitrary Levi factor. The Malcev–Harish-Chandra theorem states that any two Levi factors of \( g \) are conjugate by an automorphism \( \exp(\text{Ad}(Z)) \) of \( g \), where \( Z \) is in the nilradical \( n(g) \) of \( g \). We have \( \tau(g) = [s, \tau(g)] \oplus C_{\tau(g)}(s) \) (a direct sum of linear subspaces), where \( C_{\tau(g)}(s) \) is the centralizer of \( s \) in \( \tau(g) \). Recall also that \([g, \tau(g)] \subset n(g)\), therefore, \([s, \tau(g)] \subset [g, \tau(g)] \subset n(g)\). Moreover, \( D(\tau(g)) \subset n(g) \) for every derivation \( D \) of \( g \). For any Levi factor \( s \), we have \([g, g] = [\tau(g) + s, \tau(g) + s] = [\tau(g)] \ltimes s \subset n(g) \ltimes s \). For a more detailed discussion of the Lie algebra structure we refer to [14].
Recall that a subalgebra $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$ is said to be compactly embedded in $\mathfrak{g}$ if $\mathfrak{g}$ admits an inner product relative to which the operators $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$, $X \in \mathfrak{k}$, are skew-symmetric. This condition is equivalent to the following condition: the closure of $\text{Ad}(\exp(\mathfrak{k}))$ in $\text{Aut}(\mathfrak{g})$ is compact, see e. g. [14]. Note that for a compactly embedded subalgebra $\mathfrak{k}$, every operator $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$, $X \in \mathfrak{k}$, is semisimple and the spectrum of $\text{ad}(X)$ lies in $i\mathbb{R}$, where $i = \sqrt{-1}$. Recall also that a subalgebra $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$ is said to be compact if it is compactly embedded in itself. It is equivalent to the fact that there is a compact Lie group with a given Lie algebra $\mathfrak{k}$. It is clear that any compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is compact.

One of the main results of this paper is the following

**Theorem 1.** For any Killing field of constant length $X \in \mathfrak{g}$ on a Riemannian manifold $(M, g)$, the spectrum of the operator $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is pure imaginary, i. e. is in $i\mathbb{R}$.

This result is a partial case of Theorem 3. It is clear that it is non-trivial only for noncompact Lie algebras $\mathfrak{g}$ and only when $X$ is not a central element of $\mathfrak{g}$. On the other hand, there are examples of Killing fields of constant length $X \in \mathfrak{g}$ for noncompact $\mathfrak{g}$. Moreover, Theorems 4 and 5 give us examples when $X \in \mathfrak{n}(\mathfrak{g})$ and $\text{ad}(X)$ is non-trivial and nilpotent. In particular, $\text{ad}(X)$ is not semisimple in this case. This observation leads to the following conjecture.

**Conjecture 1.** If $\mathfrak{g}$ is semisimple, then any Killing field of constant length $X \in \mathfrak{g}$ on $(M, g)$ is a compact vector in $\mathfrak{g}$, i. e. the Lie algebra $\mathbb{R} \cdot X$ is compactly embedded in $\mathfrak{g}$.

The paper is organized as follows. In Section 2, we establish some important spectral properties of the operator $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ for any Killing vector field of constant length $X \in \mathfrak{g}$ on a given Riemannian manifold $(M, g)$. One of the main results is Theorem 2, that implies non-trivial geometric properties of $(M, g)$ in the case when the Lie algebra $\mathfrak{g}$ could be decomposed as a direct Lie algebra sum. In Section 3, we obtain some results on Killing vector field of constant length on geodesic orbit Riemannian spaces. In particular, Theorems 4 and 5 imply that any Killing vector field in the center of $\mathfrak{n}(\mathfrak{g})$ has constant length on a given geodesic orbit space $(G/H, g)$. This observation provides non-trivial examples of Killing vector field of constant length $X$ such that the operator $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.

## 2. KVFCL on General Riemannian Manifolds

In what follows, we assume that a Lie group $G$ acts effectively on a Riemannian manifold $(M, g)$ by isometries, $\mathfrak{g}$ is the Lie algebra of $G$, elements of $\mathfrak{g}$ are identified with Killing vector field on $(M, g)$ according to (1).

The following characterizations of Killing vector fields of constant length on Riemannian manifolds is very useful.

**Lemma 1** (Lemma 3 in [6]). Let $X$ be a non-trivial Killing vector field on a Riemannian manifold $(M, g)$. Then the following conditions are equivalent:

1) $X$ has constant length on $M$;
2) $\nabla_X X = 0$ on $M$;
3) every integral curve of the field $X$ is a geodesic in $(M, g)$.

**Lemma 2** (Lemma 2 in [21]). If a Killing vector field $X \in \mathfrak{g}$ has constant length on $(M, g)$, then for any $Y, Z \in \mathfrak{g}$ the equalities

$$g([Y, X], X) = 0,$$  \hfill (2)

$$g([Z, [Y, X]], X) + g([Y, X], [Z, X]) = 0$$  \hfill (3)

hold at every point of $M$. If $G$ acts on $(M, g)$ transitively, then condition (2) implies that $X$ has constant length. Moreover, the condition (3) also implies that $X$ has constant length for compact $M$ and transitive $G$. 


Now, we are going to get some more detailed results.

**Proposition 1.** Let $X \in \mathfrak{g}$ be a Killing vector field of constant length on $(M,g)$. Denote by $C_0(X)$ the centralizer of $X$ in $\mathfrak{g}$ and consider $P(X) : = \{ Y \in \mathfrak{g} \mid g(X,Y) = 0 \text{ on } M \}$. Then we have $[X,\mathfrak{g}] \subset P(X)$ and $[Z,P(X)] \subset P(X)$ for any $Z \in C_0(X)$.

**Proof.** By Lemma 2 we have $g([Y,X],X) = 0$ for any $Y \in \mathfrak{g}$, hence, $[X,\mathfrak{g}] \subset P(X)$. If $Y \in P(X)$, then $g(X,Y) = 0$. Therefore, for any $Z \in C_0(X)$ we get $g(X,[Z,Y]) = Z \cdot g(X,Y) = 0$ on $M$, i.e. $[Z,P(X)] \subset P(X)$. ■

**Theorem 2.** Let $X \in \mathfrak{g}$ be a Killing vector field of constant length on $(M,g)$. Suppose that we have a direct Lie algebra sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l$, $l \geq 2$. Then for every $i = 1, \ldots, l$, there is an ideal $u_i$ in $\mathfrak{g}_i$ such that $[X,\mathfrak{g}_i] \subset u_i$ and $g(u_i, u_j) = 0$ on $M$ for every $i \neq j$.

**Proof.** Since $X$ is of constant length, then $\mathfrak{g}_i \cdot g(X,X) = g([\mathfrak{g}_i,X],X) = 0$ for any $i$ by Lemma 2. If we take $j \neq i$, then

$$0 = \mathfrak{g}_j \cdot g([\mathfrak{g}_i,X],X) = g([\mathfrak{g}_i,X],[\mathfrak{g}_j,X]).$$

Let $\{u_i\}$, $i = 1, \ldots, l$, be a set of maximal (by inclusion) subspaces $u_i \subset \mathfrak{g}_i$, such that $[\mathfrak{g}_i,X] \subset u_i$ and $g(u_i, u_j) = 0$ for every $i \neq j$ (such a set of subspaces should not be unique in general). Since

$$0 = \mathfrak{g}_i \cdot g(u_i,u_j) = 0 = g([\mathfrak{g}_i,u_i],u_j),$$

then $[\mathfrak{g}_i,u_i] \subset u_i$ due to the choice of $u_i$. Hence, every $u_i$ is an ideal in $\mathfrak{g}_i$ and in $\mathfrak{g}$. ■

**Remark 1.** If $X = X_1 + X_2 + \cdots + X_l$, where $X_i \in \mathfrak{g}_i$, then $u_i \neq 0$ if $X_i$ is not in the center of $\mathfrak{g}_i$. In particular, if $X_i \neq 0$ and $\mathfrak{g}_i$ is simple, then $u_i = \mathfrak{g}_i$. Note, that Theorem 2 leads to a more simple proof of Theorem 1 in [20] about properties of Killing vector fields of constant length on compact homogeneous Riemannian manifolds. See also Remark 6 about geodesic orbit spaces.

**Proposition 2.** Let $X \in \mathfrak{g}$ be a Killing vector field of constant length on $(M,g)$. Then for every $V,W \in [X,\mathfrak{g}]$ we have the equality

$$g([X,V],W) + g(V,[X,W]) = 0$$

on $M$.

**Proof.** Taking in mind the polarization, it suffices to prove $g([X,V],V) = 0$ on $M$ for every $V \in [X,\mathfrak{g}]$. Take any $U$ such that $[X,U] = V$. Since $X$ has constant length, we have $g(X,[U,X]) = 0$ according to (2). Hence,

$$g([X,V],V) = g([X,U],X), [X,U]) = [X,U] \cdot g(X,[X,U]) = 0,$$

that proves the proposition. ■

In what follows, for a Killing vector field of constant length $X \in \mathfrak{g}$ on $(M,g)$, we denote by $L = L(X)$ the linear operator $\text{ad}(X): \mathfrak{g} \rightarrow \mathfrak{g}$.

**Proposition 3.** Let $X \in \mathfrak{g}$ be a Killing vector field of constant length on $(M,g)$. Then

1) $L = \text{ad}(X)$ has no non-zero real eigenvalue;
2) If $[[X,Z],Z] \in [X,\mathfrak{g}]$ (in particular, if $[[X,Z],Z] = 0$) for some $Z \in \mathfrak{g}$, then we get $[X,Z] = 0$;
3) If $a$ is an $\text{ad}(X)$-invariant subspace in $\mathfrak{g}$ such that $[a,a] \subset [X,\mathfrak{g}]$, then $[X,a] = 0$;
4) If $a$ is any abelian ideal in $\mathfrak{g}$ (one may take $a = C(n(\mathfrak{g}))$ in particular), then $[X,a] = 0$.

**Proof.** 1) Suppose the contrary, i.e. there is nontrivial $Y \in \mathfrak{g}$ such that $[X,Y] = \lambda Y$ for some real $\lambda \neq 0$. Then $[Y,[Y,X]] = -\lambda[Y,Y] = 0$ and we get

$$0 = g([Y,[Y,X]],X) + g([Y,X],[Y,X]) = g([Y,X],[Y,X]) = \lambda^2 g(Y,Y)$$

by Lemma 2, that is impossible.
2) If \([X, Z] = [X, U]\) for some \(U \in \mathfrak{g}\), then \(g([X, Z], Z) = g([X, U], X) = 0\). Therefore, \(g([Z, X], [Z, X]) = g([Z, [Z, X]], X) + g([Z, X], [Z, X]) = 0\) by Lemma 2. Hence, \([X, Z] = 0\).

3) Take any \(Z \in \mathfrak{a}\). Since \([X, Z] \in \mathfrak{a}\), we get \([X, Z] \in [X, \mathfrak{g}]\). Then we have \([X, Z] = 0\) by 2).

4) This is a partial case of 3). □

**Theorem 3.** Let \(X \in \mathfrak{g}\) be a Killing vector field of constant length on \((M, g)\). Then the following assertions hold.

1) We have an \(L\)-invariant linear space decomposition \(\mathfrak{g} = A_1 \oplus A_2\), where \(A_1 = \text{Ker}(L^2)\) and \(A_2 = \text{Im}(L^2)\). Moreover, \(A_1\) is the root space for \(L\) with the eigenvalue 0 and \(L\) is invertible on \(A_2\).

2) If \(\mathfrak{o}\) is a 2-dimensional \(L\)-invariant subspace, corresponding to a complex conjugate pair of eigenvalues \(\alpha \pm \beta i\) (\(\beta \neq 0\)), i.e. \(L(U) = \alpha \cdot U - \beta \cdot V\) and \(L(V) = \beta \cdot U + \alpha \cdot V\) for some non-trivial \(U, V \in \mathfrak{o}\), then \(\alpha = 0\), \(g(U, V) = 0\), and \(g(U, U) = g(V, V)\) on \(G/H\).

3) All eigenvalues of \(L\) have trivial real parts.

**Proof.** 1) Let us fix an arbitrary \(Z \in \mathfrak{g}\). If we put \(V = [X, Z]\) and \(W = [X, [X, Z]]\) in (4), we get
\[
g([X, [X, Z]], [X, [X, Z]]) + g([X, Z], [X, [X, Z]]) = g(L^2(Z), L^2(Z)) + g(L(Z), L^3(Z)) = 0.
\]
From this we see that \(L^3(Z) = 0\) implies \(L^2(Z) = 0\). This observation implies \(L^3(\mathfrak{g}) = L^2(\mathfrak{g}) = \text{Im}(L^2)\). In particular, we get that \(L\) is invertible on \(A_2 = \text{Im}(L^2)\).

Let us prove that \(A_1 \cap A_2 = \text{Ker}(L^2) \cap \text{Im}(L^2) = 0\). Suppose that there is a non-trivial \(W \in \text{Ker}(L^2) \cap \text{Im}(L^2)\). Then \(L^2(W) = 0\) and there is \(V \in g\) such that \(W = L^2(V)\). If \(L(W) = 0\), then we have \(L^2(V) \neq 0\) and \(L^3(V) = 0\), that is impossible. If \(L(W) \neq 0\), then we have \(L^3(L(V)) = L^2(W) = 0\). That is again impossible by the above discussion. Therefore, \(\text{Ker}(L^2) \cap \text{Im}(L^2) = A_1 \cap A_2 = 0\) and \(\mathfrak{g} = A_1 \oplus A_2\).

Since \(L\) is invertible on \(A_2 = \text{Im}(L^2)\), then \(A_1 = \text{Ker}(L^2)\) exhausts the root space for \(L\) with the eigenvalue 0.

2) Clear that \(\mathfrak{o} \subset A_2\). Since \(L(U) = \alpha \cdot U - \beta \cdot V\) and \(L(V) = \beta \cdot U + \alpha \cdot V\), then
\[
L^2(U) = (\alpha^2 - \beta^2) \cdot U - 2\alpha \beta \cdot V, \quad L^2(V) = 2\alpha \beta \cdot U + (\alpha^2 - \beta^2) \cdot V.
\]
By (4), we get
\[
g(L^2(U), L(U)) = 0, \quad g(L^2(V), L(V)) = 0, \quad g(L^2(U), L(V)) + g(L^2(V), L(U)) = 0
\]
on \(M\). These three equalities could be re-written as follows:
\[
\begin{pmatrix}
\alpha^2 - \beta^2 & -\beta(3\alpha^2 - \beta^2) \\
2\alpha \beta^2 & \beta(3\alpha^2 - \beta^2) & \alpha(\alpha^2 - \beta^2) \\
\beta(3\alpha^2 - \beta^2) & 2\alpha(\alpha^2 - 3\beta^2) & -\beta(3\alpha^2 - \beta^2)
\end{pmatrix}
\begin{pmatrix}
g(U, U) \\
g(U, V) \\
g(V, V)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]
The equality \(g(U, U) = 0\) is impossible, since \(U\) is non-trivial. Hence, we have non-trivial solution \((g(U, U), g(U, V), g(V, V))\) of the above homogeneous linear system with the determinant \(-2\alpha(\alpha^2 + \beta^2)\). Since \(\alpha^2 + \beta^2 \neq 0\), we get \(\alpha = 0\). Moreover, \(\alpha = 0\) and \(\beta \neq 0\) imply obviously \(g(U, V) = 0\), and \(g(U, U) = g(V, V)\) on \(M\).

3) Recall that \(L\) has no non-trivial real eigenvalue by 1) in Proposition 3. This observation and 2) imply that all eigenvalues of \(L\) have trivial real parts. □

In what follows, we use the notation \(A_1 = \text{Ker}(L^2)\) and \(A_2 = \text{Im}(L^2)\) as in Theorem 3.

**Remark 2.** Note that \(\mathfrak{g} = A_1 \oplus A_2 = \text{Ker}(L^2) \oplus \text{Im}(L^2)\) is the Fitting decomposition (see e.g. Lemma 5.3.11 in [14]) for the operator \(L\). It should be noted that the decomposition \(\mathfrak{g} = \text{Ker}(L) \oplus \text{Im}(L)\) is not valid at least for \(X \in C(\mathfrak{n}(\mathfrak{g})) \setminus C(\mathfrak{g})\) (see Theorem 5) since \(\text{Im}(L) \subset C(\mathfrak{n}(\mathfrak{g})) \subset \text{Ker}(L)\) in this case.
Remark 3. By 2) of Proposition 3 we have \([L(Y), Y] = [[X, Y], Y] \notin L(g)\) for any \(Y \in A_1 \setminus \text{Ker}(L)\). On the other hand, \([[X, Y], Y] \in \text{Ker}(L)\). Indeed, \(L^2(A_1) = 0\) and

\[L([[X, Y], Y]) = [X, [[X, Y], Y]] = [[X, [X, Y]], Y] + [[X, Y], [X, Y]] = 0.\]

Remark 4. It is interesting to study KVFCL \(X\) with \(\text{Ker}(L) \neq A_1\). For such \(X\) the operator \(L = \text{ad}(X)\) is not semisimple. One class of suitable examples are \(X \in C(n(g))\) for geodesic orbit spaces \((G/H, g)\) as in Theorem 5 (if there is a vector \(V \in g \setminus n(g)\) such that \([X, V] \neq 0\)). It is not clear whether there is such KVFCL \(X\) on a homogeneous Riemannian space \((G/H, g)\) with semisimple \(G\).

We will use the Jordan decomposition \(L = L_s + L_n\), where \(L_s\) and \(L_n\) are semisimple and nilpotent part of \(L = \text{ad}(X)\) respectively. Note that \(L_s\) and \(L_n\) are derivations on \(g\) that are vanished on \(\text{Ker}(L)\), see e.g. Propositions 5.3.7 and 5.3.9 in [14].

Let us consider all complex conjugate pairs of eigenvalues \(\pm \beta_j i, 0 < \beta_1 < \beta_2 < \cdots < \beta_t\), for \(L (i = \sqrt{-1})\). Note that a semisimple part \(L_s\) of the operator \(L\) has the same eigenvalues. Let us consider

\[V_j = \left\{ Y \in A_2 \mid \left(L^2 + \beta_j^2 \text{Id}\right)^m(Y) = 0 \text{ for some } m \in \mathbb{N} \right\},\]

the root space of \(L\) for the pair \(\pm \beta_j i\) (equivalently, the root space of \(L^2\) for the eigenvalue \(-\beta_j^2\)), \(1 \leq j \leq t\). It is easy to see that \(V_j = \{ Y \in A_2 \mid L_s^2(Y) = -\beta_j^2 Y \}\) for the eigenvalue \(-\beta_j^2\). In particular, \([V_i, V_j] \subset V_p \oplus V_q\), where \(V_p (V_q)\) are supposed to be trivial if \(-\beta_p^2 (\text{respectively}, -\beta_q^2)\) is not an eigenvalue of \(L^2\), \(1 \leq i, j \leq t\).

Proposition 4. For every \(Y \in V_i\) and \(Z \in V_j, i, j \geq 1\), we have \([Y, Z]^+ \in V_k, [Y, Z]^− \in V_l\), where \(\beta_k = |\beta_i - \beta_j|\) and \(\beta_l = \beta_i + \beta_j\). If \(-\beta_k^2 (\text{respectively}, -\beta_l^2)\) is not an eigenvalue of \(L^2\), then \([Y, Z]^+ = 0 (\text{respectively}, [Y, Z]^− = 0)\). In particular, \([V_i, V_j] \subset V_p \oplus V_q\), where \(V_p (V_q)\) are supposed to be trivial if \(-\beta_p^2 (\text{respectively}, -\beta_q^2)\) is not an eigenvalue of \(L^2\), and \([V_0, V_i] \subset V_i\) and \([V_i, V_j] \subset A_2\) for \(i \neq j\). In particular, \([A_1, A_2] \subset A_2\).

Proof. Straightforward calculations using (6) and (7) imply that

\[L_s^2([Y, Z]^+) = -(\beta_i - \beta_j)^2 \cdot [Y, Z]^+, \quad L_s^2([Y, Z]^−) = -(\beta_i + \beta_j)^2 \cdot [Y, Z]^−.\]

If \(U \in V_0 = A_1\) and \(Y \in V_i\), then \(L_s([U, Y]) = [L_s(U), Y] + [U, L_s(Y)] = [U, L_s(Y)]\) and \(L_s^2([U, Y]) = [U, L_s^2(Y)] = -\beta_i^2 \cdot [U, Y]\), hence, \([V_0, V_i] \subset V_i\). These arguments prove the proposition.

Proposition 5. In the above notations and assumptions, the following assertions hold.

1) \(C(n(g)) \subset \text{Ker}(L) \subset A_1\).

2) If \(J = \{ Y \in g \mid [C(n(g)), Y] = 0 \}\), then \(J\) is an ideal in \(g\), such that \(X \in J\) and \(A_2 \subset L(g) \subset [X, g] \subset J\).

3) If \(I = \{ Y \in A_1 \mid g(Y, A_2) = 0 \text{ on } M\}\), then \(I\) is an ideal in \(A_1\), such that \(X \in I\), \(L(A_1) = [X, A_1] \subset I\), and \([L(A_1), A_1] = [[X, A_1], A_1] \subset I\).

4) \(J := A_2 + [A_2, A_2]\) is an ideal in \(g\), \(J \subset J\), and \(g(I \cap C(n(g)), J) = 0 \text{ on } M\).

5) \(L(J \cap A_1) \subset J \cap I\) and \(L_s(J \cap A_1) = 0\).
Proof. 1) We get $C(n(g)) \subset \text{Ker}(L) \subset A_1$ by 4) of Proposition 3.

2) If $[C(n(g)), Y] = 0$, then for any $Z \in g$ we have (recall that $C(n(g))$ is an ideal in $g$)

$$[C(n(g)), [Z, Y]] = [[C(n(g)), Z], Y] + [Z, [C(n(g)), Y]] \subset [C(n(g)), Y] = 0,$$

hence, $J$ is an ideal in $g$. We know that $X \in J$ by 1), hence, $A_2 \subset L(g) = [X, g] \subset J$.

3) Since $g(I, A_2) = 0$ on $G/H$ and $[A_1, A_2] \subset A_2$, we get $0 = A_1 \cdot g(I, A_2) = g([A_1, I], A_2)$, therefore, $[A_1, I] \subset I$. We know that $g(X, A_2) = 0$ due to $A_2 \subset [X, g]$ and Proposition 1, hence $X \in I$. Therefore, $L(A_1) = [X, A_1] \subset I$ and $[L(A_1), A_1] = [[X, A_1], A_1] \subset I$.

4) Since $[A_1, A_2] \subset A_2$, $\tilde{J} = A_2 + [A_2, A_2]$ is an ideal in $g$. Since $[C(n(g)), A_2] = 0$, we get $\tilde{J} \subset J$. Since $g(I \cap C(n(g)), A_2) = 0$ and $\tilde{J}$ is generated by $A_2$, we get

$$g(I \cap C(n(g)), [A_2, A_2]) = A_2 \cdot g(I \cap C(n(g)), A_2) = 0$$

and $g(I \cap C(n(g)), \tilde{J}) = 0$.

5) $L(\tilde{J} \cap A_1) \subset L(A_1) \subset I$ by 3) and $L(\tilde{J} \cap A_1) \subset [X, \tilde{J}] \subset J$ since $\tilde{J}$ is an ideal in $g$. Note that $L_s(\tilde{J} \cap A_1) = 0$ follows from $A_1 = \text{Ker}(L_s)$. ■

Remark 5. Since $[A_1, A_2] \subset A_2$, then for any ideal $i$ of $g$ with the property $i \subset A_1$ we have $[i, A_2] = 0$.

Proposition 6. Suppose that $X \in g$ has constant length on $(M, g)$. Then the following assertions hold.

1) If $Y, Z \in g$ are such that $[[X, Y], Z] \in [X, g]$, then $g([[X, Y], [X, Z]]) = 0$ on $M$.

2) If $V_i$ and $V_j$ are the root spaces (5) with $i \neq j$, then $g(V_i, V_j) = 0$ on $M$.

3) If $i \neq j$ and $\beta_j \neq 2\beta_i$, then $g([V_i, V_j], V_i) = 0$ and $g([V_i, V_j], V_j) = 0$ on $M$.

Proof. 1) If $U \in g$ is such that $[[X, Y], Z] = [X, U]$, then

$$g([[Z, [Y, X]], X]) = g([[X, Y], Z], X) = g([X, U], X) = 0.$$

Then we get $g([Y, X], [Z, X]) = g([Z, [Y, X]], X) + g([Y, X], [Z, X]) = 0$ by (3).

2) For every $Y \in V_i$ and $Z \in V_j$ we get $[X, Y] \in V_i$ and $[[X, Y], Z] \in A_2 \subset [X, g]$ by Proposition 4. Therefore, $g([X, Y], [X, Z]) = 0$ on $M$ by 1). Since $L = \text{ad}(X)$ is invertible on $A_2$, we get $g(V_i, V_j) = 0$ on $M$.

3) If $\beta_j \neq 2\beta_i$, then $[V_i, V_j] \subset \oplus_{k \neq i} V_k$ by Proposition 4. Hence, $g([V_i, V_j], V_i) = 0$ by 2).

Therefore, $g([V_i, V_j], V_j) = g([V_i, V_i], V_j) + g([V_i, V_j], V_j) = V_i \cdot g(V_i, V_j) = 0$. ■

Since always $X \in \text{Ker}(L) = \text{Ker}(\text{ad}(X)) \subset A_1$, we get $A_1 \neq 0$. On the other hand, it is possible that $A_1 = g$ and $A_2 = 0$ (see Remark 7 and Proposition 11).

Proposition 7. Suppose that $X \in g$ has constant length on $(M, g)$ and $A_1 = g$. Then $X \in n(g)$.

Proof. If $A_1 = g$, then $L^2 = (\text{ad}(X))^2 = 0$ on $g$. Elements $X \in g$ with this property are called absolute zero divisors in $g$. Using the Levi decomposition $g = \tau(g) \times s$, one can show that $X \in \tau(g)$. If $X = X_{\tau(g)} + X_s$, where $X_{\tau(g)} \in \tau(g)$ and $X_s \in s$, then for any $Y \in s$ we have $(\text{ad}(X))^2(Y) = [X_s, [X_s, Y]] + Z$, where $Z = [X, [X_{\tau(g)}, Y]] + [X_{\tau(g)}, [X_s, Y]] \in \tau(g)$. Hence, $(\text{ad}(X))^2 = 0$ imply $[X_s, [X_s, Y]] = 0$ for all $Y \in s$, i.e., $X_s$ is an absolute zero divisor in $s$, that is impossible for $X_s \neq 0$. Indeed, if $U \in s$ is a non-trivial absolute zero divisor, then $U$ is a non-trivial nilpotent element in $s$. Hence, there are $V, W \in s$ such that $[W, U] = 2U$, $[W, V] = -2V$, and $[U, V] = W$ by the Morozov–Jacobson theorem (see e.g. Theorem 3 in [15]). But this implies $[U, [U, V]] = -2U$ that contradicts to $(\text{ad}(U))^2 = 0$ (see [16] for a more detailed discussion). Therefore, we get $X_s = 0$ and $X \in \tau(g)$.

Moreover, it is known that $n(g) = \{Y \in \tau(g) \mid (\text{ad}(Y))^p = 0$ for some $p \in \mathbb{N} \}$, see e.g. Remark 7.4.7 in [14]. Therefore, $X \in n(g)$.
SPECTRAL PROPERTIES OF KILLING VECTOR FIELDS OF CONSTANT LENGTH 7

3. KVFCL ON GEODESIC ORBIT SPACES

Let \((M = G/H, g)\) be a homogeneous Riemannian space, where \(H\) is a compact subgroup of a Lie group \(G\) and \(g\) is a \(G\)-invariant Riemannian metric. We will suppose that \(G\) acts effectively on \(G/H\) (otherwise it is possible to factorize by \(U\), the maximal normal subgroup of \(G\) in \(H\)).

We recall the definition of one important subclass of homogeneous Riemannian spaces.

A Riemannian manifold \((M, g)\) is called a manifold with homogeneous geodesics or a geodesic orbit manifold (shortly, GO-manifold) if any geodesic \(\gamma\) of \(M\) is an orbit of a 1-parameter subgroup of the full isometry group of \((M, g)\). Obviously, any geodesic orbit manifold is homogeneous. A Riemannian homogeneous space \((M = G/H, g)\) is called a space with homogeneous geodesics or a geodesic orbit space (shortly, GO-space) if any geodesic \(\gamma\) of \(M\) is an orbit of a 1-parameter subgroup of the group \(G\). Hence, a Riemannian manifold \((M, g)\) is a geodesic orbit Riemannian manifold, if it is a geodesic orbit space with respect to its full connected isometry group. This terminology was introduced in [17] by O. Kowalski and L. Vanhecke, who initiated a systematic study of such spaces. In the same paper, O. Kowalski and L. Vanhecke classified all GO-spaces of dimension \(\leq 6\). A detailed exposition on geodesic orbit spaces and some important subclasses could be found also in [4, 11, 21], see also the references therein. In particular, one can find many interesting results about GO-manifolds and its subclasses in [1, 2, 3, 6, 7, 8, 10, 12, 13, 18, 22, 23, 24, 27, 28].

Recall that all symmetric, weakly symmetric, normal homogeneous, naturally reductive, generalized normal homogeneous, and Clifford–Wolf homogeneous Riemannian spaces are geodesic orbit, see [10]. Besides the above examples, every isotropy irreducible Riemannian space is naturally reductive, and hence geodesic orbit, see e. g. [9].

The following simple result is very useful (\(M_x\) denotes the tangent space to \(M\) at the point \(x \in M\)).

Lemma 3 ([19], Lemma 5). Let \((M, g)\) be a Riemannian manifold and \(\mathfrak{g}\) be its Lie algebra of Killing vector fields. Then \((M, g)\) is a GO-manifold if and only if for any \(x \in M\) and any \(v \in M_x\) there is \(X \in \mathfrak{g}\) such that \(X(x) = v\) and \(x\) is a critical point of the function \(g \in M \mapsto g_y(X, X)\). If \((M, g)\) is homogeneous, then the latter condition is equivalent to the following one: for any \(Y \in \mathfrak{g}\) the equality \(g_x([Y, X], X) = 0\) holds.

Now, we recall the following remarkable result.

Proposition 8 ([19], Theorem 1). Let \((M, g)\) be a GO-manifold, \(\mathfrak{g}\) is its Lie algebra of Killing vector fields. Suppose that \(\mathfrak{a}\) is an abelian ideal of \(\mathfrak{g}\). Then any \(X \in \mathfrak{a}\) has constant length on \((M, g)\).

As is noted in [19], Proposition 8 could be generalized for geodesic orbit spaces. For the reader’s convenience, we provide also the proof of the corresponding result.

Theorem 4. Let \((M = G/H, g)\) be a geodesic orbit Riemannian space. Suppose that \(\mathfrak{a}\) is an abelian ideal of \(\mathfrak{g} = \text{Lie}(G)\). Then any \(X \in \mathfrak{a}\) (as a Killing vector field) has constant length on \((M, g)\). As a corollary, \(g(X, Y) \equiv \text{const}\) on \(M\) for every \(X, Y \in \mathfrak{a}\).

Proof. Let \(x\) be any point in \(M\). We will prove that \(x\) is a critical point of the function \(g \in M \mapsto g_y(X, X)\). Since \((M = G/H, g)\) is homogeneous, then (by Lemma 3) it suffices to prove that \(g_x([Y, X], X) = 0\) for every \(Y \in \mathfrak{g}\).

Consider any \(Y \in \mathfrak{a}\), then \(Y \cdot g(X, X) = 2g([Y, X], X) = 0\) on \(M\), since \(\mathfrak{a}\) is abelian.

Now, consider \(Y \in \mathfrak{g}\) such that \(g_x(Y, U) = 0\) for every \(U \in \mathfrak{a}\). We will prove that \(g_x([Y, X], X) = 0\). By Lemma 3, for the vector \(X(x) \in M_x\) there is a Killing field \(Z \in \mathfrak{g}\) such that \(Z(x) = X(x)\) and \(g_x([V, Z], Z) = 0\) for any \(V \in \mathfrak{g}\). In particular, \(g_x([Y, Z], Z) = 0\). Now, \(W = X - Z\) vanishes at \(x\) and we get

\[
g_x([Y, X], X) = g_x([Y, Z + W], Z + W) = g_x([Y, Z + W], Z) =
\]

\[ g_x([Y, Z], Z) + g_x([Y, W], Z) = g_x([Y, W], Z). \]

Note that \( g_x([Y, W], Z) = -g_x([W, Y], Z) = g_x([W, Y], Z) = g_x(Y, [W, Z]) = 0 \) due to \( W(x) = 0 \) (\( 0 = W \cdot g(Y, Z) \)). Therefore, \( g_x([Y, X], X) = 0 \).

Since every \( Y \in g \) could be represented as \( Y = Y_1 + Y_2 \), where \( Y_1 \in a \) and \( g_2(Y_2, a) = 0 \), then \( x \) is a critical point of the function \( y \in M \mapsto g_y(X, X) \). Since every \( x \in M \) is a critical point of the function \( y \in M \mapsto g_y(X, X) \), then \( X \) has constant length on \((M, g)\).

The last assertion follows from the equality \( 2g(X, Y) = g(X + Y, X + Y) - g(X, X) - g(Y, Y) \). ■

**Corollary 1.** Every geodesic orbit Riemannian space \((M = G/H, g)\) with non-semisimple group \( G \) has non-trivial Killing vector fields of constant length.

**Proof.** If the Lie algebra \( g = \text{Lie}(G) \) is non semisimple, then it has a non-trivial abelian ideal \( a \) (for instance, this property has the center of the nilradical \( n(g) \) of \( g \)). Now, it suffices to apply Theorem 4. ■

We recall some other important properties of geodesic orbit spaces. Any semisimple Lie algebra \( s \) is a direct Lie algebra sum of its compact and noncompact parts \((s = s_c \oplus s_{nc})\).

The following proposition is asserted in [12], a detailed proof could be found in [13].

**Proposition 9.** Let \((G/H, g)\) be a connected geodesic orbit space and let \( s \) be any Levi factor of \( G \). Then the noncompact part \( s_{nc} \) of \( s \) commutes with the radical \( r(g) \).

**Remark 6.** For a geodesic orbit space \((G/H, g)\) we have a direct Lie algebra sum \( g = (t(g) \times s_c) \oplus s_{nc} \) by Proposition 9. Moreover, we can represent \( s_{nc} \) as a direct sum of simple noncompact ideals. This decomposition is useful for applying of Theorem 2.

**Proposition 10** (C. Gordon, [12]). Let \((G/H, \rho)\) be a geodesic orbit space. Then the nilradical \( n(g) \) of the Lie algebra \( g = \text{Lie}(G) \) is commutative or two-step nilpotent.

Suppose that \( X \in g \) has constant length on a geodesic orbit space \((G/H, g)\), then \([A_1, A_2] \subset A_2 \) and \( C(n(g)) \subset A_1 \) by Propositions 4 and 5 for a given Killing field \( X \) of constant length \((A_1 = \text{Ker}(L^2) \) and \( A_2 = \text{Im}(L^2) \) as in Theorem 3). Moreover, \( n(g) \) is at most 2-step nilpotent by Proposition 10. All these considerations lead to the following

**Conjecture 2.** If \( X \in g \) has constant length on a geodesic orbit space \((G/H, g)\), then \( n(g) \subset A_1 \).

**Theorem 5.** For a geodesic orbit space \((G/H, g)\), we consider any \( X \in n(g) \). Then the following conditions are equivalent:

1) \( X \) is in the center \( C(n(g)) \) of \( n(g) \);
2) \( X \) has constant length on \((G/H, g)\).

**Proof.** 1) \( \Rightarrow \) 2). Since the center \( C(n(g)) \) is an abelian ideal in \( g \), then \( X \) has constant length due to Theorem 4.

2) \( \Rightarrow \) 1). Since \( X \in n(g) \) and \( n(g) \) is at most two step nilpotent by Proposition 10, then \([Z, [Z, X]] = 0 \) for any \( Z \in n(g) \). Now, by Lemma 2, we have

\[ g([Z, X], [Z, X]) = g([Z, [Z, X]], X) + g([Z, X], [Z, X]) = 0, \]

hence \([Z, X] = 0 \) for any \( Z \in n(g) \). Consequently, \( X \in C(n(g)) \). ■

**Corollary 2.** Under conditions of Theorem 5, any abelian ideal \( a \) in \( g \) is in \( C(n(g)) \). In particular, \( C(n(g)) \) is a maximal abelian ideal in \( g \) by inclusion.

**Proof.** It is clear that \( a \) is a nilpotent ideal in \( g \), hence \( a \subset n(g) \). By Proposition 10, \( a \) consists of Killing fields of constant length, hence, \( a \subset C(n(g)) \) by Theorem 5. ■
Remark 7. It should be recalled that there are many examples of geodesic orbit nilmanifolds [12]. Therefore, Theorems 4 and 5 give non-trivial examples $X$ of KVFCL on $(M = G/H, g)$, where $X \in C(n(g))$. For any such example, the operator $\text{ad}(X) : g \to g$ is non-semisimple, since it is nilpotent. In this case, $A_1 = g$ and $L^2 = (\text{ad}(X))^2 = 0$. For semisimple $g$, there is no counterexample for Conjecture 1.

Let us recall Problem 2 in [20]: Classify geodesic orbit Riemannian spaces with nontrivial Killing vector fields of constant length. Now, this problem is far from being resolved. We have one modest result in this direction.

Proposition 11. Let $(G/H, g)$ be a geodesic orbit space and $X \in g = \text{Lie}(G)$. Then the following conditions are equivalent:

1) $X$ has constant length on $(G/H, g)$ and $A_1 = \text{Ker}(L^2) = g$;
2) $X$ is in the center $C(n(g))$ of $n(g)$.

Proof. 1) $\Rightarrow$ 2). By Proposition 7, we get $X \in n(g)$. Hence, $X \in C(n(g))$ by Theorem 5.

2) $\Rightarrow$ 1). By Theorem 5, $X$ has constant length on $(G/H, g)$. Since $n(g)$ is an ideal in $g$, then $[X, Y] \in n(g)$ and $L^2(Y) = [X, [X, Y]] \in [C(n(g)), n(g)] = 0$ for all $Y \in g$. $\blacksquare$

Acknowledgements. The author is indebted to Prof. V.N. Berestovskii for helpful discussions concerning this paper.

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