REGULARITY OF THE VANISHING IDEAL OVER A PARALLEL COMPOSITION OF PATHS

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Abstract. Let $G$ be a graph obtained by taking $r \geq 2$ paths and identifying all first vertices and identifying all the last vertices. We compute the Castelnuovo–Mumford regularity of the quotient $S/I(X)$, where $S$ is the polynomial ring on the edges of $G$ and $I(X)$ is the vanishing ideal of the projective toric subset parameterized by $G$. The case we consider is the first case where the regularity was unknown, following earlier computations (by several authors) of the regularity when $G$ is a tree, cycle, complete graph or complete bipartite graph, but specially in light of the reduction of the computation of the regularity in the bipartite case to the computation of the regularity of the blocks of $G$. We also prove new inequalities relating the Castelnuovo–Mumford regularity of $S/I(X)$ with the combinatorial structure of $G$, for a general graph.

1. Introduction

Let $K$ be a field and denote by $S$ the polynomial ring $K[t_1, \ldots, t_s]$ with the standard grading. If $M$ is a finitely generated graded $S$-module and

\begin{equation}
0 \to F_c \xrightarrow{\phi_c} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to M
\end{equation}

is a minimal graded free resolution, the Castelnuovo–Mumford regularity of $M$ is the integer:

$$\text{reg } M = \max_{i,j} \{ j - i \mid b_{ij} \neq 0 \},$$

where $b_{ij}$ are the graded Betti numbers of $M$, defined by $F_i \cong \oplus_{j \in \mathbb{Z}} S(-j)^{b_{ij}}$. The regularity of $M$ reflects the size of the degrees of the entries of the matrices in (1), and therefore, in a certain sense, the complexity of $M$ as a graded module. In the case when $M = S/I$, with $I$ a Cohen–Macaulay homogeneous ideal, we know that (cf. [17, Proposition 4.2.3]):

$$\text{reg } S/I = \max_j \{ j - c \mid b_{ij} \neq 0 \} = \deg F_{S/I}(t) + \dim S/I,$$

where $F_{S/I}(t)$ is the Hilbert Series of the module $S/I$ in rational function form.

Recently, many authors have studied the Castelnuovo-Mumford regularity of ideals associated to some combinatorial structure. For square free monomial ideals generated in degree 2, so-called edge ideals as their generators correspond to the edges of a graph (cf. [17, Chapter 6]), the regularity can be bounded using the induced matching number of the associated graph (cf. [7, Lemma 2010].

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For chordal graphs, it has been shown that the regularity actually coincides with this graph invariant (see [2, Corollary 6.9]). Several families of binomial ideals associated with a combinatorial structure have also been studied. The class of toric ideals, i.e., the ideal of relations of the edge subring of a graph, whose generators correspond to even closed walks on the graph (cf. [17, Chapter 8]), is one such example. For a complete graph $K_n$, the regularity of its edge subring is equal to $[n/2]$, while for a complete bipartite graph $K_{a,b}$, this invariant coincides with $\min\{a,b\} - 1$ (cf. [17]). Lower and upper bounds for the regularity of toric ideals, in terms of the structure of the underlying graph, have recently been established (cf. [1]). Another class of binomial ideals which has been extensively studied in recent times is the class of binomial edge ideals. These ideals are generated by the maximal minors of a $2 \times s$ generic matrix, whose column indices correspond to the edges of a graph. The regularity of these ideals can also be expressed and bounded in terms of graph-theoretic invariants (cf. [3, 9, 10]).

For the purposes of this work, $K$ will be a finite field of cardinality $q$. In the rest of the paper all graphs will be undirected and without loops; multiple edges are allowed. The vertex set of a graph $G$ will be denoted by $V_G$ and its edge set by $E_G$. We denote the number of edges by $s$ and we fix an ordering of the set of edges given by an identification of $E_G$ with the set of variables of $K[t_1, \ldots, t_s]$. If $H$ is a subgraph of $G$ we denote by $K[E_H]$ the polynomial subring on the variables of $E_H$, under the above identification. To $G$ we associate a set $X$ defined by

$$X = \{ (x^{t_1}, x^{t_2}, \ldots, x^{t_s}) \in \mathbb{P}^{s-1} \mid x \in (K^*)^{V_G} \},$$

where, if $x = \sum_{v \in V_G} x_v v$, with $x_v \in K^*$, for all $v \in V_G$, and $t_i$ is the edge $\{v, w\}$ (with $v \neq w$), we set $x^{t_i} = x_v x_w$. As $x^{t_i} \neq 0$, for all $i$, $X$ is a subset of the projective torus $\mathbb{T}^{s-1} \subset \mathbb{P}^{s-1}$. We refer to $X$ as the projective toric subset parameterized by $G$. Denote by $I(X)$ the vanishing ideal of $X$. Observe that

$$I(\mathbb{T}^{s-1}) = (t_1^{q-1} - t_2^{q-1}, \ldots, t_{s-1}^{q-1} - t_s^{q-1}) \subset I(X).$$

The notion of parameterized projective toric subsets and the study of their vanishing ideals was introduced in [13]. Unlike in the case of the edge ideal of $G$, we know that $I(X)$ is always a Cohen–Macaulay homogeneous binomial ideal of height $s - 1$ (Cf. [13, Theorem 2.1]).

In the original definition of a parameterized projective toric subset, $G$ is assumed to be a simple graph. However, on the one hand, we note that multiple edges play no part in the invariants of $K[E_G]/I(X)$. More precisely, if $G'$ is the simple graph obtained from $G$ by removing all extra edges through any two given vertices and $X'$ is the projective toric subset parameterized by $G'$, then

$$K[E_G]/I(X') \cong K[E_G]/I(X),$$

simply because $t_j - t_i \in I(X)$, for every extra edge $t_j$ between the endpoints of $t_i$. On the other hand, allowing extra edges eases notation and simplifies statements and proofs.

As $X$ is a finite set, the value of the Hilbert function of $K[E_G]/I(X)$ is eventually equal to $|X|$, the cardinality of $X$; therefore, $\deg K[E_G]/I(X) = |X|$. A formula for the degree was first given in [13] for connected graphs and then generalized to any graph in [12, Theorem 3.2]:

$$\deg K[E_G]/I(X) = \begin{cases} \left(\frac{q}{2}\right)^{\gamma-1} (q - 1)^{n-m+\gamma-1}, & \text{if } \gamma \geq 1 \text{ and } q \text{ is odd}, \\ (q - 1)^{n-m+\gamma-1}, & \text{if } \gamma \geq 1 \text{ and } q \text{ is even}, \\ (q - 1)^{n-m-1}, & \text{if } \gamma = 0, \end{cases}$$

where, if $x = \sum_{v \in V_G} x_v v$, with $x_v \in K^*$, for all $v \in V_G$, and $t_i$ is the edge $\{v, w\}$ (with $v \neq w$), we set $x^{t_i} = x_v x_w$. As $x^{t_i} \neq 0$, for all $i$, $X$ is a subset of the projective torus $\mathbb{T}^{s-1} \subset \mathbb{P}^{s-1}$. We refer to $X$ as the projective toric subset parameterized by $G$. Denote by $I(X)$ the vanishing ideal of $X$. Observe that

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where \((q\) is the cardinality of \(K\), \(n\) is the cardinality of \(V_G\), \(m\) is the number of connected components of \(G\) and \(\gamma\) the number of those that are non-bipartite.

Using the identity \((2)\) and the fact that \(\dim K[E_G]/I(X) = 1\), we deduce that the regularity of \(K[E_G]/I(X)\) coincides with its regularity index, i.e., the minimum degree \(d\) for which the value of the Hilbert function at \(k\) is equal to the value of the Hilbert polynomial at \(k\), for every \(k \geq d\). (Cf. \([17, \text{Corollary 4.1.12}]\).) Since the Hilbert function of \(K[E_G]/I(X)\) is strictly increasing for \(0 \leq d \leq \reg K[E_G]/I(X)\) and the Hilbert polynomial is equal to \(|X| = \deg K[E_G]/I(X)\) we conclude that \(\reg K[E_G]/I(X)\) is the minimum \(d\) for which the value of the Hilbert function at \(d\) is equal to \(|X| = \deg K[E_G]/I(X)\).

In Table 1 we list cases for which this invariant is known. When \(X\) coincides with the projective torus \(\mathbb{T}^{n-1}\) (which, from \((4)\), is the case, for example, if \(G\) is a tree or an odd cycle),

\[
I(X) = (t_1^{q-1} - t_2^{q-1}, \ldots, t_r^{q-1} - t_r^{q-1}).
\]

Thus the regularity can be computed from \((2)\), (see also \([15]\)). The regularity in the case \(G = K_n\) is given in \([6, \text{Remark 3}]\). The case \(G = K_{a,b}\) is given in \([4, \text{Corollary 5.4}]\) and the case of an even cycle, \(G = C_{2k}\), in \([12, \text{Theorem 6.2}]\). In the case of a complete multipartite graph, \(G = K_{\alpha_1,\ldots,\alpha_r}\) this invariant was computed in \([11, \text{Theorem 4.3}]\). (Here \(r \geq 3\) and the \(n\) in the formula is \(|V_G| = \alpha_1 + \cdots + \alpha_r\).)

A graph \(G\) is said to be 2-connected if \(|V_G| > 2\) and, for every vertex \(v \in V_G\), the graph \(G - v\) is connected. Any graph decomposes into blocks, which consist of either maximal 2-connected subgraphs, single edges or isolated vertices. When \(G\) is bipartite, we know that \(\reg(E_G)/I(X)\) can be computed from its block decomposition. More precisely, if \(G\) is a simple bipartite graph with no isolated vertices and \(H_1,\ldots,H_r\) are the blocks of \(G\), then

\[
\reg K[E_G]/I(X) = \sum_{k=1}^{r} \reg K[E_{H_k}]/I(X_k) + (r - 1)(q - 2),
\]

where \(X_k\) is the projective toric subset parameterized by the graph \(H_k\), for each \(k = 1,\ldots,r\) (cf. \([13, \text{Theorem 7.4}]\)). This reduces the problem of computing \(\reg K[E_G]/I(X)\) for a bipartite graph to the case of 2-connected graphs. Notice that \((5)\), together with the formula for the regularity in the case of even cycles, gives the regularity for any bipartite cactus graph (a simple graph the blocks of which are edges or even cycles).
A 2-connected graph can be reconstructed from one of its cycles by adding a path by its endpoints (also known as an ear) to the cycle and successively repeating this operation (a finite number of times) to the graphs obtained (cf. [2, Proposition 3.1.1]). The simplest 2-connected graph is a cycle. The second simplest 2-connected graph is a cycle with an attached ear. This graph can also be obtained by identifying the endpoints of 3 paths, which, in turn, is also known as the parallel composition of 3 paths. Therefore the parallel composition of 3 paths is the first case of a 2-connected graph for which the regularity of $K[E_G]/I(X)$ was not known.

The aim of this work is to compute the Castelnuovo–Mumford regularity of $K[E_G]/I(X)$, when $X$ belongs to the family of projective toric subsets parameterized by a graph given as the parallel composition of $r \geq 2$ paths, as illustrated in Figure 1. (Notice that this graph may well have multiple edges if more than one $P_i$ has length equal to 1.)

![Figure 1. G, the parallel composition of paths $P_1, P_2, \ldots, P_r$.](image)

Our first main result concerns the bipartite case.

**Theorem 1.1.** Let $X$ be the projective toric subset parameterized by the parallel composition of $r \geq 2$ paths, the lengths of which, $k_1, \ldots, k_r$, have the same parity. Then

$$
\text{reg } K[E_G]/I(X) = \begin{cases} 
\left\lfloor k_1/2 \right\rfloor + \cdots + \left\lfloor k_r/2 \right\rfloor (q-2), & \text{if } k_i \text{ are odd}, \\
(k_1/2 + \cdots + k_r/2 - 1)(q-2), & \text{if } k_i \text{ are even}.
\end{cases}
$$

We prove this result in Section 3 by proving the two inequalities involved. The lower bound is a straightforward consequence of the fact that $G$ is bipartite (cf. (7) and Lemma 3.1, below). For the upper bound we divide the proof into two cases. The case of $k_i$ even is worked out by induction on $r$ and arguing using suitable coverings of $G$ (cf. Proposition 3.2). The case of $k_i$ odd is harder and relies on a characterization of the homogeneous binomials in $I(X)$ (cf. Theorem 3.3).

With Theorem 1.1 we are able to study the non-bipartite case.

**Theorem 1.2.** Let $X$ be the projective toric subset parameterized by a graph $G$ that is the parallel composition of $r \geq 2$ paths, the lengths of which have mixed parities. Then

$$
\text{reg } K[E_G]/I(X) = \text{reg } K[E_{H_1}]/I(X_1) + \text{reg } K[E_{H_2}]/I(X_2) + (q-2),
$$

where $H_1$ is the parallel composition of the paths of odd lengths, $H_2$ is the parallel composition of the paths of even lengths, and $X_1, X_2$, respectively, are the projective toric subsets they parameterize.

We point out that the formula of Theorem 1.2 includes the case when only one path has length of different parity. In this situation, the corresponding summand of the formula does not follow from Theorem 1.1, rather, it can be retrieved from the first formula of Table 1 more precisely, if $H_i$ consists of a path of length $k$ then $\text{reg } K[E_{H_i}]/I(X_i) = (k-1)(q-2)$. 


The proof of Theorem 1.2 occupies the second half of Section 3. As with our other main result we prove the two inequalities separately (cf. Lemma 3.4 and Theorem 3.5). This time, the easier inequality is the one giving the upper bound. For the lower bound inequality we need to use different techniques to those used in the proof of Theorem 1.1.

Section 2 provides the background theory and the results that are used in our proofs. We single out the new contributions of Proposition 2.5, Proposition 2.6 and Proposition 2.7, as we believe these results will prove useful in the study of the regularity for a general graph.

2. Preliminaries

Let $K$ be a finite field of cardinality $q$. As in Section 1, $G$ will denote a graph with edge set $E_G$ of cardinality $s$ (we always assume that $G$ has no isolated vertices). We fix an identification of the variables of $K[t_1, \ldots, t_s]$ with $E_G$ and denote the former by $K[E_G]$. Let $X$ be the projective toric subset parameterized by $G$, as defined in (3). If $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, $t^a$ denotes the monomial $t_1^{a_1} \cdots t_s^{a_s} \in K[E_G]$.

We start by recalling a criterion for membership in $I(X)$ of a homogeneous binomial that only involves the combinatorics of $G$. It involves checking a linear congruence at every vertex of the graph. Let $v \in V_G$ and let $t_{i_1}, \ldots, t_{i_r}$ be the edges incident to $v$ (cf. Figure 2). Then by [11, Lemma 2.3], a homogeneous binomial $t^a - t^b \in K[E_G]$ belongs to $I(X)$ if and only if, for every vertex $v \in V_G$, if $i_1, \ldots, i_r$ are the indices of the edges incident to it, the congruence

$$a_{i_1} + \cdots + a_{i_r} \equiv b_{i_1} + \cdots + b_{i_r} \pmod{q-1} \quad (6)$$

is satisfied. It follows easily from this criterion, that if $H$ is a subgraph of $G$ and $Y$ is the projective toric subset parameterized by $H$, then $I(Y) = I(X) \cap K[E_H]$.

The following lemma will be used in the proof of Theorem 3.3 below. Recall that an ear of $G$ is a path which is maximal with respect to the condition that all of its interior vertices have degree 2 in $G$.

**Lemma 2.1.** Let $t^a - t^b \in K[E_G]$ be a homogeneous binomial. Let $t_i$ and $t_j$ be edges along an ear of $G$ in a same parity position along this path. Let $\sigma : K[E_G] \to K[E_G]$ be the automorphism defined by swapping the two edges $t_i$ and $t_j$. Then

$$t^a - t^b \in I(X) \iff \sigma(t^a) - \sigma(t^b) \in I(X).$$

**Proof.** It is clear we can reduce to the case illustrated in Figure 3. Since $\sigma(t^a) - \sigma(t^b)$ is homogeneous if and only if $t^a - t^b$ is, it suffices to check the equivalence of the system of 4 linear congruences given
by the 4 vertices \( v_1, v_2, v_3 \) and \( v_4 \). Let \( E(v_i) \) denote the set of edges incident to \( v_i \) and denote by \( E_1 \) the set \( E(v_i) \setminus \{ t_i \} \), and, likewise, \( E_4 = E(v_4) \setminus \{ t_j \} \). Let

\[
A_1 = \sum_{t_i \in E_1} a_i, \quad A_4 = \sum_{t_i \in E_4} a_i, \quad B_1 = \sum_{t_i \in E_1} b_i, \quad \text{and} \quad B_4 = \sum_{t_i \in E_4} b_i.
\]

Then, we need to show that the two systems of congruences modulo \( q - 1 \)

\[
\begin{align*}
A_1 + a_i & \equiv B_1 + b_i \\
A_4 + a_i & \equiv b_i + b_k \\
A_1 + a_j & \equiv B_1 + b_j \\
A_4 + a_j & \equiv b_j + b_k \quad \text{and} \quad A_k + a_j & \equiv b_k + b_i \\
A_j + A_4 & \equiv b_j + B_4
\end{align*}
\]

are equivalent, which is clearly true.

Our approach to computing \( \text{reg} \, K[E_G]/(I(X), g) \) is to consider an Artinian quotient \( K[E_G]/(I(X), g) \), where \( g \in K[E_G] \) is a suitable monomial.

**Proposition 2.2.** Let \( g \in K[E_G] \) be a monomial.

(i) There exists a monomial order and a binomial Gröbner basis \( B \) of \( (I(X), g) \) such that \( B \cup \{ g \} \) is a Gröbner basis for the ideal \((I(X), g) \subset K[E_G]\).

(ii) A monomial \( t^a \in K[E_G] \) belongs to \((I(X), g) \) if and only if there exists a monomial \( t^b \in K[E_G] \) such that \( t^a - gt^b \) is homogeneous and belongs to \((I(X), g) \).

**Proof.** Since \( I(X) \) is generated by homogeneous binomials, the Gröbner basis obtained from such a set, after fixing any monomial order, consists of homogeneous binomials, by Buchberger’s Algorithm. Let \( t_{i_1}, \ldots, t_{i_r} \) be the variables dividing \( g \). Fix the graded reverse lexicographical order after reordering the variables in such a way that \( t_{i_1} \succ \cdots \succ t_{i_r} \) are the last variables of the ring. Let \( B \) be a binomial Gröbner basis of \((I(X), g) \) with respect to such order. To prove (i) it suffices to show that \( S(f, g) \) reduces to 0 modulo \( B \cup \{ g \} \), for every \( f \in B \). Let \( f = t^a - t^b \in B \). Assume, without loss of generality, that \( \ell(t) = t^a \). If \( t_{i_r} \) divides \( t^a \), then \( t_{i_r} \) does not divide \( t^b \) (we may assume the generating set \( S \) start with consists of irreducible binomials). This implies that \( t^b \nmid t^a \), hence \( t_{i_r} \) does not divide \( t^b \). Arguing in the same way, by induction, we conclude that none of \( t_{i_1}, \ldots, t_{i_r} \) divides \( t^a \) and thus \( \text{gcd}(g, t^a) = 1 \). Accordingly,

\[
S(f, g) = g(t^a - t^b) - t^a g = -gt^b
\]

which reduces to zero modulo \( B \cup \{ g \} \). This completes the proof of (i).

Let \( t^a \) be a monomial. One direction of the equivalence in (ii) is clear. Assume that \( t^a \in (I(X), g) \). Then, considering the Gröbner basis \( B \cup \{ g \} \) obtained in (i), \( t^a \) has zero remainder after division with \( B \cup \{ g \} \). Since the division of a monomial by a binomial is still a monomial, the division algorithm stops the first time \( g \) is used. Thus, the partial quotients of division are monomials \( t^a = h_0, h_1, \ldots, h_k \) such that \( h_i - h_{i-1} \in (I(X), g) \), for all \( i = 1, \ldots, k \) and such that \( g \) divides \( h_k \). Writing \( h_k = gt^b \), we get a homogeneous binomial \( t^a - gt^b \) which belongs to \((I(X), g) \), as required. \( \square \)
Proposition 2.3. Let \( g \in K[E_G] \) be a monomial. Then \( K[E_G]/(I(X), g) \) is zero in degree \( d \) if and only if \( d \geq \text{reg} K[E_G]/I(X) + \text{deg}(g) \).

Proof. We denote \( K[E_G]/I(X) \) by \( R \) and, by abuse of notation, \( K[E_G]/(I(X), g) \) by \( R/g \). Since \( g \) is an \( R \)-regular element and \( R \) is Cohen–Macaulay,

\[
\dim R/g = \dim R - 1 = 0.
\]

Moreover, since \( R/g \) is a quotient of a polynomial ring with the standard grading by a homogeneous ideal, its regularity index is the minimum degree \( d \) for which \( (R/g)_d = 0 \). \((R/g)_{d+k} = 0 \), for all \( k \geq 0 \). Hence we need to show that the regularity index of \( R/g \) is equal to \( \text{reg} K[E_G]/I(X) + \text{deg}(g) \). Consider the following exact sequence of graded \( K[E_G] \)-modules:

\[
0 \to R[-\text{deg}(g)] \xrightarrow{\cdot g} R \to R/g \to 0.
\]

Comparing the degree of the Hilbert series of the three terms and using the identity (2), we get \( \text{deg} F_{R/g} + 1 = \text{deg}(g) \), where \( F_{R/g} \) is the Hilbert Series of the \( K[E_G] \)-module \( R/g \) in rational function form. As \( \text{deg} F_{R/g} + 1 \) is the regularity index (cf. [17] Corollary 4.1.12), we have proved the claim. \( \square \)

We note that the following proposition can be easily derived from [13] Theorem 7.4 in the bipartite case, and from [15] Corollary 3.10 and [5] Lemma 1 in the non-bipartite case, when \( G \) is a unicyclic connected graph and the only cycle of \( G \) is odd. Here, we do not assume \( G \) is bipartite nor a unicyclic connected graph with an odd cycle.

Proposition 2.4. Let \( v \in V_G \) be a vertex of degree 1. Assume that \( |E_G| > 1 \). Consider the graph \( G' = G - v \) and denote by \( X' \) the projective toric subset parameterized by it. Then

\[
\text{reg} K[E_G]/I(X) = \text{reg} K[E_G']/I(X') + (q - 2).
\]

Proof. Let \( t_i \in E_G \) be incident to \( v \) and let \( t_j \in E_G \setminus t_i \). According to Proposition 2.3 to show that

\[
\text{reg} K[E_G]/I(X) \leq \text{reg} K[E_G']/I(X') + (q - 2)
\]

it suffices to show that for any monomial \( t^a \in K[E_G] \) of degree \( \text{reg} K[E_G']/I(X') + (q - 2) + 1 \) we have \( t^a \in (I(X), t_j) \). Let \( t^{a'} \) be such a monomial. If \( a_i \geq q - 1 \) then writing \( t^{a'} = t^{a'} t_i^{q-1} \) for some \( a' \in \mathbb{N}^s \), we get:

\[
t^{a'} = t^{a'} (t_i^{q-1} - t_j^{q-1}) + t^{a'} t_j^{q-1} \in (I(X), t_j).
\]

Assume now that \( a_i < q - 1 \). Consider \( a' \in \mathbb{N}^s \), with \( a'_i = 0 \), such that \( t^{a'} = t^{a'} t_i^{q-1} \). Then \( \text{deg} t^{a'} = \text{deg} t^a - a_i \geq \text{reg} K[E_G']/I(X') + 1 \), by our assumptions. As \( t^{a'} \) belongs to \( K[E_G'] \), using Proposition 2.3 we get \( t^{a'} \in (I(X'), t_j) \subseteq K[E_G'] \). As \( G' \) is a subgraph of \( G \) we have \( I(X') \subseteq I(X) \) and therefore \( t^{a'} \in (I(X), t_j) \).

Using the same idea, let us now show that

\[
\text{reg} K[E_G']/I(X') \leq \text{reg} K[E_G']/I(X) - (q - 2).
\]

Let \( t^{a'} \in K[E_G'] \) be a monomial of degree \( \text{reg} K[E_G']/I(X) - (q - 2) + 1 \). Then \( t^{a'} t_i^{q-2} \) belongs to \( K[E_G] \) and has degree \( \text{reg} K[E_G']/I(X) + 1 \). We deduce that \( t^{a'} t_i^{q-2} \in (I(X), t_j) \). By Proposition 2.3 there exists a monomial \( t^b \in K[E_G] \) such that \( t^{a'} t_i^{q-2} - t_j t^b \in I(X) \). However the congruence at
vertex $v$ gives $b_i = q - 2 + k(q - 1)$, for some $k \geq 0$. Let $b' \in \mathbb{N}^*$ be such that $b'_i = 0$ and $t^b = t^{b_i}t^{b_i}$. Then:
\[ t^a t_i^{q-2} - t_j t^b \in I(X) \implies t^a - t_j t_i^{k(q-1)} t^{b'} \in I(X) \implies t^a - t_j^{1+k(q-1)} t^{b'} \in I(X). \]
Since $t^a - t_j^{1+k(q-1)} t^{b'} \in K[E_G]$ and $I(X') = I(X) \cap K[E_G]$, we deduce that $t^a \in (I(X'), t_j)$. \hfill \box

Let $G$ be a connected graph and a spanning subgraph of a bipartite graph $H$. Let $Y$ be the projective toric subset parameterized by $H$. Then, by \cite{16} Lemma 2.13, if $|X| = |Y|$, it follows that
\[ \text{reg } K[E_G]/I(X) \geq \text{reg } K[E_H]/I(Y). \]
Hence if $G$ is a connected bipartite spanning subgraph of $K_{a,b}$, by \cite{4} the assumption on the cardinality of the associated parameterized projective toric subsets holds and we obtain:
\[ \text{reg } K[E_G]/I(X) \geq (\max \{a, b\} - 1)(q - 2). \]

In the remainder of this section we introduce three new inequalities involving $\text{reg } K[E_G]/I(X)$.
They will play an important role in the proofs of Theorem 1.1 and Theorem 1.2.

**Proposition 2.5.** Let $v_1$ and $v_2$ be two vertices of $G$ such that $\{v_1, v_2\}$ is a non-edge of $G$. Let $G'$ be the graph obtained by identifying $v_1$ with $v_2$ and denote by $X'$ the projective toric subset parameterized by it. Then $\text{reg } K[E_G]/I(X) \geq \text{reg } K[E_G]/I(X')$.

**Proof.** The edge sets of $G$ and $G'$ have the same cardinality. Moreover, there is an induced identification of the edges of $G'$ with the variables of the polynomial ring $K[t_1, \ldots, t_s]$ under which $K[E_G] = K[E_G']$. Since the parameterization of $X'$ is obtained by adding the restriction that the coefficient of $v_1$ in the formal sum $\sum_{v \in Y_G} x_v \cdot v$ be equal to the coefficient of $v_2$ we obtain $X' \subseteq X$ (cf. \cite{3}), and thus, $I(X) \subseteq I(X')$. Let $t_1$ be an edge. According to Proposition 2.3 to show that
\[ \text{reg } K[E_G]/I(X') \leq \text{reg } K[E_G]/I(X) \]
it suffices to prove that for any monomial $t^a$ of degree $\text{reg } K[E_G]/I(X) + 1$ we have $t^a \in (I(X'), t_1)$. Let $t^a$ be such a monomial. Then, using again Proposition 2.3 we deduce that $(t^a \in I(X), t_1)$. Since $I(X) \subseteq I(X')$ we get $(t^a \in I(X'), t_1)$. \hfill \box

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The graph obtained by identifying two vertices of $G$.}
\end{figure}
Proposition 2.6. Let $H_1, H_2 \subseteq G$ be subgraphs such that $E_G = E_{H_1} \cup E_{H_2}$ and $E_{H_1} \cap E_{H_2} \neq \emptyset$. Let $X_1$ and $X_2$ be the projective toric subsets parameterized by $H_1$ and $H_2$ and $I(X_1) \subset K[E_{H_1}]$, $I(X_2) \subset K[E_{H_2}]$ their corresponding vanishing ideals. Then
\[
\text{reg } K[E_G]/I(X) \leq \text{reg } K[E_{H_1}]/I(X_1) + \text{reg } K[E_{H_2}]/I(X_2).
\]

Proof. Let $t_i \in E_{H_1} \cap E_{H_2}$. According to Proposition 2.3, it suffices to show that any monomial $t^a \in K[E_G]$, of degree $\text{reg } K[E_{H_1}]/I(X_1) + \text{reg } K[E_{H_2}]/I(X_2) + 1$, belongs to $(I(X), t_i)$. Let us write $t^a = t^b t^c$ for some $t^b \in K[E_{H_1}]$ and $t^c \in K[E_{H_2}]$. Since $\deg(t^a) = \deg(t^b) + \deg(t^c)$, we have $\deg(t^b) \geq \text{reg } K[E_{H_1}]/I(X_1) + 1$ or $\deg(t^c) \geq \text{reg } K[E_{H_2}]/I(X_2) + 1$. By Proposition 2.3, it follows that $t^b \in (I(X_1), t_i) \subset K[E_{H_1}]$ or $t^c \in (I(X_2), t_i) \subset K[E_{H_2}]$, respectively. In both cases we conclude that $t^a \in (I(X), t_i)$. \hfill \Box

Proposition 2.7. Let $\{v_1, \ldots, v_r\}$ be an independent set of vertices of $G$. Assume that there is an edge in $G - \{v_1, \ldots, v_r\}$. Then $\text{reg } K[E_G]/I(X) \geq r(q - 2)$.

Proof. By Proposition 2.3, to show that $\text{reg } K[E_G]/I(X) \geq r(q - 2)$ it suffices to show that there exists an edge $t_i$ and a monomial $t^a \in K[E_G]$ of degree $r(q - 2)$ that does not belong to $(I(X), t_i)$. Let $t_i$ be an edge of $G - \{v_1, \ldots, v_r\}$ and, for every $i = 1, \ldots, r$, let $t_j$ be an edge incident to $v_i$. Such edges exist since we assume that $G$ has no isolated vertices. Notice also that since $\{v_1, \ldots, v_r\}$ is a set of edges $t_j_1, \ldots, t_j_r$ and let us consider the monomial:
\[
t^a = (t^{j_1}_1 \cdots t^{j_r}_r)^{r-2}
\]
and let us show that $t^a \not\in (I(X), t_i)$. Suppose the contrary holds. Then, by Proposition 2.2, there exists a monomial $t^b$ such that $t^a - t^b$ is homogeneous and belongs to $(I(X), t_i)$.

Since $t_i$ is not incident to any of the vertices of $\{v_1, \ldots, v_r\}$, evaluating the congruence at a particular vertex of this set, we conclude that the degree of $t^b$ in the edges incident to it is $\geq q - 2$. Since, by assumption, these vertices possess no common incident edges, we deduce that the degree of $t^b$ in edges incident to the vertices of $\{v_1, \ldots, v_r\}$ is $\geq r(q - 2)$. In particular, $\deg(t^b) \geq r(q - 2)$. But this implies that $t^a - t^b$ is not homogeneous, which is a contradiction. \hfill \Box

We note that Proposition 2.7 implies 4.

3. Proof of the main results

The aim of this section is to prove Theorem 1.1 and Theorem 1.2. In what follows $G$ is the parallel composition of $r \geq 2$ paths $P_1, \ldots, P_r$ of lengths $k_1, \ldots, k_r$. In a first instance, we assume that these integers have the same parity, so that $G$ is bipartite. If $r = 2$ and one of $k_1, k_2$ is greater than 1, then $G$ is an even cycle of length $k_1 + k_2$. In this case, by [12, Theorem 6.2], we know that $\text{reg } K[E_G]/I(X) = (k_1 + k_2)/(2)(q - 2)$. If $r = 2$ and $k_1 = k_2 = 1$, then $G$ is a graph on 2 vertices with 2 multiple edges. Hence the value of the regularity is the same as in the case of a tree with a single edge, which is $(s - 1)(q - 2) = 0$ (cf. Table 1). Both cases agree with the formula in Theorem 1.1

Lemma 3.1.
\[
\text{reg } K[E_G]/I(X) \geq \begin{cases} \lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor(q - 2), & \text{if } k_i \text{ are odd,} \\ (k_1/2 + \cdots + k_r/2 - 1)(q - 2), & \text{if } k_i \text{ are even.} \end{cases}
\]
Proof. If \( k_i \) are odd, then \( G \) is a connected spanning subgraph of \( K_{\rho, \rho} \), where \( \rho \) is the integer \( 1 + [k_1/2] + \cdots + [k_r/2] \). If \( k_i \) are even, then \( G \) is a connected spanning subgraph of \( K_{(\rho-r+2), \rho} \) where \( \rho \) is the integer \( k_1/2 + \cdots + k_r/2 \). Hence the claim follows from (7).

In the next two results we prove the opposite inequalities in each case. We need to fix some notation. For each \( i \in \{1, \ldots, r\} \), let \( \sigma_i = k_1 + \cdots + k_{i-1} \), so that, in particular, \( \sigma_1 = 0 \).

Let us label the edges of \( G \) as in Figure 5. For each \( i \in \{1, \ldots, r\} \), let \( f_i, g_i \in K[E_G] \) be:

\[
f_i = t_{\sigma_{i+1}} \cdot t_{\sigma_{i+3}} \cdots t_{\sigma_{i+2[k_i/2]-1}}
\]

(In other words, \( f_i \) is the product of every other edge in \( P_i \) starting with \( t_{\sigma_{i+1}} \) and \( g_i \) is the product of every other edge in \( P_i \) starting with \( t_{\sigma_{i+2}} \).) We notice that, for all \( i \neq j \),

\[
f_i g_j - f_j g_i \in I(X).
\]

Proposition 3.2. If \( k_i \) are even, then \( \operatorname{reg} K[E_G]/I(X) \leq (k_1/2 + \cdots + k_r/2 - 1)(q - 2) \).

Proof. According to Proposition 2.3 it suffices to show that any monomial \( t^a \in K[E_G] \) of degree \( (k_1/2 + \cdots + k_r/2 - 1)(q - 2) + 1 \) belongs to \( (I(X), t_1) \). We may assume \( t_1 \) does not divide \( t^a \). We will argue by induction on \( r \). For \( r = 2 \), as observed earlier, the result holds true. Assume now that \( r \geq 3 \). Let \( H \) be the subgraph of \( G \) given by \( \{t_1\} \cup P_2 \cup \cdots \cup P_r \) and \( Y \) be the projective toric subset parameterized by \( G \). By induction and [23] Theorem 7.4,

\[
\operatorname{reg} K[E_H]/I(Y) = (k_2/2 + \cdots + k_r/2)(q - 2).
\]

Set \( t^a = t^b t^c \), with \( t^b \in K[E_{P_1}] \) and \( t^c \in K[E_{H_t}] \). If \( \deg(t^c) \geq (k_2/2 + \cdots + k_r/2)(q - 2) + 1 \), then, by Proposition 2.3 \( t^c \in (I(Y), t_1) \subset (I(X), t_1) \), which implies that \( t^b \in (I(X), t_1) \). Assume that \( \deg(t^c) \leq (k_2/2 + \cdots + k_r/2)(q - 2) \). Then \( \deg(t^b) \geq (k_1/2 - 1)(q - 2) + 1 \). Consider now the subgraphs of \( G \) given by

\[
H_1 = P_1 \cup P_2 \quad \text{and} \quad H_2 = P_1 \cup P_3 \cup \cdots \cup P_r
\]

and denote by \( X_1 \) and \( X_2 \), respectively, the projective toric subsets parameterized by them. Set \( t^c = t^d t^e \) with \( t^d t^e \in K[E_{H_1}] \) and \( t^b t^e \in K[E_{H_2}] \). By the induction hypothesis,

\[
\operatorname{reg} K[E_{H_1}]/I(X_1) = (k_1/2 + k_2/2 - 1)(q - 2) \quad \text{and} \quad \operatorname{reg} K[E_{H_2}]/I(X_2) = (k_1/2 + k_3/2 + \cdots + k_r/2 - 1)(q - 2).
\]
Hence, if \( \deg(t^b t^d) \geq (k_1/2 + k_2/2 - 1)(q - 2) + 1 \), we get \( t^b t^d \in (I(X_1), t_1) \subset (I(X), t_1) \) which implies that \( t^a \in (I(X), t_1) \). Similarly, if \( \deg(t^b t^e) \geq (k_1/2 + k_3/2 + \cdots + k_r/2 - 1)(q - 2) + 1 \). Suppose that
\[
\deg(t^b t^d) \leq (k_1/2 + k_2/2 - 1)(q - 2) \quad \text{and} \\
\deg(t^b t^e) \leq (k_1/2 + k_3/2 + \cdots + k_r/2 - 1)(q - 2).
\]
Since \( \deg(t^a) = \deg(t^b t^d) + \deg(t^b t^e) - \deg(t^b) \), we deduce that
\[
\deg(t^a) \leq (k_1/2 + \cdots + k_r/2 - 1)(q - 2) - 1,
\]
which is a contradiction.  

**Theorem 3.3.** If \( k_i \) are odd, then \( \text{reg} K[E_G]/I(X) \leq ([k_1/2] + \cdots + [k_r/2])(q - 2) \).

**Proof.** We will use induction on \( k_1 + \cdots + k_r \). In the base case, \( r = 2 \) and \( k_1 = k_2 = 1 \). As we mentioned earlier, \( \text{reg} K[E_G]/I(X) = 0 \).

Assume that \( k_1 + \cdots + k_r > 3 \) and, as induction hypothesis, that the statement of the theorem holds for any \( k'_1, \ldots, k'_r \) and \( r' \geq 2 \) such that \( k'_1 + \cdots + k'_r < k_1 + \cdots + k_r \). If \( r = 2 \), then, as observed in the beginning of this section, \( G \) is an even cycle of length \( k_1 + k_2 \) and accordingly
\[
\text{reg} K[E_G]/I(X) = ((k_1 + k_2)/2 - 1)(q - 2) = ([k_1/2] + [k_2/2])(q - 2).
\]
Hence, we may assume \( r \geq 3 \). If, for some \( i, k_i = 1 \), let \( G' \) be the subgraph of \( G \) given as the parallel composition of all \( P_1, \ldots, P_r \) but \( P_i \). We note that \( G' \) is a connected spanning subgraph of \( G \) and hence, if \( X' \) is the projective toric subset parameterized by \( G' \), by the induction hypothesis, since \([k_i/2] = 0\), we get
\[
\text{reg} K[E_G]/I(X) \leq \text{reg} K[E_{G'}]/I(X') \leq ([k_1/2] + \cdots + [k_r/2])(q - 2).
\]
Thus, we may assume \( k_i \geq 3 \), for all \( i = 1, \ldots, r \). According to Proposition 2.3 to show that \( \text{reg} K[E_G]/I(X) \leq ([k_1/2] + \cdots + [k_r/2])(q - 2) \), it suffices to show that any monomial \( t^a \in K[E_G] \) of degree
\[
([k_1/2] + \cdots + [k_r/2])(q - 2) + [k_1/2] + \cdots + [k_r/2]
\]
belongs to the ideal \( (I(X), g) \subset K[E_G] \), where \( g = g_1 \cdots g_r \) and \( g_i \) were defined in (8). Let \( t^a \) be one such monomial and write it as the product of monomials, \( h_1 \cdots h_r \), where \( h_i \in K[E_{P_i}] \). By (10), we have \( \deg(h_i) \leq [k_i/2](q - 1) \), for some \( i \in \{1, \ldots, r\} \). Without loss of generality we assume \( i = 1 \).

In particular,
\[
\deg(h_2 \cdots h_r) \geq ([k_2/2] + \cdots + [k_r/2])(q - 2) + [k_2/2] + \cdots + [k_r/2].
\]
Since \( g \) is invariant under the swapping of variables corresponding to edges of \( P_1 \) in a same parity position, using Lemma 2.1 we may assume that
\[
a_1 \leq a_3 \leq \cdots \leq a_{2[k_1/2] - 1} \quad \text{and} \quad a_2 \leq a_4 \leq \cdots \leq a_{2[k_1/2]}
\]
Let \( H \) be the subgraph of \( G \) given by \( P_2 \cup \cdots \cup P_r \) and denote by \( Y \) the projective toric subset parameterized by it. By induction, \( \text{reg} K[E_H]/I(Y) = ([k_2/2] + \cdots + [k_r/2])(q - 2) \). Then, by (11), Proposition 2.3 and Proposition 2.2 there exists a monomial \( t^b \in K[E_H] \), for some \( b \in \mathbb{N}^s \) supported on the edges of \( H \), such that \( h_2 \cdots h_r - g_2 \cdots g_r t^b \in I(Y) \subset I(X) \) and hence
\[
t^a - h_1 g_2 \cdots g_r t^b \in I(X).
\]
If $a_2 \neq 0$, then from (12) we deduce that $g_1$ divides $h_1$ and we are done. If $a_1 \neq 0$, then there exists $c \in \mathbb{N}^*$ such that $h_1 = f_1 t^c$. Accordingly, $h_1 g_2 \cdots g_t t^b = f_1 g_2 \cdots g_t t^{b+c}$. Since $f_1 g_2 - f_2 g_1 \in I(X)$, we deduce that $f_1 g_2 \cdots g_t t^{b+c} - f_2 g_1 g_3 \cdots g_t t^{b+c} \in I(X)$, which, together with (13), implies that

$$t^a - f_2 g_1 g_3 \cdots g_t t^{b+c} \in I(X).$$

Consider $a' \in \mathbb{N}^*$ such that $t^{a'} = f_2 g_1 g_3 \cdots g_t t^{b+c}$. Since $h_1 = f_1 t^c$ and the monomials $f_2, g_3, \ldots, g_r, t^b$ are supported away from the edges of $P_1$, we see that, if $1 \leq i \leq k_1$, $a_i' = a_i - 1$, when $i$ is odd, and $a_i' = a_i + 1$, when $i$ is even. In particular, $a_2' \neq 0$ and, in the corresponding decomposition $t^{a'} = h'_1 \cdots h'_r$ with monomials $h'_i \in K[E_{P_1}]$, we get $\deg(h'_i) = \deg(h_1) - 1$. Repeating the previous argument, we deduce that $t^{a'} \in I(X, g)$, which, using (14) implies that $t^{a'} \in (I(X), g)$.

We are left with the case of $a_1 = a_2 = 0$. We regard $t^a$ as a monomial in $K[G']$, where $G'$ is the graph obtained as the parallel composition of $P_1 \setminus \{t_1, t_2\}$, $P_2, \ldots, P_r$.

Let $X'$ be the projective toric subset parameterized by $G'$. By the induction hypothesis

$$\reg K[E_{G'}/I(X')] = ([k_1/2] + \cdots + [k_r/2] - 1)(q-2).$$

Hence, by Proposition 2.3 and Proposition 2.2 there exists $t^d \in K[E_{G'}]$, where $d \in \mathbb{N}^*$ is supported on the edges of $G'$, such that

$$t^a - g'_1 g_2 \cdots g_r t^{a-1} t^d \in I(X'),$$

where $g'_1 = g_1/t_2 \in K[E_{G'}]$. We claim there exists $k \in \{1, 2, \ldots, q-1\}$ such that

$$t^{a'} - t_1^k t_2^k g'_1 g_2 \cdots g_r t^d \in I(X)$$

with $k = q - 1 - k$. We define $k$ using the congruence at vertex $v'_i$ of $G'$ (see Figure 6) which, according to [11] Lemma 2.3, is satisfied for the binomial in (15). This congruence is:

$$a_3 + a_{\sigma_2} + \cdots + a_{\sigma_r} \equiv q - 1 + d_3 + d_{\sigma_2} + \cdots + d_{\sigma_r} \quad (\text{mod } q - 1)$$

$$\iff a_3 - d_3 \equiv d_{\sigma_2} + \cdots + d_{\sigma_r} - (a_{\sigma_2} + \cdots + a_{\sigma_r}) \quad (\text{mod } q - 1).$$

Let $k \in \{1, 2, \ldots, q-1\}$ to be such that:

$$k \equiv a_3 - d_3 \equiv d_{\sigma_2} + \cdots + d_{\sigma_r} - (a_{\sigma_2} + \cdots + a_{\sigma_r}) \quad (\text{mod } q - 1).$$

Let us now show that (16) holds. Since $t^{a'} - t_1^k t_2^k g'_1 g_2 \cdots g_r t^d$ is homogeneous, it will suffice to check the congruences at each vertex of $G$. Since for the binomial in (15), from which we obtain this binomial,
the congruences are satisfied at all vertices of $G'$, it will be enough to check the congruences for the vertices $v_1, v_2$ and $v_3$. At $v_1$, we have:

$$a_{\sigma_2+1} + \cdots + a_{\sigma_r+1} \equiv (q - 1) - k + d_{\sigma_2+1} + \cdots + d_{\sigma_r+1} \pmod{q - 1},$$

at $v_2$, $0 \equiv (q - 1) - k + k \pmod{q - 1}$ and at $v_3$, $a_3 \equiv k + d_3 \pmod{q - 1}$, all of which hold, by virtue of (17). This completes the proof of the theorem. □

The proof of Theorem 1.1 follows from Lemma 3.1. Proposition 3.2 and Theorem 3.3.

We now turn to the proof of Theorem 1.2. In this case, $G$ is the parallel composition of paths $P_1, \ldots, P_r$ the lengths of which have mixed parity. We assume, without loss of generality, that $P_1, \ldots, P_l$ have odd lengths and $P_{l+1}, \ldots, P_r$ even lengths, for some $1 \leq l < r$. We will keep the notation for the edges of $G$ as in the beginning of this section and recall that (as in the statement of Theorem 1.2), we will be denoting by $H_1$ the parallel composition of the paths of odd lengths, by $H_2$ the parallel composition of the paths of even lengths and by $X_1, X_2$, respectively, the projective toric subsets they parameterize.

Lemma 3.4. $\text{reg } K[E_G]/I(X) \leq \text{reg } K[E_{H_1}]/I(X_1) + \text{reg } K[E_{H_2}]/I(X_2) + (q - 2)$.

Proof. Consider the cover of $G$ given by $H_1$ and $H_2'$ where $H_2'$ is given by $\{h_1\} \cup H_2$. Then $E_{H_1} \cap E_{H_2'} \neq \emptyset$ and therefore by Proposition 2.9

$$K[E_G]/I(X) \leq K[E_{H_1}]/I(X_1) + K[E_{H_2}]/I(X_2'),$$

where $X_2'$ is the projective toric subset parameterized by $H_2'$. By Proposition 2.4, we know that $\text{reg } K[E_{H_2}]/I(X_2') = \text{reg } K[E_{H_2}]/I(X_2) + (q - 2)$. Combining this with (18) completes the proof of the lemma. □

Theorem 3.5. $\text{reg } K[E_G]/I(X) \geq \text{reg } K[E_{H_1}]/I(X_1) + \text{reg } K[E_{H_2}]/I(X_2) + (q - 2)$.

Proof. We divide the proof into cases.

The case $l = 1$ and $r = 2$. In this case $G$ is a cycle of (odd) length $k_1 + k_2$. Accordingly, $X$ coincides with $T^{k_1+k_2-1}$ and, by the formula in Table 1, $\text{reg } K[E_G]/I(X) = (k_1 + k_2 - 1)(q - 2)$. On the other hand $H_1$ and $H_2$ are paths of lengths $k_1$ and $k_2$ and the projective toric subsets they parameterize are the tori $T^{k_1-1}$ and $T^{k_2-1}$ so that, again by the same formula, $\text{reg } K[E_{H_1}]/I(X_1) = (k_1 - 1)(q - 2)$ and $\text{reg } K[E_{H_2}]/I(X_2) = (k_2 - 1)(q - 2)$. We deduce that

$$\text{reg } K[E_G]/I(X) = \text{reg } K[E_{H_1}]/I(X_1) + \text{reg } K[E_{H_2}]/I(X_2) + (q - 2).$$

In the other cases, we will use vertex identifications and Proposition 2.7. For this purpose, let us denote the terminal vertices of the parallel composition yielding $G$ by $v$ and $w$.

The case $l = 1$, $k_1 = 1$ and $r - l > 1$. Consider the vertices of $P_2, \ldots, P_r$ at an odd number of edges away from $v$ (or $w$). They form an independent set of vertices of cardinality $k_2/2 + \cdots + k_r/2$. Then, by Proposition 2.7, we get

$$\text{reg } K[E_G]/I(X) \geq (k_2/2 + \cdots + k_r/2)(q - 2).$$

Now, by Theorem 1.1, the right-hand of (19) is equal to

$$0 + \text{reg } K[E_{H_2}]/I(X_2) + (q - 2) = \text{reg } K[E_{H_1}]/I(X_1) + \text{reg } K[E_{H_2}]/I(X_2) + (q - 2).$$
The case $l = 1$, $k_1 > 1$ and $r - l > 1$. Let $G'$ be the graph obtained by identifying all the vertices in the paths $P_2, \ldots, P_r$ at an even number of edges away from $v$ (or $w$) with the vertex $v$. The resulting graph $G'$ consists of an odd cycle of length $k_1$ with a set of $k_2/2 + \cdots + k_r/2$ double edges incident to one of its vertices (cf. Figure 7). Let $X'$ the projective toric subset parameterized by $G'$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{G', obtained by identifying every other vertex in $P_2, \ldots, P_r$.}
\end{figure}

The regularity of $K[EG']/I(X')$ is the same as if in $G'$ all double edges were single edges. Hence by Proposition 2.4 and the formula for the odd cycle case we get:

\[ K[EG']/I(X') = (k_1 - 1)(q - 2) + (k_2/2 + \cdots + k_r/2)(q - 2) \]

which coincides with $\text{reg} K[E_{H_1}]/I(X_1) + \text{reg} K[E_{H_2}]/I(X_2) + (q - 2)$. Since, by Proposition 2.5, $\text{reg} K[EG]/I(X) \geq \text{reg} K[EG']/I(X')$ we obtain the desired inequality.

The case $l > 1$ and $r - l = 1$. In this case we construct a graph $G'$ by identifying all vertices in $P_1, \ldots, P_l$ at an even number of edges away from $v$ with the vertex $v$. This graph consists of an odd cycle of length $k_r + 1$ (given by $P_r$ and (a choice of) an edge $\{v, w\}$) that has $l$ multiple edges between $v$ and $w$ and of a set of $[k_1/2] + \cdots [k_{r-1}/2]$ double edges incident at $v$ (cf. Figure 8).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{G', obtained by identifying with $v$ every other vertex in $P_1, \ldots, P_{r-1}$.}
\end{figure}

Arguing as above, we get:

\[ \text{reg} K[EC]/I(X) \geq \text{reg} K[EG']/I(X') = ([k_1/2] + \cdots [k_{r-1}/2])(q - 2) + k_r(q - 2) \]

The case $l > 1$ and $r - l > 1$. As in the previous case, let $G'$ be the graph obtained by identifying the vertices in $P_1, \ldots, P_l$ at an even number of edges away from $v$ with this vertex. We notice that the subgraph of $G'$ consisting of the paths $P_{l+1}, \ldots, P_r$ and (a choice of) an edge $\{v, w\}$ belongs to
the second case, above. Consequently,
\[
\text{reg } K[EG]/I(X) \geq (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_l/2 \rfloor)(q - 2) + (k_{l+1}/2 + \cdots + k_r/2)(q - 2)
\]
\[= \text{reg } K[EH_1]/I(X_1) + \text{reg } K[EH_2]/I(X_2) + (q - 2). \quad \Box
\]

The proof of Theorem 1.2 is obtained by combining Lemma 3.4 and Theorem 3.5. In Table 3 we give explicit formulas for the regularity of \( K[EG]/I(X) \) when \( G \) is a parallel composition of \( r \geq 2 \) paths of lengths \( k_1, \ldots, k_r \), of which \( k_1, \ldots, k_l \) are odd and \( k_{l+1}, \ldots, k_r \) are even.

| \( l \) | \( \text{reg } K[EG]/I(X) \) |
| --- | --- |
| 0 | \((k_1/2 + \cdots + k_r/2 - 1)(q - 2)\) |
| 1, \( r = 2 \) | \((k_1 + k_2 - 1)(q - 2)\) |
| 1, \( r > 2 \) | \((k_1 + k_2/2 + \cdots + k_r/2 - 1)(q - 2)\) |
| 0, \( r = l + 1 \) | \((\lfloor k_1/2 \rfloor + \cdots + \lfloor k_{r-1}/2 \rfloor + k_r)(q - 2)\) |
| 1, \( r > l + 1 \) | \((\lfloor k_1/2 \rfloor + \cdots + \lfloor k_{l}/2 \rfloor + k_{l+1}/2 + \cdots + k_r/2)(q - 2)\) |

Table 2. Values of \( \text{reg } K[EG]/I(X) \) when \( G \) is a parallel composition of paths.

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