Nonabelian Mixed Hodge Structure on Brill-Noether Stacks

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Abstract

A Brill-Noether stack is an algebraic very presentable stack whose homotopy type has two nontrivial homotopy groups. We consider one with a fundamental group — a reductive algebraic group-scheme $S$ and one higher homotopy group, represented by a vector space $V$. The homotopy type also defines an action of $S$ on $V$. This stack is used as coefficient space for nonabelian cohomological space on a smooth algebraic variety $X$.

We define nonabelian MHS on cohomological spaces of this type in the context of the work of C. Simpson related to MHS on the space of local systems. It is defined via an action of the multiplicative complex group on a appropriately chosen category. The exhibited structure in fact generalizes Simpson’s work. Allowing a more general type of coefficient stack.

Furthermore, the so-defined MHS, when considered at a vicinity of an object, which remains fixed under the structural action of $bC^*$, produces the local MHS on Brill-Noether Stacks as defined in our earlier work.

The nonabelian mixed Hodge structure on a Brill-Noether stack is a example of the mixed Hodge structure on a schematic homotopy type, studied by Katzarkov, Panetve and Toen. It has the advantage, due to the relative simplicity of the coefficient stack, that it could be locally written out in terms of iterated integrals.

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1 Introduction

Let $X$ be a smooth complex algebraic curve. We study the geometry of $X$ through introducing a Hodge structure on the nonabelian cohomology space $\text{Hom}(X, T)$, where the coefficient space, $T$, is an algebraic stack. To simplify the problem, one can restrict $T$ to be a very presentable geometric $n$-stack, which, in the category of topological spaces, is analogous to considering a homomorphism with to an $n$-truncated CW-complex. The case of $X$ being a curve of positive genus is additionally simplified by the fact that $X$ is a topologically $K(\pi_1, 1)$-space.

This problem was studied before by Simpson, in e.g. [17, 18], where he defined a Hodge structure on the stack $\text{Hom}(X, \kappa(G, 1))$, which parametrizes $G$-local systems on $X$ up to homotopy equivalence. Simpson proves that each $\mathbb{C}$-valued point corresponds to a Higgs bundle with a holomorphic connection $(E, \theta)$ and, on the other hand, to a representation $\sigma : \pi_1(X, x) \to G$. The ring of functions $\mathcal{O}(E)$ reflects the local geometry of $\text{Hom}(X, k(G, 1))$ at the point $(E, \theta)$ and Simpson proves, that his Hodge structure restricts to a mixed Hodge structure on $\mathcal{O}(E)$, which, under some additional restrictions, coincides with the MHS defined by Hain [12].

In this paper we suggest a MHS on $\text{Hom}(X, T)$, where $T$ is a Brill-Noether stack, i.e., a stack which has only 2 nontrivial homotopy groups: $\pi_1 = G$ and $\pi_n = V$ and $\pi_1$ acts on $\pi_n$ by a chosen representation $\rho : G \to GL(V)$. For that, to a $\mathbb{C}$-valued point of $\text{Hom}(X, T)$, which parametrizes a triple $(E, \theta, \eta)$ of a flat $G$-bundle, holomorphic connection on $E$ end a cohomology class $\eta \in H^n(X, E \times^G V)$, we define a MHS on a proalgebraic group $G_\rho$ and we define an action of $G$ on $G_\rho$, which reflects the Whitehead product on the homotopy type of $T$. The resultant MHS on $\text{Hom}(X, T)$ is compose by 2 ingredients — Simpson’s nonabelian MHS on $\text{Hom}(X, \kappa(G, 1))$ an the usual abelian Hodge theory on the cohomology group $H^n(X, E \times^G V)$.

In section 2 we provide some background information about the theory of stacks.

In section 3 contains the main result, Theorem 3.5. In a previous work we have defined local mixed Hodge structure on a Brill-Noether stack. Here we justify the name local, proving that at a given complex point of $\text{Hom}(X, BN)$, fixed under the defining $\mathbb{C}^*$-action for the Hodge structure, the the resultant filtration on the local algebra is the same as defined before.

In section 3 we discuss, also, some questions of compatibility with existant literature give a few conjectures which demonstrate that our approach can be
generalized in further study to define MHS on \( Hom(X, T) \), for any algebraic smooth variety \( X \) and any topological sheaf \( T \).

2 The Brill-Noether stack

2.1 The Eilenberg-MacLane stacks \( \kappa(G, n) \).

In this section we recall the definitions and basic properties of \( \kappa(G, n) \), following [17] and [21].

We denote by \( T \) the category of topological spaces. For us a \( n \)-stacks on the site \( T \) will be \( n \)-truncated objects in the simplicially enriched category \( St(T) = LSPr(T) \), which is obtained in [21] by taking the category of simplicial presheaves on \( T \), applying simplicial localization with respect to the local equivalences and introducing simplicial structure on the arrows. According to [17], these stacks correspond to very presentable geometric \( n \)-stacks on \( (\text{Aff})_{et} \), via a theorem of GAGA type. This means, that \( \pi_1(T, t) \) is represented by an algebraic group-scheme of finite type, and \( \pi_j(T, t) \) is represented by a vector space for \( j > 1 \).

If \( G \) is a group-sheaf, represented by an algebraic scheme \( G \), a simple example of a very presentable stack, one can define \( \kappa(G, n) \) is trivial except \( \pi_j = G \). (\( G \) must be abelian if \( n > 1 \).)

Remark. For \( n = 1 \), this is the classifying stack \( BG \) of \( G \)-torsors. For \( n > 1 \), \( \kappa(G, n) \) can be constructed as a topological realization of certain sheaf of Lie algebras.

The stacks \( \kappa(G, n) \) are classifying for the group \( G \) in the sense of the following proposition:

Proposition 2.1 (Simpson, [18]) For each \( X \in T \), \( Hom(X, \kappa(G, n)) \) is a stack, such that

\[
\pi_j(Hom(X, \kappa(G, n))) = H^{n-j}(X, G).
\]

This proposition gives a common expression for two separate notions:

1. When \( n = 1 \) and \( G \) is any sheaf of groups, \( H^1(X, G) \) is the stack parametrizing the \( G \)-torsors on \( X \), i.e., the pairs \( (E, \theta) \), where \( E \) is a flat \( G \)-bundles over \( X \), and \( \theta \) is a flat holomorphic connection.

2. If \( n \geq 0 \), and \( G = V \) is a \( \mathbb{C} \)-vector space, \( H^n(X, V) \) is the usual abelian cohomology and the \( \mathbb{C} \)-valued points of \( Hom(X, \kappa(V, n)) \) correspond to pairs \( (E, \eta) \), where \( E \) is isomorphic to the trivial vector bundle \( V = V \times X \to X \) and \( \eta \in H^n(X, V) \). In this way, we can introduce a structure of \( n \)-stack on \( H^n(X, V) \).
As an extension, $\text{Hom}(X, \kappa(V, 1))$ can be viewed as both $BV(X)$ or $H^1(X, V)$.

**Proposition 2.2** There is an isomorphism:

$$BV(X) \cong H^1(X, V)$$

**Proof.** For each flat $V$-bundle the flat connection $\theta$ can be considered as a cohomological class from $H^1_{DR}(X, V)$, which is an expression of the usual, abelian Hodge theory in degree 1. □

### 2.2 Postnikov towers

Very presentable geometric stack can be built as Postnikov towers of $\kappa(G, n)$-s. The simplest example is the Brill-Noether stack, denoted $\kappa(G, \rho, n)$, whose $\pi_1 = G$, $\pi_n = V$, the representation $\rho : G \to GL(V)$ gives the action of $\pi_1$ on $\pi_n$ and the rest of the homotopy groups being trivial. $\kappa(G, \rho, n)$ is a fibration over $\kappa(G, 1)$:

$$\kappa(V, n) \overset{i}{\longrightarrow} \kappa(G, \rho, n) \;
\tau \downarrow \;
\kappa(G, 1)$$

The map $\tau$ is truncation and $i$ is inclusion.

For each $X \in T$, $\text{Hom}(X, \kappa(G, \rho, n))$ parametrizes triples $(E, \theta, \eta)$ of a $G$-bundle $E$, flat connection $\theta$ on $E$ and a cohomology class $\eta \in H^n(X, E \times^G V)$ (cf. [17]). The diagram (1) induces a smooth fibration of geometric $n$-stacks

$$\text{Hom}(X, \kappa(G, \rho, n)) \overset{\phi}{\longrightarrow} \text{Hom}(X, \kappa(G, 1))$$

Over each $\mathbb{C}$-point $(E, \theta)$ of $\text{Hom}(X, \kappa(G, 1))$, the fiber is $H^n(X, E \times^G V)$.

**Examples:**

**A.** For any algebraic group $H$ one can introduce as a special case of Brill-Noether stack $G_H$ as an extension:

$$\kappa(Z(H), 2) \overset{i}{\longrightarrow} G_H \;
\tau \downarrow \;
\kappa(\text{Out}(H), 1)$$

$\text{Hom}(X, G_H)$ is then a 2-stack, parametrizing the 1-stacks $X \to X$ which are locally equivalent to $\text{Hom}(X, \kappa(G, 1))$.

**B.** Let $G$ be an complex algebraic group, which splits into a semidirect product $G = V \rtimes S$ of a reductive subgroup $S$ and an abelian group $V$ by the map $\rho S \to Aut(V)$. 

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Proposition 2.3 There is an equivalence of categories

\[ \text{Hom}(X, \kappa(G, 1)) \cong \text{Hom}(X, \kappa(S, \rho, 1)) \]

\[ \square \]

2.3 Universal Fibrations

In this paragraph we make a clarification on the terminology.

If we consider the \( \kappa(V, n) \) as an appropriate quotient stack we can restate Proposition 2.1 in the following form:

Proposition 2.4 There is an universal bundle \( \ast \to \kappa(V, n) \) from which every \( n \)-torsor with structural group \( V \to X \) is a pullback via certain morphism \( X \to \kappa(V, n) \).

Corollary 2.5 There is an universal fibration \( \ast \to \kappa(G, \rho, n) \), in the category of triples \( (E, \theta, \eta) \), as above. The Higgs bundles which have \( h^a(X, E \times^G V) \geq 1 \) form an open locus in \( \text{Hom}(X, \kappa(G, \rho, n)) \) consisting from the non-trivial fibers in (1).

Proof. The suggested universal fibration is a pull-back from the universal \( G \)-bundle \( \ast \to \kappa(G, 1) \) through the canonical morphism (1). Every map \( f : X \to \kappa(G, \rho, n) \) corresponds to a Higgs bundle \( (E, \theta) \), which is a pull back of the universal \( G \)-bundle by the composition of \( f \) with (1). By universality of pull-back, the corresponding to \( f \) triple, is a pull-back from \( E \). \( \square \)

Remark. From the usual abelian Hodge theory follows that:

\[ H^a(X, E \times^G V) \cong H^0(X, E \times^G V \otimes \Omega^n) = H^0(X, E \times^G V) \]

3 Nonabelian mixed Hodge structure

3.1 MHS on \( \text{Hom}(X, \kappa(G, 1)) \)

In this section we briefly remind some definitions and facts of the nonabelian Hodge theory of \( \text{Hom}(X, \kappa(G, 1)) \), as developed in [19].

The classical (abelian, pure) Hodge structure can be defined using an \( \mathbb{C}^* \) action on the underlying vector space \( A_C \), which is usually a cohomological algebra, and deriving the filtrations from the eigenspaces of the action. More formally (cf. [19]),
Definition 3.1 A non-abelian Hodge structure is a triple \((G, \Gamma, \mu)\), consisting of

1. an affine group scheme (pro-algebraic group) \(G\), defined over \(\mathbb{R}\)
2. a finitely generated subgroup \(\Gamma \subset G_{\mathbb{R}}\), which is Zariski dense in \(G\).
3. a group action \(\mu: U(1) \times G_{\mathbb{R}} \to G_{\mathbb{R}}\) via homomorphism of the proalgebraic group, such that, the map \(U(1) \times \Gamma \to G^{an}\) is continuous and the element \(C = -1\) in \(U(1)\) is a Cartan involution of \(G_{\mathbb{R}}\).

We consider the stack \(\text{Hom}(X, \kappa(G, 1))\) as a cohomological space based on 2.1.

The nonabelian Hodge structure is defined by introducing an action on the objects of the category \(\text{Hom}(X, \kappa(G, 1))(U)\). Each object in there parametrizes a pair \((E, \theta)\), where \(E\) is a Higgs bundle on \(X \times U\) and \(\theta\) is a Higgs field on \(E\), i.e., a holomorphic 1-form with coefficient in \(E\). Simpson defines a \(\mathbb{C}^*\) action on \(\text{Hom}(X, \kappa(G, 1))(U)\) by

\[
t: (E, \theta) \mapsto (E, t \theta).
\]

In this case the group \(G\) is the pro-algebraic completion of the fundamental group \(\pi_1(X, x_0)\) and the action \(U(1)\) is induced from the described geometric action of \(\mathbb{C}^*\).

To compare this theory with the MHS found by Hain, one considers any \(\mathbb{C}\)-valued point of \(\text{Hom}(X, \kappa(G, 1))\) as a representation \(\sigma: \pi_1(X, x_0) \to G\), which gives a relative prounipotent completion \(\mathcal{G}\).

Recall, that \(\sigma\) is called to be of Hodge type if \(t\) comes from an admissible complex variation of Hodge structure. Simpson proves the following proposition:

**Proposition 3.2** \(\sigma\) comes from a complex variation of Hodge structure if, and only if, it is a fixed point of the \(\mathbb{C}^*\) action on \(\text{Hom}(X, \kappa(G, 1))\).

In this case, there is an induced action on the Lie algebra of the progroup \(\mathcal{G}\) and hence, a decreasing filtration \(F^\bullet\) on \(\text{Lie}\, \mathcal{G}\), defined by:

\[
F^p = \bigoplus_{r \geq p} H^r,
\]

where \(H^r \subset \text{Lie}\, \mathcal{G}\) is the subspace of elements \(h\), such that \(t(h) = t^r\) for all \(t \in \mathbb{C}\). The weight filtration is, again as in (?), defined in the terms of the lower central series of the Lie algebra of the unipotent radical of \(\mathcal{G}\).

**Theorem 3.3 (Simpson [19])** For the trivial representation \(\pi_1(X, x_0) \to \{1_G\}\) (which is definitely fixed) the mixed Hodge structure on the prounipotent completion \(\mathcal{G}\), defined above, coincides with Hain’s (as described in (?)).
3.2 Proposed Hodge structure on the Brill-Noether stack

Let $X$ be a smooth algebraic curve and let $E$ be any quasi-coherent sheaf of rings on $X$. The d.g.a $\Omega^\bullet(X; E)$ has a natural $\mathbb{C}^*$ action:

$$t : \omega \mapsto t^r \omega, \quad \text{if } \omega \in A^{r,s}(X; E) \quad (3)$$

This action induces one on the cohomology d.g.a. $H^\bullet(X; \Omega^r(X; E))$.

Further, for each locally free $E$-module $M$, there is a $\mathbb{C}^*$-action on $A^\bullet(X; M)$ and hence on $H^\bullet(X; \Omega^r(X; M))$.

Suppose, that $T = \kappa(G, \rho, n)$ is a Brill-Noether stack, such that $\rho$ is a representation of Hodge type\(^1\). The $\mathbb{C}$-points of $Hom(X, T)$ are expressed in the terms of (eventually, nonabelian) cohomology classes of $H^\bullet(X; \Omega^r(X; E))$ and $H^\bullet(X; \Omega^r(X; M))$. This defines an $\mathbb{C}^*$-action on $Hom(X, T)$.

**Conjecture 3.4** There is a non-abelian MHS on $Hom(X, \kappa(G, \rho, n))$, in the sense of [14], defined by the action (3).

Let $[b]$ be a $\mathbb{C}$-point of $Hom(X, T)$, which is fixed under that action. Then there are induced actions on the relative prounipotent completion $\mathcal{G}$ of the $\pi_1(X, x_0)$, correspondent to $E$ and on all the relevant quotients $\mathcal{G}_\rho$ and therefore on the correspondent Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_\rho$.

**Theorem 3.5** The MHS’s on $\mathfrak{g}$ and on $\mathfrak{g}_\rho$, with Hodge filtration induced from the $\mathbb{C}^*$ action and weight filtration coming from the unipotent radical of $\mathcal{G}$ coincide with those described in section ???. The natural action $\mathfrak{g} \otimes \mathfrak{g}_\rho \to \mathfrak{g}_\rho$, induced by the Whitehead product of the homotopy type of $T$, is is given by Theorem ??.

□

3.3 Further conjectures

The described mixed Hodge structure on $Hom(X, T)$, where $X$ is a $K(\pi_1, 1)$ complex space and $T$ is a Brill-Noether stack, suggests the following generalizations:

\(^1\)The assumption that $\rho : G \to GL(V)$ must be a homomorphism of Hodge type is not essential. It can be omitted by considering mixed twistor structures. Nonetheless, even if in this case, some special points of $Hom(X, \kappa(G, \rho, n))$ will carry Mixed Hodge structure, due to additional symmetries that they may have. This approach shall be discussed in the future.
**A:** Let $T$ to be a perfect complex, i.e., the homotopy type of $T$ is analogous to a simply-connected topological space. If $X$ is still a topological $K(\pi_1, 1)$ space this means that the nonabelian part of the MHS on $Hom(X, T)$ will be trivial. Locally, we expect the following:

**Conjecture 3.6** Suppose $T$ is constructed as a Postnikov tower

$$T = T_n \to \cdots \to T_i \to T_{i-1} \to \cdots \to F_2 \to T_0 = *$$

and on each step we have a fibration:

$$\kappa(V_i, i) \to T_i \xrightarrow{\tau_i} T_{i-1}$$

Then for each $i$ there exists a proalgebraic group $U_i$ (which will be unipotent), with a mixed Hodge structure, corresponding to the cohomologies of the of the trivial principal bundle. The $U_i$-s form a mixed Hodge complex with maps $u_i \otimes u_j \to u_{i+j-1}$ coming from Whitehead product on $T$.

**B:** Let $T$ to be a very presentable $n$-stack, which is a geometrical realization of a general topological presheaf with $\pi_1 = G$ and $\pi_i = V_i$ for $i = 2 \ldots n$. Then for each $i = 2 \ldots n$ there is a smooth fibration of algebraic stacks: $T \to \kappa(G, \rho_i, i) = F_i$, where $\rho_i : G \to GL(V_i)$ is a part of the definition of $T$.

**Conjecture 3.7** There exist proalgebraic groups $G_i = U_i \rtimes G$, (for $i = 2 \ldots n$) with MHS on each of them, coming from the corresponding Brill-Noether stack $F_i$. They form a MH complex with structure maps from (3.6).

**C:** Let $X$ to be any $\mathbb{C}$-space. Then we can represent it as a fibration

$$X_1 \to X \xrightarrow{K(\pi_1, 1)}$$

where $X_1$ is simply-connected. Then MHS on $Hom(X, T)$ should combine the MHS on $Hom(K(\pi_1, 1), T)$ from before and $Hom(X_1, T)$ which can be defined by modifying the MHS of Morgan and Hain on the cohomology algebra of simply-connected manifold. Locally, on the MH complex language, this means that we will have a HS for each $Hom(\pi_i(X), \pi_j(T))$, some abelian and some not, related between each other through Whitehead product in $T$ or $X$.

A reasonable definition of a MHS in all of the cases A, B, and C should respect the MHS on the schematization of the homotopy type, according to [14].
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