Quark mass and mixing in the 3-3-1 model with neutral leptons
based on $D_4$ flavor symmetry

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The $D_4$ flavor model based on $\text{SU}(3)_C \otimes \text{SU}(3)_L \otimes \text{U}(1)_X$ gauge symmetry is updated in which the quark mixing matrix is concentrated. After spontaneous breaking of flavor symmetry, with the constraint on Higgs VEVs in the Yukawa couplings, all of quarks have consistent masses and a small deviation from the unity is obtained at the tree-level. To obtain the quark mixing matrix consistent with experimental data in 2012, the violation terms with $1'$ under $D_4$ are introduced. The realistic quark mass and mixing are derived.

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I. INTRODUCTION

One of the most interesting challenges in particle physics is to determine the origin of quark mixing, described by the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix \cite{1, 2}, which is approximately proportional to the identity. The CKM matrix elements are fundamental parameters of the standard model (SM), so their precise determination is important. There are many parametrizations of the CKM matrix \cite{3-9}, however, the CKM parametrization \cite{1, 2} and the Wolfenstein one \cite{10} are widely used. It is interesting to note that the discrete symmetry $S_3$ was very early applied for the understanding the CKM matrix \cite{11}. Recently, the discrete symmetries are useful tool for understanding quark and lepton mixing \cite{12-16}. The elements in the CKM matrix have now been determined with a high accurate level. The fit results for the magnitudes of

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all CKM elements in Ref. [17] imply:

\[
U_{\text{CKM}} = \begin{pmatrix}
0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & (4.15 \pm 0.49) \times 10^{-3} \\
0.230 \pm 0.011 & 1.006 \pm 0.023 & (40.9 \pm 1.1) \times 10^{-3} \\
(8.4 \pm 0.6) \times 10^{-3} & (42.9 \pm 2.6) \times 10^{-3} & 0.89 \pm 0.07
\end{pmatrix}.
\tag{1}
\]

From (1) it follows that the quark mixing angles are small which completely differ from the lepton mixing ones; the latter has been studied widely by many authors in recent years [18–32] (and references therein).

In our previous work [22–24, 26–32] the lepton mass and mixing are studied in detail, however, the realistic quark mixing has not been considered. In Ref. [25], we have studied the 3-3-1 model with neutral fermions based on $D_4$ group, in which the quark mixing matrix is unity at the tree-level, and the 1–2 mixing of the ordinary quarks is obtained if the soft terms violating $D_4$ symmetry with $1'$ were added. Our aim in this paper is to construct the 3-3-1 model with neutral leptons based on $D_4$ flavor symmetry having a realistic quark mixing.

As the same as in [29], all the fermion fields act as singlets under $D_4$ [29], and a new parametrization of quark mixing is proposed and at the tree-level, the quark mixing matrix is a small deviation from the unity. The realistic quark mixing is obtained at the first order of perturbation theory in the case the $D_4$ symmetry is violated with $1'$.

The rest of this work is organized as follows. In Sec. II we introduce necessary Higgs fields responsible for the charged lepton as well as neutrino mass and mixing. Sec. III is devoted for the quark mixing. We summarize our results and make conclusions in the section IV. Appendix A presents a brief of the $D_4$ theory. Appendix B provides the lepton number ($L$) and lepton parity ($P_l$) of particles in the model.

II. THE MODEL

The lepton content of the model is the same as that in [29]. In this work, we will concentrate on quark sector, where under the $[SU(3)_L, U(1)_X, U(1)_L, D_4]$ symmetries, the left- and right-handed quark fields transform as follows:

\[
Q_{3L} = (u_{3L} \ d_{3L} \ U_L)^T \sim [3, 1/3, -1/3, 1], \quad u_{3R} = [1, 2/3, 0, 1], \quad d_{3R} = [1, -1/3, 0, 1],
\]

\[
Q_{1L} = (d_{1L} \ -u_{1L} \ D_{1L})^T \sim [3', 0, 1/3, 1'], \quad u_{1R} = [1, 2/3, 0, 1'], \quad d_{1R} = [1, -1/3, 0, 1'],
\]

\[
Q_{2L} = (d_{2L} \ -u_{2L} \ D_{2L})^T \sim [3', 0, 1/3, 1''], \quad u_{2R} = [1, 2/3, 0, 1''], \quad d_{2R} = [1, -1/3, 0, 1''],
\]

\[
U_R \sim [1, 2/3, -1, 1], \quad D_{1R} \sim [1, -1/3, 1, 1'], \quad D_{2R} \sim [1, -1/3, 1, 1''].
\]
Note that the $1_1, 1_1'$ and $1''_2$ for quarks meets the requirement of anomaly cancellation condition in the 3-3-1 models since one family of quarks transforms differently from the two others. In what follows, we consider possibilities for generating the fermion masses. The scalar multiplets needed for this purpose would be introduced accordingly.

To generate masses for the charged leptons, we introduce two $SU(3)_L$ scalar triplets $\phi$ and $\phi'$ respectively lying in $1_1$ and $1''_2$ under $D_4$, with the VEVs $\langle \phi \rangle = (0 \ v \ 0)^T$ and $\langle \phi' \rangle = (0 \ v' \ 0)^T$. From the Yukawa interactions for the charged leptons, we get $m_e = h_1 v$, $m_\mu = h v - h' v'$, $m_\tau = h v + h' v'$, and the left- and right-handed charged leptons mixing matrices are obtained

$$U_L = U_R \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (2)$$

In similarity to the charged lepton sector, to generate the neutrino masses, we have additionally introduced the two scalar Higgs anti-sextets $s, \sigma$ respectively lying in $1_1, 1_1'$ and $1''_2$ under $D_4$, and one $SU(3)_L$ triplet lying in $1''_2$ under $D_4$ in which the $\rho$ contribution is regarded as a small perturbation. The neutrino mass and mixing are then consistent with experimental data given in $[17]$ in both normal and inverted hierarchical patterns. A detailed study on charged lepton and neutrino sector the reader is referred to Ref. $[29]$.

### III. QUARK MASSES AND MIXING

#### A. The tree level

Let’s us recall the two $SU(3)_L$ Higgs scalars responsible for charged lepton masses $[29]$

$$\phi = \left(\phi_1^+ \phi_2^0 \phi_3^+\right)^T \sim [3, 2/3, -1/3, 1], \quad \phi' = \left(\phi_1'^+ \phi_2^0 \phi_3'^+\right)^T \sim [3, 2/3, -1/3, 1''_2].$$

To generate masses for quarks with a minimal Higgs content, the following Higgs triplets are necessary:

$$\chi = \left(\chi_1^0 \chi_2^- \chi_3^0\right)^T \sim [3, -1/3, 2/3, 1],$$
$$\eta = \left(\eta_1^0 \eta_2^- \eta_3^0\right)^T \sim [3, -1/3, -1/3, 1],$$
$$\eta' = \left(\eta_1'^0 \eta_2'^- \eta_3'^0\right)^T \sim [3, -1/3, -1/3, 1''_2].$$

The Higgs content and Yukawa couplings in the quark sector are summarized in Table $[1]$.
The mass Lagrangian for quarks is then given by

$$- \mathcal{L}_q = h_3^d \bar{Q}_{3L}^d \phi d_{3R} + h_1^u \bar{Q}_{1L}^u \phi^* u_{1R} + h_2^u \bar{Q}_{2L}^u \phi^* u_{2R}$$

$$+ h_3^u (\bar{Q}_{1L}^u u_{2R} + \bar{Q}_{2L}^u u_{1R}) \phi^*$$

$$+ h_3^d \bar{Q}_{3L}^d \eta u_{3R} + h_1^d \bar{Q}_{1L}^d \eta^* d_{1R} + h_2^d \bar{Q}_{2L}^d \eta^* d_{2R}$$

$$+ h_3^d (\bar{Q}_{1L}^d d_{2R} + \bar{Q}_{2L}^d d_{1R}) \eta^*$$

$$+ f_3 \bar{Q}_{3L}^d \chi U_R + f_1 \bar{Q}_{1L}^u \chi^* D_R + f_2 \bar{Q}_{2L}^d \chi^* D_R + H.c.$$

The Yukawa interactions are given by

$$\begin{aligned}
- \mathcal{L}_q &= h_3^d \bar{Q}_{3L}^d \phi d_{3R} + h_1^u \bar{Q}_{1L}^u \phi^* u_{1R} + h_2^u \bar{Q}_{2L}^u \phi^* u_{2R} \\
&\quad + h_3^u (\bar{Q}_{1L}^u u_{2R} + \bar{Q}_{2L}^u u_{1R}) \phi^* \\
&\quad + h_3^d \bar{Q}_{3L}^d \eta u_{3R} + h_1^d \bar{Q}_{1L}^d \eta^* d_{1R} + h_2^d \bar{Q}_{2L}^d \eta^* d_{2R} \\
&\quad + h_3^d (\bar{Q}_{1L}^d d_{2R} + \bar{Q}_{2L}^d d_{1R}) \eta^* \\
&\quad + f_3 \bar{Q}_{3L}^d \chi U_R + f_1 \bar{Q}_{1L}^u \chi^* D_R + f_2 \bar{Q}_{2L}^d \chi^* D_R + H.c. \\
\end{aligned}$$

It is worth mentioning that the VEV of $\chi, \phi, \eta$ conserve $D_4$ while that of $\phi', \eta'$ breaks this symmetry in to $Z_2 \otimes Z_2 \ [24]$. Therefore the $D_4$ breaking in the quark sector is $D_4 \rightarrow Z_2 \otimes Z_2$. We assume that the VEVs of $\chi, \phi, \phi', \eta, \eta'$, respectively, are given as

$$\langle \chi \rangle = \begin{pmatrix} 0 & 0 \\ v_\chi \end{pmatrix}^T,$$

$$\langle \phi \rangle = \begin{pmatrix} 0 & v \end{pmatrix}^T, \quad \langle \phi' \rangle = \begin{pmatrix} 0 & v' \end{pmatrix}^T,$$

$$\langle \eta \rangle = \begin{pmatrix} u & 0 \end{pmatrix}^T, \quad \langle \eta' \rangle = \begin{pmatrix} u' & 0 \end{pmatrix}^T.$$

The mass Lagrangian for quarks is then given by

$$- \mathcal{L}_q^{mass} = h_3^d v d_{3R} \bar{Q}_{3L}^d - h_1^u v^* u_{1R} \bar{Q}_{1L}^u - h_2^u v^* u_{2R} \bar{Q}_{2L}^u - h_3^d \bar{Q}_{3L}^d \eta u_{3R}$$

$$- h_1^d \bar{Q}_{1L}^d \eta^* d_{1R} - h_2^d \bar{Q}_{2L}^d \eta^* d_{2R}$$

$$- f_3 \bar{Q}_{3L}^d \chi U_R - f_1 \bar{Q}_{1L}^u \chi^* D_R - f_2 \bar{Q}_{2L}^d \chi^* D_R + H.c.$$
The mass matrices for ordinary up-quarks and down-quarks are, respectively, obtained as follows:

\[
M_u = \begin{pmatrix}
-h^u_1 v & -h^{u'}_2 v & 0 \\
-h^{u'}_2 v & -h^d_2 v & 0 \\
0 & 0 & h^u_3 u
\end{pmatrix}, \quad M_d = \begin{pmatrix}
h^d_1 u & h^{d'}_2 u & 0 \\
h^{d'}_2 u & h^d_3 u & 0 \\
0 & 0 & h^d_4 v
\end{pmatrix}.
\]

The matrices \(M_u, M_d\) in (8) are diagonalized as

\[
V^u_L M_u V^u_R = \text{diag}(m_u, m_c, m_t), \quad V^d_L M_d V^d_R = \text{diag}(m_d, m_s, m_b),
\]

where

\[
m_u = -\frac{1}{2} \left[ (h^u_1 + h^u_2) v + \sqrt{(h^u_1 - h^u_2)^2 v^2 + (2 h^{u'}_2 v')^2} \right],
\]

\[
m_c = -\frac{1}{2} \left[ (h^u_1 + h^u_2) v - \sqrt{(h^u_1 - h^u_2)^2 v^2 + (2 h^{u'}_2 v')^2} \right], \quad m_t = h^u_3 u,
\]

\[
m_d = \frac{1}{2} \left[ (h^d_1 + h^d_2) v - \sqrt{(h^d_1 - h^d_2)^2 u^2 + (2 h^{d'}_2 u')^2} \right],
\]

\[
m_s = \frac{1}{2} \left[ (h^d_1 + h^d_2) u + \sqrt{(h^d_1 - h^d_2)^2 u^2 + (2 h^{d'}_2 u')^2} \right], \quad m_b = h^d_3 v.
\]

and

\[
U^u_L = U^u_R = \begin{pmatrix}
\frac{K}{\sqrt{K^2 + 1}} & -\frac{1}{\sqrt{K^2 + 1}} & 0 \\
\frac{1}{\sqrt{K^2 + 1}} & \frac{K}{\sqrt{K^2 + 1}} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad U^d_L = U^d_R = \begin{pmatrix}
\frac{A}{\sqrt{A^2 + 1}} & -\frac{1}{\sqrt{A^2 + 1}} & 0 \\
\frac{1}{\sqrt{A^2 + 1}} & \frac{A}{\sqrt{A^2 + 1}} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

with

\[
K = \frac{(h^u_1 - h^u_2) v + \sqrt{(h^u_1 - h^u_2)^2 v^2 + (2 h^{u'}_2 v')^2}}{2 h^{u'} v'}, \quad A = \frac{(h^d_1 - h^d_2) u - \sqrt{(h^d_1 - h^d_2)^2 u^2 + (2 h^{d'}_2 u')^2}}{2 h^{d'} u'}.
\]

The CKM matrix is defined as

\[
U_{\text{CKM}} = U^u_L U^d_L = \begin{pmatrix}
\frac{1+A K}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}} & \frac{K - A}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}} & 0 \\
\frac{A - K}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}} & \frac{1+A K}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
where $K$ and $A$ are defined in Eqs. (14) and (15). In the special case $K = A$, i.e,

$$\frac{u}{u'} = \frac{h_u^1 - h_u^2 h^{ud} v}{h_d^1 - h_d^2 h^{ud} v'},$$

the $U_{CKM}$ in Eq. (16) reduces to the identity.

In the model under consideration, the following limit is often taken into account [22–24, 33, 34]:

$$u \sim u' \sim v' \sim v.$$  \hfill (17)

On the other hand, taking into account of the discovery of the long-awaited Higgs boson at around 125 GeV by ATLAS [35] and CMS [36], we can choose the VEVs

$$u = u' = v' = v = 100 \text{ GeV} = 10^{11} \text{ eV}.$$  \hfill (18)

The matrix $U_{CKM}$ in Eq. (19) is closer to the realistic quark mixing matrix than those derived at the tree level from other discrete symmetry groups [22–24, 26–30]. Indeed, with the help of (18) and taking the experimental data on quark mass [17]

$$m_u = 2.3 \text{ MeV}, \quad m_c = 1.275 \text{ GeV}, \quad m_t = 173.5 \text{ GeV},$$

$$m_d = 4.8 \text{ MeV}, \quad m_s = 95 \text{ MeV}, \quad m_b = 4.18 \text{ GeV},$$  \hfill (19)

and the average values of the CKM matrix elements in Ref. [17] given in Eq. (1), with $|U_{ud}| = 0.97425 \pm 0.00022$, we get four solutions for $A, K$ and the Yukawa quark couplings $h_{1,2,3}^u, h_{1,2,3}^{du}, h^{ud}$ as listed in Tab. II. We see that the matrices in Tab. II are close to the realistic quark mixing matrix, i.e, the deviations of the matrix $U_{CKM}$ in Eq. (1) from the matrices in Tab. II are very small, and this is a good approximation for the realistic quark mixing matrix, which implies that the mixings among the quarks are dynamically small, and this is one of the most striking prediction of the model under consideration. As will see in Sec. III B, a violation of $D_4$ symmetry due to unnormal Yukawa interactions will disturb the tree level matrix resulting in mixing between ordinary and exotic quarks and providing the desirable quark mixing pattern.

**B. The first-order corrections**

All terms of the Yukawa interactions responsible for quarks masses in Tab. II are invariant under the $[SU(3)_L, U(1)_X, U(1)_L, D_4]$ symmetries. To obtain a realistic quark mixing, in this work, the soft terms violating $D_4$ symmetry with $1'$ are added. These terms are
The total mass matrices for the ordinary up-quarks and down-quarks then take the form:

\[ M_u = \begin{pmatrix} -h_1^u v & -h_3^u v' & -k_1^u v \\ -h_3^u v' & -h_2^u v & 0 \\ k_2^u u & 0 & h_3^u u \end{pmatrix}, \quad M_d = \begin{pmatrix} h_1^d u & h_3^d u' & k_1^d u \\ h_3^d u' & h_2^d u & 0 \\ k_2^d v & 0 & h_3^d v \end{pmatrix}. \]

We can separate the quark mass matrices in Eq. (22) into two parts as follows

\[ M_u' = M_u + \Delta M_u, \quad M_d' = M_d + \Delta M_d, \]

where \( M_u, M_d \) are given by [9] due to the contribution of the invariant terms only, and

\[ \Delta M_u = \begin{pmatrix} 0 & 0 & -k_1^u v \\ 0 & 0 & 0 \\ k_2^u u & 0 & 0 \end{pmatrix}, \quad \Delta M_d = \begin{pmatrix} 0 & 0 & k_1^d u \\ 0 & 0 & 0 \\ k_2^d v & 0 & 0 \end{pmatrix}. \]

are deviations from the contributions of the \( D_4 \) violation terms. In the case without \( D_4 \) violation, the first terms can approximately fit the data in [17] with a very small deviations as shown in Sec.

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**TABLE II:** Model parameters derived from the fit with data in [17] at the tree level.

| \( A, K \) | \( h_1^u, h_2^u, h_3^u, h'^u \) | \( h_1^d, h_2^d, h_3^d, h'^d \) | \( U_{CKM} \) |
|---|---|---|---|
| 0.655578, 1.04565 | \(-0.00610263, -0.00667037, 1.735, 0.00635717, 0.000319135, 0.000678865, 0.0418, 0.000413581\) | \([0.97425, 0.22546, 0]\) | \(0, 0, 1\) |
| 3.45036, 18.2734 | \(-0.000610076, -0.012712, 1.735, 0.000694396, 0.000880103, 0.00117895, 0.0418, 0.000241164\) | \([0.97425, 0.22547]\) | \(0, 0, 1\) |
| -0.655578, -1.04565 | \(-0.00667037, -0.00610263, 1.735, 0.00635717, 0.000319135, 0.000678865, 0.0418, 0.000413581\) | \(0.22547, 0.97425\) | \(0, 0, 1\) |
| -3.45036, -18.2734 | \(-0.012712, -0.000610004, 1.735, 0.000694397, 0.000117897, 0.000880105, 0.0418, 0.000241164\) | \(0.22547, 0.97425\) | \(0, 0, 1\) |
The second terms belong to the contributions of the $D_4$ violation in Eqs. (20) and (21). So, we can consider the contributions of $D_4$ violation as small perturbations in quark sector and terminating the theory at the first order.

At the first order of perturbation theory, the matrices $\Delta M_u, \Delta M_d$ in Eq. (24) do not give contribution to quark eigenvalues. However, they change the corresponding eigenvectors. The up- and down quark masses are thus obtained as:

$$m_i' = m_i \quad (i = u, c, t, d, s, b),$$

where $m_i$ ($i = u, c, t, d, s, b$) are given in Eqs. (11) and (12).

The unitary matrices which couple the left-handed quarks $u_L$ and $d_L$ to those in the mass bases, respectively, are

\[
U_{L}^{u} = \begin{pmatrix}
\frac{K}{\sqrt{K+1}} & \frac{-1}{\sqrt{K+1}} & k_{uv}^2\left[\frac{K^2}{(K+1)(m_u-m_t)(m_c-m_t)}\right] \\
\frac{1}{\sqrt{K+1}} & \frac{K}{\sqrt{K+1}} & \frac{-1}{(K+1)(m_u-m_t)(m_c-m_t)} \\
\sqrt{K+1}(m_u-m_t) & \sqrt{K+1}(m_c-m_t) & 1
\end{pmatrix}, \quad (26)
\]

\[
U_{L}^{d} = \begin{pmatrix}
\frac{A}{\sqrt{A+1}} & \frac{-1}{\sqrt{A+1}} & k_{uv}^2\left[\frac{A^2}{(A+1)(m_u-m_t)(m_c-m_t)}\right] \\
\frac{1}{\sqrt{A+1}} & \frac{A}{\sqrt{A+1}} & \frac{-1}{(A+1)(m_u-m_t)(m_c-m_t)} \\
\sqrt{A+1}(m_d-m_b) & \sqrt{A+1}(m_s-m_b) & 1
\end{pmatrix}, \quad (27)
\]

where $A, K$ are given in Eqs. (15) and (16). The CKM matrix at the first order of perturbation theory is now defined as

\[
U_{CKM}' = U_{L}^{u}U_{L}^{d*} = \begin{pmatrix}
U_{11} & U_{12} & U_{13} \\
U_{21} & U_{22} & U_{23} \\
U_{31} & U_{32} & U_{33}
\end{pmatrix}, \quad (28)
\]
The relations between experimental data given in [17], as follows:

\[ U_{23} = -\frac{K k_1^d v (m_u - m_c)}{(K^2 + 1)(m_u - m_t)(m_c - m_t)} + \frac{k_2^d v}{\sqrt{K^2 + 1}\sqrt{A^2 + 1}} \left( \frac{A}{m_d - m_b} - \frac{K}{m_s - m_b} \right), \]

\[ U_{31} = -\frac{k_1^d u [A^2 (m_s - m_b) + (m_d - m_b)]}{(A^2 + 1)(m_s - m_b)(m_d - m_b)} + \frac{k_2^d u}{\sqrt{K^2 + 1}\sqrt{A^2 + 1}} \left( \frac{A K}{m_u - m_t} + \frac{1}{m_c - m_t} \right), \]

\[ U_{32} = -\frac{A k_1^d u (m_d - m_b)}{(A^2 + 1)(m_s - m_b)(m_d - m_b)} + \frac{k_2^d u}{\sqrt{K^2 + 1}\sqrt{A^2 + 1}} \left( \frac{K}{m_u - m_t} + \frac{A}{m_c - m_t} \right), \]

\[ U_{33} = 1 + \frac{k_2^d k_3^d u v}{\sqrt{K^2 + 1}\sqrt{A^2 + 1}} \left[ \frac{AK}{(m_u - m_t)(m_d - m_b)} + \frac{1}{(m_c - m_t)(m_s - m_b)} \right]. \]  

(29)

It is easily shown that our model is consistent since the experimental constraints on the mixing angles and the masses of quarks can be respectively fitted with the quark Yukawa coupling parameters \( h_1^{u, 2, 3}, h_1^{d, 2, 3}, h_1^{u, d}, h_1^{d, 2}, h_1^{d, 2} \) of the all \( SU(3)_L \) triplet scalars, provided that the VEVs \( u, u', v, v' \) and quark masses are given in [18] and [19], respectively. Indeed, by taking the best fit values \( U_{11} = 0.97425, U_{13} = 4.15 \times 10^{-3}, U_{13} = 8.4 \times 10^{-3} \), and \( U_{23} = 40.9 \times 10^{-3} \) as given in [28], we get

\[ k_1^d = \frac{0.0017055692(A^2 + 1)(K^2 + 1) \left( 0.97425 - \frac{AK + 1}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}} \right)}{(4.1752 + 4.085 A^2)(5.80636 + 5.76377 K^2) \times 10^{-3} k_1^d}, \]

\[ k_2^d = \frac{\sqrt{A^2 + 1}\sqrt{K^2 + 1}(0.0409 + 0.0409 K^2 - 0.00425928 K k_2^d)}{(K + 1)(-23.9509 A + 24.4798 K)}, \]

\[ k_1^u = \frac{[1.00122 + A(-0.0993964 + 0.979594 K) + 0.101591 K](K^2 + 1)^2}{14.1096 K - 28.2192 K^3 - 14.1096 K^5 + 13.9068 A(K^2 + 1)^2}, \]

\[ k_2^u = \frac{0.0804 + \frac{\Gamma_1}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}}}{0.580636 + 0.576377 K}, \]  

(30)

where

\[ \Gamma_1 = \frac{1.70557(2.44798 + 2.39509 A^2) \left( 0.97425 - \frac{AK + 1}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}} \right) \times (-14.1096 K - 28.2192 K^3 - 14.1096 K^5 + 13.9068 A(K^2 + 1)^2)}{\sqrt{A^2 + 1}\sqrt{K^2 + 1}}, \]

\[ \Gamma_2 = (4.1752 + 4.085)[1.00122 + A(-0.0993964 + 0.979594 K) + 0.101591 K] \times \]

\[ \times (5.80636 + 5.76377 K^2)(K^2 + 1). \]  

(31)

Substituting (30) into (29) we get the dependence of \( U_{12}, U_{21}, U_{22}, U_{32} \) and \( U_{33} \) on the parameters \( A, K \) with \( A \in (1.5, 2.0) \) and \( K \in (2.0, 3.0) \) as plotted in Fig[1]. In the case \( A = 1.8 \), we have the relations between \( U_{12}, U_{21}, U_{22}, U_{32}, U_{33} \) on only \( K \) as given in Fig. [2]. If we choose \( A = 1.8 \) and \( K = 3.0 \), we get the explicit values of the model parameters, which are good consistent with the experimental data given in [17], as follows:

\[ U'_{CKM} = \begin{pmatrix} 0.97425 & 0.184369 & 0.00415 \\ -0.184269 & 0.982872 & 0.0409 \\ 0.0084 & 0.00939399 & 0.989604 \end{pmatrix}, \]  

(32)
FIG. 1: The dependence of $U_{12}, U_{21}, U_{32}$ and $U_{33}$ on the parameters $A, K$ with $A \in (1.5, 2.0)$ and $K \in (2.0, 3.0)$

and

$$
\begin{align*}
    h^u_1 &= -0.0012957, \quad h^u_2 = -0.0114773, \quad h^u_3 = 1.735, \quad h'^u = 0.0038181, \\
    h^d_1 &= 0.000737264, \quad h^d_2 = 0.000260736, \quad h^d_3 = 0.0418, \quad h'^d = 0.000382925. 
\end{align*}
$$

(33)

The results in Eq. (33) and Tab. II show that $h^u_1, h^u_2, h'^u \ll h^u_3$ and $h^d_1, h^d_2, h'^d \ll h^d_3$. It is a consequence of the fact that the top- and bottom quark mass are much larger than that of other quarks.
FIG. 2: The dependence of $U_{12}, U_{21}, U_{22}, U_{32}$ and $U_{33}$ on $K$ with $A = 1.8$ and $K \in (2.0, 3.0)$

IV. CONCLUSION

In this paper, we have proposed a new $D_4$ flavor model based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ gauge symmetry in which the quark mixing matrix is concentrated. After spontaneous breaking of flavor symmetry, with the constraint on Higgs VEVs in the invariant Yukawa couplings, all of quarks have consistent masses and a small deviation from the unity is obtained at the tree-level. To obtain the quark mixing matrix consistent with experimental data in 2012, the violation terms with $1'$ under $D_4$ are introduced. Numerical estimation shows that the Yukawa couplings in the model under consideration are consistent to those in the SM.

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Appendix A: $D_4$ group and Clebsch-Gordan coefficients

$D_4$ is the symmetry group of a square. It has eight elements divided into five conjugacy classes, with $1, 1', 1'', 1'''$ and $2$ as its five irreducible representations. Any element of $D_4$ can be formed by multiplication of the generators $a$ (the $\pi/2$ rotation) and $b$ (the reflection) obeying the relations $a^4 = e$, $b^2 = e$, and $bab = a^{-1}$. $D_4$ has the following five conjugacy classes,

$C_1 : \{a_1 \equiv e\}$,

$C_2 : \{a_2 \equiv a^2\}$,
\[ C_3 : \{a_3 \equiv a, a_4 \equiv a^3\}, \]
\[ C_4 : \{a_5 \equiv b, a_6 \equiv a^2b\}, \]
\[ C_5 : \{a_7 \equiv ab, a_8 \equiv a^3b\}. \]

The character table of \( D_4 \) is given as follows:

| Class | \( n \) | \( h \) | \( \chi_1 \) | \( \chi_1' \) | \( \chi_1'' \) | \( \chi_2 \) |
|-------|--------|--------|-----------|----------|----------|----------|
| \( C_1 \) | 1 | 1 | 1 | 1 | 1 | 2 |
| \( C_2 \) | 1 | 2 | 1 | 1 | 1 | -2 |
| \( C_3 \) | 2 | 4 | 1 | -1 | -1 | 1 | 0 |
| \( C_4 \) | 2 | 2 | 1 | 1 | -1 | -1 | 0 |
| \( C_5 \) | 2 | 2 | 1 | -1 | 1 | -1 | 0 |

where \( n \) is the order of class and \( h \) the order of elements within each class.

We have worked in real basis, in which the two-dimensional representation \( 2 \) of \( D_4 \) is real, \( 2^\ast(1^*, 2^*) = 2(1^*, 2^*) \). One possible choice of generators is given as follows

\[ 1 \ : \ a = 1, \ b = 1, \]
\[ 1' \ : \ a = 1, \ b = -1, \]
\[ 1'' \ : \ a = -1, \ b = 1, \]
\[ 1''' \ : \ a = -1, \ b = -1, \]
\[ 2 \ : \ a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]  

(A2)

Using them we calculate the Clebsch-Gordan coefficients for all the tensor products as given below.

First, let us put \( 2(1, 2) \) which means some \( 2 \) doublet such as \( x = (x_1, x_2) \sim 2 \) or \( y = (y_1, y_2) \sim 2 \) and so on, and similarly for the other representations. Moreover, the numbered multiplets such as \( (\ldots, ij, \ldots) \) mean \( (\ldots, x_iy_j, \ldots) \) where \( x_i \) and \( y_j \) are the multiplet components of different representations \( x \) and \( y \), respectively. In the following the components of representations in left-hand side will be omitted and should be understood, but they always exist in order in the components of decompositions in right-hand side:

\[ 1(1) \otimes 1(1) = 1(11), \]
\[ 1'(1) \otimes 1'(1) = 1(11), \]
\[ 1''(1) \otimes 1''(1) = 1(11), \]
\[ 1'''(1) \otimes 1'''(1) = 1(11), \]
\[ 1(1) \otimes 1'(1) = 1'(11), \]
\[ 1(1) \otimes 1''(1) = 1''(11), \]
\[ 1(1) \otimes 1'''(1) = 1'''(11), \]
\[ 1'(1) \otimes 1''(1) = 1''(11), \]
\[ 1'(1) \otimes 1'''(1) = 1'''(11), \]
\[ 1''(1) \otimes 1'''(1) = 1'''(11), \]

(A3)
\[ 1(1) \otimes 1'''(1) = 1'''(11), \quad 1'(1) \otimes 1''(1) = 1'''(11), \]
\[ 1''(1) \otimes 1'''(1) = 1'(11), \quad 1'''(1) \otimes 1'(1) = 1''(11), \quad (A4) \]
\[ 1(1) \otimes 2(1,2) = 2(11,12), \quad 1'(1) \otimes 2(1,2) = 2(11,12), \quad (A5) \]
\[ 1''(1) \otimes 2(1,2) = 2(12,11), \quad 1'''(1) \otimes 2(1,2) = 2(12,11), \quad 2(1,2) \otimes 2(1,2) = 1(11+22) \oplus 1'(11-22) \oplus 1''(12+21) \oplus 1'''(12-21). \]

In the text we usually use the following notations, for example, \( (xy)_1 \equiv (x_1 y_1 + x_2 y_2) \) which is the Clebsch-Gordan coefficients of 1 in the decomposition of 2 \( \otimes 2 \), where as mentioned \( x = (x_1, x_2) \sim 2 \) and \( y = (y_1, y_2) \sim 2 \).

The rules to conjugate the representations 1, 1', 1'', 1''' and 2 are given by
\[ 2^*(1^*,2^*) = 2(1^*,2^*), \]
\[ 1'(1^*) = 1(1^*), \quad 1''(1^*) = 1'(1^*), \quad 1'''(1^*) = 1''(1^*), \quad 1''''(1^*) = 1'''(1^*), \quad (A6) \]
where, for example, \( 2^*(1^*,2^*) \) denotes some 2* multiplet of the form \( (x_1^* , x_2^*) \sim 2^* \).

**Appendix B: The numbers**

In the following we will explicitly point out the lepton number \((L)\) and lepton parity \((P_l)\) of the model particles (notice that the family indices are suppressed):

| Particles  | \( L \) | \( P_l \) |
|------------|--------|--------|
| \( N_R, u, d, \phi_1^+, \phi_1'^+, \phi_2^0, \phi_2^0, \eta_1^0, \eta_1'^0, \eta_2^- \eta_2'^- \), \( \chi_3, \sigma_3^0, s_3^0 \) | 0 | 1 |
| \( \nu_L, l, U^*, \phi_3^+, \phi_3'^+, \eta_3^0, \eta_3'^0, \lambda_1^+, \lambda_1'^+, \sigma_1^+, \sigma_1'^+, \sigma_3^0, s_3^0 \) | -1 | -1 |
| \( \sigma_0^0, \sigma_1^+, \sigma_2^+, s_1^+, s_2^+, s_3^+ \) | -2 | 1 |

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