The Conjugacy Problem in Free Solvable Groups and Wreath Products of Abelian Groups is in $\text{TC}^0$

Alexei Miasnikov$^1$ · Svetla Vassileva$^2$ · Armin Weiß$^3$

© Springer Science+Business Media, LLC, part of Springer Nature 2018

Abstract We show that the conjugacy problem in a wreath product $A \wr B$ is uniform-$\text{TC}^0$-Turing-reducible to the conjugacy problem in the factors $A$ and $B$ and the power problem in $B$. If $B$ is torsion free, the power problem in $B$ can be replaced by the slightly weaker cyclic submonoid membership problem in $B$. Moreover, if $A$ is abelian, the cyclic subgroup membership problem suffices, which itself is uniform-$\text{AC}^0$-many-one-reducible to the conjugacy problem in $A \wr B$. Furthermore, under certain natural conditions, we give a uniform $\text{TC}^0$ Turing reduction from the power problem in $A \wr B$ to the power problems of $A$ and $B$. Together with our first result, this yields a uniform $\text{TC}^0$ solution to the conjugacy problem in iterated wreath products of abelian groups – and, by the Magnus embedding, also in free solvable groups.

Keywords Wreath products · Conjugacy problem · Word problem · $\text{TC}^0$ · Free solvable group
1 Introduction

The conjugacy problem is one of Dehn’s fundamental algorithmic problems in group theory [3]. It asks on input of two group elements (given as words over a fixed set of generators) whether they are conjugate. The conjugacy problem can be seen as a generalization of the word problem, which on input of one word asks whether it represents the identity element of the group. In recent years the conjugacy problem gained an increasingly important role in non-commutative cryptography; see for example [2, 5, 10, 24, 28]. These applications use the fact that it is easy to create elements which are conjugate, but to check whether two given elements are conjugate might be difficult even if the word problem is easy. In fact, there are groups where the word problem is decidable in polynomial time, but the conjugacy problem is undecidable [17]. Moreover, there are natural classes, like polycyclic groups, which have a word problem in uniform $\text{TC}^0$ [23], but the conjugacy problem not even known to be in $\text{NP}$. Another example for such a huge contrast is the Baumslag group, whose word problem is decidable in polynomial time, but the conjugacy problem is conjectured to be non-elementary [4].

The class $\text{TC}^0$ is a very low complexity class consisting of those problems that can be recognized by a family of constant-depth and polynomial-size Boolean circuits which may also use majority gates. We only consider ($\text{Dlogtime}$-)uniform $\text{TC}^0$ (and subsequently simply write $\text{TC}^0$ for uniform $\text{TC}^0$). The word problem of abelian groups as well as integer arithmetic (iterated addition, multiplication, division) are problems in $\text{TC}^0$. However, there are not many groups known to have conjugacy problem in $\text{TC}^0$. Indeed, without the results of this paper, the Baumslag-Solitar groups $\text{BS}_{1,q}$ [4] and nilpotent groups [21] are the only natural examples we are aware of. On the other hand, there is a wide range of groups having word problem in $\text{TC}^0$: all polycyclic groups [23] and, more generally, by a recent result all solvable linear groups [11]. Also iterated wreath products of abelian groups are known to have word problem in $\text{TC}^0$ [12].

The study of the conjugacy problem in wreath products has quite a long history: in [16] Matthews proved that a wreath product $A \wr B$ has decidable conjugacy problem if, and only if, both $A$ and $B$ have decidable conjugacy problem and $B$ has decidable cyclic subgroup membership problem (note that in [16] this is called power problem). As a consequence, she obtained a solution to the conjugacy problem in free metabelian groups. Kargapolov and Remeslennikov generalized the result by establishing decidability of the conjugacy problem in free solvable groups of arbitrary degree [9].

A few years later Remeslennikov and Sokolov [22] also generalized Matthews’ results to iterated wreath products by solving the cyclic subgroup membership problem in these groups. They also showed that the Magnus embedding [15] of free solvable groups into iterated wreath products of abelian groups preserves conjugacy – thus, giving a new proof for decidability of the conjugacy problem in free solvable groups.

Later, in [18] a polynomial time algorithm for the conjugacy problem in free solvable groups has been given and in [25] it is shown that for iterated wreath products of abelian groups Matthews’ criterion [16] can be actually checked in polynomial time.
In [20] this has been further improved to \( \text{LOGCFL} \). Recently, in [6], Matthews’ result has been generalized to a wider class of groups without giving precise complexity bounds – see the discussion in the last section.

In this work we use the same techniques as in [16, 20, 25] to give a precise complexity version of Matthews’ result. Moreover, we extend the result of [20, 25] in several directions. As in [20], at some points we need a stronger hypothesis than in [16] though: it is not sufficient to assume that the cyclic subgroup membership problem is decidable in \( \text{TC}^0 \) in order to reduce the conjugacy problem in a wreath product to the factors. Instead, we need the stronger power problem to be in \( \text{TC}^0 \): on input of two group elements \( b \) and \( c \) compute an integer \( k \) such that \( b^k = c \). More precisely, we establish the following results:

- The word problem of \( A \wr B \) is uniform-\( \text{AC}^0 \)-Turing-reducible to the word problems of \( A \) and \( B \).
- There is a uniform \( \text{TC}^0 \) Turing reduction from the conjugacy problem in \( A \wr B \) to the conjugacy problems in \( A \) and \( B \) together with the power problem in \( B \). If \( B \) is torsion-free, the power problem can be replaced by the cyclic submonoid membership problem; if \( A \) is abelian, the power problem can be replaced by the cyclic subgroup membership problem.
- The cyclic subgroup membership problem in \( B \) is \( \text{AC}^0 \)-reducible to the conjugacy problem in \( A \wr B \) and, if \( A \) is non-abelian, then also the cyclic submonoid membership problem in \( B \) is \( \text{AC}^0 \)-reducible to the conjugacy problem in \( A \wr B \).
- Suppose the orders of torsion elements of \( B \) are \( \beta \)-smooth for some \( \beta \in \mathbb{N} \). Then, the power problem in \( A \wr B \) is uniform-\( \text{TC}^0 \)-Turing-reducible to the power problems in \( A \) and \( B \). As a corollary we obtain that iterated wreath products of abelian groups have conjugacy problem in uniform \( \text{TC}^0 \). Using the Magnus embedding [15, 22], also the conjugacy problem in free solvable groups is in uniform \( \text{TC}^0 \).

Notice that images of group elements under the Magnus embedding can be computed in \( \text{TC}^0 \) (since any image under homomorphisms of finitely generated monoids can be computed in \( \text{TC}^0 \) [13]). Thus, for free solvable groups as well as for iterated wreath products of abelian groups, our results nail down the complexity of conjugacy precisely. This is because the word problem in \( \mathbb{Z} \) is already hard for \( \text{TC}^0 \) (and so the conjugacy problem in free solvable groups is \( \text{TC}^0 \)-complete). Also for wreath products \( A \wr B \) with \( A \) abelian or \( B \) torsion-free, we have a tight complexity bound because in this case there is a reduction from the cyclic subgroup membership problem (resp. cyclic submonoid membership problem) in \( B \) to the conjugacy problem in \( A \wr B \).

To solve the conjugacy problem, we first deal with the word problem. For a free solvable group of degree \( d \), we obtain a circuit of majority depth \( d \). It is not clear how a circuit of smaller majority depth could be constructed. On the other hand, [18] presents an algorithm for the word problem running in cubic time for arbitrary solvability degree. This gives rise to the question whether the depth (or the size) of circuits for the word and conjugacy problem of free solvable groups could be bounded uniformly independent of the degree. Note that a negative answer to this question would imply that \( \text{TC}^0 \neq \text{NC}^1 \).
We want to emphasize that throughout we assume that the groups are finitely generated. As wreath products we consider only restricted wreath products, that is the underlying functions are required to have finite support. (This is a natural restriction because unrestricted wreath products are uncountable, in general. In particular, there is no way to represent every element as a finite input for some algorithm – so any algorithmic problem could be defined only for a proper subset of the whole group and complexity questions depend very much on the encoding of elements.)

Outline  Section 2 introduces some notation and recalls some basic facts on complexity. Then in Section 3, we define wreath products and discuss the solution to the word problem. Sections 4 and 5, the main parts, examine the conjugacy problem in wreath products resp. iterated wreath products. In order to do so, we deal with the power problem in iterated wreath products in Section 5. Finally, in Section 6, we discuss some open problems. This work is the full version of the conference paper [19]. It contains all proofs, some more examples and a slightly stronger version of Theorem 2.

2 Preliminaries

Words  An alphabet is a (finite or infinite) set \( \Sigma \); an element \( a \in \Sigma \) is called a letter. The free monoid over \( \Sigma \) is denoted by \( \Sigma^* \); its elements are finite sequences of letters and they are called words. The multiplication of the monoid is the concatenation of words. The identity element is the empty word \( \epsilon \).

Groups  We consider a finitely generated group \( G \) together with a surjective homomorphism \( \eta : \Sigma^* \to G \) (a monoid presentation) for some finite alphabet \( \Sigma \). Throughout, all groups we consider are finitely generated even if not mentioned explicitly. In order to keep notation simple, we suppress the homomorphism \( \eta \) and consider words also as group elements. We write \( w = G w' \) as a shorthand for \( \eta(w) = \eta(w') \) and \( w \in G A \) instead of \( \eta(w) \in \eta(A) \) for \( A \subseteq \Sigma^* \) and \( w, w' \in \Sigma^* \). Whenever it is clear that we deal with group elements \( g, h \in G \), we simply write \( g = h \) for equality in \( G \). We always assume that \( \Sigma = G \Sigma^{-1} \).

We say two group elements \( g, h \in G \) are conjugate, and we write \( g \sim h \), if there exists an element \( x \in G \) such that \( g^x = x^{-1}gx = h \). Similarly, we say two words \( u \) and \( v \) in generators of \( G \) are conjugate, and we write \( u \sim_G v \), if the elements of \( G \) represented by \( u \) and \( v \) are conjugate as elements of \( G \). We denote by \( \text{ord}(g) \) the order of a group element \( g \) (i.e., the smallest positive integer \( d \) such that \( g^d = 1 \), or \( \infty \) if no such integer exists). For \( g \in G \), the cyclic subgroup generated by \( g \) is denoted by \( \langle g \rangle \). A \( d \)-fold commutator is a group element of the form \( x^{-1}y^{-1}xy \) for \((d−1)\)-fold commutators \( x \) and \( y \); a 0-fold commutator is any group element. The free solvable group of degree \( d \) is the group subject only to the relations that all \( d \)-fold commutators are trivial.

Complexity  Computation or decision problems are given by functions \( f : \Delta^* \to \Sigma^* \) for some finite alphabets \( \Delta \) and \( \Sigma \). A decision problem (or formal language) \( L \) is
identified with its characteristic function \( \chi_L : \Delta^* \to \{0, 1\} \) with \( \chi_L(x) = 1 \) if, and only if, \( x \in L \).

**Computational Problems in Group Theory** Let \( G \) be a group with finite generating set \( \Sigma \). We define the following algorithmic problems in group theory.

- The word problem \( WP(G) \) of \( G \), is the set of all words representing the identity in \( G \).
- The conjugacy problem \( CP(G) \) is the set of all pairs \((v, w)\) such that \( v \sim_G w \).
- The cyclic subgroup membership problem \( CSGMP(G) \): the set of all pairs \((v, w)\) such that \( w \in \langle v \rangle \) (i.e., there is some \( k \in \mathbb{Z} \) with \( v^k =_G w \)).
- The cyclic submonoid membership problem \( CSMMP(G) \): the set of all pairs \((v, w)\) such that \( w \in \mathbb{G}^{\ast} \{v\}^\ast \) (i.e., there is some \( k \in \mathbb{N} \) with \( v^k =_G w \)).
- The power problem \( PP(G) \): on input of some \((v, w) \in \Sigma^\ast \times \Sigma^\ast \) decide whether there is some \( k \in \mathbb{Z} \) such that \( v^k =_G w \) and, in the “yes” case, compute this \( k \) in binary representation. If \( v \) has finite order in \( G \), the computed \( k \) has to be the smallest non-negative such \( k \).

Whereas the first four of these problems are decision problems, the last one is an actual computation problem. Be aware that sometimes in literature the power problem is defined as what we refer to as cyclic subgroup membership problem.

**Circuit Classes** The class \( AC^0 \) is defined as the class of functions computed by families of circuits of constant depth and polynomial size with unbounded fan-in Boolean gates (and, or, not). \( TC^0 \) additionally allows majority gates. A majority gate (denoted by Maj) returns 1 if the number of 1s in its input is greater than or equal to the number of 0s. In the following, we always assume that the alphabets \( \Delta \) and \( \Sigma \) are encoded over the binary alphabet \( \{0, 1\} \) such that each letter uses the same number of bits. Moreover, we assume that also the empty word \( \epsilon \) has such a encoding over \( \{0, 1\} \), which is denoted by \( \epsilon \) as well (be aware of the slight ambiguity). The empty word letter is used to pad outputs of circuits to fit the full number of output bits; still we do not forbid to use it in the middle. We say a function \( f \) is \( AC^0 \)-computable (resp. \( TC^0 \)-computable) if \( f \in AC^0 \) (resp. \( f \in TC^0 \)).

In the following, we only consider \( Dlogtime \)-uniform circuit families. \( Dlogtime \)-uniform means that there is a deterministic Turing machine that decides in time \( O(\log n) \) on input of two gate numbers (given in binary) and the string \( 1^n \) whether there is a wire between the two gates in the \( n \)-input circuit and also decides of which type each gate is. Note that the binary encoding of the gate numbers requires only \( O(\log n) \) bits – thus, the Turing machine is allowed to use time linear in the length of the encodings of the gates. For more details on these definitions we refer to [26]. In order to keep notation simple we write \( AC^0 \) (resp. \( TC^0 \)) for \( Dlogtime \)-uniform \( AC^0 \) (resp. \( Dlogtime \)-uniform \( TC^0 \)) throughout. We have the following inclusions (note that even \( TC^0 \subseteq P \) is not known to be strict):

\[
AC^0 \subseteq TC^0 \subseteq \text{LOGCFL} \subseteq P.
\]

The following facts are well-known and will be used in the following without further reference:
- Barrington, Immerman, and Straubing [1] showed that \( \text{TC}^0 = \text{FO}(+, \ast, \text{Maj}) \), i.e., \( \text{TC}^0 \) comprises exactly those languages which are defined by some first order formula with majority quantifiers where positions may be compared using \( +, \ast \) and \(<\). In particular, if we can give a formula with majority quantifiers using only addition and multiplication predicates, we do not need to worry about uniformity.

- Homomorphisms can be computed in \( \text{TC}^0 \) [13]: on input of two alphabets \( \Sigma_1 \) and \( \Delta_1 \) (coded over the binary alphabet), a list of pairs \( (a, v_a) \) with \( a \in \Sigma_1 \) and \( v_a \in \Delta_1^* \), such that each \( a \in \Sigma_1 \) occurs in precisely one pair, and a word \( w \in \Sigma_1^* \), the image \( \phi(w) \) under the homomorphism \( \phi \) defined by \( \phi(a) = v_a \) can be computed in \( \text{TC}^0 \). Moreover, if \( \phi \) is length-multiplying (that is \( \phi(a) \) and \( \phi(b) \) have the same length for all \( a, b \in \Sigma_1 \)), the computation is in \( \text{AC}^0 \). Note that by padding with the empty-word letter \( \epsilon \), we can assume that all homomorphisms are length-multiplying.

- Iterated addition is the following problem: given \( n \) numbers \( a_1, \ldots, a_n \) (in binary), compute \( \sum_{i=1}^n a_i \) (as binary number). This is well-known to be in \( \text{TC}^0 \).

**Example 1** Finitely generated abelian groups have word problem in \( \text{TC}^0 \): the word problem of \( \mathbb{Z} \) is in \( \text{TC}^0 \) using iterated addition (summing up numbers 1 and \(-1\)), the word problem of finite cyclic groups is in \( \text{TC}^0 \) by iterated addition and then calculating modulo; and, finally, a word in a direct product is the identity if, and only if, it is the identity in all components.

**Example 2** Let \( (k_1, v_1), \ldots, (k_n, v_n) \) be a list of \( n \) key-value pairs \( (k_i, v_i) \) equipped with a total order on the keys \( k_i \) such that it can be decided in \( \text{TC}^0 \) whether \( k_i < k_j \). We assume that all pairs \( (k_i, v_i) \) are encoded with the same number of bits. It is a standard fact that the problem of sorting the list according to the keys is in \( \text{TC}^0 \) (i.e., the desired output is a list \( (k_{\pi(1)}, v_{\pi(1)}), \ldots, (k_{\pi(n)}, v_{\pi(m)}) \) for some permutation \( \pi \) such that \( k_{\pi(i)} < k_{\pi(j)} \) for all \( i < j \)).

We briefly describe a circuit family to do so: The first layer compares all pairs of keys \( k_i, k_j \) in parallel. The next layer for all \( i \) and \( j \) computes a predicate \( P(i, j) \) which is true if, and only if, \( |\{ \ell \mid k_\ell < k_i \}| = j \). The latter is computed by iterated addition. As a final step the \( j \)-th output pair is set to \( (k_i, v_i) \) if, and only if, \( P(i, j) \) is true.

**Reductions** Let \( K \subseteq \Delta^* \) and \( L \subseteq \Sigma^* \) be languages and \( C \) a complexity class. Then \( K \) is called \( C \)-many-one-reducible to \( L \) if there is a \( C \)-computable function \( f : \Delta^* \rightarrow \Sigma^* \) such that \( w \in K \) if, and only if, \( f(w) \in L \). In this case, we write \( K \leq_m^C L \).

A function \( f \) is \( \text{AC}^0 \)-(Turing)-reducible to a function \( g \) if there is a \( D\text{logtime} \)-uniform family of \( \text{AC}^0 \) circuits computing \( f \) which, in addition to the Boolean gates, may also use oracle gates for \( g \) (i.e., gates which on input \( x \) output \( g(x) \)). This is expressed by \( f \in \text{AC}^0(g) \) or \( f \leq_{T}^\text{AC}^0 g \). Likewise \( \text{TC}^0 \) (Turing) reducibility is defined. Note that if \( L_1, \ldots, L_k \) are in \( \text{TC}^0 \), then \( \text{TC}^0(L_1, \ldots, L_k) = \text{TC}^0 \) (see e.g. [26]).

**Remark 1** The cyclic subgroup membership problem, in particular, allows to solve the word problem: some group element is in the cyclic subgroup generated by the
identity if, and only if, it is the identity. Moreover, the cyclic subgroup membership problem for \((v, w)\) can be decided by two calls to the cyclic submonoid membership problem (for \((v, w)\) as well as for \((v^{-1}, w)\)). Also, the power problem is a stronger version of the cyclic submonoid membership problem (simply check the sign of the output of the power problem). Thus, we have

\[
\text{WP}(G) \leq_{m}^{AC^0} \text{CSGMP}(G) \leq T^{AC^0} \text{CSMMP}(G) \leq T^{AC^0} \text{PP}(G).
\]

Moreover, an algorithm for the power problem allows us to decide whether an element is of finite order (just compute the \(k\) such that \(g^k = g^{-1}\) if this is a positive number, then \(g\) is of finite order, otherwise not).

**Example 3** Let \(BS_{1,2} = \langle a, t \mid tat^{-1} = a^2 \rangle\) be the Baumslag-Solitar group. The conjugacy problem of \(BS_{1,2}\) is in \(TC^0\) by [4]. Moreover, let us show that the power problem is also in \(TC^0\): \(BS_{1,2}\) is the semi-direct product \(\mathbb{Z}[1/2] \rtimes \mathbb{Z}\) with multiplication defined by \((r, m) \cdot (s, q) = (r + 2^m s, m + q)\) – see e.g. [4]. Any word of length \(n\) over the generators can be transformed in \(TC^0\) to a pair \((r, m)\) with \(m \leq n\) and \(r\) can be written down with \(O(n)\) bits in binary. Let \((r, m)\) and \((s, q)\) be two such inputs for the power problem. We wish to decide whether there is some \(\ell\) with \((r, m)\ell = (s, q)\):

- If \(q \neq 0\), then the only possibility for \(\ell\) is \(\ell = q/m\). If this is not an integer, then there is no such \(\ell\). If it is, one needs to check whether it satisfies \((r, m)\ell = (s, q)\).
- Because \(\ell\) is bounded by the input length, this can be done in \(TC^0\) using the circuit for the word problem [4, 23].

Now let \(q = 0\). If also \(s = 0\), then the solution is \(\ell = 0\). So let \(s \neq 0\). If \(m \neq 0\), clearly there is no solution, so we are in the case \(q = m = 0\) and \(r, s \neq 0\). But now, again we simply need to compute \(\ell = s/r\) (this can be done in \(TC^0\) using Hesse’s circuit for division [7, 8]). If it is an integer, the power problem has the solution \(\ell\), otherwise, it does not have a solution.

Notice that this example shows that there are natural groups where the power problem can be solved in \(TC^0\), but – because of the exponential distortion of the subgroup \(\langle a \rangle\) – the solution to the power problem can only be returned if encoded in binary.

### 3 Wreath Products and the Word Problem

Let \(A\) and \(B\) be groups. For a function \(f : B \rightarrow A\) the *support* of \(f\) is defined as \(\text{supp}(f) = \{ b \in B \mid f(b) \neq 1 \}\). For two groups \(A\) and \(B\), the set of functions from \(B\) to \(A\) with finite support is denoted by \(A(B)\); it forms a group under point-wise multiplication. Mapping \(a \in A\) to the function

\[
a(b) = \begin{cases} 
a & \text{if } b = 1, \\
1 & \text{otherwise},
\end{cases}
\]

(1)
gives an embedding of \(A\) into \(A(B)\). In what follows we identify \(A\) with its image in \(A(B)\). The *wreath product* \(A \wr B\) of \(A\) and \(B\) is defined as the semi-direct product \(B \rtimes A(B)\), where the action of \(b \in B\) on a function \(f \in A(B)\) is defined by \(f^b(x) = f(xb)\).
\[ f(xb^{-1}). \] Note that this is also referred to as restricted wreath product. We identify \( B \) and \( A^B \) (and hence also \( A \)) with their canonical images in \( A \wr B \). Thus, for the multiplication in \( A \wr B \) we have the following rules

\[
(b, f)(c, g) = (bc, f^c g), \quad (b, f)^{-1} = (b^{-1}, (f^{-1})^{-1})
\]

for \( b, c \in B \) and \( f, g \in A^B \), where \( f^{-1} \) is the point-wise inverse (i.e., \( f^{-1}(b) = (f(b))^{-1} \)) for all \( b \in B \).

Let \( \Sigma_A \) and \( \Sigma_B \) be fixed generating sets of \( A \) and \( B \), correspondingly. Then, \( A \wr B \) is generated by \( \Sigma = \Sigma_A \cup \Sigma_B \) (using the embedding (1) of \( A \) into \( A \wr B \)). Given a word \( w \in \Sigma^* \) of length \( n \), we can group it as \( w = a_1b_1 \cdots a_mb_m \) with \( a_i \in \Sigma_A^*, b_i \in \Sigma_B^* \) and \( m \leq n \). Introducing factors \( b^{-1}b \in \Sigma_B^* \), we can rewrite this as follows:

\[
w = G \ a_1b_1 \cdots a_mb_m = G \ b_1^{-1}a_1b_1 \cdots a_mb_m = G \ b_1a_1^b \cdot a_2b_2 \cdots a_mb_m = G \ b_1b_2 \cdots a_mb_m
\]

Thus, we have \( w = G \ (b, f) \) with \( b = b_1 \cdots b_m \) and \( f = a_1^b \cdots a_m^b \). Note that, in this setting, \( a^c \) and \( \tilde{a}^c \) commute for all \( a, \tilde{a} \in A \) and \( c \neq \tilde{c} \in B \). Therefore, we can reorder this product to ensure that the exponents are distinct: whenever we have \( b_i \cdots b_m = B \ b_j \cdots b_m \) for \( i < j \), we combine the corresponding terms into a single term \( (a_i a_j)^{b_i \cdots b_m} \). Thus, we can rewrite \( f \) as the product \( \tilde{a}_1^\tilde{b}_1 \cdots \tilde{a}_k^\tilde{b}_k \), where \( \tilde{a}_1, \ldots, \tilde{a}_k \in \Sigma_A^*, \) and \( \tilde{b}_1, \ldots, \tilde{b}_k \in \Sigma_B^* \) all represent distinct elements of \( B \). Moreover, we can assume that all \( \tilde{a}_i \) represent non-trivial elements of \( A \). With this notation, we have \( f(\tilde{b}_i) = \tilde{a}_i \neq 1 \) and \( f(c) = 1 \) for \( c \notin \{\tilde{b}_1, \ldots, \tilde{b}_k\} = \text{supp}(f) \).

Furthermore, \( f \) is completely given by the set of pairs \( \{(b_1, \tilde{a}_1), \ldots, (b_k, \tilde{a}_k)\} \).

In the following, we always assume that a function \( f \in A^B \) is represented as a list of pairs \( f = ((\tilde{b}_1, \tilde{a}_1), \ldots, (\tilde{b}_k, \tilde{a}_k)) \) with \( \{\tilde{b}_1, \ldots, \tilde{b}_k\} = \text{supp}(f) \). The order of the pairs does not matter – but they are written down in some order. We also assume for an input \( w \) of length \( n \), that \( k = m = n \) and that every word \( b_1, \tilde{a}_1 \) has length \( n \). This is achieved by padding with pairs \( (\epsilon, \epsilon) \) (where \( \epsilon \) is the letter representing the empty word).

**Lemma 1** Let \( A \) and \( B \) be finitely generated groups and let \( G = A \wr B \). Then there is a family of \( \mathcal{AC}^0(\text{WP}(A), \text{WP}(B)) \) circuits which on input \( w \in \Sigma^* \) computes \((b, f) \) with \( w = G \ (b, f) \) where \( b \in \Sigma_B^* \) and \( f \) is encoded as described in the preceding paragraph.

**Proof** For an input word \( w = w_1 \cdots w_n \in \Sigma^* \), we first calculate the image under the projection \( \pi_B : a \mapsto \epsilon \) for \( a \in \Sigma_A \). Since \( \epsilon \) is a letter in our alphabet, this is a length-preserving homomorphism, and thus, can be computed in \( \mathcal{AC}^0 \) [13]. We have \( b = \pi_B(w) \). Next, define the following equivalence relation \( \approx \) on \( \{1, \ldots, n\} \):

\[
i \approx j \iff \pi_B(w_{i+1} \cdots w_n) = \pi_B(w_{j+1} \cdots w_n)
\]
After the computation of $\pi_B$ it can be checked for all pairs $i, j$ in parallel whether $i \approx j$ using $\binom{n}{2}$ oracle calls to the word problem of $B$. Let $[i]$ denote the equivalence class of $i$. Now, $b^{-1} w$ is in the (finite) direct product $\prod_{[i]} A^{\pi_B(w_{i+1} \cdots w_n)} \leq A(B)$ (this is well-defined by the definition of $\approx$). The projection to the component associated to $[i]$ is computed by replacing all $w_j$ by $\epsilon$ whenever $w_j \in \Sigma_B$ or $j \not\approx i$. As before, this computation is in $\text{AC}^0$. As a representative of $[i]$, we choose the smallest $i \in [i]$. Now, the preliminary output is the pair $(b, (f_1, \ldots, f_n))$ with

$$f_i = \begin{cases} (\pi_B(w_{i+1} \cdots w_n), \prod_{j \in [i]} w_j) & \text{if } i = \min[i], \\ (\epsilon, \epsilon) & \text{otherwise.} \end{cases}$$

Up to the calculation of $\approx$, everything can be done in $\text{AC}^0$ (checking $i = \min[i]$ amounts to $\bigwedge_{j < i} \neg (i \approx j)$). Finally, pairs $f_i = (b_i, a_i)$ with $a_i = A_1$ are replaced by $(\epsilon, \epsilon)$. This requires an additional layer of calls to the word problem of $A$.

If we assign appropriate gate numbers corresponding to the description of our circuit (e.g. concatenation of the number of the layer and the indices $i, j$), it is easy to see that it can be checked in linear time on input of two binary gate numbers whether the two gates are connected. This establishes uniformity of the circuit.

**Theorem 1** $\text{WP}(A \wr B) \in \text{AC}^0(\text{WP}(A), \text{WP}(B))$.

**Proof** This is an immediate consequence of Lemma 1 since $(b, f) =_G 1$ if, and only if, $b =_B 1$ (can be checked using the word problem of $B$) and $f = ((\epsilon, \epsilon), \ldots, (\epsilon, \epsilon))$. □

Note that Theorem 1 is a stronger version of [27] where $\text{NC}^1$ reducibility is shown.

**Definition 1** Let $d \in \mathbb{N}$. We define the left-iterated wreath product, $A^d \lhd B$, and the right-iterated wreath product $A \triangleright d B$ of two groups $A$ and $B$ inductively as follows:

$$-A^1 B = A \lhd B$$
$$-A^d B = A \triangleright (A^{d-1} B)$$

Let $S_{d,r}$ denote the free solvable group of degree $d$ and rank $r$. The Magnus embedding [15] is an embedding $S_{d,r} \to \mathbb{Z}^r \lhd S_{d-1,r}$. By iterating the construction, we obtain an embedding $S_{d,r} \to \mathbb{Z}^{rd} \lhd 1$. For the purpose of this paper, the explicit definition of the homomorphism is not relevant – it suffices to know that it is an embedding and that it preserves conjugacy [22]. The following corollary is also a consequence of [12] since a wreath product can be embedded into the corresponding block product.

**Corollary 1** Let $A$ and $B$ be finitely generated abelian groups and let $d \geq 1$. The word problems of $A \triangleright d B$ and of $A \triangleright d B$ are in $\text{TC}^0$. In particular, the word problem of a non-trivial free solvable group is $\text{TC}^0$-complete.
Note that here the groups $A$, $B$ and the number $d$ of wreath products are fixed. Indeed, if there were a single $\text{TC}^0$ circuit which worked for free solvable groups of arbitrary degree, this circuit would also solve the word problem of the free group, which is $\text{NC}^1$-hard, – thus, showing $\text{TC}^0 = \text{NC}^1$.

**Proof** The first statement follows from Theorem 1 because finitely generated abelian groups have word problem in $\text{TC}^0$ (see Example 1). The second statement then follows by the Magnus embedding [15] and the fact that homomorphisms can be computed in $\text{TC}^0$. The hardness part is simply due to the fact that a non-trivial free solvable group has an element of infinite order, i.e., a subgroup $\mathbb{Z}$, whose word problem is hard for $\text{TC}^0$.

**Remark 2** For a $\text{TC}^0$ circuit, the *majority depth* is defined as the maximal number of majority gates on any path from an input to an output gate (see e. g. [14]). Assume that $\text{WP}(A), \text{WP}(B) \in \text{TC}^0$. The circuit in the proof of Lemma 1 contains one layer of oracle gates to the word problem of $B$ followed by a layer of oracle gates to the word problem of $A$. The additional check for $b =_B 1$ in the proof of Theorem 1 can be done in parallel to the computation of Lemma 1; thus, it can be viewed as part of the layer of oracle gates for $\text{WP}(B)$. Since Lemma 1 is an $\text{AC}^0$ reduction, the majority depth of the resulting circuit is at most $m_A + m_B$ where $m_A$ (resp. $m_B$) is the majority depth of the circuit family for $\text{WP}(A)$ (resp. $\text{WP}(B)$).

Starting with the word problem of a free abelian group $\mathbb{Z}^r$, which is in $\text{TC}^0$ with majority depth one, we see inductively that a $d$-fold iterated wreath product $\mathbb{Z}^{rd} \wr 1$ – and thus the free solvable group of degree $d$ – has word problem in $\text{TC}^0$ with majority depth at most $d$. On the other hand, we do not see a method to improve this bound any further. In [12] a similar observation was stated for iterated block products (into which wreath products can be embedded). There the question was raised how the depth of the circuit for the word problem (or more general any problem recognized by the block product) is related to the number of block products in an iterated block product (the so-called block-depth).

**Question 1** Can the word problem of a free solvable group of degree $d$ be decided in $\text{TC}^0$ with majority depth less than $d$?

A negative answer to Question 1 would imply a negative answer to the analog question for iterated block products (the converse is not clear). Moreover, we want to point out that Question 1 is related to an important question in complexity theory: as outlined in [14], a negative answer would imply that $\text{TC}^0 \neq \text{NC}^1$. Nevertheless, the following observations point rather toward a positive answer of Question 1: the word problem of free solvable groups is decidable in cubic time – regardless of the solvability degree $d$ [18, 25]. Moreover, the circuit for linear solvable groups (*not* for free solvable groups with $d > 2$) from [11] can be arranged with majority depth bounded uniformly for all groups. This is because every matrix entry in a product of upper triangular matrices can be obtained as iterated addition of iterated multiplications of the entries of the original matrices (for the precise formula, see [11]). These operations have circuits of uniformly bounded depth (also for finitely generated field
extensions). Hence, only the size of the circuits, but not the depth, depends on the solvability degree.

4 The Conjugacy Problem in Wreath Products

In order to give a $\text{TC}^0$ reduction of the conjugacy problem of $A \wr B$ to the conjugacy problems of $A$ and $B$ and the power problem of $B$, we follow Matthews’ outline [16], where the same reduction was done for decidability. For deciding conjugacy of two elements $(b, f), (c, g)$ in a wreath product $A \wr B$ we will study the behavior of $f$ and $g$ on cosets of $\langle b \rangle \leq B$. For $b, d, t \in B, f \in A^B$, and $t \in T$, we define

$$\pi_{t,b}^{(d)}(f) = \begin{cases} \prod_{j=0}^{N-1} f(tb^j \cdot d^{-1}) & \text{if ord}(b) = N < \infty, \\ \prod_{j=-\infty}^{\infty} f(tb^j \cdot d^{-1}) & \text{if ord}(b) = \infty, \end{cases}$$

which is an element of $A$. We denote $\pi_{t,b}^{(1)}(f)$ by $\pi_{t,b}(f)$. The definition of the $\pi_{t,b}$ depends on the order of $b$. However, observe that even in the case when the order of $b$ is infinite the product is finite since the function $f$ is of finite support. In fact, it is the product of all possible non-trivial factors of the form $f(tb^j \cdot d^{-1})$ multiplied in increasing order of $j$. The same is true in the case when the order of $b$ is finite. So in order to compute $\pi_{t,b}^{(d)}$, we need to find all the elements of the form $tb^j \cdot d^{-1}$ at which $f$ is non-trivial, arrange them in increasing order of $j$ and concatenate the respective $a_j$.

Lemma 2 The computation of $\pi_{t,b}^{(d)}(f)$ is in $\text{TC}^0(\text{PP}(B))$. More precisely, the input is $b, d, t \in \Sigma_B^*$ and a function $f = ((b_1, a_1), \ldots, (b_n, a_n))$, the output is $\pi_{t,b}^{(d)}(f)$ given as a word over $\Sigma_A$. Moreover,

- if $B$ is torsion-free, then it is in $\text{TC}^0(\text{CSMMP}(B))$,
- if $A$ is abelian, then it is in $\text{TC}^0(\text{CSGMP}(B))$.

Proof One needs to check for all $j$ whether $t^{-1}b_j d \in \langle b \rangle$ (for $(b_j, a_j) \in f$) and if so, the respective power $k_j$ such that $t^{-1}b_j d = b^{k_j}$ has to be computed. For all $j$ this can be done in parallel using oracle gates for the power problem of $B$. The next step is to sort the tuples $(b_j, a_j)$ with $t^{-1}b_j d \in \langle b \rangle$ according to their power $k_j$. This can be done in $\text{TC}^0$ as described in Example 2. The output $\pi_{t,b}^{(d)}(f)$ is the product (in the correct order) of the respective $a_j$.

Now, let $B$ be torsion-free. The exponents $k_j$ with $t^{-1}b_j d = b^{k_j}$ are only needed in order to sort the pairs $(b_j, a_j)$. Thus, it suffices to decide for given $j$ and $j'$ whether $k_j \leq k_j'$ (where $k_j'$ is defined analogously to $k_j$). Since we assumed that $b$ has infinite order, we have $k_j \leq k_j'$ if, and only if, $(t^{-1}b_j d)^{-1}t^{-1}b_j' d = b^{-k_j}b^{k_j'} \in G \{b^k | k \in \mathbb{N}\}$ that is if, and only if, $(t^{-1}b_j d)^{-1}t^{-1}b_j' d$ is in the cyclic submonoid
generated by \( b \). Therefore, we can replace the power problem by the cyclic submonoid membership problem in the torsion-free case.

Finally, let \( A \) be abelian. In this case, the order of the factors of \( \pi_{t,b}^{(d)}(f) \) does not matter; hence, there is no need for sorting the factors. For checking \( t^{-1}b_{j}d \in \langle b \rangle \), the cyclic subgroup membership problem suffices. \( \square \)

A full system of \( \langle b \rangle \)-coset representatives is a set \( T \subseteq B \) of such that \( t \langle b \rangle \cap t' \langle b \rangle = \emptyset \) for \( t \neq t' \in T \) and \( B = T \langle b \rangle \). In [16], Matthews provides the following criterion for testing whether two elements of a wreath product are conjugate.

**Proposition 1** ([16, Prop. 3.5 and 3.6]) Let \( A \) and \( B \) be groups. Two elements \( x = (b, f) \) and \( y = (c, g) \) in \( A \wr B \) are conjugate if, and only if, there exists \( d \in B \) such that

- \( db = cd \) in \( B \) and
- if \( \text{ord}(b) \) is finite, \( \pi_{t,b}(f) \) is conjugate to \( \pi_{t,b}^{(d)}(g) \) for all \( t \in T \),
- if \( \text{ord}(b) \) is infinite, \( \pi_{t,b}(f) \) is equal to \( \pi_{t,b}^{(d)}(g) \) for all \( t \in T \),

where \( T \) is a full system of \( \langle b \rangle \)-coset representatives.

**Example 4** Let \( G = \mathbb{Z}_{2} \wr \mathbb{Z} \) be the Lamplighter group and let \( (b, f), (c, g) \in G \) with \( c, b \in \mathbb{Z} \), \( f, g \in \mathbb{Z}_{2}^{(\mathbb{Z})} \). We can view \( f \) and \( g \) as finite subsets of \( \mathbb{Z} \) (i.e., we identify \( f \) with \( \text{supp}(f) \)). Now the point-wise addition in \( \mathbb{Z}_{2}^{(\mathbb{Z})} \) becomes the symmetric difference \( \triangle \) of subsets and we obtain the multiplication rule \( (b, f)(c, g) = (b+c, (f+c)\triangle g) \) where \( f+c \) is defined as \( \{f_{1}+c, \ldots, f_{n}+c\} \) for \( f = \{f_{1}, \ldots, f_{n}\} \).

Now, \( T = \{0, \ldots, b-1\} \) if \( b \neq 0 \) and \( T = \mathbb{Z} \) if \( b = 0 \). For \( t \in T \) and \( d \in \mathbb{Z} \) we have

\[
\pi_{t,b}^{(d)}(f) = |\{fi \in f \mid fi \equiv t - d \mod b\}| \mod 2.
\]

Proposition 1 tells us that \( (b, f) \sim (c, g) \) if, and only if, \( b = c \) and there is some \( d \in \mathbb{Z} \) such that

\[
|\{fi \in f \mid fi \equiv t \mod b\}| \equiv |\{gi \in g \mid gi \equiv t - d \mod b\}| \mod 2
\]

for all \( t \in T \) (or equivalently for all \( t \in \mathbb{Z} \)).

In particular, \( (1, f) \sim (1, g) \) as soon as \( |f| \equiv |g| \mod 2 \) and \( (0, f) \sim (0, g) \) if, and only if, there is some \( x \in \mathbb{Z} \) with \( f = g + x \).

In order to derive a criterion for conjugacy, which is more suitable for working in \( \text{TC}^{0} \) or \( \text{LOGCFL} \), [20] follows the outline of [16]. For completeness, we will give a similar criterion in Proposition 2 and we will show how it follows from Proposition 1.

**Lemma 3** Let \( c, d, e, r, s \in B \) with \( d \langle c \rangle = e \langle c \rangle \) and \( r \langle c \rangle = s \langle c \rangle \). Then for every \( g \in A \langle B \rangle \), we have \( \pi_{r,c}^{(d)}(g) \sim \pi_{s,c}^{(e)}(g) \) and, if \( c \) has infinite order, we have \( \pi_{r,c}^{(d)}(g) = \pi_{s,c}^{(e)}(g) \).
Proof Since \( d \langle c \rangle = e \langle c \rangle \) and \( r \langle c \rangle = s \langle c \rangle \), there are integers \( p, q \) for which \( d = ec^p \) and \( r = sc^q \); hence,
\[
\pi_{r,c}^{(d)}(g) = \prod_k g(rc^k d^{-1}) = \prod_k g(sc^q c^p e^{-1}) = \prod_k g(sc^{k+q-p} e^{-1}).
\]
In the infinite order case, the last product in the above equation is equal to \( \prod_k g(sc^k e^{-1}) = \pi_{s,c}^{(e)}(g) \), in the finite order case it is a cyclic permutation of the factors in the product \( \prod_k g(sc^k e^{-1}) = \pi_{s,c}^{(e)}(g) \) and hence is conjugate to \( \pi_{s,c}^{(e)}(g) \).

Proof We have to show that the conditions of Proposition 2 imply the condition of Proposition 1. The proof follows the one of [16, Thm. B]. Let \( T \) be the full system of \( \langle b \rangle \)-coset representatives of Proposition 1.

First, observe that by Lemma 3 the condition of Proposition 1 is invariant under change of the system of representatives \( T \). Moreover, we can add multiple representatives of one coset to \( T \) (i.e., we do not need to require that \( t \langle b \rangle \cap t' \langle b \rangle = \emptyset \) for \( t \neq t' \in T \)) as long as \( T \langle b \rangle = B \), without changing the condition of Proposition 1. Hence, we can assume that \( \widetilde{T} \subseteq T \) and
\[
(T \setminus \widetilde{T}) \cap \widetilde{T} \langle b \rangle = \emptyset. \tag{2}
\]
Let us show that
\[
\pi_{t,b}(f) = 1 \quad \text{and} \quad \pi_{t,b}^{(d)}(g) = 1 \quad \text{for} \ t \in T \setminus \widetilde{T} \quad \text{and} \ d \in \{\beta_1^{-1} t, \ldots, \beta_m^{-1} t\}. \tag{3}
\]
Let \( t \in T \). If \( \pi_{t,b}(f) \neq 1 \), then \( tb^\ell \in \text{supp}(f) \subseteq \widetilde{T} \) for some \( \ell \in \mathbb{Z} \); hence, by (2), \( t \in \widetilde{T} \). If \( \pi_{t,b}^{(d)}(g) \neq 1 \) for some \( d \in \{\beta_1^{-1} b_k, \ldots, \beta_m^{-1} b_k\} \), then \( tb^\ell d^{-1} \in \text{supp}(g) \) for some \( \ell \in \mathbb{Z} \). Therefore, there is some \( i \in \{1, \ldots, m\} \) with \( \beta_i = tb^\ell d^{-1} = tb^\ell b_k^{-1} \beta_j \). Hence, \( tb^\ell = \beta_i b_j^{-1} b_k \in \widetilde{T} \) and, by (2), \( t = \beta_i b_j^{-1} b_k \). This shows (3).

Now consider \( x = (b, f) \) and \( y = (c, g) \). If \( b \) and \( c \) are not conjugate, then \( x \) and \( y \) are certainly not conjugate. If they are, we consider the following two cases:

(i) Suppose \( \pi_{t,b}(f) = 1 \) for all \( t \in \widetilde{T} \). By the same argument as for (3), this is the case if, and only if, \( \pi_{t,b}(f) = 1 \) for all \( t \in T \). Let \( S \) be a full system of
(c)-coset representatives. By Proposition 1, $x \sim y$ if, and only if, there is some $d \in B$ such that $db = cd$ and $\pi_{t,b}^{(d)}(g) = 1$ for all $t \in T$. Now,

$$\pi_{t,b}^{(d)}(g) = \prod_j g(tb^jd^{-1}) = \prod_j g(td^{-1}c^j) = \pi_{td^{-1},c}(g).$$

For each $t \in T$, there is some $s \in S$ with $td^{-1} \in s\langle c \rangle$ and vice-versa. By Lemma 3, $\pi_{td^{-1},c}(g) \sim \pi_{s,c}(g)$ (resp. $\pi_{td^{-1},c}(g) = \pi_{s,c}(g)$), and it follows that

$$\pi_{td^{-1},c}(g) = 1 \text{ for all } t \in T \iff \pi_{s,c}(g) = 1 \text{ for all } s \in S.$$

Thus, $x \sim y$ if, and only if, there is some $d \in B$ with $db = cd$ and $\pi_{s,c}(g) = 1$ for all $s \in S$.

Assume that $\pi_{s,c}(g) \neq 1$ for some $s \in S$. Then $se^\ell = \beta_i \in \text{supp}(g)$ for some $\ell \in \mathbb{Z}$ and so $\pi_{\beta_i,c}(g) \neq 1$. Thus, $\pi_{s,c}(g) = 1$ for all $s \in S$ and, only if, $\pi_{s,c}(g) = 1$ for all $s \in \text{supp}(g)$.

(ii) Now, suppose that $\pi_{t,b}(f) \neq 1$ for some $t \in \text{supp}(f)$ and let $x$ and $y$ be conjugate. By Proposition 1, there is some $d \in B$ such that $db = cd$ and $\pi_{t,b}(f)$ is conjugate (resp. equal) to $\pi_{t,b}^{(d)}(g)$. For this $d$ we have $\pi_{t,b}^{(d)}(g) \neq 1$.

In particular, there is some $l \in \mathbb{Z}$ with $g(tb^ld^{-1}) \neq 1$ and so $tb^ld^{-1} = \beta_i \in \text{supp}(g)$ for some $i \in \{1, \ldots , m\}$. Hence, $d \in \{\beta_1^{-1}tb^1, \ldots , \beta_m^{-1}tb^1\} \subseteq \mathbb{Z}$.

We can assume $l = 0$ because, if $d = eb^l$, then $db = cd$ if, and only if, $eb = ce$ and, by Lemma 3, for every $t' \in \tilde{T}$ we have $\pi_{t',b}^{(d)}(g) = \pi_{t',b}^{(e)}(g)$ (resp. $\pi_{t',b}^{(d)}(g) \sim \pi_{t',b}^{(e)}(g)$). Thus, for some $d \in \{\beta_1^{-1}t, \ldots , \beta_m^{-1}t\}$ with $db = cd$ we have $\pi_{t',b}(f) = \pi_{t',b}^{(d)}(g)$ (resp. $\pi_{t',b}(f) \sim \pi_{t',b}^{(d)}(g)$) for all $t' \in \tilde{T}$.

The converse implication follows immediately from (3) and Proposition 1.

\[\square\]

**Theorem 2** Let $A$ and $B$ be finitely generated groups. We have

- $\text{CP}(A \times B) \in \text{TC}^0(\text{CP}(A), \text{CP}(B), \text{PP}(B))$,
- $\text{CP}(A \times B) \in \text{TC}^0(\text{CP}(A), \text{CP}(B), \text{CSMMP}(B))$ if $B$ is torsion-free,
- $\text{CP}(A \times B) \in \text{TC}^0(\text{CP}(A), \text{CP}(B), \text{CSGMP}(B))$ if $A$ is abelian.

**Proof** By Lemma 1, we may assume that the input is given as two pairs $(b, f)$ and $(c, g)$. As before we write $\text{supp}(f) = \{b_1, \ldots , b_n\}$ and $\text{supp}(g) = \{\beta_1, \ldots , \beta_m\}$.

By Lemma 2, we can assume that $\pi_{t',b}(f) = \pi_{t',b}^{(d)}(g)$, and $\pi_{s,c}(g)$ for $d \in \{\beta_1^{-1}t, \ldots , \beta_m^{-1}t\}$, $s \in \text{supp}(g)$, and $t, t' \in \tilde{T}$ are part of the input.

Now, let us describe an $\text{AC}^0$-circuit with oracle calls to the word and conjugacy problems of $A$ and $B$ which evaluates the criterion of Proposition 2. If $A$ is non-abelian and $B$ has torsion it also uses oracle gates for $\text{PP}(B)$.

First, one call to the conjugacy problem in $B$ is performed for determining whether $b$ and $c$ are conjugate. Then, in the next stage the two cases can be distinguished by at most $|\tilde{T}|$ calls to the word problem of $A$. Now, case (i) is simply a conjunction of calls to the word problem of $A$. Case (ii) is a disjunction over all possible values for
For each value of \( d \) it is again a conjunction of one call to the word problem of \( B \) and several calls to the word problem of \( A \) (case (ii a)) or the conjugacy problem in \( A \) (case (ii b)). Cases (ii a) and (ii b) can be distinguished using the power problem in \( B \). If \( B \) is torsion-free, then the word problem suffices because in this case \( \text{ord}(b) < \infty \) if, and only if, \( b =_B 1 \). If \( A \) is abelian, then the conditions (ii a) and (ii b) are equivalent, i.e., we are always in case (ii a) and there is no need for a check whether \( \text{ord}(b) < \infty \). To be more explicit, we can write down the circuit as a formula (for the general non-abelian case):

\[
(b, f) \sim (c, g) \iff b \sim_B c \land ((i) \lor (iii)).
\]

Moreover, we have

\[
(i) \iff \bigwedge_{i=1}^{n} \pi_{b_i, b}(f) =_A 1 \land \bigwedge_{j=1}^{m} \pi_{\beta_j, c}(g) =_A 1,
\]

\[
(iii) \iff \bigwedge_{i=1}^{n} \left( \pi_{b_i, b}(f) \neq_A 1 \land \bigvee_{k=1}^{m} \left( \beta_k^{-1}b_1b =_B c\beta_k^{-1}b_i \land \left( \text{ord}(b) = \infty \land \bigwedge_{t \in \tilde{T}} \pi_{t, b}(f) =_A \pi_{t, b}^{(\beta_k^{-1}b_i)}(g) \right) \right) \right) \lor \left( \text{ord}(b) < \infty \land \bigwedge_{t \in \tilde{T}} \pi_{t, b}(f) \sim_A \pi_{t, b}^{(\beta_k^{-1}b_i)}(g) \right).\]

\[\square\]

**Corollary 2** Let \( A \) and \( B \) be finitely generated groups and \( d \geq 1 \). Then

- \( \text{CP}(A \rtimes_d B) \in \text{TC}^0(\text{CP}(A), \text{CP}(B), \text{PP}(B)) \),
- \( \text{CP}(A \rtimes_d B) \in \text{TC}^0(\text{CP}(A), \text{CP}(B), \text{CSMMP}(B)) \) if \( B \) is torsion-free.

**Proof** Immediate consequence of Theorem 2 by induction. \[\square\]

Notice that \( A \rtimes_d B \) is not abelian (for non-trivial \( A \) and \( B \)). Hence, it does not follow that \( \text{CP}(A \rtimes_d B) \in \text{TC}^0(\text{CP}(A), \text{CP}(B), \text{CSGMP}(B)) \) even if \( A \) is abelian.

The following quite trivial observation turns out to be very useful.

**Lemma 4** Let \( G \) be finitely generated by \( \Sigma \) and let the orders of its torsion elements be uniformly bounded. Suppose there is a polynomial \( p(n) \) such that for every \( w \in \Sigma^* \) which is non-torsion, the inequality \( k \leq p(||w^k||) \) is satisfied, where \( ||w^k|| \) denotes the geodesic length of the group element \( w^k \). Then \( \text{PP}(G) \in \text{AC}^0(\text{WP}(G)) \).

**Proof** Let \( D \) be a bound on the orders of torsion elements of \( G \). For input words \( v, w \in \Sigma^* \) for the power problem, simply test whether \( v^k =_G w \) for all \( k \) with \( -p(|w|) \leq k \leq \max\{p(|w|), D\} \) in parallel using the word problem of \( G \). \[\square\]
The second condition of Lemma 4 means that there is a uniform polynomial bound on the distortion of infinite cyclic subgroups. This is satisfied by abelian groups (with \( p \) being linear). Since the conjugacy problem in abelian groups is in \( \mathsf{TC}^0 \) (as it is the word problem), we obtain the following corollary of Theorem 2.

**Corollary 3** Let \( A \) and \( B \) be finitely generated abelian groups and \( d \geq 1 \). Then \( \text{CP}(A \wr^{d} B) \in \mathsf{TC}^0 \).

**The role of the power problem** The following result is a complexity analog of the “only if” part of [16, Thm. B], which only considers decidability. Note that for pure decidability, it does not matter if we consider \( \text{CSMP}(B) \), \( \text{CMMP}(B) \) or \( \text{PP}(B) \) since they can all be reduced to each other.

**Theorem 3** Let \( A \) be finitely generated and non-trivial. Then \( \text{CSGMP}(B) \leq_{m}^{\mathsf{AC}^0} \text{CP}(A \wr B) \). If, moreover, \( A \) is non-abelian, then \( \text{CMMP}(B) \leq_{m}^{\mathsf{AC}^0} \text{CP}(A \wr B) \).

Notice that Theorem 3 shows that in the cases that \( A \) is abelian or \( B \) torsion-free Theorem 2 is the best possible result one could expect. However, it is totally unclear how \( \text{PP}(B) \) could be reduced to \( \text{CP}(A \wr B) \) in \( \mathsf{TC}^0 \) (even if the answer to the power problem is guaranteed to have polynomial size). Thus, there remains the possibility that Theorem 2 could be strengthened in the general case.

**Proof** The first statement is simply due to the observation that the construction in [16, Thm. B] can be computed in \( \mathsf{AC}^0 \). We repeat the argument here: fix some \( a \in \Sigma_{A}^* \) with \( a \neq A \). For \( b, c \in \Sigma_{B}^* \), the function \( f \in A(B) \) is defined by

\[
\begin{align*}
 f(1) &= a, & f(c) &= a^{-1}, & f(\beta) &= 1 \quad \text{for } \beta \in B \setminus \{1, c\}.
\end{align*}
\]

Then by Proposition 2 (ii), \((b, 1) \sim (b, f)\) if, and only if, \( \pi_{1,b}(f) = \pi_{c,b}(f) = 1 \), which is the case if, and only if, \( c \in \langle b \rangle \). Obviously, the tuples \((b, 1)\) and \((b, f)\) can be computed in \( \mathsf{AC}^0 \).

Now, let \( A \) be non-abelian. In particular, there are elements \( a_1, a_2 \in A \) with \( a_1 a_2 \neq_A a_2 a_1 \). For \( b, c \in \Sigma_{B}^* \), we define two functions \( f, g \in A(B) \) by

\[
\begin{align*}
 f(1) &= a_1 a_2, & f(\beta) &= 1 \quad \text{for } \beta \in B \setminus \{1\}, \\
 g(1) &= a_1, g(c) &= a_2, & g(\beta) &= 1 \quad \text{for } \beta \in B \setminus \{1, c\}.
\end{align*}
\]

Note that in the case \( c = 1 \), technically \( g \) is not well-defined; however, the group element \( a_1 a_2^c \) is a valid input which can be written down (and in this case \( g(1) = g(c) = a_1 a_2 \)), so the reduction is still defined.

We have \( \pi_{1,b}(f) = a_1 a_2 \) and \( \pi_{t,b}(f) = 1 \) for \( t \notin \langle b \rangle \). For \( g \), according to Proposition 2 (iii), we have to consider \( \pi^{(1)}_{1,b}(g) \) and \( \pi^{(c)}_{1,b}(g) \). If \( b \) has finite order, then \( \pi^{(1)}_{1,b}(g) \) and \( \pi^{(c)}_{1,b}(g) \) are both one of \( a_1 a_2 \) or \( a_2 a_1 \) (which are conjugate) if, and
only if, \( c \in \langle b \rangle = G \{ b \}^* \) (because \( b \) has finite order) – otherwise \( \pi_{1,b}^{(1)}(g) = a_1 \) and \( \pi_{1,b}^{(c)}(g) = a_2 \). On the other hand if \( b \) has infinite order, we have

\[
\pi_{1,b}^{(1)}(g) = a_1a_2 \text{ if } c = B b^k \text{ with } k \geq 0, \quad a_2a_1 \text{ if } c = B b^k \text{ with } k < 0, \quad a_1 \text{ otherwise},
\]

\[
\pi_{1,b}^{(c)}(g) = a_1a_2 \text{ if } c = B b^k \text{ with } k \geq 0, \quad a_2a_1 \text{ if } c = B b^k \text{ with } k < 0, \quad a_2 \text{ otherwise}.
\]

Thus, \( \pi_{1,b}^{(d)}(g) = \pi_{1,b}^{(f)}(f) \) for some \( d \in \{1, c\} \) if, and only if, \( c = B b^k \) with \( k \geq 0 \). Therefore, by Proposition 2, \( (b, f) \sim (b, g) \) if, and only if, \( c \in G \{ b \}^* \).

5 Conjugacy and Power Problem in Left-Iterated Wreath Products

In order to solve the conjugacy problem in left-iterated wreath products, we also need to solve the power problem in wreath products. In general, we do not know whether the power problem in a wreath product is in \( \text{TC}^0 \) given that the power problem of the factors is in \( \text{TC}^0 \). The issue is that when dealing with torsion it might be necessary to compute greatest common divisors – which is not known to be in \( \text{TC}^0 \). By restricting torsion elements to have only smooth orders, we circumvent this issue. Recall that a number is called \( \beta \)-smooth for some \( \beta \in \mathbb{N} \) if it only contains prime factors less than or equal to \( \beta \).

**Lemma 5** Let \( \beta \in \mathbb{N} \). Suppose the orders of all torsion elements in \( A \) and \( B \) are \( \beta \)-smooth. Then the orders of all torsion elements in \( A \wr B \) are \( \beta \)-smooth.

**Proof** First consider a torsion element \( f \in A^{(B)} \). Since \( f(b) \) is a torsion element for all \( b \in B \), we have \( f^\ell = 1 \) for some \( \beta \)-smooth \( \ell \). Next consider some arbitrary torsion element \( (b, f) \in A \wr B \). Then \( b \) is torsion as well (since \( (b, f) \) projects to \( b \) in \( B \)), and thus \( b^m = 1 \) for some \( \beta \)-smooth \( m \). Consequently, \( (b, f)^m = A^{(B)} \). Hence, \( (b, f)^{\ell m} = 1 \) and so the order of \( (b, f) \) is \( \beta \)-smooth. \( \square \)

Note that we are not aware of any finitely generated group with word problem in \( \text{TC}^0 \) and torsion elements whose orders are not \( \beta \)-smooth for any \( \beta \). On the other hand, there are recursively presented such groups; for instance, take the infinite direct sum of cyclic groups of arbitrary order.

We say \( (t_1, \ldots, t_m) \) is a list of \( \langle b \rangle \)-coset representatives if the \( t_i \) represent pairwise distinct \( \langle b \rangle \)-cosets.

**Lemma 6** The following problems are in \( \text{TC}^0(\text{PP}(B)) \):

(i) **Input:** a function \( f = ((b_1, a_1), \ldots, (b_n, a_n)) \in A^{(B)} \) and \( b \in \Sigma_B^* \). **Output:** a list of \( \langle b \rangle \)-coset representatives \( (t_1, \ldots, t_m) \) such that \( \text{supp}(f) \subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle \).

(ii) **Input:** a function \( f = ((b_1, a_1), \ldots, (b_n, a_n)) \in A^{(B)} \), \( b \in \Sigma_B^* \) and a list of \( \langle b \rangle \)-coset representatives \( (t_1, \ldots, t_m) \). **Decide whether** \( \text{supp}(f) \subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle \).
(iii) Input: a function \( f = ((b_1, a_1), \ldots, (b_n, a_n)) \in A(B) \), \( b \in \Sigma_B^* \) and a list of \( (b) \)-coset representatives \((t_1, \ldots, t_m)\). Output: for each \( 1 \leq i \leq m \) a list \(((e_{i,1}, a_{i,1}), \ldots, (e_{i,n_i}, a_{i,n_i}))\) with \( e_{i,j} \in \mathbb{Z} \) (encoded in binary), \( e_{i,1} < \cdots < e_{i,n_i} \) and \( a_{i,j} \in \Sigma_A^* \) such that
\[
\text{supp}(f) = \{ t_i b^{e_{i,j}} \mid 1 \leq i \leq m, 1 \leq j \leq n_i \} \quad \text{and} \quad f(t_i b^{e_{i,j}}) = a_{i,j}.
\]

We assume that all the words in the output of the circuit of Lemma 6 are encoded with the same number of bits (the number of bits is a fixed polynomial in the number of input bits depending only on the group \( B \)).

**Proof** (i) The first layer of the circuit decides for all \( i, j \leq n \) in parallel whether \( b_i^{-1} b_j \in \langle b \rangle \) using oracle gates for the power problem in \( B \). In the next layer, an element \( b_i \) is included in the list of representatives \((t_1, \ldots, t_m)\) if, and only if, there is no \( j < i \) with \( b_i^{-1} b_j \in \langle b \rangle \).

(ii) One simply needs to check whether for all \( j \in \{1, \ldots, n\} \) there is some \( i \in \{1, \ldots, m\} \) such that \( t_i^{-1} b_j \in \langle b \rangle \) using the power problem of \( B \).

(iii) For all \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, n\} \) one checks whether \( t_i^{-1} b_j \in \langle b \rangle \) and, if so, computes the respective exponent \( e_{i,j} \) such that \( t_i^{-1} b_j = b^{e_{i,j}} \). For all \( i \) and \( j \) this can be done in parallel by using oracle gates to the power problem for \( B \).

The next step is to sort for all \( i \) in parallel the tuples \((b_j, a_j)\) with \( t_i^{-1} b_j \in \langle b \rangle \) according to their exponent \( e_{i,j} \). This can be done in \( \text{TC}^0 \) as described in Example 2. This yields the output lists. \( \Box \)

For the proof of Theorem 4, we need some more notation: for \( k > 0, b \in B \), and \( f \in A(B) \), we define \( f^{(b,k)} \) by \((b, f)^k = (b^k, f^{(b,k)})\). Then we have
\[
f^{(b,k)}(c) = (f^{b^{-1}} \cdots f^b f)(c) \quad \text{for } c \in B. \tag{4}
\]

**Lemma 7** Let \( e_1, \ldots, e_n \in \mathbb{Z} \) with \( e_1 < \cdots < e_n \) and \( a_1, \ldots, a_n \in A \) and \( b, t \in B \). Furthermore, let \( f(t b^{e_i}) = a_i \) for \( i = 1, \ldots, n \) and \( f(c) = 1 \) for all other \( c \in B \). Then, for \( 0 < k \leq \text{ord}(b) \), we have
\[
f^{(b,k)}(t b^\ell) = a_i \cdots a_{j-1}
\]
for \( 1 \leq i \leq j \leq n + 1 \) such that \( \max \{ e_{j-1}, e_i + k \} \leq \ell \leq \min \{ e_i + k - 1, e_j - 1 \} \). Here, we set \( e_0 = -\infty \) and \( e_{n+1} = \infty \). Note that \( a_i \cdots a_{j-1} \) is possibly the empty product.

**Proof**
\[
f^{(b,k)}(t b^\ell) = (f^{b^{-1}} \cdots f^b f)(t b^\ell)
\]
\[
= f(t b^{\ell-(k-1)} \cdots f(t b^{\ell-1}) f(t b^{\ell})
\]
\[
= \prod_{v=i}^{j-1} f(t b^{e_v}) \quad \text{(because all other } f(c) \text{ are trivial)}
\]
for \( i = \min \{ v \mid e_v \geq \ell - (k - 1) \} \) and \( j = \max \{ v \mid e_v \leq \ell \} + 1 \). Thus, \( e_{i-1} < \ell - (k - 1) \leq e_i \) and likewise \( e_{j-1} \leq \ell \leq e_j - 1 \).

Lemma 8 The following problem is in \( \text{TC}^0(\text{PP}(B)) \):

Input: a function \( f = ((b_1, a_1), \ldots, (b_n, a_n)) \in A^{(B)}, b, t \in \Sigma_B^* \) such that \( \text{supp}(f) \subseteq t(b) \) and \( k, \ell \in \mathbb{Z} \) (in binary).

Compute \( f^{(b, k)}(t b^\ell) \in \Sigma_A^* \).

Proof By Lemma 6, we can compute a representation \( ((e_1, a_1), \ldots, (e_n, a_n)) \) with \( e_1 < \cdots < e_n \) of \( f \) such that \( f(t b^e_i) = a_i \) for all \( i \) and \( f(t c) = 1 \), otherwise.

By Lemma 7, \( f^{(b, k)}(t b^\ell) \) is of the form \( a_i \cdots a_j \) for appropriate \( i \) and \( j \). The indices \( i \) and \( j \) can be found by evaluating the inequality \( \max\{e_j - 1, e_i - 1 + k\} \leq e_v \leq \min\{e_i + k - 1, e_j - 1\} \) — that is a simple Boolean combination of comparisons of integers (integers can be compared in \( \text{TC}^0 \) e.g. by subtracting them and then checking the sign).

Theorem 4 Let \( \beta \in \mathbb{N} \) and suppose the order of every torsion element in \( A \) is \( \beta \)-smooth. Then we have \( \text{PP}(A \wr B) \in \text{TC}^0(\text{PP}(A), \text{PP}(B)) \).

Roughly the proof of Theorem 4 works as follows: on input \( (b, f) \) and \( (c, g) \) first apply the power problem in \( B \) to \( b \) and \( c \). If there is no solution, then there is also no solution for \( (b, f) \) and \( (c, g) \). Otherwise, the smallest \( k \geq 0 \) with \( b^k = c \) can be computed. If \( b \) has infinite order, it remains to check whether \( (b, f)^k = (c, g) \).

Since \( k \) might be too large, this cannot be done by simply applying the word problem. Nevertheless, we only need to establish equality of functions in \( A^{(b)} \). We show that it suffices to check equality on certain (polynomially many) “test points”. In the case that \( b \) has finite order \( K \), we know that if there is a solution to the power problem it must be in \( k + K \mathbb{Z} \). Now, similar techniques as in the infinite order case can be applied to find the solution.

Proof By Lemma 1, we may assume that the input is given as two pairs \( (b, f) \) and \( (c, g) \). We aim to compute some \( k \) such that \( (b, f)^k = (c, g) \) if there exists such \( k \). We describe a circuit in several stages. It will use oracle gates for \( \text{PP}(A), \text{PP}(B) \) as well as sorting in \( \text{TC}^0 \) and integer arithmetic. As in the previous proofs it is straightforward to assign gate numbers such that on input of two gate numbers it can be decided in linear time whether there is a wire connecting them. As a first step, the power problem in \( B \) is applied to determine whether there is some \( k \) with \( b^k = c \). If the answer is “no”, then the over all answer is “no”. Otherwise, we distinguish the two cases that \( b \) is of finite order and that \( b \) is of infinite order (which can be distinguished by using the power problem).

First assume that \( b \) has infinite order. Let \( k \) be the answer for the power problem in \( b \) and \( c \), i.e., \( k \) is the unique integer with \( b^k = c \). Now, it remains to check whether \( (b, f)^k = (c, g) \). This cannot be done by simply applying the word prob-
lem because \( k \) might be exponentially large (we know that it is bounded by some exponential function because it can be computed in \( \text{TC}^0 \)). Without loss of generality, we may assume that \( k > 0 \). Indeed, if \( k < 0 \), we can replace \((c, g)\) by \((c, g)^{-1}\) and, if \( k = 0 \), we only need to check whether \( g = 0 \) in order to establish \((b, f)^k = (c, g)\).

Since \( k > 0 \), by (4), we have \((b, f)^k = (b^k, f(b^k))\) where \( f(b^k) = f^{b^k-1} \cdots f^bf\) – thus, we have to compare \( f(b^k) \) and \( g \) for equality in \( A(B) \). By Lemma 6(i), a list of \((b)-\)coset representatives \((t_1, \ldots, t_m)\) can be computed in \( \text{TC}^0(\text{PP}(B)) \) such that \( \text{supp}(f) \subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle \). Because of (4), also \( \text{supp}(f(b^k)) \subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle \). Thus, if \( \text{supp}(g) \not\subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle \) (which can be checked in \( \text{TC}^0(\text{PP}(B)) \) by Lemma 6(ii)), then \((b, f)^k \neq g\).

Because \((b, f)^k = g\) if, and only if, they agree on every \( b\)-coset, we can assume that \( \text{supp}(f) \subseteq \langle t \rangle b^t \) (for some \( t \in B \) – the general case is then simply a conjunction over all coset representatives. By Lemma 6(iii), we can compute representations \(((e_1, a_1), \ldots, (e_n, a_n))\) with \( e_1 < \cdots < e_n \) (resp. \(((e'_1, a'_1), \ldots, (e'_n, a'_n))\)) with \( e' _1 < \cdots < e'_n \) of \( f \) (resp. \( g \)) such that \( f(tb^{e'_i}) = a_i \) for all \( i \) and \( f(c) = 1 \), otherwise (and likewise for \( g \)).

Lemma 7 allows us to compare \((b, f)^k\) and \( g \) for equality. We do this in two steps: first we check for all \( tb^{e'_v} \in \text{supp}(g) \) (i.e., for \( v = 1, \ldots, n' \)) whether \( f((b, f)^k)(tb^{e'_v}) = A_g(tb^{e'_v}) \). We can find \((b, f)^k)(tb^{e'_v})\) by Lemma 8. Now it remains to check whether \((b, f)^k)(tb^{e'_v}) = A_a_i = g(tb^{e'_v})\) using oracle gates for the word problem for \( A \). For all \( v \) this can be done in parallel.

At this point, we know that \((b, f)^k\) and \( g \) agree on \( \text{supp}(g) \). The second step is to check that \( \text{supp}(f(b, f)^k) \subseteq \text{supp}(g) \). Since \( \text{supp}(f(b, f)^k) \) might be exponentially large, we have to use a different strategy than a point-wise check. Instead, we do the following for all \( 1 \leq i \leq j \leq n + 1 \) in parallel:

- Check whether \( a_i \cdots a_{j-1} = A_1 \) (can be checked with oracle gates for \( \text{WP}(A) \)).

  If not, then there are two possibilities:

  - If \( \min \{e_i + k - 1, e_j - 1\} - \max \{e_{j-1}, e_{i-1} + k\} > n \), then by Lemma 7 \( |\text{supp}(f(b, f)^k)| > n \geq |\text{supp}(g)| \); thus, we know that \((b, f)^k \neq g\).

  - Otherwise, test for all \( \ell \) satisfying \( \max \{e_{j-1}, e_{i-1} + k\} \leq \ell \leq \min \{e_i + k - 1, e_j - 1\} \) whether there is some \( v \) with \( \ell = e'_v \) (since it is a simple disjunction over equality tests of integers, it can be done in \( \text{TC}^0 \)). If there is some \( \ell \) which is not equal to any \( e'_v \), then \( \text{supp}(f(b, f)^k) \not\subseteq \text{supp}(g) \).

If none of the above cases refutes that \((b, f)^k = g\), then we know that indeed \((b, f)^k = g\).

Now, let \( b \) have finite order \( K \) and let \( 0 \leq k < K \) with \( b^K = c \) i.e., \( k \) is the solution to the power problem for \( b \) and \( c \). As remarked before, also \( K \) can be computed by using the the oracle for power problem of \( B \). We have \((b, f)^K \in A(B)\). Moreover, \( b^{k'} = c \) for \( k' \in \mathbb{Z} \) if, and only if, \( k' \equiv k \mod K \). Thus, there is a solution to the power problem if, and only if, there is some \( \ell \in \mathbb{Z} \) with \(((b, f)^K)\ell(b, f)^k = (c, g)\). In other words it remains to solve the power problem for \((b, f)^K\) and \((c, g)(b, f)^{−k} \).
We can simplify the latter element as follows

\[(c, g)(b, f)^{-k} = (c, g)\left( (b, f)^k \right)^{-1} = (c, g)\left( b^k, f(b,k) \right)^{-1} = (c, g)\left( b^{-k}, \left( f(b,k) b^{-k} \right)^{-1} \right) = \left( c b^{-k}, g b^{-k} \cdot \left( f(b,k) b^{-k} \right)^{-1} \right) = \left( 1, g \cdot \left( f(b,k)^{-1} b^{-k} \right) \right)\]

and we see that we have to solve the power problem for \(f(b,K)\) and \(g b^{-k} \cdot \left( f(b,k) b^{-k} \right)^{-1}\) in \(A^{(B)}\). Note that since the numbers \(k\) and \(K\) might be exponential in the input size, these group elements cannot be written down completely inside the polynomial size circuit.

We start as in the infinite order case: by Lemma 6(i), a list of \(\langle b \rangle\)-coset representatives \((t_1, \ldots, t_m)\) with \(\text{supp}(f) \subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle\) can be computed in \(\text{TC}^0\).

Because of (4), also \(\text{supp}(f(b,K)) \subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle\) – thus, again, if \(\text{supp}(g) \not\subseteq \{t_1, \ldots, t_m\} \cdot \langle b \rangle\) (which can be checked in \(\text{TC}^0(\text{PP}(B))\) by Lemma 6(ii)), then we already know that \((b, f)^k f(b,k) \neq (c, g)\) for any \(\ell\).

In the following, we assume again that \(\text{supp}(f)\), \(\text{supp}(g) \subseteq t \langle b \rangle\) for some \(t \in B\) – the set of solutions to the general case is the intersection over the solution sets for all coset representatives. In the end we will show how to compute this intersection. By Lemma 6(iii)6, we can compute representations \((e_1, a_1), \ldots, (e_n, a_n)\) with \(e_1 < \cdots < e_n\) (resp. \((e_1', a_1'), \ldots, (e_n', a_n')\)) with \(e_1 < \cdots < e_n\) of \(f\) (resp. \(g\)) such that \(f(t b^{e_i}) = a_i\) and \(f(c) = 1\), otherwise (and likewise for \(g\)).

We have to solve the power problem for \(f(b,K)(t b^\ell)\) and \(g(t b^{\ell+k}) \cdot f(b,k)(t b^{\ell+k})^{-1}\) for all \(\ell\). Since again there might be too many points in the support of \(f(b,K)\), we have to restrict to certain test points.

By Lemma 7, for each \(\langle b \rangle\)-coset intersecting \(\text{supp}(f)\) there are lists \(\gamma_0, \ldots, \gamma_\nu \in \mathbb{Z}\) and \(\alpha_1, \ldots, \alpha_\nu \in \Sigma_A^*\) with \(f(b,K)(t b^\ell) = \alpha_i\) for all \(\gamma_{i-1} < \ell \leq \gamma_i\) and \(\gamma_0 + K = \gamma_\nu\) (with \(v \leq 2n + 1\)). The numbers \(\gamma_i\) can be computed in \(\text{TC}^0\) like in Lemma 8. Moreover, we can compute similar lists \(\gamma'_0, \ldots, \gamma'_{\nu'} \in \mathbb{Z}, \alpha'_1, \ldots, \alpha'_{\nu'} \in \Sigma_A^*\) for \(f(b,k)\) and \(\gamma''_0, \ldots, \gamma''_{\nu''} \in \mathbb{Z}, \alpha''_1, \ldots, \alpha''_{\nu''} \in \Sigma_A^*\). Now, it suffices to solve the power problem for \(f(b,K)(t b^\ell) = \alpha_i \ell\) (where \(i \ell\) is such that \(\gamma_{i-1} < \ell \leq \gamma_i\)) and \(g(t b^\ell) \cdot f(b,k)(t b^\ell)^{-1} b^{-k}\) for all \(\ell \in \{\gamma_0, \ldots, \gamma_\nu, \gamma'_0 - k, \ldots, \gamma'_{\nu'} - k, \gamma''_0 - k, \ldots, \gamma''_{\nu''} - k\} =: \Gamma\).

This is because for increasing \(\ell\), the values \(f(b,K)(t b^\ell), g(t b^\ell) b^{-k}\), and \(f(b,K)(t b^\ell) b^{-k}\) only change at these points. The functions can be evaluated in \(\text{TC}^0(\text{PP}(B))\) by Lemma 8, then oracle gates for \(\text{PP}(B)\) are used.
For $\ell \in \Gamma$, let $K_\ell$ denote the order of $\alpha_i^{\ell}$ and let $k_\ell \in \mathbb{Z}$ such that $\alpha_i^{k_\ell} = (g(tb^\ell) \cdot f^{(b,k)}(tb^\ell)^{-1})^{b^{-k}}$ (i.e., the solution to the power problem). We obtain a system of congruences

$$x \equiv k_\ell \mod K_\ell$$

(here congruent modulo $\infty$ means equality). Since the $K_\ell$ are all $\beta$-smooth, they can be factored in $\mathsf{TC}^0$ and a solution (if there is one) of this system can be determined in $\mathsf{TC}^0$ (see e.g. [29, Lem. 27]) with the help of Hesse’s division circuit [7, 8] using the Chinese remainder theorem.

We do this also for all coset representatives in parallel. In the end, we either see that $g^{b^{-k}}((f^{(b,k)})^{b^{-k}})^{-1}$ is not a power of $f^{(b,K)}$, or we obtain a list of solutions $x_1, \ldots, x_m \in \mathbb{Z}$, which give rise to a system of congruences which can be solved like in the preceding paragraph.

**Corollary 4** Let $A$ and $B$ be finitely generated abelian groups or solvable Baumslag-Solitar groups (i.e., $\text{BS}_{1,q}$ for some $q \in \mathbb{Z} \setminus \{0\}$) and let $d \geq 1$. The conjugacy problem of $A^d \wr B$ is in $\mathsf{TC}^0$. Also, the conjugacy problem of free solvable groups is in $\mathsf{TC}^0$.

**Proof** As before, the power problem of abelian groups is in $\mathsf{TC}^0$ because of Lemma 4 and the conjugacy problem is trivially in $\mathsf{TC}^0$. Also the order of torsion elements is bounded by some constant. For solvable Baumslag-Solitar groups the power problem is in $\mathsf{TC}^0$ by Example 3 and the conjugacy problem is in $\mathsf{TC}^0$ by [4]. Moreover, Baumslag-Solitar groups are torsion-free.

By repeated application of Theorem 2, Lemma 5, and Theorem 4, we obtain the first statement of Corollary 4. The second statement follows since the Magnus embedding preserves conjugacy [22] (that means two elements are conjugate in the free solvable group if, and only if, their images under the Magnus embedding are conjugate).

**Remark 3** In [21] it is shown that also nilpotent groups have power problem and conjugacy problem in $\mathsf{TC}^0$ and that the orders of torsion elements are uniformly bounded. Thus, also iterated wreath products of nilpotent groups have conjugacy problem in $\mathsf{TC}^0$. For more details see [21].

**6 Conclusion and Open Problem**

As already discussed in Question 1, an important open problem is the dependency of the depth of the circuits for the word problem on the solvability degree.

We have seen how to solve the conjugacy problem in a wreath product in $\mathsf{TC}^0$ with oracle calls to the conjugacy problems of both factors and the power problem (resp. cyclic submonoid/subgroup membership problem) in the second factor. However, we do not have a reduction from the power problem in the second factor to the conjugacy problem in the wreath product: even if $A$ is non-abelian, we only know that the cyclic
submonoid membership problem is necessary to solve the conjugacy problem in the wreath product.

**Question 2** Is \( CP(A \wr B) \in TC^0(CP(A), CP(B), CSMMP(B)) \) in general?

For iterated wreath products we needed the power problem to be in \( TC^0 \) in order to show that the conjugacy problem is in \( TC^0 \). One reason was that we only could reduce the power problem in the wreath product to the power problems in the factors. However, we have seen that in torsion-free groups, we do not need the power problem to solve conjugacy, as the cyclic submonoid membership problem is sufficient. Therefore, it would be interesting to reduce the cyclic submonoid membership problem in a wreath product to the same problem in its factors.

**Question 3** Is \( CSMMP(A \wr B) \in TC^0(CSMMP(A), CSMMP(B)) \) or similarly is \( CSGMP(A \wr B) \in TC^0(CSGMP(A), CSGMP(B)) \)?

In [6], Gul, Sohrabi, and Ushakov generalized Matthews’ result by considering the relation between the conjugacy problem in \( F/N \) and the power problem in \( F/N' \), where \( F \) is a free group with a normal subgroup \( N \) and \( N' \) is its derived subgroup. They show that \( CP(F/N') \) is polynomial-time-Turing-reducible to \( CSMP(F/N) \) and \( CSMP(F/N) \) is Turing-reducible to \( CP(F/N') \) (no complexity bound). Moreover, they establish that \( WP(F/N') \) is polynomial-time-Turing-reducible to \( WP(F/N) \).

**Question 4** What are the precise relations in terms of complexity between \( CP(F/N') \) and \( CSGMP(F/N) \) (resp. \( WP(F/N') \) and \( WP(F/N) \))? 

**References**

1. Barrington, D.A.M., Immerman, N., Straubing, H.: On uniformity within NC\(^1\). J. Comput. Syst. Sci. 41(3), 274–306 (1990). https://doi.org/10.1016/0022-0000(90)90022-D
2. Craven, M.J., Jimbo, H.C.: Evolutionary algorithm solution of the multiple conjugacy search problem in groups, and its applications to cryptography. Groups Complex. Cryptol. 4, 135–165 (2012). https://doi.org/10.1515/gcc-2016-0012
3. Dehn, M.: Über unendliche diskontinuierliche Gruppen. Math. Ann. 71(1), 116–144 (1911). https://doi.org/10.1007/BF01456932
4. Diekert, V., Myasnikov, A.G., Weiß, A.: Conjugacy in baumslag’s group, generic case complexity, and division in power circuits. In: Latin American Theoretical Informatics Symposium, pp. 1–12 (2014)
5. Grigoriev, D., Shpilrain, V.: Authentication from matrix conjugation. Groups Complex. Cryptol. 1, 199–205 (2009)
6. Gul, F., Sohrabi, M., Ushakov, A.: Magnus embedding and algorithmic properties of groups \( F/N^{(d)} \). Trans. Amer. Math. Soc. 369(9), 6189–6206 (2017). https://doi.org/10.1090/tran/6880
7. Hesse, W.: Division is in uniform \( TC^0 \). In: Orejas, F., Spirakis, P.G., van Leeuwen, J. (eds.) ICALP 2001, Proceedings, Lecture Notes in Computer Science, vol. 2076, pp. 104–114. Springer (2001)
8. Hesse, W., Allender, E., Barrington, D.A.M.: Uniform constant-depth threshold circuits for division and iterated multiplication. JCSS 65, 695–716 (2002)
9. Kargapolov, M.I., Remeslennikov, V.N.: The conjugacy problem for free solvable groups. Algebra i Logika Sem. 5(6), 15–25 (1966)
10. Ko, K.H., Lee, S.J., Cheon, J.H., Han, J.W., Kang, J.S., Park, C.: New public-key cryptosystem using braid groups. In: Advances in cryptology—CRYPTO 2000 (Santa Barbara, CA), Lecture Notes in Computer Science, vol. 1880, pp. 166–183. Springer, Berlin (2000). https://doi.org/10.1007/3-540-44598-6_10

11. König, D., Lohrey, M.: Evaluation of circuits over nilpotent and polycyclic groups. Algorithmica (2017). https://doi.org/10.1007/s00453-017-0343-z

12. Krebs, A., Lange, K., Reifferscheid, S.: Characterizing $\text{TC}^0$ in terms of infinite groups. Theory Comput. Syst. 40(4), 303–325 (2007). https://doi.org/10.1007/s00224-006-1310-2

13. Lange, K., McKenzie, P.: On the complexity of free monoid morphisms. In: Chwa, K., Ibarra, O.H. (eds.) ISAAC 1998, Proceedings, Lecture Notes in Computer Science, vol. 1533, pp. 247–256. Springer (1998). https://doi.org/10.1007/3-540-49381-6_27

14. Maciel, A., Thérien, D.: Threshold circuits of small majority-depth. Inf. Comput. 146(1), 55–83 (1998). https://doi.org/10.1006/inco.1998.2732

15. Magnus, W.: On a theorem of Marshall Hall. Ann. of Math. (2) 40, 764–768 (1939)

16. Matthews, J.: The conjugacy problem in wreath products and free metabelian groups. Trans. Amer. Math. Soc. 121, 329–339 (1966)

17. Miller, C.F.I.: On group-theoretic decision problems and their classification, Ann. of Math. Studies, vol. 68. Princeton University Press, NJ (1971)

18. Myasnikov, A., Roman’kov, V., Ushakov, A., Vershik, A.: The word and geodesic problems in free solvable groups. Trans. Amer. Math. Soc. 362(9), 4655–4682 (2010). https://doi.org/10.1090/S0002-9947-10-04959-7

19. Myasnikov, A., Vassileva, S., Weiss, A.: The conjugacy problem in free solvable groups and wreath products of abelian groups is in $\text{TC}^0$. In: CSR 2017, pp. 217–231. Proceedings (2017). https://doi.org/10.1007/978-3-319-58747-9_20

20. Myasnikov, A., Vassileva, S., Weiss, A.: Log-Space Complexity of the Conjugacy Problem in Wreath Products, chap. 12, pp. 215–236. World Scientific (2017). https://doi.org/10.1142/9789813204058_0012

21. Myasnikov, A., Weiss, A.: $\text{TC}^0$ circuits for algorithmic problems in nilpotent groups. ArXiv e-prints (2017)

22. Remeslennikov, V., Sokolov, V.G.: Certain properties of the Magnus embedding. Algebra i logika 9(5), 566–578 (1970)

23. Robinson, D.: Parallel algorithms for group word problems. Ph.D. thesis, University of California, San Diego (1993)

24. Shpilrain, V., Zapata, G.: Combinatorial group theory and public key cryptography. Appl. Algebra Engng. Comm. Comput. 17, 291–302 (2006)

25. Vassileva, S.: Polynomial time conjugacy in wreath products and free solvable groups. Groups Complex. Cryptol. 3(1), 105–120 (2011). https://doi.org/10.1515/GCC.2011.005

26. Vollmer, H.: Introduction to Circuit Complexity. Springer, Berlin (1999)

27. Waack, S.: The parallel complexity of some constructions in combinatorial group theory, pp. 492–498. Springer-Verlag New York, Inc., New York (1990)

28. Wang, L., Wang, L., Cao, Z., Okamoto, E., Shao, J.: New constructions of public-key encryption schemes from conjugacy search problems. In: Information security and cryptology, Lecture Notes in Comput. Sci., vol. 6584, pp. 1–17. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-21518-6_1

29. Weiss, A.: A logspace solution to the word and conjugacy problem of generalized Baumslag-Solitar groups. In: Algebra and computer science, Contemporary Mathematics, vol. 677, pp. 185–212. American Mathematics Society, Providence (2016)