THE UNIVALENCE AXIOM FOR ELEGANT REEDY
PRESHEAVES

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Abstract. We show that Voevodsky's univalence axiom for intensional type
theory is valid in categories of simplicial presheaves on elegant Reedy cate-
gories. In addition to diagrams on inverse categories, as considered in previous
work of the author, this includes bisimplicial sets and $\Theta_n$-spaces.

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1. Introduction

Recently it has emerged (see e.g. [HS98, War08, AW09, AK11, vdBG12, Voe, LW13])
that just as extensional type theory admits a semantics in categories, intensional
type theory admits semantics in well-behaved model categories. This raises the
possibility of using intensional type theory as an “internal language” for model
categories and the $(\infty, 1)$-categories that they present.

One of the most significant innovations to come out of this perspective has been
Voevodsky’s univalence axiom, which identifies the identity types of a universe
type with certain types of equivalences. This suggests that the categorical ana-
logue of the univalence axiom should be the object classifiers of Rezk and Lurie
(see [Lur09, §6.1.6] and also [GK12]). Thus, we may hope to model type theory
with the univalence axiom in any $(\infty,1)$-category with object classifiers, and in
particular in any “$(\infty,1)$-topos”.

The main problem is that type theory is stricter than the categorical models.
The first model of the univalence axiom to overcome this difficulty was also due to
Voevodsky [Voe, KLV12], using the model category $\sSet$ of simplicial sets, which
presents the $(\infty,1)$-category of $\infty$-groupoids (the most basic $(\infty,1)$-topos).
In [Shu12], starting from Voevodsky’s model in sSet, I constructed a model of type theory satisfying univalence in the Reedy model category $sSet^I$, whenever $I$ is an inverse category. This paper will generalize that result to the Reedy model structure on $sSet^{C_{op}}$ whenever $C$ is an elegant Reedy category, as in [BR13]. Elegant Reedy categories include direct categories (the opposites of inverse categories), but also categories such as the simplex category $\Delta$ and Joyal’s categories $\Theta_n$.

The proof given in this paper does not depend on that of [Shu12], and is more similar in flavor to that of [KLV12]. In particular, it is purely model-category-theoretic; no knowledge of type theory is required. However, this new proof does not replace [Shu12], since it applies only to presheaves of simplicial sets, whereas that of [Shu12] applies to diagrams in any category which models type theory with univalence (such as the syntactic category of type theory itself) and also generalizes to oplax limits and gluing constructions.

Remark 1.1. Most aspects of type theory aside from univalence (e.g. $\Sigma$-types, $\Pi$-types, and identity types) are now known to admit models in all $(\infty, 1)$-toposes, and indeed in all locally presentable, locally cartesian closed $(\infty, 1)$-categories. By the coherence theorem of [LW13], it suffices to present such an $(\infty, 1)$-category by a “type-theoretic model category” in the sense of [Shu12], such as a right proper Cininski model category [Cis02,Cis06]. That this is always possible has been proven by Cisinski [Cis12] and by Gepner–Kock [GK12]. Moreover, if the $(\infty, 1)$-category is an $(\infty, 1)$-topos, then we can choose fibrations of fibrant objects representing its object classifiers, which will behave almost like univalent universes. What is missing is that such “universes” need not be strictly closed under the type-forming operations, i.e. the coercion from elements of the universe to types only respects these operations up to equivalence. It is this extra missing bit of strictness which we aim to provide here, in the special case of elegant Reedy presheaves.

We begin in §2 with some remarks on univalent universe objects, and on the methods of [KLV12] which we will adopt in this paper. We compare the constructions of such universes given in [KLV12] and in [Str], and give a third such construction which makes the relationship to $(\infty, 1)$-categorical object classifiers a bit clearer. In §3 we recall the definition of the (type-theoretically motivated) “universal spaces of equivalences” that appear in the univalence axiom, and prove a postponed lemma from §2. Then in §4 we summarize the requirements for universe objects in a model category to give rise to universes in the internal type theory, using the coherence theorem of [LW13]. Finally, in §5 we show that elegant Reedy presheaves satisfy all the accumulated requirements.

I am grateful to Peter Lumsdaine for many very helpful discussions, and for emphasizing the essential outline of the proof in [KLV12], as described in §2. I am also grateful to Bas Spitters for some helpful comments.

2. ON PROOFS OF UNIVALENCE

Naively, the construction of a univalent universe object can be broken down into four parts:

(1) Construct a particular small fibration $p: \tilde{U} \rightarrow U$.
(2) Prove that every small fibration is a (strict) pullback of $p$.
(3) Prove that $U$ is fibrant (so that it models a type in the internal type theory).
(4) Prove that the univalence axiom holds.
The proof in [Shu12] that univalence lifts to inverse diagrams follows this outline: we construct a Reedy fibration \( p \) which satisfies (2) and (3) almost by definition, and then (4) follows by a somewhat lengthy, but direct, analysis of exactly what the univalence axiom claims.

The proof of univalence for simplicial sets in [KLV12], by contrast, follows a slightly different route. We first construct a fibration \( \tilde{U} \rightarrow U \) which satisfies the following stronger version of (2):

\[
(2') \quad \text{Given the solid arrows in the following diagram, where } A \hookrightarrow B \text{ is a cofibration, } P \rightarrow B \text{ is a small fibration, and both squares of solid arrows are pullbacks:}
\]

\[
\begin{array}{ccc}
Q & \rightarrow & \tilde{U} \\
\downarrow & & \downarrow \\
A & \rightarrow & U \\
\end{array}
\]

there exist the dashed arrows rendering the diagram commutative and the third square also a pullback.

In a context (such as simplicial sets) where all objects are cofibrant, taking \( A = \emptyset \) in (2') yields (2).

Condition (2') can be rephrased in the following suggestive way: suppose for the sake of argument that there were a thing called \( \mathfrak{U} \) such that maps \( A \rightarrow \mathfrak{U} \) were precisely small fibrations over \( A \). Then the small fibration \( p: \tilde{U} \rightarrow U \) would be classified by a map \( U \rightarrow \mathfrak{U} \), and (2') asserts that this map is an acyclic fibration. (One can even make this precise by regarding \( \mathfrak{U} \) as a fibered category or groupoid.)

With (2') in hand, (3) and (4) can be reduced to statements not referring to \( U \) at all. For instance, suppose we can show:

\[
(3') \quad \text{If } i: A \hookrightarrow B \text{ is an acyclic cofibration and } P \rightarrow A \text{ a small fibration, then there exists a small fibration } Q \rightarrow B \text{ such that } P \cong i^*Q.
\]

Then given an acyclic cofibration \( i: A \hookrightarrow B \) and a map \( f: A \rightarrow U \), (3') gives us a fibration \( Q \) over \( B \) which pulls back to \( f^*\tilde{U} \) over \( A \), and by (2') we have \( g: B \rightarrow U \) with \( g^*\tilde{U} \cong Q \) and \( gi = f \). Thus, (3) follows. Intuitively, we are saying that since \( U \rightarrow \mathfrak{U} \) is an acyclic fibration, if \( \mathfrak{U} \) is fibrant then so is \( U \).

For (4), the univalence axiom for \( U \) asserts that \( PU \rightarrow Eq(\tilde{U}) \) is an equivalence, where \( PU \) denotes the path object of \( U \) and \( Eq(\tilde{U}) \) is the universal space of equivalences over \( U \times U \). (We will define this precisely in §3.) By the 2-out-of-3 property, this is equivalent to \( U \rightarrow Eq(\tilde{U}) \) being an equivalence, and therefore also to either projection \( Eq(\tilde{U}) \rightarrow U \) being an equivalence. Since these projections are always fibrations, we want them to be acyclic fibrations. But acyclic fibrations are characterized by a lifting property, and rephrasing this property for the second projection in terms of \( \mathfrak{U} \), we obtain:

\[
(4') \quad \text{Suppose given a cofibration } i: A \hookrightarrow B, \text{ a small fibration } E_2 \rightarrow B, \text{ and an equivalence } w: E_1 \Rightarrow E_2 \text{ of fibrations over } A, \text{ where } E_2 := i^*E_2. \text{ Then there exists a small fibration } E_1 \text{ over } B \text{ and an equivalence } \tilde{w}: E_1 \Rightarrow \tilde{E}_2 \text{ over } B, \text{ which yields } w \text{ when pulled back along } i.
\]

If (4') holds, then for any commutative square
with $i$ a cofibration, the given maps $A \to \text{Eq}(\tilde{U})$ and $B \to U$ respectively classify $w$ and $\overline{E}_2$ as in $(4')$. Then $(4')$ gives $\overline{E}_1$ and $\overline{w}$, condition $(2')$ yields a classifying map for $\overline{E}_1$ extending the composite $A \to \tilde{U} \to U$, and using the following lemma, we can construct a lift in the above square, so that $\text{Eq}(\tilde{U}) \to U$ is an acyclic fibration.

**Lemma 2.1.** In a suitable model category, let $E_1 \twoheadrightarrow B$ and $E_2 \twoheadrightarrow B$ be fibrations classified by maps $B \to U$, let $\overline{w}: \overline{E}_1 \to \overline{E}_2$ be a weak equivalence over $B$, let $i: A \hookrightarrow B$ be a cofibration, and suppose we have a lift of $A \to B \to U \times U$ to $\text{Eq}(U)$ which classifies $i^*(\overline{w})$. Then this lift can be extended to $B$ so as to classify $\overline{w}$.

In particular, if all objects are cofibrant (as will be the case in all our examples), then taking $A = \emptyset$ in Lemma 2.1 implies that any weak equivalence between fibrations over $B$ is classified by some map $B \to \text{Eq}(\tilde{U})$.

The proof of Lemma 2.1 is the only place where we need to know the actual definition of $\text{Eq}(\tilde{U})$. This definition is determined by the specific formulation of the univalence axiom in type theory and is somewhat technical, so we will consider it separately in §3, postponing the proof of Lemma 2.1 until then. For now, it suffices to take Lemma 2.1 as a (hopefully plausible) black box. In fact, one might argue that just as $(2')$ determines a good notion of what it means to be a universe object in a model category, the conclusion of Lemma 2.1 is a good definition of what it means for $\text{Eq}(U)$ to be a “universal space of equivalences” therein.

Let us now consider in what level of generality we can prove $(2')$, $(3')$, and $(4')$. Perhaps surprisingly, given that $(4')$ is a modification of the actual statement $(4)$ of univalence, it seems to be the easiest to prove in the most generality. The proof in [KLV12] for simplicial sets carries through almost word-for-word in a much more general context.

In a presheaf category $\text{Set}^{\mathcal{C}^{\text{op}}}$, we say a morphism $f: A \to B$ is $\kappa$-small, for some cardinal number $\kappa$, if for all $c \in \mathcal{C}$ and $b \in B_c$ we have $|f^{-1}(b)| < \kappa$.

**Theorem 2.2.** If $\text{Set}^{\mathcal{C}^{\text{op}}}$ is a presheaf category that is a simplicial model category whose cofibrations are exactly the monomorphisms, and $\kappa$ is a cardinal number larger than $|\mathcal{C}|$, then the $\kappa$-small fibrations in $\text{Set}^{\mathcal{C}^{\text{op}}}$ satisfy $(4')$.

**Proof (from [KLV12]).** Suppose given a cofibration $i: A \hookrightarrow B$, a $\kappa$-small fibration $E_2 \to B$, and an equivalence $w: E_1 \simeq E_2 := i^*E_2$ of $\kappa$-small fibrations over $A$; we want to construct $E_1$ and the dashed arrows below.
Define $E_1$ and $w$ as the following pullback in $\text{Set}^{\text{op}}/B$, where $i_*$ denotes the right adjoint of pullback $i^*$:

\[
\begin{array}{ccc}
E_1 & \rightarrow & i_*E_1 \\
\downarrow & & \downarrow \text{(w)} \\
E_2 & \rightarrow & i_*i^*E_2 \cong i_*E_2
\end{array}
\]

Since $i^*$ preserves this pullback, and $i_*$ is fully faithful, $\overline{w}$ pulls back to $w$. It is straightforward to check that $E_1 \rightarrow B$ is $\kappa$-small; it remains to show it is a fibration and that $\overline{w}$ is an equivalence.

We factor $w$ as an acyclic cofibration followed by an acyclic fibration and treat the two cases separately. In the second case, $i_*(w)$ is an acyclic fibration and thus so is $\overline{w}$. In the first case, since $E_1$ and $E_2$ are fibrations over $A$, by [Hir03, 7.6.11 and 9.5.24], we have a simplicial deformation retraction $H: \Delta^1 \otimes E_2 \rightarrow E_2$ of $E_2$ onto $E_1$ in $\text{Set}^{\text{op}}/A$, where $\otimes$ denotes the tensor for the simplicial enrichment. Now $\eta$ and $\overline{w}$ are monic, so if $P$ denotes the following pushout:

\[
\begin{array}{ccc}
E_1 & \rightarrow & E_1 \\
\downarrow & & \downarrow \text{(w)} \\
E_2 & \rightarrow & P \\
\downarrow & \downarrow & \downarrow j \\
& & E_2
\end{array}
\]

then $j$ is also a monomorphism. Since we are in a simplicial model category, the pushout product on the left of the following square is an acyclic cofibration.

\[
\begin{array}{ccc}
(\Delta^0 \otimes E_2) \cup_{\Delta^0 \otimes P} (\Delta^1 \otimes P) & \rightarrow & E_2 \\
\downarrow & & \downarrow \overline{w} \\
\Delta^1 \otimes E_2 & \rightarrow & B
\end{array}
\]

The map at the top is induced by the identity on $\Delta^0 \otimes E_2 \cong E_2$, and on $\Delta^1 \otimes P$ by a combination of $\eta H$ on $E_2$ and the constant homotopy at $\overline{w}$ on $E_1$ (which agree in $E_1$ since $H$ is a deformation retraction). Since $E_2 \rightarrow B$ is a fibration, $\overline{H}$ exists, and since $i^*(\overline{H}) = H$ is a deformation retraction into $E_1$, $\overline{H}$ is a deformation retraction into $E_1$. Thus $\overline{w}$ is the inclusion of a deformation retract, hence a weak equivalence; and $E_1 \rightarrow B$, being a retract of $E_2 \rightarrow B$, is a fibration. \hfill $\square$

I do not know any general context of this sort in which one can prove $(3')$. However, the construction in [KLV12] of a fibration $\tilde{U} \rightarrow U$ satisfying $(2')$ can also be immediately generalized somewhat.

**Theorem 2.3.** Suppose $\text{Set}^{\text{op}}$ is a presheaf category that is a cofibrantly generated model category in which the cofibrations are the monomorphisms, and that the codomains of the generating acyclic cofibrations are representable. Then there exists a $\kappa$-small fibration satisfying $(2')$. 

Proof. As in [KLV12], consider \( \kappa \)-small morphisms \( f : A \to B \) equipped with well-orderings on all fibers \( f_c^{-1}(b) \). The isomorphism classes of these over any \( B \) form a set, and the resulting presheaf on \( \text{Set}^{\text{C}^{\text{op}}} \) preserves limits and is representable by some \( \tilde{W} \to W \). By definition, \( W_c \) is the set of isomorphism classes of well-ordered morphisms into \( C(-,c) \). The hypothesis on the generating acyclic cofibrations implies that a morphism is a fibration just when all its pullbacks to representables are. Thus, if \( U \subset W \) is the subobject determined by the fibrations, the pullback \( \tilde{U} \to U \) of \( \tilde{W} \) is a \( \kappa \)-small fibration and represents well-ordered \( \kappa \)-small fibrations. Finally, if \( i : A \to B \) is monic and \( P \rightrightarrows B \) is a \( \kappa \)-small fibration, a well-ordering on the fibers of \( i^*P \) can be extended to a well-ordering on all fibers of \( P \). \( \Box \)

It is a bit unsatisfying that this proof resorts to such an unnatural device as the well-ordering principle. Fortunately, there are other proofs of the same theorem. In [Str], Streicher proposed another construction of universes in presheaf categories, which also satisfy (2′).

**Second proof of Theorem 2.3.** For any \( X \in \text{Set}^{\text{C}^{\text{op}}} \), let \( \text{el}(X) \) denote its category of elements, whose objects are pairs \((c,x)\) with \( c \in C \) and \( x \in X(c) \), and whose morphisms \((c,x) \to (c',x')\) are morphisms \( \alpha : c \to c' \) in \( C \) such that \( X(\alpha)(x') = x \). If \( Y_c \) is the representable presheaf at \( c \in C \), then \( \text{el}(Y_c) \cong C/c \). There is a functor \( \text{el} : \text{Set}^{\text{C}^{\text{op}}} \to \text{Cat} \), which of course restricts along the Yoneda embedding to the functor \( C \to \text{Cat} \) defined by \( c \mapsto C/c \).

Let \( \text{Set}_\kappa \) be a small full subcategory of \( \text{Set} \) containing a set of each cardinality \( < \kappa \). Define \( W \in \text{Set}^{\text{C}^{\text{op}}} \) by

\[
W(c) = \left( (\text{Set}_\kappa)^{(C/c)^{\text{op}}} \right)_0 ,
\]

i.e. \( W(c) \) is the set of functors \((C/c)^{\text{op}} \to \text{Set}_\kappa \). Since functors compose associatively, \( W \) is indeed a presheaf. Similarly, let \( \text{Set}_{\kappa,*} \) denote the category of pointed sets in \( \text{Set}_\kappa \), and define \( \tilde{W} \) by

\[
\tilde{W}(c) = \left( (\text{Set}_{\kappa,*})^{(C/c)^{\text{op}}} \right)_0 .
\]

The forgetful functor \( \text{Set}_{\kappa,*} \to \text{Set}_\kappa \) induces a natural transformation \( p : \tilde{W} \to W \). Moreover, given a functor \( F : (C/c)^{\text{op}} \to \text{Set}_{\kappa,*} \) in \( W(c) \), since \( C/c \) has a terminal object \( 1_c \), to lift \( F \) to \( \text{Set}_{\kappa,*} \) is equivalently to give an element of \( F(1_c) \). Thus, since \( F(1_c) \) is \( \kappa \)-small by definition, each fiber of \( p \) has cardinality \( < \kappa \), so that \( p \) is \( \kappa \)-small.

Now for any \( A \in \text{Set}^{\text{C}^{\text{op}}} \), a standard adjunction argument implies that to give a morphism \( A \to W \) is equivalently to give a functor

\[
A \otimes_C (C/-)^{\text{op}} \to \text{Set}_\kappa
\]

whose domain is the “tensor product of functors” over \( C \), defined by a suitable coend. Equivalently, this is the “weighted colimit” of the functor \( c \mapsto (C/c)^{\text{op}} \) weighted by \( A : C^{\text{op}} \to \text{Set} \).
Now, the functor $\text{el}(-)$ is simply a lax colimit, and thus it preserves (weighted) colimits since colimits commute with colimits. Thus we have

$$A \otimes_C (C/-)^{\text{op}} \cong A \otimes_C \text{el}(Y_-)^{\text{op}} \cong \text{el}(A \otimes_C Y)^{\text{op}} \cong \text{el}(A)^{\text{op}}$$

the last step by the “co-Yoneda lemma” (every presheaf is a weighted colimit of representables, weighted by itself). Thus, the set of morphisms $A \to W$ is naturally bijective with the set of functors $\text{el}(A)^{\text{op}} \to \text{Set}_\kappa$.

However, it is well-known that the category of functors $\text{Set}^{\text{el}(A)^{\text{op}}}$ is equivalent to the slice category $\text{Set}^{C^{\text{op}}}/A$. Namely, given $F : \text{el}(A)^{\text{op}} \to \text{Set}$, we define $P \in \text{Set}^{C^{\text{op}}}$ by $P(c) = \sum_{x \in A(c)} F(x)$, with the obvious action of $C$ and projection to $A$. Clearly $F$ factors through $\text{Set}_\kappa$, up to isomorphism, if and only if $P \to A$ is $\kappa$-small. Moreover, it is easy to verify that given a map $f : A \to W$, the morphism $P \to A$ corresponding to the induced functor $\text{el}(A)^{\text{op}} \to \text{Set}_\kappa$ is (up to isomorphism) the pullback of $\tilde{W} \to W$ along $f$. Thus, a map $P \to A$ of presheaves is a pullback of $\tilde{W} \to W$ if and only if it is $\kappa$-small.

Now we define $U$ to be the sub-presheaf of $W$ containing exactly those functors $(C/c)^{\text{op}} = \text{el}(Y_c)^{\text{op}} \to \text{Set}_\kappa$ whose corresponding morphism $P \to Y_c$ is a fibration, and $\tilde{U}$ the pullback of $\tilde{W}$ to $U$. Then a morphism $A \to W$ factors through $U$ if and only if the corresponding map $P \to A$ pulls back to a fibration over all representables. But as remarked in the previous proof, the assumption on the generating acyclic cofibrations implies that this is equivalent to $P \to A$ being itself a fibration.

Finally, in the situation of $(2')$, let $F : \text{el}(A)^{\text{op}} \to \text{Set}_\kappa$ be the functor corresponding to the given map $A \to U$, and let $i : A \to B$ be the given monomorphism with $i^* P \cong Q$. By the equivalence of categories mentioned above, there is a functor $G : \text{el}(B)^{\text{op}} \to \text{Set}_\kappa$ corresponding to some map $B \to U$ which pulls $\tilde{U}$ back to $P$, and hence we have an isomorphism $G \circ \text{el}(i)^{\text{op}} \cong F$. But since $i$ is a monomorphism, the functor $\text{el}(i)^{\text{op}}$ is injective on objects, and so we can modify $G$ to make this isomorphism an equality. The conclusion of $(2')$ follows.  

I would now like to give a third proof of Theorem 2.3, which I believe makes the connection to $(\infty, 1)$-categorical object classifiers rather clearer. We will need the following “exactness properties” of any Grothendieck topos.

(a) Given a family

$$
\begin{array}{ccc}
X_i & \longrightarrow & Z \\
\downarrow & & \downarrow \\
A_i & \longrightarrow & \sum_i A_i
\end{array}
$$

of commutative squares in which the bottom family of morphisms are the injections into a coproduct $(A_i \to \sum_i A_i)_{i \in I}$ and the right-hand maps are independent of $i$, then the top family of morphisms form a coproduct diagram (so that $Z \cong \sum_i X_i$) if and only if all the squares are pullbacks. A category with this property is called (infinitary) extensive [CLW93]. Extensivity is equivalent to coproducts being stable and disjoint, and implies that coproducts preserve monomorphisms and pullback squares.
(b) Given a commutative cube

\[
\begin{array}{ccc}
X & \to & Z \\
\downarrow & & \downarrow \\
Y & \to & W \\
\downarrow & & \downarrow \\
A & \to & C \\
\downarrow & & \downarrow \\
B & \to & D
\end{array}
\]

in which \(A \to B\) is a monomorphism, the bottom face is a pushout, and the left and back faces are pullbacks, then the top face is a pushout if and only if the front and right faces are pullbacks. A category with this property is called \textbf{adhesive} [LS04]. Adhesivity is equivalent to pushout squares of monomorphisms being also pullback squares and being stable under pullback [GL12], and implies that the pushout of a monomorphism is a monomorphism.

(c) Given a commutative diagram

\[
\begin{array}{ccc}
X_0 & \to & X_1 & \to & \cdots & \to & X_\alpha & \to & \cdots & \to & X_\lambda \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \cdots & & \downarrow \\
A_0 & \to & A_1 & \to & \cdots & \to & A_\alpha & \to & \cdots & \to & A_\lambda
\end{array}
\]

of transfinite sequences for \(\alpha < \lambda\), with \(\lambda\) some limit ordinal, in which the bottom row is a colimit diagram, and for each \(\alpha < \beta < \lambda\) the morphism \(A_\alpha \to A_\beta\) is a monomorphism and the square

\[
\begin{array}{ccc}
X_\alpha & \to & X_\beta \\
\downarrow & & \downarrow \\
A_\alpha & \to & A_\beta
\end{array}
\]

is a pullback, then the top row is a colimit diagram if and only if for each \(\alpha < \lambda\) the square

\[
\begin{array}{ccc}
X_\alpha & \to & X_\lambda \\
\downarrow & & \downarrow \\
A_\alpha & \to & A_\lambda
\end{array}
\]

is a pullback. I have not been able to find a name in the literature for categories with this property; I propose to call them \textbf{exhaustive}. Exhaustivity is equivalent to asking that in a transfinite composite of monomorphisms, the coprojections into the colimit are also monomorphisms and the colimit is pullback-stable [S+12]. It implies that transfinite composites of monomorphisms preserve pullbacks, and hence also monomorphisms.

\textit{Third proof of Theorem 2.3.} Recall that the hypotheses of Theorem 2.3 ensure that a morphism is a \(\kappa\)-small fibration if and only if all of its pullbacks to representables are. Let \(I\) be a generating set of cofibrations.
The proof may be described as “constructing a cofibrant replacement of $\mathcal{U}$ by the small object argument”. (This is similar to some proofs of the Brown representability theorem.) We define a transfinite sequence

\[
\begin{array}{ccccccc}
\tilde{U}_0 & \rightarrow & \tilde{U}_1 & \rightarrow & \tilde{U}_2 & \rightarrow & \cdots & \rightarrow & \tilde{U}_\alpha & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
U_0 & \rightarrow & U_1 & \rightarrow & U_2 & \rightarrow & \cdots & \rightarrow & U_\alpha & \rightarrow & \cdots
\end{array}
\]

such that

(i) Each map $\tilde{U}_\alpha \rightarrow U_\alpha$ is a $\kappa$-small fibration.
(ii) For $\alpha < \beta$, the map $U_\alpha \rightarrow U_\beta$ is monic.
(iii) For $\alpha < \beta$, the square

\[
\begin{array}{ccc}
\tilde{U}_\alpha & \rightarrow & \tilde{U}_\beta \\
\downarrow & & \downarrow \\
U_\alpha & \rightarrow & U_\beta
\end{array}
\]

is a pullback (hence $\tilde{U}_\alpha \rightarrow \tilde{U}_\beta$ is also monic).

For limit $\alpha$, we take colimits of both sequences. (Including $\alpha = 0$ as a limit, this means we begin with $\tilde{U}_0 = U_0 = \emptyset$). By induction, these are colimits of monomorphisms and all intermediate squares are pullbacks. Thus, by exhaustivity, (ii) and (iii) remain true in the colimit. As for (i), since maps out of representables preserve colimits, and $\tilde{U}_\gamma$ is the pullback of $\tilde{U}_\alpha$ to $U_\gamma$ for all $\gamma < \alpha$, any commutative square as on the left below factors as on the right for some $\gamma < \alpha$.

\[
\begin{array}{ccc}
C & \rightarrow & \tilde{U}_\alpha \\
\downarrow & & \downarrow \\
Y_c & \rightarrow & U_\alpha
\end{array} = \begin{array}{ccc}
C & \rightarrow & \tilde{U}_\gamma & \rightarrow & \tilde{U}_\alpha \\
\downarrow & & \downarrow & & \downarrow \\
Y_c & \rightarrow & U_\gamma & \rightarrow & U_\alpha
\end{array}
\]

Thus, since each $\tilde{U}_\gamma \rightarrow U_\gamma$ is a fibration, and the codomains of the generating acyclic cofibrations are representable, also $\tilde{U}_\alpha \rightarrow U_\alpha$ is a fibration.

At a successor stage, given $\tilde{U}_\alpha \rightarrow U_\alpha$ we consider the set of pairs $(i, f, p)$, where

- $i: A \rightarrow B$ is in $I$;
- $f: A \rightarrow U_\alpha$ is any morphism; and
- $p: P \rightarrow B$ lies in a small set of representatives for isomorphism classes of $\kappa$-small fibrations into $B$ which are equipped with an isomorphism $i^*P \cong f^*\tilde{U}_\alpha$. 
We define $\tilde{U}_{\alpha+1} \to U_{\alpha+1}$ to make the top and bottom squares of the following cube into pushouts:

$$
\begin{array}{c}
\sum_{(i,f,p)} f^* \tilde{U}_\alpha & \to & \tilde{U}_\alpha \\
\downarrow & & \downarrow \\
\sum_{(i,f,p)} P & \to & \tilde{U}_{\alpha+1} \\
\sum_{(i,f,p)} A & \to & \sum_{(i,f,p)} B \\
\downarrow & & \downarrow \\
\sum_{(i,f,p)} \Sigma i & \to & \Sigma \Sigma p \\
\downarrow & & \downarrow \\
Y_c & \to & U_{\alpha+1} \\
\end{array}
$$

By extensivity, $\sum i$ is monic, so by adhesivity, so is $U_\alpha \to U_{\alpha+1}$, giving (ii). Likewise, by extensivity, the left and back faces are pullbacks, so by adhesivity, so are the right and front faces, giving (iii). Finally, since maps out of representables preserve colimits, any map $Y_c \to U_{\alpha+1}$ factors through $\sum_{(i,f,p)} B$ or $U_\alpha$, and since the front and right faces are pullbacks, any commutative square of the form

$$
\begin{array}{c}
C & \to & \tilde{U}_{\alpha+1} \\
\downarrow & & \downarrow \\
Y_c & \to & U_{\alpha+1} \\
\end{array}
$$

factors through either $\sum p$ or $\tilde{U}_\alpha \to U_\alpha$, both of which are fibrations (the former by a similar argument using extensivity). Thus, we can lift in any such square with $C \to Y_c$ a generating acyclic cofibration, so $\tilde{U}_{\alpha+1} \to U_{\alpha+1}$ is a $\kappa$-small fibrations; thus (i) holds.

Now since a presheaf category is locally presentable, there exists $\lambda$ such that the domains of all morphisms in $\mathcal{I}$ are $\lambda$-compact (i.e. their covariant representable functors preserve $\lambda$-filtered colimits). For such a $\lambda$, if $i: A \to B$ is in $\mathcal{I}$, and $f: A \to \tilde{U}_\lambda$ is a morphism, and $p: P \to B$ is a $\kappa$-small fibration with $i^* P \cong f^* \tilde{U}_\lambda$, then by $\lambda$-compactness of $A$, $f$ factors through $U_\alpha$ for some $\alpha < \lambda$. By construction, $(i, f, p)$ then induces a map $g: B \to U_{\alpha+1}$ with $g^* \tilde{U}_{\alpha+1} \cong P$; so $(2')$ holds for $i \in \mathcal{I}$.

It suffices, therefore, to show that the class of monomorphisms $i$ satisfying $(2')$ is closed under pushout, transfinite composition, and retracts. For closure under pushouts, suppose given the solid arrows in the following diagram:

```
X ----------------> Z --> \tilde{U} \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu...
where the bottom square is a pushout, \( i \) (hence also \( j \)) is monic, and the other two squares of solid arrows are pullbacks. Then we can fill in the objects \( X \) and \( Y \) and the dotted arrows to make all vertical faces of the cube pullbacks; hence by adhesivity the top face is a pushout. Assuming \( i \) satisfies \((2')\), we have a map \( B \to U \) which pulls back \( \tilde{U} \) to \( Y \) compatibly; thus the universal property of pushouts induces the dashed arrows shown. Finally, stability of pushouts under pullback implies that the square involving the dashed arrows is also a pullback.

For closure under transfinite composites, suppose \( A_0 \to A_\lambda \) is a transfinite composite of monomorphisms, and suppose given the solid arrows in the following diagram making the left-hand rhombus and the outer rectangle pullbacks.

\[
\begin{array}{c}
X_0 \to X_1 \to \cdots \to X_\alpha \to \cdots \to X_\lambda \\
U \\
A_0 \to A_1 \to \cdots \to A_\alpha \to \cdots \to A_\lambda \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\tilde{U} \\
U
\end{array}
\]

Then we can fill in the \( X_\alpha \) and the dotted arrows making the other squares all pullbacks; hence by exhaustivity the top row is a colimit. Assuming each \( A_\alpha \to A_\beta \) satisfies \((2')\), we can successively extend the maps \( A_0 \to U \) and \( X_0 \to \tilde{U} \) to all \( A_\alpha \) and \( X_\alpha \), and hence in the colimit to \( A_\lambda \) and \( X_\lambda \). Finally, stability of transfinite composites under pullback implies that the induced squares are all also pullbacks.

For closure under retracts, suppose given the following solid arrows:

\[
\begin{array}{c}
Z \to X \to Z \to \tilde{U} \\
W \to Y \to W \\
C \to A \to C \\
D \to B \to D \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
U \\
U
\end{array}
\]

where the composites \( C \to A \to C \) and \( D \to B \to D \) are identities. Then we can fill in \( X \) and \( Y \) and the dotted arrows making all squares pullbacks. Assuming \( i \) satisfies \((2')\), we have a map \( B \to U \) compatibly classifying \( Y \), and then the composite \( D \to B \to U \) compatibly classifies \( W \).

I say that this proof makes the connection to object classifiers clearer because it depends mainly on the fact that the pseudo 2-functor

\[
\left( \text{Set}^{\text{c}^{\text{op}}} \right)^{\text{op}} \to \text{Cat}
\]

\[
B \mapsto \{ \text{fibrations over } B \} 
\]
preserves coproducts, pushouts of monomorphisms, and transfinite composites of monomorphisms. These can of course be regarded as “stack” conditions. Moreover, since the monomorphisms in question are cofibrations, these colimits are also homotopy colimits.

By comparison, in [Lur09, 6.1.6.3] object classifiers in an \((\infty,1)\)-category \(\mathcal{C}\) are constructed under the assumption that the \((\infty,1)\)-functor
\[
\mathcal{C}^{\text{op}} \to (\infty,1)\text{Cat}
\]
\[B \mapsto \{\text{all morphisms into } B\}\]
preserves all (homotopy) colimits. In this situation one can simply apply the \((\infty,1)\)-categorical adjoint functor theorem, but this could be unraveled more explicitly into a transfinite construction very like that in the third proof of Theorem 2.3.

3. Universal spaces of equivalences

In this section we will define the universal space of equivalences \(\text{Eq}(\tilde{U})\) and prove Lemma 2.1. The definition is exactly the categorical interpretation of Voevodsky’s type-theoretic definition of equivalences. However, in keeping with the tone of this paper, we will describe it without reference to type theory.

For all of this section, let \(\mathcal{M}\) be a locally cartesian closed, right proper, simplicial model category whose cofibrations are the monomorphisms. This is roughly what is needed for it to interpret type theory (it is somewhat stronger than being a type-theoretic model category in the sense of [Shu12]). In particular, all objects of \(\mathcal{M}\) are cofibrant.

Local cartesian closure implies that for any \(f : A \to B\), the pullback functor \(f^* : \mathcal{M}/B \to \mathcal{M}/A\) has a right adjoint, which we denote \(\Pi_f\). By adjointness, since \(f^*\) preserves cofibrations (i.e. monomorphisms), \(\Pi_f\) preserves acyclic fibrations.

Now for any fibration \(p : E \to B\) in \(\mathcal{M}\), let \(P_B E = (E \to B)^{\Delta^1}\) be the cotensor in \(\mathcal{M}/B\) of \(E \to B\) by the standard interval \(\Delta^1\). Since \(E\) is fibrant in \(\mathcal{M}/B\), \(P_B E\) is a valid path object for it, i.e. we have an acyclic cofibration \(E \Rightarrow P_B E\) and a fibration \(P_B E \to E \times_B E\) factoring the diagonal \(E \to E \times_B E\). Writing \(\pi_2 : E \times_B E \to E\) for the second projection, we define
\[
is\text{Contr}_B(E) := \Pi_{\pi_2}(P_B E)
\]
regarded as an object of \(\mathcal{M}/B\) via the composite fibration \(\Pi_{\pi_2}(P_B E) \to E\).

**Lemma 3.1.** For a fibration \(p : E \to B\), the following are equivalent:

1. \(p\) is an acyclic fibration.
2. \(\text{isContr}_B(E) \to B\) has a section.
3. \(\text{isContr}_B(E) \to B\) is an acyclic fibration.

**Proof.** By adjunction, to give a section of \(\text{isContr}_B(E) \to B\) is precisely to give a section \(s\) of \(p\) and a simplicial homotopy \(sp \sim 1_E\). By [Hir03, 7.6.11(2)], therefore, we have (i)\(\Leftrightarrow\) (ii). And certainly (iii)\(\Rightarrow\) (ii), so it will suffice to show (i)\(\Rightarrow\) (iii).

However, if \(p\) is an acyclic fibration, then both projections \(E \times_B E \to E\) are weak equivalences. Thus, by the 2-out-of-3 property, so is the diagonal \(E \to E \times_B E\). Again by the 2-out-of-3 property, therefore, \(P_B E \to E \times_B E\) is an acyclic fibration. But \(\Pi_{\pi_2} : \mathcal{M}/(E \times_B E) \to \mathcal{M}/E\) preserves acyclic fibrations, so \(\text{isContr}_B(E) \to B\) is the composite of two acyclic fibrations. \(\square\)
Now let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations and $f : E_1 \to E_2$ a map over $B$. Define $P_B f$ by the following pullback:

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow^{1_{E_1} \times f} & & \downarrow^{P_B f} \\
E_1 \times_B E_2 & \xrightarrow{f \times \id_{E_2}} & E_2 \times_B E_2
\end{array}
\]

The universal property of $P_B f$ is that to give a map $A \to P_B f$ is the same as to give a map $x : A \to E_1$, a map $y : A \to E_2$, and a simplicial homotopy $fx \sim y$ over $B$. The induced map $r : E_1 \to P_B f$ corresponds to $x = 1_{E_1}$, $y = f$, and the constant homotopy. There is of course a projection $q : P_B f \to E_1$ which, interpreted representably, remembers only the map $x$; and the composite $E_1 \xrightarrow{\sim} P_B f \xrightarrow{q} E_1$ is evidently the identity.

On the other hand, the composite $P_B f \xrightarrow{q} E_1 \xrightarrow{r} P_B f$ acts representably by taking $x, y$, and a homotopy $H : fx \sim y$ to the triple of $x, fx$, and the constant homotopy. This map is homotopic to the identity, so $r$ admits a deformation retraction, hence is a weak equivalence. (In fact, $r$ is an acyclic cofibration; this is proven in [Shu12] by mimicking the type-theoretic proof in [GG08].)

Now $P_B f$ comes with a composite fibration $P_B f \to E_1 \times_B E_2 \to E_2$, whose composite with $r$ is $f$ itself. Thus, by the 2-out-of-3 property, $f$ is a weak equivalence if and only if this fibration $P_B f \to E_2$ is acyclic, and therefore if and only if $\text{isContr}_{E_2}(P_B f) \to E_2$ has a section (in which case it is also acyclic, by Lemma 3.1).

We now define

\[\text{isEquiv}_B(f) := \Pi_{p_2} \text{isContr}_{E_2}(P_B f),\]

regarded as an object of $\mathcal{M}/B$.

**Lemma 3.2.** For a map $f$ between fibrations $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, the following are equivalent.

(i) $f$ is a weak equivalence.

(ii) $\text{isEquiv}_B(f) \to B$ has a section.

(iii) $\text{isEquiv}_B(f) \to B$ is an acyclic fibration.

**Proof.** By adjunction, to give a section of $\text{isEquiv}_B(f) \to B$ is equivalently to give a section of $\text{isContr}_{E_2}(P_B f) \to E_2$, which as we have seen above is equivalent to $f$ being a weak equivalence; thus we have (i)$\Leftrightarrow$(ii). And certainly (iii)$\Rightarrow$(ii), so it remains to show the converse. But by Lemma 3.1, if $\text{isContr}_{E_2}(P_B f) \to E_2$ has a section, then it is an acyclic fibration, and $\Pi_{p_2}$ preserves acyclic fibrations. \qed

Now, given fibrations $p_1 : E_1 \to B$ and $p_2 E_2 \to B$ as above, we define

\[\text{Equiv}_B(E_1, E_2) := \text{isEquiv}_{\text{fun}_B(E_1, E_2)}(h)\]

where $\text{fun}_B(E_1, E_2)$ is the exponential in $\mathcal{M}/B$, and $h$ is the universal morphism between the pullbacks of $E_1$ and $E_2$ to $\text{fun}_B(E_1, E_2)$. Finally, we can define the universal space of equivalences as:

\[\text{Eq}(\bar{U}) := \text{Equiv}_{U \times U}(\pi_1^* \bar{U}, \pi_2^* \bar{U}).\]
Of course, it comes with a canonical projection to $U \times U$.

Now observe that all the constructions $\text{isContr}$, $\text{isEquiv}$, and $\text{Equiv}$ are defined in terms of structure in $\mathcal{M}/B$ which is stable under pullback (up to isomorphism). Therefore, for any map $j : A \to B$, say, we have

\[ j^* \text{Equiv}_B(E_1, E_2) \cong \text{Equiv}_A(j^* E_1, j^* E_2) \]

and similarly for $\text{isEquiv}$ and $\text{isContr}$. In particular, for any $e_1, e_2 : B \Rightarrow U$ we have

\[ (e_1, e_2)^* \text{Eq}(\tilde{U}) \cong \text{Equiv}_B(e_1^* \tilde{U}, e_2^* \tilde{U}). \]

Proof of Lemma 2.1. As in Lemma 2.1, let $\overline{E}_1 \to B$ and $\overline{E}_2 \to B$ be fibrations classified by $e_1, e_2 : B \Rightarrow U$, let $\overline{w} : \overline{E}_1 \to \overline{E}_2$ be a weak equivalence over $B$, let $i : A \to B$ be a cofibration, and suppose we have a lift of $A \xrightarrow{i} \frac{\{e_1, e_2\}}{B} U \times U$ to $\text{Eq}(\tilde{U})$ which classifies $i^*(\overline{w})$. We want to extend this lift to $B$ so as to classify $\overline{w}$.

By the above remarks, we have $(e_1, e_2)^* \text{Eq}(\tilde{U}) \cong \text{Equiv}_B(\overline{E}_1, \overline{E}_2)$, and so our given lift is equivalently a lift of $i$ to $\text{Equiv}_B(\overline{E}_1, \overline{E}_2)$. Let $k : B \to \text{fun}_B(\overline{E}_1, \overline{E}_2)$ be the classifying map of $\overline{w}$; then we have the following commutative square:

\[
\begin{array}{ccc}
A & \rightarrow & \text{Equiv}_B(\overline{E}_1, \overline{E}_2) \\
| & & | \\
B & \rightarrow & \text{fun}_B(\overline{E}_1, \overline{E}_2)
\end{array}
\]

and hence also, invoking pullback-stability and the definition of $\text{Equiv}_B$, the following commutative square:

\[
\begin{array}{ccc}
A & \rightarrow & \text{isEquiv}_B(\overline{w}) \\
| & & | \\
B & = & B.
\end{array}
\]

But since $\overline{w}$ is a weak equivalence, by Lemma 3.2 the right-hand fibration in this square is acyclic. Since $i$ is a cofibration, there exists a lift as shown, and tracing backwards this gives us our desired lift $B \to \text{Eq}(\tilde{U})$. \qed

4. Modeling Type Theory

In §2 we described a general plan for obtaining a universe object and showing that it is fibrant and univalent. Of course, for the interpretation of type theory, we also need the more basic structure: category-theoretic operations corresponding to all the type-forming operations, which preserve $\kappa$-small fibrations.

The easiest case is dependent sums, which are modeled by composition of fibrations. The composite of fibrations is always a fibration; for the composite of $\kappa$-small fibrations to remain $\kappa$-small we merely need $\kappa$ to be regular.

Dependent products are most directly modeled by right adjoints to pullback. These exist in any locally cartesian closed category, but we require that the dependent product of a ($\kappa$-small) fibration along a ($\kappa$-small) fibration is again a ($\kappa$-small) fibration. The most natural way to ensure preservation of fibrations is via the adjoint condition that pullback along fibrations preserves acyclic cofibrations. If the cofibrations are the monomorphisms, then they are stable under pullback, and if
the model category is right proper, then weak equivalences are also stable under pullback along fibrations; so these two conditions together suffice.

For dependent product to preserve \( \kappa \)-smallness in a presheaf category \( \text{Set}^{\text{C}^{\text{op}}} \), we need \( \kappa \) to be a strong limit cardinal and larger than \( |\mathcal{C}| \). Thus, in conjunction with dependent sums, we need \( \kappa \) to be inaccessible and larger than \( |\mathcal{C}| \).

Finally, the central insight of homotopy type theory is that identity types are modeled by path objects. That is, for a \( \kappa \)-small fibration \( B \to A \), we factor the diagonal \( B \to B \times_A B \) into an acyclic cofibration followed by a fibration, \( B \to P_A B \to B \times_A B \). Since \( B \) and \( B \times_A B \) are fibrant in the slice model category over \( A \), in a simplicial model category we can let \( P_A B \) be the simplicial cotensor by \( \Delta^1 \) in this slice category. In a category \( \text{sSet}^{\text{C}^{\text{op}}} \) of presheaves of simplicial sets, this preserves \( \kappa \)-smallness as long as \( \kappa \) is uncountable and larger than \( |\mathcal{C}| \).

There is also the issue of coherence for all these structures, but fortunately this is taken care of by the coherence theorem of [LW13]. Thus, we can say:

**Theorem 4.1.** If \( \text{sSet}^{\text{C}^{\text{op}}} \) has a right proper, cofibrantly generated simplicial model structure whose cofibrations are the monomorphisms, then it models type theory with dependent sums, dependent products, and intensional identity types.

Moreover, if \( \kappa \) is inaccessible and larger than \( |\mathcal{C}| \), and the codomains of the generating acyclic cofibrations are representable, then \( \text{sSet}^{\text{C}^{\text{op}}} \) contains a universe object classifying \( \kappa \)-small fibrations and satisfying (2'). If this universe is fibrant (such as if (3') holds), it represents a univalent universe in the internal type theory.

Of course, with multiple inaccessibles larger than \( |\mathcal{C}| \), we can find multiple universe objects of this sort, each contained in the next. More precisely, if \( \kappa < \lambda \), then every \( \kappa \)-small fibration is also \( \lambda \)-small, so we can find a pullback square

\[
\begin{array}{ccc}
\bar{U} & \longrightarrow & \bar{U}' \\
\downarrow_{p} & & \downarrow_{p'} \\
U & \longrightarrow & U'
\end{array}
\]

where \( p \) and \( p' \) classify \( \kappa \)-small and \( \lambda \)-small fibrations respectively, along with a classifying map \( 1 \to U' \) for \( U \) itself.

In fact, every proof of Theorem 2.3 gives a little more: there is a canonical choice of such a pullback square in which \( U \to U' \) is a monomorphism. In the first proof, the inclusion \( U \hookrightarrow U' \) is obvious. In the second proof, we simply choose \( \text{Set}_\lambda \) so that it contains \( \text{Set}_\kappa \). And in the third proof, we can inductively construct monomorphisms \( U_\alpha \to U'_\alpha \) which are preserved by all the colimits, as long as we choose the sets of \( (i,f,p) \)s for \( U' \) to contain those for \( U \).

These canonical inclusions are important, because in order to model a cumulative hierarchy of universes in type theory, we need to ensure furthermore that the inclusions respect the “universe structure”. To explain what this means, suppose \( \bar{U} \to U \) is a universe. Then the local exponential

\[ U^{(1)} := (U \times U \to U)(\bar{U} \to U) \]

is the base of a universal pair of composable small fibrations. If the composite of these fibrations is again small, it is classified by some map \( \Sigma : U^{(1)} \to U \), and in order to model a type-theoretic universe by \( U \) we must choose such a map.
Similarly, if the dependent product of the universal composable pair of small fibrations is small, we can choose for it a classifying map $\Pi : U(1) \to U$. And for identity types, we consider $\tilde{\times}_{U(1)} U$, which is the base of a universal “type with two sections”. We have a small fibration $P_U \tilde{\times}_{U(1)} \tilde{U}$, where $P_U \tilde{\times}_{U(1)} \tilde{U}$ is the path object of $\tilde{U}$ in $sSet^{C^{op}} / U$, and we can choose for it a classifying map $\text{Id} : \tilde{U} \times_{U(1)} \tilde{U} \to U$.

The requirement for a cumulative hierarchy of universes is then that the inclusions $U \hookrightarrow U'$ respect this chosen structure. In [Shu12] I called such a $U \hookrightarrow U'$ a universe embedding. Fortunately, if our universes all satisfy property $(2')$, then any inclusion can be made into a universe embedding as follows.

Suppose, under the hypotheses of Theorem 4.1, that we have a monomorphism $i : U \rightarrow U'$ between universes such that $i^*(U') \cong \tilde{U}$. Then there is an induced monomorphism $U(1) \rightarrow (U')'(1)$, which pulls back the universal composable pair of $U'$-small fibrations to the analogous pair of $U$-small ones. If in addition we have chosen some classifying map $U(1) \to U$ for the universal composite of $U$-small fibrations, then composing it with $i$ we obtain another classifying map $U(1) \to U'$ for the same fibration. But now since $U'$ satisfies $(2')$, we can extend this to a compatible classifying map $(U')'(1) \to U'$ for the universal composite of $U'$-small fibrations.

Thus, given $\Sigma : U(1) \to U$, it is always possible to choose $\Sigma' : (U')'(1) \to U'$ such that $i$ commutes with $\Sigma$ and $\Sigma'$. The same technique applies to dependent products, and also to path objects as long as we choose constructions of the universal path objects relative to $U$ and $U'$ which are compatible under $i^*$ (up to isomorphism); in our simplicial model category, we can again use the cotensor with $\Delta^1$. We can furthermore induct up any sequence of universe inclusions

$$U_0 \rightarrow U_1 \rightarrow U_2 \rightarrow \cdots$$

to obtain a sequence of universe embeddings. Thus we have:

**Theorem 4.2.** Under all the hypotheses of Theorem 4.1, $sSet^{C^{op}}$ contains as many nested universe objects satisfying $(2')$ as there are inaccessible cardinals greater than $|C|$. If these universes are fibrant, they represent univalent universes in the internal type theory.

### 5. Elegant Reedy presheaves

For any simplicial category $C$, there is an injective model structure on the category $sSet^{C^{op}}$ of simplicial presheaves which is cofibrantly generated, left and right proper, and simplicial, and its cofibrations are the monomorphisms. Thus, it satisfies all the conditions of Theorems 4.1 and 4.2 except for representability of the codomains of the generating acyclic cofibrations. In fact, in general the generating acyclic cofibrations are the most mysterious part of the injective model structure.

However, there is a special class of categories $C$ for which the injective model structure can be described much more explicitly. When $C$ is an elegant Reedy category as in [BR13], the injective model structure coincides with the Reedy model structure, which is perhaps the most explicit sort of model structure that can be put on a category of simplicial presheaves. We will show that in this case, the rest of the structure follows as well, so that $sSet^{C^{op}}$ models type theory with univalence.

Recall from [Hir03, Ch. 15] or [Hov99, Ch. 5] that $C$ is a Reedy category if the following hold.
There is a well-founded relation \(<\) on the objects of \(\mathcal{C}\).
- There are two subcategories \(\mathcal{C}^+\) and \(\mathcal{C}^-\) containing all the objects of \(\mathcal{C}\).
- Every morphism \(\alpha\) of \(\mathcal{C}\) can be written uniquely as \(\alpha^+\alpha^-\), where \(\alpha^+\) lies in \(\mathcal{C}^+\) and \(\alpha^-\) lies in \(\mathcal{C}^-\).
- If \(\alpha: c \to d\) lies in \(\mathcal{C}^+\) and is not an identity, then \(c \prec d\).
- If \(\alpha: c \to d\) lies in \(\mathcal{C}^-\) and is not an identity, then \(d \prec c\).

We say \(\mathcal{C}\) is **direct** if \(\mathcal{C}^-\) contains only identities, and **inverse** if \(\mathcal{C}^+\) contains only identities.

For a presheaf \(X \in s\mathbf{Set}^{\mathcal{C}^+\text{op}}\) on a Reedy category \(\mathcal{C}\) and an object \(c \in \mathcal{C}\), the **matching object** is defined by

\[
M_c X := \lim_{\partial(C^+ \downarrow c)} \left( X|_{\partial(C^+ \downarrow c)} \right)
\]

where \(\partial(C^+ \downarrow c)\) denotes the full subcategory of the over-category \((C^+ \downarrow c)\) which omits the identity arrow of \(c\). Similarly, the **latching object** is defined by

\[
L_c X := \colim_{\partial(C^- \downarrow c)^{\text{op}}} \left( X|_{\partial(C^- \downarrow c)^{\text{op}}} \right).
\]

Then the category \(s\mathbf{Set}^{\mathcal{C}^+\text{op}}\) has a model structure, called the **Reedy model structure**, in which a morphism \(f: A \to B\) is a fibration just when each map

\[
(5.1) \quad A_c \longrightarrow B_c \times_{M_c A} M_c B
\]

is a fibration in \(s\mathbf{Set}\), a cofibration just when each map

\[
A_c \amalg_{L_c A} L_c B \longrightarrow B_c
\]

is a cofibration in \(s\mathbf{Set}\), and a weak equivalence just when it is a levelwise weak equivalence in \(s\mathbf{Set}\). This model structure is simplicial and left and right proper.

Note that if \(\mathcal{C}\) is direct, then each \(L_c A\) is initial, so the Reedy cofibrations are just the levelwise ones, and dually. For future use, we record the following:

**Lemma 5.2.** If \(A \to B\) is a Reedy cofibration, then each map \(L_c A \to L_c B\) is a cofibration, and dually.

**Proof.** Since \(\partial(c \downarrow \mathcal{C}^-)\) is an inverse category, its Reedy fibrations are levelwise. In particular, the constant diagram functor \(s\mathbf{Set} \to s\mathbf{Set}^{\partial(c \downarrow \mathcal{C}^-)^{\text{op}}}\) is right Quillen, and so the colimit functor over \(\partial(c \downarrow \mathcal{C}^-)^{\text{op}}\) is left Quillen. Hence, it suffices to show that the restriction functor \(s\mathbf{Set}^{\mathcal{C}^+\text{op}} \to s\mathbf{Set}^{\partial(c \downarrow \mathcal{C}^-)^{\text{op}}}\) preserves Reedy cofibrations. But given \(\gamma: c \to d\) in \(\partial(c \downarrow \mathcal{C}^-)\), we have \(\partial(\gamma \downarrow \partial(c \downarrow \mathcal{C}^-)) \cong \partial(d \downarrow \mathcal{C}^-)\), so this restriction preserves matching objects. \(\square\)

By [Hir03, 15.6.24], the Reedy model structure on \(s\mathbf{Set}^{\mathcal{C}^+\text{op}}\) is cofibrantly generated; the generating Reedy acyclic cofibrations are the pushout products

\[
(\Lambda^n_k \otimes Y_c) \cup_{\Lambda^n_k \otimes L_c Y} (\Delta^n \otimes L_c Y) \longrightarrow \Delta^n \otimes Y_c.
\]

Here \(Y: \mathcal{C} \to s\mathbf{Set}^{\mathcal{C}^+\text{op}}\) is the Yoneda embedding, and \(K \otimes X\) denotes the simplicial tensor (which in this case is the levelwise cartesian product). Since the functors \(\Delta^n \otimes Y_c\) are exactly the representable functors in \(s\mathbf{Set}^{\mathcal{C}^+\text{op}}\) (when regarded as the presheaf category \(\mathbf{Set}^{(\Delta^n \times \mathcal{C}^+)}\)), we can apply Theorem 2.3 to obtain universes for small Reedy fibrations satisfying \((2')\).

Note that for any regular cardinal \(\kappa\), if \(f\) and \(g\) are composable functions such that \(g\) has \(\kappa\)-small fibers, then \(f\) has \(\kappa\)-small fibers if and only if \(gf\) does. Moreover,
\(\kappa\)-small morphisms are closed under limits of size \(< \kappa\). Thus, if \(\kappa > |C|\), a Reedy fibration is \(\kappa\)-small in the sense of §2 if and only if each map (5.1) is a \(\kappa\)-small fibration in \(\text{sSet}\).

We henceforth assume \(C\) to be an \emph{elegant Reedy category}. By [BR13], this implies that the Reedy cofibrations in \(\text{sSet}^{\text{op}}\) are exactly the (levelwise) monomorphisms, i.e. that the Reedy model structure coincides with the injective one. In particular, any direct category is elegant Reedy. Thus, to obtain a model of type theory with univalent universes, it remains only to show that the universes are fibrant.

**Lemma 5.3.** If \(C\) is an elegant Reedy category, then the Reedy model structure on \(\text{sSet}^{\text{op}}\) satisfies (3′).

*Proof.* Let \(i: A \rightarrow B\) be a Reedy (i.e. levelwise) acyclic cofibration, and let \(p: P \rightarrow A\) be a small Reedy fibration. The small fibration \(Q \rightarrow B\) we want to define will be, in particular, a factorization of the composite \(pi: P \rightarrow B\). By a standard argument for Reedy diagrams, to give such a factorization is equivalent to giving, by well-founded induction on \(c \in C\), a factorization of the induced map

\[
P_c \amalg_{L_c P} L_c Q \rightarrow B_c \times_{M_c B} M_c Q
\]

(with \(L_c Q\) and \(M_c Q\) being defined inductively as we go). Equivalently, we must give an object \(Q_c\) and dashed arrows which complete the following diagram to be commutative:

\[
\begin{array}{ccc}
P_c & \rightarrow & Q_c \\
\downarrow & & \downarrow \\
A_c \times_{M_c A} M_c P & \rightarrow & B_c \times_{M_c B} M_c Q
\end{array}
\]

By assumption, the lower-left map \(P_c \rightarrow A_c \times_{M_c A} M_c P\) in (5.4) is a small fibration. Thus, it has a classifying map \(A_c \times_{M_c A} M_c P \rightarrow U\), where \(U\) is a universe for \(\kappa\)-small Kan fibrations in \(\text{sSet}\). The section \(L_c P \rightarrow P_c\) of this fibration corresponds to a dashed lifting:

\[
\begin{array}{c}
L_c P \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
\tilde{U}
\end{array}
\]

Now by induction, for all \(\alpha: c \rightarrow d\) in \(C^-\), the map \(P_d \rightarrow Q_d\) is a pullback of an acyclic cofibration along a fibration, hence also an acyclic cofibration. Thus, by Lemma 5.2, so is \(L_c P \rightarrow L_c Q\). Therefore, since \(\tilde{U}\) is fibrant, we can extend the above classifying map \(L_c P \rightarrow \tilde{U}\) to \(L_c Q\).

Now I claim that the bottom horizontal map in (5.4) is an acyclic cofibration. By induction, for all \(\alpha: d \rightarrow c\) in \(C^+\), we have a pullback square

\[
\begin{array}{c}
P_d \rightarrow Q_d \\
\downarrow \downarrow \\
A_d \rightarrow B_d
\end{array}
\]
Therefore, the right-hand square below is also a pullback (while the left-hand square is a pullback by definition).

\[
\begin{array}{ccc}
A_c \times M_e A & \longrightarrow & M_e P \\
\downarrow & & \downarrow \\
A_c & \longrightarrow & M_e A
\end{array}
\rightarrow
\begin{array}{ccc}
& & M_e Q \\
& & \downarrow \\
& & M_e B.
\end{array}
\]

Thus, the outer rectangle above is also a pullback. Since this is also the outer rectangle in the next diagram, whose right-hand square is a pullback by definition, so is its left-hand square.

\[\text{(5.5)}\]

But each \(Q_d \rightarrow B_d\) is a fibration, hence by Lemma 5.2 so is \(M_c Q \rightarrow M_c B\). Therefore, so is the middle vertical map in (5.5). This means that the left-hand square in (5.5) exhibits the bottom horizontal map in (5.4) as a pullback of the acyclic cofibration \(A_c \rightarrow B_c\) along a fibration, so it is an acyclic cofibration.

Let \(D\) be the following pushout, with induced map as shown:

\[
\begin{array}{ccc}
L_c P & \longrightarrow & L_c Q \\
\downarrow & & \downarrow \\
A_c \times M_e A & \longrightarrow & M_c Q
\end{array}
\rightarrow
\begin{array}{ccc}
& & D \\
& & \downarrow \\
& & B_c \times M_e B M_e Q
\end{array}
\]

Since every morphism in \(C\) is split epic in an elegant Reedy category, every morphism \(L_c X \rightarrow M_c X\) is a monomorphism. It follows that the maps \(L_c P \rightarrow A_c \times M_e A M_e P\) and \(L_c Q \rightarrow B_c \times M_e B M_e Q\) are also monomorphisms. We have already observed that \(L_c P \rightarrow L_c Q\) and \(A_c \times M_e A M_e P \rightarrow B_c \times M_e B M_e Q\) are monomorphisms (cofibrations), so the above pushout is a union of subobjects, and hence the induced dotted map is also a monomorphism.

Moreover, we have also observed that \(A_c \times M_e A M_e P \rightarrow B_c \times M_e B M_e Q\) is an acyclic cofibration, and so is \(A_c \times M_e A M_e P \rightarrow D\) since it is a pushout of such. Therefore, by the 2-out-of-3 property, \(D \rightarrow B_c \times M_e B M_e Q\) is also a weak equivalence, hence an acyclic cofibration.

Now recall that we have a classifying map \(A_c \times M_e A M_e P \rightarrow U\) for \(P_c\), and an extension to \(L_c Q\) of its restriction to \(L_c P\) (by way of \(U\)). Thus, we have an induced map \(D \rightarrow U\), and since \(U\) is fibrant we can extend this map to \(B_c \times M_e B M_e Q\). Let \(Q_c \rightarrow B_c \times M_e B M_e Q\) be the fibration classified by this map. Then we have all the dashed arrows making (5.4) commutative and its lower square a pullback. By pasting this on top of the left-hand square in (5.5), we see that \(P_c\) is the pullback of \(Q_c\) along \(i_c\), as desired.

Putting this together with §§2–4, we have shown:
Theorem 5.6. For any elegant Reedy category $C$, the Reedy model category $sSet^{C^{op}}$ supports a model of intensional type theory with dependent sums and products, identity types, and as many univalent universes as there are inaccessible cardinals greater than $|C|$. □

Since direct categories are elegant Reedy categories, and presheaves on a direct category are of course the same as covariant diagrams on its opposite (which is an inverse category), this generalizes (the restriction to $sSet$ of) the corresponding theorem proven in [Shu12].

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