Essentially high-order compact schemes with application to stochastic volatility models on non-uniform grids

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Abstract
We present high-order compact schemes for a linear second-order parabolic partial differential equation (PDE) with mixed second-order derivative terms in two spatial dimensions. The schemes are applied to option pricing PDE for a family of stochastic volatility models. We use a non-uniform grid with more grid-points around the strike price. The schemes are fourth-order accurate in space and second-order accurate in time for vanishing correlation. In our numerical convergence study we achieve fourth-order accuracy also for non-zero correlation. A combination of Crank-Nicolson and BDF-4 discretisation is applied in time. Numerical examples confirm that a standard, second-order finite difference scheme is significantly outperformed.

1 Introduction
We consider the following parabolic partial differential equation for
\[ u = u(x_1, x_2, t) \] in two spatial dimensions and time,
\[ du_\tau + a_1 u_{x_1x_1} + a_2 u_{x_2x_2} + b_{12} u_{x_1x_2} + c_1 u_{x_1} + c_2 u_{x_2} = 0 \quad \text{in } \Omega \times ]0, T[ =: Q_T, \]
subject to suitable boundary conditions and initial condition \( u(x_1, x_2, 0) = u_0(x_1, x_2) \) with \( T > 0 \) and \( \Omega = [x_{\text{min}}^{(1)}, x_{\text{max}}^{(1)}] \times [x_{\text{min}}^{(2)}, x_{\text{max}}^{(2)}] \subset \mathbb{R}^2 \) with \( x_{\text{min}}^{(i)} < x_{\text{max}}^{(i)} \) for \( i = 1, 2 \). The functions \( a_i = a_i(x_1, x_2, \tau) < 0, b_{12} = b_{12}(x_1, x_2, \tau), \]
\( c_i = c_i(x_1, x_2, \tau), d = d(x_1, x_2, \tau) \) map \( Q_T \) to \( \mathbb{R} \), and \( a_i (\cdot, \tau), b (\cdot, \tau), c_i (\cdot, \tau), \) and \( d (\cdot, \tau) \) are assumed to be in \( C^2(\Omega) \) and \( u (\cdot, t) \in C^6(\Omega) \) for all \( \tau \in ]0, T[ \). We define a uniform spatial grid \( G \) with step size \( \Delta x_k \) in \( x_k \) direction for \( k = 1, 2 \). Setting \( f = -du_\tau \) and applying a standard, second-order central difference ap-

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approximation leads to the elliptic problem
\[ f = A_0 - \frac{a_1(\Delta x_1)^2}{12} \frac{\partial^4 u}{\partial x_1^4} - \frac{a_2(\Delta x_2)^2}{12} \frac{\partial^4 u}{\partial x_2^4} - \frac{b_{12}(\Delta x_1)^2}{6} \frac{\partial^4 u}{\partial x_1^2 \partial x_2} \]
with \( A_0 := a_1D_1D_1U_{i_1,i_2} + a_2D_2D_2U_{i_1,i_2} + b_{12}D_1D_2U_{i_1,i_2} + c_1D_1U_{i_1,i_2} + c_2D_2U_{i_1,i_2}, \)
where \( D_k \) denotes the central difference operator in \( x_k \) direction, and \( \varepsilon \in O(h^4) \) if \( \Delta x_k \in O(h) \) for \( h > 0 \). We call a finite difference scheme high-order compact (HOC) if its consistency error is of order \( O(h^4) \) for \( \Delta x_k \in O(h) \) for \( h > 0 \), and it uses only points on the compact stencil, \( U_{k,p} \) with \( k \in \{i_1-1, i_1, i_1+1\} \) and \( p \in \{i_2-1, i_2, i_2+1\} \), to approximate the solution at \( (x_{i_1}, x_{i_2}) \in \Omega \).

2 Auxiliary relations for higher derivatives

Our aim is to replace the third- and fourth-order derivatives in (2) which are multiplied by second-order terms by equivalent expressions which can be approximated with second order on the compact stencil. Indeed, if we differentiate (1) (using \( f = -du_x \)) once with respect to \( x_k \) \( (k = 1, 2) \), we obtain relations
\[ \frac{\partial^3 u}{\partial x_1^3} = A_1, \quad \frac{\partial^3 u}{\partial x_2^3} = A_2, \]
where we can discretise \( A_i \) with second order on the compact stencil using the central difference operator. Analogously, we obtain
\[ \frac{\partial^4 u}{\partial x_1^4} = B_1 - \frac{b_{12}}{a_1} \frac{\partial^4 u}{\partial x_1^2 \partial x_2} \quad \iff \quad \frac{\partial^4 u}{\partial x_1^4} = \frac{a_1}{b_{12}} B_1 - \frac{a_1}{b_{12}} \frac{\partial^4 u}{\partial x_1^2 \partial x_2}, \]
\[ \frac{\partial^4 u}{\partial x_2^4} = B_2 - \frac{b_{12}}{a_2} \frac{\partial^4 u}{\partial x_1 \partial x_2^2} \quad \iff \quad \frac{\partial^4 u}{\partial x_2^4} = \frac{a_2}{b_{12}} B_2 - \frac{a_2}{b_{12}} \frac{\partial^4 u}{\partial x_1 \partial x_2^2}, \]
\[ \frac{\partial^4 u}{\partial x_1 \partial x_2^3} = C_1 - \frac{a_1}{a_2} \frac{\partial^4 u}{\partial x_1 \partial x_2^2} \quad \iff \quad \frac{\partial^4 u}{\partial x_1 \partial x_2^3} = \frac{a_1}{a_2} C_1 - \frac{a_1}{a_2} \frac{\partial^4 u}{\partial x_1 \partial x_2^2}, \]
where we can approximate \( B_k \) and \( C_k \) with second order on the compact stencil using the central difference operator. A detailed derivation can be found in [3] [5].

3 Derivation of high-order compact schemes

In general it is not possible to obtain a HOC scheme for (1), since there are four fourth-order derivatives in (2), but only three auxiliary equations for these in (3). Hence, we propose four different versions of the numerical schemes, where only one of the fourth-order derivatives in (2) is left as a second-order remainder term. Using (3) and (4) in (2) we obtain as Version 1 scheme
\[ f = A_0 - \frac{c_1(\Delta x_1)^2}{6} A_1 - \frac{c_2(\Delta x_2)^2}{6} A_2 - \frac{a_2(\Delta x_2)^2}{12} B_2 - \frac{b_{12}(\Delta x_2)^2}{12} C_2 \]
\[ - \frac{a_1(2a_2(\Delta x_1)^2 - a_1(\Delta x_2)^2)}{12a_2} B_1 + \frac{a_1(a_2(\Delta x_1)^2 - a_1(\Delta x_2)^2)}{12a_2} \frac{\partial^4 u}{\partial x_1^2 \partial x_2} + \varepsilon, \]

(5)
as Version 2 scheme

\[
\begin{align*}
  f &= A_0 - \frac{c_1(\Delta x)^2}{6} A_1 - \frac{c_2(\Delta x)^2}{6} A_2 - \frac{a_1(\Delta x)^2}{12} B_1 - \frac{b_{12}(\Delta x)^2}{12} C_1 \\
  &\quad - \frac{a_2}{12a_1}(2a_1(\Delta x)^2 - a_2(\Delta x)^1) B_2 + \frac{a_2}{12a_1}(a_1(\Delta x)^2 - a_2(\Delta x)^1) \frac{\partial^4 u}{\partial x^4} + \varepsilon,
\end{align*}
\]

(6)

as Version 3 scheme

\[
\begin{align*}
  f &= A_0 - \frac{c_1(\Delta x)^2}{6} A_1 - \frac{c_2(\Delta x)^2}{6} A_2 - \frac{a_1(\Delta x)^2}{12} B_1 - \frac{a_2(\Delta x)^2}{12} B_2 \\
  &\quad - \frac{b_{12}}{12} (\Delta x)^2 C_2 + \frac{b_{12}}{12a_2} (a_1(\Delta x)^2 - a_2(\Delta x)^1) \frac{\partial^4 u}{\partial x^4} + \varepsilon,
\end{align*}
\]

(7)

and, finally, as Version 4 scheme

\[
\begin{align*}
  f &= A_0 - \frac{c_1(\Delta x)^2}{6} A_1 - \frac{c_2(\Delta x)^2}{6} A_2 - \frac{a_1(\Delta x)^2}{12} B_1 - \frac{a_2(\Delta x)^2}{12} B_2 \\
  &\quad - \frac{b_{12}}{12} (\Delta x)^2 C_1 + \frac{b_{12}}{12a_1} (a_2(\Delta x)^1 - a_1(\Delta x)^2) \frac{\partial^4 u}{\partial x^4} + \varepsilon.
\end{align*}
\]

(8)

Employing the central difference operator with \( \Delta x = \Delta y = h \) for \( h > 0 \) to discretise \( A_i, \ B_i, \ C_i \), in (5–8) and neglecting the remaining lower-order term leads to four semi-discrete (in space) schemes. A more detailed description of this approach can be found in [3–5]. When \( a_1 \equiv a_2 \) or \( b_{12} \equiv 0 \) these schemes are fourth-order consistent in space, otherwise second-order.

In time, we apply the implicit BDF4 method on an equidistant time grid with stepsize \( k \in \mathcal{O}(h^2) \). The necessary starting values are obtained using a Crank-Nicolson time discretisation, where we subdivide the first timesteps with a step size \( k' \in \mathcal{O}(h^2) \) to ensure the fourth-order time discretisation in terms of \( h \).

With additional information on the solution of (11) even better results are possible. If the specific combination of pre-factors in (11) and the higher derivatives in the second-order terms is sufficiently small, the second-order term dominates the computational error only for very small step-sizes \( h \). Before this error term becomes dominant one can observe a fourth-order numerical convergence. In this case we call the scheme essentially high-order compact (EHOC).

4 Application to option pricing

In this section we apply our numerical schemes to an option pricing PDE in a family of stochastic volatility models, with a generalised square root process for the variance with nonlinear drift term,

\[
\begin{align*}
  dS_t &= \mu S_t dt + \sqrt{\sigma} S_t dW_t^{(1)}, \\
  dv_t &= \kappa v_t (\theta - v_t) dt + \sigma \sqrt{v_t} dW_t^{(2)},
\end{align*}
\]

with \( \alpha \geq 0 \), a correlated, two-dimensional Brownian motion, \( dW_t^{(1)} dW_t^{(2)} = \rho dt \), as well as drift \( \mu \in \mathbb{R} \) of the stock price \( S \), long run mean \( \theta > 0 \), mean reversion speed \( \kappa > 0 \), and volatility of volatility \( \sigma > 0 \). For \( \alpha = 0 \) one obtains the
standard Heston model, for $\alpha = 1$ the SQRN model, see [1]. Using Itô’s lemma and standard arbitrage arguments, the option price $V = V(S, v, t)$ solves
\[
\frac{\partial V}{\partial t} + \frac{vS^2}{2} \frac{\partial^2 V}{\partial S^2} + \rho \sigma vS \frac{\partial^2 V}{\partial S \partial v} + \frac{\sigma^2 v}{2} \frac{\partial^2 V}{\partial v^2} + rS \frac{\partial V}{\partial S} + \kappa \nu^\alpha (\theta - v) \frac{\partial V}{\partial v} - rV = 0, \tag{9}
\]
where $S, \sigma > 0$ and $t \in [0, T]$ with $T > 0$. For a European Put with exercise price $K$ we have the final condition $V(S, T) = \max(0, K - S)$. The transformations $\tau = T - t$, $u = e^{-\nu \tau} V/K$, $\hat{S} = \ln(S/K)$, $v = v/\sigma$ as well as $\hat{S} = \varphi(x)$ [2], lead to
\[
\varphi^3 u_{\tau} + \frac{\sigma y}{2} \left[ \varphi_x u_{xx} + \varphi^3 u_{yy} \right] - \rho \sigma y \varphi_x^2 u_{xy}
\]
\[
+ \left[ \frac{\sigma y \varphi_{xx}}{2} + \left( \frac{\sigma y}{2} - r \right) \varphi_x^2 \right] u_x - \kappa \sigma^\alpha y \frac{\theta - \sigma y}{\sigma} \varphi_x^3 u_y = 0,
\]
with initial condition $u(x, y, 0) = \max(1 - e^{\psi(x), 0})$. The function $\varphi$ is considered to be four times differentiable and strictly monotone. It is chosen in such a way that grid points are concentrated around the exercise price $K$ in the $S-v$ plane when using a uniform grid in the $x-y$ plane.

Dirichlet boundary conditions are imposed at $x = x_{\min}$ and $x = x_{\max}$ similarly as in [2],
\[
u(x_{\min}, y, \tau) = u(x_{\min}, y, 0), \quad u(x_{\max}, y, \tau) = u(x_{\max}, y, 0),
\]
for all $\tau \in [0, \tau_{\max}]$ and $y \in [y_{\min}, y_{\max}]$. At the boundaries $y = y_{\min}$ and $y = y_{\max}$ we employ the discretisation of the interior spatial domain and extrapolate the resulting ghost-points using
\[
U_{i-1} = 3U_{i,0} - 3U_{i,1} + U_{i,2} + O(h^3),
\]
\[
U_{i,M+1} = 3U_{i,M} - 3U_{i,M-1} + U_{i,M-2} + O(h^3),
\]
for $i = 0, \ldots, N$. Third-order extrapolation is sufficient here to ensure overall fourth-order convergence [4].

5 Numerical experiments

We employ the function $\varphi(x) = \sinh(c_2 x + c_1 (1 - x))/\zeta$, where $c_1 = \sinh(\zeta \hat{S}_{\min})$, $c_2 = \sinh(\zeta \hat{S}_{\max})$ and $\zeta > 0$. We use $\kappa = 1.1$, $\theta = 0.2$, $v = 0.3$, $r = 0.05$, $K = 100$, $T = 0.25$, $v_{\min} = 0.1$, $v_{\max} = 0.3$, $S_{\min} = 1.5$, $S_{\max} = 250$, $\rho = 0$, $-0.4$ and $\zeta = 7.5$. Hence, $x_{\max} - x_{\min} = y_{\max} - y_{\min} = 1$. For the Crank-Nicolson method we use $k'/h^2 = 0.4$, for the BDF4 method $k/h = 0.1$. We smooth the initial condition according to [6, 3], so that the smoothed initial condition tends towards the original initial condition for $h \to 0$. We neglect the case $\alpha = 0$ (Heston model), since a numerical study of that case has been performed in [2].

In the numerical convergence plots we use a reference solution $U_{\text{ref}}$ on a fine grid ($h = 1/320$) and report the absolute $L^2$-error compared to $U_{\text{ref}}$. The numerical convergence order is computed from the slope of the linear least square fit of the points in the log-log plot.

Figure 1(a) shows the transformation from $x$ to $S$. The transformation focuses on the region around the strike price. Figures 1(b), 1(c), 1(d) and 1(e) show that the HOC schemes lead to a numerical convergence order of about 3.5.
whereas the standard, second-order central difference discretisation (SD) leads to convergence orders of about 2.3, in the case of vanishing correlation. In all cases with non-vanishing correlation ($\rho \neq 0$) we observe only slightly improved convergence for Version 1 (V1) when comparing it to the standard discretisation. Version 2 (V2) and Version 3 (V3), however, lead to similar convergence orders as the HOC scheme, even for non-vanishing correlation. Results of Version 4 are not shown as this scheme shows instable behaviour in this example.

In summary, we obtain high-order compact schemes for vanishing correlation and achieve high-order convergence also for non-vanishing correlation for the family of stochastic volatility model. A standard, second-order discretisation is significantly outperformed in all cases.

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