Killing forms on the five-dimensional Einstein-Sasaki $Y(p, q)$ spaces

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Abstract

We present the complete set of Killing-Yano tensors on the five-dimensional Einstein-Sasaki $Y(p, q)$ spaces. Two new Killing-Yano tensors are identified, associated with the complex volume form of the Calabi-Yau metric cone. The corresponding hidden symmetries are not anomalous and the geodesic equations are superintegrable.

Keywords: Einstein-Sasaki spaces, metric cone, Killing forms.

1 Introduction

In the last time Einstein-Sasaki geometries have become of large interest in connection with many modern studies in physics. In this paper we deal with the infinite family $Y(p, q)$ of Einstein-Sasaki metrics on $S^2 \times S^3$ \cite{1,2,3,4}. Such manifolds provide supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space $Y(p, q)$ of an $S^1$-fibration over $S^2 \times S^2$ with relative prime winding numbers $p$ and $q$ is topologically $S^2 \times S^3$.

In the present paper it will be shown that the equations of the geodesic motions on the $Y(p, q)$ spaces are superintegrable. For this purpose we present the complete set of Killing-Yano tensors which play an essential role for the integrability of the equations of motion.

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In general a system could possess explicit and hidden symmetries encoded in the multitude of Killing vectors and higher rank Killing tensors respectively. The customary conserved quantities originate from symmetries of the configuration space of the system. They are represented by isometries of the metric generated by Killing vector fields. An extension of the Killing vector fields is given by the conformal Killing vector fields with flows preserving a given conformal class of metrics [5].

A natural generalization of Killing vector fields is represented by conformal Killing-Yano tensors. A conformal Killing-Yano tensor of rank $p$ on a $n$-dimensional Riemannian manifold $(M, g)$ is a $p$-form $\omega$ which satisfies:

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner \, d \omega - \frac{1}{n-p+1} X^* \wedge d^* \omega, \quad (1)$$

for any vector field $X$ on $M$. Here $\nabla$ is the Levi-Civita connection of $g$, $X^*$ is the 1-form dual to the vector field $X$ with respect to the metric $g$, $ \lrcorner $ is the operator dual to the wedge product and $d^*$ is the adjoint of the exterior derivative $d$. If $\omega$ is co-closed in (1), then we obtain the definition of a Killing-Yano tensor [5]. Killing-Yano tensors are also called Yano tensors or Killing forms. A particular class of Killing forms is represented by the special Killing forms which satisfy, for some constant $c$, the additional equation [6]:

$$\nabla_X (d \omega) = c X^* \wedge \omega, \quad (2)$$

for any vector field $X$ on $M$. Let us note that most known Killing forms are special.

Besides the antisymmetric generalizations of the Killing vectors one might also consider higher order symmetric tensors. A covariant symmetric field $K$ of rank $r$ on a Riemannian manifold $(M, g)$ is a Stäckel-Killing tensor field if the symmetrization of its covariant derivative vanishes identically

$$\nabla (\lambda K_{\mu_1,\ldots,\mu_r}) = 0.$$  

These symmetric tensors are associated with conserved quantities of degree $r$ in the momentum variables, and generate symplectic transformations in the phase space of the system [7].

These two generalizations of the Killing vectors could be related. Given two rank $r$ Killing forms $\omega_{\mu_1\ldots\mu_r}$ and $\sigma_{\mu_1\ldots\mu_r}$ it is possible to associate with them a Stäckel-Killing tensor of rank 2

$$K^{(\omega,\sigma)}_{\mu\nu} = (\omega_{\mu\lambda_2\ldots\lambda_r} \sigma^{\lambda_2\ldots\lambda_r} + \sigma_{\mu\lambda_2\ldots\lambda_r} \omega^{\lambda_2\ldots\lambda_r}). \quad (3)$$

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At the quantum level the operator $\nabla_\mu K^{\mu \nu} \nabla_\nu$ may be thought of as the equivalent of the classical conserved quantity $K_{\mu \nu} \dot{\gamma}^{\mu} \dot{\gamma}^{\nu}$ which is constant along geodesics $\gamma$. The remarkable fact is that $\nabla_\mu K^{\mu \nu} \nabla_\nu$ operating on scalar fields commutes with the Klein-Gordon operator $\Box = \nabla_\mu g^{\mu \nu} \nabla_\nu$ \[8\]. Therefore, when the Stäckel-Killing tensor $K_{\mu \nu}$ is of the form \[3\], there are no quantum anomalies thanks to an integrability condition satisfied by the Killing-Yano tensors \[7, 8, 9\].

The organization of this paper is as follows: In the next Section we present what is essentially a brief review of how Einstein-Sasaki manifolds can be constructed as $U(1)$ bundles over Einstein-Kähler manifolds. Further an Einstein-Sasaki metric may be defined as a $(2n + 1)$-dimensional Einstein metric such that the cone over it is a Kähler, Ricci-flat metric of complex dimension $n + 1$. In Section 3 we restrict to the five-dimensional $Y(p, q)$ manifolds and present the complete set of Killing forms. Finally we give our conclusions in Section 4.

2 Progression from Einstein-Kähler to Einstein-Sasaki to Calabi-Yau manifolds

A Riemannian manifold $(M_{2n+1}, g_S)$ of odd dimension $2n + 1$ is Sasakian if and only if its metric cone

$$C(M_{2n+1}) = \mathbb{R}_{>0} \times M_{2n+1}, \quad g_{\text{cone}} = dr^2 + r^2 g_S,$$

is Kähler with the complex dimension $n + 1$. Moreover a Sasakian metric $g_{ES}$ is Einstein with $\text{Ric} = 2ng_{ES}$ if and only if its metric cone is Ricci-flat and Kähler, i. e. a Calabi-Yau manifold \[10\].

On the other hand Sasakian manifolds can be constructed as principal $S^1$-bundle over a Kähler manifold $M_{2n}$. It is convenient to write the Einstein-Sasaki metric in the form \[11, 12\]

$$g_{ES} = g_{EK} + (d\alpha + \sigma)^2,$$

where $g_{EK}$ is the metric of the Einstein-Kähler manifold $M_{2n}$ with the complex dimension $n$ and

$$\eta = d\alpha + \sigma,$$

is the Sasakian 1-form of the Einstein-Sasaki metric. The form $\sigma$ satisfies

$$d\sigma = 2\Omega_{EK},$$

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where $\Omega_{EK}$ is Kähler form of the Einstein-Kähler base manifold $M_{2n}$. In consequence the Kähler form of the Calabi-Yau cone manifold can be written as

$$\Omega_{\text{cone}} = r dr \wedge (d\alpha + \sigma) + r^2 \Omega_{EK}.$$  \hfill (6)

In local holomorphic coordinates $(z^1, \ldots, z^m)$, the associated Kähler form $\Omega$ of a Kähler manifold $M_m$ of complex dimension $m$ is

$$\Omega = i g_{jk} dz^j \wedge d\bar{z}^k,$$

and the volume form (which is just the Riemannian volume form determined by the metric) is a real $(m, m)$-form

$$dV = \frac{1}{m!} \Omega^m,$$

where $\Omega^m$ is the wedge product of $\Omega$ with itself $m$ times. If the volume of a Kähler manifold is written as

$$dV = dV \wedge d\bar{V},$$

then $dV$ is the complex volume holomorphic $(m, 0)$-form of $M$ \[13, 14\].

There is a 1-1-correspondence between the special Killing $p$-forms \[2\] on a compact oriented simply connected Riemannian manifold $M$ and parallel $(p + 1)$-forms on the cone manifold $C(M)$. Taking into account that the metric cone $C(M)$ is either flat or irreducible, the existence of Killing forms on the manifold $M$ can be settled investigating the parallel forms on flat or irreducible manifolds. These forms are classified by means of holonomy groups \[15, 16\]. In the case of a Sasakian base manifold $M_{2n+1}$ the following two possibilities are of interest \[6\]:

1. The metric cone $C(M_{2n+1})$ has holonomy $U(n + 1)$ and equivalently is Kähler, $M_{2n+1}$ is Sasaki and all special Killing forms are described by the $(2k + 1)$-forms:

$$\Psi_k = \eta \wedge (d\eta)^k, \quad k = 0, 1, \ldots, n - 1.$$  \hfill (7)

Besides these Killing forms, there are $n$ closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (d\eta)^k, \quad k = 1, \ldots, n.$$
2. The holonomy of $C(M_{2n+1})$ is $SU(n+1)$, the metric cone is Kähler and in addition Ricci flat, or equivalently $M_{2n+1}$ is Einstein-Sasaki. In this situation there are two additional Killing forms of degree $n$ on the manifold $M_{2n+1}$. These additional Killing forms are connected with the additional parallel forms of the Calabi-Yau cone manifold $M_{2n+2}$ given by the complex volume form and its conjugate [6] which can also be described using the Killing spinors of an Einstein-Sasaki manifold [17].

3 $Y(p, q)$ manifolds

The starting point is the explicit local metric of the 5-dimensional $Y(p, q)$ manifold given by the line element [1, 2, 18, 19]

$$ds^2_{ES} = \frac{1 - cy}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} (d\psi - \cos \theta d\phi)^2$$

$$+ w(y) \left[ d\alpha + \frac{ac - 2y + cy^2}{6(a - y^2)} (d\psi - \cos \theta d\phi) \right]^2,$$

where

$$w(y) = \frac{2(a - y^2)}{1 - cy}, \quad q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2},$$

and $a, c$ are constants. A detailed discussion of the range of these parameters is given in [2] in connection with the regularity properties of the $Y(p, q)$ metrics. For $c = 0$ the metric takes the local form of the standard homogeneous metric on $T^{1,1}$ [20]. Otherwise the constant $c$ can be rescaled by a diffeomorphism and in what follows we assume $c = 1$.

The coordinate change $\alpha = -\frac{1}{6} \beta - \frac{1}{a} \psi', \quad \psi = \psi'$ takes the line element (8) to the following form

$$ds^2_{ES} = \frac{1 - y}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{p(y)} dy^2 + \frac{1}{36} p(y) (d\beta + \cos \theta d\phi)^2$$

$$+ \frac{1}{9} [d\psi' - \cos \theta d\phi + y(d\beta + \cos \theta d\phi)]^2,$$

with $p(y) = w(y) q(y)$.

The Sasakian 1-form of the $Y(p, q)$ space is

$$\eta = \frac{1}{3} d\psi' + \sigma,$$
with
\[ \sigma = \frac{1}{3} \left[ -\cos \theta d\phi + y(d\beta + \cos \theta d\phi) \right], \]
connected with local Kähler form \( \Omega_{EK} \) as in (5). Note also that the holomorphic \((2,0)\)-form of the Einstein-Kähler base manifold \( M_4 \) is
\[
dV_{EK} = \sqrt{1 - \frac{y}{6p(y)}} (d\theta + i \sin \theta d\phi) \wedge \left[ dy + i \frac{p(y)}{6} (d\beta + \cos \theta d\phi) \right].
\]

The form of the metric (8) with the 1-form (9) is the standard one for a locally Einstein-Sasaki metric with \( \partial_{\psi'} \) the Reeb vector field.

From the isometries \( SU(2) \times U(1) \times U(1) \) the momenta \( P_\phi, P_\psi, P_\alpha \) and the Hamiltonian describing the geodesic motions are conserved [18, 19]. \( P_\phi \) is the third component of the \( SU(2) \) angular momentum, while \( P_\psi \) and \( P_\alpha \) are associated with the \( U(1) \) factors. Additionally, the total \( SU(2) \) angular momentum given by
\[
J^2 = P_\theta^2 + \frac{1}{\sin^2 \theta} (P_\phi + \cos \theta P_\psi)^2 + P_\psi^2,
\]
is also conserved.

In what follows we are looking for further conserved quantities specific for motions in Einstein-Sasaki spaces. First of all, on the Sasakian manifold with the 1-form (4), the form (7) for \( k = 1 \)
\[
\Psi = \frac{1}{9} \left[ (1 - y) \sin \theta d\theta \wedge d\phi \wedge d\psi' + dy \wedge d\beta \wedge d\phi' + \cos \theta dy \wedge d\phi \wedge d\psi' - \cos \theta dy \wedge d\beta \wedge d\phi \right]
+ (1 - y) y \sin \theta d\beta \wedge d\theta \wedge d\phi',
\]
is a special Killing form (2). Let us note also that
\[
\Psi_k = (d\eta)^k, \quad k = 1, 2,
\]
are closed conformal Killing forms [12, 14].

On the Calabi-Yau manifold \( C(M_{2n+2}) \) the Kähler form (6) with the Sasakian 1-form (9) is
\[
\Omega_{cone} = r^2 \frac{1 - y}{6} \sin \theta d\theta \wedge d\phi + \frac{r^2}{6} dy \wedge (d\beta + \cos \theta d\phi)
+ \frac{1}{3} r dr \wedge [ y d\beta + d\psi' - (1 - y) \cos \theta d\phi].
\]
The complex volume holomorphic \((3,0)\)-form on the metric cone is \(dV_{\text{cone}} = e^{i\psi'} r^2 dV_{EK} \wedge (dr + ir \wedge \eta)\)

\[
dV_{\text{cone}} = e^{i\psi'} r^2 \sqrt{1 - \frac{y}{6p(y)}} (d\theta + i \sin \theta \, d\phi) \\
\wedge \left[ dy + i \frac{p(y)}{6} (d\beta + \cos \theta \, d\phi) \right] \\
\wedge \left[ dr + \frac{r}{3} [y \, d\beta + d\psi' - (1 - y) \cos \theta \, d\phi] \right].
\]

(12)

To find explicitly the additional Killing forms on the \(Y(p,q)\) spaces we shall consider the additional parallel forms on the metric cone related to the complex volume form (12) and its conjugate. As real forms they are given by the real and respectively imaginary part of the volume form. For this purpose we make use of the fact that for any \(p\)-form \(\omega^M\) on the space \(M_{2n+1}\) we can define an associated \((p+1)\)-form \(\omega^C\) on the cone \(C(M_{2n+1})\)

\[
\omega^C := r^p dr \wedge \omega^M + \frac{r^{p+1}}{p+1} d\omega^M.
\]

(13)

Moreover \(\omega^C\) is parallel if and only if \(\omega^M\) is a special Killing form \(\omega^\phi\) with constant \(c = -(p+1)\) \(\phi\). Therefore the 1-1-correspondence between special Killing \(p\)-forms on \(M_{2n+1}\) and parallel \((p+1)\)-forms on the metric cone \(C(M_{2n+1})\) allows us to describe the additional Killing forms on Einstein-Sasaki \(Y(p,q)\) spaces.

Extracting from the complex volume form (12) the form \(\omega^M\) on the Einstein-Sasaki space according to (13) for \(p = 2\) we get the following additional Killing 2-forms of the \(Y(p,q)\) spaces written as real forms:

\[
\Xi = \text{Re} \, \omega^M = \sqrt{1 - \frac{y}{6p(y)}} \\
\times \left( \cos \psi' \left[ -dy \wedge d\theta + \frac{p(y)}{6} \sin \theta \, d\beta \wedge d\phi \right] - \\
- \sin \psi' \left[ -\sin \theta \, dy \wedge d\phi - \frac{p(y)}{6} d\beta \wedge d\theta + \frac{p(y)}{6} \cos \theta \, d\theta \wedge d\phi \right] \right),
\]

(14)
\[ \Upsilon = \text{Im} \omega^M = \sqrt{\frac{1 - y}{6p(y)}} \]
\[ \times \left( \cos \psi' \left[ -\sin \theta \, dy \wedge d\phi - \frac{p(y)}{6} d\beta \wedge d\theta + \frac{p(y)}{6} \cos \theta \, d\theta \wedge d\phi \right] - \left( \sin \psi' \left[ -dy \wedge d\theta + \frac{p(y)}{6} \sin \theta \, d\beta \wedge d\phi \right] \right) \right) \] (15)

The Stäckel-Killing tensors associated with the Killing forms \( \Psi \), \( \Xi \), \( \Upsilon \) are constructed as in (3). The list of the non vanishing components of these Stäckel-Killing tensors is quite long and will be given elsewhere. Together with the Killing vectors \( P_\phi, P_\psi, P_\alpha \) and the total angular momentum \( J^2 \) these Stäckel-Killing tensors provide the superintegrability of the \( Y(p, q) \) geometries.

4 Conclusions

In this paper we have presented the complete set of Killing forms on five-dimensional Einstein-Sasaki \( Y(p, q) \) spaces. The multitude of Stäckel-Killing tensors associated with these Killing forms implies the superintegrability of the geodesic motions.

In connection with the third rank Killing-Yano tensors on the \( Y(p, q) \) spaces let us note an interesting geometrical interpretation of the Lax representation [21, 22, 23].

In the theory of the classical spinning particles additional non-generic supersymmetries are generated from Killing-Yano tensors. At the quantum level from Killing forms one can construct Carter-McLenagham like operators [24] which commute with the standard Dirac operator. It is also worth noting that in the full quantum theory the symmetries generated by Killing forms are not anomalous [7].

These remarkable properties of the Killing forms offer new perspectives in the investigation of the supersymmetries, separability of Hamilton-Jacobi, Klein-Gordon, Dirac equations on \( Y(p, q) \) spaces.

Acknowledgments

Support through CNCS-UEFISCDI project number PN-II-ID-PCE-2011-3-0137 is acknowledged.
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