Conservative stochastic 2-dimensional Cahn-Hilliard equation *

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Abstract

We consider the stochastic 2-dimensional Cahn-Hilliard equation which is driven by the derivative in space of a space-time white noise. We use two different approaches to study this equation. First we prove that there exists a unique solution $Y$ to the shifted equation (1.4). Then $X := Y + Z$ is the unique solution to the stochastic Cahn-Hilliard equation, where $Z$ is the corresponding O-U process. Moreover, we use the Dirichlet form approach in [AR91] to construct a probabilistically weak solution to the original equation (1.1) below. By clarifying the precise relation between the two solutions, we also get the restricted Markov uniqueness of the generator and the uniqueness of the martingale solutions to the equation (1.1). Furthermore, we also obtain exponential ergodicity of the solutions.

Keywords: stochastic quantization problem, Dirichlet forms, space-time white noise, Wick power, non-linear stochastic PDE

1 Introduction

In this paper we show the well-posedness for the conservative stochastic Cahn-Hilliard equation

\[
\begin{aligned}
    dX_t &= -\frac{1}{2} A \left( AX - : X^3 : \right) dt + B dW_t, \\
    X(0) &= z \in V^{-1}_0,
\end{aligned}
\tag{1.1}
\]

on $\mathbb{T}^2$ in the probabilistically strong sense where $A = \Delta$, $B = \text{div}$. $W_t$ is an $L^2_0(\mathbb{T}^2, \mathbb{R}^2)$-cylindrical Wiener process, which is defined in Section 3. $X^3$ denotes the Wick power, which is introduced in Section 3 and the space $V^{-1}_0$ is defined similarly as the Sobolev space of order $-1$ (see Section 2.

The Cahn-Hilliard equation is given by

\[
\partial_t u = -\Delta^2 u - \Delta f(u),
\]

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which was introduced by Cahn and Hilliard [CH58] to study the phase separation of binary alloys. Here $f$ is the derivative of a free energy and generally $f$ is chosen as $f(u) = u^3 - u$. The stochastic Cahn-Hilliard equation was first studied in [PAMS], where Petschek and Metiu performed some numerical experiments for the stochastic Cahn-Hilliard equation driven by space-time white noise. In [EM91], Elezovic and Mikelic proved the existence and uniqueness of a strong solution to the stochastic Cahn-Hilliard equation driven by trace-class noise. Then Da Prato and Debussche [DPD96] proved existence and uniqueness of solutions for space-time white noise and obtained the existence and uniqueness of an invariant measure for trace-class noise. Later there many more papers appeared in which the authors study the properties of the solutions to the stochastic Cahn-Hilliard equations driven by trace-class noise (e.g. [DG11, Sca17]).

For the conservative-type equation (1.1), the Gibbs measure $\nu$ is formally given by

$$
\nu(d\phi) = c \exp\left( -\int_{\mathbb{R}^2} \frac{1}{4} : \phi^4 : dx \right) \mu(d\phi),
$$

where $\mu$ is the Gaussian free field $\mathcal{N}(0, (\Delta)^{-1})$, $c$ is a normalization constant, and $: \phi^4 :$ is the fourth order Wick power of $\phi$. Thus the Gibbs measure $\nu$ is the restriction of $\Phi^4_2$-field to the mass-conserving subspace, i.e. $\{ f : \int f(x)dx = 0 \}$. Equation (1.1) is sometimes called time-dependent Ginzburg-Landau (TDGL) equation of conservative type or Model B, while the stochastic Allen-Cahn equation (i.e. dynamical $\Phi^4_2$-equation) is called the TDGL equation of non-conservative type or Model A (see [HH77, Fmm11]). Since equation (1.1) satisfies a conservation law, i.e. $\partial_t \int X(t,x)dx \equiv 0$ for any solution $X$ to (1.1), the class of equilibrium states is richer than for the stochastic Allen-Cahn equation.

In [PAMS] Parisi and Wu proposed a program for Euclidean quantum field theory based on getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially as solutions to non-linear stochastic differential equations. Then one can use the stochastic differential equations to study properties of the Gibbs states. This procedure is called stochastic field quantization (see [ILMS85]). The equation (1.1) can be also viewed as a stochastic quantization equation for the conservative $\Phi^4_2$-field $\nu$.

Over the years, on the stochastic quantization of the $\Phi^4_2$-field, the literature has kept on growing (see e.g. [ILMS85, AR91, DPD96, MW17, RZZ17a, RZZ17b]). The authors in these papers consider the following non-conservative stochastic quantization equation (Model A):

$$
\begin{align*}
\frac{dX_t}{t} &= (AX - : X^3 : )dt + dW_t. \\
\end{align*}
$$

First results are due to Jona-Lasinio and Mitter [ILMS85]. Using the Girsanov theorem, they constructed solutions to a modified equation on $\mathbb{T}^2$:

$$
\begin{align*}
\frac{dX_t}{t} &= (-\Delta + 1)^{-\varepsilon}(\Delta X - : X^3 : + aX) + (-\Delta + 1)^{-\frac{1}{2}}dW_t. \\
\end{align*}
$$

for $\frac{9}{10} < \varepsilon < 1$. They also proved the ergodicity for (1.3). In [AR91] Albeverio and Röckner studied (1.2) using Dirichlet forms and constructed probabilistically weak solutions to (1.2) for all $\varepsilon \in [0, 1]$. In [MR95], Mikulevicius and Rozovskii constructed martingale solutions to (1.2) but the uniqueness remained open. In [DPD96] Da Prato and Debussche considered the associated shifted equation to (1.2) on $\mathbb{T}^2$ and proved local existence and uniqueness of solutions in the probabilistically strong sense via a fixed point argument and then showed the non-explosion for almost every initial point by using the invariant measure. Recently Mourrat and Weber [MW17] showed the global existence and uniqueness for the shifted equation both on $\mathbb{T}^2$ and $\mathbb{R}^2$ for every initial point. Combining the results from the weak approach and strong approach, Röckner, Zhu and Zhu [RZZ17b] proved the restricted Markov uniqueness for the generator of (1.2) and the uniqueness of the martingale problem.
to (1.2) in [MR99] on $\mathbb{T}^2$ and $\mathbb{R}^2$. Furthermore, the ergodicity of (1.2) on $\mathbb{T}^2$ has been obtained in [HM18, RZZ17a, IW18].

For the conservative case, Funaki [Fun89] proved the existence and uniqueness of solutions to equation (1.1) on $\mathbb{R}$ and in [DZ07] Debussche and Zambotti studied equation (1.1) on $[0,1]$ with reflection. But for the higher dimensional case, even though the linear operator $\Delta^2$ gives much more regularity, the noise and hence the solutions are still so singular that the non-linear terms in (1.1) are not well-defined in the classical sense. This difficulty is similar as in equation (1.2).

To overcome this difficulty, we use two approaches to study (1.1). First we follow the idea in [DDP03, MW17a] and [RZZ17b] to split the solution to $X = Y + Z$, where $Z(t) = \int_0^t e^{-\frac{t-s}{2} A^2} BdW_s$. Similarly as in the $\Phi^4_2$ case, $Y$ has better regularity than the solution to (1.1) and satisfies the following shifted equation:

$$\begin{aligned}
\begin{cases}
\frac{dY}{dt} &= -\frac{1}{2} A^2 Y + \frac{1}{2} A \sum_{k=0}^{3} C_k^k Y^{3-k} : Z^k : \\
Y(0) &= z
\end{cases}
\end{aligned}$$

(1.4)

where $Z(t) = \int_0^t e^{-\frac{t-s}{2} A^2} BdW_s$. In this paper we obtain the existence and uniqueness of the solution to (1.4). The fixed point arguments for local well-posedness in [DDP03] and [MW17a] only hold for initial values in $C^{\frac{1}{2},+}$. Due to the singularity of the noise and the lack of a maximum principle and a uniform $L^p$-estimate, we only have a uniform $H^{-1}$-estimate (see Theorem 1.1), which is not strong enough to combine it with local well-posedness (see Remark 1.3). Instead, our argument is based on a classical compactness argument. We obtain the existence of global solutions starting from the uniform $H^{-1}$-estimate directly. Moreover we consider the solutions in $H^{-1}$ and use the $L^4$-integrability to obtain uniqueness for (1.4).

In addition, we use the method in [AR91] to construct the Dirichlet form for (1.1) (see Theorem 5.12), which is given by

$$\mathcal{E}(\varphi, \psi) = \frac{1}{2} \int \langle \nabla \varphi, \nabla \psi \rangle_{V_0} d\nu, \varphi, \psi \in \mathcal{F}C^\infty,$$

where $\mathcal{F}C^\infty$ is defined in Section 6. We note that the tangent space is chosen as $V_0^{-1}$ and the gradient operator $\nabla$ is also defined in $H^{-1}$. This is different from the Dirichlet form for (1.2), where the tangent space is chosen as $L^2$ and the gradient is the $L^2$-derivative. By the integration by parts formula for $\nu$ we also obtain the closability for the bilinear form $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form, which enables us to construct a probabilistically weak solution to (1.1). Then by clarifying the relation between this solution and the solution to (1.3), we prove that $X - Z$, where $X$ is the solution obtained by the Dirichlet form approach, also satisfies the shifted equation (1.4). It follows that the $\Phi^4_2$-field is an invariant measure for $X$. Then we obtain the Markov uniqueness in the restricted sense for the generator of the Dirichlet form restricted to $\mathcal{F}C^\infty$ and the uniqueness of probabilistically weak solutions to (1.1) having $\nu$ as an invariant measure.

We prove exponential ergodicity by two approaches. One simple and short way based on the Dirichlet form approach is presented in Remark 6.4. Using a uniform estimate, an invariant measure can also be constructed by the Krylov-Bogoliubov method. We follow an idea from [IW18] to prove the strong Feller property of the semigroup of the solution to the equation (1.1). Then we obtain exponential convergence to the unique invariant measure of the semigroup for every starting point.

Finally we comment on the motivations to study the Dirichlet form of equation (1.1). First Dirichlet form theory plays an important role in the scaling limit of particle system. It was conjectured in [GLP99] that the stochastic Cahn-Hilliard equation is the scaling limit of the Kawasaki dynamics of the Ising-Kac model, while it has been proved in [BPRS93, MW17a] that the stochastic Allen-Cahn equation is the scaling limit of Glauber dynamics of the Ising-Kac model. Until now, for the scaling
limit of the Kawasaki dynamics, even in the $1 - d$ case where no renormalization method involved, there is still no complete result (see [Ibe18]). To use Dirichlet form theory to identify the scaling limit of particle systems, the Markov uniqueness of the Dirichlet form is required. We hope that in future work we can use the restricted Markov uniqueness of the Dirichlet form obtained in this paper to study the scaling limit of the Kawasaki dynamics. Another motivation to study this Dirichlet form is to study spectral properties. As we have shown in Remark [Ibe18], the spectral gap for the stochastic Cahn-Hilliard equation is controlled by the spectral gap of the stochastic Allen-Cahn equation in finite volume. But the situation is different from infinite volume case. On the whole space $\mathbb{R}^d$, it is expected that for the polynomial potential given by $\Phi^4 - m\Phi^2$, the stochastic Allen-Cahn equation still has a spectral gap so that exponential ergodicity holds, while the stochastic Cahn-Hilliard equation looses this property. In fact, the Dirichlet form $(\Lambda, D(\Lambda))$ for equation (1.1) on $\mathbb{R}^2$ can be directly constructed as the closure of the following bilinear form

$$\Lambda(\varphi, \psi) = \int \langle \nabla f, \nabla g \rangle_{\dot{H}^{-1}}, d\nu, \forall \varphi, \psi \in FC_b^\infty(\dot{H}^{-1}),$$

(1.5)

where $\dot{H}^s$ is the homogeneous Sobolev space of order $s$, and $\nabla$ is the gradient in $\dot{H}^{-1}$. Similarly to Section 3, it is easy to check that $(\Lambda, D(\Lambda))$ is quasi-regular and obtain a probabilistically weak solution directly. Since $(-\Delta)^{-1}$ is not bounded in $\dot{H}^{-1}$, the argument in Remark [Ibe18] fails in the case of $\mathbb{R}^2$. We hope to use (1.5) to study spectral properties and functional inequalities of the stochastic Cahn-Hilliard equation on $\mathbb{R}^2$ in our future work. Moreover, from the viewpoint of particle systems, it has been proved in [BPRS99, MW17a] that the stochastic Allen-Cahn equation is the scaling limit of the Glauber dynamics of the Ising-Kac model, while the stochastic Cahn-Hilliard equation is expected to be the scaling limit of the Kawasaki dynamics of the Ising-Kac model. The spectral gap for these two kinds of particle system was studied in [LYB19]. The authors considered the model on a bounded domain with size $L$ and proved that as $L \to \infty$, the spectral gap for the Glauber dynamics remains strictly positive while the spectral gap for Kawasaki dynamics decays with a rate $L^{-2}$ (see also recent work for the continuum Sine-Gordon model [BS14]).

This paper is organized as follows: In Section 2 we collect some results related to Besov spaces. In Section 3 we study the solution to the linear equation and define the Wick power. In Section 4 we obtain the global existence and uniqueness of solutions to the shifted equation (1.4). In Section 5 we obtain existence of probabilistically weak solutions via the Dirichlet form approach. By clarifying the relation between the two solutions we obtain that the $\Phi_2^4$-field $\nu$ is an invariant measure of $X$, Markov uniqueness in the restricted sense for the generator of the Dirichlet form restricted to $FC_b^\infty$ and uniqueness of the probabilistically weak solutions to (1.1). Finally we prove the strong Feller property and exponential ergodicity of the Markov semigroup associated to the solution to (1.1) in Section 6.

2 Preliminaries

Below we recall the definition of Besov spaces. For a general introduction to the theory of Besov spaces we refer to [BCD11, IIP78, IIP16]. First we introduce the following notations. Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and we write $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $S(\mathbb{R}^d)$. Its dual, the space of tempered distributions, is denoted by $S'(\mathbb{R}^d)$. The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively.

Let $\chi, \theta \in \mathcal{D}$ be non-negative radial functions on $\mathbb{R}^d$, such that
i. the support of $\chi$ is contained in a ball and the support of $\theta$ is contained in an annulus;

ii. $\chi(z) + \sum_{j \geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.

iii. $\text{supp}(\chi) \cap \text{supp}(\theta(2^{-j} \cdot)) = \emptyset$ for $j \geq 1$ and $\text{supp}\theta(2^{-i} \cdot) \cap \text{supp}\theta(2^{-j} \cdot) = \emptyset$ for $|i - j| > 1$.

We call such a pair $(\chi, \theta)$ dyadic partition of unity, and for the existence of dyadic partitions of unity we refer to [BCD11, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j} \cdot) \mathcal{F} u).$$

**Besov spaces**

For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $u \in \mathcal{D}$ we define

$$\|u\|_{B^\alpha_{p,q}(\mathbb{R}^d)} := (\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p}^q)^{1/q},$$

with the usual interpretation as $l^\infty$ norm in case $q = \infty$. The Besov space $B^\alpha_{p,q}(\mathbb{R}^d)$ consists of the completion of $\mathcal{D}$ with respect to this norm and the Hölder-Besov space $C^\alpha$ is given by $C^\alpha(\mathbb{R}^d) = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. For $p, q \in [1, \infty)$,

$$B^\alpha_{p,q}(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d) : \|u\|_{B^\alpha_{p,q}} < \infty\},$$

$$C^\alpha(\mathbb{R}^d) \subset \{u \in S'(\mathbb{R}^d) : \|u\|_{C^\alpha(\mathbb{R}^d)} < \infty\}.$$

We point out that everything above and everything that follows can be applied to distributions on the torus (see [Sic85, SW72]). More precisely, let $S'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} := (\sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p(\mathbb{T}^d)}^q)^{1/q},$$

and the Hölder-Besov space $C^\alpha$ is given by $C^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d)$. We write $\| \cdot \|_\alpha$ instead of $\| \cdot \|_{B^\alpha_{\infty,\infty}(\mathbb{T}^d)}$ in the following for simplicity. For $p, q \in [1, \infty)$, we have

$$B^\alpha_{p,q}(\mathbb{T}^d) = \{u \in S'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} < \infty\},$$

$$C^\alpha \subset \{u \in S'(\mathbb{T}^d) : \|u\|_{\alpha} < \infty\}. \quad (2.1)$$

Here we choose Besov spaces as completions of smooth functions, which ensures that the Besov spaces are separable which has a lot of advantages for our analysis below.

**Wavelet analysis**

We will also use wavelet analysis to determine the regularity of a distribution in a Besov space. Below we briefly summarize wavelet analysis and refer to work of Meyer [Mey95], Daubechies [Dau92] and [Tri06] for more details. For every $r > 0$, there exists a compactly supported function $\varphi \in C^r(\mathbb{R})$ such that:

1. \(\langle \varphi(\cdot), \varphi(\cdot - k) \rangle = \delta_{k,0}\) for every $k \in \mathbb{Z}$;

2. There exist $\hat{a}_k, k \in \mathbb{Z}$, with only finitely many non-zero values, and such that $\varphi(x) = \sum_{k \in \mathbb{Z}} \hat{a}_k \varphi(2x - k)$ for every $x \in \mathbb{R}$;

3. For every polynomial $P$ of degree at most $r$ and for every $x \in \mathbb{R}$, $\sum_{k \in \mathbb{Z}} \int_\mathbb{R} P(y) \varphi(y - k) dy \varphi(x - k) = P(x)$.

Given such a function $\varphi$, we define for every $x \in \mathbb{R}^d$ the recentered and rescaled function $\varphi^h_x$ as follows

$$\varphi^h_x(y) := \Pi_{i=1}^d 2^{-\frac{h}{2}} \varphi(2^h(y_i - x_i)).$$
Observe that this rescaling preserves the $L^2$-norm. We let $V_n$ be the subspace of $L^2(\mathbb{R}^d)$ generated by \{\psi_x^n : x \in \Lambda_n\}, where
$$\Lambda_n := \{(2^{-n}k_1, ..., 2^{-n}k_d) : k_i \in \mathbb{Z}\}.$$ 
An important property of wavelets is the existence of a finite set $\Psi$ of compactly supported functions in $C^\infty$ such that, for every $n \geq 0$, the orthogonal complement of $V_n$ inside $V_{n+1}$ is given by the linear span of all the $\psi_x^n, x \in \Lambda_n, \psi \in \Psi$. For every $n \geq 0$
$$\{\varphi_x^n, x \in \Lambda_n \} \cup \{\psi_x^m : m \geq n, \psi \in \Psi, x \in \Lambda_m\},$$
forms an orthonormal basis of $L^2(\mathbb{R}^d)$. This wavelet analysis allows one to identify a countable collection of conditions that determine the regularity of a distribution.

Setting $\Psi_* = \Psi \cup \{\varphi\}$, by results on weighted Besov spaces (see [RZZ17], (2.2), (2.3), (2.4)) and its reference for details), we know that for $p \in (1, \infty)$, $\alpha \in \mathbb{R}$, $f \in \mathcal{C}^\alpha$
$$\|f\|_{\alpha}^p \lesssim \sum_{n=0}^{\infty} 2^{n(\alpha+1)p} \sum_{\psi \in \Psi_*} \sum_{x \in \Lambda_n} |\langle f, \psi_x^n \rangle|^p w(x)^p. \tag{2.2}$$
where $w(x) = (1 + |x|^2)^{-\frac{d}{2}}, \sigma > 0$.

**Estimates on the torus**

In this part we give estimates on the torus for later use. Set $\Lambda = (I - \Delta)^{\frac{1}{2}}$. For $s \geq 0, p \in [1, +\infty]$ we use $H^s_p$ to denote the subspace of $L^p(\mathbb{T}^d)$, consisting of all $f$ which can be written in the form $f = \Lambda^{-s}g, g \in L^p(\mathbb{T}^d)$ and the $H^s_p$ norm of $f$ is defined to be the $L^p$ norm of $g$, i.e. $\|f\|_{H^s_p} := \|\Lambda^s f\|_{L^p(\mathbb{T}^d)}$.

To study (1.1) in the finite volume case, we need several important properties of Besov spaces on the torus and we recall the following Besov embedding theorems on the torus first (c.f. [ITV18, Theorem 4.6.1], [GIP13, Lemma A.2], [IT19], Remark 3, Section 2.3.2]):

**Lemma 2.1** (i) Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B^\alpha_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^\alpha_{p_2,q_2}(\mathbb{T}^d)$.

(ii) Let $s \geq 0$, $1 < p < \infty$, $\varepsilon > 0$. Then $H^{s+\varepsilon}_p \subset B^s_{p,1}(\mathbb{T}^d) \subset B^s_{1,1}(\mathbb{T}^d)$.

(iii) Let $1 \leq p_1 \leq p_2 < \infty$ and let $\alpha \in \mathbb{R}$. Then $H^\alpha_{p_1} \subset H^\alpha_{p_2}(\mathbb{T}^d)$.

(iv) Let $0 < q \leq \infty$, $1 \leq p \leq \infty$ and $s > 0$. Then $B^s_{p,q} \subset L^p$.

Here $\subset$ means that the embedding is continuous and dense.

We recall the following Schauder estimates, i.e. the smoothing effect of the heat semigroup, for later use.

**Lemma 2.2** ([GIP13, Lemma A.7]) Let $u \in B^\alpha_{p,q}(\mathbb{T}^d)$ for some $\alpha \in \mathbb{R}, p, q \in [1, \infty]$. Then for every $\delta \geq 0$
$$\|e^{-t A^2} u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} \lesssim t^{-\delta/4}\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)}.$$ 

One can extend the multiplication on suitable Besov spaces and also have the duality properties of Besov spaces from [ITV18, Chapter 4]:

**Lemma 2.3** (i) The bilinear map $(u; v) \mapsto uv$ extends to a unique continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$ if and only if $\alpha + \beta > 0$.

(ii) Let $\alpha \in (0, 1)$, $p, q \in [1, \infty]$, $p'$ and $q'$ be their conjugate exponents, respectively. Then the mapping $(u; v) \mapsto \int uv dx$ extends to a unique continuous bilinear form on $B^\alpha_{p,q}(\mathbb{T}^d) \times B^{-\alpha}_{p',q'}(\mathbb{T}^d)$. 

6
We recall the following interpolation and multiplicative inequalities for the elements in $H^s_p$, which is required for the a-priori estimate in Section II (cf. [1118S, Theorem 4.3.1], [1231R, Lemma 2.1], [1038b, Theorem 2.80]):

Lemma 2.4 (i) Suppose that $s \in (0, 1)$ and $p \in (1, \infty)$. Then for $u \in H^1_p$

$$
\|u\|_{H^s_p} \lesssim \|u\|_{L^p(T^d)} \|u\|_{H^1_p}.
$$

(ii) Suppose that $s > 0$ and $p \in (1, \infty)$. If $u, v \in C^\infty(T^d)$ then

$$
\|\Lambda^s(uv)\|_{L^p(T^d)} \lesssim \|u\|_{L^p1(T^d)}\|\Lambda^s(v)\|_{L^p2(T^d)} + \|v\|_{L^p3(T^d)}\|\Lambda^s(u)\|_{L^p4(T^d)},
$$

with $p_i \in (1, \infty], i = 1, \ldots, 4$ such that

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
$$

(iii) Suppose that $s_1 < s_2$ and $1 \leq p, q \leq \infty$. Then for $u \in B^s_{p,q}$ and $\forall \theta \in (0, 1)$

$$
\|u\|_{L^{\theta s_1+(1-\theta)s_2}_{p,q}} \leq \|u\|_{B^{s_1}_{p,q}}^{\theta} \|u\|_{B^{s_2}_{p,q}}^{1-\theta}.
$$

We also collect some important properties for the multiplicative structure of Besov spaces from [MW17a] and [1130b].

Lemma 2.5 ([MW17a, Corollary 3.19, Corollary 3.21]) (1) For $\alpha > 0, p_1, p_2, p, q \in [1, \infty], \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, the bilinear map $(u; v) \mapsto uv$ extends to a unique continuous bilinear map from $B^{\alpha}_{p_1,q} \times B^{\alpha}_{p_2,q}$ to $B^{\alpha}_{p,q}$.

(2) For $\alpha < 0, \alpha + \beta > 0, p_1, p_2, p, q \in [1, \infty], \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, the bilinear map $(u; v) \mapsto uv$ extends to a unique continuous bilinear map from $B^{\alpha}_{p_1,q} \times B^{\beta}_{p_2,q}$ to $B^{\alpha}_{p,q}$.

Notations

Let $L$ denote the space $L^2(T^2) = [0, 1]^2$, where $T^2$ is the 2-dimensional torus and we use $\langle \cdot, \cdot \rangle$ to denote the inner product in $L$. $A$ is the Laplacian operator on $L$, that is,

$$
D(A) = H^2(T^2), A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.3}
$$

$A$ is a self-adjoint operator in $L$, with complete orthonormal system $(e_n)_n$ of eigenvectors in $L$, given by

$$
e_0(x) := 1, e_{(k_1,0)}(x) = \sqrt{2}e^{i\pi k_1 x_1}, e_{(0,k_2)}(x) = \sqrt{2}e^{i\pi k_2 x_2}, \quad e_k(x) := 2e^{i\pi (k_1 x_1 + k_2 x_2)}, k_1 k_2 \neq 0.
$$

Then we have $\Lambda e_k = -\lambda_k e_k$, where $\lambda_k = |k|^2 \pi^2, k = (k_1, k_2) \in Z^2, |k|^2 = k_1^2 + k_2^2$. We also introduce a notation for the average of $h \in S'(T^2)$:

$$
m(h) := S(\langle h, e_0 \rangle s).
$$

For any $\alpha \in \mathbb{R}$, we define

$$
V^\alpha := \{ u \in S' : \sum_k \lambda_k^\alpha \langle s', \langle u, e_k \rangle s \rangle^2 < \infty \}.
$$
For any $u, v \in V^\alpha$, define

$$\langle u, v \rangle_{V^\alpha} := m(u)m(v) + \sum_k \lambda_k^\alpha \langle u, e_k \rangle_{SS'} \langle v, e_k \rangle_{SS'}.$$ 

It’s easy to see that $(V^\alpha, \langle \cdot, \cdot \rangle_{V^\alpha})$ is a Hilbert space and $V^\alpha \simeq H_2^\alpha$. Then for any $s, \alpha \in \mathbb{R}$, we can define a bounded linear operator $(-A)^s : V^\alpha \rightarrow V^{\alpha-2s}$ by:

$$(-A)^s u = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \lambda_k^\alpha u_k e_k,$$

where $u = \sum_k u_k e_k \in V^\alpha$. In particular, we set $Q := (-A)^{-1}$ and extend it to a one-to-one bounded linear operator $\tilde{Q}$ by

$$\tilde{Q}h := Qh + m(h)e_0.$$ (2.4)

Note that

$$Qe_k = \begin{cases} \frac{1}{\lambda_k^\alpha} e_k & k \neq (0,0), \\ 0 & k = (0,0), \end{cases}$$ (2.5)

and

$$\tilde{Q}e_k = \begin{cases} \frac{1}{\lambda_k^\alpha} e_k & k \neq (0,0), \\ e_0 & k = (0,0). \end{cases}$$ (2.6)

Then we have

$$\langle u, v \rangle_{V^\alpha} := \langle \tilde{Q}^{-\alpha/2} u, \tilde{Q}^{-\alpha/2} v \rangle,$$

and $\tilde{Q}^s : V^\alpha \rightarrow V^{\alpha+2s}$ is an isomorphism for any $\alpha, s \in \mathbb{R}$, since

$$\langle \tilde{Q}^s u, \tilde{Q}^s v \rangle_{V^{\alpha+2s}} = \langle u, v \rangle_{V^\alpha}.$$

We also set

$$V^\alpha_0 := \{ h \in V^\alpha : \langle h, e_0 \rangle_{V^\alpha} = 0 \},$$

and denote $L^2_0 := V^\alpha_0$. Let $\Pi$ denote the symmetric projector of $V^\alpha$ on $V^\alpha_0$, that is,

$$\Pi : V^\alpha \rightarrow V^\alpha_0, \Pi h := h - m(h).$$ (2.7)

Moreover, we define

$$V^\alpha(T^2, \mathbb{R}^2) := \{ f = (f_1, f_2) : f_i \in V^\alpha, i = 1, 2 \},$$

and similarly

$$V^\alpha_0(T^2, \mathbb{R}^2) := \{ f = (f_1, f_2) : f_i \in V^\alpha_0, i = 1, 2 \}.$$
3 The Linear Equation and Wick Powers

We consider the O-U process

$$\begin{cases}
dZ_t = -\frac{1}{2}A^2 Zdt + BdW_t, \\
Z(0) = 0,
\end{cases} \tag{3.1}$$

where $W$ is a $U$-cylindrical Wiener process and $U := L^2_0(\mathbb{T}^2, \mathbb{R}^2)$. For $f \in L^2(\mathbb{T}^2, \mathbb{R}^2)$ we denote its component functions by $f_1, f_2 \in L^2_0(\mathbb{T}^2)$ i.e. $f(x) = (f_1(x), f_2(x)), \forall x \in \mathbb{T}^2$. There exist two independent $L^2(\mathbb{T}^2)$-cylindrical Wiener processes $W^1$ and $W^2$ such that $W = (W^1, W^2)$. Set

$$D(B) = H^1(\mathbb{T}^2, \mathbb{R}^2), B = \text{div}, D(B^*) = H^1_2(\mathbb{T}^2), B^* = -\nabla. \tag{3.2}$$

We know that

$$Z_t(x) = \int_0^t e^{-\frac{t-s}{2}A^2} B dW_s = \int_0^t \langle K(t-s, x-\cdot), dW_s \rangle_U,$$

where $K(t, x) = -\nabla_x M(t, x) = (K^1, K^2)$, and $M(t, x)$ is the kernel of $e^{-\frac{t}{2}A^2}$, that is, $M(t, x) = \sum_k e^{-\frac{t}{2}X_k} e_k(x)$.

For any function $f$ on $\mathbb{T}^2$, we can view it as a periodic function on $\mathbb{R}^2$ by defining $\tilde{f}(x) := f(x+m)$, when $x + m \in \mathbb{T}^2$, $x \in \mathbb{R}^2$, $m \in \mathbb{Z}^2$. Moreover, define

$$K^j(t, x) := -\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{t}{2}|\xi|^4})(x), j = 1, 2,$$

and $\bar{K} := (\bar{K}^1, \bar{K}^2)$. By the Poisson summation formula (see [SW72, Section VII.2]) we know that

$$K(t, x) = \sum_m \bar{K}(t, x + m), \forall t \tag{3.3}$$

and for any $f \in L^2(\mathbb{T}^2), j = 1, 2, x \in \mathbb{T}^2$

$$\partial_t e^{-\frac{t}{2}A^2} f(x) = \int_{\mathbb{T}^2} K^j(t, x-y) f(y)dy$$

$$= \int_{\mathbb{R}^2} K^j(t, x-y) f(y) 1_{\mathbb{T}^2}(y)dy$$

$$= \sum_m \int_{\mathbb{R}^2} \bar{K}^j(t, x-y+m) f(y) 1_{\mathbb{T}^2}(y)dy$$

$$= \int_{\mathbb{R}^2} \bar{K}^j(t, x-y) \sum_m 1_{\mathbb{T}^2}(y+m) f(y+m)dy$$

$$= (\bar{K}^j(t, \cdot) * \tilde{f})(x) \tag{3.4}$$

where we used (3.3) in the third inequality and $1_{\mathbb{T}^2}$ is the indicator function of $\mathbb{T}^2$. Since

$$\bar{K}^j(t, x) = -\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{t}{2}|\xi|^4})(x) = t^{-\frac{3}{2}} \bar{K}^j(1, t^{-\frac{1}{2}}x)$$

and

$$|\bar{K}^j(1, t^{-\frac{1}{2}}x)| \lesssim |\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{1}{2}|\xi|^4})(t^{-\frac{1}{2}}x)| \lesssim |1 + t^{-\frac{1}{2}}x|^{-3},$$

we have the following estimate:

$$|\bar{K}(t, x)| \lesssim t^{-\frac{3}{2}} |x|^{-3+\varepsilon}, \forall \varepsilon \in [0, 3]. \tag{3.5}$$
Lemma 3.1 $Z \in C([0,T]; C^{-\alpha})$ $\mathbb{P}$-almost-surely, for all $\alpha > 0$.

Proof By the factorization method in [DP04] we have that for $\kappa \in (0,1)$

$$Z(t) = \frac{\sin(\pi \kappa)}{\pi} \int_0^t (t-s)^{\kappa-1} \langle M(t-s, x), U(s) \rangle ds,$$

and

$$U(s, \cdot) = \int_0^s (s-r)^{-\kappa}e^{-\frac{r^2}{2}}A^2 BdW_r.$$

A similar argument as in the proof of Lemma 2.7 in [DP04] implies that it suffices to prove that for $p > 1/(2\kappa)$,

$$\mathbb{E}\|U\|_{L^{2p}(0,T; C^{-\alpha})} < \infty. \quad (3.6)$$

In fact, by (2.2) we have that

$$\mathbb{E}\|U(s)\|_{-\alpha}^{2p} \lesssim \sum_{\psi \in \Psi, n \geq 0} \sum_{x \in \Lambda_n} 2^{-2\alpha n+2np}(\mathbb{E}|U(s, \psi^n_x)|^2)^p w(x)^{2p}$$

Here we used that $(U(s, \psi^n_x))$ belongs to the first order Wiener-chaos as well as Gaussian hypercontractivity (cf. [Hai14], Section 1.4.3 and [Nel73]) in the second inequality. Moreover, we obtain that

$$\mathbb{E}|\langle U(s), \psi^n_x \rangle|^2 = \mathbb{E}|\langle U^1(s), \psi^n_x \rangle|^2 + \mathbb{E}|\langle U^2(s), \psi^n_x \rangle|^2$$

and in the second inequality we used (3.4) and we also used [Hai14], Lemma 10.17 and (3.7) to deduce that $|\tilde{K}^j * K^j(s-r, t, y)| \lesssim |s-r|^{-\frac{\kappa}{2}}|y-t|^{-4+2\varepsilon}$.

In fact, we can decompose $K$ into $K := K_\delta + \tilde{K}_\delta$, where $K_\delta$ is a compactly supported function and satisfies (3.7), and $\tilde{K}_\delta$ is a Schwartz function. Then $K*K = K_\delta*K_\delta + H$, where $H$ is a Schwartz function. By [Hai14], Lemma 10.17 we have $K_\delta*K_\delta(t, x) \lesssim t^{-\frac{\kappa}{2}}|x|^{-4+2\varepsilon}$ and $K*K$ satisfies the same inequality.

Thus, we have

$$\mathbb{E}\|U(s)\|_{-\alpha}^{2p} \lesssim \sum_{n \geq 0} 2^{(4-2\varepsilon-2\alpha)pn} s^{(1-2\kappa-\frac{\varepsilon}{2})p}.$$ 

Let $\kappa$ be so small that $2 - \alpha < \varepsilon < 2 - 4\kappa + \frac{2}{p}$, which implies that

$$4 - 2\varepsilon - 2\alpha < 0, (1 - 2\kappa - \frac{\varepsilon}{2})p > -1.$$ 

Then (3.7) follows.
Note that $BB^* = -A$. Then by Fourier expansion it is easy to see that $Z_t \sim \mathcal{N}(0, Q_t)$, i.e. for any $h \in S(T^2)$
\[ E e^{i \langle h, Z_t \rangle} = \exp\left(-\frac{1}{2} Q_t h, h \right), \]
where $Q_t = (-A)^{-1} (I - e^{-\frac{t}{2} A^2})$.

According to the definition of $V^\alpha$ and Lemma 2.1, we have $C_{\alpha} \subset V^{\alpha - \varepsilon}$ for any $\alpha, \varepsilon > 0$. Then by Lemma 3.1, $\mu_t$ is supported on $V_0^{\alpha}$ for any $\alpha > 0$ and letting $t \to \infty$, by [Bog98, Example, 3.8.13], the law of $Z_t$ converges to the Gaussian measure $\mu = \mathcal{N}(0, Q)$, which is also supported on $V_0^{\alpha}$.

In the following we are going to define the Wick powers both in the state space and the path space.

**Wick powers on $L^2(S'(T^2), \mu)$**

$\mu$ is of course also a measure supported on $S'(T^2)$. We have the well-known (Wiener-Itô) chaos decomposition
\[ L^2(S'(T^2), \mu) = \bigoplus_{n \geq 0} H_n, \]
where $H_n$ is the Wiener chaos of order $n$ (cf. [Nua13, Section 1.1.1]). Now we define the Wick powers by using approximations: for $\phi \in S'(T^2)$ define
\[ \phi_\varepsilon := \rho_\varepsilon \ast \phi, \]
with $\rho_\varepsilon$ an approximate delta function on $\mathbb{R}^2$ given by
\[ \rho_\varepsilon(x) = \varepsilon^{-2} \rho(\frac{x}{\varepsilon}) \in D, \int \rho = 1. \]
Here the convolution means that we view $\phi$ as a periodic distribution in $S'(\mathbb{R}^2)$. For every $n \in \mathbb{N}$ we set
\[ : \phi_\varepsilon^n := c_\varepsilon^{n/2} P_n (c_\varepsilon^{-1/2} \phi_\varepsilon), \]
where $P_n, n = 0, 1, ..., $ are the Hermite polynomials defined by the formula
\[ P_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{(n - 2j)!j!2^{n-2j}} x^{n-2j}, \]
and $c_\varepsilon = \int \phi_\varepsilon^2 \mu(d\phi) = \int \int G(z - y) \rho_\varepsilon(y)dy \rho_\varepsilon(z)dz$. Then
\[ : \phi_\varepsilon^n : \in H_n. \]
Here and in the following $G$ is the Green function associated with $-A$ on $T^2$. In fact by [SW72, Section 6.1, Chapter VII],
\[ G(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{\lambda_k} e_k(x) \simeq -\log |x|, \quad |x| \to 0, \]
and $G$ is continuously differentiable outside $\{0\}$.

For Hermite polynomial $P_n$ we have that for $s, t \in \mathbb{R}$
\[ P_n(s + t) = \sum_{m=0}^{n} C_m^n P_m(s) t^{n-m}, \quad (3.7) \]
where \( C^m_n = \frac{n!}{m!(n-m)!} \).

A direct calculation yields the following:

**Lemma 3.2** Let \( \alpha > 0, n \in \mathbb{N} \) and \( p > 1 \). \( \phi^\varepsilon_n \) converges to some element in \( L^p(S'(\mathbb{T}^2), \mu; C^{-\alpha}) \) as \( \varepsilon \to 0 \). This limit is called the \( n \)-th Wick power of \( \phi \) with respect to the covariance \( Q \) and denoted by \( \phi^n \).

**Proof** The proof is similar to that of [RZZ17b, Lemma 3.1] since the Green function \( G \) has the same regularity. Therefore we omit it here for simplicity. \( \square \)

**Wick powers on a fixed probability space**

Now we fix a probability space \( (\Omega, \mathcal{F}, P) \) and consider a \( U \)-cylindrical Wiener process \( W \). In the following we assume that \( \mathcal{F} \) is the \( \sigma \)-field generated by \( \{ (W_t, h), h \in U, t \in \mathbb{R}^+ \} \). We also have the well-known (Wiener-Itô) chaos decomposition

\[
L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n \geq 0} \mathcal{H}'_n,
\]

where \( \mathcal{H}'_n \) is the Wiener chaos of order \( n \) (cf. [Nua13, Section 1.1.1]). We can define Wick powers of \( Z(t) \) with respect to different covariances by approximations. Let

\[
Z_\varepsilon(t, x) = \rho_\varepsilon * Z_t = \int_0^t \langle B^* e^{-\frac{(t-s)^2}{2}} \rho_\varepsilon x, dW_s \rangle_U,
\]

where \( \rho_\varepsilon x = \rho_\varepsilon (x - \cdot) \), \( K_\varepsilon(t, x) = (\rho_\varepsilon * K^1_\varepsilon, \rho_\varepsilon * K^2_\varepsilon) \) and

\[
K^j_\varepsilon = -\sum_k (i\pi k_j) e^{-\frac{1}{2} \lambda^2_k} e_k, j = 1, 2.
\]

For any \( n \in \mathbb{N} \), we set

\[
: Z_\varepsilon^n(t) :_{Q_\varepsilon} := (c_{\varepsilon,t})^{\frac{n}{2}} P_n \left( (c_{\varepsilon,t})^{-\frac{1}{2}} Z_\varepsilon(t) \right) \in \mathcal{H}'_n,
\]

where \( P_n, n = 0, 1, \ldots \), are the Hermite polynomials and \( c_{\varepsilon,t} = \| 1_{[0,t]} K_\varepsilon \|^2_{L^2(\mathbb{R} \times \mathbb{T}^2; \mathbb{R}^2)} \).

**Lemma 3.3** ([RZZ17b, Lemma 3.4]) For \( \alpha > 0, p > 1, n \in \mathbb{N} \), \( : Z_\varepsilon^n : \) converges in \( L^p(\Omega, C([0,T]; C^{-\alpha})) \). The limit is called Wick power of \( Z(t) \) of order \( n \) with respect to the covariance \( Q \) and is denoted by \( : Z^n(t) : \).

**Proof** The kernel \( K \) is a little different from the kernel in [RZZ17b]. But (3.3) satisfies the condition in [RZZ17b, Lemma 3.2] and [ZZ18, Lemma 4.1] which leads to a similar proof as for [RZZ17b, Lemma 3.3], so we omit it here. \( \square \)

**Remark 3.4** Here we do not combine the initial value with the Wick powers as in [MW17a, RZZ17b], since we can obtain existence of solutions to the shifted equation (4.1) for any initial value in \( V^{-1}_0 \) (see Section 4).
Relations between two different Wick powers

We introduce the following probability measure. Set \( q(\phi) := \frac{1}{4} : \phi^4 : \), \( p(\phi) := : \phi^3 : \). Let

\[
\nu = c \exp(-N) \mu,
\]

where \( c \) is a normalization constant and \( N = S:\left( q : e_0 \right) \). Then according to [Sim74, Lemma V.5
and Theorem V.7] we have for every \( p \in [1, \infty) \), \( \varphi(\phi) := e^{-N} \in L^p(S'(\mathbb{T}^2), \mu) \).

The following result is about the relation between the two different Wick powers.

**Lemma 3.5** Let \( \phi \) be a measurable map from \( (\Omega, \mathcal{F}, \mathbb{P}) \) to \( C([0, T], B_{2,2}^\gamma) \) with \( \gamma > 2 \), \( \mathbb{P} \circ \phi(t)^{-1} = \nu \) for every \( t \in [0, T] \) and let \( Z(t) \) be defined as above. Assume in addition that \( y = \phi - Z \in C((0, T]; C^\beta) \) \( \mathbb{P} \)-a.s. for some \( \beta > \alpha > 0 \). Here \( C((0, T]; C^\beta) \) is equipped with the norm \( \sup_{t \in [0, T]} t^{\beta+n} \| \cdot \|_\beta \). Then for every \( t > 0, n \in \mathbb{N} \)

\[
: \phi^n(t) := \sum_{k=0}^n C_n^k y^{n-k}(t) : Z^k(t) : \quad \mathbb{P} \text{-a.s.}
\]  

(3.8)

Here the Wick power on the left hand side is the limit obtained and defined in Lemma V.2.

**Proof** By Lemma V.3 it follows that for every \( k \in \mathbb{N}, p > 1 \)

\[
: Z^k_e \to: Z^k : \quad \text{in } L^p(\Omega, C((0, T]; C^{-\alpha})), \quad \varepsilon \to 0.
\]

Since \( y_e = \phi_e - Z_e = \rho_e * y \) and \( y \in C((0, T]; C^\beta) \) \( \mathbb{P} \)-a.s., it is obvious that \( y_e \to y \) in \( C((0, T]; C^\beta-\kappa) \) \( \mathbb{P} \)-a.s. for every \( k \geq 0 \) with \( \beta - \kappa > 0 \), which combined with Lemma V.3 implies that for every \( k \in \mathbb{N}, \gamma > 0, n \in \mathbb{N} \)

\[
y_e^{n-k} : Z^k_e \to: y^{n-k} : Z^k : \quad \text{in } C((0, T]; C^{-\alpha}), \quad \varepsilon \to 0.
\]

Here \( \to \mathbb{P} \) means convergence in probability. Since \( e^{-N} \in L^p(S'(\mathbb{T}^2), \mu) \) for every \( p > 1 \), by Hölder’s inequality and Lemma V.2 we get that for every \( t > 0, p > 1 \)

\[
: \phi^n_e(t) := (y_e + Z_e)^n := c_e^{n/2} P_n(c_e^{-1/2}(y_e + Z_e))
\]

\[
= \sum_{k=0}^n C_n^k c_e^{n-k/2} P_k(c_e^{-1/2} Z_e)(c_e^{-1/2} y_e)^{n-k}
\]

\[
= \sum_{k=0}^n C_n^k : Z_e^k : y_e^{n-k},
\]

which implies the result by letting \( \varepsilon \to 0 \). \( \square \)

4 The Solution to the Shifted Equation

Now we fix a stochastic basis \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in [0, \infty)}, \mathbb{P}) \) and on it a U-cylindrical Wiener process \( W \). Define \( Z(t) = \int_0^t e^{-(t-s)} A^{1/2} B dW(s) \) as in Section 3. Now we consider the following shifted equation:

\[
\begin{align*}
\frac{dY}{dt} &= -\frac{1}{2} A^2 Y + \frac{1}{2} A \sum_{k=0}^3 C_3^k Y^{3-k} : Z^k :, \\
Y(0) &= x.
\end{align*}
\]

(4.1)
Generally we consider initial data \( x \) that are \( \mathcal{F}_0 \) measurable and belong to \( V^{-1}_t; \mathbb{P} \) a.s. To prove the existence of the solution to equation (3.1), we use a smooth approximation on each path:

\[
\begin{aligned}
\frac{dY_\varepsilon}{dt} &= -\frac{1}{2} A^2 Y_\varepsilon + \frac{1}{2} A \sum_{k=0}^{3} C_k^k Y_\varepsilon^{3-k} : Z_\varepsilon^k, \\
Y_\varepsilon(0) &= x_\varepsilon,
\end{aligned}
\]  

(4.2)

where \( Z_\varepsilon = Z * \rho_\varepsilon \), \( x_\varepsilon = x * \rho_\varepsilon \), and \( \rho_\varepsilon \) is as introduced in Section 3. Note that the solution \( Y \) to equation (3.1) and the solution \( Y_\varepsilon \) to (4.2) also satisfy:

\[
\frac{dm(Y(t))}{dt} = 0, m(Y(0)) = 0,
\]  

(4.3)

which means that \( m(Y(t)) = m(Y_\varepsilon(t)) = 0 \).

From Lemma 3.2 we know that there exists a \( \Omega' \subseteq \Omega \), \( \mathbb{P}(\Omega') = 1 \), such that for any \( \omega \in \Omega' \), \( Z(\omega) : Z^n : (\omega) \in C((0, T]: C^{\alpha}) \), \( n = 2, 3 \), \( k \leq 3 \). Since \( Z_\varepsilon(\omega) \) is smooth, by monotonicity trick in \[ \text{Remark 5.3.1 and Theorem 5.2.4}, \] there exists a unique solution \( Y_\varepsilon(\omega) \) to equation (4.2) in \( L^2(0, T; V_0^2) \cap C([0, T]; L_0^2) \) for each \( \omega \in \Omega' \). We are going to find a convergent subsequence of \( \{ Y_\varepsilon(\omega) \} \), which converge to a solution to equation (4.1) and prove uniqueness of solutions to (4.1). Then we obtain a unique \( \mathcal{F}_\tau \)-adapted solution to equation (4.1).

In this section we never distinguish \( V^\alpha, H_2^0 \) and \( B_{2,2}^3 \) since they have equivalent norms. For convenience we denote all of them by \( H^\alpha \).

**Theorem 4.1 (a-priori estimate)** If \( Y \) is a solution to equation (4.1), then there exists a constant \( C_T \) which only depends on \( T \) and \( Z(\omega) \), such that for \( \forall t \in [0, T] \)

\[
||Y||_{H-1}^2 + ||x||_{H-1} + \frac{1}{2} \int_0^t (||Y(s)||_{H^1}^2 + ||Y(s)||_{L^4}^4) \, ds \leq C_T.
\]  

(4.4)

Moreover there exist constants \( C > 0, \gamma_k > 0, k = 1, 2, 3 \), such that for every \( t \in (0, T] \)

\[
||Y||_{H-1}^2 \leq C \left( t^{-1} \sqrt{\sum_{k=1}^{3} t^{-\rho \gamma_k} \sup_{0 \leq r \leq t} (t^{\rho \gamma_k} : Z_r : ||\gamma_k ||_{\mathcal{F}_r})} \right)^{\frac{1}{2}},
\]  

(4.5)

where \( \rho > 0 \) is the small enough constant introduced in Lemma 3.3.

**Proof** Since

\[
\frac{dY}{dt} = -\frac{1}{2} A(AY - \sum_{k=0}^{3} C_k^k Y^{3-k} : Z^k),
\]

and \( m(Y) = 0 \), taking scalar product with \((-A)^{-1}Y\) we obtain that

\[
\frac{d}{dt} ||Y||_{H-1}^2 + ||Y||_{H^1}^2 + ||Y||_{L^4}^4 = -\sum_{k=1}^{3} C_k^k Y^{3-k} : Z^k : Y, 
\]

that is

\[
\frac{d}{dt} ||Y||_{H-1}^2 + ||Y||_{H^1}^2 + ||Y||_{L^4}^4 \leq ||\langle Y, Z^3 : \rangle|| + ||\langle Y^2, Z^2 : \rangle|| + ||\langle Y^3, Z \rangle||.
\]  

(4.6)
So we only need to estimate the right hand side of (L6). We only consider \(|\langle Y^3, Z \rangle|\). The other terms can be estimated similarly. Lemma 2.3 implies
\[
|\langle Y^3, Z \rangle| \lesssim \|Z\|_{-\alpha} \|Y^3\|_{B^\alpha_{1,1}}, \quad \forall \alpha > 0.
\]
Moreover, by Lemma 2.4 and Lemma 3.8
\[
\|Y^3\|_{B^\alpha_{1,1}} \lesssim \|A^{\beta_0} Y^3\|_{L^{p_0}} \lesssim \|A^{\beta_0} Y^{3/2}\|_{L^{p_1}} \|Y^{3/2}\|_{L^{q_1}},
\]
where \(\beta_0 > \alpha, p_0 > 1\) and \(\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{q_1}\).

Choose \(q_1 \leq \frac{8}{3}\) and \(p_1 > \frac{8}{5}\). Then we have
\[
\|Y^{3/2}\|_{L^{q_1}} = \|Y^{3/2}_{j+1}\|_{L^{q_1}} \lesssim \|Y^{3/2}\|_{L^4}.
\]
For \(\|A^{\beta_0} Y^{3/2}\|_{L^{p_1}}\), we have
\[
\|A^{\beta_0} Y^{3/2}\|_{L^{p_1}} \lesssim \|A^{\beta_1} Y^{3/2}\|_{L^{p_2}} \lesssim \|A Y^{3/2}\|_{L^{p_2}} \|Y^{3/2}\|_{L^{p_2}},
\]
where \(1 < p_2 < p_1 < 2\), \(\beta_0 = \beta_1 - 2(\frac{1}{p_2} - \frac{1}{p_1})\), \(\beta_1 < 1\) and we used Lemma 2.4 in the first inequality and Lemma 3.8 in the second inequality. For \(\|A Y^{3/2}\|_{L^{p_2}}\), let \(p_2 < \frac{8}{5}\). Then we have
\[
\|A Y^{3/2}\|_{L^{p_2}} \lesssim \|Y^{1/2} \nabla Y\|_{L^{p_2}} \lesssim \|Y\|_{H^1} \|Y^{1/2}\|_{L^{2^{p_2}}} \lesssim \|Y\|_{H^1} \|Y\|_{L^4} \lesssim \|Y\|_{H^1} \|Y\|_{L^4},
\]
where we used Hölder’s inequality in the second inequality. Furthermore
\[
\|Y^{3/2}\|_{L^{p_2}} \lesssim \|Y^{3/2}_{j+1}\|_{L^{p_2}} \lesssim \|Y^{3/2}_{j+1}\|_{L^4}.
\]
Combining the above estimates we get
\[
\|Y^3\|_{B^\alpha_{1,1}} \lesssim \|Y^{3-\beta_1}_{L^4}\|_{L^4}. \|Y^{\beta_1}_{H^1}\|_{L^4}.
\]
Combining this with Lemma 3.8, we have
\[
|\langle Y^3, Z \rangle| \lesssim \|Y^{3-\beta_1}_{L^4}\|_{L^4} \|Y^{\beta_1}_{H^1}\|_{L^4} t^{-\frac{4}{3}\gamma} \lesssim t^{-\frac{4}{3}\gamma} + \kappa \left(\|Y^4_{L^4}\| + \|Y^2_{H^1}\|\right),
\]
where \(\gamma = \frac{4}{3}\beta_1\) and we used Young’s inequality. Choosing \(\rho\) to be so small that \(\frac{\rho}{4}\gamma < 1\), we can conclude that there exists \(\gamma_k > 0\), \(k = 1, 2, 3\) such that \(\frac{\gamma_k}{4} < 1\) and
\[
\frac{d}{dt} \|Y\|_{H^{-1}} + \frac{1}{2} \left(\|Y\|_{H^1} + \|Y\|_{L^4}\right) \lesssim \sum_{k=1}^{3} \|Z^k : \|Y_k\|_{H^{-\alpha}} \lesssim \sum_{k=1}^{3} t^{-\frac{\gamma_k}{4}}.
\]
Hence (L3) follows. Moreover, since \(\|Y\|_{H^{-1}} \lesssim \|Y\|_{L^4}\) we have that
\[
\frac{d}{dt} \|Y\|_{H^{-1}}^2 + \frac{1}{2} \|Y\|_{H^{-1}}^2 \lesssim \sum_{k=1}^{3} \|Z^k : \|Y_k\|_{H^{-\alpha}}. \|Y_k\|_{H^{-\alpha}}.
\]
By [TW18, Lemma 3.8], we have
\[
\|Y^k_{H^{-1}}\| \lesssim t^{-1} \left(\sum_{k=1}^{3} \|Z^k: \|Y_k\|_{H^{-\alpha}}\right)^{1/2}.
\]
Since the approximating equation (1.2) obey the same a-priori estimate as (1.1), by (1.3) we deduce that the sequence \( \{Y_{\varepsilon}^2\} \) is bounded in \( L^\infty(0,T;H^{-1}) \cap L^4([0,T] \times \mathbb{T}^2) \cap L^2(0,T;H^1) \). This implies that \( \{AY_{\varepsilon}^2\} \) is bounded in \( L^2(0,T;H^{-1}) \) and \( \{Y_{\varepsilon}^3\} \) is bounded in \( L^{4/3}([0,T] \times \mathbb{T}^2) \). Moreover, Lemma 2.1 and Lemma 3.3 imply that \( \{Z_{\varepsilon}^3\} \) is bounded in \( L^p(0,T;H^{-\alpha}) \) for any \( \alpha > 0, \varepsilon > 0 \) and \( p > 1 \). Then we can prove the following lemma:

**Lemma 4.2** \( \{ \frac{dY}{dt} \} \) is bounded in \( L^p(0,T;H^{-3}) \), where \( p \in (1, \frac{4}{3}) \).

**Proof** According to the argument before, we only need to show that \( \{Y_{\varepsilon}^2Z_{\varepsilon}\} \) and \( \{Y_{\varepsilon} : Z_{\varepsilon}^2\} \) are bounded in \( L^p(0,T;H^{-1}) \) when \( p \in (1, \frac{3}{2}) \).

We omit \( \varepsilon \) if there is no confusion in this proof.

For \( Y^2Z \) we have

\[
||Y^2Z||_{B^0_{2,\infty}} \lesssim ||Y^2||_{B^0_{2,\infty}} ||Z||_{-\alpha} \lesssim ||Y^2||_{B^0_{2,1}} ||Z||_{-\alpha},
\]

where \( \beta_0 > \alpha > 0 \), we used Lemma 3.3 in the first inequality and Lemma 2.1 in the second inequality. Furthermore,

\[
||Y^2||_{B^0_{2,1}} \lesssim ||\Lambda^{\beta_1}Y^2||_{L^2} \lesssim ||\Lambda^{\beta_1}Y||_{L^{p_0}} ||Y||_{L^{q_0}},
\]

where \( \beta_1 > \beta_0, \frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{2} \), we used Lemma 2.3 in the first inequality and Lemma 2.2 in the second inequality. By Lemma 2.1, \( B^s_{q,2} \subset L^q \) for any \( q \geq 1 \) and \( s > 0 \). Since \( H^\delta \simeq B^\delta_{2,2} \subset B^\delta_{q,2} \) for \( q \geq 2 \), the Besov interpolation in Lemma 2.3 implies that

\[
||Y||_{L^{p_0}} \lesssim ||Y||_{B^0_{q,2}} \lesssim ||Y||_{1} \frac{1 - \beta_0}{2} ||Y||_{L^{q_0}} \frac{1 - \beta_0}{2} \lesssim ||Y||_{H^{1}} \frac{1 - \beta_0}{2} ||Y||_{H^{-1}} \quad (4.7)
\]

For \( ||\Lambda^{\beta_1}Y||_{L^{p_0}} \), let \( p_0 \geq 2 \). Then we use Lemma 2.1 and Sobolev interpolation to get

\[
||\Lambda^{\beta_1}Y||_{L^{p_0}} \lesssim ||Y||_{H^{\beta_2}} \lesssim ||Y||_{H^{1}} \frac{1 - \beta_2}{2} ||Y||_{H^{1}} \frac{1 - \beta_2}{2},
\]

where \( \beta_1 = \beta_2 + \frac{2}{p_0} - 1 = \beta_2 - \frac{2}{q_0} \). Thus we have

\[
||Y^2||_{B^0_{2,1}} \lesssim ||Y||_{H^{1}} \frac{1 - \beta_2}{2} ||Y||_{H^{1}} \frac{1 - \beta_2}{2} \quad (4.8)
\]

By Lemma 3.3 we deduce that

\[
||Y^2Z||_{B^0_{2,\infty}} \lesssim ||Y||_{H^{1}} \frac{1 - \beta_2}{2} ||Y||_{H^{1}} \frac{1 - \beta_2}{2} t^{-\frac{\delta}{2}}.
\]

For any \( p \in (1, \frac{4}{3}) \), let \( \beta_1 \) and \( s \) be small enough such that \( (\beta_1 + s + 3)p < 4 \). Then Young’s inequality implies that there exists \( \gamma > 0 \) such that

\[
||Y^2Z||_{B^0_{2,\infty}}^p \lesssim ||Y||_{H^{1}}^p + ||Y||_{H^{-1}}^p \frac{4}{3} (\frac{1 - \beta_2}{2}) t^{-\frac{\delta}{2}}
\]

For \( p \) small enough, \( \{Y_{\varepsilon}^2Z_{\varepsilon}\} \) is bounded in \( L^p(0,T;B^\alpha_{2,\infty}) \).

On the other hand,

\[
||Y : Z^2 : ||_{B^0_{2,\infty}} \lesssim ||Y||_{B^0_{2,\infty}} ||Z^2||_{-\alpha} \lesssim ||Y||_{H^{1}} \quad (4.9)
\]
where we used Lemma 2.3 in the first inequality and Lemma 2.1, Lemma 3.3 in the second inequality. Then by Young’s inequality
\[ \|Y : Z^2 : \|_{B_{2,\infty}}^\alpha \lesssim \|Y\|_{H^1}^2 + t^{-\rho}. \]

Choosing \( \rho \) small enough we deduce that \( \{Y_\varepsilon : Z^2_\varepsilon \} \) is bounded in \( L^2(0, T; B_{2,\infty}^\alpha) \). By Lemma 2.1 we have \( B_{2,\infty}^\alpha \subset H^2_\alpha - \delta \) for any \( \delta > 0 \). Hence \( \{Y_\varepsilon Z_\varepsilon \} \) and \( \{Y_\varepsilon : Z^2_\varepsilon \} \) are bounded in \( L^p(0, T; H^{-1}) \), \( \forall p \in (1, \frac{4}{3}) \), which implies the results.

**Theorem 4.3** For every \( x \in V_0^{-1} \), there exists at least one solution to equation (4.4) in \( C([0, T]; V_0^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; V_0^1) \).

**Proof** Since \( H^1 \subset H^3 \) compactly for any \( \delta < 1 \) (see [Tem01 Proposition 4.6]), a classical compactness argument (cf. [GRZ09 Lemma C.2] or [Tem01 Theorem 2.1, Chapter III]) implies that there exists a sequence \( \{\varepsilon_k\} \) and \( Y \in L^\infty(0, T, H^{-1}) \cap L^2(0, T; H^1) \cap L^4([0, T] \times \mathbb{T}^2) \), such that \( Y_{\varepsilon_k} \rightarrow Y \) in \( L^2(0, T; H^2) \cap C([0, T]; H^{-3}) \), \( \forall \delta < 1 \).

It is sufficient to show that for a suitable \( \delta \in (0, 1) \), the limit \( Y \) we obtained above is a solution in \( H^{-3} \).

In fact, if \( Y \) is a solution in \( H^{-3} \), i.e. for any \( h \in H^3 \)
\[ H^{-3}(Y - Y_0, h)_{H^3} = -\frac{1}{2} \int_0^t H^{-1}(A^2 h, Y_s)_{H^1} ds + \frac{1}{2} \int_0^t H^{-4}(\sum_{k=0}^{3} C^k_{3} \langle Y^3, k : Z^k_s : ; \cdot \rangle_{H^1} ds, \quad (4.9) \]
\( Y \) is in \( L^\infty(0, T, H^{-1}) \cap L^2(0, T; H^1) \cap L^4([0, T] \times \mathbb{T}^2) \). Then we take the scalar product of \( \frac{dY}{dt} \) and \( (-A^{-1}Y) \), which is just the duality in \( H^{-3} \) and \( H^3 \). Hence
\[ \frac{d}{dt} \|Y\|_{H^{-1}}^2 + \|Y\|_{H^1}^2 + \|Y\|_{L^4}^4 = -\sum_{k=1}^{3} C^k_{3} \langle Y^3, k : Z^k_s : ; \cdot \rangle. \]

Thus \( \|Y\|_{H^{-1}} \) is continuous w.r.t \( t \). Moreover, [Tem01 Lemma 1.4, Chapter III] implies that \( Y \) is weakly continuous in \( H^{-1} \). Hence \( Y \in C([0, T]; H^{-1}) \).

We still write \( \varepsilon \) instead of \( \varepsilon_k \) if there is no confusion. Since \( Y_\varepsilon \) is a solution to equation (4.2), letting \( \varepsilon \to 0 \), it is easy to see that
\[ \lim_{\varepsilon \to 0} H^{-3}(Y_\varepsilon, h)_{H^3} = H^{-3}(Y, h)_{H^3}, \quad \lim_{\varepsilon \to 0} H^{-1}(A^2 h, Y_\varepsilon)_{H^1} = H^{-1}(A^2 h, Y)_{H^1}, \]
\[ \lim_{\varepsilon \to 0} H^{-4}\langle \varepsilon^3 \rangle_{H^1} = H^{-4}\langle \varepsilon^3 \rangle_{H^1}. \]

It remains to show that for any \( h \in H^1 \)
\[ \lim_{\varepsilon \to 0} \left| \int_0^t \langle Y^3_\varepsilon(s) - Y^3(s), h \rangle ds \right| = 0, \quad (4.10) \]
\[ \lim_{\varepsilon \to 0} \left| \int_0^t \langle Y^2_\varepsilon(s)Z_\varepsilon(s) - Y^2(s)Z(s), h \rangle ds \right| = 0, \quad (4.11) \]
\[ \lim_{\varepsilon \to 0} \left| \int_0^t \langle Y^2_\varepsilon(s) - Z^2(s), -Y(s) : Z^2(s), h \rangle ds \right| = 0. \quad (4.12) \]

Since \( Y_\varepsilon \to Y \) in \( L^4([0, T] \times \mathbb{T}^2) \), which is equivalent to \( \|Y_\varepsilon\|_{L^4([0, T] \times \mathbb{T}^2)} \to \|Y\|_{L^4([0, T] \times \mathbb{T}^2)} \) and \( Y_\varepsilon \Rightarrow^m Y \), where \( \Rightarrow^m \) means convergence in Lebesgue measure \( m \) on \([0, T] \times \mathbb{T}^2 \), we have \( \|Y^3_\varepsilon\|_{L^\frac{4}{3}([0, T] \times \mathbb{T}^2)} \to \|Y^3\|_{L^\frac{4}{3}([0, T] \times \mathbb{T}^2)} \) and \( Y^3 \Rightarrow^m Y^3 \). Then (4.11) holds by uniform integrability.
For (4.14), let $R_\varepsilon = Y_\varepsilon - Y$. By the triangle inequality
\[
|\langle Y_\varepsilon^2 Z_\varepsilon - Y^2 Z, h \rangle| \lesssim |\langle R_\varepsilon(Y + Y_\varepsilon)h, Z \rangle| + |\langle Z_\varepsilon - Z, Y^2 h \rangle|.
\]
For the second term on the right hand side of the above inequality, we have
\[
|\langle Z_\varepsilon - Z, Y^2 h \rangle| \lesssim ||Z_\varepsilon - Z||_{-\alpha}||Y^2 h||_{B^\beta_{\alpha,1}} \lesssim ||Z_\varepsilon - Z||_{-\alpha}||Y^2||_{B^\alpha_{2,1}}||h||_{B^\beta_{2,1}},
\]
where we used Lemma 2.2 in the first inequality and Lemma 2.3 in the second inequality. By [Tri92, Remark 2, Section 3.2, Chapter 2] we have $H^1 \subset B^\alpha_{2,1}$ for any $\alpha < 1$. Hence
\[
|\langle Z_\varepsilon - Z, Y^2 h \rangle| \lesssim ||Z_\varepsilon - Z||_{-\alpha}||Y^2||_{B^\alpha_{2,1}}||h||_{H^1}.
\]
Combining with (4.13), we have
\[
|\langle Z_\varepsilon - Z, Y^2 h \rangle| \lesssim ||Z_\varepsilon - Z||_{-\alpha}||h||_{H^1}||Y||_{H^\beta_{1}}^\frac{3+\beta_3}{2}||Y||_{H^{-\frac{\beta_3}{2}}-\frac{s}{2}},
\]
where $\beta_3 > \alpha > 0, s > 0$. Let $\frac{3}{2} + \frac{\beta_3}{2} + \frac{s}{2} < 2$. Then Lemma 4.1 and Hölder's inequality imply that
\[
|\int_0^t \langle Z_\varepsilon - Z, Y^2 h \rangle ds| \to 0, \quad \varepsilon \to 0.
\]
Similarly
\[
||R_\varepsilon Y h, Z|| \lesssim ||R_\varepsilon Y||_{B^\beta_{2,1}}||h||_{H^1}||Z||_{-\alpha}.
\]
For $||R_\varepsilon Y||_{B^\alpha_{2,1}}$, we have
\[
||R_\varepsilon Y||_{B^\alpha_{2,1}} \lesssim ||R_\varepsilon Y||_{B^\alpha_{2,2}} \lesssim ||\Lambda^{\beta_0} R_\varepsilon||_{L^4}||Y||_{L^4} + ||\Lambda Y||_{L^4}||R_\varepsilon||_{L^4},
\]
where $\beta_0 > \alpha > 0$ and we used Lemma 2.1 in the first inequality and Lemma 2.3 in the second inequality. By Lemma 2.1 we have the Sobolev embedding $H^{\beta_0+\frac{1}{2}}_2 \subset H^{\beta}_4$. Hence
\[
||R_\varepsilon Y||_{B^\alpha_{2,1}} \lesssim ||R_\varepsilon||_{H^{\beta_0+\frac{3}{2}}}||Y||_{L^4} + ||Y||_{H^{\beta_0+\frac{3}{2}}}||R_\varepsilon||_{H^{\frac{1}{2}}_2}.
\]
By Sobolev interpolation, choosing $\delta > \frac{1}{2} + \beta_0$, we have
\[
||Y||_{H^{\beta_0+\frac{3}{2}}} \lesssim ||Y||_{H^{\delta}}^{\frac{\frac{3}{2}+\beta_0}{\delta}} ||Y||_{H^{-\frac{\beta_0}{2}}}^{\frac{\frac{3}{2}-\delta}{\delta}}.
\]
Moreover, since $\delta > \frac{1}{2} + \beta_0$, we have $||R_\varepsilon||_{H^\delta} \leq ||R_\varepsilon||_{H^{\delta}}$ and $||Y||_{H^{\frac{1}{2} + \beta_0}} \lesssim ||Y||_{H^{\delta}}$. Then we deduce that
\[
||R_\varepsilon Y||_{B^\alpha_{2,1}} \lesssim ||R_\varepsilon||_{H^{\delta}}||Y||_{L^4} + ||Y||_{H^{\delta}}^{\frac{\frac{3}{2}+\beta_0}{\delta}} ||R_\varepsilon||_{H^{\delta}} ||Y||_{H^{-\frac{\beta_0}{2}}}^{\frac{1}{\delta}}.
\]
Let $\beta_0 < \frac{1}{2}$ such that
\[
\frac{3}{4} + \frac{\beta_0}{2} + 1 < 2.
\]
Then by Hölder inequality, we get
\[
\int_0^t ||R_\varepsilon Y||_{B^\alpha_{2,1}}||h||_{H^1}||Z||_{-\alpha} ds \lesssim \left( \int_0^t ||R_\varepsilon||_{H^{\delta}}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t ||Y||_{H^1}^2 F ds \right)^{\frac{1}{2}} \left( \int_0^t ||Y||_{L^4}^4 ds \right)^{\frac{1}{4}} \to 0,
\]
where $F \in L^\infty(0, T)$. 

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Moreover, we have 

\[ |\langle Y_\varepsilon : Z_\varepsilon^2 : -Y : Z^2 : , h \rangle| \lesssim |\langle Y_\varepsilon : Z_\varepsilon^2 : - : Z^2 : , h \rangle| + |\langle R_\varepsilon : Z^2 : , h \rangle|. \]

By essentially the same argument as above, (1.12) also follows.

Then we have got a solution \( Y \) in \( C([0, T]; H^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; H^1) \). Combining this with (4.3), we have \( Y \in C([0, T]; V_0^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; V_0^1) \). \( \square \)

Now we have obtained the existence of solutions to equation (4.1). The following is the uniqueness result.

**Theorem 4.4** For every \( x \in V_0^{-1} \), there exists a unique solution to equation (4.1) in \( C([0, T]; V_0^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; V_0^1) \).

**Proof** Suppose \( u, v \) are two solutions of (4.1) with the same initial value. Let \( r = u - v \), then \( r \) satisfies:

\[
\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{2}A^2 r + \frac{1}{2}A^{\sum_{k=0}^{3} C_k} (u^{3-k} - v^{3-k}) : Z^k, \\
\end{aligned}
\]

\( r(0) = 0. \)

Similarly to (4.4) we have:

\[
\| r \|_{H^{-1}}^2 + \| r \|_{L^2}^2 \lesssim \| (r^2(u + v), Z) \| + \| (r^2, Z^2) \|. \tag{4.13}
\]

By Lemma 4.1 and Lemma 4.2 we know

\[ |(r^2, Z^2 : )| \lesssim \| r^2 \|_{B_1^{\beta_0}} t^{-\beta}, \]

where \( \beta > \alpha > 0 \). Then Lemma 4.1 and Lemma 4.2 imply that

\[ \| r^2 \|_{B_1^{\beta_0}} \lesssim \| \Lambda^{\beta_0} r^2 \|_{L^4} \lesssim \| \Lambda^{\beta_0} r \|_{L^2} \| r \|_{L^4} \lesssim \| r \|_{H^1}^{\beta_0} \| r \|_{H^{-1}}^{1-\beta_0}, \]

where \( 1 > \beta_0 > \alpha > 0 \) and we used the Sobolev interpolation and Sobolev embedding theorem in the last inequality. Then by Young’s inequality, there exists a \( \gamma_1 > 0 \) such that for any \( \varepsilon > 0 \)

\[ |(r^2, Z^2 : )| \lesssim \varepsilon \| r^2 \|_{H^1}^2 + \| r \|_{H^{-1}}^2 t^{-\beta \gamma_1}. \tag{4.14} \]

Let \( \rho \) be small enough. Then \( g := t^{-\rho \gamma_1} \in L^1(0, T) \).

For \( |(r^2(u + v), Z)| \), we similarly obtain that

\[ |(r^2(u + v), Z)| \lesssim \| r^2(u + v) \|_{B_1^{\beta_0}} \| Z \|_{-\alpha} \lesssim \left( \| ur^2 \|_{B_1^{\beta_0}} + \| vr^2 \|_{B_1^{\beta_0}} \right) t^{-\rho}. \]

For \( \| ur^2 \|_{B_1^{\beta_0}} \), we have

\[ \| ur^2 \|_{B_1^{\beta_0}} \lesssim \| \Lambda^0 (ur^2) \|_{L^{p_0}} \lesssim \| \Lambda^{\beta_0} u \|_{L^{p_1}} \| r^2 \|_{L^{q_1}} + \| \Lambda^{\beta_0} r^2 \|_{L^{p_2}} \| u \|_{L^{q_2}} := (I) + (II), \]

with \( p_0 > 1, \beta_0 > \alpha > 0, \) and \( \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}, p_i, q_i > p_0, i = 1, 2 \). Here we used Lemma 4.1 in the first inequality and Lemma 4.2 in the second inequality.

For \( (I) \), according to (4.1) we know that for any \( s > 0 \)

\[ \| r^2 \|_{L^{q_1}} = \| r \|_{L^{2q_1}}^{2 - \frac{1}{q_1} + s} \| r \|_{H^1}^{\frac{1}{q_1} - s} \| r \|_{H^{-1}}^{\frac{1}{q_1} - s}. \]
Moreover, let $p_1 \geq 4$. Then

$$\|\Lambda^{\beta_0} u\|_{L^{p_1}} \lesssim \|\Lambda^{\beta_1} u\|_{L^4} \lesssim \|u\|^{1-2\beta_1}_{L^2} \|u\|^{2\beta_1}_{H^2} \lesssim \|u\|^{1-2\beta_1}_{L^4} \|u\|^{2\beta_1}_{H^1},$$

where $\beta_1 = \beta_0 + \frac{1}{2} - \frac{2}{p_1}$ and we used Lemma $<$ in the first inequality and Sobolev interpolation in the second inequality and Besov embedding Lemma $<$ in the last inequality. Combining these estimates above we have

$$(I) \lesssim \|r\|_{H^1}^{2-\frac{1}{q_1}+s} \|r\|_{H^{-1}}^{\frac{1}{q_1}-s} \|u\|_{L^4}^{1-2\beta_1} \|u\|_{H^1}^{2\beta_1}.$$ 

Hence by Young’s inequality

$$t^{-\rho}(I) \lesssim \|r\|_{H^1}^{2} + \|r\|_{H^{-1}}^{2} \|u\|_{L^4}^{4\beta_1(1-s)} \frac{2(1-2\beta_1)}{\frac{\beta_1}{q_1}-s} \|u\|_{L^4}^{\frac{4\beta_1}{q_1}-s} \frac{2(1-\beta_1)}{\frac{\beta_1}{q_1}-s} \|u\|_{L^4}^{\frac{2(1-\beta_1)}{q_1}-s} \frac{2\rho}{\frac{\beta_1}{q_1}-s}.$$ 

Let $p_0$ be close to 1 and $\beta_0$, $s$ be small enough such that $\frac{1}{p_1} > 1 - \frac{1}{p_0} + \beta_0 + s$, which is equivalent to $\frac{2\beta_1}{q_1} - s + (1-2\beta_1)(\frac{1}{q_1} - s) < 1$. Then the Hölder inequality yields for $\rho$ small enough

$$\int_0^t \|u\|_{H^1}^{\frac{4\beta_1}{q_1}-s} \|u\|_{L^4}^{\frac{2(1-\beta_1)}{q_1}-s} \frac{2\rho}{\frac{\beta_1}{q_1}-s} d\tau \lesssim \left(\int_0^t \|u\|_{H^1}^{2} d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|u\|_{L^4}^{4\beta_1} d\tau\right)^{\frac{1}{4}}.$$ 

Then we get

$$f_1^\rho := \|u\|_{H^1}^{\frac{4\beta_1}{q_1}-s} \|u\|_{L^4}^{\frac{2(1-\beta_1)}{q_1}-s} \frac{2\rho}{\frac{\beta_1}{q_1}-s} \in L^1(0,T),$$

and for any $\varepsilon > 0$,

$$t^{-\rho}(I) \lesssim \varepsilon \|r\|_{H^1}^{2} + f_1^\rho \|r\|_{H^{-1}}^{2}.$$  

(4.15)

For (II), let $q_2 = 4$. Then we have $\frac{1}{p_2} + \frac{1}{4} = \frac{1}{p_0} \in (\frac{3}{4}, 1)$, which implies that $p_2 \in (\frac{3}{4}, 2)$. Similarly by Lemma $<$ we have

$$\|\Lambda^{\beta_0} r^2\|_{L^{p_2}} \lesssim \|\Lambda^{\beta_0} r\|_{L^{p_3}} \|r\|_{L^{q_3}},$$

where $\frac{1}{p_3} + \frac{1}{q_3} = \frac{1}{p_2}$, $p_3, q_3 > p_2$. From (14) we know that for every $s > 0$

$$\|r\|_{L^{q_3}} \lesssim \|r\|_{H^1}^{1-\frac{1}{q_3}+\frac{s}{2}} \|r\|_{H^{-1}}^{\frac{1}{q_3}-\frac{s}{2}}.$$ 

Let $p_3 \geq 2$. Then by Lemma $<$ we have

$$\|\Lambda^{\beta_0} r\|_{L^{p_3}} \lesssim \|r\|_{H^{p_2}} \lesssim \|r\|_{H^1}^{1+\beta_2} \|r\|_{H^{-1}}^{1-\beta_2},$$

where we used Sobolev interpolation in the second inequality and that $\beta_0 = \beta_2 - 1 + \frac{2}{p_2}$. Hence

$$\|\Lambda^{\beta_0} r^2\|_{L^{p_3}} \lesssim \|r\|_{H^1}^{\frac{3}{2} + \frac{\beta_2}{p_2}} \|r\|_{H^{-1}}^{\frac{1}{2} - \frac{\beta_2}{p_2}} \|r\|_{H^1}^{\frac{1}{2} - \frac{\beta_2}{p_2}} \|r\|_{H^{-1}}^{\frac{1}{2} - \frac{\beta_2}{p_2}} = \|r\|_{H^1}^{2 + \frac{\beta_2}{p_2}} \|r\|_{H^{-1}}^{\frac{1}{2} - \frac{\beta_2}{p_2}} \|u\|_{L^4}^{\frac{1}{2} - \frac{\beta_2}{p_2}}.$$ 

Thus, we have

$$(II) \lesssim \|r\|_{H^1}^{\frac{2}{p_2} - \frac{1}{p_2} + \frac{s}{2}} \|r\|_{H^{-1}}^{\frac{1}{p_2} - \frac{s}{2}} \|u\|_{L^4}.$$ 

Then by Young’s inequality we have

$$t^{-\rho}(II) \lesssim \varepsilon \|r\|_{H^1}^{2} + \|r\|_{H^{-1}}^{2} \|u\|_{L^4}^{\frac{2}{p_2} - \frac{1}{p_2} + \frac{s}{2}} \|u\|_{L^4}^{\frac{2}{p_2} - \frac{s}{2}}.$$ 

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It is easy to see that $p_2 < 2$ yields $\frac{2}{p_2 - \frac{2}{2} - \frac{\beta_0}{2}} \leq 4$ when $s, \beta_0$ are small enough. Then for small enough $\rho$ we have $f_2^\rho := \|u\|^2_{L^3} - \frac{1}{\rho^2} - \frac{\beta_0}{2} \in L^1(0, T)$.

Then we obtain that for any $\varepsilon > 0$

$$|\langle r^2 u, Z \rangle| \lesssim \varepsilon \|r\|^2_{H^1} + f^u \|r\|^2_{H^{-1}},$$

where $f^u := f_1^u + f_2^u \in L^1(0, T)$.

The same holds with $u$ replaced by $v$. Let $f = f^u + f^v \in L^1(0, T)$. Then

$$|\langle r^2(u + v), \tilde{Z} \rangle| \lesssim \varepsilon \|r\|^2_{H^1} + f \|r\|^2_{H^{-1}}.$$

Hence we get

$$\frac{d}{dt} \|r\|^2_{H^{-1}} + \|r\|^2_{H^1} \lesssim \varepsilon \|r\|^2_{H^1} + (f + g) \|r\|^2_{H^{-1}}.$$

Choose a suitable $\varepsilon > 0$ such that

$$\frac{d}{dt} \|r\|^2_{H^{-1}} \lesssim (f + g) \|r\|^2_{H^{-1}}. \quad (4.16)$$

Then by Gronwall’s inequality we have

$$\|r(t)\|^2_{H^{-1}} \lesssim \|r(0)\|^2_{H^{-1}} \exp \left( \int_0^t f(s) + g(s) ds \right) = 0.$$ 

Since $V_0^{-1}$ is a subspace of $H^{-1}$, we obtain the uniqueness.

\[ \square \]

**Remark 4.5** We emphasize that we can also obtain local well-posedness by using the fixed point argument in [DDP03] and [MW17b] with initial value in $C^{-\frac{1}{4}}$. Since we only have an $H^{-1}$-uniform estimate, the local solution cannot be extended to a global solution similarly as for the dynamical $\Phi^4_2$ equation.

## 5 Relation to the solution obtained by the Dirichlet form approach

In Section 4, we obtained a unique solution $Y$ to the shifted equation (1.1). Now it is natural to ask whether $X := Y + Z$ satisfies the original equation (1.1) and having $\nu$ as an invariant measure. In this section, we are going to obtain a probabilistically weak solution of equation (1.1) via the Dirichlet form approach and compare this solution with the solution we obtain in Section 4. This helps us to obtain the uniqueness of the corresponding quasi-regular Dirichlet forms (see Theorem 5.14). As mentioned in the introduction, this may give us some hope to study the scaling limit of the Kawasaki dynamics of the Ising-Kac model.

First, we introduce the Gelfand triple that we will work on. According to the definition of $V_0^a$ and [Hi]{Theorem 3.1}, $\mu$ is supported on $V_0^{-s}$ for any $s > 1$. So we fix a small enough $s_0 > 0$ and $V_0^{-1-s_0}$ as the state space and denote it by $E$ for convenience. By identifying $V_0^1$ and $V_0^{-1}$ via the Riesz isomorphism we have the following Gelfand triple:

$$E^* \subset V_0^{-1} \subset E \quad (5.1)$$
where \( E^* = V_0^{s_0-1} \) and the dualization between \( E^* \) and \( E \) is \( E^* \langle u, v \rangle_E := \langle Q_0 u, v \rangle_{V_0^{1-s_0}} \) for any \( u \in E^*, v \in E \). Here \( V_0^{\langle \cdot, \cdot \rangle_{V_0^{1-s_0}}} \) is given by

\[
V_0^{\langle \cdot, \cdot \rangle_{V_0^{1-s_0}}} := \sum_k s_0^k \langle u, e_k \rangle_s s_0^k \langle v, e_k \rangle_s, \quad u \in V_0^s, v \in V_0^{-s}. \tag{5.2}
\]

Then we have that \( E^* \langle u, v \rangle_E = \langle u, v \rangle_{V_0^{-1}}, \forall u \in E^*, \forall v \in V_0^{-1}. \) \( \tag{5.3} \)

Moreover, we define \( FC_{b}^\infty := \{ f(E^* \langle l_1, \cdot \rangle_E, \cdots, E^* \langle l_m, \cdot \rangle_E) : m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \cdots, l_m \in E^* \} \)

For all \( \varphi = f(E^* \langle l_1, \cdot \rangle_E, \cdots, E^* \langle l_m, \cdot \rangle_E) \in FC_{b}^\infty \), we can define the directional derivative for \( h \in V_0^{-1} \):

\[
\partial_h \varphi(z) := \lim_{t \to 0} \frac{\varphi(z + th) - \varphi(z)}{t} = \sum_{i=1}^m \partial_i f(E^* \langle l_1, \cdot \rangle_E, \cdots, E^* \langle l_m, \cdot \rangle_E) \langle l_i, h \rangle_{V_0^{-1}}.
\]

Then by the Riesz representation theorem, there exists a map \( \nabla \varphi : E \to V_0^{-1} \) such that

\[
\langle \nabla \varphi(z), h \rangle_{V_0^{-1}} = \partial_h \varphi(z), h \in V_0^{-1}.
\]

### 5.1 Solution given by Dirichlet forms

Since \( Q^{-1-s_0} : V_0^{1+s_0} \to V_0^{1-s_0} \) is the Riesz isomorphism for \( V_0^{1+s_0} \), i.e.

\[
v_0^{1+s_0} \langle h, k \rangle_{V_0^{1-s_0}} = \langle Q^{-1-s_0} h, k \rangle_{V_0^{1-s_0}},
\]

\( \mu \) is in fact a Gaussian measure on Hilbert space \( V_0^{1-s_0} \), with covariance operator \( C := Q^{2+s_0} \), that is

\[
\int_{V_0^{1-s_0}} e^{i \langle h, z \rangle_{V_0^{1-s_0}}} \mu(dz) = \langle Ch, h \rangle_{V_0^{1-s_0}}.
\]

Then we have the following integration by parts formula for \( \mu \):

**Proposition 5.1** For all \( F \in FC_{b}^\infty, h \in V_0^{3+s_0} \), we have

\[
\int \partial_h F d\mu = \int E^* \langle A^2 h, \phi \rangle_E F(\phi) \mu(d\phi). \tag{5.4}
\]

**Proof** First, by [DPZ02, Section 1.2.4] we know the reproducing kernel of \( (V_0^{1-s_0}, \mu) \) is \( V_\mu := C^{1/2} V_0^{1-s_0} = V_0^{1} \). Then by [MR92, Theorem 3.1, Chapter II] we have

\[
\int \partial_h F d\mu = \int \langle C^{-1} h, \phi \rangle_{V_0^{1-s_0}} F(\phi) \mu(d\phi)
\]

\[
= \int \langle Q^{-2-s_0} h, \phi \rangle_{V_0^{1-s_0}} F(\phi) \mu(d\phi)
\]

\[
= - \int_{V_0^{1+s_0}} \langle Ah, \phi \rangle_{V_0^{1-s_0}} F(\phi) \mu(d\phi)
\]

\[
= - \int_{E^*} \langle Q^{-1} Ah, \phi \rangle_E F(\phi) \mu(dz)
\]

\[
= \int_{E^*} \langle A^2 h, \phi \rangle_E F(\phi) \mu(dz).
\]

\[
\square
\]
Remark 5.2 In fact, by a similar argument as in [GJ12, (9.1.32)], (5.4) still holds for \( F \exp(-N) \), where \( N = \mathcal{A}(q^n : e_0) \), i.e., for all \( F \in \mathcal{F}_b^\infty \), \( h \in V_0^{3+s_0} \)

\[
\int \partial_h (F \exp(-N)) \, d\mu = \int E^* \langle A^2 h, \phi \rangle_E F(\phi) \exp(-N(\phi)) \mu(\phi)
\]

Then for the Gibbs measure \( \nu \) defined in Section 3, we have the following integration by parts formula:

**Proposition 5.3** For all \( F \in \mathcal{F}_b^\infty \), \( h \in V_0^{3+s_0} \), we have

\[
\int \partial_h F d\nu = \int \left( E^* \langle A^2 h, \phi \rangle_E - E^* \langle Ah, : \phi^3 : \rangle_E \right) F(\phi) \nu(\phi) \, d\phi,
\]

(5.5)

where : \( \phi^3 : \) has been constructed in Lemma 3.2.

**Proof** According to Proposition 5.1 and Remark 5.2,

\[
\int \partial_h F d\nu = \int \langle \partial_h F \exp(-N) \rangle d\mu \\
= c \int [\partial_h (F \exp(-N)) + F \exp(-N) \partial_h N] d\mu \\
= \int F(\phi) \left( E^* \langle A^2 h, \phi \rangle_E - \partial_h N(\phi) \right) \nu(\phi) \, d\phi
\]

By [Oka94, Theorem 4.1.1],

\[
\partial_h : \phi^n(x) := n : \phi^{n-1}(x) : (\rho \ast h)(x).
\]

Here \( \partial_h : \phi^n(x) : \) is defined as the directional derivative of the function \( \phi \rightarrow \phi^n(x) \). Then

\[
\partial_h N(\phi) = \langle : \phi^3 : ; h \ast \rho \rangle,
\]

where \( N(\phi) := \langle \frac{1}{4} : \phi^4 ; e_0 \rangle \). Letting \( \varepsilon \rightarrow 0 \), due to the closability of \( \partial \Pi_h \) in \( L^2(E, \mu) \),

\[
\partial_h N(\phi) = \langle : \phi^3 ; h \rangle = -E^* \langle Ah, : \phi^3 : \rangle_E,
\]

which implies

\[
\int \partial_h F d\nu = \int \left( E^* \langle A^2 h, \phi \rangle_E - E^* \langle Ah, : \phi^3 : \rangle_E \right) F(\phi) \nu(\phi) \, d\phi.
\]

\( \square \)

**Theorem 5.4** The bilinear form

\[
\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int \langle \nabla \varphi, \nabla \psi \rangle_{V_0^{-1}} \, d\nu, \forall \varphi, \psi \in \mathcal{F}_b^\infty,
\]

is closable on \( L^2(E, \nu) \). Its closure is a symmetric quasi-regular Dirichlet form denoted by \( (\mathcal{E}, D(\mathcal{E})) \).
Proof Let \( h_k = \sqrt{\lambda_k} e_k \). Then \( \{ h_k \}_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \) is an orthonormal basis of \( V_0^{-1} \). Then

\[
\mathcal{E}(\varphi, \psi) = \frac{1}{2} \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \int \partial_{h_k} \varphi \partial_{h_k} \psi \, d\nu, \forall \varphi, \psi \in \mathcal{F}C_b^\infty,
\]

By Proposition 5.3 we have \( \int \partial_{h_k} \varphi \, d\nu = -\int \varphi \beta_k \, d\nu \), where \( \beta_k \in L^2(\mathcal{E}, \nu) \) and

\[
\beta_k(\phi) = -E^\nu(\langle A^2 h_k, \phi \rangle) + E^\nu(\langle Ah_k, \phi^3 \rangle), \forall k \neq (0,0).
\]

According to [MR92, Proposition 3.3, Chapter II], any \( \mathcal{E}_1 \)-Cauchy sequence \( \{ u_n \} \subset \mathcal{F}C_b^\infty \), i.e. \( \lim_{n,m \to \infty} \mathcal{E}(u_n - u_m, u_n - u_m) + \| u_n - u_m \|_{L^2(\mathcal{E}, \nu)} = 0 \), has a unique limit in \( L^2(\mathcal{E}, \nu) \), which is also called \( (\mathcal{E}, \mathcal{F}C_b^\infty) \) is closable on \( L^2(\mathcal{E}, \nu) \). The closure \( (\mathcal{E}, D(\mathcal{E})) \) is a symmetric Dirichlet form. Moreover, by [MR92, Proposition 4.2, Chapter IV], it is standard to prove that the capacity of \( (\mathcal{E}, D(\mathcal{E})) \) is tight, and according to the fact that \( \mathcal{F}C_b^\infty \) is dense in \( L^2(\mathcal{E}, \nu) \) and separates the points in \( L^2(\mathcal{E}, \nu) \), this means that \( (\mathcal{E}, D(\mathcal{E})) \) is a quasi-regular Dirichlet form in the sense of Definition A.2 or [MR92, Definition 3.1]. □

**Theorem 5.5** There is a conservative Markov diffusion process on another probability space \( (\Omega_1, \mathcal{F}_1, (\mathbb{P}_1^z)_{z \in E}) \)

\[
M = (\Omega, \mathcal{F}, \mathcal{M}_t, (\mathbb{X}(t))_{t \geq 0}, (\mathbb{P}_1^z)_{z \in E}),
\]

which is properly associated with \( (\mathcal{E}, D(\mathcal{E})) \), i.e. for \( u \in L^2(\mathcal{E}, \nu) \cap \mathcal{B}_0(E) \), the transition semigroup \( P_t u(z) := E^\nu(u(\mathbb{X}(t))) \) is \( \mathcal{E} \)-quasi-continuous for all \( t > 0 \) and is a \( \nu \)-version of \( T_t u \) where \( T_t \) is the semigroup associated with \( (\mathcal{E}, D(\mathcal{E})) \).

**Proof** Since \( (\mathcal{E}, D(\mathcal{E})) \) is a quasi-regular Dirichlet form on \( L^2(\mathcal{E}, \nu) \), it is a direct consequence of Theorem A.3 and [AR91, Theorem 3.6]. □

In particular, by the construction of \( M \) (see e.g. [FOT94, Chapter 7] or [MR92, Section 3, Chapter IV]), \( M \) can be chosen as the canonical process on \( E \). Moreover, for the conservative Markov diffusion \( M \) properly associated with \( (\mathcal{E}, D(\mathcal{E})) \), we say that a set \( S \subseteq E \) is a properly \( \mathcal{E} \)-exceptional set if \( \nu(S) = 0 \) and \( \mathbb{P}_1^z(\mathbb{X}(t) \in E \setminus S, \forall t \geq 0 = 1 \) for \( z \in E \setminus S \). For the relation between \( \mathcal{E} \)-exceptional set and properly exceptional set, we have the following theorem:

**Theorem 5.6** ([FOT94, Theorem 4.1.1]) If \( N \) is an \( \mathcal{E} \)-exceptional set, then \( N \) is contained in a properly \( \mathcal{E} \)-exceptional set \( S \). \( S \) can be taken to be Borel measurable.

As in [AR91], we now derive the SPDE satisfied by \( \mathbb{X} \) in the analytically weak form:

**Theorem 5.7** There exists a map \( \mathbb{W} : \Omega \to C([0, \infty); C([0, \infty); V_0^{-1-s_0}(\mathbb{T}^2, \mathbb{R}^2))) \), and a properly \( \mathcal{E} \)-exceptional set \( S \in \mathcal{B}(E) \), such that \( \forall z \in E \setminus S, \mathbb{W} \) is a \( U \)-cylindrical Wiener process on \( (\Omega, \mathcal{M}_t, \mathbb{P}_1^z) \) and the sample paths of the associated process \( M = (\Omega, \mathcal{F}, \mathcal{M}_t, (\mathbb{X}(t))_{t \geq 0}, (\mathbb{P}_1^z)_{z \in E}) \) on \( E \) satisfy the following: for \( h \in V^{3+s_0} \),

\[
E^\nu(\langle h, \mathbb{X}(t) - \mathbb{X}(0) \rangle_E) = -\frac{1}{2} \int_0^t E^\nu(\langle A^2 h, \mathbb{X}(s) \rangle_E) \, ds
+ \frac{1}{2} \int_0^t E^\nu(\langle Ah, : \mathbb{X}(s)^3 : \rangle_E) \, ds
+ \int_0^t \langle B^* h, d\mathbb{W}_s \rangle_{V_0^{-1}(\mathbb{T}^2, \mathbb{R}^2)}, \forall t \geq 0, \mathbb{P}_1^z - a.s.,
\]

where \( B, B^* \) are defined as in (5.2). Moreover, \( \nu \) is an invariant measure for \( M \) in the sense that \( \int P_t u \, d\nu = \int u \, d\nu \) for \( u \in L^2(\mathcal{E}, \nu) \cap \mathcal{B}_0(E) \).
Proof Let \( u_h(\phi) = E^* \langle h, \phi \rangle_E, h \in V_0^{3+s_0} \) and let \((\mathcal{L}, D(\mathcal{L}))\) be the generator of \((\mathcal{E}, D(\mathcal{E}))\). For any \( v \in D(\mathcal{E}) \)

\[
\frac{1}{2} \int (\nabla u_h, \nabla v)_{V_0^{-1}} d\nu = -\frac{1}{2} \int \partial_h v(\phi) \nu(d\phi) = \frac{1}{2} \int \left( E^* \langle A^2 h, \phi \rangle_E - E^* \langle Ah, \phi^3 \rangle_E \right) v(\phi) \nu(d\phi).
\]
Hence \( v \mapsto \frac{1}{2} \int (\nabla u_h, \nabla v)_{V_0^{-1}} d\nu \) is continuous in \( L^2(E, \nu) \). By definition of the generator of \((\mathcal{E}, D(\mathcal{E}))\), \( u_h \in D(\mathcal{L}) \) and \( \mathcal{L} u_h(\phi) = -\frac{1}{2} \left( E^* \langle A^2 h, \phi \rangle_E - E^* \langle Ah, \phi^3 \rangle_E \right) \).

By the well-known Fukushima’s decomposition (see e.g. [AR91, Theorem 4.3]), we have for q.e. \( z \in E \),

\[
u h (\bar{X}_t) - \nu h (\bar{X}_0) = M^h_t + \int_0^t \mathcal{L} \nu h (\bar{X}_s) ds = M^h_t - \frac{1}{2} \int_0^t \left( E^* \langle A^2 h, \bar{X}_s \rangle_E - E^* \langle Ah, \bar{X}_s^3 \rangle_E \right) ds,
\]
where \( M^h \) is an additive functional (Definition [A2]) and also a martingale with \( \langle M^h \rangle_t = t\|h\|^2_{V_0^{-1}} \). By [AR91, Proposition 4.5],

\[
\langle M^h \rangle_t = \int_0^t \nabla \nu h (\bar{X}_s), \nabla \nu h (\bar{X}_s) \rangle_{V_0^{-1}} ds = t\|h\|^2_{V_0^{-1}}.
\]

Now we identify \( M^h \) as the component in direction \( h \) of the conservative noise. For \( f = B^* Qh \in U \), with \( h \in V_0^{-1} \), define \( \bar{W}^f := M^h_t \) and let \( D := \text{span}\{B^* Qe_k : k \in \mathbb{Z}^2 \setminus \{(0,0)\}\} \). Since \( \|B^* Qh\|^2_U = \|h\|^2_{V_0^{-1}} \), it is easy to check that \( \langle \bar{W}^f, \bar{W}^g \rangle_t = t \langle f, g \rangle_U \) for \( f, g \in D \), where \( \langle \bar{W}^f, \bar{W}^g \rangle_t \) is the bracket process of \( \bar{W}^f \) and \( \bar{W}^g \). Moreover, \( D \) is dense in \( U \) and \( \bar{W}^f \) is \( \mathbb{Q} \)-linear on \( D \), since the embedding \( U \rightarrow V_0^{-1-s}(\mathbb{T}^2, \mathbb{R}^2) \) is Hilbert-Schmidt for any \( s > 0 \). By [AR91, Theorem 6.2], there exist a map \( \bar{W} : \Omega \rightarrow C([0, \infty); V_0^{-1-s}(\mathbb{T}^2, \mathbb{R}^2)) \), and a properly \( \mathcal{E} \)-exceptional set \( S \in \mathcal{B}(E) \), i.e. \( \nu(S) = 0 \) and \( \mathbb{P}^z(\bar{X}(t) \in E \setminus S, \forall t \geq 0) = 1 \) for \( z \in E \setminus S \), such that for any \( f \in D \)

\[
V_0^{-1-s} \langle \bar{W}, f \rangle_{V_0^{1+s}} = \bar{W}^f, \mathbb{P}^z - a.s.,
\]
where \( V_0^{-1-s} \langle \cdot, \cdot \rangle_{V_0^{1+s}} \) is defined by (1.2). In particular,

\[
(B^* h, \bar{W})_{V_0^{-1}(\mathbb{T}^2, \mathbb{R}^2)} = \langle \bar{W}, (B^* Qh)_U \rangle = M^h_t,
\]
and \( \bar{W} = (\bar{W}^1, \bar{W}^2) \), where \( \bar{W}^i : \Omega \rightarrow C([0, \infty); E), i = 1, 2 \) are two independent \( L^2_0 \)-cylindrical Wiener processes under \( \mathbb{P}^z \) for any \( z \in E \setminus S \).

Moreover, it is easy to check that for the constant function \( 1 \) on \( E \), i.e. \( 1(z) = 1, \forall z \in E \),

\[
1 \in FC_b^\infty, \quad \nabla 1 \equiv 0,
\]
and

\[
-\int \mathcal{L} 1 v d\nu = \mathcal{E}(1, v) = 0, \forall v \in D(\mathcal{E}).
\]
Thus we obtain that \( \mathcal{L} 1 = 0 \) and \( T_t 1 = 1 \) in \( L^2(E, \nu) \). Then by the symmetry of the semigroup \( T_t \),

\[
\int P_t udv = \int T_t udv = \int u T_t 1 dv = \int udv, \forall u \in L^2(E, \nu) \cap \mathcal{B}_b(E).
\]
This yields that \( \nu \) is an invariant measure of \( \bar{X} \). \(\square\)
Remark 5.8 We mention that the above Dirichlet form arguments can be easily extend to the infinite volume case. The Dirichlet form $(\Lambda, D(\Lambda))$ for equation (5.7) on $\mathbb{R}^2$ can be directly constructed by the closure of the following bilinear form

$$\Lambda(\varphi, \psi) = \int \langle \nabla f, \nabla g \rangle_{\tilde{H}^{-1}}, d\nu, \forall \varphi, \psi \in \mathcal{F}C^\infty_0(\tilde{H}^{-1}),$$

(5.7)

where $\tilde{H}^s$ is the homogeneous Sobolev space of order $s$, and $\nabla$ is the gradient in $\tilde{H}^{-1}$. By a similar argument as before, it is easy to check $(\Lambda, D(\Lambda))$ is quasi-regular and one obtains again a probabilistically weak solution directly.

5.2 Relation between the two solutions

In the following we discuss the relation between $M$ constructed above and the solution of the shifted equation (4.4). For $\tilde{W}$ constructed in Theorem 6.2, define $\tilde{Z}(t) := \int_0^t e^{-(t-s)A^2/2} B d\tilde{W}_s$. We will prove that the difference $\tilde{Y} := X - \tilde{Z} \in C([0, T]; C^{-\alpha})$ is a solution to equation (4.4) by replacing $Z$ by $\tilde{Z}$. Recall that in Section 4.2, for every $W$, we constructed a corresponding strong solution $X := Y + Z$. In particular for $\tilde{W}$, we have a solution $\tilde{X}$. By the pathwise uniqueness of (4.4) in Theorem 4.2, we prove $\tilde{X} = X$. Thus the law of the solution constructed by the Dirichlet form approach is the same as the law of $X := Y + Z$ given in Section 4.2. This implies that $\nu$ is also an invariant measure of $X$.

We also mention that by Lemma 2.1, $C^{-\alpha} \subset V^{-1} \subset V^{-1-s_0}$ for $\alpha \in (0, 1), C^{-\alpha}$ is Borel-measurable subset of $V^{-1-s_0}$ and $\nu(E \cap C^{-\alpha}) = 1$.

Theorem 5.9 Let $\alpha \in (0, \frac{1}{2}), \alpha < \beta < 2 - \alpha$. There exists a properly $E$-exceptional set $S_2 \subset E$ such that for every $z \in (C^{-\alpha} \cap E) \setminus S_2$ under $\mathbb{P}^z$, $\tilde{Y} := X - \tilde{Z} \in C([0, T]; C^\beta) \cap C([0, T]; C^{-\alpha} \cap V_0^{-1})$ is a solution to the following equation:

$$\tilde{Y}(t) = \frac{1}{2} \int_0^t e^{-(t-s)A^2/2} A \sum_{l=0}^3 C^l_3 \tilde{Y}(s)^l : \tilde{Z}(s)^{3-l} : ds + e^{-\frac{1}{2}A^2} X(0).$$

(5.8)

Here $C([0, T]; C^\beta)$ is equipped with the norm $\sup_{t \in [0, T]} \frac{t^{\beta+\alpha}}{\beta} \| \cdot \|_{\beta}$. Moreover,

$$\mathbb{P}^z[\tilde{X}(t) \in (C^{-\alpha} \cap E) \setminus S_2, \forall t \geq 0] = 1 \text{ for } z \in (C^{-\alpha} \cap E) \setminus S_2.$$

(5.9)

Proof For $z \in E \setminus S$ under $\mathbb{P}^z$ we have that

$$\tilde{X}(t) = \frac{1}{2} \int_0^t e^{-(t-s)A^2/2} A : \tilde{X}(\tau)^3 : d\tau + \tilde{Z}(t) + e^{-\frac{1}{2}A^2} \tilde{X}(0),$$

where $S$ is the properly $E$-exceptional set in Theorem 5.4. Since $\nu$ is an invariant measure for $\tilde{X}$, by Lemma 2.1 and Lemma 5.2 we conclude that for every $T \geq 0, p > 1, \delta > 0$, with $2\delta - \alpha < 0$, and $p_0 > 1$ large enough

$$\int \mathbb{E}^z \int_0^T \| : \tilde{X}(\tau)^3 : \|_{p_0}^p d\tau \nu(dz) \lesssim \int \mathbb{E}^z \int_0^T \| : \tilde{X}(\tau)^3 : \|_{p_{\beta-\alpha}}^p d\tau \nu(dz)$$

$$= T \int \| : \phi^3 : \|_{p_{\beta-\alpha}}^p \nu(d\phi) \lesssim T \int \| : \phi^3 : \|_{2\beta-\alpha}^p \nu(d\phi) < \infty,$$

which implies that for $\nu - a.s. \ z \in E \setminus S, \mathbb{P}^z$-a.s.

$$: \tilde{X}(\cdot)^3 : \in L^p(0, T; C^{-\alpha}), \quad \mathbb{E}^z \int_0^T \| : \tilde{X}(\tau)^3 : \|_{p_0}^p d\tau \lesssim 1, \ \forall p > 1.$$
Here we used Lemma 2.1 to deduce the first result. The second, however, does not imply the first directly because of (2.8). By the definition of Wick power, it is easy to check that $z \mapsto \mathbb{E}^z \int_0^T \|:X(\tau)^3:\|_\alpha d\tau$ is quasi-continuous in the sense of Definition A.1 (see e.g. [MR92, Chapter 4, Exercise 2.9]). By Definition A.1, (5.10) holds $\mathbb{P}^z$-a.s. for q.e. $z \in E$. Then Theorem 5.1 implies that there exists a proper $\mathcal{E}$-exceptional set $S_1 \supset S$ such that (5.10) holds $\mathbb{P}^z$-a.s. for $z \in E \setminus S_1$.

Moreover, Lemma 2.2 implies that for $\alpha < \beta < 2 - \alpha$, $z \in (E \cap \mathcal{C}^{-\alpha}) \setminus S_1$

$$\int_0^t e^{-\alpha A^{2/\alpha} A :X(\tau)^3: :} d\tau \in C([0,\infty);\mathcal{C}^\beta) \mathbb{F}^z - a.s..$$

Now by Lemma 2.2 we conclude that for $z \in (E \cap \mathcal{C}^{-\alpha}) \setminus S_1$, $e^{-\frac{\alpha}{2} A^2 \mathbb{X}(0) \in C([0,T],\mathcal{C}^{-\alpha}) \cap C((0,T],\mathcal{C}^\beta)$. Thus,

$$\mathbb{X} - \mathbb{Z} \in C([0,T],\mathcal{C}^{-\alpha}) \cap C((0,T],\mathcal{C}^\beta) \mathbb{P}^z - a.s..$$

Since $\mathbb{P}^z \circ \mathbb{X}(t)^{-1} = \nu$, by Lemma 6.5 we conclude that under $\mathbb{P}^\nu$, by Fubini’s theorem $\mathbb{Y} := \mathbb{X} - \mathbb{Z}$ satisfies (2.8) and for $\nu$-a.e. $z \in E$ under $\mathbb{P}^\nu$, $\mathbb{Y} := \mathbb{X} - \mathbb{Z}$ satisfies (2.8). Moreover, it is easy to check that $\mathcal{C}^{-\alpha} \subset V^{-1}$ and $\mathbb{Y}(e_0) = 0$. Then we obtain that $\mathbb{Y} \in C([0,T],\mathcal{C}^{-\alpha} \cap V^{-1}) \cap C((0,T],\mathcal{C}^\beta), \mathbb{P}^\nu - a.s.$ for $\nu$-a.e. $z \in E \cap \mathcal{C}^{-\alpha}$.

In the following we prove that these results hold under $\mathbb{P}^z$ for $z$ outside a properly $\mathcal{E}$-exceptional set. First we have $\mathbb{Z} \in C([0,\infty);\mathcal{C}^{-\alpha}) \mathbb{P}^z$-a.s., which combined with $\mathbb{X} - \mathbb{Z} \in C([0,T],\mathcal{C}^{-\alpha})$ implies

$$\mathbb{P}^\nu[\mathbb{X} \in C([0,\infty),\mathcal{C}^{-\alpha})] = 1.$$

We also have

$$\mathbb{Y}(s,t_0) := \mathbb{X}(s + t_0) - \mathbb{Z}(s + t_0) = \frac{1}{2} \int_{t_0}^{t_0 + s} e^{-(t_0 + s - \tau) A^{2/\alpha} A :X(\tau)^3: :} d\tau$$

$$+ e^{-s A^{2/\alpha} (\mathbb{X}(t_0) - \mathbb{Z}(t_0))} \in C((0,\infty)^2;\mathcal{C}^\beta) \mathbb{P}^\nu - a.s..$$

Similar arguments as in the proof of Lemma 6.5 imply that $\forall s > 0, t_0 \geq 0$

$$\mathbb{P}^\nu : (\mathbb{X}(s + t_0)^3) := \sum_{l=0}^3 C_3^l \mathbb{Y}(s,t_0)^l : \mathbb{Z}(s + t_0)^{3-l} : ,$$

$$\mathbb{X} \in C([0,\infty),\mathcal{C}^{-\alpha}), \mathbb{Y} \in C((0,\infty)^2;\mathcal{C}^\beta)) = 1.$$

In the following we use $I_{t,t_0}$ to denote the equality

$$\int_0^t e^{-(t-s) A^{2/\alpha} A :X(s + t_0)^3: :} ds$$

$$= \sum_{l=0}^3 \int_0^t e^{-(t-s) A^{2/\alpha} A C_3^l \mathbb{Y}(s,t_0)^l : \mathbb{Z}(s + t_0)^{3-l} : :} ds.$$

We say that $I_{t,t_0}$ holds if the above identity holds in $\mathcal{C}^\beta$ and the $\mathcal{C}^\beta$-norms of both sides are finite. Then using Fubini’s theorem we know that

$$\mathbb{P}^\nu(I_{t,t_0} \text{ holds } \forall t \geq 0, \forall t_0 \in \mathbb{Q}^+, \mathbb{X} \in C([0,\infty);\mathcal{C}^{-\alpha}), \mathbb{Y} \in C((0,\infty)^2;\mathcal{C}^\beta)) = 1.$$

Here we used $\mathbb{X} \in C([0,\infty);\mathcal{C}^{-\alpha})$ for $\alpha < \frac{1}{3}$ to make the right hand side of $I_{t,t_0}$ meaningful. It is obvious that the right hand side of the first equality is continuous in $\mathcal{C}^\beta$ with respect to $t_0$. Since
\[f^t_{0} e^{-(t-s)A^2/2}A : \overline{X}(s + t_0)^3 : ds = \int_{t_0}^{t} e^{-(t-s)A^2/2}A : \overline{X}(s)^3 : ds\]

we know that \(f^t_{0} e^{-(t-s)A^2/2}A : \overline{X}(s + t_0)^3 : ds\) is also continuous with respect to \(t_0\) and we obtain that

\[\mathbb{P}^\prime' (I,t,t_0) = \mathbb{P}^\prime (I,t,t_0) \] \[\forall t, t_0 \geq 0, \overline{X} \in C([0, \infty); C^{-\alpha}, \overline{Y} \in C((0, \infty)^2; C^{\beta}).\]

Then we can claim that there exists a properly \(E\)-exceptional set \(S_2 \supset S_1\) such that for \(z \in (C^{-\alpha} \cap E) \setminus S_2\) under \(\mathbb{P}^z\)

\[\mathbb{P}^z (\overline{X} \in C([0, \infty); C^{-\alpha})) = 1.\]

Indeed, define

\[\Omega_0 := \{\omega : \overline{X} \in C([0, \infty); C^{-\alpha}), \overline{Z}^k \in C([0, \infty); C^{-\alpha}), k = 1, 2, 3, I_t,t_0 \text{ holds } \forall t, t_0 \in \mathbb{Q}^+\},\]

which is measurable in \(\Omega\) which is taken as the canonical space on \(E\). Let \(\Theta_t : \Omega \rightarrow \Omega, t > 0\), be the canonical shift, i.e. \(\overline{X} \circ \Theta_t = \Theta_t(\omega) = \omega(\cdot + t) = \overline{X}(\cdot + t), \omega \in \Omega\).

In the following we will show that there exists a properly \(E\)-exceptional set \(\tilde{S}_2 \supset S_1\) such that for any \(z \in E \setminus \tilde{S}_2, t \in \mathbb{Q}^+\)

\[\mathbb{P}^z (\cap_{t \in \mathbb{Q}^+} \Theta_t^{-1} \Omega_0) = \mathbb{P}^z (\Omega_0) .\] (5.11)

As seen in the proof of Theorem 4.5, by the Fukushima’s decomposition (see e.g. [ARU1] Theorem 4.3), for any \(k \in \mathbb{Z}^2, M^{\alpha_k} = (\overline{W}, e_k)\) is an additive functional in the sense of Definition 4.2. Thus we can find a properly \(E\)-exceptional set \(\tilde{S}_2 \supset S_1\) and a set \(\Lambda \subset \Omega\), such that \(\mathbb{P}^\prime (\Lambda) = 1, \forall t \in E \setminus \tilde{S}_2, \Theta_t \Lambda \subset \Lambda, \forall t > 0\), and moreover, for each \(\omega \in \Lambda, M^{\alpha_k} (\omega) = M^{\alpha_k} (\Theta_t \omega), \forall t, s \geq 0, \forall k \in \mathbb{Z}^2\)

\[\mathbb{W}_{s+t} (\omega) = \mathbb{W}_s (\omega) + \mathbb{W}_t (\Theta_t \omega) .\]

Thus \(\overline{Z} \circ \Theta_t (r) := \int_0^r e^{-(r-s)A^2/2}Bd(\overline{W} \circ \Theta_t)_s = \overline{Z}_{t+r} - e^{-rA^2/2} \overline{Z}_t\). Then we can define the Wick power : \((\overline{Z} \circ \Theta_t)^k (r)\) : by Hermite polynomials in a similar way as Lemma 5.2. In particular, for any \(\omega \in \Lambda, t, s, r, t_0 \geq 0\), by direct calculation, we obtain that for any \(k \in \mathbb{N}\)

\[(\overline{X} \circ \Theta_t) (r) = \overline{X}(t + r),\]

\[(\overline{Z} \circ \Theta_t)^k (r) := (\overline{Z}(t + r) - e^{-rA^2/2} \overline{Z}(t))^k := \sum_{l=0}^{k} C^l_k : \overline{Z}^{k-l}(t + r) : (-e^{-rA^2/2} \overline{Z}(t))^l,\] (5.12)

\[(\overline{Y} \circ \Theta_t)(s,t_0) = (\overline{X} \circ \Theta_t)(s,t_0) - (\overline{Z} \circ \Theta_t)(s + t_0) = \overline{Y}(s + t_0, t) + e^{-s+t_0}A^2 \overline{Z}(t).\]

For any \(\omega \in \Omega_0 \cap \Lambda\), the continuity of \(\overline{X} \circ \Theta_t\) and : \((\overline{Z} \circ \Theta_t)^k (r)\) : follows from the first two identities above. Moreover, since \(I_{s,t_0}\) holds for \(\omega\), we have

\[\sum_{l=0}^{k} C^l_k \overline{Y}(s + t_0, t)^{k-l} : \overline{Z}(s + t_0 + t)^l := \sum_{l=0}^{k} C^l_k \left((\overline{Y} \circ \Theta_t)(s,t_0) - e^{-s+t_0}A^2 \overline{Z}(t)\right)^{k-l} : \overline{Z}(s + t_0 + t)^l :=\]

which implies that \(I_{s,t_0}\) also holds for \(\Theta_t \omega\) for any \(t \in \mathbb{Q}^+\). Hence we have proved that

\[\Theta_t^{-1} \Omega_0 \cap \Lambda \subset \Omega_0 \cap \Lambda, \quad t \in \mathbb{Q}^+.\]

On the other hand, since (5.12) holds for all \(\omega \in \Lambda\), if \(\omega \in \Omega_0 \cap \Lambda\), by the definition of \(\Omega_0\), we conclude that \(\overline{X}(\omega)(t + \cdot), \overline{Z}(\omega)(t + \cdot) \in C([0, \infty); C^{-\alpha})\) for any \(t \in \mathbb{Q}^+\). Hence \(\overline{X}(\omega), \overline{Z}(\omega) \in \mathbb{P}^z (\Omega_0) .\)
It is also easy to check that \( \lim_{t \to 0} X(t) = X(0) \) and \( \lim_{t \to 0} Z(\omega)(t) = Z(\omega)(0) = 0 \).

Similarly we can show that : \( Z(\omega)^k \in C([0, \infty); \mathcal{C}^{-\alpha}) \). Thus we obtain that \( \bigcap_{t \in Q^+} \Theta_t^{-1} \Omega_0 \cap \Lambda \subseteq \Omega_0 \cap \Lambda \). Hence \( \bigcap_{t \in Q^+} \Theta_t^{-1} \Omega_0 \cap \Lambda = \Omega_0 \cap \Lambda \). Since \( P^\nu(\Lambda) = 1 \), \( \forall x \in E \setminus \mathcal{S}_2 \), (5.11) follows.

It is clear that \( P^\nu(\Omega_0) = 1 \) for \( \nu - a.e. \ y \in E \) and \( P(\Omega_0) \in L^2(E, \nu) \cap B_b(E) \). By the Markov property and the conservativity of \( \mathcal{X} \), we know that

\[
\mathbb{P}^\nu(\Theta_t^{-1} \Omega_0) = \mathbb{P}^\nu(\mathbb{P}^\nu(\Omega_0)) = P_t(\mathbb{P}(\Omega_0))(z) = 1, \quad \nu - a.e. z \in E
\]

which by Theorem 6.3, \( z \mapsto \mathbb{P}^\nu(\Theta_t^{-1} \Omega_0) \) has an \( \mathcal{E} \)-quasi-continuous \( \nu \)-version on \( E \). It follows that for every \( t > 0 \)

\[
\mathbb{P}^\nu(\Theta_t^{-1} \Omega_0) = 1 \quad \text{q.e.} \ z \in E,
\]

which by (6.14) implies that

\[
\mathbb{P}^\nu(\Omega_0) = 1 \quad \text{q.e.} \ z \in E.
\]

By the same argument as before, and by Theorem 5.6, there exists a properly \( \mathcal{E} \)-exceptional set \( S_2 \supset \tilde{S}_2 \) such that outside \( S_2 \) the result holds. Moreover, since for \( z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2 \) \( \mathbb{P}^\nu - a.e. \), both sides of \( \mathcal{T}_t \) belong to \( C((0, \infty) \times \mathcal{C}^2) \) as a map on \( \mathcal{C}^2 \) w.r.t \( (t, t_0) \in (0, \infty) \times (0, \infty) \), we conclude that, for \( z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2 \)

\[
\mathbb{P}^\nu(\tilde{X} \in C((0, \infty); \mathcal{C}^{-\alpha}), \mathcal{T}_t \text{ holds } \forall t, t_0 \geq 0) = 1.
\]

Now \( \tilde{Y} \) satisfies (5.5) \( \mathbb{P}^\nu \)-a.s. for \( z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2 \). Moreover, for \( z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2 \), \( \tilde{Y} \in C((0, \infty); \mathcal{C}^{-\alpha}) \cap C([0, T]; \mathcal{C}^3), \mathcal{Z} \in C([0, \infty); \mathcal{C}^{-\alpha}) \) \( \mathbb{P}^\nu \)-a.s., which combined with the definition of properly \( \mathcal{E} \)-exceptional set, implies that

\[
\mathbb{P}^\nu[\tilde{X} \in C((0, \infty); \mathcal{C}^{-\alpha} \cap E) \setminus S_2, \forall t \geq 0] = 1 \quad \text{for} \quad z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2.
\]

\[ \square \]

**Corollary 5.10** Let \( X = Y + Z \) where \( Y \) is the unique solution to \( (L, \mathcal{A}) \) on probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

Then \( \nu \) is an invariant measure of \( X \).

**Proof** By Theorem 6.4, we know that for \( z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2 \), under the measure \( \mathbb{P}^\nu \), \( \tilde{Y} := \tilde{X} - \tilde{Z} \) satisfies the shift equation (5.5) and has better regularity i.e. \( \tilde{Y} \in C((0, T]; \mathcal{C}^{-\alpha} \cap V_0^{-1}) \). By the pathwise uniqueness result in Theorem 5.3, we know that \( \mathbb{P} \circ (Y, Z)^{-1} = \mathbb{P} \circ (Y, Z)^{-1} \) for \( Y(0) = \mathcal{T} \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2 \). Thus \( \mathbb{P} \circ \tilde{X}^{-1} = \mathbb{P} \circ \tilde{X}^{-1} \), for \( \tilde{T}(0) = \tilde{Z} \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2 \). Since \( \nu \) is an invariant measure of \( \tilde{X} \) and \( \nu(\mathcal{C}^{-\alpha} \cap E) = 1 \), \( \nu \) is an invariant measure of \( X \).

\[ \square \]

**Remark 5.11** If we have a probabilistically strong solution on infinite volume case. We can obtain the above results and \( \nu \) is an invariant measure for the solution to equation \( (L, \mathcal{A}) \) on \( \mathbb{R}^2 \).

### 5.3 Markov uniqueness in the restricted sense

In this subsection we will use the pathwise uniqueness results in Theorem 6.1 to prove Markov uniqueness in the restricted sense and the uniqueness of the martingale (probabilistically weak) solutions to \( (L, \mathcal{A}) \) if the solution has \( \nu \) as an invariant measure.

Let \( \mathcal{E}^{\alpha} \) be the set of all quasi-regular Dirichlet forms \( (\mathcal{L}, D(\mathcal{L})) \) (cf. [MR92]) on \( L^2(E; \nu) \) such that \( \mathcal{F}C_b^{\infty} \subset D(L) \) and \( \mathcal{E} = \mathcal{E} \) on \( \mathcal{F}C_b^{\infty} \times \mathcal{F}C_b^{\infty} \). Here for a Dirichlet form \( (\mathcal{L}, D(\mathcal{L})) \) we denote its generator by \( (L(\mathcal{E}), D(L(\mathcal{E}))) \).

In the following we consider the martingale problem in the sense of [AR94] and probabilistically weak solutions to \( (L, \mathcal{A}) \):
\textbf{Definition 5.12} (i) A continuous strong Markov process \( M = (\Omega, \mathcal{F}, (M_t), X_t, (\mathbb{P}^z)) \) in the sense of [MR92, Chapter IV] with state space \( E \) is said to solve the martingale problem for \( (L(E), D) \) if for all \( u \in D, u(X(t)) - u(X(0)) - \int_0^t L(E)u(X(s))ds, t \geq 0 \), is an \((M_t)-\text{martingale under } \mathbb{P}^z\).

(ii) A continuous strong Markov process \( M = (\Omega, \mathcal{F}, (M_t), X_t, (\mathbb{P}^z)) \) with state space \( E \) is called a probabilistically weak solution to \((\mathcal{L}, E)\) if there exists two maps \( W : \Omega \to C([0, \infty); E) \) \( i = 1, 2 \) such that for \( \nu \)-a.e. \( z \) under \( \mathbb{P}^z \), \( W := (W^1, W^2) \) is an \( L^2(T^2, \mathbb{R}^2) \)-cylindrical Wiener process with respect to \((M_t)\) and the sample paths of the associated process satisfy \((\mathcal{L}, E)\) for all \( h \in V^{3 + \infty} \).

\textbf{Remark 5.13} If \( M \) is a probabilistically weak solution to \((\mathcal{L}, E)\), we can easily check that it also solves the martingale problem. Conversely, if \( M \) solves the martingale problem, then with the same argument in Theorem 5.7, there exists an \( L^2(T^2, \mathbb{R}^2) \)-cylindrical Wiener process \( W \) such that \((X, W)\) satisfies \((\mathcal{L}, E)\) for all \( h \in V^{3 + \infty} \). That is, these two definitions are equivalent.

To explain the uniqueness result below we also introduce the following concept:

Two strong Markov processes \( M \) and \( M' \) with state space \( E \) and transition semigroups \((p_t)_{t \geq 0}\) and \((p'_t)_{t \geq 0}\) are called \( \nu \)-equivalent if there exists \( S \in \mathcal{B}(E) \) such that (i) \( \nu(E \setminus S) = 0 \), (ii) \( \mathbb{P}^z[X(t) \in S, \forall t \geq 0] = \mathbb{P}^{z'}[X(t) \in S, \forall t \geq 0] = 1 \), \( z \in S \), (iii) \( p_t f(z) = p'_t f(z) \) for all \( f \in \mathcal{B}_b(E), t > 0 \) and \( z \in S \).

Combining Theorem 5.8 and Theorem 5.14, we obtain Markov uniqueness in the restricted sense for \((L(E), D)\) (see part (iii)) and the uniqueness of martingale (probabilistically weak) solutions to (1.1) having \( \nu \) as an invariant measure (see part (i), (ii)).

\textbf{Theorem 5.14} (i) There exists (up to \( \nu \)-equivalence) exactly one probabilistically weak solution \( M \) to \((\mathcal{L}, E)\) satisfying \( \mathbb{P}^z(X \in C([0, \infty); E)) = 1 \) for \( \nu \)-a.e. and having \( \nu \) as an invariant measure, i.e. for the transition semigroup \((p_t)_{t \geq 0}\), \( \int p_t f d\nu = \int f d\nu \) for \( f \in L^2(E; \nu) \).

(ii) \( \mathcal{E}^{X, \nu} = \mathcal{E} \). Moreover, there exists (up to \( \nu \)-equivalence) exactly one continuous strong Markov process \( M \) with state space \( E \) associated with a Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) solving the martingale problem for \((L(E), D)\).

\textbf{Proof} The proof is the same as [RZZ17b, Theorem 3.12]. We only explain the idea of proof here.

For (i), suppose there is another probabilistically weak solution \( \tilde{M} = (\tilde{\Omega}, \tilde{\mathbb{P}}^z) \) to \((\mathcal{L}, E)\). By the same argument as in the proof of Theorem 5.8, \( \tilde{X} - \int_0^t e^{-(s-t)A^2/2}BdW_s \) also satisfies the shifted equation \((\mathcal{L}, E)\) by replacing \( Z \) by \( \tilde{Z} := \int_0^t e^{-(s-t)A^2/2}BdW_s \). Similarly to the proof of Corollary 5.10, the pathwise uniqueness result in Theorem 5.8 implies that \( \mathbb{P}^{\tilde{X}} \circ \tilde{X}^{-1} = \mathbb{P}^X \circ X^{-1} \) for \( \nu \)-a.e. \( z \in E \). Thus \( \tilde{M} \) is properly associated with the same quasi-regular Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) as \( M \), then the assertion follows from [MR92, Theorem 6.4].

For (ii), the second result follows from the first result and [AR94, Theorem 3.4]. For the first result, suppose \( \tilde{X} \in \mathcal{E}^{X, \nu} \) and there exists a unique Markov process \( \tilde{M} \) associated with \( \tilde{E} \). Similarly as before, the pathwise uniqueness for the shifted equation yields the uniqueness in law. Thus the semigroup of \( \tilde{M} \) are the same as \( M \)'s. Then \( \tilde{M} \) and \( M \) generate the same Dirichlet form, i.e. \( (\mathcal{E}, D(\mathcal{E})) = (\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}})) \).

\textbf{Remark 5.15} By the same argument, we can also prove the uniqueness of the probabilistically strong stationary solution to \((\mathcal{L}, E)\). For more details we refer to [RZZ17b, Section 3.5].

\section{Ergodicity}

Let \( X = Y + Z \), where \( Y \) is the solution to equation (1.1). By the uniqueness of the solution \( Y \) we have that \( X \) is a Markov process. Let \( P_t \) be the semigroup of \( X \), i.e.

\[ P_t \Phi(x) = \mathbb{E} \Phi(X(t, x)), \quad \forall \Phi \in C_b(V_0^{-1}). \]
We recall that the $U$-cylindrical Wiener process $W$ takes values in $C([0,T],V_0^{-1-s_0}(T^2,\mathbb{R}^2)), \mathbb{P}$ – a.s., for any $s_0 > 0$. Let $D$ denote the Fréchet derivative of functions on $C([0,T],V_0^{-1-s_0}(T^2,\mathbb{R}^2))(i.e.$ with respect to the noise). We also denote the Cameron-Martin space by $CM := \{ \omega : \partial_t \omega \in L^2([0,T],L_0^2(T^2;\mathbb{R}^2)), \omega(0) = (0,0) \}$. Here we view $\partial_t \omega$ as a function on $[0,T] \times T^2$ rather than lying in the tangent space of $T^2$.

**Proposition 6.1** For a fixed $x \in V_0^{-1}$, let $X_t^x := X(t,x) = Z_t + Y(t,x)$ be a map from $C([0,T],V_0^{-1-s_0})$ to $V_0^{-1}$. For any $\omega \in CM$ its directional derivative $DX_t^x(\omega)$ is given in mild form as

$$
DX_t^x(\omega) = \frac{1}{2} \int_0^t e^{-(t-s)A^2/2} A \sum_{l=0}^2 3C_2^l Y^{2-l}(s) : Z_t^l : DX_s^x(\omega)ds + \int_0^t e^{-(t-s)A^2/2} Bd\omega_s. \tag{6.1}
$$

The proof of Proposition 6.1 can be obtained by using approximation or the implicit function theorem (see [Dri03], Theorem 19.28, [Hus95], [Ivi86], [Kra94]).

Let $D$ denote the Fréchet derivative of functions on $V_0^{-1}$. We also consider the following equation:

$$
\begin{cases}
\partial_t J_{s,t}h = -\frac{1}{2}A^2 J_{s,t}h + \frac{1}{2}A \left( \sum_{l=0}^2 3C_2^l Y^{2-l}(t) : Z_t^l : J_{s,t}h \right), \\
J_{s,s}h = h \in V_0^{-1}
\end{cases} \tag{6.2}
$$

Then $J_{0,t}h = DX(t,x)(h)$, i.e. it is the derivative of $X(t, \cdot)$ in the direction $h$. For $\omega \in CM$, by Duhamel’s principle

$$
DX_t^x(\omega) = \int_0^t J_{s,t}B\partial_s\omega(s)ds. \tag{6.3}
$$

We define the stopping time

$$
\tau^r := \inf \{ t \in (0,T) : \|t^k\| : |Z_t| > r, k = 1,2,3 \}, \tag{6.4}
$$

where $\rho > 0$ is the small enough constant introduced in Lemma 6.3.

**Proposition 6.2** For any $x \in V_0^{-1}$ with $\|x\|_{H^{-1}} \leq R$, there exist constants $C_1(R), C_2(R)$ such that for all $t \leq \tau^r$

$$
\sup_{s \leq t} \|Y_s\|_{H^{-1}} + \int_0^t \|Y_s\|_{L^4}^4ds + \int_0^t \|Y_s\|^2_{H^1}ds \leq C_1 \quad \text{and} \quad \sup_{s \leq t} \|J_{0,s}h\|_{H^{-1}} \leq C_2 \|h\|_{H^{-1}}
$$

**Proof** The first bound with constant $C_1$ follows from the proof of Theorem 4.1.

For the second bound, we note that $J_{0,t}h$ satisfies the following equation:

$$
\begin{cases}
\frac{du}{dt} = -\frac{1}{2}A^2u + \frac{1}{2}A \left( \sum_{l=0}^2 3C_2^l Y^{2-l}(t) : Z_t^l : u \right), \\
u(0) = h
\end{cases}
$$

Taking scalar product with $(-A)^{-1}u$, we obtain that

$$
\frac{d}{dt} \|u\|^2_{H^{-1}} + \|u\|^2_{H^1} = -3\langle Y^2 + 2YZ + : Z^2, u^2 \rangle,
$$

that is

$$
\frac{d}{dt} \|u\|^2_{H^{-1}} + \|u\|^2_{H^1} \leq 6|\langle YZ, u^2 \rangle| + 3|\langle Z^2, u^2 \rangle|.
$$

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Following the same argument that we used to get estimate (6.13) and using the first bound, we use Grönwall’s inequality to obtain the second bound.

Let \( \chi_r \in C^\infty(\mathbb{R}) \) such that \( \chi_r(\zeta) \in [0, 1] \) for all \( \zeta \in \mathbb{R} \), and

\[
\chi_r(\zeta) = \begin{cases} 
1, & |\zeta| \leq \frac{r}{2} \\
0, & |\zeta| \geq r
\end{cases}.
\]

Following the notation in [11], we set

\[
C^{3,-\alpha}(0, T) := C([0, T]; C^{-\alpha}) \times C((0, T]; C^{-\alpha})^2,
\]
and \( Z := \left( Z_1, Z^2, \ldots, Z^3 \right) \in C^{3,-\alpha}(0, T) \). We also define

\[
\|Z\|_t^\alpha := \max_{k=1,2,3} \left\{ \sup_{0 \leq s \leq T} s^\alpha \| Z_s^k : \| - \alpha \right\}.
\]

**Theorem 6.3 (Bismut-Elworthy-Li Formula)** Let \( x \in V_0^{-1} \), \( \Phi \in C_b^1(\mathbb{R}_0^+) \) and \( \omega \) be a process taking values in the Cameron-Martin space \( CM \) with \( \partial^\omega \) adapted. Assume that there exists a deterministic constant \( C \equiv C(t) \) such that \( \|\partial^\omega\|_{L^2(0,t;U)} \leq C \) \( \mathbb{P} \) a.s. Then we have

\[
\mathbb{E}[D\Phi(X^x_t)](D\Phi(\omega))\chi_r(\|Z\|_t^\alpha) = \mathbb{E} \left( \Phi(X^x_t)\chi_r(\|Z\|_t^\alpha) \int_0^t \partial^\omega(s) \cdot dW_s \right),
\]

where

\[
\partial^+\chi_r(\|Z\|_t^\alpha)(\omega) = \partial^\omega(\|Z\|_t^\alpha)\partial^+\|Z\|_t^\alpha(Y)
\]

\[
\partial^+\|Z\|_t^\alpha(Y) = \lim_{\delta \to 0^+} \frac{\|Z + \delta Y\|_t^\alpha - \|Z\|_t^\alpha}{\delta},
\]

\( Y = (Q_\omega(\cdot), 2ZQ_\omega(\cdot), 3: Z^2 : Q_\omega(\cdot)) \in C^{3,-\alpha}(0, t) \) and

\[
Q_\omega(\cdot) := \int_0^1 e^{(-s)A^2/2} B\partial^\omega(s)ds.
\]

**Proof** This is proved by the same calculation as that in the proof of [11], Theorem 5.4.

We use (6.6) to prove the following proposition.

**Proposition 6.4** There exists universal constants \( \theta_1 > 0 \) such that for every \( T > 0 \), \( x \in V_0^{-1} \) with \( \|x\|_{H^{-1}} \leq R \), there exists a constant \( C \equiv C(T, R) \) such that

\[
|\mathbb{P}_t \Phi(x) - \mathbb{P}_t \Phi(y)| \leq C(T, R) \frac{1}{\theta_1} \|\Phi\|_{\infty} \|x - y\|_{H^{-1}} + C \|\Phi\|_{\infty} \mathbb{P}(t \geq \tau^x_{\frac{1}{2}})
\]

for every \( y \in V_0^{-1} \), \( \|x - y\|_{H^{-1}} \leq 1 \), \( \Phi \in C_b^1(\mathbb{R}_0^+) \) and \( t \in [0, T] \).

**Proof** Let \( \Phi \in C_b^1(\mathbb{R}_0^+) \). Then

\[
|\mathbb{P}_t \Phi(x) - \mathbb{P}_t \Phi(y)| = |\mathbb{E}[\Phi(X(t,x)) - \Phi(X(t,y))]| \leq I_1 + I_2,
\]

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where
\[ I_1 := |\mathbb{E} [\Phi (X(t, x)) - \Phi (X(t, y)) \chi_t (\|Z\|_t)]| \]
\[ I_2 := |\mathbb{E} [\Phi (X(t, x)) - \Phi (X(t, y)) (1 - \chi_t (\|Z\|_t))]|. \]

For the second term we have that \( I_2 \leq 2\|\Phi\|_\infty \mathbb{P}(t \geq \tau^z) \). By the mean value theorem we get that
\[ I_1 = \left| \mathbb{E} \left( \int_0^1 D \Phi (x_{i}^z (y - x)) d\lambda \cdot \chi_t (\|Z\|_t) \right) \right|
\[ = \left| \int_0^1 \mathbb{E} \left( D \Phi (x_{i}^z (y - x)) \right) (y - x) \chi_t (\|Z\|_t) d\lambda \right|, \]
where \( z_{\lambda} := x + \lambda (y - x) \). For any \( h \in V_0^{-1} \), let \( \omega \) be such that \( B \partial_\omega (s) = J_{0,s} h \) for \( s \leq \tau^z \) and 0 otherwise. Then \( \partial_\omega (s) \) satisfies the condition in Theorem \( \text{4.5} \). Furthermore, by \( \text{4.5} \) and since \( J_{0,s} J_{s,t} = J_{0,t} \), we have \( DX_i^z (\omega) = tDX_i^z (h) \). Then we use \( \text{4.0} \) to obtain that
\[ \mathbb{E} (D \Phi (x_{i}^z)) (h) \chi_t (\|Z\|_t) = \frac{1}{t} \mathbb{E} \left( \Phi (x_{i}^z) \int_0^t \partial_\omega (s) \cdot dW_s \chi_t (\|Z\|_t) \right)
\[ - \frac{1}{t} \mathbb{E} \left( \Phi (x_{i}^z) \partial_{+} \chi_t (\|Z\|_t) (\omega) \right). \]

Then we have
\[ I_1 \leq \frac{1}{t} \|\Phi\|_\infty \int_0^1 \mathbb{E} \left| \int_0^t \partial_{s} \omega (s) dW_s \chi_t (\|Z\|_t) \right| d\lambda + \frac{1}{t} \|\Phi\|_\infty \int_0^1 \mathbb{E} \left| \partial_{+} \chi_t (\|Z\|_t) (\omega) \right| d\lambda. \]

For the first term we have
\[ \mathbb{E} \left| \int_0^t \partial_{s} \omega (s) dW_s \chi_t (\|Z\|_t) \right| \leq \mathbb{E} \left| \int_0^{t \wedge \tau^z} \partial_{s} \omega (s) \cdot dW_s \right|
\[ \leq \left( \int_0^{t \wedge \tau^z} \|\partial_{s} \omega (s)\|^2 d\lambda \right)^{1/2}
\[ \leq \left( \int_0^{t \wedge \tau^z} \|J_{0,s} h\|_{H^{-1}}^2 d\lambda \right)^{1/2}
\[ \leq C_2 t \|h\|_{H^{-1}}, \]
where we used the Cauchy-Schwarz inequality and Itô’s isometry in the second step and Proposition \( \text{4.2} \) in the last step.

By the definition of \( \partial_{+} \chi_t (\|Z\|_t) (\omega) \), we have
\[ \left| \partial_{+} \chi_t (\|Z\|_t) (\omega) \right| \leq \partial_{+} \|Z\|_t (Y) \leq \|Y\|_t \leq \|Z\|_t \|Q_\omega (t)\|_{\beta}, \]
where \( Y \) is as introduced in Theorem \( \text{4.5} \) and we used Lemma \( \text{4.5} \) in the last inequality. Moreover, we use Lemma \( \text{4.2} \) and Lemma \( \text{4.3} \) to obtain
\[ \|Q_\omega (t)\|_{\beta} \leq \int_0^t (t - s)^{-\frac{\beta + 2}{4}} \|J_{0,s} h\|_{-2} d\lambda \leq \int_0^t (t - s)^{-\frac{\beta + 2}{4}} \|J_{0,s} h\|_{H^{-1}} d\lambda \leq C_2 t^{\frac{2 - \beta}{4}} \|h\|_{H^{-1}}. \]
Choosing \( \beta \) small enough, we deduce that there exists a constant \( \theta_1 \in (0, \frac{1}{2}) \), such that
\[ I_1 \lesssim C_2 t^{\frac{1}{\theta_1}} \|\Phi\|_\infty \|h\|_{H^{-1}}. \]
Letting \( h = y - x \) we finish the proof. \( \square \)

We denote by \( \| \mu_1 - \mu_2 \|_{TV} \) the total variation distance of two probability measures \( \mu_1, \mu_2 \) on \( V_{0}^{-1} \) given by

\[
\| \mu_1 - \mu_2 \|_{TV} := \sup_{\| \Phi \|_\infty \leq 1} \left| \int \Phi d\mu_1 - \int \Phi d\mu_2 \right|.
\]

**Theorem 6.5** There exists \( \theta \in (0, 1) \) such that for any \( x, y \in V_{0}^{-1} \) with \( \| x \|_{H^{-1}} \leq R \) and \( \| x - y \|_{H^{-1}} \leq 1 \) there exists a constant \( C(R) > 0 \) satisfying

\[
\| P_t(x, \cdot) - P_t(y, \cdot) \|_{TV} \leq C(R) \| x - y \|_{H^{-1}}^\theta,
\]

for every \( t \geq 1 \).

**Proof** The proof is the same as that of [TW18, Theorem 5.8]. Therefore we omit it. \( \square \)

In order to use Krylov-Bogoliubov method to prove the existence of an invariant measure, the \( H^{-1} \) uniform estimate is not enough. We need to find a space compactly embedded in \( H^{-1} \) where the solution is bounded in probability. We make use of the integrability on a smaller space, which is compactly embeded in \( H^{-1} \). Thus we have:

**Theorem 6.6** For every \( x \in V_{0}^{-1} \), there exists a probability Borel measure \( \nu_x \) on \( V_{0}^{-1} \) such that \( \nu_x \) is an invariant measure for the semigroup \( \{ P_t, t \geq 0 \} \) on \( V_{0}^{-1} \).

**Proof** By (4.5) and a similar argument as in the proof of [TW18, Corollary 3.10] we have that

\[
\mathbb{E} \int_0^{t+1} \| Y_{t,r} \|_{H^1}^2 dr \lesssim 1 + \mathbb{E} \| Y_{t,t} \|_{H^{-1}}^2 = 1 + \mathbb{E} \| X(t, x) \|_{H^{-1}}^2.
\]

Combining this with (6.9) we deduce that for \( \alpha \in (0, 1) \),

\[
\mathbb{E} \int_0^{t+1} \| X(s, x) \|_{C_{-\alpha}}^2 ds \lesssim \mathbb{E} \int_0^{t+1} \| Y_{t,s} \|_{H^1}^2 ds + \mathbb{E} \int_0^{t+1} \| Z_{t,s} \|_{C_{-\alpha}}^2 ds \lesssim 1 + \frac{1}{1 \wedge t},
\]

where we used a similar argument as in the proof of [TW18, Theorem 2.1] in the last inequality. Then we obtain that for \( t \geq 1 \)

\[
\mathbb{E} \int_1^{t} \| X(s, x) \|_{C_{-\alpha}}^2 ds \lesssim t.
\]
Moreover, by (14.4) we have
\[ E \int_0^1 \| Y_s \|_{H^1}^2 ds \leq 1 + \| x \|_{H^{-1}}^2. \]
Thus for \( t \geq 1 \)
\[ \int_0^t E \| X(s, x) \|_{C^\omega}^2 ds \leq \int_0^1 E \| X(s, x) \|_{C^\omega}^2 ds + \int_1^t E \| X(s, x) \|_{C^\omega}^2 ds \leq 1 + \| x \|_{H^{-1}}^2 + t. \]
By Chebyshev’s inequality, for any \( K > 0 \)
\[ P(\| X(t, x) \|_{C^\omega} > K) \leq \frac{1}{K^2} E \| X(t, x) \|_{C^\omega}. \]
Thus there exists a constant \( C > 0 \), such that
\[ \int_0^t P(\| X(s, x) \|_{C^\omega} > K) \leq \frac{C}{K^2} \int_0^t E \| X(s, x) \|_{C^\omega}^2 ds \leq \frac{C}{K^2} (1 + \| x \|_{H^{-1}}^2 + t). \]
Letting \( R_t(x, \cdot) = \frac{1}{t} \int_0^t P_s(x, \cdot) ds \), for \( K^2 = \frac{C}{\varepsilon} \) we get
\[ R_t(f \in C^{-\alpha} \cap V_0^1 : \| f \|_{C^{-\alpha}} > K \varepsilon) \leq R_t(f \in V_0^1 : \| f \|_{C^{-\alpha}} > K \varepsilon) \leq (1 + \frac{1 + \| x \|_{H^{-1}}}{t}) \varepsilon. \]
By [11U06, Proposition 4.6] we know that \( \{ f \in C^{-\alpha} \cap V_0^1 : \| f \|_{C^{-\alpha}} > K \varepsilon \} \) is a compact subset of \( V_0^{-1} \) since the embedding \( C^{-\alpha} \subset V^{-1} \) is compact. This implies the tightness of \( \{ R_t \}_{t \geq 0} \) in \( V_0^{-1} \). By the Krylov-Bogoliubov existence theorem (see [DPZ98, Corollary 3.1.2]), there exists a sequence \( t_k \not\to \infty \) and a measure \( \nu_x \) such that \( R_{t_k} \to \nu_x \) weakly in \( V_0^{-1} \) and \( \nu_x \) is an invariant measure for the semigroup \( \{ P_t \}_{t \geq 0} \).

To prove the exponential mixing property, we make use of the irreducibility of \( Z \) and a uniform estimate, which is slightly different from that in the proof of [TW18, Theorem 6.3].

**Theorem 6.7** There exists a constant \( \lambda \in (0, 1) \) and \( T_0 > 0 \) such that
\[ \| P_t(x) - P_t(y) \|_{TV} \leq 1 - \lambda, \]
for every \( x, y \in V_0^{-1}, \ t \geq T_0 + 1. \)

**Proof** From (13.3) we know that for any fixed \( r > 0 \), there exist \( T_0, M > 0 \) which are independent of \( \omega, x \), such that for any initial value \( x \in V_0^{-1} \), we have that \( \{ \omega : \| Z \|_{T_0} \leq M \} \subset \{ \| Y(T_0) \|_{V_0^{-1}} \leq \frac{\varepsilon}{2} \} \cap \{ \| Z(T_0) \|_{V_0^{-1}} \leq \frac{\varepsilon}{2} \}. \)

By Theorem 6.3 for every \( a \in (0, 1) \) there exists \( r \equiv r(a) > 0 \) such that for every \( x, y \in B_r(0) \) and \( t \geq 1 \)
\[ \| P_t(x) - P_t(y) \|_{TV} \leq 1 - a, \quad (6.11) \]
where \( B_r(a) := \{ x \in V_0^{-1} : \| x - u \|_{V_0^{-1}} < r \} \). Then by (13.3) for any initial value \( x \in V_0^{-1} \), there exists \( b \equiv b(r) \in (0, 1) \) such that
\[ P(\| X(T_0) \|_{V_0^{-1}} \leq r) \geq P \left( \{ \| Y(T_0) \|_{V_0^{-1}} \leq \frac{r}{2} \} \cap \{ \| Z(T_0) \|_{V_0^{-1}} \leq \frac{r}{2} \} \right) \geq P(\| Z \|_{T_0} \leq M) \geq b, \quad (6.12) \]
where in the last step we used the irreducibility of the law of \( Z \). Here we omit the proof of the irreducibility of \( Z \), since it is the same as that of [TW18, Theorem 6.3]. Moreover, by (6.12) for any \( R > 0 \)
\[
\inf_{x \in V_0^{-1}} P_{T_0}(x, B_r(0)) \geq b.
\]
Then combining (6.11)-(6.13) and the Markov property by the same argument as in the proof of [TW18, Theorem 6.5], we can complete the proof. \( \square \)

The following corollary gives the exponential convergence to a unique invariant measure.

**Corollary 6.8** There exists a unique invariant measure \( \bar{\nu} \) for the semigroup \( \{P_t\}_{t \geq 0} \) such that
\[
\|P_t - \bar{\nu}\|_{TV} \leq (1 - \lambda)^{\frac{t}{2}}\|\delta_x - \bar{\nu}\|_{TV},
\]
for every \( x \in V_0^{-1}, t \geq T_0 + 1 \). Moreover, \( \bar{\nu} = \nu \).

**Proof** For the first result, see the proof of [TW18, Corollary 6.6]. By Corollary 5.10, \( \nu \) is an invariant measure of \( X \). Hence \( \bar{\nu} = \nu \). \( \square \)

**Remark 6.9** In the following we give a simple and short proof for exponential convergence by the theory of Dirichlet forms.

Similarly to [DPD13], by comparing the two Dirichlet forms for the Cahn-Hilliard equation and the dynamical \( \Phi^4_2 \) model, we obtain the spectral gap of equation (1.1). Indeed, since \( \nu \) is the restriction of \( \tilde{\nu} \) on \( E \), by the same arguments in [RZZ17b] and [TW18] we know that \( \tilde{\nu} \) is the invariant measure for the solution to the dynamical \( \Phi^4_2 \) model. We denote the Dirichlet form associated with the dynamical \( \Phi^4_2 \) model by \( (\mathcal{E}, D(\mathcal{E})) \), i.e.
\[
\mathcal{E}(f, g) = \frac{1}{2} \int_E \langle Df, Dg \rangle_{L^2 \nu} f, g \in D(\mathcal{E}),
\]
where \( \mathcal{E} = H^{-1-s_0} \), \( D \) denotes the gradient operator in \( L^2(T^2) \) (see [RZZ17b]). In [TW18] the exponential convergence for the dynamical \( \Phi^4_2 \) model in total variation is proved. This implies the exponential convergence in \( L^2(\mathcal{E}, \tilde{\nu}) \)-norm. By [Wan15, Theory 1.1, Example 1.1.2] this is equivalent to the Poincaré inequality
\[
\int |f|^2 d\tilde{\nu} - \left( \int fd\tilde{\nu} \right)^2 \leq C \mathcal{E}(f, f), f \in D(\mathcal{E}).
\]
From the proof of Theorem 6.4 we know that for any \( f \in D(\mathcal{E}) \subset D(\bar{\mathcal{E}}) \),
\[
\mathcal{E}(f, f) = \frac{1}{2} \sum_{k \neq (0,0)} \int |\frac{\partial f}{\partial h_k}|^2 d\nu = \frac{1}{2} \sum_{k \neq (0,0)} \lambda_k \int |\frac{\partial f}{\partial e_k}|^2 d\nu \geq \frac{1}{2} \sum_{k \neq (0,0)} \int |\frac{\partial f}{\partial e_k}|^2 d\nu = \mathcal{E}(f, f),
\]
where \( h_k = \sqrt{\sum_k e_k}, \{h_k\}_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \) is an orthonormal basis of \( V_0^{-1} \). Then by [Wan15, Theory 1.1, Example 1.1.2] we have
\[
\|P_t f - \int fd\nu\|_{L^2(\mathcal{E}, \nu)} \leq e^{-\frac{t}{2}} \|f - \int fd\nu\|_{L^2(\mathcal{E}, \nu)}.
\]
A Symmetric quasi regular Dirichlet forms and Markov Processes

In this section we recall some general Dirichlet form results from [FOT94 MR92].

Let $E$ be a Hausdorff topological space, $m$ a $\sigma$-finite measure on $E$, and let $\mathcal{B}$ the smallest $\sigma$-algebra of subsets of $E$ with respect to which all continuous functions on $E$ are measurable. Let $\mathcal{E}$ be a symmetric Dirichlet form acting in the real $L^2(m)$-space, i.e. $\mathcal{E}$ is a positive, symmetric, bilinear, closed form with domain $D(\mathcal{E})$ dense in $L^2(m)$, and such that $\mathcal{E}(\Phi(u), \Phi(u)) \leq \mathcal{E}(u, u)$, for any $u \in D(\mathcal{E})$, where $\Phi(t) = (0 \vee t) \wedge 1, t \in \mathbb{R}$. The latter condition is known to be equivalent with the condition that the associated $C_0$-contraction semigroup $T_t, t \geq 0$, is submarkovian (i.e. $0 \leq u \leq 1$ a.e. implies $0 \leq T_t u \leq 1$ a.e., for all $u \in L^2(m)$); association means that $\lim_{t \downarrow 0} \frac{1}{t} \langle u - T_t u, v \rangle_{L^2(m)} = \mathcal{E}(u, v), \forall u, v \in D(\mathcal{E})$.

Definition A.1 ([MR92, Chapter III, Definition 2.1, Definition 3.2])

(i) An increasing sequence $(F_k)_{k \in \mathbb{N}}$ of closed subsets of $E$ is called an $\mathcal{E}$-nest if $\bigcup_k \{ f \in D(\mathcal{E}) : \text{supp } f \subset F_k \}$ is dense in $D(\mathcal{E})$ w.r.t. $\mathcal{E}_1^2$). Here $\mathcal{E}_1^2$ is a norm on $E$ defined by $\mathcal{E}_1(u) := \mathcal{E}(u, u) + \|u\|_{L^2(\mathcal{E}, E)}^2, \forall u \in E$.

(ii) A subset $N \subset E$ is called $\mathcal{E}$-exceptional if $N \subset \bigcap_k F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$. We say that a property of points in $E$ holds $\mathcal{E}$-quasi everywhere (abbreviated $\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set.

(iii) An $\mathcal{E}$-q.e. defined function $f$ on $E$ is called $\mathcal{E}$-quasi-continuous if there exists an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$ such that $f |_{F_k}$ is continuous for every $k \in \mathbb{N}$.

Definition A.2 (cf. [MR92, Chap. IV, Defi. 3.1]) A symmetric Dirichlet form is called quasi-regular if the following holds:

(i) There exists an $\mathcal{E}$-nest of $E$ such that $F_k$ is compact in $E$ for any $k$.

(ii) There exists an $\mathcal{E}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi continuous $m$-versions. A real function $u$ on $E$ is called quasi continuous when there exists an $\mathcal{E}$-nest $(F_k)$ s.t. $u$ restricted to $F_k$ is continuous.

(iii) There exists $u_n \in D(\mathcal{E}), n \in \mathbb{N}$, with $\mathcal{E}$-quasi continuous $m$-versions $\tilde{u}_n$ and there exists an $\mathcal{E}$-exceptional subset $N$ of $E$ s.t. $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ separates the points of $E \setminus N$. An $\mathcal{E}$-exceptional subset of $E$ is a subset $N \subset \bigcap_k (E \setminus F_k)$ for some $\mathcal{E}$-nest $(F_k)$.

To recall the main results in [MR92] we recall the definitions of a Markov process and a right process. Here we consider only Markov processes with life time $\infty$.

Definition A.3 (cf. [MR92, Chap. IV Defi. 1.5]) A collection $\mathcal{M} := (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (\mathbb{P}^z)_{z \in E})$ is called a Markov process (with state space $E$) if it has the following properties.

(i) There exists a filtration $(\mathcal{M}_t)$ on $(\Omega, \mathcal{M})$ such that $(X_t)_{t \geq 0}$ is an $(\mathcal{M}_t)_{t \geq 0}$ adapted stochastic process with state space $E$.

(ii) For each $t \geq 0$ there exists a shift operator $\theta_t : \Omega \rightarrow \Omega$ such that $X_s \circ \theta_t = X_{s+t}$ for all $s, t \geq 0$

(iii) $\mathbb{P}^z, z \in E$, are probability measures on $(\Omega, \mathcal{M})$ such that $z \mapsto \mathbb{P}^z(A)$ is $\mathcal{B}(E)^*\text{-measurable}$ for each $A \in \mathcal{M}$ resp. $\mathcal{B}(E)$-measurable if $A \in \sigma\{X_s | s \in [0, \infty)\}$, where $\mathcal{B}(E)^* := \cap_{F \in \mathcal{F}(E)} \mathcal{B}^F(E)$ for $\mathcal{F}(E)$ denoting the family of all probability measures on $(E, \mathcal{B}(E))$ and $\mathcal{B}^F(E)$ denotes the completion of the $\sigma$-algebra $\mathcal{B}(E)$ w.r.t. a probability $\mathbb{P}$.

(iv) (Markov property) For all $A \in \mathcal{B}(E)$ and any $t, s \geq 0$

\[ \mathbb{P}^z[X_{s+t} \in A | \mathcal{M}_s] = \mathbb{P}^{X_t}[X_t \in A] \quad \mathbb{P}^z - \text{a.s., } z \in E. \]
Definition A.4 (cf. [MR92, Chap. IV Def. 1.8]) Let $M := (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (\mathbb{P}_z)_{z \in E})$ be a Markov process with state space $E$ and corresponding filtration $(\mathcal{M}_t)$. $M$ is called a right process if it has the following additional properties.

(i) (Normal property) $\mathbb{P}_z(X_0 = z) = 1$ for all $z \in E$.
(ii) (Right continuity) For each $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$.
(iii) (Strong Markov property) $(\mathcal{M}_t)$ is right continuous and for every $(\mathcal{M}_t)$-stopping time $\sigma$ and every $\nu \in \mathcal{P}(E)$

$$\mathbb{P}^\nu[X_{\sigma+t} \in A | \mathcal{M}_\sigma] = \mathbb{P}^{X_\sigma}[X_t \in A] \quad \text{P}^\nu - \text{a.s.}$$

for all $A \in \mathcal{B}(E)$, $t \geq 0$.

Theorem A.5 ([MR92, Chap. IV Thm 6.7]) Let $E$ be a metrizable Lusin space. Then a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, m)$ is quasi-regular if and only if there exists a right process $M$ associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e. the semigroup of $M$ is an $m$-version of the semigroup associated with $(\mathcal{E}, D(\mathcal{E}))$. In this case $M$ is always properly associated with $(\mathcal{E}, D(\mathcal{E}))$.

Remark A.6 The results in [MR92, Chap. IV] are more general and can be applied for general Hausdorff topological spaces and more general Markov processes. Lusin spaces are enough for our use in this thesis.

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References

[AR91] Sergio Albeverio and Michael Röckner. Stochastic Differential-Equations in Infinite Dimensions - Solutions via Dirichlet Forms. Probability Theory and Related Fields, 89(3):347–386, 1991.

[AR94] Sergio Albeverio and Michael Röckner. Dirichlet form methods for uniqueness of martingale problems and applications, volume 57 of Proceedings of Symposia in Pure Mathematics. American Mathematical Society, Providence, Rhode Island, 1994.

[BB19] Roland Bauerschmidt and Thierry Bodineau. Log-Sobolev inequality for the continuum Sine-Gordon model. arXiv.org, July 2019.

[BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. Fourier Analysis and Nonlinear Partial Differential Equations. Springer Science & Business Media, January 2011.
Vladimir Bogachev. *Gaussian Measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, September 1998.

L Bertini, E PRESUTTI, B Rudiger, and E Saada. Dynamical fluctuations at the critical point: convergence to a nonlinear stochastic PDE. *Rossi skaya Akademiya Nauk. Teoriya Veroyatnosti i ee Primeneniya*, 38(4):689–741, 1993.

John W Cahn and John E Hilliard. Free Energy of a Nonuniform System. I. Interfacial Free Energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958.

Ingrid Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

Arnaud Debussche and Giuseppe Da Prato. Strong solutions to the stochastic quantization equations. *The Annals of Probability*, 31(4):1900–1916, 2003.

Arnaud Debussche and Ludovic Goudenège. Stochastic Cahn–Hilliard Equation with Double Singular Nonlinearities and Two Reflections. *Siam Journal on Mathematical Analysis*, 43(3):1473–1494, 2011.

Giuseppe Da Prato. *Kolmogorov equations for stochastic PDEs*. Advanced Courses in Mathematics. Birkhäuser Verlag, Basel, Basel, 2004.

Giuseppe Da Prato and Arnaud Debussche. Stochastic Cahn–Hilliard equation. *Nonlinear Analysis: Theory, Methods & Applications*, 26(2):241–263, 1996.

Giuseppe Da Prato, Arnaud Debussche, and Luciano Tubaro. Irregular semi-convex gradient systems perturbed by noise and application to the stochastic Cahn–Hilliard equation. *Annales de l’Institut Henri Poincare (B) Probability and Statistics*, 40(1):73–88, February 2004.

Giuseppe Da Prato and J Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, Cambridge, 1996.

Giuseppe Da Prato and Jerzy Zabczyk. *Second order partial differential equations in Hilbert spaces*, volume 293 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, Cambridge, 2002.

B K Driver. *Analysis tools with applications*. Springer, 2003.

Arnaud Debussche and Lorenzo Zambotti. Conservative stochastic Cahn–Hilliard equation with reflection. *The Annals of Probability*, 35(5):1706–1739, 2007.

Neven Elezovic and Andro Mikelic. On the Stochastic Cahn–Hilliard Equation. *Nonlinear Analysis: Theory, Methods & Applications*, 16(12):1169–1200, 1991.

Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda. Dirichlet Forms and Symmetric Markov Processes. DE GRUYTER, Berlin, New York, 1994.

Tadahisa Funaki. Derivation of the Hydrodynamical Equation for One-Dimensional Ginzburg-Landau Model. *Probability Theory and Related Fields*, 82(1):39–93, 1989.
Tadahisa Funaki. *Lectures on random interfaces*. SpringerBriefs in Probability and Mathematical Statistics. Springer, Singapore, Singapore, 2016.

Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum of Mathematics. Pi*, 3:e6–75, 2015.

James Glimm and Arthur Jaffe. *Quantum Physics*, volume 35 of *A Functional Integral Point of View*. Springer Science & Business Media, December 2012.

Giambattista Giacomin, Joel L Lebowitz, and Errico Presutti. Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems. In *Stochastic Partial Differential Equations: Six Perspectives*, pages 107–152. Amer. Math. Soc., Providence, RI, 1999.

Benjamin Goldys, Michael Röckner, and Xicheng Zhang. Martingale solutions and Markov selections for stochastic partial differential equations. *Stochastic Processes and their Applications*, 119(5):1725–1764, 2009.

Martin Hairer. A theory of regularity structures. *Inventiones Mathematicae*, 198(2):269–504, 2014.

P C Hohenberg and B I Halperin. Theory of Dynamic Critical Phenomena. *Reviews of Modern Physics*, 49(3):435–479, 1977.

Takeyuki Hida. Brownian Motion. Springer US, New York, NY, 1980.

Giovanni Jona-Lasinio and Pronob Mitter. On the stochastic quantization of field theory. *Communications in Mathematical Physics*, 101(3):409–436, 1985.

Wei Liu and Michael Röckner. *Stochastic Partial Differential Equations: An Introduction*. Springer, October 2015.

Sheng Lin Lu and Horng-Tzer Yau. Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Communications in Mathematical Physics*, 156(2):399–433, 1993.

Yves Meyer. *Wavelets and Operators*. Cambridge University Press, January 1995.

Zhi-Ming Ma and Michael Röckner. Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Springer Berlin Heidelberg, Berlin, Heidelberg, 1992.

Remigijus Mikulevicius and Boris L. Rozovskii. Martingale problems for stochastic PDE’s. In *Stochastic Partial Differential Equations: Six Perspectives*, pages 243–325. American Mathematical Society, Providence, Rhode Island, November 1999.

Jean-Christophe Mourrat and Hendrik Weber. Convergence of the Two-Dimensional Dynamic Ising-Kac Model to $\Phi^4_2$. *Communications on Pure and Applied Mathematics*, 70(4):717–812, April 2017.
Jean-Christophe Mourrat and Hendrik Weber. Global well-posedness of the dynamic $\Phi^4$ model in the plane. *The Annals of Probability*, 45(4):2398–2476, July 2017.

Edward Nelson. The free Markoff field. *Journal of Functional Analysis*, 12(2):211–227, February 1973.

David Nualart. *The Malliavin Calculus and Related Topics*. Probability and its Applications. Springer Science & Business Media, New York, NY, December 2013.

Nobuaki Obata. *White noise calculus and Fock space*, volume 1577 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Berlin, Heidelberg, 1994.

Rolfe G Petschek and Horia Metiu. A computer simulation of the time-dependent Ginzburg–Landau model for spinodal decomposition. *The Journal of Chemical Physics*, 79(7):3443–3456, October 1983.

Georgio Parisi and Yong Shi Wu. Perturbation Theory Without Gauge Fixing. *Scientia Sinica*, 24(4):483–496, 1981.

Michael Röckner, Rongchan Zhu, and Xiangchan Zhu. Sub and supercritical stochastic quasi-geostrophic equation. *The Annals of Probability*, 43(3):1202–1273, 2015.

Michael Röckner, Rongchan Zhu, and Xiangchan Zhu. Ergodicity for the Stochastic Quantization Problems on the 2D-Torus. *Communications in Mathematical Physics*, 352(3):1061–1090, 2017.

Michael Röckner, Rongchan Zhu, and Xiangchan Zhu. Restricted Markov uniqueness for the stochastic quantization of $P(\Phi)^2$ and its applications. *Journal of Functional Analysis*, 272(10):4263–4303, 2017.

Luca Scarpa. On the stochastic Cahn-Hilliard equation with a singular double-well potential. 2017.

Winfried Sickel. Periodic Spaces and Relations to Strong Summability of Multiple Fourier Series. *Mathematische Nachrichten*, 124(1):15–44, 1985.

Barry Simon. *The $P(\phi)^2$ Euclidean (quantum) Theory*. Princeton Series in Physics. NJ: Princeton University Press, Princeton, N.J., 1974.

Elias M Stein and Guido Weiss. Introduction to Fourier Analysis on Euclidean Spaces (PMS-32). Princeton University Press, Princeton, 1972.

Roger Temam. *Navier-Stokes Equations*. Theory and Numerical Analysis. American Mathematical Soc., April 2001.

Hans Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North Holland, 1978.

Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, Basel, 1992.

Hans Triebel. *Theory of function spaces. III*, volume 100 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006.
[TW18] Pavlos Tsatsoulis and Hendrik Weber. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *Annales de l’Institut Henri Poincaré Probabilités et Statistiques*, 54(3):1204–1249, August 2018.

[Wan06] Fengyu Wang. *Functional Inequalities Markov Semigroups and Spectral Theory*. Elsevier, April 2006.

[ZZ18] Rongchan Zhu and Xiangchan Zhu. Piecewise linear approximation for the dynamical $\Phi^4_3$ model. *Science China Mathematics*, 63(2):381–410, December 2018.