The odd harmonious labelling of $n$-hair-$kC_4$-snake graph

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Abstract. Let $G(p,q)$ be graph that consists of $p = |V|$ vertices and $q = |E|$ edges, where $V$ is the set of vertices and $E$ is the set of edges of $G$. A graph $G(p,q)$ is odd harmonious if there exist an injective function $f: V \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ that induced a bijective function $f^*: E \rightarrow \{1, 3, 5, \ldots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$. The function $f$ is called harmonious labelling of graph $G(p,q)$. A hair-$kC_4$ snake graph is a graph obtain by attaching $n$ leaves to vertices of degree two in $kC_4$-snake graph. In this paper we prove that $n$-hair-$kC_4$-snake graph is odd harmonious.

Keywords: Odd harmonious labelling, snake graph, cycle graph

1. Introduction

A graph $G(V,E)$ is an ordered pair of sets $V$ and $E$ such that the elements of $E$ are unordered pairs of distinct elements of $V$. An element of $V$ and $E$ is called vertex and edge, respectively. The order of $G$ is $|V|$ and the size of $G$ is $|E|$. A graph $G(p,q)$ is graph with order $p$ and size $q$. A labelling of $G$ is an assignment of integers to element $G$ (vertices, edges, or both).

There are some types of graph labelling. One of them is odd harmonious labelling. An odd harmonious labelling of graph $G(p,q)$ is a function $f: V \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ that induces a bijective function $f^*: E \rightarrow \{1, 3, 5, \ldots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$, for each $uv \in E$ and $u, v \in V$. A graph $G(p,q)$ that admits odd harmonious labelling is called odd harmonious graph.

There are many results on the existence of odd harmonious labelling on various classes of graphs, and still many more to discover. Liang and Bai [1] proved that a cycle $C_n$ is odd harmonious if and only if $n \equiv 0 \pmod{4}$; a complete graph $K_n$ is odd harmonious if and only if $n = 2$; a complete $k$-partite graph $K_{n1,n2,\ldots,n_k}$ is odd harmonious if and only if $k = 2$; a windmill graph is odd harmonious if and only if $n = 2$; caterpillars and lobsters are odd harmonious. Alyani et al. [2] proved that $kC_n$-snake graphs for specific values of $n$, that is, for $n = 4$ and $n = 8$ are odd harmonious. Jeyanthi and Philo [3] proved that $C(m \cdot n)$ is odd harmonious if and only if either both $m, n \equiv 0 \pmod{4}$ or both $m, n \equiv 2 \pmod{4}$. Saputri et al. [4] proved that dumbbell and generalized prism graphs are odd harmonious.

In this paper, we consider a $kC_4$-snake graphs with hair that we call $n$-hair-$kC_4$-snake graph. An $n$-hair-$kC_4$-snake is a graph obtained by attaching $n$ leaf to all vertices of degree two in $kC_4$-snake graph. We show that graph $n$-hair-$kC_4$-snake is odd harmonious.
2. Methodology

To show that $\text{nhair-}k\text{C}_4\text{-snake}$ is a harmonious graph, we do three steps. First, we do exploration to find pattern of the vertex label that make all edge labels are odd and in the range of $1$ to $2q - 1$, where $q$ is the size of $\text{nhair-}k\text{C}_4\text{-snake}$ graph. Second, we define the labelling function $f$ and show that $f: V(\text{nhair-}k\text{C}_4\text{-snake}) \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$. Third, we show that the induced function $f^*: E(\text{nhair-}k\text{C}_4\text{-snake}) \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$.

3. Results

Gallian in [5] gives the definition of triangle snake. We define $\text{C}_4\text{-snake}$ using the same way as Gallian.

**Definition 1.** A $\text{C}_4\text{-snake}$ is a graph obtained from a path $u_1, u_2, \ldots, u_k$ by joining $u_i$ and $u_{i+1}$ to new vertices $v_i$ and $w_i$ for $i = 1, 2, \ldots, k - 1$, where $k$ is positive integers.

**Definition 2.** An $\text{nhair-}k\text{C}_4\text{-snake}$ is a graph obtained by attaching $n$ leaves to all vertices of degree two in $\text{C}_4\text{-snake}$ graph, where $k$ and $n$ are positive integers.

The sets of vertices and edges of $\text{nhair-}k\text{C}_4\text{-snake}$ are $V = \{u_j | 0 \leq j \leq k\} \cup \{v_j | 1 \leq j \leq k\} \cup \{w_j | 1 \leq j \leq k\} \cup \{v_{j+1} | 1 \leq j \leq k\} \cup \{v_{j+1} | 1 \leq j \leq k\} \cup \{v_{j+1} | 1 \leq j \leq k\} \cup \{v_{j+1} | 1 \leq j \leq k\} \cup \{v_{j+1} | 1 \leq j \leq k\} \cup \{v_{j+1} | 1 \leq j \leq k\}$ and $E = \{u_jv_j | 1 \leq j \leq k\} \cup \{u_jv_{j+1} | 0 \leq j \leq k - 1\} \cup \{u_jw_j | 1 \leq j \leq k\} \cup \{u_jw_{j+1} | 0 \leq j \leq k - 1\} \cup \{u_{j+1}v_{j+1} | 1 \leq j \leq k\} \cup \{u_{j+1}w_{j+1} | 1 \leq j \leq k\} \cup \{v_{j+1}v_{j+1} | 1 \leq j \leq k\} \cup \{v_{j+1}w_{j+1} | 1 \leq j \leq k\}$, where $n = 2q$.

**Theorem 1.** An $\text{nhair-}k\text{C}_4\text{-snake}$ is an odd harmonious graph.

**Proof.** Let $G$ be the graph $\text{nhair-}k\text{C}_4\text{-snake}$. Define the vertex labels as follows.

For the vertices in $k\text{C}_4\text{-snake}$ let:

$f(u_j) = 2j + 1$ for $0 \leq j \leq k$,

$f(v_j) = 6j - 6$ for $1 \leq j \leq k$,

$f(w_j) = 6j - 2$ for $1 \leq j \leq k$.

![Figure 1. The graph $\text{nhair-}k\text{C}_4\text{-snake}$](image-url)
For the vertices in nhair let:
\[ f(u_b^i) = \begin{cases} 4k + 4ki + 4i - 4, & k \text{ is even}, \quad 1 \leq i \leq n, \\ 8k + 4ki + 4i - 2, & k \text{ is odd}, \quad 1 \leq i \leq n, \end{cases} \]
\[ f(u_k^i) = \begin{cases} 3k + 4ki + 4i - 4, & k \text{ is even}, \quad 1 \leq i \leq n, \\ 2k + 4ki + 4i - 4, & k \text{ is odd}, \quad 1 \leq i \leq n, \end{cases} \]
\[ f(v_j^i) = \begin{cases} 4k - 4j + 4ki + 4i + 5, & k \text{ is even}, \quad 1 \leq j \leq k, \quad 1 \leq i \leq n, \\ 8k - 10j + 4ki + 4i + 5, & k \text{ is odd}, \quad 1 \leq j \leq k, \quad 1 \leq i \leq n, \end{cases} \]
\[ f(w_j^i) = \begin{cases} 8k - Bj + 4ki + 4i + 3, & k \text{ is even}, \quad 1 \leq j \leq k, \quad 1 \leq i \leq n, \\ 8k - 10j + 4ki + 4i + 3, & k \text{ is odd}, \quad 1 \leq j \leq k, \quad 1 \leq i \leq n. \end{cases} \]

From the vertex labelling above, we have
\[ f: u_j^i | 0 \leq j \leq k \rightarrow \{1,3,5,7, \ldots, 2k + 1\}, \]
\[ f: v_j^i | 1 \leq j \leq k \rightarrow \{0, 6, 12, 18, \ldots, 6k - 6\}, \]
\[ f: w_j^i | 1 \leq j \leq k \rightarrow \{4, 10, 16, 22, \ldots, 6k - 2\}. \]

For \(1 \leq i \leq n\)
\[ f: u_b^i | 1 \leq i \leq n \rightarrow \{8k, 12k + 4, 16k + 8, \ldots, 4k + 4kn + 4n - 4\}, \quad \text{if } k \text{ is even}, \]
\[ \{12k + 2, 16k + 6, 20k + 10, \ldots, 8k + 4kn + 4n - 2\}, \quad \text{if } k \text{ is odd}, \]
\[ f: u_k^i | 1 \leq i \leq n \rightarrow \{7k, 11k + 4, 15k + 8, \ldots, 3k + 4kn + 4n - 4\}, \quad \text{if } k \text{ is even}, \]
\[ \{6k, 10k + 4, 14k + 8, \ldots, 2k + 4kn + 4n - 4\}, \quad \text{if } k \text{ is odd}, \]
\[ f: v_j^i | 1 \leq i \leq n, 1 \leq j \leq k \rightarrow \{8k + 5, 12k + 9, 16k + 3, \ldots, 4kn - 2k + 4n + 5\}, \quad \text{if } k \text{ is even}, \]
\[ \{12k - 1, 16k + 3, 20k + 7, \ldots, 4kn - 2k + 4n + 5\}, \quad \text{if } k \text{ is odd}, \]
\[ f: w_j^i | 1 \leq i \leq n, 1 \leq j \leq k \rightarrow \{12k - 1, 16k + 3, 20k + 7, \ldots, 4kn + 4n + 3\}, \quad \text{if } k \text{ is even}, \]
\[ \{12k - 3, 16k - 1, 20k + 5, \ldots, 4kn - 2k + 4n + 3\}, \quad \text{if } k \text{ is odd}. \]

It is clear that \( f \) is an injection from \( V(\text{nhair} - kC_4 - \text{snake}) \) to \( \{0, 1, 2, \ldots, 8k + 4n + 4kn - 5\} \) for \( k \) is even and an injection from \( V(\text{nhair} - kC_4 - \text{snake}) \) to \( \{0, 1, 2, \ldots, 8k + 4n + 4kn - 2\} \) for \( k \) is odd.

The induced edge labels which defined by \( f^*(uv) = f(u) + f(v) \) where \( uv \in E(G) \) and \( u, v \in V(G) \), are:
\[ f^*(u_0v_1) = 1, \]
\[ f^*(u_0w_1) = 5, \]
\[ f^*(u_jv_j) = 8j - 5, \text{for } 1 \leq j \leq k. \]
\[ f^*(u_jw_j) = 8j - 1, \text{for } 1 \leq j \leq k. \]
\[ f^*(u_jv_{j+1}) = 8j + 1, \text{for } 1 \leq j \leq k. \]
\[ f^*(u_jw_{j+1}) = 8j + 5, \text{for } 1 \leq j \leq k. \]
\[ f^*(u_0u_b^i) = \begin{cases} 4k + 4ki + 4i - 3, & k \text{ is even}, \quad 1 \leq i \leq n, \\ 8k + 4ki + 4i - 1, & k \text{ is odd}, \quad 1 \leq i \leq n, \end{cases} \]
\[ f^*(u_ku_b^i) = \begin{cases} 5k + 4ki + 4i - 3, & k \text{ is even}, \quad 1 \leq i \leq n, \\ 4k + 4ki + 4i - 3, & k \text{ is odd}, \quad 1 \leq i \leq n. \end{cases} \]
\[
 f^*(v_jv_j') = \begin{cases} 
 4k + 2j + 4ki + 4i - 1, & \text{if } k \text{ is even}, \text{ for } 1 \leq j \leq k, \ 1 \leq i \leq n, \\
 8k - 4j + 4ki + 4i - 1, & \text{if } k \text{ is odd}, \text{ for } 1 \leq j \leq k, \ 1 \leq i \leq n, 
\end{cases}
\]

\[
 f^*(w_jw_j') = \begin{cases} 
 8k - 2j + 4ki + 4i + 1, & \text{if } k \text{ is even}, \text{ for } 1 \leq j \leq k, \ 1 \leq i \leq n, \\
 8k - 4j + 4ki + 4i + 1, & \text{if } k \text{ is odd}, \text{ for } 1 \leq j \leq k, \ 1 \leq i \leq n. 
\end{cases}
\]

From the edge labelling above, we have

\[
 f^*: \{u_0v_1\} \rightarrow \{1\},
 f^*: \{u_0w_1\} \rightarrow \{5\},
 f^*: \{v_jv_j'\} [1 \leq j \leq k] \rightarrow \{3, 11, 19, \ldots, 8k - 5\},
 f^*: \{w_jw_j'\} [1 \leq j \leq k] \rightarrow \{7, 15, 23, \ldots, 8k - 1\},
 f^*: \{u_jv_{j+1}\} [1 \leq j \leq k] \rightarrow \{9, 17, 25, \ldots, 8k + 1\},
 f^*: \{u_jw_{j+1}\} [1 \leq j \leq k] \rightarrow \{13, 21, 29, \ldots, 8k + 5\},
 f^*: \{u_0u_1\} [1 \leq i \leq n] \rightarrow \{(8k + 1, 12k + 5, \ldots, 4k + 4kn + 4n - 3), \text{ if } k \text{ is even,}
 \{12k + 3, 16k + 7, \ldots, 8k + 4kn + 4n - 1\}, \text{ if } k \text{ is odd,}
\}
\]

\[
 f^*: \{u_ku_1\} [1 \leq i \leq n] \rightarrow \{(9k + 1, 13k + 5, \ldots, 5k + 4kn + 4n - 3), \text{ if } k \text{ is even,}
 \{8k + 1, 12k + 5, \ldots, 4k + 4kn + 4n - 3\}, \text{ if } k \text{ is odd,}
\}
\]

\[
 f^*: \{v_jv_j'\} [1 \leq j \leq k, 1 \leq i \leq n] \rightarrow \{(8k + 5, 12k + 9, \ldots, 6k + 4kn + 4n - 1), \text{ if } k \text{ is even,}
 \{12k - 1, 16k + 3, \ldots, 2k + 4kn + 4n - 1\}, \text{ if } k \text{ is odd,}
\}
\]

\[
 f^*: \{w_jw_j'\} [1 \leq j \leq k, 1 \leq i \leq n] \rightarrow \{(12k + 3, 16k + 7, \ldots, 6k + 4kn + 4n + 1), \text{ if } k \text{ is even,}
 \{12k + 1, 16k + 5, \ldots, 4k + 4kn + 4n + 1\}, \text{ if } k \text{ is odd.}
\}
\]

It is clear that \( f^* \) is a bijection from \( E(n\text{hair} - kC_4\text{-snake}) \) to \( \{1, 3, 5, \ldots, 4kn + 4n + 8k - 1\} \). Therefore, \( f \) is an odd harmonious labelling of \( n\text{hair} - kC_4\text{-snake} \) graph and \( n\text{hair} - kC_4\text{-snake} \) graph is odd harmonious.

The examples of odd harmonious labelling of graph \( n\text{hair} - kC_4\text{-snake} \) for \( k \) is odd is shown in figure 2 and for \( k \) is even is shown in figure 3.

![Figure 2. An odd harmonious labelling for 4hair-3C4-snake graph.](image-url)
4. Conclusion
The graph \( rhair-kC_4 \)-snake is proven to be odd harmonious. Further investigation can be conducted to find the odd harmonious labelling of \( rhair-kC_m \)-snake graph for \( m \neq 4 \).

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