Criticality and oscillatory behavior in non-Markovian Contact Process

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A Non-Markovian generalization of one-dimensional Contact Process (CP) is being introduced in which every particle has an age and will be annihilated at its maximum age \( \tau \). There is an absorbing state phase transition which is controlled by this parameter. The model can demonstrate oscillatory behavior in its approach to the stationary state. These oscillations are also present in the mean-field approximation which is a first-order differential equation with time-delay. Studying dynamical critical exponents suggests that the model belongs to the DP universality class.

I. INTRODUCTION

Studying phase transitions in the systems far from equilibrium has been a topic of growing interest in recent years \([2]\). Specially systems with absorbing states which can not evolve further once they are trapped in one such state have been an interesting subject of research. Various models with one or more absorbing states have been studied which belong to a few universality classes and mostly to that of Directed Percolation (DP). According to the DP conjecture \([4]\), every phase transition in a system with a single absorbing state having short range interactions with no special symmetry or quenched disorder \([6]\) belongs to the DP class. There has been examples of absorbing state phase transitions without some of the DP conjecture conditions that still belong to the DP class such as systems with an infinite number of absorbing states \([7]\). There has also been another universality classes for parity conserving systems \([8]\) and as recently proposed, for systems with infinitely many absorbing states coupled to a conserved field \([9,10]\).

Although the Markovian property have been implicitly accepted in these models it is not essential for a nonequilibrium phase transition. Usually, adding some kind of memory to the system, such that the system should refer to its history in order to define its future, gives rise to some new interesting behaviors that are absent in Markovian ones \([11,12]\). However properties of non-Markovian nonequilibrium phase transition have not been studied.

In this paper a non-Markovian variant of the Contact Process (CP) \([13]\) is introduced and the critical behavior is investigated. Standard CP in its continuous time version is a lattice model in which every empty site is being occupied by a particle with rate \( \lambda n/z \) and every particle is removed with rate one, where \( z \) is the coordination number, \( n \) number of occupied nearest neighbors and \( \lambda \) a positive parameter controlling the creation rate. The system has a second-order critical point at \( \lambda_c = 3.2978 \) and it belongs to the DP class \([4]\).

In this model I introduce a memory for each particle: every particle knows when it has been created. Like standard CP, every site is being occupied by a particle at a rate proportional to the number of its occupied nearest neighbors, while every existing particle will die exactly at age \( \tau \). For large values of \( \tau \) the particles live enough to reproduce plenty of another ones and the system can remain in its active state. As \( \tau \) is decreased, each particle has less time to create another particles and for \( \tau < \tau_c \) the system will be trapped in its absorbing state with probability 1, where there is no existing particle and no new particles can be born.

Attributing age to the particles in the CP model has been suggested earlier \([14]\), but not in a way that leads to a non-Markovian model. Although some interesting alterations in the dynamical behavior of the system have been observed.

Non-Markovian property gives rise to some oscillations in the density of particles. These oscillations are also supported by the Mean-Filed approximation. Because of the non-Markovian property of the system, there is a delay parameter in the mean field equation and this reproduces the oscillatory solutions observed in the simulations. By mean-field approach, existence of the phase transition is justified and a critical age can be found, but like standard CP the critical behavior is not described completely.

In this paper, the critical behavior of the model is studied using the time-dependent Monte Carlo method. The critical dynamical exponents has been calculated, and shown to be in good agreement of those of DP.

II. MODEL

The model is defined on a 1-dimensional lattice and with continuous time. Every site is either empty or occupied by a single particle. There is a chance for a vacant site to be occupied provided there are occupied sites in its nearest neighborhood. A new particle is born in an empty site with rate \( n/2 \) (\( n \) number of occupied nearest neighbors). Every particle will die exactly at time \( \tau \) after its birth.

Obviously there must be a phase transition in the system with density of the particles as the order parameter. For small values of \( \tau \) particles die fast and eventually the
system is trapped in its absorbing state. For large \( \tau \)'s, particles have a large lifetime and reproduce sufficiently other ones to keep the system active. Fig. 1 shows a single cluster in a realization of the model for \( \tau = 3.5 \) and up to \( t = 100 \). As can be seen, every particle has a definite lifetime and here the system is in its active state.

FIG. 1. A typical space-time cluster started with a single particle in the origin for \( \tau = 3.5 \) and up to \( t = 100 \).

Like CP, here the creation process is not history-dependent, however the death process is, and thus we have a non-Markovian process. To find out that how dependent, however the death process is, and thus we the system is not enough to find out the future states. Therefore knowledge of only the present state of particles is not necessary to predict the future states.

### III. MEAN-FIELD EQUATION

Let \( \sigma_1(x) \) and \( \sigma_0(x) \) denote the state of site \( x \), and \( \rho \) be the density of particles. \( \sigma_1(x) = 1 \) if the site is occupied and 0 if it is vacant, and we have

\[
\rho = \langle \sigma_1(x) \rangle_x = 1 - \langle \sigma_0(x) \rangle_x.
\]

Therefore the rate of reproduction is

\[
\langle \sigma_1(x)\sigma_0(x+1) \rangle_x
\]

it can be written in terms of the density-vacancy correlation function (the correlation of the of occupied and vacant sites)

\[
C_{10}(\delta) = \frac{\langle \sigma_1(x)\sigma_0(x+\delta) \rangle_x - \rho(1-\rho)}{\rho(1-\rho)}
\]

Hence the rate of reproduction is

\[
r \rho(1-\rho)
\]

where

\[
r = 1 + C_{10}(\delta = 1).
\]

Obviously \( C_{10}(\delta = 1) \) is negative (a particle reduces the chance of its nearest neighbors to be vacant), and thus \( r < 1 \). In the mean-field approximation the correlation is neglected and we put \( r = 1 \). Therefore the mean-field equation will be

\[
\frac{d\rho_t}{dt} = \rho_t(1-\rho_t) - \rho_{t-\tau}(1-\rho_{t-\tau}) \quad (t > \tau)
\]

where the second term is the rate of annihilation at time \( t \), equal to the rate of creation at time \( t-\tau \). This equation is true for \( t > \tau \). I assume that all existing particles at \( t = 0 \), gradually die during the time interval \((0,\tau)\). So for \( t < \tau \) we have:

\[
\frac{d\rho_t}{dt} = \rho_t(1-\rho_t) - \rho_0/\tau \quad (t \leq \tau).
\]

This equation can be re-written in the integral form. First by integrating Eq. (7)

\[
\rho_t = \int_0^t \rho_t' (1-\rho_t') dt' - \rho_0 t/\tau + \rho_0 \quad (t \leq \tau)
\]

specially for \( t = \tau \)

\[
\rho_{t=\tau} = \int_0^\tau \rho_t' (1-\rho_t') dt'.
\]

By integrating eq. (6) and making use of eq. (9) we find

\[
\rho_t = \int_{t-\tau}^t \rho_t' (1-\rho_t') dt' \quad (t > \tau)
\]

It is a definite integral with a time dependent lower- and upper-limit. So although the integrand is non-negative, \( \rho(t) \) may have a non-monotonic behavior.

Finding stationary density is not possible in the differential equation. Setting \( d\rho_t/dt = 0 \) leads to nothing more than \( \rho_t = 1 - \rho_{t-\tau} \) or \( \rho_t = \rho_{t-\tau} \). The former is irrelevant in the steady state and the latter is correct for every value of \( \bar{\rho} \). However by setting \( \rho_t = \rho_t' = \bar{\rho} \) in the integral equation (Eq. (10) ), it turns out that

\[
\bar{\rho} = 0
\]

or

\[
\bar{\rho} = 1 - 1/\tau.
\]

Thus there is a phase transition at \( \tau = \tau_c = 1 \).
IV. OSCILLATIONS

It is being observed that the density of particles $\rho$ undergoes damped oscillations while approaching the stationary state. The solid line in the figure 2 shows one such oscillatory evolution of $\rho$. Simulations are done in a lattice of 10000 sites with periodic boundary condition. The plotted curves are averaged over 100 realizations for $\tau = 7$. Period of oscillations is slightly greater than $\tau$. This kind of oscillations are present for all values of $\tau$, but they are weaker for smaller $\tau$.

![Graph showing density of particles versus time for Non-Markovian CP in real model (solid line) and in Mean-Field approximation (dashed line) for $\tau = 7$ and $\rho_0 = 0.75$.](image)

The oscillatory behavior can be understood by paying attention to the history-dependence feature of the model. Since every particle dies at age $\tau$, the time evolution at time $t$ is coupled to the state of the system at time $t - \tau$. A high creation rate at time $t$ is equivalent to a high annihilation rate at time $t + \tau$. So an increase in the density at time $t$ can lead to a decrease in it at some time later and naturally the period of the oscillations is of order $\tau$.

These oscillations are also supported by the mean-field approximation. The mean-field equation is a delayed nonlinear first-order differential equation that is able to have oscillatory solutions which do not exist in ordinary first-order differential equations. Figure 2 (dashed line) shows one of these oscillatory solutions for $\tau = 7$. Period of oscillations in real model and Mean-Field are the same.

The delay time in the mean-field differential equation, can be eliminated by a Taylor expansion of $\rho_{1-\tau}$. It basically contains derivatives up to infinite order. In fact here we have an infinite-order differential equation which is naturally able to demonstrate many complex behaviors. As in simulations, Oscillatory behavior is sensitive to changes in the value of $\tau$. It disappears for small enough values of $\tau$ and it is less damped for larger $\tau$’s.

V. CRITICAL BEHAVIOR

In this section I present the results concerning the critical behavior of the system obtained from simulating the model. Simulation is made using the time-dependent Monte Carlo method [15]. In this method, the simulation is started with the system in a state very close to the absorbing state that is all the sites are vacant except one in origin which is occupied. The age of this single particle is initially set to 0. The sites are updated parallel and after every time increment all existing particles become older by that amount. They die after growing up to age $\tau$.

I measure the average population of particles $N(t)$, averaged over all realizations, $P(t)$, the probability of not entering the absorbing state up to time $t$, and $R^2(t)$ the mean square spreading distance. As a result of the scaling hypothesis [15], at criticality, these quantities should scale algebraically as

$$N(t) \sim t^\eta$$

$$P(t) \sim t^{-\delta}$$

$$R^2(t) \sim t^z$$

So at criticality the log-log plot of these functions should asymptotically become a straight line with the slope equal to the dynamical critical exponents. The local slopes for the survival probability $P(t)$ are defined by

$$\delta(t) = -\frac{\ln[P(t)/P(t/b)]}{\ln(b)}$$

and similarly for the other exponents. I usually use $b = 15$. Away from criticality there are either upward or downward curvatures in the log-log plot of functions versus $t$ and also in the plot of the critical exponents versus $1/t$, depending upon the super- or sub-criticality of the system. By detecting the straight line from the curved ones, the value of $\tau_c$ can be evaluated with a good precision. Having $\tau_c$, the critical exponents can also be found.

Simulations are typically done up to time 400 (although many runs enter the absorbing state earlier) with a time increment of 0.004 for continuous-time simulation. Obviously this is equal to the maximum precision possible in determining $\tau_c$. Statistical quantities have been generally averaged over 10000 different realizations of the model.
VI. CONCLUSION

In summary, a Non-Markovian version of the Contact Process is introduced. Particles have an age which determines when they are born and when they will be annihilated. Interesting oscillatory behaviors are observed in density of particles and the same oscillations are also present in the Mean-Filed approximation. Because of the non-Markovian property of the model there is a time delay in the mean-filed first-order differential equation which allows it to demonstrate oscillatory behaviors.

Applying time-dependent Monte Carlo technique, critical properties of the absorbing phase transition has been investigated and shown to belong to DP universality class. DP class is so extended to contain a non-Markovian model.

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FIG. 3. Dynamical critical exponents as functions of $1/t$. Different curves in each panel correspond to $\tau = 3.06, 3.07, 3.08$ respectively from top to bottom.

Figure (3) shows the results of the dynamical simulations. In different panels local slopes as defined in eq.(16) for different dynamical critical exponents have been depicted against $1/t$. The best estimation for the critical age based on these graphs is $\tau_c \simeq 3.07(1)$. For the critical exponents I find $\eta = 0.304(1)$, $\delta = 0.1653(1)$ and $z = 1.272(1)$. These critical exponents are in good agreement with those of Directed Percolation and thus the system belongs to the DP class.

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