The Goldberg-Sachs theorem in linearized gravity

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Abstract

The Goldberg-Sachs theorem has been very useful in constructing algebraically special exact solutions of Einstein vacuum equation. Most of the physical meaningful vacuum exact solutions are algebraically special. We show that the Goldberg-Sachs theorem is not true in linearized gravity. This is a remarkable result, which gives light on the understanding of the physical meaning of the linearized solutions.

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Solutions of the linearized Einstein vacuum equation are usually considered as approximations of solutions of the full vacuum equation. They are useful tools for describing physical systems.

It is important to understand the relation between the full vacuum equation and the linearized one; in particular it is interesting to know which properties are common, or not, to both sets of solutions. An example of a common property is the Birkhoff’s theorem; which can be enunciated in the following way: vacuum spherically symmetric solutions are static. This statement remains true in linearized gravity (i.e., if we replace the vacuum equation by the linear one); as one can check by reconstructing the linear version of the standard proofs that appear in the literature.

In this work we present an example of a property, the so called Goldberg-Sachs theorem[1], which is not common to both set of solution.

The Goldberg-Sachs theorem for the Einstein vacuum equation relates algebraic properties of the Weyl tensor with the existence of a null, geodesic, shear-free congruence in the space-time. In the search of vacuum solutions, the existence of such a congruence leads to considerable simplification in the calculation. The Schwarzschild, Kerr and Robinson-Trautman space-times, which

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are probably the most useful metrics in the study of compact objects, are all algebraically special. However our present result shows that if one has a linearized solution admitting shear free null geodesic congruence, then in general it will not be algebraically special. The proof of this consists in presenting explicit counterexamples.

The statement of Goldberg-Sachs theorem is [1] (see also [2] [3]).

**Theorem 1 (Goldberg-Sachs)** A vacuum metric admits a shear free null geodesic congruence \(l^a\) if and only if \(l^a\) is a degenerate eigendirection of the Weyl tensor

\[
C_{abc[d}l^{b]l^e} = 0. \tag{1}
\]

The condition (1) is what characterize algebraically special space-times.

Let us study this statement in the context of linearized gravity. Consider a one-parameter family of metrics \(g_{ab}(\gamma)\) with the corresponding metric, torsion free, connection \(\nabla_a\), Weyl tensor \(C_{abcd}\) and Ricci tensor \(R_{ab}\). We define the first order Weyl and Ricci tensor by

\[
C_{abcd}^{(1)} = \frac{d}{d\gamma} C_{abcd}|_{\gamma=0}, \quad R_{ab}^{(1)} = \frac{d}{d\gamma} R_{ab}|_{\gamma=0}.
\]

In the same way we can define the ‘first order’ shear of a vector field, and so on.

We assume that

\[
g_{ab}(\gamma = 0) = \eta_{ab}, \tag{2}
\]

where \(\eta_{ab}\) is the flat metric. We say that \(g_{ab}(\gamma)\) satisfies the linear vacuum equation if

\[
R_{ab}^{(1)} = 0.
\]

Let us assume that \(g_{ab}(\gamma)\) satisfies the linear vacuum equation, and also that \(g_{ab}(\gamma)\) admits a first order geodesic shear free null vector field \(l^a\). We ask the following question: Is it true that \(l^a\) is a degenerate eigendirection of the first order Weyl tensor \((C_{abc[d}l^{b]l^e}^{(1)} = 0)\)? Note that this question is a linearized version of the first implication of the Goldberg-Sachs theorem.

We prove that the answer is negative by giving an explicit one-parameter family \(g_{ab}(\gamma)\) and a vector field \(l^a\), with the following properties (we define \(l_a \equiv g_{ab}l^b\)):

(i) The vector field \(l^a\) is null, geodesic and shear free with respect to \(g_{ab}(\gamma)\), i.e., it satisfies the next equations

\[
l^a l^b g_{ab}(\gamma) = 0, \\
l^a \nabla_b l^b = \phi l^b, \\
\nabla_a (l_b) = J_a l_b + \theta g_{ab},
\]

for some scalar fields \(\phi\), \(\theta\) and some vector field \(J_a\). The last equation means that the shear is zero.
(ii) The family $g_{ab}(\gamma)$ satisfies the linear field equation
\[ R^{(1)}_{ab} = 0. \]

(iii) The vector field $l^a$ is not a degenerate eigendirection of the linear Weyl tensor
\[ C^{(1)}_{abc[d]l^e}l^d l^e \neq 0. \]

We would like to emphasize here that our example below satisfies (i) exactly; i.e., to all orders in $\gamma$. Of course in order to construct a counter-example it is enough to satisfy this condition only up to first order in $\gamma$.

A one-parameter family $g_{ab}(\gamma)$ which satisfies condition (i), can be constructed in the following way.

Let $l^a$ be a vector field that is null, geodesic, and shear-free with respect to the flat connection $\partial_a = \nabla_a(\gamma = 0)$:

\begin{align*}
l^a l^b \eta_{ab} &= 0, \quad \text{(3)} \\
l^b \partial_b l^a &= \phi l^a, \quad \text{(4)} \\
\partial_b(l^a_0) &= J(a l^0_b) = \theta \eta_{ab}, \quad \text{(5)}
\end{align*}

for some scalars fields $\phi$ and $\theta$, some vector field $J_a$, and where we have defined $l^0_a = \eta_{ab} l^b$ in order to distinguish it from $l_a = g_{ab} l^b$. In other words, $l^a$ satisfies condition (i) for $\gamma = 0$.

Consider the following one-parameter family
\[ g_{ab}(\gamma) = \eta_{ab} + \gamma(\mu \eta_{ab} + l^0_a v_b), \quad \text{(6)} \]
where $v_b$ is a smooth vector field, and $\mu$ a smooth scalar field.

We now claim: For all $v_a$ and $\mu$, the metric $g_{ab}(\gamma)$ satisfies condition (i), i.e., it preserves the null, geodetic, shear-free character of $l^a$.

First note that preservation of the null character
\[ l^a l^b g_{ab} = 0 \quad \text{(7)} \]
is immediate.

Now we compute $l^a \nabla_a l^b$. We calculate the Lie derivative $L_l$ of $l^0_a$
\[ L_l(l^0_a) = l^b \partial_b(l^0_a) + l^0_b \partial_a l^b, \]
using equations (3) and (4) we obtain
\[ L_l(l^0_a) = \phi l^0_a. \quad \text{(8)} \]
The Lie derivative can also be written in terms of the connection $\nabla_a$:
\[ L_l(l^0_a) = l^b \nabla_b l^0_a + l^0_b \nabla_a l^b = \phi l^0_a. \quad \text{(9)} \]
We use (6) to replace $l_0^a$ by 

$$l_0^a = l_a(1 + \gamma(\mu + l^c v_c))^{-1} \quad (10)$$

in equation (9) to obtain

$$l^b \nabla_b l^a = l^a \left( \frac{l^c \partial_c \gamma(\mu + l^c v_c)}{1 + \gamma(\mu + l^c v_c)} + \phi \right); \quad (11)$$

where we have used equation (7). Then $l_a$ is geodesic with respect to $g_{ab}$.

It remains to be shown that $l^a$ is shear-free with respect to $g_{ab}$. We have to prove that $\nabla (a l_b)$ is equal to some vector symmetrized with $l_a$ plus some multiple of $g_{ab}$. In order to prove this, let us note that

$$2 \nabla (a l_b) = \mathcal{L}_l g_{ab} = \gamma \eta_{ab} \mathcal{L}_l \mu + (\mathcal{L}_l v^a) l^b_0$$

$$+ (1 + \gamma \mu) \mathcal{L}_l (\eta_{ab}) + (\mathcal{L}_l l^0_0) v_b. \quad (12)$$

We will prove that each term in the right-hand side of equation (12) has the desired form.

For the first and second term one only has to use the definition (6) and the relation (10). For the third term one uses that $\mathcal{L}_l (\eta_{ab}) = 2 \partial (a l_b)$ and equation (5) (the flat shear-free condition). Finally, for the last term one uses equation (8).

Consider now a metric of the following form

$$g^{KS}_{ab}(\gamma) = \eta_{ab} + \gamma f l^a l_b, \quad (13)$$

where $f$ is some scalar field. We assume also that $g^{KS}_{ab}(\gamma)$ satisfies the linear vacuum equation (condition (ii)). This class of metrics are said to have the Kerr-Schild form. Examples of these metrics are the Schwarzschild and Kerr metrics.

Let $k^a$ be an arbitrary Killing vector of $\eta_{ab}$ (i.e. it satisfies $\mathcal{L}_k \eta_{ab} = 0$). Take the metric given by

$$g_{ab}(\gamma) = \eta_{ab} + \mathcal{L}_k g^{KS}_{ab}. \quad (14)$$

Since by hypothesis $g^{KS}_{ab}$ satisfies the linear vacuum equation, it follows that also $g_{ab}$ satisfies it (condition (ii)); and also we can see that condition (i) is fulfilled; since one can easily show that it has the form (4).

The linear Weyl tensor corresponding to $g_{ab}$ is

$$C^{(1)}_{abcd} = \mathcal{L}_k C^{KS(1)}_{abcd}, \quad (15)$$

where $C^{KS(1)}_{abcd}$ is the linear Weyl tensor of $g^{KS}_{ab}$.

One can prove that the vector $l^a$ is a degenerate eigendirection of $C_{abcd}$. But, if we choose $k^a$ such that $\mathcal{L}_k l^a$ is not proportional to $l^a$, then $l^a$ will be a
principal null direction of $C_{abcd}^{(1)}$ but not a degenerate one; as one can see from equation (15). An explicit example of this situation is given when $g_{KS}$ is the Schwarzschild metric and $k^a$ is any space translation.

Then the metric $g_{ab}$ satisfies also condition (iii), and this completes the proof.

The given counterexamples are by no means the only possible ones. In a separate work we discuss another class of linearized solutions, which include also different counterexamples to the Goldberg-Sachs theorem in linearized gravity [1].

Given an exact vacuum solution which depends smoothly on some parameter $\gamma$ whose vanishing yields Minkowski space, and which is algebraically special for all values of the parameter, then the corresponding linearized solution will also be algebraically special. The solutions we indicate in this work belong to a set of linearized solutions that can not be reached by this means. They are linearizations of some vacuum solutions which are not algebraically special. These vacuum solutions have a vector field $l^a$ which is null, shear free and geodesic only up to the first order in $\gamma$, but they are not algebraically special not even up to the the first order in $\gamma$.

We have here shown that the Goldberg-Sachs theorem is one of those properties of the Einstein vacuum equation with no analog in the linear theory. As we have mentioned in the introduction, most useful metric in the study of compact object are algebraically special; we hope that our result will contribute to understand the relevance of the algebraic special condition in the physically meaningful solutions of Einstein vacuum equation.

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References

[1] J. N. Goldberg and R. K. Sachs. A theorem on Petrov types. Acta Phys. Polon., 22:13–23, 1962. Suppl.

[2] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt. Exact solutions of Einstein’s field equations. Cambridge University Press, Cambridge, 1980.

[3] R. Penrose and W. Rindler. Spinors and Space-Time, volume 2. Cambridge University Press, Cambridge, 1986.

[4] R. P. Kerr and A. Schild. Some algebraically degenerate solutions of Einstein’s gravitational field equations. Proc. Symp. Appl. Math., 17:199–209, 1965.
[5] G. C. Debney, R. P. Kerr, and A. Schild. Solution of the Einstein and Einstein-Maxwell equation. *J. Math. Phys.*, 10:1842–1854, 1969.

[6] S. Dain and O. M. Moreschi. Particles in linear gravity and Robinson-Trautman solutions, unpublished, 2000.