Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

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Abstract We provide a unified description of the three covariant superspace approaches to $\mathcal{N} = 2$ conformal supergravity in four dimensions: (i) conformal superspace; (ii) $\text{U}(2)$ superspace; and (iii) $\text{SU}(2)$ superspace. Each of them can be used to formulate general supergravity-matter systems, although conformal superspace has the largest structure group and is intimately related to the superconformal tensor calculus. We review the structure of covariant projective multiplets and demonstrate how they are used to describe pure and matter-coupled supergravity, including locally superconformal off-shell sigma models. Higher-derivative invariants, topological invariants and super-Weyl anomalies are also briefly discussed.

Keywords

Superconformal symmetry, Supergravity, Superspace, Projective multiplets

To Daniel Butter with gratitude and admiration

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1 Introduction

Pure $\mathcal{N} = 2$ supergravity in four dimensions was constructed by Ferrara and van Nieuwenhuizen in 1976 [1], some six months after the creation of $\mathcal{N} = 1$ supergravity [2, 3]. It fulfilled Einstein’s dream of unifying gravity and electromagnetism, albeit using a symmetry principle that was not known to Einstein – local supersymmetry.

In 1979, Fradkin and Vasiliev [4] and, independently, de Wit and van Holten [5] proposed an off-shell formulation for linearised $\mathcal{N} = 2$ supergravity. Shortly thereafter, these linearised results were extended to the first off-shell formulation for $\mathcal{N} = 2$ supergravity [6, 7]. In [7] de Wit, van Holten and Van Proeyen made use of the so-called $\mathcal{N} = 2$ superconformal tensor calculus, a natural extension of the $\mathcal{N} = 1$ superconformal method [8, 9, 10, 11, 12]. Since then, the $\mathcal{N} = 2$ superconformal tensor calculus of [7] has been further developed [13, 14, 15] and applied [16, 17] to derive many important results for $\mathcal{N} = 2$ supergravity-matter systems. For comprehensive reviews of this method, see [18, 19].

In parallel with the progress achieved in [4, 5, 6, 7], there appeared several works [20, 21, 22, 23, 24, 25, 26] devoted to $\mathcal{N} = 2$ superfield supergravity. In these works, the component results were recast in a superspace setting. More importantly, these publications pursued an ambitious goal of developing superspace formulations to describe general supergravity-matter systems, including the construction of an off-shell charged hypermultiplet that can be coupled to a $U(1)$ vector multiplet.

Within the superconformal tensor calculus, hypermultiplets are either on-shell or involve a gauged central charge. As is well known, such hypermultiplet realisations cannot be used to provide an off-shell formulation for the most general locally supersymmetric sigma model. It is also known that such a sigma model formulation, if it exists, must use off-shell hypermultiplets possessing an infinite number of auxiliary fields [27, 28, 29]. The latter feature makes the off-shell hypermultiplets extremely difficult to work with at the component level, and a superspace setting is required.

The problem of constructing an off-shell charged hypermultiplet (in short, the charged hypermultiplet problem) remained unsolved until 1984. Nevertheless, the early works on $\mathcal{N} = 2$ superfield supergravity [20, 21, 22, 23, 24, 25, 26] have yielded several important results. It suffices to mention the linear multiplet action originally discovered by Sohnius in the rigid supersymmetric case [30]. Since the linear multiplet was lifted to $\mathcal{N} = 2$ supergravity [20], and then reformulated [31] within the $\mathcal{N} = 2$ superconformal tensor calculus [7, 13, 14], it has become a universal tool to construct the component actions for supergravity-matter systems.

The construction of the relaxed hypermultiplet in 1983 [32] was perhaps the pinnacle of conventional $\mathcal{N} = 2$ superspace techniques, but it did not solve the charged hypermultiplet problem.² In spite of being off-shell, this hypermultiplet is neutral and cannot couple to a $U(1)$ vector multiplet. It became apparent that the conven-

² There exist infinitely many off-shell formulations for the neutral hypermultiplet [33, 34], in addition to the relaxed hypermultiplet.
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

1.1. $\mathcal{N} = 2$ superspace $\mathbb{M}^{4|8}$ is not suitable, say, for off-shell $\sigma$-model constructions. The correct superspace setting was found in 1983–1984 independently by three groups who pursued somewhat different goals [35, 36, 37], which is:

$$\mathbb{M}^{4|8} \times \mathbb{C}P^1 = \mathbb{M}^{4|8} \times S^2.$$  \hspace{1cm} (1.1)

This superspace was introduced for the first time by Rosly [35] who used it to derive an interpretation of the $\mathcal{N} = 2$ super Yang-Mills constraints [38] as integrability conditions. Rosly and Schwarz [39] called (1.1) isotwistor superspace.

The starting point of the analysis in [35] was the observation that, given an isotwistor $v^i \in \mathbb{C}^2 \setminus \{0\}$, the set of eight spinor covariant derivatives $D_{ai}$ and $\bar{D}_{\dot{a}i}$ for $\mathbb{M}^{4|8}$ contains a subset of four operators, $D_{ai}^{(1)} := -\nu^i D_{ai}$ and $\bar{D}_{\dot{a}i}^{(1)} := -\nu^{\dot{i}} \bar{D}_{\dot{a}i}$, which strictly anticommute with each other. Therefore, one can introduce a new family of supersymmetric multiplets constrained by

$$D_{ai}^{(1)} \phi = 0, \quad \bar{D}_{\dot{a}i}^{(1)} \phi = 0, \quad \phi = \phi(z, v, \bar{v}) , \quad \bar{v}_i := \nu^i .$$  \hspace{1cm} (1.2)

In order for these constraints to be invariant under arbitrary re-scalings of $v, \phi$ should be homogeneous,

$$\phi(z, c v, \bar{c} \bar{v}) = c^{n_+} \bar{c}^{n_-} \phi(z, v, \bar{v}) , \quad c \in \mathbb{C} \setminus \{0\} \equiv \mathbb{C}^* ,$$  \hspace{1cm} (1.3)

for some parameters $n_{\pm}$ such that $n = n_+ - n_-$ is an integer. Redefining $\phi(z, v, \bar{v}) \rightarrow \phi(z, v, \bar{v}) / (v^i v^j)^{n_-}$ allows one to choose $n_- = 0$. Any superfield with the homogeneity property

$$\phi^{(n)}(z, c v, \bar{c} \bar{v}) = c^n \phi^{(n)}(z, v, \bar{v}) , \quad c \in \mathbb{C}^*$$  \hspace{1cm} (1.4)

is said to have the weight $n \in \mathbb{Z}$. This superfield lives in the superspace (1.1), since the isotwistor $v^i \in \mathbb{C}^2 \setminus \{0\}$ is defined modulo the equivalence relation $v^i \sim c v^i$, with $c \in \mathbb{C}^*$, hence it parametrises $\mathbb{C}P^1$. A weight-$n$ superfield $\phi^{(n)}(z, v, \bar{v})$ is called isotwistor if it obeys the constraints (1.2).

A new approach to $\mathcal{N} = 2$ supersymmetric field theory was put forward by Galperin et al. [36]. Using harmonic superspace $\mathbb{M}^{4|8} \times S^2$, they proposed the first off-shell formulation of charged hypermultiplet (the so-called $q^+$ hypermultiplet) and its self-couplings. Moreover, unconstrained prepotential descriptions of $\mathcal{N} = 2$ super Yang-Mills and supergravity theories were also provided. Since then the harmonic superspace approach has developed into a powerful branch of supersymmetric field theory, see [40] for a review. In the harmonic superspace approach, one deals with those isotwistor superfields $\phi^{(n)}(z, v, \bar{v})$ which are globally defined smooth functions on $\mathbb{C}P^1$. In the literature, they are known as harmonic analytic superfields.

Projective superspace $\mathbb{M}^{4|8} \times \mathbb{C}P^1$ was originally employed in [37] to provide a manifestly $\mathcal{N} = 2$ supersymmetric description for the general self-couplings of $\mathcal{N} = 2$ tensor multiplets constructed earlier [41] in terms of $\mathcal{N} = 1$ superfields. Since then, this approach has been extended to include some other interesting mul-
4 S. M. Kuzenko, E. S. N. Raptakis and G. Tartaglino-Mazzucchelli

tiplets [42, 43]. In particular, a new off-shell formulation for the charged hyper-multiplet was derived [42] and used to construct off-shell nonlinear σ-models, see [44, 45] for reviews. The name ‘projective superspace’ was coined in 1990 [43]. In the projective superspace approach, one deals with those isotwistor superfields \( \phi^{(n)}(z,v) \) which are holomorphic functions on an open domain of \( \mathbb{C}P^1 \). In the literature, they are known as projective superfields.

Both harmonic and projective superspace make use of the same superspace (1.1). Without going into technical details, which are spelled out in [46] (see also [45, 47, 48]), they differ in (i) the structure of off-shell supermultiplets used; and (ii) the supersymmetric action principle chosen. Due to these conceptual differences, the two approaches prove to be complementary to each other in many respects. In particular, harmonic superspace offers powerful prepotential formulations for \( \mathcal{N}=2 \) supergravity [49, 50] (reviewed in [40]) which are similar in spirit to the Ogievetsky-Sokatchev approach to \( \mathcal{N}=1 \) supergravity [51]. Projective superspace proves to be ideal for developing covariant geometric formulations for supergravity-matter systems with eight supercharges. The harmonic superspace approach to \( \mathcal{N}=2 \) supergravity is reviewed in this volume by Ivanov [52].

The formalism of curved projective superspace was originally developed in 2008 for \( \mathcal{N}=1 \) supergravity-matter systems in five dimensions [53, 54] using the structure of superconformal projective multiplets [55]. Shortly thereafter, these constructions were generalised to develop the projective superspace approach for \( \mathcal{N}=2 \) matter-coupled supergravity in four dimensions [56, 57, 58, 59].\(^3\) With the advent of \( \mathcal{N}=2 \) conformal superspace [63], and its applications to component reduction [64], a novel formulation of curved projective superspace has been given in [65, 66]. This approach has also been extended to a novel covariant harmonic superspace framework in [67]. All of these publications followed the philosophy of the \( \mathcal{N}=2 \) superconformal tensor calculus to realise supergravity-matter systems as conformal supergravity coupled to superconformal matter multiplets.

There are three superspace formulations for \( \mathcal{N}=2 \) conformal supergravity that have found numerous applications in the recent years, specifically: (i) \( \text{U}(2) \) superspace [26]; (ii) \( \text{SU}(2) \) superspace [56]; and (iii) conformal superspace [63]. The \( \mathcal{N}=2 \) conformal superspace of [63] is an ultimate formulation for \( \mathcal{N}=2 \) conformal supergravity in the sense that any different off-shell formulation is either equivalent to it or is obtained from it by partially fixing the gauge freedom. In particular, the \( \text{U}(2) \) and \( \text{SU}(2) \) superspaces can be derived from conformal superspace by imposing partial gauge fixing conditions.\(^4\) At the component level, \( \mathcal{N}=2 \) conformal superspace reduces to the \( \mathcal{N}=2 \) superconformal tensor calculus.

The \( \mathcal{N}=2 \) conformal superspace of [63, 64] is a natural extension of the \( \mathcal{N}=1 \) formulation [68]. Conformal superspace approaches have also been developed for extended supergravity-matter systems in three [69, 70, 71], five [72] and six [73, 74] dimensions. These references include various applications.

\(^3\) It was subsequently extended to supergravity-matter theories in two [60], three [61] and six [62] dimensions.

\(^4\) The relationship between the \( \text{U}(2) \) and \( \text{SU}(2) \) superspaces is described in [59].
Recently, new supertwistor formulations were discovered for conformal supergravity theories in diverse dimensions [75]. In the four-dimensional $\mathcal{N} = 2$ case, the supertwistor formulation is expected to be related to conformal superspace, however relevant technical details have not yet been worked out in the literature.

Our two-component spinor notation and conventions follow [76], and are similar to those adopted in [77]. The only difference is that the spinor Lorentz generators $(\sigma_{ab})_{a}^{\beta}$ and $(\tilde{\sigma}_{ab})_{a}^{\dot{\beta}}$ used in [76] have an extra minus sign as compared with [77], specifically $\sigma_{ab} = -\frac{1}{4}(\sigma_{a}\sigma_{b} - \sigma_{b}\sigma_{a})$ and $\tilde{\sigma}_{ab} = \frac{1}{4}(\tilde{\sigma}_{a}\sigma_{b} - \tilde{\sigma}_{b}\sigma_{a})$.

## 2 Rigid superconformal transformations

We denote by $z^A = (x^a, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha}}{}^i)$ the Cartesian coordinates for Minkowski superspace $\mathcal{M}^{4|8}$ and use the notation $D_A = (\partial_a, D^i_\alpha, \bar{D}^{\dot{i}}{}_{\dot{\alpha}})$ for the superspace covariant derivatives. The only non-zero graded commutation relation is

$$\{D_i^A, \bar{D}_j^B\} = -2i\epsilon^{ij}((\sigma_c)_{a}^{\alpha}{\partial}_c = -2i\epsilon^{ij} D^a_{\alpha} = (\sigma^a, \partial^i, \partial^\dot{i})_{a}^{\alpha}{\partial}_c$$ (2.1)

The $\mathcal{N} = 2$ super-Poincaré algebra has an outer automorphism group $SU(2)_R \times U(1)_R$, which is also called the $R$-symmetry group. The $SU(2)_R$ indices are raised and lowered using the antisymmetric tensor $\varepsilon^{ij} = -\varepsilon^{ji}$ and its inverse $\varepsilon_{ij}$ normalised by $\varepsilon_{12} = 1$.

### 2.1 Conformal Killing supervector fields

Superconformal transformations in $\mathcal{M}^{4|8}$ were first studied by Sohnius [78]. Our presentation follows [79].

An infinitesimal superconformal transformation $z^A \rightarrow z^A + \delta z^A$, with $\delta z^A = \xi^A = (\xi^a + i(\xi_\alpha \sigma^a \bar{\theta}^\alpha - \bar{\theta}_\alpha \sigma^a \bar{\theta}^\alpha), \xi^\alpha_i, \xi^\dot{\alpha}_{\dot{i}})$, is generated by a conformal Killing supervector field

$$\xi = \xi^b \partial_b + \xi^i_\alpha D_i^\alpha + \bar{\xi}^{\dot{i}}{}_{\dot{\alpha}} \bar{D}_{\dot{i}}{}^{\dot{\alpha}} = \xi.$$ (2.2)

The defining property of $\xi$ is

$$[\xi, D^A] = -(D^A_{\alpha} \xi^\beta) D_j^\beta.$$ (2.3)

This condition implies the relations

$$\bar{D}_i^\alpha \xi_j^\beta = 0, \quad D_i^\alpha \xi^\beta_j = 4i \epsilon^{\alpha\beta} \xi_i^\gamma \quad \Rightarrow \quad \xi^\alpha_i = -\frac{i}{8} D_i^\alpha \xi^\alpha_{\alpha}$$ (2.4)
and their complex conjugates, and therefore
\[ \hat{D}_{(\alpha} \xi_{\beta)} = 0, \quad \hat{D}^i_{(\alpha} \xi^i_{\beta)} = 0 \implies \partial_{(\alpha} (\alpha^i_{\beta)} \beta) = 0. \] (2.5)

It then follows that
\[ [\xi, D^i_{\alpha}] = -K^i_{\alpha \beta} |\xi| D^i_{\alpha \beta} - \Lambda^i_{\alpha \beta} [\xi] D^i_{\alpha}. \] (2.6)

Here we have introduced the chiral Lorentz $K^i_{\beta \gamma}[\xi]$ and super-Weyl $\sigma[\xi]$ parameters, as well as the SU(2)$_\alpha$ parameter $K^{ij}[\xi]$ defined by
\[ K_{\alpha \beta}[\xi] = \frac{1}{2} D^i_{\alpha} \xi^i_{\beta} = K_{\beta \alpha}[\xi], \quad \hat{D}^i_{\alpha} K_{\alpha \beta}[\xi] = 0, \] (2.7a)
\[ \sigma[\xi] = \frac{1}{2} \hat{D}^i_{\alpha} \xi^i_{\alpha}, \quad \hat{D}^i_{\alpha} \sigma[\xi] = 0, \] (2.7b)
\[ \Lambda^{ij}[\xi] = -\frac{i}{16} [D^i_{\alpha}, D^j_{\alpha}] \xi^{i \alpha \beta} = \Lambda^{ij}[\xi], \quad \Lambda^{ij}[\xi] = \Lambda^{ij}[\xi]. \] (2.7c)

We recall that the Lorentz parameters with vector and spinor indices are related to each other as follows: $K^{ik}[\xi] = (\sigma^{ik})_{\beta \gamma} K^i_{\beta \gamma}[\xi] - (\sigma^{ik})_{\beta \gamma} K^i_{\gamma \beta}[\xi]$. The parameters in (2.7) obey several first-order differential properties:
\[ D^i_{\alpha} \Lambda^{jk}[\xi] = \epsilon^{ijk} D^k_{\alpha} \sigma[\xi], \] (2.8a)
\[ D^i_{\alpha} K^j_{\beta \gamma}[\xi] = -\epsilon_{\alpha \beta} D^i_{\gamma} \sigma[\xi], \] (2.8b)
and therefore
\[ D^i_{\alpha} \Lambda^{jk}[\xi] = D^i_{\alpha} \Lambda^{jk}[\xi] = 0, \] (2.9a)
\[ D^i_{\alpha} D^j_{\beta} \sigma[\xi] = 0. \] (2.9b)

The most general conformal Killing supervector field has the form
\[ \bar{\xi}^{aa} = a^{aa} + \frac{1}{2} (\sigma + \bar{\sigma}) y^{aa} + K^{a \alpha}_{\beta} \Lambda^{\beta a} + K^{a \alpha}_{\beta} \Lambda^{\beta a} + 4i \hat{\epsilon} \eta^i_{\alpha} a^i - 4 \hat{\epsilon} \eta^i_{\alpha} a^i, \] (2.10a)
\[ \bar{\xi}^{ai} = \epsilon^a_{\alpha} + \frac{1}{2} \eta^i_{\alpha} a^i + \eta^i_{\alpha} a^i + \Lambda^i_{\alpha} \eta^j_{\beta} \eta^j_{\beta} a^i - 4 \hat{\epsilon} \eta^i_{\alpha} a^i \] (2.10b)

where we have introduced the complex four-vector
\[ \bar{\xi}^a = \bar{\xi}^a + 2i \bar{\xi}^a \sigma^a \hat{\theta}^i, \quad \bar{\xi}^a = \bar{\xi}^a, \] (2.11)

along with the complex bosonic coordinates $y^a = x^a + i \theta^a \sigma^a \hat{\theta}^i$ of the chiral subspace of $\mathbb{R}^{4|8}$. The constant bosonic parameters in (2.10) correspond to the space-
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

time translation ($a^{\alpha \alpha}$), Lorentz transformation ($K^{\alpha}_{\beta}$, $K_{\beta}^{\alpha}$), SU(2)$_R$ transformation ($\Lambda^{ij} = \Lambda^{ji}$), special conformal transformation ($b_{\alpha \beta}$), and combined scale and U(1)$_R$ transformations ($\sigma = \tau - 2i\phi$). The constant fermionic parameters in \eqref{2.10} correspond to the $Q$-supersymmetry ($e^{\alpha}_{\bar{\alpha}}$) and $S$-supersymmetry ($\eta^{\alpha}_{\bar{\alpha}}$) transformations. The constant parameters $K_{\alpha \beta}$, $\Lambda^{ij}$ and $\sigma$ are obtained from $K_{\alpha \beta}[\xi]$, $\Lambda^{ij}[\xi]$ and $\sigma[\xi]$, respectively, by setting $\varepsilon^4 = 0$.

It is useful to introduce a condensed notation for the superconformal parameters

$$\lambda^a = (a^A, K^{ab}, \Lambda^{ij}, \tau, \varphi, b_A), \quad a^A := (a^\alpha, e^\alpha_{\bar{\alpha}}, \xi^i_{\bar{\alpha}}), \quad b_A := (b_a, \eta^i_a, \bar{\eta}^i_{\bar{\alpha}}),$$

as well as for the generators of the superconformal group

$$X_a = (P_A, M_{ab}, I_{ij}, D, Y, K^A), \quad P_A := (P_a, Q^\alpha_{\bar{\alpha}}, \tilde{Q}^\alpha_{\bar{\alpha}}), \quad K^A := (K^a, S^\alpha_{\bar{\alpha}}, \tilde{S}^\alpha_{\bar{\alpha}}).$$

The general conformal Killing supervector field on $\mathbb{C}^{4/2}$,

$$\xi = \xi^a_{\bar{\alpha}}(y, \theta) \frac{\partial}{\partial y^a} + \xi^a_{\bar{\alpha}}(y, \theta) \frac{\partial}{\partial \theta^a_{\bar{\alpha}}} \equiv \xi^a_{\bar{\alpha}} \partial_a \tau + \xi^a_{\bar{\alpha}} \partial_{\bar{\alpha}}^a,$$

may be written in the form:

$$\xi = \lambda^a \tilde{\xi}_a(X) = a^A \tilde{\xi}_a(P) + \frac{1}{2} K^{ab} \xi_{ab}(M) + \Lambda^{ij} \xi_{ij}(J) + \tau \xi(\mathbb{D})$$

$$+ i\varphi \tilde{\xi}(\mathbb{Y}) + b_A \tilde{\xi}^A(K).$$

We read off the relevant supervector fields:

$$\xi_a(P) = \partial / \partial y^a, \quad \xi_{\bar{\alpha}}(P) = \partial_{\bar{\alpha}}, \quad \xi^a_{\bar{\alpha}}(P) = -2i(\tilde{\sigma}^a_{\bar{\alpha}} \theta^\alpha \partial / \partial y^c),$$

$$\xi_{ab}(M) = y_a \partial / \partial y^b - y_b \partial / \partial y^a + (\theta \sigma_{ab}) \gamma^i \partial_i, \quad \xi_{ij}(J) = \theta^{ij} \partial_{\bar{\alpha}},$$

$$\xi(\mathbb{D}) = y^c \partial / \partial y^c + \frac{1}{2} \theta_i^j \partial_i^j, \quad \xi(\mathbb{Y}) = \theta^i_\gamma \partial_i^j, \quad \xi^a(\mathbb{Y}) = 2y^a y^c \partial / \partial y^c - y^a \partial / \partial y^a - (\theta \sigma^a \sigma^c) \gamma^c \partial_c, \quad \xi^i_{\bar{\alpha}}(K) = i(\sigma^a)_{\bar{\alpha}} \gamma_a \partial^i, \quad \xi^a_{\bar{\alpha}}(K) = i(\sigma^a)_{\bar{\alpha}} \gamma_a \partial^i.$$

Making use of the above operators, the graded commutation relations for the superconformal algebra, $[X_a, X_{\bar{b}}] = -f_{ab}^c X_c$, can be derived keeping in mind the relation

$$\xi = \lambda^a \tilde{\xi}_a(X) \rightarrow \delta_\xi = \lambda^a \delta_X^a, \quad [\xi_1, \xi_2] \rightarrow -[\delta_{\xi_1}, \delta_{\xi_2}].$$
2.2 Superconformal algebra

Here we describe the graded commutation relations for the $\mathcal{N} = 2$ superconformal algebra $su(2,2|2)$. We start with the commutation relations for the conformal algebra:

\[
[M_{ab}, M_{cd}] = 2\eta_{[a} M_{b]cd} - 2\eta_{d[a} M_{b]c}, \quad (2.18a)
\]
\[
[M_{ab}, P_c] = 2\eta_{[a} P_{b]c}, \quad [\mathbb{D}, P_a] = P_a, \quad (2.18b)
\]
\[
[M_{ab}, K_c] = 2\eta_{[a} K_{b]c}, \quad [\mathbb{D}, K_a] = -K_a, \quad (2.18c)
\]
\[
[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab}. \quad (2.18d)
\]

The $R$-symmetry generators $\mathbb{Y}$ and $J_{ij}$ commute with all the generators of the conformal group. Amongst themselves, they obey the algebra:

\[
[J_{ij}, J_{kl}] = e^{i(j_i j^l + + e^{i(j_i j^l) k}. \quad (2.19)
\]

The superconformal algebra is then obtained by extending the translation generator to $P_a$ and the special conformal generator to $K^\alpha$. The commutation relations involving the $Q$-supersymmetry generators with the bosonic ones are:

\[
[M_{ab}, Q^\alpha_i] = (\sigma_{ab})^{\gamma} Q^\beta_\delta, \quad [M_{ab}, \bar{Q}^\beta_i] = (\bar{\sigma}_{ab})^{\gamma} \bar{Q}^\delta_i, \quad (2.20a)
\]
\[
[\mathbb{D}, Q^\alpha_i] = \frac{1}{2} Q^\alpha_i, \quad [\mathbb{D}, \bar{Q}^\beta_i] = \frac{1}{2} \bar{Q}^\beta_i, \quad (2.20b)
\]
\[
[\mathbb{Y}, Q^\alpha_i] = Q^\alpha_i, \quad [\mathbb{Y}, \bar{Q}^\beta_i] = -\bar{Q}^\beta_i, \quad (2.20c)
\]
\[
[K^\alpha, Q^\beta_i] = -i(\sigma^a)^\beta_\beta S^a_i, \quad [K^\alpha, \bar{Q}^\beta_i] = -i(\sigma^a)^\beta_\beta S^a_i. \quad (2.20d)
\]

The commutation relations involving the $S$-supersymmetry generators with the bosonic operators are:

\[
[M_{ab}, S^\alpha_i] = -(\sigma_{ab})^{\gamma} S^\beta_i, \quad [M_{ab}, \bar{S}^\alpha_i] = -(\bar{\sigma}_{ab})^{\gamma} \bar{S}^\beta_i, \quad (2.21a)
\]
\[
[\mathbb{D}, S^\alpha_i] = \frac{1}{2} S^\alpha_i, \quad [\mathbb{D}, \bar{S}^\alpha_i] = -\frac{1}{2} \bar{S}^\alpha_i, \quad (2.21b)
\]
\[
[\mathbb{Y}, S^\alpha_i] = -S^\alpha_i, \quad [\mathbb{Y}, \bar{S}^\alpha_i] = \bar{S}^\alpha_i, \quad (2.21c)
\]
\[
[S^\alpha_i, P_b] = i(\sigma_b)^\alpha_\beta P^\beta_i, \quad [\bar{S}^\alpha_i, P_b] = i(\sigma_b)^\alpha_\beta \bar{Q}^\beta_i. \quad (2.21d)
\]

Finally, the anti-commutation relations of the fermionic generators are:

\[
\{Q^\alpha_i, \bar{Q}^\beta_{i'}\} = -2i\delta_i^{i'}(\sigma^b)^\alpha_\beta P_b = -2i\delta_i^{i'}(\bar{Q}^\beta_i, \bar{Q}^\alpha_{i'}), \quad (2.22a)
\]
\[
\{S^\alpha_i, S^\beta_{i'}\} = 2i\delta_i^{i'}(\sigma^b)^\alpha_\alpha K_b = 2i\delta_i^{i'}K^\alpha_{i'}, \quad (2.22b)
\]
\[
\{S^\alpha_i, Q^\beta_{i'}\} = \delta_i^{i'}(\sigma^b)^\alpha_\beta (2\mathbb{D} - \mathbb{Y}) - 4\delta_i^{i'}M^\alpha_{i'\beta} + 4\delta_i^{i'}J^\beta_{i'}, \quad (2.22c)
\]
\[
\{\bar{S}^\alpha_i, \bar{Q}^\beta_{i'}\} = \delta_i^{i'}(\sigma^b)^{\bar{\alpha}}_{\bar{\beta}} (2\mathbb{D} + \mathbb{Y}) + 4\delta_i^{i'}M_{i'\bar{\beta}} - 4\delta_i^{i'}J_{i'}. \quad (2.22d)
\]
where $M_{\alpha\beta} = \frac{1}{2} (\sigma^{ab})_{\alpha\beta} M_{ab}$ and $\tilde{M}_{\alpha\beta} = -\frac{1}{2} (\sigma^{ab})_{\alpha\beta} M_{ab}$. Note that all remaining (anti-)commutators not explicitly listed above vanish identically.

The graded commutation relations (2.18) – (2.22) constitute the $\mathcal{N} = 2$ superconformal algebra, $\text{su}(2,2|2)$. Its generators obey the graded Jacobi identity

$$(-1)^{\varepsilon_\alpha \varepsilon_\beta} \{ X_\alpha, \{ X_\beta, X_\gamma \} \} + \text{(two cycles)} = 0 , \quad (2.23)$$

where $\varepsilon_\alpha = \varepsilon(X_\alpha)$ is the Grassmann parity of the generator $X_\alpha$. Making use of $[X_\alpha, X_\beta] = -f_{\alpha \beta} \varepsilon X_\varepsilon$, the Jacobi identities are equivalently written as

$$f_{\alpha \beta}^{\varepsilon_\beta} f_{\varepsilon_\alpha}^{\varepsilon_\beta} = 0 . \quad (2.24)$$

### 2.3 Superconformal primary multiplets

The superconformal transformation law of a primary tensor superfield (with suppressed indices) is

$$\delta_\xi U = \mathcal{K}[\xi] U ,$$

$$\mathcal{K}[\xi] = \xi + \frac{1}{2} K^{ab} [\xi] M_{ab} + \Lambda^{ij}[\xi] J_{ij} + p \sigma[\xi] + q \bar{\sigma}[\xi] . \quad (2.25)$$

Here the generators $M_{ab}$ and $J_{ij}$ act on the Lorentz and $\text{SU}(2)$ indices of $U$, respectively. The parameters $p$ and $q$ are related to the dimension (or Weyl weight) $w$ and $\text{U}(1)_R$ charge $c$ of $U$ as $p + q = w$ and $p - q = -\frac{1}{2} c$.

As an example, let us consider a chiral tensor superfield $\phi$, $\bar{D}_{\dot{a}}^a \phi = 0$. Requiring it to be primary leads to the conditions

$$\tilde{M}_{\alpha\beta} \phi = 0 , \quad J_{ij} \phi = 0 , \quad q = 0 . \quad (2.26)$$

These conditions imply that (i) $\phi$ can carry only undotted spinor indices; (ii) $\phi$ must be neutral under the group $\text{SU}(2)_R$; and (iii) the dimension $w$ and the $\text{U}(1)_R$ charge $c$ of $\phi$ are related as $c = -2w$. A chiral scalar $W$ is called reduced if it obeys the reality condition

$$D^{\dot{a}} W = \bar{D}^{\dot{a}} \bar{W} , \quad D^{\dot{a}} := D^{\dot{a}}(D_{\dot{a}}) , \quad \bar{D}_{\dot{a}} := \bar{D}_{\dot{a}}(\bar{D}^{\dot{a}}) , \quad (2.27)$$

which uniquely fixes the dimension of $W$ to be $+1$. Chiral multiplets exist both in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric cases. New types of multiplets are offered by $\mathcal{N} = 2$ supersymmetry, as will be discussed below.

An $\mathcal{O}(n)$ multiplet $H^{1: \cdots : n} = H^{1: \cdots : 1}$ obeys the analyticity constraints$^5$

$^5$ The $\mathcal{O}(n)$ multiplets are well-known in the literature on $\mathcal{N} = 2$ supersymmetric field theories. The $n = 1$ case corresponds to the on-shell Fayet-Sohnius hypermultiplet $[30, 80]$. The field strength of the $\mathcal{N} = 2$ tensor multiplet $[81]$ is described by a real $\mathcal{O}(2)$ multiplet $[20, 24, 82]$. 
In the super-Poincaré case, these constraints are consistent with \( \mathcal{H}^{1\cdots i_n} \) carrying Lorentz indices in addition to the SU(2) summation law of \( H \). If this is no longer allowed if \( \mathcal{H}^{1\cdots i_n} \) is a superconformal primary multiplet. Then, the superconformal transformation law of \( H \) is uniquely determined by the constraints (2.28) to be
\[
\delta_\xi \mathcal{H}^{1\cdots i_n} = \xi \mathcal{H}^{1\cdots i_n} + n \Lambda_{i_1}^{(1)} [\xi] \mathcal{H}^{i_1\cdots i_n} + \frac{n}{2} (\sigma[\xi] + \bar{\sigma}[\xi]) \mathcal{H}^{1\cdots i_n} .
\]
(2.29)
In the case that \( n \) is even, \( n = 2m \), this transformation law is compatible with the reality condition \( \mathcal{H}^{1\cdots i_{2m}} = \mathcal{H}_{i_1\cdots i_{2m}} = \epsilon_{i_1 j_1} \cdots \epsilon_{i_{2m} j_{2m}} \mathcal{H}^{j_1\cdots j_{2m}} \).

### 2.4 Superconformal projective multiplets

The constraints (2.28) can be given a more transparent interpretation if one makes use of an isotwistor \( \nu^i \in \mathbb{C}^2 \setminus \{0\} \) that allows one to introduce new spinor covariant derivatives,
\[
D_{\alpha}^{(1)} = v_i D_{\alpha}^i , \quad \bar{D}_{\alpha}^{(1)} = v_i \bar{D}_{\alpha}^i , \quad v_i := e_{ij} v^j ,
\]
(2.30)
and to convert \( \mathcal{H}^{i_1\cdots i_n}(z) \) into an index-free homogeneous polynomial of degree \( n \),
\[
\mathcal{H}^{(n)}(z, v) = v_{i_1} \cdots v_{i_n} \mathcal{H}^{i_1\cdots i_n}(z) .
\]
(2.31)
In accordance with (2.1), the operators (2.30) strictly anticommute with each other,
\[
\{ D_{\alpha}^{(1)}, D_{\beta}^{(1)} \} = \{ \bar{D}_{\alpha}^{(1)}, \bar{D}_{\beta}^{(1)} \} = \{ D_{\alpha}^{(1)}, \bar{D}_{\beta}^{(1)} \} = 0 ,
\]
(2.32)
and annihilate \( \mathcal{H}^{(n)} \),
\[
D_{\alpha}^{(1)} \mathcal{H}^{(n)} = 0 , \quad \bar{D}_{\alpha}^{(1)} \mathcal{H}^{(n)} = 0 .
\]
(2.33)
These constraints do not change if we replace \( \nu^i \to c \nu^i \), with \( c \in \mathbb{C} \setminus \{0\} \equiv \mathbb{C}^* \), in the definition of the operators (2.30) and the superfield (2.31). We see that the isotwistor \( \nu^i \in \mathbb{C}^2 \setminus \{0\} \) is defined modulo the equivalence relation \( \nu^i \sim c \nu^i \), with \( c \in \mathbb{C}^* \), hence it provides homogeneous coordinates for \( \mathbb{C}P^1 \). The superfield (2.31) can be interpreted to be a holomorphic tensor field on the superspace (1.1).

The superconformal transformation law (2.29) can be recast in term of \( \mathcal{H}^{(n)} \). For this it is useful to introduce a new isotwistor \( \nu' \) such that \( \nu' \) and \( \nu' \) form a basis for \( \mathbb{C}^2 \), that is \( (\nu, \nu') := \nu' \nu_i \neq 0 \).

---

General \( \mathcal{O}(n) \) multiplets, with \( n > 2 \), were introduced in [83, 33, 42]. The case \( n = 4 \) was first studied in [82].
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

$$\delta_\xi H^{(n)} = \left( \xi - \Lambda^{(2)}[\xi] \partial^{(-2)} \right) H^{(n)} + n \Sigma[\xi] H^{(n)} . \quad (2.34)$$

Here we have introduced the differential operator

$$\partial^{(-2)} := \frac{1}{(v,u)} u^i \partial \partial u^i , \quad (2.35)$$

as well as the parameters

$$\Lambda^{(2)}[\xi] := v_i v_j \Lambda^{ij}[\xi] , \quad \Lambda^{(0)}[\xi] := \frac{v_i u_j}{(v,u)} \Lambda^{ij}[\xi] ; \quad (2.36a)$$

$$\Sigma[\xi] := \Lambda^{(0)}[\xi] + \frac{1}{2} (\sigma[\xi] + \bar{\sigma}[\xi]) , \quad (2.36b)$$

which have the following properties

$$D_a^{(1)} \Lambda^{(2)}[\xi] = 0 , \quad \bar{D}_a^{(1)} \Lambda^{(2)}[\xi] = 0 ; \quad (2.37a)$$

$$D_a^{(1)} \Sigma[\xi] = 0 , \quad \bar{D}_a^{(1)} \Sigma[\xi] = 0 . \quad (2.37b)$$

The variation (2.34) obeys the analyticity constraints (2.33) due to the identity

$$\left[ \xi - \Lambda^{(2)}[\xi] \partial^{(-2)} , D_a^{(1)} \right] = -K_\alpha \beta \left( \xi \right) D_\beta^{(1)} - \left( \frac{1}{2} \sigma[\xi] + \Lambda^{(0)}[\xi] \right) D_a^{(1)} , \quad (2.38)$$

and a similar relation for $\bar{D}_a^{(1)}$.

The above discussion can be extended to more general superconformal projective multiplets $[55, 84]$. A superconformal projective multiplet of weight $n$, $Q^{(n)}(z,v)$, is a superfield that lives on $\mathbb{R}^{4|8}$ with respect to the superspace variables $z^A$, is holomorphic with respect to the isotwistor variables $v^i$ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterised by the following conditions:

(a) it obeys the analyticity constraints

$$D_a^{(1)} Q^{(n)} = 0 , \quad \bar{D}_a^{(1)} Q^{(n)} = 0 ; \quad (2.39a)$$

(b) it is a homogeneous function of $v$ of degree $n$,

$$Q^{(n)}(z,cv) = c^n Q^{(n)}(z,v) , \quad c \in \mathbb{C}^* ; \quad (2.39b)$$

(c) it possesses the superconformal transformation law

$$\delta_\xi Q^{(n)} = \left( \xi - \Lambda^{(2)}[\xi] \partial^{(-2)} \right) Q^{(n)} + n \Sigma[\xi] Q^{(n)} . \quad (2.39c)$$

By construction, the superfield $Q^{(n)}$ is independent of the isotwistor $u_i$,

$$\partial^{(2)} Q^{(n)} = 0 , \quad \partial^{(2)} := (v,u)^i v^j \partial \partial u^i . \quad (2.40)$$
One may check that the variation $\delta_\xi Q^{(n)}$, eq. (2.39c), is characterised by the same property, $\delta^{(2)}_\xi Q^{(n)} = 0$, due to the homogeneity condition (2.39b).

There exists a real structure on the space of projective multiplets [35, 36, 42]. Given a weight-$n$ projective multiplet $Q^{(n)}(\nu')$, its smile conjugate $\breve{Q}^{(n)}(\nu)$ is defined by

$$Q^{(n)}(\nu') \to \breve{Q}^{(n)}(\bar{\nu}_i) \to \breve{Q}^{(n)}(\bar{\nu}_i \to -\nu_i) =: \breve{Q}^{(n)}(\nu'),$$

with $\breve{Q}^{(n)}(\bar{\nu}_i) := Q^{(n)}(\bar{\nu}')$ the complex conjugate of $Q^{(n)}(\nu')$, and $\bar{\nu}_i$ the complex conjugate of $\nu_i$. One can show that $\breve{Q}^{(n)}(\nu)$ is a weight-$n$ projective multiplet. In particular, $\breve{Q}^{(n)}(\nu)$ obeys the analyticity constraints $\bar{D}^{(1)}_{\alpha} \breve{Q}^{(n)} = 0$ and $\bar{D}^{(1)}_{\dot{\alpha}} \breve{Q}^{(n)} = 0$, unlike the complex conjugate of $Q^{(n)}(\nu)$. One can also check that

$$\breve{Q}^{(n)}(\nu) = (-1)^n Q^{(n)}(\nu).$$

Therefore, if $n$ is even, one can define real projective multiplets, which are constrained by $\breve{Q}^{(2n)} = Q^{(2n)}$. Note that geometrically, the smile-conjugation is complex conjugation composed with the antipodal map on the projective space $\mathbb{CP}^1$.

The $\mathcal{O}^{(n)}$ multiplets, $H^{(n)}(\nu)$, are well defined on the entire projective space $\mathbb{CP}^1$. There also exist important projective multiplets that are defined only on an open domain of $\mathbb{CP}^1$. Before introducing them, let us give a few definitions. We define the north chart of $\mathbb{CP}^1$ to consist of those points for which the first component of $\nu' = (\nu, i\bar{\nu})$ is non-zero, $i\bar{\nu} \neq 0$. The north chart of $\mathbb{CP}^1$ may be parametrised by the inhomogeneous complex coordinate $\xi = i\bar{\nu}/\nu \in \mathbb{C}$. The only point of $\mathbb{CP}^1$ outside the north chart is characterised by $\nu' = (0, i\bar{\nu})$ and describes an infinitely separated point. Thus we may think of the projective space $\mathbb{CP}^1$ as $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The south chart of $\mathbb{CP}^1$ is defined to consist of those points for which the second component of $\nu' = (i\bar{\nu}, \nu)$ is non-zero, $i\nu \neq 0$. The south chart is naturally parametrized by $1/\xi$. The intersection of the north and south charts is $\mathbb{C} \setminus \{0\}$.

An off-shell (charged) hypermultiplet can be described in terms of the so-called arctic weight-$n$ multiplet $\Upsilon^{(n)}(\nu)$ which is defined to be holomorphic in the north chart of $\mathbb{CP}^1$:

$$\Upsilon^{(n)}(\nu) = (i\bar{\nu})^n \Upsilon^{[n]}(\xi), \quad \Upsilon^{[n]}(\xi) = \sum_{k=0}^{\infty} \bar{\nu}_k \xi^k. \tag{2.43}$$

Its smile-conjugate antarctic multiplet $\bar{\Upsilon}^{(n)}(\nu)$, has the explicit form

$$\bar{\Upsilon}^{(n)}(\nu) = (i\bar{\nu})^n \bar{\Upsilon}^{[n]}(\xi) = (i\nu \xi)^n \bar{\Upsilon}^{[n]}(\xi), \quad \bar{\Upsilon}^{[n]}(\xi) = \sum_{k=0}^{\infty} \bar{\nu}_k \frac{(-1)^k}{\xi^{2k}} \tag{2.44}$$

and is holomorphic in the south chart of $\mathbb{CP}^1$. The arctic multiplet can be coupled to a Yang-Mills multiplet in a complex representation of the gauge group [43]. The pair consisting of $\Upsilon^{[n]}(\xi)$ and $\bar{\Upsilon}^{[n]}(\xi)$ constitutes the so-called polar weight-$n$ multiplet.

Our last example is a real tropical multiplet $\Xi^{(2n)}(\nu)$ of weight $2n$ defined by
\[ U(2n) = \left( v^1, v^2, \ldots, v^{2n} \right), \quad U = (-1)^{k} U_{-k}. \] (2.45)

This multiplet is holomorphic in the intersection of the north and south charts of the projective space \( \mathbb{CP}^1 \).

It should be pointed out that the modern projective superspace terminology was introduced in [34].

### 2.5 Non-superconformal projective multiplets

In the original papers [42, 43], general projective multiplets were introduced for the case of \( \mathcal{N} = 2 \) Poincaré supersymmetry, while definition (2.39) corresponds to superconformal projective multiplets. To define the former, the conformal Killing supervector field \( \xi \) in (2.39) should be replaced by a Killing supervector field

\[ \Xi = \Xi^a \partial_a + \Xi_j^\alpha D_j^\alpha + \bar{\Xi}_j^\beta \bar{D}_j^\beta = \Xi. \] (2.46)

By definition, \( \Xi \) is a conformal Killing supervector field such that the parameters (2.7b) and (2.7c) vanish. Its components are obtained from (2.10) by switching several parameters off:

\[ \Xi^a = a^a_{\alpha \alpha} + \bar{K}^a_{\beta} y^\beta_{\alpha} + y^a_{\beta} K^a_{\beta} + 4i \bar{e}^a_{\alpha} \theta^\alpha, \] (2.47a)
\[ \Xi^j_{\alpha} = e^j_{\alpha} + \theta^j_{\beta} K^a_{\beta}, \] (2.47b)

where the complex four-vector \( \Xi^a \) is related to the vector component \( \Xi^a \) in (2.46) by the rule \( \Xi^a = \Xi^a + 2i \Xi \sigma^a \bar{D}^i \). The super-Poincaré transformation law of a weight-n projective multiplet \( Q^{(n)}(z, v) \) is obtained from (2.39c) by replacing \( \xi \rightarrow \Xi \):

\[ \delta_{\Xi} Q^{(n)} = \Xi Q^{(n)}. \] (2.48)

It is seen that the weight \( n \) of \( Q^{(n)} \) becomes irrelevant from the point of view of the Poincaré supersymmetry. In particular, for the arctic (2.43) and antarctic (2.44) multiplets we can use the simplified notation

\[ Y^{(n)}(v) = (\gamma_{\alpha}^n Y(\zeta), \quad \bar{Y}^{(n)}(v) = (\gamma_{\bar{\alpha}}^n \bar{Y}(\zeta). \] (2.49)
3 Rigid supersymmetric sigma models

In order to get a better understanding of the opportunities provided by the projective multiplets, in this section we briefly discuss off-shell $\mathcal{N} = 2$ supersymmetric sigma models in Minkowski superspace. We recall that the target spaces of $\mathcal{N} = 2$ supersymmetric sigma models are hyperkähler manifolds in the super-Poincaré case [85] and hyperkähler cones in the superconformal case [86, 87] (see also [88] for the mathematical framework and [89] for a discussion in dimensions $3 \leq d \leq 6$).

The $\mathcal{N} = 2$ supersymmetric action principle in projective superspace is formulated in terms of a Lagrangian $\mathcal{L}^{(2)}(\zeta, \nu)$ which is a real weight-2 projective superfield. The action is

\[ S := \frac{1}{2\pi} \oint_{\gamma} (v, dv) \int d^4 x D^{(-4)} \mathcal{L}^{(2)}(\zeta, \nu) \bigg|_{\theta = \bar{\theta} = 0}, \quad (v, dv) := v^i dv_i, \quad (3.1) \]

where $\gamma$ denotes a closed integration contour, and $D^{(-4)}$ is the fourth-order differential operator:

\[ D^{(-4)} := \frac{1}{16}(D^{(-1)})^2(D^{(-1)})^2, \quad D^{(-1)} := \frac{u_i D^i}{(v, u)}, \quad D^{(-1)} := \frac{u_i \bar{D}^i}{(v, u)}. \quad (3.2) \]

We recall that $u$ is a fixed isotwistor chosen to be arbitrary modulo the condition $(v, u) \neq 0$ along the integration contour. The action is invariant under arbitrary projective transformations of the form:

\[ (u^i, v^i) \rightarrow (u^i, v^i) \mathcal{R}, \quad \mathcal{R} = \begin{pmatrix} a & 0 \\ b & \epsilon \end{pmatrix} \in \text{GL}(2, \mathbb{C}). \quad (3.3) \]

This gauge-like symmetry implies that the action is independent of $u$, $\delta_u S = 0$. It is also invariant under $\mathcal{N} = 2$ supersymmetry transformations

\[ \delta_{\text{SUSY}} \mathcal{L}^{(2)} = (\epsilon_i^a Q^i_a + \bar{\epsilon}_i^a \bar{Q}^i_a) \mathcal{L}^{(2)}. \quad (3.4) \]

The projective superspace action was originally given in [37] in a form that differs slightly from (3.1). The latter representation appeared first in [90].

The action (3.1) is superconformal if the Lagrangian $\mathcal{L}^{(2)}$ is a superconformal weight-2 projective multiplet, see [55, 84] for the proof. As an example, we consider an off-shell nonlinear $\sigma$-model described by $n$ superconformal weight-1 arctic multiplets $\Upsilon^{(1)}$ and their smile-conjugates $\bar{\Upsilon}^{(1)}$ with Lagrangian [84]

\[ \mathcal{L}^{(2)} = i \mathcal{L}(\Upsilon^{(1)}, \bar{\Upsilon}^{(1)}), \quad (3.5a) \]

\[ 2 \mathcal{L}(\Upsilon^{(1)}, \bar{\Upsilon}^{(1)}) = \left( \frac{\partial}{\partial \Upsilon^{(1)}} + \bar{\Upsilon}^{(1)} \frac{\partial}{\partial \bar{\Upsilon}^{(1)}} \right) \mathcal{L}(\Upsilon^{(1)}, \bar{\Upsilon}^{(1)}). \quad (3.5b) \]

In order for $\mathcal{L}^{(2)}$ to be real, it suffices to choose
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

\[ \mathcal{L}(\bar{\Upsilon}^{(1)}, \tilde{\Upsilon}^{(1)}) = \mathcal{K}(\Upsilon^{(1)}, \tilde{\Upsilon}^{(1)}), \quad \Phi^{(1)} \frac{\partial}{\partial \Phi^{(1)}} \mathcal{K}(\Phi, \tilde{\Phi}) = \mathcal{K}(\Phi, \tilde{\Phi}), \]  

(3.6)

where $\mathcal{K}(\Phi, \tilde{\Phi})$ is a real analytic function of $n$ complex variables $\Phi^{(1)}$ and their complex conjugates $\tilde{\Phi}^{(1)}$. The homogeneity properties of $\mathcal{K}$ imply that it can be interpreted as the Kähler potential of a Kähler cone [88].

The Lagrangian $\mathcal{L}^{(2)}$ in the general case of (3.5) is real if $\mathcal{L}(\Phi, \tilde{\Omega})$ obeys the reality condition $\mathcal{L}(\tilde{\Phi}, -\Omega) = -\mathcal{L}(\Phi, \Omega)$, where $\mathcal{L}(\Phi, \Omega)$ denotes the complex conjugate of $\mathcal{L}(\Phi, \tilde{\Omega})$. A detailed study of the superconformal $\sigma$-model (3.6) was carried out in [84, 91]. That analysis was extended and generalised in [66] to the case of the most general superconformal $\sigma$-model (3.5).

Without loss of generality, we can assume that the integration contour $\gamma$ does not pass through the “north pole” $v^i \sim (0, 1)$. This chart is naturally parametrised by the inhomogeneous complex coordinate $\zeta$ defined by $v = v^i(1, \zeta)$. Since the action is independent of $u_i$, the latter can be chosen to be $u_i = (1, 0)$, such that $(v, u) = v^i \neq 0$. We also represent the Lagrangian in the form:

\[ \mathcal{L}^{(2)}(v, \zeta) = i v^i L \mathcal{L}(z, \zeta) = i(\bar{v})^j \zeta \mathcal{L}(z, \zeta), \quad \mathcal{L} = \mathcal{L}. \]  

(3.7)

Now, the action takes the form:

\[ S = \frac{1}{16} \int \frac{d\zeta}{\gamma} \int d^4 x \zeta \left( (D_{\zeta}^1)^2 (D_{\bar{\zeta}}^1)^2 \mathcal{L}(z, \zeta) \right)_{\theta = 0}. \]  

(3.8)

Finally, the analyticity constraints (2.39a) on $\mathcal{L} \propto \mathcal{L}^{(2)}$ are equivalent to

\[ D_{\tilde{\zeta}}^\alpha \mathcal{L}(\zeta) = \zeta \frac{\partial}{\partial \zeta} \mathcal{L}(\zeta), \quad \tilde{D}_{\bar{\zeta}}^\alpha \mathcal{L}(\zeta) = -\frac{1}{\zeta} \tilde{D}_{\bar{\zeta}}^\alpha \mathcal{L}(\zeta), \]  

(3.9)

hence the action turns into

\[ S = \frac{1}{2 \pi i} \int \frac{d\zeta}{\zeta} \int d^4 z \left. \mathcal{L}(z, \zeta) \right|_{\theta_2 = \theta_{\bar{2}} = 0}, \quad d^4 z := d^4 x d^2 \theta d^2 \bar{\theta}. \]  

(3.10)

Here the integration is carried out over the $\mathcal{N} = 1$ Minkowski superspace with Grassmann coordinates $\theta^\alpha \equiv \theta_2^\alpha$ and $\bar{\theta}_\alpha \equiv \theta_\bar{2}^\alpha$. The action is now formulated entirely in terms of $\mathcal{N} = 1$ superfields. By construction, it is off-shell $\mathcal{N} = 2$ supersymmetric! This is one of the most powerful features of the projective superspace approach. In what follows, we assume that $\theta^\alpha = 0$ and $\bar{\theta}_{\bar{2}} = 0$.

The most general off-shell $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-model in projective superspace [42] can be realised in terms of polar supermultiplets

\[ S[\Upsilon, \tilde{\Upsilon}] = \frac{1}{2 \pi i} \int \frac{d\zeta}{\zeta} \int d^4 \bar{z} \mathcal{L}(\Upsilon^I, \tilde{\Upsilon}^\bar{I}, \zeta), \]  

(3.11)

where the arctic $\Upsilon^I(\zeta)$ and antarctic $\tilde{\Upsilon}^{\bar{I}}(\zeta)$ dynamical variables are generated by an infinite set of ordinary $\mathcal{N} = 1$ superfields:
As follows from the analyticity conditions (2.39a) (see also (3.9)), \( \Phi := \Upsilon_0 \) is chiral, \( \partial_a \Phi = 0 \), \( \Sigma := \Upsilon_1 \) is complex linear, \( \partial^2 \Sigma = 0 \), while the remaining components, \( \Upsilon_2, \Upsilon_3, \ldots \), are unconstrained complex \( \mathcal{N} = 1 \) superfields. The latter superfields are auxiliary, since they appear in the action without derivatives.

Although the \( \sigma \)-model (3.11) was first introduced in 1988 [42], for some ten years it remained a purely formal construction, because the existed no technique to eliminate the auxiliary superfields contained in \( \Upsilon_I \), except in the case of Lagrangians quadratic in \( \Upsilon^I \) and \( \bar{\Upsilon}^{\bar{J}} \). This situation changed in the late 1990s when Refs. [46, 92, 93] identified a subclass of models (3.11) with interesting geometric properties.

They correspond to the special case

\[
\mathcal{L}(\Upsilon^I, \bar{\Upsilon}^{\bar{J}}, \zeta) = K(\Upsilon^I, \bar{\Upsilon}^{\bar{J}}),
\]

(3.13)

where \( K(\Phi^I, \bar{\Phi}^{\bar{J}}) \) is the Kähler potential of a Kähler manifold \( \mathcal{M} \). The Kähler invariance \( K(\Phi, \bar{\Phi}) \rightarrow K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \) of the \( \mathcal{N} = 1 \) supersymmetric \( \sigma \)-model [94],

\[
S[\Phi, \bar{\Phi}] = \int d^4|z| K(\Phi^I, \bar{\Phi}^{\bar{J}}),
\]

(3.14)

turns into

\[
K(\Upsilon, \bar{\Upsilon}) \rightarrow K(\Upsilon, \bar{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\bar{\Upsilon})
\]

(3.15)

for the model

\[
S[\Upsilon, \bar{\Upsilon}] = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta} \int d^4|z| K(\Upsilon^I, \bar{\Upsilon}^{\bar{J}}).
\]

(3.16)

A holomorphic reparametrisation of the Kähler manifold, \( \Phi^I \rightarrow \Phi'^I = f^I(\Phi) \), has the following counterpart \( \Upsilon^I(\zeta) \rightarrow \Upsilon'^I(\zeta) = f^I(\Upsilon(\zeta)) \) in the \( \mathcal{N} = 2 \) case. Therefore, the physical superfields of the \( \mathcal{N} = 2 \) theory, \( \Phi^I \) and \( \Sigma^I \), should be regarded as coordinates of a point in the Kähler manifold and a tangent vector at the same point, respectively. Thus the variables \( (\Phi^I, \Sigma^I) \) parametrise the holomorphic tangent bundle \( T \mathcal{M} \) of the Kähler manifold \( \mathcal{M} \) [46].

We assume that the auxiliary superfields in the model (3.11) have been eliminated. Then, the action (3.11) turns into

\[
S = \int d^4|z| \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}),
\]

(3.17)

for some Lagrangian \( \mathcal{L} \). In the case of the model (3.16), \( \mathcal{L} \) has the form [93]
\[ L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = K(\Phi, \bar{\Phi}) + \sum_{i=1}^{n} \mathcal{L}_{I_1 \cdots I_n}(\Phi, \bar{\Phi}) \Sigma^{I_1} \cdots \Sigma^{I_n} \bar{\Sigma}^{I_1} \cdots \bar{\Sigma}^{I_n}, \quad (3.18) \]

where \( \mathcal{L}_{I_1 \cdots I_n} \) and the coefficients \( \mathcal{L}_{I_1 \cdots I_n} \), for \( n > 1 \), are tensor functions of the Kähler metric \( g_{I_1 I_2} = \partial_I \partial_J K(\Phi, \bar{\Phi}) \), the Riemann curvature \( R_{IJKL}(\Phi, \bar{\Phi}) \) and its covariant derivatives. Each term in the action contains equal powers of \( \Sigma \) and \( \bar{\Sigma} \), since the original model \((3.16)\) is invariant under rigid \( U(1) \) transformations [92]

\[ \Upsilon(\zeta) \mapsto \Upsilon(e^{i\alpha} \zeta) \quad \Leftrightarrow \quad \Upsilon_n(z) \mapsto e^{i\alpha} \Upsilon_n(z). \quad (3.19) \]

To uncover the explicit structure of the hyperkähler target space associated with the \( \sigma \)-model \((3.17)\), we should construct a dual formulation of the theory \((3.17)\), obtained by dualising each complex linear superfield \( \Sigma^I \) and its conjugate \( \bar{\Sigma}^I \) into a chiral–antichiral pair \( \Psi_I \) and \( \bar{\Psi}_I \). In accordance with [42], this is accomplished through the use of the first-order action

\[ S_{\text{first-order}} = \int d^{4|4}z \left\{ L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I \bar{\Sigma}^I + \bar{\Psi}_I \Sigma^I \right\}. \quad (3.20) \]

Here \( \Sigma^I \) is an unconstrained complex superfield, while \( \Psi_I \) is chiral, \( \bar{D}_a \Psi_I = 0 \). This model is equivalent to \((3.17)\). Indeed, varying \( S_{\text{first-order}} \) with respect to \( \Psi_I \) gives \( \bar{D}^2 \Sigma^I = 0 \) and then \((3.20)\) reduces to the original theory, eq \((3.17)\). On the other hand, we can integrate out \( \Sigma^I \)’s and their conjugates using their equations of motion

\[ \frac{\partial}{\partial \Sigma^I} L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I = 0, \quad (3.21) \]

which can be used to express the variables \( \Sigma^I \) and their conjugates in terms of the other superfields, \( \Sigma^I = \Sigma^I(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \). This leads to the dual action

\[ S_{\text{dual}} = \int d^{4|4}z K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \]. \quad (3.22) \]

Since this \( \mathcal{N} = 2 \) supersymmetric \( \sigma \)-model is formulated in terms of chiral \( \mathcal{N} = 1 \) superfields, its Lagrangian \( K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \) is the Kähler potential of a hyperkähler manifold [95] (or simply the hyperkähler potential).

It may be shown [96] that the dual theory \((3.22)\) is invariant under the second supersymmetry transformation

\[ \delta \Phi^I = \frac{1}{2} \bar{D}^2 \left\{ \bar{\epsilon} \theta \frac{\partial K}{\partial \Psi^I} \right\}, \quad \delta \Psi_I = -\frac{1}{2} \bar{D}^2 \left\{ \bar{\epsilon} \theta \frac{\partial \bar{K}}{\partial \bar{\Phi}^I} \right\}. \quad (3.23) \]

These transformation laws follow from the linear supersymmetry \((2.48)\) of the off-shell theory \((3.11)\). If we introduce the condensed notation \( \phi^a := (\Phi^I, \Psi_I) \) and \( \bar{\phi}^{\dot{a}} := (\bar{\Phi}^I, \bar{\Psi}_I) \), as well as the symplectic matrices
\[ y^{ab} = y^{ba} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \] (3.24)

then the supersymmetry transformation (3.23) can be rewritten as
\[ \delta \phi^a = \frac{1}{2} \partial \beta \left\{ \bar{\beta} \gamma^{ab} \frac{\partial \gamma}{\partial \phi^b} \right\}, \] (3.25)

which agrees with the general results of [95]. A remarkable result of Lindström and Roček [44] is the observation that the \( \mathcal{N} = 2 \) superfield Lagrangian in (3.11) can be identified with the generating function of a twisted symplectomorphism.

In the case of the model (3.16), the hyperkähler potential has the form
\[ K(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \ldots I_n} (\Phi, \bar{\Phi}) \Psi_{I_1} \ldots \Psi_{I_n} \bar{\Psi}_{I_n} \] (3.26)

where \( \mathcal{H}^{IJ}(\Phi, \bar{\Phi}) = g^{IJ}(\Phi, \bar{\Phi}) \). By construction, \( (\Sigma^I, \bar{\Sigma}^I) \) is a tangent vector at the point \( (\Phi^I, \bar{\Phi}^I) \) of \( \mathcal{M} \), therefore \( (\Psi_I, \bar{\Psi}_I) \) is a one-form at the same point. The variables \( \phi^a = (\Phi^I, \Psi_I) \) parametrise the holomorphic cotangent bundle \( T^*\mathcal{M} \) of the Kähler manifold \( \mathcal{M} \). The hyperkähler potential (3.26) was computed for all Hermitian symmetric spaces, see [97, 98, 99] and references therein.

To conclude this section, we consider one more example of an off-shell \( \sigma \)-model, introduced in [100]. It is described by several real \( \mathcal{O}(2) \) multiplets \( H^{[2]}(v) \), where \( I = 1, \ldots, n \), which we represent as
\[ H^{[2]}(v) = i(\nabla^2) H^I(\zeta), \quad H^I(\zeta) = \Phi^I + \zeta G^I - \bar{\zeta}^2 \bar{\Phi}^I. \] (3.27)

The action functional is defined as follows
\[ S = -\frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta} \int d^4\zeta \frac{F(H(\zeta))}{\zeta^2} + c.c., \] (3.28)

where \( \gamma \) is a closed contour around the origin, and \( F(z) \) is a holomorphic function of \( n \) variables. In accordance with the analyticity conditions (2.39), the \( \mathcal{N} = 1 \) superfield \( \Phi^I \) is chiral, \( \partial_{\zeta} \Phi^I = 0 \), while the real superfield \( G^I = \bar{G}^I \) is linear, \( \partial_{\zeta} G^I = 0 \).

The contour integral in (3.28) is easy to evaluate if we take into account that
\[ F(H(\zeta)) = F(\Phi^I) + \zeta F_I(\Phi^I) G^I - \zeta^2 \left( F_I(\Phi^I) \bar{\Phi}^I - \frac{1}{2} F_{IJ} G^I G^J \right) + O(\zeta^3). \] (3.29)

Only the \( \zeta^2 \) term in this expression contributes to the contour integral. Thus we get
\[ S[\Phi, \bar{\Phi}, G] = \int d^4\zeta \left\{ K(\Phi, \bar{\Phi}) - \frac{1}{2} G_{IJ}(\Phi, \bar{\Phi}) G^I G^J \right\}, \] (3.30a)

where we have defined
\[ K(\Phi, \bar{\Phi}) = \Phi^I F_I(\Phi) + \Phi^I F_I(\bar{\Phi}) \quad \text{and} \quad g_{IJ}(\Phi, \bar{\Phi}) = F_{IJ}(\Phi) + F_{IJ}(\bar{\Phi}). \] (3.30b)
We can interpret \( K(\Phi, \bar{\Phi}) \) and \( g^{IJ}(\Phi, \bar{\Phi}) \) as the Kähler potential of a Kähler manifold and the corresponding Kähler metric. Each linear superfield \( G^i \) in (3.30a) may be dualised into into a chiral superfield \( \Psi^i \) and its conjugate \( \bar{\Psi}^i \). As a result, the action turns into

\[
S[\Phi, \Psi, \bar{\Phi}, \bar{\Psi}] = \int d^4z \mathbb{K}[\Phi, \Psi, \bar{\Phi}, \bar{\Psi}] ,
\]

(3.31a)

\[
\mathbb{K}[\Phi, \Psi, \bar{\Phi}, \bar{\Psi}] = K(\Phi, \bar{\Phi}) + \frac{1}{2} g^{IJ}(\Phi, \bar{\Phi})(\Psi^I + \bar{\Psi}^I)(\Psi^J + \bar{\Psi}^J) .
\]

(3.31b)

Since the original action (3.28) is \( \mathcal{N} = 2 \) supersymmetric, its dual (3.31a) is also \( \mathcal{N} = 2 \) supersymmetric. Since the latter is formulated in terms of \( \mathcal{N} = 1 \) superfields, (3.31b) is the Kähler potential of a hyperkähler manifold. The correspondence \( K(\Phi, \bar{\Phi}) \rightarrow \mathbb{K}[\Phi, \Psi, \bar{\Phi}, \bar{\Psi}] \) constitutes the so-called rigid c-map [101, 102].

**4 Conformal superspace**

In section 2 we have reviewed a simple approach to obtain the \( \mathcal{N} = 2 \) superconformal algebra by employing the conformal Killing supervector fields of flat superspace. The goal of this section is to construct the gauge theory of the latter, known in the literature as conformal superspace. It was introduced in [68] as a generalisation of the \( \mathcal{N} = 1 \) case analysed in [68]. This approach, which will be reviewed in the present section, is of particular importance as it provides powerful tools to construct manifestly gauge-invariant supergravity actions and to engineer general couplings of supergravity to matter.

### 4.1 Gauging the superconformal algebra in superspace

Conformal superspace is a gauge theory of the superconformal algebra. It can be identified with a pair \((\mathcal{M}^{1|8}, \nabla)\). Here \(\mathcal{M}^{1|8}\) denotes a supermanifold parametrised by local coordinates \(z^M = (x^m, \theta^\mu, \bar{\theta}^\dot{\mu})\), and \(\nabla\) is a covariant derivative associated with the superconformal algebra. We recall that the generators \(X_a\) of the superconformal algebra are given by eq. (2.13). They can be grouped in two disjoint subsets,

\[
X_a = (P_A, \bar{X}_a) , \quad X_{\underline{a}} = (M_{ab}, \mathcal{Y}, I_{ij}, \bar{D}, K^A) ,
\]

(4.1)

each of which constitutes a superalgebra:

\[
[X_a, P_B] = -f^{cB}_{\underline{a}} X_c , \quad [X_{\underline{a}}, \bar{X}_b] = -f^b_{\underline{a}} X_{\underline{c}} , \quad [X_{\underline{a}}, P_B] = -f^{a}_{\underline{b}} X_{\underline{c}} - f_{ab}^{c} P_c .
\]

(4.2a-4.2c)
Here the structure constants $f^{ABC}$ contain only one non-zero component, which is $f^{i}_{\alpha j} c = 2\delta^{i}_{j} (\sigma^{c})^{\alpha}_{\beta}$.

In order to define the covariant derivatives, $\nabla_{A} = (\nabla_{\alpha}, \bar{\nabla}^{\dot{a}}_{\dot{A}})$, we associate with each generator $X_{\alpha} = (M_{\alpha}, Y, J_{\alpha}, \bar{\xi}_{\alpha}, K^{A}) = (M_{\alpha}, Y, J_{\alpha}, \bar{\xi}_{\alpha}, \bar{\xi}_{\alpha}, \bar{\xi}_{\alpha})$ a connection one-form $\omega^{A} = (\Omega^{ab}, \Phi, \theta^{ij}, \bar{\xi}_{\alpha}, \bar{\xi}_{\alpha}) = (\Omega^{ab}, \Phi, \theta^{ij}, \bar{\xi}_{\alpha}, \bar{\xi}_{\alpha})$ which constitutes a new basis for the tangent space at each point $z \in \mathcal{M}^{4|8}$. The supermatrices $E_{A}^{M}$ and $E_{A}^{A}$ satisfy the properties $E_{A}^{M} E_{B}^{M} = \delta_{A}^{B}$ and $E_{A}^{A} E_{A}^{N} = \delta_{N}^{N}$. With respect to the basis $E_{A}^{A}$, the connection is expressed as $\omega^{A} = \bar{E}^{B} \omega_{B}$, where $\omega_{B} = E_{B}^{M} \omega_{M}$, the covariant derivative is given by

$$\nabla_{A} = E_{A} - \omega_{A} X_{\alpha} = E_{A} - \frac{1}{2} \Omega_{\alpha}^{bc} M_{bc} - i \Phi_{A} Y - \Theta_{A} J_{jk} - B_{A} \Xi - \bar{\xi}_{\alpha} \bar{K}^{\alpha}.$$ (4.3)

It can be recast as a super one-form

$$\nabla = d - \omega X_{\alpha}, \quad \nabla = E^{A} \nabla_{A}.$$ (4.4)

The translation generators $P_{\alpha}$ do not show up in (4.3) and (4.4). It is assumed that the operators $\nabla_{A}$ replace $P_{\alpha}$ and obey the graded commutation relations

$$[X_{\alpha}, \nabla_{A}] = - f^{i}_{\alpha} \bar{C}_{\beta} \nabla_{C} - f^{i}_{\alpha} \bar{\xi}_{\beta} X_{\alpha},$$ (4.5)

compare with (4.2c). In particular, the algebra of $K^{A}$ with $\nabla_{B}$ is given by

$$[K^{a}, \nabla_{b}] = 2 \delta^{a}_{b} \Xi + 2 M^{a}_{b},$$ (4.6a)

$$[S^{a}, \nabla^{i}] = \delta^{a}_{b} \delta^{i}_{\beta} (2 \Xi - Y) - 4 \delta^{i}_{\beta} M^{a}_{\beta} + 4 \delta^{i}_{\beta} J^{i}_{\beta},$$ (4.6b)

$$[\bar{S}^{\dot{a}}, \bar{\nabla}^{\dot{i}]} = \delta^{\dot{a}}_{\dot{b}} \delta^{\dot{i}}_{\beta} (2 \Xi + Y) + 4 \delta^{\dot{i}}_{\beta} M_{\beta} - 4 \delta^{\dot{i}}_{\beta} \dot{J}^{\dot{j}}_{\beta},$$ (4.6c)

$$[K^{a}, \bar{\nabla}_{b}] = - i (\sigma^{a})^{\alpha}_{\beta} \bar{\xi}^{\alpha}_{\beta}, \quad [K^{a}, \bar{\xi}^{\alpha}_{\beta}] = - i (\sigma^{a})^{\beta}_{\alpha} \bar{S}^{\alpha}_{\beta},$$ (4.6d)

$$[S^{a}, \bar{\nabla}_{b}] = i (\sigma_{b})^{\alpha}_{\beta} \bar{\xi}^{\alpha}_{\beta}, \quad [\bar{S}^{\dot{a}}, \bar{\nabla}_{b}] = i (\sigma_{b})^{\dot{a}}_{\dot{\beta}} \bar{\xi}^{\dot{a}}_{\dot{\beta}},$$ (4.6e)

where all other graded commutators vanish.

By definition, the gauge group of conformal supergravity is generated by local transformations of the form

$$\delta_{\xi} \nabla_{A} = [\mathcal{X}, \nabla_{A}],$$ (4.7a)

$$\mathcal{X} = \xi^{\beta} X_{\beta} + \Lambda_{\alpha} X_{\alpha},$$

$$= \xi^{\alpha} \bar{X} + \frac{1}{2} K^{\alpha} M_{bc} + \Sigma \Xi + i \rho Y + \theta^{ij} J_{jk} + \Lambda_{A} \bar{K}^{A},$$ (4.7b)
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity where the gauge parameters satisfy natural reality conditions. In applying eq. (4.7), we interpret that

$$\nabla_A \xi^B := E_A \xi^B + \omega_A \xi^D f_D^B,$$

$$\nabla_A \Lambda^b := E_A \Lambda^b + \omega_A \xi^D f_D^b + \omega_A \xi^D f_D^b.$$

Then it follows from (4.7) that

$$\delta K E_A \xi^B = d \xi^A + E_B \Lambda^C f_C^A + \omega^b \xi^D f_D^A + E_B \xi^D f_D^A,$$

$$\delta K \omega^a = d \Lambda^a + \omega^b \Lambda^C f_C^a + \omega^b \xi^D f_D^a + E_B \Lambda^C f_C^a + E_B \xi^D f_D^a.$$

Here we have made use of the graded commutation relations

$$[\nabla_A, \nabla_B] = - \mathcal{T}_{AB} \nabla_C - \mathcal{R}_{AB} \nabla_C,$$

where $\mathcal{T}_{AB}$ and $\mathcal{R}_{AB}$ denote the torsion and the curvature, respectively. They can be recast in terms of two-forms

$$\mathcal{T}^A := \frac{1}{2} E^C \wedge E^B \mathcal{T}_{BC}^A = d E^A - E^C \wedge \omega^b f_D^A,$$

$$\mathcal{R}^a := \frac{1}{2} E^C \wedge E^B \mathcal{R}_{BC}^a = d \omega^a - E^C \wedge \omega^b f_D^a - \frac{1}{2} \omega^c \wedge \omega^d f_E^a.$$

Making use of the graded Jacobi identity

$$0 = (-1)^{fa} [X_a, [\nabla_B, \nabla_C]],$$

we derive the action of $X_a$ on the geometric objects

$$X_a \mathcal{T}_{BC}^D = - (-1)^{fa} (\mathcal{T}_{BC}^E f_E^c - 2 f_B^c \mathcal{E} \mathcal{T}_{[E|C]}^D - 2 f_B^c \mathcal{E} f_C^d),$$

$$X_a \mathcal{R}_{BC}^d = - (-1)^{fa} (\mathcal{T}_{BC}^E f_E^c + \mathcal{R}_{BC}^c f_C^d) - 2 f_B^c \mathcal{E} \mathcal{R}_{[E|C]}^d.$$

The supergravity gauge group acts on a conformal tensor superfield $U$ (with indices suppressed) as

$$\delta U = \mathcal{K} U.$$

The torsion $\mathcal{T}_{AB}$ and the curvature $\mathcal{R}_{AB}$ are conformal tensor superfields. Of special significance are primary superfields. A tensor superfield $U$ (with suppressed indices) is said to be primary if it is characterised by the properties

$$K^A U = 0, \ \mathcal{D} U = w U, \ \mathcal{V} U = c U,$$

for some real constants $w$ and $c$, which are called the dimension (or Weyl weight) and $U(1)_R$ charge of $U$ respectively. From the algebra (2.22b), it is seen that if a
superfield is annihilated by the $S$-supersymmetry generators, then it is necessarily primary.

Let us summarise some important features of the gauging procedure. In curved superspace, the superconformal algebra (4.2) is replaced with

\[
\{X_a, X_b\} = -f_{ab}^c X_c, \quad (4.16a)
\]
\[
\{X_a, \nabla_B\} = -f_{ab}^c \nabla_C - f_{ab}^c X_c, \quad (4.16b)
\]
\[
\{\nabla_A, \nabla_B\} = -T_{ABC} \nabla_C - R_{ABc} X_c. \quad (4.16c)
\]

Here the torsion and curvature tensors obey Bianchi identities which follow from

\[
0 = (-1)^{f_{a}^{c}e_{c}}\{\nabla_A, [\nabla_B, \nabla_C]\} + \text{(two cycles)}. \quad (4.17)
\]

Unlike (4.2), which is determined by the structure constants, the graded commutation relations (4.16) involve structure functions $T_{ABC}$ and $R_{ABc}$.

### 4.2 Conventional constraints for Weyl multiplet

The framework described in the previous subsection defines a geometric set-up to obtain a multiplet of conformal supergravity containing the metric. However, in general, the resulting multiplet is reducible. To obtain an irreducible multiplet it is necessary to impose constraints on the torsion and curvatures appearing in eq. (4.10). This is a standard task in geometric superspace approaches to supergravity, and it is pedagogically reviewed in [76, 103]. One beautiful feature of the construction of [63] is the simplicity of the superspace constraints needed to obtain the Weyl multiplet of conformal supergravity. In fact, to obtain a sufficient set of constraints, one requires the algebra (4.10) to have a Yang-Mills structure. Specifically, one imposes

\[
\{\nabla^i, \nabla^j\} = -2\varepsilon^{ij} \varepsilon^{\alpha\beta} W, \quad \{\bar{\nabla}^i, \bar{\nabla}^j\} = 2\varepsilon_{ij} \varepsilon^{\alpha\beta} \bar{W}, \quad (4.18a)
\]
\[
\{\nabla^i, \bar{\nabla}^j\} = -2i\delta^{ij} \nabla^\alpha, \quad (4.18b)
\]

where the operator $\bar{W}$ is the complex conjugate of $W$. The latter takes the form

\[
W = \frac{1}{2} \mathcal{W}(M)^{ab} M_{ab} + i\mathcal{W}(S)\bar{S} + \mathcal{W}(J)_{ij} J_{ij} + \mathcal{W}(\bar{S}) S + \mathcal{W}(K)_{\alpha} K^\alpha. \quad (4.19)
\]

Having imposed the constraints (4.18a), the Bianchi identities (4.17) become non-trivial and now play the role of consistency conditions which may be used to determine the torsion and curvature. Their solution, up to mass dimension-3/2 is as follows
The conformal superspace algebra is expressed in terms of a single superfield $W_{\alpha \beta} = W_{(\alpha \beta)}$, its conjugate $\bar{W}_{\alpha \beta}$, and their covariant derivatives. This superfield is an $\mathcal{N} = 2$ extension of the Weyl tensor, and is called the super-Weyl tensor. It proves to be a primary chiral superfield of dimension 1,

$$K^C W_{\alpha \beta} = 0 , \quad \bar{\nabla}^j W_{\alpha \beta} = 0 , \quad D W_{\alpha \beta} = W_{\alpha \beta} , \quad \bar{\nabla} W_{\alpha \beta} = -2 W_{\alpha \beta} , \quad (4.21)$$

and it obeys the Bianchi identity

$$\mathcal{B} := \nabla_{\alpha \beta} W^{\alpha \beta} = \bar{\nabla}^{\alpha \beta} W_{\alpha \beta} = \bar{\mathcal{B}} , \quad (4.22a)$$

$$\nabla_{\alpha \beta} := \bar{\nabla}^{\gamma \delta} \nabla_{\gamma \delta} W_{\alpha \beta} = \bar{\nabla}^{\gamma \delta} \bar{\nabla}_{\gamma \delta} W_{\alpha \beta} = \bar{\mathcal{B}} , \quad (4.22b)$$

The real scalar superfield $\mathcal{B}$ is the $\mathcal{N} = 2$ supersymmetric generalisation of the Bach tensor. This super-Bach multiplet proves to be primary, $K^A \mathcal{B} = 0$, carries weight 2, $D^2 \mathcal{B} = 2 \mathcal{B}$, and satisfies the conservation equation [104]

$$\nabla^i \mathcal{B} = 0 \iff \bar{\nabla} j \mathcal{B} = 0 , \quad (4.23a)$$

$$\bar{\nabla}^i := \bar{\nabla}^{\gamma \delta} \bar{\nabla} \gamma \delta \mathcal{B} , \quad \bar{\nabla} j := \bar{\nabla}^{\gamma \delta} \bar{\nabla} \gamma \delta \mathcal{B} . \quad (4.23b)$$

The structure of the conformal superspace algebra leads to highly non-trivial implications. In particular, eq. (4.6c) implies that primary covariantly chiral superfields, $\bar{\nabla}^j U = 0$, can carry neither isospinor nor dotted spinor indices. Given such a superfield, $\phi_{\alpha(n)} := \phi_{\alpha_1 ... \alpha_n} = \phi_{(\alpha_1 ... \alpha_n)}$, eq. (4.6c) further implies that the U(1)$_R$ charge of $\phi_{\alpha(n)}$ is determined in terms of its dimension,

$$K^B \phi_{\alpha(n)} = 0 , \quad \bar{\nabla}^j \phi_{\alpha(n)} = 0 , \quad D \phi_{\alpha(n)} = w \phi_{\alpha(n)} , \quad \bar{\nabla} \phi_{\alpha(n)} = -2w \phi_{\alpha(n)} $$
and thus $c = -2w$.

There is a regular procedure to construct primary chiral multiplets and their conjugate antichiral ones. It is based on the use of operators

\[ \nabla^4 := \frac{1}{48} \nabla^{ij} \nabla_{ij} = -\frac{1}{48} \nabla^{\alpha\beta} \nabla_{\alpha\beta} \quad \bar{\nabla}^4 := \frac{1}{48} \bar{\nabla}^{i\bar{j}} \bar{\nabla}_{i\bar{j}} = -\frac{1}{48} \bar{\nabla}^{\dot{\alpha}\dot{\beta}} \bar{\nabla}_{\dot{\alpha}\dot{\beta}} . \] (4.25)

Let us consider a rank-$n$ spinor superfield $\psi_{\alpha(n)}$ that is $\text{SU}(2)_R$ neutral and has the following superconformal properties:

\[ K^B \psi_{\alpha(n)} = 0 \quad \Box \psi_{\alpha(n)} = (w - 2) \psi_{\alpha(n)} \quad \nabla \psi_{\alpha(n)} = 2(2 - w) \psi_{\alpha(n)} . \] (4.26)

Then its descendant

\[ \phi_{\alpha(n)} = \bar{\nabla}^4 \psi_{\alpha(n)} \] (4.27)

is a primary covariantly chiral superfield of the type (4.24).

### 4.3 Covariant projective multiplets

The concept of rigid superconformal projective multiplets, which was reviewed in subsection 2.3, naturally extends to conformal superspace. The operators (2.30) are replaced with

\[ \nabla^{(1)} = v_i \nabla_i \quad \bar{\nabla}^{(1)} = \bar{v}_i \bar{\nabla}_i , \] (4.28)

which strictly anti-commute with each other due to (4.18). We recall that the rigid superconformal projective multiplet $Q^{(n)}(z,v)$ is defined by the relations (2.39), of which the conditions (2.39a) and (2.39b) trivially extend to conformal superspace,

\[ K^A Q^{(n)} = 0 \quad \nabla^{(1)} Q^{(n)} = 0 \quad \bar{\nabla}^{(1)} Q^{(n)} = 0 , \quad Q^{(n)}(z,cv) = c^n Q^{(n)}(z,v) , \quad c \in \mathbb{C}^\ast , \] (4.29a-29b)

while the rigid superconformal transformation law (2.39c) is replaced with

\[ \delta_x Q^{(n)} = \left( \xi^A \nabla_A + \Lambda^{ij} J_{ij} + \Sigma \Box \right) Q^{(n)} , \] (4.30a)

\[ \Lambda^{ij} J_{ij} Q^{(n)} = -\left( \Lambda^{(2)} \partial^{(2)} - n \Lambda^{(0)} \right) Q^{(n)} . \] (4.30b)

Making use of the graded commutation relations (4.6b) and (4.6c) uniquely fixes the dimension of $Q^{(n)}$

\[ \Box Q^{(n)} = nQ^{(n)} . \] (4.30c)
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

We now list some projective multiplets that can be used to describe superfield dynamical variables. A complex $\mathcal{O}(m)$ multiplet, with $m = 1, 2, \ldots$, is described by a weight-$m$ projective superfield $H^{(m)}(v)$ of the form:

$$H^{(m)}(v) = v_1 \cdots v_m H^{i_1 \cdots i_m}.$$  \hspace{1cm} (4.31a)

The analyticity constraint (2.39b) is equivalent to

$$\nabla^{(i_1}_a H^{i_2 \cdots i_{m+1}} = 0, \quad \bar{\nabla}^{(i_1}_a H^{i_2 \cdots i_{m+1}} = 0.$$  \hspace{1cm} (4.31b)

If $m$ is even, $m = 2n$, we can define a real $\mathcal{O}(2n)$ multiplet obeying the reality condition $\bar{H}^{(2n)} = H^{(2n)}$, or equivalently

$$\bar{H}^{i_1 \cdots i_{2n}} = H_{i_1 \cdots i_{2n}} = \epsilon_{i_1 j_1} \cdots \epsilon_{i_{2n} j_{2n}} H^{j_1 \cdots j_{2n}}.$$  \hspace{1cm} (4.32)

For $n > 1$, the real $\mathcal{O}(2n)$ multiplet can be used to describe an off-shell (neutral) hypermultiplet.

There is a simple construction to generate covariant projective multiplets. It makes use of isotwistor superfields. By definition, a weight-$n$ isotwistor superfield $U^{(n)}(z, v)$ is a primary tensor superfield (with suppressed Lorentz indices) that has the following properties: (i) it is neutral with respect to the group $U(1)_R$; (ii) it is holomorphic with respect to the isospinor variables $v^i$ on an open domain of $\mathbb{C}^2 \setminus \{0\}$; (iii) it is a homogeneous function of $v^i$ of degree $n$,

$$U^{(n)}(cv) = c^n U^{(n)}(v), \quad c \in \mathbb{C} \setminus \{0\};$$  \hspace{1cm} (4.33a)

and (iv) it is characterised by the gauge transformation law

$$\delta U^{(n)} = \left( \xi A \nabla_A + \frac{1}{2} A^{ab} M_{ab} + A^{ij} J_{ij} + \Sigma D \right) U^{(n)};$$

$$J_{ij} U^{(n)} = \left( v_i v_j \partial^{(-2)} - \frac{n}{(v,v)} v_i v_j \right) U^{(n)}.$$  \hspace{1cm} (4.33b)

It is clear that any weight-$n$ projective multiplet is an isotwistor superfield, but not vice versa. The main property in the definition of isotwistor superfields is their transformation rules under $SU(2)_R$. In principle, the definition could be extended to consider non-primary superfields.

Let $U^{(n-4)}$ be a Lorentz-scalar isotwistor superfield such that

$$\mathbb{D} U^{(n-4)} = (n-2) U^{(n-4)}.$$  \hspace{1cm} (4.34)

Then the weight-$n$ isotwistor superfield

$$Q^{(n)} := \nabla^{(4)} U^{(n-4)}$$  \hspace{1cm} (4.35)

satisfies all the properties of a covariant projective multiplet given by eqs. (4.29) and (4.30). Here we have introduced the operator.
\[ \nabla^{(4)} = \frac{1}{16} \nabla^{(2)} \tilde{\nabla}^{(2)}, \quad \nabla^{(2)} = v_i v_j \nabla^{ij}, \quad \tilde{\nabla}^{(2)} = v_i v_j \tilde{\nabla}^{ij}. \] (4.36)

5 Component reduction and the Weyl multiplet

Within the superconformal tensor calculus, the standard Weyl multiplet of conformal supergravity is associated with the local off-shell gauging in spacetime of the superconformal group SU(2,2|2) [7, 14, 15, 16, 17], see also [18, 19] for a review. This multiplet comprises 24 + 24 physical components described by a set of independent gauge fields: the vielbein \( e_m^a \) and a dilatation connection \( b_m \); the gravitino \( (\psi_m^a, \bar{\psi}_m^a) \), associated with the gauging of \( Q \)-supersymmetry; a U(1)_R gauge field \( A_m \); and SU(2)_R gauge fields \( \phi_m^{ij} = \phi_m^{ji} \). The fields associated with the remaining generators of SU(2,2|2), specifically the Lorentz connections \( \omega_m^{cd}, \) S-supersymmetry connection \( \phi_m^{\alpha\beta} \) and the special conformal connection \( \Omega_m \), are composite fields. To ensure that the local superconformal transformations of the standard Weyl multiplet close off-shell it is necessary to add a set of covariant matter fields. These are an anti-symmetric real tensor \( T^{ab} = T^{ba} = T^{ab} + T^{ab} \), which decomposes into its imaginary (anti-)self-dual components \( T^{ab}_\perp \), a real scalar field \( D \), and the fermions \( (\Sigma^{ai}, \bar{\Sigma}_{ai}) \).

As described in the previous section, conformal superspace provides an off-shell gauging of the superconformal group SU(2,2|2) in superspace rather than spacetime. Apart from the fact that \( Q \)-supersymmetry is geometrically realised on superfields in a superspace setting, the conformal superspace and component approaches are very similar. In fact, it is straightforward to reduce the results of 4 from superspace to spacetime and obtain all the details of the standard Weyl multiplet [63].

The identification of the component gauge fields of the standard Weyl multiplet is straightforward. The vielbein \( (e_m^a) \) and gravitini \( (\psi_m^a, \bar{\psi}_m^a) \) appear as the \( \theta = 0 \) projections of the coefficients of \( dx^m \) in the supervielbein \( E^a \) one-form, \( e^a = dx^m e_m^a = E^a \), \( \psi_i^a = dx^m \psi_m^{ai} = 2E^a \), \( \bar{\psi}_i^a = dx^m \bar{\psi}_m^{ai} = 2\bar{E}^a \). (5.1)

Here we have defined the double bar projection of a superform as \( \Omega \equiv \Omega|_{\theta = \bar{\theta} = 0} \). On the other hand, a single bar next to a superfield denotes the usual bar projection \( X \equiv X|_{\theta = 0}. \) The remaining component one-forms are defined as

\[
A := \Phi, \quad \phi^{ij} := \Theta^{ij}, \quad b := B, \quad \omega^{cd} := \Omega^{cd}, \quad \phi^{\alpha\beta} := \bar{\Theta}^{\alpha\beta}, \quad \bar{\phi}^{\gamma}_c := \bar{\Theta}^{\gamma}_c, \quad f_c := \bar{\Theta}_c. \] (5.2)

The covariant matter fields \( T_{ab}, D, \) and \( (\Sigma^{ai}, \bar{\Sigma}_{ai}) \) arise as some of the components of the multiplet described by the super-Weyl tensor \( W_{ab} = (\sigma_{ab})^{\alpha\beta} W_{\alpha\beta} - (\bar{\sigma}_{ab})^{\alpha\beta} \bar{W}_{\alpha\beta} \), which satisfies the constraints (4.21) and (4.22a). In particular, it holds that

\[
T_{ab} := W_{ab}|, \quad D = \frac{1}{12} \nabla^{\alpha\beta} W_{\alpha\beta}| = \frac{1}{12} \bar{\nabla}^{\alpha\beta} \bar{W}_{\alpha\beta}|. \] (5.4a)
Covariant superfield approaches to $N = 2$ supergravity

\[ \Sigma^{\alpha i} = \frac{1}{3} \nabla^{\alpha i} \bar{W}^a \bar{W}^\alpha |, \quad \bar{\Sigma}_{a i} = -\frac{1}{3} \bar{\nabla}^\beta \bar{W}_a |. \]  

(5.4b)

The local superconformal transformations of the gauge fields listed above can be straightforwardly derived by taking the $\theta = 0$ projection of the superspace transformations (4.9). At the same time, the transformations of $T_{ab}$, $D$, and $(\Sigma^{ai}, \bar{\Sigma}_{ai})$ can be obtained by applying the transformation rule for covariant superfields, eq. (4.14) and (4.7), and the definition of the descendant fields in eq. (5.4). The resulting transformation laws are given in [105].

By taking the double bar projection of the superspace covariant derivative one-form $\bar{V}_i$, eq. (4.4), one defines a component vector covariant derivative as follows

\[ D = e^a D_a := \nabla ||, \quad e_m^a D_a = \partial_m - \frac{1}{2} \phi_{m}^{\alpha} \bar{\nabla}^{\alpha} | - \frac{1}{2} \bar{\phi}_{m}^{\alpha} \nabla^{\alpha} | - \frac{1}{2} \omega_{m}^{\alpha} \bar{\nabla}_{\alpha} | \phi_{m}^{\alpha} \bar{\phi}_{m}^{\alpha} M_{cd} - i A_m \bar{\nabla} - \phi_m \bar{\phi}_m J_{kl} - b_m \bar{\nabla} - \frac{1}{2} \phi_m^{\alpha} \bar{\phi}_m^{\alpha} J_{kl} - f_m K^c. \]  

(5.5b)

Provided we appropriately interpret the projected spinor covariant derivatives $\nabla_\alpha |$ and $\bar{\nabla}_\alpha |$ as the generators of $Q$-supersymmetry, $D$ describes a gauging in spacetime of the superconformal group $SU(2,2|2)$, precisely as in [7]. This means that local diffeomorphisms, and all other structure group transformations of the derivatives (5.5), including $Q$-supersymmetry, consistently descend from their corresponding rule in superspace. With this interpretation, the algebra of component covariant derivatives acting on a covariant field is also completely determined by the geometry of conformal superspace. All the component torsions and curvatures are simply the $\theta = 0$ projections of the superspace ones. The algebra of $D_a$ is

\[ [D_a, D_b] = -R(P)_{ab} \bar{D}_c - R(\bar{Q})_{ab} \nabla_\alpha | - R(\bar{Q})_{ab} \bar{\nabla}^{\alpha} | - \frac{1}{2} R(M)_{ab} \bar{\nabla}_{\alpha} | - R(\bar{Q})_{ab} \nabla_\alpha | - R(\bar{Q})_{ab} \bar{\nabla}^{\alpha} | - \frac{1}{2} R(M)_{ab} \bar{\nabla}_{\alpha} | - R(\bar{Q})_{ab} \nabla_\alpha | - R(\bar{Q})_{ab} \bar{\nabla}^{\alpha} | - \frac{1}{2} R(M)_{ab} \bar{\nabla}_{\alpha} | - R(\bar{Q})_{ab} \nabla_\alpha | - R(\bar{Q})_{ab} \bar{\nabla}^{\alpha} | - \frac{1}{2} R(M)_{ab} \bar{\nabla}_{\alpha} | - R(\bar{Q})_{ab} \nabla_\alpha | - R(\bar{Q})_{ab} \bar{\nabla}^{\alpha} | - \frac{1}{2} R(M)_{ab} \bar{\nabla}_{\alpha} |.

(5.6)

By using the commutator of two superspace vector derivatives $\nabla_a$, see [63], one can readily obtain all the component curvatures above. These prove to be determined by the lowest component of the super-Weyl tensor $W_{ab}$ and its descendants. We do not present the results here but stress that the conformal superspace geometry implies the following conditions on the component superconformal curvatures

\[ R(P)_{ab}^c = 0, \]  

(5.7a)

---

6 Given a covariant superfield $U$, and its lowest component $\mathcal{U} = U |$, one defines $Q_a \mathcal{U} = \nabla_a | \mathcal{U} := (\nabla_a U |)$ and $\bar{Q}^a \mathcal{U} = \bar{\nabla}^a | \mathcal{U} := (\bar{\nabla}^a U |)$. The action of the other generators $X_a$ on $\mathcal{U}$ is simply given by $X_a \mathcal{U} := (X_a U |)$.

7 All fields and curvatures introduced so far satisfy natural conjugation properties. We refer the reader to [63] and, in particular, [105] for results in our notation, with the only difference being that the field $W_{ab}$ in [105] is denoted as $T_{ab}$ here.
\[ R(Q)_{\alpha\beta}^{\gamma} (\sigma^b)_{\beta\alpha} = \frac{3}{4} \sum_j (\sigma^b)_{\beta\alpha}, \quad R(\tilde{Q})_{\alpha\beta}^{\gamma} (\tilde{\sigma}^b)_{\gamma\alpha} = \frac{3}{4} \sum_j (\tilde{\sigma}^b)_{\gamma\alpha}, \] (5.7b)

\[ R(M)_{\alpha\beta}^{\gamma} = R(\mathbb{D})_{\alpha\beta} + 3 \eta_{\alpha\beta} D - \eta^{cd} T_{\alpha\beta}^dT_{\alpha\beta}^+ , \] (5.7c)

These are the conventional constraints that render the connections \( \omega_m^{\alpha\beta} \), \( \phi^{i\alpha\beta} \), and \( f^{ma} \) composite. We refer the reader to [63, 105] for the expressions of the composite connections and the superconformal curvatures expressed in terms of the independent physical fields of the standard Weyl multiplet. Note that the conventional constraints (5.7) are not the same as the ones originally employed in [7]. This is not surprising since there is large freedom in the choice of conventional constraints whenever it is necessary to add matter fields to achieve an off-shell representation. Different papers often make different choices. For example, the geometry of [7] is obtained through a shift of the special conformal connection \( f_{ab} K^b \) proportional to \( DK_a \), see [63]. A particularly useful choice of constraints for calculations by using component fields is the “traceless” one employed in [106]

\[ R(P)_{\alpha\beta} = 0, \quad R(M)_{\alpha\beta}^{\gamma} = R(\mathbb{D})_{\alpha\beta} , \] (5.8a)

\[ R(Q)_{\alpha\beta}^{\gamma} (\sigma^b)_{\beta\alpha} = 0, \quad R(\tilde{Q})_{\alpha\beta}^{\gamma} (\tilde{\sigma}^b)_{\gamma\alpha} = 0. \] (5.8b)

6 Other superspace formulations for conformal supergravity

As pointed out in section 1, conformal superspace is not the only superspace setting to describe conformal supergravity. Here we consider two other covariant formulations that have found applications in the recent years, specifically: (i) \( U(2) \) superspace [26, 59]; and (ii) \( SU(2) \) superspace [107, 56]. They differ by their structure groups, which are \( SL(2, \mathbb{C}) \times U(2)_R \) and \( SL(2, \mathbb{C}) \times SU(2)_R \), respectively. Below we describe the relevant “degauging” procedures that lead to these geometries.

6.1 \( U(2) \) superspace

According to (4.7), under an infinitesimal special superconformal gauge transformation \( \delta_x = \Lambda_\beta K^\beta \), the dilatation connection transforms as follows

\[ \delta_x A_\alpha = -2A_\alpha . \] (6.1)

Thus, it is possible to choose a gauge condition \( B_A = 0 \), which completely fixes the special superconformal gauge freedom.\(^8\) As a result, the corresponding connection is no longer required for the covariance of \( \nabla_A \) under the residual gauge freedom and may be extracted from \( \nabla_A \).

\(^8\) There is a class of residual gauge transformations preserving the gauge \( B_A = 0 \). These generate the super-Weyl transformations of \( U(2) \) superspace, see the next subsection.
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

$$\nabla_A = \mathcal{D}_A - \tilde{F}_{AB}K^B. \quad (6.2)$$

Here the operator $\mathcal{D}_A$ involves only the Lorentz and $\text{U}(2)_R$ connections

$$\mathcal{D}_A = E_A - \frac{1}{2} \Omega_A^{bc} M_{bc} - \Phi_A^{k}j^{k} - i\Phi_A Y. \quad (6.3)$$

It obeys the graded commutation relations

$$[\mathcal{D}_A, \mathcal{D}_B] = -\tilde{F}_{AB}^C \mathcal{D}_C - \frac{1}{2} \tilde{R}_{AB}^{cd} M_{cd} - \tilde{R}_{AB}^{kl} j^{kl} - i\tilde{R}_{AB} Y. \quad (6.4)$$

The next step is to relate the special superconformal connection $\tilde{F}_{AB}$ to the torsion tensor of $\text{U}(2)$ superspace. To do this, one can make use of the relation

$$[\nabla_A, \nabla_B] = [\mathcal{D}_A, \mathcal{D}_B] - \left( \mathcal{D}_A \tilde{\mathcal{S}}_{BC} - (-1)^{AB} \mathcal{D}_B \tilde{\mathcal{S}}_{AC} \right) \mathcal{K}^C - \tilde{\mathcal{S}}_{AC}[\mathcal{K}_B, \nabla_B]$$

$$+ (-1)^{AB}\tilde{\mathcal{S}}_{BC}[\mathcal{K}_A, \nabla_A] + (-1)^{BC}\tilde{\mathcal{S}}_{AC}\tilde{\mathcal{S}}_{BD}[\mathcal{K}_D, \mathcal{K}^C]. \quad (6.5)$$

In conjunction with (4.20), this relation leads to a set on consistency conditions that are equivalent to the Bianchi identities of $\text{U}(2)$ superspace [26]. Their solution expresses the components of $\tilde{F}_{AB}$ in terms of the torsion tensor of $\text{U}(2)$ superspace and completely determines the geometry of the $\mathcal{D}_A$ derivatives [63]. Here we will present results only up to mass dimension-3/2. The outcome of the analysis is as follows:

$$\tilde{\mathcal{S}}_{ij}^\alpha = -\frac{1}{2} \varepsilon_{\alpha\beta} S_{ij}^{\beta} + \frac{1}{2} \varepsilon_{ij} Y_{\alpha\beta}, \quad (6.6a)$$

$$\tilde{\mathcal{S}}_{ij}^\beta = \frac{1}{2} \varepsilon_{\alpha\beta} \tilde{S}_{ij}^\alpha + \frac{1}{2} \varepsilon_{ij} \tilde{Y}_{\alpha\beta}, \quad (6.6b)$$

$$\tilde{\mathcal{S}}_{ij} = -\tilde{S}_{ij}^\beta G_{\alpha}^\beta - i\tilde{G}_{ij}^\alpha, \quad (6.6c)$$

$$\tilde{\mathcal{S}}_{ij} = -\frac{1}{2} (\tilde{\delta}_{ij})^\alpha G_{\alpha}^\beta + \frac{1}{12} \varepsilon_{ij} \tilde{Y}_{\alpha\beta} + \frac{1}{3} \tilde{\mathcal{D}}_{(ij}) G_{ij)}^\alpha \tilde{S}_{ij}^{\alpha\beta} \quad (6.6d)$$

$$\tilde{\mathcal{S}}_{ij} = -\frac{1}{2} (\tilde{\delta}_{ij})^\alpha \tilde{G}_{ij}^{\alpha\beta} + \frac{1}{6} \tilde{\mathcal{D}}_{(ij)} \tilde{G}_{ij)}^{\alpha\beta} \tilde{S}_{ij}^{\alpha\beta} \quad (6.6e)$$

$$\tilde{\mathcal{S}}_{ij} = -\frac{1}{2} (\tilde{\delta}_{ij})^{\alpha\beta} \left\{ - \frac{i}{12} \varepsilon_{ij} \tilde{G}_{ij}^{\alpha\beta} S_{ij}^{\alpha\beta} - \frac{1}{4} \tilde{\mathcal{D}}_{(ij)}^{\alpha\beta} Y_{\alpha\beta} + \frac{1}{3} \tilde{\mathcal{D}}_{(ij)}^{\alpha\beta} G_{ij)}^{\alpha\beta} \right\}, \quad (6.6f)$$

$$\tilde{\mathcal{S}}_{ij} = -\frac{1}{2} (\tilde{\delta}_{ij})^{\alpha\beta} \left\{ - \frac{i}{12} \varepsilon_{ij} \tilde{G}_{ij}^{\alpha\beta} S_{ij}^{\alpha\beta} - \frac{1}{4} \tilde{\mathcal{D}}_{(ij)}^{\alpha\beta} Y_{\alpha\beta} + \frac{1}{3} \tilde{\mathcal{D}}_{(ij)}^{\alpha\beta} G_{ij)}^{\alpha\beta} \right\}. \quad (6.6g)$$

The dimension-1 superfields have the following symmetry properties:

$$S_{ij}^\alpha = S_{ji}^\alpha, \quad Y_{\alpha\beta} = Y_{\beta\alpha}, \quad W_{\alpha\beta} = W_{\beta\alpha}, \quad G_{\alpha\alpha} = G_{\alpha\alpha}^\alpha. \quad (6.7)$$
and the reality conditions

\[ \overline{S}^{ij} = \overline{S}_{ij}, \quad \overline{W}_{a\beta} = \overline{W}_{a\beta}, \quad \overline{Y}_{a\beta} = \overline{Y}_{a\beta}, \quad \overline{G}_{\beta a} = G_{a\beta}, \quad \overline{G}_{\beta a}^{ij} = G_{a\beta}^{ij}. \]  

(6.8)

The U(1)_{R} charges of the complex fields are:

\[ \forall S^{ij} = 2\overline{S}^{ij}, \quad \forall Y_{a\beta} = 2\overline{Y}_{a\beta}, \quad \forall W_{a\beta} = -2\overline{W}_{a\beta}. \]  

(6.9)

The algebra obeyed by \( \mathfrak{D}_{A} \) takes the form:

\[ \{ \mathfrak{D}_{a}^{\alpha}, \mathfrak{D}_{b}^{\beta} \} = 4\varepsilon^{\alpha i}M_{a\beta} + 2\varepsilon_{a\beta}e^{ij\gamma Y_{\beta}q}M_{\gamma q} + 2\varepsilon^{ij\gamma}e_{a\beta}\overline{W}_{q\alpha}\overline{Y}_{\gamma q} \]
\[ + 2\varepsilon_{a\beta}e^{ij\gamma l}J_{kl} + 4Y_{a\beta}^{ij}, \]  

(6.10a)

\[ \{ \mathfrak{D}_{a}^{\alpha}, \mathfrak{D}_{b}^{\beta} \} = -2i\delta^{\alpha\beta}M_{a\beta} + 4\left( \delta^{\alpha\beta}G_{a\beta} + iG_{a\beta}^{ij} \right)M_{\alpha\beta} + 4\left( \delta^{\alpha\beta}G_{a\beta} + iG_{a\beta}^{ij} \right)M_{\alpha\beta} \]
\[ + 8G_{a\beta}^{ij}J_{kl} - 4i\delta^{\alpha\beta}G_{a\beta}^{ij}J_{kl} - 2\left( \delta^{\alpha\beta}G_{a\beta} + iG_{a\beta}^{ij} \right)Y_{\alpha\beta}, \]  

(6.10b)

\[ [\mathfrak{D}_{a}, \mathfrak{D}_{b}^{j}] = -i(\mathfrak{D}_{a})^{\alpha\gamma}\left( \delta^{\alpha\gamma}G_{a\beta} + iG_{a\beta}^{ij} \right)\mathfrak{D}_{b}^{j} \]
\[ + \frac{1}{2} \left( (\sigma_{a})^{\beta j} \right)^{\alpha\gamma} \left( \delta^{\alpha\gamma}G_{a\beta} - iG_{a\beta}^{ij} \right)\mathfrak{D}_{b}^{j} \]
\[ - \frac{1}{2} \mathfrak{R}_{a}^{i}e^{i}M_{cd} - 9\mathfrak{R}_{a}^{i}J_{kl} - i\mathfrak{R}_{a}^{j}Y_{\alpha\beta}. \]  

(6.10c)

The dimension-3/2 components of the curvature appearing in (6.10c) are

\[ \mathfrak{R}_{a}^{j}e^{i}M_{cd} = -i(\mathfrak{D}_{a})^{\alpha\beta} \overline{\mathfrak{R}}_{a\beta}^{j} \]
\[ + i(\mathfrak{D}_{a})^{\alpha\beta} \overline{\mathfrak{R}}_{a\beta}^{j} \]
\[ = \frac{1}{2}(\mathfrak{D}_{a})^{\alpha\beta} \left( \delta^{\alpha\beta}G_{a\beta} + iG_{a\beta}^{ij} \right) \mathfrak{D}_{b}^{j} \]
\[ + \frac{1}{2} \mathfrak{R}_{a}^{i}e^{i}M_{cd} - 9\mathfrak{R}_{a}^{i}J_{kl} - i\mathfrak{R}_{a}^{j}Y_{\alpha\beta}. \]  

(6.11a)

\[ \mathfrak{R}_{a}^{j} \]
\[ = \frac{1}{2} \left( (\sigma_{a})^{\beta j} \right)^{\alpha\gamma} \left( \delta^{\alpha\gamma}G_{a\beta} - iG_{a\beta}^{ij} \right) \mathfrak{D}_{b}^{j} \]
\[ + \frac{1}{2} \mathfrak{R}_{a}^{i}e^{i}M_{cd} - 9\mathfrak{R}_{a}^{i}J_{kl} - i\mathfrak{R}_{a}^{j}Y_{\alpha\beta}. \]  

(6.11b)

\[ \mathfrak{R}_{a}^{j} \]
\[ = \frac{1}{2} \left( (\sigma_{a})^{\beta j} \right)^{\alpha\gamma} \left( \delta^{\alpha\gamma}G_{a\beta} + iG_{a\beta}^{ij} \right) \mathfrak{D}_{b}^{j} \]
\[ + \frac{1}{2} \mathfrak{R}_{a}^{i}e^{i}M_{cd} - 9\mathfrak{R}_{a}^{i}J_{kl} - i\mathfrak{R}_{a}^{j}Y_{\alpha\beta}. \]  

(6.11c)

together with their complex conjugates. The right-hand side of (6.11a) involves the dimension-3/2 components of the torsion, which take the form

\[ \overline{\mathfrak{T}}_{a\beta}^{k} = \left( \sigma_{a\beta} \right)^{\alpha\beta} \overline{\mathfrak{T}}_{a\beta}^{k} - \left( \sigma_{a\beta} \right)^{\alpha\beta} \overline{\mathfrak{T}}_{a\beta}^{k}, \]  

(6.12a)

\[ \overline{\mathfrak{T}}_{a\beta}^{k} = \frac{1}{4} \overline{\mathfrak{T}}_{a\beta}^{k} Y_{a\beta} - \frac{1}{3} \overline{\mathfrak{T}}_{a\beta}^{k} Y_{a\beta}, \]  

(6.12b)

\[ \overline{\mathfrak{T}}_{a\beta}^{k} = \frac{1}{4} \overline{\mathfrak{T}}_{a\beta}^{k} W_{a\beta} + \frac{1}{6} \overline{\mathfrak{T}}_{a\beta}^{k} Y_{a\beta}^{k} + \frac{1}{3} \overline{\mathfrak{T}}_{a\beta}^{k} Y_{a\beta}^{k}. \]  

(6.12c)

The consistency conditions arising from solving (6.5) and the constraints satisfied by \( W_{a\beta} \) in conformal superspace lead to the following set of dimension-3/2 Bianchi identities:
Covariant superspace approaches to \( \mathcal{N} = 2 \) supergravity

\[ \mathcal{D}_{\alpha}^{(i} \xi^{jk)} = 0 , \]  
\[ \mathcal{D}_{\alpha}^{i} \bar{W}_{\beta j} = 0 , \]  
\[ \mathcal{D}_{(\alpha}^{i} Y_{\beta j)} = 0 , \]  
\[ \mathcal{D}_{(\alpha}^{ij} G_{\beta j k)} = 0 , \]  
\[ \mathcal{D}_{\alpha}^{(i} S_{jk)} = 0 , \]  
\[ \mathcal{D}_{\alpha}^{i} S_{ij} = - \mathcal{D}_{\alpha}^{i} Y_{\beta j} , \]  
\[ \mathcal{D}_{\alpha}^{i} G_{\beta j} = - \frac{1}{4} \mathcal{D}_{\beta}^{j} Y_{\alpha \beta} + \frac{1}{12} \varepsilon_{\alpha \beta} \mathcal{D}_{\beta}^{j} S^{ij} - \frac{1}{4} \varepsilon_{\alpha \beta} \bar{\mathcal{D}}^{i} \bar{W}_{\gamma \beta} \]
\[ - \frac{i}{3} \varepsilon_{\alpha \beta} \mathcal{D}_{j}^{(i} G_{\gamma k)}^{j) , \]  
and the dimension-2 constraint

\[ (\mathcal{D}_{\alpha} - 4 Y_{\alpha \beta}) W_{\alpha \beta} = (\mathcal{D}_{\alpha} - 4 Y_{\alpha \beta}) \bar{W}_{\alpha \beta} . \]

Here we have made the definitions

\[ \mathcal{D}_{\alpha} = \mathcal{D}_{(\alpha}^{i} \mathcal{D}_{\beta)}^{i} , \quad \tilde{\mathcal{D}}_{\alpha}^{i} = \mathcal{D}_{(\alpha}^{i} \mathcal{D}_{\beta)}^{i} , \]

and it is useful to also define

\[ \mathcal{D}^{ij} = \mathcal{D}^{(i} \mathcal{D}_{\alpha}^{\alpha j)} , \quad \tilde{\mathcal{D}}_{ij} = \tilde{\mathcal{D}}_{(i} \mathcal{D}_{\alpha}^{\alpha j)} . \]

In closing, we note that, upon degauging, relation (4.27) takes the form [58, 108]

\[ \phi_{\alpha (n)} = \left( \frac{1}{96} \mathcal{D}_{i}^{ij} \mathcal{D}_{ij} - \frac{1}{96} \mathcal{D}_{\alpha \beta} \mathcal{D}_{\alpha \beta} + \frac{1}{6} \mathcal{D}_{i}^{ij} \mathcal{D}_{ij} + \frac{1}{6} \bar{Y}_{\alpha \beta} \mathcal{D}_{\alpha \beta} \right) \psi_{\alpha (n)} \]
\[ \equiv \Delta \psi_{\alpha (n)} . \]

### 6.2 The super-Weyl transformations of \( U(2) \) superspace

In the previous subsection we made use of the special conformal gauge freedom to degauge from conformal to \( U(2) \) superspace. The goal of this subsection is to show that residual dilatation symmetry manifests in the latter as super-Weyl transformations.

To preserve the gauge \( B_{A} = 0 \), every local dilatation transformation with parameter \( \Sigma \) should be accompanied by a compensating special conformal one

\[ \mathcal{K}(\Sigma) = \Lambda_{B}(\Sigma) K^{A} + \Sigma \mathcal{D} \implies \delta_{\mathcal{K}(\Sigma)} B_{A} = 0 . \]
\[ \Lambda_\Lambda(\Sigma) = \frac{1}{2} \nabla_\Lambda \Sigma. \] (6.19)

As a result, we define the following transformation
\[ \delta_\Sigma \nabla_A = \delta_\Sigma D_A - \delta_\Sigma \tilde{D}_{AB} K^B = [\mathcal{K}(\Sigma), \nabla_A]. \] (6.20)

By making use of (6.6), one can obtain the following transformation laws for the U(2) superspace covariant derivatives
\[
\begin{align*}
\delta_\Sigma D^i_A &= \frac{1}{2} \Sigma D^i_A + 2(\Sigma D^i_A)M_{\gamma a} - 2(\Sigma D^i_A)J^{k i} - \frac{1}{2}(\Sigma D^i_A)\gamma, \\
\delta_\Sigma \tilde{D}_{ai} &= \frac{1}{2} \Sigma \tilde{D}_{ai} + 2(\Sigma \tilde{D}_{ai})\tilde{M}_{\gamma a} + 2(\Sigma \tilde{D}_{ai})J_{kl} + \frac{1}{2}(\Sigma \tilde{D}_{ai})\gamma, \\
\delta_\Sigma D_{\alpha \alpha} &= \Sigma D_{\alpha \alpha} + \Sigma (\Sigma D_{\alpha \alpha}) + (\Sigma D_{\alpha \alpha})\Sigma + (\Sigma D_{\alpha \alpha})\gamma, \\
\delta_\Sigma G_{\alpha \dot{\alpha} i j} &= \Sigma G_{\alpha \dot{\alpha} i j} + \frac{1}{4} [\Sigma D_{\alpha \dot{\alpha}} M_{ij}], \quad \delta_\Sigma G_{\alpha \dot{\alpha} i j} = \Sigma G_{\alpha \dot{\alpha} i j} + \frac{1}{4} [\Sigma D_{\alpha \dot{\alpha}} M_{ij}].
\end{align*}
\] (6.21)

The dimension-1 components of the torsion transform as
\[
\begin{align*}
\delta_\Sigma W_{\alpha \beta} &= \Sigma W_{\alpha \beta}, \\
\delta_\Sigma Y_{\alpha \beta} &= \Sigma Y_{\alpha \beta} - \frac{1}{2} \Sigma D_{\alpha \beta} \Sigma, \\
\delta_\Sigma S_{ij} &= \Sigma S_{ij} - \frac{1}{2} \Sigma D_{ij} \Sigma, \\
\delta_\Sigma G_{\alpha \dot{\alpha}} &= \Sigma G_{\alpha \dot{\alpha}} - \frac{1}{8} [\Sigma D_{\alpha \dot{\alpha}} M_{\gamma \delta}] \Sigma, \\
\delta_\Sigma G_{\alpha \dot{\alpha} i j} &= \Sigma G_{\alpha \dot{\alpha} i j} + \frac{1}{4} [\Sigma D_{\alpha \dot{\alpha}} M_{ij}] \Sigma.
\end{align*}
\] (6.22)

### 6.3 SU(2) superspace

It can be proven that the torsion \( G_{\alpha \dot{\alpha} i j} \) of U(2) superspace is a pure gauge degree of freedom [26, 59]. One can use super-Weyl gauge freedom (6.22e) to choose
\[
G_{\alpha \dot{\alpha}}^{ij} = 0. \quad (6.23)
\]

In this gauge, it is natural to introduce new covariant derivatives \( \mathcal{D}_A \) defined by
\[
\mathcal{D}_A = \mathcal{D}_A - i G_a \gamma. \quad (6.24)
\]

Making use of (6.10), we find that they obey the graded commutation relations
\[
\begin{align*}
\{ \mathcal{D}_A, \mathcal{D}_B \} &= 4S^{ij}M_{\alpha \beta} + 2\varepsilon^{ij} \epsilon_{\alpha \beta} \gamma^\delta M_{\gamma \delta} + 2\varepsilon^{ij} \epsilon_{\alpha \beta} \gamma^\delta M_{\gamma \delta} \\
&\quad + 2\varepsilon_{\alpha \beta} \varepsilon^{ij} S^{kl} J_{kl} + 4Y_{\alpha \beta} P_{ij}. 
\end{align*}
\] (6.25)
formation it may be gauged away. The corresponding connection is flat. Hence, by performing an appropriate local transformation upon the replacement \( \mathcal{O}_A \rightarrow \tilde{\mathcal{O}}_A \) and imposing \( (6.23) \). By examining equations \( (6.25) \) we see that the U(1)_R curvature has been eliminated and therefore the corresponding connection is flat. Hence, by performing an appropriate local U(1)_R transformation it may be gauged away

\[ \tilde{\Phi}_A = 0 . \]  (6.26)

As a result, the gauge group reduces to SL(2, \( \mathbb{C} \)) \( \times \) SU(2)_R and the superspace geometry is the so-called SU(2) superspace of \([107, 56]\). It turns out that the gauge conditions \( (6.23) \) and \( (6.26) \) allow for residual super-Weyl transformations, which are described by a parameter \( \Sigma \) constrained by

\[ [\delta_{ab}, \delta_{ij}^k]\Sigma = 0 . \]  (6.27)

The general solution of this condition is \([56]\)

\[ \Sigma = \frac{1}{2} (\sigma + \bar{\sigma}) , \quad \tilde{\delta}_i^a \sigma = 0 , \quad \forall \sigma = 0 , \]  (6.28)

where the parameter \( \sigma \) is covariantly chiral, with zero U(1)_R charge, but otherwise arbitrary. To preserve the gauge condition \( \Phi_A = 0 \), every super-Weyl transformation, see \( (6.21a) \) and \( (6.21b) \), must be accompanied by the following compensating U(1)_R transformation

\[ \delta \mathcal{O}_A = [\rho \mathcal{O}_A, \mathcal{O}_A] , \quad \rho = \frac{i}{4} (\sigma - \bar{\sigma}) . \]  (6.29)

As a result, the SU(2) geometry is left invariant by the following set of super-Weyl transformations \([56]\):

\[ \delta_{\sigma} \sigma^i = \frac{1}{2} \sigma \delta_{ab} \sigma_{ab} + (\delta^i_\sigma) \mathcal{M}_{a \bar{a}} - (\sigma_{a \bar{a}}) \mathcal{J}^{k_1} , \]  (6.30a)

\[ \delta_{\sigma} \tilde{\mathcal{O}}_{ai} = \frac{1}{2} \sigma \delta_{ab} \tilde{\mathcal{O}}_{ab} + (\delta_{a i}^\sigma) \mathcal{M}_{a \bar{a}} + (\tilde{\mathcal{O}}_{a i}^\sigma) \mathcal{J}^{k_1} , \]  (6.30b)
\[
\delta_\sigma \mathcal{D}_a = \frac{1}{2}(\sigma + \bar{\sigma})\mathcal{D}_a + \frac{i}{4}(\sigma_a)^{\alpha \beta}(\mathcal{D}_a^{\alpha \beta})\mathcal{D}_a^{\alpha \beta} + \frac{i}{4}(\sigma_a)^{\alpha \beta}(\mathcal{D}_a^{\alpha \beta})\mathcal{D}_a^{\alpha \beta}
\]

(6.30c)

\[\delta_\sigma S^{ij} = \bar{\sigma}S^{ij} - \frac{1}{4}S^{ij}\sigma,\]

(6.30d)

\[\delta_\sigma Y_{\alpha \beta} = \bar{\sigma}Y_{\alpha \beta} - \frac{1}{4}Y_{\alpha \beta}\sigma,\]

(6.30e)

\[\delta_\sigma W_{\alpha \beta} = \sigma W_{\alpha \beta},\]

(6.30f)

\[\delta_\sigma G_{\alpha \beta} = \frac{1}{2}(\sigma + \bar{\sigma})G_{\alpha \beta} - \frac{i}{4}G_{\alpha \beta}(\sigma - \bar{\sigma}).\]

(6.30g)

Here we have made use of the definitions

\[
\mathcal{D}_a = \mathcal{D}_{(\alpha \beta)}^i, \quad \mathcal{D}^{ij} = \mathcal{D}^{(i j)},
\]

(6.31)

and it is useful to also define

\[
\mathcal{D}^i_{\alpha \beta} = \mathcal{D}_{(i \alpha \beta)}^i, \quad \mathcal{D}^{ij} = \mathcal{D}_{(i j)}^{(\alpha \beta)}.
\]

(6.32)

Due to these transformations, SU(2) superspace provides a geometric description of the Weyl multiplet of \(\mathcal{N} = 2\) conformal supergravity \([56]\). It should be emphasised that the algebra of covariant derivatives (6.25) was derived originally by Grimm \([107]\). However, no discussion of super-Weyl transformations was given in \([107]\).

Let us fix a background curved superspace \((\mathcal{M}^4, \mathcal{D})\). A supervector field \(\xi = \xi^B E_B\) on this superspace is called conformal Killing if there exist a Lorentz parameter \(K^B[\xi]\), SU(2)\(_R\) parameter \(\Lambda^{ij}[\xi]\) and a chiral super-Weyl parameter \(\sigma[\xi]\) such that

\[
\left[\xi^B \mathcal{D}_B + \frac{1}{2}K^{bc}[\xi]M_{bc} + \Lambda^{ij}[\xi]J_{ij}, \mathcal{D}_A\right] + \delta_\sigma[\xi] \mathcal{D}_A = 0.
\]

(6.33)

In other words, the coordinate transformation generated by \(\xi\) is accompanied by certain Lorentz, SU(2)\(_R\) and super-Weyl transformations such that the superspace geometry does not change. It can be shown that the equation (6.33) uniquely determines the spinor components of \(\xi^B = (\xi^B, \bar{\xi}^B, \bar{\xi}^\dagger)\) and the parameters \(K^B[\xi], \Lambda^{ij}[\xi]\) and \(\sigma[\xi]\) in terms of \(\xi^B\), and the latter obeys the equation

\[
\mathcal{D}^i_{(\alpha \beta)} \bar{\xi}^\dagger_{(\beta)} = 0 \iff \mathcal{D}^i_{(\alpha \beta)} \bar{\xi}^\dagger_{(\beta)} = 0.
\]

(6.34)

The set of all conformal Killing supervector fields on \((\mathcal{M}^4, \mathcal{D})\) constitutes the superconformal algebra of \((\mathcal{M}^4, \mathcal{D})\). Given a super-Weyl invariant theory on \((\mathcal{M}^4, \mathcal{D})\) described by primary superfields \(U\), its action is invariant under the superconformal transformations

\[
\delta_\xi U = \mathcal{X}[\xi]U,
\]
Covariant superspace approaches to \( \mathcal{N} = 2 \) supergravity

\[
\mathcal{K}[\xi] = \xi^B \partial_B + \frac{1}{2} K^{bc}[\xi] M_{bc} + \Lambda^{ij}[\xi] J_{ij} + p \sigma[\xi] + q \bar{\sigma}[\xi],
\]

(6.35)

for an arbitrary conformal Killing supervector field \( \xi \). In the case that \( (\mathcal{M}^{4|8}, D) \) coincides with Minkowski superspace, \( (\mathcal{M}^{4|8}, D) \), the superconformal Killing equation (6.33) is equivalent to (2.6) and the transformation law (6.35) to (2.25).

7 Superconformal action principles

To construct supergravity-matter systems, a locally superconformal action principle is required. Here we review three types of superconformal actions in \( \mathcal{N} = 2 \) supergravity that have played important roles in the literature.

7.1 Full superspace action

The simplest locally superconformal action involves a full superspace integral:

\[
S[\mathcal{L}] = \int d^{4|8} z E \mathcal{L}, \quad d^{4|8} z := d^4 x d^4 \theta d^4 \bar{\theta}, \quad E := \text{Ber}(E_M^A),
\]

(7.1)

where \( \mathcal{L} \) is a primary real dimensionless scalar Lagrangian,

\[
K^A \mathcal{L} = 0, \quad \bar{\mathcal{L}} = \mathcal{L}, \quad \Box \mathcal{L} = 0.
\]

(7.2)

As an example, we consider a superconformal higher-derivative \( \sigma \)-model with action [104, 109, 110]

\[
S = \int d^{4|8} z E \mathcal{K}(X^I, \bar{X}^\bar{I}), \quad K^A X^I = 0, \quad \bar{\nabla}_a^\alpha X^I = 0, \quad \Box X^I = 0
\]

(7.3)

where \( \mathcal{K} \) is the Kähler potential of a Kähler manifold. The action is locally superconformal. It is also invariant under Kähler transformations

\[
\mathcal{K}(X, \bar{X}) \rightarrow \mathcal{K}(X, \bar{X}) + \Lambda(X) + \bar{\Lambda}(\bar{X}),
\]

(7.4)

with \( \Lambda(X) \) an arbitrary holomorphic function.

7.2 Chiral action

More general is the chiral action, which involves an integral over the chiral subspace

\[
\mathcal{K}(X, \bar{X}) \rightarrow \mathcal{K}(X, \bar{X}) + \Lambda(X) + \bar{\Lambda}(\bar{X}),
\]

(7.4)
Here $\mathcal{E}$ is a suitably chosen chiral measure, and $\mathcal{L}_c$ is a primary covariantly chiral Lagrangian of dimension $+2$,

$$K^A \mathcal{L}_c = 0\, , \quad \bar{\nabla}^A \mathcal{L}_c = 0\, , \quad \mathcal{D} \mathcal{L}_c = 2 \mathcal{L}_c \, .$$  \hspace{1cm} (7.6)

The precise definition of $\mathcal{E}$ in conformal superspace is somewhat technical \([63]\). In SU(2) superspace, $\mathcal{E}$ was obtained by making use of normal coordinates \([58]\).

A different definition of $S_c$ exists, which is based on the use of a primary complex superfield $\Upsilon$ with the following superconformal properties (for some constant $w$):

$$K^A \Upsilon = 0 \, , \quad \mathcal{D} \Upsilon = (w - 2) \Upsilon \, , \quad \Upsilon^A = 2(2 - w) \Upsilon \, .$$  \hspace{1cm} (7.7)

such that $\bar{\nabla}^A \Upsilon$ is nowhere vanishing, that is $(\bar{\nabla}^A \Upsilon)^{-1}$ exists. Specifically, the chiral action may be identified with the functional

$$S_c[\mathcal{L}_c] = \int \! d^{4\!|8}z \, \frac{\Upsilon}{\bar{\nabla}^A \Upsilon} \mathcal{L}_c \, ,$$  \hspace{1cm} (7.8)

which possesses the two fundamental properties: (i) it is locally superconformal under the conditions (7.6); and (ii) it is independent of $\Upsilon$,

$$\delta \Upsilon \int \! d^{4\!|8}z \, \frac{\Upsilon}{\bar{\nabla}^A \Upsilon} \mathcal{L}_c = 0 \, ,$$  \hspace{1cm} (7.9)

for an arbitrary variation $\delta \Upsilon$. Using the representation (7.8) for the chiral action (7.5), it holds that

$$\int \! d^{4\!|8}z \, \mathcal{E} \mathcal{L}_c = \int \! d^{4}x \, d^{4}\theta \, \mathcal{E} \mathcal{L}_c \, , \quad \mathcal{L}_c = \bar{\nabla}^A \Upsilon \mathcal{L}_c \, .$$  \hspace{1cm} (7.10)

There is an alternative definition of the chiral action that follows from the superform approach to the construction of supersymmetric invariants \([111, 112, 113]\). It is based on the use of the following super 4-form \([114]\):

$$\Xi_4 = -4 E^i_{\dot{\beta}} \wedge E_j^i \wedge E_k^\dot{\alpha} \wedge E_l^\dot{\alpha} \mathcal{L}_c - 2 E^i_{\dot{\beta}} \wedge E_j^\dot{\alpha} \wedge E_k^\dot{\alpha} \wedge E^a (\tilde{\sigma}_a)^{\dot{\alpha} \dot{\alpha} \dot{\beta} \dot{\beta}} \nabla \mathcal{L}_c$$

$$- i \frac{1}{2} E^i_{\dot{\beta}} \wedge E_j^i \wedge E^b \wedge E^a (\tilde{\sigma}_{ab})^{\dot{\alpha} \dot{\beta} \dot{\beta} \dot{\beta}} \nabla \mathcal{L}_c$$

$$- i \frac{1}{4} E^i_{\dot{\alpha} \dot{\beta} \dot{\beta} \dot{\beta}} \wedge E^\dot{\alpha} \wedge E^\dot{\beta} \wedge E^a (\sigma_{ab})_{\dot{\alpha} \dot{\beta} \dot{\beta} \dot{\beta}} \nabla \mathcal{L}_c$$

$$- i \frac{1}{36} \epsilon_{abcd} E^i_{\dot{\alpha} \dot{\beta} \dot{\beta} \dot{\beta}} \wedge E^j \wedge E^b \wedge E^a (\sigma_{ij})_{\dot{\alpha} \dot{\alpha} \dot{\alpha} \dot{\alpha}} \nabla \mathcal{L}_c$$

$$- i \frac{1}{24} \epsilon_{abcd} E^j \wedge E^b \wedge E^a (\nabla_i + W_{\dot{\alpha} \dot{\alpha}} W_{\dot{\alpha} \dot{\beta}})^{-1} \mathcal{L}_c \, .$$  \hspace{1cm} (7.11)

This superform is closed,
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

\[ d\Xi_4 = 0. \]  

(7.12)

It proves to be primary\(^9\)

\[ K^\bar{\beta}\Xi_4 = 0. \]  

(7.13)

The chiral action (7.5) can be recast as an integral of $\Xi_4$ over a spacetime $\mathcal{M}^4$,

\[ S_c[\mathcal{L}_c] = \int_{\mathcal{M}^4} \Xi_4, \]  

(7.14a)

where $\mathcal{M}^4$ is the bosonic body of the curved superspace $\mathcal{M}^{4|4}$ obtained by switching off the Grassmann variables. It turns out that (7.14a) leads to the following representation [63] (see also [64]):

\[ S_c = \int d^4x e \left( \nabla^4 + \bar{W}^{\bar{\alpha}\bar{\beta}}W_{\alpha\beta} - \frac{i}{12} \bar{\psi}_{d\bar{\delta}} \left( (\sigma^{d})^{\hat{\beta} \hat{\alpha}} \nabla_{\hat{\alpha}} - 6(\sigma^{d})_{\alpha\alpha} \bar{W}^{\bar{\alpha}} \nabla_{\alpha} \right) \right) + \frac{1}{4} \bar{\psi}_k \bar{\psi}_l \left( (\sigma^{cd})^{\hat{\gamma} \hat{\delta}} \nabla_{\hat{\delta}} - \frac{i}{2} \epsilon^{\hat{\delta} \hat{\gamma}} \epsilon_{\hat{\delta}} (\sigma^{cd})_{\beta \gamma} \nabla_{\hat{\beta}} \right) - \frac{1}{4} \epsilon^{abcd} (\sigma^{0})_{\hat{\beta} \hat{\alpha}} \bar{\psi}_i \bar{\psi}_j \bar{\psi}_k \bar{\psi}_l \nabla_{\hat{\alpha} \hat{\beta}} \nabla_{\hat{\gamma} \hat{\delta}} \nabla_{\hat{\delta}} \nabla_{\hat{\gamma}} - \frac{i}{4} \epsilon^{abcd} \bar{\psi}_i \bar{\psi}_j \bar{\psi}_k \bar{\psi}_l \nabla_{\hat{\alpha} \hat{\beta}} \nabla_{\hat{\gamma} \hat{\delta}} \nabla_{\hat{\delta}} \nabla_{\hat{\gamma}} \right) \mathcal{L}_c |_{\theta = 0}, \]  

(7.14b)

where $e := \det(e_m^a)$. This result agrees with the action of a chiral multiplet coupled to conformal supergravity [115].

### 7.3 Projective action

Consider a Lagrangian $\mathcal{L}^{(2)}$ that is a real weight-2 projective multiplet. Associated with $\mathcal{L}^{(2)}$ is the action

\[ S[\mathcal{L}^{(2)}] = \frac{1}{2\pi} \int_{\gamma} (v, dv) \int d^4z E \frac{\gamma^{(n)}(z)}{\gamma^{(4)}} \mathcal{L}^{(2)}, \]  

(7.15)

where $\gamma^{(n)}(z, v)$ is a primary weight-$n$ isotwistor superfield and the operator $\nabla^{(4)}$ is defined in (4.36). This action proves to have the following fundamental properties: (i) it is locally superconformal; and (ii) it is independent of $\gamma^{(n)}$,

\[ \delta_{\gamma^{(n)}} \int_{\gamma} (v, dv) \int d^4z E \frac{\gamma^{(n)}(z)}{\gamma^{(4)}} \mathcal{L}^{(2)} = 0. \]  

(7.16)

In the $n = 0$ case we can specialise $\gamma^{(0)}$ to be $W_0 \bar{W}_0$, where $W_0$ is the chiral field strength of a vector multiplet, see section 8.1, such that the descendant

\(^9\) The superform may be degauged to SU(2) superspace. Then the condition (7.13) is equivalent to the super-Weyl invariance of $\Xi_4$. 

\[ \Sigma^{ij} := \frac{1}{4} \nabla^{ij} W_0 = \frac{1}{4} \bar{\nabla}^{ij} \bar{W}_0 \]  
(7.17)

is nowhere vanishing, that is \((\Sigma^{ij} \Sigma_{ij})^{-1}\) exists. Then (7.15) turns into [59]

\[ S[L^{(2)}] = \frac{1}{2\pi} \oint_{\gamma} (v, dv) \int d^4z E \frac{W_0 \bar{W}_0}{(\Sigma_0^{(2)})^2} L^{(2)}. \]  
(7.18)

An important remark is in order. In the case of Minkowski superspace, it may be seen that the definition of the projective action (7.15) is equivalent to (3.1).

There is a remarkable relationship between the projective and the chiral actions [58, 57] derived originally in SU(2) superspace. It makes use of the vector multiplet introduced above. For every chiral Lagrangian \( L_c \) with the properties (7.6), the chiral action

\[ S_{\text{chiral}} = \int d^4x d^4\theta \delta \mathcal{L}_c + \text{c.c.} \]  
(7.19)

can be represented as a projective action

\[ S_{\text{chiral}} = \frac{1}{2\pi} \oint_{\gamma} (v, dv) \int d^4z E \frac{W_0 \bar{W}_0}{(\Sigma_0^{(2)})^2} L_c^{(2)}, \]

\[ L_c^{(2)} = -\frac{1}{4} V \left( \tilde{V}^{(2)} \frac{\mathcal{L}_c}{W} + \tilde{V}^{(2)} \frac{\bar{\mathcal{L}}_c}{\bar{W}} \right) \equiv V \Theta^{(2)}. \]  
(7.20)

Here \( V(z, v) \) is a tropical prepotential for the vector multiplet with the chiral field strength \( W_0 \), see the next section.

On the other hand, the projective action (7.18) can be rewritten as a special chiral action [58]

\[ S[L^{(2)}] = \int d^4x d^4\theta \delta W_0 \mathbb{W}, \quad \mathbb{W} = \frac{1}{8\pi} \oint (v, dv) \tilde{V}^{(-2)} \left( \frac{\mathcal{L}^{(2)}}{\Sigma_0^{(2)}} \right), \]  
(7.21)

with the operator \( \tilde{V}^{(-2)} \) being defined in (8.8). The composite superfield \( \mathbb{V} \) can be interpreted as a tropical prepotential for the vector multiplet described by the reduced chiral superfield \( \mathbb{W} \).

An important example of a dynamical system described by the projective action is provided by the off-shell sigma model (3.5), in which \( \Upsilon^{(1)} \) and \( \bar{\Upsilon}^{(1)} \) are now covariant arctic and antarctic multiplets, respectively. This most general locally superconformal sigma model was studied in detail in [66], where its component reduction was worked out.
8 Vector and tensor multiplets

Of special importance in $\mathcal{N} = 2$ supersymmetry are vector and tensor multiplets. Here we review their fundamental properties in the framework of conformal superspace.

8.1 Vector multiplet

In rigid supersymmetry, the off-shell $\mathcal{N} = 2$ vector multiplet was formulated by Grimm, Sohnius and Wess [38]. In conformal superspace, it can be described by a field strength $W$, which has the superconformal properties

$$K^A W = 0, \quad D W = W, \quad \bar{\nabla}^\alpha W = 0 \quad (8.1a)$$

and satisfies the Bianchi identity

$$\Sigma^{ij} := \frac{1}{4} \nabla^{ij} W = \frac{1}{4} \bar{\nabla}^{ij} \bar{W}. \quad (8.1b)$$

Covariantly chiral scalars satisfying the reality condition (8.1b) are called reduced chiral. The constraint (8.1b) uniquely determines the dimension of $W$.

There are several ways to realise $W$ as a gauge invariant field strength. One possibility is to introduce a curved superspace extension of Mezincescu’s prepotential [116] (see also [117]), $V_{ij} = V_{ji}$, which is a primary unconstrained real SU(2) triplet of dimension $-2$. The expression for $W$ in terms of $V_{ij}$ [118] is

$$W = \frac{1}{4} \bar{\nabla}^4 \nabla^{ij} V_{ij}, \quad (8.2)$$

where the chiral operator $\bar{\nabla}^4$ is defined in (4.25). It may be shown that that $V_{ij}$ is defined only up to gauge transformations of the form

$$\delta V^{ij} = \nabla^a \Lambda^{ij}_{\alpha a} + \bar{\nabla}^a \bar{\Lambda}^{a kj}_{\alpha i j}, \quad \Lambda^{ij}_{\alpha a} = \Lambda_{\alpha (kij)}, \quad \bar{\Lambda}^{a kj}_{\alpha i j} := \bar{\Lambda}^{a kj}_{\alpha i j}, \quad (8.3)$$

with the primary gauge parameter $\Lambda^{ij}_{\alpha a}$ being completely arbitrary modulo the algebraic condition given. The superconformal properties of $\Lambda^{ij}_{\alpha a}$ are determined by those of $V^{ij}$.

Let us show how Mezincescu’s prepotential for the vector multiplet can be introduced within standard superspace. For this a simple generalisation of the rigid supersymmetric analysis in [117] can be used. One begins with the first-order action

$$S = \frac{1}{4} \int d^4 x d^4 \theta \mathcal{L} \mathcal{W} + \text{c.c.} - \frac{1}{8} \int d^4 \bar{z} \mathcal{E} \left( \mathcal{W} \nabla^{ij} V_{ij} - \bar{\mathcal{W}} \bar{\nabla}^{ij} V_{ij} \right), \quad (8.4)$$
where $\mathcal{W}$ is a covariantly chiral superfield, and $V^{ij} = V^{ji}$ is an unconditional real SU(2) triplet acting as a Lagrange multiplier. Varying (8.4) with respect to $V_{ij}$ gives $\mathcal{W} = W$, where $W$ obeys the Bianchi identity (8.1b). As a result, the second term in (8.4) drops out and we end up with the $\mathcal{N} = 2$ super-Maxwell action

$$ S = \frac{1}{4} \int d^4x d^4\theta \, \mathcal{W} W + \text{c.c.} \quad (8.5) $$

On the other hand, because the action (8.4) is quadratic in $W$, we may easily integrate $W$ out using its equation of motion

$$ W = i W_D, \quad W_D := \frac{1}{4} \bar{\nabla}^4 \bar{\nabla}^{ij} V_{ij}. \quad (8.6) $$

This leads to the dual action

$$ S = \frac{1}{4} \int d^4x d^4\theta \, \mathcal{W} D W_D + \text{c.c.} \quad (8.7) $$

The dual field strength $W_D$ must be both reduced chiral and given by (8.6).

Within the curved projective-superspace approach of [56, 59, 57], the constraints on $W$ can be solved in terms of a covariant real weight-0 tropical prepotential $V(v^i)$, $\bar{V} = V$. The solution [58] is

$$ W = \frac{1}{8\pi} \int_{\gamma} (v, dv) \bar{\nabla}^{(-2)} V(v), \quad \bar{\nabla}^{(-2)} := \frac{1}{(v, u)^2} u_i u_j \bar{\nabla}^{ij}. \quad (8.8) $$

where $\gamma$ is an appropriately chosen contour. We recall that $v^i \in \mathbb{C}^2 \setminus \{0\}$ denotes the homogeneous coordinates for $\mathbb{C}P^1$. The right-hand side of the expression for $W$ involves a constant isotwistor $u_i$, which is chosen to obey the constraint $(v, u) := v^i u_i \neq 0$, but otherwise is completely arbitrary. Using the analyticity constraints (4.29) obeyed by $V$, one can check that $W$ is invariant under arbitrary projective transformations (3.3). The field strength (8.8) proves to be invariant under gauge transformations

$$ V \rightarrow V + \lambda + \bar{\lambda}, \quad (8.9) $$

where the gauge parameter $\lambda(v)$ is a covariant weight-0 arctic multiplet.

It is worth discussing how the Mezincescu prepotential $V_{ij}$ emerges within projective superspace, see [118] for more details. One begins with the expression for $W$ in terms of $V(v)$, eq. (8.8). In accordance with eq. (4.34), the analyticity conditions on $V$ may be solved in terms of an unconstrained isotwistor superfield $U^{(-4)}$, which is real under smile-conjugation

$$ V(v) = \frac{1}{16} \bar{\nabla}^{(2)} \bar{\nabla}^{(2)} U^{(-4)}(v) = \frac{1}{16} \bar{\nabla}^{(2)} \bar{\nabla}^{(2)} U^{(-4)}(v). \quad (8.10) $$

Using this construction, one may rewrite $W$ as
\[ W = \frac{1}{128\pi} \oint \gamma (v, \text{d}v) \tilde{\nabla}^{(-2)} \nabla^{(2)} U^{(-4)} = \frac{1}{8\pi} \tilde{\nabla}^4 \oint \gamma (v, \text{d}v) \nabla^{(2)} U^{(-4)} \] (8.11)

where the chiral operator \( \tilde{\nabla}^4 \) is defined in (4.25). This may be rewritten as

\[ W = \frac{1}{8\pi} \tilde{\nabla}^4 \nabla^{ij} \oint \gamma (v, \text{d}v) v^i v^j U^{(-4)}(v) = \frac{1}{4} \tilde{\nabla}^4 \nabla^{ij} V_{ij} , \] (8.12)

where we have defined the Mezincescu prepotential

\[ V_{ij} = \frac{1}{2\pi} \oint \gamma (v, \text{d}v) v^i v^j U^{(-4)}(v) . \] (8.13)

Given a system of \( n \) Abelian vector multiplets with chiral field strengths \( W_I \), let \( \mathcal{F}(W) \) be a holomorphic function of degree +2,

\[ W_I \frac{\partial}{\partial W_I} \mathcal{F}(W) = 2 \mathcal{F}(W) . \] (8.14)

Then the following action

\[ S = \int d^4x d^4\theta \mathcal{F}(W) + \text{c.c.} \] (8.15)

is locally superconformal. The component reduction of this model was described by Butter and Novak [64], and their results agree with [17]. The model has led to the notion of special Kähler geometry [119], see [18] for a review. A rigid supersymmetric limit of (8.15) corresponds to rigid special Kähler geometry [120, 121].

### 8.2 Tensor multiplet

In rigid supersymmetry, the massless \( \mathcal{N} = 2 \) tensor multiplet was introduced by Wess [81]. It was rediscovered by de Wit and van Holten [5]. The tensor multiplet can be described in conformal superspace by its gauge invariant field strength \( G^{ij} \), which is a real \( \mathcal{O}(2) \) multiplet. It obeys the constraints

\[ \nabla^{(i} G^{jk)} = \tilde{\nabla}^{(i} G^{jk)} = 0 , \] (8.16)

which generalise those given in [20, 24, 82]. These constraints are solved in terms of a chiral prepotential \( \Psi \) with the superconformal properties

\[ K^A \Psi = 0 , \quad \bar{D} \Psi = \Psi , \quad \bar{\nabla}^i \Psi = 0 . \] (8.17)

The solution to the tensor multiplet constraints was given in [117, 25, 122, 123]. In conformal superspace the solution is
The chiral prepotential is invariant under gauge transformations

\[ \Psi \rightarrow \Psi + i \Lambda \]  

where the gauge parameter \( \Lambda \) is a reduced chiral superfield with the properties (8.1).

Consider a system of \((n + 1)\) tensor multiplets, \( n > 0 \), and let \( G_I^{(2)} \) be their gauge-invariant field strengths, \( I = 0, 1, \ldots, n \). Its dynamics can be described by a Lagrangian of the form

\[ \mathcal{L}^{(2)} = \mathcal{L}(G_I^{(2)}), \quad G_I^{(2)} \frac{\partial}{\partial G_I^{(2)}} \mathcal{L} = \mathcal{L} \]  

where \( \mathcal{L} \) is a real homogeneous function of degree +1. Of special significance is the special choice of \( \mathcal{L} \) defined by the Lagrangian

\[ \mathcal{L}^{(2)} = \frac{1}{iG_0^{(2)}} \left( \mathcal{F}(G_I^{(2)}) - \bar{\mathcal{F}}(\bar{G_I}^{(2)}) \right) . \]  

Here and \( \mathcal{F}(\zeta') \) is a holomorphic homogeneous function of second degree, \( \mathcal{F}(c\zeta') = c^2 \mathcal{F}(\zeta') \). This model provides a manifestly supersymmetric description of the c-map \([101, 102]\). The rigid c-map is described by the model (3.28).

For a single tensor multiplet there is only one superconformal model, which is described by the Lagrangian

\[ \mathcal{L}^{(2)}_{\text{IT}} = -G^{(2)} \ln \frac{G^{(2)}}{i \Upsilon^{(1)} \bar{\Upsilon}^{(1)}} , \]  

with \( \Upsilon^{(1)} \) a weight-one arctic multiplet (both \( \Upsilon^{(1)} \) and its smile-conjugate \( \bar{\Upsilon}^{(1)} \) are pure gauge degrees of freedom). It describes an improved tensor multiplet. Historically, the improved tensor multiplet was independently constructed in the following works (submitted to the journal Nuclear Physics within a one day time difference): (i) Ref. [37] provided its construction in terms of \( \mathcal{N} = 1 \) superfields in the rigid supersymmetric case; and (ii) and Ref. [15] proposed this multiplet within the \( \mathcal{N} = 2 \) superconformal tensor calculus. The rigid supersymmetric version of (8.22) was proposed in the first projective-superspace paper [37].

The improved tensor multiplet can be coupled to weight-0 polar hypermultiplets. The corresponding locally superconformal \( \sigma \)-model [57] is

\[ \mathcal{L}^{(2)} = -G^{(2)} \ln \frac{G^{(2)}}{i \Upsilon^{(1)} \bar{\Upsilon}^{(1)}} + G^{(2)} K(\Upsilon, \bar{\Upsilon}) , \]  

where the Kähler potential is the same as in (3.13). The rigid supersymmetric limit of this \( \sigma \)-model was studied in [125]. The above \( \sigma \)-model has a dual formulation in terms of polar hypermultiplets:
Covariant superspace approaches to $\mathcal{N} = 2$ supergravity

\[ \mathcal{L}^{(2)} = i \tilde{\mathcal{F}}^{(1)} \mathcal{F}^{(1)} \mathcal{F}^{(1)} . \]  

(8.24)

The locally $\mathcal{N} = 2$ superconformal $\sigma$-models (8.23) and (8.24) have a striking resemblance to their $\mathcal{N} = 1$ counterparts, see e.g. [126].

8.3 Linear multiplet action

The linear multiplet action is a BV-type superconformal invariant based on the Lagrangian

\[ \mathcal{L}^{(2)} = V G^{(2)} . \]  

(8.25)

There are three equivalent representations for the linear multiplet action:

\[ S[V G^{(2)}] = \int d^4 x d^4 \theta \delta W \Psi + \text{c.c.} = \int d^4 \varepsilon E V_{ij} G^{ij} . \]  

(8.26)

The action is invariant under the gauge transformations for the vector and tensor multiplets. The invariance under (8.9) follows from the identity

\[ S[(\lambda + \bar{\lambda}) G^{(2)}] = 0 , \]  

(8.27)

where $\lambda$ is an arctic multiplet. The invariance under (8.19) follows from the Bianchi identity (8.1b).

We have seen that every chiral action can be represented as a projective action, eq. (7.20). On the other hand every projective action can be recast as a chiral action, eq. (7.21). These results show that the linear multiplet action (8.26) is universal.

8.4 Composite reduced chiral superfields

The above discussion has an important implication. Given a composite real weight-0 tropical multiplet $V$, the following descendant

\[ \mathcal{W} = \frac{1}{8\pi} \int_{\gamma} (v, dv) \bar{\nabla} (-2) \Psi(v) \]  

(8.28)

is a primary reduced chiral superfield, with $\gamma$ being an appropriately chosen contour. This observation has been used in [118] to derive a number of composite reduced chiral superfields.

Our first example is

\[ \Psi = \ln \frac{G^{(2)}}{i \bar{\mathcal{F}}^{(1)} \mathcal{F}^{(1)}} . \]  

(8.29)
It can be seen that the arctic multiplet \( \Upsilon^{(1)} \) and its conjugate \( \bar{\Upsilon}^{(1)} \) do not contribute to the contour integral, and so they will be ignored below. Evaluating the contour integral in (8.28) gives

\[
\mathbb{W} := -\frac{G}{8} \bar{\nabla}_{ij} \left( \frac{G^{ij}}{G^2} \right), \quad G := \sqrt{\frac{1}{2} G^{ij} G_{ij}},
\]

(8.30)

see [118] for the technical details. This composite multiplet was discovered originally (in a different but equivalent form) in [15] using the superconformal tensor calculus. It was later reconstructed in curved superspace by Müller [123] with the aid of the results in [15] and [90]. Its contour origin was explored in the globally supersymmetric case by Siegel [90].

Our second example is

\[
\mathbb{V} = \frac{H^{(2n)}}{(G^{ij})^n},
\]

(8.31)

where \( H^{(2n)} \) is a real \( O(2n) \) multiplet, see eqs. (4.31) and (4.32). Evaluating the contour integral in (8.28) gives

\[
\mathbb{W}_n = -\frac{(2n)!}{2^{2n+2} (n+1)! (n-1)!} G \bar{\nabla}_{ij} \mathcal{R}^{ij}_n,
\]

(8.32)

where

\[
\mathcal{R}^{ij}_n = \left( \delta^{ij}_{kl} - \frac{1}{2 G^2} G^{ij} G_{kl} \right) H^{kl_{i_1 \cdots i_{2n-2}} G_{i_1 i_2} \cdots G_{i_{2n-3} i_{2n-2}} G^{-2n}.
\]

(8.33)

The expression for \( \mathbb{W}_n \) has an overall structure similar to (8.30), except the argument \( \mathcal{R}^{ij}_n \) of the derivative is much more complicated.

### 9 Off-shell formulations for supergravity

Within the conformal approach to locally supersymmetric theories [9], Poincaré and AdS supergravity may be realised as conformal supergravity coupled to a compensating multiplet. Two compensating massless multiplets are typically required in the case of \( \mathcal{N} = 2 \) supergravity, see [15, 18] for comprehensive discussions. In this section we describe two off-shell formulations for \( \mathcal{N} = 2 \) supergravity.
9.1 Supergravity with vector and tensor multiplet compensators

The minimal formulation for \( \mathcal{N} = 2 \) supergravity with vector and tensor compensators [15] admits a simple superspace description. Using the techniques developed above, the gauge-invariant supergravity action can be written as

\[
S_{\text{SUGRA}} = \frac{1}{\kappa^2} \int d^4x d^4\theta \mathcal{S} \{ \Psi W - \frac{1}{4} W^2 + \bar{\Psi} \Psi W \} + \text{c.c.}
\]

where \( \kappa \) is the gravitational constant, \( \bar{\Psi} \) the cosmological constant, \( W \) is given by the expression (8.30), and \( V_{ij} \) is the Mezincescu prepotential. Within the projective-superspace approach of [56, 57, 59], this action is equivalently given by (7.15) with the following Lagrangian [57]

\[
\kappa^2 S_{\text{SUGRA}}^{(2)} = \mathcal{G}^{(2)} \ln \frac{\mathcal{G}^{(2)}}{i \Gamma^{(1)} \bar{\Gamma}^{(1)}} - \frac{1}{2} V \Sigma^{(2)} + \bar{\Psi} V G^{(2)},
\]

with \( V \) the tropical prepotential for the vector multiplet, and \( \Gamma^{(1)} \) a weight-one arctic multiplet (both \( \Gamma^{(1)} \) and its smile-conjugate \( \bar{\Gamma}^{(1)} \) are pure gauge degrees of freedom). The fact that the vector and the tensor multiplets are compensators means that their field strengths \( W \) and \( G^{ij} \) should possess non-vanishing expectation values, that is \( W \neq 0 \) and \( G \equiv \sqrt{\frac{1}{2} G^{ij} G_{ij}} \neq 0 \).

The equation of motion for the gravitational superfield [127, 118] is

\[
G - W \bar{W} = 0, \quad (9.3a)
\]

and it is consistent with the conditions \( W \neq 0 \) and \( G \neq 0 \). The equations of motion for the compensators are [118]

\[
\Sigma^{ij} - \bar{\Psi} G^{ij} = 0, \quad (9.3b)
\]

\[
\Psi W + \bar{\Psi} W = 0. \quad (9.3c)
\]

The equations (9.3b) and (9.3c) can be degauged to U(2) superspace, which results in the following equations

\[
\frac{1}{4} ( \mathcal{D}^{ij} + 4 \bar{S}^{ij} ) W = \frac{1}{4} ( \bar{\mathcal{D}}^{ij} + 4 \bar{S}^{ij} ) \bar{W} = \bar{\Psi} G^{ij}, \quad (9.4a)
\]

\[
\frac{G}{8} ( \mathcal{D}^{ij} + 4 \bar{S}^{ij} ) \frac{G_{ij}}{G^2} = \bar{\Psi} W. \quad (9.4b)
\]

The super-Weyl and local U(1)\( R \) gauge freedom can be used to impose the gauge condition \( W = \bar{W} = 1 \). The integrability conditions of these constraints are \( \mathcal{D}_A W = \mathcal{D}_A \bar{W} = 0 \) which imply
\[ G_{\alpha \alpha}^{ij} = 0, \quad \Phi_\alpha = G_\alpha. \quad (9.5) \]

After employing the redefinitions \( \mathcal{D}_\alpha = \mathcal{D}_\alpha^i \) and \( \mathcal{D}_a = \mathcal{D}_a - i G_a \bar{\Upsilon} \), the resulting geometry coincides with SU(2) superspace for which the U(1)_R connection is pure gauge and can be set to zero. The equation of motion (9.3a) implies \( G = 1 \) and \( \mathcal{D}_A G = 0 \). The latter can be shown to imply the condition \( \mathcal{D}_A G^{ij} = 0 \), which breaks the local SU(2)_R to a residual U(1) subgroup. It consists of those transformations that keep \( G^{ij} \) invariant. Integrability of the constraint \( \mathcal{D}_A G^{ij} = 0 \) implies

\[ Y_{\alpha \beta} = 0, \quad G_{\alpha \alpha} = 0, \quad S_{(i}^j G_{j)k} = 0, \quad \bar{S}_{(i}^j G_{j)k} = 0. \quad (9.6) \]

The remaining supergravity equations (9.4) turn into

\[ S^{ij} = \bar{S}^{ij} = \xi G^{ij}, \quad S^2 := \frac{1}{2} S^{ij} S_{ij} = \xi^2, \quad \mathcal{D}_A S^{ij} = 0. \quad (9.7) \]

All the remaining information about the dynamics of supergravity is encoded in the super-Weyl tensor \( W_{\alpha \beta} \).

A maximally supersymmetric solution of (9.7) is characterised by the condition \( W_{\alpha \beta} = 0 \) and the resulting superspace geometry is uniquely determined to be

\[ \{ \mathcal{D}_\alpha, \mathcal{D}_\beta^i \} = 4 S^{ij} M_{i\alpha \beta} + 2 \varepsilon_{\alpha \beta} \varepsilon^{ij} S_{kl} J_{kl}, \quad \{ \mathcal{D}_\alpha^i, \mathcal{D}^\alpha_j \} = -2 i \delta^{ij} (\sigma^e)_{\alpha \beta} \mathcal{D}_e, \quad (9.8a) \]

\[ [\mathcal{D}_a, \mathcal{D}_b^j] = \frac{1}{2} (\sigma_a)_{\beta \gamma} S^k \mathcal{D}^\gamma_k, \quad [\mathcal{D}_a, \mathcal{D}_b] = -S^2 M_{ab}. \quad (9.8b) \]

This geometry corresponds to the four-dimensional \( \mathcal{N} = 2 \) AdS superspace

\[ \text{AdS}^{4|8} = \frac{\text{OSp}(2|4)}{SO(3,1) \times SO(2)}. \quad (9.9) \]

The most general \( \mathcal{N} = 2 \) supersymmetric nonlinear \( \sigma \)-models in AdS4 were studied in [128, 129, 130, 131]. They have important distinct features as compared with the \( \mathcal{N} = 2 \) supersymmetric nonlinear \( \sigma \)-models in Minkowski space. Specifically, the target space must be a non-compact hyperkähler manifold endowed with a Killing vector field which generates an SO(2) group of rotations on the two-sphere of complex structures.

As a generalisation of (9.2), we consider the model for matter-coupled supergravity [57]

\[ \kappa^2 \mathcal{L}^{(2)} = -\frac{1}{2} V \Sigma^{(2)} + G^{(2)} \left( \frac{G^{(2)}}{i \bar{\Upsilon}^{(1)} e^{-\xi \bar{\Upsilon}^{(1)}}} + \kappa^2 K(Y^I, \bar{Y}^J) \right), \quad (9.10) \]

where \( K(\Phi, \bar{\Phi}) \) is the Kähler potential of a Kähler manifold, and \( Y^I \) are covariant weight-0 arctic multiplets.
9.2 Supergravity with vector and hyper multiplet compensators

The compensators for this supergravity formulation are a vector multiplet and a polar hypermultiplet. The Lagrangian has the form

\[ \kappa^2 \mathcal{L}^{(2)}_{\text{SUGRA}} = -\frac{1}{2} V \Sigma^{(2)} - i \tilde{T}^{(1)} e^{-\xi V^I Y^{(1)}}, \]  

(9.11)

with \( \xi \) a cosmological constant. The action is invariant under the gauge transformations

\[ \delta \lambda V = \lambda + \tilde{\lambda}, \quad \delta \lambda Y^{(1)} = \xi \lambda Y^{(1)}. \]  

(9.12)

The equation of motion for \( \tilde{T}^{(1)} \) implies that

\[ e^{-\xi V^I (\xi)} Y^{(1)}(v) = Y^I v_i, \quad V^I (\xi) = \frac{1}{2} V_0 + \sum_{k=1}^{\infty} V_k \xi^k, \]  

(9.13)

where \( Y^I \) obeys the equations

\[ \Sigma_{i j} Y^{(1)} = 0, \quad \tilde{\Sigma}_{i j} Y^{(1)} = 0, \]  

(9.14)

which defines the on-shell Fayet-Sohnius hypermultiplet. Here \( \Sigma_A \) denotes the gauge and conformal covariant derivative, which is obtained form \( V_A \) by adding a \( U(1) \) connection. We point out that \( V^I (\xi) = V_+ (\xi) + V_- (\xi) \), where \( V_- (\xi) \) is the smile-conjugate of \( V_+ (\xi) \). The equation of motion for \( V \) is

\[ \Sigma^{(2)} - \xi i \tilde{T}^{(1)} e^{-\xi V^I Y^{(1)}} = 0 \iff \Sigma^{ij} + \xi i \tilde{T}^{(1)} Y^{(1)} = 0. \]  

(9.15)

Finally, the equation of motion for the gravitational superfield \( H \) is (see \([79, 132]\) for the derivation)

\[ \bar{W} W - \frac{1}{2} \tilde{T}^I Y^I = 0. \]  

(9.16)

The supergravity-matter system (9.10) has a dual formulation [84] described by the Lagrangian

\[ \kappa^2 \mathcal{L}^{(2)} = -\frac{1}{2} V \Sigma^{(2)} - i \tilde{Y}^{(1)} e^{-\xi V - \kappa^2 K(Y, \tilde{Y}) Y^{(1)}}. \]  

(9.17)

If the cosmological constant vanishes, \( \xi = 0 \), this supergravity-matter system turns into the one introduced in [56].
10 Conformal supergravity, topological invariants and super-Weyl anomalies

In this section we describe a powerful formalism to generate locally superconformal higher-derivative invariants developed in \[104\]. Its applications include the superfield construction of the $\mathcal{N} = 2$ Gauss-Bonnet term and the general structure of super-Weyl anomalies in $\mathcal{N} = 2$ superconformal field theories. To start with, we review the $\mathcal{N} = 2$ conformal supergravity theory.

10.1 Conformal supergravity

The action for $\mathcal{N} = 2$ conformal supergravity [13] is

$$S_{CSG} = \frac{1}{4} \int d^4x d^4\theta \varepsilon W^{\alpha\beta} W_{\alpha\beta} + \text{c.c.} \quad (10.1)$$

The corresponding equation of motion is

$$\nabla_{\alpha\beta} W^{\alpha\beta} = \bar{\nabla}^{\alpha\beta} \bar{W}_{\alpha\beta} = 0 \quad (10.2)$$

and states that the super-Bach multiplet (4.22a) vanishes. This equation is obtained by varying $S_{CSG}$ with respect to a gravitational superfield $H = \bar{H}$ which is the only unconstrained prepotential of $\mathcal{N} = 2$ conformal supergravity modulo purely gauge degrees of freedom, see the discussions in [79, 104] and references therein. Performing this variation, we find

$$\delta H \int d^4x d^4\theta \varepsilon W^{\alpha\beta} W_{\alpha\beta} = 2 \int d^4 z E \delta H \nabla_{\alpha\beta} W^{\alpha\beta} \quad (10.3)$$

where the variation $\delta H$ is a real primary superfield of dimension $-2$. Since the Bach multiplet $\mathcal{B}$, eq. (4.22a), and the variation $\delta H$ are real, the functional

$$\Psi = -\frac{i}{2} \int d^4x d^4\theta \varepsilon W^{\alpha\beta} W_{\alpha\beta} + \text{c.c.} \quad (10.4)$$

is a topological invariant being proportional to the Pontryagin term. As a consequence of (4.23a), the right-hand side of (10.3) is invariant under gauge transformations of the form [79, 127, 104]

$$\delta_\Omega H = \frac{1}{12} \nabla^{ij} \Omega_{ij} + \frac{1}{12} \bar{\nabla}_{ij} \bar{\Omega}^{ij} \quad (10.5)$$

where the complex gauge parameter $\Omega_{ij} = \bar{\Omega}_{ji}$ is unconstrained and has the superconformal properties

$$K^A \Omega_{ij} = 0 \quad \mathbb{D} \Omega_{ij} = -3 \Omega_{ij} \quad \Psi \Omega_{ij} = -2 \Omega_{ij} \quad (10.6)$$
This gauge invariance expresses the fact that the action (10.1) is locally superconformal.

Any conformally flat superspace, $W_{\alpha\beta} = 0$, is a solution of the equation (10.2). It is instructive to linearise the conformal supergravity action (10.1) about such a background. From the linearised prepotential $H$, we construct the linearised super-Weyl tensor

$$\mathcal{M}_{\alpha\beta} = \bar{\nabla}^{4}\nabla_{\alpha\beta}H, \quad (10.7)$$

which is primary, $K^{C}\mathcal{M}_{\alpha\beta} = 0$, and covariantly chiral, $\bar{\nabla}^{\dot{\gamma}}\mathcal{M}_{\alpha\beta} = 0$. It proves to be invariant under the gauge transformations (10.5), $\delta_{\Omega}\mathcal{M}_{\alpha\beta} = 0$, and obeys the Bianchi identity

$$\nabla^{\alpha\beta}\mathcal{M}_{\alpha\beta} = \bar{\nabla}^{}_{\dot{\alpha}\dot{\beta}}\bar{\nabla}^{4}\ln\bar{\Phi}, \quad (10.8)$$

Thus, the action for linearised conformal supergravity is simply

$$S_{\text{LC}SG} = \frac{1}{4} \int d^{4}\!d^{4}\theta \mathcal{M}_{\alpha\beta}\bar{\nabla}^{4}\ln\bar{\Phi} + \text{c.c.} \quad (10.9)$$

If the background superspace is flat, the field strength (10.7) reduces to that described in [117], and the action (10.9) turns into the one given in [13, 117].

The model (10.9) is known to possess $\text{U}(1)$ duality invariance [133]. The formalism of $\text{U}(1)$ duality rotations has been used [133] to construct nonlinear extensions of (10.9).

### 10.2 Logarithm construction and the Gauss-Bonnet invariant

We now turn to describing the logarithm construction of [104] and its use in defining the $\mathcal{N} = 2$ supersymmetric extension of the Gauss-Bonnet term.

Let $\bar{\Phi}$ be a primary antichiral scalar with the superconformal properties:

$$K^{A}\bar{\Phi} = 0, \quad \bar{\nabla}^{i}_{\alpha}\bar{\Phi} = 0, \quad \bar{D}_{\alpha}\bar{\Phi} = w\bar{\Phi} \implies \bar{Y}_{\dot{\alpha}}\bar{\Phi} = 2w\bar{\Phi}, \quad (10.10)$$

where $w \neq 0$, but it is otherwise arbitrary. We assume $\bar{\Phi}$ to be nowhere vanishing such that $\bar{\Phi}^{-1}$ exists. Then, it may be shown that $\bar{\nabla}^{4}\ln\bar{\Phi}$ is a primary chiral superfield of dimension 2,

$$K^{A}\bar{\nabla}^{4}\ln\bar{\Phi} = 0, \quad \bar{\nabla}^{i}_{\alpha}\bar{\nabla}^{4}\ln\bar{\Phi} = 0, \quad \bar{D}_{\alpha}\bar{\nabla}^{4}\ln\bar{\Phi} = 2\bar{\nabla}^{4}\ln\bar{\Phi}, \quad (10.11)$$

By following the degauging procedure to $\text{U}(2)$ superspace, which was detailed in section 6.1, it may be shown that

$$\bar{\nabla}^{4}\ln\bar{\Phi} = \bar{\Delta}\ln\bar{\Phi} + \frac{w}{2}\left(\bar{\nabla}^{\alpha\beta}\bar{Y}_{\alpha\beta} + \bar{S}_{ij}\bar{S}^{ij} + \frac{1}{6}\bar{S}_{ij}\bar{S}^{ij}\right) \equiv \bar{\Delta}\ln\bar{\Phi} + \frac{w}{2}\bar{S}, \quad (10.12)$$
where $\tilde{\Delta}$ denotes the chiral projecting operator (6.17). It is important to note that since $\tilde{\nabla}^4\ln\Phi$ and $\tilde{\Delta}\ln\Phi$ are both manifestly chiral, $\Xi$ shares this property

$$\tilde{\mathcal{D}}_i^\alpha\Xi = 0 .$$  \hspace{1cm}  (10.13)

At the same time, we emphasise that while the right hand side of (10.12) is primary, each individual term possesses an inhomogeneous contribution under the super-Weyl transformations of $U(2)$ superspace

$$\delta_\Sigma\tilde{\Delta}\ln\Phi = 2\Sigma\tilde{\Delta}\ln\Phi + w\bar{\Delta}\Sigma , \hspace{1cm} \delta_\Sigma\Xi = 2\Sigma\Xi - 2\bar{\Delta}\Sigma .$$  \hspace{1cm}  (10.14)

In the case of $SU(2)$ superspace, these transformation laws turn into

$$\delta_\sigma\tilde{\Delta}\ln\Phi = 2\sigma\tilde{\Delta}\ln\Phi + w\bar{\Delta}\sigma , \hspace{1cm} \delta_\sigma\Xi = 2\sigma\Xi - 2\bar{\Delta}\sigma .$$  \hspace{1cm}  (10.15)

Our analysis leads to an important conclusion. Specifically, for every primary dimensionless chiral scalar $\Psi$, the following functional

$$\int d^4x d^4\theta \mathcal{E}\Psi\tilde{\nabla}^4\ln\Phi = \int d^4x d^4\theta \mathcal{E}\Psi\Xi$$  \hspace{1cm}  (10.16)

is locally superconformal. Here the expression on the right is given in $U(2)$ superspace (its form is preserved upon degauging to $SU(2)$ superspace).

In Ref. [104] the superconformal chiral action

$$S_{\chi}^- = -\int d^4x d^4\theta \mathcal{E} (W^{\alpha\beta}W_{\alpha\beta} - 2w^{-1}\tilde{\nabla}^4\ln\Phi)$$  \hspace{1cm}  (10.17)

was identified with the $\mathcal{N} = 2$ Gauss-Bonnet topological invariant. More precisely, it may be shown that, at the component level, $S_{\chi}^-$ is a combination of the Gauss-Bonnet and Pontryagin invariants. Under suitable boundary conditions on $\tilde{\Phi}$, the functional (10.17) proves to be independent of $\tilde{\Phi}$. This follows from (10.16) in conjunction with the identity in $U(2)$ superspace

$$\tilde{\mathcal{D}}_i^\alpha\sigma = 0 \implies \int d^4x d^4\theta \mathcal{E} (W^{\alpha\beta}W_{\alpha\beta} - 2w^{-1}\tilde{\nabla}^4\ln\Phi) = 0 ,$$  \hspace{1cm}  (10.18)

for any covariantly chiral scalar $\sigma$. Therefore, we obtain

$$S_{\chi}^- = -\int d^4x d^4\theta \mathcal{E} (W^{\alpha\beta}W_{\alpha\beta} - \Xi) .$$  \hspace{1cm}  (10.19)

The topological nature of (10.17) was established in [104] at the component level. A solid superspace proof is still absent.
10.3 Super-Weyl anomalies

Consider a superconformal field theory coupled to supergravity. The classical action of such a theory is invariant under the super-Weyl transformations, and it is independent of the supergravity compensators. In other words, the superconformal field theory couples to the Weyl multiplet.

In the quantum theory, integrating out the matter fields leads to an effective action that is no longer a functional of the Weyl multiplet only. There are two different contributions to the $\mathcal{N}=2$ super-Weyl anomaly. One of them is given in terms of the supergravity multiplet. In the framework of $SU(2)$ superspace, the super-Weyl variation of the effective action $\Gamma$ has the form

$$\delta_\sigma \Gamma = (c - a) \int d^4x d^4\theta \sigma \sigma^{\alpha\beta} W_{\alpha\beta} + a \int d^4x d^4\theta \sigma \Xi + \text{c.c.} , \quad (10.20)$$

for some anomaly coefficients $a$ and $c$. One can check that the super-Weyl variation (10.20) obeys the Wess-Zumino consistency condition

$$\delta_\sigma \delta_{\sigma_2} - \delta_{\sigma_2} \delta_\sigma \Gamma = 0 . \quad (10.21)$$

This property guarantees the existence of $\Gamma$. The other sector of the $\mathcal{N}=2$ super-Weyl anomaly is determined by local couplings in a superconformal field theory. According to [110, 135], it is given by

$$\delta_\sigma \Gamma = \int d^4z E \left( \sigma + \bar{\sigma} \right) K(X, \bar{X}) , \quad (10.22)$$

where the Kähler potential $K(X, \bar{X})$ is the same as in (7.3). Since the chiral scalars $X^I$ are inert under the super-Weyl transformations, the anomaly clearly satisfies the Wess-Zumino consistency condition. The right-hand side of (10.22) is not invariant under Kähler transformations. However, the $\mathcal{N}=2$ super-Weyl anomaly is invariant under a joint Kähler-Weyl transformation. A detailed analysis of the anomaly (10.22) is given in the original publications [110, 135].

The super-Weyl anomaly (10.20) is generated by the $\mathcal{N}=2$ dilaton action [134]

$$S_D = \frac{1}{4} f^2 \int d^4x d^4\theta \sigma \ln \mathcal{Z} + \int d^4x d^4\theta \left\{ (c - a) W^{\alpha\beta} W_{\alpha\beta} + a \Xi \right\} \ln \mathcal{Z} + \text{c.c.} + 2a \int d^4z E \ln \mathcal{Z} \ln \mathcal{Z} , \quad (10.23)$$

where $f$ is a constant parameter, and $\mathcal{Z}$ is the chiral field strength of a vector multiplet such that $\mathcal{Z}^{-1}$ exists. One may check that the super-Weyl variation $\delta_\sigma S_D$ coincides with the right-hand side of (10.20).

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