On the renormalisation of Newton’s constant

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The problem of obtaining a gauge independent beta function for Newton’s constant is addressed. By a specific parameterisation of metric fluctuations a gauge independent functional integral is constructed for the semiclassical theory around an arbitrary Einstein space. The effective action then has the property that only physical polarisations of the graviton contribute, while all other modes cancel with the functional measure. We are then able to compute a gauge independent beta function for Newton’s constant in $d$-dimensions to one-loop order. No Landau pole is present provided $N_g < 18$, where $N_g = d(d - 3)/2$ is the number of polarisations of the graviton. While adding a large number of matter fields can change this picture, the absence of a pole persists for the particle content of the standard model in four spacetime dimensions.
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I. INTRODUCTION

In quantum field theory it is well known that coupling constants become functions of the energy scales entering the renormalisation process. In turn this implies a modification of the classical scaling properties of a theory. Such energy dependence of a coupling \( a \) is encoded in its beta functions \( \beta_a = \mu \frac{\partial}{\partial \mu} a \), where \( \mu \) is the renormalisation scale. In quantum chromodynamics (QCD) this non-trivial scaling can be observed in the one-loop beta function \( \beta_{\alpha_s} \),

\[
\beta_{\alpha_s} = -\left(11 - \frac{2}{3} N_f\right) \frac{\alpha_s^2}{2\pi},
\]

(1.1)

where \( \alpha_s \) is the strong coupling and \( N_f \) denotes the number of flavours. For \( N_f \leq 16 \) this equation describes the weakening of the strong force as the energy scale \( \mu \) is increased. On the other hand if we have \( N_f > 16 \) the coupling constant will diverge at a finite energy. In the former case the theory is said to be asymptotically free and is well defined for all scales. In the later case the coupling has a Landau pole and the theory can only be considered as an effective one, at energies below the pole.

When considering gravity as a quantum field theory it is natural to ask whether there exists an analog to (1.1) for Newton’s constant \( G_N \). In four spacetime dimensions Newton’s constant is dimensionful which implies that the structure of the beta function differs from that of QCD, where \( \alpha_s \) is dimensionless. In particular, for \( d \) spacetime dimensions one should work with the dimensionless coupling \( G = \mu^{d-2} G_N \). The analog of (1.1) will then take the form

\[
\beta_G = (d - 2) G - b G^2,
\]

(1.2)

for some constant \( b \). Whereas the coefficient of \( G \) just reflects the dimensionality of \( G_N \) the coefficient \( b \) is a quantum correction \(^1\). Clearly \( b \) plays the same role as \( 11 - \frac{2}{3} N_f \) in the QCD; if \( b \) is positive then \( G \) will remain finite for all scales, but if \( b \) is negative \( G \) will blow up at a finite energy. Thus, although non-perturbative effects may change the naive conclusion, the sign of \( b \) is a strong indication of whether quantum gravity should be replaced by some new theory at a finite energy scale, or whether gravity may itself constitute a perfectly valid quantum field theory. The latter scenario, dubbed asymptotic safety by Weinberg \(4\), would involve a fixed point for \( G \) where \( \beta_G = 0 \) for some positive value \( G = G_* \). At the level of the one-loop beta function (1.2) this is found at \( G_* = \frac{1}{b}(d-2) \) and its existence only depends on the sign of \( b \). However, one must go beyond one-loop in order to be sure that such a fixed point exists non-perturbatively.

Although \( \beta_G \) has been calculated previously, in these calculations the value of \( b \) has been found to depend on the gauge and parameterisation (an exception are calculations with the geometrical effective action \(10\), which are gauge independent although a particular parameterisation of the fields is used). Hence these calculations will generically include unphysical degrees of freedom coming from the gauge fixing action and may not be trustworthy. This issue stems from the fact that calculations have to be performed off shell and therefore generally depend on both the gauge and the parameterisation. When working on shell such dependencies vanish. On the other hand the Einstein equations dictate that the scalar curvature is given by

\[
R = \frac{2d}{d-2} \tilde{\lambda},
\]

(1.3)

where \( \tilde{\lambda} \) is the cosmological constant. As a consequence one cannot disentangle the renormalisation of Newton’s constant from the vacuum, forcing us to work off shell to determine \( b \). Thus, it is clear that by working off shell, in a gauge dependent setup, no definitive conclusion can be reached as to the sign of \( b \). On the other hand explicit non-perturbative calculations both in four and higher dimensions indicate that a

\(^1\) \( b \) is proportional to \( \hbar \), although here we work in natural units \( \hbar = 1 = c \).
fixed point for $G$ exists \[1,2,10,17\], despite calculations being generally gauge dependent. Hence, although evidence is strong that the coupling $G$ does not blow up for some finite energy scale, it is also questionable since the dependence on the gauge parameters implies that unphysical contributions are still present.

In this paper we shall obtain gauge independent results by disentangling physical degrees of freedom at the level of the functional integral. In particular we shall obtain a semiclassical approximation to the $d$-dimensional functional integral in quantum gravity on an arbitrary Einstein space independent of the gauge. This functional integral has the property that it only receives contributions from physical fluctuations of the metric that survive on shell. We will then see that the sign of $b$ can be universally determined by a unique factor

$$b \propto \frac{2}{3} (18 - N_g), \quad \text{with} \quad N_g = \frac{d(d - 3)}{2}, \quad (1.4)$$

where $N_g$ is the number of dynamical degrees of the metric in general relativity. Thus one finds that $b$ is positive in $d = 4$ dimensions but becomes negative for integer dimension $d \geq 8$. This result is for pure gravity but is also easily generalised when matter is included.

At this point we should take a moment to clarify the meaning of (1.4). Although one may like to think of $G$ being related to an observable, this may not be the case. In particular, at least perturbatively, there is no universal definition of a running Newton’s constant for e.g. scattering processes \[18\]. What \[18\] does tell us is whether or not the ultra-violet (UV) cutoff $\Lambda$ can be removed and hence whether a continuum limit exists. For $N_g < 18$ one can remove $\Lambda \rightarrow \infty$ without any problems. However, for $N_g > 18$ one may show that for the bare Newton’s constant to be positive, and hence for the functional integral to exist, $\Lambda$ must remain finite. Thus the beta function implies the existence (non-existence) of a continuum limit for $N_g < 18$ ($N_g > 18$). The universal and gauge independent result (1.4) therefore provides evidence for the continuum limit of quantum gravity in four spacetime dimensions.

The rest of this paper is as follows. \[2\] In section \[2\] we find the form of the functional measure for the gauge fixed functional integral over geometries and use it to determine the on shell functional integral for the semi classical theory. This generalises the result of \[19\] to all dimensions $d > 2$. Turning to section \[3\] we then confront the problem of obtaining gauge independent results off shell and find a parameterisation of metric fluctuations which achieve this end. We are then able to write down a gauge independent one-loop effective action. We then use our parameterisation to derive a semiclassical renormalisation group equation for a scale dependent action $\Gamma_k$ in section \[4\]. In section \[5\] we evaluate traces appearing in the flow equation using the early time heat kernel expansion and access their universal content. We may then observe the origin of the universal factor (1.4) while also reproducing the curvature squared counter term found in [20]. Then in section \[6\] we give the explicit form of the beta function $\beta_G$. We first discuss the case near two dimensions and the limit $d \rightarrow 2$. Then we consider the general case in $d$ dimensions and discuss the physical origin of the universal factor (1.4). In section \[7\] we find the form of the divergent counter term for the vacuum energy and note that it vanishes in $d = 3$ dimensions. In section \[8\] we derive a bound on the UV scale $\Lambda$ of the theory in higher dimensions and show no bound exists for $N_g < 18$. In Section \[9\] we present the beta function with the inclusion of matter. We end with our conclusions in section \[10\].

**II. THE ON SHELL FUNCTIONAL INTEGRAL**

General relativity in $d$ dimensions describes $(d-3)d/2$ dynamical degrees of freedom coming from the $d(d+1)/2$ components of the metric, which are determined by Einstein equations, minus $d$ constraints and $d$ diffeomorphisms. In the quantum theory these degrees of freedom correspond to the $N_g$ polarisations of the graviton, schematically one has

$$N_g = \frac{d(d+1)}{2} - d - d = d(d-3)/2. \quad (2.1)$$
Thus one should expect that the additional $2d$ unphysical degrees of freedom are removed when physical quantities are computed. At the level of the functional integral such cancellations occur between metric fluctuations and the functional measure. One way to ensure this cancelation occurs is to turn off any external source terms which violate parameterisation invariance, equivalently, when working with the effective action, this can be achieved by working on shell. In this section we will determine the form of the semiclassical functional integral by expanding around a solution to the equations of motion to ensure gauge and parameterisation independence. This we do as an intermediate step to gain physical intuition and see mathematically how the unphysical states are removed when computing on shell quantities.

The first step in our calculation is to determine the form of the functional measure for quantum gravity. We assume a bare action for Euclidean quantum gravity of the Einstein-Hilbert form

$$S_{\text{grav}}[\gamma_{\mu\nu}] = \frac{1}{16\pi G_b} \int d^d x \sqrt{\gamma} \left[ 2\bar{\lambda}_b - R(\gamma_{\mu\nu}) \right] + S_{gf}[\gamma_{\mu\nu}],$$

(2.2)

where $S_{gf}$ is the gauge fixing action. Here $\gamma_{\mu\nu}$ is the metric tensor with determinant $\gamma = \det \gamma_{\mu\nu}$ and $R(\gamma_{\mu\nu})$ denotes the Ricci scalar. The action depends on two parameters $G_b$ and $\bar{\lambda}_b$ which denote the bare Newton’s constant and bare cosmological constant respectively. The functional integral is then given by

$$Z = \int \mathcal{D}\gamma_{\mu\nu}(\det Q) e^{-S_{\text{grav}}[\gamma_{\mu\nu}]}$$

(2.3)

where $\det Q$ is the determinant of the Faddeev-Popov operator.

Here we shall evaluate the functional integral (2.3) in the semiclassical approximation. To this end we first consider parameterisations of metric fluctuations $h_{\mu\nu}$ in terms of a background field $g_{\mu\nu}$ which satisfies the classical equations of motion

$$R_{\mu\nu} = \frac{2}{d-2}\bar{\lambda}_b g_{\mu\nu}.$$  

(2.4)

Two such parameterisations are the linear parameterisation

$$\gamma_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

(2.5)

and the exponential parameterisation

$$\gamma_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} + \frac{1}{2} h_{\lambda\mu} h^{\lambda\nu} + \ldots,$$

(2.6)

where in the later case $h$ is a symmetric matrix with components $[h]_{\mu\nu}^\mu = h_{\mu\nu}$. An advantage of the exponential parameterisation is that one may single out the conformal factor of the metric by

$$\gamma_{\mu\nu} = e^{\frac{1}{2}\bar{\phi}} g_{\mu\nu} \left(e^h\right)^\lambda_\nu = \frac{1}{e^{\frac{1}{2}\bar{\phi}}} \gamma_{\mu\nu}$$

(2.7)

where $\bar{\phi} = h_{\mu}^\mu$ is the trace of $h$ and $\hat{h}$ is the trace free part. From here it follows that the determinant of $\gamma$ is fixed to the background one

$$\hat{\gamma} = g$$

(2.8)

and thus $e^{\frac{1}{2}\bar{\phi}}$ is identified as the conformal factor of $\gamma_{\mu\nu}$ with respect to the background metric $g_{\mu\nu}$.

We shall need to expand the action to second order in the fluctuation around the on shell metric $g_{\mu\nu}$. To ensure the resulting Hessians are invertible we must also fix a gauge, the choice of which should fall out
of physical quantities. Here we choose a class of background field gauges, linear in the fluctuation $h_{\mu\nu}$ for either parameterisation,

$$S_{gf} = \frac{1}{32\pi G_{b\alpha}} \int d^d x \sqrt{g} g^{\mu\nu} F_\mu F_\nu, \quad F_\mu = \nabla_\lambda h^\lambda_{\mu} - \frac{1 + \rho}{d} \nabla_\mu h^\lambda_{\lambda}.$$  \hspace{1cm} (2.9)

We now follow the steps out lined in [21] to decompose the fluctuation $h_{\mu\nu}$ such that it becomes manifest that the gauge fixing action only depends on the transverse vector fields matching the number of diffeomorphisms. The first step is to adopt the transverse-traceless decomposition of the fluctuations in terms of differentially constrained fields [22],

$$h_{\mu\nu} = h^T_{\mu\nu} + \phi \frac{1}{d} g_{\mu\nu} + \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu + \nabla_\mu \nabla_\nu \bar{\psi} - \frac{1}{d} g_{\mu\nu} \nabla^2 \bar{\psi},$$ \hspace{1cm} (2.10)

$$h^T_{\mu} = 0, \quad \nabla_\mu h^T_{\mu} = 0, \quad \nabla_\mu \xi_\mu = 0.$$

One observes that $\xi_\mu$ takes the form of a transverse diffeomorphism of the metric to linear order. Thus we can identify $\xi_\mu$ as $d - 1$ of the unphysical fields corresponding to such diffeomorphisms. We then further redefine the scalar fields $\{\phi, \bar{\psi}\} \rightarrow \{\phi, \psi\}$ as

$$\bar{\phi} = \phi + \nabla^2 \bar{\psi}, \quad \bar{\psi} = \psi + \frac{\rho}{(d - 1 - \rho)\nabla^2 + R} \phi,$$ \hspace{1cm} (2.11)

for which it becomes manifest that the gauge fixing action (2.10) only depends on the transverse vector $\xi_\mu$ and the scalar $\psi$. Thus we see that $\psi$ represents the additional longitudinal diffeomorphism. One should then expect that the integral of these fields in (2.3) should be cancelled by the functional measure. The field redefinition (2.10) leads to the following Jacobians in the functional measure

$$J_h = \sqrt{\det [\Delta]} \sqrt{\det [\Delta_0]} \sqrt{\det [\Delta_1]},$$ \hspace{1cm} (2.12)

whereas the redefinition (2.11) has a trivial Jacobian. The non-trivial Jacobian (2.12) is given by determinants of the differential operators,

$$\Delta \varphi = -\nabla^2 \varphi$$

$$\Delta_0 \varphi = \left(-\nabla^2 - \frac{R}{d - 1}\right) \varphi$$

$$\Delta_1 \varphi_{\mu} = \left(-\nabla^2 \delta_{\nu}^{\mu} - R_{\nu}^{\mu}\right) \varphi_{\nu},$$ \hspace{1cm} (2.13)

acting on scalars and transverse vectors as indicated. It is important to note that the constant mode of the scalar $\psi$ should be left out of the functional measure since it cannot contribute to the fluctuation $h_{\mu\nu}$. Accordingly the constant modes in the scalar Jacobians should also be removed. Similarly one should leave out the Killing vectors $\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0$ from the vectors $\xi^\mu$ and the transverse vector Jacobian $\sqrt{\det [\Delta_1]}$, this corresponds to the zero mode of $\Delta_1$.

Next we turn to the Faddeev-Popov determinant. Here we will exploit the liberty to write the determinant as $\operatorname{det} Q = \sqrt{\det Q^T}$ such that the ghosts become second order in derivatives at the price of including a set of commutative ghosts fields $B^\mu$ in addition to the anticommuting ghosts $C^\mu_{\mu}$ and $\bar{C}^\mu_{\mu}$. We then decompose the ghosts as

$$B_{\mu} = B^T_{\mu} + \nabla_\mu B^L, \quad C^T_{\mu} = C^T_{\mu} + \nabla_\mu C^L, \quad \bar{C}^T_{\mu} = \bar{C}^T_{\mu} + \nabla_\mu \bar{C}^L.$$ \hspace{1cm} (2.14)

This transformation then leads to a Jacobian $J_{gh} = \frac{1}{\sqrt{\det [\Delta]}}$ which cancels with the corresponding factor in the metric fluctuation Jacobian $J_h$ given by (2.12) leaving just two Jacobians,

$$J_0 = \sqrt{\det [\Delta_0]}, \quad J_1 = \sqrt{\det [\Delta_1]}.$$ \hspace{1cm} (2.15)
It is these Jacobians that remove the \( d \) degrees of freedom corresponding to the constraints whereas the ghosts should cancel the integral over diffeomorphisms (i.e. \( \xi_\mu \) and \( \psi \)). Due to the decomposition \(^{2.14}\) the corresponding Faddeev-Popov determinants splits into two factors for the longitudinal and transverse parts,

\[
\det Q_L = \sqrt{\det[\Delta_L^2]} , \quad \det Q_T = \sqrt{\det[\Delta_T^2]} \tag{2.16}
\]

where we define the gauge dependent differential operator \( \Delta_L \) by

\[
\Delta_L \varphi = \left( -\nabla^2 - \frac{R}{d-1-\rho} \right) \varphi , \tag{2.17}
\]

acting on scalar fields. The path integral is then given by

\[
Z = \int D h^{\mu \nu} D \phi D \xi_\mu D \psi J_0 J_1 \det(Q_L) \det(Q_T) e^{-S_{grav}[\phi, h^i, \psi, \xi]} . \tag{2.18}
\]

To obtain the semiclassical functional integral we must then expand the action to second order in the fluctuations of the fields \{\( \phi, h^i, \psi, \xi \)\} while taking into account the contributions from the measure. The next stage is therefore to compute the second variation of the bare action and apply the on shell condition \(^{2.4}\).

First let’s show that the choice of parameterisation, such as \(^{2.6}\) or \(^{2.5}\), does not effect the on shell Hessians. Here we just assume that the metric is a function of the fluctuation \( h_A(x) \) (with e.g. \( A = \{\mu \nu\} \)) such that

\[
\gamma_{\mu \nu} = \gamma_{\mu \nu}(h_A(x)) \quad \text{with} \quad \gamma_{\mu \nu}(0) = g_{\mu \nu} \tag{2.19}
\]

which agree only up to linear order in the fluctuation. The choices \(^{2.6}\) and \(^{2.5}\) are just two examples of such parameterisations. Introducing de Witt’s condensed notation we write \( i = \{x, \mu \nu\} \) and \( a = \{x, A\} \) and take two functional derivatives with respect to the fluctuations \( h_A(x) \equiv h^i \) expressed with derivatives with respect to the metric \( \gamma_{\mu \nu}(x) \equiv \gamma^i \). Acting on e.g. \( S[\gamma^i] = S_{grav} \) we obtain,

\[
\frac{\delta^2 S[\gamma^k[h^a]]}{\delta h^a \delta h^b} = \delta \frac{\delta S[\gamma^i]}{\delta h^a} \delta \frac{\delta S[\gamma^i]}{\delta h^b} = \frac{\delta^2 \gamma^i}{\delta h^a} \frac{\delta^2 \gamma^i}{\delta h^b} + \frac{\delta \gamma^i}{\delta h^a} \frac{\delta \gamma^i}{\delta h^b} \delta S[\gamma^i] + \delta \gamma^i \delta \gamma^j \frac{\delta S[\gamma^i]}{\delta h^b} \frac{\delta S[\gamma^j]}{\delta h^b} \delta \gamma^i \delta S[\gamma^j] . \tag{2.20}
\]

On shell the first term is zero since it is proportional to the equation of motion, while the second term only depends on \( \gamma_{\mu \nu} \) to linear order of in \( h_{\mu \nu} \). This shows that by going on shell the results cannot depend on which of parameterisations is used.

The on shell Hessians \( S^{(2)}_{grav} \) has the following components, for metric fluctuation fields \( \xi_\mu, \psi \) and \( h^i_{\mu \nu} \),

\[
16 \pi G_b S_{\xi \xi}^{(2)} = \frac{1}{\alpha} \Delta_1^2 , \quad 16 \pi G_b S_{\psi \psi}^{(2)} = \left( \frac{d-1-\rho}{\alpha d^2} \right)^2 \Delta_1^2 \Delta , \quad 16 \pi G_b S_{h_{\mu \nu} h_{\mu \nu}}^{(2)} = \Delta_2 , \tag{2.21}
\]

where

\[
\frac{\delta^2 S^{(2)}_{grav}}{\delta \varphi^A(x) \delta \varphi^B(y)} = \sqrt{g} \delta^{(2)}_{\varphi^A \varphi^B} 1_{AB} \delta(x - y) \tag{2.22}
\]

with \( \varphi^A \) denoting the various fields and \( 1_{AB} \) is the identity on the corresponding field space. In \(^{2.21}\) the differential operator \( \Delta_2 \) is the Lichnerowicz Laplacian acting on symmetric transverse-traceless two tensors as

\[
\Delta_2 \varphi_{\mu \nu} = -\nabla^2 \varphi_{\mu \nu} - 2 R^\rho_\alpha_{\mu \nu} \varphi_{\beta \alpha \rho} . \tag{2.23}
\]

The field \( \phi \), whose on shell Hessian is given by

\[
16 \pi G_b S_{\phi \phi}^{(2)} = \left( \frac{(d-1)(d-2)}{2d^2} \right) \Delta_0 , \tag{2.24}
\]
needs extra attention since one observes that (2.21) is negative for positive eigenvalues of the operator $\Delta_0$. This indicates that the naive Wick rotation of the functional integral is unbounded from below. However, as pointed out by Mottola and Mazur [19] one should Wick rotate all modes with positive eigenvalue as $\phi \rightarrow i\phi$ whereas modes with negative should be Wick rotated trivially. This rule can be derived by considering the super-metric on the space of fluctuations $h_{\mu\nu}$. Hence one can write the corresponding operator as

$$16\pi G_b S^{(2)}_{\phi\phi} = \frac{(d-1)(d-2)}{2d^2} |\Delta_0|,$$

(2.25)

to get a well defined Euclidean functional integral.

Expanding the action to second order in the fluctuation, and comparing $S^{(2)}_\xi$ and $S^{(2)}_{\phi\psi}$ given in (2.21) to (2.10), one observes that the functional integrals over $\xi$ and $\psi$ cancel with the Faddeev-Popov determinants $\det Q_L$ and $\det Q_T$. All gauge dependence has therefore cancelled out. Furthermore comparing (2.25) with $J_0$ we see that all modes of $\phi$ apart from the constant mode

$$\partial_\mu \phi_0 = 0,$$

(2.26)

are cancelled by the Jacobian $J_0$ given by (2.19) (here we assume that the constant mode is the only mode for which $\Delta_0$ has a negative eigenvalue, which is true at least for the $d$-sphere). This follows since as we argued that the constant mode must be left out of $J_0$. The only contributions that remain are the $(d-2)(1+d)/2$ transverse-traceless fluctuations, the Jacobian $J_1$, which comprises $d-1$ negative degrees of freedom, and the constant mode $\phi_0$. Thus one is left with $(d-3)d/2$ local degrees of freedom corresponding to the graviton and one global degree of freedom corresponding to a constant rescaling of the metric. The functional integral then reduces to the form

$$Z = \int d\phi_0 \int Dh^4 \sqrt{\det[\Delta_1]} \exp \left[ -S_{\text{grav}}[g] - \frac{1}{32\pi G_b} \int d^d x \sqrt{g} \left( h_{\mu\nu} \Delta_2 h^{\mu\nu} - \frac{(d-1)(d-2)}{2d^2} \partial_0 \Delta_0 \phi_0 \right) \right],$$

(2.27)

this result is valid in all dimensions $d > 2$ and for all parameterisations (2.19) of the metric fluctuations. The remaining determinant can be expressed in terms of integral over auxiliary fields

$$J_1 = \sqrt{\det[\Delta_1]} = \int D\xi_\mu Dc_\mu D\bar{c}_\mu \exp \left[ -\frac{1}{32\pi G_b} \int d^d x \sqrt{g} \left( \bar{c}_\mu \Delta_1 c^\mu + \zeta_\mu \Delta_1 \zeta^\mu \right) \right]$$

(2.28)

$$\equiv \int D\xi_\mu Dc_\mu D\bar{c}_\mu e^{-S_{\text{aux}}[\bar{c}_\mu, c_\mu, \xi_\mu]},$$

where $\bar{c}_\mu, c_\mu$ are anticommuting and $\zeta_\mu$ is commuting and all are transverse. Here $S_{\text{aux}}$ is the auxiliary action that combines with the gravity action in (2.27) to form a modified bare action $S = S_{\text{grav}} + S_{\text{aux}}$. The semiclassical functional integral (2.27) generalises the $d = 4$ result of [19]. We note that, while the cancellation of the $d$-diffeomorphisms and one of the constrained degrees of freedom is explicit, the cancellation between the remaining $d-1$ unphysical transverse traceless fluctuations is implicit and implies that non-trivial cancelations between these modes and the Jacobian $J_1$ should occur. It is clear therefore that one should not leave out the Jacobians from the functional, as was done in [23], doing so implicitly includes $d$ unphysical degrees of freedom and fails to reproduce the one-loop functional integral (2.27). The importance of including such Jacobians to obtain the correct covariant measure has been stressed in [24] for the case of QED.

III. THE OFF SHELL EFFECTIVE ACTION AND GAUGE INDEPENDENCE

It is well known that the off shell effective action in quantum gravity generally depends on both the gauge and the field parameterisation (see e.g. [25]). On the other hand since we observe which modes survive in the on shell semiclassical approximation and which modes cancel in the functional measure, we understand which fields are carrying physics and which fields are only present due to us necessarily breaking diffeomorphism invariance.
In order not to be forced to expand around a solution to the bare equations of motion we will now drop the condition (2.4) and instead simply require that the background is an Einstein space,

\[ R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R. \]  

As pointed out in the introduction this is crucial if we are to be able to extract the renormalisation of Newton’s constant.

Our aim is to preserve the gauge independence of the on shell functional integral (2.27) such that physically meaningful results may be derived off shell. The problem that we face is that by working off shell the cancellations between the measure and the 2d unphysical degrees of freedom is not guaranteed. For a generic parameterisation terms proportional to the equation of motion arise in the Hessians and in the presence of such terms the cancellations observed in the previous section will not occur. In particular the Hessians have the general form,

\[ S^{(2)} = S^{(2)}|_{\text{on shell}} + X \left( R - \frac{2d}{d-2} \lambda_b \right), \]  

for any arbitrary parameterisation, where \( X \) is a matrix in field space that depends on the parameterisation and \( S^{(2)}|_{\text{on shell}} \) denotes the terms that survive on shell, given by the Hessians (2.21) and (2.25) but with \( R \neq \frac{2d}{d-2} \lambda_b \).

At this point we make two observations, if \( X \neq 0 \) the gauge fixing dependence will remain and furthermore \( S^{(2)} \) will possess negative eigenmodes depending on \( \lambda_b \). Thus, only by picking a parametrisation for which \( X = 0 \) will the gauge fixing dependence vanish and the Gaussian integrals converge (assuming positivity of \( S^{(2)} \) for the chosen background). By picking a parameterisation for which \( X \neq 0 \) the corresponding effective action will not formally exist for certain values of \( \lambda_b \) where negative eigenvalues of \( S^{(2)} \) are present, as well as being gauge dependent. We are therefore lead to the conclusion that a parameterisation for which \( X = 0 \) is the key to obtaining a physical meaningful gauge independent one-loop result. In turn this removes any dependence of the Hessians on the cosmological constant.

To find such a parametrisation we first note that by using the exponential parameterisation (2.6) the equation of motion will only appear in the Hessians for \( \psi \) and \( \phi \). In fact, if we revert to the parameterisation in terms of \( \bar{\psi} \) and \( \bar{\phi} \), (see (2.11)) the equation of motion will only appear in the Hessian for \( \bar{\phi} \) due to (2.8), this field being identified as the conformal factor of the metric. Then we may choose to parameterise this scalar degree of freedom in terms of a field \( \sigma \) related to the conformal factor by

\[ e^{\frac{\bar{\phi}}{d}} = \left( 1 + \frac{\sigma}{2} \right)^{\frac{d}{2}} = 1 + \frac{\sigma}{d} + \frac{2-d}{4d^2} \sigma^2 + \ldots \]  

such that by using (2.7) and (2.8) we find that the spacetime volume \( V(\gamma) \) is simply linear in \( \sigma \)

\[ V(\gamma) = \int d^d x \sqrt{\gamma} = \int d^d x \sqrt{g} \left( 1 + \frac{\sigma}{2} \right) = V(g) + \int d^d x \sqrt{g} \frac{\sigma}{2}. \]  

It then follows that \( X \) vanishes along with terms proportional to \( R - \frac{2d}{d-2} \lambda_b \) in all vertices \( S^{(n)} \) with \( n \geq 2 \). We then perform the transformations (2.11) but now for \( \sigma \) instead of \( \bar{\phi} \),

\[ \sigma = \bar{\phi} + \nabla^2 \bar{\psi}, \quad \bar{\psi} = \psi + \frac{\rho}{(d-1-\rho) \nabla^2 + R} \bar{\phi}. \]  

After this transformation the Hessians then reduce to the form (2.21) and (2.25) without use of the equations of motion. One may expand around an arbitrary Einstein background (3.1) and again obtain a functional integral given by (2.27).

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3 We note that the dependence on the cosmological constant can also be removed by use of specific gauge fixing conditions [26] whereby the conformal fluctuations \( \bar{\phi} \) are constrained.
As a final step we introduce a sharp infra-red (IR) regulator for the single constant mode \( \phi_0 \) since it anyway will not affect the UV physics we are interested in. The one loop effective action \( \Gamma \) can then be obtained from (2.27) in the standard way by adding a source term and performing a Legendre transform. We then set \( \langle h_{\mu \nu} \rangle = 0 \) to obtain a functional of a single metric field \( g_{\mu \nu} \). On an arbitrary Einstein space the gauge independent action \( \Gamma \) then takes the simple form

\[
\Gamma[g_{\mu \nu}] - S[g_{\mu \nu}] + \frac{1}{2} \text{STr}[\log S^{(2)}] = \frac{1}{2} \text{Tr}[\log \Delta_2] - \frac{1}{2} \text{Tr}[\log \Delta_1],
\]

(3.6)

where \( S \) is the total bare action appearing in the functional integral, with all determinants expressed in terms of auxiliary fields and ghosts, and \( \text{(STr) Tr} \) denotes the (super)-trace. The simplicity of this effective action compared other gauge fixed actions is quite appealing, in particular it is independent of any scalar modes (notwithstanding the constant mode \( \phi_0 \) which we have neglected). Most importantly (3.6) has the property that all unphysical polarisations of the graviton, that cancel on shell, also cancel off shell.

IV. SEMI-CLASSICAL FLOW EQUATION

We would like to derive the beta function for Newton’s constant using the results of the last two sections. The starting point is the functional integral (2.27) on an arbitrary Einstein background with the constant mode \( \phi_0 \) removed (in \( d > 2 \) this mode will not renormalise the Newton’s constant but should modify curvature squared corrections in \( d = 4 \)). Here we shall use the modern formalism of the non-perturbative renormalisation group \([27,29]\), even though it is not technically needed since we are working at one-loop. This approach can then be compared to the old fashioned renormalisation scheme of simply evaluating (3.6) and regulating the one loop integrals. To this end we add a regulator term,

\[
\Delta S_k[\varphi] = \frac{1}{2} \varphi \cdot \mathcal{R}_k \cdot \varphi,
\]

(4.1)

to the bare action. Here \( \mathcal{R}_k \) is an infra red regulator which suppresses low momentum modes \( p^2 < k^2 \) while vanishing for high momentum modes \( p^2 \gg k^2 \). The scale \( k \) plays the role of \( \mu \) as the scale which defines the beta functions. In our case the fields \( \varphi = \{ h, \zeta, c \} \) are all fields appearing in the bare action after exponentiating the determinants in functional measure. Then \( \varphi \) can be thought of as a vector in field space with components \( \varphi_a \) and \( \mathcal{R}_k \) as a matrix with components \( \mathcal{R}_{ab} \).

For the covariant momentum \( p^2 \) we take the eigenvalues of the differential operators of the operators \( \Delta_n \) for \( n = \{ 1, 2 \} \) (given in equations (2.13) and (2.23)). The regulator then takes a form such that

\[
S^{(2)}(\Delta_n) + \mathcal{R}_k(\Delta_n) = S^{(2)}(\Delta_n \rightarrow P_n), \quad P_n = \Delta_n + k^2 C(\Delta_n/k^2),
\]

(4.2)

where \( C(z) \) is a dimensionless scalar cutoff function which vanishes for \( z \gg 1 \) but stays finite and non-zero for \( z \rightarrow 0 \). This constitutes a type-II cutoff, in the nomenclature of \([30]\), where also the potential terms of \( \Delta_n \) are included in the regulator. Another choice, termed type I, is to replace \( \Delta_n \) by \( -\nabla^2 \) in (1.2), however we expect a non-trivial cancellation to occur between the traces of the transverse traceless fluctuations and the Jacobian \( J_1 \) such that only \((d-3)d/2\) propagating degrees of freedom remain. Since, without the regulator \( \mathcal{R}_k \), these cancellations must be between traces depending only on the operators \( \Delta_n \), that appear in \([30]\), it seems natural that the regulator should also depend solely on these operators as well. In addition it has been shown \([31]\) that only type II regulators for fermions correctly regulate modes for the Dirac operator in curved spacetime. Finally we point out that it is the properties of the Lichnerowicz Laplacian \( \Delta_2 \) that determine the stability of classical solutions to the Einstein equations and hence one should expect that this continues to be the case in the semiclassical theory.

After inserting the regulator into the path integral the flowing effective action \( \Gamma_k \) is defined by,

\[
e^{-\Gamma_k[\varphi]} = \int D\varphi e^{-S[\varphi]-\Delta S_k[\varphi-\bar{\varphi}]+(\varphi-\bar{\varphi}) \frac{d \mathcal{R}_k}{dz}},
\]

(4.3)
where $\bar{\varphi} = (\varphi)$. One observes that if $R_k$ diverges for $k^2 \gg p^2$ the regulator term becomes a Gaussian peaked around $\varphi = \bar{\varphi}$, whereas for $k^2 \to 0$ the regulator must vanish and assumes the form of the effective action. It follows that $\Gamma_k$ interpolates between the bare action $S$ for large $k$ and the effective $\Gamma$ in the limit $k \to 0$. In the semi-classical approximation the following approximation the flow effective action takes the form,

$$\Gamma_k - S[\bar{\varphi}] = \frac{1}{2} \text{Tr} \left[ \log \left( \Delta_2 + R_k(\Delta_2) \right) \right] - \frac{1}{2} \text{Tr} \left[ \log (\Delta_1 + R_k(\Delta_1)) \right], \quad (4.4)$$

where we have absorbed irrelevant constant factors of $16\pi G_b$ into the fields. Setting $R_k = 0$ we recover (3.0). Taking a derivative with respect to the Wilsonian RG time $t = \log k/k_0$ to obtain the following gauge independent flow equation,

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{\partial R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2)} - \frac{1}{2} \text{Tr} \frac{\partial R_k(\Delta_1)}{\Delta_1 + R_k(\Delta_1)}. \quad (4.5)$$

The first term in (4.5) comes from the transverse traceless modes and the second from the Jacobian term in (4.5) arising from this single mode.

The advantage of this scheme is that it dispenses with with the formally divergent path integral and generalises beyond the perturbative regime [27–29].

We now put the flow equation together expressing all quantities in units of $\bar{\varphi}$ around $k = 0$. In the semi-classical approximation the flowing effective action takes the form,

$$S[\bar{\varphi}] = \frac{1}{2} \text{Tr} \left[ \log \left( \Delta_2 + R_k(\Delta_2) \right) \right] - \frac{1}{2} \text{Tr} \left[ \log (\Delta_1 + R_k(\Delta_1)) \right],$$

where $\bar{\varphi}$ is the flowing action. It follows that $\Gamma_k$ takes the form

$$\Gamma_k - S[\bar{\varphi}] = \frac{1}{2} \text{Tr} \left[ \log \left( \Delta_2 + R_k(\Delta_2) \right) \right] - \frac{1}{2} \text{Tr} \left[ \log (\Delta_1 + R_k(\Delta_1)) \right], \quad (4.4)$$

where we have absorbed irrelevant constant factors of $16\pi G_b$ into the fields. Setting $R_k = 0$ we recover (3.0). Taking a derivative with respect to the Wilsonian RG time $t = \log k/k_0$ to obtain the following gauge independent flow equation,

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{\partial R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2)} - \frac{1}{2} \text{Tr} \frac{\partial R_k(\Delta_1)}{\Delta_1 + R_k(\Delta_1)}. \quad (4.5)$$

The first term in (4.5) comes from the transverse traceless modes and the second from the Jacobian term in (4.5) arising from this single mode.

Here we consider the following ansatz of the flow action

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} \left( 2\bar{\lambda}_k - R \right) \equiv S[g_{\mu\nu}] + \int d^d x \sqrt{g} (\delta \lambda_k - \delta \kappa_k R), \quad (4.6)$$

which takes the Einstein Hilbert form and allows us to extract the beta functions functions for the running couplings $G_k$ and $\bar{\lambda}_k$. Equivalently we define the vacuum energy density $\lambda_k$ and the inverse Newtons couplings $\kappa_k$ by

$$\lambda_k \equiv \frac{\bar{\lambda}_k}{8\pi G_k} \equiv \frac{\bar{\lambda}_0}{16\pi G_b} + \delta \lambda_k, \quad \kappa_k \equiv \frac{1}{16\pi G_k} \equiv \frac{1}{16\pi G_b} + \delta \kappa_k. \quad (4.7)$$

The UV scale $\Lambda$ can then be defined as the scale for which $\delta \lambda_{k=\Lambda} = 0 = \delta \kappa_{k=\Lambda}$. In turn these boundary definitions can be viewed as replacing the operational meaning of the ‘bare quantities’ $G_b$ and $\bar{\lambda}_0$ entering the functional integral. Thus, the flow equation defines a renormalisation scheme whereby the UV boundary condition replaces the bare action and the IR boundary condition sets the renormalisation condition in terms of renormalised quantities obtained in the limit $k \to 0$. The advantage of this scheme is that it dispenses with with the formally divergent path integral and generalises beyond the perturbative regime [27–29].

We now put the flow equation together expressing all quantities in units of $\bar{\varphi}$ in order to find the autonomous system of beta functions. In particular we drop the index $k$ from the couplings when referring to the dimensionless quantities such that $G = k^{d-2}G_k$ and $\bar{\lambda} = k^{-2} \bar{\lambda}_k$. The dimensionless flow equation then takes the form

$$\int d^d x \sqrt{g} \left( \frac{\partial_t G - (d-2)G}{16\pi G^2} R + \partial_t \lambda + d\lambda \right) = \text{Tr}[W(\Delta_2)] - \text{Tr}[W(\Delta_1)], \quad (4.8)$$

where we define the function,

$$W(z) = \frac{C(z) - zC'(z)}{z + C(z)}. \quad (4.9)$$

This gauge independent flow equation (4.8) is the main result of this section.
V. HEAT KERNELS AND UNIVERSALITY

To obtain the beta functions one computes the traces on the RHS of the flow equation (1.8) and compares the terms on each side of the equation to linear order in the curvature. In the same way one way evaluate (3.0) directly to determine the counter terms needed for the one loop effective action on an Einstein space. To evaluate the traces in both cases we therefore exploit the early time heat kernel expansion for the operators $\Delta_1$ and $\Delta_2$. For a general function $f(\Delta_n)$ one has the following expressions,

$$\text{Tr}[f(\Delta_n)] = \frac{1}{(4\pi)^2} \sum_{i=0}^\infty Q_{i/n} f B_{2i,n},$$

where the functionals $Q_m$ for $m > 0$ are given by

$$Q_m[f] = \frac{1}{\Gamma(m)} \int_0^\infty dz z^{m-1} f(z),$$

and the coefficients $B_{2i,n}$ are the heat kernel coefficients for the operators $\Delta_n$ proportional to curvature monomials with mass dimension 2i. In order to find the beta functions for $G$ and $\Lambda$ we need only the heat kernels for $i = 0, 1$. For the operators $\Delta_1$ and $\Delta_2$ these are given by

$$B_{0,1} = (d-1) \int d^d x \sqrt{g}, \quad B_{0,2} = \frac{1}{2} (d-2)(d+1) \int d^d x \sqrt{g},$$

$$B_{2,1} = \left( \frac{(d-3)(d+2)}{6d} + \frac{(d-1)}{d} \right) \int d^d x \sqrt{g} R,$$

$$B_{2,2} = \left( \frac{(d-5)(d+1)(d+2) R}{12(d-1)} - \frac{(d-2)(d+1) R}{(d-1)d} \right) \int d^d x \sqrt{g} R.$$

We then obtain

$$\text{Tr}[W(\Delta_2)] - \text{Tr}[W(\Delta_1)] = \frac{1}{(4\pi)^2} \left( (B_{0,2} - B_{0,1}) Q_{2/2}[W] + (B_{2,2} - B_{2,1}) Q_{2/3}[W] \right)$$

$$= \frac{1}{(4\pi)^2} \int d^d x \sqrt{g} \left[ N_g Q_{2/2}[W] + \frac{1}{6} (N_g - 18) Q_{2/3}[W] R \right] + ...$$

where $N_g = \frac{1}{2} d(d-3)$ is the number of propagating degrees of freedom in $d$-dimensional quantum gravity. One sees here the universal origin of the factor $N_g - 18$ coming just from the heat kernel coefficient.

We note that one could of have simply renormalised via the introduction of counter terms without the introduction of the IR regulator. That is we could evaluate (3.0) directly to obtain

$$\Gamma = S + \frac{1}{2(4\pi)^2} \left( (B_{0,2} - B_{0,1}) Q_{2/2}[\log] + (B_{2,2} - B_{2,1}) Q_{2/3}[\log] \right)$$

$$= S + \frac{1}{2(4\pi)^2} \int d^d x \sqrt{g} \left[ N_g Q_{2/2}[\log] + \frac{1}{6} (N_g - 18) Q_{2/3}[\log] R \right] + ...$$

Then one simply observes that the counter term for the $\sqrt{g} R(\gamma)$ term in (2.2) would be proportional to $N_g - 18$ and the volume term is proportional to $N_g$. Thus we see, from this point of view, why it is important to make the regulator a function of the differential operators appearing in the Hessians themselves. Any other choice, such as $\mathcal{R}_k = \mathcal{R}_k(-\nabla^2)$, would lead to non-universal heat kernel coefficients that do not appear in the one-loop effective action. One should also stress that the origin of $-11 + \frac{2}{3} N_f$ can also be traced to the heat kernel coefficient for the corresponding tensor structure in Yang-Mills (32). In addition one my also go
to the next order in the curvature expansion where in $d = 4$ the result is universal since $Q_0[W] = W(0) = 1$. Here we find explicitly the curvature squared term

$$\text{Tr}[W(\Delta_2)] - \text{Tr}[W(\Delta_1)] = \ldots + \frac{1}{(4\pi)^2} \int d^d x \sqrt{g} \left( \frac{53}{45} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{29}{40} R^2 \right) + \ldots$$

reproducing the one-loop counter term found in [20] but with the cosmological constant replaced by $\lambda_b \rightarrow R/4$.

We note the general pattern that universal factors appearing in beta functions and counter terms throughout quantum field theory can be found by use of heat kernel techniques. These factors can be found for dimensionful couplings, such as Newton’s constant, as well as dimensionless ones. The difference is that the renormalisation of dimensionless couplings do not come additionally with non-universal factors. However the sign of non-universal factors appearing in the renormalisation of dimensionful should be universal and gauge independent if they are to be physically meaningful. The fact that the universal factor found here only depends on the dimension through the number of polarisations would seem to support the view that this is indeed the case.

It should be noted, however, that the unique factors $N_g - 18$ and $N_g$ could be modified if one was to regulate the auxiliary fields differently than the transverse traceless fields. That is if we choose different regulators $C_1(z)$ and $C_2(z)$ for the separate traces in $\text{Tr}W(z)$ we would end up with different functions $W(z)$. Making such a choice would mean that the factors $N_g$ and $N_g - 18$ of (5.6) no longer be forthcoming. Equally if we choose to regulate with a type I cut off such that $R_k = R_k(-\nabla^2)$ rather than the choice $412$ these factors would again not appear in (5.10). One must conclude then that if the factor $N_g - 18$ is truly universal and physical only a subset of all possible regulator choices leads to physical results. What we suspect is that regulator schemes must break diffeomorphism invariance sufficiently mildly such that the cancellations of unphysical polarisations continues to be present.

To back up our point of view imagine we pick $C_1(z) = C_2(z)$, then the term in (5.6) proportional to the space time volume would not necessarily be proportional to $d - 3$. This being the case the vacuum energy would appear to be renormalised in $d = 3$ even though the evaluation of the unregulated effective action (5.8) would lead to no diversgencies proportional to the volume. A similar argument can be made for the factor $N_g - 18$ in the critical dimension $d = d_c$ for which $N_g(d_c) = 18$. Thus only by picking the regulator scheme as we have do we avoid such inconsistencies.

VI. BETA FUNCTION FOR NEWTON’S CONSTANT

It is now straightforward to find the beta function $\beta_G = \partial_\epsilon G$ for Newton’s constant by inserting (5.6) into (1.8). For general cutoff function $C(z)$ it reads

$$\beta_G = (d - 2)G - 2 \frac{N_g - 18}{(4\pi)^2} \frac{1}{\Gamma(\frac{d-2}{2})} \mathcal{I}_{d/2 - 1}[C] G^2$$

where the regulator dependent functional $\mathcal{I}_n[C]$ is given by the integral

$$\mathcal{I}_n[C_k] = \int_0^\infty dz z^n W(z) = \int_0^\infty dz z^{d/2 - 2} \frac{C(z) - z C'(z)}{z + C(z)}.$$  \hspace{1cm} (6.2)

The first term in (6.1) is the classical scaling arising from the dimensionful nature of $G_k$. The second term is a quantum correction proportional to the momentum integral $\mathcal{I}_{d/2 - 1}[C]$ indicating that this equation is one loop.

A. $d = 2 + \epsilon$ and the $d \rightarrow 2$ limit, a tale of three beta functions

Before continuing to the general $d$ case we take a digression into the behaviour of the beta function near two dimensions and try to understand the subtleties of the limit $d \rightarrow 2$. First we note that in the limit $d \rightarrow 2$
we have

$$\mathcal{I}_{d/2-1}(C)\big|_{d=2} = \frac{2}{d-2} + \text{finite terms} \quad (6.3)$$

where the singular term is independent of the regulator function $C_k$. However the coefficient of $G^2$ has a finite universal limit for $d \to 2$ owing to the presence of the gamma function. This allows one to extract a universal beta function in $d = 2 + \epsilon$ dimensions, given by

$$\beta_G = \epsilon G - \frac{38}{3} G^2. \quad (6.4)$$

We note that this is in agreement with previous studies \cite{33-35} using the linear parameterisation (2.5) but differs from the result obtained with the exponential parameterisation (2.6) \cite{9, 36} with $\rho = d/2 - 1$ and $\bar{\lambda}_b = 0$ which gives

$$\beta_G = \epsilon G - \frac{50}{3} G^2. \quad (6.5)$$

The reason for this is that by working off shell and setting $\bar{\lambda}_b = 0$ the exponential parameterisation gives a Hessian for the scalar mode $\phi$ becomes $S^{(2)}_{\phi\phi} \propto (d - 2) \Delta$ and thus the cancellation between these modes and $J_0$ does not occur. Instead the Hessian for the conformal modes mimics the induced action

$$S_{\text{anom}} \propto \int d^2 x \sqrt{\gamma} G^{IJ} \partial_\mu X^I \partial^\mu X^J, \quad (6.6)$$

of the conformal anomaly in $d = 2$ despite being of order $\epsilon^0 \bar{\lambda}^0$. In the equations leading to (6.5) such fluctuations are not present since all conformal fluctuations are cancelled by the functional measure. The result (6.5) can then be understood as arising from the particular way the limit $d \to 2$ is taken where the fact that the Hessian for $\bar{\phi}$ vanishes is not taken into account. The limit then paradoxically reproduces the expected result in $d = 2$ in the presence of the conformal anomaly even though the induced action (6.6) for has not been accounted for.

To clarify the situation we note that the beta function (6.4) is valid for $d \neq 2$ and that for the strict limit $d \to 2$ the Hessian for the modes $\phi$ vanishes reflecting conformal invariance. As such, we can simply adopt the additional condition $\sigma = 0$ to gauge fix the conformal invariance when the limit is taken. If we additionally include $D$ scalar fields these can be interpreted as the $D$ spacetime dimensions in string theory. That is we include an action of the form

$$S_{\text{string}} = \int d^4 x \sqrt{\gamma} G_{IJ} \partial_\mu X^I \partial^\mu X^J \big|_{d=2}, \quad (6.7)$$

with $I = 1, \ldots, D$ where we assume $G_{IJ}$ to be a flat metric over the $D$ dimensional spacetime in which the string lives. Going through the motions one then obtains a beta function for $d \to 2$ given by

$$\beta_G = -(26 - D) \cdot \frac{2}{3} G^2. \quad (6.8)$$

The difference between (6.4) and (6.8) arises from the lack of conformal fluctuations contributing to the later which therefore do not cancel the Jacobian $J_0$ which in turn leads to an additional term in the flow equation along with a contribution from the $D$ scalar fields. The beta function (6.8) is the expected result reflecting the fact that when 26 scalar fields are added to the action the conformally anomaly vanishes. Thus we reproduce the result that the bosonic string lives in $D = 26$ dimensions, in which case we do not need to bother about the conformal anomaly \cite{37}. If $D < 26$ the beta function (6.8) does not vanish and the conformally anomaly should be taken into account by including the induced action (6.6). This has been investigated recently in \cite{38} where the beta function (6.8) is found (for strictly $d = 2$) when using the exponential parameterisation. This is nonetheless consistent with our result (6.8) since the conformal fluctuations which are present there essentially play the role of one of the $D$ scalar fields.
We conclude that (6.4) is the physically meaningful gauge independent beta function $\epsilon > 0$ while in strictly $d = 2$ one needs to take into account the conformal invariance (or anomalous breaking thereof) to correctly evaluate the functional integral and find the corresponding beta function. To summarise (6.4) is the result of quantising general relativity in $d > 2$ where conformal fluctuations do not propagate and are exactly canceled by the Jacobian $J_0$. The beta function (6.5) arises when working in the limit $d \to 2$ where the conformal fluctuations mimic the effect of the induced action (6.6) such that the cancelation with $J_0$ is no longer exact. Finally the beta function (6.8) occurs when the conformal fluctuations are absent entirely, as is the case in strictly $d = 2$.

**B. $d$-dimensional beta function for Newton’s constant**

In $d > 2$ dimensions the integral $\mathcal{I}_{d/2-1}[C]$ depends on the regulator function. This must be the case since Newton’s coupling $G_k$ is dimensionful and thus $G$ will always depend on the regulator at least up to rescalings of $k$. However it is interesting to note that for an optimised cutoff [39] of the form $C_{\text{opt}} = (z_0 - z)\Theta(z_0 - z)$, where $\Theta(x)$ is a Heaviside theta function and $z_0$ is a positive constant, one obtains

$$\mathcal{I}_{d/2-1}[C_{\text{opt}}] = z_0 \frac{2}{d-2},$$

valid in all dimensions $d > 2$. Hence a property of the optimised cutoff is to set all higher order corrections in (6.2) to zero up to an arbitrary rescaling. For $z_0 \frac{2}{d-2} = (4\pi)^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)$ one then obtains the beta function

$$\beta_G = (d-2)G - \frac{2}{3}(18 - N_g) G^2,$$

where only the universal factor of $b$ remains. This form of this beta function is clearly universal up to the normalisation of the RG scale $k$ or equivalently $G$. Thus although $G$ itself is not a physical observable it is clear that the sign of the quantum correction to $\beta_G$ is universal since one has $\mathcal{I}_n[C] > 0$ for all regulator functions $C$.

For $N_g < 18$ the beta function implies $G_k$ decreases as the scale $k$ is increased. The running of $G$ stops at a UV fixed point given by,

$$G_* = \frac{d-2}{2} \cdot \frac{3(4\pi)^{\frac{d}{2}-1}\Gamma\left(\frac{d-2}{2}\right)}{18 - N_g} \frac{1}{\mathcal{I}_{d/2-1}[C]}.$$

This fixed point exists for positive $G_*$ in all dimensions for which $N_g < 18$ and describes an asymptotically safe quantum field theory at high energies. However since this is a semiclassical result this interpretation is rather premature and one would like to go beyond this approximation to confirm this conclusion. Indeed the critical exponent

$$1/\nu \equiv \frac{\partial\beta_G}{\partial G}(G_*) = d - 2,$$

which describes how $G$ approaches $G_*$ is simply the canonical one, independent of the regulator. To compute quantum corrections to $\nu$ one must go beyond the semi-classical approximation by computing $\beta_G$ to higher orders in $G$. This will be investigated in a companion paper [40] where we exploit a non-perturbative approximation to find quantum corrections to $\nu$.

**C. Paramagnetic vs Diamagnetic interactions**

From the form of the beta function (6.10) we observe that the increase of spin two degrees of freedom is responsible for the loss of asymptotic safety in higher dimensions. The term $-18$ which allows for asymptotic...
safety in $d < 8$ dimensions is universal and independent of the dimension. The origin of this term is the interactions of the metric fluctuations with the curvature of spacetime, and represents the ‘non-Abelian’ nature of gravity, in agreement with the ideas put forward in [41]. This can be seen by writing the heat kernel coefficients of the transverse-traceless tensor and transverse vectors in terms of the heat kernel coefficients $A_{2,n}$ for scalars ($n = 0$), vectors ($n = 1$), and traceless tensors ($n = 2$) without differential constraints. The heat kernels coefficients $A_{2,n}$ correspond to differential operators of the form,

$$\tilde{\Delta}_n = -\nabla^2 + U_n$$

which possess potentials $U_n$ given by

$$U_2 = \frac{2}{d(d-1)}R, \quad U_1 = -\frac{1}{d}R, \quad U_0 = -\frac{2}{d}R.$$  

(6.14)

Now one can show the following relation between the differentially constrained heat kernel coefficients $B_{2,n}$ and the unconstrained coefficients

$$B_{2,2} - B_{2,1} = A_{2,2} + A_{2,0} - 2A_{2,1}.$$  

(6.15)

Following [41] we then denote terms proportional to the potentials (6.14) as ‘paramagnetic’ and those arising from the Laplacian as ‘diamagnetic’ to obtain the expression

$$A_{2,n} - 2A_{2,1} + A_{2,0} = \left(\frac{N_g}{d_{\text{diamagnetic}}} - \frac{18}{d_{\text{paramagnetic}}}\right)\frac{R}{6}.$$  

(6.16)

This confirms the results of [41] in a gauge independent setting: the physical mechanism behind asymptotic safety is related to the metric fluctuation’s paramagnetic interaction with the curvature of spacetime encoded in the potentials $U_n$. This effect is countered in $d > 3$ dimensions by the diamagnetic interactions encoded in the Laplacians which dominate starting in $d \geq 8$ dimensions.

VII. RENORMALISATION OF THE VACUUM ENERGY

As we have seen the absence of the cosmological constant in the beta function for Newton’s constant results from demanding gauge independence and therefore its presence in gauge dependent beta functions is most likely unphysical. This can be seen as resulting from the fact that the cosmological constant is not a mass for the graviton. Rather the vacuum energy only couples to the conformal fluctuations $\sigma$ which are not themselves dynamical. Nonetheless the vacuum energy $\lambda = \frac{\lambda}{8\pi G}$ has also a $k$ dependence given by,

$$\partial_t \lambda = -d\lambda + \frac{N_g}{(4\pi)^{\frac{d}{2}}\Gamma(d/2)}I_d[C]$$

(7.1)

The quantum correction being proportional to $N_g$. This beta function has a fixed point,

$$\lambda_* = \frac{1}{d} \frac{N_g}{(4\pi)^{\frac{d}{2}}\Gamma(d/2)}I_d[C].$$

(7.2)

Note that in the absence of any propagating degrees of freedom, i.e. when taking $d = 3$, the quantum corrections to this beta function vanish. This result is just related to the divergencies of the vacuum

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4 This relation also shows that one will find the same beta functions if one does not perform the decomposition $h_{\mu\nu}$ used here and instead regulates the fields without differential constraints. Then the contribution from the ghosts produces the term $-2A_{2,1}$ while the trace free fluctuations give $A_{2,2}$ and the trace fluctuations $\delta = h^\mu_\mu$ give $A_{2,0}$. 
encountered in any quantum field theory in flat spacetime. In this case we are just looking at the graviton contribution to the vacuum energy. To see this lets write (9.4) explicitly in terms of the IR cut off scale \( k \) and the dimensionful vacuum energy \( \lambda k = k^d \lambda \), then one has

\[
\partial_t \lambda_k = d k^d \lambda_* ,
\]

introducing the bare UV scale \( \Lambda \) defined by \( \lambda_k = \Lambda \equiv \lambda_b \) we can solve this equation to find,

\[
\lambda_k = \left( k^d - \Lambda^d \right) \lambda_* + \lambda_b .
\]

In the limit \( k \to 0 \) we remove the IR regulator to obtain the observed vacuum energy \( \lambda_0 \equiv \lambda_{k=0} \). In turn we may express the bare vacuum energy in terms of \( \lambda_0 \) as

\[
\lambda_b = \lambda_0 + \Lambda^d \lambda_* .
\]

Thus the ‘beta function’ (9.4) is nothing but the statement that we must include a counter term proportional to the number of local degrees of freedom in order to get a finite renormalised vacuum energy \( \lambda_0 \).

**VIII. CONTINUUM LIMIT**

As discussed previously the existence of a continuum limit, \( \Lambda \to \infty \), relies on the sign of \( b \), the \( G^2 \) coefficient of \( \beta_G \). To understand this we note that the sign of the bare Newtons constant \( G_b \) must be positive if the functional integral is to make sense. If one is forced to take \( G_b < 0 \) to renormalise the theory one must concede that the theory cannot be fundamental and that the continuum limit does not exist. From the (6.1) one may infer the energy dependence of the Planck mass \( \kappa_k = \frac{1}{16 \pi G_k^*} \),

\[
\kappa_k = \left( k^{d-2} - \Lambda^{d-2} \right) \kappa_* + \kappa_b ,
\]

where \( \kappa_* = \frac{1}{16 \pi G_*} \) is the regulator dependent value of the fixed point for the dimensionless quantity \( \kappa = k^{2-d} \kappa_k \). Taking \( k \to 0 \) one has the renormalised Planck mass \( M_{Pl}^{-2} \equiv \frac{1}{16 \pi G_N} = \kappa_{k=0} \) where \( G_N \) is the measured value of Newtons constant. Then we have

\[
\frac{1}{16 \pi G_b} = \frac{1}{16 \pi G_N} + \Lambda^{d-2} \kappa_* .
\]

for the functional integral to make sense one must have \( G_b > 0 \) otherwise the kinetic terms will have the wrong sign and the theory will breakdown. In particular for the semiclassical theory to be valid we need

\[
\frac{1}{16 \pi G_N} > -\Lambda^{d-2} \kappa_* .
\]

Now we know that \( \kappa_* \propto (18 - N_g) \) and that the proportionality constant is positive from which we obtain the inequality for \( N_g > 18 \)

\[
\left( \frac{\Lambda}{M_{Pl}} \right)^{d-2} < 16 \pi|G_*| .
\]

Thus, while for \( 18 < N_g \) there is no maximum value for which the UV scale can take, for \( N_g > 18 \) it provides a bound. In the latter case we can say that the semiclassical theory predicts its own downfall at a finite energy scale. In the former case no bound exists and the semiclassical approximation suggests that the theory is asymptotically safe.
IX. INCLUSION OF MATTER

It is rather straightforward to include the effects of matter. This is done by adding a matter action to the bare Lagrangian and adding a corresponding regulator term (see for example [42]) or by computing the regulated one loop effective action [43]. Doing the former adds new traces to the right hand side of (4.5) proportional to the number of matter fields. In turn these give contributions to the beta functions. Here we give the result in $d = 4$ spacetime dimensions, although the result is easily generalised to arbitrary $d$,\(^5\)

$$
\beta_G = 2G - \frac{2}{3} (16 - 2N_D + 4N_M - N_s) \ G^2 ,
$$

(9.1)

up to the normalisation factor, where $N_D$ is the number of Dirac fermions, $N_M$ is the number of gauge fields and $N_s$ is the number of scalars. Plugging in the values for the standard model ($N_D = 45/2$, $N_M = 12$, $N_s = 4$) we have

$$
\beta_G = (d - 2)G - \frac{2}{3} \cdot 15 \ G^2 ,
$$

(9.2)

which continues to be anti-screening as in the pure gravity case. We therefore can conclude that the semi-classical theory of the standard model coupled to gravity is not predicting its own downfall at the quantum level and the existence of a continuum limit along the lines of asymptotic safety remains a possibility. In particular the beta function predicts asymptotic safety provided

$$
16 - 2N_D + 4N_M - N_s > 0
$$

(9.3)

otherwise $G_*$ is negative and the bound (8.4) applies. We observe that by adding large numbers of fermions or scalars the theory can break down a fraction of the Planck scale. This then puts constraints on effective theories that contain a large number of scalars or fermions.

Similarly the renormalisation of the vacuum energy at one-loop is given by

$$
\partial_t \lambda = -d \lambda + \frac{N_{\text{Bose}} - N_{\text{Fermi}}}{4\pi^2 \Gamma(d/2)} \frac{1}{\sqrt{2}} \mathcal{I}_d [C] ,
$$

(9.4)

where the number of bosons $N_{\text{Bose}}$ (including gravitons) and the number of fermions $N_{\text{Fermi}}$ enter with opposite signs. This is the expected result and entails that the vacuum energy is not renormalised at one-loop for super-symmetric theories.

\(^5\) Here we assume the fields are massless and minimally coupled to gravity. The beta function will be modified if these assumptions are dropped.
X. CONCLUSIONS

In this paper we have derived a gauge independent effective action for quantum gravity at one-loop which only depends on physical fluctuations of the metric. To achieve gauge independence we have used a specific parameterisation of the metric for which the volume element is linear in the conformal fluctuation $\sigma$. We have then used renormalisation group techniques to compute the one-loop beta function for Newton’s constant. When introducing a regulator scheme we have been careful to do so in such a manner that cancellations, which should occur when diffeomorphism invariance is present, continue to occur in the presence of the regulator. In particular this relies on choosing a type II regulator of the form $\mu^2$ while choosing the same regulator function $C(z)$ for the different fields. Different choices can lead to unphysical results. In particular the absence of the local fluctuations in $d=3$ may not be manifest if the regulator functions are chosen differently.

Here we offer an interpretation of the gauge independence of our results. A key point is that our calculations are performed on Einstein spaces for which the trace-free Einstein equations are solved. Thus we are on shell with respect to all but one of the equations of motion, this one being essentially the equation of motion for the volume element. Now, one way to ensure gauge independence in any gauge theory is to work in terms of physical quantities rather than gauge variant fields such as the metric. Since we single out the volume element by our parameterisation we are therefore singling out the corresponding physical quantity which in this case is the gauge invariant field $\phi$ given in (3.5). Thus we solve the problem of finding a physical parameterisation on an Einstein space (at least at one-loop order). This is seen explicitly since without the gauge fixing terms $S^{(2)}$ is independent of both $\xi$ and $\psi$.

We therefore believe that the general scheme put forward here is physically well motivated and should be used in approximations beyond the simple one employed here. The fact that the universal factor only depends on the dimensionality of spacetime through the number polarisations $N_g$ would seem to be a vindication of this claim and that it represents the analogy to the factor $11 - \frac{2}{3} N_f$ in QCD.

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