Non-relativistic Schrödinger theory on q-deformed quantum spaces I
Mathematical framework and equations of motion

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Abstract
The aim of these three papers (I, II, and III) is to develop a q-deformed version of non-relativistic Schrödinger theory. Paper I introduces the fundamental mathematical and physical concepts. The braided line and the three-dimensional q-deformed Euclidean space play the role of position space. For both cases the algebraic framework is extended by a time element. A short review of the elements of q-deformed analysis on the spaces under consideration is given. The time evolution operator is introduced in a consistent way and its basic properties are discussed. These reasonings are continued by proposing q-deformed analogs of the Schrödinger and the Heisenberg picture.

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1 Introduction

From the beginning of quantum field theory to the present day there is a great hope that the occurrence of infinities in the formalism of quantum field theory results from an incomplete description of space-time at very small distances [1]. Bohr and Heisenberg have been the first to suggest that quantum field theories should be formulated on a space-time lattice, since it would imply the existence of a smallest length [2]. One of the earliest and most serious attempts towards this goal was the concept of ‘quantized space-time’ due to Snyder [3, 4]. Of course there were many other researchers who took up this idea over and over again (see for example Refs. [5–8]) and each of the different approaches has its own difficulties and advantages. However, with the arrival of quantum groups and quantum spaces [9–20] in the eighties of the last century a new, very promising method to discretize space and time seems to be available [21, 22]. The observation that it leads to very realistic deformations of classical space-time symmetries nourishes the hope for a new powerful regularization schema [23–25].

In this paper attention is focused on q-deformed quantum spaces [26–31]. (For other deformations of space-time see Refs. [32–38].) Concretely, we deal with the so-called 'braided line' and the three-dimensional q-deformed Euclidean space. The braided line can be viewed as deformation of the set of real numbers, while the three-dimensional q-deformed Euclidean space is
nothing other than a non-commutative version of the classical three-dimensional Euclidean space. Essentially for us is the fact that on both spaces differential calculi exist which are recognized as q-analogs of classical translational symmetry [39–43]. The aim of our paper is to show that this algebraic framework is suitable to formulate a q-deformed version of non-relativistic Schrödinger theory.

Towards this end we first extend the coordinate algebras of braided line and three-dimensional q-deformed Euclidean space by a time element in such a way that it perfectly fits into the existing algebraic structures. Section 2 is devoted to this task. Our considerations will show us that the time element is central in the corresponding space-time algebras and is decoupled from space coordinates. In Sec. 3 these results are combined with those of our previous work in Refs. [44–50] to give a q-deformed version of analysis to the extended quantum spaces of braided line and three-dimensional q-deformed Euclidean space. In doing so, we present expressions for calculating products, partial derivatives, integrals, exponentials, translations, and braided products on the quantum spaces under consideration. We will see that due to its special algebraic properties the time element behaves like a commutative coordinate and the expressions for its derivatives, integrals, and so on are given by the classical ones. For the space coordinates, however, the situation is somewhat different, since their derivatives, integrals, and so on lead to one- and three-dimensional versions of the well-known q-calculus [51]. In this manner we obtain space-time structures in which space is discretized but time is still continuous.

Our results so far will then be used to introduce time evolution operators in a consistent way. Towards this end we start in Sec. 4 from q-deformed Taylor rules for the quantum spaces under consideration. Exploiting the algebraic properties of space and momentum variables we finally regain the well-known expressions for the time evolution operator. In Sec. 5 this result will enable us to formulate q-deformed versions of the Schrödinger and the Heisenberg picture in considerable analogy to the undeformed case. Finally, Sec. 6 closes the considerations so far by a conclusion before we will take them up in part II of this paper.

2 Algebraic set-up

This section is devoted to the algebras we are dealing throughout the paper, i.e. the braided line and the q-deformed Euclidean space in three dimensions. It is our aim to enlarge both algebras by a time element and to derive
2 ALGEBRAIC SET-UP

commutation relations for the generators of the extended algebras.

2.1 Extended braided line

Let us recall that the algebraic properties of quantum groups and quantum spaces are completely encoded in their R-matrices, for which we require to satisfy the quantum Yang-Baxter equation [17–19]:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}.$$ (1)

If we want to extend the algebra of the braided line by a time element with trivial braiding the R-matrix for this space should take the form

$$\hat{R}^{ij}_{kl} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$ (2)

where $q > 1$ and $i, j, k, l \in \{0, 1\}$. Notice that rows and columns of the above matrix are arranged in the order 00, 01, 10, and 11. We take the convention that the indices 0 and 1 respectively correspond to a time and a space coordinate. One can check that the above matrix indeed gives a solution to the Yang-Baxter equation (I am very grateful to Alexander Schmidt for doing this calculation with Mathematica.). Thus, we refer to it as the R-matrix of the so-called extended braided line.

Next, we would like to find the commutation relations for the generators of the extended braided line. Towards this end we first determine the eigenvalues of the new R-matrix in (2). They take on the values 1, $-1$, and $q$. The projectors onto the corresponding eigenspaces are then given by

$$P_+ = \frac{(\hat{R} - \text{Id})(\hat{R} - q\text{Id})}{2(1 + q)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$ (3)

$$P_- = \frac{(\hat{R} + \text{Id})(\hat{R} - q\text{Id})}{2(1 - q)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ (4)
The projector $P_-$ can be recognized as $q$-analog of an antisymmetrizer. For this reason, it determines the commutation relations for the extended braided line, i.e.

\[(P_-)^{ij}_{kl} X^k X^l = 0 \quad \Rightarrow \quad X^0 X^1 = X^1 X^0,\]  

where summation over repeated indices is understood. The other two projectors lead us to the commutation relations of the exterior algebra of the braided line, since the differentials have to fulfill

\[ (P_+)^{ij}_{kl} dX^k dX^l = 0, \quad (P_0)^{ij}_{kl} dX^k dX^l = 0, \]  

or, more concretely,

\[ dX^0 dX^0 = 0, \quad dX^1 dX^1 = 0, \quad dX^0 dX^1 = -dX^1 dX^0. \]  

Let us note that the commutation relations of the differentials can alternatively be written as

\[ dX^i dX^j = (P_-)^{ij}_{kl} dX^k dX^l = -\hat{R}^{ij}_{kl} dX^k dX^l, \]  

which implies

\[ X^i dX^j = \hat{R}^{ij}_{kl} dX^k X^l. \]  

As next step we introduce partial derivatives by

\[ d = dX^i \partial_i, \]  

where the exterior derivative $d$ obeys the usual properties of nilpotency and Leibniz rule, i.e.

\[ d^2 = 0, \]  
\[ d(fg) = (df)g + f(dg). \]  

From (12) together with (10) it can be shown that we have as Leibniz rules

\[ \partial_i X^j = \delta^j_i + \hat{R}^{jk}_{il} X^l \partial_k, \]  

where summation over repeated indices is understood.
or, more explicitly,

\begin{align}
\partial_0 X^0 &= 1 + X^0 \partial_0, \\
\partial_0 X^1 &= X^1 \partial_0, \\
\partial_1 X^0 &= X^0 \partial_1, \\
\partial_1 X^1 &= 1 + q X^1 \partial_1.
\end{align}

(14)

\begin{align}
\partial_0 X^0 &= 1 + X^0 \partial_0, \\
\partial_0 X^1 &= X^1 \partial_0, \\
\partial_1 X^0 &= X^0 \partial_1, \\
\partial_1 X^1 &= 1 + q X^1 \partial_1.
\end{align}

(15)

For the sake of completeness, it should be noted that partial derivatives satisfy the same commutation relations as quantum space coordinates, i.e.

\[ \partial_0 \partial_1 = \partial_1 \partial_0. \]

(16)

Now, we would like to enrich the algebraic structure by adding a conjugation. A consistent choice is given by

\[ \overline{X}^0 = X^0, \quad \overline{X}^1 = X^1, \]

(17)

and

\[ \overline{\partial}_0 = -\partial_0, \quad \overline{\partial}_1 = -\partial_1. \]

(18)

Applying this conjugation to the relations in (14) and (15) yields a second differential calculus. Its relations read

\begin{align}
\hat{\partial}_0 X^0 &= 1 + X^0 \hat{\partial}_0, \\
\hat{\partial}_0 X^1 &= X^1 \hat{\partial}_0, \\
\hat{\partial}_1 X^0 &= X^0 \hat{\partial}_1, \\
\hat{\partial}_1 X^1 &= 1 + q^{-1} X^1 \hat{\partial}_1,
\end{align}

(19)

\begin{align}
\hat{\partial}_0 X^0 &= 1 + X^0 \hat{\partial}_0, \\
\hat{\partial}_0 X^1 &= X^1 \hat{\partial}_0, \\
\hat{\partial}_1 X^0 &= X^0 \hat{\partial}_1, \\
\hat{\partial}_1 X^1 &= 1 + q^{-1} X^1 \hat{\partial}_1,
\end{align}

(20)

where \( \hat{\partial}_0 \equiv \partial_0 \) and \( \hat{\partial}_1 \equiv q \partial_1 \). In analogy to (13) we have

\[ \hat{\partial}_i X^j = \delta_i^j + (\hat{R}^{-1})_{ik} X^l \hat{\partial}_k. \]

(21)

Last but not least, we would like to say a few words about the quantum group coacting on the extended braided line. If we require for the coaction

\[ \beta(X^i) = T^i_j \otimes X^j, \]

(22)

to be compatible with the algebraic structure of the extended braided line
the quantum group generators have to be subject to the relations

$$\hat{R}_{kl}^{ij} T^k_m T^l_n = T^i_k T^j_l \hat{R}^{kl}_{mn},$$

(23)

from which it follows that

$$T^i_j = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{with } ab = ba.$$  

(24)

If we have $\pi = a$ and $\bar{b} = b$ the extended braided line even becomes a comodule-\(\ast\)-algebra.

### 2.2 Extended three-dimensional q-deformed Euclidean space

As in the previous subsection we start our considerations from the R-matrix. The R-matrix for the three-dimensional q-deformed Euclidean space extended by a time element is of block-diagonal form. Its building-blocks read [52]

$$
\begin{align*}
++ & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
-- & \begin{bmatrix} 3+ & 3+ \\ 3+ & 3- \end{bmatrix} \\
+3 & \begin{bmatrix} 0 & q^{-2} \\ q^{-2} & q^{-2}\lambda_+ \end{bmatrix}, \\
3+ & \begin{bmatrix} 0 & q^{-2} \\ q^{-2} & q^{-2}\lambda_+ \end{bmatrix}, \\
3- & \begin{bmatrix} 0 & q^{-2} \\ q^{-2} & q^{-2}\lambda_+ \end{bmatrix}, \\
-3 & \begin{bmatrix} 0 & q^{-2} \\ q^{-2} & q^{-2}\lambda_+ \end{bmatrix}, \\
+3 & \begin{bmatrix} 0 & q^{-2} \\ q^{-2} & q^{-2}\lambda_+ \end{bmatrix}, \\
33 & \begin{bmatrix} 0 & q^{-2} \\ q^{-2} & q^{-2}\lambda_+ \end{bmatrix}.
\end{align*}
$$

(25-28)
Notice that space coordinates are labeled by \(+\), \(3\), or \(\ast\), while the index 0 refers to the time element.

To get commutation relations for the coordinates in space and time we need to know the eigenvalues of the above \(R\)-matrix. They are given by 1, \(-q^{-4}\), \(-q^{-6}\), and \(-1\). Thus, the projectors onto the corresponding irreducible subspaces can be calculated from the identities

\[
P_+ = \frac{(\hat{R} + q^{-4}\text{Id})(\hat{R} - q^{-6}\text{Id})(\hat{R} + \text{Id})}{2(1 + q^{-4})(1 - q^{-6})},
\]

\[
P_- = \frac{(\hat{R} - \text{Id})(\hat{R} - q^{-6}\text{Id})(\hat{R} + \text{Id})}{(1 + q^{-4})(q^{-4} + q^{-6})(1 - q^{-4})},
\]

\[
P_0 = \frac{(\hat{R} - \text{Id})(\hat{R} + q^{-4}\text{Id})(\hat{R} + \text{Id})}{(q^{-6} - 1)(q^{-6} + q^{-4})(q^{-6} + 1)},
\]

\[
P' = \frac{(\hat{R} - \text{Id})(\hat{R} + q^{-4}\text{Id})(\hat{R} - q^{-6}\text{Id})}{2(q^{-4} - 1)(1 + q^{-6})}.
\]

The projectors \(P_-\) and \(P'\) lead us to the defining relations of the extended three-dimensional q-deformed Euclidean space:

\[
(P_-)_{kl}^{ij} X^k X^l = 0, \quad (P')_{kl}^{ij} X^k X^l = 0.
\]

Written out explicitly, these relations become

\[
X^0 X^+ = X^+ X^0, \quad X^0 X^- = X^- X^0, \quad X^0 X^3 = X^3 X^0, \\
X^- X^3 = q^2 X^3 X^-, \quad X^3 X^+ = q^2 X^+ X^3, \\
X^- X^+ - X^+ X^- = \lambda X^3 X^3,
\]

where \(\lambda = q - q^{-1}\). It should be mentioned that under transformations of the quantum group \(SU_q(2)\) the quantum space coordinates \(X^+, X^3\), and
$X^-$ behave like components of a three-vector, while the time coordinate $X^0$ transforms like a scalar. This situation is in complete analogy to the undeformed case.

The exterior algebra to the extended q-deformed Euclidean space in three dimensions is defined by the relations

$$(P_+)^{ij}_{kl} dX^k dX^l = 0, \quad (P_0)^{ij}_{kl} dX^k dX^l = 0. \quad (36)$$

From these relations we obtain

$$dX^A dX^B = -q^4 (\hat{R}_{SO_q(3)})^{AB}_{CD} dX^C dX^D$$

and

$$dX^0 dX^0 = 0, \quad dX^0 dX^A = -dX^A dX^0, \quad (37)$$

where $A, B \in \{+, 3, -\}$. Notice that $\hat{R}_{SO_q(3)}$ stands for the R-matrix of the three-dimensional q-deformed Euclidean space without a time element. Furthermore, we took the convention that capital letters like $A, B, \text{etc.}$ run through $(+, 3, -)$.

The commutation relations between differentials of coordinates require that the braiding between quantum space coordinates and their differentials takes the form

$$X^A dX^B = q^4 (\hat{R}_{SO_q(3)})^{AB}_{CD} dX^C X^D, \quad (39)$$

or, alternatively,

$$X^A dX^B = q^{-4} (\hat{R}^{-1}_{SO_q(3)})^{AB}_{CD} dX^C X^D. \quad (40)$$

If the time element is involved we additionally have

$$X^0 dX^A = dX^A dX^0, \quad X^A dX^0 = dX^0 X^A, \quad X^0 dX^0 = dX^0 X^0. \quad (41)$$

We recommend Ref. [53] if the reader is unfamiliar with the reasonings leading to the above relations.

Now, we have everything together to introduce partial derivatives. We start from the Leibniz rules

$$dX^A = (dX^A) \cdot X^A d., \quad \quad dX^0 = (dX^0) \cdot X^0 d., \quad (42)$$

where the point stands for an additional element. Then we substitute the
exterior derivative by \( d = dX^i \partial_i \) and obtain
\[
\begin{align*}
  dX^i \partial_i X^A &= dX^A + X^A dX^i \partial_i, \\
  dX^i \partial_i X^0 &= dX^0 + X^0 dX^i \partial_i. 
\end{align*}
\] (43)

Using relations (39) and (41) we are able to switch all differentials to the far left:
\[
\begin{align*}
  dX^i \partial_i X^A &= dX^A + dX^0 X^A \partial_0 + q^4 (\hat{R}_{SO_0(3)}^{AB})_{CD} A X^C X^D \partial_B, \\
  dX^i \partial_i X^0 &= dX^0 + dX^i X^0 \partial_i. 
\end{align*}
\] (44)

Since the differentials \( dX^i, i \in \{+, 0, -\} \), are linearly independent, it follows from the above identities that
\[
\begin{align*}
  \partial_B X^A &= \delta_B^A + q^4 (\hat{R}_{SO_0(3)}^{AB})_{CD} A X^C \partial_D, \\
  \partial_0 X^A &= X^A \partial_0, 
\end{align*}
\] (45)

and
\[
\begin{align*}
  \partial_0 X^0 &= 1 + X^0 \partial_0, \quad \partial_A X^0 = X^0 \partial_A. 
\end{align*}
\] (46)

If we apply (40) instead of (39) in the above derivation, we arrive at a second differential calculus with relations
\[
\begin{align*}
  \hat{\partial}_B X^A &= \delta_B^A + q^{-4} (\hat{R}_{SO_0(3)}^{-1}^{AB})_{CD} A X^C \hat{\partial}_D, \\
  \hat{\partial}_0 X^A &= X^A \hat{\partial}_0, 
\end{align*}
\] (47)

and
\[
\begin{align*}
  \hat{\partial}_0 X^0 &= 1 + X^0 \hat{\partial}_0, \quad \hat{\partial}_A X^0 = X^0 \hat{\partial}_A, 
\end{align*}
\] (48)

where \( \hat{\partial}_A \equiv q^6 \partial_A \) for \( A \in \{+, 3, -\} \) and \( \hat{\partial}_0 \equiv \partial_0 \). For the sake of completeness let us note that partial derivatives now obey the same commutation relations as covariant quantum space coordinates, i.e.
\[
\begin{align*}
  \partial_0 \partial_+ &= \partial_+ \partial_0, \quad \partial_0 \partial_- = \partial_- \partial_0, \quad \partial_0 \partial_3 = \partial_3 \partial_0, \\
  \partial_+ \partial_3 &= q^2 \partial_3 \partial_+, \quad \partial_3 \partial_- = q^2 \partial_- \partial_3, \\
  \partial_+ \partial_- - \partial_- \partial_+ &= \lambda \partial_3 \partial_3. 
\end{align*}
\] (49)

It is rather instructive to write the Leibniz rules out. In doing so, we obtain
\[
\partial_+ X^0 = X^0 \partial_+, 
\]
\( \partial_+ X^+ = 1 + q^4 X^+ \partial_+ , \)
\( \partial_+ X^3 = q^2 X^3 \partial_+ , \)
\( \partial_+ X^- = X^- \partial_+ , \) \hspace{1cm} (50)

\( \partial_3 X^0 = X^0 \partial_3 , \)
\( \partial_3 X^+ = q^2 X^+ \partial_3 , \)
\( \partial_3 X^3 = 1 + q^2 X^3 \partial_3 + q^2 \lambda \lambda_+ X^+ \partial_+ , \)
\( \partial_3 X^- = q^2 X^- \partial_3 + q \lambda \lambda_+ X^3 \partial_+ , \) \hspace{1cm} (51)

\( \partial_- X^0 = X^0 \partial_- , \)
\( \partial_- X^+ = X^+ \partial_- , \)
\( \partial_- X^3 = q^2 X^3 \partial_- + q \lambda \lambda_+ X^+ \partial_3 , \)
\( \partial_- X^- = 1 + q^4 X^- \partial_- + q^2 \lambda \lambda_+ X^3 \partial_3 + q \lambda^2 \lambda_+ X^+ \partial_+ , \) \hspace{1cm} (52)

\( \partial_0 X^0 = 1 + X^0 \partial_0 , \)
\( \partial_0 X^+ = X^+ \partial_0 , \)
\( \partial_0 X^3 = X^3 \partial_0 , \)
\( \partial_0 X^- = X^- \partial_0 , \) \hspace{1cm} (53)

and

\( \hat{\partial}_+ X^0 = X^0 \hat{\partial}_+ , \)
\( \hat{\partial}_+ X^- = X^- \hat{\partial}_+ , \)
\( \hat{\partial}_+ X^3 = q^{-2} X^3 \hat{\partial}_+ - q \lambda \lambda_+ X^- \hat{\partial}_3 , \)
\( \hat{\partial}_+ X^+ = 1 + q^{-4} X^+ \hat{\partial}_+ - q^{-2} \lambda \lambda_+ X^3 \hat{\partial}_3 + q^{-1} \lambda^2 \lambda_+ X^- \hat{\partial}_-, \) \hspace{1cm} (54)

\( \hat{\partial}_3 X^0 = X^0 \hat{\partial}_3 , \)
\( \hat{\partial}_3 X^- = q^{-2} X^- \hat{\partial}_3 , \)
\( \hat{\partial}_3 X^3 = 1 + q^{-2} X^3 \hat{\partial}_3 - q^{-2} \lambda \lambda_+ X^- \hat{\partial}_- , \)
\( \hat{\partial}_3 X^+ = q^{-2} X^+ \hat{\partial}_3 - q^{-1} \lambda \lambda_+ X^3 \hat{\partial}_-, \) \hspace{1cm} (55)

\( \hat{\partial}_- X^0 = X^0 \hat{\partial}_- \)
\( \hat{\partial}_- X^+ = X^+ \hat{\partial}_- , \)
\[
\hat{\partial}_- X^3 = q^{-2} X^3 \hat{\partial}_- , \\
\hat{\partial}_- X^- = 1 + q^{-4} X^- \hat{\partial}_- ,
\]
(56)
\[
\hat{\partial}_0 X^0 = 1 + X^0 \hat{\partial}_0 , \\
\hat{\partial}_0 X^+ = X^+ \hat{\partial}_3 , \\
\hat{\partial}_0 X^3 = X^3 \hat{\partial}_0 , \\
\hat{\partial}_0 X^- = X^- \hat{\partial} ,
\]
(57)
where we set, for brevity, \( \lambda_+ \equiv q + q^{-1} \).

It remains to introduce a conjugation being compatible with the algebraic structure presented so far. To this end, we need the quantum metric of the three-dimensional q-deformed Euclidean space. Its explicit form can be read off from the projector \( P_0 \), as it holds [31]
\[
(P_0)_{AB}^{CD} = \frac{1}{g_{EF} g_{EF}} g^{AB} g_{CD} .
\]
(58)
In this manner, one can verify that the non-vanishing entries of \( g_{AB} \) and \( g^{AB} \) are given by
\[
g^{+-} = g_{+-} = -q , \quad g^{-+} = g_{-+} = -q^{-1} , \quad g^{33} = g_{33} = 1 .
\]
(59)
With the three-dimensional quantum metric at hand we are able to write down the conjugation properties of coordinates and partial derivatives in a rather compact form:
\[
\overline{X^A} = X_A = g_{AB} X^B , \quad \overline{\partial_A} = -\partial^A = -g^{AB} \partial_B ,
\]
(60)
and
\[
\overline{X^0} = X^0 , \quad \overline{\partial_0} = -\partial_0 .
\]
(61)

3 Elements of q-analysis

In this section a q-deformed version of analysis is given to the extended braided line and the extended q-deformed Euclidean space in three dimensions. Especially, we present expressions for computing star products, braided products, q-translations, operator representations of partial derivatives, q-integrals, and q-exponentials. With this toolbox of essential elements of q-analysis we are in a position to formulate quantum mechanics on
the quantum spaces under consideration. Finally, it should be noted that the reasonings in this section are mainly based on the ideas developed in Refs. [17, 20, 30, 44–50, 54–58].

3.1 Extended braided line

First of all, let us mention that the product on the extended braided line is the commutative one. This observation follows from a short look at the commutation relations in Eq. (6). However, if we want to commute functions living in distinct quantum spaces things become slightly different. In this respect, let us recall that the commutation relations between generators of different quantum spaces are determined by the R-matrix or its inverse, i.e.

\[ X^i \odot_L Y^j \equiv (1 \otimes X^i)(Y^j \otimes 1) = \hat{R}_{kl}^{ij} Y^k \otimes X^l, \quad (62) \]

and alternatively

\[ X^i \odot_L Y^j \equiv (1 \otimes X^i)(Y^j \otimes 1) = (\hat{R}^{-1})_{kl}^{ij} Y^k \otimes X^l. \quad (63) \]

These relations lead us to braided products for commutative functions,

\[
\begin{align*}
  f(x^i) \odot_L g(y^j) &= q^{\hat{n}_{y^j} \odot \hat{n}_{x^1}^{x^i}} g(y^j) \otimes f(x^i), \\
  f(x^i) \odot_L g(y^j) &= q^{-\hat{n}_{y^j} \odot \hat{n}_{x^1}^{x^i}} g(y^j) \otimes f(x^i),
\end{align*}
\]

(64)

where we introduced the operator

\[ \hat{n}_{x^1} \equiv x^1 \frac{\partial}{\partial x^1}. \quad (65) \]

Notice that the partial derivative in Eq. (65) is a commutative one. At this place it should also be mentioned that throughout this paper we use the convention to write generators of quantum spaces in capitals, while commutative coordinates are written in small letters. (In the case of the braided line the identification with a commutative algebra is rather trivial. For q-deformed Euclidean space in three dimensions, however, such an identification needs some more thoughts.)

Next, we come to translations on the extended braided line. Translations on quantum spaces are described by their Hopf structures. On quantum space generators these Hopf structures become

\[ \Delta_L(X^0) = X^0 \otimes 1 + 1 \otimes X^0, \]
3 ELEMENTS OF Q-ANALYSIS

\[ \Delta_L(X^1) = X^1 \otimes 1 + \Lambda^{-1} \otimes X^1, \]
\[ S_L(X^0) = -X^0, \]
\[ S_L(X^1) = -\Lambda X^1, \]
\[ \epsilon_L(X^i) = 0, \] (66)

and

\[ \Delta_{\bar{L}}(X^0) = X^0 \otimes 1 + 1 \otimes X^0, \]
\[ \Delta_{\bar{L}}(X^1) = X^1 \otimes 1 + \Lambda \otimes X^1, \]
\[ S_{\bar{L}}(X^0) = -X^0, \]
\[ S_{\bar{L}}(X^1) = -\Lambda^{-1}X^1, \]
\[ \epsilon_{\bar{L}}(X^i) = 0, \] (67)

where \( \Lambda \) stands for a unitary scaling operator subject to

\[ \Lambda X^0 = X^0 \Lambda, \quad \Lambda X^1 = qX^1 \Lambda, \]
\[ \Lambda \partial_0 = \Lambda \partial_0, \quad \Lambda \partial_1 = q^{-1} \Lambda \partial_1. \] (68)

This scaling operator and its inverse can be viewed as generators of a Hopf algebra denoted by \( H \). The corresponding Hopf structure reads

\[ \Delta(\Lambda) = \Lambda \otimes \Lambda, \quad S(\Lambda) = \Lambda^{-1}, \quad \epsilon(\Lambda) = 1. \] (69)

To proceed any further we need the algebra morphisms \( W^{-1}_L \) and \( W^{-1}_R \) defined by

\[ W^{-1}_L : A_q \times H \rightarrow A_q, \]
\[ W^{-1}_L((X^0)^{n_0}(X^1)^{n_1} \otimes h) \equiv (x^0)^{n_0}(x^1)^{n_1} \varepsilon(h), \] (70)

\[ W^{-1}_R : H \times A_q \rightarrow A_q, \]
\[ W^{-1}_R(h \otimes (X^0)^{n_0}(X^1)^{n_1}) \equiv \varepsilon(h) (x^0)^{n_0}(x^1)^{n_1}. \] (71)

With these mappings at hand we are able to introduce the operations

\[ f(x^i \oplus_{L/\bar{L}} y^j) \equiv ((W^{-1}_L \otimes W^{-1}_L) \circ \Delta_{L/\bar{L}})(f), \] (72)
\begin{equation}
\mathcal{W}_{R}^{-1} \circ S_{L/L}(f).
\end{equation}

Repeating the same steps already applied in Ref. [48] one can show that
\begin{equation}
f(x^{i} \oplus_{L} y^{j}) = \sum_{k,l=0}^{\infty} (x^{0})^{k}(x^{1})^{l} k! \frac{1}{[[l]]_{q^{-1}}} \left( \frac{\partial}{\partial y^{0}} \right)^{k} (D_{q^{-1}}^{1})^{l} f(y^{j}),
\end{equation}
and
\begin{equation}
f(x^{i} \oplus_{\bar{L}} y^{j}) = \sum_{k,l=0}^{\infty} (x^{0})^{k}(x^{1})^{l} k! \frac{1}{[[l]]_{q^{-1}}} \left( \frac{\partial}{\partial y^{0}} \right)^{k} (D_{q^{-1}}^{1})^{l} f(y^{j}),
\end{equation}

Notice that the expressions in (74) and (75) use the so-called Jackson derivatives [59]
\begin{equation}
D_{q}^{a} \equiv f(q^{a}x^{i}) - f(x^{i}) (q^{a} - 1)x^{i}, \quad a \in \mathbb{C}.
\end{equation}
Furthermore, the so-called antisymmetric q-numbers are given by
\begin{equation}
[[n]]_{q^{a}} \equiv \sum_{k=0}^{n-1} q^{ak} = \frac{1 - q^{an}}{1 - q^{a}},
\end{equation}
and their factorials are defined by
\begin{equation}
[[n]]_{q^{a}}! \equiv [[1]]_{q^{a}}[[2]]_{q^{a}} \ldots [[n]]_{q^{a}}, \quad [[0]]_{q^{a}}! \equiv 1.
\end{equation}

Now, we would like to turn our attention to operator representations of partial derivatives. From the q-deformed Leibniz rules in (14) and (15) as well as those in (19) and (20) we can derive right and left actions of partial derivatives on the algebra of quantum space coordinates. To this end, we repeatedly apply the Leibniz rules to the product of a partial derivative with a normally ordered monomial of coordinates, until we obtain an expression with all partial derivatives standing to the right of all quantum space coordinates, i.e.
\begin{equation}
\partial^{i}(X^{0})^{k_{0}}(X^{1})^{k_{1}} = (\partial_{(1)}^{i}) \triangleright (X^{0})^{k_{0}}(X^{1})^{k_{1}} \partial_{(2)}^{j}.
\end{equation}
Taking the counit of all partial derivatives appearing on the right-hand side finally yields the left action of $\partial^i$, since we have
\[(\partial^i_{(1)} \triangleright (X^0)^{k_0}(X^1)^{k_1}) \varepsilon(\partial^i_{(2)}) = \partial^i \triangleright (X^0)^{k_0}(X^1)^{k_1}.\] (82)

Right actions of partial derivatives can be calculated in a similar way if we start from a partial derivative standing to the right of a normally ordered monomial and commute it to the left of all quantum space coordinates. Instead of (81) and (82) we have
\[(X^0)^{k_0}(X^1)^{k_1} \partial^i = \partial^i (2) ((X^0)^{k_0}(X^1)^{k_1} \triangleleft \partial^i_{(1)}),\] (83)
and
\[\varepsilon(\partial^i_{(2)}) ((X^0)^{k_0}(X^1)^{k_1} \triangleleft \partial^i_{(1)}) = (X^0)^{k_0}(X^1)^{k_1} \triangleleft \partial^i.\] (84)

These reasonings show us a method to calculate explicit formulae for the action of partial derivatives on normally ordered monomials. From these results we can finally read off the operator representations
\[
\begin{align*}
\partial_0 \triangleright f(x^i) &= \frac{\partial}{\partial x^0} f(x^i), \\
\partial_1 \triangleright f(x^i) &= D^1_q f(x^i), \\
\hat{\partial}_0 \triangleright f(x^i) &= \frac{\partial}{\partial \hat{x}^0} f(x^i), \\
\hat{\partial}_1 \triangleright f(x^i) &= D^1_{q^{-1}} f(x^i),
\end{align*}
\] (85)
and
\[
\begin{align*}
f(x^i) \triangleleft \hat{\partial}_0 &= -\frac{\partial}{\partial \hat{x}^0} f(x^i), \\
f(x^i) \triangleleft \hat{\partial}_x &= -D^1_{q^{-1}} f(x^i), \\
f(x^i) \triangleleft \partial_0 &= -\frac{\partial}{\partial x^0} f(x^i), \\
f(x^i) \triangleleft \partial_1 &= -D^1_q f(x^i).
\end{align*}
\] (86)

With these formulae at hand it follows at once that
\[df(x^i) = dx^i \partial_j \triangleright f(x^i) = 0 \iff f(x^i)|_{x^0 = a} = f(x^i)|_{x^0 = b}, \quad f(x^i)|_{x^1 = a} = f(x^i)|_{x^1 = qa},\] (89)
for all $a, b \in \mathbb{C}$. Notice that the above condition characterizes functions being constant from the point of view of q-deformation.

Next, we come to integrals on the extended braided line. (For the different approaches to introduce integrals on q-deformed spaces see also Refs. [39, 46, 50, 55, 58, 60–63].) Integrals can be recognized as operations being inverse to partial derivatives. Thus, we first try to extend the algebra of partial derivatives by introducing inverse elements. In doing so, we get as additional relations

\[
\begin{align*}
(\partial_0)^{-1} \partial_0 &= \partial_0 (\partial_0)^{-1} = 1, \\
(\partial_1)^{-1} \partial_1 &= \partial_1 (\partial_1)^{-1} = 1, \\
(\partial_0)^{-1} \partial_1 &= \partial_1 (\partial_0)^{-1}, \\
(\partial_1)^{-1} \partial_0 &= \partial_0 (\partial_1)^{-1}, \\
(\partial_0)^{-1} (\partial_1)^{-1} &= (\partial_1)^{-1} (\partial_0)^{-1}.
\end{align*}
\] (90)

As next step we search for representations of the inverse partial derivatives that fulfill the above relations. It should be obvious that they are given by

\[
\begin{align*}
(\partial_0)^{-1} &\big|_{x^0 = a}^b f(x^i) = \int_a^b dx^0 f(x^i), \\
(\partial_1)^{-1} &\big|_{x^1 = a}^b f(x^i) = (D^1_q)^{-1} \big|_{x^0 = a}^b f(x^i), \\
(\hat{\partial}_0)^{-1} &\big|_{x^0 = a}^b f(x^i) = \int_a^b dx^0 f(x^i), \\
(\hat{\partial}_1)^{-1} &\big|_{x^1 = a}^b f(x^i) = (D^1_{q^{-1}})^{-1} \big|_{x^1 = a}^b f(x^i),
\end{align*}
\] (91)

and

\[
\begin{align*}
f(x^i) &\check{\circ} (\hat{\partial}_0)^{-1} \big|_{x^0 = a}^b = -\int_a^b dx^0 f(x^i), \\
f(x^i) &\check{\circ} (\hat{\partial}_1)^{-1} \big|_{x^1 = a}^b = -(D^1_{q^{-1}})^{-1} \big|_{x^1 = a}^b f(x^i), \\
f(x^i) &\check{\circ} (\partial_0)^{-1} \big|_{x^0 = a}^b = -\int_a^b dx^0 f(x^i), \\
f(x^i) &\check{\circ} (\partial_1)^{-1} \big|_{x^1 = a}^b = -(D^1_q)^{-1} \big|_{x^1 = a}^b f(x^i),
\end{align*}
\] (93)

where $(D^i_q)^{-1}$ denotes the Jackson integral operator [64]. For the sake of
completeness we would like to give the definition of the Jackson integral. For \( a > 0, q > 1, \) and \( x^i > 0, \) it becomes

\[
(D_{q^a}^i)^{-1}|_0^0 f = -(1 - q^a) \sum_{k=1}^{\infty} (q^{-ak} x^i) f(q^{-ak} x^i),
\]

\[
(D_{q^a}^i)^{-1}|_0^\infty x^i f = -(1 - q^a) \sum_{k=0}^{\infty} (q^{ak} x^i) f(q^{ak} x^i),
\]

\[
(D_{q^{-a}}^i)^{-1}|_0^0 f = (1 - q^{-a}) \sum_{k=0}^{\infty} (q^{ak} x^i) f(q^{ak} x^i),
\]

\[
(D_{q^{-a}}^i)^{-1}|_0^\infty x^i f = (1 - q^{-a}) \sum_{k=1}^{\infty} (q^{ak} x^i) f(q^{ak} x^i),
\]

and, likewise for \( a > 0, q > 1, \) and \( x^i < 0, \)

\[
(D_{q^a}^i)^{-1}|_0^0 f = (1 - q^a) \sum_{k=1}^{\infty} (q^{-ak} x^i) f(q^{-ak} x^i),
\]

\[
(D_{q^a}^i)^{-1}|_0^-\infty x^i f = (1 - q^a) \sum_{k=0}^{\infty} (q^{ak} x^i) f(q^{ak} x^i),
\]

\[
(D_{q^{-a}}^i)^{-1}|_0^0 f = -(1 - q^{-a}) \sum_{k=0}^{\infty} (q^{ak} x^i) f(q^{ak} x^i),
\]

\[
(D_{q^{-a}}^i)^{-1}|_0^-\infty x^i f = -(1 - q^{-a}) \sum_{k=1}^{\infty} (q^{ak} x^i) f(q^{ak} x^i),
\]

In analogy to the undeformed case we have rules for integration by parts. To derive them we start from the Leibniz rules for partial derivatives. These Leibniz rules can be read off from the coproducts in (67) and (66). In this manner, we find

\[
\partial_0 \triangleright (fg) = (\partial_0 \triangleright f)g + f(\partial_0 \triangleright g),
\]

\[
\partial_1 \triangleright (fg) = (\partial_1 \triangleright f)g + (\Lambda \triangleright f)(\partial_1 \triangleright g),
\]

\[
\hat{\partial}_0 \triangleright (fg) = (\hat{\partial}_0 \triangleright f)g + f(\hat{\partial}_0 \triangleright g),
\]

\[
\hat{\partial}_1 \triangleright (fg) = (\hat{\partial}_1 \triangleright f)g + (\Lambda^{-1} \triangleright f)(\hat{\partial}_1 \triangleright g),
\]
and
\[(fg) \triangleleft \hat{\partial}_0 = f(g \triangleleft \hat{\partial}_0) + (f \triangleleft \hat{\partial}_0)g, \quad (fg) \triangleleft \hat{\partial}_1 = f(g \triangleleft \hat{\partial}_1) + (f \triangleleft \hat{\partial}_1)(g \triangleleft \Lambda), \quad (99)\]
\[(fg) \triangleright \partial_0 = f(g \triangleright \partial_0) + (f \triangleright \partial_0)g, \quad (fg) \triangleright \partial_1 = f(g \triangleright \partial_1) + (f \triangleright \partial_1)(g \triangleright \Lambda^{-1}). \quad (100)\]

Hitting the above equations with the corresponding integral operator and rearranging terms, we get
\[
(\hat{\partial}_0)^{-1} b_{x_0 = a} \triangleright (\hat{\partial}_0 \triangleright f)g = fg \big|_{x_0 = a}^b - (\hat{\partial}_0)^{-1} b_{x_0 = a} \triangleright f(\partial_0 \triangleright g), \\
(\hat{\partial}_1)^{-1} b_{x_1 = a} \triangleright (\hat{\partial}_1 \triangleright f)g = fg \big|_{x_1 = a}^b - (\hat{\partial}_1)^{-1} b_{x_1 = a} \triangleright (\Lambda \triangleright f)(\partial_1 \triangleright g), \quad (101)\]
\[
(\hat{\partial}_0)^{-1} b_{x_0 = a} \triangleright (\hat{\partial}_0 \triangleright f)g = fg \big|_{x_0 = a}^b - (\hat{\partial}_0)^{-1} b_{x_0 = a} \triangleright f(\partial_0 \triangleright g), \\
(\hat{\partial}_1)^{-1} b_{x_1 = a} \triangleright (\hat{\partial}_1 \triangleright f)g = fg \big|_{x_1 = a}^b - (\hat{\partial}_1)^{-1} b_{x_1 = a} \triangleright (\Lambda \triangleright f)(\partial_1 \triangleright g), \quad (102)\]

and
\[
(f(g \triangleleft \hat{\partial}_0) \triangleleft (\hat{\partial}_0)^{-1} b_{x_0 = a} = fg \big|_{x_0 = a}^b - (f \triangleleft \hat{\partial}_0)g \triangleleft (\hat{\partial}_0)^{-1} b_{x_0 = a}, \\
f(g \triangleleft \hat{\partial}_1) \triangleleft (\hat{\partial}_1)^{-1} b_{x_1 = a} = fg \big|_{x_1 = a}^b - (f \triangleleft \hat{\partial}_1)(g \triangleleft \Lambda) \triangleleft (\hat{\partial}_1)^{-1} b_{x_1 = a}, \quad (103)\]
\[
f(g \triangleright \partial_0) \triangleright (\hat{\partial}_0)^{-1} b_{x_0 = a} = fg \big|_{x_0 = a}^b - (f \triangleright \partial_0)g \triangleright (\hat{\partial}_0)^{-1} b_{x_0 = a}, \\
f(g \triangleright \partial_1) \triangleright (\hat{\partial}_1)^{-1} b_{x_1 = a} = fg \big|_{x_1 = a}^b - (f \triangleright \partial_1)(g \triangleright \Lambda^{-1}) \triangleright (\hat{\partial}_1)^{-1} b_{x_1 = a}. \quad (104)\]

Before we can apply these formulae it remains to write down the explicit form of the action of the scaling operator. A short glance at the identities in (68) should tell us that
\[\Lambda \triangleright f(x^i) = f(x^i) \triangleleft \Lambda^{-1} = f(x^0, qx^1), \\
\Lambda^{-1} \triangleright f(x^i) = f(x^i) \triangleleft \Lambda = f(x^0, q^{-1}x^1). \quad (105)\]

We would like to close this subsection by dealing with dual pairings and q-exponentials. In Ref. [54] it was shown that the algebra of quantum space coordinates and that of the corresponding partial derivatives are dual to each other. The dual pairings are given by
\[
\langle f(\partial_i), g(x^j) \rangle_{L,R} = (f(\partial_i) \triangleright g(x^i))|_{x^i = 0} = (f(\partial_i) \triangleright g(x^i))|_{\partial_i = 0},
\]
\[
\langle f(\hat{\partial}_i), g(x^j) \rangle_{\bar{L},\bar{R}} \equiv (f(\hat{\partial}_i) \triangleright g(x^j))|_{x^j=0} = (f(\hat{\partial}_i) \triangleleft g(x^j))|_{\partial_i=0}, \quad (106)
\]
\[
\langle f(x^i), g(\partial_j) \rangle_{\bar{L},\bar{R}} \equiv (f(x^i) \triangleright \partial g(\partial_j))|_{x^i=0} = (f(x^i) \triangleleft \partial g(\partial_j))|_{\partial_j=0},
\]
\[
\langle f(x^i), g(\hat{\partial}_j) \rangle_{\bar{L},\bar{R}} \equiv (f(x^i) \triangleright \partial g(\hat{\partial}_j))|_{x^i=0} = (f(x^i) \triangleleft \partial g(\hat{\partial}_j))|_{\partial_j=0}. \quad (107)
\]

On monomials we get
\[
\langle (\partial_0)^n_0 (\partial_1)^n_1, (X^0)^m_0 (X^1)^m_1 \rangle_{\bar{L},\bar{R}} = n_0![[n_1]]_q! \delta^{m_0,m_0} \delta^{n_1,m_1},
\]
\[
\langle (\hat{\partial}_0)^n_0 (\hat{\partial}_1)^n_1, (X^0)^m_0 (X^1)^m_1 \rangle_{\bar{L},\bar{R}} = n_0![[n_1]]_{q^{-1}}! \delta^{m_0,m_0} \delta^{n_1,m_1}, \quad (108)
\]
and
\[
\langle (X^0)^m_0 (X^1)^m_1, (\partial_0)^n_0 (\partial_1)^n_1 \rangle_{\bar{L},\bar{R}} = (-1)^{n_0+n_1} n_0![[n_1]]_q! \delta^{m_0,m_0} \delta^{n_1,m_1},
\]
\[
\langle (X^0)^m_0 (X^1)^m_1, (\hat{\partial}_0)^n_0 (\hat{\partial}_1)^n_1 \rangle_{\bar{L},\bar{R}} = (-1)^{n_0+n_1} n_0![[n_1]]_{q^{-1}}! \delta^{m_0,m_0} \delta^{n_1,m_1}. \quad (109)
\]

These equalities can easily be checked by the identities in (85)-(88).

Now, let us make contact with q-deformed exponentials. From an abstract point of view an exponential is nothing other than an object whose dualization is one of the above pairings. In this sense, the exponential is given by the expression
\[
\exp(x^i|\partial_j) \equiv \sum_a e^a \otimes f_a, \quad (110)
\]
or
\[
\exp(\partial_i|x^j) \equiv \sum_a f_a \otimes e^a, \quad (111)
\]
where \(\{e_a\}\) is a basis in the coordinate algebra and \(\{f^a\}\) a dual basis in the algebra of partial derivatives.

If we want to derive explicit formulae for q-deformed exponentials it is our task to determine a basis of the coordinate algebra being dual to a given one of the algebra of derivatives. Inserting the elements of the two bases into the expressions (110) and (111) will then provide us with formulae for q-deformed exponentials. It should be obvious that the two bases being dually paired depend on the choice of the pairing. Thus, each pairing in (108) and (109) leads to its own q-exponential:
\[
\langle f(\partial_i), g(x^j) \rangle_{\bar{L},\bar{R}} \quad \Rightarrow \quad \exp(x^i|\partial_j)_{\bar{R},\bar{L}}.
\]
\[ \langle f(\hat{\partial}_i), g(x^j) \rangle_{L,R} \Rightarrow \exp(x^i | \hat{\partial}_j)_{R,L}, \quad (112) \]
\[ \langle f(x^i), g(\partial_j) \rangle_{L,R} \Rightarrow \exp(\partial_i | x^j)_{R,L}, \quad (113) \]

From the results in (108) and (109) we can rather easily read off two dually paired bases. Proceeding in the above mentioned way we find

\[ \exp(x^i | \partial_j)_{R,L} = \sum_{n_0,n_1=0}^{\infty} \frac{1}{n_0! [n_1]!_q} (x^0)^{\alpha_0} (x^1)^{\alpha_1} \otimes (\partial_0)^{\alpha_0} (\partial_1)^{\alpha_1} \]
\[ = \exp(x^0 \otimes \partial_0) \cdot \exp_q(x^1 \otimes \partial_1), \quad (114) \]

\[ \exp(x^i | \hat{\partial}_j)_{R,L} = \sum_{n_0,n_1=0}^{\infty} \frac{1}{n_0! [n_1]!_q^{-1}} (x^0)^{\alpha_0} (x^1)^{\alpha_1} \otimes (\hat{\partial}_0)^{\alpha_0} (\hat{\partial}_1)^{\alpha_1} \]
\[ = \exp(x^0 \otimes \hat{\partial}_0) \cdot \exp_{q^{-1}}(x^1 \otimes \hat{\partial}_1), \quad (115) \]

and

\[ \exp(\partial_i | x^j)_{R,L} = \sum_{n_0,n_1=0}^{\infty} \frac{1}{n_0! [n_1]!_q^{-1}} (\partial_0)^{\alpha_0} (\partial_1)^{\alpha_1} \otimes (x^0)^{\alpha_0} (x^1)^{\alpha_1} \]
\[ = \exp(\partial_0 \otimes x^0) \cdot \exp_{q^{-1}}(\partial_1 \otimes x^1), \quad (116) \]

\[ \exp(\hat{\partial}_i | x^j)_{R,L} = \sum_{n_0,n_1=0}^{\infty} \frac{1}{n_0! [n_1]!_q} (\hat{\partial}_0)^{\alpha_0} (\hat{\partial}_1)^{\alpha_1} \otimes (x^0)^{\alpha_0} (x^1)^{\alpha_1} \]
\[ = \exp(\hat{\partial}_0 \otimes x^0) \cdot \exp_q(\hat{\partial}_1 \otimes x^1). \quad (117) \]

### 3.2 Extended three-dimensional q-deformed Euclidean space

In this subsection we collect the elements of q-analysis to the extended three-dimensional q-deformed Euclidean space. Before we can do so, we first have to answer the question how to perform calculations on an algebra of non-commutative coordinates - in the following denoted by \( A_q \). This can be accomplished by a kind of pullback that transforms operations on the non-commutative coordinate algebra to those on a commutative one. For this to become more clear, one should realize that the non-commutative coordinate algebra we are dealing with satisfies the Poincaré-Birkhoff-Witt property. It
tells us that the dimension of a subspace of homogeneous polynomials has to be the same as for commuting coordinates. This property is the deeper reason why monomials of a given normal ordering constitute a basis of the non-commutative algebra $A_q$. Due to this fact we can establish a vector space isomorphism between $A_q$ and a commutative algebra $A$ generated by ordinary coordinates $x^1, x^2, \ldots, x^n$:

$$W : A \rightarrow A_q,$$

$$W((x^1)^{i_1} \cdots (x^n)^{i_n}) \equiv (X^1)^{i_1} \cdots (X^n)^{i_n}. \quad (118)$$

This vector space isomorphism can even be extended to an algebra isomorphism by introducing a non-commutative product in $A_q$, the so-called star product [65–67]. This product is defined via the relation

$$W(f \star g) = W(f) \cdot W(g), \quad (119)$$

being tantamount to

$$f \star g \equiv W^{-1}(W(f) \cdot W(g)), \quad (120)$$

where $f$ and $g$ are formal power series in $A$.

In the case of the extended three-dimensional $q$-deformed Euclidean space it is convenient to work with the normal orderings

$$W((x^0)^{n_0}(x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-}) = (X^0)^{n_0}(X^+)^{n_+}(X^3)^{n_3}(X^-)^{n_-}, \quad (121)$$

and

$$\tilde{W}((x^0)^{n_0}(x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-}) = (X^0)^{n_0}(X^-)^{n_-}(X^3)^{n_3}(X^+)^{n_+}. \quad (122)$$

The star product corresponding to the first choice takes the form [44]

$$f(x^i) \star g(x^j) = \sum_{k=0}^{\infty} \lambda^k \frac{(x^3)^{2k}}{[[k]]_q!} q^{2(\hat{n}_{x^3} \hat{n}_{y^3} + \hat{n}_{x^-} \hat{n}_{y^+})} \times (D_{q^+})^k f(x^i) \cdot (D_{q^-})^k g(y^j) \bigg|_{y \rightarrow x}, \quad (123)$$

and likewise for the second choice,

$$\tilde{f}(x^i) \star \tilde{g}(x^j) = \sum_{k=0}^{\infty} (-\lambda)^k \frac{(x^3)^{2k}}{[[k]]_{q^{-1}}!} q^{-2(\hat{n}_{x^3} \hat{n}_{y^3} + \hat{n}_{x^-} \hat{n}_{y^+})} \times (D_{q^+})^k \tilde{f}(x^i) \cdot (D_{q^-})^k \tilde{g}(y^j) \bigg|_{y \rightarrow x}, \quad (124)$$
\[ \times (D_{q^{-4}}^+)^k \tilde{f}(x^i) \cdot (D_{q^{-4}}^-)^k \tilde{g}(y^j) \bigg|_{y \to x}. \]  

(124)

Notice that the tilde on top of the symbols for the functions shall remind us of the fact that the star product refers to the algebra homomorphism \( \tilde{W} \). Extending the three-dimensional q-deformed Euclidean space by a time element does not really change the operator expressions for the star product. This observation is a consequence of the fact that the time element is central in the algebra of quantum space coordinates.

In very much the same way as was done for the braided line we can calculate actions of partial derivatives on normally ordered monomials by applying the commutation relations in (50)-(57). By means of the algebra isomorphisms (118) these actions carry over to commutative functions, i.e. we have

\[ \partial_i \triangleright W(f) = W(\partial_i \triangleright f), \quad f \in A, \]
\[ W(f) \triangleleft \partial_i = W(f \triangleleft \partial_i), \]

(125)

or

\[ \partial_i \triangleright f \equiv W^{-1} (\partial_i \triangleright W(f)), \]
\[ f \triangleleft \partial_i \equiv W^{-1} (W(f) \triangleleft \partial_i). \]

(126)

In the work of Ref. [45] we derived operator representations of q-deformed partial derivatives by applying these ideas. The results for the q-deformed three-dimensional Euclidean space can easily be modified to include the time element \( X^0 \) and the corresponding partial derivative \( \partial_0 \). In doing so we get

\[ \partial_0 \triangleright f = \frac{\partial}{\partial x^0} f, \]
\[ \partial_+ \triangleright f = D_{q^4}^+ f, \]
\[ \partial_3 \triangleright f = D_{q^2}^3 f(q^2 x^+), \]
\[ \partial_- \triangleright f = D_{q^{-2}}^- f(q^2 x^3) + \lambda x^+(D_{q^2}^3)^2 f. \]

(127)

The expressions for the other types of actions of partial derivatives follow from the above formulae by applying the substitutions

\[ \partial_i \triangleright f \overset{q^{-1/4}}{\leftrightarrow} \hat{\partial}_\tau \triangleright \tilde{f}, \]
\[ f \triangleleft \partial_i \overset{q^{-1/4}}{\leftrightarrow} \tilde{f} \triangleleft \hat{\partial}_\tau, \]

(128)
and
\[
\partial_i \triangleright f \quad \overset{\pm}{\longleftrightarrow} \quad f \triangleright \partial_i,
\]
\[
\hat{\partial}_i \triangleright \hat{f} \quad \overset{\pm}{\longleftrightarrow} \quad \hat{f} \triangleright \hat{\partial}_i,
\]
(129)
where the symbols \(\pm \quad \overset{\pm}{\longleftrightarrow} \quad \frac{1}{\pm} \) and \(\overset{\pm}{\longleftrightarrow} \quad \overset{\pm}{\longleftrightarrow} \) respectively denote transitions via the substitutions
\[
D^{\pm}_q \to D^{\mp}_{q^{-1}}, \quad x^{\pm} \to x^{\mp}, \quad q \to q^{-1},
\]
(130)
and
\[
D^{\pm}_q \to D^{\mp}_{q^{3}}, \quad x^{\pm} \to x^{\mp}.
\]
(131)
Notice that in (128) and (129) we introduced a conjugate index with
\[
(+, 3, -, 0) = (-, 3, +, 0).
\]
(132)

Now, we come to integrals for the extended three-dimensional q-deformed Euclidean space. For this reason we enhance the algebra of partial derivatives by introducing inverse elements. The additional relations then read
\[
(\partial_i)^{-1} \partial_i = \partial_i (\partial_i)^{-1} = 1,
\]
\[
(\partial_0)^{-1} \partial_0 = \partial_0 (\partial_0)^{-1},
\]
\[
(\partial_i)^{-1} \partial_0 = \partial_0 (\partial_i)^{-1}, \quad i \in \{+, 3, -, 0\},
\]
\[
(\partial_3)^{-1} \partial_\pm = q^{\pm 2} \partial_\pm (\partial_3)^{-1},
\]
\[
(\partial_\pm)^{-1} \partial_3 = q^{\pm 2} \partial_3 (\partial_\pm)^{-1},
\]
\[
(\partial_+)^{-1} \partial_- = \partial_- (\partial_+)^{-1} - q^{-4} \lambda (\partial_3)^2 (\partial_+)^{-2},
\]
\[
(\partial_-)^{-1} \partial_+ = (\partial_-)^{-1} \partial_+ - q^{-4} \lambda (\partial_-)^{-2} (\partial_3)^2.
\]
(133)

As a next step we would like to find representations for the inverse partial derivatives. From a short glance at (127) it should become obvious that
\[
(\partial_0)^{-1} \bigg|_{x^0 = a} \triangleright f = \int_a^b dx^0 f,
\]
\[
(\partial_+)^{-1} \bigg|_{x^+ = a} \triangleright f = (D^+_q)^{-1} \bigg|_{x^+ = a} f,
\]
\[
(\partial_3)^{-1} \bigg|_{x^3 = a} \triangleright f = (D^3_q)^{-1} \bigg|_{x^3 = a} f (q^{-2} x^+).
\]
(134)

It remains to derive the representation corresponding to \((\partial_-)^{-1}\). To this end, the representation of \(\partial_-\) is divided up into a classical part and
corrections vanishing in the undeformed limit $q \to 1$, i.e.
\[ \partial_- \triangleright f = (\partial_-)_{cl} f + (\partial_-)_{cor} f, \tag{135} \]
where
\[ (\partial_-)_{cl} f = D^{-}_q f(q^2 x^3), \quad (\partial_-)_{cor} f = \lambda x^+(D^{3}_{q^2})^2 f. \tag{136} \]

Then we can proceed as follows:
\[ (\partial_-)^{-1} \triangleright f = \frac{1}{(\partial_-)_{cl} + (\partial_-)_{cor}} f = \frac{1}{(\partial_-)_{cl} (1 + (\partial_-)^{-1}_{cl}(\partial_-)_{cor})} f 
= \frac{1}{1 + (\partial_-)^{-1}_{cl}(\partial_-)_{cor}} \cdot \frac{1}{(\partial_-)_{cl}} f 
= \sum_{k=0}^{\infty} (-1)^k [((\partial_-)^{-1}_{cl}(\partial_-)_{cor})^k ((\partial_-)_{cl})^{-1} f 
= \sum_{k=0}^{\infty} q^{2k(k+1)}(-\lambda x^+)^k(D^{3}_{q^2})^{2k}(D^{-}_{q^2})^{-(k+1)} f(q^{-2(k+1)}x^3). \tag{137} \]

In complete analogy to the correspondences in (128) and (129) the other types of representations follow from the above formulae by applying the transformation rules
\[ (\partial_i)^{-1} \triangleright f \xrightarrow{q \to -1/q} (\hat{\partial}_i)^{-1} \triangleright \tilde{f}, \]
\[ f \triangleright (\partial_i)^{-1} \xrightarrow{q \to -1/q} \tilde{f} \triangleleft (\hat{\partial}_i)^{-1}, \tag{138} \]
and
\[ (\hat{\partial}_i)^{-1} \triangleright \tilde{f} \xrightarrow{q \to -1/q} \tilde{f} \triangleleft (\hat{\partial}_i)^{-1}, \]
\[ (\hat{\partial}_i)^{-1} \triangleright \tilde{f} \xrightarrow{q \to -1/q} \tilde{f} \triangleleft (\hat{\partial}_i)^{-1}. \tag{139} \]

Next, we would like to concern ourselves with $q$-translations on the three-dimensional $q$-deformed Euclidean space. We already mentioned that the Hopf structures on quantum space coordinates imply their translations. For the coproducts on quantum space coordinates we have
\[ \Delta_L(X^0) = X^0 \otimes 1 + 1 \otimes X^0, \]
\[ \Delta_L(X^-) = X^- \otimes 1 + \Lambda^{-1/2} \tau^{-1/2} \otimes X^-, \]
\[ \Delta_L(X^3) = X^3 \otimes 1 + \Lambda^{-1/2} \otimes X^3 + \lambda \Lambda \Lambda^{-1/2} L^+ \otimes X^-. \]
\[ \Delta_L(X^+) = X^- \otimes 1 + \Lambda^{-1/2} \tau^{-1/2} \otimes X^+ + q\lambda\lambda_+\Lambda^{-1/2} \tau^{1/2} L^+ \otimes X^3 + q^{-2}\lambda^2\lambda_+\Lambda^{-1/2} \tau^{1/2} (L^+)^2 \otimes X^-, \] 
\[ \text{and} \]
\[ \Delta_L(X^0) = X^0 \otimes 1 + 1 \otimes X^0, \]
\[ \Delta_L(X^+) = X^+ \otimes 1 + \Lambda^{1/2} \tau^{-1/2} \otimes X^+, \]
\[ \Delta_L(X^3) = X^3 \otimes 1 + \Lambda^{1/2} \otimes X^3 + \lambda\lambda_+\Lambda L^- \otimes X^+, \]
\[ \Delta_L(X^-) = X^- \otimes 1 + \Lambda^{1/2} \tau^{1/2} \otimes X^- + q^{-1}\lambda\lambda_+\Lambda^{1/2} \tau^{1/2} L^- \otimes X^3 + q^{-2}\lambda^2\lambda_+\Lambda^{1/2} \tau^{1/2} (L^-)^2 \otimes X^+, \]

where \( L^+, L^- \), and \( \tau \) denote generators of \( U_q(su_2) \), while \( \Lambda \) plays the role of a scaling operator with \( (A = \{+, 3, -\}) \),

\[ \Lambda X^0 = X^0\Lambda, \quad \Lambda X^A = q^4 X^A\Lambda, \]
\[ \Lambda\partial_0 = \partial_0\Lambda, \quad \Lambda\partial_A = q^{-4}\partial_A\Lambda. \] 

The corresponding antipodes take the form

\[ S_L(X^0) = -X^0, \]
\[ S_L(X^-) = -\Lambda^{1/2} \tau^{1/2} X^-, \]
\[ S_L(X^3) = -\Lambda^{1/2} X^3 + q^2\lambda\lambda_+\Lambda^{1/2} \tau^{1/2} L^+ X^-, \]
\[ S_L(X^+) = -\Lambda^{1/2} \tau^{1/2} X^+ + q\lambda\lambda_+\Lambda^{1/2} L^+ X^3 - q^4\lambda^2\lambda_+\Lambda^{1/2} \tau^{1/2} (L^+)^2 X^-, \]

and

\[ S_L(X^0) = -X^0, \]
\[ S_L(X^+) = -\Lambda^{-1/2} \tau^{1/2} X^+, \]
\[ S_L(X^3) = -\Lambda^{-1/2} X^3 + q^{-2}\lambda\lambda_+\Lambda^{-1/2} \tau^{1/2} L^- X^+, \]
\[ S_L(X^-) = -\Lambda^{-1/2} \tau^{-1/2} X^- + q^{-1}\lambda\lambda_+\Lambda^{-1/2} L^- X^3 - q^{-4}\lambda^2\lambda_+\Lambda^{-1/2} \tau^{1/2} (L^-)^2 X^. \] 

We see that coproduct and antipode become rather simple on the time element. This observation is a direct consequence of the fact that the time element is completely decoupled from position space. For the same reason
the Hopf structures on the subspace spanned by the coordinates $X^+, X^3,$
and $X^-$ are identical to those already presented in the work of Ref. [48].

It is not very difficult to modify the reasonings in Ref. [48] in a way that
they take account of the existence of the time element $X^0$. In this manner,
we can show that the above relations imply

$$f(x^i \oplus_L y^j) = \sum_{n_0} \sum_{n_+} \sum_{n_3} \sum_{n_-} \sum_{k_0} (q \lambda \lambda^i)^l
\times \frac{(x^0)^{k_0} (x^+)^{k_+} (x^3)^{k_3-l} (x^-)^{k_-+l} (y^+)^l}{k_0! ![2l]_q^2 ![k_+]_q ![k_-]_q ![k_3-l]_q ![k_-]_q ![k_3-l]_q}
\times \left( (D_{q_+}^+)^{k_+} (D_{q_3}^3)^{k_3+l} (D_{q_-}^-)^{k_-} \left( \frac{\partial}{\partial x^0} \right)^{k_0} f \right) (q^{2(k_3-l)} y^+, q^{2k_-} y^-). \quad (145)$$

and

$$\hat{U}(f(\oplus_L x^i)) = \sum_{k=0}^{\infty} (q^{-1} \lambda \lambda^i)^k q^{4k^2} \frac{(x^+ x^-)^k}{[2k]_q^2!!}
\times (D_{q^3}^3)^{2k} q^{2(n^+_3 + n^+_2 + n_3(2n^+_2 + 2n^+_3 + n^+_2))} f(-x^0, -x^+, -q^{-2k} x^3, -x^-), \quad (146)$$

where

$$[[2k]_q^2!! = [[2k]_q ![2(k - 1)]_q ![2]_q.$$

(147)

The operator $\hat{U}$ in Eq. (146) transforms a function of normal ordering
$X^+ X^3 X^-$ into another function representing the same element but now for
reversed ordering. Its explicit form was presented in the work of Ref. [45].

The expressions corresponding to the other Hopf structures are obtained
from the formulae in (145) and (146) most easily by means of the transitions

$$f(x^i \oplus_L y^j) \stackrel{\pm \rightarrow \mp}{q \rightarrow q^{1/q}} \tilde{f}(x^i \oplus_L y^j), \quad (148)$$

and

$$\hat{U}(f(\oplus_L x^i)) \stackrel{\pm \rightarrow \mp}{q \rightarrow q^{1/q}} \hat{U}^{-1}(\tilde{f}(\oplus_L x^i)). \quad (149)$$

The tilde again reminds us of the fact that the function refers to reversed
normal ordering.

Next, we would like to say a few words about braided products on the
extended q-deformed Euclidean space in three dimensions. As we know, braided products describe how elements of different quantum spaces com-
mute. In this sense they are an essential ingredient to formulate multiplication on tensor products of quantum spaces. The entries $L^i_j$ of the so-called L-matrix determine the braiding of the quantum space coordinates $X^i$, $i \in \{0, +, 3, -\}$ (if not stated otherwise summation over repeated indices is to be understood):

$$X^i \odot_L w = (L^i_j \triangleright w) \otimes X^j.$$  \hfill (150)

The explicit form of the L-matrix can be read off from the coproduct on coordinates, since it holds

$$\Delta_L(X^i) = X^i \otimes 1 + L^i_j \otimes X^j.$$  \hfill (151)

In very much the same way we have

$$X^i \odot_L w = (\bar{L}^i_j \triangleright w) \otimes X^j,$$  \hfill (152)

and

$$\Delta_L(X^i) = X^i \otimes 1 + \bar{L}^i_j \otimes X^j.$$  \hfill (153)

These considerations are consistent with the observation that the time coordinate $X^0$ shows trivial braiding. In Ref. [49] we presented operator expressions that realize braided products for the three-dimensional q-deformed Euclidean space on a commutative coordinate algebra. Due to the trivial braiding of the time coordinate these expressions carry over to the extended three-dimensional q-deformed Euclidean space without any changes.

Last but not least, we come to dual pairings and q-exponentials. We already recalled their definition in Sec. 3.1. With the results of Ref. [47] it is not very difficult to show that

$$\langle (\partial_0)^{n_0} (\partial_-)^{n_-} (\partial_3)^{n_3} (\partial_+)^{n_+}, (X^0)^{m_0} (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-} \rangle_{L, R} =$$

$$= \delta_{m_- m_0} \delta_{m_+ m_3} \delta_{m_0 m_0} [m_+]^q [m_3]^q [m_-]^q,$$  \hfill (154)

$$\langle (\partial_0)^{n_0} (\partial_3)^{n_3} (\partial_-)^{n_-}, (X^0)^{m_0} (X^-)^{m_-} (X^3)^{m_3} (X^+)^{m_+} \rangle_{L, R} =$$

$$= \delta_{m_- m_0} \delta_{m_3 m_3} \delta_{m_0 m_0} [m_+]^q [m_3]^q [m_-]^q,$$  \hfill (155)

and

$$\langle (X^0)^{m_0} (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-}, (\partial_0)^{n_0} (\partial_-)^{n_-} (\partial_3)^{n_3} (\partial_+)^{n_+} \rangle_{L, R} =$$

$$= (-1)^{n_0 + n_+ + n_3 + n_-} \delta_{m_- m_0} \delta_{m_3 m_3} \delta_{m_0 m_0}.$$
\[ \times m_0![m_+][q^3][m_3][q^2][m_-][q^4], \]

\[ \langle (X^0)^m_0(X^m_0)(X^3)^n_3(X^m_0)^n_0(\partial_0)^n_0(\partial_3)^n_3(\partial_3)^n_0 \rangle_{L,R} = \]

\[ = (-1)^{m_0+n_3+n_0-\delta_{m_-,n_0}} \delta_{m_3,n_3} \delta_{m_+,n_+} \delta_{m_0,n_0} \]

\[ \times m_0![m_+][q_3][m_3][q_2][m_-][q^4]. \]

From these pairings we can read off the explicit form of \(q\)-exponentials for the extended three-dimensional \(q\)-deformed Euclidean space:

\[ \exp(x^i|\partial_j)_{R,L} = \]

\[ = \sum_{n=0}^{\infty} \frac{(x^0)^n_0(x^m_0)^n_0(x^3)^n_3(x^m_0)^n_0(\partial_0)^n_0(\partial_3)^n_3(\partial_3)^n_0}{n_0![m_+][q^3][m_3][q^2][m_-][q^4]}, \]

\[ \exp(x^i|\partial_j)_{R,L} = \]

\[ = \sum_{n=0}^{\infty} \frac{(x^0)^n_0(x^m_0)^n_0(x^3)^n_3(x^m_0)^n_0(\partial_0)^n_0(\partial_3)^n_3(\partial_3)^n_0}{n_0![m_+][q^3][m_3][q^2][m_-][q^4]}. \]

The expressions for the other exponentials follow from these formulae by applying the transformations

\[ \exp(x^i|\partial_j)_{R,L} \leftrightarrow \exp(\partial_i|x^j)_{R,L}, \]

\[ \exp(x^i|\partial_j)_{R,L} \leftrightarrow \exp(\partial_i|x^j)_{R,L}, \]

where the symbol \(\leftrightarrow\) denotes a transition between the two expressions via one of the following substitutions:

\[ a) \quad X^i \leftrightarrow -\partial_i, \quad \partial_i \leftrightarrow X^i, \]

\[ b) \quad X^i \leftrightarrow -\partial_i, \quad \partial_i \leftrightarrow X^i. \]

### 4 Time evolution operator

In this section we discuss the question how wave functions on the quantum spaces under consideration change in time. First of all, we recall that translations on quantum spaces are generated by \(q\)-exponentials [17, 48, 50, 54, 68]. This observation leads us to \(q\)-deformed Taylor rules which take the form [50]

\[ \exp(x^i \oplus_L (\oplus_L y^j)|\partial_k)_{R,L} \delta y^| = g(x^i), \]
\[ \exp(x^i \oplus_L (\ominus_L y^j)\partial_k)_{R,L} \triangleright g(y^k) = g(x^i), \quad (162) \]

and

\[ g(y^j) \triangleright \exp(\partial_k[\ominus_R y^j] \oplus_R x^i)_{R,L} = g(x^i). \quad (163) \]

For a correct understanding of these expressions see also Ref. [48, 50].

If the q-deformed Taylor rules shall describe translations in time, only, they have to be modified as follows:

\[ [\exp(x^i \oplus_L (\ominus_L y^j)\partial_k)_{R,L} \triangleright g(y^j)]_{x^A = y^A} = g(y^j)\big|_{y^0 = x^0}, \]

\[ [\exp(x^i \oplus_L (\ominus_L y^j)\partial_k)_{R,L} \triangleright g(y^j)]_{x^A = y^A} = g(y^j)\big|_{y^0 = x^0}, \quad (164) \]

and

\[ [g(y^j) \triangleright \exp(\partial_k[\ominus_R y^j] \oplus_R x^i)_{R,L}]_{x^A = y^A} = g(y^j)\big|_{y^0 = x^0}, \]

\[ [g(y^j) \triangleright \exp(\partial_k[\ominus_R y^j] \oplus_R x^i)_{R,L}]_{x^A = y^A} = g(y^j)\big|_{y^0 = x^0}, \quad (165) \]

where \( A \) represents the indices (+, 3, −). In the above expressions we first perform a general translation and then require that the space coordinates of the translated function take on the same values as the original function. Since space and time are completely decoupled from each other the above formulae simplify to

\[ [\exp(x^0 \otimes \partial_0) \triangleright g(y^j)]_{y^0 = 0} = g(y^j)\big|_{y^0 = x^0}, \]

\[ [\exp(x^0 \otimes \partial_0) \triangleright g(y^j)]_{y^0 = 0} = g(y^j)\big|_{y^0 = x^0}, \quad (166) \]

and

\[ [g(y^j) \triangleright \exp(-\partial_0 \otimes x^0)]_{y^0 = 0} = g(y^j)\big|_{y^0 = x^0}, \]

\[ [g(y^j) \triangleright \exp(-\partial_0 \otimes x^0)]_{y^0 = 0} = g(y^j)\big|_{y^0 = x^0}. \quad (167) \]

In quantum mechanics the set of values a wave function takes on in space at a certain time completely determines the behavior of that wave
function at all later times. This requires that time derivatives acting on wave functions can be substituted by a linear operator $i^{-1}H$ that acts on space coordinates, only, and has the same algebraic properties as the time derivative $\partial_0$.

In this manner, it should be clear that for the time evolution operator we have

$$\phi(x^A, t) = U_L(t, t' = 0) \triangleright \phi(x^A, t' = 0) = U_L(t, t' = 0) \triangleright \phi(x^A, t' = 0) = \phi(x^A, t' = 0) \triangleright U_R(t, t' = 0) = \phi(x^A, t' = 0) \triangleright U_R(t, t' = 0),$$  \hspace{1cm} (168)

where

$$U_L(t, t' = 0) = U_L(t, t' = 0) \equiv \exp(-t \otimes iH), \hspace{1cm} (169)$$

$$U_R(t, t' = 0) = U_R(t, t' = 0) \equiv \exp(iH \otimes t). \hspace{1cm} (170)$$

We see that the time evolution operator is of the same form as in the undeformed case. In the remainder of this section we collect basic properties of the time evolution operators. This is mainly done for the purpose of providing consistent notation.

First of all, we are seeking operators $U_\gamma^{-1}(t, t' = 0), \gamma \in \{L, \bar{L}, R, \bar{R}\}$, with

$$U_\gamma(t, t' = 0) U_\gamma^{-1}(t, t' = 0) = 1,$$

$$U_\gamma^{-1}(t, t' = 0) U_\gamma(t, t' = 0) = 1. \hspace{1cm} (171)$$

One readily checks that

$$U_\alpha^{-1}(t, t' = 0) \equiv U_\alpha(-t, t' = 0) = \exp(t \otimes iH), \hspace{1cm} (172)$$

$$U_\beta^{-1}(t, t' = 0) \equiv U_\beta(-t, t' = 0) = \exp(-iH \otimes t), \hspace{1cm} (173)$$

where $\alpha \in \{L, \bar{L}\}$ and $\beta \in \{R, \bar{R}\}$. As a direct consequence of these identities we have

$$\phi(x^A, t' = 0) = U_\alpha^{-1}(t, t' = 0) \triangleright \phi(x^A, t) = \phi(x^A, t) \triangleright U_\beta^{-1}(t, t' = 0). \hspace{1cm} (174)$$
The operators $U^{-1}_\gamma(t, t') = 0$ describe particles traversing backwards in time, since we have

$$
\phi(x^A, -t) = U^{-1}_\alpha(t, t' = 0) \triangleright \phi(x^A, t' = 0)
= \phi(x^A, t' = 0) \triangleleft U^{-1}_\beta(t, t' = 0).
$$ (175)

Now, we are in a position to generalize the time evolution operators by

$$
U_\alpha(t, t') \equiv U_\alpha(t, t'' = 0) U^{-1}_\alpha(t', t'' = 0) = \exp(- (t - t') \otimes iH),
$$ (176)

$$
U_\beta(t, t') \equiv U^{-1}_\beta(t', t'' = 0) U_\beta(t, t'' = 0) = \exp(- iH \otimes (t' - t)).
$$ (177)

The new operators tell us how wave functions change under a time displacement $t' \rightarrow t$:

$$
\phi(x^A, t) = U_\alpha(t, t') \triangleright \phi(x^A, t') = \phi(x^A, t') \triangleleft U_\beta(t, t').
$$ (178)

To prove these equalities one can apply the identities in (168) and (174). An essential feature of the time evolution operators in (176) and (177) is the composition property,

$$
U_\alpha(t, t') = U_\alpha(t, 0) U^{-1}_\alpha(t', 0)
= U_\alpha(t, 0) U^{-1}_\alpha(t'', 0) U_\alpha(t'', 0) U^{-1}_\alpha(t', 0)
= U_\alpha(t, t'') U_\alpha(t'', t'),
$$ (179)

and

$$
U_\beta(t, t') = U^{-1}_\beta(t', 0) U_\beta(t, 0)
= U^{-1}_\beta(t', 0) U_\beta(t'', 0) U^{-1}_\beta(t'', 0) U_\beta(t, 0)
= U_\beta(t'', t') U_\beta(t, t'').
$$ (180)

Next, we would like to consider operators $U^{-1}_\gamma(t, t')$ being subject to

$$
U_\gamma(t, t') U^{-1}_\gamma(t, t') = U^{-1}_\gamma(t, t') U_\gamma(t, t') = 1.
$$ (181)

They are given by

$$
U^{-1}_\alpha(t, t') \equiv U_\alpha(t', t'' = 0) U^{-1}_\alpha(t, t'' = 0) = U_\alpha(t', t)
= U_\alpha(-t, -t') = \exp((t - t') \otimes iH),
$$ (182)
5 SCHRÖDINGER AND HEISENBERG PICTURE

\[
U^{-1}_\beta(t, t') = U^{-1}_\beta(t, t'' = 0)U_\beta(t', t'' = 0) = U_\beta(t', t) = U_\beta(-t, -t') = \exp(iH \otimes (t' - t)). \tag{183}
\]

These operators reverse the time displacement \( t' \rightarrow t \):

\[
\phi(x^A, t') = U^{-1}_\alpha(t, t') \triangleright \phi(x^A, t) = \phi(x^A, t) \triangleleft U^{-1}_\beta(t, t'). \tag{184}
\]

Last but not least we would like to mention some simplifications. Let us recall that the time coordinate shows trivial braiding. Thus, the tensor products in the expressions on the right-hand side of (176) and (177) can be omitted. Concretely, we can make the identifications

\[
\exp(t \otimes iH) = \exp(iH \otimes t) = \exp(iHt), \tag{185}
\]

and

\[
U(t, t') = U_\alpha(t, t') = U_\beta(-t, -t') = U^{-1}_\alpha(-t, -t') = U^{-1}_\beta(t, t'), \tag{186}
\]

where

\[
U(t, t') = \exp(-iH(t - t')). \tag{187}
\]

It is obvious that the operator \( U(t, t') \) becomes unitary, if the Hamiltonian \( H \) is assumed to be Hermitian:

\[
U^{-1}(t, t') = U(-t, -t') = U(t', t) = U_\dagger(t, t'). \tag{188}
\]

5 Schrödinger and Heisenberg picture

In the last section we found that the time evolution operator on the quantum spaces under consideration is of the same form as its undeformed counterpart. For this reason we should be able to introduce the Heisenberg and the Schrödinger picture on our quantum spaces along the same line of reasonings as in the undeformed case (see for example Ref. [69]).

To begin with we derive differential equations for the time evolution operators. We have

\[
\partial_0 \triangleright U_L(t, t') = \partial_0 \triangleright U_L(t, 0)U_L^{-1}(t', 0) = \partial_0 \triangleright \exp(-t \otimes iH)U_L^{-1}(t', 0) = \exp(-t \otimes iH)(-iH)U_L^{-1}(t', 0)
\]
Schrödinger and Heisenberg Picture

\[ = -iH \exp(-t \otimes iH) U_L^{-1}(t', 0) \]
\[ = -iH U_L(t, 0) U_L^{-1}(t', 0) \]
\[ = -iH U_L(t, t') \]  \hspace{1cm} (189)

and, likewise,

\[ \mathcal{U}_R(t, t') \triangleleft \hat{\partial}_0 = \mathcal{U}_R^{-1}(t', 0) \mathcal{U}_R(t, 0) \triangleleft \hat{\partial}_0 \]
\[ = \mathcal{U}_R^{-1}(t', 0) \exp(iH \otimes t) \triangleleft \hat{\partial}_0 \]
\[ = \mathcal{U}_R^{-1}(t', 0) (-iH) \exp(iH \otimes t) \]
\[ = \mathcal{U}_R^{-1}(t', 0) \mathcal{U}_R(t, 0) (-iH) \]
\[ = \mathcal{U}_R(t, t') (-iH). \]  \hspace{1cm} (190)

In this manner, we find that

\[ i \partial_0 \triangleright \mathcal{U}_L(t, t') = H \mathcal{U}_L(t, t'), \]
\[ i \hat{\partial}_0 \triangleright \mathcal{U}_L(t, t') = H \mathcal{U}_L(t, t'), \]  \hspace{1cm} (191)

and

\[ \mathcal{U}_R(t, t') \triangleright (i \hat{\partial}_0) = \mathcal{U}_R(t, t') H, \]
\[ \mathcal{U}_R(t, t') \triangleright (i \partial_0) = \mathcal{U}_R(t, t') H. \]  \hspace{1cm} (192)

The above equations, which are often referred to as Schrödinger equations of the time evolution operator, correspond to different geometries. However, from the considerations so far one can conclude that the equations in (191) and (192) are not really different from each other. Thus, the reader may think that such a distinction is unnecessary. But this is not the case, since the realization of the Hamiltonian often depends on the choice for the geometry.

It should also be mentioned that the differential equations in (191) and (192) are equivalent to the integral equations

\[ \mathcal{U}_\alpha(t, t') = 1 - i \int_0^t dt'' H \mathcal{U}_\alpha(t'', t'), \]
\[ \mathcal{U}_\beta(t, t') = 1 + i \int_0^t dt'' \mathcal{U}_\beta(t'', t') H, \]  \hspace{1cm} (193)
if we require
\[ U_\gamma(t, t) = 1. \]  
(194)

Formal solutions are given by
\[
U_\alpha(t, t') = 1 + \sum_{n=1}^{\infty} i^{-n} \int_t^{t'} dt_1 \int_{t_1}^{t_1} dt_2 \ldots \int_{t_{n-1}}^{t_{n-1}} dt_n \ H(t_1)H(t_2)\ldots H(t_n),
\]
\[
U_\beta(t, t') = 1 + \sum_{n=1}^{\infty} i^{n} \int_t^{t'} dt_1 \int_{t_1}^{t_1} dt_2 \ldots \int_{t_{n-1}}^{t_{n-1}} dt_n \ H(t_n)H(t_{n-1})\ldots H(t_1).
\]
(195)

Let us recall that in the Schrödinger picture wave functions vary with time, while observables like \(X^i\) and \(P^i\) are fixed in time. We obtain the equations of motion in the Schrödinger picture by combining (178) with the equations in (191) and (192). Proceeding in this manner yields
\[
i \partial_0 \triangleright \phi(x^A, t) = i \partial_0 \triangleright U_L(t, t') \triangleright \phi(x^A, t') = H \triangleright \phi(x^A, t),
\]
(196)
\[
i \hat{\partial}_0 \triangleright \phi(t, x^A) = i \hat{\partial}_0 \triangleright U_L(t, t') \triangleright \phi(x^A, t') = H \triangleright \phi(x^A, t),
\]
(197)
and
\[
\phi(x^A, t) \triangleright (i \partial_0) = \phi(x^A, t') \triangleright U_R(t, t') \triangleright \partial_0
\]
\[= \phi(x^A, t') \triangleright U_R(t, t')H = \phi(x^A, t) \triangleright \bar{H}, \]
(198)
\[
\phi(x^A, t) \triangleright (i \hat{\partial}_0) = \phi(x^A, t') \triangleright U_R(t, t') \triangleright (i \hat{\partial}_0)
\]
\[= \phi(x^A, t') \triangleright U_R(t, t')H = \phi(x^A, t) \triangleright \bar{H}. \]
(199)

Next, we would like to discuss the implications of these equations on the time dependence of transition amplitudes and expectation values. To this end, we first introduce sesquilinear forms on the quantum spaces under consideration. In analogy to the undeformed case they can be defined by [70]
\[
\langle f, g \rangle_{\gamma} \equiv \int_{-\infty}^{+\infty} d^n x \ f(x^A, t) \otimes g(x^B, t),
\]

\[ \langle f, g \rangle_\gamma' \equiv \int_{-\infty}^{+\infty} d^n x f(x^A, t) \otimes g(x^B, t), \quad (200) \]

where again \( \gamma \in \{ L, \bar{L}, R, \bar{R} \} \). For the integrals over the whole space we have to insert the expressions \([46, 48, 50]\)

(i) (braided line)

\[ \int_{-\infty}^{+\infty} d_L x f(x^A, t) = (D^1_{q})^{-1} \big|_{-\infty}^{\infty} f \]

\[ = - \int_{-\infty}^{+\infty} d_R x f(x^A, t), \quad (201) \]

\[ \int_{-\infty}^{+\infty} d\bar{L} x f(x^A, t) = (D^1_{q^{-1}})^{-1} \big|_{-\infty}^{\infty} f \]

\[ = - \int_{-\infty}^{+\infty} d\bar{R} x f(x^A, t), \quad (202) \]

(ii) (q-deformed Euclidean space)

\[ \int_{-\infty}^{+\infty} d^3_L x f(x^A, t) = \frac{q^{-6}}{4} (D^3_{q^2})^{-1} \big|_{-\infty}^{\infty} (D^3_{q^2})^{-1} \big|_{-\infty}^{\infty} (D^3_{q^2})^{-1} \big|_{-\infty}^{\infty} f \]

\[ = - \int_{-\infty}^{+\infty} d^3_R x f(t, x^A), \quad (203) \]

\[ \int_{-\infty}^{+\infty} d^3\bar{L} x f(x^A, t) = \frac{q^6}{4} (D^3_{q^{-2}})^{-1} \big|_{-\infty}^{\infty} (D^3_{q^{-2}})^{-1} \big|_{-\infty}^{\infty} (D^3_{q^{-2}})^{-1} \big|_{-\infty}^{\infty} f \]

\[ = - \int_{-\infty}^{+\infty} d^3\bar{R} x f(x^A, t), \quad (204) \]

However, there is one difficulty we have to overcome here. The conjugation properties of q-deformed integrals are responsible for the fact that the sesquilinear forms in \((200)\) are not symmetrical \([50, 70]\). To circumvent this problem one can take the sesquilinear forms

\[ \langle f, g \rangle_1 = \frac{i^n}{2} (\langle f, g \rangle_L + \langle f, g \rangle_R), \]

\[ \langle f, g \rangle_2 = \frac{i^n}{2} (\langle f, g \rangle_L + \langle f, g \rangle_R), \quad (205) \]
\[
\langle f, g \rangle_1' = \frac{i^n}{2} (\langle f, g \rangle_L' + \langle f, g \rangle_R'),
\]
\[
\langle f, g \rangle_2' = \frac{i^n}{2} (\langle f, g \rangle_L' + \langle f, g \rangle_R').
\]

(206)

Clearly, all information on the time development of a sesquilinear form is contained in the time dependence of its arguments. Normally, the time evolution operators are unitary, so sesquilinear forms of two wave functions should not vary with time. In complete analogy to the undeformed case we have \((i = 1, 2)\)

\[
\langle \phi, \psi \rangle_i' \equiv \int_{-\infty}^{+\infty} d_t^n x \phi(x^A, t') \otimes \psi(x^B, t'),
\]

\[
= \int_{-\infty}^{+\infty} d_t^n x \bar{\mathcal{U}}(t, t') \triangleright \phi(x^A, t') \otimes (\mathcal{U}(t, t') \triangleright \psi(x^B, t'))
\]

\[
= \int_{-\infty}^{+\infty} d_t^n x \phi(x^A, t') \langle \mathcal{U}^{-1}(t, t') \rangle \otimes (\mathcal{U}(t, t') \triangleright \psi(x^B, t'))
\]

\[
= \int_{-\infty}^{+\infty} d_t^n x \phi(x^A, t') \langle \mathcal{U}^{-1}(t, t') \rangle \otimes \psi(x^B, t')
\]

\[
= \int_{-\infty}^{+\infty} d_t^n x \phi(x^A, t') \otimes \psi(x^B, t')
\]

\[
= \langle \phi, \psi \rangle_{i, t=0}.
\]

(207)

where, for brevity, we introduced

\[
\int_{-\infty}^{+\infty} d_t^n x = \frac{i^n}{2} \left( \int_{-\infty}^{+\infty} d_L^n x + \int_{-\infty}^{+\infty} d_R^n x \right).
\]

\[
\int_{-\infty}^{+\infty} d_t^n x = \frac{i^n}{2} \left( \int_{-\infty}^{+\infty} d_L^n x + \int_{-\infty}^{+\infty} d_R^n x \right).
\]

(208)

Similar arguments lead us to

\[
\langle \phi, \psi \rangle_i' \equiv \int_{-\infty}^{+\infty} d_t^n x \phi(x^A, t') \otimes \bar{\psi}(x^B, t')
\]

\[
= \int_{-\infty}^{+\infty} d_t^n x \phi(x^A, t') \triangleright \bar{\psi}(x^B, t')
\]
\[ \int_{-\infty}^{+\infty} d^n x \phi(x^A, t' = 0) \bar{\psi}(x^B, t' = 0) \]
\[ = \langle \phi, \psi \rangle_{i, t = 0}. \tag{209} \]

We see that on the quantum spaces under consideration wave functions keep their normalization, i.e. the equalities
\[ \langle \phi, \phi \rangle_{i, x} = 1, \tag{210} \]
or
\[ \langle \phi, \phi \rangle'_{i, x} = 1, \tag{211} \]
remain unchanged as time goes by.

Next, we turn attention to matrix elements of observables and examine their time development. With the same reasonings already applied in (207) we obtain
\[ \langle \phi, \hat{O} \psi \rangle_{i, x} = \int_{-\infty}^{+\infty} d^n x \phi(x^A, t) \bar{\psi}(x^B, t) \]
\[ = \int_{-\infty}^{+\infty} d^n x U(t, t') \psi(x^A, t') \bar{\psi}(x^B, t') \]
\[ = \int_{-\infty}^{+\infty} d^n x \phi(x^A, t') U^{-1}(t, t') \psi(x^B, t') \]
\[ = \int_{-\infty}^{+\infty} d^n x \phi(x^A, 0) \bar{\psi}(x^B, 0). \tag{212} \]

Repeating the same steps for the sesquilinear forms with apostrophe we get
\[ \langle \phi \triangleleft \hat{O}' \psi \rangle'_{i, x} = \int_{-\infty}^{+\infty} d^n x \phi(x^A, t') \bar{\psi}(x^B, t') \]
\[ = \int_{-\infty}^{+\infty} d^n x \phi(x^A, 0) \bar{\psi}(x^B, 0). \tag{213} \]

The above reasonings show us that the Heisenberg picture can indeed be introduced in very much the same way as is done in the undeformed case, i.e. we define the Heisenberg picture observable by
\[ \hat{O}_H \equiv U^{-1}(t, 0) \hat{O} U(t, 0), \quad \hat{O}'_H \equiv U(t, 0) \hat{O}' U^{-1}(t, 0), \tag{214} \]
while the corresponding wave functions are independent from time:

$$\phi_H(x^A) \equiv \phi(x^A, t = 0).$$  

(215)

It should be obvious that this convention leads to the same matrix elements and expectation values as in the Schrödinger picture.

In the Heisenberg picture time evolution is assigned to observables and not to wave functions. Thus, the equations of motion do not concern wave functions but observables. Realizing that the time derivatives on the quantum spaces under considerations coincide with those on commutative spaces we regain the well-known Heisenberg equations of motion, i.e.

\[
\frac{d\hat{O}_H}{dt} = \frac{\partial U^{-1}(t, 0)}{\partial t} \hat{O}_H(t, 0) + U^{-1}(t, 0) \frac{\partial U(t, 0)}{\partial t} = iH U^{-1}(t, 0) \hat{O}_H(t, 0) - U^{-1}(t, 0) \frac{\partial U(t, 0)}{\partial t} \
= i[H, \hat{O}_H],
\]

(216)

\[
\frac{d\hat{O}_H'}{dt} = \frac{\partial U(t, 0)}{\partial t} \hat{O} U^{-1}(t, 0) + U(t, 0) \hat{O}' \frac{\partial U^{-1}(t, 0)}{\partial t} = -iH U(t, 0) \hat{O}' U^{-1}(t, 0) + U(t, 0) \hat{O}' U^{-1}(t, 0) iH \
= i[\hat{O}_H', H],
\]

(217)

where we assumed the Hamiltonian to be time-independent.

6 Conclusion

Let us end with some comments on what we have done so far. In this article we enhanced the algebras of braided line and q-deformed three-dimensional Euclidean space by adding a time element. This was done in a way being consistent with the existing algebraic framework. We were then able to apply our reasonings about constructing q-deformed analogs of classical analysis. We saw that the time element is completely decoupled from space coordinates and behaves like a commutative variable. In doing so, we arrived at mathematical structures in which space is discretized while time is still continuous. The clear distinction between space and time made it easy to develop the basics of a q-deformed analog of non-relativistic Schrödinger theory. Fortunately, we could apply the same reasonings as in the undeformed case, to which our results tend in the limit \( q \to 1 \).

Especially, we found that the time evolution operators are of the same
general form as their undeformed counterpart, i.e. they can again be obtained by exponentiation of a Hamiltonian. The Schrödinger and the Heisenberg picture could be developed in a rather straightforward way and apart from the fact that we have different q-geometries we could regain the well-known equations of motion, i.e. the Schrödinger and the Heisenberg equations. In this manner, we laid the foundations for discretized versions of non-relativistic quantum mechanics that do not lack space-time symmetries.

Based on the reasonings of part I we will continue this program in part II of our paper. In this respect, let us point out that compared to other quantum spaces, like the q-deformed Minkowski space, extended braided line and extended three-dimensional q-deformed Euclidean space provide a rather simple arena for studying the implications of q-deformation on quantum mechanics and quantum field theory.

Last but not least, we would like to say a few words about q-deformed superanalysis on the braided line, since this subject has not been treated up to now. To this end we have to consider the antisymmetrized space determined by relation \( S \). (However, we are mainly interested in the subspace that is spanned by the q-deformed Grassmann variable \( \theta^1 \) subject to \( (\theta^1)^2 = 0 \).) In the work of Refs. [53,68] it was described how to construct superanalysis on q-deformed quantum spaces. It is rather easy to apply these ideas to the antisymmetrized braided line. In what follows we give a short review of the results of this undertaking.

If we require for the differential \( d \equiv d\theta^i(\partial_\theta)_i \) to hold

\[
d^2 = 0, \\
d(fg) = (df)g - f(dg),
\]

the Leibniz rules for the two differential calculi on the antisymmetrized braided line become

\[
(\partial_\theta)_i \theta^j = \delta^j_i - \hat{R}^j_{ik} \theta^k(\partial_\theta)_k, \\
(\hat{\partial}_\theta)_i \theta^j = \delta^j_i - (\hat{R}^{-1})^j_{ik} \theta^k(\hat{\partial}_\theta)_k,
\]

where

\[
(\partial_\theta)_0 = -(\partial_\theta)_0, \quad (\partial_\theta)_1 = -q(\partial_\theta)_1.
\]

Especially, we have

\[
(\partial_\theta)_1 \theta^1 = 1 - q \theta^1(\partial_\theta)_1, \\
(\hat{\partial}_\theta)_1 \theta^1 = 1 - q^{-1} \theta^1(\hat{\partial}_\theta)_1.
\]
For supernumbers of the form
\[ f(\theta^1) = f' + f_1 \theta^1, \quad f', f_1 \in \mathbb{C}, \tag{222} \]
the actions of antisymmetric partial derivatives take the form
\[ (\partial_{\theta})_1 \triangleright f(\theta^1) = (\hat{\partial}_{\theta})_1 \triangleright f(\theta^1) = f_1, \tag{223} \]
and
\[ f(\theta^1) \tilde{\Lambda} (\partial_{\theta})_1 = f(\theta^1) \triangleleft (\hat{\partial}_{\theta})_1 = -f_1. \tag{224} \]

In complete analogy to the undeformed case integration and differentiation are the same on q-deformed antisymmetrized spaces:
\[ \int d_L \theta^1 f(\theta^1) = (\partial_{\theta})_1 \triangleright f(\theta^1) = f_1, \]
\[ \int d_{\bar{L}} \theta^1 f(\theta^1) = (\hat{\partial}_{\theta})_1 \triangleright f(\theta^1) = f_1, \tag{225} \]
\[ \int d_R \theta^1 f(\theta^1) = f(\theta^1) \triangleleft (\hat{\partial}_{\theta})_1 = -f_1, \]
\[ \int d_{\bar{R}} \theta^1 f(\theta^1) = f(\theta^1) \tilde{\Lambda} (\partial_{\theta})_1 = -f_1. \tag{226} \]

Following the ideas in Ref. [68] translations of q-deformed Grassmann variables are determined by their Hopf structures, for which we have
\[ \Delta_L(\theta^i) = \theta^i \otimes 1 + \tilde{\Lambda}^{-1} \otimes \theta^i, \]
\[ S_L(\theta^i) = -\tilde{\Lambda}^{-1} \theta^i, \]
\[ \epsilon_L(\theta^i) = 0, \tag{227} \]
and
\[ \Delta_{\bar{L}}(\theta^i) = \theta^i \otimes 1 + \tilde{\Lambda} \otimes \theta^i, \]
\[ S_{\bar{L}}(\theta^i) = -\tilde{\Lambda} \theta^i, \]
\[ \epsilon_{\bar{L}}(\theta^i) = 0, \tag{228} \]
where the unitary scaling operator \( \tilde{\Lambda} \) has to fulfill
\[ \tilde{\Lambda} \theta^i = -q^{\delta_{i1}} \theta^i \tilde{\Lambda}, \quad \tilde{\Lambda}(\partial_{\theta})_1 = -q^{-\delta_{i1}} (\partial_{\theta})_1 \tilde{\Lambda}. \tag{229} \]
The above Hopf structures then induce the operations (for the details see Ref. [68])
\[ f(\theta^1 \oplus_L \psi^1) = f(\theta^1 \oplus_L \psi^1) = f' + f_1(\theta^1 + \psi^1), \]
(230)
and
\[ f(\ominus_L \theta^1) = f(\ominus_L \theta^1) = f' - f_1\theta^1. \]
(231)

For the sake of completeness let us note that from the coproducts in (227) and (228) we can read off the explicit form of the L-matrices for the Grassmann variables \(\theta^i, i = 0, 1\). As soon as we know the action of the scaling operator \(\hat{\Lambda}\) on a given element \(w\) the L-matrices provide a simple method to calculate the braiding of q-deformed Grassmann variables with \(w\) (see also Ref. [53]).

In complete analogy to the symmetrized braided line we can introduce dual pairings for the antisymmetrized braided line. For normally ordered monomials we find as non-vanishing pairings
\[
\langle (\partial \theta)_i, \theta^j \rangle_{L,R} = \delta^j_i,
\]
\[
\langle (\hat{\partial} \theta)_i, \theta^j \rangle_{L,R} = \delta^j_i,
\]
(232)
\[
\langle \theta^j, (\partial \theta)_i \rangle_{L,R} = -\delta^j_i,
\]
\[
\langle \theta^j, (\hat{\partial} \theta)_i \rangle_{L,R} = -\delta^j_i,
\]
(233)
and
\[
\langle (\partial \theta)_0 (\partial \theta)_1, \theta^1 \theta^0 \rangle_{L,R} = 1,
\]
\[
\langle (\hat{\partial} \theta)_1 (\hat{\partial} \theta)_0, \theta^0 \theta^1 \rangle_{L,R} = 1,
\]
(234)
\[
\langle \theta^0 \theta^1, (\partial \theta)_1 (\partial \theta)_0 \rangle_{L,R} = 1,
\]
\[
\langle \theta^1 \theta^0, (\hat{\partial} \theta)_0 (\hat{\partial} \theta)_1 \rangle_{L,R} = 1.
\]
(235)

On the subspace spanned by \(\theta^1\) these pairings correspond to the exponentials
\[
\exp(\theta^1)(\partial \theta)_1)_{R,L} = 1 + \theta^1 \otimes (\partial \theta)_1,
\]
\[
\exp(\theta^1)(\hat{\partial} \theta)_1)_{R,L} = 1 + \theta^1 \otimes (\hat{\partial} \theta)_1,
\]
(236)
\[
\exp((\partial \theta)_1 |\theta^1)_{R,L} = 1 - (\partial \theta)_1 \otimes \theta^1,
\]
\[
\exp((\hat{\partial} \theta)_1 |\theta^1)_{R,L} = 1 - (\hat{\partial} \theta)_1 \otimes \theta^1,
\]
(237)
which, in turn, give rise to the q-deformed delta functions

\[
\delta^1_L(\eta_1) = \int d_L \theta^1 \exp(\theta^1 | \eta_1)_{R,L} = \eta_1, \\
\delta^1_R(\eta_1) = \int d_R \theta^1 \exp(\theta^1 | \eta_1)_{R,L} = \eta_1, \\
\delta^1_{\bar{L}}(\eta_1) = \int d_{\bar{L}} \theta^1 \exp(\theta^1 | \eta_1)_{R,L} = \eta_1, \\
\delta^1_{\bar{R}}(\eta_1) = \int d_{\bar{R}} \theta^1 \exp(\theta^1 | \eta_1)_{R,L} = \eta_1.
\] (238)

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