We formalize a systematic method of constructing forward self-similar solutions to the Navier–Stokes equations in order to characterize the late stage of decaying process of turbulent flows. (i) In view of critical scale-invariance of type 2 we exploit the vorticity curl as the dependent variable to derive and analyse the dynamically scaled Navier–Stokes equations. This formalism offers the viewpoint from which the problem takes the simplest possible form. (ii) Rewriting the scaled Navier–Stokes equations by Duhamel principle as integral equations, we regard the nonlinear term as a perturbation using the Fokker–Planck evolution semigroup. Systematic successive approximations are introduced and the leading-order solution is worked out explicitly as the Gaussian function with a solenoidal projection. (iii) By iterations the second-order approximation is estimated explicitly up to solenoidal projection and is evaluated numerically. (iv) A new characterization of nonlinear term is introduced on this basis to estimate its strength $N$ quantitatively. We find that $N = O(10^{-2})$ for the three-dimensional Navier–Stokes equations. This should be contrasted with $N = O(10^{-1})$ for the Burgers equations and $N \equiv 0$ for the two-dimensional Navier–Stokes equations. (v) As an illustration we explicitly determine source-type solutions to the multi-dimensional Burgers equations. Implications and applications of the current results are given.
1. Introduction

Self-similarity is a tool of fundamental importance in analysing partial differential equations, including the construction of solutions and the determination of their stability. It is also useful for numerical and asymptotic methods of studying partial differential equations. For general aspects of self-similarity and its applications, we refer the readers to e.g. [1,2].

In this paper we will study the so-called source-type self-similar solution to the Navier–Stokes equations. Our motivation for this is as follows. First of all, it will give us a particular self-similar solution to the Navier–Stokes equations, which characterizes the decaying process in the late stage of evolution. Second, it is likely to give useful information as to how we may handle more general solutions.

It may be in order to have a look at previous works which are related to this paper. It has been shown under mild conditions that no nontrivial smooth backward self-similar solution exists to the Navier–Stokes equations. On the other hand, it is known that nontrivial forward self-similar solutions do exist, but their explicit functional forms are not known, except for some asymptotic results. It is of interest to see how they actually behave because such solutions contain important information regarding more general solutions. This is particularly the case when the governing equations are exactly linearizable, e.g. the Burgers equations. While it is not expected that the Navier–Stokes equations are exactly soluble in general, we might still obtain insights into the nature of their solutions.

In [3] the existence of forward self-similar solutions for small data was proven by using a fixed-point theorem in a Besov space (see below). There, initial data for the self-similar solution (in three dimensions) are assumed to be homogeneous of degree $-1$ in velocity

$$
\mathbf{u}_0(\lambda \mathbf{x}) = \lambda^{-1} \mathbf{u}_0(\mathbf{x}),
$$

and the existence of a self-similar solution of the form

$$
\mathbf{u}(\mathbf{x}, t) = \frac{1}{\sqrt{t}} \mathbf{U}\left( \frac{\mathbf{x}}{\sqrt{t}} \right),
$$

has been established under the assumption that initial data are small in some Besov space. Moreover, it has been proved that the self-similar profile $\mathbf{U}$ satisfies (in their notations)

$$
\mathbf{U} = \mathbf{S}(1) \mathbf{u}_0 + \mathbf{W},
$$

where $\mathbf{S}(1)$ denotes a heat operator at time 1 and $||\mathbf{W}||_{L^3}$ is small. In [4] using a locally Hölder class in $\mathbb{R}^3 \setminus \{0\}$, the smallness assumption has been removed and it is furthermore shown that

$$
||\mathbf{U}(\mathbf{x}) - e^{\nabla^2} \mathbf{u}_0(\mathbf{x})|| \leq \frac{C(M)}{(1 + ||\mathbf{x}||)^{1+\alpha}},
$$

where $0 < \alpha < 1$ and $C(M)$ denotes a constant with some norm $M$ of $\mathbf{u}_0$. Those studies indicate that the self-similar solution is close to the heat flow in the late stage. However, studies on the determination of a specific functional form of self-similar solutions are few and far between, except for an attempt in [5]. We also note that the existence of generalized self-similar solutions (in the sense that the scaling holds only at a set of discrete values of $\lambda$) was studied subsequently (e.g. [6,7]).

Our basic strategy in practice is as follows. After recasting the dynamically scaled Navier–Stokes equations as integral equations via the Duhamel principle, we regard the nonlinear term as a perturbation using the Fokker–Planck evolution semigroup. Systematic successive approximations are then introduced and the first-order solution is worked out explicitly as the Gaussian function with a solenoidal projection. The second-order approximation is also evaluated numerically to assess the strength of the nonlinear term.

We will construct an approximate solution to the three-dimensional Navier–Stokes equations valid in the long-time limit, using the vorticity curl $\nabla \times \mathbf{\omega}$. In §4, we will see why this is the most convenient variable from the reaction of the Navier–Stokes equations under dynamic scaling.
Here we appreciate the suitability of such a choice of the unknown by comparing the source-type solutions to the Navier–Stokes equations and their self-similar solutions at criticality. By definition the source-type solution for nonlinear parabolic PDEs is a solution in a scaled space, which starts from the Dirac mass in some dependent variable and ends up like a near-identity of the Gaussian function in the long-time limit. It serves as an analogue of the fundamental solution to nonlinear PDEs.

The unknown whose $L^1$-norm is marginally divergent is suitable for describing the late-stage evolution. This is because this self-similar solution satisfies the same scaling as the Dirac mass and both of them belong to a Besov space near $L^1$. In one dimension, in the limit of $t \to 0$, we have roughly
\[
u \sim \frac{1}{x}, \quad \text{marginally } \notin L^1(\mathbb{R}^1),
\]
which suggests that the velocity is convenient in this case.

In two dimensions, it is the vorticity which is the most convenient, as can be seen from
\[
\omega \sim \frac{1}{r^2}, \quad \text{marginally } \notin L^1(\mathbb{R}^2),
\]
where $|x| = r$. Recall that those scaling properties of velocity in one dimension or vorticity in two dimensions are the same as that of the Dirac mass; $\lambda^d \delta(\lambda x) = \delta(x)$ in $d$-dimensions.

Now consider Besov spaces whose norms are given by
\[
||u||_{B^s_{pq}} \equiv \left\{ \sum_{j=1}^{\infty} (2^j ||\Delta_j(u)||_{L^p})^q \right\}^{1/q},
\]
where $1 \leq p, q \leq \infty, s \in \mathbb{R}$ and $\Delta_j(u)$ represents band-filtered velocity at frequency $2^j$. It is known that in $\mathbb{R}^d$ the Dirac delta mass is embedded as
\[
\delta(x) \in B^{-d+d/p}_{p,\infty}, \quad \text{for } p \geq 1,
\]
see e.g. [8]. In particular we have $\delta(x) \in B^0_{1,\infty}$ for any $d$.

While the velocity $u \sim \frac{1}{r} \notin L^3(\mathbb{R}^3)$, we have
\[
u \sim \frac{1}{r} \in B^0_{3,\infty}(\mathbb{R}^3),
\]
and correspondingly for $\chi = \nabla \times \omega$, the vorticity curl,
\[
\chi \sim \frac{1}{r^3} \in B^0_{1,\infty}(\mathbb{R}^3).
\]
Hence in three dimensions this $\chi$ and the Dirac mass belong to the same function class $B^0_{1,\infty}(\mathbb{R}^3)$, with $p = 1$ in (1.1).

The rest of this paper is organized as follows. In §2 after reviewing critical scale-invariance of type 2 (to be defined below) using the Burgers equation we introduce successive approximations of determining the self-similar profile. On this basis, we introduce and quantify the strength of nonlinearity. Higher-dimensional Burgers equations are also discussed. In §3, we have a brief look at the two-dimensional Navier–Stokes equations. In the main §4, we study the self-similar solutions of the three-dimensional Navier–Stokes equations using the vorticity curl as the unknown to achieve critical scale-invariance of type 2. We carry out successive approximations and determine the strength of nonlinearity as introduced above. Section 5 will be devoted to the Summary and outlook. Some further details and derivations are given in appendices.

2. Burgers equations

We review the source-type solution of the Burgers equation with an emphasis on the critical scale-invariance. Our approach is novel in the introduction of its successive approximations,
in preparation for handling the three-dimensional Navier–Stokes equations, and in employing a new method of estimating the strength of nonlinearity on this basis. Also described are the source-type solutions of the Burgers equations in \( n \)-dimensions.

(a) Critical scale-invariance

We consider the Burgers equation \([9]\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2},
\]

(2.1)

which satisfies static scale-invariance under

\[
x \rightarrow \lambda x, \quad t \rightarrow \lambda^2 t, \quad u \rightarrow \lambda^{-1} u.
\]

This means that if \( u(x, t) \) is a solution, so is \( u(\lambda x, \lambda^2 t) \equiv \lambda u(\lambda x, \lambda^2 t) \), for any \( \lambda > 0 \). It is readily checked that

\[
||u_\lambda||_{L^p} = \lambda^{(p-1)/p} ||u||_{L^p},
\]

which shows that the \( L^1 \)-norm is scale-invariant.

Let us clarify the two kinds of critical scale-invariance. Type 1 scale-invariance is achieved when we use a dependent variable whose physical dimension is the same as \( \nu \). Type 1 is deterministic in nature where the additional term arising in the governing equations under dynamic scaling is minimized in number. Type 2 instead is statistical in nature where the additional terms under dynamic scaling are maximized in number so that a divergence form is completed and the dynamically scaled equations have the Fokker–Planck operator as the linearization. In the former, the dependent variable has the same physical dimension as kinematic viscosity, whereas in the latter the argument of the Hopf characteristic functional (the independent variable) has the same physical dimension as the reciprocal of kinematic viscosity \([10]\). This approach provides a viewpoint from which the problem appears in the simplest possible form.

Critical scale-invariance of type 1 is achieved with the velocity potential \( \phi \), which is defined by \( u = \partial_x \phi \). If \( \phi(x, t) \) is a solution, so is \( \phi(\lambda x, \lambda^2 t) \). Under dynamic scaling for the velocity potential \( \phi(x, t) = \Phi(\xi, \tau), \xi = x/\sqrt{2at}, \tau = (1/2a) \log(t) \) we have

\[
\frac{\partial \Phi}{\partial \tau} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 = \frac{a}{2} \frac{\partial \Phi}{\partial \xi} + \nu \frac{\partial^2 \Phi}{\partial \xi^2},
\]

(2.2)

whose linearization has the Ornstein–Uhlenbeck operator. This is called type 1 (deterministic) scale-invariance where the number of additional terms is minimized, that is, only the drift term remains. Under dynamic scaling for velocity \( u(x, t) = (1/\sqrt{2at}) U(\xi, \tau) \), we find

\[
\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} = a \frac{\partial}{\partial \xi} (\xi U) + \nu \frac{\partial^2 U}{\partial \xi^2},
\]

(2.3)

whose linearization is the Fokker–Planck equation. Here the zooming-in parameter \( a(>0) \) has the same physical dimension as \( \nu \) and is of the same order. With this type 2 (statistical) scale-invariance the number of additional terms is maximized meaning that a divergence form is completed with the addition of \( aU \) term. As it is a second-order equation it has two independent solutions, of which we will focus on the Gaussian one. See the separate electronic supplementary material for the other non-Gaussian kinds of solutions.

Equation (2.3) is exactly soluble and its steady solution is called the source-type solution \([11,12]\):

\[
U(\xi) = \frac{U(0) \exp \left( -\frac{a\xi^2}{2\nu} \right)}{1 - (U(0)/2\nu) \int_0^\infty \exp \left( -\frac{a\eta^2}{2\nu} \right) d\eta}.
\]

(2.4)
The name has come from the time zero asymptotics
\[
\lim_{t \to 0} \frac{1}{\sqrt{2\alpha t}} U(\xi) = M \delta(x),
\]
where \(M \equiv \int_{\mathbb{R}} u_0(x) \, dx\) and \(U(0) = \sqrt{8\alpha\nu/\pi} \tanh(M/4\nu) \approx \sqrt{\alpha/2\pi} vM\) (for \(M/\nu \ll 1\)). Observe that (2.4) is a near-identity transformation of the Gaussian function. See [11–13].

It is also known that for \(u_0 \in L^1\) we have
\[
l^{\frac{1}{2}}(1-\frac{1}{r}) \left\| u(x,t) - \frac{1}{\sqrt{2\alpha t}} U(\xi) \right\|_{L^r} \to 0 \quad \text{as} \; t \to \infty,
\]
where \(1 \leq p \leq \infty\).

The simplest method for solving (2.3) without linearization is as follows. Rewrite the equation
\[
\frac{U^2}{2} = a\xi U + v \frac{dU}{d\xi} = v \exp\left(-\frac{a\xi^2}{2\nu}\right) \frac{d}{d\xi} \left(U \exp\left(\frac{a\xi^2}{2\nu}\right)\right).
\]
By changing variables to \(\bar{U} = U \exp(a\xi^2/2\nu), \eta = 1/2\nu \int_0^{\xi} \exp(-(a\zeta^2/2\nu)) \, d\zeta\), we find
\[
\frac{d\bar{U}}{d\eta} = \bar{U}^2,
\]
which is readily integrable. Alternatively we may solve equation (2.1) by regarding it as a Bernoulli equation.

It may be in order to comment on the significance of source-type solution. When we recast (2.4) as
\[
U(\xi) = -2\nu \frac{\partial}{\partial \xi} \log \left(1 - \frac{U(0)}{2\nu} \int_0^{\xi} \exp\left(-\frac{a\eta^2}{2\nu}\right) \, d\eta\right), \tag{2.5}
\]
it is reminiscent of the celebrated Cole–Hopf transform. In other words, the source-type solution encodes the vital information of the nonlinear term in the case of the Burgers equation. Note that the error-function itself \(\int_0^\xi \exp(-(a\eta^2/2\nu)) \, d\eta\) in (2.4), (2.5) is a self-similar solution to the heat equation. This suggests that studying source-type solution of the Navier–Stokes equations may give a hint on how to characterize their long-time evolution by a heat flow.

(b) Successive approximations

The operator \(L = \Delta^* \equiv \Delta + (a/\nu) \partial_\xi (\xi \cdot \cdot )\) is not self-adjoint. It is possible to find a function \(G\) such that \(L^* G(\xi) = -\delta(\xi)\) holds, where \(L^* \equiv \partial_\xi^2 - (a/\nu) \partial_\xi\) is the adjoint of \(L\). In fact \(G(\xi) \propto D(\sqrt{a/2\nu} \xi)\), where \(D(\cdot)\) denotes Dawson’s integral, defined by \(D(x) \equiv e^{-x^2} \int_0^x e^{y^2} \, dy\) [14]. However, because \(G\) decays slowly at large distances \(G(\xi) \propto 1/\xi\) as \(|\xi| \to \infty\), it cannot be used as Green’s function, at least, in the usual manner.

The inversion formula for \(\Delta^*\) can be obtained by an alternative method. Recall that based on \(\frac{1}{\eta} = \int_0^\infty e^{-at} \, dt\) \((a > 0)\), the fundamental solution to the Poisson equation in one dimension is given by
\[
(\nu \Delta)^{-1} \equiv -\int_0^\infty ds e^{\nu s \Delta} = |\xi|/2\nu *,
\]
where * denotes convolution. Likewise for the fundamental solution to the Fokker–Planck equation in one dimension we write
\[
(\nu \Delta^*)^{-1} \equiv -\int_0^\infty ds e^{\nu s \Delta^*} = \int_0^\infty d\eta g(\xi, \eta),
\]

\footnote{With a slight abuse of notation this \(G\) should be distinguished from the Gaussian function used in §3.}
where
\[ g(\xi, \eta) \equiv \frac{-1}{\sqrt{2\pi}} \text{f.p.} \int_{\sigma}^{\infty} \frac{d\sigma}{\sqrt{2\sigma^2 - a}} e^{-(1/2\nu)(\sigma^2 - a - \sigma\eta)} \, d\sigma, \]
and f.p. denotes the finite part of Hadamard, e.g. [15,16]. It can be verified by changing the variable from \( s \) to \( \sigma = \sqrt{a/(1 - e^{-2\nu})} \), using the solution of the Fokker–Planck equation
\[ e^{\nu \tau} f = \left( \frac{a}{2\pi \nu (1 - e^{-2\nu})} \right)^{1/2} \int_0^{\tau} e^{\nu (s - \tau) \Delta^*} \frac{U(s)^2}{2} \, ds \]
As we will consider the three-dimensional Navier–Stokes equations, for which methods of exact solutions are unavailable, we treat (2.3) by approximate methods as an illustration. Because the inversion of \( \Delta^* \) is unwieldy, we will seek a workaround by which we can dispense with it.

First we convert it to an integral equation by the Duhamel principle for the Fokker–Planck operator
\[ \Delta^* \]
\[ U(\tau) = e^{\nu \tau \Delta^*} U(0) - \int_0^\tau e^{\nu (\tau - s) \Delta^*} \frac{U(s)^2}{2} \, ds \]
The long-time limit \( U_1 = \lim_{\tau \to \infty} e^{\nu \tau \Delta^*} U(0) \) is given by
\[ U_1 = \left( \frac{a}{2\pi \nu} \right)^{1/2} M e^{-(a/2\nu)\xi^2} \text{ with } M = \int_{-\infty}^{\infty} U(0) \, d\xi. \]
We may consider a number of different iteration schemes. For example, the following option (1), also known as the Picard iteration, requires the inversion \( (\Delta^*)^{-1} \):

Successive approximation (1) : \( U_{n+1} = U_1 - \int_0^\infty e^{\nu s \Delta^*} \frac{U_n^2}{2} \, ds \)
and
in particular, for \( n = 1 \) : \( U_2 = U_1 - \int_0^\infty e^{\nu s \Delta^*} \frac{U_1^2}{2} \, ds \).

Note that \( U_n \) is a steady function at each step.

Alternatively, we first consider the steady equation
\[ \Delta^* U \equiv \Delta U + \frac{a}{\nu} (\xi U)_{\xi} = \frac{1}{\nu} \left( \frac{U_1^2}{2} \right)_{\xi}, \]
and then introduce iteration schemes:

Iteration scheme (2a) : \( \Delta U_{n+1} + \frac{a}{\nu} (\xi U_{n+1})_{\xi} = \frac{1}{\nu} \left( \frac{U_n^2}{2} \right)_{\xi}, \) \( (n \geq 0) \)

For \( n = 1 \) : \( \Delta U_2 + \frac{a}{\nu} (\xi U_2)_{\xi} = \frac{1}{\nu} \left( \frac{U_1^2}{2} \right)_{\xi}, \)
or

Iteration scheme (2b) : \( \Delta U_{n+1} = -\frac{a}{\nu} (\xi U_n)_{\xi} + \frac{1}{\nu} \left( \frac{U_n^2}{2} \right)_{\xi}, \) \( (n \geq 1) \)

For \( n = 1 \) : \( \Delta U_2 = -\frac{a}{\nu} (\xi U_1)_{\xi} + \frac{1}{\nu} \left( \frac{U_1^2}{2} \right)_{\xi}. \)

Note that iteration schemes (1) and (2a) coincide with each other at \( n = 1 \).
(c) Estimation of the strength of nonlinearity

For the Burgers equation, we can work out the two kinds of approximations to the second-order analytically. After straightforward algebra, they are

\[ U \approx C e^{-(a/2v)\xi^2} \left( 1 + \frac{C}{2v} \int_0^\xi e^{-(a/2v)\eta^2} d\eta \right) \]

and

\[ U \approx C e^{-(a/2v)\xi^2} + \frac{C^2}{2v} \int_0^\xi e^{-(a/2v)\eta^2} d\eta, \]

where \( C \approx \sqrt{a/2\pi vM} \). On this basis, we estimate the strength of the nonlinear term \( N(\xi) \). The source-type solution is a near identity transform of the Gaussian function. In its series expansion in the Reynolds number \( Re = M/v \) after non-dimensionalization, the nonlinear correction term has \( Re \) as its prefactor. Consider the scheme (1), or (2a) equivalently, taking \( a/2v = 1 \) without loss of generality. We have

\[ U_2 = \frac{M}{\sqrt{\pi}} e^{-\xi^2} \left( 1 + \frac{Re}{2\sqrt{\pi}} \int_0^\xi e^{-\eta^2} d\eta \right). \]

Separating out the \( Re \)-dependence, or equivalently assuming that \( Re = 1 \), we define \( N(\xi) \) by

\[ N(\xi) = \sup_{\xi} N(\xi) = \frac{1}{4}, \]

so that \( U_2 \propto 1 + ReN(\xi) \) holds. The strength of nonlinearity is given by

\[ N = \sup_{\xi} N(\xi) = \frac{1}{4}. \]

The same goes for (2b) from the above expressions for the one-dimensional Burgers equation. Altogether we find

\[ (1) \quad N = \frac{1}{4} = 0.25, \]

\[ (2b) \quad N = \frac{1}{4\sqrt{2}} \approx 0.2. \]

We conclude that the typical strength of nonlinearity is \( N = O(10^{-1}) \), irrespective of the choice of schemes.

(d) Burgers equations in several dimensions

The source-type solution is basically a near-identity function of the Gaussian form. It has been seen how the source-type solutions show up in the long-time limit in one and two spatial dimensions in [10]. Here we will take a look at cases in three and higher dimensions. From the Cole–Hopf transform, we have

\[ U_1(\xi, \tau) = -2v \frac{\partial_\xi \int_{\mathbb{R}^3} \psi_0(\lambda \eta) \exp(-a/2v)(|\xi - \eta|^2/(1 - e^{-2\tau})) d\eta}{\int_{\mathbb{R}^3} \psi_0(\lambda \eta) \exp(-a/2v)(|\xi - \eta|^2/(1 - e^{-2\tau})) d\eta}. \]

As we are going for the type 2 scale-invariance, differentiating it twice we find

\[ \partial_\xi \partial_\eta U_1(\xi, \tau) = -2v \frac{\partial_\xi \partial_\eta \partial_\xi \partial_\eta \int_{\mathbb{R}^3} \psi_0(\lambda \eta) \exp(-a/2v)(|\xi - \eta|^2/(1 - e^{-2\tau})) d\eta}{\int_{\mathbb{R}^3} \psi_0(\lambda \eta) \exp(-a/2v)(|\xi - \eta|^2/(1 - e^{-2\tau})) d\eta} + \ldots \]

\[ = -\lambda^2 \frac{\int_{\mathbb{R}^3} \partial_\xi \partial_\eta \psi_0(\lambda \eta) \exp(-a/2v)(|\xi - \eta|^2/(1 - e^{-2\tau})) d\eta}{\int_{\mathbb{R}^3} \psi_0(\lambda \eta) \exp(-a/2v)(|\xi - \eta|^2/(1 - e^{-2\tau})) d\eta} + \ldots \]
The denominator then tends to \( K_{ijk} \exp(-(a/2v)(\xi^2)) \) as \( \tau \to \infty \), where \( K_{ijk} = \int_{\mathbb{R}^3} \partial_i \partial_j \partial_k \psi_0(\eta) \, d\eta \), \((i = 1, 2, 3)\). Hence
\[
\partial_i \partial_j \partial_k U_i(\xi, \infty) = -2v \left( \frac{K_{ijk} \exp \left(-\frac{a}{2v} \xi^2 \right)}{F_{ijk}(\xi)} + \ldots \right),
\]
where the function \( F_{ijk} \) is to be determined such that \( \partial_i \partial_j \partial_k F_{ijk} \propto \exp(-(a/2v)(\xi^2)), \) (no summation implied). We can thus take
\[
F_{ijk}(\xi) = -\frac{K_{ijk}}{2v} \int_0^{\xi_1} \exp \left(-\frac{a}{2v} \xi_1^2 \right) \, d\xi \int_0^{\xi_2} \exp \left(-\frac{a}{2v} \eta^2 \right) \, d\eta \int_0^{\xi_3} \exp \left(-\frac{a}{2v} \zeta^2 \right) \, d\zeta + 1.
\]
Therefore after collecting other terms of derivatives we find in three dimensions, say, with \((i, j, k) = (1, 2, 3)\),
\[
\frac{\partial^2 U_1}{\partial \xi_2 \partial \xi_3} = K_{123} \exp \left( -\frac{a}{2v} \xi_1^2 + \xi_2^2 + \xi_3^2 \right) \frac{1 + R(\xi_1, \xi_2, \xi_3)}{(1 - R(\xi_1, \xi_2, \xi_3))^3}, \tag{2.6}
\]
where
\[
R(\xi_1, \xi_2, \xi_3) = \frac{K_{123}}{2v} \int_0^{\xi_1} \exp \left(-\frac{a}{2v} \xi_1^2 \right) \, d\xi \int_0^{\xi_2} \exp \left(-\frac{a}{2v} \eta^2 \right) \, d\eta \int_0^{\xi_3} \exp \left(-\frac{a}{2v} \zeta^2 \right) \, d\zeta,
\]
denotes the Reynolds number. Because \( R \) is small the expression (2.6) is near-Gaussian. It can be verified that \( K_{123} = \sqrt{32a^3/\pi^3 v} \tanh(M_{123}/16v) \), where \( M_{123} = \int (\partial^2 U_1/\partial \xi_2 \partial \xi_3) \, d\xi \). We can also write
\[
\frac{\partial^2 U_1}{\partial \xi_2 \partial \xi_3} = -2v \frac{\partial^3}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \log(1 - R(\xi_1, \xi_2, \xi_3)),
\]
which reflects the Cole–Hopf transform more directly, that is, \( U = \nabla_\xi \phi, \, \phi = -2v \log(1 - R(\xi)) \). See appendix A for the general form in \( n \)-dimensions.

### 3. Two-dimensional Navier–Stokes equation

We briefly recall the self-similar solution of the two-dimensional Navier–Stokes equations, where the so-called Burgers vortex appears after dynamic scaling.

**(a) Critical scale-invariance**

The Burgers vortex was originally introduced to represent the reaction of a vortex under the influence of the collective effect of surrounding vortices in the ambient medium. When we write the steady solution in velocity and vorticity using cylindrical coordinates
\[
u = (u_r, u_\theta, u_\phi) = (-ar, v(r), 2az), \quad \omega = (0, 0, \omega(r)),
\]
the solution takes the following forms:
\[
\omega(r) = \frac{a\Gamma}{2\pi v} \exp \left(-\frac{a r^2}{2v} \right)
\]
and
\[
v(r) = \frac{\Gamma}{2\pi r} \left( 1 - \exp \left(-\frac{ar^2}{2v} \right) \right),
\]
where \( \Gamma \equiv \int_{\mathbb{R}^2} \omega_0(x) \, dx \) denotes the velocity circulation.
In two dimensions, dynamic scaling transforms take the following form

\[ \xi = \frac{x}{\sqrt{2at}}, \quad \tau = \frac{1}{2a} \log t, \quad \psi(x,t) = \Psi(\xi, \tau) \]

and

\[ u(x, t) = \frac{1}{\sqrt{2at}}U(\xi, \tau), \quad \omega(x, t) = \frac{1}{2at} \Omega(\xi, \tau). \]

In the two-dimensional case, in order to achieve the critical scale-invariance of type 2, we must choose the second spatial derivative of the stream function, which is the vorticity, as the unknown. The scaled form of the vorticity equation in two dimensions reads

\[ \partial \Omega \partial \tau + U \cdot \nabla \Omega = \nu \Delta \Omega + a \nabla \cdot (\xi \Omega), \]

where \( \Omega \) satisfies the type 2 scale-invariance. It is known that the self-similar solution under scaling has a mathematically identical form to the Burgers vortex above. Indeed in the scaled variables the above expression can be written

\[ \Omega(\xi) = \frac{a\Gamma}{2\pi \nu} \exp \left( -\frac{a|\xi|^2}{2\nu} \right), \quad \xi = \frac{x}{\sqrt{2at}}. \]

Note that \((1/2at)\Omega(\xi) = (\Gamma/4\pi \nu t) \exp(-(|x|^2/4\nu t))\) is an exact self-similar decaying solution\(^2\) with the following property

\[ \lim_{t \to 0} \Omega(\cdot) = \Gamma \delta(x). \]

It also satisfies the following asymptotic property, for \( \omega(\cdot, 0) \in L^1 \),

\[ t^{1-\frac{1}{p}} \left\| \omega(x, t) - \frac{1}{2at} \Omega(\xi) \right\|_{L^p} \to 0 \quad \text{as} \quad t \to \infty, \]

where \( 1 \leq p \leq \infty \), see e.g. [17].

(b) Interpretation

Because the source-type solution coincides with the linearized solution, the inhomogeneous terms on the right-hand side of the approximations at each order vanish identically. Hence there is no way to set up successive approximations that can capture non-zero nonlinear corrections. The strength of nonlinearity is identically zero; \( N = 0 \).

4. Three-dimensional Navier–Stokes equations

We will describe two approaches for handling the scaled three-dimensional Navier–Stokes equations perturbatively. First we describe a general framework based on Green’s function. Second we describe an iterative approach which is specifically suited for calculations associated with the three-dimensional Navier–Stokes problem.

(a) Critical scale-invariance

We consider the three-dimensional Navier–Stokes equations written in four different dependent variables. Starting from the vector potential and taking a curl successively \( u = \nabla \times \psi, \omega = \nabla \times u \),

\(^2\)When \( a = 0 \) an exact decaying solution is obtained by formally replacing \( a \to 1/2t \), which is known as the Lamb–Oseen vortex.
\( \chi = \nabla \times \omega \), we have
\[
\frac{\partial \Psi}{\partial t} = \frac{3}{4\pi} \text{p.v.} \int_{\mathbb{R}^3} \frac{r \times (\nabla \times \Psi(y)) \cdot r \cdot (\nabla \times \Psi(y))}{|r|^5} \, dy + v \Delta \Psi,
\]
\[
\frac{\partial \chi}{\partial t} = \nabla \times \nabla u + v \Delta \chi,
\]
where \( r = x - y \) and p.v. denotes a principal-value integral. We also have \( \omega = -\Delta \chi \), \( \chi = -\Delta u \), because of the incompressibility condition. The derivation of (4.1)\(_1\) can be found in [18]. The final fourth equation (4.1)\(_4\) is obtained by taking the Laplacian of the velocity equations (4.1)\(_2\). For the \( \chi \) equations, we may alternatively take a curl on the vorticity equations to obtain a form of equation different from the final line in (4.1), which is useful for handling inviscid fluids (details to be found in appendix B).

Under dynamic scaling
\[
\xi = \frac{x}{\sqrt{2at}}, \quad \tau = \frac{1}{2a} \log t, \quad \Psi(x, t) = \Psi(\xi, \tau),
\]
\[
u(x, t) = \frac{1}{(2at)^{1/2}} U(\xi, \tau), \quad \omega(x, t) = \frac{1}{2at} \Omega(\xi, \tau) \quad \text{and} \quad \chi(x, t) = \frac{1}{(2at)^{3/2}} X(\xi, \tau),
\]
the three-dimensional Navier–Stokes equations in four different unknowns are transformed, respectively, to
\[
\frac{\partial \Psi}{\partial \tau} = \frac{3}{4\pi} \text{p.v.} \int_{\mathbb{R}^3} \frac{\rho \times (\nabla \times \Psi(\eta)) \cdot \rho \cdot (\nabla \times \Psi(\eta))}{|\rho|^3} \, d\eta + v \Delta \Psi + a(\xi \cdot \nabla) \Psi,
\]
\[
\frac{\partial U}{\partial \tau} + U \cdot \nabla U = -\nabla P + v \Delta U + a(\xi \cdot \nabla) U + aU,
\]
\[
\frac{\partial \Omega}{\partial \tau} + U \cdot \nabla \Omega = \Omega \cdot \nabla U + v \Delta \Omega + a(\xi \cdot \nabla) \Omega + 2a\Omega
\]
and
\[
\frac{\partial X}{\partial \tau} = \Delta(U \cdot \nabla U + \nabla P) + v \Delta X + a \nabla \cdot (\xi \otimes X),
\]
where \( \rho \equiv \xi - \eta \).

It is to be noted that the coefficient of the linear term increases in number with the increasing order of derivatives and for the variable \( \chi \) a divergence is completed in the convective term. Observe that type 1 scale-invariance is achieved with \( \Psi \) and type 2 scale-invariance with \( X \).

(b) Successive approximations

Using the Duhamel principle, we convert the scaled Navier–Stokes equations (4.2)\(_4\) into integral equations
\[
X(\xi, \tau) = e^{v \Delta^*_s} X(\xi, 0) + \int_0^\tau e^{v s \Delta^*_s} \Delta(U \cdot \nabla U + \nabla P)(\xi, \tau - s) \, ds.
\]
Here \( \Delta^*_s \equiv \Delta + (a/v) \nabla \cdot (\xi \otimes \cdot) \) and the action of whose exponential operator is given by
\[
\exp(v \tau \Delta^*_s) f(\cdot) = \left( \frac{a}{2\pi v(1 - e^{-2\tau v})} \right)^{3/2} \int_{\mathbb{R}^3} e^{3\tau x} f(e^{2\tau x}) \exp \left( -\frac{a}{2v} \frac{|\xi - y|^2}{1 - e^{-2\tau v}} \right) \, dy,
\]
for any function \( f \), as can be verified by combining the heat kernel and dynamic scaling transforms.
The inverse operator associated with the fundamental solution to the Fokker–Planck equation in three dimensions is defined by

$$(\nu \triangle^* )^{-1} \equiv - \int_0^\infty ds e^{\nu s \triangle^*} = \int \delta \eta (\xi, \eta),$$

where Green’s function is given by

$$g(\xi, \eta) \equiv - \frac{1}{(2\pi \nu)^{3/2}} \text{f.p.} \int_0^\infty \frac{d\sigma}{\sqrt{a^2 - \sigma^2}} e^{-(1/2\nu)(\sigma^{-2} - \sigma^{-2} - a^2)}.$$

This can be verified by changing the variable from $s$ to $\sigma = \sqrt{a/(1 - e^{-2as})}$ in the solution (4.3) to the Fokker–Planck equation.

We consider the steady solution $X(\xi)$ in the long-time limit of $\tau \to \infty$

$$X(\xi) = X_1(\xi) + \lim_{\tau \to \infty} \int_0^\tau e^{\nu s \triangle^*} \triangle (U \cdot \nabla U + \nabla P)(\xi, \tau - s) ds,$$

where $X_1 = PMG$ denotes the leading-order approximation (to be made more explicit in the next subsection). This is one form of integral equations we are supposed to handle.

On the other hand, steady equations are obtained by assuming $\partial/\partial \tau = 0$ in (4.2)

$$\triangle X \equiv \triangle X + \frac{a}{\nu} \nabla \cdot (\xi \otimes X) = - \frac{1}{\nu} \triangle (U \cdot \nabla U + \nabla P),$$

or

$$X = - \frac{1}{\nu}(U \cdot \nabla U + \nabla P) + \frac{a}{\nu} \triangle^{-1} \nabla \cdot (\xi \otimes X).$$

This is yet another form of the steady Navier–Stokes equations after dynamic scaling. It is noted that one of the potential problems associated with the nonlinear term has been eliminated without having a recourse to Green’s function. It is this virtually trivial fact that allows us to set up a simple successive approximation.

To summarize, the steady Navier–Stokes equations after dynamic scaling can be written as

$$X = - \frac{1}{\nu} \text{P}(U \cdot \nabla U) - \frac{a}{\nu} \triangle^{-1} \nabla \cdot (\xi \otimes X), \quad (4.4)$$

or, by $X = - \triangle U$, we can express it solely in terms of $X$ as

$$X = - \frac{1}{\nu} \text{P}(\triangle^{-1} X \cdot \nabla \triangle^{-1} X) - \frac{a}{\nu} \triangle^{-1} \nabla \cdot (\xi \otimes X). \quad (4.5)$$

This is the set of equations that we need to solve.

In passing we note the following facts before proceeding to the specific results. By the definition of scaled variables, it is easily seen that for $p \geq 1$

$$t^{1 - \frac{3}{2}} \left\| X(x, t) - \frac{X(\xi)}{(2at)^{3/2}} \right\|_{L^p} = \left\| X(\xi, \tau) - X(\xi) \right\|_{L^p}.$$

This means that if $\left\| X(\xi, \tau) - X(\xi) \right\|_{L^p} \to 0$ as $\tau \to \infty$, we have

$$t^{1 - \frac{3}{2}} \left\| X(x, t) - \frac{X(\xi)}{(2at)^{3/2}} \right\|_{L^p} \to 0 \quad \text{as} \quad t \to \infty.$$

That is about the long-time asymptotics. On the other hand, as time-zero asymptotics we have

$$\frac{X(\xi)}{(2at)^{3/2}} \to PM\delta \quad \text{as} \quad t \to 0,$$

where $\delta(\cdot)$ is the Dirac mass.

(c) Leading-order approximations

Before discussing the second-order approximation, we derive expressions of the first-order solutions in several different variables.
We will derive the basic formulas by solving the heat equation

$$\frac{\partial \Psi_1}{\partial t} = \nu \Delta \Psi_1.$$  

The first-order approximation is given by

$$\Psi_1(x,t) = \frac{1}{(4\pi \nu t)^{3/2}} \int_{\mathbb{R}^3} \psi_0(y) \exp \left( -\frac{|x-y|^2}{4\nu t} \right) dy.$$  

After applying the dynamic scaling

$$\Psi_1(x,t) = \Psi_1(\xi, \tau), \quad \xi = \frac{x}{\sqrt{2a\tau}}, \quad \tau = \frac{1}{2a} \log t,$$

the linearized equations for the vector potential read

$$\frac{\partial \Psi_1}{\partial \tau} = a\xi \cdot \nabla \Psi_1 + \nu \Delta \Psi_1.$$  

After dynamic scaling their solution is given by

$$\Psi_1(\xi, \tau) = e^{-3\tau} \left( \frac{a}{2\pi \nu (1 - e^{-2a\tau})} \right)^{3/2} \int_{\mathbb{R}^3} \psi_0(y) \exp \left( -\frac{a}{2\nu} \frac{|\xi - \eta - e^{-a\tau}|^2}{1 - e^{-2a\tau}} \right) d\eta$$

$$= \left( \frac{a}{2\pi \nu (1 - e^{-2a\tau})} \right)^{3/2} \int_{\mathbb{R}^3} \psi_0(e^{a\tau} y) \exp \left( -\frac{a}{2\nu} \frac{|\xi - y|^2}{1 - e^{-2a\tau}} \right) dy,$$

where $\Psi_1$ denotes the first-order approximation and $\psi_0$ the initial data. (The same convention applies to $X_1$ and $X_0$ in the following.) Taking a curl with respect to $\xi$ three times we find the expressions for the vorticity curl

$$X_1(\xi, \tau) = \left( \frac{a}{2\pi \nu (1 - e^{-2a\tau})} \right)^{3/2} \int_{\mathbb{R}^3} e^{3\tau} \psi_0(e^{a\tau} y) \exp \left( -\frac{a}{2\nu} \frac{|\xi - y|^2}{1 - e^{-2a\tau}} \right) dy.$$  

For well-localized initial data, we make use of the formula $\lambda^3 X_0(\lambda y) \to M\delta(y)$ where $M = \int X_0 dy$. Noting that $\mathbb{P} X_0 = X_0$ we have

$$X_1(\xi, \tau) = \left( \frac{a}{2\pi \nu (1 - e^{-2a\tau})} \right)^{3/2} \int_{\mathbb{R}^3} e^{3\tau} \psi_0(e^{a\tau} y) \exp \left( -\frac{a}{2\nu} \frac{|\xi - y|^2}{1 - e^{-2a\tau}} \right) dy$$

$$= \left( \frac{a}{2\pi \nu (1 - e^{-2a\tau})} \right)^{3/2} \int_{\mathbb{R}^3} e^{3\tau} \psi_0(e^{a\tau} y) \exp \left( -\frac{a}{2\nu} \frac{|\xi - y|^2}{1 - e^{-2a\tau}} \right) dy$$

$$\to \mathbb{P} M G \quad \text{as} \quad \tau \to \infty,$$

where $G(\xi) \equiv (a/2\pi \nu)^{3/2} \exp\left(-\frac{a}{2\nu}(\xi)^2\right)$ and $M = \int X_0 d\xi$. This is the leading-order approximation for the scaled three-dimensional Navier–Stokes equations.

The first-order (that is, the leading-order) approximation obtained above can be calculated explicitly because the Gaussian function is a radial function. Care should be taken that the leading-order approximations themselves are *not* radial because of the incompressibility.

---

1. If the initial condition satisfies the similarity condition $\lambda^3 X_0(\lambda y) = X_0(y)$ for $\forall \lambda (>0)$, we have $X_1(\xi, \tau) \to (a/2\pi \nu)^{3/2} \int_{\mathbb{R}^3} X_0(y) \exp\left(-\frac{a}{2\nu}(\xi - y)^2\right) dy = X_0 * G$. In this case, $X_0(\xi)$ is singular like $\sim |\xi|^{-3}$.  

---
condition. Indeed, in terms of the vorticity curl the first-order approximation is given by

\[ X_i = M_j \left( \frac{a}{2\pi} \right)^{3/2} (\delta_{ij} - \partial_i \partial_j) \exp \left( -\frac{a}{2r^2} \right), \quad (i = 1, 2, 3), \]

\[ = M_j \left( \frac{\mu}{\pi} \right)^{3/2} \exp(-\mu r^2) + M_j \partial_i \partial_j \frac{e^{\sqrt{\mu}r}}{4\pi r}, \]

\[ = M_j \left( \frac{\mu}{\pi} \right)^{3/2} \exp(-\mu r^2) \]

\[ - M_j \left( \frac{\delta_{ij} - \xi_i \xi_j}{r^2} \right) \left( \frac{\mu}{\pi} \right)^{3/2} e^{-\mu r^2}, \]

\[ - 2M_j \left( \frac{\delta_{ij}}{r^3} - \frac{3\xi_i \xi_j}{r^5} \right) \left( \frac{\mu}{\pi} \right)^{3/2} \left\{ \exp(\sqrt{\mu}r) - \frac{r}{2\mu} \left( \frac{\mu}{\pi} \right)^{3/2} e^{-\mu r^2} \right\}, \]

(4.6)

where \( \mu = a/(2\pi) \), \( r = |\xi| \), \( M_j = \int X_j d\xi \), (\( j = 1, 2, 3 \)) and summation is implied on repeated indices. In the second line, we computed \( \Delta^{-1} \) for the Gaussian function using \( \Delta = 1/r^2(d/dr)(r^2(d/dr)) \) and the final line by direct computations. Clearly the final expression is not radial.

Hereafter in this subsection we take \( \mu = 1 \) for simplicity. Note that \( \Delta^{-n} e^{-r^2} \) for \( n = 1, 2, 3 \) can be evaluated by quadratures and their explicit form are as follows, which can be obtained most conveniently with the assistance of computer algebra. The results are

\[ \Delta^{-1} e^{-r^2} = -\frac{1}{2r} \int_0^r e^{-s^2} ds = -\frac{\sqrt{\pi}}{4r} \text{erf}(r), \]

(4.7)

\[ \Delta^{-2} e^{-r^2} = -\frac{1}{2r} \int_0^r ds \int_0^s ds' e^{-s^2} ds'' \]

\[ = -\frac{e^{-r^2}}{8} - \frac{\sqrt{\pi}}{8}(r + (1/2)r)\text{erf}(r), \]

(4.8)

and

\[ \Delta^{-3} e^{-r^2} = -\frac{1}{2r} \int_0^r ds \int_0^s ds' \int_0^{s''} ds''' e^{-s^2}, \]

\[ = -\frac{1}{384r} \left( (4r^3 + 10r)e^{-r^2} + 4\sqrt{\pi} \left( r^4 + 3r^2 + \frac{3}{4} \right) \text{erf}(r) \right). \]

(4.9)

With these at hand the expressions for the several different unknowns, including the vorticity curl above, are (\( i = 1, 2, 3 \))

\[ B_i = \frac{1}{\pi^{3/2}} \left( M_i \Delta^{-2} e^{-r^2} - M_j \partial_i \partial_j \Delta^{-3} e^{-r^2} \right), \]

(4.10)

\[ \Psi_i = \frac{\epsilon_{ijk} M_i \xi_k}{8\pi^{3/2}} J(r), \]

(4.11)

\[ U_i = \frac{1}{\pi^{3/2}} \left( M_i \Delta^{-1} e^{-r^2} - M_j \partial_i \partial_j \Delta^{-2} e^{-r^2} \right), \]

(4.12)

\[ \Omega_i = \frac{\epsilon_{ijk} M_j \xi_k}{4\pi^{3/2}} H(r), \]

(4.13)

and

\[ X_i = \frac{1}{\pi^{3/2}} \left( M_i e^{-r^2} - M_j \partial_i \partial_j \Delta^{-1} e^{-r^2} \right), \]

(4.14)

where \( \Psi = \nabla \times B \), \( U = \nabla \times \Psi \), \( \Omega = \nabla \times U \), \( X = \nabla \times \Omega \) and \( \text{erf}(r) \equiv 2/\sqrt{\pi} \int_0^r e^{-t^2} dt \) denotes the error function. Here for convenience we have introduced two functions

\[ H(r) = \frac{\sqrt{\pi} \text{erf}(r) - 2re^{-r^2}}{r^3} \quad \text{and} \quad J(r) = \frac{r e^{-r^2} + \sqrt{\pi} \text{erf}(r)(r^2 - (1/2))}{r^3}, \]

both of which are continuous at \( r = 0 \) (figure 1). Because all the fields are incompressible we also have \( U = -\Delta B \), \( \Omega = -\Delta \Psi \), \( X = -\Delta U \).

Regarding the derivations of (4.10)–(4.14), applying \( \Delta^{-1} \) to (4.14) repeatedly we get (4.12) and (4.10), respectively. Then, taking a curl of (4.10) we get (4.11) and taking a curl of (4.12) we get (4.13). Finally, simpler forms of (4.12) and (4.14) are obtained by taking a curl of (4.11) and (4.13),
Figure 1. Comparison of $\exp(-\xi^2)$ (solid), $\pi^{3/2} \chi_j(\xi) = H(\xi)/2$ (dashed) and $J(\xi)/2$ (dotted). See the text for their definitions. Note that $H(0) = J(0) = 2/3$.

respectively. The final results in vectorial form read

$$
\Psi = \frac{M \times \xi}{8\pi^{3/2}} J(r),
$$

(4.15)

$$
U = \frac{\text{erf}(r)}{4\pi r} \left( M - \frac{(M \cdot \xi) \xi}{r^2} \right) - \frac{J(r)}{8\pi^{3/2}} \left( M - \frac{3(M \cdot \xi) \xi}{r^2} \right),
$$

(4.16)

$$
\Omega = \frac{M \times \xi}{4\pi^{3/2}} H(r)
$$

(4.17)

and

$$
X = \frac{e^{-r^2}}{\pi^{3/2}} \left( M - \frac{(M \cdot \xi) \xi}{r^2} \right) - \frac{H(r)}{4\pi^{3/2}} \left( M - \frac{3(M \cdot \xi) \xi}{r^2} \right).
$$

(4.18)

It is of interest to observe that $U \cdot \Omega = \Omega \cdot X = 0$. Using the above formula (4.13) with $M = (1, 1, 1)$ it is instructive to compare a component of the Gaussian function $\pi^{3/2} \exp(-\xi^2)$ with that of the vorticity curl

$$
X_1(\xi, 0, 0) = \frac{\sqrt{\pi} \text{erf}(\xi) - 2x e^{-\xi^2}}{2\pi^{3/2} \xi^3} \left( \frac{H(\xi)}{2\pi^{3/2}} \right).
$$

(4.19)

Figure 1 shows how $X_1(\xi, 0, 0)$ is affected by the incompressible condition (solenoidality), in particular the peak value at $\xi = 0$ is reduced by a factor of 2/3.

(d) Estimation of the strength of nonlinearity

We first describe the numerical methods employed to obtain the approximate solutions. We use a centred finite-difference scheme in a box $[-L, L]^3$ of size $L$, with discretized coordinates $(x_i, y_j, z_k) = (i \Delta h, j \Delta h, k \Delta h)$ where $i, j, k = -M, \ldots, M$ and $\Delta h = L/M$.

An important step in the determination of the second-order approximation is the inversion of the Laplacian operator. This was done by using a Poisson solver in the Intel MKL library. Parameters used are $L = 40, M = 800$, hence $\Delta h = 0.05$. The box size $L$ was chosen large enough to capture the decay of $X(\xi)$ near the boundaries and $M$ was chosen large enough to resolve spatial structure in the centre of the domain. As a code validation we calculated (4.6) numerically and compared against the analytical expression (4.18) and confirmed their agreement (figure omitted).

To evaluate the perturbation, we make use of the iteration scheme (2b) illustrated in the previous section for simplicity. Consider a series expansion of the source-solution in $Re$. In

\[ 14 \]
comparison to the leading-order, the nonlinear correction is proportional to $Re$.\footnote{After full non-dimensionalization $\tilde{X} = v x(x, t)/(2\alpha t)^{3/2}$ we have $\tilde{X}_2 = \tilde{F}^2 - F(\Delta^{-1}\tilde{F}\tilde{G} \cdot \nabla \Delta^{-1}\tilde{F}^2 \tilde{G})$, where $\tilde{M} = M/v$. As $|\tilde{M}| = Re$ it is clear that the nonlinear term has a factor of $Re$ in comparison to the leading-order approximation.} Separating out $Re$, or equivalently assuming that $Re = 1$, we define the strength of nonlinearity by the remaining factor. To be more specific the second-order solution in this case is given by

$$X_2 = X_1 - \frac{1}{v} \tilde{F}(\Delta^{-1} X_1 \cdot \nabla \Delta^{-1} X_1) = \tilde{F} \tilde{M} G - \frac{1}{v} \tilde{F}(\Delta^{-1} \tilde{F} \tilde{M} G \cdot \nabla \Delta^{-1} \tilde{F} \tilde{M} G) = \tilde{F} \tilde{M} G - Re \tilde{F}(\Delta^{-1} \tilde{F} \tilde{M} G \cdot \nabla \Delta^{-1} \tilde{F} \tilde{M} G) / |M|.$$ 

Separating out the $Re$-dependence we define $N(\xi)$ by

$$N(\xi) = \frac{|\tilde{F}(\Delta^{-1} \tilde{F} \tilde{M} G \cdot \nabla \Delta^{-1} \tilde{F} \tilde{M} G) / |M| |}{\sup_{\xi} |\tilde{F} \tilde{M} G|}.$$ 

(4.20)

and put $N = \sup_{\xi} N(\xi)$.

It turns out that the $\xi_1$-component of the second-order correction along the line $(\xi_1, 0, 0)$ is equal to zero owing to the radial symmetry of the Gaussian function. For this reason, we show in figure 2 the $\xi_1$-component of the first-order approximation $X_1$ as a function of $\xi_2$ along the line $(0, \xi_2, 0)$. It has a peak at the origin whose height is approximately 0.12. Accordingly, we show in figure 3 the $\xi_1$-component of the second-order correction due to the nonlinearity $(1/v)\tilde{F}(\Delta^{-1} X_1 \cdot \nabla \Delta^{-1} X_1)$ as a function of $\xi_2$. It has double peaks near the origin, but their value is small and is about 0.0015. Noting $Re = M/v = 1/(1/2) = 2$, by (4.19) we can estimate the strength of nonlinearity in that cross-section as

$$N \approx \frac{0.0015}{2 \times 0.12} \approx 6 \times 10^{-3}.$$ 

Actually the maximum value of the nonlinear term in the above sense in $\mathbb{R}^3$ is 0.0022, not much different from the above value. Thus the strength of nonlinearity is at most $N \approx 9 \times 10^{-3}$ and we conclude

$$N = O(10^{-2}) \quad \text{for the three-dimensional Navier–Stokes equations.}$$

It should be noted that it is much smaller than the value of $N$ for the Burgers equations, whose solutions are known to remain regular all the time. Because the difference between the Navier–Stokes and Burgers equations is the presence or absence of the incompressibility condition, it is the incompressibility that makes the value of $N$ smaller for the Navier–Stokes equations. On a practical side this also means that even if we add the second-order correction to the first-order term at low Reynolds number, say $Re = 1$, the superposed solution is virtually indistinguishable from the first-order approximation. In the final period of decay, the Navier–Stokes flows are very close to (4.17), to within 1% at $Re = 1$, which may be regarded as the building blocks for representing the late-stage of evolution, that is, as the counterpart of the Burgers vortex in two dimensions.

### 5. Summary and outlook

We have studied self-similar solutions to the fluid dynamical equations with particular focus on the so-called source-type solutions. As an illustration of successive approximation schemes we have discussed the one-dimensional Burgers equation which is exactly soluble. In this case the velocity is the most convenient choice for its analysis. We have introduced a method of quantitatively assessing the strength of nonlinearity $N$ using the source-type solutions. Similar analyses have been carried out for higher-dimensional Burgers equations. For the Burgers equations, we find $N = O(0.1)$ in any dimensions.

We then move on to consider the two- and three-dimensional Navier–Stokes equations. In two dimensions, we review the known results done using the vorticity. In three dimensions, the
Figure 2. The first-order term, i.e. the $\xi_1$-component of $X_1 = PMG$ as a function of $\xi_2$.

Figure 3. The $\xi_1$-component of the second-order correction, i.e. the nonlinear term $\nu \frac{1}{\epsilon} \text{P}(\nabla^{-1} X_1 \cdot \nabla \nabla^{-1} X_1)$ depicted as in figure 2.

most convenient choice of the unknown is the vorticity curl. We have formulated the dynamically scaled equations using that variable and set up the successive approximation schemes. We have found that the second-order correction stemming from the nonlinear term gives rise to $N \approx 0.01$, an order of magnitude smaller than that for the Burgers equations. We are led to conclude that the incompressible condition makes $N$ smaller for the Navier–Stokes equations than for the Burgers equations.

The current approach relies on perturbative treatments. It may be challenging, but worthwhile to study the functional form of the solution by non-perturbative methods for further theoretical developments. It is also of interest to seek a fully non-linear solution by numerical methods. It is noted that this is at least one order of magnitude smaller than $N$ found for the Burgers equations whose solutions are known to remain regular all the time. As an application of the source-type solution, it is useful to characterize the late stage of statistical solutions of the Navier–Stokes equations [10].
Data accessibility. All scripts used in this study are openly accessible through https://github.com/StochasticBiology/boolean-efflux.git. The data are provided in electronic supplementary material [19]. This article has no additional data.

Authors’ contributions. K.O. contributed to the development of mathematical theory and R.V. contributed to that of numerics. K.O. wrote the manuscript and both have contributed its revision. They gave final approval for publication and agree to be held accountable for the work performed therein.

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Appendix A. Source solution to the Burgers equations in n dimensions

The source-type solution to the $n$-dimensional Burgers equations takes the following form:

\[ \frac{\partial^{n-1} U_1}{\partial \xi_2 \ldots \partial \xi_n} = K_{1 \ldots n} \exp \left( -\frac{a}{2v} |\xi|^2 \right) \frac{P_n(-R(\xi))}{(1-R(\xi))^n}, \]

where $P_n(s)$’s are polynomials to be constructed below:

\[ R(\xi) = \frac{K_{1 \ldots n}}{2v} \prod_{j=1}^{n} \int_0^{\xi_j} \exp \left( -\frac{a}{2v} \xi_j^2 \right) \, d\xi_j, \]

\[ K_{1 \ldots n} = 2v \left( \frac{2a}{\pi v} \right)^{n/2} \tanh \left( \frac{a}{2v} \right)^{n/2} M_{1 \ldots n} \quad (\text{for } M_{1 \ldots n}/v \ll 1) \]

and $M_{1 \ldots n} = \int \frac{\partial^{n-1} U_1}{\partial \xi_2 \ldots \partial \xi_n} \, d\xi$.

(a) Construction

Consider a function $s(x_1, x_2, \ldots, x_n)$ separable in $n$ variables $x_1, x_2, \ldots, x_n$

\[ s(x_1, x_2, \ldots, x_n) = F(x_1)F(x_2) \ldots F(x_n), \]

where $F(\cdot)$ is a smooth function and $n \in \mathbb{N}$. Define another function $\phi$ by

\[ \phi(x_1, x_2, \ldots, x_n) = \log(1 + s), \]

then we have

\[ \frac{\partial^n \phi}{\partial x_1 \partial x_2 \ldots \partial x_n} = \frac{\partial^n s}{\partial x_1 \partial x_2 \ldots \partial x_n} \frac{P_n(s)}{(1 + s)^n}, \]

where $P_n(s)$ is a sequence of polynomials in $s$ of degree not exceeding $n - 2$. In fact, it is given by

\[ P_n(s) = (1 + s)^n \left( \frac{d}{ds} s \right)^n \frac{\log(1 + s)}{s}, \quad (\text{for } n = 1, 2, \ldots), \]

or equivalently,

\[ P_n(s) = (1 + s)^n \sum_{k=0}^{\infty} (k + 1)^{n-1} (-s)^k. \]

The first four of them are $P_1(s) = 1$, $P_2(s) = 1$, $P_3(s) = 1 - s$, $P_4(s) = 1 - 4s + s^2$. 

Proof (due to Yuji Okitani)

\[ \phi_{x_1} = s_{x_1} \frac{1}{1 + s} . \]

Noting \( s_{x_2} s_{x_1} = s_{x_3} x_2 \), we have

\[ \phi_{x_1 x_2} = s_{x_1 x_2} \frac{1}{1 + s} - s_{x_1} \frac{1}{(1 + s)^2} s_{x_2} = s_{x_1 x_2} \left( \frac{1}{1 + s} - \frac{s}{(1 + s)^2} \right) = s_{x_1 x_2} \frac{1}{(1 + s)^2} , \]

while the penultimate expression of which may also be written

\[ = s_{x_1 x_2} \left( \frac{1}{1 + s} + s \frac{d}{ds} \frac{1}{1 + s} \right) . \]

Likewise we have

\[ \phi_{x_1 x_2 x_3} = s_{x_1 x_2 x_3} \left( \frac{1}{(1 + s)^2} + s \frac{d}{ds} \frac{1}{(1 + s)^2} \right) \]

Hence in general the recursion relationship is

\[ f_{n+1}(s) = f_n(s) + s \frac{d}{ds} f_n(s) = \frac{d}{ds} (s f_n(s)) , \]

where \( f_0 = \frac{p_n}{(1 + s)^n} . \)

Alternatively, by \( \phi(s) = \sum_{k=1}^{\infty} (-1)^{k-1} (s^k/k) \), we compute

\[ \phi_{x_1} = \sum_{k=1}^{\infty} (-1)^{k-1} s^{k-1} x_1 , \]

\[ \phi_{x_1 x_2} = \sum_{k=1}^{\infty} (-1)^{k-1} ((k - 1)s^{k-2} x_2 x_1 + s^{k-1} x_1 x_2) = \sum_{k=1}^{\infty} (-1)^{k-1} ks^{k-1} x_1 x_2 , \]

and

\[ \phi_{x_1 x_2 x_3} = \sum_{k=1}^{\infty} (-1)^{k-1} (k(k - 1)s^{k-2} x_3 x_1 x_2 + ks^{k-1} x_1 x_2 x_3) = s_{x_1 x_2 x_3} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 s^{k-1} . \]

In general we find

\[ \phi_{x_1 x_2 \ldots x_n} = s_{x_1 x_2 \ldots x_n} \sum_{k=1}^{\infty} (-1)^{k-1} k^{n-1} s^{k-1} , \]

and hence

\[ f_n(s) = \sum_{k=1}^{\infty} k^{n-1} (-s)^{k-1} . \]

Using the general expression, we can estimate the nonlinearity \( N \) in \( n \)-dimensional cases. Write \( P_n(-R) = 1 + c_n R + \ldots \) for small \( R \), we have

\[ \frac{P_n(-R)}{(1 - R)^n} \approx (1 + c_n R + \ldots)(1 + nR + \ldots) \approx 1 + (c_n + n)R + \ldots \]

and

\[ R(\xi) \approx \left( \frac{a}{2 \pi v} \right)^{n/2} M_{1 \ldots n} \left( \frac{\pi v}{2\pi} \right)^{n/2} \frac{Re}{2^{n+1}} , \]

so we find

\[ \frac{\partial^{n-1} U_1}{\partial \xi_2 \ldots \partial \xi_{n-1}} \approx \left( \frac{a}{2 \pi v} \right)^{n/2} M_{1 \ldots n} \left( 1 + \frac{c_n + n}{2^{n+1}} Re + \ldots \right) . \]

As \( c_n = 2^{n-1} - n \), we deduce \( N = (c_n + n)/2^{n+1} = 1/4 \) for all \( n \geq 1 \). Hence the estimate obtained in the iteration (1) in §2(c) holds valid in any dimensions.
Appendix B. Derivation of the vorticity curl equations

Recalling vector identities

$$\nabla(A \cdot B) = A \cdot \nabla B + B \cdot \nabla A + A \times \text{rot} B + B \times \text{rot} A$$

and

$$\text{rot}(A \times B) = -A \cdot \nabla B + B \cdot \nabla A + A \text{div} B - B \text{div} A$$

and adding them and solving for $B \cdot \nabla A$, we have

$$B \cdot \nabla A = \frac{1}{2} (\nabla (A \cdot B) + \text{rot}(A \times B) - A \times \text{rot} B - A \times \text{rot} A + B \text{div} A).$$

Taking $A = \omega$, $B = u$, we find

$$(u \cdot \nabla) \omega = \frac{1}{2} (\nabla (u \cdot \omega) + \nabla \times (\omega \times u) - u \times \chi).$$

We then compute

$$\frac{\partial \chi}{\partial t} = \nabla \times \{ \nabla \times (u \times \omega) \}$$

$$= \nabla \times (\omega \cdot \nabla) u - \nabla \times (u \cdot \nabla) \omega$$

$$= \nabla \times (\omega \cdot \nabla) u - \frac{1}{2} \nabla \times \{ \nabla (u \cdot \omega) + \nabla \times (\omega \times u) - u \times \chi \}$$

$$= \nabla \times (\omega \cdot \nabla) u + \frac{1}{2} \nabla \times \nabla \times (u \times \omega) + \frac{1}{2} \nabla \times (u \times \chi),$$

Identifying the underlined part as the right-hand side, we obtain

$$\frac{\partial \chi}{\partial t} = 2 \nabla \times (\omega \cdot \nabla) u + \nabla \times (u \times \chi),$$

that is,

$$\frac{D\chi}{Dt} = \chi \cdot \nabla u + 2 \nabla \times (\omega \cdot \nabla) u.$$ 

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