An Efficient Approximation Algorithm for the Colonel Blotto Game

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Abstract

In the storied Colonel Blotto game, two colonels allocate $a$ and $b$ troops, respectively, to $k$ distinct battlefields. A colonel wins a battle if they assign more troops to that particular battle, and each colonel seeks to maximize their total number of victories. Despite the problem’s formulation in 1921, the first polynomial-time algorithm to compute Nash equilibrium (NE) strategies for this game was discovered only quite recently. In 2016, Ahmadinejad et al. (2019) formulated a breakthrough algorithm to compute NE strategies for the Colonel Blotto game, receiving substantial media coverage (e.g., Insider, 2016, NSF, 2016, ScienceDaily, 2016).

In this work, we present the first known $\epsilon$-approximation algorithm to compute NE strategies in the two-player Colonel Blotto game in runtime $\tilde{O}(\epsilon^{-k^8 \max\{a, b\}^2})$ for arbitrary settings of these parameters. Moreover, this algorithm computes approximate coarse correlated equilibrium strategies in the multiplayer (continuous and discrete) Colonel Blotto game (when there are $\ell > 2$ colonels) with runtime $\tilde{O}(\ell \epsilon^{-k^8 n^2 + \ell^2 \epsilon^{-2} k^3 n(n + k))}$, where $n$ is the maximum troop count. Before this work, no polynomial-time algorithm was known to compute exact or approximate equilibrium (in any sense) strategies for multiplayer Colonel Blotto with arbitrary parameters.

Our algorithm computes these approximate equilibria by a novel (to the author’s knowledge) sampling technique with which we implicitly perform multiplicative weights update over the exponentially many strategies available to each player.

1 Introduction

Zero-sum games see a variety of applications from politics (Behnezhad et al., 2019; Kovenock & Roberson, 2012) to machine learning (Andoni & Beaglehole, 2021; Freund & Schapire, 1996). Although these games can be exponentially large in description, in some cases they have underlying structure that permits efficient algorithms for computing their equilibria. A well-known case study for such games is the Colonel Blotto game, for which computing equilibria in general settings has been notoriously difficult.

The Colonel Blotto game was originally described by Borel in 1921 and formalizes how warring colonels should distribute soldiers over different battlefields (Borel, 1953). In the most general version of this game, two colonels have $a$ and $b$ armies that they must assign to $k$ different battlefields, each with a non-negative integer weight. A colonel wins a battle if they assign more armies to that battle than their opponent. Each colonel seeks to maximize the number of weighted battles that they win in a single assignment. In today’s times, solving this game finds applications in large swath of market competitions including advertising and auctions (Roberson, 2006), budget allocation (Kvasov, 2007), elections (Laslier & Picard, 2002), and even ecological modeling (Golman & Page, 2009).

Despite its breadth of applications and its simplicity to state, the first polynomial time algorithm to compute optimal strategies for this game had not been developed until very recently (Ahmadinejad et al., 2019). This breakthrough result received a large media response (Insider, 2016, Mewright, 2016). While polynomial-time, to the best of the author’s knowledge, the algorithm has computational complexity $O(n^{13}k^{14})$, where $n$ is the number of troops/armies and $k$ is the number of battles, which is still too large to be practical (Behnezhad et al., 2017). This is the only known algorithm to provably compute exact optimal strategies for the Colonel Blotto game with arbitrary parameters in polynomial time.

1To the best of our knowledge, the algorithm from Ahmadinejad et al. (2019) has computational complexity $O(k^{14} \max\{a, b\}^{13})$
Our Contribution

In this work, we present the first known $\epsilon$-approximation algorithm for computing optimal strategies for the Colonel Blotto game for any $\epsilon > 0$ with computational complexity $O(n^2 k^3 \epsilon^{-4})$, improving substantially over other known algorithms. We note that in contrast to other approximation algorithms, our work provides an algorithm for computing a Nash equilibrium (NE) pair of strategies for all game parameters and to any desired approximation. In addition, this algorithm is the first known algorithm to compute coarse correlated equilibrium (CCE) strategies for the multiplayer Colonel Blotto game in runtime $\tilde{O}(n^2 k^8 \epsilon^{-4} + \ell^2 nk^3(n + k)\epsilon^{-2})$. Note it is PPAD-complete to compute Nash equilibrium strategies in multiplayer games [Daskalakis et al., 2006], and so a CCE is a standard intermediate goal [Anagnostides et al., 2021; Papadimitriou & Roughgarden, 2008].

Our algorithm computes these optima by a novel sampling technique (to the author’s knowledge) that we expect will have applications for computing equilibria in other large games. Our algorithm simulates repeated play of the Colonel Blotto game where each player performs the multiplicative weights update (MWU) rule. It is well known that this will converge to an NE in two-player zero-sum games and CCE in general multiplayer games. At first glance, performing MWU may appear intractable as this algorithm requires tracking a weight for each one of the exponentially many available strategies to each player. (Note that there are $\binom{n+k-1}{k-1}$ strategies available to a player with $n$ troops and $k$ battlefields). Nonetheless, we are able to implicitly perform MWU by sampling from the distribution produced by MWU without ever performing the updates (or even writing the distribution) explicitly.

2 Related Work

The Colonel Blotto game is fairly well studied - we will mention some of the most notable equilibrium computation results for this game here. As mentioned previously, the first known algorithm to compute exact NE strategies for Colonel Blotto was introduced in [Ahmadinejad et al., 2019]. Further, there exists a variant of this algorithm with efficient performance in practice, but it has exponential worst-case computational complexity due to its application of the simplex method [Behnezhad et al., 2017].

There is also a breadth of work that constructs strategies for approximate and exact equilibrium under constrained parameter settings. Since the initial publication of this work, the authors in [Perchet et al., 2022] present an algorithm to compute approximate NE in the two-player continuous Colonel Blotto game. In [Vu et al., 2018], the authors construct approximate equilibria when the number of battles is sufficiently large. They also give an algorithm to compute the best-response strategy to a given distribution over soldiers in each battlefield using dynamic programming. In [Boix-Adserà et al., 2020; Thomas, 2017], the authors describe equilibria in the symmetric case where the number of soldiers is the same for both players. In [Roberson, 2006], the author constructs equilibrium strategies in the case that the rewards for each battlefield are the same. In [Vu & Loiseau, 2021], the authors construct equilibria under particular conditions for an extension of the Colonel Blotto game that accounts for pre-allocations and resource effectiveness.

3 Preliminaries

Game Theory

The multiplayer Colonel Blotto game is an example of a larger class of objects called multiplayer games.

Definition 3.1. A (simultaneous) multiplayer game is when $\ell$ actors (players) are each able to play an action, without knowledge of the other players action, where each action incurs a reward that is a function of the actions of all players. This game is called iterated if the game is repeated in sequential rounds. This game is called zero-sum if the rewards of all players in a single round of the game sum to 0, regardless of the strategies played by any individual player.

Definition 3.2 (Strategy). Suppose a player in a (multiplayer) game has $N$ actions available to them. One such action is called a strategy, and is represented by a standard basis vector $e_i$ for $i \in [N]$, using an arbitrary fixed ordering on the strategies.
In a multiplayer game, all players simultaneously sample strategies $s_1, \ldots, s_\ell$ for players $1, \ldots, \ell$, respectively. Here $x_i \sim \Delta^N_i$, where $N_i$ is the number of strategies available to player $i$ and $\Delta^d$ is the probability simplex in $d$ dimensions. These distributions $x_i$ need not be independent from one another. We consider the expected reward of the $i$-th player. The distributions $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_\ell$ define a joint distribution over the possible strategies of the $i$-th player’s opponents. We write this distribution as a vector $x_{-i}$ with each coordinate corresponding to the probability of a possible combination of strategies of all other players $j \neq i, j \in [\ell]$. Then, the rewards experienced by player $i$ is computed from a matrix $R_i$ in the following way.

**Definition 3.3 (Reward).** Suppose we are in a multiplayer game with $\ell$ players. The rewards for these players are determined by a set of matrices $R_1, \ldots, R_\ell$ called reward matrices. For a player $i$ and fixed strategies $s_1, \ldots, s_\ell$, the reward for that player is computed using these matrices by:

$$\text{Reward of player } i = s_i^T R_i s_{-i}$$

Suppose players sample strategies $s_1, \ldots, s_\ell$ from a distribution $x \equiv (x_1, \ldots, x_\ell)$ (not necessarily independently). The expected reward for player $i$ is computed by:

$$\text{Expected Reward of player } i = \mathbb{E}_{s \sim x} [s_i^T R_i s_{-i}]$$

When the distributions $x_1, \ldots, x_\ell$ are independent (defining a product distribution), this expected reward is exactly:

$$\text{Expected Reward of player } i = x_i^T R_i x_{-i}$$

**Game Formulation**

We formally describe the (discrete) multiplayer Colonel Blotto game (considered for two players in [Ahmadinejad et al., 2019] and the continuous variant of which was introduced in [Boix-Adserà et al., 2020]).

In this game, a group of $\ell$ players labeled $i \in [\ell]$ simultaneously assign a non-negative integer number of troops to a collection of $k$ distinct battlefields. Each battlefield $i$ has associated with it a weight $w(i) \in \mathbb{Z}^+$. We write the weights as a $k$-dimensional vector $w$. Each player $i$ has available to them $a(i) \in \mathbb{Z}^+$ troops, indexing into a vector $a$ with non-negative integer entries. A strategy in this game is a partition of the fungible $a(i)$ troops to the $k$ total battlefields. In this case, troops are allocated in integer quantities to the $k$ battlefields. Where $x_i(j)$ is the allocation of player $i$ to battle $j$, $\sum_{j=1}^k x_i(j) = a(i)$ for all $i$.

A player wins a certain battle if they assign strictly more troops to that battle than all other players. If a player wins a battle, they are awarded the weight of that battle. If the player loses a battle, they earn 0 reward. We make the standard assumption for the Colonel Blotto game that ties are broken in lexicographic order [Roberson, 2006] to ensure the game is zero-sum. The results for multiplayer Blotto hold for arbitrary tie-breaking mechanisms, provided the maximum loss is still bounded by $kw_{\text{max}}$. Each player seeks to maximize the sum of all rewards they receive in a single round of this game.

Formally, suppose there are $k$ battles, the players play strategies $x_1, \ldots, x_\ell$, and the weights are $w_1, \ldots, w_k$. Further, let $x_i(j)$ be number of troops player $i$ allocates to battle $j$ for $i \in [\ell], j \in [k]$. The total reward for a player $i$ can be computed by:

$$\text{Reward}(i) = \sum_{j=1}^k w_j \left( \mathbb{1} \{ x_i(j) \geq x_{i'}(j), \forall i' \neq i \} \cdot \mathbb{1} \{ x_i(j) > x_{i'}(j), \forall i' < i \} \right) \quad (1)$$

We term the multiplayer Colonel Blotto game with $\ell = 2$ players as the two-player Colonel Blotto game, which is the standard version of the Colonel Blotto game.

**Notation**

We write $a^v$ for some vector $v$ and value $a$ to mean $(a^{v_1}, \ldots, a^{v_d})$. We denote the element-wise product of two vectors $x, y$ to be $x \odot y$. For a finite set $S$, we denote its cardinality by $\#S$. For a vector $v$, we write $v(i) \equiv v_j$ to indicate the $i$-th entry in the vector. We denote the transpose of a vector $x$ by $x'$.

We also define the following helper functions that will be useful in simplifying notation.

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Definition 3.4 (Helper functions). Let \( f^s_i(m) \) be the total number of battles lost by player \( s \) using \( m \) troops in battle \( i \) over all rounds up to and including round \( t - 1 \) of the iterated multiplayer Colonel Blotto game. This is the number of rounds where an opponent used \( m \) troops in battle \( i \) or an opponent in earlier lexicographic order used \( m \) troops in that battle.

4 Main Algorithm

We now describe an algorithm to compute optimal strategies in the multiplayer Colonel Blotto game, and state its main run-time and correctness guarantees.

We must be specific about the sense in which the strategies we compute are optimal. The optimal strategies we seek in the two-player Blotto game compose a Nash Equilibrium.

Definition 4.1. Consider an \( \ell \)-player game where each player has \( N_i \) possible strategies. Suppose that each player \( i \) has reward matrix \( R_i \). (A multiplayer game is called zero-sum when for all strategies \( s_1, ..., s_\ell \), \( \sum_i s_i'R_is_{-i} = 0 \). Then, a product distribution \( x_1 \times \cdots \times x_\ell \) over strategies with \( x_i \in \Delta^{N_i} \) is considered an \( \epsilon \)-approximate Nash equilibrium if and only if the following condition holds:

\[
x_i'R_i x_{-i} \geq \max_{x \in \Delta^N} x'R_i x_{-i} - \epsilon, \text{ for all } i \in [\ell]
\]

In words, approximate Nash equilibrium strategies are a tuple of distributions over strategies for which no player can improve their expected outcome (by greater than \( \epsilon \)) by unilaterally deviating from the equilibrium distribution. Note that the NE strategy models the case when all players sample strategies independently (without coordination).

This leads us to our first main algorithmic result - that we can efficiently approximate NE strategies for two-player Colonel Blotto.

Theorem 4.2 (NE for Two Players). Let \( n = \max\{a, b\} \), where \( a,b \) are the soldier counts for the two player Colonel Blotto game. Let \( w \) be the weights for the \( k \) battles, with maximum entry \( w_{\max} \). There exists an algorithm to compute an \( O(\epsilon) \)-approximate Nash equilibrium for the two-player Colonel Blotto Game in \( O\left(k^8\epsilon^{-4}w_{\max}^5n^2\ln^2(n+k)\right) \) time, with high (constant) probability.

It is well known that in multiplayer games identifying Nash equilibria is PPAD-complete. This is a complexity class that is thought to be strictly harder than P. Therefore, a somewhat standard goal for identifying optimal strategies in multiplayer games is to seek a coarse correlated equilibrium.

Definition 4.3. Suppose we are in an \( \ell \)-player game with rewards \( R_i(s_1, ..., s_\ell) \) as a function of strategies \( s_1, ..., s_\ell \) from players 1, ..., \( \ell \), respectively. For a set of strategies \( s \equiv \{s_1, ..., s_\ell\} \), we refer to the vector of strategies for players other than player \( i \) as \( s_{-i} \). In other words, \( s = (s_i, s_{-i}) \) up to a permutation. An \( \epsilon \)-approximate coarse correlated equilibrium is a joint distribution \( \sigma \) over strategies \( s \) for all players that satisfies:

\[
\forall i \in [\ell], \text{ and strategies } s'_i, \quad E_{s \sim \sigma} R_i(s) \geq E_{s_{-i} \sim \sigma} R_i(s'_i, s_{-i}) - \epsilon
\]

Correlated equilibria, introduced by Aumann (Aumann 1974), are a relaxation of NE. In a CCE, the strategies of different players are allowed to be sampled in coordination with one another (as opposed to sampled independently as in NE). In contrast to NE, although a player \( i \) cannot benefit from switching to any single action \( s'_i \) before the joint strategy is sampled, once a strategy \( s_i \) is sampled from a CCE distribution (becoming known to each player), a player may improve their outcome by deviating (using the fact that her strategy is correlated with other players’). Thus, CCE apply to situations where a player must commit their strategy up front and are unable to deviate after sampling.

This leads us to our second main algorithmic result - that we can efficiently approximate CCE strategies for multiplayer Colonel Blotto.

Theorem 4.4 (CCE for Many Players). Suppose we are in the \( \ell \)-player Colonel Blotto game. Let \( a \) be a vector of soldier counts for \( \ell \) players, with \( n \triangleq \max_i a(i) \) Let \( w \) be the weights for the \( k \) battles, and
$w_{\text{max}} \triangleq \max_i w(i)$. There exists an algorithm to compute an $O(\epsilon)$-approximate coarse correlated equilibrium for the multiplayer Colonel Blotto Game in $O(k^8 \epsilon^{-4} w_{\text{max}}^3 n^2 (n+k) + \ell^4 k^8 \epsilon^{-2} w_{\text{max}} n (k+n) \ln(n+k))$ time with high (constant) probability.

Core to our algorithm is: (1) If all players in an iterated game sample strategies according to the multiplicative weights update (MWU) algorithm, then with high constant probability their time-averaged strategies converge to NE in two-player zero-sum games, and the sequence of strategies played converges to a CCE in general multiplayer games, (2) sampling from MWU can be done efficiently for the Colonel Blotto game with any number of players.

**Definition 4.5.** Suppose a player in some game has available to them $N$ strategies. Suppose we impose an arbitrary ordering on the strategies so each one gets mapped to an index $i \in [N]$. Fix some parameter $\beta \in (0,1)$. The **multiplicative weights update** (MWU) method is a method for choosing a distribution over these $N$ possible strategies so as to maximize one’s reward on a sequence of rewards. In particular, suppose a player enjoys a sequence of rewards $R_t(i)$ for $t = 1, ..., T$ where $i \in [N]$ is index for a strategy. For $M > 0$, suppose $R_t(i) \leq M$ for all $t$ and $i$. For the MWU update rule, the player initializes a set of weights to $w_{i,1} = \frac{1}{M}$ for all $i \in [N]$ at round 1. In subsequent rounds $t > 1$, the player updates these weights according to $w_{i,t+1} = w_{i,t} \odot \beta^{M-R_t(i)}$. Ultimately, the probability of sampling the strategy with index $i$ at round $t$ is $\frac{w_{i,t}}{\sum_{j \in [N]} w_{j,t}}$.

**Theorem 4.6 (Freund & Schapire (1999)).** Suppose in a two-player zero sum game with a maximum of $N$ strategies per player and where entries in the reward matrices are bounded by a value $M$, both players sample strategies $x_1, ..., x_T$ and $y_1, ..., y_T$, respectively, from MWU in rounds 1, ..., $T$. Then, the pair of strategies $(\frac{1}{T} \sum_{t=1}^{T} x_t, \frac{1}{T} \sum_{t=1}^{T} y_t)$ constitute an $O(M \sqrt{\ln N}/\sqrt{T})$-approximate Nash Equilibrium for the game with high (constant) probability.

For multiplayer Colonel Blotto, we leverage the following fact.

**Fact 4.7.** If all players in a multiplayer game sample from multiplicative weights update with $N$ strategies, rewards bounded by $M$, and $T$ rounds, the distribution over strategies defined by uniformly sampling a time $t \in [T]$ and playing strategies $x_t^{(1)}, ..., x_t^{(\ell)}$ is an $O(M \sqrt{\ln N}/\sqrt{T})$-approximate coarse correlated equilibrium with high constant probability.

Thus, to compute approximate NE and CCE we simply execute $T$ rounds of multiplicative weights update for each player as in Algorithm 1 (for $T$ to be chosen later). If there are two players, we simply need to return the time-averaged strategies. If there are more than two players, we must return all of the strategies played in the game, as these are needed to define the CCE.

### 4.1 The Sampling Procedure and its Correctness

Critically, it is computationally intractable to explicitly write down the distribution over all strategies in these Colonel Blotto games, as there are exponentially many strategies available to each player. In particular, there are $\binom{s+k-1}{k-1}$ many ways for a colonel to allocate $a$ troops to $k$ battlefields. To circumvent this issue, in each round, the players never perform the distribution update directly, but rather, samples a strategy from the implicitly updated distribution. We state this result in the following main theorem.

**Theorem 4.8 (Sampling).** Suppose a player is in the iterated multiplayer Colonel Blotto game. Then, in round $T$ of the game, a player can sample a strategy from the MWU distribution in $O(w_{\text{max}} k^2 T n^2)$ time with Algorithm 2 (described at a high level in Algorithm 4).

We describe how given the previous strategies of the opponent, a player can efficiently sample a strategy from the correct distribution - that is, one updated by multiplicative weights update. Our sampling approach consists of two sampling phases. In the first phase, we sample the number of battles that the strategy we choose will lose over all battles and all rounds of play. In the second phase, we sample a strategy uniformly at random from the strategies that lose the number of battles we sampled in the first phase. This is written at a high level here in Algorithm 2 and with implementations of each step in Algorithm 3.
Algorithm 1: Compute Equilibrium Strategies

Function IterateBlotto(battles: k, soldiers: a, weights: w):
  for s = 1, ..., ℓ do
    Sample \( x_s^{(s)} \) uniformly at random
  end

for t = 2, ..., T do
  for each player \( s \) do
    Recompute helper functions \( f_{j,i}^{t,s}(m) \) for all \( j \in [k] \) and \( m \in \{0, ..., n\} \)
    Sample from MWU Distribution
    \( x_t^{(s)} \leftarrow \text{SampleMWU}(k, a, s, \{f_{j,i}^{t,s}\}_{j=1}^k) \)
  end
end

{ Return CCE if \( ℓ > 2 \), NE if \( ℓ = 2 \) }
if \( ℓ > 2 \) then
  return \( \{ (\frac{1}{T} \sum_{t=1}^T x_t^{(1)}, ..., \frac{1}{T} \sum_{t=1}^T x_t^{(ℓ)}) \}_{t=1}^T \)
else
  return \( \left( \frac{1}{\ell} \sum_{i=1}^\ell x_t^{(i)} \right)_{i=1}^\ell \)
end

We fix \( t \) and let \( j \in \{0, ..., w_{\text{max}}kt\} \), where \( w_{\text{max}}kt \) is the total number of battles that can be lost of \( t \) rounds and \( k \) battlefields. The probabilities in this section are over the distributions induced by performing multiplicative weights update. We define the distribution:

\[
P_j^t \triangleq \Pr[\text{choosing a strategy that loses } j \text{ battles in rounds } 1, ..., t]
\]

Then, the algorithm to sample from the distribution produced by MWU is the following:

Algorithm 2: Sample from MWU Distribution (High-level)

Function SampleMWU(round \( t \), maximum weight \( w_{\text{max}} \), round \( t \), maximum weight \( w_{\text{max}} \)):
  Compute the distribution \( P_0^t, ..., P_{w_{\text{max}}kt}^t \) over total losses
  Sample \( j \in \{0, ..., w_{\text{max}}kt\} \) from \( P_0^t, ..., P_{w_{\text{max}}kt}^t \)
  Sample \( x \) uniformly from the set of strategies that lose \( j \) battles over all \( t' \in [t] \) and \( k' \in [k] \).
  return \( x \)
end

We now prove the correctness of this sampling strategy.

Lemma 4.9 (Correctness). Sampling strategies according to Algorithm 2 at round \( T \) is equivalent to sampling from the distribution generated by performing MWU up to that round.

Proof of Lemma 4.9 To compute the probability of selecting a strategy \( x \), we simply need to know how many battles that strategy loses over all rounds and all battles played so far. To derive this, we focus on one player for now and let \( n \) be the total number of available troops. Consider a particular strategy \( x \) with losses \( \ell_t(x) \) in rounds \( i \) up to and including \( t \). The possible losses that can be experienced by the player on all possible strategies are among the set \( \{0, ..., k\} \) corresponding to the total number of battles that can be lost in each round.

We ask, assuming we’ve updated the distributions over strategies for a fixed player using MWU, what is the probability in the next round that the this player selects a certain strategy \( x \)? Where \( S \) is the set of all
possible strategies, this probability is exactly:

\[
Pr[\text{selecting } x \text{ in round } t + 1] = \frac{\beta \sum_{i=1}^{t} \ell_i(x)}{\sum_{x' \in S} \beta \sum_{i=1}^{t} \ell_i(x')}
\]  

(3)

Thus, the probability of a certain strategy can be computed from the number of battles that the strategy loses over all rounds. A simple corollary is that strategies that lose the same number of battles over all rounds are chosen with equal probability. Therefore, conditioned on a certain number of total losses \( \hat{j} \), strategies are sampled uniformly at random.

We now describe how to efficiently perform the sampling steps in Algorithm 2. In particular, because the probability of selecting a strategy is exactly determined by the number of losses incurred by that strategy, we simply need to count the number of strategies that lose each of the possible number of losses.

**Lemma 4.10.** The distribution over total losses \( P_t^0, \ldots, P_t^{\hat{w}_{\max}kt} \) in round \( t \) of the iterated game can be computed in \( O(\hat{w}_{\max}k^2tn^2) \) time.

*Proof of Lemma 4.10.* Suppose for the sake of clarity, we are in the uniform Colonel Blotto game, in which the weight vector \( w \) is the vector of 1’s (the analysis is identical for more general weights). To sample the number of losses our strategy suffers over all battles in all rounds, we just need to count the number of strategies that suffer each of the possible number of losses (from 0 up to and including \( \hat{k}t \) battles). We can compute these counts using a dynamic program (DP). To see this, fix a round \( t' \) and the opponent strategies for that round.

As a warm-up, we ask: (for the first player lexicographically) how many strategies win \( 0 \leq j \leq k \) battles in a single round of play? Let \( m_i \) be the maximum number of troops used by all other players in battle \( i \), where \( i \in \{1, \ldots, k\} \). First, we note that:

\[
\#(\text{strategies that lose } j \text{ total battles in battles } \{1, \ldots, k\}) = \#(\text{allocations of } n \text{ troops in battles } \{1, \ldots, k\} \text{ that lose } j \text{ battles})
\]  

(4)

Further,

\[
\#(\text{allocations of } n \text{ troops in battles } \{i, \ldots, k\} \text{ that lose } j \text{ battles})
\]  

(6)

\[
= \sum_{m=0}^{n} \#(\text{allocations of } n - m \text{ troops in battles } \{i + 1, \ldots, k\} \text{ that lose } j - 1\{m < m_i\} \text{ battles})
\]  

(7)

Here the number of losses in the recursion is \( j \) if \( m \) troops win the battle, and \( j - 1 \) if \( m \) troops lose the battle.

To compute the weight of a strategy after \( t \) rounds of multiplicative weights update, the question we are really interested in is: how many strategies lose \( j \in [0, \ldots, tk] \) battles over all rounds \( 1, \ldots, t \)? We use the helper functions of definition 3.4 to simplify notation. Then, for any player \( s \),

\[
\#(\text{allocations of } n \text{ troops in battles } \{i, \ldots, k\} \text{ that lose } j \text{ battles in rounds } 1, \ldots, t) = \sum_{m=0}^{n} \#(\text{allocations of } n - m \text{ troops in battles } \{i + 1, \ldots, k\} \text{ that lose } j - f_s^{t,i}(m) \text{ battles in rounds } 1, \ldots, t)
\]  

(8)

Thus, the counts above can be computed with a polynomial sized DP, and each entry in the DP table can be computed in polynomial time!

We remedy the assumption that the battles are equally weighted. Suppose instead that the battles are all weighted by a vector of positive integers \( (w_1, \ldots, w_k) \) each bounded by an integer \( \hat{w}_{\max} \). Then, in the preceding analysis, we replace ”losses” with ”weighted losses”. Specifically, the loss suffered for losing a battle is in this case the weight of that battle instead of just 1. This increases the number of entries in the
DP matrix for each round by exactly a factor of $w_{\text{max}}$, but we can still compute the DP table just the same. Then, the DP table of counts (written $M$) is a $k$ by $ktw_{\text{max}}$ by $n$ table with entries:

$$M[i, j, m] = \# \text{allocations of } m \text{ troops in battles } \{i, \ldots, k\} \text{ that lose } j \text{ battles in rounds } 1, \ldots, t$$ (10)

In this case, $M[1, j, n]$ is exactly the number of strategies that lose $j$ battles over all rounds and all battles with $n$ total troops. The algorithm to compute this DP is formally described in Algorithm 3.

The number of entries in the corresponding DP table is at most $O(w_{\text{max}}kt \cdot k \cdot n)$ and each entry takes $O(n)$ time to compute. Further, the base cases are to compute the number of strategies that lose $r \leq j$ battles with $s \leq n$ troops in the last battle $k$. This can be computed directly in $O(1)$ time by storing a running sum over the $T$ rounds.

In particular, we assume all strategies use all available troops (which will always occur as using all troops can only improve a player’s outcome). Therefore, there are either 1 or 0 strategies that lose $r$ such battles, depending on whether using all the remaining troops wins more or fewer than $r$ battles. To compute this base case, at every round of the game, we compute the threshold number of troops needed by the current player to win at least $r$ battles over the $t$ rounds of the $k$-battle. Then, the number of strategies that lose $r$ battles is 0 if and only if the number of battles won on the $k$-th battle is greater than $r$. We then increment a running sum for all $m$ tallying $\sum_{t'=1}^{\ell} \mathbb{1} \left[ m \geq n_{t', k} \right] \cdot \mathbb{1} \left[ m > n_{t', k} \right] \cdot w_k$ where $n_{t', k}$ is the number of troops used by opponent $s'$ in round $t'$ of battle $k$. In total recomputing this running sum requires $O(\ell n)$ computation per round.

After using multiplicative weights update to compute strategies for $t$ rounds of the multiplayer Blotto game, the probability of losing $j$ battles over all rounds and over all battles with $n$ total troops, can be written as:

$$P^t_j \triangleq \Pr[\text{lose } j \text{ battles in rounds } 1, \ldots, t] = \frac{\beta^j \cdot \# \text{strats that lose } j \text{ battles in rounds } 1, \ldots, t}{\sum_{j' = 0}^{ktw_{\text{max}}} \beta^{j'} \cdot \# \text{strats that lose } j' \text{ battles in rounds } 1, \ldots, t}$$ (11)

$$= \frac{\beta^j M[1, j, n]}{\sum_{j' = 0}^{ktw_{\text{max}}} \beta^{j'} M[1, j', n]}$$ (12)

Therefore, the entries of $M$ are sufficient to compute the entire PMF of the distribution $P^t = (P^t_0, \ldots, P^t_{ktw_{\text{max}}})$. \qed
Algorithm 3: Solve the DP Table for Sampling MWU

\begin{algorithm}
\textbf{Function} SolveStrategyDP(player $s$, helper functions $\{f_j\}_{j=1}^k$, soldier counts $a$, weights $w$):
\begin{algorithmic}
  \State $w_{\text{max}} \leftarrow \max_i w(i)$
  \State $n \leftarrow a(s)$
  \State initialize empty table $M \in \mathbb{Z}^{k \times k \times tw_{\text{max}} \times n}$
  \State \{ Compute Base Cases \}
  \For{$j = 0, \ldots, tw_{\text{max}}$}
  \For{$m = 0, \ldots, n$}
  \State $n_{t',k}^{s'} \leftarrow \# \text{ of troops used by opponent } s' \text{ in round } t' \text{ of battle } k$
  \State $M[k,j,m] \leftarrow 1 \left[ \sum_{t'=1}^{t'} \mathbb{1}[m \geq n_{t',k}^{s'} \text{, } \forall s' \in [f]] \cdot \mathbb{1}[m > n_{t',k}^{s'} \text{, } \forall s' < s] \cdot w_k = j \right]$
  \EndFor
  \EndFor
  \State \{ Fill in DP Table \}
  \For{$i = (k-1), \ldots, 1$}
  \For{$j = 0, \ldots, tw_{\text{max}}$}
  \For{$m = 0, \ldots, n$}
  \State \{ Count strats. that lose $j$ battles in battles $\{k-i+1, \ldots, k\}$ with $m$ troops \}
  \State $M[i,j,m] \leftarrow \sum_{m'=0}^{m} M[i+1,j-f_i(m'),m-m']$
  \EndFor
  \EndFor
  \EndFor
  \State return $M$
\end{algorithmic}
\end{algorithm}

The formal sampling approach for implicitly updated strategies (written in Algorithm 5) requires (1) a pre-processing step: fill in the DP table, $M$, necessary to compute the distributions $P_j^t$, and then (2) the two-phase sampling described informally in Algorithm 2.

We also note that computing the helper functions does not affect the asymptotic efficiency of the algorithm (proved in Lemma 5.1).

We have already described how to sample the number of total losses a strategy incurs. To sample uniformly at random from strategies that won this total number of battles, we recursively sample strategies based on the number of soldiers used in each battle (Algorithm 4). For this, we will again need the counts computed in the DP table, $M$.

**Lemma 4.11.** Fix a number of losses $\hat{j}$ over all rounds $t$ and battles $k$. A player can sample uniformly among strategies that lose $\hat{j}$ battles in only $O(kn)$ time.

**Proof of Lemma 4.11.** We formally write our procedure to sample strategies uniformly among those that lose $\hat{j}$ battles in Algorithm 4 and provide a technical description here.

Beginning with the first battle ($i = 1$), for all possible troop counts $m \in [n]$, count the number of strategies that use $m$ troops in battle $i$ and lose $\hat{j}$ total battles. Then, the probability our algorithm samples $m$ troops for battle $i$ is the fraction of strategies that use $m$ troops in battle $i$ and lose $\hat{j}$ total battles. Once we have sampled the number of troops used in a battle $i$, $\hat{m}_i$, according to this distribution, we then inductively sample the number of strategies used in battle $i + 1$ subtracting the number of available troops by the number of troops used in battle $i$, and subtracting the number of total losses by the number of battles lost using $\hat{m}_i$ strategies in battle $i$.

This procedure is equivalent to sampling strategies uniformly at random because of Bayes’ rule. In particular, conditioned on a number of total losses $\hat{j}$, for a choice of troop assignments $r_1, \ldots, r_k$ to the $k$ battles:

$$
\Pr[r_1, \ldots, r_k | \text{total losses}] = \prod_{i=1}^k \Pr[r_i | r_{<i}, \hat{j} \text{ total losses}] \quad (13)
$$
Suppose, for a battle \( i \), troops \( r_{<i} \) to battles \( r_1, \ldots, r_{i-1} \) lose \( j - j_i \) total battles over all rounds and use \( n - m_i \) troops. Crucially then, the total number of strategies with allocations \( r_{<i} \) and \( j \) total losses over all battles and rounds, where \( M \) is the DP table we computed in the first sampling step, is exactly \( M[i, j, m_i] \). Suppose further \( r_i \) troops are then allocated to battle \( i \). Then, the total number of strategies that use \( r_{<i}, r_i \) in battles \( 1, \ldots, i \) is exactly \( M[i + 1, j_i - f_i^i(r_i), m_i - r_i] \), where \( f^i_i(r_i) \) is the number of battles lost with \( r_i \) troops in battle \( i \) over all rounds (Definition 3.4). Therefore,

\[
\Pr[r_i | r_{<i}, j \text{ total losses}] = \frac{\Pr[r_i | r_{<i}, j \text{ total losses}]}{\Pr[r_{<i}, j \text{ total losses}]} \\
= \frac{M[i + 1, j_i - f_i^i(r_i), m_i - r_i]}{M[i, j, m_i]} \cdot \frac{M[1, j, n]}{M[1, j, n]}
\]

Finally, because the DP is pre-computed from the previous sampling step, the runtime is the number of battles times the number of troops = \( O(kn) \), concluding the proof.

\[ \square \]

**Algorithm 4: Two-phase Sampling from the DP**

```python
Function SampleDP(table of strategy counts M, round t, max weight \( w_{\text{max}} \), helper functions \( \{f_j\}_{j=1}^k \)):
    { Compute the PMF \( P^t \) from \( M \) }
    for \( j = 0, \ldots, kw_{\text{max}} \) do
        \( P^t(j) \leftarrow \frac{\beta^j M[1, j, n]}{\sum_{j' = 0}^{kw_{\text{max}}} \beta^j M[1, j', n]} \)
    end
    Sample a number of losses \( \hat{j} \sim P^t \)
    { Sample a strategy \( x \) uniformly from strategies that experience \( \hat{j} \) losses }
    \( m_1 \leftarrow n \)
    \( j_1 \leftarrow \hat{j} \)
    for \( i = 1, \ldots, k \) do
        { Compute the PMF of \( P^i \) from \( M \) }
        for \( r' = 0, \ldots, m_i \) do
            \( P^i(r') \leftarrow \frac{M[i + 1, j_i - f_i^i(r'), m_i - r']}{M[i, j, m_i]} \)
        end
        Sample troop count \( r_i \sim P^i \)
        \( m_{i+1} \leftarrow m_i - r_i \)
        \( j_{i+1} \leftarrow j_i - f_i^i(r_i) \)
    end
    return \( x \equiv [r_1, \ldots, r_k] \) (This a one-hot vector)
```

**Algorithm 5: Sample from the MWU Distribution**

```python
Function SampleMWU(battles: \( k \), soldiers: \( a \), weights: \( w \), player \( s \) \( \in \) \( \ell \), helper functions \( \{f_j^{l,s}\}_{j=1}^k \)):
    \( M_i^{(s)} \leftarrow \text{SolveStrategyDP}(s, \{f_j^{l,s}\}_{j=1}^k, a, w) \)
    \( x_i^{(s)} \leftarrow \text{SampleDP}(M_i^{(s)}, i, \arg \max_i w(i), \{f_j^{l,s}\}_{j=1}^k) \)
    return \( x_i^{(s)} \)
```

The proof of Theorem 4.3 follows from lemmas 4.9, 4.10, and 4.11.
5 Runtime Analysis

To complete the runtime analysis, we briefly address the complexity of computing the helper functions that appear in Algorithm 1 and are defined in Definition 3.4.

Lemma 5.1. The helper functions \( \{f_i^{s,t+1}\}_{i \in [k], s \in [\ell]} \) can be computed in time \( O(\ell^2 k) \) given \( \{f_i^{s,t}\}_{i \in [k], s \in [\ell]} \).

Proof of Lemma 5.1. This is performed by storing the values of \( f_i^{s,t} \) from previous rounds and computing them using the helper functions from the previous round. Given \( f_i^{s,t}(m) \), we can compute \( f_i^{s,t+1}(m) \) by adding \( w_i \) to \( f_i^{s,t}(m) \) if player \( s \) lost that round with \( m \) troops and 0 otherwise. This can be computed in \( \ell \) comparisons to the troops used by all other players for all values of \( m \) in constant time (just compute and store the threshold number of battles needed for victory). The final runtime comes from computing \( f_i^{s,t}(m) \) for all \( s \) and all \( i \).

We now prove one of our two main approximation results for Colonel Blotto.

Proof of Theorem 4.2. We apply Algorithm 1 to compute these NE strategies. The proof follows directly by the convergence of MWU for two-player zero-sum games [Freund & Schapire, 1999]. In particular, by Theorems 4.6 and 4.8, iterated play via sampled multiplicative weights update (Algorithm 1) gives a set of strategies that compose an expected Nash equilibrium with approximation \( \epsilon \).

Remark 5.2. At the end of \( T \) rounds of the game, we return the average strategies of all players or the history of strategies used in the iterated game. Notably, these both have polynomial-sized support because \( T \) is polynomial in the parameters of the game, and we increase the support of the solution by at most one strategy per player per round. This is despite the fact that the number of possible strategies is exponential.

6 Discussion

In this work, we show how to compute approximate equilibrium strategies in two-player and multiplayer Colonel Blotto games. Further, for approximation \( \epsilon < \frac{1}{k} \), our algorithm gives strategies that lose at most 1 additional battle on average compared to the exact algorithms of [Ahmadinejad et al., 2019] with a much smaller worst-case computational complexity. We achieve this improvement through implicitly performing multiplicative weights update. In particular, we show that sampling from such distributions can be used to solve zero-sum games that are exponential in size. We expect this technique can be applied to other exponentially sized zero-sum games where explicitly writing down and updating the distributions over strategies is computationally intractable.
Runtime

We highlight this important point regarding the runtime of our algorithm - our runtime estimate in Theorem 4.2 is likely to be pessimistic. This is because we apply Theorem 4.6 black-box to this game, which holds for worst-case losses in the uncoupled dynamics of a zero-sum game. The same holds for CCE in multiplayer Blotto. We expect that in many structured games (like Colonel Blotto) the number of rounds needed for convergence may be smaller than $O(k^3 \ln \frac{n}{\epsilon^2})$. Therefore, it is possible the dependence on $k$ is more favorable.

Further, although we require $\Omega(k^3 \ln \frac{n}{\epsilon^2})$ rounds to solve a worst-case main min-max game, we can halt the optimization with a theoretical approximation guarantee (in, possibly, many fewer iterations than the worst-case). The proximity to an $\epsilon$-approximate NE is bounded by the sum of the regrets of all players to their single best strategy. The regret can be computed in hindsight in time $O(n^2 k)$, using the best-response algorithm from [Vu et al. 2018]. For example, in instance-optimal nearest neighbors search, the number of iterations needed for convergence with MWU in practice is much fewer than the worst-case bound (Andoni & Beaglehole [2021]).

Online Algorithm

An especially interesting aspect of this algorithm is that the NE and CCE arise from completely uncoupled interaction. All players in the game can sample from MWU without knowing other players’ rewards or even number of troops. Hence, the sampling algorithm presented here can be used in an online setting, where in practice the other player’s preferences and resources are often unknown.

Tie-breaking

It may also be the case that a practitioner would want to use a different tie-breaking mechanism other than the (standard) assumption that ties are broken in lexicographic order (see [Roberson, 2006]). The rewards can be modified for different tie-breaking rules, while retaining the polynomial complexity of our algorithm. As an example, if in the case of ties rewards are split evenly among the tied players, you can simply discretize the losses by $1/\ell$ to account for all possible losses (increasing the number of elements in each DP and the runtime by at most a factor of $\ell$).

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