Symmetric collective attacks for the eavesdropping of symmetric quantum key distribution

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We consider the collective eavesdropping of the BB84 and six-state protocols. Since these protocols are symmetric in the eigenstates of conjugate bases, we consider collective attacks having the same kind of symmetry. We then show how these symmetric collective attacks are sufficiently strong in order to minimize the Devetak-Winter rates. In fact, it is quite easy to construct simple examples able to reach the unconditionally-secure key-rates of these protocols.

I. INTRODUCTION

Recently, Renner [1] has shown how to reduce quantum key distribution (QKD) to the cryptoanalysis of collective attacks. This is possible by turning an arbitrary QKD protocol into a permutation invariant one, where Alice and Bob publicly agree on a random permutation which they use to reorder their classical values just at the end of the quantum communication and before any other classical processing of the data [1]. Thanks to this permutation invariance, a finite quantum de Finetti theorem [2] can be applied to the cryptographic scenario and, therefore, the most general coherent attack can be approximated by a mixture of collective attacks. As a consequence, a bound on the key-rate for all the possible collective attacks becomes automatically a bound for the most general attacks allowed by quantum mechanics. Since a natural upper bound for the eavesdropper’s information $I_{AE}$ is given by the Holevo information $\chi_{AE}$, the minimization of the Devetak-Winter rate $R_{\text{DW}} := I_{AB} - \chi_{AE}$ on the class of collective attacks provides a natural lower bound for the unconditionally-secure key-rate.

In this paper we consider the cryptoanalysis of the BB84 and the six-state protocols. Such QKD schemes can be called symmetric since they are based on the symmetric exploitation of the eigenstates of conjugate bases (mutually unbiased bases). It is then intuitive to consider collective attacks whose action is symmetric on these eigenstates, resulting in uniform contractions within the Bloch sphere. Such symmetric collective attacks are in fact a trivial extension of the symmetric individual attacks defined by Gisin et al. [5]. The naive result of this paper is that, for symmetric QKD schemes like the BB84 and six-state protocols, the minimization of the Devetak-Winter rates can be restricted to the class of symmetric collective attacks. In fact, it is very easy to find simple examples of symmetric collective attacks whose Devetak-Winter rates correspond exactly to the unconditionally-secure key-rates of these symmetric protocols.

II. THE BB84 PROTOCOL AND ITS SYMMETRIC EAVESDROPPING

In the BB84 protocol [6], two honest users (Alice and Bob) randomly choose between two conjugate bases, i.e., the $Z$-basis $\{|0\rangle, |1\rangle\}$ (the eigenstates of the Pauli operator $Z$) and the $X$-basis $\{|+, -\rangle\}$ (the eigenstates $|\pm\rangle = 2^{-1/2}(|0\rangle \pm |1\rangle)$ of the Pauli operator $X$). Alice encodes a logical bit into her basis $\sigma_A = Z \lor X$ according to the mapping $0 = |0\rangle \lor |+\rangle$ and $1 = |1\rangle \lor |-\rangle$. The signal state $|u\rangle$ with $u = \{0, 1, +, -\}$ is then sent to Bob through the noisy channel $\mathcal{E}$, who will project the output state $\rho_B(u) := \mathcal{E}(|u\rangle \langle u|)$ onto his basis $\sigma_B = Z \lor X$ in order to decode Alice’s logical bit. At the end of the quantum communication, Alice and Bob publicly agree a random permutation of their binary data (called the raw key). Then, they disclose all their bases (basis reconciliation) and keep only the compatible data, forming the so-called sifted key. Such a key is still affected by errors due to the noise of the channel and the corresponding error rate is called QBER (for quantum bit error rate). The QBER is computed during the subsequent error estimation, where the honest users publicly compare a (small) random subset of the sifted key. From the knowledge of the QBER, the honest users can bound the amount of information potentially stolen by an eavesdropper (Eve). In particular, if the QBER is below a certain security threshold, then Alice and Bob can apply procedures of error correction and privacy amplification in order to derive a final secret and error-free binary key.

In a collective attack, Eve probes each signal qubit using a fresh ancilla, which is then stored in a cell of a quantum memory coherently measured at the end of the protocol. In particular, such a coherent measurement is also optimized on every classical communication used by Alice and Bob during the protocol like, e.g., the basis reconciliation. As a consequence, Eve has an a posteriori knowledge of the basis ($Z$ or $X$) which was used for each signal qubit. On the one hand, Eve can exploit this knowledge in the final detection [7]. On the other hand, she cannot exploit it for a conditional optimization of the signal-ancilla interactions (which, of course, have been already occurred). Since the usage of the two conjugate bases is perfectly symmetric in the BB84 protocol, the optimal eavesdropping strategy
should consist in signal-ancilla interactions which are symmetric in the eigenstates of these conjugate bases.

Let us explicitly construct this kind of symmetric interaction. According to the Stinespring dilation theorem \[8\], the quantum channel \( \mathcal{E} \) acting on the signal qubit can be represented by a unitary interaction \( \hat{U} \) coupling the signal qubit with two ancillary qubits initially prepared in the vacuum state (such a representation is also minimal and unique up to partial isometries). Then, for every input \( u = \{0, 1, +, -\} \), we can write the following signal-ancilla unitary interaction

\[
\hat{U} (|u\rangle \otimes |0, 0\rangle) = |u\rangle |F_u\rangle + |u \oplus 1\rangle |D_u\rangle ,
\]

where \( u \oplus 1 = \{1, 0, -, +\} \) and the output ancillas (\( F \) and \( D \)'s) are generally not orthogonal neither normalized. Now, the condition of symmetry in the four eigenstates \( |u\rangle \) reduces the number of possible unitaries \( \hat{U} \). In particular, by imposing the conditions

\[
\langle F_u | F_u \rangle = F , \quad \langle D_u | D_u \rangle = D := 1 - F , \quad \langle F_u | D_u \rangle = 0 ,
\]

one makes \( \hat{U} \) symmetric and Eq. \( 1 \) a Schmidt form. The Stinespring dilation of Eq. \( 1 \) under the conditions of Eq. \( 2 \) describes a uniform contraction by \( F \) of the Bloch sphere \[5\]. From Eq. \( 3 \) it is clear that parameter \( F \) represents the fidelity while \( D \) is the QBER. As a consequence, Alice and Bob’s mutual information is simply given by \( I_{AB} = 1 - H(D) \) where \( H(p) = -p \log_2 p - (1 - p) \log_2 (1 - p) \) is the Shannon entropy.

Let us now consider the output state \( \rho_{E}(u) \) which is received by Eve in the complementary Alice-Eve channel \( \hat{E} : |u\rangle \langle u| \rightarrow \rho_{E}(u) \). This is equal to

\[
\rho_{E}(u) = |F_u\rangle \langle F_u| + |D_u\rangle \langle D_u| = F |f_u\rangle \langle f_u| + D |d_u\rangle \langle d_u| ,
\]

where the normalized states \( |f_u\rangle := F^{-1/2} |F_u\rangle \) and \( |d_u\rangle := D^{-1/2} |D_u\rangle \) have been introduced. In case of collective attack, this output state is subject to an optimal coherent measurement involving all the cells of the quantum memory. Since Eve has the a posteriori knowledge of the basis, her coherent measurement has to discriminate between the two states of the quantum ensemble

\[
Q = \begin{cases} 
\rho_{E}(u) , & p(u) = \frac{1}{2} \\
\rho_{E}(u \oplus 1) , & p(u \oplus 1) = \frac{1}{2} 
\end{cases} \quad \Rightarrow \quad \rho_{E} := \frac{\rho_{E}(u) + \rho_{E}(u \oplus 1)}{2} .
\]

It is known that the maximal amount of classical information (accessible information) that Eve can steal from this ensemble is upper-bounded by the Holevo information

\[
\chi_{AE} := S(\rho_{E}) - \frac{S[\rho_{E}(u)] + S[\rho_{E}(u \oplus 1)]}{2} ,
\]

where \( S(\rho) := -\text{Tr}(\rho \log_2 \rho) \) is the Von Neumann entropy. As a consequence, the secret-key rate is lower bounded by the Devetak-Winter rate \[4\]

\[
R_{DW} := I_{AB} - \chi_{AE} .
\]

Since \( \langle f_u | d_u \rangle = \langle F_u | D_u \rangle = 0 \) in Eq. \( 4 \), we have that \( S[\rho_{E}(u)] = S[\rho_{E}(u \oplus 1)] = H(D) \). By exploiting this expression and \( I_{AB} = 1 - H(D) \), the Devetak-Winter rate for a symmetric collective attack simply becomes

\[
R_{DW} = 1 - S(\rho_{E}) ,
\]

where only \( S(\rho_{E}) \) remains to be computed.

Following Gisin et al. \[5\], let us simplify the structure of the symmetric attack by imposing the additional conditions

\[
\langle F_u | F_{u \oplus 1} \rangle = F \cos x , \quad \langle D_u | D_{u \oplus 1} \rangle = D \cos y , \quad \langle F_u | D_{u \oplus 1} \rangle = 0 ,
\]

with \( x \) and \( y \) real numbers. Such conditions imply

\[
D = \frac{1 - \cos x}{2 - \cos x + \cos y} := D(x, y) ,
\]
so that $\hat{U}$ is not only symmetric but also completely determined by two angles $x$ and $y$. In particular, we can realize all the conditions in Eqs. (2) and (11) by choosing in Eq. (11) the ancilla states [3]

$$
|F_0\rangle = \begin{pmatrix} \sqrt{F} \\ 0 \\ 0 \\ 0 \end{pmatrix},
|D_0\rangle = \begin{pmatrix} 0 \\ \sqrt{D} \\ 0 \\ 0 \end{pmatrix},
|F_1\rangle = \begin{pmatrix} \sqrt{F} \cos x \\ 0 \\ 0 \\ \sqrt{F} \sin x \end{pmatrix},
|D_1\rangle = \begin{pmatrix} 0 \\ \sqrt{D} \cos y \\ 0 \\ \sqrt{D} \sin y \end{pmatrix}.
$$ (11)

Let us denote by $S(x, y)$ the symmetric collective attack specified by the interaction of Eq. (11). Then, it is easy to prove that the attack $S(x, y)$ has a Devetak-Winter rate equal to

$$
R_{DW} = 1 - 2H(D),
$$
which corresponds exactly to the unconditionally-secure key-rate of the BB84 protocol [3] (with unconditional security threshold $D \simeq 11\%$ as given by $1 - 2H(D) = 0$).

**Proof of Eq. (12).** In order to prove the rate of Eq. (12) we have to compute the entropy $S(\rho_E)$ in Eq. (8) by exploiting the properties of the attack $S(x, x)$, which are simply given by conditions of Eqs. (2) and (11) with $x = y$. By introducing the states

$$
\rho_F := \frac{1}{2} (|f_u\rangle \langle f_u| + |f_u\downarrow\rangle \langle f_u\downarrow|),
\rho_D := \frac{1}{2} (|d_u\rangle \langle d_u| + |d_u\downarrow\rangle \langle d_u\downarrow|),
$$
we can recast the average state $\rho_E$ of Eq. (5) in the form $\rho_E = F\rho_F + D\rho_D$, so that it can be equivalently seen as the average state of the quantum ensemble

$$
\mathcal{Q} = \left\{ \rho_F, p(F) = F, \rho_D, p(D) = D \right\}.
$$ (14)

From Eqs. (2) and (11) we easily derive that $\rho_F$ and $\rho_D$ are orthogonal, i.e., $\text{Tr}(\rho_F \rho_D) = 0$. As a consequence, we have

$$
\chi(\mathcal{Q}) := S(\rho_E) - [FS(\rho_F) + DS(\rho_D)] = H(D).
$$ (15)

In order to extract $S(\rho_E)$ from Eq. (15), we have to compute the two entropies $S(\rho_F)$ and $S(\rho_D)$. For computing $S(\rho_F)$ let us introduce the orthonormal set $\{|f_u\rangle, |f_u\downarrow\rangle\}$, where $|f_u\downarrow\rangle$ is an arbitrary vector defined by $\langle f_u | f_u\downarrow \rangle = 0$ and $\langle f_u | f_u\downarrow \rangle = 1$. By using Eq. (11), we can always decompose $|f_u\downarrow\rangle = \cos x |f_u\rangle + e^{i\phi} \sin x |f_u\downarrow\rangle$ with $\phi$ arbitrary phase, so that

$$
\rho_F = \left( |f_u\rangle \langle f_u\downarrow| \right) \begin{pmatrix} 1 + \cos^2 x & e^{-i\phi} \sin 2x \\ e^{i\phi} \sin 2x & \frac{\sin^2 x}{\cos^2 x} \end{pmatrix} \left( |f_u\rangle \langle f_u\downarrow| \right).
$$ (16)

By means of a suitable unitary we then get

$$
\hat{U}\rho_F\hat{U}^\dagger = \lambda |\Phi_-\rangle \langle \Phi_-| + (1 - \lambda) |\Phi_+\rangle \langle \Phi_+|,
$$
where

$$
\lambda(x) = \frac{1 - \cos x}{2},
|\Phi_\pm\rangle = \frac{e^{-i\phi} (1 + \cos 2x \pm 2 \cos x) (\csc 2x) |f_u\rangle + |f_u\downarrow\rangle}{N_\pm},
$$
and $N_\pm^2 = 1 + (1 + \cos 2x \pm 2 |\cos x|)^2 (\csc 2x)^2$. Since $\langle \Phi_+ | \Phi_-\rangle = 0$, we simply achieve $S(\rho_F) = S(\hat{U}\rho_F\hat{U}^\dagger) = H[\lambda(x)]$. In order to compute the other entropy $S(\rho_D)$, we just introduce an analogous orthonormal set $\{|d_u\rangle, |d_u\downarrow\rangle\}$ which leads to the corresponding result $S(\rho_D) = H[\lambda(y)]$.

Now, by setting $x = y$, we clearly have $S(\rho_F) = S(\rho_D) = H[\lambda(x)]$. Then, we also have $\lambda(x) = D(x, x)$ for $-\pi/2 \leq x \leq \pi/2$ and $\lambda(x) = 1 - D(x, x)$ for $\pi/2 \leq x \leq 3\pi/2$, so that we can always write $S(\rho_F) = S(\rho_D) = H(D)$. By replacing the latter result in Eq. (15) we finally get $S(\rho_E) = 2H(D)$ which leads to the rate of Eq. (12).
III. THE SIX-STATE PROTOCOL AND ITS SYMMETRIC EAVESDROPPING

In the BB84 protocol the signal states represent the four equidistant poles lying on the equator of the Bloch sphere. In order to enhance the security, one can then think to saturate the sphere by including the exploitation of the remaining two poles. This is done in the six-state protocol [10] where also the basis \{0\}, \{i\}, \{1\}, \{i\}, \{0\} the third Pauli operator Y = iXZ is exploited in both Alice’s random encoding and Bob’s random decoding. The six-state protocol is then formulated like the BB84 protocol except that now we have three bases \{X, Y, Z\} and, therefore, six possible signal states \{\{u\}; u = 0, 1, +, -, R, L\} encoding a logical qubit according to the mapping 0 = \{0\} \lor \{+\} \lor \{R\} and 1 = \{1\} \lor \{-\} \lor \{L\}.

Since the six-state protocol is a symmetric extension of the BB84 to the third Pauli operator, we consider the same extension for the symmetric attacks. This means that an arbitrary symmetric attack against the six-state protocol is defined by Eqs. (11) and (12) where now u = \{0, 1, +, -, R, L\}. The corresponding channel is again described by Eq. (3) which now corresponds to a uniform contraction by F = D of all the Bloch sphere. It is trivial to check that a symmetric collective attack against the six-state protocol is characterized by the same Devetak-Winter rate of Eq. (8), exactly as before [11].

Let us construct a simple example for the explicit computation of this rate. We can simplify the structure of the average state since we have \(\lambda \leq \frac{2}{3} \) and not \(\lambda \leq \frac{4}{9} \). As a consequence, we have

\[
D = \frac{1 - \cos x}{2 - \cos x} := D(x),
\]

and the unitary interaction \(\hat{U}\) is completely determined by a single angle \(x\). In particular, we can realize all the previous conditions by choosing the ancillas of Eq. (11) with \(y = \pi/2\), i.e.,

\[
|F_0\rangle = \begin{pmatrix} \sqrt{F} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |D_0\rangle = \begin{pmatrix} 0 \\ \sqrt{D/2} \\ 0 \\ 0 \end{pmatrix}, \quad |F_1\rangle = \begin{pmatrix} \sqrt{F} \cos x \\ 0 \\ 0 \\ \sqrt{F} \sin x \end{pmatrix}, \quad |D_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{D/2} \end{pmatrix},
\]

where also \(|D_0\rangle\) and \(|D_1\rangle\) are orthogonal. Let us denote by \(\tilde{S}(x)\) the symmetric collective attack specified by the interaction of Eq. (20). Then, it is easy to prove that \(\tilde{S}(x)\) has a Devetak-Winter rate equal to

\[
R_{DW} = 1 + \frac{3D}{2} \log_2 \frac{D}{2} + \left(1 - \frac{3D}{2}\right) \log_2 \left(1 - \frac{3D}{2}\right),
\]

which corresponds exactly to the unconditionally-secure key-rate of the six-state protocol [12] (with unconditional security threshold \(D \approx 12.6%\)).

**Proof of Eq. (21).** In order to get the result we have to compute \(S(\rho_E)\) for the simple attack \(\tilde{S}(x)\). Eve’s output state \(\rho_E(u)\) has the same form of Eq. (1). Thus, the average state \(\rho_E\) can be again recasted in terms of the states \(\rho_D\) and \(\rho_F\) of Eq. (13), in such a way to represent the same quantum ensemble \(\tilde{Q}\) of Eq. (14). As a consequence, the entropy \(S(\rho_E)\) can be again extracted from of Eq. (15), where the computation of \(S(\rho_F)\) and \(S(\rho_D)\) is now different since we have \(y = \pi/2\) and not \(y \neq \pi/2\) as before. The computation of \(S(\rho_D)\) is very easy thanks to the orthogonality which now exists between the D’s states. Since \(\langle d_u | d_{u+1} \rangle = \langle d_u | d_{u+1} \rangle = 0\), we have in fact \(S(\rho_D) = H(1/2) = 1\). The computation of \(S(\rho_F)\) is the same as before except that now the eigenvalue \(\lambda(x)\) of Eq. (15) is differently connected to the QBER \(D(x)\) of Eq. (15). It is easy to check that \(\lambda(x) = (1 - D(x))^{-1} D(x)/2\) for \(-\pi/2 \leq x \leq \pi/2\) and \(\lambda(x) = 1 - (1 - D(x))^{-1} D(x)/2\) for \(\pi/2 \leq x \leq 3\pi/2\), so that we can always write \(S(\rho_F) = H(\lambda) = H \left( \frac{(1 - D)^{-1} D/2}{(1 - D)^{-1} D/2} \right)\). By inserting the latter result and \(S(\rho_D) = 1\) into Eq. (15), one gets \(S(\rho_E) = D + H(D) + (1 - D)H \left( \frac{(1 - D)^{-1} D/2}{(1 - D)^{-1} D/2} \right)\) and, therefore, the Devetak-Winter rate

\[
R_{DW} = (1 - D) \left\{ 1 - H \left[ \frac{D}{2(1 - D)} \right] \right\} - H(D),
\]

which is equivalent to the result of Eq. (21).
IV. CONCLUSION

In conclusion, we have considered very simple collective attacks against the BB84 and six-state protocols, which are constructed by trivially extending the individual symmetric attacks of Gisin et al.\cite{5}. Such symmetric collective attacks have been proven to be sufficiently strong in order to minimize the Devetak-Winter rates of these protocols. In fact, it has been shown how to construct simple examples able to reach their unconditionally-secure key-rates. Our results can be useful in the cryptoanalysis of other QKD protocols which are based on the symmetric exploitation of the vertices of regular polygons or polyhedrons embedded in the Bloch sphere.

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