Isomorphisms of Affine Plücker Spaces

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Abstract

All isomorphisms of Plücker spaces on affine spaces with dimensions \( \geq 3 \) arise from collineations of the underlying affine spaces.

1 Introduction

Let \( L \) be a set and \( \sim \) a reflexive and symmetric binary relation on \( L \) such that \((L, \sim)\) is connected, i.e., for any \( a, b \in L \) there exists a finite sequence \( a = a_1 \sim a_2 \sim \cdots \sim a_n = b \). Following W. BENZ the pair \((L, \sim)\) is called a Plücker space [1, p. 199]. Elements \( a, b \in L \) are said to be related if \( a \sim b \).

Adjacent elements \( (a \approx b) \) are characterized by \( a \sim b \) and \( a \neq b \).

The relation \( \not\approx \) is reflexive and symmetric. However, \((L, \not\approx)\) is not necessarily a Plücker space, since it need not be connected. Nevertheless, \( L \) splits into a family of connected components with respect to \( \not\approx \), say \((L_i)_{i \in I}\). Each component \( L_j \) \((j \in I)\) gives rise to the Plücker space \((L_j, \not\approx_j)\), where \( \not\approx_j \) denotes the restriction of \( \not\approx \) to \( L_j \times L_j \). On the other hand, \( L_j \) is not necessarily connected with respect to \( \sim_j \), i.e., the restriction of \( \sim \) to \( L_j \times L_j \). Hence \((L_j, \sim_j)\) need not be a Plücker space.

Given two Plücker spaces \((L, \sim)\) and \((L', \sim')\) a bijection \( \varphi : L \to L' \) is called an isomorphism if

\[
a \sim b \iff a^\varphi \sim' b^\varphi \quad \text{for all} \quad a, b \in L.
\]  

(1)

Obviously, (1) and

\[
a \not\approx b \iff a^\varphi \not\approx' b^\varphi \quad \text{for all} \quad a, b \in L
\]  

(2)

are equivalent conditions.

All automorphisms of \((L, \sim)\) form its so-called Plücker group. Write, as above, \((L_i)_{i \in I}\) for the connected components of \( L \) with respect to \( \not\approx \). Then each automorphism of \((L_j, \not\approx_j)\) \((j \in I)\) extends to an automorphism of \((L, \sim)\) by setting \( x \mapsto x \) for all \( x \in L \setminus L_j \).
Let $A = (\mathcal{P}, \mathcal{L}, \parallel)$ be an affine space, where $\mathcal{P}$, $\mathcal{L}$ and $\parallel$ denotes the set of points, the set of lines and the parallelism, respectively. Lines $a, b \in \mathcal{L}$ are called related ($a \sim b$), if $a \cap b \neq \emptyset$. The pair $(\mathcal{L}, \sim)$ is satisfying the conditions mentioned before and will be called an affine Plücker space. We remark that for $\dim A \geq 3$ the set $\mathcal{L}$ is the set of ‘points’ of partial linear space, the affine Grassmann space on $\mathcal{L}$; cf. [4], [5], [6], [16] and [18]. However, the relation $\sim$ is not the same as the binary relation of ‘collinearity’ used in those papers, since ‘collinear points’ are represented by lines that are related or parallel.

If $\dim A \neq 2$, then $(\mathcal{L}, \not\approx)$ is a Plücker space. If $A$ is an affine plane, then $(\mathcal{L}, \not\approx)$ is not a Plücker space. The connected components $(\mathcal{L}_i)_{i \in I}$ with respect to $\not\approx$ are the pencils of parallel lines, since the relations $\not\approx$ and $\parallel$ are coinciding now. If $\mathcal{L}_j$ ($j \in I$) is a fixed pencil of parallel lines, then the relation $\not\approx_j$ is the coarsest relation on $\mathcal{L}_j$. Thus Plücker spaces on affine planes have indeed a very poor structure. The case $\dim A \leq 1$ cannot deserve interest at all.

We shall determine all isomorphisms of Plücker spaces on affine spaces $A, A'$ with dimensions $\geq 3$: Any collineation yields an isomorphism of the associated affine Plücker spaces and vice versa. If we impose additional assumptions on $A, A'$ (cf. Theorem [3]), then any bijection $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ is already an isomorphism of Plücker spaces whenever (11) is satisfied with an implication ($\Rightarrow$) rather than an equivalence ($\Leftrightarrow$).

Similar theorems for Plücker spaces on projective spaces are due to W.L. CHOW [9], H. BRAUNER [7] and the author [12]. For further results and references on Plücker spaces see, among others, [1], [2] and [13].

### 2 Isomorphisms

Let $A = (\mathcal{P}, \mathcal{L}, \parallel)$ and $A' = (\mathcal{P}', \mathcal{L}', \parallel')$ be affine spaces. If $\kappa : \mathcal{P} \rightarrow \mathcal{P}'$ is a collineation, i.e., a bijection preserving collinearity and non-collinearity of points, then $\kappa$ gives rise to a bijection

$$\varphi : \mathcal{L} \rightarrow \mathcal{L}', Q \vee R \mapsto Q'^{\kappa} \vee R'^{\kappa} \quad (Q, R \in \mathcal{P}, Q \neq R)$$

(3)

taking related lines to related lines in both directions.

We shall prove the following converse:

**Theorem 1** Let $A = (\mathcal{P}, \mathcal{L}, \parallel)$ and $A' = (\mathcal{P}', \mathcal{L}', \parallel')$ be affine spaces with $\dim A' \geq 3$. Suppose that $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ is an isomorphism of the Plücker space $(\mathcal{L}, \sim)$ onto the Plücker space $(\mathcal{L}', \sim')$. Then

$$\kappa : \mathcal{P} \rightarrow \mathcal{P}', a \cap b \mapsto a^{\varphi} \cap b^{\varphi} \quad (a, b \in \mathcal{L}, a \sim b)$$

(4)

is a well-defined collineation.
Proof. (a) We infer from $\dim A' \geq 3$ and the bijectivity of $\varphi$ that $\# L > 1$. Therefore $\dim A \geq 2$. With $Q \in P$ write $L(Q)$ for the star of lines with centre $Q$, i.e., the set of all lines in $L$ running through $Q$. Any star of lines is a maximal set of mutually related lines.

Suppose that $(L(Q))^{\varphi}$ contains a trilateral spanning a plane $E' \subset P'$, say. All lines of $(L(Q))^{\varphi}$ are mutually related. Therefore they are all contained in $E'$. By $\dim A' \geq 3$, there exists a line $a \in L$ with $a^{\varphi} \cap E' = \emptyset$. Thus

$$a^{\varphi} \not\sim x^{\varphi} \text{ for all } x \in L(Q)$$

(5)

and therefore

$$a \not\sim x \text{ for all } x \in L(Q).$$

(6)

On the other hand, there exists a line joining $Q$ with an arbitrarily chosen point of the line $a$. This contradicts (6).

Thus we have established that $(L(Q))^{\varphi}$ is a subset of a star of lines for any $Q \in P$. It is obvious now that $\kappa$ is a well-defined mapping.

(b) Given a point $Q' \in P'$ one may show as above that $(L'(Q'))^{\varphi^{-1}}$ is a subset of a star of lines. Therefore $\kappa$ is a surjection and under $\varphi$ stars of lines go over to stars of lines in both directions.

If points $Q, R \in P$ are distinct, then

$$\#(L(Q) \cap L(R)) = \#((L(Q))^{\varphi} \cap (L(R))^{\varphi}) = 1$$

(7)

and $Q^\kappa \neq R^\kappa$, whence $\kappa$ is injective.

Three mutually distinct points $Q, R, S \in P$ are collinear if, and only if,

$$\#(L(Q) \cap L(R) \cap L(S)) = 1.$$

(8)

This in turn is equivalent to the collinearity of $Q^\kappa, R^\kappa, S^\kappa \in P'$. Hence $\kappa$ is a collineation. □

Remark 1 If the order of $A'$ is greater than two or if $\dim A' \leq 2$, then any collineation $P \to P'$ is even an affinity, i.e. a collineation preserving parallelism in both directions. Otherwise, the existence of a collineation $P \to P'$ implies the existence of an affinity $P \to P'$; see [17, 32.5 and 40.4]. Hence for $\dim A' \geq 3$ we obtain all isomorphisms of $(L, \sim)$ onto $(L', \sim')$ via the Plücker group of $(L, \sim)$ and a single affinity $P \to P'$.

Remark 2 If $A = A'$, then Theorem [11] describes the Plücker group for affine spaces with dimension $\geq 3$. This generalizes a result in [11, p. 205] for real affine space.

The proof given there fails to work in case of characteristic two.
Remark 3 Suppose that \( \dim A \geq 2 \). The following construction yields all maximal sets of mutually related lines that are different from stars: Choose a point \( Q \), an incident line \( a \) and a plane \( E \) containing \( a \). Write \( L(Q, E) \) for the pencil of lines in \( E \) running through \( Q \). Next define a family \( \{ \tau_x \}_{x \in L(Q, E)} \) of translations \( \tau_x : E \to E \) such that \( \bigcap_{x \in L(Q, E)} x \tau_x = \emptyset \). Then \( \{ x \tau_x \mid x \in L(Q, E) \} \) is a maximal set of mutually related lines other than a star.

Remark 4 If \( \varphi : L \to L' \) is an isomorphism and if \( \dim A' = 2 \), then \( \dim A = 2 \) according to Theorem 1. By virtue of (2) it is easy to establish the following result: Plücker spaces on affine planes \( A \) and \( A' \) are isomorphic if, and only if, \( A \) and \( A' \) have equipotent pencils of parallel lines or, in other words, if the order of \( A \) equals the order of \( A' \).

The transposition of two distinct parallel lines of an affine plane is an example of a Plücker transformation that does not stem from a collineation.

We are now going to weaken the assumptions on \( \varphi \) in Theorem 1.

Theorem 2 Let \( A = (P, L, \|) \) and \( A' = (P', L', \|') \) be affine spaces with \( \dim A' \geq 3 \). Suppose that \( \varphi : L \to L' \) is a bijection satisfying
\[
a \sim b \implies a^\varphi \sim b^\varphi \quad \text{for all } a, b \in L.
\]

Then
\[
\lambda : P \to P', \ a \cap b \mapsto a^\varphi \cap b^\varphi \quad (a, b \in L, \ a \approx b)
\]
is a well-defined injection that preserves collinearity and non-collinearity of points. Moreover,
\[
L(Q)^\varphi = L'(Q^\lambda) \quad \text{for all } Q \in P.
\]

Proof. (a) By the proof of Theorem 1 part (a), the following assertions have been already verified: The dimension of \( A \) is \( \geq 2 \). For all \( Q \in P \) the set \( (L(Q))^\varphi \) is a subset of a star of lines, whence \( (10) \) is a well-defined mapping.

(b) Let \( Q, R \in P \) be distinct and assume that \( Q^\lambda = R^\lambda \). Choose a line \( c \in L \setminus (L(Q) \cup L(R)) \) and a point \( S \in c \) such that \( Q, R, S \) are not collinear. Therefore \( Q \cap S, R \cap S \) and \( c \) are three distinct concurrent lines. We deduce from \( (10) \) that
\[
S^\lambda = (Q \cap S)^\varphi \cap (R \cap S)^\varphi = Q^\lambda = R^\lambda.
\]
Consequently, \( Q^\lambda \in x^\varphi \) for all \( x \in L \). This is impossible due to the surjectivity of \( \varphi \). Hence \( \lambda \) is injective.

If \( \{ Q, R, S \} \subset P \) is a triangle, then \( Q \cap R, R \cap S \) and \( S \cap Q \) are three distinct lines. The injectivity of \( \varphi \) and the injectivity of \( \lambda \) force that \( \{ Q^\lambda, R^\lambda, S^\lambda \} \subset P' \) is a triangle. By definition, \( \lambda \) is a collinearity-preserving mapping.
Finally, we establish (11). Assume to the contrary that there exists a point \( Q \in \mathcal{P} \) and a line \( b \in \mathcal{L} \setminus \mathcal{L}(Q) \) with \( Q^\lambda \in b^\varphi \). Choose two distinct points \( R_1, R_2 \in b \). Then \( \{Q, R_1, R_2\} \) is a triangle, but \( \{Q^\lambda, R_1^\lambda, R_2^\lambda\} \subset b^\varphi \) is a collinear set, an absurdity. □

The aim of the following discussion is to give sufficient conditions for \( \lambda \) to be a collineation or, equivalently, a surjection. We could apply results on injective mappings of affine spaces preserving collinearity of points; see [1, 3.1–3.3], [19] and the references in [11]. However, we proceed instead in close analogy with [12]. Any star of lines in an affine space, for example, a star \( \mathcal{L}(Q) \) in \( A \), carries in a natural way the structure of a projective space, viz.

\[
A/Q := (\mathcal{L}(Q), \{\mathcal{L}(Q, \mathcal{E}) \mid Q \in \mathcal{E}, \mathcal{E} \text{ a plane}\}).
\]  

(13)

Recall the following concept due to P.V. Ceccherini [8]: A semicollineation of projective spaces is a bijection taking any three collinear points to collinear points. The existence of proper semicollineations (different from collineations) of Desarguesian projective spaces seems to be an open problem; see also [3], [14] and [15]. Semicollineations fall within the wider class of weak linear mappings that have been characterized independently in [10] and [11].

Now (11) can be improved as follows:

**Theorem 3** Let \( A, A' \) and \( \varphi \) be given as in Theorem 2. Then \( \dim A \geq 3 \). If moreover the order of \( A \) is not two and if \( Q \in \mathcal{P} \), then the restricted mapping

\[
\varphi|\mathcal{L}(Q) : \mathcal{L}(Q) \to \mathcal{L}'(Q^\lambda)
\]

is a semicollineation of \( A/Q \) onto \( A'/Q^\lambda \).

**Proof.** A cannot be the affine plane of order two, since \( 6 < \#\mathcal{L}' \).

Suppose that the order of \( A \) is not two. By (11), the mapping (13) is bijective. Let \( a, b, c \in \mathcal{L}(Q) \) be ‘collinear points’ of \( A/Q \). There exists a line \( d \in \mathcal{L} \setminus \mathcal{L}(Q) \) that is adjacent to \( a, b \) and \( c \). Hence \( a^\varphi, b^\varphi, c^\varphi \in \mathcal{L}(Q^\lambda, Q^\lambda \setminus d^\varphi) \) represent ‘collinear points’ in \( A'/Q^\lambda \) so that (14) is a semicollineation. There are non-coplanar lines through \( Q^\lambda \) representing ‘non-collinear’ points of \( A'/Q^\lambda \). Their pre-images under (14) are distinct and non-coplanar, so that \( \dim A \geq 3 \). □

If \( A \) is of order two, then (14) is in general merely a bijection.

**Theorem 4** With the settings of Theorem 2, each of the following conditions is sufficient for \( \lambda \) to be a collineation:

1. \( A \) or \( A' \) is a finite affine space.
2. \( \dim A \leq \dim A' < \infty \).

3. The order of \( A \) is different from two and every monomorphism of an underlying field \( F \) of \( A \) in an underlying field \( F' \) of \( A' \) is surjective.

4. \( A \) and \( A' \) are affine spaces of order two.

Proof. Ad 1. Since \( \varphi \) is bijective, both \( A \) and \( A' \) are finite affine spaces.

Let \( A \cong \AG(n,2) \) and \( A' \cong \AG(m,p^h) \), where \( n \geq 3, m \geq 3, h \geq 1 \) are integers and \( p \) is a prime. Choose \( Q \in \mathcal{P} \). We deduce from (11) that

\[
2^{n-1} \# \mathcal{L}(Q) = \# \mathcal{L} = \# \mathcal{L}' = p^{h(m-1)} \# \mathcal{L}(Q^\lambda) = p^{h(m-1)} \# \mathcal{L}(Q).
\]

Consequently, \( p = 2 \), \( n - 1 = h(m - 1) \) and, by \( \# \mathcal{L}(Q) = \# \mathcal{L}(Q^\lambda) \),

\[
\sum_{i=0}^{n-1} 2^i = \sum_{i=0}^{m-1} 2^{hi}.
\]

We infer that each summand on the right hand side of (16) appears exactly once on the left hand side, whence \( h = 1 \) and \( n = m \). This implies that \( \lambda \) is surjective.

If the order of \( A \) is greater than two, then (14) is a semicollineation and, by [8, 14.2], even a collineation. Therefore \( A \) and \( A' \) are of equal order. Now \( \lambda \) turns out to be surjective, because of

\[
\dim A = \dim(A/Q) + 1 = \dim(A'/Q^\lambda) + 1 = \dim A' < \infty.
\]

Ad 2. By virtue of the previous result, we may exclude affine spaces \( A, A' \) of order two from the following discussion.

Choose any point \( R' \in \mathcal{P}' \). As \( \varphi \) is surjective, there exists a line \( d \in \mathcal{L} \) with \( R' \in d^{\varphi} \). Let \( Q \in \mathcal{P} \) be off the line \( d \) and put \( E := Q \vee d \). We observe that

\[
\dim(A/Q) = \dim(A - 1) \leq \dim A' - 1 = \dim(A'/Q^\lambda) < \infty.
\]

By [8, 8.4] or [14, Theorem 2.2] the semicollineation (14) turns out to be a collineation. Consequently, \( \mathcal{L}(Q,E)^\varphi \) is a pencil of lines \( \mathcal{L}'(Q^\lambda,E') \), say. The line \( d^\varphi \not\equiv Q^\lambda \) is related to all lines of this pencil with at most one exception. This implies that the point-set \( d^\lambda \) is equal to the affine line \( d^\varphi \). Thus

\[
R' \in d^\varphi = d^\lambda \subset \text{im} \lambda.
\]

Ad 3. Choose any point \( Q \in \mathcal{P} \). By Theorem 3, the mapping (14) is a semicollineation. This implies the existence of a monomorphism \( F \to F' \); cf. [8, 5.1], [10, Theorem 5.4.1] or [11, Theorem 2]. By [8, 5.3], the mapping (14) is a collineation. From this the surjectivity of \( \lambda \) is established as above.

Ad 4. Choose any point \( R' \in \mathcal{P}' \). Since \( \varphi \) is surjective, there exists a line \( \{R_1, R_2\} \in \mathcal{L} \) with \( R' \in \{R_1, R_2\}^\varphi = \{R_1^\lambda, R_2^\lambda\} \). □
Remark 5 Let $A$ be an affine space over GF(2) with a countable basis and let $A'$ be an $m$-dimensional affine space ($3 \leq m \leq \aleph_0$) over a countable field $F'$ of arbitrary characteristic. Hence there is either no monomorphism or no surjective monomorphism of GF(2) in $F'$. As $\#\mathcal{P} = \#\mathcal{L'} = \aleph_0$, we can index all points of $\mathcal{P}$ as $Q_1, Q_2, \ldots$ and all lines of $\mathcal{L'}$ as $a'_1, a'_2, \ldots$ such that there are no repeated elements.

Let us define, by recursion, an injective sequence $\{1, 2, \ldots\} \to \mathcal{P}'$, $s \mapsto R'_s$ such that each line of $\mathcal{L}'$ contains exactly two points: We start with a point $R'_1 \in a'_1$ and put $B'_1 := \{R'_1\}$. Next assume that we are already given a set $B'_i = \{R'_1, \ldots, R'_i\}$ formed by $i \geq 1$ mutually distinct points no three of which are collinear. Write $N'_i$ for the set of all lines that arise by joining distinct points of $B'_i$. Then let $j \in \{1, 2, \ldots\}$ be the least element such that the line $a'_j$ is not in $N'_i$. Since $a'_j$ carries an infinite number of points, we can choose such a point $R'_{i+1} \in a'_j \setminus B'_i$ that no three elements of the set $B'_{i+1} := B'_i \cup \{R'_{i+1}\}$ are collinear.

Put $B' := \bigcup_{s=1}^{\infty} B'_s$. By construction, no three distinct points of $B'$ are collinear. Furthermore, given a line $a'_k \in \mathcal{L'}$ we obtain that $a'_k \in N'_{2k}$, as required.

The mapping

$$\varphi : \mathcal{L} \to \mathcal{L}', \{Q_s, Q_t\} \mapsto R'_s \lor R'_t, \quad (s, t \in \{1, 2, \ldots\}, \ s \neq t) \quad (20)$$

is a bijection satisfying (9). The associated injection $\lambda$ (see (10)) takes $Q_s$ to $R'_s$ ($s \in \{1, 2, \ldots\}$). Only two points of $a'_1$ belong to im $\lambda$, whence $\lambda$ is not surjective.

Remark 6 Let $A$, $A'$ be affine spaces with equal infinite order and $2 = \dim A' < \dim A \leq \aleph_0$. There exists a bijection $\varphi : \mathcal{L} \to \mathcal{L'}$ such that any class of parallel lines in $A$ is mapped onto a pencil of parallel lines in $A'$. Such a $\varphi$ is satisfying (9) without being an isomorphism of Plücker spaces.

Remark 7 Let $A = A'$ be an affine plane of infinite order. Choose two non-parallel lines $a, b \in \mathcal{L}$. There exists a bijection $\varphi : \mathcal{L} \to \mathcal{L}$ such that the parallel class of $a$ is mapped onto the union of the parallel classes of $a$ and $b$, whereas any other pencil of parallel lines is mapped onto a pencil of parallel lines. Then $\varphi$ is satisfying (9) without being a Plücker transformation.

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