Deformations of Vaisman manifolds

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Abstract We construct a type of transverse deformations of a Vaisman manifold, which preserves the canonical foliation. For this construction we only need a basic 1-form with certain properties. We show that such basic 1-forms exist in abundance.

Keywords: Vaisman manifolds, foliation, transverse geometry, deformation, basic form

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1 Introduction

A Vaisman manifold is a particular Hermitian manifold $(M, J, g)$ with its fundamental form $\omega(\cdot, \cdot) := g(\cdot, J\cdot)$ satisfying the relation $d\omega = \theta \wedge \omega$, for a nonzero one-form $\theta$, which is parallel with respect to the Levi-Civita connection of the metric $g$. The one-form $\theta$ is called the Lee form. We assume throughout the paper that the manifold $M$ is closed, connected, with $\dim_C M \geq 2$.

Since a parallel form is closed, Vaisman manifolds are locally conformally Kähler (LCK). Note that many of the known (LCK) manifolds are in fact Vaisman. For the main properties and examples of LCK manifolds, see e.g. [DrOr], [OrVe4].

Vaisman manifolds bear a holomorphic foliation of complex dimension 1, generated by the Lee and anti-Lee fields $\theta^\sharp$ and $J\theta^\sharp$, usually called the canonical foliation. It is locally Euclidean and transversally Kähler.

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Unlike Kähler structures, LCK structures are not stable to small deformations ([Be]). However, Vaisman structures are preserved by some particular type of deformations obtained by perturbing the Lee vector field in the Lie algebra of the closure of the group generated by its flow, see [OrVe2].

Other types of deformations of Vaisman structures are obtained by fixing the complex structure and the Lee vector field and changing the Lee form ([Be]) (these are related to deformations of Sasaki structures, see also [BoGa]).

In this paper, we adopt another strategy. We construct a deformation \((M, J_t, g_t)\) of the Vaisman structure, with \(J_0 = J, g_0 = g\) and \(t\) sufficiently small, in such a way that the canonical foliation is not affected and the deformation concerns only the transverse orthogonal complement and the transverse Kähler geometry. This type of deformation is related to the deformations of second type on Sasakian manifolds, as defined in [Be]. For our construction, we need a basic 1-form with certain natural properties and we indicate a way of producing such 1-forms (Subsection 3.2). This procedure does not cover the whole space of deformations, still it gives a way of obtaining new Vaisman structures out of a given one (see Theorem 3.8). We end with a completely worked-out application of the method to a classical Hopf surface \(\mathbb{C}^2 \setminus \{e^2 \cdot I_2\}\).

2 Vaisman manifolds

We outline the basic facts needed for Vaisman manifolds. For a more detailed exposure we refer to [DrOr] and recent papers by Verbitsky and the first author.

Let \((M, g, J)\) be a connected, smooth, Hermitian manifold of real dimension \(2n + 2\), with \(n \geq 1\). Let \(\omega(X, Y) := g(X, JY)\), with \(X, Y \in \Gamma(TM)\), denote the fundamental 2-form. We also denote by \(\nabla\) the Levi-Civita connection of \(\omega\).

**Definition 2.1:** A Hermitian manifold \((M, J, g)\) is called **Vaisman** if there exists a \(\nabla\)-parallel 1-form \(\theta\) satisfying \(d\omega = \theta \wedge \omega\).

**Remark 2.2:** The universal Riemannian cover of a Vaisman manifold is a metric cone of a Sasaki manifold (see e.g. [OrVe1]). Hence the local structure of a Vaisman manifold implies the existence of a local Sasaki structure transverse to the flow generated by the unitary vector field \(U\).

**Example 2.3:** All diagonal Hopf manifolds are Vaisman ([OrVe3]). The Vaisman compact complex surfaces are classified in [Be], see also [VeVuOr].

**Remark 2.4:** There exist compact LCK manifolds which do not admit Vaisman metrics. Such are the LCK Inoue surfaces, [Be], the Oeljeklaus-Toma manifolds, [Ot], and the non-diagonal Hopf manifolds, [OrVe3], [VeVuOr].

**Remark 2.5:** As \(\nabla \theta = 0\) implies \(d\theta = 0\), a Vaisman manifold is locally conformally Kähler (LCK) [DrOr]. The 1-forms \(\theta\) is called the **Lee form**.
The following is a fundamental observation due to I. Vaisman.

**Proposition 2.6:** ([Va]) The following equation holds on a Vaisman manifold:

\[ \omega = d\theta^c - \theta \wedge \theta^c. \] (2.1)

### 2.1 The canonical foliation of a Vaisman manifolds

Since the Lee form is parallel, it has constant norm and hence we can normalize it such that \( \|\theta\| = 1 \). We denote by \( \theta^c = \theta \circ J \) the **anti-Lee form**. Let \( U = \theta^c \) and \( V = \theta^c \) denote the respective g-dual vector fields:

\[ \theta(U) = 1, \quad \theta^c(V) = 1, \quad \theta(V) = 0, \quad \theta^c(U) = 0. \]

Note that the parallelism of \( \theta \) implies

\[ \nabla_U U = \nabla_V U = 0, \quad \nabla_U V = \nabla_V V = 0. \] (2.2)

**Remark 2.7:** On a Vaisman manifold, the Lee and anti-Lee fields are real holomorphic (\( \text{Lie}_U I = \text{Lie}_V I = 0 \)) and Killing (\( \text{Lie}_U g = \text{Lie}_V g = 0 \)), see [DrOr]. Moreover, they commute: \([U, V] = 0\). The 2-dimensional foliation \( \Sigma \) they generate is called the **canonical foliation**. Its leaves are totally geodesic.

**Remark 2.8:** Recall that a metric defined on a foliated manifold is said to be **bundle-like** if locally it can be identified with a Riemannian submersion [Re]. The Vaisman metric is thus bundle-like with respect to the canonical foliation [Va].

Consider the canonical splitting of the tangent bundle

\[ TM = Q \oplus \Sigma = Q \oplus (U) \oplus (V), \] (2.3)

where \( Q \) is the transverse g-orthogonal complement of \( \Sigma \). Let \( \pi_Q : TM \to Q \) be the canonical projection induced by the splitting. This also imply a splitting of the metric

\[ g = g^T \oplus g^\Sigma, \]

where \( g^T \) is the transverse metric, and \( g^\Sigma \) is the leafwise metric. By [Va], \( g^\Sigma \) is Euclidean and \( g^T \) the transverse metric, as in the case of a Riemannian submersion, can be locally projected on a local transversal (i.e. a local submanifold of dimension \( 2n \), transverse to the leaves of the foliation). Moreover, \( g^T \) is Kähler and hence \( \Sigma \) is **transversally Kähler**.

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\(^1\)We denote with \( \text{Lie}_X \) the Lie derivative along the vector field \( X \).
Along with the metric, we define other geometric objects that can be locally projected on submanifolds transverse to the leaves.

**Definition 2.9**: The de Rham complex of the basic (projectable) differential forms is defined as the restriction

\[ \Omega_b(M) := \{ \alpha \in \Omega(M) \mid \iota_X \alpha = 0, \text{ Lie}_X \alpha = 0 \text{ for any } X \in \Gamma(\Sigma) \}. \]

Here \( \iota \) stands for interior product.

**Definition 2.10**: The basic de Rham derivative is the restriction of the exterior derivative \( d \), namely \( d_b := d|_{\Omega_b(M)} \) (see e.g. [Ton]).

### 2.2 The transverse geometry of a Vaisman manifold

On a Vaisman manifold the complex structure \( J \) and the transverse fundamental 2-form \( \omega^T \) are projectable. In particular, \( \omega^T \) is a basic form, according to the above definition.

Since \( \Sigma \) is bundle-like, we choose an orthonormal frame \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}, U, V\} \) such that

\[
J(U) = V, \quad J(V) = -U, \\
J(e_j) = e_{n+j}, \quad J(e_{n+j}) = -e_j.
\]

The dual frame will be denoted \( \{e^{\flat}_1, \ldots, e^{\flat}_n, e^{\flat}_{n+1}, \ldots, e^{\flat}_{2n}, \theta, \theta^c\} \).

#### 2.2.1 Foliated charts on Vaisman manifolds

Consider a local foliated chart \( (U, z^j, \bar{z}^j, x, y) \). Here \( U = T \times O \), where:

- the local transversal \( T \) is a local chart \( (T, z^j, \bar{z}^j) \) on a complex Kähler manifold, \( z^j, \bar{z}^j \) are the transverse complex coordinates, \( z^j = x^j + iy^j, \bar{z}^j = x^j - iy^j \),
- \( x, y \) are the leafwise real coordinates defined by \( \frac{\partial}{\partial x} = U, \frac{\partial}{\partial y} = V \).

**Remark 2.11**: The differential forms \( dz^j, d\bar{z}^j \) are basic. Recall that \( X \in \Gamma(Q) \) is a basic vector field if \( [X, Y] \in \Gamma(Q) \), for any leafwise tangent vector field \( Y \in \Gamma(\Sigma) \) (see e.g. [Mo, Chapter 1]). It follows that the vector fields \( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j} \) dual to \( dz^j, d\bar{z}^j \) are basic.

#### 2.2.2 The complex structure in a foliated chart

The action of the transversal complex structure \( J_0 \) on the coordinate vector fields is: In the local map, the standard complex structure \( J_0 \) is defined by the relations

\[
J_0 \left( \frac{\partial}{\partial x^j} \right) = \frac{\partial}{\partial y^j}, \quad J_0 \left( \frac{\partial}{\partial y^j} \right) = -\frac{\partial}{\partial x^j},
\]
\[
J_0 \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial y}, \quad J_0 \left( \frac{\partial}{\partial y} \right) = -\frac{\partial}{\partial x}.
\]
As pointed out in the Remark 2.2, a Vaisman manifold is locally a Riemannian submersion, the fibers having a local structure of a Sasaki manifold (see e.g. [OrVe1]). The local structure of a Sasaki manifold using a local transverse Kähler potential is described in [GoKoNu] (see also [SmWaZh]). We consider a local Kähler potential associated to the transverse Kähler structure. The local function \( h \) is basic, constant along the leaves of the canonical foliation. Since the transverse Kähler structure of the Vaisman manifold is locally the same as the local Kähler structure of the Sasakian structure of the fibers, the complex structure \( J \) is expressed in terms of the potential as:

\[
J = J_0 + \frac{\partial}{\partial x} \otimes d^c (h) + \frac{\partial}{\partial y} \otimes d^c (h) \circ J_0
\]

In the coordinates of the foliated chart, the Lee and anti-Lee forms are:

\[
\theta = dx, \quad \theta^c = -\theta \circ J = dx + i \frac{\partial h}{\partial z^j} dz^j - i \frac{\partial h}{\partial \bar{z}^j} d\bar{z}^j.
\]

We introduce a new basis of local vector fields \( X_j, X_j^\perp, U, V \) in \( Q \), in which the complex structure acquire a simpler expression. Let

\[
X_j = \frac{\partial}{\partial z^j} - i \frac{\partial h}{\partial z^j} \frac{\partial}{\partial x}, \quad X_j^\perp = \frac{\partial}{\partial \bar{z}^j} + i \frac{\partial h}{\partial \bar{z}^j} \frac{\partial}{\partial x}.
\] (2.4)

A direct computation shows that

\[
J(X_j) = iX_j, \quad J(X_j^\perp) = -iX_j. \quad (2.5)
\]

We then work with the local basis of vector fields \( \{X_j, X_j^\perp, U, V\} \). The dual basis of 1-forms is \( \{dz^j, d\bar{z}^j, \theta, \theta^c\} \).

**Remark 2.12:** The local vector fields \( X_j, X_j^\perp \) and the dual one-forms \( dz^j, d\bar{z}^j \) are projectable on a local transversal.

Let \( g_{jk}^T := g(X_j, X_k) \) be the coefficients of the transverse metric in these new coordinates. We then have:

\[
g^T = g_{jk}^T dz^j \otimes d\bar{z}^k := g_{jk}^T (dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j).
\]

The associated transverse fundamental 2-form is \( \omega^T = ig_{jk}^T dz^j \wedge d\bar{z}^k \).

## 3 Deformation of a Vaisman structure using a differential basic 1-form

### 3.1 Description of the method

Let \( (M, J, g, \theta) \) be a closed Vaisman manifold, with real dimension \( 2n + 2 \). In the sequel we construct a deformation \( (M, J_t, g_t) \) of the Vaisman structure for \( t \in (-\varepsilon, \varepsilon) \), with
Let \( \{ \zeta_t \} \) be a family of differential 1-forms, which are basic with respect to \( \Sigma (\zeta_t(U) = \zeta_t(V) = 0, \text{Lie}_U \zeta_t = \text{Lie}_V \zeta_t = 0) \), with \( \zeta_0 = 0 \). Later on, we shall add new conditions on the 1-forms \( \{ \zeta_t \} \).

We fix \( U_t = U, V_t = V \). This ensures that \( \Sigma \) is not affected by the deformation.

Consider the canonical splitting \((2.3)\) of the tangent bundle. Now deform the normal bundle \( Q \) to \( Q_t = \{ v - \zeta_t(v) \mid v \in Q \} \).

and associate the morphism of vector (sub-)bundles
\[
\pi_t : Q \to Q_t, \quad \pi_t(v) = v - \zeta_t(v)V, \quad v \in Q.
\]

Then
\[
\pi_t^{-1}(w) = w + \zeta_t(w)V, \quad w \in Q_t.
\]

Define \( J_t \) on \( \Sigma \) by
\[
J_t(U) = V, \quad J_t(V) = -U.
\]

and on \( Q_t \) by \( J_t|_{Q_t} = \pi_t \circ J \circ \pi_t^{-1} \). Then
\[
J_t|_{Q_t}(w) = \pi_t \circ J \circ \pi_t^{-1}(w)
= \pi_t \circ J(w + \zeta_t(w)V)
= J(w + \zeta_t(w)V) - \zeta_t(J(w + \zeta_t(w)V))
= J(w) - \zeta_t(w)(J(w)V).
\]

Written in a compact way, the deformation \( J_t \) reads
\[
J_t = J - U \otimes \zeta_t - V \otimes (\zeta_t \circ J). \tag{3.1}
\]

**Claim 3.1:** \( J_t^2 = J^2 = -\text{Id} \). Hence \( J_t \) is an almost complex structure for all \( t \in \mathbb{R} \).

**Proof.** Since \( \zeta_t \) is a basic 1-form, we have
\[
J_t^2 = (J - U \otimes \zeta_t - V \otimes (\zeta_t \circ J))^2
= J^2 - J(U) \otimes \zeta_t - J(V) \otimes \zeta_t \circ J - U \otimes \zeta_t \circ J + V \otimes \zeta_t
= J^2 - \text{Id}.
\]

**Proposition 3.2:** If \( d\zeta_t \) are of type \((1,1)\) with respect to \( J \), then \( J_t \) is integrable for all \( t \in \mathbb{R} \).
Proof. Let $N^{J_t}$ be the Nijenhuis tensor field associated to $J_t$:

$$N^{J_t}(\cdot, \cdot) = J_t^2(\cdot, \cdot) - J_t J_t(\cdot, \cdot) - J_t(\cdot, J_t \cdot) + [J_t, J_t]$$

We show that $N^{J_t}$ vanishes. It is enough to show that it vanishes on basic vector fields. Let then $X$ and $Y$ be basic vector fields (we include here the case when $X$, $Y$ are tangent to the leaves). Then $[U, X]$, $[U, Y]$, $[V, X]$ and $[U, Y]$ are also tangent to the leaves of the foliation. As $\zeta_t$ is basic, we have

$$N^{J_t}(X, Y) = N^J(X, Y) - d\zeta_t(JX, Y)U - d\zeta_t(X, JY)V + d\zeta_t(X, JY)V - \zeta_t(X)(\text{Lie}_U J(Y)) - \zeta_t(J(X))(\text{Lie}_V J)(Y) + \zeta_t(J(Y))(\text{Lie}_V J)(X) - \zeta_t(N^J(X, Y)).$$

Recall the relations

$$\text{Lie}_U J = \text{Lie}_V J = 0.$$  \hfill (3.3)

Since $d\zeta_t$ is of type (1,1) and $J$ is integrable, by (3.3) we obtain

$$N^{J_t}(X, Y) = 0,$$

for $X$ and $Y$ basic vector fields. \hfill \blacksquare

From now on, we assume that $d\zeta_t$ are of type (1,1) w.r.t $J$.

We now choose $\theta_t = \theta$, so $\theta_t^c = \theta \circ J_t = \theta_t + \zeta_t$ is the deformation of anti-Lee 1-form. One can easily check

$$\theta_t^c(V) = 1, \quad \theta_t^c(U) = 0.$$

Remark 3.3: Recall (Proposition 2.6) that on a Vaisman manifold,

$$d\theta^c = \omega + \theta \wedge \theta^c.$$ \hfill (3.4)

Therefore $(d\theta^c)^n \wedge \theta \wedge \theta^c$ is a volume form (where $\dim \mathbb{R} M = 2n + 2$). We need that the form $(d\theta_t^c)^n \wedge \theta_t^c \wedge \theta$ remain a volume form during the deformation process (see also [BoGa, pag. 447]). This is not automatic, hence we add a condition on $\{\zeta_t\}_t$ such as to fulfill this requirement.

We have $\|\theta\| = \|\theta_t\| = \sqrt{n}$ w.r.t. the initial metric $g$. In the expansion

$$(d\theta_t^c)^n \wedge \theta \wedge \theta_t^c = \sum_{k=1}^n \binom{n}{k} (d\theta_t^c)^{n-k} \wedge \zeta_t^k \wedge \theta \wedge \theta^c + \sum_{k=1}^n \binom{n}{k} (d\theta_t^c)^{n-k} \wedge d\zeta_t^k \wedge \theta \wedge \zeta_t$$
there are $2^{n+1}$ terms in the right hand side. If we choose $\zeta_t$ such that
\[
\|\zeta_t\| < \frac{1}{(2^{n+1} - 1)n^{n+1}}, \quad \|d\zeta_t\| < \frac{1}{(2^{n+1} - 1)n^{n+1}},
\] (3.5)
then, using the Cauchy-Buniakovski-Schwarz inequality $\|\alpha^1 \wedge \alpha^2\| \leq \|\alpha^1\| \cdot |\alpha^2|$ for differential forms $\alpha^1, \alpha^2$, we get the estimate
\[
\|(d\theta^c_t)^n \wedge \theta \wedge \theta^c_t\| \geq 1 - \sum_{k=2}^{n} \binom{n}{k} \|d\theta^c_t\|^{n-k} \cdot \|\theta\| \cdot \|\theta^c_t\|
\]
\[
- \sum_{k=1}^{n} \binom{n}{k} \|d\theta^c_t\|^{n-k} \cdot \|d\zeta^k_t\| \cdot \|\theta\| \cdot \|\zeta_t\|
\]
\[
> 1 - (2^{n+1} - 1) \frac{1}{2^{n+1} - 1} > 0.
\]
Then $(d\theta^c_t)^n \wedge \theta \wedge \theta^c_t$ remains a volume form.
In conclusion, to assure (3.5) it is enough to divide any given $\zeta_t$ by a constant greater than $(2^{n+1} - 1)n^{n+1} \max\{\|\zeta_t\|, \|d\zeta_t\|\}.

In the sequel we assume that $\zeta_t$ fulfill the inequalities (3.5).

Claim 3.4: Let $\omega_t$ be the differential 2-form
\[
\omega_t = d\theta^c_t - \theta \wedge \theta^c_t = d\zeta_t - \theta \wedge \theta^c_t.
\]
Then $\omega_t$ is of type (1,1) w.r.t. $J_t$.

Proof. It will be enough to verify the claim on basic 1-forms.

Step 1: $J_t = J$ on basic differential forms. Let $\alpha$ be a basic 1-form. Then
\[
J_t(\alpha)(v) = -\alpha(Jv - \zeta_t(v)U - \zeta_t(Jv)V) = -\alpha(Jv) = J(\alpha)(v),
\]
for any $v \in \Gamma(TM)$, since $\alpha(U) = \alpha(V) = 0$.

Step 2: $d\zeta_t$ and $d\theta^c_t$ are of type (1,1) w.r.t. $J_t$. Indeed, since $\zeta_t$ are basic and $[\text{Lie}, d] = 0$, we see that $d\zeta_t$ are basic too. By assumption, $d\zeta_t$ are of type (1,1) w.r.t. the complex structure $J$. Then by Step 1 they are of type (1,1) w.r.t. $J_t$, for all $t \in \mathbb{R}$.

By (3.4), $d\theta^c_t$ is basic and of type (1,1) w.r.t. $J_t$, and hence $\omega_t$ is a sum of (1,1) 2-forms.

Now, since $\theta$ is closed, we have
\[
d\omega_t = \theta \wedge d\theta^c_t = \theta \wedge \omega_t,
\] (3.6)
and hence

Claim 3.5: The manifold $(M, \omega_t, \theta)$ is locally conformally symplectic for all $t \in \mathbb{R}$.
We define a symmetric (0,2) tensor field by
\[ g_t = \omega_t \circ (J_t \otimes \text{Id}) \]
\[ = d\theta^c \circ (J_t \otimes \text{Id}) + d\zeta_t \circ (J_t \otimes \text{Id}) + \theta^c_t \otimes \theta^c_t + \theta \otimes \theta \quad (3.7) \]
In the last equality we used the fact that \( d\theta^c, d\zeta_t \) are basic differential forms. Note that
\[ \theta^c_t \otimes \theta^c_t = \theta^c \otimes \theta^c + \zeta_t \otimes \theta^c + \theta^c \otimes \zeta_t + \zeta_t \otimes \zeta_t. \]
The tensor field \( g_t \) is related to the metric tensor \( g \) by the formula:
\[ g_t = g + d\zeta_t \circ (J \otimes \text{Id}) + \zeta_t \otimes \theta^c + \theta^c \otimes \zeta_t + \zeta_t \otimes \zeta_t. \quad (3.8) \]

**Remark 3.6:** We want \( g_t \) to be a metric such as to regard (3.7) as a deformation of the initial metric \( g \). To this end, we need to find a condition which implies the positivity of the tensor field \( g_t \). At any point \( x \in M \) consider \( S^1_vM := \{v \in T_xM \mid \|v\| = 1 \} \), where the norm is taken w.r.t. the metric \( g \). Define
\[ \mu_x := \min_{v \in S^1_vM} d\zeta(Jv, v). \]
As \( S^1_vM \) is a compact set, \( \mu_x > -\infty \). Moreover, \( \mu_x \) is a smooth function on the compact manifold \( M \), and attains its minimum at a certain point. We make the assumption that
\[ \min_{x \in M} \mu_x > -1. \quad (3.9) \]
Note that by (3.7), \( g_t(v, v) > 0 \) for all \( v \) tangent to the leaves of the canonical foliation.

For a transverse unit vector field \( v \in Q_{t,x} \), we have
\[ g_t(v, v) = d\theta^c(Jv, v) + d\zeta(Jv, v) = 1 + d\zeta(Jv, v) > 0 \]
by (3.9) and since \( d\theta^c(Jv, v) \) is the transverse part of the metric \( g \). Hence (3.9) is a sufficient condition for \( g_t \) to be positive definite.

**Remark 3.7:** Let \( c > \min_{x \in M} \mu_x \). Then, for any given \( \zeta_t \), the form \( c \cdot \zeta \) satisfies the inequality (3.9).

In the following we assume that \( \zeta_t \) satisfy the inequality (3.9).

The form \( \zeta_t \) is basic, and hence \( \text{Lie}_U \zeta_t = 0, \text{Lie}_U d\zeta_t = 0 \); then from (3.8) we obtain
\[ \text{Lie}_U g_t = 0, \quad (3.10) \]
so \( U \) stays Killing w.r.t. all the metrics \( g_t \).
A straightforward computation shows that $U$ and $V$ are vector fields metrically equivalent to $\theta$ and $\theta^c$, with respect to the metric $g_t$. Also,

$$\|\theta\|_{g_t} = \|\theta^c\|_{g_t} = 1. \tag{3.11}$$

Gathering together the above facts we obtain:

**Theorem 3.8:** Let $(M, J, g, \theta)$ be a Vaisman manifold and $J_t, g_t$ given by (3.1) and (3.8). Assume that the basic 1-forms $\zeta_t$ satisfy (3.5) and (3.9) and $d\zeta_t$ are of type $(1,1)$ w.r.t. $J$. Then $(M, J_t, g_t)$ is a Vaisman structure with Lee form $\theta$ for all $t \in \mathbb{R}$.

### 3.2 Construction of the basic forms $\zeta_t$

We start with a globally defined, basic function $\varphi$ (w.r.t. the canonical foliation). Let $d^c_b := Jd_bJ^{-1}$ be the corresponding complex operator acting on basic $k$-forms. By Step 1 in the proof of Claim 3.4, $d^c_b$ remains unchanged by the deformation of the complex structure, so this operator can be taken w.r.t. the complex structure $J$.

Let $\zeta_t := t d^c_b \varphi$.

Clearly $t d^c_b \varphi$ is a basic 1-form of type $(1,1)$, and we take $t$ small enough such that $t d^c_b \varphi$ satisfies the inequalities (3.5) and (3.9). Then, according to Theorem 3.8, the family of metrics $g_t$ defined by

$$g_t = g + t (d^c_b \varphi \circ (J \otimes \text{Id}) + d^c_b \varphi \otimes \theta^c + \theta^c \otimes d^c_b \varphi) + t^2 d^c_b \varphi \otimes d^c_b \varphi. \tag{3.12}$$

represent a deformation of the initial Vaisman metric $g$.

### 3.3 Example: deforming a diagonal Hopf surface

We describe the geometry of a classical Hopf surface as it is useful for our deformation process. For the reader’s convenience, we provide the explicit computations.

Let $\Delta$ the cyclic group generated by the holomorphic transformation $z \mapsto e^{2z}$ on $\mathbb{C}^2 \setminus \{0\}$. The quotient space $H := (\mathbb{C}^2 \setminus \{0\}) / \Delta$, is a (primary) Hopf surface.

To describe the foliated coordinates, consider the Hopf fibration $S^3 \to \mathbb{C}P^1$ and let $w := u + iv$ be the complex local coordinate on $S^2 \simeq \mathbb{C}P^1$. The standard Kähler metric on the space of leaves $S^2$ is determined by the local potential $\frac{1}{2} \log(1 + |w|^2)$. Let $x$ be the coordinate on the factor $S^1$ and $y$ the coordinate of the fibre of the Hopf fibration. Then $(w, x, y)$ are local coordinates on the foliated manifold (with $w$ the transverse coordinate).

In the local foliated chart, the standard complex structure $J_0$ is defined by the relations

$$J_0 \left( \frac{\partial}{\partial u} \right) = \frac{\partial}{\partial v}, \quad J_0 \left( \frac{\partial}{\partial v} \right) = - \frac{\partial}{\partial u},$$

$$J_0 \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial y}, \quad J_0 \left( \frac{\partial}{\partial y} \right) = - \frac{\partial}{\partial x}.$$
Define the almost complex structure $J$ on $H$ as
\[
J = J_0 + \frac{\partial}{\partial x} \otimes dx \left( \frac{1}{2} \log(1 + |w|^2) \right) + \frac{\partial}{\partial y} \otimes dx \left( \frac{1}{2} \log(1 + |w|^2) \right) \circ J_0.
\]

By Proposition 3.2, $J$ is integrable. Since $\left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] = 0$ and the Kähler potential is basic, $J$ is invariant along the leaves:
\[
\operatorname{Lie}_{\frac{\partial}{\partial x}} J = \operatorname{Lie}_{\frac{\partial}{\partial y}} J = 0.
\]

In these local coordinates, the Lee form is $\theta = dx$, and anti-Lee is
\[
\theta^c = -\theta \circ J
\]
\[
= -dx \left( J_0 + \frac{\partial}{\partial x} \otimes dx \left( \frac{1}{2} \log(1 + |w|^2) \right) \right) + \frac{\partial}{\partial y} \otimes dx \left( \frac{1}{2} \log(1 + |w|^2) \right) \circ J_0
\]
\[
= -dx \circ J_0 + d\theta^c \left( \frac{1}{2} \log(1 + |w|^2) \right)
\]
\[
= dy - \frac{i}{2} \frac{w}{1 + |w|^2} dw + \frac{i}{2} \frac{\bar{w}}{1 + |w|^2} dw.
\]

Then the fundamental 2-form reads
\[
\omega = d\theta^c - \theta \wedge \theta^c
\]
\[
= -dd^c \left( \frac{1}{2} \log(1 + |w|^2) \right) - dx \wedge \left( dy - \frac{i}{2} \frac{w}{1 + |w|^2} dw + \frac{i}{2} \frac{\bar{w}}{1 + |w|^2} dw \right)
\]
\[
= -i \frac{1}{(1 + |w|^2)^2} dw \wedge d\bar{w} + dx \wedge dy + \frac{i}{2} \frac{w}{1 + |w|^2} dx \wedge d\bar{w}
\]
\[
- \frac{i}{2} \frac{\bar{w}}{1 + |w|^2} dx \wedge dw.
\]

The corresponding metric is $g = \omega \circ (J \otimes \text{Id})$. It is positive definite since the transverse part is the standard Kähler metric on $S^2$ while on leaves it is the Euclidean metric. Using (3.13) one easily proves that
\[
\operatorname{Lie}_{\frac{\partial}{\partial x}} g = \operatorname{Lie}_{\frac{\partial}{\partial y}} g = 0,
\]
and hence the metric is bundle-like with respect to the foliated structure. Since
\[
\left\| \frac{\partial}{\partial x} \right\| = \left\| \frac{\partial}{\partial y} \right\| = 1,
\]
the metric $g$ is Vaisman.

We deform the Vaisman structure, using a smooth basic function defined on our foliation, as indicated in Subsection 3.2. Let
\[
f(w, x, y) = \frac{1}{2} \left( \frac{1 - |w|^2}{1 + |w|^2} \right)^2,
\]

\[
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and take \( \zeta_t = t \cdot f \).

**Remark 3.9:** On the manifold \( S^3 \), consider the standard parametrization

\[
[z^1, z^2] = [\cos s e^{i\Phi_1}, \sin s e^{i\Phi_2}].
\]

One can see that \( f = \frac{1}{2} \cos^2 s \) is a basic, globally defined function on \( S^3 \) (see e.g. [Slo, Example 1]); the same holds on the Hopf manifold \( H \).

The family of complex structures constructed as in Subsection 3.1 is the following:

\[
J_t = J - \frac{\partial}{\partial x} \otimes d^c \left( \frac{1}{2} t^2 \left( \frac{1 - |w|^2}{1 + |w|^2} \right)^2 \right) - \frac{\partial}{\partial y} \otimes d^c \left( \frac{1}{2} t^2 \left( \frac{1 - |w|^2}{1 + |w|^2} \right)^2 \right) \circ J.
\]

The Lee form remains fixed, \( \theta_t = \theta = dx \), and the anti-Lee 1-form is

\[
\theta_t^c = -\theta \circ J_t
\]

\[
= -dx \left( J - \frac{\partial}{\partial x} \otimes d^c \left( \frac{1}{2} \left( \frac{1 - |w|^2}{1 + |w|^2} \right)^2 \right) - \frac{\partial}{\partial y} \otimes d^c \left( \frac{1}{2} t^2 \left( \frac{1 - |w|^2}{1 + |w|^2} \right)^2 \right) \circ J \right)
\]

\[
= -dx \circ J + d^c \left( \frac{1}{2} \log(1 + |w|^2) \right)
\]

\[
= dy - \frac{i}{2} \left( \frac{w}{1 + |w|^2} - t \frac{1 - |w|^2}{(1 + |w|^2)^3} \right) d\overline{w} + \frac{i}{2} \left( \frac{\overline{w}}{1 + |w|^2} - t \frac{1 - |w|^2}{(1 + |w|^2)^3} \right) dw.
\]

Finally, we describe the deformation of the fundamental 2-form:

\[
\omega_t = d\theta_t^c - \theta \wedge \theta_t^c
\]

\[
= -i \frac{\left( (1 - t) + (2 + 4t) |w|^2 + (1 - t) |w|^4 \right)^2}{(1 + |w|^2)^8} d\overline{w} \wedge dx \wedge dy
\]

\[
+ \frac{i w(1 + \|w\|^2 - t(1 - \|w\|^2))}{2(1 + \|w\|^2)^3} dx \wedge d\overline{w}
\]

\[
- \frac{i \overline{w}(1 + \|w\|^2 - t(1 - \|w\|^2))}{2(1 + \|w\|^2)^3} dx \wedge dw.
\]

If \( M_t \) is the skew-symmetric matrix associated to \( \omega_t \) with respect to the local basis \((du, dv, dx, dy)\), then

\[
\det M_t = \frac{\left( (1 - t) + (2 + 4t) |w|^2 + (1 - t) |w|^4 \right)^2}{(1 + |w|^2)^8}.
\]
Taking \( t \in (-\frac{1}{2}, \frac{1}{2}) \) we see that all terms that appear are strictly positive and \( \det M_t \neq 0 \) for all \( w \). Hence \( \omega_t \) is non-degenerate for all \( t \in (-1/2, 1/2) \). Define the family of metrics \( g_t = \omega_t \circ (J_t \otimes \text{Id}) \).

**Claim 3.10:** \( g_t \) is positive definite.

**Proof.** Denote by \( \{\lambda_i(t, w, x, y)\}_{1 \leq i \leq n} \) the eigenvalues associated to \( g_t \) at the point of coordinates \((w, x, y)\). As \( g_t \) is not degenerate, the smooth eigenfunctions \( \lambda_i \) are nowhere vanishing, so they have constant sign. Moreover, \( \lambda_i > 0 \) since \( \lambda_i(0, w, x, y) > 0 \), hence the claim is proved.

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