The magnetic monopole and the separation between fast and slow magnetic degrees of freedom

J-E Wegrowe¹ and E Olive²

¹ LSI, Ecole Polytechnique, CEA, CNRS, Université Paris-Saclay, 91128 Palaiseau Cedex, France
² GREMAN, UMR 7347, Université François Rabelais-CNRS, Parc de Grandmont, 37200 Tours, France

E-mail: jean-eric.wegrowe@polytechnique.edu

Received 13 November 2015, revised 6 January 2016
Accepted for publication 11 January 2016
Published 12 February 2016

Abstract
The Landau–Lifshitz–Gilbert (LLG) equation that describes the dynamics of a macroscopic magnetic moment finds its limit of validity at very short times. The reason for this limit is well understood in terms of separation of the characteristic time scales between slow degrees of freedom (the magnetization) and fast degrees of freedom. The fast degrees of freedom are introduced as the variation of the angular momentum responsible for the inertia. In order to study the effect of the fast degrees of freedom on the precession, we calculate the geometric phase of the magnetization (i.e. the Hannay angle) and the corresponding magnetic monopole. In the case of the pure precession (the slow manifold), a simple expression of the magnetic monopole is given as a function of the slowness parameter, i.e. as a function of the ratio of the slow over the fast characteristic times.

Keywords: geometrical angle, magnetic monopole, magnetization dynamics, inertial regime of the magnetization

(Some figures may appear in colour only in the online journal)
the magnetization. The three concepts are coupled because the dynamics of a magnetic dipole are composed of both fast and slow dynamics, and the geometric phase is an efficient tool for the study of the separation of time-scales between slow and fast degrees of freedom [47, 48]. The influence of the fast variables on the slow motion is treated in perturbation expansions [49] in which the ratio of small and fast time scales define a slowness parameter, and the successive terms are interpreted as reaction forces of the fast variables on the slow motion [50, 51].

The magnetization \( \mathbf{M} \) of a uniformly magnetized body is usually defined as a magnetic dipole. The description of the dynamics of a classical magnetic dipole is however still problematic today [52]. Ampere’s magnetic dipole is defined by an electric charge that is moving at high speed about a microscopic ‘loop’, typically an atomic orbital. This simple model allows the gyromagnetic relation to be derived: the magnetization \( \mathbf{M} \) of the dipole then follows the angular momentum \( \mathbf{L} \) of the electric carrier, with the relation \( \mathbf{M} = \gamma \mathbf{L} \)

where \( \gamma = g q / (2 m) \) is the gyromagnetic ratio \( (m \) is the mass and \( q \) is the electric charge of the electric carrier, and \( g \) is the Landé factor).

If a static magnetic field \( H \) (oriented along \( \hat{e}_z \)) is applied, the magnetization precesses at the Larmor angular velocity \( \Omega_L \) around the axis defined by \( \hat{e}_z \). In other terms, a slow motion (precession) is added to the fast motion (moving electric carrier) that defines the magnetic dipole. In the absence of dissipation the dynamics of the dipole are reduced to a simple precessional motion. However, this reduction is valid only if the velocity of the electric charge is much higher than the precession velocity, i.e. if the typical time-scales are well separated.

Indeed, if the Larmor angular velocity is high enough and becomes of the same order as the angular velocity of the electrical carrier moving in the loop, the Amperian magnetic dipole \( \mathbf{M} \) is no longer defined by a simple expression (the exact trajectory of the punctual electric carrier should be taken into account instead of averaging over the loop) [53, 54].

However there is another way to define a magnetic dipole, namely the Gilbert’s dipole (according to D. J. Griffiths, the Gilbert dipole is a double monopole [55]). In our non-relativistic context, the Gilbert magnetic dipole is defined by its dynamical properties, based on the mechanical analogy with the spinning top [24]. This mechanical approach allowed T. H. Gilbert to derive the well-known Landau–Lifshitz–Gilbert equation (LLG), providing that the first two principal moments of inertia vanish \( I_1 = I_2 = 0 \), but not the third one \( I_3 \neq 0 \) [25].

This ad-hoc assumption is related to the electrodynamic limitation of the Amperian magnetic dipole mentioned above.

In this context, fast degrees of freedom have been taken into account as inertial variables (so that \( I_1 = I_2 = 0 \)) by enlarging the configuration space to the corresponding phase space, i.e. including the angular momentum. The corresponding generalized LLG equation then contains a supplementary term proportional to the second time-derivative of the magnetization [39–45].

In the present work, we show that the Hannay angle and the corresponding magnetic monopole are able to describe, in the adiabatic limit, the transition from the usual precession to more complex dynamics containing the inertial effects. The analysis follows the method recently proposed by Berry and Shukla in [51] for the study of the spinning top. Within this approach, the dynamics of the magnetization are interpreted as the reaction of the fast dynamics on the slow. A simple analytical result is obtained by reducing the phase space to the slow manifold.

The paper is composed as follows. Section 1 below is devoted to the mechanical definition of the adiabatic Gilbert dipole without taking into account the fast degrees of freedom. Section 2 describes the adiabatic kinetic equation. The geometric phase is presented in section 2.3, and the corresponding magnetic monopole is described in section 2.4. Section 3 studies the effect of the fast degrees of freedom. In particular, the calculation of the adiabatic dynamics of the magnetization that includes inertia is presented in section 3.1, and the calculation of the geometric phase with inertia is given in section 3.2. The case of the pure precession is studied in section 3.3, and the corresponding magnetic monopole is given in section 3.4. The conclusion is proposed in section 4.

1. The Gilbert magnetic dipole

Gilbert’s mechanical model is sketched in figure 1. A rigid cylindrical stick of length \( M_s \), with one end fixed at the origin, is pointing in a direction described by the angles \( \theta \) and \( \varphi \). The magnetization is aligned along the effective magnetic field \( H_z \) at equilibrium. Due to the application of a vertical
force oriented along the \( z \) axis, the stick is precessing around the vertical axis at angular velocity \( \psi \). The magnetic energy is \( V^e = -\mathbf{M} \cdot \mathbf{H} \) where \( \mathbf{H} \) is the effective field and \( \mathbf{M} = M e_3 \) is the magnetization (\( e_3 \) is the unit vector defined in figure 1). Furthermore, the stick is spinning around its own symmetry axis at angular velocity \( \psi \). This motion corresponds to the rotation of the electric carrier of Ampere’s dipole (see below). The phase space of this rigid rotator is defined by the angles \( \{ \theta, \varphi, \psi \} \) and the three components of the associated angular momentum \( \mathbf{L} \). The relation between the angular momentum and the angular velocity \( \Omega \) is \( \mathbf{L} = \tilde{I} \Omega \), where \( \tilde{I} \) is the inertia tensor.

In the rotating frame, or body-fixed frame \( \{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \} \), the inertial tensor is reduced to the principal moments of inertia \( \{ I_1, I_2, I_3 \} \). The symmetry of revolution of the spinning stick imposes furthermore that \( I_1 = I_2 \):

\[
\tilde{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}
\] (1)

In the fixed body frame, the angular velocity reads (see figure 1):

\[
\Omega_1 = \psi \sin \theta \sin \psi + \theta \cos \psi, \\
\Omega_2 = \psi \sin \theta \cos \psi - \theta \sin \psi, \\
\Omega_3 = \psi \cos \theta + \psi.
\] (2-4)

The kinetic equation is obtained from the angular velocity: for any vector \( \mathbf{M} \) of constant modulus carried with the rotating body, we have:

\[
\frac{d\mathbf{M}}{dt} = \tilde{\Omega} \times \mathbf{M}.
\] (5)

2. The kinetic equation

2.1. Gyromagnetic relation

Let us start with Gilbert’s hypothesis of vanishing inertia [39]: \( I_1 = I_2 = 0 \) so that \( L_1 = L_2 = 0 \). However, we have \( L_3 = I_3 \Omega_3 \neq 0 \). Since \( \tilde{L} = \tilde{L} \tilde{e}_3 \), the conservation of angular momentum \( \tilde{L} \) imposes \( L_3 \) constant (this is also valid in the case of damping [39]). Without loss of generality, we can define the modulus of the vector \( \mathbf{M} \) with the help of the constant \( \gamma \), such that

\[
M_3 = \gamma L_3 = \gamma \Omega_3 I_3
\] (6)

where \( \gamma \) defines the well-known gyromagnetic ratio.

2.2. Precession equation without damping

The effective magnetic field is defined by the canonical relation \( \mathbf{H} = -\nabla_2 V^e \) where \( \nabla_2 \equiv \partial / \partial \mathbf{M} \) is the gradient defined on the configuration space \( \Sigma \) (which is the surface of the sphere of radius \( M_3 \)). The torque exerted on the system is defined by the vectorial product \( \Gamma = \mathbf{M} \times (-\nabla_2 V^e) = \mathbf{M} \times \mathbf{H} \). By convention, we defined the direction \( \tilde{e}_z \) along the effective field \( \tilde{H} = H \tilde{e}_z \). The third Newton’s law \( d\mathbf{L}/dt = \Gamma \) gives then the kinetic equation of the magnetization:

\[
\frac{dM_3}{dt} = -\frac{M_3 H_3}{L_3} (\tilde{e}_3 \times \tilde{e}_z).
\] (7)

According to the gyromagnetic relation equation (6), we have \( M_3 L_3 = \gamma H_3 \) and equation (7) is nothing but the well-known equation of the precession of the magnetization without damping \( \frac{dM_3}{dt} = \gamma (\tilde{M} \times \tilde{H}) \). Furthermore, since \( \frac{d\tilde{e}_3}{dt} = \tilde{\Omega} \times \tilde{e}_3 \), the kinetic equation reads \( \Omega \times \tilde{e}_3 = \frac{M_3 H_3}{L_3} (\tilde{e}_3 \times \tilde{e}_z) \). Inserting the precession angular velocity \( \Omega_\varphi = \psi \), we have:

\[
\Omega_\varphi = \psi = \frac{M_3 H_3}{L_3} = -\gamma H_3,
\] (8)

which is the definition of the Larmor angular velocity, as expected for a precessing magnetic moment.

2.3. The geometric phase

The geometric phase is the phase difference acquired over the course of a precession loop. The precession time \( t_\psi \) (i.e. the slow characteristic time of our problem) is the time at which the slow over the fast angular momentum, or equivalently of the slow over the fast time-scale:

\[
\text{the slow over the fast angular momentum, or equivalently of the slow over the fast time-scale}:
\]

\[
G = \frac{L_3}{\sqrt{I_0 M_3 H_3}} = \frac{1}{\gamma H_3} \sqrt{\frac{M_3 H_3}{I_0}} \equiv \frac{1}{2\pi t_0}.
\] (11)

The last term in the right-hand side of equation (11) defines the fast characteristic time \( t_0 = \sqrt{I_0/(M_3 H_3)} \) of the motion.

The expression of \( \Delta \Psi_0 \) now reads:

\[
\Delta \Psi_0 = 2\pi \left( \frac{I_1 G^2 + \cos \theta}{I_3} \right).
\] (12)
The first term in the right hand side can be defined as the dynamical angle, while the second term $2\pi \cos \theta$ can be defined as the geometric phase (see however the discussion in reference [51]). Note that the factor $(4\pi/\hbar)G^2 = \Omega_3/\gamma_3\hbar$ also defines a time ratio $t/f$, where $t_f = 2\pi/\Omega_3$ is another possible fast characteristic time of the movement. This parameter will be discussed below. The expression equation (12) is completed in section 4 below, in the case of inertia, with an expansion as a series of power of $\cos \theta$.

2.4. The magnetic monopole

From the viewpoint of the geometric phase, the Gilbert’s magnetic dipole is defined by the two magnetic monopoles $\pm B_{\text{eff}}$ that radiate from the center of a sphere of radius $R$ through both north (+) and south (-) hemispheres. The parameter $R$ is defined by Ampère’s magnetic dipole $\vec{M} = \gamma \vec{L}$ that is generated by the electric carrier of charge $q$ and mass $m$ rotating inside the loop of radius $R$. The phase $\Delta \Psi_0$ then allows to link the mechanical definition of Gilbert’s magnetic dipole to Ampère’s magnetic dipole. If we define the radial field $B_{\text{eff}} = B_{\text{eff}} \hat{r}_3$ by a potential vector $\vec{A} = \hat{r}_3 \times \vec{B}_{\text{eff}}$, the circulation of $\vec{A}$ around a closed loop of radius $R$ defines a phase [1]

$$\Delta \Psi_0 \equiv \oint \vec{A} \cdot d\vec{l} = \oint B_{\text{eff}} \hat{r}_3 \cdot dS = \pi R^2 B_{\text{eff}},$$

which is the geometric phase calculated above. Equation (13) and equation (10) gives the expression of $B_{\text{eff}}$:

$$B_{\text{eff}} = \frac{2}{R^2} \left( \frac{M_3}{\gamma^2 I_3 H_e} + \cos \theta \right)$$

On the other hand, in the framework of the Ampère’s model of the ‘molecular currents’, a microscopic magnetic moment is defined by the Bohr magneton $M_3 = \mu_B = \gamma \hbar$ generated by an electron of mass $m$ and charge $q$ moving in a loop of Bohr radius $R$. The gyromagnetic ratio is $\gamma = q/(2m)$ and the moment of inertia associated to the loop of radius $R$ is $I_3 = mR^2$. Furthermore, the flux $\Phi_0 = \int H_e dS$ of the external magnetic field (by convention along $Oz$) $\vec{H} = H_e \hat{z}$ through the microscopic hemisphere of radius $R$ is also quantified, with the well-known quantized flux :

$$\Phi_0 = H_e \pi R^2 \cos \theta = \frac{\hbar}{q}$$

Equation (14) then reads :

$$B_{\text{eff}} = \frac{4}{R^2} \cos \theta$$

This expression defines the classical counterpart of the magnetic monopole [1, 3, 4, 56, 57]. Note that the corresponding geometric phase equation (12) reduces to : $\Delta \Psi_0 = 4\pi \cos \theta$.

3. The effect of inertia

3.1. Inertial equation of the magnetization without damping

The scalar gyromagnetic relation equation (6) used above in the framework of the mechanical (or Gilbert’s) model of the magnetic dipole coincides with the usual vectorial definition $\vec{M} = \gamma \vec{L}$ of the gyromagnetic relation if the inertial effects are neglected $I_1 = I_2 = 0$. If we take into account inertial effects, $I_1 = I_2 \neq 0$, the gyromagnetic relation $\vec{M} = \gamma \vec{L}$ is no longer valid in this form. The generalized equation is obtained, by cross- multiplication of equation (5) with the vector $\vec{M} = M_e \hat{e}_3$.

$$\dot{\Omega} = \frac{M_3}{M_e^2} \times \frac{dM_3}{dt} + \Omega_3 \hat{e}_3,$$

or:

$$L = \frac{I_3}{M_e^2} \left( \frac{dM_3}{dt} \right) + L_3 \hat{e}_3.$$  

Newton’s law $d\hat{L} / dt = \vec{M} \times \vec{H}_{\text{eff}}$ becomes, with the constant $L_3 = M_3 / \gamma$

$$\frac{d\hat{e}_3}{dt} = \gamma H_e \hat{e}_3 \times \left( \hat{e}_3 - \frac{\gamma}{\gamma M_3} \frac{d^2 \hat{e}_3}{dt^2} \right).$$

Equations (20) becomes

$$\theta'' \sin \theta - \theta' \cos \theta = G \theta' - 2\theta' \cos \theta \sin \theta - \sin \theta$$

$$\Theta'' \sin \theta = G \Theta' - 2\Theta' \cos \theta$$

This equation is the dynamical equation of the magnetization generalized to inertial effects (in the absence of damping). These equations allow the adiabatic movement to be studied below in terms of the geometric phase. The generalized equation including Gilbert damping has been studied in previous reports [40, 42, 44].

3.2. The geometric phase with inertia

The number $\Delta \Psi$ of rotation around the $\hat{e}_3$ axis performed by the magnetization vector during the (dimensionless) time $\tau = t/(2\pi \Omega_3)$ of one precession is :

$$\Delta \Psi = \int_0^{\tau} \Psi d\tau = \int_0^{\tau} (\Omega_3 - \varphi' \cos \theta) d\tau,$$

where $\Omega_3$ is the dimensionless angular velocity $t_3 \Omega_3$. Due to the conservation of the angular momentum component $L_3$, $\Omega_3$ is constant which implies

$$\Delta \Psi = \Omega_3 \tau_3 = \int_0^{\tau} \varphi' \cos \theta d\tau = \Omega_3 \tau_3 - 2\pi + \int_0^{\tau} \varphi' (1 - \cos \theta d\tau).$$
The Hannay angle $\Delta \psi_H$ is

$$\Delta \psi_H = \int_0^\tau \varphi' - (1 - \cos \theta) d\tau$$

(24)

which is the solid angle swept by the axis in one precession cycle.

3.3. Pure precession: an exact solution

Following [51] we seek for the slow manifold, i.e. the set of initial conditions in the phase space for which the particular solution of the equations of motion equation (21) corresponds to pure precession, which means precession in the absence of nutation. It therefore corresponds to $\theta' = 0$, from which inserted in equation (21) a gives

$$G \varphi' = \varphi'^2 \cos \theta - 1.$$  

(25)

The dynamics of pure precession therefore give two corresponding precessional velocities, a slow one $\varphi'_\perp$ and a fast one $\varphi'_\parallel$, which are given by

$$\varphi'_\pm = \frac{G}{2 \cos \theta} \left( 1 \pm \sqrt{1 + \frac{4 \cos \theta}{G^2}} \right).$$

(26)

The square root in this equation shows that the pure precession requires $\cos \theta > -G^2/4$. Therefore, pure precession without nutation is possible for $|G| > 2$ for any inclination angle $\theta$, whereas for $|G| < 2$, pure precession is only possible for inclination angles such that $\cos \theta > -G^2/4$. We now consider the slow precession velocity $\varphi'_\perp$ given by equation (26). For such slow pure precession it is possible to derive exact results from equation (22). In this case $\varphi'$ and $\theta$ are constant, and since $\varphi'_\perp$ is negative whatever the sign of $\cos \theta$, the precession time reads $\tau_s = 2\pi/|\varphi'_\perp| = -2\pi/\varphi'_\perp$. Combined with $\tilde{\Omega}_3 = \omega_0 \Omega_3 = G \mu / I_3$, equation (22) gives

$$\frac{\Delta \Psi}{2\pi} = -\frac{I_3}{G} \frac{G}{\varphi'_\perp} + \cos \theta.$$  

(27)

Using from equation (25)

$$\frac{G}{\varphi'_\perp} = G^2 \left( \frac{\varphi'_\perp \cos \theta}{G} - 1 \right)$$

and using the slow precession velocity from equation (26)

$$\frac{\varphi'_\perp \cos \theta}{G} = \frac{1}{2} \left( 1 - \sqrt{1 + \frac{4 \cos \theta}{G^2}} \right)$$

Equation (27) gives

$$\frac{\Delta \Psi}{2\pi} = \frac{I_3}{G^2} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4 \cos \theta}{G^2}} \right) + \cos \theta$$

$$= \frac{I_3}{G^2} + \cos \theta \left( 1 + \frac{I_1}{I_3} \right) - \frac{I_3 \cos^2 \theta}{G^2} \times \left( 1 - \frac{2 \cos \theta}{G^2} + 5 \frac{\cos^2 \theta}{G^2} + \ldots \right).$$

(28)

This expression generalizes equation (12) of section 3 to the inertial regime for the pure precession. This is of course the same expression as that obtained for the spinning top in [51]. In this framework, the first term $G^2 (I_1 / I_3)$ of the expansion was the dynamical phase. The question that was discussed in [51], was about the nature of the second term $\cos \theta$. There was an ambiguity about associating it to the dynamical phase or to the geometric phase. It appears below that, in the framework of the ‘Bohr magneton’ approach used in section 3.4 for the magnetic monopole, the two first terms in the right hand side of equation (28) are identical. Indeed, according to the II-D, we have $\cos \theta = G^2 (I_1 / I_3) = \Omega_f i / I_f$ and equation (28) reads:

$$\frac{\Delta \Psi}{2\pi} = \cos \theta \left( \frac{1}{2} + \sqrt{1 + \frac{4 \cos \theta}{G^2}} \right) + \cos \theta$$

$$= \cos \theta \left( 2 + \left( \frac{\cos \theta}{G^2} \right) - \left( \frac{\cos \theta}{G^2} \right)^2 \right)$$

$$+ \left( \frac{\cos \theta}{G^2} \right)^3 - \frac{5 \left( \cos \theta \right)^4 + \ldots \right).$$

(29)

The geometric phase $\Delta \Psi$ is a function of the precession angle $\theta$ and the slowness parameter $G$. Note that if we remove the dynamical angle $2 \cos \theta$, the development is a function of a single parameter $\cos \theta / G^2$ only.

3.4. The classical magnetic monopole for pure precession

The generalization of the magnetic monopole equation (14) is $B_{\text{eff}} = \frac{\Delta \Psi}{2\pi}$ so that

$$B_{\text{eff}} = \frac{1}{R^2} \left[ \cos \theta \left( 1 + \sqrt{1 + \frac{4 \cos \theta}{G^2}} \right) + 2 \cos \theta \right]$$

$$= \frac{2 \cos \theta}{R^2} \left( \frac{\cos \theta}{G^2} - \left( \frac{\cos \theta}{G^2} \right)^2 \right)$$

$$+ \frac{2 \left( \cos \theta \right)^3 - 6 \left( \cos \theta \right)^4 + \ldots \right).$$

(30)

This equation gives the influence of the inertia (i.e. the fast magnetic degrees of freedom) on the magnetic monopole, in the case of the pure precession.

4. Conclusion

Magnetization dynamics have been investigated beyond the usual assumption of the total separation of time scales between slow and fast magnetic degrees of freedom, for the adiabatic limit. We have exploited the analogy with the spinning top by pushing the mechanical model of the magnetic dipole beyond Gilbert’s assumption. Fast degrees of freedom are introduced with the angular momentum $L$ and its time variation (with non-zero first and second principal moment of inertia $I_1 = I_2 = 0$).
The problem is investigated from the viewpoint of the geometric phase which allows the magnetic monopole to be defined naturally. The effect of inertia is then taken into account, and an analytical expression is obtained in the case of the pure precession, for which the nutation vanishes.

In the case of pure precession with precession angle \( \theta \), the calculation of the geometric phase shows that, beyond a dynamical phase of the form \( 2 \cos \theta /G^2 \), the Hannay angle is a simple function of the parameter \( \cos \theta /G^2 \), where \( G = J/(2\pi \hbar) \) is the slowness parameter (i.e. the ratio of the slow characteristic time of the precession over the fast characteristic time).

The magnetic monopole (defined as the radial magnetic field produced from a punctual center), is derived directly from the geometric phase. In the usual case without inertia (\( \cos \theta /G^2 \to 0 \)), the Bohr magneton approach gives a very simple expression of the magnetic monopole as a function of the precession angle \( B_{\text{eff}} = 4 \cos \theta /R^2 \). In the case of pure precession, the correction due to the action of the fast degrees of freedom is given as a simple expression \( B_{\text{eff}} = \rho \theta \left( 1 + \sqrt{1 + 4 \cos \theta /G^2} + 2 \cos \theta \right) \). Note that in an experimental context, the magnetic monopole \( B_{\text{eff}} \) is constant because it is related to a given material, and the precession angle \( \theta \) depends the parameter \( G \).

This result suggests that the pure precession—i.e. the slow manifold for the dynamics of the magnetization [51]—should not be a purely formal concept, but could correspond to the actual motion of the magnetization for the ultrafast precession of the magnetization, that would correspond to the minimum power dissipated by the system (in comparison with the motion that includes nutation oscillations superimposed to the precession). This point should however still be clarified in further studies.

Acknowledgments

J-EW is grateful to M V Berry for helpful comments.

References

[1] Berry M V 1984 Quantal phase factors accompanying adiabatic changes Proc. R. Soc. A 392 45
[2] Aharonov Y and Stern A 1992 Origin of the geometrical forces accompanying Berry’s geometrical potentials Phys. Rev. Lett. 69 3593
[3] Hannay J H 1985 Angle variable holonomy in adiabatic excusion of an integrable Hamiltonian J. Phys. A : Math. Gen. 18 221
[4] Berry M V 1985 Classical adiabatic adiabatic and quantal adiabatic phase J. Phys. A : Math. Gen. 18 15
[5] Bruno P 2005 Berry phase effects in magnetism Magneticity Goes Nano (Matter and Material vol 26) ed S Blügel et al (Jülich: Forschungszentrum Jülich) (http://hdl.handle.net/2128/560)
[6] Berry M V and Shukla P 2012 Classical dynamics with curl forces, and motion driven by time-dependent flux J. Phys. A : Math. Theor. 45 305201
[7] Sonin E B 2010 Spin currents and spin superfluidity Adv. Phys. 59 181–255
[8] Nagaosa N, Sinova J, Onoda S, MacDonald A H and Ong N P 2010 Anomalous hall effect Rev. Mod. Phys. 82 1539
[9] Xiao D, Chang M C and Niu Q 2010 Berry phase effects on electronic properties Rev. Mod. Phys. 82 1959
[10] Nagaosa N, Yu X Z and Tokura Y 2012 Gauge fields in real and momentum spaces in magnets: monopoles and skyrmions Phil. Trans. R. Soc. A 370 5806
[11] Nagasawa F, Frustaglia D, Saarikoski H, Richter K and Nitta J 2013 Control of the spin geometric phase in semiconductor quantum rings Nat. Commun. 4 2526
[12] Bruno P 2004 Nonquantized dirac monopoles and strings in the berry phase of anisotropic spin systems Phys. Rev. Lett. 93 247202
[13] Shibata J, Tatara G and Kohno H 2005 Effect of spin current on uniform ferromagnetism : domain nucleation Phys. Rev. Lett. 94 076601
[14] Barnes S E and Maekawa S 2005 Current-spin coupling for ferromagnetic domain walls in fine wires Phys. Rev. Lett. 95 107204
[15] Kurebayashi H et al 2014 An antidamping spin–orbit torque originating from the Berry curvature Nat. Nanotechnol. 9 211
[16] Aharonov Y and Casher A 1984 Topological quantum effects for neutral particles Phys. Rev. Lett. 53 319
[17] Aharonov Y, Pearle P and Vaidman L 1988 Comment on proposed Aharonov–Casher effect: another example of an Aharonov–Bohm effect from classical lag Phys. Rev. A 37 4052
[18] Hertel R, Wulfhekel W and Kirscher J 2004 Domain-wall induced phase shifts in spin waves Phys. Rev. Lett. 93 257202
[19] Hertel R 2013 Curvature-induced magnetochirality SPIN 3 1340009
[20] Kovalev A A and Tserkovnyak Y 2012 Thermomagnonic spin transfer and Peltier effects in insulating magnets Europhys. Lett. 97 67002
[21] Onose Y, Ideue T, Katsura H, Shiomori Y, Nagaosa N and Tokura Y 2010 Observation of the Magnon Hall effect Science 329 297
[22] Matsumoto R and Murakami S 2011 Rotational motion of magnons and the thermal Hall effect Phys. Rev. B 84 184406
[23] Landau L D and Lifshitz L M 1935 Phys. Z. Sowjetunion 8 153
[24] Gilbert T H 1956 Formulations, fundations and applications of the phenomenological theory of ferromagnetism PhD Dissertation Illinois Institute of Technology appendix B
[25] Gilbert T L 2004 A phenomenological theory of damping in ferromagnetic materials IEEE Trans. Mag. 40 3443
[26] Bar yakhtar V G and Ivanov B A 2015 The Landau–Lifshitz equation: 80 years of history, advances, and prospects Low Temp. Phys. 41 663
[27] Bertotti G, Mayergoyz I and Serpico C 2009 Nonlinear Magnetization Dynamics in Nanosystems (Amsterdam: Elsevier)
[28] Lakshmanan M 2011 The fascinating world of the Landau–Lifshitz–Gilbert equation: an overview Phil. Trans. R. Soc. A 369 1280–300
[29] Pylypowsky O V, Kravchuk V P, Sheka D D, Makarov D, Schmidt O G, Gaididei Y 2015 Coupling of chiralities in spin and physical spaces: the Möbius ring as a case study Phys. Rev. Lett. 114 197204
[30] Goussev A, Robbins J M and Slastikov V 2014 Domain wall motion in thin ferromagnetic nanotubues: analytic results Europhys. Lett. 105 67006
[31] Stöhr J and Siegmann H C 2006 Magnetism, From Fundamentals to Nanoscale Dynamics (Berlin: Springer)
[32] Goussev A, Lund R G, Robbins J M and Slastikov V O 2013 Sonnenberg domain wall motion in magnetic nanowires: an asymptotic approach Proc. R. Soc. A 469 2160–78
[33] Miltat J, Alburquerque G and Thiaville A 2002 An introduction to micromagnetics in the dynamics regime *Spin Dynamics in Confined Magnetic Structures I* ed B Hillebrands and K Ounadjela (Berlin: Springer)

[34] Brown W F 1963 Thermal fluctuations of a single-domain particle *Phys. Rev.* **130** 1677–86

[35] Coffey W T and Kalmykov Y P 2012 *The Langevin Equation* (World Scientific Series in Contemporary Chemical Physics vol 27) 3rd edn (Singapore: World Scientific)

[36] Aron C, Barci D G, Cugliandolo L F, Arenas Z G and Lozano G S 2014 Magnetization dynamics: path-integral formalism for stochastic Landau–Lifshitz–Gilbert equation *J. Stat. Mech.* P09008

[37] Skrotskiì G V 1984 The Landau–Lifshitz equation revisited *Usp. Fiz. Nauk.* **144** 681–6

[38] Maamache M 1996 Exact solution and geometric angle for the classical spin system *Phys. Scr.* **54** 21–3

[39] Wegrowe J-E and Ciornei M-C 2012 Magnetization dynamics, gyromagnetic relation, and inertial effects *Am. J. Phys.* **80** 607

[40] Ciornei M-C, Rubí J M and Wegrowe J-E 2011 Magnetization dynamics in the inertial regime: nutation predicted at short time scales *Phys. Rev. B* **83** 020410

[41] Faehnle M, Steiauf D and Illg C 2011 Generalized Gilbert equation including inertial damping : derivation from an extended breathing Fermi surface model *Phys. Rev. B* **84** 172403

[42] Olive E, Lansac Y and Wegrowe J-E 2012 Beyond ferromagnetic resonance: the inertial regime of the magnetization *Appl. Phys. Lett.* **100** 192407

[43] Bhattacharjee S, Nordström L and Fransson J 2012 Atomistic spin dynamic method with both damping and moment of inertia effects included from first principles *Phys. Rev. Lett.* **108** 057204

[44] Olive E, Lansac Y, Meyer M, Hayoun M and Wegrowe J-E 2015 Deviation from Landau–Lifshitz–Gilbert equation in the inertial regime of the magnetization *J. Appl. Phys.* **117** 213904

[45] Thonig D, Hnck J and Eriksson O 2015 Gilbert-like damping caused by time retardation in atomistic magnetization dynamics *Phys. Rev. B* **92** 104403

[46] Li Y, Barras A-L, Aufrert S, Ebels U and Bailey W E 2015 Inertial terms to magnetization dynamics in ferromagnetic thin films *Phys. Rev. B* **92** 140413

[47] Aharonov Y, Ben-Reuven E, Popescu S and Rohrlich D 1991 Born–Oppenheimer revisited *Nucl. Phys. B* **350** 818

[48] Robbins J M 1997 *Topological Phase Effects* (Encyclopaedia of Applied Physics Editors) ed G Trigg (New York: Wiley) pp 549–84

[49] van Kampen N G 1985 Elimination of fast variables *Phys. Rep.* **124** 69–160

[50] Berry M V and Shukla P 2010 High-order classical adiabatic reaction forces: slow manifold for a spin model *J. Phys. A: Math. Theor.* **43** 045102

[51] Berry M V and Shukla P 2011 Slow manifold and Hannay angle in the spinning top *Eur. J. Phys.* **32** 115–27

[52] Griffiths D J and Hnizdo V 2013 Mansuripur’s paradox *Am. J. Phys.* **81** 570

[53] Landau L D and Lifshitz E M 1987 *Classical Theory of Fields* (Course of Theoretical Physics) 4th edn (Oxford: Butterworth-Heinemann) chapter 5, 45 (Larmor’s theorem)

[54] McDonlad K T 2010 Radiation by a time-dependent current loop (http://puhep1.princeton.edu/kirkmcd/examples/currentloop.pdf)

[55] Griffiths D J 1992 Dipoles at rest *Am. J. Phys.* **60** 979

[56] Griffiths D J 2011 Dynamic dipoles *Am. J. Phys.* **79** 867

[57] Garg A 2010 Berry phase near degeneracies : beyond the simplest case *Am. J. Phys.* **78** 661