THE $(g,K)$-MODULE STRUCTURES OF PRINCIPAL SERIES
REPRESENTATIONS OF $Sp(3,R)$.

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ABSTRACT. We describe explicitly the whole structures of the $(g,K)$-modules of principal series representations of $Sp(3,R)$. We apply this result to determine the holonomic system characterizing those Whittaker functions.

1. Introduction

In the investigation of a representation $\pi$ of a reductive Lie group $G$, it is a standard method to pass from the original $\pi$ to its associated $(g,K)$-module. Also in many applications, it is important to understand various ‘good’ realizations of $\pi$ in some function spaces. These functions can be sometimes described as solutions of some differential equations, say, the Casimir equations derived from the $(g,K)$-module structure of the given representation $\pi$.

For some ‘small’ semisimple Lie groups $G$, the $(g,K)$-module structures of the standard representations are completely described. For example, the description of them for $SL(2,R)$ is found in standard textbooks, and there are rather complete results for some groups of real rank 1, e.g. $SU(n,1)$ in [7] and $Spin(1,2n)$ in [17]. But, for Lie groups of higher rank there are few references as far as the author knows. It seems to be difficult to describe the whole $(g,K)$-module structures even for standard representations of classical groups of higher rank. However, in this paper we consider the case of the real symplectic group $Sp(3,R)$ of rank 3, and solve this problem generalizing the result of the paper [15] for $Sp(2,R)$ of rank 2.

Before describing our situation for $Sp(3,R)$, let us explain the problem in a more precise form. Let $G$ be a real semisimple Lie group and $\mathfrak{g}$ the Lie algebra of $G$. Fix a maximal compact subgroup $K$ of $G$. Since any standard $(\mathfrak{g},K)$-modules are realized as subspaces of $L^2(K)$ as $K$-modules, we investigate the $K$-module structure of standard $(\mathfrak{g},K)$-modules by the Peter-Weyl theorem. Because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, in order to describe the action of $\mathfrak{g}$ or $\mathfrak{g}_C = \mathfrak{g} \otimes_R C$ it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_C$. Therefore, the investigation of the action of $\mathfrak{p}$ or $\mathfrak{p}_C$ is essential to give the explicit $(g,K)$-module structure of a standard representation. To investigate the action of $\mathfrak{p}_C$, we compute the linear map $\Gamma_{\tau,i}$ defined as follows. Let $(\pi,H_\pi)$ be a standard representation of $G$ with the subspace $H_{\pi,K}$ of $K$-finite vectors. For a $K$-type $(\tau,V_\tau)$ of $\pi$, and a nonzero $K$-homomorphism $\eta: V_\lambda \to H_{\pi,K}$, we define a linear map $\tilde{\eta}: \mathfrak{p}_C \otimes_C V_\lambda \to H_{\pi,K}$ by $X \otimes v \mapsto X \cdot \eta(v)$. Then $\tilde{\eta}$ is a $K$-homomorphism with $\mathfrak{p}_C$ endowed with the adjoint action $Ad$ of $K$. Let $V_\tau \otimes_C \mathfrak{p}_C \cong \bigoplus_{i \in I} V_{\tau_i}$ be the decomposition into a direct sum of irreducible $K$-modules and $\iota_i$ an injective $K$-homomorphism from $V_{\tau_i}$ to $V_\tau \otimes_C \mathfrak{p}_C$ for each $i$. We define a linear map $\Gamma_{\tau,i}: Hom_K(V_\tau,H_{\pi,K}) \to Hom_K(V_{\tau_i},H_{\pi,K})$ by $\eta \mapsto \tilde{\eta} \circ \iota_i$. These linear maps $\Gamma_{\tau,i}$ $(i \in I)$ characterize the action of $\mathfrak{p}_C$. Our purpose of this paper is to give the explicit expressions of $\Gamma_{\tau,i}$ when $\pi$ is a principal series representation of $G = Sp(3,R)$. This is described in Theorem 5.4.

As an application of the explicit expressions of $\Gamma_{\tau,i}$, we get a system of differential equations satisfied by some types of spherical functions. Here we consider only the case of the Whittaker functions. In order to introduce this application, let us recall the general setting of the theory of the spherical functions. Fix a closed subgroup $R$ of $G$. Take a character $\xi$ of $R$ and consider its $C^\infty$-induction $C^\infty Ind^G_R(\xi)$. For an irreducible admissible representation $(\pi,H_\pi)$ of $G$ with the subspace $H_{\pi,K}$ of $K$-finite vectors, we consider the subspace $Hom_{(\mathfrak{g}_C,K)}(H_{\pi,K},C^\infty Ind^G_R(\xi))$...
of intertwining operators. Consider the restriction of elements in this space to specific $K$-type as follows. Let $(\tau, V_{\tau})$ be a multiplicity one $K$-type of $\pi$ and let $i: V_{\tau} \to H_{\pi}$ be a nonzero $K$-homomorphism. For $\Phi \in \text{Hom}_{(g, C)}(H_{\pi, K}, C^{\infty}\text{Ind}_{R}^{G}(\xi))$, we can define the function $\phi_{\pi, \tau}$ contained in the space $C_{\text{loc}}^{\infty}(R(G/K))$ of $V_{\tau}$-valued smooth functions on $G$ satisfying $f(rgk) = \xi(r)\tau^{*}(k)^{-1}f(g)$ for all $(r, g, k) \in R \times G \times K$ by $\Phi(\mu(v))(g) = \langle v, \phi_{\pi, \tau} \rangle$ ($g \in G$, $v^{*} \in V_{\tau}$). Here $\tau^{*}$ means the contragradient representation of $\tau$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing of $V_{\tau} \times V_{\tau}^{*}$. When $R$ is a maximal unipotent subgroup of $G$ and $\xi$ is a unitary character of $R$, the space $\text{Hom}_{(g, C)}(H_{\pi, K}, C^{\infty}\text{Ind}_{R}^{G}(\xi))$ is called the space of Whittaker functionals and $\phi_{\pi, \tau}$ is called a Whittaker function.

In the case of $G = Sp(2, \mathbb{R})$, the explicit formulas of Whittaker functions for various standard representations are given in the papers [3], [5], [12], [13], [14] as well as the generalized Whittaker functions in the papers [2], [11]. In these papers, the explicit formulas of spherical functions are given as solutions of the system of differential equations which are obtained from shift operators and the Casimir element. Since the Casimir element is represented by a composite of the shift operators, the utilization of the shift operator is essential. Our operator $\Gamma_{\tau,i}$ is compatible with the shift operator and we give the holonomic systems of differential equations characterizing Whittaker functions in Theorem 6.12. Therefore, we can compute explicit formulas of Whittaker functions of principal series representations of $G = Sp(3, \mathbb{R})$ by using the result of this paper. We hope that this interesting possibility will be considered in future work.

We give the contents of this paper. In Section 2 we recall the structure of $Sp(3, \mathbb{R})$ and define a principal series representation, that is, a standard representation obtained by a parabolic induction with respect to the minimal parabolic subgroup $P_{\text{min}}$. In Section 3 we introduce the monomial basis of a finite dimensional irreducible representation of $\mathfrak{g}_{C} \simeq \mathfrak{gl}(3, \mathbb{C})$ and investigate adjoint representation of $K$ on $\mathfrak{p}_{C}$. Here the monomial basis is an alias of Gelfand Zetlin basis, which is twisted dual of the crystal basis of Kashiwara or the canonical basis of Lusztig. In Section 4 we give the explicit expressions of $\iota_{i}: V_{\tau, i} \to V_{\tau} \otimes_{C} \mathfrak{p}_{C}$. Section 5 is the main body of this paper and we give the matrix representation of $\Gamma_{\tau,i}$ with respect to the induced basis from the monomial basis in Theorem 5.6. In Section 6 we introduce some examples of $\Gamma_{\tau,i}$ and give the holonomic systems of differential equations characterizing Whittaker functions in Theorem 6.12.

This is an enhanced version of the Master’s thesis [10]. The author would like to express his gratitude to Takayuki Oda for valuable advice on this work. He also would like to thank Miki Hirano to provide a reference for the computation of Clebsch-Gordan coefficients.

2. Preliminaries

2.1. Groups and algebras. We denote by $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the real number field and the complex number field, respectively. Let $1_{n}$ (resp. $O_{n}$) be the unit (resp. the zero) matrix in the space $M_{n}(\mathbb{R})$ of real matrices of size $n$.

The real symplectic group $G = Sp(3, \mathbb{R})$ of degree three is defined by

$$G = Sp(3, \mathbb{R}) = \{ g \in Sl(6, \mathbb{R}) \mid ^{t}gJ_{3}g = J_{3} \}$$

$$J_{3} = \begin{pmatrix} O_{3} & 1_{3} \\ -1_{3} & O_{3} \end{pmatrix},$$

which is connected, semisimple, and split over $\mathbb{R}$. Here $^{t}g$ and $g^{-1}$ mean the transpose and the inverse of $g$, respectively. Let $\theta: G \ni g \mapsto ^{t}g^{-1} \in G$ be a Cartan involution of $G$. Then

$$K = \{ g \in G \mid \theta(g) = g \}$$

$$= \left\{ g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \bigg| A + \sqrt{-1}B \in U(3) \right\}.$$
if we define a map $\leq_1$ isomorphism $k$.

Now we take a basis of $\beta$ be the set of compact and non-compact positive roots, respectively. If we denote the root space $\theta$ if we denote the differential of $\theta$ again by $\theta$, then we have $\theta(X) = -^tX$ for $X \in \mathfrak{g}$. Let $\mathfrak{t}$ and $\mathfrak{p}$ be the $+1$ and the $-1$ eigenspaces of $\theta \in \mathfrak{g}$, respectively, that is,

$$
\mathfrak{t} = \left\{ \begin{array}{c} X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} & \in \mathfrak{g} \\
A, B \in M_3(\mathbb{R}), ^tA = -A, ^tB = B \end{array} \right\},
$$

$$
\mathfrak{p} = \left\{ \begin{array}{c} X = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} & \in \mathfrak{g} \\
A, B \in M_3(\mathbb{R}), ^tA = A, ^tB = B \end{array} \right\}.
$$

Then $\mathfrak{t}$ is the Lie algebra of $K$ which is isomorphic to the unitary algebra

$$
\mathfrak{u}(3) = \{ A + \sqrt{-1}B \in M_3(\mathbb{C}) \mid A, B \in M_3(\mathbb{R}), ^tA = -A, ^tB = B \}
$$

of degree three, and $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. We fix an isomorphism $\kappa : \mathfrak{u}(3) \rightarrow \mathfrak{t}$ which is given by the inverse of

$$
\mathfrak{t} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in \mathfrak{u}(3).
$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_\mathbb{C} = \mathfrak{l} \otimes \mathbb{C}$ the complexification of $\mathfrak{l}$.

For $1 \leq i, j \leq 3$, we denote by $E_{ij}$ the matrix unit in $M_3(\mathbb{R})$ with entry 1 at $(i, j)$-th component and 0 at other entries. Take a compact Cartan subalgebra $\mathfrak{h} = \bigoplus_{i=1}^3 \mathbb{R}T_i$ where $T_i = \kappa(\sqrt{-1}E_{ii}) \in \mathfrak{t}$. For $1 \leq i \leq 3$, define a linear form $\beta_i$ on $\mathfrak{h}_\mathbb{C}$ by $\beta_i(T_j) = \sqrt{-1}\delta_{ij}$, $1 \leq j \leq 3$. Here $\delta_{ij}$ is the Kronecker’s delta. Then the set $\Delta$ of the roots for $(\mathfrak{h}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$ is given by

$$
\Delta = \Delta(\mathfrak{h}_\mathbb{C}, \mathfrak{g}_\mathbb{C}) = \{ \pm 2\beta_i (1 \leq i \leq 3), \pm \beta_j \pm \beta_k (1 \leq j < k \leq 3) \},
$$

and the subset $\Delta^+ = \{ 2\beta_i (1 \leq i \leq 3), \beta_j \pm \beta_k (1 \leq j < k \leq 3) \}$ form a positive root system. Let

$$
\Delta^+_c = \{ \beta_j - \beta_k (1 \leq j < k \leq 3) \},
$$

$$
\Delta^+_c = \{ 2\beta_i (1 \leq i \leq 3), \beta_j + \beta_k (1 \leq j < k \leq 3) \}
$$

be the set of compact and non-compact positive roots, respectively. If we denote the root space for $\beta \in \Delta$ by $\mathfrak{g}_\beta$, then $\mathfrak{t}_\mathbb{C} \simeq \mathfrak{gl}(3, \mathbb{C})$ and $\mathfrak{p}_\mathbb{C}$ have the decompositions

$$
\mathfrak{t}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\beta \in \Delta_e} \mathfrak{g}_\beta, \quad \Delta_e = \Delta^+_c \cup (-\Delta^+_c),
$$

$$
\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-, \quad \mathfrak{p}_+ = \bigoplus_{\beta \in \Delta^+_c} \mathfrak{g}_{\pm \beta}.
$$

Now we take a basis of $\mathfrak{t}_\mathbb{C}$ and $\mathfrak{p}_\pm$ consisting of root vectors. If we denote the extension of the isomorphism $\kappa$ to their complexifications again by $\kappa$, then we have $\kappa(E_{ij}) \in \mathfrak{g}_{\beta_i - \beta_j}$ for each $1 \leq i \neq j \leq 3$ and thus the set $\{ \kappa(E_{ij}) \mid 1 \leq i, j \leq 3 \}$ forms a basis of $\mathfrak{t}_\mathbb{C}$. On the other hand, if we define a map

$$
p_{\pm} : \{ X \in M_3(\mathbb{C}) \mid X = ^tX \} \ni X \mapsto \begin{pmatrix} X & \pm \sqrt{-1}X \\ \pm \sqrt{-1}X & -X \end{pmatrix} \in \mathfrak{p}_{\pm},
$$

then the element $X_{\pm ij} = p_{\pm}((E_{ij} + E_{ji})/2)$ is a root vector in $\mathfrak{g}_{\pm(\beta_i + \beta_j)}$ for $1 \leq i \leq j \leq 3$ and the set $\{ X_{\pm ij} \mid 1 \leq i \leq j \leq 3 \}$ gives a basis of $\mathfrak{p}_{\pm}$.

Put $\mathfrak{a} = \bigoplus_{i=1}^3 \mathbb{R}H_i$ with $H_1 = \text{diag}(1, 0, 0, -1, 0, 0)$, $H_2 = \text{diag}(0, 1, 0, 0, -1, 0)$, $H_3 = \text{diag}(0, 0, 1, 0, 0, -1)$. Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. For each $1 \leq i \leq 3$, we
define a linear form \( e_i \) on \( a \) by \( e_i(H_j) = \delta_{ij} \) for \( 1 \leq j \leq 3 \). The set \( \Sigma \) of the restricted roots for \((a, g)\) is given by

\[
\Sigma = \Sigma(a, g) = \{ \pm 2e_i \ (1 \leq i \leq 3), \ \pm e_j \pm e_k \ (1 \leq j < k \leq 3) \},
\]

and the subset \( \Sigma^+ = \{ 2e_i \ (1 \leq i \leq 3), \ e_j \pm e_k \ (1 \leq j < k \leq 3) \} \) forms a positive root system. For each \( \alpha \in \Sigma \), we denote the restricted root space by \( g_{\alpha} \) and choose a restricted root vector \( E_{\alpha} \) in \( g_{\alpha} \) as follows.

\[
E_{2e_i} = \left( \begin{array}{cc} O_3 & E_{ii} \\ O_3 & O_3 \end{array} \right), \quad 1 \leq i \leq 3,
E_{e_j + e_k} = \left( \begin{array}{cc} O_3 & E_{jk} + E_{kj} \\ O_3 & O_3 \end{array} \right), \quad \text{for each } j < k,
E_{e_j - e_k} = \left( \begin{array}{cc} E_{jk} & O_3 \\ -O_3 & -E_{kj} \end{array} \right), \quad 1 \leq j < k \leq 3,
\]

and \( E_{-\alpha} = \theta(E_{\alpha}) \) for \( \alpha \in \Sigma^+ \). If we put \( n = \bigoplus_{\alpha \in \Sigma^+} g_{\alpha} \) then \( g \) has an Iwasawa decomposition \( g = n \oplus a \oplus t \). Also we have \( G = N_{\min} A_{\min} K \), where \( A_{\min} = \exp(a) \) and \( N_{\min} = \exp(n) \).

### 2.2. Definition of principal series representations of \( G \).

Let \( P_{\min} = M_{\min} A_{\min} N_{\min} \) be the minimal parabolic subgroup of \( G \), where

\[
M_{\min} = Z_K(A_{\min}) = \{ \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_i \in \{ \pm 1 \} \ (1 \leq i \leq 3) \} \simeq \{ \pm 1 \}^{\oplus 3}.
\]

For \( \nu \in \text{Hom}_R(a, C) \), we define a coordinate \( (\nu_1, \nu_2, \nu_3) \in C^3 \) by \( \nu_i = \nu(H_i), \ 1 \leq i \leq 3 \). Then the half sum \( \rho = \frac{1}{2} \left( \sum_{\alpha \in \Sigma^+} \alpha \right) = 3e_1 + 2e_2 + e_3 \) of the positive roots has coordinate \((\rho_1, \rho_2, \rho_3) = (3, 2, 1)\). We define a quasicharacter \( e^\nu : A_{\min} \to C^\times \) by

\[
e^\nu(\text{diag}(a_1, a_2, a_3, a_1^{-1}, a_2^{-1}, a_3^{-1})) = a_1^{\nu_1} a_2^{\nu_2} a_3^{\nu_3}.
\]

Moreover, we fix a character \( \sigma \) of \( M_{\min} \). \( \sigma \) is realized by \((\sigma_1, \sigma_2, \sigma_3) \in \{ 0, 1 \}^{\oplus 3} \) such that

\[
\sigma(\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3)) = \varepsilon_1^{\sigma_1} \varepsilon_2^{\sigma_2} \varepsilon_3^{\sigma_3} \in C \text{ for } \varepsilon_i \in \{ \pm 1 \}, \ 1 \leq i \leq 3.
\]

With these data \((\sigma, \nu)\), the parabolic induction

\[
\pi_{(\sigma, \nu)} = \text{Ind}_{P_{\min}}^G(\sigma \otimes e^{\nu + \rho} \otimes 1_{N_{\min}})
\]

is a Hilbert representation of \( G \) by the right regular action on the Hilbert space \( H_{(\sigma, \nu)} \) which is the completion of

\[
H_{(\sigma, \nu)}^\infty = \left\{ f : G \to C \text{ smooth} \mid f(manx) = \sigma(m)e^{\nu + \rho}(a)f(x) \text{ for } m \in M_{\min}, \ a \in A_{\min}, \ n \in N_{\min}, \ x \in G \right\}
\]

with an inner product

\[
(f_1, f_2) = \int_K f_1(k)f_2(k)dk \quad \text{for} \quad f_1, f_2 \in H_{(\sigma, \nu)}^\infty.
\]

Here \( dk \) is a Haar measure of \( K \).

### 3. The monomial basis for simple \( \mathfrak{gl}(3, C) \)-modules and the adjoint representation of \( \mathfrak{f}_C \simeq \mathfrak{gl}(3, C) \) on \( p_{\pm} \)

In this section, we recall some basic facts about the representations of \( \mathfrak{f}_C \simeq \mathfrak{gl}(3, C) \), and evaluate the adjoint representations of \( \mathfrak{f}_C \) on \( p_{\pm} \).
3.1. **The highest weight theory for** \( \mathfrak{gl}(n, \mathbb{C}) \). We recall the highest weight theory for \( \mathfrak{gl}(n, \mathbb{C}) \). A **weight** of length \( n \) is an integral vector \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{Z}^n \). The weight \( \gamma \) is called **dominant** if \( \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \). It is well-known that every irreducible finite dimensional representation \((\tau, V)\) of \( \mathfrak{gl}(n, \mathbb{C}) \) has a **weight space decomposition**

\[
V = \bigoplus V(\gamma), \quad V(\gamma) = \{ v \in V \mid E_i v = \gamma_i v, \ 1 \leq i \leq n \}.
\]

There is a dominant weight \( \lambda \) which satisfies \( \lambda \geq \gamma \) in the lexicographical order for any weight \( \gamma \) such that \( V(\gamma) \neq 0 \). Such dominant weight is called the **highest weight** and the representation \((\tau, V)\) is labeled by the highest weight, i.e., \((\tau_\lambda, V_\lambda)\).

3.2. **Gelfand-Tsetlin patterns.** Recall that a **Gelfand-Tsetlin pattern** (which simply we call **G-pattern**) of type \( m_3 = (m_{13}, m_{23}, m_{33}) \) is a triangular array

\[
M = \begin{pmatrix} m_3 \\ m_2 \\ m_1 \end{pmatrix} = \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} \\ m_{11} \end{pmatrix}
\]

of integers satisfying the conditions

\[(3.1) \quad m_{13} \geq m_{12} \geq m_{23} \geq m_{22} \geq m_{33}, \quad m_{12} \geq m_{11} \geq m_{22}.\]

The weight \( \gamma^M = (\gamma_1^M, \gamma_2^M, \gamma_3^M) \) of a G-pattern \( M \) is defined from the equations

\[
\gamma_1^M = m_{11}, \quad \gamma_1^M + \gamma_2^M = m_{12} + m_{22}, \quad \gamma_1^M + \gamma_2^M + \gamma_3^M = m_{13} + m_{23} + m_{33}.
\]

For a G-pattern \( M \), we define

\[
M \left( \begin{array}{ccc} i_{13} & i_{23} & i_{33} \\ i_{12} & i_{22} & \end{array} \right) = \left( \begin{array}{ccc} m_{13} + i_{13} & m_{23} + i_{23} & m_{33} + i_{33} \\ m_{12} + i_{12} & m_{22} + i_{22} \\ m_{11} + i_{11} \end{array} \right).
\]

If the vector \((i_{13}, i_{23}, i_{33})\) is zero, we omit the top row in the left hand side of the above defining equality. So the left hand side is written as

\[
M \left( \begin{array}{cc} i_{12} & i_{22} \\ i_{11} \end{array} \right).
\]

A convenient symbol is \( M[k] \), which is defined by

\[
M \left( \begin{array}{c} k - k \\ 0 \end{array} \right).
\]

We note that G-patterns \( M_1 \) and \( M_2 \) have the same type and weight if and only if \( M_1[k] = M_2 \) for some \( k \in \mathbb{Z} \).

We define some functions of G-patterns. We set

\[
\delta(M) = \gamma_2^M - m_{23} = m_{12} + m_{22} - m_{11} - m_{23}.
\]

Let \( \chi_+(M) \) and \( \chi_-(M) \) be the characteristic functions of the sets \( \{ M \mid \delta(M) > 0 \} \) and \( \{ M \mid \delta(M) < 0 \} \), respectively. More generally we introduce functions \( \chi_\pm^{(i)}(M) \) by

\[
\chi_+^{(i)}(M) = \begin{cases} 1, & \delta(M) > i \\ 0, & \delta(M) \leq i \end{cases}, \quad \chi_-^{(i)}(M) = \begin{cases} 1, & \delta(M) < -i \\ 0, & \delta(M) \geq -i \end{cases}.
\]

Then we have \( \chi_+(M) = \chi_+^{(0)}(M) \) and \( \chi_-(M) = \chi_-^{(0)}(M) \).

We introduce ‘piecewise-linear’ functions \( C_1(M), \bar{C}_1(M) \) and \( C_2(M) \) by

\[
C_1(M) = \begin{cases} m_{11} - m_{22}, & \text{if } \delta(M) \geq 0 \\ m_{12} - m_{23}, & \text{if } \delta(M) \leq 0 \end{cases}, \quad \bar{C}_1(M) = \begin{cases} m_{23} - m_{22}, & \text{if } \delta(M) \geq 0 \\ m_{12} - m_{11}, & \text{if } \delta(M) \leq 0 \end{cases},
\]

and

\[
C_2(M) = C_1(M) \bar{C}_1(M).
\]
Another expressions of $C_1(M)$ and $\bar{C}_1(M)$ are
\[
C_1(M) = \min\{m_{11} - m_{22}, \ m_{12} - m_{23}\}, \quad \bar{C}_1(M) = \min\{m_{23} - m_{22}, \ m_{12} - m_{11}\}.
\]

3.3. The monomial basis in the sense of Gelfand-Zelevinsky. We recall the definition of the monomial basis in the sense of Gelfand-Zelevinsky.

For a weight subspace $V_\lambda(\gamma)$ and a dominant weight $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$, we set
\[
V_\lambda(\gamma, \nu) = \{v \in V_\lambda(\gamma) \mid E_i^{\nu_i-\nu_{i+1}+1}v = 0, \ 1 \leq i \leq n - 1\}.
\]
A basis $B$ in $V_\lambda$ is called proper if each of subspaces $V_\lambda(\gamma, \nu)$ (for all possible $\gamma, \nu$) is spanned by its subset $B \cap V_\lambda(\gamma, \nu)$. It is known that the representation $(\tau_\lambda, V_\lambda)$ of $\mathfrak{gl}(3, \mathbb{C})$ has a proper basis, which is unique up to scalar multiple. Gelfand and Zelevinsky normalized the scalar factor somehow to get the formulas in Proposition 3.1. The normalized proper basis is called the monomial basis because it is twisted dual of the crystal basis of Kashiwara or the canonical basis of Lusztig. For the representation $(\tau_{m_3}, V_{m_3})$, the monomial basis in $V_{m_3}(\gamma)$ is parameterized by the G-patterns of type $m_3$ whose weights are $\gamma$. We denote the monomial basis of $V_{m_3}$ by $\{f(M)\}_{M \in G(m_3)}$. Here $G(m_3)$ is the set of the G-patterns of type $m_3$.

The action of $\mathfrak{t}_C \simeq \mathfrak{gl}(3, \mathbb{C})$ is given by following formulas.

**Proposition 3.1.** (Gelfand-Zelevinsky) The action of simple root vectors on the monomial basis $\{f(M)\}_{M \in G(m_3)}$ of $V_{m_3}$ are given as follows.
\[
\begin{align*}
E_{12} f(M) &= (m_{12} - m_{11}) f \left( M \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) \right) + (m_{23} - m_{22}) \chi_+(M) f \left( M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right) [1] \\
E_{21} f(M) &= (m_{11} - m_{22}) f \left( M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right) + (m_{12} - m_{23}) \chi_-(M) f \left( M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right) [1] \\
E_{23} f(M) &= (m_{13} - m_{12}) f \left( M \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \right) + \{m_{13} - m_{12} - \delta(M)\} \chi_-(M) f \left( M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right) [1] \\
E_{32} f(M) &= (m_{22} - m_{33}) f \left( M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right) + \{m_{22} - m_{33} + \delta(M)\} \chi_+(M) f \left( M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right) [1]
\end{align*}
\]

In the right hand side of above formulas, we put $f(M') = 0$ if $M'$ is a triangular array which does not satisfy the condition (3.1) of G-patterns.

In the later section, we need the action of root vectors on the monomial basis of $V_{m_3}$. So we compute the action of $E_{13}$ and $E_{31}$. Since $E_{13} = [E_{12}, E_{23}]$ and $E_{31} = [E_{32}, E_{21}]$, we obtain
\[
\begin{align}
E_{13} f(M) &= [E_{12}, E_{23}] f(M) = (m_{13} - m_{12}) f \left( M \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \right) - \bar{C}_1(M) f \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right), \\
E_{31} f(M) &= [E_{32}, E_{21}] f(M) = (m_{33} - m_{22}) f \left( M \left( \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \right) \right) + C_1(M) f \left( M \left( \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \right) \right).
\end{align}
\]

Here $[,]$ is a Lie bracket, i.e., $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{gl}(3, \mathbb{C})$.

The monomial basis has a interesting symmetry property. For each G-pattern $M$, we define the dual pattern $\hat{M}$ by
\[
\hat{M} = \left( \begin{array}{ccc}
-m_{33} & -m_{23} & -m_{13} \\
-m_{22} & -m_{12} & -m_{11}
\end{array} \right).
\]

If $M$ is a G-pattern of type $m_3$ and weight $\gamma^M$ then $\hat{M}$ is a G-pattern of type $\hat{m}_3 = (-m_{33}, -m_{23}, -m_{13})$ and weight $-\gamma^M = (-\gamma^\hat{M}, -\gamma^\hat{M}, -\gamma^\hat{M})$.

**Proposition 3.2.** Let $\omega$ be the automorphism of $\mathfrak{gl}(3, \mathbb{C})$ defined by $\omega(E_{ii}) = -E_{ii}$ and $\omega(E_{ik}) = E_{kj}$ for $i, j, k \in \{1, 2, 3\}$ such that $|j - k| = 1$. Let $T_{m_3} : V_{m_3} \to V_{\hat{m}_3}$ be the linear map defined by $T_{m_3}(f(M)) = f(\hat{M})$ for $M \in G(m_3)$. Then $X \circ T_{m_3} = T_{m_3} \circ \omega(X)$.

For the existence, uniqueness and properties of the monomial basis, we refer to the paper [1].

In the later sections, we give the explicit expressions of various $K$-homomorphisms in terms of the monomial basis. However, the monomial basis have the ambiguity of scalar multiple. Thus, we have to fix the monomial basis for simple $K$-modules.
Definition 3.3. A simple $K$-module $V_\lambda$ equipped with the fixed monomial basis $\{f(M)\}_{M \in G(\lambda)}$ is called a simple $K$-module with the marking $\{f(M)\}_{M \in G(\lambda)}$.

3.4. The adjoint representations of $\mathfrak{tc}$ on $\mathfrak{p}_\pm$. We denote by $e_i$ the unit vector of degree three with its $i$-th component 1 and the remaining component 0. It is known that both of $\mathfrak{p}_\pm$ become $K$-modules via the adjoint action of $K$. Concerning this, we have the following lemma.

Lemma 3.4. We have isomorphisms $i_{\mathfrak{p}_+}: \mathfrak{p}_+ \to V_{2e_1}$ and $i_{\mathfrak{p}_-}: \mathfrak{p}_- \to V_{-2e_3}$ by the correspondences between their basis

$$(X_{+11}, X_{+12}, X_{+13}, X_{+22}, X_{+23}, X_{+33})$$

$$(X_{-33}, -X_{-23}, -X_{-22}, -X_{-13}, -X_{-12}, X_{-11})$$

Proof. By direct computation, we have the following tables of the adjoint actions of Cartan subalgebra and simple root vectors of $\mathfrak{tc}$ on the basis $\{X_{\pm ij}\}_{1 \leq i \leq j \leq 3}$ of $\mathfrak{p}_\pm$.

| $\kappa(E_{11})$ | $\kappa(E_{22})$ | $\kappa(E_{33})$ | $\kappa(E_{12})$ | $\kappa(E_{23})$ | $\kappa(E_{32})$ | $\kappa(E_{13})$ | $\kappa(E_{31})$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $X_{+11}$        | $X_{+11}$        | 0                | 0                | 0                | 2$X_{+12}$       | 0                | 0                |
| $X_{+12}$        | $X_{+12}$        | $X_{+12}$        | 0                | $X_{+11}$        | $X_{+22}$        | 0                | $X_{+13}$        |
| $X_{+13}$        | $X_{+13}$        | 0                | $X_{+13}$        | 0                | $X_{+23}$        | 0                | $X_{+12}$        |
| $X_{+22}$        | 0                | 2$X_{+22}$       | 0                | $2X_{+12}$       | 0                | 0                | 2$X_{+23}$       |
| $X_{+23}$        | 0                | $X_{+23}$        | $X_{+13}$        | 0                | $X_{+22}$        | $X_{+33}$        | $X_{+12}$        |
| $X_{+33}$        | 0                | 0                | $2X_{+33}$       | 0                | 0                | 2$X_{+23}$       | 0                |

TABLE 1. The adjoint actions of $\mathfrak{tc}$ on the basis $\{X_{+ij}\}_{1 \leq i \leq j \leq 3}$ of $\mathfrak{p}_+$

| $\kappa(E_{11})$ | $\kappa(E_{22})$ | $\kappa(E_{33})$ | $\kappa(E_{12})$ | $\kappa(E_{23})$ | $\kappa(E_{32})$ | $\kappa(E_{13})$ | $\kappa(E_{31})$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $-X_{-11}$       | -2$X_{-11}$      | 0                | 0                | 2$X_{-12}$       | 0                | 0                | 2$X_{-13}$       |
| $-X_{-12}$       | $-X_{-12}$       | $X_{-12}$        | 0                | $X_{-22}$        | $-X_{-11}$       | $X_{-13}$        | 0                |
| $-X_{-13}$       | $-X_{-13}$       | 0                | $X_{-13}$        | 0                | $X_{-23}$        | 0                | 0                |
| $-X_{-22}$       | 0                | 2$X_{-22}$       | 0                | 0                | 2$X_{-12}$       | 2$X_{-23}$       | 0                |
| $-X_{-23}$       | 0                | $X_{-23}$        | $X_{-23}$        | 0                | $X_{-13}$        | $X_{-33}$        | $X_{-22}$        |
| $-X_{-33}$       | 0                | 0                | $2X_{-33}$       | 0                | 0                | 2$X_{-23}$       | 0                |

TABLE 2. The adjoint actions of $\mathfrak{tc}$ on the basis $\{X_{-ij}\}_{1 \leq i \leq j \leq 3}$ of $\mathfrak{p}_-$

Comparing the actions in above tables with the actions in Proposition 3.1, we have the assertion.

Remark 3.5. The above lemma tells that $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are simple $K$-modules with the markings $\{X_{+ij}\}_{1 \leq i \leq j \leq 3}$ and $\{X_{-ij}\}_{1 \leq i \leq j \leq 3}$, respectively. From now on we always take these markings for $\mathfrak{p}_\pm$.

4. Clebsch-Gordan coefficients for the representations of $\mathfrak{gl}(3, \mathbb{C})$ with respect to the monomial basis

In the later sections, we need irreducible decompositions of the tensor products $V \otimes_{\mathbb{C}} \mathfrak{p}_+$ and $V \otimes_{\mathbb{C}} \mathfrak{p}_-$ as $K$-modules for a $K$-type $(\tau, V)$ of $\pi_{(\sigma, \nu)}$. Since $\mathfrak{p}_+ \simeq V_{2e_1}$, $\mathfrak{p}_- \simeq V_{-2e_3}$ and $\mathfrak{tc} \simeq \mathfrak{gl}(3, \mathbb{C})$, it suffices to consider the irreducible decomposition of $V_{\lambda} \otimes_{\mathbb{C}} V_{2e_1}$ and $V_{\lambda} \otimes_{\mathbb{C}} V_{-2e_3}$ as $\mathfrak{gl}(3, \mathbb{C})$-modules for arbitrary dominant weight $\lambda$. In this section, we take the marking $\{f(M)\}_{M \in G(\lambda)}$ for a simple $K$-module $V_{\lambda}$.
4.1. The irreducible decomposition of $V_\lambda \otimes C V_{e_1}$. Generically the tensor product $V_\lambda \otimes C V_{e_1}$ has three irreducible components: $V_{\lambda+e_1}$, $V_{\lambda+e_2}$ and $V_{\lambda+e_3}$. If $\lambda+e_i$ ($i = 2, 3$) is not dominant, the corresponding irreducible component does not occur.

For $1 \leq i \leq 3$, let $i_1^\lambda$ be a non-zero generator of $\operatorname{Hom}_K(V_{\lambda+e_i}, V_\lambda \otimes C V_{e_1})$, which is unique up to scalar multiple if $V_{\lambda+e_i}$ is non-zero. Our purpose of this subsection is to give explicit expressions of these injectors $i_1^\lambda$, $i_2^\lambda$ and $i_3^\lambda$ in terms of the monomial basis. For this purpose, we prepare following equations of the functions of G-patterns.

**Lemma 4.1.** (i) We have another expressions of $C_1(M)$, $\bar{C}_1(M)$ and $C_2(M)$, which is suitable for computation:

\begin{align*}
(4.1) \quad C_1(M) &= m_{11} - m_{22} + \delta(M)\chi_-(M) \\
&= m_{12} - m_{23} - \delta(M)\chi_+(M), \\
(4.2) \quad \bar{C}_1(M) &= m_{23} - m_{22} + \delta(M)\chi_-(M) \\
&= m_{12} - m_{11} - \delta(M)\chi_+(M), \\
(4.3) \quad C_2(M) &= (m_{12} - m_{23})(m_{12} - m_{11}) - (m_{12} - m_{22})\delta(M)\chi_+(M) \\
&= (m_{11} - m_{22})(m_{23} - m_{22}) + (m_{12} - m_{22})\delta(M)\chi_-(M).
\end{align*}

(ii) We have relations of the values of the functions $\delta$ and $\chi^{(r)}_\pm$ for another G-patterns as follows:

\begin{align*}
(4.4) \quad \delta \left( M \left( \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 1
\end{array} \right) \right) &= \delta(M) + d, \\
(4.5) \quad \chi^{(r)}_+(M) \left( \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 1
\end{array} \right) &= \chi^{(r-d)}_+(M), \\
(4.6) \quad \chi^{(r)}_-(M) \left( \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 1
\end{array} \right) &= \chi^{(r+d)}_-(M).
\end{align*}

Here $d = \delta \left( \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 1
\end{array} \right)$.

(iii) We have shift relations of $\chi^{(r)}_\pm(M)$ as follows:

\begin{align*}
(4.7) \quad (\delta(M) - r)\chi^{(r)}_+(M) &= (\delta(M) - r)\chi^{(r-1)}_+(M), \\
(4.8) \quad (\delta(M) + r)\chi^{(r)}_- (M) &= (\delta(M) + r)\chi^{(r-1)}_-(M), \\
(4.9) \quad \chi^{(r)}_+(M) + \chi^{(r-1)}_- (M) &= 1, \\
(4.10) \quad \chi^{(r)}_+(M)\chi^{(r-2)}_+(M) = 0 & \text{ if } r_1 + r_2 > -2, \\
(4.11) \quad \chi^{(r)}_+(M)\chi^{(r_1)}_+(M) = \chi^{(r_1)}_+(M) & \text{ if } r_1 > r_2, \\
(4.12) \quad \chi^{(r)}_-(M)\chi^{(r_2)}_-(M) = \chi^{(r_2)}_-(M) & \text{ if } r_1 > r_2.
\end{align*}

(iv) We have convenient relations of $C_1(M)\chi^{(r)}_\pm(M)$ and $\bar{C}_1(M)\chi^{(r)}_\pm(M)$ as follows:

\begin{align*}
(4.13) \quad C_1(M)\chi^{(r)}_+(M) &= (m_{11} - m_{22})\chi^{(r)}_+(M) & (r \geq -1), \\
(4.14) \quad C_1(M)\chi^{(r)}_-(M) &= (m_{12} - m_{23})\chi^{(r)}_-(M) & (r \geq -1), \\
(4.15) \quad \bar{C}_1(M)\chi^{(r)}_+(M) &= (m_{23} - m_{22})\chi^{(r)}_+(M) & (r \geq -1), \\
(4.16) \quad \bar{C}_1(M)\chi^{(r)}_-(M) &= (m_{12} - m_{11})\chi^{(r)}_-(M) & (r \geq -1).
\end{align*}

**Proof.** We can easily check these equations by direct computation.

The explicit expressions of the injectors $i_1^\lambda$, $i_2^\lambda$ and $i_3^\lambda$ are given as follows.
Proposition 4.2. For 1 ≤ i ≤ 3, the image of the monomial basis by the injector \( i_{e_i}^\lambda : V_{\lambda + e_i} \to V_{\lambda} \otimes C V_{e_i} \) is given by the form

\[
i_{e_i}^\lambda(f(M)) = \sum_{0 \leq j \leq k \leq 1} \left\{ \sum_{l=0}^{r_{i[jk]}} c_{i[jk;l]}^\lambda(M) f\left(M \left( \begin{array}{c} \delta_k \varepsilon_l \\ -j \\
-j \end{array} \right) \left[ -l \right] \right) \right\} \otimes f\left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right)
\]

for a G-pattern \( M \) of type \( \lambda + e_i \). In the right hand side of the above equation, we put \( f(M') = 0 \) if \( M' \) is a triangular array which does not satisfy the condition (3.1) of G-patterns.

The explicit expressions of the coefficients are given by following formulas.

**Formula 1:** The coefficients of the injector \( i_{e_1}^\lambda : V_{\lambda + e_1} \to V_{\lambda} \otimes C V_{e_1} \) are given as follows:

\[
(c_{1;11;0}^\lambda(M) = (m_{13} - m_{12})(m_{22} - m_{33}), \quad c_{1;11;1}^\lambda(M) = -\bar{E}(M), \quad c_{1;11;1}^\lambda(M) = -\bar{E}(M), \quad c_{1;01;2}^\lambda(M) = -C_2(M)\chi_+(M), \quad c_{1;00;0}^\lambda(M) = -(m_{13} - m_{12})(m_{13} - m_{22} + 1).
\]

Here

\[
\bar{E}(M) = C_1(M)\{m_{13} - m_{33} + 1 - C_1(M)\}, \quad \bar{F}(M) = -C_2(M) - \chi_+(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33} + 1)\delta(M)\}.
\]

**Formula 2:** The coefficients of the injector \( i_{e_2}^\lambda : V_{\lambda + e_2} \to V_{\lambda} \otimes C V_{e_2} \) are given as follows:

\[
(c_{2;11;0}^\lambda(M) = m_{22} - m_{33}, \quad c_{2;11;1}^\lambda(M) = -\bar{D}(M)\chi_-(M), \quad c_{2;21;0}^\lambda(M) = -(m_{22} - m_{33}), \quad c_{2;21;1}^\lambda(M) = \bar{C}_1(M), \quad c_{2;20;0}^\lambda(M) = -(m_{23} - m_{33}), \quad c_{2;20;0}^\lambda(M) = -\bar{C}_1(M)\chi_-(M).
\]

Here \( \bar{D}(M) = -(m_{22} + m_{33} + \delta(M) \).

**Formula 3:** The coefficients of the injector \( i_{e_3}^\lambda : V_{\lambda + e_3} \to V_{\lambda} \otimes C V_{e_3} \) are given as follows:

\[
(c_{3;11;0}^\lambda(M) = 1, \quad c_{3;01;0}^\lambda(M) = -1, \quad c_{3;01;1}^\lambda(M) = -\chi_+(M), \quad c_{3;00;0}^\lambda(M) = 1.
\]

Proof. For 1 ≤ i ≤ 3, let \( i_{e_i}^\lambda \) be a linear map which is defined by the equations in the statement of this proposition. In order to prove that \( i_{e_i}^\lambda \) is a \( K \)-homomorphism, it suffices to check the actions of the basis \( E_{mn} (1 \leq m \leq 3) \) of Cartan subalgebra and simple root vectors \( E_{n,n+1}, E_{n+1,n} (n = 1, 2) \) by direct computation.

Since

\[
E_{mn}(f(M_1) \otimes f(M_2)) = (E_{mn}f(M_1)) \otimes f(M_2) + f(M_1) \otimes (E_{mn}f(M_2))
\]

we easily check \( E_{mn} \circ i_{e_i}^\lambda(f(M)) = i_{e_i}^\lambda \circ E_{mn}(f(M)) \) for 1 ≤ m ≤ 3 and a G-pattern \( M \) of type \( \lambda + e_i \).

Therefore the essential computation is those of simple root vectors. We have to confirm that \( E_{mn} \circ i_{e_i}^\lambda = i_{e_i}^\lambda \circ E_{mn} \) for four simple root vectors \( E_{mn} (|m - n| = 1) \) and each \( i = 1, 2, 3 \). We set \( c_{[i,j;k;l]}^\lambda(M) = 0 \) if \( l > r_{i[jk]} \) or \( l < 0 \).

First, we compute the image of the monomial basis \( f(M) \) by \( i_{e_i}^\lambda \circ E_{mn} \). By using the equations in Proposition 3.1, we have

\[
i_{e_i}^\lambda(E_{12}f(M)) = (m_{12} - m_{11})i_{e_i}^\lambda(f(M, \varepsilon_1^{10})) + (m_{23} - m_{22})\chi_+(M)i_{e_i}^\lambda(f(M, \varepsilon_1^{10}) \left[-1\right])
\]
\[
\begin{align*}
&= (m_{12} - m_{11}) \sum_{0 \leq j \leq k \leq 1} \left\{ \sum_{l=0}^{r_{[j,k]}} c^{\lambda}_{i,j,k;l} (M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1-j \end{smallmatrix} \right)) f \left( M \left( \begin{smallmatrix} -e_k & 0 \\ 0 & 1-j \end{smallmatrix} \right) \right) \right\} \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix} \right) \\
&\quad + (m_{23} - m_{22}) \chi_+ (M) \sum_{0 \leq j \leq k \leq 1} \left\{ \sum_{l=0}^{r_{[j,k]}} c^{\lambda}_{i,j,k;l} (M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)) [-l-1] \right\} \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix} \right) \\
&\quad \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)
\end{align*}
\]

where

\[
A^{\lambda}_{i,j,k;l} (M) = (m_{12} - m_{11}) c^{\lambda}_{i,j,k;l} (M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1-j \end{smallmatrix} \right)) \\
+ (m_{23} - m_{22}) \chi_+ (M) c^{\lambda}_{i,j,k;l-1} (M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)) [-l-1] .
\]

Similarly, we obtain the following equations:

\[
\begin{align*}
i^{\lambda}_{e_1} \left( E_{21} f (M) \right) &= \sum_{0 \leq j \leq k \leq 1} \left\{ \sum_{l=0}^{r_{[j,k]}} A^{\lambda}_{21;i,j,k;l} (M) f \left( M \left( \begin{smallmatrix} -e_k & 0 \\ 0 & 1-j \end{smallmatrix} \right) \right) \right\} \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix} \right), \\
i^{\lambda}_{e_1} \left( E_{23} f (M) \right) &= \sum_{0 \leq j \leq k \leq 1} \left\{ \sum_{l=0}^{r_{[j,k]}} A^{\lambda}_{23;i,j,k;l} (M) f \left( M \left( \begin{smallmatrix} -e_k & 0 \\ 0 & 1-j \end{smallmatrix} \right) \right) \right\} \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix} \right), \\
i^{\lambda}_{e_1} \left( E_{32} f (M) \right) &= \sum_{0 \leq j \leq k \leq 1} \left\{ \sum_{l=0}^{r_{[j,k]}} A^{\lambda}_{32;i,j,k;l} (M) f \left( M \left( \begin{smallmatrix} -e_k & 0 \\ 0 & 1-j \end{smallmatrix} \right) \right) \right\} \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix} \right),
\end{align*}
\]

where

\[
A^{\lambda}_{21;i,j,k;l} (M) = (m_{11} - m_{22}) c^{\lambda}_{i,j,k;l} (M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1-j \end{smallmatrix} \right)) \\
+ (m_{12} - m_{23}) \chi_-(M) c^{\lambda}_{i,j,k;l-1} (M \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)) [-l-1] ,
\]

and

\[
A^{\lambda}_{23;i,j,k;l} (M) = (m_{13} - m_{12}) c^{\lambda}_{i,j,k;l} (M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)) \\
+ \{m_{13} - m_{12} - \delta(M)\} \chi_- (M) c^{\lambda}_{i,j,k;l-1} (M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)) [-l-1] .
\]

Next, we compute the image of the monomial basis \( f (M) \) by \( E_{mn} \circ i^{\lambda}_{e_1} \) as follows:

\[
E_{12} \circ i^{\lambda}_{e_1} \left( f (M) \right) = \sum_{0 \leq j \leq k \leq 1} \sum_{l=0}^{r_{[j,k]}} c^{\lambda}_{i,j,k;l} (M) E_{12} \left\{ f \left( M \left( \begin{smallmatrix} -e_k & 0 \\ 0 & 1-j \end{smallmatrix} \right) \right) \right\} \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix} \right)
\]

\[
= \sum_{0 \leq j \leq k \leq 1} \left\{ \sum_{l=0}^{r_{[j,k]}} (m_{12} - m_{11} - l + j) c^{\lambda}_{i,j,k;l} (M) f \left( M \left( \begin{smallmatrix} -e_k & 0 \\ 0 & 1-j \end{smallmatrix} \right) \right) \right\} \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix} \right) \\
+ (m_{23} - m_{22} + k - l - \delta_{21}) \chi_+ \left( M \left( \begin{smallmatrix} -e_k & 0 \\ 0 & 1-j \end{smallmatrix} \right) \right) \right\}
\]

where
\[ B_{[12;ij;k;l]}^\lambda(M) = (m_{12} - m_{11} - l + j) c_{[i;j;k;l]}^\lambda(M) \]
\[ + (m_{23} - m_{22} + k - l + 1 - \delta_{2l}) \chi_{(k - j - \delta_{2l})}^*(M) c_{[i;j;k;l]}^\lambda(M) \]
\[ + \left\{ \begin{array}{ll}
  c_{[i;01;l]}^\lambda(M) & \text{if } (j, k) = (1, 1), \\
  0 & \text{otherwise},
\end{array} \right. \]

Here we use the relation \[4.35\] in Lemma \[4.5.1\].

Similarly, we obtain the following equations:
\[ E_{21} \circ i_{e_i}^\lambda(f(M)) = \sum_{0 \leq j \leq k \leq l} \left\{ \sum_{l=0}^{r_{[i;j;k;l]}} B_{[12;ij;k;l]}^\lambda(M) f \left( M \left( \begin{array}{cc}
 - e_i & 0 \\
 0 & -e_i \end{array} \right) \right) \right\} \otimes f \left( \begin{array}{cc}
 1 & 0 \hfill \\
 0 & 0 \hfill \\
 \end{array} \right), \]
where
\[ B_{[21;ij;k;l]}^\lambda(M) = (m_{11} - m_{22} + k - l - j) c_{[i;j;k;l]}^\lambda(M) \]
\[ + (m_{12} - m_{23} - l + 1 + \delta_{2i}) \chi_{(k - j + \delta_{2i})}^*(-M) c_{[i;j;k;l]}^\lambda(M) \]
\[ + \left\{ \begin{array}{ll}
  c_{[i;11;l]}^\lambda(M) & \text{if } (j, k) = (0, 1), \\
  0 & \text{otherwise},
\end{array} \right. \]
\[ E_{23} \circ i_{e_i}^\lambda(f(M)) = \sum_{0 \leq j \leq k \leq l} \left\{ \sum_{l=0}^{r_{[i;j;k;l]}} B_{[23;ij;k;l]}^\lambda(M) f \left( M \left( \begin{array}{cc}
 - e_i & 0 \\
 0 & -e_i \end{array} \right) \right) \right\} \otimes f \left( \begin{array}{cc}
 1 & 0 \hfill \\
 0 & 0 \hfill \\
 \end{array} \right), \]
where
\[ B_{[23;ij;k;l]}^\lambda(M) = (m_{13} - m_{12} + l - \delta_{1l}) c_{[i;j;k;l]}^\lambda(M) \]
\[ + \{ m_{13} - m_{12} - \delta(M) + k - j + l - 1 - \delta_{11} - \delta_{2i} \} \]
\[ \times \chi_{(k - j + \delta_{2i})}^*(-M) c_{[i;j;k;l]}^\lambda(M) \]
\[ + \left\{ \begin{array}{ll}
  c_{[i;00;l]}^\lambda(M) & \text{if } (j, k) = (0, 1), \\
  0 & \text{otherwise},
\end{array} \right. \]
and
\[ E_{32} \circ i_{e_i}^\lambda(f(M)) = \sum_{0 \leq j \leq k \leq l} \left\{ \sum_{l=0}^{r_{[i;j;k;l]}} B_{[32;ij;k;l]}^\lambda(M) f \left( M \left( \begin{array}{cc}
 - e_i & 0 \\
 0 & -e_i \end{array} \right) \right) \right\} \otimes f \left( \begin{array}{cc}
 1 & 0 \hfill \\
 0 & 0 \hfill \\
 \end{array} \right), \]
where
\[ B_{[32;ij;k;l]}^\lambda(M) = (m_{22} - m_{33} + l + \delta_{3i}) c_{[i;j;k;l]}^\lambda(M) \]
\[ + \{ m_{22} - m_{33} + \delta(M) + j - 2k + l + 1 + \delta_{2i} + \delta_{3i} \} \]
Here we use the relations in Lemma 4.1 (ii).

In order to complete the proof, we check the equations (4.17)

$$A^\lambda_{mn;ij;kl} (M) = B^\lambda_{mn;ij;kl} (M)$$

by direct computation.

First, we check the equations (4.17) for $i = 3$, that is, the case of formula 3.

- the proof of $E_{12} \circ i^\lambda_{e_3} = i^\lambda_{e_3} \circ E_{12}$.

We have

$$A^\lambda_{12;3,11;0} (M) = m_{12} - m_{11},$$
$$A^\lambda_{12;3,11;1} (M) = (m_{23} - m_{22}) \chi_+ (M),$$
$$A^\lambda_{12;3,01;0} (M) = -(m_{12} - m_{11}),$$
$$A^\lambda_{12;3,01;1} (M) = -(m_{12} - m_{11}) \chi_+ (M) \chi_+ (M \left( \begin{smallmatrix} 0_0 \\ 1 \end{smallmatrix} \right) - (m_{23} - m_{22}) \chi_+ (M),$$
$$A^\lambda_{12;3,01;2} (M) = -(m_{23} - m_{22}) \chi_+ (M) \chi_+(M \left( \begin{smallmatrix} 0_0 \\ 1 \end{smallmatrix} \right) [-1]),$$
$$A^\lambda_{12;3,00;0} (M) = m_{12} - m_{11},$$
$$A^\lambda_{12;3,00;1} (M) = (m_{23} - m_{22}) \chi_+ (M),$$

and

$$B^\lambda_{12;3,11;0} (M) = (m_{12} - m_{11} + 1) - 1,$$
$$B^\lambda_{12;3,11;1} (M) = (m_{23} - m_{22} + 1) \chi_+ (M) - \chi_+ (M),$$
$$B^\lambda_{12;3,01;0} (M) = -(m_{12} - m_{11}),$$
$$B^\lambda_{12;3,01;1} (M) = -(m_{12} - m_{11} - 1) \chi_+ (M) - (m_{23} - m_{22} + 1) \chi^{(1)}_+ (M),$$
$$B^\lambda_{12;3,01;2} (M) = -(m_{23} - m_{22}) \chi^{(1)}_+(M) \chi_+ (M),$$
$$B^\lambda_{12;3,00;0} (M) = m_{12} - m_{11},$$
$$B^\lambda_{12;3,00;1} (M) = (m_{23} - m_{22}) \chi_+ (M).$$

By direct computation, we have

$$A^\lambda_{12;3,01;2} (M) - B^\lambda_{12;3,01;2} (M) = (m_{23} - m_{22}) \left( \chi^{(1)}_+(M) \chi_+ (M) - \chi_+ (M) \chi_+ (M \left( \begin{smallmatrix} 0_0 \\ 1 \end{smallmatrix} \right) [-1]) \right)$$
$$= (m_{23} - m_{22}) (\chi^{(1)}_+(M) - \chi^{(1)}_+(M)) = 0,$$

$$A^\lambda_{12;3,01;1} (M) - B^\lambda_{12;3,01;1} (M) = -(m_{12} - m_{11}) \chi_+ (M \left( \begin{smallmatrix} 0_0 \\ 1 \end{smallmatrix} \right)) - (m_{23} - m_{22}) \chi_+ (M)$$
$$+ (m_{12} - m_{11} - 1) \chi_+ (M) + (m_{23} - m_{22} + 1) \chi^{(1)}_+ (M)$$
$$= (\delta(M) - 1) \chi_+ (M) - (\delta(M) - 1) \chi^{(1)}_+ (M) = 0.$$

Hence we obtain the equations (4.17) for $(j, k, l) = (0, 1, 1)$ and $(0, 1, 2)$. Here we use the relations (4.5) and (4.7).

It is trivial that the equations (4.17) hold for other $(j, k, l)$.

- the proof of $E_{21} \circ i^\lambda_{e_3} = i^\lambda_{e_3} \circ E_{21}$.

We have

$$A^\lambda_{21;3,11;0} (M) = m_{11} - m_{22},$$
\[ A_{[21;3,11;1]}^{\lambda} (M) = (m_{12} - m_{23}) \chi_-(M), \]
\[ A_{[21;3,01;0]}^{\lambda} (M) = - (m_{11} - m_{22}), \]
\[ A_{[21;3,01;1]}^{\lambda} (M) = - (m_{11} - m_{22}) \chi_+ (M \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) - (m_{12} - m_{23}) \chi_-(M), \]
\[ A_{[21;3,01;2]}^{\lambda} (M) = - (m_{12} - m_{23}) \chi_-(M) \chi_+ (M \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} [-1]), \]
\[ A_{[21;3,00;0]}^{\lambda} (M) = m_{11} - m_{22}, \]
\[ A_{[21;3,00;1]}^{\lambda} (M) = (m_{12} - m_{23}) \chi_-(M), \]

and
\[ B_{[21;3;11;0]}^{\lambda} (M) = m_{11} - m_{22}, \]
\[ B_{[21;3;11;1]}^{\lambda} (M) = (m_{12} - m_{23}) \chi_-(M), \]
\[ B_{[21;3;01;0]}^{\lambda} (M) = - (m_{11} - m_{22} + 1) + 1, \]
\[ B_{[21;3;01;1]}^{\lambda} (M) = - (m_{11} - m_{22}) \chi_+ (M) - (m_{12} - m_{23}) \chi_- (M), \]
\[ B_{[21;3;01;2]}^{\lambda} (M) = - (m_{12} - m_{23} - 1) \chi_- (M) \chi_+ (M), \]
\[ B_{[21;3;00;0]}^{\lambda} (M) = m_{11} - m_{22}, \]
\[ B_{[21;3;00;1]}^{\lambda} (M) = (m_{12} - m_{23}) \chi_-(M). \]

We have
\[ A_{[21;3,01;1]}^{\lambda} (M) = - (m_{11} - m_{22}) \chi_- \chi_- (M) - (m_{12} - m_{23}) (1 - \chi_+^2 (M)) \]
\[ = - m_{12} + m_{23} + \delta(M) \chi_- (M) \]
\[ = - m_{12} + m_{23} + \delta(M) \chi_+ (M), \]
\[ B_{[21;3,01;1]}^{\lambda} (M) = - (m_{11} - m_{22}) \chi_+ (M) - (m_{12} - m_{23}) (1 - \chi_+ (M)) \]
\[ = - m_{12} + m_{23} + \delta(M). \]

Hence we obtain \( A_{[21;3,01;1]}^{\lambda} (M) = B_{[21;3,01;1]}^{\lambda} (M) \). Here we use the relations \((4.5)\), \((4.7)\) and \((4.9)\).

We have
\[ A_{[21;3,01;2]}^{\lambda} (M) = - (m_{12} - m_{23}) \chi_- (M) \chi_- (M) = 0, \]
\[ B_{[21;3,01;2]}^{\lambda} (M) = - (m_{12} - m_{23} - 1) \chi_- (M) \chi_+ (M) = 0. \]

Hence we obtain \( A_{[21;3,01;2]}^{\lambda} (M) = B_{[21;3,01;2]}^{\lambda} (M) \). Here we use the relations \((4.5)\) and \((4.10)\).

It is trivial that the equations \((4.17)\) hold for other \((j,k,l)\).

- the proof of \( E_{23} \circ i_{e_3}^\lambda = i_{e_3}^\lambda \circ E_{23} \).

We have
\[ A_{[23;3,11;0]}^{\lambda} (M) = (m_{13} - m_{12}), \]
\[ A_{[23;3,11;1]}^{\lambda} (M) = \{m_{13} - m_{12} - \delta(M)\} \chi_-(M), \]
\[ A_{[23;3,01;0]}^{\lambda} (M) = - (m_{13} - m_{12}), \]
\[ A_{[23;3,01;1]}^{\lambda} (M) = - (m_{13} - m_{12}) \chi_+ (M \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) - \{m_{13} - m_{12} - \delta(M)\} \chi_-(M), \]
\[ A_{[23;3,01;2]}^{\lambda} (M) = - \{m_{13} - m_{12} - \delta(M)\} \chi_-(M) \chi_+ (M \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [-1]), \]
\[ A_{[23;3,00;0]}^{\lambda} (M) = m_{13} - m_{12}, \]
\[ A_{[23;3,00;1]}^{\lambda} (M) = \{m_{13} - m_{12} - \delta(M)\} \chi_-(M), \]
and
\[
B^{\lambda}_{[23;3,11;0]}(M) = m_{13} - m_{12},
\]
\[
B^{\lambda}_{[23;3,11;1]}(M) = \{m_{13} - m_{12} - \delta(M)\} \chi_{-}(M),
\]
\[
B^{\lambda}_{[23;3,01;0]}(M) = -(m_{13} - m_{12}),
\]
\[
B^{\lambda}_{[23;3,01;1]}(M) = -(m_{13} - m_{12} + 1) \chi_{+}(M) - \{m_{13} - m_{12} - \delta(M) + 1\} \chi_{-}^{-1}(M) + 1,
\]
\[
B^{\lambda}_{[23;3,01;2]}(M) = -\{m_{13} - m_{12} - \delta(M) + 2\} \chi_{-}^{-1}(M) \chi_{+}(M),
\]
\[
B^{\lambda}_{[23;3,00;0]}(M) = m_{13} - m_{12},
\]
\[
B^{\lambda}_{[23;3,00;1]}(M) = \{m_{13} - m_{12} - \delta(M)\} \chi_{-}(M).
\]

We have
\[
A^{\lambda}_{[23;3,01;1]}(M) = -(m_{13} - m_{12})(\chi_{+}^{-1}(M) + \chi_{-}(M)) + \delta(M) \chi_{-}(M)
= -m_{13} + m_{12} + \delta(M) \chi_{-}(M),
\]
\[
B^{\lambda}_{[23;3,01;1]}(M) = -(m_{13} - m_{12} + 1)(\chi_{+}(M) + \chi_{-}^{-1}(M)) + 1 + \delta(M) \chi_{-}^{-1}(M)
= -m_{13} + m_{12} + \delta(M) \chi_{-}(M).
\]

Hence \( A^{\lambda}_{[23;3,01;1]}(M) = B^{\lambda}_{[23;3,01;1]}(M) \). Here we use the relations \((4.5)\) and \((4.9)\).

We have
\[
A^{\lambda}_{[23;3,01;2]}(M) = -(m_{13} - m_{12} - \delta(M)) \chi_{-}(M) \chi_{+}^{-1}(M)
= 0,
\]
\[
B^{\lambda}_{[23;3,01;2]}(M) = -(m_{13} - m_{12} - \delta(M) + 2) \chi_{-}^{-1}(M) \chi_{+}(M)
= 0.
\]

Hence \( A^{\lambda}_{[23;3,01;2]}(M) = B^{\lambda}_{[23;3,01;2]}(M) \). Here we use the relations \((4.5)\) and \((4.10)\).

It is trivial that the equations \((4.17)\) hold for other \((j,k,l)\).

• the proof of \( E_{32} \circ i^{\lambda}_{e_{3}} = i^{\lambda}_{e_{3}} \circ E_{32} \).
\[
A^{\lambda}_{[32;3,11;0]}(M) = m_{22} - m_{33},
\]
\[
A^{\lambda}_{[32;3,11;1]}(M) = \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M),
\]
\[
A^{\lambda}_{[32;3,01;0]}(M) = -(m_{22} - m_{33}),
\]
\[
A^{\lambda}_{[32;3,01;1]}(M) = -(m_{22} - m_{33}) \chi_{+}(M^{(o \circ 1)}) - \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M),
\]
\[
A^{\lambda}_{[32;3,01;2]}(M) = -\{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M) \chi_{+}(M^{(o \circ 1)} [-1]),
\]
\[
A^{\lambda}_{[32;3,00;0]}(M) = (m_{22} - m_{33}),
\]
\[
A^{\lambda}_{[32;3,00;1]}(M) = \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M),
\]

and
\[
B^{\lambda}_{[32;3,11;0]}(M) = m_{22} - m_{33},
\]
\[
B^{\lambda}_{[32;3,11;1]}(M) = \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M),
\]
\[
B^{\lambda}_{[32;3,01;0]}(M) = -(m_{22} - m_{33}),
\]
\[
B^{\lambda}_{[32;3,01;1]}(M) = -(m_{22} - m_{33} + 1) \chi_{+}(M) - \{m_{22} - m_{33} + \delta(M) - 1\} \chi_{+}^{(1)}(M),
\]
\[
B^{\lambda}_{[32;3,01;2]}(M) = -\{m_{22} - m_{33} + \delta(M)\} \chi_{+}^{(1)}(M) \chi_{+}(M),
\]
\[
B^{\lambda}_{[32;3,00;0]}(M) = (m_{22} - m_{33} + 1) - 1,
\]
\[
B^{\lambda}_{[32;3,00;1]}(M) = \{m_{22} - m_{33} + \delta(M) + 1\} \chi_{+}(M) - \chi_{+}(M).
We have
\[
A^\lambda_{[32;3,01;1]}(M) - B^\lambda_{[32;3,01;1]}(M) \\
= -(m_{22} - m_{33})\chi_+^{(1)}(M) - \{m_{22} - m_{33} + \delta(M)\} \chi_+(M) \\
+ (m_{22} - m_{33} + 1)\chi_+(M) + \{m_{22} - m_{33} + \delta(M) - 1\} \chi_+^{(1)}(M) \\
= -\delta(M - 1)\chi_+(M) + (\delta(M - 1)\chi_+^{(1)}(M) = 0.
\]
Hence we obtain the equation (4.17) for \((j, k, l) = (0, 1, 1)\). Here we use the relations (4.5) and (4.7).

We have
\[
A^\lambda_{[32;3,01;2]}(M) - B^\lambda_{[32;3,01;2]}(M) \\
= -\{m_{22} - m_{33} + \delta(M)\} \left(\chi_+(M)\chi_+\left(M\left(\begin{smallmatrix}a & -1 \\ -1 & b \end{smallmatrix}\right)\right) - \chi_+^{(1)}(M)\chi_+(M)\right) \\
= -\{m_{22} - m_{33} + \delta(M)\} \left(\chi_+^{(1)}(M) - \chi_+^{(1)}(M)\right) = 0.
\]
Hence we obtain the equation (4.17) for \((j, k, l) = (0, 1, 2)\). Here we use the relations (4.5) and (4.11).

It is trivial that the equations (4.17) hold for other \((j, k, l)\).

In these computations, we use the relations in Lemma 4.1 frequently. So we use these relations without notice in the proof of formula 1 and formula 2. Next, we check the equations 4.17 for \(i = 2\), that is, the case of formula 2.

- the proof of \(E_{12} \circ i^\lambda_{e_2} = i^\lambda_{e_2} \circ E_{12} \).

We have
\[
A^\lambda_{[12;2,11;0]}(M) = (m_{12} - m_{11})(m_{22} - m_{33}), \\
A^\lambda_{[12;2,11;1]}(M) = -(m_{12} - m_{11})\tilde{D}(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right))\chi_-(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right)) \\
+ (m_{23} - m_{22})\chi_+(M)(m_{22} - m_{33} + 1), \\
A^\lambda_{[12;2,11;2]}(M) = -(m_{23} - m_{22})\chi_+(M)\tilde{D}(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right))\chi_-(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right)), \\
A^\lambda_{[12;2,01;0]}(M) = -(m_{12} - m_{11})(m_{22} - m_{33}), \\
A^\lambda_{[12;2,01;1]}(M) = (m_{12} - m_{11})\tilde{C}_1(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right)) - (m_{23} - m_{22})\chi_+(M)(m_{22} - m_{33} + 1), \\
A^\lambda_{[12;2,01;2]}(M) = (m_{23} - m_{22})\chi_+(M)\tilde{C}_1(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right)), \\
A^\lambda_{[12;2,00;0]}(M) = -(m_{12} - m_{11})(m_{23} - m_{22}), \\
A^\lambda_{[12;2,00;1]}(M) = -(m_{12} - m_{11})\tilde{C}_1(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right))\chi_-(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right)) \\
- (m_{23} - m_{22})\chi_+(M)(m_{23} - m_{22} - 1), \\
A^\lambda_{[12;2,00;2]}(M) = -(m_{23} - m_{22})\chi_+(M)\tilde{C}_1(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right))\chi_-(M\left(\begin{smallmatrix}a & 0 \\ 0 & b \end{smallmatrix}\right)), \\
\]
and
\[
B^\lambda_{[12;2,11;0]}(M) = (m_{12} - m_{11} + 1)(m_{22} - m_{33}) - (m_{22} - m_{33}), \\
B^\lambda_{[12;2,11;1]}(M) = -(m_{12} - m_{11})\tilde{D}(M)\chi_-(M) \\
+ (m_{23} - m_{22})\chi_+^{(-1)}(M)(m_{22} - m_{33} + \tilde{C}_1(M), \\
B^\lambda_{[12;2,11;2]}(M) = -(m_{23} - m_{22} - 1)\chi_+^{(-1)}(M)\tilde{D}(M)\chi_-(M), \\
B^\lambda_{[12;2,01;0]}(M) = -(m_{12} - m_{11})(m_{22} - m_{33}), \\
B^\lambda_{[12;2,01;1]}(M) = (m_{12} - m_{11} - 1)\tilde{C}_1(M) - (m_{23} - m_{22})\chi_+(M)(m_{22} - m_{33}).
\]
Hence we obtain

\[
B_{[12;2,00,2]}^\lambda (M) = - (m_{23} - m_{22} - 2)\chi_+^{-1}(M)\bar{C}_1(M)\chi_-(M).
\]

We have

\[
A_{[12;2,11;1]}^\lambda (M) = - (m_{12} - m_{11})(-m_{22} + m_{33} + \delta(M) - 1)(1 - \chi_+(M))
\]
\[
+ (m_{23} - m_{22})\chi_+(M)(m_{22} - m_{33} + 1)
\]
\[
= - (m_{12} - m_{11})(\bar{D}(M) - 1) - (m_{12} - m_{11} + m_{22} - m_{33} + 1)\delta(M)\chi_+(M),
\]

\[
B_{[12;2,11;1]}^\lambda (M) = - (m_{12} - m_{11})(-m_{22} + m_{33} + \delta(M))(1 - \chi_+^{-1}(M))
\]
\[
+ (m_{23} - m_{22})\chi_+^{-1}(M)(m_{22} - m_{33} + 1) + (m_{12} - m_{11} - \delta(M)\chi_+(M))
\]
\[
= - (m_{12} - m_{11})(\bar{D}(M) - 1) - (m_{12} - m_{11} + m_{22} - m_{33} + 1)\delta(M)\chi_+^{-1}(M)
\]
\[
- \delta(M)\chi_+(M)
\]
\[
= - (m_{12} - m_{11})(\bar{D}(M) - 1) - (m_{12} - m_{11} + m_{22} - m_{33} + 1)\delta(M)\chi_+^{-1}(M).
\]

Hence we obtain \(A_{[12;2,11;1]}^\lambda (M) = B_{[12;2,11;1]}^\lambda (M).\)

We have

\[
A_{[12;2,11;2]}^\lambda (M) = - (m_{23} - m_{22})\bar{D}(M (0,0) [-1]) \chi_+(M)\chi_+^{-1}(M) = 0,
\]

\[
B_{[12;2,11;2]}^\lambda (M) = - (m_{23} - m_{22} - 1)\bar{D}(M)\chi_+^{-1}(M)\chi_-(M) = 0.
\]

Hence we obtain \(A_{[12;2,11;2]}^\lambda (M) = B_{[12;2,11;2]}^\lambda (M).\)

We have

\[
A_{[12;2,01,1]}^\lambda (M) = (m_{12} - m_{11})\{m_{12} - m_{11} - 1 - (\delta(M) - 1)\chi_+(M)\}
\]
\[
- (m_{23} - m_{22})\chi_+(M)(m_{22} - m_{33} + 1)
\]
\[
= (m_{12} - m_{11})(m_{12} - m_{11} - 1)
\]
\[
- \chi_+(M)((m_{23} - m_{22})(m_{22} - m_{33} + 1) + (m_{12} - m_{11})(\delta(M) - 1)),
\]

\[
B_{[12;2,01,1]}^\lambda (M) = (m_{12} - m_{11} - 1)(m_{12} - m_{11} - \delta(M)\chi_+(M))
\]
\[
- (m_{23} - m_{22})\chi_+(M)(m_{22} - m_{33})
\]
\[
= (m_{12} - m_{11})(m_{12} - m_{11} - 1)
\]
\[
- \chi_+(M)((m_{23} - m_{22})(m_{22} - m_{33}) + (m_{12} - m_{11} - 1)\delta(M)).
\]

Therefore

\[
A_{[12;2,01,1]}^\lambda (M) - B_{[12;2,01,1]}^\lambda (M) = -\chi_+(M)((m_{23} - m_{22}) - (m_{12} - m_{11}) + \delta(M)) = 0.
\]

Hence we obtain \(A_{[12;2,01,1]}^\lambda (M) = B_{[12;2,01,1]}^\lambda (M).\)

We have

\[
A_{[12;2,01,2]}^\lambda (M) = (m_{23} - m_{22})\chi_+(M)(m_{23} - m_{22} - 1 + (\delta(M) - 1)\chi_+^{-1}(M))
\]
\[
= (m_{23} - m_{22})(m_{23} - m_{22} - 1)\chi_+(M),
\]

\[
B_{[12;2,01,2]}^\lambda (M) = (m_{23} - m_{22} - 1)(m_{23} - m_{22})\chi_+(M).
\]

Hence we obtain \(A_{[12;2,01,2]}^\lambda (M) = B_{[12;2,01,2]}^\lambda (M).\)
We have
\[
A_{[12;2,0;0;1]}^\lambda (M) = -(m_{12} - m_{11})(m_{12} - m_{11} - 1)\chi_+^{-1}(M) \\
- (m_{23} - m_{22})(m_{23} - m_{22} - 1)\chi_+(M) \\
= -(\bar{C}_1(M))(\bar{C}_1(M) - 1)(\chi_+^{-1}(M) + \chi_+(M)) \\
= -(\bar{C}_1(M))(\bar{C}_1(M) - 1),
\]
\[
B_{[12;2,0;0;1]}^\lambda (M) = -(m_{12} - m_{11} - 1)(m_{12} - m_{11})\chi_-(M) \\
- (m_{23} - m_{22})(m_{23} - m_{22} - 1)\chi_+^{-1}(M) \\
= -(\bar{C}_1(M))(\bar{C}_1(M) - 1)(\chi_-(M) + \chi_+^{-1}(M)) \\
= -(\bar{C}_1(M))(\bar{C}_1(M) - 1).
\]
Hence we obtain \(A_{[12;2,0;0;1]}^\lambda (M) = B_{[12;2,0;0;1]}^\lambda (M)\).

We have
\[
A_{[12;2,0;0;2]}^\lambda (M) = -(m_{23} - m_{22})\bar{C}_1 (M \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [-1]) \chi_+(M)\chi_+^{-1}(M) = 0,
\]
\[
B_{[12;2,0;0;2]}^\lambda (M) = -(m_{23} - m_{22} - 2)\bar{C}_1(M)\chi_+^{-1}(M)\chi_-(M) = 0.
\]
Hence we obtain \(A_{[12;2,0;0;2]}^\lambda (M) = B_{[12;2,0;0;2]}^\lambda (M)\).

It is trivial that the equations (4.1.7) hold for \((j, k, l) = (1, 1, 0), (0, 1, 0)\) and \((0, 0, 0)\).

- the proof of \(E_{21} \circ i_{e_2}^\lambda = i_{e_2}^\lambda \circ E_{21}\).

We have
\[
A_{[21;2,1;1;0]}^\lambda (M) = (m_{11} - m_{22})(m_{22} - m_{33}) \\
A_{[21;2,1;1;1]}^\lambda (M) = -(m_{11} - m_{22})\bar{D}(M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} ) \chi_-(M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} ) \\
+ (m_{12} - m_{23})\chi_-(M)(m_{22} - m_{33} + 1), \\
A_{[21;2,1;1;2]}^\lambda (M) = -(m_{12} - m_{23})\chi_-(M)\bar{D}(M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} [-1]) \chi_-(M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} [-1]) , \\
A_{[21;2,1;2;0;1]}^\lambda (M) = -(m_{11} - m_{22})(m_{22} - m_{33}), \\
A_{[21;2,1;2;0;1;1]}^\lambda (M) = (m_{11} - m_{22})\bar{C}_1 (M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} ) -(m_{12} - m_{23})\chi_-(M)(m_{22} - m_{33} + 1), \\
A_{[21;2,1;2;0;1;2]}^\lambda (M) = (m_{12} - m_{23})\chi_-(M)\bar{C}_1 (M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} [-1]), \\
A_{[21;2,1;2;0;0;0;0]}^\lambda (M) = -(m_{11} - m_{22})(m_{23} - m_{22}), \\
A_{[21;2,1;2;0;0;0;1]}^\lambda (M) = -(m_{11} - m_{22})\bar{C}_1 (M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} ) \chi_-(M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} ) \\
- (m_{12} - m_{23})\chi_-(M)(m_{23} - m_{22} - 1), \\
A_{[21;2,1;2;0;0;0;2]}^\lambda (M) = -(m_{12} - m_{23})\chi_-(M)\bar{C}_1 (M \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} [-1]) \chi_-(M \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} [-1]) ,
\]
and
\[
B_{[21;2,1;1;0]}^\lambda (M) = (m_{11} - m_{22})(m_{22} - m_{33}), \\
B_{[21;2,1;1;1]}^\lambda (M) = -(m_{11} - m_{22} - 1)\bar{D}(M)\chi_-(M) \\
+ (m_{12} - m_{23} + 1)\chi_+(M)(m_{22} - m_{33}), \\
B_{[21;2,1;1;2]}^\lambda (M) = -(m_{12} - m_{23})\chi_+(M)\bar{D}(M)\chi_-(M), \\
B_{[21;2,1;2;0;1;0]}^\lambda (M) = -(m_{11} - m_{22} + 1)(m_{22} - m_{33}) + (m_{22} - m_{33}), \\
B_{[21;2,1;2;0;1;1]}^\lambda (M) = (m_{11} - m_{22})\bar{C}_1 (M)
Therefore

\[ A_{[21,2;11,1]}^\lambda (M) = - (m_{11} - m_{22})(-m_{22} + m_{33} + \delta(M) + 1)\chi_-(M) \\
+ (m_{12} - m_{23})\chi_-(M)(m_{22} - m_{33} + 1) \\
= - (m_{11} - m_{22})(\delta(M) + 1)\chi_-(M) \\
+ (m_{22} - m_{33})\{(m_{11} - m_{22})\chi_-(M) + (m_{12} - m_{23})\chi_-(M)\} \\
= - (m_{11} - m_{22} - 1)\delta(M)\chi_-(M) \\
+ (m_{22} - m_{33})\{(m_{11} - m_{22} - 1)\chi_-(M) + (m_{12} - m_{23})\chi_-(M)\}. \]

Therefore

\[ A_{[21,2;11,1]}^\lambda (M) - B_{[21,2;11,1]}^\lambda (M) \\
= (m_{22} - m_{33})\{(\delta(M) + 1)\chi_-(M) - (\delta(M) + 1)\chi_-(M)\} = 0. \]

Hence we obtain \( A_{[21,2;11,1]}^\lambda (M) = B_{[21,2;11,1]}^\lambda (M) \).

We have

\[ A_{[21,2;11,2]}^\lambda (M) = - (m_{12} - m_{23})\chi_-(M)(-m_{22} + m_{33} + \delta(M))\chi_-(M) \\
= - (m_{12} - m_{23})\bar{D}(M)\chi_-(M)\chi_-(M), \]

\[ B_{[21,2;11,2]}^\lambda (M) = - (m_{12} - m_{23})\bar{D}(M)\chi_-(M)\chi_-(M). \]

Hence we obtain \( A_{[21,2;11,2]}^\lambda (M) = B_{[21,2;11,2]}^\lambda (M) \).

We have

\[ A_{[21,2;01,1]}^\lambda (M) = (m_{11} - m_{22})\{m_{22} - m_{22} + (\delta(M) + 1)\chi_-(M)\} \\
- (m_{12} - m_{23})\chi_-(M)(m_{22} - m_{33} + 1), \]

\[ B_{[21,2;01,1]}^\lambda (M) = (m_{11} - m_{22})(m_{22} - m_{22} + \delta(M)\chi_-(M)) \\
- (m_{12} - m_{23} + 1)\chi_-(M)(m_{22} - m_{33}) - \bar{D}(M)\chi_-(M). \]

Therefore

\[ A_{[21,2;01,1]}^\lambda (M) - B_{[21,2;01,1]}^\lambda (M) \\
= (m_{11} - m_{22})\chi_-(M) - (m_{12} - m_{23})\chi_-(M) + (m_{22} - m_{33})\chi_-(M) + \bar{D}(M)\chi_-(M) = 0. \]

Hence we obtain \( A_{[21,2;01,1]}^\lambda (M) = B_{[21,2;01,1]}^\lambda (M) \).
We have
\[ A_{[21,2;00;1]}^\lambda (M) = (m_{12} - m_{23})\chi_-(M) \{ m_{12} - m_{11} - (\delta(M) + 1)\chi_+^{-1}(M) \} = (m_{12} - m_{23})(m_{12} - m_{11})\chi_-(M), \]
\[ B_{[21,2;00;1]}^\lambda (M) = (m_{12} - m_{23})(m_{12} - m_{11})\chi_-(M). \]
Hence we obtain \( A_{[21,2;01;2]}^\lambda (M) = B_{[21,2;01;2]}^\lambda (M). \)

We have
\[ A_{[21,2;00;1]}^\lambda (M) = - (m_{11} - m_{22})\tilde{C}_1 (M \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \chi_-(M) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - (m_{12} - m_{23})\chi_-(M)(m_{23} - m_{22} - 1) \]
\[ = \begin{cases} 0 & \text{if } \delta(M) > -1, \\ -(m_{12} - m_{23})(m_{23} - m_{22} - 1) & \text{if } \delta(M) = -1, \\ -(m_{11} - m_{22})(m_{12} - m_{11} + 1) - (m_{12} - m_{23})(m_{23} - m_{22} - 1) & \text{if } \delta(M) < -1, \end{cases} \]
\[ B_{[21,2;00;1]}^\lambda (M) = - (m_{11} - m_{22} - 1)\tilde{C}_1 (M)\chi_-(M) - (m_{12} - m_{23} + 1)\chi_-(M)(m_{23} - m_{22}) \]
\[ = \begin{cases} 0 & \text{if } \delta(M) > -1, \\ -(m_{11} - m_{22} - 1)(m_{12} - m_{11}) & \text{if } \delta(M) = -1, \\ -(m_{11} - m_{22} - 1)(m_{12} - m_{11}) - (m_{12} - m_{23} + 1)(m_{23} - m_{22}) & \text{if } \delta(M) < -1. \end{cases} \]
Hence we obtain \( A_{[21,2;00;1]}^\lambda (M) = B_{[21,2;00;1]}^\lambda (M). \)

We have
\[ A_{[21,2;00;2]}^\lambda (M) = - (m_{12} - m_{23})\chi_-(M)(m_{12} - m_{11})\chi_-(M) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} [-1] \]
\[ = - (m_{12} - m_{23})(m_{12} - m_{11})\chi_-(M)\chi_-(M), \\ B_{[21,2;00;2]}^\lambda (M) = - (m_{12} - m_{23})\chi_-(M)(m_{12} - m_{11})\chi_-(M). \]
Hence we obtain \( A_{[21,2;00;2]}^\lambda (M) = B_{[21,2;00;2]}^\lambda (M). \)

It is trivial that the equations \( \text{(14.17)} \) hold for \((j, k, l) = (1, 1, 0), (0, 1, 0)\) and \((0, 0, 0)\).

- the proof of \( E_{23} \circ i_{\psi_2}^\lambda = i_{\psi_2}^\lambda \circ E_{23} \).

We have
\[ A_{[23,2;11;0]}^\lambda (M) = (m_{13} - m_{12})(m_{22} - m_{33}), \\ A_{[23,2;11;1]}^\lambda (M) = - (m_{13} - m_{12})\tilde{D} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \chi_-(M) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \{ m_{13} - m_{12} - \delta(M) \} \chi_-(M)(m_{22} - m_{33} + 1), \\ A_{[23,2;11;2]}^\lambda (M) = - \{ m_{13} - m_{12} - \delta(M) \} \chi_-(M)\tilde{D} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [-1] \chi_-(M) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [-1], \\ A_{[23,2;01;0]}^\lambda (M) = - (m_{13} - m_{12})(m_{22} - m_{33}), \\ A_{[23,2;01;1]}^\lambda (M) = (m_{13} - m_{12})\tilde{C}_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \{ m_{13} - m_{12} - \delta(M) \} \chi_-(M)(m_{22} - m_{33} + 1), \\ A_{[23,2;01;2]}^\lambda (M) = \{ m_{13} - m_{12} - \delta(M) \} \chi_-(M)\tilde{C}_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [-1], \\ A_{[23,2;00;0]}^\lambda (M) = - (m_{13} - m_{12})(m_{23} - m_{22}), \\ A_{[23,2;00;1]}^\lambda (M) = - (m_{13} - m_{12})\tilde{C}_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \chi_-(M) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \{ m_{13} - m_{12} - \delta(M) \} \chi_-(M)(m_{23} - m_{22} - 1), \\ A_{[23,2;00;2]}^\lambda (M) = - \{ m_{13} - m_{12} - \delta(M) \} \chi_-(M)\tilde{C}_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [-1] \chi_-(M) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [-1]. \]
and

\[
B_{[23;2,11;0]}^\lambda (M) = (m_{13} - m_{12})(m_{22} - m_{33}), \\
B_{[23;2,11;1]}^\lambda (M) = -(m_{13} - m_{12} + 1)\tilde{D}(M)\chi_-(M) + \{m_{13} - m_{12} - \delta(M) - 1\}\chi_-^{(1)}(M)(m_{22} - m_{33}), \\
B_{[23;2,11;2]}^\lambda (M) = -\{m_{13} - m_{12} - \delta(M)\}\chi_-^{(1)}(M)\tilde{D}(M)\chi_-(M), \\
B_{[23;2,01;0]}^\lambda (M) = -(m_{13} - m_{12})(m_{22} - m_{33}), \\
B_{[23;2,01;1]}^\lambda (M) = (m_{13} - m_{12} + 1)\tilde{C}_1(M) - \{m_{13} - m_{12} - \delta(M) - 1\}\chi_-^{(1)}(M)(m_{22} - m_{33}) - (m_{23} - m_{22}), \\
B_{[23;2,00;0]}^\lambda (M) = -(m_{13} - m_{12} + 1)\tilde{C}_1(M)\chi_-(M) + \{m_{13} - m_{12} - \delta(M) - 1\}\chi_-^{(1)}(M)(m_{23} - m_{22}), \\
B_{[23;2,00;1]}^\lambda (M) = -\{m_{13} - m_{12} - \delta(M)\}\chi_-^{(1)}(M)\tilde{C}_1(M)\chi_-(M), \\
B_{[23;2,00;2]}^\lambda (M) = -\{m_{13} - m_{12} - \delta(M)\}\chi_-^{(1)}(M)\tilde{C}_1(M)\chi_-(M).
\]

We have

\[
A_{[23;2,11;1]}^\lambda (M) = -(m_{13} - m_{12})(-m_{22} + m_{33} + \delta(M) - 1)\chi_-^{(1)}(M) + \{m_{13} - m_{12} - \delta(M)\}\chi_-(M)(m_{22} - m_{33} + 1), \\
B_{[23;2,11;1]}^\lambda (M) = -(m_{13} - m_{12} + 1)(-m_{22} + m_{33} + \delta(M))\chi_-(M) + \{m_{13} - m_{12} - \delta(M) - 1\}\chi_-^{(1)}(M)(m_{22} - m_{33}).
\]

Therefore

\[
A_{[23;2,11;1]}^\lambda (M) - B_{[23;2,11;1]}^\lambda (M) = (m_{13} - m_{12} - m_{22} + m_{33})\delta(M) + 1)(\chi_-(M) - \chi_-^{(1)}(M)) = 0.
\]

Hence we obtain

\[
A_{[23;2,11;1]}^\lambda (M) = B_{[23;2,11;1]}^\lambda (M).
\]

We have

\[
A_{[23;2,11;2]}^\lambda (M) = -\{m_{13} - m_{12} - \delta(M)\}\chi_-(M)(-m_{22} + m_{33} + \delta(M))\chi_-^{(1)}(M), \\
B_{[23;2,11;2]}^\lambda (M) = -\{m_{13} - m_{12} - \delta(M)\}\tilde{D}(M)\chi_-^{(1)}(M)\chi_-(M).
\]

Hence we obtain

\[
A_{[23;2,11;2]}^\lambda (M) = B_{[23;2,11;2]}^\lambda (M).
\]

We have

\[
A_{[23;2,01;1]}^\lambda (M) = (m_{13} - m_{12})\{m_{23} - m_{22} + (\delta(M) + 1)\chi_-(M)\} + \{m_{13} - m_{12} - \delta(M)\}\chi_-(M)(m_{22} - m_{33} + 1), \\
B_{[23;2,01;1]}^\lambda (M) = (m_{13} - m_{12} + 1)(m_{23} - m_{22} + \delta(M))\chi_-(M) + \{m_{13} - m_{12} - \delta(M)\}\chi_-^{(1)}(M)(m_{22} - m_{33}) - (m_{23} - m_{22}) + \{m_{13} - m_{12} - \delta(M) - 1\}\chi_-^{(1)}(M)(m_{22} - m_{33} + 1).
\]

Hence we obtain

\[
A_{[23;2,01;1]}^\lambda (M) = B_{[23;2,01;1]}^\lambda (M).
\]
We have
\[
A^\lambda_{[32;2,01;2]}(M) = \{m_{13} - m_{12} - \delta(M)\} \chi_- (M) \{m_{12} - m_{11} - (\delta(M) + 1) \chi^{-1}_+(M)\} = \{m_{13} - m_{12} - \delta(M)\} (m_{12} - m_{11}) \chi_- (M),
\]
\[
B^\lambda_{[32;2,01;2]}(M) = \{m_{13} - m_{12} - \delta(M)\} \tilde{C}_1 (M) \chi_- (M) = \{m_{13} - m_{12} - \delta(M)\} (m_{12} - m_{11}) \chi_- (M).
\]
Hence we obtain \( A^\lambda_{[32;2,01;2]}(M) = B^\lambda_{[32;2,01;2]}(M) \).

We have
\[
A^\lambda_{[32;2,00;1]}(M) = - (m_{13} - m_{12})(m_{12} - m_{11} + 1) \chi^{-1}_-(M) - \{m_{13} - m_{12} - \delta(M)\} \chi_- (M) (m_{23} - m_{22} - 1),
\]
\[
B^\lambda_{[32;2,00;1]}(M) = - (m_{13} - m_{12} + 1)(m_{12} - m_{11}) \chi_- (M) - \{m_{13} - m_{12} - \delta(M) - 1\} \chi^{-1}_-(M) (m_{23} - m_{22}).
\]
Therefore
\[
A^\lambda_{[32;2,00;1]}(M) - B^\lambda_{[32;2,00;1]}(M) = (m_{13} - m_{12} + m_{23} - m_{22}) (\delta(M) + 1) (\chi_- (M) - \chi^{-1}_-(M)) = 0.
\]
Hence we obtain \( A^\lambda_{[32;2,00;1]}(M) = B^\lambda_{[32;2,00;1]}(M) \).

It is trivial that the equations (14.17) hold for \((j, k, l) = (1, 1, 0), (0, 1, 0)\) and \((0, 0, 0)\).

- the proof of \( E_{32} \circ i^\lambda_{e_2} = i^\lambda_{e_2} \circ E_{32} \).

We have
\[
A^\lambda_{[32;2,11;0]}(M) = (m_{22} - m_{33})(m_{22} - m_{33} - 1),
\]
\[
A^\lambda_{[32;2,11;1]}(M) = - (m_{22} - m_{33}) \tilde{D} (M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) \chi_-(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) + \{m_{22} - m_{33} + \delta(M)\} \chi_+ (M) (m_{22} - m_{33}),
\]
\[
A^\lambda_{[32;2,11;2]}(M) = - \{m_{22} - m_{33} + \delta(M)\} \chi_+ (M) \tilde{D} (M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) \chi_-(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}),
\]
\[
A^\lambda_{[32;2,01;0]}(M) = - (m_{22} - m_{33})(m_{22} - m_{33} - 1),
\]
\[
A^\lambda_{[32;2,01;1]}(M) = (m_{22} - m_{33}) \tilde{C}_1 (M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) - \{m_{22} - m_{33} + \delta(M)\} \chi_+ (M) (m_{22} - m_{33}),
\]
\[
A^\lambda_{[32;2,01;2]}(M) = \{m_{22} - m_{33} + \delta(M)\} \chi_+ (M) \tilde{C}_1 (M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}),
\]
\[
A^\lambda_{[32;2,00;0]}(M) = - (m_{22} - m_{33})(m_{23} - m_{22} + 1),
\]
\[
A^\lambda_{[32;2,00;1]}(M) = - (m_{22} - m_{33}) \tilde{C}_1 (M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) \chi_-(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) - \{m_{22} - m_{33} + \delta(M)\} \chi_+ (M) (m_{23} - m_{22}),
\]
\[
A^\lambda_{[32;2,00;2]}(M) = - \{m_{22} - m_{33} + \delta(M)\} \chi_+ (M) \tilde{C}_1 (M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) \chi_-(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}),
\]
and

\[ B_{[32,2,11;0]}^\lambda (M) = (m_{22} - m_{33} - 1)(m_{22} - m_{33}), \]
\[ B_{[32,2,11;1]}^\lambda (M) = - (m_{22} - m_{33}) \tilde{D}(M) \chi_-(M) \]
\[ + \{m_{22} - m_{33} + \delta(M)\} \chi_+^{(-1)}(M) (m_{22} - m_{33}), \]
\[ B_{[32,2,11;2]}^\lambda (M) = - \{m_{22} - m_{33} + \delta(M) + 1\} \chi_+^{(-1)}(M) \tilde{D}(M) \chi_-(M), \]
\[ B_{[32,2,01;0]}^\lambda (M) = - (m_{22} - m_{33} - 1)(m_{22} - m_{33}), \]
\[ B_{[32,2,01;1]}^\lambda (M) = (m_{22} - m_{33}) \tilde{C}_1(M) \]
\[ - \{m_{22} - m_{33} + \delta(M) - 1\} \chi_+(M) (m_{22} - m_{33}), \]
\[ B_{[32,2,01;2]}^\lambda (M) = (m_{22} - m_{33} + \delta(M)) \chi_+(M) \tilde{C}_1(M), \]
\[ B_{[32,2,00;0]}^\lambda (M) = - (m_{22} - m_{33})(m_{23} - m_{22}) - (m_{22} - m_{33}), \]
\[ B_{[32,2,00;1]}^\lambda (M) = - (m_{22} - m_{33} + 1) \tilde{C}_1(M) \chi_-(M) \]
\[ - \{m_{22} - m_{33} + \delta(M) + 1\} \chi_+^{(-1)}(M) (m_{23} - m_{22}) + \tilde{C}_1(M), \]
\[ B_{[32,2,00;2]}^\lambda (M) = - \{m_{22} - m_{33} + \delta(M) + 2\} \chi_+^{(-1)}(M) \tilde{C}_1(M) \chi_-(M). \]

We have

\[ A_{[32,2,11;1]}^\lambda (M) = - (m_{22} - m_{33})(-m_{22} + m_{33} + \delta(M))(1 - \chi_+(M)) \]
\[ + \{m_{22} - m_{33} + \delta(M)\} \chi_+(M) (m_{22} - m_{33}) \]
\[ = (m_{22} - m_{33})\{\tilde{D}(M) + 2\delta(M) \chi_+(M)\}, \]
\[ B_{[32,2,11;1]}^\lambda (M) = - (m_{22} - m_{33})(-m_{22} + m_{33} + \delta(M))(1 - \chi_+^{(-1)}(M)) \]
\[ + \{m_{22} - m_{33} + \delta(M)\} \chi_+^{(-1)}(M) (m_{22} - m_{33}) \]
\[ = (m_{22} - m_{33})\{\tilde{D}(M) + 2\delta(M) \chi_+^{(-1)}(M)\} \]
\[ = (m_{22} - m_{33})\{\tilde{D}(M) + 2\delta(M) \chi_+(M)\}. \]

Hence we obtain \( A_{[32,2,11;1]}^\lambda (M) = B_{[32,2,11;1]}^\lambda (M). \)

We have

\[ A_{[32,2,11;2]}^\lambda (M) = - \{m_{22} - m_{33} + \delta(M)\} \tilde{D} \left( M \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \right) \chi_+(M) \chi_+^{(-1)}(M) = 0, \]
\[ B_{[32,2,11;2]}^\lambda (M) = - \{m_{22} - m_{33} + \delta(M) + 1\} \tilde{D}(M) \chi_+^{(-1)}(M) \chi_-(M) = 0. \]

Hence we obtain \( A_{[32,2,11;2]}^\lambda (M) = B_{[32,2,11;2]}^\lambda (M). \)

We have

\[ A_{[32,2,01;1]}^\lambda (M) = (m_{22} - m_{33})\{m_{12} - m_{11} - (\delta(M) - 1) \chi_+(M)\} \]
\[ - \{m_{22} - m_{33} + \delta(M)\} \chi_+(M) (m_{22} - m_{33}) \]
\[ = (m_{22} - m_{33})\{(m_{12} - m_{11}) - \chi_+(M)(m_{22} - m_{33} + 2\delta(M) - 1)\}, \]
\[ B_{[32,2,01;1]}^\lambda (M) = (m_{22} - m_{33})\{m_{12} - m_{11} - \delta(M) \chi_+(M)\} \]
\[ - \{m_{22} - m_{33} + \delta(M) - 1\} \chi_+(M)(m_{22} - m_{33}) \]
\[ = (m_{22} - m_{33})\{(m_{12} - m_{11}) - \chi_+(M)(m_{22} - m_{33} + 2\delta(M) - 1)\}. \]

Hence we obtain \( A_{[32,2,01;1]}^\lambda (M) = B_{[32,2,01;1]}^\lambda (M). \)
We have
\[ A^\lambda_{[32;2,01;2]}(M) = \{m_{22} - m_{33} + \delta(M)\} \chi_+(M)(m_{23} - m_{22} + (\delta(M) - 1)\chi_-(M)) = \{m_{22} - m_{33} + \delta(M)\}(m_{23} - m_{22})\chi_+(M), \]
\[ B^\lambda_{[32;2,01;2]}(M) = \{m_{22} - m_{33} + \delta(M)\}(m_{23} - m_{22})\chi_+(M). \]
Hence we obtain \( A^\lambda_{[32;2,01;2]}(M) = B^\lambda_{[32;2,01;2]}(M). \)
We have
\[ A^\lambda_{[32;2,00;1]}(M) = - (m_{22} - m_{33})(m_{12} - m_{11})\chi_-^{-1}(M) \]
\[ - (m_{22} - m_{33} + \delta(M))(m_{23} - m_{22})\chi_+(M) = - (m_{22} - m_{33})\bar{C}_1(M)\chi_-^{-1}(M) - (m_{22} - m_{33} + \delta(M))\bar{C}_1(M)\chi_+(M) \]
\[ = - (m_{22} - m_{33})\bar{C}_1(M)(\chi_-^{-1}(M) + \chi_+(M)) - \bar{C}_1(M)\delta(M)\chi_+(M) \]
\[ = - (m_{22} - m_{33})\bar{C}_1(M) - \bar{C}_1(M)\delta(M)\chi_+(M), \]
\[ B^\lambda_{[32;2,00;1]}(M) = - (m_{22} - m_{33} + 1)\bar{C}_1(M)\chi_-(M) \]
\[ - (m_{22} - m_{33} + \delta(M) + 1)\bar{C}_1(M)\chi_+^{-1}(M) + \bar{C}_1(M) \]
\[ = - (m_{22} - m_{33} + 1)\bar{C}_1(M)(\chi_-^{-1}(M) + \chi_+^{-1}(M)) + \bar{C}_1(M) \]
\[ = - \bar{C}_1(M)\delta(M)\chi_+^{-1}(M) \]
\[ = - (m_{22} - m_{33})\bar{C}_1(M) - \bar{C}_1(M)\delta(M)\chi_+(M). \]
Hence we obtain \( A^\lambda_{[32;2,00;1]}(M) = B^\lambda_{[32;2,00;1]}(M). \)
We have
\[ A^\lambda_{[32;2,00;2]}(M) = - (m_{22} - m_{33} + \delta(M))\bar{C}_1(M^{-1} \cdots [-1]) \chi_+(M)\chi_-^{-1}(M) = 0, \]
\[ B^\lambda_{[32;2,00;2]}(M) = - (m_{22} - m_{33} + \delta(M) + 2)\bar{C}_1(M)\chi_+^{-1}(M)\chi_-(M) = 0. \]
Hence we obtain \( A^\lambda_{[32;2,00;2]}(M) = B^\lambda_{[32;2,00;2]}(M). \)
It is trivial that the equations (4.17) hold for \((j, k, l) = (1, 1, 0), (0, 1, 0)\) and \((0, 0, 0)\).

At last, we check the equations [4.17] for \(i = 1\), that is, the case of formula 1 by direct computation.
• the proof of \( E_{12} \circ i^\lambda_{e_1} = i^\lambda_{e_1} \circ E_{12} \).
We have
\[ A^\lambda_{[12;1,11;0]}(M) = (m_{12} - m_{11})(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ A^\lambda_{[12;1,11;1]}(M) = - (m_{12} - m_{11})E(M^{0}_1) + (m_{23} - m_{22})\chi_+(M)(m_{13} - m_{12} + 1)(m_{22} - m_{23} + 1), \]
\[ A^\lambda_{[12;1,11;2]}(M) = - (m_{23} - m_{22})\chi_+(M)E(M^{0}_1)[-1], \]
\[ A^\lambda_{[12;1,01;0]}(M) = -(m_{12} - m_{11})(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ A^\lambda_{[12;1,01;1]}(M) = (m_{12} - m_{11})E(M^{0}_1) - (m_{23} - m_{22})\chi_+(M)(m_{13} - m_{12} + 1)(m_{22} - m_{23} + 1), \]
\[ A^\lambda_{[12;1,01;2]}(M) = -(m_{12} - m_{11})C_2(M^{0}_1)\chi_+(M^{0}_1) + (m_{23} - m_{22})\chi_+(M)E(M^{0}_1)[-1], \]
\[ A^\lambda_{[12;1,01;3]}(M) = -(m_{23} - m_{22})\chi_+(M)C_2(M^{0}_1)[-1] \chi_+(M^{0}_1)[-1], \]
\[ A_{(1,1:0,0,0)}^\lambda (M) = -(m_{12} - m_{11})(m_{13} - m_{12})(m_{13} - m_{22} + 1), \]
\[ A_{(1,1:1,0,0)}^\lambda (M) = (m_{12} - m_{11})C_2 \left( M \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) \]
\[ \quad - (m_{23} - m_{22})\chi_+(M) (m_{13} - m_{12} + 1)(m_{13} - m_{22}), \]
\[ A_{(1,1:0,0,2)}^\lambda (M) = (m_{23} - m_{22})\chi_+(M)C_2 \left( M \left( \begin{array}{c} 0 \\ 0 \end{array} \right) [-1] \right), \]

and
\[ B_{(1,1:1,1,0)}^\lambda (M) = (m_{12} - m_{11} + 1)(m_{13} - m_{12})(m_{22} - m_{33}) - (m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B_{(1,1:1,1,1)}^\lambda (M) = -(m_{12} - m_{11})\tilde{E}(M) + (m_{23} - m_{22} + 1)\chi_+(M) (m_{13} - m_{12})(m_{22} - m_{33}) + \tilde{F}(M), \]
\[ B_{(1,1:1,1,2)}^\lambda (M) = -(m_{23} - m_{22})\chi_+(M)\tilde{E}(M) - C_2(M)\chi_+(M), \]
\[ B_{(1,1:0,1,0)}^\lambda (M) = -(m_{12} - m_{11})(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B_{(1,1:0,1,1)}^\lambda (M) = -(m_{12} - m_{11} - 1)\tilde{E}(M) - (m_{23} - m_{22} + 1)\chi_+^{(1)}(M) (m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B_{(1,1:0,1,2)}^\lambda (M) = -(m_{12} - m_{11} - 2)C_2(M)\chi_+(M) + (m_{23} - m_{22})\chi_+^{(1)}(M)\tilde{F}(M), \]
\[ B_{(1,1:0,1,3)}^\lambda (M) = -(m_{23} - m_{22} - 1)\chi_+^{(1)}(M)C_2(M)\chi_+(M), \]
\[ B_{(1,1:0,0,0)}^\lambda (M) = -(m_{12} - m_{11})(m_{13} - m_{12})(m_{13} - m_{22} + 1), \]
\[ B_{(1,1:0,0,1)}^\lambda (M) = (m_{12} - m_{11} - 1)C_2(M) - (m_{23} - m_{22})\chi_+(M) (m_{13} - m_{12})(m_{13} - m_{22} + 1), \]
\[ B_{(1,1:0,0,2)}^\lambda (M) = (m_{23} - m_{22} - 1)\chi_+(M)C_2(M). \]

We have
\[ A_{(1,1:1,1,1)}^\lambda (M) - B_{(1,1:1,1,1)}^\lambda (M) \]
\[ = -(m_{12} - m_{11})\{(m_{12} - m_{23}) + (m_{13} - m_{33} - m_{12} + m_{22} + 1)\chi_+(M)\} \]
\[ + (m_{13} - m_{33} - m_{12} + m_{22} + 1)(m_{23} - m_{22})\chi_+(M) \]
\[ - (m_{13} - m_{12})(m_{22} - m_{33})\chi_+(M) - \tilde{F}(M) \]
\[ = -(m_{13} - m_{33} - m_{12} + m_{22} + 1)\delta(M)\chi_+(M) \]
\[ - (m_{13} - m_{12})(m_{22} - m_{33})\chi_+(M) - (m_{12} - m_{11})(m_{12} - m_{23}) - \tilde{F}(M) \]
\[ = -\{(m_{12} - m_{11})(m_{12} - m_{23}) - (m_{12} - m_{22})\delta(M)\chi_+(M)\} \]
\[ - \chi_+(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33} + 1)\delta(M)\} - \tilde{F}(M) \]
\[ = 0. \]

Hence we obtain \( A_{(1,1:1,1,1)}^\lambda (M) = B_{(1,1:1,1,1)}^\lambda (M) \). Here we use the relations
\[ (4.18) \quad \tilde{E}(M \left( \begin{array}{c} 0 \\ 1 \end{array} \right)) = \tilde{E}(M) + (m_{12} - m_{23}) + (m_{13} - m_{33} - m_{12} + m_{22} + 1)\chi_+(M). \]

We have
\[ A_{(1,1:1,1,2)}^\lambda (M) = -(m_{23} - m_{22})(\tilde{E}(M) + C_1(M))\chi_+(M) \]
\[ = -(m_{23} - m_{22})\chi_+(M)\tilde{E}(M) - C_1(M)(m_{23} - m_{22})\chi_+(M) \]
\[ = B_{(1,1:1,1,2)}^\lambda (M). \]

Hence we obtain \( A_{(1,1:1,1,2)}^\lambda (M) = B_{(1,1:1,1,2)}^\lambda (M) \). Here we use the relation
\[ (4.19) \quad \tilde{E}(M \left( \begin{array}{c} 0 \\ 0 \end{array} \right) [-1])\chi_+(M) = (\tilde{E}(M) + C_1(M))\chi_+(M). \]
We have
\[ A^\lambda_{[12;1;01;1]}(M) - B^\lambda_{[12;1;01;1]}(M) \]
\[ = (m_{12} - m_{11}) (m_{12} - m_{23}) - (m_{13} - m_{12})(m_{22} - m_{33}) \chi^+_+(M) \]
\[ + (m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) \chi_+(M) \]

Hence we obtain
\[ A^\lambda_{[12;1;01;1]}(M) = B^\lambda_{[12;1;01;1]}(M). \]

Here we use the relations
\[ \bar{F}(M(0 \circ 0)) = \bar{F}(M) + (m_{12} - m_{23}) - (m_{13} - m_{12})(m_{22} - m_{33}) \chi^+_+(M) \]
\[ + (m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) \chi_+(M). \]

We have
\[ A^\lambda_{[12;1;01;2]}(M) = (m_{23} - m_{22}) \bar{F}(M(0 \circ 0)) [1] \chi_+(M), \]
\[ B^\lambda_{[12;1;01;2]}(M) = (m_{23} - m_{22}) \bar{F}(M)_+ \]

By the relation
\[ \bar{F}(M(0 \circ 0)) [1] \chi_+(M) = \bar{F}(M)_+ \]
\[ \bar{F}(M(0 \circ 0)) [1] \chi_+(M) \]

we have
\[ A^\lambda_{[12;1;01;2]}(M) - B^\lambda_{[12;1;01;2]}(M) = 0. \]

Hence we obtain
\[ A^\lambda_{[12;1;01;2]}(M) = B^\lambda_{[12;1;01;2]}(M). \]

We have
\[ A^\lambda_{[12;1;01;3]}(M) = - (m_{23} - m_{22}) \chi_+(M) C_1(M)(m_{23} - m_{22} - 1) \chi^+_+(M) \]
\[ - C_1(M) \chi_+(M) C_1(M)(m_{23} - m_{22} - 1) \chi^+_+(M) \]
\[ = B^\lambda_{[12;1;01;3]}(M). \]

Hence we obtain
\[ A^\lambda_{[12;1;01;3]}(M) = B^\lambda_{[12;1;01;3]}(M). \]

We have
\[ A^\lambda_{[12;1;00;1]}(M) - B^\lambda_{[12;1;00;1]}(M) \]
\[ = (m_{12} - m_{11}) (m_{12} - m_{23}) \chi_+(M) - (m_{12} - m_{23}) + C_2(M) \]
\[ - (m_{23} - m_{22}) \chi_+(M)(m_{12} - m_{22}) \]
\[ = C_2(M) - (m_{12} - m_{23})(m_{12} - m_{11}) + (m_{12} - m_{22}) \delta(M) \chi_+(M) \]
\[ = 0. \]

Hence we obtain
\[ A^\lambda_{[12;1;00;1]}(M) = B^\lambda_{[12;1;00;1]}(M). \]

We use the relations
\[ C_2(M(0 \circ 0)) = C_2(M) + (m_{12} - m_{22}) \chi_+(M) - (m_{12} - m_{23}). \]
We have
\[ A^{\lambda}_{12;1;00;2} (M) = (m_{11} - m_{22})(m_{23} - m_{22})(m_{23} - m_{22} - 1)\chi_+(M)C_2(M) \]
\[ = B^{\lambda}_{12;1;00;2} (M). \]

Hence we obtain \( A^{\lambda}_{12;1;00;2} (M) = B^{\lambda}_{12;1;00;2} (M). \)

It is trivial that the equations (4.17) hold for \((j, k, l) = (1, 1, 0), (0, 1, 0)\) and \((0, 0, 0)\).

- the proof of \( E_{21} \circ i_{e_1}^{\lambda} = i_{e_1}^{\lambda} \circ E_{21} \)

We have
\[ A^{\lambda}_{21;1;11;0} (M) = (m_{11} - m_{22})(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ A^{\lambda}_{21;1;11;1} (M) = - (m_{11} - m_{22})E (M ( 00 \choose 01)) \]
\[ + (m_{12} - m_{23})\chi_-(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1), \]
\[ A^{\lambda}_{21;1;11;2} (M) = - (m_{12} - m_{23})\chi_-(M)E (M ( 00 \choose 01) [-1]), \]
\[ A^{\lambda}_{21;1;01;0} (M) = - (m_{11} - m_{22})(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ A^{\lambda}_{21;1;01;1} (M) = (m_{11} - m_{22})\bar{F} (M ( 00 \choose 01)) \]
\[ - (m_{12} - m_{23})\chi_-(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1), \]
\[ A^{\lambda}_{21;1;01;2} (M) = - (m_{11} - m_{22})\bar{C}_2 (M ( 00 \choose 01)) \chi_+ (M ( 00 \choose 01)) \]
\[ + (m_{12} - m_{23})\chi_-(M)\bar{F} (M ( 00 \choose 01) [-1]), \]
\[ A^{\lambda}_{21;1;01;3} (M) = - (m_{12} - m_{23})\chi_-(M)\bar{C}_2 (M ( 00 \choose 01) [-1]) \chi_+ (M ( 00 \choose 01) [-1]), \]
\[ A^{\lambda}_{21;1;00;0} (M) = - (m_{11} - m_{22})(m_{13} - m_{12})(m_{13} - m_{22} + 1), \]
\[ A^{\lambda}_{21;1;00;1} (M) = (m_{11} - m_{22})\bar{C}_2 (M ( 00 \choose 01)) \]
\[ - (m_{12} - m_{23})\chi_-(M)(m_{13} - m_{12} + 1)(m_{13} - m_{22}), \]
\[ A^{\lambda}_{21;1;00;2} (M) = (m_{12} - m_{23})\chi_-(M)\bar{C}_2 (M ( 00 \choose 01) [-1]), \]
and
\[ B^{\lambda}_{21;1;11;0} (M) = (m_{11} - m_{22})(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{21;1;11;1} (M) = - (m_{11} - m_{22} - 1)\bar{E}(M) \]
\[ + (m_{12} - m_{23})\chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{21;1;11;2} (M) = - (m_{12} - m_{23} - 1)\chi_-(M)\bar{E}(M), \]
\[ B^{\lambda}_{21;1;01;0} (M) = - (m_{11} - m_{22} + 1)(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{21;1;01;1} (M) = (m_{11} - m_{22})\bar{F}(M) \]
\[ - (m_{12} - m_{23})\chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33}) - \bar{E}(M), \]
\[ B^{\lambda}_{21;1;01;2} (M) = - (m_{11} - m_{22} - 1)\bar{C}_2(M)\chi_+(M) \]
\[ + (m_{12} - m_{23} - 1)\chi_-(M)\bar{F}(M), \]
\[ B^{\lambda}_{21;1;01;3} (M) = - (m_{12} - m_{23} - 2)\chi_-(M)\bar{C}_2(M)\chi_+(M), \]
\[ B^{\lambda}_{21;1;00;0} (M) = - (m_{11} - m_{22})(m_{13} - m_{12})(m_{13} - m_{22} + 1), \]
\[ B^{\lambda}_{21;1;00;1} (M) = (m_{11} - m_{22} - 1)\bar{C}_2(M) \]
\[ - (m_{12} - m_{23})\chi_-(M)(m_{13} - m_{12})(m_{13} - m_{22} + 1). \]
\[ \begin{align*}
B_{21;1,00;2}^\lambda (M) &= (m_{12} - m_{23} - 1) \chi_-(M) C_2(M).
\end{align*} \]

We have
\[ \begin{align*}
A_{21;1,11;1}^\lambda (M) - B_{21;1,11;1}^\lambda (M) &= (m_{11} - m_{22})(m_{12} - m_{23}) \chi_-(M) + \bar{E}(M) \chi_+^{(-1)}(M) \\
&+ (m_{13} - m_{33} - m_{12} + m_{22} + 1)(m_{12} - m_{23}) \chi_-(M) - \bar{E}(M) \\
&= (m_{12} - m_{23})(m_{13} - m_{33} + 1 - m_{12} + m_{11}) \chi_-(M) + \bar{E}(M) \chi_+^{(-1)}(M) - 1) \\
&= \bar{E}(M)(\chi_-(M) + \chi_+^{(-1)}(M) - 1) = 0.
\end{align*} \]

Hence we obtain \( A_{21;1,11;1}^\lambda (M) = B_{21;1,11;1}^\lambda (M) \). Here we use the relations
\[ \begin{align*}
(4.23) \quad (m_{11} - m_{22}) \bar{E}(M ( \begin{smallmatrix} 0 & 0 \\ -1 & 1 \end{smallmatrix} )) &= (m_{11} - m_{22}) \{ \bar{E}(M) - (m_{12} - m_{23}) \chi_-(M) \} \\
&- \bar{E}(M) \chi_+^{(-1)}(M).
\end{align*} \]

We have
\[ \begin{align*}
A_{21;1,11;2}^\lambda (M) &= -(m_{12} - m_{23}) \chi_-(M)(m_{12} - m_{23} - 1)(m_{13} - m_{33} + 1 + m_{12} - m_{11}) \\
&= B_{21;1,11;2}^\lambda (M).
\end{align*} \]

Hence we obtain \( A_{21;1,11;2}^\lambda (M) = B_{21;1,11;2}^\lambda (M) \).

We have
\[ \begin{align*}
A_{21;1,01;1}^\lambda (M) - B_{21;1,01;1}^\lambda (M) &= (m_{11} - m_{22})\{ -(m_{13} - m_{12})(m_{22} - m_{33}) \chi_+^{(-1)}(M) \\
&+ (m_{12} - m_{12} + 1)(m_{22} - m_{33} + 1) \chi_-(M) -(m_{13} - m_{23} - m_{33} + m_{22} + 1) \} \\
&-(m_{12} - m_{23}) \chi_-(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) \\
&+(m_{12} - m_{23}) \chi_+^{(-1)}(M)(m_{13} - m_{12})(m_{22} - m_{33} + 1) \\
&= -\delta(M) \chi_-(M) (m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) \\
&+ \delta(M) \chi_+^{(-1)}(M)(m_{13} - m_{12})(m_{22} - m_{33}) \\
&-(m_{11} - m_{22})(m_{13} - m_{23} - m_{33} + m_{22} + 1) + \bar{E}(M) \\
&= -(m_{11} - m_{22})(m_{13} - m_{23} - m_{33} + m_{22} + 1) \\
&+ (m_{13} - m_{33} - m_{12} + m_{22} + 1) \delta(M) \chi_-(M) + \bar{E}(M) \\
&= 0.
\end{align*} \]

Hence we obtain \( A_{21;1,01;1}^\lambda (M) = B_{21;1,01;1}^\lambda (M) \). Here we use the relations
\[ \begin{align*}
(4.24) \quad \bar{F}(M ( \begin{smallmatrix} 0 & 0 \\ -1 & 1 \end{smallmatrix} )) &= \bar{F}(M) - (m_{13} - m_{12})(m_{22} - m_{33}) \chi_+^{(-1)}(M) \\
&+ (m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) \chi_-(M) \\
&-(m_{13} - m_{23} - m_{33} + m_{22} + 1),
\end{align*} \]
\[ \begin{align*}
(4.25) \quad \bar{E}(M) &= (m_{11} - m_{22})(m_{13} - m_{23} - m_{33} + m_{22} + 1) \\
&+ (m_{13} - m_{33} - m_{12} + m_{22} + 1) \delta(M) \chi_-(M).
\end{align*} \]

We have
\[ \begin{align*}
A_{21;1,01;2}^\lambda (M) &= -(m_{11} - m_{22})(m_{11} - m_{22} - 1)(m_{23} - m_{22}) \chi_+^{(-1)}(M) \\
&-(m_{12} - m_{23})(m_{12} - m_{23} - 1)(m_{12} - m_{11}) \chi_-(M) \\
&= -C_2(M)(C_1(M) - 1) \chi_+^{(-1)}(M) - C_2(M)(C_1(M) - 1) \chi_-(M) \\
&= -C_2(M)(C_1(M) - 1),
\end{align*} \]
Hence we obtain \( A_{21;1,01;2}^\lambda (M) = B_{21;1,01;2}^\lambda (M) \). Here we use the relation
\[
\hat{F}(M)\chi^{(r)}(M) = -C_2(M)\chi^{(r)}(M) \quad (r \geq -1).
\]

We have
\[
A_{21;1,01;3}^\lambda (M) = -(m_{12} - m_{23})C_2(M \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) [-1]) \chi_-(M)\chi^{(-1)}(M) = 0,
\]
\[
B_{21;1,01;3}^\lambda (M) = -(m_{12} - m_{23} - 2)C_2(M)\chi^{(-1)}(M)\chi_+(M) = 0,
\]
Hence we obtain \( A_{21;1,01;3}^\lambda (M) = B_{21;1,01;3}^\lambda (M) \).

We have
\[
A_{21;1,00;1}^\lambda (M) - B_{21;1,00;1}^\lambda (M) = (m_{11} - m_{22})\{(m_{12} - m_{22})\chi_-(M) - (m_{23} - m_{22})\} + C_2(M)
\]
\[
= -(m_{12} - m_{23})(m_{12} - m_{22})\chi_-(M)
\]
\[
= -(m_{11} - m_{22})(m_{23} - m_{22}) - (m_{12} - m_{22})\delta(M)\chi_-(M) + C_2(M)
\]
\[
= 0.
\]

Hence we obtain \( A_{21;1,00;1}^\lambda (M) = B_{21;1,00;1}^\lambda (M) \). Here we use the relations
\[
C_2(M \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) ) = C_2(M) + (m_{12} - m_{22})\chi_-(M) - (m_{23} - m_{22}).
\]

We have
\[
A_{21;1,00;2}^\lambda (M) = (m_{12} - m_{23})(m_{12} - m_{23} - 1)(m_{12} - m_{11})\chi_-(M)
\]
\[
= B_{21;1,00;2}^\lambda (M).
\]

Hence we obtain \( A_{21;1,00;2}^\lambda (M) = B_{21;1,00;2}^\lambda (M) \).

It is trivial that the equations (4.17) hold for \((j,k,l) = (1,1,0), (0,1,0)\) and \((0,0,0)\).

• the proof of \( E_{23} \circ i_{e_1}^\lambda = i_{e_1}^\lambda \circ E_{23} \)

We have
\[
A_{23;1,11,0}^\lambda (M) = (m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33}),
\]
\[
A_{23;1,11,1}^\lambda (M) = -(m_{13} - m_{12})\hat{E} \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right)
\]
\[
+ \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33} + 1),
\]
\[
A_{23;1,11,2}^\lambda (M) = - \{m_{13} - m_{12} - \delta(M)\} \hat{E} \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) [-1] \right),
\]
\[
A_{23;1,01,0}^\lambda (M) = -(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33}),
\]
\[
A_{23;1,01,1}^\lambda (M) = (m_{13} - m_{12})\hat{E} \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right)
\]
\[
- \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33} + 1),
\]
\[
A_{23;1,01,2}^\lambda (M) = - (m_{13} - m_{12})C_2 \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right) \chi_+ \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right)
\]
\[
+ \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)\hat{E} \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) [-1] \right),
\]
\[
A_{23;1,01,3}^\lambda (M) = - \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)C_2 \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) [-1] \right) \chi_+ \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) [-1] \right),
\]
\[
A_{23;1,00,0}^\lambda (M) = -(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{13} - m_{22} + 1),
\]
\[
A_{23;1,00,1}^\lambda (M) = (m_{13} - m_{12})C_2 \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right)
\]
\[
- \{m_{13} - m_{12} - \delta(M)\} \chi_- \left( M \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right)(m_{13} - m_{12})(m_{13} - m_{22}).
\]
Hence we obtain

\[ A^\lambda_{[23;1:00;2]}(M) = \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)C_2(M) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

and

\[ B^\lambda_{[23;1:11;0]}(M) = (m_{13} - m_{12} - 1)(m_{13} - m_{12})(m_{22} - m_{33}), \]

\[ B^\lambda_{[23;1:11;1]}(M) = -(m_{13} - m_{12})\tilde{E}(M) \]
\[ + \{m_{13} - m_{12} - \delta(M) - 1\} \chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33}), \]

\[ B^\lambda_{[23;1:11;2]}(M) = -\{m_{13} - m_{12} - \delta(M)\} \chi_-(M)\tilde{E}(M), \]

\[ B^\lambda_{[23;1:01;0]}(M) = -(m_{13} - m_{12} - 1)(m_{13} - m_{12})(m_{22} - m_{33}), \]

\[ B^\lambda_{[23;1:01;1]}(M) = -(m_{13} - m_{12})\tilde{E}(M) \]
\[ - \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33}) \]
\[ - \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)(m_{13} - m_{22} + 1), \]

\[ B^\lambda_{[23;1:01;2]}(M) = -(m_{13} - m_{12} + 1)C_2(M)\chi_+(M) \]
\[ + \{m_{13} - m_{12} - \delta(M) + 1\} \chi_-(M)\tilde{E}(M) + C_2(M), \]

\[ B^\lambda_{[23;1:01;3]}(M) = -(m_{13} - m_{12} - 2\chi_-(M))C_2(M)\chi_+(M), \]

\[ B^\lambda_{[23;1:00;0]}(M) = -(m_{13} - m_{12} - 1)(m_{13} - m_{12})(m_{13} - m_{22} + 1), \]

\[ B^\lambda_{[23;1:00;1]}(M) = -(m_{13} - m_{12})C_2(M) \]
\[ - \{m_{13} - m_{12} - \delta(M) - 1\} \chi_-(M)(m_{13} - m_{12})(m_{13} - m_{22} + 1), \]

\[ B^\lambda_{[23;1:00;2]}(M) = \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)C_2(M). \]

We have

\[ A^\lambda_{[23;1:11;1]}(M) - B^\lambda_{[23;1:11;1]}(M) = -(m_{13} - m_{12})(m_{13} + m_{23} - m_{33} - 2m_{12} + m_{11})\chi_-(M) \]
\[ + (m_{13} - m_{12})\chi_-(M)(m_{13} - m_{12} - \delta(M) + m_{22} - m_{33}) \]
\[ = 0. \]

Hence we obtain \( A^\lambda_{[23;1:11;1]}(M) = B^\lambda_{[23;1:11;1]}(M). \) Here we use the relation

\[ (4.28) \quad \tilde{E}(M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = \tilde{E}(M) + (m_{13} + m_{23} - m_{33} - 2m_{12} + m_{11})\chi_-(M). \]

By the relation

\[ (4.29) \quad \tilde{E}(M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \chi_-(M) = \tilde{E}(M)\chi_-(M), \]

Hence we obtain

\[ A^\lambda_{[23;1:11;2]}(M) = B^\lambda_{[23;1:11;2]}(M). \]

We have

\[ A^\lambda_{[23;1:01;1]}(M) - B^\lambda_{[23;1:01;1]}(M) \]
\[ = (m_{13} - m_{12})\left\{(m_{13} - m_{12} - \delta(M))\chi_-(M) \right. \]
\[ - (m_{13} - m_{12})(m_{22} - m_{33})(\chi_-(M) - \chi_-(-M)) + (m_{13} - m_{22} + 1) \}
\[ - \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33} + 1) \]
\[ + \{m_{13} - m_{12} - \delta(M)\} \chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33}) \]
\[ + (m_{13} - m_{12})(m_{13} - m_{22} + 1) \]
\[ = (m_{13} - m_{12})(m_{22} - m_{33})\delta(M)(\chi_-(M) - \chi_-(M)) \]
\[ = 0. \]
Hence we obtain $A_{\lambda}^{[23;1;01;1]} (M) = B_{\lambda}^{[23;1;01;1]} (M)$. Here we use the relation
\[(4.30) \quad F(M (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})) = F(M) + (m_{13} - m_{12} - \delta(M))\chi_-(M) - (m_{13} - m_{12})(m_{22} - m_{33})(\chi^{(-1)}_+ - \chi_-(M)) + (m_{13} - m_{22} + 1).
\]

We have
\[
A_{\lambda}^{[23;1;01;2]} (M) = -(m_{13} - m_{12})(m_{11} - m_{22})(m_{23} - m_{22})\chi^{(-1)}_+ (M)
- \{m_{13} - m_{12} - \delta(M)\}{m_{12} - m_{23}}(m_{12} - m_{11})\chi_-(M)
= -(m_{13} - m_{12})C_2(M)\chi^{(-1)}_+ (M) - \{m_{13} - m_{12} - \delta(M)\}C_2(M)\chi_-(M)
= -(m_{13} - m_{12})C_2(M) + \delta(M)C_2(M)\chi_-(M),
\]
\[
B_{\lambda}^{[23;1;01;2]} (M) = -(m_{13} - m_{12} - \delta(M) + 1)C_2(M)\chi^{(-1)}_+ (M) + C_2(M)
= -(m_{13} - m_{12} + 1)C_2(M) + C_2(M)\delta(M)\chi^{(-1)}_+ (M) + C_2(M)
= -(m_{13} - m_{12})C_2(M) + \delta(M)C_2(M)\chi_-(M).
\]

Hence we obtain $A_{\lambda}^{[23;1;01;2]} (M) = B_{\lambda}^{[23;1;01;2]} (M)$. Here we use the relation \((4.26)\).

We have
\[
A_{\lambda}^{[23;1;01;3]} (M) = -(m_{13} - m_{12} - \delta(M))C_2 (M (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})) \chi_-(M)\chi^{(-1)}_+ (M) = 0,
\]
\[
B_{\lambda}^{[23;1;01;3]} (M) = -(m_{13} - m_{12} - \delta(M) + 2)C_2(M)\chi^{(-1)}_+ (M)\chi_-(M) = 0.
\]

Hence we obtain $A_{\lambda}^{[23;1;01;3]} (M) = B_{\lambda}^{[23;1;01;3]} (M)$.

We have
\[
A_{\lambda}^{[23;1;00;1]} (M) - B_{\lambda}^{[23;1;00;1]} (M)
= (m_{13} - m_{12})(m_{12} - m_{22} + 1 + \delta(M))\chi_-(M)
- \{m_{13} - m_{12} - \delta(M)\}C_2(M)(m_{13} - m_{12})(m_{13} - m_{22})
+ \{m_{13} - m_{12} - \delta(M) - 1\}\chi_-(M)(m_{13} - m_{12})(m_{13} - m_{22} + 1)
= 0.
\]

Hence we obtain $A_{\lambda}^{[23;1;00;1]} (M) = B_{\lambda}^{[23;1;00;1]} (M)$. Here we use the relation
\[(4.31) \quad C_2 (M (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})) = C_2(M) + (m_{12} - m_{22} + 1 + \delta(M))\chi_-(M).
\]

We have
\[
A_{\lambda}^{[23;1;00;2]} (M)
= \{m_{13} - m_{12} - \delta(M)\}C_2(M)
= B_{\lambda}^{[23;1;00;2]} (M).
\]

Hence we obtain $A_{\lambda}^{[23;1;00;2]} (M) = B_{\lambda}^{[23;1;00;2]} (M)$.

It is trivial that the equations \((4.17)\) hold for \((j, k, l) = (1, 1, 0), (0, 1, 0)\) and \((0, 0, 0)\).

• the proof of $E_{32} \circ i_{\omega_1}^\lambda = i_{\omega_1}^\lambda \circ E_{32}$

We have
\[
A_{\lambda}^{[32;1;11;0]} (M) = (m_{22} - m_{33})(m_{13} - m_{12})(m_{22} - m_{33} - 1),
\]
\[
A_{\lambda}^{[32;1;11;1]} (M) = -(m_{22} - m_{33})\bar{E} (M (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}))
+ \{m_{22} - m_{33} + \delta(M)\}\chi_+ (M)(m_{13} - m_{12} + 1)(m_{22} - m_{33}),
\]
\[
A_{\lambda}^{[32;1;11;2]} (M) = -(m_{22} - m_{33} + \delta(M))\chi_+(M)\bar{E} (M (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) [-1]),
\]
\[ A^{\lambda}_{[32;1,01;0]}(M) = -(m_{22} - m_{33})(m_{13} - m_{12})(m_{22} - m_{33} - 1), \]
\[ A^{\lambda}_{[32;1,01;1]}(M) = (m_{22} - m_{33})\bar{F}(M)_{(0,1)} \]
\[ - \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33}), \]
\[ A^{\lambda}_{[32;1,01;2]}(M) = -(m_{22} - m_{33})C_{2}(M)_{(0,0)} \chi_{+}(M)(0,1) \]
\[ + \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)\bar{F}(M)_{(0,1)}[-1], \]
\[ A^{\lambda}_{[32;1,01;3]}(M) = -\{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)C_{2}(M)_{(0,0)}[-1] \chi_{+}(M)(0,1)[-1], \]
\[ A^{\lambda}_{[32;1,00;0]}(M) = -(m_{22} - m_{33})(m_{13} - m_{12})(m_{13} - m_{22} + 2), \]
\[ A^{\lambda}_{[32;1,00;1]}(M) = (m_{22} - m_{33})C_{2}(M)_{(0,0)} \]
\[ - \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)(m_{13} - m_{12} + 1)(m_{13} - m_{22} + 1), \]
\[ A^{\lambda}_{[32;1,00;2]}(M) = \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)C_{2}(M)_{(0,0)}[-1], \]

and
\[ B^{\lambda}_{[32;1,11;0]}(M) = (m_{22} - m_{33} - 1)(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{[32;1,11;1]}(M) = -(m_{22} - m_{33})\bar{E}(M) \]
\[ + \{m_{22} - m_{33} + \delta(M) - 1\} \chi_{+}(M)(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{[32;1,11;2]}(M) = -\{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)\bar{E}(M), \]
\[ B^{\lambda}_{[32;1,01;0]}(M) = -(m_{22} - m_{33} - 1)(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{[32;1,01;1]}(M) = (m_{22} - m_{33})\bar{F}(M) \]
\[ - \{m_{22} - m_{33} + \delta(M) - 2\} \chi_{+}^{(1)}(M)(m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{[32;1,01;2]}(M) = -(m_{22} - m_{33} + 1)C_{2}(M)_{(1,0)} \chi_{+}(M) \]
\[ + \{m_{22} - m_{33} + \delta(M) - 1\} \chi_{+}^{(1)}(M)\bar{F}(M), \]
\[ B^{\lambda}_{[32;1,01;3]}(M) = -\{m_{22} - m_{33} + \delta(M)\} \chi_{+}^{(1)}(M)C_{2}(M)_{(1,0)} \chi_{+}(M), \]
\[ B^{\lambda}_{[32;1,00;0]}(M) = -(m_{22} - m_{33})(m_{13} - m_{12})(m_{13} - m_{22} + 1) - (m_{13} - m_{12})(m_{22} - m_{33}), \]
\[ B^{\lambda}_{[32;1,00;1]}(M) = (m_{22} - m_{33} + 1)C_{2}(M) \]
\[ - \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)(m_{13} - m_{12})(m_{13} - m_{22} + 1) + \bar{F}(M), \]
\[ B^{\lambda}_{[32;1,00;2]}(M) = \{m_{22} - m_{33} + \delta(M) + 1\} \chi_{+}(M)C_{2}(M)_{(1,0)}- C_{2}(M)_{(1,0)} \chi_{+}(M). \]

We have
\[ A^{\lambda}_{[32;1,11;1]}(M) - B^{\lambda}_{[32;1,11;1]}(M) \]
\[ = -(m_{22} - m_{33})(m_{13} - m_{33} - m_{12} + m_{22} + \delta(M)) \chi_{+}(M) \]
\[ + \{m_{22} - m_{33} + \delta(M)\} \chi_{+}(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33}) \]
\[ - \{m_{22} - m_{33} + \delta(M) - 1\} \chi_{+}(M)(m_{13} - m_{12})(m_{22} - m_{33}) \]
\[ = 0. \]

Hence we obtain \( A^{\lambda}_{[32;1,11;1]}(M) = B^{\lambda}_{[32;1,11;1]}(M) \). Here we use the relation
\[ E(M)_{(0,1)}(0,1) \chi_{+}(M) = \bar{E}(M) \chi_{+}(M). \]

By the relation
\[ E(M)_{(0,1)}[-1] \chi_{+}(M) = \bar{E}(M) \chi_{+}(M), \]
we obtain

\[ A^\lambda_{[32,1,11:2]}(M) = B^\lambda_{[32,1,11:2]}(M). \]

We have

\[
A^\lambda_{[32,1,01:1]}(M) - B^\lambda_{[32,1,01:1]}(M) = (m_{22} - m_{33} - (m_{13} - m_{12})(m_{22} - m_{33} - 1)\chi^\lambda_+ (M)
+ (m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1)\chi_+(M) + (\delta(M) - 1)\chi_+(M)
- (m_{22} - m_{33} + \delta(M))\chi_+(M)(m_{13} - m_{12} + 1)
+ (m_{22} - m_{33} + (\delta(M) - 2)\chi_+(M)(m_{13} - m_{12}))
= (m_{22} - m_{33})(m_{13} - m_{12})(\delta(M) - 1)(\chi^\lambda_+(M) - \chi_+(M))
= 0.
\]

Hence we obtain \( A^\lambda_{[32,1,01:1]}(M) = B^\lambda_{[32,1,01:1]}(M). \) Here we use the relation

\[
(4.34) \quad \bar{F}(M(\delta_0^{\lambda})))) = \bar{F}(M) - (m_{13} - m_{12})(m_{22} - m_{33} - 1)\chi^\lambda_+(M)
+ ((m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) + \delta(M) - 1)\chi_+(M).
\]

We have

\[
A^\lambda_{[32,1,01:2]}(M) - B^\lambda_{[32,1,01:2]}(M) = (m_{22} - m_{33} + \delta(M))\bar{F}(M(\delta_0^{\lambda})))) - \bar{F}(M)\chi^\lambda_+(M)
+ \bar{F}(M)\chi^\lambda_+(M) + (m_{22} - m_{33} + 1)C_2(M)\chi_+(M) - (m_{22} - m_{33})C_2(M^{\lambda_0})\chi^\lambda_+(M)
= (m_{22} - m_{33} + \delta(M))(m_{13} - m_{22} + 1)\chi^\lambda_+(M)
+ (\delta(M) - 1)C_2(M)\chi_+(M) + \bar{F}(M)\chi^\lambda_+(M) - (m_{22} - m_{33})C_2(M^{\lambda_0})\chi^\lambda_+(M)
= \chi^\lambda_+(M)\{C_2(M) + (m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33} + 1)\delta(M) + \bar{F}(M)\chi^\lambda_+(M)\}
= 0.
\]

Hence we obtain \( A^\lambda_{[32,1,01:2]}(M) = B^\lambda_{[32,1,01:2]}(M). \) Here we use the relations

\[
(4.35) \quad \bar{F}(M(\delta_0^{\lambda}))))[-1] \chi_+(M) = \bar{F}(M)\chi^\lambda_+(M) + (m_{13} - m_{22} + 1)\chi^\lambda_+(M)
+ C_2(M)(\chi^\lambda_+(M) - \chi_+(M)),
\]

\[
C_2(M^{\lambda_0})\chi^\lambda_+(M) = (C_2(M) + m_{12} - m_{22} + 1 - \delta(M))\chi^\lambda_+(M).
\]

We have

\[
A^\lambda_{[32,1,01:3]}(M) = -\{m_{22} - m_{33} + \delta(M)\}\chi_+(M)(m_{11} - m_{22})(m_{23} - m_{22})\chi^\lambda_+(M)
= B^\lambda_{[32,1,01:3]}(M).
\]

Hence we obtain \( A^\lambda_{[32,1,01:3]}(M) = B^\lambda_{[32,1,01:3]}(M). \)

We have

\[
A^\lambda_{[32,1,00:1]}(M) - B^\lambda_{[32,1,00:1]}(M) = (m_{22} - m_{33})(m_{12} - m_{22} + 1 - \delta(M))\chi_+(M) - C_2(M)
- \{m_{22} - m_{33} + \delta(M)\}\chi_+(M)(m_{13} - m_{22} + 1) - \bar{F}(M)
= -C_2(M) - \chi_+(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33} + 1)\delta(M)\} - \bar{F}(M)
= 0.
\]
Hence we obtain $A^\lambda_{[32;1;00;1]}(M) = B^\lambda_{[32;1;00;1]}(M)$. Here we use the relation
\[(4.36)\]
\[C_2(M^{(0,1)}) = C_2(M) + (m_{12} - m_{22} + 1 - \delta(M))\chi_+(M).\]

We have
\[A^\lambda_{[32;1;00;2]}(M) = B^\lambda_{[32;1;00;2]}(M).\]

Hence we obtain $A^\lambda_{[32;1;00;2]}(M) = B^\lambda_{[32;1;00;2]}(M)$.

It is trivial that the equations (4.17) hold for $(j, k, l) = (1, 1, 0)$, $(0, 1, 0)$ and $(0, 0, 0)$.

\[\square\]

4.2. Irreducible decompositions of $V_\lambda \otimes_C V_{2e_1}$ and $V_\lambda \otimes_C V_{-2e_1}$. For a vector space $W$, we denote by id$_W$ the identity map of $W$. We denote $\lambda \pm (e_i + e_j)$ $(1 \leq i \leq j \leq 3)$ by $\lambda[\pm ij]$ for the sake of simplicity. Generically the tensor product $V_\lambda \otimes_C V_{2e_1}$ has six irreducible components: $V_{\lambda[\pm ij]}$, $1 \leq i \leq j \leq 3$. Here some components may vanish.

For $1 \leq i \leq j \leq 3$, let $\tau^j_{e_i + e_j}$ be a non-zero generator of Hom$_K(V_{\lambda[\pm ij]}, V_\lambda \otimes_C V_{2e_1})$, which is unique up to scalar multiple if $V_\lambda \otimes_C V_{2e_1}$ has a non-zero $\tau_{\lambda[\pm ij]}$-isotypic component.

**Lemma 4.3.** We define a linear map $P_{e_1} : V_{e_1} \otimes C V_{e_1} \rightarrow V_{2e_1}$ by
\[
\begin{align*}
P_{e_1}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 \end{pmatrix}), & P_{e_1}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}), \\
P_{e_1}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}), & P_{e_1}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}), \\
P_{e_1}(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}), & P_{e_1}(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}), \\
P_{e_1}(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}), & P_{e_1}(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}) & = f(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}).
\end{align*}
\]

Then $P_{e_1}$ is a non-zero generator of Hom$_K(V_{e_1} \otimes C V_{e_1}, V_{2e_1})$.

**Proof.** Since $V_{e_1} \otimes C V_{e_1} \simeq V_{2e_1} \oplus V_{e_1 + e_2}$, in order to prove this lemma, it suffices to check $P_{e_1} \circ \tau^j_{e_1 + e_2} = 0$ and that $P_{e_1} \circ \tau^j_{e_1}$ agree with the identity map up to scalar multiple. To consider the case $\lambda = e_1$ in Proposition 4.2, we obtain the explicit expressions of $\tau^j_{e_1}$ and $\tau^j_{e_2}$ in terms of the monomial basis as follows:
\[
\begin{align*}
\tau^j_{e_1}(\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 \end{pmatrix}) & = -6f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 \end{pmatrix}) \otimes f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}), \\
\tau^j_{e_1}(\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 \end{pmatrix}) & = -6f(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \otimes f(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \\
\tau^j_{e_1}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 \end{pmatrix}) & = -6f(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \otimes f(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}).
\end{align*}
\]
By direct computation, we can easily check $P_{e_1} \circ \tau_{e_1} = -6 \text{id}_{V_{2e_1}}$ and $P_{e_1} \circ \tau_{e_2} = 0$. \hfill \Box

By a composition of the projectors in Lemma 4.3 and the injectors in Proposition 4.2, we obtain following formulas.

**Proposition 4.4.** For $1 \leq i \leq j \leq 3$ and a $G$-pattern $M$ of type $\lambda[ij]$, the image of the monomial basis $f(M)$ by the injector $i_{e_i+e_j}^\lambda : V_{\lambda[ij]} \to V_\lambda \otimes_C V_{2e_1}$ is given by

$$i_{e_i+e_j}^\lambda (f(M)) = \sum_{0 \leq k \leq l \leq 2} \left\{ \sum_{m=0}^{\tau_{i,j,k,l}} c_{i,j,k,l,m}^\lambda (M) f \left( M \left( \frac{-e_i-e_j}{-k} \right) \right) \right\} \otimes f \left( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

In the right hand side of the above formula, we put $f(M') = 0$ if $M'$ is a triangular array which does not satisfy the condition (3.1) of $G$-patterns.

The explicit expressions of the coefficients are given by the following formulas.

**Formula 1:** The coefficients of the injector $i_{2e_1}^\lambda : V_{\lambda[11]} \to V_\lambda \otimes_C V_{2e_1}$ are given as follows: $(\tau_{[1122], \tau_{[1112]}, \tau_{[1111]}, \tau_{[1102]}, \tau_{[1101]}, \tau_{[1100]}}) = (2, 3, 2, 4, 3, 2)$ and

- $c_{[1122]}^\lambda (M) = (m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{22} - m_{33} - 1),$
- $c_{[1121]}^\lambda (M) = -2(m_{13} - m_{12})(m_{22} - m_{33})(E(M) - C_1(M));$
- $c_{[1122]}^\lambda (M) = E(M)(C_1(M) - 1)(m_{13} - m_{33} - C_1(M));$
- $c_{[1121]}^\lambda (M) = -2(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{22} - m_{33} - 1),$
- $c_{[1122]}^\lambda (M) = 2(m_{13} - m_{12})(m_{22} - m_{33}) \left\{ F \left( M \left( \frac{-100}{-1} \right) \right) + E(M) \right\},$
- $c_{[1122]}^\lambda (M) = -2 \left( E(M)F \left( M \left( \frac{-100}{-1} \right) \right) \right) + (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(C_1(M) + 1)\chi_+(M)\right\},$
- $c_{[1123]}^\lambda (M) = 2(C_1(M) - 1)\tilde{C}_1(M)\tilde{E}(M)\chi_+(M),$
\[ c_{[1,0;1]}^1 (M) = -2(m_{13} - m_{12}) \left\{ (m_{13} - m_{22}) \bar{F}(M) + (m_{22} - m_{33}) C_2 \left( M - \frac{b}{a} \right) \right\}, \]
\[ c_{[1,0;2]}^1 (M) = 2 \left\{ C_2 \left( M - \frac{b}{a} \right) \bar{F}(M) \right\} + \chi_+(M) (m_{13} - m_{12}) + 1)(m_{13} - m_{22} - 1) C_2(M), \]
\[ c_{[1,2]}^1 \left( M \right) = -2C_2(M)(C_1(M) - 1)(C_1(M) - 1), \]
\[ c_{[1,0;3]}^1 (M) = -2C_2(M)(C_1(M) - 1)(C_1(M) - 1) \lambda_+ (M), \]
\[ c_{[1,0;2]}^1 (M) = (m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{13} - m_{22} + 1)(m_{13} - m_{22}), \]
\[ c_{[1,0;1]}^1 (M) = -2(m_{13} - m_{12}) (m_{13} - m_{22}) C_2(M), \]
\[ c_{[1,0;0]}^1 (M) = C_2(M)(C_1(M) - 1)(C_1(M) - 1). \]

**Formula 2:** The coefficients of the injector \( i_{2e_2}^\lambda \): \( V_\lambda[+22] \to V_\lambda \otimes C V_{2e_1} \) are given as follows:
\[ (r_{[22;22]}, r_{[22;12]}, r_{[22;02]}, r_{[22;01]}, r_{[22;00]}) = (2, 2, 2, 2, 2) \] and
\[ c_{[22;22;0]}^\lambda (M) = (m_{22} - m_{33})(m_{22} - m_{33} - 1), \]
\[ c_{[22;22;1]}^\lambda (M) = -2(m_{22} - m_{33}) \left\{ \bar{D}(M) \chi_-(M) + (\bar{D}(M) + 2) \chi_-^{(1)} (M) \right\}, \]
\[ c_{[22;22;2]}^\lambda (M) = \bar{D}(M)(\bar{D}(M) + 1) \chi_-(M), \]
\[ c_{[22;22;0]}^\lambda (M) = -2(m_{22} - m_{33})(m_{22} - m_{33} - 1), \]
\[ c_{[22;22;1]}^\lambda (M) = 2(m_{22} - m_{33}) \left\{ \bar{C}_1(M) + (\bar{D}(M) + 1) \chi_-(M) \right\}, \]
\[ c_{[22;22;2]}^\lambda (M) = -2 \bar{C}_1(M) \bar{D}(M) \chi_-(M), \]
\[ c_{[22;11;0]}^\lambda (M) = -2(m_{22} - m_{33})(m_{23} - m_{22}), \]
\[ c_{[22;11;1]}^\lambda (M) = 2(\bar{D}(M)(m_{23} - m_{22} - 1) \chi_-(M) - (m_{22} - m_{33}) \bar{C}_1(M) + 1) \chi_-^{(1)} (M), \]
\[ c_{[22;11;2]}^\lambda (M) = 2 \bar{C}_1(M) \bar{D}(M) \chi_-^{(1)} (M), \]
\[ c_{[22;02;0]}^\lambda (M) = (m_{22} - m_{33})(m_{22} - m_{33} - 1), \]
\[ c_{[22;02;1]}^\lambda (M) = -2(m_{22} - m_{33}) \bar{C}_1(M), \]
\[ c_{[22;02;2]}^\lambda (M) = \bar{C}_1(M)(\bar{C}_1(M) - 1), \]
\[ c_{[22;01;0]}^\lambda (M) = 2(m_{22} - m_{33})(m_{23} - m_{22}), \]
\[ c_{[22;01;1]}^\lambda (M) = 2 \bar{C}_1(M) \left\{ (m_{22} - m_{33}) \chi_-(M) - (m_{23} - m_{22} - 1) \right\}, \]
\[ c_{[22;01;2]}^\lambda (M) = -2 \bar{C}_1(M)(\bar{C}_1(M) - 1) \chi_-(M), \]
\[ c_{[22;00;0]}^\lambda (M) = (m_{23} - m_{22})(m_{23} - m_{22} - 1), \]
\[ c_{[22;00;1]}^\lambda (M) = \bar{C}_1(M) \left\{ (m_{23} - m_{22} - 2) \chi_-(M) + (m_{23} - m_{22}) \chi_-^{(1)} (M) \right\}, \]
\[ c_{[22;00;2]}^\lambda (M) = \bar{C}_1(M)(\bar{C}_1(M) - 1) \chi_-^{(1)} (M). \]

**Formula 3:** The coefficients of the injector \( i_{2e_1}^\lambda \): \( V_\lambda[+33] \to V_\lambda \otimes C V_{2e_1} \) are given as follows:
\[ (r_{[33;22]}, r_{[33;12]}, r_{[33;11]}, r_{[33;02]}, r_{[33;01]}, r_{[33;00]}) = (0, 1, 0, 2, 1, 0) \] and
\[ c_{[33;22;0]}^\lambda (M) = 1, \]
\[ c_{[33;12;0]}^\lambda (M) = -2 \chi_+(M), \]
\[ c_{[33;11;0]}^\lambda (M) = 1, \]
\[ c_{[33;02;0]}^\lambda (M) = 1, \]
\[ c_{[33;02;2]}^\lambda (M) = \chi_-^{(1)} (M), \]
\[ c_{[33;01;1]}^\lambda (M) = -2 \chi_+(M), \]
\[ c_{[33;00;0]}^\lambda (M) = 1. \]
**Formula 4:** The coefficients of the injector $i^\lambda_{e_1+e_5}: V_{\lambda[+12]} \rightarrow V_\lambda \otimes C V_{2e_5}$ are given as follows

\[(r_{[13;22]}, r_{[13,12]}, r_{[13,11]}, r_{[13,02]}, r_{[13,01]}, r_{[13,00]}) = (2, 2, 2, 3, 2, 2)\] and

\[
c^\lambda_{[12,22,0]}(M) = (m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1),
\]

\[
c^\lambda_{[12,22,1]}(M) = - (m_{22} - m_{33}) \{ \tilde{E}(M) + \chi_-(M)(m_{13} - m_{12})(\tilde{D}(M) + 1) \},
\]

\[
c^\lambda_{[12,22,2]}(M) = \tilde{D}(M)\tilde{E}(M)\chi_-(M),
\]

\[
c^\lambda_{[12,12,0]}(M) = - 2(m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1),
\]

\[
c^\lambda_{[12,12,1]}(M) = (m_{22} - m_{33}) \{ \tilde{E}(M) + \tilde{F}(M) + (m_{13} - m_{12}) \{ \tilde{C}_1(M) + 1 + \tilde{D}(M)(1 - \chi_+(M)) \} \},
\]

\[
c^\lambda_{[12,12,2]}(M) = - \tilde{C}_1(M)\tilde{E}(M) - C_2(M) (1 - D(M) + \delta(M)\chi_+(M)),
\]

\[
c^\lambda_{[12,11,0]}(M) = (m_{13} - m_{12})(m_{22} - m_{33})(2m_{22} - m_{13} - m_{23} - 2),
\]

\[
c^\lambda_{[12,11,1]}(M) = \tilde{E}(M)(m_{23} - m_{22}) + C_2(M)(m_{22} - m_{33} + 1)
\]

\[+ (m_{13} - m_{12})\chi_-(M) \{ \tilde{D}(M)(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(\tilde{C}_1(M) + 1) \},
\]

\[
c^\lambda_{[12,11,2]}(M) = C_2(M)\chi_-(M)(m_{13} - m_{33} + 2 - \tilde{C}_1(M) - \tilde{D}(M)),
\]

\[
c^\lambda_{[12,02,0]}(M) = (m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1),
\]

\[
c^\lambda_{[12,02,1]}(M) = - (m_{22} - m_{33}) \{ \tilde{F}(M) + (m_{13} - m_{12})(\tilde{C}_1(M) + \chi_+(M)) \},
\]

\[
c^\lambda_{[12,02,2]}(M) = (\tilde{C}_1(M) + \chi_+(M) - 1)\tilde{F}(M) + (m_{22} - m_{33} + 1)C_2(M)\chi_+(M),
\]

\[
c^\lambda_{[12,02,3]}(M) = - C_2(M)(\tilde{C}_1(M) - 1)\chi_+(M),
\]

\[
c^\lambda_{[12,01,0]}(M) = - (m_{13} - m_{12})(m_{22} - m_{33})(2m_{22} - m_{13} - m_{23} - 2),
\]

\[
c^\lambda_{[12,01,1]}(M) = (m_{13} - m_{12})\tilde{C}_1(M) \{ (m_{22} - m_{33})(1 - \chi_+(M)) - (m_{13} - m_{22} + 1) \}
\]

\[+ (m_{23} - m_{22})\tilde{F}(M) - (m_{22} - m_{33} + 1)C_2(M),
\]

\[
c^\lambda_{[12,01,2]}(M) = 2C_2(M)(\tilde{C}_1(M) - 1),
\]

\[
c^\lambda_{[12,00,0]}(M) = (m_{13} - m_{12})(m_{13} - m_{22} + 1)(m_{23} - m_{22}),
\]

\[
c^\lambda_{[12,00,1]}(M) = (m_{13} - m_{12})(m_{13} - m_{22} + 1)\tilde{C}_1(M)\chi_-(M) - (m_{23} - m_{22} - 1)C_2(M),
\]

\[
c^\lambda_{[12,00,2]}(M) = - C_2(M)(\tilde{C}_1(M) - 1)\chi_-(M).
\]

**Formula 5:** The coefficients of the injector $i^\lambda_{e_1+e_3}: V_{\lambda[+13]} \rightarrow V_\lambda \otimes C V_{2e_3}$ are given as follows

\[(r_{[13,22]}, r_{[13,12]}, r_{[13,11]}, r_{[13,02]}, r_{[13,01]}, r_{[13,00]}) = (1, 2, 1, 3, 2, 1)\] and

\[
c^\lambda_{[13,22,0]}(M) = (m_{13} - m_{12})(m_{22} - m_{33}),
\]

\[
c^\lambda_{[13,22,1]}(M) = - \tilde{E}(M),
\]

\[
c^\lambda_{[13,12,0]}(M) = - 2(m_{13} - m_{12})(m_{22} - m_{33}),
\]

\[
c^\lambda_{[13,12,1]}(M) = \tilde{E}(M) + \tilde{F}(M) - (m_{13} - m_{12})(m_{22} - m_{33})\chi_+(M),
\]

\[
c^\lambda_{[13,12,2]}(M) = (\tilde{E}(M) - C_2(M))\chi_+(M),
\]

\[
c^\lambda_{[13,11,0]}(M) = (m_{13} - m_{12})(2m_{22} - m_{13} - m_{33} - 1),
\]

\[
c^\lambda_{[13,11,1]}(M) = C_2(M) - \tilde{E}(M),
\]

\[
c^\lambda_{[13,02,0]}(M) = (m_{13} - m_{12})(m_{22} - m_{33}),
\]

\[
c^\lambda_{[13,02,1]}(M) = (m_{13} - m_{12})(m_{22} - m_{33})\chi_+(M) - \tilde{F}(M),
\]
Homomorphisms

\[ c_{[13,02,2]}^\lambda (M) = C_2(M)\chi_+(M) - \bar{F}(M)\chi_+^{(1)}(M), \]
\[ c_{[13,02,3]}^\lambda (M) = C_2(M)\chi_+^{(1)}(M), \]
\[ c_{[13,01,0]}^\lambda (M) = - (m_{13} - m_{12})(2m_{22} - m_{13} - m_{23} - 1), \]
\[ c_{[13,01,1]}^\lambda (M) = \bar{F}(M) - C_2(M) + (m_{13} - m_{12})(m_{13} - m_{22} + 1)\chi_+(M), \]
\[ c_{[13,01,2]}^\lambda (M) = - 2C_2(M)\chi_+(M), \]
\[ c_{[13,00,0]}^\lambda (M) = - (m_{13} - m_{12})(m_{13} - m_{22} + 1), \]
\[ c_{[13,00,1]}^\lambda (M) = C_2(M). \]

**Formula 6:** The coefficients of the injector \( i_{e_2 + e_3}^{\lambda} : V_{\lambda} \to V_{\lambda} \otimes_c V_{2e_1} \) are given as follows:
\( (r_{[23,22]}, r_{[23,12]}, r_{[23,11]}, r_{[23,02]}, r_{[23,01]}, r_{[23,00]}) = (1, 1, 1, 2, 1, 1) \) and

\[ c_{[23,22,0]}^\lambda (M) = m_{22} - m_{33}, \]
\[ c_{[23,22,1]}^\lambda (M) = - \bar{D}(M)\chi_-(M), \]
\[ c_{[23,12,0]}^\lambda (M) = - 2(m_{22} - m_{33}), \]
\[ c_{[23,12,1]}^\lambda (M) = \bar{C}_1(M) - (m_{22} - m_{33}) + \delta(M)\chi_-(M), \]
\[ c_{[23,11,0]}^\lambda (M) = 2m_{22} - m_{23} - m_{33}, \]
\[ c_{[23,11,1]}^\lambda (M) = - (\bar{C}_1(M) + \bar{D}(M))\chi_-(M), \]
\[ c_{[23,02,0]}^\lambda (M) = m_{22} - m_{33}, \]
\[ c_{[23,02,1]}^\lambda (M) = - \{ \bar{C}_1(M) - (m_{22} - m_{33})\chi_+(M) \}, \]
\[ c_{[23,01,2]}^\lambda (M) = - \bar{C}_1(M)\chi_+(M), \]
\[ c_{[23,01,0]}^\lambda (M) = - (2m_{22} - m_{23} - m_{33}), \]
\[ c_{[23,00,0]}^\lambda (M) = - (m_{23} - m_{22}), \]
\[ c_{[23,00,1]}^\lambda (M) = - \bar{C}_1(M)\chi_-(M). \]

**Proof.** For \( 1 \leq i \leq j \leq 3 \), let \( i_{e_i + e_j}^{\lambda} : V_{\lambda} \to V_{\lambda} \otimes_c V_{2e_1} \) be the composite of three \( K \)-homomorphisms

\[ i_{e_i}^{\lambda} \otimes_c \text{id}_{V_{e_i}} : V_{\lambda} \otimes_c V_{e_i} \to V_{\lambda} \otimes_c V_{2e_1}, \]
\[ i_{e_j}^{\lambda} \otimes_c \text{id}_{V_{e_j}} : V_{\lambda} \otimes_c V_{e_j} \to V_{\lambda} \otimes_c V_{e_1} \]
\[ \text{id}_{V_{\lambda}} \otimes_c P_{e_1} : V_{\lambda} \otimes_c V_{e_1} \otimes_c V_{e_1} \to V_{\lambda} \otimes_c V_{2e_1}. \]

Then \( i_{e_i + e_j}^{\lambda} \) is an element of \( \text{Hom}_K(V_{\lambda} \otimes_c V_{2e_1}) \). By direct computation, we confirm that \( i_{e_i + e_j}^{\lambda} \) is non-zero and obtain the explicit expression of this \( K \)-homomorphism.

\[ i_{e_i + e_j}^{\lambda} (f(M)) = (\text{id}_{V_{\lambda}} \otimes_c P_{e_1}) \circ (i_{e_j}^{\lambda} \otimes_c \text{id}_{V_{e_1}}) \circ i_{e_i}^{\lambda + e_j} (f(M)) \]
\[ = \sum_{0 \leq k \leq t \leq 1} \left( \sum_{m=0}^{r_{[1,k,l]}} c_{[1,k:l]}^\lambda (M) (\text{id}_{V_{\lambda}} \otimes_c P_{e_1}) \circ (i_{e_j}^{\lambda} \otimes_c \text{id}_{V_{e_1}}) \left( f \left( M \left( \begin{smallmatrix} -e_i & e_i \\ 0 & -k \end{smallmatrix} \right) \right) \otimes f \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \right) \]
\[ = \sum_{0 \leq k \leq t \leq 1} \left( \sum_{m=0}^{r_{[1,k,l]}} c_{[1,k:l]}^\lambda (M) \left( \sum_{0 \leq p \leq q \leq 1} \left( \sum_{r=0}^{r_{[p,q,r]}} c_{[p,q,r]}^\lambda \left( M \left( \begin{smallmatrix} -e_i \\ 0 & -p \end{smallmatrix} \right) \right) \otimes P_{e_1} \left( f \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \right) \right) \right) \]
\[ \times f \left( M \left( \begin{smallmatrix} -e_i & e_i \\ 0 & -t \end{smallmatrix} \right) \right) \otimes P_{e_1} \left( f \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \]
\[ = \sum_{0 \leq s \leq t \leq 2} \left( \sum_{u=0}^{R_{[s,t]}^{\lambda}} c_{[s,t:u]}^\lambda (M) f \left( M \left( \begin{smallmatrix} -e_i & e_i \\ 0 & -t \end{smallmatrix} \right) \right) \right) \otimes f \left( \begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix} \right), \]
where
\[
    c^\lambda_{[ij;st];u}(M) = \sum_{0 \leq p \leq q \leq 1,} \sum_{0 \leq m \leq r_{[ik]},} c^\lambda_{[ij;kl];m}(M) c^\lambda_{[ij;pq];r}(M) M \left( \frac{-\epsilon_i}{-k} \right) [-m],
\]
and
\[
    R_{[ij;st]} = \max \{ r_{[ik]}, r_{[pq]} \mid 0 \leq k \leq l \leq 1, 0 \leq p \leq q \leq 1, p + k = s, q + l = t \}.
\]
Now we simplify each coefficient by direct computation.

First, we compute the case of formula 1.

- the case \((s, t) = (2, 2)\).

  We have
  \[
  c^\lambda_{[11;22];0}(M) = c^\lambda_{[11;11];0}(M) c^\lambda_{[11;11];0}(M) M \left( \frac{-100}{-1} \right)
  = (m_{13} - m_{12})(m_{22} - m_{33})(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1),
  \]
  \[
  c^\lambda_{[11;22];l}(M) = c^\lambda_{[11;11];l}(M) c^\lambda_{[11;11];l}(M) M \left( \frac{-100}{-1} \right) + c^\lambda_{[11;11];l}(M) c^\lambda_{[11;11];0}(M) M \left( \frac{-100}{-1} \right) [-1]
  = -(m_{13} - m_{12})(m_{22} - m_{33}) E \left( M \left( \frac{-100}{-1} \right) \right) - E(M)(m_{13} - m_{12})(m_{22} - m_{33})
  = 2(m_{13} - m_{12})(m_{22} - m_{33})(E(M) - C_1(M)),
  \]
  \[
  c^\lambda_{[11;22];2}(M) = c^\lambda_{[11;11];2}(M) c^\lambda_{[11;11];2}(M) M \left( \frac{-100}{-1} \right) [-1]
  = E(M) E \left( M \left( \frac{-100}{-1} \right) [-1] \right)
  = E(M)(C_1(M) - 1)(m_{13} - m_{33} - C_1(M)).
  \]

  Here we use the relations:
  \[
  E \left( M \left( \frac{-100}{-1} \right) \right) = E(M) - 2C_1(M),
  \]
  \[
  C_1 \left( M \left( \frac{-100}{-1} \right) \right) = C_1(M), \quad C_1 \left( M \left( \frac{-100}{-1} \right) \right) = C_1(M) + 1,
  \]
  \[
  C_1(M[-1]) = C_1(M) - 1, \quad C_1(M[-1]) = C_1(M) - 1.
  \]

- the case \((s, t) = (1, 2)\).

  We have
  \[
  c^\lambda_{[11;12];0}(M) = c^\lambda_{[11;11];0}(M) c^\lambda_{[11;01];0}(M) M \left( \frac{-100}{-1} \right) + c^\lambda_{[11;01];0}(M) c^\lambda_{[11;11];0}(M) M \left( \frac{-100}{-1} \right)
  = 2(m_{13} - m_{12})(m_{22} - m_{33})(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1),
  \]
  \[
  c^\lambda_{[11;12];1}(M) = c^\lambda_{[11;11];1}(M) c^\lambda_{[11;01];1}(M) M \left( \frac{-100}{-1} \right) + c^\lambda_{[11;01];1}(M) c^\lambda_{[11;11];0}(M) M \left( \frac{-100}{-1} \right) [-1]
  \]
  \[
  + c^\lambda_{[11;01];0}(M) c^\lambda_{[11;11];1}(M) M \left( \frac{-100}{-1} \right) + c^\lambda_{[11;11];1}(M) c^\lambda_{[11;01];0}(M) M \left( \frac{-100}{-1} \right) [-1]
  = (m_{13} - m_{12})(m_{22} - m_{33}) E \left( M \left( \frac{-100}{-1} \right) \right) + E(M)(m_{13} - m_{12})(m_{22} - m_{33})
  \]
  \[
  \quad + (m_{13} - m_{12})(m_{22} - m_{33}) E \left( M \left( \frac{-100}{-1} \right) \right) + E(M)(m_{13} - m_{12})(m_{22} - m_{33})
  = 2(m_{13} - m_{12})(m_{22} - m_{33}) E \left( M \left( \frac{-100}{-1} \right) \right) + E(M),
  \]
  \[
  c^\lambda_{[11;12];2}(M) = c^\lambda_{[11;11];1}(M) c^\lambda_{[11;01];1}(M) M \left( \frac{-100}{-1} \right) [-1] + c^\lambda_{[11;01];1}(M) c^\lambda_{[11;11];1}(M) M \left( \frac{-100}{-1} \right) [-1]
  \]
  \[
  \quad + c^\lambda_{[11;01];0}(M) c^\lambda_{[11;11];1}(M) M \left( \frac{-100}{-1} \right) + c^\lambda_{[11;11];1}(M) c^\lambda_{[11;01];0}(M) M \left( \frac{-100}{-1} \right) [-1].
  \]
Here we use the relations:

\[
\begin{align*}
+ c_{1;11;0}^\lambda (M) c_{1;01;2}^\gamma (M) \left( M \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ -1 \end{pmatrix} \right) + c_{1;01;2}^\mu (M) c_{1;11;0}^\lambda \left( M \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \\
= -\bar{E}(M)\bar{F} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) - \bar{F}(M)\bar{E} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \\
- (m_{13} - m_{12})(m_{22} - m_{33})C_2 \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \chi_+ \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \\
- C_2(M)\chi_+(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) \\
= -2\bar{E}(M)\bar{F} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) - (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)\chi_+(M) \\
+ C_2(M)\chi_+(M)(m_{13} - m_{33} + 1 - m_{12} + m_{22}) \\
- (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(C_1(M) + 1)\chi_+(M) \\
- C_2(M)\chi_+(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) \\
= -2 \left\{ \bar{E}(M)\bar{F} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \right\} + (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(C_1(M) + 1)\chi_+(M),
\end{align*}
\]

\[
c_{11;12;3}^\lambda (M) = c_{1;11;1}^\lambda (M) c_{1;01;2}^\gamma (M) \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) + c_{1;01;2}^\mu (M) c_{1;11;1}^\lambda \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \\
= \bar{E}(M)C_2 \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \chi_+ \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \\
+ C_2(M)\chi_+(M)\bar{E} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \left[ -1 \right] \\
= 2(C_1(M) - 1)\bar{C}_1(M)\bar{E}(M)\chi_+(M).
\]

Here we use the relations:

\[
\begin{align*}
(4.40) & \quad \bar{F} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) - \bar{F}(M) = \bar{E} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) - \bar{E}(M), \\
(4.41) & \quad \bar{F}(M)\bar{E} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \left[ -1 \right] \\
& = \bar{E}(M)\bar{F} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \left[ -1 \right] + (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)\chi_+(M) \\
& \quad - C_2(M)\chi_+(M)(m_{13} - m_{33} + 1 - m_{12} + m_{22}), \\
(4.42) & \quad C_2 \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \chi_+ \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) = C_1(M)(C_1(M) + 1)\chi_+(M), \\
(4.43) & \quad C_2 \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \chi_+ \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \left[ -1 \right] = (C_1(M) - 1)\bar{C}_1(M)\chi_+(M), \\
(4.44) & \quad C_2(M)\chi_+(M)\bar{E} \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \left[ -2 \right] = (C_1(M) - 1)\bar{C}_1(M)\bar{E}(M)\chi_+(M).
\]

• the case $(s, t) = (1, 1)$.

We have

\[
c_{11;11;0}^\lambda (M) = c_{1;11;0}^\lambda (M) c_{1;00;0}^\gamma (M) \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) + c_{1;00;0}^\mu (M) c_{1;11;0}^\lambda \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \\
= -(m_{13} - m_{12})(m_{22} - m_{33})(m_{13} - m_{12} - 1)(m_{13} - m_{22} + 1) \\
- (m_{13} - m_{12})(m_{13} - m_{22} + 1)(m_{13} - m_{12} - 1)(m_{22} - m_{33}) \\
- 2(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{13} - m_{22} + 1),
\]

\[
c_{11;11;1}^\lambda (M) = c_{1;11;0}^\lambda (M) c_{1;00;1}^\gamma (M) \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) + c_{1;00;1}^\mu (M) c_{1;11;0}^\lambda \left( M \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \right) \left[ -1 \right]
\]
\[ 
+ c^\lambda_{[1;00;0]}(M) c^\lambda_{[1;11;1]} \left( M \begin{pmatrix} 1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) + c^\lambda_{[1;11;1]}(M) c^\lambda_{[1;00;0]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) 
= (m_{13} - m_{12})(m_{22} - m_{33})C_2 \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) 
+ C_2(M)(m_{13} - m_{12})(m_{22} - m_{33} + 1) 
+ (m_{13} - m_{12})(m_{13} - m_{22} + 1) \tilde{E} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr 0 \end{pmatrix} \right) 
+ \tilde{E}(M)(m_{13} - m_{12})(m_{13} - m_{22}) 
= (m_{13} - m_{12}) \left\{ (m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1) \right. 
+ C_2(M)(m_{22} - m_{33} + 1) 
+ (m_{13} - m_{22} + 1)(\tilde{E}(M) - C_1(M)) 
+ \left. \tilde{E}(M)(m_{13} - m_{22}) \right\} 
= (m_{13} - m_{12}) \left\{ 2(m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1) 
+ 2(m_{13} - m_{22})\tilde{E}(M) + \tilde{E}(M) - C_1(M)(m_{13} - m_{33} + 1 - \tilde{C}_1(M)) \right\} 
= 2(m_{13} - m_{12}) \left\{ (m_{22} - m_{33})C_1(M)\tilde{C}_1(M) + 1 + (m_{13} - m_{22})\tilde{E}(M) \right\}, 
\]
\[ 
c^\lambda_{[1;11;2]}(M) = c^\lambda_{[1;11;1]}(M) c^\lambda_{[1;00;1]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) + c^\lambda_{[1;00;1]}(M) c^\lambda_{[1;11;1]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr 0 \end{pmatrix} \right) 
= -\tilde{E}(M)C_2 \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) - C_2(M)\tilde{E} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr 0 \end{pmatrix} \right) 
= -2\tilde{E}(M)(C_1(M) - 1)\tilde{C}_1(M). 
\]

Here we use the relations:
\[ 
(4.45) \quad \tilde{E} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr 0 \end{pmatrix} \right) = \tilde{E}(M) - C_1(M), 
\]
\[ 
(4.46) \quad C_2(M)\tilde{E} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr 0 \end{pmatrix} \right) = (C_1(M) - 1)\tilde{C}_1(M)\tilde{E}(M), 
\]
\[ \text{[1.38] and [1.39].} \]

- the case \((s, t) = (0, 2)\).

We have
\[ 
c^\lambda_{[1;02;0]}(M) = c^\lambda_{[1;01;0]}(M) c^\lambda_{[1;01;0]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) 
= (m_{13} - m_{12})(m_{22} - m_{33})(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1), 
\]
\[ 
c^\lambda_{[1;02;1]}(M) = c^\lambda_{[1;01;0]}(M) c^\lambda_{[1;01;1]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) + c^\lambda_{[1;01;1]}(M) c^\lambda_{[1;01;0]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) 
= -(m_{13} - m_{12})(m_{22} - m_{33}) \left\{ F \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) + F(M) \right\}, 
\]
\[ 
c^\lambda_{[1;02;2]}(M) = c^\lambda_{[1;01;0]}(M) c^\lambda_{[1;01;2]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) + c^\lambda_{[1;01;1]}(M) c^\lambda_{[1;01;1]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) 
+ c^\lambda_{[1;01;2]}(M) c^\lambda_{[1;01;0]} \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) 
= (m_{13} - m_{12})(m_{22} - m_{33})C_2 \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) \chi_+ \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) 
+ F(M)F \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) + C_2(M)\chi_+(M)(m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1) 
= (m_{13} - m_{12} + 1)(m_{22} - m_{33} + 1)C_2(M)\chi_+(M) 
+ (m_{13} - m_{12})(m_{22} - m_{33})(C_1(M) + 1)(\tilde{C}_1(M) + 1)\chi_+(M) 
+ F(M)F \left( M \begin{pmatrix} -1 \cr 0 \cr 0 \cr -1 \end{pmatrix} \right) \chi_+(M). 
\]
Here we use the equations:

(4.47) \( C_2 \left( M \left( \begin{smallmatrix} -\frac{1}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \chi_+ \left( M \left( \begin{smallmatrix} -\frac{1}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) = (C_1(M) + 1)(\bar{C}_1(M) + 1)\chi_+^{(1)}(M), \)

(4.5), (4.11) and (4.39).

- the case \((s, t) = (0, 1)\).

We have

\[
c_{\lambda^{[11,02;3]}(M)} = c_{\lambda^{[1,01;1]}(M)} c_{\lambda^{[1,01;2]}(M)} \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \left[ -1 \right] + c_{\lambda^{[1,01;1]}(M)} c_{\lambda^{[1,01;1]}(M)} \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \left[ -2 \right] 
- \bar{F}(M)C_2 \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \chi_+ \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \left[ -1 \right] 
- C_2(M)\chi_+(M)\bar{F} \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \left[ -2 \right] 
= -C_2(M) \left( \chi_+^{(1)}(M)\bar{F}(M) + \chi_+(M)\bar{F} \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \right), 
\]

\[
c_{\lambda^{[11,02;4]}(M)} = c_{\lambda^{[1,01;2]}(M)} c_{\lambda^{[1,01;2]}(M)} \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \left[ -2 \right] 
= C_2(M)\chi_+(M)C_2 \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \chi_+ \left( M \left( \begin{smallmatrix} -\frac{100}{0} & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \left[ -2 \right] 
= C_2(M)(C_1(M) - 1)(\bar{C}_1(M) - 1)\chi_+^{(1)}(M). 
\]
\[ c_{[11,01;2]}^\lambda (M) = c_{[1;01;1]}^\lambda (M) c_{[1;00;1]}^\lambda \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) + c_{[1;01;1]}^\lambda (M) c_{[1;01;2]}^\lambda \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) \right) \\
+ c_{[1;01;2]}^\lambda (M) c_{[1;00;0]}^\lambda \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) + c_{[1;00;0]}^\lambda (M) c_{[1;01;2]}^\lambda \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \\

= \bar{F}(M)C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) - [1] + C_2(M) \bar{F} \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \chi_+ \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \\
+ C_2(M)\chi_+ (M)(m_{13} - m_{12} + 1)(m_{13} - m_{22} - 1) \\
+ (m_{13} - m_{12})(m_{13} - m_{22} + 1)C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \chi_+ \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \\

= \bar{F}(M)C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) - [1] \\
+ C_2(M)\{\bar{F}(M) + m_{12} - m_{22} - 1 + \delta(M) - \chi_+(M)(m_{13} - m_{12} + \delta(M))\} \\
+ C_2(M)\chi_+ (M)\{(m_{13} - m_{12} + 1)(m_{13} - m_{22} - 1) \\
+ (m_{13} - m_{12})(m_{13} - m_{22} + 1)\} \\

= 2\bar{F}(M)C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) - [1] - (m_{12} - m_{22} - 1 + \delta(M))C_2(M)(1 - \chi_+(M)) \\
+ C_2(M)\{m_{12} - m_{22} - 1 + \delta(M) - \chi_+(M)(m_{13} - m_{12} + \delta(M))\} \\
+ C_2(M)\chi_+ (M)\{(m_{13} - m_{12} + 1)(m_{13} - m_{22} - 1) \\
+ (m_{13} - m_{12})(m_{13} - m_{22} + 1)\} \\

= 2\{\bar{F}(M)C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) - [1] \}
+ C_2(M)\chi_+ (M)(m_{13} - m_{12} + 1)(m_{13} - m_{22} - 1), \\

\[ c_{[11,01;3]}^\lambda (M) = c_{[1;00;1]}^\lambda (M) c_{[1;01;2]}^\lambda \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) + c_{[1;01;2]}^\lambda (M) c_{[1;00;0]}^\lambda \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \\

= -C_2(M)C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) \chi_+ \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \\
- C_2(M)\chi_+ (M)C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \chi_+ \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) \\

= -2C_2(M)(C_1(M) - 1)(C_1(M) - 1)\chi_+(M). \\

Here we use the relations:

\[(4.48) \quad \bar{F} \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) = \bar{F}(M) + (m_{22} - m_{33} + \delta(M))\chi_+(M),\]

\[(4.49) \quad C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right) = C_2(M) + (m_{12} - m_{22} + 1 - \delta(M))\chi_+(M),\]

\[= (C_1(M) + \chi_+(M))(C_1(M) + \chi_+(M)),\]

\[(4.50) \quad C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) = C_2(M),\]

\[(4.51) \quad \bar{F} \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) = \bar{F}(M) + m_{12} - m_{22} - 1 + \delta(M) \\
- \chi_+(M)(m_{13} - m_{12} + \delta(M)),\]

\[(4.52) \quad C_2 \left( M \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right) \right) = C_2(M) - (m_{12} - m_{22} - 1 + \delta(M))(1 - \chi_+(M)),\]

\[(4.53) \quad \bar{F}(M)(1 - \chi_+(M)) = -C_2(M)(1 - \chi_+(M)),\]

\[(4.5) \text{ and } (4.39).\]

- the case \((s, t) = (0, 0)\).
We have
\[ c_{[11,00,0]}^\lambda(M) = c_{[1,00,0]}^\lambda(M) c_{[1,00,0]}^\lambda(M - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) = (m_{13} - m_{12})(m_{13} - m_{22} + 1)(m_{13} - m_{12} - 1)(m_{13} - m_{22}), \]
\[ c_{[11,00,1]}^\lambda(M) = c_{[1,00,0]}^\lambda(M) c_{[1,00,1]}^\lambda(M - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) + c_{[1,00,1]}^\lambda(M) c_{[1,00,0]}^\lambda(M - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})(-1)]\]
\[ = -(m_{13} - m_{12})(m_{13} - m_{22} + 1)C_2\left(M - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) \]
\[ = -C_2(M)(m_{13} - m_{12})(m_{13} - m_{22} - 1) \]
\[ = -2(m_{13} - m_{12})(m_{13} - m_{22})C_2(M), \]
\[ c_{[11,00,2]}^\lambda(M) = c_{[1,00,1]}^\lambda(M) c_{[1,00,1]}^\lambda(M - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})(-1)]\]
\[ = C_2(M)C_2\left(M - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)(-1)]\]
\[ = C_2(M)(C_1(M) - 1)(C_1(M) - 1). \]

Here we use the relations \ref{4.39} and \ref{4.50}.

Next, we compute the case of formula 2.

- the case \((s, t) = (2, 2)\).
  We have
  \[ c_{[22,22,0]}^\lambda(M) = c_{[2,11,0]}^\lambda(M) c_{[2,11,0]}^\lambda(M) = (m_{22} - m_{33})(m_{22} - m_{33} - 1), \]
  \[ c_{[22,22,1]}^\lambda(M) = c_{[2,11,0]}^\lambda(M) c_{[2,11,1]}^\lambda(M) = c_{[2,11,1]}^\lambda(M) c_{[2,11,0]}^\lambda(M - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix})(-1)]\]
  \[ = -m_{22} - m_{33}\bar{D}\left(M - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right) \chi_{-}\left(M - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right) = \bar{D}(M)\chi_{-}(M)(m_{22} - m_{33}) \]
  \[ = -(m_{22} - m_{33})\{\bar{D}(M)\chi_{-}(M) + (\bar{D}(M) + 2)\chi_{-}(M)\}, \]
  \[ c_{[22,22,2]}^\lambda(M) = c_{[2,11,1]}^\lambda(M) c_{[2,11,1]}^\lambda(M - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix})(-1)]\]
  \[ = \bar{D}(M)\chi_{-}(M)\bar{D}(M)\chi_{-}(M)(m_{22} - m_{33}) \]
  \[ = \bar{D}(M)\left(\bar{D}(M) + 1\right)\chi_{-}(M)\chi_{-}(M)(m_{22} - m_{33}) \]
  \[ = \bar{D}(M)(\bar{D}(M) + 1)\chi_{-}(M). \]

Here we use the relations
\[ \bar{D}\left(M - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right) = \bar{D}(M) + 2, \]
\ref{4.6}, \ref{4.12} and \ref{4.62}.

- the case \((s, t) = (1, 2)\).
  We have
  \[ c_{[22,12,0]}^\lambda(M) = c_{[2,11,0]}^\lambda(M) c_{[2,01,0]}^\lambda(M) = (m_{22} - m_{33})(m_{22} - m_{33} - 1) \]
  \[ = -m_{22} - m_{33}(m_{22} - m_{33} - 1) - (m_{22} - m_{33})(m_{22} - m_{33} - 1) \]
  \[ = -2(m_{22} - m_{33})(m_{22} - m_{33} - 1), \]
  \[ c_{[22,12,1]}^\lambda(M) = c_{[2,11,0]}^\lambda(M) c_{[2,01,1]}^\lambda(M) = c_{[2,01,1]}^\lambda(M) c_{[2,11,1]}^\lambda(M - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix})(-1)]\]
  \[ + c_{[2,01,1]}^\lambda(M) c_{[2,11,0]}^\lambda(M - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix})(-1)]\]
  \[ = C_2(M)\bar{D}(M) + 1\chi_{-}(M). \]
Here we use the relations
\[(4.55)\]
\[\bar{C}_1 \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} = \bar{C}_1(\bar{M}) + \bar{C}_1(\bar{M} + 1) = \bar{C}_1(\bar{M}) \] 
\[(4.56)\]
\[\bar{C}_1 \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} = \bar{C}_1(\bar{M}) + \text{chi}_-(\bar{M}),\]
\[(4.57)\]
\[\bar{C}_1 \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} \] 
\[\chi_- (\bar{M}) = \bar{C}_1(\bar{M}) \chi_- (\bar{M}),\]
\[(4.61)\] and \[(4.62)\].

- the case \((s, t) = (1, 1)\). We have

\[c_{[21;11;0]}^{\lambda} (M) = c_{[21;11;0]}^{\lambda} (M) c_{[20;0;0]}^{\lambda} \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} + c_{[20;0;0]}^{\lambda} (M) c_{[21;11;0]}^{\lambda} \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} \]
\[= -(m_{22} - m_{33})m_{23} - m_{22} - (m_{23} - m_{22})(m_{22} - m_{33}) \]
\[= 2(m_{22} - m_{33})(m_{23} - m_{22}),\]

\[c_{[21;11;1]}^{\lambda} (M) = c_{[21;11;0]}^{\lambda} (M) c_{[20;0;1]}^{\lambda} \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} + c_{[20;0;1]}^{\lambda} (M) c_{[21;11;0]}^{\lambda} \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} - 1 \]
\[= -(m_{22} - m_{33}) \bar{C}_1 \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} \] 
\[- \bar{C}_1(\bar{M}) \chi_-(\bar{M}) (m_{22} - m_{33} + 1) + \bar{D}(\bar{M}) \chi_-(\bar{M})(m_{23} - m_{22} - 1) + (m_{23} - m_{22}) \bar{D}(\bar{M}) \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} \] 
\[= -(m_{22} - m_{33})(m_{12} - m_{11} + 1) \chi^{(1)}_- (\bar{M}) - (m_{12} - m_{11}) \chi_-(\bar{M})(m_{22} - m_{33} + 1) \]
\[+ (-m_{22} + m_{33} + \delta(\bar{M})) \chi_-(\bar{M})(m_{23} - m_{22} - 1) \]
\[+ (m_{23} - m_{22})(m_{23} - m_{22} + \delta(\bar{M})) \chi^{(1)}_-(\bar{M}) \]
\[= 2(m_{22} - m_{33})(m_{12} - m_{11} + 1) \chi^{(1)}_- (\bar{M}) \]
\[+ 2(-m_{22} + m_{33} + \delta(\bar{M})) \chi_-(\bar{M})(m_{23} - m_{22} - 1) \]
\[+ (m_{23} - m_{33})(\delta(\bar{M}) + 1) \chi^{(1)}_-(\bar{M}) - \chi_-(\bar{M})) \]
\[= 2\bar{D}(\bar{M})(m_{23} - m_{22} - 1) \chi_-(\bar{M}) - (m_{22} - m_{33})(\bar{C}_1(\bar{M}) + 1) \chi^{(1)}_-(\bar{M})),\]

\[c_{[21;11;2]}^{\lambda} (M) = c_{[21;11;1]}^{\lambda} (M) c_{[20;0;1]}^{\lambda} \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} + c_{[20;0;1]}^{\lambda} (M) c_{[21;11;1]}^{\lambda} \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} \]
\[= \bar{D}(\bar{M}) \chi_-(\bar{M}) \bar{C}_1 \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} \] 
\[\chi_-(\bar{M}) \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \\ \end{pmatrix} \]
Here we use the relations

\[ (4.58) \quad \tilde{D}(M \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) = \tilde{D}(M) + 1, \]

\[ (4.4), (4.6), (4.12), (4.16), (4.56) \text{ and } (4.62). \]

- the case \((s, t) = (0, 2)\).

We have

\[ c^\lambda_{[22,02;0]}(M) = c^\lambda_{[2,01;0]}(M) c^\lambda_{[2,01;0]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ = (m_{22} - m_{33})(m_{22} - m_{33} - 1) \]

\[ c^\lambda_{[22,02;1]}(M) = c^\lambda_{[2,01;0]}(M) c^\lambda_{[2,01;1]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + c^\lambda_{[2,01;1]}(M) c^\lambda_{[2,01;0]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ = - (m_{22} - m_{33})\tilde{C}_1 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \tilde{C}_1(M)(m_{22} - m_{33}) \]

\[ = - (m_{22} - m_{33})\tilde{C}_1(M), \]

\[ c^\lambda_{[22,02;2]}(M) = c^\lambda_{[2,01;1]}(M) c^\lambda_{[2,01;1]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ = \tilde{C}_1(M)\tilde{C}_1(M) - 1. \]

Here we use the relations

\[ (4.59) \quad \tilde{C}_1(M \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) = \tilde{C}_1(M), \]

and \((4.39)\).

- the case \((s, t) = (0, 1)\).

We have

\[ c^\lambda_{[22,01;0]}(M) = c^\lambda_{[2,01;0]}(M) c^\lambda_{[2,00;0]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + c^\lambda_{[2,00;0]}(M) c^\lambda_{[2,01;0]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ = (m_{22} - m_{33})(m_{23} - m_{22}) + (m_{23} - m_{22})(m_{22} - m_{33}) \]

\[ = 2(m_{22} - m_{33})(m_{23} - m_{22}), \]

\[ c^\lambda_{[22,01;1]}(M) = c^\lambda_{[2,01;0]}(M) c^\lambda_{[2,00;1]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + c^\lambda_{[2,00;1]}(M) c^\lambda_{[2,01;0]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ + c^\lambda_{[2,01;1]}(M) c^\lambda_{[2,00;0]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + c^\lambda_{[2,00;0]}(M) c^\lambda_{[2,01;1]}(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ = (m_{22} - m_{33})\tilde{C}_1 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \tilde{C}_1 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \tilde{C}_1(M)\tilde{C}_1(M)(m_{22} - m_{33}) + 1 \]

\[ - \tilde{C}_1(M)(m_{23} - m_{22} - 1) - (m_{23} - m_{22})\tilde{C}_1(M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ = (m_{22} - m_{33})\tilde{C}_1(M)\tilde{C}_1(M) + \tilde{C}_1(M)\tilde{C}_1(M)(m_{22} - m_{33}) + 1 \]

\[ - \tilde{C}_1(M)(m_{23} - m_{22} - 1) - (m_{23} - m_{22})(\tilde{C}_1(M) - \chi^{(-1)}(M)) \]

\[ = 2(m_{22} - m_{33})\tilde{C}_1(M) - 2(m_{23} - m_{22})\tilde{C}_1(M) + \tilde{C}_1(M)(1 + \chi(M) + \chi^{(-1)}(M)) \]
Here we use the relations (4.10), (4.39) and (4.60).

• the case \((s, t) = (0, 0)\).
We have
\[
c_{[22,00;0]}^\lambda (M) = c_{[22,00;0]}^\lambda (M) c_{[22,00;0]}^\lambda (M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]}
= (m_{23} - m_{22})(m_{23} - m_{22} - 1)
\]
\[
c_{[22,00;1]}^\lambda (M) = c_{[22,00;0]}^\lambda (M) c_{[22,00;1]}^\lambda (M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]} + c_{[22,00;1]}^\lambda (M) c_{[22,00;0]}^\lambda (M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]}
= (m_{23} - m_{22})\bar{C}_1 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]} \chi_- \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]} + \bar{C}_1 (M) \chi_- (M)(m_{23} - m_{22} - 2)
= \bar{C}_1 (M)\{(m_{23} - m_{22} - 2)\chi_- (M) + (m_{23} - m_{22})\chi_-^{(1)} (M)\},
\]
\[
c_{[22,00;2]}^\lambda (M) = c_{[22,00;1]}^\lambda (M) c_{[22,00;1]}^\lambda (M) \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]}
= \bar{C}_1 (M)\chi_- (M)\bar{C}_1 \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]} \chi_- \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{[-1]}
= \bar{C}_1 (M)(\bar{C}_1 (M) - 1)\chi_-^{(1)} (M).
\]

Here we use the relations (4.6), (4.10), (4.12), (4.39) and (4.60).

Next, we compute the case of formula 3.

• the case \((s, t) = (2, 2)\).
We have
\[
c_{[33,22;0]}^\lambda (M) = c_{[33,11;0]}^\lambda (M) c_{[33,11;0]}^\lambda \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}^{[-1]} = 1.
\]

• the case \((s, t) = (1, 2)\).
We have
\[
c_{[33,12;0]}^\lambda (M) = c_{[33,11;0]}^\lambda (M) c_{[33,01;0]}^\lambda \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}^{[-1]} + c_{[33,01;0]}^\lambda (M) c_{[33,11;0]}^\lambda \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}^{[-1]}
= -1 - 1 = -2,
\]
\[
c_{[33,12;1]}^\lambda (M) = c_{[33,11;0]}^\lambda (M) c_{[33,01;1]}^\lambda \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}^{[-1]} + c_{[33,01;1]}^\lambda (M) c_{[33,11;0]}^\lambda \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}^{[-1]}
= -\chi_+ \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}^{[-1]} - \chi_+(M) = -2\chi_+(M).
\]
Here we use the relation (4.5).

- the case \((s, t) = (1, 1)\).
  We have
  \[
  c^\lambda_{[3;1;1;0]}(M) = c^\lambda_{[3;1;0;0]}(M) c^\lambda_{[3;0;0;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) + c^\lambda_{[3;0;0;0]}(M) c^\lambda_{[3;1;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) \\
  = 1 + 1 = 2.
  \]

- the case \((s, t) = (0, 2)\).
  We have
  \[
  c^\lambda_{[3;3;2;2]}(M) = c^\lambda_{[3;0;1;0]}(M) c^\lambda_{[3;0;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) = 1.
  \]
  \[
  c^\lambda_{[3;3;0;2;2]}(M) = c^\lambda_{[3;0;1;0]}(M) c^\lambda_{[3;0;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) + c^\lambda_{[3;0;1;1]}(M) c^\lambda_{[3;0;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} [-1]) \\
  = \chi_+(M \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) + \chi_+(M) = \chi_+^{(1)}(M) + \chi_+(M) \\
  = \chi_+^{(1)}(M)
  \]
  Here we use the relation (4.5).

- the case \((s, t) = (0, 1)\).
  We have
  \[
  c^\lambda_{[3;3;2;2]}(M) = c^\lambda_{[3;0;1;0]}(M) c^\lambda_{[3;0;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) + c^\lambda_{[3;0;0;0]}(M) c^\lambda_{[3;0;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) \\
  = -1 - 1 = -2,
  \]
  \[
  c^\lambda_{[3;3;0;2;2]}(M) = c^\lambda_{[3;0;1;0]}(M) c^\lambda_{[3;0;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) + c^\lambda_{[3;0;1;1]}(M) c^\lambda_{[3;0;1;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} [-1]) \\
  = -\chi_+(M) - \chi_+(M \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) = -2\chi_+(M).
  \]
  Here we use the relation (4.5).

- the case \((s, t) = (0, 0)\).
  We have
  \[
  c^\lambda_{[3;3;0;0;0]}(M) = c^\lambda_{[3;0;0;0]}(M) c^\lambda_{[3;0;0;0]}(M \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) = 1.
  \]

Next, we compute the case of formula 4.

- the case \((s, t) = (2, 2)\).
  We have
  \[
  c^\lambda_{[12;2;2;0]}(M) = c^\lambda_{[1;1;1;0]}(M) c^\lambda_{[2;1;1;0]}(M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \\
  = (m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1),
  \]
  \[
  c^\lambda_{[12;2;2;2]}(M) = c^\lambda_{[1;1;1;0]}(M) c^\lambda_{[2;1;1;1]}(M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) + c^\lambda_{[1;1;1;1]}(M) c^\lambda_{[2;1;1;0]}(M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} [-1]) \\
  = -(m_{13} - m_{12})(m_{22} - m_{33})D(M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \chi - (M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \\
  - E(M)(m_{22} - m_{33})}
Here we use the relations:

\[(4.61)\]
\[
\bar{D} \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \bar{D}(M) + 1,
\]
\[(4.62)\]
\[
\bar{D}(M[-1]) = \bar{D}(M) - 1,
\]
and \[(4.6)\].

- the case \((s, t) = (1, 2)\).

We have

\[
c_{12;12;0}^\lambda(M) = c_{1;11;0}^\lambda(M) c_{[2;01;0]}^\lambda \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) + c_{[1;01;0]}^\lambda(M) c_{[2;11;0]}^\lambda \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - (m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1)
\]
\[
= -(m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1)
\]
\[
= -2(m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1),
\]

\[
c_{12;12;1}^\lambda(M) = c_{1;11;0}^\lambda(M) c_{[2;01;1]}^\lambda \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) + c_{[1;01;1]}^\lambda(M) c_{[2;11;0]}^\lambda \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - (m_{13} - m_{12})(m_{22} - m_{33})C_1 \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)
\]
\[
+ \bar{F}(M)(m_{22} - m_{33}) + (m_{13} - m_{12})(m_{22} - m_{33}) \bar{D} \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \chi_-(M \begin{pmatrix} -1 \end{pmatrix})
\]
\[
+ E(M)(m_{22} - m_{33}) - (m_{22} - m_{33}) \{ \bar{E}(M) + \bar{F}(M) + (m_{13} - m_{12})[C_1(M) + 1 + \bar{D}(M)(1 - \chi_+(M))] \}.
\]

\[
c_{12;12;2}^\lambda(M) = c_{1;11;1}^\lambda(M) c_{[2;01;1]}^\lambda \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - (m_{13} - m_{12})(m_{22} - m_{33})C_1 \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - E(M)\bar{C}_1 \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - \bar{F}(M)\bar{D} \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \chi_-(M \begin{pmatrix} -1 \end{pmatrix})
\]
\[
- C_2(M)\chi_+(M)(m_{22} - m_{33} + 1)
\]
\[
= - E(M)\bar{C}_1(M) - C_2(M)(1 - \bar{D}(M) + \delta(M)\chi_+(M)),
\]

\[
c_{12;12;3}^\lambda(M) = c_{[1;01;2]}^\lambda(M) c_{[2;11;1]}^\lambda \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) - C_2(M)\chi_+(M)\bar{D} \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \chi_-(M \begin{pmatrix} -1 \end{pmatrix})
\]
\[
= 0.
\]
Here we use the relations:

\[(4.63)\quad \tilde{D}\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = \tilde{D}(M),\]
\[(4.64)\quad \tilde{C}_1\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = \tilde{C}_1(M) + 1,\]
\[(4.6), \(4.19), \(4.10), \(4.62), \(4.26)\text{ and } (4.39).\]

- the case \((s, t) = (1, 1)\).

We have
\[
c^\lambda_{\{1,1:0\}}(M) = c^\lambda_{\{1,1:0\}}(M) c^\lambda_{\{2,0:0\}}(M) + c^\lambda_{\{1,0:0\}}(M) c^\lambda_{\{2,1:0\}}(M)
\]
\[= -(m_{13} - m_{12})(m_{22} - m_{33})(m_{23} - m_{22} + 1)
-(m_{13} - m_{12})(m_{13} - m_{22} + 1)(m_{22} - m_{33})
=(m_{13} - m_{12})(m_{22} - m_{33})(2m_{22} - m_{13} - m_{23} - 2),\]

\[
c^\lambda_{\{1,1:1\}}(M) = c^\lambda_{\{1,1:1\}}(M) c^\lambda_{\{2,0:1\}}(M) + c^\lambda_{\{1,0:1\}}(M) c^\lambda_{\{2,1:1\}}(M)
\]
\[+ c^\lambda_{\{1,1:1\}}(M) c^\lambda_{\{2,0:0\}}(M) - [1]
+c^\lambda_{\{1,0:1\}}(M) c^\lambda_{\{2,1:1\}}(M)
\]
\[= -(m_{13} - m_{12})(m_{22} - m_{33})\tilde{C}_1\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\chi_-(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right))\]
\[+ C_2(M)\left(m_{22} - m_{33} + 1\right) + E(M)\left(m_{23} - m_{22}\right)
\]
\[+ (m_{13} - m_{12})(m_{13} - m_{22} + 1)\tilde{D}\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)\chi_-(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right))
\]
\[= E(M)\left(m_{23} - m_{22}\right) + C_2(M)\left(m_{22} - m_{33} + 1\right)
\]
\[+ (m_{13} - m_{12})\chi_-\left(M\left\{m_{13} - m_{22} + 1\right)\left(M - (m_{22} - m_{33}))(C_1(M) + 1)\right\},\]

\[
c^\lambda_{\{1,2:1\}}(M) = c^\lambda_{\{1,2:1\}}(M) c^\lambda_{\{2,0:1\}}(M) - [1]
+c^\lambda_{\{1,0:1\}}(M) c^\lambda_{\{2,1:1\}}(M)
\]
\[= \tilde{E}(M)\tilde{C}_1\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\chi_-(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right))\]
\[+ C_2(M)\tilde{D}\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)\chi_-(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right))
\]
\[= C_2(M)\left(m_{13} - m_{33} + 2 - C_1(M) - \tilde{D}(M)\right).\]

Here we use the relations:

\[(4.65)\quad \tilde{D}\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = \tilde{D}(M),\]
\[(4.6), \(4.62), \(4.64)\text{ and } (4.39).\]

- the case \((s, t) = (0, 2)\).

We have
\[
c^\lambda_{\{1,2:0\}}(M) = c^\lambda_{\{1,0:1\}}(M) c^\lambda_{\{2,0:1\}}(M)
\]
\[= (m_{13} - m_{12})(m_{22} - m_{33})(m_{23} - m_{33} - 1),\]
\[
c^\lambda_{\{1,2:1\}}(M) = c^\lambda_{\{1,0:1\}}(M) c^\lambda_{\{2,1:1\}}(M) + c^\lambda_{\{1,0:1\}}(M) c^\lambda_{\{2,0:1\}}(M)
\]
\[= -(m_{13} - m_{12})(m_{22} - m_{33})\tilde{C}_1\left(M\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) - E(M)\left(m_{22} - m_{33}\right)
\]
\[= -(m_{22} - m_{33})\left\{(m_{13} - m_{12})(\tilde{C}_1(M) + \chi_+(M)) + E(M)\right\},\]
Here we use the relations:

\[
\tilde{C}_1 \left(M \left( \begin{smallmatrix} -100 \\ 0 \\ -1 \end{smallmatrix} \right) \right) = \tilde{C}_1(M) + \chi_+(M),
\]

and \((4.39)\).

• the case \((s, t) = (0, 1)\).

We have

\[
c^\lambda_{[12,02;2]}(M) = c^\lambda_{[1,01;2]}(M) c^\lambda_{[2,01;0]} \left( M \left( \begin{smallmatrix} -100 \\ 0 \\ -1 \end{smallmatrix} \right) \right) + c^\lambda_{[1,01;1]}(M) c^\lambda_{[2,01;1]} \left( M \left( \begin{smallmatrix} -100 \\ 0 \\ -1 \end{smallmatrix} \right) \right) - C_2(M) \chi_+(M)(m_{22} - m_{33} + 1) + \tilde{F}(M) \tilde{C}_1 \left( M \left( \begin{smallmatrix} -100 \\ 0 \\ -1 \end{smallmatrix} \right) \right) - \left( \tilde{C}_1(M) + \chi_+(M) - 1 \right) \tilde{F}(M) + (m_{22} - m_{33} + 1) C_2(M) \chi_+(M),
\]

\[
c^\lambda_{[12,02;3]}(M) = c^\lambda_{[1,01;2]}(M) c^\lambda_{[2,01;1]} \left( M \left( \begin{smallmatrix} -100 \\ 0 \\ -1 \end{smallmatrix} \right) \right) - C_2(M) \chi_+(M) \tilde{C}_1 \left( M \left( \begin{smallmatrix} -100 \\ 0 \\ -1 \end{smallmatrix} \right) \right) - C_2(M)(\tilde{C}_1(M) - 1) \chi_+(M).
\]
Here we use the relations:

\[ (4.67) \quad \tilde{C}_1 \left( M \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \tilde{C}_1(M), \]

(4.6), (4.7), (4.8), (4.10), (4.15), (4.26), (4.39) and (4.67).

- the case \((s, t) = (0, 0)\).

We have

\[
c^\lambda_{[12;00;0]}(M) = c^\lambda_{[1;00;0]}(M) \cdot c^\lambda_{[2;00;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = (m_{13} - m_{12})(m_{13} - m_{22} + 1)(m_{23} - m_{22}),
\]

\[
c^\lambda_{[12;00;1]}(M) = c^\lambda_{[1;00;0]}(M) \cdot c^\lambda_{[2;00;1]}(M) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + c^\lambda_{[1;00;1]}(M) \cdot c^\lambda_{[2;00;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = (m_{13} - m_{12})(m_{13} - m_{22} + 1)\tilde{C}_1(M) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} - C_2(M)(m_{23} - m_{22} - 1)
\]

\[
c^\lambda_{[12;00;2]}(M) = c^\lambda_{[1;00;1]}(M) \cdot c^\lambda_{[2;00;1]}(M) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = - C_2(M)(\tilde{C}_1(M) - 1)\chi_-(M).
\]

Here we use the relations (4.6), (4.39) and (4.67).

Next, we compute the case of formula 5.

- the case \((s, t) = (2, 2)\).

We have

\[
c^\lambda_{[13;22;0]}(M) = c^\lambda_{[1;11;0]}(M) \cdot c^\lambda_{[3;11;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (m_{13} - m_{12})(m_{22} - m_{33}),
\]

\[
c^\lambda_{[13;22;1]}(M) = c^\lambda_{[1;11;1]}(M) \cdot c^\lambda_{[3;11;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = - \tilde{E}(M).
\]

- the case \((s, t) = (1, 2)\).

We have

\[
c^\lambda_{[13;12;0]}(M) = c^\lambda_{[1;11;0]}(M) \cdot c^\lambda_{[3;01;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + c^\lambda_{[1;01;0]}(M) \cdot c^\lambda_{[3;11;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = - (m_{13} - m_{12})(m_{22} - m_{33}) - (m_{13} - m_{12})(m_{22} - m_{33})
\]

\[
+ C_2(m_{22} - m_{33}),
\]

\[
c^\lambda_{[13;12;1]}(M) = c^\lambda_{[1;11;0]}(M) \cdot c^\lambda_{[3;01;1]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + c^\lambda_{[1;01;1]}(M) \cdot c^\lambda_{[3;11;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
+ C_2(m_{22} - m_{33}) \chi_+(M) + \tilde{F}(M) + \tilde{E}(M),
\]

\[
c^\lambda_{[13;12;2]}(M) = c^\lambda_{[1;11;1]}(M) \cdot c^\lambda_{[3;01;1]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + c^\lambda_{[1;01;2]}(M) \cdot c^\lambda_{[3;11;0]}(M) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
\[ E(M) \chi_+ \left( M \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) \begin{pmatrix} -1 \end{pmatrix} - C_2(M) \chi_+(M) \]
\[ = (E(M) - C_2(M)) \chi_+(M). \]

Here we use the relation (4.5).

- the case \((s, t) = (1, 1)\).
  We have
  \[ c_{[13,11;0]}(M) = c_{[1,11;0]}(M) c_{[3,00;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} + c_{[1,00;0]}(M) c_{[3,11;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = (m_{13} - m_{12})(m_{22} - m_{33}) - (m_{13} - m_{12})(m_{13} - m_{22} + 1) \]
  \[ = (m_{13} - m_{12})(2m_{22} - m_{13} - m_{33} - 1), \]
  \[ c_{[13,11;1]}(M) = c_{[1,00;1]}(M) c_{[3,11;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} + c_{[1,11;1]}(M) c_{[3,00;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = C_2(M) - E(M). \]

- the case \((s, t) = (0, 2)\).
  We have
  \[ c_{[13,02;0]}(M) = c_{[1,01;0]}(M) c_{[3,01;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} = (m_{13} - m_{12})(m_{22} - m_{33}), \]
  \[ c_{[13,02;1]}(M) = c_{[1,01;0]}(M) c_{[3,01;1]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} + c_{[1,01;1]}(M) c_{[3,01;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = (m_{13} - m_{12})(m_{22} - m_{33}) \chi_+ \begin{pmatrix} -1 \end{pmatrix} - F(M) \]
  \[ = (m_{13} - m_{12})(m_{22} - m_{33}) \chi_+^{(1)}(M) - F(M), \]
  \[ c_{[13,02;2]}(M) = c_{[1,01;2]}(M) c_{[3,01;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \end{pmatrix} + c_{[1,01;1]}(M) c_{[3,01;1]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = C_2(M) \chi_+(M) - F(M) \chi_+ \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = C_2(M) \chi_+(M) - F(M) \chi_+^{(1)}(M), \]
  \[ c_{[13,02;3]}(M) = c_{[1,01;2]}(M) c_{[3,01;1]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \end{pmatrix} \]
  \[ = C_2(M) \chi_+(M) \chi_+ \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = C_2(M) \chi_+(M) \chi_+^{(1)}(M). \]

Here we use the relations (4.2) and (4.11).

- the case \((s, t) = (0, 1)\).
  We have
  \[ c_{[13,01;0]}(M) = c_{[1,01;0]}(M) c_{[3,00;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} + c_{[1,00;0]}(M) c_{[3,01;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = - (m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{12})(m_{13} - m_{22} + 1) \]
  \[ = - (m_{13} - m_{12})(2m_{22} - m_{13} - m_{33} - 1), \]
  \[ c_{[13,01;1]}(M) = c_{[1,00;1]}(M) c_{[3,01;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} \]
  \[ + c_{[1,01;1]}(M) c_{[3,00;0]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} + c_{[1,00;0]}(M) c_{[3,01;1]}(M) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \end{pmatrix} \]
  \[ = C_2(M) - F(M) \chi_+^{(1)}(M). \]
We have

\[ c_{[3,0,0]}(M) = c_{[1,0,0]}(M) c_{[3,0,0,0]}(M) \left( M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) = -(m_{13} - m_{12})(m_{13} - m_{22} + 1), \]

\[ c_{[3,0,0,1]}(M) = c_{[1,0,0,1]}(M) c_{[3,0,0,0]}(M) \left( M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) = C_2(M). \]

At last, we compute the case of formula 6.

- the case \((s, t) = (2, 2)\).
  We have
  \[
  c_{[2,12,0]}(M) = c_{[2,11,0]}(M) c_{[3,0,1]}(M) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) = m_{22} - m_{33},
  \]
  \[
  c_{[2,12,1]}(M) = c_{[2,11,1]}(M) c_{[3,0,1]}(M) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) = -D(M)X_-(M).
  \]

- the case \((s, t) = (1, 2)\).
  We have
  \[
  c_{[2,12,0]}(M) = c_{[2,11,0]}(M) c_{[3,0,1]}(M) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) + \delta(M)X_-(M),
  \]
  \[
  c_{[2,12,1]}(M) = c_{[2,11,1]}(M) c_{[3,0,1]}(M) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) \left( M \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) = 0.
  \]

Here we use the relations \((4.3), (4.9)\) and \((4.10)\).

- the case \((s, t) = (1, 1)\).
  We have
  \[
  c_{[2,11,0]}(M) = c_{[2,11,0]}(M) c_{[3,0,0]}(M) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) + \delta(M)X_-(M)X_+(M),
  \]
  \[
  c_{[2,11,1]}(M) = c_{[2,11,1]}(M) c_{[3,0,0]}(M) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) \left( M \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( M \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right) = 0.
  \]
\[(m_{22} - m_{33}) - (m_{23} - m_{22}) = 2m_{22} - m_{23} - m_{33},\]

\[c_{[23,11]}(M) = c_{[23,11]}(M) c_{[3,00]}(M) \left( M \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) [-1] + c_{[23,11]}(M) c_{[3,00]}(M) \left( M \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right) = -C_1(M) \chi_-(M) - \bar{D}(M) \chi_-(M) = -(\bar{C}_1(M) + \bar{D}(M)) \chi_-(M).\]

- the case \((s, t) = (0, 2)\).
  We have
  \[c_{[23,02]}(M) = c_{[23,02]}(M) c_{[3,01]}(M) \left( M \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) = m_{22} - m_{33},\]

- the case \((s, t) = (0, 1)\).
  We have
  \[c_{[23,01]}(M) = c_{[23,01]}(M) c_{[3,00]}(M) \left( M \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) + c_{[23,01]}(M) c_{[3,00]}(M) \left( M \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right) = -(m_{22} - m_{33}) + (m_{23} - m_{22}) = -(2m_{22} - m_{23} - m_{33}),\]

- the case \((s, t) = (0, 0)\).
  We have
  \[c_{[23,00]}(M) = c_{[23,00]}(M) c_{[3,00]}(M) \left( M \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) = -(m_{23} - m_{22}),\]

\[c_{[3,00]}(M) \chi_+(M) + (m_{23} - m_{22}) \chi_+(M) \chi_+(M) = 2\bar{C}_1(M),\]

\[c_{[3,00]}(M) \chi_-(M) \chi_+(M) \chi_+(M) = 0.\]

\[(1.5), (1.9), (1.10) \text{ and } (1.15).\]

- the case \((s, t) = (0, 0)\).
  We have
  \[c_{[23,00]}(M) = c_{[23,00]}(M) c_{[3,00]}(M) \left( M \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) = -(m_{23} - m_{22}),\]

\[c_{[3,00]}(M) \chi_-(M) \chi_+(M) = -\bar{C}_1(M) \chi_-(M).\]

For \(1 \leq i \leq j \leq 3\), we define a linear map \(i_{-e_i-e_j}^\lambda : V_{\lambda[-ij]} \to V_{\lambda} \otimes_C V_{-2e_i}\) by

\[i_{-e_i-e_j}^\lambda = (T_\lambda \otimes_C T_{2e_i}) \circ i_{e_{-i} + e_{-j}}^\lambda \circ T_{-e_i-e_j}.\]
By Proposition 3.2, we see that
\[ X \circ i^\lambda_{e_i-e_j} = i^\lambda_{e_i-e_j} \circ \omega^2(X), \quad X \in \mathfrak{g}. \]

Since \( \omega^2 = \text{id}_\mathfrak{g} \), \( i^\lambda_{e_i-e_j} \) is a non-zero generator of \( \text{Hom}_K(V_\lambda[-ij], V_\lambda \otimes_C V_{-2e_1}) \), which is unique up to scalar multiple. So, we obtain the following proposition from Proposition 4.4 easily.

**Proposition 4.5.** For \( 1 \leq i \leq j \leq 3 \) and \( G \)-pattern \( M \) of type \( \lambda[-ij] \), the image of the monomial basis \( f(M) \) by the injector \( i^\lambda_{e_i-e_j} : \bigoplus \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \right
For a G-pattern $M$ of type $\lambda$, let $S_\lambda(M)$ be a column vector of degree $d_\lambda = \dim V_\lambda$ with its $l(N)$-th component $s(M,N)$ ($N \in G(\lambda)$), i.e.,

$$S_\lambda(M) = \begin{pmatrix} s(M, (\lambda_1^1 \lambda_2^1 \lambda_3^1)) \\ \vdots \\ s(M,N), \cdots, s(M,(\lambda_1^3 \lambda_2^3 \lambda_3^3)) \end{pmatrix}.$$ 

Moreover we denote by $(S_\lambda(M))$ the subspace of $H(\sigma,\nu)$ generated by the functions in the entries of the vector $S_\lambda(M)$, i.e., $(S_\lambda(M)) = \bigoplus_{N \in G(\lambda)} Cs(M,N)$ ($\simeq V_\lambda$). We take the marking $\{s(M,N)\}_{N \in G(\lambda)}$ for the simple $K$-module $(S_\lambda(M))$.

**Proposition 5.1.** As a unitary representation of $K$, it has an irreducible decomposition:

$$H(\sigma,\nu) \simeq \bigoplus_{\lambda \in L^+} (V_\lambda^* [\sigma]) \otimes_C V_\lambda.$$

The direct sum

$$\bigoplus_{M \in G_\sigma(\lambda)} (S_\lambda(M))$$

is the $\tau_\lambda$-isotypic component $\langle V_\lambda^*[\sigma] \rangle \otimes_C V_\lambda$ in $H(\sigma,\nu)$.

5.2. **General setting.** The $K$-finite part $H(\sigma,\nu,K)$ of $H(\sigma,\nu)$ is also a $k$-module. Because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, in order to describe the action of $\mathfrak{g}$ or $\mathfrak{g}_C$ it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_C = \mathfrak{p}_+ \oplus \mathfrak{p}_-.$

For a $K$-type $(\tau_\lambda, V_\lambda)$ of $(\pi(\sigma,\nu), H(\sigma,\nu), K)$ and a nonzero $K$-homomorphism $\eta: V_\lambda \to H(\sigma,\nu, K)$, we define linear map $\tilde{\eta}: \mathfrak{p}_C \otimes_C V_\lambda \to H(\sigma,\nu,K)$ by $X \otimes v \mapsto X \cdot \eta(v)$. Then $\tilde{\eta}$ is $K$-homomorphism with $\mathfrak{p}_C$ endowed with the adjoint action $\text{Ad}$ of $K$.

Since

$$V_\lambda \otimes_C \mathfrak{p}_+ \simeq V_\lambda \otimes_C V_{2e_1} \simeq \bigoplus_{1 \leq i \leq j \leq 3} V_{\lambda[i+j]},$$

there are six injective $K$-homomorphisms

$$I^\lambda_{+ij} = (\text{id}_{V_\lambda} \otimes_C i^\lambda_{+e_i + e_j}) : V_{\lambda[i+j]} \to V_\lambda \otimes_C \mathfrak{p}_+, \quad 1 \leq i \leq j \leq 3$$

for general $\lambda$. Then we define $C$-linear maps

$$\Gamma^\lambda_{+ij}: \text{Hom}_K(V_\lambda, H(\sigma,\nu,K)) \to \text{Hom}_K(V_{\lambda[i+j]}, H(\sigma,\nu,K)), \quad 1 \leq i \leq j \leq 3$$

by $\eta \mapsto \tilde{\eta} \circ I^\lambda_{+ij}$.

Now we settle two purposes of this paper:

(i): Describe the injective $K$-homomorphism $I^\lambda_{+ij}$ in terms of the monomial basis.

(ii): Determine the matrix representations of the linear homomorphisms $\Gamma^\lambda_{+ij}$ with respect to the induced basis defined in the next subsection.

We have already accomplished the first purpose in Lemma 3.4, Proposition 4.4 and 4.5. We accomplish the second purpose in the subsection 5.5. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H(\sigma,\nu)[\tau_\lambda]$ and $H(\sigma,\nu)[\tau_\lambda[\pm ij]]$. Here $H(\sigma,\nu)[\tau]$ is $\tau$-isotypic component of $H(\sigma,\nu)$. 


5.3. The canonical blocks of elementary functions. Let \( \eta: \lambda \rightarrow H_{(\sigma,\nu),K} \) be a non-zero \( K \)-homomorphism, where \( \lambda \) is a simple \( K \)-module with the marking \( \{f(M)\}_{M \in G(\lambda)} \). Then we identify \( \eta \) with the column vector of degree \( d_\lambda = \dim \lambda \); whose \( l(N) \)-th component is \( \eta(f(N)) \) for \( N \in G(\lambda) \), i.e.,

\[
\begin{align*}
\eta(f(N)) &= \left( \eta\left( f\left( \lambda_1^{\lambda_2^N} \right) \right), \ldots, \eta\left( f\left( \lambda_1^{\lambda_2^N + \lambda_3^N} \right) \right) \right) \\
&= \left( \eta\left( f\left( \lambda_1^{\lambda_2^N} \right) \right), \ldots, \eta\left( f\left( \lambda_1^{\lambda_2^N + \lambda_3^N} \right) \right) \right)
\end{align*}
\]

By this identification, we identify \( S_\lambda(M) \) with the injective \( K \)-homomorphism

\[
V_\lambda \ni f(N) \mapsto s(M, N) \in H_{(\sigma,\nu),K}, \quad N \in G(\lambda)
\]

for \( M \in G(\lambda) \). We note that \( \{S_\lambda(M)\}_{M \in G(\lambda)} \) is a basis of \( \text{Hom}_K(V_\lambda, H_{(\sigma,\nu),K}) \) and we call it the induced basis from the monomial basis.

We define a certain matrix of elementary functions corresponding to the induced basis \( \{S_\lambda(M)\}_{M \in G(\lambda)} \) of \( \text{Hom}_K(V_\lambda, H_{(\sigma,\nu),K}) \) for each \( K \)-type \( \tau_\lambda \) of our principal series representation \( \pi_{(\sigma,\nu)} \).

**Definition 5.2.** For \( M \in G(\lambda) \), let \( d_\lambda^* \) and \( l^*(M) \) be the orders of the set \( G(\lambda) \) and \( \{N \in G(\lambda) \mid l(M) \leq l(N)\} \), respectively. The \( d_\lambda \times d_\lambda^* \) matrix \( S_\sigma(\lambda) \) whose \( l^*(M) \)-th column is \( S_\lambda(M) \) for \( M \in G(\lambda) \) is called the canonical block of elementary functions for \( \tau_\lambda \)-isotypic component.

5.4. The \( p_{\pm} \)-matrix corresponding to \( I_{\pm ij}^\lambda \). In this subsection, we define \( p_{\pm} \)-matrix \( C_{\pm ij}^\lambda \) of size \( d_{\lambda[\pm ij]} \times d_\lambda \) corresponding to \( I_{\pm ij}^\lambda \) with respect to the monomial basis.

**Definition 5.3.** We define a \( p_{\pm} \)-matrix \( C_{\pm ij}^\lambda \) of size \( d_{\lambda[\pm ij]} \times d_\lambda \) as follows.

(i) For \( 1 \leq i \leq j \leq 3 \), we define a \( p_{\pm} \)-matrix \( C_{\pm ij}^\lambda \in M(d_{\lambda[\pm ij]} \times d_\lambda, C) \otimes p_{\pm} \) by

\[
C_{\pm ij}^\lambda = \left\{ \begin{array}{ll}
\bigg\{ \sum_{m=0}^{l-1} L_{\pm ij}^\lambda (\binom{0}{0}^\lambda [-m]) \bigg\} \otimes X_{+12} + \bigg\{ \sum_{m=0}^{l-1} L_{\pm ij}^\lambda (\binom{0}{0}^\lambda [-m]) \bigg\} \otimes X_{+13} + \bigg\{ \sum_{m=0}^{l-1} L_{\pm ij}^\lambda (\binom{0}{0}^\lambda [-m]) \bigg\} \otimes X_{+23},
\end{array} \right.
\]

Here \( L_{\pm ij}^\lambda (\binom{0}{0}^\lambda [-m]) \) is a matrix of size \( d_{\lambda[\pm ij]} \times d_\lambda \) whose \( l(M) \)-th row is given by

\[
\begin{array}{c}
\sum_{m=0}^{d_{\lambda[\pm ij]}} c_{\lambda[ij;kl;m]}(M),
\sum_{m=0}^{d_{\lambda[\pm ij]}} c_{\lambda[ij;kl;m]}(M)
\end{array}
\]

for \( M \in G(\lambda[\pm ij]) \).

(ii) For \( 1 \leq i \leq j \leq 3 \), we define a \( p_{\pm} \)-matrix \( C_{\pm ij}^\lambda \in M(d_{\lambda[\pm ij]} \times d_\lambda, C) \otimes p_{\pm} \) by

\[
C_{\pm ij}^\lambda = \left\{ \begin{array}{ll}
\bigg\{ \sum_{m=0}^{l-1} L_{\pm ij}^\lambda (\binom{0}{0}^\lambda [-m]) \bigg\} \otimes X_{-12} + \bigg\{ \sum_{m=0}^{l-1} L_{\pm ij}^\lambda (\binom{0}{0}^\lambda [-m]) \bigg\} \otimes X_{-13} + \bigg\{ \sum_{m=0}^{l-1} L_{\pm ij}^\lambda (\binom{0}{0}^\lambda [-m]) \bigg\} \otimes X_{-23},
\end{array} \right.
\]
Here $L_{ij}^\lambda \begin{pmatrix} 1 & 0 \\ k & 0 \end{pmatrix} [-m]$ is a matrix of size $d_{\lambda |ij}\times d_{\lambda}$ whose $l(M)$-th row is given by

$$
\begin{cases}
(0, \ldots, 0, c_{[j-4-4i;klm]}(M), 0, \ldots, 0) & \text{if } M \begin{pmatrix} a_{ij} \oplus c_{i} \\ k \end{pmatrix} [-m] \in G(\lambda) \\
0 & \text{otherwise}
\end{cases}
$$

for $M \in G(\lambda |ij)$.

Now we define $c_{\pm ij}^\lambda S_{\lambda}(M) \in (H_{(\sigma, \nu), K})^{\oplus d_{\lambda |ij}} \simeq C^{d_{\lambda |ij}} \otimes C H_{(\sigma, \nu), K}$ by the action

$$(L \otimes X)(v \otimes f) = L(v) \otimes Xf,$$

$L \otimes X \in \text{Hom}_{C}(C^{d_{\lambda}}, C^{d_{\lambda |ij}}) \otimes C p_\pm, \ v \otimes f \in C^{d_{\lambda}} \otimes C H_{(\sigma, \nu), K}$

for

$$S_{\lambda}(M) \in (H_{(\sigma, \nu), K})^{\oplus d_{\lambda}} \simeq C^{d_{\lambda}} \otimes C H_{(\sigma, \nu), K},$$

$C_{\pm ij}^\lambda \in M(d_{\lambda |ij}, d_{\lambda}, C) \otimes C p_\pm \simeq \text{Hom}_{C}(C^{d_{\lambda}}, C^{d_{\lambda |ij}}) \otimes C p_\pm$.

By the definition of $c_{\pm ij}^\lambda S_{\lambda}(M)$, we note that the vector $c_{\pm ij}^\lambda S_{\lambda}(M)$ is identified with the image of $S_{\lambda}(M)$ by $\Gamma_{\pm ij}^\lambda$.

5.5. The contiguous relations. To compute the matrix representation of $\Gamma_{\pm ij}^\lambda$ with respect to the induced basis, we prepare the following lemmas.

**Lemma 5.4.** The root vectors $X_{\pm ij}$ ($0 \leq i \leq j \leq 3$) in $p_\pm$ have the following expressions according to the Iwasawa decomposition of $g$.

$$X_{+ij} = \begin{cases}
2\sqrt{-1}E_{2e_i} + H_i + \kappa(E_{ii}), & i = j \\
(E_{e_i-e_j} + \sqrt{-1}E_{e_i+e_j}) + \kappa(E_{jj}), & i < j
\end{cases}$$

$$X_{-ij} = \begin{cases}
-2\sqrt{-1}E_{2e_i} + H_i - \kappa(E_{ii}), & i = j \\
(E_{e_i-e_j} - \sqrt{-1}E_{e_i+e_j}) - \kappa(E_{ij}), & i < j
\end{cases}$$

**Proof.** These are obtained by direct computation. \( \square \)

**Lemma 5.5.** The coefficients $c_{[ij;kl;m]}^\lambda(M)$ in Proposition 4.4 have the following relations:

$$k_{ij}(M)c_{[ij;kl;m]}^\lambda(M \begin{pmatrix} e_i \oplus c_i \\ 0 \end{pmatrix} [m]) = (m_{12} - m_{23} + 1)\chi^{(-1)}(M)c_{[ij;12;m-1]}^\lambda(M \begin{pmatrix} e_i \oplus c_i \\ 0 \end{pmatrix} [m]) + (m_{11} - m_{22} + 1)c_{[ij;12;m]}^\lambda(M \begin{pmatrix} e_i \oplus c_i \\ 0 \end{pmatrix} [m]) + (C_1(M) + 1)c_{[ij;11;m-1]}^\lambda(M \begin{pmatrix} e_i \oplus c_i \\ 0 \end{pmatrix} [m]) + (m_{33} - m_{22} - 1)c_{[ij;11;m]}^\lambda(M \begin{pmatrix} e_i \oplus c_i \\ 0 \end{pmatrix} [m])$$

for $1 \leq i \leq j \leq 3$, $m \in \mathbb{Z}$ and $M \in G(\lambda)$. Here

$$c_{[ij;kl;m]}^\lambda(M \begin{pmatrix} e_i \oplus c_i \\ 0 \end{pmatrix} [m]) = 0 \quad \text{if } r_{[ij;kl]} < m \text{ or } m < 0.$$

and

$$k_{11}(M) = -2m_{11} + 2m_{13}, \quad k_{12}(M) = -2m_{11} + m_{13} + m_{23} - 2,$$

$$k_{22}(M) = -2m_{11} + 2m_{23} - 2, \quad k_{13}(M) = -2m_{11} + m_{13} + m_{33} - 3,$$

$$k_{23}(M) = -2m_{11} + 2m_{23} - 2, \quad k_{33}(M) = -2m_{11} + m_{13} + m_{33} - 3.$$
k_{33}(M) = -2m_{11} + 2m_{33} - 4, \quad k_{23}(m) = -2m_{11} + m_{23} + m_{33} - 4.

Proof. In order to prove the assertion, it suffices to confirm the equations (4.1), (4.4) and (4.6), the equations (5.1) are equivalent to the equations (5.7).

We prove the equations (5.2) by direct computation.

Since \( (i, j) = (1, 1) \), the equations (5.1) are equivalent to the equations (5.2) for \( 1 \leq i \leq j \leq 3, m \in \mathbb{Z} \) and \( M \in G(\lambda[i+ij]) \).

We prove the equations (5.2) by direct computation.

- the proof of the case of the \((i, j) = (1, 1)\).

Since \( r_{[11;22]}, r_{[11;12]}, r_{[11;11]} = (2, 3, 2) \), we have to confirm the following equations:

\[
\begin{align*}
(5.3) & \quad (m_{11} - m_{22} + 1)c^{\lambda}_{[11;12;0]}(M) + (m_{33} - m_{22} + 1)c^{\lambda}_{[11;11;0]}(M) \\
& = (-2m_{11} + 2m_{13})c^{\lambda}_{[11;22;0]}(M),
\end{align*}
\]

\[
\begin{align*}
(5.4) & \quad (m_{12} - m_{23})\chi^{-1}(M)c^{\lambda}_{[11;12;0]}(M) + (m_{11} - m_{22})c^{\lambda}_{[11;12;1]}(M) \\
& + C_{1}(M)c^{\lambda}_{[11;11;0]}(M) + (m_{33} - m_{22})c^{\lambda}_{[11;11;1]}(M) \\
& = (-2m_{11} + 2m_{13})c^{\lambda}_{[11;22;1]}(M),
\end{align*}
\]

\[
\begin{align*}
(5.5) & \quad (m_{12} - m_{23} - 1)\chi^{-1}(M)c^{\lambda}_{[11;12;1]}(M) + (m_{11} - m_{22} - 1)c^{\lambda}_{[11;12;2]}(M) \\
& + (C_{1}(M) - 1)c^{\lambda}_{[11;11;1]}(M) + (m_{33} - m_{22} - 1)c^{\lambda}_{[11;11;2]}(M) \\
& = (-2m_{11} + 2m_{13})c^{\lambda}_{[11;22;2]}(M),
\end{align*}
\]

\[
\begin{align*}
(5.6) & \quad (m_{12} - m_{23} - 2)\chi^{-1}(M)c^{\lambda}_{[11;12;2]}(M) + (m_{11} - m_{22} - 2)c^{\lambda}_{[11;12;3]}(M) \\
& + (C_{1}(M) - 2)c^{\lambda}_{[11;11;2]}(M) = 0,
\end{align*}
\]

\[
\begin{align*}
(5.7) & \quad (m_{12} - m_{23} - 3)\chi^{-1}(M)c^{\lambda}_{[11;12;3]}(M) = 0.
\end{align*}
\]
We have
\[(m_{11} - m_{22} + 1)c_{11;12;0}^\lambda (M) + (m_{33} - m_{22} + 1)c_{11;11;0}^\lambda (M)\]
\[= -2(m_{11} - m_{22} + 1)(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{22} - m_{33} - 1)\]
\[-2(m_{33} - m_{22} + 1)(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{13} - m_{22} + 1)\]
\[= -2(m_{11} - m_{13})(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{22} - m_{33} - 1)\]
\[= (-2m_{11} + 2m_{13})c_{11;22;0}^\lambda (M).\]

Hence we obtain the equation [5.3].

We have
\[(m_{12} - m_{23})\chi^{(-1)}_{11;12;0} (M)c_{11;12;0}^\lambda (M) + (m_{11} - m_{22})c_{11;11;1}^\lambda (M)\]
\[= -2(m_{12} - m_{23})\chi^{(-1)}_{11;12;0} (M)(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{22} - m_{33} - 1)\]
\[+ 2(m_{11} - m_{22})(m_{13} - m_{12})(m_{22} - m_{33})\left\{\tilde{F} \left( M \left( \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right) + \tilde{E}(M) \right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-(m_{12} - m_{23})(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1)(1 - \chi_+(M))\right.\]
\[+ (m_{11} - m_{22})\left\{-C_1(M)(\tilde{C}_1(M) + 1)\right.\]
\[-\chi_+(M)\left\{(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1) + (m_{13} - m_{33})\delta(M)\right\} + \tilde{E}(M)\right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-(m_{13} - m_{12} - 1)(m_{12} - m_{23} - \delta(M)\chi_+(M))(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1)\right.\]
\[-(m_{11} - m_{22})C_1(M)(\tilde{C}_1(M) + 1) - (m_{11} - m_{22})\chi_+(M)(m_{13} - m_{33})\delta(M)\right.\]
\[+ (m_{11} - m_{22})\tilde{E}(M)\right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-C_1(M)(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1)\right.\]
\[-(m_{11} - m_{22})C_1(M)(\tilde{C}_1(M) + 1) - C_1(M)\chi_+(M)(m_{13} - m_{33})\delta(M) + (m_{11} - m_{22})\tilde{E}(M)\right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-C_1(M)(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1)\right.\]
\[-(m_{11} - m_{22})C_1(M)(\tilde{C}_1(M) + 1) - (m_{11} - m_{22})\tilde{E}(M)\right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-C_1(M)(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1)\right.\]
\[-(m_{11} - m_{22})\tilde{E}(M)\right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-C_1(M)(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1)\right.\]
\[-(m_{11} - m_{22})\tilde{E}(M)\right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-C_1(M)(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1)\right.\]
\[-(m_{11} - m_{22})\tilde{E}(M)\right\} + (m_{13} - m_{22})\tilde{E}(M)\right\},
\[C_1(M)c_{11;11;0}^\lambda (M) + (m_{33} - m_{22})c_{11;11;1}^\lambda (M)\]
\[= -2C_1(M)(m_{13} - m_{12})(m_{13} - m_{12} - 1)(m_{22} - m_{33})(m_{13} - m_{22} + 1)\]
\[+ 2(m_{33} - m_{22})(m_{13} - m_{12})\left\{(m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1) + (m_{13} - m_{22})\tilde{E}(M)\right\}\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-C_1(M)(m_{13} - m_{12} - 1)(m_{13} - m_{22} + 1)\right.\]
\[+ (m_{22} - m_{33})(\tilde{C}_1(M) + 1)\right\} - (m_{13} - m_{22})\tilde{E}(M)\right\}.

Therefore
\[(m_{12} - m_{23})\chi^{(-1)}_{11;12;0} (M)c_{11;12;0}^\lambda (M) + (m_{11} - m_{22})c_{11;12;1}^\lambda (M)\]
\[+ C_1(M)c_{11;11;0}^\lambda (M) + (m_{33} - m_{22})c_{11;11;1}^\lambda (M)\]
\[= 2(m_{13} - m_{12})(m_{22} - m_{33})\left\{-C_1(M)(m_{13} - m_{12} - 1)(m_{13} - m_{33})\right.\]
\[-(m_{11} - m_{22})\tilde{E}(M)\right\}.
Hence we obtain the equation (5.4). Here we use the relations

\[ (m_{13} - m_{33})\delta(M)\chi_+(M) + (m_{11} - m_{33})(\tilde{C}_1(M) + 1) + (m_{11} - m_{13})\tilde{E}(M) \]

\[ = 2(m_{13} - m_{12})(m_{22} - m_{33}) \left\{ - C_1(M) \{(m_{13} - m_{11})/(m_{13} - m_{33}) \right. \]

\[ - (m_{13} - m_{33})(m_{12} - m_{11} - \delta(M)\chi_+(M) + 1) + (m_{11} - m_{33})(\tilde{C}_1(M) + 1) \right\} \]

\[ + (m_{11} - m_{13})\tilde{E}(M) \}

\[ = 2(m_{13} - m_{12})(m_{22} - m_{33}) \left\{ (m_{11} - m_{13})C_1(M)(m_{13} - m_{33} - \tilde{C}_1(M) - 1) \right. \]

\[ + (m_{11} - m_{13})\tilde{E}(M) \} \]

\[ = 4(m_{11} - m_{13})(m_{13} - m_{12})(m_{22} - m_{33})(\tilde{E}(M) - C_1(M)) \]

\[ = (-2m_{11} + 2m_{13})c^{\lambda}_{[11;22;1]}(M). \]

Hence we obtain the equation (5.4). Here we use the relations

\[(5.8) \quad \tilde{F}\left(M \left( \begin{array}{c} -1 \cr 0 \cr -1 \end{array} \right) \right) = -C_1(M)(\tilde{C}_1(M) + 1) \]

\[- \chi_+(M)\{(m_{13} - m_{12} - 1)(m_{22} - m_{33} - 1) + (m_{13} - m_{33})\delta(M)\}, \]

(1.2) and (4.9).

We have

\[ (m_{12} - m_{23} - 1)\chi^{(-1)}_-(M)c^{\lambda}_{[11;12;1]}(M) + (m_{11} - m_{22} - 1)c^{\lambda}_{[11;12;2]}(M) \]

\[ = 2(m_{12} - m_{23} - 1)\chi^{(-1)}_-(M)(m_{13} - m_{12})(m_{22} - m_{33}) \left\{ - C_1(M)(\tilde{C}_1(M) + 1) + \tilde{E}(M) \right\} \]

\[ - 2(m_{11} - m_{22} - 1)\left\{ \tilde{E}(M)\tilde{F}\left(M \left( \begin{array}{c} -1 \cr 0 \cr -1 \end{array} \right) \right) + \tilde{E}(M) \right\} \]

\[ + (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1)\chi_+(M) \} \]

\[ = 2(m_{12} - m_{23} - 1)\chi^{(-1)}_-(M)(m_{13} - m_{12})(m_{22} - m_{33}) \left\{ - C_1(M)(\tilde{C}_1(M) + 1) + \tilde{E}(M) \right\} \]

\[ - 2(m_{11} - m_{22} - 1)\left\{ \tilde{E}(M)\left\{ - (C_1(M) - 1)\tilde{C}_1(M) - \chi_+(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) \right. \]

\[ + (m_{13} - m_{33})\delta(M)\right\} \} + (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1)\chi_+(M) \} \]

\[ = 2(C_1(M) - 1)\chi^{(-1)}_-(M)(m_{13} - m_{12})(m_{22} - m_{33}) \left\{ - C_1(M)(\tilde{C}_1(M) + 1) + \tilde{E}(M) \right\} \]

\[ - 2\tilde{E}(M)\left\{ - (C_1(M) - 1)\tilde{C}_1(M)(m_{11} - m_{22} - 1) \right. \]

\[ - (C_1(M) - 1)\chi_+(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33})\delta(M)\} \} \]

\[ - 2(C_1(M) - 1)(m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1)\chi_+(M) \]

\[ = -2(C_1(M) - 1)\left\{ (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1)(\chi^{(-1)}_-(M) + \chi_+(M)) \right. \]

\[ - \tilde{E}(M)\left\{ \tilde{C}_1(M)(m_{11} - m_{22} - 1) + (m_{13} - m_{12})(m_{22} - m_{33})(\chi^{(-1)}_-(M) + \chi_+(M)) \right. \]

\[ + (m_{13} - m_{33})\delta(M)\chi_+(M) \} \right\} \}

\[ = -2(C_1(M) - 1)\left\{ (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(\tilde{C}_1(M) + 1) \right. \]

\[ - \tilde{E}(M)\left\{ \tilde{C}_1(M)(m_{11} - m_{22} - 1) + (m_{13} - m_{12})(m_{22} - m_{33}) \right. \]

\[ + (m_{13} - m_{33})\delta(M)\chi_+(M) \} \right\} \}, \]

\[ (C_1(M) - 1)c^{\lambda}_{[11;11;1]}(M) + (m_{33} - m_{22} - 1)c^{\lambda}_{[11;11;2]}(M) \]
\[
= 2(C_1(M) - 1)(m_{13} - m_{12})((m_{22} - m_{33})C_1(M)(C_1(M) + 1) + (m_{13} - m_{22})E(M)) \\
- 2(m_{33} - m_{22} - 1)E(M)(C_1(M) - 1)C_1(M) \\
= -2(C_1(M) - 1)\left\{ - (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(C_1(M) + 1) \\
- E(M)((m_{13} - m_{12})(m_{13} - m_{22}) + (m_{22} - m_{33} + 1)C_1(M) \right\}.
\]

Therefore
\[
(m_{12} - m_{23} - 1)\chi^{(-1)}(M)c_{11;12;1}^{\lambda}(M) + (m_{11} - m_{22} - 1)c_{11;12;2}^{\lambda}(M) \\
+ (C_1(M) - 1)c_{11;11;1}^{\lambda}(M) + (m_{33} - m_{22} - 1)c_{11;11;2}^{\lambda}(M) \\
= 2(C_1(M) - 1)E(M)\left\{ C_1(M)(m_{11} - m_{33}) + (m_{13} - m_{12})(m_{13} - m_{33}) \\
+ (m_{13} - m_{33})\delta(M)\chi_+(M) \right\} \\
= 2(C_1(M) - 1)E(M)\left\{ C_1(M)(m_{11} - m_{33}) + (m_{13} - m_{11})(m_{13} - m_{33}) \\
- (m_{13} - m_{33})(m_{12} - m_{11} - \delta(M)\chi_+(M)) \right\} \\
= 2(m_{13} - m_{11})E(M)(C_1(M) - 1)(m_{13} - m_{33} - C_1(M)) \\
= (-2m_{11} + 2m_{13})c_{11;12;2}^{\lambda}(M).
\]

Hence we obtain the equation (5.5). Here we use the relations
\[
(5.9) \quad \bar{F}\left( M \left( \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right) = -(C_1(M) - 1)\bar{C}_1(M) \\
- \chi_+(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33})\delta(M)\},
\]
(4.17), (4.13), (4.14), (4.10) and (5.8).

We have
\[
(m_{12} - m_{23} - 2)\chi^{(-1)}(M)c_{11;12;2}^{\lambda}(M) + (m_{11} - m_{22} - 2)c_{11;12;3}^{\lambda}(M) \\
+ (C_1(M) - 2)c_{11;11;2}^{\lambda}(M) \\
= -2(m_{12} - m_{23} - 2)\chi^{(-1)}(M)\left\{ \bar{E}(M)\bar{F}\left( M \left( \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right) \\
+ (m_{13} - m_{12})(m_{22} - m_{33})C_1(M)(C_1(M) + 1)\chi_+(M) \right\} \\
+ 2(m_{11} - m_{22} - 2)(C_1(M) - 1)\bar{C}_1(M)\bar{E}(M)\chi_+(M) \\
- 2(C_1(M) - 2)\bar{E}(M)(C_1(M) - 1)\bar{C}_1(M) \\
= 2(C_1(M) - 2)\chi^{(-1)}(M)\bar{E}(M)(C_1(M) - 1)\bar{C}_1(M) \\
+ 2(C_1(M) - 2)(C_1(M) - 1)\bar{C}_1(M)\bar{E}(M)\chi_+(M) \\
- 2(C_1(M) - 2)\bar{E}(M)(C_1(M) - 1)\bar{C}_1(M) \\
= 2(C_1(M) - 2)(C_1(M) - 1)\bar{C}_1(M)\bar{E}(M)(\chi^{(-1)}(M) + \chi_+(M) - 1) = 0.
\]

Hence we obtain the equation (5.6). Here we obtain the relations (4.9), (4.10), (4.13), (4.14) and (5.8).

We have
\[
(m_{12} - m_{23} - 3)\chi^{(-1)}(M)c_{11;12;3}^{\lambda}(M) \\
= 2(m_{12} - m_{23} - 3)(C_1(M) - 1)\bar{C}_1(M)\bar{E}(M)\chi^{(-1)}(M)\chi_+(M) = 0.
\]
Hence we obtain the equation (5.7). Here we use the relation (1.10).

- the proof of the case of the \((i, j) = (2, 2)\).

Since \(r_{[22,22]}^j r_{[22,12]} r_{[22,11]} = (2, 2, 2)\), we have to confirm the following equations:

\[(5.10)\]
\[
(m_{11} - m_{22} + 1)c_{[22,12;0]}^\lambda (M) + (m_{33} - m_{22} + 1)c_{[22,11;0]}^\lambda (M)
= (-2m_{11} + 2m_{23} - 2)c_{[22,22;0]}^\lambda (M),
\]

\[(5.11)\]
\[
(m_{12} - m_{23} + 1)\chi_-(M)c_{[22,12;1]}^\lambda (M) + (m_{11} - m_{22} - 1)c_{[22,12;2]}^\lambda (M)
+ (C_1(M) + \chi_-(M) + \chi_-(M)) c_{[22,11;0]}^\lambda (M) + (m_{33} - m_{22} - 1)c_{[22,11;1]}^\lambda (M)
= (-2m_{11} + 2m_{23} - 2)c_{[22,22;1]}^\lambda (M),
\]

\[(5.12)\]
\[
(m_{12} - m_{23} + 1)\chi_-(M)c_{[22,12;2]}^\lambda (M) + (m_{11} - m_{22} - 1)c_{[22,12;2]}^\lambda (M)
+ (C_1(M) - 1 + \chi_-(M) + \chi_-(M)) c_{[22,11;1]}^\lambda (M) + (m_{33} - m_{22} - 1)c_{[22,11;2]}^\lambda (M)
= (-2m_{11} + 2m_{23} - 2)c_{[22,22;2]}^\lambda (M),
\]

\[(5.13)\]
\[
(m_{12} - m_{23})\chi_-(M)c_{[22,12;2]}^\lambda (M) + (C_1(M) - 2 + \chi_-(M) + \chi_-(M)) c_{[22,11;2]}^\lambda (M) = 0.
\]

We have
\[
(m_{11} - m_{22} + 1)c_{[22,12;0]}^\lambda (M) + (m_{33} - m_{22} + 1)c_{[22,11;0]}^\lambda (M)
= -2(m_{11} - m_{22} + 1)(m_{22} - m_{33})(m_{22} - m_{33} - 1)
- 2(m_{33} - m_{22} + 1)(m_{22} - m_{33})(m_{23} - m_{22})
= -2(m_{11} - m_{23} + 1)(m_{22} - m_{33})(m_{22} - m_{33} - 1)
= (-2m_{11} + 2m_{23} - 2)c_{[22,22;0]}^\lambda (M).
\]

Hence we obtain the equation (5.10).

We have
\[
(m_{12} - m_{23} + 2)\chi_-(M)c_{[22,12;0]}^\lambda (M) + (m_{11} - m_{22})c_{[22,12;1]}^\lambda (M)
+ (C_1(M) + \chi_-(M) + \chi_-(M)) c_{[22,11;0]}^\lambda (M) + (m_{33} - m_{22})c_{[22,11;1]}^\lambda (M)
= -2(m_{12} - m_{23} + 2)\chi_-(M)(m_{22} - m_{33})(m_{22} - m_{33} - 1)
+ 2(m_{11} - m_{22})(m_{22} - m_{33})\{C_1(M) + (\bar{D}(M) + 1)\chi_-(M)\}
- 2(C_1(M) + \chi_-(M) + \chi_-(M)) (m_{22} - m_{33})(m_{23} - m_{22})
+ 2(m_{33} - m_{22})\{\bar{D}(M)(m_{23} - m_{22} - 1)\chi_-(M) - (m_{22} - m_{33})(C_1(M) + 1)\chi_-(M)\}
= 2(m_{22} - m_{33}) \left\{ - (m_{12} - m_{23} + 2)(m_{22} - m_{33} - 1)\chi_-(M)
+ (m_{11} - m_{22})(m_{23} - m_{22} + (\delta(M) + \bar{D}(M) + 1)\chi_-(M))
- (m_{11} - m_{22} + (\delta(M) + 1)\chi_-(M) + \chi_-(M)) (m_{23} - m_{22})
- \{\bar{D}(M)(m_{23} - m_{22} - 1)\chi_-(M) - (m_{22} - m_{33})(m_{12} - m_{11} + 1)\chi_-(M)\} \right\}.
\]
Hence we obtain the equation (5.11). Here we use the relations (4.1), (4.2), (4.8) and (4.16). 

We have

\[
(C_1(M) - 1 + \chi^{(i)}(M) + \chi(M)) c_{[22,11,1]}^\lambda(M) + (m_{11} - m_{22} - 1) c_{[22,12;2]}^\lambda(M) \\
+ (C_1(M) - 1 + \chi^{(i)}(M) + \chi(M)) c_{[22,11,1]}^\lambda(M) + (m_{33} - m_{22} - 1)c_{[22,12;2]}^\lambda(M) \\
= 2(m_{12} - m_{23} + 1)\chi^{(i)}(M)(m_{22} - m_{33})(\tilde{C}_1(M) + (\tilde{D}(M) + 1)\chi(M)) \\
- 2(m_{11} - m_{22} - 1)\tilde{C}_1(M)\tilde{D}(M)\chi(M) \\
+ 2C_1(M)\tilde{D}(M)(m_{23} - m_{22} - 1)\chi(M) \\
- 2((C_1(M) + 1)(m_{22} - m_{33})(\tilde{C}_1(M) + 1) - \tilde{D}(M)(m_{23} - m_{22} - 1))\chi^{(i)}(M) \\
+ 2(m_{33} - m_{22} - 1)\tilde{C}_1(M)\tilde{D}(M)\chi^{(i)}(M) \\
= 2(m_{12} - m_{23} + 1)(m_{22} - m_{33})(m_{12} - m_{11} + \tilde{D}(M) + 1)\chi^{(i)}(M) \\
- 2(m_{11} - m_{22} - 1)(m_{12} - m_{11})\tilde{D}(M)\chi(M) \\
+ 2(m_{12} - m_{23})\tilde{D}(M)(m_{23} - m_{22} - 1)\chi(M) \\
- 2((m_{12} - m_{23} + 1)(m_{22} - m_{33})(m_{12} - m_{11} + 1) - \tilde{D}(M)(m_{23} - m_{22} - 1))\chi^{(i)}(M) \\
+ 2(m_{33} - m_{22} - 1)(m_{12} - m_{11})\tilde{D}(M)\chi^{(i)}(M) \\
= 2(m_{11} - m_{23})(m_{22} - m_{33})\tilde{D}(M)\chi^{(i)}(M) - 2(-m_{22} + m_{33} + \delta(M) + 1)\tilde{D}(M)\chi^{(i)}(M) \\
- 2(m_{11} - m_{23})\delta(M) + 1)\tilde{D}(M)\chi^{(i)}(M) \\
= -2(m_{11} - m_{23} + 1)(-m_{22} + m_{33} + \delta(M) + 1)\tilde{D}(M)\chi^{(i)}(M) \\
= (-2m_{11} + 2m_{23} - 2)c_{[22,22;2]}^\lambda(M).
\]

Hence we obtain the equation (5.12). Here we use the relations (4.1), (4.2), (4.8), (1.12), (4.16) and (1.14).

We have

\[
(m_{12} - m_{23})\chi^{(i)}(M)c_{[22,12;2]}^\lambda(M) + (C_1(M) - 2 + \chi^{(i)}(M) + \chi(M)) c_{[22,11;2]}^\lambda(M)
\]
Hence we obtain the equation (5.13). Here we use the relations (4.14) and (4.12).

- the proof of the case of the \((i, j) = (3, 3)\).

Since \(r_{[33;22]}, r_{[33;12]}, r_{[33;11]}\) = \((0, 1, 0)\), we have to confirm the following equations:

\[
(m_{11} - m_{22} + 1)c_{[33;12,0]}^\lambda(M) + (m_{33} - m_{22} - 1)c_{[33;11,0]}^\lambda(M) = (-2m_{11} + 2m_{33} - 4)c_{[33;22,0]}^\lambda(M),
\]

(5.14)

\[
(m_{12} - m_{23})\chi_-(M)c_{[33;12,0]}^\lambda(M) + (m_{11} - m_{22})c_{[33;12,1]}^\lambda(M) + C_1(M)c_{[33;11,0]}^\lambda(M) = (-2m_{11} + 2m_{33} - 4)c_{[33;22,1]}^\lambda(M),
\]

(5.15)

\[
(m_{12} - m_{23} - 1)\chi_-(M)c_{[33;12,1]}^\lambda(M) = 0.
\]

(5.16)

We have

\[
(m_{11} - m_{22} + 1)c_{[33;12,0]}^\lambda(M) + (m_{33} - m_{22} - 1)c_{[33;11,0]}^\lambda(M) = -2(m_{11} - m_{22} + 1) + 2(m_{33} - m_{22} - 1)
= -2m_{11} + 2m_{33} - 4 = (-2m_{11} + 2m_{33} - 4)c_{[33;22,0]}^\lambda(M).
\]

Hence we obtain the equation (5.14).

We have

\[
(m_{12} - m_{23})\chi_-(M)c_{[33;12,0]}^\lambda(M) + (m_{11} - m_{22})c_{[33;12,1]}^\lambda(M) + C_1(M)c_{[33;11,0]}^\lambda(M) = -2(m_{11} - m_{23}) + (1 - \chi_+(M)) - 2(m_{11} - m_{22})\chi_+(M) + 2C_1(M)
= -2(m_{12} - m_{23} - \delta(M)\chi_+(M) - C_1(M)) = 0.
\]

Hence we obtain the equation (5.15). Here we use the relations (4.1) and (4.9).

We have

\[
(m_{12} - m_{23} - 1)\chi_-(M)c_{[33;12,1]}^\lambda(M) = -2(m_{12} - m_{23} - 1)\chi_-(M)\chi_+(M) = 0.
\]

Hence we obtain the equation (5.16). Here we use the relation (4.10).

- the proof of the case of the \((i, j) = (1, 2)\).

Since \(r_{[12;22]}, r_{[12;12]}, r_{[12;11]}\) = \((2, 2, 2)\), we have to confirm the following equations:

\[
(m_{11} - m_{22} + 1)c_{[12;12,0]}^\lambda(M) + (m_{33} - m_{22} + 1)c_{[12;11,0]}^\lambda(M) = (-2m_{11} + m_{13} + m_{23})c_{[12;22,0]}^\lambda(M),
\]

(5.17)

\[
(m_{12} - m_{23} + 1)\chi_-(M)c_{[12;12,0]}^\lambda(M) + (m_{11} - m_{22})c_{[12;12,1]}^\lambda(M) + (C_1(M) + \chi_-(M))c_{[12;11,0]}^\lambda(M) + (m_{33} - m_{22})c_{[12;11,1]}^\lambda(M)
= (-2m_{11} + m_{13} + m_{23})c_{[12;22,1]}^\lambda(M),
\]

(5.18)
\begin{align}
(5.19) \quad & (m_{12} - m_{23}) \chi_-(M) c^\lambda_{[12,12,1]}(M) + (m_{11} - m_{22} - 1)c^\lambda_{[12,12,2]}(M) \\
& + (C_1(M) - 1 + \chi_-(M)) c^\lambda_{[12,11,1]}(M) + (m_{33} - m_{22} - 1)c^\lambda_{[12,11,2]}(M) \\
& = (-2m_{11} + m_{13} + m_{23})c^\lambda_{[12,22,2]}(M),
\end{align}

\begin{align}
(5.20) \quad & (m_{12} - m_{23} - 1)\chi_-(M) c^\lambda_{[12,12,2]}(M) + (C_1(M) - 2 + \chi_-(M)) c^\lambda_{[12,11,2]}(M) = 0.
\end{align}

We have
\begin{align*}
(m_{11} - m_{22} + 1)c^\lambda_{[12,12,0]}(M) + (m_{33} - m_{22} + 1)c^\lambda_{[12,11,0]}(M) \\
& = -2(m_{11} - m_{22} + 1)(m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1) \\
& + (m_{33} - m_{22} + 1)(m_{13} - m_{12})(m_{22} - m_{33})(2m_{22} - m_{13} - m_{23} - 2) \\
& = -(2m_{11} - m_{13} - m_{23})(m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1) \\
& = (-2m_{11} + m_{13} + m_{23})c^\lambda_{[12,22,0]}(M).
\end{align*}

Hence we obtain the equation \((5.17)\).

We have
\begin{align*}
(m_{11} - m_{22})c^\lambda_{[12,12,1]}(M) + (m_{33} - m_{22})c^\lambda_{[12,11,1]}(M) \\
& = (m_{11} - m_{22})(m_{22} - m_{33})\{\bar{E}(M) \\
& - C_2(M) - \chi_+(M)\{(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33} + 1)\delta(M)\} \\
& + (m_{13} - m_{12})\{\bar{C}_1(M) + 1 + \bar{D}(M)(1 - \chi_+(M)))\} \\
& + (m_{33} - m_{22})\{\bar{E}(M)(m_{23} - m_{22}) + C_2(M)(m_{22} - m_{33} + 1) \\
& + (m_{13} - m_{12})\chi_-(M)\{\bar{D}(M)(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(\bar{C}_1(M) + 1)\}\} \\
& = (m_{22} - m_{33})\{(2m_{11} - m_{13} - m_{23})\bar{E}(M) + (m_{13} - m_{11})C_1(M)(m_{13} - m_{33} + 1 - \bar{C}_1(M)) \\
& - (m_{11} - m_{33} + 1)C_2(M) - (m_{11} - m_{22})(m_{13} - m_{33} + 1)\delta(M)\chi_+(M) \\
& + (m_{13} - m_{12})\{(m_{11} - m_{22})\{\bar{C}_1(M) + 1 + \bar{D}(M)(1 - \chi_+(M)) - (m_{22} - m_{33})\chi_+(M)\} \\
& - \chi_-(M)\{\bar{D}(M)(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(\bar{C}_1(M) + 1)\}\} \\
& = (m_{22} - m_{33})\{(2m_{11} - m_{13} - m_{23})\bar{E}(M) \\
& + (m_{13} - m_{11} - \bar{C}_1(M) - \delta(M)\chi_+(M))C_1(M)(m_{13} - m_{33} + 1) \\
& + (m_{13} - m_{12})\{(m_{11} - m_{22})\{\bar{C}_1(M) + 1 + \bar{D}(M) - \delta(M)\chi_+(M)\} \\
& - \chi_-(M)\{\bar{D}(M)(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(\bar{C}_1(M) + 1)\}\} \\
& = (m_{22} - m_{33})\{(2m_{11} - m_{13} - m_{23})\bar{E}(M) + (m_{13} - m_{12})\{C_1(M)(m_{13} - m_{33} + 1)
\end{align*}
\begin{align*}
+ (m_{11} - m_{22}) \{ C_1(M) + 1 + \bar{D}(M) - \delta(M) \chi_+(M) \} \\
- \chi_-(M) \{ \bar{D}(M)(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(C_1(M) + 1) \}\}. \\
\end{align*}

Therefore
\begin{align*}
(m_{12} - m_{23} + 1) \chi_-(M)c_\lambda^{12;12;0} (M) + (m_{11} - m_{22}) c_\lambda^{12;12;1} (M) \\
+ (C_1(M) + \chi_-(M))c_\lambda^{12;11;0} (M) + (m_{33} - m_{22}) c_\lambda^{12;11;1} (M) \\
= -2(m_{12} - m_{23} + 1) \chi_-(M)(m_{13} - m_{12})(m_{22} - m_{33})(m_{22} - m_{33} - 1) \\
+ (C_1(M) + \chi_-(M))(m_{13} - m_{12})(m_{22} - m_{33})(2m_{22} - m_{13} - m_{23} - 2) \\
+ (m_{22} - m_{33}) \{ (2m_{11} - m_{13} - m_{23}) \bar{E}(M) + (m_{13} - m_{12}) \{ C_1(M)(m_{13} - m_{33} + 1) \\
+ (m_{22} - m_{33}) \{ (2m_{11} - m_{13} - m_{23}) \bar{E}(M) \\
+ (m_{13} - m_{12}) \{ -2(m_{12} - m_{23} + 1)(m_{22} - m_{33} - 1) \chi_-(M) \\
+ \{ m_{11} - m_{22} + (\delta(M) + 1) \chi_-(M) \}(2m_{22} - m_{13} - m_{23} - 2) \\
+ (m_{11} - m_{22} + \delta(M) \chi_-(M))(m_{13} - m_{33} + 1) + (m_{11} - m_{22}) \\
\times \{ m_{23} - m_{22} + \delta(M) \chi_-(M) + 1 + (-m_{22} + m_{33} + \delta(M)) - \delta(M)(1 - \chi_-(M)) \} \\
- \chi_-(M) \{( -m_{22} + m_{33} + \delta(M))(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(m_{12} - m_{11} + 1) \} \} \} \} \\
= (m_{22} - m_{33}) \{ (2m_{11} - m_{13} - m_{23}) \bar{E}(M) \\
+ (2m_{11} - m_{13} - m_{23})(m_{13} - m_{12}) \chi_-(M)(-m_{22} + m_{33} + \delta(M) + 1) \} \\
= (2m_{11} - m_{13} - m_{23}) \{ \bar{E}(M) + (m_{13} - m_{12}) \chi_-(M)(\bar{D}(M) + 1) \} \\
= (-2m_{11} + m_{13} + m_{23})c_\lambda^{12;22;1} (M).
\end{align*}

Hence we obtain the equation (6.18). Here we use the relations (4.11), (4.2) and (4.14).

We have
\begin{align*}
(m_{12} - m_{23}) \chi_-(M)c_\lambda^{12;12;0} (M) \\
= (m_{12} - m_{23}) \chi_-(M)(m_{22} - m_{33}) \{ \bar{E}(M) + \bar{F}(M) \\
+ (m_{13} - m_{12}) \{ C_1(M) + 1 + \bar{D}(M)(1 - \chi_+(M)) \} \} \\
= (m_{22} - m_{33}) \chi_-(M) \{ (m_{12} - m_{23}) \bar{E}(M) - C_2(M) \} \\
+ (m_{13} - m_{12}) C_1(M)(C_1(M) + 1 + \bar{D}(M)) \}
\end{align*}

\begin{align*}
(m_{11} - m_{22} - 1)c_\lambda^{12;12;2} (M) \\
= -(m_{11} - m_{22} - 1) \{ C_1(M) \bar{E}(M) + C_2(M) \{ 1 - \bar{D}(M) + \delta(M) \chi_+(M) \} \} \\
= -(m_{11} - m_{22} - 1) \{ C_1(M) \bar{E}(M) + C_2(M)(m_{22} - m_{33} + 1 - \delta(M) \chi_-(M)) \} \\
(C_1(M) - 1 + \chi_-(M)) c_\lambda^{12;11;1} (M) \\
= (C_1(M) - 1 + \chi_-(M)) \{ \bar{E}(M)(m_{23} - m_{22}) + C_2(M)(m_{22} - m_{33} + 1) \\
+ (m_{13} - m_{12}) \chi_-(M)(\bar{D}(M)(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(C_1(M) + 1)) \}.
\[
(m_{11} - m_{22} - 1)\{ \bar{C}_1(M) \bar{E}(M) - \delta(M) \chi_-(M) \bar{E}(M) + C_2(M)(m_{22} - m_{33} + 1) \}
+ (\delta(M) + 1)\chi_-(M) \{ \bar{E}(M)(m_{23} - m_{22}) + C_2(M)(m_{22} - m_{33} + 1) \}
+ (m_{13} - m_{12})C_1(M)\chi_-(M) \{ \bar{D}(M)(m_{13} - m_{22} + 1) - (m_{22} - m_{33})(\bar{C}_1(M) + 1) \}
\]

\[
(m_{33} - m_{22} - 1)c_{12;11;1}^{\lambda} c_{12;12;2}^{\lambda} \bar{C}_1(M) \chi_-(M)(m_{13} - m_{33} + 2 - \bar{C}_1(M) - \bar{D}(M))
\]

\[
= - (m_{22} - m_{33} + 1)\chi_-(M) \{ (m_{12} - m_{11})\bar{E}(M) + (1 - \bar{D}(M))C_2(M) \}
\]

Therefore

\[
(m_{11} - m_{22} - 1)c_{12;12;2}^{\lambda} (M) + (C_1(M) - 1 + \chi_-(M))c_{12;11;1}^{\lambda} (M)
+ (m_{12} - m_{23})\chi_-(M) c_{12;12;1}^{\lambda} (M) + (m_{33} - m_{22} - 1)c_{12;11;2}^{\lambda} (M)
\]

\[
= \chi_-(M) \{ - (m_{11} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{11} - m_{33})\delta(M)C_2(M)
+ (m_{13} - m_{33} + 1)(m_{13} - m_{12})C_1(M)\bar{D}(M) - (m_{12} - m_{23})(m_{22} - m_{33})C_2(M)
+ (m_{22} - m_{33} + 1)\bar{D}(M)C_2(M) \}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

\[
= \chi_-(M) \{ - (2m_{11} - m_{13} - m_{23})\bar{E}(M)\bar{D}(M) + (m_{12} - m_{23})(m_{12} - m_{11})\}
\]

Hence we obtain the equation 5.19. Here we use the relations (4.10), (4.12), (4.9), (4.26), (4.16) and (4.14).

We have

\[
(m_{12} - m_{23} - 1)\chi_-(M)c_{12;12;2}^{\lambda} (M)
\]

\[
= (m_{12} - m_{23} - 1)\chi_-(M) \{ - \bar{C}_1(M)\bar{E}(M) - C_2(M)(1 - \bar{D}(M) + \delta(M)\chi_+(M)) \}
\]

\[
= - (C_1(M) - 1)\chi_-(M)(m_{13} - m_{33} + 2 - \bar{C}_1(M) - \bar{D}(M)),
\]

\[
(C_1(M) - 2 + \chi_-(M)) c_{12;11;2}^{\lambda} (M)
\]

\[
= (C_1(M) - 2 + \chi_-(M)) C_2(M)\chi_-(M)(m_{13} - m_{33} + 2 - \bar{C}_1(M) - \bar{D}(M))
\]

\[
= (C_1(M) - 1)C_2(M)\chi_-(M)(m_{13} - m_{33} + 2 - \bar{C}_1(M) - \bar{D}(M)).
\]

Therefore

\[
(m_{12} - m_{23} - 1)\chi_-(M)c_{12;12;2}^{\lambda} (M) + (C_1(M) - 2 + \chi_-(M))c_{12;11;2}^{\lambda} (M) = 0.
\]
Hence we obtain the equation (5.20). Here we use the relations (4.10), (4.12) and (4.14).

- the proof of the case of the \((i, j) = (1, 3)\).

Since \(r_{[13, 22]}, r_{[13, 12]}, r_{[13, 11]} = (1, 2, 1)\), we have to confirm the following equations:

\[
\begin{align*}
(5.21) & \quad (m_{11} - m_{22} + 1)c_{\lambda_{[13, 12; 0]}(M)}^\lambda + (m_{33} - m_{22})c_{\lambda_{[13, 11; 0]}(M)}^\lambda \\
& = (-2m_{11} + m_{13} + m_{33} - 1)c_{\lambda_{[13, 22; 0]}(M)}^\lambda, \\
(5.22) & \quad (m_{12} - m_{23})\chi^{(-1)}_-(M)c_{\lambda_{[13, 12; 0]}(M)}^\lambda + (m_{11} - m_{22})c_{\lambda_{[13, 12; 1]}(M)}^\lambda \\
& + C_1(M)c_{\lambda_{[13, 11; 0]}(M)}^\lambda + (m_{33} - m_{22} - 1)c_{\lambda_{[13, 11; 1]}(M)}^\lambda \\
& = (-2m_{11} + m_{13} + m_{33} - 1)c_{\lambda_{[13, 22; 1]}(M)}^\lambda, \\
(5.23) & \quad (m_{12} - m_{23} - 1)\chi^{(-1)}_-(M)c_{\lambda_{[13, 12; 1]}(M)}^\lambda + (m_{11} - m_{22} - 1)c_{\lambda_{[13, 12; 2]}(M)}^\lambda \\
& + (C_1(M) - 1)c_{\lambda_{[13, 11; 1]}(M)}^\lambda = 0, \\
(5.24) & \quad (m_{12} - m_{23} - 2)\chi^{(-1)}_-(M)c_{\lambda_{[13, 12; 2]}(M)}^\lambda = 0.
\end{align*}
\]

We have

\[
\begin{align*}
(m_{11} - m_{22} + 1)c_{\lambda_{[13, 12; 0]}(M)}^\lambda + (m_{33} - m_{22})c_{\lambda_{[13, 11; 0]}(M)}^\lambda \\
& = -2(m_{11} - m_{22} + 1)(m_{13} - m_{12})(m_{22} - m_{33}) \\
& + (m_{33} - m_{22})(m_{13} - m_{12})(2m_{22} - m_{13} - m_{23} - 1) \\
& = (-2m_{11} + m_{13} - m_{23} + 1)(m_{13} - m_{12})(m_{22} - m_{33}) \\
& = (-2m_{11} + m_{13} + m_{33} - 1)c_{\lambda_{[13, 22; 0]}(M)}^\lambda.
\end{align*}
\]

Hence we obtain the equation (5.21).

We have

\[
\begin{align*}
(m_{12} - m_{23})&\chi^{(-1)}_-(M)c_{\lambda_{[13, 12; 0]}(M)}^\lambda + (m_{11} - m_{22})c_{\lambda_{[13, 12; 1]}(M)}^\lambda \\
& = -2(m_{12} - m_{23})\chi^{(-1)}_-(M)(m_{13} - m_{12})(m_{22} - m_{33}) \\
& + (m_{11} - m_{22})\{E(M) + F(M) - (m_{13} - m_{12})(m_{22} - m_{33})\chi_+(M)\} \\
& = -2(m_{12} - m_{23})(m_{13} - m_{12})(m_{22} - m_{33})(1 - \chi_+(M)) + (m_{11} - m_{22})\{E(M) - C_2(M) \\
& - \chi_+(M)\{2(m_{13} - m_{12})(m_{22} - m_{33}) + (m_{13} - m_{33} + 1)\delta(M)\}\} \\
& = -2(m_{12} - m_{12})(m_{22} - m_{33})(m_{12} - m_{23} - \delta(M)\chi_+(M)) \\
& + (m_{11} - m_{22})(E(M) - C_2(M) - C_1(M)) \\
& = -2(m_{13} - m_{12})(m_{22} - m_{33})C_1(M) \\
& + (m_{11} - m_{22})(E(M) - C_2(M)) - C_1(M)(m_{13} - m_{33} + 1)\delta(M)\chi_+(M) \\
& = -C_1(M)\{(m_{13} - m_{33} + 1)\delta(M)\chi_+(M) + 2(m_{13} - m_{12})(m_{22} - m_{33})\} \\
& + (m_{11} - m_{22})(E(M) - C_2(M)), \\
C_1(M)c_{\lambda_{[13, 11; 0]}(M)}^\lambda + (m_{33} - m_{22} - 1)c_{\lambda_{[13, 11; 1]}(M)}^\lambda \\
& = C_1(M)(m_{13} - m_{12})(2m_{22} - m_{13} - m_{33} - 1) + (m_{33} - m_{22} - 1)(C_2(M) - E(M)).
\end{align*}
\]

Therefore

\[
(m_{12} - m_{23})\chi^{(-1)}_-(M)c_{\lambda_{[13, 12; 0]}(M)}^\lambda + (m_{11} - m_{22})c_{\lambda_{[13, 12; 1]}(M)}^\lambda
\]
Hence we obtain the equation $(5.22)$. Here we use the relations $(4.1)$, $(4.2)$, $(4.9)$ and $(4.13)$.

We have

$$(m_{12} - m_{23} - 1)\chi^{(-1)}(M)c_{[13,12;1]}(M) = (m_{12} - m_{23} - 1)\chi^{(-1)}(M)\{\bar{E}(M) + \bar{F}(M) - (m_{13} - m_{12})(m_{22} - m_{33})\chi(M)\}$$

Therefore

$$(m_{12} - m_{23} - 1)\chi^{(-1)}(M)c_{[13,12;1]}(M) + (m_{11} - m_{22} - 1)c_{[13,12;2]}(M)$$

$$+ (C_1(M) - 1)c_{[13,11;1]}(M)$$

$$= (m_{12} - m_{23} - 1)(1 - \chi(M))(\bar{E}(M) - C_2(M))$$

$$+ (m_{11} - m_{22} - 1)(\bar{E}(M) - C_2(M))\chi(M)$$

$$+ (C_1(M) - 1)(C_2(M) - \bar{E}(M))$$

$$= (m_{12} - m_{23} - \delta(M)\chi(M) - 1)(\bar{E}(M) - C_2(M))$$

$$- (C_1(M) - 1)(\bar{E}(M) - C_2(M)) = 0.$$ 

Hence we obtain the equation $(5.23)$. Here we use the relations $(4.1)$, $(4.9)$, $(4.10)$ and $(4.26)$.

We have

$$(m_{12} - m_{23} - 2)\chi^{(-1)}(M)c_{[13,12,2]}(M) = (m_{12} - m_{23} - 2)(\bar{E}(M) - C_2(M))\chi^{(-1)}(M)\chi(M) = 0.$$

Hence we obtain the equation $(5.24)$. Here we use the relation $(4.10)$.

- the proof of the case of the $(i,j) = (2,3)$.

Since $(r_{[23,22]},r_{[23,12]},r_{[23,11]}) = (1,1,1)$, we have to confirm the following equations:

$$(5.25)$$

$$(m_{11} - m_{22} + 1)c_{[23,12,0]}(M) + (m_{33} - m_{22})c_{[23,11,0]}(M)$$

$$= (-2m_{11} + m_{23} + m_{33} - 2)c_{[23,22,0]}(M),$$

$$(5.26)$$

$$(m_{12} - m_{23} + 1)\chi(M)c_{[23,12,0]}(M) + (m_{11} - m_{22})c_{[23,12,1]}(M)$$

$$+ (C_1(M) + \chi(M))c_{[23,11,0]}(M) + (m_{33} - m_{22} - 1)c_{[23,11,1]}(M)$$

$$= (-2m_{11} + m_{23} + m_{33} - 2)c_{[23,22,1]}(M).$$
Hence we obtain the equation (5.27).

We have

\[
(m_{11} - m_{22} + 1)c_{[23;12;0]}^\lambda (M) + (m_{33} - m_{22})c_{[23;11;0]}^\lambda (M) \\
= -2(m_{11} - m_{22} + 1)(m_{22} - m_{33}) + (m_{33} - m_{22})(2m_{22} - m_{23} - m_{33}) \\
= (-2m_{11} + m_{23} + m_{33} - 2)(m_{22} - m_{33}) \\
= (-2m_{11} + m_{23} + m_{33} - 2)c_{[23;22;0]}^\lambda (M).
\]

Hence we obtain the equation (5.25).

We have

\[
(m_{12} - m_{23} + 1)\chi_-(M)c_{[23;12;0]}^\lambda (M) + (m_{11} - m_{22})c_{[23;12;1]}^\lambda (M) \\
+ (C_1(M) + \chi_-(M))c_{[23;11,0]}^\lambda (M) + (m_{33} - m_{22} - 1)c_{[23;11;1]}^\lambda (M) \\
= -2(m_{11} - m_{23} + 1)c_{[23;12;0]}^\lambda (M)(m_{22} - m_{33}) \\
+ (m_{11} - m_{22})(C_1(M) - m_{22} - m_{33} + \delta(M)\chi_-(M)) \\
+ (C_1(M) + \chi_-(M))(2m_{22} - m_{23} - m_{33}) - (m_{33} - m_{22} - 1)(C_1(M) + \bar{D}(M))\chi_-(M) \\
= 2m_{11} - m_{23} + 1)(-m_{22} + m_{33} + \delta(M)\chi_-(M) + (m_{22} - m_{33} + 1)\bar{D}(M)\chi_-(M) \\
= (-2m_{11} + m_{23} + m_{33} - 2)c_{[23;22;1]}^\lambda (M).
\]

Hence we obtain the equation (5.26). Here we use the relations (4.1), (4.2) and (4.16).

We have

\[
(m_{12} - m_{23})\chi_-(M)c_{[23;12;1]}^\lambda (M) + (C_1(M) - 1 + \chi_-(M))c_{[23;11;1]}^\lambda (M) \\
= (m_{12} - m_{23})\chi_-(M)(C_1(M) - m_{22} - m_{33} + \delta(M)\chi_-(M)) \\
- (C_1(M) - 1 + \chi_-(M))(C_1(M) + \bar{D}(M))\chi_-(M) \\
= (m_{12} - m_{23})\chi_-(M)(C_1(M) - m_{22} + m_{33} + \delta(M)) - (m_{12} - m_{23})(C_1(M) + \bar{D}(M))\chi_-(M) \\
= 0.
\]

Hence we obtain the equation (5.27). Here we use the relations (4.16) and (4.12).

\[\square\]

**Theorem 5.6.** For \(1 \leq i \leq j \leq 3\), we have an following equation with the matrix representation \(R(\Gamma_{\lambda}^\lambda) \in M(d_{\alpha_{\lambda}}^\lambda, d_{\alpha_{\lambda}}^\lambda, C)\) of \(\Gamma_{\lambda}^\lambda\) with respect to the induced basis \(\{S_{\lambda}(M)\}_{M \in G_\alpha(\lambda)}\): 

\[
C_{\lambda_{\pm ij}}^\lambda S_{\sigma}(\lambda) = S_{\sigma}(\lambda [\pm ij]) \cdot R(\Gamma_{\lambda}^\lambda).
\]

Here \(C_{\lambda_{\pm ij}}^\lambda S_{\sigma}(\lambda)\) is a matrix of the size \(d_{\lambda_{\pm ij}}^\lambda \times d_{\alpha_{\lambda}}^\lambda\), whose \(l^\sigma(M)\)-th column is \(C_{\lambda_{\pm ij}}^\lambda S_{\lambda}(M)\) for \(M \in G_\alpha(\lambda)\).

The explicit expression of the matrix \(R(\Gamma_{\lambda}^\lambda)\) is given as follows.

(i) \(p_+\)-side:
\[ R(\Gamma_{+}^{\lambda}_{ij}) \quad (1 \leq i \leq j \leq 3) \] is a matrix of size \( d_{\lambda+ij}^\sigma \times d_{\lambda}^\sigma \), whose \( l^\sigma (M) \)-th column is given by

\[
(\nu_1 + \rho_1 + \gamma_1 M + k_{ij}(M)) \sum_{m=0}^{r_{ij;22}} c_{ij;22,m}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m] u_{\lambda+ij}^{\sigma} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

\[ + \sum_{m=0}^{r_{ij;02}} h_{ij;m}(\nu_2, M) u_{\lambda+ij}^{\sigma} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

\[ + (\nu_3 + \rho_3 + \gamma_3 M) \sum_{m=0}^{r_{ij;00}} c_{ij;00,m}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m] u_{\lambda+ij}^{\sigma} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

for \( M \in G_\sigma (\lambda) \). Here

\[
h_{ij;m}(\nu_2, M) = (\nu_2 + \rho_2 + \gamma_2 M) c_{ij;02,m}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

\[
+ (m_{22} - m_{33} + 1 + \delta(M)) \chi_{+}^{(\lambda-1)}(M) c_{ij;01,m-1}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

\[
+ (m_{22} - m_{33} + 1) c_{ij;01,m}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

\[
\left( c_{ij;kl,m}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m] = 0 \text{ if } r_{ij;kl} < m \text{ or } m < 0 \right)
\]

and \( u_{\lambda}^{\sigma}(M) \) is a column vector of degree \( d_{\lambda}^\sigma \) which is defined by

\[
u_{ij}^{\sigma}(M) = \left\{ \begin{array}{ll}
\frac{r^\sigma (M) - 1}{0, \ldots, 0, 1, 0, \ldots, 0} & \text{if } M \in G_\sigma (\lambda), \\
0 & \text{otherwise}.
\end{array} \right.
\]

(ii) \( p \)-side:

\[ R(\Gamma_{-}^{\lambda}_{ij}) \quad (1 \leq i \leq j \leq 3) \] is a matrix of size \( d_{\lambda-ij}^\sigma \times d_{\lambda}^\sigma \), whose \( l^\sigma (M) \)-th column is given by

the form

\[
(\nu_1 + \rho_1 - \gamma_1 M + k_{4-i-4-j}(M)) \sum_{m=0}^{r_{4-j-4-i;22}} c_{4-j-4-i;22,m}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m] u_{\lambda-ij}^{\sigma} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

\[ + \sum_{m=0}^{r_{4-i-4-j;02}} h_{4-i-4-j;m}(\nu_2, M) u_{\lambda-ij}^{\sigma} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

\[ + (\nu_3 + \rho_3 - \gamma_3 M) \sum_{m=0}^{r_{4-i-4-j;00}} c_{4-i-4-j;00,m}^{\lambda} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m] u_{\lambda-ij}^{\sigma} \left( M \left( \begin{smallmatrix} \frac{e_i + e_j}{2} \\ 0 \\ \frac{e_i + e_j}{2} \end{smallmatrix} \right) \right) [m]
\]

for \( M \in G_\sigma (\lambda) \).

\textbf{Proof.} For \( M, N \in G(\lambda) \), we define

\[
\Delta(M, N) = \begin{cases} 
1 & \text{if } M = N, \\
0 & \text{otherwise}.
\end{cases}
\]

Since

\[
(5.29) \quad s(M, N)(1_6) = \langle f(M)^*, f(N) \rangle = \Delta(M, N),
\]

we see that the value at \( 1_6 \in G \) of the vector \( S_{\lambda}(M) \) is the \( l(M) \)-th unit column vector \( t(0, \ldots, 0, 1, 0, \ldots, 0) \) of degree \( d_{\lambda} \). Therefore, we note that it suffices to evaluate the both of the equation \( (5.28) \) at \( 1_6 \in G \).
First, we compute \( \{X_{p+q}s(M,N)\}(16) \) for \( 1 \leq p \leq q \leq 3 \). Since \( \{s(M,N)\}_{N \in G(\lambda)} \) is the monomial basis of \( \langle S_\lambda(M) \rangle \), we obtain

\[
\{\kappa(E_{pp})s(M,N)\}(16) = \gamma_p^N s(M,N)(16) = \gamma_p^M \Delta(M,N), \quad 1 \leq p \leq 3,
\]

\[
\{\kappa(E_{21})s(M,N)\}(16)
= (n_{11} - n_{22})s(M,N (0^0_{-1})) + (n_{12} - n_{23})\chi(N)s(M,N (0^0_{-1}) [1])(16)
= (m_{11} - m_{22} + 1)\Delta(M (0^0_{-1}), N) + (m_{12} - m_{23} + 1)\chi^{(-1)}(M)\Delta(M (0^0_{-1}) [1], N),
\]

\[
\{\kappa(E_{31})s(M,N)\}(16)
= (n_{33} - n_{22})s(M,N (0^0_{-1})) + C_1(N)s(M,N (0^0_{-1}) [1])(16)
= (m_{33} - m_{22} - 1)\Delta(M (0^0_{-1}), N) + (C_1 + 1)\Delta(M (0^0_{-1}) [1], N),
\]

\[
\{\kappa(E_{32})s(M,N)\}(16)
= (n_{22} - n_{33})s(M,N (0^0_{-1})) + (n_{22} - n_{33} + \delta(N))\chi(N)s(M,N (0^0_{-1}) [1])(16)
= (m_{22} - m_{33} + 1)\Delta(M (0^0_{-1}), N) + \{m_{22} - m_{33} + 1 + \delta(M)\}\chi^{(-1)}(N)\Delta(M (0^0_{-1}) [1], N).
\]

by Proposition \( \ref{Eps}\), and the equations \( \ref{delta}, \ref{chi} \). Moreover, we obtain

\[
\{E_\alpha s(M,N)\}(16) = 0, \quad \alpha \in \Sigma^+,
\]

\[
\{H_p s(M,N)\}(16) = (\nu_p + \rho_p)s(M,N)(16), \quad 1 \leq p \leq 3
\]

by the definition of principal series representation. By above computations and Iwasawa decomposition in Lemma \( \ref{Iwasawa}\), we obtain

\[
\{X_{p+q}s(M,N)\}(16) = (\nu_p + \rho_p + \gamma_p^M)\Delta(M,N), \quad 1 \leq p \leq 3,
\]

\[
\{X_{12}s(M,N)\}(16) = (m_{11} - m_{22} + 1)\Delta(M (0^0_{-1}), N)
+ (m_{12} - m_{23} + 1)\chi^{(-1)}(M)\Delta(M (0^0_{-1}) [1], N),
\]

\[
\{X_{13}s(M,N)\}(16) = (m_{33} - m_{22} - 1)\Delta(M (0^0_{-1}), N) + (C_1 + 1)\Delta(M (0^0_{-1}) [1], N),
\]

\[
\{X_{23}s(M,N)\}(16) = (m_{22} - m_{33} + 1)\Delta(M (0^0_{-1}), N)
+ \{m_{22} - m_{33} + 1 + \delta(M)\}\chi^{(-1)}(N)\Delta(M (0^0_{-1}) [1], N).
\]

Let \( u_\lambda(N) \) be a column vector of degree \( d_\lambda \) which is defined by

\[
u_\lambda(N) = \begin{cases} 
\frac{l(N)-1}{0} & \text{if } M \in G(\lambda), \\
\frac{d_\lambda-l(N)}{0} & \text{otherwise}.
\end{cases}
\]

We denote by \( X_{p+q}S_\lambda(M) \) the vector of degree \( d_\lambda \) whose \( l(N) \)-th component is \( X_{p+q}s(M,N) \). Then we obtain

\[
\{X_{p+q}S_\lambda(M)\}(16) = (\nu_p + \rho_p + \gamma_p^M)u_\lambda(M) \quad \text{for } 1 \leq p \leq 3,
\]

\[
\{X_{12}S_\lambda(M)\}(16) = (m_{12} - m_{23} + 1)\chi^{(-1)}(M)u_\lambda(M (0^0_{-1}) [1])
+ (m_{11} - m_{22} + 1)u_\lambda(M (0^0_{-1})),
\]

\[
\{X_{13}S_\lambda(M)\}(16) = (C_1 + 1)u_\lambda(M (0^0_{-1}) [1]) + (m_{33} - m_{22} - 1)u_\lambda(M (0^0_{-1})),
\]

\[
\{X_{23}S_\lambda(M)\}(16) = (m_{22} - m_{33} + 1 + \delta(M))\chi^{(-1)}(M)u_\lambda(M (0^0_{-1}) [1])
+ (m_{22} - m_{33} + 1)u_\lambda(M (0^0_{-1})).
\]

Let us compute \( \{C^\lambda_{ij}S_\lambda(M)\}(16) \).
Since

\[ L_{+ij}^\lambda \left( \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \right) [-m] \cdot u_\lambda(N) = c_{[ij;kl;m]}^\lambda \left( N \left( \begin{pmatrix} 0&1 \\ 0 & k \end{pmatrix} \right) [m] \right) u_{\lambda+[ij]} \left( N \left( \begin{pmatrix} 0&0 \\ 0 & k \end{pmatrix} \right) [m] \right) \]

for \( N \in G(\lambda) \), we obtain

\[ (5.30) \]

\[ \left\{ \left( \sum_{m=0}^{\tilde{N}_{[ij;2]}^r} L_{+ij}^\lambda \left( \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \right) [-m] \right) \otimes X_{+11} \right\} S_\lambda(M) \] \( (16) \)

\[ = \left( \sum_{m=0}^{\tilde{N}_{[ij;2]}^r} L_{+ij}^\lambda \left( \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \right) [-m] \right) \cdot \{ X_{+11} S_\lambda(M) \} \]

\[ = (\nu_1 + \rho_1 + \gamma_1^M) \sum_{m=0}^{\tilde{N}_{[ij;2;2]}^r} c_{[ij;2;2;m]}^\lambda \left( M \left( \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right) [m] \right) u_{\lambda+[ij]} \left( M \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) [m] \right), \]

\[ (5.31) \]

\[ \left\{ \left( \sum_{m=0}^{\tilde{N}_{[ij;12]}^r} L_{+ij}^\lambda \left( \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \right) [-m] \right) \otimes X_{+12} \right\} S_\lambda(M) \] \( (16) \)

\[ = \left( \sum_{m=0}^{\tilde{N}_{[ij;12]}^r} L_{+ij}^\lambda \left( \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \right) [-m] \right) \cdot \{ X_{+12} S_\lambda(M) \} \]

\[ = (m_{12} - m_{23} + 1) \chi_\lambda^{-1}(M) \sum_{m=0}^{\tilde{N}_{[ij;12;2]}^r} c_{[ij;12;2;m]}^\lambda \left( M \left( \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right) [m+1] \right) u_\lambda \left( M \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) [m+1] \right) \]

\[ + (m_{11} - m_{22} + 1) \sum_{m=0}^{\tilde{N}_{[ij;12;2]}^r} c_{[ij;12;2;m]}^\lambda \left( M \left( \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right) [m] \right) u_\lambda \left( M \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) [m] \right), \]

\[ \sum_{m=0}^{\tilde{N}_{[ij;12;2]}^r} \left\{ \chi_\lambda^{-1}(M) c_{[ij;12;2;m]}^\lambda \left( M \left( \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right) [m] \right) u_\lambda \left( M \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) [m] \right) \right\} \]

\[ + (m_{11} - m_{22} + 1) c_{[ij;12;2;m]}^\lambda \left( M \left( \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right) [m] \right) \left\{ u_\lambda \left( M \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) [m] \right) \right\} \]

\[ = (C_1(M) + 1) \sum_{m=0}^{\tilde{N}_{[ij;11]}^r} c_{[ij;11;m]}^\lambda \left( M \left( \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right) [m+1] \right) u_\lambda \left( M \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) [m+1] \right) \]

\[ + (m_{33} - m_{22} - 1) \sum_{m=0}^{\tilde{N}_{[ij;12;2]}^r} c_{[ij;12;2;m]}^\lambda \left( M \left( \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right) [m] \right) u_\lambda \left( M \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right) [m] \right). \]

\[ (5.32) \]
By direct computation from the equations (5.34) and (5.35), we obtain
\[ (5.33) \]
Similarly,
\[ (5.34) \]
\[ (5.35) \]
\[ (5.36) \]
By the equations (5.30), (5.31), (5.32) and Lemma 5.5, we obtain
\[ (5.33) \]
\[ (5.34) \]
\[ (5.35) \]
\[ (5.36) \]
\[ r_{[ij],[02]} = \sum_{m=0}^{r_{[ij],[00]}} h_{[ij];m}(\nu_2, M) u_{\lambda+[ij]} \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) [m]. \]

Similarly,

\[ (5.37) \quad \left\{ \left( \sum_{m=0}^{r_{[ij],[00]}} L_{+ij}^\lambda \left( \begin{array}{c} 0 \\ 0 \\ -m \end{array} \right) \otimes X_{+33}^\lambda \right) S_\lambda(M) \right\} \quad (16) \]

\[ = \left( \sum_{m=0}^{r_{[ij],[00]}} L_{+ij}^\lambda \left( \begin{array}{c} 0 \\ 0 \\ -m \end{array} \right) \right) \cdot \{ X_{+33} S_\lambda(M) \} \quad (16) \]

\[ = (\nu_3 + \rho_3 + \gamma_3^M) \sum_{m=0}^{r_{[ij],[00]}} \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) u_{\lambda+[ij]} \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) [m]. \]

Summing up the equations \(5.33\), \(5.36\) and \(5.37\), we obtain

\[ (5.38) \quad \{ C_{11} S_\lambda(M) \} \quad (16) \]

\[ = (\nu_1 + \rho_1 + \gamma_1^M + k_{ij}(M)) \sum_{m=0}^{r_{[ij],[22]}} c_{[ij];22;m}^\lambda \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) u_{\lambda+[ij]} \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) [m] \]

\[ + \sum_{m=0}^{r_{[ij],[02]}} h_{[ij];m}(\nu_2, M) u_{\lambda+[ij]} \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) [m] \]

\[ + (\nu_3 + \rho_3 + \gamma_3^M) \sum_{m=0}^{r_{[ij],[00]}} \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) u_{\lambda+[ij]} \left( M \left( \begin{array}{c} a_i+b_j \\ 0 \\ 0 \end{array} \right) \right) [m]. \]

By the remark at the beginning of this proof, we obtain the assertion of the case of (i). The case of (ii) is treated similarly. \(\square\)

6. Examples of contiguous relations and their applications

Here are some examples of the contiguous relations at the peripheral \(K\)-types. Moreover, as their applications, we determine the holonomic systems, whose solutions are Whittaker functions.

6.1. Whittaker functions. For a unitary character \(\xi\) of \(N_{\text{min}}\), we denote the derivative of \(\xi\) by the same letter. Since

\[ n/[n,n] \simeq g_{e_1-e_2} \oplus g_{e_2-e_3} \oplus g_{2e_3}, \]

\(\xi\) is specified by three real numbers \(c_{12}, c_{23}\) and \(c_3\) such that

\[ \xi(E_{e_1-e_2}) = 2\pi \sqrt{-1} c_{12}, \quad \xi(E_{e_2-e_3}) = 2\pi \sqrt{-1} c_{23} \quad \text{and} \quad \xi(E_{2e_3}) = 2\pi \sqrt{-1} c_3. \]

When \(c_{12} c_{23} c_3 \neq 0\), a unitary character \(\xi\) of \(N_{\text{min}}\) is called non-degenerate.

For a finite dimensional representation \((\tau, V)\) of \(K\) and a non-degenerate unitary character \(\xi\) of \(N_{\text{min}}\), we consider the space \(C_{\xi,\tau}^\infty(N_{\text{min}} \backslash G/K)\) of smooth functions \(\varphi: G \to V_\tau\) with the property

\[ \varphi(n g k) = \xi(n) \tau(k)^{-1} \varphi(g), \quad (n, g, k) \in N_{\text{min}} \times G \times K. \]

Here we remark that any functions \(\varphi \in C_{\xi,\tau}^\infty(N_{\text{min}} \backslash G/K)\) is determined by its restriction \(\varphi|_{A_{\text{min}}}\) to \(A_{\text{min}}\) from the Iwasawa decomposition \(G = N_{\text{min}} A_{\text{min}} K\) of \(G\). \(\varphi|_{A_{\text{min}}}\) is called the \(A_{\text{min}}\)-radial part of \(\varphi\). Also let \(C_{\text{Ind}}^\infty_{N_{\text{min}}} (\xi)\) be the \(C^\infty\)-induced representation from \(\xi\) with the representation space

\[ C_{\xi}^\infty(N_{\text{min}} \backslash G) = \{ \varphi \in C^\infty(G) \mid \varphi(n g) = \xi(n) \varphi(g), \quad (n, g) \in N_{\text{min}} \times G \}. \]
on which $G$ acts by right translation. Then the space $C_{ξ,τ}^∞(N_{min}\backslash G/K)$ is isomorphic to $\text{Hom}_K(V^*, Cξ^∞(N_{min}\backslash G))$ via the correspondence between $i \in \text{Hom}_K(V^*, Cξ^∞(N_{min}\backslash G))$ and $F[i] \in C_{ξ,τ}^∞(N_{min}\backslash G/K)$ given by the relation $i(v^*)(g) = \langle v^*, F[i](g) \rangle$ for $v^* \in V^*$ and $g \in G$ with canonical pairing $\langle \cdot, \cdot \rangle$ on $V^* \times V$.

Let $(π, H_π)$ be an irreducible admissible representation of $G$, and take a $K$-type $(τ^*, V^*)$ of $π$ with an injective $K$-homomorphism $η: V^* \to H_π$. Then, for each element $T$ in the intertwining space $I_{ξ,τ} = \text{Hom}_{K}(g \cdot K, \pi, \tau) \backslash H_π$, consisting of all $K$-finite vectors. Now we put $\text{Wh}(π, ξ, τ) = \bigcup_{η \in \text{Hom}_K(V^*, H_π)} \{\Phi(T, η) \in C_{ξ,τ}^∞(N_{min}\backslash G/K) \mid T \in I_{ξ,τ} \}$ and call $\text{Wh}(π, ξ, τ)$ the space of Whittaker functions for $(π, ξ, τ)$. We consider the Whittaker functions for the irreducible principal series representation $π = π_{(σ, ν)}$.

### 6.2. The ±-chilarity matrices

We define the ±-chilarity matrices as follows.

**Definition 6.1.** The ±-chilarity matrices $m_i(C_±)$ for $1 \leq i \leq 3$ are defined by

$$m_1(C_±) = \begin{pmatrix} X_{±11} & X_{±12} & X_{±13} \\ X_{±12} & X_{±22} & X_{±23} \\ X_{±13} & X_{±23} & X_{±33} \end{pmatrix}, \quad m_2(C_±) = \begin{pmatrix} M_{±11} & -M_{±12} & M_{±13} \\ -M_{±12} & M_{±22} & -M_{±23} \\ M_{±13} & -M_{±23} & M_{±33} \end{pmatrix},$$

and $m_3(C_±) = \det(m_1(C_±))$. Here $M_{±ij}$ is $(i,j)$-minor of the matrix $m_1(C_±)$ for each $1 \leq i \leq j \leq 3$, that is,

$$M_{±11} = X_{±22} X_{±23}, \quad M_{±22} = X_{±11} X_{±13}, \quad M_{±33} = X_{±11} X_{±12},$$

$$M_{±12} = X_{±12} X_{±23}, \quad M_{±13} = X_{±12} X_{±22}, \quad M_{±23} = X_{±12} X_{±22}.$$

Then we can find the following lemma immediately from the definition of the chilarity matrices.

**Lemma 6.2.** For each $1 \leq i \leq 3$, the element $C_{2i} = \text{Tr}(m_i(C_+ m_i(C_-))$ in $U(\mathfrak{g}_C)$ is invariant under the adjoint action of $K$, that is,

$$C_{2i} \in U(\mathfrak{g}_C)^K = \{X \in U(\mathfrak{g}_C) \mid \text{Ad}(k)X = X, \ k \in K\}.$$

**Remark 6.3.** In the case of $Sp(n, \mathbb{R})$, we can define $C_{2i}$ for each $1 \leq i \leq n$ belonging to $U(\mathfrak{g}_C)^K$ similarly. The operator $C_{2i}$ is essentially same as the so-called Maass shift operator in the classical literature [9]. Also, the chilarity matrices are used to construct the Caselli elements for a symmetric pair in [8], recently.

Now we consider a system of differential equations which are satisfied by the $A_{min}$-radial part of each element in $\text{Wh}(π, ξ, τ)$ when $τ^*$ is a multiplicity one $K$-type of $π_{(σ, ν)}$. Let $λ$ be the highest weight of $τ^*$. The elements $C_2, C_4$ and $C_6$ in $U(\mathfrak{g}_C)^K$ defined in Lemma 6.2 are acting on the space $C_ξ^∞(N_{min}\backslash G)$ as differential operators and acting on the space $H_{(σ, ν)}$ as scalar operators. Therefore, each element $Φ$ in $\text{Wh}(π, ξ, τ)$ satisfies the following system of differential equations

$$C_{2i}Φ = χ_{2i, σ, ν, λ}Φ \quad (1 \leq i \leq 3),$$

where $χ_{2i, σ, ν, λ}$ is the scalar value for the action of the operator $C_{2i}$. In the later subsections, we compute those values for peripheral $K$-types.
6.3. The chirality operators in the cases of $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1)$. In this subsection, we consider the cases of $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1)$. We set $M_l = \left( \begin{smallmatrix} l & l \\ -l & -l \end{smallmatrix} \right) \in G((l, l, l))$ for $l \in \mathbb{Z}$. Let $\varepsilon_{\sigma}$ be the integer defined by

$$\varepsilon_{\sigma} = \begin{cases} 0 & \text{if } (\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), \\ 1 & \text{if } (\sigma_1, \sigma_2, \sigma_3) = (1, 1, 1). \end{cases}$$

Since $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1)$, there are multiplicity one $K$-types $\tau_{(l,l,l)}$ ($l \equiv \varepsilon_{\sigma} \mod 2$) of $\pi(\sigma, \nu)$. Let us compute the values of $\chi_{2i, \sigma, \nu}(l, l, l)$ $(1 \leq i \leq 3)$.

**Lemma 6.4.** (i) Let $\sigma$ be a character of $M_{\min}$ such that $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1)$. For $l \equiv \varepsilon_{\sigma} \mod 2$, the following equations hold:

$$\begin{align*}
C_{+11}^{(l-2,l-2,l-2)} & \cdot \sigma_{\nu}(l-2, l-2, l-2) = \sigma_{\nu}(l-2, l-2, l-2) \cdot R(\Gamma_{+11}^{(l-2,l-2,l-2)}), \\
C_{+22}^{(l-2,l-2,l-2)} & \cdot \sigma_{\nu}(l-2, l-2, l-2) = \sigma_{\nu}(l-2, l-2, l-2) \cdot R(\Gamma_{+22}^{(l-2,l-2,l-2)}), \\
C_{+33}^{(l-2,l-2,l-2)} & \cdot \sigma_{\nu}(l-2, l-2, l-2) = \sigma_{\nu}(l-2, l-2, l-2) \cdot R(\Gamma_{+33}^{(l-2,l-2,l-2)}), \\
C_{-33}^{(l-2,l-2,l-2)} & \cdot \sigma_{\nu}(l-2, l-2, l-2) = \sigma_{\nu}(l-2, l-2, l-2) \cdot R(\Gamma_{-33}^{(l-2,l-2,l-2)}), \\
C_{-22}^{(l-2,l-2,l-2)} & \cdot \sigma_{\nu}(l-2, l-2, l-2) = \sigma_{\nu}(l-2, l-2, l-2) \cdot R(\Gamma_{-22}^{(l-2,l-2,l-2)}), \\
C_{-11}^{(l-2,l-2,l-2)} & \cdot \sigma_{\nu}(l-2, l-2, l-2) = \sigma_{\nu}(l-2, l-2, l-2) \cdot R(\Gamma_{-11}^{(l-2,l-2,l-2)}).
\end{align*}$$

Here

$$\begin{align*}
C_{+11}^{(l-2,l-2,l-2)} & = 12 \begin{pmatrix} X_{+11} & X_{+12} & X_{+13} \\ X_{+22} & X_{+23} & X_{+23} \\ X_{+33} & X_{+33} & X_{+33} \end{pmatrix}, \\
R(\Gamma_{+11}^{(l-2,l-2,l-2)}) & = 12 \begin{pmatrix} \nu_1 + l + 1 \\ \nu_2 + l \\ \nu_3 + l - 1 \end{pmatrix}, \\
C_{+22}^{(l-2,l-2,l-2)} & = 2 \begin{pmatrix} X_{+22} & -2X_{+12} & X_{+11} \\ X_{+23} & -X_{+13} & -X_{+12} \\ X_{+33} & 0 & -2X_{+13} \end{pmatrix}, \\
R(\Gamma_{+22}^{(l-2,l-2,l-2)}) & = 2 \begin{pmatrix} \nu_2 + l \\ \nu_3 + l - 1 \\ 0 \end{pmatrix}, \\
C_{+33}^{(l-2,l-2,l-2)} & = \begin{pmatrix} X_{+33} \\ -2X_{+23} \\ X_{+22} \\ 2X_{+13} \\ -2X_{+12} \\ X_{+11} \end{pmatrix}, \\
R(\Gamma_{+33}^{(l-2,l-2,l-2)}) & = \begin{pmatrix} \nu_3 + l - 1 \\ \nu_2 + l - 2 \\ \nu_1 + l - 3 \end{pmatrix}, \\
C_{-33}^{(l,l,l)} & = 12 \begin{pmatrix} X_{-33} \\ -X_{-23} \\ X_{-22} \\ X_{-13} \\ -X_{-12} \\ X_{-11} \end{pmatrix}, \\
R(\Gamma_{-33}^{(l,l,l)}) & = 12 \begin{pmatrix} \nu_3 - l + 1 \\ \nu_2 - l + 2 \\ \nu_1 - l + 3 \end{pmatrix}. 
\end{align*}$$
The elements \( C_{2i} \) (\( i = 1, 2, 3 \)) are represented by the \( p_{\pm} \)-matrices \( C_{\pm k l}^\lambda \) as follows:

\[
C_2 = \frac{1}{12} \cdot C_{(l,l,l-2)}^{(l,l,l-2)} \cdot C_{(l,l,l)}^{(l,l,l)}, \quad C_4 = \frac{1}{192} \cdot C_{(l,l,l-2)}^{(l,l,l-2)} \cdot C_{(l,l,l-2)}^{(l,l,l-2)} \cdot C_{(l,l,l)}^{(l,l,l)},
\]

\[
C_6 = \frac{1}{20736} \cdot C_{(l,l,l-2)}^{(l,l,l-2)} \cdot C_{(l,l,l-2)}^{(l,l,l-2)} \cdot C_{(l,l,l-2)}^{(l,l,l-2)} \cdot C_{(l,l,l)}^{(l,l,l)}.
\]

Proof. From Theorem 5.6 in the case of \((\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1)\), we obtain the assertion by direct computation. \( \square \)

From above lemma, we obtain differential equations which Whittaker functions satisfy.

**Proposition 6.5.** Let \( \sigma \) be a character of \( M_{\text{min}} \) such that \((\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0) \) or \((1, 1, 1)\), and \( T \) be an element of the space \( I_{\xi, \pi(\sigma, \nu)} \). For \( l \equiv \epsilon_\sigma \mod 2 \), we define a function \( \phi_{T,I} \in C^\infty(N_{\text{min}} \backslash G) \) by the equation \( \Phi(T, S_{(l,l,l)}(M_I)) = \phi_{T,I} \otimes f(M_I)^* \). Then \( \phi_{T,I} \) satisfies the following differential equations:

\[
C_{2i} \phi_{T,I} = \chi_{2i,\sigma,\nu,(l,l,l)} \phi_{T,I} \quad (i = 1, 2, 3).
\]

Here

\[
(6.1) \quad \chi_{2,\sigma,\nu,(l,l,l)} = \{\nu_2^2 - (l - 3)^2\} + \{\nu_2^2 - (l - 2)^2\} + \{\nu_3^2 - (l - 1)^2\},
\]

\[
(6.2) \quad \chi_{4,\sigma,\nu,(l,l,l)} = \{\nu_2^2 - (l - 2)^2\} + \{\nu_2^2 - (l - 2)^2\} + \{\nu_3^2 - (l - 2)^2\} + \{\nu_3^2 - (l - 1)^2\},
\]

\[
+ \{\nu_2^2 - (l - 1)^2\} + \{\nu_3^2 - (l - 1)^2\},
\]

\[
(6.3) \quad \chi_{6,\sigma,\nu,(l,l,l)} = \{\nu_2^2 - (l - 3)^2\} + \{\nu_2^2 - (l - 1)^2\} + \{\nu_2^2 - (l - 1)^2\} + \{\nu_2^2 - (l - 1)^2\}.
\]

Proof. Let \( \chi_{2i,\sigma,\nu,(l,l,l)} \) (\( i = 1, 2, 3 \)) be the complex numbers defined by the equations (6.1), (6.2) and (6.3). From Lemma 6.4 and \( S_{(l,l,l)}(M_I) = s(M_I, M_I) \), we see that

\[
C_{2i} s(M_I, M_I) = \frac{1}{12} \cdot s(M_I, M_I) \cdot R(\Gamma^{(l,l,l-2)}_{+33}) \cdot R(\Gamma^{(l,l,l-1)}_{-33}) = \chi_{2i,\sigma,\nu,(l,l,l)} s(M_I, M_I).
\]

Similarly, we obtain \( C_{2i} s(M_I, M_I) = \chi_{2i,\sigma,\nu,(l,l,l)} s(M_I, M_I) \) (\( i = 2, 3 \)). Since \( \phi_{T,I} = T(s(M_I, M_I)) \), we obtain the assertion from these equations. \( \square \)

**6.4. The chirality operators in the case of** \((\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1)\). In this subsection, we consider the cases of \((\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1)\). Let \( \epsilon_\sigma \), \( \delta_{2i} \in \{0, 1\} \) (\( i = 1, 2, 3 \)) be the integers defined by

\[
\epsilon_\sigma = \begin{cases} 0 & \text{if } (\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1), \\ 1 & \text{otherwise} \end{cases}
\]

\[
\delta_{2i} = \begin{cases} 0 & \text{if } \epsilon_\sigma - \sigma_i = 0, \\ 1 & \text{otherwise} \end{cases}
\]

In the cases of \((\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1)\), there are multiplicity one \( K \)-types \( \tau_{(l,l,l)} \) and \( \tau_{(l,l,l-1)} \) (\( l \equiv \epsilon_\sigma \mod 2 \)) of \( \pi_{(\sigma, \nu)} \). Let us compute the values \( \chi_{2i,\sigma,\nu,(l,l,l)} \), \( \chi_{2i,\sigma,\nu,(l,l,l-1)} \) (\( 1 \leq i \leq 3 \)).
Lemma 6.6. (i) Let $\sigma$ be a character of $M_{\min}$ such that $(\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0)$, $(1, 1, 1)$. For $l \equiv \varepsilon_\sigma \mod 2$, the following equations hold:

$$C_{l+1,l-2,-2}^{(l,l+2,2,-2)} \sigma_l((l + 1, l - 2, -2)) = \sigma_l((l + 1, l, l - 2)) \cdot R(\Gamma_{l+22}^{(l,l+2,2,-2)}),$$

$$C_{l+23}^{(l,l+1,2)} \sigma_l((l + 1, l, l - 2)) = \sigma_l((l + 1, l, l)) \cdot R(\Gamma_{l+23}^{(l,l+1,2)}),$$

$$C_{l+33}^{(l,l-1)} \sigma_l((l, l, l - 3)) = \sigma_l((l, l, l - 1)) \cdot R(\Gamma_{l+33}^{(l,l-1)}),$$

$$C_{l+12}^{(l,l-1,-2)} \sigma_l((l - 1, l - 2, l - 2)) = \sigma_l((l - 1, l - 1, l - 2)) \cdot R(\Gamma_{l+12}^{(l,l-1,-2)}),$$

$$C_{l+13}^{(l,l-1)} \sigma_l((l, l, l - 1)) = \sigma_l((l + 1, l, l)) \cdot R(\Gamma_{l+13}^{(l,l-1)}),$$

$$C_{l+12}^{(l,l-1,-3)} \sigma_l((l - 1, l - 2, l - 3)) = \sigma_l((l - 1, l - 2, l - 2)) \cdot R(\Gamma_{l+13}^{(l,l-1,-3)}),$$

$$C_{l+23}^{(l,l-1,-2)} \sigma_l((l, l - 1, l - 2)) = \sigma_l((l, l - 1, l - 1)) \cdot R(\Gamma_{l+23}^{(l,l-1,-2)}),$$

$$C_{l+33}^{(l,l-1)} \sigma_l((l, l, l - 3)) = \sigma_l((l, l, l - 2)) \cdot R(\Gamma_{l+33}^{(l,l-1)}),$$

$$C_{-l+12}^{(l,l+1,2)} \sigma_l((l + 1, l, l)) = \sigma_l((l - 1, l - 1, l - 2)) \cdot R(\Gamma_{l+12}^{(l,l+1,2)}),$$

$$C_{-l+13}^{(l,l+1)} \sigma_l((l + 1, l, l)) = \sigma_l((l, l, l - 1)) \cdot R(\Gamma_{l+13}^{(l,l+1)}),$$

$$C_{-l+13}^{(l,l+1)} \sigma_l((l, l, l - 2)) = \sigma_l((l - 2, l - 2, l - 2)) \cdot R(\Gamma_{l+13}^{(l,l+1)}),$$

The explicit expressions of $C_{l+ij}^\lambda$ and $R(\Gamma_{l+ij}^\lambda)$ in the above equations are given as follows:

$$C_{l+22}^{(l,l+2,2,-2)} =$$

$$\begin{pmatrix}
X_{+22} + 2X_{+12} & 0 & X_{+11} & 0 & 0 & 0 & 0 & 0 & 0 \\
X_{+23} & -X_{+13} & -X_{+12} & 0 & X_{+11} & 0 & 0 & 0 & 0 \\
X_{+33} & 0 & -2X_{+13} & 0 & 0 & X_{+11} & 0 & 0 & 0 \\
0 & X_{+22} & 0 & -2X_{+12} & 0 & 0 & X_{+11} & 0 & 0 \\
0 & X_{+23} & -X_{+22} & -X_{+13} & X_{+12} & 0 & 0 & 0 & 0 \\
0 & 0 & X_{+22} & 0 & -2X_{+12} & 0 & 0 & X_{+11} & 0 \\
0 & X_{+33} & -X_{+23} & 0 & -X_{+13} & X_{+12} & 0 & 0 & 0 \\
0 & 0 & X_{+23} & 0 & -X_{+13} & -X_{+12} & 0 & 0 & X_{+11} \\
0 & 0 & X_{+33} & 0 & 0 & -2X_{+13} & 0 & 0 & X_{+11} \\
0 & 0 & 0 & X_{+23} & -X_{+22} & 0 & -X_{+13} & X_{+12} & 0 \\
0 & 0 & 0 & X_{+33} & -2X_{+23} & X_{+22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X_{+23} & -X_{+22} & 0 & -X_{+13} & X_{+12} \\
0 & 0 & 0 & 0 & X_{+33} & -X_{+23} & 0 & 0 & -X_{+13} & X_{+12} \\
0 & 0 & 0 & 0 & 0 & X_{+33} & -2X_{+23} & X_{+22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_{+33} & -2X_{+23} & X_{+22} 
\end{pmatrix}.$$
\[ C_{++3}^{(l+1,l-2)} = t \begin{pmatrix} X_{++3} & 0 & 0 \\ -2X_{++23} & 0 & 0 \\ X_{++22} & 0 & 0 \\ 0 & X_{++33} & 0 \\ 2X_{++13} & -2X_{++23} & 0 \\ 0 & -2X_{++23} & X_{++33} \\ -2X_{++12} & X_{++22} & 0 \\ 0 & X_{++22} & -2X_{++23} \\ 0 & 0 & X_{++22} \\ 0 & 2X_{++13} & 0 \\ X_{++11} & -2X_{++12} & 0 \\ 0 & -2X_{++12} & 2X_{++13} \\ 0 & 0 & -2X_{++12} \\ 0 & X_{++11} & 0 \\ 0 & 0 & X_{++11} \end{pmatrix}, \]

\[ C_{++3}^{(l,l-3)} = \begin{pmatrix} X_{++33} & -2X_{++23} & X_{++22} & 0 \\ X_{++33} & -2X_{++23} & X_{++22} & 0 \\ 0 & X_{++33} & -2X_{++23} & X_{++22} \\ 0 & 0 & 0 & X_{++33} \end{pmatrix} \begin{pmatrix} 2X_{++13} & -2X_{++12} & 0 & X_{++11} \\ 0 & 2X_{++13} & -2X_{++12} & 0 \\ X_{++11} & 0 & 0 \\ 0 & 0 & 0 & 2X_{++13} \end{pmatrix}, \]

\[ C_{+23}^{(l-1,l-2)} = 3 \begin{pmatrix} X_{++12} & -X_{++11} & 0 \\ X_{++13} & 0 & -X_{++11} \\ X_{++22} & -X_{++12} & 0 \\ 0 & X_{++13} & -X_{++12} \\ X_{++23} & -X_{++13} & 0 \\ X_{++33} & 0 & -X_{++13} \\ 0 & X_{++23} & -X_{++22} \\ 0 & X_{++33} & -X_{++23} \end{pmatrix}, \]

\[ C_{+13}^{(l,l-1)} = C_{+13}^{(l-2,l-2,l-3)} = 2 \begin{pmatrix} -X_{++13} & X_{++12} & -X_{++11} \\ -X_{++23} & X_{++22} & -X_{++12} \\ -X_{++33} & X_{++23} & -X_{++13} \end{pmatrix}, \]

\[ C_{+23}^{(l-1,l-2,l-1)} = \begin{pmatrix} -X_{++23} & X_{++22} & X_{++13} & -2X_{++12} & -X_{++12} & 0 & X_{++11} & 0 \\ -X_{++33} & X_{++23} & 0 & -X_{++13} & X_{++13} & -X_{++12} & 0 & X_{++11} \\ 0 & 0 & X_{++33} & X_{++23} & 2X_{++23} & -X_{++22} & -X_{++13} & X_{++12} \end{pmatrix}, \]

\[ t \begin{pmatrix} -X_{++23} & -X_{++33} & 0 & 0 & 0 & 0 & 0 & 0 \\ X_{++22} & X_{++23} & 0 & 0 & 0 & 0 & 0 & 0 \\ X_{++13} & 0 & -X_{++23} & -X_{++33} & 0 & 0 & 0 & 0 \\ -2X_{++12} & -X_{++13} & X_{++22} & X_{++23} & 0 & 0 & 0 & 0 \\ -X_{++12} & X_{++13} & X_{++22} & 2X_{++23} & -X_{++23} & -X_{++33} & 0 & 0 \\ 0 & -X_{++12} & 0 & -X_{++22} & X_{++23} & X_{++23} & 0 & 0 \\ 0 & 0 & X_{++13} & 0 & 0 & 0 & -X_{++33} & 0 \\ X_{++11} & 0 & -2X_{++12} & -X_{++13} & 0 & 0 & X_{++23} & 0 \\ 0 & 0 & -X_{++12} & 0 & X_{++13} & 0 & 2X_{++23} & -X_{++33} \\ 0 & X_{++11} & 0 & X_{++12} & -2X_{++12} & -X_{++13} & -X_{++22} & X_{++23} \\ 0 & 0 & 0 & 0 & -X_{++12} & X_{++13} & -X_{++22} & 2X_{++23} \\ 0 & 0 & 0 & 0 & 0 & -X_{++12} & 0 & -X_{++22} \\ 0 & 0 & X_{++11} & 0 & 0 & 0 & -X_{++13} & 0 \\ 0 & 0 & 0 & 0 & X_{++11} & 0 & X_{++12} & -X_{++13} \\ 0 & 0 & 0 & 0 & 0 & X_{++11} & 0 & X_{++12} \end{pmatrix}, \]
$$C_{-22}^{(l+1,l-2)} = \begin{pmatrix}
3X_{-22} & -2X_{-12} & 0 & X_{-11} & 0 & 0 & 0 & 0 & 0 & 0
6X_{-23} & -2X_{-13} & -2X_{-12} & 0 & X_{-11} & 0 & 0 & 0 & 0 & 0
3X_{-33} & 0 & -2X_{-13} & 0 & 0 & X_{-11} & 0 & 0 & 0 & 0
0 & X_{-22} & 0 & -2X_{-12} & 0 & 0 & 3X_{-11} & 0 & 0 & 0
0 & 4X_{-23} & -2X_{-22} & -4X_{-13} & 0 & 0 & 0 & 2X_{-11} & 0 & 0
0 & 2X_{-23} & X_{-22} & -2X_{-13} & -2X_{-12} & 0 & 0 & 3X_{-11} & 0 & 0
0 & 3X_{-33} & -2X_{-23} & 0 & -2X_{-13} & 2X_{-12} & 0 & 0 & X_{-11} & 0
0 & X_{-33} & 2X_{-23} & 0 & -2X_{-13} & -2X_{-12} & 0 & 0 & 3X_{-11} & 0
0 & 0 & X_{-33} & 0 & 0 & -2X_{-13} & 0 & 0 & 0 & 3X_{-11}
0 & 0 & 0 & 2X_{-23} & -1X_{-22} & 0 & -6X_{-13} & 2X_{-12} & 0 & 0
0 & 0 & 0 & 3X_{-33} & -2X_{-23} & X_{-22} & 0 & -2X_{-13} & 2X_{-12} & 0
0 & 0 & 0 & 2X_{-33} & 0 & -2X_{-22} & 0 & -4X_{-13} & 4X_{-12} & 0
0 & 0 & 0 & 0 & X_{-33} & -2X_{-23} & 0 & 0 & -2X_{-13} & 6X_{-12}
0 & 0 & 0 & 0 & 0 & 0 & 3X_{-33} & -2X_{-23} & -X_{22} & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{-33} & -2X_{-23} & 3X_{-22}
\end{pmatrix},$$

$$C_{-33}^{(l+1,l,l)} = 6 \begin{pmatrix}
4X_{-33} & 0 & 0 & 0
-4X_{-23} & 0 & 0 & 0
4X_{-22} & 0 & 0 & 0
0 & 4X_{-33} & 0 & 0
3X_{-13} & -X_{-23} & -X_{-33} & 0
-2X_{-13} & -2X_{-23} & 2X_{-33} & 0
-3X_{-12} & X_{-22} & X_{-23} & 0
X_{-12} & X_{-22} & -3X_{-23} & 0
0 & 0 & 4X_{-22} & 0
0 & 4X_{-13} & 0 & 0
2X_{-11} & -2X_{-12} & -2X_{-13} & 0
-X_{-11} & -X_{-12} & 3X_{-13} & 0
0 & 0 & -4X_{-12} & 0
0 & 4X_{-11} & 0 & 0
0 & 0 & 4X_{-11} & 0
\end{pmatrix},$$

$$C_{-33}^{(l,l,l-1)} = 24 \begin{pmatrix}
3X_{-33} & 0 & 0 & 0
-2X_{-23} & X_{-33} & 0 & 0
X_{-22} & -2X_{-23} & 0 & 0
0 & 3X_{-22} & 0 & 0
2X_{-13} & 0 & X_{-33} & 0
-X_{-12} & X_{-13} & -X_{-23} & 0
0 & -2X_{-12} & X_{-22} & 0
X_{-11} & 0 & 2X_{-13} & 0
0 & X_{-11} & -2X_{-12} & 0
0 & 0 & 3X_{-11} & 0
\end{pmatrix},$$

$$C_{-12}^{(l,l-1,l-2)} = \begin{pmatrix}
-X_{-12} & -X_{-13} & -X_{-22} & -X_{-23} & -2X_{-23} & -X_{-33} & 0 & 0
X_{-11} & 0 & X_{-12} & -X_{-13} & X_{-13} & 0 & -X_{-23} & -X_{-33}
0 & 0 & X_{-11} & 0 & 2X_{-12} & X_{-12} & X_{-13} & X_{-23}
\end{pmatrix},$$

$$C_{-13}^{(l+1,l,l)} = C_{-13}^{(l-1,l-2,l-2)} = 2 \begin{pmatrix}
-X_{-13} & -X_{-23} & -X_{-33} & X_{-12} & X_{-22} & X_{-23}
-X_{-11} & -X_{-12} & -X_{-13}
\end{pmatrix}. $$
the case of $(\sigma_1, \sigma_2, \sigma_3) = (1, 0, 0), (0, 1, 1)$.

\[ R(\Gamma_{+22}^{(l+1,l-2,-2)}) = 2 \begin{pmatrix} \nu_2 + l & \nu_1 + l - 2 & 0 \\ \nu_3 + l - 1 & 0 & \nu_1 + l - 2 \\ 0 & \nu_3 + l - 1 & -(\nu_2 + l - 1) \end{pmatrix}, \]

\[ R(\Gamma_{+33}^{(l+1,l,l-2)}) = \begin{pmatrix} \nu_3 + l - 1 & \nu_2 + l - 2 & \nu_1 + l - 4 & 0 \end{pmatrix}, \]

\[ R(\Gamma_{+33}^{(l,l,l-3)}) = \begin{pmatrix} \nu_3 + l - 1 & \nu_2 + l - 2 & \nu_1 + l - 4 \end{pmatrix}, \]

\[ R(\Gamma_{+23}^{(l+1,l-2,l-2)}) = 3 \begin{pmatrix} \nu_2 + l \\ \nu_3 + l - 1 \end{pmatrix}, \quad R(\Gamma_{+13}^{(l,l,l-1)}) = -2(\nu_1 + l), \]

\[ R(\Gamma_{+13}^{(l-2,l-2,l-3)}) = -2(\nu_1 + l - 2), \quad R(\Gamma_{+23}^{(l-1,l-1,l-2)}) = -\begin{pmatrix} \nu_3 + l - 1 & \nu_2 + l - 2 \end{pmatrix}, \]

\[ R(\Gamma_{+23}^{(l-2,l-2,l-3)}) = \begin{pmatrix} \nu_2 + l - 1 & \nu_2 + l - 2 & \nu_1 + l - 3 & 0 \\ 0 & -(\nu_3 + l - 1) & 0 & \nu_1 + l - 3 \end{pmatrix}, \]

\[ R(\Gamma_{-22}^{(l+1,l,l,-2)}) = 2 \begin{pmatrix} 3(\nu_2 - l) & 3(\nu_3 - l + 1) & 0 & 0 \\ \nu_1 - l + 2 & 0 & 3(\nu_3 - l + 1) & 2(\nu_3 - l + 1) \\ 0 & \nu_1 - l + 2 & \nu_2 - l + 4 & -2(\nu_2 - l) \end{pmatrix}, \]

\[ R(\Gamma_{-33}^{(l+1,l,l,-2)}) = 6 \begin{pmatrix} 4(\nu_3 - l + 1) \\ 4(\nu_2 - l + 2) \\ 2(\nu_1 - l + 4) \end{pmatrix}, \quad R(\Gamma_{-33}^{(l,l,l-1)}) = 24 \begin{pmatrix} \nu_1 - l + 1 & \nu_2 - l + 2 & 3(\nu_1 - l + 4) \end{pmatrix}, \]

\[ R(\Gamma_{-12}^{(l+1,l,l,-2)}) = -\begin{pmatrix} \nu_2 - l & \nu_3 - l + 1 \end{pmatrix}, \quad R(\Gamma_{-13}^{(l+1,l,l)}) = -2(\nu_1 - l), \]
\[ R(\Gamma_{-13}^{(l-1,l-2,l-2)}) = -2(\nu_1 - l + 2), \quad R(\Gamma_{-23}^{(l,l,l-1)}) = 3 \begin{pmatrix} \nu_3 - l + 1 \\ \nu_2 - l + 2 \end{pmatrix}, \]

\[
R(\Gamma_{-23}^{(l,l,l-1)}) = \begin{pmatrix} -2(\nu_2 - l) & -2(\nu_3 - l + 1) \\ -(\nu_2 - l + 4) & 3(\nu_3 - l + 1) \\ -8(\nu_1 - l + 3) & 0 \end{pmatrix},
\]

\[
R(\Gamma_{-23}^{(l-1,l-2,l-2)}) = \begin{pmatrix} -2(\nu_2 - l) & -2(\nu_3 - l + 1) \\ -(\nu_2 - l + 4) & 3(\nu_3 - l + 1) \\ -8(\nu_1 - l + 3) & 0 \end{pmatrix}.
\]

- the case of \((\sigma_1, \sigma_2, \sigma_3) = (0, 1, 0), (1, 0, 1)\).

\[
R(\Gamma_{+22}^{(l+1,l-2,-2)}) = 2 \begin{pmatrix} \nu_2 + l + 1 & \nu_1 + l - 1 & 0 \\ \nu_3 + l - 1 & 0 & 0 \\ 0 & \nu_1 + l - 1 & \nu_2 + l - 3 \end{pmatrix},
\]

\[
R(\Gamma_{+33}^{(l+1,l,l-2)}) = \begin{pmatrix} 0 & \nu_3 + l - 1 & \nu_2 + l - 3 & \nu_1 + l - 3 \\ \nu_3 + l - 1 & \nu_1 + l - 3 \\ \nu_3 + l - 1 & \nu_2 + l - 3 \end{pmatrix},
\]

\[
R(\Gamma_{+12}^{(l-1,l-2,-2)}) = \begin{pmatrix} -3(\nu_1 + l) & \nu_3 + l - 1 \\ 0 & -3(\nu_1 + l - 1) \end{pmatrix},
\]

\[
R(\Gamma_{+13}^{(l+1,l,l-1)}) = 2(\nu_2 + l),
\]

\[
R(\Gamma_{+13}^{(l-2,l-2,l-3)}) = 2(\nu_2 + l - 2), \quad R(\Gamma_{+23}^{(l,l,l-1)}) = 2(\nu_3 + l - 1),
\]

\[
R(\Gamma_{-22}^{(l+1,l,l-1)}) = 2 \begin{pmatrix} \nu_2 - l - 1 & 3(\nu_3 - l + 1) & \nu_3 - l + 1 & 0 \\ 0 & \nu_1 - l + 1 & 3(\nu_3 - l + 1) & (\nu_2 - l - 3) \end{pmatrix},
\]

\[
R(\Gamma_{-33}^{(l+1,l,l,l)}) = 6 \begin{pmatrix} 4(\nu_3 - l + 1) \\ \nu_2 - l \\ \nu_2 - l + 4 \\ 4(\nu_1 - l + 3) \end{pmatrix}, \quad R(\Gamma_{-33}^{(l,l,l,l)}) = 24 \begin{pmatrix} \nu_3 - l + 1 \\ 3(\nu_2 - l + 3) \\ \nu_1 - l + 5 \end{pmatrix}.
\]

- the case of \((\sigma_1, \sigma_2, \sigma_3) = (0, 0, 1), (1, 1, 0)\).

\[
R(\Gamma_{+22}^{(l+1,l-2,-2)}) = 2 \begin{pmatrix} -3(\nu_1 + l - 1) & 0 \\ \nu_2 + l & \nu_1 + l - 1 \\ \nu_3 + l & 0 \\ 0 & \nu_3 + l \end{pmatrix},
\]

\[
R(\Gamma_{+33}^{(l+1,l,l-2)}) = \begin{pmatrix} 0 & \nu_3 + l & \nu_2 + l - 2 & \nu_1 + l - 3 \end{pmatrix},
\]

\[
R(\Gamma_{+33}^{(l,l,l-3)}) = \begin{pmatrix} \nu_3 + l - 2 & \nu_2 + l - 4 & \nu_1 + l - 5 \end{pmatrix},
\]

\[
R(\Gamma_{+12}^{(l-1,l-2,l-2)}) = -3 \begin{pmatrix} \nu_1 + l \\ \nu_2 + l - 1 \end{pmatrix}, \quad R(\Gamma_{+13}^{(l,l,l,l)}) = -2(\nu_3 + l),
\]

\[
R(\Gamma_{+13}^{(l-2,l-2,l-3)}) = -2(\nu_3 + l - 2), \quad R(\Gamma_{+23}^{(l,l-1,l,l)}) = \begin{pmatrix} \nu_2 + l - 1 & \nu_1 + l - 2 \end{pmatrix}.
\]
(ii) The elements $C_{2i}$ $(i = 1, 2, 3)$ are represented by the $p\pm$-matrices $C_{\pm kl}^\lambda$ as follows:

\[
\begin{pmatrix}
C_2 & 0 & 0 \\
0 & C_2 & 0 \\
0 & 0 & C_2
\end{pmatrix} = \frac{1}{24} \left\{ C_{(l+1,l-2)}^{(l+1,l,l)} \cdot C_{-33}^{(l+1,l,l)} + 3 C_{+13}^{(l+l,l-1)} \cdot C_{-13}^{(l+l,l,l)} \right\},
\]

\[
= \frac{1}{72} \left\{ C_{+33}^{(l+l,l-3)} \cdot C_{-33}^{(l,l,l-1)} - 16 C_{+23}^{(l+l,l-1,l-2)} \cdot C_{-23}^{(l+l,l,l-1)} \right\},
\]

\[
\begin{pmatrix}
C_4 & 0 & 0 \\
0 & C_4 & 0 \\
0 & 0 & C_4
\end{pmatrix} = \frac{1}{1152} \left\{ C_{+33}^{(l+l,l-2)} \cdot C_{+22}^{(l+l,l-2,2)} \cdot C_{-22}^{(l+l,l-2,2)} \cdot C_{-33}^{(l+l,l,l)} \\
-64 C_{+13}^{(l+l,l-1)} \cdot C_{+23}^{(l+l,l-1,-2)} \cdot C_{-23}^{(l+l,l-1,-2)} \cdot C_{-13}^{(l+l,l,l)} \right\},
\]

\[
= \frac{1}{72} \left\{ C_{+33}^{(l+l,l-2)} \cdot C_{+23}^{(l+l,l-2,2)} \cdot C_{-23}^{(l+l,l-2,2)} \cdot C_{-33}^{(l+l,l,l)} + 3 C_{+23}^{(l+l,l-2)} \cdot C_{+12}^{(l+l,l-2,3)} \cdot C_{-12}^{(l+l,l-2,3)} \cdot C_{-23}^{(l+l,l,l)} \right\},
\]

\[
\begin{pmatrix}
C_6 & 0 & 0 \\
0 & C_6 & 0 \\
0 & 0 & C_6
\end{pmatrix} = \frac{1}{144} \left\{ C_{+13}^{(l+l,l,l-2)} \cdot C_{+23}^{(l+l,l,l-2)} \cdot C_{+12}^{(l+l,l,l-2)} \cdot C_{-12}^{(l+l,l,l-2)} \cdot C_{-23}^{(l+l,l,l)} \cdot C_{-33}^{(l+l,l,l)} \right\},
\]

\[
= \frac{1}{144} \left\{ C_{+13}^{(l+l,l,l-2)} \cdot C_{+23}^{(l+l,l,l-2)} \cdot C_{+12}^{(l+l,l,l-2)} \cdot C_{-12}^{(l+l,l,l-2)} \cdot C_{-23}^{(l+l,l,l)} \cdot C_{-33}^{(l+l,l,l)} \right\} \times C_{+13}^{(l+l,l-2,2)} \cdot C_{+23}^{(l+l,l-2,2)} \cdot C_{-23}^{(l+l,l-2,2)} \cdot C_{-13}^{(l+l,l,l)} \cdot C_{-23}^{(l+l,l,l)},
\]

Proof. From Theorem 5.6 in the case of $(\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1)$, we obtain the assertion by direct computation.\]
Then \( G((l+1,l,l)) = \{ M_{l,j}^{(1)} \mid j = 1, 2, 3 \} \) and \( G((l,l,l-1)) = \{ M_{l,j}^{(2)} \mid j = 1, 2, 3 \} \). For \( l \equiv \varepsilon_\sigma \) mod 2, let \( M_{l}^{(1)} \) and \( M_{l}^{(2)} \) be the unique element of \( G_\sigma((l+1,l,l)) \) and \( G_\sigma((l,l,l-1)) \), respectively.

**Proposition 6.7.** Let \( \sigma \) be a character of \( M_{\min} \) such that \( (\sigma_1,\sigma_2,\sigma_3) \) is neither \((0,0,0)\) nor \((1,1,1)\), and \( T \) be an element of the space \( T_{\xi,\pi_{(\varepsilon_\sigma)}} \).

(i) For \( l \equiv \varepsilon_\sigma \) mod 2, we define functions \( \Phi_{T,l;i,j}^{(1)} \in C^\infty_\xi(N_{\min}\backslash G) \) \( (j = 1,2,3) \) by the equation

\[
\Phi(T,S_{(l+1,l,l)}(M_{l,j}^{(1)})) = \sum_{j=1}^{3} \Phi_{T,l;j}^{(1)} \otimes f(M_{l,j}^{(1)})^*.
\]

Then \( \Phi_{T,l;1}^{(1)} \), \( \Phi_{T,l;2}^{(1)} \) and \( \Phi_{T,l;3}^{(1)} \) satisfy following differential equations:

\[
\begin{align*}
C_{2i} \Phi_{T,l;3}^{(1)} &= \chi_{2i,\nu,(l+1,l,l)} \Phi_{T,l;1}^{(1)}, \\
D_{i1}^{(-,+)} \Phi_{T,l;1}^{(1)} + D_{i2}^{(-,+)} \Phi_{T,l;2}^{(1)} + D_{i3}^{(-,+)} \Phi_{T,l;3}^{(1)} &= \tilde{x}_{2i,\nu,(l+1,l,l)} \Phi_{T,l;i}^{(1)},
\end{align*}
\]

for \( i, j = 1, 2, 3 \). Here

\[
\begin{pmatrix}
D_{11}^{(-,+)} & D_{12}^{(-,+)} & D_{13}^{(-,+)} \\
D_{21}^{(-,+)} & D_{22}^{(-,+)} & D_{23}^{(-,+)} \\
D_{31}^{(-,+)} & D_{32}^{(-,+)} & D_{33}^{(-,+)}
\end{pmatrix} = m_1(C_+) m_1(C_-),
\]

\[
\chi_{2i,\nu,(l+1,l,l)} = \{ \nu_1^2 - (l-3)^2 \} + \{ \nu_2^2 - (l-2)^2 \} + \{ \nu_3^2 - (l-1)^2 \} - 2l + 1,
\]

\[
\chi_{4,\nu,(l+1,l,l)} = \{ \nu_1^2 - (l-2)^2 \} + \{ \nu_2^2 - (l-1)^2 \} \{ \nu_3^2 - (l-2)^2 \} - 2l - 1,
\]

\[
\chi_{6,\nu,(l+1,l,l)} = \{ \nu_1^2 - (l-1)^2 \} \{ \nu_2^2 - (l-1)^2 \} \{ \nu_3^2 - (l-1)^2 \} - 2l - 1.
\]

(ii) For \( l \equiv \varepsilon_\sigma \) mod 2, we define functions \( \Phi_{T,l;i,j}^{(2)} \in C^\infty_\xi(N_{\min}\backslash G) \) \( (j = 1,2,3) \) by the equation

\[
\Phi(T,S_{(l,l,l-1)}(M_{l,j}^{(2)})) = \sum_{j=1}^{3} \Phi_{T,l;j}^{(2)} \otimes f(M_{l,j}^{(2)})^*.
\]

Then \( \Phi_{T,l;1}^{(2)} \), \( \Phi_{T,l;2}^{(2)} \) and \( \Phi_{T,l;3}^{(2)} \) satisfy following differential equations:

\[
\begin{align*}
C_{2i} \Phi_{T,l;3}^{(2)} &= \chi_{2i,\nu,(l+1,l,l-1)} \Phi_{T,l;1}^{(2)}, \\
-D_{i1}^{(-,+)} \Phi_{T,l;1}^{(2)} + D_{i2}^{(-,+)} \Phi_{T,l;2}^{(2)} - D_{i3}^{(-,+)} \Phi_{T,l;3}^{(2)} &= (-1)^i \tilde{x}_{2i,\nu,(l+1,l,l-1)} \Phi_{T,l;4-i}^{(2)},
\end{align*}
\]

for \( i, j = 1, 2, 3 \). Here

\[
\begin{pmatrix}
D_{11}^{(-,+)} & D_{12}^{(-,+)} & D_{13}^{(-,+)} \\
D_{21}^{(-,+)} & D_{22}^{(-,+)} & D_{23}^{(-,+)} \\
D_{31}^{(-,+)} & D_{32}^{(-,+)} & D_{33}^{(-,+)}
\end{pmatrix} = m_1(C_+) m_1(C_-),
\]

\[
\chi_{2,\nu,(l,l,l-1)} = \{ \nu_1^2 - (l-3)^2 \} + \{ \nu_2^2 - (l-2)^2 \} + \{ \nu_3^2 - (l-1)^2 \} + 2l - 7,
\]

\[
\chi_{4,\nu,(l,l,l-1)} = \{ \nu_1^2 - (l-2)^2 \} + \{ \nu_2^2 - (l-1)^2 \} \{ \nu_3^2 - (l-2)^2 \} - 2l - 5,
\]

\[
\chi_{6,\nu,(l,l,l-1)} = \{ \nu_1^2 - (l-1)^2 \} \{ \nu_2^2 - (l-1)^2 \} \{ \nu_3^2 - (l-1)^2 \} - 2l - 5.
\]
Proof. From the Lemma 6.6 and the equations
\[ m_1(C_+)m_1(C_-) = \frac{1}{4} c^{(l,l,l-1)}_{+13} c^{(l,l,l)}_{-13}, \]
\[ m_1(C_-)m_1(C_+) \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) = \frac{1}{4} \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) c^{(l,l,l)}_{-13} c^{(l,l,l-1)}_{+13}, \]
\[ S_{(l+1,l,l)}(M_{1}(\sigma;1)) = \begin{pmatrix} s(M_{1}(\sigma;1), M_{1}(\sigma;1)) \\ s(M_{1}(\sigma;1), M_{1}(\sigma;1)) \\ s(M_{1}(\sigma;1), M_{1}(\sigma;1)) \end{pmatrix}, \]
\[ S_{(l,l,l-1)}(M_{1}(\sigma;2)) = \begin{pmatrix} s(M_{1}(\sigma;2), M_{1}(\sigma;2)) \\ s(M_{1}(\sigma;2), M_{1}(\sigma;2)) \\ s(M_{1}(\sigma;2), M_{1}(\sigma;2)) \end{pmatrix}, \]
we obtain
\[ C_{2i} \phi_{T,i,j}^{(1)} = \chi_{2,i,\sigma,\nu,(l+1,l,l)} \phi_{T,i,j}^{(1)} \]
\[ C_{2j} \phi_{T,i,j}^{(2)} = \chi_{2,j,\sigma,\nu,(l+1,l,l)} \phi_{T,i,j}^{(2)} \]
\[ (D_{11}^{(+,-)}, D_{12}^{(+,-)}, D_{13}^{(+,-)}) = \begin{pmatrix} \sigma(M_{1}(\sigma;1), M_{1}(\sigma;1)) \\ \sigma(M_{1}(\sigma;1), M_{1}(\sigma;1)) \\ \sigma(M_{1}(\sigma;1), M_{1}(\sigma;1)) \end{pmatrix} = \tilde{\chi}_{2,\sigma,\nu,(l+1,l,l)} \begin{pmatrix} s(M_{1}(\sigma;2), M_{2}(\sigma;2)) \\ s(M_{1}(\sigma;2), M_{2}(\sigma;2)) \\ s(M_{1}(\sigma;2), M_{2}(\sigma;2)) \end{pmatrix}. \]
From these equations, we obtain the assertion. \( \square \)

Remark 6.8. Since \( V_{(l+1,l,l)} \) and \( V_{(l,l,l-1)} \) are three dimensional, the differential equations obtained from \( C_{2i} (i = 1, 2, 3) \) do not suffice to characterize the Whittaker functions.

6.5. Differential equations. To obtain the explicit actions of the operators \( C_{2i}, D_{jk}^{(+,-)} (1 \leq i, j, k \leq 3) \), we may express these operators in the normal order modulo \([n, n]\) with respect to the Iwasawa decomposition of \( g \), according to the following lemma.

Lemma 6.9. (i) Let \( \phi_{T,i,j}^{(1)}, \phi_{T,i,j}^{(2)} \) be the elements of \( C_{\xi}^{\infty}(N_{\min}\backslash G) \) defined in Proposition 6.5 and 6.7. The explicit expressions of the action of \( t_{C} \) on these functions given as follows.

- the case of \( (\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1) \).
  \[ \kappa(E_{ii}) \phi_{T,i,j}^{(1)} = l \phi_{T,i,j}^{(1)} \quad (1 \leq i \leq 3), \]
  \[ \kappa(E_{jj}) \phi_{T,i,j}^{(1)} = 0 \quad (1 \leq j \neq k \leq 3). \]

- the case of \( (\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1) \).
  \[ \kappa(E_{ij}) \phi_{T,i,j}^{(1)} = (l + \delta_{ij}) \phi_{T,i,j}^{(1)} \quad (1 \leq i, j \leq 3), \]
  \[ \kappa(E_{mn}) \phi_{T,i,j}^{(1)} = \delta_{nk} \phi_{T,i,m}^{(1)} \quad (1 \leq k \leq 3, 1 \leq m \neq n \leq 3), \]

and

\[ \kappa(E_{ij}) \phi_{T,i,j}^{(2)} = (l - \delta_{4-i,j}) \phi_{T,i,j}^{(2)} \quad (1 \leq i, j \leq 3), \]
\[ \kappa(E_{mn}) \phi_{T,i,j}^{(2)} = (-1)^{m+n+1} \delta_{4-m,n} \phi_{T,i,m}^{(2)} \quad (1 \leq k \leq 3, 1 \leq m \neq n \leq 3). \]

(ii) Let \( \phi \in C_{\xi}^{\infty}(N_{\min}\backslash G) \). For \( X \in U(n_{C}), Y \in U(a_{C}) \) and \( a \in A_{\min} \), we have the equation
\[ (A_{1}^{-1}a_{1}X)Y \phi(a) = \xi(X)(Y \phi)(a). \]
In particular, for \( a = \text{diag}(a_{1}, a_{2}, a_{3}, a_{1}^{-1}, a_{2}^{-1}, a_{3}^{-1}) \) \( a \in A_{\min} \), we have
\[ H_{i} \phi(a) = a_{1} \frac{\partial}{\partial a_{i}} \phi(a) \quad (1 \leq i \leq 3), \]
\[ E_{e_{1}-e_{2}} \phi(a) = 2\pi \sqrt{-1} a_{12} \frac{a_{1}}{a_{2}} \phi(a), \]
\[ E_{e_2-e_3}(a) = 2\pi \sqrt{-1}c_{23}^{a_2/a_3} \phi(a), \quad E_{2e_3}(a) = 2\pi \sqrt{-1}c_{3}^{a_3^2} \phi(a), \]

and \( E_\alpha \phi(a) = 0 \) for \( \alpha \in \Sigma^\dagger \{ e_1 - e_2, e_2 - e_3, 2e_3 \} \).

**Proof.** From the definition of \( \phi_{T,l} \), \( \phi_{T,l}^{(1)} \), \( \phi_{T,l}^{(2)} \) \( (i = 1, 2, 3) \) and Lemma 3.4, we obtain the statement (i). The statement (ii) is obvious from the definition of \( C_\xi(T, N_{\text{min}} \mid G) \). \( \square \)

Moreover, we have the following lemma which is required to get the expressions of the elements in \( U(\mathfrak{g}_C) \) in normal order. In the following, we denote \( X \equiv Y \) for two elements \( X \) and \( Y \) in \( U(\mathfrak{g}_C) \) when \( X - Y \in [n, n]U(\mathfrak{g}_C) \).

**Lemma 6.10.** (i) The root vectors \( X_{\pm ij} \) in \( \mathfrak{p}_\pm \) have the following expressions:

\[
X_{+ij} = \begin{cases} 
H_i + \kappa(E_{ii}) & (i = j = 1, 2), \\
E_{e_i-e_j} + \kappa(E_{ji}) & (i, j) = (1, 2), (2, 3),
\end{cases}
\]

\[
X_{-ij} = \begin{cases} 
H_i - \kappa(E_{ii}) & (i = j = 1, 2), \\
E_{e_i-e_j} - \kappa(E_{ij}) & (i, j) = (1, 2), (2, 3),
\end{cases}
\]

and \( X_{\pm 33} \equiv \pm 2\sqrt{-1}E_{2e_3} + H_3 \pm \kappa(E_{33}) \), \( X_{+13} \equiv \kappa(E_{31}) \), \( X_{-13} \equiv -\kappa(E_{13}) \).

(ii) Each \((i, j)\)-minor \( M_{+ij} \) in the matrix \( m_2(C_+) \) has the following expression.

\[
M_{+11} \equiv (H_2 - 1)X_{+33} + X_{+33}X_{-23} - E_{e_2-e_3}X_{+23} - X_{+23}\kappa(E_{32}) - X_{+13}\kappa(E_{31}),
\]

\[
M_{+22} \equiv (H_1 - 1)X_{+33} + X_{+33}X_{-23} - E_{e_2-e_3}X_{+23} - X_{+13}\kappa(E_{31}),
\]

\[
M_{+33} \equiv (H_1 - 1)X_{+23} + X_{+23}X_{-23} - E_{e_2-e_3}X_{+23} - X_{+23}\kappa(E_{31}),
\]

\[
M_{+12} \equiv E_{e_1-e_2}X_{+33} + X_{+33}\kappa(E_{21}) - X_{+23}\kappa(E_{31}),
\]

\[
M_{+23} \equiv (H_1 - 1)X_{+23} + X_{+23}X_{-23} - E_{e_2-e_3}X_{+23} - X_{+23}\kappa(E_{31}),
\]

\[
M_{+13} \equiv E_{e_1-e_2}X_{+23} + X_{+23}\kappa(E_{21}) - X_{+23}\kappa(E_{31}).
\]

(iii) Each \((i, j)\)-minor \( M_{-ij} \) in the matrix \( m_2(C_-) \) has the following expression.

\[
M_{-11} \equiv (H_2 - 1)X_{-33} - X_{-33}\kappa(E_{22}) - E_{e_2-e_3}X_{-23} + X_{-23}\kappa(E_{23}),
\]

\[
M_{-22} \equiv (H_1 - 1)X_{-33} - X_{-33}\kappa(E_{22}) + X_{-23}\kappa(E_{13}),
\]

\[
M_{-33} \equiv (H_1 - 1)X_{-23} - X_{-23}\kappa(E_{22}) - E_{e_2-e_3}X_{-23} + X_{-23}\kappa(E_{13}),
\]

\[
M_{-12} \equiv E_{e_1-e_2}X_{-33} - X_{-33}\kappa(E_{12}) + X_{-23}\kappa(E_{13}),
\]

\[
M_{-23} \equiv (H_1 - 1)X_{-23} - X_{-23}\kappa(E_{12}) + X_{-23}\kappa(E_{13}),
\]

\[
M_{-13} \equiv E_{e_1-e_2}X_{-23} - X_{-23}\kappa(E_{12}) + X_{-23}\kappa(E_{13}).
\]

**Proof.** The statement (i) is obvious from Lemma 3.4. The statements (ii), (iii) are obtained by direct computation using tables in the proof of Lemma 3.4. \( \square \)

By using above lemma and tables in the proof of Lemma 3.4, we have the following expressions of the elements \( D_{ij}^{(\pm, \pm)} \) \( (1 \leq i, j \leq 3) \), \( C_{2k} \) \( (1 \leq k \leq 2) \), \( m_3(C_\pm) \) in normal order:

For \( 1 \leq i \leq 3 \), we have

\[
D_{1i}^{(\pm, \pm)} \equiv (H_1 - 4)X_{+1i} - X_{+1i}\kappa(E_{11}) + E_{e_1-e_2}X_{+2i} + X_{-2i}\kappa(E_{21}) + X_{-3i}\kappa(E_{31}),
\]

\[
D_{2i}^{(\pm, \pm)} \equiv E_{e_1-e_2}X_{-1i} + X_{-1i}\kappa(E_{21}) + (H_2 - 3 + \delta_{1i})X_{-2i} + X_{-2i}\kappa(E_{22}) + E_{e_2-e_3}X_{-3i} + X_{-3i}\kappa(E_{32}) - \delta_{2i}X_{11},
\]

\[
D_{3i}^{(\pm, \pm)} \equiv -X_{+1i}\kappa(E_{31}) + E_{e_2-e_3}X_{-2i} + X_{-2i}\kappa(E_{32})
\]

\[
+ (H_3 - 1 - \delta_{3i} + 2\sqrt{-1}E_{2e_3})X_{-3i} + X_{-3i}\kappa(E_{33}) - \delta_{3i}(X_{-11} + X_{-22}),
\]

and

\[
D_{1i}^{(-, \pm)} \equiv (H_1 - 4)X_{+1i} - X_{+1i}\kappa(E_{11}) + E_{e_1-e_2}X_{+2i} - X_{+2i}\kappa(E_{12}) - X_{+3i}\kappa(E_{13}),
\]
\[ D_{2i}^{(+,+)} \equiv \begin{array}{l} E_{e_1-e_2}X_{+1i} - X_{+1i}\kappa(E_{12}) + (H_2 - 3 + \delta_{11})X_{+2i} - X_{+2i}\kappa(E_{22}) \\ + E_{e_2-e_3}X_{+3i} - X_{+3i}\kappa(E_{23}) - \delta_{2i}X_{+1i}, \end{array} \]

\[ D_{3i}^{(+,+)} \equiv -X_{+1i}\kappa(E_{13}) + E_{e_2-e_3}X_{+2i} - X_{+2i}\kappa(E_{23}) \\ + (H_3 - 1 - \delta_{3i} - 2\sqrt{-1}E_{2e_3}X_{+3i} - X_{+3i}\kappa(E_{33}) - \delta_{3i}(X_{+1i} + X_{+2i}). \]

Here we denote \( X_{\pm ji} = X_{\pm ij} \) \((1 \leq i < j \leq 3)\).

\[ C_2 \equiv (H_1 - 6)X_{-11} + X_{-11}\kappa(E_{11}) + (H_2 - 4)X_{-22} + X_{-22}\kappa(E_{22}) \\ + (H_3 + 2\sqrt{-1}E_{2e_3} - 2)X_{-33} + X_{-33}\kappa(E_{33}) + 2E_{e_1-e_2}X_{-12} \\ + 2X_{-12}\kappa(E_{21}) + 2E_{e_2-e_3}X_{-23} + 2X_{-23}\kappa(E_{32}) + 2X_{-13}\kappa(E_{31}). \]

\[ C_4 \equiv (H_2 - 1)\{ (2\sqrt{-1}E_{2e_3} + H_3)M_{-11} + M_{-11}\kappa(E_{33} - 2) \} \\ + \{(2\sqrt{-1}E_{2e_3} + H_3)M_{-11} + M_{-11}\kappa(E_{33} - 2) \} \kappa(E_{22} - 2) \\ - E_{e_1-e_2}^2M_{-33} - 2E_{e_1-e_2}M_{-33}\kappa(E_{21}) - M_{-33}\kappa(E_{21})^2 \\ + 2E_{e_1-e_2}\{ (2\sqrt{-1}E_{2e_3} + H_3)M_{-12} + M_{-12}\kappa(E_{33} - 2) \} \kappa(E_{21}) \\ - 2\{ (2\sqrt{-1}E_{2e_3} + H_3)M_{-22} + M_{-22}\kappa(E_{33} - 2) \} - 2E_{e_2-e_3}M_{-23} \\ - 2(E_{e_2-e_3}M_{-12} + M_{-12}\kappa(E_{32}) - M_{-13})\kappa(E_{31}) + 2M_{-33} - 2M_{-23}\kappa(E_{32}) \\ + 2(H_1 - 1)(E_{e_2-e_3}M_{-23} + M_{-23}\kappa(E_{32}) - M_{-33}) \\ + 2(E_{e_2-e_3}M_{-23} + M_{-23}\kappa(E_{32}) - M_{-33})(\kappa(E_{11}) - 2) \\ - 2(E_{e_1-e_2}M_{-23} + M_{-23}\kappa(E_{21})\kappa(E_{31}) \\ + 2E_{e_1-e_2}(E_{e_2-e_3}M_{-13} + M_{-13}\kappa(E_{32})) - 2E_{e_2-e_3}M_{-23} \\ - 2(H_2 - 3)M_{-33} - 2M_{-33}\kappa(E_{22}) - 2M_{-23}\kappa(E_{32}) \\ + 2(E_{e_2-e_3}M_{-13} + M_{-13}\kappa(E_{32}))\kappa(E_{21}) \\ - 2\{ (H_2 - 2)M_{-13} + M_{-13}\kappa(E_{22}) \} \kappa(E_{31}), \]

\[ m_3(C_+) \equiv (H_1 - 2)M_{+11} + M_{+11}\kappa(E_{11}) - E_{e_1-e_2}M_{+12} - M_{+12}\kappa(E_{21}) + M_{+13}\kappa(E_{31}), \]

\[ m_3(C_-) \equiv (H_1 - 2)M_{-11} - M_{-11}\kappa(E_{11}) - E_{e_1-e_2}M_{-12} + M_{-12}\kappa(E_{12}) - M_{-13}\kappa(E_{13}). \]

From above expressions and Lemma 6.9(i), We can summarize the explicit actions of the operators \( C_{2i} \) \((1 \leq i \leq 3)\) and \( D_{jk}^{(\pm, \pm)} \) \((1 \leq j, k \leq 3)\) on the functions in Proposition 6.3, 6.7.

**Proposition 6.11.** The operators \( C_{2i} \) \((1 \leq i \leq 3)\) and \( D_{jk}^{(\pm, \pm)} \) \((1 \leq j, k \leq 3)\) acting on the functions \( \phi_{T, i} \) \((1 \leq i \leq 3)\) in Proposition 6.3, 6.7 are given as follows.

- **the case of** \( \{ \sigma_1, \sigma_2, \sigma_3 \} = (0, 0, 0), (1, 1, 1). \)

\[ C_{2\phi_{T, i}} = \{(H_1 + l - 6)(H_1 - l) + (H_2 + l - 4)(H_2 - l) \\ + (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3}) + 2E_{e_1-e_2}^2 + 2E_{e_2-e_3}^2 \} \phi_{T, i}. \]
\[ C_4 \phi_{T,l} = \left\{ (H_2 + l - 3)(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3}) - E_{e_2-e_3}^2 \right\} \]
\[ \times \left\{ (H_2 - l - 1)(H_3 - l - 2\sqrt{-1}E_{2e_3}) - E_{e_2-e_3}^2 \right\} \]
\[ + (H_1 + l - 5)(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_1 - l - 1)(H_3 - l - 2\sqrt{-1}E_{2e_3}) \]
\[ + \{(H_1 + l - 5)(H_2 + l - 4) - E_{e_1-e_2}^2\}\{(H_1 - l - 1)(H_2 - l) - E_{e_1-e_2}^2\} \]
\[ + 2E_{e_1-e_2}^2(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3}) \]
\[ + 2E_{e_1-e_2}^2E_{e_2-e_3}^2(H_1 + l - 5)(H_1 - l - 1)\phi_{T,l} \]

\[ C_6 \phi_{T,l} = \left\{ (H_1 + l - 4)(H_2 + l - 3)(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3}) \right\} \]
\[ - E_{e_2-e_3}^2(H_1 + l - 4) - E_{e_1-e_2}^2(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3}) \]
\[ \times \left\{ (H_1 - l - 2)(H_2 - l - 1)(H_3 - l - 2\sqrt{-1}E_{2e_3}) \right\} \]
\[ - E_{e_2-e_3}^2(H_1 - l - 2) - E_{e_1-e_2}^2(H_3 - l - 2\sqrt{-1}E_{2e_3}) \}\phi_{T,l}. \]

- the case of \((\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1)\),

\[ D^{(+-)}_{11}(\phi_{T,l})_{T,l} + D^{(+-)}_{12}(\phi_{T,l})_{T,l} + D^{(+-)}_{13}(\phi_{T,l})_{T,l} \]
\[ = \{(H_1 + l - 3)(H_1 - l - 3) + E_{e_1-e_2}^2\}\phi_{T,l}^{(1)} + E_{e_1-e_2}(H_1 + H_2 - 4)\phi_{T,l}^{(1)} \]
\[ + E_{e_1-e_2}E_{e_2-e_3}\phi_{T,l}^{(1)} \]

\[ D^{(+-)}_{21}(\phi_{T,l})_{T,l} + D^{(+-)}_{22}(\phi_{T,l})_{T,l} + D^{(+-)}_{23}(\phi_{T,l})_{T,l} \]
\[ = E_{e_1-e_2}(H_1 + H_2 - 6)\phi_{T,l}^{(1)} + \{(H_2 + l - 2)(H_2 - l - 2) + E_{e_1-e_2}^2 + E_{e_2-e_3}^2\} \phi_{T,l}^{(1)} \]
\[ + E_{e_2-e_3}(H_2 + H_3 - 2 + 2\sqrt{-1}E_{2e_3})\phi_{T,l}^{(1)} \]

\[ D^{(+-)}_{31}(\phi_{T,l})_{T,l} + D^{(+-)}_{32}(\phi_{T,l})_{T,l} + D^{(+-)}_{33}(\phi_{T,l})_{T,l} \]
\[ = + E_{e_1-e_2}E_{e_2-e_3}\phi_{T,l}^{(1)} + E_{e_2-e_3}(H_2 + H_3 - 4 + 2\sqrt{-1}E_{2e_3})\phi_{T,l}^{(1)} \]
\[ + \{(H_3 + l - 1 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 1 - 2\sqrt{-1}E_{2e_1}) + E_{e_2-e_3}^2\} \phi_{T,l}^{(1)} \]

\[ C_2^{(1)} \phi_{T,l}^{(1)} = \left\{ (H_1 + l - 6 + \delta_{11})(H_1 - l - \delta_{11}) + (H_2 + l - 4 + \delta_{21})(H_2 - l - \delta_{21}) \right\} \]
\[ + (H_3 + l - 2 + \delta_{31} + 2\sqrt{-1}E_{2e_3})(H_3 - l - \delta_{31} - 2\sqrt{-1}E_{2e_3}) \]
\[ + 2E_{e_1-e_2}^2 + E_{e_2-e_3}^2 \phi_{T,l}^{(1)} \]
\[ - 2\delta_{11}\{2\phi_{T,l}^{(1)} - E_{e_1-e_2}\phi_{T,l}^{(1)}\} - 2\delta_{21}\{E_{e_1-e_2}\phi_{T,l}^{(1)} + \phi_{T,l}^{(1)} - E_{e_2-e_3}\phi_{T,l}^{(1)}\} \]
\[ - 2\delta_{31}E_{e_2-e_3}\phi_{T,l}^{(1)} \]

\[ C_4^{(1)} \phi_{T,l}^{(1)} = \left\{ (H_2 + l - 3 + \delta_{21})(H_3 + l - 2 + \delta_{31} + 2\sqrt{-1}E_{2e_3}) - E_{e_2-e_3}^2 \right\} \]
\[ \times \left\{ (H_2 - l - 1 - \delta_{21})(H_3 - l - \delta_{31} - 2\sqrt{-1}E_{2e_3}) - E_{e_2-e_3}^2 \right\} \]
\[ + (H_1 + l - 5 + \delta_{11})(H_3 + l - 2 + \delta_{31} + 2\sqrt{-1}E_{2e_3}) \]
\[ \times \left\{ (H_1 - l - 1 - \delta_{11})(H_3 - l - \delta_{31} - 2\sqrt{-1}E_{2e_3}) \right\} \]
\[ + \{(H_1 + l - 5 + \delta_{11})(H_2 + l - 4 + \delta_{21}) - E_{e_1-e_2}^2 \}
\[ \times \{(H_1 - l - 1 - \delta_{11})(H_2 - l - \delta_{21}) - E_{e_1-e_2}^2 \} \]
\[ + 2E_{e_1-e_2}^2(H_3 + l - 2 + \delta_{31} + 2\sqrt{-1}E_{2e_3})(H_3 - l - \delta_{31} - 2\sqrt{-1}E_{2e_3}) \]
\[ + 2E_{e_1-e_2}^2E_{e_2-e_3}^2 + 2E_{e_2-e_3}^2(H_1 + l - 5 + \delta_{11})(H_1 - l - 1 - \delta_{11}) \}\phi_{T,l}^{(1)} \]
\[ + 2\delta_{11}\left\{ - \{(H_2 + l - 4)(H_2 - l)ight.
\]
\[ + (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3}) + 3E_{e_1-e_2}^2 + 2E_{e_2-e_3}^2\right\}\phi_{T,l;1}^{(1)}
\]
\[ + E_{e_1-e_2}\{(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3})
\]
\[ - (H_1 - l - 1)(H_2 - l - 1) + H_1 + H_2 - 6 + E_{e_1-e_2}^2 + E_{e_2-e_3}^2\}\phi_{T,l;2}^{(1)}
\]
\[ - E_{e_1-e_2}E_{e_2-e_3}(H_1 + H_2 + H_3 - l - 6 - 2\sqrt{-1}E_{2e_3})\phi_{T,l;3}^{(1)}\right\}\]
\[ + 2\delta_{21}\left\{ - E_{e_1-e_2}\{(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3})
\]
\[ - (H_1 + l - 4)(H_2 + l - 4) - (H_1 + H_2 - 4) + E_{e_1-e_2}^2 + E_{e_2-e_3}^2\}\phi_{T,l;1}^{(1)}
\]
\[ - \{(H_1 + l - 5)(H_1 - l - 1) + E_{e_1-e_2}^2 + 2E_{e_2-e_3}^2\}\phi_{T,l;2}^{(1)}
\]
\[ + E_{e_2-e_3}\{(H_1 + l - 5)(H_1 - l - 1)
\]
\[ - (H_2 - l - 1)(H_3 - l - 1 - 2\sqrt{-1}E_{2e_3}) + E_{e_1-e_2}^2 + E_{e_2-e_3}^2\}\phi_{T,l;3}^{(1)}\right\}\]
\[ + 2\delta_{31}\left\{ E_{e_1-e_2}E_{e_2-e_3}(H_1 + H_2 + H_3 + l - 7 + 2\sqrt{-1}E_{2e_3})\phi_{T,l;1}^{(1)}
\]
\[ + E_{e_2-e_3}\{(H_2 + l - 2)(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})
\]
\[ - (H_1 + l - 5)(H_1 - l - 1) - E_{e_1-e_2} - E_{e_2-e_3}^2\}\phi_{T,l;2}^{(1)}\right\}\]
\[ + C_0\phi_{T,l;i}^{(1)} = \{(H_1 + l - 4 + \delta_{11})(H_2 + l - 3 + \delta_{21})(H_3 + l - 2 + \delta_{3i} + 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_1 + l - 4 + \delta_{11}) - E_{e_1-e_2}^2(H_3 + l - 2 + \delta_{3i} + 2\sqrt{-1}E_{2e_3})\}
\[ \times \{(H_1 - l - 2 - \delta_{1i})(H_2 - l - 1 - \delta_{2i})(H_3 - l - \delta_{3i} - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_1 - l - 2 - \delta_{1i}) - E_{e_1-e_2}^2(H_3 - l - \delta_{3i} - 2\sqrt{-1}E_{2e_3})\}\phi_{T,l;i}^{(1)}
\]
\[ + 2\delta_{1i}\left\{ - 2E_{e_1-e_2}^2((H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3}) + E_{e_2-e_3}^2\}\phi_{T,l;1}^{(1)}
\]
\[ - E_{e_1-e_2}((H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_1 - l - 2)(H_2 - l - 2)(H_3 - l - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_1 - l - 2)
\]
\[ - E_{e_2-e_3}^2(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3})\}\phi_{T,l;2}^{(1)}
\]
\[ + E_{e_1-e_2}E_{e_2-e_3}\{(H_1 - l - 2)(H_2 - l - 1)(H_3 - l - 1 - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_1 - l - 2) - E_{e_1-e_2}^2(H_3 - l - 1 - 2\sqrt{-1}E_{2e_3})\}\phi_{T,l;3}^{(1)}\right\}\]
\[ + 2\delta_{2i}\left\{ E_{e_1-e_2}\{(H_1 + l - 3)(H_2 + l - 3)
\]
\[ \times (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_1 + l - 3)(H_3 - l - 2 + 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3})\}\phi_{T,l;1}^{(1)}
\]
\[ - 2E_{e_2-e_3}^2(H_1 + l - 4)(H_1 - l - 2)\phi_{T,l;2}^{(1)}
\]
\[ + E_{e_2-e_3}(H_1 + l - 4)((H_1 - l - 2)(H_2 - l - 1)(H_3 - l - 1 - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_1 - l - 2) - E_{e_1-e_2}^2(H_3 - l - 1 - 2\sqrt{-1}E_{2e_3})\}\phi_{T,l;3}^{(1)}\right\}\]
\[ + 2\delta_{3i}\left\{ - E_{e_1-e_2}E_{e_2-e_3}\{(H_1 + l - 3)(H_2 + l - 3)(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})
\]
\[ \times (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_1-e_2}^2(H_1 + l - 3)(H_3 - l - 2 - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_1-e_2}^2(H_3 + l - 2 - 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3})\}\phi_{T,l;1}^{(1)}
\]
\[ - 2E_{e_2-e_3}^2(H_1 + l - 4)(H_1 - l - 2)\phi_{T,l;2}^{(1)}
\]
\[ + E_{e_2-e_3}(H_1 + l - 4)((H_1 - l - 2)(H_2 - l - 1)(H_3 - l - 1 - 2\sqrt{-1}E_{2e_3})
\]
\[ - E_{e_2-e_3}^2(H_1 - l - 2) - E_{e_1-e_2}^2(H_3 - l - 1 - 2\sqrt{-1}E_{2e_3})\}\phi_{T,l;3}^{(1)}\right\}\]
\[ -E_{e_2-e_3}^2 (H_1 + l - 3) - E_{e_1-e_2}^2 (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3}) \phi_{T,l;1}^{(1)} + E_{e_2-e_3}^2 (H_1 + l - 4)(H_2 + l - 2)(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3}) \]
\[ -E_{e_2-e_3}^2 (H_1 + l - 4) - E_{e_1-e_2}^2 (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3}) (H_1 - l - 2) \phi_{T,l;2}^{(1)} \],

and

\[ -D_{11}^{(-,+)} (H_1 + l - 3) - D_{12}^{(+,-)} (H_1 + l - 3) - D_{13}^{(-,+)} \phi_{T,l;1}^{(2)} \]
\[ = - E_{e_1-e_2} E_{e_2-e_3} \phi_{T,l;1}^{(2)} + E_{e_1-e_2} (H_1 + H_2 - 4) \phi_{T,l;2}^{(2)} \]
\[ - \{ (H_1 + l - 3)(H_1 + l - 3) + E_{e_1-e_2}^2 \} \phi_{T,l;3}^{(2)} \]
\[ - D_{21}^{(-,+)} (H_1 + l - 3) - D_{22}^{(+,-)} \phi_{T,l;2}^{(2)} - D_{23}^{(-,+)} \phi_{T,l;1}^{(2)} \]
\[ = - E_{e_1-e_2} E_{e_2-e_3} (H_2 + H_3 - 2 + 2\sqrt{-1}E_{2e_3}) \phi_{T,l;1}^{(2)} \]
\[ + \{ (H_2 - l - 2)(H_2 + l - 2) + E_{e_1-e_2}^2 + E_{e_2-e_3}^2 \} \phi_{T,l;2}^{(2)} - E_{e_1-e_2} (H_1 + H_2 - 6) \phi_{T,l;3}^{(2)} \]
\[ - D_{31}^{(-,+)} (H_3 - l - 1 - 2\sqrt{-1}E_{2e_3})(H_3 + l - 1 - 2\sqrt{-1}E_{2e_3}) + E_{e_2-e_3}^2 \phi_{T,l;1}^{(2)} \]
\[ + E_{e_2-e_3} (H_2 + H_3 - 4 - 2\sqrt{-1}E_{2e_3}) \phi_{T,l;2}^{(2)} - E_{e_1-e_2} E_{e_2-e_3} \phi_{T,l,3}^{(2)} \]

\[ C_2 \phi_{T,l;i}^{(2)} = \{ (H_1 + l - 6 - \delta_{3i})(H_1 - l + \delta_{3i}) + (H_2 + l - 4 - \delta_{2i})(H_2 - l + \delta_{2i}) \]
\[ + (H_3 + l - 2 - \delta_{1i} + 2\sqrt{-1}E_{2e_3})(H_3 - l + \delta_{1i} - 2\sqrt{-1}E_{2e_3}) \]
\[ + 2E_{e_1-e_2} + 2E_{e_2-e_3} \} \phi_{T,l;i}^{(2)} \]
\[ - 2\delta_{1i} \phi_{T,l;i}^{(2)} - E_{e_2-e_3} \delta_{1i} \phi_{T,l;2}^{(2)} - 2\delta_{2i} \{ E_{e_2-e_3} \phi_{T,l;1}^{(2)} + \phi_{T,l;2}^{(2)} - E_{e_1-e_2} \phi_{T,l;3}^{(2)} \} \]
\[ - 2\delta_{3i} E_{e_1-e_2} \phi_{T,l;2}^{(2)} \],

\[ C_4 \phi_{T,l;i}^{(2)} = \{ (H_2 + l - 3 - \delta_{2i})(H_3 + l - 2 - \delta_{1i} + 2\sqrt{-1}E_{2e_3}) - E_{e_2-e_3}^2 \}
\[ \times \{ (H_2 + l - 1 + \delta_{2i})(H_3 - l + \delta_{1i} - 2\sqrt{-1}E_{2e_3}) - E_{e_2-e_3}^2 \}
\[ + (H_1 + l - 5 + \delta_{3i})(H_3 + l - 2 - \delta_{1i} + 2\sqrt{-1}E_{2e_3}) \]
\[ \times (H_1 + l - 1 + \delta_{3i})(H_3 - l + \delta_{1i} - 2\sqrt{-1}E_{2e_3}) \]
\[ + \{ (H_1 + l - 5 - \delta_{3i})(H_2 + l - 4 - \delta_{2i}) - E_{e_1-e_2}^2 \}
\[ \times \{ (H_1 + l - 1 - \delta_{3i})(H_2 - l + \delta_{2i}) - E_{e_1-e_2}^2 \}
\[ + 2E_{e_1-e_3} (H_3 + l - 2 - \delta_{1i} + 2\sqrt{-1}E_{2e_3})(H_3 - l + \delta_{1i} - 2\sqrt{-1}E_{2e_3}) \]
\[ + 2E_{e_2-e_3} \{ (H_1 + l - 5 - \delta_{3i})(H_1 - l - 1 + \delta_{3i}) + E_{e_1-e_2}^2 \} \phi_{T,l;i}^{(2)} \]
\[ + 2\delta_{1i} \left\{ - \{(H_1 + l - 5)(H_1 - l - 1) + (H_2 + l - 3)(H_2 - l - 1) \]
\[ + 2E_{e_1-e_2} + 3E_{e_2-e_3} \phi_{T,l;1}^{(2)} \]
\[ + E_{e_2-e_3} \{ (H_1 + l - 5)(H_1 - l - 1) \]
\[ - (H_2 - l)(H_3 - l - 2\sqrt{-1}E_{2e_3}) \]
\[ - (H_2 + H_3 - 2 - 2\sqrt{-1}E_{2e_3}) + E_{e_1-e_2}^2 + E_{e_2-e_3}^2 \phi_{T,l;2}^{(2)} \]
\[ + E_{e_1-e_2} E_{e_2-e_3} (H_1 + H_2 + H_3 - l - 3 - 2\sqrt{-1}E_{2e_3}) \phi_{T,l;3}^{(2)} \} \]
\[ + 2\delta_{2i}\left\{ -E_{e_2-e_3}(H_1 + l - 5)(H_1 - l - 1) \\
- (H_2 + l - 3)(H_3 + l - 3 + 2\sqrt{-1}E_{2e_3}) \\
+ H_2 + H_3 - 4 + 2\sqrt{-1}E_{2e_3} + E_{e_2-e_3}^2\phi_{T,l;1}^{(2)} \\
- \{(H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3}) \\
+ 2E_{e_2-e_3}\phi_{T,l;2}^{(2)} \\
+ E_{e_2-e_3}(- (H_1 - l)(H_2 - l) \\
+ (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3}) \\
+ E_{e_2-e_3}^2\phi_{T,l;3}^{(2)} \right\} \]

\[ + 2\delta_{3i}\left\{ -E_{e_1-e_2}E_{e_2-e_3}(H_1 + H_2 + H_3 + l - 9 + 2\sqrt{-1}E_{2e_3})\phi_{T,l;1}^{(2)} \\
+ E_{e_1-e_2}((H_1 + l - 5)(H_2 + l - 5) \\
- (H_3 + l - 2 + 2\sqrt{-1}E_{2e_3})(H_3 - l - 2\sqrt{-1}E_{2e_3}) \\
- E_{e_2-e_3}(H_1 - l - 2 + \delta_{3i})\phi_{T,l;2}^{(2)} \right\}, \]

\[ C_{6\phi_{T,l;i}^{(2)}} = \{(H_1 + l - 4 - \delta_{3i})(H_2 + l - 3 - \delta_{3i})(H_3 + l - 2 - \delta_{3i} + 2\sqrt{-1}E_{2e_3}) \\
- E_{e_2-e_3}^2(H_1 + l - 4 - \delta_{3i}) - E_{e_1-e_2}(H_3 + l - 2 - \delta_{3i} + 2\sqrt{-1}E_{2e_3}) \} \\
\times \{(H_1 - l + \delta_{3i})(H_2 - l - 1 + \delta_{3i})(H_3 - l + \delta_{3i} - 2\sqrt{-1}E_{2e_3}) \\
- E_{e_1-e_2}(H_3 - l + \delta_{3i} - 2\sqrt{-1}E_{2e_3}) - E_{e_2-e_3}(H_1 - l - 2 + \delta_{3i})\phi_{T,l;i}^{(2)} \}
\]
\[-E_{e_2-e_3}^2 (H_1 + l - 4) - E_{e_1-e_2}^2 (H_3 + l - 3 + 2\sqrt{-1}E_{2 e_3}) \phi_{T,l,1}^{(2)}
+ E_{e_1-e_2}(H_1 + l - 4)(H_2 + l - 4)(H_3 + l - 2 + 2\sqrt{-1}E_{2 e_3})
- E_{e_2-e_3}^2 (H_1 + l - 4) - E_{e_1-e_2}^2 (H_3 + l - 2 + 2\sqrt{-1}E_{2 e_3})\]
\times (H_3 - l - 2 - 2\sqrt{-1}E_{2 e_3}) \phi_{T,l,2}^{(2)}.

Proof. The actions of the operators $C_{2i}$ ($1 \leq i \leq 2$), $D_{j,k}^{(\pm, \pm)}$ ($1 \leq j, k \leq 3$) and $m_3(C_{\pm})$ are obtained by direct computation from the expressions in normal order and Lemma 6.9(i).

We can summarize the explicit actions of the operators $C_{2i}$ ($1 \leq i \leq 3$) and $D_{j,k}^{(\pm, \pm)}$ ($1 \leq j, k \leq 3$) on the functions in Proposition 6.5, 6.7. The action of $C_6 = m_3(C_+)m_3(C_-)$ is obtained by the composite of following actions of $m_3(C_+)$ and $m_3(C_-)$:

- the case of $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1)$.

\[m_3(C_+) \phi_{T,l-2}^{(1)} = \{(H_1 + l - 4)(H_2 + l - 3)(H_3 + l - 2 + 2\sqrt{-1}E_{2 e_3})
- E_{e_2-e_3}^2 (H_1 + l - 4) - E_{e_1-e_2}^2 (H_3 + l - 2 + 2\sqrt{-1}E_{2 e_3}) \phi_{T,l-2}^{(1)}
- 2\delta_{11}(E_{e_1-e_2}(H_3 + l - 2 + 2\sqrt{-1}E_{2 e_3}) \phi_{T,l-2;2}^{(1)} - E_{e_1-e_2}E_{e_2-e_3} \phi_{T,l-2;3}^{(1)}
- 2\delta_{2i}E_{e_2-e_3}(H_1 + l - 4) \phi_{T,l-2;3}^{(1)}
- 2\delta_{3i}E_{e_1-e_2}E_{e_2-e_3} \phi_{T,l-1}^{(1)} - E_{e_2-e_3}(H_1 - l - 2) \phi_{T,l;2}^{(1)}\}
\]

- the case of $(\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1)$.

\[m_3(C_+) \phi_{T,l-2;2}^{(2)} = \{(H_1 + l - 4 - \delta_{3i})(H_2 + l - 3 - \delta_{3i})(H_3 + l - 2 - \delta_{3i} + 2\sqrt{-1}E_{2 e_3})
- E_{e_2-e_3}^2 (H_1 + l - 4 - \delta_{3i}) - E_{e_1-e_2}^2 (H_3 + l - 2 - \delta_{3i})\]
\times (H_3 - l - 2 - 2\sqrt{-1}E_{2 e_3}) \phi_{T,l-2;2}^{(2)}
- 2\delta_{11}(E_{e_1-e_2}(H_3 + l - 2 + 2\sqrt{-1}E_{2 e_3}) \phi_{T,l-2;2}^{(2)} - E_{e_1-e_2}E_{e_2-e_3} \phi_{T,l-2;3}^{(2)}
- 2\delta_{2i}E_{e_2-e_3}(H_1 + l - 4) \phi_{T,l-2;3}^{(2)}
- 2\delta_{3i}E_{e_1-e_2}E_{e_2-e_3} \phi_{T,l-1}^{(2)} - E_{e_2-e_3}(H_1 - l - 2) \phi_{T,l;2}^{(2)}\}
\]
To state an explicit form of a holonomic system of partial differential equations satisfied by the $A$-radial part of each element in $\text{Wh}(\tau_{(\sigma, \nu)}, \xi, \tau)$ for peripheral $K$-type $\tau^*$, we introduce the coordinates $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ on $A$ defined by

$$x_1 = \left(\frac{\pi c_{12} a_1}{a_2}\right)^2, \quad x_2 = \left(\frac{\pi c_{23} a_2}{a_3}\right)^2, \quad x_3 = 4\pi c_3 a_3^2,$$

$$y_1 = 2\pi c_{12} a_1, \quad y_2 = 2\pi c_{23} a_2, \quad y_3 = 4\pi c_3 a_3^2,$$

for $\text{diag}(a_1, a_2, a_3, a_1^{-1}, a_2^{-1}, a_3^{-1}) \in A$. We use the coordinate $x$ for the case of $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0), (1, 1, 1)$, and use the coordinate $y$ for the case of $(\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0), (1, 1, 1)$. Then we have following theorem.

**Theorem 6.12.** Let $T$ be an element of the space $I_{\xi, \tau_{(\sigma, \nu)}}$.

(i) If $\sigma$ is a character of $M_{\min}$ such that $(\sigma_1, \sigma_2, \sigma_3) = (0, 0, 0)$ or $(1, 1, 1)$, then there exist multiplicity one $K$-types $\tau_{(l, l, l)}$ ($l \equiv \epsilon_\sigma \mod 2$). For $l \equiv \epsilon_\sigma \mod 2$, the Whittaker function $\Phi(T, S_{(l, l, l)}(M_j)) = \phi_{T, l} \otimes f(M_j)$ satisfies the following holonomic system of partial differential equations of rank 48:

$$
\{(2\partial x_1 + l - 6)(2\partial x_1 - l) + (-2\partial x_1 + 2\partial x_2 + l - 4)(-2\partial x_1 + 2\partial x_2 - l) + (-2\partial x_2 + 2\partial x_3 + l - 2 - x_3)(-2\partial x_2 + 2\partial x_3 - l + x_3) - 8x_1 - 8x_2 - \chi_{2, \sigma, \nu, (l, l, l)}\} \phi_{T, l} = 0,
$$

(ii) If $\sigma$ is a character of $M_{\min}$ such that $(\sigma_1, \sigma_2, \sigma_3) \neq (0, 0, 0)$ or $(1, 1, 1)$, there exists multiplicity one $K$-types $\tau_{(l+1, l+1, l)}$, $\tau_{(l, l, l-1)}$ ($l \equiv \epsilon_\sigma \mod 2$).

For $l \equiv \epsilon_\sigma \mod 2$, Whittaker function $\Phi(T, S_{(l+1, l+1, l)}(M_j^{(1)})) = \sum_{j=1}^{3} \phi_{T, l}^{(1)} \otimes f(M_j^{(1)})$ satisfy the following holonomic system of partial differential equations of rank 48:

$$
\{(2\partial x_1 + l - 4)(-2\partial x_1 + 2\partial x_2 + l - 3)(-2\partial x_2 + 2\partial x_3 + l - 2 - x_3) + 4x_2(2\partial x_1 + l - 4) + 4x_1(-2\partial x_2 + 2\partial x_3 + l - 2 - x_3) \}
\times \{(2\partial x_1 - l - 2)(-2\partial x_1 + 2\partial x_2 - l - 1)(-2\partial x_2 + 2\partial x_3 - l + x_3) + 4x_2(2\partial x_1 - l - 2) + 4x_1(-2\partial x_2 + 2\partial x_3 - l + x_3) \} \chi_{6, \sigma, \nu, (l, l, l)} \phi_{T, l} = 0.
$$
\[\begin{align*}
&\sqrt{-1}y_1(\partial_{y_1} - 6)\phi_{T,l,1}^{(1)} \\
&+ \{(-\partial_{y_1} + \partial_{y_2} + l - 2)(-\partial_{y_1} + \partial_{y_2} - l - 2) - y_1^2 - y_2^2 - \tilde{\chi}_{2,\sigma,\nu,(l+1,l,l)}\} \phi_{T,l,2}^{(1)} \\
&+ \sqrt{-1}y_2(-\partial_{y_1} + 2\partial_{y_3} - 2 + y_3)\phi_{T,l,3}^{(1)} = 0, \\
&- y_1y_2\phi_{T,l,1}^{(1)} + \sqrt{-1}y_2(-\partial_{y_1} + 2\partial_{y_3} - 4 - y_3)\phi_{T,l,2}^{(1)} \\
&+ \{(-\partial_{y_2} + 2\partial_{y_3} + l - 1 - y_3)(-\partial_{y_2} + 2\partial_{y_3} - l - 1 + y_3) - y_2^2 - \tilde{\chi}_{2,\sigma,\nu,(l+1,l,l)}\} \phi_{T,l,3}^{(1)} = 0,
\end{align*}\]

\[\begin{align*}
&\left\{(\partial_{y_1} + l - 6 + \delta_1)(\partial_{y_1} - l - \delta_1) + (-\partial_{y_1} + \partial_{y_2} + l - 4 + \delta_2)(-\partial_{y_1} + \partial_{y_2} - l - \delta_2) \\
&+ (-\partial_{y_2} + 2\partial_{y_3} + l - 2 + \delta_3i - y_3)(-\partial_{y_2} + 2\partial_{y_3} - l - \delta_3i + y_3) \\
&- 2y_1^2 - 2y_2^2 - \chi_{2,\sigma,\nu,(l+1,l,l)}\right\}\phi_{T,l,i}^{(1)} \\
&- 2\delta_1\left\{\sqrt{-1}y_1\phi_{T,l,1}^{(1)} + \phi_{T,l,2}^{(1)} - \sqrt{-1}y_2\phi_{T,l,3}^{(1)} - 2\delta_3i\sqrt{-1}y_2\phi_{T,l,2}^{(1)} = 0,\right\}
\end{align*}\]

\[\begin{align*}
&\left\{(-\partial_{y_1} + \partial_{y_2} + l - 3 + \delta_2)(-\partial_{y_2} + 2\partial_{y_3} + l - 2 + \delta_3i - y_3) + y_2^2\right\} \\
&\times \left\{(-\partial_{y_1} + \partial_{y_2} + l - 1 - \delta_2)(-\partial_{y_2} + 2\partial_{y_3} - l - \delta_3i + y_3) + y_2^2\right\} \\
&+ (\partial_{y_1} + l - 5 + \delta_1)(-\partial_{y_2} + 2\partial_{y_3} + l - 2 + \delta_3i - y_3) \\
&\times (\partial_{y_1} + l - 1 - \delta_1)(-\partial_{y_2} + 2\partial_{y_3} - l - \delta_3i + y_3) \\
&+ \{(-\partial_{y_1} + \partial_{y_2} + l - 4 + \delta_2)\phi_{T,l,i}^{(1)} \\
&\times \{(-\partial_{y_1} + \partial_{y_2} + l - 4 + \delta_2)\phi_{T,l,i}^{(1)} \\
&+ \sqrt{-1}y_1\{(\partial_{y_1} + \partial_{y_2} - l - 2 + y_3)(-\partial_{y_2} + 2\partial_{y_3} - l + y_3) \\
&- (\partial_{y_1} - l - 1)(-\partial_{y_1} + \partial_{y_2} - l - 1) + \partial_{y_2} - 6 - y_1^2 - y_2^2\} \phi_{T,l,2}^{(1)} \\
&+ y_1y_2(2\partial_{y_3} - l - 6 + y_3)\phi_{T,l,3}^{(1)}\right\} \\
&+ 2\delta_1\left\{\right\}
\end{align*}\]
\[
\left\{ \left( \partial_{y_1} + l - 4 + \delta_{1i} \right) \left( \partial_{y_1} + \partial_{y_2} + l - 3 + \delta_{2i} \right) \left( \partial_{y_2} + 2 \partial_{y_3} + l - 2 + \delta_{3i} - y_3 \right) \\
+ y_2^2 \left( \partial_{y_1} + l - 4 + \delta_{1i} \right) + y_1^2 \left( \partial_{y_2} + 2 \partial_{y_3} + l - 2 + \delta_{3i} - y_3 \right) \right\} \\
\times \left\{ \left( \partial_{y_1} - l - 2 - \delta_{1i} \right) \left( \partial_{y_1} + \partial_{y_2} - l - 1 - \delta_{2i} \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l - \delta_{3i} + y_3 \right) \\
+ y_2^2 \left( \partial_{y_1} - l - 2 - \delta_{1i} \right) + y_1^2 \left( \partial_{y_2} + 2 \partial_{y_3} - l - \delta_{3i} + y_3 \right) \right\} - \chi_{6, \sigma, \nu, (l+1, l, l)}^{(1)} \phi_{T, l, i}^{(1)} \right\}
\]

\[
2 \delta_{1i} \left\{ 2 y_1^2 \left( \left( \partial_{y_2} + 2 \partial_{y_3} + l - 2 - y_3 \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l + y_3 \right) - \dot{y}_2^2 \right) \phi_{T, l, i}^{(1)} \\
- \sqrt{-1} y_1 \left\{ \left( \partial_{y_1} + \partial_{y_2} + l - 2 - y_3 \right) \left( \partial_{y_1} - l - 2 \right) \left( \partial_{y_1} + \partial_{y_2} - l - 2 \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l + y_3 \right) \\
+ y_2^2 \left( \partial_{y_1} + 2 \partial_{y_3} + l - 2 - y_3 \right) \left( \partial_{y_1} - l - 2 \right) \right\} \phi_{T, l, i}^{(1)} \\
+ y_2^2 \left( \partial_{y_1} + 2 \partial_{y_3} - l - 1 + y_3 \right) \phi_{T, l, i}^{(1)} \right\}
\]

\[
2 \delta_{2i} \left\{ \sqrt{-1} y_1 \left\{ \left( \partial_{y_1} + l - 3 \right) \left( \partial_{y_1} + \partial_{y_2} - l - 3 \right) \\
- \left( \partial_{y_2} + 2 \partial_{y_3} - l + y_3 \right) \left( \partial_{y_1} - 2 \partial_{y_3} + l + y_3 \right) \\
+ y_2^2 \left( \partial_{y_1} + l - 3 \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l + y_3 \right) \phi_{T, l, i}^{(1)} \phi_{T, l, i}^{(1)} \\
+ 2 y_2^2 \left( \partial_{y_1} + l - 4 \right) \left( \partial_{y_1} - l - 2 \right) \phi_{T, l, i}^{(2)} \\
- \sqrt{-1} y_2 \left( \partial_{y_1} + 2 \partial_{y_3} - l - 2 \right) \left( \partial_{y_1} + \partial_{y_2} - l - 1 \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l - 1 + y_3 \right) \\
+ y_2^2 \left( \partial_{y_1} + l - 2 \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l - 1 + y_3 \right) \phi_{T, l, i}^{(1)} \phi_{T, l, i}^{(1)} \right\}
\]

\[
2 \delta_{3i} \left\{ y_1 y_2 \left\{ \left( \partial_{y_1} + l - 3 \right) \left( \partial_{y_1} + \partial_{y_2} + l - 3 \right) \left( \partial_{y_1} + 2 \partial_{y_3} + l - 2 - y_3 \right) \\
+ y_2^2 \left( \partial_{y_1} + l - 3 \right) + y_1^2 \left( \partial_{y_2} + 2 \partial_{y_3} + l - 2 - y_3 \right) \phi_{T, l, i}^{(1)} \right\} \\
+ y_2^2 \left( \partial_{y_1} + 2 \partial_{y_3} + l - 2 \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l + y_3 \right) \phi_{T, l, i}^{(1)} \\
\right\} \phi_{T, l, i}^{(1)} = 0,
\]

for i = 1, 2, 3.

For l \equiv \varepsilon \mod 2, Whittaker functions \( \Phi(T, S_{(l,l-1)}(M_l^{(\sigma, 2)})) = \sum_{j=1}^3 \phi_{T, l, j}^{(2)} \otimes f(M_l^{(2)})^* \)
satisfy the following holonomic system of partial differential equations of rank 48:

\[
y_1 y_2 \phi_{T, l, i}^{(2)} + \sqrt{-1} y_1 (\partial_{y_2} - 4) \phi_{T, l, i}^{(2)} \\
\right\}
\]

\[
- \left\{ \left( \partial_{y_1} - l - 3 \right) \left( \partial_{y_1} + l - 3 \right) - y_1^2 \left( \partial_{y_1} + l - 3 \right) \right\} \phi_{T, l, i}^{(2)} = 0,
\]

\[
- \sqrt{-1} y_2 (\partial_{y_1} + 2 \partial_{y_3} - 2 - y_3) \phi_{T, l, i}^{(2)} \\
+ \left\{ \left( \partial_{y_1} + \partial_{y_2} - l - 2 \right) \left( \partial_{y_1} + \partial_{y_2} - l - 2 \right) \left( \partial_{y_1} + \partial_{y_2} + l - 2 \right) \right\} \phi_{T, l, i}^{(2)} \\
- \sqrt{-1} y_1 (\partial_{y_2} - 6) \phi_{T, l, i}^{(2)} = 0,
\]

\[
- \left\{ \left( \partial_{y_2} + 2 \partial_{y_3} - l - 1 + y_3 \right) \left( \partial_{y_2} + 2 \partial_{y_3} + l - 1 - y_3 \right) \left( \partial_{y_2} + 2 \partial_{y_3} - l + y_3 \right) \right\} \phi_{T, l, i}^{(2)} \\
+ \sqrt{-1} y_2 (\partial_{y_1} + 2 \partial_{y_3} - 4 + y_3) \phi_{T, l, i}^{(2)} + y_1 y_2 \phi_{T, l, i}^{(2)},
\]

\]
\[
\left\{ (\partial y_1 + l - 6 - \delta_{3i})(\partial y_1 - l + \delta_{3i}) \\
+ (-\partial y_1 + \partial y_2 + l - 4 - \delta_{2i})(-\partial y_1 + \partial y_2 - l + \delta_{2i}) \\
+ (-\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i} - y_3)(-\partial y_2 + 2\partial y_3 - l + \delta_{1i} + y_3) \\
- 2\phi_{T,l;1}^2 - 2\phi_{T,l;2}^2 - \chi_{2,\sigma,\nu,(l,l,l-1)}(\partial y_1 - l + \delta_{3i}) \phi_{T,l;i} \right. \\
- 2\delta_{2i}\{\sqrt{-1}y_2\phi_{T,l;1}^2 + \phi_{T,l;2}^2 - \sqrt{-1}y_1\phi_{T,l;3}^2\} - 2\delta_{3i}\sqrt{-1}y_1\phi_{T,l;2}^2 = 0,
\]

\[
\left\{ \{-\partial y_1 + \partial y_2 + l - 3 - \delta_{2i}\}(-\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i} - y_3) + y_2^2 \right. \\
\times \{(-\partial y_1 + \partial y_2 - l - 1 + \delta_{2i})(-\partial y_1 + \partial y_2 - l + \delta_{1i} + y_3) + y_2^2 \} \\
+ (\partial y_1 + l - 5 - \delta_{3i})(-\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i} - y_3) \\
\times (\partial y_1 - l - 1 + \delta_{3i})(-\partial y_2 + 2\partial y_3 - l + \delta_{1i} + y_3) \\
+ \{((\partial y_1 + \partial y_2 + l - 4 - \delta_{2i}) \phi_{T,l;1}^2 + 2\partial y_3 + l - 2 - \delta_{1i}) + y_1\}^2 \\
\times \{(\partial y_1 - l - 1 + \delta_{3i})(\partial y_2 - l + \delta_{2i}) + y_1\}^2 \\
- 2\phi_{T,l;1}^2(\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i} - y_3)(-\partial y_2 + 2\partial y_3 - l + \delta_{1i} + y_3) \\
- 2\phi_{T,l;2}^2((\partial y_1 + l - 5 - \delta_{3i})(\partial y_1 - l - 1 + \delta_{3i}) - y_1^2 - \chi_{4,\sigma,\nu,(l,l,l-1)} \phi_{T,l;i} \right. \\
+ 2\delta_{1i}\{\{-\partial y_1 + \partial y_2 + l - 3\}(-\partial y_1 + \partial y_2 - l - 1) \\
- 2\phi_{T,l;1}^2 - 3y_2^2 \phi_{T,l;2}^2 \\
+ 2\phi_{T,l;3}^2 \phi_{T,l;3} \phi_{T,l;3} \} \\
- \{-\partial y_1 + \partial y_2 + l - 3\}(-\partial y_1 + \partial y_2 - l - 3)(-\partial y_1 + \partial y_2 - l - 1) \\
- 2\phi_{T,l;1}^2 - 3y_2^2 \phi_{T,l;2}^2 \\
\times \{(-\partial y_1 + \partial y_2 - l - 2 + \delta_{2i})(-\partial y_1 + \partial y_2 - l + \delta_{2i}) - 2y_1^2 - y_2^2 \phi_{T,l;2}^2 \\
+ \sqrt{-1}y_1\{-\partial y_1 - l\}(-\partial y_1 + \partial y_2 - l) \\
\times \{(-\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i}) - y_3\}(-\partial y_2 + 2\partial y_3 - l + \delta_{1i} + y_3) - y_1^2 - y_2^2 \phi_{T,l;3}^2 \} \\
+ 2\phi_{T,l;1}^2 \phi_{T,l;1}^2 \\
+ \sqrt{-1}y_1\{-\partial y_1 - l\}(\partial y_1 + \partial y_2 - l - 5) \\
\times \{-\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i}\}(-\partial y_2 + 2\partial y_3 - l + \delta_{1i} + y_3) + y_1^2 + y_2^2 \phi_{T,l;2}^2 \} = 0,
\]

\[
\left\{ (\partial y_1 + l - 4 - \delta_{3i})(-\partial y_1 + \partial y_2 + l - 3 - \delta_{2i})(-\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i} - y_3) \\
+ y_2^2(\partial y_1 + l - 4 - \delta_{3i}) + y_2^2(-\partial y_2 + 2\partial y_3 + l - 2 - \delta_{1i} - y_3) \right. \\
\times \{((\partial y_1 + \partial y_2 + l - 4 - \delta_{2i}) \phi_{T,l;1}^2 + 2\partial y_3 + l - 2 - \delta_{1i}) + y_1\}^2 \\
+ \{((\partial y_1 + \partial y_2 + l - 4 - \delta_{2i}) \phi_{T,l;1}^2 + 2\partial y_3 + l - 2 - \delta_{1i}) + y_1\}^2 \\
+ y_1^2(\partial y_2 + 2\partial y_3 + l + \delta_{1i} + y_3) + y_2^2(\partial y_1 + l - 4 - \delta_{3i}) \phi_{T,l;i} \right. \\
- \chi_{6,\sigma,\nu,(l,l,l-1)} \phi_{T,l;i} \} = 0.
\]
\[
+ 2\delta_{ii} \left\{ 2y_2^2 ((\partial_{y_1} + l - 4)(\partial_{y_1} - l - 2) - y_1^2) \phi_{T,i;1}^{(2)}
- \sqrt{-1}y_2((\partial_{y_1} + l - 4)(\partial_{y_1} - l - 2)
\times (-\partial_{y_1} + \partial_{y_2} - l)(-\partial_{y_2} + 2\partial_{y_3} - l + y_3)
+ y_2^2(\partial_{y_1} + l - 4)(-\partial_{y_2} + 2\partial_{y_3} - l + y_3)
+ y_2^2(\partial_{y_1} + l - 4)(\partial_{y_1} - l - 2)\phi_{T,i;2}^{(2)}
+ y_1y_2((\partial_{y_1} - l - 1)(-\partial_{y_1} + \partial_{y_2} - l - 1)(-\partial_{y_2} + 2\partial_{y_3} - l + y_3)
+ y_2^2(-\partial_{y_2} + 2\partial_{y_3} - l + y_3) + y_2^2(\partial_{y_1} - l - 1)\phi_{T,i;3}^{(2)} \right\} \\
+ 2\delta_{ii} \left\{ \sqrt{-1}y_2((\partial_{y_1} + l - 4)(\partial_{y_1} + \partial_{y_2} + l - 3)
\times (-\partial_{y_2} + 2\partial_{y_3} + l - 3 - y_3)(\partial_{y_1} - l - 2)
+ y_2^2(\partial_{y_1} + l - 4)(\partial_{y_1} - l - 2)
+ y_2^2(-\partial_{y_2} + 2\partial_{y_3} + l - 3 - y_3)(\partial_{y_1} - l - 2)\phi_{T,i;1}^{(2)}
+ 2y_1^2(-\partial_{y_2} + 2\partial_{y_3} + l - 2 - y_3)(-\partial_{y_2} + 2\partial_{y_3} - l + y_3)\phi_{T,i;2}^{(2)}
- \sqrt{-1}y_1(-\partial_{y_2} + 2\partial_{y_3} + l - 2 - y_3)
\times ((\partial_{y_1} - l - 1)(-\partial_{y_1} + \partial_{y_2} - l - 1)(-\partial_{y_2} + 2\partial_{y_3} - l + y_3)
+ y_1^2(-\partial_{y_2} + 2\partial_{y_3} - l + y_3) + y_2^2(\partial_{y_1} - l - 1)\phi_{T,i;3}^{(2)} \right\} \\
+ 2\delta_{ii} \left\{ - y_1y_2((\partial_{y_1} + l - 4)(\partial_{y_1} + \partial_{y_2} + l - 3)(-\partial_{y_2} + 2\partial_{y_3} + l - 3 - y_3)
+ y_2^2(\partial_{y_1} + l - 4) + y_1^2(-\partial_{y_2} + 2\partial_{y_3} + l - 3 - y_3))\phi_{T,i;1}^{(2)}
+ \sqrt{-1}y_1((\partial_{y_1} + l - 4)(-\partial_{y_1} + \partial_{y_2} + l - 4)(-\partial_{y_2} + 2\partial_{y_3} + l - 2 - y_3)
+ y_2^2(\partial_{y_1} + l - 4) + y_1^2(-\partial_{y_2} + 2\partial_{y_3} + l - 2 - y_3))\phi_{T,i;2}^{(2)} \right\} = 0,
\]
for \(i = 1, 2, 3\).

Proof. We obtain the assertion from Proposition 6.5, 6.7, and 6.11. \qed

From the results of Kostant (6.8.1), it follows that the dimension of the intertwining space \(I_{\xi,\tau}\) is 48. Hence, for a multiplicity one \(K\)-type \(\tau\), the dimension of the space \(\text{Wh}(\pi, \xi, \tau)\) of Whittaker functions is also 48. Therefore, every solution of the holonomic systems in the Theorem 6.12 gives an element of \(\text{Wh}(\pi, \xi, \tau)|_{A_{\min}}\).

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