Particle Density in Zero Temperature Symmetry Restoring Phase Transitions in Four-Fermion Interaction Models

ZHOU Bang-Rong

Department of Physics, Graduate School of the Chinese Academy of Sciences, Beijing 100039, China
CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China

(Received May 19, 2003)

Abstract By means of critical behaviors of the dynamical fermion mass in four-fermion interaction models, we show by explicit calculations that when $T = 0$ the particle density will have a discontinuous jumping across the critical chemical potential $\mu_c$ in 2D and 3D Gross–Neveu (GN) model and these physically explain the first-order feature of the corresponding symmetry restoring phase transitions. For the second-order phase transitions in the 3D GN model when $T \to 0$ and in 4D Nambu–Jona-Lasinio (NJL) model when $T = 0$, it is proven that the particle density itself will be continuous across $\mu_c$, but its derivative over the chemical potential $\mu$ will have a discontinuous jumping. The results give a physical explanation of implications of the tricritical point $(T, \mu) = (0, \mu_c)$ in the 3D GN model. The discussions also show effectiveness of the critical analysis approach of phase transitions.

PACS numbers: 11.10.Wx, 12.40.-y, 11.30.Qc
Key words: Gross–Neveu and Nambu–Jona-Lasinio models, symmetry restoration at zero temperature and high density, particle number density, first- and second-order phase transitions

1 Introduction

Four-fermion interaction models$^{[1-2]}$ are good laboratories to research dynamical symmetry restoring phase transitions at high temperature and high density.$^{[3-5]}$ It has been proven that when temperature $T$ goes to zero, the feature of the phase transitions in these models strongly depend on the dimension $D$ of space-time.$^{[6-8]}$ It is shown that the symmetry restoring phase transitions at high fermion chemical potential $\mu$ and $T = 0$ are first order when $D = 2$ and $D = 3$ and are second order when $D = 4$ and the momentum cutoff of the loop integrations is large enough. The above conclusions can come from analyses of critical behaviors of the dynamical fermion mass $m$ as the order parameter of symmetry breaking. The critical behaviors of the order parameter can show the essential characteristics, however, they do not directly explain the physical realization of the above phase transitions. The latter is just what we want to explore in this paper. Since particle number density is a thermodynamical quantity directly relative to the chemical potential $\mu$, we will calculate it in various cases and combine the results with the critical behaviors of the order parameter obtained in Refs. $[7]$ and $[8]$ so as to further expound physical characteristics of the above first-and second-order phase transitions.

The total particle number density $n_\rho(T, \mu, m)$ in $D$-dimensional space-time should be the difference between fermion and antifermion densities and at finite $T$ and finite $\mu$ it can be expressed by$^{[9]}$

\[
n_\rho(T, \mu, m) = \int_{-\infty}^{\infty} \frac{d^{D-1}p}{(2\pi)^{D-1}} \left[ \frac{1}{e^{\beta(\sqrt{p^2 + m^2} - \mu)} + 1} - \frac{1}{e^{\beta(\sqrt{p^2 + m^2} + \mu)} + 1} \right] = \frac{2}{\Gamma((D-1)/2)(4\pi)^{(D-1)/2}} \int_0^{2T^{D-1}} dp \left[ \frac{1}{e^{\beta(\sqrt{p^2 + m^2} - \mu)} + 1} - \frac{1}{e^{\beta(\sqrt{p^2 + m^2} + \mu)} + 1} \right] \]
\[
= \frac{2T^{D-1}}{\Gamma((D-1)/2)(4\pi)^{(D-1)/2}} \int_0^\infty dx \left[ \frac{x^{D-2}}{e^{\sqrt{2(x^2 + y^2 - \alpha y)} + 1}} - \frac{x^{D-2}}{e^{\sqrt{2(x^2 + y^2 + \alpha y)} + 1}} \right],
\] (1)

where $m$ is the fermion mass and in a four-fermion interaction model with spontaneous symmetry breaking, it will be replaced by the dynamical fermion mass $m \equiv m(T, \mu)$ caused by the bilinear fermion condensates.$^{[7]}$ We will discuss the cases of $D = 2, 3,$ and $4$, respectively, in Secs. 2, 3, and 4, then come to our conclusions in Sec. 5.

2 $D = 2$

First let us consider the case of $m = 0$. We may find out from Eq. (1) that

---

*The project supported by National Natural Science Foundation of China
Equation (9) will lead to the physical characteristic of the first order phase transition at symmetry breaking to the density of free fermion with $m$. Equation (7) indicates that the jumping of the order parameter will vary in the following form, 

$$n_2(T, \mu, m) = \frac{m}{\pi} \int_1^{\infty} dz \frac{z}{\sqrt{z^2 - 1}} \left[ \frac{1}{e^{\frac{z}{\sqrt{z^2}} + \alpha}} - \frac{1}{e^{\frac{z}{\sqrt{z^2}} - \alpha}} \right].$$

We will concern ourselves only with the results in the limit of the thermodynamical particle density $n$. Substituting Eq. (6) into Eq. (5), we will reach the conclusion that

$$n_2(T = 0, \mu, m) = \begin{cases} 0 & \text{when } \alpha \leq 1 \\ \frac{m}{\pi} \int_1^{\infty} dz \frac{z}{\sqrt{z^2 - 1}} & \text{when } \alpha > 1 \\ \theta(\mu - m) / \sqrt{\mu^2 - m^2} & \text{when } \mu > m(0). \end{cases}$$

Equation (5) is applicable to the general case of free fermions with $m \neq 0$. Obviously, $n_2(T = 0, \mu, m)$ usually varies continuously when $\mu$ crosses $m$. However, in a four-fermion interaction model with spontaneous symmetry breaking and restoration, there could be different cases. It has been proven that in a 2D GN model, when $T = 0$ the order parameter $m$ will vary in the following form,

$$m(T = 0, \mu) = m(0) \theta(\mu - m(0)),$$

where $m(0)$ is the dynamical fermion mass at $T = \mu = 0$ and just the critical chemical potential $\mu_c$ at $T = 0$. Substituting Eq. (6) into Eq. (5), we will reach the conclusion that

$$n_2(T = 0, \mu, m) = \begin{cases} 0 & \text{when } \mu \leq m(0) \\ \mu / \pi & \text{when } \mu > m(0). \end{cases}$$

Equation (7) indicates that the jumping of the order parameter $m$ from $m(0)$ to 0 across $\mu_c = m(0)$ leads to the jumping of the thermodynamical particle density $n_2(T = 0, \mu, m)$ from 0 to $\mu / \pi$, i.e. from the zero value with spontaneous symmetry breaking to the density of free fermion with $m = 0$ after the symmetries are restored. This clearly shows physical characteristic of the first order phase transition at $T = 0$ and $\mu = \mu_c$ in 2D GN model.

3 $D = 3$

In this case, we will first obtain from Eq. (2) that

$$n_3(T, \mu, m) = \frac{T^2}{2\pi} \int_0^{\infty} dx \left[ \frac{x}{e^{\frac{x}{\sqrt{x^2 + y^2}} + \alpha}} - \frac{x}{e^{\frac{x}{\sqrt{x^2 + y^2}} - \alpha}} \right] = m^2 \int_1^{\infty} dz \frac{1}{e^{\frac{yz - \alpha}{\sqrt{y^2 + 1}}} - e^{\frac{yz + \alpha}{\sqrt{y^2 + 1}}} + 1}$$

$$= - \frac{m^2}{2\pi y} \int_1^{\infty} dz \frac{\partial}{\partial z} \left[ \ln(1 + e^{-yz + r}) - \ln(1 + e^{-yz - r}) \right]$$

$$= \frac{Tm}{2\pi} \ln \left[ \frac{1 + e^{-y + r}}{1 + e^{-y - r}} \right] + \frac{T^2}{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sinh(kr) e^{-ky}$$

$$= \frac{Tm}{2\pi} \ln \left[ \frac{1 + e^{(\mu - m)/r}}{1 + e^{(\mu + m)/r}} \right] + \frac{T^2}{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left[ e^{k(\mu - m)/r} - e^{-k(\mu + m)/r} \right], \quad r = \beta \mu. \tag{8}$$

Equation (9) will lead to the $T \to 0$ limit of $n_3(T, \mu, m)$,

$$n_3(T = 0, \mu, m) = \frac{1}{4\pi} \theta(\mu - m)(\mu^2 - m^2). \tag{10}$$

Assuming that $y = m/T \ll 1$ (e.g. near a critical point of a second-order phase transition), then we can expand $n_3(T, \mu, m)$ expressed by Eq. (8) in power of $y$, keeping only the terms up to the order of $y^2$, and obtain

$$n_3(T, \mu, m) \simeq n_3(T, \mu, m = 0) - \frac{m^2}{4\pi} \frac{e^r - 1}{e^r + 1}, \tag{11}$$
where

\[ n_3(T, \mu, m = 0) = \frac{T^2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \sinh(k\beta \mu) = \frac{\mu^2}{4\pi} + \frac{T^2 \pi}{12} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} e^{-k\beta \mu}. \]  

(12)

Noting that different from the case of \( D = 2 \), the total particle density is temperature-dependent and when \( T = 0 \) we have

\[ n_3(T = 0, \mu, m = 0) = \frac{\mu^2}{4\pi}. \]  

(13)

Equation (10) is a general expression for free fermions with \( m \neq 0 \) at \( T = 0 \). However, for 3D GN model with symmetry restoring phase transition when \( T = 0 \), we have the order parameter \( m \) whose critical behavior is similarly expressed by Eq. (6).\(^7\) Hence we obtain

\[ n_3(T = 0, \mu, m) = \begin{cases} 0 & \text{when } \mu \leq m(0), \\ \frac{\mu^2}{4\pi} & \text{when } \mu > m(0), \end{cases} \]  

(14)

i.e., the zero temperature particle density \( n_3(T = 0, \mu, m) \) will jump from 0 with spontaneous symmetry breaking to \( \mu^2/4\pi \) of massless fermions (symmetries being restored) when \( \mu \) crosses the critical point \( m(0) \) of the phase transition. Such jumping of the particle density definitely indicates the first order feature of the phase transition.

On the other hand, when \( T \neq 0 \), near a critical chemical potential \( \mu_c \), the order parameter \( m \) will continuously vary in the form\(^7\)

\[ m^2 \simeq 2T \left( \sinh \frac{\mu}{T} \right) (\mu_c - \mu), \quad \text{when } T \neq 0. \]  

(15)

Substituting Eq. (15) into Eq. (11) we will obtain

\[ n_3(T, \mu, m) = n_3(T, \mu, m = 0) - \begin{cases} \frac{T}{2\pi} \left( e^\mu - 1 \right) \left( -\sinh \frac{\mu}{T} \right) (\mu_c - \mu) & \text{when } \mu \lesssim \mu_c, \\ 0 & \text{when } \mu > \mu_c, \end{cases} \]  

(16)

and

\[ \frac{\partial n_3(T, \mu, m)}{\partial \mu} \simeq \frac{\partial n_3(T, \mu, m = 0)}{\partial \mu} + \begin{cases} \frac{T}{2\pi} \left( e^\mu - 1 \right) \sinh \frac{\mu}{T} & \text{when } \mu \lesssim \mu_c, \\ 0 & \text{when } \mu > \mu_c. \end{cases} \]  

(17)

It is seen from Eqs. (16) and (17) that at \( T \neq 0 \) when \( \mu \) crosses over \( \mu_c \), \( n_3(T, \mu, m) \) continuously increases from the value less than \( n_3(T, \mu, m = 0) \) up to \( n_3(T, \mu, m = 0) \), however the derivative \( \partial n_3(T, \mu, m)/\partial \mu \) has a discontinuous jumping at \( \mu = \mu_c \). This shows the second order feature of the phase transition. We emphasize that this feature can be maintained when \( T \) is finite and very small as long as \( \mu_c - \mu \) is also very small so that \( m^2/T^2 \ll 1 \) is kept. The limiting point will be \((T = 0, \mu = m(0))\), which is the same as the first order phase transition point when \( T = 0 \). The above discussions give a physical explanation of implications of the tricritical point \((T, \mu) = (0, m(0))\) in 3D GN model, i.e., the particle density will jump from 0 to \( \mu^2/4\pi \) when the point \((0, m(0))\) is approached along the \((T = 0, \mu)\) axis but will continuously increase from a lower value to \( \mu^2/4\pi \) when the point is approached along the \(T - \mu\) critical curve of the second order phase transitions.

4 \( D = 4 \)

For 4D NJL model, when \( m = 0 \) we will have

\[ n_4(T, \mu, m = 0) = \frac{1}{2\pi^2} \int_0^{\infty} dp pp^2 \left[ \frac{1}{\cosh(p + \mu)} - \frac{1}{\cosh(p - \mu)} \right]. \]  

(18)

The equivalent differential equation is

\[ \frac{\partial^2 n_4(T, \mu, 0)}{\partial \mu^2} = \frac{T}{\pi^2} [\ln(1 + e^{\beta \mu}) - \ln(1 + e^{-\beta \mu})] = \frac{\mu}{\pi^2} \]  

(19)

with the boundary conditions

\[ n_4(T, \mu = 0, 0) = 0, \quad \frac{\partial n_4(T, \mu, 0)}{\partial \mu} \bigg|_{\mu=0} = \frac{T^2}{6}. \]  

(20)

The solution of Eq. (19) satisfying the conditions in Eq. (20) is that

\[ n_4(T, \mu, 0) = \frac{\mu^3}{6\pi^2} + \frac{T^2}{6} \mu \frac{T - o}{\mu^3} \frac{\mu^3}{6\pi^2}. \]  

(21)
When \( m \neq 0 \) and \( T \to 0 \) \((y \to \infty)\), we may obtain from Eq. (2) that
\[
n_4(T = 0, \mu, m) = \lim_{y \to \infty} \frac{m^3}{2\pi^2} \int_1^{\infty} dz z \sqrt{z^2 - 1} \left[ \frac{1}{e^{y(z+\alpha)} + 1} \right]
\]
\[
= \begin{cases} 
0 & \text{when } \alpha \leq 1, \\
\frac{m^3}{2\pi^2} \int_1^{\infty} dz z \sqrt{z^2 - 1} & \text{when } \alpha > 1.
\end{cases}
\]
\[
= \theta(\mu - m) \frac{\mu^3}{6\pi^2} \left(1 - \frac{m^2}{\mu^2}\right)^{3/2}.
\]

Equation (22) is a general result of free fermions and usually does not involve any spontaneous symmetry breaking and restoration. Now consider the case in which symmetry restoring phase transition is assumed to occur. We should identify \( m \) in Eq. (22) with the dynamical fermion mass. It has been proven that in a 4D NJL model, the dynamical fermion mass \( m \) has the following behavior,[8]
\[
m = m(0) \quad \text{when } \mu \leq m(0),
\mu^2 - \mu_0^2 - m^2 = \frac{m^2}{2} \ln \frac{\Lambda^2 + m^2}{(\mu + \sqrt{\mu^2 - m^2})^2}
\]
\[
\text{when } m(0) < \mu < \mu_0,
\]
where \( \Lambda \) is the 4-dimensional Euclidean momentum cutoff and the critical chemical potential \( \mu_0 \) is defined by
\[
\mu_0^2 = \frac{1}{2} m^2(0) \ln \left[ \frac{\Lambda^2}{m^2(0)} + 1 \right].
\]
We will assume that
\[
\mu_0^2 \leq \frac{\Lambda^2}{4\pi},
\]
so that all the phase transitions will be second-order.[8] Combining Eq. (22) with Eq. (23) we may find that when \( \mu \leq m(0) \), \( n_4(T = 0, \mu, m) = 0 \) and when \( m(0) < \mu < \mu_0 \), \( n_4(T = 0, \mu, m) \) will increase in the form \( \mu^3(1 - m^2/\mu^2)^{3/2}/6\pi^2 \) (noting that \( \partial m/\partial \mu < 0 \) in this region). As soon as \( \mu \) arrives at \( \mu_0 \), where symmetries will be restored, \( m \) reduces to zero and \( n_4(T = 0, \mu, m) \) becomes \( \mu^3/6\pi^2 \), which is just the number density of massless fermions given by Eq. (21). It should be emphasized that, since \( m \) reduces to zero at \( \mu_0 \), continuously, \( n_4(T = 0, \mu, m) \) changes from 0 into \( \mu^3/6\pi^2 \) also in continuous form when \( \mu \) varies from \( m(0) \) to \( \mu_0 \), and this is just the characteristic of a second-order phase transition. The above result can be outlined as
\[
n_4(T = 0, \mu, m) = \begin{cases} 
\frac{1}{6\pi^2} (\mu^2 - m^2)^{3/2}, & \text{when } m(0) < \mu < \mu_0, \\
\frac{\mu^3}{6\pi^2}, & \text{when } \mu > \mu_0.
\end{cases}
\]
We can also prove that the derivative of \( n_4(T = 0, \mu, m) \) over \( \mu \) will be discontinuous when \( \mu \) crosses \( \mu_0 \). To this end, we will use the critical behavior of the squared order parameter \( m^2 \),
\[
m^2 \simeq \frac{(\mu_0^2 - \mu^2)}{\ln(\Lambda/2\mu) - 1/2} \simeq 2(\mu_0 - \mu) b(\mu_0), \quad \text{when } \mu \lesssim \mu_0,
\]
where
\[
b(\mu_0) = 1 \left/ \left( \frac{\Lambda}{2\mu_0} - 1/2 \right) \right. > 0
\]
owing to Eq. (25). Substituting Eq. (27) into Eq. (26), we get
\[
n_4(T = 0, \mu, m) = \begin{cases} 
\frac{1}{6\pi^2} [\mu^2 - 2(\mu_0 - \mu)b(\mu_0)]^{3/2}, & \text{when } \mu \lesssim \mu_0, \\
\frac{\mu^3}{6\pi^2}, & \text{when } \mu > \mu_0,
\end{cases}
\]
and furthermore,
\[
\frac{\partial n_4(T = 0, \mu, m)}{\partial \mu} = \begin{cases} 
\frac{\mu^2}{2\pi^2} + \frac{\mu_0 b(\mu_0)}{2\pi^2}, & \text{when } \mu \lesssim \mu_0, \\
\frac{\mu^2}{2\pi^2}, & \text{when } \mu > \mu_0.
\end{cases}
\]
Equation (29) indicates that $\partial n_4(T = 0, \mu, m) / \partial \mu$ has a jumping over $\mu = \mu_c$ and is discontinuous. The above results verify that the discussed symmetry restoring phase transition in the 4D NJL model is second order indeed.

5 Conclusions

By combining the calculated fermion number densities as a thermodynamical quantity with the known critical behaviors of the dynamical fermion mass in the four-fermion interaction models, we have shown that when temperature $T = 0$, across over the critical chemical potential $\mu_c$, the particle number density will have a discontinuous jumping in 2D and 3D GN model and these physically explain the first order feature of corresponding zero temperature symmetry restoring phase transitions. The second order phase transitions in 3D GN model when $T \neq 0$ and in 4D NJL model when $T = 0$ and $\mu_c^2 \leq \Lambda^2 / 4e$ are illustrated by the fact that, through a critical chemical potential, the densities themselves are continuous but their derivatives with respect to chemical potential $\mu$ have a discontinuous jumping. These results clearly show physical difference between the first order and second order symmetry restoring phase transition in these models, especially give a physical explanation of implications of the tricritical point in the 3D GN model. The whole discussions also indicate the full effectiveness of critical analysis of the order parameter for phase transition problem.

References

[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.
[2] D.J. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.
[3] D.A. Kirzhnits and A.D. Linde, Phys. Lett. B42 (1972) 471; S. Weinberg, Phys. Rev. D7 (1973) 2887; D9 (1974) 3357; L. Dolan and R. Jackiw, ibid. D9 (1974) 3320.
[4] A.D. Linde, Rep. Prog. Phys. 42 (1979) 389; R.H. Brandenberger, Rev. Mod. Phys. 57 (1985) 1.
[5] B. Rosenstein, B.J. Warr, and S.H. Park, Phys. Rep. 205 (1991) 59; B.R. Zhou, Phys. Rev. D57 (1998) 3171; B.R. Zhou, Commun. Theor. Phys. (Beijing, China) 32 (1999) 425; B.R. Zhou, Phys. Lett. B444 (1998) 455; B.R. Zhou, Commun. Theor. Phys. (Beijing, China) 33 (2000) 451.
[6] U. Wolff, Phys. Lett. B157 (1985) 303; F. Karsch, J. Kogut, and H.W. Wyld, Nucl. Phys. B280 (1987) 289; K.G. Klimentko, Z. Phys. C37 (1988) 457; A. Chodos and H. Minakata, Phys. Lett. A191 (1994) 39; Nucl. Phys. B490 (1997) 687; T. Inagaki, T. Kouno, and T. Muta, Int. J. Mod. Phys. A10 (1995) 2241; A. Chodos, F. Cooper, W. Mao, H. Minakata, and A. Singh, Phys. Rev. D61 (2000) 045011; S. Hands, S. Kim, and J.B. Kogut, Nucl. Phys. B442 (1995) 364; S. Hands, Nucl. Phys. A642 (1998) 228c; J.B. Kogut and C.G. Strouthos, Phys. Rev. D63 (2001) 054502.
[7] B.R. Zhou, Commun. Theor. Phys. (Beijing, China) 40 (2003) 67.
[8] B.R. Zhou, Commun. Theor. Phys. (Beijing, China) 40 (2003) 669.
[9] J.I. Kapusta, *Finite-Temperature Field Theory*, Cambridge University Press, England (1989).