Universality of tunnelling particles in Hawking radiation

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Abstract

The complex path (or Hamilton–Jacobi) approach to Hawking radiation corresponds to the intuitive picture of particles tunnelling through the horizon and forming a thermal radiation. This method computes the tunnelling rate of a given particle from its equation of motion and equates it to the Boltzmann distribution of the radiation from which the Hawking temperature is identified. In agreement with the original derivation by Hawking and the other approaches, it has been checked case by case that the temperature is indeed universal for a number of backgrounds and the tunnelling of particles from spin 0 to 1 (and in some cases with spin 3/2 and 2). In this letter we give a general proof that the temperature is indeed equal for all (massless and massive) particles with spin from 0 to 2 on an arbitrary background (limited to be Einstein for spin greater than 1) in any number of dimensions. Moreover we propose a general argument to extend this result to any spin greater than 2.
1 Introduction

In his seminal paper [1] Hawking proved that black holes emit a thermal radiation at a temperature $T$ due to quantum mechanical effects. The intuitive picture of this radiation is the following: pairs of virtual particles created near a black hole horizon through vacuum fluctuations become real once one of them cross the horizon while the other extracts energy from the black hole. This idea has lead to two different approaches of the Hawking radiation: the complex path (or Hamilton–Jacobi) method due to Shankaranarayanan–Srinivasan–Padmanabhan [2–4] (see also [5]), and the null geodesic method (or Parikh–Wilczek) method [6] (see [7] for a review). Both methods are not restricted to black hole radiation but can also be applied for any black hole with a thermal horizon for other background which can be thermal, such as the Rindler or de Sitter spaces. Moreover they can also be used to define the Hawking temperature in situations where the traditional methods are not defined [7].

The first method computes the tunnelling rate of a particle of a given spin $s$ by solving its equations of motion in the black hole background through a WKB approximation, and then equates this rates to the probability given by the Boltzmann distribution at temperature $T$.

The main drawback of this method is that the computations depend on the spin and the intermediate steps do not show any reason for the temperature to be universal. Nonetheless it has been checked explicitly case by case for many backgrounds that the tunnelling of particles with different spins (mostly $s = 0, 1/2, 1$, but also $s = 3/2, 2$ in some instances) always yields the same temperature, see [8–16] for a selected sample and references therein for more details.

The goal of this letter is to prove that the Hawking temperature is universal for the tunnelling of neutral massless and massive particles with spin ranging from 0 to 2 for a generic background in any dimension (restricted to be Einstein for $s = 3/2, 2$). This is achieved by showing that, in the WKB approximation, the equation of motion for a spin $s \leq 2$ particle reduces to the Hamilton–Jacobi equation of a scalar field. We then give a general argument to extend this result to massive particle of any spin $s > 2$. Moreover we stress that our proof is fully covariant, in contrast with the former computations which were not explicitly covariant since the fields and the background metric were written in components.

The limitation on the background for spins $s > 1$ and the need of non-minimal coupling are related to the well-known difficulty propagating higher-spin particles on a curved background [17–19]. An interesting question would be to analyse the subleading quantum corrections and the deviation from thermality due to the backreaction of the radiation and to see whether they differ for the different types of particles. The generalization of our argument to background with gauge fields under which the particles are charged is expected to be straightforward, even if one can expect difficulties already for $s = 1$ due to inconsistencies in the coupling of spin $s \geq 1$ to electromagnetic fields [20, 21].

In section 2 we review the complex path method for a scalar field and we show in section 3 how the higher-spin cases reduce to this case.
2 Complex path method for the scalar field

One considers a background (case of interests being black holes, the Rindler space, etc.) in $d$ dimensions described by a fixed metric $g_{\mu \nu}$. For definiteness the background metric is taken to be a solution of the Einstein equations with a cosmological constant

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} = 0, \quad (1)$$

where $\Lambda$ is the cosmological constant, $R_{\mu \nu} = R_{\rho \mu \rho \nu}$ is the Ricci tensor obtained by contracting the Riemann tensor, and $R = g^{\mu \nu} R_{\mu \nu}$ is the Ricci scalar. As we will see, this restriction to Einstein spacetimes for deriving the Hawking temperature only concerns spins higher than 1. The Laplacian on this background is defined by

$$\Delta = g^{\mu \nu} \nabla_\mu \nabla_\nu \quad (2)$$

where $\nabla_\mu$ is the covariant derivative with the Levi–Civita connection. The tunnelling rate $\Gamma$ for a particle is given by

$$\Gamma = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{|\phi_{\text{out}}|^2}{|\phi_{\text{in}}|^2} \quad (3)$$

where $P_{\text{in}}$ ($P_{\text{out}}$) is the tunnelling probability for an ingoing (outgoing) particle and $\phi_{\text{in}}$ ($\phi_{\text{out}}$) is the associated solution to the equation of motion. Assuming that the radiation is thermal\(^1\) this rate can be equated to the Boltzmann distribution through the detailed balance

$$\Gamma = e^{-\frac{E_{\text{tot}}}{T}} \quad (4)$$

where $E_{\text{tot}}$ is the total energy (including kinetic, rotational, electromagnetic, etc.) carried by the particle tunnelling, and measured by a freely falling observer in the vicinity of the external horizon.

From this point we consider a free (massive or massless) spin 0 scalar field $\phi$. The equation of motion for a scalar field in a curved background with non-minimal coupling

$$\left(-\Delta + \frac{m^2}{\hbar^2} + \xi R\right) \phi = 0 \quad (5)$$

where $m^2$ can be zero. This equation can be solved at leading order in $\hbar$ through the WKB approximation

$$\phi = \phi_0 \, e^{iS/\hbar}, \quad (6)$$

where $\phi_0$ is a constant wave function. Inserting this ansatz into (5) provides, at leading order in $\hbar$, the Hamilton–Jacobi equation on curved space

$$g^{\mu \nu} \partial_\mu S \partial_\nu S + m^2 = 0, \quad (7)$$

and allows to identify $S$ with the classical action and one can note that the non-minimal coupling term is subleading (such terms are also present for higher spins and will not contribute at leading order). It can be solved quite generically with the following ansatz [5, 10]

$$S_{\text{out}} = -Et + W(r_0) + F(x^i) + K, \quad S_{\text{in}} = -Et - W(r_0) + F(x^i) + K, \quad (8)$$

where $t$ is the time, $r_0$ the radial location of the horizon and $x^i$ denotes any other coordinate, and $K$ is a complex constant. One needs to ensure that the ingoing probability is one in the classical limit because the horizon necessarily absorbs the particle. This manifests differently

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\(^1\)This hypothesis is not strictly correct due to backreaction of the radiation on the geometry [6], but we will ignore this effect for our purpose.
depending on the choice of coordinates. It may occur that the inverse of the radial velocity has no pole for an ingoing classical particle, implying that this imaginary part vanishes. If this is not the case then one needs to find the relation between the ingoing and outgoing actions such that this condition holds. Both situations amount to setting $\text{Im } K = \text{Im } W(r_0)$ and one finally obtains the tunnelling rate

$$\Gamma = e^{-E_{tot}/T} = e^{-2(\text{Im } S_{out} - \text{Im } S_{in})/\hbar} = e^{-4\text{Im } W(r_0)/\hbar},$$

(9)

which yields the temperature:

$$T = \frac{\hbar E_{tot}}{4\text{Im } W(r_0)}.$$ 

(10)

In order to make contact with the well-known formula of the Hawking radiation, one can show for general rotating black holes (including the Schwarzschild back hole as a limiting case) that the expression for $W(r_0)$ is proportional to the surface gravity $\kappa$:

$$\text{Im } W(r_0) = \frac{\pi E_{tot}}{2\kappa},$$

(11)

and the final result agrees with the well know Hawking temperature formula [1]

$$T = \frac{\hbar \kappa}{2\pi}.$$ 

(12)

The reason is that $W(r_0)$ is defined by an integral over $r$ with a pole at the horizon due to the presence of the metric components in the denominator: evaluating the integral yields a residue (imaginary) proportional to the surface gravity. In the case of Schwarzschild one finds

$$f(r) = 1 - \frac{2M}{r}, \quad r_0 = 2M \quad \Rightarrow \quad \kappa = \frac{f'(r_0)}{2} = \frac{1}{4M}, \quad T = \frac{\hbar}{8\pi M}.$$ 

(13)

### 3 Tunnelling of higher-spins

In this section – which contains our new results – we will extend the results of the previous section and show that the equations of motion for higher-spin particles reduce to the Hamilton–Jacobi equation (7) of a scalar field in the leading order of the WKB approximation. This is sufficient to establish that the temperature will be given by (10) and thus identical for all spins.\footnote{This point involves different subtleties and making a precise statement is very coordinate-dependent. We refer the reader to the literature for more details [5, 22–25].}

#### Spin 1/2

The equation of motion for a spin 1/2 fermion $\psi$ is

$$\left(\not\!D - \frac{m}{\hbar}\right)\psi = 0$$ 

(14)

where $\not\!D \equiv \gamma^\mu \nabla_\mu$ and $\gamma^\mu$ are the Dirac matrices. The multiplication of (14) with $\not\!D$ gives the second order partial derivative equation:

$$-\Delta \psi + \frac{1}{4} R\psi + \frac{m^2}{\hbar^2} \psi = 0.$$ 

(15)

As for the scalar field, the WKB approximation for this equation can be investigated using the following ansatz

$$\psi = \psi_0 e^{iS/\hbar},$$ 

(16)

where $\psi_0$ is constant. Putting this ansatz in (15), we deduce an equation for $S$, and keeping only the leading order terms in $\hbar$, it reduces to the the scalar Hamilton–Jacobi equation (7).\footnote{For this it is important that the evaluation of the action in (10) depends only on the properties of the background and not on the type of the particle.}
Spin 1 The equation of motion for a massive vector field \(A_\mu\) can be derived from the standard Proca Lagrangian, and may be written in the form

\[
0 = -\Delta A_\mu + \nabla_\nu \nabla_\mu A^\nu + \frac{m^2}{\hbar^2} A_\mu.
\]  

(17a)

Up to straightforward manipulations, this equation is equivalent to

\[
-\Delta A_\mu + R_{\mu\nu} A^\nu + \frac{m^2}{\hbar^2} A_\mu = 0
\]

(18)

together with the constraint

\[
\nabla_\mu A^\mu = 0
\]

(19)

which can be imposed at the dynamical level as a consequence of the equation of motion for \(m^2 \neq 0\), or through a gauge transformation

\[
\delta A_\mu = \nabla_\mu \alpha,
\]

(20)

for vanishing mass, the scalar field \(\alpha\) being the gauge parameter. Remember that this constraint is necessary for ensuring that the correct degrees of freedom propagate (the spin 1) while the extraneous ones are removed (the spin 0 part). In the leading order of the WKB approximation

\[
A_\mu = A_{0\mu} e^{iS/\hbar}
\]

(21)

the equation (18) corresponds to the scalar Hamilton–Jacobi equation (7).

Note that for the two previous cases it was not necessary to use the fact that the background metric is a solution of the Einstein equation (1). Hence the universality of Hawking temperature for spin \(s = 0, 1/2, 1\) is valid for any background, irrespective of the theory of gravity or the matter content under consideration, with the exception of gauge couplings.\(^4\)

Spin 3/2 The massive Rarita–Schwinger field is described by a (bi-)spinor-valued vector field \(\psi_\mu\) whose equation of motion is:

\[
\gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho - \frac{m}{\hbar} \gamma^{\mu\nu} \psi_\nu = 0.
\]

(22)

Some lengthy but simple manipulations [26] show that \(\psi_\mu\) obeys the Dirac equation

\[
\left(\slashed{\nabla} - \frac{m}{\hbar}\right) \psi_\mu = 0,
\]

(23)

together with the condition

\[
\psi = 0.
\]

(24)

Note that these conditions result from the equation of motion (22) if

\[
m^2 \neq 0, m_0^2, \quad m_0^2 = \frac{d - 2}{2(d - 1)} \hbar^2 \Lambda,
\]

(25)

or from the gauge invariance under the following transformation otherwise:

\[
\delta \psi_\mu = \left(\nabla_\mu - \frac{m_0}{(d - 2)} \gamma_\mu\right) \epsilon,
\]

(26)

\(^4\)Indeed the coupling to the gauge field in the covariant derivative comes with a factor \(\hbar^{-1}\). On the other coupling to other scalar and fermions fields can come only with positive power of \(\hbar\), implying that these terms do not contribute at the leading order of the WKB approximation.
where $\epsilon$ is a spinor-valued gauge parameter. As discussed for the spin 1, the constraint ensures that only the spin 3/2 part of the field propagates. However it can be imposed only if the background satisfies the Einstein equation (1) for the spin 3/2. Finally the equation (23) can be multiplied with $\nabla$ which leads to

$$-\Delta\psi_{\rho} + \gamma_{\mu\nu} R_{\mu\nu\rho} \psi_{\sigma} + \frac{R}{4} \psi_{\rho} + \frac{m^2}{\hbar^2} \psi_{\rho} = 0,$$

and inserting the WKB ansatz

$$\psi_{\mu} = \psi_{0\mu} e^{iS/\hbar}$$

inside the equation (27) brings it to the form of (7) at the leading order in $\hbar$.

**Spin 2** The massive spin 2 field is usually described by a symmetric tensor of rank 2, $h_{\mu\nu}$, whose equation of motion may be written as [27]

$$-\Delta h_{\mu\nu} + g_{\mu\nu} \Delta h - \nabla_{\mu} \nabla_{\nu} h - g_{\mu\nu} \nabla^{\rho} \nabla^{\sigma} h_{\rho\sigma} + \nabla^{\rho} \nabla_{\mu} h_{\nu\rho} + \nabla^{\rho} \nabla_{\nu} h_{\mu\rho} - \frac{2\xi}{d} R h_{\mu\nu} - \frac{1 - 2\xi}{d} R h_{\mu\nu} + \frac{m^2}{\hbar^2} (h_{\mu\nu} - hg_{\mu\nu}) = 0$$

(29)

where $\xi$ is an arbitrary parameter parametrizing the non-minimal coupling (the latter is necessary in order to get the correct constraints on the propagating degrees of freedom below). Then the equation (29) can be simplified to

$$-\Delta h_{\mu\nu} - 2R_{\rho\sigma} h_{\rho\sigma} - \frac{2(\xi - 1)}{d} R h_{\mu\nu} + \frac{m^2}{\hbar^2} h_{\mu\nu} = 0$$

(30)

together with the constraints

$$h = 0, \quad \nabla^{\mu} h_{\mu\nu} = 0$$

(31)

if the background satisfies the Einstein equation (1). In the case where the condition

$$m^2 \neq m_0^2 \equiv -\frac{4\hbar^2(1 - \xi)}{d - 2} \Lambda$$

(32)

holds then the constraints (31) result from the equation of motion (29) [27]. Otherwise if $m^2 = m_0^2$ then they can be imposed through a gauge transformation

$$\delta h_{\mu\nu} = \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu}.$$  

(33)

Note that this include the case of the graviton propagating on a curved space which corresponds to $m^2 = 0$ and $\xi = 1$ [28]. In the WKB approximation

$$h_{\mu\nu} = h_{0\mu\nu} e^{iS/\hbar}$$

(34)

the equation (29) is again equivalent to (7).

**Higher spins** More generally one can consider a massive particle of arbitrary integer spin $s > 2$ (the case of half-integer is a straightforward extension) represented by a field $\phi_{\mu_1 \cdots \mu_s}$ symmetric in all indices for which the equation of motion is

$$-\Delta \phi_{\mu_1 \cdots \mu_s} + f(R) \phi_{\mu_1 \cdots \mu_s \nu_1 \cdots \nu_s} + \frac{m^2}{\hbar^2} \phi_{\mu_1 \cdots \mu_s} = 0$$

(35)

5To our knowledge the gauge transformation (33) has not been discussed elsewhere for generic $\xi$. 

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after elimination of the auxiliary fields and imposing the constraints \[29–31\]

\[\nabla^\mu \phi_{\mu \nu \cdots \mu s} = 0, \quad g^{\mu \nu} \phi_{\mu \nu \cdots \mu s} = 0, \quad (36)\]

where \(f(R)\) is a function of the Riemann tensor and its contractions, arising both from anti-commutation of covariant derivatives and from non-minimal coupling terms (which ensures causality and unitarity \[32\]). Introducing the WKB ansatz

\[\phi_{\mu_1 \cdots \mu s} = \phi_{0, \mu_1 \cdots \mu s} e^{iS/h} \quad (37)\]

yields the Hamilton–Jacobi equation \((7)\). The reason is that curvature terms cannot have factors of \(h\) because they do not contain derivatives as the Laplacian or built-in factors as the mass term. Any other term would be eliminated by the constraints (which are necessary for the theory to exist and be consistent).

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