\( \mathcal{N}=4 \) Supersymmetric \( d = 1 \) Sigma Models on Group Manifolds

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Abstract

We construct manifestly \( \mathcal{N} = 4 \) supersymmetric off-shell superfield actions for the HKT \( d = 1 \) sigma models on the group manifolds U(2) and SU(3), using the harmonic \( d = 1 \) approach. The underlying \((4,4,0)\) and \((4,4,0) \oplus (4,4,0)\) multiplets are described, respectively, by one and two harmonic analytic superfields \( q^+ \) satisfying the appropriate nonlinear harmonic constraints. The invariant actions in both cases are bilinear in the superfields. We present the corresponding superfield realizations of the U(2) and SU(3) isometries and show that in fact they are enlarged to the products U(2)×SU(2) and SU(3)×U(2). We prove the corresponding invariances at both the superfield and component levels and present the bosonic \( d = 1 \) sigma model actions, as integral over \( t \) in the U(2) case and over \( t \) and SU(2) harmonics in the SU(3) case. In the U(2) case we also give a detailed comparison with the general harmonic approach to HKT models and establish a correspondence with a particular action of the off-shell nonlinear multiplet \((3,4,1)\). A possible way of generalizing U(2) model to the matrix U(2n) case is suggested.

PACS: 11.30.Pb, 11.15.-q, 11.10.Kk, 03.65.-w
Keywords: Supersymmetry, geometry, superfield
1 Introduction

The $\mathcal{N} = 4$ supersymmetric $d = 1$ sigma models based on the multiplets with the off-shell content $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ are known to lead to the HKT (“Hyper-Kähler with torsion”) geometry in the bosonic target space and, as a particular case, the HK (“Hyper-Kähler”) geometry [2] - [17]. The intrinsic geometries of supersymmetric sigma models are displayed in the clearest way in the appropriate superfield formulations, with all supersymmetries being manifest and off-shell [18] - [21]. For HKT $d = 1$ sigma models with a target space of real dimension $4n$ such a general formulation was proposed in our paper [22], where it was shown that they are described by $2n$ analytic harmonic superfields $q^{+a}, a = 1, \ldots 2n$, (with the appropriate reality conditions) subjected to the nonlinear harmonic constraint

$$D^{++} q^{+a} = L^{+3a}(q^{+}, u^{\pm}), \quad (1.1)$$

where $L^{+3a}$ is an arbitrary analytic function carrying the harmonic charge $+3$. The general superfield action is the following integral over the total $\mathcal{N} = 4, d = 1$ harmonic superspace,

$$S = -\frac{1}{8} \int dt du d^4\theta L(q^{+}, q^{-}, u^{\pm}), \quad q^{-a} := D^{-+} q^{+a}. \quad (1.2)$$

The analytic function $L^{+3a}(q^{+}, u)$ and the general function $L(q^{+}, q^{-}, u^{\pm})$ are two independent HKT “prepotentials” fully characterizing the given HKT geometry. The general expressions for the target metric and torsion were given and a few interesting particular cases were discussed. It was shown that the general HK geometries arise for the simplest choice

$$L = q^{+a} q^{-b} \Omega_{ab}, \quad L^{+3a}(q^{+}, u) = \frac{\partial L^{+4}(q^{+}, u)}{\partial q^{-a}}, \quad (1.3)$$

where $\Omega_{ab} = -\Omega_{ba}, \Omega_{ab}^{bc} = \delta^{c}_{a}$, is a constant symplectic metric, while $L^{+4}$ is the renowned Hyper-Kähler potential [20], [19]. If $L^{+3a}(q^{+}, u)$ in (1.3) is arbitrary, i.e. is not subject to the second condition but $L$ remains quadratic, the relevant geometries are strong HKT, that is, such that the corresponding torsion is closed. This class of HKT geometries is the same as in the $d = 2 \mathcal{N} = (4, 0)$ heterotic sigma models and the analytic prepotential $L^{+3a}(q^{+}, u)$ coincides with the one introduced in [13]. The case of general HKT potentials $L^{+3a}$ and $L$ amounts to the “weak” HKT geometry, in which there is no closedness condition on the torsion two-form.

A more detailed geometric analysis of the two-potential formulation of the HKT geometry was undertaken in a recent paper [25].

While for the HK manifolds a few concrete examples of $\mathcal{N} = 4, d = 1$ sigma models were presented (see, e.g., [24], [26] - [28]), not too many explicit examples of this kind are known for the HKT case. It still remains to find the precise form of the potentials for the HKT manifolds known in the literature. On the other hand, there exists a wide class of homogeneous group-manifold strong HKT metrics associated with those groups which admit a quaternionic structure

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1In this notation [1], the first two numerals stand for the number of physical bosonic and fermionic fields and the third one for the number of auxiliary fields.

2Our notation is the same as in [22], [23], [24]. The reader can find there all the necessary information and further references related to the $d = 1$ harmonic superspace formalism.

3It is comparatively easy [25] to figure out these potentials for one of the inhomogeneous strong HKT examples given in [10] (the “Taub-NUT with torsion” model).
The full list of such groups was given in [7], and it is tempting to explicitly construct the relevant \( \mathcal{N} = 4 \) HKT sigma models. The simplest non-trivial series from this list are

\begin{equation}
\begin{aligned}
(\text{a}) & \quad \text{SU}(2n) \times \text{U}(1); \\
(\text{b}) & \quad \text{SU}(2n + 1),
\end{aligned}
\end{equation}

with the dimensions \( 4n^2 \) and \( 4n(n + 1) \). Following reasoning of [22], we can expect that the corresponding superfield Lagrangians are always given by bilinears \( \sim q^+ q^- \) and the whole non-triviality of one or another case is encoded in the structure of the potential \( L^{+3a} \).

In the present paper we construct the HKT sigma models for the simplest representatives of the series (1.4), viz. the group manifolds \( \text{SU}(2) \times \text{U}(1) \) and \( \text{SU}(3) \). The first case is worked out in detail in sections 2 and 3. We start by giving the full superfield formulation of the relevant sigma model and then consider the component bosonic limit of the superfield action. The latter is the action of the \( d = 1 \) sigma model with the target space \( S^3 \times S^1 \), where \( S^3 \sim \text{SU}(2)_L \times \text{SU}(2)_R / \text{SU}(2)_{\text{diag}} \) and \( \text{SU}(2)_{\text{diag}} \) is none other than the standard harmonic \( \text{SU}(2) \) group. The fact that the full symmetry of the model is the product \( \text{U}(1) \times \text{SU}(2) \times \text{SU}(2) \) is non-trivial: this property does not immediately follow from the form of the superfield action. We also gauge the \( \text{U}(1) \) symmetry of the model by a non-propagating \( \mathcal{N} = 4 \) gauge multiplet along the line of ref. [24]. The outcome is a particular model of the nonlinear \((3, 4, 1)\) multiplet with the \( S^3 \) sigma model in the bosonic sector. We present both superfield and bosonic component actions for this case. In section 3 we perform the detailed comparison of the \( \text{U}(2) \) model with the general harmonic approach to HKT models developed in [22] and find complete agreement. We explicitly construct the corresponding closed torsion, as well as the triplet of quaternionic complex structures which are covariantly constant with respect to the relevant Bismut connection. The non-trivial symmetric Obata connection is also explicitly constructed, such that the complex structures are covariantly constant with respect to it. The Riemann curvatures of both Bismut and Obata connections are shown to vanish. In section 4 we proceed to the \( \text{SU}(3) \) case. It is much more complicated than the \( \text{U}(2) \) case. This time, we deal with two independent \( q^+ \) superfields one of which is conveniently represented in the \((N^{++}, \omega)\) basis, like in the previous case. We construct the corresponding \( \text{SU}(3) \) covariant superfield constraints and show that the invariant superfield action is bilinear in the involved superfields, in full agreement with the general assertion of ref. [22]. Further we pass to the bosonic limit and show that it reveals one more symmetry, \( \text{U}(2) \), an essential part of which is the harmonic \( \text{SU}(2) \) as in the previous \( \text{U}(2) \) system. The extra \( \text{U}(2) \) is shown to commute with \( \text{SU}(3) \), so the full symmetry of the model is \( \text{SU}(3) \times \text{U}(2) \). The relevant \( d = 1 \) nonlinear sigma model is associated with the 8-dimensional coset \([\text{SU}(3) \times \text{U}(2)] / \text{U}(2)_{\text{diag}}\). We solve the bosonic sector of the initial superfield constraints, find the bosonic action as an integral over \( t \) and harmonics and explicitly do the harmonic integration in a part of this action. In Appendix A, without performing the integration over harmonics, we independently demonstrate that the bosonic action possesses an \( \text{SU}(3) \times \text{U}(2) \) invariance. In the last section 5 we suggest a possible way of extending our considerations to generic \( n \) in (1.4b). Some useful harmonic integrals are collected in Appendices B and C.
2 SU(2) × U(1) group manifold

2.1 Superfield constraints and action

This case is described by two real harmonic analytic superfields \( N^{++} = \tilde{N}^{++} \) and \( \omega = \tilde{\omega} \) subjected to the harmonic constraints

\[
\begin{align*}
D^{++}N^{++} + (N^{++})^2 &= 0, \\
D^{++}\omega - N^{++} &= 0.
\end{align*}
\]

This set of constraints is invariant under the following infinitesimal SU(2) transformations:

\[
\delta N^{++} = \lambda^{++} - 2\lambda^+ - N^{++} + \lambda^{--}(N^{++})^2, \quad \delta \omega = \lambda^{+-} - \lambda^{--}N^{++},
\]

where

\[
\lambda^{\pm\pm} = \lambda^{(ik)}u_i^\pm u_k^\mp, \quad \lambda^{+-} = \lambda^{(ik)}u_i^+ u_k^-,
\]

and \( \lambda^{(ik)} \) are constant symmetric parameters of SU(2). These harmonic projections satisfy the relations

\[
D^{++}\lambda^{++} = 0, \quad D^{++}\lambda^{+-} = \lambda^{++}, \quad D^{++}\lambda^{--} = 2\lambda^{+-}.
\]

The invariant sigma-model type action is written as an integral over the whole harmonic \( \mathcal{N} = 4 \) superspace

\[
S_{su(2)} = -\frac{1}{4} \int dt du d^4 \theta N^{++} D^{-+}\omega
\]

(the particular normalization factor was chosen for further convenience). Using the fact that the integral of an analytic superfield over the full superspace is vanishing, as well as the constraints (2.1) together with the evident relation \( D^{-+}\lambda^{--} = 0 \), it is easy to check that the superfield action (2.5) is indeed invariant under the SU(2) transformations (2.2) up to a total harmonic derivative in the integrand. Though the action is bilinear in the involved superfields, in components it yields a nontrivial nonlinear SU(2) × U(1) group manifold sigma-model action. The extra U(1) acts as pure constant shifts of the superfield \( \omega \)

\[
\delta \omega = \lambda,
\]

and evidently leaves invariant both the constraints (2.1) and the action (2.5).

Combining the superfields \( \omega \) and \( N^{++} \) into yet another analytic superfield \( q^{+a} \) as

\[
q^{+a} := u^{+a} \omega - u^{-a} N^{++}, \quad \omega = -(u^- \cdot q^+), \quad N^{++} = -(u^+ \cdot q^+)
\]

the constraints (2.1) and the action (2.5) can be cast into the standard form [22]

\[
D^{++}q^{+a} = \mathcal{L}^{+3a}, \quad \mathcal{L}^{+3a} := u^{-a}(u^+ \cdot q^+)^2,
\]

\[
S_{su2} = -\frac{1}{8} \int dt du d^4 \theta q^{+a} D^{-+}q^+_a.
\]

Based on the results of [22], this reformulation immediately implies that the bosonic target space of the system under consideration is “strong HKT”, that is a HKT manifold with a

\[\text{[We use the same conventions as in [22, 24]. In particular, } \tilde{\text{ denotes the generalized conjugation [19, 21] preserving the analytic harmonic subspace and } D^{\pm\pm} = \partial^{\pm\pm} - 2i\theta^\pm \tilde{\theta}^\pm \partial_t + \theta^\pm \partial_{\theta^\pm} + \tilde{\theta}^\pm \partial_{\tilde{\theta}^\pm}, \tilde{\theta}^\pm = \tilde{\theta}^\pm, \quad \tilde{\theta}^\pm = -\theta^\pm.}\]
closed torsion 3-form. At the component level, this property is well known and is common to all group manifold HKT models [7]. In the present case we deal with the simplest non-trivial example of such sigma model corresponding to the four-dimensional target space \( SU(2) \times U(1) \). In what follows, it will be more convenient to deal with the superfields \( \omega \) and \( N^{++} \).

We also note that one more evident symmetry of the constraints (2.1) and the actions (2.5), (2.9) is the standard harmonic \( SU(2)^H \) realized on the harmonic variables as

\[
\delta_H u_i^\pm = \tau_i^{(k)} u_k^\pm .
\]

It induces linear \( SU(2) \) rotations of all component fields with respect to their doublet indices. We will see later that its presence ensures \( U(1) \times SU(2) \times SU(2) \) symmetry of the nonlinear sigma model appearing in the bosonic sector of the action (2.5).

In fact, in the present case \( SU(2)^H \) is a combination of the standard automorphism group of \( \mathcal{N} = 4 \) supersymmetry in the \( q^{+a} \) representation and of the so called Pauli-G"ursey \( SU(2)^{PG} \), which commutes with supersymmetry and acts on the doublet index of the superfield \( q^{+a} \). Such a non-uniqueness of the automorphism \( SU(2) \) group was mentioned in [21]. It is with respect to such “shifted” \( SU(2)^H \) that the superfield projections defined in (2.7) are singlets. Though the action (2.9) (and (2.5), up to a total harmonic derivative) is formally invariant under \( SU(2)^{PG} \) taken alone, the constraints (2.8) (or (2.1)) are covariant only with respect to the “shifted” \( SU(2)^H \).

### 2.2 Bosonic component action and its symmetries

In the limit where all fermionic fields are suppressed, the analytic superfields \( N^{++} \) and \( \omega \) have the following \( \theta \) expansion

\[
N^{++} = n^{++} + \theta^+ \bar{\theta}^+ \sigma , \quad \omega = \omega_0 + \theta^+ \bar{\theta}^+ \sigma^{-2} ,
\]

where all component fields are defined on the coordinate set \((t, u_i^\pm)\). The harmonic superfield constraints (2.1) imply the following ones for the component fields

\[
\begin{align*}
\partial^{++} n^{++} + (n^{++})^2 &= 0 , \\
\partial^{++} \sigma - 2i n^{++} + 2n^{++} \sigma &= 0 , \\
\partial^{++} \omega_0 - n^{++} &= 0 , \\
\partial^{++} \sigma^{-2} - 2i \dot{\omega}_0 - \sigma &= 0 .
\end{align*}
\]

After rather simple manipulations which make use of the completeness condition for harmonics, \( u_i^+ u_k^- - u_k^+ u_i^- = \varepsilon_{ik} \), one solves all these equations, except for the last one, as

\[
\begin{align*}
\omega_0 &= b + \ln \left( 1 + a^{+-} \right) , \\
\sigma &= \frac{1}{(1 + a^{+-})^2} \left\{ c + 2i \left[ \dot{a}^{+-} - a^{ik} u_i^+ u_k^- \right] \right\} ,
\end{align*}
\]

where the fields \( b, c \) and \( a^{ik} \) depend only on \( t \), but not on harmonics. After substituting these expressions into the remaining equation for \( \sigma^{-2} \) and taking the harmonic integral of both sides, we obtain a relation between the fields \( b(t) \) and \( c(t) \)

\[
c \int du \frac{1}{(1 + a^{+-})^2} + 2i \frac{d}{dt} \left[ b + \int du \ln(1 + a^{+-}) - \int du \frac{1}{1 + a^{+-}} \right] = 0 .
\]
Taking into account the last equation, we are left with a set of four $d = 1$ fields $b(t), a^{(ik)}(t)$ parametrizing (as it will become clear soon) the target $\text{SU}(2) \times \text{U}(1)$ group manifold. The harmonic integrals entering (2.14) can be explicitly carried out, giving us a simple expression for the field $c(t)$ in terms of these basic fields. We dropped in (2.14) the term

$$-2i \int du \frac{(a^i \dot{a}_i^k) u^+_i u^-_k}{(1 + a^{+-})^2}, \quad (2.15)$$

which is vanishing upon integration over harmonics.

The bosonic component action following from the superfield one (2.5) is

$$S_{\text{bos}} = \frac{i}{2} \int dt du \left( n^{++} \dot{\sigma}^2 + \sigma \dot{\omega}_0 \right) := \int dt du \mathcal{L}(t, u) = \int dt L(t). \quad (2.16)$$

After substituting the solutions (2.13) into $\mathcal{L}(t, u)$, integrating by parts with respect to $\partial_t$ and $\partial^{++}$, as well as using the constraint for $\sigma^{-2}$, the Lagrangian $\mathcal{L}(t, u)$ can be presented in the following form

$$\mathcal{L}(t, u) = \dot{b}^2 + ic \dot{a}^{+-} \frac{\dot{a}^{+-}}{(1 + a^{+-})^3} - (\dot{a}^{+-})^2 \frac{3 + a^{+-}}{(1 + a^{+-})^3}$$

$$+ a^i \dot{a}_i^k u^+_i u^-_k \left[ \frac{\dot{b}}{(1 + a^{+-})^2} + \frac{2 \dot{a}^{+-}}{(1 + a^{+-})^3} \right]. \quad (2.17)$$

It remains to calculate the harmonic integrals in (2.14) and (2.17). Using the formulas (B.2) - (B.5), after some algebra (2.14) can be reduced to

$$c = -2i \dot{b} \left( 1 + \frac{a^2}{2} \right), \quad \dot{b} := b + \frac{1}{2} \ln \left( 1 + \frac{a^2}{2} \right). \quad (2.18)$$

Next, as explained in Appendix A, the first term in the square bracket in (2.17) gives zero contribution to the harmonic integral in (2.16). The harmonic integral of the second term can be represented as

$$\sim a^i \dot{a}_i^k \dot{a}^{mn} \int du \frac{u^+_m u^-_n u^+_i u^-_k}{(1 + a^{+-})^3},$$

and

$$\int du \frac{u^+_m u^-_n u^+_i u^-_k}{(1 + a^{+-})^3} = a_{mn} a_{(ik)} f_1(a^2) + (\varepsilon_{mi} \varepsilon_{nk} + \varepsilon_{ni} \varepsilon_{mk}) f_2(a^2),$$

which is the most general structure compatible with the $\text{SU}(2)$ covariance. Substituting the second expression into the first one, we immediately find that the latter is also vanishing. Thus only the first line in (2.17) contributes. Using the formulas (B.12) - (B.14), it is easy to compute the remaining harmonic integrals and to present the final answer for the Lagrangian $L(t)$ in (2.16) as

$$L = (\dot{b})^2 + \frac{1}{2} \left( (\dot{a} \cdot \dot{a}) \frac{1}{1 + \frac{a^2}{2}} - \frac{1}{2} (\dot{a} \cdot a)^2 \frac{1}{(1 + \frac{a^2}{2})^2} \right). \quad (2.19)$$
Here, the field $\tilde{b}$ is invariant under the nonlinear realization of SU(2) acting on the second (sigma-model) piece of (2.19), while $a_{(ik)}$ are just Goldstone fields supporting this nonlinear realization. Thus (2.19) describes a nonlinear $d = 1$ sigma model on the group manifold $SU(2) \times U(1)$, parametrized by the $d = 1$ fields $a_{(ik)}(t)$ and $\tilde{b}(t)$. The U(1) factor acts as constant shifts of $\tilde{b}(t)$, while nonlinear SU(2) transformations of $a_{ik}$ can be found by considering the component-field realization of the SU(2) symmetry originally defined on the superfields $\omega$ and $N^{++}$.

Starting from the transformation of $n^{++}$,
\[ \delta n^{++} = \lambda^{++} - 2\lambda^{+-}n^{++} + \lambda^{--}(n^{++})^2, \]  
and using the first constraint in (2.12), one obtains an equation which determines $\delta a^{(ik)}$:
\[ \delta a^{++}(1 + a^{+-}) - \delta a^{+-}a^{++} = (1 + a^{+-})^2\lambda^{++} - 2\lambda^{+-}a^{++}(1 + a^{+-}) + \lambda^{--}(a^{++})^2. \]  
After some algebra, making use of the completeness relation for harmonics, one finds
\[ \delta a^{ik} = \lambda^{ik} + \frac{1}{2}(\lambda \cdot a)a^{(ik)} + \lambda^{il}a^{ik}, \]  
It is easy to check that the Lie bracket of these transformations is again of the same form, i.e. we deal with a realization of SU(2) on itself by left (or right) multiplications. The fields $a^{(ik)}(t)$ just provide a particular parametrization of the SU(2) group element.

Analogously, starting from the SU(2) transformation of $\omega_0$,
\[ \delta \omega_0 = \lambda^{+-} - \lambda^{--}n^{++} = \lambda^{+-} - \lambda^{--}\frac{a^{++}}{1 + a^{+-}}, \]  
and rewriting this variation in terms of the solution for $\omega_0$ in (2.13),
\[ \delta \omega_0 = \delta b + \frac{\delta a^{+-}}{1 + a^{+-}}, \]  
on one finds the SU(2) transformation law of the field $b(t)$:
\[ \delta b = -\frac{1}{2}(\lambda \cdot a). \]  
Now it is easy to show that the field $\tilde{b}(t)$ defined in (2.18) and entering the bosonic Lagrangian (2.19) is indeed inert under SU(2)
\[ \delta \tilde{b} = 0. \]  

We finally note that in fact the nonlinear sigma model part of the Lagrangian (2.19) is invariant under two independent $SU(2)$ symmetries, one being just (2.22), while another is the linearly realized SU(2) induced by the SU(2)$_H$ rotations of the harmonic variables (2.10):
\[ \delta_H a^{ik} = 2\tau^{(il}a^{k)}_{l}. \]  
This SU(2)$_H$ does not commute with (2.22). However, one can choose another basis for the generators of these two SU(2) groups, such that the new SU(2) commutes with (2.22). This redefined SU(2) is given by the following transformations
\[ \delta a^{ik} = -\tau^{ik} - \frac{1}{2}(\tau \cdot a)a^{(ik)} + \tau^{(il}a^{k)^{l}}, \]
where the signs chosen ensure that the relevant Lie bracket parameter is composed in the same way as in the cases (2.22) and (2.27). We denoted the transformation parameter in (2.28) by the same letter as in (2.27), hoping that this will not give rise to confusion. The mutually commuting SU(2) transformations (2.22) and (2.28) in fact amount to the left and right shifts on the same SU(2) group manifold.

It is easy to see that the diagonal SU(2) subgroup in the product of these two commuting nonlinearly realized SU(2) is just SU(2)$_H$ (2.27), so the bosonic sigma-model in (2.19) describes a nonlinear realization of this product on the coset manifold SU(2) $\times$ SU(2)/SU(2)$_H$, i.e., the 3-sphere $S^3$. The field $b$ transforms under the transformations (2.28) as

$$\delta b = \frac{1}{2}(\tau \cdot a),$$

while the redefined field $\tilde{b}$ is inert as in the case of $\lambda$-transformations. At the superfield level, the $\tau$ transformations are generated by transformations of the same form as (2.2), modulo the common sign minus before their r.h.s. and the appropriate addition of the linear harmonic-induced SU(2)$_H$ transformations of $N^{++}$ and $\omega$. Thus the full symmetry of the model under consideration, both on the superfield and the component levels, is the direct product U(1)$\times$SU(2)$\times$SU(2)$_5$.

It is worth pointing out a crucial difference between the left and right SU(2) symmetries. The left one (2.2) (as well as the constant shift (2.6)) commutes with supersymmetry and so defines the triholomorphic (or “translational”) isometry of the model under consideration. It is realized as a subgroup of general analytic reparametrizations of the superfields $q^+$ and $a$. The right SU(2) involves the R-symmetry SU(2)$_H$ as its essential part and so does not commute with supersymmetry. For what follows it is instructive to give the realization of the triholomorphic (left) SU(2) in terms of $q^{+a}$:

$$\delta \lambda q^{+a} = u^+[\lambda^{+} + \lambda^{--}(u^{+} \cdot q^{+})] - u^{-}a[\lambda^{++} + 2\lambda^{+-}(u^{+} \cdot q^{+}) + \lambda^{--}(u^{+} \cdot q^{+})^2],$$

$$\partial_{+b} \delta \lambda q^{+a} = -u^{+}a u^{+}_{b} \lambda^{--} + 2u^{-}a[u^{+}_{b} \lambda^{+} + u^{+}_{b}(u^{+} \cdot q^{+})\lambda^{--}],$$

with $\partial_{+b} := \frac{\partial}{\partial q^{+b}}$.

At this point, let us recall the existence of two distinguished connections on $S^3$ as a textbook example of a parallelizable manifold (see, e.g., [29]). One of them is the standard Levi-Civita torsionless connection and it corresponds to treating $S^3$ as a symmetric Riemannian coset space SO(4)/SO(3). Another connection involves a closed torsion and has zero curvature, which corresponds to the identification of $S^3$ with the non-symmetric SU(2) group manifold itself. This torsionful connection is what is called a “Bismut connection” (see [25] and references therein) and it is the one ensuring the SU(2)$\times$U(1) manifold to be HKT. Just with respect to the Bismut connection is the triplet of quaternionic complex structures covariantly constant. In more detail, these geometric issues are discussed in section 3.

It is the appropriate place here to note that the 4-dimensional SO(4)$\times$U(1) invariant metric in (2.19) admits an equivalent representation as the conformally-flat “Hopf manifold” metric

$$\sim \frac{(\dot{x}_4)^2 + \dot{x}_m \dot{x}_m}{(x_4 + c)^2 + x_m x_m}, \quad m = 1, 2, 3,$$

Formally, the right SU(2) could be promoted to U(2), with U(1) being realized by shifts undistinguishable from the left U(1) ones (2.6).
where \( c \neq 0 \) is a constant (in fact it can be fixed as \( c = 1 \) by a simple field redefinition). The SO(4) isometry acts as a rotation group of the four-dimensional Euclidean space \( \mathbb{R}^4 = (x_4 + c, x_m) \), with the 3-dimensional rotations \( \text{SO}(3) \subset \text{SO}(4) \) forming stability subgroup, while the U(1) (or R(1)) isometry as the common constant rescaling of the \( \mathbb{R}^4 \) coordinates. The coset \( \text{SO}(4)/\text{SO}(3) \) transformations act inhomogeneously on the coordinates \( x_m \), the same being true for the transformation of \( x_4 \) under rescaling. In both cases the inhomogeneities are proportional to \( c \). It is rather straightforward to establish the equivalence relation between the coordinate sets \((\tilde{b}(t), a^{ik}(t))\) and \((x_4(t), x_m(t))\). We prefer here to deal with the coordinates \((\tilde{b}(t), a^{ik}(t))\) because they manifest the product structure \( S^1 \times S^3 \) of the target space. The corresponding superfield description precisely matches the general case of the strong HKT \( \mathcal{N} = 4, d = 1 \) sigma models [22] and so can serve as a prototype for the analogous description of general group-manifold \( \mathcal{N} = 4, d = 1 \) models. It can be shown that the metric (2.32) naturally appears in an alternative description of the \( \mathcal{N} = 4 \) \( U(2) \) model by a linear \( q^{+a} \) multiplet, with vanishing \( L^{+3a} \) and more complicated \( L \). Such a formulation and its relation to the one given here will be considered elsewhere (see Sect. 7 of [23] for the relevant discussion).

### 2.3 Reduction to nonlinear \((3, 4, 1)\) multiplet

The last topic of the present section is the gauging of the \( U(1) \) symmetry (2.6) along the line of ref. [24]. We promote the constant parameter \( \lambda \) to an arbitrary analytic superfield parameter, \( \lambda \rightarrow \lambda(\zeta, u) \), \( \omega' = \omega + \lambda, N^{++'} = N^{++} \), and introduce two abelian harmonic connections \( V^{\pm\pm}, V^{\pm\pm'} = V^{\pm\pm} - D^{\pm\pm} \lambda \), such that \( V^{++} \) is analytic, \( V^{++} = V^{++}(\zeta, u) \), while \( V^{--} \) is related to \( V^{++} \) by the covariant “flatness” condition,

\[
D^{++}V^{--} - D^{--}V^{++} = 0. \tag{2.33}
\]

The constraints (2.1) and the superfield action (2.5) are covariantized as

\[
D^{++}N^{++} + (N^{++})^2 = 0, \tag{2.34}
\]

\[
S_{su(2)}^{gauge} = -\frac{1}{4} \int dt du d^4 \theta N^{++} (D^{--}\omega + V^{--}). \tag{2.35}
\]

The modified constraints and the action are, respectively, covariant and invariant under all rigid \( SU(2) \) transformations discussed in this section. While checking this, an essential use of the flatness condition (2.33) is needed.

Let us show that the system so constructed describes the appropriate \( SU(2) \) invariant sigma model of the single nonlinear multiplet \( N^{++} \). To this end, we choose the \( \mathcal{N} = 4 \) supersymmetric gauge

\[
\omega = 0. \tag{2.36}
\]

In this gauge, the second constraint in (2.34) just amounts to identifying

\[
V^{++} = N^{++}, \tag{2.37}
\]

while the action (2.35) becomes

\[
S_{su(2)}^{gauge} = -\frac{1}{4} \int dt du d^4 \theta N^{++} V^{--}. \tag{2.38}
\]
Using (2.33), $V^{--}$ can be expressed through $V^{++} = N^{++}$ by the well-know formula [21] involving the harmonic distributions

$$V^{--} = \int du_1 \frac{N^{++}(1)}{(u^+ u^+_1)^2},$$

(2.39)

where $V^{--}$ is taken at the harmonic “point” $u^+_1$, while $N^{++}$ in the r.h.s. is taken at the point $u^+_1$. Then the action (2.38) can be written solely in terms of $N^{++}$ (in the central basis) as

$$S^\text{gauge}_{su(2)} = -\frac{1}{4} \int dt d\theta du_1 du_2 \frac{N^{++}(1)}{(u^+_1 u^+_2)^2} N^{++}(2).$$

(2.40)

The component bosonic action can be obtained in a simple form with the alternative choice of the Wess-Zumino gauge for $V^{\pm\pm}$

$$V^{++} \rightarrow 2i\theta^+ \bar{\theta}^+ A(t), \quad V^{--} \rightarrow 2i\theta^- \bar{\theta}^- A(t), \quad A' = A + \dot{\lambda}(t).$$

(2.41)

In this gauge, the fourth of the component constraints (2.12) is modified as

$$\partial^{++} \sigma^2 - 2i\dot{\omega}_0 - \sigma + 2iA = 0,$$

(2.42)

while the bosonic action (2.16) is modified as

$$S_{bos} \rightarrow S_{bos}' = \frac{i}{2} \int dt du \left[ n^{++}\dot{\sigma}^2 + \sigma(\dot{\omega}_0 - A) \right] = \int dt du L'(t, u) = \int dt L'(t).$$

(2.43)

After some simple algebra, one finds that the only modification of the bosonic Lagrangian (2.19) is the replacement

$$\dot{\tilde{b}} \rightarrow \dot{\tilde{b}} - A.$$

(2.44)

Keeping in mind that under local U(1) transformation $\tilde{b}' = \tilde{b} + \lambda(t)$, one can always choose the gauge $\tilde{b} = 0$, after which the relevant bosonic Lagrangian will reduce to

$$L' = A^2 + \frac{1}{2} \left[ (\dot{a} \cdot \dot{a}) \frac{1}{1 + \frac{a^2}{2}} - \frac{1}{2} (\dot{a} \cdot a)^2 \frac{1}{1 + \frac{a^2}{2}} \right].$$

(2.45)

The field $A$ proves to be just the auxiliary field of the nonlinear $(3, 4, 1)$ multiplet described by the superfield $N^{++}$. It fully decouples in the bosonic action (2.45) leaving us with the $S^3$ non-linear sigma model for 3 physical bosonic fields.

So, applying the general gauging procedure of ref. [24] to the superfield Lagrangian (2.5) (or (2.9)) of the U(2) HKT model, we recovered a particular Lagrangian of the nonlinear $(3, 4, 1)$ multiplet, with the $S^3$ non-linear sigma model in the bosonic sector.

---

6 In the description through the standard (harmonic-independent) $N' = 4, d = 1$ superfields the nonlinear $(3, 4, 1)$ multiplet was studied in [30]. The case with the bosonic $S^3$ action (2.19) was obtained as a particular case of the corresponding general superfield action. Using a duality procedure at the component level, the bosonic SU(2)$\times$U(1) invariant action (2.19) was recovered through the dualization of the auxiliary field $A$ into the physical scalar $b$. 
3 U(2) model as an example of HKT geometry in the harmonic approach

Here we recover the metric and the torsion associated with the U(2) group manifold sigma model directly from the general geometric formalism of nonlinear (4, 4, 0) multiplets pioneered in [22] and further elaborated on in [25]. Since we will be interested in the bosonic target geometry, we omit fermions and put $q_i^a = f_i^a$ altogether (in particular, in (2.30), (2.31)). The bosonic fields $\sigma$ and $\sigma^{-2}$ defined in (2.11) are of use only while deriving the bosonic sigma model action from the superfield one. In the general formalism of ref. [22] these additional quantities play no role, as they are eliminated there, from the very beginning, by the harmonic superfield constraint.

The basic object of the geometric formalism is the “bridge” from the target $\lambda$ frame parametrized by the coordinates $f^+(t, u)$ and $f^{-a}(t, u) := \partial^{-a} f^+(t, u)$ and harmonics $u^\pm_i$ to the $\tau$ frame parametrized by the harmonic-independent coordinates, in our case the $d = 1$ fields $b(t)$ and $a^R(t)$ defined in the previous subsection. The bridge is a $2 \times 2$ complex matrix $M_{a}^b$ subject to the equation

$$\partial^{++} M_{a}^b + E_{a}^c M_{c}^b = 0, \tag{3.1}$$

with

$$E_{a}^{2c} = \partial_{++} L_{a}^{c} = -2u^{-c}u^+(u^+ \cdot f^+). \tag{3.2}$$

The underlined doublet indices refer to the $\tau$ frame, and on them some harmonic-independent $\tau$ group acts. In the present case the latter is realized as SU(2) rotations with the parameters $\lambda_{a}^{b}, \lambda_{a}^{c} = 0$ (these parameters are the same as in (2.2), (2.30) and (2.31)), accompanied by some rescaling, see eqs. (3.3), (3.4) below. On the non-underlined indices another realization of the same SU(2) is defined, as a particular isometry subgroup of general harmonic analyticity-preserving target space reparametrizations.

Eq. (3.1) also implies that

$$\partial^{--} M_{a}^b + E_{a}^{-2c} M_{c}^b = 0, \tag{3.3}$$

where the non-analytic connection $E_{a}^{-2c}$ is related to $E_{a}^{2c}$ by the “harmonic flatness condition”

$$\partial^{++} E_{a}^{-2c} = \partial^{--} E_{a}^{+2c} + E_{d}^{+2c} E_{a}^{-2d} - E_{a}^{+2d} E_{d}^{-2c}. \tag{3.4}$$

Fortunately, eq. (3.1) has the simple solution

$$M_{a}^b = \delta_{a}^{b}(1 + a^+) - a^+ u^- u^b, \quad (M^{-1})_{b}^{a} = \frac{1}{1 + a^+ + a^- u^- u^b}. \tag{3.5}$$

Now, the connection $E_{a}^{-2c}$ can be easily restored from (3.3):

$$E_{a}^{-2c} = -\left[\delta_{b}^{c} n^{--} + u^{-c} u^b \left(n^{++} n^{--} - 2a^+ \frac{a^-}{1 + a^+}\right)\right], \quad n^{\pm} := (u^a f^a) = \frac{a^{\pm}}{1 + a^+}. \tag{3.6}$$

This expression can be checked to satisfy eq. (3.3). It is also straightforward to check that under the isometry (2.2), (2.30) and (2.31) the bridge $M_{a}^b$ has the correct geometric transformation law (as anticipated above)

$$\delta_{\lambda} M_{a}^b = -\partial_{+a} \delta_{\lambda} q^{++} M_{c}^b + \omega_{\lambda}^{b} M_{c}^b, \tag{3.7}$$

In fact, the same expression for $E_{a}^{-2c}$ can be directly derived by solving (3.3).
where \( \partial_{\tau a} \delta \lambda q^+ c \) is given in (2.31) and

\[
\omega^\pm_\perp = -\lambda(\underline{\kappa}) + \frac{1}{2} \underline{M}(\lambda \cdot a),
\] (3.8)

are the harmonic-independent parameters of the \( \tau \) group the generical definition of which is given in [22].

Having bridges at hand, we can calculate some “semi-vierbeins” which are building blocks of all geometric quantities in the approach of [22]. They transform the tangent space objects to the world objects and, being specialized to the case under consideration, are defined by

\[
E_{(ik)}^{+a} = \partial_{(ik)} f^+ a M_{ab}^+ = c_{(ik)}^{ia} u^+_i, \quad E_4^{+a} = \partial_b f^+ a M_{ab}^4 = \epsilon_{ia}^{4a} u^+_i.
\] (3.9)

Simple calculations yield

\[
\epsilon_{(ik)}^{ia} = \frac{1}{2} \left( \delta^i_l \delta^k_k + \delta^i_k \delta^k_l \right) \frac{a^{(ik)}}{2 + a^2} \left[ \varepsilon_{ia} + a_{(ai)} \right], \quad \epsilon_{ia}^{4a} \equiv \varepsilon_{ia} + a_{(ai)},
\] \[
\epsilon_{(ik)}^{ik} = \frac{1}{2} \left( \delta^i_l \delta^k_k + \delta^i_k \delta^k_l \right) - \frac{1}{2} a^{(ik)} \varepsilon_{ia}, \quad \epsilon_{ia}^4 = \frac{1}{2 + a^2} \left[ \varepsilon_{ia} + a_{(ai)} \right].
\] (3.10)

The vierbein coefficients satisfy the standard orthogonality conditions. It can be checked that, under the triholomorphic SU(2) isometry, they transform as covariant and contravariant tensors with respect to non-underlined indices, with the infinitesimal parameters \( \partial_{(ik)} \delta a^{(jl)} \), and as spinors with the parameters \( \tilde{a} \) with respect to the underlined doublet indices \( a, b, \ldots \). On the indices \( \tilde{i}, \tilde{k}, \ldots \) the standard automorphism SU(2) group acts.

Now we are prepared to compute the basic \( \tau \) frame geometric objects of the U(2) model by specializing the general expressions of ref. [22], [25] to this case.

**Metric.** The target metric components are calculated by the general formula

\[
g_{ia \ jb} = \epsilon_{ia}^{ja} \epsilon_{j\underline{b}}^{\underline{a}} G_{[\underline{a} \underline{b}]} ,
\] (3.11)

where

\[
G_{[\underline{a} \underline{b}]} = \int du \mathcal{F}_{[cd]} (M^{-1})_{[\underline{a}}^c (M^{-1})_{d\underline{b}]}, \quad \mathcal{F}_{[cd]} = \partial_{[c} \partial_{d]} \mathcal{L}.
\] (3.12)

In our case \( \mathcal{L} = q^+ a \varepsilon_{ab} q^b \), so \( \mathcal{F}_{[cd]} = \varepsilon_{cd} \). Using the harmonic integral formulas from Appendix B, it is easy to find

\[
G_{[ab]} = \varepsilon_{ab} \frac{1}{1 + \frac{1}{2} a^2}.
\] (3.13)

Substituting this into (3.11), we find

\[
g_{(ia) \ (jb)} = \frac{1}{2 + a^2} \left[ (\varepsilon_{ij} \varepsilon_{ab} + \varepsilon_{aj} \varepsilon_{ib}) - \frac{1}{1 + \frac{1}{2} a^2} a_{(ia)} a_{(jb)} \right],
\]

\[
g_{4 \ (j\underline{b})} = 0, \quad g_{4 \ 4} = 2,
\] (3.14)

which precisely matches with the sigma-model Lagrangian (2.19).

\footnote{We changed some signs as compared to [22].}
From now on, it will be convenient to pass to the 3-vector notation,

\[
a_{(ik)} = \frac{i}{\sqrt{2}} x_m (\sigma_m)_{ik}, \quad a_{(ik)} (\sigma_m)^{ik} = -i\sqrt{2} x_m, \quad a^2 = x^2. \tag{3.15}
\]

All world tensor indices \((ia)\) are replaced by the 3-vector ones by attaching the matrix factors \(\frac{i}{\sqrt{2}}(\sigma_m)^{ia}\). Then the metric takes the form\(^9\)

\[
g_{mn} = \frac{1}{1 + \frac{1}{2}x^2} \left( \delta_{mn} - \frac{1}{4} \frac{1}{2} x_m x_n \right), \quad g^{mn} = \left( 1 + \frac{1}{2} x^2 \right) \left( \delta_{mn} + \frac{1}{2} x_m x_n \right). \tag{3.16}
\]

**Torsion.** The torsion in the tangent space notation is expressed as

\[
C_{ia\, k \, b \, c} = \varepsilon_{ik} \nabla_{ka} G_{[bc]} + \varepsilon_{ib} \nabla_{ib} G_{[ac]} \tag{3.17}
\]

where

\[
\nabla_{ia} G_{[bc]} = \int du \left( M^{-1} \right)^a_{\underline{a}} \left( M^{-1} \right)^c_{\underline{c}} \left( M^{-1} \right)^b_{\underline{b}} \left( \partial_{-a} F_{cb} u_{\underline{b}}^+ + D_{+a} F_{cb} u_{\underline{b}}^- \right) \tag{3.18}
\]

and

\[
D_{+a} F_{cb} = \nabla_{+a} F_{cb} + E^{-d}_{ac} F_{db} + E^{-d}_{ab} F_{cd}, \quad \nabla_{+a} = \partial_{+a} + E^{-d}_{ad} \partial_{-d}, \quad D^{+a} E^{-d}_{ac} = E^{+d}_{ac}, \quad E^{-d}_{ad} = \partial_{+a} \partial_{-b} L^{+d}. \tag{3.19}
\]

In our case \(G_{[bc]} := \varepsilon_{ac} G \) and \(F_{[cd]} = \varepsilon_{cd} \), so (3.18) is simplified to

\[
\nabla_{ia} G = \int du \left( M^{-1} \right)^a_{\underline{a}} (\det M)^{-1} E_a^+ u_{\underline{a}}^- \quad E_a^- := E^{-d}_{ad}. \tag{3.20}
\]

Next, \(E^{+d}_{ab} = 2u^{-d}_a u^+_b + \) and, by solving the harmonic equation in (3.19), we find

\[
E^{-d}_{ab} = -\frac{4}{3} \delta^d_{(a} u^b_{)} + \varepsilon^{de}_{(c} u^+_{a} u^+_{b)}, \quad \Rightarrow \quad E_a^- = -2u^a. \tag{3.21}
\]

It is also easy to compute

\[
(\det M) = (1 + a^{+ -})^2. \tag{3.22}
\]

Substituting (3.21) and (3.22) in (3.20), we obtain

\[
\nabla_{ia} G = -\int du \frac{1}{(1 + a^{+ -})^3} (\varepsilon_{ia} + 2u^+_{\underline{a}} u^-_{\underline{a}}). \tag{3.23}
\]

Calculating the harmonic integral with the help of eqs. (B.7) and (B.12) from Appendix B, we finally obtain

\[
\nabla_{ia} G = -\frac{1}{(1 + \frac{1}{2} a^2)^3} \left[ \varepsilon_{ia} + a_{(ia)} \right] \tag{3.23}
\]

and

\[
C_{ia\, k \, b \, c} = -\frac{1}{(1 + \frac{1}{2} a^2)^3} \left[ \varepsilon_{ia} \varepsilon_{cb} (\varepsilon_{ka} + a_{ka}) + \varepsilon_{ib} \varepsilon_{ac} (\varepsilon_{ib} + a_{ib}) \right]. \tag{3.24}
\]
It is easy to check the full antisymmetry of this expression with respect to permutation of the pairs of indices. Next we project this expression to the world-index representation by contracting it with the “semi-vierbeins” (3.10). We find that all its components containing the index 4 vanish and only the projections on the 3-vector subspace are non-zero. Substituting the triplet indices by the vector ones by the rule (3.15), we finally obtain the only non-zero torsion component as

$$C_{mns} = \sqrt{2} \frac{1}{(1 + \frac{1}{2}x^2)^2} \varepsilon_{mns}.$$  (3.25)

It is evidently closed as it should be for the group-manifold HKT models [7].

**Bismut connection.** The Bismut connection on $4q$ dimensional HKT manifold is defined as

$$\hat{\Gamma}_{MS}^N = \Gamma_{MS}^N + \frac{1}{2} g^{NP} C_{PMS}, \quad N, M, \ldots = 1, \ldots, 4q,$$  (3.26)

where $\Gamma_{MS}^N$ is the standard symmetric Levi-Civita connection and $C_{PMS}$ is the torsion tensor. With respect to it, the triplet of the corresponding quaternionic complex structures is covariantly constant. Since in our case (with $q = 1$) $C_{4mn} = 0$, the only non-vanishing components of the Bismut connection are

$$\hat{\Gamma}_{ms}^n = \Gamma_{ms}^n + \frac{1}{\sqrt{2}} \frac{1}{(1 + \frac{1}{2}x^2)^2} \varepsilon_{pms}.$$  (3.27)

The connection $\Gamma_{ms}^n$ for the metric defined in (3.16) is easily calculated to be

$$\Gamma_{ms}^n = -\frac{1}{2} \frac{1}{1 + \frac{1}{2}x^2} \left( x_m \delta_{ns} + x_s \delta_{nm} \right).$$  (3.28)

The triplet of complex structures in the tangent space representation is given by

$$(I_A)^{ia}_{ijk} = -i(\sigma_A)^i_{ab} \delta^a_k,$$  (3.29)

where $\sigma_A, A = 1, 2, 3$, are Pauli matrices. Transforming it to the world indices by contracting with the vierbeins (3.10) and then passing to the indices 4, $m$ through the correspondence (3.15), we explicitly find

$$(I_A)^4_4 = 0, \quad (I_A)^4_m = \frac{1}{\sqrt{2}} \frac{1}{1 + \frac{1}{2}x^2} \left( \delta_{mA} - \frac{1}{\sqrt{2}} \varepsilon_{Am} x_p \right),$$

$$(I_A)^m_4 = -\sqrt{2} \left( \delta_{Am} + \frac{1}{2} x_A x_m - \frac{1}{\sqrt{2}} \varepsilon_{Ap} x_p \right),$$

$$(I_A)^m_n = \varepsilon_{Amn} - \frac{1}{\sqrt{2}} \delta_{Am} x_m + \frac{1}{\sqrt{2}} \frac{1}{1 + \frac{1}{2}x^2} x_n \left( \delta_{Am} - \frac{1}{\sqrt{2}} \varepsilon_{Ap} x_p + \frac{1}{2} x_A x_m \right).$$  (3.30)

Despite the somewhat involved form of these expressions, it can be checked that they form the algebra of imaginary quaternions and possess the correct tensor transformation rules under the SU(2) isometry

$$\delta_{x} x_m = \lambda_m + \frac{1}{2} (\lambda \cdot x) x_m - \frac{1}{\sqrt{2}} \varepsilon_{mns} \lambda_n x_s.$$  (3.31)
After some effort it can be also checked that they are covariantly constant with respect to \( \hat{\Gamma}^{m}_{ns} \):

\[
\partial_{s}(I_{4A})^{m}_{p} - \hat{\Gamma}^{m}_{sp} (I_{4A})^{p}_{4} = 0, \quad \partial_{s}(I_{A})_{4}^{m} + \hat{\Gamma}^{m}_{su} (I_{4A})^{u}_{4} = 0,
\]

\[
\partial_{s}(I_{A})^{m}_{n} - \hat{\Gamma}^{m}_{sn} (I_{A})^{p}_{p} + \hat{\Gamma}^{m}_{su} (I_{4A})^{u}_{s} = 0.
\]

(3.32)

One more important property of the Bismut connection \( \hat{\Gamma}^{m}_{ns} \) (specific just for the group-manifold HKT models [7]) is that its Riemann curvature vanishes (the property pertinent to the parallelized manifolds [29]),

\[
R^{m}_{nsp}(\hat{\Gamma}) = \partial_{s}\hat{\Gamma}^{m}_{pn} - \partial_{p}\hat{\Gamma}^{m}_{sn} + \hat{\Gamma}^{m}_{su}\hat{\Gamma}^{u}_{pn} - \hat{\Gamma}^{m}_{pu}\hat{\Gamma}^{u}_{sn} = 0.
\]

(3.33)

This property can also be directly checked using the explicit expression (3.27) and (3.28).

\textit{Obata connection}. Besides the Bismut connection, one more important geometric object of HKT models is the Obata connection [31]. It is a symmetric connection with respect to which the triplet of complex structures is still covariantly constant (but not the metric, as distinct from the Levi-Civita connection). For HK manifolds it coincides with the Levi-Civita connection, but for HKT manifolds it does not.

In the harmonic approach to HKT geometry, Obata connection in the tangent space representation was defined in [25]. It is the following deformation of the Bismut connection

\[
\tilde{\Gamma}^{i}_{ablc} = \Gamma^{i}_{ablc} + \frac{1}{2}(C^{i}_{ablc} + \Delta C^{i}_{ablc}) = \hat{\Gamma}^{i}_{ablc} + \frac{1}{2}\Delta C^{i}_{ablc},
\]

(3.34)

where

\[
\Delta C^{i}_{ablc} = -2\varepsilon^{ijkl} \nabla_{kc} G_{[a[kl]}.
\]

(3.35)

The full symmetry of \( \tilde{\Gamma}^{i}_{ablc} \) in the last two pairs of indices can be proved using the cyclic identity [22]

\[
\nabla_{kc} G_{[a[kl]} + \text{cycle} (a, b, c) = 0.
\]

Now it is straightforward to compute the world-index form \( \tilde{\Gamma}^{M}_{NS} \), \( M, N, S = (4, m), (4, n), (4, s) \), of the Obata connection for our model. One should substitute \( \nabla_{kc} G_{[a[kl]} \rightarrow \varepsilon_{a2} \nabla_{kc} G \), use eq. (3.23), further contract (3.34) with the proper vierbeins (3.10) and finally pass to the 3-vector notation. The non-vanishing components of \( \tilde{\Gamma}^{M}_{NS} \) prove to be

\[
\tilde{\Gamma}^{m}_{ns} = \Gamma^{m}_{ns}, \quad \tilde{\Gamma}^{m}_{4n} = \Gamma^{m}_{4n} = \delta^{m}_{n}, \quad \tilde{\Gamma}^{4}_{ns} = -\frac{1}{2}g_{ns}, \quad \tilde{\Gamma}^{4}_{44} = 1,
\]

(3.36)

where we took into account that \( g^{44} = \frac{1}{2} \). Now it is a matter of somewhat tiring (though direct) computation to check the covariant constancy of the complex structure \( (I_{A})^{M}_{N} \) defined in (3.30), (3.31):

\[
\partial_{p}(I_{A})^{M}_{N} - \tilde{\Gamma}^{S}_{pN}(I_{A})^{M}_{S} + \Gamma^{M}_{pS}(I_{A})^{S}_{N} = 0, \quad \tilde{\Gamma}^{S}_{4N}(I_{A})^{M}_{S} - \tilde{\Gamma}^{M}_{4S}(I_{A})^{S}_{N} = 0.
\]

(3.37)

It is also of interest to calculate the curvature of the Obata connection. Surprisingly, it turns out to vanish:

\[
R^{M}_{NSP}(\tilde{\Gamma}) = 0.
\]

(3.38)
According to ref. \cite{6}, the vanishing of the curvature of the Obata connection for some HKT manifold signals that the three corresponding complex structures are simultaneously integrable and so there exists a coordinate frame where all these can be chosen constant. In the harmonic approach, this amounts to the conjecture \cite{25} that there exists a redefinition of the original $q^+$ superfields such that they satisfy a linear harmonic constraint\footnote{$\mathcal{N} = 4$ HKT models with linear harmonic constraints were addressed in \cite{23} and, e.g., in \cite{32}.}. This interesting issue will be studied elsewhere (recall the related discussion in section 2.2 above).

4 \quad SU(3) group manifold

4.1 Superfield consideration

In this case, beside the superfields $N^{++}, \omega$ with four physical bosonic fields parametrizing $SU(2) \times U(1)$, we need one more analytic superfield in order to accommodate four extra bosonic fields which complete the $SU(2) \times U(1)$ group manifold to that of $SU(3)$. The natural choice is the complex analytic superfield $(q^+, \bar{q}^+)$, $(\bar{q}^+)= -q^+$ where “tilde” stands for the generalized conjugation defined in \cite{19}, \cite{21} and becoming the ordinary conjugation on the component fields. Thus in the SU(3) case we operate with the following set of analytic $\mathcal{N} = 4, d = 1$ superfields

$$N^{++}, \omega, q^+, \bar{q}^+.$$ \hfill (4.1)

The extra fields appear as first terms in the harmonic expansion of $q^+, \bar{q}^+$, $q^+ = f_i u_i^+ + \ldots$, $q^+ = \bar{f}_i u_i^+ + \ldots$, $\bar{f}^i = (f_i)$, i.e. they are doublets with respect to the automorphism $SU(2)$. The corresponding new constant group parameters are defined in a similar way

$$\xi^\pm = \xi^i u^\pm_i, \quad \bar{\xi}^\pm = \bar{\xi}^i u^\pm_i, \quad \xi^i = (\bar{\xi}_i), \quad (\bar{\xi}^\pm) = -\xi^\pm.$$ \hfill (4.2)

Using the trial and error method, we have eventually found the following self-consistent set of the coset $SU(3)/U(2)$ transformations for $q^+, \bar{q}^+, N^{++}$:

$$\begin{align*}
\delta q^+ &= \xi^+ - \xi^- N^{++} + \alpha \xi^- q^+ \bar{q}^+ + 2\alpha_0 \xi^- (q^+)^2, \\
\delta \bar{q}^+ &= \bar{\xi}^+ - \bar{\xi}^- N^{++} - \bar{\alpha} \bar{\xi}^- q^+ \bar{q}^+ - 2\alpha_0 \xi^-(q^+)^2, \\
\delta N^{++} &= \alpha (\xi^+ - \xi^- N^{++})q^+ - \bar{\alpha} (\xi^+ - \xi^- N^{++})q^+ - \alpha \bar{\alpha} [\xi^- (q^+)^2 q^+ + \bar{\xi}^- (q^+)^2 \bar{q}^+]. \quad (4.3)
\end{align*}$$

Here, $\alpha = \alpha_0 + i\alpha_1$ and $\alpha_1$ is, for the time being, an undetermined free real parameter. These transformations close on the $SU(2) \times U(1)$ ones:

$$\begin{align*}
(\delta_2 \delta_1 - \delta_1 \delta_2) q^+ &= \delta_{br} q^+ = 3i\alpha_0 \phi_{br} q^+ - 2\alpha_0 \left[ (\lambda_{br}^{++} - \lambda_{br}^{--} N^{++}) q^+ + \alpha \lambda_{br}^{--} (q^+)^2 q^+ \right], \\
\delta_{br} N^{++} &= 2\alpha_0 \left[ \lambda_{br}^{++} - 2\lambda_{br}^{+-} N^{++} + \lambda_{br}^{--} (N^{++})^2 \right].
\end{align*} \quad (4.4)$$

where

$$\phi_{br} = -i[\xi(i)] \bar{\xi}(2) - (1 \leftrightarrow 2)], \lambda_{br}^{++} = \lambda_{br}^{ik} u^+_i u^+_k, \lambda_{br}^{+-} = \lambda_{br}^{ik} u^+_i u^-_k, \lambda_{br}^{--} = \lambda_{br}^{ik} u^-_i u^-_k, \lambda_{br}^{ik} = [\xi(i) \bar{\xi}(k) - (1 \leftrightarrow 2)]. \quad (4.5)$$

Note the necessary modification of the SU(2) transformation rule for $N^{++}$ as compared to the pure SU(2)$\times U(1)$ case. Also note the presence of a non-trivial U(1) phase transformation in
the closure on \( q^+ \) (the closure transformations of \( \bar{q}^+ \) can be obtained by conjugation of \( \delta_{br} q^+ \)). The non-zero real parameter \( \alpha_0 \) can be fixed at any value via a simultaneous rescaling of \( q^+ \) and of the coset parameters \( \xi^i \).

It remains to quote the corresponding transformation properties of the superfield \( \omega \). Its coset SU(3)/U(2) transformation reads

\[
\delta \omega = \gamma \xi^- q^+ - \bar{\gamma} \bar{\xi}^- \bar{q}^+ , \quad \gamma = \alpha_0 + i \gamma_1 ,
\]

where \( \gamma_1 \) is yet another undetermined real parameter. The closure of these transformations is the same as on the superfields \( q^+, \bar{q}^+, N^{++}, \bar{N}^{++} \), with

\[
\delta_{br} \omega = -\gamma_1 \phi_{br} + 2 \alpha_0 \left[ \lambda_{br}^+ - \lambda_{br}^- N^{++} + i(\alpha_1 - \gamma_1) \lambda_{br}^- \right] \left( q^+ \bar{q}^+ \right) .
\]

The parameters \( \alpha_1 \) and \( \gamma_1 \) cannot be fixed from considering the closure of the above transformations: they form an \( su(2) \) algebra irrespective of the choice of these parameters. Surprisingly, they are fixed when constructing the invariant action. But before discussing this issue, let us write down the relevant set of the harmonic constraints generalizing and extending \( (2.11) \). This set is as follows

\[
D^{++} q^+ + N^{++} q^+ - \alpha (q^+)^2 q^+ = 0 ,
\]

\[
D^{++} \bar{q}^+ + N^{++} \bar{q}^+ + \bar{\alpha} (q^+)^2 \bar{q}^+ = 0 ,
\]

\[
D^{++} N^{++} + (N^{++})^2 + (\alpha \bar{\alpha}) (q^+ \bar{q}^+)^2 = 0 ,
\]

\[
D^{++} \omega - N^{++} + i(\alpha_1 - \gamma_1) q^+ \bar{q}^+ = 0 .
\]

One can check that the set \((4.8)\) is covariant under the SU(3) transformations \((4.3), (4.6)\) and \((4.4), (4.7)\).

It is interesting that the set of constraints \((4.8)\) for the choice \( \alpha_1 = -\gamma_1 \) can be cast in a simpler suggestive form. Introducing

\[
\Phi^{++} := N^{++} - \alpha \ q^+ \bar{q}^+ , \quad \bar{\Phi}^{++} = N^{++} + \bar{\alpha} q^+ \bar{q}^+ ,
\]

we can rewrite \((4.8)\) as

\[
(D^{++} + \Phi^{++}) q^+ = 0 , \quad D^{++} \Phi^{++} + (\Phi^{++})^2 = 0 ,
\]

\[
(D^{++} + \bar{\Phi}^{++}) \bar{q}^+ = 0 , \quad D^{++} \bar{\Phi}^{++} + (\bar{\Phi}^{++})^2 = 0 ,
\]

\[
D^{++} \omega - \Phi^{++} - \bar{\alpha} q^+ \bar{q}^+ = 0 , \quad \text{or} \quad D^{++} \omega - \bar{\Phi}^{++} + \alpha q^+ \bar{q}^+ = 0 .
\]

The constraints \((4.10)\) supplemented by the condition

\[
\Phi^{++} - \bar{\Phi}^{++} = -2 \alpha_0 \ q^+ \bar{q}^+ ,
\]

can be treated as the basic ones. The constraint \((4.12)\) serves just for expressing the superfield \( \omega \) (up to the harmonic-independent part) from the known \( \Phi^{++}, q^+ \) and the conjugated superfields. Note that the set \( q^+, \Phi^{++} \) is closed under the SU(3)/U(2) transformations \((4.3)\)

\[
\delta \Phi^{++} = -2 \alpha_0 \left( \xi^+ q^+ - \xi^- \bar{\Phi}^{++} q^+ \right) , \quad \delta q^+ = \xi^+ - \xi^- \Phi^{++} + 2 \alpha_0 \xi^- (q^+)^2
\]

and hence under the U(2) ones as well:

\[
\delta q^+ = \alpha_0 \left[ 3i \phi q^+ - 2 \left( \lambda^+ - \lambda^- \Phi^{++} \right) q^+ \right] , \quad \delta \Phi^{++} = 2 \alpha_0 \left[ \lambda^{++} - 2 \lambda^- \Phi^{++} + \lambda^- \left( \Phi^{++} \right)^2 \right] .
\]
Thus the set \((q^+, \Phi^{++}, u^\pm)\) can be interpreted as a kind of complex analytic subspace in the harmonic extension of the target SU(3) group manifold (likewise, in the U(2) case the set \((N^{++}, u^\pm)\) can be treated as some analytic subspace of the harmonic extension of the U(2) group manifold).

The invariant action should be an extension of the action (2.5) and, following the general reasoning of ref. [22], admit a formulation in the full harmonic superspace as an expression bilinear with respect to the superfields involved and linear in the harmonic derivative \(D^-\). Combining (2.5) with \(\bar{q}^+D^-q^+\), we find that requiring it to be invariant, up to a total harmonic derivative, under the transformations (4.3) and (4.6) (and, hence, under their closure), uniquely fixes the ratio of these two terms in such a way that

\[
S_{su(3)} = -\frac{1}{4} \int dt du d^4 \theta \left( N^{++} D^- \omega - 2\alpha_0 \bar{q}^+ D^- q^+ \right),
\]

and, what is even more surprising, fixes the constants \(\alpha_1\) and \(\gamma_1\) in terms of the single normalization constant \(\alpha_0\) as

\[
\alpha = \alpha_0(1 \pm i\sqrt{3}), \quad \gamma = \alpha_0(1 \mp i\sqrt{3}),
\]

so that

\[
\alpha \bar{\alpha} = 4\alpha_0^2.
\]

While checking the invariance, an essential use of the harmonic constraints (4.8) was made.

It is instructive to give some technical details of the proof of the SU(3) invariance of the action (4.16). It will be enough to prove the invariance under the \(\xi\) transformations, as the rest of SU(3) is contained in their closure. Moreover, it suffices to consider the holomorphic \(\xi_i\) parts of these transformations as the antiholomorphic ones \(\bar{\xi}_i\) are obtained through the tilde-conjugation. So we start from the transformations

\[
\delta N^{++} = \left[ \xi^+ - \xi^- (N^{++} + \alpha \bar{\alpha} q^+ q^+) \right] q^+, \quad \delta \omega = \gamma \xi^- \bar{q}^+,
\]

\[
\delta q^+ = \xi^+ - \xi^- (N^{++} - \alpha \bar{\alpha} q^+ q^+), \quad \delta \bar{q}^+ = -2\alpha_0 \xi^- (\bar{q}^+)^2,
\]

and the following ansatz for the superfield Lagrangian

\[
L = L_1 + \kappa L_2, \quad L_1 = N^{++} D^- \omega, \quad L_2 = \bar{q}^+ D^- q^+,
\]

where \(\kappa\) is some real parameter. The variations of \(L_1\) and \(L_2\), up to total time and harmonic derivatives and upon using the harmonic constraints (4.8) along with the property that the full superspace integral of an analytic expression vanishes, are reduced to

\[
\delta L_1 \Rightarrow - (\alpha + \gamma) \xi^- \bar{q}^+ D^- N^{++} + i \frac{\alpha}{2} (\alpha_1 - \gamma_1) \xi^- (\bar{q}^+)^2 D^- q^+,
\]

\[
\delta L_2 \Rightarrow - \xi^- \bar{q}^+ D^- N^{++} + \left( \frac{\alpha}{2} - 2\alpha_0 \right) \xi^- (\bar{q}^+)^2 D^- q^+.
\]

The superfield structures in these variations are independent, so the conditions for the vanishing of \(\delta L\) are

\[
\kappa + \alpha + \gamma = 0, \quad i \frac{\alpha}{2} (\alpha_1 - \gamma_1) + \kappa \left( \frac{\alpha}{2} - 2\alpha_0 \right) = 0.
\]

The separate vanishing of the real and imaginary parts of these equations uniquely yields

\[
\alpha_1 = -\gamma_1, \quad \kappa = -2\alpha_0, \quad \alpha_1^2 = 3\alpha_0^2,
\]
that precisely matches with (4.16) - (4.18).

It is also easy to directly prove the invariance of (4.16) under the U(2) ⊂ SU(3) transformations and under the transformations of the second U(2) commuting with SU(3) (the second SU(2) transformations are a fixed combination of the first SU(2) ones and of those of the harmonic SU(2), while the second U(1) factor acts just as a constant shift of ω, see eqs. (4.49) - (4.51) below).

4.2 Bosonic limit and its symmetries

Here we solve the purely bosonic part of the constraints (4.8) and find the realization of the SU(3) transformations on the physical bosonic variables.

The bosonic core of the involved superfields is as follows

\[ N^{++} = n^{++} + \theta^+ \bar{\theta}^+ \sigma, \quad \omega = \omega_0 + \theta^+ \bar{\theta}^+ \sigma^{-2}, \quad q^+ = f^+ + \theta^+ \bar{\theta}^+ \mu^-, \quad \bar{q}^+ = \bar{f}^- - \theta^+ \bar{\theta}^+ \bar{\mu}^-, \]  

(4.24)

where all fields in the r.h.s are defined on the manifold \((t, u_i^\pm)\), e.g., \(f^+ = f^+(t, u)\), etc. The basic bosonic constraints are obtained from the superfield ones by the direct replacement

\[(q^+, \bar{q}^+, N^{++}, \omega) \rightarrow (f^+, \bar{f}^+, n^{++}, \omega_0),\]  

(4.25)

The constraints on these fields literally mimic (4.8), in which one should just put \(\theta = 0\) and make the replacements (4.25). It will be more convenient to start from the equivalent constraints (4.10) - (4.12), in which case we obtain

\[(\partial^{++} + \phi^{++}) f^+ = 0, \quad \partial^{++} \phi^{++} + (\phi^{++})^2 = 0, \]  

(4.26)

\[(\partial^{++} + \bar{\phi}^{++}) \bar{f}^+ = 0, \quad \partial^{++} \bar{\phi}^{++} + (\bar{\phi}^{++})^2 = 0, \]  

(4.27)

\[\phi^{++} - \bar{\phi}^{++} = -2\alpha_0 f^+ \bar{f}^+, \]

\[\partial^{++} \omega_0 - \phi^{++} - \alpha f^+ \bar{f}^+ = 0, \quad \text{or} \quad \partial^{++} \omega_0 - \bar{\phi}^{++} + \alpha f^+ \bar{f}^+ = 0, \]  

(4.28)

where \(\phi^{++} = \Phi^{++}_{\theta=0}\). The \(\phi^{++}\)-constraints in (4.26) and (4.27) can be solved as

\[\phi^{++} = \frac{A^{++}}{1 + A^{++}}, \quad \bar{\phi}^{++} = \frac{\bar{A}^{++}}{1 + \bar{A}^{++}}, \quad \partial^{++} A^{++} = \partial^{++} \bar{A}^{++} = 0, \]  

(4.30)

whence

\[A^{++} = A^{(ik)} u^+_i u^+_k, \quad \bar{A}^{++} = \bar{A}^{(ik)} u^+_i u^+_k, \quad \phi^{++} = \partial^{++} \ln(1 + A^{++}), \quad \bar{\phi}^{++} = \partial^{++} \ln(1 + \bar{A}^{++}). \]  

(4.31)

Then the constraints for \(f^+\) and \(\bar{f}^+\) in (4.26) and (4.27) are solved as

\[f^+ = \frac{f^+ u^+_i}{1 + A^{++}}, \quad \bar{f}^+ = \frac{\bar{f}^+ u^+_i}{1 + \bar{A}^{++}}. \]  

(4.32)

The complex field \(A^{ik}\) can be divided into real and imaginary parts

\[A^{ik} = a^{ik} + i b^{ik}. \]  

(4.33)

The real field \(a^{ik}\) is just an analog of the field \(a^{ik}\) of the U(2) model and it parametrizes the SU(2) part of the SU(3) manifold. The doublet fields \(f^k, \bar{f}^k\), together with the appropriate
analog of the singlet field $b$, complement this SU(2) manifold to the whole SU(3). Using the algebraic constraint (4.28), $b^k$ can be expressed in terms of $a^ik$ and $f^i, \bar{f}^i$:

$$b^k = \frac{2\alpha_0}{2 + a^2} \left[ \ell^k - \ell^j (a^k_j) + \frac{1}{2} (\ell \cdot a) a^k \right], \quad \ell^k := i f^i \bar{f}^k. \tag{4.34}$$

Its useful corollaries are

$$b \cdot a = \alpha_0 \ell \cdot a, \quad b^2 = \alpha_0 (\ell \cdot b) = \frac{2\alpha_0^2}{2 + a^2} \left[ \ell^2 + \frac{1}{2} (\ell \cdot a)^2 \right]. \tag{4.35}$$

The constraint (4.28), in terms of the harmonic projections, can be also written as

$$A^{++} (1 + \bar{A}^{--}) - \bar{A}^{++} (1 + A^{--}) = -2\alpha_0 f^i \bar{f}^k u^+_i u^+_k = -2\alpha_0 \ell^k u^+_i u^+_k. \tag{4.36}$$

In terms of the ordinary fields $A^{ik}, \bar{A}^{ik}$, it amounts to

$$A^{ik} - \bar{A}^{ik} + A^{(it} \bar{A}^{k)} = -2\alpha_0 f^i \bar{f}^k. \tag{4.37}$$

Note also the relation

$$A^{++} A^{--} - (A^{--})^2 = \frac{1}{2} A^2 \tag{4.38}$$

(and the analogous relation for the complex-conjugated fields). Some useful relations following from the constraint (4.37) are

$$f^i \bar{f}^k A^{ik} = \frac{1}{2\alpha_0} (A \cdot \bar{A} - A^2), \quad f^i \bar{f}^k \bar{A}^{ik} = -\frac{1}{2\alpha_0} (A \cdot \bar{A} - A^2), \tag{4.39}$$

$$f^i \bar{f}^k A^m_k = \frac{1}{4\alpha_0} \left[ \varepsilon^{im} (A^2 - A \cdot \bar{A}) + 2 A^k (i A_k^m) + A^m (A \cdot \bar{A}) - \bar{A}^m A^2 \right], \tag{4.40}$$

$$f^i \bar{f}^k \bar{A}_k^m = -\frac{1}{4\alpha_0} \left[ \varepsilon_{im} (\bar{A}^2 - A \cdot \bar{A}) - 2 A^k (i A_k^m) + \bar{A}^m (A \cdot \bar{A}) - A^m A^2 \right], \tag{4.41}$$

$$(2 + A^2)(2 + \bar{A}^2) = B_+ B_-, \quad B_\pm := 2 + A \cdot \bar{A} \pm 2\alpha_0 \bar{f}^i f_i, \tag{4.42}$$

$$(2 + A \cdot \bar{A})^2 - (2 + A^2)(2 + \bar{A}^2) = 4\alpha_0^2 (\bar{f}^i f_i)^2, \tag{4.43}$$

$$(\bar{f}^i f_i) \partial_i (\bar{f}^k f_k) = \frac{1}{4\alpha_0^2} \left[ \partial_i (A \cdot \bar{A})(2 + A \cdot \bar{A}) - (\dot{A} \cdot A)(2 + \bar{A}^2) - (A \cdot \bar{A})(2 + \bar{A}^2) \right]. \tag{4.44}$$

Using the superfield transformation laws (4.14) and (4.15), we can find the SU(3)/U(2) transformations of the “central basis” fields $A^{ik}$ and $f^i, \bar{f}^i$:

$$\delta A^{ik} = \alpha_0 \left[ (\xi^j f_i^i) A^{ik} - 2\xi^j (f^k)^j \right], \quad \delta \bar{A}^{ik} = \alpha_0 \left[ (\xi^j \bar{f}_i^i) \bar{A}^{ik} + 2\xi^j (\bar{f}^k)^j \right], \tag{4.45}$$

$$\delta f^k = \xi^k - A^{kl} \xi_l + \alpha_0 (\xi^l f^k)^l f^k, \quad \delta \bar{f}^k = \bar{\xi}^k - \bar{A}^{kl} \xi_l + \alpha_0 (\xi^l \bar{f}^k)^l \bar{f}^k, \tag{4.46}$$

as well as their U(2) transformations

$$\delta A^{ik} = 2\alpha_0 \left[ \lambda^{ik} + \frac{1}{2} (\lambda \cdot A) A^{ik} + \lambda (l A)^k \right], \quad \delta \bar{A}^{ik} = 2\alpha_0 \left[ \lambda^{ik} + \frac{1}{2} (\lambda \cdot \bar{A}) \bar{A}^{ik} + \lambda (l \bar{A})^k \right], \tag{4.47}$$

$$\delta f^i = \alpha_0 [3i\phi f^i + (\lambda \cdot A) f^i], \quad \delta \bar{f}^i = \alpha_0 [-3i\phi \bar{f}^i + (\lambda \cdot \bar{A}) \bar{f}^i]. \tag{4.48}$$
An interesting point is that, like in the previously considered U(2) case, there exists another SU(2) which commutes with the SU(3) transformations given above. It is realized by the transformations

\[ \delta A^{ik} = 2\alpha_0 \left[ -\tau^{ik} - \frac{1}{2} (\tau \cdot A) A^{ik} + \tau^{(il) A^{ik}} \right], \]

\[ \delta \bar{A}^{ik} = 2\alpha_0 \left[ -\tau^{ik} - \frac{1}{2} (\tau \cdot \bar{A}) \bar{A}^{ik} + \tau^{(il) \bar{A}^{ik}} \right], \]

\[ \delta f^i = \alpha_0 \left[ -(\tau \cdot A) f^i + 2\tau^{(il) f^i} \right], \]

\[ \delta \bar{f}^i = \alpha_0 \left[ -(\tau \cdot \bar{A}) \bar{f}^i + 2\tau^{(il) \bar{f}^i} \right]. \]

They have the same closure as the SU(2) transformations (4.47), (4.48) and the analogous ones considered in subsection 2.2. The diagonal SU(2) in the product of (4.47), (4.48) and (4.49), (4.50) is just the harmonic group SU(2)_H (with the parameters rescaled by 2\alpha_0). It is easy to check the covariance of the constraint (4.37) under these transformations and to be convinced that the superfield constraints (4.10) - (4.13) are also covariant. This last property immediately follows from the fact that the generators of the extra SU(2) are proper linear combinations of those generating the SU(2) part of the transformations (4.15) and of the generators of the harmonic SU(2). By the same reasoning, the action (4.16) is invariant. Moreover, we can extend the extra SU(2) to U(2) by noting that the superfield constraints and the action (4.16) are invariant under an extra U(1) acting as constant shifts of the superfield \( \omega \) and its first bosonic component \( \omega_0 \):

\[ \omega' = \omega + \tau_0 \quad \Leftrightarrow \quad \omega'_0 = \omega_0 + \tau_0. \]

So we come to the conclusion that the invariance group of the system we are considering is the product SU(3)_L×U(2). Only SU(3)_L×U(1) in this product commutes with \( \mathcal{N} = 4 \) supersymmetry and so defines the triholomorphic isometries [11].

In analogy with the SU(2)×U(1) case, the corresponding bosonic nonlinear sigma model should be associated with the eight-dimensional coset manifold \([SU(3)_L \times U(2)_R]/U(2)_{\text{diag}}\), where \( U(2)_{\text{diag}} \) involves SU(2)_H [12]. If \( g(\varphi) \) is an element of SU(3) and \( \varphi^A, A = 1, \ldots, 8 \), are local coordinates on SU(3), the nonlinear sigma model Lagrangian can be constructed from the \( d = 1 \) pullbacks of the current

\[ J_A = g^{-1} \partial_A g. \]

A metric on SU(3) respecting at least SU(3)_L invariance then reads

\[ g_{AB} = \text{tr}(J_A X J_B), \]

where \( X \) is a 3×3 matrix which should be chosen as \( X = \text{diag}(\kappa, \kappa, \chi) \) in order to preserve the SU(3)_L×U(2)_R symmetry. Here \( \kappa \) and \( \chi \) are some numerical parameters. At \( \kappa = \chi \), i.e. for \( X = \kappa I_{3\times3} \), the symmetry is enhanced to the product SU(3)_L×SU(3)_R, while for \( \kappa \neq \chi \) SU(3)_R is broken to U(2). Just the latter option is expected to be valid in our \( \mathcal{N} = 4 \) SU(3) model, with the ratio of the parameters \( \chi \) and \( \kappa \) strictly fixed. In order to check all this, we need the explicit expression for the bosonic invariant action as an integral over \( t \). This form of the bosonic action (with the harmonic integrals explicitly done) will be presented elsewhere.

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[11] We inquired whether one can extend (4.49), (4.50) to another (right) SU(3) which would (a) commute with the first (left) SU(3) and (b) preserve the important constraint (4.37). We failed to find such an extension.

[12] The U(1) factor in U(2)_diag acts as homogeneous phase transformations of \( f^i \) and \( \bar{f}^i \) (or of the superfields \( \eta^+, \bar{\eta}^+ \), in the original setting).
4.3 Solving further constraints

Our ultimate purpose is to find the bosonic component Lagrangian. Besides the constraints on the fields $\omega_0, f^i, \bar{f}^i$ and $n^{++}$, we now need also those for the remaining bosonic components in the $\theta$ expansions (4.24), because all such components are involved in the bosonic action. The harmonic equations for the extra fields $\sigma, \omega_0, \sigma^{-2}$ and $\mu^-, \bar{\mu}^-$ following from the superfield constraints read

\[
(\partial^{++} + 2\phi^{++})\tilde{\sigma} - 2i\dot{\phi}^{++} = 0, \quad (\partial^{++} + 2\bar{\phi}^{++})\tilde{\sigma} + 2i\dot{\bar{\phi}}^{++} = 0, \tag{4.54}
\]

\[
\partial^{++}\omega_0 - \frac{1}{2}\beta_{(+)}\phi^{++} - \frac{1}{2}\beta_{(-)}\bar{\phi}^{++} = 0, \tag{4.55}
\]

\[
\partial^{++}\sigma^{-2} - 2i\dot{\omega}_0 - \frac{1}{2}(\tilde{\sigma} - \bar{\tilde{\sigma}}) + i\alpha_1(\mu^-\bar{f}^+ - \bar{\mu}^-f^+) = 0, \tag{4.56}
\]

\[
(\partial^{++} + \phi^{++})\mu^- + \sigma f^+ - 2i\dot{f}^+ = 0, \quad (\partial^{++} + \bar{\phi}^{++})\bar{\mu}^- + \bar{\sigma}\bar{f}^+ + 2i\dot{\bar{f}}^+ = 0, \tag{4.57}
\]

where

\[
\tilde{\sigma} = \sigma - \alpha(\mu^-\bar{f}^+ - \bar{\mu}^-f^+), \quad \bar{\tilde{\sigma}} = -\sigma - \bar{\alpha}(\mu^-\bar{f}^+ - \bar{\mu}^-f^+), \quad \beta_{(\pm)} = 1 \pm i\frac{\alpha_1}{\alpha_0}, \tag{4.58}
\]

and we took advantage of the constraint (4.28).

Using (4.31), it is easy to solve eq. (4.55):

\[
\omega_0 = b + \frac{1}{2}\beta_{(+)}\ln(1 + A^{+-}) + \frac{1}{2}\beta_{(-)}\ln(1 + \bar{A}^{+-}), \tag{4.59}
\]

where $b = b(t)$ is a real harmonic-independent $d = 1$ field. Also, the general solution of eqs. (4.54) is

\[
\tilde{\sigma} = \frac{1}{(1 + A^{+-})^2}\left\{c + 2i\left[\hat{A}^{+-} + A^i\hat{A}_i^k u^+_k u^-_i\right]\right\}, \text{and c.c.}, \tag{4.60}
\]

where $c = c(t)$ is a complex harmonic-independent $d = 1$ field. Note that (4.60) and the definition (4.58) allow one to find

\[
\mu^-\bar{f}^+ - \bar{\mu}^-f^+ = -\frac{1}{2\alpha_0}(\tilde{\sigma} + \bar{\tilde{\sigma}}). \tag{4.61}
\]

Further, taking the harmonic integral of eq. (4.56) and making use of the formulas from Appendix, we find one relation between the fields $c$ and $b$:

\[
\beta_{(+)}\frac{1}{2 + A^2}(c + i\hat{A} \cdot A) - \beta_{(-)}\frac{1}{2 + \bar{A}^2}(\bar{c} - i\hat{A} \cdot \bar{A}) + 2ib = 0. \tag{4.62}
\]

Like in the case of the $SU(2) \times U(1)$ model, to compute the component action there is no need to explicitly solve eq. (4.56). However, one still needs to have the solution of eqs. (4.57). It is convenient to redefine

\[
\mu^- = \frac{1}{1 + A^{+-}}\bar{\mu}^-, \quad \partial^{++}\bar{\mu}^- - 2i\dot{f}^i u^+_i + f^i u^+_i \left(\tilde{\sigma} + \frac{2i}{1 + A^{+-}}\hat{A}^{+-}\right) = 0. \tag{4.63}
\]
After some work, the solution is found to be

\[ \hat{\mu}^- = 2i\hat{f}^i u_i^- + \omega^- c(t) - 2i\hat{A}^{(jk)}(f^i u_i^- A_{(jk)} \frac{1}{2 + A^2} + f^i u_i^+ \Psi_{(jk)}^-), \]

\[ \hat{\mu}^- = -2i\hat{f}^i u_i^- + \bar{\omega}^- \bar{c}(t) + 2i\hat{A}^{(jk)}(\bar{f}^i u_i^- \bar{A}_{(jk)} \frac{1}{2 + A^2} + \bar{f}^i u_i^+ \bar{\Psi}_{(jk)}^-). \]

where

\[ \omega^- = -\frac{2}{(2 + A^2)} \left( f^k u_k^- - f^k u_k^+ \frac{A^-}{1 + A^+} \right), \quad \bar{\omega}^- = -\frac{2}{(2 + A^2)} \left( \bar{f}^k u_k^- - \bar{f}^k u_k^+ \frac{\bar{A}^-}{1 + A^+} \right), \]

\[ \Psi_{ik}^- = \frac{1}{(2 + A^2)(1 + A^-)} \left( u_i^- u_k^- [2 + \frac{1}{2} A^2 - (A^+)^2] + [2u_{(i}^+ u_{k)}^- A^+ - u_i^+ u_k^+ A^-] A^- \right). \]

Note the useful formula

\[ \hat{A}^i{}^k \Psi_{ik}^- = \frac{1}{(2 + A^2)(1 + A^-)} \left( \hat{A}^- - (2 + A^2) - (\hat{A} \cdot A) A^- \right). \]

Now, using the explicit expressions for \( \mu^-, \bar{\mu}^- \) given above and the relation (4.61), we can establish one more relation between fields \( c(t) \) and \( b(t) \), in addition to (4.62)

\[ \frac{1}{2 + A^2} (c + i\hat{A} \cdot A) + \frac{1}{2 + A^2} (\bar{c} - i\hat{A} \cdot \bar{A}) = 2i\alpha_0 D, \]

where

\[ D := \frac{1}{2 + (A \cdot A) - 2\alpha_0 \hat{f}^i f_i} \left[ f^i \hat{f}_i - f^i \hat{f}_i + \frac{1}{2\alpha_0} (\hat{A} \cdot \bar{A} - \hat{A} \cdot A) \right], \quad \bar{D} = -D. \]

The set of eqs. (4.62) and (4.69) can be solved for \( c \) and \( \bar{c} \) as

\[ c + i(\hat{A} \cdot A) = -i(2 + A^2) \left[ \hat{b} - \alpha_0 \beta_{(-)} D \right], \quad \bar{c} - i(\hat{A} \cdot \bar{A}) = i(2 + \bar{A}^2) \left[ \hat{b} + \alpha_0 \beta_{(+)} \bar{D} \right]. \]

In what follows, we will also use

\[ \hat{c} := c(b = 0) = -i \left[ (\hat{A} \cdot A) - \alpha_0 (2 + A^2) \beta_{(-)} D \right], \quad \hat{\bar{c}} = i \left[ (\hat{A} \cdot \bar{A}) + \alpha_0 (2 + \bar{A}^2) \beta_{(+)} \bar{D} \right]. \]

To close this subsection, we present the SU(3) transformation laws of the field \( b(t) \). They can be found by comparing the \( \theta = 0 \) part of the transformations (4.6), (4.7) with the direct variation of \( \omega_0 \) in (4.59):

\[ \delta(\xi) b = -\alpha_0 \left[ \beta_{(-)} (\xi^i \hat{f}_i + \bar{\xi}_i f^i) \right], \quad \delta(\phi, \lambda) b = \alpha_1 \phi - \frac{\alpha_0}{2} \left[ \beta_{(+)} (\lambda \cdot A) + \beta_{(-)} (\lambda \cdot \bar{A}) \right]. \]

It is easy to construct the covariant derivative \( D b \). We first notice that under the \( \xi_i \) and \( \lambda \) transformations:

\[ \delta(\xi) B_- = \alpha_0 (\xi^i \hat{f}_i + \bar{\xi}_i f^i) B_-, \quad \delta(\xi) D = -\frac{1}{2} (\xi^i \hat{f}_i - \bar{\xi}_i f^i), \]

\[ \delta(\lambda) B_- = \alpha_0 (\lambda \cdot A + \lambda \cdot \bar{A}) , \quad \delta(\lambda) D = \frac{1}{2} (\lambda \cdot \hat{A} - \lambda \cdot \bar{A}). \]
where $B_-$ was defined in (4.42). Then

$$D_t b = \dot{b} + i\alpha_1 D + \frac{1}{2} B_-^{-1} \dot{B}_-, \quad \delta(\xi) D_t b = \delta(\phi, \lambda) D_t b = 0.$$  (4.76)

From this definition it follows:

$$(D_t b)^2 = (\dot{b})^2 + 2i\alpha_1 \dot{b} D + bB_-^{-1} \dot{B}_- - \alpha_1^2 D^2 + \frac{1}{4} B_-^{-2} (\dot{B}_-)^2 + i\alpha_1 D B_-^{-1} \dot{B}_-.$$  (4.77)

Like in the $U(2)$ case, the covariant derivative $D_t b$ is simplified after the redefinition

$$\tilde{b} = b + \frac{1}{2} \ln B_-, \quad D_t b = \tilde{b} + i\alpha_1 D.$$  (4.78)

Finally, we note that the extra $U(1)$ symmetry (4.51) acts as a constant shift of the field $b$ (or of $\tilde{b}$):

$$b' = b + \tau_0.$$  (4.79)

The group $U(1) \subset SU(3)$ also acts as a shift of $b$, but it acts as well on $f^i, \bar{f}^i$ as a phase transformation. The extra U(1) affects only the field $b(t)$ $\textsuperscript{13}$. It is also worth noting that under the extra SU(2) symmetry (4.49), (4.50) the field $b$ transforms as

$$\delta_\tau b = \frac{\alpha_0}{2} \left[ \beta_+(\tau \cdot A) + \beta_-(\tau \cdot \bar{A}) \right].$$

The covariant derivative $D_t b$ is invariant under the $\tau$ transformations too, $\delta_\tau D_t b = 0$.

### 4.4 The bosonic invariant action

The invariant action (4.16) consists of two parts, which can be written as

$$S_{su(3)} = S(N, \omega) + \alpha_0 S(q).$$  (4.80)

After integrating over Grassmann coordinates and passing to the bosonic limit, we obtain

$$S(N, \omega) \rightarrow S_1 = \frac{i}{2} \int dt du \left( n^+ \sigma^{-2} + \sigma \dot{\omega}_0 \right), \quad S(q) \rightarrow S_2 = i \int dt du \left( \mu^- \tilde{f}^+ + \mu^- \tilde{f}^+ \right).$$  (4.81)

and so

$$S_{bos}^{su(3)} = S_1 + \alpha_0 S_2.$$  (4.82)

In Appendix A we will present the proof that (4.82) is indeed SU(3) invariant like its full superfield prototype (4.16).

To compute the bosonic action as a $t$-integral, one needs to substitute the explicit expressions of the harmonic fields that were found in the previous subsection and then do the harmonic integrals. The latter task is rather difficult, because, as opposed to the $U(2)$ case, we face here integrals involving in the denominator products of the harmonic factors $(1 + A^{+-})$ and $(1 + \bar{A}^{+-})$ with $A^{ik} \neq \bar{A}^{ik}$. Some basic integrals of this kind are calculated in Appendix B. We

\textsuperscript{13}The homogeneous part of the product of these two $U(1)$ transformations belongs to the stability subgroup $U(2)_{\text{diag}}$ of the coset $[SU(3)\times U(2)]/U(2)_{\text{diag}}$, while the remainder amounts to a shift of $b$ as the 8-th coset parameter.
postpone the purely technical task of restoring the full bosonic component action to the next publication. Here we limit our consideration only to the $b$-dependent part of the bosonic action (4.82).

To find this part, we should collect those terms in (4.82) which involve the field $b$. The full set of such terms is as follows

$$
S(b) = \int dt \left\{ 2\dot{b} \left( \frac{\dot{A} \cdot A}{2 + A^2} + \frac{\dot{A} \cdot \bar{A}}{2 + \bar{A}^2} \right) - \frac{1}{2} \dot{b} \left[ (2 + A^2) \int du \frac{\dot{A}^{+-}}{(1 + A^{+-})^2(1 + \bar{A}^{+-})} \right] + (2 + \bar{A}^2) \int du \frac{\dot{A}^{+-}}{(1 + A^{+-})^2(1 + \bar{A}^{+-})} \right\} + 2i\alpha_1 \dot{b} D + (\dot{b})^2 
$$

Here, the first two lines are the contribution from $S_1$, while the third line comes from $S_2$. The explicit expressions for the relevant harmonic integrals are collected in Appendix B. Using them and integrating by parts in the last line of (4.83), we finally obtain

$$
S(b) = (\dot{b})^2 + 2i\alpha_1 \dot{b} D + \dot{b} B^{-1} \dot{B}_- ,
$$

in accordance with the formula (4.77). The $b$-independent terms completing $S(b)$ to the total expression (4.77) should come from the remainder of the bosonic action (4.82).

5 Summary and outlook

The basic aim of the present paper was to start the construction of the harmonic superfield actions for the group manifolds with quaternionic structure as nice explicit examples of the “strong” HKT $\mathcal{N} = 4$ supersymmetric $d = 1$ sigma models the general formulation of which was given in [22]. We limited our attention to the simple cases of the 4-dimensional group manifold $U(2)$ and the 8-dimensional one $SU(3)$. For both cases we have found the relevant nonlinear harmonic constraints and shown that the relevant invariant actions are bilinear in the superfields involved, as it should be for the strong HKT models in agreement with the reasoning of [22]. It was found that the full internal symmetry of the $U(2)$ and $SU(3)$ models are, respectively, $U(2) \times SU(2)$ and $SU(3) \times U(2)$, with the standard harmonic $SU(2)_H$ symmetry forming the diagonals in these products. For the $U(2)$ case we computed the full bosonic action, which is none other than the $S^3 \times S^1$ one, and showed that the gauging of its $U(1)$ isometry, both at superfield and at component levels, yields a particular case of the nonlinear $(3, 4, 1)$ multiplet action, with pure $S^3$ bosonic target. For the $SU(3)$ case we presented the bosonic action in the space $(t, u^\pm_0)$, independently checked its $SU(3) \times U(2)$ invariance and gave explicitly an important part of it by doing the integration over harmonic variables. For the $U(2)$ case we also performed the detailed comparison with the general harmonic approach to HKT geometries (of refs. [22] and [25]) and found excellent agreement with this approach. We explicitly constructed the closed torsion for this case, as well as the Bismut and Obata connections, with respect to which the corresponding triplet of quaternionic complex structures is covariantly constant. The Riemann curvatures of both these connections were checked to vanish.
It may be possible to generalize the considerations of section 2 to bigger groups. One could consider harmonic superfields $N^{++}$ and $\Omega$ which are $n$ by $n$ matrices, satisfying the constraints

$$D^{++}\Omega + N^{++}\Omega = 0, \quad D^{++}N^{++} + (N^{++})^2 = 0.$$ (5.1)

These constraints are covariant under the following transformation laws

$$\delta \Omega = (-\Lambda^{+-} + \rho + N^{++}\Lambda^{--})\Omega, \quad \delta N^{++} = \Lambda^{+-} - \{\Lambda^{+-}, N^{++}\} + [\rho, N^{++}] + N^{++}\Lambda^{--}N^{++},$$ (5.2)

where $\rho$ and $\Lambda^{\pm\pm}$ are $n \times n$ matrices. Elements of $\rho$ are constants, and elements of $\Lambda^{\pm\pm}$ are triplets of the harmonic $SU(2)$, $\Lambda^{\pm\pm} = \Lambda_{ij}^\pm u^\pm_i u^\pm_j$. These transformations form an algebra

$$[\delta, \delta'](\Omega, N^{++}) = \delta''(\Omega, N^{++}), \quad \rho'' = -[\rho, \rho'] - [\Lambda^{+-}, \Lambda'^{+-}] + \frac{1}{2}[\Lambda^{++}, \Lambda'^{+-}] + \frac{1}{2}[\Lambda^{--}, \Lambda'^{+-}],$$

$$\Lambda''^{++} = -\frac{1}{2}[\Lambda^{++}, \Lambda'^{+-}] + \frac{1}{2}[\Lambda^{--}, \Lambda'^{+-}] + [\rho', \Lambda^{+-}] - [\rho, \Lambda^{+-}].$$ (5.3)

The bosonic part of the superfields $N^{++}$ and $\Omega$ should parametrize the group $SU(2n) \times U(1)$, and the transformations in (5.2) correspond to infinitesimal $SU(2n) \times U(1)$ translations. These fields generalize those in section 2. However, we have not been able as yet to write an invariant action generalizing (2.5). We leave this problem for further study.

It remains as well to explicitly derive the complete component action for the $SU(3)$ model, and to recover the basic geometric objects of this model from the general harmonic HKT formalism of refs. [22] and [25], as it was done for the $U(2)$ model in section 3.

Acknowledgments

The authors thank Andrei Smilga for his interest in this work and valuable discussions. The work of E.I. was supported by the RFBR grant, project No 18-02-01046, and a grant of the Heisenberg-Landau program. He thanks the Laboratoire de Physique, ENS-Lyon, for the kind hospitality extended to him a few times in the course of this work. A part of this research was completed, when E.I. was visiting SUBATECH, University of Nantes. He is indebted to the Directorate of SUBATECH for its kind hospitality.

Appendix A

In this Appendix we prove the $SU(3)$ invariance of the bosonic action (4.82). The proof follows the same line as the check of the invariance of the superfield action (4.16): we consider only the coset $SU(3)/U(2)$ transformations and leave in them only holomorphic parts $\sim \xi_i$. We do not assume in advance the relations (4.17), (4.18) except for $\gamma_1 = -\alpha_1$. Then the component
field transformations we are interested in read

\[ \delta f^+ = \xi^+ - \xi^- \phi^+ , \quad \delta f^- = -2\alpha_0 \xi^- (\tilde{f}^+)^2 , \]
\[ \delta n^{++} = \alpha (\xi^+ - \xi^- n^{++}) \tilde{f}^+ - \alpha \bar{\alpha} \xi^- f^+ (\tilde{f}^+)^2 , \]
\[ \delta \phi^{++} = 0 , \quad \delta \bar{\phi}^{++} = 2\alpha_0 (\xi^+ - \xi^- \bar{\phi}^{++}) \tilde{f}^+ , \]
\[ \delta \sigma = -\alpha \xi^+ \bar{\mu}^- + \alpha \xi^- (n^{++} \bar{\mu}^- - \bar{\sigma} \tilde{f}^+) - \alpha \bar{\alpha} \xi^- [(\tilde{f}^+)^2 \mu^- - 2f^+ \tilde{f}^+ \bar{\mu}^-] , \]
\[ \delta \omega_0 = \bar{\alpha} \xi^- \tilde{f}^+ , \quad \delta \sigma^{-2} = -\bar{\alpha} \xi^- \bar{\mu}^- , \]
\[ \delta \mu^- = -\xi^- [\sigma + \alpha (f^+ \bar{\mu}^- - \tilde{f}^+ \bar{\mu}^-)] , \quad \delta \bar{\mu}^- = -4\alpha_0 \xi^- \tilde{f}^+ \bar{\mu}^- . \quad (A.1) \]

These are directly deduced by substituting the truncated superfields \((4.24)\) into the transformation laws \((4.3), (4.6)\) and \((4.14)\). Substituting \(\xi^+ = \partial^{++} \xi^-\), integrating by parts with respect to \(\partial^{++}\) and \(\partial_t\), using the constraint \((4.55)\) alongside with the second constraint in \((4.57)\), as well as the relations \((4.58)\) and \((4.61)\), we are able to cast the variation \(\delta S_1\) in the form

\[ \delta S_1 = i \int dt du \xi^+ \left\{ \frac{\alpha}{2} \left[ \beta_+ \partial_0 \sigma - \beta_- \sigma \bar{\partial}_0 \right] + \bar{\alpha} \left( \bar{n}^{++} \mu + \sigma \tilde{f}^+ \right) \right. \]
\[ \left. - \alpha \bar{\mu}^- (n^{++} - 2i\alpha_1 f^+ \tilde{f}^+) \right\} . \quad (A.2) \]

Analogously,

\[ \delta S_2 = i \int dt du \xi^- \left\{ - 4\alpha_0 \bar{\mu}^- \tilde{f}^+ + \bar{\mu}^- \phi^{++} - \sigma \tilde{f}^+ \right. \]
\[ \left. - \alpha (f^+ \bar{\mu}^- - \tilde{f}^+ \mu^-) \tilde{f}^+ + 2\alpha_0 \bar{\mu}^- (\tilde{f}^+)^2 \right\} . \quad (A.3) \]

For \(\delta S_{su(3)}^{bos} = \delta S_1 + \alpha_0 \delta S_2\) we obtain, up to total \(t\)-derivative and making use of \((4.61)\),

\[ \delta S_{su(3)}^{bos} = i \int dt du \xi^- \left\{ \tilde{f}^+ \bar{f}^+ \mu^- (\alpha \bar{\alpha} - 4\alpha_0^2) + \tilde{f}^+ f^+ \bar{\mu}^- [i\alpha_1 \alpha_0 \beta_+ + \alpha \bar{\alpha} - \alpha \alpha_0] \right. \]
\[ \left. + \tilde{f}^+ f^+ \bar{\mu}^- [\alpha \alpha_0 - 4\alpha_0^2 - i\alpha_1 \alpha_0 \beta_+] \right\} . \quad (A.4) \]

All terms in \((A.4)\) are independent, so in order to have \(\delta S_{su(3)}^{bos} = 0\), their coefficients should vanish separately,

\( (a) \, \alpha \bar{\alpha} - 4\alpha_0^2 = 0 , \quad (b) \, i\alpha_1 \alpha_0 \beta_+ + \alpha \bar{\alpha} - \alpha \alpha_0 = 0 , \quad (c) \, \alpha \alpha_0 - 4\alpha_0^2 - i\alpha_1 \alpha_0 \beta_+ = 0 . \quad (A.5) \]

It is direct to check that the conditions \((A.5b)\) and \((A.5c)\) are identically satisfied as a consequence of the single condition \((A.5a)\),

\[ \alpha \bar{\alpha} - 4\alpha_0^2 = 0 \iff \alpha_1^2 = 3\alpha_0^2 ; \quad (A.6) \]

which is just the relation already found in \((4.18)\).

Thus we have independently proved that the bosonic action \((4.82)\) is SU(3) invariant under the condition \((A.6)\) and so is guaranteed to define some \(d = 1\) sigma model on the SU(3) group manifold. It is also straightforward to show its invariance under the transformations of the U(2) group commuting with SU(3). It is worth pointing out that checking all these invariances does not require doing explicitly the harmonic integrations in \((4.82)\).
Appendix B

In this Appendix we present the calculation of some harmonic integrals. The general method consists in expanding the harmonic integrands in power series in \( a^+ = a^{ik} u^+_i u^-_k \) and then doing the harmonic integrals for each term in this expansion using the general formula

\[
\int du (a^+)^{2n} = (-1)^n \frac{1}{2n+1} \left( \frac{a^2}{2} \right)^n, \quad a^2 = a^{ik} a_{(ik)}.
\] (B.1)

The integral of any odd power of \( a^+ \) is vanishing. In this way, we obtain, e.g.,

\[
\int du \ln(1 + a^+) = -1 + \frac{1}{2} \ln \left( 1 + \frac{a^2}{2} \right) + \frac{\arctan \sqrt{\frac{a^2}{2}}}{\sqrt{\frac{a^2}{2}}},
\] (B.2)

\[
\int du \frac{1}{1 + a^+} = \frac{\arctan \sqrt{\frac{a^2}{2}}}{\sqrt{\frac{a^2}{2}}},
\] (B.3)

\[
\int du \frac{a^+}{(1 + a^+)^2} = - \left( \partial_\beta \int du \frac{1}{1 + \beta a^+} \right)_{\beta=1} = \frac{\arctan \sqrt{\frac{a^2}{2}}}{\sqrt{\frac{a^2}{2}}} - \frac{1}{1 + \frac{a^2}{2}},
\] (B.4)

\[
\int du \frac{1}{(1 + a^+)^2} = \int du \left[ \frac{1}{1 + a^+} - \frac{a^+}{(1 + a^+)^2} \right] = \frac{1}{1 + \frac{a^2}{2}},
\] (B.5)

\[
\int du \frac{a^+}{(1 + a^+)^3} = - \frac{1}{2} \left( \partial_\beta \int du \frac{1}{(1 + \beta a^+)^2} \right)_{\beta=1} = \frac{a^2}{2} \frac{1}{(1 + \frac{a^2}{2})^2},
\] (B.6)

\[
\int du \frac{1}{(1 + a^+)^3} = \int du \left[ \frac{1}{(1 + a^+)^2} - \frac{a^+}{(1 + a^+)^3} \right] = \frac{1}{(1 + \frac{a^2}{2})^2}.
\] (B.7)

It is also easy to calculate the integrals of the type

(a) \( I_{(ik)} = \int du \frac{u^+_i u^-_k}{(1 + a^+)^2} \),  (b) \( I'_{(ik)} = \int du \frac{u^+_i u^-_k}{1 + a^+} \). (B.8)

Keeping in mind that SU(2) acting on the doublet indices cannot be broken in the process of harmonic integration, these integrals should be of the form

\[
I_{(ik)} = a_{(ik)} f(a^2), \quad I'_{(ik)} = a_{(ik)} f'(a^2). \] (B.9)

Contracting \( I_{(ik)} \) with \( a^{ik} \) and using (B.4), we find

\[
I_{(ik)} = a_{(ik)} \frac{1}{a^2} \left( \frac{\arctan \sqrt{\frac{a^2}{2}}}{\sqrt{\frac{a^2}{2}}} - \frac{1}{1 + \frac{a^2}{2}} \right). \] (B.10)
This expression is obviously non-singular at \( a^2 = 0 \). Using (B.9), it is easy to show, e.g., that 
\[ a^l \partial^k_i I_{(ik)} = 0, \]
which just means vanishing of the expression (2.15) and of the first term in the square bracket in (2.17). The explicit form of the integral \( I'_{(ik)} \) can also be easily found

\[ I'_{(ik)} = a_{(ik)} \frac{1}{a^2} \left( 1 - \frac{\arctan \sqrt{\frac{a^2}{2}}}{\sqrt{\frac{a^2}{2}}} \right). \]  

(B.11)

It is non-singular at \( a^2 = 0 \), like \( I_{(ik)} \).

Using similar reasonings, one can compute

\[ J_{(ik)} = \int du \frac{u^+_i u^-_k}{(1 + a^{+-})^3} = \frac{1}{2} a_{(ik)} \frac{1}{(1 + \frac{a^2}{2})^2}, \]  

(B.12)

and

\[ J_{(ik)(mn)} = \int du u^+_i u^-_k u^+_m u^-_n (1 + a^{+-})^3 = a_{(ik)} a_{(mn)} g(a^2) + (\varepsilon_{im}\varepsilon_{kn} + \varepsilon_{km}\varepsilon_{in}) \tilde{g}(a^2), \]  

(B.13)

where

\[ g(a^2) = \frac{1}{2 (1 + \frac{a^2}{2})^2}, \quad \tilde{g}(a^2) = -\frac{1}{4 (1 + \frac{a^2}{2})}. \]  

(B.14)

Let us also give an example of splitting some function of \( a^{+-} \) into harmonic independent and harmonic dependent parts. As such function we choose \( 1/(1 + a^{+-}) \) and write

\[ \frac{1}{1 + a^{+-}} = \lambda(a^2) + \partial^{++} \psi^{--}(a^{+-}, a^2). \]  

(B.15)

Our task is to find \( \lambda(a^2) \) and \( \psi^{--}(a^{+-}, a^2) \).

Integrating both sides of (B.15) over harmonics and using (B.3), we find

\[ \lambda(a^2) = \frac{\arctan \sqrt{\frac{a^2}{2}}}{\sqrt{\frac{a^2}{2}}} \]  

(B.16)

Then we represent \( \psi^{--} \) as

\[ \psi^{--} = a^{--} f(a^2, a^{+-}), \]  

(B.17)

and substitute it into (B.15), denoting \( a^{+-} := z \). We find

\[ \lambda + [(z^2 + \frac{1}{2} a^2) f]' = \frac{1}{1 + z}, \]  

(B.18)

whence

\[ (z^2 + \frac{1}{2} a^2) f = -\lambda z + c + \ln(1 + z), \]  

(B.19)

where \( c = c(a^2) \) is an integration constant. Now we should show that this constant can be chosen in such a way that the right-hand side will be \( O(z^2 + \frac{1}{2} a^2) \). We rewrite

\[ \ln(1 + z) = \frac{1}{2} \ln(1 - z^2) + \frac{1}{2} \left[ \ln(1 + z) \frac{z}{z} - \ln(1 - z) \frac{z}{z} \right] z. \]  

(B.20)
Both the first term and the expression inside the square brackets in (B.20) are functions of $z^2$ and we denote
\[
\frac{\ln(1 + z) - \ln(1 - z)}{z} := g(z^2) = 2 + \frac{2}{3} z^2 + \frac{2}{5} z^4 + \ldots.
\] (B.21)

Now we denote $z^2 + \frac{1}{2} a^2 := y$ and rewrite
\[
\ln(1 - z^2) = \ln \left(1 + \frac{1}{2} a^2\right) + \ln \left(1 - \frac{y}{1 + \frac{1}{2} a^2}\right), \quad g(z^2) = g(y - \frac{1}{2} a^2).
\] (B.22)

We wish the r.h.s. in (B.19) to start with $y$. This implies, first,
\[
c = -\frac{1}{2} \ln \left(1 + \frac{1}{2} a^2\right).
\] (B.23)

Second, we need to cancel the contribution from $g(-\frac{1}{2} a^2)$ which can be written as
\[
\frac{1}{2i \sqrt{\frac{a^2}{2}}} \left[ \ln \left(1 + i \sqrt{\frac{a^2}{2}}\right) - \ln \left(1 - i \sqrt{\frac{a^2}{2}}\right) \right] z
\] (B.24)

Using the well known formula
\[
\arctan x = \frac{1}{2i} \ln \frac{1 + ix}{1 - ix},
\]
we observe that (B.24) exactly cancels the $\lambda$ term in (B.19). So we can divide both sides of (B.19) by $y$ and write the nonsingular solution for the function $f$ as
\[
f(a^2, z) = \frac{\ln \left(1 - \frac{y}{1 + \frac{1}{2} a^2}\right)}{y} + \frac{1}{2y} \left[ g(z^2) - g(-\frac{1}{2} a^2) \right].
\] (B.25)

**Appendix C**

In this Appendix we present some results of calculation of harmonic integrals depending on the fields $A^{ik}$ and $\bar{A}^{ik}$. To compute the harmonic integrals in (1.83) and those appearing in the other pieces of the total bosonic action we apply the well-known Feynman formula
\[
\frac{1}{\mathcal{A}^p \mathcal{B}^q} = \int_0^1 dx \frac{x^{p-1}(1-x)^{q-1}}{[x \mathcal{A} + (1-x) \mathcal{B}]^{p+q}} \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}.
\] (C.1)

We find:
\[
\int du \frac{1}{(1 + A^-)(1 + A^+)} = -\frac{1}{2\sqrt{\Delta}} \ln \frac{B_-}{B_+},
\] (C.2)
\[
\int du \frac{1}{(1 + A^-)^2(1 + A^+)} = \frac{1}{2\Delta} \left[ \frac{1}{2\sqrt{\Delta}} (\bar{A}^2 - A \cdot \bar{A}) \ln \frac{B_-}{B_+} - 2 \frac{A^2 - A \cdot \bar{A}}{2 + A^2} \right],
\] (C.3)
\[
\int du \frac{1}{(1 + A^-)^2(1 + A^+)} = \frac{1}{2\Delta} \left[ \frac{1}{2\sqrt{\Delta}} (A^2 - A \cdot \bar{A}) \ln \frac{B_-}{B_+} - 2 \frac{\bar{A}^2 - A \cdot \bar{A}}{2 + A^2} \right],
\] (C.4)
\[ \int du \frac{u^+_{(i} u^-_{k)}}{(1 + A^+)^2(1 + A^-)} = \frac{1}{2\Delta} A_{ik} \left( \frac{2 + A \cdot \bar{A}}{2 + A^2} + \frac{2 + \bar{A}^2}{4\sqrt{\Delta}} \ln \frac{B_-}{B_+} \right) \]

\[ -\frac{1}{2\Delta} A_{ik} \left( 1 + \frac{2 + A \cdot \bar{A}}{4\sqrt{\Delta}} \ln \frac{B_-}{B_+} \right), \quad (C.5) \]

\[ \int du \frac{u^+_{(i} u^-_{k)}}{(1 + A^+)^2(1 + A^-)} = \frac{1}{2\Delta} \bar{A}_{ik} \left( \frac{2 + A \cdot \bar{A}}{2 + A^2} + \frac{2 + A^2}{4\sqrt{\Delta}} \ln \frac{B_-}{B_+} \right) \]

\[ -\frac{1}{2\Delta} A_{ik} \left( 1 + \frac{2 + A \cdot \bar{A}}{4\sqrt{\Delta}} \ln \frac{B_-}{B_+} \right). \quad (C.6) \]

In these formulas,

\[ B_{\pm} := 2 + A \cdot \bar{A} \pm 2\alpha_0 \bar{f}^i f_i, \quad \Delta := \alpha_0^2 (\bar{f}^i f_i)^2. \quad (C.7) \]

The \( f^i = \bar{f}^i = 0 \) and \( a^{ik} = 0 \) limits.

In the limit \( f^i = \bar{f}^i = 0 \), \( A^{ik} = \bar{A}^{ik} = a^{ik} \) the expressions above go over to the corresponding expressions from Appendix A. Less trivial is the limiting case \( a^{ik} = 0 \), \( A^{ik} = -\bar{A}^{ik} = ib^{ik} = -\alpha_0 \bar{f}^i (f^k) \). In this limit

\[ (B.1) \Rightarrow -\frac{1}{\sqrt{\Delta}} \ln \frac{1 - \frac{\alpha_0}{2} (\bar{f} f)}{1 + \frac{\alpha_0}{2} (f \bar{f})}, \]

\[ (B.2) = (B.3) \Rightarrow -\frac{1}{2\sqrt{\Delta}} \ln \frac{1 - \frac{\alpha_0}{2} (\bar{f} f)}{1 + \frac{\alpha_0}{2} (f \bar{f})} + \frac{1}{2} \left[ 1 - \frac{1}{\alpha_0^2 (f \bar{f})^2} \ln \frac{1 - \frac{\alpha_0}{2} (\bar{f} f)}{1 + \frac{\alpha_0}{2} (f \bar{f})} \right], \]

\[ (B.4) = -(B.5) \Rightarrow -ib_{ik} \frac{1}{\Delta} \left[ \frac{1}{1 - \frac{\alpha_0}{2} (f \bar{f})^2} + \frac{1}{\sqrt{\Delta}} \ln \frac{1 - \frac{\alpha_0}{2} (\bar{f} f)}{1 + \frac{\alpha_0}{2} (f \bar{f})} \right]. \]

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