Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations

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Abstract

Dynamic phenomena in social and biological sciences can often be modeled by employing reaction-diffusion equations. Frequently in applications, their control plays an important role when avoiding population extinction or propagation of infectious diseases, enhancing multicultural features, etc.

When addressing these issues from a mathematical viewpoint one of the main challenges is that, because of the intrinsic nature of the models under consideration, the solution, typically a proportion or a density function, needs to preserve given lower and upper bounds (taking values in \([0,1]\)). Controlling the system to the desired final configuration then becomes complex, and sometimes even impossible.

In the present work, we analyze the controllability to constant steady-states of spatially homogeneous semilinear heat equations, with constraints in the state, and using boundary controls, which is indeed a natural way of acting on the system in the present context. The nonlinearities considered are among the most frequent: monostable and bistable ones. We prove that controlling the system to a constant steady-state may become impossible when the diffusivity is too small (or when the domain is large), due to the existence of barrier functions. When such an obstruction does not arise, we build sophisticated control strategies combining the dissipativity of the system, the existence of traveling waves and some connectivity of the set of steady-states. This connectivity allows building paths that the controlled trajectories can follow, in a long time, with small oscillations, preserving the natural constraints of the system. This kind of strategy was successfully implemented in one-space dimension, where phase plane analysis techniques allowed to decode the nature of the set of steady-states. These techniques fail in the present multi-dimensional setting. We employ a fictitious domain technique, extending the system to a larger ball, and building paths of radially symmetric solution that can then be restricted to the original domain. The results are illustrated by numerical simulations of these models that find several applications, such as the extinction of minority languages or the survival of rare species in sufficiently large reserved areas.

Keywords: Controllability, Reaction-diffusion, Constraints, Mathematical Biology

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1. Introduction

In this work we consider the boundary controllability of homogeneous monostable and bistable reaction diffusion equations under natural state constraints coming from the physical model.

1.1. Motivation

Reaction-diffusion equations appear frequently in nature in a wide variety of phenomena, such as:

- *Population dynamics* and invasion of species (see the pioneering work of Kolmogorov [1]).
- *Neuroscience*, where models for neuronal impulses exhibit traveling waves (see [2]).
- *Chemical Reactions*, when modeling concentrations of chemicals that react and diffuse (see [3]).
- *Evolutionary game theory* where one can also find reaction-diffusion systems (see [4, 5]).
- *Magnetic systems* in material science and their phase transitions (see [6]).
- *Linguistics*, we refer to [7] where the authors consider reaction-diffusion for analyzing language shift by means of a traveling wave.

The combination of diffusion and the nonlinear interaction gives rise to a wide variety of phenomena: steady-states, Turing patterns in morphogenesis (see [8, 9]) and traveling waves (see [10]), for instance.

The nonlinearities of the models that we are going to discuss fall in two important types (Figure 1): monostable, key in models of population growth, and bistable, which is important in non-equilibrium phase transitions [11].

1. Monostable. The first ODE models in population dynamics exhibit an exponential growth of the population,

\[
\begin{align*}
  u_t &= \beta u, \\
  u(0) &= u_0,
\end{align*}
\]

![Figure 1: Monostable nonlinearity (left), bistable nonlinearity (right).](image-url)
where $\beta > 0$ represents the reproduction rate. Monostability comes from a correction to the exponential growth by assuming that the population can at most grow up to a certain level \[ (1) \]

$$u_t = \beta u \left( 1 - \frac{u}{\kappa} \right),$$

$$u(0) = u_0,$$

where $\kappa > 0$ is the capacity of the environment, related to the finite amount of resources that are available for the population. For $0 < u_0 < \kappa$ the solution of (1) tends to $\kappa$ as $t \to \infty$.

We study the associated reaction-diffusion model, in which the nonlinearity and the diffusivity are spatially homogeneous since we focus on the most fundamental aspects of the dynamics and the controllability properties. Non-homogeneous diffusion or spatial drifts can lead to different behaviors, in [13] one can find an extension of the results here discussed with the new phenomenology arising due to the presence of heterogeneous drifts.

The pioneering works of Fisher and Kolmogorov [14, 1] discovered a moving profile $s(x - ct)$, called traveling wave, for the Cauchy problem associated with the now so-called Fisher-KPP equation:

$$
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= u(1-u)(1-\theta) \quad (x,t) \in \mathbb{R}^N \times (0, +\infty), \\
u(x,0) &= u_0(x) \in [0,1] \quad x \in \mathbb{R}^N.
\end{align*}
$$

(2)

2. Bistable. This type of nonlinearity also appears in different contexts, for example, in magnetic systems [6], where $u$ stands for a proportion of spin state. The nonlinearity also appears in population dynamics in the so-called Allee effect (see [15]), where the negative values of $f$ indicate that the density is too low for the population to increase. In economics and linguistics, one can find nonlinearities of this type. For example, in evolutionary game theory, a two-strategy game (see [16, 17, 18]), here $u$ stands for the proportion of players using one strategy. The replicator dynamics (see [19]) with spatial diffusion of coordination game leads to bistable nonlinearities [4, 5]:

$$
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= u(1-u)(u - \theta) \quad (x,t) \in \mathbb{R}^N \times (0, +\infty), \\
u(x,0) &= u_0(x) \in [0,1] \quad x \in \mathbb{R}^N.
\end{align*}
$$

(2)

where $\theta \in (0,1)$. In the context of gene traits evolution, $u$ also stands for a proportion of individuals and can be seen as a solution of a bistable reaction-diffusion equation (see for instance [3, 20, 21, 22]).

Traveling waves for bistable systems also exist and enjoy exponential convergence for specific initial data (see [10]). This idea was used in [23] for approximately controlling to a traveling wave via the interaction with the parameter $\theta$.

Moreover, in relation with optimal control of traveling wave solutions, we refer [24], and in the context of bistable equations [25].

In this work, we consider models of the types mentioned above in bounded domains, which, in turn, are more natural in the context of applications. We are interested in various dynamic phenomena, specifically the ones related to the construction of constrained controls.
Among the different control mechanisms, the boundary control is of particular interest since it represents to modify the quantity of interest in the boundary of a domain whose accessibility might not be possible. In applications, this process could refer to different phenomena. Examples could be the release of certain substances from the boundary of a domain, the modification of the proportion of individuals of a specific genetic trait, or the enforcement of players of a game to play a particular strategy. All these applications need any dynamic control to satisfy the nature of the quantities represented in the model.

However, from the control perspective, the classical methodology, for instance, the control with minimum $L^2$ norm [26] (see also [27, 28]) should not be applied since it can violate the state constraints as we observe in Figure 2. The models mentioned above make sense when the state quantities are positive or between 0 and 1.

If these constraints are violated during the control process, we lose the physical meaning of the model. The controllability under state constraints of such equations has already been treated in [29] for the one-dimensional case.

1.2. Statement of the problem

We study the controllability towards certain steady-states under state constraints of reaction-diffusion equations. Let $\Omega \subset \mathbb{R}^N$ be a given bounded $C^2$-regular domain such that $|\Omega| = 1$, and consider the following reaction-diffusion equation where we allow ourselves to act with a control function $a$ on the whole boundary:

$$
\begin{cases}
  u_t - \mu \Delta u = f(u) & (x, t) \in \Omega \times (0, T], \\
  u(x, t) = a(x, t) & (x, t) \in \partial \Omega \times (0, T], \\
  0 \leq u(x, t) \leq 1 & (x, t) \in \Omega \times [0, T],
\end{cases}
$$

(3)
where $\mu > 0$ is the diffusivity constant. We will discuss two types of nonlinearities $f : [0, 1] \to \mathbb{R}$, monostable and bistable (see Figure 1).

A function $f \in C^1(\mathbb{R})$ is called monostable if:

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad f(s) > 0 \text{ if } 0 < s < 1,$$

and called bistable if:

$$f(0) = f(\theta) = f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0, \quad f'(\theta) > 0,$$

$$f(s) < 0 \text{ if } 0 < s < \theta, \quad f(s) > 0 \text{ if } \theta < s < 1.$$

For any $T > 0$, $u_0 \in L^\infty(\Omega; [0, 1])$ and $a \in L^\infty(\partial\Omega \times [0, T]; [0, 1])$, the problem (3) has a unique weak solution $u \in L^\infty(\Omega \times [0, T]) \cap C([0, T]; H^{-1}(\Omega))$. This follows by transposition and a fixed point argument (see [30, Ch.13] and [31]).

In the monostable case typically we require that $u \geq 0$ since it models a density of population.

Moreover, in the bistable case $u$ stands for a proportion of individuals or a concentration of mass, therefore the model carries the natural constraint:

$$0 \leq u(x, t) \leq 1.$$

We have to build a control strategy that respects this constraint for all time. This can be guaranteed by the comparison principle [32] [33] by imposing that:

$$0 \leq a(x, t) \leq 1.$$

The constraint in the control implies that the state fulfills automatically the limitations prescribed. It has to be noticed that this equivalence between state constraints and control constraints does not hold for any system; this is the case of the wave equation where the maximum principle does not hold (see [34]). The solution of (3) is globally defined for any initial data taking values in $u_0(x) \in [0, 1]$ since $w \equiv 0$ (respectively $w \equiv 1$) is a subsolution (supersolution).

**Definition 1.1.** We say that Equation (3) is controllable to $w \in L^\infty(\Omega, [0, 1])$ for an initial datum $u_0 \in L^\infty(\Omega, [0, 1])$ if there exists some $T > 0$ (possibly infinite) and a control $a \in L^\infty(\partial\Omega \times [0, T])$ such that the solution of (3) with initial data $u_0$ and control $a$ is equal to $w$ at some time $T > 0$, i.e. $u(T; a, u_0) = w$.

Note that the relevant constraint $0 \leq u \leq 1$ on the controlled state is built in the present definition.

Moreover, notice that this definition differs from the standard notion of controllability (see [35]) in the sense that our time $T$ is not fixed a priory.

Our main goal is to discuss the controllability of (3) to the following constant steady-states:

$$w \equiv \alpha \quad \alpha \in \{0, 1\},$$
in the monostable case, and

\[ w \equiv \alpha \quad \alpha \in \{0, \theta, 1\}, \]

in the bistable case.

1.3. Main features of the problem and main results

Our first novel result is of negative nature. If we fix our target to be the function \( w \equiv 0 \), we shall prove that the presence of state constraints can lead to a lack of controllability to this state for \( \mu \) small. The main cause of the lack of controllability is the existence of a barrier, a non-trivial solution to the problem

\[
\begin{align*}
-\mu \Delta v &= f(v) \quad x \in \Omega, \\
v &= 0 \quad x \in \partial \Omega, \\
0 < v < 1 \quad x \in \Omega.
\end{align*}
\] (4)

The existence and nonexistence for the problem (4) has been studied for instance in [36].

If the initial datum is above the nontrivial solution, the comparison principle [32, 33] ensures that the solution of the parabolic problem, for any control function \( 0 \leq a(x, t) \leq 1 \), will be above (4). This leads to an obstruction to the controllability to \( w \equiv 0 \).

In this case, we say that the solution of (4) is a barrier function for the initial data \( u_0 \).

Furthermore, the barrier function not only prevents to reach the steady state \( w \equiv 0 \). If we set our target to be \( w \equiv \theta \) we observe that a barrier to reach \( w \equiv 0 \) is also a barrier to reach \( w \equiv 0 \), since these non-trivial solutions have its maximum above \( \theta \) (see Proposition 3.4). We will see that the existence of non-trivial solutions depends basically on a relation between \( \mu \) and \( \Omega \).

A way to study the existence and non-existence of (4) is by means of finding critical points for the energy functional associated to (4):

\[
J(u) = \int_\Omega \frac{\mu}{2} |\nabla u|^2 - F(u) \, dx,
\]

where \( F(u(x)) := \int_0^{u(x)} f(s) \, ds \). One can see that if \( \mu \) is sufficiently big the convex part dominates and \( w \equiv 0 \) is the unique solution. However, if \( \mu \) is small enough, the term \( \int_\Omega F(u) \, dx \) can be dominant. If the second term is dominant, then there exist a function \( v \in H^1_0(\Omega), v \neq 0 \) which is a critical point of the functional. We will see that this critical point corresponds to a barrier function.

In addition, it has to be noticed that this obstruction depends on being above the nontrivial solution. Therefore, there might be other initial data for which the target can be reached. We will determine a set of initial data for which controllability will hold regardless of \( \mu \).

Observe that this situation occurs in the case of the monostable nonlinearity, since we have that \( F(u) \geq 0 \). For the bistable case, the same paradigm is found under the assumption that \( F(1) > 0 \).

A nontrivial solution with Dirichlet condition equal to one would have the same effect for approaching \( w \equiv 1 \). However, we will see that such non-trivial solution does not exist by using a comparison argument with the traveling wave solutions.
When using a control function that is between 0 and 1, both $w \equiv 1$ and $w \equiv 0$ are respectively super and subsolutions, and this implies that we can only approach them in infinite time.

In the bistable case, nontrivial solutions with boundary value $\theta$ may exist. In this case, the simple strategy of setting the boundary control equal to $\theta$ for large time, stabilize, and finally apply a local control does not work. This is the reason why we have to study how to overpass these nontrivial solutions in a way that we can ensure we are close to $w \equiv \theta$.

The previous discussion points out that at least two regimes can be studied depending on the existence of barriers. Let us introduce the following quantity:

**Definition 1.2.** We define

$$\mu^*(\Omega, f) := \sup \{ \mu \in \mathbb{R}^+ \text{for which there is a solution of (4)} \}$$

Analogously, we define $\mu^*_\theta(\Omega, f)$ as the supremum value such that a non-trivial solution of the problem (4) exists with Dirichlet boundary conditions equal to $\theta$.

We will devote more attention to estimates on the thresholds of non-trivial solutions in Section 3.1.

The staircase method presented in [31], roughly speaking, tells us that the problem of controlling to steady-states respecting constraints can be addressed by finding a control such that it can steer the system to a steady-state path-connected to the target. The key point is to find admissible continuous paths of steady-states connecting to the target. This strategy was used in [29] for the one-dimensional bistable semilinear heat equation. We will introduce both results in Section 2.

In the multidimensional case, the construction of the admissible path is done in this work by considering a ball containing our domain and constructing the continuous path there. Then, by restricting the path to our original domain $\Omega$, we obtain the desired result. Note that for this procedure to work, we need to be able to construct the path for arbitrary large balls. In this way, we are able to consider every possible geometry of $\Omega$.

With the construction of the path, we show that nontrivial solutions with boundary value $\theta$ do not constitute a fundamental obstruction to reach $w \equiv \theta$.

For proving boundary controllability to the desired steady-states the following ingredients will be crucial:

- Dissipativity of the system for certain values of $\mu$.
- Comparison with the restriction of traveling wave solutions.
- The existence of an admissible continuous path of steady-states for every domain $\Omega$ connecting $w \equiv 0$ with $w \equiv \theta$ for the bistable case.

Our main results are Theorem 1.1 for monostable equations and Theorem 1.2 for bistable ones:

**Theorem 1.1.** Let $f$ be monostable.

Let $\Omega \subset \mathbb{R}^N$ be a $C^2$-regular domain of measure 1. The system (3) can be controlled:
• in infinite time to $w \equiv 1$ for any admissible initial condition and any $\mu > 0$.

• in infinite time to $w \equiv 0$ :
  - for any initial condition iff $\mu^*(\Omega, f) < \mu$,
  - for $u_0$ with $\|u_0\|_{L^\infty}$ small enough if $0 < \mu < \frac{F'(0)}{\lambda_1(\Omega)}$.

where $\mu^*(\Omega, f)$ defined in [1.3] is a positive constant and $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian.

Remark 1.1. In the monostable case, there might exist a non-trivial solution with Dirichlet condition equals to 0 even if the solution $u \equiv 0$ is still locally stable. In this situation, we might still approach $w \equiv 0$ for certain initial data. However, when $w \equiv 0$ becomes unstable it is not possible to approach $w \equiv 0$ without violating the state constraints.

Remark 1.2. For the case of the Fisher-KPP nonlinearity $f(s) = s(1-s)$ the situation in which a positive non-trivial solution around 0 exists together with the fact that $w \equiv 0$ is stable does not occur. See proofs in Appendix A.

Remark 1.3. Physically it makes sense also to remove the upper constraint in the state, i.e. to ask only for $u \geq 0$. In this situation, one can see that $w \equiv 1$ can be reached in finite time.

Now we state the main result for bistable nonlinearities:

**Theorem 1.2.** Let $f$ be bistable. Let $\Omega \subset \mathbb{R}^N$ be a $C^2$-regular domain of measure 1.

1. If $F(1) > 0$, there exist a time $T \in (0, +\infty)$, and a set $A \subset L^\infty(\Omega; [0, 1])$ such that the solution of the system (3) can be controlled by means of a function $a \in L^\infty(\partial \Omega \times [0, T]; [0, 1])$
   - in infinite time to $w \equiv 0$:
     - for any initial data $u_0$ iff $\mu > \mu^*(\Omega, f)$,
     - for any $\mu > 0$ if $u_0 \in A$,
   - in finite time to $w \equiv \theta$:
     - for any initial data $u_0$ iff $\mu > \mu^*(\Omega, f)$,
     - for any $\mu > 0$ if $u_0 \in A$,
   - in infinite time to $w \equiv 1$ for any admissible initial data $u_0$ and for any $\mu > 0$.

Furthermore $\mu^*(\Omega, f)$ defined in [1.3] is a positive constant.

2. If $F(1) = 0$, there exist a function $a \in L^\infty(\partial \Omega \times [0, T]; [0, 1])$ such that the solution of system (3) is driven in finite time to $w \equiv \theta$ and in infinite time to $w \equiv 0$ and $w \equiv 1$ for any $\mu > 0$.

Remark 1.4. In Proposition [3.1] and Proposition [3.2] we give upper and lower bounds for $\mu^*(\Omega, f)$. For instance for the bistable, in the one-dimensional case we have that $rac{\mu^*(\Omega, f)}{\max_{x \in (0,1)} \frac{F'(x)}{x^2}} \leq \frac{F'(1)}{8(F(0)-F(\theta))} \leq \frac{\mu^*(\Omega, f)}{\pi^2}$. 

8
Remark 1.5. We will see that $\mathcal{A}$ is not small, indeed we have that $L^\infty(\Omega;[0,\theta]) \subset \mathcal{A}$. Moreover, note that, whenever we are able to go to $w \equiv 0$ in infinite time, we are able to go to $w \equiv \theta$ in finite time and vice versa.

Remark 1.6. Either for the case such that $F(1) > 0$ or $F(1) = 0$, whenever $\Omega$ is a ball and the initial data is radial, $a$ can be constructed so that does not depend on space $a(x,t) = a(t)$.

Remark 1.7. The fact of having constraints implies that there exists a minimal controllability time [31] (see also [37]).

Remark 1.8 (Big domains). We have adopted the diffusivity $\mu$ as a relevant parameter. However, this approach is completely analogous to fix $\mu$ and consider dilations and contractions of our domain $\Omega$, namely consider $\sqrt{\frac{T}{\mu}}\Omega$ for $\mu > 0$. Indeed, after a space rescaling of (3) we obtain:

$$
\begin{aligned}
\begin{cases}
u_t - \Delta u = f(u) & (x,t) \in \left(\sqrt{\frac{T}{\mu}}\Omega \times (0,T]\right), \\
u(x,t) = a(x,t) & (x,t) \in \left(\sqrt{\frac{T}{\mu}}\partial\Omega \times (0,T]\right), \\
0 \leq u(x,t) \leq 1 & (x,t) \in \left(\sqrt{\frac{T}{\mu}}\Omega \times [0,T]\right).
\end{cases}
\end{aligned}
$$

Hence if $\mu$ is small, it is analogous to consider the equation with diffusivity 1 in a domain of measure $\sqrt{\frac{T}{\mu}}$. The existence of non-trivial solutions occurs for big dilations of a fixed domain. This paradigm corresponds to the biological concept of the minimum area requirement for species to survive [38].

The structure of the paper is the following:

• First, in Section 2 we recall the main results on asymptotic dynamics of the semilinear heat equation [39, 40], the staircase method [31], and the approach taken in dimension one, [29].

• Afterwards, in Section 3 we proceed to provide some technical lemma that are needed for the proofs of the theorems above.

• Then, in Section 4 we give the proof of the Theorems 1.1 and 1.2.

• In Section 5 we provide numerical simulations of the construction of the path, an implementation of the quasistatic control in Ipopt, and we give a numerical example of the control and controlled state in minimal time.

• Finally, in Section 6 we set the conclusions and discuss future perspectives on the topic.
2. Preliminary results

2.1. Asymptotic behavior

The semilinear heat equation

\[ \begin{align*}
  &u_t - \mu \Delta u = f(u) \quad (x,t) \in \Omega \times (0,T], \\
  &u = 0 \quad (x,t) \in \partial \Omega \times (0,T], \\
  &u(0) = u_0 \in L^\infty(\Omega) \quad x \in \Omega,
\end{align*} \]  

(6)

where \( f \) is globally Lipschitz, is a gradient dynamical system on the metric space \( C_0(\Omega) \),

\[ u_t = -\nabla_u J[u], \]

where \( J[u] = \int_\Omega \frac{1}{2} |\nabla u|^2 - F(u) \, dx \). \( J \) acts as a strict Lyapunov functional. Due to this fact, the solution of the semilinear heat equation, whenever its trajectory is globally defined and bounded, it approaches the set of steady-states:

**Theorem 2.1** (Theorem 9.4.2 in [40]). Let \( S_E := \{ v \in S \text{ such that } J(v) = E \} \) and \( J \) be coercive. Then the solution \( u \) of (6) and \( S \) satisfy:

- \( J(u(t)) \to E \),
- \( S_E \neq \emptyset \),
- \( d(u(t), S_E) \to 0 \text{ as } t \to \infty \),

where \( d \) is the distance in \( C_0(\Omega) \cap H_0^1(\Omega) \).

By the LaSalle invariance principle, if the set of steady-states consists of discrete points, the solution converges to one of them. This is the situation of (6) (see [39]). However, whenever the nonlinearity depends on the space, \( f(x,u) \), one cannot guarantee in general that the set is discrete. Moreover, there can exist bounded trajectories that are nonconvergent for \( N \geq 2 \) (see [41, 42]). In the case that \( f \) is analytic, the convergence to steady-states is guaranteed [43, 44, 45]. This is a feature of \( N \geq 2 \), since in the one dimensional case, the convergence of bounded trajectories is guaranteed for any \( C^2 \) nonlinearity by Matano’s result [46].

2.2. Staircase method

The primary tool for understanding how we can reach the steady-state \( w \equiv \theta \) will be the staircase method.

The following Theorem in [31] ensures that if we find an admissible continuous path of steady-states between two steady-states, then we are able to find a control function that steers one steady-state to the other. Consider

\[ \begin{align*}
  &v_t - \mu \Delta v = f(v) \quad (x,t) \in \Omega \times (0,T], \\
  &v = a(x,t) \quad (x,t) \in \partial \Omega \times (0,T], \\
  &0 \leq v(0,x) = v_0(x) \quad x \in \Omega,
\end{align*} \]  

(7)
where $\Omega$ is a bounded domain with boundary $C^2$. We say that $v_1$ and $v_0$ are path connected steady-states if there exist a continuous function (with respect to $\|\cdot\|_{L^\infty}$) from $[0,1]$ to the set of steady-states $\gamma : [0,1] \to S$ such that $\gamma(0) = v_0$ and $\gamma(1) = v_1$. Denote by $\bar{v}^s := \gamma(s)$.

**Theorem 2.2** (Theorem 1.2 in [31]). Let be $v_0$ and $v_1$ be path connected bounded steady-states. Assume there exists $\nu > 0$ such that:

$$v^s \geq \nu \text{ a.e. on } \partial \Omega$$

for any $s \in [0,1]$. Then, if $T$ is large enough, there exist a control function $a \in L^\infty(\partial \Omega \times (0,T);[0,1])$ such that the problem (7) with initial datum $v_0$ and control $a$ admits unique a solution verifying $v(T, \cdot) = v_1$ and $a \geq 0$ on $(0,T) \times \partial \Omega$.

The proof is based on extracting a finite sequence of points of the continuous path of steady-states and apply local controllability between them for going from one to another until the target is reached. This is the reason why it is called the staircase.

2.3. One dimensional approach

For proving the controllability to $w \equiv \theta$ under the prescribed state constraints, one strategy is to find an admissible path of steady-states [29]. The construction of a path of steady-states for controlling the one-dimensional semilinear heat equation was firstly considered in [47]. For finding the admissible path, one sees the steady-states for the problem:

$$\begin{cases}
  u_t - u_{xx} = f(u), \\
  u(0,t) = a_1(t), \quad u(L,t) = a_2(t), \\
  0 \leq u(x,0) \leq 1,
\end{cases}$$

as an ODE system:

$$\frac{d}{dx} \begin{pmatrix} v \\ v_x \end{pmatrix} = \begin{pmatrix} v_x \\ -f(v) \end{pmatrix},$$

for initial conditions:

$$\begin{pmatrix} v(s)(0) \\ v_x(s)(0) \end{pmatrix} = \begin{pmatrix} s\alpha + (1-s)\theta \\ s\beta \end{pmatrix}.$$  \(8\)

The key point is to find an admissible invariant region for the dynamics [8] that ensures that all steady-states will be admissible. Remind that an invariant region in the phase plane is a set $\Gamma \subset \mathbb{R}^2$ such that for every $(v^0,v_x^0) \in \Gamma$ the solution of (8) with initial data $(v^0,v_x^0)$ remains inside $\Gamma$ for all $x \geq 0$. Moreover, if the invariant region $\Gamma$ is such that $\forall (v,v_x) \in \Gamma$ we have that $v \in [0,1]$ then it will be an admissible invariant region.

Then by solving the ODE, we will obtain the required boundary conditions. Moreover, by continuous dependence on the initial data, we have that the path is continuous. Figure [3] illustrates the procedure to build the path of steady-states and the invariant region for [8].
Remark 2.1. In the one dimensional approach the authors \[29\] work in the framework of variable \(L\) instead of considering the diffusivity \(\mu\).

3. Technical Lemmas

3.1. Estimates on the existence of non-trivial solutions

In this subsection, we study the existence of non-trivial solutions of the boundary value problems around our steady-states of interest.

**Proposition 3.1** (An upper bound for \(\mu^*\)). Let \(f : \mathbb{R} \rightarrow \mathbb{R}\), assume that \(f\) is bounded. Assume furthermore that \(f(0) = 0\) Consider:

\[
\begin{align*}
-\mu \Delta v &= f(v) \quad x \in \Omega, \\
v &= 0 \quad x \in \partial \Omega, \\
1 > v > 0 \quad x \in \Omega.
\end{align*}
\]

Then,

\[
\mu^* \leq \max_{s \in [0,1]} \frac{f(s)}{\lambda_1(\Omega)},
\]

where \(\lambda_1(\Omega)\) is the first eigenvalue of the Dirichlet Laplacian.

**Proposition 3.2** (A lower bound for \(\mu^*\)). Assume that \(f(0) = f(\theta) = f(1) = 0\), and that \(f'(0) < 0\), \(f'(1) < 1\), \(f'(<\theta) > 0\). Moreover consider \(F(v) = \int_0^v f(s) ds\) and assume that \(F(1) > 0\). Consider \(\Omega \subset \mathbb{R}^N\) to have measure 1 and the following problem:

\[
\begin{align*}
-\mu \Delta u &= f(u) \quad x \in \Omega, \\
u &= 0 \quad x \in \partial \Omega, \\
u > 0 \quad x \in \Omega.
\end{align*}
\]
Denote by $B_\Omega$ a ball of maximal measure inside $\Omega$, $B_\Omega \subset \Omega$. Then, for any $\mu > 0$ fulfilling

$$\mu < \frac{2\delta^2 \Gamma \left( \frac{N}{2} + 1 \right)^{2/N} \left( F(\theta) + (1 - \delta)^N (F(1) - F(\theta)) \right)}{\pi (1 - (1 - \delta)^N)} m(B_\Omega)^{2/N},$$

there exists a solution of the problem (A.1), where $\delta > 0$ fulfills

$$\delta < 1 - \left( \frac{-F(\theta)}{F(1) - F(\theta)} \right)^{1/N},$$

and $\Gamma$ is the gamma function. This provides a lower bound of $\mu^*$. 

Remark 3.1. One can obtain a similar threshold for the monostable nonlinearity following the same argument of the proof.

Remark 3.2. The proof is based on finding a test function $v_\delta$ and on seeing for which choice of $\mu > 0$ and $\delta > 0$, we can guarantee that $J[v_\delta] < 0$. Since $J[w \equiv 0] = 0$ for all $\mu > 0$ we have that for the previous choice of $\mu$, there will exist a non-trivial solution.

We postpone the discussion in the Appendix.

Proposition 3.3 (Order in the thresholds). For $f$ bistable, when $F(1) > 0$ we have that:

$$\mu_\theta^* \geq \mu^*.$$

Moreover, for $\mu < \mu^*$, we denote by $v_\theta$ and $v_0$ the maximum non-trivial solution bounded by 1 of the elliptic problem with Dirichlet boundary conditions equal to $\theta$ and 0 respectively, then:

$$v_\theta \geq v_0.$$

Proof. The result follows from the elliptic comparison principle together with the fact that any non-trivial solution of the boundary value problem has its maximum above $\theta$. \qed

Remark 3.3 (Bounds on $\mu_\theta^*$). When we study non-trivial solutions for the problem with Dirichlet boundary conditions equals to $\theta$, we can reduce our analysis to the study of a monostable nonlinearity, since after the change of variables $u = v - \theta$, the nonlinearity ends up being $f(v + \theta)$, which is monostable. The first non-trivial solution that appears is not going to change sign, because otherwise, such oscillating solution $w$ will have a positive (or negative) part which will be a subsolution (or a supersolution) of the problem by extending it with $\theta$ outside the positive (or negative) part, leading to the existence of a non-trivial solution that does not change sign.

Remark 3.4. Note that the upper bound proved for $\mu^*$ and $\mu_\theta^*$ is the same, in general, to our knowledge, we do not know if $\frac{\lambda_1(\Omega)}{F'(\theta)}$ is smaller or bigger than $\mu^*$.

Remark 3.5. Note that the stability of the stationary solution $w \equiv \theta$ becomes more and more unstable as the diffusivity decreases. However, the number of eigenfunctions that are unstable is only finite.
**Proposition 3.4** (Maximum of positive solutions). Let \( u \) be a solution to:

\[
\begin{cases}
    -\mu \Delta u = f(u) & x \in \Omega, \\
    u = 0 & x \in \partial \Omega, \\
    u > 0 & x \in \Omega,
\end{cases}
\]

with \( f \) being bistable then the maximum of \( u \) in \( \Omega \) is above \( \theta \):

\[
\max_{x \in \Omega} u(x) > \theta.
\]

**Proof.** The proof follows by contradiction. Assume that the maximum of \( u \) is lower or equal than \( \theta \), then the energy estimate gives us the contradiction:

\[
0 < \int_{\Omega} \mu |\nabla u|^2 \, dx = \int_{\Omega} u f(u) < 0,
\]

where the strict inequality in the left hand side comes from the assumption that the solution is not trivial and the right hand side inequality comes from the fact that \( f \) is negative in \((0, \theta)\).

\(\square\)

**Remark 3.6.** Note that the fact that the maximum of a non-trivial solution with boundary value 0 is above \( \theta \) implies that we cannot reach the steady state \( w \equiv 0 \) asymptotically neither the steady \( w \equiv \theta \) if we start with an initial value above this non-trivial solution.

Now we turn our attention to the existence or nonexistence of elliptic solutions with Dirichlet boundary 1. The presence of traveling waves for the one-dimensional case has been widely studied [1, 10, 3]. One can easily see that the same functions constantly extended in the \( N-1 \) remaining space dimensions are also solutions of the Cauchy problem in \( \mathbb{R}^N \).

**Proposition 3.5** (Traveling waves and convergence to \( w \equiv 1 \) for any domain). The existence of a decreasing traveling wave implies that for any initial admissible condition and every domain, the solution can be asymptotically driven towards \( w \equiv 1 \).

If \( f \) is monostable or \( f \) is bistable with \( F(1) > 0 \) then there is a unique solution of the boundary value problem

\[
\begin{cases}
    -\mu \Delta u = f(u) & \text{in } \Omega, \\
    1 > u > 0 & \text{in } \Omega, \\
    u = 1 & \text{in } \partial \Omega
\end{cases}
\]

and for any domain \( \Omega \), we have that any initial admissible condition can be asymptotically driven towards the steady-state \( w \equiv 1 \).

**Proof.** It is known that the problem:

\[
\begin{cases}
    u_t - \Delta u = f(u) & (x,t) \in \mathbb{R}^N \times (0, +\infty), \\
    0 \leq u(0, x) \leq 1 & x \in \mathbb{R}^N,
\end{cases}
\]

is...
has a traveling wave solution. Furthermore, the traveling wave profile takes values in \([0, 1]\), and it is a monotone function decreasing in the direction of the velocity vector \([3, 10]\). The idea is to use a section of the traveling wave as a parabolic subsolution to our problem. We rescale our domain to \(\sqrt{\mu} \Omega\) to have diffusivity 1. Now we come back to our new (parabolic) problem:

\[
\begin{aligned}
    u_t - \Delta u &= f(u) & (x,t) &\in \sqrt{\mu} \Omega \times (0, T], \\
    u(x,t) &= 1 & (x,t) &\in \sqrt{\mu} \partial \Omega \times (0, T], \\
    0 < u(x,0) < 1 & x \in \sqrt{\mu} \Omega.
\end{aligned}
\]

(9)

Since the traveling wave profile is monotone decreasing, we can consider a section of the traveling wave such that is below to \(u(x,0)\) in \(\sqrt{\mu} \Omega\), let us denote by \(TW(x)\) the maximum profile of traveling wave that satisfies:

\[
TW(x) \leq u(x,0) \quad \forall x \in \sqrt{\mu} \Omega.
\]

Now we note that the following problem:

\[
\begin{aligned}
    u_t - \Delta u &= f(u) & (x,t) &\in \sqrt{\mu} \Omega \times (0, T], \\
    u(x,t) &= TW(x-ct) & (x,t) &\in \sqrt{\mu} \partial \Omega \times (0, T], \\
    u(x,0) &= TW(x) & x &\in \sqrt{\mu} \Omega,
\end{aligned}
\]

(10)

is a subsolution of (9), then by the parabolic comparison principle we have that the solution of (9) will be above (10) and therefore the solution of (9) will converge to \(w \equiv 1\).

\[\square\]

3.2. Radial solutions

In this subsection, we discuss radial solutions of semilinear PDEs via ODE methods. The reason to do so is that the construction of the path towards \(w \equiv \theta\) for the bistable case will rely on extending our domain \(\Omega\) to a ball \(\Omega \subset B_R\), construct the path for this ball and then restrict to our original domain \(\Omega\).

Consider the following elliptic PDE:

\[
\begin{aligned}
    -\mu \Delta u &= f(u) & x &\in B_r \subset \mathbb{R}^N, \\
    u(0) &= a, \\
    Du(0) &= 0,
\end{aligned}
\]

(11)

where \(f\) is globally Lipschitz. It is well known that the solutions of a semilinear elliptic equation in a ball are radially symmetric [50]. We rewrite the (11) in radial coordinates and absorbing \(\mu\) in the nonlinearity.
the following is obtained:

\[
\begin{aligned}
&\begin{cases}
  u_{rr}(r) + \frac{N-1}{r} u_r(r) = -f(u(r)) & r \in [0, R_m), \\
  u(0) = a, \\
  u_r(0) = 0.
\end{cases}
\end{aligned}
\]

(12)

The following Lemmas are devoted to the existence, uniqueness, continuous dependence with respect to parameters, and global definition of (12) for certain ranges of \(a\). The local existence and uniqueness follow from standard contraction argument [51].

**Lemma 3.1** (Local Existence and uniqueness). There exist a unique solution for \(R_m\) small enough to (12).

**Proof.** First we proof the wellposedness of (12), since the term \(\frac{1}{r}\) is not integrable. We proceed by multiplying by \(r^{N-1}\) and integrating to obtain something of the form:

\[
u(r) = a + \int_0^r \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} f(u(\sigma)) d\sigma ds\]

Now define the map:

\[
Tu = a + \int_0^r \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} f(u(\sigma)) d\sigma ds
\]

and we can show that it is a contraction for \(r\) small enough:

\[
\|Tu - Tv\|_{\infty} \leq \int_0^r \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} \|f(u(\sigma)) - f(v(\sigma))\|_{\infty} d\sigma ds
\]

\[
\leq L \|u - v\|_{\infty} \int_0^r \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} d\sigma ds
\]

\[
\leq Lr \|u - v\|_{\infty}
\]

Choosing \(r\) small enough we have the contraction and hence the solution is unique.

From now on we assume \(f\) to be bistable with \(F(1) \geq 0\). The following lemma is key for the admissibility of the path of steady-states.

**Lemma 3.2** (Invariant region). Assume that \(F(1) \geq 0\). Then there exists an admissible region in the phase space \(\Gamma\) that is positively invariant.

**Proof.**

\[
E(u, v) = \frac{1}{2} v^2 + F(u)
\]

where \(F(u) = \int_0^u f(s) ds\). Define the following region:

\[
D := \{(u, v) \in \mathbb{R}^2 \text{ such that } E(u, v) \leq 0\}
\]

Let \(\theta_1\) be defined as:

\[
\theta_1 = \min_{s > 0} \{F(s) = 0\}
\]
Note that the region defined by
\[
\Gamma := \{(u, v) \in [0, \theta_1] \times \mathbb{R} \text{ such that } |v| \leq \sqrt{-2F(u)}\}
\]
Note that \( \Gamma \subset D \).

Take \((u_0, 0) \in \Gamma\), then the solution of (12) with initial datum \((u_0, 0)\) satisfies:
\[
\frac{d}{dr} E(u, v) = vv_r + f(u)v = -N - \frac{1}{r}v^2 < 0
\]
So \((u, v) \in \Gamma\) for all \(r > 0\).

Therefore radial solutions are globally defined and admissible for any \((u_0, 0) \in \Gamma\).

Remark 3.7 ((\(\theta, 0) \in \Gamma\)). Note that \(F(\theta) < 0\) hence \(\sqrt{-2F(u)}\) is well defined.

Remark 3.8 (Stationary Traveling waves and the invariant region). If \(F(1) = 0\) we obtain a region that is defined from \((0, 0)\) up to \((1, 0)\) corresponding to the constant stationary solution \(u(x) = 1\). The traveling waves in the one dimensional case, the one satisfying:
\[
\lim_{x \to -\infty} TW(x) = 1, \quad \lim_{x \to +\infty} TW(x) = 0
\]
and the symmetric one satisfying
\[
\lim_{x \to -\infty} TW(x) = 0, \quad \lim_{x \to +\infty} TW(x) = 1,
\]
in the case that \(F(1) = 0\) are stationary, and define the aforementioned invariant region in the phase plane.

Lemma 3.3 (Continuous dependence). The solution of the initial value problem
\[
\begin{aligned}
\frac{d}{dr} \begin{pmatrix} u \\ u_r \\ \end{pmatrix} &= \begin{pmatrix} u_r \\ -N - 1 \frac{1}{r}u_r - f(u) \\ \end{pmatrix}, \\
\begin{pmatrix} u(0) \\ u_r(0) \\ \end{pmatrix} &= \begin{pmatrix} a \\ 0 \\ \end{pmatrix},
\end{aligned}
\]
for \(r \in [0, R_m]\) is continuous with respect to the initial condition \(a\).

Proof. Note that \(\xi(r) = u(r)^2 + u_r(r)^2\) satisfies the following differential inequality:
\[
\begin{aligned}
\frac{d}{dr} \xi(r) &= 2uu_r + 2u_r \left( N - 1 \frac{1}{r}u_r - f(u) \right) \\
&\leq 2uu_r(1 + L) - 2N - 1 \frac{1}{r}u_r^2 \\
&\leq (1 + L)\xi(r)
\end{aligned}
\]
Applying Gronwall’s inequality the result follows.
4. Proofs of Theorem 1.1 and Theorem 1.2

4.1. Proof of Theorem 1.1

4.1.1. Towards \( w \equiv 1 \)

The existence of a traveling wave implies the convergence to \( w \equiv 1 \) for \( a(x,t) = 1 \) for every \( \mu > 0 \), Proposition 3.5.

4.1.2. Towards \( w \equiv 0 \)

Discussion on the existence of a non-trivial solution already done previously in Proposition 3.1 implies that we cannot converge to \( w \equiv 0 \) for every \( \mu > 0 \). Then we separate the discussion in three possible regimes that depend on the value of \( \mu \):

1. There does not exist any non-trivial solution with boundary value 0. Then setting \( a(t,x) = 0 \) the equation will go asymptotically to \( w \equiv 0 \).
2. There exists a non-trivial solution with boundary value 0, but \( w \equiv 0 \) is locally stable. In this situation, if our initial datum is close enough to zero, it can be stabilized to \( w \equiv 0 \).
3. \( w \equiv 0 \) is locally unstable. When \( w \equiv 0 \) becomes unstable, it is impossible to stabilize around it since it cannot exist a close positive steady-state, and the constraints do not allow the control to be negative.

4.2. Proof of the Theorem 1.2, \( F(1) > 0 \)

The technical point is the controllability to \( w \equiv \theta \). For this reason a scheme of the proof is given before:

1. We construct an admissible continuous path of steady states from \( w \equiv 0 \) to \( w \equiv \theta \).
2. Given an initial data \( u_0 \) for which we are able to approach \( w \equiv 0 \), we set the control to be \( a = 0 \) for a large time.
3. Then we shall prove that the we can approach the path of steady states. This is done by proving that a minimal solution with respect to some boundary condition belongs to the path of steady states constructed before.
4. We apply local controllability to the path and follow the staircase method to reach \( w \equiv \theta \).

In the previous strategy the lack of controllability to \( w \equiv \theta \) is encoded in the fact of not being able to approach \( w \equiv 0 \).

In the following we will define a set \( \mathcal{A} \) for which we know that we will approach \( w \equiv 0 \). This set is constructed using comparison arguments.
4.2.1. The set $A$

In the Theorem, we give some insight into which kind of initial data for the parabolic problem is controllable. For this reason, we need to define some concepts for constructing the set $A$:

The subset $A$ is constructed using the parabolic comparison principle with steady-states that are radial for balls which include our domain $\Omega$, for this reason, we introduce some definitions before defining $A$.

**Definition 4.1** (Admissible Radial Steady States ($S_R$)). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We say that a steady state $v$ is admissible radial if there exist a radius $R > 0$ and a center $x_0 \in \mathbb{R}^N$ where there is an extension of the steady state $v$ to $\tilde{v}$ (with $\tilde{v}|_\Omega = v$) in the ball $B_R(x_0)$ satisfying:

$$
\begin{cases}
-\mu \Delta \tilde{v} = f(\tilde{v}) & x \in B_R(x_0), \\
\tilde{v} = b & x \in \partial B_R(x_0), \\
0 \leq \tilde{v} \leq 1 & x \in B_R(x_0).
\end{cases}
$$

The following subset of steady-states will be useful for discussing for which initial data we can control when there is a barrier, a solution of (4):

**Definition 4.2** (Comparison set). For $\mu < \mu^*(\Omega, f)$, let us denote by $S_c \subset S_R$ the set of admissible radial steady-states such that for any $v \in S_c$ their corresponding radial extension $\tilde{v}$ into a ball of radius $R$ satisfies:

$$
\frac{1}{2} \tilde{v}_r(R)^2 + F(b) < \frac{1}{2} (v_{\min,R})_r(R)^2,
$$

and $\|\tilde{v}\|_{L^\infty} < \|v_{R,\min}\|_{L^\infty}$, where $v_{R,\min}$ is the minimum solution with respect to the $L^\infty$ norm of:

$$
\begin{cases}
-\mu \Delta v_{R,\min} = f(v_{R,\min}) & x \in B_R(x_0), \\
v_{R,\min} = 0 & x \in \partial B_R(x_0), \\
v_{R,\min} > 0 & x \in B_R(x_0).
\end{cases}
$$

**Remark 4.1.** Note that a solution of (14) will exist because for $\mu < \mu^*(\Omega, f)$ the non-trivial solution in $\Omega$ extended by 0 outside of $\Omega$ generates a subsolution for (14).

Finally we define:

**Definition 4.3.** Let $A \subset L^\infty(\Omega; [0, 1])$ be defined by the functions $u_0 \in L^\infty(\Omega; [0, 1])$ such that there exists $v \in S_c$ fulfilling:

$$
u_0(x) \leq v(x) \quad \forall x \in \Omega.
$$

**Remark 4.2.** By Proposition 3.4, the minimum non-trivial solution around 0 must have its maximum above $\theta$, since the steady state $\theta$ is radial this implies that the set $\{u_0 \in L^\infty(\Omega) \text{ such that } 0 \leq u_0(x) \leq \theta \} \subset A$. 

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4.2.2. Towards \( w \equiv 0 \)

- If \( w \equiv 0 \) is the only steady-state with boundary value 0, then for any initial condition, the system will converge to it. Estimates on the threshold for nonexistence are given in Proposition 3.1 and Remark 3.3.

- If a non-trivial solution exists, then for any initial data above this non-trivial solution, the solution will stay above it for all \( t \geq 0 \). Estimates for the existence are in Proposition 3.2 and Remark 3.3.

- The remaining part is in the subsequent subsections construction in the construction of the path of steady-states. If we can control in finite time to \( w \equiv \theta \) and we have a continuous path of steady-states connecting \( w \equiv \theta \) with \( w \equiv 0 \) we are able to approach any steady-state belonging to the path and as close to \( w \equiv 0 \) with respect to the \( L^\infty \) norm as we want in finite time (consequence of \[31\] and the local stability of \( w \equiv 0 \)). However, \( w \equiv 0 \) cannot be reached in finite time due to the comparison principle.

4.2.3. Towards \( w \equiv 1 \)

Proposition \[3.5\] ensures us that setting the control to be \( a(x,t) = 1 \) in the boundary we will asymptotically reach the steady state \( w \equiv 1 \).

4.2.4. Towards \( w \equiv \theta \)

We discuss the control strategy of the parabolic problem for different cases depending on \( \mu \) and \( \Omega \):

Case 1 \( w \equiv \theta \) is the unique solution of the elliptic problem with boundary value \( \theta \). In this situation we know that setting \( a(x,t) = \theta \) the equation is going to converge to \( w \equiv \theta \) in \( C^0 \cap H^1 \). We wait until the state is very close to \( w \equiv \theta \) and we apply local controllability from \[31\] to arrive there in finite time.

For the other two cases,

Case 2 \( w \equiv \theta \) is not the unique solution of the elliptic problem with boundary value \( \theta \) but there does not exist a nontrivial solution with boundary value 0,

Case 3 \( w \equiv \theta \) is not the unique solution of the elliptic problem with boundary value \( \theta \) but there exists a barrier,

we need the construction of the path.

**Claim 4.1** (Construction of the path). For every \( \mu > 0 \) and every \( \Omega \) there exists an admissible continuous path of steady-states connecting \( w \equiv 0 \) and \( w \equiv \theta \).

**Proof.**
1. Since our domain \( \Omega \) is bounded we can find a ball with a big enough radius such that \( \Omega \subset B_R \) Figure [4]

2. Construction of the path of steady-states on \( B_R \):
Let $R > 0$ be an arbitrary positive number. Consider the one parameter family of solutions of the following Cauchy problem where $\mu$ has been absorbed by the nonlinearity:

$$\begin{cases}
    u^{(a)}_r(r) + \frac{N-1}{r} u^{(a)}_\theta(r) = -f(u^{(a)}(r)) & r \in [0, R], \\
    u^{(a)}(0) = a \in [0, \theta], \\
    u^{(a)}_\theta(0) = 0.
\end{cases}$$

Applying Lemma 3.2 we obtain that the solutions are globally defined and applying Lemma 3.3 we obtain the continuous dependence with respect to initial data which implies the continuity of the path.

Figure 5 shows the admissible invariant region and the connected path of steady-states connecting $w \equiv 0$ to $w \equiv \theta$.

3. Restriction to the original domain $\Omega$. Once we have the path of steady-states for any ball in $\mathbb{R}^N$ we restrict our family of steady-states on $\Omega$.

In this way we obtain the path for any $\Omega$ bounded and any $\mu$.  

Claim 4.1 has constructed a path from $w \equiv 0$ to $w \equiv \theta$, now the concern is how can we control to
some part of this path to be able to apply the Staircase method \[31\]. In Claim 4.2 we proceed to check that the minimal solutions under certain boundary conditions belong to the path of steady-states:

**Claim 4.2.** Consider a ball $B_{R}(x_{0})$ such that $\Omega \subset B_{R}(x_{0})$. Consider the following problem on $B_{R}(x_{0})$:

\[
\begin{aligned}
-\mu \Delta u &= f(u) & x \in B_{R}(x_{0}), \\
u &= \epsilon & x \in \partial B_{R}(x_{0}), \\
0 < u < \epsilon & x \in B_{R}(x_{0}).
\end{aligned}
\]  

(15)

Then:

a) the problem (15) has a minimal solution.

b) Let $a(x)$ be the restriction of such minimal solution to $\Omega$, then: The a minimal solution of:

\[
\begin{aligned}
-\mu \Delta u &= f(u) & x \in \Omega, \\
u &= a(x) & x \in \partial \Omega, \\
0 < u < 1 & x \in \Omega,
\end{aligned}
\]  

(16)

is radial with respect to $x_{0}$.

**Proof.**

a) The existence of a solution of (15) follows from sub and supersolutions, taking $u = 0$ and $\pi = \epsilon$ respectively.

The existence of a minimal solution is proved noticing that if two solutions $u_{1}$ and $u_{2}$ of (15) cross $\phi(x) = \min\{u_{1}(x), u_{2}(x)\}$ is a supersolution of (15), since $u = 0$ is always a subsolution we have the existence of a radial solution below $u_{1}$ and $u_{2}$. A minimal solution $u$ of (15) exists by the Zorn’s Lemma.

b) By contradiction, assume that the minimal solution $v$ of (16), is not the restriction of the minimal solution $u$ of (15) then $\psi(x) = \min\{u(x), v(x)\}$ is a supersolution for (15) and this will contradict the fact that $u$ is minimal.

\[\square\]

Case 2 $w \equiv 0$ is the unique steady state, so the dynamics is converging to it. We wait a long time so that the state is very close to $w \equiv 0$.

In turn, if the measure of the domain is small enough so that $w \equiv 0$ is the unique solution with boundary 0 setting $a(x, t) = 0$ we approach $w \equiv 0$ in the $C_{0}(\Omega) \cap H^{1}_{0}(\Omega)$ norm for every $u_{0}$.

Therefore, it exists a time $t_{1}$ such that the solution at $t_{1}$ is below the minimal solution of Claim 4.2 $v$, $u(t_{1}) < v$. 

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At this point, we set boundary \( a_v(x) \) corresponding to the boundary value of the minimal solution of 4.2. We know that by the gradient structure of the semilinear heat equation, we will approach the set of steady-states with boundary value \( a_v(x) \). Since there exists a minimal solution, by the parabolic comparison principle, we will converge to it. There exists a time \( t_2 > t_1 \) such that the solution \( u(t_2) \) will be very close to \( v \) in the \( C_0(\Omega) \cap H_0^1(\Omega) \) norm, then we apply controllability to attach this minimal solution. The results in [31, Lemma 8.3] ensure that we do not violate the constraints in this process.

The application of the staircase method in the path of Claim 4.1 ensures that we can reach \( w \equiv \theta \) in finite time.

We summarize the control strategy:

- \( a(x,t) = 0 \) from \( t \in [0,t_1] \), where \( t_1 \) is the time needed until the solution \( u \) is below a minimal solution \( v \) that belongs to the admissible path.

- \( a(x,t) = a_v(x) \) from \( t \in (t_1,t_2] \), where \( a_v(x) \) is as Claim 4.2 and where \( t_2 \) is the time needed until the solution \( u \) is close enough to the minimal solution \( v \) for applying local controllability without violating the constraints.

- \( a(x,t) \) resulting from local controllability to \( v \).

- Application of the Staircase method.

Case 3 There exists a non-trivial solution with boundary value 0.

If a non-trivial solution with boundary 0 exists and we start above it we will never reach \( w \equiv \theta \). However, even if a non-trivial solution with boundary 0 exists we might start from an initial data such that the steady state \( \theta \) is reached. Assume that for the initial datum \( u_0 \in A \). We want to show first that the solution of the following problem can be driven to \( w \equiv 0 \) in the \( L^\infty \) norm.

\[
\begin{cases}
  u_t - \mu \Delta u = f(u) & (x,t) \in \Omega \times [0, +\infty) \\
  u = a(x,t) & x \in \partial \Omega \times [0, +\infty) \\
  u_0 \in A
\end{cases}
\]

Consider the extension to a ball containing \( \Omega \) with the corresponding radius and origin from the stationary solution in \( S_c \).

\[
\begin{cases}
  \tilde{u}_t - \mu \Delta \tilde{u} = f(\tilde{u}) & (x,t) \in B_R(x_0) \times [0, +\infty) \\
  \tilde{u} = \tilde{a}(t) & (x,t) \in \partial B_R(x_0) \times [0, +\infty) \\
  \tilde{u}_0 = \begin{cases} u_0 & \text{if } x \in \Omega \\ \hat{v} & \text{otherwise} \end{cases}
\end{cases}
\]

where \( \hat{v} \) denotes the corresponding radial solution in \( S_c \). We set \( a(x,t) = 0 \) and \( \tilde{a}(t) = 0 \). The stationary solution \( \hat{v} \) is a supersolution for both problems (18) and for (17) considering its restriction to the smaller
domain Ω. Moreover, the restriction to Ω of the solution of (18) is a parabolic supersolution of problem (17) since \( \tilde{u}(x, t)|_{\partial\Omega} > 0 \).

Moreover we have that the solution to (18) converges to \( w \equiv 0 \) because there is no steady-state a side of \( w \equiv 0 \) between \( w \equiv 0 \) and \( \tilde{v}(x) \) by definition. Note that the condition (13) is an energetic condition for the radial ODE. Since the radial equations dissipate, we know that in the boundary, we have less or equal energy than at every point inside, Lemma 3.2. For this reason, to ask a stationary solution to have less energy in the boundary than the minimum non-trivial solution, and to have a lower maximum, implies that there is no other solution with boundary value 0 between \( w \equiv 0 \) and this stationary solution.

Therefore we obtain that the solution of \( u \) tends to \( w \equiv 0 \) and the same procedure from the previous case can be applied.

4.2.5. Remarks

Remark 4.3. In fact \( \mathcal{A} \) is not the whole subset of admissible initial data for which one can expect controllability. The result on \( \mathcal{A} \) is obtained by the parabolic comparison principle, but every steady-state locally has a stable manifold. High-frequency oscillations around those steady-states create functions that are not in comparison with any steady-state in \( \mathcal{A} \), but they converge to one of them.

Remark 4.4. We can conclude that it is equivalent to say that \( u_0 \) admissible is controllable in infinite time to \( w \equiv 0 \) than to say that \( u_0 \) is controllable in finite time to \( w \equiv \theta \).

4.3. Proof of the Theorem 1.2, \( F(1) = 0 \)

In this case the scheme is very similar with only one difference, that the traveling waves for the Cauchy problem are stationary. In particular this will imply the following claim:

Claim 4.3. If \( F(1) = 0 \), there is a unique solution of:

\[
\begin{cases}
-\Delta v = f(v) & x \in \Omega, \\
v = a & x \in \partial\Omega,
\end{cases}
\]

for \( a \in \{0, 1\} \)

Hence Claim 4.3 ensures that there is no barrier. The construction of the path of steady-states works very similarly also due to the fact that the traveling waves will generate an invariant region of negative energy where \( w \equiv \theta \) will be inside and \( w \equiv 1 \) and \( w \equiv 0 \) will be at the boundary of the region. Moreover, the restriction of the traveling waves gives other admissible paths of continuous steady-states for going from \( w \equiv 0 \) to \( w \equiv 1 \) and vice-versa. We proceed to prove the claim:

Proof. The proof follows by contradiction. By simplicity, assume \( a = 0 \), the other case follows with the same argument. Assume it exists a solution of (19). Extend the solution by 0 in a ball \( B_R \) for \( R \) big enough such that \( \Omega \subset B_R \). Since \( w \equiv 1 \) is a supersolution, we have that in the ball \( B_R \), there exists a
non-trivial solution with boundary value 0. This solution is radial since we are working in a ball, and we can write it as a radial ODE. We see that the boundary of this solution must satisfy that \( v_r^2 > 0 \), which implies that the energy at \( R \) is positive. By the dissipation of the radial ODE, we know that the energy at the origin of the ODE (or center of the ball) is bigger or equal than this energy. From here, it leads to contradiction because the ODE cannot cross the horizontal axis of the phase plane \( v_r = 0 \), between 0 and 1 because it is a region of negative energy.

5. Numerical illustration

5.1. Minimal controllability time

In this subsection we are going to compute the control with the minimal time. The fact of adding state constraints to the problem induces a minimum controllability time (see [31, 37]), in our case the state constraints are bilateral instead of unilateral.

\[ I[a] = T, \]

will be minimized under the dynamic constraints

\[
\begin{align*}
u_r - \mu u_{rr} - \mu \frac{N - 1}{r} u_r &= u(1 - u)(u - \theta), \\
u(t, R) &= a(t), \\
u_r(t, 0) &= 0, \\
u(0, r) &= 0,
\end{align*}
\]

and:

\[
0 \leq u(t, r) \leq 1 \quad \forall (t, r) \in [0, T] \times [0, R],
\]

\[
\theta - \epsilon \leq u(T, r) \leq \theta + \epsilon \quad \forall r \in [0, R].
\]

For the numerical scheme, finite differences where employed using an implicit scheme with a fixed point for the nonlinearity. In Figure 6 one can see the control function showing a bang-bang behavior.

The numerical simulations point that the control in minimal time is bang-bang.

5.2. Visualization of the continuous path for large \( R \)

Figure 7 shows several captures of the continuous path of steady-states for \( R = 30 \). In Figure 8, the trace of the continuous path is shown. One can observe its oscillations which also are natural in the 1-d case. Notice that the appearance of non-trivial solutions with boundary value \( \theta \) by the comparison principle can create barriers so that the monotonous path cannot work. In the one dimensional case the oscillations are higher due to the lack of dissipativity of the ODE system generating the elliptic solutions. Indeed, locally around \( \theta \), one can observe in the 1D case the harmonic oscillator.
Figure 6: Minimal controllability time from $u_0 \equiv 0$ to $u(T_{\text{min}}) = w \equiv \theta$ in a ball of measure 1 with diffusivity $\mu = 0.0611$, $\epsilon = 0.01$.

Figure 7: Some steady-states forming part of the continuous path of admissible steady-states connecting to $u = \theta = 0.33$ for $R = 30$ and $N = 2$. 
5.3. Quasistatic control

In this subsection numerical approximations in IPOPT are performed in order to observe an approximation of the quasistatic control strategy used in [47] in which the authors stabilize the resulting trajectory of setting the boundary as the trace of the path of steady-states.

The discrete version of the following cost functional is minimized

\[ I[a] = \int_0^T a_t(t)^2 dt, \]

under the dynamic constraints

\[
\begin{cases} 
  u_r - u_{rr} - \frac{N-1}{r} u_r = u(1-u)(u-\theta), \\
  u(t, R) = a(t), \\
  u_r(t, 0) = 0, \\
  u(0, r) = 0, 
\end{cases}
\]

and:

\[
0 \leq u(t, r) \leq 1 \quad \forall (t, r) \in [0, T] \times [0, R], \\
\theta - \epsilon \leq u(T, r) \leq \theta + \epsilon \quad \forall r \in [0, R], \\
\|a_t\|_\infty < \epsilon,
\]

where the last constraint can be required since we already know that the system is controllable to \( \theta \) if we start with an initial datum small enough.

Figure 9 shows the controlled state against the time. Figure 10 shows time snapshots of Figure 9 in the state space and in the phase space, in the later one elliptic solutions are also shown.

Figure 8: Trace of the continuous path of admissible steady-states.
Figure 9: Time evolution of the controlled state in radial coordinates.

Figure 10: States of the controlled solution in different equispaced times. In red earlier times and darker later ones (left). In red the parabolic controlled state at different equispaced times in the phase plane. In Black the elliptic solution which has the same condition at the origin (right).
6. Conclusions and perspectives

In this work, we have seen that the presence of state constraints can lead to the existence of non-trivial solutions which act as obstructions to the controllability for certain initial data (barrier functions). We have seen how the shape of the domain and the diffusivity play an essential role for discussing when barriers for reaching $w \equiv 0$ exist.

Moreover, the existence of traveling waves for the corresponding Cauchy problem in the whole space helps us to determine the nonexistence of barrier functions for reaching $w \equiv 1$.

In the bistable case, for reaching the intermediate equilibrium $w \equiv \theta$, the staircase method has been crucial. The construction of the corresponding path relies on two ideas. First, we enlarge the domain to a ball, and second, when the elliptic problem in the ball is understood as an ODE problem, we observe that there exists a positively invariant region in the phase space containing our target.

Furthermore, a discussion of the set of initial conditions that lead to controllability has been done.

When constraints in the state are present, we can encounter different situations that summarized hereafter:

1. There does not exist any continuous path of admissible steady-states connecting the initial steady-state and the target. However, we are able to control from one to another because the target the only steady state. This is the case discussed employing a comparison with the traveling waves with the convergence to $w \equiv 1$.

2. There does not exist any continuous path of admissible steady-states connecting an initial steady-state with the target, and we are not able to control. The emergence of a barrier illustrates this case. A non-trivial solution touching the boundary of the admissible states makes impossible any control strategy to succeed by the comparison principle.

3. There is a continuous path of admissible steady-states from our initial steady-state and the target. This implies the controllability. Moreover, we emphasize that:
   - For such domain, there can be non-trivial solutions that can act as a barrier. However, the fact that we have an attainable path for our initial data ensures that we will not be in comparison with this solution in case it exists.
   - The stability of the target does not matter. The path of steady-states ensures that we can control towards an unstable equilibrium.

We can summarize our conclusions for the Bistable case with the following diagrams of Figure 11 and 12 where the possible transitions to the constant steady-states is depicted.
Figure 11: Connectivity map for $F(1) > 0$. In red, it is shown an admissible continuous path of steady-states (for any $\Omega$ and any $\mu > 0$) connecting stationary solutions. In green, it is an admissible and continuous path of steady-states connecting two stationary solutions, but in this case, its existence depends on $\Omega$ and $\mu$. In black, traveling waves for the Cauchy problem are shown. The Traveling wave from $w \equiv 0$ to $w \equiv 1$ is unique while the traveling waves from $w \equiv \theta$ to $w \equiv 1$ or to $w \equiv 0$ are infinitely many.

Figure 12: Connectivity map for $F(1) = 0$. In red, an admissible continuous path of steady-states (for any $\Omega$ and any $\mu > 0$) connecting stationary solutions is shown. The traveling wave from $w \equiv 0$ to $w \equiv 1$ is unique and stationary, giving a continuous path of admissible steady-states connecting $w \equiv 0$ and $w \equiv 1$. In black, non-stationary traveling waves for the Cauchy problem are shown. The traveling waves from $w \equiv \theta$ to $w \equiv 1$ or to $w \equiv 0$ are infinitely many.

Further perspectives and problems are:

- To consider spatial heterogeneity. More realistic models carry more spatial dependences.

$$\begin{cases}
    u_t - \text{div}(A(x)\nabla u) + \langle b(x), \nabla u \rangle = f(u, x) & (x,t) \in \Omega \times (0,T], \\
    u = a(x,t) & (x,t) \in \partial\Omega \times (0,T], \\
    0 \leq u(x,0) \leq 1.
\end{cases}$$

For instance, the carrying capacity or the diffusion can vary depending on the space or having space-dependent drifts. This is tackled in [13] for the case of spatially heterogeneous drifts,

$$\begin{cases}
    u_t - \Delta u + \left( \frac{\nabla N(x)}{N(x)}, \nabla u \right) = f(u) & (x,t) \in \Omega \times (0,T], \\
    u = a(x,t) & (x,t) \in \partial\Omega \times (0,T], \\
    0 \leq u(x,0) \leq 1.
\end{cases}$$

where the authors extend the results of the present work. The presence of heterogeneity leads, for example, can lead to obstructions for reaching $w \equiv 1$. Furthermore, an extended version of the
staircase method [31] is needed for controlling towards $w \equiv \theta$ for a small varying heterogeneity on the drift.

- To fully characterize the set of initial data for which we can control to $w \equiv \theta$. Which is the exact division in the initial datum set among the initial datums for which the system is controllable and the ones that it is not? Further study of the stable manifolds of the steady-states is required.

- The proof given here is based on the staircase method [31], this strategy requires, by construction, a long time for achieving the target. The construction of the path guarantees that for specific initial data, the set of controls that drive from $u_0$ to the target $w \equiv \theta$ is not empty for $T$ big. But this geometrical construction does not give us any insight into controls that are not close to the path. For example, how can the dynamical control associated with the minimal controllability time be?

- Following the previous point, many perspectives are open, for instance, can we build a control that sends our state to the stable manifold of $w \equiv \theta$? How much time do we need to reach this manifold?

- To extend these results to systems of several coupled semilinear PDEs. More realistic ecological models or chemical reactions will carry a higher number of species with different relationships in the nonlinearity [3, 20]. For example

\[
\begin{aligned}
\partial_t u_1 - \mu_1 \Delta u_1 &= f_1(u_1, u_2, u_3) \quad (x, t) \in \Omega \times (0, T], \\
\partial_t u_2 - \mu_2 \Delta u_2 &= f_2(u_1, u_2, u_3) \quad (x, t) \in \Omega \times (0, T], \\
\partial_t u_3 - \mu_3 \Delta u_3 &= f_3(u_1, u_2, u_3) \quad (x, t) \in \Omega \times (0, T], \\
u_1 &\equiv a(x, t) \quad (x, t) \in \partial \Omega \times [0, T], \\
\frac{\partial}{\partial \nu} u_j &= 0 \quad (x, t) \in \partial \Omega \times (0, T) \quad j = 2, 3, \\
0 \leq u_i(x, 0) &\leq 1 \quad i = 1, 2, 3.
\end{aligned}
\]

- Due to technical reasons regarding the construction of the path, we have considered a control acting in the whole boundary. A future perspective is to construct the path taking only a control in a part of the boundary $\eta \subset \partial \Omega$:

\[
\begin{aligned}
\partial_t u - \mu \Delta u &= f(u) \quad (x, t) \in \Omega \times (0, T], \\
u(x, t) &= a(x, t) \quad (x, t) \in \eta \times (0, T], \\
\frac{\partial}{\partial \nu} u(x, t) &= 0 \quad (x, t) \in \partial \Omega \setminus \eta \times (0, T], \\
0 \leq u(x, t) &\leq 1 \quad (x, t) \in \Omega \times [0, T].
\end{aligned}
\]

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Appendix A. Estimates on the thresholds

Proposition Appendix A.1 (A lower bound for $\mu^*$).

Assume that $f(0) = f(\theta) = f(1) = 0$, and that $f'(0) < 0$, $f'(1) < 1$, $f'(\theta) > 0$. Moreover consider $F(v) = \int_0^v f(s) ds$ and assume that $F(1) > 0$. Consider $\Omega \subset \mathbb{R}^N$ be a bounded set of measure 1 and boundary $C^2$, consider also the following problem:

\[
\begin{aligned}
-\mu \Delta u &= f(u) & x \in \Omega, \\
u &= 0 & x \in \partial \Omega, \\
u > 0 & x \in \Omega.
\end{aligned}
\]  

(A.1)

Denote by $B_\Omega$ a ball of maximal measure inside $\Omega$, $B_\Omega \subset \Omega$. Then, for any $\mu > 0$ fulfilling

$$
\mu < \frac{2\delta \Gamma \left( \frac{N}{2} + 1 \right)^{2/N} (F(\theta) + (1 - \delta)^N (F(1) - F(\theta)))^{2/N}}{\pi (1 - (1 - \delta)^N)}
$$

there exists a solution of the problem (A.1), where $\delta > 0$ fulfills

$$
\delta < 1 - \left( \frac{-F(\theta)}{F(1) - F(\theta)} \right)^{1/N}
$$

and $\Gamma$ is the gamma function. This implies a lower bound for $\mu^*$.

Remark Appendix A.1. Roughly speaking, the proposition says that if there exists a ball big enough inside the domain under consideration, then there is multiplicity of solutions.

Proof. We know that $w \equiv 0$ is a solution of the Euler-Lagrange equations of the corresponding functional.

$$
I[u] = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{\mu} \int_\Omega F(v) dx
$$

The strategy of the proof is the following, as in the other Theorem Appendix A.3, we proof that the functional admits a minimizer independently of $\mu$. Then we construct a family of functions $v_\delta \in H_0^1(B_\Omega)$ (which is trivially included in $H_0^1(\Omega)$ by extending its functions by zero up to the boundary $\partial \Omega$). This family is zero in the boundary and increase linearly to 1 (where $F(1) > 0$) and $\delta$ is related with the measure of the set in which the function is not constant.
The idea is first ensure under which conditions on $\delta$ we have that:

$$\int_{\Omega} F(v(x)) > c > 0.$$  

Once we have this, we choose $\mu$ in order to dominate the term:

$$\frac{1}{2} \int_{\Omega} |\nabla v|^2 dx,$$

that will depend only on the $\delta$ chosen before and thus we constructed $v$ such that:

$$I[v] < 0.$$

The ball is a star-shaped set which means that we can consider another ball contained inside it which is defined by:

$$(1 - \delta)B_{\Omega} := \left\{ x \in \mathbb{R}^N \text{ s.t. } \frac{x}{1 - \delta} \in B_{\Omega} \right\},$$

for $1 > \delta > 0$. We define $v_\delta$ in the following way, let $R$ be the radius of $B_{\Omega}$, $(1 - \delta)R$ will be the radius of $(1 - \delta)B_{\Omega}$:

$$v_\delta(x) = \begin{cases} 1 & \text{if } x \in (1 - \delta)B_{\Omega}, \\
-\frac{1}{\delta R} (||x||_2 - R) & \text{if } x \in B_{\Omega}\setminus(1 - \delta)B_{\Omega}. \end{cases}$$

Note that $v_\delta \in H^1_0(B_{\Omega})$ and extending it to be zero in $\Omega\setminus B_{\Omega}$ we have a function in $H^1_0(\Omega)$. Then we have that:

$$|\nabla v_\delta|^2 = \begin{cases} 0 & \text{in } (1 - \delta)B_{\Omega}, \\
\frac{\pi}{\delta^2} \left(m(B_{\Omega})\Gamma\left(\frac{N}{2} + 1\right)\right)^{-2/N} & \text{in } B_{\Omega}\setminus(1 - \delta)B_{\Omega},\end{cases}$$

where $\Gamma$ denotes the gamma function, and the term $\pi \left(m(B_{\Omega})\Gamma\left(\frac{N}{2} + 1\right)\right)^{-2/N}$ comes from the volume of
a $N$ dimensional sphere $m(B_1) = \frac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} R^N_{B_1}$. Moreover we have that:

\[ m((1 - \delta)B_1) = (1 - \delta)^N m(B_1), \]

\[ m(B_1 \setminus (1 - \delta)B_1) = (1 - (1 - \delta)^N) m(B_1), \]

we want to find a pair $(\mu, \delta)$ for which:

\[ I[v] = \int_{\Omega} \frac{1}{2} |\nabla v|^2 - \frac{1}{\mu} f(v)dsdx < 0. \]

For doing so, first we choose $\delta > 0$ to be small enough such that:

\[ \int_{B_1} \int_0 f(s)dsdx > c > 0, \]

we split the space integral in two parts:

\[ \int_{B_1} \int_0 f(s)dsdx = \int_{B_1 \setminus (1 - \delta)B_1} \int_0 f(s)dsdx + \int_{(1 - \delta)B_1} \int_0 f(s)dsdx \]

\[ \geq \int_{B_1 \setminus (1 - \delta)B_1} F(\theta)dx + F(1)m((1 - \delta)B_1) \]

\[ = F(\theta)m(B_1 \setminus (1 - \delta)B_1) + F(1)m((1 - \delta)B_1) \]

\[ = m(B_1) \left[ F(\theta)(1 - (1 - \delta)^N) + F(1)(1 - \delta)^N \right] \]

\[ = m(B_1) \left[ F(\theta) + (1 - \delta)^N (F(1) - F(\theta)) \right] \]

So, it will suffice if we ensure that:

\[ F(\theta) + (1 - \delta)^N (F(1) - F(\theta)) > 0 \]

which corresponds to ask that:

\[ \delta < 1 - \left( \frac{-F(\theta)}{F(1) - F(\theta)} \right)^{1/N} \]  \hspace{1cm} (A.2)

We fix $\delta > 0$ fulfilling (A.2) and now our goal is to choose $\mu$ small enough so that the space integral on $F(v(x))$ dominates the gradient part.

\[ I[v_3] = \int_{\Omega} \frac{1}{2} |\nabla v_3|^2 - \frac{1}{\mu} F(v_3(x))dx \]

\[ = \int_{B_1} \frac{1}{2} |\nabla v_3|^2 - \frac{1}{\mu} F(v_3(x))dx \]

\[ \leq \int_{B_1 \setminus (1 - \delta)B_1} \frac{1}{2} |\nabla v_3|^2 dx - \frac{1}{\mu} \left( F(\theta) + (1 - \delta)^N (F(1) - F(\theta)) \right) m(B_1) \]

\[ = m(B_1) \left( \frac{1}{2} ((1 - (1 - \delta)^N)) \frac{\pi}{\delta^2} \left( m(B_1) \Gamma \left( \frac{N}{2} + 1 \right) \right)^{-2/N} - \frac{1}{\mu} \left( F(\theta) + (1 - \delta)^N (F(1) - F(\theta)) \right) \right) \]

so, it will be sufficient if:

\[ \frac{1}{2} ((1 - (1 - \delta)^N)) \frac{\pi}{\delta^2} \left( m(B_1) \Gamma \left( \frac{N}{2} + 1 \right) \right)^{-2/N} - \frac{1}{\mu} \left( F(\theta) + (1 - \delta)^N (F(1) - F(\theta)) \right) < 0 \]
which corresponds to:

\[
\mu < \frac{2\delta^2 \Gamma \left( \frac{N}{2} + 1 \right)^{2/N} \left( F(\theta) + (1 - \delta)^N (F(1) - F(\theta)) \right)}{\pi \left( 1 - (1 - \delta)^N \right)} m(B) \]

Remark Appendix A.2. Notice that the structure of the proof of Proposition 3.2 also works for the monostable case. When bounding by above the integral of the primitive, we will have \( F(1) \) instead of \( F(1) - F(\theta) \) because the primitive in the monostable case is monotone.

Consider a bounded domain \( \Omega \) of measure 1 and \( C^2 \) boundary, consider the following problem:

\[
\begin{aligned}
-\mu \Delta u &= f(u) & x \in \Omega, \\
u \in C^2(\Omega), & u > 0 & x \in \Omega, \\
u &= 0 & x \in \partial \Omega,
\end{aligned}
\]

(A.3)

where \( \mu > 0 \).

**Proposition Appendix A.2 (Coercivity of \( J \)).** Assume that:

\[
\limsup_{|s| \to \infty} \frac{1}{\mu} \frac{f(s)}{s} < \lambda_1(\Omega)
\]

(A.4)

Then \( J \) is coercive.

**Proof.** For simplicity we will take care only on the case in which \( s \to +\infty \), the case \( s \to -\infty \) follows similarly. Since

\[
\limsup_{s \to \infty} \frac{1}{\mu} \frac{f(s)}{s} < \lambda_1(\Omega)
\]

we know that there exist \( R > 0 \) such that

\[
\frac{1}{\mu} \frac{f(s)}{s} < \lambda_1 \quad \forall s \geq R
\]

using also the fact that \( f(0) = 0 \) we can write:

\[
\int_0^R \frac{1}{\mu} f(s) ds = \int_0^R \frac{1}{\mu} f(s) ds + \int_R^\infty \frac{1}{\mu} f(s) s ds \leq \frac{p}{2} u^2 + C(R, f, \mu)
\]

for a certain \( p < \lambda_1(\Omega) \). So

\[
J[u] = \int_\Omega \frac{1}{2} \nabla u^2 - F(u) dx \geq \int_\Omega \frac{1}{2} \nabla u^2 - \frac{p}{2} u^2 - C(R, f, \mu) dx
\]

\[
\geq c ||u||^2_{H^1(\Omega)} - C(R, f, \mu)
\]

where \( c > 0 \) because \( p < \lambda_1(\Omega) \). So \( J \) is coercive.

**Theorem Appendix A.1 (Existence of a minimizer).** Under the assumptions of Proposition Appendix A.2, \( J \) has a minimizer.
Proof. Note that the Lagrangian \( L(p, z, x) = \frac{1}{2}|p|^2 - F(z) \) is convex with respect to the variable \( p \). Therefore \( I \) is weakly lower-semicontinuous (Theorem 1 Ch.8 pp.468 in [50]) and the functional has a minimizer \( u \in H^1_0(\Omega) \).

Remark Appendix A.3. The hypothesis [A.4] is only needed for proving that the functional is coercive. In our case, it does not matter because, in the application of the monostable nonlinearity, we require \( f(s) \) to have a positive zero, which means that we can extend \( f \) by zero afterward.

Note that for both the monostable case and the bistable case we can redefine \( f \) to be constant before \( s = 0 \) (\( f(s) = 0 \) for \( s \leq 0 \)) and after \( s = 1 \) (\( f(s) = 0 \) for \( s \geq 1 \)). In general we are interested on minimizers such that \( 0 \leq v(x) \leq 1 \), after redefining (if needed) \( f \) by

\[
\tilde{f}(s) = \begin{cases} 
  f(s) & \text{if } 1 \geq s \geq 0 \\
  0 & \text{otherwise}
\end{cases}
\]

and we define the functional:

\[
J : H^1_0(\Omega) \longrightarrow \mathbb{R} \\
v \longmapsto I[v] := \int_{\Omega} \frac{1}{2} |\nabla v|^2 - \tilde{F}(v)dx
\]

where \( \tilde{F}(v) = \frac{1}{\mu} \int_0^s \tilde{f}(s)ds \).

Indeed \( J[0] = 0 \). Thanks to the redefinition of \( f \), we ensure that the minimizer satisfies \( 0 < v < 1 \) and satisfies the Euler-Lagrange equations:

\[
\begin{cases} 
  -\mu \Delta u = \tilde{f}(u) & x \in \Omega, \\
  u > 0 & x \in \Omega, \\
  u = 0 & x \in \partial \Omega,
\end{cases}
\]

since \( 0 < u < 1 \), \( \tilde{f} = f \) in this range and therefore is a solution of our original problem as well.

Making use of the variational structure of the problem, we will be able to give estimates from below for \( \mu^* \) in any dimensions.

The following proposition was proved in [52, Theorem II.1], the proof given below is making use of a variational argument rather than comparison.

**Proposition Appendix A.3 (One lower bound of \( \mu^* \) for monostable nonlinearities).** Assume that

\[
\lim_{s \to 0^+} \frac{1}{\mu} \frac{f(s)}{s} > \lambda_1(\Omega)
\]

and

\[
\limsup_{s \to \infty} \frac{1}{\mu} \frac{f(s)}{s} < \lambda_1(\Omega)
\]

1Here we provide a weaker version of the original theorem and moreover the proof given is a variational argument. In [52] one can find also a proof that does not rely on a variational argument.
Then, for all \( \mu < \frac{f'(0)}{\lambda_1(\Omega)} \) there exist a solution of the problem \((A.3)\). Therefore:

\[
\mu^* \geq \frac{f'(0)}{\lambda_1(\Omega)}
\]

**Proof.** The main idea of the proof is simple: To prove the existence of a local minimizer for \( I \) that takes values in \( 0 \leq v \leq 1 \) and to show that 0 is not a minimizer. Zero is not a minimizer. Let \( e_1 \) be the first eigenfunction of \(-\Delta\). We know that this function is positive. Set \( v = \epsilon e_1 \) for \( \epsilon > 0 \) to be chosen later on.

By hypothesis we assumed that:

\[
\lim_{s \to 0^+} \frac{1}{\mu} \frac{f(s)}{s} > \lambda_1(\omega)
\]

this means that there exists \( r \in \mathbb{R}^+ \) such that

\[
\frac{1}{\mu} \frac{f(s)}{s} > \lambda_1(\omega) \quad \forall s \in [0, r]
\]

choose \( \epsilon \) small enough such that \( \epsilon e_1 < r \) and now evaluate the functional:

\[
I[\epsilon e_1] = \int_\Omega \frac{\epsilon^2}{2} |\nabla e_1|^2 - \int_0^{\epsilon e_1} \frac{1}{\mu} \frac{f(s)}{s} sdsdx \lesssim \int_\Omega \frac{\epsilon^2}{2} |\nabla e_1|^2 - p \frac{\epsilon^2 e_1^2}{2} dx,
\]

for some \( p > \lambda_1(\Omega) \). Then integrating by parts \( |\nabla e_1|^2 \) and using the fact that \(-\Delta e_1 = \lambda_1 e_1\):

\[
I[\epsilon e_1] \leq \frac{\epsilon^2}{2} \int_\Omega (\lambda_1(\Omega) - p) e_1^2 dx < 0
\]

Therefore we know that for all \( \frac{1}{\mu} \leq \frac{\lambda_1(\Omega)}{f'(0)} \) there exists a positive solution, this means that

\[
\mu^* \geq \frac{f'(0)}{\lambda_1(\Omega)}
\]

The following Proposition is a lower bound for this splitting in the case in which the nonlinearities are concave and twice differentiable.

**Proposition Appendix A.4** (An upper bound of \( \mu^* \) for monostable nonlinearities). Let \( f \) be twice differentiable such that \( f'(0) > 0 \) and concave, i.e. \( f''(t) \leq 0 \). Then if:

\[
\frac{1}{\mu} \leq \frac{\lambda_1(\Omega)}{f'(0)}
\]

there cannot be any positive solution to Problem \((A.3)\). Therefore,

\[
\mu^* \leq \frac{f'(0)}{\lambda_1(\Omega)} > 0
\]

**Proof.** Multiply the equation by \( v \) and integrate over the domain and integrate by parts:

\[
\int_\Omega |\nabla v|^2 - \frac{1}{\mu} \int_\Omega f(v)v = 0,
\]

by the Poincar inequality,

\[
\int_\Omega \lambda_1(\Omega)v^2 - \frac{1}{\mu} \int_\Omega f(v)v \leq 0.
\]
Now consider the Taylor formula on $f$

$$f(v) = f(0) + f'(0)v + \int_0^v f''(t)(v - t)dt$$

Due to the fact that $f(0) = 0$ we end up with

$$\int_\Omega \left( \lambda_1(\Omega) - \frac{1}{\mu}f'(0) \right) v^2 - \lambda v \int_0^v f''(t)(v - t)dt dx \leq 0$$

since $v > 0$ we have that $v - t \geq 0$ moreover by assumption $f''(t) \leq 0$ we obtain that the second term is bigger or equal than zero. Hence we can conclude that a necessary condition to have a positive solution is:

$$\int_\Omega \left( \lambda_1(\Omega) - \frac{1}{\mu}f'(0) \right) v^2 \leq 0$$

which concludes the proof.

\[ \square \]

**Proposition Appendix A.5** ($\mu^*$ for monostable concave nonlinearities). When $f$ does not depend on $x$ we have:

$$\mu^* = \frac{f'(0)}{\lambda_1(\Omega)} > 0$$

**Remark Appendix A.4** (Uniqueness of positive solutions for concave nonlinearities). When $f$ is concave and a positive solution exists it is unique [36, 53].

**Remark Appendix A.5** (Non uniqueness of positive solutions for non concave nonlinearities). When $f$ is not concave we might not have uniqueness. Assuming that $f'(0) > 0$, in this case if $\mu^* > \frac{f'(0)}{\lambda_1(\Omega)}$ we have that for all $\mu$ such that $\mu^* > \mu > \frac{f'(0)}{\lambda_1(\Omega)}$ there exists a second positive solution. This is proven using topological degree arguments [36].

Now we turn our attention to bistable nonlinearities. The structure of the proofs estimates will be similar.

**Proposition Appendix A.6** (An upper bound for $\mu^*$). Let $f : \mathbb{R} \to \mathbb{R}$, assume that $f$ is bounded. Assume furthermore that $f(0) = 0$. Consider:

$$\begin{cases} 
-\mu \Delta v = f(v) & x \in \Omega \\
v = 0 & x \in \partial \Omega \\
1 > v > 0 & x \in \Omega 
\end{cases}$$

Then,

$$\frac{\max_{s \in [0,1]} f(s)}{\lambda_1(\Omega)} \geq \mu^*$$

**Proof.** For proving it we will make use of the parabolic problem:

$$\begin{cases} 
v_t - \mu \Delta v = f(v) & (x, t) \in \Omega \times (0, T] \\
v = 0 & (x, t) \in \partial \Omega \times (0, T] \\
1 > v(x, 0) > 0 & x \in \Omega
\end{cases} \quad (A.5)$$
by a comparison argument we have that the solution of this problem is:

\[ f(v) = \frac{f(v)}{v} v \leq P v \]

since \( \frac{f(v)}{v} \leq P \) for some \( P \). Then we know that the solution of problem (A.5) is lower than the solution to:

\[
\begin{cases}
y_t - \mu \Delta y = P y \\ y = 0 \\ 1 > y(x,0) > 0
\end{cases}
\]

Consider the eigenfunctions of the Laplacian \( -\Delta e_n = \lambda_n e_n \). One can express the solution of the problem above by the separation of variables technique: Let \( c_n(t) = (y(t), e_n)_{L^2(\Omega)}, and take its time derivative:

\[
\frac{d}{dt} c_n(t) = (y_t, e_n)_{L^2(\Omega)} = (\mu \Delta y + Py, e_n)_{L^2(\Omega)}
\]

\[
= (y, \mu \Delta e_n)_{L^2(\Omega)} + P(y, e_n)_{L^2(\Omega)} = (P - \mu \lambda_n) c_n(t)
\]

So,

\[
y(t, x) = \sum_{n=1}^{\infty} (v(x,0), e_n)_{L^2(\Omega)} e^{(P - \mu \lambda_n)t} e_n
\]

By the parabolic comparison principle that:

\[
\|v(t)\|_{L^2}^2 \leq \|g(t)\|_{L^2}^2 = \sum_{n=1}^{\infty} \|v(x,0), e_n\|_{L^2(\Omega)}^2 e^{2(P - \mu \lambda_n(\Omega))t}
\]

\[
\leq \|v_0\|_{L^2}^2 e^{2(P - \mu \lambda_1(\Omega))t}
\]

If \( P < \mu \lambda_1(\Omega) \), then \( v(t) \) will go to zero independently of the initial datum.

For concluding, we come back to the elliptic problem to see a contradiction with the existence of a positive solution.

Assume that \( v(x,0) \) is stationary and solves the semilinear elliptic problem. By the estimate just shown above, we know that differentiating with respect to time we should get zero from one side, and the estimate in the right-hand side from the parabolic problem.

\[
0 = 2v(\mu \Delta v + f(v)) \leq 2(P - \mu \lambda_1)\|v(x,0)\|_{L^2}^2 e^{2(P - \mu \lambda_1)t}
\]

But the quantity in the left is negative if \( v(x,0) \) is not identically zero. So the proof is concluded.

This estimate is:

\[
\max_{s \in [0,1]} \frac{f(s)}{\lambda_1(\Omega)} \geq \mu^*
\]

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