SEPARATE HOLOMORPHIC EXTENSION ALONG LINES AND HOLOMORPHIC EXTENSION FROM THE SPHERE TO THE BALL

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ABSTRACT. We give positive answer to a conjecture by Agranovsky. A continuous function on the sphere which has separate holomorphic extension along the complex lines which pass through three non aligned interior points, is the trace of a holomorphic function in the ball.
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1. Introduction

The problem of describing families of discs which suffice for testing analytic extension of a function $f$ from the sphere $\partial \B^2$ to the ball $\B^2$ has a long history. For $f$ continuous on $\partial \B^2$, Agranovsky-Valski [4] use all the lines, Agranovki-Semenov [3] the lines through an open subset $D' \subset \B^2$, Rudin [10] the lines tangent to a concentric subsphere $B_1^2$, Baracco–Tumanov-Zampieri the lines tangent to any strictly convex subset $D' \subset \B^2$. There are many other contributions such as [2], [11], [8] just to mention a few. It is a challenging attempt to reduce the number of parameters in the testing families. However, one encounters an immediate constraint: lines which meet a single point $z_0 \in \B^2$ do not suffice. Instead, two interior points or a single boundary point suffice: Agranovsky [1] and Baracco [5]. However, in these last two results, the reduction of the testing families is compensated by an assumption of extra initial regularity: $f$ is assumed to be real analytic. Globevnik [7] shows that, for two points, $C^\infty$-regularity still suffices, but $C^k$ does not. This suggests that holomorphic extension is a good balance between reduction of testing families and improvement of initial regularity. And in fact, it is showed here, that for $f \in C^0$ three not on the same line points suffice. Here is our result.

**Theorem 1.1.** Let $f$ be a continuous function on the sphere $\partial \B^2$ which extends holomorphically along any complex line in $\B^2$ which encounters the set consisting of 3 points not on the same line. Then, $f$ extends holomorphically to $\B^2$. 

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The proof follows in Section 2 below. It shows that, the result should hold for a ball of general dimension \( \mathbb{B}^n \). In this case, \( n + 1 \) points in generic position should suffice. We first introduce some terminology. A straight disc \( A \) is the intersection of a straight complex line with \( \mathbb{B}^2 \); \( \mathbb{P}T^*\mathbb{C}^2 \) is the cotangent bundle with projectivized fibers, and \( \pi \) the projection on the base space; \( \mathbb{P}T^*_{\partial \mathbb{B}^2} \mathbb{C}^2 \) the projectivized conormal bundle to \( \partial \mathbb{B}^2 \) in \( \mathbb{C}^2 \). It is readily seen that the straight discs \( A \) of the ball are the geodesics of the Kobayashi metric, or, equivalently, the so-called “stationary discs” (cf. Lempert [9]). These are the discs endowed with a meromorphic lift \( A^* \subset \mathbb{P}T^*\mathbb{C}^2 \) with a simple pole attached to \( \mathbb{T}^*_{\partial\mathbb{B}^2} \mathbb{C}^2 \), that is, satisfying \( \partial A^* \subset \mathbb{P}T^*_{\partial\mathbb{B}^2} \mathbb{C}^2 \). We fix three points \( P_j \), \( j = 1, 2, 3 \) in \( \mathbb{B}^2 \) and consider a set, indexed by \( j \), of \((2)\)-parameter families of straight discs \( A^j \) passing through \( P_j \). We define \( M_j \) to be the union of the lifts of the family with index \( j \). The set \( M_j \) is generically a CR manifold with CR dimension 1 except at the points that project over \( P_j \); we denote by \( M^\text{reg}_j \) the complement of this set. The boundary of \( M_j \) concides with \( \mathbb{P}T^*_{\partial\mathbb{B}^2} \mathbb{C}^2 \) which is maximal totally real in \( \mathbb{P}T^*\mathbb{C}^2 \). Here is the central point of our construction. Though the function \( f \), in the beginning of the proof, is not extendible to \( \mathbb{B}^2 \) as a result of the separate extensions to the \( A \)'s, nevertheless it is naturally lifted to a function \( F \) on \( M_j \) by gluing the bunch of separate holomorphic extensions to the lifts \( A^* \)'s. This is defined by

\[
F(z, [\zeta]) = f_{A(z, [\zeta])}(z),
\]

where \( A(z, [\zeta]) \) is the unique stationary disc whose lift \( A^*_{(z, [\zeta])} \) passes through \( (z, [\zeta]) \). The crucial point here is that the \( A \)'s may overlap on \( \mathbb{C}^2 \) but the \( A^* \)'s do not in \( \mathbb{P}T^*\mathbb{C}^2 \). The function \( F \) is therefore well defined and CR on \( M^\text{reg}_j \).

2. Proof of Theorem 1.1

The proof consists of several steps. We start by collecting some easy computations. We identify \( \mathbb{P}T^*\mathbb{C}^2 \cong \mathbb{C}^2 \times \mathbb{CP}_1 \cong \mathbb{C}^3 \) with coordinates \((z_1, z_2) \in \mathbb{C}^2 \) and \( z_3 = \frac{\bar{z}_2}{z_1} \in \mathbb{CP}_1 \). Let \( M_0 \) be the collection of the lifts of the discs through 0.

**Lemma 2.1.** Let \( A^*_0 \) be the (unique) disc of \( M_0 \) which projects over the \( z_1 \)-axis. Then, \( A^*_0 \), identified to a disc of \( \mathbb{C}^3 \), has two holomorphic lifts to \( T^*\mathbb{C}^3 \) attached to \( T^*_{M_0} \mathbb{C}^3 \). Their components are parametrized by \( z_1 \mapsto (0, -\frac{1}{z_1}, 1) \) and \( z_1 \mapsto (0, \frac{1}{z_1}, \frac{1}{z_1}) \) respectively.

**Proof.** First, we notice that for any \( z = (z_1, z_2) \in \mathbb{B}^2 \) the disc \( \tau \mapsto \tau \frac{z}{\|z\|} \) is the only passing through \( z \) and 0. The lift attached to the
projectivized conormal bundle of this disc is the constant \( \bar{z} \). We have
\[
M_0 = \{(z; \bar{z}) \mid z \in \mathbb{B}^2 \setminus \{0\} \cup \{(0; \bar{\zeta}) \mid \forall \zeta \in \mathbb{C}P_1\}\}.
\]
Clearly \( M_0 \) (or more precisely \( M_0^{\text{reg}} \)) has equation \( r : z_3 - \frac{\bar{z}}{z_1} = 0 \). In particular the lift of \( A_0 \) to \( \mathbb{P}T^* \mathbb{C}^2 \) is \( A_0^*(\tau) = ((\tau, 0); [1, 0]) \) which in coordinates is expressed by \( A_0^*(\tau) = (\tau, 0, 0) \). Since \( M_0 \) is Levi flat, the space of holomorphic lifts contained in \( T^* M_0 \) has dimension two. For instance a basis for the space of lifts is given by
\[
(\text{2.1}) \quad \omega_1(z_1, z_2) = \partial \text{Re} r = \left( \frac{z_2}{z_1}, -\frac{1}{z_1}, 1 \right) \quad \text{and} \quad \omega_2(z_1, z_2) = \partial \text{Im} r = \frac{1}{i} \left( -\frac{z_2}{z_1}, \frac{1}{z_1}, 1 \right).
\]
In particular, along \( A_0^* \) the conormal bundle to \( M_0 \) is generated by \( \omega_1(z_1, 0) = (0, 1, 0, 1) \) and \( \omega_2(z_1, 0) = (0, 0, 1, 1) \). As one can readily note both sections are holomorphic along \( A_0^* \) and they are exactly the lifts of \( A_0^* \) to the conormal bundle of \( T^*_M \mathbb{C}^3 \).

\[\square\]

Remark 2.2. Note that if in the lemma above we consider the union of the lifts of discs passing through the point \( P_{\zeta_0} = (\zeta_0, 0) \) the manifold resulting \( M_{\zeta_0} \) still contains \( A_0^* \) (i.e. the \( z_1 \) axis) and along the boundary of \( A_0^* \) we have \( TM_0|_{\partial A_0^*} = TM_{\zeta_0}|_{\partial A_0^*} \) and thus also \( T^*_M \mathbb{C}^3|_{\partial A_0^*} = T^*_M \mathbb{C}^3|_{\partial A_0^*} \). From this equality we have that if \( \bar{\omega}_1, \bar{\omega}_2 \) is a basis of lifts of \( A_0^* \) to the conormal bundle to \( M_{\zeta_0} \), then this is related to the basis \( \omega_1, \omega_2 \) by
\[
(\text{2.2}) \quad \text{Span}\{\bar{\omega}_1, \bar{\omega}_2\}|_{\partial A_0^*} = \text{Span}\{\omega_1, \omega_2\}|_{\partial A_0^*}.
\]
Combination of (2.2) with the fact that singularity of \( \bar{\omega}_1, \bar{\omega}_2 \) must now be located at \( \zeta_0 \) yields a choice of holomorphic basis as \( \bar{\omega}_1(z_1) = \left( 0, -\frac{1}{(z_1 - \zeta_0)}, \frac{1}{(1-z_1 \zeta_0)} \right) \) and \( \bar{\omega}_2(z_1) = \left( 0, \frac{1}{(z_1 - \zeta_0)}, \frac{1}{(1-z_1 \zeta_0)} \right) \).

Before the proof of our main theorem we need a preliminary crucial result

Theorem 2.3. Let \( P_1, P_2 \in \mathbb{B}^2 \) be two distinct points inside the ball and let \( f : \partial \mathbb{B}^2 \to \mathbb{C} \) be a continuous function such that \( f \) extends holomorphically along every complex straight line passing through either \( P_1 \) or \( P_2 \). Then for any such disc \( A \), except the one passing through both points, the lifted function \( F \) extends holomorphically to a neighborhood of \( A^* \setminus \pi^{-1}(P_j) \) where \( j \) is 1 or 2 according to \( P_1 \in A \) or \( P_2 \in A \).

Proof. It is not restrictive to assume that the disc \( A \) is the \( z_1 \) axis, that \( P_2 = (0, z_2) \) and that \( P_1 = (\zeta_0, 0) \). We note that \( M_1 \) and \( M_2 \) intersect transversally along the boundary of \( A^* \). Let \( P = (\zeta, 0) \) be a point of
the boundary of \( A \) and \( P^* = (\zeta, 0, 0) \) be the corresponding point on \( A^* \). \( P^* \) lies in the common boundary of \( M_1 \) and \( M_2 \). Let \( v_\zeta \) be a tangent vector to \( T_{P^*}M_2 \setminus T_{P^*}E \) which points inside \( M_2 \). The equivalence class \([v_\zeta]\) in the vector spaces quotient \( \frac{T_{P^*}C^3}{T_{P^*}M_1} \) is called the pointing direction of \( M_2 \) with respect to \( M_1 \). We say in this case that \( F \) extends at \( P^* \) in direction \([v_\zeta]\). Let \( Q^* = (\zeta_Q, 0, 0) \) be a point of \( A^* \) (\( \zeta_Q \neq 0 \)). Following [13] by effect of the extension of \( F \) at \( P^* \) in direction \([v_\zeta]\) we have extension at \( Q^* \) in direction \([w_\zeta]\) \( \in \frac{T_{Q^*}C^3}{T_{Q^*}M_1} \). The relation of \([w_\zeta]\) with the initial \([v_\zeta]\) is expressed by means of contraction with the holomorphic basis of lifts \( \tilde{\omega}_1, \tilde{\omega}_2 \):

\[ \text{Re} \langle \tilde{\omega}_1(\zeta), v_\zeta \rangle = \text{Re} \langle \tilde{\omega}_1(\zeta_Q), w_\zeta \rangle \quad \text{and} \quad \text{Re} \langle \tilde{\omega}_2(\zeta), v_\zeta \rangle = \text{Re} \langle \tilde{\omega}_2(\zeta_Q), w_\zeta \rangle. \]

In other words the directions of \( CR \) extendibility, which are vectors in the normal bundle \( \frac{T_{C^3}}{T_{M_1}} \), are constant in the system dual to \( \{\tilde{\omega}_1, \tilde{\omega}_2\} \).

We first compute the pointing direction of \( M_2 \) at the point \( P^* \). To this end we first compute the disc passing through \( P \) and \( P^* \)

\[ A_{P^*P}(\tau) = \left( \frac{|z_2|^2\zeta}{1 + |z_2|^2}, \frac{z_2}{1 + |z_2|^2}\right) + \frac{\tau}{1 + |z_2|^2}(\zeta, -z_2); \]

note that \( A_{P^*P}(1) = P \). The lift component of \( A_{P^*P} \) is

\[ A^*_{P^*P} = [|z_2|^2\hat{\zeta}\tau + \hat{\zeta}, z_2\tau - \bar{z}_2], \]

and dividing the second component by the first we get that the \( A^*_{P^*P} \)'s coordinates in \( C^3 \) are

\[ \left( \frac{|z_2|^2\zeta}{1 + |z_2|^2} + \frac{\tau}{1 + |z_2|^2}\zeta, \frac{z_2}{1 + |z_2|^2} - \frac{\tau z_2}{1 + |z_2|^2}, \frac{\bar{z}_2(\tau - 1)}{1 + |z_2|^2}\zeta(|z_2|^2\tau + 1) \right). \]

The pointing direction of \( M_2 \) at \( P \) is

\[ v_\zeta = -\partial_\tau A^*_{P^*P}(1) = \frac{-1}{1 + |z_2|^2}(\zeta, -z_2, \bar{z}_2, \zeta). \]

We have

\[ \text{Re} \langle \tilde{\omega}_1(\zeta), v_\zeta \rangle = \frac{-1}{1 + |z_2|^2} \text{Re} \frac{z_2}{\bar{\zeta} - \bar{\zeta}_0} \]

and

\[ \text{Re} \langle \tilde{\omega}_2(\zeta), v_\zeta \rangle = \frac{-1}{1 + |z_2|^2} \text{Im} \frac{z_2}{\zeta - \zeta_0}. \]

The first members of (2.3) and (2.5) express the components in the normal bundle to \( M_1 \) of \( w_\zeta \) with respect to the dual basis of \( \omega_1(\zeta_Q), \omega_2(\zeta_Q) \). By letting \( \zeta \) vary in \( \partial A \) we see that \([w_\zeta]\) sweeps all the directions in
Therefore, by the edge of the wedge theorem, $F$ extends holomorphically to a neighborhood of $Q^*$ and, by propagation, to a neighborhood of any other point of $A^*$ except from the point over $P_1$ where the $CR$ singularity is located.

□

We are ready for the proof of Theorem 1.1

End of Proof of Theorem 1.1

Let $A_0$ be the disc passing through $P_1$ and $P_3$. Then in particular $P_2 \notin A_0$. Applying the theorem above we get that $F$ extends holomorphically to a neighborhood of $A_0^* \setminus \{P_1\}$. By repeating the same argument we see that $F$ extends to a neighborhood of $A_0^* \setminus \{P_3\}$. Therefore $F$ extends to a full neighborhood of $A_0^*$. For any other straight line $A$ through $P_1$ we can say that $F$ extends holomorphically to a neighborhood of $A^* \setminus P_1$. By applying the continuity principle to the family of discs formed by $A_0^*$ and all the discs through $P_1$, we get that $F$ extends holomorphically to a set of the form $V \times \mathbb{C}P_1^C$, where $V$ is a neighborhood of $P_1$. Therefore $F$ does not depend on the second argument and it is therefore naturally defined on the projection of the collection of all the $A^*$’s, that is, on $\mathbb{B}^2$.

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