**BICATEGORIES OF ACTION GROUPOIDS**

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**Abstract.** We prove that the 2-category of action Lie groupoids localised in the following three different ways yield equivalent bicategories: localising at equivariant weak equivalences à la Pronk, localising using surjective submersive equivariant weak equivalences and anafunctors à la Roberts, and localising at all weak equivalences. These constructions generalise the known case of representable orbifold groupoids. We also show that any weak equivalence between action Lie groupoids is isomorphic to the composition of two particularly nice forms of equivariant weak equivalences.

1. **Introduction**

The study of topological or Lie groupoids up to Morita equivalence, generated by a topologised/differential geometric version of categorical weak equivalence, appears in many contexts. One standard way to do so, as in Pronk [21], is to define a bicategory of fractions of Lie groupoids \( \text{LieGpoid}[W^{-1}] \) in which all of the Morita equivalences become invertible 1-cells. In this bicategory, a 1-cell from \( \mathcal{G} \) to \( \mathcal{H} \) is defined by a span of functors \( \mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H} \) where the functor \( \mathcal{K} \rightarrow \mathcal{G} \) is a weak equivalence, and a 2-cell is defined as an equivalence class of diagrams with natural transformations. There is also a “smaller” localisation using anafunctors à la Roberts [25, 27]. By additionally requiring the weak equivalences used in the span to be surjective submersive on objects, we obtain a bicategory with similar 1-cells, but whose 2-cells are actual natural transformations chosen to represent the equivalence classes defining the 2-cells of Pronk. Roberts proves that this bicategory is equivalent to the bicategory of fractions of Pronk. Understanding conditions under which a class of 1-cells of a general bicategory admits a localisation is a current field of study; see also [1, 24]. The localisation of topological groupoids is made explicit in [7], where the authors discuss Lie groupoids but details are only provided for the topological case. In [29], a detailed development of localisations of diffeological groupoids using Pronk’s and Roberts’ approaches is presented, comparing and connecting them to the theory of bibundles in the diffeological groupoid context established in [28], as well as to stacks over diffeological spaces.

An important class of groupoids comes from Lie group actions. Having an action groupoid allows for the application of an equivariant functor such as cohomology. The question then becomes whether the invariant respects Morita equivalence. Functoriality ensures that our chosen invariant is unchanged under equivariant weak equivalences, but a priori there is no mechanism for checking more general weak equivalences. Thus we wish to know whether
Morita equivalent action groupoids are also Morita equivalent via equivariant weak equivalences. More precisely: if we consider the full sub-2-category of action groupoids, can we create a bicategory of fractions inverting the equivariant weak equivalences? If so, is this equivalent to the full sub-bicategory we get by inverting more general weak equivalences and then restricting to action groupoids?

If the answer to both questions is yes, we have a mechanism for transferring equivariant techniques to groupoids: choose a groupoid that is Morita equivalent to an action groupoid, apply some equivariant functor, and check that the result is Morita invariant. This strategy has been successfully applied, for example, to orbifold groupoids Morita equivalent to an action groupoid: in [22, Proposition 5.13] they define Bredon cohomology under a mild condition on the coefficient systems.

In this paper, instead of focusing on orbifolds, we generalise our group actions to those which have any subset of the following properties: free, locally free, transitive, effective, compact, discrete, proper, and being Morita equivalent to a proper étale Lie groupoid. (There is, of course, much redundancy when multiple properties are taken together.) Given action groupoids satisfying any selection of these properties, we localise at equivariant weak equivalences, and we show that the resulting bicategory is equivalent to the localisation at all weak equivalences; see Theorem 5.7. This yields affirmative answers to the two questions above for these action groupoids. Moreover, we show that we can construct this localisation using the method of Roberts, giving us a smaller and more concrete category to work with; see Proposition 4.12. This is a first step towards generalising [22] to more general action groupoids, and defining Bredon cohomology as a Morita invariant in a more general context. As an additional step towards understanding when equivariant functors might be Morita invariant, we also show that the decomposition of equivariant weak equivalences used in [22, Proposition 3.5] also applies in our more general setting. This allows us to break down equivariant weak equivalences into two specific types: projections and inclusions. This decomposition has proved useful in other contexts such as topological complexity [2].

The paper is structured as follows: Section 2 contains background information on Lie groupoids, and Section 3 background on the localisation of \textit{LieGpoid} at the class of weak equivalences, as well as at surjective submersive weak equivalences. Sections 4 and 5 contains our main results about action groupoids, answering the two questions posed above. In particular, in Section 4 we construct the localisation of action groupoids satisfying any subset of properties from a fixed list (see [3] and Proposition 4.10) and show that equivariant weak equivalences with these chosen properties can be decomposed into an equivariant projection and an equivariant inclusion (see Theorem 4.6). In Section 5 we show that this is equivalent to the full sub-bicategory of such action groupoids (see Theorem 5.7).

This paper is written with a wide audience in mind, and so some details (for instance, proofs of smoothness of maps) are spelled out. Throughout, we refer to [1, 13, 20] for categorical definitions such as bicategory, pseudofunctor, 2-category, and 2-commutative, and all of their constituent parts.

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2. The 2-Category of Lie Groupoids

In this section we discuss the 2-category of Lie groupoids and its properties, with special attention to the notion of weak equivalence, which gives rise to the ubiquitous notion of Morita equivalence.

We begin by clarifying our setting. Throughout this paper, “smooth” means infinitely-differentiable, and all manifolds are smooth and without boundary. By a “curve”, we mean a smooth map \( p: I := (-\varepsilon, \varepsilon) \to M \) where \( M \) is a smooth manifold and \( \varepsilon > 0 \); by “shrinking \( I \)”, we mean taking \( \varepsilon' \in (0, \varepsilon] \) and redefining \( I := (-\varepsilon', \varepsilon') \).

Throughout this work, we will check whether a smooth map is a submersion by looking at properties of curves; in particular, whether they admit local lifts. This allows us to avoid explicitly dealing with tangent bundles.

**Definition 2.1.** A function \( f: M \to N \) satisfies the **local curve lifting (LCL) condition** if for any curve \( p: I \to N \) and \( x \in M \) satisfying \( f(x) = p(0) \), after possibly shrinking \( I \), there exists a (smooth) lift \( q: I \to M \) of \( p \) (restricted to the redefined \( I \)) through \( x \) with respect to \( f \). Explicitly, \( q \) satisfies \( f \circ q = p \) and \( q(0) = x \).

The LCL condition allows us to identify submersions and diffeomorphisms as follows.

**Lemma 2.2.** Let \( M \) and \( N \) be manifolds.

1. A smooth surjection \( f: M \to N \) is a surjective submersion if and only if it satisfies the LCL condition.
2. A smooth map \( f: M \to N \) is a diffeomorphism if and only if \( f \) is bijective and satisfies the LCL condition. Moreover, the LCL condition can be relaxed in this case to finding a lift through any point.

We will also make use of the following well-known fact about fibred products of manifolds, which follows from the Transversality Theorem [15, Theorem 6.30].

**Lemma 2.3.** Let \( f: M \to P \) and \( g: N \to P \) be smooth maps between manifolds, in which \( f \) is a surjective submersion. Then \( M_f \times_g N \) is a manifold and \( \text{pr}_2 \) is a surjective submersion.

With this set, we consider our basic context of Lie groupoids.

**Definition 2.4.** [17] The 2-category of **Lie groupoids**, denoted \( \text{LieGpoid} \), has objects which are groupoids \( \mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0) \) in which all structure maps are smooth: source and target maps \( s, t: \mathcal{G}_1 \to \mathcal{G}_0 \) (which are additionally required to be submersive); the unit map \( u: \mathcal{G}_0 \to \mathcal{G}_1 \); and the inverse map \( \text{inv}: \mathcal{G}_1 \to \mathcal{G}_1 \) (where we indicate the inverse \( \text{inv}(g) \) by \( g^{-1} \)). The 1-cells are smooth functors; that is the maps on both arrows and objects are smooth. The 2-cells are natural transformations between smooth functors that are defined by a smooth map.

For the remainder of this paper, unless stated otherwise, we assume that all of our groupoids are Lie groupoids and that all functors and natural transformations are smooth.
In the study of Lie groupoids, especially orbifolds, actions of groupoids, and stacks, the notion of Morita equivalence is paramount. We recall its definition:

**Definition 2.5.** [17] A functor \( \varphi : \mathcal{G} \to \mathcal{H} \) is a **weak equivalence** (sometimes called an **equivalence** or an **essential equivalence** in the literature) if it satisfies the following two conditions:

1. **Smooth Essential Surjectivity:** The induced map
   \[ ES_\varphi : \mathcal{G}_0 \times \mathcal{H}_1 \to \mathcal{H}_0 : (x, h) \mapsto s(h) \]
   is a surjective submersion.
2. **Smooth Fully Faithfulness:** The induced map
   \[ FF_\varphi : \mathcal{G}_1 \to \mathcal{H}_1 : g \mapsto (s(g), t(g), \varphi(g)) \]
   is a diffeomorphism.

We will denote a weak equivalence with the symbol \( \cong \Rightarrow \), and denote the class of all weak equivalences in \( \text{LieGpoid} \) by \( W \). We say that two Lie groupoids \( \mathcal{G} \) and \( \mathcal{H} \) are **Morita equivalent** if there is a generalised morphism \( \mathcal{G} \cong \leftarrow \varphi \Rightarrow \mathcal{K} \cong \rightarrow \psi \mathcal{H} \) in which both \( \varphi \) and \( \psi \) are weak equivalences.

We have the following properties of weak equivalences.

**Lemma 2.6.** Let \( \varphi : \mathcal{G} \to \mathcal{H} \) and \( \varphi' : \mathcal{H} \to \mathcal{K} \) be functors in \( \text{LieGpoid} \).

1. If any two of \( \varphi, \varphi' \), and \( \varphi' \circ \varphi \) are weak equivalences, then so is the third.
2. If \( \varphi \) is smoothly fully faithful and \( \varphi_0 \) is a surjective submersion, then \( \varphi \) is a weak equivalence.
3. A functor \( \varphi : \mathcal{G} \to \mathcal{H} \) in \( \text{LieGpoid} \) is smoothly fully faithful if and only if for any functors \( \psi, \psi' : \mathcal{K} \to \mathcal{G} \) and natural transformation \( \eta : \varphi \circ \psi \Rightarrow \varphi \circ \psi' \), there exists a unique natural transformation \( \eta' : \psi \Rightarrow \psi' \) such that \( \eta = \varphi \eta' \).

In fact, the weak equivalences as in Item 2 are important enough to this paper to warrant a definition.

**Definition 2.7.** A weak equivalence that is a surjective submersion on objects is called a **surjective submersive weak equivalence**, often denoted using \( \cong \Rightarrow \). (They are also called Morita fibrations in [9].) We denote the class of surjective submersive weak equivalences by \( sW \).

**Proof of Lemma 2.6.** Item 1 is the so-called 3-for-2 property, as proved in [23, Lemma 8.1].

Item 2 appears in [9, Subsection 6.1]; alternatively, the reader can check that \( ES_\varphi \) satisfies the LCL condition and then apply Lemma 2.2.

For Item 3 suppose \( \varphi \) is smoothly fully faithful. Fix functors \( \psi, \psi' : \mathcal{K} \to \mathcal{G} \) and natural transformation \( \eta : \varphi \circ \psi \Rightarrow \varphi \circ \psi' \). Define \( \eta' : \mathcal{K}_0 \to \mathcal{G}_1 \) by \( \eta'(z) := FF_\varphi^{-1}(\psi(z), \psi'(z), \eta(z)) \); this is well-defined and smooth. The fact that \( \varphi \eta' = \eta \) follows from the construction, and uniqueness follows from smooth fully faithfulness of \( \varphi \).
Conversely, suppose for any functors \( \psi, \psi' : \mathcal{K} \to \mathcal{G} \) and natural transformation \( \eta : \varphi \circ \psi \Rightarrow \varphi \circ \psi' \), there exists a unique natural transformation \( \eta' : \psi \Rightarrow \psi' \) such that \( \eta = \varphi \eta' \). Let \( \mathcal{K} \) be the trivial Lie groupoid of a point \(* \Rightarrow *\), and fix \((x, x', h) \in \mathcal{G}_0^2 \times (s, t) \mathcal{H}_1\). Set \( \psi : \mathcal{K} \to \mathcal{G} \) to be the functor sending the point to \( x \), and set \( \psi' : \mathcal{K} \to \mathcal{G} \) the functor sending the point to \( x' \). The natural transformation \( \eta : \varphi \circ \psi \Rightarrow \varphi \circ \psi' \) sending the point to \( h \) factors uniquely as \( \varphi \eta' \). Thus, \( \mathbf{F} \varphi(\eta'(*)) = (x_1, x_2, h) \), and \( \mathbf{F} \varphi \) is bijective.

Now we show that \( \mathbf{F} \varphi \) satisfies the LCL condition. Fix a curve \( p = (x_\tau, x'_\tau, h_\tau) : I \to \mathcal{G}_0^2 \times (s, t) \mathcal{H}_1 \). Let \( \mathcal{K} \) be the trivial Lie groupoid \( I \Rightarrow I \); let \( \psi, \psi' : \mathcal{K} \to \mathcal{G} \) be the functors defined by \( \psi_0 = x_\tau \) and \( \psi'_0 = x'_\tau \); and let \( \eta : \varphi \circ \psi \Rightarrow \varphi \circ \psi' \) defined by the natural transformation \( h_\tau \). There is a unique \( \eta' : \psi \Rightarrow \psi' \) defined by \( \eta'(\tau) = g_\tau \in \mathcal{G}_1 \) such that \( \varphi \eta' = \eta \), and so \( \varphi(g_\tau) = h_\tau \). Thus \( \mathbf{F} \varphi(g_\tau) = p \), showing that \( g_\tau \) gives the required lift. By Item 2 of Lemma 2.2, it follows that \( \mathbf{F} \varphi \) is a diffeomorphism, establishing smooth fully faithfulness of \( \varphi \). This proves Item 3.

We next consider pullbacks of Lie groupoids.

**Definition 2.8.** Let \( \varphi : \mathcal{G} \to \mathcal{K} \) and \( \psi : \mathcal{H} \to \mathcal{K} \) be functors. The **strict pullback** of \( \varphi \) and \( \psi \) is the groupoid \( \mathcal{G}_\varphi \times_\psi \mathcal{H} \), whose object and arrow spaces are the corresponding fibred products of the object and arrow spaces of \( \mathcal{G} \) and \( \mathcal{H} \), respectively, with two projection functors \( \text{pr}_1 \) and \( \text{pr}_2 \).

The strict pullback may not be a Lie groupoid in general. The following proposition provides a sufficient condition for when it is a Lie groupoid.

**Proposition 2.9.** Let \( \varphi : \mathcal{G} \to \mathcal{K} \) and \( \psi : \mathcal{H} \to \mathcal{K} \) be functors in which \( \varphi \in \mathbf{sW} \). Then \( \mathcal{G}_\varphi \times_\psi \mathcal{H} \) is a Lie groupoid and \( \text{pr}_2 \in \mathbf{sW} \).

**Proof.** By Lemma 2.3, the object and arrow spaces of \( \mathcal{G}_\varphi \times_\psi \mathcal{H} \) are manifolds. We verify that the source map of the pullback groupoid is a surjective submersion using the LCL condition. Fix a curve \( p = (x_\tau, y_\tau) : I \to \mathcal{G}_0 \times_\psi \mathcal{H}_0 \) and let \((g_0, h_0) \in \mathcal{G}_1 \times_\psi \mathcal{H}_1 \) such that \( s(g_0, h_0) = (x_\tau, y_\tau) \). After shrinking \( I \), there is a lift \( h_\tau : I \to \mathcal{H}_1 \) of \( y_\tau \) through \( h_0 \) such that \( s(h_\tau) = y_\tau \). Then \( \psi_1(h_\tau) \) defines a curve \( I \to \mathcal{K}_1 \), and since \( \varphi_0 \) is a surjective submersion, after shrinking \( I \) again there is a lift \( x'_\tau : I \to \mathcal{G}_0 \) of this curve through \( t(g_0) \) with \( \varphi(x'_\tau) = \psi(h_\tau) \). The curve \( g_\tau := \mathbf{F} \varphi^{-1}(x_\tau, x'_\tau, \psi(h_\tau)) : I \to \mathcal{G}_1 \) is a lift of \( x_\tau \) through \( g_0 \) such that \( s(g_\tau) = x_\tau \). Thus \( (g_\tau, h_\tau) : I \to \mathcal{G}_1 \times_\psi \mathcal{H}_1 \) is well-defined, and is the desired lift of \((x_\tau, y_\tau)\) through \((g_0, h_0)\) with \( s(g_\tau, h_\tau) = (x_\tau, y_\tau) \) verifying the LCL condition for the source map of the pullback groupoid. By Item 2 of Lemma 2.2, the source map is a surjective submersion, from which it follows that the target map is as well. Thus \( \mathcal{G}_\varphi \times_\psi \mathcal{H} \) is a Lie groupoid.

Next we show that \( \text{pr}_2 \) is in \( \mathbf{sW} \). By Lemma 2.3, the map \( (\text{pr}_2)_0 : (\mathcal{G}_\varphi \times_\psi \mathcal{H})_0 = \mathcal{H}_0 \) is a surjective submersion. Since \( \mathbf{F} \varphi \) is a diffeomorphism, it follows that \( \mathbf{F} \text{pr}_2 \) is bijective. Let \( p = ((x_\tau, y_\tau), (x'_\tau, y'_\tau), h_\tau) : I \to (\mathcal{G}_\varphi \times_\psi \mathcal{H})_0^2 \times (s, t) \mathcal{H}_1 \) be a curve. Define the curve \( g_\tau := \mathbf{F} \varphi^{-1}((x_\tau, x'_\tau), \psi(h_\tau)) \). Then \((g_\tau, h_\tau)\) defines the desired lift of the curve \( p \) and \( \mathbf{F} \text{pr}_2 \) satisfies the LCL condition. Hence \( \mathbf{F} \text{pr}_2 \) is a surjective
submersion. By Item 2 of Lemma 2.2, $\text{FF}_{pr_2}$ is a diffeomorphism. By Item 3 of Lemma 2.6, $pr_2$ is in $\mathcal{sW}$.

We will also be using the weak pullback of groupoids.

**Definition 2.10.** [17] Let $\varphi: \mathcal{G} \to \mathcal{K}$ and $\psi: \mathcal{H} \to \mathcal{K}$ be functors in $\text{LieGpoid}$. The *weak pullback* of $\varphi$ and $\psi$ is the groupoid $\mathcal{G}_{\varphi, \psi} \times \mathcal{H}$, whose object space is

$$
(\mathcal{G}_{\varphi, \psi} \times \mathcal{H})_0 := \left\{ (x, k, y) \in \mathcal{G}_0 \times \mathcal{K}_1 \times \mathcal{H}_0 \mid \varphi(x) \overset{k}{\smile} \psi(y) \right\},
$$

and arrow space is

$$
(\mathcal{G}_{\varphi, \psi} \times \mathcal{H})_1 := \left\{ (g, k, h) \in \mathcal{G}_1 \times \mathcal{K}_1 \times \mathcal{H}_1 \mid \varphi(s(g)) \overset{k}{\smile} \psi(s(h)) \right\}.
$$

The weak pullback comes equipped with two projection functors $pr_1$ and $pr_3$ to $\mathcal{G}$ and $\mathcal{H}$, respectively, and the natural transformation $\text{PR}_2: \varphi \circ pr_1 \Rightarrow \psi \circ pr_3$.

In general, the weak pullback may not be a Lie groupoid. The following proposition, which is [18] Proposition 5.12(iv)], gives a sufficient condition for when it is a Lie groupoid.

**Proposition 2.11.** Let $\varphi: \mathcal{G} \to \mathcal{K}$ and $\psi: \mathcal{H} \to \mathcal{K}$ be functors in which $\varphi$ is a weak equivalence. Then $\mathcal{G}_{\varphi, \psi} \times \mathcal{H}$ is a Lie groupoid and $pr_3 \in \mathcal{sW}$.

The following lemma shows how surjective submersive weak equivalences interact with natural transformations. This property is called “co-fully faithfulness” by Roberts [27, Definition 2.12] and Pronk-Scull [24, Definition 5.1].

**Lemma 2.12.** Given $\varphi: \mathcal{G} \to \mathcal{H}$ be in $\mathcal{sW}$, for any functors $\psi, \psi': \mathcal{H} \to \mathcal{K}$ and natural transformation $\eta: \psi \circ \varphi \Rightarrow \psi' \circ \varphi$, there exists a unique natural transformation $\eta': \psi \Rightarrow \psi'$ such that $\eta = \eta' \varphi$.

**Proof.** Fix functors $\psi, \psi': \mathcal{H} \to \mathcal{K}$ and natural transformation $\eta: \psi \circ \varphi \Rightarrow \psi' \circ \varphi$. Define $\eta': \mathcal{H}_0 \to \mathcal{K}_1$ by $\eta'(y) := \eta(x)$, where $x \in \varphi_0^{-1}(y)$: since $\varphi_0$ is surjective, $\varphi_0^{-1}(y)$ is non-empty. Suppose $\varphi_0(x_1) = \varphi_0(x_2)$. Since $\varphi$ is a weak equivalence, there exists an arrow $g = \text{FF}_{\varphi}^{-1}(x_1, x_2, u_{\varphi(x_1)})$ from $x_1$ to $x_2$. Naturality gives the following commutative diagram

$$
\begin{array}{ccc}
\psi \circ \varphi(x_1) & \overset{\eta_1}{\longrightarrow} & \psi' \circ \varphi(x_1) \\
\psi \circ \varphi(g) \downarrow & & \downarrow \psi' \circ \varphi(g) \\
\psi \circ \varphi(x_2) & \overset{\eta_2}{\longrightarrow} & \psi' \circ \varphi(x_2).
\end{array}
$$

Since $\varphi(g) = u_{\varphi(x_1)}$, we have $\eta_1 = \eta_2$, and so $\eta'$ is well-defined. By construction $\eta = \eta' \varphi$.

To show that $\eta'$ is smooth, fix a curve $p = y_I: I \to \mathcal{H}_0$. After shrinking $I$, there exists a curve $x_I: I \to \mathcal{G}_0$ such that $y_I = \varphi(x_I)$, since $\varphi$ is a surjective submersion. Since $\eta'(y_I) = \eta(x_I)$, we conclude that $\eta'(y_I)$ is a curve in $\mathcal{K}_1$. By Boman’s Lemma [5], $\eta'$ is smooth. The naturality of $\eta'$ follows from that of $\eta$. Finally, uniqueness follows from the construction. 


The following identifies weak equivalences using a property called “J-locally split” in [27, Definition 3.22]; in our case, $J = sW$.

**Lemma 2.13.** A functor $\varphi : \mathcal{G} \to \mathcal{H}$ is a weak equivalence if and only if it is smoothly fully faithful and $sW$-locally split: i.e. there exist a functor $\psi : \mathcal{K} \to \mathcal{H}$ in $sW$, a functor $\sigma : \mathcal{K} \to \mathcal{G}$, and a natural transformation $\eta : \varphi \circ \sigma \Rightarrow \psi$.

**Proof.** Suppose $\varphi$ is a weak equivalence. Choose $\mathcal{K} := \mathcal{G}_{\varphi}^{\perp} \mathcal{H}$ and $\psi = \text{pr}_3$, $\sigma = \text{pr}_1$ and $\eta = \text{PR}_2$. Then $\mathcal{K}$ is a Lie groupoid and $\psi \in sW$ by Proposition [2.11].

Conversely, suppose $\varphi$ is smoothly fully faithful, and there exist $\psi : \mathcal{K} \to \mathcal{H}$ in $sW$, a functor $\sigma : \mathcal{K} \to \mathcal{G}$, and a natural transformation $\eta : \varphi \circ \sigma \Rightarrow \psi$. We will verify that $\text{ES}_\varphi$ is a surjective submersion.

Let $y \in \mathcal{H}_0$. Since $\psi_0$ is surjective, there exists $z \in \mathcal{K}_0$ such that $\psi(z) = y$. Then $(\sigma(z), \eta(z)^{-1}) \in \mathcal{G}_{0,\varphi} \times_\mathcal{H}_1$ and $\text{ES}_\varphi(\sigma(z), \eta(z)^{-1}) = y$. Thus $\text{ES}_\varphi$ is surjective. To show $\text{ES}_\varphi$ is a surjective submersion, fix a curve $p = y_\gamma : I \to \mathcal{H}_0$. Let $(x_0, h_0) \in \mathcal{G}_{0,\varphi} \times_\mathcal{H}_1$ such that $\text{ES}_\varphi(x_0, h_0) = y_0$. Since $\psi_0$ is a surjective submersion, after shrinking $I$, there is a curve $z_\gamma : I \to \mathcal{K}_0$ such that $\psi(z_\gamma) = y_\gamma$. Since $\varphi$ is smoothly fully faithful, $\text{FF}_\varphi$ is a diffeomorphism and so we define $g_0 \in \mathcal{G}_1$ by $\text{FF}_\varphi^{-1}(\sigma(z_0), x_0, h_0 \cdot \eta(z_0))$. Since source maps of Lie groupoids are surjective submersions, we can lift the curve $\sigma(z_\gamma)$ after shrinking $I$ to get a curve $g_\gamma : I \to \mathcal{G}_1$ through $g_0$ such that $s(g_\gamma) = \sigma(z_\gamma)$. Then the desired lift of the curve $p$ through $(x_0, h_0)$ is defined by $(t(g_\gamma), \varphi(g_\gamma) \cdot \eta(z_\gamma)^{-1})$. Thus $\text{ES}_\varphi$ satisfies the LCL condition and is a surjective submersion, completing the verification that $\varphi$ is a weak equivalence. \[\square\]

### 3. Localising Lie Groupoids at Weak Equivalences

In this section we recall how to construct a localised bicategory which inverts weak equivalences in the 2-category $\text{LieGpoid}$, giving us a formal mechanism for working with Morita equivalence classes of groupoids. We will use a bicategory of fractions construction in which the objects are still the Lie groupoids of $\text{LieGpoid}$, but the 1- and 2-cells are adjusted. In particular, the arrows of the bicategory of fractions will be given by so-called “spans” of arrows of $\text{LieGpoid}$, so that we add inverse arrows for any weak equivalences and make all weak equivalences (and hence all Morita equivalences) into isomorphisms of the localised bicategory.

We outline two constructions of this localisation: the first is the bicategory of fractions defined by [21], and a second related but smaller construction based on so-called anafunctors by [27].

We begin by recalling the construction and properties of the localized bicategory of [21].

**Definition 3.1.** The localized bicategory $\text{LieGpoid}[W^{-1}]$ is defined by:

- The objects of $\text{LieGpoid}[W^{-1}]$ are the same as the objects of $\text{LieGpoid}$.
- The arrows are defined by spans of arrows: a **generalised morphism** between Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ is a Lie groupoid $\mathcal{K}$ and two functors $\mathcal{G} \xrightarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}$ in which $\varphi$ is a weak equivalence.
• The identity generalised morphism of \( G \) in \( \text{LieGpoid}[W^{-1}] \) is given by \( G \xleftarrow{} G \).
• Let \( G \xleftarrow{} K \xrightarrow{} H \) and \( H \xleftarrow{} L \xrightarrow{} I \) be generalised morphisms. Define their composition to be the generalised morphism
  \[
  G \xleftarrow{} K \xrightarrow{w} L \xrightarrow{} I.
  \]
• 2-cells between generalised morphisms are detailed below in Definition 3.2.

It follows from Proposition 2.11 and Item 1 of Lemma 2.6 that the composition of two generalised morphisms is a generalised morphism. The composition of generalised morphisms is an associative operation up to a canonical isomorphism.

We think of a generalised morphism \( G \xleftarrow{} K \xrightarrow{} H \) as replacing \( G \) with a weakly equivalent Lie groupoid \( K \) which admits the left functor \( K \xrightarrow{} H \). Thus we consider weakly equivalent groupoids to be interchangeable. Weakly equivalent groupoids are always Morita equivalent, as there is a generalised morphism between them using an identity morphism as one leg. A Morita equivalence \( G \xleftarrow{} K \xrightarrow{} H \) is invertible in \( \text{LieGpoid}[W^{-1}] \), with inverse defined by \( H \xleftarrow{} K \xrightarrow{} G \).

There may be many different choices of groupoids weakly equivalent to \( G \), and we want to recognise when two choices of generalised morphism carry the same geometric information. Thus we define the following 2-cells.

**Definition 3.2.** Given two generalised morphisms \( G \xleftarrow{} K \xrightarrow{} H \) and \( G \xleftarrow{} K' \xrightarrow{} H \), we consider a generalised morphism of the form \( K \xleftarrow{} L \xrightarrow{} K' \) in which both functors are weak equivalences, along with two natural transformations \( \eta_1 : \alpha \circ \alpha \Rightarrow \varphi' \circ \alpha' \) and \( \eta_2 : \psi \circ \alpha \Rightarrow \psi' \circ \alpha' \) making the following diagram 2-commute:

\[
\begin{array}{ccc}
G & \xleftarrow{\phi} & K \\
\downarrow {\eta_1} & & \downarrow \psi \\
\xleftarrow{\varphi} & \xrightarrow{\alpha} & \xleftarrow{\psi} \\
& \xleftarrow{\alpha'} & \xrightarrow{\psi'} \\
& & \xrightarrow{\eta_2} \\
K' & \xrightarrow{\psi'} & H
\end{array}
\]

We denote a diagram (1) by the quadruple \((\alpha, \alpha', \eta_1, \eta_2)\). A 2-cell from \( G \xleftarrow{} K \xrightarrow{} H \) to \( G \xleftarrow{} K' \xrightarrow{} H \) is an equivalence class of such diagrams of form (1): \((\alpha, \alpha', \eta_1, \eta_2)\) is equivalent to another such diagram \((\beta, \beta', \mu_1, \mu_2)\) if there exists a third generalised morphism \( L \xleftarrow{} N \xrightarrow{} M \) and natural transformations \( \nu_1 : \alpha \circ \gamma \Rightarrow \beta \circ \gamma' \) and \( \nu_2 : \alpha' \circ \gamma \Rightarrow \beta' \circ \gamma' \) such that

\[
(\mu_1 \gamma') \circ (\varphi \nu_1) = (\varphi' \nu_2) \circ (\eta_1 \gamma) \quad \text{and} \quad (\mu_2 \gamma') \circ (\psi \nu_1) = (\psi' \nu_2) \circ (\eta_2 \gamma).
\]
We denote the 2-cell given by the equivalence class by \([\alpha, \alpha', \eta_1, \eta_2]\). The identity 2-cell of a generalised morphism \(G \leftrightarrow K \rightarrow H\) is given by \([\text{id}_K, \text{id}_K, \text{ID}_\phi, \text{ID}_\psi]\) where ID represents the identity natural transformation.

Vertical composition of 2-cells in \(\text{LieGpoid}[\mathcal{W}^{-1}]\) is performed using a weak pullback, and horizontal compositions are defined using whiskering operations. Unitors for this bicategory are defined using projection maps. Explicit descriptions of 2-cell compositions in \(\text{LieGpoid}[\mathcal{W}^{-1}]\) can be found in [21, Subsection 2.3] or [24, Section 3].

The bicategory \(\text{LieGpoid}[\mathcal{W}^{-1}]\) inverts all weak equivalences and satisfies the universal property of a localisation.

**Proposition 3.3.** [21, Section 2] The bicategory \(\text{LieGpoid}[\mathcal{W}^{-1}]\) satisfies the universal property of a localisation: any functor from \(\text{LieGpoid}\) to another bicategory which takes 1-cells in \(\mathcal{W}\) to invertible 1-cells will factor through this localised bicategory \(\text{LieGpoid}[\mathcal{W}^{-1}]\).

On a practical level, this bicategory can be hard to work with on the 2-cell level, since the 2-cells are defined as equivalence classes of diagrams. We now describe an alternate localised bicategory \(\text{AnaLieGpoid}\) developed in [25, 26, 27] which is “smaller” than \(\text{LieGpoid}[\mathcal{W}^{-1}]\) described in the previous section. The fact that this smaller construction applies in the category of Lie groupoids is well-known to experts but tracking down exact references has proved difficult.

The starting point to understanding the difference between \(\text{LieGpoid}[\mathcal{W}^{-1}]\) and \(\text{AnaLieGpoid}\) is in looking at the subclass \(\mathcal{S}\mathcal{W}\) of \(\mathcal{W}\) comprising surjective submersive weak equivalences. A priori we will create a localisation which inverts only this subclass \(\mathcal{S}\mathcal{W}\). It will turn out that the resulting localised bicategories are equivalent.

We start creating \(\text{AnaLieGpoid}\) using objects of \(\text{LieGpoid}\) as before, but look only at generalised morphisms which use a surjective submersive weak equivalence as their left leg. These are called anafunctors in [27].

**Definition 3.4.** The category \(\text{AnaLieGpoid}\) is defined as follows:

- objects are the same as the objects of \(\text{LieGpoid}\).
- a 1-cell is given by an anafunctor, a generalised morphism \(G \leftrightarrow K \rightarrow H\) such that \(\varphi \in \mathcal{S}\mathcal{W}\).
- The identity generalised morphism of Definition 3.1 is an anafunctor, and so it defines the identity anafunctor of \(G\).
- Composition is defined using the strict pullback: Let \(G \leftrightarrow K \rightarrow H\) and \(H \leftrightarrow \mathcal{L} \rightarrow \mathcal{I}\) be anafunctors. Define their composition to be the anafunctor

\[
G \leftrightarrow K \psi \times \mathcal{L} \longrightarrow \mathcal{I}.
\]
- the 2-cells are defined by natural transformations (not equivalence classes of diagrams): Given two anafunctors \( G \leftarrow \varphi K \rightarrow \psi H \) and \( G' \leftarrow \varphi' K' \rightarrow \psi' H \) a 2-cell between them is a natural transformation \( \eta \) making the following diagram 2-commute.

- The identity 2-cell of an anafunctor \( G \leftarrow \varphi K \rightarrow \psi H \) is given by the natural transformation

  \[
  \iota_{G \leftarrow K \rightarrow H} : (K \times \varphi K) \rightarrow H_1 : (y_1, y_2) \mapsto \psi(\mathbb{F} \varphi^{-1}((y_1, y_2), u\varphi(y_1))).
  \]

It follows from Proposition 2.9 Item 1 of Lemma 2.6 and the fact that the composition of surjective submersions is again a surjective submersion that the composition of two anafunctors is an anafunctor. Similar to the case of generalised morphisms, composition of anafunctors is an associative operation, again up to a canonical isomorphism.

We have chosen a canonical representative of a 2-cell defined by a particular natural transformation which will be the 2-cell in our new bicategory \( \text{AnaLieGpoid} \). Thus the 2-cells are actual natural transformations rather than equivalence classes as in \( \text{LieGpoid}[W^{-1}] \); these anafunctor 2-cells can be drawn in the diagram form (1) of the representatives of 2-cells of \( \text{LieGpoid}[W^{-1}] \) of Definition 3.2 by adding the left side of the diagram with the trivial natural transformation. However, in \( \text{AnaLieGpoid} \), the vertical and horizontal compositions are not the usual composition of natural transformations. We do not need the details of these compositions until Section 5, so we will defer the relevant details.

Both of the constructions we have just outlined create localisations of \( \text{LieGpoid} \) at \( W \). To compare \( \text{AnaLieGpoid} \) to \( \text{LieGpoid}[W^{-1}] \), we again apply a result of Roberts; be warned that what is called a “weak equivalence” in his paper [27] is defined there to be a functor that is \( sW \)-locally split and representably fully faithful. However, by Item 3 of Lemma 2.6 and Lemma 2.13, this is equivalent to smooth essential surjectivity and smooth fully faithfulness, and so our notion of weak equivalence coincides with his. Thus we have [27, Theorem 3.24]:

**Proposition 3.5.** The inclusion \( \text{AnaLieGpoid} \rightarrow \text{LieGpoid}[W^{-1}] \) is an equivalence of bicategories, where this inclusion takes a 2-cell to its equivalence class.

Proposition 3.5 implies that any generalised morphism \( G \leftarrow \varphi K \rightarrow \psi H \) admits a 2-cell from itself to an anafunctor; in the proof, this is the anafunctor \( G \leftarrow \varphi K \rightarrow \psi \text{id}_G \times \varphi K \rightarrow \psi \text{pr}_3 H \). It also implies that there is a 2-cell from a composition of generalised morphisms in \( \text{LieGpoid}[W^{-1}] \), defined by the weak pullback, to the corresponding composition in \( \text{AnaLieGpoid} \) using the strict pullback. This can be constructed explicitly as a vertical composition using the following, which we use in Section 5.
Proposition 3.6. Let $\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}$ and $\mathcal{H} \xleftarrow{\chi} \mathcal{L} \xrightarrow{\omega} \mathcal{I}$ be generalised morphisms where $\chi \in s\mathcal{W}$ so that the second generalised morphism is an anafunctor. There is a 2-cell from the composition defined using the weak pullback in $\text{LieGpoid}[W^{-1}]$ of Definition 3.1 to the composition defined using the strict pullback in $\text{AnaLieGpoid}$ defined in Definition 3.4.

Proof. We define a 2-cell between the two compositions using the following strictly commutative diagram:

Here, inc is the inclusion functor defined on objects by inc$_0$: $(x, y) \mapsto (x, u_\psi(x), y)$ and on arrows by inc$_1$: $(k, \ell) \mapsto (k, u_{\psi}(k), \ell)$. We check that inc is a weak equivalence. It is straightforward to check that FF$_{\text{inc}}$ is bijective. To show it is a diffeomorphism, we check the LCL condition: a curve $p: I \to (\mathcal{K}_x \times \mathcal{L})^2_{\text{inc}2 \times (s,t)} (\mathcal{K}_y \times \mathcal{L})^1_1$ has the form $p(t) = ((x_t, y_t), (x'_t, y'_t), (k_t, u_{x_t}, \ell_t))$ and so $p = \text{FF}_\text{inc}(k_t, \ell_t)$ and $(k_t, \ell_t)$ gives the desired lift. By Item 2 of Lemma 2.2 FF$_{\text{inc}}$ is a diffeomorphism.

Let $(x, h, y) \in (\mathcal{K}_x \times \mathcal{L})^0_0$. Since $\chi_0$ is surjective, there exists $y' \in \mathcal{L}^0_0$ such that $\chi(y') = \psi(x)$. Define $\ell := \text{FF}_\chi^{-1}(y, y', h^{-1})$. Then ES$_\text{inc}((x, y'), (u_x, h, \ell)) = (x, h, y)$, and so ES$_\text{inc}$ is surjective.

To show that ES$_\text{inc}$ is a surjective submersion, we again use the LCL condition: let $p = (x_t, h_t, y_t): I \to (\mathcal{K}_x \times \mathcal{L})^0_0$ be a curve and suppose we have a point $((x'_t, y'_0), (k_0, h_0, \ell_0)) \in ((\mathcal{K}_x \times \mathcal{L})^0_0)_{\text{inc}2 \times (s,t)} (\mathcal{K}_x \times \mathcal{L})^1_1$ sent by ES$_\text{inc}$ to $(x_0, h_0, y_0)$. By definition, we must have that $t(k_0) = x_0$, $t(\ell_0) = y'_0$, and $\chi(\ell_0)h_0\psi(k_0)^{-1} = u_x(y'_0)$. Since $\mathcal{K}$ is a Lie groupoid, its source map is a surjective submersion and so after shrinking $I$, there is a curve $k_t: I \to \mathcal{K}_s$ through $k_0$ with $s(k_t) = x_t$; denote the target $t(k_t) = x'_t$, and note that $u_x(x'_t) = t(\psi(k_0)) = t(\chi(\ell_0)h_0) = t(\ell_0)$. Next, since $\chi_0$ is a surjective submersion we can lift $t(\psi(k_0)) = \psi(x'_t)$ through $t(\ell_0)$ to get $y'_t$. Thus we have $(y_t, y'_t, \psi(k_t)h_t^{-1}) \in \mathcal{L}^2_0 \times (s,t) \mathcal{H}_1$ and so we can define $\ell_t = \text{FF}_\chi^{-1}(y_t, y'_t, \psi(k_t)h_t^{-1})$. Then $q = (x_t, y'_t, (k_t, h_t, \ell_t))$ is the desired lift of $p$ with ES$_\text{inc}(q) = p$.

By Item 1 of Lemma 2.6 since $\varphi$ is a weak equivalence, $\varphi \circ \text{pr}_1$ is a weak equivalence (in both instances it appears in the diagram above) provided pr$_1$ is. But by Proposition 2.11 pr$_1: \mathcal{K}_x \times \mathcal{L} \to \mathcal{K}$ is a weak equivalence since $\chi$ is, and by Proposition 2.9 pr$_1: \mathcal{K}_x \times \mathcal{L}$ is a weak equivalence (in fact, it is in sW) since $\chi$ is in sW. Thus the diagram above represents an equivalence between the two generalised morphisms. 

\qed
Our main interest in this paper is in Lie groupoids which come from the smooth action of a Lie group on a manifold. For the rest of the paper, we will focus on these, and so all group actions are assumed to be Lie group actions henceforth.

Recall that the action groupoid of a Lie group action $G \ltimes X$ is defined to be the Lie groupoid with object space $X$; arrow space $G \times X$; source and target given by the second projection map and the action map, resp.; multiplication $(g_1, g_2 x)(g_2, x) = (g_1 g_2, x)$; unit at $x$ given by $(1_G, x)$; and inverse $(g, x)^{-1} = (g^{-1}, gx)$.

We are interested in looking at action groupoids with various special properties which commonly come up in contexts such as the study of orbifolds, symplectic geometry, and bundle theory. Recall that a group action is free if all stabilisers are trivial, locally free if there is a neighbourhood $U$ of $1_G$ in $G$ such that the restriction of the action to $U$ is free, transitive if for each pair $x, y \in X$ there exists $g \in G$ such that $gx = y$, and effective if for each $g \neq 1_G \in G$ there exists $x \in X$ such that $gx \neq x$. We will apply these adjectives to both the action and the corresponding action groupoid. We will also refer to the action as compact (resp. discrete) if the corresponding group is compact (resp. discrete). A Lie groupoid $\mathcal{G}$ is proper if the map $\mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$: $g \mapsto (s(g), t(g))$ is a proper map. In particular, if $\mathcal{G}$ is an action groupoid, then we say that the corresponding action is proper. Finally, an étale groupoid is a Lie groupoid whose source (and hence target) is a local diffeomorphism, and an orbifold groupoid is a Lie groupoid that is Morita equivalent to a proper étale groupoid.

Remark 4.1. There are subtle differences in how “orbifold groupoid” is defined in the literature. Our definition above matches that of Pronk-Scull [22, Definition 2.7]. However, others refer to only proper étale groupoids as “orbifold groupoids”; see, for instance, [3, 12]. Further, some authors restrict their attention to effective orbifolds [19, 18]. It follows from the Slice Theorem and associated Tube Theorem [10, Theorems 2.3.3, 2.4.1] that a proper and locally free group action corresponds to an orbifold groupoid as in the definition above. Conversely, if an action groupoid is Morita equivalent to a proper étale groupoid, then it is proper and locally free. This follows from the fact that weak equivalences, and hence Morita equivalences, preserve stabilisers and properness (see, for instance, [17, Subsection 2.7], [8, Proposition 5.1.5], or [30, Proposition 2.2]).

We will localise a specified sub-2-category of Lie groupoids whose objects are action groupoids that satisfy a desired set of properties $\mathcal{P}$; this is Proposition 4.12. Our selection of properties comes from those defined above. Specifically, $\mathcal{P}$ is any subset (possibly empty) of the following list of properties (acknowledging that some combinations are redundant; see Remark 4.1):

$$
\mathcal{P} \subseteq \left\{ \text{free, locally free, transitive, effective, compact, discrete, proper, is an orbifold groupoid} \right\}.
$$

Remark 4.2. Several of the properties in $\mathcal{P}$ are known to be Morita invariants; that is, preserved by weak equivalences, and hence Morita equivalences: free, locally free, transitive, proper, and being an orbifold groupoid. See, for instance, [8, Theorem 4.3.1, Proposition 4.3.2].
5.1.5] or [30, Proposition 2.2]. This fact will be important in some proofs below. In fact, we could add the property of being Morita equivalent to any class of groupoids into $\mathcal{P}$, and all of the following results will continue to hold.

Similar results to Proposition[4.12] already appear in the literature for a few specific subclasses of Lie groupoids. For instance, in [21], Pronk localises étale Lie groupoids using the bicategory of fractions outlined in Section 3, and in [25], Roberts localises Lie groupoids, proper Lie groupoids, étale Lie groupoids, and étale proper Lie groupoids using the method of anafunctors of Section 3.

We generalise these scattered results below by considering action groupoids satisfying the properties $\mathcal{P}$. In fact, we go further. We begin by considering action groupoids with so-called equivariant functors between them.

**Definition 4.3.** Let $\text{ActGpd}_\mathcal{P}$ be the sub-2-category of $\text{LieGpoid}$ whose objects are action Lie groupoids $\mathcal{G} = G \ltimes X$ satisfying $\mathcal{P}$, 1-cells are equivariant functors (a functor $\varphi: G \ltimes X \to H \ltimes Y$ is equivariant if there exists a Lie group homomorphism $\tilde\varphi: G \to H$ and a smooth map $\varphi_0: X \to Y$ such that the functor satisfies

$$\varphi_1(g,x) = (\tilde\varphi(g), \varphi_0(x))$$

for all $(g,x) \in G \times X$), and 2-cells are natural transformations. Denote by $\mathcal{W}_\mathcal{P}$ the class of equivariant weak equivalences and $\mathcal{SW}_\mathcal{P}$ the class of equivariant surjective submersive weak equivalences.

Equivariant surjective submersive weak equivalences take on a very special form.

**Lemma 4.4.** Let $G \ltimes X$ and $H \ltimes Y$ be action groupoids and let $\varphi: G \ltimes X \to H \ltimes Y$ be in an equivariant surjective submersive weak equivalence. There is a closed normal subgroup $K \trianglelefteq G$ that acts freely and properly on $X$ and an equivariant isomorphism of Lie groupoids $\psi: K\backslash G \ltimes K\backslash X \rightarrow\leftarrow H \ltimes Y$ satisfying $\varphi = \psi \circ \pi$ where $\pi: K \ltimes X \to K\backslash G \ltimes K\backslash X$ is the quotient functor. Moreover, if $G \ltimes X$ satisfies $\mathcal{P}$ (excluding the property “effective”) then $\psi$ is a morphism in $\text{ActGpd}_\mathcal{P}$.

**Proof.** It follows from the smooth fully faithfulness of $\varphi$ that the group homomorphism $\tilde\varphi: G \to H$ is an epimorphism. Let $K = \ker \tilde\varphi$. By the First Isomorphism Theorem for groups, the continuity of $\varphi$, and [10, Corollaries 1.10.10, 1.11.5], there is an induced Lie group isomorphism $\tilde\psi: K\backslash G \rightarrow H$.

It also follows from the smooth fully faithfulness of $\varphi$ that $K$ acts freely on $X$; these in turn imply that $K \ltimes X$ is Lie groupoid isomorphic to the submersion groupoid $X_{\varphi \times \varphi}X \rightarrow X$ via the map $(s,t): K \times X \to K \ltimes X \times X$. Since submersion groupoids are proper, $K$ acts on $X$ freely and properly. It follows from the Quotient Manifold Theorem (see, for instance, [10, Theorem 1.11.4]) that $K\backslash G \ltimes K\backslash X$ is an action Lie groupoid whose action is induced by the residual action of $G$ on $K\backslash G$.

Equivariance of $\varphi$ yields the identity $\varphi(kx) = \varphi(x)$ for all $k \in K$ and $x \in X$, from which it follows that $\psi_0([x]) := \varphi(x)$ is well-defined and smooth for all $[x] \in K\backslash X$. Injectivity of
ψ₀ follows from the smooth fully faithfulness of φ, and surjective submersivity of ψ₀ follows from that of φ. The smooth functor ψ := (ψ, ψ₀) is a Lie groupoid isomorphism, as desired.

By Remark 4.2 if G × X is free, locally free, transitive, proper, or is an orbifold groupoid, then so is H × Y. If G is compact or discrete, then it is immediate that H satisfies the same property. It follows that if G × X is in \textbf{ActGpd}_p, then so is ψ.

An equivariant weak equivalence in \textbf{W}_p also has a special form: it decomposes into the composition of an equivariant inclusion functor with an equivariant surjective submersive weak equivalence, which are in \textbf{W}_p and \textbf{sW}_p, resp. This decomposition was originally observed in [22] in the proper étale case. The discrete case of the decomposition already appears in [6, Theorem 5.4] in the context of discrete dynamical systems.

Before proving the decomposition, we first show that the inclusion functor appearing in the decomposition is a weak equivalence. Recall that if K is a closed Lie subgroup of G and K × X is a Lie group action, then the anti-diagonal action of K on G × X is free and proper and so the quotient G ×_K X is a manifold by the Quotient Manifold Theorem [10, Theorem 1.11.4].

**Lemma 4.5.** Given a closed subgroup K ≤ G and an action groupoid K × X, the induced inclusion functor i: K × X → G × (G ×_K X) sending (k, x) to (k, [1_G, x]) is an equivariant weak equivalence. If either action groupoid is free, locally free, transitive, proper, or an orbifold groupoid, then so is the other. Moreover, if K × X is effective; then so is G × (G ×_K X); if G is compact or discrete, then K has the same property.

**Proof.** The manifold G ×_K X comes equipped with a G-action g[g', x] := [gg', x], for which i is an equivariant functor with respect to the K- and G-actions. Since the quotient map G × X → G ×_K X is a principal K-bundle, it follows that i is a weak equivalence. If either groupoid is free, locally free, transitive, proper, or an orbifold groupoid, so is the other by Remark 4.2.

If the G-action is not effective, then there is some g' ∈ G contained in every stabiliser Stab_G([g, x]) for [g, x] ∈ G ×_K X. Since Stab_G([g, x]) = gStab_K(x)g⁻¹ for every [g, x] ∈ G ×_K X it follows that g' ∈ Stab_K(x) for every x ∈ X, in which case the K-action is not effective. Finally, if G is compact or discrete, it is immediate that K also shares this property.

We now establish the decomposition of an equivariant weak equivalence.

**Theorem 4.6.** Let G × X and H × Y satisfy properties \textbf{P} (except for “effective”), and let φ: G × X → H × Y be an equivariant weak equivalence in which \bar φ: G → H is proper. Then φ factors as i ◦ π where π ∈ \textbf{sW}_p, and i ∈ \textbf{W}_p is an inclusion functor of the form as in Lemma 4.5.

**Proof.** Claim 1: ker(\bar φ)\nobreak {\G} is a Lie group isomorphic to im(\bar φ).

Indeed, ker(\bar φ)\nobreak {\G} is a Lie group [10, Corollary 1.11.5 and Proposition 1.11.8], and \bar φ descends to a smooth bijective homomorphism \bar φ: ker(\bar φ)\nobreak {\G} → im(\bar φ). Since \bar φ is proper, it is closed,
and so \( \text{im}(\tilde{\varphi}) \) is a closed subgroup of \( H \), and hence a Lie subgroup \([10] \text{ Corollary 1.10.7}\). By \([10] \text{ Corollary 1.10.10}\), \( \tilde{\varphi} \) is a Lie group isomorphism.

**Claim 2:** \( \varphi_0(X) \) is an injectively immersed submanifold diffeomorphic to \( \ker(\tilde{\varphi})\backslash X \).

It follows from the equivariance of \( \varphi \) and the injectivity of \( \mathbf{FF}_\varphi \) that \( \ker(\tilde{\varphi}) \) acts freely on \( X \). Since \( \tilde{\varphi} \) is proper, \( \ker(\tilde{\varphi}) \) is a compact submanifold of \( G \). Thus \( \ker(\tilde{\varphi})\backslash X \) is a manifold \([10] \text{ Theorem 1.11.4}\). Since \( \varphi_0 \) is \( \ker(\tilde{\varphi}) \)-invariant, it descends to a smooth surjection \( \psi: \ker(\tilde{\varphi})\backslash X \to \varphi_0(X) \). Now suppose \( x, x' \in X \) such that \( \psi([x]) = \psi([x']) \). Then \( \varphi_0(x) = \varphi_0(x') \), and since \( \mathbf{FF}_\varphi \) is a diffeomorphism, there exists a (unique) \( k \in \ker(\tilde{\varphi}) \) such that \( x = k \cdot x' \). It follows that \( \psi \) is a smooth bijection onto \( \varphi_0(X) \).

Fix a curve \( p = y_r: I \to \varphi_0(X) \). Shrinking \( I \), since \( \mathbf{ES}_\varphi \) is surjective submersive, there is a lift \( q = (x_r, (h_r, y_r)): I \to X_{\varphi_0\times I}(H \times Y) \) of \( p \). By the smooth fully faithfulness of \( \varphi \), the curve \( h_r \) is contained in \( \text{im}(\tilde{\varphi}) \). By Claim 1, we identify \( \text{im}(\tilde{\varphi}) \) with \( \ker \varphi^G \), and since \( G \to \ker(\varphi) \backslash G \) is a principal (\( \ker(\varphi) \))-bundle, after shrinking \( I \) again, there is a lift \( g_t \) of \( h_t \) to \( G \). But then \( y_t = \psi((s(g_t))_{\ker(\varphi)}) \), which proves that \( \psi \) is an immersion. This proves Claim 2.

By Claim 2 and Lemma 4.4, \( \pi := (\tilde{\varphi}, \varphi_0): G \times X \to \ker(\varphi)\backslash G \times \varphi_0(X) \) is an equivariant weak equivalence. It is straightforward to check that \( (\tilde{\varphi}, \psi) \) is an isomorphism of Lie groupoids between \( K^{G \times K}\backslash X \) and \( \text{im}(\tilde{\varphi}) \times \varphi_0(X) \); we identify these. Let \( i = (i_{\text{im}(\tilde{\varphi})}, i_{\text{im}(\varphi_0)}) \), where the two components are inclusions of the images into \( H \) and \( Y \), resp. By Claim 1, we have the following factorisation

\[
G \times X \xrightarrow{\pi} \text{im}(\tilde{\varphi}) \times \varphi_0(X) \xrightarrow{i} H \times Y.
\]

To obtain the desired decomposition, by Lemma 4.5 it suffices to show that \( Y \) is \( H \)-equivariantly diffeomorphic to \( H \times_{\text{im}(\tilde{\varphi})} \varphi_0(X) \).

Define \( \chi: H \times_{\text{im}(\tilde{\varphi})} \varphi_0(X) \to Y \) to be the smooth map given by \( \chi([h, \varphi_0(x)]) := h \cdot \varphi_0(x) \). Suppose \( \chi([h, \varphi_0(x)]) = \chi([h', \varphi_0(x')]) \). Since \( \mathbf{FF}_\varphi \) is a diffeomorphism, there exists a unique \( g \in G \) such that \( \mathbf{FF}_\varphi(g, x) = (x, x', (h')^{-1}h, \varphi_0(x')) \), and so \( x' = g \cdot x \) and \( \tilde{\varphi}(g) = (h')^{-1}h \). Thus \( [h, \varphi_0(x)] = [h', \varphi_0(x')] \), from which it follows that \( \chi \) is injective. For a fixed \( y \in Y \), since \( \mathbf{ES}_\varphi \) is surjective, there exists \( (x, (h, y)) \in X_{\varphi_0 \times I}(H \times Y) \) with \( (h, y) : y \to \varphi(x) \) and thus \( y = h^{-1} \varphi(x) = \chi(h^{-1}, \varphi(x)) \). Thus \( \chi \) is bijective.

Let \( p = y_r: I \to Y \) be a curve. Since \( \mathbf{ES}_\varphi \) is a surjective submersion, after shrinking \( I \), there is a lift \( q = (x_r, (h_r, y_r)) \) of \( p \) to \( X_{\varphi_0 \times I}(H \times Y) \). The curve \( [h_r^{-1}, \varphi(x_r)] \) has image \( p \) via \( \chi \), and thus \( \chi \) is an immersion. Since immersive bijections are diffeomorphisms, this shows that \( \varphi \) decomposes into \( i \circ \pi \) as desired.

It remains to show that the domain of \( i \) is an action groupoid satisfying \( \mathcal{P} \) (except for “effective”). But this follows from the preservation of these properties as stated in Lemmas 4.4 and 4.5.

We now use the recipe of Roberts \([27]\) to produce a localisation \( \text{AnaLieGpoid}_\mathcal{P} \) of \( \text{ActGpd}_\mathcal{P} \) at \( W_\mathcal{P} \), which has the equivariant anafunctors as 1-cells. Using \([27] \text{ Theorem} \)
3.24], we show that this bicategory is equivalent to $\text{ActGpd}_P[W^{-1}]$. Finally, we show that both of these bicategories are equivalent to $\text{LieGpoid}[W^{-1}]_P$, the full sub-bicategory of $\text{LieGpoid}[W^{-1}]$ whose objects are action groupoids satisfying $P$.

We begin by constructing our localisation of $\text{ActGpd}_P$ at $W_P$.

**Definition 4.7.** A $P$-anafunctor is a generalised morphism

\[ G \ltimes X \leftarrow \varphi \rightarrow \psi \rightarrow K \ltimes Y \rightarrow H \ltimes Z \]

in which all groupoids involved are action groupoids satisfying $P$, with $\psi$ equivariant and $\varphi \in sW_P$.

We will compose $P$-anafunctors using the strict pullback. So we need to verify that $\text{ActGpd}_P$ is also closed under strict pullbacks.

**Lemma 4.8.** Let $G = G \ltimes X$, $H = H \ltimes Y$, and $K = K \ltimes Z$, and let $\varphi: G \to K$ and $\psi: H \to K$ be equivariant functors in $\text{ActGpd}_P$ (with their defining group maps $\varphi, \psi$) such that $\varphi \in sW_P$. The strict pullback groupoid $G_\varphi \times_\psi H$ is an action groupoid of a $(G_\varphi \times_\psi H)$-action that satisfies properties $P$, and $\text{pr}_1$ and $\text{pr}_2$ are equivariant with respect to the restricted projection homomorphisms from $G_\varphi \times_\psi H$ with $\text{pr}_2 \in sW_P$.

**Proof.** By Proposition 2.9, $G_\varphi \times_\psi H$ is a Lie groupoid and $\text{pr}_2 \in sW$. Since $G_\varphi \times_\psi H$ is a closed subgroup of the Lie group $G \times H$, it is a Lie subgroup [10, Corollary 1.10.7]. It is straightforward to check that $G_\varphi \times_\psi H$ is isomorphic to $(G_\varphi \times_\psi H) \ltimes (X_{\varphi_0} \times_{\psi_0} Y)$. The restricted projection functors $\text{pr}_1$ and $\text{pr}_2$ from $G_\varphi \times_\psi H$ are equivariant with respect to the restricted projection functors on $G_\varphi \times_\psi H$.

If $H$ is free, locally free, transitive, proper, or an orbifold groupoid, then so is $G_\varphi \times_\psi H$ since $\text{pr}_2$ is a weak equivalence, and these properties are Morita invariants; see Remark 4.2.

If $G$ and $H$ are compact/discrete, then so is $G_\varphi \times_\psi H$. An examination of the stabilisers of the $(G_\varphi \times_\psi H)$-action on $X_{\varphi_0} \times_{\psi_0} Y$ yields that if the $G$- and $H$-actions are effective, then so is the $(G_\varphi \times_\psi H)$-action. \hfill $\square$

**Remark 4.9.** Lemma 4.8 in fact does not require $K$ to satisfy $P$; this property was not used in the proof.

Thus we know that we can define the composition of $P$-anafunctors using the strict pullback and get another $P$-anafunctor. We can now construct a bicategory localising $\text{ActGpd}_P$ following the anafunctor method of Section 3, provided $sW_P$ is a so-called “bi-faithfully singleton strict pretopology”, using the terminology of Roberts [27, Definitions 2.9, 2.12]. We have already verified that $sW_P$ satisfies the conditions this entails: All identity arrows are in $sW_P$, which is immediate. $sW_P$ must be closed under strict pullback, which is Lemma 3.8. $sW_P$ must be closed under composition, which follows from Item [1] of Lemma 2.6 and the fact that surjective submersions and equivariant maps are closed under composition. Finally, elements of $sW_P$ inherit “representably fully faithfulness” (equivalent to smoothly fully faithfulness, which is Item [3] of Lemma 2.6) and co-faithfully faithfulness (Lemma 2.12) from $\text{AnaLieGpoid}$. Thus, by [27, Theorem 3.20] we have:
Proposition 4.10. There is a bicategory $\text{AnaActGpd}_P$ whose objects are those of $\text{ActGpd}_P$, arrows are $P$-anafunctors, and 2-cells are the natural transformations of Definition 3.4.

To compare $\text{AnaActGpd}_P$ to $\text{ActGpd}_P[W_P^{-1}]$ using Roberts’ setup, we have to confirm that “weak equivalences” as defined by Roberts are the same as ours here. Again, “representable fully faithfulness” is the same as smooth fully faithfulness by Item 3 of Lemma 2.6. We check the $sW_P$-locally split condition using the following lemma showing that $\text{ActGpd}_P$ admits weak pullbacks, from which the required $sW_P$-locally split condition follows.

Lemma 4.11. Let $\mathcal{G} = G \ltimes X$ and $\mathcal{H} = H \ltimes Y$ be objects in $\text{ActGpd}_P$, and let $\varphi: \mathcal{G} \to \mathcal{K}$ and $\psi: \mathcal{H} \to \mathcal{K}$ be functors with $\varphi \in W$. Then the weak pullback $\mathcal{G}_\varphi \times_\psi \mathcal{H}$ is isomorphic to an action groupoid of a $(G \times H)$-action on its object space $Z$ that satisfies properties $P$, and $\text{pr}_1$ and $\text{pr}_3$ are equivariant with respect to the projection homomorphisms $G \times H \to G$ and $G \times H \to H$ respectively.

Proof. By Proposition 2.11, $\mathcal{G}_\varphi \times_\psi \mathcal{H}$ is a Lie groupoid and $\text{pr}_3 \in sW$. Let $Z$ be its object space. The action of $G \times H$ on $Z$ given by

$$(g, h, (x, k, y)) \mapsto (g x, \psi(h, y) k \varphi(g, x)^{-1}, hy)$$

yields an action groupoid canonically isomorphic as a Lie groupoid to $(G \times H) \ltimes Z$ to $\mathcal{G}_\varphi \times_\psi \mathcal{H}$, and $\text{pr}_1$ and $\text{pr}_3$ are equivariant with respect to the projection homomorphisms $G \times H \to G$ and $G \times H \to H$ respectively.

If $\mathcal{H}$ is free, locally free, transitive, proper, or an orbifold groupoid, then so is $\mathcal{G}_\varphi \times_\psi \mathcal{H}$ by Remark 4.2. If $G$ and $H$ are compact/discrete, then so is $G \times H$. Finally, an examination of the stabilisers of the $(G \times H)$-action on $Z$ immediately yields that if the $G$- and $H$- actions are effective, then so is the $(G \times H)$-action. □

That elements of $W_P$ are $sW_P$-locally split now follows from Lemma 2.13 and Lemma 4.11.

It now follows from [27, Theorem 3.24] that:

Proposition 4.12. The inclusion $\text{AnaActGpd}_P \to \text{ActGpd}_P[W_P^{-1}]$ is an equivalence of bicategories, where this inclusion takes a 2-cell to its equivalence class.

5. The Equivalence of $\text{AnaActGpd}_P$ and $\text{LieGpoid}[W^{-1}]_P$

In the previous section, we constructed a bicategory $\text{AnaActGpd}_P$ out of the action groupoids satisfying the chosen properties $P$, with 1-cells given by $P$-anafunctors. In this section, we prove Theorem 5.7 which states that $\text{AnaActGpd}_P$ and $\text{ActGpd}_P[W_P^{-1}]$ are equivalent to the full sub-bicategory $\text{LieGpoid}[W^{-1}]_P$ of $\text{LieGpoid}[W^{-1}]$ whose objects are exactly the action groupoids satisfying $P$. This allows us to replace any generalised morphism between two such groupoids with one from either $\text{AnaActGpd}_P$ or $\text{ActGpd}_P[W_P^{-1}]$: these all admit 2-cells between each other in $\text{LieGpoid}[W^{-1}]$.

The objects of $\text{AnaActGpd}_P$ and $\text{LieGpoid}[W^{-1}]_P$ are the same, and every 1-cell of $\text{AnaActGpd}_P$ is a particular kind of generalised morphism, thus defining a 1-cell in
Definition 5.1. Define \( I_P : \text{AnaActGpd}_P \to \text{LieGpoid}[W^{-1}]_P \) to be the assignment sending objects to themselves, sending a \( P \)-anafunctor to itself as a generalised morphism, and sending a 2-cell between \( P \)-anafunctors to its equivalence class as a 2-cell between generalised morphisms.

The goal of this section is to show that this inclusion is a pseudofunctor which induces an equivalence of bicategories. Thus we have to check that the inclusion \( I_P \) respects compositions and unitors, and that it is essentially surjective and fully faithful: any generalised morphism between two objects of \( \text{LieGpoid}[W^{-1}]_P \) admits a 2-cell from itself to a \( P \)-anafunctor, and that any 2-cell between \( P \)-anafunctors can be represented by a unique 2-cell from \( \text{AnaActGpd}_P \).

We begin with the generalised morphisms. Our strategy will be to show that any generalised morphism is equivalent to a \( P \)-anafunctor induced by a bibundle. The theory of bibundles offers another method of localising \( \text{LieGpoid} \) with a more geometric flavour; see [11, 13, 16] for details. We do not require the full theory here, but simply borrow the necessary concepts.

Proposition 5.2. Any generalised morphism

\[
\mathcal{G} = G \times X \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} H \times Z = \mathcal{H}
\]

between objects in \( \text{ActGpd}_P \) admits a 2-cell from itself to a \( P \)-anafunctor \( \mathcal{G} \xleftarrow{\chi} \mathcal{L} \xrightarrow{\omega} \mathcal{H} \).

Proof. Given the generalised morphism \( \mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H} \), one can construct an anafunctor

\[
\mathcal{G} \xleftarrow{\chi} \mathcal{L} \xrightarrow{\omega} \mathcal{H}
\]

in which \( \mathcal{L} \) is the action groupoid \( \mathcal{G} \rtimes \mathcal{L}_0 \times \mathcal{H} \) as constructed in the proof of [8] Theorem 4.6.3. Since \( \mathcal{G} = G \times X \) and \( \mathcal{H} = H \times Z \), it follows that \( \mathcal{L} \) is a \((G \times H)\)-action groupoid. Thus all that remains is to verify that \( \mathcal{L} \) inherits the properties in \( P \).

Since \( \chi \) is a weak equivalence, \( \mathcal{L} \) is free, locally free, transitive, proper, or an orbifold groupoid if \( \mathcal{G} \) is, by Remark 4.2. If \( G \) and \( H \) are compact/discrete, then so is \( G \times H \). Finally, if the actions of \( G \) and \( H \) on \( X \) and \( Y \), resp., are effective, then an examination of the stabilisers of the \((G \times H)\)-action of \( \mathcal{L} \) reveals that this action is effective as well. \( \square \)

We now want to prove a result similar to Proposition 5.2 for 2-cells: that any 2-cell between \( P \)-anafunctors in \( \text{LieGpoid}[W^{-1}] \) has a unique representative that is a 2-cell in \( \text{AnaActGpd}_P \). The proof of this will require several lemmas, following the outline of the proof of a similar result of Pronk-Scull (see [24 Section 5]), but with some necessary modifications: equivariant surjective submersive weak equivalences are not preserved under natural transformations, so we cannot follow Pronk-Scull verbatim. Existence of the 2-cell representative from \( \text{AnaActGpd}_P \) is the content of the first lemma below, which is a modified version of [24 Lemma 5.2].
Lemma 5.3. Let $\mathcal{G} = G \times X$ and $\mathcal{H} = H \times Y$ be objects of $\text{ActGpd}_P$. Suppose we have two $P$-anafunctors, the top and bottom of the diagram below, with a 2-cell connecting them in $\text{LieGpoid}[W^{-1}]_P$ represented by the following diagram:

\[
\begin{array}{c}
\mathcal{G} \overset{\varphi}{\twoheadleftarrow} \mathcal{K} \overset{\psi}{\twoheadrightarrow} \mathcal{H} \\
\downarrow \eta_1 \quad \downarrow \eta_2 \quad \downarrow \alpha' \quad \downarrow \alpha \\
\mathcal{L} \quad \mathcal{L}' \end{array}
\]

This 2-cell is represented by the following 2-cell from $\text{AnaActGpd}_P$:

\[
\begin{array}{c}
\mathcal{G} \overset{\varphi}{\twoheadleftarrow} \mathcal{K} \overset{\psi}{\twoheadrightarrow} \mathcal{H} \\
\downarrow \varphi' \quad \downarrow \alpha'' \quad \downarrow \psi' \quad \downarrow \alpha''' \quad \downarrow \nu \\
\mathcal{K}' \quad \mathcal{K}''
\end{array}
\]

Proof. By Lemma 4.8, $\mathcal{K}_\varphi \times \mathcal{K}'_{\varphi'}$ is an action groupoid of a Lie group action of $K := (G \times H)_{\varphi \times \psi}(G \times H)$.

Define $\widetilde{\mathcal{L}} := (\mathcal{K}_\varphi \mathcal{X}_{\varphi'} \mathcal{K}')_{\text{pr}_1} \mathcal{X}_{\alpha} \mathcal{L}$, and consider the following diagram, in which Proposition 2.11 justifies the decorations on the arrows:

\[
\begin{array}{c}
\mathcal{G} \overset{\varphi}{\twoheadleftarrow} \mathcal{K} \overset{\psi}{\twoheadrightarrow} \mathcal{H} \\
\downarrow \varphi' \quad \downarrow \alpha'' \quad \downarrow \psi' \quad \downarrow \alpha''' \quad \downarrow \nu \\
\mathcal{K}' \quad \mathcal{K}''
\end{array}
\]

By Item 3 of Lemma 2.6, the natural transformation

\[ (\eta_1 \text{pr}_3) \circ (\varphi \text{PR}_2) \circ (\text{ID}_{\varphi' \text{op} \text{pr}_2} \text{pr}_1) : \varphi' \circ \text{pr}_2 \circ \text{pr}_1 \Rightarrow \varphi' \circ \alpha' \circ \text{pr}_3 \]

factors as $\varphi' \mu$ for a unique natural transformation $\mu : \text{pr}_2 \circ \text{pr}_1 \Rightarrow \alpha' \circ \text{pr}_3$, making the lower triangle in the above diagram 2-commute. It follows from the definition of $\mu$ that

\[ (\eta_1 \text{pr}_3) \circ (\varphi \text{PR}_2) = (\varphi' \mu) \circ (\text{ID}_{\varphi' \text{op} \text{pr}_1} \text{pr}_1). \]
By Lemma 2.12, the natural transformation

\[(\psi'\mu^{-1}) \circ (\eta_2\text{pr}_3) \circ (\psi\text{PR}_2) : \psi \circ \text{pr}_1 \circ \text{pr}_1 \Rightarrow \psi' \circ \text{pr}_2 \circ \text{pr}_1\]
factors as \(\nu\text{pr}_1\) for a unique natural transformation \(\nu : \psi \circ \text{pr}_1 \Rightarrow \psi' \circ \text{pr}_2\). It follows from the definition of \(\nu\) that

\[(\eta_2\text{pr}_3) \circ (\psi\text{PR}_2) = (\psi'\mu) \circ (\nu\text{pr}_1)\).

This shows that the diagram (5) is indeed a representative of a 2-cell, is in \(\text{AnaActGpd}_P\), and in the same equivalence class as the diagram (4).

Thus we have shown that any 2-cell between \(P\)-anafunctors is represented by a 2-cell from \(\text{AnaActGpd}_P\). We now need to prove the uniqueness of such a representative. We begin with a technical lemma.

**Lemma 5.4.** Given two representatives of a 2-cell which both come from \(\text{AnaActGpd}_P\) and have the form

\[
\begin{array}{c}
\varphi \\
\downarrow_{\text{pr}_1} \\
\varphi' \\
\downarrow_{\text{pr}_2} \\
\varphi''
\end{array}
\]

the generalised morphism

\[
\mathcal{K}_{\varphi} \times_{\varphi'} \mathcal{K}' \leftarrow_{\alpha} \mathcal{M} \rightarrow_{\alpha'} \mathcal{K}_{\varphi} \times_{\varphi'} \mathcal{K}'
\]

inducing the equivalence between the two representatives (see Definition 3.2) can be chosen so that \(\beta_0\) is a surjective submersion.

**Proof.** Since the two equivalences are in the same equivalence class, there exists a generalised morphism

\[
\mathcal{K}_{\varphi} \times_{\varphi'} \mathcal{K}' \leftarrow_{\alpha} \mathcal{L} \rightarrow_{\alpha'} \mathcal{K}_{\varphi} \times_{\varphi'} \mathcal{K}'
\]

and natural transformations

\[\nu : \text{pr}_1 \circ \alpha \Rightarrow \text{pr}_1 \circ \alpha' \quad \text{and} \quad \nu' : \text{pr}_2 \circ \alpha \Rightarrow \text{pr}_2 \circ \alpha'\]

such that

\[
(\text{ID}_{\varphi\text{op}\text{pr}_1}\alpha') \circ (\varphi\nu) = (\varphi'\nu') \circ (\text{ID}_{\varphi\text{op}\text{pr}_1}\alpha) \quad \text{and} \quad (\mu\alpha') \circ (\psi\nu) = (\psi'\nu') \circ (\eta\alpha).
\]

Define \(\mathcal{M} := (\mathcal{K}_{\varphi} \times_{\varphi'} \mathcal{K}')_{\varphi\text{op}\text{pr}_1} \times_{\varphi\text{op}\text{pr}_1} \alpha \mathcal{L}\). By Item 1 of Lemma 2.6 and Proposition 2.9, \(\mathcal{M}\) is a Lie groupoid. It follows from Item 3 of Lemma 2.6 and the fact that \(\varphi \circ \text{pr}_1 = \varphi' \circ \text{pr}_2\) that the two natural transformations between functors \(\mathcal{M} \rightarrow \mathcal{G}\)

\[
(\varphi \circ \text{pr}_1) \circ \text{pr}_1 = \varphi \circ (\text{pr}_1 \circ \alpha) \circ \text{pr}_2 \quad \text{and} \quad (\varphi' \circ \text{pr}_2) \circ \text{pr}_1 = (\varphi' \circ \text{pr}_1) \circ \text{pr}_1 = (\varphi \circ \text{pr}_1 \circ \alpha) \circ \text{pr}_2 = \varphi' \circ (\text{pr}_2 \circ \alpha) \circ \text{pr}_2\]

and

\[
(\varphi' \circ \text{pr}_2) \circ \text{pr}_1 = (\varphi' \circ \text{pr}_1) \circ \text{pr}_1 = (\varphi \circ \text{pr}_1 \circ \alpha) \circ \text{pr}_2 = \varphi' \circ (\text{pr}_2 \circ \alpha) \circ \text{pr}_2.
\]
factor as \( \varphi \omega \) and \( \varphi' \omega' \) for some natural transformations \( \omega: \text{pr}_1 \circ \text{pr}_1 \Rightarrow \text{pr}_1 \circ (\alpha' \circ \text{pr}_2) \) and \( \omega': \text{pr}_2 \circ \text{pr}_1 \Rightarrow \text{pr}_2 \circ (\alpha' \circ \text{pr}_2) \). Thus we have the following 2-commutative diagram:

To show that this diagram is an equivalence in \( \text{LieGpoid}[W^{-1}]_{\mathcal{P}} \) between the original diagrams, we need to check Equations (2). The first equation is straightforward from the definitions of \( \omega \) and \( \omega' \) and the first equation of (6). The second follows from the smooth fully faithfulness of \( \varphi, \varphi', \text{pr}_1 \circ \alpha, \) and \( \text{pr}_2 \circ \alpha; \) the naturality of \( \nu \) and \( \nu' \); Item 3 of Lemma 2.6; and the second equation of (6).

Since \( \varphi \circ \text{pr}_1 \) and its composition with \( \alpha \) are weak equivalences, so are \( \text{pr}_1, \alpha' \circ \text{pr}_2: \mathcal{M} \rightarrow \mathcal{K}_{\varphi} \times _{\varphi'} \mathcal{K}' \), with \((\text{pr}_1)_0\) a surjective submersion. Thus the generalised morphism that we require is

\[
\mathcal{K}_{\varphi} \times _{\varphi'} \mathcal{K}' \xleftarrow{\sim \text{pr}_1} \mathcal{M} \xrightarrow{\sim \alpha' \circ \text{pr}_2} \mathcal{K}_{\varphi} \times _{\varphi'} \mathcal{K}'
\]

where \( \beta = \text{pr}_1 \).

We now prove uniqueness of the 2-cell representative in Lemma 5.3.

**Lemma 5.5.** If the two diagrams below represent the same 2-cell in \( \text{LieGpoid}[W^{-1}]_{\mathcal{P}} \),

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\varphi} & \mathcal{K} \\
\downarrow \text{pr}_1 & \downarrow \psi & \downarrow \text{pr}_1 \\
\mathcal{K}_{\varphi} \times _{\varphi'} \mathcal{K}' & \xrightarrow{\sim} & \mathcal{M} \\
\downarrow \text{pr}_2 & \downarrow \eta & \downarrow \eta' \\
\mathcal{K}' & \xrightarrow{\psi'} & \mathcal{H}
\end{array}
\]

then \( \eta = \eta' \).

**Proof.** Since \( \eta \) and \( \eta' \) are in the same equivalence class, by Lemma 5.4, there exists a generalised morphism

\[
\mathcal{K}_{\varphi} \times _{\varphi'} \mathcal{K}' \xleftarrow{\sim \gamma} \mathcal{L} \xrightarrow{\sim \gamma'} \mathcal{K}_{\varphi} \times _{\varphi'} \mathcal{K}'
\]

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and natural transformations

\[ \mu \colon \text{pr}_1 \circ \gamma \Rightarrow \text{pr}_1 \circ \gamma' \quad \text{and} \quad \mu' \colon \text{pr}_2 \circ \gamma \Rightarrow \text{pr}_2 \circ \gamma' \]

inducing the equivalence relation between them, in which \( \gamma \) is a surjective submersive weak equivalence. It follows from Lemma 2.12 and the middle four exchange (a standard coherence relation for bicategories; see, for instance, \[14, (2.1.9)\]) that \( \eta \gamma = \eta' \gamma' \). Since \( \gamma \in sW \), Lemma 2.12 implies that \( \eta = \eta' \), and thus the two 2-cells are equal in \( \text{AnaActGpd} \). □

Lastly, we will verify that the inclusion \( I_P \) is a pseudofunctor and respects the operations in the localised bicategories.

**Proposition 5.6.** The assignment \( I_P \colon \text{AnaActGpd}_P \to \text{LieGpoid}[W^{-1}]_P \) is a pseudofunctor.

**Proof.** To begin, we must show that for each pair of action groupoids \( G := G \ltimes X \) and \( H := H \ltimes Y \), \( I_P \) induces a functor \( \text{AnaLieGpoid}_P(G, H) \to \text{LieGpoid}[W^{-1}]_P(G, H) \) between the categories of 1-cells between \( G \) and \( H \), with 2-cells between those. In particular, \( I_P \) must respect vertical composition and unit 2-cells.

Suppose the diagrams which are being vertically composed are in fact in \( \text{AnaLieGpoid}_P \), and so \( \varphi, \varphi', \) and \( \varphi'' \) are surjective submersive, \( \mathcal{L}_1 = \mathcal{K}_\varphi \times \mathcal{K}' \), \( \mathcal{L}_2 = \mathcal{K}'_\varphi \times \mathcal{K}'' \), the vertical maps from \( \mathcal{L}_i \) are projection maps, and \( \mu_1 \) and \( \mu_2 \) are trivial. We consider the vertical composition in the two categories: in \( \text{LieGpoid}[W^{-1}]_P \) the vertical composition is defined by the diagram

\[
\begin{array}{c}
\mathcal{G} \\
\downarrow \varphi \text{ pr}_2 \\
(K_\varphi \times K') \text{ pr}_2 \times (K'_\varphi \times K'') \\
\downarrow \psi
\end{array}
\]

where \( \kappa = (\eta_2 \text{ pr}_3) \circ (\psi \text{ PR}_2) \circ (\eta_1 \text{ pr}_1) \) defining the vertical composition as in \[21\] Subsection 2.3 or \[24\] Section 3.

In \( \text{AnaActGpd}_P \), we define the vertical composition by

\[
\begin{array}{c}
\mathcal{G} \\
\uparrow \varphi \\
K_\varphi \times K'' \\
\uparrow \psi
\end{array}
\]

where \( \lambda \) is the unique natural transformation such that \( \lambda \text{ pr}_{13} = (\eta_2 \text{ pr}_{23}) \ast (\eta_1 \text{ pr}_{12}) \) where \( \text{pr}_{ij} = (\text{pr}_i, \text{pr}_j) \) is the projection of \( K_\varphi \times K'_\varphi \times K'' \) and \( \ast \) denotes horizontal composition of natural transformations. Such a \( \lambda \) exists by Lemma 2.12 since \( \text{pr}_{13} \in sW \). Thus it suffices
to show for a fixed \((y, y', y'')\) \(\in \mathcal{K}_\varphi \times_\varphi \mathcal{K}' \times_\varphi \mathcal{K}''\) that \(\lambda \text{pr}_{13}(y, y', y'') = \nu \text{pr}_{13}(y, y', y'')\), where again \(\nu\) is the \(\nu\) of Lemma \(5.3\).

In order to show that our inclusion respects vertical composition, we need to show that these two diagrams are in the same equivalence class. To accomplish this, we will apply Lemma \(5.3\) to the first diagram above, and show that the resulting natural transformation again \(\nu\) to show for a fixed \(2\)

To show that our inclusion respects vertical composition, we need to show that the first coherence condition reduces to showing that the vertical composition of the \(I\)-anafunctors yields equivalent results. Fix a \(\mathcal{P}\)-anafunctor \(\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \rightarrow \mathcal{H}\), and let \(\Delta : \mathcal{K} \rightarrow \mathcal{K} \times_\varphi \mathcal{K}\) be the diagonal map. Then \(\text{pr}_1 \circ \Delta = \text{pr}_2 \circ \Delta\), and \(\iota_{\mathcal{G} \times_\varphi \mathcal{H} \Delta}\) is trivial. Via \(\mathcal{K} \times_\varphi \mathcal{K} \xleftarrow{\varphi} \mathcal{K} \rightarrow \mathcal{K}\) it follows that the identity 2-cell of \(\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \rightarrow \mathcal{H}\) in \(\text{AnaLieGpoid}_\mathcal{P}\) is equivalent to the identity 2-cell in \(\text{LieGpoid}[W^{-1}]_\mathcal{P}\), and we conclude that \(I\) induces a functor \(\text{AnaLieGpoid}_\mathcal{P}(\mathcal{G}, \mathcal{H}) \rightarrow \text{LieGpoid}[W^{-1}]_\mathcal{P}(\mathcal{G}, \mathcal{H})\).

Since the identity generalised morphism of a Lie groupoid \(\mathcal{G}\) is the same as the identity anafunctor, \(I\) trivially preserves identity 1-cells.

By Proposition \(3.6\) for each pair of \(\mathcal{P}\)-anafunctors \(\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \rightarrow \mathcal{H}\) and \(\mathcal{H} \xleftarrow{\chi} \mathcal{L} \rightarrow \mathcal{I}\) there is a 2-cell in \(\text{LieGpoid}[W^{-1}]_\mathcal{P}\) from the composition as generalised morphisms to the composition as anafunctors, represented by \((\text{inc}, \text{id}_{\mathcal{K} \times \mathcal{L}}, \text{id}_{\varphi \circ \text{pr}_1 \circ \text{inc}}, \text{id}_{\text{pr}_3 \circ \omega \circ \text{inc}}\). We now check the first of three coherence conditions (namely, [14 (4.1.3)] or (M.1) of [14 page 30]), which indicates that the various compositions of \(\mathcal{P}\)-anafunctors yields equivalent results. Fix three \(\mathcal{P}\)-anafunctors

Then the first coherence condition reduces to showing that the vertical composition of the 2-cells induced by the inclusions \(\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P} \rightarrow \mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P}\) and \(\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P} \rightarrow \mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P}\) is equal to the vertical composition of the 2-cells induced by the inclusions \(\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P} \rightarrow \mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P}\) and \(\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P} \rightarrow \mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P}\). These two 2-cells are
where

\[ Q_1 := (\mathcal{M}_\psi \times \chi \omega \times \xi \mathcal{P})_{id} \mathcal{M}_\psi \times \chi \omega \times \xi \mathcal{P} \]

and

\[ Q_2 := (\mathcal{M}_\psi \times \chi \omega \times \xi \mathcal{P})_{\text{id} \mathcal{M}_\psi \times \chi \omega \times \xi \mathcal{P}} \]

(note that we have suppressed some of the notation). The equivalence is established by the quadruple \((j_1, j_2, \nu_1, \nu_2)\) with \(\nu_1\) and \(\nu_2\) trivial and where the generalised morphism

\[ Q_1 \xrightarrow{\sim j_1} \mathcal{M}_\psi \times \chi \omega \times \xi \mathcal{P} \xrightarrow{\sim j_2} Q_2 \]

is defined by

\[ j_1(m, n, p) = ((m, n, u_\omega(s(n)), p), u_{\text{id} \mathcal{M}_\psi \times \chi \omega \times \xi \mathcal{P}}(s(m, n, p)), (m, n, p)) \]

and

\[ j_2(m, n, p) = ((m, u_\chi(s(n)), n, p), u_{\text{inc} \mathcal{P}}(s(m, n, p)), (m, n, p)). \]

Indeed, the natural transformations of Equations \[2\] all reduce to trivial ones.

Finally, we check that \(I_P\) respects the unitors from each bicategory. For right unitors, the relevant coherence condition (see [14 (4.1.4)] or (M.2) of [4, page 30]) reduces to checking
that the following two diagrams represent the same 2-cell.

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\varphi \circ \text{pr}_1} & \mathcal{K} \\
\downarrow \varphi & & \downarrow \varphi \\
\mathcal{L} & \xrightarrow{\text{incpr}_1} & \mathcal{K}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\psi \circ \text{pr}_1} & \mathcal{K} \\
\downarrow \psi & & \downarrow \psi \\
\mathcal{L} & \xrightarrow{\text{opr}_2} & \mathcal{K}
\end{array}
\end{array}
\end{array}
\]

where

\[
\mathcal{L} := (\mathcal{K}_{\psi \times \text{id}_H} \mathcal{H})_{\text{id}_{\mathcal{K}} \times H} \times_{\text{pr}_1} ((\mathcal{K}_{\psi \times \text{id}_H} \mathcal{H})_{\varphi \times \text{id}} \times \mathcal{K})
\]

and

\[
\rho = (\rho^{\text{ana}}_{\mathcal{G}, \mathcal{H}}(\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}) \text{pr}_3) \circ (\text{pr}_3 \text{PR}_2).
\]

The equivalence is given by the quadruple \((j_1, j_2, \nu_1, \nu_2)\) where \(j_1: \mathcal{K} \rightarrow \mathcal{K}_{\psi \times \text{id}_H} \mathcal{H}\) sends \(k \in \mathcal{K}_1\) to \((k, u_{s(\psi(k))}, \psi(k))\) and \(j_2: \mathcal{K} \rightarrow \mathcal{L}\) \(k\) to \(((k, \psi(k)), u_{s(k, u_{s(\psi(k))}, \psi(k))}, ((k, \psi(k)), k))\), and both \(\nu_1\) and \(\nu_2\) are trivial. Again, the natural transformations of Equations 2 are all trivial. The computation for left unitors using the corresponding coherence condition (M.2) on page 30 of [4] is similar.

Combining the results above yields the desired equivalence of bicategories.

**Theorem 5.7.** The pseudofunctor \(I_P: \text{AnaActGpd}_P \rightarrow \text{LieGpoid}[W^{-1}]_P\) is an equivalence of bicategories. Consequently, \(\text{AnaActGpd}_P, \text{ActGpd}[W^{-1}]_P\) and \(\text{LieGpoid}[W^{-1}]_P\) are all equivalent bicategories.

**Proof.** By Proposition [5.6] \(I_P\) is a pseudofunctor. Since \(I_P\) is surjective on objects, it suffices to show that for two action groupoids \(\mathcal{G} = G \times X\) and \(\mathcal{H} = H \times Y\) satisfying \(P\), the restriction of \(I_P\) to the category of \(P\)-anafunctors from \(\mathcal{G}\) to \(\mathcal{H}\), which maps into the category of all generalised morphisms between them, is an equivalence of categories. Essential surjectivity follows from Proposition [5.2] and fully faithfulness follows from Lemmas [5.3] and [5.5].

We have the following immediate corollary of Theorem [4.6] and Proposition [5.2].

**Corollary 5.8.** For any subset of properties of \(P\) which does not include “effective”, given a generalised morphism \(\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \rightarrow \mathcal{H}\) between objects \(\mathcal{G} := G \times X\) and \(\mathcal{H} := H \times Y\) of...
ActGpd_p, there is a 2-cell from the generalised morphism to a P-anafunctor $\mathcal{G} \xleftarrow{\chi} \mathcal{L} \rightarrow \omega$ $\mathcal{H}$ in which $\chi$ is of the form $\pi$ as described in Lemma 4.4.

Note that if the generalised morphism in the corollary is a Morita equivalence, then $\omega$ also will decompose in the way described. We exclude effective actions specifically from this result because they are not preserved under equivariant weak equivalences, as the following example shows.

Example 5.9. Let $G$ be the four-element dihedral group, with elements $(e, e), (e, \tau), (\tau, e)$ and $(\tau, \tau)$. This acts on the set $X$ consisting of four points laid out in the cardinal directions, $N, S, E, W$: $(\tau, e)$ reflects so that $N$ and $S$ switch and $E, W$ are fixed, and $(e, \tau)$ reflects so that $E$ and $W$ switch and $N, S$ are fixed, and $(\tau, \tau)$ rotates by half a turn and has no fixed points. This is an effective action.

The subgroup $K = \langle (\tau, \tau) \rangle$ acts freely, so we can take the quotient by $K$. Then $K \backslash X$ consists of two points $[N] = [S]$ and $[E] = [W]$. Both of the projected points have isotropy $\mathbb{Z}/2 = K \backslash G$, and this action is not effective.

References

[1] Omar Abbad and Enrico M. Vitale, “Faithful calculus of fractions”, Cah. Topol. Géom. Différ. Catég., 54 (2013), 221–239.

[2] A. Angel, Helen Colman, M. Grant and John Oprea, “Morita Invariance of Equivariant Lusternik-Schnirelmann Category and Invariant Topological Complexity”, Theory and Application of Categories, 35 (2020), 179–195.

[3] Alejandro Adem, Johann Leida, and Yongbin Ruan, “Orbifolds and Stringy Topology”, Cambridge Tracts in Mathematics, 171, Cambridge University Press, Cambridge, 2007.

[4] Jean Bénabou, “Introduction to bicategories” In: Reports of the Midwest Category Seminar, Lecture Notes in Mathematics 47, Springer, 1967, 1–77.

[5] Jan Boman, “Differentiability of a function and of its compositions with functions of one variable”, Math. Scand., 20 (1967), 249–268.

[6] Alejandro Cabrera, Matias del Hoyo, and Enrique Pujals, “Discrete dynamics and differentiable stacks”, Rev. Mat. Iberoam, 36 (2020), 2121–2146.

[7] Bohui Chen, Cheng-Yong Du, and Rui Wang, "The groupoid structure of groupoid morphisms", J. Geom. Phys., 145 (2019), 26 pp.

[8] Matias del Hoyo, "Lie groupoids and their orbispaces", Port. Math., 70 (2013), 161–209.

[9] Matias del Hoyo, Rui Loja Fernandes, "Riemannian metrics on differentiable stacks", Math. Z. 292 (2019), 103-132.

[10] Johannes J. Duistermaat and Johan A. C. Kolk, “Lie Groups”, Springer, 2000.

[11] André Haefliger, “Groupoïdes d’holonomie et classifiants”, Astérisque 116 (1984), 70–97.

[12] André Henriques and David S. Metzler, “Presentations of noneffective orbifolds”, Trans. Amer. Math. Soc. 356 (2004) 2481–2499.

[13] Michel Hillsum and Georges Skandalis, “Morphismes K-orientés d’espaces de feuilles et functorialité en théorie de Kasparov”, Ann. Acien. Ec. Norm. Sup. 20 (1987), 325–390.

[14] Niles Johnson and Donald Yau, “2-Dimensional Categories”, Oxford University Press, Oxford, 2021.

[15] John M. Lee, “Introduction to smooth Manifolds”, 2nd Ed., Graduate Texts in Mathematics 218, Springer, 2013.

[16] Eugene Lerman, “Orbifolds as stacks?”, Enseign. Math. (2), 56 (2010), 315–363.

[17] Ieke Moerdijk, “Orbifolds as groupoids: an introduction”, Orbifolds in mathematics and physics (Madison, WI, 2001), 205-222, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, USA, 2002.
18] Ieke Moerdijk and Janez Mrčun, “Introduction to foliations and Lie Groupoids”, Cambridge University Press, 2003.
[19] Ieke Moerdijk and Dorette Pronk, “Orbifolds, Sheaves and Groupoids”, K-Theory, 12 (1997), 3–21.
[20] nLab, [https://ncatlab.org/nlab/show/HomePage](https://ncatlab.org/nlab/show/HomePage)
[21] Dorette Pronk, “Etendues and stacks as bicategories of fractions”, Compositio Math. 102 (1996), 243–303.
[22] Dorette Pronk and Laura Scull, “Translation groupoids and orbifold cohomology”, Canad. J. Math, 62 (2010), 614–645.
[23] Dorette Pronk and Laura Scull, Correction to “Translation groupoids and orbifold cohomology”, Canad. J. Math, 6 (2017), 851–853.
[24] Dorette Pronk and Laura Scull, “Bicategories of fractions revisited”, Theory Appl. Categ. 34 (2022), 913–1014.
[25] David M. Roberts, “Internal categories, anafunctors, and localisations”, Theory Appl. Cat., 26 (2012), 788–829.
[26] David M. Roberts, “On certain 2-categories admitting localisation by bicategories of fractions”, Appl. Categ. Structures, 24 (2016), 373–384.
[27] David M. Roberts, “The elementary construction of formal anafunctors”, Categ. Gen. Algebr. Struct. Appl. 15 (2021), 183–229.
[28] Nesta van der Schaaf, “Diffeological Morita equivalence”, Cah. Topol. Géom. Différ. Catég., LXII (2021), 177–238.
[29] Jordan Watts, “Bicategories of differological groupoids”, (preprint).
[30] Nguyen Tien Zung, “Proper groupoids and momentum maps: linearization, affinity, and convexity”, Ann. Sci. École Norm. Sup., 39, (2006), 841–869.

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