Mathematical modeling and numerical algorithm for the gas displacement by water in an inhomogeneous porous medium

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Abstract. The problems associated with the development of gas fields bounded by aquifer systems are considered in the paper in order to increase gas recovery and determine the key parameters of the reservoir for further development. To conduct a comprehensive study of the process under consideration, a computer model has been developed that is described by a differential equation with variable coefficients under corresponding initial and boundary conditions and moving boundaries. An arbitrarily defined region is converted to a standard one using the dummy region method. Functional dependences of the parameters of porosity and permeability are constructed by the method of local approximation given at some defined points in the reservoir for each nodal point. An algorithm is developed for solving the problem using the methods of longitudinal-transverse scheme and the flow option of the sweep method. The results are shown as isolines. In this paper, an attempt is made to prove that in the practice of fluid production it is advisable to introduce development technology based on inhomogeneous formation parameters. Numerical experiments have shown that using this technology wide opportunities appear for finding highly effective design solutions for fields development.

1. Introduction
At effective design and development of gas and oil fields, it is necessary to solve the problems of different profiles. It makes it possible to predict a change in the field pressure, i.e. to numerically solve the corresponding hydrodynamic problems. This leads to a numerical solution of fluid (oil, gas, and water) filtration problem with an arbitrary region, corresponding initial and boundary conditions.

The first steps in the development of the methods of filtration theory were to create an algorithm and software for solving prognostic tasks, i.e. to solve corresponding hydrodynamic problems, mathematical study of which is reduced to the consideration of a boundary-value problem described by corresponding equations of the theory of filtration in multi-related regions with inhomogeneous boundary conditions. These problems were successfully solved by analytical, approximate-analytical, variational and numerical methods.

Due to the large variety of oil and gas fields, and most importantly - due to the complexity of their geological structure, the situation in the computer era has changed dramatically. The methods of finite differences are especially widely used in conducting large-scale mathematical experiments on modern computers. This suggests that today the role of exploratory research to substantiate new technologies in deposits development and the adoption of various technological decisions is growing.

To solve a substantial scientific and technical problem (for example, an increase in gas and oil output, a cone rise of bottom water, etc.), not only a certain set of applied software of different level of
difficulty is required, but it is important to understand what features are related to the essence of original physical problems, and what features are introduced by the choice of a mathematical model.

Groundwater monitoring is considered an important task in hydrodynamic work. It makes possible to judge the filtration paths and rates at water rise in the well locations and probable flooding, the degree of development stability in the site under consideration, etc.

Therefore, research on the development effectiveness of water-gas zones and gas deposits with bottom water is relevant. Firstly, this is due to the fact that considerable gas reserves are confined in the indicated zones and deposits. Secondly, the gas industry is characterized by significant flooding of produced products. To date, a considerable number of studies have been published in which the patterns of wells and fields flooding by contour and bottom waters have been revealed.

Nevertheless, at present there are objective reasons to conduct research in order to find the ways to increase the developing efficiency of water-gas zones and gas deposits with bottom water. This is due, firstly, to the fact that well-known theory and practice of developing the type of deposits under consideration is based on the use of the systems of vertical production and injection wells. Secondly, this is a result of significant progress in the creation and use of numerical algorithms and software, as well as the production of modern powerful computers. It became possible to set up large-scale mathematical experiments on development elements, taking into account the key determining parameters and factors. The results of many years of research are summarized by authors to substantiate new principles and technologies for the development of oil and gas fields. These results to a large extent are associated with the current state in oil and gas industry and the achievements of scientific and technological progress [1,2].

Due to the widespread occurrence of horizontal wells in oil and gas production systems, research on the problem of steady and unsteady inflow to this type of wells has significantly increased. The research provides the methodological basis for interpreting the results obtained while studying horizontal wells under unsteady conditions and for coning in the development of oil and gas or floating oil and gas deposits by a system of horizontal wells [3,4].

[6] provides the information on mathematical foundations of theoretical gas dynamics. The principles of constructing various gas-dynamic models are stated - from integral conservation laws to specific formulas that describe a particular gas flow. Group-theoretical foundations of differential equations derivation describing the classes of partial solutions are given. The solution of many specific problems is done by the methods of qualitative analysis. To facilitate the perception of the material, the text is completed with graphic illustrations.

As you know, when solving practical problems regarding the phenomenon occurring in the reservoir, the types of the considered areas of a real field are arbitrary, and geological and geophysical data are obtained using core analysis at some given points in the reservoir. Therefore, in such cases, it is necessary to consider the solution to the problem with inhomogeneous coefficients in a porous medium.

To achieve this goal, it is necessary to fulfill the following:
– to recognize a randomly given region and to convert it to a standard one;
– to build a functional dependence of the porosity and permeability parameters, i.e. at each nodal point of the grid to determine the values of these parameters;
– to solve the problem with variable coefficients.

2. Statement of the problem

Let there be a productive layer of an arbitrary form of a region initially saturated with fluid (gas), confined to edgewater. The reservoir is developed using randomly located wells, with coordinates $(x_i, y_i)$, in the mode of specified volume flow rates in time $q_i$. It is necessary to determine the time change of reservoir pressure and the position of the moving boundaries. For this, it is necessary to integrate the nonlinear partial differential equation with variable coefficients of parabolic type:

in the gas zone
\[
\frac{\partial}{\partial x} \left[ \frac{k(x,y)}{\mu_g} p_1(x,y,t) \frac{\partial p_1(x,y,t)}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \frac{k(x,y)}{\mu_g} p_1(x,y,t) \frac{\partial p_1(x,y,t)}{\partial y} \right] = \sigma m(x,y) \frac{\partial p_1(x,y,t)}{\partial t} + F(x,y,t), \quad (x,y) \in G_1, \ t > 0;
\]

in the fluid zone

\[
\frac{\partial}{\partial x} \left[ \frac{k(x,y)}{\mu_w} \frac{\partial p_2(x,y,t)}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \frac{k(x,y)}{\mu_w} \frac{\partial p_2(x,y,t)}{\partial y} \right] = \beta_w m(x,y) \frac{\partial p_2(x,y,t)}{\partial t}, \quad (x,y) \in G_2, \ t > 0;
\]

under initial conditions

\[
p_1(x,y,t) = p_1^0(x,y), \quad t = 0, \ (x,y) \in G_1,
\]

\[
p_2(x,y,t) = p_2^0(x,y), \quad t = 0, \ (x,y) \in G_2
\]

under boundary conditions

\[
\frac{\partial p_2(x,y,t)}{\partial l_1} = 0, \quad (x,y) \in \Gamma_2
\]

Conditions of pressure continuity and flow continuity are fulfilled in moving boundaries

\[
p_{1i}(x,y,t)|_{(x,y) \in \partial R(x,y,t) \cap \partial G^0} = p_{2i}(x,y,t)|_{(x,y) \in \partial R(x,y,t) \cap \partial G^0},
\]

\[
k(x,y) \frac{\partial p_1(x,y,t)}{\partial n}|_{(x,y) \in \partial R(x,y,t) \cap \partial G^0} = k(x,y) \frac{\partial p_2(x,y,t)}{\partial n}|_{(x,y) \in \partial R(x,y,t) \cap \partial G^0},
\]

The law of flow is

\[
\frac{dR(x,y,t)}{dt} = -\frac{1}{\sigma m(x,y)} \frac{k(x,y)}{\mu_w} \frac{\partial p_2(x,y,t)}{\partial n} |_{(x,y) \in \partial R(x,y,t) \cap \partial G^0},
\]

\[
R(x,y,t)|_{t=0} = R^0(x,y);
\]

here - \( p(x,y,t) \) - is the pressure; \( \mu_g, \mu_w \) - coefficient of dynamic viscosity of gas and fluid, respectively; \( k(x,y) \) - permeability coefficient; \( m(x,y) \) - porosity coefficient; \( \beta_w^* \) - coefficient of elastic capacity in the watercut; \( t \) - time; \( p_1^0, p_2^0 \) - initial reservoir pressure.

\[
F = \sum_{i=1}^{n_s} q_i(t) \delta(x - x_i, y - y_i), \quad q_i(t) = \int_{S_i} k(x,y) \frac{p_1}{\mu_g} \frac{\partial p_2}{\partial l_2} dS, \quad (x,y) \in S_i
\]

are the flow rates of gas wells reduced to atmospheric pressure and reservoir temperature. \( \delta = \begin{cases} 1, & x = x_i, y = y_i \\ 0, & x \neq x_i, y \neq y_i \end{cases} \) - Dirac delta function; \( p_{at} \) - atmospheric pressure; \( l_1, l_2 \) - the normals to the contours \( \Gamma_2, S \), respectively; \( n_s \) - the number of wells. \( R^0(x,y) \) - a curve set initially, showing the initial position of the moving boundary surface. \( R(x,y,t) \) - the position of the moving boundary surface; \( n \) - the normal to the line of the moving boundary surface.

For convenience, equations (1) and (2), can be written in the following form
\[
\frac{\partial}{\partial x} \left[ K(x, y) A(p) \frac{\partial P(x, y, t)}{\partial x} \right] + \frac{\partial}{\partial y} \left[ K(x, y) A(p) \frac{\partial P(x, y, t)}{\partial y} \right] =
\]
\[
= M(x, y) \frac{\partial P(x, y, t)}{\partial t} + F(x, y, t), \quad (x, y) \in (G_1 \cup G_2), \quad t > 0;
\]
\[
P(x, y, t) = P^0(x, y), \quad t = 0, \quad (x, y) \in G_1 \cup G_2.
\]
\[
\frac{\partial P(x, y, t)}{\partial l} = 0, \quad (x, y) \in \Gamma_2
\]
\[
dR(x, y, t) \frac{dt}{d\tau} = - \frac{1}{m(x, y) \sigma} k(x, y) \frac{\partial P(x, y, t)}{\partial n} \bigg|_{(x, y) \in R(x, y, t) + 0},
\]
\[
R(x, y, t) \bigg|_{t=0} = R^0(x, y);
\]

Here –
\[
P(x, y, t), A(p), K(x, y), M(x, y) = \begin{cases} 
    p_1(x, y, t), & m(x, y) \sigma, \quad (x, y) \in G_1 \\
    p_2(x, y, t), & m(x, y) \beta, \quad (x, y) \in G_2
\end{cases}
\]

3. Numerical algorithm for solving the problem

The arbitrary shape of the boundaries of considered region complicates the construction of a computational algorithm. To eliminate this gap, we supplement the auxiliary region \( G_0 \), hereinafter referred to as a dummy region, with a given arbitrary region to a region \( \Omega \) with boundary \( \Gamma \). That is, we take \( \Omega = G + G_0 \) instead of considered region \( G = G_1 + G_2 \). At a given boundary \( \Gamma_2 \), we set the ordinary fitting condition for the problems with discontinuous coefficients and extend the coefficients and the right-hand side of the equation to the dummy region using some given small parameter. As a result, we obtain a standard rectangular region with a standard boundary condition.

To construct a numerical algorithm for solving the problem using finite-difference methods [5,7], first proceed to dimensionless variables. For this, introduce some characteristic quantities

\[
\tilde{x} = \frac{x}{L_x}, \quad \tilde{y} = \frac{y}{L_y}, \quad \tilde{k} = \frac{k}{k_x}, \quad \tilde{\mu} = \frac{\mu}{\mu_x} \quad \tilde{p} = \frac{P}{\mu_x}, \quad \tilde{\tau} = \frac{t}{t_x}
\]

where \( L_x, L_y, k_x, \mu_x, p_x, t_x \) are some given constants. Omitting the bar accent over the letters and making some calculations, we obtain a dimensionless problem, which has the form similar to problem (6) - (9) in the region \( \Omega = [0 \leq x \leq 1; \ 0 \leq y \leq 1] \).

Cover the given region \( \Omega = [0 \leq x \leq 1; \ 0 \leq y \leq 1] \) with a uniform grid

\[
\tilde{\omega}_{xy} = \{(x_i = i h_x, h_x = 1 / N_x, i = 0, \ldots, N_x; \ y_j = j h_y, h_y = 1 / N_y, \ j = 0, \ldots, N_y)\}
\]

For the time step we take the grid

\[
\tilde{\omega}_\tau = \{(k = \tau, \ tau = 1 / T, \ k = 0, T)\}
\]
Using the algorithm for constructing the functional dependencies of the parameters, using the local approximation method [8], we create the value of the permeability and porosity fields at each grid point of the region under consideration.

By entering the notation

\[ W_x = K \frac{\partial P}{\partial x}, \quad W_y = K \frac{\partial P}{\partial y} \]

and approximating the dimensionless problem using the longitudinal-transverse Samarsky scheme [5], we obtain a chain of one-dimensional difference problems of the form

\[
\begin{align*}
A^k_{i+1/2,j} W_{i+1/2,j}^{k+1/2} - A^k_{i-1/2,j} W_{i-1/2,j}^{k+1/2} &= \frac{h_x}{0.5\tau} M_{i,j} P_{i,j}^{k+1/2} + \frac{h_x}{0.5\tau} \Phi_{i,j}^{k+1/2}, \\
\Phi_{i,j}^{k+1/2} &= -M_{i,j} P_{i,j}^{k+1/2} + 0.5\tau F_{i,j}^{k+1/2} - (A^k_{i,j+1/2} W_{i,j+1/2}^{k+1/2} - A^k_{i,j-1/2} W_{i,j-1/2}^{k+1/2}) / h_y, \\
W_{i+1/2,j}^{k+1/2} &= \frac{1}{h_x} K_{i+1/2,j} (P_{i+1/2,j}^{k+1/2} - P_{i,j}^{k+1/2}), \\
A_{0,j}^{k+1/2} W_{0,j}^{k+1/2} &= 0, \quad A_{N_1,j}^{k+1/2} W_{N_1,j}^{k+1/2} = 0, \\
R_{1/2}^{k+1/2} &= R_{1/2}^k - C_{1/2}^k W_{1/2}^{k+1/2} \\
\end{align*}
\]

\[
\begin{align*}
A^k_{i,j+1/2} W_{i,j+1/2}^{k+1/2} - A^k_{i,j-1/2} W_{i,j-1/2}^{k+1/2} &= \frac{h_y}{0.5\tau} M_{i,j} P_{i,j}^{k+1/2} + \frac{h_y}{0.5\tau} \Phi_{i,j}^{k+1/2}, \\
\Phi_{i,j}^{k+1/2} &= -M_{i,j} P_{i,j}^{k+1/2} + 0.5\tau F_{i,j}^{k+1/2} - (A^k_{i+1/2,j} W_{i+1/2,j}^{k+1/2} - A^k_{i-1/2,j} W_{i-1/2,j}^{k+1/2}) / h_x, \\
W_{i,j+1/2}^{k+1/2} &= \frac{1}{h_y} K_{i,j+1/2} (P_{i+1/2,j}^{k+1/2} - P_{i,j}^{k+1/2}), \\
A_{0,j}^{k+1/2} W_{0,j}^{k+1/2} &= 0, \quad A_{N_2,j}^{k+1/2} W_{N_2,j}^{k+1/2} = 0, \\
R_{1/2}^{k+1/2} &= R_{1/2}^k - C_{1/2}^k W_{1/2}^{k+1/2} \\
\end{align*}
\]

The resulting chain of one-dimensional difference problems (10) - (11) is solved by the method of streaming sweep.

4. Computational experiments
To verify the reliability of the results obtained using the above computational algorithms, we will choose a circular region as a test example, confined to edgewater. In the center of the region there is one pumping well with a constant flow rate \( q_0 = 10 \text{ m}^3/\text{day} \) and a thickness 10 m. The example is solved to compare the results obtained with the results of two-dimensional problem of one-dimensional flat-radial filtration under the same initial data.

Table 1 shows the values of the pressure field obtained in both statements for a time \( t=180 \) days; due to the symmetry in two-dimensional statement the results of one quarter are given. The iteration number when refining a nonlinear term does not exceed two steps for the whole time of solution.

Based on comparative results, we note that the results obtained by the developed algorithms for the numerical determination of the pressure field and the position of the moving boundaries of two phases
in a two-dimensional statement correctly reflect the physical picture of the flow process. The change in the moving boundaries is shown in Fig. 1.

Table 1. Change in pressure field

| One-dimensional statement | 0.930 | 0.931 | 0.933 | 0.935 | 0.936 | 0.937 | 0.937 |
|---------------------------|-------|-------|-------|-------|-------|-------|-------|
| Two-dimensional statement |       |       |       |       |       |       |       |
| y \ x                     | 0.50  | 0.55  | 0.65  | 0.75  | 0.85  | 0.95  | 1.0   |
| 0.50                      | 0.931 | 0.933 | 0.933 | 0.933 | 0.934 | 0.935 | 0.936 |
| 0.55                      | 0.933 | 0.933 | 0.934 | 0.934 | 0.935 | 0.936 |
| 0.65                      | 0.933 | 0.933 | 0.934 | 0.935 | 0.936 | 0.936 |
| 0.75                      | 0.934 | 0.934 | 0.935 | 0.936 | 0.936 |
| 0.85                      | 0.935 | 0.934 | 0.936 | 0.936 |
| 0.95                      | 0.936 | 0.936 | 0.936 |
| 1.0                       | 0.936 |       |       |       |       |       |       |

Figure 1. Region view of the test example

Now consider the problem using an example of one real deposit, the region view of which is shown in Fig. 2 under the following initial data: length, width, height of the deposit: = 11 km, 5.5 km, 32 m, respectively; $p_0 = 550$ MPa; $\mu_r, \mu_w = 0.01, 0.1$ cPz, respectively; $k(x, y) = 0.1$ darcy; $m(x, y) = 0.09$; $\beta_w = 0.000001$. The coordinates of 20 randomly located production wells, with identical flow rates equal to 69.7 thousand cubic meters per day, are given in Table 2.
Table 2. Well coordinates

| №   | № of wells | X  | Y  | №   | № of wells | X  | Y  |
|-----|------------|----|----|-----|------------|----|----|
| 1   | 2          | 27 | 16 | 11  | 16         | 32 | 16 |
| 2   | 3          | 23 | 8  | 12  | 17         | 34 | 15 |
| 3   | 4          | 34 | 14 | 13  | 18         | 37 | 14 |
| 4   | 9          | 12 | 20 | 14  | 19         | 31 | 14 |
| 5   | 20         | 38 | 13 | 15  | 21         | 36 | 12 |
| 6   | 23         | 29 | 13 | 16  | 22         | 32 | 12 |
| 7   | 25         | 30 | 16 | 17  | 24         | 34 | 13 |
| 8   | 28         | 34 | 11 | 18  | 26         | 28 | 15 |
| 9   | 14         | 32 | 15 | 19  | 27         | 39 | 13 |
| 10  | 15         | 35 | 14 | 20  | 29         | 23 | 17 |

With given values of the permeability and porosity coefficients at the borehole points, a functional dependence is constructed for all the nodal points of the deposit using the developed algorithm and the local approximation method [12].

Figure 3 shows the results of calculations of the pressure field at a) constant values of these coefficients and b) variable values for a time $t = 180$ days, respectively.

![Figure 3](image_url)

**Figure 3.** Isolines of the pressure fields a) with constant coefficients; b) with variable coefficients

Analysis of calculation results and comparison of different parameter values allow us to note the following characteristic aspects. The performed studies quantitatively confirm that in the case of
constant values of porosity and permeability parameters, the change in the pressure values does not very closely correspond to the real pressure values, which can lead to incorrect conclusions and negative results. Therefore, the work performed calculations with variable values of these coefficients in the considered area, that is, close to real results.

5. Conclusion
In conclusion, it can be noted that in order to obtain more accurate results, we first need to build the functional dependences of the coefficients given at some specific points in the considered area and perform calculations with variable values of the permeability and porosity coefficients. In order to eliminate this drawback, calculations were carried out with variable values of the coefficients of permeability and porosity. A comparison of the results obtained at different values indicates that the results obtained with variable coefficients more accurately describe the pattern of the process than the ones with constant coefficients. The obtained development indices are the initial ones on the way to find the best development option. In this study, an attempt was made to prove that it is advisable to introduce the development technology based on inhomogeneous formation parameters into the practice of gas production under water pressure conditions. Mathematical experiments have shown that there are wide opportunities for finding highly effective design solutions in field development using this technology.

6. References
[1] Zakirov S N et al 2000 Improvement of technologies for the development of oil and gas fields (Moscow: Grail) p 642
[2] Zakirov S N et al 2004 New principles and technologies for the development of oil and gas fields (Moscow:): p 520
[3] Chernykh V A 2000 Hydro-gas-dynamics of horizontal gas wells (Moscow: VNIIGAZ) p 189
[4] Mirzadzhanzade A Kh et al 2003 Basics of gas production technology (Moscow: Nedra) p 880
[5] Samarsky A A Theory of difference schemes 1977 (Moscow: Nauka)
[6] Degtyarev L I and Favorsky A P 1969 Stream version of the sweep method for difference problems with highly variable coefficients ZhVM and MF 9 1 211
[7] Marchuk G I 1980 Methods of Computational Mathematics (Moscow: Nauka)
[8] Pirnazarova T E and Alimova I I 2018 Modern technologies in oil and gas industry Proceedings of the international scientific and technical conference in 2 volumes - Ufa, UGNTU Publishing House