Communication strength of boxes violating monogamy relations

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In any theory satisfying the no-signalling principle correlations generated among spatially separated parties in a Bell-type experiment are subject to certain constraints known as monogamy relations. Violation of such a relation implies that correlations arising in the experiment must be signalling, and, as such, they can be used to send classical information between parties. Here, we study the amount of information that can be sent using such correlations. To this aim, we first provide a framework associating them with classical channels whose capacities are then used to quantify the usefulness of these correlations in sending information. Finally, we determine the minimal amount of information that can be sent using signalling correlations violating the monogamy relation associated to the chained Bell inequalities.

Introduction. In recent years a lot of research has been devoted to probabilistic nonsignalling theories [1, 2]. They are formulated in terms of boxes, that is, families of probability distributions describing correlations generated in a Bell-type experiment by spatially separated observers. The boxes are required to satisfy the no-signalling principle which means that expectation values seen by some of the observers cannot depend on the measurement choices made by the remaining ones (see e.g. Ref. [3]). A particular example of a theory obeying the no-signalling principle is quantum mechanics. It was realized, however, that there exist nonsignalling theories which lead to higher violations of Bell inequalities than it is allowed by quantum mechanics [4]. This discovery raised a debate as to whether such supra-quantum nonsignalling correlations can be found in Nature (see, e.g., Refs. [5]).

One of the most interesting feature of the nonsignalling correlations is that they are monogamous [6–9]. Consider for instance a three-partite scenario in which Alice and Bob violate the Clauser-Horne-Shimony-Holt (CHSH) [10] or the chained [11] Bell inequality up to its maximal algebraic value. Then, each of Alice’s or Bob’s observers appearing in it cannot be correlated with an arbitrary observable measured by Eve [8]. This fact found important applications in cryptography based on nonsignalling principle [12] and randomness amplification [13] — tasks that are impossible in classical world.

Let us now consider a box violating some monogamy relation. Then, this box must be signalling. One can ask the following question: how can the box be used to send information from some parties to the other parties and, moreover, how much communication can be sent? In this Letter we answer these questions for three-partite boxes which violate monogamy relations for the CHSH and the chained Bell inequalities. Although it is believed that Nature satisfies nonsignalling condition these results are important because they give insight into the structure of monogamy relations. We also present a very simple proof of the monogamy relations introduced in [8].

Preliminaries. Before presenting our results, we need to introduce some notation and terminology. Imagine that three parties A, B, and E perform a Bell-type experiment in which A and B can measure one of M observables, denoted A_i and B_j, respectively, while the external observer E measures a single observable, which we also denote by E. We assume that all these observables have two outcomes ±1, denoted a, b, and e. The correlations that are generated in such an experiment are described by a set of probabilities \{p(A_i,B_j,E)\} = p(a,b,e|A_i,B_j), where p(a,b,e|A_i,B_j) is the probability of obtaining a, b, e when A_i, B_j, and E are measured by A, B, and E, respectively. In what follows we arrange these probabilities in vectors denoted \vec{p} and refer to them as boxes. We then say that the distribution \{p(A_i,B_j,E)\} obeys the no-signalling principle (it is nonsignalling) if any of its marginals describing a subset of parties is independent of the measurement choices made by the remaining parties. Then, by \langle XY \rangle_Z we denote the standard bipartite expectation values, which in general are conditioned on the third party’s measurement choice [e.g., \langle A_i | E \rangle_{B_j} = \sum a,b,e = \pm 1 aep(A_i,B_j,E)]; if \{p(A_i,B_j,E)\} is nonsignalling, then clearly \langle XY \rangle_{Z'} = \langle XY \rangle_Z, for any choice of X, Y, and Z ≠ Z'.

Simple derivation of a monogamy relation for the CHSH Bell inequality. For clarity we begin our considerations with the simplest scenario of M = 2.

The key ingredient of our framework is a simple proof of the monogamy relation obeyed by any nonsignalling probability distribution \{p(a,b,e|A_i,B_j)\} [6, 8]:

\[ |I_{AB}| + 2|\langle B_0 | E \rangle| \leq 4, \tag{1} \]

where \( I_{AB} \) stands for the Bell expression giving rise to the well-known CHSH Bell inequality [10]

\[ I_{AB} := \langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_1 B_1 \rangle - \langle A_0 B_1 \rangle \leq 2 \tag{2} \]

The inequality (1) compares the nonlocality shared by A and B, as measured by the violation of (2), to the (classical) correlations that the external party E can establish with outcomes of \( B_0 \). It should be noticed that it remains valid if in the last correlator, \( B_0 \) is replaced by any \( A_i \) or \( B_j \) (for clarity, however, we proceed with a fixed measurement \( B_0 \)). Also, without any loss of generality we can assume that both \( I_{AB} \) and \( \langle B_0 | E \rangle \) are positive; if this is not the case, we redefine observables \( A_0, A_1 \) and/or \( E \) in the following way: \( A_0 \rightarrow -A_0, A_0 \rightarrow -A_1, \) and/or \( E \rightarrow -E \). Consequently, in what follows we omit the absolute values in (1).
In order to prove (1), let us first make the following observation. Suppose that for some random variables $X$, $Y$ and $Z$ taking values $±1$ there exists the joint probability distribution $p(XYZ)$. Then, the latter fulfils the following inequalities

$$(−1)^i⟨XY⟩_Z + (−1)^j⟨YZ⟩_X + (−1)^k⟨XZ⟩_Y ≤ 1,$$  \hspace{1cm} (3)

with $i, j, k = 0, 1$ such that $i + j + k = 1$, where addition is modulo two. To prove (3), it suffices to check it for the deterministic distributions $p(XYZ)$.

Now, one notices that each triple of observables $A_i$, $B_j$ and $E$ is jointly measurable and therefore, for any pair $i, j$, there exists the joint probability distribution $p(A_iB_jE)$ which must satisfy (3). This gives rise to the following four inequalities

$$⟨A_0B_0⟩_E + ⟨A_0B_0⟩_{A_0} − ⟨A_0⟩_{B_0} ≤ 1,$$ \hspace{1cm} (4)

$$⟨A_1B_0⟩_E + ⟨B_0⟩_{A_1} − ⟨A_1⟩_{B_0} ≤ 1,$$ \hspace{1cm} (5)

$$⟨A_1B_1⟩_E − ⟨B_1⟩_{A_1} − ⟨A_1⟩_{B_1} ≤ 1,$$ \hspace{1cm} (6)

$$−⟨A_0B_1⟩_E + ⟨B_1⟩_{A_0} + ⟨A_1⟩_{B_1} ≤ 1.$$ \hspace{1cm} (7)

By summing these up and using the fact that in a nonsignalling theory $⟨XY⟩_Z = ⟨YZ⟩_X$ for any $Z ≠ Z'$, one obtains (1).

Signalling boxes as classical channels. Let us assume that correlators $⟨B_0E⟩_{A_0}$ and $⟨B_0E⟩_{A_1}$ are equal (later we will show how this assumption can be relaxed). Then the monogamy relation (1) is well defined. It bounds the possible correlations achievable in any no-signalling theory between outcomes of measurements performed by the three parties $A$, $B$ and $E$. If it is violated by some probability distribution $p$, then the latter must be signalling. In other words, if $p$ violates (1), then values of some bipartite correlators become dependent on the measurement choice made by the third party. This dependence allows one to use such signalling boxes to send information from a single party to the remaining two parties. To illustrate this idea, suppose that a box $p$ violates the relation (1) by $Δ > 0$, that is, $R(p) = I_{AB} + ⟨B_0E⟩ = 4 + 2Δ$. Then, by adding the inequalities (4)-(7), one concludes that

$$⟨A_0⟩_{B_0} − ⟨A_0⟩_{B_1} + ⟨A_1⟩_{B_0} − ⟨A_1⟩_{B_1} + ⟨B_1⟩_{A_1} − ⟨B_1⟩_{A_0} ≥ Δ.$$ \hspace{1cm} (8)

Consequently, in at least one of the three pairs $S_{B→AB}^0 = \{(A_0E)_{B_0}, (A_0E)_{B_1}\}$, $S_{B→AE}^1 = \{(A_1E)_{B_0}, (A_1E)_{B_1}\}$, or $S_{A→BE}^1 = \{(B_1E)_{A_1}, (B_1E)_{A_0}\}$, the correlators must differ. In particular, in one of them the difference must not be lower than $Δ/3$. The correlators in $S_{B→AB}^0$ correspond to signalling from $B$ to the pair $A$ and $E$, while those in $S_{A→BE}^1$ to signalling from $A$ to $B$ and $E$.

Let us now assume, without any loss of generality, that

$$⟨A_0⟩_{B_0} − ⟨A_0⟩_{B_1} > 0,$$ \hspace{1cm} (9)

which can be rewritten as $p − q > 0$, where $p = p(A_0E = 1 | B_0)$ and $q = p(A_0E = 1 | B_1)$. It then follows from (9) that the probability that the parties $A$ and $E$ obtain the same results while measuring $A_0$ and $E$, respectively, depends on whether the remaining party measures $B_0$ or $B_1$. This gives rise to a binary asymmetric channel, denoted $C_{B→AB}^0$, with the input and output alphabets $\{B_0, B_1\}$ and $\{0, 1\}$, respectively, and the transition probabilities given by (see Fig. 1)

$$p(A_0E = 1 | B_0) = 1 − p,$$ \hspace{1cm} (10)

$$p(A_0E = 1 | B_1) = 1 − q.$$ \hspace{1cm} (11)

The capacity of a binary asymmetric channel with the transition probabilities (10) can be explicitly written as

$$C(p, q) = H(pq) − H(pq) + \log_2 \left( 1 + 2 \frac{H(pq) − H(pq)}{q−p} \right)$$ \hspace{1cm} (12)

with $H(p)$ being the standard binary entropy.

Analogously, one associates classical channels to the other two pairs of correlators $S_{A→BE}^1$ and $S_{B→AE}^0$. As a result, any box violating the monogamy relation (1) gives rise to three channels $C_{A→BE}^1$ and $C_{B→AE}^0$ of capacities $C_{A→BE}^1 = C(p^1_A, q^1_A)$ and $C_{B→AE}^0 = C(p^0_B, q^0_B)$, where $p^1_X = (1 + x^1_X)/2$ and $q^0_X = (1 + y^0_X)/2$ are probabilities corresponding to the correlators $x^1_B = ⟨A_1E⟩_{B_0}$ and $y^0_B = ⟨A_1E⟩_{B_1}$ for $i = 0, 1$, and $x^1_A = ⟨B_1E⟩_{A}$, and $y^0_A = ⟨B_1E⟩_{A_0}$.

It should finally be noticed that a box violating (1) might also feature signalling from one or two parties to a single one; still, by definition, $E$ cannot signal to $A$ and $B$. Such situations could, however, make our considerations difficult to handle and in order to avoid them, in what follows we restrict our attention to a subclass of boxes whose all one-partite expectation values $⟨X⟩_{YZ}$ with $X, Y, Z = A_1, B_1, E$ are zero. Let us stress, nevertheless, that this assumption does not influence at all what we have said so far as for any box violating (1) there exists another one with exactly the same two-body correlators (and giving rise to exactly the same channels and the same violation of (1)) whose all one- and three-partite expectation values vanish. Precisely, given a probability distribution $\{p(A_iB_jE)\}$, the box $\{p'(A_iB_jE)\}$ with $p'(A_iB_jE) = (1/2)[p(A_iB_jE) + p(A_iB_jE)]$, where $A_1 = −A_1$ etc., has the same two-body correlators as $\{p(A_iB_jE)\}$ and all its one-partite and three-partite mean values are zero. Below we then restrict our attention to boxes having only bipartite correlators non-vanishing. They form a convex set denoted by $\mathcal{P}$. Let also $\mathcal{P}_Δ$ be the subset of $\mathcal{P}$ composed of boxes $\vec{p}$ for which $R(\vec{p}) = 4 + Δ$ with $Δ ∈ [0, 2]$. 

![FIG. 1. A binary classical channel that can be associated to one of the pair of correlators $S_{A→BE}^1$ and $S_{B→AE}^0$ (with $i = 0, 1$)](https://example.com/figure1.png)
Communication strength of boxes violating (1). Our aim now is to explore the communication strength of boxes violating (1) in terms of capacities of the three associated channels. To this aim, we will first determine a set of constraints on elements of \( P_\Delta \) that fully characterizes correlators giving rise to these channels. It follows from (3) that \( p(A_0B_1E) \) and \( p(A_1B_1E) \) obey the following inequalities

\[
\begin{align*}
(A_1B_1)_E + (B_1E)_{A_1} - (A_1B)_E &\leq 1, \\
-(A_0B)_E - (B_1E)_{A_0} - (A_0B_1)_E &\leq 1.
\end{align*}
\]

(13) (14)

Putting each of them instead of (6) and (7), we obtain four non-equivalent sets of four inequalities of the form (4)-(7). By adding them in each of these sets and assuming that (1) is violated by \( \Delta \in (0, 2] \), we arrive at the following inequalities

\[
\begin{align*}
x_B^0 - y_B^0 + x_B^1 - y_B^1 + x_A^1 - y_A^1 &\geq \Delta, \\
x_B^0 + y_B^0 + x_B^1 - y_B^1 - x_A^1 - y_A^1 &\geq \Delta, \\
x_B^0 - y_B^0 + x_B^1 + y_B^1 + x_A^1 + y_A^1 &\geq \Delta, \\
x_B^0 + y_B^0 + x_B^1 + y_B^1 - x_A^1 + y_A^1 &\geq \Delta.
\end{align*}
\]

(15) (16) (17) (18)

In Appendix A we show that for a given \( \Delta \in [0, 2] \), these inequalities and the trivial conditions \(-1 \leq (XY)_Z \leq 1 \) are the only restrictions on the two-partite correlators \( x_{A_i}^i, y_{A_i}^i, x_{B_i}^i, y_{B_i}^i \), which for further purposes we arrange in a vector \( \vec{c} \). In other words, for any vector of correlators \( \vec{c} \) satisfying (15)-(18) there always exists a probability distribution \( \vec{p} \) that realizes \( \vec{c} \) and violates (1) by \( \Delta \). On the level of correlators, this observation gives us a complete characterization of signalling in boxes violating the monogamy (1).

Having the above constraints, we are now in position to study the communication properties of boxes violating (1). More precisely, we will determine the minimal amount (nonzero) of information that can be sent from at least one party to the remaining two parties using a box \( \vec{p} \) such that \( R(\vec{p}) = 4 + \Delta \). We notice that for a given 0 < \( \Delta \leq 2 \) one might find a box for which e.g. \( C_{B\rightarrow AE}^0 > 0 \) and \( C_{B\rightarrow AE}^1 = 0 \). And, at the same time, there exists a box for which \( C_{B\rightarrow AE}^0 > 0 \) and \( C_{B\rightarrow AE}^1 = 0 \), yet they both give rise to the same violation of (1). For this reason we consider the following quantity that depends on the three capacities

\[
C_\Delta = \min_{P_\Delta} \max\{C_{A\rightarrow BE}^1, C_{B\rightarrow AE}^0, C_{B\rightarrow AE}^1, C_{B\rightarrow AE}^1\},
\]

(19)

where, due to what has been previously said, the minimization over \( P_\Delta \) can be replaced by a minimization over the polytope \( Q_\Delta \) of all vectors \( \vec{c} \) satisfying (15)-(18) and the trivial conditions \(-1 \leq (XY)_Z \leq 1 \). The quantity \( C_\Delta \) tells us that at least one of the three associated channels to any box from \( P_\Delta \) has capacity at least \( C_\Delta \).

Clearly, \( C_0 = 0 \) and in the case when (1) is violated maximally, i.e., for \( \Delta = 2 \), \( C_2 \) can be computed almost by hand and amounts to \( C_2 = 0.158 \) (see App. B for the proof). For all the intermediate values 0 < \( \Delta < 2 \) the problem of determining \( C_\Delta \) becomes difficult to handle analytically. Still it can be efficiently computed numerically. This is because the capacity (12) is a convex function in both arguments [14] and so is the function \( \max\{C_{A\rightarrow BE}^1, C_{B\rightarrow AE}^0, C_{B\rightarrow AE}^1\} \) due to the well-known property that a function resulting from a pointwise maximization of convex functions is also convex. Then, the minimization in (19) is executed over a convex polytope.

The results of our numerical computations are plotted in Fig. 2. We find that the obtained values of \( C_\Delta \) can be realized by boxes obeying the conditions \( x_B^0 = x_B^1 = \Delta/2 \) and \( x_A^1 = -y_A^0 = y_B^0 \), and the value of the remaining free parameter \( x_A^0 \) is set by the condition \( C((1+\Delta)/2, (1+x_A^0)/2) = C((1+x_A^0)/2, (1-x_A^0)/2) \). An exemplary box \{p(A_1B_1E)\} realizing \( C_\Delta \) and satisfying the above conditions is given by

\[
p(A_1B_1E) = \frac{1}{4} \left[ 1 + A_1E \left( \frac{\Delta}{2} \delta_{j,0} + x_A^1 \delta_{j,1} \right) \right] \delta_{ij,0} \delta_{A_jB_j,0} + \delta_{ij,1} \delta_{A_jB_j,-1},
\]

(20)

where \( \delta_{m,n} \) denotes the Kronecker delta, and \( x_A^1 \) is the solution of the above equation. One can see that for this box all one-partite and three-partite expectation values vanish. Moreover, it restriction \{p(A_1B_1E)\} to the parties A and B is equivalent to the so-called Popescu-Rohrlich box [4].

Let us conclude by noting that one can also drop the assumption that the correlators \( \langle B_0E \rangle_{A_0} \) and \( \langle B_0E \rangle_{A_1} \) are equal, in which case the monogamy relation (1) reads

\[
|I_{AB}| + |\langle B_0E \rangle_{A_0} + \langle B_0E \rangle_{A_1}| \leq 4.
\]

(21)

Then, following the above methodology one can associate another classical channel to the pair \( S_{A\rightarrow BE}^0 = \{\langle B_0E \rangle_{A_0}, \langle B_0E \rangle_{A_1} \} \). Our numerics shows, however, that an addition of this channel in the definition of \( C_\Delta \) does not change its value; in particular, the box (20) realizes \( C_\Delta \) and has the property that \( \langle B_0E \rangle_{A_0} = \langle B_0E \rangle_{A_1} \).

Generalization to the chained Bell inequality. The above considerations can be applied to a monogamy relation for the generalization of the CHSH Bell inequality to any number of dichotomic measurements at both sites—the chained Bell inequality [11]. To recall the latter and the corresponding monogamy, let us assume that now \( A \) and \( B \) have \( M \) dichotomic measurements at their disposal denoted \( A_k \) and \( B_k \).
The chained Bell inequality reads [11]:

\[ I_{AB}^M := \sum_{k=0}^{M-1} \left( \langle A_k B_k \rangle + \langle A_{k+1} B_k \rangle \right) \leq 2M - 2, \tag{22} \]

and, as shown in Ref. [8], it obeys the following simple monogamy relation for any nonsignalling correlations

\[ |I_{AB}^M| + 2\langle B_0 E \rangle \leq 2M. \tag{23} \]

Here, \( E \) stands for Eve’s measurement and we use the convention that \( A_M = -A_0 \). As before, we can assume that both \( I_{AB}^M \) and \( \langle B_0 E \rangle \) are nonnegative, and hence, in what follows we omit the absolute values in (23).

To proceed with our considerations we first note that analogously to (1), the monogamy (23) can be derived from (3). Precisely, as any three observables \( A_i, B_j, E \) are jointly measurable, due to (3) the following set of 2M inequalities

\[ \langle A_0 B_0 \rangle_E + \langle B_0 E \rangle_{A_0} - \langle A_0 E \rangle_{B_0} \leq 1, \tag{24} \]

\[ \langle A_1 B_0 \rangle_E + \langle B_0 E \rangle_{A_1} - \langle A_1 E \rangle_{B_0} \leq 1 \tag{25} \]

and

\[ \langle A_{i+j} B_i \rangle_E - (-1)^i \langle B_i E \rangle_{A_{i+j}} + (-1)^j \langle A_{i+j} E \rangle_{B_i} \leq 1 \tag{26} \]

with \( i = 1, \ldots, M - 1 \) and \( j = 0, 1 \) must hold. By adding them and assuming that the no-signalling principle is fulfilled, one obtains (23).

It is of importance to point out that the inequalities (24), (25), and (26) form a unique minimal set of inequalities of the form (3) that, via the above proof, give rise to the monogamy (23). To be more explicit, note that any such set must consists of at least 2M inequalities because there are that many different correlators in the Bell inequality (22). Then, each of these correlators must appear in any such 2M-element set with the same sign as in (22). As one directly checks, this is enough to conclude that the only 2M-element set is the one above.

Let us now assume as before that all correlators appearing in the monogamy relation (23) do not depend on the the third party’s measurements, in particular, \( \langle B_0 E \rangle_{A_i} = \langle B_0 E \rangle_{A_j} \) for any \( i \neq j \). Let then \( \mathcal{P}_\Delta^M \) be the convex set of boxes for which \( \mathcal{R}_M(p_B) = 2M + \Delta \) with \( \Delta \in [0, 2] \). If \( \Delta > 0 \) there must be some signalling between \( A, B, \) and \( E \) in a box \( p_B \in \mathcal{P}_\Delta^M \). In particular, it follows from (24), (25), and (26) that

\[ \sum_{i=1}^{M-1} (x_A^i - y_A^i) + \sum_{i=1}^{M-1} (x_B^i - y_B^i) + x_0 - y_0 \geq \Delta, \tag{27} \]

where \( x_A^i = \langle B_0 E \rangle_{A_i} \), \( y_A^i = \langle B_0 E \rangle_{A_{i+1}} \), \( x_B^i = \langle A_0 E \rangle_{B_{i+1}} \), and finally \( x_B^0 = \langle A_0 E \rangle_{B_0} \) and \( y_B^0 = \langle A_0 E \rangle_{B_{M-1}} \). Since \( \Delta > 0 \), this implies that in some of the following 2M - 1 pairs (perhaps all) \( S_{A_{i} \rightarrow BE} = \{ x_A^i, y_A^i \} \) with \( i = 1, \ldots, M - 1 \), and \( S_{B_{j} \rightarrow AE} = \{ x_B^j, y_B^j \} \) with \( i = 0, \ldots, M - 1 \) and \( B_{M-1} \equiv B_{M-1} \), the correlators must differ; recall that for the nonsignalling correlations, correlators belonging to each \( S_{A_{i} \rightarrow BE} \) or \( S_{B_{j} \rightarrow AE} \) are equal. In the first case this means that there is signalling from \( A \) to \( B_E \), while in the second one, from \( B \) to \( A_E \).

Now, analogously to the case \( M = 2 \), to each pair of correlators \( S_{A_{i} \rightarrow BE}^i \) and \( S_{B_{j} \rightarrow AE}^j \), can be associated a binary classical channel of capacity \( C(p_A^i, q_A^i) \) and \( C(p_B^j, q_B^j) \), respectively, where \( p_X^i = (1 + x_X^i)/2 \) and \( q_X^i = (1 + y_X^i)/2 \) with \( X = A, B \). We then quantify the communication strength of boxes from \( \mathcal{P}_\Delta^M \) by

\[ C_\Delta^M = \min_{p_B^M} \max_{i=1, \ldots, M-1} \{ C(p_A^i, q_A^i), C(p_B^j, q_B^j) \}, \tag{28} \]

which for \( M = 2 \) reduces to \( C_\Delta \).

Similarly to the case \( M = 2 \), (27) is not the only inequality bounding the values of the above correlators. In fact, each of 2(M - 1) inequalities in (26) remains satisfied if the signs in front of the second and the third correlator are swapped. By concatenating such swaps, one obtains \( 4^{M-1} \) sets of 2M inequalities and each set when summed up produces an analogous inequality to (27). All the resulting inequalities read

\[ \sum_{i=1}^{M-1} (-1)^{a_i} (x_A^i - y_B^i) + \sum_{i=1}^{M-2} (-1)^{b_i} (x_B^{i+1} - y_A^i) \]

\[-(-1)^c(y_A^{M-1} + y_B^0) + x_B + x_B^0 \geq \Delta \tag{29} \]

with \( a_i, b_i, c \in \{0, 1\} \) for \( i = 1, \ldots, M - 1 \). Although we cannot prove it as in the case \( M = 2 \), we conjecture that all possible values of the correlators in \( S_{A_{i} \rightarrow BE}^i \) and \( S_{B_{j} \rightarrow AE}^j \) that satisfy inequalities (29) can always be realized with some signalling probability distribution \( p_B \) for which \( \mathcal{R}_M(p_B) = 2M + \Delta \). In general, by minimizing the right-hand side of (28) over the correlators \( x_A^i, y_A^i, x_B^i \) and \( y_B^i \), satisfying (29) and the trivial conditions \( -1 \leq \langle XY \rangle \leq 1 \) instead of \( \mathcal{P}_\Delta^M \) leads to a lower bound on \( C_\Delta^M \).

In general, it is a hard task to compute \( C_\Delta^M \). Still, one can easily bound it from below by noting \( C_\Delta^M \) majorizes any of the capacities appearing in (28). Moreover, by consulting (27), one finds that at least one pair in \( S_{A_{i} \rightarrow BE}^i \) or \( S_{B_{j} \rightarrow AE} \) is \( S_{B_{j} \rightarrow AE} \), satisfies \( x_B - y_B \geq \Delta/(2M - 1) \). In terms of probabilities this reads \( p_B^0 - q_B^0 \geq \Delta/(4M - 2) \). Now, the lower bound on \( C_\Delta^M \) is given by the minimum of \( C(p_B^0, q_B^0) \) given the above constraint on \( p_B^0 \) and \( q_B^0 \). Using (12) we conclude that for a given value of \( \Delta \), the capacity attains the minimum for \( p_B^0 = 1 - q_B^0 \), for which the corresponding binary channel becomes symmetric whose capacity reads \( 1 - H(p_B^0) \). Therefore,

\[ C(p_B^0, q_B^0) \]

is minimized by \( p_B^0 = [1 + \Delta/(4M - 2)]/2 \) and \( q_B^0 = [1 - \Delta/(4M - 2)]/2 \), which leads to

\[ C_\Delta^M \geq 1 - H(1 + \Delta/(4M - 2))/2. \tag{30} \]

For large \( M \) the above lower bound tends to zero and for \( M = 2 \) and \( M = 3 \) it is plotted in Fig. 2.

**Conclusion.** In this work we have shown how signalling correlations violating monogamy relations can be utilized to send classical information between spatially separated observers. We have also proposed a quantity that allows one to
quantify the communication strength of such boxes. We believe that our results shed some new light on our understanding of monogamy properties of nonsignalling correlations. On the other hand, they allow us to get some insight into the structure of signalling in correlations that are not monogamous.

Let us finally notice that our analysis suggests that there is some trade-off between capacities of the three introduced channels $C_{1B \rightarrow AE}$ and $C_{1A \rightarrow BE}$. Namely, one can satisfy Ineqs. (15)-(18) with a signalling box for which two channels are of zero capacities, but then the third capacity must be high. In order to lower it, it is necessary to increase the capacity of one of the two remaining channels. It seems interesting to determine an analytical relation linking these capacities.

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APPENDIX A: CONDITIONS FOR CORRELATORS $\vec{c}$

We will now prove that for a particular violation $\Delta$, the inequalities (15)-(18) along with the trivial conditions $-1 \leq \langle XY \rangle \leq 1$ constitute the only restrictions on the two-body correlators $\vec{c} = (x_A^1, y_A^0, x_B^0, y_B^0, x_B^1, y_B^1)$ in the sense that for any such $\vec{c}$ satisfying (15)-(18), there is a box $\{p(A,B_j|E)\}$ realizing these correlators and violating (1) by $\Delta$.

Before passing to the proof, let us first introduce some additional notions. Let again $B$ be the convex set of all tripartite boxes $\{p(A,B_j|E)\}$ whose all one and three-partite expectation values vanish. Notice that such boxes are fully characterized by twelve two-body correlators $\langle A_j B_j|E \rangle, \langle A_i E|B_j \rangle$, and $\langle B_j E|A_i \rangle$, with $i, j = 0, 1, 2$.

In particular, we want to find, and, in particular, the vertices of $P$ belong to either $P_0$ or $P_2$.

Let finally $\phi : B \rightarrow \mathbb{R}^6$ be a vector-valued function associating a vector of six correlators $\vec{c} = (x_A^1, y_A^0, x_B^0, y_B^0, x_B^1, y_B^1)$ to any element of $B$. With the aid of this mapping we can associate to $P$ the following polytope

$$Q_\Delta = \{ \vec{p} \in \mathbb{R}^7 | \vec{p} \in P_\Delta \} \quad (33)$$

On the other hand, let us introduce the polytope $\tilde{Q}_\Delta$ to be a convex set of vectors of the form $\vec{c}(\Delta)$ with $\vec{c}$ satisfying the inequalities (15)-(18) for some fixed $\Delta$ along with the trivial conditions. By definition, $Q_\Delta \subseteq \tilde{Q}_\Delta$ for any $\Delta$ and our aim now is to prove that $Q_\Delta = \tilde{Q}_\Delta$. In particular, we want to show that any $\vec{c} \in Q_\Delta$ with some fixed $\Delta \geq 0$ can always be completed to a full probability distribution $\vec{p} \in P_\Delta$ violating (1) by $\Delta$.

With the above goal we define two additional polytopes

$$Q_v = \{ (\phi(\vec{p}), M(\vec{p}) - 4) \in \mathbb{R}^7 | \vec{p} \in P \} \quad (34)$$

and

$$\tilde{Q}_v = \bigcup_{\Delta \in [0,2]} \tilde{Q}_\Delta. \quad (35)$$

Direct numerical computation shows that, analogously to $P$, the vertices of $Q_v$ belong to either $Q_0$ or $Q_2$. In the same way one shows that the vertices of both polytopes $Q_v$ and $\tilde{Q}_v$ overlap, which implies that $Q_v = \tilde{Q}_v$. Using then the definition of these sets and the fact that the mapping $\vec{p} \rightarrow (\phi(\vec{p}), M(\vec{p}))$ is linear, one obtains that $Q_\Delta = \tilde{Q}_\Delta$ for any $\Delta$.

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APPENDIX B: ANALYTICAL COMPUTATION OF $C_2$

Here we determine analytically the capacity $C_\Delta$ in the case when the monogamy relation (1) is violated maximally, i.e., for $\Delta = 2$. From Ineqs. (15)-(18) it immediately follows that $x_B^0 = x_B^1 = 1$, $y_B^0 = x_A^1$, and $y_B^1 = -y_A^1$, and the problem of determining $C_2$ considerably simplifies to

$$C_2 = \min_{-1 \leq \alpha, \beta \leq 1} \max\{\tilde{C}(1, \alpha), \tilde{C}(1, \beta), \tilde{C}(\alpha, -\beta)\},$$  

(36)

where we have substituted $y_B^0 = \alpha$ and $y_B^1 = \beta$ and have denoted $\tilde{C}(\alpha, \beta) = C((1 + x)/2, (1 + y)/2)$ with $C$ defined in Eq. (12). To compute the above, it is useful to notice that the function $\tilde{C}$ satisfies $\tilde{C}(\alpha, \beta) = \tilde{C}(\alpha, -\beta) = \tilde{C}(\beta, -\alpha)$, and that it is convex in both arguments (cf. Ref. [14]). The latter implies in particular that for any $\alpha \leq 0$, $\tilde{C}(1, \alpha) \geq \tilde{C}(\alpha, \beta)$ and also $\tilde{C}(1, \alpha) \geq \tilde{C}(\alpha, -\beta)$ with $-1 \leq \beta \leq 1$. This observation suggests dividing the square $-1 \leq \alpha, \beta \leq 1$ into four ones (closed) whose facets are given by $\alpha = 0$ and $\beta = 0$, and determining $C_2$ in each of them. In fact, whenever $\alpha \leq 0$ or $\beta \leq 0$,

$$C_2 = \min_{\alpha, \beta} \max\{\tilde{C}(1, \alpha), \tilde{C}(1, \beta)\},$$

(37)

and by direct checking one obtains $C_2 = 0.322$. In order to find $C_2$ in the last region given by $\alpha \geq 0$ and $\beta \geq 0$, one first notices $\tilde{C}(1, \alpha) \geq \tilde{C}(1, \beta)$ if, and only if $\alpha \leq \beta$. This, along with the fact that $\tilde{C}(\alpha, -\beta) = \tilde{C}(\beta, -\alpha)$ means that we can restrict our attention to the case $\alpha \leq \beta$, for which

$$C_2 = \min_{\alpha \leq \beta} \max\{\tilde{C}(1, \alpha), \tilde{C}(\alpha, -\beta)\}. $$

(38)

In the last step we notice that for any $0 \leq \beta \leq 1$, $\tilde{C}(\alpha, -\beta)$ and $\tilde{C}(1, \alpha)$ are, respectively, monotonically increasing and decreasing functions of $\alpha$. Additionally, for any $0 \leq \alpha \leq 1$, $\tilde{C}(\alpha, -\beta)$ is a monotonically increasing function of $\beta$. Then, for $\alpha = 1$, $\tilde{C}(1, -1) = 1$, while $\tilde{C}(1, 1) = 0$ (recall that we assume that $\alpha \leq \beta$), and for $\alpha = 0$, $\min_{\beta \geq 0} \tilde{C}(\alpha, -\beta) = 0$ and $\tilde{C}(1, 0) > 0$. All this means that both functions $\tilde{C}(1, \alpha)$ and $\tilde{C}(\alpha, -\beta)$ intersect, implying that $C_2$ lies on the line given by $\tilde{C}(1, \alpha) = \tilde{C}(\alpha, -\beta)$. Finally, as already mentioned, $\tilde{C}(\alpha, -\beta)$ is a monotonically decreasing function of $\beta$ which together with $\alpha \leq \beta$ means that $\alpha = \beta$ has to be taken. One then arrives at the condition that $\tilde{C}(1, \alpha) = \tilde{C}(\alpha, -\alpha)$, which has a solution when for $\alpha = 0.469$ giving $C_2 = 0.158$. By comparing both minima, we finally obtain that $C_2 = 0.158$. 

