Common best proximity points in complex valued b-metric spaces

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Abstract: In this paper, we obtain some results on common best proximity points for non-self mappings between two subsets of complex valued b-metric spaces. For this purpose, first we generalize some well-known results that were proved in classic metric spaces on complex valued b-metric space by some new definitions. Second we present a type of contractive condition and develop a common best proximity point theorem for non-self mappings in complex-valued b-metric spaces. Our results are supported by some examples.

Subjects: Complex Variables; Functional Analysis; Mathematical Analysis

Keywords: common best proximity point; complex valued b-metric space; weakly dominate proximally; L-contractive condition

1. Introduction and preliminaries

Fixed point theory focuses on solving the equation $Tx = x$, where $T$ is a self-mapping defined on a subset of a metric space or other suitable spaces. If it is assumed that, $T$ is not a self-mapping then the equation $Tx = x$ does not have a solution. Consequently, the significant aim is determining an element $x$ that is in close proximity to $Tx$ in some sense. Eventually, the target is finding an element $x$ in a metric space satisfying the following conditions that $d(x, Tx) = d(A, B)$ and $d(x, Sx) = d(A, B)$ where $d$ is a metric function and $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. Now, if $T, S : A \rightarrow B$ are two non-self mappings, then the equations $Sx = x$ and $Tx = x$ are likely to have no solution, the solution known as a common fixed point of the mappings $S$ and $T$ (see, Ahmad, Azam, & Saejung, 2014; Klin-eam & Suanoom, 2013; Mukheimer, 2014; Rouzkard & Imdad, 2012; Sintunavarat & Kumam, 2012). So, the purpose is finding an element $x$ in $A$ such that $d(x, Sx) = d(A, B)$ and $d(x, Tx) = d(A, B)$ where $x$ is called the common best proximity point of mappings $S$ and $T$ in a metric space (see,
Amini-Harandi, 2014; Sadiq Basha, 2012, 2013). Azam, Fisher, and Khan (2011) introduced the notion of complex-valued metric space which is a generalization of the classical metric space and established the existence of common fixed point theorems for mappings satisfying contraction conditions (see Azam et al., 2011, Theorem 4). Rao, Swamy, and Prasad (2013) introduced the notion of complex-valued b-metric spaces. The purpose of this article is generalizing some well-known results about common best proximity points that were established in the classical metric space to the complex valued b-metric space by some new definitions, presenting a type of contractive condition and developing a common best proximity point theorem for non-self mappings in the complex valued b-metric space.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$z_1 \preceq z_2$ if and only if $\Re(z_1) \leq \Re(z_2), \Im(z_1) \leq \Im(z_2)$.

It follows that $z_1 \preceq z_2$ if and only if one of the following conditions is satisfied:

(i) $\Re(z_1) = \Re(z_2), \Im(z_1) < \Im(z_2)$,
(ii) $\Re(z_1) < \Re(z_2), \Im(z_1) = \Im(z_2)$,
(iii) $\Re(z_1) < \Re(z_2), \Im(z_1) < \Im(z_2)$,
(iv) $\Re(z_1) = \Re(z_2), \Im(z_1) = \Im(z_2)$.

In particular, we will write $z_1 \triangleleft z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied where we denote $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$0 \leq z_1 \triangleleft z_2 \implies |z_1| < |z_2|$,
$z_1 \preceq z_2, z_2 \triangleleft z_3 \implies z_1 \prec z_3$.

Definition 1 (Azam et al., 2011) Let $X$ be a nonempty set. Suppose that the mapping $d:X \times X \rightarrow \mathbb{C}$, satisfies:

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then $d$ is called a complex-valued metric on $X$, and $(X, d)$ is called a complex-valued metric space.

Example 1 Let $X = \mathbb{C}$. Define the mapping $d:X \times X \rightarrow \mathbb{C}$ for all $z_1, z_2 \in X$, by

$d(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|$.

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Clearly, the pair $(X, d)$ is a complex-valued metric space.

Definition 2 (Rao et al., 2013) Let $X$ be a nonempty set and $s \geq 1$ be a given real number. Suppose that the mapping $d:X \times X \rightarrow \mathbb{C}$, satisfies:

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, z) \leq s(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

Then $d$ is called a complex-valued b-metric on $X$, and $(X, d)$ is called a complex-valued b-metric space (with constant $s$).
Example 2 Let \( X = \mathbb{C} \). Define the mapping \( d : X \times X \to \mathbb{C} \) for all \( z_1, z_2 \in X \), by

\[
d(z_1, z_2) = |x_1 - x_2|^2 + |y_1 - y_2|^2.
\]

where \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \). Clearly, the pair \((X, d)\) is a complex valued b-metric space with \( s = 2 \).

Definition 3 (Rao et al., 2013) Let \((X, d)\) be a complex valued b-metric. Consider the following.

(a) A point \( x \in X \) is called interior point of a set \( A \subseteq X \) whenever there exists \( 0 < r \in \mathbb{C} \) such that \( B(x, r) = \{ y \in X : d(x, y) < r \} \subseteq A \).

(b) A point \( x \in X \) is called a limit point of a set \( A \subseteq X \) whenever, for every \( 0 < r \in \mathbb{C} \), \( B(x, r) \cap (A - X) \neq \emptyset \).

(c) A subset \( A \subseteq X \) is called open whenever each element of \( A \) is an interior point of \( A \).

(d) A subset \( A \subseteq X \) is called closed whenever each limit point of \( A \) belongs to \( A \).

(e) A sub-basis for a Hausdorff topology on \( X \) is a family \( F = \{ B(x, r) : x \in X \) and \( 0 < r \} \).

Definition 4 (Choudhury, Metiya, & Maity, 2014) Let \( A \) be a subset of \( \mathbb{C} \). If there exists \( u \in \mathbb{C} \) such that \( z \leq u \) for all \( z \in A \), then \( A \) is bounded above and \( u \) is an upper bound. Similarly, if there exists \( l \in \mathbb{C} \) such that \( l \leq z \), for all \( z \in A \), then \( A \) is bounded below and \( l \) is a lower bound.

Definition 5 (Choudhury et al., 2014) For a \( A \subseteq \mathbb{C} \) which is bounded above if there exists an upper bound \( s \) of \( A \) such that, for every upper bound \( u \) of \( A \), \( s \leq u \), then the upper bound \( s \) is called \( \sup A \).

Similarly, for a subset \( A \subseteq \mathbb{C} \) which is bounded below if there exists a lower bound \( t \) of \( A \) such that for every lower bound \( l \) of \( A \), \( l \leq t \), then the lower bound \( t \) is called \( \inf A \).

Suppose that \( A \subseteq \mathbb{C} \) is bounded above. Then there exists \( q = u + iv \in \mathbb{C} \) such that \( z = x + iy \leq q = u + iv \), for all \( z \in A \). It follows that \( x \leq u \) and \( y \leq v \), for all \( z = x + iy \in A \); i.e. \( S = \{ xz = x + iy \in A \} \) and \( T = \{ yz = x + iy \in A \} \) are two sets of real numbers which are bounded above. Hence both \( \sup S \) and \( \sup T \) exist. Let \( \bar{x} = \sup S \) and \( \bar{y} = \sup T \). Then \( z = \bar{x} + i\bar{y} \) is \( \sup A \).

Similarly, if \( A \subseteq \mathbb{C} \) is bounded below, then \( z^\ast = x^\ast + iy^\ast \) is \( \inf A \), where \( x^\ast = \inf S = \inf \{ x^2 = x + iy \in A \} \) and \( y^\ast = \inf T = \inf \{ y^2 = x + iy \in A \} \).

Any subset \( A \subseteq \mathbb{C} \) which is bounded above has supremum. Equivalently, any subset \( A \subseteq \mathbb{C} \) which is bounded below has infimum.

Definition 6 (Azam et al., 2011; Rao et al., 2013) Let \((X, d)\) be a complex-valued b-metric space. Let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \).

(i) If for every \( c \in \mathbb{C} \), with \( 0 < c \) there is \( n_0 \in \mathbb{N} \) such that \( d(x_n, x) < c \), for all \( n > n_0 \), then \( \{x_n\} \) is said to be convergent, \( \{x_n\} \) converges to \( x \), \( x \) is the limit point of \( \{x_n\} \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) as \( n \to \infty \).

(ii) If for every \( c \in \mathbb{C} \), with \( 0 < c \) there is \( N \in \mathbb{N} \) such that for all \( n > N \), \( d(x_n, x_{n+m}) < c \), for all \( m \in \mathbb{N} \), then \( \{x_n\} \) is said to be Cauchy sequence.

(iii) If every Cauchy sequence is convergent in \((X, d)\), then \((X, d)\) is called a complete complex-valued b-metric space.

Remark 1 In a b-metric space \((X, d)\), the following assertions hold:

(i) a convergent sequence has a unique limit;
(ii) each convergent sequence is Cauchy;
(iii) in general, a b-metric is not continuous.
The following Lemmas prove like Lemmas 3 and 2 in Azam et al. (2011), respectively.

**Lemma 1** Let \((X, d)\) be a complex valued \(b\)-metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if \(|d(x_n, x_{n+m})| \to 0\) as \(n \to \infty\).

**Lemma 2** Let \((X, d)\) be a complex-valued \(b\)-metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) if and only if \(|d(x_n, x)| \to 0\) as \(n \to \infty\).

Given nonempty subsets \(A\) and \(B\) of complex-valued \(b\)-metric space \((X, d)\). Then \(\{d(x, y): x \in A, y \in B\} \subseteq \mathbb{C}\) is always bounded below by \(z_0 = 0 + i0\) and hence inf\(\{d(x, y): x \in A, y \in B\}\) exists. Here we define

\[
d(A, B) = \inf\{d(x, y): x \in A \text{ and } y \in B\},
\]

\(A_0 = \{x \in A: d(x, y) = d(A, B) \text{ for some } y \in B\},\)

\(B_0 = \{y \in B: d(x, y) = d(A, B) \text{ for some } x \in A\}.

From the above definition, it is clear that for every \(x \in A_0\) there exists \(y \in B_0\) such that \(d(x, y) = d(A, B)\) and conversely, for every \(y \in B_0\) there exists \(x \in A_0\) such that \(d(x, y) = d(A, B)\).

**Definition 7** Given non-self mapping \(S: A \to B\) and \(T: A \to B\), an element \(x \in X\) is called a common best proximity point of the mappings if it satisfies the condition

\[
d(x, Sx) = d(A, B) = d(x, Tx).
\]

**Definition 8** Let \((A, B)\) be a pair of nonempty subsets of a complex-valued \(b\)-metric space \((X, d)\) with \(A_0 \neq \emptyset\). Then that pair \((A, B)\) is said to have the weak P-property if and only if

\[
\begin{align*}
d(x_1, y_1) &= d(A, B) \\
d(x_2, y_2) &= d(A, B)
\end{align*}
\]

implies \(d(x_1, x_2) \leq d(y_1, y_2),\) \(\text{for all } x_1, x_2 \in A_0 \text{ and } y_1, y_2 \in B_0.\)

**Definition 9** Let \(A\) and \(B\) be two non-empty subsets of a complex-valued \(b\)-metric space \((X, d)\). The mappings \(S: A \to B\) and \(T: A \to B\) are said to commute proximally if they satisfy the condition that

\[
[d(u, Sx) = d(v, Tx) = d(A, B)] \Rightarrow Sv = Tu.
\]

**Definition 10** Let \(A\) and \(B\) be two non-empty subsets of a complex-valued \(b\)-metric space \((X, d)\) with \(s \geq 1\). Non-self mappings \(S, T: A \to B\) are said to satisfy a \(L\)-contractive condition if there exist non-negative numbers \(a_i\) where \(i = 1, \ldots, 4\) and \(s(a_1 + a_2) + a_3 + s(s + 1)a_4 < 1\), then for each \(x, y \in A,\)

\[
d(Sx, Sy) \leq a_1d(Tx, Ty) + a_2d(Tx, Sx) + a_3d(Ty, Sy) + a_4[d(Ty, Sx) + d(Sy, Tx)].
\]

**Definition 11** Let \((X, d)\) be a complex-valued \(b\)-metric space. A mapping \(T: A \to B\) is said to dominate a mapping \(S: A \to B\) proximally if there exists a non-negative real number \(\alpha < \frac{1}{s}\) such that for all \(u_1, u_2, v_1, v_2, x_1, x_2 \in A,\)

\[
d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2) \Rightarrow d(u_1, u_2) \leq \alpha d(v_1, v_2).
\]

**Definition 12** Let \((X, d)\) be a complex-valued \(b\)-metric space. A mapping \(T: A \to B\) is said to weakly dominate a mapping \(S: A \to B\) proximally if there exists a non-negative real number \(\alpha < \frac{1}{s}\) such that for all \(u_1, u_2, v_1, v_2, x_1, x_2 \in A,\)
d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2)
\Rightarrow d(u_1, u_2) \leq \alpha \omega_{u_1, u_2, v_1, v_2},

where \(\omega_{u_1, u_2, v_1, v_2} = \text{Re}\omega_{u_1, u_2, v_1, v_2} + i\text{Im}\omega_{u_1, u_2, v_1, v_2}\) and

\[
\begin{align*}
\text{Re}\omega_{u_1, u_2, v_1, v_2} &= \max \left\{ \frac{\text{Re}d(v_1, v_2) + \text{Re}d(v_1, u_2)}{2s}, \frac{\text{Re}d(v_2, v_1) + \text{Re}d(v_2, u_1)}{2s} \right\}, \\
\text{Im}\omega_{u_1, u_2, v_1, v_2} &= \max \left\{ \frac{\text{Im}d(v_1, v_2) + \text{Im}d(v_1, u_2)}{2s}, \frac{\text{Im}d(v_2, v_1) + \text{Im}d(v_2, u_1)}{2s} \right\}.
\end{align*}
\]

If \(T\) dominates \(S\) then \(T\) weakly dominates \(S\). But the converse is not true.

**Example 3** Let us consider the complex valued b-metric space \((X, d)\) with \(s = 2\), where \(X = \mathbb{C}\) and let \(d:X \times X \rightarrow \mathbb{C}\) be given as

\[d(z_1, z_2) = |x_1 - x_2|^2 + |y_1 - y_2|^2,\]

where \(z_1 = x_1 + iy_1\) and \(z_2 = x_2 + iy_2\). Let \(A\) and \(B\) be two subsets of \(X\) given by

\[A = \{ z \in \mathbb{C} : \text{Re}(z) = -1, 0 \leq \text{Im}(z) \leq 1 \},\]
\[B = \{ z \in \mathbb{C} : \text{Re}(z) = 1, 0 \leq \text{Im}(z) \leq 1 \}.

So we have that \(A_0 = A, B_0 = B\) and \(d(A, B) = 4 + 0i\). Let \(T, S:A \rightarrow B\) be defined as

\[Tz = -x + iy\] for each \(z = x + iy \in A\)

and

\[Sx = \begin{cases} 1 + i & 0 \leq y < 1 \\ 1 + i & y = 1 \end{cases}\]

for each \(z = x + iy \in A\). If we suppose that \(v_1 = x_1 = -1 + \frac{12}{11}i, v_2 = x_2 = -1 + i, u_1 = -1 + \frac{1}{2}i, u_2 = -1 + \frac{1}{5}i\), it implies that

\[d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2).
\]

Clearly, \(0 + \left( \frac{1}{12} \right)^2 i = d(u_1, u_2) \nless a \omega_1 d(v_1, v_2) = a \left( 0 + \left( \frac{1}{11} \right)^2 i \right)\) for each non-negative real number \(a < 1\). But obviously, we have that for \(a = \frac{1}{12}\), \(T\) weakly dominates \(S\) proximally.

2. **Common best proximity point by weakly dominate proximally property**

**Theorem 1** Let \((X, d)\) be a complete complex-valued b-metric space with \(s \geq 1\), \(A\) and \(B\) be two non-empty subsets of \(X\). Assume that \(A_0\) and \(B_0\) are nonempty and \(A_0\) is closed. Let \(S:A \rightarrow B\) and \(T:A \rightarrow B\) be two non-self mappings that satisfy the following conditions:

(a) \(T\) weakly dominates \(S\) proximally
(b) \(S\) and \(T\) commute proximally
(c) \(S\) and \(T\) are continuous
(d) \(S(A_0) \subseteq B_0\)
(e) \(S(A_0) \subseteq T(A_0)\)
Then there exists a unique element \( x \in A \) such that

\[
d(x, Tx) = d(A, B) \quad \text{and} \quad d(x, Sx) = d(A, B).
\]

**Proof**  Let \( x_0 \) be a fixed element in \( A \). Since \( S(A_0) \subseteq T(A_0) \), then there exists an element \( x_1 \in A_0 \) such that

\[
Sx_0 = Tx_0.
\]

Then by continuing this process we can choose \( x_n \in A_0 \) such that there exists \( x_{n+1} \in A_0 \) satisfying

\[
Sx_n = Tx_{n+1} \quad \text{for each } n \in N
\]

since \( S(A_0) \subseteq B_0 \), there exists an element \( u_n \in A \) such that

\[
d(Sx_n, u_n) = d(A, B) \quad \text{for each } n \in N.
\]  

(2)

By choosing \( x_n \) and \( u_n \), it follows that

\[
d(Sx_n, u_n) = d(Sx_{n+1}, u_{n+1}) = d(A, B) = d(Tx_n, u_{n-1}) = d(Tx_{n+1}, u_n)
\]  

(3)

Since \( T \) weakly dominates \( S \) proximally then we have

\[
d(u_n, u_{n+1}) \leq \alpha \alpha_n \quad \text{for all } n \in N,
\]

where \( \alpha < 1 \) and

\[
\Re \alpha \alpha_n u_{n+1} u_n = \alpha \max \left\{ \Re d(u_{n-1}, u_n), \Re d(u_{n-1}, u_n), \frac{\Re d(u_{n-1}, u_{n+1}) + \Re d(u_{n}, u_n)}{2s} \right\}
\]

and

\[
\Im \alpha \alpha_n u_{n+1} u_n = \alpha \max \left\{ \Im d(u_{n-1}, u_n), \Im d(u_{n-1}, u_n), \frac{\Im d(u_{n-1}, u_{n+1}) + \Im d(u_{n}, u_n)}{2s} \right\}
\]

We focus on \( \Re d(u_n, u_{n+2}) \) and conclude for \( \Im d(u_n, u_{n+1}) \) and finally for \( d(u_n, u_{n+1}) \).

\[
\Re d(u_n, u_{n+2}) \leq \alpha \max \left\{ \frac{\Re d(u_{n-1}, u_n)}{2s}, \frac{\Re d(u_{n-1}, u_{n+1}) + \Re d(u_n, u_{n+1})}{2s} \right\}
\]

We will prove that \( \{u_n\} \) is a Cauchy sequence. We distinguish two cases.

**Case I.** Suppose that

\[
\Re d(u_n, u_{n+1}) \leq \alpha \Re d(u_{n-1}, u_n)
\]

so we get that

\[
\Re d(u_n, u_{n+1}) \leq \alpha^n \Re d(u_0, u_1).
\]  

(4)
Let $m, n \in \mathbb{N}$ and $m > n$, we have
\[
\text{Re} \ d(u_n, u_m) \leq s \left[ \text{Re} \ d(u_n, u_{n+1}) + \text{Re} \ d(u_{n+1}, u_m) \right]
\leq s \left[ \text{Re} \ d(u_n, u_{n+1}) + s^2 \left[ \text{Re} \ d(u_{n+1}, u_{n+2}) + \text{Re} \ d(u_{n+2}, u_m) \right] \right]
\leq s \left[ \text{Re} \ d(u_n, u_{n+1}) + s^2 \text{Re} \ d(u_{n+1}, u_{n+2}) + \ldots \right.
\left. + s^{m-n-1} \left[ \text{Re} \ d(u_{m-2}, u_{m-1}) + \text{Re} \ d(u_{m-1}, u_m) \right] \right]
\leq s \left[ \text{Re} \ d(u_n, u_{n+1}) + s^2 \text{Re} \ d(u_{n+1}, u_{n+2}) + \ldots \right.
\left. + s^{m-n-1} \text{Re} \ d(u_{m-2}, u_{m-1}) + s^{m-n} \text{Re} \ d(u_{m-1}, u_m) \right].
\]

By (4) and $s \alpha < 1$, we conclude that
\[
\text{Re} \ d(u_n, u_m) \leq \left( sa^n + s^2 a^{n+1} + \ldots + s^{m-n} a^{m-1} \right) \text{Re} \ d(u_0, u_1)
= sa^n(1 + sa + \ldots + (sa)^{m-n-1}) \text{Re} \ d(u_0, u_1)
\leq \frac{sa^n}{1 - sa} \text{Re} \ d(u_0, u_1) \to 0 \text{ as } m, n \to \infty.
\]

**Case II.** Assume that
\[
\text{Re} \ d(u_n, u_{n+1}) \leq \frac{a}{2} \text{Re} \ d(u_{n-1}, u_n).
\]

Put $h = \frac{\sqrt{a}}{1 - \frac{a}{2}} < 1$, (note that $sh < 1$), so we have that
\[
\text{Re} \ d(u_n, u_{n+1}) \leq h^n \text{Re} \ d(u_0, u_1).
\]

Like above, for any $m > n$ where $m, n \in \mathbb{N}$ we have
\[
\text{Re} \ d(u_n, u_m) \to 0 \text{ as } m, n \to \infty.
\]

Similarly, we can conclude that for any $m > n$ where $m, n \in \mathbb{N}$
\[
\text{Im} \ d(u_n, u_m) \to 0 \text{ as } m, n \to \infty,
\]

This implies that for any $m > n$, where $m, n \in \mathbb{N}$
\[
d(u_n, u_m) \to 0 \text{ as } m, n \to \infty.
\]

Then $\{u_n\}$ is a Cauchy sequence and since $X$ is complete and $A_0$ is closed, there exists $u \in A_0$ such that $u_n \to u$. By hypothesis, mappings $S$ and $T$ are commuting proximally and by (3) we have that
\[
Tu_n = Su_{n-1}, \quad \text{for every } n \in \mathbb{N}.
\]

Since $T$ and $S$ are continuous it implies that
\[
Tu = \lim_{n \to \infty} Tu_n = \lim_{n \to \infty} Su_{n-1} = Su.
\]

As $Su \in S(A_0) \subseteq B_0$, there exists an $x \in A_0$ such that
\[
d(x, Su) = d(A, B) = d(x, Tu).
\]

Since $S$ and $T$ commute proximally, $Sx = Tx$. Also, $Sx \in S(A_0) \subseteq B_0$, there exists a $z \in A_0$ such that
\[
d(x, Su) = \frac{d(x, Sx) + d(Sx, Su)}{2} = d(x, Sx) = d(x, Tz) = d(x, Tu).
\]

\[\text{(5)}\]
Since \( T \) weakly dominates \( S \) then from (5) and (6), we can conclude that
\[
d(x, z) ≤ α \omega_{x, z, x_1} = α(\Re d(x, z) + i \Im d(x, z)) = α d(x, z).
\]

It follows that \( x = z \), therefore we have that
\[
d(x, Sx) = d(A, B) = d(x, Tx).
\]

We now show that \( S \) and \( T \) have unique common best proximity point. For this, assume that \( x^* \) in \( A \) is a second common best proximity point of \( S \) and \( T \), then
\[
d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*).
\]

Since \( T \) weakly dominates \( S \) proximally then from (7) and (8), we have
\[
d(x, x^*) ≤ ad(x, x^*).
\]

Consequently, \( x = x^* \) and \( S \) and \( T \) have a unique common best proximity point. \( \square \)

**Example 4** Let us consider the complex-valued b-metric space \( (X, d) \) with \( s = 2 \), where \( X = \mathbb{C} \) and let \( d: X \times X \to \mathbb{C} \) be given as
\[
d(z_1, z_2) = |x_1 - x_2|^2 + |y_1 - y_2|^2,
\]
where \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \). Let \( A \) and \( B \) be two subsets of \( X \) given by
\[
A = \{ z \in \mathbb{C} : \Re(z) = -1, \ 0 ≤ \Im(z) ≤ 1 \} \\
\cup \{ z \in \mathbb{C} : \Re(z) = 1, \ 0 ≤ \Im(z) ≤ 1 \},
\]
\[
B = \{ z \in \mathbb{C} : \Re(z) = -2, \ 0 ≤ \Im(z) ≤ 1 \} \\
\cup \{ z \in \mathbb{C} : \Re(z) = 2, \ 0 ≤ \Im(z) ≤ 1 \}.
\]

Then \( A \) and \( B \) are closed and bounded subsets of \( X \) such that
\[
d(A, B) = 1, \ A_0 = A, \ B_0 = B.
\]

Let \( T, S: A \to B \) be defined as
\[
Tz = 2|x| + iy \quad \text{for each } z = x + iy \in A
\]

and
\[
Sz = 2|x| + \frac{iy}{2} \quad \text{for each } z = x + iy \in A.
\]

Therefore \( T \) and \( S \) satisfy the properties mentioned in Theorem 1. Hence the conditions of Theorem 1 are satisfied and \( 1 + 0i \) is the unique common best proximity point of \( S \) and \( T \).

By Theorem 1, we obtain the following results in the fixed point theorem.

**Corollary 1** Let \( (X, d) \) be a complex-valued b-metric space with \( s ≥ 1 \). Let \( S, T: X \to X \) be continuous mappings and \( T \) commutes with \( S \). Further let \( S \) and \( T \) satisfy \( S(X) \subseteq T(X) \) and there exists a constant \( α < \frac{1}{s} \) such that for every \( x, y \in X \).
\[ d(Sx, Sy) \leq \alpha \omega_{Sx, Sy, Tx, Ty}, \]

where
\[
\begin{align*}
\text{Re} \omega_{Sx, Sy, Tx, Ty} &= \max \left\{ \frac{\text{Re} d(Tx, Ty) + \text{Re} d(Tx, Sx)}{2s}, \frac{\text{Re} d(Ty, Sx)}{2s} \right\}, \\
\text{Im} \omega_{Sx, Sy, Tx, Ty} &= \max \left\{ \frac{\text{Im} d(Tx, Ty) + \text{Im} d(Tx, Sx)}{2s}, \frac{\text{Im} d(Ty, Sx)}{2s} \right\}.
\end{align*}
\]

Then \( S \) and \( T \) have a unique common fixed point.

If \( T \) is assumed to be identity mapping in Corollary 1, then we have the following result.

**Corollary 2** Let \( (X, d) \) be a complex-valued b-metric space with \( s \geq 1 \), \( S \) be a continuous self-mapping on \( X \) and there exists a constant \( \alpha < \frac{1}{s} \) such that for every \( x, y \in X \)

\[ d(Sx, Sy) \leq \alpha \omega_{Sx, Sy, x, y}, \]

where
\[
\begin{align*}
\text{Re} \omega_{Sx, Sy, x, y} &= \max \left\{ \frac{\text{Re} d(x, y) + \text{Re} d(x, Sx)}{2s}, \frac{\text{Re} d(y, Sx)}{2s} \right\}, \\
\text{Im} \omega_{Sx, Sy, x, y} &= \max \left\{ \frac{\text{Im} d(x, y) + \text{Im} d(x, Sx)}{2s}, \frac{\text{Im} d(y, Sx)}{2s} \right\}.
\end{align*}
\]

Then \( S \) has a unique fixed point.

### 3. Common best proximity point for L-contractive condition mappings

**Theorem 2** Let \( (X, d) \) be a complete complex valued b-metric space with \( s \geq 1 \), \( A \) and \( B \) be two non-empty subsets of \( X \) and the pair \((A, B)\) satisfies the weak P-property. Moreover assume that \( A_0 \) and \( B_0 \) are non-empty and \( A_0 \) is closed. Let \( S, T : A \rightarrow B \) be two non-self mappings satisfying the following conditions:

\( (a) \) \( S \) and \( T \) commute proximally;
\( (b) \) \( S \) and \( T \) are continuous;
\( (c) \) \( S(A_0) \subseteq B_0 \) and \( S(A_0) \subseteq T(A_0) \);
\( (d) \) \( S \) and \( T \) satisfy L-contractive condition.

Then, there exists a unique point \( x \in A \) such that

\[ d(x, Tx) = d(A, B) = d(x, Sx). \]

**Proof** Let \( x_0 \) be a fixed element in \( A_0 \). Since \( S(A_0) \subseteq T(A_0) \), then there exists an element \( x_1 \in A_0 \) such that \( Sx_0 = Tx_1 \). Then by continuing this process we can choose \( x_n \in A_0 \) such that there exists \( x_{n+1} \in A_0 \) satisfying

\[ Sx_n = Tx_{n+1} \quad \text{for each} \quad n \in \mathbb{N}. \quad (9) \]

Since \( S(A_0) \subseteq B_0 \) there exists an element \( u_n \in A_0 \) such that

\[ d(Sx_n, u_n) = d(A, B) \quad \text{for each} \quad n \in \mathbb{N}. \quad (10) \]

Further, it follows from the choice \( x_n \) and \( u_n \) that

\[ d(Sx_n, u_n) = d(A, B) = d(Sx_{n+1}, u_{n+1}). \]
Using the weak P-property and L-contractive condition, we have
\[
d(u_n, u_{n+1}) \leq d(Sx_n, Sx_{n+1}) \\
\leq \alpha_1 d(Tx_n, Tx_{n+1}) + \alpha_2 d(Tx_n, Sx_n) + \alpha_3 d(Tx_{n+1}, Sx_{n+1}) \\
+ \alpha_4 d(Tx_{n+1}, Tx_n) + d(Sx_{n+1}, Tx_n)
\]
\[
\leq \alpha_1 d(Sx_{n-1}, Sx_n) + \alpha_2 d(Sx_{n-1}, Sx_n) + \alpha_2 d(Sx_{n-1}, Sx_n) + \alpha_4 d(Sx_{n-1}, Sx_n) \\
+ \alpha_4 d(Sx_{n-1}, Sx_n) + d(Sx_n, Sx_{n+1}).
\]
Consequently, it implies that
\[
d(u_n, u_{n+1}) \leq h d(Sx_{n-1}, Sx_n) \leq \cdots \leq h^n d(Sx_0, Sx_1),
\]
where \( h = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} < 1 \), (note that \( s(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) < 1 \)).

Let \( m, n \in N \) and \( m > n \), we have
\[
d(u_n, u_m) \leq \sum_{k=n}^{m-1} d(u_k, u_{k+1}) \leq \sum_{k=n}^{m-1} s^k d(u_n, u_{n+1}) + \sum_{k=n}^{m-1} s^{k+1} d(u_{n+1}, u_{n+2}) + \cdots \\
+ s^{m-1} d(u_{m-1}, u_{m-2}) + s^m d(u_{m-1}, u_{m}).
\]
By (11) and \( sh < 1 \), we conclude that
\[
d(u_n, u_m) \leq (sh^n + s^2 h^{n+1} + \cdots + s^{m-n} h^{m-1}) d(Sx_0, Sx_1) \\
= sh^n(1 + sh + \cdots + (sh)^{m-n-1}) d(Sx_0, Sx_1) \\
\leq \frac{sh^n}{1 - sh} d(Sx_0, Sx_1) \to 0 \text{ as } m, n \to \infty.
\]
Therefore, \( \{u_n\} \) is a Cauchy sequence and there exists \( u \in A_0 \) such that \( u_n \to u \) as \( n \to \infty \). Also, we have that
\[
d(Sx_n, u_n) = d(A, B) = d(Tx_n, u_{n-1}).
\]
Since \( S \) and \( T \) commute proximally, we get that
\[
Tu_n = Su_{n-1}.
\]
Thus, it follows that \( Tu = Su \), because \( S \) and \( T \) are continuous. Since \( Su \in S(A_0) \subseteq B_\infty \), there exists \( x \in A_0 \) such that
\[
d(x, Su) = d(A, B) = d(x, Tu).
\]
Therefore, \( Tx = Sx \), because \( S \) and \( T \) commute proximally. Since \( Sx \in S(A_0) \subseteq B_\infty \), there exists \( z \in A_0 \) it implies that
\[
d(z, Sx) = d(A, B) = d(z, Tx).
\]
By L-contractive condition, we get that
Therefore, \( Su = Sx \). From (12) and (13), we have the weak P-property of the pair \((A, B)\) implies
\[
d(x, z) = d(Sx, Su).
\]
Suppose that \( x^* \) is another common best proximity point of the mappings \( S \) and \( T \) so that
\[
d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*). \tag{16}
\]
Since \( S \) and \( T \) commute proximally, then \( Sx = Tx \) and \( Sx^* = Tx^* \). So we have
\[
d(Sx, Sx^*) \preceq \alpha_1 d(Tx, Tx^*) + \alpha_2 d(Tx, Sx) + \alpha_3 d(Tx^*, Sx^*)
\]
\[
+ \alpha_4 [d(Tx^*, Sx) + d(Tx, Sx^*)]
\]
\[
= (\alpha_1 + 2\alpha_4) d(Sx, Sx^*),
\]
which implies that \( Sx = Sx^* \). Since the pair \((A, B)\) satisfies weak P-property, from (15) and (16) we have that
\[
d(x, x^*) \preceq d(Sx, Sx^*).
\]
Eventually, we have that \( x = x^* \). Hence \( S \) and \( T \) have a unique common best proximity point. \( \square \)

**Example 5** Let \((X, d)\) be a complex-valued b-metric space defined as in Example 4 and \( A, B \) be two subsets of \( X \) given by
\[
A = \{ z \in \mathbb{C} : \text{Re}(z) = 0, \ 0 \leq \text{Im}(z) \leq 1 \},
\]
\[
B = \{ z \in \mathbb{C} : \text{Re}(z) = 1, \ 0 \leq \text{Im}(z) \leq 1 \}.
\]
Let \( T : A \to B \) be defined as
\[
T(0 + iy) = 1 + iy \quad \text{for each} \ 0 \leq y \leq 1
\]
and
\[
S(0 + iy) = 1 + i \frac{y}{4} \quad \text{for each} \ 0 \leq y \leq 1.
\]
Then \((A, B)\) is a pair of nonempty closed and bounded subsets of \( X \) such that \( A_0 = A, B_0 = B \) and \( d(A, B) = 1 + 0i \). It is verified that the \((A, B)\) satisfies the weak P-property. Also \( T \) and \( S \) satisfy the properties mentioned in Theorem 2. Hence the conditions of Theorem 2 are satisfied and it is seen that \( 0 = 0 + 0i \) is the unique common best proximity point of \( S \) and \( T \).

If we suppose that \( S \) and \( T \) are self-mappings on \( X \), then Theorem 2 implies the following common fixed point theorem, that generalizes and complements the results of Hardy and Rogers (1973), Jungck (1976), Reich (1971a, 1971b) and others in complex-valued b-metric spaces.
**Corollary 3** Let \((X, d)\) be a complete complex valued \(b\)-metric space with \(s \geq 1\). Moreover, assume that \(S, T : X \rightarrow X\) are two self mappings satisfying the following conditions:

(a) \(S\) and \(T\) commute;
(b) \(S(X) \subseteq T(X)\);
(c) \(T\) and \(S\) are continuous;
(d) \(S\) and \(T\) satisfy \(L\)-contractive condition.

Then \(S\) and \(T\) have a unique common fixed point.