Instantons in $\sigma$ model and tau functions

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Abstract

We show that a number of multiple integrals may viewed as tau functions of various integrable hierarchies. The instanton contributions in the two-dimensional O(3) $\sigma$ model is an example of such an approach.

1 Introduction

The purpose of this paper is to interpretate the the contribution of instantons in Euclidean Green function of the O(3) non-linear $\sigma$ model (or the continuum classical Heisenberg ferromagnet in two space dimensions) in terms of tau functions of integrable hierarchies. This model can be described by the action

$$S = \frac{1}{2f} \int \sum_{a=1}^{3} (\partial_\mu \sigma^a (x))^2$$ (1.1)

with $\sum_{a=1}^{3} \sigma^a (x) \sigma^a (x) = 1$ ; $\mu = 0,1$.

The model similar to a Yang-Mills theory and posesses exact multi-instanton solutions. The Euclidean Green functions can be represented in the form

$$\frac{\int \phi (\sigma) \exp (-S) \prod_x d\sigma (x)}{\int \exp (-S) \prod_x d\sigma (x)}$$ (1.2)

Here $\phi (\sigma)$ is an arbitrary functional of $\sigma$. If we parametrize $\sigma (x)$ with use of the complex function $\omega = \sigma^1 + i \sigma^2$ obtained from the field $(\sigma^1, \sigma^2, \sigma^3)$ and the complex variable $z = x_0 + ix_1$ instead of the time and space coordinate $x_0, x_1$, then the instanton is the solution of the equation $\delta S = 0$ with the topological charge $q > 0$ is given

$$\omega (z) = c (z - a_1) ... (z - a_q) (z - b_1) ... (z - b_q)$$ (1.3)

where $c, a_i, b_i$ are arbitrary complex parameters.

2 The instanton contribution and the $\tau$ function

In [2] the following integral was obtained as the contribution of instantons in Euclidean Green function in the form

$$Z_q (\phi) = \sum_{q \geq 0} \frac{K^q}{(q!)^2} \int \phi (a, b, c) \prod_{i < j \leq q} |a_i - a_j|^2 \prod_{i < j \leq q} |b_i - b_j|^2 \prod_{i < j \leq q} \frac{1}{|a_i - b_j|^2} \prod i d^2 a_i d^2 b_i \frac{d^2 c}{1 + |c|^2}$$ (2.1)

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where $K$ is a real constant and $\phi(a, b, c)$ is an arbitrary function of $c, a_i, b_i$.

Compare with the $\tau$ function of the two-component KP (about two-component KP see [3]) in terms of free fermionic formalism

$$\psi^{(\alpha)}(z) = \sum_{i \in \mathbb{Z}} \psi_i^{(\alpha)} z^i, \quad \psi^{(\alpha)}_i(z) = \sum_{i \in \mathbb{Z}} \psi_i^{(\alpha)} z^{-i-1}$$

(2.2)

where $\alpha$ is a sort of fermions ($\alpha = 1, 2$) and anti-commutators companions $\psi_i^{(\alpha)}$, $\psi_i^{(\beta)}$ are

$$\left[\psi_i^{(\alpha)}, \psi_j^{(\beta)}\right]_+ = 0 \quad \left[\psi_i^{(\alpha)}, \psi_j^{(\beta)}\right]_- = 0 \quad \left[\psi_i^{(\alpha)}, \psi_j^{(\beta)}\right]_+ = \delta_{\alpha, \beta} \delta_{i, j}$$

(2.3)

The fermionic states with occupied level up to $N^{(\alpha)}(M^{(\beta)})$ satisfy the conditions

$$\langle \Lambda^{(\alpha)} | \Lambda^{(\beta)} \rangle = \delta_{\alpha, \beta} \delta_{N^{(\alpha)}, M^{(\beta)}}$$

(2.4)

The $\tau$ function of the two-component KP (with $n_1 = N^{(1)}, n_2 = N^{(2)}$) is given by

$$\tau(n_1, n_2, t^{(1)}, t^{(2)}) = \langle n_1, n_2 | \Gamma(t^{(1)}, t^{(2)}) e^{\int \frac{K}{2} \sum_{i,j} \psi_i^{(1)}(\alpha) \psi_j^{(2)}(\alpha) V(\bar{a}_i, a_j, t^{(1)}) d^2 a_i d^2 a_j} | n_2, n_1 \rangle$$

(2.5)

and express in terms of currents $J^{(\alpha)}_i$ by

$$\psi^{(\alpha)}(z) \psi^{(\alpha)}(z) := \psi^{(\alpha)}(z) \psi^{(\alpha)}(z) - \langle 0 | \psi^{(\alpha)}(z) \psi^{(\alpha)}(z) | 0 \rangle = \sum_{i \in \mathbb{Z}} J^{(\alpha)}_i z^{i-1}$$

(2.7)

or for the fermionic components we set for $n \in \mathbb{Z}$

$$J^{(\alpha)}_n = \sum_{i \in \mathbb{Z}} \psi_i^{(\alpha)} \psi_{i+n}^{(\alpha)}$$

(2.8)

then we have the commutation relation

$$\left[J^{(\alpha)}_n, J^{(\beta)}_m\right] = n \delta_{\alpha, \beta} \delta_{n, m, 0}$$

(2.9)

As a result we obtain $\tau$ function of the two-component KP

$$\tau(n_1, n_2, t^{(1)}, t^{(2)}) = \sum_{q \geq 0} \frac{K^q}{(q!)^2} \prod_{1 \leq i \leq q} \frac{1}{(a_i - a_j)^2} \prod_{1 \leq i \leq q} \frac{1}{|b_i - b_j|^2} \prod_i \Phi_{n_1, n_2}(a_i, b_i, t^{(1)}, t^{(2)}) d^2 a_i d^2 b_i$$

(2.10)

where the function

$$\Phi_{n_1, n_2}(a, b, t^{(1)}, t^{(2)}) = \left(\frac{a}{b}\right)^{n_1} \left(\frac{b}{a}\right)^{n_2} e^{V(a, t^{(1)}) - V(b, t^{(1)}) - V(a, t^{(2)}) + V(b, t^{(2)})}$$

(2.11)

and

$$V(z, t) = \sum_{t > 0} t z^t$$

(2.12)

with the corresponding sets of the two-component KP times $t^{(1)}_i, t^{(2)}_i$ for $i > 0$. Thus for

$$\phi(a, b, c) = f(c) \prod_{i=1}^q \Phi_{n_1, n_2}(a_i, b_i, t^{(1)}_i, t^{(2)}_i)$$

(2.13)

with an arbitrary function $f(c)$ we observe

$$Z(\phi) = C \cdot \tau(n_1, n_2, t^{(1)}, t^{(2)})$$

(2.14)

where $C$ is some constant. Namely we have $C = \int \frac{f(c)}{1 + |c|^{2/3}}$.
3 Analysis of the instanton contributions and the \( \tau \) functions

More exactly in our case taking into account the instantons having arbitrary topological number we obtain for Green functions in the steepest descent approximation \([2]\):

\[
Z(\phi) = \frac{\sum_{q \geq 0} K_q \int \phi(a, b, c) \prod_{1<i<j \leq q} |a_i - a_j|^2 \prod_{1<i<j \leq q} |b_i - b_j|^2 \frac{1}{\prod_{1<i<j \leq q} |a_i - b_j|^2} \prod_{i=1}^q d^2ad^2b_1 d^2c}{\sum_{q \geq 0} K_q \int \prod_{1<i<j \leq q} |a_i - a_j|^2 \prod_{1<i<j \leq q} |b_i - b_j|^2 \frac{1}{\prod_{1<i<j \leq q} |a_i - b_j|^2} \prod_{i=1}^q d^2ad^2b_1},
\]

where \( \phi(a, b, c) = \phi(\omega) \) and \( \omega \) is given by \([1,3]\). In the case \( \phi(a, b, c) = f(c) \prod_{i}^q \Phi_{n_i} (a_i, b_i, t^{(1)}, t^{(2)}) \)
we obtain

\[
Z(\phi) = \frac{C \cdot \tau(n_1, n_2, t^{(1)}, t^{(2)})}{\tau(0, 0, 0, 0)} \tag{3.1}
\]

The denominator in \((3.1)\) and \((3.2)\) coincides with the partition function \( \Xi \) of the neutral classical Coulomb system (CCS) in the grand canonical ensemble with the definite temperature \( T \) (\( T=1 \) see \([2]\)).

\[
\tau(0, 0, 0, 0) = \Xi \tag{3.2}
\]

The constant \( K \) plays the role of fugacity of the Coulomb system. The expression \([2,1]\) also coincides with the correlation function of the CCS (at \( T=1 \)). Let us consider the instanton contribution \( G_{\text{inst}}(x, y) \) in the Green function

\[
G(x, y) = \langle \Delta_x \log |\omega(x)| \Delta_y \log |\omega(y)| \rangle \tag{3.3}
\]

where

\[
x^{(a)}_n = -\frac{t}{nx^n} - \frac{\tau}{ny^n} \tag{3.4}
\]

then we achieve

\[
\Delta_x \Delta_y \frac{\partial}{\partial t} \left( \prod_i^q \Phi_{0,0}(a_i, b_i, t^{(1)}, t^{(2)}) \right)_{|t=0} = \Delta_x \log |\omega(x)| \Delta_y \log |\omega(y)| = \rho(x)\rho(y). \tag{3.5}
\]

One can interpret \( \rho(x) \) as the charge density. We see

\[
G_{\text{inst}}(x, y) = \langle \Delta_x \log |\omega(x)| \Delta_y \log |\omega(y)| \rangle_{\text{inst}} = \langle \rho(x)\rho(y) \rangle_{\text{CCS}} \tag{3.6}
\]

Similarly to the previous way we can obtain the instanton contribution in the more general Green function corresponding to the functional

\[
\phi(\omega) = \Delta_{x_1} \log |\omega(x_1)| \Delta_{x_2} \log |\omega(x_2)| \ldots \Delta_{x_m} \log |\omega(x_m)| \tag{3.7}
\]

by

\[
G_{\text{inst}}(x_1, x_2, \ldots x_m) = \langle \rho(x_1)\rho(x_2) \ldots \rho(x_m) \rangle_{\text{CCS}} = \frac{C \Delta_{x_1} \Delta_{x_2} \ldots \Delta_{x_m} \frac{\partial}{\partial t} \left( \tau(0, 0, t^{(1)}, t^{(2)}) \right)_{|t=0}}{\tau(0, 0, 0, 0)} \tag{3.8}
\]
4  Bilinear identity for the τ function

In this section we define more general τ functions in comparison with [2,3]:

\[ \tau(n_1, n_2, n, t^{(1)}, t^{(2)}) = \langle n_1, n_2 | \Gamma(t^{(1)}, t^{(2)}) g | n_2 - n, n_1 + n \rangle, \quad (4.1) \]

where

\[ g = e^{K \frac{\tau}{2} \int \psi^{(1)}(a) \psi^{(2)}(a) da} \]

and \( \Gamma(t^{(1)}, t^{(2)}) \) is given by \([2,3]\). Particularly the interesting for us τ function

\[ \tau(n_1, n_2, t^{(1)}, t^{(2)}) = \tau(n_1, n_2, 0, t^{(1)}, t^{(2)}) \]

The bilinear identity is valid in the following form (see [3]). For \( n_1 - n_1' \geq n' - n \geq n_2' - n_2 \geq 2 \), we have

\[ \sum_{a=1}^{2} \int \frac{dz}{2\pi i z} \langle n_1, n_2 | \Gamma(t^{(1)}, t^{(2)}) \psi^{(a)}(z) g | n_2 - n - 1, n_1 + n \rangle (4.3) \]

\[ \times \langle n_1', n_2' | \Gamma(t^{(1)}, t^{(2)}) \psi^{(a)}(z) g | n_2' - n' + 1, n_1' + n' \rangle = 0 \]

and the integration is taken along a small contour at \( z = \infty \) so that \( \int \frac{dz}{2\pi iz} = 1 \).

Rewrite this \([3,5]\) we obtain

\[ \int \frac{dz}{2\pi iz} (-1)^{n_2+n_3} z^{n_1-1-n_1'} e^{V(z,t^{(1)}-t^{(2)})} \]

\[ \times \tau(n_1 - 1, n_2, n + 1, t^{(1)} - \theta(z^{-1}) t^{(2)}) \tau(n_1' + 1, n_2', n' - 1, t^{(1)} + \theta(z^{-1}) t^{(2)}) \]

\[ + \int \frac{dz}{2\pi iz} (-1)^{n_2-n_2'} e^{V(z,t^{(2)}-t^{(1)})} \]

\[ \times \tau(n_1 - 1, n_2 - 1, n, t^{(1)} - t^{(2)} - \theta(z^{-1}) t^{(2)}) \tau(n_1', n_2', n', t^{(1)} + \theta(z^{-1}) t^{(2)}) \]

where \( \theta(z^{-1}) = (z^{-1}, z^{-1}, \ldots, z^{-1}) \).

An example of \([3,6]\), we have the following bilinear equations for \( f = \tau(n_1, n_2, 0, t^{(1)}, t^{(2)}) = \tau(n_1, n_2, t^{(1)}, t^{(2)}) \),

\[ g = \tau(n_1 - 1, n_2 + 1, 1, t^{(1)}, t^{(2)}) \]

and \( g^* = \tau(n_1 + 1, n_2 - 1, -1, t^{(1)}, t^{(2)}) \):

\[ \left( D_{t^{(1)}} - D_{t^{(2)}}^2 \right) f \cdot g = 0, \quad \left( D_{t^{(1)}}^2 - D_{t^{(2)}}^2 \right) g^* \cdot f = 0, \quad (4.5) \]

\[ \left( D_{t^{(2)}} - D_{t^{(1)}}^2 \right) f \cdot g = 0, \quad \left( D_{t^{(2)}}^2 + D_{t^{(1)}}^2 \right) g^* \cdot f = 0, \]

\[ D_{t^{(1)}} D_{t^{(2)}} f \cdot f - 2g \cdot g^* = 0, \]

where Hirota operator as usual \( D_\sigma \tau = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} \tau(x + \varepsilon) \tau(x - \varepsilon) = \sigma_x \tau - \sigma \tau_x \)

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