Ground State Entanglement in a Combination of Star And Ring Geometries Of Interacting Spins

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We compare a star and a ring network of interacting spins in terms of the entanglement they can provide between the nearest and the next to nearest neighbor spins in the ground state. We then investigate whether this entanglement can be optimized by allowing the system to interact through a weighted combination of the star and the ring geometries. We find that such a weighted combination is indeed optimal in certain circumstances for providing the highest entanglement between two chosen spins. The entanglement shows jumps and counterintuitive behavior as the relative weighting of the star and the ring interactions is varied. We give an exact mathematical explanation of the behavior for a five qubit system (four spins in a ring and a central spin) and an intuitive explanation for larger systems. For the case of four spins in a ring plus a central spin, we demonstrate how a four qubit GHZ state can be generated as a simple derivative of the ground state. Our calculations also demonstrate that some of the multi-particle entangled states derivable from the ground state of a star network are sufficiently robust to the presence of nearest neighbor ring interactions.

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I. INTRODUCTION

The entanglement present in natural spin systems has been a subject of serious interest in recent years. It is believed that this entanglement can even have consequences on the macroscopic properties of such systems. The entanglement is found to exhibit interesting behavior near the points of quantum phase transitions. Ground states of some finite systems can serve as a convenient template for generating multipartite entangled states. Creating the state requires only cooling the system down.

A number of spin structures can be investigated to determine their entanglement properties in the ground state, in particular a ring, as well as other lattice structures. However, 1D chains and lattices of various dimensions are not the only physical systems whose fabrication is possible with current technology. It is possible to extend the line of research on entanglement in spin systems to other than spin chains. In particular, various technologies have evolved which can make any member of an array of qubits interact with any other member. One such structure is the star in which a number of spins interact through one central spin. Apart from our work on the ground state entanglement in a spin-star, recently interesting non-Markovian dynamics, quantum cloning, and quantum gates in such a system has been investigated by other authors.

In this paper, we first briefly compare the star model and the ring model for spins interacting via the $XX$ interaction, which is physically realizable, to find out whether one model or the other is better for establishing entanglement between two spins (in the ground state). Then, as the main focus of the paper, we study entanglement in a model in which the spins interact simultaneously through both means. We study both nearest and next to nearest neighbour

![FIG. 1: The star (a), ring (b) and star-ring combination (c) spin models illustrated with $N = 5$ spins placed in a ring with one spin at the center. The lines connect interacting pairs of spins.]
entanglement between spins. The system where the spins can interact both with their neighbours and a central spin (a combination of the ring and star) also represents a logical continuation of the work on entangled rings [1, 4], especially that of Wang [4]. It simply considers the consequences of adding a central spin to a ring of spins, so that the spins can also interact via this central spin. Note that this is different to a star network in which spins interact only with a central spin. In addition, it is also a natural extension of our earlier work on a star network of spins [22]. In practice, in a spin star, unwanted interactions between the outer spins would exist due to their physical proximity and thus our current investigation considers how much these unwanted interactions would modify the properties of the unpolluted star system (such as the ability of the star system to produce interesting multi-particle entangled states as the ground state).

Our work is also motivated by the fact that entanglement can show interesting behavior at points of quantum phase transitions [21], such as scaling [10, 11, 15, 10] and macroscopic jumps [12]. Very often frustration due to competing non-commuting terms in a Hamiltonian is the cause of the curious behavior of entanglement at a quantum phase transition [12, 29]. By combining a star system Hamiltonian and a ring system Hamiltonian, we precisely intend to create such a frustration between competing ordering tendencies. The system of our current paper is finite, and the two parts of the Hamiltonian, namely the star system part and the ring system part, do not commute, so we do not expect a quantum phase transition in our system [21]. Nonetheless, as we will show, the competition between two different parts of our Hamiltonian leads to sharp changes (“jumps” in the same sense as Ref. [12]) in the entanglement as the relative strength of the two terms is varied. In addition to sharp changes, we will also show that the magnitude of entanglement changes in a counterintuitive manner as the relative strength of the two terms is varied. We will provide a heuristic explanation for the observed behavior of entanglement. Moreover, we will explicitly solve a system of four spins in a ring interacting with a common central spin and show that this system can be used to produce a Greenberger-Horne-Zeilinger (GHZ) state [30] as a simple derivative of the ground state, a feat not achieved yet, to our knowledge, by any other spin Hamiltonian.

Figure 1 depicts schematically the star and ring spin models. For both models the qubits in the outer ring will be referred to as the ‘outer qubits’. Figure 1(a) depicts the star model, in which the outer qubits interact only with a central qubit. Figure 1(b) depicts the ring model, in which the outer qubits interact with their nearest neighbours in the ring, while Figure 1(c) illustrates a model where qubits interact both with their nearest neighbours and with a central spin. The outer qubits are labelled 1 to N, while the central qubit in the star model is labelled 0. The Hamiltonians for the two models are

\[
H_{\text{ring}} = \sum_{i=1}^{N} \left( \sigma_x^i \sigma_x^{i+1} + \sigma_y^i \sigma_y^{i+1} \right)
\]

\[
H_{\text{star}} = \sum_{i=1}^{N} \left( \sigma_z^0 \sigma_x^i + \sigma_y^0 \sigma_y^i \right)
\]

with periodic boundary conditions i.e. N + 1 = 1.

The measure we will use for entanglement between two qubits is the entanglement of formation [34, 35]. Specifically, we will use the concurrence [35, 36], of which the entanglement of formation is a monotonic function. To determine the concurrence between two qubits, we firstly trace out the other qubits in the model and then calculate the concurrence

\[
\rho_{12} = \begin{pmatrix}
    \varphi & w & z & y \\
    w & \varphi & z & x \\
    z & z & \varphi & y \\
    y & x & y & \varphi
\end{pmatrix}
\]

in the basis \{ |00⟩, |01⟩, |10⟩, |11⟩ \}. The concurrence for such a density matrix is given by

\[
C = 2 \max \{|z| - \sqrt{\varphi y}, 0\}.
\]

(1)

In [22] we used this formula to obtain an analytic formula for the concurrence between any two outer spins in the star model

\[
C = 2 \max \{1/2N, 0\} = 1/N \text{ for } N \text{ odd}
\]

\[
C = 2 \max \{1/2N - 1/(2N^2 - 2N), 0\} = 1/N - 1/(N^2 - N) \text{ for } N \text{ even}
\]

(2)
FIG. 2: (a) This figure plots the nearest neighbour concurrence. (b) This figure plots the next to nearest neighbour concurrence. The solid line is the ring and the dashed line is the star (for the star, nearest and next to nearest neighbors would be defined in an ad-hoc basis, in terms of physical adjacency, rather than in terms of interactions). The thin lines depict the antiferromagnetic interaction and the thick lines depict the ferromagnetic interaction. Note that for the star, the thick and thin lines are coincident.

To compare with the above, we numerically evaluate the concurrence for the ring for a range of values of \( N \). Wang [4] has ascertained that for even \( N \) the concurrence is independent of the sign of the coupling constant \( J \) which multiplies the Hamiltonians (the Hamiltonians being \( JH_{\text{star}} \) and \( JH_{\text{ring}} \). We note that this is also true for both even and odd \( N \) for the star model (the formulae in [2] do not depend on \( J \)).

The one major advantage that the star has, just by sheer virtue of its geometry, is that the entanglement between any two outer spins is the same as there is perfect permutation symmetry in the model. This fact tells us that the star model will most probably be superior in comparison to the ring model for sharing entanglement between spins that are not necessarily physically adjacent (i.e., nearest neighbors in the sense of a ring).

In section II we directly compare the star and ring models for sharing entanglement between neighbouring spins. Then in section III we investigate how successful a model in which both interactions occur is at sharing entanglement. An insight into the reason for the behaviour of the combination is given by the energy level crossings which is highlighted in section IV. An full explanation for the case of \( N = 4 \) is presented in section V and section VI makes some generalisations for any \( N \). Finally we present our conclusions in section VIII.

II. COMPARING THE STAR AND RING SPIN MODELS

In this section we compare how well the ground state of the star and ring models share entanglement between neighbouring spins. For both models we find the density matrix numerically for the ground state for \( 2 \leq N \leq 6 \) and select two spins by tracing out remaining spins from the density matrix. The concurrence between the two spins is then used as a measure of the entanglement between them.

Figure 2(a) plots the nearest neighbour entanglement of two outer spins. We observe that the entanglement of the star is not affected by whether the interaction is ferromagnetic or antiferromagnetic. Note that for \( N = 2 \) and \( N = 3 \) all the spins are nearest neighbours. Also note that for even \( N \), the ring is not affected by the sign of \( J \), which agrees with Wang in [4].

Figure 2(b) plots the next to nearest neighbour entanglement. The most noticeable feature of this graph is that the ring entanglement drops to zero and completely vanishes. In fact, for \( N = 2 \) and \( N = 3 \) the points on the graph really represent nearest-neighbour entanglement. Therefore we can venture to conjecture that there is no next to nearest entanglement in the ring model. On the other hand, the star displays exactly the same behavior for any pair of spins, as would be expected from the symmetry of the model.

In summary then, for nearest neighbour interactions the ring appears (in the limited range of \( N \) considered) to have slightly higher entanglement than the star, but not very dissimilar for \( N \) odd, and furthermore the star model shares entanglement between states for next to nearest neighbours whereas the ring shares none at all. Finally we also note that the ferromagnetic interaction is the best for sharing entanglement in the ring (and equally good in the star as the antiferromagnetic interaction).
III. COMBINING THE STAR AND RING SPIN MODELS

In the previous section, we compared the star and ring models for establishing entanglement between spins. We observed that the ring was superior at establishing nearest-neighbour entanglement while the star was better for establishing next-to-nearest neighbour entanglement. It is then natural to wonder about the nature of entanglement between spins in a model in which the spins could interact both with the nearest neighbour spins (like in the ring) and with a central spin (like in a star). This is what we investigate in rest of the paper. In this section, we will find out whether the combination of star and ring interactions can produce an entanglement (between nearest neighbor pairs or non-nearest neighbor pairs) which is higher than that of the star alone or the ring alone. The Hamiltonian for such a system could be written

$$H = J \left[ cH_{\text{star}} + (1 - c)H_{\text{ring}} \right]$$

where $c$ is a parameter $0 \leq c \leq 1$ which interpolates between the two extremes of the ring ($c = 0$) and the star ($c = 1$).

Our interest is in how the concurrence varies with $c$ and whether it is maximal for a non-extremal value of $c$. While there are well known analytical methods for solving for the ground state for both the extremal values of $c$ (one can solve the ground state of $H_{\text{ring}}$ by using the Bethe Ansatz [37] and we have already presented the ground state of $H_{\text{star}}$ in Ref. [22]), there does not seem to be any easy to apply analytic technique for obtaining the ground state for arbitrary $c$. So we have obtained the ground state for the Hamiltonian in (3) numerically and calculated the concurrence between two of the spins lying on the ring. Figure 3 plots the concurrence against $c$ for $N = 4, 5, 6$ and 7 for both nearest-neighbor (solid line) and next to nearest neighbor (dashed line) concurrence. These values of $N$ were chosen for computational convenience. There are a number of interesting observations that can be made at once from these plots:

- For odd $N$, there appears to be an initial rise in nearest-neighbor entanglement as $c$ increases from 0.
- For any $N$, the maximum next-to-nearest neighbor entanglement occurs at a value of $c$ less than 1, i.e. before the network becomes entirely star-like.

- There are a number of sharp changes in the entanglement.

- In general there seems to be a point around $c \approx 0.7$ where all entanglement drops to zero.

This suggests that a combination of the two models can maximize either the nearest neighbor or the next-to-nearest neighbor entanglement. We use ‘either . . . or’ because the two maxima do not occur at the same value of $c$. Thus, though a ring interaction ($c = 0$) does not, by itself, favor next to nearest neighbor entanglement (its value being zero), it can be mixed with the star to actually increase the next to nearest neighbor entanglement of the star system. This is a counterintuitive feature. Moreover, for odd $N$, a proportion of the star interaction seems to increase the nearest neighbor entanglement, though the star is expected to remove the special status of nearest neighbors of a ring system. Furthermore we observe that there are sharp jumps in entanglement, which include drops to zero at an or for a range of intermediate values of $c$. These features are consequently quite surprising and we will devote majority of the rest of the paper to seeking their explanation.

Having looked at the concurrence, we now also take a brief look at the localizable entanglement in the combination of the star and the ring models. The localisable entanglement has been introduced by Verstraete et al. [38] and studied further by others [39], and is the average amount of entanglement that can be established between two spins by performing local measurements on the other individual spins. The authors of Ref. [38] showed that all classical correlation functions provide lower bounds to this localizable entanglement. In Figure 4 we have plotted the $XX$ and $ZZ$ correlations (which provide lower bounds on the localizable entanglement) together with the concurrence that was plotted in Figure 3. There are two main points of interest. Firstly the correlations (and hence the lower bound on the localizable entanglement) are non-zero even in areas where the concurrence is zero, in particular around $c \approx 0.7$. 

![Graphs showing XX and ZZ correlations for different N as c varies from 0 (ring) to 1 (star).](attachment:graph.png)
Thus these areas are not uninteresting in terms of entanglement. In fact, in these regimes of $c$, one can search for interesting multiparticle entangled states (two particle entanglement being zero does not imply that the system of spins in not in a multiparticle entangled state). Secondly, we note that the localizable entanglement is quite high in general for most values of $c$, both for nearest neighbors and next to nearest neighbors. For example, for $N = 4$ it ranges from 0.7 for the ring to 0.6 for the star, dropping to a low of 0.4 for intermediate values of $c$. Therefore the localizable entanglement in this model is significant in magnitude.

**IV. ENERGY LEVEL CROSSINGS AND THE SHARP CHANGES IN ENTANGLEMENT**

Some light can be shed on the reason behind one of the results described in the previous section (namely the sharp changes in concurrence as $c$ is varied) by studying the energy levels of the system and how they vary with $c$. Energy level diagrams are plotted in Figure 5. Superposed on the diagrams are the nearest and next to nearest neighbor concurrences. The figure clearly shows that the jumps in the concurrence are due to crossings of the lowest energy levels. Thus the sharp transitions in entanglement are in one to one correspondence with a sudden qualitative change in the ground state of the system which happens due to level crossings. The sudden change in the ground state energy level changes the two spin reduced density matrices and thereby the concurrence. Such changes are also the cause of quantum phase transitions, which occur in infinite systems. Here we note the similarity, though our system is finite. In other words, the cause of quantum phase transitions (competing Hamiltonian terms causing energy level crossings or infinitesimal avoided level crossings), when applied to our finite system, also causes sharp transitions in entanglement.
V. THE VARIATION OF EIGENSTATES FROM STAR TO RING

An initial impression of how the ground states of the model change as \( c \) varies can be obtained by studying how similar they are to the pure ring \((c = 0)\) and pure star \((c = 1)\) ground states. The ground states can be compared by calculating the fidelity \( F \) of the two density matrices for the two cases. The fidelity is a measure of ‘how close’ two states \( \rho_1 \) and \( \rho_2 \) are. It ranges in value from 0 to 1 and is equal to one if and only if \( \rho_1 \) and \( \rho_2 \) are equal. It is defined by

\[
F(\rho_1, \rho_2) = \left| \frac{\text{tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}}{\sqrt{\text{tr} \rho_1 \text{tr} \rho_2}} \right|^2
\]

In Figure 6, the fidelity of the ground state as \( c \) varies with the pure star and pure ring ground states has been plotted for \( N = 4, 5 \) and 6. The fidelity with the ring ground state has been labelled by \( O_r \) and with the star ground state labelled by \( O_s \). If we denote the ground state by \( \rho_g \) then

\[
O_r = F[\rho_g(c), \rho_g(c = 0)]
\]

\[
O_s = F[\rho_g(c), \rho_g(c = 1)]
\]

What is illustrated by Figure 6 is that the eigenstates remain ‘ring-like’ and ‘star-like’ for much of the range of \( c \) either side of \( c \approx 0.7 \). This is shown by the fact that \( O_r \) is near to one for much of \( c < 0.7 \) and \( O_s \) is near to one for much of \( c > 0.7 \) (the other overlap \( O_p \) appearing in Figure 6 is introduced and discussed in section VII).

Note that for the case of \( N = 5 \) the overlap with the ring \( O_r \) may seem very low, as it is approximately 0.2 rather than the 1 that might be expected. This is because the ground state of the pure ring is 8-fold degenerate. This degeneracy is lifted for \( c \neq 0 \), and therefore its overlap with states for which \( c \neq 0 \) will be smaller. In the bottom left plot in Figure 6 is the same graph, but this time \( O_r \) is the fidelity not with the pure ring, but with the state for which \( c = 0.01 \). This was done to lift the degeneracy in the ‘pure ring’ state used to calculate the fidelity.

The \( N = 5 \) case then confirms the same pattern as was observed for \( N = 4 \) and \( N = 6 \). This is that the ring and star ground states are quite stable, because the ground state stays close to that of the ring (near \( c = 0 \)) or the star (near \( c = 1 \)) for a considerable range of values of \( c \).

In summary, these initial numerical investigations for \( N = 4, 5, 6 \) and 7 indicate that there is interesting structure present in a combination of the star and ring models and that it deserves further study. The next section describes a detailed study of the cases for \( N = 4 \).

VI. \( N = 4 \) IN DETAIL

In this section, we give a mathematical explanation for the behavior of the entanglement in the ground state for the case \( N = 4 \) by presenting the exact eigenstates for this case (a more physical explanation will be presented in the discussions section). The energy level diagram in Figure 6 for \( N = 4 \) indicates that there are two different energy levels which, depending on the value of \( c \), are the ground state. The energy level which is the ground state for most of the time, except for \( c \) near 0.7, is the ground state for both the pure ring and pure star. We call this energy level I, and the other energy level, which is the ground state only for a short while near \( c = 0.7 \), we call energy level II. Our aim in this section is to analyze how the concurrence varies for these two energy levels and to determine why one energy level takes over from the other as the ground state for a certain range of \( c \). We will begin by setting out for reference a number of relevant states and their actions under the Hamiltonians being studied. After reviewing the two energy levels of interest for the extremes of the star and ring models, we will describe the behavior of the energy levels for \( 0 < c < 1 \).

Firstly, we define some convenient states which will be frequently used:

\[
|A\rangle = \frac{1}{\sqrt{2}} (|0101\rangle + |1010\rangle)
\]

\[
|B\rangle = \frac{1}{\sqrt{2}} (|0011\rangle + |0110\rangle + |1100\rangle + |1001\rangle)
\]

\[
|C_1\rangle = \frac{1}{\sqrt{2}} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)
\]

\[
|C_2\rangle = \frac{1}{\sqrt{2}} (|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)
\]

\[
|C_3\rangle = \frac{1}{\sqrt{2}} (|0001\rangle - |0010\rangle + |0100\rangle - |1000\rangle)
\]

\[
|C_4\rangle = \frac{1}{\sqrt{2}} (|0111\rangle - |1011\rangle + |1101\rangle - |1110\rangle)
\]

\[
|D\rangle = \frac{1}{\sqrt{2}} (|0101\rangle - |1010\rangle)
\]

Note that these states are rotationally invariant. This is because the ring is rotationally invariant. In fact so is the star, although the star also has the stronger permutation symmetry. As the eigenstates of the star are made up from angular momentum eigenstates, the following relations will also be useful...
FIG. 6: This figure plots the fidelity of the ground state at a certain value of $c$ with three states. $O_p$ is the fidelity with a speculated state composed of singlets. $O_r$ is the fidelity with the ground state of the pure ring. $O_s$ is the fidelity with the ground state of the pure star. Also plotted here are the concurrences for reference, marked by $C_g$.

\[ |j = 2, m = 0 \rangle = \frac{1}{\sqrt{6}} \left( \sqrt{2} |A \rangle + 2 |B \rangle \right) \]
\[ |j = 1, m = 1 \rangle = |C_3 \rangle \]
\[ |j = 1, m = 0 \rangle = |D \rangle \]
\[ |j = 1, m = -1 \rangle = |C_1 \rangle \]

Table I shows how these states (plus the central spin) are affected by the star and ring Hamiltonians individually. Having set out the relevant states we now review the state vectors for energy levels I and II at the extremes of the pure ring ($c = 0$) and pure star ($c = 1$). The eigenstates for the pure $XX$ ring for $N = 4$ are given by Wang in [4]. Note that in our model there is a central spin which is uncoupled from the outer spins for $c = 0$. Consequently it doubles the degeneracy of all the ring eigenstates. Energy level I is the ground state for the pure ring. It has energy $-4J\sqrt{2}$ and it is a mixture of
\[ |0 \rangle \frac{1}{\sqrt{2}} (|A \rangle - |B \rangle) \]
\[ |1 \rangle \frac{1}{\sqrt{2}} (|A \rangle - |B \rangle) \]

Energy level II for the pure ring is a mixture of
\[ |0 \rangle |C_3 \rangle \]
\[ |1 \rangle |C_1 \rangle \]
In the case of the pure star the energy eigenstates are given in [22]. Energy level I for the pure star is a mixture of
\[
\frac{1}{\sqrt{2}} \left( |0\rangle \langle C_3| - |1\rangle \langle 1| \left( \sqrt{2} |A\rangle + 2 |B\rangle \right) \right)
\]
\[
\frac{1}{\sqrt{2}} \left( |0\rangle \langle 0| \left( \sqrt{2} |A\rangle + 2 |B\rangle \right) - |1\rangle \langle 1| C_1 \right) \right)
\]
(6)

Energy level II for the pure star is a mixture of
\[
\frac{1}{\sqrt{2}} (|0\rangle \langle C'_1| - |1\rangle \langle D|)
\]
\[
\frac{1}{\sqrt{2}} (|0\rangle \langle D| - |1\rangle \langle C'_1|)
\]

A few remarks on degeneracy: From Table I it is apparent that some states have been omitted here. For example, $|1\rangle \langle C'_3|$ and $|0\rangle \langle C'_4|$ are also eigenstates for the pure ring at this point of energy level II. However, from the energy level diagram in Figure 5 it can be seen that for $c \neq 0$ the energy level splits and some states move up towards $E = 0$ as $c$ approaches 1. Therefore, in the mixtures above we have only included states that feature throughout all of energy level II. Similarly, at $c = 1$ (the star end) there are additional states present in energy level II due to degeneracy in $j$. However these additional states diverge from energy level II for $c < 1$ and therefore they have also been omitted above.

Given these extremes, we can form an impression of how the ring state must mutate into the star state as $c$ changes from 0 to 1. Our next step is to write down a general expression for the state of each energy level which can cover a range of values of $c$. To do this, we split the graph into three regions - a 'ring' region, a 'star' region, and an 'intermediate' region. These regions are separated by the energy level crossings, causing discontinuities in the entanglement. Figure 4 illustrates these regions.

### A. The ring and star regions

In the ground state the ring and star regions correspond to energy level I. It is straightforward to write down a general expression for the two degenerate eigenstates in energy level I:
\[
\gamma |0\rangle \langle C_3| + |1\rangle \langle 1| \left( \alpha |A\rangle + \beta |B\rangle \right)
\]
\[
|0\rangle \langle 0| \left( \alpha |A\rangle + \beta |B\rangle \right) + \gamma |1\rangle \langle 1| C_1 \right) \right)
\]
(7)

where $\alpha$, $\beta$ and $\gamma$ are all functions of $c$. From Figures 3 and 4 we see that at $c = 0$ (the ring), $\alpha = \frac{1}{\sqrt{2}}$, $\beta = -\frac{1}{\sqrt{2}}$ and $\gamma = 0$. At $c = 1$ (the star), $\alpha = -\sqrt{\frac{2}{3}}$, $\beta = -\sqrt{\frac{2}{3}}$ and $\gamma = \frac{1}{\sqrt{2}}$. The coefficients $\alpha$, $\beta$ and $\gamma$ have been calculated numerically as functions of $c$ and are plotted in Figure 4.

| $H_{\text{star}}$ | $H_{\text{ring}}$ |
|------------------|------------------|
| $|0\rangle A$ | $2\sqrt{2} |1\rangle C_1 \rangle$ |
| $|1\rangle A$ | $2\sqrt{2} |0\rangle C_4 \rangle$ |
| $|0\rangle B$ | $4 |1\rangle C_1 \rangle$ |
| $|1\rangle B$ | $4 |0\rangle C_3 \rangle$ |
| $|0\rangle C_1$ | $4 |1\rangle \{0000\}$ |
| $|1\rangle C_1$ | $2\sqrt{2} |j = 2, m = 0\rangle$ |
| $|0\rangle C_3$ | $2\sqrt{2} |1\rangle C_2 \rangle$ |
| $|1\rangle C_3$ | $4 |0\rangle |1111\rangle$ |
| $|0\rangle C'_1$ | $0$ |
| $|1\rangle C'_1$ | $2\sqrt{2} |0\rangle D \rangle$ |
| $|0\rangle C'_3$ | $2\sqrt{2} |1\rangle D \rangle$ |
| $|1\rangle C'_3$ | $0$ |
| $|0\rangle D$ | $2\sqrt{2} |1\rangle C'_1 \rangle$ |
| $|1\rangle D$ | $2\sqrt{2} |0\rangle C'_3 \rangle$ |

TABLE I: This table displays the action of the star and ring Hamiltonian on some important states
FIG. 7: This figure plots the coefficients $\alpha, \beta, \gamma, \alpha'$ and $\gamma'$ involved in the expressions for the ground states of $N = 4$ for various values of the parameter $c$.

The reduced density matrix for nearest neighbors that arises when we take an equal mixture of the above two states is

$$\frac{\beta^2}{4} \langle |00\rangle \langle 00| + |11\rangle \langle 11| \rangle + \left| \psi^+ \left( \frac{\alpha}{\sqrt{2}}, \frac{\beta}{2} \right) \right\langle \psi^+ \left( \frac{\alpha}{\sqrt{2}}, \frac{\beta}{2} \right) \rangle + \left| \psi^+ \left( \frac{\beta}{2}, \frac{\alpha}{\sqrt{2}} \right) \right\rangle \langle \psi^+ \left( \frac{\beta}{2}, \frac{\alpha}{\sqrt{2}} \right) \rangle + \frac{1}{4} \langle 2 | \Psi^+ \rangle \langle \Psi^+ | + |00\rangle \langle 00| + |11\rangle \langle 11| \rangle$$

where we have defined $|\psi^+(u,v)\rangle = (u|01\rangle + v|10\rangle)$. This gives a concurrence

$$C = 2 \max \left\{ 0, \frac{\gamma^2}{4} + \frac{\alpha\beta}{\sqrt{2}} - \frac{1}{4} \left( \gamma^2 + \beta^2 \right) \right\}$$

Note that in the star region, $\alpha$ and $\beta$ are of the same sign, which essentially makes the concurrence proportional to $\alpha - \beta/2\sqrt{2}$, and thus it decreases with decreasing $c$ because the $\alpha/\beta$ ratio decreases. It vanishes when the ratio falls below $1/2\sqrt{2}$. In the ring region, $\alpha$ and $\beta$ are of opposite sign and $|\alpha\beta| > \gamma^2$, which gives concurrence as $-\frac{\beta}{\sqrt{2}}(\alpha + \beta/2\sqrt{2}) - \gamma^2/2$. In the ring region $\alpha + \beta/2\sqrt{2}$ is positive and the concurrence decreases with increasing $c$ because $\gamma^2$ increases.

For next-to-nearest neighbors, the reduced density matrix is

$$\frac{\beta^2}{4} \langle |00\rangle \langle 00| + |11\rangle \langle 11| \rangle + \beta^2 |\Psi^+\rangle \langle \Psi^+ | + \frac{1}{4} \langle 2 | \Psi^+ \rangle \langle \Psi^+ | + |00\rangle \langle 00| + |11\rangle \langle 11| \rangle$$

and this gives a concurrence

$$C = 2 \max \left\{ 0, \frac{1}{2} (\beta^2 - \alpha^2) \right\}$$
This expression for the concurrence neatly explains how the next to nearest neighbor concurrence for \( N = 4 \) varies with \( c \) in the ring region \(|\alpha| \approx |\beta|\) and the next to nearest neighbor concurrence is low, whereas in the star region \(|\beta| > |\alpha|\) and thus the concurrence is higher.

### B. The intermediate region

Finally we consider the intermediate region, in which energy level II is the ground state. Given the two extremes at the star and ring ends for energy level II above, we postulate the state is a mixture of

\[
\frac{\gamma^2}{4} \left[ (|11\rangle \langle 11| + |00\rangle \langle 00| + 2 |\Psi^-\rangle \langle \Psi^-|) + \frac{\alpha^2}{2} (|01\rangle \langle 01| + |10\rangle \langle 10|) \right]
\]

which gives concurrence \( C = 0 \). The next-to-nearest reduced density matrix is given by

\[
\frac{\gamma^2}{4} \left[ (|11\rangle \langle 11| + |00\rangle \langle 00| + 2 |\Psi^+\rangle \langle \Psi^+|) + \frac{\alpha^2}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|) \right]
\]

which also gives concurrence \( C = 0 \). Hence we have shown that this energy level always has zero entanglement for both nearest and next to nearest neighbors.

In summary, we have analysed the form of the state vectors of the energy levels I and II at the extremes of the star and ring models. We have then interpolated the states to give general forms which cover the range \( 0 < c < 1 \). Using these general forms for the states we have obtained analytic formulae for the concurrence which match the numerical results.

### C. The production of GHZ and other multiparticle entangled states

As a final thought in this section, we note that, the intermediate region, in which concurrence turns out to be zero, is not entirely uninteresting. In fact, it can even be regarded as the most interesting region of the model because it allows the production of a four particle GHZ (Greenberger-Horne-Zeilinger) state. Suppose we apply a magnetic field and the measurement on the central spin are crucial. However, we can regard the GHZ state as a “simple derivative” of our ground state in the intermediate region. Even when the production of the GHZ state is unsuccessful, the state of the outer spins is projected to yet another type of interesting multiparticle entangled state namely \(|C_0\rangle\) or \(|C_0'\rangle\). This state has the property that the concurrence between any two spins is \( 2/N \), which is the maximum possible entanglement in a collection of \( N \) spins in which all pairs of spins are equally entangled \( \frac{\sqrt{2}}{2} \) (in the present case, \( N = 4 \)).

The analysis also shows the robustness of the process of multiparticle entangled state production from ground states of spin stars \[22\]. Any star geometry with a sufficiently large number of outer spins placed in a ring will have them physically close and will thereby add unwanted ring interactions (interactions of an outer spin with its neighbors). Our calculations here show that throughout the star region (as long as the ring interactions are not too strong), we can produce the states \(|C_0\rangle\) or \(|C_0'\rangle\) which have the property that the concurrence between any two spins is \( 2/N \). This happens with a probability \(|\gamma|^2\) (about 0.49) in the star region when the degeneracy of the ground states is lifted by a magnetic field and a measurement is performed on the central spin. Thus in the same way as described for the pure star in Ref.\[22\], multi-particle entangled states can be produced in a star polluted with some degree of ring interaction.
VII. GROUND STATES FOR GENERAL $N$

In this section we attempt to give a general explanation of the form of the ground state that applies to general values of $N$. We start from the observation that, if two spins interact with each other via the $XX$ interaction then the ground state is the singlet state. That is, the energy is minimised when the two spins are antiparallel. As a general hypothesis then, we suppose that, in the combined star and ring model ground state, the $XX$ interaction generally tries to put adjacent spins into a singlet state. Below we apply this hypothesis to even and odd $N$ separately and consider the effect that this would have on the ground state.

Firstly we consider the case $N = 4$. If we expect the $XX$ interaction to form singlets, then when in a ring we might expect the superposition of singlet states depicted graphically in Figure 8. Here and henceforth (in the figures 8-10) the sign + in the superposition should be quite generally interpreted to mean superposition with a general phase. We have always varied these phases to optimize the overlap of our test states (those in figures 8-10) with the actual ground states.

To determine how close this superposition is to the ground state of the star-ring combination the fidelity between the ground state and this superposition was calculated and is plotted in the top left graph in Figure 6 where the fidelity is marked by $O_p$. As can be seen, the fidelity is very close to 1 on the ring-side, giving credence to the intuitive state. As $c$ increases, the central spin is brought in to interact with the other spins. There is consequently an odd number of spins and the system cannot form singlets. The system is said to be ‘frustrated’. We would expect this to decrease the nearest-neighbour entanglement and this is exactly what happens while energy level I is the ground state - the nearest neighbour entanglement decreases as $c$ increases.

Next we apply the same hypothesis to the case of odd $N$, taking $N = 5$ as an illustrative example. In this case, for $c = 0$ i.e. pure ring, one of the spins is unpaired, or frustrated. As $c$ is increased, allowing interactions through the central spin, the frustrated spin may pair up with the central qubit in some manner and become less frustrated. In Figure 10 we depict a superposition of states which represents this idea that the central spin pairs up with an ‘outer’ qubit, taking account of the rotational symmetry of the model.

This model in Figure 10 could explain the variation of the entanglement for odd $N$ with $c$ - as $c$ increases the central spin somehow allows all the spins to pair up and form singlets, relieving the frustration. If nearest-neighbour spins are ‘singlet-like’ then are more likely to be entangled and thus the nearest-neighbour entanglement rises, as is indeed the case. To provide some confirmation of this hypothesis the ground state of the star-ring combination can be compared.
to the superposition depicted in Figure 10 by calculating the fidelity between them. This fidelity, marked by $O_p$, is plotted in the top right plot in Figure 10. This graph shows that this speculated state is indeed very close the ground state, and in fact, near $c = 0.7$, it is the ground state. Interestingly, the point at which it exactly becomes the ground state is also the point at which a different energy level takes over, leading to the sudden drop in nearest-neighbour entanglement.

We have shown that the hypothesis that the $XX$ interaction encourages spins to form singlets not only provides an explanation for the behaviour of the concurrence, but also gives a close approximation to the ground states as evidence by having a fidelity with the ground state approaching 1.

### A. Discussion

In this section we first highlight some of the salient observations in the paper:

- In section VI it was pointed out that discontinuities in concurrence occurred when the ground state changed.
- The behavior of the concurrence as $c$ varied in the regions were concurrence varies smoothly is not due to the crossings of energy levels. The example of $N = 6$ in section VI showed that even though the energy level was the ground state for $0.4 < c < 1$ the patterns observed in the concurrence, such as a peak in the next to nearest-neighbour concurrence for $c$ slightly greater than 0.7 still occur.
- Section VI indicated that the ground state stays ‘ring-like’ and ‘star-like’ for some time near $c = 0$ and $c = 1$ respectively.
- Section VII gave a reasonable explanation for why the concurrence increases as $c$ was increased from $c = 0$ for $N$ odd and decreased for $N$ even based on frustration.
- In the intermediate region, at least for $N = 4$, a GHZ state can be produced as a simple derivative of the ground state.
- Some interesting multiparticle entangled states which can be produced in a star system of spins can also be produced even when a significant proportion of ring interaction is present on top of the star interaction.

In Section VII we have already given an explanation for the apparently counter-intuitive rise of the nearest neighbor entanglement on increasing $c$ from the ring side for odd $N$. We now attempt to provide an explanation for the rest of the behavior of the entanglement in the system that we have numerically observed. This includes the apparently counter-intuitive rise of the next to nearest neighbor entanglement as $c$ is decreased in the star region and the vanishing of both nearest and next to nearest neighbor entanglement around $c = 0.7$. We will provide the explanation only in the case of $N = 4$, and assume that an analogous argument holds for other $N$. In the case of $N = 4$, we have already provided a mathematical explanation of the behavior of entanglement (in section VI) by accepting certain numerically observed patterns for the behavior of $\alpha, \beta$ and $\gamma$. It is these patterns that we will now explain by taking for granted the numerically observed facts that the coefficients $\alpha, \beta$ and $\gamma$ always remain real and they smoothly change to interpolate between the ground state of the star and the ring as $c$ varies.

In the ground state at the star ($c = 1$) end, the outer spins are in a mixture of fully symmetric states. This is because the star Hamiltonian converts states $|1\rangle |C_1\rangle$ to $|0\rangle \frac{1}{\sqrt{6}} (|2\rangle |A\rangle + 2 |B\rangle)$ (and $|0\rangle |C_3\rangle$ to $|1\rangle \frac{1}{\sqrt{6}} (|2\rangle |A\rangle + 2 |B\rangle)$) and vice versa. The star interaction thus tends to symmetrize the state of the outer spins (i.e., impose the same sign on $\alpha$ and $\beta$) and impose an opposite sign to $\gamma$ with respect to $\alpha$ and $\beta$ in the ground state. On the other hand, the ring interaction converts two subparts of a symmetric state (namely $|A\rangle$ and $|B\rangle$) to each other, and thus tends to impose a sign difference between these states in the ground state. Indeed the ground state at the ring end ($c = 0$) consists of the state $\frac{1}{\sqrt{2}}(|A\rangle - |B\rangle)$ irrespective of the state of the central spin. To smoothly interpolate between the ground states of the star and the ring by a single ground state, then, $\alpha$ and $\beta$ have to change from being of the same sign to being of opposite signs as one proceeds from the star end to the ring end. In order to do this, one of them (either $\alpha$ or $\beta$) has to retain its sign and thereby remain opposite in sign to $\gamma$, while the other has to go through zero and reverse its sign. In the star region, the effect of the star interaction is strong, and in this region the energy is lowest if $\beta$, rather than $\alpha$, is maintained to be opposite in sign from $\gamma$. This is simply because the energy of a state of the type $\gamma |0\rangle |C_4\rangle + |1\rangle (\alpha |A\rangle + \beta |B\rangle)$ with positive $\gamma$ and negative $\alpha$ and $\beta$ is lower for a larger proportion of $|B\rangle$ rather than for a larger proportion of $|A\rangle$. The value of $\alpha$, which should reverse sign, will thus move towards zero as $c$ is decreased from the star end. This decreases the proportion of $|A\rangle$ in the ground state. As $|A\rangle$ has nearest neighbors in opposite states, it constructively contributes to the entanglement of nearest neighbors. If it decreases, so does the nearest neighbor entanglement. As far as next to nearest neighbor is concerned, though, $|A\rangle$ contributes only $|00\rangle$ or
that entanglement can only stem from the presence of states $|01\rangle$ or $|10\rangle$. So $|A\rangle$ does not contribute positively to next to nearest neighbor entanglement. On the other hand, it contributes a fraction of unentangled states $|00\rangle\langle 00|$ and $|11\rangle\langle 11|$ to the mixed state of the next to nearest neighbor qubits, which reduces the entanglement in the state. Thus when the fraction of $|A\rangle$ decreases, the part of the state which contributes to next to nearest neighbor entanglement increases due to normalization, thereby increasing this entanglement. This explains the apparently counterintuitive rise of the next to nearest neighbor entanglement in the star region as $c$ decreases.

The decrease in the proportion of $|A\rangle$ in the state, however, increases the energy due to the ring part of the interaction, as this part of the interaction “prefers” (i.e., lowers the energy of) states in which nearest neighbors are oppositely aligned. The energy of the state thus continues to increase as $c$ decreases. For $0.7 \leq c \leq 1$, however, it still continues to be the lowest energy state because of the dominance of the star interaction in this region. Around $c \approx 0.7$ the buildup of energy due to decreasing proportion of $|A\rangle$ is not sustainable, and a different pair of states overtake as the ground state. These states, which signal the start of the intermediate region, are of the form $\gamma|0\rangle|C'_4\rangle + \alpha|1\rangle|D\rangle$ and $\alpha'|0\rangle|D\rangle - \gamma'|1\rangle|C'_7\rangle$. This has a significant proportion of $|D\rangle$, which, because of its similar nature as $|A\rangle$, lowers the energy due to the ring part of the interaction. The state of the outer spins corresponding to this state is a mixture of a state $|C'_4\rangle$, $|C'_7\rangle$ and $|D\rangle$. Both $|D\rangle$, and an equal mixture of $|C'_4\rangle$ and $|C'_7\rangle$, individually have zero nearest neighbor and next to nearest neighbor entanglement, which explains the dropping of all entanglement to zero for a region after $c \approx 0.7$ (the intermediate region).

The state in the intermediate region still has a significant absolute value of $\gamma'$, which plays a role in lowering the energy due to the star part of the interaction. However, as we approach the end of the intermediate region by decreasing $c$, the star interaction becomes altogether less important, and then it is more important to have a state of the form $\alpha|A\rangle + \beta|B\rangle$ with $\alpha$ and $\beta$ of comparable absolute values but opposite in sign to maximally lower the energy due to the dominant “ring” part of the interaction. At this value of $c$, the energy level which smoothly interpolates between the ground states of the star and the ring has assumed precisely such a form (except for a very small extra fraction of $\gamma|1\rangle|C'_1\rangle$ or $\gamma|0\rangle|C'_3\rangle$), and becomes the ground state once again. This indicates the start of the ring region when $c$ is decreased.

VIII. CONCLUSIONS

This paper has demonstrated that the model in which the outer spins can interact through a combination of star and ring type interactions possesses a number of surprising features which make it interesting to study. In this paper we have shown that both nearest neighbor (for odd $N$) and next to nearest neighbor (for all $N$) entanglement in the ground state have their maxima for a Hamiltonian which is neither a pure ring nor a pure star in its interactions. We have drawn attention to the link between dramatic changes in entanglement and the change in the ground state due to crossing over of energy levels. The case of four outer spins was analyzed in detail and interpolated ground states given for all values of $c$. By hypothesizing a tendency of the interaction to form singlet states we have found an explanation for the behavior of the entanglement in the ring region that applies to general values of $N$. We have found that we can produce a GHZ state as a simple derivative of the ground state for $N = 4$. We have also shown that the multi-particle entangled states producible from a pure star are also producible from in a star system polluted with a significant proportion of ring interaction.

We believe the concept of a combination of interactions by both models is relevant because although it is unlikely to be a naturally occurring structure, experimental implementations of quantum computing, for example using quantum dots, may allow artificial structures to be created where the topology is in fact the combination we have been describing. In that case our results will be useful, especially is situations where there are untunable (fixed) interactions.

There are a number of potential avenues for future working stemming from the material described here. It would be satisfying to be able to expand the range of $N$ considered to try to spot broader trends and patterns. Indeed to fully describe the model a complete analytical solution would be desirable although this is most likely very difficult to find. The concept of a network of spins interacting through a combination of two different topologies could perhaps be extended to other structures and dimensions. Recently, it has been shown that spin systems can be used for studying non-Markovian dynamics [31], optimal quantum cloning [32] (where a spin star can be used) and quantum computation [42]. It would be interesting to investigate the dynamical consequences of spins interacting in a combination of star and ring geometries.
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