COUNTING MAXIMAL NEAR PERFECT MATCHINGS IN QUASIRANDOM AND DENSE GRAPHS

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ABSTRACT. A maximal ε-near perfect matching is a maximal matching which covers at least $(1 - \varepsilon)|V(G)|$ vertices. In this paper, we study the number of maximal near perfect matchings in generalized quasirandom and dense graphs. We provide tight lower and upper bounds on the number of ε-near perfect matchings in generalized quasirandom graphs. Moreover, based on these results, we provide a deterministic polynomial time algorithm that for a given dense graph $G$ of order $n$ and a real number $\varepsilon > 0$, returns either a conclusion that $G$ has no ε-near perfect matching, or a positive non-trivial number $\ell$ such that the number of maximal ε-near perfect matchings in $G$ is at least $n^\ell n$. Our algorithm uses algorithmic version of Szemerédi Regularity Lemma, and has $O(f(\varepsilon)n^{5/2})$ time complexity. Here $f(\cdot)$ is an explicit function depending only on $\varepsilon$.

Keywords: maximal matching, perfect matching, quasirandom graph, regularity
MSC numbers: 05C70, 05C80, 05C85

1. Introduction

For a simple graph $G = (V, E)$, a matching $\mathcal{M}$ of $G$ is a subset of $E(G)$ such that the edges in $\mathcal{M}$ do not have common end vertices. We say $\mathcal{M}$ is a perfect matching if $|\mathcal{M}| = |V(G)|/2$. The problem of computing the total number of perfect matchings in a graph, has been extensively studied by mathematicians and computer scientists. It is known that the number of perfect matchings in a bipartite graph is equivalent to the permanent of its adjacency matrix. See [3] for a recent survey on several theorems and open problems on permanent of matrices and its algebraic properties. The evaluation of the permanent has attracted the attention of researchers for almost two centuries, however, despite many attempts, an efficient algorithm for general matrices has proved elusive. Indeed, Ryser’s algorithm [27] remains the most efficient for computing the permanent exactly, even though it uses as many as $\Theta(n2^n)$ arithmetic operations. A notable breakthrough was achieved about 60 years ago with the publication of Kasteleyn’s algorithm for counting perfect matchings in planar graphs [22], which uses just $O(n^3)$ arithmetic operations.

It turns out that computing the number of perfect matchings in a bipartite graph (computing permanent of a $\{0,1\}$-matrix) falls into the \#P-complete complexity
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class \[37\], and thus, modulo a basic complexity theoretic conjecture, cannot be solved (exactly) in polynomial time. This remains true even for 3-regular bipartite graphs \[9\], and for bipartite graphs with minimum vertex degree at least \( n/2 \) \[5\].

Using the so-called Pfaffian orientations, the perfect matchings in a planar graph can be counted in polynomial time \[13, 22, 33\]. A generalization of this approach yields a polynomial time algorithm for graphs of bounded genus \[17, 34\]. Furthermore, we can count the perfect matchings in a graph of bounded treewidth \[2\]. Basically, most of the positive results are concerned with sparse graphs. For other graph classes, less is known, but \#P-completeness is known for chordal and chordal bipartite graphs \[25\].

Ever since the introduction of the \#P complexity class by Valiant \[37\], the focus on these problems shifted to finding approximate solutions. Jerrum, Sinclair, and Vigoda \[21\] in a breakthrough obtained a fully polynomial time randomized approximation scheme (FPRAS) for the permanent of matrices with nonnegative entries. In other words, they designed a randomized algorithm that for any given \( \varepsilon > 0 \), outputs a \( 1 + \varepsilon \) multiplicative approximation of the permanent, in time polynomial in \( n \) and \( 1/\varepsilon \). This approach focuses on rapidly mixing Markov chains to obtain appropriate random samples. Many randomized approximation schemes for various counting problems were derived in this way – see e.g., \[18, 20, 29\] for several nontrivial applications. Unfortunately, Jerrum, Sinclair and Vigoda’s \[21\] approach seems too complicated to be used in practice and the approach does not appear to extend to nonbipartite graphs, since odd cycles are problematic. For this reason, a simpler Markov chain was proposed in \[10, 11\]. In \[11\], counting all perfect matchings in some particular classes of bipartite graphs was examined. Recently, Dyer and Müller \[12\] extended the analyses in \[11\] to hereditary classes of nonbipartite graphs.

There are only a few results concerning approximately counting perfect matchings in general graphs. Jerrum and Sinclair \[19\] considered this problem in general graphs. Their Markov chain method requires exponential time complexity for general graphs. More precisely, their method requires time polynomial in the ratio of number of near perfect matchings and number of perfect matchings, which may be exponential in the size of graph. This condition is satisfied for graphs with \( 2n \) vertices and minimum degree at least \( n \), therefore providing a FPRAS for this class of graphs. There have been other approaches to tackle the problem. Chien \[6\] presents a determinant-based algorithm for the number of perfect matchings in a general graph. His estimator requires \( O(\varepsilon^{-2}3^{n/2}) \) trials to obtain a \( (1 \pm \varepsilon) \)-approximation of the correct value with high probability on a graph with \( 2n \) vertices, and a polynomial number \( (O(\varepsilon^{-2}n\omega(n))) \) of trials on random graphs, where \( \omega(n) \) is any function tending to infinity. Refer to \[16\] for a simpler algorithm with experimental results.

There are results concerning counting total number of matchings (not only perfect matchings) in graphs and random graphs. Vadhan in \[36\] showed that the problems of counting matchings remain hard when restricted to planar bipartite graphs of
bounded degree or regular graphs of constant degree. Therefore, approximating this number has been studied by researchers. For example, Bayati et al. [4] construct a deterministic fully polynomial time approximation scheme (FPTAS) for computing the total number of matchings in a bounded degree graph. Additionally, for an arbitrary graph, they construct a deterministic algorithm for computing approximately the number of matchings within running time $\exp(O(\sqrt{n} \log^2 n))$, where $n$ is the number of vertices. Patel and Regts [26] recently provided an alternative deterministic algorithm to approximately count matchings in bounded degree graphs. This is the same result as in [4], using a completely different method. Zdeborová and Mézard [38] considered this problem on sparse random graphs, in fact, their result is the computation of the entropy, i.e. the leading order of the logarithm of the number of solutions, of matchings with a given size.

In terms of lower bounds, Schrijver [28] shows that any $d$-regular bipartite graph with $2n$ vertices has at least

$$\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n$$

perfect matchings. More generally, let $m_k(G)$ denote number of matchings of size $k$ in graph $G$. Friedland, Krop and Markström [15] conjectured the following lower bound on $m_k(G)$ where $G$ is a $d$-regular bipartite graph

$$m_k(G) \geq \binom{n}{k}^2 \left(\frac{d-n/k}{d}\right)^{n(d-n/k)}(dn/k)^{n^2/k}$$

The conjecture was proved in [8] and extended to irregular bipartite graphs in [23].

Given the difficulty of counting number of perfect matchings, in particular beyond bipartite graphs, we turn our attention to near perfect matchings. In this paper, we focus on counting the number of maximal near perfect matchings in graphs. A matching $\mathcal{M}$ in $G$ is maximal if the graph induced by the vertices which are not in $\mathcal{M}$ is empty. Counting maximal matchings is $\mathbb{NP}$-complete even in bipartite graphs with maximum degree five [36]. To the best of our knowledge there is no result concerning approximating the number maximal matchings. A maximal $\varepsilon$-near perfect matching is a maximal matching that covers at least $(1-\varepsilon)|V(G)|$ vertices. Let $\mathbb{NM}(G,\varepsilon)$ denote the number of maximal $\varepsilon$-near perfect matchings in graph $G$. Our first result is an approximation on the number of near perfect matchings in $\varepsilon$-regular graphs.

**Theorem 1.1.** Given $\varepsilon > 0$ and a bipartite $\varepsilon$-regular graph $G$ with density $p$. Then there exists $n_0 = n_0(\varepsilon, p)$, such that for $|V(G)| = 2n > n_0$, we have

$$(1 - 3\sqrt{\varepsilon})n \log p n \leq \log \mathbb{NM}(G, \sqrt{\varepsilon}) \leq (1 + 3\sqrt{\varepsilon})n \log p n.$$
(V_i, V_j) is \( \varepsilon \)-regular with density \( p_{ij} \), and \( G[V_i] \) is \( \varepsilon \)-close (in the sense of cut metric) to a random graph \( G(n, p_i) \) for every \( i \). Here \( p_{i,j} \) is the \((i, j)\)-entry of \( P \), and \( p_i \) is \((i, i)\)-entry of \( P \).

Given \( G \in \mathcal{Q}(n^{(m)}, P, \varepsilon) \), define the quotient graph \( H = G/m \) as a weighted graph, such that \( V(H) = [m] \), and the edge weight \( u(ij) = p_{ij} \), the vertex weight \( u(i) = p_i \). We let \( ij \in E(H) \) if \( p_{ij} \neq 0 \). Let \( w : [m]^2 \to [0, 1] \) be a function, we consider the following linear equations on \( H \).

\[
(1) \quad \sum_{1 \leq j \leq m, ij \in E(H)} w(ij) = 1 \quad \text{for every } 1 \leq i \leq m,
\]

We have the following result on the number of maximal near perfect matchings in general quasirandom graphs.

**Theorem 1.2.** Suppose we have an integer \( m \geq 2 \), and a \( m \times m \)-matrix \( P \). Then there exists \( n_0 > 0 \) and \( c > 0 \), such that if \( n > n_0 \) and \( \varepsilon < c \), for every graph \( G \in \mathcal{Q}(n^{(m)}, P, \varepsilon) \), let \( H \) be the quotient graph of \( G \), we have

1. If the linear system (1) of \( H \) does not have any solution, \( G \) does not have maximal \( \sqrt{\varepsilon} \)-near perfect matchings.
2. If the linear system (1) of \( H \) has solutions, then

\[
(1 - 4\sqrt{\varepsilon}) \frac{m}{2} n \log n \leq \log \text{NM}(G, \sqrt{\varepsilon}) \leq (1 + 7\sqrt{\varepsilon}) \frac{m}{2} n \log n.
\]

Based on the algorithmic version of Szemerédi regularity lemma and the results we obtain for quasirandom graphs, we provide a deterministic polynomial-time algorithm on approximating the number of maximal near perfect matchings in dense graphs. A graph \( G \) is called dense if \(|E(G)| \geq \alpha |V(G)|^2 \) for some fixed \( \alpha \). Given a dense graph \( G \), our algorithm provide a non-trivial lower bound on the number of maximal near perfect matchings in \( G \).

**Number of Max Near Perfect Matchings Dense.**

**Input:** A graph \( G \) of order \( n \) and a real number \( \varepsilon > 0 \).

**Output:** Either a conclusion that \( G \) does not contain a maximal \( \varepsilon \)-near perfect matching, or a (non-trivial) real number \( \ell \) such that \( \text{NM}(G, \varepsilon) > n^\ell n \).

In particular, the lower bound is obtained by the following theorem.

**Theorem 1.3.** Let \( G \) be a dense graph on \( n \) vertices. Then

\[
\log \text{NM}(G, \sqrt{\varepsilon}) \geq (1 - 4\sqrt{\varepsilon}) \sup_{w(e) \in \mathcal{E}} \sum_{e \in E_4} \frac{w(e)n}{K} \log \frac{w(e)n}{K} + \sum_{e \in E_3} \frac{w(e)n}{K} \log p_e \frac{w(e)n}{K}.
\]
The value of $w$ and the set $\mathcal{S}$ are determined by a linear programming, and the values of $K, p$, the sets $E_3, E_4$ are determined by the algorithm, we will discuss it in details in Section 4.

The paper is organized as follows. In the next section, we give basic definitions and properties in graph theory, and the theoretical background used in the paper. In Section 3, we discuss the matchings in generalized quasirandom graphs. In Section 4, we consider the problem for the dense graphs, and provide an approximation algorithm.

2. Preliminaries

We will use standard definitions and notation in graph theory. Given a graph $G$, a matching $\mathcal{M}$ of $G$ is a subset of $E(G)$ such that the edges in $\mathcal{M}$ do not have common end vertices. We say $\mathcal{M}$ is a perfect matching if $|\mathcal{M}| = |V(G)|/2$, and $\mathcal{M}$ is maximal if there does not exist another matching $\mathcal{M}_1 \neq \mathcal{M}$ such that $\mathcal{M} \subseteq \mathcal{M}_1$. Given a graph $G$ and a real number $\varepsilon > 0$, we say a matching $\mathcal{M}$ is $\varepsilon$-near perfect if $|\mathcal{M}| \geq (1 - \varepsilon)|V(G)|/2$. Given a simple graph $G$, let $NM(G, \varepsilon)$ be the number of maximal $\varepsilon$-near perfect matchings in $G$. We use $[n]$ to denote the set of integers $\{1, \ldots, n\}$. All the logarithms in the paper are taken base $e$.

Szemerédi Regularity Lemma \[31\] is one of the most powerful tools in modern graph theory. Szemerédi first used this lemma in his celebrated theorem on the existence of long arithmetic progressions in dense subset of integers \[30\]. The lemma gives us the rough structure of dense graphs. Roughly speaking, Given any dense graph $G$ and the error $\varepsilon > 0$, one can partition the vertex set of $G$ into constant (only depending on $\varepsilon$) parts, and the subgraph between each two parts except an $\varepsilon$ fraction performs like a random graph. To make this precise, we need some definitions.

Given a simple graph $G$ and $X, Y \subseteq V(G)$. Let $e(X, Y)$ be the number of edges between $X, Y$ then the edge density between $X$ and $Y$ is defined as $d(X, Y) = e(X, Y)/(|X||Y|)$. A pair of vertex subsets $(X, Y)$ is $\varepsilon$-regular if for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ that satisfy $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have $|d(X', Y') - d(X, Y)| < \varepsilon$. A pair of vertex set $(X, Y)$ is $\varepsilon$-regular with density $p$, if for every $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have $|d(X', Y') - p| \leq \varepsilon$. Note that under this definition, the edge density between $X$ and $Y$ is not necessarily $p$.

We say a vertex partition $\mathcal{P} = \{V_1, \ldots, V_K\}$ is equitable if $||V_i| - |V_j|| \leq 1$ for every $1 \leq i < j \leq K$. An equitable vertex partition $\mathcal{P}$ with $K$ parts is $\varepsilon$-regular if all but at most $\varepsilon K^2$ pairs of parts $(V_i, V_j)$ are $\varepsilon$-regular.

Theorem 2.1 (Szemerédi Regularity Lemma \[31\]). For every $\varepsilon > 0$ and every integer $m$, there exists an integer $M = M(m, \varepsilon)$ such that every simple graph $G$ has an $\varepsilon$-regular partition into $K$ parts, where $m \leq K \leq M$. 
To obtain a Szemerédi partition, there are many known polynomial time algorithms, for example, [1]. Recently, Tao [32] provided a probabilistic algorithm which produces an \( \varepsilon \)-regular partition with high probability in constant time (depending on \( \varepsilon \)). In this paper, we will use a more recent deterministic PTAS due to Fox et al. [14].

**Theorem 2.2** ([14]). There exists an \( O_{\varepsilon,\alpha,k}(n^2) \) time algorithm, which, given \( \varepsilon > 0 \), and \( 0 < \alpha < 1 \), an integer \( k \), and a graph \( G \) on \( n \) vertices that admits an \( \varepsilon \)-Szemerédi partition with \( k \) parts, outputs a \((1 + \alpha)\varepsilon\)-Szemerédi partition of \( G \) into \( k \) parts.

Our algorithm to compute a lower bound on the number of maximal near perfect matchings in dense graphs is based on estimating the number of maximal near perfect matchings in quasirandom graphs. Quasirandom graphs are graphs which share many properties with random graphs. The notion of quasirandomness was first introduced in seminal papers by Chung, Graham and Wilson [7] and independently by Thomason [35]. In this paper, we will use a slightly different notion of quasirandomness.

Given a simple graph \( G \) and \( \varepsilon > 0 \), we say \( G \) is \((\varepsilon,p)\)-quasirandom, denoted by \( G \in \mathcal{Q}(n,p,\varepsilon) \), if for every \( X,Y \subseteq V(G) \) that satisfy \( X \cap Y = \emptyset \) and \( |X| \geq \varepsilon|V(G)| \), \(|Y| \geq \varepsilon|V(G)| \), we have \(|d_G(X,Y) - d_{K_{n,p}}(X,Y)| < \varepsilon \), where \( K_{n,p} \) is an edge weighted complete graph on \( V(G) \) with edge weight \( p \). Here for a weighted graph, we define \( e(X,Y) = \sum_{e \in E(X,Y)} w(e) \), and the edge density \( d(X,Y) = e(X,Y) / (|X||Y|) \). We say a graph \( G \) is generalized quasirandom, if there is an equitable vertex partition of \( V(G) \), such that the graphs induced on each part, and between every two different parts, are "random like". To be more precise, let \( P \) be a symmetric \( m \times m \)-matrix, such that \( 0 \leq p_{ij} \leq 1 \), where \( p_{ij} \) is the \((i,j)\)-entry of \( P \). A graph \( G \in \mathcal{Q}(n^{(m)},P,\varepsilon) \) if \( V(G) = \bigsqcup_{i=1}^m V_i \) such that \(|V_i| = n \) and \((V_i, V_j)\) is \( \varepsilon \)-regular with density \( p_{ij} \), and \( G[V_i] \) is \((\varepsilon,p_i)\)-quasirandom, where \( p_i \) is the \((i,i)\)-entry of \( P \).

3. **Matchings in generalized quasirandom graphs**

3.1. **Matchings in quasirandom graphs.** By the definition of \( \varepsilon \)-regular, we have the following lemma.

**Lemma 3.1.** Suppose \(|X| = |Y| = n \) and \((X,Y)\) is \( \varepsilon \)-regular with density \( p \). Then

\[
\left| \{v \in X \mid (1-\delta)pn \leq d(v) \leq (1+\delta)pn \} \right| \geq (1-2\varepsilon)n,
\]

where \( \delta = \varepsilon/p \).

**Proof.** Let \( X' \subseteq X \) such that for every \( v \in X' \), we have \( d(v) > (1+\delta)pn \). Thus \( e(X',Y)((1+\delta)pn|X'| = (p+\varepsilon)|X'||Y| \). On the other hand, if \(|X'| \geq \varepsilon n \), since \((X,Y)\) is \( \varepsilon \)-regular with density \( p \), this gives us \( e(X',Y) \leq (p+\varepsilon)|X'||Y| \), contradiction. \( \square \)

We are now going to prove Theorem 1.1. We suggest that reader consults Algorithm 1 while reading the proof.
Proof of Theorem 1.1. Suppose $V(G)$ has a bipartition $X, Y$, with $|X| = |Y| = n$, and the edge density between $X, Y$ is $p$. Since $(X,Y)$ is $\varepsilon$-regular, it is $\sqrt{\varepsilon}$-regular. Let $\mathcal{T}(X)$ be the set of typical vertices in $X$, that is, set of vertices $v$ such that $(1 - \delta)pn \leq d(v) \leq (1 + \delta)pn$, where $\delta = \sqrt{\varepsilon}/p$.

We first consider the lower bound. Count the number of near perfect matchings greedily in $k + t$ phases, the values of $k$ and $t$ will be determined later. In Phase 1, let $X_1 := X$, and we pick an arbitrary vertex $v \in \mathcal{T}(X_1)$. Pick a vertex $u \in N(v)$ arbitrarily. Let $X_2 := X - v, G := G - \{u, v\}$ and $\mathcal{T}(X_1) := \mathcal{T}(X_1) - v$. Keep doing this procedure $\sqrt{\varepsilon}n$ times, that is, we stop Phase 1 after removing $\sqrt{\varepsilon}n$ vertices from $X_1$. We denote the remaining vertices in $X_1$ by $X_2$, and move to Phase 2.

In Phase 2, let

$$\mathcal{T}(X_2) := \{v \in X_2 | (1 - \delta)pn(1 - \sqrt{\varepsilon}) \leq d(v) \leq (1 + \delta)pn(1 - \sqrt{\varepsilon})\}.$$ 

We pick a vertex $v \in \mathcal{T}(X_2)$ arbitrarily and pick $u \in N(v)$. Let $X_3 := X_2 - v, G := G - \{u, v\}$ and $\mathcal{T}(X_2) := \mathcal{T}(X_2) - v$. We keep doing this procedure $\sqrt{\varepsilon}n$ times, then we let $X_3 := X_2$ and move to Phase 3, and we similarly let $\mathcal{T}(X_3) := \{v \in X_3 | (1 - \delta)pn(1 - 2\sqrt{\varepsilon}) \leq d(v) \leq (1 + \delta)pn(1 - 2\sqrt{\varepsilon})\}$.

Suppose after applying Phase $k$, we have $|X_{k+1}| \leq c\sqrt{\varepsilon}n$ in Phase $k+1$, where $c = \frac{1}{(1 - \delta)p}$. In Phase $k+1$, we pick a vertex in $\mathcal{T}(X_{k+1})$ and remove it as well as one of its neighbor as we did before. But now, instead of repeating this procedure $\sqrt{\varepsilon}n$ times, we do it $(1 - \delta)p|X_{k+1}|$ times. Then we move to Phase $k+2$. We run the algorithm in Phase $k+2 (1 - \delta)p|X_{k+2}|$ times and then move to the next phase. We stop the algorithm after Phase $k + T$, if in Phase $k + T + 1$ we have $|X_{k+T+1}| \leq \sqrt{\varepsilon}n$. See Algorithm 1 for the algorithm.

The algorithm is well-defined, since in Phase 1 for each $i$, the graph on $(X_i,Y_i)$ is $\sqrt{\varepsilon}$-regular with density $p$. By Lemma 3.1 we can always define the set $\mathcal{T}(X_i)$. We ignore the floor and ceiling function here to simplify the computation. By the way we define $k$ and $t$, we have

$$k = \frac{1}{\sqrt{\varepsilon}} - c, \quad t = \frac{\log(1 - \delta)p}{\log(1 - (1 - \delta)p)}.$$ 

Note that the collection of edges we removed in each steps in the algorithm gives us a $\sqrt{\varepsilon}$-near perfect matching in $G$. Therefore,

$$NM(G, \sqrt{\varepsilon}) \geq \prod_{i=0}^{k-1} \frac{((1 - \delta)p(n - i\sqrt{\varepsilon}n))!}{((1 - \delta)p(n - i\sqrt{\varepsilon}n) - \sqrt{\varepsilon}n)!} \prod_{i=0}^{t-1} ((1 - \delta)pc\sqrt{\varepsilon}n(1 - (1 - \delta)p)^i)!.$$ 

The first product counts the number of possible different collections of edges we removed from first $k$ phases, and the second product counts the number of different
collections of edges we removed from the last $t$ phases. By a complicated but standard computation (see Lemma A.1 in Appendix for the computation), we have
\[
\log \text{NM}(G, \sqrt{\varepsilon}) \geq (1 - 3\sqrt{\varepsilon})n \log pn.
\]

Now we consider the upper bound. Note that all the vertices in $\mathcal{X}(X_1)$ has at most $(1 + \delta)pn$ neighbors, we have $\text{NM}(G, \sqrt{\varepsilon}) \leq ((1 + \delta)pn)^{(1 - 2\sqrt{\varepsilon})}n^{2\sqrt{\varepsilon}n}$. Therefore,
\[
\log \text{NM}(G, \sqrt{\varepsilon}) \leq (1 + 3\sqrt{\varepsilon})n \log pn,
\]
finishes the proof.

\[\square\]

**Algorithm 1:** Near Perfect Matchings in Quasirandom Bipartite Graphs

**Input**: bipartite $\varepsilon$-regular graph $G = (X, Y)$ with density $p$.

\[c = \frac{1}{(1 - \delta)p}, k = \frac{1}{\sqrt{\varepsilon}} - c, t = \frac{\log(1 - \delta)p}{\log(1 - (1 - \delta)p)}, \text{NM} = 1, X_0 = X;\]

for $i = 1$ to $k$
do

\[X_i = X_{i-1};\]

\[\mathcal{X}(X_i) = \{v \in X_i \mid (1 - \delta)pn(1 - (i - 1)\sqrt{\varepsilon}) \leq d(v) \leq (1 + \delta)pn(1 - (i - 1)\sqrt{\varepsilon})\};\]

for $j = 1$ to $\sqrt{\varepsilon}n$
do

Pick $v$ from $\mathcal{X}(X_i)$ and pick $u$ from $N(v)$;

\[X_i = X_i - v, G = G - \{u, v\}, \mathcal{X}(X_i) = \mathcal{X}(X_i) - v;\]

\[\text{NM} = \text{NM} \times [(1 - \delta)pn(1 - (i - 1)\sqrt{\varepsilon}) - j + 1];\]

end

end

for $i = 1$ to $t$
do

\[X_{k+i} = X_{k+i-1};\]

\[\mathcal{X}(X_{k+i}) = \{v \in X_{k+i} \mid (1 - \delta)pn(1 - ((k + i) - 1)\sqrt{\varepsilon}) \leq d(v) \leq (1 + \delta)pn(1 - ((k + i) - 1)\sqrt{\varepsilon})\};\]

for $j = 1$ to $(1 - \delta)p|X_{k+i}|$
do

Pick $v$ from $\mathcal{X}(X_{k+i})$ and pick $u$ from $N(v)$;

\[X_{k+i} = X_{k+i} - v, G = G - \{u, v\}, \mathcal{X}(X_{k+i}) = \mathcal{X}(X_{k+i}) - v;\]

\[\text{NM} = \text{NM} \times ((1 - \delta)pc\sqrt{\varepsilon}n(1 - (1 - \delta)p)^i - j + 1);\]

end

end

Return: $\text{NM}$

By applying the same greedy procedure, we also obtain a good approximation for the quasirandom graphs. The proof is similar to the proof of Theorem 1.1 and we omit further details.
Theorem 3.2. Suppose $\varepsilon > 0$ and $G \in Q(n, p, \varepsilon)$. Then there exists $n_0 = n_0(\varepsilon, p)$, such that if $|V(G)| = n > n_0$, we have
\[ (1 - 3\sqrt{\varepsilon}) \frac{1}{2} n \log pn \leq \log NM(G, \sqrt{\varepsilon}) \leq (1 + 3\sqrt{\varepsilon}) \frac{1}{2} n \log pn. \]

3.2. Matchings in generalized quasirandom graphs. In this subsection, we will focus on generalized quasirandom graphs.

For a graph $H$ of order $m$, where $V(H) = [m]$, let $w : [m]^2 \to [0, 1]$ be a symmetric function such that $w(ij) = 0$ when $ij \notin E(H)$. Now consider the following linear equations.
\[
\sum_{j=1}^{m} w(ij) = 1 \quad \text{for every } 1 \leq i \leq m,
\]
We write $w(i) := w(ii)$. The following example shows that, the system of linear equations (2) may have exactly one solution, or infinitely many solutions, or no solutions, see Figure 1. In all the examples, we assume $w(i) = 0$ for every vertex $i$.

![Figure 1](image_url)

**Figure 1.** Linear equation (2) of $G_1$ does not have any solutions, $G_2$ has exactly one solution, and $G_3$ has infinitely many solutions.

Given a graph $H$, suppose the linear equation (2) does not have any solutions, then for each edge $e$ and vertex $v$ in $H$, we assign values $w(i) \in [0, 1]$ and $w(ij) \in [0, 1]$ arbitrarily if $ij \in E(H)$. Define the error $\mathcal{E}(H, w)$ as follows
\[ \mathcal{E}(H, w) = \sum_{i=1}^{m} \left| 1 - \sum_{j=1}^{m} w(ij) \right|. \]
Let $\mathcal{E}(H) = \min_{w : E(H) \to [0, 1]} \mathcal{E}(H, w)$. Suppose $|E(H)| = h$ and let $x = (e_1, \ldots, e_h)$. We can rewrite (2) as
\[ Ax = b^T, \]
where $b = (1, \ldots, 1)$ is a $(1 \times m)$-vector and $A$ is a $(m \times h)$-matrix. Let
\[ Q := \begin{bmatrix} A & I_{m \times m} \\ -A & I_{m \times m} \end{bmatrix} \]
where $I$ is the identity matrix. Let $c = (b, -b)$ be a $(1 \times 2m)$-vector, and let $y = (x, y_1, \ldots, y_m)$. Then $\mathcal{E}_r(H)$ is the solution of the linear programming

$$
\begin{align*}
\min & \sum_{i=1}^{m} y_i, \\
Q y^T & \geq c^T, \\
0 & \leq y \leq 1.
\end{align*}
$$

(3)

Note that $A$ is a $(0,1)$-matrix and the sum of each column of $A$ is 2. Then for a fixed $m$, there are finitely many possible matrices $A$. This fact implies the following lemma.

**Lemma 3.3.** Given a graph $H$ with $V(H) = [m]$. If the system of linear equations $[2]$ does not have any solution on $H$, then there exists a constant $c = c(m) > 0$, such that $\mathcal{E}_r(H) \geq c$.

For graph $G \in Q(n^{(m)}, P, \varepsilon)$, let $p_{ij}$ be the $(i, j)$-entry of $P$ and $p_i$ be the $(i, i)$-entry of $P$. Suppose $H$ is the quotient graph of $G$, that is, $V(H) = [m]$ and edge $ij$ in $H$ has weight $p_{ij}$ if $p_{ij} \neq 0$, vertex $i$ has weight $p_i = p_{ii}$. We are going to prove Theorem 1.2.

**Proof of Theorem 1.2.** Suppose $V(G) = V_1 \sqcup \cdots \sqcup V_m$ and $|V_1| = \cdots = |V_m| = n$. We consider linear equation $[2]$ on the quotient graph $H$.

**Case 1:** Linear system $[2]$ does not have any solution.

By Lemma 3.3, given $m$ there is a constant $c(m) > 0$ such that $\mathcal{E}_r(H) \geq c(m)$. Let $\varepsilon < c^2(m)$. Suppose $G$ has a maximal $\sqrt{\varepsilon}$-near perfect matching $M$. Then for every $1 \leq i \leq m$, except at most $\sqrt{\varepsilon} n$ vertices, all the vertices in $V_i$ are covered by edges in $M$. Now we consider the quotient graph $H$, for every $ij \in E(H)$, define $w(ij) = \frac{|M \cap E(G[V_i, V_j])|}{n}$ and $w(i) = \frac{|M \cap E(G[V_i])|}{n}$ for every $i \in [m]$. $M$ being a $\sqrt{\varepsilon}$-near perfect matching means $\mathcal{E}_r(H, w) < \sqrt{\varepsilon} < c(m)$, which is a contradiction.

**Case 2:** Linear system $[2]$ has a unique solution.

Now for every $e \in E(H)$ and $v \in V(H)$, we have a solution $w$ for the linear system $[2]$. For every $1 \leq i \leq m$, partition the vertex set $V_i$ into at most $m$ parts $V_{i,0}, V_{i,1}, \ldots, V_{i,i-1}, V_{i,i+1}, \ldots, V_{i,m}$, satisfying $|V_{i,j}| = w(ij)n$ and $|V_{i,0}| = w(i)n$. Suppose $N(V_i)$ is the number of different ways of partitioning $V_i$, define $w(i0) = w(i)$, then we have

$$
N(V_i) = \frac{n!}{\prod_{0 \leq j \leq m, j \neq i} w(ij)n!}.
$$

Note that $G[V_{i,j}, V_{j,i}]$ and $G[V_{i,0}]$ is quasirandom for every $ij \in E(H)$ and every $i \in [m]$. Applying Theorems 1.1 and 3.2 on $G[V_{i,j}, V_{j,i}]$ for every $ij \in E(H)$ and
\( G[V_i,0] \) for every \( i \in [m] \) with \( w(i) \neq 0 \) gives
\[
\log \text{NM}(G[V_i,j, V_j,i], \sqrt{\varepsilon}) \geq (1 - 4\sqrt{\varepsilon})w(ij)n \log w(ij)n,
\]
\[
\log \text{NM}(G[V_i,0], \sqrt{\varepsilon}) \geq (1 - 4\sqrt{\varepsilon})w(i)n \log w(i)n.
\]

Therefore,
\[
\text{NM}(G, \sqrt{\varepsilon}) \geq \prod_{w(ij) \neq 0} \text{NM}(G[V_i,j, V_j,i], \sqrt{\varepsilon}) \prod_{w(i) \neq 0} \text{NM}(G[V_i,0], \sqrt{\varepsilon}) \prod_{i=1}^{m} N(V_i),
\]
which means
\[
\log \text{NM}(G, \sqrt{\varepsilon}) \geq \sum_{w(ij) \neq 0} \log \text{NM}(G[V_i,j, V_j,i], \sqrt{\varepsilon}) + \sum_{w(i) \neq 0} \log \text{NM}(G[V_i,0], \sqrt{\varepsilon}) + \sum_{i=1}^{m} \log N(V_i)
\]
\[
\geq (1 - 4\sqrt{\varepsilon}) \frac{m}{2} n \log n.
\]

We now consider the upper bound. Let \( M \) be an arbitrary \( \sqrt{\varepsilon} \)-near perfect matching in \( G \). For the quotient graph \( H \), define \( w'(ij) = \frac{|M \cap E(G[V_i,V_j])|}{n} \) for every \( ij \in E(H) \), and \( w'(i) = \frac{2|\{M \cap E(G[V_i])\}|}{n} \). It is easy to see that \( \mathcal{E}(H,w') < \sqrt{\varepsilon} \).

Therefore,
\[
\text{NM}(G, \sqrt{\varepsilon}) \leq \prod_{w'(ij) \neq 0} \text{NM}(G[V_i,j, V_j,i], \sqrt{\varepsilon}) \prod_{w'(i) \neq 0} \text{NM}(G[V_i,0], \sqrt{\varepsilon}) \prod_{i=1}^{m} N(V_i) \cdot n^{\sqrt{\varepsilon} n}.
\]

Note that \( w' \) may not be a solution of (2). Then we have
\[
\log \text{NM}(G, \sqrt{\varepsilon}) \leq \sum_{w'(ij) \neq 0} \log \text{NM}(G[V_i,j, V_j,i], \sqrt{\varepsilon}) + \sum_{w'(i) \neq 0} \log \text{NM}(G[V_i,0], \sqrt{\varepsilon})
\]
\[
+ \sum_{i=1}^{m} \log N(V_i) + \sqrt{\varepsilon} n \log n
\]
\[
\leq (1 + 6\sqrt{\varepsilon}) \frac{m}{2} n \log n.
\]

**Case 3**: Linear system (2) has infinitely many solutions.

In this case, there exists a positive integer \( t \), and variables \( x_1, \ldots, x_t \in [m]^2 \), such that if we fix the value of \( w(x_1), \ldots, w(x_t) \), the system of linear equations (2) has a unique solution. Let \( \mathbf{x} = (x_1, \ldots, x_t) \), and define \( \text{NM}(G, \varepsilon, w(\mathbf{x})) \) to be the number of maximal \( \varepsilon \)-near perfect matchings \( \mathcal{M} \) in \( G \), such that for every \( i \in [t] \), suppose
$x_i$ corresponds $ij$ in $[m]^2$ with $i \neq j$ (or $i = j$), then $|\mathcal{M} \cap G[V_i, V_j]| = w(x_i)n$ (or $|\mathcal{M} \cap G[V_i]| = w(x_i)n/2$).

Roughly speaking, the number of maximal $\varepsilon$-near perfect matchings in $G$ is about

$$\int_{[0,1]^t} \text{NM}(G, \varepsilon, w(x)) \, dx.$$ 

In order to avoid double counting, we should be more careful here, since an $\varepsilon$-near perfect matching in $\text{NM}(G, \varepsilon, w(x))$ will also be counted in $\text{NM}(G, \varepsilon, w(x) + \varepsilon/m^2)$. Let $l = 1/\sqrt{\varepsilon}$, we have

$$\text{NM}(G, \sqrt{\varepsilon}) \geq \sum_{i_1, \ldots, i_t = 0}^t \text{NM}(G, \sqrt{\varepsilon}, (i_1\sqrt{\varepsilon}, \ldots, i_t\sqrt{\varepsilon})).$$

Then applying the results in Case 2 yields

$$\log \text{NM}(G, \sqrt{\varepsilon}) \geq (1 - 4\sqrt{\varepsilon}) \frac{m}{2} n \log n.$$ 

Considering the upper bound, given an arbitrary $\sqrt{\varepsilon}$-near perfect matching $\mathcal{M}$, similarly as we did before, define $w'(ij) = \frac{|M \cap E(G[V_i, V_j])|}{n}$ and $w'(i) = \frac{2|M \cap E(G[V_i])|}{n}$. This gives $\mathcal{E}(H, w') < \sqrt{\varepsilon}$, and therefore, for $n^t \geq (l + 1)^t m$, we have

$$\log \text{NM}(G, \sqrt{\varepsilon}) \leq t \log(l + 1) + \log \text{NM}(G, \sqrt{\varepsilon}, w'(x))$$

$$\leq (1 + 7\sqrt{\varepsilon}) \frac{m}{2} n \log n,$$

which completes the proof. \hfill \Box

4. Matchings in dense graphs

In this section, we analyze the properties of large dense graphs. Suppose $G$ has $n$ vertices. After applying Szemerédi Regularity Lemma, we have an equitable partition $\mathcal{P} = \{V_1, \ldots, V_K\}$. Situation here is more complicated than the one in Section 3, since there can be large matchings between irregular pairs and pairs with low edge densities. We will use the following algorithm by Micali and Vazirani [24] to get the size of the maximum matching in graph $G$ and in graph $G[V_i, V_j]$ when $(V_i, V_j)$ is irregular or $\varepsilon$-regular but has low edge density.

**Theorem 4.1** ([24]). Given a graph $G$, there is a polynomial time algorithm which outputs the size of the maximum matching in $G$, and the running time is $O(\sqrt{|V||E|})$.

Suppose $H$ is the quotient graph $G/\mathcal{P}$. Let $E_1 \subseteq E(H)$ be the set of edges corresponding to the irregular pairs in $G$, $E_2 \subseteq E(H)$ be the set of edges corresponding to the $\varepsilon$-regular pairs with edge density at most $n^{\sqrt{\varepsilon} - 1}$, $E_3 \subseteq E(H)$ be the set of edges corresponding to the $\varepsilon$-regular pairs with edge density in $[n^{\sqrt{\varepsilon} - 1}, n^{-\sqrt{\varepsilon}}]$ in $G$,
and let $E_4 \subseteq E(H)$ be the set of edges corresponding to the $\epsilon$-regular pairs with edge density at least $n^{-\sqrt{\epsilon}}$. For every $ij \in E_1$, let $m_{ij}$ be size of the maximum matching in $G[V_i, V_j]$, and let $r_{ij} = Km_{ij}/n$. For every $i \in V(H)$, let $m_i$ be the size of maximum matching in $G[V_i]$, and let $r_i = 2Km_i/n$.

Let $Q$ be the graph obtained from $G$ by removing edges inside each $V_i$ and edges between irregular pairs. Suppose $\mathcal{M}(Q)$ is the set of maximal matchings in $Q$ which can be extended to $\sqrt{\epsilon}$-near perfect matchings in $G$. We write $M(Q) = |\mathcal{M}(Q)|$. The following inequalities gives us a way to find $M(Q)$ and maximize $M(Q)$.

\begin{align}
0 \leq w(e) \leq 1, & \quad \text{for every } e \in E_2 \cup E_3 \cup E_4, \\
0 \leq w(e) \leq r_e, & \quad \text{for every } e \in E_1, \\
\sum_{j \neq i} w(ij) \geq 1 - r_i - \sqrt{\epsilon}, & \quad \text{for every } i \in V(H).
\end{align}

(4)

It is easy to see that if $G$ has $\sqrt{\epsilon}$-near perfect matchings, inequality (4) has solutions. Define $\mathcal{S}$ to be the set of feasible solutions of (4), and let

\begin{align}
(5) \quad s := \sup_{w(e) \in \mathcal{S}} \sum_{e \in E_4} \frac{w(e)n}{K} \log \frac{w(e)n}{K} + \sum_{e \in E_3} \frac{w(e)n}{K} \log p_e \frac{w(e)n}{K}.
\end{align}

By Theorems 1.1 and 3.2, we have $M(Q) \geq (1 - 4\sqrt{\epsilon})s$, which means $NM(G, \sqrt{\epsilon}) \geq (1 - 4\sqrt{\epsilon})s$, and this proves Theorem 1.3.

With all tools in hand, we are going to state the algorithm **Number of Max Near Perfect Matchings Dense**. Given a graph $G$ of order $n$ and a real number $\epsilon > 0$, we do the following:

**Algorithm 2**: **Number of Max Near Perfect Matchings Dense**

**Step 1.** Apply the algorithm in Theorem 4.1 on $G$. If $G$ does not contain any $\epsilon$-near perfect matchings, output 0. Otherwise, do the following steps.

**Step 2.** Take $\tau = 3\epsilon^2/2$, and $\alpha = 1/2$, $h = 1/\epsilon$. Apply the algorithm in Theorem 2.2 with integer $k$ taking values from $h$ to $M(h, \tau)$. Then the algorithm will output an $\epsilon^2$-Szemerédi partition into $K$ parts, with $h \leq K \leq M(h, \tau)$.

**Step 3.** Apply the algorithm in Theorem 4.1 at most $K^2$ times, to compute the size of maximum matchings to obtain $r_i$ and $r_{ij}$. Solve the inequalities (4) and compute the value of $s$ in (5). Let $\ell = (1 - 4\epsilon)s$, then output $n^{\ell n}$.

The above algorithm provides a lower bound for the number of maximal near perfect matchings, and its running time is $O(n^{5/2})$. Unfortunately, the lower bound we obtain is not tight. Let us illustrate on this by an example.
Suppose $G$ is a dense graph of order $n$ together with a Szemerédi partition $P = V_1, \ldots, V_K$, each of size $n/K$, where $K \geq 2/\varepsilon$ and suppose $K \equiv 2 \mod 4$. Induced graphs between all the pairs $(V_i, V_j)$ are $\varepsilon$-regular except $K/2 \leq \varepsilon K^2$ irregular pairs $(V_i, V_{i+1})$ for $i = 1, 3, 5, \ldots, K/2$. Graphs $G[V_i]$ are empty for $1 \leq i \leq (K+2)/2$ and graphs $G[V_i, V_{i+1}]$ are complete bipartite for $i = 1, 3, 5, \ldots, K/2$.

Now, it is easy to see that the number of perfect matchings in $G$ is $n^{n/2}$. After we remove edges between irregular pairs, we remove $K/2(nK^2) < \varepsilon n^2$ edges. Then the number of extend-able maximal matchings in the obtained graph (the output of the above algorithm) is $n^{n/4}$, we lose a factor $n^{n/4}$.

Acknowledgements

The authors would like to thank Andrei Bulatov, Bojan Mohar and Fan Wei for many helpful discussions. We are also thankful to Heng Guo for pointing out an inaccuracy in the introduction, and bringing reference [26] to our attention after the first version of this paper appeared on arXiv.

Appendix A.

Lemma A.1. Given $\varepsilon > 0$, $\delta = \sqrt{\varepsilon}/p$ and $c = 1/(1-\delta)p$. Suppose

$$k = \frac{1}{\sqrt{\varepsilon}} - c, \quad t = \frac{\log(1-\delta)p}{\log(1-(1-\delta)p)},$$

and

$$\text{NM}(G, \sqrt{\varepsilon}) \geq \prod_{i=0}^{k-1} \frac{(1-\delta)p(n-i\sqrt{\varepsilon}n))!}{((1-\delta)p(n-i\sqrt{\varepsilon}n)-\sqrt{\varepsilon}n)!} \prod_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)!.$$ 

Then we have $\log \text{NM}(G, \sqrt{\varepsilon}) \geq (1-3\sqrt{\varepsilon})n \log pn$.

Proof. We have

$$\log \text{NM}(G, \sqrt{\varepsilon}) \geq \sum_{i=0}^{k-1} (1-\delta)p(n-i\sqrt{\varepsilon}n) \log ((1-\delta)p(n-i\sqrt{\varepsilon}n))$$

$$- \sum_{i=0}^{k-1} ((1-\delta)p(n-i\sqrt{\varepsilon}n)-\sqrt{\varepsilon}n) \log ((1-\delta)p(n-i\sqrt{\varepsilon}n)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$

$$- \sum_{i=0}^{t-1} ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n) \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i)-\sqrt{\varepsilon}n)$$

$$+ \sum_{i=0}^{t-1} (1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i \log ((1-\delta)p\sqrt{\varepsilon}n(1-(1-\delta)p)^i))$$
\begin{align*}
&= (1 - \delta)p\frac{nk(1 + c\sqrt{\varepsilon})}{2} \log(1 - \delta)pn + \sum_{i=0}^{k-1} (1 - \delta)pn(1 - i\sqrt{\varepsilon}) \log(1 - i\sqrt{\varepsilon}) \\
&\quad - \left((1 - \delta)p\frac{nk(1 + c\sqrt{\varepsilon})}{2} - k\sqrt{\varepsilon}n\right) \log(1 - \delta)pn \\
&\quad - \sum_{i=0}^{k-1} \left((1 - \delta)pn(1 - i\sqrt{\varepsilon}) - \sqrt{\varepsilon}n\right) \log \left((1 - i\sqrt{\varepsilon}) - \frac{\sqrt{\varepsilon}}{(1 - \delta)p}\right) \\
&\quad + \frac{1 - (1 - (1 - \delta)p)^{t+1}}{(1 - \delta)p} (1 - \delta)pc\sqrt{\varepsilon}n \log(1 - \delta)pn \\
&\quad + \sum_{i=0}^{t-1} (1 - \delta)pc\sqrt{\varepsilon}n(1 - (1 - \delta)p)^i \log \left(c\sqrt{\varepsilon}(1 - (1 - \delta)p)^i\right) \\
&\geq k\sqrt{\varepsilon}n \log(1 - \delta)pn + (1 - (1 - \delta)p)c\sqrt{\varepsilon}n \log(1 - \delta)pn \\
&\quad + \sum_{i=0}^{k-1} \sqrt{\varepsilon}n \log \left((1 - i\sqrt{\varepsilon}) - c\sqrt{\varepsilon}\right) + \frac{1}{2} \sum_{i=0}^{t-1} \sqrt{\varepsilon}n(1 - (1 - \delta)p)^i \log \varepsilon \\
&\geq (1 - 2\sqrt{\varepsilon})n \log(1 - \delta)pn + k\sqrt{\varepsilon}n \log \sqrt{\varepsilon} + (c - 1)\sqrt{\varepsilon}n \log \sqrt{\varepsilon} \\
&= (1 - 2\sqrt{\varepsilon})n \log(1 - \delta)\sqrt{\varepsilon}pn > (1 - 3\sqrt{\varepsilon})n \log pn.
\end{align*}

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