Generalized gauge transformation with $PT$-symmetric non-unitary operator and classical correspondence of non-Hermitian Hamiltonian for a periodically driven system

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We in this paper demonstrate that the $PT$-symmetric non-Hermitian Hamiltonian for a periodically driven system can be generated from a kernel Hamiltonian by a generalized gauge transformation. The kernel Hamiltonian is Hermitian and static, while the time-dependent transformation operator has to be $PT$ symmetric and non-unitary in general. Biorthogonal sets of eigenstates appear necessarily as a consequence of non-Hermitian Hamiltonian. We obtain analytically the wave functions and associated non-adiabatic Berry phase $\gamma_n$ for the $n$th eigenstate. The classical version of the non-$PT$-symmetric Hamiltonian becomes a complex function of canonical variables and time. The corresponding kernel Hamiltonian is derived with $PT$ symmetric canonical-variable transfer in the classical gauge transformation. Moreover, with the change of position-momentum to angle-action variables it is revealed that the non-adiabatic Hannay’s angle $\Delta \theta_H$ and Berry phase satisfy precisely the quantum-classical correspondence, $\gamma_n = (n + 1/2)\Delta \theta_H$.

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I. INTRODUCTION

In standard quantum theory, observable quantities are associated with Hermitian operators, and time evolution is generated by a Hermitian Hamiltonian\(^1\). The Hermiticity (or more precisely self-adjointness\(^2\)) of the Hamiltonian ensures both the reality of the energy spectrum and more importantly the unitarity of time evolution. Surprisingly, Bender and Boettcher showed in 1998 that the non-Hermitian Hamiltonians can still possess real and positive eigenvalues\(^3\). They claimed that the reality of the spectrum is due to parity-time ($PT$) symmetry of Hamiltonian. The appearance of the $PT$-symmetry was initially considered as a mathematically interesting finding\(^4\)–\(^6\), with which many new Hamiltonians can be constructed instead of the Hermiticity\(^6\). The predicted properties of $PT$-symmetric Hamiltonians\(^5\)–\(^7\) have been observed at the classical level in a wide variety of laboratory experiments involving superconductivity\(^8\)–\(^9\), optics\(^10\)–\(^13\), microwave cavities\(^14\), atomic diffusion\(^15\), nuclear magnetic resonance\(^16\). F. Klauck et al. demonstrated two-particle quantum interference in a $PT$-symmetric system\(^17\).

The connection between $PT$ symmetry and positive spectra is simply illustrated by adding a $PT$-symmetric non-Hermitian term to the Hamiltonian of a harmonic oscillator\(^3\). If the harmonic oscillator is extended into the complex domain more intricate behavior is discovered\(^18\). The $PT$-symmetric oscillator has been investigated extensively in recent years. In 2005, Cem Yuce studied the exactly solvable $PT$ symmetric harmonic oscillator problem\(^19\). In 2013, Carl M. Bender et al. observed $PT$ phase transition in a two-oscillator model\(^20\), and studied twofold transitions as well\(^21\). Two years later, Alireza Beygi et al. examined coupled oscillator chain with partial $PT$ symmetry\(^22\). Subsequently Andreas Fring and Thomas Frith presented exact analytical solutions for a two-dimensional time-dependent non-Hermitian system by coupling two harmonic oscillators with infinite dimensional Hilbert space in the broken $PT$-regime\(^23\). And then they provided a time-dependent Dyson map and metric for the two-dimensional, non-Hermitian oscillator in 2020\(^24\). The dynamics of the average displacement of the mechanical oscillator were investigated in different regimes for the $PT$-symmetric, optomechanical system in 2021\(^25\).

In the present paper we study the gauge equivalence of a $PT$ symmetric Hamiltonian with a Hermitian Hamiltonian for a periodically driven system. Since the Schrödinger equation is invariant under the $PT$ transformation for $PT$ symmetric non-Hermitian Hamiltonian, we can generate various Hamiltonians by means of the $PT$ symmetric but non-unitary transformation operators, different from the ordinary quantum mechanics, in which the transformation...
operators have to be unitary. Moreover classical correspondence of $PT$-symmetric non-Hermitian is formulated explicitly in the viewpoint of gauge equivalence. The quantum–classical correspondence was demonstrated for the first time by Schrödinger that the center of wave packet in coherent state evolves in time following exactly the corresponding motion of classical particle\[26\]. In fact the quantum-classical correspondence has resulted in many impressive aspects showing the unite nature of quantum and classical worlds. For two dimensional harmonic-oscillator, the quantum wave packet of coherent states corresponds exactly to the transformation geometry of classical dynamics\[27\]. Y. F. Chen et al. manifested the connection between topological structures of quantum stationary coherent-states and bundles of classical Lissajous orbits in 2018\[28\]. V. Gattus and S. Karamitsos show that the uncertainties of certain coupled classical-systems and their quantum counterparts converge in the classical limit\[29\]. Quantum-classical correspondence is also demonstrated by comparing the statistical moments of eigenfunctions in closed systems with chaotic dynamics\[30\].

When a classical or quantum system undergoes a cyclic evolution governed by slow change in its parameter space, it acquires a topological phase factor known as the geometric or Berry phase\[31\], which reveals the gauge structure in quantum mechanics\[34\]. Hannay’s angle is the classical counterpart of this additional quantum phase\[32, 33\]. The quantum geometric-phase and the classical Hannay’s angle come, as a matter of fact, from the same nature in the semiclassical level according to Berry\[33\]. Recently, L. Latmiral and F. Armata addressed the quantum-classical comparison of phase measurements in optomechanics for composite systems\[34\]. A. Rückriegel and R. A. Duine study the geometric phases that occur in semi-classical magnetic dynamics\[35\] within the framework developed by Hannay for classical integrable systems.

We propose a gauge transformation method to solve the time-dependent non-Hermitian Hamiltonian with $PT$ symmetric but non-unitary operator. Quantum wave functions are obtained analytically along with the non-adiabatic Berry phase for the periodically driven system. Then classical counterpart of the non-Hermitian Hamiltonian is solved in terms of the gauge transformation in classical mechanics. The explicit relation of Hannay’s angle and Berry phase is presented for the $PT$-symmetric non-Hermitian Hamiltonian.

The paper is organized as follows. We explore in Sec. II the $PT$-symmetric non-Hermitian Hamiltonian consisting of periodically driven $SU(1, 1)$ generators. The kernel Hamiltonian, which is Hermitian and static is derived by means of generalized gauge transformation with $PT$-symmetric non-unitary operator. The biorthogonal sets of eigenfunctions are obtained along with associated Berry phases. In Sec. III, we present the corresponding variation of classical Hamiltonian in the gauge transformation of classical mechanics. And the Hannay’s angle is found to have the precise correspondence with the Berry phase\[36, 37\]. The conclusion is given in the last section.

II. $PT$-SYMMETRIC NON-HERMITIAN HAMILTONIAN AND NON-UNITARY TRANSFORMATION OPERATOR

The $PT$-symmetric non-Hermitian Hamiltonian that we consider is written as

$$\hat{H}(t) = \Omega \hat{S}_z + G \left( \hat{S}_+ e^{i\phi(t)} - \hat{S}_- e^{-i\phi(t)} \right)$$

in which $\phi(t) = \omega t$ with $\omega$ being the driving frequency, $\Omega$ and $G$ are the frequency parameters. The $SU(1, 1)$ generators $\hat{S}_z$ and $\hat{S}_+ = (\hat{S}_-)^\dagger$ satisfy the communication relation\[38\]

$$[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad [\hat{S}_+, \hat{S}_-] = -2\hat{S}_z,$$

where natural unit $\hbar = 1$ is used throughout. The Hamiltonian Eq.(1) is obviously non-Hermitian \[6\]

$$\hat{H}(t) \neq \hat{H}^\dagger(t) = \Omega \hat{S}_z - G \left( \hat{S}_+ e^{i\phi(t)} - \hat{S}_- e^{-i\phi(t)} \right).$$

The $PT$ transformation operator in quantum mechanics can be defined as

$$\hat{\Theta} = \hat{U} K, \quad \hat{\Theta}^{-1} = K \hat{U}^\dagger$$

where $\hat{U}$ is the usual unitary operator for the space-time reflection and $K$ denotes the operation of complex conjugate. The position and momentum operators become $\hat{\Theta} \hat{x} \hat{\Theta}^{-1} = -\hat{x}$, $\hat{\Theta} \hat{p} \hat{\Theta}^{-1} = \hat{p}$ under $PT$ transformation. And the boson creation operator $\hat{a}^\dagger = \sqrt{\frac{\hbar}{2}} \left( \hat{x} - i\hat{p} \right)$ and annihilation operator $\hat{a} = \sqrt{\frac{\hbar}{2}} \left( \hat{x} + i\hat{p} \right)$ in harmonic oscillator become

$$\hat{\Theta} \hat{a}^\dagger \hat{\Theta}^{-1} = -\hat{a}^\dagger, \hat{\Theta} \hat{a} \hat{\Theta}^{-1} = -\hat{a}.$$
The $SU(1, 1)$ Lie algebra has a realization in terms of boson creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ such that

\[
\hat{S}_z = \frac{1}{2} (\hat{a}^\dagger \hat{a} + \frac{1}{2}), \hat{S}_+ = \frac{1}{2} (\hat{a}^\dagger)^2, \hat{S}_- = \frac{1}{2} (\hat{a})^2.
\]

(4)

Thus, under the $PT$ transformation the $SU(1, 1)$ generators transform as

\[
\hat{\Theta} \hat{S}_z \hat{\Theta}^{-1} = \hat{S}_z, \quad \hat{\Theta} \hat{S}_+ \hat{\Theta}^{-1} = \hat{S}_+, \quad \hat{\Theta} \hat{S}_- \hat{\Theta}^{-1} = \hat{S}_-.
\]

(5)

The commutation relation Eq.(2) is $PT$-transformation invariant as well as the Hamiltonian

\[
\hat{\Theta} \hat{H} (t) \hat{\Theta}^{-1} = \hat{H} (t).
\]

The time-dependent Schrödinger equation is covariant under $PT$ transformation

\[
i \frac{\partial}{\partial t} |\psi' (t)\rangle = \hat{H} (t) |\psi' (t)\rangle,
\]

where $|\psi' (t)\rangle = \hat{\Theta} |\psi (t)\rangle$.

### A. Generalized gauge transformation with non-unitary operator

Since the Schrödinger equation is invariant under $PT$ transformation, it is always possible to construct a proper transformation such that the Hamiltonian in the new gauge can be easily diagonalized. We propose a gauge transformation method to solve the time-dependent non-Hermitian Hamiltonian system with a transformation operator considered as

\[
\hat{R} (t) = e^{-i\frac{2}{\eta} (\hat{S}_+ e^{i\phi(t)} + \hat{S}_- e^{-i\phi(t)})}, \quad \hat{R}^{-1} (t) = e^{i\frac{2}{\eta} (\hat{S}_+ e^{i\phi(t)} + \hat{S}_- e^{-i\phi(t)})},
\]

(6)

in which the real parameter $\eta$ is to be determined. The operator Eq.(6) is $PT$-symmetric

\[
\hat{\Theta} \hat{R} (t) \hat{\Theta}^{-1} = \hat{R} (t),
\]

in order to maintain the $PT$ symmetry of the transformed Hamiltonian. However the operator $\hat{R} (t)$ is non-unitary with $\hat{R}^\dagger \neq \hat{R}^{-1}$ different from the ordinary quantum mechanics, in which the transformation operator has to be unitary. Occasionally it is Hermitian ($\hat{R}^\dagger (t) = \hat{R} (t)$) in this particular model but is unnecessary in general. The time-dependent Schrödinger equation becomes

\[
i \frac{\partial}{\partial t} |\psi' (t)\rangle = \hat{H}' (t) |\psi' (t)\rangle.
\]

(7)

where $\hat{R} (t) |\psi (t)\rangle = |\psi' (t)\rangle$. The Hamiltonian in the new gauge is

\[
\hat{H}' (t) = \hat{R} (t) \hat{H} (t) \hat{R}^{-1} (t) - i \frac{\partial}{\partial t} \hat{R}^{-1} (t).
\]

(8)

With the help of $SU(1, 1)$ commutation relation Eq.(2) and the Baker-Hausdorff-Campbell formula we have the following transformations

\[
\hat{R} (t) \hat{S}_+ \hat{R}^{-1} (t) = \hat{S}_+ \cos^2 \left(\frac{\eta}{2}\right) - \hat{S}_z e^{-i\phi(t)} \sin (\eta) + \hat{S}_- e^{-2i\phi(t)} \sin^2 \left(\frac{\eta}{2}\right),
\]

\[
\hat{R} (t) \hat{S}_- \hat{R}^{-1} (t) = \hat{S}_- \cos^2 \left(\frac{\eta}{2}\right) + \hat{S}_z e^{i\phi(t)} \sin (\eta) + \hat{S}_+ e^{2i\phi(t)} \sin^2 \left(\frac{\eta}{2}\right),
\]

\[
\hat{R} (t) \hat{S}_z \hat{R}^{-1} (t) = \hat{S}_z \cos (\eta) + \frac{1}{2} \sin (\eta) \left(\hat{S}_+ e^{i\phi(t)} - \hat{S}_- e^{-i\phi(t)}\right).
\]
and

\[ i\dot{R}(t) \frac{\partial}{\partial t} R^{-1}(t) = 2 \frac{d\phi}{dt} \hat{S}_z \sin\left(\frac{\eta}{2}\right) - \frac{d\phi}{2dt} \sin(\eta) \left(\hat{S}_+ e^{i\phi(t)} - \hat{S}_- e^{-i\phi(t)}\right) \]  

(9)

Substituting transformation relations Eq. (9) into the Hamiltonian Eq. (8) in the new gauge yields

\[
\hat{H}' = \left[ \Omega \cos(\eta) - 2G \sin(\eta) - 2\omega \sin^2\left(\frac{\eta}{2}\right) \right] \hat{S}_z \\
+ \left[ G \cos(\eta) + \frac{\Omega + \omega}{2} \sin(\eta) \right] \left( \hat{S}_+ e^{i\phi(t)} - \hat{S}_- e^{-i\phi(t)} \right),
\]

which is \textit{PT} symmetric. The parameter \( \eta \) can be determined from the following auxiliary equation

\[
G \cos(\eta) + \frac{\omega + \Omega}{2} \sin(\eta) = 0,
\]

which leads to

\[
\sin(\eta) = \pm \frac{2G}{\Delta},
\]

with

\[
\Delta = \sqrt{(\omega + \Omega)^2 + 4G^2}.
\]

The Hamiltonian in new gauge Eq. (8) then becomes time independent and Hermitian

\[
\hat{H}' = 2\Gamma \hat{S}_z,
\]

(10)

with an effective frequency \( \Gamma \)

\[
\Gamma = \pm \frac{\left(\omega + \Omega\right)^2 - 4G^2}{2\Delta} - \frac{\omega}{2}.
\]

Let \(|n\rangle\) be the eigenstate of \(2\hat{S}_z = (\hat{a}^\dagger \hat{a} + 1/2)\) such that \(\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle\), the energy eigenvalues of \(\hat{H}'\) are that of a simple Harmonic oscillator

\[
E_n = \left(n + \frac{1}{2}\right) \Gamma.
\]

The \textit{PT}-symmetric non-Hermitian Hamiltonian has pure real-eigenvalues known as the unbroken \textit{PT} symmetry\textsuperscript{[20]}. We have shown that the \textit{PT}-symmetric non-Hermitian Hamiltonian can be transformed into a Hermitian one in terms of the generalized gauge transformation.

Recently, it was demonstrated that a \textit{PT}-symmetric non-Hermitian Hamiltonian can be transformed into the corresponding Hermitian one with the vielbein formalism\textsuperscript{[40]}.

\textbf{B. Biorthogonal sets of eigenstates and non-adiabatic Berry phase}

The special solution of the Schrödinger equation in the original gauge for the eigenstate \(|n\rangle\) is

\[
|\psi_n(t)\rangle_r = e^{-iE_n t} \hat{R}^{-1}(t) |n\rangle,
\]

(11)

in which the subscript "\(r\)" denotes "ket" state since these states resulted from non-unitary transformation are not orthonormal. The corresponding "bra" state is defined by

\[
|\psi_n(t)\rangle_l = e^{-iE_n t} \hat{R}(t) |n\rangle.
\]
The orthonormal condition is
\[ \langle \psi_n(t) | \psi_m(t) \rangle_r = \delta_{nm}. \tag{12} \]
According to Ref.\[41\], we can define a metric operator $\hat{\chi}$ relating the "ket" and "bra" states by
\[ |\psi_n(t)\rangle_l = \hat{\chi} |\psi_n(t)\rangle_r, \tag{13} \]
in which the metric operator of Hilbert space for the present model is
\[ \hat{\chi} = \hat{R}^2(t). \tag{14} \]
With the metric operator orthogonality condition Eq. (12) becomes
\[ \langle\langle \psi_n(t) | \psi_m(t) \rangle \rangle \equiv \langle \psi_n(t) | \hat{\chi} | \psi_m(t) \rangle = \delta_{nm} \tag{15} \]
without using the two sets of basis. The equation of motion for the metric operator $\hat{\chi}$ is given explicitly in Ref.\[41\] for a full description of the non-Hermitian system dynamics.

With the transformation operator $\hat{R}$ the explicit form of Berry phase is found as
\[ \gamma_n(T) = \pi \left( n + \frac{1}{2} \right) \left( 1 + \frac{\omega + \Omega}{\Delta} \right). \tag{16} \]

As a matter of fact the Hamiltonian Eq. (10) becomes in the representation of coordinate and momentum operators
\[ \hat{H}' = \frac{\Gamma}{2} \left( \hat{X}^2 + \hat{P}^2 \right), \tag{17} \]
in which $\hat{X} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger)$ and $\hat{P} = \frac{1}{\sqrt{2\Delta}} (\hat{a} - \hat{a}^\dagger)$ as usual. The eigenstate wave-functions are given by
\[ \psi_n(X, \Gamma) = \sqrt{n!} \frac{\Gamma^\frac{1}{2}}{(2\pi n)^\frac{1}{2}} e^{-\Gamma X^2/2} H_n(\Gamma^\frac{1}{2} X), \]
where $H_n(\Gamma^\frac{1}{2} X)$ is the Hermit polynomial expressed as
\[ H_n\left(\Gamma^\frac{1}{2} X\right) = (-1)^n e^{\Gamma X^2} \frac{d^n}{d \left( \Gamma^\frac{1}{2} X \right)^n} e^{-\Gamma X^2}. \]
The special solution of the time-dependent Schrödinger equation (7) for $n$th eigenvalue is obviously
\[ \psi_n'(X, \Gamma, t) = \psi_n'(X, \Gamma) e^{-i E_n t}. \]

The wave function of original Schrödinger equation for the time-dependent Hamiltonian $\hat{H}(t)$ Eq. (1) can be obtained from $\psi_n'(X, \Gamma, t)$ with the inverse transformation operator such that
\[ \psi_n(X, \Gamma, t) = \hat{R}^{-1}(X, P, t) \psi_n'(X, \Gamma, t), \]
where the transformation operator in the coordinate representation is given by
\[ \hat{R}^{-1}(t) = e^{\frac{i}{2} \int_t^0 \left[ (X^2 + \frac{\partial_2^2}{\partial X^2}) \cos \phi(t) - i(2X \frac{\partial_2}{\partial X} + 1) \sin \phi(t) \right] dt}. \]
III. CLASSICAL COUNTERPART OF NON-HERMITIAN HAMILTONIAN AND CANONICAL TRANSFORMATION

The Hannay’s angle is a classical analogue of the quantum Berry phase\[^{[33, 42]}\]. Therefore, it is of interest to see the classical correspondence of the non-Hermitian Hamiltonian and the related Berry phase. In terms of the boson operator representation of the SU(1, 1) generators Eq. (4), the classical version of the \(\text{PT}\)-symmetric non-Hermitian Hamiltonian Eq. (1) becomes

\[
H(x, p, t) = \left[\frac{\Omega}{4} + i\frac{G}{2} \sin \phi(t)\right] x^2 + \left[\frac{\Omega}{4} - i\frac{G}{2} \sin \phi(t)\right] p^2 - iG \cos \phi(t) xp. \tag{18}
\]

Surprisingly, but naturally, the classical Hamiltonian is a complex function of canonical variable \(x, p\) and time \(t\). It is obviously \(\text{PT}\) symmetric, noticing that the \(\text{PT}\) transformation operator Eq. (3) is also valid in the classical mechanics including the complex conjugation operation.

A. Time-dependent canonical transformation

The classical kernel Hamiltonian can be found with the help of a time-dependent canonical transformation\[^{[37]}\] corresponding to the quantum transformation Eq. (6)

\[
x = \left[\cos \frac{\eta}{2} - i\alpha \sin \frac{\eta}{2}\right] X + i\beta P \sin \frac{\eta}{2},
\]

\[
p = i\beta X \sin \frac{\eta}{2} + \left[\cos \frac{\eta}{2} + i\alpha \sin \frac{\eta}{2}\right] P, \tag{19}
\]

in which \(\alpha = \sin \phi(t)\), \(\beta = \cos \phi(t)\) and the parameter \(\eta\) is to be determined. The above transformation Eq. (19) is obviously \(\text{PT}\) symmetric.

It is well known that two Lagrangians, which differ by a total time-derivative of space-time function, give rise to the same equation of motion. This is called the generalized gauge transformation in classical mechanics\[^{[39]}\]. We require that the phase-space Lagrangians

\[
L'(X, P) = P\dot{X} - H'(X, P, t)
\]

in new variables and

\[
L(x, p) = p\dot{x} - H(x, p, t)
\]

in the original variables differ by a total time-derivative of the generating function

\[
L'(X, P) = L(x, p) + \frac{dF(X, P, t)}{dt}.
\]

So that \(L'(X, P)\) and \(L(x, p)\) are gauge equivalent. The generating function is constructed as

\[
F(X, P, t) = \left(-i\frac{\beta}{2} \sin \eta - \alpha\beta \sin^2 \frac{\eta}{2}\right) \frac{X^2}{2} + \beta^2 XP \sin^2 \frac{\eta}{2} + \left(-i\frac{\beta}{2} \sin \eta + \alpha\beta \sin^2 \frac{\eta}{2}\right) \frac{P^2}{2}.
\]

We choose the parameter value \(\eta\) such that

\[
\sin \eta = -\frac{2G}{\Delta}
\]

the Hamiltonian in the new gauge becomes exactly the classical counterpart of the quantum version Eq. (17)

\[
H'(X, P) = \frac{\Gamma}{2} \left(X^2 + P^2\right). \tag{20}
\]
B. Action-angle variable and Hannay’s angle

If an integrable classical-Hamiltonian \( H \) describing a bound motion depends on slowly varying parameters the action variable \( I \) of the motion is conserved according to Hannay\[32\]. The frequency of quasi-periodic motion \( \Gamma = \frac{\partial H'(I, \Theta)}{\partial I} \) does not vanish, and then the action variable \( I \) is an adiabatic invariant. According to the canonical equations

\[
\dot{\Theta} = \frac{\partial H'(I, \Theta)}{\partial I} = \Gamma, \\
\dot{I} = -\frac{\partial H'(I, \Theta)}{\partial \Theta} = 0,
\]

the Hamiltonian \( H'(X, P) \) becomes in the action-angle variables

\[ H'(I, \Theta) = I \Gamma \].

The Hannay’s angle in classical mechanics is introduced with the canonical transformation from the position-momentum variables \((X, P)\) to the action-angle variables \((I, \Theta)\) such that

\[ X(I, \Theta) = \sqrt{2I} \sin \Theta, \quad P(I, \Theta) = \sqrt{2I} \cos \Theta. \] (21)

Substituting the \( X(I, \Theta) \) and \( P(I, \Theta) \) defined in Eq.\ref{21} into Eq.\ref{19}, we have the canonical transformation from the original variables \((x, p)\) to the action-angle variables \((I, \Theta)\),

\[
x(I, \Theta) = \sqrt{2I} \left[ \left( \cos \frac{\eta}{2} - i \alpha \sin \frac{\eta}{2} \right) \sin \Theta + i \beta \sin \frac{\eta}{2} \cos \Theta \right], \\
p(I, \Theta) = \sqrt{2I} \left[ i \beta \sin \frac{\eta}{2} \sin \Theta + \left( \cos \frac{\eta}{2} + i \alpha \sin \frac{\eta}{2} \right) \cos \Theta \right].
\]

The Hannay’s angle of the time-dependent system with the Hamiltonian \( H(x, p, t) \) in the original gauge is evaluated in one-period \( T = 2\pi/\omega \) as\[33\]

\[
\Delta \theta_H = -\frac{\partial}{\partial I_c} \langle p(I, \Theta) d\phi x(I, \Theta) \rangle_\Theta = -\frac{\partial}{\partial I} \frac{1}{2\pi} \int_0^{2\pi} d\Theta \int_0^{2\pi} p(I, \Theta) d\phi x(I, \Theta) \\
= \pi \left( \frac{\pm \omega + \Omega}{\Delta} - 1 \right) \] (22)

where \( \langle \ldots \rangle_\Theta \) denotes the average over the angle variable \( \Theta \).

We reveal a precise relation\[36, 37\]

\[ \gamma_n(T) = -(n + 1/2) \Delta \theta_H \]

between the non-adiabatic Berry phase Eq.\ref{16} and Hannay angle Eq.\ref{22} for the \( n \)th eigenstates.

IV. CONCLUSION

The time-dependent Schrödinger equation is invariant under the \( PT \) transformation for the \( PT \)-symmetric non-Hermitian Hamiltonian. Based on the invariance we propose a generalized gauge transformation with \( PT \)-symmetric, however non-unitary operator different from the ordinary quantum mechanics. It is demonstrated that the non-Hermitian Hamiltonian can be generated from a kernel Hamiltonian, which is Hermitian and static, by the gauge transformation. Thus the time-dependent non-Hermitian Hamiltonian is solved analytically. We obtain explicit quantum wave-functions \( \psi_n(X, \Gamma, t) \) and associated nonadiabatic Berry phase \( \gamma_n \). The classical counterpart of the Non-Hermitian Hamiltonian can be transformed by the classical gauge transformation to an exactly same form as the quantum Kernel-Hamiltonian. The quantum-classical correspondence is then established explicitly for the \( PT \)-symmetric non-Hermitian Hamiltonian system. Moreover the classical Hannay’s angle \( \Delta \theta_H \) is derived by the canonical-variable transfer to action-angle variables. We reveal a precise relation of quantum Berry phase \( \gamma_n \) and classical Hannay angle \( \Delta \theta_H \).
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Conflict of Interest

The authors declare no conflict of interest.

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