Natural families in evolution algebras

Yolanda Cabrera Casado
University of Málaga
In this book a new and outstanding type of genetic algebra was introduced to model the non-Mendelian genetics: the so-called evolution algebras.
Mathematical formulation of the Mendel's second law

1 Gene
2 Chromosomes
2 Alleles

Parent 1
Gene for eye color
Gene for production of nails

Parent 2
Gene for temperature regulation
Gene for fur texture
Gene for production of fur

Offspring’s chromosome pair
Two alleles → N, g.

Three genotypes → NN, Ng, gg.

Algebra → B={NN, Ng, gg}

**Example:** Zygotic algebra (Reed 1997)

|   | NN          | Ng                                      | gg           |
|---|-------------|-----------------------------------------|--------------|
| NN| $\frac{1}{4}$ | $\frac{1}{2}(NN + Ng)$                  | $\frac{1}{2}(gg + Ng)$ |
| Ng| $\frac{1}{2}(NN + Ng)$ | $\frac{1}{4}(NN + gg) + \frac{1}{2}Ng$ | gg           |
| gg| Ng          |                                         |              |

This is not an evolution algebra.
In the asexual inheritance,

- $a_ia_j$ does not make sense biologically ($a_ia_j = 0$) $i \neq j$.
- $a_ia_i = a_i^2 = \sum_{k=1}^{n} \gamma_{ki}a_k$. Interpreted as self-replication.

It is called evolution algebra.
Simultaneously orthogonalizable

**Corollary 1:** Fix a basis $B$ of a vector space $V$ of finite dimension over a field $\mathbb{K}$ with $\text{char}(\mathbb{K}) \neq 2$ and assume that $\mathcal{F} = \{\langle \cdot, \cdot \rangle_i \}_{i \in I \cup \{0\}}$ is a family of inner products on $V$ whose matrices in $B$ are $M_{i,B}$. Further assume that $M_{0,B}$ is nonsingular. Then $\mathcal{F}$ is simultaneously orthogonalizable if and only if the collection of matrices $\{M_{i,B}M_{0,B}^{-1}\}_{i \in I}$ is commutative and each one of them is diagonalizable.

---

Yolanda Cabrera Casado, Cristóbal Gil Canto, Dolores Martín Barquero and Cándido Martín González, Simultaneous orthogonalization of inner products over arbitrary fields. https://arxiv.org/pdf/2012.06533.pdf.
Structure Inner Products

**Definition:** If $A$ is a commutative algebra over a field $\mathbb{K}$ the product in $A$ can be written in the form

$$xy = \sum_{i \in I} \langle x, y \rangle_i e_i$$

where $\{e_i\}_{i \in I}$ is any fixed basis of $A$ and the inner products $\langle \cdot, \cdot \rangle_i : A \times A \to \mathbb{K}$ provide the coordinates of $xy$ relative to the basis. So $A$ is an evolution algebra if and only if the set of inner products $\langle \cdot, \cdot \rangle_i$ is simultaneously orthogonalizable.
Some General Properties

Evolution algebras are **commutative** and hence **flexible**.

The **direct sum** of evolution algebras is an evolution algebra.

Evolution algebras are **not** power associative, Jordan, alternative or associative algebras.

The quotient algebra $A/I$ with $I$ ideal of $A$ is an evolution algebra.

Moussa Ouattara and Souleymane Savadogo, Power-associative evolution algebras
https://arxiv.org/abs/1812.09986.
Theorem: Let $A$ be an evolution algebra and let $B = \{ e_i \mid i \in \Lambda \}$ be a natural basis of $A$ with structure matrix $M_B = (\omega_{ij})$.

- If $B' = \{ f_i \mid i \in \Lambda \}$ is a natural basis of $A$ with $P_{B'B} = (p_{ij})$ the change of basis matrices, then
  \[ M_B (P \ast P) = 0 \]
  for every $i \neq j$ with $i, j \in \Lambda$. Moreover
  \[ M_{B'} = P_{B'B}^{-1} M_B P_{B'B}^{(2)} \]
  where $P_{B'B}^{(2)} = (p_{ij}^2)$.

- Assume that $P = (p_{ij}) \in \text{CFM}_\Lambda(K)$ is invertible and satisfies the first above relation. Define $B' = \{ f_i \mid i \in \Lambda \}$, where
  \[ f_i = \sum_{j \in \Lambda} p_{ji} e_j \]
  for every $i \in \Lambda$. Then $B'$ is a natural basis and the second above relation is satisfied.
**Definition:** Let $A$ be an evolution algebra. We say that $A$ has a unique natural basis if the only change of basis matrices are $S_n \times (\mathbb{K}^\times)^n$. 
**Definition:** Let $A$ be an evolution algebra of dimension $n$. We say that $A$ has *Property (2LI)* if for any different vectors $e_i, e_j$ of a natural basis, the set $\{e_i^2, e_j^2\}$ is linearly independent.

**Corollary:** Let $A$ be a non-degenerate evolution algebra over $\mathbb{K}$. Then the following assertions are equivalent:

1. $A$ has a unique natural basis.
2. There exists a natural basis $B$ such that for any 2 different vectors $u$ and $v$ of $B$, $u^2$ and $v^2$ are linearly independent.
Fact: There are ideals (and hence subalgebras) of an evolution algebra which are not evolution algebras.

Example: Let $A$ be the evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ and product $e_1^2 = e_2 + e_3; \ e_2^2 = e_1 + e_2$ and $e_3^2 = -(e_1 + e_2)$. Let

$$I = \{ae_1 + (\alpha + \beta)e_2 + \beta e_3 : \alpha, \beta \in \mathbb{K}\}$$

Then $I$ is an ideal without a natural basis. Therefore:

$I$ is an ideal (and hence a subalgebra) but $I$ is not an evolution algebra.

Definition: An evolution subalgebra (respectively ideal) of an algebra $A$ is a subalgebra (respectively ideal) provided with a natural basis.
Fact: Evolution subalgebra does not need to be an ideal.

Example: Let $A$ be an evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ such that $e_1^2 = e_2$, $e_2^2 = e_1$ and $e_3^2 = e_3$. Then, the subalgebra $A'$ generated by $e_1 + e_2$ and $e_3$ is an evolution subalgebra but it is not an ideal as $e_1(e_1 + e_2) \notin A'$. 
Fact: Not every basis of an evolution subalgebra can be extended to a natural basis of the whole algebra.

Example: Let $A$ be an evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ and multiplication given by $e_1^2 = e_3$, $e_2^2 = e_1 + e_2$ and $e_3^2 = e_3$. Let $I$ be the evolution ideal generated by $e_1 + e_2$ and $e_3$.

These definitions are less restrictive.
Example: $B = \{e_1, e_2, e_3\}$ natural basis of $A$ with product:

$$e_1^2 = -5e_1, \quad e_2^2 = 2e_1 - 3e_3, \quad e_3^2 = -2e_2.$$
The graph associated to an evolution algebra depends on the chosen basis.

- Isomorphic evolution algebras $\nRightarrow$ isomorphic graphs.

**Example:** Let $A$ be the evolution algebra with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_1 + e_2$ and $e_2^2 = 0$. Consider the natural basis $B' = \{e_1 + e_2, e_2\}$. Then the graphs associated to the bases $B$ and $B'$ are, respectively

E: \[ \begin{array}{c}
\circ \rightarrow \\
V_1 \rightarrow V_2
\end{array} \]

F: \[ \begin{array}{c}
\circ \\
V_1 \\
\circ
\end{array} \]
Let $A$ evolution algebra with structure matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & -6 \\
\end{pmatrix}
$$

$$D^1(1) = \{ k \in \Lambda \mid e_1^2 = \sum_k \omega_k e_k \text{ with } \omega_k \neq 0 \} = \{2, 4\}.$$
Let $A$ be an evolution algebra with structure matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -6
\end{pmatrix}
$$

$D^1(1) = \{k \in \Lambda \mid e_1^2 = \sum_k \omega_{k1} e_k \text{ with } \omega_{k1} \neq 0\} = \{2, 4\}$.

$D^2(1) = \bigcup_{k \in D^1(1)} D^1(k) = \{2, 5\}$. 
Let $A$ evolution algebra with structure matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 \\
10 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$D^1(1) = \{ k \in \Lambda \mid e_1^2 = \sum_k \omega_{k1} e_k \text{ with } \omega_{k1} \neq 0 \} = \{2, 4\}$.

$D^2(1) = \bigcup_{k \in D^1(1)} D^1(k) = \{2, 5\}$.

$D^3(1) = \{2, 4, 6\}$.

$D^4(1) = \{2, 5\}$.

$D(1) = \bigcup_{m \in \mathbb{N}} D^m(1) = \{2, 4, 5, 6\}$. 
Definition: An evolution algebra $A$ is **non-degenerate** if it has a natural basis $B = \{e_i \mid i \in \Lambda\}$ such that $e_i^2 \neq 0$ for every $i \in \Lambda$.

Lemma

$A$ is non-degenerate $\iff$ $\text{ann}(A) = 0$

\[
\text{ann}(A) := \{x \in A \mid xA = 0\}
\]

\[
\text{ann}(A) = \text{lin}\{e_i \in B \mid e_i^2 = 0\}
\]

Does not depend on the basis

It is an evolution ideal of $A$
**Remark:** Let $A$ be an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis.

- $A/{\text{ann}}(A)$ is not necessarily a non-degenerate evolution algebra.
- $A_1 := \text{lin}\{e_i \in B \mid e_i^2 \neq 0\}$ is not necessarily a subalgebra of $A$. 
Absorption property. Properties

\[
\text{ann}(A/\text{ann}(A)) = \bar{0} \quad \text{?} \quad \text{No}
\]

Absorption property

\[xA \subseteq I \text{ implies } x \in I\]

\[\text{ann}(A/I) = \bar{0}\]

Proposition
Absorption property. Properties

\[ \text{ann}(A/\text{ann}(A)) = \overline{0} \quad \text{No} \]

Absorption radical

Intersection of all absorption ideals

**Proposition**

\[
\begin{align*}
\text{rad}(A) &= 0 \\
\text{ann}(A) &= 0 \\
A &\quad \text{Non-degenerate}
\end{align*}
\]

\[
\text{rad}(A/\text{rad}(A)) = \overline{0}
\]
**Definition:** An algebra $A$ is simple if $A^2 \neq 0$ and 0 is the only proper ideal.

**Proposition:** Every simple evolution algebra, then $A$ is non-degenerate.

**Remark:** An evolution algebra $A$ whose associated graph has sinks cannot be simple.
Theorem: Let $A$ be a non-zero evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis. The following conditions are equivalent:

- $A$ is simple.
- If $\text{lin}\{e_i^2 \mid i \in \Lambda'\}$ is an ideal for a non-empty subset $\Lambda' \subseteq \Lambda$, then $A = \text{lin}\{e_i^2 \mid i \in \Lambda'\}$.
- $A = \langle e_i^2 \rangle = \text{lin}\{e_j^2 \mid j \in D(i)\}$ for every $i \in \Lambda$.
- $A = \text{lin}\{e_i^2 \mid i \in \Lambda\}$ and $\Lambda = D(i)$ for every $i \in \Lambda$. 
In terms of structure matrix:

**Corollary:** Let $A$ be a $n$-dimensional evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of $A$. Then $A$ is simple if and only if $|M_B(A)| \neq 0$ and $B$ cannot be reordered in such a way that the corresponding structure matrix is:

$$
\begin{pmatrix}
W_{m \times m} & U_{m \times (n-m)} \\
0_{(n-m) \times m} & Y_{(n-m) \times (n-m)}
\end{pmatrix}
$$
Definition: An evolution algebra is called **reducible** if \( A = \bigoplus_{\gamma \in \Gamma} A_\gamma \) where \( A_\gamma \) is a non-zero evolution subalgebra. If \( A_\gamma \) is irreducible, then \( A = \bigoplus_{\gamma \in \Gamma} A_\gamma \) is an **optimal decomposition** of \( A \).

Proposition: Let \( A \) be an evolution algebra. The following assertions are equivalent:

- There exists a family of evolution subalgebras \( \{ A_\gamma \}_{\gamma \in \Gamma} \) such that \( A = \bigoplus_{\gamma \in \Gamma} A_\gamma \).
- There exists a family of evolution ideals \( \{ I_\gamma \}_{\gamma \in \Gamma} \) such that \( A = \bigoplus_{\gamma \in \Gamma} I_\gamma \).
- There exists a family of ideals \( \{ I_\gamma \}_{\gamma \in \Gamma} \) such that \( A = \bigoplus_{\gamma \in \Gamma} I_\gamma \).
Infinite-dimensional case:

**Theorem:** Let $A$ be a non-degenerate evolution algebra with a natural basis $B = \{e_i \mid i \in \Lambda\}$. Assume that $A = \bigoplus_{\gamma \in \Gamma} I_{\gamma}$, where each $I_{\gamma}$ is an ideal of $A$. Then, there exists a disjoint decomposition of $\Lambda = \dot{\cup}_{\gamma \in \Gamma} \Lambda_{\gamma}$ such that

$$I_{\gamma} = \text{lin}\{e_i \mid i \in \Lambda_{\gamma}\}.$$
Finite-dimensional case:

**Corollary:** A non-degenerate finite dimensional evolution algebra \( A \) with natural basis \( B = \{ e_i \mid i = 1, \ldots, n \} \) is reducible if and only if \( B \) can be ordered in such a way the corresponding structure matrix is

\[
\begin{pmatrix}
W_{m \times m} & 0_{(n-m) \times (n-m)} \\
0_{(n-m) \times m} & Y_{(n-m) \times (n-m)}
\end{pmatrix}
\]

In terms of graph:

**Corollary:** Let \( A \) be a non-degenerate evolution algebra. Then \( A \) is irreducible if and only if the associated graph is connected.
Reducible evolution algebras

**Theorem:** The optimal direct sum decomposition of an evolution algebra $A$ does exist and it is unique whenever the algebra is non-degenerate.

- **Finite-dimensional case:** If $A$ is a finite dimensional evolution algebras (degenerated or not), we get an optimal decomposition through the fragmentation process (decomposition of $E$ into its connected components).
Orthogonality

Definitions:

- Two elements \( u \) and \( v \) are **orthogonal** if \( uv = 0 \).
- A family of vectors \( C \) is an **orthogonal family** if any pair of elements are orthogonal.
- A family of pairwise orthogonal and linearly independent vectors of an evolution algebra which have the extension property will be called an **extending natural family**.
- Any (linear) subspace \( E \) of an evolution algebra generated by an extending natural family is an **extending evolution subspace**. Such a family will be called an **extending natural basis** of \( E \).
- The **evolution rank** of \( E \) is defined by \( \text{erk}(E) = \dim(E^2) \).
**Definition:** Let $A$ be an evolution algebra, $B = \{e_i\}_{i \in \Lambda}$ a natural basis and $u = \sum_{i \in \Lambda} \alpha_i e_i$ an element of $A$. The **support of $u$ relative to $B$**, denoted $\text{supp}_B(u)$, is defined as the set $\text{supp}_B(u) = \{i \in \Lambda \mid \alpha_i \neq 0\}$. If $X \subseteq A$, we put $\text{supp}_B(X) = \bigcup_{x \in X} \text{supp}_B(x)$.

**Theorem:** Let $A$ be an evolution $K$-algebra with natural basis $B = \{e_i\}_{i \in \Lambda}$ and let $u \in A$. Set $\text{supp}(u) = \{i_1, \ldots, i_r\}$. Then

- If $u^2 \neq 0$, then $u$ is a natural vector if and only if $\text{rk}(\{e_{i_1}^2, \ldots, e_{i_r}^2\}) = 1$.
- If $u^2 = 0$, then $u$ is a natural vector if and only if $e_{i_1}^2 = \cdots = e_{i_r}^2 = 0$.
Theorem: Let $A$ be an evolution algebra and let $r = \dim A^2$. Then:

$$A = \text{ann}(A) \oplus E_1 \oplus \ldots \oplus E_r,$$

where $E_1, \ldots, E_r$ are extending evolution subspaces of $A$ satisfying $\text{erk}(E_i) = 1$ for all $i$ and if $i \neq j$, $E_i E_j = 0$, $\dim (E_i^2 + E_j^2) = 2$. Moreover, if $A$ is non-degenerate, the decomposition is unique.
Corollary:
Let $A$ be an evolution algebra and let $B = B_0 \cup B_1 \cup \cdots \cup B_r$ and $B' = B'_0 \cup B'_1 \cup \cdots \cup B'_r$ be two natural bases of $A$ given by two decompositions as in previous theorem, where $B_0$ and $B'_0$ are bases of $\text{ann}(A)$. Then, we can reorder the elements of $B$ and $B'$ so that the change of basis matrix has the following block form

$$
\begin{pmatrix}
* & * & * & \cdots & * \\
0 & * & 0 & \cdots & 0 \\
0 & 0 & * & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & * 
\end{pmatrix}.
$$
Proposition: Let $A$ be an evolution algebra and let $E$ be an extending evolution subspace of $A$ with evolution rank one and such that $E \cap \text{ann}(A) = \{0\}$. Let $C$ be a linearly independent orthogonal family of $E$. Then $C$ can be extended to a natural basis of $E$, which can be extended to a natural basis of $A$, if and only if $u^2 \neq 0$ for all $u \in C$. 
Gian Carlo Rota, Discrete Thoughts (1953).

The lack of real contact between mathematics and biology is either a tragedy, a scandal or a challenge, it is hard to decide which.