A REFINEMENT OF KOOl-THOMAS INVARIANTS VIA EQUIVARIANT K-THEORETIC INVARIANTS

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Abstract. In this article we are defining a refinement of Kool-Thomas invariants of local surfaces via the equivariant K-theoretic invariants proposed by Nekrasov and Okounkov. Kool and Thomas defined the reduced obstruction theory for the moduli of stable pairs \( P_{\chi}(X, i^* \beta) \) as the degree of the virtual class \([P_{\chi}(S, \beta)]^{\text{red}}\) after we apply \( \tau([pt])^m \in H^*(P_{\chi}(X, i^* \beta), \mathbb{Z}) \). \( \tau([pt]) \) contains the information of the incidence of a point and a curve supporting a \((F, s)\).

The K-theoretic invariants proposed by Nekrasov and Okounkov is the equivariant holomorphic Euler characteristic of \( O_{\text{vir}}P_{\chi}(X, i^* \beta) \otimes K^1_{\text{vir}} \). We introduce two classes \( \gamma(O_s) \) and \( \bar{\gamma}(O_s) \) in the Grothendieck group of vector bundles on the moduli space of stable pairs of the local surfaces that contains the information of the incidence of a curve with a point. We define two invariants \( P_{X, \beta, \chi}(s_1, \ldots, s_m) \) and \( \bar{P}_{X, \beta, \chi}(s_1, \ldots, s_m) \) each corresponds to \( \gamma(O_s) \) and \( \bar{\gamma}(O_s) \). We found that the contribution of \( P_{\chi}(S, \beta) \subset P_{G} \) to \( P_{X, \beta, \chi}(s_1, \ldots, s_m) \) and to \( \bar{P}_{X, \beta, \chi}(s_1, \ldots, s_m) \) are the same. Moreover, if we evaluate this contribution at \( t=1 \) we get the Kool-Thomas invariants.

The generating function of this contribution contain the same information as the generating function of the refined curve counting invariants defined by Göttsche and Shende in [8]. After a change of variable there exist a coefficient \( N_{[S, \mathcal{L}]}^{\delta}(y) \) of the generating function of the refined curve counting that counts the number of \( \delta \)-nodal curve in \( \mathbb{P}^d \subset \mathcal{L} \). We conjecture that after the same change of variable the corresponding coefficient \( M_{[S, \mathcal{L}]}^{\delta}(y) \) coming from the generating function of the contribution of \( P_{\chi}(S, \beta) \) to \( \bar{P}_{X, \beta, \chi}(s_1, \ldots, s_m) \) is identical with \( N_{[S, \mathcal{L}]}^{\delta}(y) \).

Keywords: Kool-Thomas invariants, K-theoretic invariants, Göttsche Shende invariants

1. Introduction

Fix a nonsingular projective surface \( S \) and a sufficiently ample line bundle \( \mathcal{L} \) on \( S \). A \( \delta \)-nodal curve \( C \) on \( S \) is a 1 dimensional subvariety of \( S \) which has nodes at \( \delta \) points and is regular outside these singular points. For any scheme \( Y \), let \( Y^{[n]} \) be the Hilbert scheme of \( n \)-points i.e. \( Y^{[n]} \) parametrizes subschemes \( Z \subset Y \) of length \( n \). Given a family of curves \( \mathcal{C} \rightarrow B \) over a base \( B \), we denote by \( \text{Hilb}^n(C/B) \) the relative Hilbert scheme of points. Kool, Thomas and Shende showed that some linear combinations \( n_{r,C} \) of the Euler characteristic of \( C^{[n]} \) counts the number of curves of arithmetic genus \( r \) mapping to \( C \). Applying this to the family \( \mathcal{C} \rightarrow \mathbb{P}^d \) where \( \mathbb{P}^d \subset \mathcal{L} \), the number of \( \delta \)-nodal curves is given by a coefficient of the generating function of the Euler characteristic of \( \text{Hilb}(C/\mathbb{P}^d) \) after change of variable [11]. By replacing euler characteristic with Hirzebruch \( \chi_y \)-genus, Göttsche and Shende give a refined counting of \( \delta \)-nodal curves.
Pandharipande and Thomas showed that a stable pair \((F, s)\) on a surface \(S\) is equivalent to the pair \((C, Z)\) of a curve \(C\) on \(S\) supporting the sheaf \(F\) with \(Z \subset C\) a subscheme of finite length. Thus the moduli space of stable pairs on a surface \(S\) is a relative Hilbert scheme of points corresponding to a family of curves on \(S\).

The study of the moduli space of stable pairs on Calabi-Yau threefold \(Y\) is an active area of research. This moduli space gives a compactification of the moduli space of nonsingular curves in \(Y\). To get an invariant of the moduli space Behrend and Fantechi introduce the notion of perfect obstruction theory. With this notion we can construct a class in the Chow group of dimension 0 that is invariant under some deformations of \(Y\)\(^1\).

The homological invariants of the stable pair moduli space \(\mathcal{P}_X(X, i, \beta)\) of the total space \(X\) of \(K\) of some smooth projective surface \(S\) contain the information of the number of \(\delta\)-nodal curves in a hyperplane \(\mathbb{P}^d \subset |L|\). Notice that \(X\) is Calabi-Yau. There exist a morphism of schemes \(\text{div} : \mathcal{P}_X(X, i, \beta) \rightarrow |L|\) that maps a point \((F, s) \in \mathcal{P}_X(X, i, \beta)\) to a divisor \(\text{div}(\pi_* F)\) that support \(\pi_* F\) on \(S\) where \(\pi : X \rightarrow S\) is the structure morphism of \(X\) as a vector bundle over \(S\). Using descendents, Kool and Thomas translate the information of the incidence of a curve with a point into cutting down the moduli space by a hypersurface pulled back from \(|L|\) so that after cutting down, we have a moduli space that parameterize Hilbert scheme of curves in \(\mathbb{P}^d\)\(^2\).

The famous conjecture of Maulik, Nekrasov Okounkov and Pandharipande states that the invariants corresponding to the moduli space of stable pairs have the same information as the invariants defined from the moduli space of stable maps and the Hilbert schemes.

The next development in the theory of PT invariants is to give a refinement of the homological invariant. The end product of this homological invariant is a number. A refinement of this invariant would be a Laurent polynomial in a variable \(t\) such that when we evaluate \(t\) at 1 we get the homological invariant.

There are several methods that have been introduced to give a refinement for DT invariants, for example both motivic and \(K\)-theoretic definitions. In this thesis we use the \(K\)-theoretic definition which has been proposed by Nekrasov and Okounkov in \(^15\) where we compute the holomorphic Euler characteristic of the twisted virtual structure sheaf of the corresponding moduli space. In the case when \(S = \mathbb{P}^3\) or \(S = \mathbb{P}^1 \times \mathbb{P}^1\) Choi, Katz and Klemm have computed a \(K\)-theoretic invariant of the moduli space of stable pairs in the paper \(^2\). Their computation does not include any information about the incidence of subschemes of \(S\).

2. Equivariant Chow Groups and \(K\)-theory

In this section we will describe the notation and definition we use regarding equivariant Chow Groups and \(K\)-theory.

2.1. Equivariant Chow Groups. In this section we review the definition of equivariant Chow groups given in \(^8\)\(^4\). We will use \(g\) to denote the dimension of our group \(G\) as a scheme over \(\mathbb{C}\).

Given \(i \in \mathbb{Z}\). Let \(X\) be a \(G\)-scheme with \(\text{dim} X = d\). Let \(V\) be \(G\)-vector space of dimension \(l\). Assume that there exists an open subscheme \(U \subset V\) and a principal \(G\)-bundle \(\pi : U \rightarrow U_G\). By giving \(X \times V\) a diagonal action of \(G\), assume furthermore that there exist a principal \(G\)-bundle \(\pi_X : X \times U \rightarrow (X \times U)/G\). We will use \(X \times_G U\)
to denote \((X \times U)/G\). Assume also that \(V \setminus U\) has codimension greater than \(d - i\), then the equivariant Chow group is defined as
\[
A_G^i(X) := A_{i+d}(X \times_G U).
\]
The definition is independent up to isomorphism of the choice of a representation as long as \(V \setminus U\) is of codimension greater than \(d - i\).

For a \(G\)-equivariant map \(f : X \to Y\) with property \(P\) where \(P\) is either proper, flat, smooth, or regular embedding the \(G\)-equivariant map \(f \times 1 : X \times X \to Y \times X\) has the property \(P\) since all of these properties are preserved by a flat base change. Moreover, the corresponding morphism \(f_G : X \times_G U \to Y \times_G U\) also has property \(P\).

In fact, these properties are local on the target in the Zariski topology and for any trivialization \((V_i, \varphi_i)_{i \in \Lambda}\) of \(\pi : U \to U_G\), the restriction of \(f_G\) on \(\pi_X(X \times \pi^{-1}(V_i))\) is isomorphic to \(f \times 1\). So from the definition, for a flat \(G\)-map \(f : X \to Y\) of codimension \(l\) we can define pullback map \(f^* : A_G^l(Y) \to A_G^l(X)\) for equivariant Chow groups. Similarly, for regular embedding \(f : X \to Y\) of codimension \(d\) we have a Gysin homomorphism \(f^* : A_G^l(Y) \to A_{i-d}(X)\) and for proper \(G\)-map \(f : X \to Y\) we can define pushforward \(f_* : A_G^l(X) \to A_G^l(Y)\) for equivariant Chow groups.

For \(G = T_1\) and an \(l + 1\)-dimensional weight space \(V_\chi\) we have a principal \(G\)-bundle \(\pi_U : V_\chi \setminus \{0\} \to P(V_\chi^*)\). There exist a principal \(G\)-bundle \(\pi_X : X \times X \to X \times_G U\). Since \(\text{codim} V_\chi \setminus U = l + 1\), for each \(i \in \mathbb{Z}\) we can take \(A_{i+l}(X \times_G U)\) to represent \(A_G^i(X)\) if \(l + i \geq d\). We can also fix \(\chi\) to be \(-1\) to cover all \(i\).

Thus we fix the following notation. For each positive integer \(l\) let \(V_l\) be a \(T_1\)-space of weight \(-1\) with coordinate \(x_0, \ldots, x_l\). Then \(V_{l-1}\) is the zero locus of the last coordinate of \(V_l\). We use \(U_l\) to denote \(V_l \setminus \{0\}\) and \(X_l\) to denote \(X \times_G U_l\) and \(\pi_{X,l} : X \times U_l \to X_l\) the corresponding principal bundle. Thus we have the following direct system

\[
\cdots \longrightarrow X_{l-1} \stackrel{j_{X,l-1}}{\longrightarrow} X_l \stackrel{j_{X,l}}{\longrightarrow} X_{l+1} \stackrel{j_{X,l+1}}{\longrightarrow} \cdots
\]

There is a projection from \(\xi : V_{l+1} \to V_l\) by forgetting the last coordinate such that \(j_l : V_l \to V_{l+1}\) is the zero section of \(\xi\). By removing the fiber of \(p := (0 : 0 : \ldots : 0 : 1) \in P(V_{l+1})\), the corresponding projection \(\xi : X_{l+1} \setminus \pi_X^1(p) \to X_l\) is a line bundle over \(X_l\) such that \(j_{X,l} : X_l \to X_{l+1} \setminus \pi_X^1(p)\) is the zero section. Note that \(\text{dim} \pi_X^1(\xi) = \text{dim} X = d\). Thus for \(i \geq d - l\) the restriction map \(A_{i+l+1}(X_{l+1}) \to A_{i+l}(X_{l+1}) = \pi_{X,l+1}(X_{l+1} \setminus \pi_X^1(\xi))\) is an isomorphism. In general this restriction is a surjection.

Since \(j_{X,l} : X_l \to X_{l+1} \setminus \pi_X^1(p)\) is the zero section of \(\xi\), the Gysin homomorphism \(j_{X,l}^* : A_k(X_{l+1} \setminus \pi_X^1(\xi)) \to A_k(X_l)\) is an isomorphism. Since \(j_l\) is a regular embedding we have a Gysin homomorphism \(j_l : A_{k+l}(X_{l+1}) \to A_k(X_l)\) which is the composition of the above homomorphisms.

The direct system (2.1) induces an inverse system

\[
\cdots \longleftarrow A^*_s(X_{l-1}) \stackrel{j_{X,l-1}}{\longrightarrow} A^*_s(X_l) \stackrel{j_{X,l}}{\longrightarrow} A^*_s(X_{l+1}) \longrightarrow \cdots
\]

deepth of abelian groups. Let \((\lim_i A(X_i), \lambda_i)\) be the inverse limit of the above inverse system. From the definition of equivariant Chow groups, \(A_G^i(X) = A_{i+n}(X_n)\) for \(i \geq d - n\) so that we can identify \(\prod_{i=d-n}^d A_G^i(X)\) with the group \(\prod_{i=d-n}^d A_i(X_n)\). Recall that \(\left(\prod_{i=d} A_G^i(X), \nu_i\right)\) where \(\nu_n : \prod_{i=d} A_G^i(X) \to \prod_{i=d-n} A_G^i(X)\) is defined by
(a_d, a_{d-1}, \ldots) \mapsto (a_d, \ldots, a_{d-n}) is the inverse limit of the inverse system defined by the projection \( p_{X_n} : \prod_{i=d-n}^d A^G(X_n) \to \prod_{i=d-n}^d A^G(X), \) \((a_d, \ldots, a_{d-n}, a_{d-n-1}) \mapsto (a_d, \ldots, a_{d-n}) .\) After indentifying \( \prod_{i=d-n}^d A^G_i(X_n) \) with \( \prod_{i=d-n}^d A_i(X_n), \) \( p_{X_n} \) and \( j_{X,n} \) are the same homomorphism. The composition of the projections \( \xi_n : A_*(X_n) \to \prod_{i=d-n}^d A(X_n) \) with \( \lambda_n : \lim A_*(X_i) \to A_*(X_n) \) are homomorphisms \( \xi_i : \lim A_*(X_i) \to \prod_{i=d-n}^d A^G_i(X) \) satisfying \( p_{X_n+1} \circ \xi_n = p_{X,n} \) so that by the universal property of inverse limit we have a group homomorphism \( \xi : \lim A_*(X_i) \to \prod_{i=-\infty}^d A^G_i(X) \) satisfying \( p_{X,n} \circ \xi = \xi_n .\) We will use the following proposition later.

**Proposition 1.** \( \xi : \lim A_*(X_i) \to \prod_{i=-\infty}^d A^G_i(X) \) is an isomorphism.

### 2.2. Equivariant K-theory

Let \( G \) acts on a scheme \( X .\) Let \( Vec_G(X) \) (resp. \( Coh_G(X) \)) be the category of equivariant vector bundle (resp. equivariant coherent sheaves) on scheme \( X .\) We will use \( K^G(X) \) (resp. \( G^G(X) \)) to denote the group generated by equivariant vector bundle (resp. coherent sheaves) modulo short exact sequence. More generally for any exact category \( \mathcal{N} \) we can construct the group \( K_0\mathcal{N} \) generated by objects of \( \mathcal{N} \) modulo short exact sequence.

We will sketch the construction of pushforward map \( f_* : K^G(X) \to K^G(Y) \) in some special cases induced by direct image functor. For more details, readers should consult chapter 2 of [24] or section 7 and 8 of [20]. Since taking pullback is exact on vector bundles, the definition of \( f^* : K^G(Y) \to K^G(X) \) is straightforward.

First we need the following Lemma.

**Lemma 2.** Let \( \mathcal{N}_X \) be a full subcategory of \( Coh_G(X) \) satisfying the following conditions:

1. \( \mathcal{N}_X \) contains \( Vec_G(X) \)
2. \( \mathcal{N}_X \) is closed under extension
3. Each objects of \( \mathcal{N}_X \) has a resolution by a bounded complex of elements in \( Vec_G(X) \)
4. \( \mathcal{N}_X \) is closed under kernels of surjections.

Then

1. \( \mathcal{N}_X \) is exact and the inclusion \( Vec_G(X) \subset \mathcal{N}_X \) induce the group homomorphism \( i : K^G(X) \to K_0(\mathcal{N}_X) \) by mapping the class \( [\mathcal{P}]_{Vec_G(X)} \) of any locally free sheaf \( \mathcal{P} \) to its class \( [\mathcal{P}]_{\mathcal{N}_X} \) in \( K_0(\mathcal{N}_X) \)
2. all resolutions of \( \mathcal{F} \) by equivariant locally free sheaves

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{P}_n & \rightarrow & \mathcal{P}_{n-1} & \rightarrow & \cdots & \rightarrow & \mathcal{P}_1 & \rightarrow & \mathcal{P}_0 & \rightarrow & \mathcal{F} & \rightarrow & 0
\end{array}
\]

define the same element \( \chi(\mathcal{F}) := \sum_{i=0}^n (-1)^i [\mathcal{P}_i] \) in \( K^G(X) .\) Furthermore, \( \chi \) define a group homomorphism \( \chi : K_0(\mathcal{N}_X) \to K^G(X) \) which is the inverse of \( i : K^G(X) \to K_0(\mathcal{N}_X) .\)

**Corollary 3.** Let \( f : X \to Y \) be a finite \( G \)-morphism such that \( f_* : Vec_G(X) \to Coh_G(Y) \) factors through a subcategory \( \mathcal{N}_Y \subset Coh_G(Y) \) satisfying all 4 conditions of lemma 2 above. Then there exist a group homomorphism \( f_* : K^G(X) \to K^G(Y) \) such that \( f_*[\mathcal{E}] = \chi(f_*\mathcal{E}) \) for any locally free sheaf \( \mathcal{E} \) on \( X .\)
Proposition 4 (Projection Formula). Let \( f : X \to Y \) be a morphism satisfying the condition in corollary\(^3\) or the projection \( \varphi : \mathbb{P}_Y(V) \to Y \) where \( V \) is a \( G \)-equivariant vector bundle. Then for any \( x \in K^G(X) \) and \( y \in K^G(Y) \) we have
\[
f_* (x.f^* y) = (f_* x).y \in K^G(Y).
\]

Proposition 5 (Base change formula).

(1) Consider the following cartesian diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\bar{g}} & X \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{g} & X
\end{array}
\]
such that \( f \) and \( f' \) are \( G \)-regular embeddings of codimension \( r \). Then \( g^* \circ f_* = f_* \circ \bar{g}^*: K^G(X) \to K^G(Y) \).

(2) Let \( A \) be a smooth projective variety and let \( p : A \times Y \to Y \) be the projection to the second factor. Let \( g : Y \to Y' \) be any morphism and consider the following cartesian diagram
\[
\begin{array}{ccc}
A \times Y & \xrightarrow{\bar{g}} & A \times Y' \\
\downarrow{p} & & \downarrow{p} \\
Y & \xrightarrow{g} & Y
\end{array}
\]
Then the pushforward maps \( p_* : K^G(A \times Y) \to K^G(Y) \) and \( p_* : K^G(A \times Y) \to K^G(Y) \) are well defined and \( p_* \circ \bar{g}^* = g^* \circ p_* : K^G(A \times Y) \to K^G(Y) \).

Let \( d : D \to A \times Y \) be a \( G \)-closed embedding such that \( D \) is flat over \( Y \) and let \( d' : D' \to A \times Y' \) be the corresponding pullback so that we have the following cartesian diagram
\[
\begin{array}{ccc}
\bar{D} & \xrightarrow{\bar{g}} & D \\
\downarrow{d} & & \downarrow{d} \\
A \times Y & \xrightarrow{\bar{g}} & A \times Y.
\end{array}
\]
Then \( \bar{g}^* [\mathcal{O}_D] = [\mathcal{O}_{\bar{D}}] \in K^G(A \times Y) \).

Let \( i : X \to Y \) be a \( G \)-equivariant closed embedding and let \( U = Y \setminus X \) with open embedding \( j : U \to Y \). Then there exist group homomorphism \( i_* : G^G(X) \to G^G(Y) \) and \( j^* : G^G(Y) \to G^G(U) \). These two homomorphism is related as follows:

Lemma 6. The following complex of abelian groups is exact
\[
G^G(X) \xrightarrow{i_*} G^G(Y) \xrightarrow{j^*} G^G(U) \to 0.
\]

Proof. This is Theorem 2.7 of [23]. \( \square \)

We call a class \( \beta \in G^G(Y) \) is supported on \( X \) if \( \beta \) is in the image of \( i_* \). Equivalently \( \beta \) is supported on \( X \) if \( j^* \beta = 0 \).

Let \( \text{Coh}^G_Y(X) \) be the abelian group of coherent sheaves supported on \( X \). Note that \( \mathcal{F} \in \text{Coh}^G_Y(X) \) is not necessarily an \( \mathcal{O}_X \)-module. Let \( G^G_X(Y) \) be the corresponding Grothendieck group. The pushforward functor \( i_* : \text{Coh}^G_X(X) \to \text{Coh}^G_Y(Y) \)
factors through $\text{Coh}^X_Y(Y)$ so that there exist a group homomorphism $\bar{i} : G^G(X) \to G^G_X(Y)$, $[\mathcal{F}] \to [i_* \mathcal{F}]$. There exist an inverse of $\bar{i}$ described as follows. Let $\mathcal{F} \in \text{Coh}^X_Y(Y)$ and let $\mathcal{I}$ be the ideal of $X$. Then there exist positive integer $n$ such that $\mathcal{I}^n \mathcal{F} = 0$ so that we have a filtration

$$\mathcal{F} \supseteq \mathcal{I} \mathcal{F} \supseteq \mathcal{I}^2 \mathcal{F} \supseteq \ldots \supseteq \mathcal{I}^{n-1} \mathcal{F} \supseteq \mathcal{I}^n \mathcal{F} = 0.$$  

Note that each $\mathcal{I}^r \mathcal{F}/\mathcal{I}^{r+1} \mathcal{F}$ is an $\mathcal{O}_X$-module. One can show that $[\mathcal{F}] \to \sum_{r=0}^{n-1} [\mathcal{I}^r \mathcal{F}/\mathcal{I}^{r+1} \mathcal{F}]$ defines a group homomorphism $\bar{i}^{-1} : G^G_X(Y) \to G^G(X)$.

**Lemma 7.** $\bar{i} : G^G(X) \to G^G_X(Y)$ is an isomorphism.

Given a cartesian diagram

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{i} & \overline{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

with $i$, $f$ are closed embeddings and a coherent sheaf $\mathcal{E}$ on $X$ such that $f_* \mathcal{E}$ has a finite resolution by a complex of locally free sheaves. Then we can define a group homomorphism $f^{[\mathcal{E}]} : G^G(\overline{Y}) \to G^G(\overline{X})$, described as follows. Let $\mathcal{F}$ be a coherent sheaf on $Y$ supported on $\overline{Y}$. For each $y \in \overline{Y}$, the stalk of $\text{Tor}^1_Y(f_* \mathcal{E}, \mathcal{F})$ on $y$ is $\text{Tor}^{\mathcal{O}_Y}_1((f_* \mathcal{E})_y, \mathcal{F}_y)$ so that $\text{Tor}^1_Y(f_* \mathcal{E}, \mathcal{F})$ is supported on $\overline{X}$. For any exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on $\overline{Y}$ we have a long exact sequence

$$\text{Tor}^{\mathcal{O}_Y}_1(f_* \mathcal{E}, \mathcal{F}') \to \text{Tor}^{\mathcal{O}_Y}_{1}(f_* \mathcal{E}, \mathcal{F}) \to \text{Tor}^{\mathcal{O}_Y}_1(f_* \mathcal{E}, \mathcal{F}) \to \text{Tor}^{\mathcal{O}_Y}_1(f_* \mathcal{E}, \mathcal{F}')$$

so that

$$\sum_{i \geq 0} (-1)^i [\text{Tor}^1_Y(f_* \mathcal{E}, \mathcal{F})] = \sum_{i \geq 0} (-1)^i [\text{Tor}^1_Y(f_* \mathcal{E}, \mathcal{F}')] + \sum_{i \geq 0} (-1)^i [\text{Tor}^1_Y(f_* \mathcal{E}, \mathcal{F}'')] \in G^G(\overline{X}).$$

Thus there exist a group homomorphism $f^{[\mathcal{E}]} : G^G(\overline{Y}) \to G^G(\overline{X})$. By Lemma 7 we can define $f^{[\mathcal{E}]}$ as the composition $\bar{i}^{-1} \circ f^{[\mathcal{E}]}$.

**Lemma 8.** Let $f : X \to Y$ be a closed embedding and a coherent sheaf $\mathcal{E}$ on $X$ such that $f_* \mathcal{E}$ has a finite resolution by locally free sheaves. For any closed embedding $i : \overline{Y} \to Y$, there exist a group homomorphism $f^{[\mathcal{E}]} : G^G(Y) \to G^G(\overline{Y} \cap X)$ that maps $[\mathcal{F}]$ to $\sum_{i \geq 0} (-1)^i [\text{Tor}^1_Y(f_* \mathcal{E}, \mathcal{F})]_{Y \cap X}$. Furthermore, $i_* f^{[\mathcal{E}]}([\mathcal{F}]) = \sum_{i \geq 0} (-1)^i [\text{Tor}^1_Y(f_* \mathcal{E}, \mathcal{F})]_{Y \cap X}$.

Let $G$ be the torus $T_1$ and let $X$ be a $G$-scheme. Recall that by Proposition 1 there exist an isomorphism $\xi : \text{lim} A_*(X_n) \to \prod_{n \to -\infty} A^G(X)$. In this section we want to recall some results of the corresponding $\text{lim} K(X_n)$.

From the direct system $\mathcal{X}$ we have the inverse system

$$\begin{array}{ccc}
\ldots & \xleftarrow{\rho_{X_{i+1}}} & K(X_{i+1}) & \xleftarrow{\rho_{X_i}} & K(X_i) & \xleftarrow{\rho_{X_{i-1}}} & \ldots
\end{array}$$

We denote the inverse limit of the above inverse system as $\text{lim} K(X)_{\mathcal{I}}$ and use $\rho_{X_{i}}$ to denote the canonical morphism $\text{lim} K(X)_{\mathcal{I}} \to K(X_i)$. The pullback functor induced
from the projection map $pr_X : X \times U_l \to X$ and the equivalence between $Vec_G(X \times U_l)$ and $Vec(X_l)$ induces group homomorphisms $\kappa_{X,l} : K^G(X) \to K(X_l)$. It's easy to show that $\kappa_{X,l} = j_{X,l}^* \circ \kappa_{X,l+1}$ so that we have a unique group homomorphism $\kappa_X : K^G(X) \to \lim K(X_l)$ such that $\kappa_{X,l} = \rho_{X,l} \circ \kappa_X$. In this section, to distinguish between the ordinary and the equivariant version of pullback and pushforward map, we will use superscript $\oslash$ to denote the equivariant version, for example we will use $f^G_*$ to denote the pullback in the equivariant setting.

There is a canonical way to define pullback map $\widetilde{f^*} : \lim K(Y) \to \lim K(X)$ and pushforward map $\widetilde{f_*} : \lim K(X_l) \to \lim K(Y_l)$ for $\lim K(X_l)$.

**Lemma 9.** Let $f : X \to X$ be a $G$ morphism.

1. If $f : X \to Y$ is a finite $G$-morphism satisfying the condition in corrolary $\mathfrak{A}$ Assume also that for all $l$, $(f \times id_l)$ also satisfies the condition in corrolary $\mathfrak{A}$. Then there exist a group homomorphism $\widetilde{f_*} : \lim K(X_l) \to \lim K(Y_l)$ satisfying the identity $\kappa_Y \circ \widetilde{f_*} = \widetilde{f_*} \circ \kappa_X$.

2. If $f : X \to Y$ is the structure morphism $\mathbb{P}_Y(V) \to Y$ where $V$ is a $G$-equivariant vector bundle. Then there exist a group homomorphism $\widetilde{f_*} : \lim K(X) \to \lim K(Y)$ satisfying the identity $\kappa_Y \circ \widetilde{f_*} = \widetilde{f_*} \circ \kappa_X$.

3. If $f : X \to Y$ is a $G$-morphism that can be factorized into $p \circ i$ where $i : X \to Z$ is a finite morphism satisfying the condition 1. and $p$ satisfies condition 2. then the group homomorphism $\widetilde{f_*} := \widetilde{f_*} \circ \widetilde{i_*} : \lim K(X) \to \lim K(Y)$ is independent of the factorization.

For each equivariant vector bundle $\mathcal{E}$ on $X$, its pullback $\mathcal{E}$ to $X \times U_n$ correspond to a vector bundle $\mathcal{E}_n$ on $X_n$ such that $\pi^* \mathcal{E}_n = \mathcal{E}$. By the identification $A^G_j(X) = A_j(X_n)$, $c^G_j(\mathcal{E}) : A^G_j(X) \to A^G_{j-1}(X)$ is given by $c^G_j(\mathcal{E}_n) : A_j(X_n) \to A_{j-1}(X_n)$. Since Chern class commutes with pullback this definition is well defined. Furthermore, $c^G_j(\mathcal{E})$ is an element of $A^G_j(X)$.

In the non equivariant case, each vector bundle $\mathcal{E}$ of rank $r$ has Chern roots $x_1, \ldots, x_r$ such that $c^G_j(\mathcal{E}) = e_i(x_1, \ldots, x_r)$ where $e_i$ is the $i^{th}$ symmetric polynomial. Furthermore, its Chern character is defined as $ch(\mathcal{E}) = \sum c^i \mathcal{E}$. From this definition, we have the following formula of Chern character in terms of Chern classes

$$ch(\mathcal{E}) = r + c^1(\mathcal{E}) + \frac{1}{2} \left( c^1(\mathcal{E})^2 - 2c^2(\mathcal{E}) \right) + ..$$

where $P_j(c^1(\mathcal{E}), \ldots, c^j(\mathcal{E}))$ is a polynomial of order $j$ with $c^i(\mathcal{E})$ has weight $i$.

In $\mathfrak{P}$, Edidin and Graham define an equivariant Chern character map $ch^G : K^G(X) \to \prod_{i=0}^\infty A^G_i(X)$ by the following formula

$$ch^G(\mathcal{E}) = \sum_{i=0}^\infty P_i(c^1_G(\mathcal{E}), \ldots, c^i_G(\mathcal{E}))$$

One can show that $ch^G$ is a ring homomorphism. Let $\tilde{ch} : K^G(X) \to \lim A^*(X_n)$ denote the composition $\alpha \circ ch^G$. 


For each \( n \) there is a Chern character map \( ch_n : K(X_n) \to A^*(X_n) \) which commutes with refined Gysin homomorphisms. By the universal property of inverse limits we have a ring homomorphism \( \overline{ch} : \lim \rightarrow K(X_n) \to \lim \rightarrow A^*(X_n) \). Since each \( ch_n \) is a ring homomorphism, \( \overline{ch} \) is also a ring homomorphism. Furthermore the following diagram commutes

\[
\begin{array}{ccc}
K^G(X) & \xrightarrow{ch^G} & \prod_{i=0}^{\infty} A^i_G(X) \\
\downarrow \kappa & & \downarrow \sigma \\
\lim K(X_n) & \xrightarrow{ch} & \lim A^*(X_n)
\end{array}
\]

**Lemma 10.** For all \( x \in \lim A_*(X_n) \) and for any \( \beta \in K^G(X) \) we have \( \xi \left( \overline{ch}(\beta)(x) \right) = ch^G(\beta)(\xi x) \).

### 3. Kool-Thomas invariants

In this section we will review the definition of Kool-Thomas invariants and its relation to curve counting.

Let \( X \) be a projective smooth variety of dimension 3. A pair \((\mathcal{F}, s)\) where \( \mathcal{F} \) is a coherent sheaf of dimension 1 and \( s \) is a section of \( \mathcal{F} \) is called stable if the following two conditions holds:

1. \( \mathcal{F} \) is pure
2. The cokernel \( Q \) of \( s \) is of dimension 0.

Let \( X \) be a smooth projective 3-fold and let \( \chi \) be an integer and \( \beta \) be a class in \( H_2(X, \mathbb{Z}) \), there exists a projective scheme \( \mathcal{P}_\chi(X, \beta) \) parametrizing pairs \((\mathcal{F}, s)\) satisfying the above conditions with scheme theoretic support of \( \mathcal{F} \) is of class \( \beta \) and holomorphic Euler characteristic \( \chi(\mathcal{F}) \) equals to \( \chi[13] \). We will use \( \mathcal{P} \) to denote \( \mathcal{P}_\chi(X, \beta) \) whenever the context is clear. There exist a universal sheaf \( \mathcal{F} \) and universal section \( \xi : \mathcal{O}_{\mathcal{P} \times X} \to \mathcal{F} \) on the product space \( X \times \mathcal{P} \). We denote by \( p \) and \( q \) the projection from \( \mathcal{P}_\chi(X, \beta) \times X \) to the factor \( \mathcal{P}_\chi(X, \beta) \) and \( X \) respectively. Note that in general \( \mathcal{P}_\chi(X, i_*\beta) \) is singular.

If \( G \) acts on \( X \) there is a natural \( G \) action on \( \mathcal{P} \). Moreover, if \( G \) acts diagonally on \( X \times X \) i.e. \( \sigma_{p \times X} : G \times p \times X \to \mathcal{P} \times X \), \( (g.p, x) \to (g.p, g.x) \), then the universal sheaf \( \mathcal{F} \) is an equivariant sheaf and \( \xi : \mathcal{O}_{\mathcal{P} \times X} \to \mathcal{F} \) is an equivariant morphism of sheaves.

Given a map \( \phi : E^* \to L_Y \), where \( E^* = \{ E^{-1} \to E^0 \} \) is a two term complex of vector bundles and \( L_Y \) is the cotangent complex of \( Y \), such that the induced map \( \phi^0 \), \( \phi^{-1} \) is isomorphism and epimorphism respectively, Behrend and Fantechi construct a class \([Y]_{vir} \in A_{4kE^0 -rkE^{-1}}(Y)\) called virtual fundamental class[11]. The map \( \phi \) is called a perfect obstruction theory of \( Y \) and \( vd := \text{rk}E^0 - \text{rk}E^{-1} \) is the virtual dimension of \( Y \). The perfect obstruction theory gives a virtual class even though the space \( Y \) is badly singular.

Let \( \mathbb{I}^* = \{ \mathcal{O}_{\mathcal{P} \times X} \to \mathcal{F} \} \) be the universal complex on \( \mathcal{P} \times X \). Pandharipande and Thomas have shown that \( R\mathbb{I}^* (R\text{Hom}(\mathbb{I}^* , \mathbb{I}^*)_0 \otimes \omega_X)[2] \) is a two term complex of locally free sheaves and there exist a map \( \phi : R\mathbb{I}^* (R\text{Hom}(\mathbb{I}^* , \mathbb{I}^*)_0 \otimes \omega_X)[2] \to \mathbb{L}_\mathcal{P} \) satisfying the above conditions. We will use \( \mathbb{E}^* = \{ E^{-1} \to E^0 \} \) to denote the complex \( R\mathbb{I}^* (R\text{Hom}(\mathbb{I}^* , \mathbb{I}^*)_0 \otimes \omega_p)[2] \) on \( \mathcal{P} \). The virtual dimension of \( \mathcal{P}_\chi(X, \beta) \) is then
A REFINEMENT OF KOOL-THOMAS INVARIANTS VIA EQUIVARIANT K-THEORETIC INVARIANTS

-χ(\mathcal{R}Hom(I^\bullet, I^\bullet)_0) = \int_\beta c_1(X). If X is Calabi-Yau the dualizing sheaf ω_X ≃ \mathcal{O}_X so that by Serre duality vd = rkE^0 - rkE^{-1} = 0. If vd = 0 then P_{X,\beta,\chi} = \int_\beta[P^\bullet]_\chi 1 ∈ \mathbb{Z}

and is invariant along a deformation of X. P_{X,\beta,\chi} is called Pandharipande-Thomas invariant or PT-invariant.

One technique to compute PT-invariants is using the virtual localization formula by Graber and Pandharipande. If G = \mathbb{C}^* acts on \mathcal{P}_X(X, \beta) then \mathbb{L}_{\mathcal{P}_X(X, \beta)} has a natural equivariant structure. Let \mathcal{P}^G be the fixed locus of \mathcal{P}, then E^* has a sub-bundle (E^*|_{\mathcal{P}^G})^{fix} which has weight 0 and a sub-bundle (E^*|_{\mathcal{P}^G})^{mov} with non zero weight such that E^*|_{\mathcal{P}^G} = (E^*|_{\mathcal{P}^G})^{fix} \oplus (E^*|_{\mathcal{P}^G})^{mov}. Graber and Pandharipande showed that there exists a canonical morphism \phi : (E^*|_{\mathcal{P}^G})^{fix} → \mathcal{L}_{\mathcal{P}^G} that induces a perfect obstruction theory for \mathcal{P}^G. So that we have the virtual fundamental class \([\mathcal{P}^G]_{vir}\) of \mathcal{P}^G. Graber and Pandharipande gives a formula that relates \([\mathcal{P}^G]_{vir}\) with \([\mathcal{P}]_{vir}\) as follows:

\[ [\mathcal{P}]_{vir} = i_* \left( \frac{[\mathcal{P}^G]_{vir}}{e(N_{vir})} \right) ∈ A^*_G \otimes \mathbb{Q}[t, t^{-1}] \]

where e(N_{vir}) is the top Chern class of the vector bundle N_{vir} = ((E^*|_{\mathcal{P}^G})^{mov})^{\wedge}

and t is the first Chern class of the equivariant line bundle with weight 1.

Let S be a nonsingular projective surface with canonical bundle ω_S and let X be the total space of ω_S i.e. X = Spec(Sym(ω_S^\vee)). Then there is a closed embedding i of S into X as the zero section. Let \pi : X → S be the structure morphism. Since ω_X ≃ π^\ast ω_S ⊗ π^\ast ω_S^\vee ≃ \mathcal{O}_X, X is Calabi-Yau. Let X = \mathbb{P}(X ⊕ \mathcal{O}_X^N), then X is an open subscheme of \tilde{X} and let j : X → \tilde{X} be the inclusion and \tilde{π} : \tilde{X} → S be the structure morphism of \tilde{X} as a projective bundle over S. Since S is projective, \tilde{i} := j ∘ i : S → \tilde{X}

is a closed embedding.

Let \beta ∈ H_2(S, \mathbb{Z}) be an effective class and \chi ∈ \mathbb{Z}. By [10] there is a projective scheme \mathcal{P}_X(\tilde{X}, \tilde{i}_*\beta) parametrizing stable pairs (F, s) with χ(F) = \chi and the cycle [C_F] of the supporting curve is in class β. By removing the pairs (F, s) with supporting curve C_F which intersect the closed subscheme \tilde{X} \times X, we have an open subscheme \mathcal{P}_X(X, i_*\beta) that parametrize stable pairs (F, s) with \mathcal{F} supported on X and let \tilde{j} : \mathcal{P}_X(X, i_*\beta) → \mathcal{P}_X(\tilde{X}, \tilde{i}_*\beta) be the inclusion. Let \mathcal{F} be the universal sheaf on \mathcal{P}_X(\tilde{X}, \tilde{i}_*\beta) × \tilde{X} and \mathcal{S} : \mathcal{O}_{\mathcal{P}_X(\tilde{X}, \tilde{i}_*\beta) × \tilde{X}} → \mathcal{F} be the universal section, then their restriction F, S to \mathcal{P}_X(X, i_*\beta) × X is the universal sheaf and the universal section corresponding to the moduli space \mathcal{P}_X(X, i_*\beta). Notice that \((\text{id}_{\mathcal{P}_X(X, i_*\beta) × X})^*\mathcal{F} = \tilde{j} \times i^*\mathcal{F}\) on \mathcal{P}_X(X, i_*\beta) × X. We also use F to denote \((\text{id}_{\mathcal{P}_X(X, i_*\beta) × X})^*\mathcal{F}\) on \mathcal{P}_X(X, i_*\beta) × \tilde{X}.

There exists an action of G = \mathbb{C}^* on \tilde{X} by scaling the fiber such that X is an invariant open subscheme. It follows that there exist a canonical action of G on \mathcal{P}_X(X, i_*\beta) and on \mathcal{P}_X(X, i_*\beta). Since X is an invariant open subscheme, \mathcal{P}_X(X, i_*\beta) is also invariant in \mathcal{P}_X(\tilde{X}, \tilde{i}_*\beta). Thus \mathcal{F} and \mathcal{F} are equivariant sheaves and \mathcal{S} and \mathcal{S} are equivariant morphism of sheaves.
Consider the following diagrams

\[
\begin{array}{c}
\mathcal{P}_X(X, i_* \beta) \times X \\
p \downarrow \quad \downarrow q \\
\mathcal{P}_X(X, i_* \beta) \\
\end{array}
\]

(3.1)

Let \( \mathbb{I}^* \) be the complex \( \mathcal{O}_{\mathcal{P}_X(X, i_* \beta) \times X} \xrightarrow{\mathcal{S}} \mathbb{F} \) in \( D(\mathcal{P}_X(X, i_* \beta) \times X) \). Let \( \mathcal{E}^* \) be the complex

\[
Rp_* \left( \mathcal{R}Hom(\mathbb{I}^*, \mathbb{I}^*)_0 \otimes \omega_X \right) [2].
\]

Maulik, Pandharipande and Thomas has shown that the above complex define a perfect obstruction theory on \( \mathcal{P}_X(X, i_* \beta) \) \cite{12}.

Notice that \( \omega_X \cong \mathcal{O}_X \otimes t^* \) so that by Serre’s duality we have an isomorphism \( (\mathcal{E}^*)^\vee \rightarrow \mathcal{E}^*[-1] \otimes t \) and \( \mathcal{E} \) is a symmetric equivariant obstruction theory.

Let \( \mathcal{P}_X(S, \beta) \) be the scheme parameterizing stable pairs \( (\mathcal{F}, s) \) on \( S \) such that the support \( \text{Supp} \mathcal{F} \) of \( \mathcal{F} \) is in class \( \beta \) and \( \mathcal{F} \) has Euler characteristic \( \chi(\mathcal{F}) = \chi \). On \( \mathcal{P}_X(S, \beta) \times S \) there exists a universal sheaves \( F \) and universal section \( S \). With the closed embedding \( i := i|_{\mathcal{P}_X(S, \beta)} \times i: \mathcal{P}_X(S, \beta) \times S \rightarrow \mathcal{P}_X(S, \beta) \times X \),

\[
\mathcal{O}_{\mathcal{P}_X(S, \beta) \times X} \rightarrow \hat{i}_* \mathcal{O}_{\mathcal{P}_X(S, \beta) \times S} \xrightarrow{i_* \mathcal{S}} \hat{i}_* \mathbb{F}
\]

is a family of pairs over \( \mathcal{P}_X(S, \beta) \). This family induces a closed embedding \( \mathcal{P}_X(S, \beta) \rightarrow \mathcal{P}_X(X, i_* \beta) \). Indeed, \( \mathcal{P}_X(S, \beta) \) is a connected component of \( \mathcal{P}_X(X, i_* \beta)^G \).

Let \( \mathbb{I}^*_S \) denote the complex \( [\mathcal{O}_{\mathcal{P}_X(S, \beta) \times S} \rightarrow \mathbb{F}] \) and \( \mathbb{I}^* \) denotes the complex \( [\mathcal{O}_{\mathcal{P}_X(S, \beta) \times X} \rightarrow \hat{i}_* \mathbb{F}] \). Proposition 3.4 of \cite{12} gives us the decomposition of \( \mathbb{I}^*|_{\mathcal{P}_X(S, \beta)} \) into its fixed and moving part as follows:

\[
(\mathcal{E}^*|_{\mathcal{P}_X(S, \beta)})^{fix} \cong Rp_* \mathcal{R}Hom(\mathbb{I}^*_S, \mathbb{F})^\vee \\
(\mathcal{E}^*|_{\mathcal{P}_X(S, \beta)})^{mov} \cong Rp_* \mathcal{R}Hom(\mathbb{I}^*_S, \mathbb{F})[1] \otimes t^*
\]

We will use \( \mathcal{E}^* \) to denote \( (\mathcal{E}^*|_{\mathcal{P}_X(S, \beta)})^{fix} \). \( (\mathcal{E}^*|_{\mathcal{P}_X(S, \beta)})^{fix} \) defines a perfect obstruction theory on \( \mathcal{P}_X(S, \beta) \) with virtual dimension \( v = \beta^2 + n \).

If there is a deformation of \( S \) such that the class \( \beta \) is no longer algebraic, then the virtual fundamental class will be zero because the the virtual class is deformation invariant. If we restrict the deformation inside the locus when \( \beta \) is always algebraic we get the reduced obstruction theory. In \cite{12}, Kool and Thomas also construct a reduced obstruction theory on \( \mathcal{P}_X(X, i_* \beta) \) of virtual dimension \( h^{0,2}(\mathcal{S}) \).

Proposition 3.4 of \cite{12} gives us the decomposition of \( \mathcal{E}^*|_{\mathcal{P}_X(S, \beta)} \) into fixed part and moving part as follows:

\[
\left((\mathcal{E}^*|_{\mathcal{P}_X(S, \beta)})^{fix}\right)^\vee = \text{Cone} \left(R\hat{p}_* \mathcal{R}Hom(\mathbb{I}^*_S, \mathbb{F}) \xrightarrow{\psi} H^2(\mathcal{O}_S) \otimes \mathcal{O}_{\mathcal{P}_X(S, \beta)}[-1]\right)
\]

\[
\left((\mathcal{E}^*|_{\mathcal{P}_X(S, \beta)})^{mov}\right) = Rp_* R\mathcal{R}Hom(\mathbb{I}^*_S, \mathbb{F})[1] \otimes t
\]

where \( \psi \) is the composition

\[
R\hat{p}_* \mathcal{R}Hom(\mathbb{I}^*_S, \mathbb{F}) \xrightarrow{\text{tr}} R\hat{p}_* \mathcal{R}Hom(\mathbb{F}, \mathbb{F})[1] \xrightarrow{\text{tr}} R\hat{p}_* \mathcal{O}[1] \xrightarrow{R^2 \hat{p}_* \mathcal{O}[1]}
\]
We will use $\mathcal{E}_{\text{red}}^{\bullet}$ to denote \( \mathbb{E}_{\text{red}}^{\bullet}/\mathbb{P}_x(S, \beta) \). $\mathcal{E}_{\text{red}}^{\bullet}$ defines a perfect obstruction theory on $\mathbb{P}_x(S, \beta)$ of virtual dimension $v_{\text{red}} = \beta^2 + n + h^0(S)$.

For a cohomology classes $\sigma_i \in H^*(X, \mathbb{Z})$, $i = 1, \ldots, m$ Kool and Thomas assign a class $\tau(\sigma_i) := p_*(ch_2(F)^{q^*} \sigma_i) \in H^*(\mathbb{P}_x(X, i_\beta))$ where $ch_2(F)$ is the second Chern character of $\mathbb{F}$ and define the reduced invariants as

$$\mathcal{P}^{\text{red}}(X, \sigma_1, \ldots, \sigma_m) := \int_{\mathbb{P}_x(X, i_\beta)^{\gamma_{\text{vir}}}} \frac{1}{e(N_{\text{vir}})} \prod_{i=1}^m \tau(\sigma_i).$$

Assume that $b_1(S) = 0$ so that $\text{Hilb}_2 = \mathcal{L}$. It was shown that if for all $i$, $\sigma_i$ is the pullback of the Poincaré dual of the $(pt) \in H^2(S, \mathbb{Z})$ represented by a closed point then

$$\mathcal{P}^{\text{red}}(X, [pt]^m) = \int_{\mathbb{P}_x(X, i_\beta)^{\gamma_{\text{vir}}}} \frac{1}{e(N_{\text{vir}})}$$

where $j'$ is the refined Gysin homomorphism corresponding to the following cartesian diagram

$$\begin{align*}
\mathbb{P}^c & \times |\mathcal{L}| \mathbb{P}_x(X, i_\beta) \\
\mathbb{P}^c & \downarrow j \downarrow |\mathcal{L}|
\end{align*}$$

where $j$ is a regular embedding $\mathbb{P}^c \subset |\mathcal{L}|$ of a sublinear system and $e = \dim |\mathcal{L}| - m$.

### 3.1. $\delta$-nodal Curve Counting via Kool-Thomas Invariants.

Recall that a line bundle $\mathcal{L}$ on a surface $S$ is $n$-very ample if for any subscheme $Z$ with length $\leq n + 1$ the natural morphism $H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}_Z)$ is surjective.

We assume that $b_1(S) = 0$ and let $\mathcal{L}$ be $(2\delta + 1)$-very ample line bundle on $S$ with $H^1(\mathcal{L}) = 0$. We also assume that the first Chern class $c_1(\mathcal{L}) = \beta \in H^2(S, \mathbb{Z})$ of $\mathcal{L}$ satisfies the condition that the the morphism $\cup : H^1(T_S) \to H^2(\mathcal{O}_S)$ is surjective; in particular then $H^2(\mathcal{L}) = 0$ also. Given a curve $C$ not necessarily reduced and connected, we let $g(C)$ to denote its arithmetic genus, defined by $1 - g(C) = \chi(\mathcal{O}_C)$. If $C$ is reduced its geometric genus $g(C)$ is defined to be the $g(C)$ the genus of its normalization. And let $h$ denote the arithmetic genus of curves in $|\mathcal{L}|$, so that $2h - 2 = \beta^2 - c_1(S, \beta)$.

Proposition 2.1 of [11] and Proposition 5.1 of [12] tells us that the general $\delta$-dimensional linear system $\mathbb{P}^\delta \subset |\mathcal{L}|$ only contains reducible and irreducible curves. Moreover $\mathbb{P}^\delta$ contains finitely many $\delta$-nodal curves with geometric genus $h - \delta$ and other curves has geometric genus $\geq h - \delta$.

Kool and Thomas also define

$$\mathcal{P}^{\text{red}}(S, [pt]^m) := \int_{\mathbb{P}_x(S, \beta)^{\gamma_{\text{vir}}}} \frac{1}{e(N_{\text{vir}})} \tau([pt]^m).$$

They compute $\mathcal{P}^{\text{red}}(S, [pt]^m)$ in [13] and $\mathcal{P}^{\text{red}}(S, [pt]^m)$ is given by the following expression

$$\left(\frac{1}{t}\right)^{n+\chi(\mathcal{L})-1} \int S^{[1]}_{\overline{\mathcal{L}}^{[n]}(1)} c_n(\mathcal{L}^{[n]}(1)) t^{n+\chi(\mathcal{L})-1-m} c_n(\mathcal{L}^{[n]}(1))$$

where $\mathcal{L}^{[n]}$ is the vector bundle of rank $n$ on $S^{[n]}$ with fiber $H^0(\mathcal{L}_Z)$ for a point $Z \in S^{[n]}$ and $\mathcal{L}^{[n]}(1) = \mathcal{L}^{[n]} \otimes \mathcal{O}(1)$.
Under the above assumption, only the contribution from \( P_\chi(S, \beta) \) counts for \( P_{\chi, \beta}^{red}(X, [pt]^m) \), so \( P_{\chi, \beta}^{red}(X, [pt]^m) = P_{\chi, \beta}^{red}(S, [pt]^m) \). Define the generating function for \( P_{\chi, \beta}^{red}(X, [pt]^m) \) as

\[
\sum_{\chi \in \mathbb{Z}} P_{\chi, \beta}^{red}(X, [pt]^m)q^\chi
\]

then define \( \overline{q} = q^{1-i}(1 + q)^{2i-2} \) then the coefficient of \( \overline{q}^{h-\delta} \) is \( n_\delta(L)i^{\delta-h}c_1(S) \) where \( n_\delta(L) \) is the number of \( \delta \)-nodal curves in \( \mathbb{P}^\delta \).

\( n_\delta(L) \) has been studied for example in [7] and [11]. In [11], it is shown that after the same change of variable \( n_\delta(L) \) can be computed as the coefficient of \( \overline{q}^{h-\delta} \) of the generating function

\[
\sum_{\chi \in \mathbb{Z}} e(\text{Hilb}^n(C/P^\delta))q^{i+1-h}
\]

where \( e(\text{Hilb}^n(C/P^\delta)) \) is the Euler characteristic of the relative Hilbert scheme of points. Moreover \( e(\text{Hilb}^n(C/P^\delta)) \) can be computed as

\[
\int_{S[i_1] \times \mathbb{P}^\delta} c_i(L^{[n]}(1))c_\bullet(T_{S[i_1]}(\mathcal{O}(1)^{\otimes \delta+1}))c_\bullet(L^{[n]}(1)).
\]

In [11], we have to assume that \( L \) is sufficiently ample and \( H^i(L) = 0 \) for \( i > 0 \) so that \( \text{Hilb}^n(C/P^\delta) \) are smooth. While in [12], \( P_{\chi, \beta}^{red}(S, [pt]^m) \) can be defined under the assumption that \( H^2(L) = 0 \) for all \( L \) with \( c_1(L) = 0 \). We can think \( n_\delta(L) \) as a generalization of the one studied in [11]. In particular, we can think \( n_\delta(L) \) as a virtual count of \( \delta \)-nodal curves for not necessarily ample line bundle \( L \).

4. Equivariant K-theoretic PT invariants of local surfaces

In this section we will recall the K-theoretic invariants proposed by Nekrasov and Okounkov in [10] and introduce a class that will account for the incidence of the supporting curve of a stable pairs and a point. The definition of this class is motivated by the definition of points insertions in [12].

Given a perfect obstruction theory \( \phi : E_\bullet \to \mathbb{L}_Y \) the K-theoretic version of the virtual class is given in [9] as follows:

\[
\mathcal{O}_Y^{vir} := \sum_{i} (-1)^i [\text{Tor}^i_{\mathcal{O}_{E_1}}(\mathcal{O}_Y, \mathcal{O}_D)]_{Y} \in G(Y)
\]

where \( D \) is the cone \( D \subset E_1 \) that gives the virtual class \([Y]^{vir}\). We call \( \mathcal{O}_Y^{vir} \) the virtual structure sheaf of \( Y \). Note that \( \mathcal{O}_Y^{vir} \) is not a sheaf but a class in the Grothendieck group of coherent sheaves on \( Y \). If \( \phi \) is an equivariant perfect deformation theory, \( D \) is an invariant subscheme of \( E_1 \) and we can construct \( \mathcal{O}_Y^{vir} \in \mathcal{C}^G(Y) \). If \( Y \) is proper over \( \mathbb{C} \), the virtual fundamental class and virtual structure sheaf are related by the following virtual Riemann-Roch formula by Fantechi and Göttsche in [10]

\[
(4.1) \quad \chi(\mathcal{O}_Y^{vir}) = \int_{[Y]^{vir}} \text{td}(T_Y^{vir})
\]

where \( T_Y^{vir} := [E_0] - [E_1] \in K(Y) \). We call \( T_Y^{vir} \) the virtual tangent bundle and the dual of it’s determinant \( K_{Y,vir} := (\det E_0)^{-1} \otimes \det E_1 = \det E_0 \otimes (\det E_1)^{-1} \in \text{Pic}(Y) \) the virtual canonical bundle.
If $vd = 0$, by equation (1.1) we have
\begin{equation}
\chi(\mathcal{O}_{Y}^{\text{vir}}) = \int_{[Y]^{\text{vir}}} 1 \in \mathbb{Z}
\end{equation}
so that we can use either virtual structure sheaf or virtual fundamental class to define a numerical invariant. If there exist an isomorphism $\theta : E^{*} \rightarrow (E^{*})^{\vee}$ [1] then $\text{rk}E^{*} = \text{rk}((E^{*})^{\vee}) = -\text{rk}E^{*}$ so that $vd = 0$.

One advantage of working equivariantly is that to compute $\chi(\mathcal{O}_{Y}^{\text{vir}})$, we can use the virtual localization formula for the Grothendieck group of coherent sheaves from [19] by Qu. Let $G = \mathbb{C}^{\times}$ act on $Y$ and $\phi : \mathbb{E}^{*} \rightarrow L_{Y}$ be an equivariant perfect obstruction theory. Similar to the virtual localization formula by Graber and Pandaripande, it states that, the virtual structure sheaf equals a class coming from the fixed locus. On $Y^{G}$ we can decompose $\mathbb{E}^{*}$ into $(\mathbb{E}^{*})^{\text{fix}} \oplus (\mathbb{E}^{*})^{\text{mov}}$ where $(\mathbb{E}^{*})^{\text{fix}}$ is a two term complex with zero weight and $(\mathbb{E}^{*})^{\text{mov}}$ is a two term complex with non zero weight. Let $i : Y^{G} \rightarrow Y$ be the closed embedding and let $N^{\text{vir}} = ((\mathbb{E}^{*})^{\text{mov}})^{\vee}$. Then the virtual localization formula can be stated as
\begin{equation}
i_{*} \left( \frac{\mathcal{O}_{Y}^{\text{vir}}}{\Lambda^{*}(N^{\text{vir}})^{\vee}} \right) = \mathcal{O}_{Y}^{\text{vir}} \in G^{G}(Y) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}(t)
\end{equation}
where for a two term complex $F^{*} = [F^{-1} \rightarrow F^{0}]$, $\Lambda^{*}F^{*} = \sum_{i=0}^{\infty} (-1)^{i} \Lambda^{i}F^{*}$ with $r_{i} = \text{rk}F^{-i}$. On the fixed locus, the Grothendieck group of coherent sheaves is isomorphic to the tensor product $G(Y^{G}) \otimes_{\mathbb{Z}} K^{G}(pt)$ which is easier to work with.

In [15], Nekrasov and Okounkov propose that we should choose a square root of $K^{\text{vir}}$ and work with the twisted virtual structure sheaf [20]
\[
\mathcal{O}_{Y}^{\text{vir}} := K_{Y,vir}^{\hat{t}} \otimes \mathcal{O}_{Y}^{\text{vir}}.
\]
To get a refinement of (1.2), we have to consider the action of the symmetry group of $Y$ so that $\chi(\mathcal{O}_{Y}^{\text{vir}})$ is a function with the equivariant parameter as variables. For example let $Y$ be the moduli space of stable pairs on a toric 3-folds $X$ and $(\mathbb{C}^{\times})^{3}$ acts on $Y$. Choi, Katz and Klemm have calculated $\chi(\mathcal{O}_{Y}^{\text{vir}})$ where $X$ is the total space of the canonical bundle $K_{S}$ for $S = \mathbb{P}^{2}$ and $S = \mathbb{P}^{1} \times \mathbb{P}^{1}$ in [2]. They have shown that the generating function with coefficients $\chi(\mathcal{O}_{Y}^{\text{vir}})$ calculates a refinement of BPS invariants.

To incorporate $K_{Y,vir}^{\hat{t}}$ in our computation we will consider a double cover $G'$ of $G$ so that $t^{\frac{1}{2}}$ is a representation of $G'$. Explicitly let $\zeta : G' := \mathbb{C}^{*} \rightarrow \mathbb{C}^{*} \rtimes G$, $z \mapsto z^{2}$ be the double cover. Then $G'$ acts on $Y$ via $\zeta$ by defining $\sigma_{Y^{G'}} : G' \times Y \rightarrow Y$, $(g, y) \mapsto \sigma_{Y}(\zeta(g^{2})), y$ where $\sigma : G \times Y \rightarrow Y$ is the morphism defining the action of $G$ on $Y$. Also via $\zeta$ any $G$-equivariant sheaf $\mathcal{F}$ on $Y$ is a $G'$-equivariant sheaf by pulling back the equivariant structure via $\zeta$. This gives an exact functor $\text{Coh}^{G}(Y) \rightarrow \text{Coh}^{G'}(Y)$ and a group homomorphism $\zeta : G^{G}(Y) \rightarrow G^{G'}(Y)$. Moreover $\zeta$ is a morphism of $K^{G}(pt)$-modules. For example, the primitive representation $t$ of $G$ has weight 2 at $G'$ module. We can take the primitive representation of $G'$ as the canonical square root of $t$ and denote it by $t^{\frac{1}{2}}$.

Next we have to compute the restriction of $K_{Y,vir}^{\hat{t}}$ on the fixed locus. Notice that $Y^{G'} = Y^{G}$. Assume that there exist an isomorphism $\theta : E^{*} \rightarrow (E^{*})^{\vee}$ [1] $\otimes_{t}$. By the
argument of Richard Thomas in [22], it shows that on $Y^G$, $K_{Y,\text{vir}}^{1/2}$ has a canonical equivariant structure.

We decompose $E^*|_{Y^G}$ into its weight spaces so that

$$E^*|_{Y^G} = \bigoplus_{i \in \mathbb{Z}} F^i t^i$$

where $F^i$ are two-term complex of non-equivariant vector bundle which only finitely many of them are nonzero and $t$ is a representation of $G$ of weight 1. $E^*$ can be computed as the determinant of its class in $K^G(Y)$. The isomorphism $\theta$ implies that $[[F^i]] = [F^{-i-1}[-1]]$ in $K^G(Y)$. Thus $K_{Y,\text{vir}}$ is a square twisted by a power of $t$, explicitly

$$K_{Y,\text{vir}} = \bigotimes_{i \geq 0} \det(F^i t^i) \otimes t^{r_0 + r_1 + ...}$$

where $r_i = \text{rk} F^i$. Thus the canonical choice for $K_{Y,\text{vir}}|_{Y^G}$ is

$$\bigotimes_{i \geq 0} \det(F^i t^i) \otimes t^{(r_0 + r_1 + ...)} \in K^G(Y^G) \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z}[t^\frac{1}{2}, t^{-\frac{1}{2}}].$$

Recall that $N^{\text{vir}}$ is the moving part of the dual of $E^*|_{Y^G}$ so that in our case $(N^{\text{vir}})^\vee = \bigotimes_{i \geq 0} F^i t^i$.

After choosing a square root of $K_{Y,\text{vir}}$, and that the square root has an equivariant structure, by equation (4.3) we then have

$$i_* \left( \Omega^*_{Y^G} \otimes K_{Y,\text{vir}}^{1/2} \big|_{Y^G} \right)^\vee = \mathcal{O}^\text{vir}_Y \in K^G(Y) \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Q}(t^\frac{1}{2})$$

If $Y$ is compact we can apply the right derived functor $R\Gamma$ to both sides of the above equation and we have

$$R\Gamma \left( Y^G, \Omega^*_{Y^G} \otimes K_{Y,\text{vir}}^{1/2} \big|_{Y^G} \right)^\vee = R\Gamma \left( Y, \mathcal{O}^\text{vir}_Y \right) \in \mathbb{Q}(t^\frac{1}{2}).$$

Thomas has proved the above identity in without using equation (4.3). Furthermore Thomas has shown that

$$R\Gamma \left( Y^G, \Omega^*_{Y^G} \otimes K_{Y,\text{vir}}^{1/2} \big|_{Y^G} \right)^\vee_{t=1} = \int_{[Y]^{\text{vir}}} \frac{1}{e((N^{\text{vir}})^\vee)} \in \mathbb{Q}$$

In the case that we are interested on, the moduli space $Y$ is not compact. Thus we will use the left hand side of equation (4.3) to define our invariants.

### 4.1. Equivariant $K$-theoretic invariants.

Let $Y$ be the moduli space of stable pairs on the canonical bundle $X := \text{Spec} (\text{Sym} \omega_X^2)$ of a smooth projective surface i.e. $Y = \mathcal{P}_z(X, i, \beta)$ for some $z \in \mathbb{Z}$ and $\beta \in H_2(S, \mathbb{Z})$ where $i : S \to X$ is the zero section. We will use $\pi$ to denote the structure map $X \to S$ of $X$ as a vector bundle over $S$. Note that $\mathcal{P}_z(X, i, \beta)$ is a quasiprojective scheme over $\mathbb{C}$. In particular, $\mathcal{P}_z(X, i, \beta)$ is separated and of finite type.

Let $G = \mathbb{C}^*$ act on $X$ by scaling the fiber of $\pi$. Consider the following diagram:
A REFINEMENT OF KOOL-THOMAS INVARIANTS VIA EQUIVARIANT K-THEORETIC INVARIANT

Since $\mathcal{P}_X(X, i_*, \beta)$ has an equivariant perfect obstruction theory $\phi: \mathbf{E}^* \to \mathbb{L}_{\mathcal{P}_X(X, i_*, \beta)}$ where $\mathbf{E}^*$ is the complex $RP_* (\mathbf{R} \text{Hom}(I^*, I^*)_0 \otimes \omega_p)[2]$ with $\omega_p = q^* \omega_X$ and since $X$ is Calabi-Yau $\omega_X \cong \mathcal{O} \otimes t^*$ Serre duality gives us the isomorphism

$$ (\mathbf{E}^*)^\ast \cong \mathbf{E}^*[-1] \otimes t. $$

So that by Proposition 2.6 of [22] we have an equivariant line bundle $K^\pm_{\mathcal{P}_X(X, i_*, \beta), \text{vir}}$ on $\mathcal{P}_X(X, i_*, \beta)$. We want to study how to define a class that contains the information about the incidence between a $K$-theory class in $K^T(X)$ and the class of the universal sheaf $\mathcal{F}$. From another direction we also want to give a refinement for the Kool-Thomas invariants. In [12], Kool and Thomas take the cup product of the second Chern character of the universal sheaf $\mathcal{F}$ with the cohomology class of points coming from $X$. Informally we could think that as taking the intersection between the universal supporting curve and the points of $X$.

In this article we are exploring two approaches. In the first approach we are trying to imitate the definition of descendents used in the article [12]. In [12] the authors are cupping the cohomology class coming from $X$ with the second Chern class of $\mathcal{F}$. Since we are unfamiliar on how to define Chern classes as a $K$-theory class, we are considering to take the class of the structure sheaf of the supporting scheme $\mathcal{O}_{C_Y}$ and take the tensor product of $\mathcal{O}_{C_Y}$ with the the class coming from $X$ through the projection $q: \mathcal{P}_X(X, i_*, \beta) \times X \to X$. In the second approach we use the $K$-theory class on $\mathcal{P}_X(X, i_*, \beta) \times S$ of the structure sheaf of the divisor $\text{div} \pi_\ast \mathcal{F}$ and take the tensor product of $\mathcal{O}_{\text{div} \pi_\ast \mathcal{F}}$ with the class coming from $S$ through the projection $q_S: \mathcal{P}_X(X, i_*, \beta) \times S \to S$.

The following proposition is an equivariant version of Proposition 2.1.0 in [10] which we will use to define the $K$-theory class.

**Proposition 11.** Let $f: Y \to T$ be a smooth projective $G$-map of relative dimension $n$ with $G$-equivariant $f$-very ample line bundle $\mathcal{O}_Y(1)$. Let $\mathcal{F}$ be a $G$-equivariant sheaf flat over $T$. Then there is a resolution of $\mathcal{F}$ by a bounded complex of $G$-equivariant locally free sheaves:

$$ 0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \ldots \to \mathcal{F}_0 \to \mathcal{F} $$

where all morphisms are $G$-equivariant such that $R^i f_* \mathcal{F}_\nu$ is locally free for $\nu = 0, \ldots, n$ and $R^i f_* \mathcal{F}_\nu = 0$ for $i \not= n$ and $\nu = 0, \ldots, n$.

**Proof.** The equivariant structure of all sheaves constructed in the proof of Proposition 2.1.10 in [10] can be defined canonically. $\square$

If $\mathcal{O}_C$ is flat over $\mathcal{P}_X(X, i_*, \beta)$ then $\mathcal{O}_C$ define a $K$-theory class in $\mathcal{P}_X(X, i_*, \beta) \times X$. To push the tensor product down to a $K$-theory class in $\mathcal{P}_X(X, i_*, \beta)$, we push forward $\mathcal{O}_C$ to $\mathcal{P}_X(X, i_*, \beta) \times X$ where $X$ is $\mathbb{P}(K_S \otimes \mathcal{O}_S)$ the projective completion of $X$. Since $\mathcal{O}_F$ is proper relative to $\mathcal{P}_X(S, \beta)$ the push forward $i_* \mathcal{O}_C$ by the open
embedding $i : \mathcal{P}_X(X, \iota_s) \times X \to \mathcal{P}_X(X, i_s) \times \bar{X}$. Then Proposition \[\ref{eq:1}\] implies that $\mathcal{O}_{\mathcal{C}_0}$ has a resolution by a finite complex of locally free sheaf $F^\bullet$ on $\mathcal{P}_X(X, i_s) \times \bar{X}$ so that we can take $[\mathcal{O}_{\mathcal{C}_0}] := \sum_i (-1)^i [F^i]$.

The class $[\mathcal{O}_{\mathcal{C}_0}]$ is independent of the resolution.

In section \[\ref{sec:2}\] we have described the ring homomorphism $f^* : K^G(Y) \to K^G(Y)$ for any morphism of schemes $f : Y \to \bar{Y}$. We also described the group homomorphism $f_* : K^G(Y) \to K^G(Y)$ when $f$ is the structure morphism of a projective bundle or when $f$ is finite and $f_* \mathcal{F}$ has a resolution by locally free sheaves.

Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{P}_X(X, i_s) \times \bar{X} & \xrightarrow{\tilde{\pi}} & \bar{X} \\
\downarrow{\tilde{\pi}} & & \downarrow{\bar{\pi}} \\
\mathcal{P}_X(X, i_s) & \xrightarrow{\bar{\pi}} & \bar{X}
\end{array}
\]

(4.7)

Let $\bar{\pi} : \bar{X} \to S$ be the structure morphism of $\bar{X}$ as a projective bundle over $S$. We assign for each class $\alpha \in K^T(X)$ a class $\gamma(\alpha)$ in $K^T(\mathcal{P}_X(X, i_s))$ as follows. The pullback map $\pi^* : K^T(S) \to K^T(X)$ is an isomorphism. Thus there exist a unique class $\alpha \in K^T(S)$ such that $\pi^* \tilde{\pi} = \alpha$. We define $\gamma(\alpha) := \bar{\pi}_* \left( [\mathcal{O}_{\mathcal{C}_0}] \cdot [\tilde{\pi}^* \pi^* \tilde{\pi}^* \tilde{\pi}] \right)$. By Proposition \[\ref{prop:1}\] $[\mathcal{O}_{\mathcal{C}_0}] \in K^T(\mathcal{P}_X(X, i_s) \times \bar{X})$ and since $\bar{X}$ is smooth and projective over $\mathbb{C}$, $\bar{\pi}_*$ can be defined as the composition of $i_*$ and $r_*$ where $i$ is a regular embedding and $r$ is the structure morphism $\mathbb{P}^n_{\mathcal{P}_X(X, i_s)} \to \mathcal{P}_X(X, i_s)$. Thus the class $\gamma(\alpha)$ is well defined. In particular for every subscheme $Z \subset X$, $\gamma(\mathcal{O}_Z)$ is an element in $K^T(\mathcal{P}_X(X, i_s))$.

For the second approach, $\text{div} \pi_* \mathcal{F}$ is a Cartier divisor on $\mathcal{P}_X(X, i_s) \times S$ so that we have a line bundle $\mathcal{O}(\text{div} \pi_* \mathcal{F})$ and exact sequence

\[
\begin{array}{c}
0 \longrightarrow \mathcal{O}(-\text{div} \pi_* \mathcal{F}) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\text{div} \pi_* \mathcal{F}} \longrightarrow 0.
\end{array}
\]

Thus the $K$-theory class of $\mathcal{O}_{\text{div} \pi_* \mathcal{F}}$ is $1 - [\mathcal{O}(\text{div} \pi_* \mathcal{F})]$.

Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{P}_X(X, i_s) \times S & \xrightarrow{q_S} & S \\
\downarrow{\tilde{\pi}} & & \downarrow{\bar{\pi}} \\
\mathcal{P}_X(X, i_s) & \xrightarrow{\bar{\pi}} & \bar{X}
\end{array}
\]

(4.8)

Similar to the first approach we assign for each $\alpha \in K^T(X)$ the class $\check{\gamma}(\alpha) := \tilde{\pi}_* \left( [\mathcal{O}_{\text{div} \pi_* \mathcal{F}}] \cdot q_S^* \tilde{\alpha} \right)$.

In this article we only working for the case when $\alpha$ is represented by the class of the pullback of a closed point $s \in S$. Instead of $\gamma(\pi^* [\mathcal{O}_s])$ we will use $\gamma([\mathcal{O}_s])$ to denote this class. We also assume that $b_1(S) = 0$ so that $\text{Hilb}_\beta$ is simply $|L|$ for a line bundle $L$ on $S$ with $c_1(L) = \beta$. In this article, we want to study the following invariants

\[
R\Gamma \left( \mathcal{P}_G^G, \frac{\mathcal{O}_{\text{vir}}^G}{\Lambda^\bullet (N^\text{vir})^\vee} \otimes K_{\mathcal{P}_G, \text{vir}}^+ \otimes \prod_{i=1}^m \beta_i \Big|_{\mathcal{P}_G} \right) \in \mathbb{Q}(t^\bullet)
\]

(4.9)

where $\beta_i$ is either $\gamma(\mathcal{O}_s_i)$ or $\check{\gamma}(\mathcal{O}_s_i)$ with $\mathcal{O}_s_i$ are the classes of the structure sheaves of closed points $s_i \in S$. In a special case that we have worked out in this article in
order to make the invariant coincide with Kool-Thomas invariant when we evaluate it at \( t = 1 \) we have to replace \( \gamma(\mathcal{O}_s) \) by \( \frac{\gamma(\mathcal{O}_s)}{t^{n_s - n_s - 1}} \) and \( \bar{\gamma}(\mathcal{O}_s) \) with \( \frac{\bar{\gamma}(\mathcal{O}_s)}{t^{n_s - n_s - 1}} \). Thus we define the following invariants

\[
P_{X,\beta,\chi}(s_1, \ldots, s_m) := R\Gamma \left( \mathcal{P}, \frac{\mathcal{O}_{\mathcal{P}}^{vir}}{\Lambda^* (N_{vir})^*} \otimes K_{\mathcal{P},vir}^{1/2} \otimes \prod_{i=1}^m \frac{\gamma(\mathcal{O}_{s_i})}{t^{n_s - n_s - 1}} \right)
\]

when \( \mathcal{O}_s \) is flat and

\[
\bar{P}_{X,\beta,\chi}(s_1, \ldots, s_m) := R\Gamma \left( \mathcal{P}, \frac{\mathcal{O}_{\mathcal{P}}^{vir}}{\Lambda^* (N_{vir})^*} \otimes K_{\mathcal{P},vir}^{1/2} \otimes \prod_{i=1}^m \frac{\bar{\gamma}(\mathcal{O}_{s_i})}{t^{n_s - n_s - 1}} \right)
\]

4.2. Vanishing of the contribution of pairs supported on thickening of \( S \) in \( X \).

In this subsection we will prove that under the assumption that all curve that pass through all the \( m \) points are reduced and irreducible, the contribution to the invariants \( P_{X,\beta,\chi}(s_1, \ldots, s_m) \) and \( \bar{P}_{X,\beta,\chi}(s_1, \ldots, s_m) \) of curves not supported on \( S \) is zero.

Proposition 2.1 of \([11]\) tells us that if \( \mathcal{L} \) is a \( 2\delta + 1 \)-very ample line bundle on \( S \) then the \( \delta \)-dimensional general sublinear system \( \mathbb{P}^\delta \subset |\mathcal{L}| \) only contain reduced curves. Proposition 5.1 of \([12]\) also implies that these curves are also irreducible. Thus our assumption that all curves passing through all \( m \) points are reduced and irreducible is more likely to happen. If for all \( s_i \), \( \mathcal{O}_{s_i} \) are in the same class, our assumption does not depend on a particular set of \( s_i \) but only on the number of points.

First we work for \( P_{X,\beta,\chi}(s_1, \ldots, s_m) \).

Let \( \bar{\pi}^P : \mathcal{P}_X(X, i_*, \beta) \times \bar{X} \to \mathcal{P}_X(X, i_*, \beta) \times S \) be the pullback of \( \bar{\pi} \) and let \( i : C \to \mathcal{P}_X(X, i_*, \beta) \times X \) be the closed embedding of the universal curve. As the composition of projective morphisms is projective then the composition \( \bar{\pi}^P \circ i \) is also projective. Notice the above composition equals to the composition \( C \to \mathcal{P}_X(X, i_*, \beta) \times X \to \mathcal{P}_X(X, i_*, \beta) \times S \) which is affine. Thus we can conclude that \( \bar{\pi}^P \circ i \) is a finite morphism. We denote this morphism by \( \rho \).

Recall the morphism \( \text{div} : \mathcal{P}_X(X, i_*, \beta) \to |\mathcal{L}| \) from section \([3]\) that maps the stable pairs \( (\mathcal{F}, s) \) to the supporting curve \( C_x \in |\mathcal{L}| \) of \( \mathcal{F} \). Let \( D \subset |\mathcal{L}| \times S \) be the universal divisor and let \( \mathcal{D}_P \subset \mathcal{P} \times S \) be the family of divisors that correspond to the morphism \( \text{div} : \mathcal{P}_X(X, i_*, \beta) \to |\mathcal{L}| \) and let \( j : \mathcal{D}_P \to \mathcal{P} \times S \) be the closed embedding. Equivalently \( \mathcal{D}_P = \text{div}^{-1} \mathcal{D} \).

**Lemma 12.** \( \rho \) factors through \( j \).

**Proof.** The ideal \( I \) in \( \mathcal{O}_{\mathcal{P}_X(X, i_*, \beta)} \times S \) corresponding to the divisor \( \mathcal{D}_P \) is flat over \( \mathcal{P}_X(X, i_*, \beta) \) and \( \rho \) factorize through \( j \) if the composition \( I \to \mathcal{O}_{\mathcal{P}_X(X, i_*, \beta)} \times S \to \rho_* \mathcal{O}_C \) is zero. By Nakayama’s Lemma it is sufficient to check whether the composition is zero for each \( p \in \mathcal{P}_X(X, i_*, \beta) \). Or equivalently, we can check whether \( \rho \) factorize through \( j \) at each point \( p \in \mathcal{P}_X(X, i_*, \beta) \).

Let \( \rho^P : \mathcal{C}_p \to \{p\} \times S = S \) be the restriction of \( \rho \) to the point \( p \in \mathcal{P}_X(X, i_*, \beta) \) and let \( W \subset S \) be the scheme theoretic support of \( \rho^P_* \mathcal{O}_{\mathcal{C}_p} \). Notice that \( |W| = \text{Supp}(\rho_* \mathcal{O}_{\mathcal{C}_p}) \) is a curve. We claim that \( W \) is a Cartier Divisor. We will show that \( W \) is a subscheme of \( \text{div} \mathcal{F} = \text{div} \rho_* \mathcal{O}_{\mathcal{C}_p} \) so that \( \rho^P \) factorize through \( j \). Let \( \sigma : \mathcal{O}_S \to \rho^P_* \mathcal{O}_{\mathcal{C}_p} \) be the morphism of sheaves corresponding to the morphism \( \rho^P : \mathcal{C}_p \to S \). Then \( \mathcal{O}_W \) is the image of \( \sigma \) so that we have an injection \( \mathcal{O}_W \to \rho^P_* \mathcal{O}_{\mathcal{C}_p} \to \rho^P_* \mathcal{F}_p \). By Proposition 1 of \([6]\) we have \( \text{div} \rho^P_* \mathcal{F}_p = \text{div} \mathcal{O}_W + D \) where \( D \) is some effective divisor. Since \( W \) is a
Cartier divisor then \( \text{div} \mathcal{O}_W = W \). So we can conclude that \( W \) is a subscheme of \( \text{div} \mathcal{F} \).

It remains to show that \( W \) is a Cartier divisor. Let \( I \subset \mathcal{O}_S \) be the ideal sheaf of \( W \). It is sufficient to show that \( I_x \) is a free \( \mathcal{O}_{S,x} \)-module of rank 1 for every \( x \in X \). For \( U = S \setminus W \), the inclusion \( I \subset \mathcal{O}_S \) is an isomorphism so that if \( x \notin W \), \( I_x \) is isomorphic to \( \mathcal{O}_{S,x} \). Since \( S \) is nonsingular, \( \mathcal{O}_{S,x} \) is a domain so that it is sufficient to show that \( I_x \) is generated by one element \( f \in \mathcal{O}_{S,x} \).

By Proposition 11 the subcategory of flat coherent sheaves on \( D \) satisfies all conditions in Lemma 2 so that by Corollary 3 we have a group homomorphism

\[
\rho^*: \mathcal{O}_C \to \mathcal{O}_S \text{ isomorphic to a submodule of } R \quad \text{denote the inclusion}
\]

\[
\rho^* \mathcal{O}_C \text{ is a Cohen-Macaulay } \mathcal{O}_{S,x}\text{-module. By Proposition IV.11 of } [21], \text{ any prime } p \subset \mathcal{O}_{S,x} \text{ such that } \mathcal{O}_{S,x}/p \text{ is isomorphic to a submodule of } (\rho^* \mathcal{O}_C)_x \text{ must be generated by a single irreducible element } g \in \mathcal{O}_{S,x}. \text{ There are finitely many of such } p \text{ and we denote them by } p_1, \ldots, p_k. \text{ Let } g_i \text{ generate } p_i. \text{ By Proposition IV.11 of } [21], \text{ } I_x \text{ is the intersection } \bigcap_{i=1}^k q_i \text{ where } q_i \text{ is an ideal of } \mathcal{O}_{S,x} \text{ such that } p_i^{n_i} \subset q_i \subset p_i \text{ for some positive integer } n_i. \text{ Since } \mathcal{O}_{S,x} \text{ is a domain, } q_i \text{ must be generated by a single element } g_i^{m_i} \text{ for some positive integer } m_i. \text{ Thus we conclude that } I_x \text{ is generated by a single element } \prod_{i=1}^k g_i^{m_i}. \quad \square

Let \( R \subset \mathcal{P}_X(X, i_* \beta)^G \) be a connected component different from \( \mathcal{P}_X(S, \beta) \). We denote the inclusion \( R \subset \mathcal{P}_X(X, i_* \beta) \) by \( \iota \). For every \( (F, s) \in R \) the supporting curve \( C \subset X \) is not supported by \( S \) but \( F \) is supported on an infinitesimal thickening of \( S \) in \( X \). So we have the following diagram where all square are Cartesian

\[
\begin{array}{ccc}
C_R & \longrightarrow & C \\
\downarrow \iota^R & & \downarrow \\
R \times \tilde{X} & \longrightarrow & \mathcal{P}_X(X, i_* \beta) \times \tilde{X} \\
\downarrow \pi^R & & \downarrow \pi \\
R \times S & \longrightarrow & \mathcal{P}_X(X, i_* \beta) \times S \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
R & \longrightarrow & \mathcal{P}_X(X, i_* \beta)
\end{array}
\]

(4.10)

By base change formula [5] and projection formula [4] we have

\[
\iota^* \gamma(\mathcal{O}_S) = (\tilde{\beta}^R \circ \pi^R)_* (\iota_X^* \mathcal{O}_C \cup \tilde{q}_* \tilde{\pi}^* \mathcal{O}_S)
\]

\[
= \tilde{\beta}_{X*} (\tilde{\pi}^R_\mathcal{O}_C \cup \tilde{q}_* \tilde{\pi}^* \mathcal{O}_S).
\]

(4.11)

Now we restrict \( \rho \) from [12] to \( R \subset \mathcal{P}_X(X, i_* \beta) \). By the Lemma [12] we can write \( \rho^R \) as the composition \( j^R \circ \lambda^R \). So now we have the following diagram

\[
\begin{array}{ccc}
\mathcal{C}_R & \longrightarrow & \mathcal{D}_R \\
\downarrow \lambda^R & & \downarrow \tilde{\beta}^R \\
R \times S & \longrightarrow & S \\
\downarrow \tilde{\pi} & & \downarrow \\
R & \longrightarrow & S
\end{array}
\]

By Proposition [11] the subcategory of flat coherent sheaves on \( \mathcal{D}_R \) satisfies all conditions in Lemma [2] so that by Corollary [3] we have a group homomorphism
λ^R : K^G(S_R) \rightarrow K^G(D_R) that maps [F] to χ(λ^R[F]). By the same argument we can conclude the existence of the group homomorphism j^R : K^G(D_R) \rightarrow K^G(R \times S).

Recall the definition of the ring homomorphism κ : K^G(Y) \rightarrow lim K(Y) from Section 2. Although we have not proved that π^R \circ i^R_*[\mathcal{O}_C] = j^R_* \circ λ^R_*[\mathcal{O}_C], by Lemma 9 we still have κ_{R \times S} \circ π^R_* \circ i^R_* = κ_{R \times S} \circ j^R_* \circ λ^R_*.

Lemma 13.

\[ \kappa_R(\gamma(\mathcal{O}_s)|_R) := \kappa_R(\tilde{p}_R^* (\pi^R_* \circ i^R_*[\mathcal{O}_{C_R}] \otimes \iota_2^* \omega_2^* [\mathcal{O}_s])) \]

\[ = \kappa_R(\tilde{p}_R^* ((j^R_* \circ λ^R_*[\mathcal{O}_C]) \otimes \iota_2^* \omega_2^* [\mathcal{O}_s])). \]

We will use \hat{\gamma}(\mathcal{O}_s)|_R to denote \hat{p}_R^* ((j^R_* \circ λ^R_*[\mathcal{O}_{C_R}] ) \otimes \iota_2^* \omega_2^* [\mathcal{O}_s]) and [\mathcal{O}_{C_R}] to denote \lambda^R_*[\mathcal{O}_{C_R}].

Lemma 14.

\[ RT\left(R, \frac{\mathcal{O}^{vir}_R \otimes K^\frac{1}{2} \otimes R}{\Lambda^* (N^{vir}_R)^{\otimes R}} \bigotimes_{i=1}^m \gamma(\mathcal{O}_s)|_R \right) = RT\left(R, \frac{\mathcal{O}^{vir}_R \otimes K^\frac{1}{2}}{\Lambda^* (N^{vir}_R)^{\otimes R}} \bigotimes_{i=1}^m \hat{\gamma}(\mathcal{O}_s)|_R \right). \]

Proof. The Chern character map ch^G : \mathbb{Q}(t^\frac{1}{2}) \rightarrow \mathbb{Q}((t)) is an injection since \epsilon \rightarrow e^{\frac{1}{2} \epsilon} is invertible in \mathbb{Q}((t)). By virtual Riemann-Roch theorem of [5], Lemma 10 and Lemma 13 we have

\[ ch^G RT\left(R, \frac{\mathcal{O}^{vir}_R \otimes K^\frac{1}{2} \otimes R}{\Lambda^* (N^{vir}_R)^{\otimes R}} \bigotimes_{i=1}^m \hat{\gamma}(\mathcal{O}_s)|_R \right). \]

The injectivity of ch^G : \mathbb{Q}(t^\frac{1}{2}) \rightarrow \mathbb{Q}((t)) implies the lemma.

The above lemma also holds if we replace K^\frac{1}{2} by any class α \in K^G(R).

By the above lemma we can replace \gamma(\mathcal{O}_s) with \hat{\gamma}(\mathcal{O}_s) = \hat{p}_*(\rho_*[\mathcal{O}_C], \omega_2^* [\mathcal{O}_s]). The advantage of using \hat{\gamma}(\mathcal{O}_s) will become clear later.

Lemma 15. Let \mathcal{L} be a globally generated line bundle on S. Let dim |\mathcal{L}| = n and \mathcal{D} \subset |\mathcal{L}| \times S be the universal divisor. Then for any point s ∈ S the fiber product \mathcal{D} × |\mathcal{L}|_s (|\mathcal{L}| \times \{s\}) is a hyperplane \mathbb{P}^{n-1} \subset |\mathcal{L}| \times \{s\}.

Proof. Let \mathcal{L} be globally generated line bundle on S and let f : S → Spec \mathbb{C} be the structure morphism. Then S × |\mathcal{L}| = Proj Sym(f^* \mathcal{O}(\mathcal{L})^\vee) and the canonical morphism \xi : f^* f_\mathcal{L} \rightarrow \mathcal{L} is surjective. Let \xi^\vee : \mathcal{L}^\vee \rightarrow f^* (f_\mathcal{L})^\vee be the dual of \xi. Let e_i be the basis of f_\mathcal{L} and let \epsilon_i^\vee ∈ (f_\mathcal{L})^\vee defined as \epsilon_i^\vee(e_j) = 1 if i = j and 0 if i ≠ j. Then \xi^\vee sends a local section ψ of \mathcal{L}^\vee to \xi^\vee(ψ) : \sum_i a_i e_i → a_i ψ(e_i) \epsilon_i^\vee.

Sections of f^* (f_\mathcal{L})^\vee are linear combinations v of {\epsilon_i^\vee} with coefficient in \mathcal{O}_S and sections of Sym(f^* (f_\mathcal{L})^\vee are polynomials P in {\epsilon_i^\vee} with coefficient in \mathcal{O}_S. There is a canonical graded morphism \phi : f^* (f_\mathcal{L})^\vee ⊗ Sym(f^* (f_\mathcal{L})^\vee) → Sym(f^* (f_\mathcal{L})^\vee), that sends v ⊗ P to the products of the polynomials v.P. The composition of \xi^\vee ◦ id_{Sym(f^* (f_\mathcal{L})^\vee)} with \phi sends ψ ⊗ P to \xi^\vee(ψ).P. Let θ be this composition. This composition is injective since \xi^\vee is injective. This composition correspond to the morphism σ : \mathcal{L}^\vee ⊗ \mathcal{O}(-1) → \mathcal{O} on S × |\mathcal{L}| which is injective because θ is injective.
and Proj construction preserve injective morphism. The cokernel \( \sigma \) is the structure sheaf of the universal divisor \( D \subset S \times |\mathcal{L}| \).

For any closed point \( s \in S \), we want to show that the restriction of \( \sigma \) to \( |\mathcal{L}| \) is still injective. In this case \( D \times |\mathcal{L}| \cap (|\mathcal{L}| \times \{s\}) \) is an effective divisor with ideal \( \mathcal{O}(-1) \) so that \( D \times |\mathcal{L}| \cap (|\mathcal{L}| \times \{s\}) \) is a hyperplane \( \mathbb{P}^{n-1} \). Since \( \xi \) is surjective, its restriction to \( s \) is also surjective. Any element \( \alpha \in L^\prime|_s \) is the restriction of a local section \( \psi \in L' \).

Thus if \( \alpha \) is not zero there exist \( \psi \in L' \) such that its restriction to \( s \) is equal to \( \psi_1 \) such that \( \psi(e_j)|_s = \psi(e_i)|_s \) is not zero. We can conclude that \( \xi'|_s \) is injective. Because \( \sigma|_s : \psi|_s \otimes P|_s \to \xi'|(\psi)|_s P|_s \) we can conclude that \( \sigma|_s \) is injective. □

We will use \( \mathbb{P}_{n-1} \) to denote \( D \times |\mathcal{L}| \cap (|\mathcal{L}| \times \{s\}) \).

**Lemma 16.** Let \( c_1(\mathcal{L}) = \beta \) and let \( \mathcal{P} = \mathcal{P}_\chi(X, i_s \beta) \). Then all squares in the following diagram are Cartesian.

\[
\begin{array}{ccc}
\mathcal{D}_\mathcal{P} \times \mathcal{D} \mathbb{P}^{n-1} & \rightarrow & \mathbb{P}^{n-1} \\
\downarrow \downarrow & & \downarrow \\
\mathcal{P} \times \{s\} & \rightarrow & |\mathcal{L}| \times \{s\} \\
\downarrow \downarrow & & \downarrow \\
\mathcal{P} \times S & \rightarrow & |\mathcal{L}| \times S \\
\downarrow \downarrow & & \downarrow \\
\mathcal{D}_\mathcal{P} & \rightarrow & \mathcal{D} \\
\end{array}
\]

(4.12)

**Lemma 17.** If \( \beta \in G^T(\mathcal{P}) \) is supported on \( V \subset \mathcal{P} \) then \( \beta \hat{\gamma}(\mathcal{O}_s) \) is supported on \( V \times \mathcal{P} W_s \) where \( W_s := \mathcal{D}_\mathcal{P} \times \mathcal{P} \times S(\mathcal{P} \times \{s\}) \).

**Proof.** Recall the morphism \( \hat{p} \) from diagram (4.10) and \( \hat{h}, \hat{h} \) from (4.12). Since \( \hat{p} \circ h = \text{id}_\mathcal{P} \) we can conclude that \( \hat{\gamma}(\mathcal{O}_s) = h^* j_* [\mathcal{O}_C] = h^* [j_*, \mathcal{O}_C] \). Let \( E' \) be a finite resolution of \( j_*, \mathcal{O}_C \) by locally free sheaves. It suffices to prove the statement for the case when \( \beta \) is the class of a coherent sheaf \( \mathcal{F} \) on \( V \). By Lemma 8 we have

\[
[\mathcal{F}], \hat{\gamma}(\mathcal{O}_s) = j_* k_* j[\mathcal{O}_C](\mathcal{F})
\]

where \( j[\mathcal{O}_C] \) is the refined Gysin homomorphism and \( k \) is the closed embedding \( V \times_{\mathcal{P} \times S} W_s \to W_s \) where \( W_s = \mathcal{D}_\mathcal{P} \times \mathcal{P} \mathcal{P}^{n-1} \). □

**Lemma 18.** Given \( m \) points \( s_1, \ldots, s_m \in S \) in general position such that all curves in \( |\mathcal{L}| \) that passes through all \( m \) points are reduced and irreducible, then for any component \( R \subset P^G \) different from \( \mathcal{P}_\chi(S, \beta) \) we have \( \iota_* \mathcal{O}_R \rightarrow \prod_{i=1}^m \hat{\gamma}(\mathcal{O}_{s_i}) = 0 \).

**Proof.** Let \( \beta_l = \iota_* \mathcal{O}_R \rightarrow \prod_{i=1}^m \hat{\gamma}(\mathcal{O}_{s_i}) \). By Lemma 17 \( \beta_l \) is supported on \( R \times \mathcal{P} W_s = R \times |\mathcal{L}| P^n_{s_1} \). Our assumptions implies that for any \( 1 \leq l \leq m \), \( r_{s_1}^{P^n_{s_1}} \) is not contained in \( P^n_{s_1} \). In particular, \( r_{s_1}^{P^n_{s_1}} = P^{n-m} \) and by induction we can conclude that \( \beta_m \) is supported on \( R \times |\mathcal{L}| P^{n-m} \). Note that all curves in \( P^{n-m} \) are reduced and irreducible.

We will show that for any \( (\mathcal{F}, s) \in R \), \( \text{div}(\mathcal{F}, s) \) is not in \( P^{n-m} \). Let \( C_{\mathcal{F}} \) be the curve on \( X \) supporting an element \( (\mathcal{F}, s) \in R \). Note that the reduced subscheme \( C_{\mathcal{F}}^{\text{red}} \) of \( C_{\mathcal{F}} \) is a curve on \( S \) so that if \( C_{\mathcal{F}} \) is reduced and irreducible then \( C_{\mathcal{F}} = C_{\mathcal{F}}^{\text{red}} \).
is a curve on $S$ and $(F, s)$ can’t be in $R$. If $C_F$ is not irreducible, then the support of $\pi_*O_{C_F}$ is not irreducible so that $\text{div}(F, s)$ is not in $\mathbb{P}^{n-m}$. So we are left with the case when $C_F$ is irreducible. Let $C$ be the reduced subscheme of $C_F$. Let $Spec A \subset S$ be an open subset such that $K_\mathcal{E}$ is a free line bundle over $Spec A$. We can write $C = Spec A/(f)$ for an irreducible element $f \in A$ and $X|_{Spec A} = Spec A[x]$. Then $O_{C_F}$ can be written as $M := \mathcal{O}_{Spec A}/(f^m)x^j$ for some positive integers $r, n_i$ and $\text{div} M$ is described by the ideal $(f^{m_i} n_i)$. Since $C_F$ is not supported on $S$, then $\sum n_i \geq 2$ and $\text{div} M$ is not reduced. Thus in this case $\text{div}(F, s)$ is not in $\mathbb{P}^{n-m}$.

Since $\text{div}(R)$ is disjoint from $\mathbb{P}^{n-m}$, we can conclude that $R \setminus |L| \mathbb{P}^{n-m}$ is empty. By lemma \ref{lem:b_m} $\beta_m$ is zero. 

Following the proof of Lemma \ref{lem:17} and Lemma \ref{lem:18} and by replacing $[\mathcal{O}_C]$ with $[\mathcal{O}_D]$ we can prove that the contribution to $P_{X, \beta, \chi}(s_1, \ldots, s_m)$ of the component $R \in \mathcal{P}^G$ where $R \neq P_{X}(S, \beta)$ is zero when $s_1, \ldots, s_m$ is in general position and all curves on $S$ that pass through all $m$ points are reduced and irreducible.

Actually we have a stronger result for $P_{X, \beta, \chi}(s_1, \ldots, s_m)$. By Proposition \ref{prop:22} for any point $s \in S$, $\bar{\gamma}(\mathcal{O}_s)$ is $1 - [\text{div}^r \mathcal{O}(-1)]$. In particular it’s independent from the chosen point.

**Proposition 19.** Given a positive integer $\delta$, let $S$ be a smooth projective surface with $b_1(S) = 0$. Let $L$ be a $2\delta + 1$-very ample line bundle on $S$ with $c_1(L) = \beta$ and $H^0(L) = 0$ for $i > 0$. Let $X = K_S$ be the canonical line bundle over $S$. Then for any connected component $R$ of $P_{X}(X, i, \beta)\mathbb{C}^{	imes}$ different from $P_{X}(S, \beta)$ and for $m \geq H^0(L) - 1 - \delta$, we have

$$
\mathcal{R}^\Gamma \left( R, \frac{\mathcal{O}_{R}^{vir}}{\Lambda^* (N_{vir}^*)} K_{vir}^\frac{1}{2} \mathcal{O}_{\mathcal{X}} \otimes \prod_{i=1}^{m} \bar{\gamma}(\mathcal{O}_{s_i}) \right) = 0
$$

where $s_1, \ldots, s_m$ are closed points of $S$ which can be identical. We then can conclude that

$$
P_{X, \beta, \chi}(s_1, \ldots, s_m) = \mathcal{R}^\Gamma \left( P_{X}(S, \beta), \frac{\mathcal{O}_{P_{X}(S, \beta)}^{vir}}{\Lambda^* (N_{vir}^*)} K_{vir}^\frac{1}{2} \mathcal{O}_{\mathcal{X}} \otimes \prod_{i=1}^{m} \bar{\gamma}(\mathcal{O}_{s_i}) \right).
$$

The same result also holds for $P_{X, \beta, \chi}(s_1, \ldots, s_m)$ under additional assumption that the structure sheaf $\mathcal{O}_{C_E}$ of the universal supporting curve $C_E$ is flat over $P_{\mathcal{X}}(X, i, \beta)$ and $s_1, \ldots, s_m$ are closed points in $S$ in general position such that all curves in $|L|$ passing through all the given $m$ points are irreducible.

### 4.3. The contribution of $P_{X}(S, \beta)$

The component $P_{X}(S, \beta)$ of $P_{X}(X, i, \beta)\mathbb{C}^\times$ parametrize stable pairs $(F, s)$ supported on $S \subset X$ where $S$ is the zero section. The restriction of $\mathcal{X}$ to $P_{X}(S, \beta) \times X$ is $\mathcal{X}_s := \{ \mathcal{O}_{P_{X}(S, \beta) \times X} \times \mathcal{F} \}$, where $\mathcal{F}$ is the universal sheaf restricted to $P_{X}(S, \beta) \times X$, so that the restriction of $\mathcal{E}^*$ to $P_{X}(S, \beta)$ is $R_p \mathcal{R}Hom(\mathcal{I}_S^* \mathcal{E}^* \otimes t^*)[2]$. Thomas and Kool showed that on $\mathcal{P}_{X}(S, \beta)$, the decomposition of $\mathcal{E}^*|_{P_{X}(S, \beta)}$ into fixed and moving part is

$$
(\mathcal{E}^*)^{mov} \simeq R_p \mathcal{R}Hom(\mathcal{I}_S^* \mathcal{E}^* \otimes t^*) \quad (\mathcal{E}^*)^{fix} \simeq (R_p \mathcal{R}Hom(\mathcal{I}_S^* \mathcal{F}))^{\vee}
$$

where $\mathcal{I}_S^* = \{ \mathcal{O}_{P_{X}(S, \beta) \times S} \rightarrow \mathcal{F} \}$. $(\mathcal{E}^*)^{fix}$ gives $P_{X}(S, \beta)$ a perfect obstruction theory. We will use $\mathcal{E}^*$ to denote $(\mathcal{E}^*)^{fix}$. From equation \ref{eq:13} and \ref{eq:16} we have $(\mathcal{E}^*)^{mov} \simeq (\mathcal{E}^*)^{\vee}[1] \otimes t^*$. 


Proposition 20. On \( \mathcal{P}_\chi(S, \beta) \) we have

\[
\frac{K_{\text{vir}}^+}{\Lambda^* (N_{\text{vir}}^*)^\vee} = \left(-t^{\frac{1}{2}}\right)^{v} \Lambda_{-t} \mathcal{E}^*
\]

where \( vd = rk \mathcal{E}^* \) and \( \Lambda_{-t} \mathcal{E}^* = \sum_{j=0}^{rk \mathcal{E}^*} (-t)^j \Lambda^j \mathcal{E}^0 \) for \( \mathcal{E}^* = [E^{-1} \rightarrow E^0] \).

Proof. By equation (4.13) and (4.6) we have

\[
K_{\text{vir}}^\vee |_{\mathcal{P}_\chi(S, \beta)} = \det \mathcal{E}^\vee \det \left( (\mathcal{E}^*)^\vee \otimes t^* \right)^\vee = \det \mathcal{E}^* \det \mathcal{E}^* t^v
\]

where \( v = rk \mathcal{E}^* \). Thus we can take \( K_{\text{vir}}^+ |_{\mathcal{P}_\chi(S, \beta)} = \det \mathcal{E}^* t^{\frac{1}{2}v} \). Let \( \mathcal{E}^* = [E^{-1} \rightarrow E^0] \) so that \( (\mathcal{E}^*)^\vee [1] \otimes t^* = [(E^0)^\vee \otimes t^* \rightarrow (E^{-1})^\vee \otimes t^*] \) in the place of \(-1\) and 0. Let \( r_i = rk \mathcal{E}^i \) for \( i = -1 \) and \( i = 0 \). Thus in \( K^G(\mathcal{P}_\chi(S, \beta)) \) we have

\[
\frac{K_{\text{vir}}^+}{\Lambda^* (N_{\text{vir}}^*)^\vee} = \left(-t^{\frac{1}{2}}\right)^{vd} \Lambda_{-t} \mathcal{E}^*
\]

The calculation of the contribution from this component is given in the next section. We recall Corollary 25 here.

Under the assumption of Proposition 19, the formula for \( \tilde{P}_{X, \chi, \chi}(s_1, \ldots, s_m) \) is

\[
(-1)^{vd} \int_{[\mathcal{P}_\chi(S, \beta)]^{[n]}} \frac{X_4 \left( \mathcal{O} \right)}{X_4 \left( \mathcal{O} \right)} \left( \frac{t^{-1/2} - \sqrt{1/2} e^{-H(t^{-1/2} - 1/2)}}{t^{-1/2} - \sqrt{1/2}} \right)^m \HH^m
\]

where \( vd \) is the virtual dimension of \( \mathcal{P}_\chi(S, \beta) \) and \( \mathcal{O} \) is the dual of the pullback by the morphism \( \text{div} : \mathcal{P}_\chi(X, i, \beta) \rightarrow \mathcal{L} \) of the tautological line bundle and \( H = c_1(\mathcal{O}) \) for any vector bundle \( E \) of rank \( r \) with Chern roots \( x_1, \ldots, x_r \),

\[
X_4(E) = \prod_{i=1}^r \frac{x_i \left( t^{-1/2} - \sqrt{1/2} e^{-x_i(t^{-1/2} - 1/2)} \right)}{1 - e^{-x_i(t^{-1/2} - 1/2)}}.
\]

We have the same formula for \( P_{X, \beta, \chi}(s_1, \ldots, s_m) \) whenever \( P_{X, \beta, \chi}(s_1, \ldots, s_m) \) can be defined. This is because the restriction of \( \gamma(\mathcal{O}_n) \) and \( \bar{\gamma}(\mathcal{O}_n) \) to \( \mathcal{P}_\chi(S, \beta) \) are identical.

We can observe from the above formula that \( P_{X, \beta, \chi}(s_1, \ldots, s_m) \) is independent from the choosen points. It’s natural to ask if without assuming that \( s_1, \ldots, s_m \) are in general positions such that all curves passing through all these points are reduced and irreducible the above proposition still holds.

5. Refinement of Kool-Thomas Invariants

5.1. Reduced obstruction theory of moduli space of stable pairs on surface. In Section 4 we have reviewed the construction of reduced obstruction theory by Kool and Thomas in [12]. In this section we will review the description of it’s restriction to \( \mathcal{P}_\chi(S, \beta) \) as a two term complex of locally free sheaves following Appendix A of [12]. The Appendix is written by Martijn Kool, Richard P. Thomas and Dmitri Panov.
Pandharipande and Thomas showed that $\mathcal{P}_\chi(S,\beta)$ is isomorphic to the relative Hilbert scheme of points $\text{Hilb}^n(C/\text{Hilb}_\beta(S))$ where $C \to \text{Hilb}_\beta(S)$ is the universal family of curves $C$ in $S$ in class $\beta \in H^1(S,\mathbb{Z})$ and $\chi = n+1-h$ where $h$ is the arithmetic genus of $C$. Notice that for $n = 1$, $\mathcal{P}_\chi(S,\beta) = \text{Hilb}^1(C/\text{Hilb}_\beta(S)) = \text{Hilb}_\beta(S)$.

We will review first the description of $\mathcal{P}_\chi(S,\beta)$ as the zero locus of a vector bundle on a smooth scheme. We assume that $b_1(S) = 0$ for simplicity and also because we are only working for this case in this article. The following construction does not need this assumption.

For $n = 0$, pick a sufficiently ample line divisor $A$ on $S$ such that $\mathcal{L}(A) = \mathcal{L} \otimes \mathcal{O}(A)$ satisfies $H^i(\mathcal{L}(A)) = 0$ for $i > 0$. Let $\gamma = \beta + [A]$. Then $\text{Hilb}_\gamma(S) = [\mathcal{L}(A)] = [\text{Hilb}(A)^{\chi-1}]$ has the right dimension. The map that send $C \in [\mathcal{L}]$ to $C + A \in [\mathcal{L}(A)]$ defines a closed embedding $\text{Hilb}_\beta(S) \to \text{Hilb}_\gamma(S)$.

Let $\mathcal{D} \subset H_\gamma(S) \times S$ be the universal divisor and let $\tilde{p}$ and $q_S$ be the projections $H_\gamma(S) \times S \to H_\gamma(S)$ and $H_\gamma(S) \times S \to S$ respectively. Let $s_D \in H^0(\mathcal{O}(\mathcal{D}))$ be the section defining $\mathcal{D}$ and restrict it to $H_\gamma(S) \times A$ and consider the section

$$\zeta := s_D|_{\pi_S^{-1}A} \in H^0(H_\gamma(S) \times A, \mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A}) = H^0(H_\gamma(S), \pi_{H*}(\mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A}))$$

where for a point $D \in H_\gamma(S)$ we have $\zeta|_D = s_D|_A \in H^0(A, \mathcal{L}(A))$ where $s_D$ is the section of $\mathcal{L}(A)$ defining $D$. $s_D|_A = 0$ if and only if $A \subset D$ i.e. $D = A + C$ for some effective divisor $C$ with $\mathcal{O}(C) \otimes \mathcal{O}(A) = \mathcal{L}(A)$. Thus the zero locus of $\zeta$ is the image of the closed embedding $\text{Hilb}_\beta(S) \to \text{Hilb}_\gamma(S)$. If $H^2(\mathcal{L}) = 0$ then $F = \pi_{H*}(\mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A})$ is a vector bundle of rank $\chi(\mathcal{L}(A)) - \chi(\mathcal{L}) = h^0(\mathcal{L}(A)) - h^0(\mathcal{L}) + h^1(\mathcal{L})$ on $\text{Hilb}_\gamma(S)$ since $R^i\pi_{H*} (\mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A}) = 0$ for $i > 0$. Consider the following diagram

$$\begin{array}{ccc}
F^*_{\text{red}} &=& \left\{ F^* \xrightarrow{d_{\mathcal{O}^*}} \Omega_{H_\gamma(S)}|_{H_\beta(S)} \right\} \\
\mathbb{L}_{H_\beta(S)} &=& \left\{ I/J^2|_{H_\gamma(S)} \xrightarrow{d} \Omega_{H_\gamma(S)}|_{H_\beta(S)} \right\}. \\
\end{array}$$

The above morphism is a perfect obstruction theory for $\text{Hilb}_\beta(S)$.

Next, we embed $\text{Hilb}^0(C/\text{Hilb}_\beta(S))$ into $S[n] \times \text{Hilb}_\beta(S)$. Let $Z \subset S[n] \times \text{Hilb}_\beta(S) \times S$ be the pullback of the universal length $n$ subscheme of $S[n] \times S$. Let $C \subset S[n] \times \text{Hilb}_\beta(S) \times S$ be the pullback of the universal divisor of $\text{Hilb}_\beta \times S$ and let $\pi : S[n] \times \text{Hilb}_\beta(S) \times S \to S[n] \times \text{Hilb}_\beta(S)$ be the projection. Then $C$ correspond to a section $s_C$ of the line bundle $\mathcal{O}(C)$ on $S[n] \times \text{Hilb}_\beta(S) \times S$. A point $(Z, C) \in S[n] \times \text{Hilb}_\beta(S)$ is in the image of $\text{Hilb}^n(C/\text{Hilb}_\beta(S))$ if $Z \subset C$. We denote by $\mathcal{O}(C)^[n]$ the vector bundle $\pi_* (\mathcal{O}(C)|_Z)$ of rank $n$. Let $\sigma_C$ be the pushforward of $s_C$ so that $\sigma_C|_Z = s_C|_Z \in H^0(\mathcal{L}|_Z)$. Thus a point $(Z, C) \in S[n] \times \text{Hilb}_\beta(S)$ is in the image of $\text{Hilb}^n(C/\text{Hilb}_\beta(S))$ if and only if $\sigma_C|_{(Z, C)} = s_C|_Z = 0$. Thus we get a perfect relative obstruction theory :

$$E^* = \left\{ (\mathcal{O}(C)^[n])^* \xrightarrow{d_{\mathcal{O}^*}^*} \Omega_{S[n]} \right\}$$

$$\mathbb{L}_{\text{Hilb}^n(C/\text{Hilb}_\beta(S))/\text{Hilb}_\beta(S)} = \left\{ J/J^2 \xrightarrow{d} \Omega_{S[n]} \right\}$$
where $J$ is the ideal describing $\text{Hilb}^n(C/\text{Hilb}_\beta(S))$ as a subscheme of $S^{(n)} \times \text{Hilb}_\beta(S)$. Notice that in general $|\mathcal{L}|$ is not of the right dimension.

Appendix A of [12] shows how to combine the above obstruction theories to define an absolute perfect obstruction theory for $\text{Hilb}^n(C/\text{Hilb}_\beta(S))$. To do this we have to consider the embedding of $\text{Hilb}^n(C/\text{Hilb}_\beta(S))$ into $S^{(n)} \times \text{Hilb}_\beta(S)$. $E^*$ is the restriction of $[(\mathcal{O}(D - A)^{(n)})* \to \Omega_{S^{(n)}}]$ to $\text{Hilb}^n(C/\text{Hilb}_\beta(S))$. It was shown that the complex $E^*_{\text{red}}$ that corresponds to the combined obstruction theory sits in the following exact triangle

$$F^*_{\text{red}} \to E^*_{\text{red}} \to E^*.$$ 

Also in Appendix A of [12], it was shown that the combination of the above obstruction theories has the same $K$-theory class with the reduced obstruction theory $E^*_{\text{red}}$. Thus we can conclude that the $K$-theory class of $E^*_{\text{red}}$ is

$$(5.1) \quad [\Omega_{S^{(n)} \times \text{Hilb}_\beta(S)}] - [(\mathcal{O}(D - A)^{(n)})*] - [E^*]$$

Moreover, Theorem A.7 of [12] gives the virtual class corresponding to the reduced obstruction theory $[\mathcal{P}_\chi(S, \beta)]^{\text{red}}$ as the class $c_n(\mathcal{O}(D - A)^{(n)}) \cdot c_{\text{top}}(F) \cap [S^{(n)} \times \text{Hilb}_\beta(S)].$

### 5.2. Point insertion and linear subsystem

In this section we assume that $h^{0,1}(S) = 0$ i.e. $\text{Pic}_\beta = \{L\}$ and $\text{Hilb}_\beta(S) = |\mathcal{L}|$.

Let $D \subset S \times |\mathcal{L}|$ be the universal curve. Pandharipande and Thomas showed in [17] that $\mathcal{P}_\chi(S, \beta)$ is isomorphic to the relative Hilbert scheme of points $\text{Hilb}^n(D \to |\mathcal{L}|)$. There is an embedding of $\text{Hilb}^n(D \to |\mathcal{L}|)$ into $S^{(n)} \times |\mathcal{L}|$ and the projection $\text{Hilb}^n(D \to |\mathcal{L}|) \to |\mathcal{L}|$ gives a morphism $\text{div} : \mathcal{P}_\chi(S, \beta) \to |\mathcal{L}|$ that maps $(F, s) \in \mathcal{P}_\chi(S, \beta)$ to the supporting curve $C_F \in |\mathcal{L}|$ of $F$.

Fix $\chi \in \mathbb{Z}$ and let $\mathcal{C}$ be the universal curve supporting the universal sheaf $\mathcal{F}$ on $S \times \mathcal{P}_\chi(S, \beta)$. Consider the following diagram

$$\begin{array}{ccc}
\mathcal{P}_\chi(S, \beta) \times S & \xrightarrow{q^S} & S \\
\downarrow \bar{\rho} & & \\
\mathcal{P}_\chi(S, \beta)
\end{array}$$

Of course when $n = 1$, $\mathcal{P}_\chi(S, \beta)$ is $|\mathcal{L}|$ and $\mathcal{C} = \mathcal{D}$.

Here we will compute explicitly the class $\gamma(\mathcal{O}_s)$ restricted to $\mathcal{P}_\chi(S, \beta) \to \mathcal{P}_\chi(X, i_* \beta)^G$. Note that $G$ acts trivially on $S$ and on $\mathcal{P}_\chi(S, \beta)$. Let $\mathcal{C} \subset \mathcal{P}_\chi(S, \beta) \times \bar{X}$ be the support of the universal sheaf. Note that $\mathcal{C}$ is supported on $\mathcal{P}_\chi(S, \beta) \times S$ where $S$ is the zero section of the bundle $X \to S$. Thus $\pi \circ i : \mathcal{C} \to \mathcal{P}_\chi(S, \beta) \times S$ is a closed embedding. By equation (1.11), $\gamma(\mathcal{O}_s) = \bar{\rho}_*([\mathcal{O}_{\mathcal{C}}] \otimes q^S_\ast [\mathcal{O}_s])$. Notice that $G$ acts on $\mathcal{O}_s$ and $\mathcal{O}_C$ trivially.

**Proposition 21.** Let $s \in S$ be a point with structure sheaf $\mathcal{O}_s$. Let $[\mathcal{O}_s]$ be its class in $K(S)$. Then

$$\bar{\rho}_*([\mathcal{O}_{\mathcal{C}}] \cdot q^S_\ast [\mathcal{O}_s]) = 1 - [\text{div}^* \mathcal{O}(-1)].$$

where $\mathcal{O}(-1)$ is the tautological line bundle on $|\mathcal{L}|$.

**Proof.** First consider the following diagram

\[
\begin{array}{ccc}
\mathcal{P}_\chi(S, \beta) \times S & \xrightarrow{q^S} & S \\
\downarrow \bar{\rho} & & \\
\mathcal{P}_\chi(S, \beta)
\end{array}
\]
\[
\begin{align*}
|\mathcal{L}| \times S \xrightarrow{q_\mathcal{L}} S.
\end{align*}
\]

We will show that \( \hat{p}_* (q_\mathcal{L}^* [\mathcal{O}_s], [\mathcal{O}_\mathcal{D}]) = 1 - [\mathcal{O}(-1)] \). Since \( q_\mathcal{L} \) is a flat morphism \( q_\mathcal{L}^*[\mathcal{O}_s] = [q_\mathcal{L}^* \mathcal{O}_s] = k_* [\mathcal{O}_{|\mathcal{L}| \times \{z\}}] \) where \( k \) is the inclusion \( k : |\mathcal{L}| \times \{z\} \to |\mathcal{L}| \times S \). \( \mathcal{C} \) is the universal divisor with \( \mathcal{L}^* \boxtimes \mathcal{O}(-1) \) as the defining ideal. By the projection formula \( q_\mathcal{L}^*[\mathcal{O}_s].[\mathcal{O}_\mathcal{D}] \) is equal to
\[
k_* [\mathcal{O}_{|\mathcal{L}| \times \{z\}}] . (1 - [\mathcal{L}^* \boxtimes \mathcal{O}(-1)]) = k_* ([k^* \mathcal{O}_{|\mathcal{L}| \times S}] - [k^* q_\mathcal{L}^* \mathcal{L}^* \otimes k^* p^* \mathcal{O}(-1)]) .
\]

\( k^* q_\mathcal{L}^* \mathcal{L}^* = q_\mathcal{L}^* \mathcal{L}^* \vert_{|\mathcal{L}| \times \{s\}} \) where \( q_s = q_{s\vert_{|\mathcal{L}| \times \{s\}}} \) and \( k^* \hat{p}^* \mathcal{O}(-1) = \mathcal{O}(-1) \) since \( \hat{p} \circ k \) is the identity morphism. Thus we conclude that
\[
\hat{p}_* (q_\mathcal{L}^*[\mathcal{O}_s].[\mathcal{O}_\mathcal{D}]) = \hat{p}_* k_* ([\mathcal{O}_{|\mathcal{L}| \times \{s\}}] - [\mathcal{O}(-1)]) = 1 - [\mathcal{O}(-1)].
\]

Now we are working on \( \mathcal{P}_X(S, \beta) \). Consider the following Cartesian diagram

\[
\begin{array}{c}
\text{div}^{-1} \mathcal{D} \\
\downarrow \\
\text{div} \end{array}
\begin{array}{c}
\mathcal{D} \\
\downarrow \\
\text{div}^{-1} \mathcal{D}
\end{array}
\begin{array}{c}
\mathcal{P}_X(S, \beta) \times S \\
\downarrow \\
\mathcal{P}_X(S, \beta)
\end{array}
\begin{array}{c}
\text{div} \end{array}
\begin{array}{c}
|\mathcal{L}| \times S \\
\downarrow \\
|\mathcal{L}|
\end{array}
\]

\( \text{div}^{-1} \mathcal{D} \) is the family of effective Cartier divisors corresponding to the morphism \( \text{div} : \mathcal{P}_X(S, \beta) \to |\mathcal{L}| \). For each point \( p \in \mathcal{P}_X(S, \beta) \), \( \text{div}^{-1} \mathcal{D}_p \) is the corresponding curve \( \mathcal{C}_p \) supporting the sheaf \( \mathcal{F}_p \). We conclude that \( \mathcal{C} \) and \( \text{div}^{-1} \mathcal{C} \) are the same families of divisors on \( S \) so that we have a short exact sequence

\[
0 \to \text{div}^* (\mathcal{L}^* \boxtimes \mathcal{O}(-1)) \to \mathcal{O}_{\mathcal{P}_X(S, \beta) \times S} \to \mathcal{O}_\mathcal{C} \to 0
\]

and \( [\mathcal{O}_\mathcal{C}] = \text{div}^* [\mathcal{O}_\mathcal{D}] \). Thus we have
\[
\hat{p}_* (p^* (\mathcal{C}_p) q_\mathcal{L}^*[\mathcal{O}_s]) = \hat{p}_* (p^* (\mathcal{O}_\mathcal{D}) \cdot \text{div}^* q_\mathcal{L}^*[\mathcal{O}_s])
\]
\[
= \text{div}^* \hat{p}_* ([\mathcal{C}_p] q_\mathcal{L}^*[\mathcal{O}_s])
\]
\[
= \text{div}^* (1 - [\mathcal{O}(-1)])
\]

We also have similar result for \( \mathcal{P}_X(S, \beta) \) if we replace \( \mathcal{O}_\mathcal{C} \) with \( \mathcal{O}_{\text{div} \pi_* \mathcal{F}} \).

**Proposition 22.** Let \( \mathcal{O}_s \) be the structure sheaf of the points \( s \in S \). Then \( \hat{p}_* ([\mathcal{O}_{\text{div} \pi_* \mathcal{F}}] q_\mathcal{L}^*[\mathcal{O}_s]) = 1 - \text{div}^* (\mathcal{O}(-1)) \) where \( \mathcal{O}(-1) \) is the tautological bundle of \( |\mathcal{L}| \) and \( \hat{p} \). \( q_\mathcal{L} \) are morphism from diagram 4.8.

**Proof.** From the definition of the morphism \( \text{div} \), \( \text{div} \pi_* \mathcal{F} \) is exactly \( \text{div}^{-1} \mathcal{D} \). Thus we can use exactly the same proof as the previous Proposition.

Later we will drop \( \text{div}^* \) from \( \text{div}^* \mathcal{O}(-1) \) for simplicity.
5.3. Refinement of Kool-Thomas invariants. Assume that \( b_1(S) = 0 \).
From Proposition \([21]\) and Proposition \([22]\), the contribution of \( \mathcal{P}_\chi(S, \beta) \) to \( P_{X, \beta, \chi}(s_1, \ldots, s_m) \) and to \( \bar{P}_{X, \beta, \chi}(s_1, \ldots, s_m) \) are equal. Consider the contribution of \( \mathcal{P}_\chi(S, \beta) \) to \( P_{X, \beta, \chi}(s_1, \ldots, s_m) \) invariants, i.e.

\[
\Xi = \text{R} \Gamma \left( \mathcal{P}_\chi(S, \beta), \frac{\mathcal{O}_\chi^{\text{vir}}(\mathcal{N}^{\text{vir}}) \otimes K_{\chi}}{\Lambda^*(\mathcal{N}^{\text{vir}})^\vee} \prod_{t=1}^m \gamma_t(\mathcal{O}_{s_t}) \right).
\]

On \( \text{Hilb}_\beta(S) \times S \) we have the following exact sequence

\[
\begin{array}{c}
0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(\mathcal{L}) \longrightarrow \mathcal{O}_C(\mathcal{L}) \longrightarrow 0
\end{array}
\]

which induces the exact sequence

\[
\begin{array}{c}
H^1(\mathcal{O}_C(\mathcal{L})) \longrightarrow H^2(\mathcal{O}_S) \longrightarrow H^2(\mathcal{L}).
\end{array}
\]

If \( H^2(\mathcal{L}) = 0 \) then \( \hat{\phi} \) is surjective. Observe that \( R\pi_H \mathcal{O}_C(\mathcal{L}) \) is the complex \( \mathcal{E}^* \) from Subsection 2.2.1 when \( \chi = 2 - h \) or equivalently when \( n = 1 \). For \( n > 1 \), it was shown in Appendix A of \([12]\) that \( \mathcal{E}^* \) sits in the exact triangle

\[
R\pi_H \mathcal{O}_C(\mathcal{L}) \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{E}^*.
\]

Thus if \( h^2(\mathcal{O}_S) > 0 \) then \( \mathcal{E}^* \) contain a trivial bundle so that \([\mathcal{P}_\chi(S, \beta)]^{\text{vir}}\) vanish. In particular, by virtual Riemann-Roch the contribution of \( \mathcal{P}_\chi(S, \beta) \) is zero.

If \( H^2(\mathcal{O}_S) = 0 \), \( \mathcal{E}_{\text{red}}^* \) and \( \mathcal{E}^* \) are quasi isomorphic. Let \( P \) be the moduli space \( \mathcal{P}_\chi(S, \beta) \). By the virtual Riemann-Roch theorem and by Lemma \([22]\) we then have

\[
\chi^G(\Xi) = \left( -t^{\frac{1}{2}} \right)^{\text{td}} \int_{P[\text{red}]} \chi \left( \Lambda_{-t}\mathcal{E}_{\text{red}}^* \left( \Lambda_{-\frac{1}{2}} \mathcal{O}(1) \right)^m \right). \text{td}(T_P^{\text{red}})
\]

where \( T_P^{\text{red}} \) is the derived dual of \( \mathcal{E}_{\text{red}}^* \) and \( \left( -t^{\frac{1}{2}} \right)^{\text{td}} \) should be understood as \( \left( -e^{\frac{i}{2}} \right)^{\text{td}} \) where \( t \) is the equivariant first Chern class of \( t \). Observe that \( \chi^G(\Xi) \) can be computed whenever \( H^2(\mathcal{L}) = 0 \) without assuming \( h^2(\mathcal{O}_S) = 0 \). Thus for \( S \) with \( b_1(S) = 0 \) and a line bundle \( \mathcal{L} \) with \( H^2(\mathcal{L}) = 0 \), we define \( F_{\mathcal{L}, \pi, \chi_\alpha} = \chi^G(\Xi) \).

The \( K \)-theory class of \( \mathcal{E}_{\text{red}}^* \) is given by equation (5.1). Since \( \mathcal{O}(\mathcal{C}) = \mathcal{L} \otimes \mathcal{O}(1) \), by the projection formula we have \( F = H^0(\mathcal{L} \mid A) \otimes \mathcal{O}(1) \). From the exact sequence

\[
\begin{array}{c}
0 \longrightarrow \mathcal{O}(\mathcal{C}) \longrightarrow \mathcal{O}(\mathcal{C} + A) \longrightarrow \mathcal{O}_{\pi^{-1}\mathcal{A}(\mathcal{C} + \pi^{-1}\mathcal{A})} \longrightarrow 0
\end{array}
\]

on \( P \), and since \( H^{\geq 0}(\mathcal{L} \mid A) = 0 \), we conclude that

\[
F = \mathcal{O}(1)^{\otimes \chi(\mathcal{L}(A))} - \chi(\mathcal{L})
\]

And again by projection formula we have \( \mathcal{O}(\mathcal{C})^{[n]} = \mathcal{L}^{[n]} \otimes \mathcal{O}(1) \). By Theorem A.7 of \([12]\) we then can compute \( F_{\mathcal{L}, \pi, \chi_\alpha} \) as

\[
\begin{array}{c}
\left( -t^{\frac{1}{2}} \right)^n \int_{S^{[n]}} H^\chi(\mathcal{L}(A)) - \chi(\mathcal{L}) \mathcal{O}(\mathcal{D} - A)^{[n]} \\
\chi \left( \Lambda_{-t}\mathcal{E}_{\text{red}}^* \left( \Lambda_{-\frac{1}{2}} \mathcal{O}(1) \right)^m \right). \text{td}(T_P^{\text{red}})
\end{array}
\]
where $H = c_1(\mathcal{O}(1))$ and $n = \chi + h - 1$.

**Theorem 23.** $P_{S,L,m,\chi}|_{t=1} = (1)^{-1} \int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \frac{c_t(\mathcal{T}^{|S[n]}|c_t(\mathcal{O}(1) )^\chi)}{c_t(\mathcal{L}[n] \otimes \mathcal{O}(1))}$ where $\varepsilon = \chi(\mathcal{L}) - 1 - m$. Thus we can relate Kool-Thomas invariants with our invariants as follows:

$$P^{red}_{S,L,m,\chi}|_{t=1} = (1)^{-1} m + 1 - \chi(\mathcal{O}_S) P_{S,L,m,\chi}|_{t=1}. $$

**Proof.** Let $X_t(T^{|S|}_P) := \text{ch}(\Lambda_{-1}(\mathcal{L})^\chi) \text{ch}(T^{|S|}_P)$ and let $d := \text{rk} \mathcal{L}^\chi = n + \chi(\mathcal{L}) - 1$ be the virtual dimension of $P$ so that we can rewrite (5.4) as

$$(1)^{-m} \left( -t^\frac{1}{2} \right)^{d-m} \int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \chi(\Lambda_{-1}(\mathcal{O}(1)))^m.$$

By Proposition 5.3 of \cite{5} we can write

$$X_t(T^{|S|}_P) = \sum_{l=0}^d (1-t)^{d-l} X^l$$

where $X^l = c_l(T^{|S|}_P) + b_l$ where $b_l \in A^{*}(P)$. Then we can write $P_{S,L,m,\chi}$ as

$$(1)^{-m} \left( -t^\frac{1}{2} \right)^{d-m} \int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \chi(\Lambda_{-1}(\mathcal{O}(1)))^m.$$

Note that $\text{ch}(\Lambda_{-1}(\mathcal{O}(1)))^m = H^m + O(H^{m+1})$ so that

$$\int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \chi^l \text{ch}(\Lambda_{-1}(\mathcal{O}(1)))^m = 0$$

for $l > d - m$. Thus the summation ranges from $l = 0$ to $l = d - m$. In this range the power of $(1 - t)$ is positive except when $l = d - m$ in which the power of $(1 - t)$ is zero. Thus we can conclude that $P_{S,L,m,\chi}|_{t=1}$ equals to

$$(1)^{-m} \left( -t^\frac{1}{2} \right)^{d-m} \int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \chi^d \text{ch}(\Lambda_{-1}(\mathcal{O}(1)))^m.$$

Since $b_{d-m} \in A^{*}(P)$ and $c_{d-m}(T^{|S|}_P) \in A^{*}(P)$ we have

$$\int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) b_{d-m} \text{ch}(\Lambda_{-1}(\mathcal{O}(1)))^m = 0$$

and

$$\int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) c_{d-m}(T^{|S|}_P) H^k = 0$$

for $k > m$ and we can conclude that

$$P_{S,L,m,\chi}|_{t=1} = (1)^{-m} \left( -t^\frac{1}{2} \right)^{d-m} \int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \chi^d \text{ch}(\Lambda_{-1}(\mathcal{O}(1)))^m.$$

From (5.1) and (5.3) we have

$$T^{|S|}_P = T(S[n]) + O(1)^{\chi(A)} - \mathcal{O} - \mathcal{L}[n] \otimes \mathcal{O}(1) - O(1)^{\chi(A)}$$

and

$$c_{d-m}(T^{|S|}_P) = \text{Coeff}_{d-m} \left[ c_t(TS[n]) c_t(\mathcal{O}(1))^{\chi(\mathcal{L})} \right].$$

Finally we conclude that

$$P_{S,L,m,\chi}|_{t=1} = (1)^{-m} \left( -t^\frac{1}{2} \right)^{d-m} \int_{S[n],p} c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \frac{c_t(TS[n]) c_t(\mathcal{O}(1))^{\chi(\mathcal{L})}}{c_t(\mathcal{L}[n] \otimes \mathcal{O}(1))}$$
Let \(X_y(x) \in \mathbb{Q}[\![x, y]\!]\) defined by
\[
X_y(x) := \frac{x \left( y^{-\frac{1}{2}} - y^{\frac{1}{2}} e^{-x(y^{-\frac{1}{2}} - y^{\frac{1}{2}})} \right)}{1 - e^{-x(y^{-\frac{1}{2}} - y^{\frac{1}{2}})}}.
\]

For a vector bundle \(E\) on a scheme \(Y\) of rank \(r\) with Chern roots \(x_1, \ldots, x_r\), we will use \(X_y(E)\) to denote
\[
\prod_{i=1}^{r} \frac{x_i \left( y^{-\frac{1}{2}} - y^{\frac{1}{2}} e^{-x_i(y^{-\frac{1}{2}} - y^{\frac{1}{2}})} \right)}{1 - e^{-x_i(y^{-\frac{1}{2}} - y^{\frac{1}{2}})}}.
\]

Observe that \(X_y\) is additive on an exact sequence of vector bundles. Thus we can extend \(X_y\) to \(K(Y)\). For a class \(\beta \in K(Y)\) we can write \(\beta = \sum [E_i^+] - \sum [E_j^-]\) for vector bundles \(E_i^+, E_j^-\) and we can define \(X_y(\beta) = \frac{\prod_{i} X_y(E_i^+)}{\prod_j X_y(E_j^-)}\). For a proper nonsingular scheme \(Y\) with tangent bundle \(T_Y\)
\[
\int_Y X_y(T_Y) = \left(\frac{1}{y} \right)^d \sum_i (-1)^{p+q} y^q h^{p,q}(Y)
\]
where \(h^{p,q}(Y)\) are the Hodge number of \(Y\) i.e. \(\int_Y X_y(T_Y)\) is the normalized \(\chi_y\) genus.

**Theorem 24.**
\[
(5.6) \quad P_{S, L, m, \chi} = (-1)^{ed} \int_{[P]^{red}} \frac{X_y(TS[n])}{X_1(L[n] \otimes O(1))} X_1(O(1))^{\delta + 1} \left( \frac{t^{1/2} - (1/2)e^{-H(t^{1/2} - t^{1/2})}}{t^{-1/2} - t^{1/2}} \right)^m \]
where \([P]^{red}\) is \(c_n(L[n] \otimes O(1)) \cap [S[n] \times \mathbb{P}(L)^{-1}]\).

**Proof.** \(P_{S, L, m, \chi}\) equals to (5.5), and we can rewrite it as
\[
P_{S, L, m, \chi} = (-1)^{ed} \int_{[P]^{red}} \prod_{i=1}^{2n+\chi(L)-1} \frac{\phi_{-1}(\alpha_i)}{t^{1/2}} \left( \frac{1 - e^{-H}}{t^{-1/2} - t^{1/2}} \right)^m
\]
where \(\phi_{-1}(x) = \frac{x(1-e^{-x})}{1-x}\) and \(\alpha_i\) are the Chern roots of \(T(S[n] \times \mathbb{P}(L)^{-1})\) and \(\beta_i\) are the Chern roots of \(L[n] \otimes O(1)\). Let’s define
\[
\bar{\phi}_{-1}(x) := \frac{\phi_{-1}(x)}{t^{1/2}} = \frac{x \left( t^{-1/2} - t^{1/2} e^{-x} \right)}{1 - e^{-x}} = \sum_{i \geq 0} \bar{\phi}_i x^i.
\]
Note that this power series starts with \(t^{-1/2} - t^{1/2}\). By substituting \(x\) with \(x \left( t^{-1/2} - t^{1/2} \right)\) and dividing it by \(t^{-1/2} - t^{1/2}\) we have the power series
\[
X_{-1}(x) = \frac{x \left( t^{-1/2} - t^{1/2} e^{-x(t^{-1/2} - t^{1/2})} \right)}{1 - e^{-x(t^{-1/2} - t^{1/2})}} = \sum_{i \geq 0} \xi_i x^i.
\]
such that $\xi_0 = 1$ and $\xi_i = \phi_i \left( t^{-1/2} - t^{1/2} \right)^{-1}$. Thus by substituting $x$ in

$$\Pi_{i=1}^{2n+\chi(\mathcal{L})-1} \frac{\phi_i(\alpha_i)}{t^{1/2}} \left( 1 - e^{-H} \right)^m \frac{t^{-1/2} - t^{1/2}}{t^{-1/2} - t^{1/2}}$$

with $x \left( t^{-1/2} - t^{1/2} \right)$ whenever $x = \alpha_i, \beta_i, H$ and dividing it by

$$\left( t^{-1/2} - t^{1/2} \right)^{n+\chi(\mathcal{L})-1}$$

so that the coefficients of $x^{n+\chi(\mathcal{L})-1}$ in

$$\Pi_{i=1}^{2n+\chi(\mathcal{L})-1} X_{t}(\beta_i q) \left( 1 - e^{-Hq(t^{-1/2}-t^{1/2})} \right)^m \frac{t^{-1/2} - t^{1/2}}{t^{-1/2} - t^{1/2}}$$

and

$$\Pi_{i=1}^{2n+\chi(\mathcal{L})-1} \phi_{i-}(\alpha_i q) \left( 1 - e^{-Hq} \right)^m \frac{t^{-1/2} - t^{1/2}}{t^{-1/2} - t^{1/2}}$$

are the same. Since $[TP^\chi(\mathcal{L})^{-1}] = [\oplus_{i=1}^{\chi(\mathcal{L})} \mathcal{O}(1)] - [\mathcal{O}_{P^\chi(\mathcal{L})-1}]$, $P_{S,\mathcal{L},m,\chi}$ equals

$$(-1)^{\nu_d} \int_{[P]^{\nu_d}} \frac{X_{t}(TS[n]) X_{t}(\mathcal{O}(1))^{\chi(\mathcal{L})}}{X_{t}(\mathcal{L}[n] \otimes \mathcal{O}(1))} \left( 1 - e^{-H(t^{-1/2}-t^{1/2})} \right)^m \frac{t^{-1/2} - t^{1/2}}{t^{-1/2} - t^{1/2}}$$

$$\left( t^{-1/2} - t^{1/2} e^{-H(t^{-1/2}-t^{1/2})} \right)^m H^m$$

□

In the following Corollary we want to complete the computation of $P_{X,\beta,\chi}(s_1, \ldots, s_m)$.

**Corollary 25.** Given a positive integer $\delta$, let $S$ be a smooth projective surface with $b_1(S) = 0$. Let $\mathcal{L}$ be a $2\delta+1$-very ample line bundle on $S$ with $c_1(\mathcal{L}) = \beta$ and $H^i(\mathcal{L}) = 0$ for $i > 0$. Let $X = K_S$ be the canonical line bundle over $S$. Then for $m = \chi(\mathcal{L}) - 1 - \delta$ points $s_1, \ldots, s_m$ which is not necessarily different

$$\tilde{P}_{X,\beta,\chi}(s_1, \ldots, s_m) =$$

$$(-1)^{\nu_d} \int_{[P]^{\nu_d}} \frac{X_{t}(TS[n]) X_{t}(\mathcal{O}(1))^{\delta+1}}{X_{t}(\mathcal{L}[n] \otimes \mathcal{O}(1))} \left( t^{-1/2} - t^{1/2} e^{-H(t^{-1/2}-t^{1/2})} \right)^m \frac{t^{-1/2} - t^{1/2} e^{-H(t^{-1/2}-t^{1/2})}}{t^{-1/2} - t^{1/2}}$$

where $[P]^{\nu_d} = c_n(\mathcal{L}[n] \otimes \mathcal{O}(1)) \cap [S[n] \times \mathbb{P}^{\chi(\mathcal{L})-1}]$ for $m \geq H^0(\mathcal{L}) - 1 - \delta$.

If additionally $\mathcal{O}_S$ is flat over $P_{X,\beta,\chi}$ and $s_1, \ldots, s_m$ are closed points of $S$ in general position such that all curves on $S$ that pass through all $m$ points are reduced and irreducible then $P_{X,\beta,\chi}(s_1, \ldots, s_m)$ is given by the same formula.

**Proof.** By Proposition [19] $\tilde{P}_{X,\beta,\chi}(s_1, \ldots, s_m) = P_{S,\mathcal{L},m,\chi}$. Similarly for $P_{X,\beta,\chi}(s_1, \ldots, s_m)$.
In [8][9], for every smooth projective surface $S$ and line bundle $\mathcal{L}$ on $S$, Göttsche and Shende defined the following power series

$$D^{S,\mathcal{L}}(x, y, w) := \sum_{n \geq 0} w^n \int_{S[x]} X_{-y}(T S[n]) \frac{c_n \left( \mathcal{L}^{[n]} \otimes e^x \right)_{X_{-y}}}{X_{-y}(\mathcal{L}^{[n]} \otimes e^x)} \in \mathbb{Q}[x, y, w]$$

where $e^x$ denotes a trivial line bundle with nontrivial $\mathbb{C}^*$ action with equivariant first Chern class $x$. Motivated by this power series we define a generating function

$$(5.7) \quad P_{S,\mathcal{L},m} := \sum_{n \geq 0} (-w)^n P_{S,\mathcal{L},n+1}$$

where $h$ is the arithmetic genus of the curve $C$ in $S$ with $O(C) \simeq \mathcal{L}$ so that for the pair $(\mathcal{F}, s) \in \mathbb{P}_x(S, \beta)$, $n = \chi - 1 + h$.

By Theorem 24 after substituting $t$ by $y$ we can rewrite $P_{S,\mathcal{L},m}$ as

$$\text{Coeff}_{x^i} \left[ D^{S,\mathcal{L}}(x, y, w) X_{-y}(x)^{\delta+1} \left( \frac{y^{-1/2} - y^{1/2} e^{-x}(y^{-1/2} - y^{1/2})}{y^{-1/2} - y^{1/2}} \right)^m \right]$$

Note that

$$Q_{S,\mathcal{L},m} := \text{Coeff}_{x^i} \left[ D^{S,\mathcal{L}}(x, y, w) X_{-y}(x)^{\delta+1} \right]$$

is equation (2.1) of [3] and

$$\left( \frac{y^{-1/2} - y^{1/2} e^{-x}(y^{-1/2} - y^{1/2})}{y^{-1/2} - y^{1/2}} \right)^m$$

is a power series starting with 1.

In [3], Göttsche and Shende defined the power series $N^{i}_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y)$ by the following equation:

$$(5.8) \quad \sum_{i \in \mathbb{Z}} N^{i}_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y) \left( \frac{w}{(1-y^{-1/2}w)(1-y^{1/2}w)} \right)^{i+1-g} = Q_{S,\mathcal{L},m}$$

Motivated by this we also define $M^{i}_{\chi(\mathcal{L})-1-m,[S,\mathcal{L}]}(y)$ as

$$(5.9) \quad \sum_{i \in \mathbb{Z}} M^{i}_{\chi(\mathcal{L})-1-m,[S,\mathcal{L}]}(y) \left( \frac{w}{(1-y^{-1/2}w)(1-y^{1/2}w)} \right)^{i+1-g} = P_{S,\mathcal{L},m}$$

Let’s define \( \frac{1}{Q} = \frac{1-y^{-1/2}w(1-y^{1/2}w)}{w} = w + w^{-1} - y^{-1/2} - y^{1/2} \) and recall a conjecture from [8].

Conjecture 26 (Conjecture 55 of [8]).

$$\left( \frac{w(Q)}{Q} \right)^{1-g(\mathcal{L})} D^{S,\mathcal{L}}(x, y, w(Q)) \in \mathbb{Q}[y^{-1/2}, y^{1/2}][x, xQ]$$

Motivated by the conjecture above we define another power series

$$\tilde{D}^{S,\mathcal{L}}(x, y, Q) := \left( \frac{w(Q)}{Q} \right)^{1-g(\mathcal{L})} D^{S,\mathcal{L}}(x, y, w(Q)).$$
Proposition 27. Assume Conjecture [26] For $\chi(L) - 1 \geq k \geq 0$ we have
1. $M^i_{\chi(L) - 1 - k, [S, L]}(y) = 0$ and $N_i^{i, \chi(L) - 1 - k}(y) = 0$ for $i > \chi(L) - 1 - k$ and for $i \leq 0$.
2. $M^i_{\chi(L) - 1 - k, [S, L]}(y)$ and $N_i^{i, \chi(L) - 1 - k}(y)$ are Laurent polynomials in $y^{1/2}$.
3. Furthermore $M^i_{\chi(L) - 1 - k, [S, L]}(y) = N_i^{i, \chi(L) - 1 - k, [S, L]}(y)$. Moreover
   \[
   \sum_{i \in \mathbb{Z}} M^i_{\delta, [S, L]}(y)(s)^i = \tilde{D}^{S, L}(x, y, \frac{s}{x})|_{x=0} = \sum_{\delta \geq 0} N^\delta_{\delta, [S, L]}(y)s^\delta
   \]

Proof. After substituting $w$ by $w(Q)$ we rewrite equation (5.8) and (5.9)
\[
\sum_{i \in \mathbb{Z}} N^i_{\delta, [S, L]}(y)x^{\delta-i}(xQ)^i = [\tilde{D}^{S, L}(x, y, Q)X_y(x)^{\delta+1}]_{x^\delta}
\]

By Conjecture [26]
\[
\sum_{i \in \mathbb{Z}} N^i_{\delta, [S, L]}(y)x^{\delta-i}(xQ)^i, \sum_{i \in \mathbb{Z}} M^i_{\delta, [S, L]}(y)x^{\delta-i}(xQ)^i \in \mathbb{Q}[y^{-1/2}, y^{1/2}][x, Q]
\]
so that the only possible power of $Q$ that could appear is $i = 0, \ldots, \delta$. We can directly conclude that $N^i_{\delta, [S, L]}(y)$ are Laurent polynomial in $y^{1/2}$. Set $s = xQ$.

By Conjecture [26] we can write $\tilde{D}^{S, L}(x, y, Q)$ as power series of $x$ and $s$, i.e.
\[
\tilde{D}^{S, L}(x, y, \frac{s}{x}) \in \mathbb{Q}[y^{-1/2}, y^{1/2}]\mathbb{Q}[Q, s]
\]

And since
\[
X_y(x = 0) = 1
\]
\[
\left(\frac{y^{1/2} - y^{1/2}e^{-x(y^{-1/2}-y^{1/2})}}{y^{1/2} - y^{1/2}}\right)|_{x=0} = 1
\]
we can conclude that
\[
\sum_{i \in \mathbb{Z}} M^i_{\delta, [S, L]}(y)(s)^i = \tilde{D}^{S, L}(x, y, \frac{s}{x})|_{x=0}
\]
\[
= \sum_{\delta \geq 0} N^\delta_{\delta, [S, L]}(y)s^\delta
\]

If $H^i(L) = 0$ for $i > 0$ and $L$ is $\delta$-very ample, then $N^\delta_{\delta, [S, L]}(y)$ is the refinement defined by Goettsche and Shende in [8] of $n_\delta(L)$ that computes the number of $\delta$-nodal curves in $L$. Theorem [24] and Theorem [26] gives geometric argument for the equality $M^i_{\delta, [S, L]}(y)|_{y=1} = N^i_{\delta, [S, L]}(y)|_{y=1}$. Without assuming the conjecture above we would like to know if Proposition [27] still true.

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