Robust Dynamic Event-Triggered Coordination with a Designable Minimum Inter-Event Time

James Berneburg Cameron Nowzari

Abstract—This paper revisits the classical multi-agent average consensus problem for which many different event-triggered control strategies have been proposed over the last decade. Many of the earliest versions of these works conclude asymptotic stability without proving that Zeno behavior, or deadlocks, do not occur along the trajectories of the system. More recent works that resolve this issue either: (i) propose the use of a dwell-time that forces inter-event times to be lower-bounded away from 0 but sacrifice asymptotic convergence in exchange for practical convergence (or convergence to a neighborhood); (ii) guarantee non-Zeno behaviors and asymptotic convergence but do not provide a positive minimum inter-event time guarantee; or (iii) are not fully distributed. Additionally, the overwhelming majority of these works provide no form of robustness analysis. Instead, this work for the first time presents a fully distributed, robust, dynamic event-triggered algorithm, for general directed communication networks, for which a desired positive minimum inter-event time can be chosen by each agent in a distributed fashion. Simulations illustrate our results.

I. INTRODUCTION

Systems composed of individually controlled agents are increasingly common and a very active area of research. Such systems are designed for the coordination of unmanned air vehicles, distributed reconfigurable sensor networks, and attitude alignment for satellites, etc; see [1] and [2] and their references. These are often intended to fulfill some coordinated task, but require distributed control to be scaled with large systems. In this case, communication limitations, such as wireless bandwidth, mean that agents cannot be assumed to have continuous access to others’ states. Therefore, some works have considered communication to be a limited resource, where agents instead only share their state information at certain, discrete instances of time rather than continuously or periodically. The question, then, is: exactly when and how should agents decide to share their information with others in the network to guarantee that the intended function of the network is preserved?

One answer to this question comes in the form of event-triggered coordination, where communication at specific instances of time when some event condition on the state is satisfied. Examples of this are the triggering mechanism in [3], which causes an event when an error state grows to a state defined threshold, and the one in [4], which causes an event when a clock state reaches a threshold value. Event-triggered coordination is conveniently described by the formalism of hybrid systems [5]. It allows for continuous flow of the states and also discontinuous jumps, so some works ([4], [6]) model events as jumps where the control is updated.

One potential problem in event triggered coordination is the Zeno phenomenon, where the number of events triggered goes to infinity in a finite time period. This would not be implementable on hardware which has a maximum operating frequency. A way to prevent this problem is to guarantee that a positive minimum inter-event time (MIET) exists. If one is simply forced, then stability may not be preserved. As noted in [7], the existence of a positive MIET is important, not just to preventing Zeno behavior, but also to ensuring that the event triggering mechanism does not become unimplementable because it requires events to be triggered arbitrarily quickly.

Existing work for event-triggered mechanisms include [8] and [6]. The authors of [8] develop a general method for achieving exponential stabilization using event-triggering based on Lyapunov functions with specific qualities. To prevent Zeno solutions, it ensures a positive MIET, although it does not explicitly calculate it. In [6], the hybrid systems formalism of [5] is applied to event-triggering mechanisms, providing a general framework for event-triggering mechanisms, and it discusses methods for showing that Zeno solutions do not occur. The authors of [7] introduce event-triggered control and the closely related self-triggered control, where the controller uses state information to schedule events ahead of time. They show several different scenarios for these control strategies, and discuss their properties.

The preceding is relevant for event-triggered coordination in general, but we are concerned with applying event-triggering in a distributed manner, so that each agent has its own trigger mechanism which is based only on information available to that agent. This distributed event-triggered control is more practical for large scale systems of multiple agents for which centralized coordination is infeasible. Self-triggered strategies can also be applied in such distributed cases, such as in [9].

In order to study distributed event-triggered strategies, we use a simple but widely applicable example of a network problem involving multiple agents: consensus. Consensus problems are when multiple agents, each with its own dynamics and limited access to the other agents’ states, are intended to be stabilized such that all the agents’ states are equal. Applications include distributed computing, networks of sensors, flocking, rendezvous, attitude alignment, and synchronization of coupled oscillators; see [10], [2] and [1] and their references. Event triggered strategies for consensus problems have been studied for a long time, with some of the earliest works being [11], [12], [13] in 2009 and 2010. See [14] for a detailed survey of works applying event-triggered control to
this consensus problem.

The work in [15] pertains to the consensus problem for agents with single integrator dynamics. It develops centralized event- and self-triggered strategies that lead to consensus, and then modifies them to be distributed. Unfortunately, although the centralized event-triggered strategy is able to provide a positive lower bound on inter-event times, the distributed strategies are unable to guarantee the prevention of Zeno solutions. Similarly, the results in [16] are unable to exclude Zeno behavior. The authors of [9] are able to avoid the Zeno phenomenon for a related consensus problem. A local clock is used which is reset to an upper bound that depends on local states. This self-triggering strategy has an enforced positive MIET and is used to guarantee convergence for ternary controllers, but only to a neighborhood of consensus.

On the other hand, in [17], [18], [19], [20], convergence to consensus is guaranteed, and Zeno solutions are excluded by periodically sampling the condition for an event, rather than continuously monitoring it. However, the algorithms in [17] and [20] require global knowledge of the communication network to choose the sampling period, and all require that the sampling be synchronized across all agents, which is impractical. Some works, such as [21], [22], are able to guarantee consensus by sampling the event-trigger condition, without such global synchronization. However, [21] still requires synchronization between pairs of agents, and both require knowledge of the entire system.

The authors of [23] avoid Zeno behavior by including an explicit function of time in the trigger function, but their algorithm also requires knowledge of global parameters. Additionally, both [24] and [3] present distributed event-triggered strategies for the consensus problem that prevent Zeno solutions and ensure convergence to consensus. They both do so in a manner similar to [23]; the former includes an explicit function of time in the trigger mechanism, and its results apply more generally to agents with linear dynamics. The latter, on the other hand, uses a dynamic triggering mechanism, by including a dynamic virtual state in the triggering mechanism. It is based on the dynamic event trigger of [25], which, in turn, is augmenting the event trigger mechanism of [20] with a dynamic state. Because [23], [24] and [3] only ensure that the times between events for an agent are strictly positive for all time, all of these strategies are unable to guarantee a positive lower bound on those times. Similarly, in [27], Zeno solutions are excluded without a positive MIET, but its main contribution is in providing a set of algorithms which can be chosen from for better performance.

The distributed event-triggered strategy in [28], on the other hand, is able to guarantee convergence for a system of agents with identical linear dynamics with a positive MIET enforced, even in the presence of bounded communication delays. Unfortunately, it requires global parameters in order to design each agent’s controller, so it is not fully distributed.

However, the authors of [4] are able to show the existence of a positive MIET for a fully distributed event-triggered strategy that guarantees asymptotic convergence with an event trigger that employs a dynamic virtual state. This work considers events to be triggered by multiple agents monitoring the same quantity at the same time, requiring a sort of synchronization. Instead, we assume agents alone are responsible for deciding when to broadcast information to their neighbors, which is known as an agent-triggered strategy. On the other hand, the algorithm in [29], employs a similar strategy to achieve a positive MIET guarantee which can be chosen up to a specified maximum, without requiring synchronization. However, the results are limited to undirected communication networks.

Furthermore, a common weakness in many event-triggered strategies is a lack of robustness in the trigger times, because their results rely on events being triggered the instant an inequality is violated. Many of the event-triggered algorithms presented here do not address whether or not their algorithms are robust to slight delays in the times when events are triggered, for example. This lack of robustness, in general, poses a potential problem if implemented, for example, on a system that evaluates the trigger condition periodically.

The authors of [30] note another type of robustness that is lacking. They find that, although some algorithms may guarantee a positive MIET, referred to as an event-separation property, that time can become zero in the presence of arbitrarily small disturbances. They show which classifications of event triggering mechanisms have MIETs which are robust to disturbances by defining such robustness. Informally, a system is said to have the robust global event-separation property if it maintains a positive MIET for all relevant initial conditions in the presence of a bounded disturbance, provided the bound is small enough. If a system lacks this property, it indicates that it may trigger an undesirably large amount of events in a short period of time.

Statement of contributions: The contributions of this paper are threefold. First, we provide the first known fully distributed solution to the problem originally conceived in [11], [12], [13] that guarantees a positive MIET, by extending the results in [29] to directed graphs. Second, we provide a distributed way for designing the triggering functions such that each agent is able to independently prescribe their guaranteed MIET, which has important implications on implementability of solutions on physical platforms. Finally, we investigate the robustness of our algorithm against missed event times and state disturbances, and provide a modified algorithm that is capable of handling these forms of practical uncertainties. Simulations illustrate our results.

II. Preliminaries

The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $||v||$. An $n$-dimensional column vector with every entry equal to 1 is denoted by $1_n$, and an $n$-dimensional column vector with every entry equal to 0 is denoted by $0_n$. The minimum eigenvalue of a square matrix $A$ is given by $\text{eigmin}(A)$ and its maximum eigenvalue is given by $\text{eigmax}(A)$. The distance of $x$ from the set $\mathcal{A}$, which is the $\min_a ||x-a||$, where $a \in \mathcal{A}$.
is denoted by $\|x\|_A$. Given a vector $v \in \mathbb{R}^N$, we denote by $\text{diag}(v)$ the $N \times N$ diagonal matrix with the entries of $v$ along its diagonal.

Young’s inequality is
\[
xy \leq \frac{a}{2} x^2 + \frac{1}{2a} y^2, \tag{1}
\]
for $a > 0$ and $x, y \in \mathbb{R}$.

By $V^{-1}(C)$ where $C \subset \mathbb{R}^m$, we denote the set of points 
\[
\{s \in \mathbb{R}^n : V(s) \in C\},
\]
for a function $V : \mathbb{R}^n \to \mathbb{R}^m$. By $\mathbb{R}_{\geq 0}$ we denote the set of nonnegative real numbers, and by $\mathbb{Z}_{\geq 0}$ we denote the set of nonnegative integers. The closure of a set $U \subset \mathbb{R}^n$ is denoted by $\overline{U}$. The domain of a mapping $f$ is denoted by dom $f$ and its range is denoted by range $f$.

**Graph Theory:** An unweighted graph $G = (V, \mathcal{E}, A)$ has a set of vertices $V = \{1, 2, ..., N\}$, a set of edges $\mathcal{E} \subset V \times V$, and an adjacency matrix $A \in \mathbb{R}^{N \times N}$ with each entry $a_{ij} \in \{0, 1\}$, where $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. For a digraph (directed graph), edge $(i, j)$ is distinct from edge $(j, i)$. A path between vertex $i$ and vertex $j$ is a finite sequence of edges $(i, k), (k, l), (l, m), \ldots, (n, j)$. A digraph is strongly connected if there exists a path between any two vertices. A weighted digraph is one where each edge $(i, j) \in \mathcal{E}$ has a weight $w_{ij} > 0$ to it. For an edge $(i, j)$, $j$ is an out neighbor of $i$ and $i$ is an in neighbor of $j$. The in-degree, $d_{i}^{\text{in}}$, for a vertex $i$ is the sum of all the weights for the edges that correspond to its in neighbors, and the out-degree, $d_{i}^{\text{out}}$, is the same for the out neighbors. A weight-balanced digraph is a digraph where $d_{i}^{\text{in}} = d_{i}^{\text{out}} = d_{i}$ for each vertex $i$. A weighted digraph has a weighted adjacency matrix $A$ where the $ij$th element is the weight for edge $(i, j)$. For a weight-balanced digraph, the degree matrix $D^{\text{out}} = D$ is a diagonal matrix with $d_{i}$ as the $i$th diagonal element, and the Laplacian is $D^{\text{out}} - A$.

**Hybrid Systems:** A hybrid system $\mathcal{H} = (C, f, D, G)$ is a tuple composed of a flow set $C \subset \mathbb{R}^n$, where the system state $x \in \mathbb{R}^n$ continuously changes according to $\dot{x} = f(x)$, and a jump set $D \subset \mathbb{R}^n$, where $x$ discretely jumps to $x^+ \in G(x)$, where $f$ maps $\mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ is set valued [5, Definition 2.2]. While $x \in C$, the system can flow continuously and while $x \in D$, the system can jump discontinuously.

A compact hybrid time domain is a subset $E_{\text{compact}} \subset \mathbb{R} \times \mathbb{N}$ for which $E_{\text{compact}} = \bigcup_{j=1}^{T_{\text{end}}} \{ (t_j, t_{j+1}], j \}$, for a finite sequence of times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_j$, and a hybrid time domain is a subset $E \subset \mathbb{R} \times \mathbb{N}$ such that $\forall (T, J) \in E, E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain [5, Definition 2.3]. The hybrid time domain is used to keep track of both the elapsed continuous time $t$ and the number of discontinuous jumps $j$.

See the appendix for more definitions and results relating to hybrid systems.

**III. PROBLEM STATEMENT**

Consider a group of $N$ agents whose communication topology is described by a directed, weight-balanced, and strongly connected graph $G$ with edges $\mathcal{E}$ and Laplacian matrix $L$. Each agent is able to receive information from its out neighbors and send information to its in neighbors, and each weight of the graph is a gain applied to the information sent from one agent to another.

The state of each agent $i$ at time $t \geq 0$ is given by $x_i(t)$ with single-integrator dynamics
\[
\dot{x}_i(t) = u_i(t),
\]
where $u_i$ is the input for agent $i$. It is well known that the input
\[
u_i(t) = - \sum_{j \in \mathcal{N}_{\text{out}}^i} w_{ij}(x_i(t) - x_j(t))
\]
drives all agent states to the average of the initial conditions $\mathbf{1}$, which is defined as
\[
\bar{x} \triangleq \frac{1}{N} \sum_{i=1}^{N} x_i(0).
\]
Note that under the control law (3), the average $\bar{x}(t)$ is an invariant quantity. Defining $x = [x_1, x_2, \ldots, x_N]^T$ and $u = [u_1, u_2, \ldots, u_N]^T$ as the vectors containing all the state and input information about the network of agents, respectively, we can describe all inputs together by
\[
u(t) = -Lx(t).
\]
However, in order to implement this control law, each agent must have continuous access to the state of each of its out neighbors. Instead, we assume that each agent $i$ can only measure its own state $x_i$ and must receive neighboring state information through wireless communication. We consider event-triggered communication and control where each agent only broadcasts its state to its neighbors at discrete instances of time. More formally, letting $\{t^i_1\}_{t^i_1 \in \mathbb{Z}_{\geq 0}} \subset \mathbb{R}_{\geq 0}$ be the sequence of times at which agent $i$ broadcasts its state to its neighbors $j \in \mathcal{N}_{\text{out}}^i$, the agents instead implement the control law
\[
u(t) = -L\hat{x}(t),
\]
where $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_N]^T$ is the vector of the last broadcast state of each agent. More specifically, given the sequence of broadcast times $\{t^i_1\}_{t^i_1 \in \mathbb{Z}_{\geq 0}}$ for agent $i$, we have
\[
\hat{x}_i(t) = x_i(t^i_k) \quad \text{for} \quad t \in [t^i_k, t^i_{k+1}).
\]
Note that the input (4) still ensures that the average of all agent states is an invariant quantity because $\dot{\bar{x}} = \frac{1}{N} \mathbf{1}_N^T \dot{x} = \frac{1}{N} \mathbf{1}_N^T (-L\bar{x}) = 0$, which follows from the weight-balanced property of the graph.

At any given time $t \geq 0$, we define
\[
v_i(t) \triangleq (x_i(t), \bar{x}_i(t), \{\bar{x}_j(t)\}_{j \in \mathcal{N}_{\text{out}}^i})
\]
as all the dynamic variables locally available to agent $i$. The problem of interest, formalized below, is then to obtain a
triggering condition based on this information such that the sequence of broadcasting times \( \{t_\ell\}_{\ell \in \mathbb{Z}_{\geq 0}} \) guarantees that the system eventually reaches the average consensus state.

**Problem III.1 (Distributed Event Triggered Consensus)**

Given the directed, weight-balanced, and strongly connected graph \( G \) with dynamics \( 2 \) and input \( 4 \), find a triggering condition for each agent \( i \), which depends only on locally available information \( v_i \), such that \( x_i \to \bar{x} \) for all \( i \in \{1, \ldots, N\} \).

This problem was first formulated in \( 11 \), \( 12 \) in 2009. Since then, there have been many attempts at a solution, and, as stated above, there exist solutions the correctly solve this problem in both the undirected \( 17 \), \( 23 \), \( 15 \), \( 16 \), \( 3 \), \( 29 \) and directed \( 18 \), \( 19 \), \( 27 \) cases. Unfortunately, although the above referenced papers provide theoretical solutions to this problem, we are not aware of a single solution which can be implemented on physical systems. For details on the history of this problem and attempted solutions we refer the interested reader to \( 14 \), but we summarize the main points here.

The earliest solutions to this problem, such as \( 15 \), \( 16 \), ignored the possibility of Zeno phenomena which invalidates their correctness. In particular, these solutions to Problem III.1 did not rule out the possibility of Zeno behavior meaning that it was possible for a sequence of broadcasting times \( \{t_\ell\}_{\ell \in \mathbb{Z}_{\geq 0}} \) to converge to some finite time \( t_\ell \to T > 0 \). This is clearly troublesome since all theoretical analysis then falls apart after \( t > T \), invalidating the asymptotic convergence results.

More recently, the community has acknowledged the importance of ruling out Zeno behavior to guarantee that all the sequences of times \( t_\ell \to \infty \) as \( \ell \to \infty \) for all \( i \in \{1, \ldots, N\} \). While enforcing this additional constraint on the sequences of broadcasting times guarantees that theoretically the solutions will converge to the average consensus state, there are still some important practical issues that must be considered.

More specifically, even if it can be guaranteed that Zeno behaviors do not occur, and the inter-event times are strictly positive for all agents \( i \),

\[
t_\ell^{i+1} - t_\ell^i > 0,
\]

unfortunately this is still not enough to guarantee that the solution can be implemented across physical platforms. This is because although the inter-event times are technically positive, they can become arbitrarily small to the point that no physical hardware exists that can keep up with the speed of actions required by the event-triggered algorithm. The solution in \( 3 \) has this problem. This is inherently different from guaranteeing a strictly positive MIET \( \tau \), where \( t_\ell^{i+1} - t_\ell^i \geq \tau > 0 \), which is the focus of our work here.

Specifically, we consider the case where each agent \( i \) has some maximum rate \( \frac{1}{\tau_i} \) at which it can take actions (e.g., broadcasting information, computing control inputs). That is, each agent \( i \) cannot broadcast twice in succession in less than \( \tau_i \) seconds. In other words, each agent \( i \in \{1, \ldots, N\} \) is limited by hardware in terms of how fast they are able to take actions,

\[
t_\ell^{i+1} - t_\ell^i \geq \tau_i,
\]

for all \( \ell \in \mathbb{Z}_{\geq 0} \).

Note that there are also solutions that guarantee a MIET, but make other sacrifices to do so. For example, the solution in \( 23 \) is able to guarantee a MIET under certain conditions, but convergence is only to a neighborhood of consensus. Additionally, the algorithms in \( 23 \), \( 20 \) are able to enforce a MIET, but only by using global parameters of the system to design the algorithm, which is impractical in cases where the parameters may change or otherwise be difficult to measure. The algorithm in \( 4 \) is fully distributed and has a positive MIET, but it still requires pair of agents to trigger events at the same time, which necessitates synchronization. With all this in mind, we reformatulate Problem III.1 such that solutions to the problem can be implemented on physical platforms given that each agent \( i \) is capable of processing actions at a frequency of up to \( \frac{1}{\tau_i} \).

**Problem III.2 (Distributed Event Triggered Consensus with Designable MIET)**

Given the directed, weight-balanced, and strongly connected graph \( G \) with dynamics \( 2 \), input \( 4 \), and the minimum periods \( (\tau_1, \ldots, \tau_N) \) for each agent, find a triggering condition for each agent \( i \), which depends only on local information \( v_i \), such that \( x_i \to \bar{x} \) and

\[
\min_{\ell \in \mathbb{Z}_{\geq 0}} t_\ell^{i+1} - t_\ell^i \geq \tau_i,
\]

for all \( i \in \{1, \ldots, N\} \).

Although \( 29 \) provides a solution to a similar problem for undirected graphs, to the best of our knowledge, Problem III.2 has not yet been solved for a directed graph, and so we provide the first complete solution to it. Later in Section VII we study the robustness of the proposed algorithm with respect to various forms of disturbances.

### A. Hybrid Systems Formulation

In order to solve Problem III.2 we first reformulate it using tools from Hybrid Systems. This gives us the advantage of being able to properly keep track of the physical state of the network \( x \) separately from so-called virtual states that pertain to the memory stored by the agents. In this case we can treat the last broadcast state \( \bar{x} \) as an additional virtual state of the system that gets updated only at event times. To help us establish a specified positive MIET later, we introduce an additional virtual state \( \chi_i \) for each agent \( i \in \{1, \ldots, N\} \) and collect these components in the vector \( \chi = [\chi_1, \chi_2, \ldots, \chi_N]^T \).

The extended state vector for a single agent \( i \) is then given by

\[
q_i(t) = \begin{bmatrix} x_i(t) \\ \bar{x}_i(t) \\ \chi_i(t) \end{bmatrix},
\]
and the extended state of the entire system is \( q = [q_1, q_2, \ldots, q_N]^T \in \mathbb{R}^{3N} \). Letting \( \chi_i \) be an internal dynamic variable only available to agent \( i \), with a slight abuse of notation we redefine the information locally available to agent \( i \) by augmenting it with the additional variable \( \chi_i \),

\[
v_i(t) \triangleq (q_i(t), \{\tilde{x}_j(t)\}_{j \in N_i^{\text{out}}}).
\]

The exact dynamics of each \( \chi_i \) will be designed later as part of our algorithm. However, the intuitive idea of the virtual state \( \chi_i \geq 0 \) is a type of timer or “clock-like” variable that begins at some value \( \chi_i > 0 \) and decreases towards 0, at which time agent \( i \) broadcasts its state \( x_i \) to its neighbors, updating \( \tilde{x}_i = x_i \), and the clock-like state is reset to its maximum value \( \chi_i = \chi_i \). This variable will drive the timing of events. Thus, the triggering condition is defined as

\[
h_i(v_i) \triangleq \chi_i = 0.
\]

More specifically, the sequence of event times \( \{t^i_\ell\}_{\ell \in \mathbb{Z}_{\geq 0}} \) is given by

\[
t_{\ell + 1} = \min\{t \geq t_{\ell} | x_i(t) = 0\},
\]

for all \( \ell \in \mathbb{Z}_{\geq 0} \).

Note that we want the dynamics of \( \chi_i \) to be determined by only the information \( v_i \) that is available to agent \( i \). Thus, let \( \gamma_i(v_i) \triangleq \chi_i \) be the function describing the dynamics of \( \chi_i \) that is to be designed in Section IV.

With the role of our clock-like variable defined, we can formally define our hybrid system

\[
H = (C, f, D, G).
\]

The flow set is given by

\[
C = \{ q \in \mathbb{R}^{3N} : \chi_i \geq 0 \ \forall i = 1, 2, \ldots, N \},
\]

The dynamics of each agent \( i \) while the system is flowing is

\[
\dot{q}_i = f_i(v_i) \triangleq \begin{bmatrix}
-(L\tilde{x})_i \\
0 \\
\gamma_i(v_i)
\end{bmatrix}, \quad \text{for } q \in C,
\]

where \( (L\tilde{x})_i \) denotes the \( i \)th row of the column vector \( L\tilde{x} \).

The dynamics of the whole system is then given by

\[
\dot{q} = f(q) = \begin{bmatrix}
f_1(v_1) \\
\vdots \\
f_N(v_N)
\end{bmatrix}, \quad \text{for } q \in C.
\]

The jump set is given by \( D = \bigcup_{i=1}^N D_i \), where

\[
D_i = \{ q \in \mathbb{R}^{3N} : \chi_i \leq 0 \}
\]

denotes the portion of the jump set that corresponds to agent \( i \) triggering an event, i.e., \( q \in D_i \) means that agent \( i \) can trigger an event.

For \( q \in D_i \), we begin by considering the following local jump map

\[
g_i(q) = \begin{bmatrix} q_i^+ \\ x_i \\ q_N \end{bmatrix} \triangleq \begin{bmatrix} q_i \\ \vdots \\ x_i \\ \chi_i \\ q_N \end{bmatrix}.
\]

More specifically, letting \( t^i_\ell \) be the time at which agent \( i \) triggers its \( \ell \)th event \( q(t^i_\ell) \in D_i \), this map leaves the physical state unchanged \( x^+_i = x_i \), updates its ‘last broadcast state’ to its current state \( \tilde{x}^+_i = x_i \), and resets the clock-like variable \( \chi_i = \chi_i \). Note also that this leaves all other agents’ states unchanged \( q^+_j = q_j \) for all \( j \neq i \).

Since \( q \in D \) doesn’t specify which agent \( i \) triggers an event, the jump map must be described by a set-valued map \( G : \mathbb{R}^{3N} \rightrightarrows \mathbb{R}^{3N} \), where

\[
G(q) = \{ g_1(q), \ldots, g_N(q) \}.
\]

Now, we reformulate Problem III.2 in a more structured manner by using the hybrid system \( H \). Note that it is important that the dynamics of each \( \chi_i \) depend only on local information, to ensure that the resulting algorithm is fully distributed.

**Problem III.3 (Distributed Event Triggered Consensus with Designable MIET)** Given the directed, weight-balanced, and strongly connected graph \( \mathcal{G} \) and the hybrid system \( H \) in (6), find the dynamics of \( \chi_i \), \( \gamma_i(v_i) \), such that \( x_i \to \bar{x} \) and

\[
\min_{\ell \in \mathbb{Z}_{\geq 0}} t^i_{\ell+1} - t^i_\ell \geq \tau_i,
\]

for all \( i \in \{1, \ldots, N\} \).

**IV. Dynamic Event-Triggered Algorithm Design**

In order to solve Problem III.3, we perform a Lyapunov analysis to design \( \gamma \triangleq \chi \), the dynamics of \( \chi \). Inspired by (3), we use a Lyapunov function with two components: \( V_P \) depends on the physical aspects of the system, while \( V_C \) depends on the cyber aspects introduced by wireless communication. For convenience, let \( e \triangleq x - \bar{x} \) denote the vector containing the error for each agent’s state, which is the difference between the actual state and the last broadcast state. We begin by considering

\[
V_P(q) = \frac{1}{2}(x - \bar{x})^T(x - \bar{x}) = \frac{1}{2}||x - \bar{x}||^2,
\]

and

\[
V_C(q) = \frac{1}{2} \sum_{i=1}^N \chi_i e_i^2 \triangleq \frac{1}{2}e^T \mathcal{X} e,
\]
where we define $X = \text{diag}(\chi)$. Note $V_C \geq 0$ because $\chi_i \geq 0$ for all $i \in \{1, \ldots, N\}$. We then consider the Lyapunov function
\[ V(q) = V_P(q) + V_C(q). \] (12)
Note that $V(q) \geq 0$ is continuously differentiable for all $q \in \mathbb{R}^{3N}$. Moreover, $V(q) = 0$ when all agents have reached their target state and each agent’s error $e_i$ or clock-like variable $\chi_i$ is equal to 0. We define
\[ A = \{ q \in \mathbb{R}^{3N} : V(q) = 0 \} \]
\[ = \{ q \in \mathbb{R}^{3N} : V_C(q) = 0 \text{ and } ||x - \bar{x}||^2 = 0 \} \]
as this target set we want our hybrid system to reach.

Now we will examine the evolution of $V$ along the trajectories of our algorithm to see under what conditions it is nonincreasing, and design $\gamma$ accordingly.

Recalling $x = \bar{x} + e$ and defining $\Gamma = \text{diag}(\gamma),
\begin{align*}
V_P &= -(x - \bar{x})^T L \bar{x} \\
V_C &= -e^T \mathcal{X} L \bar{x} + \frac{1}{2} e^T \Gamma e.
\end{align*}
Because the graph is weight-balanced, $\bar{x}^T L = 0_N$. Therefore,
\[ V_P = -x^T L \bar{x} = -\bar{x}^T L \bar{x} - e^T \mathcal{X} L \bar{x} + \frac{1}{2} e^T \Gamma e. \]

Expanding this out yields
\begin{equation}
\hat{V} = \sum_{i=1}^{N} \left( -\frac{1}{2} \sum_{j \in \mathcal{N}^{in}_i} w_{ij}(\hat{x}_i - \hat{x}_j)^2 - e_i(L \hat{x})_i \right) - e_i \chi_i(L \hat{x})_i + \frac{1}{2} \gamma_i e_i^2 \right).
\end{equation}

For convenience in notation, let $\hat{z} = L \bar{x}$ and $\hat{\varphi}_i = \sum_{j \in \mathcal{N}^{in}_i} w_{ij}(\hat{x}_i - \hat{x}_j)^2$.

Rewriting (14), we have
\[ V_i = \sum_{i=1}^{N} \left( -\frac{1}{2} \hat{\varphi}_i - e_i \hat{z}_i - e_i \chi_i \hat{z}_i + \frac{1}{2} \gamma_i e_i^2 \right). \]
We are now interested in designing $\gamma_i$ for each agent $i \in \{1, \ldots, N\}$ such that $V(q) < 0$ for all $q \not\in A$.

In the case that $e_i = 0$, we automatically have $V_i = -\frac{1}{2} \hat{\varphi}_i < 0$. In the case that $e_i \neq 0$, choosing
\[ \gamma_i < \frac{\hat{\varphi}_i}{e_i^2} + 2(\chi_i + 1) \frac{\hat{z}_i}{e_i} \]
eq 0, choosing
\[ \gamma_i < \frac{\hat{\varphi}_i}{e_i^2} + 2(\chi_i + 1) \frac{\hat{z}_i}{e_i} \]
ensures that $V_i < 0$. Since $\gamma_i$ denotes the dynamics of the clock-like state $\chi_i$ for agent $i$, we constrain it such that $\gamma_i = \dot{\chi}_i$ is negative so that $\chi_i$ remains in the interval $[0, \chi_i]$.

Therefore, in order to satisfy (15), we define
\[ \gamma_i(v_i) = \begin{cases} 
\min\{\gamma_i, 0\} - \varepsilon_i & \text{for } e_i \neq 0, \\
-\varepsilon_i & \text{for } e_i = 0,
\end{cases} \]
where $\varepsilon_i > 0$ is a design parameter that essentially sets the minimum speed at which the clock-like variable counts down towards 0. This parameter will be helpful later to force a specified MIET. This choice of the clock-like dynamics $\gamma_i$ is continuous in $q$ for constant $\bar{x}$ and ensures that $V < 0$ as long the extended network state doesn’t belong to the target state, that is, $q \not\in A$.

Algorithm Synthesis

Here we summarize all the components of our synthesized distributed dynamic event-triggered coordination algorithm and formally describe it. From the viewpoint of a single agent $i \in \{1, \ldots, N\}$, the synthesized communication and control strategy is informally described as follows.

The control input at any given time $t \geq 0$ is
\[ u_i(t) = -(L \hat{x}(t))_i = - \sum_{j \in \mathcal{N}^{in}_i} (\hat{x}_i(t) - \hat{x}_j(t)), \]
where $\hat{x}(t)$ is updated according to (4). The sequence of event times $\{t_i\}_{t \geq 0}$ at which agent $r$ broadcasts its state to neighbors is given by each time the clock-like variable reaches zero, i.e.,
\[ t_i^k = \min\{t \geq t_i^{k-1} : \chi_i(t) = 0\}. \]

The design parameters are $\chi_i > 0$ and $\varepsilon_i > 0$. Both design parameters have a similar effect on the clock-like dynamics, with $\varepsilon_i$ changing its speed and $\chi_i$ changing the ‘distance’ it must travel between events. Increasing $\varepsilon_i$ or decreasing $\chi_i$ enables the clock to reach its lower threshold from its upper one faster, potentially resulting in more events but faster convergence, while decreasing the MIET.

The algorithm is formally presented in Table I

| Initialization; at time $t = 0$ each agent $i \in \{1, \ldots, N\}$ performs: |
| 1: Initialize $\hat{x}_i = x_i$ |
| 2: Initialize $\chi_i = \chi_i$ |
| At all times $t$ each agent $i \in \{1, \ldots, N\}$ performs: |
| 1: if $\chi_i = 0$ then |
| 1a: set $\hat{x}_i = x_i$ (broadcast state information to neighbors) |
| 1b: set $\chi_i = \chi_i$ (reset clock-like variable) |
| 2: set $u_i = -\sum_{j \in \mathcal{N}^{in}_i} w_{ij}(\hat{x}_i - \hat{x}_j)$ (update control signal) |
| 3: else |
| 3a: propagate $\chi_i$ according to its dynamics $\gamma_i$ in (16) |
| 4: end if |
| 5: propagate $\chi_i$ according to its dynamics $\gamma_i$ in (16) |
| 6: end if |
| 7: if new information $\hat{x}_k$ is received from some neighbor(s) $k \in \mathcal{N}^{out}_i$ then |
| 8: set $u_i = -\sum_{j \in \mathcal{N}^{out}_i} w_{ij}(\hat{x}_k - \hat{x}_j)$ (update control signal) |
| 9: end if |

TABLE I

DISTRIBUTED Dynamic EVENT-Triggered COORDINATION ALGORITHM.

V. MAIN RESULTS

Here we present the main results of the paper by discussing the properties of our algorithm. We begin by finding the
guaranteed positive minimum inter-event time (MIET) for each agent.

**Theorem V.1 (Positive MIET)** Given the hybrid system \( \mathcal{H} \), if each agent \( i \) implements the distributed dynamic event-triggered coordination algorithm presented in Table 7 with \( \chi_i > 0 \) and \( \varepsilon_i > 0 \), then the inter-event times for agent \( i \) are lower-bounded by

\[
T_i \geq \sqrt{\frac{w_{\min,i}}{\varepsilon_i ||W_i||^2}} \left( \atan \left[ \frac{||W_i||^2}{\varepsilon_i w_{\min,i}} \right] (\chi_i + 1) \right) - \atan \left[ \frac{||W_i||^2}{\varepsilon_i w_{\min,i}} \right] > 0, \tag{17}
\]

where \( W_i \) is a vector of the weights with agent \( i \)'s out neighbors, so \( ||W_i||^2 = \sum_{j \in N_i} w_{ij}^2 \), and \( w_{\min,i} = \min_{j \in N_i} w_{ij} \). That is,

\[
t_{\ell+1}^i - t_{\ell}^i \geq T_i
\]

for all \( i \in \{1, \ldots, N\} \) and \( \ell \in \mathbb{Z}_{\geq 0} \).

**Proof:** See the appendix. \( \square \)

Next, we present our main convergence result. To the best of our knowledge, this is the first work to design a fully distributed event-triggered communication and control algorithm that guarantees asymptotic convergence to the average consensus state with a lower bound on the agent-specific MIET that can be chosen by the designer. Note that, by selecting design parameters, MIETs can be guaranteed up to a maximum of \( T_{i,\max} = \frac{w_{\min,i}}{||W_i||^2} \) for all agents. If this is not large enough, \( \tau_i \geq T_{i,\max} \), then a global gain can be adjusted as explained later.

**Theorem V.2 (Asymptotic Convergence)** Given the hybrid system \( \mathcal{H} \), if each agent \( i \) implements the distributed dynamic event-triggered coordination algorithm presented in Table 7 with agent \( i \) triggering events when \( \chi_i = 0 \) and with \( \chi_i > 0 \) and \( \varepsilon_i > 0 \), then all trajectories of the system are guaranteed to asymptotically converge to the average consensus state, i.e.,

\[
x_i(t) \rightarrow \bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j(0)
\]

as \( t \rightarrow \infty \) for all \( i \in \{1, \ldots, N\} \).

**Proof:** See the appendix. \( \square \)

**Minimum Inter-Event Time Design**

The design parameters \( \chi_i \) and \( \varepsilon_i \) can be chosen to achieve a specific MIET for agent \( i \). According to Problem III.2, we must be able to guarantee that the lower-bound on the MIET \( T_i \) as provided in Theorem V.1 is greater than or equal to the prescribed \( \tau_i \).

From Theorem V.1, we see that the lower bound on the MIET depends on the design parameters \( \varepsilon_i \) and \( \chi_i \). Note that the equation for the MIET in Theorem V.1 is an increasing function of \( \chi_i \) and a decreasing function of \( \varepsilon_i \). However, we also notice that selecting these parameters can only affect \( T_i \) so much. Therefore, we define

\[
T_{i,\max} = \frac{w_{\min,i}}{||W_i||^2}
\]

as the maximum achievable MIET by choosing these parameters. Consequently, if \( \tau_i \leq T_{i,\max} \) for all \( i \in \{1, \ldots, N\} \), then it is easy to see how (17) in Theorem V.1 can directly be used to choose the design parameters appropriately for each agent. In the case that there exists some agent(s) \( j \) such that \( \tau_j > T_{j,\max} \), we must actually redefine the original controller (4) by introducing a scaling factor to essentially slow down the convergence of the entire network to accommodate the slow agents \( i \) with large \( \tau_i \). This is discussed in Remark V.3 below.

**Remark V.3 (Arbitrary MIET Selection)** The preceding discussion seems to imply that there is a maximum achievable MIET for each agent. However, this is because we have not yet considered that the weights of the graph might be adjusted in the control input (4). Note that there exist distributed methods of choosing these gains for an existing strongly connected digraph so that it will be weight-balanced [33], [34]. However, to quantify this in a simple manner, let us consider the graph defined by the Laplacian \( L' = KL \), where \( K > 0 \) is a gain applied to each agent’s input, so that \( L' \) is another weight-balanced Laplacian for the same set of vertices and set of edges. In this case, (18) becomes

\[
T_{i,\max}' = \frac{w_{\min,i}}{||W_i||^2}.
\]

This indicates that, if the maximum achievable MIET is not high enough for some agent \( i \), e.g., \( T_{i,\max} < \tau_i \), then the gain \( K \) can be chosen smaller than 1 allowing for the selection of arbitrarily large MIETs. However, this must be done with care, because the same gain \( K \) must be applied to the input of each agent, to preserve the weight-balanced property of the graph, and because this will ultimately result in slower convergence.

**VI. Robustness**

A problem with many event triggered algorithms is a lack of robustness guarantees. In particular, we consider the effects of two different types of disturbances that are generally problems in event-triggered controllers and show how our algorithm is robust against them. More specifically, we show that our algorithm has a robust positive MIET against arbitrary state disturbances and that with a slight modification it can also be made robust against imperfect event detection.

**Robustness Against State Disturbances**

We first analyze the robustness of our algorithm against state disturbances. As noted in [30], simply guaranteeing a positive
MIET may not be practical if the existence of arbitrarily small disturbances can remove this property, resulting again in solutions that might require the agents to take actions faster than physically possible to still ensure convergence. Therefore, it is desirable for our algorithm to exhibit robust global event-separation, as defined in \[30\], which essentially means that the algorithm guarantees a positive MIET for all initial conditions even in the presence of state disturbances.

Instead of the deterministic dynamics (2), consider
\[
\dot{x}_i(t) = u_i(t) + w_i(t),
\]
where \(w_i(t)\) is an arbitrary, unknown, additive state disturbance applied to each agent’s state. Remarkably, it is easy to see in this case that the algorithm presented in Table \[\ref{table:algorithm}\] results in a system with the global event-separation property, meaning that the lower bound on the MIET (17) still holds for the disturbed system (19). This property holds because the derivation of the formula for the MIET makes no assumption about these states and depends only on the clock-like state \(\chi_i\), so disturbances to \(x\) and \(\hat{x}\) have no effect on the calculated MIET (17). This result is formalized next and its proof is analogous to that of Theorem \[\ref{thm:robust-positive-miet}\].

**Corollary VI.1 (Robust Positive MIET)** The positive MIET (17) is robust to state disturbances of the form (19).

It is worth noting here that this result does not even require the disturbance \(w_i(t)\) to be bounded. However, this does not guarantee convergence all the way to consensus in the presence of disturbances, simply that the positive MIET will be preserved. Note that the conditions of convergence would of course depend on the properties of the disturbance \(w_i(t)\) but it is easy to show that for zero-mean, slowly changing disturbances all agent states will asymptotically converge to their initial average in expectation.

**Robustness Against Imperfect Event Detection**

In addition to robustness against state disturbances, another important source of uncertainty that cannot be overlooked in event-triggered control system is imperfect event detection. Event-triggered controllers are generally designed and analyzing assuming very precise timing of different actions are possible while continuously monitoring the event conditions. This is not only impractical but problematic if not considered in the event-triggered control design.

For our algorithm presented in Table \[\ref{table:algorithm}\] convergence is no longer guaranteed for arbitrarily small delays in event times. That is, if the event condition \(\chi_i(t^\star) = 0\) is satisfied for some agent \(i\) but for whatever reason is not detected or acted upon until time \(t^\star + t_c\) for some arbitrarily small delay \(t_c > 0\), the Lyapunov function \(V\) given by (12) may increase when the next jump finally occurs which invalidates our certificate of convergence.

Fortunately, the algorithm can be slightly modified to instead exhibit robustness against these imperfect timings on event detection. Instead of each agent \(i\) triggering an event precisely when \(\chi_i = 0\) using the trigger function (5), each agent can instead trigger an event anytime \(\chi_i \leq \chi_i\) using the trigger function
\[
\tilde{h}(v_i) \triangleq \chi_i - \chi_i \leq 0
\]
where \(\chi_i \in (0, \chi_i)\) is a design parameter that captures how tolerant the missed event detection needs to be. In particular, whereas the original algorithm presented requires the event to be triggered exactly when the clock-like variable precisely hits 0, we now allow an event to be triggered anywhere in a window to preserve the convergence properties of the algorithm. This is formalized next.

**Theorem VI.2 (Robust Convergence with MIET)** Given the hybrid system \(\mathcal{H}_i\), if each agent \(i\) implements the distributed dynamic event-triggered coordination algorithm presented in Table \[\ref{table:algorithm}\] with triggering function (20), \(\chi_i > \chi_i > 0\), and \(\varepsilon_i > 0\), then all trajectories of the system are guaranteed to asymptotically converge to the average consensus state, i.e.,
\[
x_i(t) \to \bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j(t) \quad \text{as } t \to \infty \text{ for all } i \in \{1, \ldots, N\}.\]
Additionally, the inter-event times for agent \(i\) are lower-bounded by
\[
\tilde{T}_i \triangleq \sqrt{\frac{\hat{w}_{\min,i}}{\varepsilon_i \|W_i\|^2}} \left( \mathrm{atan} \left( \frac{\|W_i\|^2}{2 \hat{w}_{\min,i} \varepsilon_i} (\chi_i + 1) \right) - \mathrm{atan} \left( \frac{\|W_i\|^2}{2 \hat{w}_{\min,i} \varepsilon_i} (\bar{\chi}_i + 1) \right) \right) > 0. \tag{21}
\]
That is,
\[
t^i_{\ell+1} - t^i_{\ell} \geq \tilde{T}_i
\]
for all \(i \in \{1, \ldots, N\}\) and \(\ell \in \mathbb{Z}_{\geq 0}\). Moreover, \(\tilde{T}_i\) is a robust positive MIET for disturbances of the form (19).

**Proof:** See the appendix.

**Remark VI.3 (Trigger Robustness)** The implication of Theorem \[\ref{thm:robust-event-convergence}\] is the following. Intuitively, rather than agent \(i\) waiting for the clock-like variable \(\chi_i\) to hit exactly zero and respond immediately, it simply begins triggering an event when \(\chi_i\) hits \(\chi_i > 0\) instead, and as long as the event can be detected and fully responded to by the time \(\chi_i\) hits 0, the algorithm will work as intended. However, note that this imposes a trade off because triggering earlier will result in a shorter guaranteed MIET as is also shown in the result of Theorem \[\ref{thm:robust-event-convergence}\].

As a simple example of why one might use the robust version of the algorithm presented in Theorem \[\ref{thm:robust-event-convergence}\] one could imagine a sampled-data implementation of the problem considered in this paper. Until now, we have simply assumed that each agent measures its own state continuously...
for simplicity, but, in practice, this must be done at discrete time instances, which can result in events being triggered imperfectly. However, as noted in Remark VI.3, convergence may still be guaranteed as long as each agent triggers the event before its clock becomes negative. This is formalized in the following corollary.

**Corollary VI.4 (Sampled State Implementation)** Given the hybrid system \( \mathcal{H} \), if each agent \( i \) implements the distributed dynamic event-triggered coordination algorithm presented in Table 2 except that each agent \( i \) triggers events when \( \chi_i \in [0, \chi_i] \), with agent \( i \) sampling its own state \( x_i \) with a maximum time between samples of \( T_{s,i} \) and with \( \nabla_i > 0 \), \( \chi_i \in (0, \chi_i) \), and \( \varepsilon_i > 0 \), then all trajectories of the system are guaranteed to asymptotically converge to the average consensus state, if the following condition holds

\[
T_{s,i} \leq \sqrt{\frac{W_{\text{min},i}}{\varepsilon_i ||W_i||^2}} \left( \frac{\sqrt{||W_i||^2 + \chi_i + 1}}{\varepsilon_i W_{\text{min},i}} \right) - \frac{\sqrt{||W_i||^2}}{\varepsilon_i W_{\text{min},i}}. \tag{22}
\]

**Proof:** If the maximum time between samples is \( T_{s,i} \), then the latest agent \( i \) will trigger an event is at time \( t' + T_{s,i} \), where \( t' \) is the most recent time such that \( \chi_i(t') = \chi_i \). The right hand side of the inequality \( (22) \) gives the minimum amount of time for \( \chi_i \) to reach zero from \( \chi_i \) by the same argument provided in the proof of Theorem VI.1. Therefore, if this inequality holds, then \( \chi_i(t' + T_{s,i}) \geq 0 \) and by Theorem VI.2, all trajectories of the system are guaranteed to asymptotically converge to the average consensus state.

---

**VII. SIMULATIONS**

To demonstrate our distributed event-triggered control strategy, we perform various simulations using \( N = 5 \) agents and a directed graph whose Laplacian is given by \( L' = KL \), where

\[
L = \begin{bmatrix}
2 & -1 & 0 & -1 & 0 \\
0 & 2 & 0 & -2 & 0 \\
-2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & -3 \\
0 & -1 & -2 & 0 & 3
\end{bmatrix}.
\]

All simulations, unless otherwise specified, use the same initial conditions of \( \bar{x} = x = [-1, 0, 2, 2, 1]^T \) in order to explore the effects of the different design parameters. For simplicity, we set \( \chi_i = \chi \) and \( \varepsilon_i = \varepsilon \), for all \( i \in \{1, \ldots, N\} \), so that all agents have identical design parameters. For these simulations, we choose nominal values for each parameter; \( K = 1 \), \( \chi = 1 \), and \( \varepsilon = 1 \). Considering each agent’s difference from the average as an output, similar to \( \mathcal{H}_2 \), we adopt the square of the \( H_2 \)-norm of the system as a cost performance metric

\[
C \triangleq \int_{t=0}^{t=\infty} \sum_{i=1}^{N} (x(t) - \bar{x})^2 dt.
\]

Each simulation was run until \( \sum_{i=1}^{N} (x(t) - \bar{x})^2 \leq 10^{-4} \).

The first simulations were run with these nominal parameters. Figure 1 shows these results. Figure 1 (a) shows the positions of the agents over time, demonstrating that they converge to initial average position, indicated by the dashed line and the evolution of the Lyapunov function, which can be seen to be nonincreasing, as expected. Figure 1(b) shows the evolution of the clock-like state, \( \chi_5 \), for agent 5, and when each agent triggers an event, demonstrating the asynchronous, aperiodic nature of event triggering. The number of events, cost, observed MIET, and calculated lower bound on the MIET were 40, 1.6216, 0.2552 and 0.0897, respectively. Note that the observed MIET is indeed higher than the calculated lower bound. Figure 1(c) shows the effect of applying an additive white Gaussian noise disturbance, i.e. \( \bar{x} = u + w \), where each element of \( w \) is an independent and identically distributed Gaussian process, with zero mean and a variance of 0.1. This suggests that the expected value of each agent’s state is the current average position, although that average can now change with time. In this case, the calculated lower bound on the MIET was 0.0897, which is less than the observed MIET of 0.1824, showing that the bound on the MIET was preserved, as expected.

Next, to show the effect of each design parameter on each algorithm’s performance, each parameter was varied, one at a time, while all other parameters remained at their nominal values. Figure 2 (a), (b), and (c) show the results of varying \( \chi \), \( \varepsilon \), and \( K \), respectively. Each subplot shows the effect of varying the design parameter on one of 3 statistics: the total number of events triggered, the measured cost, and the calculated MIET. The choice of parameters \( \chi \) and \( \varepsilon \) can be seen to be a trade off between cost (speed of convergence) and MIET, with the higher values of \( \chi \) and lower values of \( \varepsilon \) increasing both the cost and the MIET. For \( \chi \), both the cost and the MIET can be seen to exhibit asymptotic behavior. Note that decreasing \( \varepsilon \) also shows this asymptotic behavior, although this cannot be seen from the graphs. The number of events triggered during the simulation does not seem to have a clear correlation to either parameter, and it varies depending on the initial conditions. Decreasing \( K \) is shown to increase the MIET, allowing for arbitrarily high ones as \( K \) approaches 0, but this comes with a steep penalty because both the number of events and the cost are drastically increased. This is expected, because decreasing \( K \) means that the entire system is slowed down.

**VIII. CONCLUSIONS**

This paper has used the multi-agent average consensus problem to present a dynamic agent-focused event-triggered mechanism which ensures stabilization and prevents Zeno solutions by allowing for a chosen minimum inter-event time for each agent. The algorithm is fully distributed in that it not only requires no global parameters, but the correctness of the algorithm can also be guaranteed by each agent individually. That is, no global conditions (besides connectivity of the graph) need to even be checked to ensure the overall system
asymptotically converges. Additionally, it provides robustness against missed event times, guaranteeing convergence as long as events are triggered within a certain amount of time.

While this work has presented an algorithm that distributed agents can implement to guarantee asymptotic convergence, further research is needed to study the transient properties or our proposed and related algorithms. More specifically, our algorithm is guaranteed to asymptotically converge but provides no guarantees on the benefits with respect to traditional implementation methods in terms of different performance metrics, such as amount of communication. We plan to rigorously quantify these types of trade-offs in future works.

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A. Hybrid Systems Results

Lemma A.1 (Lemma 5.10 in [5]) A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous if and only if the graph of $M$ is closed.

Definition A.2 (Definition 2.4 in [5]) A function $\varphi : E \to \mathbb{R}^n$ is a hybrid arc if $E$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \varphi(t, j)$ is locally absolutely continuous on the interval $I^j = \{t : (t, j) \in E\}$.

Definition A.3 (Definition 2.6 in [5]) A hybrid arc $\varphi$ is a solution to the hybrid system $(C, F, D, G)$ if $\varphi \in \mathcal{C} \cup D$, and

(i) for all $j \in \mathbb{N}$ such that $I^j \triangleq \{t : (t, j) \in \text{dom} \varphi\} \neq \emptyset$ and $t \in \text{int} I^j$,

\[
\varphi(t, j) + F(\varphi(t, j)) = \phi(t, j) \quad \text{for almost all } t \in I^j
\]

(ii) for all $(t, j) \in \text{dom} \varphi$ such that $(t, j + 1) \in \text{dom} \varphi$,

\[
\varphi(t, j + 1) \in G(\varphi(t, j)).
\]

Definition A.4 (Definition 2.7 in [5]) A solution $\varphi$ is maximal if there does not exist another solution $\varphi'$ to $\mathcal{H}$ such that $\text{dom} \varphi'$ is a strict subset of $\text{dom} \varphi$ and $\varphi'(t, j) \neq \varphi(t, j) \forall (t, j) \in \text{dom} \varphi$.

Theorem A.5 (Theorem 6.8 in [5]) If a hybrid system $\mathcal{H}$ satisfies the following assumption, then it is nominally wellposed. $C$ and $D$ are closed subsets of $\mathbb{R}^n$;

(i) $F : \mathbb{R}^n \to \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to $C$, $C$ is a subset of the domain of $F$, and $F(q)$ is convex for every $q \in C$;

(ii) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to $D$ and $D$ is a subset of the domain of $G$.

A solution $\varphi$ is complete if $\text{dom} \varphi$ is unbounded.

Definition A.6 (Weak Invariance [5]) Given a hybrid system $\mathcal{H}$, a set $S \subseteq \mathbb{R}^n$ is said to be

- **weakly forward invariant** if, for every $q \in S$, $\exists$ a complete solution $\varphi$ to $\mathcal{H}$ with initial condition $q$ whose range is a subset of $S$;

- **weakly backward invariant** if for every $q \in S$ and every $\tau > 0$, there exists at least one maximal solution $\varphi$ to $\mathcal{H}$ with initial condition $q$ in $S$ such that for some $(t^*, j^*)$ in the domain of $\varphi$, $t^* + j^* > \tau$, it is the case that $\varphi(t^*, j^*) = q$ and $\varphi(t, j) \in S \forall (t, j)$ in the domain of $\varphi$ with $t + j \leq t^* + j^*$,

- **weakly invariant** if it is both weakly forward invariant and weakly backward invariant.

APPENDIX

The necessary results from hybrid systems for the rigorous proof are presented here, and the proof of of Theorems [V1] [VI.2] and [V2] follow.
For a solution \( \varphi \) to a hybrid system \( \mathcal{H} \), \( t(j) \) denotes the least time \( t \) such that \((t, j)\) is in its domain and \( j(t) \) denotes the least index \( j \) such that \((t, j)\) is in its domain \cite{5}. Given \( V : \mathbb{R}^n \to \mathbb{R} \), any functions \( u_C, u_D : \mathbb{R}^n \to [-\infty, \infty] \), and a set \( U \subset \mathbb{R}^n \), it is said that the growth of \( V \) along solutions to \( \mathcal{H} \) is bounded by \( u_C, u_D \) on \( U \) if for any solution \( \varphi \) to \( \mathcal{H} \) with its range in \( U \),

\[
V(\varphi(t, j)) - V(\varphi(t, j)) = \int_{t}^{t+1} u_C(\varphi(s, j(s))) ds + \sum_{j=\hat{j}+1}^{\hat{j}+1} u_D(\varphi(t(j), j)) \tag{23}
\]

for all \((t, j), (\hat{t}, \hat{j})\) in the domain of \( \varphi \) such that \((t, j) \prec (\hat{t}, \hat{j})\) \cite{5}. Intuitively, \( u_D \) acts as a bound for the growth of \( V \) during system flow, while \( u_D \) acts as a bound for its growth during system jumps.

**Theorem A.7 (Invariance Principle for Hybrid Systems) **Consider a continuous function \( V : \mathbb{R}^n \to \mathbb{R} \), any functions \( u_C, u_D : \mathbb{R}^n \to [-\infty, \infty] \), and a set \( U \subset \mathbb{R}^n \) such that \( u_C(q), u_D(q) \leq 0 \) \( \forall q \in U \) and such that the growth of \( V \) along solutions to \( \mathcal{H} \) is bounded by \( u_C, u_D \) on \( U \). Let a complete, bounded solution to \( \mathcal{H}, \varphi^* \), be such that the closure of its range in \( U \). Then, for some \( r \in V(U) \), \( \varphi^* \) approaches the nonempty set that is the largest weakly invariant subset of

\[
S = V^{-1}(r) \cap U \cap \left[ u_C^{-1}(0) \cup (u_D^{-1}(0) \cap \mathbb{R}^n) \right]. \tag{24}
\]

**B. Proofs**

**Proof of Theorem 7.1 and the MIET Result of Theorem 7.2**

The proof will be shown for the existence of the MIET given in Theorem 7.1 because Theorem 7.1 is a special case of the former with \( \chi_i = 0 \) for each agent \( i \). To facilitate the proof, let \( \hat{x}_i \in \mathbb{R}^{N_i} \) be a column vector of \( \hat{x}_i - \hat{x}_j \), \( j \in N_i^{out} \), and \( \hat{W}_i \in \mathbb{R}^{N_i^{out} \times N_i^{out}} \) be a column vector of \( w_{ij} \), \( j \in N_i^{out} \), such that \( \hat{W}_i \hat{x}_i = \sum_{j \in N_i^{out}} w_{ij} (\hat{x}_i - \hat{x}_j) = \hat{x}_i \). Intuitively, \( \hat{W}_i \) is a vector of the relative distances between agent \( i \) and its out neighbors, while \( \hat{W}_i \) is a vector of the weights of the graph for those relative distances. Additionally, let \( \hat{W}_i = \text{diag}(\hat{W}_i) \), so \( \hat{W}_i \hat{x}_i = \sum_{j \in N_i^{out}} w_{ij} (\hat{x}_i - \hat{x}_j)^2 = \phi_i \), for \( i = 1, 2, \ldots, N_i \). This enables us to write these two summations as matrix multiplications and to better see the relationship between them.

For any agent \( i \in \{1, \ldots, N_i\} \), the lowest inter-event time \( t_{i+1} - t_i \) for all \( \ell \in \mathbb{Z}^{\geq 0} \) is given by the time it takes the clock state \( \chi_i \) to go from \( \chi_i \) to \( \chi_i \). Therefore, to calculate a uniform local MIET, one must find a lower bound on \( \chi_i = \gamma_i \), which depends on no state other than \( \chi_i \).

Recall the clock dynamics defined by \cite{10} and \cite{15}. For now, we assume \( \gamma_i = 0 \). Applying Young’s inequality \cite{12} with \( b_i > 0 \), we can write

\[
\gamma_i - \epsilon_i \geq \frac{\chi_i}{\gamma_i} W_i \hat{W}_i \hat{x}_i - \frac{b_i(\chi_i + 1)^2}{b_i \epsilon_i^2} - (W_i \hat{W}_i \chi_i)^2 - \epsilon_i, \tag{26}
\]

Now, we wish to choose \( b_i \) such that the state dependent terms are greater than 0:

\[
\frac{\chi_i}{\gamma_i} W_i \hat{W}_i \hat{x}_i - \frac{b_i(\chi_i + 1)^2}{b_i \epsilon_i^2} - (W_i \hat{W}_i \chi_i)^2 - \epsilon_i \geq 0. \tag{25}
\]

Note that \( W_i \hat{W}_i \hat{x}_i = 0 \) if \( W_i \hat{W}_i \chi_i = 0 \), and so the inequality is satisfied for \( \chi_i = W_i \hat{x}_i \). Therefore, we consider \( b_i \neq 0 \) and write

\[
b_i \geq \frac{\chi_i}{\gamma_i} W_i \hat{x}_i W_i \hat{x}_i \gamma_i = \frac{\chi_i}{\gamma_i} W_i \hat{x}_i W_i \hat{x}_i \gamma_i.
\]

The right hand side can be upper bounded as follows

\[
\frac{\chi_i}{\gamma_i} W_i \hat{x}_i W_i \hat{x}_i \gamma_i \leq \frac{\varepsilon_{\text{max}}(W_i \gamma_i)}{\varepsilon_{\text{min}}(W_i)}|\hat{x}_i| \frac{|\hat{x}_i|}{\varepsilon_{\text{min}}(W_i)}
\]

Therefore, let \( b_i = \frac{\varepsilon_{\text{max}}(W_i \gamma_i)}{\varepsilon_{\text{min}}(W_i)} \), and we have satisfied the inequality \cite{25}. Note that, because \( W_i \) is a diagonal matrix of the weights \( w_{ij} \), \( j \in N_i^{out} \),

\[
\varepsilon_{\text{min}}(W_i) = \min_{j \in N_i^{out}} w_{ij}.
\]

Additionally, it can be shown that one eigenvalue of \( W_i \hat{x}_i \) is \(|\hat{x}_i|\) and all others are equal to zero. Therefore,

\[
\varepsilon_{\text{max}}(W_i \hat{x}_i) = \varepsilon_{\text{min}}(W_i) = \sum_{j \in N_i^{out}} w_{ij}^2.
\]

This choice of \( b_i \) indicates that

\[
\gamma_i - \epsilon_i \geq -b_i(\chi_i + 1)^2 - \epsilon_i. \tag{26}
\]

Recalling the definition of \( \gamma_i \) given in \cite{10}, we can say that \( \gamma_i = \gamma_i - \epsilon_i \), if \( \gamma_i - \epsilon_i \leq -\epsilon_i \) so that no clipping occurs. Because \( -b_i(\chi_i + 1)^2 - \epsilon_i < \epsilon_i \), we can use \cite{26} to write

\[
\gamma_i \geq -b_i(\chi_i + 1)^2 - \epsilon_i. \tag{26}
\]

Note that, if \( \epsilon_i = 0 \), we have \( \gamma_i = -\epsilon_i \), which also satisfies this inequality. Letting \( \psi_i = -b_i(\psi_i + 1)^2 - \epsilon_i \) and \( \psi_i(0) = \chi_i \), we have \( \psi_i \leq \chi_i \). Solving the differential equation for \( \psi_i(t) \) and finding the time \( T_i \) it takes to reach \( \psi_i(T_i) = \chi_i \) gives us \cite{17}.

**Proof of Theorems 7.2 and the Convergence Result of 7.2**

Here we present the proof of our main convergence result. The proof will be shown for convergence of the algorithm in Theorem 7.2 because Theorem 7.2 is a special case of the former with \( \chi_i = 0 \) for each agent \( i \). Ultimately we wish to apply Theorem A.7 to our hybrid system.

Intuitively, we are interested in showing that \( V \) is decreasing along the trajectories of our system and that \( V \to 0 \) as \( t \to \infty \), as this means the system has reached the target set \( q \in \mathcal{A} \). While the state is flowing \( (q \in \mathcal{C}) \), we have already shown that the clock defined by \cite{10} and \cite{15} ensures that \( V < 0 \)
∀q ∉ A. When the system jumps (q ∈ D), we are interested in bounding

$$V(g(q)) - V(q) = \frac{1}{2} \left( (x^+ - \bar{x})^T (x^+ - \bar{x}) + e^+ T e^+ - (x^- - \bar{x})^T (x^- - \bar{x}) + e^- T e^- \right),$$

for q ∈ D. The jump set D is when an agent’s clock is less than its threshold χi, at which point that agent broadcasts its state, setting its error to 0 without changing its position xi. The flow set ends at χi = 0, so the lowest value the clock can have at a jump is 0. Therefore,

$$V(g(q)) - V(q) = \frac{1}{2} \left( e^+ T e^+ - e^- T e^- \right) \leq 0.$$

This means that while the system is flowing but not in the target state (q ∈ C \ A), we have V < 0. When the state jumps (q ∈ D), the value of V decreases or remains unchanged. Combining this with the fact that ẋ is constant and with Theorem VI.1 to ensure that t → 0 without exhibiting Zeno behavior guarantees that V → 0, which concludes the result.

More formally, we first show that H is nominally well-posed using Theorem A.5. The flow set C and the jump set D are both closed subsets of ℝ³N, so the first condition of Theorem A.5 is satisfied. However, note that, although f is continuous in q if χi is held constant, it is not necessarily continuous in q, in general, because of the function γi. Indeed, f is not outer semicontinuous. Therefore, we wish to find a set valued mapping F ⊃ f, which still describes our algorithm, such that F is outer semicontinuous. For each agent i, γi is continuous, except where \( \hat{\phi}_i = 0 \) and \( e_i = 0 \), and the limit approaching this point does not exist. Therefore, for F, we let \( \hat{\chi}_i = \hat{\Gamma}_i(e_i) \supseteq \gamma_i \), where \( \hat{\Gamma}_i \) is a set valued mapping with a closed graph. Not that \( \gamma_i \) is upper bounded because it has been clipped to \( -\varepsilon_i \), and, as established in the proof of Theorem VI.2, \( \gamma_i \) is lower bounded by \(-b_i(\chi_i + 1)^2 - \varepsilon_i \). Now, we evaluate \( \min_{\chi_i \in [0, \gamma_i]} \left( -b_i(\chi_i + 1)^2 - \varepsilon_i \right) = -b_i(\chi_i + 1)^2 - \varepsilon_i \). Therefore, let

$$\hat{\gamma}_i = \begin{cases} \gamma_i, & \text{for } \hat{\phi}_i = 0, \ e_i = 0 \\ \Gamma_i, & \text{otherwise} \end{cases},$$

which ensures that the graph of F is closed. Therefore, F is outer semicontinuous by Theorem A.1. Additionally, F is locally bounded and convex for each value of q. This hybrid system (C, F, D, G) now satisfies the first condition of Theorem A.5.

The set valued mapping G(q) is outer semicontinuous, because its graph is closed, and it is locally bounded, satisfying the third condition of Theorem A.5. Therefore, according to Theorem A.5, the hybrid system H is nominally well-posed.

Let

$$u_D = \begin{cases} 0 & \text{for } q \in D \\ -\infty & \text{otherwise} \end{cases}. \quad (27)$$

The function \( u_D \) acts as an upper bound on the rate of change of V(q) for each jump. It is defined as \(-\infty\) outside the jump set, because it has no meaning outside the jump set.

Therefore, let

$$u_C = \sum_{i=1}^{N} \left( -\frac{1}{2} \hat{\gamma}_i - e_i \hat{\gamma}_i \right) - e_i \gamma_i \hat{\gamma}_i + \frac{1}{2} (\gamma_i - \varepsilon_i) e_i^2, \quad (28)$$

for q ∈ C and \( u_C = -\infty \) outside. Recall that each \( \gamma_i \) was defined so that this expression will be negative for q \notin A and that \( \gamma_i \leq \gamma_i - \varepsilon_i \). The function uC acts as an upper bound on the rate of change of V(q) during flow. Note that we have shown that the growth of V(q) along any solution is bounded by uC, uD.

Let \( U(\bar{x}(0)) \subset \mathbb{R}^{3N} \triangleq \{ q \in \mathbb{R}^{3N} : \frac{1}{N} \sum_{i=1}^{N} x_i = \bar{x}(0) \} \). Note that, because the average position of the agents remains constant along all solutions, U is invariant and any solution \( \varphi \) such that \( \varphi(0) \in U(\bar{x}(0)) \) remains in \( U(\bar{x}(0)) \) for as long as it is defined.

Therefore, by Theorem A.7, every complete, bounded solution \( \varphi \) such that \( \varphi(0) \in U(\bar{x}(0)) \) approaches the largest weakly invariant subset of (24). This will be shown to be A.

\( u_C = 0 \) only when \( ||e|| = 0 \) and the system has reached average consensus. Note A = V⁻¹(0), \( \hat{x} = u = 0 \) in A, and \( G(q) = q \) for all q ∈ A. Therefore, all solutions \( \varphi \) with \( \varphi(t, j) \in A \) for \( (t, j) \in \text{dom } \varphi \) remain in \( A \) for \( t \) s. t. \( t^* + j^* > t + j \). This satisfies both conditions of Definition A.6 and so A is weakly invariant. It will now be shown that the largest weakly invariant subset of S cannot include any points outside of A.

Note that our algorithm deals with a specific subset of G(q), \( G'(q) \triangleq \{ G \mid \forall i \text{ s. t. } \chi_i \leq \bar{\chi}_i \} \). Although G was defined for simplicity, it did not rule out the possibility of each agent i triggering an event while \( \chi_i > \bar{\chi}_i \) if there is another agent j such that \( \chi_j < \bar{\chi}_j \), which cannot occur for the algorithm presented in Table IV. \( G'(q) \) is outer semicontinuous and the system \( H' = (C, F, D, G') \) is nominally well-posed by the same arguments as before.

The only additional points in S, given by (24), are in \( G'(u_D^{-1}(0)) \cap u_D^{-1}(0) \), which is the set of points inside the jump set which can be reached by jumping. No maximal solution can remain here for an unbounded number of jumps; it must leave the jump set when all agents meeting the trigger condition have triggered an event, and there are only a finite number of agents. Therefore, the points in \( G'(u_D^{-1}(0)) \cap u_D^{-1}(0) \) \setminus A are not part of a weakly invariant subset of S, because they fail the second condition in Definition A.6.

This implies A is the largest weakly invariant subset of S, and so all complete, bounded solutions to \( H' \) starting in \( U(\bar{x}(0)) \) converge to A by Theorem A.7.

Finally, we must show that any maximal solution \( \varphi \) such that \( \bar{x}(0) = x \) and \( \chi_i = \bar{\chi}_i \) for \( i = 1, 2, \ldots, N \) is complete.
and bounded.

Because $C \cup D = \mathbb{R}^{3N}$, a maximal solution cannot cease to exist without being unbounded. Therefore, showing that a solution is bounded also implies that it is complete. The dynamics of $\chi_i$ ensure that $\chi_i \in [0, \chi_i]$, for $i = 1, 2, \ldots, N$ for all time. Because $\hat{x}$ is the last broadcast positions of the agents, $\hat{x}_i$ only takes on the same values as $x_i$. Therefore, to show that solutions are bounded, it must now be shown that $x$ is bounded.

Because $\dot{V} \leq 0$ for every solution with starting each clock $i$ in $[0, \chi]$, showing that $||x|| \to \infty$ implies $V \to \infty$ is sufficient to show that every such solution is bounded. That is, $V$ must be shown to be radially unbounded with respect to $x$. Recall $V_p = \frac{1}{2}(x - \bar{x})^T(x - \bar{x})$ and that $\bar{x}$ is constant, so $||x|| \to \infty$ implies $V \to \infty$.

Therefore, every solution to $\mathcal{H}'$ such that $x(0) \in \mathbb{R}^N$, $\hat{x}(0) = x(0)$, and $\chi_i = \chi$, for $i = 1, 2, \ldots, N$ is complete and bounded and converges to $\mathcal{A}$. ■