ANALYSIS OF FULLY DISCRETE MIXED FINITE ELEMENT SCHEME FOR STOCHASTIC NAVIER-STOKES EQUATIONS WITH MULTIPLICATIVE NOISE

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Abstract. This paper is concerned with stochastic incompressible Navier-Stokes equations with multiplicative noise in two dimensions with respect to periodic boundary conditions. Based on the Helmholtz decomposition of the multiplicative noise, semi-discrete and fully discrete time-stepping algorithms are proposed. The convergence rates for mixed finite element methods based time-space approximation with respect to convergence in probability for the velocity and the pressure are obtained. Furthermore, with establishing some stability and using the negative norm technique, the partial expectations of the $H^1$ and $L^2$ norms of the velocity error are proved to converge optimally.

Key words. Stochastic Navier-Stokes equations, multiplicative noise, Wiener process, Itô stochastic integral, mixed finite element, stability, error estimates

AMS subject classifications. 65N12, 65N15, 65N30,

1. Introduction. In this paper, we consider the following time-dependent stochastic incompressible Navier-Stokes equations:

\begin{align}
(1.1a) \quad &du = [\nu \Delta u - (u \cdot \nabla)u - \nabla p]dt + G(u)dw \\ 
(1.1b) \quad &\text{div } u = 0 \\ 
(1.1c) \quad &u(0) = u_0
\end{align}

where $T > 0$ denotes time, $\nu > 0$ is the viscosity of the fluid, $u$ and $p$ denote respectively the velocity and the pressure of the problem \[1.1\] which are spatially periodic with period $L > 0$, and $D = (0, L)^2 \subset \mathbb{R}^2$ is a period of the periodic domain with boundary $\partial D$ and $u_0$ denotes a given initial datum. Here we assume that \(\{W(t); t \geq 0\}\) is an $[L^2(D)]^2$-valued $Q$-Wiener process. The noise is not divergence-free (i.e., $\text{div } G(u) \neq 0$).

The stochastic system \[1.1\] can take into account noise term in the sense of physical or numerical uncertainties and thermodynamical fluctuations. In \[2\], Bensoussan and Temam started to study the stochastic Navier-Stokes in mathematical investigation. The paper \[10\] by Flandoli and Gatarek developed a fully stochastic theory to prove the existence of a martingale solution. This paper \[20\] investigated the ergodic properties for the stochastic Navier-Stokes equations with degenerate noise. In the last twenty years, there is a large amount of literature about the analysis of problem \[1.1\]. We refer to \[1, 19, 3, 4, 10, 12, 14, 15\] and the references therein for detailed discussions of the stochastic incompressible Navier-Stokes equations.

The paper \[7\] by Brzeźniak, et al. proposed two fully discrete finite element schemes for the stochastic Navier-Stokes equations with multiplicative noise. By using the compactness argument, the authors analyzed the convergence for the velocity field to weak martingale solutions in 3D and to strong solutions in 2D. In \[11\], Carelli and Prohl studied implicit and semi-implicit fully schemes for the stochastic Navier-Stokes problem. The result in \[10\] is convergence of rate (almost) $\frac{1}{4}$ in time and linear convergence in space for the velocity. However, the convergence of the pressure was not given. In work \[3\], the authors proposed an iterative splitting scheme for stochastic...
Navier-Stokes equations and established a strong convergence in probability in the 2D case. In [4], the authors studied another time-stepping semi-discrete scheme and derived strong $L^2$ convergence for the velocity. In [17], Hausenblas and Randrianasolo proposed a time semi-discrete scheme of stochastic 2D Navier-Stokes equations with penalty-projection method. As noted in [17], the result is convergence of rate (almost) $\frac{1}{2}$ in time for the velocity and the pressure. In paper [14], Feng and Qiu developed a fully discrete mixed finite element scheme of the time-dependent stochastic Stokes equations with multiplicative noise and established strong convergence with rates not only for the velocity but also for the pressure. The paper [14], by Feng, et al. proposed a new fully discrete mixed finite element scheme of the time-dependent stochastic Stokes equations with multiplicative noise and obtained optimal strong convergence with rates for both the velocity and the pressure. In a very recent paper [5], Breit and Dodgson considered a fully discrete time-space finite element scheme and proved strong convergence with rates for the velocity. The result in [5] is convergence in time and linear convergence in space. The error estimate of the velocity field $u$ and its time-space numerical solution $u_h^n$ reads as: assume that

$$Lk \leq (-\log h)^{-1}$$

for some $L > 0$, then for any $\alpha < \frac{1}{2}$

\begin{equation}
\begin{aligned}
\mathbb{E}\left[ \Omega_{k,h} \left( \max_{1 \leq n \leq N} \|u(t_n) - u_h^n\|_{L^2}^2 + k \sum_{n=1}^N \|\nabla(u(t_n) - u_h^n)\|_{L^2}^2 \right) \right] \\
\leq C(k^{2\alpha} + h^2),
\end{aligned}
\end{equation}

where $\Omega_{k,h} \subset \Omega$ with $\mathbb{P}(\Omega \setminus \Omega_{k,h}) \to 0$ as $k, h \to 0$.

The primary goal of this paper is twofold. Our first goal is to develop an optimally convergent fully discrete finite element scheme with inf-sup stability. Our main idea, which is partly used in references [14] [15], is to use the Helmholtz decomposition for the driving multiplicative noise at each time step, and then solve velocity and pressure. We propose new semi-discrete and fully discrete time-stepping algorithms for problem (1.1) and prove the convergence of the velocity and the pressure for the fully discrete scheme for the stochastic Navier-Stokes equations. The second goal is to prove strong optimal $H^1$ convergence first, and then to obtain $L^2$ convergence of the fully discrete scheme with the negative norm technique. To the best of our knowledge, it is the first time that strong optimal $L^\infty - H^1 / L^2$ convergence of the discrete solution $(u_h^n, p_h^n)$ to a fully discrete system of the stochastic Navier-Stokes equations has been established.

The highlight of this paper (see section 4) is to derive the error estimates for the numerical solution as follows: for any $\alpha < \frac{1}{2}$

\begin{align}
\mathbb{E}\left[ 1_{\Omega_{k,h} \cap \Omega_{n,h} \cap \Omega_n \cap \Omega_k} \left( \|\nabla(u(t_n) - u_h^n)\|_{L^2}^2 \right) \right] & \leq C(k^{2\alpha} + h^{2-4\alpha}), \\
\mathbb{E}\left[ 1_{\Omega_t \cap \Omega_h} \left( \left\| \int_0^t p(s) \, ds - k \sum_{n=1}^m p_n^n \right\|_{L^2}^2 \right) \right] & \leq C(k^{2\alpha} + h^{2-2\alpha}), \\
\mathbb{E}\left[ 1_{\Omega_{k,h} \cap \Omega_{n,h} \cap \Omega_n \cap \Omega_k} \left( \|u(t_n) - u_h^n\|_{L^2}^2 \right) \right] & \leq C(k^{2\alpha} + h^{2-4\alpha} + h^{4-8\alpha}).
\end{align}

The remainder of this paper is organized as follows. In Section 2 we introduce some function and space notation for problem (1.1) and obtain a few preliminary results. In Section 3 we propose the semi-discrete scheme for problem (1.1) and derive some optimal error estimates for both the velocity and pressure approximations. In Section 4 we prove some optimal error estimates for the fully discrete scheme for problem (1.1) with the negative norm technique. In Section 5 some numerical results are given to validate the theoretical error estimates.
2. Preliminaries.

2.1. Notation and assumptions. Standard function and space notation will be adopted in this paper. Let $H^m(D)$ ($m \geq 0$) denote the standard Sobolev space, and $\| \cdot \|_{H^m}$ denotes its norm. Let $H^m_{\text{per}}(D)$ be the subspace of $H^m(D)$ consisting of $\mathbb{R}^2$-valued periodic function. Let $(\cdot, \cdot) := (\cdot, \cdot)_D$ denote the standard $L^2$-inner product. We also let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ be a stochastic basis with a complete right continuous filtration. For a given random variable $v$ defined on $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$, let $\mathbb{E}[v]$ denote the expected value of $v$. Let $X$ denote a normed vector space $X$ with norm $\| \cdot \|_X$. Define the following Bochner space 

$$L^p(\Omega, X) := \{ v : \Omega \to X; \mathbb{E}[\|v\|_X^p] < \infty \}$$

and the norm

$$\|v\|_{L^p(\Omega, X)} := \left( \mathbb{E}\left[\|v\|_X^p\right] \right)^{\frac{1}{p}}, \quad 1 < p < \infty.$$ 

We also define some special space notation as follows:

$$\mathcal{V} := [H^1_{\text{per}}(D)]^2; \quad \mathcal{W} := \{ q \in L^2_{\text{per}}(D); (q, 1)_D = 0 \}, \quad \mathcal{V}_0 := \{ v \in \mathcal{V}; \text{div} \, v = 0 \} \in D \}. $$

Let $\mathcal{K} := \mathcal{L}(\mathcal{V}_0)$ denote the Banach space of linear operators from $L^2_{\text{per}}(D)$ to $[L^2_{\text{per}}(D)]^2$ with finite Hilbert-Schmidt norms denoted by $\| \cdot \|_{\mathcal{K}}$. As it is noted [22] that the stochastic integral $\int_0^T \varphi(s)dW(s)$ is an $(\mathfrak{F}_t)$-martingale and the following Itô’s isometry holds:

$$(2.1) \quad \mathbb{E}\left[ \left\| \int_0^T \varphi(s)dW(s) \right\|_{L^2_2}^2 \right] = \mathbb{E}\left[ \int_0^T \|\varphi(s)\|_2^2 \, ds \right].$$

In this paper we assume that $G : [0, T] \times [L^2_{\text{per}}(D)]^2 \to L^2(\Omega, \mathcal{K})$ satisfies the following conditions:

$$(2.2a) \quad \|G(v) - G(w)\|_{L^2(\mathcal{K}, L^2_2)} \leq C\|v - w\|_{L^2_2}, \quad \forall v, w \in [L^2_{\text{per}}(D)]^2,$$

$$(2.2b) \quad \|G(v)\|_{L^2(\mathcal{K}, L^2_2)} \leq C(1 + \|v\|_{H^2})_2, \quad \forall v \in [H^1_{\text{per}}(D)]^2, i = 1, 2,$$

$$(2.2c) \quad \|D^iG(v)\|_{L^2(\mathcal{K}, L^2(\{H^2_D(\mathcal{V}); L^2_2(\mathcal{V})\}, L^2_2))} \leq C, \quad \forall v \in [L^2_{\text{per}}(D)]^2, j = 1, 2.$$

We introduce the Helmholtz projection [18] $P_H : [L^2_{\text{per}}(D)]^2 \to \mathcal{V}$ defined by $P_H v = \eta$ for every $v \in [L^2_{\text{per}}(D)]^2$, where $(\eta, \xi) \in \mathcal{V} \times [H^1_{\text{per}}(D)]^2/\mathbb{R}$ is a unique decomposition such that

$$v = \eta + \nabla \xi,$$

and $\xi \in [H^1_{\text{per}}(D)]^2/\mathbb{R}$ satisfies the following problem:

$$(2.3) \quad (\nabla \xi, \nabla q) = (v, \nabla q), \quad \forall q \in \mathcal{V}.$$

2.2. Definition of weak solutions. In this subsection we first recall the weak solution definition for problem [14], and refer to [9] [13] [15]. We then introduce some regularity of the velocity and the pressure.

**Definition 2.1.** Assume that $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ is a given stochastic basis and $u_0$ is an $\mathfrak{F}_0$-measurable random variable. Then $(u, p)$ is called a weak pathwise solution to problem [14] if
(i) the velocity and the pressure \((u, p)\) is \(\mathcal{F}_t\)-adapted and
\[
\begin{align*}
u &\in C([0, T]; [L^2_{\text{per}}(D)]^2) \cap L^2([0, T; \mathcal{V}]), &\quad \mathbb{P}\text{-a.s.,} \\
p &\in W^{-1, \infty}(0, T; \mathcal{W}), &\quad \mathbb{P}\text{-a.s.}
\end{align*}
\]
(ii) the problem (1.1) satisfies
\[
\begin{align*}
(2.4a) \quad (u(t), v) + \int_0^t [a(u(s), v) + b(u(s), u(s), v)] \, ds - d\left(v, \int_0^t p(s) \, ds\right) &= (u_0, v) + \left(\int_0^t G(u(s)) \, dW(s), v\right) \quad \forall v \in \mathcal{V}, \\
(2.4b) \quad d(u, q) = 0, \quad \forall q \in \mathcal{W},
\end{align*}
\]
holds \(\mathbb{P}\text{-a.s. for all } t \in (0, T]\). Where the bilinear forms \(a(\cdot, \cdot)\) and \(d(\cdot, \cdot)\) are defined
\[
a(v, w) := \nu \left(\nabla v, \nabla w\right), \quad \forall v, w \in \mathcal{V}, \\
d(v, q) := \left(\nabla \cdot v, q\right), \quad \forall v \in \mathcal{V}, q \in \mathcal{W},
\]
and the nonlinear form \(b(\cdot, \cdot, \cdot)\) is defined as follows:
\[
b(w, u, v) := (w \cdot \nabla u, v) + \frac{1}{2} \left((\nabla \cdot w)u, v\right), \quad \forall w, u, v \in \mathcal{V}.
\]

Using the similar idea in [15], problem (2.4) can be considered as a mixed formulation for problem (1.1). Thus, we introduce a new pressure \(r := p - \xi(u) \, dW\), where we apply the Helmholtz decomposition \(G(u) = \eta(u) + \nabla \xi(u)\), where \(\xi(u) \in \mathcal{V}, \mathbb{P}\text{-a.s.}\) such that
\[
(\nabla \xi(u), \nabla \phi) = (G(u), \nabla \phi), \quad \forall \phi \in \mathcal{V}.
\]
By the elliptic regularity [18], we have
\[
\begin{align*}
(2.5) \quad \|\nabla \xi(u)\|_{L^2} &\leq C \|G(u)\|_{L^2}, \\
(2.6) \quad \|\nabla \xi(u)\|_{L^2/\mathbb{R}} &\leq C \|\nabla \cdot G(u)\|_{L^2}.
\end{align*}
\]

**Definition 2.2.** Assume that \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a given stochastic basis and \(u_0\) is an \(\mathcal{F}_0\)-measurable random variable. Then \((u, p)\) is called a weak pathwise solution to problem (1.1) if
(i) the velocity and the pressure \((u, r)\) is \(\mathcal{F}_t\)-adapted and
\[
\begin{align*}
u &\in C([0, T]; [L^2_{\text{per}}(D)]^2) \cap L^2([0, T; \mathcal{V}]), &\quad \mathbb{P}\text{-a.s.,} \\
r &\in W^{-1, \infty}(0, T; \mathcal{W}), &\quad \mathbb{P}\text{-a.s.}
\end{align*}
\]
(ii) the problem (1.1) satisfies
\[
\begin{align*}
(2.7a) \quad (u(t), v) + \int_0^t [a(u(s), v) + b(u(s), u(s), v)] \, ds - d\left(v, \int_0^t r(s) \, ds\right) &= (u_0, v) + \left(\int_0^t \eta(u(s)) \, dW(s), v\right) \quad \forall v \in \mathcal{V}, \\
(2.7b) \quad d(u, q) = 0, \quad \forall q \in \mathcal{W},
\end{align*}
\]
holds $\mathbb{P}$-a.s. for all $t \in (0, T]$, where $v(u(s)) := G(u(s)) - \nabla \xi(u(s))$.

Remark 1. By using similar techniques in [12, 13], we note that $\int_0^t p ds, \int_0^t r ds \in L^2(\Omega, L^2(0, T, W))$.

The next Lemma follows from [5].

Lemma 2.3. (i) Let $u_0 \in L^l(\Omega, [L^2_{adm}(D)]^2)$ for some $l \geq 2$ and let $G$ satisfy (2.2). Then there exists a constant $C > 0$, such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|u(t)\|^2_{L^2_x} + \int_0^T \|\nabla u(s)\|^2_{L^2_x} ds \right]^{\frac{1}{2}} \leq C.$$  

(ii) Let $u_0 \in L^l(\Omega, \mathcal{V}_0)$ for some $l \geq 2$ and let $G$ satisfy (2.2). Then there exists a constant $C > 0$, such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\nabla u(t)\|^2_{L^2_x} + \int_0^T \|\nabla^2 u(s)\|^2_{L^2_x} ds \right]^{\frac{1}{2}} \leq C.$$  

(iii) Let $u_0 \in L^l(\Omega, [H^2(D)]^2 \cap \mathcal{V}_0) \cap L^l(\Omega, \mathcal{V}_0)$ for some $l \geq 2$ and let $G$ satisfy (2.2). Then there exists a constant $C > 0$, such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\nabla^2 u(t)\|^2_{L^2_x} + \int_0^T \|\nabla^3 u(s)\|^2_{L^2_x} ds \right]^{\frac{1}{2}} \leq C.$$  

We finish this section by establishing some regularity of pressure of various spatial norms. For the reader's convenience, we here give theirs proofs.

Lemma 2.4. (i) Let $u_0 \in L^l(\Omega, L^2(D))$ for some $l \geq 2$ and let $G$ satisfy (2.2). Then there exists a constant $C > 0$, such that

$$\mathbb{E} \left[ \left\| \int_0^t p(s) ds \right\|^2_{L^2_x} \right]^{\frac{1}{2}} \leq C,$$  

$$\mathbb{E} \left[ \left\| \int_0^t \nabla p(s) ds \right\|^2_{L^2_x} \right]^{\frac{1}{2}} \leq C.$$  

where $(u, p)$ is the weak pathwise solution to (1.1), cf. Definition 2.1.

(ii) Let $u_0 \in L^l(\Omega, L^2(D))$ for some $l \geq 2$ and let $G$ satisfy (2.2). Then there exists a constant $C > 0$, such that

$$\mathbb{E} \left[ \left\| \int_0^t r(s) ds \right\|^2_{L^2_x} \right]^{\frac{1}{2}} \leq C,$$  

$$\mathbb{E} \left[ \left\| \int_0^t \nabla r(s) ds \right\|^2_{L^2_x} \right]^{\frac{1}{2}} \leq C.$$  

where $(u, v)$ is the weak pathwise solution to (1.1), cf. Definition 2.2.

Proof. For (2.8), from (2.4), we have

$$d \left( v, \int_0^t p(s) ds \right) = (u(t), v) - (uo, v) + \int_0^t [a(u(s), v) + (b(u(s), u(s), v)] ds - \left( \int_0^t G(u(s)) \cdot dW(s), v \right).$$
Using the Young’s inequality, the Poincaré inequality and the Hölder inequality, one finds that
\[
d\left(\int_0^t p(s)\,ds\right) \leq C(\|u_0\|_{L^2} + \|u(t)\|_{L^2})\|\nabla v\|_{L^2} + C\int_0^t \|\nabla u(s)\|_{L^2}^2 \|\nabla v\|_{L^2}
\]
\[
+ C\int_0^t (\|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2)\,ds \|\nabla v\|_{L^2}
\]
\[
+ C\left\| \int_0^t G(u(s))\,dW(s) \right\|_{L^2}^2.
\]
By the well-known inf-sup condition \[13\], it follows that
\[
\beta\left\| \int_0^t p(s)\,ds \right\|_{L^2} \leq C(\|u_0\|_{L^2} + \|u(t)\|_{L^2}) + C\int_0^t \|\nabla u(s)\|_{L^2}^2 \,ds
\]
\[
+ C\int_0^t (\|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2)\,ds
\]
\[
+ C\left\| \int_0^t G(u(s))\,dW(s) \right\|_{L^2}^2.
\]
Taking the expectation, using Itô’s isometry, \((2.2b)\) and Lemma \[2.3\] which lead to the desired result.

For \((2.9)\), from \((2.4a)\) and using the Hölder inequality, the Young’s inequality and the Poincaré inequality, we get
\[
\left(\int_0^t \nabla p(s)\,ds, v\right) = \left( (u(t), v) - (u_0, v) + \int_0^t \left( a(u(s), v) + b(u(s), u(s), v) \right)\,ds - \left( \int_0^t G(u(s))\,dW(s), v \right) \right.
\]
\[
\leq C(\|u_0\|_{L^2} + \|u(t)\|_{L^2})\|v\|_{L^2} + C\int_0^t \|\nabla^2 u(s)\|_{L^2}^2 \|v\|_{L^2}
\]
\[
+ C\int_0^t \left( \|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2 + \|\nabla^2 u(s)\|_{L^2}^2 \right)\,ds \|v\|_{L^2}
\]
\[
+ C\left\| \int_0^t G(u(s))\,dW(s) \right\|_{L^2}^2 \|v\|_{L^2}.
\]
With the definition of \(L^2\) norm, we obtain
\[
\left\| \int_0^t \nabla p(s)\,ds \right\|_{L^2} = \sup_{v \neq 0 \in L^2} \frac{\int_0^t \nabla p(s)\,ds, v}{\|v\|_{L^2}}
\]
\[
\leq C(\|u_0\|_{L^2} + \|u(t)\|_{L^2}^2) + C\int_0^t \|\nabla^2 u(s)\|_{L^2}^2 \,ds
\]
\[
+ C\int_0^t \left( \|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2 + \|\nabla^2 u(s)\|_{L^2}^2 \right)\,ds
\]
\[
+ C\left\| \int_0^t G(u(s))\,dW(s) \right\|_{L^2}^2.
\]
Taking the expectation, using Itô’s isometry and \((2.2b)\) and Lemma \[2.3\] we get the desired result.
Similarly, using \((2.23)\) and the definition of \(\eta(u)\), \((2.25)\) and Lemma \(2.6\) the results \((2.10) - (2.11)\) hold. The proof is complete. \(\square\)

3. Semi-discrete time-stepping scheme. In this section we establish semi-discrete time-stepping scheme for the mixed formulation \((2.7)\). Then we analyze the error estimates for the velocity and the pressure.

Let \(N\) be a positive integer and \(0 = t_0 < t_1 < \ldots < t_N = T\) be an uniform partition of \([0, T]\), with \(k = t_{i+1} - t_i\) for \(i = 0, \ldots, N - 1\), Set \(u^0 := u_0\). Our semi-discrete time-stepping scheme for \((1.1)\) is defined as follows:

Algorithm 1:

Step I: Find \(\xi(u^{n-1}) \in L^2(\Omega, V)\) by solving

\[
(3.1) \quad \left( \nabla \xi(u^{n-1}), \nabla \phi \right) = (G(u^{n-1}), \nabla \phi), \quad \forall \phi \in V, \ a.s.
\]

Step II: Denote \(\eta(u^{n-1}) := G(u^{n-1}) - \nabla \xi(u^{n-1})\), and find \((u^n, r^n) \in L^2(\Omega, V) \times L^2(\Omega, W)\) by solving

\[
\begin{align*}
(3.2a) \quad & (u^n, v) + ka(u^n, v) - k d(v, r^n) + kb(u^n, u^n, v) \\
& = (u^{n-1}, v) + (\eta(u^{n-1})\Delta W_n, v) \quad \forall v \in V, \ a.s.,
\end{align*}
\]

\[
(3.2b) \quad d(u^n, q) = 0 \quad \forall q \in W, \ a.s.,
\]

where \(\Delta W_n := W(t_n) - W(t_{n-1}) \sim N(0, kQ)\).

Step III: Denote \(p^n := r^n + k^{-1}\xi(u^{n-1})\Delta_n W\).

The following lemmas establish some stability results for the discrete processes \((u^n, r^n); 0 \leq n \leq N\).

**Lemma 3.1.** Let \(1 \leq p < \infty\) be a natural number. Assume \(u^0 \in L^2(\Omega, V_0)\) with \(\|u^0\|_{L^2} \leq C\). Then there exists a sequence \(\{(u^n, r^n); 1 \leq n \leq N\}\), which for all \(\omega \in \Omega\), solves Algorithm 1 and the following stability properties hold:

\[
(3.3) \quad \mathbb{E} \left[ \max_{1 \leq n \leq N} \|u^n\|_{L^2}^{2p} + \nu k \sum_{n=1}^{N} \|u^n\|_{L^2}^{2p-1} \|\nabla u^n\|_{L^2}^2 \right] \leq C.
\]

\[
(3.4) \quad \mathbb{E} \left[ \max_{1 \leq n \leq N} \|\nabla u^n\|_{L^2}^p + \nu k \sum_{n=1}^{N} \|\nabla u^n\|_{L^2}^{p-1} \|\nabla^2 u^n\|_{L^2}^2 \right] \leq C.
\]

\[
(3.5) \quad \mathbb{E} \left[ \sum_{n=1}^{N} \|\nabla (u^n - u^{n-1})\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2 \right] \leq C,
\]

\[
(3.6) \quad \mathbb{E} \left[ \left( \sum_{n=1}^{N} \|\nabla (u^n - u^n)\|_{L^2}^2 \right)^4 + \left( k \sum_{n=1}^{N} \|\nabla^2 u^n\|_{L^2}^2 \right)^4 \right] \leq C,
\]

\[
(3.7) \quad \mathbb{E} \left[ k \sum_{n=1}^{N} \|r^n\|_{L^2}^2 \right] \leq C,
\]

\[
(3.8) \quad \mathbb{E} \left[ k \sum_{n=1}^{N} \|\nabla r^n\|_{L^2}^2 \right] \leq C.
\]

**Proof.** Since the proofs of \((3.3) - (3.6)\) were derived in \(10\). The proofs of \((3.7) - (3.8)\)
are similar to \cite{14, 15}. For the reader’s convenience, we here give their proofs.

\begin{equation}
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\end{equation}

(3.9) \[ k \sum_{n=1}^{N} d(v, r^n) = (u^N, v) - (u^0, v) + k \sum_{n=1}^{N} [a(u^n, v) + b(u^n, u^n, v)] - \sum_{n=1}^{N} (\eta(u^{n-1}) \Delta W_n, v). \]

Using the Poincaré’s inequality and the Young’s inequality on the right hand side, one finds that

\[ k \sum_{n=1}^{N} d(v, r^n) \leq C(\|u^0\|_{L^2} + \|u^n\|_{L^2}) \|\nabla v\|_{L^2} + Ck \sum_{n=1}^{N} \|\nabla u^n\|_{L^2} \|\nabla v\|_{L^2} + Ck \sum_{n=1}^{N} \|\nabla u^n\|_{L^2} \|\nabla v\|_{L^2} \]

By the inf-sup condition, we get

\[ \beta k \sum_{n=1}^{N} \|r^n\|_{L^2} \leq C(\|u^0\|_{L^2} + \|u^n\|_{L^2}) + Ck \sum_{n=1}^{N} \|\nabla u^n\|_{L^2} + Ck \sum_{n=1}^{N} \|\nabla u^n\|_{L^2} \]

With the definition of \( \eta(u^{n-1}) \) and \( \text{(2.5)} \), it follows that

\begin{equation}
9
\end{equation}

(3.10) \[ \mathbb{E}[\|\eta(u^{n-1})\|_c] \leq \mathbb{E}[\|G(u^{n-1})\|_{L^2} + \|\nabla \xi(u^{n-1})\|_{L^2}] \leq C \mathbb{E}[\|G(u^{n-1})\|_{L^2}]. \]

Hence, taking the expectation and using Itô’s isometry and \( \text{(3.3) - (3.4)} \), which lead to the desired result.

For \( \text{(3.8)} \), setting \( v = \nabla r^n \) in \( \text{(3.2a)} \), by using the Poincaré’s inequality and the Young’s inequality, one finds that

\[ k \sum_{n=1}^{N} (\nabla r^n, \nabla r^n) = (u^N, \nabla r^n) - (u^0, \nabla r^n) + k \sum_{n=1}^{N} [a(u^n, r^n) + b(u^n, u^n, \nabla r^n)] - \sum_{n=1}^{N} (\eta(u^{n-1}) \Delta W_n, \nabla r^n) \]

\[ \leq C(\|u^0\|_{L^2} + \|u^n\|_{L^2}) \|\nabla r^n\|_{L^2} + C \sum_{n=1}^{N} k \|\nabla^2 u^n\|_{L^2} \|\nabla r^n\|_{L^2} \]

\[ + C \sum_{n=1}^{N} k(\|\nabla u^n\|_{L^2}^2 + \|\nabla^2 u^n\|_{L^2}^2) \|\nabla r^n\|_{L^2}. \]
With a standard calculation, we have
\[
\sum_{n=1}^{N} k \left\| \nabla v^n \right\|_{L^2}^2 \leq C \left( \left\| v^n \right\|_{L^2}^2 + \left\| u^n \right\|_{L^2}^2 \right) + Ck \sum_{n=1}^{N} \left\| \nabla^2 u^n \right\|_{L^2}^2
\]
\[+ Ck \sum_{n=1}^{N} \left( \left\| u^n \right\|_{L^2}^2 + \left\| \nabla^2 u^n \right\|_{L^2}^2 \right).
\]

Taking the expectation, using Itô’s isometry, (2.2b), (3.3) and (3.4), we get the desired result. The proof is complete.

**Lemma 3.2.** Assume \( u^0 \in L^{2q}(\Omega, H^2) \) for some \( 1 \leq q < \infty \). Then there exists a sequence \( \{ u^n; 1 \leq n \leq N \} \), which for all \( \omega \in \Omega \), solves Algorithm 1 and satisfies the following bounds:

\[
\mathbb{E} \left[ \max_{1 \leq n \leq N} \left\| Au^n \right\|_{L^2}^{2p} + k \sum_{n=1}^{N} \left\| Au^n \right\|_{L^2}^{2p-2} \left\| A^\frac{1}{2} u^n \right\|_{L^2}^2 \right] \leq C,
\]

\[
\mathbb{E} \left[ \max_{1 \leq n \leq N} \left\| u^n \right\|_{L^2}^{2p-2} \left\| A^\frac{1}{2} u^n \right\|_{L^2}^2 \right] \leq C,
\]

\[
\mathbb{E} \left[ \left( \sum_{n=1}^{N} \left\| A(u^n - v^n) \right\|_{L^2}^2 \right)^4 + \left( k \sum_{n=1}^{N} \left\| A^\frac{1}{2} u^n \right\|_{L^2}^2 \right)^4 \right] \leq C,
\]

where \( A : V \cap [H^2(D)]^d \to V_0 \) denotes the Stokes operator (cf. [23]).

**Proof.** For the first assertion, taking \( v = A^2 u^n \in V_0 \) and \( q = 0 \) in (3.2), we get

\[
\frac{1}{2} \left( \left\| Au^n \right\|_{L^2}^2 - \left\| Au^{n-1} \right\|_{L^2}^2 + \left\| A(u^n - u^{n-1}) \right\|_{L^2}^2 \right) + k\nu \left\| A^\frac{1}{2} u^n \right\|_{L^2}^2
\]
\[= k\nu \left( u^n, u^n, A^2 u^n \right) + (A\eta(u^{n-1})W_n, A(u^n - u^{n-1}))
\]
\[+ (A\eta(u^{n-1})\Delta W_n, A(u^{n-1})).
\]

By Lemma 2.1.20 in [21] and the Young’s inequality, the first term on the right hand of (3.14) can be estimated by

\[
kb(u^n, u^n, A^2 u^n) \leq Ck \left\| \nabla^3 u^n \right\|_{L^2} \left\| \nabla u^n \right\|_{L^2} \left\| u^n \right\|_{L^2} \leq \frac{k\nu}{2} \left\| \nabla^3 u^n \right\|_{L^2}^2 + Ck \left\| \nabla u^n \right\|_{L^2}^2 \left\| u^n \right\|_{L^2}^2 \leq \frac{k\nu}{2} \left\| \nabla^3 u^n \right\|_{L^2}^2 + Ck \left( \left\| \nabla u^n \right\|_{L^2}^{10} + \left\| u^n \right\|_{L^2}^{10} \right).
\]

The last term on the right hand of (3.14) vanishes when taking its expectation. Applying the Young’s inequality and the tower property for conditional expectations to
the second term on the right hand. Summing up then leads to

\[(3.15) \quad E\left[\|Au^n\|^2_{L^2_x}\right] + E\left[\sum_{n=1}^{N} \|A(u^n - u^{n-1})\|^2_{L^2_x}\right] + E\left[\nu \sum_{n=1}^{N} k\|A^2u^n\|^2_{L^2_x}\right]
\]
\[\leq CE\left[\|Au^0\|^2_{L^2_x}\right] + CE\left[\sum_{n=1}^{N} k\left(\|\nabla u^n\|^{10}_{L^2_x} + \|u^n\|^{10}_{L^2_x}\right)\right]
\[+ E\left[\sum_{n=1}^{N} E\left[\|P_H\eta(u^{n-1})\|^2_{L(\mathcal{K},H^2)}\|\Delta_n W\|^2_{\mathcal{K}\mid \Theta_{n-1}}\right]\right]
\[\leq CE\left[\|Au^0\|^2_{L^2_x}\right] + CE\left[\sum_{n=1}^{N} k\left(\|\nabla u^n\|^{10}_{L^2_x} + \|u^n\|^{10}_{L^2_x}\right)\right] + E\left[\sum_{n=1}^{N} k\|Au^{n-1}\|^2_{L^2_x}\right].\]

Using Lemma 3.1 and the discrete Gronwall’s lemma leads to

\[(3.16) \quad \max_{1 \leq n \leq N} E\left[\|Au^n\|^2_{L^2_x}\right] + E\left[\nu \sum_{n=1}^{N} k\|A^2u^n\|^2_{L^2_x}\right] \leq C.\]

To derive the first inequality in (3.11), using the Young’s inequality and Lemma 3.1 one finds that

\[(3.17) \quad E\left[\max_{1 \leq n \leq N} \|Au^n\|^2_{L^2_x}\right] \leq E\left[\|Au^0\|^2_{L^2_x}\right] + CE\left[\sum_{n=1}^{N} k\left(\|\nabla u^n\|^{10}_{L^2_x} + \|u^n\|^{10}_{L^2_x}\right)\right]
\[+ CE\left[\sum_{n=1}^{N} \|AP_H\eta(u^{n-1})\Delta W_n\|^2_{L^2_x}\right]
\[+ CE\left[\max_{1 \leq n \leq N} \sum_{l=1}^{n} (AP_H\eta(u^{l-1})\Delta W_n, Au^{l-1})\right].\]

The second term on the right hand side may be controlled by Lemma 3.1, the third term may be estimated by the tower property for conditional expectations, the fourth term is bounded with using the Burkholder-Davis-Gundy inequality. Thus, (3.11) holds for \(p = 2\). For \(p \geq 3\), by the similar line in (7), we may derive the desired result. Here we skip it.

For the second assertion, taking \(v = A^{-1}u^n \in V_0\) and \(q = 0\) in (3.2), one finds that

\[(3.18) \quad \frac{1}{2} (\|u^n\|^2_{L^2} - \|u^{n-1}\|^2_{L^2} + \|u^n - u^{n-1}\|^2_{L^2}) + k\nu\|u^n\|^2_{L^2_x}
\[= kb(u^n, u^n, A^{-1}u^n) + (A^{-\frac{1}{2}}\eta(u^{n-1})W_n, A^{-\frac{1}{2}}(u^n - u^{n-1}))
\[+ (A^{-\frac{1}{2}}\eta(u^{n-1})\Delta W_n, A^{-\frac{1}{2}}(u^n - u^{n-1})).\]

By the Young’s inequality and the Hölder inequality, the first term on the right hand side of (3.18) can be bounded

\[kb(u^n, u^n, A^{-1}u^n)
\[\leq Ck\|u^n\|_{L^2_x} \|\nabla u^n\|_{L^2_x} \|u^n - u^{n-1}\|_{-1} + Ck\|u^n\|_{L^2_x} \|\nabla u^n\|_{L^2_x} \|u^n\|_{-1}
\[\leq \frac{1}{4} \|u^n - u^{n-1}\|^2_{L^2} + Ck^2\|u^n\|^2_{L^2_x} \|\nabla u^n\|^2_{L^2_x} + Ck\|u^n\|_{L^2_x} \|\nabla u^n\|_{L^2_x} \|u^n\|_{-1}.\]
The last term on the right hand of (3.18) vanishes when taking its expectation. Using the Young’s inequality and the tower property for conditional expectations to the second term on the right hand. Summing up then leads to

\[
\begin{align*}
\text{(3.19)} \quad & E \left[ \left\| u^n \right\|_{L_2}^2 \right] + E \left[ \sum_{n=1}^{N} \left\| u^n - u^{n-1} \right\|_{L_2}^2 \right] + E \left[ \nu \sum_{n=1}^{N} k \left\| u^n \right\|_{L_2}^4 \right] \\
& \leq C E \left[ \sum_{n=1}^{N} k \left\| u^n \right\|_{L_2}^2 \left\| \nabla u^n \right\|_{L_2}^2 \right] + C E \left[ \sum_{n=1}^{N} k \left\| u^n \right\|_{L_2} \left\| \nabla u^n \right\|_{L_2} \left\| u^{n-1} \right\| \right] \\
& \quad + E \left[ \sum_{n=1}^{N} E \left[ \left\| P_H \eta (u^{n-1}) \right\|_{L_2}^2 \right] \left\| \Delta_n W \right\|_{L_2} \right] \\
& \leq C \left( E \left[ \sum_{n=1}^{N} k \left\| u^n \right\|_{L_2}^4 \right] \right)^{\frac{1}{2}} \left( E \left[ \sum_{n=1}^{N} k \left\| \nabla u^n \right\|_{L_2}^4 \right] \right)^{\frac{1}{2}} \\
& \quad + C \left( E \left[ \max_{1 \leq n \leq N} \left\| u^n \right\|_{L_2}^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \max_{1 \leq n \leq N} \left\| \nabla u^n \right\|_{L_2}^2 \right] \right)^{\frac{1}{2}} + E \sum_{n=1}^{N} k \left\| u^{n-1} \right\|_{L_2}^2.
\end{align*}
\]

Using Lemma 3.1 and the discrete Gronwall’s lemma, one finds that

\[
\begin{align*}
\text{(3.20)} \quad & E \left[ \left\| u^n \right\|_{L_2}^2 \right] + E \left[ \nu \sum_{n=1}^{N} k \left\| u^n \right\|_{L_2}^2 \right] \leq C.
\end{align*}
\]

To obtain the first inequality in (3.12), using the Young’s inequality and Lemma 3.1, it follows that

\[
\begin{align*}
\text{(3.21)} \quad & E \left[ \max_{1 \leq n \leq N} \left\| u^n \right\|_{L_2}^2 \right] \leq E \left[ \left\| u^0 \right\|_{L_2}^2 \right] + C E \left[ \sum_{n=1}^{N} k \left\| \nabla u^n \right\|_{L_2}^4 \right] + C E \left[ \max_{1 \leq n \leq N} \left\| u^n \right\|_{L_2}^4 \right] \\
& \quad + C E \left[ \sum_{n=1}^{N} \left\| A^{-\frac{1}{2}} P_H \eta (u^{n-1}) \Delta W_n \right\|_{L_2}^2 \right] \\
& \quad + C E \left[ \max_{1 \leq n \leq N} \sum_{l=1}^{n} (A^{-\frac{1}{2}} P_H \eta (u^{l-1}) \Delta W_n, A^{-\frac{1}{2}} u^{l-1}) \right].
\end{align*}
\]

The second term and the third term on the right hand side may be controlled by Lemma 3.1. The fourth term may be estimated by the tower property for conditional expectations, the fifth term is bounded with using the Burkholder-Davis-Gundy inequality. Thus, (3.11) holds for \( p = 2 \). For \( p \geq 3 \), using the similar line in [7], we skip it.

For the third assertion, using the similar line in [7], summing over the index \( n = 1 \) in (3.14) and taking the power four, it follows that

\[
\begin{align*}
\text{(3.22)} \quad & \left( \sum_{n=1}^{N} \left\| A (u^n - u^{n-1}) \right\|_{L_2}^2 \right)^4 + \left( \sum_{n=1}^{N} \nu \left\| A^{\frac{3}{2}} u^n \right\|_{L_2}^4 \right)^4 \\
& \leq C \left( \sum_{n=1}^{N} k \left\| \nabla u^n \right\|_{L_2}^{10} + \left\| u^n \right\|_{L_2}^{10} \right)^4 + C \left( \sum_{n=1}^{N} \left\| A P_H \eta (u^{n-1}) \Delta W_n \right\|_{L_2}^2 \right)^4 \\
& \quad + C \left( \sum_{n=1}^{N} (A P_H \eta (u^{n-1}) \Delta W_n, A u^{n-1}) \right)^4 + \| A u^0 \|_{L_2}^8.
\end{align*}
\]
Taking the expectation, the second term and the third term on the right hand can be bounded as in [7], it follows that

\[
\mathbb{E}\left[\left(\sum_{n=1}^{N} \|A(u^n - u^{n-1})\|_{L^2}^2\right)^4 + \left(\sum_{n=1}^{N} k\nu\|A^2 u^n\|_{L^2}^2\right)^4\right]
\leq C\mathbb{E}\left[\sum_{n=1}^{N} k\|\nabla u^n\|_{L^2}^4 + \|u^n\|_{L^2}^4\right] + C\mathbb{E}\left[\sum_{n=1}^{N} k\|u^{n-1}\|_{L^2}^8\right]
+ C\mathbb{E}\left[\|Au^0\|_{L^2}^8\right].
\]

Thanks to Lemma 3.1, the desired result (3.13) holds. The proof is complete. \(\square\)

Following [10], for \(\epsilon > 0\), we define the following sample sets

\[
\Omega_k^\epsilon = \left\{ \omega \in \Omega \left| \max_{1 \leq n \leq N} \|\nabla u^n\|_{L^2}^4 \leq -\epsilon \log k \right\}
\]

such that

\[
P(\Omega_k^\epsilon) \geq 1 - \frac{\mathbb{E}\left[\omega \in \Omega \left| \max_{1 \leq n \leq N} \|\nabla u^n\|_{L^2}^4 \right. \right]}{-\epsilon \log k} \geq 1 + \frac{C}{\epsilon \log k},
\]

and

\[
\Omega_\tau^\epsilon = \left\{ \omega \in \Omega \left| \max_{1 \leq n \leq N} \left(\|Au^n\|_{L^2}^4, \max_{t_{n-1} \leq s \leq t_n} \|u(s)\|_{H_2}^4\right) \leq -\epsilon \log k \right\}
\]

such that

\[
P(\Omega_\tau^\epsilon) \geq 1 - \frac{\mathbb{E}\left[\omega \in \Omega \left| \max_{1 \leq n \leq N} \left(\|Au^n\|_{L^2}^4, \max_{t_{n-1} \leq s \leq t_n} \|u(s)\|_{H_2}^4\right) \right. \right]}{-\epsilon \log k} \geq 1 + \frac{C}{\epsilon \log k}.
\]

By using the similar line in [10] [5], the following theorem states and derives the optimal order error estimate for \(\{u^n; 1 \leq n \leq N\}\) of various spatial norms.

**Theorem 3.3.** Assume that (2.2) holds and that \(u_0 \in L^8(\Omega, \mathcal{V}_0)\) is an \(\mathbb{F}_0\)-measurable random variable. Let \((u, r)\) be the unique strong solution to (2.7) in the sense of Definition 2, Assume that

\[
\mathbb{E}\left[\|u\|_{C^\alpha(0,T;L^4)}^2\right] \leq C,
\mathbb{E}\left[\|u\|_{C^\alpha(0,T;\mathcal{V}_1)}^2\right] \leq C,
\mathbb{E}\left[\|u\|_{C^\alpha(0,T;\mathcal{H}_2)}^2\right] \leq C
\]

for some \(\alpha \in (0, \frac{1}{2})\). Then, provided that \(0 < k < k_0\) with \(k_0\) sufficiently small, the following error estimates hold:

\[
\mathbb{E}\left[I_{\Omega_k} \left(\max_{1 \leq n \leq N} \|u(t^n) - u^n\|_{L^2}^2 + \nu k \sum_{n=1}^{N} \|\nabla (u(t^n) - u^n)\|_{L^2}^2\right) \right] \leq Ck^{2\alpha - \epsilon},
\]

\[
\mathbb{E}\left[I_{\Omega_\tau} \left(\max_{1 \leq n \leq N} \|\nabla (u(t^n) - u^n)\|_{L^2}^2 + \nu k \sum_{n=1}^{N} \|A(u(t^n) - u^n)\|_{L^2}^2\right) \right] \leq Ck^{2\alpha - \epsilon},
\]

where \(C\) is a positive constant independent of \(k\).
By adding and subtracting suitable terms, we rewrite the nonlinear term every $n$ for (3.30), setting $e^n_u := u(t_n) - u^n$, subtracting (2.7a) from (3.2a) satisfies the following error equation:

\begin{align}
(3.31) \quad (e^n_u - e^{n-1}_u, v) + \int_{t_{n-1}}^{t_n} a(u(s), v) - a(u^n, v) \, ds \\
= \int_{t_{n-1}}^{t_n} b(u^n, u^n, v) - b(u(s), u(s), v) \, ds \\
+ \int_{t_{n-1}}^{t_n} (\eta(u(s)) - \eta(u^{n-1}))dW, v \quad \forall v \in \mathcal{V}_0.
\end{align}

For (3.30), setting $v = Ae^n_u$ in (3.31), one finds that

\begin{align}
(3.32) \quad \frac{1}{2}(\|\nabla e^n_u\|_{L^2}^2 - \|\nabla e^{n-1}_u\|_{L^2}^2 + \|\nabla (e^n_u - e^{n-1}_u)\|_{L^2}^2) + kv\|Ae^n_u\|_{L^2}^2 \\
= \int_{t_{n-1}}^{t_n} a(u(s) - u(t^n), Ae^n_u)ds + \int_{t_{n-1}}^{t_n} b(u^n, u^n, e^n_u) - b(u(s), u(s), Ae^n_u)ds \\
+ \int_{t_{n-1}}^{t_n} (\nabla(\eta(u(s)) - \eta(u^{n-1})))dW, \nabla e^n_u \\
= D_1 + D_2 + D_3.
\end{align}

With the Poincaré inequality and the Young’s inequality, the term $D_1$ can be bounded by

\begin{align}
(3.33) \quad D_1 = \int_{t_{n-1}}^{t_n} a(u(s) - u(t^n), Ae^n_u)ds \\
\leq C \int_{t_{n-1}}^{t_n} \|A(u(s) - u(t^n))\|_{L^2}^2 ds + \frac{k\nu}{16}\|Ae^n_u\|_{L^2}^2
\end{align}

By adding and subtracting suitable terms, we rewrite the nonlinear term $D_2$ as follows:

\begin{align}
(3.34) \quad D_2 = \int_{t_{n-1}}^{t_n} b(u^n, u^n, Ae^n_u) - b(u(s), u(s), Ae^n_u)ds \\
= \int_{t_{n-1}}^{t_n} \left(b(u^n - u(t_n), u^n, Ae^n_u) + b(u(t_n) - u(s), u^n, Ae^n_u) \\
+ b(u(s), u(t_n) - u(s), Ae^n_u) + b(u(s), u^n - u(t_n), Ae^n_u)\right)ds \\
= D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4}.
\end{align}
With the Young’s inequality and the Sobolev inequality, we get

\[(3.35) \quad D_{2,1} = \int_{t_{n-1}}^{t_n} b(u^n - u(t_n), u^n, Ae_u^n) ds \]

\[\leq \frac{k\nu}{16} \|Ae_u^n\|_{L^2}^2 + Ck\|\nabla e^n\|_{L^2}^2 \|Au^n\|_{L^2}^2,\]

\[(3.36) \quad D_{2,2} = \int_{t_{n-1}}^{t_n} b(u(t_n) - u(s), u^n, Ae_u^n) ds \]

\[\leq \frac{k\nu}{16} \|Ae_u^n\|_{L^2}^2 + C\int_{t_{n-1}}^{t_n} \|\nabla (u(t_n) - u(s))\|_{L^2}^2 \|Au^n\|_{L^2}^2 ds,\]

\[(3.37) \quad D_{2,3} = \int_{t_{n-1}}^{t_n} b(u(s), u(t_n) - u(s), Ae_u^n) ds \]

\[\leq \frac{k\nu}{16} \|Ae_u^n\|_{L^2}^2 + C\int_{t_{n-1}}^{t_n} \|\nabla u(t_n) - u(s)\|_{L^2}^2 \|u(s)\|_{H^2}^2 ds,\]

\[(3.38) \quad D_{2,4} = \int_{t_{n-1}}^{t_n} b(u(s), u^n - u(t_n), Ae_u^n) ds \]

\[\leq \frac{k\nu}{16} \|Ae_u^n\|_{L^2}^2 + Ck\|\nabla e^n\|_{L^2}^2 \max_{t_{n-1} \leq s \leq t_n} \|u(s)\|_{H^2}^2.\]

Inserting estimates \(3.33), (3.35)-(3.38)\ into \(3.32)\, applying the summation operator \(\sum_{n=1}^{N}\) and taking the expectation, using \(3.28)\ and Lemma \(3.1)\ it follows that

\[(3.39) \quad E\left[\Omega_t \left( \max_{1 \leq n \leq N} \|\nabla e_u^n\|_{L^2}^2 + \sum_{n=1}^{N} \|\nabla(e_u^n - e_u^{n-1})\|_{L^2}^2 + \frac{1}{2} \sum_{n=1}^{N} k\nu \|Ae_u^n\|_{L^2}^2 \right) \right] \]

\[\leq Ck^{2\alpha} + CE\left[\Omega_t \left( \sum_{n=1}^{N} k\|\nabla e_u^n\|_{L^2}^2 \left( \|Au^n\|_{L^2}^2 + \max_{t_{n-1} \leq s \leq t_n} \|u(s)\|_{H^2}^2 \right) \right) \right] \]

\[+ E\left[\Omega_t \left( \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\nabla[\eta(u(s)) - \eta(u^{n-1})]dW, \nabla e_u^n) \right) \right].\]

Using \(\Omega_t, \Omega_t, \Omega_t, \Omega_t\) and \(\Omega_t\, the last term \(D_3\) can be estimated by

\[(3.40) \quad E\left[\Omega_t \left( \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\nabla[\eta(u(s)) - \eta(u^{n-1})]dW, \nabla e_u^n) \right) \right] \]

\[\leq Ck^{2\alpha} + \frac{1}{2} E\left[\Omega_t \left( \max_{1 \leq n \leq N} \|\nabla e_u^n\|_{L^2}^2 \right) \right] + CE\left[\Omega_t \left( \sum_{n=1}^{N} k\|\nabla e_u^{n-1}\|_{L^2}^2 \right) \right] \]

\[+ \frac{1}{2} E\left[\Omega_t \left( \sum_{n=1}^{N} \|\nabla(e_u^n - e_u^{n-1})\|_{L^2}^2 \right) \right].\]
Then combining (3.39) with (3.30) leads to

\[
E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \max_{1 \leq n \leq N} \| \nabla e_u^n \|_{L^2}^2 + \sum_{n=1}^{N} \| \nabla (e_u^n - e_u^{n-1}) \|_{L^2}^2 + \sum_{n=1}^{N} k \nu \| A e_u^n \|_{L^2}^2 \right) \right] \\
\leq C k^{2 \alpha} + \overline{C}_1 \log k^{- \epsilon} E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \sum_{n=1}^{N} k \| \nabla e_u^n \|_{L^2}^2 \right) \right] + \overline{C}_2 E \left[ \sum_{n=1}^{N} k \| \nabla e_u^{n-1} \|_{L^2}^2 \right] \\
+ C k E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \sum_{n=1}^{N} \| \nabla e_u^n \|_{L^2}^2 \right) \max_{t_m \leq s \leq t_n} \| u(s) \|_{H^2}^2 \right] \\
+ C k E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \sum_{n=1}^{N} \| \nabla e_u^n \|_{L^2}^2 \right) \right].
\]

The terms \( \Theta_1 \) and \( \Theta_2 \) may be controlled by Lemmas 2.3, 3.1 and 3.2. If \( 0 < k \leq k_0, k_* := \frac{1}{2C_1 \log k_0^{- \epsilon}} < \frac{1}{C_1 \log k_0^{- \epsilon}}, \) since \( 1 \leq \frac{1}{1 - C_1 \log(k-\epsilon)k} \leq 2, \) it follows that

\[
E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \max_{1 \leq n \leq N} \| \nabla e_u^n \|_{L^2}^2 + \sum_{n=1}^{N} \| \nabla (e_u^n - e_u^{n-1}) \|_{L^2}^2 + \sum_{n=1}^{N} k \nu \| A e_u^n \|_{L^2}^2 \right) \right] \\
\leq C k^{2 \alpha} + \overline{C}_1 \log k^{- \epsilon} E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \sum_{n=1}^{N} \| \nabla e_u^{n-1} \|_{L^2}^2 \right) \right] \\
+ \overline{C}_2 \left( 1 - C_1 k \log k^{- \epsilon} \right) E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \sum_{n=1}^{N} k \| e_u^{n-1} \|_{L^2}^2 \right) \right] \\
\leq C k^{2 \alpha} + 2 (\overline{C}_1 \log k^{- \epsilon} + \overline{C}_2) E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \sum_{n=1}^{N} k \| \nabla e_u^{n-1} \|_{L^2}^2 \right) \right].
\]

By using the discrete Gronwall inequality, the result (3.30) holds. The proof is complete.

**Remark 2.** From Lemma 2.3, it is easy to see that the first and second inequalities of the condition (3.28) hold. Using the similar technique of [5], the third inequality of the condition (3.28) can be satisfied.

The last result of this section is stated in the following theorems which give an optimal error estimate for the pressure \( \{ r^n; 1 \leq n \leq N \} \) and \( \{ p^n; 1 \leq n \leq N \} \).

**Theorem 3.4.** Let the assumptions of Theorem 3.3 be satisfied. Let \( \{ r^n; 1 \leq n \leq N \} \) be the pressure approximation defined by Algorithm 1. Then the following error estimate holds for \( m = 1, 2, \cdots, N \)

\[
E \left[ \mathbf{1}_{\Omega_{\tau}} \left( \int_0^{t_m} r(s) \, ds - k \sum_{n=1}^{m} r^n \right)^2 \right] \leq C k^{2 \alpha - \epsilon},
\]

where \( C \) is a positive constant independent of \( k \).
Proof. Summing (3.4a) over $1 \leq n \leq m(\leq N)$, we get

\begin{equation}
(u^m, v) + k \sum_{n=1}^{m} a(u^n, v) - k \sum_{n=1}^{m} d(v, r^n) + k \sum_{n=1}^{m} b(u^n, u^n, v) \\
= (u^0, v) + \sum_{n=1}^{m} (\eta(u^{n-1}) \Delta W_n, v) \quad \forall v \in \mathbb{V}, \text{ a.s.}
\end{equation}

Subtracting (2.7a) (with $t = t_n$) from (3.4b) and noting that $u^0 = u(0)$, we obtain

\begin{equation}
d(v, k \sum_{n=1}^{m} r^n - \int_{0}^{t_n} r(s) \, ds) = \nu \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} a(u(s) - u^n, v) \, ds \\
+ \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} b(u(s), u(s), v) - b(u^n, u^n, v) \, ds \\
+ \sum_{n=1}^{m} \left( \int_{t_{n-1}}^{t_n} (\eta(u^{n-1}) - \eta(u(s))) \, dW(s), v \right) + (u(t^n) - u^m, v) \\
= \nu \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (\nabla u(s) - \nabla u(t^n) + \nabla u(t^n) - \nabla u^n, \nabla v) \, ds \\
+ \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} b(u(s), u(s), v) - b(u^n, u^n, v) \, ds \\
+ \sum_{n=1}^{m} \left( \int_{t_{n-1}}^{t_n} (\eta(u^{n-1}) - \eta(u(s))) \, dW(s), v \right) + (u(t^n) - u^m, v).
\end{equation}

By using the Poincaré inequality, the Hölder inequality and the inf-sup condition, one finds that

\begin{equation}
\beta \left\| k \sum_{n=1}^{m} r^n - \int_{0}^{t_n} r(s) \, ds \right\|_{L^2_x} \leq C \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \|\nabla u(s) - \nabla u(t^n)\|_{L^2_x} \, ds + \nu \sum_{n=1}^{m} \|\nabla u(t^n) - \nabla u^n\|_{L^2_x} \\
+ C \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \|u - u(t_n)\|_{L^2_x} \|\nabla u\|_{L^2_x} + \|\nabla u(t_n)\|_{L^2_x} \|\nabla (u - u(t_n))\|_{L^2_x} \, ds \\
+ C \sum_{n=1}^{m} \left( \|\nabla e_u^n\|_{L^2_x} \max_{1 \leq i \leq m} \|\nabla u(t_i)\|_{L^2_x} + \max_{1 \leq i \leq m} \|\nabla u^i\|_{L^2_x} \|\nabla e_u^n\|_{L^2_x} \right) \\
+ C \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \left( \eta(u^{n-1}) - \eta(u(s)) \right) \, dW(s) \|_{L^2_x} + C \|u(t^n) - u^m\|_{L^2_x}.
\end{equation}
Making use of the Lemmas 2.3, 3.1 and Theorem 3.3, the result (3.43) holds. Theorem 3.5. Let the assumptions of Theorem 3.3 be satisfied. Let \( \{p^n; 1 \leq n \leq N\} \) be the pressure approximation defined by Algorithm 1. Then the following error estimate holds for \( m = 1, 2, \ldots, N \)

\[
\mathbb{E}[\sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (\eta(u^{n-1}) - \eta(u(s))) \, dW(s) \bigg| L^2_{x}] \leq C k^{3\alpha - \gamma},
\]

where \( C \) is a positive constant independent of \( k \).

4. Fully discrete mixed finite element scheme. In this section we propose and analyze a fully discrete time-stepping scheme for the mixed formulation (2.4). The error estimates in strong norms for both the velocity and pressure approximations are obtained. Furthermore, we derive strong optimal \( H^1 \) convergence first, and then obtain \( L^2 \) convergence of the fully discrete scheme with the negative norm technique.

Suppose that \( \mathcal{T}_h \) is a quasi-uniform family of triangulation of the periodic domain \( D \subset \mathbb{R}^2 \). We define three finite element spaces as follows:

\[
\mathcal{V}_h = \{ v_h \in H^1_{\text{per}}(D) ; v_h|_K \in P_2(K)^2 \, \forall \, K \in \mathcal{T}_h \},
\]

\[
\mathcal{W}_h = \{ q_h \in L^2_{\text{per}}(D); q_h|_K \in P_1(K) \, \forall \, K \in \mathcal{T}_h \},
\]

\[
\mathcal{S}_h = \{ w_h \in H^1_{\text{per}}(D); w_h|_K \in P_l(K)^2 \, \forall \, K \in \mathcal{T}_h \},
\]

where \( P_l(K) \) (\( l = 1, 2 \)) denotes the set of polynomials of degree less than or equal to \( l \) over the element \( K \in \mathcal{T}_h \).

In addition, we consider the weakly discrete divergence-free subspace \( \mathcal{V}_{0h} \subset \mathcal{V}_h \)

\[
\mathcal{V}_{0h} = \{ v_h \in \mathcal{V}_h; d(q_h, v_h) = 0, \forall q_h \in \mathcal{W}_h \}.
\]

As it is noted \( \mathbb{H} \) that the finite element space pair \( (\mathcal{V}_h, \mathcal{W}_h) \) is stable in the sense that the following discrete inf-sup condition holds, i.e., there exists an \( h \)-independent positive constant \( \gamma \) such that

\[
\sup_{v_h \in \mathcal{V}_h} \frac{d(v_h, q_h)}{\|\nabla v_h\|_{L^2_x}} \geq \gamma \|q_h\|_{L^2_x} \quad \forall \, q_h \in \mathcal{W}_h.
\]
We define the $L^2(D)$ projections $\rho_h : L^2_{per}(D) \to W_h$, $\Pi_h : L^2_{per}(D)^2 \to V_h$ and the $L^2$ Ritz-projection $\sigma_h : H^1_{per}(D)^2 \to S_h$ such that

\[
(\varphi - \rho_h \varphi, \psi_h) = 0, \quad \forall \psi_h \in W_h, \\
(v - \Pi_h v, w_h) = 0, \quad \forall w_h \in V_h, \\
(\nabla(\phi - \sigma_h \phi), \nabla \chi_h) = 0, \quad \forall \chi_h \in S_h.
\]

The following approximation properties are well-known\footnote{\cite{BB04, BB13, BB17}}.

\begin{align}
(4.3) \quad & \| \varphi - \rho_h \varphi \|_{L^2} + h \| \nabla (\varphi - \rho_h \varphi) \|_{L^2} \leq C h^s \| \varphi \|_{H^s} \quad \forall \varphi \in H^s_{per}(D), \\
(4.4) \quad & \| v - \Pi_h v \|_{L^2} + h \| \nabla (v - \Pi_h v) \|_{L^2} \leq C h^s \| v \|_{H^s} \quad \forall v \in H^s_{per}(D)^2, \\
(4.5) \quad & \| \phi - \sigma_h \phi \|_{L^2} + h \| \nabla (\phi - \sigma_h \phi) \|_{L^2} \leq C h^s \| \phi \|_{H^s} \quad \forall \phi \in H^s_{per}(D)^2 / \mathbb{R},
\end{align}

where $C$ is a positive constant independent of $h$.

Our fully discrete finite element algorithm for (2.7) is defined as follows.

**Algorithm 2:**

Set $u^n_0 \in L^2(\Omega, V_h)$, for $n = 1, \ldots, N$, we define the following steps:

**Step I:** Find $\xi(u_h^{n-1}) \in L^2(\Omega, S_h)$ by solving

\[
(4.6) \quad (\nabla \xi(u_h^{n-1}), \nabla \phi) = (G(u_h^{n-1}), \nabla \phi), \quad \forall \phi \in S_h \text{ a.s.}
\]

**Step II:** Denote $\eta(u_h^{n-1}) := G(u_h^{n-1}) - \nabla \xi(u_h^{n-1})$, and find $(u^n_h, r^n_h) \in L^2(\Omega, V_h) \times L^2(\Omega, W_h)$ by solving

\begin{align}
(4.7a) \quad & (u^n_h, v) + k a(u^n_h, v) - k d(v, r^n_h) + k b(u^n_h, u^n_h, v) \\
& \quad = (u^{n-1}_h, v) + (\eta(u_h^{n-1}) \Delta W_n, v) \quad \forall v \in V_h, \text{ a.s.}, \\
(4.7b) \quad & d(u^n_h, q) = 0 \quad \forall q \in W_h, \text{ a.s.}
\end{align}

**Step III:** Denote $p^n_h := r^n_h + k^{-1} \xi(u_h^{n-1}) \Delta_n W$.

We now give the following stabilities for $u^n_h$ and $r^n_h$, but omit their proofs because they are similar to semi-discrete scheme given in\footnote{\cite{BB04, BB13}}.

**Lemma 4.1.** Let $1 \leq q < \infty$ be a natural number. Assume $u^n_0 \in L^q(\Omega, V_{0h})$ with $\| u^n_0 \|_{L^q_2} \leq C$. Let $\{(u^n_h, r^n_h, p^n_h); 1 \leq n \leq N\}$ be a solution to **Algorithm 2**, then there hold

\begin{align}
(4.8a) \quad & \mathbb{E} \left[ \max_{1 \leq n \leq N} \| u^n_h \|_{L^p_2}^p + \nu k \sum_{n=1}^N \| u^n_h \|_{L^2}^{2p-1} \| \nabla u^n_h \|_{L^2}^2 \right] \leq C, \\
(4.8b) \quad & \mathbb{E} \left[ k \sum_{n=1}^N \| r^n_h \|_{L^2}^2 \right] \leq C.
\end{align}

**Lemma 4.2.** Let $1 \leq p < \infty$ be a natural number. Assume $u^n_0 \in L^q(\Omega, V_{0h})$ with $\| u^n_0 \|_{L^p_2} \leq C$. Then there exists a sequence $\{u^n_h\}_{n \geq 1}$ of $\mathcal{V}$-valued random variables,
which for all $\omega \in \Omega$, solves Algorithm 2 and has the following stability estimates:

\[(4.9a)\quad \text{E} \left[ \max_{1 \leq n \leq N} \| \nabla u^n_h \|_{L^2}^{2p} + \nu k \sum_{n=1}^{N} \| \nabla u^n_h \|_{L^2}^{2p-1} \right] \leq C, \]

\[(4.9b)\quad \text{E} \left[ \sum_{n=1}^{N} \| \nabla (u^n_h - u^{n-1}_h) \|_{L^2}^{2} \right] \leq C, \]

\[(4.9c)\quad \text{E} \left[ \left( \sum_{n=1}^{N} \| \nabla (u^n_h - u^{n-1}_h) \|_{L^2}^{2} \right) \right]^{4} + \left( k \sum_{n=1}^{N} \| \nabla^2 u^n_h \|_{L^2}^{2} \right) \leq C. \]

For $\epsilon > 0$, we introduce the sample set

\[(4.10)\quad \Omega_{h}^\epsilon = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \| \nabla u^n \|_{L^2}^{2} + \| u^n_h \|_{L^2}^{2} \leq -\epsilon \log(h^2 + k) \right\} \]

such that

\[(4.11)\quad \text{P}(\Omega_{h}^\epsilon) \geq 1 - \frac{\text{E} \left[ \omega \in \Omega \mid \max_{1 \leq n \leq N} \left( \| \nabla u^n \|_{L^2}^{2} + \| u^n_h \|_{L^2}^{2} \right) \right]}{-\epsilon \log(h^2 + k)} \geq 1 + \frac{C}{\epsilon \log(h^2 + k)}. \]

We are now in a position to state and prove the first main theorem of this section.

**Theorem 4.3.** Set $u^0 = u_0$ and let $\{u^n; 1 \leq n \leq N\}$ and $\{u^n_h; 1 \leq n \leq N\}$ be the solutions of Algorithm 1 and Algorithm 2, respectively. Then, provided that $0 < k < k_0$ and $0 < h < h_0$ with $k_0$ and $h_0$ sufficiently small, the following error estimate holds:

\[(4.12)\quad \text{E} \left[ I_{\Omega_{h}^\epsilon} \left( \max_{1 \leq n \leq N} \| u^n - u^n_h \|_{L^2}^{2} + k \sum_{n=1}^{N} \| \nabla (u^n - u^n_h) \|_{L^2}^{2} \right) \right] \leq C(h^{2-2\epsilon} + k^{1-\epsilon}). \]

**Remark 3.** The proof of Theorem 4.3 is similar to the proofs of [14] [13]. But we use different indicator function [10].

**Proof.** For every $n \geq 1$, let $e^{n,h}_u := u^n - u^n_h$ and $e^{n,h}_r := r^n - r^n_h$, it is easy to check that $(e^{n,h}_u, e^{n,h}_r)$ satisfies the following error equations:

\[(4.13a)\quad (e^{n,h}_u - e^{n-1,h}_u, v_h) + k a(e^{n,h}_u, v_h) + k d(v_h, e^{n,h}_u) + k b(u^n, u^n, v_h) - k b(u^n_h, u^n_h, v_h) = (\eta(u^{n-1}) - \eta(u^{n-1}_h)) \Delta W_n, v_h, \quad \forall v_h \in V_h, \]

\[(4.13b)\quad d(e^{n,h}_r, q_h) = 0, \quad \forall q_h \in W_h. \]

Setting $v_h = \Pi_h e^{n,h}_u$ and $q_h = \rho_h e^{n,h}_r$, we have

\[(4.14)\quad (e^{n,h}_u - e^{n-1,h}_u, \Pi_h e^{n,h}_u) + k a(e^{n,h}_u, \Pi_h e^{n,h}_u) - k d(\Pi_h e^{n,h}_u, e^{n,h}_r) + k b(u^n, u^n, \Pi_h e^{n,h}_u) - k b(u^n_h, u^n_h, \Pi_h e^{n,h}_u) = (\eta(u^{n-1}) - \eta(u^{n-1}_h)) \Delta W_n, \Pi_h e^{n,h}_u) \]
By using the identity $a \cdot (a - b) = \frac{1}{2}(|a|^2 - |b|^2 + |a|^2 - |b|^2)$, we gain

\begin{equation}
\frac{1}{2} (\| \Pi_h e_u^{n,h} \|^2_{L^2} - \| \Pi_h e_u^{n-1,h} \|^2_{L^2} + \| \Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h} \|^2_{L^2}) \\
+ k \nu \| \nabla \Pi_h e_u^{n,h} \|^2_{L^2} = k a (u^n - u_h^n, \Pi_h e_u^{n,h}) + k d (\Pi_h e_u^{n,h}, e_p^{n,h}) \\
+ k b (u_h^n, u^{n}, \Pi_h e_u^{n,h}) - k b (u^n, u^{n}, \Pi_h e_u^{n,h}) \\
+ (|\eta (u^{n-1}) - \eta (u_h^{n-1})| \Delta W_n, \Pi_h e_u^{n,h}) \\
= \sum_{i=1}^{4} I_i.
\end{equation}

For terms $I_1$ and $I_2$, thanks to the Young's inequality, (4.3) and (4.4), we obtain

\begin{align*}
I_1 & \leq \frac{\nu k}{8} \| \nabla \Pi_h e_u^{n,h} \|^2_{L^2} + C k h^2 \| \nabla^2 u^n \|^2_{L^2}, \\
I_2 & \leq \frac{\nu k}{8} \| \nabla \Pi_h e_u^{n,h} \|^2_{L^2} + C k h^2 \| \nabla u^n \|^2_{L^2}.
\end{align*}

For nonlinear term $I_3$, we rewrite as follows:

\begin{align*}
I_3 &= -k b (u^n - \Pi_h u^n, u^n, \Pi_h e_u^{n,h}) - k b (\Pi_h e_u^{n,h}, \Pi_h e_u^{n,h}, u^n) \\
&\quad - k b (\Pi_h e_u^{n,h}, \Pi_h e_u^{n,h}, u^n - \Pi_h u^n) + k b (\Pi_h u^n, \Pi_h e_u^{n,h}, u^n - \Pi_h u^n) \\
&= \sum_{i=1}^{4} I_{3,i}.
\end{align*}

Using the Poincaré inequality, the Young’s inequality, the embedding inequality and (6.3), one finds that

\begin{align*}
I_{3,1} & \leq C k \| u^n - \Pi_h u^n \|_{L^2} \| \nabla (u^n - \Pi_h u^n) \|_{L^2} \| \nabla u^n \|_{L^2} \| \nabla \Pi_h e_u^{n,h} \|_{L^2} \\
& \leq \frac{k \nu}{16} \| \nabla \Pi_h e_u^{n,h} \|^2_{L^2} + C k h^3 \| \nabla u^n \|^2_{L^2} \| \nabla u^n \|^2_{L^2}, \\
I_{3,2} & \leq k \| \Pi_h e_u^{n,h} \|^2_{L^2} \| \nabla \Pi_h e_u^{n,h} \|_{L^2} \| \nabla \Pi_h e_u^{n,h} \|_{L^2} \| u^n \|_{L^2} \\
& \leq \frac{k \nu}{16} \| \nabla \Pi_h e_u^{n,h} \|^2_{L^2} + C k \| \Pi_h e_u^{n,h} \|^2_{L^2} \| \nabla u^n \|^2_{L^2}, \\
I_{3,3} & \leq k \| \Pi_h e_u^{n,h} \|^2_{L^2} \| \nabla \Pi_h e_u^{n,h} \|_{L^2} \| \nabla \Pi_h e_u^{n,h} \|_{L^2} \| u^n - \Pi_h u^n \|_{L^2} \\
& \leq \frac{k \nu}{16} \| \nabla \Pi_h e_u^{n,h} \|^2_{L^2} + C k h^3 \| \nabla u^n \|^2_{L^2} \| \nabla u^n \|^2_{L^2}, \\
I_{3,4} & \leq k \| \nabla \Pi_h u^n \|_{L^2} \| \nabla \Pi_h e_u^{n,h} \|_{L^2} \| u^n - \Pi_h u^n \|_{L^2} \\
& \leq \frac{k \nu}{16} \| \nabla \Pi_h e_u^{n,h} \|^2_{L^2} + C k h^3 \| \nabla u^n \|^2_{L^2} \| \nabla^2 u^n \|^2_{L^2}.
\end{align*}
Inserting estimates (4.16)–(4.21) into (4.15), we arrive at

\begin{equation}
\frac{1}{2} \left( \| \Pi_h e_u \|_{L^2}^2 - \| \Pi_h e_u - \Pi_h e_u \|_{L^2}^2 + \| \Pi_h e_u - \Pi_h e_u \|_{L^2}^2 \right)
+ \frac{k
u}{2} \| \nabla \Pi_h e_u \|_{L^2}^2 \leq C k h^2 \| \nabla^2 u_n \|_{L^2}^2 + C k h^2 \| \nabla \eta \|_{L^2}^2
+ C k h^3 \| \nabla u_n \|_{L^2}^2 \| \nabla u_n \|_{L^2}^2 + C k \| \Pi_h e_u \|_{L^2}^2 \| \nabla u_n \|_{L^2}^2
+ C k h^3 \| e_u \|_{L^2}^2 \| \nabla u_n \|_{L^2}^2 + C k h^3 \| \nabla u_n \|_{L^2}^2 \| \nabla^2 u_n \|_{L^2}^2
+ (|\eta(u_n^{-1}) - \eta(u_n^{-1})| \Delta W_n, \Pi_h e_u \|_{L^2}^2) \right)
\end{equation}

Taking the expectation and applying the summation operator $\sum_{n=1}^N$, one finds that

\begin{equation}
E \left[ \Omega_h \left( \frac{1}{2} \| \Pi_h e_u \|_{L^2}^2 + \frac{N}{2} \| \Pi_h e_u - \Pi_h e_u \|_{L^2}^2 + \sum_{n=1}^N \frac{k \nu}{2} \| \nabla \Pi_h e_u \|_{L^2}^2 \right) \right]
\leq \frac{1}{2} E \left[ \| \Pi_h e_u \|_{L^2}^2 \right] + C h^2 E \left[ \sum_{n=1}^N \| \nabla^2 u_n \|_{L^2}^2 \right] + C h^2 E \left[ \sum_{n=1}^N \| \nabla \eta \|_{L^2}^2 \right]
+ C h^3 E \left[ \sum_{n=1}^N \| \nabla^2 u_n \|_{L^2}^2 \| \nabla u_n \|_{L^2}^2 \right] + C k h^2 E \left[ \sum_{n=1}^N \| e_u \|_{L^2}^2 \| \nabla u_n \|_{L^2}^2 \right] + C h^3 E \left[ \sum_{n=1}^N \| \nabla u_n \|_{L^2}^2 \| \nabla^2 u_n \|_{L^2}^2 \right]
+ C \log(h^2 + k)^{-c} E \left[ \sum_{n=1}^N \| \Pi_h e_u \|_{L^2}^2 \right] + E \left[ \sum_{n=1}^N \left( \eta(u_n^{-1}) - \eta(u_n^{-1}) \right) \Delta W_n, \Pi_h e_u \right] \right].
\end{equation}

Now we explain how to estimate in expectation for $\Lambda_i$ ($i = 1, \ldots, 4$). Making use of the Lemmas 3.1 and 4.1, the terms $\Lambda_i$ ($i = 1, 3$) are uniformly bounded

$\Lambda_1 \leq \left( E \max_{1 \leq m \leq N} \| \nabla u^m \|_{L^2}^2 \right)^\frac{1}{2} \left( E \sum_{n=1}^N k \| \nabla u_n \|_{L^2}^2 \right)^\frac{1}{2},$

$\Lambda_3 \leq \left( E \max_{1 \leq m \leq N} \| u^m \|_{L^2}^4 \right)^\frac{1}{4} \left( E \sum_{n=1}^N k \| \nabla u_n \|_{L^2}^8 \right)^\frac{1}{4}$
$+ \left( E \max_{1 \leq m \leq N} \| u^m \|_{L^2}^4 \right)^\frac{1}{4} \left( E \sum_{n=1}^N k \| \nabla u_n \|_{L^2}^8 \right)^\frac{1}{4}. $
Combining (4.24)–(4.25) into (4.23), we get

$$E \left[ \sum_{n=1}^{N} \left| e_{n,h}^{n} \right|_{L^2_x}^2 \| \nabla (u^n - u^{n-1}) \|_{L^2_x}^2 \right]$$

$$\leq \left( E \max_{1 \leq m \leq N} \| u^m \|_{L^2_x} \right)^{1/2} \left( E \left[ \sum_{n=1}^{N} \| \nabla (u^n - u^{n-1}) \|_{L^2_x}^8 \right] \right)^{1/2}$$

$$+ \left( E \max_{1 \leq m \leq N} \| e_{n,h}^m \|_{L^2_x} \right)^{1/2} \left( E \left[ \sum_{n=1}^{N} \| \nabla (u^n - u^{n-1}) \|_{L^2_x}^8 \right] \right)^{1/2},$$

and the term $\Lambda_4$ is uniformly bounded as follows:

$$\Lambda_4 \leq \left( E \max_{1 \leq m \leq N} \| \nabla u^m \|_{L^2_x}^4 \right)^{1/2} \left( E \sum_{n=1}^{N} k \| \nabla^2 u^n \|_{L^2_x}^4 \right)^{1/2}.$$

For term $I_4$, using Itô’s isometry and the Young’s inequality, we have (4.24)

$$E \left[ 1_{C_n} \left( \sum_{n=1}^{N} \left[ (u^n - u^{n-1}) \| \Delta W_n, \Pi_h e_{n,h} \right] \right) \right]$$

$$= E \left[ 1_{C_n} \left( \sum_{n=1}^{N} \left( (u^n - u^{n-1}) \| \Delta W_n, \Pi_h e_{n,h} - \Pi_h e_{n-1,h} \right) \right) \right]$$

$$\leq E \left[ 1_{C_n} \left( k \sum_{n=1}^{N} \| (u^n - u^{n-1}) \| \Delta W_n \| \Pi_h e_{n,h} - \Pi_h e_{n-1,h} \|_{L^2_x} \right) \right]$$

$$\leq \frac{1}{4} E \left[ 1_{C_n} \left( \| \Pi_h e_{n,h} - \Pi_h e_{n-1,h} \|_{L^2_x}^2 \right) \right] + E \left[ 1_{C_n} \left( k \sum_{n=1}^{N} \| (u^n - u^{n-1}) \|_{L^2_x}^2 \right) \right].$$

With the definition of $\eta$ and using (4.21), (4.22), (4.24), (4.25) and (4.26), one finds that (4.25)

$$\| \eta(u^n - u^{n-1}) - \eta(u_{h}^{n-1}) \|_{L^2_x}^2$$

$$\leq C \| e_{n-1} \|_{L^2_x}^2 + Ch^2 \| \eta(u^{n-1}) \|_{H^2}^2$$

$$\leq Ch^2 \| \nabla \eta \|_{L^2_x}^2 + C \| e_{n-1} \|_{L^2_x}^2$$

$$\leq Ch^2 \| \nabla u^{n-1} \|_{L^2_x}^2 + Ch^4 \| \nabla^2 u^{n-1} \|_{L^2_x}^2 + C \| \Pi_h e_{n-1,h} \|_{L^2_x}^2.$$

Combining (4.24)–(4.25) into (4.23), we get (4.26)

$$E \left[ 1_{C_n} \left( \frac{1}{2} \| \Pi_h e_{n,h} \|_{L^2_x}^2 + \frac{1}{4} \sum_{n=1}^{N} \| \Pi_h e_{n,h} - \Pi_h e_{n-1,h} \|_{L^2_x}^2 + \frac{\gamma k}{2} \| \nabla e_{n,h} \|_{L^2_x}^2 \right) \right]$$

$$\leq C(h^2 + k) + C_3 \log(h^2 + k) E \left[ \sum_{n=1}^{N} k \| \Pi_h e_{n,h} \|_{L^2_x}^2 \right]$$

$$+ C_4 E \left[ \sum_{n=1}^{N} k \| \Pi_h e_{n-1,h} \|_{L^2_x}^2 \right].$$
If $0 < h < h_0$ and $0 < k \leq k_0$, $k^* := \frac{1}{2C_3 \log(h_0^2 + k_0)} < \frac{1}{C_3 \log(h_0^2 + [k_0]_*)}$, since $1 \leq \frac{1}{1 - C_3 k \log(h^2 + k)} \leq 2$, it follows that

\begin{equation}
(4.27) \quad \begin{aligned}
\mathbb{E} & \left[ \mathbf{1}_{\Omega_h} \left( \frac{1}{2} \| \Pi_k e_{u,h} \|_{L^2}^2 + \frac{1}{4} \sum_{n=1}^N \| \Pi_k e_{n,h} - \Pi_k e_{n-k,h} \|_{L^2}^2 + \sum_{n=1}^N \frac{k\nu}{2} \| \nabla e_{n,h} \|_{L^2}^2 \right) \right] \\
& \leq C(h^2 + k) + \frac{C_3 \log(h^2 + k)^{-\epsilon}}{1 - C_3 k \log(h^2 + k)^{-\epsilon}} \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \sum_{n=1}^N k \| \Pi_k e_{n-1,h} \|_{L^2}^2 \right) \right] \\
& \quad + \frac{C_4}{1 - C_3 k \log(h^2 + k)^{-\epsilon}} \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \sum_{n=1}^N k \| \Pi_k e_{n+1,h} \|_{L^2}^2 \right) \right] \\
& \leq C(h^2 + k) + 2(C_3 \log(h^2 + k)^{-\epsilon} + C_4) \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \sum_{n=1}^N k \| \Pi_k e_{n-1,h} \|_{L^2}^2 \right) \right].
\end{aligned}
\end{equation}

Then (4.12) follows from an application of the discrete Gronwall inequality and the triangle inequality. The proof is complete. 

The second result of this section is the following error estimate for the pressure approximation $\{r^n, 1 \leq n \leq N\}$ and $\{\eta^n; 1 \leq n \leq N\}$.

**Theorem 4.4.** Let the assumptions of Theorem 3.3 be satisfied. Let $\{r^n; 1 \leq n \leq N\}$ be the pressure approximation defined by Algorithm 2. Then the following error estimate holds for $m = 1, 2, \ldots, N$

\begin{equation}
(4.28) \quad \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \left\| \sum_{n=1}^N (r^n - r^n_h) \right\|_{L^2}^2 \right) \right] \leq C(h^{2-2\epsilon} + k^{1-\epsilon}),
\end{equation}

where $C$ is a positive constant independent of $h$ and $k$.

**Proof.** Summing (4.27) over $1 \leq n \leq m(\leq N)$ and subtracting the resulting equation from (4.44), we have

\begin{equation}
(4.29) \quad (e_{n,h}^m, v_h) + k \sum_{n=1}^m a(e_{n,h}^m, v_h) - k \sum_{n=1}^m d(v_h, e_{n,h}^m) \\
+ k \sum_{n=1}^m [b(u^n_h, u^n_h, v_h) - b(u^n, u^n, v_h)] \\
= (e_{u}^0, v_h) + \sum_{n=1}^m \left( [\eta(u^{n-1}) - \eta(u_{n-1}^h)] W_n, v_h \right), \quad \forall v_h \in \mathcal{V}_h, \ a.s.
\end{equation}

Using the Poincaré inequality, the Hölder inequality and the embedding inequality, it
follows that

\begin{equation}
(4.30) \quad d(v_h, k \sum_{n=1}^{m} e_r^{n,h}) = (e_u^{0,h} - e_u^{n,h}, v_h) - k \sum_{n=1}^{m} a(e_u^{n,h}, v_h)
+ k \sum_{n=1}^{m} [b(u_h^{n}, u_h^{n}) - b(u^n, u^n) + b(u_h^{n}, v_h)]
+ \sum_{n=1}^{m} (\eta(u^{n-1}) - \eta(u^{n-1}) \Delta W_n, v_h)
\leq C \left( \|e_u^{0,h}\|_{L^2} + \|e_u^{n,h}\|_{L^2} + \sum_{n=1}^{m} k \|\nabla e_u^{n,h}\|_{L^2}
+ \sum_{n=1}^{m} k \|\nabla e_u^{n,h}\|_{L^2} \|\nabla u^n\|_{L^2} + \sum_{n=1}^{m} k \|\nabla u^n_h\|_{L^2} \|\nabla e_u^{n,h}\|_{L^2}
+ \sum_{n=1}^{m} \|\eta(u^{n-1}) - \eta(u^{n-1}) \Delta W_n\|_{L^2} \|\nabla v\|_{L^2} \right).
\end{equation}

Applying the discrete inf-sup condition (4.2), we obtain

\begin{equation}
(4.31) \quad \gamma \left\| \sum_{n=1}^{m} e_r^{n,h} \right\|_{L^2} \leq C \left( \|e_u^{0,h}\|_{L^2} + \|e_u^{n,h}\|_{L^2} + \sum_{n=1}^{m} k \|\nabla e_u^{n,h}\|_{L^2}
+ \sum_{n=1}^{m} k \|\nabla e_u^{n,h}\|_{L^2} \|\nabla u^n\|_{L^2} + \sum_{n=1}^{m} k \|\nabla u^n_h\|_{L^2} \|\nabla e_u^{n,h}\|_{L^2}
+ \sum_{n=1}^{m} \|\eta(u^{n-1}) - \eta(u^{n-1}) \Delta W_n\|_{L^2} \right).
\end{equation}

With (4.10) and taking the expectation, one finds that

\begin{equation}
(4.32) \quad \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \left\| \sum_{n=1}^{m} e_r^{n,h} \right\|_{L^2} \right) \right]
\leq C \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \|e_u^{0,h}\|_{L^2} \right) \right] + C \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \|e_u^{n,h}\|_{L^2} \right) \right] + C \mathbb{E} \left[ \sum_{n=1}^{m} k \|\nabla e_u^{n,h}\|_{L^2} \right]
+ C \mathbb{E} \left[ \sum_{n=1}^{m} k \|\nabla e_u^{n,h}\|_{L^2} \left( \max_{1 \leq n \leq m} \|\nabla u^n\|_{L^2} \right) \right]
+ C \mathbb{E} \left[ \sum_{n=1}^{m} k \|\nabla e_u^{n,h}\|_{L^2} \left( \max_{1 \leq n \leq m} \|\nabla u^n_h\|_{L^2} \right) \right]
+ C \mathbb{E} \left[ \|\sum_{n=1}^{m} \eta(u^{n-1}) - \eta(u^{n-1}) \Delta W_n\|_{L^2} \right].
\end{equation}
By a standard calculation, it follows that

\[(4.33)\]

\[E[1_{\Omega_h}(\|k \sum_{n=1}^{m} e_{n\cdot} |_{L^2_x})] \leq CE[1_{\Omega_h}(\|e_{n\cdot} |_{L^2_x})] + CE[1_{\Omega_h}(\| e_{n\cdot} |_{L^2_x})] + CE[1_{\Omega_h}(\sum_{n=1}^{m} k\| e_{n\cdot} |_{L^2_x})] \]

\[+ C(\max_{1 \leq n \leq m} \| \nabla u |_{L^2_x})]^{\frac{1}{2}} (\max_{1 \leq n \leq m} \| \nabla u |_{L^2_x})]^{\frac{1}{2}} \]

\[+ C(\sum_{n=1}^{m} [\eta(u^{n-1}) - \eta(u_{n\cdot}^{n-1})] \Delta W_n |_{L^2_x}). \]

With using Lemma 3.1 and (4.9), the last term in (4.33) can be bounded by (4.24) which gives the desired result (4.28). The proof is complete.

**Theorem 4.5.** Let the assumptions of Theorem 3.3 be satisfied. Let \{p_n; 1 \leq n \leq N\} be the pressure approximation defined by Algorithm 2. Then the following error estimate holds for \(m = 1, 2, \ldots, N\)

\[(4.34)\]

\[E[1_{\Omega_h}(\|k \sum_{n=1}^{N} (p_n - p_{n\cdot}) |_{L^2_x})] \leq C(h^{2-2\epsilon} + k^{1-\epsilon}), \]

where \(C\) is a positive constant independent of \(h\) and \(k\).

For \(\epsilon > 0\), we introduce the following sample set

\[(4.35)\]

\[\Omega_{h,h} = \{ \omega \in \Omega | \max_{1 \leq n \leq N} (\| Au |_{L^2_x}^4 + \| \nabla u_{n\cdot}^4 |_{L^2_x}^4) \leq (h^2 + k)^{-\epsilon} \} \]

such that

\[(4.36)\]

\[P(\Omega_{h,h}) \geq 1 - \frac{E[\omega \in \Omega | \max_{1 \leq n \leq N} (\| Au |_{L^2_x}^4 + \| \nabla u_{n\cdot}^4 |_{L^2_x}^4)]}{(h^2 + k)^{-\epsilon}} \geq 1 - \frac{C}{(h^2 + k)^{-\epsilon}}. \]

The next Theorem states and proves strong optimal \(H^1\) convergence for the velocity approximation.

**Theorem 4.6.** Set \(u^0 = u_0\) and let \{u_n; 1 \leq n \leq N\} and \{u_{n\cdot}; 1 \leq n \leq N\} be the solutions of Algorithm 1 and Algorithm 2, respectively. Then, provided that \(0 < k < k_0\) and \(0 < h < h_0\) with \(k_0\) and \(h_0\) sufficiently small, the following error estimate holds:

\[(4.37)\]

\[E[1_{\Omega_{h,h}}(\max_{1 \leq n \leq N} \| \nabla (u - u_{n\cdot}) |_{L^2_x}^4 + k \sum_{n=1}^{N} \| A(u_n - u_{n\cdot}) |_{L^2_x}^2)] \leq C(h^{2-4\epsilon} + k^{1-2\epsilon}), \]

where \(C\) is a positive constant independent of \(h\) and \(k\).
Proof. Taking $v_h = A_h \Pi_h e_{u}^{n,h} \in V_{0h}$ and $q_h = 0$ in (4.38), we have

$$\frac{1}{2}(\|\nabla \Pi_h e_{u}^{n,h}\|_{L^2}^2 - \|\nabla \Pi_h e_{u}^{n-1,h}\|_{L^2}^2 + \|\nabla \Pi_h e_{u}^{n,h} - \nabla \Pi_h e_{u}^{n-1,h}\|_{L^2}^2)$$

$$+ \kappa \nu \|A_h \Pi_h e_{u}^{n,h}\|_{L^2}^2 = k \kappa (A_h (u^n - u^n_h), A_h \Pi_h e_{u}^{n,h}) + k b(u^n_h, u^n_h, A_h \Pi_h e_{u}^{n,h})$$

$$- k b(u^n, u^n, A_h \Pi_h e_{u}^{n,h}) + ([\eta(u^n-1) - \eta(u^n_h-1)] \Delta W_n, A_h \Pi_h e_{u}^{n,h})$$

$$= II_1 + II_2 + II_3.$$

For term $II_1$, thanks to the Young’s inequality and [4.4], we obtain

$$II_1 \leq \frac{\nu k}{8} \|A_h \Pi_h e_{u}^{n,h}\|_{L^2}^2 + C k h^2 \|\nabla^3 u^n\|_{L^2}^2.$$  

For nonlinear term $II_2$, we can decomposed as follows:

$$II_2 = -k b(u^n - u^n_h, u^n, A_h \Pi_h e_{u}^{n,h}) - k b(u^n_h, u^n - u^n_h, A_h \Pi_h e_{u}^{n,h})$$

$$= II_{2,1} + II_{2,2}.$$

Using the Poincaré inequality, the Young’s inequality, the embedding inequality and [4.4], one finds that

$$II_{2,1} \leq C k \|u^n - u^n_h\|_{L^2}^2 \|\nabla (u^n - u^n_h)\|_{L^2}^2 \|A u^n\|_{L^2} \|A_h \Pi_h e_{u}^{n,h}\|_{L^2}^2$$

$$\leq \frac{k \nu}{8} \|A_h \Pi_h e_{u}^{n,h}\|_{L^2}^2 + C k \|u^n - u^n_h\|_{L^2}^2 \|\nabla (u^n - u^n_h)\|_{L^2}^2 \|A u^n\|_{L^2}^4,$$

$$II_{2,2} \leq k \|\nabla u^n\|_{L^2} \|\nabla^2 (u^n - u^n_h)\|_{L^2} \|A_h (u^n - u^n_h)\|_{L^2}^2 \|A_h \Pi_h e_{u}^{n,h}\|_{L^2}^2$$

$$\leq \frac{k \nu}{8} \|A_h \Pi_h e_{u}^{n,h}\|_{L^2}^2 + C k \|\nabla (u^n - u^n_h)\|_{L^2}^2 \|\nabla u^n\|_{L^2}^4 \|\nabla^2 u^n\|_{L^2}^2 + C k h^2 \|\nabla^3 u^n\|_{L^2}^2.$$

Inserting estimates (4.39)–(4.41) into (4.38), we have

$$\frac{1}{2}(\|\nabla \Pi_h e_{u}^{n,h}\|_{L^2}^2 - \|\nabla \Pi_h e_{u}^{n-1,h}\|_{L^2}^2 + \|\nabla \Pi_h e_{u}^{n,h} - \nabla \Pi_h e_{u}^{n-1,h}\|_{L^2}^2)$$

$$+ \kappa \nu \|A_h \Pi_h e_{u}^{n,h}\|_{L^2}^2 \leq C k h^2 \|\nabla^3 u^n\|_{L^2}^2 + C k \|u^n - u^n_h\|_{L^2}^2$$

$$+ C k \|\nabla (u^n - u^n_h)\|_{L^2}^2 \|A u^n\|_{L^2}^4 + C k \|\nabla (u^n - u^n_h)\|_{L^2}^2 \|\nabla u^n\|_{L^2}^4$$

$$+ ([\eta(u^n-1) - \eta(u^n_h-1)] \Delta W_n, A_h \Pi_h e_{u}^{n,h}).$$
For term II₃, using the Itô’s isometry and the Young’s inequality, we have
\begin{equation}
\mathbb{E}\left[\sum_{n=1}^{N} \left(\nabla u^{n} - \nabla u^{n-1}\right) \cdot \nabla \left(\Pi_{h} e_u^{n,h} - \Pi_{h} e_u^{n-1,h}\right)\right]
\end{equation}

Taking the expectation and applying the summation operator \(\sum_{n=1}^{N}\), one finds that (4.43)

\begin{align*}
\mathbb{E}&\left[\sum_{n=1}^{N} \left(\nabla u^{n} - \nabla u^{n-1}\right) \cdot \nabla \left(\Pi_{h} e_u^{n,h} - \Pi_{h} e_u^{n-1,h}\right)\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{N} \left(\nabla u^{n} - \nabla u^{n-1}\right) \cdot \nabla \left(\Pi_{h} e_u^{n,h} - \Pi_{h} e_u^{n-1,h}\right)\right]
\end{align*}

Now we explain how to estimate in expectation for \(A_5\) and \(A_6\). Using the Lemmas 3.1 and Lemma 4.2, the terms \(A_5\) and \(A_6\) are uniformly bounded

\begin{align*}
A_5 &\leq \left(\mathbb{E} \max_{1 \leq m \leq N} \|\nabla u^m\|^4_{L^2}\right)^{\frac{1}{4}} \left(\mathbb{E} \sum_{n=1}^{N} \|A(u^n - u^{n-1})\|^8_{L^2}\right)^{\frac{1}{8}} \\
&\quad + \left(\mathbb{E} \max_{1 \leq m \leq N} \|\nabla u^m_{h}\|^4_{L^2}\right)^{\frac{1}{4}} \left(\mathbb{E} \sum_{n=1}^{N} \|A(u^n - u^{n-1})\|^8_{L^2}\right)^{\frac{1}{8}},
\end{align*}

and

\begin{align*}
A_6 &\leq \left(\mathbb{E} \max_{1 \leq m \leq N} \|\nabla u^m\|^4_{L^2}\right)^{\frac{1}{4}} \left(\mathbb{E} \sum_{n=1}^{N} \|\nabla (u^n - u^{n-1})\|^8_{L^2}\right)^{\frac{1}{8}} \\
&\quad + \left(\mathbb{E} \max_{1 \leq m \leq N} \|\nabla u^m_{h}\|^4_{L^2}\right)^{\frac{1}{4}} \left(\mathbb{E} \sum_{n=1}^{N} \|\nabla (u^n - u^{n-1})\|^8_{L^2}\right)^{\frac{1}{8}}.
\end{align*}

For term \(I I_3\), using the Itô’s isometry and the Young’s inequality, we have
\begin{equation}
\mathbb{E}\left[\sum_{n=1}^{N} \left(\nabla u^{n} - \nabla u^{n-1}\right) \cdot \nabla \left(\Pi_{h} e_u^{n,h} - \Pi_{h} e_u^{n-1,h}\right)\right]
\end{equation}
By the definition of $\eta$ and using (4.44), (2.24), (2.26), (3.25) and (4.47), it follows that

\begin{align*}
(4.45) & \quad \|\nabla \eta(u_n-1) - \eta(u_n-1)^{-1}\|_2^2 \\
& \leq C\|\nabla e_n^u-1\|_2^2 + C\|u_n-1\|_{H^2}^2 \\
& \leq C\|\nabla G(u_n-1)\|_2^2 + C\|\nabla e_n^u-1\|_2^2 \\
& \leq C\|\nabla^2 u_n-1\|_2^2 + C\|\nabla^3 u_n-1\|_2^2 + C\|\nabla \Pi u_n-1,h\|_2^2.
\end{align*}

Combining (4.44) into (4.43), we get

\begin{align*}
(4.46) & \quad E[\Omega_h \cap \Omega_h \left(\frac{1}{2} \|\nabla \Pi_u e_n^u\|_2^2 + \frac{1}{4} \sum_{n=1}^N \|\nabla (\Pi u_n^u - \Pi u_n-1,h)^2\|_2^2 \\
& + \frac{\kappa}{2} \sum_{n=1}^N \|A_h \Pi u_n^u\|_2^2\right) \leq C(h^{2-2\epsilon} + k^{1-2\epsilon}) + CE[\Omega_h \cap \Omega_h \left(\sum_{n=1}^N \kappa \|\nabla \Pi u_n-1,h\|_2^2\right)]]
\end{align*}

Then the (4.37) follows from an application of the discrete Gronwall inequality and the triangle inequality.

For $\epsilon > 0$, we introduce the following sample set

\begin{align*}
(4.47) & \quad \Omega_{k,h} = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \left(\|\nabla^2 u_n\|_2^2 + \|\nabla u_n^u\|_2^2\right) \leq -\epsilon \log(h^4 + k) \right\}
\end{align*}

such that

\begin{align*}
(4.48) & \quad P(\Omega_{k,h}) \geq 1 - \frac{E[\omega \in \Omega \max_{1 \leq n \leq N} \left(\|\nabla^2 u_n\|_2^2 + \|\nabla u_n^u\|_2^2\right) \leq C\log(h^4 + k)}{\epsilon \log(h^4 + k)} \geq 1 + \frac{C}{\epsilon \log(h^4 + k)}.
\end{align*}

For $\kappa_0 > 0$, the following sample set is defined as

\begin{align*}
(4.49) & \quad \Omega_{\kappa_0} = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \|u_n - u_n^h\|_2^2 \leq \kappa_0(h^{2-2\epsilon} + k^{1-2\epsilon}) \right\}.
\end{align*}

The following Theorems give and derive strong optimal $L^2$ convergence of the scheme for the velocity approximation by using the negative norm technique.

**Theorem 4.7.** Set $u^0 = u_0$ and let $\{u_n; 1 \leq n \leq N\}$ and $\{u_n^h; 1 \leq n \leq N\}$ be the solutions of **Algorithm 1** and **Algorithm 2**, respectively, and provided that $0 < k < \kappa_0$ and $0 < h < h_0$ with $\kappa_0$ and $h_0$ sufficiently small, the following error estimate holds:

\begin{align*}
(4.50) & \quad \mathbb{E}\left[I_{\Omega_{k,h} \cap \Omega_{\kappa_0}}(\max_{1 \leq n \leq N} \|u_n - u_n^h\|_2^2 + k \sum_{n=1}^N \|u_n - u_n^h\|_2^2) \right] \leq C(\kappa_0)(h^{4-7\epsilon} + k^{1-3\epsilon}),
\end{align*}

where $C(\kappa_0)$ is a positive constant independent of $h$ and $k$.

**Proof.** Setting $u_h = A_h^{-1}\Pi u_n^u \in V_h$ and $q_h = 0$ in (4.43), we gain

\begin{align*}
(4.51) & \quad \frac{1}{2} \|A_h \nabla \Pi u_n^u\|_2^2 - \|A_h \nabla \Pi u_n-1,h\|_2^2 + \|A_h \nabla \Pi u_n^u - A_h \nabla \Pi u_n-1,h\|_2^2 \\
& + k \mu \|\Pi u_n^u\|_2^2 = a(u_n, u_n^h, A_h^{-1}\Pi u_n^u) + k b(u_n^h, u_n, A_h^{-1}\Pi u_n^u) \\
& - k b(u_n-1, u_n^h, A_h^{-1}\Pi u_n^u) + ([\eta(u_n-1) - \eta(u_n^h)] \Delta W_n, A_h^{-1}\Pi u_n^u) \\
& = III_1 + III_2 + III_3.
\end{align*}
For term $III_1$, thanks to the Young’s inequality, (4.33) and (4.34), we obtain

$$III_1 \leq \frac{\nu k}{8} \| \Pi_h e_u^{n,h} \|^2_{L^2} + C k h^4 \| \nabla^2 u^n \|^2_{L^2}.$$  

For nonlinear term $III_2$, we can decomposed as follows:

$$III_2 = -k b(u^n - u_h^n, u^n, A_h^{-1} \Pi_h e_u^{n,h}) - k b(u_h^n, u^n - u_h^n, A_h^{-1} \Pi_h e_u^{n,h})$$

$$= III_{2,1} + III_{2,2}.$$  

Using the Poincaré inequality, the Young’s inequality and the embedding inequality, one finds that

$$III_{2,1} \leq \frac{k \nu}{8} \| \Pi_h e_u^{n,h} \|^2_{L^2} + C k \| \nabla u^n \|^2_{L^2} \| u^n - \Pi_h u^n \|^2_{L^2}$$

$$+ C k \| \nabla^2 u_h^n \|^2_{L^2} \| \Pi_h e_u^{n,h} \|^2_{L^2} + C k \| \Pi_h e_u^{n,h} \|^2_{L^2} \| \nabla(u^n - u_h^n) \|^2_{L^2},$$

$$III_{2,2} \leq \frac{k}{4} \| \Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h} \|^2_{L^2} + \frac{k \nu}{8} \| \Pi_h e_u^{n,h} \|^2_{L^2}$$

$$+ C k \| u^n - \Pi_h u^n \|^2_{L^2} \| \Pi_h e_u^{n-1,h} \|^2_{L^2} + C k \| \nabla^2 u_h^n \|^2_{L^2} \| \Pi_h e_u^{n-1,h} \|^2_{L^2}$$

$$+ C k^2 \| \nabla u_h^n \|^2_{L^2} \| \nabla(u^n - u_h^n) \|^2_{L^2} + (\| \Pi_h e_u^{n,h} \|^2_{L^2} \| \nabla(u^n - u_h^n) \|^2_{L^2}).$$

Inserting estimates (4.52)–(4.54) into (4.51), we have

$$\frac{1}{2} \left( \| \Pi_h e_u^{n,h} \|^2_{L^2} - \| \Pi_h e_u^{n-1,h} \|^2_{L^2} + \| \Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h} \|^2_{L^2} \right)$$

$$+ \frac{k \nu}{2} \| \Pi_h e_u^{n,h} \|^2_{L^2} \leq C k h^4 \| \nabla^2 u^n \|^2_{L^2} + C k \| \nabla^2 u_h^n \|^2_{L^2} \| \Pi_h e_u^{n,h} \|^2_{L^2}$$

$$+ C k \| \Pi_h e_u^{n,h} \|^2_{L^2} \| \nabla(u^n - u_h^n) \|^2_{L^2} + C k^2 \| \nabla u_h^n \|^2_{L^2} \| \nabla(u^n - u_h^n) \|^2_{L^2}$$

$$+ (\| \Pi_h e_u^{n,h} \|^2_{L^2} \| \nabla(u^n - u_h^n) \|^2_{L^2}).$$

Taking the expectation and applying the summation operator $\sum_{n=1}^N$, one finds that

$$\mathbb{E} \left[ 1_{\Omega_{k,h}^{n} \cap \Omega_{k,h}^{n-1} \cap \Omega_n \cap \Omega_0} \left( \frac{1}{2} \| \Pi_h e_u^{n,h} \|^2_{L^2} + \frac{1}{2} \sum_{n=1}^N \| \Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h} \|^2_{L^2} + \sum_{n=1}^N \frac{k \nu}{4} \| \Pi_h e_u^{n,h} \|^2_{L^2} \right) \right]$$

$$\leq \frac{1}{2} \mathbb{E} \left[ \| \Pi_h e_u^{0,h} \|^2_{L^2} \right] + C h^4 \mathbb{E} \left[ k \sum_{n=1}^N \| \nabla u^n \|^2_{L^2} \right]$$

$$+ C \log(h^4 + k)^{-1} \mathbb{E} \left[ 1_{\Omega_{k,h}^{n} \cap \Omega_{k,h}^{n-1} \cap \Omega_n \cap \Omega_0} \left( k \sum_{n=1}^N \| \Pi_h e_u^{n,h} \|^2_{L^2} \right) \right]$$

$$+ C k (h^{2-4e} + k^{1-2e}) + C \mathbb{E} \left[ \sum_{n=1}^N \| \nabla^2(u_h^n - u_h^{n-1}) \|^2_{L^2} \| u^n - u_h^n \|^2_{L^2} \right]$$

$$+ C(k_0)(h^{2-4e} + k^{1-2e}) (h^{2-2e} + k^{1-e}) + C k^2 \mathbb{E} \left[ \sum_{n=1}^N \| \nabla(u_h^n - u_h^{n-1}) \|^2_{L^2} \| \nabla(u^n - u_h^n) \|^2_{L^2} \right]$$

$$+ \mathbb{E} \left[ 1_{\Omega_{k,h}^{n} \cap \Omega_{k,h}^{n-1} \cap \Omega_n \cap \Omega_0} \left( \sum_{n=1}^N (A_h^{-\frac{1}{2}} (\eta(u^{n-1}) - \eta(u_h^{n-1})) \Delta W_n, A_h^{-\frac{1}{2}} (\Pi_h e_u^{n,h})) \right) \right].$$
Now we explain how to estimate in expectation for $\Lambda_7$ and $\Lambda_8$. Making use of the Lemma 4.2 and Lemma 4.1, the terms $\Lambda_7$ and $\Lambda_8$ are uniformly bounded

\[
\Lambda_7 \leq \left( E \max_{1 \leq m \leq N} \|u^m\|_{L^2}^{\frac{1}{2}} \right)^2 \left( E \sum_{n=1}^{N} \|\nabla^2 (u_n^m - u_{n-1}^m)\|_{L^2}^{\frac{1}{2}} \right)^\frac{1}{2} \\
+ \left( E \max_{1 \leq m \leq N} \|u^m\|_{L^2}^{\frac{1}{2}} \right)^2 \left( E \sum_{n=1}^{N} \|\nabla^2 (u_n^m - u_{n-1}^m)\|_{L^2}^{\frac{1}{2}} \right)^\frac{1}{2},
\]

\[
\Lambda_8 \leq \left( E \max_{1 \leq m \leq N} \|\nabla u^m\|_{L^2}^{\frac{1}{2}} \right)^2 \left( E \sum_{n=1}^{N} \|\nabla (u^n - u^{n-1})\|_{L^2}^{\frac{1}{2}} \right)^\frac{1}{2} \\
+ \left( E \max_{1 \leq m \leq N} \|\nabla u^m\|_{L^2}^{\frac{1}{2}} \right)^2 \left( E \sum_{n=1}^{N} \|\nabla (u^n - u^{n-1})\|_{L^2}^{\frac{1}{2}} \right)^\frac{1}{2}.
\]

For term $III_3$, using the Itô’s isometry and the Young’s inequality, we have

\[
E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( \sum_{n=1}^{N} (A_{h,k}^{\frac{1}{2}} [\eta(u^{n-1}) - \eta(u_{h,n-1}^k)] \Delta W_n, \Pi_i A_{h,k}^{\frac{1}{2}} e_u^{n,h}) \right) \right]
\]

\[
= \left[ E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( \sum_{n=1}^{N} (A_{h,k}^{\frac{1}{2}} [\eta(u^{n-1}) - \eta(u_{h,n-1}^k)] \Delta W_n, \Pi_i A_{h,k}^{\frac{1}{2}} e_u^{n,h} - \Pi_i A_{h,k}^{\frac{1}{2}} e_u^{n-1,h}) \right) \right) \right]
\]

\[
\leq \left[ E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( k \sum_{n=1}^{N} \|A_{h,k}^{\frac{1}{2}} [\eta(u^{n-1}) - \eta(u_{h,n-1}^k)] \Delta W_n \|_{L^2} \|\Pi_i A_{h,k}^{\frac{1}{2}} e_u^{n,h} - \Pi_i A_{h,k}^{\frac{1}{2}} e_u^{n-1,h}\|_{L^2} \right) \right) \right]
\]

\[
\leq \frac{1}{4} E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( \|\Pi_i e_u^{n,h} - \Pi_i e_{u}^{n-1,h}\|_{L^2} \right) \right]
\]

\[
+ E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( k \sum_{n=1}^{N} \|\eta(u^{n-1}) - \eta(u_{h,n-1}^k)\|_{L^2} \right) \right] .
\]

By the definition of $\eta$ and using (2.1), (2.2), (2.5), (3.25) and (4.4), one finds that

\[
\|\eta(u^{n-1}) - \eta(u_{h,n-1}^k)\|_{L^2} \leq C \|\eta(u^{n-1}) - \eta(u_{h,n-1}^k)\|_{L^2}
\]

\[
\leq C \|e_{u}^{n-1}\|_{L^2} + C h^2 \|\eta(u^{n-1})\|_{L^2} + C \|e_{u}^{n-1}\|_{L^2} + C \|\Pi_i e_{u}^{n-1,h}\|_{L^2}.
\]

Combining (4.57) with (4.58), we get

\[
E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( \frac{1}{2} \|\Pi_i e_{u}^{n,h}\|_{L^2} + \frac{1}{2} \sum_{n=1}^{N} \|\Pi_i e_{u}^{n,h} - \Pi_i e_{u}^{n-1,h}\|_{L^2} + \sum_{n=1}^{N} \frac{k}{4} \|\Pi_i e_{u}^{n,h}\|_{L^2} \right) \right]
\]

\[
\leq C(\kappa_0)(h^2 + k^2) + C \log(h^2 + k) + C \left[ E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( \sum_{n=1}^{N} \|\Pi_i e_{u}^{n,h}\|_{L^2} \right) \right] \right]
\]

\[
+ E \left[ 1_{\Omega_{h,k} \cap \Omega_{h,k} \cap \Omega_{h,k}} \left( k \sum_{n=1}^{N} \|\Pi_i e_{u}^{n-1,h}\|_{L^2} \right) \right] .
\]
Using the similar line in the proof of Theorem 4.3 with applying the discrete Gronwall inequality and the triangle inequality, the result (4.50) holds. The proof is complete.

For \( \kappa > 0 \), we introduce the following sample set

\[
\Omega_\kappa = \left\{ \omega \in \Omega \left| \max_{1 \leq n \leq N} \| \nabla (u^n - u^n_h) \|_{L^2}^2 \leq \kappa (h^{2-4\epsilon} + k^{1-2\epsilon}) \right. \right\}.
\]

**Theorem 4.8.** Set \( u^0 = u_0 \) and let \( \{ u^n; 1 \leq n \leq N \} \) and \( \{ u^n_k; 1 \leq n \leq N \} \) be the solutions of Algorithm 1 and Algorithm 2, respectively. Then, provided that \( 0 < k < k_0 \) and \( 0 < h < h_0 \) with \( k_0 \) and \( h_0 \) sufficiently small, the following error estimate holds:

\[
\mathbb{E} \left[ 1_{\Omega_{\kappa}} \cap \Omega_{\kappa} \cap \Omega_{\kappa} \cap \Omega_{\kappa} \cap \Omega_{\kappa} \right] \left( \| u^n - u^n_h \|_{L^2}^2 \right) \leq C(\kappa_0, \kappa)(h^{4-8\epsilon} + k^{1-4\epsilon}),
\]

where \( C(\kappa_0, \kappa) \) is a positive constant independent of \( h \) and \( k \).

**Proof.** Taking \( v_h = \Pi_h e^{n,h}_u \in v_{0h} \) and \( q_h = 0 \), we have

\[
\mathbb{E} \left[ 1_{\Omega_{\kappa}} \cap \Omega_{\kappa} \right] \leq C(\kappa_0, \kappa)(h^{4-8\epsilon} + k^{1-4\epsilon}),
\]

where \( C(\kappa_0, \kappa) \) is a positive constant independent of \( h \) and \( k \).

For term \( IV_1 \), thanks to the Young’s inequality, (4.3) and (4.4), we obtain

\[
IV_1 \leq \frac{\nu k}{4} \| \nabla u^n \|_{L^2}^2 + C k h^4 \| \nabla^3 u^n \|_{L^2}^2.
\]

For nonlinear term \( IV_2 \), using the Poincaré inequality, the Young’s inequality and the embedding inequality, one finds that

\[
IV_2 = -k b(u^n - u^n_h, u^n_h, \Pi_h e^{n,h}_u) - k b(u^n_h, u^n - u^n_h, \Pi_h e^{n,h}_u) \\
\leq \frac{k \nu}{4} \| \nabla u^n \|_{L^2}^2 + C k \| \nabla^2 u^n \|_{L^2}^2 \| u^n - u^n_h \|_{L^2}^2 \\
+ C k \| \nabla (u^n - u^n_h) \|_{L^2}^4.
\]

Inserting estimates (4.63) into (4.62), we have

\[
IV_1 \leq \frac{\nu k}{4} \| \nabla u^n \|_{L^2}^2 + C k h^4 \| \nabla^3 u^n \|_{L^2}^2.
\]

For nonlinear term \( IV_2 \), using the Poincaré inequality, the Young’s inequality and the embedding inequality, one finds that

\[
IV_2 = -k b(u^n - u^n_h, u^n_h, \Pi_h e^{n,h}_u) - k b(u^n_h, u^n - u^n_h, \Pi_h e^{n,h}_u) \\
\leq \frac{k \nu}{4} \| \nabla u^n \|_{L^2}^2 + C k \| \nabla^2 u^n \|_{L^2}^2 \| u^n - u^n_h \|_{L^2}^2 \\
+ C k \| \nabla (u^n - u^n_h) \|_{L^2}^4.
\]

Inserting estimates (4.63) and (4.64) into (4.62), we have

\[
IV_1 \leq \frac{\nu k}{4} \| \nabla u^n \|_{L^2}^2 + C k h^4 \| \nabla^3 u^n \|_{L^2}^2.
\]

For nonlinear term \( IV_2 \), using the Poincaré inequality, the Young’s inequality and the embedding inequality, one finds that

\[
IV_2 = -k b(u^n - u^n_h, u^n_h, \Pi_h e^{n,h}_u) - k b(u^n_h, u^n - u^n_h, \Pi_h e^{n,h}_u) \\
\leq \frac{k \nu}{4} \| \nabla u^n \|_{L^2}^2 + C k \| \nabla^2 u^n \|_{L^2}^2 \| u^n - u^n_h \|_{L^2}^2 \\
+ C k \| \nabla (u^n - u^n_h) \|_{L^2}^4.
\]
Taking the expectation and applying the summation operator $\sum_{n=1}^{N}$, it follows that

\begin{equation}
\mathbb{E} \left[ 1_{\eta_{h}, \eta_{h}^{k} \cap \Omega} \left( \frac{1}{2} \left\| \Pi_{h} e_{u}^{n,h} \right\|_{L_{2}}^{2} + \frac{1}{2} \sum_{n=1}^{N} \left\| \Pi_{h} e_{u}^{n,h} - \Pi_{h} e_{u}^{n-1,h} \right\|_{L_{2}}^{2} + \sum_{n=1}^{N} \frac{k_{\nu}}{2} \left\| \nabla \Pi_{h} e_{u}^{n,h} \right\|_{L_{2}}^{2} \right) \right] 
\leq \frac{1}{2} \mathbb{E} \left[ \left\| \Pi_{h} e_{u}^{0,h} \right\|_{L_{2}}^{2} \right] + C h \mathbb{E} \left[ \frac{N}{2} \sum_{n=1}^{N} \left\| \nabla^{2} u^{n} \right\|_{L_{2}}^{2} \right] 
+ C \left( \eta_{0}, \eta \right) (h^{2-4} + k^{1-2}) \mathbb{E} \left[ 1_{\Omega} \left( \frac{1}{2} \sum_{n=1}^{N} \left\| \nabla (u^{n} - u_{h}^{n}) \right\|_{L_{2}}^{2} \right) \right].
\end{equation}

Using the Lemma 3.2 and Lemma 4.1, the term $\Lambda_{9}$ is uniformly bounded. Then we obtain

\begin{equation}
\mathbb{E} \left[ 1_{\eta_{h}, \eta_{h}^{k} \cap \Omega} \left( \frac{1}{2} \left\| \Pi_{h} e_{u}^{n,h} \right\|_{L_{2}}^{2} + \frac{1}{2} \sum_{n=1}^{N} \left\| \Pi_{h} e_{u}^{n,h} - \Pi_{h} e_{u}^{n-1,h} \right\|_{L_{2}}^{2} + \sum_{n=1}^{N} \frac{k_{\nu}}{2} \left\| \nabla e_{u}^{n,h} \right\|_{L_{2}}^{2} \right) \right] 
\leq C \left( \eta_{0}, \eta \right) (h^{1-8} + k^{1-4}) + C \mathbb{E} \left[ 1_{\Omega} \left( \frac{1}{2} \sum_{n=1}^{N} \left\| \nabla e_{u}^{n,h} \right\|_{L_{2}}^{2} \right) \right].
\end{equation}

By applying the discrete Gronwall inequality and the triangle inequality, the result (4.61) holds. The proof is complete. □

Theorems 3.3, 3.4, 3.6, 4.1, 4.3 and Theorem 4.8 and the triangle inequality infer the global error estimates, which are the main results of this paper.

**Theorem 4.9.** Under the assumptions of Theorems 3.3, 3.4, 3.6, 4.1, 4.3 and 4.8.
and Theorem 4.3, there hold the following error estimates:

\begin{equation}
\mathbb{E}\left[ \int_{\Omega}^{\omega} \left( \left\| \nabla (u(t_n) - u_h^n) \right\|^2 \right) \right] \leq C \left( k^{2-\epsilon} + h^{2-4\epsilon} \right),
\end{equation}

\begin{equation}
\mathbb{E}\left[ \int_{\Omega}^{\omega} \left( \left\| \int_0^t r(s) \, ds - k \sum_{n=1}^m r_h^n \right\|^2 \right) + \left\| \int_0^t p(s) \, ds - k \sum_{n=1}^m p_h^n \right\|^2 \right] \leq C \left( k^{2-\epsilon} + h^{2-2\epsilon} \right),
\end{equation}

\begin{equation}
\mathbb{E}\left[ \int_{\Omega}^{\omega} \left( \left\| u(t_n) - u_h^n \right\|^2 \right) \right] \leq C (\kappa_0, \kappa) \left( k^{2-4\epsilon} + h^{4-8\epsilon} \right),
\end{equation}

where \( C, C(\kappa_0, \kappa) \) are two positive constants independent of \( h \) and \( k \).

Remark 4. The crucial point which makes the error analysis interesting and distinct from the deterministic case is the low regularity in time. As far as the spatial regularity is concerned, we can obtain similar optimal error estimates to the deterministic case. From the numerical results of Section 5, the estimates \( 4.68 - 4.70 \) are optimal order.

5. Numerical results. In this section, we give some 2D numerical results to confirm the theoretical error estimates of our Algorithm 2. We set \( D = (0, L)^2 \) with \( L = 1 \), a deterministic constant force term, the initial condition \( u_0 = (0, 0) \) and \( G(u(t)) = ((u_1(t)^2 + 1)^{\frac{1}{2}}, (u_2(t)^2 + 1)^{\frac{1}{2}}) \). The \( W \) in \( 1.1 \) is chosen as a finite-dimensional \( \mathbb{R}^M \)-Wiener process such that

\[ W(t_n) - W(t_{n-1}) = \sum_{j=1}^M \sum_{k=1}^M \lambda_{j,k} g_{j,k} \xi_j \xi_k, \]

where \( \lambda_{j,k} = \frac{1}{N \epsilon_{j,k}} \), \( \xi_{j,k} \sim N(0, 1) \) and \( g_{j,k}(x, y) = 5 \sin(j \pi x) \sin(k \pi y) \).

We take the following parameters: \( M = 10, \nu = 1 \) and \( T = 1 \). The Monte Carlo method with \( N_p = 1200 \) realizations is utilized to compute the expectation. Since the exact solution of the problem \( 1.1 \) is unknown, we denote the time/spatial errors of the numerical solutions by

\[ E A u_k^N := \left( \mathbb{E} \left[ \left\| u_h^{N,k} - u_{h,k/2}^N \right\|^2 \right] \right)^{\frac{1}{2}}, \quad E A u_N^k := \left( \mathbb{E} \left[ \left\| u_h^{N,k} - u_{h,k/2}^N \right\|^2 \right] \right)^{\frac{1}{2}}, \]

\[ E B u_k^N := \left( \mathbb{E} \left[ \left\| \nabla (u_h^{N,k} - u_{h,k/2}^N) \right\|^2 \right] \right)^{\frac{1}{2}}, \quad E B u_N^k := \left( \mathbb{E} \left[ \left\| \nabla (u_h^{N,k} - u_{h,k/2}^N) \right\|^2 \right] \right)^{\frac{1}{2}}, \]

\[ E p_k^N := \left( \mathbb{E} \left[ \left\| \frac{1}{2} \sum_{n=1}^N p_h^n - \frac{1}{2} \sum_{n=1}^N p_h^n \right\|^2 \right] \right)^{\frac{1}{2}}, \quad E p_N^k := \left( \mathbb{E} \left[ \left\| \frac{1}{2} \sum_{n=1}^N p_h^n - \frac{1}{2} \sum_{n=1}^N p_h^n \right\|^2 \right] \right)^{\frac{1}{2}}, \]

where \( u_h^{N,k} \) is the one path simulation at \( t^N = T \) computed by using space mesh size \( h \) and time mesh size \( k \).

Figure 1 shows the errors of the time discretizations of the velocity and the pressure using different time mesh sizes. It is evident that the numerical results validate the half order for the time discretization as theoretical error estimates. Figure 2 displays the errors of the spatial discretizations of the velocity and the pressure using different space mesh sizes. It is easy to see that the numerical results check the first/second order for the spatial discretization as proved in Theorem 4.9.
Fig. 5.1. Time errors of the numerical results at $T = 1$ with $h = 2^{-7}$.

Fig. 5.2. Spatial errors of the numerical results at $T = 1$ with $k = 2^{-9}$.

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