Matroids and Codes with the Rank Metric

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Abstract
We study the relationship between a $q$-analogue of matroids and linear codes with the rank metric in the vector space of matrices with entries in a finite field. We prove a Greene type identity for the rank generating function of these matroidal structures and the rank weight enumerator of these linear codes. As an application, we give a combinatorial proof of a MacWilliams type identity for Delsarte rank-metric codes.

Keywords: rank-metric code; matroid; polymatroid; Greene’s identity; MacWilliams identity

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1 Introduction
Let $S$ be a finite set and let $\rho : 2^S \to \mathbb{Z}$ be a function. An ordered pair $M = (S, \rho)$ is called a matroid if

(1) If $X \subseteq S$, then $0 \leq \rho(X) \leq |X|$. 
(2) If $X, Y \subseteq S$ and $X \subseteq Y$, then $\rho(X) \leq \rho(Y)$. 
(3) If $X, Y \subseteq S$, then $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.

Matroids are constructed from a lot of combinatorial or algebraic structures including graphs and matrices (cf. [29, 20]). Let $\mathbb{F}_q$ be a finite field of $q$ elements. For an $[n,k]$ code $C$ over $\mathbb{F}_q$ and any subset $X \subseteq S := \{1,2,\ldots,n\}$, define

$$\rho(X) := \dim C \setminus (S-X) (= \dim C - \dim C(S-X)),$$

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where, for any $Y \subseteq S$, $C \setminus Y$ denotes the punctured code by $Y$ and

$$C(Y) := \{ x \in C : \text{supp}(x) \subseteq Y \}.$$ 

Then we find that $M_C := (S, \rho)$ satisfies the above conditions and so $M_C$ is a matroid.

The (Whitney) rank generating function of a matroid $M = (S, \rho)$ is defined by

$$R(M; x, y) := \sum_{X \subseteq S} x^{\rho(S) - \rho(X)} y^{|X| - \rho(X)}.$$ 

In 1976, Greene ([14]) proved one celebrated result, known as Greene’s identity, on the relationship between the Hamming weight enumerator $W_C(x, y)$ of an $[n, k]$ code $C$ over $\mathbb{F}_q$ and the rank generating function of the corresponding matroid $M_C$ as follows:

$$W_C(x, y) = y^{n - \dim C} (x - y)^{\dim C} R(M_C; \frac{qy}{x - y}, \frac{x - y}{y}).$$

As an application of this result, he gave a elegant combinatorial proof of the MacWilliams identity for the Hamming weight enumerator (cf. [29]). After this paper, a number of paper in the link on matroids and codes have been published, for instance, generalizations of the identity to higher weight enumerators ([2, 3, 23]), and to some enumerators for other classes of codes ([5, 6, 26, 28]). In particular, Britz ([3]) proved the equivalence between the set of higher weight enumerators of a linear code over a finite filed and the rank generating function associated to the code. In [4] and [16], the authors studied the critical problem, known as a classical problem in matroid theory, from coding theoretical approach.

Delsarte introduced a rank-metric code as a set of matrices of given size over a finite field in [8]. He mainly studied the rank-metric code based on his research on association schemes. In [9, 10], Gabidulin gave a different definition of rank-metric codes such as a set of vectors in a vector space over an extension field. The rank-weight of a codeword is defined by the rank of the associated matrix. Ravagnani proved that Gabidulin codes can be regarded as a special case of Delsarte codes and he compared the duality theories of these codes in [22]. Some useful applications of rank-metric codes are widely known, for instance, to space-time codes ([27]), to network coding ([24]), and to cryptography ([11]).

In addition, some of the results in classical coding theory are generalized to rank-metric codes. For instance, a Singleton bound on the minimum rank distance of the codes are proven in [8]. Some constructions of the codes which attain the bound, called maximum rank distance (MRD) codes, are known ([7, 8, 9, 11, 17]). The MacWilliams type identity for rank weight enumerators of rank-metric codes are proposed as several expressions: an algebraic expression ([8]), identity for rank weight enumerators of Gabidulin rank-metric codes closed to the original form ([12, 13]), and moments of rank distributions of Delsarte rank-metric codes ([22]).

The layout of this paper is as follows. In Section 2, we firstly introduce some basic notion on Delsarte rank-metric codes and new matroidal structures related rank-metric codes. And we give a Greene type identity for rank-metric codes. Section 3 contains an application of the Greene type identity to the MacWilliams identity for Delsarte rank-metric codes.

Most of the basic terminology in coding theory and matroid theory used in this paper will be found in standard texts (cf. [15, 19, 20, 29]).
2 \((q, r)\)-Polymatroids and Rank Generating Functions

We denote by \(\text{Mat}(n \times m, \mathbb{F}_q)\) the \(\mathbb{F}_q\)-vector space of \(n \times m\) matrices with entries in \(\mathbb{F}_q\). A \textit{Delsarte rank-metric code} \(C\) of size \(n \times m\) over \(\mathbb{F}_q\) is an \(\mathbb{F}_q\)-linear subspace of \(\text{Mat}(n \times m, \mathbb{F}_q)\). Throughout this paper, set \(E := \mathbb{F}_q^n\). For convenience, we shall write for \(D \leq E\) if \(D\) is a subspace of \(E\). For any subspace \(J \leq E\), define

\[
C(J) := \{ M \in C | \text{col}(M) \subseteq J \},
\]

\[
J^\perp := \{ y \in \mathbb{F}_q^n | x \cdot y = 0, \forall x \in J \},
\]

where \(\text{col}(M)\) denotes the column space of \(M\) over \(\mathbb{F}_q\).

**Lemma 1** \(C(J)\) is an \(\mathbb{F}_q\)-linear subspace of \(\text{Mat}(n \times m, \mathbb{F}_q)\).

**Proof.** For any \(M, N \in C(J)\), let \(u_1, \ldots, u_m\) and \(v_1, \ldots, v_m\) be the column vectors of \(M\) and \(N\), respectively. For any \(\alpha, \beta \in \mathbb{F}_q\), take a vector \(x \in \text{col}(\alpha M + \beta N)\). Then \(x\) can be written as

\[
x = a_1(\alpha u_1 + \beta v_1) + \cdots + a_m(\alpha u_m + \beta v_m)
\]

for some \(a_1, \ldots, a_m \in \mathbb{F}_q\). Since \(\alpha(a_1 u_1 + \cdots + a_m u_m) \in J\) and \(\beta(a_1 v_1 + \cdots + a_m v_m) \in J\), we have that \(x \in J\). 

We denote by \(\Sigma(E)\) the set of all subspaces of \(E\) and denote by \(\Sigma(J)\) the set of all subspaces of any subspace \(J \in \Sigma(E)\).

Now we shall introduce a \((q, r)\)-polymatroid as a \(q\)-analogue of \(k\)-polymatroids (cf. [21]).

**Definition 2** A \((q, r)\)-polymatroid is an ordered pair \(P = (E, \rho)\) consisting of a vector space \(E := \mathbb{F}_q^n\) and a function \(\rho : \Sigma(E) \to \mathbb{Z}^+ \cup \{0\}\) having the following properties:

(R1) If \(A \leq E\), then \(0 \leq \rho(A) \leq r \dim A\).

(R2) If \(A, B \leq E\) and \(A \subseteq B\), then \(\rho(A) \leq \rho(B)\).

(R3) If \(A, B \leq E\), then \(\rho(A + B) + \rho(A \cap B) \leq \rho(A) + \rho(B)\).

We find easily that \(\rho(\{0\}) = 0\) from (R1) and a \((q, 1)\)-polymatroid is a \(q\)-analogue of matroids.

**Proposition 3** Let \(C\) be a Delsarte rank-metric code in \(\text{Mat}(n \times m, \mathbb{F}_q)\), and let \(\rho : \Sigma(E) \to \mathbb{Z}^+ \cup \{0\}\) be the function defined by

\[
\rho(J) := \dim_{\mathbb{F}_q} C - \dim_{\mathbb{F}_q} C(J^\perp),
\]

for any subspace \(J\) of \(E\). Then \(P_C := (E, \rho)\) is a \((q, m)\)-polymatroid.
Proof. Clearly the function $\rho$ satisfies (R1) and (R2). Now we prove that (R3) holds. Let $J_1$ and $J_2$ are subspaces of $E$. From Lemma 1 we have that

$$\dim C(J_1^\perp) + \dim C(J_2^\perp) = \dim C(J_1^\perp \cap C(J_2^\perp) + \dim(C(J_1^\perp) + C(J_2^\perp)).$$

Moreover, it follows that

$$C(J_1^\perp) \cap C(J_2^\perp) = C((J_1 \cap J_2)^\perp),$$

$$C(J_1^\perp) + C(J_2^\perp) \subseteq C(J_1^\perp + J_2^\perp) = C((J_1 \cap J_2)^\perp),$$

because of $\text{col}(M_1 + M_2) \subseteq \text{col}(M_1) + \text{col}(M_2)$ for any $M_1, M_2 \in \text{Mat}(n \times m, \mathbb{F}_q)$. Therefore we have that

$$\rho(J_1) + \rho(J_2) = (\dim_{\mathbb{F}_q} C - \dim_{\mathbb{F}_q} C(J_1^\perp)) + (\dim_{\mathbb{F}_q} C - \dim_{\mathbb{F}_q} C(J_2^\perp))$$

$$\geq (\dim_{\mathbb{F}_q} C - \dim_{\mathbb{F}_q} C((J_1 + J_2)^\perp)) + (\dim_{\mathbb{F}_q} C - \dim_{\mathbb{F}_q} C((J_1 \cap J_2)^\perp))$$

$$= \rho(J_1 + J_2) + \rho(J_1 \cap J_2).$$

Thus $P_C = (E, \rho)$ is a $(q, m)$-polymatroid.

The rank generating function of a $(q, r)$-polymatroid $P = (E, \rho)$ is defined by

$$R_P(X_1, X_2, X_3, X_4) := \sum_{D \in \Sigma(E)} f_P^D(X_1, X_2) g^{\dim D}(X_3, X_4),$$

where

$$f_P^D(X, Y) := X^{\rho(E) - \rho(J)} Y^{r \dim J - \rho(J)} ,$$

for any subspace $J \in \Sigma(E)$, and

$$g^l(X, Y) := \prod_{i=0}^{l-1} (X - q^i Y),$$

for any nonnegative integer $l \in \mathbb{Z}^+ \cup \{0\}$.

The following lemma is essential.

Lemma 4 Let $P = (E, \rho)$ be a $(q, r)$-polymatroid. If $A$ and $B$ are subspaces of $E$ such as $A \subseteq B$, then

$$0 \leq \rho(A) - \rho(B) - r(\dim A - \dim B).$$

Proof. Let $D$ be a subspace of $E$ generated by $B - A$. From (R1), we have that

$$\rho(D) \leq r \dim D = r(\dim B - \dim A).$$

From (R3), we also have that

$$\rho(A) + \rho(D) \geq \rho(A + D) + \rho(A \cap D) \geq \rho(B).$$

By combining the above two inequalities, the lemma follows.
Proposition 5 For any $(q, r)$-polymatroid $P = (E, \rho)$ and any subspace $J$ of $E$, define

$$\rho^*(J) := \rho(J^\perp) + r \dim J - \rho(E).$$

Then $P^* = (E, \rho^*)$ is a $(q, r)$-polymatroid.

Proof. $J^\perp \subseteq E$ implies that $\rho(J^\perp) \leq \rho(E)$. So we have that $0 \leq \rho^*(J) \leq r \dim J$. Thus (R1) holds.

Let $J_1$ and $J_2$ are subspaces of $E$. Assume that $J_1 \subseteq J_2$. Then it follows from Lemma 4 that

$$\rho^*(J_2) - \rho^*(J_1) = (\rho(J_2^\perp) + r \dim J_2 - \rho(E)) - (\rho(J_1^\perp) + r \dim J_1 - \rho(E))$$

$$= \rho(J_2^\perp) - \rho(J_1^\perp) + r(\dim J_2 - \dim J_1)$$

$$= \rho(J_2^\perp) - \rho(J_1^\perp) + r(\dim J_1^\perp - \dim J_2^\perp)$$

$$\geq 0.$$ 

Hence (R2) holds.

From the definition of $\rho^*$, it follows that

$$\rho^*(J_1 \cap J_2) = \rho((J_1 \cap J_2)^\perp) + r \dim (J_1 \cap J_2) - \rho(E),$$

$$\rho^*(J_1 + J_2) = \rho((J_1 + J_2)^\perp) + r \dim (J_1 + J_2) - \rho(E),$$

$$\rho^*(J_1) = \rho(J_1^\perp) + r \dim J_1 - \rho(E),$$

$$\rho^*(J_2) = \rho(J_2^\perp) + r \dim J_2 - \rho(E).$$

From the inequality in (R3) for $\rho$, we have that

$$\rho^*(J_1 \cap J_2) + \rho^*(J_1 + J_2) - \rho^*(J_1) - \rho^*(J_2)$$

$$= \{\rho(J_1^\perp + J_2^\perp) + \rho(J_1^\perp \cap J_2^\perp) - (\rho(J_1^\perp) + \rho(J_2^\perp))\}$$

$$+ r\{\dim (J_1 \cap J_2) + \dim (J_1 + J_2) - (\dim J_1 + \dim J_2)\}$$

$$\leq 0.$$ 

Then (R3) holds. □

For a rank generating function $R_P(X_1, X_2, X_3, X_4)$ of a $(q, r)$-polymatroid $P = (E, \rho)$, define

$$\hat{R}_P(X_1, X_2, X_3, X_4) := \sum_{D \in \Sigma(E)} f^D_P(X_1, X_2) g^{\dim D^\perp}(X_3, X_4).$$

The following result is a duality on rank generating functions of $(q, r)$-polymatroids.

Theorem 6 Let $P = (E, \rho)$ be a $(q, r)$-polymatroid. Then

$$R_{P^*}(X_1, X_2, X_3, X_4) = \hat{R}_P(X_2, X_1, X_3, X_4).$$
Proof. It follows from the definition of rank generating functions that

\[ R_P^*(X_1, X_2, X_3, X_4) = \sum_{D \in \Sigma(E)} f^D_P(X_1, X_2) g^{\dim D}(X_3, X_4) \]

\[ = \sum_{D \in \Sigma(E)} X_1^{\rho^*(E)} X_2^{r \dim D - \rho^*(D)} \prod_{i=0}^{\dim D-1} (X_3 - q^i X_4) \]

\[ = \sum_{D \in \Sigma(E)} X_1^{\rho^*(E) - (\rho(D^+) + r \dim D - \rho(D))} X_2^{r \dim D - \rho(D)} \prod_{i=0}^{\dim D-1} (X_3 - q^i X_4) \]

\[ = \sum_{D' \in \Sigma(E)} X_2^{\rho(E) - \rho(D')} X_1^{r \dim D' - \rho(D')} \prod_{i=0}^{n - \dim D' - 1} (X_3 - q^i X_4) \]

\[ = \hat{R}_P(X_2, X_1, X_3, X_4). \]

\[ \square \]

The trace product of matrices \( M, N \in \text{Mat}(n \times m, \mathbb{F}_q) \) is defined by

\[ \langle M, N \rangle := \text{Tr}(MN^T), \]

where \( \text{Tr} \) denotes the trace of a matrix. Then it is easy to find that the trace product can be calculate from the standard inner products of each row vectors as follows:

**Lemma 7** Let \( M_i \) and \( N_i \) be the \( i \)-th column vectors of \( M \) and \( N \), respectively. Then we have that

\[ \langle M, N \rangle = \sum_{i=1}^{m} M_i \cdot N_i, \]

where \( M_i \cdot N_i \) denotes the standard inner product of vectors \( M_i \) and \( N_i \).

Let \( C \) be a Delsarte rank-metric code in \( \text{Mat}(n \times m, \mathbb{F}_q) \). The dual code of \( C \) is defined by

\[ C^\perp := \{ N \in \text{Mat}(n \times m, \mathbb{F}_q) : \langle M, N \rangle = 0 \text{ for all } M \in C \}. \]

Then the following result is well-known (see, Lemma 5 in [22]).

**Lemma 8** ([22]) For any Delsarte rank-metric codes \( C, D \in \text{Mat}(n \times m, \mathbb{F}_q) \), it follows that

1. \( (C^\perp)^\perp = C \);
2. \( \dim_{\mathbb{F}_q}(C^\perp) = nm - \dim_{\mathbb{F}_q}(C) \);
(3) \((C \cap D)^\perp = C^\perp + D^\perp\), and \((C + D)^\perp = C^\perp \cap D^\perp\).

Set \(V := \text{Mat}(n \times m, \mathbb{F}_q)\). Then it turns out easily that, for any subspace \(R\) of \(\mathbb{F}_q^n\),

1. \(C(R) = V(R) \cap C\), for any Delsarte rank-metric code \(C \subseteq V\);
2. \(|V(R)| = |R|^m = q^m \dim R\).

**Lemma 9** For any subspace \(R\) of \(\mathbb{F}_q^n\),

\[(V(R))^\perp = V(R^\perp)\].

**Proof.** For any matrix \(M \in V(R^\perp)\), Lemma \([\square]\) implies that

\[\langle M, N \rangle = \text{Tr}(MN^T) = \sum_{i=1}^m M_i \cdot N_i = 0,\]

for all \(N \in V(R)\). Thus we have that \(M \in (V(R))^\perp\) and so \(V(R^\perp) \subseteq (V(R))^\perp\).

Conversely, take a matrix \(M \in (V(R))^\perp\). Set

\(R = \{x_1, \ldots, x_{|R|}\}\).

Let \(M(x_i, j)\) be the \(n \times m\) matrix over \(\mathbb{F}_q\) with \(x_i\) as the \(j\)-th column vector and 0 elsewhere. Then it turns out that \(M(x_i, j) \in V(R)\). Thus we have that

\[0 = \langle M, M(x_i, j) \rangle = \sum_{l=1}^m M_l \cdot (M(x_i, j))_l = M_j \cdot x_i.\]

This implies that \(M_j \in R^\perp\), for all \(j = 1, 2, \ldots, m\). Therefore it follows that \(M \in V(R^\perp)\) and so \(V(R^\perp) \supseteq (V(R))^\perp\). \(\square\)

Let \(C\) be a Delsarte rank-metric code in \(\text{Mat}(n \times m, \mathbb{F}_q)\). The *dual space* \(C^*\) of \(C\) is defined by

\[C^* := \text{Hom}_{\mathbb{F}_q}(C, \mathbb{F}_q)\]

Since \(C\) is a finite free \(\mathbb{F}_q\)-module, there is a (non-natural) isomorphism \(C \cong C^*\) (see, for instance, [18]).

The following exact sequence is an analogue of the basic exact sequence on Proposition 2 in [23].

**Proposition 10** Let \(C\) be a Delsarte rank-metric code in \(\text{Mat}(n \times m, \mathbb{F}_q)\). For any subspace \(R\) of \(\mathbb{F}_q^n\), there is an exact sequence as \(\mathbb{F}_q\)-modules,

\[0 \to C^\perp(R) \xrightarrow{\text{inc}} V(R) \xrightarrow{f} C^* \xrightarrow{\text{res}} C(R^\perp)^* \to 0,\]

where \(f\) is an \(\mathbb{F}_q\)-homomorphism defined by

\[f : V \to C^*, \quad M \mapsto (\hat{M} : N \mapsto \langle M, N \rangle),\]

and the maps \(\text{inc}\) and \(\text{res}\) denote the inclusion map and the restriction map, respectively.
Proof. The inclusion map \( \text{inc} \) is a natural injection, and the restriction map \( \text{res} \) is surjective, since \( \mathbb{F}_q \) is an injective module over itself.

Take \( M \in \mathcal{C}^\perp(R) \). Then it follows that
\[
\hat{M} : N \mapsto \langle M, N \rangle = 0,
\]
for all \( N \in \mathcal{C} \). This implies that \( M \in \ker f \) and so \( \text{Im}(\text{inc}) \subseteq \ker f \). Conversely, take \( M \in \ker f \). Then we have that \( \hat{M}(N) = \langle M, N \rangle = 0 \) for all \( N \in \mathcal{C} \) and so \( M \in \mathcal{C}^\perp \cap \mathcal{V}(R) = \mathcal{C}^\perp(R) \). Therefore we have that \( \text{Im}(\text{inc}) \supseteq \ker f \). Thus it follows that \( \text{Im}(\text{inc}) = \ker f \).

Next we prove that \( \ker(\text{res}) = \text{Im} f \). Take \( M \in \mathcal{V}(R) \). Then it also follows from Lemma \([7]\) that \( \hat{M}(N) = \langle M, N \rangle = 0 \) for all \( N \in \mathcal{C}(R^\perp) \). This implies that \( \hat{M} \in \ker(\text{res}) \) and so \( \text{Im} f \subseteq \ker(\text{res}) \). Conversely, take \( \lambda \in \ker(\text{res}) \). The map \( f \) is surjective. Thus there exists \( M \in \mathcal{V} \) such that \( \lambda = M \). Therefore we have that \( \hat{M}(N) = \langle M, N \rangle = 0 \) for all \( N \in \mathcal{C}(R^\perp) \), and so \( M \in (\mathcal{C}(R^\perp))^\perp \). From Lemma \([\mathcal{E}]\) we have that
\[
(\mathcal{C}(R^\perp))^\perp = (\mathcal{C} \cap \mathcal{V}(R^\perp))^\perp = \mathcal{C}^\perp + \mathcal{V}(R^\perp),
\]
and so \( M \in \mathcal{C}^\perp + \mathcal{V}(R) \). Then \( M = M_1 + M_2 \) for some \( M_1 \in \mathcal{C}^\perp \) and \( M_2 \in \mathcal{V}(R) \). Since \( \hat{M}_1(N) = 0 \) for all \( N \in \mathcal{C} \), it turns out that \( \lambda = \hat{M} = \hat{M}_2 \in \text{Im} f \). Thus it follows that \( \text{Im} f = \ker(\text{res}) \).

Therefore the proposition follows. \( \square \)

**Proposition 11.** For any Delsarte rank-metric code \( \mathcal{C} \) in \( \text{Mat}(n \times m, \mathbb{F}_q) \), let \( P_\mathcal{C} = (E, \rho) \) be the \((q, m)\)-polymatroid obtained from \( \mathcal{C} \) such as discussed in Proposition \([\mathcal{O}]\). Then \( P_\mathcal{C}^* = P_{\mathcal{C}^\perp} \).

**Proof.** Take any subspace \( J \subseteq E \). Set
\[
\tau(J) := \dim_{\mathbb{F}_q} \mathcal{C}^\perp - \dim_{\mathbb{F}_q} \mathcal{C}^\perp(J^\perp).
\]
From the definition of dual \((q, m)\)-polymatroids, we have that
\[
\rho^*(J) = \rho(J^\perp) + m \dim J - \rho(E) = (\dim_{\mathbb{F}_q} \mathcal{C} - \dim_{\mathbb{F}_q} \mathcal{C}(J)) + m \dim J - \dim_{\mathbb{F}_q} \mathcal{C} = m \dim J - \dim_{\mathbb{F}_q} \mathcal{C}(J).
\]
By applying \( J^\perp \) for \( R \) in Proposition \([\mathcal{O}]\) we have that
\[
|\mathcal{C}^\perp(J^\perp)| \cdot |\mathcal{C}^*| = |\mathcal{V}(J^\perp)| \cdot |\mathcal{C}(J)^*| \iff |\mathcal{C}^\perp(J^\perp)| \cdot |\mathcal{C}| = |\mathcal{V}(J^\perp)| \cdot |\mathcal{C}(J)| \iff q^{\dim_{\mathbb{F}_q} \mathcal{C}^\perp(J^\perp)} \times q^{\dim_{\mathbb{F}_q} \mathcal{C}} = q^{\dim_{\mathbb{F}_q} \mathcal{C}(J)} \iff \dim_{\mathbb{F}_q} \mathcal{C}(J) = \dim_{\mathbb{F}_q} \mathcal{C} = m \dim J^\perp + \dim_{\mathbb{F}_q} \mathcal{C}(J).
\]
From the above equation and Lemma \([\mathcal{L}2]\), it follows that
\[
\tau(J) = (nm - \dim_{\mathbb{F}_q} \mathcal{C}) - (m \dim J^\perp + \dim_{\mathbb{F}_q} \mathcal{C}(J) - \dim_{\mathbb{F}_q} \mathcal{C}) = m \dim J - \dim_{\mathbb{F}_q} \mathcal{C}(J) = \rho^*(J). \square
\]
Definition 12 (cf. [13, 22]) Let $\mathcal{C}$ be a Delsarte rank-metric code in $\text{Mat}(n \times m, F_q)$. The rank distribution of $\mathcal{C}$ is the collection $\{A_i(\mathcal{C})\}_{i \in \mathbb{Z}_{\geq 0}}$, where $$A_i(\mathcal{C}) := |\{M \in \mathcal{C} : \text{rank}(M) = i\}|.$$ The rank weight enumerator of $\mathcal{C}$ is defined by $$W^R_{\mathcal{C}}(x, y) := \sum_{i=0}^{n} A_i(\mathcal{C}) x^{n-i} y^i.$$ For any Delsarte rank-metric code $\mathcal{C}$ in $\text{Mat}(n \times m, F_q)$ and any subspace $R \subseteq F_q^n$, set $$A_C(R) := |\{M \in \mathcal{C} : \text{col}(M) = R\}|,$$ $$B_C(R) := |\{M \in \mathcal{C} : \text{col}(M) \subseteq R\}|.$$ Then it turns out easily that $B_C(R) = \sum_{T \in \Sigma(R)} A_C(T)$ and $B_C(R) = |\mathcal{C}(R)|$. From the Möbius inversion formula (see, for instance, [1]), we have the following equation.

Lemma 13 $$A_C(R) = \sum_{T \in \Sigma(R)} (-1)^{\dim R - \dim T} q^{\binom{\dim R - \dim T}{2}} B_C(T),$$ where $\binom{a}{b}$ denotes the binomial coefficient of integers $a \geq b \geq 0$.

Now we prove our main result in this paper. The following equation is a kind of Greene type identities for $(q, r)$-polymatroids and Delsarte rank-metric codes.

Theorem 14 Let $\mathcal{C}$ be a Delsarte rank-metric code in $\text{Mat}(n \times m, F_q)$, and let $P_\mathcal{C} = (E, \rho)$ be the $(q, m)$-polymatroid obtained from $\mathcal{C}$ such as discussed in Proposition 3. Then $$W^R_{\mathcal{C}}(x, y) = y^{n-\dim C/m} R_{P_\mathcal{C}}\left(q y^{1/m}, \frac{1}{y^{1/m}}, x, y\right).$$
Proof. By Lemma 13 and the $q$-binomial Theorem, we obtain that

$$W_C^R(x, y) = \sum_{i=0}^n A_i(C)x^{n-i}y^i$$

$$= \sum_{R \in \Sigma(E)} A_C(R)x^{n-\dim R}y^{\dim R}$$

$$= \sum_{R \in \Sigma(E)} \left( \sum_{T \in \Sigma(R)} (-1)^{\dim R-\dim T} q^{(\dim R-\dim T)/2} B_C(T) \right) x^{n-\dim R}y^{\dim R}$$

$$= \sum_{T \in \Sigma(E)} B_C(T) \left( \sum_{T \in \Sigma(E)} (-1)^{\dim R-\dim T} q^{(\dim R-\dim T)/2} \right) T^{\dim T} - \dim J y^{\dim J + \dim T}$$

$$= \sum_{T \in \Sigma(E)} B_C(T) \sum_{J \leq T} (-1)^{\dim J} q^{(\dim J)/2} x^{n-\dim J - \dim T} y^{\dim J + \dim T}$$

$$= \sum_{T \in \Sigma(E)} B_C(T) \sum_{J \in \Sigma(T^+)} (-1)^{\dim J} q^{(\dim J)/2} x^{n-\dim J - \dim T} y^{\dim J + \dim T}$$

$$= \sum_{T \in \Sigma(E)} B_C(T) y^{\dim T} \prod_{j=0}^{\dim T^+ - 1} (x - q^j y)$$

$$= \sum_{T^+ \in \Sigma(E)} B_C((T^+)^+) y^{\dim T^+} \prod_{j=0}^{\dim T^+ - 1} (x - q^j y)$$

$$= \sum_{T^+ \in \Sigma(E)} |C((T^+)^+)| y^{\dim T^+} \prod_{j=0}^{\dim T^+ - 1} (x - q^j y)$$

$$= \sum_{T \in \Sigma(E)} q^{\dim q C(T^+)} y^{\dim T} \prod_{j=0}^{\dim T - 1} (x - q^j y)$$

$$= \sum_{T \in \Sigma(E)} q^{\dim q C - \rho(T)} y^{\dim T} \prod_{j=0}^{\dim T - 1} (x - q^j y)$$

$$= y^{\dim q C/m} \sum_{T \in \Sigma(E)} \left( \frac{1}{y^{1/m}} \right)^{\dim T - \rho(T)} (qy^{1/m})^{\rho(E) - \rho(T)} \prod_{j=0}^{\dim T - 1} (x - q^j y)$$

$$= y^{\dim q C/m} \sum_{T \in \Sigma(E)} f_{\rho C}^{\dim T} \left( \frac{1}{y^{1/m}} \right) g^\dim T(x, y).$$
3 Application to the MacWilliams Identity

In this section, we shall present a MacWilliams type identity for Delsarte rank-metric codes as an application of Theorem 13.

**Proposition 15** Let $C$ be a Delsarte rank-metric code in $\text{Mat}(n \times m, \mathbb{F}_q)$. Then

$$W^R_C(x, y) = \frac{1}{|C|} \sum_{S \in \Sigma(E)} A_C(S) \sum_{j=0}^{n} \sum_{l=0}^{j} \left[ n - \dim S \right] \left[ n - j + l \right] q^{\binom{j}{2} - \rho(j)} x^j x^{-j}.$$

where $\binom{a}{b}_q$ denote the Gaussian binomial coefficient (or $q$-binomial coefficient) of non-negative integers $a$ and $b$.

**Proof.** By combining Theorem 13, Proposition 11 and Theorem 6, we have that

$$W^R_C(x, y) = y^{n - \dim q \mathcal{C}} \sum_{D \in \Sigma(E)} f_D \left( \frac{1}{y^{1/m}}, \frac{1}{y^{1/m}} \right) g_{\dim D}(x, y) = y^{\dim q \mathcal{C}} \sum_{D \in \Sigma(E)} f_D \left( \frac{1}{y^{1/m}}, y^{1/m} \right) g_{\dim D}(x, y) = y^{\dim q \mathcal{C}} \sum_{D \in \Sigma(E)} \left( q^{1/m} \right)^{\dim D - \rho(j)} \left( \frac{1}{y^{1/m}} \right) \prod_{j=0}^{n - \dim D - 1} (x - q^2 y) = \frac{1}{|C|} \sum_{D \in \Sigma(E)} B_C(D) (q^m y)^{n - \dim D} \prod_{j=0}^{\dim D - 1} (x - q^2 y) = \frac{1}{|C|} \sum_{D \in \Sigma(E)} A_C(S) \sum_{S \in \Sigma(D)} \left[ n - \dim S \right] \left[ n - j + l \right] q^{\binom{j}{2} - \rho(j)} x^j x^{-j}.$$

The equation in Proposition 15 can be viewed as a kind of MacWilliams type identities for Delsarte rank-metric codes. Now we prove the equivalence between the above equation and the MacWilliams type identity for Gabidulin rank-metric codes proposed in [12, 13] by using the concepts of $q$-products and $q$-derivative for homogeneous polynomials.
Definition 16 \((\text{[12,13]}\)) Let \(a(x, y; m) = \sum_{i=0}^{t} a_i(m) x^{-i} y^i\) and \(b(x, y; m) = \sum_{j=0}^{s} b_j(m) x^{s-j} y^j\) be two homogeneous polynomials in \(x\) and \(y\) of degree \(r\) and \(s\) respectively with coefficients \(a_i(m)\) and \(b_j(m)\) for \(i, j \geq 0\) in turn are real functions of \(m\), and are assumed to be zero unless otherwise specified. The \(q\)-product \(c(x, y; m)\) of \(a(x, y; m)\) and \(b(x, y; m)\) is defined to be the homogeneous polynomial of degree \((r + s)\) as follows:

\[
c(x, y; m) := a(x, y; m) \ast b(x, y; m) = \sum_{u=0}^{r+s} c_u(m)x^{r+s-u}y^u,
\]

where

\[
c_u(m) := \sum_{i=0}^{u} q^{iu} a_i(m) b_{u-i} (m - i).
\]

For \(n \geq 0\), the \(n\)-th \(q\)-power of \(a(x, y; m)\) is defined recursively:

\[
a(x, y; m)^{[0]} := 1, \quad \text{and} \quad a(x, y; m)^{[n]} := a(x, y; m)^{[n-1]} \ast a(x, y; m), \quad n = 1, 2, \ldots,
\]

Lemma 17 For any \(n \geq 0\) and \((0 \leq l \leq n)\),

1. \((q^m y)^{[n]} = (q^m y)^n\),
2. \((x - y)^{[l]} \ast (q^m y)^{[n-l]} = (x - y)^{[l]} \times (q^m y)^{n-l}\).

Proof. (1) We shall prove this equation by induction on \(n\). When \(n = 0\), it follows that \((q^m y)^{[0]} = 1 = (q^m y)^0\). Suppose that the equation is true for \(n - 1\), that is, \((q^m y)^{[n-1]} = (q^m y)^{n-1}\). Then, from the definition of \(q\)-products, we have that

\[
(q^m y)^{[n]} = (q^m y)^{[n-1]} \ast (q^m y) = (q^m y)^{n-1} \ast q^m y = (q^{n-1} \times q^{m(n-1)} \times q^{m-n+1}) y^n = (q^m y)^n.
\]

(2) From Lemma 1 in \([12]\), it follows that

\[
(x - y)^{[l]} = \sum_{u=0}^{l} \left[ \begin{array}{c} l \\ u \end{array} \right] q (-1)^u q(x) x^{l-u} y^u,
\]

where

\[
\left[ \begin{array}{c} l \\ u \end{array} \right] q := \prod_{i=0}^{u-1} \frac{1 - q^{l-i}}{1 - q^{l+1}}.
\]

Then we have that

\[
(x - y)^{[l]} \ast (q^m y)^{[n-l]} = \left( \sum_{u=0}^{l} \left[ \begin{array}{c} l \\ u \end{array} \right] q (-1)^u q(x) x^{l-u} y^u \right) \ast (q^m y)^{n-l}) = \sum_{u=0}^{n} c_u(m)x^{n-u} y^u,
\]
where
\[ c_u(m) = 0, \quad u = 0, 1, \ldots, n - l - 1, \]
\[ c_{n-l+j}(m) = q^{m(n-l)} [l]_q (-1)^j q^{\binom{j}{2}}, \quad j = 0, 1, \ldots, l. \]

Therefore it follows that
\[
(x - y)^[l] * (q^m y)^[n-l] = q^{m(n-l)} y^{n-l} \times \sum_{u=0}^{l} [l]_q (-1)^u q^{\binom{u}{2}} x^{l-u} y^u \\
= (q^m y)^{n-l} \times (x - y)^[l].
\]

\[ \square \]

**Definition 18** ([12, 13]) The \( q \)-transform of a homogeneous polynomial
\[ a(x, y; m) = \sum_{r=0}^{r} a_i(m) x^{r-i} y^i \]
is defined by the homogeneous polynomial
\[ a(x, y; m) := \sum_{i=0}^{r} a_i(m) y^{[i]} \star x^{[r-i]}. \]

The following lemma is essential:

**Lemma 19**

1. \[ \left[ \begin{array}{c}
    a \\
    b
  \end{array} \right]_q = \left[ \begin{array}{c}
    a - 1 \\
    b
  \end{array} \right]_q + q^{a-b} \left[ \begin{array}{c}
    a - 1 \\
    b - 1
  \end{array} \right]_q = q^b \left[ \begin{array}{c}
    a - 1 \\
    b
  \end{array} \right]_q + \left[ \begin{array}{c}
    a - 1 \\
    b - 1
  \end{array} \right]_q; \]
2. \[ \left[ \begin{array}{c}
    a \\
    b \\
    c
  \end{array} \right]_q = \left[ \begin{array}{c}
    a \\
    b - c
  \end{array} \right]_q \left[ \begin{array}{c}
    a - b + c \\
    c
  \end{array} \right]_q; \]
3. \[ \left[ \begin{array}{c}
    a \\
    b \\
    c
  \end{array} \right]_q = \sum_{i=0}^{n} q^{i(a-b-n+i)} \left[ \begin{array}{c}
    n \\
    i
  \end{array} \right]_q \left[ \begin{array}{c}
    a - n \\
    b - i
  \end{array} \right]_q, \quad n = 0, 1, \ldots, a; \]
4. \[ \binom{a+b}{2} = \binom{a}{2} + ab + \binom{b}{2}. \]

By combining Proposition 15 and Lemmas 17 and 19, we have the following MacWilliams type identity for a rank weight enumerator of a Delsarte rank-metric code.

**Theorem 20** Let \( C \) be a Delsarte rank-metric code in Mat\((n \times m, \mathbb{F}_q)\). Then
\[ W_C^R (x, y) = \frac{1}{|C|} W_C^R (x + (q^m - 1)y, x - y). \]

**Proof.** By Corollary 1 in [13], we have that
\[ (x - y)^[l] * (x + (q^m - 1)y)^{n-[l]} = \sum_{j=0}^{n} P_j(i; m, n) y^j x^{n-j}, \]
where
\[
P_j(i; m, n) := \sum_{l=0}^{j} \left[ \begin{array}{c} n - i \\ j - l \end{array} \right] \left[ \begin{array}{c} n - i \\ l \end{array} \right] (-1)^l q_l j^{(n-i) \prod_{u=0}^{j-l-1} (q^m - q^u)}.
\]

In addition, from Proposition [15] it is sufficient to prove that
\[
\sum_{l=0}^{j} \left[ \begin{array}{c} n - i \\ j - l \end{array} \right] \left[ \begin{array}{c} n - j + l \\ l \end{array} \right] (-1)^l q_l j^{m(j-l)} = P_j(i; m, n).
\]

By using Lemma [19] we have that
\[
\text{L.H.S.} = \sum_{l=0}^{j} \left[ \begin{array}{c} n - i \\ j - l \end{array} \right] \left\{ \sum_{u=0}^{i} q^{u(n-j-i+u)} \left[ \begin{array}{c} i \\ u \end{array} \right] \left[ \begin{array}{c} n - j + l - i \\ l - u \end{array} \right] \right\} (-1)^l q_l j^{m(j-l)}
\]
\[
= \sum_{u=0}^{i} \left[ \begin{array}{c} i \\ u \end{array} \right] q^{u(n-j-i+u)} \sum_{l=0}^{j} \left[ \begin{array}{c} n - i \\ j - l \end{array} \right] \left[ \begin{array}{c} n - j + l - i \\ l - u \end{array} \right] (-1)^l q_l j^{m(j-l)}
\]
\[
= \sum_{u=0}^{i} \left[ \begin{array}{c} i \\ u \end{array} \right] q^{u(n-j-i+u)} \left[ \begin{array}{c} n - i \\ j - u \end{array} \right] \sum_{l=0}^{j} \left[ \begin{array}{c} j - u \\ l - u \end{array} \right] (-1)^l q_l j^{m(j-l-u)}
\]
\[
= \sum_{u=0}^{i} \left[ \begin{array}{c} i \\ u \end{array} \right] q^{u(n-j-i+u)} (-1)^u q_l j^{u} \sum_{l=0}^{j} \left[ \begin{array}{c} j - u \\ l \end{array} \right] (-1)^l q_l j^{m(j-l-u)}
\]
\[
= \sum_{u=0}^{i} \left[ \begin{array}{c} i \\ u \end{array} \right] q^{u(n-j-i+u)} (-1)^u q_l j^{u} \prod_{l=0}^{j-u-1} (q^m - q^{i+u})
\]
\[
= \sum_{u=0}^{i} \left[ \begin{array}{c} i \\ u \end{array} \right] q^{u(n-i)} (-1)^u q_l j^{u} \prod_{l=0}^{j-u-1} (q^{m-u} - q^{i})
\]
\[
= P_j(i; m, n).
\]

\[
\square
\]

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