Hardy’s uncertainty principle and unique continuation properties for abstract Schrödinger equations

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Abstract

In this paper, Hardy’s uncertainty principle and unique continuation properties of abstract Schrödinger equations in vector-valued $L^2$ classes are obtained.

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1. Introduction, definitions

In this paper, the unique continuation properties of the abstract Schrödinger equations

$$i\partial_t u + \Delta u + Au + V(x, t) u = 0, \ x \in \mathbb{R}^n, \ t \in [0, T],$$

(1.1)

are studied, where $A$ is a linear operator, $V(x, t)$ is a given potential operator function in a Hilbert space $H$, subscript $t$ indicates the partial derivative with respect to $t$, $n$ is the dimension of space variable $x$, $\Delta$ denotes the Laplace operator in $\mathbb{R}^n$ and $u = u(x, t)$ is the $H$-valued unknown function. This linear result was then applied to show that two regular solutions $u_1$ and $u_2$ of non-linear Schrödinger equations

$$i\partial_t u + \Delta u + Au = F(u, \bar{u}), \ x \in \mathbb{R}^n, \ t \in [0, T],$$

(1.2)

and for very general non-linearities $F$, must agree in $\mathbb{R}^n \times [0, T]$, when $u_1 - u_2$ and its gradient decay faster than any quadratic exponential at times 0 and $T$.

Hardy’s uncertainty principle and unique continuation properties for Schrödinger equations studied e.g in [4-7] and the references therein. In contrast to the mentioned above results we will study the unique continuation properties of abstract Schrödinger equations with operator potentials. Abstract differential equations studied e.g. in [2, 12-19, 22, 24, 25]. Since the Hilbert space $H$ is arbitrary and $A$ is a possible linear operator, by choosing $H$ and $A$ we can obtain numerous classes of Schrödinger type equations and its systems which occur in a wide variety of physical systems. Our main goal is to obtain sufficient conditions on a solution $u$, the operator $A$, potential $V$ and the behavior of the solution at two different times, $t_0$ and $t_1$ which guarantee that $u(x, t) \equiv 0$ for $x \in \mathbb{R}^n$, $t \in [0, T]$. If we choose the abstract space $H$ a concrete Hilbert space, for example $H = L^2(\Omega)$, $A = L$, where $\Omega$ is a domain in $\mathbb{R}^n$ with sufficiently smooth
boundary and $L$ is elliptic operator then, we obtain the unique continuation properties of following Schrödinger equation

$$\partial_t u = i [\Delta u + Lu + V(x,t) u], \quad x \in \mathbb{R}^n, \quad y \in \Omega, \quad t \in [0,T]. \quad (1.3)$$

Moreover, let we choose $H = L^2(0,1)$ and $A$ to be differential operator with generalized Wentzell-Robin boundary condition defined by

$$D(A) = \left\{ u \in W^{2,2}(0,1), \quad B_j u = Au(j) + \sum_{i=0}^{1} \alpha_{ij} u^{(i)}(j), \quad j = 0, 1 \right\}, \quad (1.4)$$

$$Au = au^{(2)} + bu^{(1)} + cu,$$

where $\alpha_{ij}$ are complex numbers, $a = a(y)$, $b = b(y)$, $c = c(y)$ are complex-valued functions and $V(x,t)$ is a integral operator defined by

$$V(x,y,t) u = \int_0^1 K(x,y,\tau,t) u(x,y,\tau,t) d\tau,$$

where, $K = K(x,y,\tau,t)$ is complex valued bounded function. Then, we get the unique continuation properties of the Wentzell-Robin type boundary value problem (BVP) for the following Schrödinger equation

$$\partial_t u = \left[ \Delta u + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + cu + \int_0^1 K(x,y,\tau,t) u(x,y,\tau,t) d\tau \right], \quad (1.5)$$

$$x \in \mathbb{R}^n, \quad y \in (0,1), \quad t \in [0,T],$$

$$B_j u = Au(x,j,t) + \sum_{i=0}^{1} \alpha_{ij} u^{(i)}(x,j,t) = 0, \quad j = 0, 1. \quad (1.6)$$

Note that, the regularity properties of Wentzell-Robin type BVP for elliptic equations were studied e.g. in [11, 12] and the references therein. Moreover, if put $H = l^2$ and choose $A$ as a infinite matrix $[a_{mj}]$, $m,j = 1, 2, \ldots, \infty$, then we obtain the unique continuation properties of the following system of Schrödinger equation

$$\partial_t u_m = i \left[ \Delta u_m + \sum_{j=1}^{N} (a_{mj} + b_{mj}(x,t)) u_j \right], \quad x \in \mathbb{R}^n, \quad t \in (0,T). \quad (1.7)$$

Let $E$ be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm
\[
\|f\|_{L^p} = \|f\|_{L^p(\Omega; E)} = \left( \int_\Omega \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.
\]

For \( p = 2 \) and \( H \) Hilbert space we get Hilbert space of \( H \)-valued functions with inner product of two elements \( f, g \in L^2(\Omega; H) \):

\[
(f, g)_{L^2(\Omega; H)} = \int_\Omega (f(x), g(x))_H \, dx.
\]

Let \( C(\Omega; E) \) denote the space of \( E \)-valued, bounded uniformly continuous functions on \( \Omega \) with norm

\[
\|u\|_{C(\Omega; E)} = \sup_{x \in \Omega} \|u(x)\|_E.
\]

\( C^m(\Omega; E) \) will denote the spaces of \( E \)-valued bounded uniformly strongly continuous and \( m \)-times continuously differentiable functions on \( \Omega \) with norm

\[
\|u\|_{C^m(\Omega; E)} = \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} \|D^\alpha u(x)\|_E.
\]

\( O_R = \{x \in \mathbb{R}^n, \ |x| < R\} \) for \( R > 0 \). Let \( \mathbb{N} \) denote the set of all natural numbers, \( \mathbb{C} \) denote the set of all complex numbers.

Let \( E_1 \) and \( E_2 \) be two Banach spaces. \( L(E_1, E_2) \) will denote the space of all bounded linear operators from \( E_1 \) to \( E_2 \). For \( E_1 = E_2 = E \) it will be denoted by \( L(E) \).

A linear operator \( A \) is said to be positive in a Banach space \( E \) with bound \( M > 0 \) if \( D(A) \) is dense on \( E \) and \( \left\| (A + sI)^{-1} \right\|_{L(E)} \leq M |s|^{-1} \) for any \( s \in (-\infty, 0) \), where \( I \) is the identity operator in \( E \).

Let \([A, B] \) be a commutator operator, i.e.

\[
[A, B] = AB - BA
\]

for linear operators \( A \) and \( B \).

Sometimes we use one and the same symbol \( C \) without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say \( \alpha \), we write \( C_{\alpha} \).

### 2. Free abstract Scrödinger equation

First of all, we generalize the result G. H. Hardy (see e.g [20], p.131) about uncertainty principle for Fourier transform:

**Lemma 2.1.** Let \( f(x) \) be \( H \)-valued function for \( x \in \mathbb{R}^n \) and

\[
\|f(x)\|_H = O \left( e^{-\frac{|x|^2}{\beta^2}} \right), \quad \|\hat{f}(\xi)\|_H = O \left( e^{-\frac{4|\xi|^2}{\alpha^2}} \right), \quad x, \xi \in \mathbb{R}^n \text{ for } \alpha \beta < 4.
\]
Then \( f(x) \equiv 0 \). Also, if \( \alpha \beta = 4 \) then \( \| f(x) \|_H \) is a constant multiple of \( e^{-\frac{|x|^2}{\beta^2}} \).

**Proof.** Indeed, by employing Phragmen–Lindelöf theorem to Hilberts space valued analytic function class and by reasoning as in [8] we obtain the assertion.

Consider the Cauchy problem for free abstract Schrödinger equation

\[
i \partial_t u + \Delta u + Au = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, 1],
\]

\[
u(x, 0) = f(x).
\]

The above result can be rewritten in terms of the solution of the (2.1) – (2.2) on \( \mathbb{R}^n \times (0, \infty) \) as:

Assume

\[
\| u(x, 0) \|_H = O \left( e^{-\frac{|x|^2}{\beta^2}} \right), \quad \| u(x, T) \|_H = O \left( e^{-\frac{|x|^2}{\alpha^2}} \right) \quad \text{for} \quad \alpha \beta < 4T.
\]

Then \( u(x, t) \equiv 0 \). Also, if \( \alpha \beta = 4T \), then \( u \) has as an initial data a constant multiple of \( e^{-\left(\frac{|\xi|^2}{\alpha^2} + \frac{\beta t}{\alpha^2}\right)} \).

**Lemma 2.2.** Assume \( A \) is a positive operator in Hilbert space \( H \) and \( iA \) generates a semigroup \( U(t) = e^{iAt} \). Then for \( f \in W^{k,2} (\mathbb{R}^n; H) \) there is a generalized solution of (2.1) expressing as

\[
u(x, t) = F^{-1} \left[ e^{iA\xi t} \hat{f} (\xi) \right], \quad A\xi = A + |\xi|^2.
\]

**Proof.** By applying the Fourier transform to the problem (2.1) – (2.2) we get

\[
i \partial_t \hat{u}(\xi, t) + A\xi \hat{u}(\xi, t) = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, 1],
\]

\[
\hat{u}(\xi, 0) = \hat{f}(\xi), \quad \xi \in \mathbb{R}^n,
\]

It is clear to see that the solution of the equation (2.4) – (2.5) can be expressed as

\[
\hat{u}(\xi, t) = e^{iA\xi t} \hat{f}(\xi).
\]

Hence, we obtain (2.3).

Let \( X = L^2 (\mathbb{R}^n; H), \ Y = W^{k,2} (\mathbb{R}^n; H), \ B = L^\infty (\mathbb{R}^n; L (H)), \)

\[
B = L^\infty (\mathbb{R}^n; B (H)) \quad \text{and} \quad \mu(t) = \frac{1}{\alpha (1 - t) + \beta t}.
\]

Consider the following abstract Schrödinger equation

\[
\partial_t u = i [\Delta u + Au + V(x, t) u], \quad x \in \mathbb{R}^n, \quad t \in [0, 1],
\]
where \( A \) is a linear operator in \( H \) and \( V(x,t) \) is a given potential operator function in \( H \).

Our main result in this paper is the following

**Theorem 1.** Assume the following condition are satisfied:

1. \( A \) is a symmetric operator in \( H \) and \( V(x,t) \in L(H) \) for \((x,t) \in \mathbb{R}^n \times [0,1]; \)
2. either, \( V(x,t) = V_1(x) + V_2(x,t) \), where \( V_1(x) \in L(H) \) for \( x \in \mathbb{R}^n \) and

\[
\sup_{t \in [0,1]} \left\| e^{|x|^2 \mu^2(t)} V_2(.,t) \right\|_B < \infty
\]
or

\[
\lim_{R \to \infty} \|V\|_{L^1(0,1;L^\infty(\mathbb{R}^n/B_R);L(H))} = 0;
\]
3. \( \alpha, \beta > 0, \alpha \beta < 2 \) and \( u \in C([0,1];X) \) is a solution of the equation (2.6) and

\[
\left\| e^{\frac{|x|^2}{\mu^2}} u(.,0) \right\|_X < \infty, \left\| e^{\frac{|x|^2}{\mu^2}} u(.,1) \right\|_X < \infty.
\]

Then \( u(x,t) \equiv 0 \).

As a direct consequence of Theorem 1 we get the following Hardy’s uncertainty principle reslult for the non-linear equations (1.2).

**Theorem 2.** Let \( u_1, u_2 \in C([0,1];Y^k) \), \( k \in \mathbb{Z}^+ \) be strong solutions of the equation (1.2) with \( k > \frac{n}{2} \). Moreover, assume \( F \in C^k(C^2, \mathbb{C}) \) and \( F(0) = \partial_u F(0) = \partial_x F(0) = 0 \). If there are \( \alpha, \beta > 0 \) with \( \alpha \beta < 2 \) such that

\[
e^{-\frac{|x|^2}{\mu^2}} (u_1(.,0) - u_2(.,0)) \in X, \ e^{-\frac{|x|^2}{\mu^2}} (u_1(.,1) - u_2(.,1)) \in X
\]
then \( u_1 \equiv u_2 \).

One of the results we get is the following one.

**Theorem 3.** Assume all conditions of Theorem 1 are satisfied. Then

\[
\left\| e^{|x|^2 \mu^2(t)} u(.,t) \right\|_{\frac{1}{\mu^2}L^\infty} \text{ is logarithmically convex in } [0,1]\text{ and there is } N = N(\alpha, \beta)
\]
such that

\[
e^{N(M_1 + M_2 + M_1^2 + M_2^2)} \left\| e^{\frac{|x|^2}{\mu^2}} u(.,0) \right\|_X \left\| e^{\frac{|x|^2}{\mu^2}} u(.,1) \right\|_X \leq \left( 1 - t \right)^{\frac{1}{2} - \frac{1}{\mu^2}} \sup_{t \in [0,1]} \left\| e^{\frac{|x|^2}{\mu^2}} V_2(.,t) \right\|_B.
\]

Moreover,

\[
\sqrt{t(1-t)} \left\| e^{|x|^2 \mu^2(t)} \nabla u \right\|_{L^2(\mathbb{R}^n \times [0,1];H)} \leq e^{N(M_1 + M_2 + M_1^2 + M_2^2)} \left\| e^{\frac{|x|^2}{\mu^2}} u(.,0) \right\|_X + \left\| e^{\frac{|x|^2}{\mu^2}} u(.,1) \right\|_X.
\]
Here, we prove the following result for abstract parabolic equations with variable coefficients.

Consider the Cauchy problem for parabolic equations with variable operator coefficients
\[
\partial_t u = \Delta u + Au + V(x,t)u, \quad x \in R^n, \quad t \in [0,1],
\] (2.7)
where \(A\) is a linear operator and \(V(x,t)\) is the given potential operator function in \(H\). By employing Theorem 1 we obtain

**Theorem 4.** Assume \(A\) is a symmetric operator in \(H\), \(V(x,t) \in L(H)\) for \((x,t) \in R^n \times [0,1]\) and either, \(V(x,t) = V_1(x) + V_2(x,t)\), where \(V_1 \in L(H)\) and
\[
\sup_{t \in [0,1]} \left\| e^{\frac{|x|^2}{2\mu(t)}} V_2(.,t) \right\|_B < \infty
\]
or
\[
\lim_{R \to \infty} \left\| V \right\|_{L^1(0,1;L^\infty(R^n/O_R);L(H))} = 0.
\]
Moreover, suppose \(u \in L^\infty(0;X) \cap L^2(0,1;Y)\) is a solution of (2.9) and
\[
\left\| f \right\|_X < \infty, \quad \left\| e^{\frac{|x|^2}{2\mu(1)}} u(.,1) \right\|_X < \infty
\]
for some \(\delta < 1\). Then, \(f(x) \equiv 0\) for \(x \in R^n\).

**3. Estimates for solutions**

We need the following lemmas for proving the main results. Consider the abstract Schrödinger equation
\[
\partial_t u = (a + ib) [\Delta u + Au + V(x,t)u + F(x,t)], \quad x \in R^n, \quad t \in [0,1],
\] (3.0)
where \(a, b\) are real numbers, \(A\) is a linear operator, \(V(x,t)\) is a given potential operator function in \(H\) and \(F(x,t)\) is a given \(H\)-valued function.

Let
\[
\Phi(A,V) v = a \Re ((A + V)v)_H - b \Im ((A + V)v)_H,
\]
for \(v = v(x,t) \in H(A)\).

**Lemma 3.1.** Assume \(a > 0, b \in \mathbb{R}\), \(A\) is a symmetric operator in \(H\). Moreover, there is a constant \(C_0 > 0\) so that
\[
\left| \Phi(A,V) v(x,t) \right| \leq C_0 \mu(x,t) \left\| v(x,t) \right\|_H^2,
\] (3.V)
for \(x \in R^n, \ t \in [0,T], \gamma \geq 0, \ T \in [0,1] \) and \(v \in H(A)\), where \(\mu\) is a positive function in \(L^1(0,T;L^\infty(R^n))\).
Then the solution $u$ of (3.0) belonging to $L^\infty(0, 1; X) \cap L^2(0, 1; Y^1)$ satisfies the following estimate

$$e^{MT} \left\| e^{\phi(\cdot,T)} u(\cdot,T) \right\|_X \leq M_T \left\| e^{\gamma |x|^2} u(\cdot,0) \right\|_X + \sqrt{a^2 + b^2} \left\| e^{\phi(t)} F \right\|_{L^1(0,T;X)},$$

where

$$\phi(x,t) = \frac{\gamma a |x|^2}{a + 4\gamma (a^2 + b^2)t}, \quad M_T = \|\mu\|_{L^1(0,T;L^\infty(\mathbb{R}^n))}.$$

**Proof.** Let $v = e^{\varphi} u$ where $\varphi$ is a real-valued function to be chosen later. The function $v$ verifies

$$\partial_t v = Sv + Kv + (a + ib) [(A + V) + e^{\varphi} F] \text{ in } \mathbb{R}^n \times [0,1],$$

where $S, K$ are symmetric and skew-symmetric operators given by

$$S = a \left( \Delta + |\nabla \varphi|^2 \right) - ib (2\nabla \varphi \cdot \nabla + \Delta \varphi) + \partial_t \varphi,$$

$$K = ib \left( \Delta + |\nabla \varphi|^2 \right) - a (2\nabla \varphi \cdot \nabla + \Delta \varphi).$$

By differentiating inner product in $X$, we get

$$\partial_t \|v\|_X^2 = 2 \Re (Sv,v)_X + 2 \Re (Kv,v)_X +$$

$$2 \Re ((a + ib) e^{\varphi} F, v)_X + 2 \Re ((a + ib) (A + V) v, v)_X, \quad t \geq 0.$$  

A formal integration by parts gives that

$$\Re (Sv,v)_X = -a \int_{\mathbb{R}^n} |\nabla v|^2_H dx + \int_{\mathbb{R}^n} \left( a |\nabla \varphi|^2 + \partial_t \varphi \right) \|v\|_H^2 dx +$$

$$b \Im \int_{\mathbb{R}^n} (2\nabla \varphi \cdot \Delta v, v)_H dx,$$

$$\Re (Kv,v)_X = (-2a \nabla \varphi \cdot (\nabla v, v))_X - a \Re \int_{\mathbb{R}^n} (2\nabla \varphi \cdot \Delta v, v)_H dx, \quad (3.2)$$

$$\Re ((a + ib) (A + V) v, v)_X = a \Re \int_{\mathbb{R}^n} ((A + V) v, v)_H dx$$

$$- b \Im \int_{\mathbb{R}^n} ((A + V) v, v)_H dx = \int_{\mathbb{R}^n} \Phi (A,V) v dx,$$

$$\Re ((a + ib) e^{\varphi} F, v)_X = a \Re \int_{\mathbb{R}^n} (e^{\varphi} F, v)_H dx = ae^{\varphi} \Re (F,v)_X.$$
By using the Cauchy-Schwarz’s inequality, by condition (3.V), in view of (3.1) and (3.2) we obtain

$$\partial_t \|v\|^2 \leq ae^\varphi \|F (t, .)\|_{\mathcal{X}} \|v\|_{\mathcal{X}} + C_0 \|\mu (\cdot, t)\|_{L^\infty (\mathbb{R}^n)} \|v\|^2_{\mathcal{X}},$$

where $a$, $b$ and $\varphi$ are such that

$$(a - b) \Delta \varphi \leq 0, \quad \left( a + \frac{b^2}{a} \right) \|\nabla \varphi\|^2 + \partial_t v \leq 0 \text{ in } \mathbb{R}^{n+1}. \tag{3.3}$$

The remainig part of the proof is obtained by reasoning as in [7, Lemma 1].

When $\varphi (x, t) = q (t) \psi (x)$, it suffices that

$$\left( a + \frac{b^2}{a} \right) q^2 (t) \|\nabla \psi\|^2 + q' (t) \psi (x) \leq 0. \tag{3.4}$$

If we put $\psi (x) = |x|^2$ then (3.4) holds, when

$$q' (t) = -4 \left( a + \frac{b^2}{a} \right) q^2 (t), \quad q (0) = \gamma, \quad \gamma \geq 0. \tag{3.5}$$

Let

$$\psi_R (x) = \begin{cases} |x|^2, & |x| < R \\ \infty, & |x| > R \end{cases}.$$

Regularize $\psi_R$ with a radial mollifier $\theta_\rho$ and set

$$\varphi_{\rho, R} (x, t) = q (t) \theta_\rho \ast \psi_R (x), \quad \psi_{\rho, R} (x, t) = e^{\varphi_{\rho, R} u},$$

where $q (t) = \gamma a \left[ a + 4 \gamma (a^2 + b^2) t \right]^{-1}$ is the solution to (3.5). Because the right hand side of (3.2) only involves the first derivatives of $\varphi$, $\psi_R$ is Lipschitz and bounded at infinity,

$$\theta_\rho \ast \psi_R (x) \leq \theta_\rho \ast |x|^2 = C (\rho) \rho^2$$

and (3.3) holds uniformly in $\rho$ and $R$, when $\varphi$ is replaced by $\varphi_{\rho, R}$. Hence, it follows that the estimate

$$e^{MT} \left\|e^{\varphi (T)} u (T)\right\|_{\mathcal{X}} \leq M_T \left\|e^{\gamma |x|^2} u (0)\right\|_{\mathcal{X}} + \sqrt{a^2 + b^2} \|e^{\varphi_{\rho, R}} F\|_{L^1 (0, T; \mathcal{X})}$$

holds uniformly in $\rho$ and $R$. The assertion is obtained after letting $\rho$ tend to zero and $R$ to infinity.

**Remark 3.1.** It should be noted that if $H = \mathbb{C}$, $A = 0$ and $V (x, t)$ is a complex valued function, then the abstract condition (3.V) can be replised by

$$M_T = \|a \text{ Re } V - b \text{ Im } V\|_{L^1 (0, T; L^\infty (\mathbb{R}^n))} < \infty.$$  

Moreover, if $A$ and $V (x, t)$ for $x \in \mathbb{R}^n$, $t \in [0, T]$ are bounded operators in $H$, then by using Cauchy-Schwarz’s inequality the assumption (3.V) replaced as

$$|\Phi (A, V) u| \leq \sqrt{a^2 + b^2} \| (A + V) u\|_H \|u\|_H \leq \sqrt{a^2 + b^2} \|A + V\|_{L (H)} \|u\|^2.$$
Let

\[ Q(t) = (f, f)_X, \quad D(t) = (Sf, f)_X, \quad N(t) = D(t) Q^{-1}(t), \]
\[ \partial_t S = S_t \quad \text{and} \quad |\nabla \upsilon|^2_{\mathcal{H}} = \sum_{k=1}^{n} \left\| \frac{\partial \upsilon}{\partial x_k} \right\|^2_{\mathcal{H}}. \]

**Lemma 3.2.** Assume \( S = S(t) \) is a symmetric, \( K = K(t) \) is a skew-symmetric operators in \( \mathcal{H} \), \( G(x, t) \) is a positive function and \( f(x, t) \) is a reasonable function. Then,

\[ Q''(t) = 2 \partial_t \operatorname{Re}(\partial_t f - Sf - Kf, f)_X + 2 (S_t f + [S, K] f, f)_X + \]
\[ \|\partial_t f - Sf - Kf\|_X^2 - \|\partial_t f - Sf - Kf\|_X^2 \] \hspace{1cm} (3.6)

and

\[ \partial_t N(t) \geq Q^{-1}(t) \left[ (S_t f + [S, K] f, f)_X - \frac{1}{2} \|\partial_t f - Sf - Kf\|_X^2 \right]. \]

Moreover, if

\[ \|\partial_t f - Sf - Kf\|_H \leq M_1 \|f\|_H + G(x, t), \quad S_t + [S, K] \geq -M_0 \]

for \( x \in \mathbb{R}^N \), \( t \in [0, 1] \) and

\[ M_2 = \sup_{t \in [0, 1]} \|G(., t)\|_{L^2(\mathbb{R}^n)} (\|f(., t)\|_X)^{-1} < \infty. \]

Then \( Q(t) \) is logarithmically convex in \( [0, 1] \) and there is a constant \( M \) such that

\[ Q(t) \leq e^{M(M_0 + M_1 + M_2^2 + M_2^2)} Q^{1-t}(0) Q^t(1), \quad 0 \leq t \leq 1. \]

**Proof.** The lemma is verifying in a similar way as in [7, Lemma 2] by replacing the inner product and norm of \( L^2(\mathbb{R}^n) \) with inner product and norm of the space \( L^2(\mathbb{R}^n; \mathbb{H}) \).

**Lemma 3.3.** Assume \( a, \gamma > 0, b \in \mathbb{R} \), \( A \) is a symmetric operator in \( \mathcal{H} \). Let,

\[ |\Phi(A, V) u(x, t)| \leq C_0 \mu(x, t) \|u(x, t)\|_H^2 \]

for \( x \in \mathbb{R}^n \), \( t \in [0, 1] \) and \( u \in H(A) \), where \( \mu \) is a positive function in \( L^1(0, T; L^\infty(\mathbb{R}^n)) \). Moreover, suppose

\[ \sup_{t \in [0, 1]} \|V(., t)\|_B \leq M_1, \quad \left\| e^{\gamma |x|^2} u(., 0) \right\|_X < \infty, \quad \left\| e^{\gamma |x|^2} u(., 1) \right\|_X < \infty \]

and

\[ M_2 = \sup_{t \in [0, 1]} \left\| \frac{e^{\gamma |x|^2} F(., t)}{\|u\|_X} \right\|_X < \infty. \]
Then, for solution $u \in L^\infty (0, 1; X) \cap L^2 (0, 1; Y^1)$ of the equation (3.0), $e^{\gamma |x|^2} u (., t)$ is logarithmically convex in $[0, 1]$ and there is a constant $N$ such that
\[
\left\| e^{\gamma |x|^2} u (., t) \right\|_X \leq e^{NM(a,b)} \left\| e^{\gamma |x|^2} u (., 0) \right\|_X^{1-t} \left\| e^{\gamma |x|^2} u (., 1) \right\|_X^t \tag{3.7}
\]
where
\[
M (a,b) = (a^2 + b^2) \left( \gamma M_1^2 + M_2^2 \right) + \sqrt{a^2 + b^2} (M_1 + M_2)
\]
when $0 \leq t \leq 1$.

**Proof.** Let $f = e^{\gamma \varphi} u$, where $\varphi$ is a real-valued function to be chosen. The function $f (x)$ verifies
\[
\partial_t f = S f + K f + (a + ib) (V f + e^{\gamma \varphi} F) \text{ in } R^n \times [0, 1], \tag{3.8}
\]
where $S, K$ are symmetric and skew-symmetric operator, respectively given by
\[
S = a \left( \Delta + A + \gamma^2 |\nabla \varphi|^2 \right) - ib \gamma (2 \nabla \varphi. \nabla + \Delta \varphi) + \gamma \partial_t \varphi, \tag{3.9}
\]
\[
K = ib \left( \Delta + A + \gamma^2 |\nabla \varphi|^2 \right) - a \gamma (2 \nabla \varphi. \nabla + \Delta \varphi).
\]
A calculation shows that,
\[
S_t + [S, K] = \gamma \partial_t^2 \varphi + 4 \gamma^2 a \nabla \varphi. \nabla \partial_t \varphi - 2ib \gamma (2 \nabla \partial_t \varphi. \nabla + \Delta \varphi) - \\
\gamma (a^2 + b^2) \left[ 4 \nabla. (D^2 \varphi \nabla) - 4 \gamma^2 D^2 \varphi \nabla \varphi + \Delta^2 \varphi \right]. \tag{3.10}
\]
If we put $\varphi = |x|^2$, then (3.10) reduce the following
\[
S_t + [S, K] = -\gamma (a^2 + b^2) \left[ 8 \Delta - 32 \gamma^2 |x|^2 \right].
\]
Moreover,
\[
(S_t f + [S, K] f, f) = \gamma (a^2 + b^2) \int_{R^n} \left( 8 |\nabla f|^2 + 32 \gamma^2 \|f\|_H^2 \right) dx. \tag{3.11}
\]
This identity, the condition on $V$ and (3.8) imply that
\[
\|\partial_t f - S f - K f\|_X \leq \sqrt{a^2 + b^2} (M_1 \|f\|_X + e^{\gamma \varphi} \|F\|_X), \quad S_t + [S, K] \geq 0. \tag{3.12}
\]
If we knew that the quantities and calculations involved in the proof of Lemma 3.2 (similar as [7, Lemma 2]) were finite and correct, when $f = e^{\gamma |x|^2} u$ we would have the logarithm convexity of $Q (t) = \left\| e^{\gamma |x|^2} u (., t) \right\|_X$ and the estimate (3.7) from Lemma 3.2. But this fact is verifying by reasonong as in [7, Lemma 3].

Let
\[
\eta = \sqrt{t(1-t) e^{\gamma |x|^2}}, \quad Z = L^2 ([0, 1] \times R^n; H). \tag{3.13}
\]
Lemma 3.4. Assume that $a$, $b$, $u$, $A$ and $V$ are as in Lemma 3.3 and $\gamma > 0$. Then,

$$\|\eta \nabla u\|_Z + \|\eta \beta |x|u\|_Z \leq N \left[(1 + M_1) \sup_{t \in [0,1]} \|e^{\gamma |x|^2} u(.,t)\|_X + \sup_{t \in [0,1]} \|e^{\gamma |x|^2} F(.,t)\|_Z\right],$$

where $N$ is bounded number, when $\gamma$ and $a^2 + b^2$ are bounded below.

Proof. The integration by parts shows that

$$\int_{\mathbb{R}^n} \left( |\nabla f|^2_H + 4\gamma^2 |x|^2 \|f\|_H \right) dx = \int_{\mathbb{R}^n} \left[ e^{2\gamma |x|^2} \left( |\nabla u|^2_H - 2n\gamma \right) \|u\|^2_H \right] dx,$$

when $f = e^{\gamma |x|^2} u$, while integration by parts, the Cauchy-Schwarz's inequality and the identity, $n = \nabla \cdot x$, give that

$$\int_{\mathbb{R}^n} \left( |\nabla f|^2_H + 4\gamma^2 |x|^2 \|f\|_H \right) dx \geq 2\gamma n \|f\|_X^2.$$

The sum of the last two formulae gives the inequality

$$2 \int_{\mathbb{R}^n} \left( |\nabla f|^2_H + 4\gamma^2 |x|^2 \|f\|_H \right) dx \geq \int_{\mathbb{R}^n} e^{\gamma |x|^2} |\nabla f|^2_H dx. \quad (3.13)$$

Integration over $[0, 1]$ of $t(1-t)$ times the formula (3.6) for $Q(t)$ and integration by parts, shows that

$$2 \int_0^1 t(1-t) \left( S_t f + [S, K] f, f \right)_X dt + \int_0^1 Q(t) dt \leq Q(1) + Q(0) + \quad (3.14)$$

$$\int_0^1 1 - 2t \text{Re} (\partial_t f - Sf - Kf, f)_X dx + \int_0^1 t(1-t) \|\partial_t f - Sf - Kf\|_X^2 dt.$$

Assuming again that the last two calculations are justified for $f = e^{\gamma |x|^2}$. Then (2.11) – (2.14) imply the assertion.

4. Appell transformation in abstract function spaces

Let

$$\eta(x,t) = \frac{(\alpha - \beta)|x|^2}{4(a + ib)\alpha (1-t) + \beta t}, \quad \nu(s) = \left[ \gamma \alpha \beta \mu^2(s) + \frac{(\alpha - \beta)a}{4(a^2 + b^2)} \mu(s) \right].$$
Lemma 4.1. Assume $A$ and $V$ are as in Lemma 3.3 and $u = u(x,s)$ is a solution of the equation
\[
\partial_s u = (a + ib) [\Delta u + Au + V(y,s)u + F(y,s)], \quad y \in \mathbb{R}^n, \quad s \in [0,1].
\]
Let $a + ib \neq 0, \gamma \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}_+$. Set
\[
\tilde{u}(x,t) = \left(\sqrt{\alpha\beta\mu(t)}\right)^\frac{\gamma}{2} u \left(\sqrt{\alpha\beta x\mu(t)}, \beta t\mu(t)\right) e^{\eta}.
\]
(4.1)

Then, $\tilde{u}(x,t)$ verifies the equation
\[
\partial_t \tilde{u} = (a + ib) \left[\Delta \tilde{u} + A\tilde{u} + \tilde{V}(x,t)u + \tilde{F}(x,t)\right], \quad x \in \mathbb{R}^n, \quad t \in [0,1]
\]
with
\[
\tilde{V}(x,t) = \alpha\beta\mu^2(t)\mu \left(\sqrt{\alpha\beta x\mu(t)}, \beta t\mu(t)\right),
\]
\[
\tilde{F}(x,t) = \left(\sqrt{\alpha\beta\mu(t)}\right)^{\gamma+2} \left(\sqrt{\alpha\beta x\mu(t)}, \beta t\mu(t)\right).
\]
Moreover,
\[
\left\|e^{\gamma|x|^2} \tilde{F}(.,t)\right\|_X = \alpha\beta\mu^2(t) e^{\nu|y|^2} \left\|F(s)\right\|_X \quad \text{and} \quad \left\|e^{\gamma|x|^2} \tilde{u}(.,t)\right\|_X = e^{\nu|y|^2} \left\|u(s)\right\|_X
\]
when $s = \mu(t)$ and $\gamma \in \mathbb{R}$.

**Proof.** If $u$ is a solution of the equation
\[
\partial_s u = (a + ib) [\Delta u + Au + Q(y,s)], \quad y \in \mathbb{R}^n, \quad s \in [0,1]
\]
(4.2)
then, the function $u_1(x,t) = u(\sqrt{r}x, rt + \tau)$ verifies
\[
\partial_t u_1 = (a + ib) \left[\Delta u_1 + A\tilde{u} + rQ(\sqrt{r}x, rt + \tau)\right], \quad x \in \mathbb{R}^n, \quad s \in [0,1]
\]
and $u_2(x,t) = t^{-\frac{\gamma}{2}} u \left(\frac{x}{\tau}, \frac{1}{\tau}\right) e^{\frac{\gamma|x|^2}{\alpha\beta + \gamma|y|^2}}$ is a solution to
\[
\partial_t u_2 = -(a + ib) \left[\Delta u_2 + A\tilde{u} + t^{-\frac{\gamma}{2}} Q(\sqrt{\tau}x, \tau t + \tau) e^{\frac{\gamma|x|^2}{\alpha\beta + \gamma|y|^2}}\right], \quad y \in \mathbb{R}^n, \quad s \in [0,1].
\]
These two facts and the sequel of changes of variables below verifies the Lemma, when $\alpha > \beta$, i.e.
\[
u \left(\sqrt{\frac{\alpha\beta}{\alpha - \beta}}x, \frac{\alpha\beta}{\alpha - \beta} t - \frac{\beta}{\alpha - \beta}\right)
\]
is a solution to the same non-homogeneous equation but with right-hand side
\[
\frac{\alpha\beta}{\alpha - \beta} Q \left(\sqrt{\frac{\alpha\beta}{\alpha - \beta}}x, \frac{\alpha\beta}{\alpha - \beta} t - \frac{\beta}{\alpha - \beta}\right).
\]
The function,
\[
\frac{1}{(\alpha - t)^{\frac{2}{n}} u}\left(\frac{\sqrt{\alpha \beta x}}{\alpha - \beta (\alpha - t)} \frac{\alpha \beta}{\alpha - \beta (\alpha - t)} \beta \alpha - \beta - \beta \alpha - \beta \right) e^{\frac{|x|^2}{4(\alpha - t)}}
\]
verifies (4.2) with right-hand side
\[
\frac{\alpha \beta}{(\alpha - \beta) (\alpha - t)^{\frac{2}{n} + 2}} Q\left(\frac{\sqrt{\alpha \beta x}}{\alpha - \beta (\alpha - t)}, \frac{\alpha \beta}{\alpha - \beta (\alpha - t)} \beta \alpha - \beta - \beta \alpha - \beta \right) e^{\frac{|x|^2}{4(\alpha + t)(\alpha - t)}}.
\]
Replacing \((x, t)\) by \((\sqrt{\alpha - \beta x}, (\alpha - \beta) t)\) we get that
\[
\mu \frac{2}{n} (t) u \left(\frac{\sqrt{\alpha \beta \mu(t) x}}{\alpha - \beta \mu(t)} \frac{\alpha \beta \mu(t)}{\alpha - \beta \mu(t)} \beta \alpha - \beta - \beta \alpha - \beta \right) e^{(\alpha - \beta) \frac{|x|^2 \mu(t)}{4(\alpha + t)}}
\]
is a solution of (4.2) but with right-hand
\[
\mu \frac{2}{n} + 2 (t) Q\left(\sqrt{\alpha \beta \mu(t) x}, \frac{\alpha \beta \mu(t)}{\alpha - \beta \mu(t)} \beta \alpha - \beta - \beta \alpha - \beta \right) e^{(\alpha - \beta) \frac{|x|^2 \mu(t)}{4(\alpha + t)}}.
\]
Finally, observe that
\[
s = \beta t \mu(t) = \frac{\alpha \beta \mu(t)}{\alpha - \beta} - \frac{\beta}{\alpha - \beta}
\]
and multiply (4.3) and (4.4) we obtain the assertion for \(\alpha > \beta\). The case \(\beta > \alpha\) follows by reversing by changes of variables, \(s' = 1 - s\) and \(t' = 1 - t\).

5. Variable coefficients. Proof of Theorem 3

We are ready to prove Theorem 3. Let
\[
B = L^1 (0, 1; L^\infty (R^n; L (H))) , \ B \ (R) = L^1 (0, 1; L^\infty (R^n/O_R; L (H))).
\]

Proof of Theorem 3. We may assume that \(\alpha \neq \beta\). The case \(\alpha = \beta\) follows from the latter by replacing \(\beta\) by \(\beta + \delta\), \(\delta > 0\), and letting \(\delta\) tend to zero. We may also assume that \(\alpha < \beta\). Otherwise, replace \(u\) by \(\bar{u}(1 - t)\). Assume \(a > 0\). Set \(W = \Delta + A + V_1\) and let \(U u_0 = e^{t(a + ib)W} u_0\) denote \(C ([0, 1]; X)\) solution to the problem
\[
\partial_t u = (a + ib) (\Delta u + A u + V_1 (x) u) , \ x \in R^n , \ t \in [0, 1], \ u (x, 0) = u_0 (x).
\]

By virtue of Duhamel principle there is a solution of
\[
\partial_t u = (a + ib) \Delta u + A u + V (x, t) u , \ x \in R^n , \ t \in [0, 1], \ u (x, 0) = u_0 (x).
\]

expressing as

\[ u(x, t) = e^{(a+ib)W}u_0 + i \int_0^t e^{-i(t-s)W}V_2(x, s)u(x, s) \, ds \quad (5.2) \]

for \( x \in \mathbb{R}^n, \ s \in [0, 1]. \)

For \( 0 \leq \varepsilon \leq 1, \) set

\[ F_\varepsilon (x, t) = \frac{i}{\varepsilon + i} e^{\varepsilon tW}V_2(x, t)u(x, t) \quad (5.3) \]

and

\[ u_\varepsilon (x, t) = e^{(\varepsilon+i)W}u_0 + (\varepsilon + i) \int_0^t e^{(\varepsilon+i)(t-s)W}F_\varepsilon(x, s)u(x, s) \, ds. \quad (5.4) \]

Then, \( u_\varepsilon(x, t) \in L^\infty (0, 1; X) \cap L^2 \left( \mathbb{R}^n; Y_1^1 \right) \) and satisfies

\[ \partial_t u_\varepsilon = (\varepsilon + i)(Wu + F_\varepsilon) \quad \text{in} \ \mathbb{R}^n \times [0, 1], \]

\[ u_\varepsilon(., 0) = u_0(\cdot). \]

The identities

\[ e^{(z_1+z_2)W} = e^{z_1W}e^{z_2W}, \ \text{when Re} \ z_1, \ \text{Re} \ z_2 \geq 0, \]

(5.2), (5.3) and (5.4) shows that

\[ u_\varepsilon(x, t) = e^{\varepsilon tW}u(x, t), \ \text{for} \ t \in [0, 1]. \quad (5.5) \]

In particular, the equality \( u_\varepsilon(x, 1) = e^{\varepsilon W}u(x, 1) \) and Lemma 3.1 with \( a+ib = \varepsilon, \ \gamma = \frac{1}{\beta}, \ F \equiv 0 \) and the fact that \( u_\varepsilon(0) = u(0) \) imply that

\[ \left\| \frac{|\varepsilon|^2}{\varepsilon^{\beta+1}e^{\varepsilon}}u_\varepsilon(., 1) \right\|_X \leq e^{\varepsilon\|V_1\|_B} \left\| \frac{|\varepsilon|^2}{\varepsilon^{\beta+1}}u(., 1) \right\|_X, \quad \left\| \frac{|\varepsilon|^2}{\varepsilon^{\alpha+1}}u_\varepsilon(., 0) \right\|_X = \left\| \frac{|\varepsilon|^2}{\varepsilon^{\alpha+1}}u(., 0) \right\|_X. \]

A second application of Lemma 3.1 with \( a+ib = \varepsilon, \ F \equiv 0, \) the value of \( \gamma = \mu^2(t) \) and (5.2) show that

\[ \left\| \frac{|\varepsilon|^2}{\varepsilon^{\alpha+1}e^{\varepsilon}}F_\varepsilon(., t) \right\| X \leq e^{\varepsilon\|V_1\|_B} \left\| \frac{|\varepsilon|^2}{\varepsilon^{\beta+1}}V_2(., t) \right\|_B \|u(., t)\|_X, \ t \in [0, 1]. \]

Setting, \( \alpha_\varepsilon = \alpha + 2\varepsilon \) and \( \beta_\varepsilon = \beta + 2\varepsilon, \) the last three inequalities give that

\[ \left\| \frac{|\varepsilon|^2}{\varepsilon^{\alpha_\varepsilon}}u_\varepsilon(., 1) \right\|_X \leq e^{\varepsilon\|V_1\|_B} \left\| \frac{|\varepsilon|^2}{\varepsilon^{\beta_\varepsilon}}u(., 1) \right\|_X, \quad (5.6) \]
Because $\alpha < \beta$ where $a \Longleftrightarrow b$, $F \equiv 0$, $\gamma = 0$, and (5.2), (5.5) implies that

$$\|F_\varepsilon(.,t)\|_X \leq e^{\varepsilon\|V_1\|_B} \|V_2(.,t)\|_B \|u(.,t)\|_X, \quad t \in [0,1].$$

(5.7)

A third application of Lemma 3.1 with $a + ib = b$, $F \equiv 0$, $\gamma = 0$, and (5.2), (5.5) implies that

$$\|F_\varepsilon(.,t)\|_X \leq e^{\varepsilon\|V_1\|_B} \|V_2(.,t)\|_B \|u(.,t)\|_X, \quad t \in [0,1].$$

(5.8)

Set $\gamma_\varepsilon = \frac{1}{\alpha_\varepsilon \beta_\varepsilon}$ and let

$$\tilde{u}_\varepsilon(.,t) = \left(\sqrt{\alpha_\varepsilon \beta_\varepsilon} \mu_\varepsilon(t)\right)^{\frac{\gamma}{2}} u \left(\sqrt{\alpha_\varepsilon \beta_\varepsilon} \mu_\varepsilon(t), \beta_\varepsilon t \mu_\varepsilon(t)\right) e^{\eta}$$

be the function associated to $u_\varepsilon$ in Lemma 4.1, where $a + ib = \varepsilon + i$ and $\alpha$, $\beta$ are replaced respectively by $\alpha_\varepsilon$, $\beta_\varepsilon$ and

$$\mu_\varepsilon(t) = \frac{1}{\alpha_\varepsilon (1-t) + \beta_\varepsilon t}.$$
On the other hand,

\[ N_1^{-1} \|u(.,0)\|_X \leq \|u(.,t)\|_X \leq N_1 \|u(.,0)\|_X, \quad t \in [0,1], \quad (5.13) \]

where

\[ N_1 = \sup_{t \in [0,1]} \|\text{Re} V_2(.,t)\|_B. \]

The energy method imply that

\[ \partial_t \|\tilde{u}_\varepsilon(.,t)\|_X^2 \leq 2\varepsilon \left( \|\tilde{V}_1(.,t)\|_B \|\tilde{u}_\varepsilon(.,t)\|_X^2 + 2 \|\tilde{F}_\varepsilon(.,t)\|_X \|\tilde{u}_\varepsilon(.,t)\|_X \right). \quad (5.14) \]

Let \( 0 = t_0 < t_1 < \ldots < t_m = 1 \) be a uniformly distributed partition of \([0,1]\), where \( m \) will be chosen later. The inequalities (5.14), (5.9), the inequality in (5.11), the second inequality in (5.10), (5.8) and (5.13) imply that there is \( N_2 \), which depends on \( \frac{\alpha}{\beta}, \|V_1\|_B \) and \( \sup_{t \in [0,1]} \|\text{Re} V_2(.,t)\|_B \) such that

\[ \|\tilde{u}_\varepsilon(.,t_i)\|_X \leq e^{\frac{2\varepsilon}{\alpha}} \|\tilde{u}_\varepsilon(.,t)\|_X + N_2 \sqrt{t_i - t_{i-1}} \|u(.,0)\|_X \quad (5.15) \]

for \( t \in [t_{i-1},t_i] \) and \( i = 1,2,\ldots,m \). Choose now \( m \) so that

\[ N_2 \max_i \sqrt{t_i - t_{i-1}} \leq \frac{1}{4N_1}. \quad (5.16) \]

Because, \( \lim_{\varepsilon \to 0} \|\tilde{u}_\varepsilon(.,t)\|_X = \|u(.,s)\|_X \) when \( s = \beta t \mu (t) \) and (5.13), there is \( \varepsilon_0 \) such that

\[ \|\tilde{u}_\varepsilon(.,t_i)\|_X \geq \frac{1}{4N_1} \|u(.,0)\|_X, \quad \text{when } 0 < \varepsilon \leq \varepsilon_0, \quad i = 1,2,\ldots,m \quad (5.17) \]

and now, (5.15) - (5.17) show that

\[ \|\tilde{u}_\varepsilon(.,t)\|_X \geq \frac{1}{4N_1} \|u(.,0)\|_X, \quad \text{when } 0 < \varepsilon \leq \varepsilon_0, \quad t \in [0,1]. \quad (5.18) \]

It is now simple to verify that (5.18), the first inequality in (5.10), (5.7) and (5.13) imply that

\[ \sup_{t \in [0,1]} \frac{\|e^{\gamma_\varepsilon \|x\|^2} \tilde{F}_\varepsilon(.,t)\|_X}{\|\tilde{u}_\varepsilon(.,t)\|_X} \leq \frac{4\beta}{\alpha} M_2(\varepsilon), \quad \text{when } 0 < \varepsilon \leq \varepsilon_0 \quad (5.19) \]

where

\[ M_2(\varepsilon) = e^{\varepsilon \|V_2(.,t)\|_B + \varepsilon \|V_1\|_B} \sup_{t \in [0,1]} \|e^{\gamma_\varepsilon \|x\|^2} u(.,t)\|_B. \]

By using Lemma 3.3, (5.12), (5.9) and (5.19) to show that \( \|e^{\gamma_\varepsilon \|x\|^2} \tilde{u}_\varepsilon(.,t)\|_X \) is logarithmically convex in \([0,1]\) and that

\[ \|e^{\gamma_\varepsilon \|x\|^2} \tilde{u}_\varepsilon(.,t)\|_X \leq e^{NM(\alpha,b)} \|e^{\gamma \|x\|^2} \tilde{u}_\varepsilon(0)\|_X e^{1-t \|e^{\gamma \|x\|^2} \tilde{u}_\varepsilon(1)\|_X} \quad (5.20) \]
when $0 < \varepsilon \leq \varepsilon_0$, $t \in [0, 1]$ and $N = N(\alpha, \beta)$. Then, Lemma 3.4 gives that
\[
\|\eta \nabla \tilde{u}_\varepsilon\|_Z + \|\eta |x| \tilde{u}_\varepsilon\|_Z \leq
\]
\[
N(1 + M_1) \left[ \sup_{t \in [0, 1]} \left\| e^{\beta |x|^2} \tilde{u}_x (., t) \right\|_X + \sup_{t \in [0, 1]} \left\| e^{\beta |x|^2} \tilde{F}_x (., t) \right\|_Z \right] \leq
\]
\[
Ne^{N(M_0 + M_1 + M_2(\varepsilon) + M_1^2 + M_2^2(\varepsilon))} \left[ \left\| e^{\beta |x|^2} u (., 0) \right\|_X + \left\| e^{\beta |x|^2} u (., 1) \right\|_X \right],
\]
when $0 < \varepsilon \leq \varepsilon_0$, the logarithmic convexity and regularity of $u$ follow from the limit of the identity in (5.11), the final limit relation between the variables $s$ and $t$, $s = \beta t \mu (t)$ and letting $\varepsilon$ tend to zero in (5.20) and the above inequality. By reasoning as in [4, Lemma 6] we obtain:

**Lemma 5.1.** Assume $A$ is a symmetric operator in $H$ and $V (x, t)$ is a potential operator function in $H$ such that
\[
\|V\|_B \leq \varepsilon_0 \text{ for a } \varepsilon_0 > 0.
\]

Let $u \in C ([0, 1] : X)$ be a solution of the equation
\[
\partial_t u = i [\Delta u + Au + V (x, t) u + F (x, t)] , \quad x \in R^n , \quad t \in [0, 1].
\]

Then,
\[
\sup_{t \in [0, 1]} \left\| e^{\lambda x} u (., t) \right\|_X \leq N \left[ \left\| e^{\lambda x} u (., 0) \right\|_X + \left\| e^{\lambda x} u (., 1) \right\|_X + \left\| e^{\lambda x} F (., t) \right\|_{L^1 ([0, 1] : X)} \right],
\]
where $\lambda \in R^n$ and $N > 0$ is constant.

**Theorem 5.1.** Assume $A$ is a symmetric operator in $H$ and $V (x, t)$ is a potential operator function in $H$ such that
\[
V \in B \text{ and } \lim_{R \to \infty} \|V\|_{B(R)} = 0.
\]

Suppose $\alpha$, $\beta$ are positive numbers and
\[
\left\| e^{\frac{|x|^2}{\alpha^2}} u (., 0) \right\|_X < \infty, \quad \left\| e^{\frac{|x|^2}{\alpha^2}} u (., 1) \right\|_X < \infty.
\]

Let $u \in C ([0, 1] : X)$ be a solution of the equation
\[
\partial_t u = i [\Delta u + Au + V (x, t) u] , \quad x \in R^n , \quad t \in [0, 1].
\]

Then, there is a $N = N(\alpha, \beta)$ such that
\[
\sup_{t \in [0, 1]} \left\| e^{\frac{|x|^2}{\alpha^2}} u (., t) \right\|_X + \left\| \sqrt{t(1 - t)} e^{\frac{|x|^2}{\alpha^2}(t)} \nabla u \right\|_{L^2 ([0, 1] : H)} \leq
\]

17
The function (5.21) and let
\[ \hat{u}(x, t) = \left(\sqrt{\frac{\alpha \beta}{\alpha \beta}}(t)\right) \hat{\lambda} u \left(\sqrt{\frac{\alpha \beta}{\alpha \beta}}(t), \beta t \mu(t)\right) e^\gamma. \tag{5.21} \]
The function (5.21) is a solution of
\[ \partial_t u = i \left[ \Delta u + A u + V(x, t) u \right], \quad x \in \mathbb{R}^n, \ t \in [0, 1] \]
with
\[ \hat{V}(x, t) = \alpha \beta \mu^2(t) V \left(\sqrt{\frac{\alpha \beta}{\alpha \beta}}(t), \beta t \mu(t)\right), \]
\[ \sup_{t \in [0, 1]} \left\| \hat{V}(\cdot, t) \right\|_B \leq \max \left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right) \sup_{t \in [0, 1]} \left\| V(\cdot, t) \right\|_B, \quad \lim_{R \to \infty} \left\| \hat{V}(\cdot, t) \right\|_{B(R)} = 0 \]
and
\[ \left\| e^{\gamma |x|^2} \hat{u}(\cdot, t) \right\|_X = \left\| e^{\mu^2(t) |x|^2} u(\cdot, s) \right\|_X, \tag{5.22} \]
\[ \left\| \hat{u}(\cdot, t) \right\|_X = \left\| u(\cdot, s) \right\|_X \text{ when } s = \beta t \mu(t). \]
Choose \( R > 0 \) such that \( \left\| \hat{V}(\cdot, t) \right\|_{B(R)} \leq \varepsilon_0 \) we get
\[ \partial_t \hat{u} = i \left[ \Delta \hat{u} + A \hat{u} + \hat{V}_R(x, t) u + \hat{F}_R(x, t) \right], \quad x \in \mathbb{R}^n, \ t \in [0, 1], \]
with
\[ \hat{V}_R(x, t) = \chi_{\mathbb{R}^n / U_R} \hat{V}(x, t), \quad \hat{F}_R(x, t) = \chi_{U_R} \hat{V}(x, t) \hat{u}. \]
Then using the Lemma 5.1 we obtain
\[ \sup_{t \in [0, 1]} \left\| e^{\lambda x} \hat{u}(\cdot, t) \right\|_X \leq \]
\[ N \left[ \left\| e^{\lambda x} \hat{u}(\cdot, 0) \right\|_X + \left\| e^{\lambda x} \hat{u}(\cdot, 1) \right\|_X + e^{\lambda |R|} \left\| \hat{V}(\cdot, t) \right\|_{B(t \in [0, 1]} \sup_{t \in [0, 1]} \left\| u(\cdot, t) \right\|_X \right]. \]
Replace \( \lambda \) by \( \lambda \sqrt{\gamma} \) in the above inequality, square both sides, multiply all by \( e^{-\frac{|x|^2}{2}} \) and integrate both sides with respect to \( \lambda \) in \( \mathbb{R}^n \). This and the identity
\[ \int_{\mathbb{R}^n} e^{2\sqrt{\gamma} \lambda x} \frac{|\lambda|^2}{2} d\lambda = \left(\frac{2\pi}{\sqrt{\gamma}}\right)^n e^{2\gamma |x|^2} \]

Proof. Assume that \( u(y, s) \) verifies the equation
\[ \partial_s u = i \left[ \Delta u + A u + V(y, s) u + F(y, s) \right], \quad y \in \mathbb{R}^n, \ s \in [0, 1]. \]
Set \( \gamma = (\alpha \beta)^{-1} \) and let
\[ \hat{u}(x, t) = \left(\sqrt{\frac{\alpha \beta}{\alpha \beta}}(t)\right) \hat{\lambda} u \left(\sqrt{\frac{\alpha \beta}{\alpha \beta}}(t), \beta t \mu(t)\right) e^\gamma. \tag{5.21} \]
imply the inequality
\[
\sup_{t \in [0,1]} \| \tilde{u}(.,t) \|_X \leq N \left[ \left\| e^{2|\varepsilon|^2} \tilde{u}(.,0) \right\|_X + \left\| e^{2|\varepsilon|^2} \tilde{u}(.,1) \right\|_X + \left\| e^{2\gamma R^2} \tilde{V}(.,t) \right\|_B \sup_{t \in [0,1]} \| \tilde{u}(.,t) \|_X \right].
\] (5.23)

This inequality and (5.22) imply that
\[
\sup_{t \in [0,1]} \| \tilde{u}(.,t) \|_X \leq
\] 
\[
N \left[ \left\| e^{2|\varepsilon|^2} \tilde{u}(.,0) \right\|_X + \left\| e^{2|\varepsilon|^2} \tilde{u}(.,1) \right\|_X + \sup_{t \in [0,1]} \| V(.,t) \|_B \sup_{t \in [0,1]} \| u(.,t) \|_X \right]
\] for some new constant \( N \).

To prove the regularity of \( u \) we proceed as in (5.2)–(5.4). The Duhamel formula shows that
\[
u(\varepsilon, x, t) = e^{i\varepsilon W} u_0 + \int_0^t e^{i(t-s)W} V_2(x, s) u(x, s) ds, \quad x \in \mathbb{R}^n, \quad t \in [0,1].
\] (5.24)

For \( 0 \leq \varepsilon \leq 1 \), set
\[
\tilde{F}(\varepsilon, x, t) = \frac{i}{\varepsilon + i} e^{(\varepsilon + i)(\Delta + A)} \tilde{V}(x, t) \tilde{u}(x, t),
\] (5.25)

and
\[
\tilde{u}(\varepsilon, x, t) = e^{(\varepsilon + i)t(\Delta + A)} u_0 + (\varepsilon + i) \int_0^t e^{(\varepsilon + i)(t-s)(\Delta + A)} \tilde{F}(x, s) u(x, s) ds,
\] (5.26)

\[ x \in \mathbb{R}^n, \quad t \in [0,1]. \]

The identities
\[
e^{(z_1 + z_2)(\Delta + A)} = e^{z_1(\Delta + A)} e^{z_2(\Delta + A)} \text{ for } \text{Re} z_1, \text{Re} z_2 \geq 0
\]

and (5.24)–(5.26) show that
\[
\tilde{u}(\varepsilon, x, t) = e^{t(\Delta + A)} \tilde{u}(x, t) \text{ for } t \in [0,1]. \] (5.27)

From Lemma 3.1 with \( a + ib = \varepsilon \), (5.27) and (5.25) we get that
\[
\sup_{t \in [0,1]} \left\| e^{\gamma |\varepsilon|^2} \tilde{u}(.,t) \right\|_X \leq \sup_{t \in [0,1]} \left\| e^{\gamma |\varepsilon|^2} \tilde{u}(.,t) \right\|_X,
\] (4.28)

\[
\sup_{t \in [0,1]} \left\| e^{\gamma |\varepsilon|^2} \tilde{F}(.,t) \right\|_X \leq \sup_{t \in [0,1]} \left\| e^{\gamma |\varepsilon|^2} \tilde{F}(.,t) \right\|_X,
\]
where
\[ \gamma_\varepsilon = \frac{\gamma}{1 + 4 \gamma \varepsilon}, \quad \tilde{V}_0 = \sup_{t \in [0,1]} \| \tilde{V} \|_B. \]

Then, Lemma 3.4, (5.28) and (5.23) show that
\[ \left\| e^{\gamma_\varepsilon |x|^2} u(.,t) \right\|_{L^2(R^n \times [0,1];H)} + \left\| \sqrt{\varepsilon} (1-t) e^{\gamma_\varepsilon |x|^2} \nabla u \right\|_{L^2(R^n \times [0,1];H)} \leq \]
\[ N e^{\tilde{V}_0} \left[ \left\| e^{\frac{|x|^2}{2\varepsilon}} u(.,0) \right\|_X + \left\| e^{\frac{|x|^2}{2\varepsilon}} u(.,1) \right\|_X + \sup_{t \in [0,1]} \| u(.,t) \|_X \right], \]

where
\[ V_0 = \sup_{t \in [0,1]} \| V(x,t) \|_B. \]

The Theorem 5.1 follows from this inequality, from (5.21) – (5.23) and letting \( \varepsilon \) tend to zero.

6. A Hardy type abstract uncertainty principle. Proof of Theorem 1.

The assertion about the Carleman inequality in Lemma 6.1 below is the following monotonicity or frequency function argument related to Lemma 3.2. When \( u \in C([0,1]; X) \) is a free solution to the free abstract Schrödinger equation
\[ \partial_t u - i (\Delta u + Au) = 0, \quad x \in R^n, \quad t \in [0,1], \]
satisfies
\[ \left\| e^{\gamma |x|^2} u(.,0) \right\|_X < \infty, \quad \left\| e^{\gamma |x|^2} u(.,1) \right\|_X < \infty \]
and
\[ f = e^\varphi u, \quad Q(t) = (f(.,t), f(.,t))_X, \]

where
\[ \varphi(x,t) = \mu |x + R t(1-t)|^2 - \frac{R^2 t(1-t)}{8\mu}, \quad \sigma(\varepsilon, t) = \frac{(1 + \varepsilon) t(1-t)}{16\mu}. \]

Then, \( \log Q(t) \) is logarithmically convex in \([0,1]\), when \( 0 < \mu < \gamma \).

The formal application of the above argument to a \( C([0,1];X) \) solution of the equation
\[ \partial_t u - i [\Delta u + Au + V(x,t) u] = 0, \quad x \in R^n, \quad t \in [0,1], \quad \text{(6.1)} \]
implies a similar result, when \( V \) is a bounded potential, though the justification of the correctness of the assertions involved in the corresponding formal application of Lemma 3.2 were formal. In fact, we can only justify these assertions,
when the potential $V$ verifies the first condition in Theorem 1 or when we can obtain the additional regularity of the gradient of $u$ in the strip, as in Theorem 5.1. Here, we choose to prove Theorem 1 using the Carleman inequality in Lemma 6.1 in place of the above convexity argument. The reason for our choice is that it is simpler to justify the correctness of the application of the Carleman inequality to a $C([0,1]; X)$ solution to (6.1) than the corresponding monotonicity or logarithmic convexity of the solution.

**Lemma 6.1.** Assume $A$ is a symmetric operator in $H$ and $V(x,t)$ is a potential operator function in $H$ such that

$$V \in B \text{ and } \lim_{R \to \infty} \|V\|_{B(R)} = 0.$$  

The estimate

$$R \sqrt{\frac{\varepsilon}{8\mu}} \left\| e^{\sigma - \varepsilon} v \right\|_{L^2(R^{n+1}, H)} \leq \left\| e^{\sigma - \varepsilon} [\partial_t u - i (\Delta u + Au)] v \right\|_{L^2(R^{n+1}, H)}$$

holds, when $\varepsilon > 0$, $\mu > 0$, $R > 0$ and $v \in C^\infty_0 (R^{n+1}; H)$.

**Proof.** Let $f = e^{\sigma - \varepsilon} v$. Then,

$$e^{\sigma - \varepsilon} [\partial_t u - i (\Delta u + Au)] v = \partial_t f + S f - K f.$$  

From (3.8) – (3.10) with $\gamma = 1$, $a + ib = i$ and $\varphi(x,t) = \kappa(x,t) - \sigma(\varepsilon,t)$ we have

$$S = -4\mu i (x + Rt(1-t)e_1) \cdot \nabla - 2\mu i + 2\mu R(1-2t)(x_1 + Rt(1-t)) - \sigma,$$

$$K = i (\Delta + A) + 4\mu^2 i |x + Rt(1-t)e_1|^2, \quad S_x + [S, K] = -8\mu \Delta + 32\mu^3 |x + Rt(1-t)e_1|^2 - 4\mu R(x_1 + Rt(1-t)) +$$

$$+ 2\mu R^2 (1 - 2t)^2 + \frac{(1 + \varepsilon) R^2}{8\mu} - 4i \mu R(1 - 2t) \partial_{x_1}$$

and

$$(S,f + [S, K]f, f)_X = 32\mu^3 \int_{\mathbb{R}^n} |x + Rt(1-t)e_1 - \frac{R}{16\mu^2} e_1|^2 \|f\|^2_H\, dx +$$

$$+ \frac{R^2}{8\mu} \int_{\mathbb{R}^n} \|f\|^2_H\, dx + 8\mu \int_{\mathbb{R}^n} \|\nabla x f\|^2_H\, dx + 8\mu \int_{\mathbb{R}^n} \|i \partial_{x_1} f - R \left(\frac{1}{2} - t\right) f\|^2_H\, dx +$$

$$\geq \frac{R^2}{8\mu} \int_{\mathbb{R}^n} \|f\|^2_H\, dx.$$  

21
Following the standard method to handle \( L_2 \)-Carleman inequalities, the symmetric and skew-symmetric parts of \( \partial_t - S - K \), as a space-time operator, are respectively \(-S\) and \( \partial_t - K \), and \([-S, \partial_t - K] = S_\ell + [S, K] \). Thus,
\[
\| \partial_t f - S f - K f \|_{L^2(R^{n+1}; H)}^2 = \| \partial_t f - K f \|_{L^2(R^{n+1}; H)}^2 + \| S f \|_{L^2(R^{n+1}; H)}^2 - 2 \text{Re} \int_{R^n \times -\infty}^{\infty} \int (S f, \partial_t f - K f)_H \, dx \, dt \geq \int_{R^n \times -\infty}^{\infty} ((-S, \partial_t - K f, f)_H \, dx \, dt = (6.3)
\]
\[
\int_{-\infty}^{\infty} (S f + [S, K] f, f)_H \, dt,
\]
and the Lemma 6.1 follows from (5.2) and (5.3).

**Proof of Theorem 1.** Let \( u \) be as in Theorem 1 and \( \tilde{u}, \tilde{V} \) the corresponding functions defined in Lemma 4.1, when \( a + ib = i \). Then, \( \tilde{u} \in C([0,1]; X) \) is a solution of the equation
\[
\partial_t u - i \left[ \Delta u + Au + \tilde{V} u \right] = 0, \quad x \in R^n, \quad t \in [0,1]
\]
and
\[
\| e^{\gamma |x|^2} \tilde{u} (., 0) \|_X < \infty, \quad \| e^{\gamma |x|^2} \tilde{u} (., 1) \|_X < \infty \quad \text{for} \quad \gamma = \frac{1}{\alpha \beta}, \quad \gamma > \frac{1}{2}.
\]
The proofs of Theorem 3 show that in either case
\[
N_\gamma = \sup_{t \in [0,1]} \left[ \| e^{\gamma |x|^2} \tilde{u} (., t) \|_{L^2(R^n \times [0,1]; H)} + \| \sqrt{t} (1 - t) e^{\gamma |x|^2} \nabla \tilde{u} \|_{L^2(R^n \times [0,1]; H)} \right] < \infty.
\]
For given \( R > 0 \), choose \( \mu \) and \( \epsilon \) such that
\[
\frac{(1 + \epsilon) \frac{1}{\mu}}{2 (1 - \epsilon)^{\frac{1}{3}}} \leq \frac{\gamma}{1 + \epsilon} \quad (6.5)
\]
and let \( \eta_M \) and \( \theta_R \) be smooth functions verifying, \( \theta_M (x) = 1 \), when \( |x| \leq M \), \( \theta_M (x) = 0 \), when \( |x| > 2M \), \( M \geq 2R \), \( \eta_R \in C_0^\infty (0,1) \), \( 0 \leq \eta_R (t) \leq 1 \), \( \eta_R (t) = 1 \) for \( t \in [\frac{1}{R}, 1 - \frac{2}{R}] \) and \( \eta_R (t) = 0 \) for \( t \in [0, \frac{1}{2R}] \cup [1 - \frac{1}{2R}, 1] \). Then, \( v (x,t) = \eta_R (t) \theta_M (x) \tilde{u} (x,t) \) is compactly supported in \( R^n \times (0,1) \) and
\[
\partial_t v - i \left[ \Delta v + Av + \tilde{V} v \right] = \eta_R (t) \theta_M (x) \tilde{u} (x,t) - (2 \nabla \theta_M \cdot \nabla \tilde{u} + \tilde{u} \Delta \theta_M ) \eta_R.
\]
The terms on the right hand side of (6.6) are supported, where
\[
\mu |x + Rt(1 - t)|^2 \leq \gamma |x|^2 + \frac{\gamma}{\epsilon}.
\]
\[
\mu |x + R t(1 - t)e_1|^2 \leq \gamma |x|^2 + \frac{\gamma}{\varepsilon} R^2.
\]

Apply now Lemma 6.1 to \( v \) with the values of \( \mu \) and \( \varepsilon \) chosen in (6.5). This, the bounds for \( \mu |x + R t(1 - t)e_1|^2 \) in each of the parts of the support of

\[
\partial_t v - i \left[ \Delta v + A v + \tilde{V} v \right]
\]

and the natural bounds for \( \nabla \theta_M, \Delta \theta_M \) and \( \eta'_R \) show that there is a constant \( N_\varepsilon \) such that

\[
R \| e^{\kappa - \sigma} v \|_{L^\infty(R^n \times [0,1];H)} \leq
N_\varepsilon \left\| \tilde{V} \right\|_B \| e^{\kappa - \sigma} v \|_{L^2(R^n \times [0,1];H)} + N_\varepsilon R e^\bar{\gamma} \sup_{t \in [0,1]} \| e^{\gamma|x|^2} \tilde{u}(.,t) \|_X + \quad (6.7)
\]

\[
N_\varepsilon M^{-1} e^{\bar{\gamma} R^2} \left\| e^{\gamma|x|^2} (\| \tilde{u} \|_H + \| \nabla \tilde{u} \|_H) \right\|_{L^2(R^n \times \sigma_R)}
\]

where

\[
\sigma_R = \left[ \frac{1}{2R}, 1 - \frac{1}{2R} \right].
\]

The first term on the right hand side of (6.7) can be hidden in the left hand side, when \( R \geq 2N_\varepsilon \left\| \tilde{V} \right\|_B \), while the last tends to zero, when \( M \) tends to infinity by (6.4). This and the fact that \( v = \tilde{u} \) in \( O_{\frac{(1 - \varepsilon)^2 R'}{4}} \times \left[ \frac{1 - \varepsilon^2}{2}, \frac{1 + \varepsilon^2}{2} \right] \), where

\[
\kappa - \sigma \geq \frac{R^2}{16\mu} \left( 4 \mu^2 (1 - \varepsilon)^6 - (1 + \varepsilon)^3 \right)
\]

and (6.5) show that

\[
e^{c(\gamma, \varepsilon)} \left\| \tilde{u} \right\|_{L^2(O_{\frac{1}{2}} \times \left[ \frac{1 - \varepsilon^2}{2}, \frac{1 + \varepsilon^2}{2} \right];H)} \leq N_{\gamma, \varepsilon}, \quad (6.8)
\]

when \( R \geq 2N_\varepsilon \left\| \tilde{V} \right\|_B \). At the same time

\[
N^{-1} \left\| \tilde{u}(.,0) \right\|_X \leq \left\| \tilde{u}(.,t) \right\|_X \leq N \left\| \tilde{u}(.,1) \right\|_X \quad (6.9)
\]

for \( 0 \leq t \leq 1 \) and \( N = e^{\sup_{t \in [0,1]} \| \tilde{V} \|_B} \). Moreover, from (6.4) we get

\[
\left\| \tilde{u}(.,t) \right\|_X \leq N \left\| \tilde{u}(.,t) \right\|_{L^2(O_{\frac{1}{2}};H)} + e^{-\frac{\gamma R^2}{4\mu}} N_\gamma \quad \text{when} \ 0 \leq t \leq 1 \quad (6.10)
\]

Then, (6.8) – (6.10) show that there is a constant \( N_{\gamma, \varepsilon, \bar{V}} \), which such that

\[
e^{c(\gamma, \varepsilon) R^2} \left\| \tilde{u}(.,0) \right\|_X \leq N_{\gamma, \varepsilon, \bar{V}}.
\]

For \( R \to \infty \) we obtain \( u \equiv 0 \).
Proof of Theorem 2.
First of all we show the following Carleman inequality

**Lemma 6.2.** Assume $A$ is a symmetric operator in $H$ and $V(x,t)$ is a potential operator function in $H$ such that

$$V \in B$$

and

$$\lim_{R \to \infty} \|V\|_{B(R)} = 0.$$ 

The estimate

$$R \sqrt{\frac{8}{S\mu}} e^{-\sigma + \chi} \left\| e^{-\sigma + \chi} [\partial_t u - \Delta u - Au] v \right\|_{L^2(R^{n+1}; H)}$$

holds, when $\varepsilon > 0$, $\mu > 0$, $R > 0$ and $v \in C^\infty_0 \left( R^{n+1}; H \right)$, where

$$\chi(t) = \frac{R^2 t (1 - t) (1 - 2t)}{6}.$$ 

**Proof.** Let $f = e^{\chi + \sigma} v$. Then,

$$e^{\chi + \sigma} [\partial_t u - (\Delta u + Au)] v = \partial_t f - S f - K f.$$ 

From (3.8) − (3.10) with $\gamma = 1$, $a + ib = 1$ and $\varphi(x,t) = \kappa(x,t) + \chi(t) - \sigma(\varepsilon, t)$ we have

$$S = \Delta + A + 4\mu^2 |x + R(1 - t) e_1|^2 + 2\mu i +$$

$$2\mu R (1 - 2t) (x_1 + R(1 - t)) - \sigma + \left( t^2 - t + \frac{1}{6} \right) R^2,$$

$$K = -4\mu (x + R(1 - t)e_1) \cdot \nabla - 2\mu n,$$

$$S_t + [S, K] = -8\mu \Delta + 32\mu^3 |x + R(1 - t)e_1|^2 +$$

$$4\mu R (4\mu (1 - 2t - 1) ((x_1 + R(1 - t)) + (2t - 1)) R^2 + \frac{(1 + \varepsilon) R^2}{8\mu}$$

and

$$(S_t + [S, K] f, f)_{x} = 32\mu^3 \int_{R^n} \left| x + R(1 - t)e_1 + \frac{(4\mu (1 - 2t - 1) R}{16\mu^2} e_1 \right|^2 \| f \|^2_H dx +$$

$$8\mu \int_{R^n} \| \nabla f \|^2_H dx + \frac{\varepsilon R^2}{8\mu} \int_{R^n} \| f \|^2_H dx \geq \frac{\varepsilon R^2}{8\mu} \int_{R^n} \| f \|^2_H dx.$$ 

Then from (6.12) a similar way as Lemma 6.1 we obtain the estimate (6.11).
Proof of Theorem 4. Assume that \( u \) verifies the conditions in Theorem 4 and let \( \tilde{u} \) be the Appel transformation of \( u \) defined in Lemma 4.1 with \( a + ib = 1, \alpha = 1 \) and \( \beta = 1 + \frac{4}{3} \). \( \tilde{u} \in L^\infty (0, 1; X) \cap L^2 (0, 1; Y^1) \) is a solution of the equation

\[
\partial_t u = \Delta u + Au + \tilde{V}u, \ x \in \mathbb{R}^n, \ t \in [0, 1]
\]

with \( \tilde{V} \) a bounded potential in \( \mathbb{R}^n \times [0, 1] \) and \( \gamma = \frac{1}{23} \). Then, we have

\[
\left\| e^{\gamma|x|^2} \tilde{u} (., 0) \right\|_X = \left\| \tilde{u} (., 0) \right\|_X, \ \left\| e^{\gamma|x|^2} \tilde{u} (., 1) \right\|_X = \left\| \tilde{u} (., 1) \right\|_X.
\]

From Lemma 3.3 and Lemma 3.4 with \( a + ib = 1 \), we have

\[
\sup_{t \in [0,1]} \left\| e^{\gamma|x|^2} \tilde{u} (., t) \right\|_X + \left\| \sqrt{t(1-t)} e^{\gamma|x|^2} \nabla \tilde{u} \right\|_{L^2(\mathbb{R}^n \times [0,1]; H)} \leq
\]

\[
e^{(M_1 + M_2^2)} \left[ \left\| e^{\gamma|x|^2} \tilde{u} (., 0) \right\|_X + \left\| e^{\gamma|x|^2} \tilde{u} (., 1) \right\|_X \right],
\]

where

\[
M_1 = \left\| \tilde{V} \right\|_B.
\]

The proof is finished by setting \( v(x, t) = \theta_M(x) \eta_R(t) \tilde{u}(x, t) \), by using Carleman inequality (6.11) and in similar argument that we used to prove Theorem 1.

7. Unique continuation properties for the system of Schrödinger equation

Consider the Cauchy problem for the system of Schrödinger equation

\[
\frac{\partial u_m}{\partial t} = i \left[ \Delta u_m + \sum_{j=1}^{N} a_{mj} u_j + \sum_{j=1}^{N} b_{mj} u_j \right], \ x \in \mathbb{R}^n, \ t \in (0, T), \quad (7.1)
\]

where \( u = (u_1, u_2, ..., u_N), u_j = u_j (x, t), a_{mj} \) are complex numbers and \( b_{mj} = b_{mj} (x, t) \) are complex valued functions. Let \( l_2 = l_2 (N) \) and \( l_2^* = l_2^* (N) \) (see [23, § 1.18]). Let \( A \) be the operator in \( l_2 (N) \) defined by

\[
D (A) = \left\{ u = \{u_j\}, \ \| u \|_{l_2^* (N)} = \left( \sum_{j=1}^{N} 2^{s_j} |u_j|^2 \right)^{1/2} < \infty \right\},
\]

\[
A = [a_{mj}], \ a_{mj} = g_m 2^{s_j}, \ m, j = 1, 2, ..., N, \ N \in \mathbb{N}
\]

and

\[
D (V (x, t)) = \{ u = \{u_j\} \in l_2^* \},
\]

25
\[ V(x, t) = [b_{mj}(x, t)], b_{mj}(x, t) = g_m(x, t) 2^{sj}, \ m, j = 1, 2, ..., N. \]

Let
\[ X_2 = L^2(R^n; l_2), Y_{s, 2} = H_{s, 2}(R^n; l_2). \]

From Theorem 1 we obtain the following result

**Theorem 7.1.** Assume \( a_{mj} = a_{jm} \) and
\[ \sup_{t \in [0, 1]} \left\| e^{i|x|^2 t} g_m(., t) \right\|_{L^\infty(R^n)} < \infty. \]

Let \( \alpha, \beta > 0 \) and \( \alpha \beta < 2 \). Assume \( u \in C([0, 1]; l_2) \) be a solution of the equation (7.1) and
\[ \left\| e^{i|x|^\alpha u(., 0)} \right\|_{X_2} < \infty, \left\| e^{i|x|^\beta u(., T)} \right\|_{X_2} < \infty. \]

Then \( u(x, t) \equiv 0 \).

**Proof.** It is easy to see that \( A \) is a symmetric operator in \( l_2 \) and other conditions of Theorem 1 are satisfied. Hence, from Theorem 1 we obtain the conclusion.

**8. Unique continuation properties for anisotropic Schrödinger equation**

The regularity property of BVP for elliptic equations were studied e.g. in [1, 2]. Let \( \Omega = R^n \times G, G \subset R^d, d \geq 2 \) is a bounded domain with \((d-1)\)-dimensional boundary \( \partial G \). Let us consider the following problem

\[ \partial_t u = i \left[ \Delta_x u + \sum_{|\alpha| \leq 2m} a_{\alpha}(y) D^\alpha_y u(x, y, t) + \int_G K(x, y, \tau, t) u(x, y, \tau) d\tau \right], \]
\[ x \in R^n, \ y \in \Omega, \ t \in [0, T], \]
\[ (8.1) \]

\[ B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D^\beta_y u(x, y, t) = 0, x \in R^n, \ y \in \partial G, \ j = 1, 2, ..., m, \]
\[ (8.2) \]

where \( a_{\alpha}, a_{i\beta}, b_{j\beta} \) are the complex valued functions, \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \beta = (\beta_1, \beta_2, ..., \beta_n), \mu_i < 2m, K = K(x, y, \tau, t) \) is a complex valued bounded function in \( \Omega \times G \times [0, T] \) and
\[ D^k_x = \frac{\partial^k}{\partial x^k}, \ D_j = -i \frac{\partial}{\partial y_j}, \ D_y = (D_1, ..., D_n), \ y = (y_1, ..., y_n), \]

26
\[ \zeta' = (\zeta_1, \zeta_2, ..., \zeta_{n-1}) \in R^{n-1}, \quad \alpha' = (\alpha_1, \alpha_2, ..., \alpha_{n-1}) \in Z^n, \]

\[ A (y_0, \zeta', D_y) = \sum_{|\alpha'| + j \leq 2m} a_{\alpha'} (y_0) \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} ... \zeta_{n-1}^{\alpha_{n-1}} D_y^j \text{ for } y_0 \in \bar{G} \]

\[ B_j (y_0, \zeta', D_y) = \sum_{|\beta'| + j \leq m_j} b_{j\beta'} (y_0) \zeta_1^{\beta_1} \zeta_2^{\beta_2} ... \zeta_{n-1}^{\beta_{n-1}} D_y^j \text{ for } y_0 \in \partial G. \]

**Theorem 8.1.** Let the following conditions be satisfied:

1. \( G \in C^2, \ a_\alpha \in C (\bar{G}) \) for each \(|\alpha| = 2m \) and \( a_\alpha \in L_\infty (G) \) for each \(|\alpha| < 2m; \)
2. \( b_{j\beta} \in C^{2m-m_j} (\partial G) \) for each \( j, \beta \) and \( m_j < 2m, \ \sum_{j=1}^{m} b_{j\beta} (y') \sigma_j \neq 0, \) for \(|\beta| = m_j, \ y' \in \partial G, \) \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in R^n \) is a normal to \( \partial G; \)
3. \( y \in \bar{G}, \ \zeta \in R^n, \ \lambda \in S (\varphi_0) \) for \( 0 \leq \varphi_0 < \pi, \ |\zeta| + |\lambda| \neq 0 \) let \( \lambda + \sum_{|\alpha| = 2m} a_\alpha (y) \zeta^\alpha \neq 0; \)
4. \( \lambda + A (y_0, \zeta', D_y) \partial (y) = 0, \)

\[ B_j (y_0, \zeta', D_y) \partial (0) = h_j, \ j = 1, 2, ..., m \]

has a unique solution \( \partial \in C_0 (R_+) \) for all \( h = (h_1, h_2, ..., h_n) \in C^n \) and for \( \zeta' \in R^{n-1}; \)

\[ \sup_{t \in [0, 1]} \| e^{-x^2 t} K (., y, t) \|_{L^\infty (\Omega \times G)} < \infty \text{ for } y \in \bar{G}; \]

(6) Let \( \alpha, \ beta > 0, \ \alpha \beta < 2. \) Assume \( u \in C ([0, 1], l_2) \) be a solution of the equation (8.1) - (8.2) and

\[ \left\| e^{\frac{i} {\sqrt{2}} u (. , 0)} \right\|_{L^2 (\Omega)} < \infty, \ \left\| e^{\frac{i} {\sqrt{2}} u (. , T)} \right\|_{L^2 (\Omega)} < \infty. \]

Then \( u (x, y, t) \equiv 0. \)

**Proof.** Let us consider operators \( A \) and \( V (x, t) \) in \( H = L^2 (G) \) that are defined by the equalities

\[ D (A) = \{ u \in W^{2m, 2} (G), \ B_j u = 0, \ j = 1, 2, ..., m \}, \ A u = \sum_{|\alpha| \leq 2m} a_\alpha (y) D_y^\alpha u (y), \]

\[ V (x, t) u = \int_G K (x, y, \tau, t) u (x, y, \tau, t) d \tau. \]
Then the problem (8.1) – (8.2) can be rewritten as the problem (1.1), where \( u(x) = u(x, .) \), \( f(x) = f(x, .) \), \( x \in \sigma \) are the functions with values in \( H = L^2(G) \). By virtue of [1] operator \( A + \mu \) is positive in \( L^2(G) \) for sufficiently large \( \mu > 0 \). Moreover, in view of (1)–(5) all conditions of Theorem 1 are hold. Then Theorem1 implies the assertion.

9. The Wentzell-Robin type mixed problem for Boussinesq equations

Consider the problem (1.5) – (1.6). Let
\[
\sigma = R^n \times (0, 1).
\]
Suppose \( \nu = (\nu_1, \nu_2, ..., \nu_n) \) are nonnegative real numbers. In this section, we present the following result:

**Theorem 9.1.** Suppose the following conditions are satisfied:

1. \( a(\cdot) \in W^{1, \infty}_c(0, 1) \), \( \alpha(\cdot) \geq \delta > 0 \), \( b(\cdot) \), \( c(\cdot) \in L^\infty(0, 1) \);
2. \[ \sup_{t \in [0, 1]} \| e^{\frac{|x|^2}{\alpha^2} t} V(\cdot, y, t) \|_{L^\infty(\sigma)} < \infty \] for \( y \in [0, 1] \);
3. Let \( \alpha, \beta > 0 \) and \( \alpha \beta < 2 \). Assume \( u \in C([0, T]; L^2(\sigma)) \) be a solution of the equation (1.5) – (1.6) and
\[ \left\| e^{\frac{|x|^2}{\beta^2}} u(\cdot, 0) \right\|_{L^2(\sigma)} < \infty, \left\| e^{\frac{|x|^2}{\alpha^2}} u(\cdot, T) \right\|_{L^2(\sigma)} < \infty. \]

Then \( u(x, y, t) \equiv 0 \).

**Proof.** Let us consider operators \( A \) in \( H = L^2(G) \) that are defined by the equalities
\[
D(A) = \{ u \in W^{2m, 2}(G) \mid B_j u = 0, j = 1, 2, ..., m \}, \quad Au = \sum_{|\alpha| \leq 2m} a_\alpha(y) D_y^\alpha u(y).
\]

Then the problem (8.1) – (8.2) can be rewritten as the problem (1.1), where \( u(x) = u(x, .) \), \( f(x) = f(x, .) \), \( x \in \sigma \) are the functions with values in \( H = L^2(G) \). By virtue of [10, 11] the operator \( A \) generates analytic semigroup in \( L^2(0, 1) \). Hence, by virtue of (1)–(5) all conditions of Theorem 1 are satisfied. Then Theorem1 implies the assertion.

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28
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