CYCLOTOMIC DOUBLE AFFINE HECKE ALGEBRAS

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With an appendix by Hiraku Nakajima and Daisuke Yamakawa

To Ivan Cherednik on his 65th birthday with admiration

ABSTRACT. We show that the partially spherical cyclotomic rational Cherednik algebra (obtained from the full rational Cherednik algebra by averaging out the cyclotomic part of the underlying reflection group) has four other descriptions: (1) as a subalgebra of the degenerate DAHA of type A given by generators; (2) as an algebra given by generators and relations; (3) as an algebra of differential-reflection operators preserving some spaces of functions; (4) as equivariant Borel-Moore homology of a certain variety. Also, we define a new $q$-deformation of this algebra, which we call cyclotomic DAHA. Namely, we give a $q$-deformation of each of the above four descriptions of the partially spherical rational Cherednik algebra, replacing differential operators with difference operators, degenerate DAHA with DAHA, and homology with K-theory, and show that they give the same algebra. In addition, we show that spherical cyclotomic DAHA are quantizations of certain multiplicative quiver and bow varieties, which may be interpreted as K-theoretic Coulomb branches of a framed quiver gauge theory. Finally, we apply cyclotomic DAHA to prove new flatness results for various kinds of spaces of $q$-deformed quasiinvariants.

1. INTRODUCTION

Let $N \geq 0, l \geq 0$ be integers, $c_0, \ldots, c_{l-1}, h, k$ be parameters, and $c = (c_0, \ldots, c_{l-1})$. Let $\mathbb{H}_{N}^{l,cyc}(c, h, k)$ be the cyclotomic rational Cherednik algebra attached to the complex reflection group $W = S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$. Let $p$ be the symmetrizer of the subgroup $(\mathbb{Z}/l\mathbb{Z})^N$, and $\mathbb{H}_{N}^{l,psc}(c, h, k) := p\mathbb{H}_{N}^{l,cyc}(c, h, k)p$ be the corresponding partially spherical subalgebra.

In this paper we give a geometric interpretation of $\mathbb{H}_{N}^{l,psc}$ as the equivariant Borel-Moore homology of a certain variety $\mathcal{R} = \mathcal{R}(N, l)$ equipped with a group action. This allows us to define a natural $q$-deformation $\mathbb{H}_{N}^{q,l}$ of $\mathbb{H}_{N}^{l,psc}$ in terms of the equivariant K-theory of $\mathcal{R}(N, l)$, which we call the cyclotomic double affine Hecke algebra (DAHA).

The existence of this $q$-deformation may seem somewhat surprising from the viewpoint of classical algebraic theory of DAHA [Ch1], since typically DAHA are attached to crystallographic reflection groups (Weyl groups), while the group $W$ is not crystallographic for $l \geq 3$. Yet, we also give a purely algebraic definition of cyclotomic DAHA. Namely, we characterize the cyclotomic DAHA as the subalgebra of the usual Cherednik’s DAHA for $GL_N$ generated by certain elements, and also as the subalgebra
preserving certain spaces of functions. Finally, we present cyclotomic DAHA by generators and relations. These three descriptions also make sense in the trigonometric limit \( q \to 1 \) (for partially spherical cyclotomic rational Cherednik algebras). We note that for \( l = 1 \), cyclotomic DAHA essentially appeared in [BF].

We also connect cyclotomic DAHA with multiplicative quiver and bow varieties. Namely, we show that the spherical cyclotomic DAHA \( e\text{HH}^l_N(Z,1,t)\) is commutative, and its spectrum for generic parameters is isomorphic to the algebra of regular functions on the multiplicative quiver variety for the cyclic quiver of length \( l \) with dimension vector \((N,\ldots,N)\); hence \( e\text{HH}^l_N(Z,q,t)\) is a quantization of this variety. In particular, we show that this multiplicative quiver variety is connected, and that \( \text{HH}^l_N(Z,1,t) \) is an Azumaya algebra of degree \( N! \) over this variety. We also show that if \( t \) is not a root of unity then the algebra \( e\text{HH}^l_N(Z,1,t)\) is an integrally closed Cohen-Macaulay domain isomorphic to the center \( Z(\text{HH}^l_N(Z,1,t)) \), while \( \text{HH}^l_N(Z,1,t)\) is a Cohen-Macaulay module over this algebra.

Finally, we provide some applications of cyclotomic DAHA to the theory of quasi-invariants. Namely, we show that natural \( q \)-deformations of various classes of spaces of quasi-invariants are flat, and therefore free modules over the algebra of symmetric polynomials. We also introduce a new type of quasi-invariants (namely, twisted quasi-invariants) and their \( q \)-deformation, and prove the freeness property for them.

We note that the degenerate cyclotomic DAHA were studied in a way similar to ours by R. Kodera and H. Nakajima in [KN]. In fact, their paper was one of the starting points for our work.

The paper is organized as follows. In Section 2 we develop the theory of partially spherical cyclotomic rational Cherednik algebras as subalgebras in trigonometric (degenerate) DAHA, and give their presentation. In Section 3 we define cyclotomic DAHA as subalgebras of DAHA, and study their properties. We also give a presentation of cyclotomic DAHA, which allows us to find various bases in them and prove flatness results. In Section 4 we give a geometric description of cyclotomic DAHA and their degenerate versions in terms of equivariant K-theory and Borel-Moore homology, and apply it to proving flatness of these algebras. In Section 5 we relate the spherical subalgebra of cyclotomic DAHA at \( q = 1 \) with certain multiplicative quiver and bow varieties; the latter are isomorphic to the \( K \)-theoretic Coulomb branch of framed quiver gauge theories of affine type \( A \). We also study the properties of the spectrum of the spherical cyclotomic DAHA for \( q = 1 \). In Section 6 we give applications of cyclotomic DAHA to proving flatness of \( q \)-deformation of various spaces of quasi-invariants. Finally, Appendix A, written by H. Nakajima and D. Yamakawa, explains the relations between multiplicative bow varieties and (various versions of) multiplicative quiver varieties for a cyclic quiver.

Acknowledgements. The work of P.E. was partially supported by the NSF grant DMS-1502244. The work of M.F. has been partially funded by the Russian Academic Excellence Project '5-100'. We are grateful to J. Stokman for useful discussions and reference [BF]; to O. Chalykh for his comments cited in Remarks 3.26, 5.19 to J. F. van
Diejen and S. Ruijsenaars for Remark 3.25; to E. Rains for explaining the connection with [GKV] and [H1, H2] (Remark 3.12); to B. Webster for sharing [W] with us prior to its publication and explaining its results (see Remark 2.21); to J. Kamnitzer for explaining the KLR-type construction of the convolution algebras of Section 4.1 to us; and to H. Nakajima for explaining to us the results of [KN] (see Remark 2.21). Also, Sections 5.4 and 5.5 are due to H. Nakajima’s patient explanations.

2. Degenerate cyclotomic DAHA

2.1. Notation. In this paper, we will consider many different algebras depending on parameters. So let us clarify our conventions.

First of all, if an algebra depends on parameters, we will list the parameters explicitly when they are given numerical values, and omit them when they are indeterminates (i.e., we work over a commutative base algebra generated by them). Also, throughout the paper, we will use the following notation, to be defined below.

- $\mathbb{H}^{\text{deg}}_{N}(\hbar, k)$: the degenerate (or trigonometric) DAHA, Definition 2.1
- $\mathbb{H}^{\text{ cyc}}_{N}(c, \hbar, k)$, $c := (c_0, ..., c_{l-1})$: the cyclotomic rational Cherednik algebra for the group $S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$, Definition 2.16
- $\mathbb{H}^{\text{psc}}_{N}(c, \hbar, k)$ := $p\mathbb{H}^{\text{ cyc}}_{N}(c, \hbar, k)p$: the partially spherical cyclotomic rational Cherednik algebra, Subsection 2.8
- $\mathbb{H}^{\text{rat}}_{N}(\hbar, k)$, the rational Cherednik algebra for $S_N$, Example 2.17
- $\mathbb{H}^{\text{deg}}_{N, deg}$: the geometric version of the degenerate cyclotomic DAHA, $H^{C^x \times T(W)_{o} \times \mathbb{P} \times C^x}(\mathcal{R})$, Section 4
- $\mathbb{H}^{\text{deg}}_{N}$: the geometric version of the cyclotomic DAHA, $K^{C^x \times T(W)_{o} \times \mathbb{P} \times C^x}(\mathcal{R})$, Section 4

2.2. Degenerate (trigonometric) DAHA. Let $\hbar, k$ be variables.

Definition 2.1. The degenerate (or trigonometric) double affine Hecke algebra (DAHA) $\mathbb{H}^{\text{deg}}_{N}$ is generated over $\mathbb{C}[\hbar, k]$ by $\pi^{\pm 1}, s_0, ..., s_{N-1}, y_1, ..., y_N$ with defining relations

\[
\begin{align*}
\pi_i^2 &= 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i}, \quad \pi_i s_j = s_js_i \text{ if } i - j \neq \pm 1, \quad \pi_i s_i = s_{i+1}\pi_i, \\
[y_i, y_j] &= 0, \quad \pi y_i = y_{i+1}\pi \text{ if } i \neq N, \quad \pi y_N = (y_1 - \hbar)\pi, \\
\pi s_i y_i = y_{i+1}s_i + k, & \quad \text{if } i \neq 0, \quad s_0y_N = (y_1 - \hbar)s_0 + k,
\end{align*}
\]
\[ [s_i, y_j] = 0 \text{ if } i - j \neq \pm 1, \]

where addition is mod \( N \).

**Proposition 2.2.** The algebra \( \mathcal{HH}_{N,\deg} \) is generated by \( S_N \ltimes \mathbb{Z}^N \) (generated by \( s_i \) and invertible commuting elements \( X_1, \ldots, X_N \)) and elements \( y_1, \ldots, y_N \) with commutation relations

\[
\begin{align*}
s_i y_i &= y_{i+1} s_i + k, \\
[y_i, y_j] &= 0, \\
[y_i, X_j] &= k X_j s_{ij}, \quad i > j, \\
[y_i, X_j] &= k X_i s_{ij}, \quad i < j, \\
[y_i, X_i] &= h X_i - k \sum_{r<i} X_r s_{ir} - k \sum_{r>i} X_i s_{ir} 
\end{align*}
\]

(and the relations of \( S_N \ltimes \mathbb{Z}^N \)). Namely, the transition between the two definitions is given by the formulas

\[ \pi = X_1 s_1 \ldots s_{N-1}, \quad s_0 = X_N^{-1} X_1 s_{1N}. \]

**Proof.** The proposition is standard, and the proof is by a direct computation; see e.g. [Su], Section 2. \( \square \)

**Remark 2.3.** The commutation relations between \( y_i \) and \( X_j \) in Proposition 2.2 can be replaced by the relations

\[
(2.1) \quad [y_i, \prod_j X_j] = h \prod_j X_j \quad \text{or} \quad \sum_i [y_i, X_j] = h X_j,
\]

and

\[
(2.2) \quad [y_2, X_1] = k X_1 s_1.
\]

Indeed, the relation for \( [y_i, X_j], i \neq j \) can be obtained from (2.2) by the action of \( S_N \), and then the relation for \( [y_i, X_i] \) can be obtained by using one of the relations (2.1).

Note that \( \mathcal{HH}_{N,\deg} \) is a bigraded algebra: \( \deg(X_i) = (1, 0), \deg(y_i) = \deg(h) = \deg(k) = (0, 1), \deg(s_i) = (0, 0) \), and the homogeneous spaces \( \mathcal{HH}_{N,\deg}[r, s] \) are finite dimensional. Moreover, the PBW theorem for the degenerate DAHA, which follows from the existence of its polynomial representation (see Subsection 2.3) implies that \( \mathcal{HH}_{N,\deg} \) is a free bigraded module over \( \mathbb{C}[h, k] \).

Also, we have the specialization \( \mathcal{HH}_{N,\deg}(h, k) \) of the degenerate DAHA at \( h, k \in \mathbb{C} \), which is defined by the same generators and relations, where \( h, k \) are numerical. The bigraded on \( \mathcal{HH}_{N,\deg} \) induces a grading on this specialization defined by \( \deg(X_i) = 1 \), \( \deg(y_i) = \deg(s_j) = 0 \) and an increasing filtration \( F^* \) compatible with this grading, defined by \( \deg(X_i) = \deg(s_j) = 0, \deg(y_i) = 1 \).
2.3. The polynomial representation. Let $D_i$ be the rational Dunkl operators

$$D_i := \hbar \partial_i - \sum_{j \neq i} \frac{k}{X_i - X_j}(1 - s_{ij}),$$

where $\partial_i$ is the derivative with respect to $X_i$, and $s_{ij}$ is the permutation of $i$ and $j$ (so $s_i = s_{i,i+1}$). Define the trigonometric Dunkl operators by the formula

$$D_{i}^{\text{trig}} := X_i D_i - k \sum_{j < i} s_{ij}.$$

The following well known proposition is due to Cherednik ([Ch2]; see also [Su], Proposition 3.1).

**Proposition 2.4.** We have a representation $\rho$ of $\mathbb{H}H_{N,\deg}(\hbar, k)$ on $P := \mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}]$, defined by

$$\rho(s_i) = s_i \text{ for } i \neq 0, \quad \rho(\pi) = X_1 s_1 \ldots s_{N-1},$$

$$\rho(s_0) = s_{1N} X_1^{-1} X_N,$$

$$\rho(y_i) = D_{i}^{\text{trig}}.$$

**Proof.** The relations involving only $\pi$ and $s_i$ are easy. Let us prove that $[\rho(y_i), \rho(y_m)] = 0$ for $i < m$. Using that $[D_i, X_m] = k s_{im}$ and $[D_i, D_m] = 0$, we get

$$[\rho(y_i), \rho(y_m)] = k X_i s_{im} D_m - k X_m s_{im} D_i - k [X_i D_i, s_{im}] + k^2 \sum_{i < j} [s_{ij}, s_{im} + s_{jm}].$$

The first three summands cancel, and the last summand is zero, as desired.

Let us prove the commutation relations between $\rho(\pi)$ and $\rho(y_i)$. For $i < N$ we have

$$\rho(\pi) \rho(y_i) = X_1 s_1 \ldots s_{N-1} (X_i D_i - k \sum_{j < i} s_{ij}) =$$

$$X_1 (X_{i+1} D_{i+1} - k \sum_{j < i} s_{j+1,i+1}) s_1 \ldots s_{N-1} =$$

$$(X_{i+1} D_{i+1} - k \sum_{j < i} s_{j+1,i+1}) X_1 s_1 \ldots s_{N-1} - k X_{i+1} s_{1,i+1} s_1 \ldots s_{N-1} = \rho(y_{i+1}) \rho(\pi).$$

Also $\rho(\pi^n) = X_1 \ldots X_n$, so $\rho(\pi^n) \rho(y_i) = (\rho(y_i) - \hbar) \rho(\pi^n)$, which implies that the relation $\pi y_N = (y_1 - \hbar) \pi$ is preserved.

Let us show that the relation $s_i y_i = y_{i+1} s_i + k$ for $i \neq 0$ is preserved. We have

$$\rho(s_i) \rho(y_i) = s_{i,i+1} (X_i D_i - k \sum_{j < i} s_{ij}) =$$

$$(X_{i+1} D_{i+1} - k \sum_{j < i} s_{i+1,j}) s_{i,i+1} = \rho(y_{i+1}) \rho(s_i) + k.$$
The relation \( s_0 y_N - (y_1 - h) s_0 + k \) is obtained from the previous relation by conjugation by \( \pi \). Finally, the relation \([s_i, y_j] = 0\) for \( i - j \neq \pm 1\) is easy for \( i \neq 0\), and for \( i = 0\) is obtained from the case \( i = 1\) by conjugation by \( \pi \). The proposition is proved. \( \square \)

**Definition 2.5.** The representation \( \mathbf{P} \) of \( \mathcal{H}_N(h, k) \) is called the polynomial representation.

It is easy to see that the polynomial representation is faithful when \( h \neq 0\). Moreover, replacing \( h \partial_i \) with momentum variables \( p_i \), we can make it faithful in the limit \( h = 0 \) (see [EM], 2.10).

Let \( \mathcal{D}_N \) be the algebra of differential operators in \( X_1, ..., X_N \) with poles at \( X_i = 0 \) and \( X_i = X_j \). Then the algebra \( \mathbb{C} S_N \ltimes \mathcal{D}_N \) acts naturally on \( \mathbf{P}[\Delta^{-1}] \), where

\[
\Delta := \prod_{i<j}(X_i - X_j),
\]

and \( \rho \) defines an inclusion

\[
\mathcal{H}_N(h, k) \hookrightarrow \mathbb{C} S_N \ltimes \mathcal{D}_N.
\]

We will use this inclusion to view \( \mathcal{H}_N(h, k) \) as a subalgebra of \( \mathbb{C} S_N \ltimes \mathcal{D}_N \). Note that the filtration \( F^* \) on \( \mathcal{H}_N(h, k) \) is induced under this inclusion by the order filtration on differential operators.

Let \( \psi_{\kappa} \) be the automorphism of the algebra \( \mathbb{C} S_N \ltimes \mathcal{D}_N \) fixing \( X_i \) and sending \( s_i \) to \( -s_i \) and \( \partial_i \) to \( \partial_i + \sum_{j \neq 1} \frac{\kappa}{X_i - X_j} \), i.e., conjugation by \( |\Delta|^{\kappa} \text{sign}(\Delta) \) on the real locus.

**Lemma 2.6.** The algebra \( \psi_{2k}(\mathcal{H}_N(h, k)) \subset \mathbb{C} S_N \ltimes \mathcal{D}_N \) preserves \( \mathbf{P} \). In other words, \( \mathcal{H}_N(h, k) \) preserves \( |\Delta|^{2k\kappa} \text{sign}(\Delta) \mathbf{P} \).

**Proof.** We need to check that

\[
\psi_{2k}(D_1) = D_1 - \sum_{j \neq 1} \frac{2k}{X_1 - X_j} = \\
\partial_1 - \sum_{j \neq 1} \frac{k}{X_1 - X_j} (1 + s_{ij}) + \sum_{j \neq 1} \frac{2k}{X_1 - X_j} = \partial_1 + \sum_{j \neq 1} \frac{k}{X_1 - X_j} (1 - s_{ij})
\]

preserves \( \mathbf{P} \), which is straightforward. \( \square \)

Let \( e = \frac{1}{N!} \sum_{s \in S_N} s \) be the symmetrizer of \( S_N \). The algebra \( e \mathcal{H}_N(h, k) e \) is called the spherical subalgebra of \( \mathcal{H}_N(h, k) \); it has the polynomial representation \( e \mathbf{P} = \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}] \mathbb{S}_N \). Let \( H := e(\sum_i y_i^2) e \in e \mathcal{H}_N(h, k) e \). This element acts on the polynomial representation by the trigonometric Calogero-Moser Hamiltonian.

**Lemma 2.7.** The algebra \( A := e \mathcal{H}_N(h, k) e \) is generated by \( H \) and \( \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}] \mathbb{S}_N \).
Proof. The algebra $A$ has a filtration $F^\bullet$ given by $\deg(X_i) = \deg(s_j) = 0$, $\deg(y_i) = 1$, and $\text{gr}(A) = \mathbb{C}[X_i^{\pm 1}, \ldots, X_N^{\pm 1}, p_1, \ldots, p_N]^{S_N}$ with Poisson bracket corresponding to the symplectic form $\sum_i dp_i \wedge dX_i/X_i$. Since the symbol of $H$ is $H_0 := \sum_i p_i^2$, it suffices to check that $\text{gr}(A)$ is Poisson generated by $\mathbb{C}[X_i^{\pm 1}, \ldots, X_N^{\pm 1}]^{S_N}$ and $H_0$. For this it suffices to show that the Poisson algebra $B$ generated by these elements contains $F_{r,s} := \sum_i X_i^r p_i^s$, where $r$ is any integer and $s$ a nonnegative integer, as such elements generate $\text{gr}(A)$ as a commutative algebra (by a theorem of H. Weyl). The Poisson bracket is generated by the same elements.

We only need to show that $\text{gr}(A)$ is Poisson generated by $\mathbb{C}[X_i^{\pm 1}, \ldots, X_N^{\pm 1}]^{S_N}$ and $H_0$. For this, it suffices to note that

$$F_{r,s} = \frac{1}{2r} \{ \sum_i p_i^2, F_{r,s-1} \};$$

and $F_{r,0} = \sum_i X_i^r \in \mathbb{C}[X_i^{\pm 1}, \ldots, X_N^{\pm 1}]^{S_N} \subset B$. So it remains to show that $F_{0,s} \in B$. For this, it suffices to note that

$$F_{0,s} = \frac{1}{s+1} \{ F_{-1,s+1}, F_{1,0} \}.$$

\[
\square
\]

2.4. Degenerate cyclotomic DAHA.

Definition 2.8. Let $l \in \mathbb{Z}_{\geq 0}$, $z_1, \ldots, z_l \in \mathbb{C}$, and $z = (z_1, \ldots, z_l)$. The degenerate cyclotomic DAHA is the subalgebra $HH_{N,\deg}(z, h, k)$ of $HH_{N,\deg}(h, k)$ generated by $s_i$, $i = 1, \ldots, N-1$, $y_i$, $i = 1, \ldots, N$, $\pi$, and the element

$$\pi_- := \pi^{-1} \prod_{i=1}^l (y_i - z_i).$$

Similarly, if $z_i$, $h$, and $k$ are variables, we define $HH_{N,\deg}^l$ to be the subalgebra of $HH_{N,\deg}[z_1, \ldots, z_l]$ generated by the same elements.

Note that by this definition

$$HH_{N,\deg}^0(h, k) = HH_{N,\deg}(h, k), \quad HH_{N,\deg}^l(z, h, k) \subset HH_{N,\deg}^l(z, h, k) \quad \text{if } l' \geq l$$

and $z' \supset z$ as a multiset.

For $u \in \mathbb{C}$, let $\phi_u$ be the automorphism of $\mathbb{C}S_N \rtimes D_N$ which preserves $X_i, s_i$, and sends $\partial_i$ to $\partial_i + uX_i^{-1}$ (i.e., conjugation by $(\prod_i X_i)^u$).

Proposition 2.9. The algebras $\phi_{z_i}(HH_{N,\deg}^l(z, 1, k))$, $i \in [1, l]$, preserve the subspace $P_+ := \mathbb{C}[X_1, \ldots, X_N] \subset P$. In other words, the algebra $HH_{N,\deg}^l(z, 1, k)$ preserves $(\prod_j X_j)^s P_+$ for all $i$.

Proof. We only need to show that $\pi^{-1} \phi_{z_i}((y_1 - z_1) \ldots (y_l - z_l))$ preserves $P_+$. Note that $y_1 = X_1 D_1$, so $\phi_{z_i}(y_1) = y_1 + z_i$. Thus, we need to check that $\pi^{-1} y_1 \prod_{j \neq i} (y_1 - z_j + z_l)$ preserves $P_+$. But this holds since $\pi^{-1} y_1 = s_{N-1} \ldots s_1 D_1$ preserves $P_+$ (and, of course, so does $y_1$). \[
\square
\]
A similar result holds when \( z_i, k \) are variables.

**Theorem 2.10.** (i) \( HH_{N,\text{deg}}(1, k) \) is the algebra of all elements of the algebra \( S_N \ltimes D_N \) which preserve \( P \) and \(|\Delta|^{2k}\text{sign}(\Delta)P\).

(ii) Suppose \( z_i - z_j \) are not integers and \( k \in \mathbb{C} \) is Weil generic \( \mathbb{C} \) (i.e., outside a countable set). Then the algebra \( HH_{N,\text{deg}}(z, 1, k) \) is the subalgebra of all elements of \( HH_{N,\text{deg}}(1, k) \) which preserve \((\prod_j X_j)^j P_+\) for all \( i \).

(iii) Under the assumption of (ii), the algebra \( HH_{N,\text{deg}}(z, 1, k) \) is the subalgebra of all elements of \( S_N \ltimes D_N \) which preserve \( P \), \(|\Delta|^r\text{sign}(\Delta)P\), and \((\prod_j X_j)^j P_+\) for all \( i \).

Theorem [2.10] is proved in the next two subsections.

2.5. **The case \( N = 1 \).** Let us first prove Theorem [2.10] for \( N = 1 \). Let \( B_l(z) := HH_l(1, z) \) (the parameter \( k \) does not enter in this case). The algebra \( B_l(z) \) is generated by \( X = X_1 \), the Euler element \( E := X\partial = y_1 \), and

\[
L := \pi_\ldots = X^{-1}(X\partial - z_1)...(X\partial - z_l).
\]

Then Theorem [2.10] for \( N = 1 \) reduces to the following statement.

**Proposition 2.11.** If \( z_i - z_j \) are not integers for \( i \neq j \) then \( B_l(z) \) is the algebra of all differential operators on \( \mathbb{C}^* \) which preserve \( X^{z_i}\mathbb{C}[X] \) for all \( i = 1, \ldots, l \).

**Proof.** It is clear that every element of \( B_l(z) \) preserves \( X^{z_i}\mathbb{C}[X] \), since so do \( X, E \) and \( L \). So we need to show that any operator \( M \) which preserves \( X^{z_i}\mathbb{C}[X] \) belongs to \( B_l(z) \).

Suppose that \( M \) is of degree \( r \), i.e., \( M = X^r g(E) \), where \( g \) is some polynomial. If \( r \geq 0 \), then \( M \in B_l(z) \), as \( E \in B_l(z) \). So it remains to consider the case \( r = -q \), where \( q > 0 \). Then \( M \) has to annihilate \( X^{z_i}, \ldots, X^{z_{i+q-1}} \) for all \( i \) (which are all different thanks to the condition on the \( z_i \)), so the degree of \( g \) is at least \( ql \). On the other hand, if we take \( g(y) = P(y)P(y - 1)...P(y - q + 1)h(y) \) where \( P(y) := (y - z_1)...(y - z_l) \), then \( M := X^{-q}g(E) = L^qh(E) \), so \( M \in B_l(z) \). This shows that \( M \) always belongs to \( B_l(z) \) (as we can subtract \( L^qh(E) \) to make the degree of the polynomial \( < ql \)). The proposition is proved.

**Remark 2.12.** We have \( E = XL + z_1 \) for \( l = 1 \) and \( E = \frac{1}{2}([L, X] + z_1 + z_2 - 1) \) for \( l = 2 \), so one may ask if \( B_l(z) \) is in fact generated by \( L, X \) (i.e., if \( E \) can be expressed via \( L, X \)). It is not hard to show that this is indeed the case for generic \( z_i \). But if \( l \geq 3 \) and \( z_i \) are special, then the algebra \( B_l(z) \) may not be generated by \( X, L \). Indeed, let \( z_1 = 0, z_2 = 1, z_3 = 2 \), and the other \( z_i \) be arbitrary. Then \( B_l(z) \) preserves \( \mathbb{C}[X] \) and \( X^2\mathbb{C}[X] \), so has a 2-dimensional representation \( V = \mathbb{C}[X]/(X^2) \) where \( L \) acts by 0 and \( X \) acts nilpotently. Hence, if \( B'_l(z) \subset B_l(z) \) is the subalgebra generated by \( L, X \) then every element of \( B'_l(z) \) has only one eigenvalue on \( V \). On the other hand, \( E \) has eigenvalues 0, 1 on \( V \), so \( E \notin B'_l(z) \).

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1We don’t know if Theorem [2.10] (ii),(iii) actually fails for any value of \( k \).
We will also need the following “unsymmetrized” version of Proposition 2.11 for \( l = 2 \). Let \( A(k) \) be the rational Cherednik algebra with parameter \( k \) attached to the group \( \mathbb{Z}/2\mathbb{Z} \), i.e., generated by \( x, s \in \mathbb{Z}/2\mathbb{Z} \) such that \( sx = -xs \), and \( D = \partial - \frac{k}{x}(1-s) \).

**Lemma 2.13.** \( A(k) \) is the algebra of all elements of \( \mathbb{C}Z/2\mathbb{Z} \rtimes \mathcal{D}(\mathbb{C}^*) \) which preserve \( \mathbb{C}[x] \) and \( |x|^{2k}\text{sign}(x)\mathbb{C}[x] \).

**Proof.** It is clear that \( A(k) \) preserves these spaces, since so do its generators. So it remains to show that any element \( M \) preserving these spaces is in \( A(k) \). We may assume that \( M \) is homogeneous. Note that \( E := \frac{1}{2}(xD + Dx - 1 + 2k) = x\partial \in A(k) \).

So if \( M \) is of nonnegative degree, then \( M = x^r(g_1(E)(1-s) + g_2(E)(1+s)) \), where \( g_1, g_2 \) are some polynomials, hence \( M \in A(k) \) automatically. Now suppose \( \text{deg}(M) = -q < 0 \), i.e., \( M = x^{-q}(g_1(E)(1-s) + g_2(E)(1+s)) \). The operator \( M \) has to annihilate \( x^m \) and \( \text{sign}(x)|x|^{2k}x^m \) for \( 0 \leq m \leq q-1 \), so \( g_1(u) \) is divisible by \( \prod_{j=0}^{q-1}(u-j - (1 + (-1)^j)k) \), while \( g_2(u) \) is divisible by \( \prod_{j=0}^{q-1}(u-j - (1 - (-1)^j)k) \). On the other hand, it is easy to see by acting on monomials \( x^p \) that

\[
x^{-q}\prod_{j=0}^{q-1}(E - j - (1 + (-1)^j)k) \cdot (1-s) = D^q(1-s),
\]

while

\[
x^{-q}\prod_{j=0}^{q-1}(E - j - (1 - (-1)^j)k) \cdot (1+s) = D^q(1+s),
\]

which are both in \( A(k) \). This implies the lemma. \( \square \)

2.6. **Proof of Theorem 2.10.** Let us prove (i). By Lemma 2.13 and Lemma 2.6 \( HH_{N,\text{deg}}(1, k) \) is the space of elements of \( S_N \rtimes \mathcal{D}_N \) which upon formal completion at a generic point of each hyperplane \( X_i = X_j \) (in the sense of [BE]) lie in the formal completion of \( A(k) \otimes \mathbb{W}_{N-1} \), where \( \mathbb{W}_{N-1} \) is the Weyl algebra of \( N-1 \) variables. So (i) follows from the results of [BE] (see also the appendix to [BE]).

It is clear that given (i), statements (ii) and (iii) are equivalent, so let us prove (ii) (assuming \( l > 0 \), as the case \( l = 0 \) is trivial). First consider the case \( k = 0 \). In this case, the result follows from the following lemma.

**Lemma 2.14.** Let \( L \in S_N \rtimes \mathcal{D}(\mathbb{C}^*)^{\otimes N} \) be an element preserving the space \((\prod_j X_j)^{\sigma_1} P_+\) for all \( i \). Then \( L \in S_N \rtimes B_{l}(z)^{\otimes N} \).

**Proof.** Let \( L = \sum_{\sigma \in S_N} \sigma L_\sigma \), where \( L_\sigma \) are differential operators. Consider a generic point \( x \) in the hyperplane \( X_1 = 0 \), and let \( E_i \) be the formal completion of the \( \mathbb{C}[X_1, ..., X_N] \)-module \((\prod_j X_j)^{\sigma_1} P_+\) near the \( S_N \)-orbit of \( x \). Then \( E_i = \oplus_{\sigma \in S_N} E_{i,\sigma} \), where \( E_{i,\sigma} \) is the completion of \((\prod_j X_j)^{\sigma_1} P_+\) at the point \( x \). It is clear that \( L \) preserves \( E_i \) for all \( i \). This implies that for each \( \sigma \), \( L_\sigma \) preserves \( E_{i,1} \). Hence \( L_\sigma \) preserves \((\prod_j X_j)^{\sigma_1} P_+\) for all \( i \) and \( \sigma \). Thus we may assume without loss of generality that \( L \in \mathcal{D}(\mathbb{C}^*)^{\otimes N} \) is a differential operator.
It is clear from taking completions that $L$ preserves the space $X_i^* \mathbb{C}[X_1, X_2, \ldots, X_N]$ for all $i$. Therefore, by Proposition 2.11 for any $v \in \mathbb{C}[X_2^* \mathbb{C}, \ldots, X_N^* \mathbb{C}]$, $\psi \in \mathbb{C}[X_2^* \mathbb{C}, \ldots, X_N^* \mathbb{C}]^*$ the differential operator $(\text{Id} \otimes \psi)(L(\text{Id} \otimes v)) \in \mathcal{D}(\mathbb{C}^*)$ in fact belongs to $B_l(z)$. Let $\{a_i\}$ be a basis of $B_l(z)$, and $\{a_j'\}$ its extension to a basis of $\mathcal{D}(\mathbb{C}^*)$. We can uniquely write $L$ as

$$L = \sum_i a_i \otimes L_i + \sum_j a_j' \otimes L_j'$$

Thus we have

$$\sum_i \psi(L_i v) a_i + \sum_j \psi(L_j' v) a_j' \in B_l(z).$$

Hence $\psi(L_j' v) = 0$ for all $j$. Since this holds for all $\psi$, we have $L_j' v = 0$. Since this holds for all $v$, we have $L_j' = 0$. Thus $L = \sum_i a_i \otimes L_i \in B_l(z) \otimes \mathcal{D}(\mathbb{C}^*)^{\otimes N-1}$, i.e., the first component of $L$ lies in $B_l(z)$. A similar argument applies to all the other components. Thus, $L \in B_l(z)^{\otimes N}$, as desired. \qed

Now, consider the case $k \neq 0$. By Proposition 2.9 elements of $\mathbb{H}_{N, \text{deg}}^l(1, k)$ preserve the spaces $(\prod_j X_j)^* \mathbb{P}_+$. Thus we only have to show that $\mathbb{H}_{N, \text{deg}}^l(1, k)$ is “big enough”, i.e., coincides with the algebra $A_N(z, k)$ of all the elements preserving the spaces $(\prod_j X_j)^* \mathbb{P}_+$. But this follows for Weil generic $k$ from the case $k = 0$ by a standard deformation argument (the algebra can only “get bigger” if we deform its generators). More precisely, recall that we have a grading on the algebra $S_N \ltimes \mathcal{D}_N$ defined by $\text{deg}(X_i) = 1$, $\text{deg}(s_i) = 0$, $\text{deg}(\partial_i) = -1$ inherited by $\mathbb{H}_{N, \text{deg}}^l(1, k)$, $A_N(z, k)$, and $\mathbb{H}_{N, \text{deg}}^l(z, 1, k)$, and the filtration $F^*$ by order of differential operators, and it is not hard to see that for each $s$, $F^* A_N(z, k)$ is a finitely generated graded $\mathbb{C}[X_1, \ldots, X_N]$-module. Thus, for each $r, s$, the space $F^s A_N(z, k)[r]$ is finite dimensional. Hence, for each $r, s$ the set of $k \in \mathbb{C}$ for which $F^s \mathbb{H}_{N, \text{deg}}^l(z, 1, k)[r] \neq F^s A_N(z, k)[r]$ is finite.

2.7. Comparison to the cyclotomic rational Cherednik algebra for $N = 1$. Let us now consider the cyclotomic rational Cherednik algebra $\mathbb{H}^{l, \text{cyc}}_1$ of rank 1 with parameters $\hbar$ and $c = (c_0, \ldots, c_{l-1})$ (see [EG2, EM]). By definition, this algebra is generated over $\mathbb{C}[\hbar, c_0, \ldots, c_{l-1}]$ by $x$ and the cyclotomic Dunkl operator

$$D_{\text{cyc}} := \hbar \partial - x^{-1} \sum_{i=0}^{l-1} c_i \sigma^i,$$

where $\sigma(x) = \zeta x$, $\zeta = e^{2\pi i/l}$. This algebra is bigraded by $\text{deg}(x) = (1, 0)$, $\text{deg}(D_{\text{cyc}}) = (-1, 1)$, $\text{deg}(\sigma) = (0, 0)$, $\text{deg}(\hbar) = \text{deg}(c) = (0, 1)$, and by the PBW theorem (see [DO, EG1]) is a free bigraded module over $\mathbb{C}[c, \hbar]$. We also have the algebra $\mathbb{H}^{l, \text{cyc}}_1(c, \hbar)$ with numerical parameters, i.e., the specialization of $\mathbb{H}^{l, \text{cyc}}_1$, and it carries a grading and a compatible filtration $F^*$. 

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We have $\mathbb{H}^{l,cyc}(c, 1) \subset \mathbb{C}Z/l\mathbb{Z} \rtimes \mathcal{D}$, where $\mathcal{D} = D_1$ is the algebra of differential operators on $\mathbb{C}^*$, and the filtration $F^\bullet$ on $\mathbb{H}^{l,cyc}(c, 1)$ is induced by the order filtration on differential operators.

Let $p$ be the symmetrizer of $\mathbb{Z}/l\mathbb{Z}$. Then we have a spherical subalgebra $B_l(c) := p\mathbb{H}^{l,cyc}_1(c, 1)p$. This algebra acts naturally on $\mathbb{C}[x^{\pm 1}]$, where $X = x^l$.

**Proposition 2.15.** If $z_i - z_j$ are not integers then $B_l(z) = B_l(c)$ inside $\text{End}\mathbb{C}[x^{\pm 1}]$, where $c_i$ are related to $z_j$ by the linear inhomogeneous change of variables

$$z_i = \frac{1}{l}(l - i + \sum_j c_j \zeta^{ij}).$$

**Proof.** Fix $c$ and let us find values of $u$ for which $X^u \mathbb{C}[X^{\pm 1}]$ is preserved by $B_l(c)$. The condition is that there exists $1 \leq i \leq l$ such that $D_{cyc}(x^{i-l}(x^l)^u) = 0$, where we treat $(x^l)^u$ as a $\mathbb{Z}/l\mathbb{Z}$-invariant. This gives the equation

$$i - l + lu - \sum_j c_j \zeta^{ij} = 0,$$

i.e.

$$u = z_i := \frac{1}{l}(l - i + \sum_j c_j \zeta^{ij}).$$

This yields the desired change of variable.

Now, $B_l(c)$ preserves the subspaces $X^{z_i}\mathbb{C}[X^{\pm 1}]$, so by Proposition 2.11 we have an inclusion $B_l(c) \subset B_l(z)$. To show that this is actually an equality, it suffices to observe that the element $L$ of $B_l(z)$ is proportional to $D_{cyc}^l p$.

### 2.8. Comparison to cyclotomic Cherednik algebra for general $N$.

Let us now extend the result of the previous subsection to general $N$.

**Definition 2.16.** The cyclotomic rational Cherednik algebra for the group $S_N \rtimes (\mathbb{Z}/l\mathbb{Z})^N$, $\mathbb{H}^{l,cyc}_N$, is the algebra generated over $\mathbb{C}[c_0, \ldots, c_{l-1}, \hbar, k]$ by the group $S_N \rtimes (\mathbb{Z}/l\mathbb{Z})^N$, elements $x_i$, and the cyclotomic Dunkl operators, also called Dunkl-Opdam operators, [DO, Definition 3.2]:

$$D_{i,cyc} = \hbar \partial_i - \frac{1}{x_i} \sum_{j=0}^{l-1} c_j \sigma_i^j - k \sum_{r \neq i, m} \frac{1}{x_i - \zeta^m x_r} (1 - s_{ir} \sigma_i^m \sigma_r^{-m}),$$

for $i = 1, \ldots, N$, where $\sigma_i$ is $\sigma$ acting in the $i$-th component.

As in the rank 1 case, this algebra is bigraded by $\text{deg}(x_i) = (1, 0)$, $\text{deg}(D_{i,cyc}) = (-1, 1)$, $\text{deg}(\sigma_i) = (0, 0)$, $\text{deg}(\hbar) = \text{deg}(c) = \text{deg}(k) = (0, 1)$, and by the PBW theorem (see [DO]) it is a free bigraded module over $\mathbb{C}[c, \hbar, k]$. We also have the algebra $\mathbb{H}^{l,cyc}_N(c, \hbar, k)$ with numerical parameters, i.e., the specialization of $\mathbb{H}^{l,cyc}_N$, and it carries a grading and a compatible filtration $F^\bullet$. 

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Let $p$ be the symmetrizer of the subgroup $(\mathbb{Z}/l\mathbb{Z})^N$, and $\mathbb{H}^{l,psc}_N(c, h, k) = p\mathbb{H}^{l,cyc}_N(c, h, k)p$ be the corresponding partially spherical subalgebra.

**Example 2.17.** Let $l = 1$. Then $\mathbb{H}^{l,psc}_N(c, h, k)$ does not depend on $c$ (up to a natural isomorphism), and is the rational Cherednik algebra $\mathbb{H}^{l,cyc}_N(h, k)$, generated by $X_i, D_i$, and $s \in S_N$ such that $X_i, D_i$ are permuted by $S_N$ and satisfy the relations

$$[X_i, X_j] = [D_i, D_j] = 0,$$

$$[D_i, X_j] = ks_{ij}, \quad [D_i, X_i] = h - k \sum_{j \neq i} s_{ij}.$$

Also in this case $p = 1$.

**Theorem 2.18.** Suppose $z_i - z_j$ are not integers and $k$ is Weil generic. Then we have a natural isomorphism $\mathbb{H}^{l,psc}_N(c, 1, k) \cong \mathbb{H}^{l}_{N,deg}(z, 1, k)$, where $z_i$ are expressed via $c_j$ by formula (2.3). This isomorphism preserves the order filtration for differential operators.

**Proof.** We have a natural faithful action of $\mathbb{H}^{l,psc}_N(c, 1, k)$ on $P$. Moreover, it is easy to see that $\mathbb{H}^{l,psc}_N(c, 1, k)$ satisfies the conclusion of Theorem [2.10](iii) (this follows by taking formal completions at generic points of reflection hyperplanes, as in [E1], and using Proposition [2.15]). Therefore, by Theorem [2.10] $\mathbb{H}^{l,psc}_N(c, 1, k) = \mathbb{H}^{l}_{N,deg}(z, 1, k)$. \hfill $\Box$

One of our main results is the following theorem.

**Theorem 2.19.** (i) We have a natural isomorphism of bigraded algebras $\theta : \mathbb{H}^{l,psc}_N \cong \mathbb{H}^{l}_{N,deg}$, where $z_i$ are expressed via $c_j$ and $h$ by the homogenization of formula (2.3):

$$(2.4) \quad z_i = \frac{1}{l}(h(l - i) + \sum_j c_j \xi^{ij}).$$

(ii) $\mathbb{H}^{l}_{N,deg}$ is a free bigraded $\mathbb{C}[z_1, ..., z_i, h, k]$-module, and $\mathbb{H}^{l}_{N,deg}(z, h, k)$ are specializations of this algebra.

(iii) The isomorphism $\theta$ induces an isomorphism $\theta_{z, h, k} : \mathbb{H}^{l,psc}_N(c, h, k) \cong \mathbb{H}^{l}_{N,deg}(z, h, k)$ for all $z_i, h, k \in \mathbb{C}$, which is compatible with the grading and the filtration.

**Proof.** Let $\text{gr}(\mathbb{H}^{l}_{N,deg}(z, 1, k)) = \mathbb{H}^{l}_{N,deg}(0, 0, 0)$ be the associated graded algebra of $\mathbb{H}^{l}_{N,deg}(z, 1, k)$ with respect to the filtration by order of differential operators. Then $\text{gr}(\mathbb{H}^{l}_{N,deg}(z, 1, k))$ contains $S_N \ltimes A^{\otimes N}$, where $A$ is the algebra of functions on the $A_{l-1}$-singularity, generated by $X, XP, and X^{-1}(XP)^l$ (where $P$ is the symbol of $\partial$). On the other hand, $\text{gr}(\mathbb{H}^{l,psc}_N(c, 1, k))$ clearly coincides with $S_N \ltimes A^{\otimes N}$. Since by Theorem [2.18] $\mathbb{H}^{l,psc}_N(c, 1, k) = \mathbb{H}^{l}_{N,deg}(z, 1, k)$ for Weil generic parameters, we conclude that $\text{gr}(\mathbb{H}^{l}_{N,deg}(z, h, k)) = S_N \ltimes A^{\otimes N}$ for all $z_i, h, k$. This implies Theorem 2.19. \hfill $\Box$
**Remark 2.20.** 1. Since the degenerate DAHA has a $\mathbb{G}_a$-action given by $y_i \mapsto y_i + a$ and trivial on other generators, the algebra $\mathcal{HH}^l_{N,\text{deg}}(z, 1, k)$ does not change under the transformation $z_i \mapsto z_i + a$ (i.e., it depends only on the differences $z_i - z_{i+1}$). Under the isomorphism of Theorem 2.19, this symmetry transforms into the symmetry $c_0 \mapsto c_0 + la$.

2. Another proof of Theorem 2.19(ii) is given in the next subsection.

**Remark 2.21.** The isomorphism of spherical subalgebras of $\mathbb{H}^l_{N,\text{psc}}$ and $\mathcal{HH}^l_{N,\text{deg}}$ is due to Kodera and Nakajima [KN]. More precisely, the parameters of the cyclotomic rational Cherednik algebra in [KN] are related to ours by the formula $h^{KN} = -h$, $c^{KN}_m = c_m(1 - \zeta^m)$ for $m \neq 0$ (and $c^{KN}_0$ is not used in [KN]). Also in [KN], one has $\sum_{m=0}^{l-1} c_m = 0$ and thus $z_i = 0$, which is not really a restriction due to the symmetry $z_i \mapsto z_i + a$, $c_0 \mapsto c_0 + la$.

Also, this isomorphism is closely related to the new presentation of the full cyclotomic DAHA $\mathbb{H}^l_{N,\text{psc}}(c, h, k)$ given in [W] (see [W], equation (4.1)). Namely, the element $e'x^{\ell-1}\sigma e'$ in [W] (which in terms of the pictures is wrapping around the cylinder $\ell - 1$ times, and a little further so as to cross the seam $\ell$ times) matches up with our element $\pi$, and similarly $e'y^{\ell-1}\tau e'$ matches with our element $\pi$ (with $\ell$ in [W] equal to our $l$).

### 2.9. A presentation of $\mathcal{HH}^l_{N,\text{deg}}$ by generators and relations.

Let us give a presentation of $\mathcal{HH}^l_{N,\text{deg}}$ by generators and relations. As generators we will use the elements of $S_N$, $y_1, \ldots, y_N$, $X_1, \ldots, X_N$, and the elements

$$D_i^{(l)} := s_{1i}D_{i1}^{(l)}s_{1i}, \text{ where } D_{i1}^{(l)} := X_1^{-1}(y_1 - z_1)\ldots(y_1 - z_l).$$

Obviously, these elements generate $\mathcal{HH}^l_{N,\text{deg}}$, so we only need to write down the relations.

First of all, the elements $s \in S_N$, $y_i$ and $X_i$ satisfy the relations of Proposition 2.22 except that $X_i$ are no longer invertible.

We also claim that

$$[D_i^{(l)}, D_j^{(l)}] = 0.$$  

Indeed, it suffices to check it for $i = 1, j = 2$. Then we have

$$D_1^{(l)}D_2^{(l)} = X_1^{-1}(y_1 - z_1)\ldots(y_1 - z_l)s_{12}X_1^{-1}(y_2 - z_1)\ldots(y_2 - z_l) = X_1^{-1}s_{12}(y_2 + ks_{12} - z_1)\ldots(y_2 + ks_{12} - z_l)X_1^{-1}(y_2 - z_1)\ldots(y_2 - z_l) = X_1^{-1}X_2^{-1}s_{12}(y_1 - z_1)\ldots(y_1 - z_l)(y_2 - z_1)\ldots(y_2 - z_l).$$

This expression commutes with $s_{12}$, which implies the statement.

We also see by direct computation that

$$[y_1, D_1^{(l)}] = -hD_1^{(l)} + k\sum_{i>1} s_{1i}D_{1i}^{(l)}$$

and

$$[y_j, D_1^{(l)}] = -ks_{1j}D_1^{(l)}, \quad j > 1.$$  

Finally, we write down the commutation relations between $D_i^{(l)}$ and $X_j$. 

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Lemma 2.22. We have

\[ [D_1^{(l)}, X_1] = \sum_{r=1}^{l} \prod_{i=1}^{r-1} (y_1 - z_i + h - k \sum_{j>1} s_{1j})(y_1 - z_i), \]

and for \( m > 1 \)

\[ [D_1^{(l)}, X_m] = k \sum_{r=1}^{l} \prod_{i=1}^{r-1} (y_1 - z_i + h - k \sum_{j>1} s_{1j})s_{1m} \prod_{i=r+1}^{l} (y_1 - z_i). \]

Proof. The first relation holds because

\[ [D_1^{(l)}, X_1] = P(y_1 + h - k \sum_{j>1} s_{1j}) - P(y_1), \]

where \( P(y) = (y - z_1)...(y - z_i) \). To prove the second relation, note that

\[ [D_1^{(l)}, X_m] = X_1^{-1}(P(y_1) - P(y_1 - k s_{1m}))X_m = \]

\[ k \sum_{r=1}^{l} X_1^{-1} \prod_{i=1}^{r-1} (y_1 - z_i)s_{1m} \prod_{i=r+1}^{l} (y_1 - z_i - k s_{1m})X_m = \]

\[ k \sum_{r=1}^{l} X_1^{-1} \prod_{i=1}^{r-1} (y_1 - z_i)X_1s_{1m} \prod_{i=r+1}^{l} (y_1 - z_i) = \]

\[ k \sum_{r=1}^{l} \prod_{i=1}^{r-1} (y_1 - z_i + h - k \sum_{j>1} s_{1j})s_{1m} \prod_{i=r+1}^{l} (y_1 - z_i), \]

as desired. \( \square \)

Note that the commutation relations between \( D_1^{(l)} \) and \( X_m \) for \( i > 1 \) can now be obtained by conjugating the relations of Lemma 2.22 by \( S_N \).

Proposition 2.23. Let \( l \geq 1 \). Let \( M_X \) be a monomial in \( X_i \), \( M_D \) a monomial in \( D_i^{(l)} \), \( M_y \) a monomial in \( y_i \) with degrees of all the \( y_i \) at most \( l - 1 \), and \( s \in S_N \). Then the elements of the form \( M_X \, M_y \, s \, M_D \) form a basis in \( \mathcal{H}_{N,\text{deg}}^l \); in particular, \( \mathcal{H}_{N,\text{deg}}^l \) is a free \( \mathbb{C}[h, k, z_1, ..., z_l] \)-module.

Proof. It is easy to see by looking at the polynomial representation that these elements are linearly independent, so we only need to establish the spanning property. Since the generators are monomials of this form, it suffices to show that any (unordered) monomial in \( s, X_i, y_i, D_i^{(l)} \) can be reduced to a linear combination of such standard monomials.

Let us introduce a filtration by setting \( \deg(S_N) = 0, \deg(y_i) = 2, \deg(X_i) = \deg(D_i^{(l)}) = l \). Using the above commutation relations, we can reduce any monomial to ordered form by adding corrections of lower degree. Further, by using the relation

\[ X_1 D_1^{(l)} = P(y_1) \]
and its conjugates, we can reduce powers of $y_i$ to $0, \ldots, l-1$. This implies the statement.

Thus, we obtain the following proposition.

**Proposition 2.24.** The degenerate cyclotomic DAHA $HH_{N,\text{deg}}^l$ is generated by $S_N$ and elements $y_i, X_i, D_i, i = 1, \ldots, N$, with the following defining relations:

$$
s_i y_i = y_{i+1} s_i + k,
[y_i, y_j] = 0,
s X_i = X s(i) s, s \in S_N, [X_i, X_j] = 0,
[y_i, X_1] = k X_1 s_{1i}, i > 1,
[y_1, X_1] = h X_1 - k \sum_{i>1} X_1 s_{1i},
s D_i = D s(i) s, [D_i, D_j] = 0,
[y_j, D_1] = -k s_{1j} D_1, j > 1,
[y_1, D_1] = -h D_1 + k \sum_{i>1} s_{1i} D_1,
$$

$$
[D_1, X_1] = \prod_{r=1}^{l-1} (y_1 - z_i + h - k \sum_{j>1} s_{1j}) (h - k \sum_{j>1} s_{1j}) \prod_{i=r+1}^{l} (y_1 - z_i),

[D_1, X_m] = k \prod_{r=1}^{l-1} (y_1 - z_i + h - k \sum_{j>1} s_{1j}) s_{1m} \prod_{i=r+1}^{l} (y_1 - z_i), m > 1,

X_1 D_1 = (y_1 - z_1) \ldots (y_1 - z_l).
$$

**Proof.** This follows from Proposition 2.23 (with $D_i = D_i^{(l)}$).

**Remark 2.25.** It is easy to check using this presentation that we have an involution $\phi$ on $HH_{N,\text{deg}}^l$ given by

$$
\phi(h) = -h, \phi(k) = -k, \phi(X_i) = D_i, \phi(D_i) = X_i, \phi(s_{ij}) = s_{ij},
\phi(y_i) = y_i + h - k \sum_{j \neq i} s_{ij}.
$$

The existence of this involution is also clear from the isomorphism of $HH_{N,\text{deg}}^l$ with $HH_{N}^l$[psc], since the latter algebra is well known to have such an involution (coming from the corresponding involution of the cyclotomic rational Cherednik algebra exchanging the coordinates with the Dunkl operators).

The proof of Proposition 2.23 in fact shows that for any $l \geq 0$ ordered products of $M_X, M_y, s, M_D$ in any of the 24 possible orders are a spanning set for $HH_{N,\text{deg}}^l$, and
those of them with degrees of $y_i$ at most $l - 1$ are a basis for $l \geq 1$. This implies that we also have another basis of this algebra, formed by monomials $M_X M_D s M_y$ without restriction on the degree of $y_i$, but with the restriction that for each $i$ either $X_i$ is missing in $M_X$ or $D_i$ is missing in $M_D$. Indeed, if this restriction is not satisfied, we may use the relation $X_1 D_1 = P(y_1)$ and its permutations to lower the number of $X_i$ and $D_i$, and it is easy to see by looking at the polynomial representation that monomials with this restriction are linearly independent. Thus we obtain the following proposition.

**Proposition 2.26.** The elements $M_X M_D$ which miss either $X_i$ or $D_i$ for each $i$ form a basis of $H^N_{\text{deg}}$ as a left or right module over the degenerate affine Hecke algebra $H^N_{\text{deg}}$ generated by $s \in S_N$ and $y_i$, $1 \leq i \leq n$; in particular, $H^N_{\text{deg}}$ is a free module over this subalgebra.

Note that the basis of Proposition 2.26 is labeled by $N$-tuples of integers, $(m_1, ..., m_N)$. Namely, if $M_X M_D$ contains $X_i^p$ then we set $m_i = p$, and if it contains $D_i^p$ then we set $m_i = -p$.

We note that Propositions 2.23 and 2.26 also follow from Theorem 2.19. Yet another, geometric proof of Proposition 2.26 will be given in Section 4.

3. CYCLOTOMIC DAHA

3.1. DAHA and formal DAHA. Recall the definition of Cherednik’s double affine Hecke algebra (DAHA), [Ch1]. Let $q, t \in \mathbb{C}^*$, and $t = t^2$. 

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Definition 3.1. The DAHA $HH_N(q,t)$ is generated by invertible elements $X_i, Y_i$, $i = 1, \ldots, N$, and $T_i$, $i = 1, \ldots, N-1$, with relations\footnote{This algebra really depends on $t$ rather than $t = t^2$, but it is traditional to use the parameter $t$, implying that a square root of this parameter has been chosen, see \cite{Ch}. While somewhat clumsy, this convention turns out to be more natural from the viewpoint of Macdonald theory.}

\begin{align*}
(T_i - t)(T_i + t^{-1}) &= 0, \quad (R1) \\
T_i T_{j+1} T_i &= T_{j+1} T_i T_{j+1}, \quad (R2) \\
T_i T_j &= T_j T_i \ (|i - j| \geq 2), \quad (R3) \\
T_i X_i T_i &= X_{i+1}, \quad (R4) \\
T_i X_j &= X_j T_i \ (j \neq i, i + 1), \quad (R5) \\
T_i Y_i T_i &= Y_{i+1}, \quad (R6) \\
T_i Y_j &= Y_j T_i \ (j \neq i, i + 1), \quad (R7) \\
X_i^{-1} Y_i^{-1} X_i Y_i &= T_i^2, \quad (R8) \\
Y_i \tilde{X} &= q^{\epsilon_i} Y_i, \quad (R9) \\
X_i \tilde{Y} &= q^{-\epsilon_i} \tilde{Y} X_i, \quad (R10) \\
[X_i, X_j] &= 0, \quad (R11) \\
[Y_i, Y_j] &= 0. \quad (R12)
\end{align*}

where $\tilde{X}:= \prod_i X_i$ and $\tilde{Y} := \prod_i Y_i$.

We can define the element

$$T_0 := T_{i-1} \ldots T_{N-1} \ldots T_1^{-1} X_1^{-1} X_N$$

which together with $T_i$, $i = 1, \ldots, N - 1$ generates the affine Hecke algebra of type $A_{N-1}$ in the Coxeter presentation (i.e., relations (R1), (R2) are satisfied for all $i, j \in \mathbb{Z}/N\mathbb{Z}$).

Similarly one defines the algebra $HH_N$ over $\mathbb{C}[q^{\pm 1}, t^{\pm 1}]$.

We will also consider a formal version of DAHA over $\mathbb{C}[[\varepsilon]]$, in which $q = e^{\varepsilon h}$ and $t = e^{-\varepsilon k}$ for $k \in \mathbb{C}$. Namely, set $T_i = s_i e^{-\varepsilon k_i}/2$, $Y_i = e^{\varepsilon u_i}$, and let $HH_N^\text{formal}(h, k)$ be the $\varepsilon$-adically complete algebra generated over $\mathbb{C}[[\varepsilon]]$ by $s_i$, $X_i$ and $y_i$ with the relations of Definition 3.1. We can also treat $h, k$ as indeterminates, working over $\mathbb{C}[h, k]$.

Note that using (R6), relation (R8) can be written as

$$X_i^{-1} T_{i-1} Y_i^{-1} T_{i-1} X_i Y_i T_i T_1 = T_1^2,$$

or

$$X_i T_i Y_i = T_i Y_i T_i X_i T_i. \quad (R8a)$$

This shows that $HH_N$ has the Cherednik involution $\varphi$ defined by $\varphi(q) = q^{-1}$, $\varphi(t) = t^{-1}$, $\varphi(X_i) = Y_i^{-1}$, $\varphi(Y_i) = X_i^{-1}$, $\varphi(T_i) = T_i^{-1}$. 

\footnote{This algebra really depends on $t$ rather than $t = t^2$, but it is traditional to use the parameter $t$, implying that a square root of this parameter has been chosen, see \cite{Ch}. While somewhat clumsy, this convention turns out to be more natural from the viewpoint of Macdonald theory.}
3.2. The quasiclassical limit of the formal DAHA.

**Proposition 3.2.** The algebra $\mathbb{H}^\text{formal}_N(h,k)/(\varepsilon)$ is isomorphic to the trigonometric DAHA $\mathbb{H}_N, \deg((h,k))$, and $\mathbb{H}^\text{formal}_N(h,k)$ is a flat deformation of $\mathbb{H}_N, \deg((h,k))$.

**Proof.** To prove the first statement, we need to show that the DAHA relations of Definition 3.1 degenerate to the relations of the trigonometric DAHA.

Clearly, relation (R1) yields $s_i^2 = 1$, and relations (R2, R3) yield $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ and $s_is_j = s_js_i$ for $1 \leq i, j \leq N-1$ and $|i-j| \geq 2$. Relations (R4, R5) give $s_iX_j = X_js_i$ if $j \neq i, i + 1$, and $s_iX_i = X_{i+1}s_i$. Relation (R6) gives a trivial relation in zeroth order, but in the first order it gives

$$-ks_i + s_iy_is_i = y_{i+1},$$

which yields

$$s_iy_i = y_{i+1}s_i + k.$$

Relation (R7) gives

$$[s_i, y_j] = 0, \ j \neq i, i + 1.$$

Relation (R8) yields

$$y_2 - X_1^{-1}y_2X_1 = -ks_1,$$

which is equivalent to

$$[y_2, X_1] = kX_1s_1.$$

Relations (R9, R10) yield

$$[y_i, \prod_j X_j] = h \prod_j X_j, \ [\sum_i y_i, X_j] = hX_j.$$

Finally, relations (R11, R12) yield

$$[X_i, X_j] = 0, \ [y_i, y_j] = 0.$$

It is easy to see that these relations are exactly the relations of $\mathbb{H}_N, \deg((h,k))$ given in Proposition 2.2 (see Remark 2.3).

The second statement of the Proposition follows from the first one and the PBW theorem for DAHA ([Ch1]).

3.3. The polynomial representation of DAHA.

**Proposition 3.3.** ([Ch1]) We have an action of $\mathbb{H}_N(q,t)$ on $\mathbb{P}$ given by

$$\rho(X_i) = X_i,$$

$$\rho(T_i) = ts_i + \frac{t - t^{-1}}{X_i/X_{i+1} - 1}(s_i - 1),$$

$$\rho(Y_i) = t^{N-1}\rho(T_i^{-1}...T_{N-1}^{-1})\omega\rho(T_1...T_{i-1}),$$

where $(\omega f)(X_1, ..., X_N) := f(qX_N, ..., X_{N-1}).$
The same formulas define a representation of $\mathcal{HH}^\text{formal}(h, k)$. The representation $\rho$ of DAHA on $\mathbb{P}$ is called the \textit{polynomial representation} of DAHA.

**Proposition 3.4.** The quasiclassical limit (i.e., reduction modulo $\varepsilon$) of the polynomial representation of $\mathcal{HH}^\text{formal}_N(h, k)$ coincides with the polynomial representation of the trigonometric DAHA given by Proposition 2.4.

*Proof.* The proof is by a direct calculation. \qed

### 3.4. The cyclotomic DAHA

Let $\pi \in \mathcal{HH}_N(q, t)$ be the element given by the formula

$$
\pi = X_1T_1...T_{N-1}.
$$

Let $Z_1, ..., Z_l \in \mathbb{C}^*$, and $Z = (Z_1, ..., Z_l)$.

**Definition 3.5.** The subalgebra $\mathcal{HH}_N^l(Z, q, t)$ of $\mathcal{HH}_N(q, t)$ is generated by $T_i$, $i = 1, ..., N - 1$, $Y_i$, $i = 1, ..., N$, $\pi$, and the element

$$
\pi_- := \pi^{-1} \prod_{i=1}^{l} (Y_1 - Z_i).
$$

Let $z_1, ..., z_l \in \mathbb{C}$, and $Z_i = q^{z_i}$ (for some choice of branches).

**Proposition 3.6.** The algebra $\mathcal{HH}_N^l(Z, q, t)$ preserves the space $\left(\prod_j X_j\right)^u \mathbb{P}_+$ for all $i$.

*Proof.* We only need to check that $\pi_-$ preserves this space. For this, it is enough to prove that for any $u \in \mathbb{C}$, the element $X_1^{-1}(Y_1 - q^u)$ preserves the space $\left(\prod_j X_j\right)^u \mathbb{P}_+$. To this end, note that

$$
\rho(T_i^{-1}) = t^{-1}s_i + \frac{(t^{-1} - t)X_i}{X_{i+1} - X_i}(s_i - 1).
$$

Now consider

$$
\rho(Y_1) = t^{N-1}\rho(T_1^{-1})...\rho(T_{N-1}^{-1})\omega =
$$

\begin{equation}
(1 + \frac{(1 - t)X_1}{X_2 - X_1}(1 - s_{12}))...(1 + \frac{(1 - t)X_1}{X_N - X_1}(1 - s_{1N}))\tau_1,
\end{equation}

where $\tau_j$ replaces $X_j$ with $qX_j$ and keeps $X_i$ fixed for $i \neq j$. By opening the brackets, this product can be written as a sum of $2^N$ terms (as in each factor, we can take the first or the second summand). If we take the first summand from all factors, we get $\tau_1$, and $X_1^{-1}(\tau_1 - q^u)$ clearly preserves $\left(\prod_j X_j\right)^u \mathbb{P}_+$. So it suffices to show that for each of the remaining $2^N - 1$ terms $T$, the operator $X_1^{-1}T$ preserves $\left(\prod_j X_j\right)^u \mathbb{P}_+$. But all of these terms have a factor $X_1$ on the left (as so does the second summand in each factor in (3.1)), which implies the desired statement. \qed
Corollary 3.7. Let $HH_N^{\text{formal}}(z, h, k)$ be the formal version of $HH_N^l(Z, q, t)$, namely the subalgebra of $HH_{N}^{\text{formal}}(h, k)$ generated by $T_i$, $i = 1, \ldots, N - 1$, $y_i$, $i = 1, \ldots, N$, $\pi$, and the element
\[
\varepsilon^{-l} \pi^{-1} \prod_{i=1}^{l} (Y_i - e^{\varepsilon z_i}) = \pi^{-1} \prod_{i=1}^{l} e^{\varepsilon y_i} - e^{\varepsilon z_i}.
\]

Proof. Recall that by Proposition 3.2, $HH_N^{\text{formal}}(h, k)$ is a flat deformation of $HH_N^{\text{deg}}(h, k)$, and by Proposition 3.4 the same applies to the polynomial representations of these algebras. Thus the result follows by a deformation argument from Proposition 3.6 and the fact that a similar statement holds in the trigonometric case (Theorem 2.10(ii)). Namely, the algebra $HH_N^{\text{formal}}(z, 1, k)$ is a priori “at least as big” as $HH_N^{\text{deg}}(z, 1, k)$, as its generators are deformations of generators of $HH_N^{\text{deg}}(z, 1, k)$. At the same time, the subalgebra of elements of $HH_N^{\text{formal}}(z, 1, k)$ preserving $(\prod X_j)^{z_i}P_+$ for all $i$ is “at most as big” as $HH_N^{\text{deg}}(z, 1, k)$, as by Theorem 2.10(ii), this condition cuts out $HH_N^{\text{deg}}(z, 1, k)$ inside $HH_N^{\text{deg}}(1, k)$. But by Theorem 3.6 the former subalgebra is contained in the latter one. This implies the corollary. □

Theorem 3.8. For any $z_1, \ldots, z_l$, the algebra $HH_N^{l,\text{formal}}(z, h, k)$ is a flat deformation of $HH_N^{l,\text{deg}}(z, h, k)$.

Proof. Since $HH_N^{l,\text{formal}}(z, h, k)$ is generated by deformations of generators of $HH_N^{l,\text{deg}}(z, h, k)$, it suffices to prove this statement for Weil generic $z_1, h, k$, but in this case it follows from Corollary 3.7. □

Another proof of Theorem 3.8 is obtained from the presentation of $HH_N^l$ given below.

Definition 3.9. The algebra $HH_N^l(Z, q, t)$ is called the cyclotomic DAHA and $HH_N^{l,\text{formal}}(z, h, k)$ is called the formal cyclotomic DAHA.

As usual, one can similarly define versions of these algebras where the parameters are indeterminates. Note also that by this definition $HH_N^0(q, t) = HH_N(Z, q, t)$, and $HH_N^l(Z', q, t) \subset HH_N^l(Z, q, t)$ if $l' \geq l$ and $Z' \supset Z$ as a multiset.

Thus, the cyclotomic DAHA is a $q$-deformation of the partly spherical cyclotomic rational Cherednik algebra. More precisely, it follows from Theorem 2.19 and Theorem 3.8 that for any $z_1, \ldots, z_l, h, k$, the formal cyclotomic DAHA $HH_N^{l,\text{formal}}(z, h, k)$ is a flat deformation of $HH_N^{l,\text{formal}}(c, h, k)$, where $c$ is related to $z$ by equation (2.4). In particular, the cyclotomic DAHA is interesting already for $l = 1$, as it provides a $q$-deformation of the rational Cherednik algebra $HH_N^l(h, k)$ attached to $S_N$ and its permutation representation.
Remark 3.10. Since the DAHA has a $\mathbb{G}_m$-action given by $Y_i \mapsto aY_i$ and trivial on other generators, the algebra $\mathcal{HH}_N^1(Z, q, t)$ does not change under the transformation $Z_i \mapsto aZ_i$ (i.e., it depends only on the ratios $Z_i/Z_{i+1}$).

Example 3.11. Let $N = 1$. Then there is no dependence on $t$, and $\mathcal{HH}_1(q, t) = \mathcal{HH}_1(q)$ is the quantum torus algebra with invertible generators $X, Y$ and relation $YX = qXY$. Let $q$ be not a root of unity. The polynomial representation is $\mathbb{P} = \mathbb{C}[X^\pm 1]$ with $Y$ acting by shift, $(Yf)(X) = f(qX)$, so that $\mathcal{HH}_1(q)$ is the algebra of polynomial $q$-difference operators. The subalgebra $\mathcal{HH}_1^1(Z, q, t) = \mathcal{HH}_1^1(Z, q)$ inside $\mathcal{HH}_1(q)$ is generated by $X, Y^{\pm 1}$, and

$$L := X^{-1}(Y - qz^1)...(Y - qz^k),$$

where $Z_i = q^{z_i}$. We claim that if $Z_i/Z_j \notin q^{z_i}$ then $\mathcal{HH}_1^1(Z, q)$ is exactly the subalgebra of all difference operators preserving $X^{z_i}\mathbb{C}[X]$ for all $i$. Indeed, if we set $\deg(Y) = 0$, $\deg(X) = 1$, then any difference operator of nonnegative degree is in both subalgebras, while an operator $M$ of degree $-d < 0$ has the form $X^{-d}g(Y)$, where $g$ is a Laurent polynomial, and applying this operator to $X^{z_i+j}$, $j < d$, we must get zero, so we get $g(q^{z_i+j}) = 0$, $i = 1, ..., l$, $j = 0, ..., d - 1$. Thus $M = L^d h(Y)$ for some polynomial $h$.

Remark 3.12. We expect that Corollary 3.7 holds in the non-formal setting, i.e., $\mathcal{HH}_N^1(Z, q, t)$ can be characterized as the algebra of elements of $\mathcal{HH}_N^1(Z, q, t)$ preserving the spaces $(\prod X_j)^{z_i} \mathbb{P}_+$, as in Proposition 3.6. For $N = 1$ this is demonstrated in Example 3.11. Moreover, recall that Ginzburg, Kapranov, and Vasserot ([GKV]) characterized DAHA (for any Weyl group $W$) as the algebra of difference-reflection operators $L = \sum_{w \in W} L_w w$ (where $L_w$ are difference operators) satisfying some residue conditions. These conditions are equivalent to the conditions that $L$ preserves $\mathbb{P}$ and $\Delta_{q,t} \mathbb{P}$, where $\Delta_{q,t}$ is an appropriate meromorphic function. Therefore, we expect that $\mathcal{HH}_N^1(Z, q, t)$ can be characterized as the algebra of difference operators preserving the spaces $\mathbb{P}$, $\Delta_{q,t} \mathbb{P}$, and $(\prod X_j)^{z_i} \mathbb{P}_+$ for $i = 1, ..., l$.

We note that this approach to DAHA-type algebras in the more general elliptic setting is developed in the ongoing work [R2]. In the one-variable case $N = 1$ such (spherical) algebras generated by a given set of difference-reflection operators have been studied in [R1].

3.5. The case $l = 1$. Let us study the algebra $\mathcal{HH}_N^1$ for $l = 1$ and give its presentation. These results can be derived from the case of general $l$ considered below, but the special case $l = 1$ is especially nice, and it is instructive to do it separately first.

By rescaling $Y_i$ by the same scalar, we may assume without loss of generality that $Z_1 = 1$. Then the algebra $\mathcal{HH}_N^1(q, t) := \mathcal{HH}_N^1(1, q, t)$ is generated inside $\mathcal{HH}_N(q, t)$ by $T_i, X_i, Y_i^{\pm 1}$, and $X_1^{-1}(Y_1 - 1)$.

This algebra actually appeared a long time ago in the paper [BF]. Let us describe it in more detail, following [BF]. (We note that our conventions are slightly different from those of [BF]).
Define the Dunkl elements $D_i := T_{i-1}^{-1} \cdots T_1^{-1} D_i T_1^{-1} \cdots T_{i-1}^{-1} \in HH_N^1(q, t)$, where $D_i = X_i^{-1}(Y_i - 1)$. It is easy to check that

$$D_i = T_i D_{i+1} T_i, \quad [T_i, D_j] = 0 \text{ for } j \neq i, i + 1.$$

**Lemma 3.13.** We have $[D_1, D_2] = 0$.

**Proof.** By definition, we have $D_2 = T_1^{-1} D_1 T_1^{-1}$, so our job is to show that

$$T_1^{-1} D_1 T_1^{-1} D_1 = D_1 T_1^{-1} D_1 T_1^{-1}.$$

In other words, we must show that

$$T_1^{-1} X_1^{-1}(Y_1 - 1) T_1^{-1} X_1^{-1}(Y_1 - 1) = X_1^{-1}(Y_1 - 1) T_1^{-1} X_1^{-1}(Y_1 - 1) T_1^{-1}.$$

Since $T_1 X_1 T_1 X_1 = X_2 X_1 = X_1 X_2 = X_1 T_1 X_1 T_1$, it suffices to prove two identities in $HH_N(q, t)$:

(3.2) $T_1^{-1} X_1^{-1} Y_1 T_1^{-1} X_1^{-1} Y_1 = X_1^{-1} Y_1 T_1^{-1} X_1^{-1} Y_1 T_1^{-1}$

and

(3.3) $T_1^{-1} X_1^{-1} T_1^{-1} X_1^{-1} Y_1 + T_1^{-1} X_1^{-1} Y_1 T_1^{-1} X_1^{-1}$

$$= X_1^{-1} T_1^{-1} X_1^{-1} Y_1 T_1^{-1} + X_1^{-1} Y_1 T_1^{-1} X_1^{-1} T_1^{-1}.$$

Identity (3.2) actually holds already in the braid group. Indeed, we have

$$T_1^{-1} X_1^{-1} Y_1 T_1^{-1} X_1^{-1} Y_1 = T_1^{-1} X_1^{-1} T_1^{-1} Y_2 T_1^{-2} X_1^{-1} Y_1 = T_1^{-1} X_1^{-1} T_1^{-1} X_1^{-1} Y_2 Y_1 =$$

$$X_1^{-1} T_1^{-1} X_1^{-1} Y_2 T_1^{-2} X_1^{-1} Y_1 = X_1^{-1} T_1^{-1} X_1^{-1} Y_1 T_1^{-1} =$$

$$X_1^{-1} T_1^{-1} Y_2 T_1^{-2} X_1^{-1} Y_1 T_1^{-1} = X_1^{-1} Y_1 T_1^{-1} X_1^{-1} Y_1 T_1^{-1}.$$

It remains to establish (3.3). Note that because $T_1^{-1} = T_1 - t + t^{-1}$, (3.3) is equivalent to

(3.4) $T_1 X_1^{-1} T_1^{-1} X_1^{-1} Y_1 + T_1^{-1} X_1^{-1} Y_1 T_1^{-1} X_1^{-1}$

$$= X_1^{-1} T_1^{-1} X_1^{-1} Y_1 T_1 + X_1^{-1} Y_1 T_1^{-1} X_1^{-1} T_1^{-1}.$$

On the other hand, (3.4) holds already in the group algebra of the braid group (i.e., termwise). Indeed, we have

$$T_1^{-1} X_1^{-1} Y_1 T_1^{-1} X_1^{-1} = T_1^{-1} X_1^{-1} T_1^{-1} Y_2 T_1^{-2} X_1^{-1} =$$

$$T_1^{-1} X_1^{-1} T_1^{-1} X_1^{-1} Y_2 = X_1^{-1} T_1^{-1} X_1^{-1} T_1^{-1} Y_2 =$$

and

$$T_1 X_1^{-1} T_1^{-1} X_1^{-1} Y_1 = X_1^{-1} T_1^{-1} X_1^{-1} T_1 Y_1 = X_1^{-1} T_1^{-1} X_1^{-1} Y_2 T_1^{-1} =$$

$$X_1^{-1} T_1^{-1} Y_2 T_1^{-2} X_1^{-1} T_1^{-1} = X_1^{-1} Y_1 T_1^{-1} X_1^{-1} T_1^{-1}.$$

□

**Corollary 3.14.** One has $[D_i, D_j] = 0$ for all $i, j$. 

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Proof. For any $j > 1$, we have
\[
[D_1, D_j] = [D_1, T_{j-1}^{-1}T_2^{-1}D_2T_2^{-1}...T_{j-1}^{-1}] = 0,
\]
since $D_1$ commutes with every factor by Lemma 3.13. Hence for $i < j$,
\[
[D_i, D_j] = [T_i^{-1}...T_1^{-1}D_1T_1^{-1}...T_{i-1}^{-1}, D_j] = 0,
\]
gain because $D_j$ commutes with every factor. 

Thus, $T_1$ and $D_i$ generate the “positive part” of an affine Hecke algebra.

When $q \to 1$ and $t = q^{-k}$, where $k$ is fixed, the algebra $\mathcal{HH}_N^q(1, k)$ degenerates to the rational Cherednik algebra $\mathcal{HH}_N^{\text{rat}}(1, k)$ for $S_N$ (associated to the permutation representation), and $X_i, \frac{D_i}{q-1}$ degenerate to the standard generators of $\mathcal{HH}_N^{\text{rat}}(1, k)$. Thus, let us compute the commutation relations between $D_i$ and $X_j$ which deform the corresponding relations of $\mathcal{HH}_N^{\text{rat}}(1, k)$.

**Lemma 3.15.** One has
\[
X_1D_2 = D_2T_1^2X_1 + (t - t^{-1})T_1^{-1}
\]
and
\[
D_1X_2 = X_2T_1^{-2}D_1 - (t - t^{-1})T_1.
\]

**Proof.** We have
\[
D_2X_1^{-1} = T_1^{-1}X_1^{-1}Y_1T_1^{-1}X_1^{-1} - T_1^{-1}X_1^{-1}T_1^{-1}X_1^{-1}
\]
Thus, by the proof of Lemma 3.13 we obtain
\[
D_2X_1^{-1} = X_1^{-1}T_1^{-1}X_1^{-1}Y_1T_1 - X_1^{-1}T_1^{-1}X_1^{-1}T_1^{-1} =
\]
\[
X_1^{-1}D_2T_1^2 + X_1^{-1}T_1^{-1}X_1^{-1}T_1 - X_1^{-1}T_1^{-1}X_1^{-1}T_1^{-1} =
\]
\[
X_1^{-1}D_2T_1^2 + (t - t^{-1})X_1^{-1}T_1^{-1}X_1^{-1}.
\]
Now the first relation of the lemma is obtained by multiplying both sides by $X_1$ on the left and on the right. To obtain the second relation of the lemma from the first one, it suffices to multiply the first relation by $T_1$ on both sides, and apply the commutation relations between $D_i, X_i$ and $T_j$. 

**Corollary 3.16.** If $i < j$, one has
\[
X_iD_j = D_jT_{j-1}...T_{i+1}T_i^2T_{i+1}...T_{j-1}X_i + (t - t^{-1})T_{j-1}^{-1}...T_i^{-1}...T_{j-1}^{-1}.
\]
If $i > j$, one has
\[
D_jX_i = X_iT_{i-1}^{-1}...T_{j+1}^{-1}T_j^{-2}T_{j+1}...T_{i-1}D_j - (t - t^{-1})T_{i-1}^{-1}...T_j...T_{i-1}.
\]

Proof. First consider the case $i = 1$. Then

$$X_1D_j = X_1T_{j-1}^{-1}...T_{2}^{-1}D_2T_{2}^{-1}...T_{j-1}^{-1} = T_{j-1}^{-1}...T_2^{-1}X_1D_2T_2^{-1}...T_{j-1}^{-1}.$$ 

By the first relation of Lemma 3.15, this implies

$$X_1D_j = T_{j-1}^{-1}...T_2^{-1}D_2T_2^{-1}X_1T_2^{-1}...T_{j-1}^{-1} + (t - t^{-1})T_{j-1}^{-1}...T_2^{-1}T_1^{-1}T_2^{-1}...T_{j-1}^{-1} =$$

$$D_jT_{j-1}...T_2T_1T_2^{-1}...T_{j-1}^{-1}X_1 + (t - t^{-1})T_{j-1}^{-1}...T_2^{-1}T_1^{-1}T_2^{-1}...T_{j-1}^{-1},$$

as claimed. Now consider the general case. We have

$$X_iD_j = T_{i-1}...T_1X_1T_1...T_{i-1}D_j = T_{i-1}...T_1X_1D_jT_1...T_{i-1} =$$

$$T_{i-1}...T_1D_jT_{j-1}...T_2T_1T_2^{-1}...T_{j-1}^{-1}X_1T_1...T_{i-1}+$$

$$(t - t^{-1})T_{i-1}...T_1T_{j-1}^{-1}...T_2^{-1}T_1^{-1}T_2^{-1}...T_{j-1}^{-1}T_1...T_{i-1} =$$

$$D_jT_{j-1}...T_1T_iT_2^{-1}...T_{j-1}^{-1}X_1 + (t - t^{-1})T_{j-1}^{-1}...T_2^{-1}T_1^{-1}T_2^{-1}...T_{j-1}^{-1},$$

(here we repeatedly used the braid relations between the $T_i$). This proves the first relation of the corollary.

The second relation is proved similarly, using the second relation of Lemma 3.15. □

Finally, let us generalize the commutation relations between $D_i$ and $X_i$ (which are the only relations containing $q$). It turns out that it is convenient to write instead the commutation relations between $D_i$ and $\sum_{j=1}^{N} X_j$.

Lemma 3.17. We have

$$[D_i, \sum_{j=1}^{N} X_j] = (1 - q^{-1})(D_iX_i + 1)T^{-1}_{i-1}...T^{-2}_{i-1}. $$

Proof. First assume $i = 1$. Looking at the polynomial representation and using that $\sum_{j=1}^{N} X_j$ is symmetric, we find that

$$[D_1, \sum_{j=1}^{N} X_j] = (1 - q^{-1})(D_1X_1 + 1).$$

The general case is now obtained by multiplying both sides by $T^{-1}_{i-1}...T^{-1}_{1}$ on the left and by $T^{-1}_{1}...T^{-1}_{i-1}$ on the right. □

We obtain the following theorem.
Theorem 3.18. (i) Let $HH_N^{1+}(q,t)$ be the subalgebra of $HH_N(q,t)$ generated by $T_i$, $1 \leq i \leq N-1$, $X_i, D_i$, $1 \leq i \leq N$. Then the defining relations for $HH_N^{1+}(q,t)$ are

$$
(T_i - t)(T_i + t^{-1}) = 0,
$$
$$
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},
$$
$$
T_i T_j = T_j T_i \ (|i-j| \geq 2),
$$
$$
T_i X_i T_i = X_{i+1}, \ [T_i, X_j] = 0 \text{ for } j \neq i, i+1.
$$
$$
D_i = T_i D_i T_i, \ [T_i, D_j] = 0 \text{ for } j \neq i, i+1.
$$
$$
[X_i, X_j] = 0,
$$
$$
[D_i, D_j] = 0,
$$
$$
X_i D_j = D_j T_{j-1} \cdots T_{i+1} T_i^2 T_{i+1} \cdots T_{j-1} X_i + (t - t^{-1}) T_{j-1} \cdots T_{i+1} T_i^2 T_{i+1} \cdots T_{j-1}, \quad i < j,
$$
$$
D_j X_i = X_i T_{i-1} \cdots T_{j+1} T_{j+1} \cdots T_{i-1} D_j - (t - t^{-1}) T_{i-1} \cdots T_{j+1} T_{j+1} \cdots T_{i-1}, \quad i > j,
$$

$$
[D_i, \sum_{j=1}^{N} X_j] = (1 - q^{-1})(D_i X_i + 1) T_{i-1} \cdots T_{i-2} \cdots T_{i-1}.
$$

(ii) (the PBW theorem) For any values of parameters, the elements $\prod_i X_i^{m_i} \cdot T_w \cdot \prod_i D_i^{n_i}$ form a basis of $HH_N^{1+}(q,t)$.

(iii) The algebra $HH_N^1(q,t)$ is obtained from $HH_N^{1+}(q,t)$ by inverting the element $Y_1 := 1 + X_1 D_1$.

Proof. We have shown in Lemma 3.14, Corollary 3.16, Lemma 3.17 that the claimed relations between $T_i$, $X_i$, $D_i$ are satisfied. These relations allow us to order any monomial as claimed in (ii). Since $HH_N^{1,\text{formal}}$ is a flat deformation of the rational Cherednik algebra $HH_N^{1,\text{rat}}(h,k)$, the relations in (i) are defining, and the monomials in (ii) are linearly independent. This implies both (i) and (ii).

Now, by definition, $HH_N^1(q,t)$ is generated inside $HH_N(q,t)$ by $HH_N^{1+}(q,t)$ and $Y_1^{-1}$, where $Y_1 = 1 + X_1 D_1$. This implies (iii). \qed

The following proposition shows that there is a symmetry between $X_i$ and $D_i$. Let us regard $q$ and $t$ as variables.

Proposition 3.19. There is an involutive automorphism $\phi$ of $HH_N^1$ such that $\phi(X_i) = D_i$, $\phi(D_i) = X_i$, $\phi(T_i) = T_i^{-1}$, $\phi(t) = t^{-1}$, $\phi(q) = q^{-1}$. This automorphism preserves the subalgebra $HH_N^{1+}$.

Proof. It is easy to see that all the relations of Theorem 3.18 except the last one are invariant under $\phi$. It is sufficient to impose the last relation for $i = 1$, so let us consider
this case. The relation has the form
\[
q^{-1}(D_1X_1 + 1) - (X_1D_1 + 1) = (t - t^{-1}) \sum_{i>1} (X_iT_i^{-1}...T_1^{-1}T_2...T_{i-1}D_1 + T_{i-1}...T_1)(X_1D_1 + 1),
\]

i.e.,
\[
D_1X_1 + 1 = q(1 + (t - t^{-1}) \sum_{i>1} T_{i-1}...T_1)(X_1D_1 + 1).
\]

Using braid relations, we see that \(T_i...T_1 = T_1...T_iT_1\), so this can be also written as
\[
D_1X_1 + 1 = q(1 + (t - t^{-1}) \sum_{i>1} T_1...T_{i-1}T_1)(X_1D_1 + 1).
\]

Now let \(J_N := T_1...T_{N-1}...T_1\) be the Jucys-Murphy element. Then, using the quadratic relation for \(T_{N-1}\), we get
\[
J_N = (t - t^{-1})T_1...T_{N-1}...T_1 + J_{N-1}.
\]

This yields
\[
J_N = 1 + (t - t^{-1}) \sum_{i>1} T_1...T_{i-1}T_1,
\]

so our relation can be simplified to
\[
(3.5) \quad D_1X_1 + 1 = qJ_N(X_1D_1 + 1).
\]

The invariance of this relation under \(\phi\) reduces to the identity \(J_N\phi(J_N) = 1\), which is obvious from the definitions of \(\phi\) and \(J_N\). This shows that \(\phi\) is a well defined automorphism of \(HH^1_{N,+}\), which is obviously involutive.

We also see from (3.5) that \(\phi(Y_1) = qJ_NY_1\), which implies that \(\phi\) extends to \(HH^1_N\). \(\square\)

**Corollary 3.20.** (i) The last defining relation of \(HH^1_{N,+}(q,t)\) may be replaced with
\[
[X_i, \sum_{j=1}^N D_j] = (1 - q)(X_iD_i + 1)T_{i-1}...T_1^2...T_{i-1}.
\]

(ii) The last defining relation of \(HH^1_{N,+}(q,t)\) may be replaced with (3.5).

**Proof.** (i) follows immediately from Theorem 3.18 and (ii) from its proof. \(\square\)
3.6. **A commutative subalgebra in $HH_N$.** In this subsection, we will construct a commutative subalgebra inside $HH_N$. In the case $l = 1$, this subalgebra will reduce to the subalgebra $C[D_1, ..., D_N]$ constructed in the previous subsection (so we will obtain an alternative construction of this subalgebra).

Let $f \in C[X]$ be any polynomial. Define the elements

$$Y_i(f) := Y_i T_{i-1}^{-1} ... T_{i-1}^{-1} f(X_1^{-1}) T_1 ... T_{i-1} \in HH_N.$$ 

Also let $e$ be the symmetrizer of the finite Hecke algebra generated by the $T_i$. (To define $e$, we need to invert $|N|_t$!)

**Lemma 3.21.** (i) *The elements $Y_i(f)$ are pairwise commuting:*

$$[Y_i(f), Y_j(f)] = 0.$$ 

(ii) *For $r \geq 1$ the element $M_r(f) = (\sum_{i=1}^{N} e_r(Y_1(f), ..., Y_N(f))))e$ (where $e_r$ is the $r$-th elementary symmetric function) commutes with $T_i$, i.e. $M_r(f) = eM_r(f)$.*

**Proof.** (i) We may specialize the variables to numerical values. Let us compute the action of $Y_i(f)$ in the polynomial representation. Using Proposition 3.3, we get

$$\rho(Y_i(f)) = t^{N-1} \rho(T_1^{-1} ... T_{N-1}^{-1}) \omega f(X_1) \rho(T_1 ... T_{i-1}).$$

Let $g(X)$ be a meromorphic function such that $g(q^{-1} X) = g(X) f(X)$. Let us extend the polynomial representations to meromorphic functions, and conjugate it by the function $G(X_1, ..., X_N) := g(X_1) ... g(X_N)$ This gives a representation $\rho_G$ such that

$$\rho_G(Y_i) = G \rho(Y_i) G^{-1} = t^{N-1} \rho(T_1^{-1} ... T_{N-1}^{-1}) g G^{-1} \rho(T_1 ... T_{i-1}) =$$

$$t^{N-1} \rho(T_1^{-1} ... T_{N-1}^{-1}) \omega \frac{G(q^{-1} X_1)}{G(X_1)} \rho(T_1 ... T_{i-1}) =$$

$$t^{N-1} \rho(T_1^{-1} ... T_{N-1}^{-1}) \omega f(X_1^{-1}) \rho(T_1 ... T_{i-1}) = \rho(Y_i(f)).$$

Thus $\rho(Y_i(f))$ are pairwise commuting, hence so are $Y_i(f)$.

(ii) It suffices to show that $M_r(f)$ maps symmetric polynomials to symmetric ones. But this follows from the fact that $\rho(M_r(f)) = \rho_G(M_r)$, where $M_r = \sum_{i=1}^{N} e_r(Y_1, ..., Y_N)$, and $M_r$ act in the polynomial representation by Macdonald difference operators. \( \square \)

Now let $f(X) = (X - Z_1)...(X - Z_l)$. Let us apply the Cherednik involution $\varphi$ to the elements $Y_i(f)$. Namely, let

$$D_i^{(l)} := \varphi(Y_i(f)) = T_{i-1}^{-1} ... T_1^{-1} D_i^{(l)} T_1 ... T_{i-1}^{-1} \in HH_N,$$

where, $D_i^{(l)} := X_1^{-1}(Y_1 - Z_1) ... (Y_1 - Z_l)$.

**Corollary 3.22.** (i) *The elements $D_i^{(l)}$, $i = 1, ..., N$, are pairwise commuting.*

(ii) *The elements $M_r(f) := \varphi(M_r(f))$ commute with $T_i$.*

**Proof.** This follows from Lemma 3.21. \( \square \)
Example 3.23. Let \( l = 1 \) and \( f(X) = X - 1 \). Then we get \( D_i^{(1)} = D_i \). Thus, we recover the \( q \)-deformed Dunkl operators of [BF] described in the previous subsection and therefore obtain another proof of Corollary 3.14.

Corollary 3.22(ii) implies that the elements \( M_r(f) \) act on symmetric functions by some commuting symmetric difference operators \( M_r(f) =: M_r^{(l)} \). Thus for each \( l \) we obtain a family of quantum integrable systems \( \{ M_1^{(l)}, ..., M_N^{(l)} \} \) depending on \( l \) parameters \( Z_1, ..., Z_l \) (and also \( q, t \)). This system is a \( q \)-deformation of the cyclotomic Calogero-Moser system.

Example 3.24. ([BF], Lemma 5.3) We have
\[
M_1^{(1)} = \sum_{j=1}^{N} \left( \prod_{i \neq j} \frac{X_i - tX_j}{X_i - X_j} \right) \frac{1}{X_j} (\tau_j - 1).
\]
Thus, this operator defines a quantum integrable system.

Remark 3.25. J. F. Van Diejen and S. Ruijsenaars have explained to us that the system defined by the operator \( M_1 := M_1^{(1)} \) from Example 3.24 may be obtained as a limit of the system from [DE] defined by the Hamiltonian (3.13a). Namely, conjugating by the Gaussian \( \exp(\sum_i (\log X_i)^2) \) and rescaling \( X_i \) and \( M_1 \), we can reduce the operator \( M_1 \) to the form
\[
M_1' = \sum_{j=1}^{N} \left( \prod_{i \neq j} \frac{X_i - tX_j}{X_i - X_j} \right) \tau_j - \sum_{j=1}^{N} X_j^{-1} = M - \sum_{j=1}^{N} X_j^{-1},
\]
where \( M \) is the first Macdonald operator. (Here we use the identity at the beginning of [DE] p. 1621). On the other hand, let us multiply the Hamiltonian [DE] (3.13a) by \( \hat{t}_0 \) and then put \( \hat{t}_1 = \hat{t}_0^{-1} \) and send \( \hat{t}_0 \) to 0. Conjugating the resulting operator by an appropriate function, and again using the identity at the beginning of [DE] p. 1621, one obtains the operator \( M_1' \).

Remark 3.26. It is interesting to compute the joint eigenfunctions of \( D_1^{(l)}, ..., D_N^{(l)} \) and symmetric joint eigenfunctions of \( M_1^{(l)}, ..., M_N^{(l)} \). For \( l = 1 \), this is done in [BF]. Let us sketch how this can be done for any \( l \) in the nonsymmetric case (without working out any details).

For simplicity, assume that \( Z_1, ..., Z_l = (-1)^l \) (this can be assumed without loss of generality, as we can simultaneously rescale the \( Z_i \)). Then \( f(0) = 1 \), so for \( q > 1 \) we can set \( g(X) = \prod_{m=1}^{\infty} f(q^{-m}X^{-1}) \). Given a collection \( \Lambda = (\lambda_1, ..., \lambda_N) \) of eigenvalues, let \( F(X_1, ..., X_N, \Lambda) \) be the joint eigenfunction of \( Y_i \) (i.e., the nonsymmetric Macdonald function). Then
\[
\tilde{F}(X_1, ..., X_N, \Lambda) := g(X_1)...g(X_N)F(X_1, ..., X_N, \Lambda)
\]
is a joint eigenfunction of $Y(f)$. So the joint eigenfunctions of $D^{(l)}_i$ can be obtained by applying Cherednik’s difference Fourier transform (Ch1) to $\tilde{F}(X_1, ..., X_N, \Lambda)$:

$$\tilde{F}(X_1, ..., X_N, \Lambda) = \mathcal{F}_\text{Cherednik}(\tilde{F}(X_1, ..., X_N, \Lambda)).$$

For $l = 1$ this should recover the formulas of [BF].

Also, the following observation was made by O. Chalykh when we sent him a preliminary version of this paper. Consider the case where $Z_1 = ... = Z_t = 0$ (this violates our restriction that $Z_i \in \mathbb{C}^*$, but this restriction is not essential here). In this case, $f(X) = X^l$, and one can take $g(X)$ to be the $l$-th power of the Gaussian $g(X) = \exp(\frac{(\log X)^2}{2\log q})$ (this function is not single-valued but this is not important if we restrict to the locus $q, X \in \mathbb{R}_+$). Therefore, the symmetrized version of formula (3.6) turns into formula (7.5) in O. Chalykh’s appendix to [CE] (up to changing $q$ to $q^{-1}$). This shows that the Hamiltonians of the twisted Macdonald-Ruijsenaars model of Theorem 7.1 in the appendix to [CE] (for type $A_{N-1}$) are a special case of the commuting Hamiltonians $e_r(D^{(l)}_1, ..., D^{(l)}_N)$ when $Z_1 = ... = Z_t = 0$.

3.7. Presentation of cyclotomic DAHA by generators and relations. Let us give a presentation of $HHI_N^t(Z, q, t)$ by generators and relations. We will use as generators the elements $T_i$, $i = 1, ..., N - 1$, and $X_i, Y_i^{\pm 1}, D^{(l)}_i$, $i = 1, ..., N$.

First of all, the elements $T_i, X_i, Y_i^{\pm 1}$ satisfy the relations of DAHA, (R1-R12) from Definition 3.1. More precisely, we need to rewrite relations (R8-R10) to account for non-invertibility of $X_i$. Namely, from (R8-R10) we get

$$X_iY_j = Y_jX_iT^{-1}_{j-1}...T^{-1}_{i+1}T^2_{i+1}...T_{j-1}, \quad i < j. \tag{3.7}$$

Similarly, we have

$$Y_iX_j = T^{-1}_{j-1}...T^{-1}_{i+1}T^2_{i+1}...T_{j-1}X_jY_i, \quad i < j. \tag{3.8}$$

Finally, we have

$$Y_iT^{-1}_{i-1}...T^{-2}_{i-1}X_i = qX_iT_i...T^2_{N-1}...T_iY_i. \tag{3.9}$$

Secondly, we have similar relations between $D^{(l)}_j$ (instead of $X_j$) and $T_i, Y_i$. Namely, we know from Corollary 3.22 that

$$[D^{(l)}_i, D^{(l)}_j] = 0, \tag{3.10}$$

and it follows from the definition that

$$T^{-1}_{i}D^{(l)}_iT^{-1}_{i} = D^{(l)}_{i+1}, \quad [T_j, D^{(l)}_i] = 0 \text{ for } |i - j| \geq 2. \tag{3.11}$$

Also since $X^{-1}_1Y_2 = Y_2T^{-2}_1X^{-1}_1$, we have

$$D^{(l)}_1Y_2 = Y_2T^{-2}_1D^{(l)}_1. \tag{3.12}$$

This implies that

$$D^{(l)}_iY_j = Y_jT^{-1}_{j-1}...T^{-1}_{i+1}T^{-2}_{i+1}...T_{j-1}D^{(l)}_i, \quad i < j. \tag{3.13}$$
Similarly, we have
\[ D_2^{(l)}T_1^2Y_1 = Y_1D_2^{(l)}, \]
which implies that
\[ D_j^{(l)}T_{j-1}...T_{i+1}T_i^2T_{i+1}...T_{j-1}^{-1}Y_i = Y_iD_j^{(l)}, \quad i < j. \]

Finally, we have
\[ D_i^{(l)}Y_1 = qT_1...T_{N-1}^2T_1Y_1D_i^{(l)}, \]
which gives
\[ D_i^{(l)}Y_iT_{i-1}^{-1}...T_{i-1}^{-2}...T_{N-1}^2...T_iY_iD_i^{(l)}. \]

Finally, we write commutation relations between \( X_i \) and \( D_j^{(l)} \). First of all, we have
\[ X_iD_i^{(l)} = (Y_1 - Z_1)...(Y_1 - Z_l), \quad D_i^{(l)}X_i = (qJ_NY_1 - Z_1)...(qJ_NY_1 - Z_l), \]
where \( J_N = T_1...T_{N-1}^2...T_1 \). Also we have
\[
[D_i^{(l)}, X_2] = \sum_{r=1}^{l} X_1^{-1}(Y_1 - Z_1)...(Y_1 - Z_{r-1})[Y_1, X_2](Y_1 - Z_{r+1})...(Y_1 - Z_l) = \\
\sum_{r=1}^{l} X_1^{-1}(Y_1 - Z_1)...(Y_1 - Z_{r-1})(T_1^{-2} - 1)X_2Y_1(Y_1 - Z_{r+1})...(Y_1 - Z_l) = \\
(t^{-1} - t) \sum_{r=1}^{l} X_1^{-1}(Y_1 - Z_1)...(Y_1 - Z_{r-1})T_1^{-1}X_2Y_1(Y_1 - Z_{r+1})...(Y_1 - Z_l) = \\
(t^{-1} - t) \sum_{r=1}^{l} X_1^{-1}(Y_1 - Z_1)...(Y_1 - Z_{r-1})X_1T_1Y_1(Y_1 - Z_{r+1})...(Y_1 - Z_l) = \\
= (t^{-1} - t) \sum_{r=1}^{l} (qJ_NY_1 - Z_1)...(qJ_NY_1 - Z_{r-1})T_1Y_1(Y_1 - Z_{r+1})...(Y_1 - Z_l).
\]
Therefore, we have
\[ [D_i^{(l)}, X_2] = \\
(t^{-1} - t) \sum_{r=1}^{l} (qJ_NY_1 - Z_1)...(qJ_NY_1 - Z_{r-1})Y_2T_1^{-2}(Y_2T_1^{-2} - Z_{r+1})...(Y_2T_1^{-2} - Z_l)T_1.
\]

Now the commutation relations between \( D_i^{(l)} \) and \( X_j \) can be obtained from this by applying the elements \( T_m \).

Thus, we have the following proposition. Let \( HH_{N_i}^{l,+} \) be the “unlocalized” version of the cyclotomic DAHA, generated by \( T_i, X_i, D_i^{(l)}, \) and \( Y_i \) (without \( Y_i^{-1} \)).
Proposition 3.27. Let $M_X$ be a monomial in $X_i$, $M_D$ a monomial in $D^{(l)}_i$, $M_Y$ a monomial in $Y_i$ with powers of $Y_i$ being $\leq l - 1$. Then the monomials of the form $M_X Y_i T_s M_D$, $s \in S_N$ form a basis in $HH^l_N$.

Proof. It is easy to see from considering the polynomial representation that these monomials are linearly independent. Therefore, it remains to establish the spanning property. We consider a filtration with $\deg(T_i) = 0$, $\deg(X_i) = \deg(D^{(l)}_i) = l$, $\deg(Y_i) = 2$. We can use the above commutation relations to order any monomial without increasing its degree. Then we can reduce the degrees of $Y_i$ below $l$ by using the relation $X_i D^{(l)}_i = (Y_i T_{i-1} ... T_1 - Z_i) ... (Y_i T_{i-1} ... T_1 - Z_i)$, and its conjugates by the $T_i$. Namely, we have the relation

$$X_i D^{(l)}_i = (Y_i T_{i-1} ... T_1 - Z_i) ... (Y_i T_{i-1} ... T_1 - Z_i),$$

which can be used to express the monomial $Y_i^l$ as

$$Y_i^l = X_i D^{(l)}_i T_{i-1} ... T_1^2 ... T_{i-1} + ...$$

where ... is a linear combination of ordered monomials of degrees $\leq l$ in $Y_i$, $T_i$ and monomials of degree $l$ involving $Y_i$ in degree $\leq l$ and some $Y_j$ with $j < i$. For example, for $i = 2$, $l = 2$ we get

$$X_2 D^{(l)}_2 = (Y_2 T_1^{-2} - Z_1)(Y_2 T_1^{-2} - Z_2),$$

so we get

$$Y_2 T_1^{-2} Y_2 = X_2 D^{(l)}_2 T_1^2 + \text{ lower degree terms}.$$ 

But

$$Y_2 T_1^{-2} Y_2 = Y_2^2 + (t^{-1} - t) Y_2 Y_1 T_1,$$

so

$$Y_2^2 = X_2 D^{(l)}_2 T_1^2 + (t - t^{-1}) Y_2 Y_1 T_1 + \text{ lower degree terms}.$$ 

This implies the required spanning property. □

Theorem 3.28. (i) The algebra $HH^l_N$ is generated by $T_i$, $X_i$, $D_i := D^{(l)}_i$, and $Y_i$ with the following defining relations:

1. relations (R1-R7), (R11), (R12) of DAHA;

2. the relations (3.7) - (3.16).

The algebra $HH^l_N$ is defined by the same generators and relations, adding the condition that $Y_i$ are invertible.

Proof. We have shown that these relations hold. Moreover, it was shown in the proof of Proposition 3.27 that using these relations, we can reduce every monomial to a linear combination of basis monomials. This implies that the relations are defining. □
Proposition 3.29. There is an involutive automorphism $\phi$ of $HH^1_N$ such that $\phi(X_i) = D_i$, $\phi(D_i) = X_i$, $\phi(T_i) = T^{-1}_i$, $\phi(t) = t^{-1}$, $\phi(q) = q^{-1}$ and
\[\phi(Y_i) = qT_i...T^2_{N-1}...T_1Y_iT^{-1}_{N-1}...T_1^{-2}...T^{-1}_{i-1}\]
This automorphism preserves the subalgebra $HH^1_{N^+}$.

Proof. We check that $\phi$ preserves the defining relations. Relations (R1-R3) go to themselves. Relations (R4,R5) get exchanged with (3.11). Relations (R6,R7) go to themselves once (R1-R3) are imposed. Relations (R11) get exchanged with (3.10) and (R12) with themselves once (R1-R3,R6,R7) are imposed. Relation (3.7) gets exchanged with (3.12), (3.8) with (3.13), (3.9) with (3.14). Relations (3.15) go to themselves. So it remains to see that relation (3.16) is preserved.

Let $K_N = T_2...T^2_{N-1}...T_2$. We have
\[\sum_{r=1}^l X^{-1}_2(Y^2_2T^{-2}_r - Z_1)...(Y^2_2T^{-2}_r - Z_{r+1})|Y^2_2T^{-2}_r, X_1|\]
\[\sum_{r=1}^l X^{-1}_2(Y^2_2T^{-2}_r - Z_1)...(Y^2_2T^{-2}_r - Z_{r+1})|Y^2_2T^{-2}_r, X_1|\]
\[(t^{-1} - t) \sum_{r=1}^l X^{-1}_2(Y^2_2T^{-2}_r - Z_1)...(Y^2_2T^{-2}_r - Z_{r+1})|Y^2_2T^{-2}_r, X_1|\]
\[(t^{-1} - t) \sum_{r=1}^l X^{-1}_2(Y^2_2T^{-2}_r - Z_1)...(Y^2_2T^{-2}_r - Z_{r+1})|Y^2_2T^{-2}_r, X_1|\]
Thus,
\[(3.17) \quad [D_2, X_1] = (t^{-1} - t) \sum_{r=1}^l (qK_N Y_2 - Z_1)...(qK_N Y_2 - Z_{r+1})|qK_N Y_2|\]
Now it is easy to see that $\phi$ maps (3.16) to (3.17) (as $Y_1$ commutes with $K_N$ and $Y_2$, hence with $K_N Y_2$).

Remark 3.30. It is easy to see that as $q \to 1$, the involution $\phi$ degenerates to the involution $\phi$ on $H^2_{N,deg}$ constructed in Remark 2.25.

Corollary 3.31. The algebra $HH^1_{N^+}(Z_1, ..., Z_1, q, t)$ is a free module over $\mathbb{C}[X_1, ..., X_N] \otimes \mathbb{C}[D_1, ..., D_N]$ of rank $N! \cdot N^N$, where the first factor acts by left multiplication and the second one by right multiplication.

Proof. This follows immediately from Proposition 3.27.

\[32\]
The proof of Proposition 3.27 in fact shows that for any \( l \geq 0 \) ordered products of \( M_X, M_Y, T_s, M_D \) in any of the 24 possible orders are a spanning set for \( HH^i_N \), and those of them with degrees of \( Y_i \) at most \( l - 1 \) are a basis for \( l \geq 1 \). This implies that we also have another basis of this algebra, formed by monomials \( M_X M_D T_s M_Y \) without restriction on the degree of \( Y_i \), but with the restriction that for each \( i \) either \( X_i \) is missing in \( M_X \) or \( D_i \) is missing in \( M_D \). Indeed, if this restriction is not satisfied, we may use the relation \( X_1 D_1 = f(Y_1) \) and its permutations to lower the number of \( X_i \) and \( D_i \), and it is easy to see by looking at the polynomial representation that monomials with this restriction are linearly independent. Thus we obtain the following proposition.

**Proposition 3.32.** The elements \( M_X M_D \) which miss either \( X_i \) or \( D_i \) for each \( i \) form a basis of \( HH^i_N \) as a left or right module over the positive part of the affine Hecke algebra \( H^+_N \) generated by \( T_s, s \in S_N \) and \( Y_i, Y_i^{-1} \), and a basis of \( HH^i_N \) as a left or right module over the affine Hecke algebra \( H_N \) generated by \( T_s, s \in S_N \) and \( Y_i, Y_i^{-1} \); in particular, \( HH^i_N \) is a free module over \( H^+_N \) and \( HH^i_N \) is a free module over \( H_N \).

Note that the basis of Proposition 3.32 is labeled by \( N \)-tuples of integers, \((m_1, \ldots, m_N)\). Namely, if \( M_X M_D \) contains \( X_i^p \) then we set \( m_i = p \), and if it contains \( D_i^p \) then we set \( m_i = -p \).

Another, geometric proof of Proposition 3.32 (for a geometric version of \( HH^i_N \)) will be given in Section 4.

### 4. Geometric realization

#### 4.1. A variety of triples.
We consider a quiver with a set of vertices \( I \) and a set of arrows \( \Omega \). Let \( V = \bigoplus_{i \in I} V_i, W = \bigoplus_{i \in I} W_i \) be \( \Omega \)-graded finite dimensional \( \mathbb{C} \)-vector spaces; \( d_i := \dim V_i \). Given a length \( \ell \) sequence \( i = (i_1, \ldots, i_\ell) \in I^\ell \) and a length \( \ell \) sequence \( a = (a_1, \ldots, a_\ell) \) of positive integers such that \( \sum_{n: i_n = i} a_n = d_i \) for any \( i \in I \), we choose an \( I \)-graded flag in \( V: V = V^0 \supset V^1 \supset \ldots \supset V^\ell = 0 \) such that \( V^{n-1}/V^n \) is an \( a_n \)-dimensional vector space supported at the vertex \( i_n \) for any \( n = 1, \ldots, \ell \).

We set \( K = \mathbb{C}((z)) \supset \mathbb{C}[z] = \mathcal{O} \). We consider the following flag of \( I \)-graded lattices in \( V_K = V \otimes K \): \( \ldots \supset L_{-1} \supset L_0 \supset L_1 \supset \ldots \), where \( L_{r+\ell} = zL_r \) for any \( r \in \mathbb{Z} \); \( L_0 = V_O \); \( L_n/L_\ell = V^n \subset V = L_0/L_\ell \) for any \( n = 1, \ldots, \ell \). Let \( GL(V) := \prod_{i \in I} GL(V_i) \), and let \( \mathcal{P} \subset GL(V) \supset \mathcal{O} \subset GL(V) \) be the stabilizer of the flag \( L \). Then \( GL(V)\mathcal{O}/\mathcal{P} \) is the set of points of the ind-projective moduli space \( \mathcal{F}_\ell \) of flags of \( I \)-graded lattices \( \ldots \supset M_{-1} \supset M_0 \supset M_1 \supset \ldots \) in \( V_K \) such that \( M_{r+\ell} = zM_r \) for any \( r \in \mathbb{Z} \), and \( M_{n-1}/M_n \) is an \( a_n \)-dimensional vector space supported at the vertex \( i_n \) for any \( n = 1, \ldots, \ell \). This is a partial affine flag variety of the reductive group \( GL(V) \). Note that the set of connected components \( \pi_0(\mathcal{F}_\ell) \) is naturally identified with \( \mathbb{Z}^\ell \) (the virtual graded dimension of \( M_0 \)).

Let \( \mathcal{R} \) be the moduli space of the following data (cf. [L Section 1]):

(a) \( M_\bullet \in \mathcal{F}_\ell \);
(b) a $\mathcal{K}$-linear homomorphism $p_i: W_{i,K} \to V_{i,K}$ for any $i \in I$;
(c) a $\mathcal{K}$-linear homomorphism $b_{i,j}: V_{i,K} \to V_{j,K}$ for any $i \to j \in \Omega$; such that
(1) $b := \sum_{i \to j \in \Omega} b_{i,j}$ takes $L_r$ to $L_{r+1}$ and $M_r$ to $M_{r+1}$ for any $r \in \mathbb{Z}$;
(2) $p := \sum_{i \in I} p_i$ takes $W_\mathcal{O}$ to $L_0 \cap M_0$.

Note that when $\ell = 1$, we have $\mathcal{P} = GL(V)_{\mathcal{O}}$, and $\mathcal{R}$ is nothing but
the variety of triples $\mathcal{R}_{GL(V),N}$ associated in [BFNa, 2(i)] to a $GL(V)$-module $N = \bigoplus_{i \to j \in \Omega} \text{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \text{Hom}(W_i, V_i)$. The definition of equivariant
Borel-Moore homology $H^\bullet_{GL(V)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}_{GL(V),N})$ (respectively, equivariant $K$-theory
$K^{GL(V)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}_{GL(V),N})$) and the construction of convolution product on it in [BFNa, Sections 2.3] work without any changes in our situation, and produce the convolution algebras $H^\bullet_{\mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$ and $K^{\mathcal{P} \times \mathbb{C}^\times}(\mathcal{R})$. Moreover, if we choose a Cartan torus $T(W_i) \subset GL(W_i)$ and set $T(W) := \prod_{i \in I} T(W_i) \subset \prod_{i \in I} GL(W_i) =: GL(W)$ (a flavor
symmetry group), we obtain the convolution algebras $H^\bullet_{\mathcal{P} \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R})$ and $K^{\mathcal{P} \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R})$. Here the first factor $\mathbb{C}^\times$ acts by dilations in the fibers of the projection

\[(4.1) \quad \varpi: \mathcal{R} \to \mathcal{F}_\ell.\]

The following easy result will be important in the future.

**Lemma 4.1.** The algebra $H^\bullet_{\mathcal{C}^\times \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R})$ is free as a module over the equivariant
point cohomology $H^\bullet_{\mathcal{C}^\times \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(pt)$. Similarly, the algebra $K^{\mathcal{C}^\times \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R})$ is
free as a module over $K^{\mathcal{C}^\times \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(pt)$.

**Proof.** It is enough to show that $\mathcal{R}$ has an algebraic cell decomposition, which is invariant
under the maximal torus of the group $\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times$. For this we need to choose an Iwahori subgroup $\mathcal{I}$ of $GL(V)_{\mathcal{K}}$ which is contained in $\mathcal{P}$. Then the $\mathcal{I}$-orbits on $\mathcal{F}_\ell$ are affine spaces; hence their preimages in $\mathcal{R}$ are (infinite-dimensional)
affine spaces as well, which are clearly invariant under $\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{I} \times \mathbb{C}^\times$, hence
under some maximal torus of $\mathbb{C}^\times \times T(W)_{\mathcal{O}} \times \mathcal{P} \times \mathbb{C}^\times$. \(\square\)

In case $W = W' \oplus W''$, we denote the variety of triples corresponding to $W'$ (resp.
$W''$) by $\mathcal{R}'$ (resp. $\mathcal{R}''$), and we have an evident closed embedding $z: \mathcal{R}' \hookrightarrow \mathcal{R}$. The argument of [BFNa, Lemma 5.11] goes through word for word in our situation and proves that

\[(4.2) \quad z^* : H^\bullet_{\mathcal{C}^\times \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}) \hookrightarrow H^\bullet_{\mathcal{C}^\times \times T(W')_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}') \otimes H^\bullet_{T(W'')}(pt)\]
and

\[(4.3) \quad z^* : K^{\mathcal{C}^\times \times T(W)_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}) \hookrightarrow K^{\mathcal{C}^\times \times T(W')_{\mathcal{O}} \times \mathbb{C}^\times}(\mathcal{R}') \otimes K_{T(W'')}(pt)\]

are the convolution algebra homomorphisms.
4.2. Jordan quiver. In what follows we consider a special case of the construction of Section 4.1 where $I$ consists of a single vertex, and $\Omega$ consists of a single loop, $W = \mathbb{C}^I$ with a basis $e_1, \ldots, e_l$ (hence the diagonal torus $T(W) \subset GL(W)$), and $V = \mathbb{C}^N$ with a basis $v_1, \ldots, v_N$. Moreover, $\ell = N$, $a_n = 1$ for any $n = 1, \ldots, N$, and $V^*$ is a complete flag $V^* := \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_{N-n}$. We fix a flag of lattices $L_i \subset V((z))$, $i \in \mathbb{Z}$, such that $L_0 = V[[z]]$, $L_{j+N} = zL_j$, and $L_j = zV[[z]] \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_{N-j}$, $j = 0, \ldots, N - 1$.

The space of triples of Section 4.1 is the moduli space of the following data:

(a) a sequence of $\mathbb{C}[[z]]$-lattices $M_i \subset V((z))$, $i \in \mathbb{Z}$, such that $M_i \supsetneq M_{i+1}$ and $M_{j+N} = zM_j$;
(b) a $\mathbb{C}((z))$-linear map $b: V((z)) \to V((z))$; and
(c) a $\mathbb{C}((z))$-linear map $p: W((z)) \to V((z))$;

such that

1. $b$ strongly preserves $L$ and $M_i$, i.e., $bL_i \subset L_{i+1}$ and $bM_i \subset M_{i+1}$; and
2. $pW[[z]] \subset L_0 \cap M_0$.

The basis of $V$ gives rise to the diagonal torus $T(V) \subset GL(V)$, and we denote by $y_1, \ldots, y_N$ the generators of $H^\bullet_{T(V)}(pt)$. Also, we denote by $z_1, \ldots, z_l$ (resp. $\hbar, -k$) the generators of $H^\bullet_{T(V)}(pt)$ (resp. $H^\bullet_{C^x}(pt)$ for the loop rotation $\mathbb{C}^x$, $H^\bullet_{C^x}(pt)$ for the dilation $\mathbb{C}^x$). We shall denote the corresponding generators of $K_{T(V)}(pt)$, $K_{T(W)}(pt)$, $K_{C^x}(pt)$, $K_{C^x}(pt)$ for $Y_1, \ldots, Y_N; Z_1, \ldots, Z_l; q; t$. The algebra $H^\bullet_{C^x \times T(W) \sigma \times \mathbb{P} \times \mathbb{C}^x}(\mathcal{R})$ (resp. $K^\bullet_{C^x \times T(W) \sigma \times \mathbb{P} \times \mathbb{C}^x}(\mathcal{R})$) will be denoted $\mathcal{H}^l_{N, \text{deg}}$ (resp. $\mathcal{H}^l_{N, \text{deg}}$). We shall also denote $H^{l, 0}_{N, \text{deg}}$ by $\mathcal{H}^l_{N, \text{deg}}$, and $H^{l, 0}_{N}$ by $\mathcal{H}^l_{N}$. According to (4.2) and (4.3) we have algebra embeddings $z^*: \mathcal{H}^l_{N, \text{deg}} \hookrightarrow \mathcal{H}^l_{N, \text{deg}}[z_1, \ldots, z_l]$ and $z^*: \mathcal{H}^l_{N} \hookrightarrow \mathcal{H}^l_{N, \text{deg}}[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$.

Note that for $W = 0$ the variety of triples $\mathcal{R}$ is nothing but the affine Steinberg variety of $GL(V)$, and $\mathcal{F}^l$ is nothing but the affine flag variety of $GL(V)$. It is well known that $\mathcal{H}^l_{N, \text{deg}} \simeq \mathcal{H}^l_{N, \text{deg}}$ (see e.g. [OY]), and one can check that $\mathcal{H}^l_{N} \simeq \mathcal{H}^l_{N}$, cf. [VV] (see [BFNa], Remark 3.9(2)). More precisely, for $n = 0, \ldots, N - 1$, we denote by $\mathbb{P}^1_n \subset \mathcal{F}^l$ the projective line formed by all the flags $M_*$ of lattices (see Section 4.1) such that $M_m = L_m$ for $m \not\equiv n \pmod{N}$. Each $\mathbb{P}^1_n$ contains the base point $L_* \in \mathcal{F}^l$, and we denote by $\mathbb{A}^1_n \subset \mathbb{P}^1_n$ the complement. The restriction of the projection $\pi: \mathcal{R} \to \mathcal{F}^l$ to $\mathbb{A}^1_n$ is a (profinite dimensional) vector bundle, and the closure of $\pi^{-1}(\mathbb{A}^1_n)$ in $\mathcal{R}$ is still a vector bundle over $\mathbb{P}^1_n$, to be denoted by $\mathbb{P}^1_n$. We define $s_n \in \mathcal{H}^l_{N, \text{deg}}$ by $1 + \llbracket \mathbb{P}^1_n \rrbracket$ for $n = 0, \ldots, N - 1$ (the fundamental cycle of $\mathbb{P}^1_n$). Also, let $\pi$ (resp. $\pi^{-1}$) be a point-orbit of the Iwahori group $\mathcal{P}$ in $\mathcal{F}^l$ consisting of the flag $M_*$ such that $M_m = L_m$ for any $m \in \mathbb{Z}$. Finally, we define $\pi^\pm \in \mathcal{H}^l_{N, \text{deg}}$ as $[\pi^{-1}(\mathbb{A}^1_n)]$. Now the desired isomorphism $\mathcal{H}^l_{N, \text{deg}} \cong \mathcal{H}^l_{N, \text{deg}}$ takes the generators of $\mathcal{H}^l_{N, \text{deg}}$ to the same named elements of $\mathcal{H}^l_{N, \text{deg}}$, except $\hbar \mapsto -\hbar$.

Furthermore, we define $T_n \in \mathcal{H}^l_{N, \text{deg}}$ as $-1 - [\mathcal{O}_{\mathbb{P}^1_n}(-2)]$ for $n = 0, \ldots, N - 1$, and we define $\pi^\pm \in \mathcal{H}^l_{N, \text{deg}}$ as $[\mathcal{O}_{\pi^{-1}(\mathbb{A}^1_n)}]$. The desired isomorphism $\mathcal{H}^l_{N} \cong \mathcal{H}^l_{N}$ takes the generators of $\mathcal{H}^l_{N}$ to the same named elements of $\mathcal{H}^l_{N}$, except $q \mapsto q^{-1}$. 

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Then we have
\[ \pi_\mathcal{H}^l, \pi_\mathcal{N} \text{ and } \pi \text{ takes this } \pi \text{ to the one in } \pi_\mathcal{H}^l, \pi_\mathcal{N}, \text{ resp. } \pi_\mathcal{H}^l, \pi_\mathcal{N} \]  
and
\[ \pi_\mathcal{H}^l, \pi_\mathcal{N} \text{ and } \pi \text{ takes this } \pi \text{ to the one in } \pi_\mathcal{H}^l, \pi_\mathcal{N}, \text{ resp. } \pi_\mathcal{H}^l, \pi_\mathcal{N} \]  
We preserve the name $\pi$ for $[ \mathcal{O}_{\mathcal{H}}^{-1}(\mathcal{N}) ]$ in $\pi_\mathcal{H}^l, \pi_\mathcal{N}$ (resp. $[ \mathcal{O}_{\mathcal{H}}^{-1}(\mathcal{N}) ]$ in $\pi_\mathcal{H}^l, \pi_\mathcal{N}$) since $z^*$ takes this $\pi$ to the one in $\pi_\mathcal{H}^l, \pi_\mathcal{N}$ (resp. in $\pi_\mathcal{H}^l, \pi_\mathcal{N}$). For the same reason we preserve the names $s_n$ (resp. $T_n$, $n = 1, \ldots, N - 1$, for the corresponding elements of $\pi_\mathcal{H}^l, \pi_\mathcal{N}$ (resp. $\pi_\mathcal{H}^l, \pi_\mathcal{N}$). Finally, we set $z_m = z_m + k - \hbar$, and $Z_m = Z_m q^{-1} t_1 - 1$. So the following diagrams commute:

\[ \begin{array}{ccc}
\mathcal{H}^l_{\mathcal{H}, \deg} & \longrightarrow & \pi_\mathcal{H}^l, \pi_\mathcal{N}, \deg
\
\downarrow & & \downarrow z^*
\
\mathcal{H}^l_{\mathcal{H}, \deg}[z_1, \ldots, z_l] & \sim & \pi_\mathcal{H}^l, \pi_\mathcal{N}, \deg[z_1, \ldots, z_l]
\end{array} \]  

\[ \begin{array}{ccc}
\mathcal{H}^l_{\mathcal{H}, \deg} & \longrightarrow & \pi_\mathcal{H}^l, \pi_\mathcal{N}, \deg
\
\downarrow & & \downarrow z^*
\
\mathcal{H}^l_{\mathcal{H}, \deg}[Z_1^{\pm 1}, \ldots, Z_l^{\pm 1}] & \sim & \pi_\mathcal{H}^l, \pi_\mathcal{N}, \deg[Z_1^{\pm 1}, \ldots, Z_l^{\pm 1}]
\end{array} \]  

Note that in diagram (4.6) we have bigradings defined by $\deg(X_i) = (1, 0)$, $\deg(y_i) = \deg(z, h, k) = (0, 1)$, which are preserved by all the maps.

The following theorem is independently obtained in [W Lemma 4.2].

**Theorem 4.2.** The map $\xi : \mathcal{H}^l_{\mathcal{H}, \deg} \rightarrow \pi_\mathcal{H}^l, \pi_\mathcal{N}, \deg$ in diagram (4.6) is an isomorphism of bigraded algebras.

**Proof.** Recall from Lemma 4.1 that $\pi_\mathcal{H}^l, \pi_\mathcal{N}$ is a free $\mathbb{C}[z_1, \ldots, z_l, h, k]$-module. Thus it suffices to show that $\xi_0$ is an isomorphism, where $\xi_0$ is the specialization of $\xi$ at $\hbar = k = 0, z = 0$. It is clear that $\xi_0$ is injective, hence so are $\xi$ and all its specializations $\xi_{z, h, k}$. Since the bigraded components $\mathcal{H}^l_{\mathcal{H}, \deg}[r, s]$ are finite dimensional, it suffices to show that the specialization $\xi_{z, h, k}$ is an isomorphism for Weil generic $z, h, k$. 


We will now use the following easy lemma about unital rings.

**Lemma 4.3.** Let $B$ be a unital ring and $A \subset B$ a unital subring. If $e \in A$ is an idempotent such that $AeA = A$ and $eBe = eB$, then $A = B$.

**Proof.** Since $AeA = A$, we have $AeB = AeAB = AB = B$. Similarly $BeA = B$. Thus $AeBeA = BeA = B$. But since $eBe = eB$ and $AeA = A$, we have $AeBeA = AeAeA = AeA = A$. Thus $A = B$. □

Now let $e$ be the symmetrizer of $S_N$, and let us apply Lemma 4.3 to $A = \mathcal{H}H^t_{N,\deg}(z,1,k)$ and $B = \mathcal{H}H^t_{N,\deg}(z,-1,k)$. Using Theorem 2.19, we see that the condition $AeA = A$ is satisfied for generic $z,k$, namely when the corresponding parameters of the cyclotomic rational Cherednik algebras are not aspherical (see e.g. [BE], Subsection 4.1). In fact, it suffices to consider the case $z = 0, k = 0$, when this is easy, since $\mathcal{H}H^t_{N,\deg}(0,1,0) = S_N \rtimes (\mathcal{D}(\mathbb{C})^{Z_{12}})^{\otimes N}$ is a simple algebra (by [Mo, Theorem 2.3]). Also, by [BFN], Proposition 3.24], the theorem holds for spherical subalgebras, i.e., $e_0^e$ is an isomorphism. Hence so are $e_0^e$ and all its specializations $e_0^e$, which yields the condition $eAe = eBe$ for all $z, h, k$. Thus Lemma 4.3 applies and Theorem 4.2 follows. □

We expect that an analog of Theorem 4.2 also holds in the K-theoretic setting, i.e., for diagram (4.7). Let us prove a formal version of this statement. Let $q = e^{th}, t = e^{-eh}, Z_i = e^{zh_i}$. Let $\mathcal{H}H^t_{N,formal}$ be the corresponding formal completion of $\mathcal{H}H^t_{N}$ (obtained by viewing equivariant K-theory as a formal deformation of equivariant Borel-Moore homology). Diagram (4.7) furnishes a map $\hat{\xi} : \mathcal{H}H^t_{N,formal} \to \mathcal{H}H^t_{N,formal}$.

**Corollary 4.4.** The map $\hat{\xi}$ is an isomorphism.

**Proof.** By Lemma 4.1, $\mathcal{H}H^t_{N,formal}$ is a flat formal deformation of $\mathcal{H}H^t_{N,\deg}$ over $\mathbb{C}[[e]]$. Finally, $\hat{\xi}|_{e=0} = \xi$. This implies the corollary. □

In particular, this gives another, geometric proof of the facts that $\mathcal{H}H^t_{N,\deg}$ is a free bigraded $\mathbb{C}[z_1, ..., z_l, h, k]$-module, and the algebra $\mathcal{H}H^t_{N,formal}(z, h, k)$ is its flat formal deformation.

We also have

**Theorem 4.5.** Let $e$ be the symmetrizer of the finite Hecke algebra generated by $T_i$, $1 \leq i \leq N - 1$. Then the natural map

$$e_0^e : e_0^e \mathcal{H}H^t_{N}(Z, q, t)e \to e_0^e \mathcal{H}H^t_{N}(Z, q, t)e$$

is an isomorphism when $q = t = 1$ and $Z_i = 1$ for all $i$.

**Proof.** Same as the proof of [BFN], Proposition 3.24]. □
Theorem 4.6. Let \( \widehat{HH}_N^l, \widehat{\mathcal{H}}_N^l \) be the completions of \( HH_N^l, \mathcal{H}_N^l \) at \( q = t = Z_i = 1 \) (as modules over \( \mathbb{C}[Z_1^{\pm 1}, ..., Z_l^{\pm 1}, q^{\pm 1}, t^{\pm 1}] \)). Then the map

\[
\widehat{\xi} : \widehat{HH}_N^l \to \widehat{\mathcal{H}}_N^l
\]

is an isomorphism.

Proof. The proof is analogous to the proof of Theorem 4.2 using Theorem 4.5. Namely, the identity \( A_eA = A \) for \( A = HH_N^l(Z, q, t) \) is established for generic \( q \) and \( Z = 1, t = 1 \) using the fact that in this case \( A \) is a simple algebra by [Mo, Theorem 2.3]. □

5. Cyclotomic DAHA and multiplicative quiver and bow varieties

5.1. Multiplicative quiver varieties. Let \( t \in \mathbb{C}^* \) be not a root of unity, and \( Z_1, ..., Z_l \in \mathbb{C}^* \) be such that \( Z_i/Z_j \) is not an integer power of \( t \) for \( i \neq j \). Let \( Q_l \) be the cyclic quiver \( \hat{A}_{l-1} \) with vertices \( 1, ..., l \) and an additional “Calogero-Moser vertex” 0 attached to the vertex 1. Let \( \mathcal{M}_N^l(Z, t) \) be the multiplicative quiver variety for \( Q_l \) with dimension vector \( d_1 = ... = d_l = N \) and \( d_0 = 1 \), see [CBS]. Namely, given complex vector spaces \( V_i, i = 1, ..., l \), with \( \dim V_i = N \), \( \mathcal{M}_N^l(Z, t) \) is the variety of collections of linear maps \( X_i : V_{i+1} \to V_i \) and \( D_i : V_i \to V_{i+1} \) (where addition is mod \( l \)) satisfying the equations

\[
Z_i(1 + X_iD_i) = Z_{i-1}(1 + D_{i-1}X_{i-1}), 2 \leq i \leq l
\]

and

\[
Z_1(1 + X_1D_1)T = Z_l(1 + D_lX_l),
\]

where \( T : V_1 \to V_1 \) is an operator conjugate to \( \text{diag}(t^{-1}, ..., t^{-1}, t^{n-1}) \), modulo simultaneous conjugation (i.e., the corresponding categorical quotient).

Example 5.1. Let \( l = 1 \). Then there is no dependence on \( Z_1 \), and \( \mathcal{M}_N^l(t) \) is the variety of pairs \( (X, D) \) of \( N \) by \( N \) matrices such that

\[
(1 + XD)T = (1 + DX),
\]

where \( 1 + XD \) is invertible, and \( T \) is as above, modulo simultaneous conjugation.

Let \( X := X_1...X_l, D := D_l...D_1, Y := Z_1(1 + X_1D_1) \). Consider the operators \( L_+ := Z_1...Z_lXD, L_- := Z_1...Z_lDX \).

Lemma 5.2. We have

\[
L_+ = (Y - Z_1)...(Y - Z_l),
\]

\[
L_- = (YT - Z_1)...(YT - Z_l).
\]

Proof. We prove the formula for \( L_+ \); the formula for \( L_- \) is proved in a similar way.

It suffices to prove by induction in \( r \) that

\[
Z_1...Z_rX_1...X_rD_r...D_1 = (Y - Z_1)...(Y - Z_r).
\]
The base $r = 0$ is obvious. For $r > 0$, we have, using the induction assumption:

\[ Z_1...Z_rX_1...X_rD_r...D_1 = X_1...X_{r-1}(Z_rX_rD_r)X_{r-1}^{-1}...X_1^{-1}(Y - Z_1)...(Y - Z_{r-1}). \]

But

\[
X_1...X_{r-1}(Z_rX_rD_r)X_{r-1}^{-1}...X_1^{-1} = \\
X_1...X_{r-1}(Z_{r-1}D_{r-1}X_{r-1} + Z_{r-1} - Z_r)X_{r-1}^{-1}...X_1^{-1} = \\
= Z_{r-1} - Z_r + X_1...X_{r-2}(Z_{r-1}X_{r-1}D_{r-1})X_{r-2}^{-1}...X_1^{-1}.
\]

Thus,

\[
X_1...X_{r-1}(Z_rX_rD_r)X_{r-1}^{-1}...X_1^{-1} = Z_1 - Z_r + Z_1X_1D_1 = Y - Z_r.
\]

This implies the induction step. \hfill \Box

Thus we have

\[
(5.3) \quad XD = (Z_1^{-1}Y - 1)...(Z_l^{-1}Y - 1), \quad DX = (Z_1^{-1}YT - 1)...(Z_l^{-1}YT - 1),
\]

\[
(5.4) \quad YX = XYT, \quad YTD = DY.
\]

**Lemma 5.3.** We have an isomorphism

\[ \Phi : \mathcal{M}_N^1(Z_1, ..., Z_l, t) \rightarrow \mathcal{M}_N^1(Z_l, ..., Z_1, t^{-1}) \]

given by $\Phi(X_i) = D_{l+1-i}$, $\Phi(D_i) = X_{l+1-i}$, $\Phi(Z_i) = Z_{l+1-i}$.

**Proof.** It is easy to check that the relations defining these varieties are matched by $\Phi$. \hfill \Box

**5.2. Quadruple varieties and their connection to multiplicative quiver varieties.** Let $\mathcal{M}_N^1(Z, t)$ be the variety of quadruples of matrices $(X, D, Y, T)$ satisfying \([5.3], [5.4]\) such that $Y$ is invertible, modulo simultaneous conjugation (i.e., the categorical quotient). We have a natural map

\[ \psi : \mathcal{M}_N^1(Z, t) \rightarrow \mathcal{M}_N^1(Z, t) \]

sending $X_i, D_i, 1 \leq i \leq l$ to $(X, D, Y, T)$.

**Proposition 5.4.** Any collection $(X, D, Y, T) \in \mathcal{M}_N^1(Z, t)$ acts irreducibly on $\mathbb{C}^N$.

**Proof.** Assume the contrary. Then there is an invariant subspace or quotient $V$ for $(X, D, Y, T)$ of dimension $1 \leq n \leq N - 1$ on which $T$ acts by $t^{-1}$. So by Lemma 5.2 on $V$ we have

\[
XD = (Z_1^{-1}Y - 1)...(Z_l^{-1}Y - 1), \quad DX = ((tZ_1)^{-1}Y - 1)...((tZ_l)^{-1}Y - 1),
\]

\[
tXY = XY,
\]

and

\[
(5.5) \quad YD = tDY.
\]

Equation \([5.5]\) implies that $D$ cannot be invertible (otherwise taking determinants of both sides gives a contradiction). Thus there is a nonzero vector $v \in V$ such that
$Dv = 0$. Since (again by \([5.3]\)) $\text{Ker} D$ is $Y$-stable, we can choose $v$ so that $Yv = \lambda v$
for some $\lambda \neq 0$. Since $XDv = 0$, we have $\lambda = Z_j$ for some $j$.

We may assume that $V$ is irreducible for the action of $(X, D, Y, T)$. Then $V$ has a
basis $v, Xv, \ldots, X^{n-1}v$ with $YX^iv = t^{-i}\lambda X^iv$, and

$$DX^i v = (Z_1^{-1}\lambda t^{-i} - 1)(Z_2^{-1}\lambda t^{-i} - 1)X^i v.$$ 

In particular, since $X^nv = 0$, we must have $\lambda = t^n Z_m$ for some $m$. Thus, we have
$t^n Z_m = Z_j$, which contradicts our assumption on the $Z_i$. \hfill \Box

By Schur’s lemma, Proposition \([5.4]\) implies that any operator $A$ commuting with
$X, D, Y, T$ has to be a scalar, so $M'_N(Z, t)$ is, in fact, the ordinary quotient of the set
of quadruples $(X, D, Y, T)$ satisfying \([5.3], [5.4]\) by the free action of $PGL_N(C)$.

**Corollary 5.5.** Every collection of endomorphisms $A_i : V_i \rightarrow V_i$ which commute with
$(X_1, \ldots, X_l, D_1, \ldots, D_l)$ satisfying \([5.1], [5.2]\) is a scalar (the same at all vertices).

**Proof.** Suppose we have such a collection. Then $A_1$ commutes with $(X, D, Y, T)$, so
by Proposition \([5.4]\), it has to be a scalar. So by shifting $A_i$ by the same scalar we may
assume that $A_1 = 0$. Our job is to show that $A_i = 0$ for all $i$. Assume the contrary, i.e.
that $A_s \neq 0$ for some $s$.

Let $V'_i = \text{Im} A_i$. Then $V'_i = 0$, so there exist $1 \leq i < j \leq l$ such that $V'_{i+1} = V'_{i+2} = \ldots = V'_{j} \neq 0$, while $V'_i = V'_{j+1} = 0$. The collection of nonzero spaces $V'_s$, $i < s \leq j$ is
invariant under the operators $X_s, D_s$, and these operators satisfy on $V'_s$ the equations

$$Z_{i+1}(1 + X_{i+1}D_{i+1}) = Z_i, \quad Z_{i+2}(1 + X_{i+2}D_{i+2}) = Z_{i+1}(1 + D_{i+1}X_{i+1}), \ldots,$$

$$Z_j = Z_{j-1}(1 + D_{j-1}X_{j-1}).$$

(if $j = i + 1$, then we get just one equation $Z_i = Z_{i+1}$). If $j \geq i + 2$, this implies
that any nonzero vector $v \in V'_i$ is an eigenvector of the operator $Z_{i+1}(1 + X_{i+1}D_{i+1})$
with eigenvalue $Z_i$. Since $Z_{i+1} \neq Z_i$, this implies that $D_{i+1}v$ is an eigenvector of the
operator

$$Z_{i+1}(1 + D_{i+1}X_{i+1}) = Z_{i+2}(1 + X_{i+2}D_{i+2})$$

with eigenvalue $Z_i$. Continuing like this, we find that $Z_{j-1}(1 + D_{j-1}X_{j-1})$ has eigenvector
$D_{j-1} \ldots D_{i+1}v$ with eigenvalue $Z_i$, hence $Z_i = Z_j$. This is a contradiction, which
proves the corollary. \hfill \Box

Corollary \([5.5]\) implies that $M'_N(Z, t)$ is also an ordinary quotient. A similar argument shows that equations \([5.1], [5.2]\) define a smooth complete intersection, i.e., the
multiplicative quiver variety $M'_N(Z, t)$ is smooth (in fact, both of these statements
follow from the results of [CBS]).

Let $M'_N(Z, t)^o$ be the open subset of $M'_N(Z, t)$ where $X$ is invertible. On this set, $D$
is redundant (i.e., expresses in terms of $X$ and $Y$), and the only equation we are left with is

$$YX = XYT.$$
Thus, $M_N^I(t)^\circ = M_N^I(Z, t)^\circ$ is independent of the $Z_i$ and is the multiplicative Calogero-Moser space considered in [Ob1] (the phase space of the Ruijsenaars integrable system). In particular, as explained in [Ob1], $M_N^I(t)^\circ$ is smooth and connected.

Now let us study the properties of the map $\psi$. Note that if $l = 1$, $\psi$ is tautologically an isomorphism. Moreover, we claim that $\psi$ is an isomorphism $\psi^{-1}(M_N^I(t)^\circ) \to M_N^I(t)^\circ$.

Indeed, if $X$ is isomorphic. Moreover, we claim that $\psi$ is an isomorphism $\psi^{-1}(M_N^I(t)^\circ) \to M_N^I(t)^\circ$.

Proof. Our job is to show that the map

$$\psi^*: \mathcal{O}(M_N^I(Z, t)) \to \mathcal{O}(M_N^I(Z, t))$$

is surjective. By the Fundamental Theorem of invariant theory, $\mathcal{O}(M_N^I(Z, t))$ is generated by the elements $\text{Tr}(w)$, where $w$ are words in $X, D, Y^{\pm 1}$, and $T^{\pm 1}$. So it suffices to show that $\mathcal{O}(M_N^I(Z, t))$ is generated by the elements $\psi^*\text{Tr}(w)$. We know that this algebra is generated by expressions $\text{Tr}(u)$, where $u$ is any cyclic word (i.e., closed path) consisting of $D_i, X_i, (1 + X_iD_i)^{\pm 1}$ and $T^{\pm 1}$, so it suffices to show that any such cyclic word can be expressed as a cyclic word $w$ in $X, D, Y^{\pm 1}$, and $T^{\pm 1}$.

Let $\Lambda$ be the deformed multiplicative preprojective algebra of the Calogero-Moser quiver $Q_l$ defined in [CBS]; thus, $\mathcal{O}(M_N^I(Z, t))$ is the representation variety of $\Lambda$ for the dimension vector $(d_i)$. Let $e_1 \in \Lambda$ be the idempotent of the vertex 1.

Lemma 5.7. One has $\Lambda = \Lambda e_1 \Lambda$.

Proof. Assume the contrary. Then we have a nonzero $\Lambda$-module $V$ such that $V_1 = 0$ (namely, any nonzero module over $\Lambda/\Lambda e_1 \Lambda$), and the argument in the proof of Corollary 5.24 gives a contradiction with the condition $Z_i \neq Z_j$. \qed

Lemma 5.7 implies that the closed path $u$ in question may be assumed to pass through the vertex 1. So it remains to show that using the relations of $\Lambda$, one may reduce $u$ to a product $w$ of $X, D, Y^{\pm 1}, T^{\pm 1}$.

To this end, we may assume that the path $u$ begins and ends at the vertex 1. The proof is by induction in the length $\ell$ of $u$. The base of induction ($\ell = 0$) is clear. Let us make the induction step from $\ell - 1$ to $\ell$. Suppose that $u$ ends with $D_1$ (the case of $X_l$ is similar). If all the factors in $u$ are $D_i$ then $u$ is a power of $D$, and we are done. So let $m$ be the number of factors $D_i$ in $u$ until the first $X_i$, counting from the end. We may assume that $m < l$, since otherwise we can split away the factor $D$ at the end of $u$ and pass to smaller length. If $m = 1$, we can split away the factor $X_1D_1 = Z_1^{-1}Y - 1$ at the end and reduce to smaller length. On the other hand, if $m > 1$, then

$$u = \bar{u}X_mD_mD_{m-1}\ldots D_1$$

for some word $\bar{u}$. By adding smaller length words, we may replace $u$ with

$$u' := \bar{u}(1 + X_mD_mD_{m-1}\ldots D_{m-1}Z_{m-1}^{-1}D_{m-1}\ldots D_1)$$

Thus, $M_N^I(t)^\circ = M_N^I(Z, t)^\circ$ is independent of the $Z_i$ and is the multiplicative Calogero-Moser space considered in [Ob1] (the phase space of the Ruijsenaars integrable system). In particular, as explained in [Ob1], $M_N^I(t)^\circ$ is smooth and connected.
By adding words of smaller length and rescaling we can replace $u'$ by

$$u'' := uD_{m-1}X_{m-1}D_{m-1} \cdots D_1,$$

which is a word of the same type as $u$ but with $m$ replaced by $m - 1$. Proceeding in this way, we will eventually reach the case $m = 1$, which has already been considered. This implies the claim. □

Let $M'_N(Z, t)_*$ be the closure of $\psi^{-1}(M'_N(t))$ and $M'_N(Z, t)_*$ be the closure of $M'_N(t)^\circ$.

**Proposition 5.8.** The map $\psi : M'_N(Z, t)_* \rightarrow M'_N(Z, t)_*$ is an isomorphism of smooth connected affine varieties.

**Proof.** This follows from Proposition 5.6 □

We will also see that the multiplicative quiver variety $M'_N(Z, t)$ is connected, i.e., $M'_N(Z, t)_* = M'_N(Z, t)$ (Theorem 5.17).

### 5.3. Connection to cyclotomic DAHA.

Now let us connect multiplicative quiver varieties with cyclotomic DAHA. Let $e_N$ be the symmetrizer of the finite Hecke algebra of $S_N$ generated by $T_i$, and consider the spherical subalgebra $e_N \HH^d_N(Z, 1, t)e_N$. This is a subalgebra of the commutative domain $e_N \HH^d_N(1, t)e_N$ (see Ob1 Theorem 5.1(1),(2)), so it is also a commutative domain. Consider the module $\HH^d_N(Z, 1, t)e_N$ over this algebra. Let $\mathbb{M}^d_N(Z, t) = \text{Spec}(e_N \HH^d_N(Z, 1, t)e_N)$.

**Proposition 5.9.** For any $Z_i$ the algebra $e_N \HH^d_N(Z, 1, t)e_N$ is finitely generated and Cohen-Macaulay (i.e., $\mathbb{M}^d_N(Z, t)$ is an irreducible Cohen-Macaulay variety) and the module $\HH^d_N(Z, 1, t)e_N$ is Cohen-Macaulay. In particular, $\HH^d_N(Z, 1, t)e_N$ is projective of rank $N!$ on the smooth locus $\mathbb{M}^d_N(Z, t)_\text{smooth}$ of $\mathbb{M}^d_N(Z, t)$.

**Proof.** The proof is analogous to the proof of [Ob1 Theorem 5.1(2),(3)]. Namely, the statements follow from the fact that by Proposition 5.11 $\HH^{d+1}_N(Z, 1, t)e_N$ and $e_N \HH^{d+1}_N(Z, 1, t)e_N$ are free modules of finite rank over the subalgebra $\mathbb{C}[X_1, ..., X_N]^{S_N} \otimes \mathbb{C}[D_1, ..., D_N]^{S_N}$, and $\HH^d_N(Z, 1, t)e_N$, $e_N \HH^d_N(Z, 1, t)e_N$ are obtained from these by inverting the element $Y_1 ... Y_N e_N$. □

**Proposition 5.10.** For any $Z_i$, the variety $\mathbb{M}^d_N(Z, t)$ is smooth outside of a set of codimension two.

**Proof.** Consider the open set $\mathbb{M}^d_N(Z, t)_X = \mathbb{M}^d_N(t)_X \subset \mathbb{M}^d_N(Z, t)$ where $\prod_i X_i$ is invertible. On this set, the localization of the spherical cyclotomic DAHA to $\mathbb{M}^d_N(t)_X$ is the usual spherical DAHA with $q = 1$, so $\mathbb{M}^d_N(t)_X$ is smooth by the result of Ob1. Similarly, consider the open set $\mathbb{M}^d_N(Z, t)_D = \mathbb{M}^d_N(t)_D \subset \mathbb{M}^d_N(Z, t)$ where $D_i$ are invertible. By Proposition 3.29 the localization of the spherical cyclotomic DAHA to $\mathbb{M}^d_N(t)_D$ is also isomorphic to the usual spherical DAHA, as there is an involution $\phi$ of the cyclotomic DAHA exchanging $X_i$ and $D_i$. Thus $\mathbb{M}^d_N(t)_D$ is also smooth, by Ob1.
But it is easy to see that the complement of $\mathcal{M}_N^l(t)_X \cup \mathcal{M}_N^l(t)_D$ has codimension at least two.

\begin{corollary}
Corollary 5.11. The variety $\mathcal{M}_N^l(Z, t)$ is normal.
\end{corollary}

\begin{proof}
The proof is similar to the proof of [Ob1, Theorem 5.1(2)]. Namely, the statement follows from Proposition 5.9 and Proposition 5.10, since by the Serre criterion, a Cohen-Macaulay variety smooth outside of a set of codimension 2 is normal.
\end{proof}

Let $Z(\mathcal{H}H_N^l(Z, 1, t))$ be the center of $\mathcal{H}H_N^l(Z, 1, t))$.

\begin{proposition}
Proposition 5.12. For any $Z_i$: (i) The natural map $\mathcal{H}H_N^l(Z, 1, t) \to \text{End}_{e_N, \mathcal{H}H_N^l(Z, 1, t)e_N}(\mathcal{H}H_N^l(Z, 1, t)e_N)$ is an isomorphism.

(ii) The natural map $Z(\mathcal{H}H_N^l(Z, 1, t)) \to e_N\mathcal{H}H_N^l(Z, 1, t)e_N$ given by $z \mapsto ze_N$ is an isomorphism.
\end{proposition}

\begin{proof}
The proof is analogous to the proof of [Ob1, Theorem 5.1(4),(5)], replacing $Y_i$ with $D_i$ and the Cherednik involution with the involution $\phi$, and using [Ob1, Theorem 5.1].
\end{proof}

We can now define a regular map $\xi : \mathcal{M}_N^l(Z, t) \to \mathcal{M}_N^l(Z, t)_*$ as follows. Given $\chi \in \mathcal{M}_N^l(Z, t)_{\text{smooth}}$, consider the representation $I(\chi) := \mathcal{H}H_N^l(Z, 1, t)e_N \otimes e_N, \mathcal{H}H_N^l(Z, 1, t)e_N \chi$. By Proposition 5.9, this representation has dimension $N!$ and is the regular representation of the finite Hecke algebra generated by $T_i$. Let $e_N$ be the symmetrizer of the subalgebra generated by $T_2, ..., T_N$, and let $V(\chi) := e_N I(\chi)$. This is an $N$-dimensional space, and it carries an action of the operators $X := X_1, D = D_1 = X_1^{-1}(Y_1 - Z_1)...(Y_1 - Z_l), Y := Y_1$ and $T := T_1 ... T_N$.

\begin{proposition}
Proposition 5.13. The operators $X, D, Y, T$ satisfy equations (5.3), (5.4); i.e., define a point of $\mathcal{M}_N^l(Z, t)$.
\end{proposition}

\begin{proof}
Relations (5.4) follow from relation (3.5). The first relation of (5.3) is easy, and the second one follows from the first one and (3.3).
\end{proof}

\begin{remark}
On the open set where $X_1$ is invertible, Proposition 5.13 reduces to the result of [Ob1].
\end{remark}

Proposition 5.13 allows us to set $\xi(\chi) := (X, D, Y, T)$, which defines the map $\xi$ on the smooth locus $\mathcal{M}_N^l(Z, t)_{\text{smooth}}$. By Corollary 5.11, $\xi$ then uniquely extends from the smooth locus to the whole variety $\mathcal{M}_N^l(Z, t)$. Also it is clear that this map lands in $\mathcal{M}_N^l(Z, t)_*$ (as $\mathcal{M}_N^l(Z, t)$ is irreducible).

Thus, altogether we obtain a map $\kappa := \psi^{-1} \circ \xi : \mathcal{M}_N^l(Z, t) \to \mathcal{M}_N^l(Z, t)_*$.

\begin{prop}
Proposition 5.15. $\kappa$ is an isomorphism.
\end{prop}

\begin{proof}
Consider the restriction $\kappa_X$ of $\kappa$ to the open set $\mathcal{M}_N^l(t)_X \subset \mathcal{M}_N^l(Z, t)$ where $X_i$ are invertible. As shown above, $\kappa_X$ is an isomorphism $\mathcal{M}_N^l(t)_X \to \mathcal{M}_N^l(Z, t)_X$ onto the open set $\mathcal{M}_N^l(Z, t)_X \subset \mathcal{M}_N^l(Z, t)_*$ where $X$ is invertible. Similarly, by using
Consider the morphism \( \kappa^* : \mathcal{O}(\mathcal{M}_N^l(Z,t)_*) \rightarrow \mathcal{O}(\mathcal{M}_N^l(Z,t)) \). Obviously, it is an inclusion which becomes an isomorphism after passing to the fields of fractions. Let \( F \in \mathcal{O}(\mathcal{M}_N^l(Z,t)) \). Then \( F \) is a rational function on \( \mathcal{M}_N^l(Z,t)_* \). As shown above, this function is regular on \( \mathcal{M}_N^l(Z,t)_X \cup \mathcal{M}_N^l(Z,t)_D \subset \mathcal{M}_N^l(Z,t) \), an open subset whose complement has codimension \( \geq 2 \). But we know that \( \mathcal{M}_N^l(Z,t)_* \) is smooth, hence normal. Thus, \( F \) extends to a regular function on the whole \( \mathcal{M}_N^l(Z,t)_* \). This implies that \( \kappa^* \) and hence \( \kappa \) is an isomorphism.

Thus, we obtain the following theorem.

**Theorem 5.16.** Under the above assumptions on the \( Z_i \), the following statements hold.

(i) The variety \( \mathcal{M}_N^l(Z,t) \) is smooth.

(ii) The module \( \mathcal{HH}_1^l(Z,1,t)e_N \) over \( \mathcal{Z}(\mathcal{HH}_1^l(Z,1,t)) \cong e_N\mathcal{HH}_1^l(Z,1,t)e_N \) is projective of rank \( N! \).

(iii) \( \mathcal{HH}_1^l(Z,1,t) \) is a split Azumaya algebra over \( \mathcal{Z}(\mathcal{HH}_1^l(Z,1,t)) \) of rank \( N! \), namely the endomorphism algebra of the vector bundle \( \mathcal{HH}_1^l(Z,1,t)e_N \). Thus, all irreducible representations of \( \mathcal{HH}_1^l(Z,1,t) \) have dimension \( N! \) and are parametrized by points of \( \mathcal{M}_N^l(Z,t) \).

Thus, we see that irreducible representations of \( \mathcal{HH}_1^l(Z,1,t) \) are parametrized by points of a connected component of the multiplicative quiver variety. In fact, it turns out that this is the only connected component. Namely, we have

**Theorem 5.17.** The variety \( \mathcal{M}_N^l(Z,t) \) is connected, i.e., \( \mathcal{M}_N^l(Z,t) = \mathcal{M}_N^l(Z,t)_* \).

Thus, \( \kappa : \mathcal{M}_N^l(Z,t) \rightarrow \mathcal{M}_N^l(Z,t)_* \) is an isomorphism.

**Proof.** See Subsection 5.6 below. \( \square \)

**Remark 5.18.** 1. Recall that the multiplicative quiver variety \( \mathcal{M}_N^l(Z,t) \) carries a Poisson structure (symplectic for generic parameters and generically for any parameters), coming from the quasi-Hamiltonian reduction procedure ([VdB]). Our results imply that the algebra \( e_N\mathcal{HH}_1^l(Z,q,t)e_N \) (where \( q = e^\varepsilon \) and \( \varepsilon \) is a formal parameter) is a deformation quantization of this Poisson variety (namely, the matching of Poisson brackets may be checked on the open set where \( X \) is invertible, using the results of [ObI]).

2. We expect that the results of this subsection can be lifted to the quantum level. Namely, we expect that the algebra \( e_N\mathcal{HH}_1^l(Z,q,t)e_N \) is isomorphic to the quantization of the multiplicative quiver variety \( \mathcal{M}_N^l(Z,t) \) defined by D. Jordan in [J], via a quantization of the map \( \kappa \).\(^3\) We note that this is known in the degenerate setting, see [Ob2, EGGO].

\(^3\)We note that for \( l = 1 \) we would need a slightly less localized version than that of [J], not requiring \( X \) to be invertible, which should produce not the full DAHA but its subalgebra.
Remark 5.19. The functions $\text{Tr}(D^r)$, $r = 1, \ldots, N$, form a classical integrable system on the symplectic variety $\mathcal{M}_N^{r}(Z,t)$. This system is the classical limit of the quantum integrable system $\{D_1^r + \ldots + D_N^r, r = 1, \ldots, N\}$ in the (spherical) cyclotomic DAHA discussed in Subsection 3.6. These classical integrable systems have been studied independently by O. Chalykh and M. Fairon (CF).

5.4. Multiplicative bow varieties. In this section we follow the notations of [NT].

Given a bow diagram with a balanced dimension vector as in [NT, 6.1] we consider another nonzero complex number.

5.5. K-theoretic Coulomb branch. Recall [BFNa, Section 2] that given a representation $\mathcal{N}$ of a reductive group $G$ one can consider the variety of triples $\mathcal{R}$. According to [BFNa, Remark 3.9(3)], the equivariant $K$-theory $K^{G_0}(\mathcal{R})$ is a commutative ring with respect to convolution. Moreover, if the $G$-action on $\mathcal{N}$ extends to an action of a larger group $\tilde{G}$ containing $G$ as a normal subgroup, then the equivariant $K$-theory $K^{\tilde{G}_0}(\mathcal{R})$ is a commutative ring with respect to convolution, a deformation of $K^{G_0}(\mathcal{R})$ over $\text{Spec}(K^{G_0}(\mathcal{R}))$ where $G_F = \tilde{G}/G$ is the flavor symmetry group. The affine variety $\text{Spec}(K^{\tilde{G}_0}(\mathcal{R}))$ is denoted $\mathcal{M}_G^{\times}(\mathcal{G}, \mathcal{N})$ and called the $K$-theoretic Coulomb branch.

A framed oriented quiver representation gives rise to a representation $\mathcal{N} = \bigoplus_{i \to j} \text{Hom}(V_i, V_j) \oplus \bigoplus_i \text{Hom}(W_i, V_j)$ of $G = GL(V) := \prod_i GL(V_i)$. Choosing a maximal torus $T(W_i) \subset GL(W_i)$, we consider the natural action of $\tilde{G} := \mathbb{C}^\times \times G \times \prod_i T(W_i)$ on $\mathcal{N}$ where $\mathbb{C}^\times$ acts by dilation on the component $\mathcal{N}_{\text{hor}} := \bigoplus_{i \to j} \text{Hom}(V_i, V_j)$ of $\mathcal{N}$. If $W := \bigoplus_i W_i \neq 0$, the action of scalars $\mathbb{C}^\times \subset T(W) := \prod_i T(W_i)$ coincides with the action of scalars $\mathbb{C}^\times \subset G$. Hence if $W \neq 0$, the action of $\tilde{G}$ on $\mathcal{N}$ factors through the

\[^{4}\text{It is different from the one considered in Section 4.1.}\]
action of $\tilde{G} := \mathbb{C}^\times \times (G \times T(W))/\mathbb{C}^\times$. If $W = 0$, we denote $\tilde{G} := \hat{G}$. The corresponding $K$-theoretic Coulomb branch $\mathcal{M}_C^\times(\hat{G}, N)$ of a framed quiver gauge theory will be denoted by $\mathcal{M}_C^\times$ for short.

Similarly to [NT, Theorem 6.18] one can construct an isomorphism $\mathcal{M}_C^\times \cong M_t^\times(v, w)$ from the $K$-theoretic Coulomb branch of a framed quiver gauge theory of affine type $A_n-1$ with dimension vectors $v, w$. Here $Z, t$ in the RHS are parameters corresponding to the equivariant flavor symmetry parameters in the LHS, cf. [NT, 6.8.2]. The proof of the above isomorphism in particular shows that $M_t^\times(v, w)$ is connected (and $M_C^\times$ is connected similarly to [BFNa, Corollary 5.22]).

5.6. Example. We consider a special case when a bow diagram has 1 cross and $l$ circles (we allow $l = 0$), that is $n = 1$, $w_0 = l$, and $\dim V_{k}^0 = N$ for any $k = 0, \ldots, l$. Then according to [NT, Lemma 3.1], $A_0 : V_0^l \rightarrow V_0^0$ is an isomorphism. We identify $V_{l}^0 \equiv V_0^0$ with the help of $A_0$, and then the definition of $M_t^\times(Z, t)$ of Section 5.4 becomes nothing but the definition of $M_t^1(N, l) \simeq M_t^1(Z, t)$. In particular, we conclude that $M_t^1(Z, t)$ is connected, which proves Theorem 5.17.

Remark 5.20. The relations between general multiplicative bow varieties and (various versions of) multiplicative quiver varieties for a cyclic quiver are explained in Appendix A.

6. APPLICATION TO q-DEFORMED m-QUASIINVARIANTS

6.1. q-deformed quasiinvariants. Let $m$ be a nonnegative integer, and $q \in \mathbb{C}^\times$.

Definition 6.1. ([C]) We call $F \in \mathbb{C}[[X_{-1}^\pm, \ldots, X_N^\pm]]$ a q-deformed $m$-quasiinvariant if $(1 - s_{ij})F$ is divisible by $\prod_{p=-m}^{m}(X_i - q^pX_j)$ for any $i < j$.

The algebra of q-deformed $m$-quasiinvariant Laurent (or trigonometric) polynomials will be denoted by $Q_{m,q}^{\text{trig}}$.

Let $Q_{m,q} \subset Q_{m,q}^{\text{trig}}$ be the graded algebra of q-deformed quasiinvariants inside $\mathbb{C}[X_1, \ldots, X_N]$. By the Hilbert basis theorem, $Q_{m,q}$ is a finitely generated module over the ring of symmetric polynomials $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$. Note that $Q_{m,1} = Q_m$, the usual space of $m$-quasiinvariants defined by Chalykh and Veselov [CV], i.e., polynomials $F$ such that $(1 - s_{ij})F$ is divisible by $(X_i - X_j)^{2m+1}$.

Theorem 6.2. For all except countably many values of $q$, the algebra $Q_{m,q}$ has the same Hilbert series as $Q_m$ and is Cohen-Macaulay, i.e., a free module over $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$.

In other words, Theorem 6.2 says that any quasiinvariant polynomial can be $q$-deformed.

Note that the Hilbert series of $Q_m$ is known (see [FeV], [BEG]).

Remark 6.3. Theorem 6.2 was conjectured by P.E. and E. Rains on the basis of a computer calculation.
Theorem 6.2 is proved in the next subsection.

Remark 6.4. The algebra of $q$-deformed trigonometric quasiinvariants $Q_{m,q}^{\text{trig}}(R)$ may be defined for any reduced root system $R$ with Weyl group $W$ and a $W$-invariant multiplicity function $m : R \to \mathbb{Z}_+$, see [C]. Namely, it is the algebra of regular functions $F$ on the corresponding torus $T$ such that for each $\alpha \in R_+$ the function $F(X) - F(s_\alpha X)$ is divisible by $\prod_{i=1}^{m_{\alpha}}(e^\alpha - q^i)$. Moreover, this algebra is Cohen-Macaulay for generic $q$. To see this, note that by using the exponential map $\exp : \text{Lie}(T) \to T$ and rescaling in $\text{Lie}(T)$, we may identify formal neighborhoods of closed points of $\text{Spec}(Q_{m,q}^{\text{trig}}(R))$ with those of $\text{Spec}(Q_{m,q}^{\text{trig}}(R))$, the usual trigonometric (a.k.a. non-homogeneous) quasiinvariants for $R$, see e.g. [ER], Remark 6.4. But the algebra $Q_{m,q}^{\text{trig}}(R)$ is Cohen-Macaulay, see [ER], Proposition 6.5. Hence so is $Q_{m,q}^{\text{trig}}(R)$, as desired.

Note that this result also holds for $q = 1$ since in this case formal neighborhoods are the same as for usual (rational) quasiinvariants $Q_m(R)$.

Theorem 6.2 is a refinement of this result for $R = A_{N-1}$, as $Q_{m,q}^{\text{trig}}$ is a localization of $Q_{m,q}$.

6.2. Proof of Theorem 6.2. The Cohen-Macaulayness statement of Theorem 6.2 holds for $q = 1$ (i.e., for $Q_m$) by the results of [EG2], [BEG] (conjectured earlier in [FV]). Thus it suffices to show that $Q_{m,q}^{\text{trig}}$ is a flat deformation of $Q_m$ when $q = e^\varepsilon$ and $\varepsilon$ is a formal parameter.

Let $e$ be the symmetrizer of the finite Hecke algebra generated by $T_i$, and consider the action of the spherical subalgebra $e\mathcal{H}N(q,t)e$ on $\mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}]^{S_N}$. This action is by $q$-difference operators. Consider the element $Y_1 + \ldots + Y_N$. It is a central element of the affine Hecke algebra generated by $T_i$ and $Y_i$, hence commutes with $e$. Thus

$$(Y_1 + \ldots + Y_N)e \in e\mathcal{H}N(q,t)e.$$

We will need the following lemma, due to Cherednik ([CH1]).

Lemma 6.5. The element $M := (Y_1 + \ldots + Y_N)e$ acts on $\mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}]^{S_N}$ by the first Macdonald operator

$$M := \sum_{j=1}^N \left( \prod_{i \neq j} \frac{X_i - tX_j}{X_i - X_j} \right) \tau_j,$$

where $\tau_iX_j = q^{\delta_{ij}}X_j$.

Proof. Denote the $q$-difference operator by which $(Y_1 + \ldots + Y_N)e$ acts by $L$. Clearly, we have $L = \sum_j f_j \tau_j$, where $f_j$ are rational functions such that $s_rf_j = f_j$ for $r \neq j, j + 1$, and $s_jf_j = f_{j+1}$. Thus it suffices to show that

$$f_1 = \prod_{i \neq 1} \frac{X_i - tX_1}{X_i - X_1}.$$
To this end, treat \( n \) as a variable, and note that
\[
L(X_1^n + \ldots + X_N^n) = \sum_j f_j(X_1, \ldots, X_N)(q^n X_j^n + \sum_{i \neq j} X_i^n),
\]
which we can treat as a polynomial in \( q^n, X_1^n, \ldots, X_N^n \) with coefficients in \( \mathbb{C}(X_1, \ldots, X_N) \).

An easy direct computation using this formula shows that
\[
\text{Lemma 6.6. If } q \text{ but we give a proof for reader’s convenience). Assume that \( \rho(\gamma) \text{ as desired.} \quad \square
\]

Thus, only the second summand contributes to the coefficient of \( q^n X_1^n \).

On the other hand, we have
\[
\rho(Y_i) = t^{i-1} \prod_{j=i+1}^N (1 + (1-t)X_j(1-s_{ij}))(1 + (1-t^{-1})X_j(1-s_{ij})).
\]

An easy direct computation using this formula shows that
\[
(Y_1 + \ldots + Y_n)(X_1^n + \ldots + X_N^n) =
\]
\[
\left(\sum_{i=1}^N t^{i-1}\right)(X_1^n + \ldots + X_N^n) + (q^n - 1) \sum_{i=1}^N t^{i-1} \prod_{j=i+1}^N (1 + (1-t)X_j(1-s_{ij}))X_i^n
\]
Thus, only the second summand contributes to the coefficient of \( q^n \), and in the second summand only \( i = 1 \) contributes to the coefficient of \( X_1^n \). Moreover, in this contribution, the transposition terms (involving \( s_{ij} \)) don’t contribute. Thus
\[
f_1(X_1, \ldots, X_n) = \prod_{j=2}^N (1 + (1-t)X_j) = \prod_{j=2}^N \frac{X_j - tX_1}{X_j - X_1},
\]
as desired.

We will also need the following lemma, which is a special case of Proposition 2.1 of \( \square \) (but we give a proof for reader’s convenience). Assume that \( q \) is not a root of unity.

**Lemma 6.6.** If \( t = q^{-m} \) then the operator \( M \) preserves \( Q_{m,q}^{\text{rig}} \).

**Proof.** Let \( r > l \), and \( F \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}] \). A direct computation shows that
\[
(1 - s_{rl})(MF)(X_1, \ldots, X_N) =
\]
\[
\prod_{j \neq r} \frac{X_j - q^{-m}X_r}{X_j - X_r} (F(\ldots X_l \ldots qX_r \ldots) - F(\ldots qX_r \ldots X_l \ldots))
\]
\[
- \prod_{j \neq l} \frac{X_j - q^{-m}X_l}{X_j - X_l} (F(\ldots X_r \ldots qX_l \ldots) - F(\ldots qX_l \ldots X_r \ldots)).
\]
If \( F \in Q_{m,q}^{\text{rig}} \) is a quasivariant then \( F(\ldots X_l \ldots qX_r \ldots) - F(\ldots qX_r \ldots X_l \ldots) \) vanishes for \( X_l = q^p X_r \) with \( -m - 1 \leq p \leq m - 1 \), while \( F(\ldots X_l \ldots qX_r \ldots) - F(\ldots qX_r \ldots X_l \ldots) \) vanishes for \( X_l = q^p X_r \) with \( -m + 1 \leq p \leq m + 1 \). Hence, both terms in the formula for \( (1 - s_{rl})MF \) vanish when \( X_l = q^p X_r \) with \( -m \leq p \leq m \), \( p \neq 0 \) and are defined for \( p = 0 \) (vanishing in the extremal cases \( p = m, -m \) follows from vanishing of the

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tions. Thus, \((1 - s,\iota)MF(X_1, \ldots, X_N)\) is a polynomial which is antisymmetric in \(X_i, X_r\), so divisible by \(X_r - X_i\). This takes care of the case \(p = 0\).

Now note that since the algebra \(eHH_N(q, t)e\) acts on \(\mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}]\) by \(q\)-difference operators, this action can be extended tautologically to non-symmetric rational functions \(\mathbb{C}(X_1^{\pm 1}, \ldots, X_N^{\pm 1})\) (by the same difference operators).

**Corollary 6.7.** If \(t = q^{-m}\) for \(m \in \mathbb{Z}_{\geq 0}\) then the spherical cyclotomic DAHA \(eHH_N(q, t)e\) (with \(Z_1 = 1\)) preserves the subspace of \(q\)-deformed \(m\)-quasiinvariants \(Q_{m,q} \subset \mathbb{C}(X_1^{\pm 1}, \ldots, X_N^{\pm 1})\).

**Proof.** By Proposition 3.6, the difference-reflection operators \(\rho(L), L \in HH_1^1(q, t)\) do not create poles at \(X_i = 0\). Therefore, the action of \(eHH_N^1(q, t)e \subset eHH_N(q, t)e\) on \(\mathbb{C}(X_1, \ldots, X_N)\) preserves the subspace \(\mathbb{C}(X_1, \ldots, X_N)_{\text{reg}}\) of functions regular at the hyperplanes \(X_i = 0, i = 1, \ldots, N\).

We now claim that the algebra \(eHH_N(q, t)e\) preserves \(Q_{m,q}^{\text{rig}}\). It suffices to prove this in the formal setting \(q = e^\varepsilon, t = e^{-m\varepsilon}\). By Lemma 6.5 and Lemma 6.6, the element \(M\) preserves \(Q_{m,q}^{\text{rig}}\). Also, \(\mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}]_{SN}\) and \(\tilde{Y} := \prod_i Y_i\) preserve \(Q_{m,q}^{\text{rig}}\). But we claim that the element \(\tilde{H} := e^{-2(M - N - \varepsilon(\tilde{Y} - 1))}\) and \(\mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}]_{SN}\) (topologically) generate \(eHH_N^{\text{formal}}(1, k)e\). Indeed, the quasiclassical limit of \(\tilde{H}\) is the trigonometric Calogero-Moser operator \(H = e \sum_i y_i^2 e = \sum_i y_i^2 e\), and by Lemma 2.7 \(H\) and \(\mathbb{C}[X_1^{\pm 1}, \ldots, X_N^{\pm 1}]_{SN}\) generate \(eHH_N^{\text{deg}}(1, k)e\). This implies the claim.

Thus, \(eHH_N^1(q, t)e\) preserves \(\mathbb{C}(X_1, \ldots, X_N)_{\text{reg}} \cap Q_{m,q}^{\text{rig}} = Q_{m,q}\), as desired.

Now we can complete the proof of the theorem. Since every polynomial divisible by \(\prod_{i<j} \prod_{p=-m}^m (X_i - q^p X_j)\) is automatically a \(q\)-deformed \(m\)-quasiinvariant, the leading coefficient of the Hilbert polynomial of \(Q_{m,q}\) is the same as that of \(Q_m\). On the other hand, working in the formal setting \((q = e^\varepsilon)\) and reducing Corollary 6.7 modulo \(\varepsilon\), we find that \(Q_{m,q}/(\varepsilon)\) is a submodule of the module \(Q_m\) over the spherical rational Cherednik algebra \(eHH_N^1(1, m)e\). But as shown in [BEG], \(Q_m = \bigoplus_\lambda eM_m(\lambda) \otimes \pi_\lambda\), where \(\lambda\) runs over partitions of \(N\), \(\pi_\lambda\) is the Specht module for \(S_N\) corresponding to \(\lambda\), and \(M_m(\lambda)\) is the Verma module over the rational Cherednik algebra \(HH_N^{\text{rat}}(1, m)\) attached to \(\lambda\). Since the \(eHH_N^{\text{rat}}(1, m)e\)-modules \(eM_m(\lambda)\) are irreducible (see [BEG]), any proper \(eHH_N^{\text{rat}}(1, m)e\)-submodule of \(Q_m\) would have a strictly smaller leading coefficient of the Hilbert series than that of \(Q_m\). This implies that \(Q_{m,q}/(\varepsilon) = Q_m\), i.e. \(Q_{m,q}\) is a flat deformation of \(Q_m\) (i.e., has the same Hilbert series). By standard abstract nonsense, this applies also to numerical values of \(q\), excluding a countable set. The theorem is proved.

### 6.3. Generalization: \(q\)-deformed cyclotomic quasiinvariants

Let us now use the cyclotomic DAHA to generalize Theorem 6.2 to the cyclotomic case.

Let \(m \geq 0, l \geq 1, q \in \mathbb{C}^*\), and \(q = q^l\). Let us first define the algebra of \(q\)-deformed cyclotomic trigonometric quasiinvariants \(Q_{m,q}^{l,\text{trig}}\). We introduce variables \(x_i\) such that
$X_i = x_i^l$. We define $Q_{m,q}^l \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ to be the subalgebra of Laurent polynomials $F$ such that for every $i, j, r$, the Laurent polynomial

$$F(..., x_i, ..., x_j, ...) - F(..., \zeta^r x_j, ..., \zeta^{-r} x_i, ...)$$

is divisible by $\prod_{p=-m}^{m} (x_i - \zeta^p q^p x_j)$.

Now let $m, m_1, \ldots, m_{l-1}$ be nonnegative integers. For $0 \leq r \leq l - 1$ let $p_r$ be the homogeneous projector $\mathbb{C}[x] \to x^r \mathbb{C}[x]$, and $p_r^{(i)}$ denotes $p_r$ acting with respect to $x_i$. By analogy with [BC], let $Q_{m,m_1,\ldots,m_{l-1},q}^l \subset Q_{m,q}^l$ be the graded space of all $q$-deformed cyclotomic trigonometric quasiinvariants $F$ inside $\mathbb{C}[x_1, \ldots, x_N]$ such that

$$p_r^{(i)} F \text{ is divisible by } x_i^{r+m_1 l} \text{ for } i = 1, \ldots, N \text{ and } r = 1, \ldots, l - 1. \tag{6.1}$$

**Definition 6.8.** The space $Q_{m,m_1,\ldots,m_{l-1},q}^l$ is called the space of $q$-deformed cyclotomic $(m, m_1, \ldots, m_{l-1})$-quasiinvariants.

By the Hilbert basis theorem, $Q_{m,m_1,\ldots,m_{l-1},q}^l$ is a finitely generated module over the ring of symmetric polynomials $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$. Note that $Q_{m,m_1,\ldots,m_{l-1}}^l := Q_{m,m_1,\ldots,m_{l-1},1}^l$ is the usual space of $(m, m_1, \ldots, m_{l-1})$-quasiinvariants for the complex reflection group $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$ defined in [BC]. Note also that similarly to [BC], $Q_{m,m_1,\ldots,m_{l-1},q}^l$ is not necessarily an algebra.

Our main result about cyclotomic $q$-deformed quasiinvariants is the following theorem, which is a generalization of Theorem 6.2.

**Theorem 6.9.** For all except countably many values of $q$, the space $Q_{m,m_1,\ldots,m_{l-1},q}^l$ has the same Hilbert series as $Q_{m,m_1,\ldots,m_{l-1},1}^l$, and is a free module over $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$.

In other words, every quasiinvariant for $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$ can be $q$-deformed. We note that the Hilbert series of $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$ is computed in [BC].

Theorem 6.9 is proved in the next subsection.

6.4. **Proof of Theorem 6.9.** The proof of Theorem 6.9 is parallel to the proof of Theorem 6.2 using the results of [BC]. In fact, most of the technical statements we’ll need have already been obtained in the proof of Theorem 6.2.

The freeness statement of Theorem 6.9 holds for $Q_{m,m_1,\ldots,m_{l-1}}^l$ (i.e., for $q = 1$) by the results of [BC]. Thus it suffices to show that $Q_{m,m_1,\ldots,m_{l-1},q}^l$ is a flat deformation of $Q_{m,m_1,\ldots,m_{l-1}}^l$ when $q = e^{\varepsilon/l}$ and $\varepsilon$ is a formal parameter.

To this end, recall from [BC] that the space $Q_{m,m_1,\ldots,m_{l-1}}^l$ carries an action of the spherical cyclotomic Cherednik algebra $e\mathsf{H}_N^{l,\mathrm{psc}}(c, 1, m)e$, where $e$ is the symmetrizer for $S_N$ and $c_i$ are certain linear functions of $m_j$ (in fact, this action is the main tool in [BC] for proving that $Q_{m,m_1,\ldots,m_{l-1}}^l$ is a free $\mathbb{C}[X_1, \ldots, X_N]$-module). Therefore, by

---

5. Theorem 6.2 is a special case of Theorem 6.9 for $l = 1$, but it is convenient for us to treat this special case first, and then pass to the general case.
Theorem 2.19: This space carries an action of $eHH_N^l(z, 1, m)e$, and it is easy to compute that $z_i = m_i/l$, $i = 1, \ldots, l-1$ and $z_l = 0$.

The main idea of the proof is to show that this representation can be $q$-deformed to a representation of the spherical cyclotomic DAHA $eHH_N^l(Z, q, t)e$ on $Q_{m,m_1,\ldots,m_{l-1}}$, where $t = q^{-m}$ and $Z_i = q^{m_i}$. Then, similarly to the proof of Theorem 6.2 the result will follow by looking at the leading coefficient of the Hilbert series and using [BC, Theorem 8.2], which gives a decomposition of the $eHH_N^l(z, 1, m)e$-module $Q_{m,m_1,\ldots,m_{l-1}}$ into a direct sum of irreducible modules.

Finally, let us show that the representation of $eHH_N^l(z, 1, m)e$ on the space $Q_{m,m_1,\ldots,m_{l-1}}$ admits a $q$-deformation. Recall from the proof of Theorem 6.2 that the algebra $eHH_N^l(Z, q, t)e$ acts on $\mathbb{C}(x_1, \ldots, x_N)$ by difference operators. This action can be straightforwardly extended to the field extension $\mathbb{C}(x_1, \ldots, x_N)$ by using the same formulas, where now $\tau_i x_j = q^{\delta_{ij}} x_j$. Hence, the subalgebra $eHH_N^l(Z, q, t)e \subset eHH_N^l(Z, q, t)e$ acts on $\mathbb{C}(x_1, \ldots, x_N)$.

Since $X_i - q^p X_j = \prod_{r=0}^{l-1}(x_i - e^{-r}\zeta^r q^p x_j)$, the argument in the proof of Theorem 6.2 implies that $eHH_N^l(Z, q, t)e$ preserves the subspace $Q_{m,q}^{\text{trig}} \subset \mathbb{C}(x_1, \ldots, x_N)$. Hence, so does the subalgebra $eHH_N^l(Z, q, t)e$. Also, by Proposition 3.6 the algebra $eHH_N^l(Z, q, t)e$ preserves the space $(x_1 \ldots x_N)^m \mathbb{C}(X_1, \ldots, X_N)_{\text{reg}}$, where, as before, the subscript “reg” means functions regular at the generic points of the hyperplanes $x_i = 0$. Therefore, the algebra $eHH_N^l(Z, q, t)e$ preserves quasiinvariance conditions (6.1).

Thus, we see that the algebra $eHH_N^l(Z, q, t)e$ preserves the space $Q_{m,m_1,\ldots,m_{l-1}}$. Moreover, it is easy to see that the classical limit of this representation as $q \to 1$ is exactly the representation of $eHH_N^l(z, 1, m)e$ on the space $Q_{m,m_1,\ldots,m_{l-1}}$ constructed in [BC]. This completes the proof of Theorem 6.9.

6.5. Twisted quasiinvariants. Let $a_1, \ldots, a_N \in \mathbb{C}$, $m \in \mathbb{Z}_+$. Let $Q_m(a_1, \ldots, a_N)$ be the space of polynomials $F \in \mathbb{C}[X_1, \ldots, X_N]$ such that the function

$$\tilde{F}(X_1, \ldots, X_N) := \left( \prod_i X_i^{a_i} \right) F(X_1, \ldots, X_N)$$

(regarded as a function of $X_i > 0$) is $m$-quasiinvariant, in the sense that $(1 - s_{ij})\tilde{F}$ vanishes to order $2m + 1$ at $X_i = X_j$ for all $i < j$. Note that $Q_m(a_1, \ldots, a_N) = Q_m(a_1 - a, \ldots, a_N - a)$, and $Q_m(a, a) = Q_m$ for all $a$. Obviously, $Q_m(a_1, \ldots, a_N)$ is a graded $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$-module. By the Hilbert basis theorem, this module is finitely generated.

**Definition 6.10.** We will call $Q_m(a_1, \ldots, a_N)$ the module of twisted quasiinvariants.

**Theorem 6.11.** If $a_i - a_j \notin \mathbb{Z} \setminus \{0\}$ for all $i < j$, then $Q_m(a_1, \ldots, a_N)$ is a free $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$-module (of rank $N!$).

**Example 6.12.** Let $N = 2$. Then $Q_m(a, 0)$ is a free module over $R := \mathbb{C}[X_1, X_2]^{S_2}$ for any $a \in \mathbb{C}$. We show this by induction in $m$. The base case $m = 0$ is obvious. Let
Then it is easy to show that the lowest degree $d$ of a nonzero element $P_{a,m}$ in $Q_m(a,0)$ is $a$ if $a = 0, 1, \ldots, m - 1$, and $m$ otherwise. Moreover, we can uniquely choose $P_{a,m}$ so that $P_{a,m}(X, X) = X^d$; for example, if $0 \leq a \leq m$ is an integer, then $P_{a,m} = X^a_2$, and

$$P_{a,1}(X_1, X_2) = \frac{(a-1)X_1 + (a+1)X_2}{2a}$$

for $a \neq 0$. Therefore,

$$Q_m(a, 0) = R P_{a,m} + (X_1 - X_2)^2 Q_{m-1}(a, 0).$$

Indeed, given a nonzero homogeneous $F \in Q_m(a,0)$ such that $F(X, X) = \beta X^r$, consider $F' := F - 2^{d-r} \beta (X_1 + X_2)^{r-d} P_{a,m}$. Then $F' \in Q_m(a,0)$ and is divisible by $X_1 - X_2$, so it is in $(X_1 - X_2)^2 Q_{m-1}(a,0)$.

Consider first the case when $0 \leq a \leq m - 1$ is an integer. Then by the induction assumption, the module $Q_{m-1}(a,0)$ is free (of rank 2), so it is generated by $X_2^a$ and some homogeneous polynomial $f_{a,m}$ of degree $2m - 1 - a$. So by (6.2), $Q_m(a,0)$ is generated by $X_2^a$ and $f_{a,m+1} := (X_1 - X_2)^2 f_{a,m}$, which validates the induction step.

Now consider the case $a \neq 0, \ldots, m - 1$. Then $Q_{m-1}(a,0)$ is free by the induction assumption, so it is generated by some homogeneous polynomials of $P_{a,m-1}$, $T_{a,m-1}$ of degrees $m - 1$ and $m$, respectively, such that $T_{a,m}(X, X) = 0$ (as one can easily check that one always has such generators). It is easy to see that there exists $\alpha_{a,m} \in \mathbb{C}$ such that

$$P_{a,m} := T_{a,m-1} + \frac{1}{2} \alpha_{a,m}(X_1 + X_2) P_{a,m-1} \in Q_m(a, 0),$$

and $T_{a,m} := (X_1 - X_2)^2 P_{a,m-1} \in Q_m(a, 0)$. Moreover, by (6.2), these elements generate $Q_m(a,0)$. This completes the induction step.

This argument also implies that the Hilbert series of $Q_m(a,0)$ is

$$\frac{t^{a+2m+1} - t^a}{(1-t)(1-t^2)(1-t^3)},$$

if $0 \leq a \leq m - 1$ is an integer, and $\frac{t^m}{(1-t)^m}$ otherwise.

**Example 6.13.** In spite of Example 6.12 for $N \geq 3$ the condition on the $a_i$ cannot be dropped. Indeed, for $N = 3$, the computer calculation shows that the Hilbert series of $Q_2(1,0,0)$ has the form

$$h(t) = t^2(1 + t + 2t^2 + 3t^3 + 5t^5 + 7t^6 + 10t^7 + 15t^8 + 20t^9 + 26t^{10} + \ldots) = \frac{t^2 + t^6 + t^7 + 2t^9 + t^{10} - t^{12} + \ldots}{(1-t)(1-t^2)(1-t^3)}$$

and the minus sign in the numerator shows that $Q_2(1,0,0)$ cannot be a free module over symmetric polynomials. Indeed, if this module were free, the numerator would have been the Hilbert polynomial of the generators.

**Proof.** (of Theorem 6.11) By permuting $a_i$ we may assume that $$(a_1, \ldots, a_N) = (z_1, \ldots, z_1, \ldots, z_l, \ldots, z_l),$$
where \( z_i \neq z_j \) and \( z_i \) occurs \( N_i \) times, where \( N = N_1 + \ldots + N_l \). By simultaneously shifting \( a_i \) we may assume that \( z_i = 0 \).

Assume first that \( z_r = \frac{r}{l} + m_r, 1 \leq r \leq l - 1 \). Let \( \chi \) be the character of \((\mathbb{Z}/l\mathbb{Z})^N\) given by \( \chi(\sigma_i) = \zeta^{a_i} \). Denote by \( Q_{m,m_1,\ldots,m_{l-1}}^{lX} \) the \( \chi \)-eigenspace of \((\mathbb{Z}/l\mathbb{Z})^N\) in \( Q_{m,m_1,\ldots,m_{l-1}}^l \). Then it is easy to see that \( F \in Q_{m,m_1,\ldots,m_{l-1}}^{lX} \) if and only if it has the form

\[
F(x_1, \ldots, x_N) = X_1^{a_1} \ldots X_N^{a_N} f(x_1, \ldots, x_N),
\]

where \( X_i = x_i^l \), and satisfies the quasiinvariance condition saying that \( (1 - s_{ij})F \) vanishes to order \( 2m + 1 \) at \( X_i = X_j \) (for \( X_i > 0 \)), i.e., if and only if \( f \in Q_m(a_1, \ldots, a_N) \).

Let \( G := S_N \rtimes (\mathbb{Z}/l\mathbb{Z})^N \). By [BC] Theorem 8.2,

\[
Q_{m,m_1,\ldots,m_{l-1}}^{lX} = \oplus_{\tau \in \text{Irrep}(G): \tau \chi \neq 0} e_G M_{c,k}(\tau^*) \otimes \tau^X,
\]

where \( e_G \) is the symmetrizer for \( G \), \( M_{c,k}(\tau) \) denotes the Verma module over the cyclotomic rational Cherednik algebra \( \mathbb{H}^I_N(c,1,k) \) for appropriate \( c,k \), and the superscript \( \chi \) denotes the \( \chi \)-eigenspace. Here the grading on \( Q_{m,m_1,\ldots,m_{l-1}}^{lX} \) is obtained from the grading on the right hand side of (6.3) (given by the scaling element \( h \) of the rational Cherednik algebra) by dividing by \( l \) and shifting by \( m \frac{N(N-1)}{2} + \frac{N}{2} \).

Thus, \( Q_m(a_1, \ldots, a_N) \) is a free module over \( \mathbb{C}[X_1, \ldots, X_N]^{S_N} \), with Hilbert series independent of the numbers \( m_r \). Since this holds for a Zariski dense set of vectors \((z_1, \ldots, z_{l-1})\) (namely, \( z_r = \frac{r}{l} + m_r, m_r \in \mathbb{Z}_+ \)), this holds for Weil generic \((z_1, \ldots, z_{l-1})\).

It remains to show that the statement holds if \( z_i - z_j \) is not a nonzero integer for \( 1 \leq i < j \leq l \). Using Theorem 2.19 and formula (6.3), we see that \( Q_{m,m_1,\ldots,m_{l-1}}^{lX} \) is a module over the spherical subalgebra \( e \mathbb{H}_N^{I_{\text{deg}}}(z,1,m)e \), where \( z_r = \frac{r}{l} + m_r \) for \( 1 \leq i \leq l - 1 \) and \( z_l = 0 \). Interpolating to arbitrary complex values of \( z_i \), we get that for any \( z_1, \ldots, z_{l-1} \in \mathbb{C} \), the algebra \( e \mathbb{H}_N^{I_{\text{deg}}}(z,1,m)e \) with \( z := (z_1, \ldots, z_{l-1}, 0) \) acts on \( Q_m(a_1, \ldots, a_N) \). If \( z_i - z_j \) is not a nonzero integer, then the category \( \mathcal{O} \) for the algebra \( e \mathbb{H}_N^{I_{\text{deg}}}(z,1,m)e \) is semisimple (see [BC] Theorem 6.6). Moreover, by a deformation argument, \( Q_m(a_1, \ldots, a_N) \) must contain representation (6.3). Since every irreducible representation in \( \mathcal{O} \) has full support, this implies that \( Q_m(a_1, \ldots, a_N) \) coincides with (6.3). This proves the theorem.

Formula (6.3) allows us to easily compute the character of \( Q_m(a_1, \ldots, a_N) \) as a graded \( S_{N_1} \times \ldots \times S_{N_l} \)-module. Namely, given irreducible representations \( \pi_r \) of \( S_{N_r} \), we have from (6.3):

\[
\text{Hom}_{\Pi_r S_{N_r}}(\pi_1 \otimes \ldots \otimes \pi_l, Q_m(a_1, \ldots, a_N)) = e_G M_{c,k}(\tau(\pi_1, \ldots, \pi_l, \chi)^*),
\]

where \( \tau(\pi_1, \ldots, \pi_l, \chi) = \text{Ind}_{(\mathbb{Z}/l\mathbb{Z})^N}^{\mathbb{Z}/l\mathbb{Z}} S_{N_1} \chi \), and the grading is modified as explained above. Therefore, we have

**Proposition 6.14.** If \((a_1, \ldots, a_N) = (z_1, \ldots, z_1, \ldots, z_l, \ldots, z_l) \) and \( z_i - z_j \) are not nonzero integers, then the Hilbert series of the graded vector space \( \text{Hom}_{\Pi_r S_{N_r}}(\pi_1 \otimes \ldots \otimes \pi_l, Q_m(a_1, \ldots, a_N)) \)
Example 6.15. 1. If \( \pi \) is an irreducible representation of \( S_n \), \( \text{cont}(\pi) \) is the content of the Young diagram of \( \pi \), and \( h_\pi(t) \) is the Hilbert series of the graded space \( (\pi \otimes \mathbb{C}[X_1, ..., X_n])^{S_n} \), i.e.,

\[
h_\pi(t) = \frac{K_\pi(t)}{(1 - t)(1 - t^n)},
\]

where \( K_\pi(t) \) is the Kostka polynomial associated to \( \pi \).

2. Let \( N = 2, l = 2, N_1 = N_2 = 1 \). We get from Proposition 6.14 that the Hilbert series of \( Q_m(a_1, a_2) \) is \( h(t) = t^{3m} \), i.e., we recover the formula of Example 6.12 for the case of generic \( a \).

3. Let \( N = 3, l = 3, N_1 = N_2 = N_3 = 1 \). Then we get from Proposition 6.14 that the Hilbert series of \( Q_m(a_1, a_2, a_3) \) is \( h(t) = t^{3m} \).

4. Let \( N = 3, l = 2, N_1 = 1, N_2 = 2 \). The space \( Q_m(a, 0, 0) \) splits into the direct sum

\[
Q_m(a, 0, 0) = Q_m(a, 0, 0)_+ \oplus Q_m(a, 0, 0)_-,
\]

the symmetric and antisymmetric part under \( s_{23} \). Denoting the Hilbert series of these spaces by \( h_+ \) and \( h_- \), we get from Proposition 6.14

\[
h_+(t) = \frac{t^{2m}}{(1 - t)^2(1 - t^2)},
\]

\[
h_-(t) = \frac{t^{4m+1}}{(1 - t)^2(1 - t^2)}.
\]

6.6. \( q \)-deformed twisted quasivariants. Keep the notation of the previous subsection, and let \( q > 0 \). Define the module of \( q \)-deformed twisted quasivariants \( Q_{m,q}(a_1, ..., a_N) \) to be the space of polynomials \( F \in \mathbb{C}[X_1, ..., X_N] \) such that the function

\[
\tilde{F}(X_1, ..., X_N) := (\prod_i X_i^{a_i})F(X_1, ..., X_N)
\]

(regarded as a function of \( X_i > 0 \)) is a \( q \)-deformed \( m \)-quasivariant, in the sense that \((1 - s_{ij})\tilde{F} \) is divisible by \( \prod_{p=-m}^m (X_i - q^p X_j) \) in the ring of analytic functions. If \( q \neq 1 \), this is equivalent to saying that \((1 - s_{ij})\tilde{F} \) vanishes if \( X_i \) is divisible by \( X_j \), \(-m \leq p \leq m \). Note that

\[
Q_{m,1}(a_1, ..., a_N) = Q_m(a_1, ..., a_N), \quad Q_{m,q}(a_1, ..., a_N) = Q_{m,q}(a_1 - a, ..., a_N - a),
\]

and \( Q_{m,q}(a, ..., a) \) for all \( a \).
Obviously, $Q_{m,q}(a_1, \ldots, a_N)$ is a graded $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$-module. By the Hilbert basis theorem, this module is finitely generated.

**Theorem 6.16.** If $a_i - a_j$ are not nonzero integers then for all but countably many $q$, the Hilbert series of $Q_{m,q}(a_1, \ldots, a_N)$ coincides with the one for $Q_m(a_1, \ldots, a_N)$ (i.e., $Q_{m,q}(a_1, \ldots, a_N)$ is a flat $q$-deformation of $Q_m(a_1, \ldots, a_N)$). Moreover, $Q_{m,q}(a_1, \ldots, a_N)$ is a free $\mathbb{C}[X_1, \ldots, X_N]^{S_N}$-module (of rank $N!$).

**Proof.** The proof is parallel to the proof of Theorem 6.9 using Theorem 6.11. Namely, the second statement follows from the first one, and it suffices to prove the first statement in the formal setting, $q = e^\varepsilon$. In this case, one shows as in the proof of Theorem 6.9 that the action of $e^H \tilde{H}^i_{N,\text{deg}}(z,1,m)e$ on $Q_m(a_1, \ldots, a_N)$ constructed in the proof of Theorem 6.11 can be $q$-deformed to an action of $e^H \tilde{H}^i_{N,\text{formal}}(z,1,m)e$ on $Q_{m,q}(a_1, \ldots, a_N)$. The rest of the argument is the same as in the proof of Theorem 6.9. □

**Remark 6.17.** We expect that $Q_{m,q}(a_1, \ldots, a_N)$ is a flat deformation of $Q_m(a_1, \ldots, a_N)$ for all values of $a_i$, in particular when the module $Q_m(a_1, \ldots, a_N)$ is not necessarily free. This has been confirmed by a computer calculation in low degrees in Example 6.13. Also, we expect that Theorems 6.2, 6.9, 6.16 hold for all nonzero values of $q$.

**Appendix A. More general multiplicative quiver and bow varieties**

by Hiraku Nakajima and Daisuke Yamakawa

In this appendix, we study more general multiplicative quiver and bow varieties, as multiplicative analog of results in [NT].

Multiplicative bow varieties are defined as in Section 5.4 corresponding to more general bow diagrams with dimension vectors not necessarily satisfying the balanced condition:

\begin{equation}
(A.1) \quad \begin{array}{c}
\begin{array}{c}
V_{i-1} \quad V_i \quad V'_i \quad V''_i \quad V_{i+1}
\end{array} \\
\begin{array}{c}
x_i \quad h_{j+1} \quad h_j \quad x_{i+1}
\end{array}
\end{array}
\end{equation}

Recall that the balanced condition is $V_i = V'_i = V''_i$ for all $i$. We put $A$, $B$, $a$, $b$ at $\times$, and $C$, $D$ at $\bigcirc$. See [NT, Figure 1]. We only consider the stability parameter $\nu^{\mathbb{R}} = 0$ for brevity, as generalization to arbitrary $\nu^{\mathbb{R}}$ is straightforward.

The definition of multiplicative quiver varieties is more delicate when $\dim W \neq 1$, hence will be given in Section A.3.

**A.1. Hanany-Witten transition.** We first introduce Hanany-Witten transition for multiplicative bow varieties.
Consider the following part of bow data:

\[
\begin{array}{c}
B_1 \xrightarrow{A} B_2 \xrightarrow{B_3} V_3 \\
V_1 \xrightarrow{D} V_2 \xrightarrow{C} V_3
\end{array}
\]

where \( t, Z, Z' \) are fixed nonzero scalars. We consider another bow data with the same \( V_1, V_3, B_1, B_3 \):

\[
\begin{array}{c}
B_1 \xrightarrow{A^n} B_2 \xrightarrow{B_3} V_3 \\
V_1 \xrightarrow{D^n} V_2 \xrightarrow{C^n} V_3
\end{array}
\]

We take quotients by \( GL(V_2), GL(V_2^n) \) respectively. (But not by \( GL(V_1), GL(V_3) \).)

**Proposition A.1.** There exists a \( GL(V_1) \times GL(V_3) \)-equivariant isomorphism between two varieties above.

The isomorphism is explicitly given during the proof.

**Proof.** We consider a three term complex

\[
V_2 \xrightarrow{\alpha = [D \, A]} V_1 \oplus V_3 \oplus \mathbb{C} \xrightarrow{\beta = [AB_2C \, (B_3 - Z') \, a]} V_3,
\]

where \( \beta \alpha = 0 \) follows from one of the defining equation. We claim that \( \alpha \) is injective. We consider \( S = \text{Ker} \alpha \). Then \( A(S) = 0 = b(S) \) and \( B_2(S) = (1 - C(1 + DC)^{-1}D)(S) = S \). Therefore the condition (S1) implies \( S = 0 \).

Let

\[
V_2^n := \text{Coker} \, \alpha.
\]

We define new bow data as

- \( A^n, D^n, a^n \) are composition of inclusions of \( V_1, V_3, \mathbb{C} \), to \( V_1 \oplus V_3 \oplus \mathbb{C} \) and the projection \( V_1 \oplus V_3 \oplus \mathbb{C} \rightarrow V_2^n \) respectively.
- \( b^n = b CB_1 \), and \( C^n \) is a homomorphism induced from \( -B_3^{-1} \beta \).
- \( B_2^n = tZ(1 + D^nC^n)^{-1} = tZ(1 - D^n(1 + C^nD^n)^{-1}C^n) \).

The last definition requires checking of the invertibility of \( 1 + D^nC^n \). Let us consider \( 1 + C^nD^n \). From the definition it is \( 1 - B_3^{-1}(B_3 - Z') = Z'B_3^{-1} \). This is the second of the defining equation. Thus \( 1 + C^nD^n \) is invertible, and hence \( 1 + D^nC^n \) is also invertible. (In fact, \( (1 + D^nC^n)^{-1} = 1 - D^n(1 + C^nD^n)^{-1}C^n \).) Hence the above \( B_2^n \) is well defined.
In order to check the remaining defining equation, we consider

$$V_1 \xrightarrow{\alpha^n} V_1 \oplus V_3 \oplus \mathbb{C} \xrightarrow{\beta^n = [A^n, D^n, a^n]} V_2^n.$$  

This is a complex if and only if the last defining equation holds. Observe that $\beta^n$ is nothing but the natural projection. We also have

$$\alpha^n = \begin{bmatrix} tZ - B_1 \\ -tZZ' - 1 B_3 C_a \end{bmatrix} = \begin{bmatrix} D B_1 \\ tZZ' - 1 A B_2 C \end{bmatrix} = \alpha C B_1,$$

where we have used $C B_1 = tZZ' - 1 B_2 C$, $C^n B_2 = tZZ' - 1 B_3 C^n$. Therefore $\beta^n \alpha^n = 0$.

Let us check the condition (S1) for new data. Take a subspace $S \subset V_1$ such that $B_1(S) \subset S$, $A^n(S) = 0 = B^n(S)$. Observe that $A^n(S) = 0$ means $S \oplus 0 \oplus 0 \subset \text{Im} \alpha$. Let us consider $\tilde{S} = \alpha^{-1}(S \oplus 0 \oplus 0)$. Then $D(\tilde{S}) = S$ and $A(\tilde{S}) = 0 = b(\tilde{S})$. Therefore

$$\alpha(Z' - B_2)(\tilde{S}) = \alpha B_2 C D(\tilde{S}) = \alpha B_2 C(S) = \alpha C B_1(S) = \alpha^n(S) = \begin{bmatrix} (Z - B_1)(S) \\ 0 \\ 0 \end{bmatrix}.$$  

The condition $B_1(S) \subset S$ implies $B_2(\tilde{S}) \subset \tilde{S}$. Hence $\tilde{S} = 0$ thanks to (S1) for the original data. We have $S = 0$ as well.

Let us check the condition (S2). Suppose we have a subspace $T \subset V_2^n$ such that $B_2^n(T) \subset T$, $\text{Im} A^n + \text{Im} a^n \subset T$. We take its inverse image $\tilde{T} = (\beta^n)^{-1}(T)$ in $V_1 \oplus V_3 \oplus \mathbb{C}$. By the second assumption, it contains $V_1 \oplus \{0\} \oplus \mathbb{C}$. Hence $\tilde{T}$ is a form of $V_1 \oplus \tilde{T} \oplus \mathbb{C}$ for $\tilde{T} \subset V_3$. We also have $\text{Im} \alpha \subset \tilde{T}$. Hence $A(V_2) \subset \tilde{T}$. As $B_2^n = tZ(1 - D^n(Z' - 1 B_3) C^n)$, the condition $B_2^n(T) \subset T$ implies $0 \oplus \beta(\tilde{T}) \oplus 0 \subset \tilde{T}$, i.e., $A B_2 C(V_1) + (B_3 - Z')(\tilde{T}) + a(\mathbb{C}) \subset \tilde{T}$. Hence $\tilde{T} = V_3$ thanks to (S2) for the original data. We have $T = V_2^n$ as well.

The inverse construction is clear. The original vector space $V_2$ is recovered from the new data as $\text{Ker} \beta^n$. Note also $\beta^n$ is surjective thanks to (S2). Then $A, b, D$ are given as restrictions of projections $V_1 \oplus V_3 \oplus \mathbb{C}$ to $V_3 \oplus \mathbb{C}$, $V_1$ to $\text{Ker} \beta^n$ respectively, $a$ is $-B_3 C a^n$, and $C$ is $\alpha^n B_1^{-1}$ by (A.2). Finally we set $B_2 = Z'(1 + CD)^{-1}$.

The conditions (S1),(S2) for $(A, B_2, B_3, a, b)$ follow from the conditions (S1),(S2) for $(A^n, B_1, B_2^n, a^n, b^n)$. We leave the details to the reader as an exercise. \hfill \square

Once this isomorphism is established, the remaining arguments of [NT §7] only use dimension vectors, hence work for the multiplicative case. To state the result, we recall invariants (see [NT §7.3]): Let $N_{x_i}$ be the difference (left minus right) of entries of the dimension vector at $x_i$. Then set $N(x_i, x_{i+1}) = N_{x_i} - N_{x_{i+1}}$ plus the number of $\mathcal{O}$ between $x_i \rightarrow x_{i+1}$. It is invariant under Hanany-Witten transition. Now by [NT Prop. 7.20] we have
**Proposition A.2.** If \( N(x_i, x_{i+1}) \geq 0 \) for any \( i \), the multiplicative bow variety is isomorphic to another multiplicative bow variety with a cobalanced dimension vector by successive applications of Hanany-Witten transitions of Proposition A.1.

Recall the cobalanced condition is \( N_{x_i} = 0 \) for all \( i \).

Recall (Section 7.5) a K-theoretic Coulomb branch for a framed quiver gauge theory of an affine type with dimension vectors \( \mathbf{v}, \mathbf{w} \) is isomorphic to a multiplicative bow variety with the balanced condition. In this case, the condition \( N(x_i, x_{i+1}) \geq 0 \) for all \( i \) is equivalent to the dominance condition \( \mathbf{w} - C \mathbf{v} \in \mathbb{Z}_{\geq 0}^n \), as the number of \( \mathcal{O} \) between \( x_i \to x_{i+1} \) is \( \mathbf{w}_i \).

### A.2. Multiplicative bow varieties with cobalanced dimension vector.

Now we study a multiplicative bow variety with cobalanced condition. Consider the following part of a bow diagram:

\[
\begin{array}{cccccc}
V_1 & \xleftarrow{D_1} & V_2 & \xrightarrow{A_2} & V_3 & \xrightarrow{A_3} & V_4 & \xrightarrow{D_4} & V_5 \\
& \xleftarrow{C_1} & \downarrow{b_2} & \xleftarrow{b_3} & \xrightarrow{a_3} & \xrightarrow{a_4} & \downarrow{C_2} & \rightleftharpoons & \\
& & C & & C & & & & \\
\end{array}
\]

We assume \( \dim V_2 = \dim V_3 = \dim V_4 \) by the cobalanced condition. By [Ta Lemma 2.18], \( A_2, A_3 \) are isomorphisms. So we normalize \( A_2 = A_3 = \text{id} \) by the action of \( GL(V_3) \times GL(V_4) \). Then the defining equation becomes

\[
1 + C_1 D_1 = Z_2 B_2^{-1} = Z_2 (B_3 + a_3 b_2)^{-1} = Z_2 (1 + B_3^{-1} a_3 b_2)^{-1} B_3^{-1} = t^{-1} Z_2 (1 + B_3^{-1} a_3 b_2)^{-1} (1 + B_4^{-1} a_4 b_3)^{-1} B_4^{-1} = t^{-2} Z_4^{-1} Z_2 (1 + B_3^{-1} a_3 b_2)^{-1} (1 + B_4^{-1} a_4 b_3)^{-1} (1 + D_4 C_4).
\]

This can be regarded as the defining equation of a multiplicative quiver variety with

\[
(V_1 \leftarrow V_2 \xrightarrow{D_4} V_5),
\]

where we do not take the quotient by \( \mathbb{C}^\times \times \mathbb{C}^\times \) at bottom vertices. We can further make

\[
(V_1 \leftarrow V_2 \xrightarrow{D_4} V_5),
\]

where \( \tilde{a}_3 = (1 + B_3^{-1} a_4 b_3) B_3^{-1} a_3, \tilde{a}_4 = B_4^{-1} a_4 \). For additive quiver varieties \((A.3)\) and \((A.4)\) give isomorphic varieties. But in the multiplicative case, \((A.3)\) gives an open
subvariety in \((A.4)\) as the invertibility of both \((1 + B_3^{-1}a_3b_2), (1 + B_4^{-1}a_4b_3)\) is stronger than the invertibility of \((1 + a_3b_2 + a_4b_3)\). In fact, we have an additional requirement that \((1 + a_4b_3)\) is invertible. Note that the difference between \((A.3)\) and \((A.4)\) disappears when we only have one \(C\).

A.3. Multiplicative quiver varieties. Before generalizing the definition of multiplicative quiver varieties given in Section 5.1, let us recall the notion of quasi-Hamiltonian spaces and fusion/reduction procedure introduced in [AMM].

Let \(G\) be a complex reductive group with Lie algebra \(\mathfrak{g}\) and fix a non-degenerate \(\text{Ad}_G\)-invariant symmetric bilinear form \((\ , \ )\) on \(\mathfrak{g}\). Let \(\theta\) (resp. \(\overline{\theta}\)) be the left (resp. right) invariant Maurer–Cartan form on \(G\).

A quasi-Hamiltonian \(G\)-space is a smooth \(G\)-variety \(M\) equipped with a \(G\)-invariant two-form \(\omega\) on \(M\) and a \(G\)-equivariant morphism \(\mu: M \to G\) (where \(G\) acts on \(G\) by conjugation) satisfying the following three axioms:

1. **(QH1)** \(12d\omega = -\mu^*(\theta \wedge [\theta \wedge \theta])\).
2. **(QH2)** \(2i(\xi^*)\omega = \mu^*(\theta + \overline{\theta}, \xi)\) for all \(\xi \in \mathfrak{g}\). Here \(\xi^*\) is the fundamental vector field: \(\xi^*_x = \frac{d}{dt}\exp(t\xi) \cdot x\bigg|_{t=0}\) (\(x \in M\)).
3. **(QH3)** \(\text{Ker}\omega_x = \{ \xi^*_x \mid \xi \in \text{Ker}(\text{Ad}_{\mu(x)} + 1)\}\) for all \(x \in M\).

The map \(\mu\) is called the group-valued moment map.

**Example A.3.** Any conjugacy class \(C\) of \(G\) has a structure of quasi-Hamiltonian \(G\)-space whose group-valued moment map is just the inclusion \(C \hookrightarrow G\). This is a multiplicative analogue of a coadjoint orbit of \(g^*\).

Also, the double \(G \times G\) is a quasi-Hamiltonian \(G \times G\)-space, which is a multiplicative analogue of \(T^*G\).

**Example A.4.** In [VdB] Van den Bergh introduced a multiplicative analogue of the Hamiltonian \(GL(V) \times GL(W)\)-space \(T^*\text{Hom}(V, W)\), where \(V, W\) are finite-dimensional \(C\)-vector spaces. It is defined to be the quasi-Hamiltonian \(GL(V) \times GL(W)\)-space

\[ \mathcal{B}(V, W) = \{ (X, Y) \in \text{Hom}(V, W) \times \text{Hom}(W, V) \mid \det(1 + XY) \neq 0 \}, \]

where the two-form and the group-valued moment map are given by

\[ \omega = \frac{1}{2} \text{Tr} \left( (1 + XY)^{-1}dX \wedge dY \right) - \frac{1}{2} \text{Tr} \left( (1 + YX)^{-1}dY \wedge dX \right), \]

\[ \mu(X, Y) = (1 + YX)^{-1}, 1 + XY). \]

This is called the Van den Bergh space.

Next we introduce the internal fusion of quasi-Hamiltonian spaces. Let \(H\) be another complex reductive group equipped with a non-degenerate \(\text{Ad}_H\)-invariant symmetric bilinear form on its Lie algebra \(\mathfrak{h}\).
Theorem A.5 ([AMM]). Let \((M, \omega, (\mu_1, \mu_2, \nu))\) be a quasi-Hamiltonian \(G \times G \times H\)-space. Let \(G \times H\) act on \(M\) through the diagonal embedding \((g, h) \mapsto (g, g, h)\). Then \(M\) equipped with the two-form
\[
\omega_{\text{fus}} := \omega + \frac{1}{2}(\mu_1^* \theta \wedge \mu_2^* \overline{\theta})
\]
and the map
\[
(\mu_{\text{fus}}, \nu) := (\mu_1 \cdot \mu_2, \nu) : M \to G \times H
\]
is a quasi-Hamiltonian \(G \times H\)-space.

The quasi-Hamiltonian \(G \times H\)-space \((M, \omega_{\text{fus}}, (\mu_{\text{fus}}, \nu))\) is called the \textit{internal fusion} and denoted by \(M_{\text{fus}}\).

The internal fusion procedure is associative in the following sense: If \(M\) is a quasi-Hamiltonian \(G \times G \times G \times H\)-space, then two quasi-Hamiltonian \(G \times H\)-spaces \(M_{(12)3}\) obtained by first fusioning the first two \(G\)-factors and \(M_{1(23)}\) obtained by first fusioning the last two \(G\)-factors are identical. Therefore, if \(I\) is a non-empty totally ordered finite set and \(M\) is a quasi-Hamiltonian \(G_I \times H\)-space with group-valued moment map \((\mu_i)_{i \in I}, \nu\), then we can define its internal fusion \(M_{\text{fus}}\) as a quasi-Hamiltonian \(G \times H\)-space in a canonical way so that its group-valued moment map is \((\mu, \nu)\), where \(\mu := \prod_{i \in I} \mu_i\).

A multiplicative analogue of the Marsden–Weinstein theorem is the following theorem:

Theorem A.6 ([AMM]). Let \((M, \omega, (\mu, \nu))\) be a quasi-Hamiltonian \(G \times H\)-space and \(C\) be a conjugacy class of \(G\). If the \(G\)-action on \(\mu^{-1}(C)\) is free, then \(\mu^{-1}(C)\) is a smooth subvariety of \(M\), and if furthermore the action has a geometric quotient \(\mu^{-1}(C)/G\), then \(\omega\) and \(\nu\) induce a quasi-Hamiltonian \(H\)-structure on \(\mu^{-1}(C)/G\).

The above quasi-Hamiltonian \(H\)-space \(\mu^{-1}(C)/G\) is called the quasi-Hamiltonian reduction of \(M\) by \(G\) along \(C\) and denoted by \(M \sslash C\).

Note that in the above situation if \(H\) is abelian then \(M \sslash C G\) is symplectic. The following fact is also useful.

Theorem A.7. Let \((M, \omega, (\mu, \nu))\) be a quasi-Hamiltonian \(G \times H\)-space. Assume that \(H\) is abelian and \(M\) has a good quotient \(\pi : M \to M/G\). Then for any open \(U \subset M/G\) and \(f \in \Gamma(U, \mathcal{O}_{M/G})\), there exists a unique vector field \(v_f \in \Gamma(\pi^{-1}(U), \Theta_M)\) such that
\[
\iota(v_f)\omega = d\pi^* f, \quad \iota(v_f)\mu^* \theta = 0.
\]
Each \(v_f\) is \(G\)-invariant and preserves \(\omega\) and \(\mu\). Also, \(\pi^* \{f, g\} = v_f(\pi^* g)\) \((f, g \in \mathcal{O}_{M/G})\) defines an \(H\)-invariant Poisson structure on \(M/G\).

See [AMM, Proposition 4.6], where \(H\) is assumed to be trivial but the arguments in the proof work in the general case.

Since each \(v_f\) preserves \(\mu\), for any conjugacy class \(C \subset G\) the closed subvariety \(M \sslash C G := \mu^{-1}(C)/G \subset M/G\) is Poisson. We call it the quasi-Hamiltonian reduction
of $M$ by $G$ along $\mathcal{C}$. On the other hand, if there exists an open $U \subset \mu^{-1}(\mathcal{C})/G$ such that $\mu(\pi^{-1}(U)) \subset \mathcal{C}$ and $G$ acts freely on $\pi^{-1}(U)$, then Theorem A.6 shows that $U$ is a quasi-Hamiltonian $H$-space, in particular symplectic. By the definition, the Poisson structure on $U$ given by Theorem A.7 coincides with one induced from the symplectic structure.

Now we introduce the multiplicative quiver varieties. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. Let $V = \bigoplus_{i \in Q_0} V_i$ and $W = \bigoplus_{i \in Q_0} W_i$ be two finite-dimensional $Q_0$-graded $\mathbb{C}$-vector spaces with dimension vectors $\mathbf{v} = (v_i)_{i \in Q_0}$ and $\mathbf{w} = (w_i)_{i \in Q_0}$, respectively. For each $i \in Q_0$ fix a decomposition of $W_i$ into one-dimensional pieces

$$W_i = \bigoplus_{j=1}^{w_i} W_{i,j}, \quad \dim W_{i,j} = 1,$$

and let $T(W) \subset GL(W) := \prod_{i \in Q_0} GL(W_i)$ be the associated maximal torus. Define

$$\tilde{B}_Q(V, W) = \prod_{h \in Q_1} B(V_{s(h)}, V_{t(h)}) \times \prod_{i \in Q_0} \prod_{j=1}^{w_i} B(W_{i,j}, V_i),$$

which is a quasi-Hamiltonian $\mathbb{G} \times T(W)$-space, where

$$\mathbb{G} := \prod_{h \in Q_1} (GL(V_{s(h)}) \times GL(V_{t(h)})) \times \prod_{i \in Q_0} GL(V_i)^{w_i}.$$

Using the double $\overline{Q}$ of $Q$ with involution $*: Q_1 \to Q_1^t$ (so $Q_1 = Q_1 \sqcup Q_1^t$ and $s \circ * = t$, $t \circ * = s$), we denote an element of $\tilde{B}_Q(V, W)$ by $(C, a, b)$, where $C = (C_h)_{h \in \overline{Q}_1}$, $a = (a_{i,j})$, $b = (b_{i,j})$ and

$$(C_h, C_h^*) \in B(V_{s(h)}, V_{t(h)}) \quad (h \in Q_1),$$

$$(a_{i,j}, b_{i,j}) \in B(W_{i,j}, V_i) \quad (i \in Q_0, j = 1, \ldots, w_i).$$

(In other places, $C_h$ is denoted by $C_i$ or $D_i$ according to whether $h$ is in $Q_1$ or not.) Fix a total ordering $<$ on $\overline{Q}_1$ and define a quasi-Hamiltonian $GL(V) \times T(W)$-space by

$$B_Q(V, W) = \tilde{B}_Q(V, W)_{\text{fus},<},$$

where the internal fusion is taken with respect to the diagonal embedding $GL(V) \hookrightarrow \mathbb{G}$ so that the resulting group-valued moment map $(\mu, \nu) = ((\mu_i), (\nu_{i,j}))$ is

$$\mu_i(B, a, b) = \prod_{h \in \overline{Q}_1, t(h) = i}^{<} (1 + C_h C_{h^*})^{\epsilon(h)} \prod_{j=1}^{w_i} (1 + a_{i,j} b_{i,j}),$$

$$\nu_{i,j}(C, a, b) = (1 + b_{i,j} a_{i,j})^{-1},$$

where $\epsilon(h) = 1$ if $h \in Q_1$ and $\epsilon(h) = -1$ if $h \in Q_1^t$. 

**Definition A.8.** For \( \gamma = (\gamma_i)_{i \in Q_0} \in (\mathbb{C}^*)^{Q_0} \), put \( \gamma_V = (\gamma_i 1_{V_i})_{i \in Q_0} \in GL(V) \) and define

\[
\mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) = \mu^{-1}(\gamma_V)/GL(V),
\]

where the quotient is taken as the affine GIT quotient. We call \( \mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) \) the *multiplicative quiver variety*.

By Theorem A.7, \( \mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) = B_Q(V,W)/\gamma_V GL(V) \) is a Poisson variety. Also, as in the case of additive quiver varieties, the \( GL(V) \)-action on the open subset \( \mu^{-1}(\gamma_V)^s \subset \mu^{-1}(\gamma_V) \) consisting of stable points has stabilizer \( \mathbb{C}^* 1_V \) everywhere (where the definition of stability is exactly the same as in the additive case), and Theorem A.6 implies that the open subset \( \mathcal{M}_\gamma^{\times reg}(\mathbf{v},\mathbf{w}) := \mu^{-1}(\gamma_V)^s/GL(V) \) of \( \mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) \) is a quasi-Hamiltonian \( T(W) \)-space. We also call \( \mathcal{M}_\gamma^{\times reg}(\mathbf{v},\mathbf{w}) \) the multiplicative quiver variety.

**Example A.9.** The multiplicative quiver variety \( \mathcal{M}_N^I(Z,t) \) introduced in Section 5.1 coincides with \( \mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) \) with \( Q = \hat{A}_{l-1} \) (the orientation is \( 1 \to l \to \cdots \to 2 \to 1 \)) and

\[
\mathbf{v} = (N,N,\ldots,N), \quad \mathbf{w} = (1,0,\ldots,0), \quad \gamma = \left( \frac{Z_l}{Z_1}, \frac{Z_1}{Z_2}, \ldots, \frac{Z_{l-1}}{Z_l} \right).
\]

**Remark A.10.** Definition A.8 is close to the definition of framed multiplicative quiver variety introduced in [Y]. Let us identify all \( W_{i,j} \) with \( \mathbb{C} \). Then the framed multiplicative quiver variety is obtained by replacing \( B_Q(V,W) \) with its internal fusion for the diagonal subgroup \( \mathbb{C}^* \subset T(W) \) in the definition of \( \mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) \). It is the same as \( \mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) \) as a variety, but the Poisson brackets are different in general.

**Remark A.11.** There is another multiplicative analogue of quiver variety. It is obtained by replacing \( \tilde{B}_Q(V,W) \) with the quasi-Hamiltonian \( G \times GL(W) \)-space

\[
\tilde{B}_Q'(V,W) := \prod_{h \in Q_1} B(V_{s(h)},V_{t(h)}) \times \prod_{i \in Q_0} B(W_i, V_i)
\]

in the definition of \( \mathcal{M}_\gamma^\times(\mathbf{v},\mathbf{w}) \). But note that in general the two-form on \( \tilde{B}_Q'(V,W)_{\text{fus},<} \) does not induce a Poisson structure on the resulting variety. It is why we decompose each \( W_i \) into one-dimensional pieces. Of course if \( \dim W_i \leq 1 \) for all \( i \in Q_0 \) then the two definitions coincide.

**Example A.12.** Consider \( Q = A_{l-1} \) with \( \mathbf{w} = (\ell,0,\ldots,0) \). The usual quiver variety \( \mathcal{M}_\zeta(\mathbf{v},\mathbf{w}) \), if it is not \( \emptyset \), is a semisimple coadjoint orbit of \( gl(\ell) \) for generic \( \zeta \). The multiplicative quiver variety in the definition in Remark A.11 is a conjugacy class of \( GL(\ell) \). (See e.g., [CBS, §8].) On the other hand, Definition A.8 gives its open subset, the intersection with a big Bruhat cell of \( GL(\ell) \). To see this, take a decomposition \( W = \bigoplus_{j=1}^\ell W_j \) into one-dimensional spaces, and write \( a = (a_j), b = (b_j) \). Then \( \tilde{a} = (\tilde{a}_1,\ldots,\tilde{a}_\ell), \tilde{b} = (\tilde{b}_1,\ldots,\tilde{b}_\ell) \) with \( \tilde{a}_j = (1+a_1b_1)\ldots(1+a_{j-1}b_{j-1})a_j, \tilde{b}_j = b_j \)
satisfy the defining equation in Remark A.11 as

\[ 1 + \sum_{j=1}^{\ell} \tilde{a}_j \tilde{b}_j = (1 + \tilde{a}_1) \cdots (1 + \tilde{a}_\ell). \]

The isomorphism to a conjugacy class is given by the group-valued moment map \((1 + \tilde{b} \tilde{a})^{-1}\). If an element \(1 + \tilde{b} \tilde{a}\) is coming from \((a_j), (b_j)\), we have additional constraint \(\det(1 + a_j b_j) \neq 0\) for \(j = 1, \ldots, \ell\), which is equivalent to require that every leading principal minor of \(1 + \tilde{b} \tilde{a}\) is nonzero.

Now we understand the correct definition of multiplicative quiver varieties, and the argument in Section A.2 gives the following.

**Theorem A.13.** Consider a multiplicative bow variety with a cobalanced dimension vector. It is isomorphic to a multiplicative quiver variety \(\mathcal{M}^\times(\mathbf{v}, \mathbf{w})\) such that \(\mathbf{v}_j\) (resp. \(\mathbf{w}_j\)) is the dimension of vector spaces (resp. the number of \(\times\)) between \(h_j\) and \(h_{j+1}\), and \(\gamma_j = t^{w_j} Z_j / Z_{j+1}\).

Combining this with Proposition A.2, we obtain an isomorphism between a \(K\)-theoretic Coulomb branch and a multiplicative quiver variety when the dominance condition is satisfied. We conjecture that this is an isomorphism of Poisson varieties. More generally we conjecture

**Conjecture A.14.** (1) The space consisting of \(A_{i-1}^{-1}, B_{i-1}, B'_i, a_i, b_{i-1}\) with (i), (a), (S1), (S2) in Section 5.4 is a quasi-Hamiltonian \(GL(V_{i-1}) \times GL(V_i) \times \mathbb{C}^*\)-space with the group-valued moment map \(B_{i-1}^{-1}, B'_i, \det B_{i-1} \det B'_i\). Therefore a multiplicative bow variety is a quasi-Hamiltonian reduction.

(2) The Hanany-Witten transition is an isomorphism of quasi-Hamiltonian \(GL(V_1) \times GL(V_3) \times \mathbb{C}^*\)-spaces.

(3) The isomorphism of Theorem A.13 is an isomorphism of quasi-Hamiltonian \(T(W)\)-spaces.

**Remark A.15.** Both homological and \(K\)-theoretic Coulomb branches have a torus action induced from \(\text{Hom}(\pi_1(G), \mathbb{Z})\). See \[4(iii)(c)] and the original physics literature therein. It is expected that the torus action extends to a nonabelian group action for homological Coulomb branches \[4(iii)(d)]\, and proved for \(ADE\) quiver gauge theories in [BFN], Remark 3.12]. On the other hand, it is not true for \(K\)-theoretic Coulomb branches as Example A.12 shows.

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