A comparison of the extended Kompaneets equations with the Ross-McCray equation

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Abstract. The Kompaneets and the Ross-McCray equations can be used to describe the Comptonization process. However, the classical Kompaneets equation fails to describe down-Comptonization while Ross-McCray equation can be applied to describe down-Comptonization but inappropriate for a blackbody equilibrium. Fortunately, the Kompaneets equation extended by frequency and the Kompaneets equation extended by momentum can solve both problems. The different physical connotations behind the four equations bring about formal differences that allow the spectral evolution under different conditions to show different characteristics. In order to compare the differences between the four equations, the evolution of our common radiation spectra in astrophysics is numerically calculated in this paper. Besides, the differences between the four diffusion equations are discussed from their physical significance to the evolutionary phenomena. According to the results, the four equations evolve at different rates during down-Comptonization, from fast to slow in order of the Kompaneets equation extended by frequency, the Kompaneets equation extended by momentum, the classical Kompaneets equation and the Ross-McCray equation. The regression of the four equations is consistent during up-Comptonization. The Ross-McCray equation eventually converges to a different end state at near equilibrium. These results shed light on analytical development of Comptonization process.

1. Introduction

Radiative transfer processes, which occur when radiation passes through a plasma medium, have been an important topic in astrophysics and radiation physics. With the rapid development of ray astronomy, there is an urgent need for further research of the radiative transfer theory concerning X-rays. With the energy exchange between the radiation field and the plasma medium, "Comptonization process" will be triggered, where the spectral shape, intensity, spectral line position, intensity ratio of each emission line, polarization characteristics of the emitted light are significantly changed. To understand the physical properties within the source and interpret the observations, it is necessary to discuss the original radiation mechanism and consider the radiative transfer during plasma propagation. Although the change in frequency is minimal during Comptonization due to inelastic collision (a photon with an electron), the cumulative effect of multiple scattering cannot be ignored on the celestial scale, especially when the Comptonization process is particularly important.
Comptonization is a very common phenomenon in astrophysics. Consider a thermal electron with temperature $T_e$, if the average photon energy $\hbar \nu$ is much less than the electron energy $kT_e$, i.e., $\hbar \nu \ll kT_e$, the photon will gain energy during Comptonization. This process is called up-Comptonization, which is often found in radio and infrared astronomy. When $\hbar \nu \gg kT_e$, the electron gains energy from the high energy photons (e.g., X-rays, $\gamma$-rays) known as down-Comptonization.

In 1957, Kompaneets established the kinetic diffusion equation for the photon distribution function $n(\nu)$, which quantitatively describes the variation of the photon gas spectral distribution with time during the Comptonization process towards thermal equilibrium[1]. Since the Kompaneets equation uses the condition $\hbar \nu \ll kT_e \ll m_ec^2$ in its derivation, it is only applicable to the up-Comptonization process. However, with the development of X-ray and $\gamma$-ray astronomy, the interest is often in the down-Comptonization processes, e.g., changes in the radiation spectrum caused by X-ray radiation near dense stars, passing through the envelope of "cold" accreting gas. In order to solve the down-Comptonization process, Ross, Weaver and McCray proposed the Ross-McCray equation in 1978[2]. Whereas, the Ross-McCray equation does not guarantee an equilibrium state under the Planck distribution. The extended Kompaneets equation applicable to the down-Comptonization process under $kT_e \ll \hbar \nu \ll m_ec^2$ was obtained by Junhan You and Junfeng Chen in 1992[3]. In 2015, another extended form of the Kompaneets equation was derived by Xu Zhang, Xurong Chen using different methods[4]. In this paper, we will start from the classical Kompaneets equation, derive the Kompaneets equation extended by frequency and the Kompaneets equation extended by momentum, and compare their evolution with the Ross-McCray equation.

Contemporarily, based on the classical Kompaneets equation, further generalisations of the Kompaneets equation have been made under relativistic conditions. Progress has also been made in the steady-state and time-containing solutions of the Kompaneets equation, as well as in Comptonization models for finite non-uniform media and polarised radiation field[5-11]. Discussing the derivation of different forms of Comptonization diffusion equations is useful to capture the differences in the physical nature behind the different scattering equations. Meanwhile, the evolutionary effects of the Comptonization equations are different in different media environments. Therefore, in order to clarify the applicability of each equation, one is ought to numerically calculate and compare the evolution of spectral lines for different spectral shapes and electron gas temperatures.

2. Kinetic equations describing the Comptonization process

The Comptonization process occurring in photon gas with electron gas is usually described by the kinetic diffusion equation. This chapter will start with the classical Kompaneets equation and then derive the Kompaneets equation extended by frequency and the Kompaneets equation extended by momentum. A comparison with the Ross-McCray equation is also made and the physical differences between these four equations are discussed.

2.1. Classical Kompaneets equation

Consider a mixture of a thermal electron gas and a photon gas that has not reached thermal equilibrium. Since collisions between electrons are extremely easy to occur and quickly reach thermal equilibrium, velocity distribution can be assumed as Maxwell distribution:

$$f(p) = f_0 \exp \left[ -\frac{1}{kT} \frac{p^2}{2m_e} \right] d^3p \quad (1)$$

It is clear that after Compton scattering of photons and electrons, thermal equilibrium will be eventually established. In fact, the photon distribution function $n(\nu)$ variation before reaching equilibrium is the main point should be focused.

The $n(\nu)$ represents the number of particles in the photon gas for each photon state of frequency $\nu$. The number of photons per unit volume near $n(\nu)$ is $\frac{\hbar \nu^2 d\nu}{c^3} n(\nu)$. Assuming a uniform distribution of photons in space, $n(\nu)$ denotes the average distribution function of a uniformly distributed photon gas in infinite space.
The variation of the photon distribution function with time during Comptonization \( \frac{\partial n}{\partial t} \) can be expressed in terms of the kinetic diffusion equation. Consider the collision of an electron with momentum \( p \) with a photon of frequency \( \nu \). The momentum of the electron after the collision is denoted \( p' \) and the frequency of the photon is denoted \( \nu' \). In the non-relativistic case, we have:

\[
\left( \frac{hv}{c} \right) n + p = \left( \frac{h\nu'}{c} \right) n' + p'
\]

\( \text{(2)} \)

\[
h\nu + \frac{p^2}{2m_e} = h\nu' + \frac{p'^2}{2m_e}
\]

\( \text{(3)} \)

where \( n \) and \( n' \) are the unit vectors of the direction of photon motion before and after scattering, respectively. The number of photons before and after the collision is noted as \( n \) and \( n' \), i.e., \( n \equiv n(\nu) \) and \( n' \equiv n(\nu') \). Compton scattering and its inverse process \((p, \nu, n) \leftrightarrow (p', \nu', n')\) have the same collision probability \( dW \), treating the dilute electron gas as a classical gas and the photon gas as a bosonic gas, the number of photon leaps is given by the second quantization of bosons as:

\[
(1 + n')nE_e f(p) d^3p dW
\]

\( \text{(4)} \)

where \( N_e \) is the electron number density per unit space and \( f(p) \) is the normalised electron distribution function, i.e., \( N_e f(p) d^3p dW \) represents the number of electrons in the momentum interval per unit space. It is worth pointing out that the non-linear terms \( n \) and \( n' \) denote the stimulated radiation process and they are crucial for the final equilibrium state of the spectral line evolution in the Comptonization process. The resulting change in photon distribution due to Compton scattering can be written as:

\[
\left( \frac{\partial n}{\partial t} \right) = -N_e \int d^3p \left[ (1 + n')nE_e f(p) - (1 + n)nE_e f(p') \right] dW
\]

\( \text{(5)} \)

Substituting Maxwell's distribution in the above equation:

\[
f(p) = f_0 \exp \left[ -\frac{p^2}{kT} \right] d^3p
\]

\( \text{(6)} \)

and the Plunk distribution of photon gas:

\[
n(\nu) = \left( \frac{e^{h\nu/kT} - 1}{e^{h\nu/kT} + 1} \right)^{-1}
\]

\( \text{(7)} \)

one obtains \( \frac{\partial n}{\partial t} = 0 \), which indicates that Compton scattering will no longer cause a change in \( n(\nu) \) when the gas mixture reaches thermal equilibrium.

Assuming that the electron is non-relativistic, based on \( kT_e \ll m_e c^2 \) and \( h\nu \ll m_e c^2 \), the change in energy per scattered photon is very small. On this basis, one can replace the frequency \( \nu \) by \( x \equiv \frac{h\nu}{kT_e} \), the dimensionless photon energy in units of \( kT_e \), and let the frequency increment \( \Delta = \nu' - \nu \), so we have \( \Delta \ll \nu \). At this point, expanding \( f(p') \) and \( n' \) with \( \Delta \) to second order minima yields:

\[
\left( \frac{\partial n}{\partial t} \right) = \left[ \frac{\partial n}{\partial x} + n(1 + n) \right] \frac{N_e h}{kT_e} \int d^3p \int f(p) \Delta dW
\]

\[
\text{+} \left[ \frac{\partial^2 n}{\partial x^2} + 2(1 + n) \frac{\partial n}{\partial x} + n(1 + n) \right] \frac{N_e h^2}{2 kT_e} \int d^3p \int f(p) \Delta^2 dW
\]

\( \text{(8)} \)

Next, calculate the second integral on the right-hand side of the above equation, such that:

\[
I = h^2 \int d^3p \int f(p) \Delta^2 dW
\]

\( \text{(9)} \)

From equation (2) and (3), preserving only the first order minima up to \( \Delta \), the expression for \( \Delta \) is derived:

\[
h\Delta = -\frac{hc p \cdot (n - n') + (hv)^2 (1 - n \cdot n')}{m_e c^2 \left[ 1 + \frac{hv}{m_e c^2} (1 - n \cdot n') - \frac{p \cdot n'}{m_e c^2} \right]}
\]

\( \text{(10)} \)
For non-relativistic electrons in low-frequency radiation fields, the $\hbar\nu \ll m_e c^2, k T_e \ll m_e c^2$, i.e., the last two terms of the denominator in equation (10) can be omitted. Since the classical Kompaneets equation discusses only the up-Comptonization process, i.e., $h\nu \ll k T$ , noting that the electron momentum $p = m_e \nu = m_e \sqrt{\frac{k T}{m_e}} \ll m_0 c$, the ratio of the second term in the numerator to the first term follows:

$$\frac{h\nu}{pc} = \frac{h\nu}{m_e c^2 k T_e} \ll 1$$

Hence, the equation (10) can be simplify by ignoring the second term in the numerator:

$$h\Delta \approx -\frac{h c}{m_e c^2} p \cdot (n - n')$$

Substituting (12) and Maxwell's distribution (1) into (9) yields:

$$I = \left(\frac{h\nu}{m_e c^2}\right)^2 \int d^3 p f(p)|p\cdot(n - n')|^2 dW$$

where the leap probability $dW = c d\sigma_T = c^2 \frac{r_0^2}{2}(1 + \cos^2 \theta)2\pi \sin \theta d\theta$, the electron radius $r_0 = \frac{e^2}{m_e c^2}$. $\theta$ is the scattering angle of the photon.

Now we fix $n - n'$, take $n - n'$ as the axis, write the angle between $p$ and $n - n'$ as $\Theta$, and integrate over $p$ to obtain:

$$I = \left(\frac{h\nu}{m_e c^2}\right)^2 \int |n - n'|^2 dW \int f(p)p^4 \cos^2 \Theta \sin \Theta dp d\theta d\phi$$

Substituting the electron Maxwell distribution and the Thomson scattering cross section we have:

$$I = \left(\frac{h\nu}{m_e c^2}\right)^2 m_e k T_e \int |n - n'|^2 dW$$

$$= \left(\frac{h\nu}{m_e c^2}\right)^2 2m_e k T_e \int (1 - \cos \Theta) dW$$

From the leap probability $dW = c d\sigma_T = c^2 \frac{r_0^2}{2}(1 + \cos^2 \Theta)2\pi \sin \theta d\theta$, $dW$ is symmetric with respect to $\theta$ and $\pi - \theta$, so that:

$$I_{ct} = \left(\frac{h\nu}{m_e c^2}\right)^2 2m_e k T_e \int (1 - \cos \Theta) dW$$

$$= \left(\frac{h\nu}{m_e c^2}\right)^2 2m_e k T_e \int dW$$

$$= 2c\sigma_T \frac{(k T_e)^3}{m_e c^2} x^2$$

We have obtained the integral value of $I_{ct}$, the second term of equation (8) with respect to $\Delta^2$. We now need to calculate the integral value of the first term in (8), which is more difficult to calculate directly, so we use the conservation of the total photon number before and after scattering to solve the complete expression.

For the Comptonization process between an isotropic photon gas and an electron gas, the photon flux in frequency space is defined as $j$. It is clear that there is only a radial component of the photon flux $j$. The equation for the continuity of the photon flux in frequency space is expressed in terms of $x$ as:

$$\left(\frac{\partial n}{\partial t}\right)_c = -\nabla \cdot j = -x^{-2} \frac{\partial (x^2 j)}{\partial x} = -\frac{2}{x} j - \frac{\partial j}{\partial x}$$

(17)
Comparing equation (8) with equation (17), the second order derivative $\frac{\partial^2 n}{\partial x^2}$ in (8) is linear, that is, the coefficients of $\frac{\partial^2 n}{\partial x^2}$ do not contain $n$. Moreover, $n$ is the highest order derivative, whereas the highest order derivative in (17) is $\frac{\partial j}{\partial x}$. It can be deduced that $j$ must be the first order derivative $\frac{\partial n}{\partial x}$ with some of the $n$ in the form of a sum of functions.

Additionally, to ensure that $\left( \frac{\partial n}{\partial x} \right)_c = 0$ when applying the Planck distribution (7), the remaining term of $j$ can only be $n(n + 1)$, so the flux must be of the form[12]:

$$j = g(x) \left[ \frac{\partial n}{\partial x} + n(n + 1) \right]$$  \hspace{1cm} (18)

Substituting equation (18) into (17) gives:

$$\left( \frac{\partial n}{\partial t} \right)_c = -g(x) \left[ \frac{\partial^2 n}{\partial x^2} + 2(n + 1) \frac{\partial n}{\partial x} \right] - \left[ \frac{\partial g}{\partial x} + \frac{2g}{x} \right] \left[ \frac{\partial n}{\partial x} + n(n + 1) \right]$$  \hspace{1cm} (19)

The coefficients of $\frac{\partial^2 n}{\partial x^2}$ in both equations (19) and (8) should be the same, so we have the following expression:

$$g_{cl}(x) = -\frac{N_e}{2} (kT_e)^{-2} I_{cl} = -N_e \sigma_T c \frac{kT_e}{m_ec^2} x^2$$  \hspace{1cm} (20)

Substituting (20) into (17), we end up with the classical Kompaneets equation:

$$\left( \frac{\partial n}{\partial t} \right)_{c, cl} = N_e \sigma_T c \frac{kT_e}{m_ec^2} \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^4 \frac{\partial n}{\partial x} + n(n + 1) \right)$$  \hspace{1cm} (21)

2.2. Kompaneets equation extended by frequency

The classical Kompaneets equation describes the radiative transfer process in an inelastic scattering process, in the form of a diffusion equation, but only for up-Comptonization processes. The derivation of the Kompaneets equation extended by frequency is similar to that of the classical Kompaneets equation; their difference lies in the different retention terms for $\Delta$ under the qualification: the classical Kompaneets equation makes the value of the quantity in the denominator square brackets approximately equal to 1 with the condition $h\nu \ll kT_e \ll m_ec^2$, while omitting the second term of the numerator. However, the Kompaneets equation extended by frequency drops the condition $h\nu \ll kT_e$ and retains only the non-relativistic conditions $h\nu \ll m_ec^2$ and $h\nu \ll kT_e$, so that both numerator terms of equation (10) need to be retained:

$$h\Delta \approx -\frac{hc}{m_ec^2} \mathbf{p} \cdot (\mathbf{n} - \mathbf{n}') - \frac{(h\nu)^2}{m_ec^2} (1 - \mathbf{n} \cdot \mathbf{n}')$$

Substituting (22) into (9) gives:

$$I = \int d^3p \int f(p) \left[ -\frac{hc}{m_ec^2} \mathbf{p} \cdot (\mathbf{n} - \mathbf{n}') - \frac{(h\nu)^2}{m_ec^2} (1 - \mathbf{n} \cdot \mathbf{n}') \right]^2 dW$$

$$= I_1 + I_2 + I_3$$

where

$$I_1 = \left( \frac{h\nu}{m_ec} \right)^2 \int d^3p \int f(p) dW |\mathbf{p} \cdot (\mathbf{n} - \mathbf{n}')|^2$$

$$I_2 = \left( \frac{h^2\nu^2}{m^2ec^2} \right)^2 \int d^3p \int f(p) dW (1 - \mathbf{n} \cdot \mathbf{n}')^2$$

$$I_3 = 2 \left( \frac{(h\nu)^3}{m^2ec^3} \right) \int d^3p \int f(p) dW \mathbf{p}(\mathbf{n} - \mathbf{n}')(1 - \mathbf{n} \cdot \mathbf{n}')^2$$

The result for $I_1$ is given in (16). The integration of $I_3$ with respect to momentum is symmetric about $\theta$ and results in zero. First integrate over the momentum part of $I_2$, which results in 1 due to normalization, and then we integrate over $\theta$:
Adding the three terms together:

\[
I_2 = \left( \frac{\hbar^2 \nu^2}{m_e c^2} \right)^2 \int dW (1 - \mathbf{n} \cdot \mathbf{n'})^2
\]

\[
= \left( \frac{\hbar^2 \nu^2}{m_e c^2} \right)^2 \int dW [1 - 2\mathbf{n} \cdot \mathbf{n'} + (\mathbf{n} \cdot \mathbf{n'})^2]
\]

\[
= \left( \frac{\hbar^2 \nu^2}{m_e c^2} \right)^2 \int dW [1 + (\mathbf{n} \cdot \mathbf{n'})^2]
\]

\[
= \left( \frac{\hbar^2 \nu^2}{m_e c^2} \right)^2 \pi c T_0^2 \int (1 + \cos^2 \theta)^2 \sin \theta \, d\theta
\]

\[
= \frac{\hbar^2 \nu^2}{m_e c^2} \frac{7}{5} \sigma_T c
\]

\[
= \frac{7}{5} \sigma_T c (kT_e)^3 (m_e c^2)^2 x^4
\]  

(27)

Carrying out a derivation similar to that of the classical Kompaneets equation and repeating (17) to (20) yields:

\[
g_{c\nu}(x) = -N_e \sigma_T c \frac{kT_e}{m_e c^2} x^2 \left[ 1 + \frac{7}{10} \frac{kT_e}{m_e c^2} x^4 \right]
\]  

(29)

Substituting (29) into the expression for \( \frac{\partial n}{\partial t} \), we have the Kompaneets equation extended by frequency:

\[
\left( \frac{\partial n}{\partial t} \right)_{c\nu} = N_e \sigma_T c \frac{kT_e}{m_e c^2} x^2 \partial x \left\{ x^4 \left[ 1 + \frac{7}{10} \frac{kT_e}{m_e c^2} x^4 \right] \frac{\partial n}{\partial x} + n(n + 1) \right\}
\]  

(30)

The Kompaneets equation extended by frequency retains the terms under \( \hbar \nu \sim kT_e \) and \( \hbar \nu \gg kT_e \) compared to the classical Kompaneets equation, thus enabling a better treatment of the down-Comptoniazation process. This is reflected formally by the addition of a new term \( \frac{7}{10} m_e c^2 x^2 \). The essential difference between the two different Kompaneets equations is that the expansion of the Boltzmann equation is retained to a different small amount.

2.3. Kompaneets equation extended by momentum

Define the increment of electron momentum before and after scattering as \( \Delta \mathbf{p} = \mathbf{p}' - \mathbf{p} \) and \( \gamma \) is the angle between \( \mathbf{p} \) and \( \Delta \mathbf{p} \). From the conservation of momentum and energy we have:

\[
\hbar \Delta = -\frac{1}{2m_e} (2|\mathbf{p}| |\Delta \mathbf{p}| \cos \gamma + |\Delta \mathbf{p}|^2)
\]  

(31)

\[
\Delta \mathbf{p} = \frac{\hbar \nu}{c} (\mathbf{n} - \mathbf{n'}) - \frac{\hbar \Delta}{c} \mathbf{n'}
\]  

(32)

Substituting the above two equations into (8), omitting the second-order and higher minima of \( |\Delta \mathbf{p}| \), yields the Boltzmann equation expanded in terms of momentum. For the integral \( l \):

\[
l = \int d^3 \mathbf{p} \int f(p) \left[ \frac{1}{2m_e} 2|\mathbf{p}| |\Delta \mathbf{p}| \cos \gamma \right]^2 dW
\]  

(33)

continue to use the relation in (32), converting \( \Delta \mathbf{p} \) back to \( \Delta \) we get:
\[-\frac{1}{2m_e}2|p||\Delta p|\cos\gamma =
\]

\[
h\Delta + \frac{1}{2m_ec^2}[(hv)^2(n-n')^2 - 2(n-n') \cdot n'(hv)\Delta + (\Delta)^2]
\]

(34)

We then obtain the square of the first term \(I_1\), the square of the second term \(I_2\) and the cross term \(I_3\) of (33), respectively:

\[
I_1 = (h\Delta)^2 \int d^3p \int f(p) \, dW
\]

(35)

\[
I_2 = \int d^3p \int f(p) \, dW \frac{1}{4(m_e c^2)^2}[(hv)^2(n-n')^2 - 2(n-n') \cdot n'(hv)\Delta + (\Delta)^2]^2
\]

(36)

\[
I_3 = \int d^3p \int f(p) \, dW \frac{1}{m_e c^2}h\Delta[(hv)^2(n-n')^2 - 2(n-n') \cdot n'(hv)\Delta + (\Delta)^2]
\]

(37)

Of these, the first has a form with which we are familiar from earlier, and now to simplify the second and third:

\[
I_2 = \int d^3p \int f(p) \, dW \frac{1}{4(m_e c^2)^2}2(hv)^2(1 - n \cdot n')^2
\]

\[
\cdot n'(hv) \left[-\frac{hc}{m_e c^2}p \cdot (n - n') - \frac{(hv)^2}{m_e c^2}(1 - n \cdot n') \right] + \left[-\frac{hc}{m_e c^2}p \cdot (n - n') - \frac{(hv)^2}{m_e c^2}(1 - n \cdot n') \right]^2
\]

(38)

Based on the previous two parts of the work, we can estimate the magnitudes corresponding to the results of the above integrals:

\[
I_2 \sim \frac{(kT_e)^4}{(m_e c^2)^2} \sigma_T c \left\{ x^2 + x \left( \frac{kT_e}{m_e c^2} x + \frac{kT_e}{m_e c^2} x^2 \right) + \left[ \frac{kT_e}{m_e c^2} x + \frac{kT_e}{m_e c^2} x^2 \right]^2 \right\}
\]

(39)

Under non-relativistic conditions, i.e., \(h\nu \ll m_e c^2\), \(kT_e \ll m_e c^2\) and \(h\nu \sim kT_e\), we have \(x \sim 1\) and the integral is preserved up to \(\frac{(kT_e)^4}{(m_e c^2)^2} \sigma_T c\) magnitude, giving:

\[
I_2 = \int d^3p \int f(p) \, dW \frac{1}{4(m_e c^2)^2}2(hv)^2(1 - n \cdot n')^2
\]

(40)

Obtaining \(I_3\) in the same way, we arrive at three integral results:

\[
l_1 = 2\sigma_T c(kT_e)^2 \left[ \frac{kT_e}{m_e c^2} x^2 + \frac{7}{10} \left( \frac{kT_e}{m_e c^2} \right)^2 x^4 \right]
\]

(41)

\[
l_2 = 2\sigma_T c(kT_e)^2 \frac{7}{10} \left( \frac{kT_e}{m_e c^2} \right)^2 x^4
\]

(42)

\[
l_3 = 2\sigma_T c(kT_e)^2 \left[ \frac{14}{5} \left( \frac{kT_e}{m_e c^2} \right)^2 x^3 - \frac{7}{5} \left( \frac{kT_e}{m_e c^2} \right)^2 x^4 \right]
\]

(43)

Summing the three terms, we obtain the integral value \(I_{ep}\) by momentum promotion:
\[ I_{ep} = 2\sigma_T c (kT_e)^2 \left[ \frac{kT_e}{m_e c^2} x^2 + \frac{14}{5} \left( \frac{kT_e}{m_e c^2} \right)^2 x^3 \right] \quad (44) \]

The coefficients of the photon number conservation component are kept consistent, giving:

\[ g_{ep}(x) = -N_e \sigma_T c \frac{kT_e}{m_e c^2} x^2 \left[ 1 + \frac{14}{5} \frac{kT_e}{m_e c^2} x \right] \quad (45) \]

Finally, the Kompaneets equation extended by momentum is obtained:

\[ \left( \frac{\partial n}{\partial t} \right)_{c, ep} = N_e \sigma_T c \frac{kT_e}{m_e c^2} x^2 \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( x^4 \left( 1 + \frac{14}{5} \frac{kT_e}{m_e c^2} x \right) \frac{\partial n}{\partial x} + n(n + 1) \right) \quad (46) \]

The difference between the two extended Kompaneets equations lies in the different small quantities chosen to expand the respective Boltzmann equation, and hence the different higher order small quantities omitted. The added term for the Kompaneets equation extended by photon frequency is proportional to \( x^2 \), and the added term for the equation extended by electron momentum is proportional to \( x \).

### 2.4. Ross-McCray equation

Ross-McCray equation calculates the change in photon number density \( N_e \) in Comptonization from the Fokker-Planck equation:

\[ \frac{\partial N_e}{\partial t} = -\frac{\partial}{\partial \varepsilon} \left[ A(\varepsilon) N_e \right] + \frac{1}{2} \frac{\partial^2}{\partial \varepsilon^2} \left[ B(\varepsilon) N_e \right] \quad (47) \]

Here the stimulated scattering is neglected, \( N_e \) is the number of photons per unit space per unit photon energy interval \( d\varepsilon \). \( A(\varepsilon) \) is the increment of photon energy per unit time, and \( B(\varepsilon) \) is the average of the squared increment:

\[ A(\varepsilon) = \left( \frac{d\varepsilon}{dt} \right) \quad B(\varepsilon) = \frac{d}{dt} \left( \langle \Delta\varepsilon \rangle^2 \right) \quad (48) \]

Taking the Klein-Nishina differential scattering cross section and integrating the equation (48) over the full scattering angle under the electron Maxwell distribution, retaining the terms for the up-Comptonization and down-Comptonization conditions, gives:

\[ A(\varepsilon) = \frac{n_e \sigma_T c}{\varepsilon} \left[ \frac{4kT_e}{m_e c^2} - \frac{\varepsilon}{m_e c^2} + \frac{21}{5} \left( \frac{\varepsilon}{m_e c^2} \right)^2 \right] \quad (49) \]

\[ B(\varepsilon) = \frac{n_e \sigma_T c}{\varepsilon^2} \left[ \frac{2kT_e}{m_e c^2} + \frac{7}{5} \left( \frac{\varepsilon}{m_e c^2} \right)^2 \right] \quad (50) \]

where \( n_e \) is the electron number density and \( \sigma_T \) is the Thomson scattering cross section. Defining the photon occupation number \( n \equiv \frac{c^3 h^3}{8\pi \varepsilon^2} N_e \) and substituting the above two equations into (47), we have the Fokker-Planck equation:

\[ \frac{\partial n}{\partial t} = \frac{n_e \sigma_T c}{m_e c^2 \varepsilon^2} \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( x^4 \left( kT_e + \frac{7n^2}{10m_e c^2} \right) \frac{\partial n}{\partial x} \right) \quad (51) \]

Further organizing (51) by substituting \( x \equiv \frac{h\nu}{kT_e} \) yields the Ross-McCray equation:

\[ \left( \frac{\partial n}{\partial t} \right)_{crm} = N_e \sigma_T c \frac{kT_e}{m_e c^2} x^2 \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( x^4 \left( n + \frac{1}{10m_e c^2} \right) \frac{\partial n}{\partial x} \right) \quad (52) \]

The Fokker-Planck equation preserves \( \Delta\varepsilon \) to second-order minima in the low-frequency radiation field with non-relativistic case, which makes its form consistent with that of the Kompaneets equation. Without the constraint of \( h\nu \ll kT_e \), the Ross-McCray equation can be used to describe the down-Comptonization process. It is worth pointing out that the process of stimulated radiation is neglected in equation (47), so that the non-linear term of \( N_e \) is absent from the right hand side, making the Ross-McCray equation a linear equation. There is a fundamental physical difference between the Ross-McCray equation and the Kompaneets equation: the Kompaneets equation starts from the Boltzmann equation and takes into account stimulated radiation and stimulated absorption, resulting in an equation...
form containing non-linear terms. The Ross-McCray equation follows from the Fokker-Planck equation, ignoring the stimulated radiation process, and the resulting equation is linear. At low light intensities, the non-linear terms of the Kompaneets equation can be neglected and the Ross-McCray equation evolves similarly to the Kompaneets equation. This is because stimulated radiation has a smaller probability of occurring at weak light intensities and the photon Compton scattering evolution is independent of the absolute magnitude of the light intensity. However, the stimulated radiation effect is necessary for the photon gas and electron gas to converge to blackbody radiation equilibrium. When the photon distribution function is the Planck distribution $n(\nu) = \left(1 + \frac{\nu}{\nu_T} - 1\right)^{-1}$, substituting it into the four equations, we find that the Kompaneets equation results in $\frac{dn}{dt}_c = 0$, i.e., the evolution due to Compton scattering stops, while the Ross-McCray equation is not 0. This is because the absence of stimulated radiation causes the Ross-McCray equation to be unable to guarantee an equilibrium state under the Planck distribution, especially when the stimulated radiation from the strong light field is significant or close to the evolutionary final state.

3. Comparison of the evolution of kinetic equations

When the astronomical radiation passes through the plasma medium, the energy exchange between the radiation field and the plasma medium will make the intensity, spectral shape, spectral line profile, and spectral line position of the emitted light change. Further, the specific spectral shapes under different conditions evolve differently during Comptonization. The four Comptonization diffusion equations have different forms and physical connotations, and their dominant Comptonization processes necessarily differ. In order to visualize the differences between the four equations, it is necessary to compare the evolution of the various initial spectra under different Comptonization conditions. This chapter will compare the similarities and differences of the four diffusion equations under different conditions by means of numerical calculations. The differences in the end states of Comptonization are also discussed.

3.1. Evolution of the initial spectrum

The Kompaneets equations are usually solved using numerical calculations. In order to compare the differences between the four equations, we can substitute the photon distribution function of the initial case into the diffusion equation to obtain the initial spectral evolution trend of the formulation for comparison.

Integrating both sides of the Kompaneets equation (21) over time yields the amount of change in the photon distribution function:

$$\Delta n = \tau_c \frac{kT_e}{m_e c^2} \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ x^4 \left[ \frac{\partial n}{\partial x} + n(n + 1) \right] \right\}$$  \hspace{1cm} (53)

where $\tau_c$ is the optical depth. When the electrons are uniformly distributed along the propagation path, we have:

$$\tau_c = N_e \sigma_T ct$$  \hspace{1cm} (54)

The photon distribution function is a function of frequency versus time. For the initial case photon distribution function $n(x, t = 0)$, substitution into (54) gives the distribution function variation $\Delta n(x)$. In practice, however, we directly observe the spectrum as a spectrum of radiation intensity rather than a photon distribution function, so the distribution function needs to be converted to radiation intensity, and the two are related as:

$$I_\nu = \frac{2(h\nu)^3}{(hc)^2} n(\nu) = \frac{2(kT_e)^3}{(hc)^2} x^3 n(x) = I_0 (kT_e)^3 x^3 n(x)$$  \hspace{1cm} (55)

where $I_0 = \frac{2}{(hc)^2}$. Substituting the photon distribution function and the electron gas temperature gives an expression for $I_\nu$ with respect to $x$ for the initial case. For finding differences in the four diffusion equations and
determining trends in the evolution of the spectral lines, formal analysis is too difficult, so we adopted numerical calculations here for graphical analysis.

3.2. Numerical calculation and comparison of the evolution of spectral lines
We have selected four spectral shapes typical of astrophysics: the blackbody spectrum, the Gaussian spectrum, the power-law spectrum and the bremsstrahlung spectrum. The four equations are solved numerically to calculate the evolution of the spectral lines, mainly by varying the relevant spectral parameters and the electron temperature. The numerical calculation of the partial differential equation is carried out using the finite difference method. Taking the frequency space grid with logarithmic type density can make the density of each part uniform when the span of frequency order of magnitude is large, which ensures the accuracy and convergence of calculation under large span. According to the photon conservation property of the Kompaneets equation, we count the photon number before and after the evolution and check the rate of photon number change as a basis for the validity of the numerical calculation results, which is guaranteed by the photon number conservation equation (continuity equation) introduced during the derivation of the Kompaneets equation. For the Ross-McCray equation, although the derivation does not explicitly use the photon number conservation equation, it still reflects the universal Compton scattering process. The calculations show that photon number conservation is also satisfied in most cases and therefore this criterion is also applicable. The total photon number is related to the photon distribution function as follows:

\[ N = \int \frac{8\pi v^2}{c^3} n(v) \, dv \] (56)

The calculation process takes the low frequency end and the high frequency end cut-off, i.e., \( n(x_{\text{min,max}}) = 0 \) at both ends. To ensure the validity of the numerical calculation, i.e., the photon number conservation relationship before and after the evolution, the electron temperature \( T_e \) is used to estimate the required frequency range, allowing the spectral intensity to converge to zero at both ends. In this way the effects caused by the cut-off can be eliminated. It is worth pointing out that the photon distribution function \( n(x) \) in diffusion equations such as the Kompaneets equation is independent of the spatial distribution and the equation does not contain a spatial diffusion term. If the radiation field is spatially inhomogeneous, the photon distribution is a function of position \( n = n(x, r) \) and it requires the addition of a spatial diffusion term: \( \nabla \cdot (D \nabla n) \), where \( D \) is the diffusion coefficient. At this point, the classical Kompaneets equation becomes:

\[ \left( \frac{\partial n}{\partial t} \right)_{c.c.I} = N_e \sigma_{Tc} \frac{kT_e}{m_e c^2} \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^4 \left[ \frac{\partial n}{\partial x} + n(n+1) \right] \right) + \nabla \cdot (D \nabla n) \] (57)

In practical calculations, we use a model in which a thermal electron gas with Maxwell distribution at a temperature of \( T_e \) is uniformly distributed within a ball of sufficient radius, while the centre of the ball generates a large number of peripherally propagating photons satisfying a specific spectral distribution, which undergo Compton scattering with the electrons during the transit. After a period of evolution, by counting the energy of all photons, the spectrum after Comptonization is obtained. We introduce a diffusion timescale \( T \) to represent the average diffusion time of the photon. When the optical depth \( \tau_s = N_e \sigma_{Tc} > 1 \), the average diffusion time of photons inside the ball is \( T \approx \frac{Nl}{c} = \tau_s \frac{R}{c} \), where \( N \) is the average scattering number, \( l \) is the mean free path and \( R \) is the radius of the ball. The spectrum of photons arriving at the surface can be approximated by simply choosing the time scale \( T \) corresponding to \( R \) during the numerical calculation. In order to account for all possible scattering, a large diffusion timescale is chosen so that the average number of photon scatterings \( N = N_e \sigma_{Tc} c T \gg 1 \). In the following evolution of the four spectral lines, we have chosen \( N_e = 2 \times 10^{16} \text{cm}^{-3} \), diffusion timescale \( T = 1.0 \times 10^{-2} \text{s} \).

3.2.1. Evolution of the blackbody spectrum
Consider the blackbody spectrum with an initial temperature of \( T_b \):
\[ n(x) = \frac{1}{\exp \left( \frac{T_e}{T_b} x \right) - 1} \]  \hspace{1cm} (58)

We give the up-Comptonization process for \( kT_b = 1\, \text{KeV} \), \( kT_e = 10\, \text{KeV} \), the down-Comptonization process for \( kT_b = 10\, \text{KeV}, kT_e = 5\, \text{KeV} \), and the blackbody equilibrium process for \( kT_b = kT_e = 5\, \text{KeV} \), respectively:

Since a suitable frequency range has been chosen, the total photon number loss rate before and after the evolution of the spectral lines does not exceed 1%. As a consequence, the photon number can be considered to be conserved. We found that during up-Comptonization, the position of the peak shifts towards the high frequency end and the peak decreases, with an increase in light intensity in the high frequency section. All four equations return to the curve of the classical Kompaneets equation, which is consistent with our expectation that the classical Kompaneets can describe the up-Comptonization process more accurately. During down-Comptonization, the position of the peak moves towards the low frequency end and the peak rises, with the low frequency part rising and the high frequency part falling relative to the initial spectrum. The four equations evolve in relatively similar trends, but show significant speed differences, from fast to slow: the Kompaneets equation extended by frequency, the Kompaneets equation extended by momentum, the classical Kompaneets equation and the Ross-McCary...

Figure 1. The evolution of the black body spectrum; (a), (b) and (c) represent the up-Comptonization process, the down-Comptonization process and the blackbody equilibrium, respectively.
equation, where the Ross-McCary equation shows an unusual enhancement in the high-frequency part. In equilibrium, the evolution dominated by the three Kompaneets equations ceases and the spectral lines do not change, while the Ross-McCary equation is anomalously elevated in the high frequency band.

3.2.2. Evolution of the Gaussian spectrum

Consider a Gaussian spectrum with a central frequency of $x_0$:

$$n(x) = A x^{-3} \exp \left[ -\frac{4 \ln 2}{(\Delta x)^2} (x - x_0)^2 \right]$$  \hspace{1cm} (59)

We take the spectral intensity factor $A = 0.01$ and the FWHM $\Delta x = x_0 / 10$ to give the up-Comptonization process for $\hbar \nu_0 = 1KeV$, $kT_e = 5KeV$ and the down-Comptonization process for $\hbar \nu_0 = 10KeV$, $kT_e = 0.1KeV$:

![Figure 2. The evolution of the Gaussian spectrum; (a) and (b) represent the up-Comptonization process and the down-Comptonization process, respectively.](image)

Similarly, a suitable frequency range is chosen so that the rate of photon number loss before and after the evolution does not exceed 1%. Both Comptonization processes show a peak reduction and a broadening of the spectrum, which is due to the inelastic collision of photons initially concentrated at $\hbar \nu_0$ with electrons, spreading towards the ends of the frequency space. During the up-Comptonization of the Gaussian spectrum, the position of the peak moves towards the high frequency end, the peak decreases, the spectral lines broaden significantly and the four equations finally return to agreement. During the down-Comptonization process, the peak decreases and the position of the peak moves towards the lower frequency end, resulting in a Compton redshift. This is because when a photon is in a "cold" electron gas, the average effect of photon and electron collisions causes the photon to gradually lose energy, resulting in a red shift in the spectrum. There are also differences in the rate of evolution of the four equations, the order being consistent with the down-Comptonization process of the blackbody spectrum, and similarly, the Ross-McCary equation shows anomalous enhancement in the high frequency part.

3.2.3. Evolution of the power-law spectrum

Consider a power law spectrum with power index $\alpha$:

$$n(x) = A x^{-3} x^{-\alpha}$$  \hspace{1cm} (60)

We take the spectral intensity factor $A = 1.0 \times 10^{-3}$, the spectral index $\alpha = 0.7$ and choose the photon frequency range $\hbar \nu_{\text{min}} = 0.01KeV$ and $\hbar \nu_{\text{max}} = 1000KeV$ cut-off respectively.
As the initial power-law spectrum taken is non-zero at the low and high frequency ends, it leads to a higher photon number loss in the limit of the cut-off at both ends, but the spectral evolution of the intermediate frequency part away from the ends is still valid. Figure 3 shows that during Comptonization there is a decrease in the intensity of the low and high frequency parts and an overall increase in the intensity of the central spectral lines. The up-Comptonization process occurs at the low-frequency end, where the low-frequency photons are subjected to up-Comptonization by thermal electron, and the number of low-frequency photons decreases, as described by the three Kompaneets equations. The high frequency part undergoes a down-Comptonization process: the high frequency end of the power-law spectrum steepens and the high energy photons move rapidly towards the low energy end, while the results of the three Kompaneets equations begin to diverge, showing differences in the rate of enhancement of the spectral lines and the rate of receding, with the Kompaneets equation extended by frequency being the fastest and the classical Kompaneets equation being the slowest. Although the frequency enhancement of the Ross-McCary equation in the middle part is consistent with the three Kompaneets equations, its results are stronger at the low frequency end and the evolution tends to flatten out at the high frequency end, a difference that implies that the evolution of the Ross-McCary equation tends to a different final state.

3.2.4. Evolution of the thermal bremsstrahlung spectrum

Finally, we consider the thermal bremsstrahlung spectrum with photon distribution function \( n(x) \):

\[
n(x) = x^{-3} \cdot \bar{g}_{ff} \cdot \exp \left[ -x \cdot \left( \frac{T_e}{T_{ff}} \right) \right]
\]

where \( T_{ff} \) is the temperature of the thermal plasma and \( \bar{g}_{ff} \) is the average Gaunt factor at temperature \( T_{ff} \), which we approximate here by taking \( \bar{g}_{ff} = 1 \). Taking the frequency range \( 0.01 \text{KeV} < h\nu < 1000\text{KeV} \), the following up-Comptonization process with \( kT_{ff} = 1\text{KeV} \), \( kT_e = 10\text{KeV} \) and down-Comptonization process with \( kT_{ff} = 30\text{KeV} \), \( kT_e = 1\text{KeV} \) are shown, respectively.
Figure 4. The evolution of the bremsstrahlung spectrum; (a) and (b) represent the up-Comptonization process and the down-Comptonization process, respectively.

Because the cut-off at the low frequency end is not zero, the thermal bremsstrahlung spectrum also has a high photon loss rate, but the evolution at the middle and high end frequencies is still valid. During up-Comptonization, the four equations regress in agreement. In down-Comptonization, the four equations begin to diverge at the high-frequency end, and the order of their evolutionary rates is consistent with the down-Comptonization process of the power-law spectrum. It is noteworthy that the Ross-McCary equation evolves more gently at the high frequency end and shows an unusual enhancement. Similarly, the Ross-McCary equation tends to evolve to a different final state during the down-Comptonization of the thermal bremsstrahlung spectrum.

4. Conclusion
In summary, the comparison between the comptonization process description equations, i.e., Kompaneets equations (classical, extended by frequency and momentum) and the Ross-McCray equation, are investigated based on both theoretical deduction and numerical simulation for four cases. Specifically, the Kompaneets equation and its extended forms as well as Ross-McCray equation are first derived step by step with the discussion of the assumptions and applying conditions. Subsequently, in order to give a better demonstration, numerical simulations are carried out for blackbody, Gaussian, power-law and thermal bremsstrahlung spectrum. According to the results, there is a clear difference in the rate of evolution speed during down-Comptonization.

In detail, the order from fast to slow is the Kompaneets equation extended by frequency, the Kompaneets equation extended by momentum, the classical Kompaneets equation and the Ross-McCary equation. Moreover, the Ross-McCary equation shows a particular evolutionary trend, with its end-state equilibrium tending towards the higher frequency end. These results offer a guideline for further analysis in comptonization process.

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