LIFTS OF $C_\infty$ AND $L_\infty$-MORPHISMS TO $G_\infty$-MORPHISMS

GRÉGORY GINOT\textsuperscript{(a)}, GILLES HALBOUT\textsuperscript{(b)}

\textsuperscript{(a)} Laboratoire Analyse Géométrie et Applications, Université Paris 13 et Ecole Normale Supérieure de Cachan, France
\textsuperscript{(b)} Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, Strasbourg, France

Abstract. Let $g_2$ be the Hochschild complex of cochains on $C^\infty(\mathbb{R}^n)$ and $g_1$ be the space of multivector fields on $\mathbb{R}^n$. In this paper we prove that given any $G_\infty$-structure (i.e. Gerstenhaber algebra up to homotopy structure) on $g_2$, and any $C_\infty$-morphism $\varphi$ (i.e. morphism of commutative, associative algebra up to homotopy) between $g_1$ and $g_2$, there exists a $G_\infty$-morphism $\Phi$ between $g_1$ and $g_2$ that restricts to $\varphi$. We also show that any $L_\infty$-morphism (i.e. morphism of Lie algebra up to homotopy), in particular the one constructed by Kontsevich, can be deformed into a $G_\infty$-morphism, using Tamarkin’s method for any $G_\infty$-structure on $g_2$. We also show that any two of such $G_\infty$-morphisms are homotopic.

0-Introduction

Let $M$ be a differential manifold and $g_2 = (C^\bullet(A,A), b)$ be the Hochschild cochain complex on $A = C^\infty(M)$. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of $g_2$ is the graded Lie algebra $g_1 = \Gamma(M, \wedge \cdot TM)$ of multivector fields on $M$. There is also a graded Lie algebra structure on $g_2$ given by the Gerstenhaber bracket. In particular $g_1$ and $g_2$ are also Lie algebras up to homotopy ($L_\infty$-algebra for short). In the case $M = \mathbb{R}^n$, using different methods, Kontsevich ([Ko1] and [Ko2]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms “up to homotopy” ($L_\infty$-morphisms) from $g_1$ to $g_2$. Kontsevich’s proof uses graph complex and is related to multizeta functions whereas Tamarkin’s construction uses the existence of Drinfeld’s associators. In fact Tamarkin’s $L_\infty$-morphism comes from the restriction of a Gerstenhaber algebra up to homotopy homomorphism ($G_\infty$-morphism) from $g_1$ to $g_2$. The $G_\infty$-algebra structure on $g_1$ is induced by its classical Gerstenhaber algebra structure and a far less trivial $G_\infty$-structure on $g_2$ was proved to exist by Tamarkin [Ta] and relies on a Drinfeld’s associator. Tamarkin’s $G_\infty$-morphism also restricts into a commutative, associative up to homotopy morphism ($C_\infty$-morphism for short). The $C_\infty$-structure on $g_2$ (given by

\textbf{Keywords} : Deformation quantization, star-product, homotopy formulas, homological methods.

\textbf{AMS Classification} : Primary 16E40, 53D55, Secondary 18D50, 16S80.
restriction of the $G_\infty$-one) highly depends on Drinfeld’s associator, and any two choices of a Drinfeld associator yields a priori different $C_\infty$-structures. When $M$ is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). A connection between the two approaches has been given in [KS] but the morphisms given by Kontsevich and Tamarkin are not the same. The aim of this paper is to show that, given any $G_\infty$-structure on $g_1$ and any $C_\infty$-morphism $\varphi$ between $g_1$ and $g_2$, there exists a $G_\infty$-morphism $\Phi$ between $g_1$ and $g_2$ that restricts to $\varphi$. We also show that any $L_\infty$-morphism can be deformed into a $G_\infty$-one.

In the first section, we fix notation and recall the definitions of $L_\infty$ and $G_\infty$-structures. In the second section we state and prove the main theorem. In the last section we show that any two $G_\infty$-morphisms given by Tamarkin’s method are homotopic.

**Remark**: In the sequel, unless otherwise is stated, the manifold $M$ is supposed to be $\mathbb{R}^n$ for some $n \geq 1$. Most results could be generalized to other manifolds using techniques of Kontsevich [Ko1] (also see [TS], [CFT]).

### 1-$C_\infty$, $L_\infty$ and $G_\infty$-structures

For any graded vector space $g$, we choose the following degree on $\wedge^\bullet g$ : if $X_1, \ldots, X_k$ are homogeneous elements of respective degree $|X_1|, \ldots, |X_k|$, then

$$|X_1 \wedge \cdots \wedge X_k| = |X_1| + \cdots + |X_k| - k.$$

In particular the component $g = \wedge^1 g \subset \wedge^\bullet g$ is the same as the space $g$ with degree shifted by one. The space $\wedge^\bullet g$ with the deconcatenation cobracket is the cofree cocommutative coalgebra on $g$ with degree shifted by one (see [LS], Section 2). Any degree one map $d^k : \wedge^k g \to g$ ($k \geq 1$) extends into a derivation $d^k : \wedge^\bullet g \to \wedge^\bullet g$ of the coalgebra $\wedge^\bullet g$ by cofreeness property.

**Definition 1.1.** A vector space $g$ is endowed with a $L_\infty$-algebra (Lie algebras “up to homotopy”) structure if there are degree one linear maps $m^1, \ldots, m^k$ given by Kontsevich and Tamarkin that restricts to $C_\infty$-morphisms given by Kontsevich [Ko1] (also see [KS], Section 2). Any degree one map $d^1 : \wedge^1 g \to g$ extends into a derivation $d^1 : \wedge^\bullet g \to \wedge^\bullet g$ of the coalgebra $\wedge^\bullet g$ by cofreeness property.

For more details on $L_\infty$-structures, see [LS]. It follows from the definition that a $L_\infty$-algebra is a differential coalgebra structure on $\wedge^\bullet g$ and that the map $m^1 : g \to g$ is a differential. If $m^1, \ldots, m^k : \wedge^k g \to g$ are 0 for $k \geq 3$, we get the usual definition of a $(\wedge$-algebra. If $m^1, \ldots, m^k : \wedge^k g \to g$ are 0 for $k \geq 3$, we get the usual definition of a $(\wedge$-algebra.

For any graded vector space $g$, we denote $g^{\otimes n}$ the quotient of $g^{\otimes n}$ by the image of all shuffles of length $n$ (see [GK] or [GH] for details). The graded vector space $\oplus_{n \geq 0} g^{\otimes n}$ is a quotient coalgebra of the tensor coalgebra $\oplus_{n \geq 0} g^{\otimes n}$. It is well known that this coalgebra is the cofree Lie coalgebra on the vector space $g$ (with degree shifted by minus one).

**Definition 1.2.** A $C_\infty$-algebra (commutative and associative “up to homotopy” algebra) structure on a vector space $g$ is given by a collection of degree one linear maps $m^k : g^{\otimes k} \to g$ such that if we extend them to maps $\oplus g^{\otimes k} \to \oplus g^{\otimes k}$, then $d \circ d = 0$ where $d$ is the derivation $d = m^1 + m^2 + m^3 + \cdots$.

In particular a $C_\infty$-algebra is an $A_\infty$-algebra.
For any space $g$, we denote $\wedge^\bullet g^{\otimes \bullet}$ the graded space

$$\wedge^\bullet g^{\otimes \bullet} = \bigoplus_{m \geq 1} g^{\otimes p_1} \wedge \cdots \wedge g^{\otimes p_n}.$$ 

We use the following grading on $\wedge^\bullet g^{\otimes \bullet}$: for $x_1^1, \ldots, x_n^p \in g$, we define

$$|x_1^1 \otimes \cdots \otimes x_n^p \wedge \cdots \wedge x_1^1 \otimes \cdots \otimes x_n^p| = \sum_{i_1}^p |x_1^{i_1}| + \cdots + \sum_{i_n}^p |x_n^{i_n}| - n.$$ 

Notice that the induced grading on $\wedge^\bullet g \subset \wedge^\bullet g^{\otimes \bullet}$ is the same than the one introduced above. The cobracket on $\bigoplus g^{\otimes \bullet}$ and the coproduct on $\wedge^\bullet g$ extend to a cobracket and a coproduct on $\wedge^\bullet g^{\otimes \bullet}$ which yield a Gerstenhaber coalgebra structure on $\wedge^\bullet g^{\otimes \bullet}$. It is well known that this coalgebra structure is cofree (see [Gi], Section 3 for example).

**Definition 1.3.** A $G_\infty$-algebra (Gerstenhaber algebra “up to homotopy”) structure on a graded vector space $g$ is given by a collection of degree one maps

$$m^{p_1,\ldots,p_n} : g^{\otimes p_1} \wedge \cdots \wedge g^{\otimes p_n} \to g$$

indexed by $p_1,\ldots,p_n \geq 1$ such that their canonical extension: $\wedge^\bullet g^{\otimes \bullet} \to \wedge^\bullet g^{\otimes \bullet}$ satisfies $d \circ d = 0$ where

$$d = \sum_{m \geq 1, \ p_1+\cdots+p_n=m} m^{p_1,\ldots,p_n}.$$ 

Again, as the coalgebra structure of $\wedge^\bullet g \subset \wedge^\bullet g^{\otimes \bullet}$ is cofree, the map $d$ makes $\wedge^\bullet g^{\otimes \bullet}$ into a differential coalgebra. If the maps $m^{p_1,\ldots,p_n}$ are 0 for $(p_1,p_2,\ldots) \neq (1,0,\ldots), (1,1,0,\ldots)$ or $(2,0,\ldots)$, we get the usual definition of (differential if $m^1 \neq 0$) Gerstenhaber algebra.

The space of multivector fields $g_1$ is endowed with a graded Lie bracket $[\cdot,\cdot]_S$ called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product: $(\alpha, \beta) \mapsto \alpha \wedge \beta$.

Setting $d_1 = m_1^{1,1} + m_1^2$, where $m_1^{1,1} : \wedge^2 g_1 \to g_1$, and $m_2^2 : g_1^{\otimes 2} \to g_1$ are the extension of the Schouten bracket and the exterior product, we find that $(g_1, d_1)$ is a $G_\infty$-algebra.

In the same way, one can define a differential Lie algebra structure on the vector space $g_2 = C(A,A) = \bigoplus_{k \geq 0} C^k(A,A)$, the space of Hochschild cochains (generated by differential $k$-linear maps from $A^k$ to $A$), where $A = C_\infty(M)$ is the algebra of smooth differential functions over $M$. Its bracket $[-,-]_G$, called the Gerstenhaber bracket, is defined for $D, E \in g_2$, by

$$[D,E]_G = \{D|E\} - (-1)^{|E||D|}\{E|D\},$$

where

$$\{D|E\}(x_1, \ldots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E|-i} D(x_1, \ldots, x_i, E(x_{i+1}, \ldots, x_{i+e}), \ldots).$$

The space $g_2$ has a grading defined by $|D| = k \Leftrightarrow D \in C^{k+1}(A,A)$ and its differential is $b = [m, -]_G$, where $m \in C^2(A,A)$ is the commutative multiplication on $A$. 

LIFT OF $C_\infty$ AND $L_\infty$ MORPHISMS TO $G_\infty$ MORPHISMS
Tamarkin (see [Ta] or also [GH]) stated the existence of a $G_\infty$-structure on $\mathfrak{g}_2$ (depending on a choice of a Drinfeld associator) given by a differential $d_2 = m_2^{1,1} + m_2^{2} + \cdots + m_2^{p_1,\ldots,p_n} + \cdots$, on $\wedge^2 \mathfrak{g} \otimes \mathfrak{g}$ satisfying $d_2 \circ d_2 = 0$. Although this structure is non-explicit, it satisfies the following three properties:

\begin{enumerate}[(a)]  
  \item $m_2^{1}$ is the extension of the differential $b$
  \item $m_2^{1,1}$ is the extension of the Gerstenhaber bracket $[-,-]_G$
  and $m_2^{1,\ldots,1} = 0$
  \item $m_2^2$ induces the exterior product in cohomology and the collection of the $(m^k)_{k \geq 1}$ defines a $C_\infty$-structure on $\mathfrak{g}_2$.
\end{enumerate}

\begin{definition}
A $L_\infty$-morphism between two $L_\infty$-algebras $(\mathfrak{g}_1, d_1 = m_1^1 + \ldots)$ and $(\mathfrak{g}_2, d_2 = m_2^1 + \ldots)$ is a morphism of differential coalgebras

$$\varphi : (\wedge^\bullet \mathfrak{g}_1, d_1) \to (\wedge^\bullet \mathfrak{g}_2, d_2).$$

\end{definition}

Such a map $\varphi$ is uniquely determined by a collection of maps $\varphi^n : \wedge^n \mathfrak{g}_1 \to \mathfrak{g}_2$ (again by cofreeness properties). In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are respectively the graded Lie algebra $(\Gamma(M, \wedge TM), [-,-]_S)$ and the differential graded Lie algebra $(C(A,A), [-,-]_G)$, the formality theorems of Kontsevich and Tamarkin state the existence of a $L_\infty$-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\varphi^1$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

\begin{definition}
A morphism of $C_\infty$-algebras between two $C_\infty$-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\oplus \mathfrak{g}_1^{\otimes k}, d_1) \to (\oplus \mathfrak{g}_2^{\otimes k}, d_2)$ of codifferential coalgebras.

A $C_\infty$-morphism is in particular a morphism of $A_\infty$-algebras and is uniquely determined by maps $\partial^k : \mathfrak{g}^{\otimes k} \to \mathfrak{g}$.

\end{definition}

\begin{definition}
A morphism of $G_\infty$-algebras between two $G_\infty$-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\wedge \mathfrak{g}_1^{\otimes k}, d_1) \to (\wedge \mathfrak{g}_2^{\otimes k}, d_2)$ of codifferential coalgebras.

There are coalgebras inclusions $\wedge \mathfrak{g} \to \wedge \mathfrak{g}^{\otimes \bullet}, \oplus \mathfrak{g}^{\otimes \bullet} \to \wedge \mathfrak{g}^{\otimes \bullet}$ and it is easy to check that any $G_\infty$-morphism between two $G_\infty$-algebras $(\mathfrak{g}, \sum m^{p_1,\ldots,p_n}), (\mathfrak{g}', \sum m'^{p_1,\ldots,p_n})$ restricts to a $L_\infty$-morphism $(\wedge \mathfrak{g}, \sum m^{1,\ldots,1}) \to (\wedge \mathfrak{g}', \sum m'^{1,\ldots,1})$ and a $C_\infty$-morphism $(\oplus \mathfrak{g}^{\otimes \bullet}, \sum m^k) \to (\oplus \mathfrak{g}^{\otimes \bullet}, \sum m'^k)$. In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are as above, Tamarkin’s theorem states that there exists a $G_\infty$-morphism between the two $G_\infty$ algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (with the $G_\infty$ structure he built) that restricts to a $C_\infty$ and a $L_\infty$-morphism.

\section{Main theorem}

We keep the notations of the previous section, in particular $\mathfrak{g}_2$ is the Hochschild complex of cochains on $C^\infty(M)$ and $\mathfrak{g}_1$ its cohomology. Here is our main theorem.
**Theorem 2.1.** Given any $G_\infty$-structure $d_2$ on $\mathfrak{g}_2$ satisfying the three properties of (1.1), and any $C_\infty$-morphism $\varphi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\varphi^1$ is the Hochschild-Kostant-Rosenberg map, there exists a $G_\infty$-morphism $\Phi : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ that restricts to $\varphi$.

Also, given any $L_\infty$-morphism $\gamma$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $\gamma^1$ is the Hochschild-Kostant-Rosenberg map, there exists a $G_\infty$-structure $(\mathfrak{g}_1, d'_1)$ on $\mathfrak{g}_1$ and $G_\infty$-morphism $\Gamma : (\mathfrak{g}_1, d'_1) \to (\mathfrak{g}_2, d_2)$ that restricts to $\gamma$. Moreover there exists a $G_\infty$-morphism $\Gamma' : (\mathfrak{g}_1, d'_1) \to (\mathfrak{g}_1, d_1)$.

In particular, Theorem 2.1 applies to the formality map of Kontsevich and also to any $C_\infty$-map derived (see [Ta], [GH]) from any $B_\infty$-structure on $\mathfrak{g}_2$ lifting the Gerstenhaber structure of $\mathfrak{g}_1$.

Let us first recall the proof of Tamarkin’s formality theorem (see [GH] for more details):

1. First one proves there exists a $G_\infty$-structure on $\mathfrak{g}_2$, with differential $d_2$, as in (1.1).
2. Then, one constructs a $G_\infty$-structure on $\mathfrak{g}_1$ given by a differential $d'_1$ together with a $G_\infty$-morphism $\Phi$ between $(\mathfrak{g}_1, d'_1)$ and $(\mathfrak{g}_2, d_2)$.
3. Finally, one constructs a $G_\infty$-morphism $\Phi'$ between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_1, d'_1)$.

The composition $\Phi \circ \Phi'$ is then a $G_\infty$-morphism between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$, thus restricts to a $L_\infty$-morphism between the differential graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

We suppose now that, in the first step, we take any $G_\infty$-structure on $\mathfrak{g}_2$ given by a differential $d_2$ and we suppose we are given a $C_\infty$-morphism $\varphi$ and a $L_\infty$-morphism $\gamma$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ satisfying $\gamma^1 = \varphi^1 = \varphi_{\text{HKR}}$ the Hochschild-Kostant-Rosenberg quasi-isomorphism.

**Proof of Theorem 2.1:**

The Theorem will follow if we prove that steps 2 and 3 of Tamarkin’s construction are still true with the extra conditions that the restriction of the $G_\infty$-morphism $\Phi$ (resp. $\Phi'$) on the $C_\infty$-structures is the $C_\infty$-morphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ (resp. id).

Let us recall (see [GH]) that the constructions of $\Phi$ and $d'_1$ can be made by induction. For $i = 1, 2$ and $n \geq 0$, let us set

$$V_i^{[n]} = \bigoplus_{p_1 + \cdots + p_k = n} \mathfrak{g}_i^{\otimes p_1} \wedge \cdots \wedge \mathfrak{g}_i^{\otimes p_k}$$

and $V_i^{[\leq n]} = \sum_{k \leq n} V_i^{[k]}$. Let $d_2^{[n]}$ and $d_2^{[\leq n]}$ be the sums

$$d_2^{[n]} = \sum_{p_1 + \cdots + p_k = n} d_2^{p_1, \ldots, p_k} \quad \text{and} \quad d_2^{[\leq n]} = \sum_{p \leq n} d_2^{[p]}.$$

Clearly, $d_2 = \sum_{n \geq 1} d_2^{[n]}$. In the same way, we denote $d'_1 = \sum_{n \geq 1} d'_1^{[n]}$ with

$$d'_1^{[n]} = \sum_{p_1 + \cdots + p_k = n} d'_1^{p_1, \ldots, p_k} \quad \text{and} \quad d'_1^{[\leq n]} = \sum_{1 \leq k \leq n} d'_1^{[k]}.$$

We know from Section 1 that a morphism $\Phi : (\wedge \mathfrak{g}_1^{\otimes p_1}, d'_1) \to (\wedge \mathfrak{g}_2^{\otimes p_1}, d_2)$ is uniquely determined by its components $\Phi^{p_1, \ldots, p_k} : \mathfrak{g}_1^{\otimes p_1} \wedge \cdots \wedge \mathfrak{g}_1^{\otimes p_k} \to \mathfrak{g}_2$. Again, we have $\Phi = \sum_{n \geq 1} \Phi^{[n]}$ with

$$\Phi^{[n]} = \sum_{p_1 + \cdots + p_k = n} \Phi^{p_1, \ldots, p_k} \quad \text{and} \quad \Phi^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi^{[k]}.$$
We want to construct the maps $d'_{1}^{[n]}$ and $\Phi^{[n]}$ by induction with the initial condition
\[
d'_{1}^{[1]} = 0 \quad \text{and} \quad \Phi^{[1]} = \varphi_{\text{HKR}},
\]
where $\varphi_{\text{HKR}} : (g_1, 0) \to (g_2, b)$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism (see [HKR]) defined, for $\alpha \in g_1$, $f_1, \cdots, f_n \in A$, by
\[
\varphi_{\text{HKR}} : \alpha \mapsto ((f_1, \ldots, f_n) \mapsto (\alpha, df_1 \land \cdots \land df_n)).
\]

Moreover, we want the following extra conditions to be true:
\[
\Phi^{k \geq 2} = \varphi^k, \quad d'_{1}^{2} = d'_{1}^{2}, \quad d'_{1}^{k \geq 3} = 0. \quad (2.3)
\]

Now suppose the construction is done for $n-1$ ($n \geq 2$), i.e., we have built maps $(d'_{1}^{[i]})_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$ satisfying conditions (2.3) and
\[
\Phi^{[\leq n-1]} \circ d'_{1}^{[\leq n-1]} = d'_{2}^{[\leq n-1]} \circ \Phi^{[\leq n-1]} \quad \text{on } V_{1}^{[\leq n-1]} \quad \text{and} \quad d'_{1}^{[\leq n-1]} \circ d'_{1}^{[\leq n-1]} = 0 \quad \text{on } V_{1}^{[\leq n]}.
\]

In [GH], we prove that for any such $(d'_{1}^{[i]})_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$, one can construct $d'_{1}^{[n]}$ and $\Phi^{[n]}$ such that condition (2.4) is true for $n$ instead of $n-1$. To complete the proof of Theorem 2.1 (step 2), we have to show that $d'_{1}^{[n]}$ and $\Phi^{[n]}$ can be chosen to satisfy conditions (2.3). In the equation 2.4, the terms $d'^{k}_{1}$ and $\Phi^{k}$ only act on $V_{1}^{k}$. So one can replace $\Phi^{n}$ with $\varphi^{n}$, $d'_{2}^{2}$ with $d'_{2}^{2}$ (or $d'_{1}^{i}, i \geq 3$ with 0) provided conditions (2.4) are still satisfied on $V_{1}^{n}$. The other terms acting on $V_{1}^{n}$ in the equation (2.4) only involve terms $\Phi^{m} = \varphi^{m}$ and $d'_{1}^{m}$. Then conditions (2.4) on $V_{1}^{[\ldots, 1]}$ are the equations that should be satisfied by a $C_{\infty}$-morphisms between the $C_{\infty}$-algebras $(g_1, d'_{1}^{1,1})$ and $(g_2, \sum_{k \geq 1} d'_{2}^{k})$ restricted to $V_{1}^{n}$. Hence by hypothesis on $\varphi$ the conditions hold.

Similarly, the construction of $\Phi'$ can be made by induction. Let us recall the proof given in [GH]. Again a morphism $\Phi' : (\land_{\mathbf{g}_1 \otimes \mathbf{g}_2}^*, d_1) \to (\land_{\mathbf{g}_1 \otimes \mathbf{g}_2}^*, d_1)$ is uniquely determined by its components $\Phi'^{p_{1}, \cdots, p_{k}} : \mathbf{g}_{1}^{\otimes p_{1}} \land \cdots \land \mathbf{g}_{1}^{\otimes p_{k}} \to \mathbf{g}_{1}$. We write $\Phi' = \sum_{n \geq 1} \Phi'^{[n]}$ with
\[
\Phi'^{[n]} = \sum_{p_{1} + \cdots + p_{k} = n} \Phi'^{p_{1}, \cdots, p_{k}} \quad \text{and} \quad \Phi'^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi'^{[k]}.
\]

We construct the maps $\Phi'^{[n]}$ by induction with the initial condition $\Phi'^{[1]} = \text{id}$. Moreover, we want the following extra conditions to be true:
\[
\Phi'^{n} = 0 \quad \text{for } n \geq 2. \quad (2.5)
\]

Now suppose the construction is done for $n-1$ ($n \geq 2$), i.e., we have built maps $(\Phi'^{[i]})_{i \leq n-1}$ satisfying conditions (2.5) and
\[
\Phi'^{[\leq n-1]} d'_{1}^{[\leq n]} = d'_{1}^{[\leq n]} \Phi'^{[\leq n-1]} \quad \text{on } V_{1}^{[\leq n]}.
\]
In [GH], we prove that for any such \((\Phi^{[i]}_r)_{i \leq n-1}\), one can construct \(\Phi'^{[n]}_r\) such that condition (2.6) is true for \(n\) instead of \(n-1\) in the following way: making the equation \(\Phi' d_1 = d'_1 \Phi'\) on \(V_1^{[n+1]}\) explicit, we get
\[
\Phi'^{[\leq n]}_r d_1^{[\leq n+1]} = d'_1^{[\leq n+1]} \Phi'^{[\leq n]}_r.
\] (2.7)

If we now take into account that \(d_1^{[i]} = 0\) for \(i \neq 2\), \(d'_1^{[1]} = 0\) and that on \(V_1^{[n+1]}\) we have \(\Phi'^{[k]}_r d_1^{[i]} = d'_1^{[\leq k]} \Phi'^{[i]}_r = 0\) for \(k + l > n + 2\), the identity (2.7) becomes
\[
\Phi'^{[\leq n]}_r d_1^{[2]} = \sum_{k=2}^{n+1} d'_1^{[k]} \Phi'^{[\leq n-k+2]}_r.
\]

As \(d'_1^{[2]} = d_1^{[2]}\), (2.7) is equivalent to
\[
d_1^{[2]} \Phi'^{[\leq n]}_r - \Phi'^{[\leq n]}_r d_1^{[2]} = \left[ d_1^{[2]}, \Phi'^{[\leq n]}_r \right] = - \sum_{k=3}^{n+1} d'_1^{[k]} \Phi'^{[\leq n-k+2]}_r.
\]

Notice that \(d_1^{[2]} = m_1^{1,1} + m_1^{2,1}\). Then the construction will be possible when the term \(\sum_{k=3}^{n+1} d'_1^{[k]} \Phi'^{[\leq n-k+2]}_r\) is a coboundary in the subcomplex of \((\text{End}(\bigwedge g_1^{\otimes \bullet}), [d_1^{[2]}, -])\) consisting of maps which restrict to zero on \(\oplus_{n \geq 2} g_1^{\otimes n}\). It is always a cocycle by straightforward computation (see [GH]) and the subcomplex is acyclic because both \((\text{End}(\bigwedge g_1^{\otimes \bullet}), [d_1^{[2]}, -])\) and the Harrison cohomology of \(g_1\) are trivial according to Tamarkin [Ta] (see also [GH] Proposition 5.1 and [Hi] 5.4).

In the case of the \(L_\infty\)-morphism \(\gamma\), the first step is similar: the fact that \(\gamma\) is a \(L_\infty\)-map enables us to build a \(G_\infty\)-structure \((g_1, d'_1)\) on \(g_1\) and a \(G_\infty\)-morphism \(\Gamma : (g_1, d'_1) \rightarrow (g_2, d_2)\) such that:
\[
\Gamma^{1, \ldots, 1} = \gamma^{1, \ldots, 1}, \quad d'^{1,1}_1 = d^{1,1}_1, \quad d'^{1,1, \ldots, 1}_1 = 0.
\] (2.8)

For the second step, we have to build a map \(\Gamma'\) satisfying the equation
\[
d_1^{[2]} \Gamma'^{[\leq n]}_r - \Gamma'^{[\leq n]}_r d_1^{[2]} = \left[ d_1^{[2]}, \Gamma'^{[\leq n]}_r \right] = - \sum_{k=3}^{n+1} d'_1^{[k]} \Gamma'^{[\leq n-k+2]}_r
\]
on \(V_1^{[n+1]}\) for any \(n \geq 1\). Again, because Tamarkin has proved that the complex \((\text{End}(\bigwedge g_1^{\otimes \bullet}), [d_1^{[2]}, -])\) is acyclic (we are in the case \(M = \mathbb{R}^n\)), the result follows from the fact that \(\sum_{k=3}^{n+1} d'_1^{[k]} \Gamma'^{[\leq n-k+2]}_r\) is a cocycle. The difference with the \(C_\infty\)-case is that the \(\Gamma^{1, \ldots, 1}\) could be non zero.

3-The difference between two \(G_\infty\)-maps

In this section we investigate the difference between two different \(G_\infty\)-formality maps. We fix once for all a \(G_\infty\)-structure on \(g_2\) (given by a differential \(d_2\)) satisfying the conditions (1.1) and a morphism of \(G_\infty\)-algebras \(T : (g_1, d_1) \rightarrow (g_2, d_2)\) such that \(T^1 : g_1 \rightarrow g_2\) is \(\varphi_{\text{HKR}}\). Let \(K : (g_1, d_1) \rightarrow (g_2, d_2)\) be any other \(G_\infty\)-morphism with \(K^1 = \varphi_{\text{HKR}}\) (for example any lift of a Kontsevich formality map or any \(G_\infty\)-maps lifting another \(C_\infty\)-morphism).
\textbf{Theorem 3.1.} There exists a map $h : \wedge^* g_1 \to \wedge^* g_2$ such that
\[T - K = h \circ d_1 + d_2 \circ h.\]

In other words the formality $G_\infty$-morphisms $K$ and $T$ are homotopic.

The maps $T$ and $K$ are elements of the cochain complex $\left(\text{Hom}(\wedge^* g_1, \wedge^* g_2), \delta\right)$ with differential given, for all $f \in \text{Hom}(\wedge^* g_1, \wedge^* g_2)$, $|f| = k$, by
\[\delta(f) = d_2 \circ f - (-1)^k f \circ d_1.\]

We first compare this cochain complex with the complexes $\left(\text{End}(\wedge^* g_1), [d_1; -]\right)$ and $\left(\text{End}(\wedge^* g_2), [d_2; -]\right)$ (where $[-; -]$ is the graded commutator of morphisms). There are morphisms
\[T_* : \text{End}(\wedge^* g_1) \to \text{Hom}(\wedge^* g_1, \wedge^* g_2), \quad T^* : \text{End}(\wedge^* g_2) \to \text{Hom}(\wedge^* g_1, \wedge^* g_2)\]
defined, for $f \in \text{End}(\wedge^* g_2)$ and $g \in \text{Hom}(\wedge^* g_1, \wedge^* g_2)$, by
\[T_*(f) = T \circ f, \quad T^*(g) = g \circ T.\]

\textbf{Lemma 3.2.} The morphisms
\[T_* : \left(\text{End}(\wedge^* g_1), [d_1; -]\right) \to \left(\text{Hom}(\wedge^* g_1, \wedge^* g_2), \delta\right) \leftrightarrow \left(\text{End}(\wedge^* g_2), [d_2; -]\right) : T^*\]
of cochain complexes are quasi-isomorphisms.

Remark: This lemma holds for every manifold $M$ and any $G_\infty$-morphism $T : (g_1, d_1) \to (g_2, d_2)$.

\textit{Proof}. First we show that $T_*$ is a morphism of complexes. Let $f \in \text{End}(\wedge^* g_2)$ with $|f| = k$, then
\[T_*([d_1; f]) = T \circ d_1 \circ f - (-1)^k T \circ f \circ d_1 = d_2 \circ (T \circ f) - (-1)^k (T \circ f) \circ d_1 = \delta(T_*(f)).\]

Let us prove now that $T_*$ is a quasi-isomorphism. For any graded vector space $g$, the space $\wedge^* g$ has the structure of a filtered space where the $m$-level of the filtration is $F^m(\wedge^* g) = \bigoplus_{p_1 + \ldots + p_n = m} g^{p_1} \wedge \ldots \wedge g^{p_n}$. Clearly the differential $d_1$ and $d_2$ are compatible with the filtrations on $\wedge^* g_1$ and $\wedge^* g_2$, hence $\text{End}(\wedge^* g_1)$ and $\text{Hom}(\wedge^* g_1, \wedge^* g_2)$ are filtered cochain complex. This yields two spectral sequences (lying in the first quadrant) $E^{*,*}$ and $\tilde{E}^{*,*}$ which converge respectively toward the cohomology $H^*(\text{End}(\wedge^* g_1))$ and $H^*(\text{Hom}(\wedge^* g_1, \wedge^* g_2))$. By standard spectral sequence techniques it is enough to
prove that the map \( T^0_* : E_0^\bullet \to \tilde{E}_0^\bullet \) induced by \( T_* \) on the associated graded is a quasi-isomorphism.

The induced differentials on \( E_0^\bullet \) and \( \tilde{E}_0^\bullet \) are respectively \([d_1^1, -] = 0 \) and \( d_1^2 \circ (-) - (-) \circ d_1^1 = b \circ (-) \) where \( b \) is the Hochschild coboundary. By cofreeness property we have the following two isomorphisms

\[
E_0^\bullet \cong \text{End}(g_1), \quad \tilde{E}_0^\bullet \cong \text{Hom}(g_1, g_2).
\]

The map \( T^0_* : E_0^\bullet \to \tilde{E}_0^\bullet \) induced by \( T_* \) is \( \varphi_{HKR} \circ (-) \). Let \( p : g_2 \to g_1 \) be the projection onto the cohomology, i.e. \( p \circ \varphi_{HKR} = \text{id} \). Let \( u : g_1 \to g_2 \) be any map satisfying \( b(u) = 0 \) and set \( v = p \circ u \in \text{End}(g_1) \). One can choose a map \( w : g_1 \to g_2 \) which satisfies for any \( x \in g_1 \) the following identity

\[
\varphi_{HKR} \circ p \circ u(x) - u(x) = b \circ w(x).
\]

It follows that \( \varphi_{HKR}(v) \) has the same class of homology as \( u \) which proves the surjectivity of \( T^0_* \) in cohomology. The identity \( p \circ \varphi_{HKR} = \text{id} \) implies easily that \( T^0_* \) is also injective in cohomology which finish the proof of the lemma for \( T_* \).

The proof that \( T^* \) is also a quasi-isomorphism is analogous.

**Proof of Theorem 3.1:**

It is easy to check that \( T - K \) is a cocycle in \( \text{Hom}(\wedge g_1^\bullet, \wedge g_2^\bullet, \delta) \). The complex of cochain \( \text{End}(\wedge g_1^\bullet, [d_1, -]) \cong \text{Hom}(\wedge g_1^\bullet, g_1, [d_1, -]) \) is trigraded with \( ||1 \) being the degree coming from the graduation of \( g_1 \) and any element \( x \) lying in \( g_1^{p_1} \wedge \cdots \wedge g_2^{p_q} \) satisfies \(|x|_2 = q - 1, |x|_3 = p_1 + \cdots p_q - q \). In the case \( M = \mathbb{R}^n \), the cohomology \( H^\bullet \left( \text{End}(\wedge g_1^\bullet), [d_1, -] \right) \) is concentrated in bidegree \((|1|_2, |1|_3) = (0, 0) \) (see [Ta], [Hi]). By Lemma 3.2, this is also the case for the cochain complex \( \text{Hom}(\wedge g_1^\bullet, \wedge g_2^\bullet, \delta) \). Thus, its cohomology classes are determined by complex morphisms \((g_1, 0) \to (g_2, d_2^1)\) and it is enough to prove that \( T \) and \( K \) determine the same complex morphism \((g_1, 0) \to (g_2, d_2^1 = b)\) which is clear because \( T^1 \) and \( K^1 \) are both equal to the Hochschild-Kostant-Rosenberg map.

**Remark.** It is possible to have an explicit formula for the map \( h \) in Theorem 3.1. In fact the quasi-isomorphism coming from Lemma 3.2 can be made explicit using explicit homotopy formulae for the Hochschild-Kostant-Rosenberg map (see [Ha] for example) and deformation retract techniques (instead of spectral sequences) as in [Ka]. The same techniques also apply to give explicit formulae for the quasi-isomorphism giving the acyclicity of \( \text{End}(\wedge g_1^\bullet, [d_1; -]) \) in the proof of theorem 3.1 (see [GH] for example).
References

[BFFLS1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Quantum mechanics as a deformation of classical mechanics, Lett. Math. Phys. 1 (1975), 521–530.

[BFFLS2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantization, I and II, Ann. Phys. 111 (1977), 61–151.

[CFT] A. S. Cattaneo, G. Felder, L. Tomassini, From local to global deformation quantization of Poisson manifolds, Duke Math. J. 115 (2002), 329–352.

[Gi] G. Ginot, Homologie et modèle minimal des algèbres de Gerstenhaber, Ann. Math. Blaise Pascal 11 (2004), 91–127.

[GH] G. Ginot, G. Halbout, A formality theorem for Poisson manifold, Lett. Math. Phys. 66 (2003), 37–64.

[GK] V. Ginzburg, M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994), 203–272.

[Ha] G. Halbout, Formule d'homotopie entre les complexes de Hochschild et de de Rham, Compositio Math. 126 (2001), 123–145.

[Hi] V. Hinich, Tamarkin’s proof of Kontsevich’s formality theorem, Forum Math. 15 (2003), 591–614.

[HKR] G. Hochschild, B. Kostant and A. Rosenberg, Differential forms on regular affine algebras, Transactions AMS 102 (1962), 383-408.

[Ka] C. Kassel, Homologie cyclique, caractère de Chern et lemme de perturbation, J. Reine Angew. Math. 408 (1990), 159–180.

[Ko1] M. Kontsevich, Formality conjecture. Deformation theory and symplectic geometry, Math. Phys. Stud. 20 (1996), 139–156.

[Ko2] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003), 157–216.

[KS] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and the Deligne conjecture, Math. Phys. Stud. 21 (2000), 255–307.

[Kos] J. L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie in “Elie Cartan et les mathématiques d’aujourd’hui”, Astérisque (1985), 257-271.

[LS] T. Lada, J. D. Stasheff, Introduction to SH Lie algebras for physicists, Internat. J. Theoret. Phys. 32 (1993), 1087-1103.

[Ta] D. Tamarkin, Another proof of M. Kontsevich’s formality theorem, math. QA/9803025.

[TS] D. Tamarkin, B Tsygan, Noncommutative differential calculus, homotopy BV algebras and formality conjectures, Methods Funct. Anal. Topology 6 (2000), 85–97.

(a) Laboratoire Analyse Géométrie et Applications, Université Paris 13, Centre des Mathématiques et de leurs Applications, ENS Cachan, 61, av. du Président Wilson, 94230 Cachan, France
E-mail: ginot@cmla.ens-cachan.fr

(b) Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS 7, rue René Descartes, 67084 Strasbourg, France
E-mail: halbout@math.u-strasbg.fr