INFINITESIMAL HECKE ALGEBRAS OF $\mathfrak{so}_N$

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Abstract. In this article we classify all infinitesimal Hecke algebras of $\mathfrak{g} = \mathfrak{so}_N$. We establish isomorphism of their universal versions and the $W$-algebras of $\mathfrak{so}_{N+2m+1}$ with a 1-block nilpotent element of the Jordan type $(1, \ldots, 1, 2m + 1)$. This should be considered as a continuation of [LT], where the analogous results were obtained for the cases of $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sp}_{2n}$.

Introduction

In this paper we consider infinitesimal Hecke algebras of $\mathfrak{so}_N$. Although their theory runs along similar lines as for the cases of $\mathfrak{gl}_N$ and $\mathfrak{sp}_{2N}$, they have not been investigated before.

We obtain the classification result in Theorem 1.6 (compare to [EGG, Theorem 4.2]), compute the Poisson center of the corresponding Poisson algebras in Theorem 4.2 (compare to [DT, Theorems 5.1 and 7.1]), compute the first non-trivial central element in Theorem 6.1 (compare to [DT, Theorem 3.1]) and derive the isomorphism with the corresponding $W$-algebras in Theorems 5.3, 5.4 (compare to [LT, Theorems 2.2 and 3.1]).

Together with [LT], this covers all basic cases of the infinitesimal Hecke algebras on the one side and the classical $W$-algebras with a 1-block nilpotent element, on the other. However, we would like to emphasize that the theory of infinitesimal/continuous Hecke algebras is much more complicated in general and has not been developed yet.

This paper is organized as follows:

- In Section 1, we recall the definitions of the continuous and infinitesimal Hecke algebras of type $(\mathfrak{g}, \mathfrak{V})$ (respectively $(\mathfrak{g}, \mathfrak{V})$). We formulate Theorems 1.4 and 1.6 which classify all such algebras for the cases of $(\mathfrak{so}_N, \mathfrak{V}_N)$ and $(\mathfrak{so}_N, \mathfrak{V}_N)$, respectively.
- In Section 2, we prove Theorem 1.4.
- In Section 3, we prove Theorem 1.6 by computing explicitly the corresponding integral.
- In Section 4, we compute the Poisson center of the classical analogue $H^\mathfrak{v}_\mathfrak{z}(\mathfrak{so}_N, \mathfrak{V}_N)$.
- In Section 5, we introduce the universal length $m$ infinitesimal Hecke algebras $H_m(\mathfrak{so}_N, \mathfrak{V}_N)$. Theorem 5.3 (and its Poisson counterpart Theorem 5.4) establish an abstract isomorphism between algebras $H_m(\mathfrak{so}_N, \mathfrak{V}_N)$ and the $W$-algebras $U(\mathfrak{so}_{N+2m+1}, e_m)$.
- In Section 6, we find a non-trivial central element of $H^\mathfrak{v}_\mathfrak{z}(\mathfrak{so}_N, \mathfrak{V}_N)$, called the Casimir element of $H^\mathfrak{v}_\mathfrak{z}(\mathfrak{so}_N, \mathfrak{V}_N)$. This can be used to establish the isomorphism of Theorem 5.3 explicitly.

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1 We assume that $N \geq 3$. 
1. Basic definitions

1.1. Continuous Hecke algebras.

We recall the definition of the continuous Hecke algebras of \((G, V)\), following \[EGG\].

Given a reductive algebraic group \(G\), its algebraic representation \(V\) and a skew-symmetric \(G\)-equivariant \(\mathbb{C}\)-linear map \(\kappa : V \times V \to \mathcal{O}(G)^*\), we set

\[\mathcal{H}_\kappa(G, V) := \mathcal{O}(G)^* \times TV/([x, y] - \kappa(x, y))\ x, y \in V.\]

Consider an algebra filtration on \(\mathcal{H}_\kappa(G, V)\) by setting \(\text{deg}(V) = 1\) and \(\text{deg}(\mathcal{O}(G)^*) = 0\).

**Definition 1.1.** \[EGG\] We say that \(\mathcal{H}_\kappa(G, V)\) satisfies the PBW property if the natural surjective map \(\mathcal{O}(G)^* \times SV \to \mathcal{H}_\kappa(G, V)\) is an isomorphism, where \(S\) denotes the symmetric algebra. We call these \(\mathcal{H}_\kappa(G, V)\) the continuous Hecke algebras of \((G, V)\).

According to \[EGG\] Theorem 2.4, \(\mathcal{H}_\kappa(G, V)\) satisfies the PBW property if and only if \(\kappa\) satisfies the Jacobi identity:

\[(z - z^g)\kappa(x, y) + (y - y^g)\kappa(z, x) + (x - x^g)\kappa(y, z) = 0, \quad \forall x, y, z \in V.\]

Define the closed subscheme \(\Phi \subset G\) by the equation \(\wedge^3(1 - g \mid V) = 0\). The set of closed points of \(\Phi\) is the set \(S\) of elements of \(G\) such that \(\text{rk}(1 - g \mid V) \leq 2\). We have:

**Proposition 1.2.** \[EGG\] Proposition 2.8 If the PBW property holds for \(\mathcal{H}_\kappa(G, V)\), then \(\kappa(x, y)\) is supported on the scheme \(\Phi\) for all \(x, y \in V\).

The classification of all \(\kappa\) which satisfy \((\dagger)\) was obtained in \[EGG\] for the following cases:

- for the pairs \((G, \mathfrak{g} \oplus \mathfrak{h}^*)\) with \(\mathfrak{h}\) being an irreducible faithful \(G\)-representation of real or complex type (see \[EGG\] Theorem 3.5]),
- for the pair \((\text{Sp}_{2n}, V_{2n})\) (see \[EGG\] Theorem 3.14)).

For general continuous Hecke algebras, such a classification is not known at the moment. However, a particular family of those was established in \[EGG\] Theorem 2.13:

**Proposition 1.3.** For any \(\tau \in (\mathcal{O}(\text{Ker} \rho)^* \otimes \wedge^2 V^*)^G\) and \(v \in (\mathcal{O}(\Phi)^* \otimes \wedge^2 V^*)^G\), the pairing \(\kappa_{\tau,v}(x, y) := \tau(x, y) + v(1 - g) (1 - g)\) satisfies the Jacobi identity.

Our first result is a full classification of all \(\kappa\) satisfying \((\dagger)\) for the case of \((\text{SO}_N, V_N)\), which is similar to the aforementioned classification for \((\text{Sp}_{2n}, V_{2n})\). However, it turns out that \(\Phi\) is not reduced in this case and so we need a more detailed argument.

**Theorem 1.4.** The PBW property holds for \(\mathcal{H}_\kappa(\text{SO}_N, V_N)\) if and only if there exists an \(\text{SO}_N\)-equivariant distribution \(c \in \mathcal{O}(S)^*\), such that \(\kappa(x, y) = ((g - g^{-1})x, y)c\) for all \(x, y \in V_N\).

The proof of this theorem is presented in Section 2.

1.2. Infinitesimal Hecke algebras.

For any triple \((\mathfrak{g}, V, \kappa)\) of a Lie algebra \(\mathfrak{g}\), its representation \(V\) and a \(\mathfrak{g}\)-equivariant \(\mathbb{C}\)-bilinear pairing \(\kappa : \wedge^2 V \to U(\mathfrak{g})\), we define

\[H_\kappa(\mathfrak{g}, V) := U(\mathfrak{g}) \times TV/([x, y] - \kappa(x, y))\ x, y \in V.\]

Endow this algebra with a filtration by setting \(\text{deg}(V) = 1\), \(\text{deg}(\mathfrak{g}) = 0\).

**Definition 1.5.** \[EGG\] Section 4 We call this algebra the infinitesimal Hecke algebra of \((\mathfrak{g}, V)\) if it satisfies the PBW property, that is the natural surjective map \(U(\mathfrak{g}) \times SV \to \text{gr} H_\kappa(\mathfrak{g}, V)\) is an isomorphism.
Any such algebra gives rise to a continuous Hecke algebra

\[ H_\kappa(g,V) := \mathcal{O}(G)^* \otimes_{U(g)} H_\kappa(g,V), \]

where \( U(g) \) is identified with a subalgebra \( \mathcal{O}(G)_{1_G} \subset \mathcal{O}(G)^* \), consisting of all algebraic distributions set-theoretically supported at \( 1_G \in G \).

In particular, having a full classification of the continuous Hecke algebras of type \((G,V)\) yields a corresponding classification for \((\text{Lie}(G),V)\). Such classifications were determined explicitly for the cases of \((g,V) = (\mathfrak{gl}_n,V_\bullet \oplus V_\bullet^*)\) in [EGG] Theorem 4.2.

To formulate our classification of infinitesimal Hecke algebras \( H_\kappa(\mathfrak{so}_N,V_N) \), we define:

\[ \gamma_{2j+1}(x,y) \in \mathfrak{so}_N \simeq \mathbb{C}[\mathfrak{so}_N] \]

by

\[ (x,A(1 + \tau^2 A^2)^{-1}) \det(1 + \tau^2 A^2)^{-1/2} = \sum_{j \geq 0} \gamma_{2j+1}(x,y)(A)^{2j}, \quad A \in \mathfrak{so}_N, \]

\[ r_{2j+1}(x,y) \in U(\mathfrak{so}_N) \] to be the symmetrization of \( \gamma_{2j+1}(x,y) \in \mathfrak{so}_N \).

The following theorem will be proved in Section 3:

**Theorem 1.6.** The PBW property holds for \( H_\kappa(\mathfrak{so}_N,V_N) \) if and only if \( \kappa = \sum_{j=0}^{k} \zeta_j r_{2j+1} \) for some non-negative integer \( k \) and parameters \( \zeta_0, \ldots, \zeta_k \in \mathbb{C} \).

This classification is very similar to the analogous results for the pairs \((\mathfrak{gl}_n,V_\bullet \oplus V_\bullet^*)\) and \((\mathfrak{sp}_{2n},V_{2n})\). We denote the corresponding algebra by \( H_\kappa(\mathfrak{so}_N,V_N) \) for \( \kappa \) of the above form.

**Remark 1.7.** (a) For \( \zeta_0 \neq 0 \) we have \( H_{\zeta_0}(\mathfrak{so}_N,V_N) \simeq U(\mathfrak{so}_{N+1}) \). Thus, for an arbitrary \( \zeta \) we can regard \( H_{\zeta}(\mathfrak{so}_N,V_N) \) as a deformation of \( U(\mathfrak{so}_{N+1}) \).

(b) This theorem does not hold for \( N = 2 \), since only half of the infinitesimal Hecke algebras are of the form given in the theorem (algebras \( H_\kappa(\mathfrak{so}_2,V_2) \) are the same as \( H_\kappa(\mathfrak{gl}_1,V_1 \oplus V_1^*) \)).

### 1.3. \( W \)-algebras.

Here we recall the definitions of finite \( W \)-algebras following [GG] (see also [LT] Section 1.6]).

Let \( g \) be a finite dimensional simple Lie algebra over \( \mathbb{C} \) and \( e \in g \) be a nonzero nilpotent element. We identify \( g \) with \( g^* \) via the Killing form \((\ ,\ )\). Let \( \chi \) be the element of \( g^* \) corresponding to \( e \) and \( \mathfrak{z}_\chi \) be the stabilizer of \( \chi \) in \( g \) (which is the same as the centralizer of \( e \) in \( g \)). Fix an \( sl_2 \)-triple \((e,h,f)\) in \( g \). Then \( \mathfrak{z}_\chi \) is ad\((h)\)-stable and the eigenvalues of ad\((h)\) on \( \mathfrak{z}_\chi \) are nonnegative integers. Consider the ad\((h)\)-weight grading on \( g = \bigoplus_{\xi \in \mathfrak{z}} g(\xi) \), that is, \( g(\xi) := \{ \xi \in g \mid [h,\xi] = \xi \} \). Equip \( g(-1) \) with the symplectic form \( \omega_\chi(\xi,\eta) := \langle \chi, [\xi,\eta] \rangle \). Fix a Lagrangian subspace \( l \subset g(-1) \) and set \( m := \bigoplus_{\xi \in l} g(\xi) \oplus l \subset g \). \( m' := \{ \xi - \langle \chi,\xi \rangle \xi \mid \xi \in m \} \subset U(g) \).

**Definition 1.8.** [PI], [GG] The \( W \)-algebra associated with \( e \) (and \( l \)) is the algebra \( U(g,e) := (U(g)/U(g)m')^{ad_m} \) with multiplication induced from \( U(g) \).

Let \( \{ F^* \} \) denote the PBW filtration on \( U(g) \), while \( U(g)(i) := \{ x \in U(g) \mid [h,x] = ix \} \). Define \( F_k U(g) = \sum_{i+j+k} (F^*_i U(g) \cap U(g)(i)) \) and equip \( U(g,e) \) with the induced filtration, denoted \( \{ F^*_e \} \) and referred to as the Kazhdan filtration.

One of the key results of [GG],[PI] is a description of the associated graded algebra \( gr_{F^*_e} U(g,e) \). Recall that the affine subspace \( S_e := \chi + (g/[g,f])^* \subset g^* \) is called the Slodowy slice. As an affine subspace of \( g \), the Slodowy slice \( S_e \) coincides with \( e + c \), where \( c = \ker(g(0)) \). So we can identify \( \mathbb{C}[S_e] \cong \mathbb{C}[c] \) with the symmetric algebra \( S闪烁\). According to [GG] Section 3, algebra \( \mathbb{C}[S_e] \) inherits a Poisson structure from \( \mathbb{C}[g^*] \) and is also graded with \( \deg(\chi \cap g(i)) = i + 2 \).

**Theorem 1.9.** [GG] Theorem 4.1 The filtered algebra \( U(g,e) \) does not depend on the choice of \( l \) (up to a distinguished isomorphism) and \( gr_{F^*_e} U(g,e) \cong \mathbb{C}[S_e] \) as graded Poisson algebras.
Lemma 2.1. For all \( x, y, z \in V_N \) and the result follows. Thus we can assume

\[ h(x, y, z; g) := (z - a^g)(x^g - x'^{g-1}) + (y - y^g)(z^g - z'^{g-1}) + (x - x^g)(y^g - y'^{g-1}), \]

Proof. For any \( g \in S \) we have \( V = V^g \oplus (V^g)^\perp \), where \( V^g := \text{Ker}(1 - g) \) is a codimension \( \leq 2 \) subspace of \( V \). If one of the vectors \( x, y, z \) belongs to \( V^g \), then all the three summands are zero and the result follows. Thus we can assume \( x, y, z \in (V^g)^\perp \). Since \( \text{dim}((V^g)^\perp) \leq 2 \) they must be linearly dependent; without loss of generality we can assume \( z = \alpha x + \beta y, \ \alpha, \beta \in \mathbb{C} \). Then

\[ h(x, y, z; g) = \alpha \left( (x - x^g)(x^g - x'^{g-1}) + (y - y^g)(y^g - y'^{g-1}) + (x - x^g)(y^g - y'^{g-1}) \right) + \beta \left( (y - y^g)(x^g - x'^{g-1}) + (y - y^g)(y^g - y'^{g-1}) + (x - x^g)(y^g - y'^{g-1}) \right). \]

Since \( x^g - x'^{g-1} = (x^g, x) - (x, x^g) = 0 \) and \( y^g - y'^{g-1} = -(y^g - y'^{g-1}) \), the first sum is zero. Analogously, the second sum is zero. The result follows.

Since \( c \) is scheme-theoretically supported at \( S \) and \( h(x, y, z; g) = 0 \) for all \( x, y, z \in V_N, g \in S \), we get \( h(x, y, z; g)c = 0 \), which implies \((\dagger)\).

Lemma 2.2. (a) The space \((\wedge^2 V^*_N \otimes \mathcal{O}(\Phi)^* )^{SO_N}\) is either zero or one-dimensional,

(b) If \((\wedge^2 V^*_N \otimes \mathcal{O}(\Phi)^* )^{SO_N} \neq 0\), then there exists \( \kappa' \in (\wedge^2 V^*_N \otimes \mathcal{O}(\Phi)^* )^{SO_N} \), not satisfying \((\dagger)\).

The following fact was communicated to us by Steven Sam:

Claim 2.3. As a \( \mathfrak{g}l_N \)-representation we have \( E \simeq \wedge^4 V_N \).

Let us first deduce Lemma 2.2 from this Claim.

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\( ^2 \) So that any element of \((\wedge^2 V^*_N \otimes \mathcal{O}(\Phi)^* )^{SO_N} \) satisfying \((\dagger)\) should be in the image of \( \phi \).
Proof of Lemma 2.2.

(a) The following facts are well-known (see [FH Section 19]):
- \( \wedge^4 V_N \simeq \wedge^{N-4} V_N \) as \( \mathfrak{so}_N \)-modules (since \( V_N \simeq V_N^* \) via the pairing),
- the \( \mathfrak{so}_{2n+1} \)-representations \( \{ \wedge^k V_{2n+1} \}_{k=0}^n \) are irreducible and pairwise non-isomorphic,
- the \( \mathfrak{so}_{2n} \)-representation \( \wedge^4 V_{2n} \) decomposes as \( \wedge^4 V_{2n} \simeq \wedge^6 V_{2n} \oplus \wedge^2 V_{2n} \oplus \wedge^3 V_{2n} \), and \( \mathfrak{so}_{2n} \)-representations \( \{ \wedge^0 V_{2n}, \ldots, \wedge^{n-1} V_{2n}, \wedge^n V_{2n}, \wedge^{2n} V_{2n} \} \) are irreducible and pairwise non-isomorphic.

Combining these facts with Claim 2.3, we get \( (\wedge^2 V_{2n+1} \otimes E^*)^{SO_{2n+1}} = 0 \), while
\[
\dim((\wedge^2 V_{2n}^* \otimes E^*)^{SO_{2n}}) = \begin{cases} 1, & n = 3 \\ 0, & n \neq 3 \end{cases}.
\]

(b) For \( N = 6 \), any nonzero element of \( (\wedge^2 V_6^* \otimes E^*)^{SO_6} \) corresponds to the composition
\[
\wedge^2 V_6 \xrightarrow{\varphi} \wedge^4 V_6 \simeq E.
\]

Let \( M_4 \subset \mathbb{C}[X]_2 \) be a subspace spanned by the Pfaffians of all \( 4 \times 4 \) principal minors. This subspace is \( \mathfrak{gl}_6 \)-invariant and \( M_4 \simeq \wedge^4 V_6 \) as \( \mathfrak{gl}_6 \)-representations. Claim 2.3 and simplicity of the spectrum of the \( \mathfrak{gl}_6 \)-module \( \mathbb{C}[so_6] \) imply \( M_4 \subset \operatorname{Rad}(I) \), \( M_4 \cap I = 0 \), so that \( M_4 \) corresponds to the copy of \( \wedge^4 V_6 \subset \operatorname{Rad}(I)/I \) from Claim 2.3.

Choose an orthonormal basis \( \{ y_i \}_{i=1}^6 \) of \( V_6 \), so that any element \( A \in \mathfrak{so}_6 \) is skew-symmetric with respect to it. We denote the corresponding Pfaffian by \( \text{Pf}_{i:j} \) (with a correctly chosen sign).

We define \( \kappa'(y_i \otimes y_j) \in U(so_6) \) as a symmetrization of \( \text{Pf}_{i:j} \). Identifying \( U(so_6) \) with \( so_6 \)-modules, we easily see that \( \kappa' : \wedge^2 V_6 \to U(so_6) \) is \( so_6 \)-invariant. Moreover, \( \psi(\kappa') \neq 0 \).

However, \( \kappa' \) does not satisfy the Jacobi identity. Indeed, let us define \( \hat{\kappa}' : V_6 \otimes V_6 \to U(so_6) \) by \( \hat{\kappa}'(y_i \otimes y_j) = \text{Pf}_{i:j} \). Then for any three different indices \( i, j, k \), the corresponding expressions \( \{ P_{i:j,k}, x_k \}, \{ P_{j:k,i}, x_i \}, \{ P_{k:i,j}, x_j \} \) coincide up to a sign and are nonzero. So their sum is also non-zero, implying that \( (\hat{1}) \) fails for \( \kappa' \).

This completes the proof of the lemma.

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- **Proof of Claim 2.3 [due to Steven Sam]**

  **Step 1: Description of \( \operatorname{Rad}(I) \).**

  Let us define \( \text{Pf}_{ijkl} \in \mathbb{C}[X]_2 \) to be the Pfaffians of the principal \( 4 \times 4 \) minors corresponding to rows/columns \#\( i, j, k, l \). Since the corresponding determinants vanish on \( S \), we get \( \text{Pf}_{ijkl} \in \operatorname{Rad}(I) \). A beautiful classical result states that those elements generate \( \operatorname{Rad}(I) \), in fact:

  **Theorem 2.4.** [W Theorem 6.4.1(b)] *The ideal \( \operatorname{Rad}(I) \) is generated by all \( \{ \text{Pf}_{ijkl} \} \).*

  **Step 2: Decomposition of \( \mathbb{C}[X] \) as a \( \mathfrak{gl}_N \)-module.**

  Let \( I \) be the set of all length \( \leq N \) Young diagrams \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq 0) \). It is known that this set parameterizes all irreducible finite dimensional polynomial \( \mathfrak{gl}_N \)-representations. For \( \lambda \in I \), we denote the corresponding irreducible \( \mathfrak{gl}_N \)-representation by \( L_\lambda \). Let us define \( I^e \) as a subset of \( I \), consisting of all Young diagrams with even columns.

  The following result describes the decomposition of \( \mathbb{C}[so_N] \) into irreducibles:

  **Theorem 2.5.** [AR] *As \( \mathfrak{gl}_N \)-modules \( \mathbb{C}[so_N] \simeq S(\wedge^2 V_N) \simeq \bigoplus_{\lambda \in I^e} L_\lambda. \)

  For any \( \lambda \in I^e \), let \( J_\lambda \subset \mathbb{C}[X] \) be the ideal generated by \( L_\lambda \subset \mathbb{C}[X] \), while \( I^e_\lambda \subset I^e \) is a subset of the diagrams containing \( \lambda \). The arguments of [AR] (this is also proved in [D] Theorem 5.1) imply that \( J_\lambda \simeq \bigoplus_{\mu \in I^e_\lambda} L_\mu \) as \( \mathfrak{gl}_N \)-modules.

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3 To make a compatible choice of signs, define \( \text{Pf}_{i:j} \) as the derivative of the total Pfaffian \( \text{Pf} \) along \( E_{ij} - E_{ji} \).
Step 3: \( \text{Rad}(I) \) and \( I \) as \( \mathfrak{gl}_N \)-representations.

Since the space \( M_4 \subset \mathbb{C}[X] \), spanned by \( \text{Pf}_{ijkl} \), is \( \mathfrak{gl}_N \)-invariant and is isomorphic to \( \wedge^4 V_N \), the results of the previous step imply that as a \( \mathfrak{gl}_N \)-module \( \text{Rad}(I) \simeq \bigoplus_{\mu \in I^e_{(1^4)}} L_\mu \).

Let us now describe the subspace \( N_3 \subset \mathbb{C}[\mathfrak{so}_N]_{3} \), spanned by the determinants of all \( 3 \times 3 \) minors. This subspace is \( \mathfrak{gl}_N \)-invariant. Actually we have:

**Lemma 2.6.** As a \( \mathfrak{gl}_N \)-representation \( N_3 \simeq L_{(2,2,1,1)} \oplus L_{(3,3)} \).

**Proof.** According to Step 1, we have \( \mathbb{C}[\mathfrak{so}_N]_3 \simeq L_{(3^6)} \oplus L_{(2,2,1,1)} \oplus L_{(3,3)} \). Since the space of \( 3 \times 3 \) minors identically vanishes when \( N = 2 \), and the Schur functor \( (3,3) \) does not, it rules \( L_{(3,3)} \) out. Also, the space of \( 3 \times 3 \) minors is nonzero for \( N = 4 \), while the Schur functor \( (1^6) \) vanishes, so \( N_3 \not\simeq L_{(1^6)} \). Since partition (\( 1^6 \)) corresponds to the subspace \( M_6 \subset \mathbb{C}[\mathfrak{so}_N] \) of \( 6 \times 6 \) Pfaffians, it suffices to prove that \( M_6 \subset N_3 \). The latter is sufficient to verify for \( N = 6 \), that is, the Pfaffian Pf of a \( 6 \times 6 \) matrix is a linear combination of its \( 3 \times 3 \) determinants.

Let \( \det_{ij}^{\mu\nu} \) be the determinant of the \( 3 \times 3 \) minor, obtained by intersecting rows \#\( i,j,k \) and columns \#\( p,q,s \). The following identity is straightforward:

\[
-4 \text{Pf} = -\det_{123}^{456} + \det_{124}^{356} - \det_{125}^{346} + \det_{126}^{345} - \det_{134}^{256} + \det_{135}^{246} - \det_{136}^{245} - \det_{145}^{236} + \det_{146}^{235} - \det_{156}^{234}.
\]

This completes the proof of the lemma. \( \square \)

The results of Step 2 imply that \( \text{Rad}(I) \simeq \bigoplus_{\mu \in I^e_{(2^2,1^2)}} L_\mu \) as \( \mathfrak{gl}_N \)-modules.

The claim follows now from the aforementioned descriptions of \( \mathfrak{gl}_N \)-modules \( \mathbb{C}[\mathfrak{so}_N]_I, \text{Rad}(I) \). This completes the proof of the claim and hence of the theorem as well. \( \blacksquare \)

### 3. Proof of Theorem 1.6

Let us introduce some notations:

- \( K := \text{SO}_N(\mathbb{R}) \) (the maximal compact subgroup of \( G = \text{SO}_N(\mathbb{C}) \)),
- \( s_\theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & \cdots & 0 \\ \sin \theta & \cos \theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \in K, \quad \theta \in [-\pi, \pi] \),
- \( S_\theta := \{ g s_\theta g^{-1} | g \in K \} \subset K \),
- \( S_\mathbb{R} := S \cap K = \bigcup_{\theta \in [0,\pi]} S_\theta \), so that \( S_\mathbb{R}/K \) gets identified with \( S^1/\mathbb{Z}_2 \).

According to Theorem 1.4, there exists a \( \mathbb{Z}_2 \)-invariant \( c \in \mathbb{O}(S^1)^* \), which is a linear combination of the delta-function \( \delta_0 \) (at 0 in \( S^1 \)) and its even derivatives \( \delta_0^{(2k)} \), such that

\[
\kappa(x,y) = \int_{-\pi}^{\pi} c(\theta) \left( \int_{S_\theta} ((g^{-1}g)x,y) \, dg \right) \, d\theta, \quad \forall x,y \in V_N.
\]

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4 The conceptual proof of this fact is as follows. Note that determinants of \( 3 \times 3 \) minors of \( A \in \mathfrak{so}_6 \) are just the matrix elements of \( \wedge^3 A \), and it acts on \( \wedge^3 V_6 = \wedge^3 V_6 \oplus \wedge^3 V_6 \). It is straightforward to see that the trace of \( \wedge^3 A \) on \( \wedge^3 V_6 \) is nonzero. This provides a cubic invariant for \( \mathfrak{so}_6 \), which is unique up to scaling (multiple of Pf).

5 Note that \( S_\theta \) and \( S_{-\theta} \) coincide for \( N \geq 3 \). That explains why \( \theta \in [0,\pi] \) instead of \( x \in [-\pi, \pi] \).

6 Here we integrate over the whole circle \( S^1 \) instead of \( S^1/\mathbb{Z}_2 \), but we require \( c(\theta) = c(-\theta) \).
For \( g \in S_\mathbb{R} \) we define a 2-dimensional subspace \( V_g \subset V_N \) by \( V_g := \text{Im}(1 - g) \). To evaluate
the above integral, choose length 1 orthogonal vectors \( p, q \in V_g \), such that the restriction of \( g \)
to \( V_g \) is given by the matrix
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
in the basis \( \{p, q\} \).

Let us define \( J_{p,q} := q \otimes p' - p \otimes q' \in \mathfrak{s}_\mathfrak{o}_N(\mathbb{R}) \). Then we have:

\( (g - g^{-1})x, y = 2 \sin \theta \cdot (x, J_{p,q}y) \),

\( g = \exp(\theta J_{p,q}) \), since
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
= \exp \left( \theta \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)
\]

As a result, we get
\[
\kappa(x, y) = \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} (x, J_{p,q}y) \left( \int_{-\pi}^{\pi} 2c(\theta) \sin \theta \cdot e^{\theta J_{p,q}} d\theta \right) dq dp,
\]
where \( S^{N-1} \) is the unit sphere in \( \mathbb{R}^N \) centered at the origin and \( S^{N-2}(p) \) is the unit sphere in \( \mathbb{R}^{N-1}(p) \subset \mathbb{R}^N \), the hyperplane orthogonal to the line passing through \( p \) and the origin.

Since \( c(\theta) \) is an arbitrary linear combination of the delta-function and its even derivatives, the above integral is a linear combination of the integrals:
\[
\int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} (x, J_{p,q}y) \cdot J_{p,q}^{2k+1} dq dp, \quad k \geq 0.
\]

This is a standard integral (see \cite{EGG}, Section 4.2) for the analogous calculations. Identifying \( U(\mathfrak{s}_\mathfrak{o}_N) \) with \( S(\mathfrak{s}_\mathfrak{o}_N) \) via the symmetrization map, it suffices to compute the integral
\[
I_{m;x,y}(A) = \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} (x, J_{p,q}y) \cdot \text{tr}(AJ_{p,q})^m dq dp, \quad A \in \mathfrak{s}_\mathfrak{o}_N(\mathbb{R}).
\]

To compute this expression, we introduce
\[
F_m(A) := \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} \text{tr}(AJ_{p,q})^{m+1} dq dp = \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} (2(Aq,p))^{m+1} dq dp,
\]
so that the former integral can be expressed in the following way:
\[
dF_m(A)(x \otimes y - y \otimes x') = -2(m + 1)I_{m;x,y}(A).
\]

Now we compute \( F_m(A) \). First, note that
\[
G_m(A, \zeta) := \int_{p \in \mathbb{R}^N} \int_{q \in \mathbb{R}^{N-1}(p)} (2(Aq,p))^{m+1} e^{-\zeta(p,p) - \zeta(q,q)} dq dp = \int_0^\infty \int_0^\infty e^{-\zeta r_1^2 - \zeta r_2^2} \int_{|p| = r_1} \int_{|q| = r_2} (2(Aq,p))^{m+1} dq dp dr_2 dr_1 = \int_0^\infty \int_0^\infty e^{-\zeta r_1^2 - \zeta r_2^2} r_1^{m+N-1} r_2^{m+N-1} dr_2 dr_1 \cdot F_m(A) = K_{m+N}(\zeta)K_{m+N-1}(\zeta)F_m(A),
\]
where
\[
K_l(\zeta) = \int_0^\infty e^{-\zeta r^l} dr = \begin{cases}
\frac{k!}{2^{k+1-l} \sqrt{\pi}} & l = 2k + 1 \\
\frac{(2k+1)!! \sqrt{\pi}}{2^{k+1} \zeta^{k+1/2}} & l = 2k
\end{cases}
\]
As a result, we get
\[
G_m(A, \zeta) = \frac{\sqrt{\pi}(m + N - 1)!}{2^{m+N+1} \zeta^{m+N+1/2}} F_m(A).
\]

\footnote{Generally speaking, the integration should be taken over the Grassmannian \( G_2(\mathbb{R}^N) \). However, it is easier to integrate over the Stiefel manifold \( V_2(\mathbb{R}^N) \), which is a principal \( O(2) \)-bundle over \( G_2(\mathbb{R}^N) \).}
On the other hand, we have:

\[
\sum_{m=-1}^{\infty} \frac{1}{(m+1)!} G_m(A, \zeta) = \int_{p \in \mathbb{R}^N} \int_{q \in \mathbb{R}^{N-1}(p)} e^{2(Aq,p)} e^{-\zeta(p,p)} \zeta(q,q) \, dq \, dp =
\]

\[
\int_{p \in \mathbb{R}^N} e^{-\zeta(p,p)} \int_{q \in \mathbb{R}^{N-1}(p)} e^{-2(Ap,p) - \zeta(q,q)} \, dq \, dp = \int_{p \in \mathbb{R}^N} e^{-\zeta(p,p) + \zeta(Ap,Ap)} \, dq \, dp \cdot (\pi/\zeta)^{N-1} =
\]

\[
(\pi/\zeta)^{N-1} \int_{p \in \mathbb{R}^N} e^{(i(\zeta - \frac{\pi}{2} A^2))} \, dp = \pi^{N-1} / \zeta^{N-2} \det(1 + \zeta A^2)^{-1/2} = \pi^{N-1} / \zeta^{N-2} \det(1 + \zeta^{-2} A^2)^{-1/2}.
\]

Hence, \( F_m(A) \) is equal to a constant times the coefficient of \( \tau^{m+1} \) in \( \det(1 + \tau^2 A^2)^{-1/2} \), expanded as a power series in \( \tau \). Differentiating \( \det(1 + \tau^2 A^2)^{-1/2} \) along \( B \in \mathfrak{so}_N \), we get

\[
\frac{\partial}{\partial B} \left( \det(1 + \tau^2 A^2)^{-1/2} \right) = -\frac{\tau^2 \text{tr}(BA(1 + \tau^2 A^2)^{-1})}{\det(1 + \tau^2 A^2)^{1/2}}.
\]

Setting \( B = x \otimes y^t - y \otimes x^t \) gives \( 2\tau^2(x, A(1 + \tau^2 A^2)^{-1}y) \det(1 + \tau^2 A^2)^{-1/2} \), as desired.

4. POISSON CENTER OF ALGEBRAS \( H_{\zeta}^{cl}(\mathfrak{so}_N) \)

Following [DT], we introduce the Poisson algebras \( H_{\zeta}^{cl}(\mathfrak{so}_N, V_N) \), where \( \zeta = (\zeta_0, \ldots, \zeta_k) \) is a deformation parameter. As algebras these are \( S(\mathfrak{so}_N \oplus V_N) \) with a Poisson bracket \( \{\cdot, \cdot\} \) modeled after the commutator \([\cdot, \cdot]\) from the definition of \( H_{\zeta}(\mathfrak{so}_N, V_N) \), that is \( \{x, y\} = \sum_j \zeta_j \gamma_{2j+1}(x, y) \).

We prefer the following short formula for \( \{\cdot, \cdot\} : V_N \times V_N \to \mathbb{C}[\mathfrak{so}_N] \cong S(\mathfrak{so}_N) \):

\[
(*) \{x, y\} = \text{Res}_{z=0} \zeta(z^{-2})(x, A(1+z^2 A^2)^{-1}y) \det(1+z^2 A^2)^{-1/2} z^{-1} \, dz, \quad \forall \ x, y \in V_N, A \in \mathfrak{so}_N,
\]

where \( \zeta(z) \) is the generating function of the deformation parameters: \( \zeta(z) := \sum_{i \geq 0} \zeta_i z^i \).

In fact, we can view algebras \( H_{\zeta}(\mathfrak{so}_N, V_N) \) as quantizations of the algebras \( H_{\zeta}^{cl}(\mathfrak{so}_N, V_N) \). The latter algebras still carry some important information. In particular, our Corollary [4] is needed to carry out the argument of Theorem 5.4. The main result of this section is a computation of the Poisson center of \( H_{\zeta}(\mathfrak{so}_N, V_N) \).

First, let us recall the corresponding result in the non-deformed case (\( \zeta = 0 \)), when the corresponding algebra is just \( S(\mathfrak{so}_N \times V_N) \) with a Lie Poisson bracket. In order to state the result, we need some more notations:

- for \( A \in \mathfrak{gl}_N \) we define \( p_i(A) \) via \( \det(I_N + tA) = \sum_{j=0}^N t^j p_j(A) \),
- define \( b_i(A) = I_N, b_k(A) = \sum_{j=0}^k (-1)^j p_j(A) A^{k-j} \), \( k > 0 \),
- define \( a_N := \mathfrak{so}_N \times V_N \), we identify \( a_N^* \) with \( a_N \) via the natural pairing,
- define \( \psi_k : a_N^* \to V_B \) by \( \psi_k(A, v) = (v, b_{2k}(A)v) \), where \( A \in \mathfrak{so}_N, v \in V_N, k \geq 0 \),
- in the case \( N = 2n+1 \), \( \psi_n \) is actually the square of a polynomial function \( \tilde{\psi}_n \), which can be realized explicitly as the Pfaffian of the matrix \( \begin{pmatrix} A & v \\ -v^t & 0 \end{pmatrix} \in \mathfrak{so}_{2n+2} \),
- identifying \( \mathbb{C}[a_N^*] \cong S(a_N) \), let \( \tau_k \in S(a_N) \) (respectively \( \tilde{\tau}_{n+1} \in S(a_{2n+1}) \)) be the elements corresponding to \( \psi_{k-1} \) (respectively \( \tilde{\psi}_{n} \)).

The following result is due to [R] Sections 3.7, 3.8:
We choose a basis
For that choice of
Similarly to the cases of \(\mathfrak{gl}_n, \mathfrak{sp}_{2n}\), this theorem has a straightforward generalization to the case of a nontrivial deformation \(\zeta\). In fact, for any deformation parameter \(\zeta = (\zeta_0, \ldots, \zeta_k)\) there exist \(c_j \in \mathfrak{z}_{\text{Pois}}(\mathfrak{so}_N)\), such that the Poisson center \(\mathfrak{z}_{\text{Pois}}(H^\zeta_N(\mathfrak{so}_N, V_N))\) is still a polynomial algebra in \(\left\lfloor \frac{n+1}{2} \right\rfloor\) generators \(\{\tau_j + c_j\}_{j=1}^{n}\) (and also \(\hat{\tau}_{n+1}\) for \(N = 2n + 1\)).

This is established in the following theorem:

**Theorem 4.2.** Define \(c_i \in \mathbb{C}[\mathfrak{so}_N]^\text{SO}_N \simeq \mathfrak{z}_{\text{Pois}}(\mathfrak{so}_N)\) via
\[
c(t) := \text{Res}_{z=0} \frac{\zeta(z^{-2})}{\det(1 + t^2 A^2)^{1/2}} \frac{1}{\det(1 + z^2 A^2)^{1/2}} \frac{dz}{1 - t^2 z^2},
\]

\((a) \mathfrak{z}_{\text{Pois}}(H^\zeta_N(\mathfrak{so}_{2n}, V_{2n}))\) is a polynomial algebra in free generators \(\{\tau_1 + c_1, \ldots, \tau_n + c_n\}\),
\((b) \mathfrak{z}_{\text{Pois}}(H^\zeta_N(\mathfrak{so}_{2n+1}, V_{2n+1}))\) is a polynomial algebra in free generators \(\{\tau_1 + c_1, \ldots, \tau_n + c_n, \hat{\tau}_{n+1}\}\).

This result is analogous to [DT, Theorems 5.1 and 7.1] and its proof utilizes the same ideas.

Let us introduce some more notations before proceeding to the proof:

- We choose a basis \(\{x_i\}_{i=1}^N\) of \(V_N\) in such a way, that \((x_i, x_j) = \delta_{N+1-i,j}\).
- \(J = (J_{ij})_{i,j=1}^N\) is the corresponding anti-diagonal symmetric matrix, so that \(J_{ij} = \delta_{N+1-i,j}\).
- For that choice of \(J\), we have \(A = (a_{ij}) \in \mathfrak{so}_N\) if and only if \(a_{ij} = -a_{N+1-j,N+1-i}\), \(\forall i,j\).
- Let \(\mathfrak{h}_N\) denote the Cartan subalgebra of \(\mathfrak{so}_N\) consisting of the diagonal matrices.
- Define \(c(i,j) := E_{i,j} - E_{N+1-j,N+1-i} \in \mathfrak{so}_N\), \(i,j \leq N\) (in particular, \(c(i,N+1-i) = 0\) \(\forall i\)).
- We set \(c_i := c(i,i), 1 \leq i \leq N := \left\lfloor \frac{N}{2} \right\rfloor\), so that \(\{c_i\}_{i=1}^N\) form a basis of \(\mathfrak{h}_N\).
- Define symmetric polynomials \(\sigma_i \in \mathbb{C}[z_1, \ldots, z_i]^{\mathfrak{h}_N}\) via \(\prod_{i=1}^l (1 + t^2 z_i) = \sum_{i=0}^l t^i \sigma_i(z_1, \ldots, z_i)\).

**Proof of Theorem 4.2.**

First, it suffices to prove that elements \(\tau_i + c_i\) (and \(\hat{\tau}_{n+1}\) for \(N = 2n + 1\)) are Poisson central. In this case they are lifts of the central generators of the deformed algebra and the statement follows from Proposition 4.1.1 by a deformation argument.

Since \(\{\tau_i, \mathfrak{so}_N\} = 0\) in the case \(\zeta = 0\), we still have \(\{\tau_i, \mathfrak{so}_N\} = 0\) for any \(\zeta\). This implies \(\{\tau_i + c_i, \mathfrak{so}_N\} = 0\) as \(c_i \in \mathfrak{z}_{\text{Pois}}(\mathfrak{so}_N)\). As a result we just need to verify
\[
(2) \quad \{c_i, x_q\} = \{\tau_i, x_q\}, \quad \forall 1 \leq q \leq N.
\]

Recalling \(\psi(A, v) = (v, b_{2s}(A)v) = \sum_{k,l} x_k x_l b_{2s}(A)_{N+1-k,l}\), we get:
\[
\{\tau_{s+1}, x_q\} = \sum_{k,l} \left\{b_{2s}(A)_{N+1-k,l}, x_q\right\} x_k x_l + \sum_{k,l} b_{2s}(A)_{N+1-k,l} \{x_k, x_q\} x_l + \sum_{k,l} b_{2s}(A)_{N+1-k,l} x_k \{x_l, x_q\}.
\]

The first summand is zero due to Proposition 4.1.1. On the other hand, \(AJ + JA^t = 0\) implies \((A^2)_{N+1-k,l} = (A^2)_{N+1-l,k}\) and \(p_{2j+1}(A) = 0\) for all \(j \geq 0\). Hence:
\[
b_{2s}(A) = A^{2s} + p_2(A) A^{2s-2} + p_4(A) A^{2s-4} + \ldots + p_{2s}(A), \quad b_{2s}(A)_{n+1-k,l} = b_{2s}(A)_{n+1-l,k}.
\]

Combining this with \(c_{s+1, x_q} = \sum_{p \neq N+1-q} \frac{\partial c_{s+1}}{\partial c_{(p,q)}} x_p\), we see that (2) is equivalent to:
\[
(3) \quad \frac{\partial c_{s+1}}{\partial c_{(p,q)}} = -2 \sum_{l} b_{2s}(A)_{N+1-p,l} \text{Res}_{z=0} \frac{\zeta(z^{-2})}{\det(1 + z^2 A^2)^{1/2}} \frac{(x_l, A(1 + z^2 A^2)^{-1} x_q) dz}{z}, \quad \forall p, q \leq N.
\]

Because both sides of (3) are \(\text{SO}_N\)-invariant, it suffices to verify (3) for \(A = \mathfrak{h}_N\), that is for:

- \(A = \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)\) in the case \(N = 2n\),
\( A = \text{diag}(\lambda_1, \ldots, \lambda_n, 0, -\lambda_n, \ldots, -\lambda_1) \) in the case \( N = 2n + 1 \).

For \( p \neq q \) both sides of (3) are zero. For \( p = q \leq n \), the only nonzero summand on the right hand side of (3) is the one corresponding to \( i = N + 1 - q \). In this case:

\[
b_{2s}(A)_{N+1-q,N+1-q} = \lambda_q^{2s} - \sigma_1(\lambda_1^2, \ldots, \lambda_n^2)\lambda_q^{2s-2} + \ldots + (-1)^s \sigma_s(\lambda_1^2, \ldots, \lambda_n^2) = (-1)^s \frac{\partial \sigma_{s+1}(\lambda_1^2, \ldots, \lambda_n^2)}{\partial \lambda_q^2},
\]

while \((x_{N+1-q}, A(1 + z^2A^2)^{-1}x_q) = \frac{\lambda_q}{1 + z^2\lambda_q^2}\) and \(\det(1 + z^2A^2)^{1/2} = \prod_{i=1}^{n}(1 + z^2\lambda_i^2)\).

For \( p = q \geq \lceil \frac{N+1}{2} \rceil \), we get the same equalities with \( \lambda_i \leftrightarrow -\lambda_i \). As a result (3) is equivalent to:

\[
\frac{\partial c_{s+1}(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_q^2} = (-1)^{s+1} \frac{\partial \sigma_{s+1}(\lambda_1^2, \ldots, \lambda_n^2)}{\partial \lambda_q^2} \text{Res}_{z=0} \frac{\zeta(z)}{(1 + z^2\lambda_q^2)\prod_{i=1}^{n}(1 + z^2\lambda_i^2)}.
\]

We thus need to verify the following identities for \( c(t) \):

\[
\frac{\partial c(t)}{\partial \lambda_q^2} = \frac{\partial}{\partial \lambda_q^2} \left[ (1 + t^2\lambda_q^2) \text{Res}_{z=0} \frac{\zeta(z)}{(1 + z^2\lambda_q^2)\prod_{i=1}^{n}(1 + z^2\lambda_i^2)} \right].
\]

It is straightforward to check that

\[\text{Res}_{z=0} \frac{\zeta(z)}{\prod_{i=1}^{n}(1 + z^2\lambda_i^2)} \frac{z^{-1}dz}{1 - t^2z^2} = c(t)(A)\]

satisfies these equations.

This proves that \( \tau_i + c_i \in \mathfrak{Pois}(H^c_\zeta(\mathfrak{so}_N, V_N)) \) for all \( 1 \leq i \leq n \). For \( N = 2n + 1 \), we also get a Poisson-central element \( \tau_{n+1} + c_{n+1} \). Since \( c_{n+1} = 0 \), we have

\[\tau_{n+1}^2 = \tau_{n+1} \in \mathfrak{Pois}(H^c_\zeta(\mathfrak{so}_{2n+1}, V_{2n+1})) \Rightarrow \tau_{n+1} \in \mathfrak{Pois}(H^c_\zeta(\mathfrak{so}_{2n+1}, V_{2n+1})).\]

This completes the proof of the theorem. \( \blacksquare \)

**Definition 4.3.** The element \( \tau_i' = \tau_i + c_i \) is called the Poisson Casimir element of \( H^c_\zeta(\mathfrak{so}_N, V_N) \).

As a straightforward consequence of Theorem 4.2 we get:

**Corollary 4.4.** We have \( \tau_i' = \tau_i + \sum_{j=0}^{k}(-1)^{j+1} \zeta_j \text{tr} S^{2j+2} A \).

5. The Key Isomorphism

Let us first introduce the notion of the universal infinitesimal Hecke algebra of \( (\mathfrak{so}_N, V_N) \):

**Definition 5.1.** Define the universal length \( m \) infinitesimal Hecke algebra \( H_m(\mathfrak{so}_N, V_N) \) as

\[H_m(\mathfrak{so}_N, V_N) := U(\mathfrak{so}_N) \otimes T(V_N)[\zeta_0, \ldots, \zeta_{m-1}]/([A, x] - A(x), [x, y] - \sum_{j=0}^{m-1} \zeta_j r_{2j+1}(x, y) - r_{2m+1}(x, y)),\]

where \( A \in \mathfrak{so}_N \), \( x, y \in V_N \) and \( \{\zeta_i\}_{i=0}^{m-1} \) are central. The filtration is induced from the grading on \( T(\mathfrak{so}_N \oplus V_N)[\zeta_0, \ldots, \zeta_{m-1}] \) with \( \text{deg}(\mathfrak{so}_N) = 2 \), \( \text{deg}(V_N) = 2m + 2 \) and \( \text{deg}(\zeta_i) = 4(m - i) \).

The algebra \( H_m(\mathfrak{so}_N, V_N) \) is free over \( \mathbb{C}[\zeta_0, \ldots, \zeta_{m-1}] \) and \( H_m(\mathfrak{so}_N, V_N)/(\zeta_i - c_i)_{i=0}^{m-1} \) is the usual infinitesimal Hecke algebra \( H_{c_\zeta}(\mathfrak{so}_N, V_N) \) for \( c_\zeta = c_0\tau_1 + \ldots + c_{m-1}\tau_{2m-1} + r_{2m+1} \).

**Remark 5.2.** For an \( \mathfrak{so}_N \)-equivariant pairing \( \eta : \wedge^2 V_N \rightarrow U(\mathfrak{so}_N)[\zeta_0, \ldots, \zeta_{m-1}] \), such that \( \text{deg}(\eta(x, y)) \leq 4m + 2 \), the algebra \( U(\mathfrak{so}_N) \otimes T(V_N)[\zeta_0, \ldots, \zeta_{m-1}]/([A, x] - A(x), [x, y] - \eta(x, y)) \) satisfies a PBW property if and only if \( \eta(x, y) = \sum_{i=0}^{m} \eta_i r_{2i+1}(x, y) \) with \( \eta_i \in \mathbb{C}[\zeta_0, \ldots, \zeta_{m-1}] \) degree \( \leq 4(m - i) \) polynomials (this is completely analogous to our Theorem 7.6).
The main goal of this section it to establish an abstract isomorphism between the algebras $H_m(\mathfrak{so}_N, V_N)$ and the $W$-algebras $U(\mathfrak{so}_{N+2m+1}, e_m)$, where $e_m \in \mathfrak{so}_{N+2m+1}$ is a nilpotent element of the Jordan type $(1^N, 2m+1)$. We will make a particular choice of such element $e_m$.

- $e_m := \sum_{j=1}^m E_{N+j, N+j+1} - \sum_{j=1}^m E_{N+m+j, N+m+j+1}$.

Recall the Lie algebra inclusion $\iota : q \to U(\mathfrak{g}, e)$ from [LT, Section 1.6], where $q := \mathfrak{z}_\mathfrak{g}(e, h, f)$.

For $(q, e) = (\mathfrak{so}_{N+2m+1}, e_m)$, we have $q \simeq \mathfrak{so}_N$. We will also denote the corresponding centralizer of $e_m \in \mathfrak{so}_{N+2m+1}$ and the Slodowy slice by $\mathfrak{z}_{N,m}$ and $S_{N,m}$, respectively.

**Theorem 5.3.** For $m \geq 1$, there is a unique isomorphism $\bar{\Theta} : H_m(\mathfrak{so}_N, V_N) \sim U(\mathfrak{so}_{N+2m+1}, e_m)$ of filtered algebras, such that $\bar{\Theta} |_{\mathfrak{so}_N} = \iota |_{\mathfrak{so}_N}$.

We will only sketch the proof as most arguments are the same as in [LT, Theorem 2.2].

**Sketch of the proof of Theorem 5.3.**
As a vector space, $\mathfrak{z}_{N,m} \cong \mathfrak{so}_N \oplus V_N \oplus \mathbb{C}^m$, where $\mathfrak{so}_N \cong q = \mathfrak{z}_{N,m}(0)$, $V_N \subset \mathfrak{z}_{N,m}(2m)$ and $\mathbb{C}^m$ has a basis $\{\xi_0, \ldots, \xi_{2m-1}\}$ with $\xi_i \in \mathfrak{z}_{N,m}(4m - 4i - 2)$. Here $\xi_{m-j} = e_{ij}^{2m-1} \in \mathfrak{so}_N$ for $1 \leq j \leq m$, $V_N$ is embedded via $x_i \mapsto E_{i,N+m+1} - E_{N+1,i}$, while $\mathfrak{so}_N$ is embedded as a top-left $N \times N$ block of $\mathfrak{so}_{N+2m+1}$.

One of the key ingredients in the proof of [LT, Theorem 2.2] was an additional $\mathbb{Z}$-grading $Gr$ on the corresponding $W$-algebras. In both cases of $(q, e) = (\mathfrak{sl}_{n+m}, e_m), (\mathfrak{sp}_{2n+2m}, e_m)$ such a grading was induced from the weight-decomposition with respect to $\text{ad}(h), h \in q$.

If $N = 2n$ same argument works for $q = \mathfrak{so}_{N+2m+1}$ as well. Namely, consider $h \in q \simeq \mathfrak{so}_{2n}$ to be the diagonal matrix $I_n^e := \text{diag}(1, \ldots, 1, -1, \ldots, -1)$. The operator $\text{ad}(i(I_n^e))$ acts on $\mathfrak{z}_{N,m}$ with zero eigenvalues on $\mathbb{C}^m$, with even eigenvalues on $\mathfrak{so}_N$ and with eigenvalues $\{\pm 1\}$ on $V_N$.

However, there is no appropriate $h \in q$ in the case of $N = 2n + 1$. Instead, such a grading originates from the adjoint action of an element

$$g_0 := (-1, \ldots, -1, 1, \ldots, 1) \in O(N + 2m + 1).$$

This element defines a $\mathbb{Z}_2$-grading on $U(\mathfrak{so}_{N+2m+1})$, which naturally descends to a $\mathbb{Z}_2$-grading $Gr$ on the $W$-algebra $U(\mathfrak{so}_{N+2m+1}, e_m)$. The induced $\mathbb{Z}_2$-grading $Gr'$ on $U(\mathfrak{so}_{N+2m+1}, e_m)$ satisfies the desired properties, that is, $\deg(\mathbb{C}^m) = 0$, $\deg(\mathfrak{so}_N) = 2$, $\deg(V_N) = 1$.

As a result, the algebra $U(\mathfrak{so}_{N+2m+1}, e_m)$ is equipped with a Kazhdan filtration and a $\mathbb{Z}_2$-grading $Gr$. The rest of the proof proceeds in the same way as in [LT]. The only remaining fact to verify is the corresponding isomorphism at the Poisson level, which is Theorem 5.4.

Let us introduce some more notations:
- Let $\bar{i} : \mathfrak{so}_N \oplus V_N \oplus \mathbb{C}^m \sim \mathfrak{z}_{N,m}$ denote the isomorphism from the proof of Theorem 5.3.
- Let $H_m(\mathfrak{so}_N, V_N)$ be the Poisson counterpart of $H_m(\mathfrak{so}_N, V_N)$ (compare to algebras $H_m^cl(\mathfrak{so}_N, V_N)$).
- Define $P_j \in \mathbb{C}[\mathfrak{so}_{N+2m+1}]$ by $\text{det}(I_N^e + 2m + 1 + tA) = \sum_{j=0}^{N+2m+1} P_j(A)t^j$.
- Define $\{\Theta_i\}_{i=0}^{m-1} \in \mathfrak{z}_{N,m} \cong \mathfrak{c}[S_{N,m}]$ by $\Theta_i := P_{2(m-i)}|_{S_{N,m}}$.

The following result can be considered as a Poisson version of Theorem 5.3.

**Theorem 5.4.** The formulas

$$\bar{\Theta}^cl(A) = \bar{i}(A), \quad \bar{\Theta}^cl(y) = (-1)^m \frac{1}{2} \cdot \bar{i}(y), \quad \bar{\Theta}^cl(\xi_k) = (-1)^{m-k} \bar{\Theta}_{k}$$

8 In this section, we view $\mathfrak{so}_N$ as corresponding to the pair $(V_N, (\cdot, \cdot))$, where $(\cdot, \cdot)$ is represented by the symmetric matrix $J^{(1)} = (J^{(1)}_{ij})$ with $J^{(1)}_{ij} = \delta_{i,j}^l, J^{(1)}_{i,N+k+l} = 0, J^{(1)}_{N+k,N+i} = \delta_{k,l}^{2m+2}, \forall i, j \leq N, k, l \leq 2m+1$.

9 Actually, as exhibited by the case of $\mathfrak{sp}_{2n+2m}$, it suffices to have a $\mathbb{Z}_2$-grading.
The right hand side of (5) can be written as 

\[ H_m^{cl}(\mathfrak{so}_N, V_N) \cong S(3N, m) \cong \mathbb{C}[S_{N,m}] \] of Poisson algebras.

The proof of this theorem proceeds along the same lines as for \( \mathfrak{sp}_{2N} \) (see [11], Theorem 3.1). The only ingredient needed is Corollary 1.3.

### 6. Casimir element

In this section we construct the first nontrivial central element of the algebras \( H_\zeta(\mathfrak{so}_N, V_N) \).

In the non-deformed case (for \( \zeta = 0 \)) we have \( t_1 := (v, v) \in Z(H_0(\mathfrak{so}_N, V_N)) \). Similarly to Corollary 1.3 this element can be deformed to a central element of \( H_\zeta(\mathfrak{so}_N, V_N) \) by adding an element of \( Z(U(\mathfrak{so}_N)) \).

In order to formulate the result, we introduce some more notations:
- Define \( \omega_s := \frac{\pi^{1/2} s (s + 1)}{2^s s!} \) and \( \mu_s := \pi^{s-1/2} (s + 1)! \omega_s^{-1} \), \( \nu_s := -\frac{\mu}{s + 1} \).
- Define a sequence \( \{\zeta^m_j\}_{j=0}^m \) of parameters recursively via \( \zeta^m_j := 2\nu_{2j+1} \sum_{l=1}^{m+1-j} (-1)^{j+l+1} \binom{2j+2l}{2l} a_{j+l-1} \).
- Define a sequence of parameters \( \{a_j\}_{j=0}^m \) recursively via \( a_j := 2\nu_{2j+1} \sum_{l=1}^{m+1-j} (-1)^{j+l+1} \binom{2j+2l}{2l} a_{j+l-1} \).
- Define a polynomial \( g(z) := \sum_{j=1}^{m+1} a_j z^j \).
- Define \( A(z)(x, y) := (x, A(1 + z^2 A^2)^{-1} y) \) and \( B(z) := \det(1 + z^2 A^2)^{-1/2} \).
- Let \( [z^m]f(z) \) denote the coefficient of \( z^m \) in the series \( f(z) \).
- Define \( C \in Z(U(\mathfrak{so}_N)) \) as the symmetrization of \( \text{Res}_{z=0} g(z^2) \det(1 + z^2 A^2)^{-1/2} z^{-1} dz \).

Then we have:

**Theorem 6.1.** The element \( t_1 := t_1 + C \) is a central element of \( H_\zeta(\mathfrak{so}_N, V_N) \).

**Definition 6.2.** We call \( t_1 = t_1 + C \) the Casimir element of \( H_\zeta(\mathfrak{so}_N, V_N) \).

**Remark 6.3.** The same formula provides a central element of the algebra \( H_m(\mathfrak{so}_N, V_N) \), where \( C \in Z(U(\mathfrak{so}_N))(\zeta_0, \ldots, \zeta_{m-1}) \).

One can use the above Theorem to establish explicitly the isomorphism \( \Theta \) of Theorem 5.3 in the same way as this has been achieved in [11] Section 4.6 for the \( \mathfrak{gl}_n \) case.

**Proof of Theorem [6.1]**

We need to prove that \( t_1 \) commutes both with \( \mathfrak{so}_N \) and \( V_N \). The former is obvious:

\[ [t_1, \mathfrak{so}_N] = 0 \in H_0(\mathfrak{so}_N, V_N) \Rightarrow [t_1, \mathfrak{so}_N] = 0 \in H_\zeta(\mathfrak{so}_N, V_N) \Rightarrow [t_1, \mathfrak{so}_N] = 0 \in H_\zeta(\mathfrak{so}_N, V_N). \]

Let us now verify \( [t_1 + C, x] = 0 \) for any \( x \in V_N \).

Identifying \( U(\mathfrak{so}_N) \) with \( S(\mathfrak{so}_N) \) via the symmetrization map and recalling (*) we get:

\[ \left[ \sum_i x_i^2, x \right] = \sum_i x_i \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} (x_i, J_{p,q} x) \left( \int_{-\pi}^\pi \frac{2c(\theta)}{\cos \theta \cdot v} d\theta \right) \right] dq dp + \sum_i \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} \left( \int_{-\pi}^\pi \frac{2c(\theta)}{\cos \theta \cdot v} d\theta \right) (x_i, J_{p,q} x) x_{i} dq dp. \]

Since \( x_i (x_i, J_{p,q} x) = J_{p,q} x \) and \( v e^{th_\zeta} = e^{th_\zeta}(\cos \theta \cdot v - \sin \theta \cdot J_{p,q} v) \), \( \forall v \in V_N \):

\[ (5) \quad [t_1, x] = \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} \left( \int_{-\pi}^\pi \frac{2c(\theta)}{\cos \theta \cdot v} d\theta \right) \left( (x, J_{p,q} x)(\sin \theta \cdot x + (1 + \cos \theta) \cdot J_{p,q} x) \right) dq dp. \]

The right hand side of (5) can be written as \( [x, C'] \), where

\[ C' := \int_{p \in S^{N-1}} \int_{q \in S^{N-2}(p)} \left( \int_{-\pi}^\pi c(\theta)(-2 - 2 \cos \theta)e^{th_\zeta} \right) dq dp. \]
Thus, it suffices to prove $C' = C$.

The following has been established during the proof of Theorem 1.6

$$\int_{p \in S_{N-1}} \int_{q \in S_{N-2}(p)} J_{p,q}^s dq dp = F_{s-1} = \mu_{s-1}[z^s]B(z),$$

(6)

$$\int_{p \in S_{N-1}} \int_{q \in S_{N-2}(p)} (x, J_{p,q})J_{p,q}^s dq dp = I_{s, x, y} = \nu_s[z^{s-1}]A(z)(x, y).$$

(7)

Let $c(\theta) = c_0 \delta_0 + c_2 \delta_0'' + c_4 \delta_0^{(4)} + \ldots$ be the distribution from (11), where $\delta_k^{(k)}$ is the $k$-th derivative of the delta-function. Since

$$\int_{-\pi}^{\pi} 2c(\theta) \sin \theta \delta_j d\theta = 2 \sum_{j \geq 1} c_j \sum_{l=1}^{\lfloor j/2 \rfloor} (-1)^{l+1} \binom{j}{2l-1} J_{p,q}^{j-2l+1},$$

formulas (11) and (7) imply

$$[x, y] = \text{Res}_{z=0} \zeta(z^{-2})A(z)(x, y)z^{-1}dz,$$

where $\zeta(z^{-2}) = \sum_{j \geq 0} \bar{\zeta}(z^{-2})^j$ and $\bar{\zeta} = 2\nu_{2j+1} \sum_{l \geq 1} (-1)^{l+1} \binom{2j+2l}{2l-1} c_{2j+2l}$.

Comparing with $[x, y] = \text{Res}_{z=0} \zeta(z^{-2})A(z)(x, y)z^{-1}dz$, we get $\zeta(z^{-2}) = \zeta(z^{-2})$ and so $c_{2s+2} = a_s$, where $a_{s, r} := 0$. On the other hand,

$$\int_{-\pi}^{\pi} c(\theta)(-2 \cos \theta - 2)c_j \delta_j d\theta = 2 \sum_{j \geq 0} c_j \left( -2J_{p,q}^j + \sum_{l=1}^{\lfloor j/2 \rfloor} (-1)^{l+1} \binom{j}{2l} J_{p,q}^{j-2l} \right).$$

Combining this equality with (11), we find:

$$C' = \text{Res}_{z=0} g(z^{-2})B(z)z^{-1}dz = C.$$  

This completes the proof of the theorem. $\square$

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