LINEAR INDEPENDENCE OF RATIONALLY SLICE KNOTS

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Abstract. A knot in $S^3$ is rationally slice if it bounds a disk in a rational homology ball. We give an infinite family of rationally slice knots that are linearly independent in the knot concordance group. In particular, our examples are all infinite order. All previously known examples of rationally slice knots were order two.

1. Introduction

The knot concordance group, denoted by $\mathcal{C}$, consists of knots in $S^3$ modulo smooth concordance. It is well known that the connected sum operation endows $\mathcal{C}$ with the structure of an abelian group, and the identity is the equivalence class of slice knots, that is, knots which bound smoothly embedded disks in $B^4$.

If one only requires the concordance to be smoothly embedded in a rational homology cobordism between two 3-spheres, then we obtain the rational concordance group, denoted by $\mathcal{C}_{\mathbb{Q}}$. Similarly, a knot is called a rationally slice knot if it represents the identity in $\mathcal{C}_{\mathbb{Q}}$, or equivalently, if it bounds a smoothly embedded disk in a rational homology ball. Note that two concordant knots are rationally concordant. Hence we obtain the following natural surjective map:

$$\psi: \mathcal{C} \to \mathcal{C}_{\mathbb{Q}}.$$  

Cochran, based on work of Fintushel-Stern [FSS84], showed that the figure-eight knot is rationally slice. This implies that $\text{ker} \, \psi \geq \mathbb{Z}/2\mathbb{Z}$ since the figure-eight knot is negative-amphichiral and not slice. Moreover, Cha [Cha07] Theorem 4.16 extended this result by showing that

$$\text{ker} \, \psi \geq (\mathbb{Z}/2\mathbb{Z})^\infty.$$  

It is natural to ask if $\text{ker} \, \psi$ contains an infinite order element (see e.g. [Ush11, Problem 1.11]). In this article, we answer this question by using the involutive knot Floer package of Hendricks-Manolescu [HM17].

Theorem 1.1. The group $\text{ker} \, \psi$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$.

As mentioned above, if a non-slice knot is concordant to a negative-amphichiral knot, then the knot represents an order two element in $\mathcal{C}$ (in fact, the converse of this statement was asked by Gordon [Hau78, Problem 16], and to the best of the authors’ knowledge, remains open).

Moreover, a knot $K$ is called strongly negative-amphichiral if there is an orientation-reversing involution $\tau: S^3 \to S^3$ such that $\tau(K) = K$. If $\tau$ is a orientation-reversing involution on $S^3$, then by Smith theory the fixed point set of $\tau$ is either $S^2$ or $S^0$. For the former case, the knot is isotopic to $J\# - J$ for some knot $J$ where $-J$ is the reverse of the mirror image of $J$. In particular, it is slice. For the case where the fixed point set of $\tau$ is $S^0$, Kawauchi [Kaw09, Section 2] showed that the knot is rationally slice. In conclusion, if a knot is concordant to a strongly negative-amphichiral knot, then the knot is rationally slice. Hence Theorem 1.1 can be interpreted as that the converse of this statement is far from being true.

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Also, recall that Levine [Lev69a, Lev69b] constructed a surjective homomorphism
\[ \phi: C \to \mathcal{AC} \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty, \]
where \( \mathcal{AC} \) is the \textit{algebraic concordance group}. He proved that the corresponding homomorphism is an isomorphism for the higher odd-dimensional knot concordance group (recall that the higher even-dimensional knot concordance group is trivial [Ker65]). Moreover, he showed that if a knot has vanishing Levine-Tristram signature function, then the knot represents a torsion element in \( \mathcal{AC} \) [Lev69a, Section 22], and Cha-Ko [CK02, Theorem 1.1] showed that the Levine-Tristram signature function vanishes for rationally slice knots. Hence every higher dimensional rationally slice knot represents a finite order element in the knot concordance group. In particular, Theorem 1.1 illustrates a significant difference between the classical knot concordance group and the higher dimensional knot concordance group. Recall that this fact was first proved by Casson-Gordon [CG78, CG86], where they showed that the Levine’s homomorphism is not an isomorphism in the classical dimension. We summarize the above discussion as follows.

**Corollary 1.2.** There is a family of rationally slice knots which generates a \( \mathbb{Z}^\infty \)-subgroup of \( \ker \phi \).

The proof of Theorem 1.1 uses the involutive knot Floer package defined by Hendricks-Manolescu [HM17]. The involutive knot Floer complex is obtained by considering an action \( \iota_K \) on the knot Floer complex \( CFK_{\mathbb{F}[U,V]} \) of [OS04] and [Ras03]. We refer to the pair \((CFK_{\mathbb{F}[U,V]}(K), \iota_K)\) as the \( \iota_K \)-complex of a knot \( K \). Moreover, Zemke [Zem19b] (see also [Zem19a, Theorem 1.5]) showed that up to an algebraic equivalence called \textit{local equivalence}, the \( \iota_K \)-complex of a knot is a concordance invariant. We use a slightly coarser equivalence relation called \textit{almost local equivalence}, which is motivated by [DHST18] (see also [DHST21]), to show the following. Note that the following theorem implies Theorem 1.1 immediately since the \((p,−1)\)-cable of a rationally slice knot is rationally slice.

**Theorem 1.3.** If \( K_n \) is the \((2n−1,−1)\)-cable of the figure-eight knot for \( n \geq 2 \), then the \( \iota_K \)-complex of a non-trivial linear combination of \( K_n \) is never locally equivalent to the \( \iota_K \)-complex of the unknot.

The explicit computation is done by first computing the knot Floer complexes of the knots \( K_n \) over \( \mathbb{F}[U,V]/(U,V) \) using bordered Floer homology [LOT18], interpreted in terms of immersed curves as in [HRW16, HRW18, HW19]. Then we lift this computation to \( \mathbb{F}[U,V] \) by using \( \partial^2 = 0 \) and determine the absolute grading by using computations of [Pet13]. Lastly, in Section 4 we use formal properties of \( \iota_K \) to obtain some partial computation of the \( \iota_K \)-complex. It turns out that the partial computation is enough to obtain Theorem 1.3.

**Remark 1.4.** The odd cabling parameter in Theorem 1.3 is essential for the proof, as it guarantees a certain asymmetry in the almost local equivalence class; an even cabling parameter, i.e., the \((2n,−1)\)-cable for the figure-eight knot, yields a trivial almost local equivalence class.

We also remark that it was previously known that any non-trivial linear combination of \( K_n \) is not ribbon (in fact not homotopy ribbon), which is implicit in [KW18, Theorem 1.2] where they use Miyazaki’s result [Miy94, Theorem 8.5.1 and 8.6] to show that any non-trivial linear combination of \( S^3 \times I \) from \( K \) to the unknot \( U \). After performing a \( ±1 \) surgery along the concordance, we see that \( S^3_{±1}(K) \) is rational homology cobordant to \( S^3_{±1}(U) \cong S^3 \); so if \( K \) is rationally slice, then both \( ±1 \) surgeries on \( K \) bound rational homology balls. Hence the \( \tau \)-invariant [OS03], \( \varepsilon \)-invariant [Hom14], \( \Upsilon \)-invariant [OSS17], \( \Upsilon^2 \)-invariant [KL18], \( s_n \)-invariant [HW16], and \( \varphi_j \)-invariants [DHST21] all vanish for \( K \) and its mirror. (On the other hand, it is not known if \( s \)-invariant [Ras10], \( s_n \)-invariant [Wu09, Lob09, Lob12],
\(J\)-invariant \([LL19]\), or \(s^\#\)-invariant \([KM13]\) vanish for rationally slice knots.) As mentioned earlier, \(K\) has vanishing Levine-Tristram \([Lev69b]\, [Tri69]\) signature function and represents a finite order element in the algebraic concordance group.

Regardless, we show that there exist rationally slice knots with arbitrarily large concordance unknotting number. We show this by using the \(\iota_K\)-connected knot Floer homology, which is an analogue of the connected Heegaard Floer homology of \([HHL13]\), and a lower bound on the unknotting number from \([AE20, \text{Theorem 1.1}]\) (see also \([Zem19b]\)). Recall that the concordance unknotting number \(u_c(K)\), is defined to be

\[
u_c(K) = \min\{u(K') \mid K \text{ and } K' \text{ are concordant}\},
\]

where \(u(K')\) is the unknotting number of \(K'\).

**Corollary 1.5.** If \(K_n\) is the \((2n - 1, -1)\)-cable of the figure-eight knot for \(n \geq 2\), then

\[
u_c(K_n) \geq n.
\]

The 4-ball genus of \(K\) is the minimal genus of a smooth orientable surface in \(B^4\) bounding \(K\), and the 4-dimensional clasp number of \(K\), denoted by \(c_4(K)\), is the minimal number of transverse double points of a smoothly immersed disk in \(B^4\) bounding \(K\). Then we have

\[
u_c(K) \geq c_4(K) \geq g_4(K).
\]

It is natural to ask the following question.

**Question 1.6.** Does there exist a family of rationally slice knots with arbitrarily large \(g_4\) or \(c_4\)?

We learned that Allison N. Miller recently obtained an affirmative answer to Question 1.6 by using Casson-Gordon invariants.

Even though in this article we only focus on the difference between the concordance group \(C\) and the rational concordance group \(C_Q\), one can also consider \(C_Z\) and \(C_{Z/2Z}\), which are defined in a similar way. Again, we have the following natural surjective maps:

\[
\psi_1 : C \to C_Z \quad \text{and} \quad \psi_2 : C \to C_{Z/2Z}.
\]

**Question 1.7.** Is the group \(\ker \psi_1\) or \(\ker \psi_2\) non-trivial?

**Remark 1.8.** If \(K\) is a knot in \(S^3\) which bounds a slice disk \(D\) in a rational homology ball \(W\), then for each spin\(^c\)-structure \(s\) on \(W\), we get concordance maps \(F_{W,D,s} : CFK_{F[U,V]}(K) \to F[U,V]\) and \(F_{W,-D,s} : F[U,V] \to CFK_{F,U,V}(K)\). These maps would satisfy the homotopy-commutativity conditions \(F_{W,D} \circ \iota_K \sim F_{W,D}\) and \(\iota_D \circ F_{W,-D} \sim F_{W,-D}\) if \(s\) is self-conjugate, i.e. \(s = \overline{s}\). One can always find such an \(s\) if \(W\) is a \(Z/2Z\)-homology sphere, in which case one can take \(s\) to be the unique spin structure on \(W\). Hence, Theorem 1.3 in fact shows that any non-trivial linear combination of \(K_n\) represents a non-trivial element in not only \(C_Z\), but also in \(C_{Z/2Z}\). On the other hand, \(K_n\) represents the identity in \(C_{Z/pZ}\) for any odd \(p\).

**Organization**

The paper is organized as follows. In Section 2 we recall the definitions and some facts about the knot Floer complex and the involutive knot Floer complex. In Section 3 we compute the knot Floer complex for \(K_n\), and in Section 4 we do a partial computation of the involutive knot Floer complex for \(K_n\) and prove Theorem 1.3. Lastly, in Section 5 we define \(\iota_K\)-connected knot Floer homology and prove Corollary 1.5.

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2. Background

We assume the reader is familiar with knot Floer homology, defined in [OS04] and [Ras03]. Much of our notation comes from [Zem19a], particularly Section 2 of that paper. We briefly recall the construction, primarily to establish notation. Throughout, \( \mathbb{F} = \mathbb{Z}/2\mathbb{Z} \).

Let \( \mathcal{H} = (\Sigma, \alpha, \beta, w, z) \) be a double-pointed Heegaard diagram compatible with a knot \( K \subset S^3 \). Consider the two variable polynomial ring \( \mathbb{F}[U, V] \). This ring is bigraded by \( \text{gr} = (\text{gr}_U, \text{gr}_V) \) where

\[
\text{gr}(U) = (-2, 0) \quad \text{and} \quad \text{gr}(V) = (0, -2).
\]

Let \( \mathcal{R} = \mathbb{F}[U, V] \) or \( \mathbb{F}[U, V]/(UV) \), where \((UV)\) is the ideal generated by \( UV \).

We now define the knot Floer complex (over \( \mathcal{R} \)), denoted \( \text{CFK}_\mathcal{R} \). Consider the chain complex \( \text{CFK}_\mathcal{R}(\mathcal{H}) \) freely generated over \( \mathcal{R} \) by intersection points \( T_\alpha \cap T_\beta \subseteq \text{Sym}^3 \Sigma \) with differential

\[
\partial x = \sum_{y \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta} \sum_{\phi \in \pi_2(x, y)} \sum_{\mu(\phi) = 1} \text{#}(\mathcal{M}(\phi)) \, U^{\mu w(\phi)} V^{\mu z(\phi)} y.
\]

The following relations give this chain complex a relative bigrading:

\[
\text{gr}_U(x) - \text{gr}_U(y) = \mu(\phi) - 2n_w(\phi)
\]

\[
\text{gr}_V(x) - \text{gr}_V(y) = \mu(\phi) - 2n_z(\phi),
\]

where \( \phi \in \pi_2(x, y) \). Setting \( V = 1 \) on \( \text{CFK}_\mathbb{F}[U, V] \), forgetting \( \text{gr}_V \), and taking homology recovers \( \text{HF}^-(S^3) \cong \mathbb{F}[U] \). We pin down the absolute \( U \)-grading by setting \( \text{gr}_U(1) = 0 \). The absolute \( V \)-grading is determined analogously, reversing the roles of \( U \) and \( V \).

The chain homotopy type of this chain complex is a knot invariant, denoted \( \text{CFK}_\mathcal{R}(K) \). It is often convenient to consider the Alexander grading

\[
A = \frac{1}{2} (\text{gr}_U - \text{gr}_V).
\]

The \( U \)-grading \( \text{gr}_U \) is often called the Maslov grading. Note that

\[
\text{gr}_U(\partial x) = \text{gr}_U(x) - 1
\]

\[
\text{gr}_V(\partial x) = \text{gr}_V(x) - 1
\]

\[
A(\partial x) = A(x).
\]

See Figure 1 for an example of a graphical depiction of the knot Floer complex. Note that horizontal arrows of length \( n \) encode terms in the differential with coefficients of the form \( U^n \), vertical arrows of length \( n \) encode terms in the differential with coefficients of the form \( V^n \), and diagonal arrows encode terms with coefficients with nonzero \( U \) and nonzero \( V \) exponents. (See Figure 2 for an example with diagonal arrows.)

The minus and hat flavors of knot Floer homology are defined as

\[
\text{HF}^-(K) := H_*(\text{CFK}_\mathbb{F}[U, V](K)/(V))
\]

\[
\text{HF}^-(K) := H_*(\text{CFK}_\mathbb{F}[U, V](K)/(U, V)).
\]

Consider the knot Floer complex over \( \mathbb{F}[U, V] \) equipped with a basis \( B \). The \( U \)-derivative of an element \( U^N V^J x \) for \( x \in B \) is \( iU^{N-1}V^J x \). We write

\[
D_U : \text{CFK}_\mathbb{F}[U, V] \to \text{CFK}_\mathbb{F}[U, V]
\]

for the \( \mathbb{F}[V] \)-linear map given by the \( U \)-derivative. We define the \( V \)-derivative \( D_V \) analogously, namely \( D_V(U^N V^J x) = jU^N V^{J-1} x \) for \( x \in B \). Let

\[
\Phi = [\partial, D_U] \quad \text{and} \quad \Psi = [\partial, D_V].
\]
It is straightforward to verify that $\Phi$ and $\Psi$ are $\mathbb{F}[U, V]$-equivariant chain maps and that
\[
\begin{align*}
gr_U(\Phi(x)) &= gr_U(x) + 1 \\
gr_V(\Phi(x)) &= gr_V(x) - 1 \\
gr_U(\Psi(x)) &= gr_U(x) - 1 \\
gr_V(\Psi(x)) &= gr_V(x) + 1.
\end{align*}
\]
Up to chain homotopy, the maps $\Phi$ and $\Psi$ are independent of the choice of basis $B$ [Zem19a, Corollary 2.9].

Example 2.1. Consider $\text{CFK}_F[U, V]$ of the figure-eight knot, as in Figure 1, which is generated by $a, b, c, d, e$. We have:

|   | $\partial$ | $\Phi$ | $\Psi$ | $\Psi \circ \Phi$ | $gr_U$ | $gr_V$ | $A$ |
|---|------------|--------|--------|-------------------|--------|--------|-----|
| $a$ | 0          | 0      | 0      | 0                 | 0      | 0      | 0   |
| $b$ | $Uc + Vd$ | $c$    | $d$    | $e$               | 0      | 0      | 0   |
| $c$ | $Ve$      | 0      | $e$    | 0                 | 1      | $-1$   | 1   |
| $d$ | $Ue$      | $e$    | 0      | 0                 | $-1$   | 1      | $-1$ |
| $e$ | 0          | 0      | 0      | 0                 | 0      | 0      | 0   |

Hendricks and Manolescu [HM17] define a chain map
\[
\iota_K : \text{CFK}_F[U, V] \to \text{CFK}_F[U, V],
\]
which satisfies the following properties:

1. The map $\iota_K$ is $\mathbb{F}[U, V]$-skew-equivariant, which means that
   \[
   \iota_K(Ux) = V\iota_K(x) \quad \text{and} \quad \iota_K(Vx) = U\iota_K(x).
   \]
2. The map $\iota_K$ is skew-graded, which means that
   \[
   gr_U(\iota_K(x)) = gr_V(x) \quad \text{and} \quad gr_V(\iota_K(x)) = gr_U(x).
   \]
3. Lastly, $\iota_K^2 \simeq 1 + \Psi \Phi$.

Here, $\simeq$ indicates that two chain maps are chain homotopic via $\mathbb{F}[U, V]$-equivariant chain homotopy.

The aforementioned results motivate the following abstract definitions.

Definition 2.2. An $\iota_K$-complex $(C, B, \iota)$ consists of a free, finitely generated, bigraded chain complex $C$ over $\mathbb{F}[U, V]$ with basis $B$ such that

1. the localization $(U, V)^{-1}C$ has homology isomorphic to $(U, V)^{-1}\mathbb{F}[U, V]$,
2. the map $\iota : C \to C$ is a $\mathbb{F}[U, V]$-skew-equivariant, skew-graded chain map,
3. $\iota^2 \simeq 1 + \Psi \Phi$.

Definition 2.3. Two $\iota_K$-complexes $(C_1, B_1, \iota_1)$ and $(C_2, B_2, \iota_2)$ are homotopy equivalent, denoted $(C_1, B_1, \iota_1) \simeq (C_2, B_2, \iota_2)$, if there exist $\mathbb{F}[U, V]$-equivariant graded chain maps
\[
f : C_1 \to C_2 \quad \text{and} \quad g : C_2 \to C_1
\]
such that
\[
f \iota_1 \simeq \iota_2 f \quad \text{and} \quad g \iota_2 \simeq \iota_1 g.
\]
and \( fg \simeq \text{id} \) and \( gf \simeq \text{id} \).

Here, \( \simeq \) indicates that two chain maps are chain homotopic via \( F[U, V] \)-skew-equivariant chain homotopy.

We will often omit the basis \( B \) from our notation; up to homotopy equivalence, this does not cause any ambiguity. Again, the chain homotopy type of \( \iota_K \)-complex \( (\text{CFK}_{F[U, V]}(K), \iota_K) \) is a knot invariant, and we call the pair the \( \iota_K \)-complex of a knot \( K \). In general, we will be interested in \( \iota_K \)-complexes up to homotopy equivalence or often an even weaker notion of equivalence, such as local equivalence or almost local equivalence, defined below.

**Definition 2.5.** Given two \( \iota_K \)-complexes \( (C_1, \iota_1) \) and \( (C_2, \iota_2) \), the products \( \times_1 \) and \( \times_2 \):
\[
(C_1, \iota_1) \times_1 (C_2, \iota_2) = (C_1 \otimes F[U, V] C_2, t_1 \otimes t_2 + (\Phi_1 \otimes \Psi_2) \circ (\iota_1 \otimes \iota_2))
\]
\[
(C_1, \iota_1) \times_2 (C_2, \iota_2) = (C_1 \otimes F[U, V] C_2, t_1 \otimes t_2 + (\Psi_1 \otimes \Phi_2) \circ (\iota_1 \otimes \iota_2))
\]
The products \((C_1, \iota_1) \times_1 (C_2, \iota_2)\) and \((C_1, \iota_1) \times_2 (C_2, \iota_2)\) are homotopy equivalent \( \iota_K \)-complexes by \cite{Zem19a} Lemmas 2.13 and 2.14.

The \( \iota_K \)-complex of the connected sum of two knots is given by the product (in the above sense) of their respective \( \iota_K \)-complexes \cite{Zem19a} Theorem 1.1:
\[
(\text{CFK}_{F[U, V]}(K_1 \# K_2), \iota_{K_1 \# K_2}) \simeq (\text{CFK}_{F[U, V]}(K_1), \iota_{K_1}) \times_1 (\text{CFK}_{F[U, V]}(K_2), \iota_{K_2})
\]
\[
\simeq (\text{CFK}_{F[U, V]}(K_1), \iota_{K_1}) \times_2 (\text{CFK}_{F[U, V]}(K_2), \iota_{K_2}).
\]

The following equivalence relation is particularly well-suited for studying knot concordance:

**Definition 2.6.** Given two \( \iota_K \)-complexes \( (C_1, \iota_1) \) and \( (C_2, \iota_2) \), a bigraded chain map
\[
f : C_1 \to C_2
\]
is called a local map if
\[
(1) \ f \iota_1 \simeq \iota_2 f, \\
(2) \ f \text{ induces an isomorphism on } H_*((U, V)^{-1}C).
\]
If there exist local maps \( f : C_1 \to C_2 \) and \( g : C_2 \to C_1 \), then we say that \((C_1, \iota_1)\) and \((C_2, \iota_2)\) are locally equivalent, and the maps \( f \) and \( g \) are local equivalences.

Zemke \cite{Zem19b} (see also \cite{Zem19a} Theorem 1.5) shows that a concordance between \( K_0 \) and \( K_1 \) induces local maps
\[
f : \text{CFK}_{F[U, V]}(K_0) \to \text{CFK}_{F[U, V]}(K_1) \quad \text{and} \quad g : \text{CFK}_{F[U, V]}(K_1) \to \text{CFK}_{F[U, V]}(K_0).
\]
Hence if \( K_0 \) and \( K_1 \) are concordant, their \( \iota_K \)-complexes are locally equivalent.

The set of \( \iota_K \)-complexes modulo local equivalence, with operation induced by either \( \times_1 \) or \( \times_2 \), forms a group \cite{Zem19a} Proposition 2.6, which we denote \( \mathcal{J}_K \). By \cite{Zem19a} Theorem 1.5, we have that
\[
C \to \mathcal{J}_K \\
[K] \mapsto [(\text{CFK}_{F[U, V]}(K), \iota_K)]
\]
is a well-defined group homomorphism.

It can sometimes be cumbersome to work with \( \iota_K \)-complexes. Let \((U, V)\) denote the ideal generated by \( U \) and \( V \). Motivated by \cite{DHST18} (see also \cite{DHST21}), we make the following more relaxed definition:

**Definition 2.6.** An almost \( \iota_K \)-complex \((C, \iota)\) consists of a free, finitely generated, bigraded chain complex \( C \) over \( F[U, V] \) such that
\[
(1) \ \text{the localization } (U, V)^{-1}C \text{ has homology isomorphic to } (U, V)^{-1}F[U, V],
(2) \ \text{the map } \iota : C/(U, V) \to C/(U, V) \text{ is an } F \text{-linear, skew-graded chain map},
\]
(3) $t^2 \simeq 1 + \Psi \Phi \mod (\mathcal{U}, \mathcal{V})$.

Note that the above definition is obtained by adding “mod $(\mathcal{U}, \mathcal{V})$” to every statement about $t$ in Definition 2.2. The almost $\iota_\mathcal{K}$-complex obtained from the $\iota_\mathcal{K}$-complex of $K$ is called the almost $\iota_\mathcal{K}$-complex of $K$. Moreover, we may define homotopy equivalence of two almost $\iota_\mathcal{K}$-complexes as in Definition 2.2 “mod $(\mathcal{U}, \mathcal{V})$”. Following this approach, we analogously modify Definition 2.5:

**Definition 2.7.** Given two almost $\iota_\mathcal{K}$-complexes $(C_1, \iota_1)$ and $(C_2, \iota_2)$, a bigraded chain map $f : C_1 \to C_2$

is called an almost local map if

1. $f \iota_1 \simeq \iota_2 f \mod (\mathcal{U}, \mathcal{V})$,
2. $f$ induces an isomorphism on $H_0((\mathcal{U}, \mathcal{V})^{-1}C_i)$.

If there exist almost local maps $f : C_1 \to C_2$ and $g : C_2 \to C_1$, then we say that $(C_1, \iota_1)$ and $(C_2, \iota_2)$ are almost locally equivalent, and the maps $f$ and $g$ are almost local equivalences.

**Example 2.8.** Consider the following two $\iota$-complexes: $C_1 = \mathbb{F}[\mathcal{U}, \mathcal{V}]$ generated by $x$ in grading $(0, 0)$ with $\iota_1 = \text{id}$, and $C_2$ generated by $a, b, c, d, e$, and $e$ with

|   | $\partial$ | $\iota_\mathcal{K}$ | $\text{gr}_\mathcal{U}$ | $\text{gr}_\mathcal{V}$ | $A$ |
|---|------------|----------------|----------------|----------------|-----|
| $a$ | $0$ | $a + \mathcal{U}^2 \mathcal{V}^2 c$ | $0$ | $0$ | $0$ |
| $b$ | $\mathcal{U}^3 c + \mathcal{V}^3 d$ | $b + a$ | $0$ | $0$ | $0$ |
| $c$ | $\mathcal{V}^3 e$ | $d$ | $5$ | $-1$ | $3$ |
| $d$ | $\mathcal{U}^3 e$ | $e$ | $-1$ | $5$ | $-3$ |
| $e$ | $0$ | $e$ | $4$ | $4$ | $0$ |

Then the map $f : C_1 \to C_2$ sending $x$ to $a$ and the map $g : C_2 \to C_1$ sending $a$ to $x$ and $b, c, d, e$ to 0 provide almost-local equivalences between $(C_1, \iota_1)$ and $(C_2, \iota_2)$. However, the two $\iota$-complexes are not locally equivalent; for example, the Hendricks-Manolescu $V_0$-invariant of the two complexes are different. Note that $C_2$ is locally equivalent to the $\iota_\mathcal{K}$-complex of $-2T_{6,7} \# T_{6,13}$; see [HHTSZ20, Section 4.1].

We leave it as an exercise for the reader to verify that the proof that $\mathcal{J}_\mathcal{K}$ is a group [Zem19a, Proposition 2.6] readily adapts to show that almost $\iota_\mathcal{K}$-complexes modulo almost local equivalence, under the analogous operation, form a group. In analogy to [DHST18, Definition 3.15], we denote this group by $\hat{\mathcal{J}}_\mathcal{K}$. There is a forgetful homomorphism

$$\mathcal{J}_\mathcal{K} \to \hat{\mathcal{J}}_\mathcal{K}.$$  

(We do not actually need the group $\hat{\mathcal{J}}_\mathcal{K}$ in this paper.)

The proof of our main theorem uses the following simple observation: a local map $f : (C_1, \iota_1) \to (C_2, \iota_2)$ of $\iota_\mathcal{K}$ complexes induces an almost local map $\hat{f} : (C_1, \iota_1) \to (C_2, \iota_2)$. Equivalently, if there is no almost local map $(C_1, \iota_1) \to (C_2, \iota_2)$, then $(C_1, \iota_1)$ and $(C_2, \iota_2)$ are not locally equivalent. We summarize the above discussions in the following.

**Lemma 2.9.** If two knots $K_0$ and $K_1$ are concordant, then their almost $\iota_\mathcal{K}$-complexes are almost locally equivalent.

Lastly, we conclude this section with the following useful algebraic lemma.

**Definition 2.10.** A chain complex $C$ over $\mathbb{F}[\mathcal{U}, \mathcal{V}]$ is reduced if $\text{im} \partial \subset (\mathcal{U}, \mathcal{V})$.

**Lemma 2.11.** Let $f$ and $g$ be graded chain maps between reduced knot Floer complexes. Suppose that $f \simeq g \mod (\mathcal{U}, \mathcal{V})$. Then

$$f = g \mod (\mathcal{U}, \mathcal{V}).$$

The analogous statement holds for skew-homotopies as well.
Proof. Let \( f, g : C_1 \to C_1 \). If \( f \simeq g \mod (\mathcal{U}, \mathcal{V}) \), then there exists a homotopy \( H \) such that
\[
f = g + \partial H + H \partial \mod (\mathcal{U}, \mathcal{V}).
\]
If \( C_1 \) and \( C_2 \) are reduced, then \( \partial H + H \partial = 0 \mod (\mathcal{U}, \mathcal{V}) \). The skew-homotopy proof is identical. \( \square \)

3. The knot Floer complex of cables of the figure-eight knot

The goal of this section is to compute the knot Floer complex of \((2n-1,-1)\)-cables of the figure-eight knot for \( n \geq 2 \). Our main tool will be Hanselman and Watson’s cabling formula in terms of immersed curves [HW19]. Recall that the immersed curve associated to a knot complement determines the knot Floer complex of the knot over \( \mathbb{F}[\mathcal{U}, \mathcal{V}]/(\mathcal{UV}) \). We will use the fact that \( \partial^2 = 0 \) to lift this computation to \( \mathbb{F}[\mathcal{U}, \mathcal{V}] \). We use [Pet13] to determine the absolute gradings.

**Proposition 3.1.** Fix an integer \( n \geq 2 \). The knot Floer complex of the \((2n-1,-1)\)-cable of the figure-eight knot takes the form described in the following table, where \( 1 \leq i \leq n - 2 \):

| \( i \) | \( \partial \) | \( \text{gr}_\mathcal{U} \) | \( \text{gr}_\mathcal{V} \) | \( A \) |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 2 | \( U^n c + U \mathcal{V} d + \mathcal{V}^m e \) | 0 | 0 | 0 |
| 3 | \( V f \) | 2n - 1 | -1 | \( n \) |
| 4 | \( U^{n-1} f + \mathcal{V}^{n-1} g \) | 1 | 1 | 0 |
| 5 | \( U g \) | -1 | 2n - 1 | -n |
| 6 | 0 | 2n - 2 | 0 | \( n - 1 \) |
| 7 | 0 | 2n - 2 | -n + 1 |
| 8 | \( U^{n+i} c_{0,i} + U \mathcal{V} d_{0,i} + \mathcal{V}^{n-i} e_{0,i} \) | 0 | 0 | 0 |
| 9 | \( V f_{0,i} \) | 2n + 2i - 1 | -1 | \( n + i \) |
| 10 | \( U^{n+i-1} f_{0,i} + \mathcal{V}^{n-i-1} g_{0,i} \) | 1 | 1 | 0 |
| 11 | \( U g_{0,i} \) | -1 | 2n - 2i - 1 | -n + i |
| 12 | 0 | 2n + 2i - 2 | 0 | \( n + i - 1 \) |
| 13 | 0 | 2n - 2i - 2 | -n + i + 1 |
| 14 | \( U^{n-i} c_{1,i} + U \mathcal{V} d_{1,i} + \mathcal{V}^{n+i} e_{1,i} \) | 0 | 0 | 0 |
| 15 | \( V f_{1,i} \) | 2n - 2i - 1 | -1 | \( n - i \) |
| 16 | \( U^{n-i-1} f_{1,i} + \mathcal{V}^{n+i-1} g_{1,i} \) | 1 | 1 | 0 |
| 17 | \( U g_{1,i} \) | -1 | 2n + 2i - 1 | -n - i |
| 18 | 0 | 2n - 2i - 2 | 0 | \( n - i - 1 \) |
| 19 | 0 | 2n + 2i - 2 | -n - i + 1 |
| 20 | \( U^{2n-1} c_{0,n-1} + \mathcal{V} e_{0,n-1} \) | 0 | 0 | 0 |
| 21 | \( V f_{0,n-1} \) | 4n - 3 | -1 | 2n - 1 |
| 22 | \( U^{2n-1} f_{0,n-1} \) | -1 | 1 | -1 |
| 23 | \( f_{0,n-1} \) | 0 | 4n - 4 | 0 | 2n - 2 |
| 24 | \( U c_{1,n-1} + \mathcal{V}^{2n-1} e_{1,n-1} \) | 0 | 0 | 0 |
| 25 | \( V^{2n-1} g_{1,n-1} \) | 1 | -1 | 1 |
| 26 | \( U g_{1,n-1} \) | -1 | 4n - 3 | -2n + 1 |
| 27 | \( g_{1,n-1} \) | 0 | 4n - 4 | -2n + 2 |

See Figure 2 for a graphical depiction of the knot Floer complex of the \((7,-1)\)-cable of the figure-eight knot.

The following lemma will be of use for computing the absolute gradings in Proposition 3.1

**Lemma 3.2.** For \( p \geq 2 \), we have that \( HFK^- \) of the \((p,1)\)-cable of the figure-eight knot contains summands of the following form for \( 2 \leq i \leq p - 1 \):

\[
\mathcal{U}^{2n-1} \mathcal{V}^{2i} + \mathcal{V}^{2n-1} \mathcal{U}^{2i-1}.
\]
Note that the table in the above lemma is not a complete description of $HFK^-$ of the cable, but it is sufficient for what we need.

Proof. Petkova [Pet13] computes $HFK^-$ for $(p, pm + 1)$-cables of thin knots. In particular, given a $1 \times 1$ square in the knot Floer complex (equivalently, a “figure-eight” component as in the far left of Figure 4), she shows that $(p, 1)$-cabling produces two summands of the form $\mathbb{F}[U]/(U^i)$ for $2 \leq i \leq p - 2$ and $i = p$, one summand of the form $\mathbb{F}[U]/(U^{p-1})$, and many summands of the form $\mathbb{F}[U]/(U^{p-i})$ for $1 \leq i \leq p - 1$. The Maslov and Alexander gradings for the summands given in the table are consistent with this description.
\[F[U]/(U)\]. See Figure 3 for a graphical depiction of the summands produced by cabling a \(1 \times 1\) square.

\[
\begin{array}{cccccc}
  & ax_4 & b_1y_3 & b_jy_4 & b_{p-1}y_1 & b_{p-1}y_3 \\
  U^p & U^{p-i} & U^{p-j} & U & U \\
  & ax_3 & b_{2p-i-1}y_3 & b_{2p-j-1}y_4 & b_py_1 & b_py_3 \\
\end{array}
\]

\[
\begin{array}{cccccc}
  & b_1y_4 & ax_1 & U^p & b_{i+1}y_2 \\
  U^{p-i} & U & & U^{p-i} & U^{p-i-1} \\
  & b_{2p-2}y_4 & b_{1y_2} & b_{2p-2}y_2 & b_{2p-i-1}y_1 & b_{2p-i-2}y_2 \\
\end{array}
\]

(a) \hspace{2cm} (b)

**Figure 3.** Left, a graphical depiction of the differential in \[Pet13\] Section 5. Right, simplifications of the bottom two summands from (a). Here, \(1 \leq i \leq p-2\) and \(2 \leq j \leq p-2\).

For \(2 \leq i \leq p-2\) and \(i = p\), one can check, using the gradings given in \[Pet13\] Section 5, that the two \(F[U]/(U^i)\) summands are in opposite parity grading. For \(2 \leq j \leq p\), we will consider the \(F[U]/(U^i)\) summand in even grading. These summands are generated by the elements listed in Lemma 3.2. Their Maslov and Alexander gradings are readily computed using the gradings given in \[Pet13\] Section 5. (Since \(\tau\) of the figure-eight knot is zero and we are considering \((p, 1)\)-cables, Petkova’s shifting constant \(c\) is zero; hence Petkova’s \(A'\) is equal to \(A\). Recall that the Maslov grading \(M\) is equal to \(N + 2A\) in Petkova’s notation and note that \(l = n = t = \tau = 0\).) \(\square\)

We are now ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** We will use \[HW19\] Theorem 1, which describes a way to compute the immersed curve associated to the \((p, q)\)-cable of a knot \(K\) in terms of a simple three step process:

1. Draw \(p\) copies of the immersed curve for \(K\), each scaled vertically by a factor of \(p\), staggered in height such that each copy of the curve is \(q\) units lower than the previous copy.
2. Connect the loose ends of the successive copies of the curve.
3. Translate the pegs horizontally so that they lie on the same vertical line.

We will apply this procedure to the immersed curve \(\gamma\) associated to the figure-eight knot. The immersed curve \(\gamma = \{\gamma_0, \gamma_1\}\) consists of a horizontal line \(\gamma_0\) together with the “figure-eight” curve \(\gamma_1\) depicted in the far left of Figure 4. We label the intersecting point of \(\gamma_0\) and the vertical line as \(a\). The horizontal line \(\gamma_0\) is unaffected by \((2n - 1, -1)\)-cabling, so we will be interested in what \(\gamma_1\) looks like after cabling. Note that \(\gamma_1\) does not have loose ends, so we may skip step (2) above.

Fix \(n \geq 2\) and let \(p = 2n - 1\). The Hanselman-Watson algorithm is illustrated in Figure 4 for \(n = 4\). We begin by drawing \(p\) copies of \(\gamma_1\), each scaled by a factor of \(p\) and staggered in height such that each copy is one unit higher than the previous copy. We label these curves, from left to right, as

\[\gamma_{1,n-1}, \gamma_{1,n-2}, \ldots, \gamma_{1,1}, \gamma_{1,0}\].

We abuse notation and use the same labels for the resulting curves after applying step (3) above.
Recall that intersection points of the immersed curve with the vertical line correspond to generators of $\text{CFK}_{F[U,V]/(UV)}$ (which are the same as the generators of $\text{CFK}_{F[U,V]}$), and that segments of the curve to the left (respectively right) of $k$ marked points on the vertical line correspond to terms in the differential with coefficients $V^k$ (respectively $U^k$). See [HRW18, Section 4], especially Proposition 47. (For a more general discussion, see also [KWZ20].)

The curve $\gamma_{1,n-1}$ intersects the vertical line in four points, which we label from top to bottom as

$$c_{1,n-1}, b_{1,n-1}, g_{1,n-1}, e_{1,n-1}.$$  

(The labeling is chosen to match the statement of Proposition 3.1) There is a bigon from $b_{1,n-1}$ to $c_{1,n-1}$ to the right of one marked point on the vertical line, and a bigon from $b_{1,n-1}$ to $e_{1,n-1}$ to the left of $2n - 1$ marked points on the vertical line, indicating that

$$\partial b_{1,n-1} = Uc_{1,n-1} + V^{2n-1}e_{1,n-1} \mod (UV).$$

Similarly, we see that

$$\partial c_{1,n-1} = Vg_{1,n-1} \mod (UV) \quad \text{and} \quad \partial e_{1,n-1} = Ug_{1,n-1} \mod (UV).$$

Each of the curves $\gamma_{1,i}$, $1 \leq i \leq n-2$, intersects the vertical line in six points, which we label from top to bottom as

$$c_{1,i}, f_{1,i}, d_{1,i}, b_{1,i}, g_{1,i}, e_{1,i}.$$  

The curves $\gamma_i$, $\gamma_{0,i}$, $1 \leq i \leq n-2$, and $\gamma_{0,n-1}$ are labeled analogously, with the corresponding subscript. It is straightforward to see that modulo $(UV)$, the differential is as given in the statement of Proposition 3.1. The generator $a$ corresponds to the horizontal line $\gamma_0$.

The parity of the Maslov grading of each generator is readily determined by the fact that knot Floer homology categorifies the Alexander polynomial and the fact that $a$ is in Maslov grading.
zero. The generators $b, f, g$ (with any possible subscript) are all in even Maslov grading, and the generators $c, d, e$ (again with any possible subscript) are all in odd Maslov grading.

We now compare our answer with Lemma 3.2. Note that the present proposition studies the $(p, -1)$-cable of the figure-eight knot, while Lemma 3.2 studies the $(p, 1)$-cable. Recall that $CFK_{\mathbb{F}[U,V]}(-K) = CFK_{\mathbb{F}[U,V]}(K)^*$, where the dual is taken over the ground ring. (The same statement holds over $\mathbb{F}[U,V]/(UV)$.) A basis for $CFK_{\mathbb{F}[U,V]}(K)$ naturally induces a basis for $CFK_{\mathbb{F}[U,V]}(K)^*$, where if $x$ is basis element for $CFK_{\mathbb{F}[U,V]}(K)$ in grading $(gr_U, gr_V)$, then $x^*$ has grading $(-gr_U, -gr_V)$.

Recall that $HF^K_-(K) = H_*(CFK_{\mathbb{F}[U,V]}(K)/(V))$. Using immersed curves, we have computed the mod $(UV)$ differential; hence we have computed $HF^K_-$ using immersed curves. Dualizing, we see that the generators in even Maslov grading of summands of $HF^K_-$ of the form $\mathbb{F}[U]/(U^i)$ for $2 \leq i \leq p$ are:

| summand | generator |
|---------|-----------|
| $\mathbb{F}[U]/(U^{n+j})$ | $b^*_{0,j}$ |
| $\mathbb{F}[U]/(U^n)$ | $b^*$ |
| $\mathbb{F}[U]/(U^{n-k})$ | $b^*_{1,k}$ |

where $1 \leq j \leq n - 1$ and $1 \leq k \leq n - 2$. (Recall that $p = 2n - 1$.) Since these are the unique generators of such summands in even Maslov grading, by comparing with the table in Lemma 3.2 we deduce the following identifications:

$$
\begin{align*}
ax_3 & \leftrightarrow b^*_{0,n-1} \\
b_{p+j-1}y_3 & \leftrightarrow b^*_{0,j-n} \\
b_{p+n-1}y_3 & \leftrightarrow b^* \\
b_{p+k-1}y_3 & \leftrightarrow b^*_{1,-k+n}.
\end{align*}
$$

where $n + 1 \leq j \leq 2n - 2$ and $2 \leq k \leq n - 1$. (In fact, since all of the gradings in the table in Lemma 3.2 are identical, the precise identifications do not matter; we list them for the sake of completeness.) It follows that that $b^*_{0,j}, b^*$, and $b^*_{1,k}$ all have $gr_U = gr_V = 0$, hence so do $b_{0,j}, b$, and $b_{1,k}$ for $1 \leq j \leq n - 1$ and $1 \leq k \leq n - 2$. By symmetry of the knot Floer complex, it follows that $b_{1,n-1}$ also has $gr_U = gr_V = 0$. Knowing the absolute gradings for all of the $b$-type generators (with any subscript, including the empty subscript) determines all of the absolute gradings, since every summand (other than $a$) contains a $b$-type generator.

We have described the differential modulo $UV$ and the absolute gradings. We now use the fact that $\partial^2 = 0$ to deduce the differential over the ring $\mathbb{F}[U, V]$.

We first consider the generator $b$. We have

$$
\begin{align*}
\partial b &= U^n c + V^ne \mod (UV) \\
\partial c &= Vf \mod (UV) \\
\partial e &= Ug \mod (UV).
\end{align*}
$$

In order for $\partial^2 b = 0$ over $\mathbb{F}[U, V]$, we need to cancel the terms $U^nVf$ and $UV^ng$ that appear when we na"{i}vely lift from $\mathbb{F}[U, V]/(UV)$ to $\mathbb{F}[U, V]$ by adding diagonal arrows to our complex, since taking a quotient mod $(UV)$ is equivalent to ignoring all diagonal arrows. We first consider the term $U^nVf$. Since the exponent on $V$ is one, it follows that the term that cancels $U^nVf$ must arise from a path from $b$ to $U^nVf$ consisting of one diagonal arrow and one horizontal arrow. By inspection on the gradings of generators, we see that any diagonal arrow starting from $b$ can only go to generators $UVd$ and $UVd_{i,j}$ for $i = 0, 1, j = 1, \ldots, n - 2$. However the boundaries $\partial d, \partial d_{0,1}, \ldots, \partial d_{0,n-2}, \partial d_{1,1}, \ldots, \partial d_{1,n-2}$ are clearly $\mathbb{F}[U, V]$-linearly independent. Thus the only
Lemma 4.1. With notation as in Proposition 3.1, we have for $1 \leq i \leq n - 2$.

Similarly, we can conclude that

$$
\partial b_0,i = U^{n+i}c_{i,j} + V^{n-i}e_{0,i} + U\partial d_{0,i}
$$

$$
\partial b_1,i = U^{n-i}c_{1,i} + V^{n+i}e_{1,i} + U\partial d_{1,i}
$$

for $1 \leq i \leq n - 2$. Furthermore, it is a straightforward exercise to verify that gradings and $\partial^2 = 0$ obstruct the existence of any other diagonal arrows (up to a possible change of basis). This concludes the proof of the proposition.

\[\square\]

4. Involutive knot Floer homology and cables of the figure-eight knot

Let $K_n$ denote $(2n - 1, -1)$-cable of the figure-eight knot for $n \geq 2$. The goal of this section is to show that the knots $K_n$ are linearly independent in the concordance group. We will compute part of $i_K$ for the knots $K_n$. Combined with the notion of almost local equivalence, this partial computation will be sufficient to show linear independence.

We begin with the partial computation of $i_K$ for the knots $K_n$.

Lemma 4.1. With notation as in Proposition 3.1, we have

$$
i_K(a) = a \mod (U, V)
$$

$$
i_K(b) = b + a \mod (U, V)
$$

$$
i_K(f) = g \mod (U, V)
$$

$$
i_K(g) = f \mod (U, V).
$$

Proof. We first show that $i_K(f) = g \mod (U, V)$ and $i_K(g) = f \mod (U, V)$. By Lemma 2.11 we have that $i^2_K(f) = f \mod (U, V)$. Note that $f$ is the unique basis element in grading $(2n - 2, 0)$ and $g$ is the unique basis element in grading $(0, 2n - 2)$. Since $i_K$ is skew-graded, it follows that $i_K(f) = g \mod (U, V)$ and $i_K(g) = f \mod (U, V)$.

We next show that $i_K(a) = a \mod (U, V)$. Notice that any cycle besides $a$ in grading $(0, 0)$ lies in the ideal $(U, V)$. Since $i_K$ is a skew-graded chain map, it follows that $i_K(a) = a \mod (U, V)$.

We now show that $i_K(b) = b + a \mod (U, V)$. The generator $b$ also has grading $(0, 0)$, so we know that $i_K(b)$ is an $\mathbb{F}$-linear combination of

$$
a, b, b_{0,1}, \ldots, b_{0,n-1}, b_{1,1}, \ldots, b_{1,n-1},
$$

$$
U^n f_{1,n+2}, U^n f_{1,n-3}, \ldots, U^n f_{1,1}, U^n + f, U^n f_{0,1}, U^n f_{0,2}, \ldots, U^{2n-2} f_{0,n-1},
$$

$$
V g_{0,n-2}, V^2 g_{0,n-3}, \ldots, V^n g_{0,1}, \ldots, V^n g_{1,1}, V^n g_{1,2}, \ldots, V^{2n-2} g_{1,n-1}.
$$

We claim that terms $b_{i,j}$ cannot appear in $i_K(b)$. Since $i_K$ is a $\mathbb{F}[U, V]$-skew-equivariant, skew-graded chain map, we have $\partial i_K(b) = i_K(\partial b) = 0 \mod (U^n, U V, V^n)$. However, if we consider the integer

$$
k = \min\{j \mid \text{either } b_{0,j} \text{ or } b_{1,j} \text{ appears in } i_K(b)\},
$$

then we have $\partial i_K(b) \neq 0 \mod (U^{n-k+1}, U V, V^{n-k+1})$, a contradiction. This proves our claim.

We will show that, up to homotopy, we may ignore any terms in $i_K(b)$ containing $f_{i,j}$ or $g_{i,j}$.

We begin by showing that terms of the form $U^i f_{0,i}, 1 \leq i \leq n - 1$, can be homotoped away. Consider $i_K$ and $i'_K$ such that $i_K + i'_K$ is zero on all basis elements except for $b$ where $(i_K + i'_K)(b) = U^{n+i-1} f_{0,i}$, for some $1 \leq i \leq n - 1$. We will find a skew-homotopy $H$ such that

$$
i_K + i'_K = \partial H + H \partial.
$$

(4.1)
Let \( H(e) = U^{i-1} f_{0,i} \) and \( H \) of any other basis element be zero. Then it is straightforward to verify that
\[
\partial H(b) + H \partial(b) = H(U^n c + U V d + V^n e) = U^n H(e) = U^{n+i-1} f_{0,i}.
\]
Since \( f_{i,j} \) is a cycle and no multiple of \( e \) occurs in the boundary of any basis element besides \( b \), it is clear that both sides of (4.1) are zero for all basis elements other than \( b \).

We now show that terms of the form \( U^{n+i-1} f_{0,i} + V^{n-i-1} g_{0,i} \), 1 \( i \leq n - 2 \), can be homotoped away. Consider \( \iota_K \) and \( \iota'_K \) such that \( \iota_K + \iota'_K \) is zero on all basis elements except for \( b \) where \( (\iota_K + \iota'_K)(b) = U^{n+i-1} f_{0,i} + V^{n-i-1} g_{0,i} \), for some \( 1 \leq i \leq n - 2 \) we will find a skew-homotopy \( H \) such that
\[
\iota_K + \iota'_K = \partial H + H \partial.
\]
Let \( H(b) = d_{0,i} \) and \( H \) of any other basis element be zero. It is straightforward to verify that \( H \) has the desired properties. The same argument applies to show that the following terms can be homotoped away:
\[
U^{n-1} f + V^{n-1} g, \quad \text{via} \quad H(b) = d,
\]
\[
V^{n+i-1} g_{1,i}, \quad \text{for} \ 1 \leq i \leq n - 1, \quad \text{via} \quad H(c) = V^{i-1} g_{1,i},
\]
\[
U^{n-i-1} f_{1,i} + V^{n-i-1} g_{1,i}, \quad \text{for} \ 1 \leq i \leq n - 2, \quad \text{via} \quad H(b) = d_{1,i}.
\]
Thus, up to homotopy, there are four possibilities for \( \iota_K(b) \):
1. \( \iota_K(b) = b \),
2. \( \iota_K(b) = b + U^{n-1} f \),
3. \( \iota_K(b) = b + a \),
4. \( \iota_K(b) = b + a + U^{n-1} f \).

Note that the possibilities \( \iota_K(b) = 0, U^{n-1} f, a + U^{n-1} f \) are ruled out. This is because the first two would imply that \( b = \iota_K^2(b) = \iota_K^3(\iota_K(b)) = 0 \mod (U, V) \) and the last two would imply that \( b + a = \iota_K^2(\iota_K(b) + a) = 0 \mod (U, V) \).

Recall that \( \iota_K^2 \simeq 1 + \Psi \Phi \). If \( n \) is odd, then \( (1 + \Psi \Phi)(b) = b + U^{n-1} f \), and if \( n \) is even, then \( (1 + \Psi \Phi)(b) = b + V^{n-1} g \). Note that (4.2) shows that these are homotopic, so without loss of generality, we may assume that \( (1 + \Psi \Phi)(b) = b + U^{n-1} f \).

Suppose that \( \iota_K(b) = b \). Then \( \iota_K^2(b) = b \). We claim that there does not exist a homotopy \( H \) such that
\[
\partial H(b) + H \partial(b) = \iota_K^2(b) + (1 + \Psi \Phi)(b) = U^{n-1} f.
\]
Consider the above equation modulo the ideal \( (U^n, V^n, U V) \). Since \( \partial b = U^n c + U V d + V^n e \) and \( H \) is \( \mathbb{F}[U, V] \)-equivariant, the left hand side of the above equation reduces to \( \partial H(b) \). But \( U^{n-1} f \notin \im \partial \), a contradiction. Hence \( \iota_K(b) \neq b \).

Now suppose that \( \iota_K(b) = b + U^{n-1} f \). Since \( \iota_K(f) = g \mod (U, V) \), it follows that
\[
\iota_K^2(b) = b + U^{n-1} f + \iota_K(U^{n-1} f) = b + U^{n-1} f + V^{n-1} g \mod (U^n, V^n, U V).
\]
By (4.2), up to homotopy, we may ignore the term \( U^{n-1} f + V^{n-1} g \). Now apply the same argument as in the \( \iota_K(b) = b \) case. Thus, \( \iota_K(b) \neq b + U^{n-1} f \).

Therefore, we have that \( \iota_K(b) = b + a \) or \( \iota_K(b) = b + a + U^{n-1} f \). In either case,
\[
\iota_K(b) = b + a \mod (U, V),
\]
as desired. \( \Box \)

We now proceed to prove Theorem 1.3 by showing linear independence of the almost \( \iota_K \)-complexes of the knots \( K_n \).
Proof of Theorem 7.3: Let $C_n$ denote the almost $\iota_K$-complex of $K_n$. For simplicity, we will write the involution $\iota_K$ on $C_n$ by $\iota_K$. By Lemma 2.9, it is enough to show that if we have

$$M \bigotimes_{i=1}^{m_i} C_{m_i}^{a_i} \sim N \bigotimes_{i=1}^{n_i} C_{n_i}^{b_i},$$

(4.3)

where $m_i, n_i \geq 2, m_i > m_{i+1}, n_i > n_{i+1}, a_i, b_i \geq 1$, then the left hand side and the right hand side agree, i.e. $M = N$ and $a_i = b_i, m_i = n_i$ for all $i$. Here, $(C_1, \iota_1) \sim (C_2, \iota_2)$ means that the two almost $\iota_K$-complexes are almost locally equivalent.

Before beginning the proof, we provide the following outline of our strategy:

1. Suppose that the left and right hand sides of (4.3) are almost locally equivalent via an almost local equivalence $f$.
2. Use the fact that $f$ is an isomorphism on $H_*((\mathcal{U}, \mathcal{V})^{-1}C_i)$ to partially determine $f(a \otimes a \otimes \cdots \otimes a)$ mod $(\mathcal{U}, \mathcal{V})$.
3. Use the preceding step and fact that $\omega f = f \omega$ mod $(\mathcal{U}, \mathcal{V})$ to partially determine $f(b \otimes a \otimes a \otimes \cdots \otimes a)$.
4. Use the preceding step and the fact that $f$ is a chain map to reach a contradiction, showing that $f$ cannot exist.

Without loss of generality, suppose that $m_1 > n_1, M \geq 1$, and $N \geq 0$. We will show that there is no almost local map

$$f: M \bigotimes_{i=1}^{m_i} C_{m_i}^{a_i} \to N \bigotimes_{i=1}^{n_i} C_{n_i}^{b_i}.$$  

More precisely, we will show that if $f$ induces an isomorphism on localized homology and $f \iota_K \simeq \iota_K f$ mod $(\mathcal{U}, \mathcal{V})$, then $f$ cannot be a chain map. Suppose such $f$ exists, then since $a \otimes a \otimes \cdots \otimes a$ is a cycle in $\bigotimes_{i=1}^{M} C_{m_i}^{a_i}$ which generates $H_*((\mathcal{U}, \mathcal{V})^{-1} \bigotimes_{i=1}^{M} C_{m_i}^{a_i})$, $f(a \otimes a \otimes \cdots \otimes a)$ is also a cycle in $\bigotimes_{i=1}^{N} C_{n_i}^{b_i}$ which generates $H_*((\mathcal{U}, \mathcal{V})^{-1} \bigotimes_{i=1}^{N} C_{n_i}^{b_i})$. We claim that

$$\langle f(a \otimes a \otimes \cdots \otimes a), a \otimes a \otimes \cdots \otimes a \rangle = 1.$$  

Here, the left hand side denotes the coefficient of $a \otimes a \otimes \cdots \otimes a$ when we express $f(a \otimes a \otimes \cdots \otimes a)$ as an $\mathbb{F}[\mathcal{U}, \mathcal{V}]$-linear combination of elements of the form $\otimes x_i$, where each $x_i$ is a basis element in Proposition 3.1.

To prove the claim, suppose that $\langle f(a \otimes a \otimes \cdots \otimes a), a \otimes a \otimes \cdots \otimes a \rangle = 0$ and consider the chain map $g_i : C_n \to \mathbb{F}[\mathcal{U}, \mathcal{V}]$ which maps $a$ to 1 and all other generators to 0. Then $g_i$ is an $\mathbb{F}[\mathcal{U}, \mathcal{V}]$-linear chain map and $(\mathcal{U}, \mathcal{V})^{-1}g_i$ is a quasi-isomorphism (although $g_i$ is not an almost local map), so $g = \bigotimes_{i=1}^{N} g_i^{b_i}$ is also a $\mathbb{F}[\mathcal{U}, \mathcal{V}]$-linear chain map and $(\mathcal{U}, \mathcal{V})^{-1}g$ is a quasi-isomorphism. Since $f(a \otimes a \otimes \cdots \otimes a)$ does not contain the term $a \otimes a \otimes \cdots \otimes a$, we deduce that $g(f(a \otimes a \otimes \cdots \otimes a)) = 0$, which is a contradiction since $f(a \otimes a \otimes \cdots \otimes a)$ generates the homology of $(\mathcal{U}, \mathcal{V})^{-1} \bigotimes_{i=1}^{N} g_i^{b_i}$. So our claim is proven.

Now consider $b \otimes a \otimes a \otimes \cdots \otimes a$. Let $\omega = 1 + \iota_K$. We have that

$$\omega(b \otimes a \otimes a \otimes \cdots \otimes a) = (1 + \iota_K)(b \otimes a \otimes a \otimes \cdots \otimes a)$$

$$= b \otimes a \otimes a \otimes \cdots \otimes a + \iota_K(b \otimes a \otimes a \otimes \cdots \otimes a)$$

$$= b \otimes a \otimes a \otimes \cdots \otimes a + (b + a) \otimes a \otimes a \otimes \cdots \otimes a \mod (\mathcal{U}, \mathcal{V})$$

$$= a \otimes a \otimes \cdots \otimes a \mod (\mathcal{U}, \mathcal{V}).$$

where the third equality uses Lemma 4.1 and the fact that $\Psi(a) = 0$; note that the connected sum formula for involutive knot Floer homology is given by $\iota_K \# K_2 = (\mathrm{id} \otimes \mathrm{id} + \Phi \otimes \Psi) \circ (\iota_K \otimes \iota_{K_2})$. Hence
\begin{align*}
\langle \omega f(b \otimes a \otimes a \otimes \cdots \otimes a), a \otimes a \otimes \cdots \otimes a \rangle &= \langle f \omega(b \otimes a \otimes a \otimes \cdots \otimes a), a \otimes a \otimes \cdots \otimes a \rangle \\
&= \langle f(a \otimes a \otimes \cdots \otimes a), a \otimes a \otimes \cdots \otimes a \rangle \\
&= 1, \quad (4.4)
\end{align*}
where the first equality follows from the fact that our complexes are reduced. This proves the case when \( N = 0 \), since \( 1 + \iota_{\text{unknot}} \) is the zero map. Thus, from now on, we will assume that \( N > 0 \).

We claim that \( f(b \otimes a \otimes a \otimes \cdots \otimes a) \) must contain at least one element of the form

\[ x = \bigotimes_i x_i \]

where each \( x_i \) is either \( a, b, \) or \( b_{j,k} \) for some \( j, k \) and at least one \( x_i \) is not \( a \), when we represent it in terms of linear combinations of tensor products of basis elements of each \( C_{n_i} \). (From now on, we will reserve the notation \( x \) for elements of this form.) To prove the claim, suppose that \( f(b \otimes a \otimes a \otimes \cdots \otimes a) \) does not contain any element of the form \( x \). Since any generator \( z \) of each \( C_{n_i} \) satisfies \( \text{gr}_U(z) + \text{gr}_V(z) \geq 0 \), and the equality is satisfied only when \( z = a, b, b_{j,k}, c_{0,n-1}, c_{1,n-1} \), we can write \( f(b \otimes a \otimes \cdots \otimes a) \mod (U, V) \) as a linear combination of elements of the form \( z = \bigotimes_i z_i \) where \( z_i \) is either \( a, b, b_{j,k}, c_{0,n-1}, c_{1,n-1} \), and at least one of \( z_i \) is either \( c_{0,n-1} \) or \( c_{1,n-1} \). But since the only generators of \( C_{n_i} \) that lie in bigrading \( (1, -1) \) and \( (-1, 1) \) are \( c_{1,n-1} \) and \( e_{0,n-1} \), respectively, we must have \( i_K(c_{1,n-1}) = e_{0,n-1} \mod (U, V) \) and \( i_K(e_{0,n-1}) = c_{1,n-1} \mod (U, V) \). Hence we see that \( \omega f(b \otimes a \otimes \cdots \otimes a) \) cannot contain the term \( a \otimes a \otimes \cdots \otimes a \). This contradicts (4.4) so our claim is proven.

We now use the fact that \( f \) is a chain map. We have that \( \partial x \) is a linear combination of elements of the form

\[ y = \bigotimes_i y_i \]

where all but one \( y_i \) is

\[ a, \ b, \ b_{j,k} \]

and exactly one \( y_i \) is

\[ U^\ell c, \ UVd, \ V^\ell e, \ U^\ell c_{j,k}, \ UVd_{j,k}, \ V^\ell e_{j,k}. \]

From now on, we will reserve the notation \( y \) for elements of this form.

We now make the following two observations:

1. For a fixed element \( y \), there is a unique element \( x \) such that \( y \) appears in \( \partial x \). More precisely,
   - (a) if \( y_i = a \), then \( x_i = a \),
   - (b) if \( y_i = b \), then \( x_i = b \),
   - (c) if \( y_i = b_{j,k} \), then \( x_i = b_{j,k} \),
   - (d) if \( y_i = U^\ell c, \ UVd, \) or \( V^\ell e \), then \( x_i = b \),
   - (e) if \( y_i = U^\ell c_{j,k}, \ UVd_{j,k}, \) or \( V^\ell e_{j,k} \), then \( x_i = b_{j,k} \).
2. If \( z \) is not of the form \( x \), then \( y \) does not appear in \( \partial z \).

Note that

\[ \partial(b \otimes a \otimes a \otimes \cdots \otimes a) = 0 \mod (U^{m_1}, V^{m_1}, UV). \]

We have that \( f(b \otimes a \otimes a \otimes \cdots \otimes a) \) contains at least one element of the form \( x \). It is straightforward to verify that for any element of the form \( x \)

\[ \partial x \neq 0 \mod (U^{m_1}, V^{m_1}, UV), \]

since \( m_1 > n_i \) for all \( i \); namely, for any \( b_{j,k} \) (respectively \( b \)) appearing in \( x \), there is at least one term in \( \partial b_{j,k} \) (respectively \( \partial b \)) of the form \( U^\ell c_{j,k} \) or \( V^\ell e_{j,k} \) (respectively \( U^\ell c \) or \( V^\ell e \)) for some \( \ell < m_1 \). Furthermore, it follows from items (1) and (2) above that the differential of no other term appearing in \( f(b \otimes a \otimes a \otimes \cdots \otimes a) \) can cancel the non-vanishing terms in \( \partial x \mod (U^{m_1}, V^{m_1}, UV) \).
It follows that \( f \) cannot be a chain map. Hence no almost local map

\[
f: \bigotimes_{i=1}^{M} C_{m_i}^{\otimes a_i} \to \bigotimes_{i=1}^{N} C_{n_i}^{\otimes b_i}
\]
can exist, and thus

\[
\bigotimes_{i=1}^{M} C_{m_i}^{\otimes a_i} \quad \text{and} \quad \bigotimes_{i=1}^{N} C_{n_i}^{\otimes b_i}
\]
are not almost locally equivalent. It follows that the knots \( K_n, n \geq 2 \), are linearly independent. \( \square \)

5. The Concordance Unknotting Number and Cables of the Figure-Eight Knot

The goal in this section is to prove Corollary 1.5. The proof will rely on the notion of the connected \( \iota_K \)-complex of a knot, which imports the ideas of [HHL18] (which deal with \( \iota \)-complexes of rational homology spheres) to the setting of \( \iota_K \)-complexes of knots. The arguments below are essentially identical to those in [HHL18, Section 3] (see also [Zho20, Section 6]); we include them here for completeness.

**Definition 5.1.** Let \((C,\iota)\) be an (almost) \( \iota_K \)-complex. A (almost) self-local equivalence \( f \) is an (almost) local equivalence \( f : C \to C \).

We define a pre-order \( \preceq \) (that is, a reflexive and transitive binary relation) on the set of (almost) self-local equivalences. Let \( f, g \) be (almost) self-local equivalences of an (almost) \( \iota_K \)-complex \((C,\iota)\). We say that \( f \preceq g \) if \( \ker f \subseteq \ker g \). A (almost) self-local equivalence \( f \) is maximal if \( f \preceq g \) implies \( \ker f = \ker g \).

Maximal (almost) self-local equivalences have the following useful properties:

**Lemma 5.2.** Let \( f \) be a maximal (almost) self-local equivalence of \((CFK_{F[U,V]}(K),\iota_K)\) and let \( g : CFK_{F[U,V]}(K) \to CFK_{F[U,V]}(K) \) be a (almost) local map. Then \( g|_{\im f} \) is injective.

**Proof.** Note that since \( \ker f \subseteq \ker(gf) \), we have that \( f \preceq gf \). Since \( f \) is maximal, this implies that \( \ker f = \ker(gf) \). Therefore, \( g|_{\im f} \) is injective. \( \square \)

**Lemma 5.3.** Let \( f \) be a maximal (almost) self-local equivalence of \((CFK_{F[U,V]}(K),\iota_K)\). The isomorphism type of \( \im f \) is independent of \( f \).

**Proof.** Let \( f \) and \( g \) be maximal (almost) self-local equivalences of \((CFK_{F[U,V]}(K),\iota_K)\). We will show that \( \im f \cong \im g \). By Lemma 5.2, we see that \( f|_{\im g} \) and \( g|_{\im f} \) are injective. Since we have injective, bigraded, \( F[U,V] \)-equivariant chain maps between \( \im f \) and \( \im g \), it follows that \( \im f \) and \( \im g \) are isomorphic, since all chain complexes involved are finitely generated over \( F[U,V] \). \( \square \)

The above lemma ensures the following is well-defined:

**Definition 5.4.** The \( \iota_K \)-connected knot Floer complex of \( K \), denoted \( CFK_{\iota_K\text{-conn}}(K) \), is the image of any maximal self-local equivalence of the \( \iota_K \)-complex of \( K \). The almost \( \iota_K \)-connected knot Floer complex of \( K \) is the image of any maximal almost self-local equivalence of the almost \( \iota_K \)-complex of \( K \). Let

\[
HFK_{\iota_K\text{-conn}} := H_*(CFK_{\iota_K\text{-conn}}(K)/(\mathcal{V})).
\]

The next proposition underscores the utility of the connected complex in the study of concordance:
Proof of Proposition 5.7. The (almost) $\iota_K$-connected knot Floer complex of $K$ is an invariant of the (almost) local equivalence class of $(\text{CFK}_{\mathcal{U},\mathcal{V}}(K),\iota_K)$. In particular, the (almost) $\iota_K$-connected knot Floer complex of $K$ is a subcomplex of $\text{CFK}_{\mathcal{U},\mathcal{V}}(K')$ for any knot $K'$ concordant to $K$.

Proof. This is straightforward from the definitions. Indeed, let $f: C_1 \to C_1$ and $g: C_2 \to C_2$ be maximal (almost) self-local equivalences, where $C_1$ and $C_2$ are finitely generated over $\mathbb{F}[\mathcal{U},\mathcal{V}]$. If $(C_1,\iota_1)$ and $(C_2,\iota_2)$ are (almost) locally equivalent via $h: C_1 \to C_2$ and $k: C_2 \to C_1$, then maximality of $f$ implies that $(kgh)_{\text{lim}f}$ is injective, hence $gh_{\text{lim}f}$ is injective. Similarly, $fk_{\text{lim}g}$ is injective. Hence if $f$ and $im\,g$ are isomorphic.

We now relate the $\iota_K$-connected knot Floer complex to the concordance unknotting number. The main idea is identical to [DHST21] Theorem 1.14. Let $M$ be a finitely generated $\mathbb{F}[\mathcal{U}]$-module and let $\text{Tor} M$ be the $\mathbb{F}[\mathcal{U}]$-torsion submodule of $M$. Define the torsion order of $M$ to be

$$\text{Ord}_\mathcal{U}(M) = \min\{n \in \mathbb{N} \mid \mathcal{U}^n(\text{Tor} M) = 0\},$$

and the torsion order of $K$ to be

$$\text{Ord}_\mathcal{U}(K) = \min\{n \in \mathbb{N} \mid \mathcal{U}^n(\text{Tor} \text{HFK}^-(K)) = 0\}.$$

Recall from [AE20] Theorem 1.1 (see also [Zem19b]) that

$$u(K) \geq \text{Ord}_\mathcal{U}(K).$$

(5.1)

Corollary 5.6. The torsion order of $\text{HFK}^{-}_{\iota_K\text{-conn}}(K)$ provides the following bound on concordance unknotting number:

$$u_r(K) \geq \text{Ord}_\mathcal{U}(\text{HFK}^{-}_{\iota_K\text{-conn}}(K)).$$

Proof. The result follows from (5.1) and Proposition 5.5.

Lastly, we need the following proposition:

Proposition 5.7. Let $K_n$ denote the $(2n-1,-1)$-cable of the figure-eight knot for $n \geq 2$. Then

$$\mathcal{U}^{n-1} \cdot \text{HFK}^{-}_{\iota_K\text{-conn}}(K_n) \neq 0.$$
In particular, \( f(b) \neq 0 \mod (U, V) \).

We also have that \( \partial b = 0 \mod (U^n, U V, V^n) \). Hence

\[
\partial f(b) = 0 \mod (U^n, U V, V^n).
\]  

(5.3)

We claim that

\[
\langle f(b), b \rangle = 1,
\]  

(5.4)

where \( \langle f(b), b \rangle \) denotes the coefficient of \( b \) when \( f(b) \) is expressed in terms of the basis in Proposition 3.1. To see why, observe that \( f(b) \) lies on the bigrading \((0, 0)\), so if the claim is false, then we can write it as a linear combination

\[
f(b) = \epsilon_0 a + \sum_{i=1}^{n-1} (\epsilon_{0,i} b_{0,i} + \epsilon_{1,i} b_{1,i}) + z
\]

for some coefficients \( \epsilon_0, \epsilon_{0,i}, \epsilon_{1,i} \in \mathbb{F} \) and some \( z \) contained in the \( \mathbb{F} \)-linear span of

\[
U^{n+i-1} f_{0,i}, U^{n-i-1} f_{1,i}, V^{n-i-1} g_{0,i}, V^{n+i-1} g_{1,i}.
\]

Since \( \partial z = 0 \), (5.3) gives

\[
0 = \partial f(b) = \sum_{i=1}^{n-1} (U^{n-i} \epsilon_{0,i} a_0 + V^{n-i} \epsilon_{1,i} c_{1,i}) \mod (U^n, U V, V^n).
\]

Hence \( \epsilon_{0,i} = \epsilon_{1,i} = 0 \) for all \( i \), i.e. \( f(b) = 0 \) or \( a \mod (U, V) \). But then we get \( \omega f(b) = 0 \mod (U, V) \), which contradicts (5.2).

By inspection of the differential in Proposition 3.1 (namely, that the unique arrow coming into any \( U \) or \( V \) power of \( c \) is a length \( n \) horizontal arrow from \( b \)), we see that (5.4) implies that

\[
\langle \partial f(b), c \rangle = U^n
\]

or equivalently

\[
\langle f(\partial b), c \rangle = U^n.
\]

Hence

\[
\langle f(U^n c + U V d + V^n e), c \rangle = U^n.
\]

By \( \mathbb{F}[U, V] \)-equivariance of \( f \), we see that

\[
\langle f(c), e \rangle = 1.
\]

We now claim that \( U^{n-1} \cdot [f(c)] \neq 0 \in \text{HF}^\text{conn}_k(K_n) = \text{im} f \). Since \( c \) is a cycle in \( C_n / (V) \), so is \( f(c) \). Let \( \partial_U \) denote the induced differential on \( C_n / (V) \). Since \( \langle f(c), e \rangle = 1 \) and the unique horizontal arrow coming into any \( U \) power of \( c \) is of length \( n \), it follows that \( U^{n-1} f(c) \mod (V) \) is not in the image of \( \partial_U \). Hence \( U^{n-1} \cdot [f(c)] \neq 0 \in \text{HF}^\text{conn}_k(K_n) \) as desired.

Since \( f \) was a maximal almost self-local equivalence, it follows that \( U^{n-1} \cdot \text{HF}^\text{conn}_k(K_n) \neq 0 \), completing the proof.

\[ \square \]

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