Gradient descent GAN optimization is locally stable

Abstract

Despite their growing prominence, optimization in generative adversarial networks (GANs) is still a poorly-understood topic. In this paper, we analyze the “gradient descent” form of GAN optimization (i.e., the natural setting where we simultaneously take small gradient steps in both generator and discriminator parameters). We show that even though GAN optimization does not correspond to a convex-concave game, even for simple parameterizations, under proper conditions, equilibrium points of this optimization procedure are still locally asymptotically stable for the traditional GAN formulation. On the other hand, we show that the recently-proposed Wasserstein GAN can have non-convergent limit cycles near equilibrium. Motivated by this stability analysis, we propose an additional regularization term for gradient descent GAN updates, which is able to guarantee local stability for both the WGAN and for the traditional GAN, and also shows practical promise in speeding up convergence and addressing mode collapse.

1 Introduction

Since their introduction a few years ago, Generative Adversarial Networks [Goodfellow et al., 2014] (GANs) have gained prominence as one of the most widely-used methods for training deep generative models. GANs have been successfully deployed for tasks such as photo super-resolution, object generation, video prediction, language modeling, vocal synthesis, and semi-supervised learning, amongst many others [Ledig et al., 2016, Wu et al., 2016, Mathieu et al., 2015, Nguyen et al., 2016, Denton et al., 2015, Im et al., 2016].

At the core of the GAN methodology is idea of jointly training two networks: a generator network, meant to produce samples from some distribution (that ideally will mimic examples from the data distribution), and a discriminator network, which attempts to differentiate between samples from the data distribution and the one produced by the generator. This problem is typically written as a min-max optimization problem of the following form:

$$\min_G \max_D \left( \mathbb{E}_{x \sim p_{\text{data}}} \left[ \log D(x) \right] + \mathbb{E}_{z \sim p_{\text{latent}}} \left[ \log(1 - D(G(z))) \right] \right).$$  \hspace{1cm} (1)

Note that for the purposes of this paper, we will shortly consider a more general form of the optimization problem, which also includes the recent Wasserstein GAN (WGAN) [Arjovsky et al., 2017] formulation.

Despite their prominence, the actual task of optimizing a GAN network remains a challenging problem, both from a theoretical and a practical standpoint. Although the original GAN paper included some analysis on the convergence properties of the approach [Goodfellow et al., 2014], this analysis assumed that updates occurred in pure function space, allowed arbitrarily powerful generator and discriminator networks, and modeled the resulting optimization objective as a convex-concave game, therefore yielding well-defined global convergence properties. Furthermore, this analysis assumed that the discriminator network is fully optimized between generator updates, an assumption
that does not mirror the practice of GAN optimization. Indeed, for GAN optimization as used in practice, there exist a number of well-documented failure modes such as mode collapse or vanishing gradient problems.

Our Contributions. In this paper, we consider the “gradient descent” formulation of GAN optimization, the setting where both the generator and discriminator are updated simultaneously via simple (stochastic) gradient updates; that is, there are no inner and outer optimization loops, and neither the generator nor the discriminator are assumed to be optimized to convergence. Despite the fact that, as we show, this does not correspond to a convex-concave optimization problem (even for simple linear generator and discriminator representations), we show that:

Under suitable conditions on the representational power of the discriminator and the generator, the resulting GAN dynamical system is locally exponentially stable.

That is, for some region around an equilibrium point of the updates, the gradient updates will converge to this point at an exponential rate. Interestingly, our conditions can be satisfied by the traditional GAN but not by the WGAN, and we indeed show that WGANs can have non-convergent limit cycles in the gradient descent case.

Our theoretical analysis also suggests a natural method for regularizing GAN updates by adding an additional regularization term on the norm of the discriminator gradient. We show that the addition of this term leads to locally exponentially stable equilibria for all classes of GANs, including WGANs. The additional penalty is highly related to (but also notably different from) recent proposals for practical GAN optimization, such as the unrolled GAN [Metz et al., 2016] and the improved Wasserstein GAN training [Gulrajani et al., 2017]. In practice, the approach is simple to implement, and preliminary experiments show that this helps avert mode collapse and leads to faster convergence.

2 Background and related work

GAN optimization and theory. Although the theoretical analysis of GANs has been far outpaced by their practical application, there have been some notable results in recent years, in addition to the aforementioned work in the original GAN paper. For the most part, this work is entirely complementary to our own, and studies a very different set of questions. Arjovsky and Bottou [2017] provides important insights into instability that arises when the supports of the generated distribution and the true distribution are disjoint. In contrast, in this paper we delve into an equally important question of whether the updates are stable even when the generator is in fact very close to the true distribution (and we answer in the affirmative). Arora et al. [2017], on the other hand, explores questions relating to the sample complexity and expressivity of the GAN architecture and their relation to the existence of an equilibrium point. However, it is still unknown as to whether, given that an equilibrium exists, the GAN update procedure will converge locally.

From a more practical standpoint, there have been a number of papers that address the topic of optimization in GANs. Several methods have been proposed that introduce new objectives or architectures for improving the (practical and theoretical) stability of GAN optimization [Arjovsky et al., 2017, Poole et al., 2016]. A wide variety of optimization heuristics and architectures have also been proposed to address challenges such as mode collapse [Salimans et al., 2016, Metz et al., 2016, Che et al., 2016, Radford et al., 2015]. Our own proposed regularization term falls under this same category, and hopefully provides some context for understanding some of these methods. Specifically, our regularization term (motivated by stability analysis) captures a degree of “foresight” of the generator in the optimization procedure, similar to the unrolled GANs procedure [Metz et al., 2016]. Indeed, we show that our gradient penalty is closely related to 1-unrolled GANs, but also provides more flexibility in leveraging this foresight. Finally, gradient-based regularization has been explored for GANs, with one of the most recent works being that of Gulrajani et al. [2017], though their penalty is on the discriminator rather than the generator as in our case.

Finally, in recent weeks there have been several pre-prints that also address similar issues as this paper. Of particular similarity to the methodology we propose here are the works by Roth et al. [2017] and Mescheder et al. [2017]. The first of these two present a regularization method that is quite similar to our own, which regularizes the generator updates based upon the norm of the discriminator gradient, though the focus there is on the precise weighting of the gradient terms to account for various measures of distribution divergence. The second work noted above has some strong similarities with...
our approach in that they also look at the dynamical system of GAN optimization; however, the authors there do not establish or disprove stability, and instead note the presence of zero eigenvalues (which we will treat in some depth) as a motivation for certain alternative optimization methods. Thus, we feel the works as a whole are quite complementary, and signify the growing interest in GAN optimization issues.

**Stochastic approximation algorithms and analysis of nonlinear systems.** The technical tools we use to analyze the GAN optimization dynamics in this paper come form the fields of stochastic approximation algorithm and the analysis of nonlinear differential equation, notably the “ODE method” for analyzing convergence properties of dynamical systems [Borkar and Meyn, 2000]. Consider a general stochastic process driven by the updates

$$\theta_{t+1} = \theta_t + \alpha_t(h(\theta_t) + \epsilon_t)$$

(2)

for vector $\theta_t \in \mathbb{R}^n$, step size $\alpha_t > 0$, $h : \mathbb{R}^n \to \mathbb{R}^n$ and $\epsilon_t$ a martingale difference sequence.\footnote{Stochastic gradient descent on an objective $f(\theta)$ can be expressed in this framework by taking $h(\theta) = \nabla_\theta f(\theta)$.} Under fairly general conditions, namely: 1) bounded second moments of $\epsilon_t$, 2) Lipschitz continuity of $h$, and 3) summable but not square-summable step sizes, the stochastic approximation algorithm converges to an equilibrium point of the (deterministic) ordinary differential equation $\dot{\theta}(t) = h(\theta(t))$.

Thus, to understand stability of the stochastic approximation algorithm, it suffices to understand the stability and convergence of the deterministic differential equation. Though such analysis is typically used to show global asymptotic convergence of the stochastic approximation algorithm to an equilibrium point (assuming the related ODE also is globally asymptotically stable), it can also be used to analyze the local asymptotic stability properties of the stochastic approximation algorithm around equilibrium points.\footnote{Note that the local analysis does not show that the stochastic approximation algorithm will necessarily converge to an equilibrium point, but still provides a valuable characterization of how the algorithm will behave around these points.} This is the technique we follow throughout this entire work, though for brevity we will focus entirely on the analysis of the continuous time ordinary differential equation, and appeal to these standard results to imply the similar properties regarding the discrete updates.

Given the above consideration, the focus of this paper will be on proving stability of the dynamical system around equilibrium points. Considering the ODE given by the GAN update equations

$$\dot{\theta}(t) = h(\theta(t))$$

3) summable but not square-summable step sizes, the stochastic approximation algorithm converges fairly general conditions, namely: 1) bounded second moments of $\epsilon_t$, 2) Lipschitz continuity of $h$, and 3) summable but not square-summable step sizes, the stochastic approximation algorithm converges to an equilibrium point of the (deterministic) ordinary differential equation $\dot{\theta}(t) = h(\theta(t))$.

Thus, to understand stability of the stochastic approximation algorithm, it suffices to understand the stability and convergence of the deterministic differential equation. Though such analysis is typically used to show global asymptotic convergence of the stochastic approximation algorithm to an equilibrium point (assuming the related ODE also is globally asymptotically stable), it can also be used to analyze the local asymptotic stability properties of the stochastic approximation algorithm around equilibrium points.\footnote{Note that this is a slightly different usage of the term equilibrium as typically used in the GAN literature, where it refers to a Nash equilibrium of the min max optimization problem. These two definitions (assuming we mean just a local Nash equilibrium) are equivalent for the ODE corresponding to the min-max game, but we use the dynamical systems meaning throughout this paper, that is, any point where the gradient update is zero.}

Thus, an important contribution of this paper is a proof of this seemingly simple fact: under some conditions, the Jacobian of the dynamical system $J = \frac{\partial h(\theta)}{\partial \theta}$ evaluated at an equilibrium point is Hurwitz (has all strictly negative eigenvalues, $\text{Re}(\lambda_i(J)) < 0$, $\forall i = 1, \ldots, n$), then the ODE will converge to $\theta^*$ for some non-empty region around $\theta^*$ at an exponential rate. This means that the system is locally asymptotically stable, or more precisely, locally exponentially stable (see Definition A.1 in Appendix A). Therefore, the system will not converge to any other equilibrium point in a local neighborhood around the considered equilibrium point. We strengthen our result further by considering a slightly more realistic scenario with similar conditions in which we allow a subspace of equilibria around a considered equilibrium point. Even though this results in a Jacobian with zero eigenvalues, we show that it is sufficient to analyze a reduced system without such zero eigenvalues.

In addition to this, we provide a stability analysis that is based on Lyapunov’s stability theorem (described in Appendix A). The crux of the idea is that to prove convergence it is sufficient to identify a non-negative “energy” function for the linearized system which always decreases with
time (specifically, the energy function will be a distance from the manifold of equivalent equilibrium points). Most importantly, this analysis provides insights into the dynamics that lead to GAN convergence.

3 GAN optimization dynamics

This section comprises the main results of this paper, showing that under proper conditions the gradient descent updates for GANs (that is, updating both the generator and discriminator locally and simultaneously), is locally exponentially stable around “good” equilibrium points (where “good” will be defined shortly). This requires that the GAN loss be strictly concave, which is not the case for WGANs, and we indeed show that the updates for WGANs can cycle indefinitely. This leads us to propose a simple regularization term that is able to guarantee exponential stability for any concave GAN loss, including the WGAN, rather than requiring strict concavity.

3.1 The generalized GAN setting

For the remainder of the paper, we consider a slightly more general formulation of the GAN optimization problem than the one presented earlier, given by the following min/max problem

$$\min_{G} \max_{D} V(G,D) = (E_{x \sim p_{data}} [f(D(x))] + E_{z \sim p_{latent}} [f(-D(G(z)))]$$

(3)

where $G : \mathcal{Z} \rightarrow \mathcal{X}$ is the generator network, which maps from the latent space $\mathcal{Z}$ to the input space $\mathcal{X}$; $D : \mathcal{X} \rightarrow \mathbb{R}$ is the discriminator network, which maps from the input space to a classification of the example as real or synthetic; and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a concave function. We can recover the traditional GAN formulation [Goodfellow et al., 2014] by taking $f$ to be the (negated) logistic loss $f(x) = -\log(1 + \exp(-x))$ (note this slightly differs from the standard formulation in that in this case the discriminator outputs the real-values “logs” and the loss function would implicitly scale this to a probability), or we can recover the Wasserstein GAN by simply taking $f(x) = x$.

Assuming the generator and discriminator networks to be parameterized by some set of parameters $\theta_D$ and $\theta_G$ respectively, we analyze the simple stochastic gradient descent approach to solving this optimization problem. That is, we take simultaneous gradient steps in both $\theta_D$ and $\theta_G$, which in our “ODE method” analysis leads to the following differential equation

$$\dot{\theta}_D = \nabla_{\theta_D} V(\theta_G, \theta_D), \quad \dot{\theta}_G := \nabla_{\theta_G} V(\theta_G, \theta_D).$$

(4)

A note on alternative updates Rather than updating both the generator and discriminator accordingly to the min-max problem above, Goodfellow et al. [2014] also proposed a modified update for just the generator that minimizes a the different objective, $V'(G, D) = -E_{z \sim p_{latent}} [f(D(G(z)))]$ (the negative sign is pulled out from inside $f$). In fact, all the analysis we consider in this paper applies equally to this case (or any convex combination of both updates), as the ODE of the update equations have the same Jacobians at equilibrium.

3.2 Why is proving stability hard for GANs?

Before presenting our main results, we first highlight why understanding the local stability of GANs is non-trivial, even when the generator and discriminator have simple forms. As stated above, GAN optimization consists of a min-max game, and gradient descent algorithms will converge if the game is convex-concave: the objective is convex in the term being minimized and concave in the term being maximized. Indeed, this was a crucial assumption in the convergence proof in the original GAN paper. However, as we describe below, for virtually any parameterization of the real GAN generator and discriminator, even if both representations are linear, the GAN objective will not be a convex-concave game.

Proposition 3.1. The GAN objective in Equation 3 can be a concave-concave objective i.e., concave with respect to both the discriminator and generator parameters, for a large part of the discriminator space, including regions arbitrarily close to the equilibrium.

To see why, consider a simple GAN objective over one dimensional data and latent space with linear generator and discriminator, i.e. $D(x) = \theta_D x + \theta_D'$ and $G(z) = \theta_G z + \theta_G'$. Then the GAN objective is given by

$$V(G, D) = (E_{x \sim p_{data}} [f(\theta_D x + \theta'_D)]) + E_{z \sim p_{latent}} [f(-\theta_D (\theta_G z + \theta'_G) - \theta'_D)].$$
Because \( f \) is concave, by inspection we can see that \( V \) is concave in \( \theta_D \) and \( \theta'_D \); but it is also concave (not convex) in \( \theta_G \) and \( \theta'_G \), for the same reason. Thus, the optimization involves concave minimization, which in general is a difficult problem. To prove that this is not a peculiarity of the above linear discriminator system, in Appendix B we show similar observations for a more general parametrization, and also for the case where \( f''(x) = 0 \) (which happens in the case of WGANs).

Thus, a major question remains as to whether or not GAN optimization is stable at all (most concave maximization is not). Indeed, there are several well-known properties of GAN optimization that may make it seem as though gradient descent optimization may not work in theory. For instance, it is well-known that at the optimal location \( p_y = p_{\text{data}} \), the optimal discriminator will output zero on all examples, which in turn means that any generator distribution will be optimal for this generator. This would seem to imply that the system can not be stable around such an equilibrium.

However, as we will show, the gradient descent GAN optimization procedure is locally asymptotically stable, even for natural parameterizations of generator/discriminator pairs (which still make up concave-concave optimization problems). Furthermore, at equilibrium, although the zero-discriminator property means that the generator is not stable “independently”, the joint dynamical system of generator and discriminator is locally asymptotically stable around certain equilibrium points.

### 3.3 Local stability of general GAN systems

This section contains our first technical result, establishing that GAN optimization is locally stable under proper local conditions. Although the proofs are all deferred to the appendix, the elements that we do emphasize here are the conditions that we identified for local stability to hold. Indeed, because the proof rests on these conditions (some of which are fairly strong), we want to highlight them as much as possible, as they themselves also convey valuable intuition as to what is required for GAN convergence.

To formalize our conditions, we denote the support of a distribution with probability density function (p.d.f) \( p \) by supp\((p)\) and the p.d.f of the generator \( \theta_G \) by \( p_{\theta_G} \). Let \( B(\cdot) \) denote the Euclidean \( L_2 \)-ball of radius of \( \epsilon \). Let \( \lambda_{\text{max}}(\cdot) \) and \( \lambda_{\text{min}}(\cdot) \) denote the largest and the smallest non-zero eigenvalues of a non-zero positive semidefinite matrix. Let Col\((\cdot)\) and Null\((\cdot)\) denote the column space and null space of a matrix respectively. Finally, we define two key matrices that will be integral to our analyses:

\[
K_{DD} \triangleq \mathbb{E}_{p_{\text{data}}}[(\nabla_{\theta_D} D_{\theta_D}(x)) (\nabla_{\theta_D} D_{\theta_D}(x))^T]_{\theta_D} \\
K_{DG} \triangleq \int_X \nabla_{\theta_D} D_{\theta_D}(x) \nabla_{\theta_G} p_{\theta_G}(x) dx \bigg|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*}
\]

Here, the matrices are evaluated at an equilibrium point \((\theta_D^*, \theta_G^*)\) (which we will characterize shortly). The significance of these terms is that, as we will see, \( K_{DD} \) is the negative Hessian of the GAN objective at equilibrium with respect to the discriminator parameters, and \( K_{DG} \) is the off-diagonal term in this Hessian, corresponding to the discriminator and generator parameters (ignoring the loss function \( f \)).

We now discuss conditions under which we can guarantee exponential stability. In our first condition we define the kind of “good” equilibria we care about. In particular, we assume that all equilibria in a small neighborhood around a considered equilibrium correspond to a generator which matches the true distribution and a discriminator that is identically zero on the support of this distribution. As described below, implicitly, this also assumes that the discriminator and generator representations are powerful enough to guarantee that there are no “bad” equilibria in a small local neighborhood of this equilibrium.

The assumption that the generator matches the true distribution is a rather strong assumption, as it limits us to the “realizable” case, where the generator is capable of creating the underlying data distribution. Furthermore, this means the discriminator is (locally) powerful enough that for any other generator distribution it is not at equilibrium (i.e., discriminator updates are non-zero). Since we do not typically expect this to be the case, we also provide an alternative non-realizable assumption that is also sufficient for our results. However, in both the realizable and non-realizable cases the restriction on the discriminator remains. This implicitly requires that the generator representation be (locally) rich enough so that when the discriminator is not identically zero, the generator is not at
equilibrium (i.e., generator updates are non-zero). Finally, note that these conditions do not disallow bad equilibria outside of this neighborhood which may potentially even be unsteady.

**Assumption I.** For any equilibrium point \((\theta^*_D, \theta^*_G)\), \(p_{\theta^*_G} = p_{\text{data}}\) and \(D_{\theta^*_G}(x) = 0, \forall x \in \text{supp}(p_{\text{data}})\).

**Assumption I. (Non-realizable)** The discriminator is linear in its parameters \(\theta_D\) and furthermore, for any equilibrium point \((\theta^*_D, \theta^*_G)\), \(D_{\theta^*_G}(x) = 0, \forall x \in \text{supp}(p_{\text{data}}) \cup \text{supp}(p_{\theta^*_G})\).

This second assumption is largely a weakening of Assumption I, as the condition on the discriminator remains, but there is no requirement that the generator give rise to the true distribution. However, the requirement that the discriminator be linear in the parameters (not in its input), is an additional restriction that seems unavoidable in this case for technical reasons. Further, note that the fact that \(D_{\theta^*_G}(x) = 0\) and that the generator/discriminator are both at equilibrium, still means that although it may be that \(p_{\theta^*_G} \neq p_{\text{data}}\), these distributions are (locally) indistinguishable as far as the discriminator is concerned. Indeed, this is a nice characterization of “good” equilibria, that the discriminator cannot differentiate between the real and generated samples.

The next assumption is straightforward, making it necessary that the loss \(f\) be strictly concave (as we will show, for non-strictly concave losses, there need not be local asymptotic convergence).

**Assumption II.** The function \(f\) satisfies \(f''(0) < 0\), and \(f'(0) \neq 0\).

The goal of our next assumption will be to allow systems with multiple equilibria in the neighborhood of a considered equilibrium, though in a limited sense. To state our assumption, we first define the following property:

**Property I.** A function \(g : \Theta \to \mathbb{R}\) is said to satisfy Property I at \(\theta^* \in \Theta\) if for any \(\theta \in \text{Null}(\nabla^2 g(\theta)|_{\theta^*})\), the function is locally constant along \(\theta\) at \(\theta^*\) i.e., \(\exists \epsilon > 0\) such that for all \(\epsilon' \in (-\epsilon, \epsilon)\), \(g(\theta^* + \epsilon\theta) = g(\theta^*)\).

In other words, we require that along any direction either the second derivative of \(g\) must be non-zero or all derivatives must be zero. For example, consider \(g(x, y) = x^2 + y^2\) at the origin. The function is flat along \(y\) and along any other direction at an angle \(\alpha \neq 0\) with the \(y\) axis, the second derivative is \(2\sin^2 \alpha\).

For the GAN system, we will require this property for two important convex functions whose Hessians are proportional to \(K_{DD}\) and \(K^2_{DG} K_{DG}\). We provide more intuition for these functions below.

**Assumption III.** At an equilibrium \((\theta^*_D, \theta^*_G)\), the functions \(\mathbb{E}_{p_{\text{data}}} [D^2_{\theta^*_G}(x)]\) and \(\left\| \mathbb{E}_{p_{\text{data}}} [\nabla_{\theta_D} \nabla_{\theta_D} (x)] - \mathbb{E}_{p_{\theta_G}} [\nabla_{\theta_D} \nabla_{\theta_D} (x)] \right\|^2_{\theta_D = \theta^*_D}\) must satisfy Property I in the discriminator and generator space respectively.

Now, for any \(\theta_D\), the first function above is a measure of how far away \(\theta_D\) is from an all-zero state. Note that at equilibrium this function is zero. We will see later that the curvature of this function at \(\theta^*_D\) is representative of the curvature of \(V(\theta_D, \theta^*_G)\) in the discriminator space, given that \(f''(0) < 0\).

In fact these quantities are also related to \(K_{DD}\) as shown below:

\[
\nabla^2_{\theta_D} \mathbb{E}_{p_{\text{data}}} [D^2_{\theta_D}(x)]|_{\theta^*_D} = \frac{2}{f''(0)} \nabla^2_{\theta_D} V(\theta_D, \theta^*_G)|_{\theta^*_G} = 2K_{DD}
\]

As a result of this, if the curvature is flat along a direction \(u\) (which also means that \(K_{DD} u = 0\)), we can show that we can perturb \(\theta^*_D\) slightly along \(u\) and still be an equilibrium discriminator as defined in Assumption I i.e., \(\forall x \in \text{supp}(p_{\theta^*_G})(x), D_{\theta^*_G}(x) = 0\). We prove this formally in Lemma C.2.

We can characterize the second function similarly as a measure of how far away \(\theta_G\) is from the true distribution. Again, note that at equilibrium, this function is zero. The curvature of this function at \(\theta^*_G\) is representative of the curvature of the magnitude of the discriminator update on the optimal discriminator as a function of \(\theta_G\), given that \(f'(0) \neq 0\). It is also related to \(K_{DG}\) as shown below:

\[
\nabla^2_{\theta_G} \left( \left\| \mathbb{E}_{p_{\text{data}}} [\nabla_{\theta_D} \nabla_{\theta_D} (x)] - \mathbb{E}_{p_{\theta_G}} [\nabla_{\theta_D} \nabla_{\theta_D} (x)] \right\|^2 \right)|_{(\theta^*_G, \theta^*_D)} = 2K_{DG}
\]
The intuition behind this relation is that, the farther $\theta_G$ is from the true distribution, the more suboptimal should $\theta_D^*$ be. The more suboptimal that $\theta_D^*$ is, the larger the magnitude of update on $\theta_D$. Now, for any direction $v$ along which the curvature is flat (i.e., $K_{DG}v = 0$), we can perturb $\theta_G^*$ slightly along that direction such that $\theta_G$ remains an equilibrium generator as defined in Assumption I i.e., $p_{\theta_G} = p_{\text{data}}$. Again, we prove this formally in Lemma C.2.

Note that had we assumed $K_{DD}, K_{DG}K_{DG} > 0$, it would mean that locally there is only a unique equilibrium.

As a final assumption, we require that all the generators in a sufficiently small neighborhood of the equilibrium have distributions with the same support as the true distribution. Then,

**Assumption IV.** $\exists \epsilon > 0$ such that $\forall \theta_G \in B_{\epsilon G}(\theta_G^*)$, $\text{supp}(p_{\theta_G}) = \text{supp}(p_{\text{data}})$.

Later in Appendix C.1, we show that we can replace this assumption with a more realistic smoothness condition on the discriminator.

We now state our result.

**Theorem 3.1.** The dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 4 is locally exponentially stable with respect to an equilibrium point $(\theta_D^*, \theta_G^*)$ under the Assumptions I, II, III, IV. Furthermore, the rate of convergence is governed only by the eigenvalues $\lambda$ of the Jacobian $J$ of the system at equilibrium with a strict negative real part upper bounded as:

- If $\text{Im}(\lambda) = 0$, then $\text{Re}(\lambda) \leq \frac{2f''(0)f''(0)\lambda_{\text{max}}^+(K_{DD})\lambda_{\text{min}}^+(K_{DG}^TK_{DD}^T)\lambda_{\text{min}}^+(K_{DG}^TK_{DG})}{4f''(0)\lambda_{\text{min}}^+(K_{DD})f''(0)\lambda_{\text{max}}^+(K_{DD})+f''(0)\lambda_{\text{min}}^+(K_{DG}^TK_{DD}^T)\lambda_{\text{min}}^+(K_{DG}^TK_{DG})}$
- If $\text{Im}(\lambda) \neq 0$, then $\text{Re}(\lambda) \leq f''(0)\lambda_{\text{min}}^+(K_{DD})$

The vast majority of our proofs are deferred to the appendix, but we briefly describe the intuition here. It is straightforward to show that the Jacobian $J$ of the system at equilibrium is:

$$J = \begin{bmatrix} J_{DD} & J_{DG} \\ -J_{DG}^T & J_{GG} \end{bmatrix} = \begin{bmatrix} 2f''(0)K_{DD} & f''(0)K_{DG} \\ -f''(0)K_{DG}^T & 0 \end{bmatrix}$$

Recall that we wish to show this is Hurwitz. First note that $J_{DD}$ (the Hessian of the objective with respect to the discriminator) is negative semi-definite if and only if $f''(0) < 0$ Next, a crucial observation is that $J_{GG} = 0$ i.e., the Hessian term w.r.t. the generator vanishes because for the all-zero discriminator, all generators result in the same objective. Fortunately, this means at equilibrium we do not have non-convexity in $\theta_G$ precluding local stability. Then, we make use of the crucial Lemma G.2 we prove in the appendix, showing that any matrix of the form $[\begin{bmatrix} -Q & P \\ -P^T & 0 \end{bmatrix}]$ is Hurwitz provided that $Q$ is strictly negative definite, and $P$ has full column rank.

However, this property holds only when $K_{DD}$ is positive definite and $K_{DG}$ is full column rank which is not necessarily the case according to Assumption III. We show that the rank deficiency is due to a subspace of equilibria around $(\theta_D^*, \theta_G^*)$ and consequently, we can analyze the stability of the system projected to an subspace orthogonal to these equilibria (Theorem A.4). Additionally, we also prove stability using Lyapunov’s stability Theorem A.1 by showing that the squared $L_2$ distance to the subspace of equilibria always either decreases or only instantaneously remains constant.

**3.4 Additional results**

Next we highlight some additional results, including a simple setting under which the assumptions hold, and a proof that the WGAN (for which $f'' = 0$) need not be not asymptotically stable.

**Linear Quadratic Gaussian GANs** In order to illustrate our assumptions in Theorem 3.1, consider a simple GAN that learns an $n$-dimensional Gaussian distribution $\mathcal{N}(\mu, \Sigma)$ (where $\Sigma > 0$). Let the latent variable be drawn from the standard normal, $\mathcal{N}(0, I_n)$. Consider a quadratic discriminator $D(x) = x^T W_2 x + w_2^T x$, and a linear generator $G(z) = Az + b$. Here, we restrict $A$ to the space of symmetric matrices. We call the resulting system LQ (linear-quadratic). Let $\Sigma^{1/2}$ be the unique real positive definite matrix such that $(\Sigma^{1/2})^2 = \Sigma$. Then we have the following:
Theorem 3.2. In LQ, \( A = \Sigma^{1/2}, b = \mu \) and \( W_2 = 0, w_1 = 0 \) corresponds to an equilibrium that is locally exponentially stable provided \( f''(0) < 0 \) and \( f'(0) \neq 0 \).

WGANs are not necessarily asymptotically stable. So far we have seen positive results about the stability of GANs when \( f''(0) < 0 \). We now consider the case where \( f(x) = x \) i.e., the Wasserstein GAN and so \( f''(x) = 0 \); we show that the equilibrium is not necessarily asymptotically stable in this case. We consider a specific case of the LQ WGAN that learns a zero mean gaussian distribution, and show that there exists points near certain equilibria such that if the system is initialized to that point, it will periodically come back to that initial point rather than converge to the equilibrium. The proof of this result is in Appendix E, but we also visualize a simple two-dimensional example in Section 4, illustrating periodic cycles for WGAN optimization.

Theorem 3.3. The LQ WGAN system for learning a zero mean Gaussian distribution \( \mathcal{N}(0, \Sigma) \) (\( \Sigma \succ 0 \)) is not asymptotically stable at the equilibrium corresponding to \( A = \Sigma^{1/2}, b = 0 \) and \( W_2 = 0, w_1 = 0 \).

3.5 Stabilizing optimization via gradient-based regularization

Motivated by the considerations above, in this section we propose a regularization penalty for the generator update, which uses a term based upon the gradient of the discriminator. Crucially, the regularization term does not change the parameter values at the equilibrium point, and at the same time enhances the local stability of the optimization procedure, both in theory and practice. Specifically, we propose the regularized update

\[
\theta_G := \theta_G - \alpha \nabla_{\theta_G} \left( \hat{V}(D_{\theta_D}, G_{\theta_G}) + \eta \| \nabla_{\theta_D} \hat{V}(D_{\theta_D}, G_{\theta_G}) \|^2 \right) \tag{5}
\]

Although these update equations do require that we differentiate with respect to a function of another gradient term, such “double backprop” terms (see e.g., [Drucker and Le Cun, 1992]) are easily computed by modern automatic differentiation tools.

Local Stability The intuition of this regularizer is perhaps most easily understood by considering how it changes the Jacobian at equilibrium (though there are other means of motivating the update as well, discussed further in Appendix F.1). We show that the Jacobian of the new update is

\[
\begin{bmatrix}
-J_{DD} & J_{DG} \\
-J_{DG}^T (I - 2\eta J_{DD}) & -2\eta J_{DG}^T J_{DG}
\end{bmatrix}
\]

Although there are now non-antisymmetric diagonal blocks, the block diagonal terms are now negative definite. As we show in the following theorem, as long as we choose \( \eta \) small enough so that \( I - 2\eta J_{DD} \succeq 0 \), this guarantees the updates are locally asymptotically stable for any concave \( f \).

Theorem 3.4. The dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 5, the system is locally exponentially stable at the equilibrium, under the same conditions as in Theorem 3.1, if \( \eta < \frac{1}{2\lambda_{\max}(-J_{DD})} \). Further, under similar conditions, the WGAN system is locally exponentially stable at the equilibrium. The rate of convergence for the WGAN is governed only by the eigenvalues \( \lambda \) of the Jacobian at equilibrium with a strict negative real part upper bounded as:

- If \( \text{Im}(\lambda) = 0 \), then \( \Re(\lambda) \leq -\frac{2f^2(0)\eta \lambda_{\min}^{(+)}(K_{DG}^T K_{DG})}{4f^2(0)\eta^2 \lambda_{\max}^2(K_{DG}^T K_{DG})+1} \)
- If \( \text{Im}(\lambda) \neq 0 \), then \( \Re(\lambda) \leq -\eta f^2(0)\lambda_{\min}^{(+)}(K_{DG}^T K_{DG}) \)

In addition to stability properties, this regularization term has natural connections to an important issue that arises in GAN optimization. Namely, in mode collapse, GANs may enter an irrecoverable failure state where the generator incorrectly assigns all its probability mass to a small region in space. This arises because a globally optimal strategy for the generator is to push all its mass towards the single point that the discriminator is the most confident about being a real data point. To overcome this the generator needs more “foresight” i.e., it must know that when it collapses all mass, the discriminator will subsequently label the collapsed point as fake data. Our penalty indeed encodes this foresight, because the discriminator’s ability to outdo the generator is quantified by the magnitude of the discriminator’s gradient. Thus, the generator seeks a state where it can spread data.
out enough, to make sure the discriminator has no obvious countermeasure (i.e., no big gradients). h
the discriminator into a

Finally, as we show in Appendix F.2, our penalty term and 1-unrolled GANs have very similar
structure because intuitively both provide a one-step lookahead to the generator. More precisely, we
can arrive at 1-unrolled updates if we simplify our updates further and replace $\theta_D$ by an “unrolled”
$\theta_D + \eta \nabla_{\theta_D} \hat{V}(D_{\theta_D}, G_{\theta_G})$. Although the resulting update is powerful, the step size $\eta$ is then
constrained to be small. Our method on the other hand, allows for larger $\eta$ which provides a way of
leveraging the lookahead more flexibly. In practice, we see that our method can be as powerful as the
more complex and slower 10-unrolled GANs.

4 Experimental results

We very briefly present experimental results that demonstrate that our regularization term also has
substantial practical promise. In Figure 1, we compare our gradient regularization to 10-unrolled
GANs on the same architecture and dataset (a mixture of seven gaussians) as in Metz et al. [2016].
The system quickly spreads out all the points instead of first exploring only a few modes and then
redistributing its mass over all the modes gradually. Note that the conventional GAN updates enters
mode collapse for this setup. We see similar results (see Figure 3 here, and Figure 4 in the Appendix
for a more detailed figure) in the case of a stacked MNIST dataset using a DCGAN [Radford et al.,
2015] i.e., three random digits from MNIST are stacked together so as to create a distribution over
1000 modes. Finally, Figure 2, presents streamline plots for a 2D system where both the true and the
latent distribution is uniform over $[-1, 1]$ and the discriminator is $D(x) = w_2 x^2$ while the generator
is $G(z) = a z$. Observe that while the WGAN system goes in orbits as expected, the original GAN
system converges. With our updates, both these systems converge quickly to the true equilibrium.

| Iteration 0 | Iteration 3000 | Iteration 8000 | Iteration 50000 | Iteration 70000 |
|-------------|---------------|---------------|----------------|---------------|

Figure 1: Gradient regularized GAN, $\eta = 0.5$ (top row) vs. 10-unrolled with $\eta = 10^{-4}$ (bottom row)

5 Conclusion

In this paper, we presented a theoretical analysis of the local asymptotic stability of GAN optimization
under proper conditions. We further showed that the recently-proposed WGAN is not asymptotically
stable under the same conditions, but we introduced a gradient-based regularizer which stabilizes
both traditional GANs and the WGANs, and can improve convergence speed in practice.

The results here provide substantial insight into the nature of GAN optimization, perhaps even offering
some clues as to why these methods have worked so well despite not being convex-concave. Our
results also offer some connections to previously-proposed methods like unrolled GANs. However,
we also emphasize that there are substantial limitations to the analysis, and directions for future work.
Perhaps most notably, the analysis here only provides an understanding of what happens locally,
close to an equilibrium point. For non-convex architectures this may be all that is possible, but it
seems plausible that much stronger global convergence results could hold for simple settings like the
linear quadratic GAN (indeed, as the streamline plots show, we observe this in practice for simple
domains). Second, the analysis here doesn’t show the equilibrium points necessarily exist, but only
illustrates convergence if there do exist points that satisfy certain criteria: the existence question has
been addressed by previous work [Arora et al., 2017], but much more analysis remains to be done here. GANs are rapidly becoming a cornerstone of deep learning methods, and the theoretical and practical understanding of these methods will prove crucial in moving the field forward.

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A Preliminaries

In this section, we present preliminaries from non-linear systems theory [Khalil, 1996]. In particular, we formally define local stability of dynamic systems, and then present an important theorem that helps us study stability of non-linear systems. Finally, we present a modification of this result that will be crucial in proving stability of GANs under our assumptions.

Consider a system consisting of variables $\theta \in \mathbb{R}^n$ whose time derivative is defined by $h(\theta)$ i.e.,

$$\dot{\theta} = h(\theta). \quad (6)$$

Without loss of generality let the origin be an equilibrium point of this system. That is, $h(0) = 0$. Let $\theta(t)$ denote the state of the system at some time $t$. Then, we have the following definition of local stability:

**Definition A.1 (Stability).** (Definition 4.1 from Khalil [1996]) The origin of the system in Equation 6 is

- stable if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that
  $$\|\theta(0)\| < \delta \implies \|\theta(t)\| < \epsilon, \forall t \geq 0.$$
- unstable if not stable.
• asymptotically stable if it is stable and \( \delta > 0 \) can be chosen such that
\[
\|\theta(0)\| < \delta \implies \lim_{t \to \infty} \theta(t) = 0
\]

• exponentially stable if it is asymptotically stable and \( \delta, k, \lambda > 0 \) can be chosen such that
\[
\|\theta(0)\| < \delta \implies \|\theta(t)\| \leq k\|\theta(0)\| \exp(-\lambda t)
\]

The system is stable if for any chosen ball around the equilibrium (of radius \( \epsilon \)), one can initialize the system anywhere within a sufficiently small ball around the equilibrium (of radius \( \delta(\epsilon) \)) such that the system always stays within the \( \epsilon \) ball. Note that such a system may either converge to equilibrium or orbit around equilibrium perenially within the \( \epsilon \) ball. In contrast, a system is unstable if there are initializations that are arbitrarily close to the equilibrium which can escape an \( \epsilon \)-ball. Finally, asymptotic stability is a stronger notion of stability, which implies that there is a region around the equilibrium such that any initialization within that region will converge to the equilibrium (in the limit \( t \to \infty \)). For example, as we saw, GANs are always stable; however, WGANs are stable but not asymptotically stable.

**Extension to multiple equilibria.** Note that since a GAN system might have multiple arbitrarily close equilibria, or a subspace of equilibria, we will define asymptotic stability to imply convergence to any of the equilibria in the neighborhood of a considered equilibrium. That is, \( \lim_{t \to \infty} \theta(t) = \theta^* \) where \( \theta^* \) is either the considered equilibrium point at the origin or any other equilibrium point that is within some small neighborhood around origin.

We now present Lyapunov’s stability theorem which is used to prove locally asymptotic stability of a given system. The basic idea is that a system is asymptotically stable if we can find a scalar “energy” function \( V(\theta) \) (also called a Lyapunov function) that i) is positive definite which means, \( V(\theta) \) positive everywhere and zero at the equilibrium ii) its time derivative \( \dot{V}(\theta) \) is strictly negative around the equilibrium.

**Theorem A.1 (Lyapunov function).** *(Theorem 4.1 from Khalil [1996])* Let \( B_{\epsilon}(0) \) be a small region around the origin of the system in Equation 6. Let \( V : B_{\epsilon}(0) \to \mathbb{R} \) be a continuously differentiable function such that

- it is positive definite i.e., \( V(0) = 0 \) and \( V(\theta) > 0 \) for \( \theta \in B_{\epsilon}(0) - \{0\} \)
- \( \dot{V}(\theta) \leq 0 \) for \( \theta \in B_{\epsilon}(0) - \{0\} \)

Then, the origin is stable. Moreover, if

\[
\dot{V}(\theta) < 0, \ \forall \theta \in B_{\epsilon}(0) - \{0\}
\]

then the origin is asymptotically stable.

We next present an important tool that simplifies the study of stability of of non-linear systems. The result is that one can “linearize” any non-linear system near an equilibrium and analyze the stability of the linearized system to comment on the local stability of the original system.

**Theorem A.2 (Linearization).** *(Theorem 4.5 from Khalil [1996])* Let \( \mathbf{J} \) be the Jacobian of the system in Equation 6 at its origin i.e.,

\[
\mathbf{J} = \frac{\partial h(\theta)}{\partial \theta} |_{\theta=0}.
\]

Then,

- The origin is locally exponentially stable if \( \mathbf{J} \) is Hurwitz i.e., \( \text{Re}(\lambda) < 0 \) for all eigenvalues \( \lambda \) of \( \mathbf{J} \).
- The origin is unstable if \( \text{Re}(\lambda) > 0 \) for all eigenvalues \( \lambda \) of \( \mathbf{J} \).

The key idea in the proof for this result is that the system can be written as \( h(\theta) = \mathbf{J}\theta + g_1(\theta) \), where \( g_1(\theta) \), the remainder of the linear approximation is bounded as \( \|g_1(\theta)\| \leq O(\|\theta\|^2) \) sufficiently
close to equilibrium. Now, it turns out that when \( J \) is Hurwitz, one can find a quadratic Lyapunov function for the original system whose rate of decrease is also quadratic in \( \theta \). Since, \( ||g_1(\theta)|| \) is only a quadratic remainder term, one can show that the remainder term only adds a cubic term to the change in the Lyapunov function. This is however smaller than a quadratic change near the equilibrium, and therefore the quadratic Lyapunov function for the linearized system works as a Lyapunov function for the original system too.

In all our analyses, we will linearize our system and show that the Jacobian is Hurwitz. However, it is often useful to identify the quadratic Lyapunov function for the (linearized) system. Unfortunately, for some of the Jacobians we will encounter, it is hard to come up with a quadratic Lyapunov function that always strictly decreases. Instead, we will identify a function that either strictly decreases or sometimes remains constant but only instantaneously. While Lyapunov’s stability theorem does not help us conclude anything about stability for this case, the following corollary of LaSalle’s theorem (we do not state the theorem here) is indeed sufficient to prove asymptotic stability in this case.

**Theorem A.3 (Corollary of LaSalle’s invariance principle).** Corollary 4.1 from Khalil [1996]. Let \( B_\epsilon(0) \) be a small region around an equilibrium \( 0 \) of the system in Equation 6. Let \( V : B_\epsilon(0) \to \mathbb{R} \) be a continuously differentiable function such that

- \( V(\theta) = 0 \) if and only if \( \dot{\theta} = 0 \) and \( V(\theta) > 0 \) for \( \theta \in B_\epsilon(0) - \{0\} \) such that \( \dot{\theta} \neq 0 \).
- \( \dot{V}(\theta) \leq 0 \) for \( \theta \in B_\epsilon(0) - \{0\} \)
- Let \( S = \{ \theta \in B_\epsilon(0) \mid \dot{V}(\theta) = 0 \} \). There is no trajectory that identically stays in \( S \) except for the trajectories at equilibrium points.

then the system is locally asymptotically stable with respect to 0 and other equilibria in its neighborhood.

Finally, we prove a theorem that help us deal with analyzing the stability of a special kind of non-linear systems, specifically those with multiple equilibria in a local neighborhood of a considered equilibrium.

**Theorem A.4.** Consider a non-linear system of parameters \((\theta, \gamma)\),

\[
\dot{\theta} = h_1(\theta, \gamma), \quad \dot{\gamma} = h_2(\theta, \gamma)
\]

with an equilibrium point at the origin. Let there exist \( \epsilon \) such that for any \( \gamma \in B_\epsilon(0) \), \((0, \gamma)\) is an equilibrium point. Then, if

\[
J = \left. \frac{\partial h_1(\theta, \gamma)}{\partial \theta} \right|_{(0,0)}
\]

is a Hurwitz matrix, the non-linear system in Equation 7 is exponentially stable.

**Proof.** The proof for this statement is quite similar to the proof of the original theorem for linearization. The high level idea is that if \( J \) is exponentially stable, then there exists a quadratic Lyapunov function that is always decreasing for the system \( \dot{\theta} = J\theta \). Then, we show that the same quadratic function works for the original non-linear system too in a small neighborhood around equilibrium for which the non-linear remainder terms are sufficiently small.

Let,

\[
h_1(\theta, \gamma) = J\theta + g_1(\theta, \gamma).
\]

The first crucial step is to show that for any constant \( c > 0 \), for a sufficiently small neighborhood around the equilibrium, we will have \( ||g_1(\theta, \gamma)|| \leq c||\theta|| \). To show this, consider the Taylor series expansion for the remainder \( g_1(\theta, \gamma) \) around equilibrium. Clearly, the expansion would not have a constant term because \( h(0, \gamma) = 0 \). It would not have a linear term in \( \theta \) because that is accounted for already. Finally, it will not have any term that is purely a function of \( \gamma \), because \( h(0, \gamma) = 0 \) in a small neighborhood around equilibrium (since \((0, \gamma)\) are all equilibria). Therefore, we can write:

\[
g_1(\theta, \gamma) = \theta g_2(\gamma) + g_3(\theta, \gamma)
\]
where $g_2(\gamma)$ only consists of linear or higher degree terms in $\gamma$ and $g_3(\theta, \gamma)$ consists only of terms that are quadratic or higher degree terms in $\theta$ (and any arbitrary degree of $\gamma$). Therefore, we have that:

$$\lim_{\gamma \to 0} g_2(\gamma) = 0, \quad \lim_{\theta \to 0} g_3(\theta, \gamma) = 0$$

Then, for an arbitrarily chosen small constant $c$, for a sufficiently close neighborhood around the equilibrium, we can say that $\|g_2(\gamma)\| \leq c/2$ and $\|g_3(\theta, \gamma)\| \leq c\|\theta\|/2$. Thus,

$$\|g_1(\theta, \gamma)\| \leq \|\theta\|\|g_2(\gamma)\| + \|g_3(\theta, \gamma)\| \leq c\|\theta\|$$

Now, by Theorem 4.6 in Khalil [1996], we have that for any positive definite symmetric matrix $Q$, there exists a positive definite matrix $P$ such that $J^T P + JP = -Q$. Then, if we choose $V(x) = \theta^T P \theta$ as the quadratic Lyapunov function for the linearized system in Equation 8, the rate of its decrease is given by $-\theta^T Q \theta$ which is negative at all points except at $\theta = 0$.

Now, if we use the same Lyapunov function, the rate of decrease near the origin for the original system would be $\dot{V}(x) = -\theta^T Q \theta + \theta^T P g_1(\theta, \gamma)$. If we choose a sufficiently small neighborhood such that $\|g_1(\theta, \gamma)\| \leq \frac{1}{2\|P\|_2} \lambda_{\text{min}}(Q)\|\theta\|$, then we have that,

$$\dot{V}(x) \leq -\lambda_{\text{min}}(Q)\|\theta\|^2 + \frac{1}{2\|P\|_2} \lambda_{\text{min}}(Q)\|P\|_2\|\theta\|^2 = -\frac{1}{2} \lambda_{\text{min}}(Q)\|\theta\|^2 < 0$$

Now, as long as we ensure that the trajectory of the system remains in the neighborhood around origin for which $\|g_1(\theta, \gamma)\| \leq \frac{1}{2\|P\|_2} \lambda_{\text{min}}(Q)\|\theta\|$ and $\|\gamma\| < \epsilon$, this system would then exponentially converge to one of the equilibria near origin. Let us call this neighborhood $S$ i.e., within this neighborhood of $\gamma$ and $\theta$, the Lyapunov function strictly decreases for the non-linear system.

This brings us to the second crucial part of this proof, which is to ensure that we always stay in $S$. Let $S$ contain a ball of radius $d$. We will show that for sufficiently close initializations which are within a ball of radius $d/2$, the displacement of $\gamma$ is at most $d/2$. Since $\theta$ only approaches origin, this means that the system never exited $S$.

Let us consider the Taylor series expansion of $h_2(\theta, \gamma)$. First of all, there is no constant term. Next, there is no term that is purely a function of $\gamma$ because $h_2(0, \gamma) = 0$. Then, we can say that:

$$h_2(\theta, \gamma) = g_4(\theta, \gamma)$$

In a small neighborhood around equilibrium, there exists a fixed constant $c'$ such that $\|g_4(\theta, \gamma)\|_2 \leq c'$. Then, $h_2(\theta, \gamma) \leq c'\|\theta\|$.

![Diagram](image-url)

**Figure 5:** Illustration of Theorem A.4. $S$ is the neighborhood within which $\theta$ converges exponentially to 0 to a point on the $\gamma$-axis which corresponds to an equilibrium. However, all initializations within $S$ may not preserve the trajectory within $S$ as illustrated by the dashed trajectory. We identify a smaller ball within $S$ such that any initialization within that ball ensures exponential convergence of $\theta$. 

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Now, if the trajectory indeed always remained in $\mathcal{S}$, we know that $\|\theta(t)\| = \|\theta(0)\| \exp (-c''t)$ for some constant $c'' > 0$. Assume we initialize $\theta(0)$ within a radius of $\frac{c''d}{c''t_0}$. The rate at which $\gamma$ changes at any point is,

$$||\gamma|| \leq c'\|\theta(0)\| \exp (-c''t)$$

Then, the maximum displacement in $\gamma$ can be,

$$\int_{t=0}^{\infty} c'\|\theta(0)\| \exp (-c''t) dt = c'\|\theta(0)\| \leq \frac{d'}{2}$$

Thus, the trajectory always lies in $\mathcal{S}$, which implies exponential convergence along $\theta$ to a point where $\theta = 0$ and $||\gamma|| < \epsilon$. Thus the system exponentially converges to an equilibrium.

\[\square\]

**B  GANs are not concave-convex near equilibrium**

In this section, we consider a more general system than the one considered in the main paper to demonstrate that GANs are not concave-convex near equilibrium. In particular, consider the following discriminator and generator pair learning a distribution in 1-D:

$$D_w(x) = \sum_{i=0}^{d_D} w_i x^i$$

$$G_a(z) = \sum_{j=0}^{d_G} a_j z^j$$

where $d_D \geq 1$ and $d_G \geq 1$. Let the distribution learned be arbitrary. On the other hand, let the latent distribution be the standard normal. Then, the gradient of the objective with respect to the generator parameters is:

$$\frac{\partial V(G, D)}{\partial a_j} = -E_{z \sim N(0,1)} \left[ f' \left( -\sum_{i=0}^{d_D} w_i (G_a(z))^i \right) \cdot \left( \sum_{i=1}^{d_D} i w_i (G_a(z))^{i-1} \right) \cdot z^j \right]$$

The second derivative is,

$$\frac{\partial^2 V(G, D)}{\partial a_j^2} = -E_{z \sim N(0,1)} \left[ f'' \left( -\sum_{i=0}^{d_D} w_i (G_a(z))^i \right) \cdot \left( \sum_{i=2}^{d_D} i(i-1) w_i (G_a(z))^{i-2} \right) \cdot z^{2j} \right]$$

$$+ E_{z \sim N(0,1)} \left[ f'' \left( -\sum_{i=0}^{d_D} w_i (G_a(z))^i \right) \cdot \left( \sum_{i=1}^{d_D} i w_i (G_a(z))^{i-1} \right) \cdot z^{2j} \right]^2$$

Now, consider the case where $f''(x) < 0$. For points where $w_1 \neq 0$ but $w_i = 0$ for all $i \neq 1$, the term above simplifies to:

$$E_{z \sim N(0,1)} \left[ f'' \left( -w_0 (G_a(z)) \right) \cdot (w_1 z^j)^2 \right]$$

which is clearly negative. On the other hand, consider the case where $f''(x) = 0$ for all $x \in \mathbb{R}$. Then, if $d_D > 2$, we can consider $w_2 \neq 0$ while $w_i = 0$ for all $i \neq 2$. The second derivative simplifies to:

$$-E_{z \sim N(0,1)} \left[ f' \left( -w_2 (G_a^2(z)) \right) 2w_2 z^{2j} \right].$$

If $f'(x) > 0$ for all $x$ (which is true in the case of WGANs), then in the region $w_2 > 0$ the above term is negative i.e., the GAN objective is concave.
C  Local exponential stability of GANs

In this section, we provide the full proof for our result about the local stability of GANs through the following lemmas.

**Lemma C.1.** For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 4, the Jacobian at an equilibrium point \((\theta_D^*, \theta_G^*)\), under the Assumptions I and IV is:

\[
\mathbf{J} = \begin{bmatrix} \mathbf{J}_{DD} & \mathbf{J}_{DG} \\ -\mathbf{J}_{GD}^T & \mathbf{J}_{GG} \end{bmatrix} = \begin{bmatrix} 2f''(0)K_{DD} & f'(0)K_{DG} \\ -f'(0)K_{DG} & 0 \end{bmatrix}
\]

where

\[
K_{DD} \triangleq \mathbb{E}_{p_{data}}[(\nabla_{\theta_D} D_{\theta_D}(x))(\nabla_{\theta_D} D_{\theta_D}(x))^T]_{\theta_D} \geq 0
\]

and

\[
K_{DG} \triangleq \int_X \nabla_{\theta_D} D_{\theta_D}(x)\nabla_{\theta_G} p_{\theta_G}(x)dx \Bigg|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*}
\]

**Proof.** To derive the Jacobian, we begin with a subtly different algebraic form of the GAN objective in Equation 3 by replacing the term \(\mathbb{E}_{z \sim p_{data}}[f(-D_{\theta_D}(G_{\theta_G}(z)))]\) with \(\mathbb{E}_{\theta_G}[f(-D_{\theta_D}(x))] = \int_X p_{\theta_G}(x)f(-D_{\theta_D}(x))\). Effectively, we separate the discriminator and the generator’s effects in this term. This is crucial because we will proceed with all of our analysis in this form. Observe that the system then becomes,

\[
V(D_{\theta_D}, G_{\theta_G}) = \mathbb{E}_{p_{data}}[f(D_{\theta_D}(x))] + \mathbb{E}_{\theta_G}[f(-D_{\theta_D}(x))]
\]

\[
\dot{\theta}_D = \mathbb{E}_{p_{data}}[f'(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{\theta_G}[f'(-D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)]
\]

\[
\dot{\theta}_G = -\int_X \nabla_{\theta_G} p_{\theta_G}(x)f(-D_{\theta_D}(x))dx
\]

Let \(n_D\) be the number of discriminator parameters and \(n_G\) the number of generator parameters. Then the first \(n_D \times n_D\) block in \(\mathbf{J}\), which we will denote by \(\mathbf{J}_{DD}\) is:

\[
\mathbf{J}_{DD} \triangleq \nabla^2_{\theta_D} V(G_{\theta_G}, D_{\theta_D})|_{(\theta_D^*, \theta_G^*)} = \frac{\partial \dot{\theta}_D}{\partial \theta_D} \Bigg|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*} = \frac{\partial \dot{\theta}_G}{\partial \theta_G} \Bigg|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*}
\]

\[
= \frac{\partial (\mathbb{E}_{p_{data}}[f'(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{data}}[f'(-D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)])}{\partial \theta_D} \Bigg|_{\theta_D=\theta_D^*} = \langle \mathbb{E}_{p_{data}}[f''(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{data}}[f''(-D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] \rangle_{\theta_D=\theta_D^*}
\]

\[
= \langle \mathbb{E}_{p_{data}}[f''(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{data}}[f''(-D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] \rangle_{\theta_D=\theta_D^*} + \mathbb{E}_{p_{data}}[f''(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{data}}[f''(-D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)]
\]

\[
= \langle \mathbb{E}_{p_{data}}[f''(0)\nabla_{\theta_D} D_{\theta_D}(x)] \rangle_{\theta_D=\theta_D^*} + \mathbb{E}_{p_{data}}[f''(0)\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{data}}[f''(0)\nabla_{\theta_D} D_{\theta_D}(x)]
\]

\[
= 2f''(0) \mathbb{E}_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)]_{\theta_D=\theta_D^*}
\]

The subsequent \(n_D \times n_G\) matrix, which we will denote by \(\mathbf{J}_{DG}\) is:

\[
\mathbf{J}_{DG} \triangleq \frac{\partial \nabla_{\theta_G} V(G_{\theta_G}, D_{\theta_D})}{\partial \theta_G} \Bigg|_{(\theta_D^*, \theta_G^*)} = \frac{\partial \dot{\theta}_D}{\partial \theta_G} \Bigg|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*} = \frac{\partial \dot{\theta}_G}{\partial \theta_G} \Bigg|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*}
\]
Then, we show how to consider a rotation of the system and then project to a space that is orthogonal.

To prove that the system is stable we will need to show that this matrix is Hurwitz. We show later that the Jacobian of the projected system is Hurwitz.

However, we argue below how we can address such a case. We first begin with a simple observation that the null space of the matrices involved in the above lemma correspond to a subspace of equilibria. Then from the Theorem A.4 that we have proved in Appendix A, it is sufficient to show that the Jacobian of the projected system is Hurwitz.

\[ \frac{\partial}{\partial \theta_G} \left|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*} \right. \]

It is easy to see that the lower \( n_G \times n_D \) matrix is \(-J_{DG}^T\):

\[ \begin{align*}
\frac{\partial \theta_G}{\partial \theta_D} \left|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*} \right. &= \frac{\partial \theta_G}{\partial \theta_D} \left|_{\theta_G=\theta_G^*} \right. \\
&= -\frac{\partial}{\partial \theta_D} \int_X f(D\theta_D(x)) \nabla \theta_G p_{\theta_G}(x) dx \left|_{\theta_G=\theta_G^*} \right. \\
&= -\left. \frac{\partial}{\partial \theta_D} f(D\theta_D(x)) \nabla \theta_G p_{\theta_G}(x) dx \right|_{\theta_G=\theta_G^*} = -J_{DG}^T
\end{align*} \]

Furthermore, the lower \( n_G \times n_G \) matrix \( J_{GG} \) turns out to be zero. Here, we will use an implication of Assumption IV. More specifically, generators \( \theta_G \) that are within a sufficiently small radius \( \epsilon_G \) around the equilibrium have the same support and therefore i) \( D\theta_G^*(x) = 0 \) for \( x \) in this support. Furthermore for all generators within a radius \( \epsilon_G/2 \), any perturbation of the generator is not going to change the support, and therefore ii) \( \nabla \theta_G p_{\theta_G}(x) = 0 \) for \( x \) that is not in this support.

Now, to show that \( J_{GG}^T \) is indeed zero, we take any vector \( v \) that is a perturbation in the generator space and show that \( v^T J_{GG} = 0 \). Here, we will use the limit definition of the derivative along a particular direction \( v \) as shown below.

\[ v^T \frac{\partial \theta_G}{\partial \theta_G} \left|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*} \right. = v^T \left. \frac{\partial \theta_G}{\partial \theta_D} \right|_{\theta_G=\theta_G^*} \frac{\partial \theta_D}{\partial \theta_G} \left|_{\theta_D=\theta_D^*} \right. = -\lim_{\epsilon \to 0} \frac{\int_X f(D\theta_D^*(x)) \nabla \theta_G p_{\theta_G}(x) dx}{\epsilon} \]

\[ = -f(0) \lim_{\epsilon \to 0} \frac{\int_{\text{supp}(p_{\theta_G^*})} \nabla \theta_G \nabla p_{\theta_G}(x) dx}{\epsilon} \]

\[ = -f(0) \lim_{\epsilon \to 0} \frac{\int_{\text{supp}(p_{\theta_G^*})} \nabla \theta_G \nabla p_{\theta_G}(x) dx}{\epsilon} \]

\[ = -f(0) \lim_{\epsilon \to 0} \frac{\nabla \theta_G \nabla p_{\theta_G}(x) dx}{\epsilon} = 0 \]

To prove that the system is stable we will need to show that this matrix is Hurwitz. We show later in Lemma G.2 that when i) \( J_{DD} \prec 0 \) and furthermore ii) \( J_{DG} \) is full column rank, then \( J \) is indeed Hurwitz. However we only have that \( J_{DD} \preceq 0 \) (because \( f''(0) < 0 \)). If these two conditions need to be met, we will need that \( V(\theta_D, \theta_G) \) is strongly concave with respect to \( \theta_D \) and \( \| \nabla \theta_G^T \nabla \theta_G V(\theta_D, \theta_G) \|^2 \) is strongly convex with respect to \( \theta_G \), thereby excluding the case where there are multiple equilibria.

However, we argue below how we can address such a case. We first begin with a simple observation that the null space of the matrices involved in the above lemma correspond to a subspace of equilibria. Then, we show how to consider a rotation of the system and then project to a space that is orthogonal to this subspace of equilibria. Then from the Theorem A.4 that we have proved in Appendix A, it is sufficient to show that the Jacobian of the projected system is Hurwitz.
In the following discussion, we will use the term “equilibrium discriminator” to denote a discriminator that is identically zero on the support and “equilibrium generator” to denote a generator that matches the true distribution. Note that for an equilibrium discriminator, the generator updates are zero and vice versa for an equilibrium generator.

**Lemma C.2.** For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 4, under Assumptions I and III, there exists $\epsilon_D, \epsilon_G > 0$ such that for all $\epsilon' \leq \epsilon_D$ and $\epsilon' \leq \epsilon_G$, and for any unit vectors $u \in \text{Null}(K_{DD}), v \in \text{Null}(K_{DG}), (\theta_D^* + \epsilon_D u, \theta_G^* + \epsilon_G v)$ is an equilibrium point (as defined in Assumption I).

**Proof.** Note that $2K_{DD}$ is the Hessian of the function $E_{p_{data}}[D^2_{\theta_D}(x)]$ at equilibrium:

$$
\nabla^2_{\theta_D} E_{p_{data}}[D^2_{\theta_D}(x)]|_{\theta_D^*} = 2 \left. \frac{\partial E_{p_{data}}[D_{\theta_D}(x)\nabla_{\theta_D} D(x)]}{\partial \theta_D} \right|_{\theta_D^*}
$$

$$
= 2 \left( E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)\nabla^T_{\theta_D} D(x)] + E_{p_{data}}[\nabla^2_{\theta_D} D(x)] \right)
$$

$$
= 2 \left. \left( E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)\nabla^T_{\theta_D} D(x)] \right) \right|_{\theta_D^*} = 2K_{DD}
$$

Then, by Assumption III, $E_{p_{data}}[D^2_{\theta_D}(x)]$ is locally constant along any unit vector $u \in \text{Null}(K_{DD})$. That is, for sufficiently small $\epsilon$, if $\theta_D = \theta_D^* + \epsilon u$, $E_{p_{data}}[D^2_{\theta_D}(x)]$ equals the value of the function at equilibrium, which is 0 because $D_{\theta_D}(x) = 0$ (according to Assumption I). Thus, we can conclude that for all $x$ in the support of $p_{data}$, $D_{\theta_D}(x) = 0$. In other words, $\theta_D$ is an equilibrium discriminator which when paired with any generator results in zero updates on the generator.

Similarly, $2K_{DG}^T K_{DG}$ is the Hessian of the function $\left\| E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right\|^2$ at equilibrium:

$$
\nabla_{\theta_G} \left\| E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right\|^2
$$

$$
= -2 \left( E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right)^T \int_{x} \nabla_{\theta_G} p_{\theta_G}(x) \nabla_{\theta_D} D_{\theta_D}(x) dx
$$

$$
\Rightarrow \nabla^2_{\theta_G} \left\| E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right\|^2|_{\theta_D^*, \theta_G^*}
$$

$$
= 2 \left( \int_{x} \nabla_{\theta_G} p_{\theta_G}(x) \nabla_{\theta_D} D_{\theta_D}(x) dx \right)^T \int_{x} \nabla_{\theta_G} p_{\theta_G}(x) \nabla_{\theta_D} D_{\theta_D}(x) dx |_{\theta_D^*, \theta_G^*}
$$

$$
- 2 \left( E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right)^T \int_{x} \nabla_{\theta_G} p_{\theta_G}(x) \nabla_{\theta_D} D_{\theta_D}(x) dx |_{\theta_D^*, \theta_G^*}
$$

$$
= 2K_{DG}^T K_{DG}
$$

Then, by Assumption III, $\left\| E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right\|^2$ is locally constant along any unit vector $v \in \text{Null}(K_{DG})$. That is, for sufficiently small $\epsilon'$, if $\theta_G = \theta_G^* + \epsilon' v$, $\left\| E_{p_{data}}[\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right\|^2$ equals the value of the function at equilibrium, which is 0 because $p_{\theta_G^*} = p_{data}$ (according to Assumption I).
Now, this means that at \((\theta_D^*, \theta_G)\), the discriminator update is zero. Furthermore, the generator update is zero too, because

\[
\hat{\theta}_G = -f(0) \int_{\supp(p_{\text{data}})} \nabla_{\theta_G} p_{\theta_G}(x) dx = -f(0) \int_{\supp(p_{\text{data}})} p_{\theta_G}(x) dx = -f(0) \nabla_{\theta_G} 1 = 0
\]

Therefore, \((\theta_D^*, \theta_G)\) is an equilibrium point and from Assumption I we can conclude that \(p_{\theta_G} = p_{\text{data}}\). Thus, \(\theta_G^*\) is an equilibrium generator i.e., when paired with any equilibrium discriminator, the discriminator updates are zero.

In summary, for all slight perturbations along \(u \in \text{Null}(K_{DD})\), \(v \in \text{Null}(K_{DG})\) we have established that the discriminator and generator individually satisfy the requirements of an equilibrium discriminator and generator pair, and therefore the system itself is in equilibrium for these perturbations. \(\square\)

Now, we show how to rotate and project the system to get a Hurwitz Jacobian matrix.

**Lemma C.3.** For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 4, consider the eigenvalue decompositions \(K_{DD} = U_D A_D U_D^T\) and \(K_{DG} = U_G A_G U_G^T\). Let \(U_D = [T_D, T'_D]\) and \(U_G = [T'_G, T''_G]\) such that \(\text{Col}(T'_D) = \text{Null}(K_{DD})\) and \(\text{Col}(T''_G) = \text{Null}(K_{DG})\). Consider the projections, \(\gamma_D = T_D \theta_D\) and \(\gamma_G = T_G \theta_G\). Then, the block in the Jacobian at equilibrium that corresponds to the projected system has the form:

\[
J' = \begin{bmatrix}
J_{DD}' & J_{DG}' \\
-J_{DG}' & 0
\end{bmatrix} = \begin{bmatrix}
2f''(0)T_D K_{DD} T_D^T & f'(0)T_D K_{DG} T_G^T \\
-f'(0)T_G K_{DG} T_D^T & 0
\end{bmatrix}
\]

Under Assumption II, we have that \(J'_{DD} < 0\) and \(J'_{DG}\) is full column rank.

**Proof.** Note that the columns of \(U_D\) and \(U_G\) correspond to eigenvectors, and furthermore, the rows of \(T'_D\) and \(T''_G\) are the eigenvectors that correspond to zero eigenvalues. Thus, the above lemma considers a projection of the system to a space orthogonal to the local subspace of equilibria.

We first address a corner case where either \(T_D\) or \(T_G\) is empty. In the case that \(T_D\) is empty, it means that all discriminators in a neighborhood of the considered equilibrium are identically zero on the support of the true distribution (as proved in Lemma C.2). Then, for any generator, the discriminator update would be zero (because moving the discriminator in any direction locally does not result in a change in the objective). At the same time, the generator update would be zero too because these are all equilibrium discriminators. This means that the considered point is surrounded by a neighborhood of equilibria. Then, the system is trivially exponentially since any sufficiently close initialization is already at equilibrium.

Similarly when \(T_G\) is empty it means that are generators in a small neighborhood have the same distribution, namely the true underlying distribution (as proved in Lemma C.2). Then, the generator update for any discriminator would be zero (changing the generator slightly in any direction does not change the objective). Furthermore, since these are equilibrium generators, the discriminator updates would be zero too, for any discriminator. Thus, again we are situated in a neighborhood of equilibria and the system is trivially at equilibrium.

Now we handle the general case. First note that, the Jacobian block of the projected variables must be

\[
\left( \begin{bmatrix} T_D & T_G \end{bmatrix} J \begin{bmatrix} T_D \end{bmatrix}^T \begin{bmatrix} T_G \\ \end{bmatrix}^T \right) = \begin{bmatrix} 2f''(0)T_D K_{DD} T_D^T & f'(0)T_D K_{DG} T_G^T \\
-f'(0)T_G K_{DG} T_D^T & 0
\end{bmatrix}
\]

where \(J\) is the Jacobian of the original system which we derived in Lemma C.1. Now note that, \(T_D K_{DD} T_D^T = T_D U_D A_D U_D^T T_D^T = A_D^{(+)}\) which is a diagonal matrix with only positive values. Therefore, since \(f''(0) < 0\), \(J'_{DD} < 0\).

Next, in a similar manner we can show that \(T_G K_{DG} T_G^T = A_G^{(+)}\), which is a diagonal matrix with only positive values. Thus, \(K_{DG} T_G^T\) is full rank. The non-trivial step here is to show that the matrix \(T_D K_{DG} T_G^T\) which has fewer rows is full column rank too. This will follow if we showed...
that for any $u$ such that $u^T K_{DD} = 0$, $u^T K_{DG} = 0$ too. That is, the left null space of $K_{DD}$ is a subset of the left null space of $K_{DG}$ and therefore projecting to the row span of $K_{DD}$ does not hurt the row rank of $K_{DG}$.

To see why this is true, observe that from Lemma C.2 for any small perturbation along such a $u$, since we are always at an equilibrium discriminator, it must be that $u^T \nabla_{\theta_D} \theta_D(x) = 0$. Then,

$$u^T K_{DG} = \left. \int_{\mathcal{X}} u^T \nabla_{\theta_D} \theta_D(x) \nabla_{\theta_D} p_{\theta_D}(x) dx \right|_{\theta_D = \theta_D^*, \theta_G = \theta_G^*} = 0$$

Therefore, since $f'(0) \neq 0$, this means $f'(0)T_D K_{DG} T_G^T$ is full column rank.

The main theorem then follows from the above lemmas.

**Theorem 3.1.** The dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 4 is locally exponentially stable with respect to an equilibrium point $(\theta_D^*, \theta_G^*)$ under the Assumptions I, II, III, IV. Furthermore, the rate of convergence is governed only by the eigenvalues $\lambda$ of the Jacobian $J$ of the system at an equilibrium with a strict negative real part upper bounded as:

- If $\text{Im}(\lambda) = 0$, then $\text{Re}(\lambda) \leq \frac{2f''(0)f'(0)\lambda_{\min}(K_{DD})\lambda_{\max}(K_{DG}^T K_{DD})}{4f''(0)\lambda_{\max}(K_{DD}) + f'(0)\lambda_{\min}(K_{DG}^T K_{DD})}$
- If $\text{Im}(\lambda) \neq 0$, then $\text{Re}(\lambda) \leq f''(0)\lambda_{\min}(K_{DD})$

**Proof.** We have from Lemma C.2 that the considered equilibrium point lies in a subspace of equilibria in a small neighborhood. Then, we have from Lemma C.3 that the Jacobian block corresponding to the subspace orthogonal to this, satisfies properties from Lemma G.2 which make it Hurwitz. We can then conclude exponential stability of the system from Theorem A.4. The eigenvalue bounds presented in the theorem follow from Lemma G.2.

Finally, we show that we can indeed find a Lyapunov function that satisfies LaSalle’s principle for the projected linearized system.

**Fact C.1.** For the linearized projected system with the Jacobian $J'$, we have that $1/2\|\gamma_D - \gamma_D^\star\|^2 + 1/2\|\gamma_G - \gamma_G^\star\|^2$ is a Lyapunov function such that for all non-equilibrium points, it either always decreases or only instantaneously remains constant.

**Proof.** Note that the Lyapunov function is zero only at the equilibrium of the projected system. Furthermore, it is straightforward to verify that the rate at which this changes is given by $f''(0)[(\gamma_D - \gamma_D^\star)^T T_D^T K_{DD} T_D (\gamma_D - \gamma_D^\star)]$. Clearly this is zero only when $\gamma_D = \gamma_D^\star$ because $T_D^T K_{DD} T_D$ is positive definite; otherwise it is strictly negative. Now, when this rate is indeed zero, we have that $\gamma_D^\star = f''(0)T_D K_{DG} T_G (\gamma_G - \gamma_G^\star)$ because the other term in the update which is proportional to $K_{DD}(\gamma_D - \gamma_D^\star)$ is zero. Now, again, this term is zero only when $\gamma_G = \gamma_G^\star$ because $T_D K_{DG} T_G^T$ is full column rank. Thus, when we are not at equilibrium which means $\gamma_G \neq \gamma_G^\star, \gamma_D \neq 0$, it does not identically stay in the manifold $\gamma_D = 0$ on which the energy does not decrease.

**C.1 Realizable case with a relaxed assumption**

In this section, we show how we relax the assumption that all generators in a small neighborhood around equilibrium share the same support (which was primarily made in the main paper for the sake of a simpler discussion). The motivation is that Assumption IV may typically hold if the support covers the whole space $\mathcal{X}$; but when the true distribution has support in some smaller disjoint parts of the space $\mathcal{X}$, nearby generators may correspond to slightly displaced versions of this distribution with a different support. Perhaps a fairer requirement from the system would be to hope that the union of the supports of the generator and the ones in its neighborhood do not cover too large a
space, and furthermore, the equilibrium discriminator is zero in the union of all these supports. This property would be satisfied if we restricted ourselves to smooth (i.e., 1-Lipschitz) discriminators. We mathematically state this assumption as follows:

**Assumption IV (Relaxed)** There exists sufficiently small $\epsilon_G > 0$ such that for all $x \in \bigcup_{\theta_G \in B_{\epsilon_G}(0)} \text{supp}(p_{\theta_G})$, $D_{\theta_G}^* (x) = 0$

We now show that the asymptotic stability holds.

**Proof.** Most of the original proof holds as it is because all we needed was that the equilibrium discriminator be identically zero on the true support. However, when we proved that the generator’s Hessian $J_{GG} = 0$ at equilibrium, we used a more intricate fact. More specifically, we said that for a generator within a radius of $\epsilon_G/2$ from equilibrium (where $\epsilon_G$ is as defined in the original version of Assumption IV), i) the support of $p_{\theta_G}$ is the same as $p_{\text{data}}$ and therefore $D_{\theta_G}^* (x) = 0$ for all $x$ in the true support and ii) for all $x$ not in the true support, and for any generator $\theta_G \in B_{\epsilon_G/2}(\theta_G^*)$, $\nabla_{\theta_G} p_{\theta_G}(x) = 0$

In this case, we can say that for a radius $\epsilon_G/2$, any perturbation of the generator ensures that the support is contained in the combined support $\bigcup_{\theta_G \in B_{\epsilon_G}(0)} \text{supp}(p_{\theta_G})$. Then, for all $x$ in the combined support $i)$ $D_{\theta_G}^* (x) = 0$ and ii) for all $x$ not in the combined support and for any generator $\theta_G \in B_{\epsilon_G/2}(\theta_G^*)$, $\nabla_{\theta_G} p_{\theta_G}(x) = 0$

Thus, in this case too, we can show that $J_{GG} = 0$:

$$V^T \left. \frac{\partial \hat{\theta}_G}{\partial \theta_G} \right|_{\theta_D^*=\theta_D^*, \theta_G^*=\theta_G^*} = V^T \left. \frac{\partial \hat{\theta}_G}{\partial \theta_G} \right|_{\theta_D^*=\theta_D^*, \theta_G^*=\theta_G^*} = - \lim_{\epsilon \to 0} \frac{\int_x f(-D_{\theta_G}^*(x)) \nabla_{\theta_G} p_{\theta_G}(x) dx}{\epsilon}$$

$$= -f(0) \lim_{\epsilon \to 0} \frac{\int_{\bigcup_{\theta_G \in B_{\epsilon_G}(0)} \text{supp}(p_{\theta_G})} \nabla_{\theta_G} p_{\theta_G}(x) dx}{\epsilon}$$

$$= -f(0) \lim_{\epsilon \to 0} \frac{\nabla_{\theta_G} \int_{\bigcup_{\theta_G \in B_{\epsilon_G}(0)} \text{supp}(p_{\theta_G})} p_{\theta_G}(x) dx}{\epsilon}$$

$$= -f(0) \lim_{\epsilon \to 0} \frac{\nabla_{\theta_G} 1}{\epsilon} = 0$$

The rest of the proof follows as it did.  

**C.2 The non-realizable case**

In this section, we extend our results about local stability of GANs to the case in which the true distribution cannot be represented by any generator in the generator space. While this is a hard problem in general, we consider a specific case in which the discriminator is linear in its parameters and show that the system is locally stable at any equilibrium and its surrounding equilibria (none of which may correspond to the true distribution). More formally, consider a discriminator of the form:

$$D_{\theta_D}(x) = \theta_D^T \Phi(x)$$

where $\Phi$ is any feature mapping. For example, $\Phi(x)$ could be a polynomial basis or the representation learned by a neural network (which we assume is not trained during the updates near equilibrium). Thus, the objective in this case is:

$$V(D_{\theta_D}, G_{\theta_G}) = \mathbb{E}_{p_{\text{data}}} [f(\theta_D^T \Phi(x))] + \mathbb{E}_{p_{\theta_G}} [f(\theta_D^T \Phi(x))]$$

We consider a generator space that does not necessarily contain the true distribution, but however contains a generator $\theta_G^*$ that is an equilibrium point when paired with a discriminator that is zero on
the support of the true data and the generated data. It must be noted that \( \theta_D^* = 0 \) is not necessarily the only equilibrium discriminator. Especially, if \( \phi \) lies in a lower dimensional manifold, there could be a subspace of all-zero discriminators. Now, for such a generator to exist, we need:

\[
\nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})|_{(\theta_D^*, \theta_G^*)} = 0
\]

\[
\implies E_{p_{data}}[\phi(x)] = E_{p_{\theta_G}}[\phi(x)]
\]

In other words, we want the means of the generated distribution and the true distribution in the representation \( \phi \) to be identical. For a given generator space, this essentially is a restriction on the representation \( \phi \) that has been learned/choosen for the discriminator. If \( \phi \) was a richer representation, we may never find an equilibrium generator.

We now prove Theorem 3.1 for the non-realizable case. Our main idea is identical to that of the proof in the realizable case. However, we need to be careful in a number of steps. We first prove a result similar to Lemma C.1.

**Lemma C.4.** For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 4, the Jacobian at an equilibrium point \((\theta_D^*, \theta_G^*)\), under the Assumptions I (for the non-realizable case) and IV is:

\[
\begin{bmatrix}
J_{DD} & J_{DG} \\
-J_{DG} & J_{GG}
\end{bmatrix} = \begin{bmatrix}
2f''(0)K_{DD} & f'(0)K_{DG} \\
-f'(0)K_{DG} & 0
\end{bmatrix}
\]

where

\[
2K_{DD} \triangleq E_{p_{data}}[(\nabla_{\theta_D} D_{\theta_D}(x))(\nabla_{\theta_D} D_{\theta_D}(x))^T] + E_{p_{\theta_G}}[(\nabla_{\theta_D} D_{\theta_D}(x))(\nabla_{\theta_D} D_{\theta_D}(x))^T]|_{\theta_D^*} \geq 0
\]

and

\[
K_{DG} \triangleq \int_X \nabla_{\theta_D} D_{\theta_D}(x)\nabla_{\theta_G} p_{\theta_G}(x)dx|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*}
\]

**Proof.** Recall that,

\[
V(D_{\theta_D}, G_{\theta_G}) = E_{p_{data}}[f(D_{\theta_D}(x))] + E_{p_{\theta_G}}[f(-D_{\theta_D}(x))]
\]

\[
\dot{\theta}_D = E_{p_{data}}[f'(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{\theta_G}}[f'(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)]
\]

\[
\dot{\theta}_G = -\int_X \nabla_{\theta_G} p_{\theta_G} f(-D_{\theta_D}(x))dx
\]

First we show that \(J_{DD}\) has a similar form which is still negative semi-definite when \(f''(0) < 0:\)

\[
J_{DD} = \nabla_{\theta_D}^2 V(G_{\theta_G}, D_{\theta_D})|_{(\theta_D^*, \theta_G^*)} = \frac{\partial \dot{\theta}_D}{\partial \theta_D}|_{\theta_D=\theta_D^*, \theta_G=\theta_G^*} = \frac{\partial \dot{\theta}_D}{\partial \theta_D}|_{\theta_D=\theta_D^*} = \left( \frac{\partial}{\partial \theta_D} \left( E_{p_{data}}[f'(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] - E_{p_{\theta_G}}[f'(-D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)] \right) \right)|_{\theta_D=\theta_D^*}
\]

\[
= \left( E_{p_{data}}[f''(D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)\nabla_{\theta_D}^T D_{\theta_D}(x)] + E_{p_{\theta_G}}[f''(-D_{\theta_D}(x))\nabla_{\theta_D}^2 D_{\theta_D}(x)] \right)|_{\theta_D=\theta_D^*}
\]

\[
+ \left( E_{p_{\theta_G}}[f''(-D_{\theta_D}(x))\nabla_{\theta_D} D_{\theta_D}(x)\nabla_{\theta_D}^T D_{\theta_D}(x)] - E_{p_{\theta_G}}[f''(-D_{\theta_D}(x))\nabla_{\theta_D}^2 D_{\theta_D}(x)] \right)|_{\theta_D=\theta_D^*}
\]

\[
= \left( E_{p_{data}}[f''(0)\nabla_{\theta_D} D_{\theta_D}(x)\nabla_{\theta_D}^T D_{\theta_D}(x)] + E_{p_{data}}[f'(0)\nabla_{\theta_D}^2 D_{\theta_D}(x)] \right)|_{\theta_D=\theta_D^*}
\]

\[
+ \left( E_{p_{\theta_G}}[f''(0)\nabla_{\theta_D} D_{\theta_D}(x)\nabla_{\theta_D}^T D_{\theta_D}(x)] - E_{p_{\theta_G}}[f'(0)\nabla_{\theta_D}^2 D_{\theta_D}(x)] \right)|_{\theta_D=\theta_D^*}
\]
In Theorem 3.2. \( \nabla \) The most crucial step here is that we were able to ignore the terms corresponding to \( \nabla_{\theta_D}^2 \theta_{D}(x) \) because the discriminator is linear in its parameters i.e., \( \nabla_{\theta_D} \theta_{D}(x) = \Phi(x) \) and thus the Hessian is zero. All other terms in the Jacobian are identical to the realizable case because we assume that at equilibrium the discriminator must be identically zero.

Now, we again show that the equilibrium point in consideration lies in a subspace of equilibria.

**Lemma C.5.** Under Assumptions I (Non-realizable), III, and IV there exists \( \epsilon_D, \epsilon_G > 0 \) such that for all \( \epsilon'_{D} \leq \epsilon_D \) and \( \epsilon'_{G} \leq \epsilon_G \), and for any unit vectors \( u \in \text{Null}(K_{DD}) \), \( v \in \text{Null}(K_{DG}) \), \( (\theta^*_D + \epsilon'_{D} u, \theta^*_G + \epsilon'_{G} v) \) is an equilibrium point.

**Proof.** Note that \( 4K_{DD} \) is the Hessian of the function \( \mathbb{E}_{p_{\text{data}}}[D_{\theta_D}^2(x)] + \mathbb{E}_{p_{\emptyset_G}}[D_{\theta_D}^2(x)] \) at equilibrium (the derivation is similar to that in Lemma C.2).

Since this is the sum of two positive semi-definite matrices, any vector in the null space of \( \text{Null}(K_{DD}) \) is also in the null space of the Hessian of \( \mathbb{E}_{p_{\text{data}}}[D_{\theta_D}^2(x)] \). Then, by Assumption III, \( \mathbb{E}_{p_{\text{data}}}[D_{\theta_D}^2(x)] \) is locally constant along any unit vector \( u \in \text{Null}(K_{DD}) \). That is, for sufficiently small \( \epsilon \), if \( \theta_D = \theta^*_D + \epsilon u, \mathbb{E}_{p_{\text{data}}}[D_{\theta_D}^2(x)] \) equals the value of the function at equilibrium, which is 0 because \( D_{\theta_D}(x) = 0 \) (according to Assumption I). Thus, we can conclude that for all \( x \) in the support of \( p_{\text{data}} \), \( D_{\theta_D}(x) = 0 \). Now from Assumption IV, the support of generators in a small neighborhood is identical to the support of the true distribution, therefore these discriminators are equilibrium discriminators i.e., when paired with any generator, the generator updates are zero.

Similarly, \( 4K_{DG}^T K_{DG} \) is the Hessian of the function \( \mathbb{E}_{p_{\text{data}}}[\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{\emptyset_G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \) at equilibrium. Then, by Assumption III, \( \mathbb{E}_{p_{\text{data}}}[\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{\emptyset_G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \) is locally constant along any unit vector \( v \in \text{Null}(K_{DG}) \). That is, for sufficiently small \( \epsilon' \), if \( \theta_G = \theta^*_G + \epsilon' v, \mathbb{E}_{p_{\text{data}}}[\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{\emptyset_G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \) equals the value of the function at equilibrium. Now, since this function is proportional to the magnitude of the equilibrium discriminator’s update, it equals zero at equilibrium. Now, observe that

\[
\mathbb{E}_{p_{\text{data}}}[\nabla_{\theta_D} D_{\theta_D}(x)] - \mathbb{E}_{p_{\emptyset_G}}[\nabla_{\theta_D} D_{\theta_D}(x)] \right|_{\theta_D=\theta^*_D} = \mathbb{E}_{p_{\text{data}}}[(\Phi(x)] - \mathbb{E}_{p_{\emptyset_G}}[\Phi(x)] = 0
\]

is independent of the discriminator variables. This means that for these generators, the discriminator update must be zero for any equilibrium discriminator. (Here, we have used the fact that the discriminator is linear in its parameters.)

In summary, for all slight perturbations along \( u \in \text{Null}(K_{DD}), v \in \text{Null}(K_{DG}) \) we have established that the discriminator and generator individually satisfy the requirements of an equilibrium discriminator and generator pair, and therefore the system is itself is in equilibrium for these perturbations. \( \square \)

It turns out that given these two lemmas, Lemma C.3 follows as it did earlier, and therefore the main theorem follows too.

**D. Linear Quadratic GAN – Gaussian example**

We now derive the Jacobian of the linear quadratic system that was described in the main paper. Since the system consists of parameters arranged in the form of matrices, we will need vectorization calculus [Magnus et al., 1995] to arrange these parameters as a vector and differentiate them with respect to them.

**Theorem 3.2.** In LQ, \( A = \Sigma^{1/2}, b = \mu \) and \( W_2 = 0, w_1 = 0 \) corresponds to an equilibrium that is locally exponentially stable provided \( f''(0) < 0 \) and \( f'(0) \neq 0 \).
We first calculate the derivative of the discriminator updates with respect to the discriminator itself.

Then we calculate the derivative of the discriminator updates with respect to the generator parameters.

The updates in Equation 4 for LQ can be written as:

$$\frac{\partial \text{vec}(\mathbf{W}_2)}{\partial \text{vec}(\mathbf{W}_2)} \bigg|_{\mathbf{b}=\mu, \mathbf{W}_2=0, \mathbf{w}_1=0, \mathbf{A}=\Sigma^{1/2}} = \frac{\partial}{\partial \text{vec}(\mathbf{W}_2)} \left( \text{vec}(\mathbf{W}_2) \bigg|_{\mathbf{b}=\mu, \mathbf{A}=\Sigma^{1/2}, \mathbf{w}_1=0} \right) \bigg|_{\mathbf{W}_2=0}$$

$$= \frac{\partial}{\partial \text{vec}(\mathbf{W}_2)} \mathbb{E}_{x \sim \mathcal{N}(\mu, \Sigma)}[\text{vec}(\mathbf{x})^T(f'(\mathbf{x}^T \mathbf{W}_2 \mathbf{x}) - f'(-\mathbf{x}^T \mathbf{W}_2 \mathbf{x}))] \bigg|_{\mathbf{W}_2=0}$$

$$= 2f''(0)\mathbb{E}_{x \sim \mathcal{N}(\mu, \Sigma)}[(\mathbf{x} \otimes \mathbf{x})(\mathbf{x} \otimes \mathbf{x})]^T$$

$$\frac{\partial \text{vec}(\mathbf{W}_2)}{\partial \mathbf{w}_1} \bigg|_{\mathbf{b}=\mu, \mathbf{W}_2=0, \mathbf{w}_1=0, \mathbf{A}=\Sigma^{1/2}} = \frac{\partial}{\partial \mathbf{w}_1} \left( \text{vec}(\mathbf{W}_2) \bigg|_{\mathbf{b}=\mu, \mathbf{A}=\Sigma^{1/2}, \mathbf{w}_2=0} \right) \bigg|_{\mathbf{w}_1=0}$$

$$= \frac{\partial}{\partial \mathbf{w}_1} \mathbb{E}_{x \sim \mathcal{N}(\mu, \Sigma)}[\text{vec}(\mathbf{x}^T(f'(\mathbf{w}_1^T \mathbf{x}) - f'(-\mathbf{w}_1^T \mathbf{x})))] \bigg|_{\mathbf{w}_1=0}$$

$$= 2f''(0)\mathbb{E}_{x \sim \mathcal{N}(\mu, \Sigma)}[\mathbf{x}(f'(\mathbf{w}_1^T \mathbf{x}) - f'(-\mathbf{w}_1^T \mathbf{x}))] \bigg|_{\mathbf{w}_1=0}$$

$$= 2f''(0)\mathbb{E}_{x \sim \mathcal{N}(\mu, \Sigma)}[\mathbf{x}^T]$$

Then we calculate the derivative of the discriminator updates with respect to the generator parameters. Note that we will be using the constant matrix $\mathbf{T}_{n,n}$ which is a matrix of zeros and ones defined in...
vectorization algebra; this matrix is the vectorization equivalent of the transpose operator. That is, for any square matrix $V \in \mathbb{R}^n$, $T_{n,n} vec(V) = vec(V^T)$.

\[
\frac{\partial vec(W_2)}{\partial vec(A)}_{b=\mu, W_2=0, w_1=0, A=\Sigma^{1/2}} = \frac{\partial}{\partial vec(A)} \left( vec(W_2)_{b=\mu, W_2=0, w_1=0} \right)_{A=\Sigma^{1/2}} \\
= - \frac{\partial}{\partial vec(A)} vec \left( E_{z \sim \mathcal{N}(0, I_n)}[(Az + \mu)(Az + \mu)^T f'(0)] \right)_{A=\Sigma^{1/2}} \\
= - \frac{\partial}{\partial vec(A)} vec \left( E_{z \sim \mathcal{N}(0, I_n)}[(Azz^T A^T + Az + \mu)(Azz^T A^T + Az + \mu)^T f'(0)] \right)_{A=\Sigma^{1/2}} \\
= - \frac{\partial}{\partial vec(A)} vec(\Sigma^{1/2} f'(0))_{A=\Sigma^{1/2}} = -(I_n^2 + T_{n,n})(\Sigma^{1/2} \times I_n) f'(0)
\]

\[
\frac{\partial vec(W_2)}{\partial b}_{b=\mu, W_2=0, w_1=0, A=\Sigma^{1/2}} = \frac{\partial}{\partial b} \left( vec(W_2)_{b=\mu, W_2=0, A=\Sigma^{1/2}} \right)_{w_1=0, A=\Sigma^{1/2}} \\
= - \frac{\partial}{\partial b} vec \left( E_{z \sim \mathcal{N}(0, I_n)}[(\Sigma^{1/2} z + b)(\Sigma^{1/2} z + b)^T f'(0)] \right) \\
= - f'(0) \frac{\partial}{\partial b} vec(bl)_{b=\mu} \\
= - f'(0)(\mu \otimes I_n + I_n \otimes \mu)
\]

\[
\frac{\partial \tilde{w}_1}{\partial vec(A)} = \frac{\partial}{\partial vec(A)} \left( \tilde{w}_1_{w_1=0, W_2=0, b=\mu} \right)_{A=\Sigma^{1/2}} \\
= - \frac{\partial}{\partial vec(A)} E_{z \sim \mathcal{N}(0, I_n)}[(Az + b)f'(0)] = 0
\]

\[
\frac{\partial \tilde{w}_1}{\partial b} = \frac{\partial}{\partial b} \left( \tilde{w}_1_{w_1=0, W_2=0, A=\Sigma^{1/2}} \right)_{b=\mu} = - \frac{\partial}{\partial b} E_{z \sim \mathcal{N}(0, I_n)}[(\Sigma^{1/2} z + b)f'(0)] \\
= - I f'(0)
\]

Recall that the Jacobian can then be written as:

\[
\begin{bmatrix}
J_{DD} & J_{DG} \\
-J_{DG}^T & 0
\end{bmatrix}
\]

where

\[
J_{DD} = \begin{bmatrix}
\frac{\partial vec(W_2)}{\partial vec(A)}_{eqbm} & \frac{\partial vec(W_2)}{\partial \tilde{w}_1}_{eqbm} \\
\frac{\partial \tilde{w}_1}{\partial vec(A)}_{eqbm} & \frac{\partial \tilde{w}_1}{\partial \tilde{w}_1}_{eqbm}
\end{bmatrix} = \\
= \begin{bmatrix}
E_{x \sim \mathcal{N}(\mu, \Sigma)}[(x \otimes x)(x \otimes x)^T] & E_{x \sim \mathcal{N}(\mu, \Sigma)}[xx^T] \\
E_{x \sim \mathcal{N}(\mu, \Sigma)}[(x \otimes x)(x \otimes x)^T] & E_{x \sim \mathcal{N}(\mu, \Sigma)}[xx^T]
\end{bmatrix} 2f''(0)
\]

and

\[
J_{DG} = \begin{bmatrix}
\frac{\partial vec(W_2)}{\partial vec(A)}_{eqbm} & \frac{\partial vec(W_2)}{\partial b}_{eqbm} \\
\frac{\partial \tilde{w}_1}{\partial vec(A)}_{eqbm} & \frac{\partial \tilde{w}_1}{\partial b}_{eqbm}
\end{bmatrix} = \\
= - \begin{bmatrix}
(I_n^2 + T_{n,n})(\Sigma^{1/2} \otimes I_n) & \mu \otimes I_n + I_n \otimes \mu \\
0 & I_n
\end{bmatrix} f'(0)
\]

26
We can show that $J_{DD}$ is a negative definite because it is a moment matrix with a negative multiplicative factor. This is proved in Theorem D.1. Recall that as long as $f''(0) < 0$, $f'(0) \neq 0$ and $J_{DG}$ is full column rank (in this case full rank because $J_{DG}$ is a square matrix), the matrix has eigenvalues whose real components are strictly negative.

To show that $J_{DG}$ is full column rank, first observe that the last few columns corresponding to $b$ are linearly independent because, if $y$ belongs to its null space, then

$$\begin{bmatrix} \mu \otimes I_n + I_n \otimes \mu \\ I \end{bmatrix} y = \begin{bmatrix} \mu \otimes I_n + I_n \otimes \mu \\ y \end{bmatrix} y = 0,$$

which implies that $y = 0$.

To verify whether the first few columns corresponding to $A$ are linearly independent or not, consider any $V \neq 0$. Then, we want to verify whether the following term is always non-zero or not:

$$(I_{n^2} + T_{n,n})(\Sigma^{1/2} \otimes I_n)vec(V) = (I_{n^2} + T_{n,n})vec(I_n, V(\Sigma^{1/2})^T) = vec(V(\Sigma^{1/2})^T + \Sigma^{1/2}V^T),$$

which is equivalent to testing whether $V(\Sigma^{1/2})^T + \Sigma^{1/2}V^T$ is non-zero. Note that because $A$ is always restricted to be symmetric, ideally in the block $J_{DG}$ we should care only about fewer columns. In particular, we should consider only those columns that corresponding to the diagonal and lower/upper triangular elements of $A$. Instead of explicitly dropping these, we will restrict $V$ to the space of symmetric matrices.

Now, assuming $V$ is symmetric, we will show that if $V(\Sigma^{1/2})^T + \Sigma^{1/2}V^T = 0$, then $V = 0$. Recall that $\Sigma^{1/2} = UA^{1/2}U^T$. Then,

$$V(\Sigma^{1/2})^T = -\Sigma^{1/2}V^T$$

$$\implies VUA^{1/2}U^T = -UA^{1/2}U^TV$$

$$U^TVUA^{1/2}U^TVU = -A^{1/2}U^TVVU$$

Observe that the left hand side is positive semi-definite while the right hand side is negative semi-definite. Therefore these terms must be equal to zero, which would then imply that $VV = 0$ i.e., $V = 0$. Thus the Jacobian is indeed Hurwitz.

In summary, this means that Assumption III holds trivially because there are no zero eigenvalues for the matrices involved in the Jacobian. This further means that there are no other equilibria in a small neighborhood around the considered equilibrium. Therefore, Assumption I is also satisfied. Finally, since the support of the distribution is $\mathbb{R}^n$, Assumption IV is also trivially satisfied. Thus, if Assumption II holds, the system is exponentially stable.

\[\square\]

We now prove that $J_{DD}$ is negative definite.

**Theorem D.1.** The matrix

$$\begin{bmatrix} \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)(x \otimes x)^T] \\ \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)x^T] \end{bmatrix}$$

is positive definite.

**Proof.** Let $U$ be any arbitrary matrix and $v$ be an arbitrary vector. Then,

$$\begin{bmatrix} vec(U) \\ v \end{bmatrix}^T \begin{bmatrix} \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)(x \otimes x)^T] & \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)x^T] \\ \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)x^T] & \mathbb{E}_{x \sim N(\mu, \Sigma)}[xx^T] \end{bmatrix} \begin{bmatrix} vec(U) \\ v \end{bmatrix} = \begin{bmatrix} vec(U) \\ v \end{bmatrix}^T \begin{bmatrix} \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)(x \otimes x)^T] & \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)x^T] \\ \mathbb{E}_{x \sim N(\mu, \Sigma)}[(x \otimes x)x^T] & \mathbb{E}_{x \sim N(\mu, \Sigma)}[xx^T] \end{bmatrix} \begin{bmatrix} vec(U) \\ v \end{bmatrix} = 0.$$
\[
\begin{align*}
&= E_{x \sim \mathcal{N}(\mu, \Sigma)} \left[ \left\| \left[ x \otimes x \right]^T \left[ \text{vec}(U) \right] \right\|^2 \right] \\
&= E_{x \sim \mathcal{N}(\mu, \Sigma)} \left[ \left( \mathbf{x}^T \mathbf{U}x + \mathbf{x}^T \mathbf{v} \right)^2 \right]
\end{align*}
\]

Now, \((\mathbf{x}^T \mathbf{U}x + \mathbf{x}^T \mathbf{v})^2 = 0\) forms a quadratic \(n-1\)-dimensional hypersurface in \(n\) dimensions, and therefore is of measure zero. For all other points, \((\mathbf{x}^T \mathbf{U}x + \mathbf{x}^T \mathbf{v})^2 \neq 0\) and therefore the above expectation is strictly positive.

\[\square\]

**E WGANs are not necessarily asymptotically stable**

We now provide the complete proof for our claim that there are systems for which the Wasserstein GAN is not asymptotically stable.

**Theorem 3.3.** The LQ WGAN system for learning a zero mean Gaussian distribution \(\mathcal{N}(0, \Sigma)\) (\(\Sigma > 0\)) is not asymptotically stable at the equilibrium corresponding to \(\mathbf{A} = \Sigma^{1/2}, \mathbf{b} = 0\) and \(\mathbf{W}_2 = 0, \mathbf{w}_1 = 0\).

**Proof.** In order to show that the system is not asymptotically stable, we show that there are initializations of the system that are arbitrarily close to the equilibrium such that the system goes orbits around the equilibrium forever. For simplicity, we first prove this for the one-dimensional gaussian \(\mathcal{N}(0, \sigma)\) and later extend it to the multi-dimensional case. Let the quadratic discriminator be \(D(x) = w_2^2 x + w_1 x\) and the linear generator be \(a \mathbf{z} + \mathbf{b}\). Then the WGAN objective in Equation 3 for the LQ system is:

\[
V(G, D) = E_{x \sim \mathcal{N}(0, \sigma)}[w_2 x^2 + w_1 x] - E_{x \sim \mathcal{N}(0, 1)}[w_2 (a x + b)^2 + w_1 (a x + b)]
\]

The updates in Equation 4 for LQ simplify as follows:

\[
\begin{align*}
\dot{w}_2 &= \sigma^2 - a^2 - b^2 \\
\dot{w}_1 &= -b \\
\dot{a} &= 2w_2 a \\
\dot{b} &= 2w_2 b + w_1
\end{align*}
\]

The system has two equilibria, \(w_2 = 0, w_1 = 0, a = \pm \sigma, b = 0\). We will assume that the system is initialized with \(w_1 = b = 0\), which means that the system will forever have \(w_1 = b = 0\) because the respective updates are zero too. Hence, we only need to focus on the variables \(w_2\) and \(a\).

Now, it can be shown that if \(a\) is initialized to \(a_0 \geq 0\), \(a\) never becomes negative (and similarly for \(a \leq 0\)). Therefore, we will focus on the equilibrium where \(a = \sigma\), and assuming \(a \geq 0\) examine how the distance from the equilibrium \(w_2^2 + w_1^2 + (a - \sigma)^2\) changes with time. The rate of change of this quantity is given by \(2(w_2 \dot{w}_2 + w_1 \dot{w}_1 + (a - \sigma) \dot{a}) = 2w_2 (a - \sigma)^2\). Observe that when \(w_2 > 0\), this term is non-negative i.e., the system never gets closer to the equilibrium. Thus, when the system is in the “bad” half-space \(w_2 > 0\), the only hope for it to converge is to exit this half-space so that \(w_2\) becomes negative. However, we show that there exists initializations that are close to the equilibrium such that even if it does exit the bad half-space it eventually re-enters it, going in a perpetual loop.

More specifically, let \((w_2(t), a(t))\) denote the system at time \(t\). Let the initialization satisfy \(w_2(0) = 0\) and \(a(0) \in (0, \sigma)\). We will now analyze the trajectory of this system. First note that \(\dot{w}_2(0) > 0\), which means the system enters the bad half-space after immediately \(t > 0\). Thus, if the system had to converge to the considered equilibrium, it would have to reach \(w_2 = 0\) again at some time \(T\). First observe that at this time \(a(T) > \sigma\) because we need \(w_2(T) < 0\) at this time. (In fact we can say
that $a(T) - \sigma \geq \sigma - a(0)$ because we know that the radius never decreased until time $T$. Now, we claim that the system simply retraces back its path along $a$ and reaches $a(0)$ at time $2T$. More clearly, we claim that the system at time $T + t$ can be described in terms of what it was at time $T - t$ as $(w_2(T + t), a(T + t)) = (-w_2(T - t), a(T - t))$.

To prove this observe that this statement is true for $t = 0$ because $w_2(T) = 0$. Then we only need to show that at any $t$, if $(w_2(T + t), a(T + t)) = (-w_2(T - t), a(T - t))$, then $w_2(T + t) = w_2(T - t)$ and $\dot{a}(T + t) = -\dot{a}(T - t)$. This is indeed true because $w_2(T + t) = \sigma^2 - a^2(T + t) = \sigma^2 - a^2(T - t) = w_2(T - t)$ and $\dot{a}(T + t) = 2w_2(T + t)a(T + t) = 2(-w_2(T - t))a(T - t) = -\dot{a}(T - t)$. Therefore, applying $t = T$, we get $(w_2(2T), a(2T)) = (-w_2(0), a(0)) = (0, a(0))$ i.e., the system has looped back to its original state by following its old path mirrored across the line $w_2 = 0$. Since this holds for initializations that are arbitrarily close to the equilibrium (i.e., $a(0)$ can be arbitrarily close to $\sigma$), the system is not asymptotically stable.

We extend this argument to the higher dimensional case as follows. Again, we initialize the system so that $w_1 = 0$ and $b = 0$, then we can only focus on the updates on $\dot{W}_2$ and $\dot{A}$:

\[
\dot{W}_2 = \Sigma - AA^T \\
\dot{A} = (W_2 + W_2^T)A
\]

As before, we initialize $W_2 = 0$. We will also consider a more sophisticated initialization compared to $a \in (0, \sigma)$. Since $\Sigma$ is positive definite, let $\Sigma = UAU^T$. We initialize $A = UA_A(0)U^T$ such that $A_A(0)$ has at least one diagonal element that is positive but strictly less than the corresponding diagonal element in $A^{1/2}$ (where $A^{1/2} \succ 0$).

Now, we first establish that all the updates and the variables in the system remain in the eigenspace defined by $U$. That is, at any point in time $t$, the variables can be expressed as $W_2(t) = UA_W(t)U^T$ and $A(t) = UA_A(t)U^T$ for some real diagonal matrices $A_W(t)$ and $A_A(t)$. Clearly, this is true for time $t = 0$. Assuming this is true for arbitrary time $t$, observe that the updates are

\[
\dot{W}_2(t) = U(A - A_A(t)^2)U^T \\
\dot{A}(t) = 2UA_W(t)U^TA_A(t)U^T = 2UA_W(t)A_A(t)U^T
\]

Thus this is true for any time $t$. Therefore, we can analyze the system in terms of $A_A$, $A_W$, and the constant $A$ as though there are $n$ independent 1-dimensional Gaussian systems. Then, the orbiting systems from the 1-dimensional updates must manifest here too. More specifically, these cycles would correspond to the diagonal in $A_A$ which was initialized to be less than $A_A^{1/2}$.

\[\Box\]

### F Gradient-based regularization

Below, we first provide an alternative mathematical intuition to our regularization term. Our intuition is based on differentiating an arg-max term.

#### F.1 Intuition based on arg-max differentiation

In an ideal world, an optimizer would hope to have access to a function $\theta_D^*(\theta_G) = \max_{\theta_G} V(D_{\theta_D}, G_{\theta_G})$, which is basically the optimal discriminator as a function of the generator; given this, the optimizer should be able to update the generator with respect to that. Then, the update can be shown to be the following (for clarity we use the superscript $t$ and $t + 1$ to denote the current and the updated parameters):

\[
\theta_G^{(t+1)} := \theta_G^{(t)} - \alpha \nabla_{\theta_G} V(D_{\theta_D^{(t)}}, G_{\theta_G}) - \alpha \left. \left( \frac{\partial \theta_D^*(\theta_G)}{\partial \theta_G} \right)^T \right|_{\theta_D = \theta_D^{(t)}(\theta_G^{(t)})} \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G^{(t)}}) \bigg|_{\theta_D = \theta_D^{(t)}(\theta_G^{(t)})}
\]

Observe that the second term is zero because, for the optimal discriminator $\nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G^{(t)}}) = 0$. However, in practice, we would not be at the optimal discriminator and therefore this term may be non-zero. Our hypothesis is that, instead of ignoring this term like it is done for the conventional updates,
We can compare our updates in Equation 12 with the above. While both have two similar terms, 

\[
0 = \nabla_{\theta_D} V(\theta_D, \theta_G^{(t)}) |_{\theta_D = \theta_D^{\star} \theta_G^{(i)}} \implies 
\]

\[
0 = \frac{\partial}{\partial \theta_G} \left( \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) |_{\theta_D = \theta_D^{\star} \theta_G^{(i)}} \right) |_{\theta_G = \theta_G^{(i)}} 
= \frac{\partial \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_G} \bigg|_{\theta_D = \theta_D^{\star} \theta_G^{(i)}, \theta_G = \theta_G^{(i)}} + \nabla_{\theta_D} V(D_{\theta_D}, D_{\theta_G^{(i)})} |_{\theta_D = \theta_D^{\star} \theta_G^{(i)}} \frac{\partial \theta_D^*}{\partial \theta_G} \bigg|_{\theta_G = \theta_G^{(i)}} 
\] 

(10)

In the second step above, we apply the chain rule. Rearranging, we get:

\[
\frac{\partial \theta_D^*}{\partial \theta_G} \bigg|_{\theta_G = \theta_G^{(i)}} = - \left( \nabla_{\theta_D} V(D_{\theta_D}, D_{\theta_G^{(i)})} |_{\theta_D = \theta_D^{\star} \theta_G^{(i)}} \right)^{-1} \frac{\partial \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_G} \bigg|_{\theta_D = \theta_D^{\star} \theta_G^{(i)}, \theta_G = \theta_G^{(i)}} 
\] 

(11)

Since we hope the objective to be concave in the discriminator parameters, we can approximate the Hessian as \(\nabla_{\theta_D}^2 V(D_{\theta_D}, D_{\theta_G^{(i)})} = -I/\eta\). Plugging this into Equation 9 and replacing the optimal discriminator with an arbitrary discriminator, we get the following update rule which is equivalent to the original one presented in Equation 5.

\[
\theta_G^{(t+1)} = \theta_G^{(t)} - \alpha \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) |_{\theta_G = \theta_G^{(i)}} - \alpha \eta \left( \frac{\partial \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_G} \right)^T \bigg|_{\theta_G = \theta_G^{(i)}} \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) |_{\theta_G = \theta_G^{(i)}} 
\] 

(12)

**F.2 Relation to 1-unrolled updates**

We now show our regularization term and the 1-unrolled updates have similar structure. We begin by simplifying the 1-unrolled updates. The key idea of a 1-unrolled update is to allow the generator to explicitly foresee how the discriminator would react to its update, and optimize accordingly.

\[
\theta_G^{(t+1)} := \theta_G^{(t)} - \alpha \nabla_{\theta_D} V(D_{\theta_D} + \eta \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}), G_{\theta_G}) |_{\theta_G = \theta_G^{(i)}} 
= \theta_G^{(t)} - \alpha \nabla_{\theta_D} V(D_{\theta_D} + \eta \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}), G_{\theta_G}) |_{\theta_G = \theta_G^{(i)}} - \alpha \nabla_{\theta_D} V(D_{\theta_D} + \eta \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}), G_{\theta_G}) |_{\theta_G = \theta_G^{(i)}} 
= \theta_G^{(t)} - \alpha \nabla_{\theta_D} V(D_{\theta_D} + \eta \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}), G_{\theta_G}) |_{\theta_G = \theta_G^{(i)}} 
- \alpha \eta \left( \frac{\partial \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_G} \right)^T \bigg|_{\theta_G = \theta_G^{(i)}} \nabla_{\theta_D} V(D_{\theta_D}', G_{\theta_G}) |_{\theta_D' = \theta_D + \eta \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})} 
\] 

In the first step, we compute gradient with respect to \(\theta_G\) as the sum of the gradients with respect to the two instances of \(\theta_G\) that occur in \(V(D_{\theta_D} + \eta \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}), G_{\theta_G})\). In the second step, we apply the chain rule on the second gradient.

We can compare our updates in Equation 12 with the above. While both have two similar terms, a crucial difference is that in the latter, every occurrence of the discriminator parameters (except one) has an additional unrolled update, namely \(\eta \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})\). Clearly, this should provide more power to the latter; however in practice, we observe that our technique can be more powerful than 1-unrolled or even 10-unrolled updates (which are in fact much slower to run). The reason is that the unrolled updates constrain \(\eta\) to be small, typically of the order \(10^{-4}\) which is the step size. It would
not be possible to increase $\eta$ to greater magnitudes as it would be equivalent to a coarse step size in the unrolling. Our method on the other hand, allows for larger $\eta$ because the discriminator is retained as it is; in some sense, our penalty provides a way of extracting and leveraging the unrolled update more flexibly.

### F.3 Local stability of gradient-regularized GANs

We now present the Jacobian of the system at equilibrium in the presence of the gradient penalty. Recall that the penalty basically adds an extra $-\nabla_{\theta_G} \| \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) \|^2$ to the generator’s update.

**Lemma F.1.** For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 5, the Jacobian at an equilibrium point $(\theta^*_D, \theta^*_G)$, under the Assumptions I and IV is:

$$ J = \begin{bmatrix} J_{DD} & J_{DG} \\ -J_{DG}^T(I + 2\eta J_{DD}) & -2\eta J_{DG}^T J_{DG} \end{bmatrix} $$

where $J_{DD}$ and $J_{DG}$ are terms in the Jacobian corresponding to the original updates, as described in Theorem 3.1.

**Proof.** Note that the only change to the Jacobian would be in the rows corresponding to the generator parameters. Therefore, we will focus only on the additional terms in these rows.

The additional term added to $-J_{DG}^T$ is:

$$ -\frac{\partial \eta \nabla_{\theta_G} \| \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) \|^2}{\partial \theta_D} \bigg|_{\theta^*_D, \theta^*_G} = -\eta \left( \frac{\partial \nabla_{\theta_D} \| \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) \|^2}{\partial \theta_G} \bigg|_{\theta^*_D, \theta^*_G} \right)^T $$

$$ = -\eta \left( \frac{\partial (2\nabla_{\theta_D}^2 V(D_{\theta_D}, G_{\theta_G}) \nabla_{\theta_G} V(D_{\theta_D}, G_{\theta_G}))}{\partial \theta_G} \bigg|_{\theta^*_D, \theta^*_G} \right)^T $$

$$ = -2\eta \left( \frac{\partial \nabla_{\theta_D}^2 V(D_{\theta_D}, G_{\theta_G}) \nabla_{\theta_G} V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_G} \bigg|_{\theta^*_D, \theta^*_G} + \frac{\partial \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) \nabla_{\theta_D}^2 V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_D} \bigg|_{\theta^*_D, \theta^*_G} \right)^T $$

$$ = -2\eta J_{DG}^T J_{DD} $$

Now, the additional term added to $J_{GG}$ is:

$$ -\frac{\partial \eta \nabla_{\theta_G} \| \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) \|^2}{\partial \theta_G} \bigg|_{\theta^*_D, \theta^*_G} $$

$$ = -\eta \frac{\partial}{\partial \theta_G} \left( 2 \frac{\partial \nabla_{\theta_G} V(D_{\theta_D}, G_{\theta_G}) \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_D} \bigg|_{\theta^*_D, \theta^*_G} \right) $$

$$ = -2\eta \left( \frac{\partial \nabla_{\theta_D}^2 V(D_{\theta_D}, G_{\theta_G}) \nabla_{\theta_G} V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_G} \bigg|_{\theta^*_D, \theta^*_G} + \frac{\partial \nabla_{\theta_D} V(D_{\theta_D}, G_{\theta_G}) \nabla_{\theta_D}^2 V(D_{\theta_D}, G_{\theta_G})}{\partial \theta_D} \bigg|_{\theta^*_D, \theta^*_G} \right)^T $$

$$ = -2\eta J_{DG}^T J_{DG} $$

Now, we will prove stability of the regularized system for conventional GANs. Observe that Lemmas C.2 regarding the subspace of equilibria holds in this case too. Again, we can project the system as follows:
Lemma F.2. For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 5, consider the eigenvalue decompositions $K_{DD} = U_D A_D U_D^T$ and $K_{DG}^T K_{DG} = U_G A_G U_G^T$. Let $U_D = [T_D^T, T_D'^T]$ and $U_G = [T_G^T, T_G'^T]$ such that $\text{Col}(T_D^T) = \text{Null}(K_{DD})$ and $\text{Col}(T_G^T) = \text{Null}(K_{DG})$. Consider the projections, $\gamma_D = T_D \theta_D$ and $\gamma_G = T_G \theta_G$. Then, the block in the Jacobian at equilibrium that corresponds to the projected system has the form:

$$J' = \begin{bmatrix} J'_{DD} & J'_{DG} \\ -J'_{DG} & J'_{GG} \end{bmatrix} = \begin{bmatrix} T_D J_{DD} T_D^T & T_D J_{DG} T_G^T \\ -T_G J_{DG}(1 + 2\eta J_{DD}) T_D^T & -2\eta T_G J_{DG}^T J_{DG} T_G^T \end{bmatrix}$$

Under Assumption II, we have that $J'_{DD} \prec 0$ and $J'_{DG}$ is full column rank and $J'_{GG} \prec 0$.

It is straightforward to extend the proof of Lemma C.3 to prove this lemma. Now, recall from Theorem A.4 that if we show $J'$ is Hurwitz the original system is exponentially stable. In the non-regularized system, we showed this by making use of the structure of the matrix. For this system, we will design a quadratic Lyapunov function that strictly decreases at non-equilibria points.

Lemma F.3. For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 5, if $\eta < \frac{1}{2\lambda_{\text{max}}(-J_{DD})}$, the linearization of the system projected to a subspace orthogonal to the subspace of equilibria is exponentially stable with the Lyapunov function $x^T P x$ where,

$$P = \begin{bmatrix} T_D (I + 2\eta J_{DD}) T_D^T & 0 \\ 0 & I \end{bmatrix}$$

and $x^T$ is $[\gamma_D^T \gamma_G^T] - [\gamma_D^* \gamma_G^*] T$. The function strictly decreases with time except at the equilibrium $[\gamma_D^* \gamma_G^*]^T$.

Proof. Note that when $\eta < \frac{1}{2\lambda_{\text{max}}(-J_{DD})}$, $P = P^T > 0$ therefore the Lyapunov function is indeed positive definite. Furthermore, note that the rate of decrease is given by $x^T Q x$ where $Q = (J'^T P + P J')$. To show that this is strictly decreasing, we only need to show that $J'^T P + PJ' < 0$.

First of all, note that $Q =$

$$\begin{bmatrix} T_D & T_G \\ T_G^T & T_D \end{bmatrix} \begin{bmatrix} J_{DD}(I + 2\eta J_{DD}) & -(I + 2\eta J_{DD}) J_{DG} \\ J_{DG}(I + 2\eta J_{DD}) & -2\eta J_{DG}^T J_{DG} \end{bmatrix} + \begin{bmatrix} (I + 2\eta J_{DD}) J_{DD} & (I + 2\eta J_{DD}) J_{DG} \\ -(I + 2\eta J_{DD}) J_{DG} & -2\eta J_{DG}^T J_{DG} \end{bmatrix} \begin{bmatrix} T_D \\ T_G \end{bmatrix}^T$$

which is equal to the diagonal matrix:

$$\begin{bmatrix} T_D [J_{DD}(I + 2\eta J_{DD}) + (I + 2\eta J_{DD}) J_{DG}] T_D^T & 0 \\ 0 & -4\eta T_G J_{DG}^T J_{DG} T_G^T \end{bmatrix}$$

Note that by our choice of $\eta$, $(I + 2\eta J_{DD}) > 0$. Therefore, $J_{DD}$ and $I + 2\eta J_{DD}$ share the same set of eigenvectors. Thus, the null space of $J_{DD}$ and the term $J_{DD}(I + 2\eta J_{DD}) + (I + 2\eta J_{DD}) J_{DD}$ are the same. In other words, the top-left block above is a diagonal matrix with strictly negative eigenvalues. Similarly, we also know that $-2\eta T_G J_{DG}^T J_{DG} T_G^T$ is a diagonal matrix with negative values. Hence, the above matrix is negative definite. \qed

F.3.1 Exponential stability of gradient-regularized WGAN

We now proceed to the Wasserstein GAN scenario. First we lay down equivalent assumptions for the WGAN under which we can guarantee exponential stability in the regularized case. Note that even under these conditions, the unregularized update does not ensure asymptotic stability.

First, we note that due to the linearity of the loss function, it is not necessary that the discriminator be only identically zero on the support for the system to be at equilibrium — it could also be constant on the support. Thus, we relax Assumption I for this case to accommodate this.

Assumption I. (WGAN, Realizable) If $(\theta_0^*_G, \theta_0^*_D)$ is an equilibrium point, we assume that $p_{\theta_0^*_G} = p_{\text{data}}$ and $D_{\theta_0^*_D}(x) = c$, $\forall x \in \text{supp}(p_{\text{data}})$ for some $c \in \mathbb{R}$.

Next, we state an assumption equivalent to Assumption III. Recall that earlier we wanted $E_{p_{\text{data}}}[D_{\theta_D}^2(x)]$ to satisfy Property I in the discriminator space. Instead of this function, we will now...
require that the magnitude of the generator updates satisfy Property I in the discriminator space. Note that the Hessian of this function at equilibrium is $K_{DG} K_{DG}^T$.

**Assumption III. (WGAN)** At an equilibrium $(\theta_D^*, \theta_G^*)$, the functions
\[
\left\|
\int x \nabla \theta_G^p \theta_G^c (x) D \theta_D (x)
\right\|^2_{\theta_G=\theta_G^*}
\] and
\[
\left\|
E_{p_{\text{data}}} [\nabla \theta_D D \theta_D (x)] - E_{p_{\theta_G}} [\nabla \theta_D D \theta_D (x)]
\right\|^2_{\theta_D=\theta_D^*}
\]
must satisfy Property I in the discriminator and generator space respectively.

We retain Assumption IV as it is. These are the only three assumptions we will need.

We will now begin with a lemma similar to Lemma C.2

**Lemma E.4.** For the dynamical system defined by the WGAN objective in Equation 3 and the updates in Equation 5, under Assumptions I and III under the WGAN case, there exists $\epsilon_D, \epsilon_G > 0$ such that for all $\epsilon_D' \leq \epsilon_D$ and $\epsilon_G' \leq \epsilon_G$, and for any unit vectors $u \in \text{Null}(K_{DG}), v \in \text{Null}(K_{DG})$, $(\theta_D^* + \epsilon_D' u, \theta_G^* + \epsilon_G' v)$ is an equilibrium point.

**Proof.** Note that $2K_{DG} K_{DG}^T$ is the Hessian of the function $\left\| \int x \nabla \theta_G^p \theta_G^c (x) D \theta_D (x) \right\|^2$ at equilibrium, namely the magnitude of the generator update. Then, by Assumption III, this function is locally constant along any unit vector $u \in \text{Null}(K_{DG})$. That is, for sufficiently small $\epsilon$, if $\theta_D = \theta_D^* + \epsilon u$, the function value is equal to the value at equilibrium which is zero, because by definition at equilibrium the generator update is zero. Now at $(\theta_D^*, \theta_G^*)$, the discriminator update is zero too since the generator matches the true distribution. Then by Assumption I, it means that $D \theta_D$ is identical to the true support. Then, it is an equilibrium discriminator such that the update for any generator would be zero.

Similarly, as we saw $2K_{DG}^T K_{DG}$ is the Hessian of the function $\left\| E_{p_{\text{data}}} [\nabla \theta_D D \theta_D (x)] - E_{p_{\theta_G}} [\nabla \theta_D D \theta_D (x)] \right\|^2$ at equilibrium, namely the magnitude of the discriminator update. We also saw that for sufficiently small $\epsilon'$, if $\theta_G = \theta_G^* + \epsilon' v$, this function is zero. Thus, at $(\theta_D^*, \theta_G^*)$, the discriminator update is zero. Furthermore, the generator update is zero too because the discriminator is constant throughout the support. Thus, $(\theta_D^*, \theta_G^*)$ is an equilibrium point and from Assumption I we can conclude that $p_{\theta_G} = p_{\text{data}}$. Thus, it is an equilibrium generator such that the update for any discriminator would be zero.

In summary, for all slight perturbations along $u \in \text{Null}(K_{DG}), v \in \text{Null}(K_{DG})$ we have established that the discriminator and generator individually satisfy the requirements of an equilibrium discriminator and generator pair, and therefore the system is itself is in equilibrium for these perturbations. □

Now, we show that this system can again be projected to a subspace orthogonal the equilibrium subspace such that the resulting Jacobian of the reduced system is Hurwitz. While earlier we chose $T_D$ based on the matrix $K_{DD}$ now we will choose it based on $K_{DG}^T$.

**Lemma E.5.** For the dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 5, consider the eigenvalue decompositions $K_{DG} K_{DG}^T = U_D A_D U_D^T$ and $K_{DG}^T K_{DG} = U_G A_G U_G^T$. Let $U_D = [T_D^T, T_D^T]$ and $U_G = [T_G^T, T_G^T]$ such that $\text{Col}(T_D^T) = \text{Null}(K_{DD})$ and $\text{Col}(T_G^T) = \text{Null}(K_{DG})$. Consider the projections, $\gamma_D = T_D \theta_D$ and $\gamma_G = T_G \theta_G$. Then, the block in the Jacobian at equilibrium that corresponds to the projected system has the form:
\[
J' = 
\begin{bmatrix}
J'_{DD} & J'_{DG} \\
-J'_{DG} & J'_{GG}
\end{bmatrix} = 
\begin{bmatrix}
0 & T_D J_{DG} T_D^T \\
-T_G J_{DG} T_D^T & -2\eta T_G J_{DG} T_D T_G^T
\end{bmatrix}
\]

Furthermore $J'_{GG} \prec 0$ and $J'_{DG}$ is full column rank.

**Proof.** Observe that the form of $J'$ follows from Lemma E.1 by substituting $J_{DD} = 0$. Furthermore, like we have seen before, observe that $T_G J_{DG}^T J_{DG} T_D^T$ is a diagonal matrix with positive eigenvalues and therefore $J'_{GG} \prec 0$. Similarly, $J_{DG}^T T_D$ is a full column rank matrix because we have projected it to the subspace orthogonal to its null space. However, we need to show that $T_G J_{DG}^T T_D$ which may have fewer rows, did not reduce in its rank. This is indeed true, since this is effectively a projection onto the subspace orthogonal to its left null space. □
We now compile the above lemmas to prove our main result.

**Theorem 3.4.** The dynamical system defined by the GAN objective in Equation 3 and the updates in Equation 5, the system is locally exponentially stable at the equilibrium, under the same conditions as in Theorem 3.1, if \( \eta < \frac{1}{2x_{\text{max}}(-J_{DG})} \). Further, under similar conditions, the WGAN system is locally exponentially stable at the equilibrium. The rate of convergence for the WGAN is governed only by the eigenvalues \( \lambda \) of the Jacobian at equilibrium with a strict negative real part upper bounded as:

\[
\begin{align*}
\text{If } \text{Im}(\lambda) = 0, & \text{ then } \text{Re}(\lambda) \leq -\frac{2f^2(0)\eta\lambda_{\text{min}}^+(K_{DG}^TK_{DG})}{4f^2(0)\eta^2\lambda_{\text{max}}(K_{DG}^TK_{DG})+1} \\
\text{If } \text{Im}(\lambda) \neq 0, & \text{ then } \text{Re}(\lambda) \leq -\eta f^2(0)\lambda_{\text{min}}^+(K_{DG}^TK_{DG})
\end{align*}
\]

**Proof.** The first part of the theorem statement for the conventional GAN follows from Lemma F.1, F.2, F.3.

To prove the second part it is sufficient to show that the projected Jacobian of the linearized system in Lemma F.5 is Hurwitz, from which exponential stability of the original system follows from Theorem A.4. The fact that this is Hurwitz follows as usual from Lemma G.2 after we flip the discriminator and generator variables:

\[
\begin{bmatrix}
J_{GG}' & -J_{DG}'^T \\
J_{DG}' & 0
\end{bmatrix}.
\]

The Jacobian is thus Hurwitz because \( J_{GG}' \) is negative definite and \( -J_{DG}'^T \) is full column rank. Now, for the eigenvalue bounds we have from Lemma G.2 that:

\[
\begin{align*}
\text{If } \text{Im}(\lambda) = 0, & \text{ then } \text{Re}(\lambda) \leq -\frac{2f^2(0)\eta\lambda_{\text{min}}^+(K_{DG}^TK_{DG})\lambda_{\text{min}}^+(K_{DG}^TK_{DG})}{4f^2(0)\eta^2\lambda_{\text{max}}(K_{DG}^TK_{DG})+1} \\
\text{If } \text{Im}(\lambda) \neq 0, & \text{ then } \text{Re}(\lambda) \leq -\eta f^2(0)\lambda_{\text{min}}^+(K_{DG}^TK_{DG})
\end{align*}
\]

However, this can be further simplified to arrive at the given bound by noting that all the non-zero eigenvalues of any matrix \( AB \) is also equal to the non-zero eigenvalues of the matrix \( BA \). Therefore, we can replace every occurrence of \( K_{DG}^TK_{DG} \) with \( K_{DG}^TK_{DG} \) in the above inequality.

Additionally, we show that we can find a Lyapunov function that satisfies LaSalle’s principle for the projected linearized system.

**Fact F.1.** For the linearized projected system with the Jacobian \( J' \), we have that \( 1/2\|\gamma_D - \gamma_D^*\|^2 + 1/2\|\gamma_G - \gamma_G^*\|^2 \) is a Lyapunov function such that for all non-equilibrium points, it either always decreases or only instantaneously remains constant.

**Proof.** Note that the Lyapunov function is zero only at the equilibrium of the projected system. Furthermore, it is straightforward to verify that the rate at which this changes is given by \(-2\eta(\gamma_G - \gamma_G^*)^T T_G K_{DG}^T K_{DG} T_G^T (\gamma_G - \gamma_G^*) \) which is non-positive. Clearly this is zero only when \( \gamma_G = \gamma_G^* \) because \( T_G K_{DG}^T K_{DG} T_G^T \) is positive definite. When this rate is indeed zero, we have that for the linearized system, \( \dot{\gamma}_G = T_G K_{DG} T_D (\gamma_D - \gamma_D^*) \) because the other term becomes zero. Again, for the system to identically stay on the manifold \( \gamma_G = \gamma_G^* \) we need \( \dot{\gamma}_G = 0 \), which happens only when \( \gamma_D = \gamma_D^* \) because \( T_G K_{DG} T_D \) is full column rank. When that is the case, we are at equilibrium.

\( \square \)

**G. Eigenvalue bounds**

In this section, we prove one of the most useful lemmas that we used in our proofs, that matrices of the form \([-Q \quad P; -P^T \quad 0]\) are Hurwitz when \( Q \succ 0 \) and \( P \) is full column rank. We also prove eigenvalue bounds for such a matrix. To do so, we begin with a simple fact:
Lemma G.1. For \( Q \succeq 0 \) be a real symmetric matrix. If \( a^T Q a = c \), then \( a^T Q^T Q a \in [\lambda_{min}(Q)c, \lambda_{max}(Q)c] \).

**Proof.** Let \( Q = U \Lambda U^T \) be the eigenvalue decomposition of \( Q \). Let \( x = U a \). Then, \( c = x^T \Lambda x \) or in other words, \( c = \sum x_i^2 \lambda_i \). Similarly, \( a^T Q^T Q a = \sum x_i^2 \lambda_i^2 \) which differs from \( c \) by a multiplicative factor within \([\lambda_{min}(Q), \lambda_{max}(Q)]\). \( \square \)

We now prove our main result.

**Lemma G.2.** Let

\[
J = \begin{bmatrix}
-Q & P \\
-P^T & 0
\end{bmatrix},
\]

where \( Q \) is a symmetric real positive definite matrix and \( P \) is a full column rank matrix. Then, \( \text{Re}(\lambda) < 0 \) for every eigenvalue \( \lambda \) of \( J \). In fact,

- When \( \text{Im}(\lambda) = 0 \),
  \[
  \text{Re}(\lambda) \leq -\frac{\lambda_{min}(Q) \lambda_{min}(P^T P)}{\lambda_{max}(Q) \lambda_{min}(Q) + \lambda_{min}(P^T P)}
  \]
- When \( \text{Im}(\lambda) \neq 0 \),
  \[
  \text{Re}(\lambda) \leq -\frac{\lambda_{min}(Q)}{2}
  \]

**Proof.** We consider a generic eigenvector equation and equate the real and complex parts together so as to arrive at our bounds. Consider the following eigenvector equation:

\[
\begin{bmatrix}
-Q & P \\
-P^T & 0
\end{bmatrix}
\begin{bmatrix}
a_1 + i a_2 \\
b_1 + i b_2
\end{bmatrix} = (\lambda_1 + i \lambda_2)
\begin{bmatrix}
a_1 + i a_2 \\
b_1 + i b_2
\end{bmatrix},
\]

where \( a_i, b_i, \lambda_i \) are all real-valued. We assume that the vector is normalized i.e., \( a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1 \).

So, in case \( \lambda_2 = 0 \), we assume that \( a_1^2 = b_1^2 = 1 \). We want to show that \( \lambda_1 < 0 \). Let us first rewrite the above equation as follows:

\[
\begin{bmatrix}
-Qa_1 + Pb_1 + i(-Qa_2 + Pb_2) \\
-P^T a_1 + i(-P^T a_2)
\end{bmatrix} = \begin{bmatrix}
\lambda_1 a_1 - \lambda_2 a_2 + i(\lambda_1 a_2 + \lambda_2 a_1) \\
\lambda_1 b_1 - \lambda_2 b_2 + i(\lambda_1 b_2 + \lambda_2 b_1)
\end{bmatrix}
\]

We can then equate the real and imaginary parts.

\[
\begin{align*}
-Qa_1 + Pb_1 &= \lambda_1 a_1 - \lambda_2 a_2 \tag{13} \\
-P^T a_1 &= \lambda_1 b_1 - \lambda_2 b_2 \tag{14} \\
-Qa_2 + Pb_2 &= \lambda_1 a_2 + \lambda_2 a_1 \tag{15} \\
-P^T a_2 &= \lambda_1 b_2 + \lambda_2 b_1 \tag{16}
\end{align*}
\]

We now multiply the above equations by \( a_1^T, b_1^T, a_2^T, b_2^T \) respectively and add them:

\[
\begin{align*}
a_1^T (-Qa_1 + Pb_1) - b_1^T P^T a_1 &= a_1^T (\lambda_1 a_1 - \lambda_2 a_2) + b_1^T (\lambda_1 b_1 - \lambda_2 b_2) \\
+ a_2^T (-Qa_2 + Pb_2) - b_2^T P^T a_2 &= a_2^T (\lambda_1 a_2 + \lambda_2 a_1) + b_2^T (\lambda_1 b_2 + \lambda_2 b_1)
\end{align*}
\]

As a result, only square terms and \( \lambda_1 \) terms remain:

\[-a_1^T Qa_1 - a_2^T Qa_2 = \lambda_1 (a_1^T a_1 + a_2^T a_2 + b_1^T b_1 + b_2^T b_2) = \lambda_1 \]

**Proof for \( \lambda_1 < 0 \).** Now observe that \(-a_1^T Qa_1 - a_2^T Qa_2 \leq 0 \) because \( Q > 0 \). If \(-a_1^T Qa_1 - a_2^T Qa_2 < 0 \), it would immediately imply that \( \lambda_1 < 0 \).

On the other hand, consider the situation when \( a_1 = 0 \) and \( a_2 = 0 \). We will show that this case would not occur. First of all, this would force \( \lambda_1 = 0 \) to ensure the above equality. By applying the Equations 14 and 16, we can conclude that \( \lambda_2 b_2 = 0 \) and \( \lambda_2 b_1 = 0 \). Since one of \( b_1, b_2 \neq 0 \), this implies that \( \lambda_2 = 0 \) too. Now, by applying Equation 13 and 15, we have that \( Pb_1 = 0 \) and \( Pb_2 = 0 \). Since one of \( b_1, b_2 \neq 0 \), this implies that \( P \) is not a full column rank matrix, which is a contradiction of our assumption. Therefore, it cannot be the case that both \( a_1 = 0 \) and \( a_2 = 0 \).
Stricter bound. Now, we prove our bounds on $\lambda_1$. (Note that an easy lower bound follows as $\lambda_1 \geq -\lambda_{\text{max}}(Q)$ but we are interested in an upper bound). In order to prove the upper bound, we multiply Equations 13 and Equations 15 by $-a_2^T$ and $a_1^T$ respectively and sum them up, and Equations 14 and Equations 16 by $-b_2^T$ and $b_1^T$ respectively and sum them up.

$$a_2^T Qa_1 - a_2^T Pb_1 - a_1^T Qa_2 + a_1^T Pb_2 = -a_2^T \lambda_1 a_1 + a_1^T \lambda_2 a_1 + a_1^T \lambda_2 a_2 + a_2^T \lambda_1 a_2$$

$$\Rightarrow -a_2^T Pb_1 + a_2^T Pb_2 = \lambda_2(\|a_2\|^2 + \|a_1\|^2)$$

$$b_2^T P^T a_1 - b_1^T P^T a_2 = -b_2^T \lambda_1 b_1 + b_2^T \lambda_2 b_2 + b_1^T \lambda_1 b_2 + b_1^T \lambda_2 b_1$$

$$\Rightarrow b_2^T P^T a_1 - b_1^T P^T a_2 = \lambda_2(\|b_2\|^2 + \|b_1\|^2)$$

As a consequence,

$$\lambda_2(\|a_2\|^2 + \|a_1\|^2) = \lambda_2(\|b_2\|^2 + \|b_1\|^2)$$

From the above we have that either $\lambda_2 = 0$ or $\|b_2\|^2 + \|b_1\|^2 = \|a_2\|^2 + \|a_1\|^2 = 1/2$. Now, if $\lambda_2 \neq 0$, since $-a_1^T Qa_1 - a_2^T Qa_2 = \lambda_1$, we immediately get a bound $\lambda_1 \leq -\lambda_{\text{min}}(Q)/2$.

In the former case, since the imaginary part of the eigenvalue is zero i.e., $\lambda_2 = 0$, the imaginary part of the eigenvector must be zero too i.e., $a_2 = b_2 = 0$. Then, we have the equations:

$$-Qa_1 + Pb_1 = \lambda_1 a_1$$

$$-P^T a_1 = \lambda_1 b_1$$

Rearranging and squaring the first equation we get:

$$b_1^T P^T Pb_1 = a_1^T (\lambda_1 I + Q)^T (\lambda_1 I + Q) a_1$$

$$= a_1^T (\lambda_1^2 I + 2\lambda_1 Q + Q^T Q) a_1$$

Then,

$$\lambda_{\text{min}}(P^T P) \|b_1\|^2 \leq \lambda_1^2 \|a_1\|^2 - 2\lambda_1^2 + a_1^T Q^T Q a_1$$

$$\lambda_{\text{min}}(P^T P) \leq (\lambda_1^2 + \lambda_{\text{min}}(P^T P)) \|a_1\|^2 - 2\lambda_1^2 + a_1^T Q^T Q a_1$$

$$\leq (-\lambda_1^3 - \lambda_{\text{min}}(P^T P) \lambda_1) \frac{1}{\lambda_{\text{min}}(Q)} - 2\lambda_1^2 - \lambda_1 \lambda_{\text{max}}(Q)$$

$$\leq - \frac{\lambda_1}{\lambda_{\text{min}}(Q)} \left( \lambda_1^2 + 2\lambda_{\text{min}}(Q) \lambda_1 + \lambda_{\text{max}}(Q) \lambda_{\text{min}}(Q) + \lambda_{\text{min}}(P^T P) \right).$$

In the first step, we make use of the fact that $\lambda_1 = -a_1^T Qa_1$. In the second step, we use $\|a_1\|^2 + \|b_1\|^2 = 1$. In the third step, we use Lemma G.1 i.e., $a_1^T Q^T Q a_1 \leq -\lambda_1 \lambda_{\text{max}}(Q)$. We also use the fact that since $\lambda_1 = -a_1^T Qa_1$, $\|a_1\|^2 \leq \frac{-\lambda_1}{\lambda_{\text{min}}(Q)}$.

How do we upper bound $\lambda_1$ using this inequality? Observe that on the right hand side, when $\lambda_1 = 0$, the term quadratic in $\lambda_1$ is positive. However, the overall term is zero. As $\lambda_1$ decreases until $\lambda_1 = -\lambda_{\text{min}}(Q)$, the quadratic would decrease, while the overall term would increase to a positive non-zero value. Thus, we can upper bound $\lambda_1$, by fixing the quadratic at the value it takes at $\lambda_1 = 0$. If this yields an bound greater than $-\lambda_{\text{min}}(Q)$, then it is a conservative estimate. Otherwise, we can say that $-\lambda_{\text{min}}(Q)$ is an upper bound on $\lambda_1$. However, we show that the former is indeed the case:

$$\lambda_{\text{min}}(P^T P) \leq - \frac{\lambda_1}{\lambda_{\text{min}}(Q)} \left( \lambda_{\text{max}}(Q) \lambda_{\text{min}}(Q) + \lambda_{\text{min}}(P^T P) \right)$$

$$\lambda_1 \leq - \frac{\lambda_{\text{min}}(Q) \lambda_{\text{min}}(P^T P)}{\lambda_{\text{max}}(Q) \lambda_{\text{min}}(Q) + \lambda_{\text{min}}(P^T P)}$$
Observe that this upper bound is strictly greater than $-\lambda_{\min}(Q)$ because the factor excluding $-\lambda_{\min}(Q)$ is a positive fraction. (If this were not true, we would simply say $\lambda_1 \leq -\lambda_{\min}(Q)$).

Now, we provide a similar upper bound result, though only partially, for eigenvalues of matrices that have the same structural properties as the Jacobian of our regularized system. Note that we only have upper bounds only for eigenvalues that are complex.

**Theorem G.3.** Let

$$J = \begin{bmatrix} -Q & P \\ -P^T(I - \eta Q) & -2\eta P^TP \end{bmatrix},$$

where $Q$ is a real symmetric positive definite matrix and $P$ is a full column rank matrix. Let $\eta < \frac{1}{\lambda_{\max}(Q)}$.

Then, if $\text{Im}(\lambda) \neq 0$ for any eigenvalue $\lambda$ of $J$,

$$\text{Re}(\lambda) \leq -\frac{1}{2} - \frac{\eta \lambda_{\max}(Q)}{1 - \eta \lambda_{\min}(Q)} \left( \lambda_{\min}(Q) + \eta \lambda_{\min}(P^TP) \right).$$

**Proof.** Consider the following eigenvector equation:

$$\begin{bmatrix} -Q & P \\ -P^T(I - \eta Q) & -2\eta P^TP \end{bmatrix} \begin{bmatrix} a_1 + ia_2 \\ b_1 + ib_2 \end{bmatrix} = (\lambda_1 + i\lambda_2) \begin{bmatrix} a_1 + ia_2 \\ b_1 + ib_2 \end{bmatrix},$$

where $u_1, v_1, \lambda_1$ are all real-valued. We want to show that $\lambda_1 < 0$. Let us first rewrite the above equation as follows:

$$\begin{bmatrix} -Qa_1 + Pb_1 + i(-Qa_2 + Pb_2) \\ -P^T(I - \eta Q)a_1 - \eta P^TPb_1 + i(-P^T(I - \eta Q)a_2 - 2\eta P^TPb_2) \end{bmatrix} = \begin{bmatrix} \lambda_1 a_1 - \lambda_2 a_2 + i(\lambda_1 a_2 + \lambda_2 a_1) \\ \lambda_1 b_1 - \lambda_2 b_2 + i(\lambda_1 b_2 + \lambda_2 b_1) \end{bmatrix}$$

We can then equate the real and imaginary parts.

$$-Qa_1 + Pb_1 = \lambda_1 a_1 - \lambda_2 a_2 \quad (17)$$

$$-P^T(I - \eta Q)a_1 - 2\eta P^TPb_1 = \lambda_1 b_1 - \lambda_2 b_2 \quad (18)$$

$$-Qa_2 + Pb_2 = \lambda_1 a_2 + \lambda_2 a_1 \quad (19)$$

$$-P^T(I - \eta Q)a_2 - 2\eta P^TPb_2 = \lambda_1 b_2 + \lambda_2 b_1 \quad (20)$$

$$-P^T(I - \eta Q)a_2 - 2\eta P^TPb_2 = \lambda_1 b_2 + \lambda_2 b_1 \quad (21)$$

We now multiply the above equations by $a_1^T, b_1^T, a_2^T, b_2^T$ respectively and add them:

$$a_1^T(-Qa_1 + Pb_1) - b_1^T(P^TPa_1 - 2\eta b_1 P^TPb_1) + a_2^T(-Qa_2 + Pb_2) - b_2^T(P^TPa_2 - 2\eta b_2 P^TPb_2) + \eta b_1^T P^T Q a_1 + \eta b_2^T P^T Q a_2 = \lambda_1$$

As a result, we get:

$$-a_1^T Q a_1 - a_2^T Q a_2 - 2\eta b_1 P^T P b_1 - 2\eta b_2 P^T P b_2 + \eta b_1^T P^T Q a_1 + \eta b_2^T P^T Q a_2 = \lambda_1$$

Above, we can substitute for $Pb_1$ and $Pb_2$ in $\eta b_1^T P^T Q a_1 + \eta b_2^T P^T Q a_2$ using the previous equations.

$$-a_1^T Q a_1 - a_2^T Q a_2 - 2\eta b_1 P^T P b_1 - 2\eta b_2 P^T P b_2$$

$$+ \eta(\lambda_1 a_1^T Q a_1 - \lambda_2 a_2^T Q a_2 + a_1^T Q^T Q a_1)$$

$$+ \eta(\lambda_1 a_2^T Q a_2 + \lambda_2 a_1^T Q a_1 + a_2^T Q^T Q a_2) = \lambda_1$$
\[
\frac{-a_1^T Q a_1 - a_2^T Q a_2 - \eta b_1 P^T P b_1 - \eta b_2 P^T P b_2 + \eta a_1^T Q^T Q a_1 + \eta a_2^T Q^T Q a_2}{1 - \eta (a_1^T Q a_1 + a_2^T Q a_2)} = \lambda_1
\]

We could do the above only because \( \eta < \frac{1}{\lambda_{\text{max}}(Q)} \) and therefore the denominator \( 1 - \eta (a_1^T Q a_1 + a_2^T Q a_2) \neq 0 \).

In order to prove our upper bound, we first note the following inequality:

\[
|\lambda_1| \geq \frac{(1 - \eta \lambda_{\text{max}}(Q)) \lambda_{\text{min}}(Q)(\|a_1\|^2 + \|a_2\|^2) + \eta \lambda_{\text{min}}(P^T P)(\|b_1\|^2 + \|b_2\|^2)}{1 - \eta \lambda_{\text{min}}(Q)(\|a_1\|^2 + \|a_2\|^2)} \tag{22}
\]

We now multiply the first and third equations by \(-a_2^T P b_1 + a_1^T P b_2 = \lambda_2(\|a_2\|^2 + \|a_1\|^2)\)

\[
b_2^T P^T (I - \eta Q) a_1 - b_1^T P^T (I - \eta Q) a_2 = \lambda_2(\|b_2\|^2 + \|b_1\|^2)
\]

Using the above,

\[
\lambda_2(\|b_2\|^2 + \|b_1\|^2) - \lambda_2(\|a_2\|^2 + \|a_1\|^2) = -\eta b_2^T P^T Q a_1 + \eta b_1^T P^T Q a_2
\]

\[
= -\eta a_1^T Q^T (\lambda_1 a_2 + \lambda_2 a_1 + Q a_2) + \eta a_1^T Q^T (\lambda_1 a_2 - \lambda_2 a_2 + Q a_1)
\]

\[
= -\eta a_1^T Q^T a_1 - \eta a_2^T Q^T a_2
\]

Then, either \( \lambda_2 = 0 \) or when \( \lambda_2 \neq 0 \), we have \( \|b_2\|^2 + \|b_1\|^2 = \|a_2\|^2 + \|a_1\|^2 - \eta \lambda_2 a_1^T Q^T a_1 - \eta \lambda_2 a_2^T Q^T a_2 \). This translates to the inequality:

\[
(1 - \eta \lambda_{\text{max}}(Q))(\|a_2\|^2 + \|a_1\|^2) \leq \|b_2\|^2 + \|b_1\|^2 \leq (1 - \eta \lambda_{\text{min}}(Q))(\|a_2\|^2 + \|a_1\|^2),
\]

By adding \( \|a_2\|^2 + \|a_1\|^2 \) everywhere and using the fact that \( \|a_2\|^2 + \|a_1\|^2 + \|b_2\|^2 + \|b_1\|^2 = 1 \),

the above inequality becomes:

\[
\frac{1}{2 - \eta \lambda_{\text{min}}(Q)} \leq \|a_2\|^2 + \|b_1\|^2 \leq \frac{1}{2 - \eta \lambda_{\text{max}}(Q)}
\]

The above inequalities yield an immediate lower bound on the magnitude in the latter case by plugging them in Equation 22:

\[
|\lambda_1| \geq \frac{(1 - \eta \lambda_{\text{max}}(Q)) \lambda_{\text{min}}(Q)}{\|a_1\|^2 + \|a_2\|^2} + \frac{\eta \lambda_{\text{min}}(P^T P)(\|b_1\|^2 + \|b_2\|^2)}{1 - \eta \lambda_{\text{min}}(Q)(\|a_1\|^2 + \|a_2\|^2)}
\]

\[
\geq \frac{(1 - \eta \lambda_{\text{max}}(Q)) \lambda_{\text{min}}(Q)}{2(1 - \eta \lambda_{\text{min}}(Q))} + \frac{\eta \lambda_{\text{min}}(P^T P)(1 - \eta \lambda_{\text{max}}(Q))}{2(1 - \eta \lambda_{\text{min}}(Q))}
\]

\[
\geq \frac{11 - \eta \lambda_{\text{max}}(Q)}{2} \left( \lambda_{\text{min}}(Q) + \eta \lambda_{\text{min}}(P^T P) \right)
\]

Since, we know \( \lambda_1 \) is negative, this implies an upper bound on \( \lambda_1 \).