The magnetization curve of the two-dimensional spin-1/2 $J_1$-$J_2$ Heisenberg model is investigated by using the Chern-Simons theory under a uniform mean-field approximation. We find that the magnetization curve is monotonically increasing for $J_2/J_1 < 0.267949$, where the system under zero external field is in the antiferromagnetic Néel phase. For larger ratios of $J_2/J_1$, various plateaus will appear in the magnetization curve. In particular, in the disordered phase, our result supports the existence of the $M/M_{\text{sat}} = 1/2$ plateau and predicts a new plateau at $M/M_{\text{sat}} = 1/3$. By identifying the onset ratio $J_2/J_1$ for the appearance of the 1/2-plateau with the boundary between the Néel and the spin-disordered phases in zero field, we can determine this phase boundary accurately by this mean-field calculation. Verification of these interesting results would indicate a strong connection between the frustrated antiferromagnetic system and the quantum Hall system.

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Due to quantum and frustration effects, rich physics can appear in the frustrated quantum spin systems at zero external field. Exciting behavior was also observed recently in several cases with an external magnetic field. For instance, the recently discovered two-dimensional (2D) $S = 1/2$ spin-gap material SrCu$_2$(BO$_3$)$_2$, which can be described by the Shastry-Sutherland model, exhibits several plateaus at $M/M_{\text{sat}} = 1/3$, 1/4, and 1/8 in its magnetization curve, where $M$ ($M_{\text{sat}} = 1/2$) is the (saturating) magnetization per site. The origin of these plateaus and the nature of the corresponding spin states are under intense debate. Recently, by mapping onto spinless fermions carrying one quantum of statistical flux and under a mean-field approximation, Misguich et al. show that the original spin model can be related to a generalized Hofstadter problem, where the spin excitation gaps that produce the observed magnetic order and of the mechanisms which create the quantum Hall effect for the fermions on a lattice. For realistic values of the exchange constants, they obtain an excellent quantitative fit to the observed magnetization curve, which demonstrates the success of their approach.

Another prototype of a realistic frustrated two-dimensional system, which has been recently realized experimentally in Li$_2$VO$_2$SiO$_4$ and Li$_2$VOGeO$_4$ compounds, is the so-called $J_1$-$J_2$ Heisenberg model with the Hamiltonian

$$H = \sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - B \sum_i S_z^i,$$

where the exchange couplings $J_{ij}$ are equal to $J_1$ when $i,j$ are nearest neighbors on the square lattice; $J_{ij}$ are equal to $J_2$ when $i,j$ are connected by a diagonal bond. The external magnetic field $B$ is applied along the $z$-axis. Both couplings are antiferromagnetic, i.e. $J_{1,2} > 0$, and the spins $S_i = 1/2$. This model has been the object of intense investigation through years.

At $B = 0$, for small $J_2/J_1$ where the frustration is weak, the system exhibits Néel ordering described by a wave vector $(\pi, \pi)$. When $J_2/J_1$ is large enough, the ground state is dominated by the next-nearest-neighbor interactions and has a collinear order described by $(\pi, 0)$ or $(0, \pi)$. There is also a general consensus on the disappearance of the magnetic ordering at $0.38 < J_2/J_1 < 0.6$, while the identification of the ground state is still a subject of much controversy.

Just like the case of the Shastry-Sutherland model, the plateaus in the magnetization curve are predicted for the $J_1$-$J_2$ model, while the situation is even more controversial. Strong evidence for a plateau at $M/M_{\text{sat}} = 1/2$ in the region $0.5 \lesssim J_2/J_1 \lesssim 0.65$ has been recently reported by Honecker et al. However, Flederjohann and Mitter do not find a plateau at $M/M_{\text{sat}} = 1/2$ in the region of the quantum disorder phase; but instead they find some indications for a plateau-like structure at $M/M_{\text{sat}} = 2/3$. Most of the previous studies on the plateaus are obtained by numerical calculations on a small clusters (the typical number of lattice sites in these works is about $6 \times 6 = 36$). As discussed in Ref. 22, the plateau structures in the magnetization curve can depend sensitively on the system size. Therefore, these results may be plagued with the finite-size effects. For example, some of the predicted plateaus may be an artifact of the special lattice geometry, and the boundary conditions used may frustrate the order which tends to develop. Thus the precise determination of the positions and widths of the plateaus in the frustrated Heisenberg model is indeed a very delicate problem, and a better theoretical understanding of the magnetic order and of the mechanisms which create the
plateaus is needed.

To avoid the possible finite-size effects, we apply the Chern-Simons (CS) theory for the magnetization plateaus to the 2D spin-1/2 $J_1$-$J_2$ Heisenberg model. Because of its success for the Shastry-Sutherland model, this approach should give us reasonable results in the present case. We find that the magnetization curve is monotonically increasing for $J_2/J_1 < 0.267949$, where the system at zero field is in the antiferromagnetic Néel phase. Besides, various plateaus will appear in the magnetization curve for larger ratios of $J_2/J_1$. In particular, in the disordered phase, our result supports the existence of the $M/M_{\text{sat}} = 1/2$ plateau and predicts a new plateau at $M/M_{\text{sat}} = 1/3$. Furthermore, we note that, by identifying the onset value of $J_2/J_1$ for the appearance of the 1/2-plateau with the critical value of the phase transition between the Néel and the spin-disordered phases in zero field, this phase boundary, which was determined earlier by heavy numerical means, can be reproduced accurately by this mean-field calculation.

The Hamiltonian in Eq. (1) can be rewritten as

$$H = H_{xy} + H_z - B \sum_i S_i^z,$$

$$H_{xy} = \frac{1}{2} \sum_{i,j} J_{ij} \left( S_i^+ S_j^- + S_i^- S_j^+ \right),$$

$$H_z = \sum_{i,j} J_{ij} S_i^z S_j^z,$$

where $H_{xy}$ and $H_z$ are the spin-flip and the Ising parts of the Hamiltonian at zero field. According to Ref. 8, $H_{xy}$ and $H_z$ are treated in different ways. First, the Ising part is approximated by a simple uniform mean-field decoupling,

$$H_z \simeq 4(J_1 + J_2)M \sum_i S_i^z - 2(J_1 + J_2)M^2 N,$$

where $N$ is the number of lattice sites. Thus $H_z$ gives a contribution to the total energy with a simple dependence on the magnetization. Second, for $H_{xy}$, one can exactly map the spin operators to spinless fermions attached with a flux tube carrying one flux quantum of statistical CS magnetic field, where $S_i^z + 1/2$ corresponds to the occupation number $n_i$ of site $i$. Under a mean-field treatment such that the flux tubes are smeared out into a uniform background magnetic field, the flux per square plaquette $\phi$ is then tied to the density of fermions and thus to the magnetization $M$ of the spin system:

$$\phi = 2\pi \langle n \rangle = M + \frac{1}{2}.$$  \hspace{1cm} (4)

Because of this flux, each energy band splits to subbands with a complicated structure. Therefore, the present spin system can be identified with a Hofstadter problem for fermions moving on a square lattice with nearest-neighbor and next-nearest-neighbor hoppings. For a given $M$ [or $\phi = 2\pi(M + 1/2)$], the mean-field ground state is obtained by filling the lowest energy subbands with fermions until their density satisfies $\langle n \rangle = \phi/2\pi$. The one-body problem from $H_{xy}$ can be straightforwardly analyzed for rational values of $\phi/2\pi$. For $\phi/2\pi = p/q$ ($p$ and $q$ are mutually prime integers), there are $q$ subbands and the ground state is the Slater determinant with the lowest $p$ subbands being completely filled. The energy of the filled subbands leads to another contribution to the total energy. Hence the total energy per site $E(M)$ as a function of the magnetization becomes

$$E(M) = \frac{1}{N} \sum_{\alpha=1,\ldots,p} \sum_{\vec{k}} \epsilon_{\vec{k}}^{(\alpha)} + 2(J_1 + J_2)M^2. \hspace{1cm} (5)$$

Here $\epsilon_{\vec{k}}^{(\alpha)}$ is the eigenenergy of the $\alpha$-th subband with the wavevector $\vec{k}$ being restricted to the magnetic Brillouin zone. The magnetization can be obtained as a function of $B$ by minimizing $E(M) - BM$. It is clear from Eq. (5) that, without the contribution from $H_{xy}$, the magnetization $M$ will be linearly proportional to $B$, and there is no magnetization plateau. Therefore, the appearance of magnetization plateaus is related to certain features of the Hofstadter spectrum.

The Hofstadter diagrams for $J_2/J_1=0.2$ and 0.3 are shown in Fig. 4, where the lower bold line marks the Fermi level (highest occupied state) and the upper one marks the lowest unoccupied level. A jump of the Fermi energy as a function of $M$ in Fig. 1(b) leads to discontinuity of the slope of the function $E(M)$. These jumps for various $M$ give rise to plateaus in the magnetization curve. They are closely related to the occurence of band-crossing when the value of $J_2/J_1$ is varied. For example, in Fig. 1(a) before the upper two subbands at $\phi/2\pi = 2/3$ touch at $J_2/J_1 = 0.267949$, the Fermi energy is continuous and there is no magnetization plateau at $M/M_{\text{sat}} = 1/3$ (see Fig. 4). However, in Fig. 1(b), the “pockets” enclosed by the bold lines are separated at $\phi/2\pi = 2/3$. It can be seen that the Fermi level to the right of the contact point is below (above) the band gap before (after) band-crossing. As discussed earlier, the Fermi level marks the position of the $p$-th subband [see Eq. (4)]. Therefore, apparently a subband associated with some flux slightly larger than $2/3$ is shifted above the energy gap after band-crossing. This shift of a fine subband due to the crossing of the broader subbands at $\phi/2\pi = 2/3$ was studied earlier in the context of the Hofstadter spectrum. It is closely related to the jump of the integer-valued Hall conductances of the broader subbands. The emergence of the magnetization plateaus for the spin system thus has an interesting connection with the change of the integer-valued Hall conductances induced by band-crossing.

To justify the present approach, it is important to check whether the plateau states are robust against fluctuations around the mean-field solutions. It was showed that the Gaussian fluctuations of the CS gauge field are massless only when the TKNN integer describing the
Thus results near the saturation. Vertical lines mark the energy bands as a function of the statistical flux $\phi/2\pi$ per square plaquette. Total Hall conductances $\sigma_T$ above the Fermi level from $M/M_{\text{sat}} = 0$ ($\phi/2\pi = 1/2$) to $M/M_{\text{sat}} = 1$ ($\phi/2\pi = 1$) are indicated. The Hall conductance at $\phi/2\pi = 2/3$ changes from 1 to $-2$ when the upper two subbands touch at $J_2/J_1 = 0.267949$.

![FIG. 1: Hofstadter spectra for a square lattice with nearest-neighbor and next-nearest-neighbor hoppings for $J_2/J_1 = 0.2$ (a) and 0.3 (b). Vertical lines mark the energy bands as a function of the statistical flux $\phi/2\pi$ per square plaquette. Total Hall conductances $\sigma_T$ above the Fermi level from $M/M_{\text{sat}} = 0$ ($\phi/2\pi = 1/2$) to $M/M_{\text{sat}} = 1$ ($\phi/2\pi = 1$) are indicated. The Hall conductance at $\phi/2\pi = 2/3$ changes from 1 to $-2$ when the upper two subbands touch at $J_2/J_1 = 0.267949$.](image)

FIG. 1: Hofstadter spectra for a square lattice with nearest-neighbor and next-nearest-neighbor hoppings for $J_2/J_1 = 0.2$ (a) and 0.3 (b). Vertical lines mark the energy bands as a function of the statistical flux $\phi/2\pi$ per square plaquette. Total Hall conductances $\sigma_T$ above the Fermi level from $M/M_{\text{sat}} = 0$ ($\phi/2\pi = 1/2$) to $M/M_{\text{sat}} = 1$ ($\phi/2\pi = 1$) are indicated. The Hall conductance at $\phi/2\pi = 2/3$ changes from 1 to $-2$ when the upper two subbands touch at $J_2/J_1 = 0.267949$.

quantized Hall coefficient of the fermions on the frustrated lattice becomes unity. In that case, the Gaussian fluctuations induce instabilities for the mean-field ground states. We have computed the TKNN integers numerically (for example, see Fig. 1) and found the Gaussian fluctuations to be massive. Therefore, the plateaus are not destroyed by quantum fluctuations.

Magnetization curves for various $J_2/J_1$ ratios are shown in Fig. 2. We note that the saturation field $B_{\text{sat}}$ can be computed exactly by identifying the energy $E_f$ of the fully polarized state with the (exact) minimum energy $E_{1s}^{\text{min}}$ of the states with one spin flipped, $E_f = E_{1s}^{\text{min}}$. Thus $B_{\text{sat}}/J_1 = 4$ for $J_2/J_1 < 1/2$; $B_{\text{sat}}/J_1 = 2 + 4J_2/J_1$ for $J_2/J_1 > 1/2$. Our findings agree with these exact results near the saturation.

In the Néel phase, it is expected that the spins cant gradually from the antiparallel configuration toward the parallel configuration until the magnetization saturates at the saturation field $B_{\text{sat}}$. The magnetization curve obtained from the present approach is consistent with this expectation: it is featureless all the way to full saturation when $J_2/J_1$ is small (see the curves for $J_2/J_1 = 0$ and 0.2 in Fig. 2). Upon increasing $J_2/J_1$, plateaus emerge and the magnetization curves become more complex. In particular, because of the band-crossing at $\phi/2\pi = 2/3$ when $J_2/J_1 = 0.267949$ (see Table I of Ref. 28), a plateau at $M/M_{\text{sat}} = 1/3$ is found [see Eq. (4)]. This is an unexpected result, especially for $0.267949 < J_2/J_1 < 0.38$ where the ground state in the absence of an external field is in the Néel phase. In the previous finite-size studies, there is no indication for the appearance of this plateau. However, the system sizes and the boundary conditions they used forbid the appearance of the $M/M_{\text{sat}} = 1/3$ plateau, therefore the possibility of this plateau cannot be ruled out. Furthermore, when $J_2/J_1 = 0.382683$, another band-crossing in the Hofstadter spectrum occurs at $\phi/2\pi = 3/4$ (see Table I of Ref. 28), and a plateau at $M/M_{\text{sat}} = 1/2$ ensued. It is interesting to find that the critical value for the appearance of the $M/M_{\text{sat}} = 1/2$ plateau agrees quite well with the critical point of the quantum phase transition at zero field between the Néel and the quantum disorder phases. This reinforces our confidence on the present CS mean-field approach. As mentioned before, while the appearance of a plateau in the quantum disorder phase had been predicted, the value of the plateau is still under debate. The controversy may come from the subtle finite-size effects in their investigations. Since the present CS theory is free from the finite-size effects, we give a strong support for the existence of the $M/M_{\text{sat}} = 1/2$ plateau.

![FIG. 2: Magnetization curves for the $J_1$-$J_2$ Heisenberg model calculated using uniform CS mean-field. The curves from left to right are for $J_2/J_1 = 0, 0.2, 0.3, 0.4, 0.5$, and 0.7, respectively.](image)
More complex structures in the magnetization curves appear when $J_2/J_1$ is further increased. For example, when $J_2/J_1 = 1/2$, a series of plateaus at $M/M_{sat} = n/(n+2)$ is found, which corresponds to the band-crossing at some magic numbers $\phi/2\pi = (n+1)/(n+2)$. Irregular plateau structures are found for even higher values of $J_2/J_1$ (see the $J_2/J_1 = 0.7$ curve in Fig. 3). This behavior is quite similar to the case of the triangular lattice, where many plateaus are predicted under the uniform CS mean-field approximation. In the case of the triangular lattice, it is shown that, when the nonuniform mean-field solutions are used, only the main plateau ($M/M_{sat} = 1/3$ in that case) survives and other mini-plateaus disappear. We believe that the same situation will happen in the present study of the $J_1$-$J_2$ model. That is, when going beyond the uniform mean-field approximation, many of the mini-plateaus in the magnetization process may disappear and only the main plateaus with simple fractions of $M/M_{sat}$ survive. Moreover, as shown in Fig. 3, the spin gap (the plateau at $M/M_{sat} = 0$) opens at $J_2/J_1 = 0$, instead of opening at the correct critical coupling $J_2/J_1 \sim 0.38$. This feature comes from the fact that the Neél state of the square-lattice antiferromagnet is not correctly described by the present uniform mean-field approximation.

In conclusion, the magnetization curve of the 2D spin-1/2 $J_1$-$J_2$ Heisenberg model is studied by using the Chern-Simons theory under a uniform mean-field approximation. In the disordered phase, our result supports the existence of the $M/M_{sat} = 1/2$ plateau and predicts a new plateau at $M/M_{sat} = 1/3$. Moreover, various plateaus appear in the magnetization curve both in the disordered and the collinear phases, which could be an artifact of our mean field approximation. More work is needed to be conclusive at this point. We note that it is experimentally accessible to confirm our results. As mentioned before, the 2D spin-1/2 $J_1$-$J_2$ Heisenberg model with $J_2/J_1 \approx 1$ has been recently realized experimentally in Li$_2$VOSiO$_4$ ($J_1+J_2 \approx 8.2$ K) and Li$_2$VOGeO$_4$ ($J_1+J_2 \approx 8.2$ K) compounds. If the $g$-factor is taken to be 2, the corresponding saturation fields will be approximately 18 T for Li$_2$VOSiO$_4$ and 13 T for Li$_2$VOGeO$_4$. In both cases, these values of the magnetic fields can be reached experimentally. Thus the full magnetization curve could be mapped out to test our results. Verification of the magnetization plateaus would indicate a strong connection between the frustrated antiferromagnetic system and the quantum Hall system.

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