WHAT IS THE COST OF ADDING A CONSTRAINT IN LINEAR LEAST SQUARES?

Ramakrishna Kakarala and Jun Wei
Omnivision Technologies, Inc.

ABSTRACT
Although the theory of constrained least squares (CLS) estimation is well known, it is usually applied with the view that the constraints to be imposed are unavoidable. However, there are cases in which constraints are optional. For example, in camera color calibration, one of several possible color processing systems is obtained if a constraint on the row sums of a desired color correction matrix is imposed; in this example, it is not clear a priori whether imposing the constraint leads to better system performance. In this paper, we derive an exact expression connecting the constraint to the increase in fitting error obtained from imposing it. As another contribution, we show how to determine projection matrices that separate the measured data into two components: the first component drives up the fitting error due to imposing a constraint, and the second component is unaffected by the constraint. We demonstrate the use of these results in the color calibration problem.

Index Terms— Least squares, constraints, color correction, Cramer-Rao bound

1. INTRODUCTION
This paper addresses the role of linear constraints in determining fitting error for linear least squares (LS) problems. Specifically, we aim to theoretically determine the increase in fitting error due to imposing a linear constraint, and to determine the geometric relationship between constrained and unconstrained estimators. Our motivation arises from a practical problem in color calibration, as we now describe.

The color correction stage of an image signal processing system is used to convert an image sensor’s color response into device-independent color values [1]. This stage is typically performed after Bayer color interpolation by applying a 3 × 3 matrix, known as the color correction matrix (CCM), to input red, green, and blue (RGB) values. The corrected values, written with subscript c, are then related to the input values as follows:

\[
\begin{pmatrix}
R_c \\
G_c \\
B_c
\end{pmatrix} = \begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33}
\end{bmatrix} \begin{pmatrix}
R \\
G \\
B
\end{pmatrix}
\]

(1)

The problem of calibrating a CCM involves fitting a set of measured RGB vectors \(m_i, i = 1, \ldots, k\), to a corresponding set of reference values \(r_i\). For example, the \(k = 24\) color checker chart shown in Figure 1 is used to measure RGB values from the sensor. These measurements are then compared to published reference values. Letting \(M\) denote the \(3 \times k\) matrix whose columns are measured RGB values, and \(R\) the corresponding reference matrix, calibration requires minimizing over \(3 \times 3\) matrices \(C\) the fitting error \(\|CM - R\|^2\), where the norm is Frobenius. This is a standard LS problem.

However, in some systems, a CCM is applied after white balance. In this case, to preserve input neutral gray tones for which \(R = G = B\), CCM matrices must be normalized so that their rows sum to 1.0 [3]. Letting \(1 = [1, 1, 1]^T\), we write a constrained linear least squares (CLS) problem as follows:

\[
\begin{align*}
\text{minimize} & \quad \|CM - R\|^2 \\
\text{subject to} & \quad C1 = 1
\end{align*}
\]

(2)

This row-sum constraint is not always used, as alternative color processing pipelines are possible without it [1]. It is clear that imposing any constraint on \(C\) should increase fitting error, for constraints reduce the freedom in choosing \(C\). It is useful to have a mathematical model of fitting error due to linear constraints, which we provide in this paper. Note that our goal is to analyze the CLS problem and contribute to its theory; the problem of CCM calibration
is used only for illustrative purposes. In particular, there exist CCM calibration algorithms besides least squares [3], which we do not consider here.

The general theory of solving CLS problems is well established [5] Ch. 16],[6] Ch. 1]. We review this theory, and point out simplifications that explicitly separate the role of the unconstrained estimator from the constraint fitting error. We determine the subspace on which both the constrained and the unconstrained estimators agree, and provide an exact result for the error increase due to applying the constraint. We show how the simplified results apply to the CCM estimation problem, comparing our approach to others [7],[8].

2. LS AND CLS ESTIMATION

This section formulates LS and CLS problems as generally as possible, for real-valued vectors and matrices. Let $M$ be a full-rank measurement matrix of arbitrary size (not necessarily $3 \times k$ as in [1]). $R$ a corresponding reference, and $W$ a positive definite weighting matrix for errors. Define the objective function as

$$
\|CM - R\|_W^2 = \text{trace}\left\{(CM - R)(CM - R)^T\right\}. \quad (3)
$$

The unique minimizing solution is:

$$
C_u = RWM^T(MWM^T)^{-1}. \quad (4)
$$

Henceforth, we use the subscript $u$ to indicate the solution to an unconstrained least-squares problem.

We now introduce linear constraints, formulating a general case that allows a variety of uses. Let $K$ denote a full row-rank matrix whose purpose is to impose a constraint on $C$ from the left, and similarly let $F$ be a full column-rank constraint matrix applied on the right, and finally let $G$ a matrix of requirements. We write the general set of linear constraints as $KCF = G$. Note that neither $K$ nor $F$ need be square; for example, in the case where $C$ is $3 \times 3$, we might have $K = [1,1,1] = 1^T$, and $F = 1$. The general CLS problem is now stated as follows:

minimize $\|CM - R\|_W^2$, subject to $KCF = G$. \quad (5)

Henceforth, we denote the solution to this problem by $\hat{C}$.

We would expect that, if it so happened that the unconstrained estimate $C_u$ already met the constraint, i.e., $KCF = G$, then $\hat{C} = C_u$ (the constrained estimate is the same as the unconstrained estimate). Let $\Delta_c = KCF - G$ denote the error in meeting the constraint. Intuitively, we would expect that, the smaller $\Delta_c$ is, the closer the constrained estimate $\hat{C}$ is to $C_u$.

Let us now derive a simple expression for $\hat{C}$. The standard theory [5] Ch. 16],[6] Ch. 1] shows how to proceed when $C$ is a vector of parameters, rather than a matrix as we have formulated in [5]. We could, if we wish, reuse this theory by “vectorizing” (3) using the vec operator and the Kronecker product $\otimes$. Vectorization means the constraint in (5) is

$$
\text{vec}(F^T \otimes K)\text{vec}(C) = \text{vec}(G). \quad (6)
$$

However, vectorization loses the intuition that comes from directly formulating the problem [5] using $C$ in matrix format, in the way $C$ is meant to be applied, as in eqn. (1). To preserve intuition, and to introduce useful simplifications, we follow [5],[6] but work directly in matrix formulations.

Letting $\Lambda$ be a matrix of Lagrange multipliers of the same size as $G$, we write the augmented function corresponding to (5) as follows:

$$
\|CM - R\|_W^2 + \text{trace}\left\{\Lambda^T(KCF - G)\right\}. \quad (7)
$$

Differentiating with respect to $C$ yields (using the rule $\partial\text{trace}(AXB)/\partial X = A^T B^T$)

$$
2\left(\hat{C}M - R\right)WM^T + K^T\Lambda F^T = 0. \quad (8)
$$

Solving for $C$ yields

$$
\hat{C} = \left(RWM^T - \frac{1}{2}K^T\Lambda F^T\right)(MWM^T)^{-1}. \quad (9)
$$

Let $J = MWM^T$; this matrix corresponds to the Fisher information matrix used in parameter estimation theory for the least squares problem [9]. It is invertible due to our assumption that $M$ is full-rank and $W$ is positive definite. Setting $K\hat{C}F = G$ and solving yields

$$
\Lambda = 2\left(KK^T\right)^{-1}\Delta_c\left[F^T J^{-1}F\right]^{-1}. \quad (10)
$$

To write the solution, we construct for simplicity a left inverse of $F$, denoted $F_\ell$, by the matrix

$$
F_\ell = (F^T J^{-1}F)^{-1} F^T J^{-1}. \quad (11)
$$

Note that $F_\ell F = I$. Similarly, let $K_{R\ell}$ denote a right inverse of $K$; since $K$ is assumed to have full row rank, we use $K_{R\ell} = K^T\left(KK^T\right)^{-1}$. Then we have our desired solution to the CLS problem:

$$
\hat{C} = C_u - K_{R\ell}\Delta_c F_\ell. \quad (12)
$$

We see the optimal constrained estimator $\hat{C}$ corrects the unconstrained estimator $C_u$ by “modulating” the constraint error $\Delta_c = KCF - G$ with the inverses of the constraint matrices. It follows directly that this estimator meets the constraint:

$$
K\hat{C}F = KCF - KK_{R\ell}\Delta_c F_\ell F = G. \quad (13)
$$

2.1. A comparison

We compare this derivation with an alternative approach to “white-point preserving” CCM estimation that is proposed by
Finlayson and Drew \cite{Finlayson:1997}. They consider the case where \( W = I, \) \( K = I, \) and \( F, G \) are vectors denoted \( f, g, \) respectively. They turn the CLS problem \cite{Gorman:1980} into an unconstrained problem by writing \( C = D + E, \) where \( D \) is any matrix such \( Df = g \) (formed, for example by placing the ratios of \( g \) to \( f \) on the diagonal), and \( E \) is a matrix such that \( Ef = 0 \). Letting \( V \) denote a (rectangular) matrix whose columns form a basis for the hyperplane perpendicular to \( f, \) we set \( E = NV^T \) for a suitable rectangular matrix \( N. \) (Note that if \( f \) is \( n \times 1, \) then both \( V \) and \( E \) are \( n \times (n - 1) \) matrices.) Now, we minimize over the unconstrained entries of \( N \) the fitting error:

\[
\| (D + NV^T)M - R \|^2.
\] (14)

The solution is shown \cite{Finlayson:1997} to be:

\[
\hat{C} = D + (R - DM)M^TV (V^TM M^TV)^{-1} V^T.
\] (15)

As we discuss below, \cite{Gorman:1980} produces exactly the same estimate as our solution \cite{Gorman:1980}.

3. MAIN RESULTS

We enumerate our main novel results in this section.

1. The estimator \cite{Gorman:1980} is the unique minimum of the CLS problem \cite{Gorman:1980}. The Appendix contains a constructive proof; alternatively, we may prove using convex optimization theory \cite{Boyd:2004} Ch. 10.

2. We show in the Appendix that the excess fitting error due to the constraint is:

\[
\| \hat{C}M - R \|_W^2 = \| C_u M - R \|_W^2 + \| K_R \Delta_c F_c M \|_W^2.
\] (16)

Note that the excess (second term on right) is independent of reference \( R. \)

3. What component of the measurement \( M \) contributes to the excess fitting error \( \| K_R \Delta_c F_c M \|_W^2 \) in \cite{Gorman:1980}? To answer this, let us introduce two orthogonal projections \( P_{\perp} = FF^T \) and \( P_{\parallel} = I - P_{\perp}. \) Then \( P_{\perp} P_{\perp} = P_{\perp} \) and \( P_{\parallel} P_{\parallel} = P_{\parallel}; \) also, \( P_{\parallel} P_{\perp} = 0. \) Therefore,

\[
M = P_{\parallel} M + P_{\perp} M. \quad (17)
\]

Since \( F_c P_{\perp} = F_{\ell}, \) and \( F_c P_{\parallel} = 0, \) we write the excess as

\[
K_R \Delta_c F_c M = K_R \Delta_c F \ell P_{\perp} M. \quad (18)
\]

Hence, only \( P_{\perp} M \) contributes to increasing fitting error. Note that both \( P_{\parallel} \) and \( P_{\perp} \) depend only on the constraint matrix \( F, \) and are independent of \( K. \)

4. From \( F_c P_{\parallel} = 0 \) and \cite{Gorman:1980}, we see that, on \( P_{\parallel} M, \) the constrained estimator \( \hat{C} \) and the unconstrained estimator \( C_u \) must agree

\[
\hat{C}P_{\parallel} M = C_u P_{\parallel} M. \quad (19)
\]

5. Let us write the fitting problem as a constrained parameter vector estimation problem, using boldface to indicate vectorized matrix of the same letter, i.e., \( x = \text{vec}(X). \) With \( \eta \) representing Gaussian noise, \( B = M^T \otimes I, \) and \( A = F^T \otimes K, \) we write

\[
r = Bc + \eta,
\]

subject to \( Ac = g. \) (20)

Gorman and Hero \cite{Gorman:1980} show that imposing constraints on \( c \) reduces the Cramer-Rao lower bound (CRLB) on estimator variance over the unconstrained case. Specifically, with \( J_u = B^T \Sigma^{-1} B \) representing the Fisher matrix of the unconstrained estimator, the bound when constraints are imposed is:

\[
\text{Var}\{\hat{c}\} \geq J_u^{-1} := J_u^{-1} - J_u^{-1} A^T [AJ_u^{-1}A^T]^{-1} AJ_u^{-1}.
\] (21)

Interestingly, even as constraints increase fitting error, they reduce CRLB. Furthermore, as Gross\cite{Gross:1990} pg 94 shows, constraints also reduce mean squared error. We may say that imposing constraints increases the precision of specifying the optimal estimator, but reduces the accuracy of that estimator.

6. The solution \cite{Gorman:1980} produces exactly the same matrix as \cite{Gorman:1980}, regardless of the choice of \( D \) and \( V. \) This follows from the uniqueness of \cite{Gorman:1980} as the optimum solution of \cite{Gorman:1980}. We may also prove the equality constructively by letting \( D = f_{\ell} \) in \cite{Gorman:1980}, and noting that we may write \( (I - f_{\ell}I) = NV^T \) for a suitable rectangular matrix \( N. \)

4. EXAMPLE IN COLOR CALIBRATION

For illustrative purposes, we apply the results of the previous section to color calibration. We consider two optional constraints on CCM matrices. The first is the row-sum constraint described in §1 that \( C1 = 1. \) This is obtained by setting \( K = 1.0, \) \( F = 1.0. \) The second requires that the sum of all entries of \( C \) is 3.0. We write this as \( C^T C1 = 3.0, \) which means \( K = 1^T, \) \( F = 1, \) and \( G = 3.0. \) Note that the row-sum constraint implies the total-sum constraint.

Let us compare the cost of the two constraints. Since row-sum is stricter, we would expect its fitting error to be higher than for total-sum; correspondingly, we would also expect the CRLB to be lower for row-sum than for total-sum.

For experimental purposes, we use an image of a measured chart\footnote{https://www.imatest.com/wp-content/uploads/2011/11/Colorcheck_1_raw_1004W.jpg}. The image is white balanced to the white patch of the chart. The coordinates of each of the 24 patches in the chart are extracted using an automated tool\footnote{https://github.com/colour-science/colour-checker-detection}. The mean patch RGB values after white balance are used to construct a \( 3 \times 24 \) matrix \( M. \) Correspondingly, we set \( R \) from sRGB values
Table 1. Comparison of fitting error and CRLB matrix norms, with row-sum constraint stricter than total-sum. Here $W = I$ and percent changes are calculated with respect to the unconstrained case on the top row.

| Constraint | Fitting Err $\|CM - R\|_W$ | CRLB $\|J^{-1}\|$ |
|------------|------------------------|---------------------|
| None       | 0.173                  | 20.025              |
| total-sum  | 0.191 (+10.4 %)        | 20.009 (-0.08%)     |
| row-sum    | 0.202 (+16.8%)         | 19.972 (-0.26%)     |

In Table 4 we compare fitting errors and CRLB matrices for both constraints. Results confirm the expectations described above, that fitting error increases with stricter constraints (in this example, by up to 17%, a significant amount), while CRLB decreases.

For the row-sum constraint, Figure 2 uses a chart format to illustrate the raw input measurements, the measurements after white balance, and the projected measurements on which the constrained and unconstrained estimators disagree $(P_\perp M)$ and agree $(P_M M)$. The values of $P_M M$ are gamut limited by clipping at 0. Comparing Fig 2(c) with (a), we see that hue is largely preserved, especially for red and green colors. Note also that $P_\perp M$ consists entirely of shades of gray, on which the constraint is active. The corresponding plots for the total sum error are exactly the same, since $P_M$ and $P_\perp$ depend only $F$, which is 1 for both constraints.

Fig. 2. The input measurements in (a) are white balanced in (b). For row-sum constraint, (c) and (d) show the gamut-limited charts on which the constrained and unconstrained estimators agree and disagree, respectively.

5. SUMMARY AND CONCLUSIONS

In this paper, we derive exact expressions for the increase in fitting error due to imposing a constraint in linear least squares problems. We provide descriptions of the subspaces on which the constrained and unconstrained estimators agree. We illustrate these results with an example from color calibration.

6. APPENDIX

6.1. Solution is unique

We use the constructive approach in [5] to prove that the solution [12] is the unique minimizer. Let $C$ be such that $C \neq \hat{C}$ and $KCF = G$. Then, we write

$$\|CM - R\|_W^2 = \|CM - \hat{C}M + \hat{C}M - R\|_W^2,$$

$$= \|CM - \hat{C}M\|_W^2 + \|\hat{C}M - R\|_W^2 + 2 \text{trace} \left\{ (CM - \hat{C}M)W (\hat{C}M - R)^T \right\}.$$

Writing the rightmost term (inside trace) as

$$(CM - \hat{C}M)W (\hat{C}M - R)^T = (C - \hat{C})M W (M^T \hat{C}^T - R^T).$$

From (8), we find that

$$MW \left( M^T \hat{C}^T - R^T \right) = -\frac{1}{2} FA^T K.$$

Substituting this back into the trace (22) leads to

$$\text{trace} \left\{ (C - \hat{C})FA^T K \right\} = \text{trace} \left\{ K \left( C - \hat{C} \right) FA^T \right\}.$$

But $K (C - \hat{C}) F = 0$, and consequently, (23) vanishes. This leaves, since $M$ is assumed to have full rank, that $CM - \hat{C}M \neq 0$, and therefore:

$$\|CM - R\|_W^2 = \|CM - \hat{C}M\|_W^2 + \|\hat{C}M - R\|_W^2 > \|\hat{C}M - R\|_W^2.$$

6.2. Excess fitting error due to constraint

Using [12], we calculate the fitting error with the constrained estimator as follows:

$$\|\hat{C}M - R\|_W^2 = \|CuM - R - K \Delta_c F_t M\|_W^2,$$

$$= \|CuM - R\|_W^2 + \|K \Delta_c F_t M\|_W^2,$$

$$+ 2 \text{trace} \left\{ (CuM - R) \left( K \Delta_c F_t M \right)^T \right\}.$$

The term in the trace may be expanded as follows. Let $Q = K \Delta_c F_t$, and use [4] to obtain [16]:

$$\left( CuM - R \right) \left( QM \right)^T = CuMW M^T Q^T - RW M^T Q^T = RW M^T (MW M^T)^{-1} (MW M^T) Q^T - RW M^T Q^T = 0.$$ (28)
7. REFERENCES

[1] Gaurav Sharma, *Color imaging handbook*, CRC Press, Boca Raton, 2003.

[2] “Color Checker chart,” [https://en.wikipedia.org/wiki/ColorChecker](https://en.wikipedia.org/wiki/ColorChecker) [Online; accessed 29-June-2021].

[3] Simone Bianco, Arcangelo Bruna, Filippo Naccari, and Raimondo Schettini, “Color correction pipeline optimization for digital cameras,” *Journal of Electronic Imaging*, vol. 22, pp. 3014–, 04 2013.

[4] Fufu Fang, Han Gong, Michal Mackiewicz, and Graham Finlayson, “Colour correction toolbox,” in *Proceedings of 13th AIC Congress 2017*. Korea Society of Color Studies, Jeju, Korea, October 2017.

[5] Stephen Boyd and Lieven Vandenberghe, *Introduction to Applied Linear Algebra*, Cambridge University Press, Cambridge, UK, 2018.

[6] Takeshi Amemiya, *Advanced Econometrics*, Harvard University Press, Cambridge, MA, 1985.

[7] Graham Finlayson and Mark S. Drew, “Constrained least-square regression in color spaces,” *Journal of Electronic Imaging*, vol. 6, no. 4, pp. 484–493, 1997.

[8] Stephen Wolf, “Color correction matrix for digital still and video imaging systems,” Tech. Rep. TM-04-406, National Telecommunications and Information Administration, 2003.

[9] John D. Gorman and Alfred O. Hero, “Lower bounds for parametric estimation with constraints,” *IEEE Transactions on Information Theory*, vol. 36, no. 6, pp. 1285–1301, 1990.

[10] Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[11] Jurgen Gross, *Linear Regression*, Springer-Verlag, Berlin, 2003.

[12] Danny Pascale, “RGB coordinates of the macbeth colorchecker,” Tech. Rep., The BabelColor Company, 2006.