REGULAR SUBGRAPHS OF UNIFORM HYPERGRAPHS

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Abstract. We prove that for every integer \( r \geq 2 \), an \( n \)-vertex \( k \)-uniform hypergraph \( H \) containing no \( r \)-regular subgraphs has at most \( (1+o(1))(n-1)^k \) edges if \( k \geq r+1 \) and \( n \) is sufficiently large. Moreover, if \( r \in \{3,4\} \), \( r \mid k \) and \( k, n \) are both sufficiently large, then the maximum number of edges in an \( n \)-vertex \( k \)-uniform hypergraph containing no \( r \)-regular subgraphs is exactly \( (n-1)^k \), with equality only if all edges contain a specific vertex \( v \). We also ask some related questions.

1. Introduction

What are the graphs containing no \( r \)-regular subgraphs? For \( r = 2 \), the answer is easy, they are forests. However, the question becomes much harder when \( r \) is larger than two. Complete characterizations of graphs with no \( r \)-regular subgraphs seem impossible even for the case \( r = 3 \). So it is natural to ask how many edges can a graph with no \( r \)-regular subgraphs have. Pyber [12] showed that there exists a constant \( c_r \) such that all \( n \)-vertex graphs with at least \( c_r n \log(n) \) edges have an \( r \)-regular subgraph. On the other hand, Pyber, Rödl and Szemerédi [13] proved that there exists a graph with \( \Omega(n \log \log n) \) edges having no \( r \)-regular subgraphs for any \( r \geq 3 \). The gap between the two bounds still remains open.

It is also natural to consider the same question for hypergraphs, both uniform and non-uniform hypergraphs. Mubayi and Verstraëte [10] proved that for every even integer \( k \geq 4 \), there exists \( n_k \) such that for \( n \geq n_k \), each \( n \)-vertex \( k \)-uniform hypergraph \( H \) with no 2-regular subgraphs has at most \( \binom{n-1}{k-1} \) edges, and equality holds if and only if \( H \) is a full \( k \)-star, that is, a \( k \)-uniform hypergraph consists of all possible edges of size \( k \) containing a given vertex. For non-uniform hypergraphs, it is easy to see that an \( n \)-vertex hypergraph \( H \) with no \( r \)-regular subgraphs has at most \( 2^{n-1} + r - 2 \) edges. One example for the equality is a full star, that is a hypergraph consisting of all possible edges containing a given vertex, with additional \( r - 2 \) smallest edges not containing the given vertex. The author and Kostochka [8] proved that if \( n \geq 425 \) and \( n > r \), hypergraph with no \( r \)-regular subgraphs contains \( 2^{n-1} + r - 2 \) edges only if \( H \) is a full star with \( r - 2 \) additional edges. Also similar question can be considered for linear hypergraphs. Dellamonica et al. [3] showed that the maximum number of edges in a linear 3-uniform hypergraph with no two-regular subgraphs is at least \( cn \log n \) and at most \( Cn^{3/2}(\log(n))^3 \) for some constant \( c, C \). They asked whether every linear 3-uniform hypergraph with no 3-regular subgraphs has at most \( o(n^2) \) edges. In Section 6 we confirm that this is true.

In this paper, we consider \( k \)-uniform hypergraphs with no \( r \)-regular subgraphs. The two main results of this paper are the following two theorems.
Theorem 1.1. Let $k, r$ be two integers with $r \geq 2, k \geq r + 1$. Then there exists $n_k$ such that for $n > n_k$ any $n$-vertex $k$-uniform hypergraph $H$ with no $r$-regular subgraph has at most $(1 + o(1))\left(\frac{n}{k-1}\right)^{\frac{r-1}{k-1}}$ edges. Moreover, if $k \geq 2r + 1$ and $|H| \geq (1 - n^{-\frac{2}{3r}})\left(\frac{n}{k-1}\right)^{\frac{r-1}{k-1}}$, then there exists a vertex $v$ which belongs to at least $(1 - n^{-\frac{1}{3r}})\left(\frac{n}{k-1}\right)^{\frac{r-1}{k-1}}$ edges.

Theorem 1.2. Let $k, r$ be two integers with $r \in \{3, 4\}, k \geq 2r + 2$ and $r$ divides $k$. Then there exists $n_k$ such that for $n > n_k$ any $n$-vertex $k$-uniform hypergraph $H$ with no $r$-regular subgraphs has at most $\left(\frac{n}{k-1}\right)^{\frac{r-1}{k-1}}$ edges. Moreover, the equality holds if and only if $H$ is a full-$k$-star.

Our proofs of theorems develop ideas in [10]. In Section 3 and 4 we prove Theorem 3.1 and Theorem 4.1 which together implies Theorem 1.1. In Section 5, we prove Theorem 1.2. In Section 6, we show some examples which somewhat explain necessity of each conditions in each theorem and we also pose some further questions.

2. Preliminaries

We say $H$ has an $r$-regular subgraph if there exists a collection of edges in $E(H)$ which all together cover each vertex in a nonempty set exactly $r$-times and no other vertices. We write $V(H)$ and $E(H)$ for the set of vertices and the set of edges in a hypergraph $H$, respectively. We denote the size of $H$ by $|H| := |E(H)|$. Also log denotes log$_2$ and $s$-set denotes a set of size $s$.

First, we introduce the following simple observation which we use several times in the paper.

Observation 2.1. For $t > 1$ and $n \geq 2k$, if an $n$-vertex $k$-uniform hypergraph $H$ has at least $t\left(\frac{n}{k-1}\right)^{\frac{r-1}{k-1}}$ edges, then $H$ contains a matching of size $\max\{2, \left\lceil \frac{t}{k} \right\rceil\}$.

Proof. If $t \leq 2k$, it is obvious by Erdős-Ko-Rado theorem. Assume $t > 2k$. We greedily choose disjoint edges from $H$. If we choose $\ell < \left\lceil \frac{t}{k} \right\rceil$ disjoint edges, the number of edges intersect at least one of them is at most $\ell k\left(\frac{n}{k-1}\right)^{\frac{r-1}{k-1}} < t\left(\frac{n}{k-1}\right)^{\frac{r-1}{k-1}}$. Thus we can choose additional edge disjoint from all previous ones. We can do this until we get $\left\lceil \frac{t}{k} \right\rceil$ disjoint edges. \qed

Now we introduce the notion of sunflower. Erdős and Rado [6] introduced the following notion of sunflower in connection with some problems in Number Theory. It is also called a $\Delta$-system.

Definition 2.2. A family of $p$ sets is a $p$-sunflower if the intersections of any two sets in the family are all the same. Let $q(k, p)$ be the least integer $q$ such that every $k$-uniform family of $q$ sets contains a $p$-sunflower.

They also showed that $q(k, p)$ exists. It means that if a $k$-uniform hypergraph has no $p$-sunflower, then the number of edges in the hypergraph is bounded by $q(k, p)$. In particular, they proved the following.

Theorem 2.3. [6]

$$(p - 1)^k \leq q(k, p) \leq (p - 1)^k k!$$
They also conjectured that \( q(k, p) \leq c_p^k \) for some constant \( c_p \). Abbott, Hanson, and Sauer [1] and later Füredi and Kahn (see [5]) improved the upper bound of Theorem 2.3. The most recent result on the topic is the following result by Kostochka.

**Theorem 2.4 (Kostochka [9]).** For \( p \geq 3 \) and \( \alpha > 1 \), there exists \( D(p, \alpha) \) such that \( q(k, p) \leq D(p, \alpha) k! \left( \frac{\log \log k}{\alpha \log k} \right)^2 k \).

Essentially, Theorem 2.4 implies that there exists a constant \( c(p) \) such that \( q(k, p) \leq \frac{k^h}{(\log \log k)^{k/2}} \) for \( k \) at least \( c(p) \). By using Theorem 2.4 we prove the following lemma which is a variation of Lemma 1 in [10].

**Lemma 2.5.** There exists a constant \( c(r) \) such that the following holds. Let \( k, r \) be integers and \( H \) be a \( k \)-uniform hypergraph on \( n \) vertices containing no \( r \)-regular subgraphs with maximum degree \( \Delta = \Delta(H) \). If \( |H| \geq c(r)k^k \), then

\[
\frac{6n^{k/(k-1)} \Delta^{(k-2)/(k-1)}}{(\log \log m)^{k(k-1)}}.
\]

**Proof.** Let \( m = \left\lfloor \frac{|H|}{k\Delta} \right\rfloor \geq c(r) \), and suppose \( |H| > c(r)k\Delta \) and

\[
|H| > \frac{6n^{k/(k-1)} \Delta^{(k-2)/(k-1)}}{(\log \log m)^{k(k-1)}}
\]

for a contradiction. This implies that

\[
m^{k-1} = \left( \frac{|H|}{k\Delta} \right)^{k-1} \geq \frac{6^{k-1} n^{k}\Delta^{2}}{k^{k-1} n^{k-1}(\log \log m)^{1/2}} \geq \frac{1}{k\Delta(\log \log m)^{1/2}} \frac{6^{k-1} n^{k}}{k^{k-2}}.
\]

So, we get

\[
(k\Delta)^m \geq \left( \frac{6^{k-1} n^{k}}{m^{k-1} k^{k-2}(\log \log m)^{1/2}} \right)^m > \frac{k^{2m} m^m}{(\log \log m)^{m/2}} \left( \frac{3n}{mk} \right)^{mk} > \frac{k^{2m} m^m}{(\log \log m)^{m/2}} \left( \frac{n}{mk} \right).
\]

Now we count matchings of size \( m \) in \( H \). We may greedily pick disjoint edges \( e_1, e_2, \ldots, e_m \) so that in each step we exclude all edges intersecting previously chosen edges. Then we exclude at most \( k\Delta \) more edges in each step. Thus we conclude that the number of matchings of size \( m \) in \( H \) is at least

\[
\frac{1}{m!} \prod_{i=0}^{m-1} (|H| - k\Delta i) \geq \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} (1 - \frac{k\Delta i}{|H|}) \geq \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} (1 - \frac{i}{m}) \geq (k\Delta)^m.
\]

Because the number of \( mk \)-sets in \( V(H) \) is \( \binom{n}{mk} \), (2.1) and (2.2) together assert that there are at least \( k^{2m} m^m / (\log \log m)^{m/2} \geq q(m, r) \) distinct matchings \( M_1, M_2, \ldots, M_{q(m, r)} \) covering the same set \( M \) of size \( mk \). Consider the following auxiliary hypergraph \( \mathcal{H} \) with

\[
V(\mathcal{H}) = \{ e \in H \}, E(\mathcal{H}) = \{ M_i : i = 1, \ldots, q(m, r) \}.
\]

Note that a vertex in \( \mathcal{H} \) is an edge in \( H \), and an edge in \( \mathcal{H} \) is a matching of size \( m \) in \( H \). By Theorem 2.4 there are at least \( r \) distinct matchings \( M_{i_1}, \ldots, M_{i_r} \), which together form an \( r \)-sunflower in \( \mathcal{H} \). Let \( M' \) be \( M_{ij} \cap M_{j'} \), for all \( j \neq j' \). Then \( M_{ij} - M' \) for \( j = 1, 2, \ldots, r \) are
r disjoint matchings covering the same set \( M = \bigcup_{e \in M'} e \). Thus \( \bigcup_{j=1}^r (M_{ij} - M) \) gives us an \( r \)-regular subgraph of \( H \), it is a contradiction. \( \square \)

We also use the following theorem by Pikhurko and Verstraëte in several places.

**Theorem 2.6.** [11] For \( k \geq 3 \), if \( H \) is an \( n \)-vertex \( k \)-uniform hypergraph with at least \( \frac{2}{3}\binom{n-1}{k-1} \) edges, then \( H \) contains two pairs of sets \( \{A, B\}, \{C, D\} \) so that

\[
A \cap B = C \cap D = \emptyset, A \cup B = C \cup D.
\]

Now we introduce new hypergraphs \( H(k, \ell) \), and \( H'(k, \ell) \) which will be useful proving several claims later.

**Definition 2.7.** For \( k > \ell \), \( X \) is a set of \( 2(k-\ell) \) vertices and \( Y = \{u_1, u_2, \ldots, u_\ell, v_1, v_2, \ldots, v_\ell\} \). \( \{A, B\} \) is an equipartition of \( X \). We define \( H(k, \ell) \) be the \( 2k \)-vertex \( k \)-uniform hypergraph satisfying the following,

\[
E(H(k, \ell)) = \{e \cup Z : |e \cap \{u_i, v_i\}| = 1 \text{ for all } i = 1, 2, \ldots, \ell, |e| = \ell, Z \in \{A, B\}\}.
\]

We call each of \( A \) and \( B \) a stationary part, and vertices in them stationary vertices. Also we call vertices in \( Y \) dynamic vertices.

**Definition 2.8.** For \( k > \ell \), \( X \) is a set of \( 2(k-\ell) \) vertices and \( Y = \{u_1, u_2, \ldots, u_\ell, v_1, v_2, \ldots, v_\ell\} \). \( \{A, B\} \) and \( \{C, D\} \) are two distinct equipartitions of \( X \). We define \( H'(k, \ell) \) be the \( 2k \)-vertex \( k \)-uniform hypergraph satisfying the following,

\[
E(H'(k, \ell)) = \{e \cup C : |e \cap \{u_i, v_i\}| = 1 \text{ for all } i = 1, 2, \ldots, \ell, |e| = \ell, Z \in \{A, B, C, D\}\}.
\]

We call each of \( A, B, C \) and \( D \) a stationary part, and vertices in them stationary vertices. Also we call vertices in \( Y \) dynamic vertices.

Indeed, \( H(k, \ell) \) is a \( k \)-uniform hypergraph which resembles the complete \((\ell+1)\)-partite \((\ell+1)\)-uniform hypergraph with all parts size two. Because of the resemblance, its Turan number is related to the Turan number of \((\ell+1)\)-partite \((\ell+1)\)-graph. The lemma below is proved by Erdős, and we use it to find the Turan number of \( H(k, \ell) \).

**Lemma 2.9.** [11] Let \( S \) be a set of \( N \) elements \( y_1, y_2, \ldots, y_N \) and let \( A_i \) for \( 1 \leq i \leq n \) be subsets of \( \binom{S}{k} \). If \( \sum_{i=1}^n |A_i| \geq \frac{wN}{w^d} \) for some \( w \) and \( n \geq 8w^d \), then there are 2 distinct \( A_{i_1}, \ldots, A_{i_2} \) so that

\[
A_{i_1} \cap A_{i_2} \geq \frac{N}{2w^2}.
\]

**Proposition 2.10.** Let \( k, \ell \) be integers with \( k > \ell \). Then for \( n > 2k \), any \( k \)-uniform hypergraph \( H \) with \( 2n^{k-2-\ell} \) edges contains a copy of \( H(k, \ell) \) as a subgraph. Moreover, if \( k \geq \ell + 3 \), then it also contains a copy of \( H'(k, \ell) \).
Proof. We use induction on $\ell$. For $\ell = 0$, assume we have an $n$-vertex $k$-uniform hypergraph $H$ with $2n^{k-1}$ edges. By Observation 2.11 and the fact $n^{k-1} > (n-1)/(k-1)$, we get $H(k,0)$ which is a matching of size two for any $k$. If $k \geq 3$ and $\ell = 0$, then Theorem 2.6 implies that $H$ contains $H'(k, \ell)$, which consists of two pairs of disjoint edges with the same union.

For $k = 2$, $\ell = 1$, Turan number for the cycle of length 4 gives us the conclusion about $H(k, \ell)$. Assume now that every $n$-vertex $k$-uniform hypergraph with $2n^{k-2-\ell}$ edges contains a copy of $H(k, \ell - 1)$ for $n > 2(k - 1)$ and $k \geq 3, \ell \geq 1$. If an $n$-vertex $k$-uniform hypergraph $H$ with $n > 2k$ contains at least $2n^{k-2-\ell}$ edges, we let $x_1, x_2, \ldots, x_n$ be the vertices of $H$ and $y_1, \ldots, y_N$ with $N = \binom{n}{r-1}$ be the all $(r-1)$-sets in $V(H)$. Let $A_i = \{y_i : y_i \cup x_i \in H\}$. Because

$$\sum_{i=1}^{n} |A_i| \geq 2kn^{k-2-\ell} > \frac{nN}{2(k-1)!},$$

we may apply Lemma 2.20 with

$$N = \binom{n}{k-1}, w = \frac{n^{-\ell}}{2(k-1)!}.$$ 

For $n > 2k, n \geq 8w^2 = \frac{2n^{2-\ell}}{(k-1)!}$ is satisfied, thus there are two sets $A_{i_1}, A_{i_2}$ for which

$$|A_{i_1} \cap A_{i_2}| \geq \frac{1}{2} \binom{n}{k-1}(2kn^{-\ell})^2 > 2n^{k-1-2-\ell+1}.$$ 

By induction hypothesis, $A_{i_1} \cap A_{i_2}$ contains $H'$, a copy of $H(k-1, \ell - 1)$. Then $\{z \in e : e \in E(H'), z \in \{x_{i_1}, x_{i_2}\}\}$ is a copy of $H(k, \ell)$. Thus $H$ must contain a copy of $H(k, l)$. We get the conclusion for $H'(k, \ell)$ in the same logic. 

Since $H(k, \ell)$ contains $2^\ell$ distinct matchings of size 2 covering the same ground set, if a hypergraph contains a copy of $H(k, \ell)$, then it contains an $r$-regular subgraphs for all $1 \leq r \leq 2^\ell$. Also, $H'(k, \ell)$ contains an $r$-regular subgraphs for all $1 \leq r \leq 2^{\ell+1}$.

3. Approximate size of $H$

In this section, we prove the following Theorem 3.1 by showing that most of the edges in $H$ contain only one vertex of high degree. Note that we only consider the case when $r \geq 3$ because the case of $r = 2$ is already done in [10]. We let $\ell := \lceil \log r \rceil$, $0 < \alpha \leq \frac{1}{2}$ be a number which we decide later, and $D := n^{k-1-\alpha(\log \log n)}^{\frac{1}{2(\alpha - 1)}}$. We let $T$ denote the set of vertices of $H$ of degree at least $D$ and set $t := |T|$. Since $tD \leq k|H|$, 

$$t < D^{k-1}|H|,$$

(3.1)

We also define $H_i := \{e \in H : |e \cap T| = i\}$ for $i \leq k$, and $G := \{e \in H_1 : \#f \in H_1 : e \setminus T = f \setminus T\}$. Then, it is obvious that $|G| \leq \binom{n}{k-1}$. Note that $r + 1$ is always at least $2^{\ell-1} + 2$.

**Theorem 3.1.** Let $H$ be an $n$-vertex $k$-uniform hypergraph with no $r$-regular subgraph with $k > r \geq 3, n > c(r)^{k^2}2^{\ell k^{10k}}$. If $k > 2^{\ell-1} + 2$, then

$$|H| < \binom{n-1}{k-1} + 4k^2n^{k-1-\frac{1}{2r^2-2}}.$$
If \( k = 2^{\ell-1} + 2 \), then
\[
|H| \leq \binom{n-1}{k-1} + \frac{4k^2n^{k-1}}{(\log \log n)\alpha^{k-1}}.
\]

**Proof.** First we suppose the conclusion does not hold. We may assume we have a counterexample \( H \) such that \(|H|\) is equal to the stated upper bound by deleting some edges if necessary. Also, \( \log \log n > k^{10k} \) and (3.1) implies \( t \leq D^{-1}k|H| \leq kn^\alpha(\log \log n)^{-\frac{1}{2(k-1)}} \).

**Claim 3.2.**
\[
|H_0| < \max\{kn^{k-1-\alpha+\frac{1}{2k^2}}, 6n^{k-1+\frac{(k-2)\alpha}{k-1}}(\log \log n)^{-\frac{1}{2(k-1)}}\}.
\]

**Proof.** First, we estimate \(|H_0|\). Since edges in \( H_0 \) do not intersect \( T \), the maximum degree of \( H_0 \) is less than \( D \). We apply Lemma 2.5 to \( H_0 \), then we get either
\[
|H| \leq kc(r)\Delta(H_0) \leq kn^{k-1-\alpha+\frac{1}{2k^2}}
\]
or
\[
|H| \leq \frac{6n^{k/(k-1)}D(k-2)/(k-1)}{(\log \log \frac{|H|}{kD})^{\alpha/(k-1)}} \leq \frac{6n^{k/(k-1)}D(k-2)/(k-1)}{(\log \log n^{\alpha/2})^{\alpha/(k-1)}} \leq 6n^{k-1+\frac{(k-2)\alpha}{k-1}}(\log \log n)^{-\frac{1}{2(k-1)}}.
\]

Last, every edge in \( H \setminus (H_0 \cup H_1) \) contains two vertices of \( T \) and \( k-2 \) vertices of \( V(H) \), so
\[
|H \setminus (H_0 \cup H_2)| < \binom{|T|}{2}n^{k-2} \leq k^2n^{k-2+2\alpha}(\log \log n)^{-\frac{1}{2(k-1)}}.
\]

\( \square \)

**Claim 3.3.**
\[
|H_1| < |G| + 2k^2n^{k-1-2-\ell+2+2\alpha}(\log \log n)^{-\frac{1}{2(k-1)}}.
\]

**Proof.** Note that \( k \geq 2^{\ell-1} + 2 \geq \ell + 2 \). We consider \( H_1 \setminus G \). Then every edge \( e \) in \( H_1 \setminus G \) satisfies \( |e \cap (V(H) \setminus T)| = k-1 \) and \( (k-1)\)-set \( e \setminus T \) lies in at least two edges of \( H \). For each pair \( \{u, u'\} \subset B \), we consider the \((k-1)\)-uniform hypergraph
\[
H_{u,u'} = \{e' \in \binom{A}{k-1} : e' \cup \{u\} \in E(H_1 \setminus G), e' \cup \{u'\} \in E(H_1 \setminus G)\}.
\]
By the definition of \( G \), every edge in \( H_1 \setminus G \) belongs to \( H_{u,u'} \) for at least one pair \( \{u, u'\} \). However, if \( H_{u,u'} \) contains a copy of \( H'(k-1, \ell-2) \), then the copy together with \( u, u' \) form a copy of \( H'(k, \ell-1) \) in \( H \), which gives us an \( r \)-regular subgraph of \( H \). Thus Proposition 2.10 and the fact that \( k-1 = 2^{\ell-1} + 1 \geq \ell + 2 + 3 \) for \( \ell \geq 2 \) implies \(|H_{u,j}| \leq 2n^{k-1-2-\ell+2} \). Thus
\[
|H| \leq \sum_{u,u'}|H_{u,u'}| \leq \binom{|T|}{2}2n^{k-1-2-\ell+2} < n^{k-1-2-\ell+2+2\alpha}(\log \log n)^{-\frac{1}{2(k-1)}}.
\]

\( \square \)
If $k > 2^\ell - 1 + 2$, we choose $\alpha = \frac{1}{2(k-2)} + 2^{-\ell}$, then we get
\[
\max\{k-1-\alpha + \frac{1}{2k^2}, k-1+\frac{1}{k-1}-(k-2)\alpha, k-1-2^{\ell+2}+2\alpha, k-2+2\alpha\} < k-1 - \frac{1}{2r^2 - 2}.
\]
Hence, $|H \setminus G| = |H_0| + |H \setminus G| + |H \setminus (H_0 \cup H_1)| < (6 + 2k^2 + k^2)n^{k-1-\frac{1}{2r^2-2}} \leq 4k^2n^{k-1-\frac{1}{2r^2-2}}$. Since $n \geq c(r)k^22^{k10^k}$, we conclude
\[
|H| < \left(\frac{n-1}{k-1}\right) + 4k^2n^{k-1-\frac{1}{2r^2-2}}.
\]
If $k = 2^\ell - 1 + 2$, then we choose $\alpha = 2^{-\ell+1} = \frac{1}{k-2}$, then we get $|H \setminus G| = |H_0| + |H_1 \setminus G| + |H \setminus (H_0 \cup H_1)| \leq \frac{4k^2n^{k-1}}{(\log \log n)\log(n^{k-1})}$ and we conclude
\[
|H| < \left(\frac{n-1}{k-1}\right) + \frac{4k^2n^{k-1}}{(\log \log n)\log(n^{k-1})}.
\]
\[\square\]

Remark 3.4. If $k \geq 2^\ell + 3$, then we may choose $\alpha := \frac{3^{2^\ell+4}}{2^3(3^{2^\ell+5})}, D := n^{k-1-\alpha}$ and go through the argument above. Then we can conclude $|H \setminus G| \leq 4k^2n^{k-1-\frac{1}{2^3(3^{2^\ell+5})}}$ for any $n$-vertex $k$-uniform hypergraph $H$ with no $r$-regular subgraph when $n > c(r)k^22^{k10^k}$. In order to get Theorem 4.1, we assume $\alpha := \frac{3^{2^\ell+4}}{2^3(3^{2^\ell+5})}, D := n^{k-1-\alpha}, \ell := \lceil \log r \rceil$ throughout the paper.

4. Asymptotic structure of $H$

In this section, we want to show that the asymptotic structure of $H$ is close to a full-$k$-star. We let $G$ be as we define in the previous section, and $\alpha, \ell, D$ be as in Remark 3.4. We also define
\[
G' := \{e \setminus T : e \in G\}, \ G_x := \{e \setminus \{x\} : x \in e, e \in G\}.
\]
In order to prove Theorem 4.1, we count the copies of $H(k-1, \ell + 1)$ in $G'$ and show that there exists a vertex $v$ such that almost all copies of $H(k-1, \ell + 1)$ consist of $(k-1)$-sets in $G_v$. Let
\[
\beta := k^{3k}n^{-\frac{1}{2^3(3^{2^\ell+5})}}.
\]

Theorem 4.1. Let $H$ be an $n$-vertex $k$-uniform hypergraph with no $r$-regular subgraph with $n > c(r)k^22^{k10^k}, k \geq 2^\ell + 3$, and $|H| \geq \binom{n-1}{k-1} - k^2n^{k-1-\frac{1}{2^3(3^{2^\ell+5})}}$. There exists a vertex $v$ in $H$ such that
\[
|G_v| \geq (1-\beta)|G'|.
\]

We take a $k$-uniform hypergraph $H$ with no $r$-regular subgraphs satisfying $|H| \geq \binom{n-1}{k-1} - k^2n^{k-1-\frac{1}{2^3(3^{2^\ell+5})}}$. Then by Remark 3.4, we know
\[
|G| = |G'| \geq \binom{n-1}{k-1} - 5k^2n^{k-1-\frac{1}{2^3(3^{2^\ell+5})}}.
\]

We pick $v$ such that
\[
|G_v| = \max_{x \in V(H)} |G_x|.
\]
For a contradiction, we assume $|G_x| < (1 - \beta)|G'|$. For each $(k-1)$-set $e$ in $G'$, we define $g(e) := x$ if $e \in G_x$. Let $R_i(G')$ be the set of all pairs $\{f_1, f_2\}$ of $(k-1)$-sets in $G'$ so that $|f_1 \cap f_2| = i$, and let $R'_i(G')$ be the subset of $R_i(G')$ of all pairs $\{f_1, f_2\}$ such that $g(f_1) \neq g(f_2)$, and let

$$R'(G') := R'_i(G') \cup R'_{i+1}(G').$$

For a hypergraph $F$, we define $P(F)$ to be the set of copies of $H(k-1, \ell + 1)$ in $F$.

$$P(F) := \{ H' : H' \subset F, H' \simeq H(k-1, \ell + 1) \}.$$

Let $P_0(G')$ be the set of copies of $H(k-1, \ell + 1)$ so that the copy does not contain a pair $\{f_1, f_2\}$ in $R'(G')$ in the way that all vertices in $f_1 \cap f_2$ are dynamic vertices in the copy of $H(k-1, \ell + 1)$. Let $P_1(G')$ be the set of copies of $H(k-1, \ell + 1)$ so that the copy contains at least one pair $\{f_1, f_2\}$ in $R'(G')$ in the way that all vertices in $f_1 \cap f_2$ are dynamic vertices of the copy of $H(k-1, \ell + 1)$.

Let $K$ be the complete $(k-1)$-graph on $V(G')$. To count the number of copies of $H(k-1, \ell + 1)$ in $K$, we choose two disjoint $(k-1)$-sets, and choose $\ell + 1$ vertices from one part and match them with other $\ell$ vertices on the other part. In this manner, one copy of $H(k-1, \ell + 1)$ is counted exactly $2^{\ell+1}$ times which is the number of pairs in $H(k-1, \ell + 1)$. Thus we get

$$|P(K)| = \frac{1}{2^{\ell+2}} \binom{k - 1}{\ell + 1} \binom{k - 1}{k - \ell - 2}! \binom{n - 1}{k - 1} \binom{n - k - 1}{k - 1}.$$

Thus (4.1) implies

$$|K \setminus G'| \leq 5k^2 n^{k-1 - \frac{1}{2^{\ell+1}(3 \cdot 2^{\ell+5})}}.$$

First we show an lower bound on $|P(G')|$. 

**Claim 4.2.** $|P(G')| > (1 - \beta)|P(K)|$

**Proof.** It is enough to show $|P(K) \setminus P(G')| < \beta |P(K)|$. In order to count $P(K) \setminus P(G')$, take a $(k-1)$-set $e$ in $K \setminus G'$ and a $(k-1)$-set $e'$ in $K$ disjoint from $e$. Then there are at most $\binom{k-1}{\ell+1} \binom{k-1}{\ell-2}!$ copies of $H(k-1, \ell + 1)$ containing both $e, e'$. By (4.3) and the fact that $5 \cdot 2^{\ell+2}k^2(k-1)! n^{-\frac{1}{2^{\ell+1}(3 \cdot 2^{\ell+5})}} < \beta$, we get

$$|P(K) \setminus P(G')| \leq \binom{k-1}{\ell+1} \binom{k-1}{k-\ell-2}! |K \setminus G'| \binom{n-1}{k-1} < \beta |P(K)|.$$

Now we estimate $|P(G')|$ to show a contradiction.

**Claim 4.3.** $|P_i(G')| \leq \frac{1}{2} \beta |P(K)|$

**Proof.** First, we count the number of pairs in $R'_i(G')$ with $i \in \{\ell, \ell + 1\}$. We take two disjoint $(k-1-i)$-sets $e_1, e_2$, and two distinct vertices $x, y \in T$. Let $p(e_1, e_2, x, y)$ be the collection of $i$-sets $h$ so that $f_1 = e_1 \cup h, f_2 = e_2 \cup h$ and $g(f_1) = x, g(f_2) = y$. If $p(e_1, e_2, x, y)$ contains a copy of $H(i, \ell - 1)$, then it contains $2^\ell \geq r$ distinct matchings of size two covering the same ground set giving an $r$-regular subgraph, a contradiction. Thus $p(e_1, e_2, x, y)$ does not contain $H(i, \ell - 1)$, so it has at most $2n^{\frac{1}{k-i-1}}$ edges by Proposition 2.10. There are at most $\left(\binom{n-1}{k-i-1}\right)^2$ choices for $\{e_1, e_2\}$ and $\binom{|T|}{2}$ choices for $\{x, y\}$. So,
\[ |R'_1(G')| \leq \sum_{\{e_1, e_2, x, y\}} 2n^{i-\frac{1}{2}} \leq 2n^{i-\frac{1}{2}} \left( \binom{n-1}{k-i-1} \right)^2 \left( \binom{\ell}{2} \right). \]

For each pair in \( R'_1(G') \), we can complete a copy of \( H(k-1, \ell+1) \) by adding \( i \) more vertices from outside and choosing \( \ell+1-i \) vertices each from \( e_1, e_2 \) to play a role of dynamic vertices and match those dynamic vertices. Thus each pair in \( R'_1(G') \) is contained in at most \( (\ell+1)! \binom{k-i-1}{\ell+1-i} \binom{n-1}{i} \) copies of \( H(k-1, \ell+1) \). Thus,

\[
|P_1(G')| = \sum_{i=\ell}^{\ell+1} |R'_1(G')|(\ell+1)! \left( \binom{k-i-1}{\ell+1-i} \right)^2 \binom{n-1}{i} \leq \sum_{i=\ell}^{\ell+1} k^2(\ell+1)! \binom{n-1}{i} \leq k^4 k! n^{2k-2} \frac{1}{2^{\ell+1}} \leq k^4 k! n^{2k-2} \frac{1}{2^{\ell+1}} < \frac{1}{2} \beta |P(K)|.
\]

Before we estimate \( |P_0(G')| \), we prove the following claim.

**Claim 4.4.** Let \( H' \) be a copy of \( H(k-1, \ell + 2) \) in \( P_0(G') \). Then there exists a vertex \( x \in T \) so that every \( (k-1)-set \) \( e \) in \( H' \) is contained in \( G_x \).

**Proof.** We consider a graph \( G_{H'} \) such that

\[ V(G_{H'}) := \{ f \in H' \}, \]

\[ E(G_{H'}) = \{ f_1, f_2 : \{ f_1, f_2 \} \in H', g(f_1) = g(f_2), f_1 \land f_2 \text{ belongs to the dynamic part of } H' \}. \]

Let \( A, B \) be two stationary parts in \( H' \). Consider two \( (k-1) \)-sets \( f_1, f_2 \) in \( H' \) such that \( |f_1 \land f_2| = |f_1 \land f_2| = 1 \) and both \( f_1, f_2 \) contains \( A \), and \( f_1 \land f_2 = \{ x \} \). We consider an edge \( e \in H' \) such that \( e = (f_1 \land A) \cup B \). Then, \( e \) does not share any stationary vertices with \( f_1 \) or \( f_2 \). \( |f_1 \land e| = \ell + 1, \) and \( |f_2 \land e| = \ell \). Thus \( e \) is adjacent to both \( f_1 \) and \( f_2 \) in \( G_{H'} \). So all \( (k-1) \)-sets in \( H' \) containing \( A \) are in the same component in \( G_{H'} \), all \( f \)\( s \) containing \( B \) are in the same component in \( G_{H'} \) by the same logic. Also there are edges between an \( f_1 \) and \( e \), so \( G_{H'} \) is connected. On the other hand, if two \( (k-1) \)-sets \( f_1, f_2 \) are adjacent in \( G_{H'} \), then \( g(f_1) = g(f_2) \) because of the definition of \( P_0(G') \). This fact and connectedness of \( G_{H'} \) together imply that there exists a \( x \in T \) such that every edge in \( H' \) belongs to \( G_x \). \( \square \)

**Claim 4.5.** \( |P_0(G')| < (1 - \frac{3}{2} \beta)|P(K)| \)

**Proof.** A copy of \( H(k-1, \ell + 1) \) in \( G' \) consists of \( 2^{\ell+1} \) pairs of two disjoint \( (k-1) \)-sets all in \( G_x \) for some \( x \in T \). To count the number of copies of \( H(k-1, \ell + 1) \), we choose two disjoint \( (k-1) \)-sets \( e, e' \) with \( e(g(e)) = g(e') \), and choose \( \ell+1 \) elements from one and match them with \( \ell+1 \) vertices in the other side. This can be done in \( \binom{k-1}{\ell+1} \binom{k-1}{k-\ell-2} \) ways. Also, each \( H(k-1, \ell + 1) \) is counted \( 2^{\ell+1} \) times from each pairs in this counting. So,

\[
|P_0(G')| \leq \frac{1}{2^{\ell+1}} \binom{k-1}{\ell+1} \binom{k-1}{k-\ell-2} \sum_{x \in T} \binom{|G_x|}{2}.
\]

(4.4)
By convexity, the right side of (4.3) is maximized when \(|G_v|=(1-\beta)|G'|\) and \(|G_u|=\beta|G'|\) for another vertex \(u \in T\) and \(|G_x| = 0\) for other \(x\). And \((|G'|)^2 \leq \frac{1}{2} \left( \frac{n-1}{k-1} \right)^2 \leq \frac{1}{2} (1 + \frac{k}{n}) \left( \frac{n-1}{k-1} \right)^{n-k-1}\). Because of (4.2) and the fact \(\frac{k}{n} + 2\beta^2 < \frac{1}{2}\),

\[
|P_0(G')| \leq ((1-\beta)^2 + \beta^2)(1 + \frac{k}{n})|P(K)| \leq (1 - \frac{3}{2}\beta)|P(K)|.
\]

\(\square\)

In total,

\[
|P(G')| = |P_0(G')| + |P_1(G')| < (1 - \frac{3}{2}\beta)|P(K)| + \frac{1}{2}\beta|P(K)| \leq (1 - \beta)|P(K)|.
\]

However, it contradicts to Claim 4.2. Therefore, we conclude \(|G_v| \geq (1-\beta)|G'|\). Note that \(2^r + 3 \geq 2r + 1, n \geq c(r)^2 k^{100k^2}\), and \(n^{-2}\frac{2^r+3}{2} \leq n^{-\frac{1}{2}\frac{2}{r+1}} \leq n^{-\frac{1}{2}\frac{1}{r+1}}\), thus Theorem 3.1 Remark 3.3 and Theorem 4.1 together imply Theorem 1.1.

5. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. We assume \(r \in \{3, 4\}, k = kr, n \geq c(r)^2 k^{100k^2}\) and \(k' \geq 2^{r+1} v^r\). If an \(n\)-vertex \(k\)-uniform hypergraph \(H\) with no \(r\)-regular subgraphs contains at least \((\frac{n-1}{k-1})\) edges, we may suppose \(|H| = \left( \frac{n-1}{k-1} \right)\) by deleting some edges if necessary. If we show that \(H\) has to be a full-\(k\)-star, then it completes the theorem because a full-\(k\)-star with one more edge \(e\) always contains an \(r\)-regular subgraph when \(r \mid k\) by the following Observation 5.1. Let \(H^* = H - \{v\}\). By Theorem 4.1 there exists a vertex \(v\) with \(|H^*| \leq \beta \left( \frac{n-1}{k-1} \right)\). We will show \(|H^*| = 0\). Suppose \(|H^*| > 0\) for a contradiction.

Observation 5.1. If \(\{e'_1, e'_2, \ldots, e'_k\}\) is a partition of \(e \in H^*\) into \(r\) sets of size \(k'\) and \(g \subset V(G) - \{v\} - e\) is a \((k' - 1)\)-set, then there exists \(j\) such that \((e \setminus e'_j) \cup g \cup \{v\}\) is not an edge of \(H^*\).

Proof. Suppose not. Then \(e, (e \setminus e'_j) \cup g \cup \{v\}, (e \setminus e'_j) \cup g \cup \{v\}, \ldots, (e \setminus e'_j) \cup g \cup \{v\}\) together form an \(r\)-regular subgraph, a contradiction. Thus there is a choice \(j\) such that \((e \setminus e'_j) \cup g \cup \{v\}\) is not an edge of \(H^*\). \(\square\)

Here is a brief description of what we do in this section. By the Observation 5.1 if \(H^*\) is not empty, every edge in \(H^*\) guarantees many non-edges containing \(v\), so there must be many non-edges containing \(v\) in terms of \(|H^*|\). However, the number of non-edges containing \(v\) is same with \(|H^*|\) since \(|H| = \left( \frac{n-1}{k-1} \right)\), so it cannot be too big in terms of \(H^*\). So, we count the pairs of \(\{e, f\}\) with \(e \in H^*\) and an non-edge \(f\) containing \(v\) with certain properties, which we define later as \(wedges\), to derive a contradiction.

Let \(s_i := \frac{2^{(k-1)\left(\frac{n-1}{k-1}\right)}}{\binom{n}{k}}\). For an \(i\)-set \(e\) in \(V(H)\) not containing \(v\), we define \(d'_i(e)\), \(i\)-deficiency of \(e\), which is the number of non-edges of \(H\) containing both \(e\) and \(v\),

\[
d'_i(e) := |\{e' \notin E(H): e' = e \cup \{v\} \cup e'', e'' \in \left( V(H) \setminus e \setminus \{v\} \right) \}|.
\]

Claim 5.2. For \(i \leq k - 2\), there are pairwise disjoint \(i\)-sets \(e'_1, e'_2, \ldots, e'_{2k}\) so that \(d'_i(e'_j) \leq s_i\).
Proof. Let $F$ be the family of $i$-sets in $V(H_v)$ whose $i$-deficiency is at most $s_i$, and let $\overline{F}$ be the rest of the $i$-sets in $V(H_v)$.

$$(k-1) \binom{n-1}{i} |H^*| = \sum_e d'_e(e) \geq 0 \cdot |F| + s_i |\overline{F}|.$$  

Because $|\overline{F}| = \binom{n-1}{i} - |F|$, this implies

$$|F| \geq \left(\frac{n-1}{k-1}\right) \binom{n-1}{i} \binom{|H^*|}{s_i} = \frac{1}{2} \left(\frac{n-1}{i}\right)^2 \geq 2k^2 \left(\frac{n-1}{i-1}\right).$$

The last inequality holds since $n > k^4$. By Observation 2.1 $F$ has a matching of size at least $2k \geq k + r$, and edges in the matching are what we want.

From Claim 5.2 we take $e_1^{k-r-1}, e_2^{k-r-1}, \ldots, e_{r+1}^{k-r-1}$. Let

$$(5.1) \quad W := \{A \subset \left(\binom{V(H_v)}{r}\right) : \exists i : e_i^{k-r-1} \cup \{v\} \cup A \notin H \text{ and } w \notin e_i^{k-r-1}\}.$$  

By claim 5.2 $|W| < [(k + r + 1)s_{k-r-1}]$. We may assume $|W| = [(k + r + 1)s_{k-r-1}]$ by adding more elements. Define

$$G^0 := \{e \in H^* : e \text{ contains at least } r \text{ disjoint } r\text{-sets not in } W\}.$$  

$$G^1 = H^* \backslash G^0.$$  

Claim 5.3. $|G^1| \leq |W|^k |r| + 1 |(\frac{7}{4} \binom{n-1}{r^2-r-1} + (k + r^2 - r + 1)s_{k-r^2+r-1}).$$  

Proof.  
If $e \in G^1$, then it must contain an $(k-r+1)$-set $A$ such that all $r$-sets contained in $A$ are in $W$. Since this $A$ is covered by at least one choice of disjoint matching of $r$-sets in $W$ with size $k' - r + 1$, the number of possible $A$ is at most $|W|^k |r| + 1$.

Consider such a $(k-r+1)$-set $A$. Assume there are distinct $(r^2 - r)$-sets $Y_1, Y_2, \ldots, Y_{m_A}$ such that $A \cup Y_i \in G^1$ with

$$m_A > \left(\frac{7}{4} \binom{n-1}{r^2-r-1} + (k + r^2 - r + 1)s_{k-r^2+r-1}\right).$$

For each of $e_1^{k-r^2+r-1}, \ldots, e_{k-r^2+r-1}^{k-r^2+r-1}$ we obtain from Claim 5.2 we delete all $(r^2 - r)$-sets $f$ from $\{Y_1, Y_2, \ldots, Y_{m_A}\}$ such that $f \cup e_j^{k-r^2+r-1} \cup \{v\} \notin H$ for some $j$. Since there are at most $s_{k-r^2+r-1}$ such $(r^2 - r)$-sets for each $e_j^{k-r^2+r-1}$, we delete at most $(k + r^2 - r + 1)s_{k-r^2+r-1}$ sets. Thus we still have at least

$$m - (k + r^2 - r + 1)s_{k-r^2+r-1} = p > \left(\frac{7}{4} \binom{n-1}{r^2-r-1}\right)$$

sets $Y_1', \ldots, Y_p'$ not deleted. By Theorem 2.6, $\{Y_1', Y_2', \ldots, Y_p'\}$ contains four sets $Z_1, Z_2, Z_3, Z_4$ such that $Z_1 \cup Z_2 = Z_3 \cup Z_4$ and $Z_1 \cap Z_2 = Z_3 \cap Z_4 = \emptyset$. Since $\{e_1^{k-r^2+r-1}, \ldots, e_{k-r^2+r-1}^{k-r^2+r-1}\}$ is a collection of disjoint sets, and $|A \cup Z_1 \cup Z_2 \cup Z_3 \cup Z_4| = k + r^2 - r$, $A \cup Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ intersect at most $k + r^2 - r$ of them. $e_1^{k-r^2+r-1}, \ldots, e_{k-r^2+r-1}$, and $e$ intersect at most $k$ of those. So
there must be at least one $e_q^{k-r^2+r-1}$ which does not intersect any of $A, Z_1, Z_2, Z_3$ and $Z_4$. Then for $i = 1, 2, 3, 4$, $\{v\} \cup e_q^{k-r^2+r-1} \cup Z_i$ is an edge in $H$ since $Z_i$ is not deleted in the process. Thus

$$A \cup Z_1, \ldots, A \cup Z_4, \{v\} \cup e_q^{k-r^2+r-1} \cup Z_1, \ldots, \{v\} \cup e_q^{k-r^2+r-1} \cup Z_4$$

gives us a 4-regular subgraph of $H$ since $A$ and $Z_i$ for $i = 1, 2, 3, 4$ are not intersecting any of $e_q^{k-r^2+r-1}$. Also removing $A \cup Z_1, \{v\} \cup e_q^{k-r^2+r-1} \cup Z_2$ gives us an 3-regular subgraph. Thus, we get a contradiction. Therefore $m < \frac{n}{4} (\frac{n-1}{r_2-r-1}) + (k + r^2 - r + 1)s_{k-r^2+r-1}$. Hence $|G|^1 \leq \sum_A m_A \leq |W|^{k-r+1}(\frac{n}{4} \frac{n-1}{r_2-r-1}) + (k + r^2 - r + 1)s_{k-r^2+r-1}$ and we get the conclusion. □

**Claim 5.4.** $|G|^1 \leq 0.01|H^*|$. Hence $|G^0| > 0.99|H^*|$.

**Proof.** Suppose $0.01|H^*| \leq |G^1|$. Note that $|W| \leq \frac{2k^{r+1}k^r}{(k-r-1)}$. By Claim 5.3 we have

$$0.01|H^*| \leq |W|^{k-r+1}(\frac{7}{4} \frac{n-1}{r_2-r-1}) + (k + r^2 - r + 1)s_{k-r^2+r-1}$$

CASE 1. $\frac{7}{4} \frac{n-1}{r_2-r-1} \geq (k + r^2 - r + 1)s_{k-r^2+r-1}$ Then we get $0.01|H^*| \leq \frac{k^{2k}|H^*|^{k-r+1}}{n(k-r-1)(k-r+1)-r^2+r+1}$.

This implies

$$|H^*| \geq \frac{k^{2k}}{100} n^{(k-r-1)(k-r+1)-r^2+r+1} \geq \frac{1}{k^{3k}} n^{k-1} > \beta n^{k-1}.$$ 

It is a contradiction since $|H^*| \leq \beta n^{k-1}$.

CASE 2. $\frac{7}{4} \frac{n-1}{r_2-r-1} \leq (k + r^2 - r + 1)s_{k-r^2+r-1}$ Then we get $0.01|H^*| \leq \frac{k^{3k}|H^*|^{k-r+2}}{n(k-r-1)(k-r+1)+k-r^2+r-1}$. 

This implies

$$|H^*| \geq \frac{k^{3k}}{100} n^{(k-r-1)(k-r+1)+k-r^2+r-1} \geq \frac{1}{k^{3k}} n^{k-1} \geq \frac{1}{k^{3k}} n^{k-1}.$$ 

It is bigger than $\beta n^{k-1}$ because $k' - r + 1 \geq 2^{k-1}(3 \cdot 2^\ell + 5)$, and it is a contradiction. Thus in any case we get $|G^1| \leq 0.01|H^*|$. Hence $|G^0| > 0.99|H^*|$. □

To derive a contradiction, we define wedge and count the wedges in $H$.

**Definition 5.5.** A pair of $k$-sets $\{e, f\}$ is a wedge if it satisfies the following:

(1) $v \notin e \in H$ and $e$ contains at least $r$ disjoint $r$-sets not in $W$;
(2) $v \in f \notin H$ and $|f| = k$;
(3) $|e \cap f| = k' - k$;
(4) $e - f$ have at least one $r$-set outside $W$, and $e \cap f$ contains at least $r - 1$ disjoint $r$-sets outside $W$.

Let $\Lambda(H)$ denote the number of wedges in $H$. Now we fix $e \in H$ as in (1) above. Since $|e \setminus W| \geq r$, in $e$ there $r$ disjoint $r$-sets not in $W$.

**Claim 5.6.**

$$\Lambda(H) \geq 0.98|H^*| \left( \frac{k-r^2}{k'-r} \right) \left( \frac{n}{k' - 1} \right).$$
Proof. Let $U = \{U_1, U_2, \ldots, U_r\}$ be $r$-disjoint $r$-sets not in $W$ but in $e$. There are $(\binom{k - r}{k' - r} \cdot \cdots \cdot \binom{k' - r}{k' - r})$ ways to extend this to a partition of $\{e_1', e_2', \ldots, e_r'\}$ of $e$ into $r$ sets of size $k'$ such that $U_i \subset e_i'$. Once we have this equipartition $\{e_1', e_2', \ldots, e_r'\}$, we take a $(k' - 1)$-set $h$ lying in $V(H) \setminus (e \cup \{v\})$ and let $f_j = (e \setminus e_j') \cup h \cup v$. Then one of $f_1, \ldots, f_r$ must be not in $H_v$ by the Observation 5.1, giving us a wedge. In this way of counting wedges, a missing edge $f_j \cup h \cup v$ is counted at most $(\binom{k - r}{k' - r} \cdot \cdots \cdot \binom{k' - r}{k' - r})$ times. Thus, the number of missing edges $f$ intersecting $e$ at $k - k'$ vertices and containing $v$ is at least

$$\binom{n - k - 1}{k' - 1} = \binom{k - r}{k' - r} \binom{n - k - 1}{k' - 1}$$

So, if we count $\Lambda(H)$ from the $e$'s point of view, then we have

$$\Lambda(H) \geq \binom{k - r}{k' - r} \binom{n - k - 1}{k' - 1} |G^0|$$

$$\geq 0.99|H^*| \binom{k - r}{k' - r} \binom{n - k - 1}{k' - 1} \geq 0.98|H^*| \binom{k - r}{k' - r} \binom{n}{k' - 1}$$

where the last inequality holds since $n$ is sufficiently larger than $k$. \hfill $\Box$

Now we count $\Lambda(H)$ from the $f$'s point of view.

Claim 5.7.

$$\Lambda(H) \leq \frac{1.76r}{k'!} \left( \frac{k - 1}{k'} \right) \left( \frac{n}{k' - 1} \right) |H^*|.$$  

Proof. We take a $k$-set $f \notin H$ with $v \in f$, and choose a subset $g$ of $f$ with size $k - k'$. We want to extend this $(k - k')$-set to $e \in G_0$. Let $F(g)$ be the $k'$-uniform hypergraph consisting all possible $e \setminus g$ so that $\{f, e\}$ forms a wedge. Let $F_0(g) \subset F$ be the $k'$-uniform hypergraph whose edges contain no $r$-sets in $W$, and $F_1(g) \subset F$ be the $k'$-uniform hypergraph whose edges contain at least one $r$-set in $W$.

Note that an $k'$-set $h$ in $F_1(g)$ has at least one $r$-set $A$ not in $W$ by the definition of wedge. Moreover, $h$ has one $r$-set in $W$, and another $r$-set not in $W$ so that two sets are disjoint. In order to get such two disjoint sets, we take an $r$-set $B \subset h$ in $W$. Since $k' \geq 3r$, we can take another $r$-set $C$ disjoint from both $A$ and $B$. Then either $\{A, C\}$ or $\{B, C\}$ is two disjoint sets we want depending on whether $C$ is in $W$ or not.

For $F_1(g)$, to each $k'$-set $h \in F_1(g)$, associate a $(k' - r)$-set $h' \subset h$ such that $h \setminus h' \notin W$ and there is an $r$-set in $W$ inside $h'$. If there are distinct $h_1, h_2, \ldots, h_m$ with $h_1 = h_2 = \cdots = h_m = h'$, then there are distinct $r$-sets $y_1, y_2, \ldots, y_m \notin W$ such that $h_i = h \cup y_i$. If $m \geq \frac{7}{4}(n - 1)$ then by Theorem 2.6 there exists $y_a, y_b, y_c, y_d$ with $a \neq b \neq c \neq d$ so that $y_a \cap y_b = y_c \cap y_d = \emptyset, y_a \cup y_b = y_c \cup y_d$.

Since $g \cup y_a$ $y_b$ covers at most $k + r - 1$ vertices, by Claim 5.2, there exists $i$ for which $e_i^{k - r - 1}$ has no vertex of $g$ $y_a$ $y_b$. Now

$$\{g \cup h' \cup y_i : \gamma \in \{a, b, c, d\}\} \cup \{e_i^{k - r - 1} \cup \{v\} \cup y_a : \gamma \in \{a, b, c, d\}\}$$

together give us both a 3-regular subgraph and a 4-regular subgraph. Thus

$$m \leq \frac{7}{4}\left(\frac{n - 1}{r - 1}\right).$$
So, $|F_1(g)|$ is at most $\frac{7}{4}(n-1)$ times the number of $(k' - r)$-sets of $V(H)$ containing an $r$-set in $W$. Because the number of $(k' - r)$-sets containing an $r$-set in $W$ is at most $|W|\binom{n-1}{k'-2r}$ and $|H^*| < \beta\binom{n-1}{k'-1}$, this is at most

$$|F_1(g)| \leq \frac{7}{4}\binom{n-1}{r-1}|W|\binom{n-1}{k'-2r} \leq \frac{7}{2}\frac{(n-1)}{(k'-r-1)}|H^*|\binom{n-1}{k'-2r} \leq \beta\binom{n-1}{k'-1}.$$  

For each $h \in F_0(g)$, we take $h' \subseteq h$ with $|h'| = k' - r$. Suppose there are $h_1, h_2, \ldots, h_m \in F_0(g)$ each of which contains $h'$ and $x_i = h_i - h'$. If $m \geq \frac{4}{7}\binom{n-k'+r-1}{r-1}$, by Theorem 2.6 we have

$$x_0, x_1, x_2, x_3, x_4 \notin W$$

Since $x_0, x_1, x_2, x_3, x_4 \notin W$, we may find $e_p^{k'-r}$ which does not contain any vertices in $g \cup h' \cup x_0 \cup x_1 \cup x_2 \cup x_3 \cup x_4$. Now

$$\{g \cup h' \cup x_0 : \gamma \in \{a, b, c, d\}\} \cup \{e_p^{k'-r} \cup \{v\} \cup x_0 : \gamma \in \{a, b, c, d\}\}$$

This implies $|F_0(g)| \binom{k'}{k'-r} \leq \frac{7}{4}\binom{n-1}{r-1}\binom{n-1}{k'-1} \leq \frac{7}{4}\binom{n-1}{k'-1}.$

Thus

$$|F_0(g)| \leq \frac{7}{4}\binom{n-1}{k'-1} \leq \frac{7}{4}\binom{n-1}{k'-1} \leq \frac{7}{4}\binom{n-1}{k'-1}.$$  

Since $n > c(r)^k k^{100k^2}$, we know $\beta < 0.001$,

$$\Lambda(H) \leq \sum_{v \in f, f \notin H, v \notin g, g \in \{k-k'\}} |F_0(g)| + |F_1(g)| \leq |H^*|\binom{k-1}{k'}\left(\frac{7}{4}\binom{n-1}{k'-1} + \beta\binom{n-1}{k'-1}\right)$$

$$\leq \frac{1.76r}{k'}\binom{k-1}{k'}\binom{n-1}{k'-1}|H^*|$$

Thus, we get

$$0.98|H^*|\binom{k-r}{k'-r}\binom{n-1}{k'-1} \leq \Lambda(H) \leq \frac{1.76r}{k'}\binom{k-1}{k'}\binom{n-1}{k'-1}|H^*|$$

This implies

$$0.556k' \leq \frac{r \cdot (rk'-1)(rk'-2) \cdots ((r-1)k')}{k'(k'-1) \cdots (k'-r+1)\binom{r}{k'-r+1}} \leq \frac{r \cdot (k'-r)\binom{r}{k'-r+1}}{(r-1)k'-r+1} \leq 1.1r^{r+1}e^r.$$  

The last inequality holds since $k' > 2r^{r+1}e^r > 2r^{2r+1}$ and $e^{\frac{1}{2r}} < 1.1$. However, it is a contradiction since $0.556k' \geq 0.556 \cdot 2r^{r+1}e^r > 1.1r^{r+1}e^r$. Therefore $H^*$ must be an empty, this proves Theorem 1.2.
6. What happens if $r$ is big or $r \nmid k$?

In the same spirit as Theorem 1.2 we propose the following conjecture.

**Conjecture 6.1.** For $r$, there exists $k_r$, $n_k$ such that for all $k > k_r$, and $n > n_k$, and $r \mid k$, if $H$ is a $k$-uniform hypergraph with no $r$-regular subgraph, then

$$|H| \leq \binom{n-1}{k-1}$$

and equality holds if and only if $H$ is a full-$k$-star.

The proof of Theorem 1.2 does not extend for the case $r > 4$ because the author does not know how to generalize Theorem 2.6 for more pairs of disjoint edges. However, if the following conjecture is true, then we can prove Conjecture 6.1.

**Conjecture 6.2.** For every positive integer $r$, there exists $k_r$, $n_k$ and $g(r)$ with the following holds. For $k \geq k_r, n \geq n_k$, an $n$-vertex $k$-uniform hypergraph $H$ contains more than

$$g(r)\binom{n-1}{k-1}$$

edges, then it contains distinct edges $A_1, B_1, \cdots, A_r, B_r$ so that $A_i \cap B_i = \emptyset$ for all $i = 1, 2, \cdots, r$ and $A_1 \cup B_1 = A_2 \cup B_2 = \cdots = A_r \cup B_r$.

Note that this conjecture is known to be true for $r = 1, 2$. For $r = 1$, it’s Erdős-Ko-Rado Theorem, and when $r = 2$ that Füredi [7] proved $k_2 = 3, g(2) \leq \frac{7}{2}$ and later Pikhurko and Verstraëte [11] improved it to $g(2) \leq \frac{7}{2}$.

In Theorem 1.2 we assume $k \geq 2r^{r+2}e^r$ and $r \mid k$. What happens if the conditions do not hold? First, let’s see what happens if $k$ is not big enough in terms of $r$. The author believe that full-$k$-star might be the only extremal example even when $k \geq 2r$ and $r \mid k$. However if $r = k$ then the extremal example is no longer only full-$k$-stars. Also, if $r > k$, then $|H|$ can be bigger than $\binom{n-1}{k-1}$. It is straightforward to check the following example.

**Example 6.3.** Take an $n$-vertex full-$k$-star $H$. We take a non-edge $e$ of $H$, and and edge $e'$ of $H$ such that $|e \cap e'| = k-1$. Then $H \setminus \{e'\} \cup \{e\}$ does not have $r$-regular subgraph when $r = k$, and $H \cup \{e\}$ does not have $r$-regular subgraph when $r = k+1$.

As the example, if $r$ is bigger than $k$, even $r = k+1$ does not implies $|H| \leq \binom{n-1}{k-1}$ any more. However, as we can see in Section 3 $|H| \leq (1+o(1))\binom{n-1}{k-1}$ still holds if $k \geq 2^{\lceil \log r \rceil} - 1 + 2$. Thus, for $r = 2^l$ and $k \geq r^2 + 3 = 2^{l-1} + 2$, the asymptotic of the number of edges in hypergraphs with no $r$-regular subgraphs is still $(1+o(1))\binom{n-1}{k-1}$ even though $r \geq k$. However, the following example shows that this becomes false if $r$ is much bigger.

**Example 6.4.** For an integer $c > 1$, Take an $n$-vertex $k$-uniform hypergraph $H$ such that $E(H) = \{e : e \in (V(H))^k, |e \cap \{x_1, x_2, \cdots, x_c\}| = 1\}$. Then $|H| = c\binom{n-1}{k-1} \sim c\binom{k-1}{k-1}$. However, $H$ does not contain any $r$-regular subgraphs when $r > c\binom{k-1}{k-2}$.

**Proof.** Suppose $H$ contains an $r$-regular subgraph $R$, then $R$ must cover some vertices in $\{x_1, x_2, \cdots, x_c\}$. Assume it covers $\{x_1, x_2, \cdots, x_{c'}\}$. Since it must cover those vertices exactly $r$-times, $|R| = c'r$. Then $V(R) = \frac{kr}{c} = c'k$. Then a vertex $x$ in $V(R)$ can be covered only by edges $e$ with $|e \cap (V(R) \setminus \{x_1, \cdots, x_c\}| = k - 1$. So, degree of $x$ is at most $c'(\binom{k-1}{k-2}) < r$, a contradiction. \[\Box\]
Hence, it is natural to ask the following question. Note that, such $r(k)$ must exist and $k \leq r(k) \leq 2^{(2k-2)/k} + 1$ by Theorem 3.1 and Example 6.4.

**Question 6.5.** What is the minimum $r = r(k)$ such that
\[
\limsup_{n \to \infty} \max \frac{|H|}{\binom{n-k-1}{k-1}} > 1.
\]
where the maximum is taken over all $n$-vertex $k$-uniform hypergraphs with no $r$-regular subgraphs.

Now we consider the case where $r$ does not divide $k$ while $k$ is bigger than $r$. In [10], Mubayi and Verstraëte conjectured the following.

**Conjecture 6.6.** [10] For every integer $k$ with $2 \nmid k$, there exists an integer $n_k$ such that for $n \geq n_k$, if $H$ is an $n$-vertex $k$-uniform hypergraph with no 2-regular subgraph then $|H| \leq \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$. Equality holds if and only if $H$ is a full-$k$-star together with a maximal matching disjoint from the full-$k$-star.

In the same spirit, we may add more edges to full-$k$-star when $r \geq 3$, $r < k$, $r \nmid k$. In order to construct an example, we need the following concept.

In 1973, Brown, Erdős and Sós [2], proposed a study for a new parameter, $f_k(n,a,b)$, the largest number of edges in an $k$-uniform hypergraph on $n$ vertices that contains no $b$ edges spanned by $a$ vertices. Determining $f_k(n,a,b)$ for general $(k,a,b)$ is very difficult. Note that finding value of $f_3(n,6,3)$ is known as the famous $(6,3)$-problem. In [2], they showed the following.

**Theorem 6.7.** [2] If $a > k$ and $b > 1$, then $f_k(n,a,b) > c_{a,b}n^{\frac{ak-b}{b-1}}$.

Now we consider the following construction.

**Construction 6.8.** Let $k = k'd, r = r'd$ be integers with $k \geq 3, r' \geq 3$ such that $k'$ and $r'$ are relatively prime. Consider a $(k-1)$-uniform $n-2$-vertex hypergraph $H'$ with $f_{k-1}(n-2, 2k-2, r')$ edges such that $H'$ does not contain any $r'$ edges spanning at most $2k-2$ vertices. Especially, $|H'|$ contains at least $c_{k',r'}n^{\frac{r'-2}{r'-1}(k-1)}$ edges.

Now we consider two vertices $x, y$ disjoint from $V(H')$ and the hypergraph $H_{k,r}$ with
\[
V(H_{k,r}) = V(H') \cup \{x, y\},
\]
\[
E(H_{k,r}) = \{e \in \binom{V(H')}{k} : x \in e\} \cup \{e \cup \{y\} : e \in E(H')\}.
\]

Then $H_{k,r}$ contains at least $\binom{n-2}{k-1} + c_{k,r}n^{\frac{r-2}{r-1}(k-1)}$ edges, and $H_{k,r}$ contains no $r$-regular subgraphs.

**Proof.** Assume that $H_{k,r}$ contains an $r$-regular subgraph $R$. Let $H_x$ be the full-$k$-star in $H_{k,r}$ and $H^*$ be the hypergraph consisting edges not containing $x$. Since both $H_x$ and $H^*$ are subgraph of two distinct full-$k$-star, each of them does not contain any $r$-regular subgraph. Thus $R$ must intersect both $H_x$ and $H^*$, thus $R$ must cover both $x$ and $y$. Since $R$ covers $x$ exactly $r$ times,
\[
|R| = |R \cap H_x| + |R \cap H^*| = r + |R \cap H^*| \geq r + 1.
\]
However, because $R$ induces an $r$-regular subgraph
\[
r|V(R)| = kr + k|R \cap H^*|.
\]
Since \(k', r'\) are relatively prime, \(|R \cap H^*|\) must be a multiple of \(r'\), moreover \(|R \cap H^*| \leq r\) because \(y\) must be covered exactly \(r\)-times. Hence \(|V(R)| = \frac{|R|}{r} \leq 2k\). Now we consider \(\{e - y : e \in R \cap H^*\}\). It is a set of at least \(r'\) edges of \(H'\) covering at most \(2k - 2\) vertices, a subst of \(V(R) - \{x, y\}\). It is a contradiction to the definition of \(H'\). Thus \(H_{k,r}\) does not contain any \(r\)-regular subgraph. 

Hence, there is an \(n\)-vertex \(k\)-uniform hypergraph \(H\) with no \(r\)-regular subgraphs which contains quite more edges than \(\binom{n-1}{k-1}\) if \(r\) does not divides \(k\). Hence we propose the following question.

**Question 6.9.** Determine the least value of \(h(k, r)\) such that there exists a contact \(c_{k,r}\) so that every \(n\)-vertex \(k\)-uniform hypergraph \(H\) with no \(r\)-regular subgraph satisfies

\[
|H| \leq \binom{n - 1}{k - 1} + c_{k,r}n^{h(k,r)}.
\]

The author suspects that \(h(k, r)\) is related to the value of \(gcd(k, r)\) based on the fact that the value we get from Construction 6.8 is related to \(k, r, \) and \(gcd(k, r)\).

Also, considering linear hypergraphs is another direction of studying regular subgraphs. The following question is proposed in [3].

**Question 6.10.** [3] For an integer \(r\), let \(f_{k,r}(n)\) be the maximum number of edges in a linear \(n\)-vertex \(k\)-uniform hypergraphs with no \(r\)-regular subgraphs. Is \(f_{3,3}(n) = o(n^2)\)?

Especially, authors of [3] asked if sufficiently large Steiner triple system contains a 3-regular subgraphs. In [14], Verstraëte observed that Lemma 2.5 together with the fact that all linear \(k\)-uniform hypergraph has maximum degree at most \(\frac{n - 1}{k - 1}\) trivially implies the following.

**Corollary 6.11.** For any integers \(k, r \geq 3\) and sufficiently large \(n\),

\[
f_{k,r}(n) < 6n^2(\log \log(n))^{-\frac{1}{2(k-1)}}.
\]

Thus this answers Question 6.10 and it implies that for an integer \(r\), every \(n\)-vertex Steiner system contains an \(r\)-regular subgraph if \(n\) is sufficiently large.

**Acknowledgement**

I am indebted to Alexandr V. Kostochka for careful reading and very helpful comments. I am grateful to Jacques Verstraëte for letting me know that Lemma 2.5 trivially implies Corollary 6.11 I also thank Joonkyung Lee for useful discussion.

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