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Some new findings on the survival Rényi entropy and application of COVID-19 data

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Motivated by the connotation of survival Rényi entropy and its related dynamic version, we introduce them in terms of their lower bounds and mean residual life function. Moreover, we illustrate the relation between survival Rényi entropy and some of measures of information. Furthermore, the hazard rate order implies ordering of dynamic survival Rényi entropy. Our models are considered a more comprehensive version of generalized order statistics and give some properties and characterization results. Finally, a non-parametric estimation of survival Rényi entropy is included based on real COVID-19 data and simulated data.

Introduction

Shannon’s function measure of information (Shannon [1]) is generalized by Rényi entropy. Rényi [2] proposed a measure of order α of the non-negative random variable (R.V.) X with probability density function (PDF) f(x) as follows

\[ R(X) = \frac{1}{1-\alpha} \log \int_{0}^{\infty} f^{\alpha}(x) dx, \]

where \( \alpha \neq 0 \). When \( \alpha \to 1 \), it is easy to observe that (1) will tend to

\[ S(X) = -\int_{0}^{\infty} f(x) \log f(x) dx, \]

which is Shannon’s measure of uncertainty. Suppose that the R.V. X has cumulative distribution function (CDF) F(x). In survival analysis and reliability theory, under a system working at time t, \( X_t = [X - t; X > t] \), \( t > 0 \), is the residual lifetime. Therefore, Abraham and Sankaran [3] defined residual Rényi entropy (RRE) as

\[ R(X; t) = R(X; t) = \frac{1}{1-\alpha} \log \int_{t}^{\infty} F^{\alpha}(x) dx, \]

where \( \alpha \neq 0, t > 0 \) and \( \hat{F}(x) = 1 - F(x) \). Findings of Rényi’s entropy for residual lifetime have been elaborated in literature such as Nanda and Maiti [4], Nanda and Paul [5] and so on. For ordered R.V.’s, Zarezhad and Asadi [6] investigated the features of RRE of ordered variables. Asha and Chacko [7] described the RRE of ith record values.

The literature and research that deal with information measures concentrate on their survival form, which depends on CDF F(x). Sunoj and Linu [8] was the first to introduce the survival Rényi’s entropy (SRE) and dynamic survival Rényi’s entropy (DSRE), respectively, as follows

\[ cR(X) = \frac{1}{1-\alpha} \log \int_{0}^{\infty} \hat{F}^{\alpha}(x) dx, \]

and

\[ cR(X; t) = \frac{1}{1-\alpha} \log \int_{t}^{\infty} \frac{\hat{F}^{\alpha}(x)}{F^{\alpha}(t)} dx. \]

where \( \alpha \neq 0, t > 0 \). They present some properties and characterization results of DSRE. Moreover, they study the weighted and conditional situation of DSRE. Different types of measure of information have been studied for ordered variables. Thapliyal and Taneja [9] discussed the dynamic residual Rényi entropy and DSRE for order statistics and give some characterization results for them. Furthermore, weighted multiscale SRE was proposed by Zhou and Shang [10].

As a matter can be noted from the literature and research that considered Rényi entropy and its related measures, compared with the other measures presented in the information theory, that they depend...
on $a$ as a power, which makes them not flexible to deal with. Therefore, we present lower bounds of SRE and DSRE, and some contain $a$ but not as a power and study the behavior of those bounds. In other words, the existence of $a$ as a power inside the logarithm in SRE and DSRE, can be manipulated by removing $a$ or change its position which gives us the lower bound that depends on a well-known value. Moreover, we have obtained the bounds of SRE with respect to the flexible and well-known measures Shannon entropy and cumulative residual entropy. Throughout this paper, we present some new properties and results of SRE and DSRE. Moreover, real-life data connected with the COVID-19 virus is applied for the non-parametric estimation of SRE. The paper is organized as follows: Section “Survival Rényi’s entropy properties” deals with SRE with respect to its lower bounds and mean residual life function. Besides, giving examples of some well-known distributions. In the same way, DSRE is considered in Section “Dynamic survival Rényi entropy” and gives some hazard rate ordering. In Section “Survival Rényi entropy of generalized order statistics”, we propose SRE and DSRE for a wider subclass of generalized order statistics (GOS) and derive some properties and characterization results of DSRE. Finally, we present two different empirical estimators for SRE and give some applications.

Survival Rényi’s entropy properties

In this section, we examine the features of SRE and DSRE that are not mentioned before in the literature. We can deal with SRE by proposing its lower bound that depends on the mean and the expectation by removing $a$ or change its position to make it flexible to study as follows.

**Theorem 1.** Assume that the non-negative R.V. $X$ is arising with an absolutely continuous CDF $F(x)$. From (4), if $1 \neq a > 0$ and $\tilde{F}(x)$ is integrable, then the lower bound of the SRE depending on the mean is obtained by

$$c(R(X)) > \frac{1}{1-a} \log \left( \int_0^\infty \tilde{F}(x)dx \right) = \frac{1}{1-a} \log (E(X)) = T_1(X)$$

$$> \frac{1}{1-a} \log \left( a \int_0^\infty \tilde{F}(x)dx \right) = \frac{1}{1-a} \log (aE(X)) = T_2(X).$$

(6)

**Proof.** We have two cases:

1. If $0 < a < 1$, then $\tilde{F}(x) < \tilde{F}(x)$. Moreover, we can see that $\log \left( \int_0^\infty \tilde{F}(x)dx \right) > \log \left( a \int_0^\infty \tilde{F}(x)dx \right)$.

2. If $a > 1$, then $\tilde{F}(x) > \tilde{F}(x)$. Moreover, we can see that $\frac{1}{1-a} \log \left( \int_0^\infty \tilde{F}(x)dx \right) > \frac{1}{1-a} \log \left( a \int_0^\infty \tilde{F}(x)dx \right)$.

Thus, the result follows. □

Fig. 1 gives two examples of **Theorem 1** for Weibull distribution with CDF $F(x) = e^{-x^\lambda}$, $\lambda, n > 0$, $x > 0$ and Pareto distribution with CDF $F(x) = x^{-\nu}$, $\nu > 0$, $x > 1$, respectively.

Fig. 1. $cR(X)$, $T_1(X)$ and $T_2(X)$ for Weibull distribution with parameters $\lambda = 3, n = 5$ (left panel) and Pareto distribution with parameter $\nu = 5$ (right panel).

**Proof.** Since the logarithm function is concave, and by using Jensen’s inequality, we have $\log (E(X)) \geq E(\log(X))$ and the proof follows. □

Fig. 2 gives an example of **Theorem 2** for Weibull distribution.

Now, we will discuss the relation between SRE and the other measures. The following theorems give the lower and upper bounds of SRE depending on the entropy and cumulative residual entropy, respectively.

**Theorem 2.** Assume that the non-negative R.V. $X$ is arising with an absolutely continuous CDF $F(x)$. From (4), if $0 < a < 1$ and $\tilde{F}(x)$ is integrable, then the lower bound of the SRE depending on the expectation is given by

$$c(R(X)) \geq \frac{1}{1-a} \log \left( \int_0^\infty \tilde{F}(x)dx \right) = \frac{1}{1-a} \log (E(X)) = T_1(X)$$

$$\geq \frac{1}{1-a} \int_0^\infty (\log(\tilde{X})) f(x)dx = \frac{1}{1-a} E(\log(X)) = T_3(X).$$

(7)

**Proof.** From log-sum inequality, we have

$$\int_0^\infty f(x) \log \frac{f(x)}{F^a(x)}dx \geq \log \frac{1}{\int_0^\infty F^a(x)dx}$$

$$= -\log \int_0^\infty F^a(x)dx = -(1-a)cR(X).$$

Thus, the result follows. □

Fig. 2. $cR(X)$, $T_1(X)$ and $T_3(X)$ for Weibull distribution with parameters $\lambda = 1, n = 2$.

**Proof.** From log-sum inequality, we have

$$\int_0^\infty f(x) \log \frac{f(x)}{F^a(x)}dx \geq \log \frac{1}{\int_0^\infty F^a(x)dx}$$

$$= -\log \int_0^\infty F^a(x)dx = -(1-a)cR(X).$$

However,

$$\int_0^\infty f(x) \log \frac{f(x)}{F^a(x)}dx = \int_0^\infty f(x) \log f(x)dx - \int_0^\infty f(x) \log F^a(x)dx$$

$$= -S(X) - \int_0^1 \log x^a dx$$

$$= -S(X) + a.$$

(10)
Thus, from (9) and (10), we get (8). □

Fig. 3 gives an example of Theorem 3 for Weibull distribution.

**Theorem 4.** Suppose that the non-negative R.V. $X$ has an absolutely continuous CDF $F(x)$. Then the lower and upper bounds of SRE, defined in (4), with respect to cumulative residual entropy, can be expressed as

$$cR(X) \geq (\leq \frac{1}{1-a}E(X)\left(cS(X) + \int_{0}^{\infty} F(x) \log F^{\alpha}(x) dx + E(X) \log E(X) \right),$$

where $0 < a < 1 (a > 1)$, $cS(X) = -\int_{0}^{\infty} F(x) \log F(x)\log dx$ is the cumulative residual entropy proposed by Rao et al. [11].

**Proof.** From log-sum inequality, we have

$$\int_{0}^{\infty} F(x) \log \frac{F(x)}{F^{\alpha}(x)} dx \geq \left[ \int_{0}^{\infty} F(x) dx \right] \log \frac{\int_{0}^{\infty} F(x) dx}{\int_{0}^{\infty} F^{\alpha}(x) dx}$$

$$\geq \left[ E(X) \log \frac{\int_{0}^{\infty} F(x) dx}{\int_{0}^{\infty} F^{\alpha}(x) dx} \right]$$

$$= E(X) \log \frac{\int_{0}^{\infty} F^{\alpha}(x) dx}{\int_{0}^{\infty} F(x) dx}$$

$$= E(X) \log \frac{\int_{0}^{\infty} F(x) dx}{\int_{0}^{\infty} F^{\alpha}(x) dx}.$$  \hspace{1cm} (11)

However,

$$\int_{0}^{\infty} F(x) \log \frac{F(x)}{F^{\alpha}(x)} dx = \int_{0}^{\infty} F(x) \log F(x) dx - \int_{0}^{\infty} F(x) \log F^{\alpha}(x) dx$$

$$= -cS(X) - \int_{0}^{\infty} F(x) \log F^{\alpha}(x) dx.$$  \hspace{1cm} (13)

Thus, from (12) and (13), we get (11). □

In the following, we will express SRE in terms of mean residual life function (MRL)

$$m(t) = E Y + t | Y > t = \int_{t}^{\infty} \frac{F(x)}{F(t)} dx.$$  \hspace{1cm} (14)

**Theorem 5.** Let $X$ be a non-negative R.V. with an absolutely continuous CDF $F(x)$. Then the SRE, defined in (4), can be expressed in terms of MRL as follows

$$cR(X) = \frac{1}{1-a} \log ((1-a)E(mr(X) F^{\alpha-1}(x)) + \mu).$$  \hspace{1cm} (15)

where $\mu = E(X)$. 

**Proof.** From Rajesh and Sunoj [12], the proof can be simply obtained with considering that the following integration is derived as

$$\int_{0}^{\infty} F^{\alpha}(x) dx = (\mu + \int_{0}^{\infty} \left( \frac{d}{dx} (mr(x) F(x)) \right) F^{\alpha-1}(x) dx) + \mu$$

$$= (1-a)E(mr(X) F^{\alpha-1}(x)) + \mu.$$  \hspace{1cm} (16)

Then, substituting (16) in (4) we obtain (15). □

**Dynamic survival Rényi entropy**

The residual lifetime through an investigation period takes great attention in survival analysis and reliability fields. Therefore, the measures of information are dynamic. In this section, we propose some properties of DSRE. In the same manner, as SRE, we can deal with DSRE by proposing its lower bound that depends on the MRL by removing $a$ or change its position to make it flexible to study as follows.

**Theorem 6.** Suppose that the non-negative R.V. $X$ has an absolutely continuous CDF $F(x)$. From (5), if $1 \neq a > 0$ and $F(x)$ is integrable, then the lower bound of DSRE with respect to MRL is given by

$$cR(X; t) > \frac{1}{1-a} \log \left( \int_{0}^{\infty} \frac{F(x)}{F(t)} dx \right) = \frac{1}{1-a} \log(mr(X)) = DT_{1}(X)$$

$$> \frac{1}{1-a} \log \left( a \int_{0}^{\infty} \frac{F(x)}{F(t)} dx \right) = \frac{1}{1-a} \log(a mr(X)) = DT_{2}(X).$$  \hspace{1cm} (17)

Fig. 4 gives an example of Theorem 3 for Weibull distribution.

**Theorem 7.** Suppose that the non-negative R.V. $X$ has an absolutely continuous CDF $F(x)$. Then DSRE, defined in (5), can be expressed in terms of MRL as follows

$$cR(X; t) = \frac{1}{1-a} \log \left( (1-a)E(mr(X) F^{\alpha-1}(x) | X > t) + mr(t) \right).$$  \hspace{1cm} (18)

**Proof.** From Rajesh and Sunoj [12], we can write (5) as

$$cR(X; t) = \frac{1}{1-a} \log \left( \int_{0}^{\infty} \frac{F^{\alpha}(x)}{F(t)} dx \right)$$

$$= \frac{1}{1-a} \log \left( \int_{t}^{\infty} \frac{d}{dx} (mr(x) F(x)) F^{\alpha-1}(x) dx \right)$$

$$= \frac{1}{1-a} \log \left( mr(t) + (1-a) \int_{t}^{\infty} \frac{mr(x) F^{\alpha-1}(x) f(x) dx}{F(t)} \right),$$

and the result follows. □

Let $X$ be a non-negative R.V. with CDF $F$. Thus, the failure (hazard) rate function for any time $t$ can be defined as $h(t) = \frac{d}{dt} \frac{f(t)}{F(t)}$ of $X$. In the following, by using the DSRE, we illustrate the hazard rate ordering.

**Proposition 1.** Let $X$ be a non-negative R.V. with mean residual life function $mr(t)$ and hazard rate function $h(t)$. When $X$ is decreasing (increasing) failure rate. Then, from (5), we have

1. For $0 < a < 1$, $cR(X; t) \geq (\leq) \frac{1}{1-a} \log ah(t).$  \hspace{1cm} (19)

2. For $a > 1$, $cR(X; t) \geq (\leq) \frac{1}{1-a} \log ah(t).$  \hspace{1cm} (20)

**Proof.** From (5), by differentiating $cR(X; t)$ with respect to $t$, we have

$$1-a) cR(X; t) = \frac{1}{1-a} \left[ \int_{0}^{\infty} \frac{F^{\alpha}(x)}{F(t)} dx \right] \left[ -1 + ah(t)e^{-1-a \int cR(X; t)} \right],$$

and the result follows. □

**Theorem 8.** Let $X$ and $Y$ be two non-negative continuous R.V.’s with CDF’s $F(t)$ and $G(t)$ and failure rate function $h_{F}(t)$ and $h_{G}(t)$, respectively. If $h_{F}(t) \leq h_{G}(t)$, $t \geq 0$, then

$$cR(X; t) \geq (\leq) cR(Y; t), 0 < a < 1 (a > 1).$$  \hspace{1cm} (22)
Proof. Let $h_T(t) \leq h_C(t)$, which implies that $\left( \frac{F(x)}{G(x)} \right)^a \geq \left( \frac{G(x)}{G(t)} \right)^a$, $\forall a > 0$. Therefore,
\[
\log \int_{t}^{\infty} \left( \frac{F(x)}{G(t)} \right)^a \, dx \geq \log \int_{t}^{\infty} \left( \frac{G(x)}{G(t)} \right)^a \, dx.
\]
(23)
For $0 < a < 1(\alpha > 1)$, we have
\[
\frac{1}{1 - a} \log \int_{t}^{\infty} \left( \frac{F(x)}{G(t)} \right)^a \, dx \geq \left( \leq \frac{1}{1 - a} \log \int_{t}^{\infty} \left( \frac{G(x)}{G(t)} \right)^a \, dx.
\]
Thus, $cR(X; t) \geq (\leq) cR(Y; t)$. □

Survival Rényi entropy of generalized order statistics

Kamps [13] presented the idea of GOS which combined the R.V.'s that are ascendingly ordered. A wider subclass of GOS was developed by Kamps and Cramer [14] in which the parameters of this model are pairwise different. Furthermore, we refer to it as pairwise different GOS (denoted by PD-GOS). The PDF and CDF of PD-GOS are given, respectively, by
\[
f_{X(x; r, k, \bar{m})}(x) = c(r-1) \sum_{i=1}^{r} \frac{p_i(r)}{x_i} (1 - F(x))^{\gamma_i},
\]
(24)
\[
F_{X(x; r, k, \bar{m})}(x) = 1 - c(r-1) \sum_{i=1}^{r} \frac{p_i(r)}{x_i} (1 - F(x))^{\gamma_i},
\]
(25)
where
\[
p_i(r) = \prod_{j=1}^{i} \frac{1}{x_j - x_{j+1}} \quad \gamma_i = k + n - i + \sum_{j=1}^{i-1} m_j > 0, \quad c_{r-1} = \prod_{j=1}^{r} y_j, \quad n \in \mathbb{N}, \quad k \geq 1,
\]
\[
\bar{m} = (m_1, \ldots, m_{r-1}) \in \mathbb{R}^{r-1}, \quad 1 \leq i \leq r \leq n.
\]
From (4) and (25), the SRE and DSRE of PD-GOS are given, respectively, by
\[
cR(X; n, r, k, \bar{m}) = \frac{1}{1 - a} \log \left( \left( \frac{c_{r-1}}{y_{r}} \right)^{y_{r}} \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right) \, dx
\]
(26)
\[
cR(Y; n, r, k, \bar{m}) = \frac{1}{1 - a} \log \left( \left( \frac{c_{r-1}}{y_{r}} \right)^{y_{r}} \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right) \, dx.
\]
(27)
Similarly, from Section “Survival Rényi’s entropy properties”, we can deal with SRE of PD-GOS by proposing its lower bound by removing $a$ from the power. Therefore, suppose that the non-negative R.V. $X$ has an absolutely continuous CDF $F(x)$. From (26) and (27), if $1 \neq a > 0$ and $F(x)$ is integrable, then the lower bound of SRE and DSRE of PD-GOS are given, respectively, by
\[
cR(X; n, r, k, \bar{m}) = \frac{1}{1 - a} \left( a \log c_{r-1} + \log \int_{0}^{\infty} \left( \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right) \, dx \right)
\]
(28)
\[
> \frac{1}{1 - a} \left( \log c_{r-1} + \log \int_{0}^{\infty} \left( \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right) \, dx \right)
\]
(29)
Fig. 5 gives an example of (28) and (29) for Pareto distribution of sequential order statistics such that $y_i = (n - i + 1)a_i$, $a_i = \frac{1}{a}, \forall i = 1, 2, \ldots, n$.

Next, based on PD-GOS, we present some characterization results of DSRE.

Theorem 9. Suppose that the non-negative R.V. $X$ has an absolutely continuous CDF $F(x)$. Then, from (27), the DSRE of PD-GOS denoted by $cR(X; n, r, k, \bar{m})$, $t \geq 0$ characterizes the distribution.

Proof. Suppose that $cR(X; n, r, k, \bar{m}) = cR(Y; n, r, k, \bar{m})$, $1 \leq r \leq n$, $t \geq 0$. From (27), we have
\[
(1 - a) cR(X; n, r, k, \bar{m}) = \int_{0}^{\infty} \left( \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right) \, dx - a \log \left( \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right).
\]
(30)
Differentiating (30) with respect to $t$, we get
\[
(1 - a) cR'(X; n, r, k, \bar{m}) = \int_{0}^{\infty} \left( \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right) \, dx - a \log \left( \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} \right).
\]
(31)
\[
= -e^{(1-a)cR(X; n, r, k, \bar{m})} + \frac{\sum_{i=1}^{r} p_i(r) (F(x))^{\gamma_i-1} f(t)}{\sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i}}
\]
(32)
where
\[
h_{n,r,k,\bar{m}}(f) = \sum_{i=1}^{r} \frac{p_i(r)}{y_i} (F(x))^{\gamma_i} f(t).
\]
Fig. 5. $c_R(X(n,r,k,\delta);t)$, $T_1(X(n,r,k,\delta))$, and $DT_1(X(n,r,k,\delta))$ for Pareto distribution with parameter $\nu = 4$ and $n = 20$, $r = 14$, $t = 4$.

Fig. 6. Histogram of genuine cases of COVID-19 infection.

Fig. 7. Empirical estimators of real data, $0 < \alpha < 1$.

Using

$$h_{(n,r,k,\delta)}(t) = \frac{1 + m'_r(x,k,\delta)(t)}{m_r(X(n,r,k,\delta)(t)},$$

where

$$m_r(X(n,r,k,\delta)(t)) = \frac{\int_0^\infty \hat{F}_X(x,r,k,\delta)(\gamma) d\gamma}{\hat{F}_X(X(n,r,k,\delta)(t))},$$

(33) $m_r(X(n,r,k,\delta)(t)) = \frac{\int_0^\infty \hat{F}_X(x,r,k,\delta)(\gamma) d\gamma}{\hat{F}_X(X(n,r,k,\delta)(t))},$
thus, we get

\[ m_r'_{X(n; r, k, \dot{m})}(t) = \frac{1}{\alpha} \left( (1 - \alpha)e^{t} R(X(n; r, k, \dot{m}); t) \right) m_r'_{X(n; r, k, \dot{m})}(t) - \alpha + m_r'_{X(n; r, k, \dot{m})}(t) \]  

\[ \times e^{(1 - \alpha)(1 - \alpha) t} \]  

Assume that \( c R(X(n; r, k, \dot{m}); t) = c R(Y(n; r, k, \dot{m}); t) = z(t) \). Thus, \((35)\), becomes

\[ m_r'_{X(n; r, k, \dot{m})}(t) = \xi(t, m_r'_{X(n; r, k, \dot{m})}(t)), m_r'_{Y(n; r, k, \dot{m})}(t) = \xi(t, m_r'_{Y(n; r, k, \dot{m})}(t)) \]  

where

\[ \xi(t, y) = \frac{(1 - \alpha)y z'(t) - \alpha + ye^{(1 - \alpha)zt}}{\alpha} \].
From Theorem (2.1) and Lemma (2.2) in Gupta and Kirmani [15], we get $\frac{d}{dt} h_{(n,r,k,\tilde{m})}(t) = m_r F_{X(r,k,\tilde{m})}(t)$, $\forall \ 1 \leq r \leq n, \ t \geq 0$, which proves the characterization.

**Theorem 10.** Let $X(r,k,\tilde{m})$ be the $r$th PD-GOS with survival function $F_{X(r,k,\tilde{m})}(x)$ defined in (25). Then the relationship

$$(1 - \alpha) c R'(X(r,k,\tilde{m}); t) = C h_{(n,r,k,\tilde{m})}(t),$$

holds if and only if

$$h_{(n,r,k,\tilde{m})}(t) = \frac{1}{C t + L},$$

where $h_{(n,r,k,\tilde{m})}(t)$ is defined in (32), $C$ and $L$ are constants.

**Proof.** Let $(1 - \alpha) c R'(X(r,k,\tilde{m}); t) = C h_{(n,r,k,\tilde{m})}(t)$. Using (31), we get

$$(a - C) h_{(n,r,k,\tilde{m})}(t) = e^{-(1-a) R r \times R X(r,k,\tilde{m})}(t).$$

Therefore,

$$\frac{d}{dt} h_{(n,r,k,\tilde{m})}(t) = C \frac{d}{dt} \log \left( \sum_{r=1}^{n} \frac{p_r(t)}{\gamma_r} \left( F_{X(r,k,\tilde{m})}(t) \right)^{\gamma_r} \right).$$

Differentiating (36) with respect to $t$, and after simple calculation, we get

$$\frac{d}{dt} \log h_{(n,r,k,\tilde{m})}(t) = C \frac{d}{dt} \log \left( \sum_{r=1}^{n} \frac{p_r(t)}{\gamma_r} \left( F_{X(r,k,\tilde{m})}(t) \right)^{\gamma_r} \right).$$

Integrating (37) with respect to $t$, we obtain

$$\log h_{(n,r,k,\tilde{m})}(t) = C \log \left( \sum_{r=1}^{n} \frac{p_r(t)}{\gamma_r} \left( F_{X(r,k,\tilde{m})}(t) \right)^{\gamma_r} \right) + T,$$

where $T$ is the constant of integration. In the sequel, differentiating (38) with respect to $t$, we have

$$\frac{d}{dt} \left( \frac{1}{h_{(n,r,k,\tilde{m})}(t)} \right) = C.$$
where \( X_1 \leq X_2 \leq \cdots \leq X_n \) are the order statistic of the random sample and the empirical SRE converges to SRE of \( X \) i.e. \( c(R_X) \to c(X) \) almost surely as \( n \to \infty \). To estimate SRE, we consider the first empirical estimator \( c(R_F) \) as follows

\[
c_{R_F}(F_n) = \frac{1}{1-a} \log \frac{1}{n} \sum_{j=1}^{n-1} (X_{(j+1)} - X_{(j)}) (1 - F_n(x))^a.
\]

where

\[
F_n(x) = \sum_{i=1}^{\lfloor x \rfloor} \frac{1}{n} I(x_{(i)}, x_{(i+1)})(x) + I(x_{\lfloor x \rfloor}, x_{\lfloor x \rfloor})(x), \quad x \in \mathbb{R},
\]

\[I_p(x) \text{ is the indicator function, i.e., } I_p(x) = 1, x \in D, I_p(x) = 0, x \notin D.
\]

Furthermore, the kernel-smoothed estimator (second empirical estimator) \( c(R_F) \) is

\[
c_{R_F}(F_n) = \frac{1}{1-a} \log \frac{1}{n} \int X_{(j+1)} (1 - F_n(x))^a \, dx
\]

\[
= \frac{1}{1-a} \log \frac{1}{n} \sum_{j=1}^{n-1} (X_{(j+1)} - X_{(j)}) (1 - F_n(x))^a.
\]

where

\[
F_n(X_{(j)}) = \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{X_{(j)} - X_i}{\theta} \right).
\]

\[h(x) = \int_{-\infty}^{x} K(t) \, dt \quad \theta \quad \text{is a bandwidth parameter, and } K(x) \text{ is the standard normal density function, for more details see Nadaraya [16].}
\]

To evaluate the mean of the empirical SRE given in (43) and (44) of exponential distribution with CDF \( F(x) = 1 - e^{-\theta x}, \theta > 0, x \geq 0 \), denoted by \( EXP(\theta) \). Recalling, the sample spacings \( W_j = X_j - X_{j-1}, j = 1, 2, \ldots, n-1 \), are independent with \( EXP(\theta(n-j)) \). Therefore, from Jensen’s inequality and the logarithm function is concave, we get

\[
\mathbb{E}(c(R_F))
\]

\[
= \frac{1}{1-a} \log \frac{1}{n} \sum_{j=1}^{n-1} (X_{(j+1)} - X_{(j)}) (1 - F_n(x))^a
\]

\[
\leq (2a-1-1) \log \frac{1}{n} \sum_{j=1}^{n-1} \mathbb{E}(X_{(j+1)} - X_{(j)}) (1 - F_n(x))^a
\]

\[
= \frac{1}{1-a} \log \frac{1}{n} \sum_{j=1}^{n-1} (1 - F_n(x))^a = \mathbb{E}(c(R_F)), 0 < a < 1(a > 1).
\]

\[
\mathbb{E}(c(R_F))
\]

\[
= \frac{1}{1-a} \log \frac{1}{n} \sum_{j=1}^{n-1} (X_{(j+1)} - X_{(j)}) (1 - F_n(x))^a
\]

\[
\leq (2a-1-1) \log \frac{1}{n} \sum_{j=1}^{n-1} \mathbb{E}(X_{(j+1)} - X_{(j)}) (1 - F_n(x))^a
\]

\[
= \frac{1}{1-a} \log \frac{1}{n} \sum_{j=1}^{n-1} (1 - F_n(x))^a = \mathbb{E}(c(R_F)), 0 < a < 1(a > 1).
\]

We can study and compare the means of \( c(R_F) \) and \( c(R_F) \) from their lower and upper bounds \( \mathbb{E}(c(R_F)) \) and \( \mathbb{E}(c(R_F)) \) respectively.

**Tables 1 and 2** present the approximate (alternative) means of the empirical SRE from EXPM(1), by using different values of sample size \( n = 5, 10, 20, 30, 50, 70, 100 \). In Table 1, we conclude the following

1. For fixed \( n \), \( c(R_F) \) decreases when \( a \) increases.
2. For fixed \( 0 < a < 1 \), \( c(R_F) \) increases when \( n \) increases. Furthermore, for fixed \( a > 1 \), \( c(R_F) \) decreases when \( n \) increases.

**Real-life data**

In this subsection, we demonstrate our empirical estimators in actual and simulated data for EXPM(1) distribution.
Example 1. To investigate the spreading patterns of the COVID-19 pandemic, Kasilingam et al. [17] employs exponential model to identify countries with early containment till March 26, 2020. The data show the percentage of infected cases in 42 countries, as listed below.

1.56, 8.51, 2.17, 0.37, 1.09, 9.84, 4.95, 3.18, 11.37, 2.81, 6.22, 1.87, 0.00, 0.00, 9.05, 2.44, 1.38, 4.17, 3.74, 1.37, 2.33, 7.80, 2.10, 0.47, 2.54, 4.92, 0.09, 0.18, 1.72, 1.02, 0.62, 2.34, 0.50, 2.37, 3.65, 0.59, 5.76, 2.14, 0.88, 0.95, 4.17, 2.25.

We utilize the test of Kolmogorov–Smirnov (K–S) to examine the fitting of the information for $EX(P(1))$ distribution which infers that the K–S statistic is 0.076282 with $p$-value 0.9674. In this way, it is conceded to fit the data by $EXP(1)$ distribution, moreover, see Fig. 6. Under $EXP(1)$ distribution, Table 3 give the theoretical value of SRE $cR(\alpha)$ and the empirical estimators $cR_1(F_n)$ and $cR_2(F_n)$ of different values of $\alpha$ for the previous real-life data. Therefore, we conclude the following

1. For $0 < \alpha < 1$, $cR_1(F_n)$ and $cR_2(F_n)$ become nearer to $cR(\alpha)$ when $\alpha$ decreases.
2. For $\alpha > 1$, $cR_1(F_n)$ and $cR_2(F_n)$ become nearer to $cR(\alpha)$ when $\alpha$ increases.
3. The difference of the values between $cR_1(F_n)$ and $cR_2(F_n)$ are small and they almost have the same effect.

Figs. 7 and 9 display the real-life and mimicked data, respectively, when $\alpha > 1$. Therefore, by increasing $n$ and decreasing $\alpha$, the empirical estimators approach the theoretical value. Moreover, Figs. 8 and 10 give the real-life and mimicked data, respectively, when $\alpha > 1$. Thus, by increasing $n$ and $\alpha$, the empirical estimators become nearer to the theoretical value.

Conclusion

In this work, the lower bounds with examples of SRE and DSRE were proposed. Those lower bounds are represented in terms of a closed-form which makes them easy to deal with. Some stochastic ordering for DSRE depending on the hazard rate was obtained. Some characterization and results of DSRE, based on PD-GOS, were determined. A non-parametric estimation presented two empirical estimators of SRE was considered. We inferred that the suggested estimators are influenced by the value of $\alpha$ and the sample size. In addition, they have almost the same value. In future work, we can extend this work to various types of measures such as Tsallis entropy; for more details, see [18–24].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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