RICCI FLOW AND NONNEGATIVITY OF CURVATURE

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Abstract. In this paper, we prove a general maximum principle for the time dependent Lichnerowicz heat equation on symmetric tensors coupled with the Ricci flow on complete Riemannian manifolds. As an application we construct complete manifolds with bounded nonnegative sectional curvature of dimension greater than or equal to four such that the Ricci flow does not preserve the nonnegativity of the sectional curvature, even though the nonnegativity of the sectional curvature was proved to be preserved by Hamilton in dimension three. The example is the first of this type. This fact is proved through a general splitting theorem on the complete family of metrics with nonnegative sectional curvature, deformed by the Ricci flow.

§0 Introduction.

The Ricci flow has been proved to be an effective tool in the study of the geometry and topology of manifolds. One of the good properties of the Ricci flow is that it preserves the ‘nonnegativity’ of the curvature. In dimension three, Hamilton [H1] proves that on compact manifolds the Ricci flow preserves the nonnegativity of the Ricci curvature and the sectional curvature. Using this property and the quantified version, curvature pinching estimate, it was proved in [H1] that the normalized Ricci flow converges to a Einstein metric if the initial metric admits positive Ricci curvature. In particular, it implies that a simply-connected compact three-manifold is diffeomorphic to the three sphere if it admits a metric with positive Ricci curvature. One can refer [Ch] for an updated survey and [P2] for some recent development on the Ricci flow on three manifolds. Later in [H2] it was proved that the Ricci flow also preserves the nonnegativity of the curvature operator in high dimension on compact manifolds. In the Kähler case, Bando and Mok [B, M] proved that the flow also preserves the nonnegativity of the holomorphic bisectional curvature. The Ricci flow on complete manifold was initiated in [Sh2]. In [Sh3] Shi generalized the above mentioned result of Bando and Mok to the complete Kähler manifolds with bounded curvature. Interesting applications were also obtained therein.

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In this paper, we shall study the topological consequences of the assumption that Ricci flow preserves the nonnegativity of the sectional curvature on complete Riemannian manifolds. The basic method is to study the heat equation, time dependent, deformation of the Busemann function via the optimal tensor maximum principle proved in [NT3]. The maximum principle of this type was first proved by Hamilton for compact manifolds [H2]. The proof of [H2] can be generalized to the complete noncompact manifolds with bounded curvature with additional assumption that the tensor satisfying certain heat equation is uniformly bounded on the space-time. See for example [NT2, Proposition 1.1]. But in order to study the deformation of continuous functions, in our case, the Busemann functions, one can not expect uniform point-wise control on its Hessian since it is not even differentiable. Therefore one needs a maximum principle assuming only integral bounds on the tensor considered. This is the main technical difficulty. This difficulty was resolved in [NT3] and an optimal maximum principle was established there for the time-independent heat equation. The tensor maximum principle proved here in Theorem 2.1 is a time dependent version of the one in [NT3] (cf. Theorem 2.1 of [NT3]).

By studying the deformation of the Busemann function, we shall prove that on a simply-connected complete Riemannian manifolds with bounded nonnegative sectional curvature, if the Ricci flow preserves the nonnegativity of the sectional curvature, then the manifold splits as the product of a compact manifold with nonnegative sectional curvature with a complete manifold which is diffeomorphic to the Euclidean space. As a corollary, we give examples of complete Riemannian manifolds with bounded nonnegative sectional curvature of dimension $\geq 4$ such that the Ricci flow does not preserve the nonnegativity of the sectional curvature. As far as we know, this is the first example of this kind. Noticing that in dimension three the Ricci flow does preserve the nonnegativity of the sectional curvature by [H1] on compact manifolds and complete manifolds with bounded curvature. Another application of our approach is a classification of complete manifolds with bounded nonnegative curvature operator, a result which has been previously established in [N] using different methods without assuming the boundedness of the curvature (see also [CY] for the compact case). The use of the heat equation deformation of Busemann functions to study the structure of complete manifolds was initiated in [NT3]. Therefore this paper can be viewed as a continuation of the pervious work. The difference between this one and [NT3] is that we have to consider the heat equation with metrics evolved by the Ricci flow in order to have nice heat equation for the the Hessian of its solution. Therefore we have to derive the heat kernel estimate of Li-Yau type (cf. [LY]) for the time dependent heat equation. The estimate of this type was considered before in [Sa] by Saloff-Coste. However, the heat equation we are considering does not belong to the classes considered in [Sa] (see Remark 1.1 for more details). Therefore we devote the first section in establishing the heat kernel estimate as well as the Harnack inequality for the time dependent heat equation, following the approach of Grigor’yan in [Gr1]. The result itself might has its own interests.

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§1 Time-dependent heat equation.

Let \((M, g^0_{ij}(x))\) be a complete Riemannian manifold (of dimension \(n\)) with bounded curvature tensor. We denote \(k_0\) to be the upper bound of \(|R_{ijkl}|^2\), the curvature tensor of \(g^0\). By [Sh2, Theorem 1.1, p. 224] we know that there exists a constant \(T(n, k_0) > 0\) such that the Ricci flow

\[
\frac{\partial}{\partial t} g_{ij}(x, t) = -2R_{ij}(x, t)
\]

has solution on \(M \times [0, T]\). Moreover, there exists \(A'_{m} = A'_{m}(n, m, k_0)\) such that

\[
\|\nabla^m R_{ijkl}\|^2(x, t) \leq \frac{A'_m}{t^m}.
\]

In particular,

\[
\|R_{ijkl}\|(x, t) \leq \sqrt{A_0}.
\]

Moreover, \(g_{ij}(x, t)\) has nonnegative curvature operator if the initial metric \(g_{ij}(x, 0)\) has the nonnegative curvature operator. We are going to study the initial value problem of the heat equation

\[
\left(\Delta - \frac{\partial}{\partial t}\right) v = 0.
\]

with initial value \(v(x, 0) = u(x)\). Here \(\Delta v = g^{ij}(x, t)v_{ij}\) with \(v_{ij}\) denoting the Hessian of \(v\). Namely \(\Delta\) is time-dependent. The following lemma is well-known to experts. For example, it was known and used in [CH] by Chow and Hamilton in their study of the linear trace Harnack inequality for the Ricci flow.

**Lemma 1.1.** Let \(v(x, t)\) be a solution to (1.4). Then the complex Hessian \(v_{ij}(x, t)\) satisfies

\[
\left(\frac{\partial}{\partial t} - \Delta\right) v_{ij} = 2R_{ipjq}v_{pq} - R_{ip}v_{pj} - R_{pj}v_{ip}.
\]

**Proof.** Direct calculation, using formulae on page 274 of [H1], one has that

\[
(v_{ij})_t = (v_t)_{ij} + (\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij}) v_k.
\]

On the other hand, the commutator calculation shows that

\[
v_{ijkk} = v_{kkij} + (-\nabla_s R_{ij} + \nabla_i R_{js} + \nabla_j R_{is}) v_s + R_{is}v_{sj} + R_{js}v_{is} - 2R_{isjk}v_{sk}.
\]

Now using (1.4) we have \((v_t)_{ij} = v_{kkij}\). Then lemma follows from (1.6) and (1.7).
Corollary 1.2. Denote $\eta$ be the symmetric tensor $v_{ij}$. Denote $\|\eta\|^2$ the norm of $v_{ij}$ with respect to $g_{ij}(x, t)$. Then $\exp(-2\sqrt{A_0 t})\|\eta\|(x, t)$ is a subsolution of (1.4).

Proof. Direct calculation shows that
\[
\left(\Delta - \frac{\partial}{\partial t}\right)\|\eta\|^2 \geq -4R_{ipjq}\eta_{pq}\eta_{ij} + 4R_{ip}\eta_{pk}\eta_{ik} + 2\|\nabla \eta\|^2 - 4R_{ip}\eta_{pk}\eta_{ik}
\]
\[
\geq 2\|\nabla \eta\|^2 - 4\sqrt{A_0}\|\eta\|^2.
\]

Here we have used Lemma 1.1. The claim of the corollary follows easily.

In the following we collect some fundamental results on solution (subsolutions) of (1.4). Our basic assumption is (1.3). For the purpose of the later section we also assume $T \leq 1$ and $g_{ij}(x, 0)$ has nonnegative Ricci curvature. By (1.1) and (1.3) we know that
\[
(1.8) \quad C(n, A_0)g_{ij}(x, 0) \leq g_{ij}(x, t) \leq g_{ij}(x, 0).
\]

Since $g_{ij}(x, 0)$ has nonnegative Ricci curvature, by (1.8), for any $0 \leq t \leq T$, we still have the following Neumann type Poincaré inequality.

Lemma 1.2. Let $(M, g_{ij}(x, t))$ be a solution to the Ricci flow such that the initial metric $g_{ij}(x, 0)$ has nonnegative Ricci curvature. For any domain $\Omega \subset B_0(y, R)$ and any Lipschitz function $\varphi$ on $\bar{\Omega}$ vanishes on $\partial \Omega$
\[
(1.9) \quad \int_{\Omega} |\nabla \varphi|^2 \geq \frac{b}{R^2} \left(\frac{V_x(R)}{|\Omega|}\right)^\beta \int_{\bar{\Omega}} \varphi^2
\]
for some positive constants $\beta, b$ which only depends on $n$ and $A_0$. Here $|\nabla \varphi|^2$ is calculated using $g_{ij}(x, t)$, while $|\Omega|$ and $V_x(R)$ are calculated using $g_{ij}(x, 0)$.

Proof. The lemma follows easily from Theorem 1.4 of [Gr1]. The point is that only the weak form Neumann-Poincaré inequality and the volume doubling property are needed in the proof of Theorem 1.4 of [Gr1]. Since $g_{ij}(x, 0)$ has nonnegative Ricci curvature these two properties hold for $(M, g_{ij}(x, 0))$. On the other hand, the metric $g_{ij}(x, t)$ is equivalent to $g_{ij}(x, 0)$. Therefore these two sufficient properties preserve.

The next result is a mean value inequality. The proof is just a modification of the one for the time-independent heat equation case in [Gr1]. Note that it is known from [H1] that the scalar curvature $\mathcal{R}(x, t)$ of $g_{ij}(x, t)$ is nonnegative, under the assumption that $g_{ij}(x, 0)$ has nonnegative Ricci, therefore scalar, curvature.

Theorem 1.1. Let $(M, g(t))$ be as in Lemma 1.2. Let $w(x, t)$ be a smooth function satisfying
\[
(1.10) \quad \left(\Delta - \frac{\partial}{\partial t}\right) w \geq 0
\]
on $\prod \sqrt{t}$ with $t \leq T$, where $\prod R = B_0(x, R) \times (0, R^2)$ and $B_\tau(x, \sqrt{t})$ is the ball of radius $\sqrt{t}$ with respect to $g_{ij}(x, \tau)$. Then

$$w_+^2(x, t) \leq \frac{C(n, A_0, T)}{V_x(\sqrt{t})t} \int_0^t \int_{B_0(x, \sqrt{t})} w_+^2(y, \tau) \, dy \, d\tau$$

Here $w_+$ is the positive part of $v$.

**Proof.** Here we basically follow the argument of the proof of Theorem 3.1 in [Gr1]. The key to the argument is the fact that $g_{ij}(x, t)$ satisfying the Neumann-Poincaré inequality (1.9) and the volume double property. We have these two properties if we assume that the initial metric has nonnegative Ricci curvature. To make the iteration argument work using (1.9) and the volume double property. We have these two properties if we assume that the initial metric has nonnegative Ricci curvature. To make the iteration argument work using Lemma 1.2 we need also to prove that the Lemma 3.1 of [Gr1] still holds. In fact, for any initial metric has nonnegative Ricci curvature. To make the iteration argument work using (1.9) and the volume double property. We have these two properties if we assume that the initial metric has nonnegative Ricci curvature. To make the iteration argument work using Lemma 1.2 we need also to prove that the Lemma 3.1 of [Gr1] still holds. In fact, for any $R \leq \sqrt{t}$, let $\phi(x, t)$ be a cut-off function on $B_0(x, R)$ such that $\phi(x, 0) = 0$. For $\theta > 0$, let $w_\theta = (w - \theta)_+$. Multiplying $w_\theta \phi^2$ on both side of (1.10) we have that

$$\int_{\{w \geq \theta\}} w_\theta \phi^2 \leq \int_{\{w \geq \theta\}} (\Delta w) \phi^2$$

$$= -2 \int_{\{w \geq \theta\}} \nabla w_\theta, \nabla \phi \phi + \int_{\{w \geq \theta\}} |\nabla w_\theta|^2 \phi^2$$

$$= -\int_{\{w \geq \theta\}} |\nabla (w_\theta \phi)|^2 + \int_M |\nabla \phi|^2 w_\theta^2.$$

Integrating the time variable and noticing that $\phi \in C^\infty_0(B_0(x, R))$ we have that

$$\int_0^t \int_{B_0(x, R)} w_\theta(w_\theta) \phi^2 \leq -\int_0^t \int_{B_0(x, R)} |\nabla (w_\theta \phi)|^2 + \int_0^t \int_{B_0(x, R)} |\nabla \phi|^2 w_\theta^2.$$

The left hand side equals to

$$\frac{1}{2} \int_0^t \int_{B_0(x, R)} (w_\theta^2)_{\tau} \phi^2 = \frac{1}{2} \int_{B_0(x, R)} w_\theta^2 \phi_0^2 + \int_0^t \int_{B_0(x, R)} w_\theta^2 \left(-\phi_\tau \phi + \frac{1}{2} \mathcal{R}(y, \tau) \phi^2\right).$$

Combining the above two inequalities and using the fact $\mathcal{R} \geq 0$ we have that

$$\int_{B_0(x, R)} w_\theta^2 \phi^2(y, t) \, dy + \int_0^t \int_{B_0(x, R)} |\nabla (w_\theta \phi)|^2 \leq 2 \int_0^t \int_{B_0(x, R)} w_\theta^2 \left(|\nabla \phi|^2 + |\phi_\tau|\right).$$

Similarly, one can prove Lemma 3.2 of [Gr1], noticing that Lemma 1.2 holds for metric $g_{ij}(x, t)$. Then the iteration scheme in [Gr1] can be applied to complete the proof of the theorem.

For the Harnack inequality, let $v$ be a positive solution to (1.4) on $\prod_{8R}$ where $\prod R = B_0(x, R) \times (0, R^2)$. 

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**Theorem 1.2.** Let \((M, g_{ij}(x, t))\) be as in Lemma 1.2. Then there exists a constant \(\gamma = \gamma(n, A_0) > 0\) such that

\[
v(x, 64R^2) \geq \gamma \sup_{B_0(x, R) \times (3R^2, 4R^2)} v.
\]

**Proof.** The proof follows similarly as the proof of Theorem 4.1 in [Gr1]. Since Lemma 4.2-4.4 in [Gr1] are robust enough to be adapted to current situation we only need to establish the following result which corresponds to Lemma 4.1 of [Gr1]. We can assume that \(\sup_{\Pi} v = 1\).

**Lemma 1.3.** Let \(v(x, t)\) be a positive solution to (1.4) in \(\Pi_{2R}\) and set

\[
H = \{(x, t) \in \Pi_R : v(x, t) > 1\}, \quad \tilde{\Pi}_R = B_0(x, R) \times (3R^2, 4R^2).
\]

Then for any \(\delta > 0\) there exists \(\epsilon = \epsilon(\delta, A_0, n)\) such that if

\[
|H| \geq \delta|\Pi_R|,
\]

then

\[
\inf_{\Pi_R} v \geq \epsilon.
\]

Here \(|H|\) and \(|\tilde{\Pi}_R|\) are measured with respect to the metric \(g_{ij}(x, 0)\).

**Proof.** We have similar situation as in the proof of Theorem 1.1. The argument follows closely as in [Gr1]. Let \(h = \log(1/v)\). It is easy to have that \((\frac{\partial}{\partial t} - \Delta) h = -|\nabla h|^2\). For a cut-off function \(\phi(x)\), we have that

\[
\frac{\partial}{\partial t} \left( \int_{B_0(x, R)} h_+ \phi^2 \right) = \int_{B_0(x, R)} (h_+) t \phi^2 - \int_{B_0(x, R)} h_+ \phi^2 \mathcal{R}
\]

\[
\leq \int_{B_0(x, R)} (h_+) t \phi^2
\]

\[
\leq \int_{B_0(x, R)} (\Delta h_+) \phi^2 - |\nabla h_+|^2 \phi^2
\]

\[
\leq -\frac{1}{2} \int_{B_0(x, R)} |\nabla h_+|^2 \phi^2 + 2 \int_{B_0(x, R)} |\nabla \phi|^2.
\]

This is the (4.3) of [Gr1]. The rest of the proof follows verbatim as in the proof of [Gr1, Lemma 4.1].

One has the following immediate corollary of the above theorem.
Corollary 1.2. Let $v(x,t)$ be a weak positive solution to (1.4) on $M \times [0,T]$. Then

$$\frac{v(y,s)}{v(x,t)} \leq \exp \left( C \left( \frac{r^2(x,y)}{t-s} + \frac{t}{s} + 1 \right) \right).$$

Here $C = C(\gamma) > 0$.

Proof. This was proved, for example in [Mo, page 110-112].

Theorem 1.3. Let $(M, g_{ij}(t))$ be as the above. Let $H(x,y,t)$ be the minimal positive heat kernel of the heat equation (1.4). Then there exist positive constants $C_1, C_2$ and $D$ such that

$$C_1 \frac{1}{V_0(x,\sqrt{t})} \exp \left( -D \frac{r^2(x,y)}{t} \right) \leq H(x,y,t) \leq C_2 \frac{1}{V_0(x,\sqrt{t})} \exp \left( -D \frac{r^2(x,y)}{t} \right).$$

Here $V_0(x,a)$ and $r(x,y)$ denote the volume of $B_0(x,a)$ and distance between $x$ and $y$, with respect to $g_{ij}(x,0)$, respectively. $D > 4$ is an absolute constant. $C_i = C_i(n,D)$.

Proof. Let $H(x,y,t)$ be the minimal positive heat kernel of $\frac{\partial}{\partial t} - \Delta$. It is easy to see that for any $t > 0$,

$$\int_M H(x,y,t) \, dy_0 \leq 1.$$ 

Here $dy_0$ is the volume element with respect to the metric at time $t = 0$. Fix a point $z \in M$ and let $u(x,t) = H(x,z,t)$. Then there exists a point $y \in B_0(z,2\sqrt{t})$ such that

$$u(y,2t) \leq \frac{1}{V_0(z,2\sqrt{t})}.$$

Applying the Harnack and the volume doubling property we have that

$$u(z,t) \leq \frac{C(n)}{V_0(z,\sqrt{t})}.$$ 

Therefore we have the upper bound for $H(x,x,t)$. The upper bound in (1.17) follows from a general result of Grigor’yan [Gr2, Theorem 1.1]. The lower bound can be obtained using the argument in [Gr1, page 73]. Let $\phi$ be a cut-off function such that $\phi = 1$ on $B_0(y,\frac{1}{2}\sqrt{t})$ and $\phi = 0$ outside $B_0(y,\sqrt{t})$. Now define

$$w(x,s) = \int_M H(x,y,s)\phi(y) \, dy_0$$

for $s \geq 0$ and $w(x,s) \equiv 1$ for $s \leq 0$. Then $w(x,s)$ is a solution to the heat equation on $B_0(y,\frac{\sqrt{t}}{2}) \times (-\infty, \infty)$. Here we have extend the metric to be $g_{ij}(x,0)$ for $s \leq 0$. Applying
the Harnack inequality (1.16) we have that

\[ 1 = u(y, 0) \leq C(n)u(y, \frac{\sqrt{t}}{2}) \]

\[ = C(n) \int_M H(y, z, \frac{t}{2})\phi(z) \, dz_0 \]

\[ \leq C(n) \int_{B_0(y, \sqrt{t})} H(y, z, \frac{t}{2}) \, dz_0 \]

\[ \leq C(n) \int_{B_0(y, \sqrt{t})} H(y, y, t) \, dz_0 \]

\[ \leq C(n)H(y, y, t)V_0(y, \sqrt{t}). \]

This gives the lower bound for \( H(x, x, t) \). The general form in (1.17) is just another application of the Harnack inequality, or Corollary 1.2.

Remark 1.1. In [Sa], the above Theorem 1.2 and Theorem 1.3 were proved for the parabolic operator of type \( \frac{\partial}{\partial t} - L \), with

\[ Lf = m^{-1} \text{div}(mA(\nabla f)), \]

where \( m \) is a measure independent of \( t \), \( A \) is a measurable section of \( \text{End}(T_M) \) which is uniformly equivalent to the identity. The time dependent Laplacian operator can only expressed in the above form with time dependent measure \( \sqrt{\det(g_{ij}(x, t))}dx_1 \wedge \cdots \wedge dx_n \). Therefore one can not just apply the results of [Sa] directly. This also is the reason that we need the scalar curvature of \( g_{ij}(x, t) \) is nonnegative to make the argument work. One could also prove the above theorems following the approach of Moser as in [Sa].

§2 A maximum principle for tensors and its applications.

In this section we shall prove a maximum principle for the symmetric tensors satisfying (1.5) under the assumption that \((M, g_{ij}(x, t))\) has bounded nonnegative sectional curvature. Since the argument is very close to that in [NT3] we will be sketchy here.

Let \( \eta_{ij} \) be a symmetric tensor satisfying (1.5). The basic assumption on \( \eta \) is that there exists a constant \( a > 0 \) such that

\[ (2.1) \quad \int_M \| \eta \|(x, 0) \exp(-ar^2(x)) \, dx < \infty \]

and

\[ (2.2) \quad \liminf_{r \to \infty} \int_0^T \int_{B_0(0, r)} \| \eta \|^2(x, t) \exp(-ar^2(x)) \, dx \, dt < \infty. \]

Here \( \| \eta \|(x, t) \) is the norm of \( \eta_{ij}(x, t) \) with respect to metrics \( g_{ij}(x, t) \). But \( B_0(0, r) \) is the ball with respect to the initial metric \( g_{ij}(x, 0) \) and \( r(x) \) is the distance from \( x \) to a fixed point \( o \in M \) with respect to the initial metric. Due to the fact that the maximum
principle for the heat equation does not hold on complete manifolds in general, one needs some growth conditions on the solutions to make it true. The condition (2.2) is optimal by comparing to the example given in [J, page 211-213]. The classical example there is a solution to the heat equation on $\mathbb{R} \times [0, \infty)$, which has zero initial data. The violation of the uniqueness implies the failure of the maximum principle for the solutions. The example has growth, as $|x| \to \infty$, just faster than $\exp(ar^2(x))$. The condition (2.1) is needed to ensure that the equation (1.5) does have a solution indeed. It is also in the sharp form.

Before we state our result, let us first fix some notations. Let $\phi : [0, \infty) \to [0, 1]$ be a smooth function so that $\phi \equiv 1$ on $[0, 1]$ and $\phi \equiv 0$ on $[2, \infty)$. For any $x_0 \in M$ and $R > 0$, let $\phi_{x_0, R}$ be the function defined by 

$$\phi_{x_0, R}(x) = \phi \left( \frac{r(x, x_0)}{R} \right).$$

Again, $r(x, y)$ denotes the distance function of the initial metric. Let $f_{x_0, R}$ be the solution of 

$$\left( \frac{\partial}{\partial t} - \Delta \right) f = -f$$

with initial value $\phi_{x_0, R}$. Then $f_{x_0, R}$ is defined for all $t$ and is positive and bounded for $t > 0$. In fact 

$$f_{x_0, R}(x, t) = e^{-t} \cdot \int_M H(x, y, t) \phi_{x_0, R}(y) dy_0.$$

We shall establish the following maximum principle.

**Theorem 2.1.** Let $(M, g_{ij}(x, t))$ be a complete noncompact Riemannian manifolds satisfying (1.1)-(1.3), with nonnegative sectional curvature. Let $\eta(x, t)$ be a symmetric tensor satisfying (1.5) on $M \times [0, T]$ with $0 < T < \frac{1}{\text{vol}}$ such that $||\eta||$ satisfies (2.1) and (2.2). Suppose at $t = 0$, $\eta_{ij} \geq -bg_{ij}(x, 0)$ for some constant $b \geq 0$. Then there exists $0 < T_0 < T$ depending only on $T$ and $a$ so that the following are true.

(i) $\eta_{ij}(x, t) \geq -bg_{ij}(x, t)$ for all $(x, t) \in M \times [0, T_0]$.

(ii) For any $T_0 > t' \geq 0$, suppose that there is a point $x'$ in $M^m$ and there exist constants $\nu > 0$ and $R > 0$ such that the sum of the first $k$ eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\eta_{ij}$ satisfies 

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k \phi_{x', R}$$

for all $x$ at time $t'$. Then for all $t > t'$ and for all $x \in M$, the sum of the first $k$ eigenvalues of $\eta_{\alpha\beta}(x, t)$ satisfies 

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k f_{x', R}(x, t - t').$$

**Proof.** The complex version of Theorem 2.1 was proved in [NT3, Theorem 2.1]. The key step of the argument is to construct the barrier

$$h(x, t) = \int_M H(x, y, t) ||\eta||(y, 0) dy_0.$$
and $h_R(x, t)$ below to control $\|\eta\|(x, t)$ on big annulus. It is easy to see that $h(x, t)$ is a solution to (1.4). Using Lemma 1.4, the assumption (2.2) and the maximum principle of [NT1] we have that

\begin{equation}
\exp(-2\sqrt{A_0t})\|\eta\|(x, t) \leq h(x, t).
\end{equation}

Let $A_0(o, r_1, r_2)$ denote the annulus $B_0(o, r_2) \setminus B_0(o, r_1)$. For any $R > 0$, let $\sigma_R$ be a cut-off function which is 1 on $A_0(o, \frac{R}{4}, 4R)$ and 0 outside $A_0(o, \frac{R}{8}, 8R)$. We define

$$h_R(x, t) = \int_M H(x, y, t)\sigma_R(y)\|\eta\|(y, 0)dy_0.$$ 

Then $h_R$ satisfies the heat equation with initial data $\sigma_R\|\eta\|$. Since the proof of Lemma 2.3 of [NT3] only uses the heat kernel upper bound it remains to be true due to Theorem 1.3. In particular, $h(x, t) \leq \exp(2\sqrt{A_0T})(h_R(x, t) + \tau(R))$ on $A_0(o, \frac{R}{2}, 2R)$ with $\tau(R) \to 0$ as $R \to \infty$. And $h_R(x, t) \to 0$ as $R \to \infty$ on any compact subset of $M$. Now using

$$\exp(2\sqrt{A_0T})(h_R(x, t) + \tau(R))$$

as the barrier the proof of Theorem 2.1 follows verbatim as the corresponding result in [NT3]. Notice that the key inequality (2.14) in [NT3], still holds under the assumption that $g_{ij}(x, t)$ has nonnegative sectional curvature (see also the proof of Theorem 2.2 following).

The similar maximum principle for the scalar heat equations is relatively easy to prove. They also require an assumption as (2.2). The time dependent case was first proved in [NT1] following the original argument for the time-independent case in [L]. As an application we have the following approximation result.

**Theorem 2.2.** Let $(M, g_{ij}(x, t))$ be as above. Let $u(x)$ be a Lipschitz continuous convex function satisfying

\begin{equation}
|u|(x) \leq C \exp(ar^2(x))
\end{equation}

for some positive constants $C$ and $a$. Let $v(x, t)$ be the solution to the time-dependent heat equation (1.4). There exists $T_0 > 0$ depending only on $a$ and there exists $T_0 > T_1 > 0$ such that the following are true.

(i) For $0 < t \leq T_0$, $v(\cdot, t)$ is a smooth convex function (with respect to $g_{ij}(x, t)$).

(ii) Let

$$\mathcal{K}(x, t) = \{w \in T_x^{1,0}(M) | v_{ij}(x, t)w^i = 0, \text{ for all } j\}$$

be the null space of $v_{ij}(x, t)$. Then for any $0 < t < T_1$, $\mathcal{K}(x, t)$ is a distribution on $M$. Moreover the distribution is invariant in time as well as under the parallel translation.

In order to prove the above theorem we need the following approximation result due to Greene-Wu [GW3, Proposition 2.3].
Lemma 2.1. Let $u$ be a convex function on $M$. Assume that $u$ is Lipschitz with Lipschitz constant $1$. For any $b > 0$, there is a $C^\infty$ convex function $w$ such that

(i) $|w(x) - w(y)| \leq r(x, y)$;
(ii) $|w - u| \leq b$ on $M$; and
(iii) $w_{\alpha \bar{\beta}} \geq -bg_{\alpha \bar{\beta}}$ on $M$.

Proof of Theorem 2.2. Once we have Lemma 2.1 and Theorem 2.1, the proof follows as the proof of Theorem 2.1, Corollary 2.1 and Theorem 3.1 of [NT 3]. The key fact is that under the assumption $K_{ijij} \geq 0$, for a choice of the orthogonal frame such that for the tensor $\eta_{ij}$ diagonalized at a fixed point $(x_0, t_0)$ with its eigenvalues $\lambda_i$ of $\eta$ ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$,

$$
\sum_{i,j=1}^{k} \left[ 2R_{ipjq} \eta_{pq} - R_{ip} \eta_{pj} - R_{pj} \eta_{ip} \right] g^{ij}
$$

$$
= 2 \left( \sum_{i=1}^{k} \frac{1}{n} R_{ip}\lambda_p - \sum_{i=1}^{k} R_{ii}\lambda_i \right)
$$

$$
= 2 \left( \sum_{i=1}^{k} \sum_{p=1}^{n} R_{ip}\lambda_p - \sum_{i=1}^{k} \sum_{p=1}^{n} R_{ip}\lambda_i \right)
$$

$$
= 2 \left( \sum_{i=1}^{k} \sum_{p=k+1}^{m} \lambda_p R_{ip} - \sum_{i=1}^{k} \sum_{p=k+1}^{m} R_{ip}\lambda_i \right)
$$

$$
= 2 \left( \sum_{i=1}^{k} \sum_{p=k+1}^{m} R_{ip} (\lambda_p - \lambda_i) \right)
$$

$\geq 0$.

The following is the main result on the structure of solutions to the Ricci flow preserving the nonnegativity of the sectional curvature.

Theorem 2.3. Let $(M, g_{ij}(x, t))$ be solution to the (1.1) satisfying (1.3) with nonnegative sectional curvature. Assume also that $M$ is simply-connected. Then $M$ splits isometrically as $M = N \times M_1$, where $N$ is a compact manifold with nonnegative sectional curvature. $M_1$ is diffeomorphic to $\mathbb{R}^k$. For the restriction of metric $g_{ij}(x, t)$ on $M_1$ with $t > 0$, there is a strictly convex exhaustion functions on $M_1$. Moreover, the soul of $M_1$ is a point and the soul of $M$ is $N \times \{o\}$, where $o$ is the soul of $M_1$.

Proof. Let $B$ be the Busemann function on $M$, with respect to the initial metric $g_{ij}(x, 0)$. As it was proved in [CG, GW2] that $B$ is a convex Lipschitz function with constant 1. Also it is an exhaustion function on $M$. In fact $B(x) \geq cr(x)$ when $r(x)$ is sufficient large, for some $C > 0$, where $r(x)$ is the distance function to a fixed point $o \in M$. Let $v(x, t)$ be the solution of (1.4) with $v(x, 0) = B(x)$. Under the assumption that $K_{ijij} \geq 0$ is preserved under the Ricci flow (1.1), we know that $v(x, t)$ is convex by Theorem 2.2. Applying
Theorem 2.2 again we know that the null space of $v_{ij}(x,t)$ is a parallel distribution on $M$. By the simply-connectedness of $M$ and the De Rham’s decomposition theorem we know that $M$ splits as $M = M' \times M''$, where on $M_1$, $(v_{ij}(x,t)) > 0$ and $v_{ij} \equiv 0$ on $N$. Since $v(x,t)$ is strictly convex and exhaustive on $M_1'$, by Theorem 3 (a) of [GW2] we know that $M_1'$ is diffeomorphic to $\mathbb{R}^{k'}$, where $k' = \dim(M_1')$. We claim that $N'$ is compact. Otherwise, $v$ is not constant since it is exhaustive on $N'$ since $v$ is an exhaustion function on $M$ by Corollary 1.4 of [NT3]. Using $v_{ij} \equiv 0$ on $N'$, the gradient of $v$ is a parallel vector field, which gives the splitting of $N'$ as $N' = N'' \times \mathbb{R}$, such that $v$ is constant on $N''$. By the exhaustion of $v$ again we conclude that $N''$ is compact. Also $v$ is a linear function on the flat factor $\mathbb{R}$. But we already know that $v$ is exhaustive, which implies that $v \to +\infty$ on both ends of $\mathbb{R}$. This is a contradiction. This proves that $N'$ is compact. Let $N = N'$ and $M = M_1'$ we have the splitting for $t > 0$. It is also clear that there exists strictly convex exhaustion function on $M_1$. As for the splitting at $t = 0$ we can obtain by the limiting argument. First we have the isometric splitting $M = N \times M_1$ as above for some fixed $t_1 > 0$. On the other hand, by [H2, Lemma 8.2] we know that the distribution given by the null space of $v_{ij}$ is also invariant in time. Therefore, the splitting $M = N \times M_1$ also holds for $0 < t \leq t_1$. Now just taking limit as $t \to 0$ we have the metric splitting of $g_{ij}(x,0)$ on $M$ as $N \times M_1$.

As a consequence of the fact that there exist strictly convex exhaustion function on $M_1$, we know that the soul of $M_1$ is a point. The reason is that first the restriction of $v(x,t)$ to its soul will be constant since the soul is a compact totally geodesic submanifold. On the other hand it is strictly convex if the soul, which is a totally geodesic submanifold, has positive dimension. The contradiction implies that the soul of $M_1$ is a point for $t > 0$. For the case $t = 0$ the result follows by the homotopy consideration. Assume that the soul is not a point. Denote the soul by $S(M_1)$. Then since $S(M_1)$ is the homotopy retraction of $M_1$ we know that $H_s(M_1) = \mathbb{Z}$, where $s = \dim(S(M_1)) \geq 1$. On the other hand since we already know that $M_1$ is diffeomorphic to $\mathbb{R}^k$. Thus $H_s(M_1) = \{0\}$, which is a contradiction. Therefore we know that the soul of $M_1$ with respect to the initial metric is also a point. The claim that the soul of $M$ is just $N \times \{a\}$ follows from the following simple lemma.

**Lemma 2.2.** Let $N$ be a compact Riemannian manifolds with nonnegative sectional curvature. Let $M_1$ be a complete noncompact Riemannian manifold with nonnegative sectional curvature. Let $M = N \times M_1$. Then the soul of $M$, $S(M) = N \times S(M_1)$, where $S(M_1)$ is the soul of $M_1$.

**Remark 2.1.** Combining with Theorem 5.2 of [NT3], this in particular implies that if the $M$ is a complete Kähler manifolds with nonnegative sectional curvature, whose universal cover does not contain the Euclidean factor, then the soul of $M$ is either a point or the compact factor which is a compact Hermitian symmetric spaces. In particular, the result holds if the Ricci curvature of $M$ is positive somewhere.

**Proof of Lemma 2.2.** For any point $z \in M$ we write $z = (x,y)$ according to the product. First of all, it is easy to see that $N \times S(M_1)$ is totally geodesic. It is also totally convex since any geodesic $\gamma(s)$ on $M$ can be written as $(\gamma_1(s), \gamma_2(s))$, where $\gamma_i(s)$ are geodesics in the factor. Therefore, due to the fact $S(M_1)$ is totally convex we know that $\gamma(s)$ lies inside $N \times S(M_1)$ if its two end points do.
Let $\gamma(s)$ be any geodesic ray issued from $p \in M$. (Let $p = (x_0, y_0)$ according to the product.) Since $N$ is compact we have that for the projection $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, $\gamma_1(s) = p$ and $\gamma_2(s)$ is a ray in $M_1$. Let $B^\gamma$ be the Busemann function with respect to $\gamma$. We claim that $B^\gamma(x, y) = B^{\gamma_2}(y)$, where $B^{\gamma_2}(y)$ is the Busemann function of $\gamma_2$ in $M_1$. Once we have the claim we conclude that the level set of $B^\gamma$ is just $N \times$ the level set of $B^{\gamma_2}$ in $M_1$ and the half space $H^\gamma = \{ z \in M \mid B^\gamma(z) \leq 0 \}$, as proved in [LT, Proposition 2.1], $H^\gamma = N \times H^{\gamma_2}$. Since this is true for any ray we have that $C = \bigwedge_\gamma H^\gamma$ is given by $N \times C_{M_1}$, where $C_{M_1}$ denote the corresponding totally convex compact subset in $M_1$. As in [CG], if the compact totally convex subset $C$ has non-empty boundary we define $C^a = \{ p, d(p, \partial C) \geq a \}$. It is easy to see that $C_a = N \times C_{M_1}^a$. In particular, this implies that the soul of $M$ is just $N \times S(M_1)$ since the soul of $M$ is constructed by retracting $C^a$ iteratively.

Now we verify the claim $B^\gamma(x, y) = B^{\gamma_2}(y)$. By the definition we have that

\[
B^\gamma(x, y) = \lim_{s \to \infty} s - d((x, y), \gamma(s))
\]

\[
= \lim_{s \to \infty} s - \sqrt{d_N^2(x, x_0) + d_{M_1}^2(y, \gamma_2(s))}
\]

\[
= \lim_{s \to \infty} (s - d_{M_1}(y, \gamma_2(s))) + \left( \sqrt{d_{M_1}^2(y, \gamma_2(s))} - \sqrt{d_N^2(x, x_0) + d_{M_1}^2(y, \gamma_2(s))} \right)
\]

\[
= \lim_{s \to \infty} (s - d_{M_1}(y, \gamma_2(s))) - \frac{d_N(x, x_0)}{\sqrt{d_{M_1}^2(y, \gamma_2(s)) + d_N^2(x, x_0) + d_{M_1}^2(y, \gamma_2(s))}}
\]

\[
= \lim_{s \to \infty} (s - d_{M_1}(y, \gamma_2(s)))
\]

\[
= B^{\gamma_2}(y).
\]

This completes the proof of the lemma.

Since the Ricci flow preserves the nonnegativity of the curvature operator if the curvature is uniformly bounded (cf. [H2]) we have the following corollary on the structure of complete simply-connected Riemannian manifolds with nonnegative curvature operator.

**Corollary 2.1.** Let $M$ be a complete simply-connected Riemannian manifold with bounded nonnegative curvature operator. Then $M$ is a product of a compact Riemannian manifold with nonnegative curvature operator with a complete noncompact manifold which is diffeomorphic to $\mathbb{R}^k$. In the case of dimension three, the same result holds if one only assumes that the sectional curvature is nonnegative.

**Remark 2.2.** The compact factor in the above result has been classified by Gallot and Meyer [GaM] (also in [CY] by Chow and Yang) to be the product of compact symmetric spaces, Kähler manifolds biholomorphic to the complex projective spaces and the manifolds homeomorphic to spheres.

The above result was proved earlier in [N] by Noronha without assuming the boundedness of the curvature tensor. Our method here has this restriction since we have to use the short time existence result of Shi in [Sh2] on the Ricci flow. For dimension three, in [Sh1] the result was proved even for nonnegative Ricci curvature case. However, it replies on the previous deep results of Hamilton and Schoen-Yau.
As another application of Theorem 2.3 we give examples of complete Riemannian manifolds with nonnegative sectional curvature on which the Ricci flow does not preserve the nonnegativity of the sectional curvature. These manifolds can be constructed as follows. Let $G = SO(n+1)$ with the standard bi-invariant metric and $H = SO(n)$ be its close subgroup. Then $H$ has action on $G$ (as translation) as well as its standard action on $P = \mathbb{R}^n$ (as rotation). Let $M = G \times P/H$. Topologically $M$ is just the tangent bundle over $S^n$ since $H \to G \to G/H = S^n$ is just the corresponding principle bundle over $S^n$. These examples were constructed in [CG] to illustrate their structure theorem therein. About these examples the following are known (cf. [CG]): The metric on $M$ has nonnegative sectional curvature due to the fact that the metric is constructed as the base of a Riemannian submersion; There is also another Riemannian submersion from $T(S^n)$ to $S^n$ with fiber given by $\pi(g \times P)$, where $\pi$ is the first submersion map from $G \times P$ to $M$ (in general, there exists a Riemannian submersion from $M$ to its soul according to a result of Perelman [P1]); The fiber (which is given by $\pi(g \times P)$) of this submersion $\pi : M \to S^n$ is totally geodesic; The fibers are not flat. Namely the metric on each tangent space $T_p(S^n)$ is not the standard flat metric; $M$ has the unique soul $S(M) = \pi(G \times 0)$ and the metric on $M$ is not of product even locally.

**Corollary 2.2.** For the example manifolds above, the Ricci flow does not preserve the nonnegativity of the sectional curvature.

**Proof.** First $M$ is simply-connected by the exact sequence of the fibration $F \to M \to S^n$ with $F = \mathbb{R}^n$. Assume that the Ricci flow preserves the nonnegativity of the sectional curvature. By Theorem 2.3, we know that $M = N \times M_1$, where $M_1$ is diffeomorphic to $\mathbb{R}^k$. This contradicts to the fact that the metric on $M$ is not locally product (for most cases, it already contradicts to the fact that the tangent bundle $T(S^n)$ is non-trivial topologically). In order to apply Theorem 2.3 we need to verify the curvature of the initial metric is uniformly bounded. In the following we focus on the case $M = SO(3) \times P/H$. The general case follows from a similar consideration.

As we know from [CG, page 442 and CE, page 146-147], the metric is given such that $\pi : SO(3) \times \mathbb{R}^2 \to T(S^2)$ is a Riemannian submersion, where $SO(3)$ is the equipped with the bi-invariant metric. Since the Riemannian submersion increases the curvature, we know that the metric constructed in this way has nonnegative sectional curvature. The metric can also be described using the second submersion $\pi_* : T(S^2) \to S^2$ such that for any point in the fiber if the tangent direction is horizontal we use the metric from $S^2$ and for the vertical direction we use the metric given by

$$dr^2 + \frac{r^2}{1+r^2}d\theta^2.$$

Here $(r, \theta)$ is the polar coordinates for $\mathbb{R}^2$. This expression was claimed in [CE, page 146]. For the sake of the completeness we indicate the calculation here. Similar to the situation considered in [C, CGL] we can use $\frac{\partial}{\partial s}$ to denote the component of the Killing vector field of action $SO(2)$ in $SO(3)$. The normalized Killing vector field is given by

$$W = \frac{1}{\sqrt{1+r^2}} \left( \frac{\partial}{\partial \theta} + \frac{\partial}{\partial s} \right).$$
Since

\[ \mathcal{H} \left( \frac{\partial}{\partial \theta} \right) = \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta} W > W \]

\[ = \frac{\partial}{\partial \theta} - \frac{r^2}{1+r^2} W \]

the metric on the base of \( \frac{\partial}{\partial \theta} \) is given by

\[ \| \frac{\partial}{\partial \theta} \|^2_M = \| \mathcal{H}(\frac{\partial}{\partial \theta}) \|^2 = \frac{r^2}{1+r^2}. \]

Here \( \mathcal{H}(\frac{\partial}{\partial \theta}) \) denotes the horizontal lift (projection) of \( \frac{\partial}{\partial \theta} \). This description make it easy to verify that the curvature is uniformly bounded. In order to calculate the curvature we need the formula of [O’N] on the submersion. The Corollary 1 of [O’N, page 465] says that

\[ (a) \ K(P_{vw}) = \hat{K}(P_{vw}) - \frac{T_v, T_w, v, w}{\|v \wedge w\|^2} \]

\[ (b) \ K(P_{xv}) \|x\|^2 \|v\|^2 = \langle (\nabla_v T)_{xv}, x > + \|A_{xv}\| - \|T_v x\|^2 \]

\[ (c) \ K(P_{xy}) = K_*(P_{x* y*}) - \frac{3\|A_{xy}\|^2}{\|x \wedge y\|^2}, \text{ where } x* = \pi_*(x), \]

where \( x, y \) are horizontal and \( v, w \) are vertical. Here \( A \) and \( T \) are the second fundamental form type tensor for the Riemannian submersion \( \pi_* : T(S^2) \to S^2 \). \( \hat{K} \) is the curvature of the fiber and \( K_* \) is the curvature of the base. Since the fiber is totally geodesic, \( T \equiv 0 \), we have the simplified formula

\[ (a) \ K(P_{vw}) = \hat{K}(P_{vw}) \]

\[ (b) \ K(P_{xv}) \|x\|^2 \|v\|^2 = \|A_{xv}\|^2 \]

\[ (c) \ K(P_{xy}) = K_*(P_{x* y*}) - \frac{3\|A_{xy}\|^2}{\|x \wedge y\|^2}, \text{ where } x* = \pi_*(x). \]

By (c) and the nonnegativity of \( K(P_{xy}) \) we have that \( K(P_{xy}) \) is uniformly bounded. The curvature of the fiber can be calculated directly. In fact in terms of the polar coordinates on the fiber it is given by

\[ \frac{3}{(1+r^2)^2}. \]

Therefore we have that \( K(P_{vw}) \) is also uniformly bounded. The only thing need to be checked is the mixed curvature \( K(P_{xv}) \). By the definition of \( A \) we know that

\[ A_{xv} = \mathcal{H} \nabla_x V \]

where \( \mathcal{H} \) is the horizontal projection and \( V \) is any arbitrary extension of \( v \). For a unit horizontal vector \( E \) we have

\[ < A_{xv}, E > = - < v, \nabla_x E >. \]
Therefore it is enough to show that the right hand side is bounded. Since, by the first submersion consideration using the quotient, we know that $K(P_{xy})$ is nonnegative. Therefore by (c) of (2.7),

$$\|A_x y\|^2 \leq \frac{1}{3} K_*(P_{x,y}) \|x \wedge y\|^2.$$ 

This shows that $| \langle v, \nabla_x E \rangle |$ is uniformly bounded.

For the sake of the completeness we also include a proof of the fact that the fiber of $\pi'$ is totally geodesic since there is no written proof in the literature. Recall that $M = SO(n+1) \times P/SO(n)$. Here $SO(n)$ is viewed as close subgroup of $SO(n+1)$ by the inclusion:

$$A \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$ 

We have the involution $\zeta$ which is given by

$$\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}.$$ 

$\zeta$ acts on $SO(n+1)$ by $A \rightarrow \zeta A \zeta$. It is easy to see that the fixed point set of $\zeta$ is $SO(n)$. Now we consider the action of $\zeta$ on $SO(n+1) \times P$ as $(g, x) \rightarrow (\zeta g \zeta, x)$. It is easy to see that this action is commutative with the action of $SO(n)$ since for any $h \in SO(n)$ $\zeta h = h \zeta$. Therefore the action descends to $M$. It is easy to see that the fixed point of this action is $\pi(e, P)$. This implies that the fiber $\pi(e, P)$ is totally geodesic since it is the fixed point set of an isometry. The other fiber can be verified similarly since for any point $p \in S^n$ there is also a involution fixes that point.

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