A generalization of the Zerilli master variable for a dynamical spherical spacetime

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Abstract
The evolution of polar perturbations on a spherical background spacetime is analyzed. The matter content is assumed to be a massless scalar field. This provides a nontrivial dynamics to the background and the linearized equations of motion become much more involved than in the vacuum case. The analysis is performed in a Hamiltonian framework, which makes explicit the dynamical role of each of the variables. After performing a number of canonical transformations, it is possible to completely decouple the different perturbative degrees of freedom into constrained, pure-gauge and gauge-invariant variables. In particular, two master variables are obtained: one corresponding to the polar mode of the gravitational wave, whereas the other encodes the complete physical information about the perturbative matter degree of freedom. The evolution equations for these master variables are obtained and simplified.

Keywords: perturbation theory, master variables, spherical symmetry

1. Introduction
The study of linearized perturbations of known background solutions of general relativity has had very important contributions to our current understanding of different gravitational scenario. As in other field theories with constraints, one of the main problems of this approach is the identification of physical degrees of freedom. There are usually two approaches one can follow for such a purpose. On the one hand, it is possible to impose convenient gauge-fixing conditions and work on a certain gauge. On the other hand, one can construct gauge-invariant variables so that any physical result is unambiguous and valid in any gauge.
The Hamiltonian formalism of general relativity gives a very clear and transparent notion of the gauge dependence. In particular, the Hamiltonian is a linear combination of first-class constraints, which are the generators of gauge transformations. In the context of perturbation theory, such a Hamiltonian formalism was pioneered by Moncrief [1] to study the non-spherical perturbations of the Schwarzschild black hole. In that reference it was shown that, for this solution, it is possible to perform several canonical transformations explicitly in such a way that the initial twelve perturbative variables (the six components of the perturbed spatial metric in combination with their corresponding conjugate momenta) are reorganized into two physical pairs, which encode the complete physical information of the gravitational wave, and four gauge pairs. In each gauge pair, one of the variables is constrained to vanish on shell, whereas its conjugate variable is nonphysical. Furthermore, the two physical pairs obey unconstrained evolution equations and are equivalent to the Regge–Wheeler [2] and Zerilli [3] master variables. In this way, the physical degrees of freedom are explicitly decoupled from the gauge degrees of freedom and the dynamical behavior of the system is completely described by the two physical pairs. This technique was also applied to other specific solutions of Einstein equations like Reissner–Nordström [4, 5], Oppenheimer–Snyder [6] or Friedmann–Robertson–Walker [7, 8].

Regarding spherically symmetric background metrics, it is well known that the perturbations can be classified into two different sectors (axial and polar) depending their polarity. At linear order these two sectors decouple, but at second and higher orders they interact as, for instance, the coupling between two axial modes give rise to both axial and polar modes [9]. For the Schwarzschild metric the Regge–Wheeler [2] and Zerilli [3] variables encode respectively the axial and polar physical degrees of freedom. Both are master scalars since they obey unconstrained evolution equations and the complete perturbed metric can be reconstructed in terms of them. These two variables were initially defined on a fixed perturbative gauge but, as commented above, were later obtained by Moncrief on a generic gauge.

A very convenient framework to study perturbations around generic (possibly dynamical) spherically symmetric spacetimes was presented by Gerlach and Sengupta [10, 11]. This is a very geometrical framework, where the four dimensional manifold is decomposed as the product between a two dimensional Lorentzian manifold with boundary and the unit twosphere. The construction of perturbative gauge-invariant variables is explicitly performed and, for the axial case, a master scalar variable is constructed. This master scalar obeys an unconstrained wave equation and can be coupled to any kind of matter. Therefore, it can be considered as the generalization of the Regge–Wheeler variable to dynamical spacetimes. On the contrary, for the polar sector, there is no known master variable valid for any spherically symmetric background. Nonetheless, some particular results for specific metrics have been obtained. For instance, on a vacuum background, the gauge-invariant combinations of the linearized stress-energy tensor have also been included in [12, 13]. On the other hand, in [14] a polar master scalar was defined, for a vacuum background, which was later generalized to nonlinear electrodynamics [15] for any background solution.

The present paper is motivated by the search of a unique gauge-invariant master variable encoding the polar gravitational wave for any matter model. In principle it is not clear that it can be constructed but, apart from its conceptual relevance, such a polar master variable would be of great use for several applications. For instance the numerical resolution of an unconstrained equation is, in principle, much easier and precise than the resolution of a coupled system of several equations which, in addition, are subjected to certain constraints. In addition, it could be particularly useful in the very hard problem of matching of polar gravitational waves through moving surfaces, for instance the surface of a supernova.
explosion [16–18]. This problem becomes even harder beyond first-order perturbation theory [19–22].

In a previous paper [23] the Gerlach–Sengupta axial master scalar was reobtained, making use of the Hamiltonian techniques explained above, for a dynamical spherical background spacetime with a matter content of a scalar field. Here the analysis of that paper is reproduced for the polar sector, as an example of a dynamical spacetime for which no polar master variable is known.

The rest of the paper is organized as follows. In section 2 the Hamiltonian framework for linearized perturbations on a generic background is briefly reviewed. Section 3 introduces the notation and the equations of motion corresponding to the particular background we will be dealing with: a spherically symmetric spacetime with a massless scalar field. In section 4 the polar part of the perturbative variables are decomposed into tensor spherical harmonic and a number of canonical transformations are performed in order to decouple the gauge and the physical degrees of freedom. In section 5 the evolution equation for the master variables are presented and simplified. Finally, section 6 discusses the main conclusions.

2. General relativistic perturbation theory on a Hamiltonian framework

2.1. Hamiltonian framework for general relativity

Let us assume general relativity coupled with a massless scalar field $\Phi$. In order to perform a Hamiltonian analysis of this system, the usual 3+1 decomposition of the spacetime is performed. Greek indices will be used for four-dimensional objects and Latin indices for three-dimensional ones. It is possible to choose coordinates $(t, x^i)$ adapted to the foliation so that the three-dimensional metric $g_{ij}$, defined by projecting the four-dimensional one $(4)g_{\mu\nu}$ to the spatial slices, is given as

$$g_{ij} := (4)g_{ij}.$$  

Furthermore, the lapse function $\alpha$ and the shift vector $\beta_i$ are defined as follows,

$$\alpha^{-2} := -(4)g^{tt}, \\
\beta_i := (4)g^{t\,i}.$$  

The action for the system under consideration is then given by,

$$S = \int dt \int d^3x \left( \Pi^{ij} \dot{g}_{ij} + \Pi \dot{\Phi} - \alpha \mathcal{H} - \beta^i \mathcal{H}_i \right),$$  

where $\Pi^{ij}$, which is related to the extrinsic curvature, is the conjugate momentum of the spatial metric $g_{ij}$ whereas $\Pi$ is the conjugate momentum of the scalar field. As it is well known, the lapse and the shift are Lagrange multipliers associated to the Hamiltonian $\mathcal{H}$ and momentum constraint $\mathcal{H}_i$ respectively. These take the following form in terms of the basic variables:

$$\mathcal{H} = \frac{1}{\mu_k} \left[ \Pi^i \Pi_{ij} - \frac{1}{2} \left( \Pi^{ij} \right)^2 \right] - \frac{1}{2} \left( \mu_k \mathcal{H} \right) + \frac{1}{2} \left( \frac{\Pi^2}{\mu_k} + \mu g^{ij} \Phi_{;i} \Phi_{;j} \right),$$  

$$\mathcal{H}_i = -2D_j \Pi^j \Phi_{;i} + \Pi \Phi_{;i},$$  

where $\mu_k := \sqrt{\det g_{ij}}$ and $D_j$ is the covariant derivative associated to $g_{ij}$. 

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2.2. Linearized perturbations

Let us begin by defining a one-parameter family of spacetimes \((\mathcal{M}(\epsilon), \tilde{g}_{\mu\nu}(\epsilon))\), where \(\epsilon\) is a dimensionless parameter. The \(\epsilon = 0\) member of the family is referred to as the background spacetime. The idea behind linear perturbation theory is to perform a linearization around a background, which is a known exact solution of the Einstein equations. This is done by performing a Taylor expansion on the parameter \(\epsilon\) of all quantities that appear in the equations. Then, terms higher than second order in \(\epsilon\) are dropped. For convenience, we define the operator

\[
\delta^2 F(\epsilon) := \left. \frac{d^n F(\epsilon)}{d\epsilon^n} \right|_{\epsilon=0},
\]

and, making use of it, introduce the following notation for the perturbative variables:

\[
C := \delta \alpha, \quad B^i := \delta \left( \beta^i \right),
\]

\[
h_{ij} := \delta \left( g_{ij} \right), \quad p^i := \delta \left( \Pi^i \right).
\]

As it was shown in [1, 24], the second variation of the action,

\[
\frac{1}{2} \delta^2 S = \int d^4x \left[ \pi^{ij} \psi_{ij} + \delta \Pi \delta \Phi - C \delta (H) - B^i \delta (H_i) - \frac{\alpha}{2} \delta^2 (H) - \frac{\beta^i}{2} \delta^2 (H_i) \right],
\]

provides an action functional for the linearized perturbations. (The explicit form of the first and second variations of the Hamiltonian and momentum constraints can be found in [23].) That is, the variation of this last action with respect to different perturbative variables (7) gives the linearized Einstein equations. In particular, the variation with respect to the perturbation of the lapse \(C\) and the shift \(B^i\) leads to the constraints obeyed by these linearized variables,

\[
\delta (H) = 0, \quad \delta (H_i) = 0.
\]

These are first-class constraints and, therefore, the generators of the perturbative gauge transformations. The idea of this paper is to perform canonical transformations of the perturbative variables so that each of these four constraints is simply expressed as one of the new variables. In this way, the different dynamical sectors would get decoupled. From the eighteen variables under consideration [the sixteen that appear in (7), in combination with the two functions \((\delta \Phi, \delta \Pi)\) encoding the scalar degree of freedom], four of them would be constrained to vanish. The four canonical conjugate variables of the constrained ones will be pure gauge, whereas the four functions \((C, B^i)\) are Lagrange multipliers with vanishing conjugate momentum and thus non-dynamical. Finally, the remaining six variables will be automatically gauge invariant. These latter six variables stand for the three physical degrees of freedom of this problem: two corresponding to the gravitational wave and one to the perturbations of the matter scalar field. This separation between the gauge and the physical sectors is not obvious and can only be performed for highly symmetric backgrounds.

3. Spherical background

We will consider a spherically symmetric background, which can be decomposed as \(\mathcal{M} = M^2 \times S^2\), where \(M^2\) is a two dimensional background with boundary and \(S^2\) the unit two-sphere. Arbitrary coordinates \((t, \rho)\) on \(M^2\) and spherical coordinates \(x^i = (\theta, \phi)\) on \(S^2\).
are chosen. The lower-case Latin indices stand for coordinates on the two-sphere. Due to the
symmetry, the lapse can only be a function of the coordinates on $M^2$, that is, $\alpha = \alpha(t, \rho)$;
whereas the shift vector has vanishing angular components $\beta^i = (\beta(t, \rho), 0, 0)$. In this way,
the four-dimensional background metric takes the following form:

$$\left( ds^2 \right)_4 = -a^2 dt^2 + a^2 (d\rho + \beta dt)^2 + r^2 d\Omega^2.$$  (10)

Furthermore the following three variables are defined, which completely encode the
information contained in the background moments $\Pi^{ij}$ and $\Pi$:

$$\Pi_1 := \frac{\alpha^2 \Pi^{\rho\rho}}{\mu_k}, \quad \Pi_2 := \frac{2r^2 \Pi^{\Phi\Phi}}{\mu_k}, \quad \Pi_3 := \frac{\Pi}{\mu_k}. \quad \quad \quad (11)$$

For completeness, here the symmetry-reduced background constraints are provided,

$$\frac{\mathcal{H}}{\mu_k} = \Pi_1 \left( \frac{\Pi_1}{2} - \Pi_2 \right) - \frac{1}{2} \left( \Pi_3^2 + \Phi'^2 \right) = 0, \quad \quad \quad (12)$$

$$\frac{1}{a} \frac{\mathcal{H}'}{\mu_k} = -\frac{2}{r} \left( r^2 \Pi_1' \right) + \frac{2r'}{r} \Pi_2 + \Pi_1 \Phi' = 0, \quad \quad \quad (13)$$

where prime stands for the derivative with respect to $\rho$ divided by the function $a$:

$$f' = \frac{f}{a}. \quad \quad \quad (14)$$

These constraints generate the following evolution equations:

$$\frac{1}{a} \left[ a, (\beta\alpha)_{,\rho} \right] = \frac{a}{2} (\Pi_1 - \Pi_2), \quad \quad \quad (15)$$

$$\frac{1}{a} \left( r, \beta r_{,\rho} \right) = -\frac{r}{2} \Pi_1, \quad \quad \quad (16)$$

$$\frac{1}{a} \left( \Phi, \beta \Phi_{,\rho} \right) = \Pi_3, \quad \quad \quad (17)$$

$$\frac{1}{a} \left( \Pi_{1,\rho} - \beta \Pi_{1,\rho} \right) = \frac{3\Pi_1^2}{4} + \frac{1}{r^2} - \frac{r'}{r} \left( \frac{a^2 r}{\alpha^2} \right)' + \frac{1}{4} \left( \Pi_3^2 + \Phi'^2 \right). \quad \quad \quad (18)$$

$$\frac{1}{a} \left( \Pi_{2,\rho} - \beta \Pi_{2,\rho} \right) = \frac{1}{2} \left( \Pi_1^2 + \Pi_2^2 - \Pi_1 \Pi_2 \right) + \frac{2ar^2}{ar} - \frac{2(\alpha r)^2}{ar} + \frac{1}{2} \left( \Pi_3^2 - \Phi'^2 \right). \quad \quad \quad (19)$$

$$\frac{1}{a} \left( \Pi_{3,\rho} - \beta \Pi_{3,\rho} \right) = \frac{\Pi_3 (\Pi_1 + \Pi_2)}{2} + \frac{\left( \alpha^2 r \Phi' \right)'}{ar^2}. \quad \quad \quad (20)$$

The case studied previously by Moncrief is vacuum in Schwarzschild coordinates, which is
recovered by choosing $\Phi = \Pi_1 = \Pi_2 = \Pi_3 = 0$. This restriction greatly simplifies the above
equations of motion and in particular both the Hamiltonian (12) and momentum constraints
(13) are trivially fulfilled.
4. Polar perturbations

4.1. Expansion in harmonics

In order to take advantage of the background spherical symmetry we will make use of the tensor spherical harmonics in order to decompose the perturbations. Properties and precise definitions of the harmonics that will be used here can be found in [25]. Following their behavior under a parity transformation, different tensor spherical harmonics can be divided into two groups: axial harmonics, with a polarity \((-1)^{l+1}\), and polar harmonics, with a polarity \((-1)^l\). Since at a linearized level these two groups of harmonics decouple, it is possible to consider the axial and the polar problem independently. As commented in the introduction, in [23] the axial case was developed, whereas here we will focus on the polar case. The decomposition of the polar part of the different perturbations is given as follows,

\[
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} a^2 \left( H_l^m \right)^m_i Y_l^m \, dp^2 + 2 \left( h_1 \right)^m_i \, dp \, Z_l^m \, dx^a \\
+ r^2 \left[ K_l^m \, t_{ab} Y_l^m + G_l^m \, Z_l^m \, ab \right] \, dx^a \, dx^b, 
\]

\[
\frac{1}{\mu_k} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a^2 \left( P_l^m \right)^m_i Y_l^m \, dp^2 + 2 \left( P_0 \right)^m \, dp \, Z_l^m \, dx^a \\
+ r^2 \left[ \left( P_k \right)^m_i Y_l^m + \left( P_0 \right)^m \, Z_l^m \, ab \right] \, dx^a \, dx^b, 
\]

\[
C = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{-\alpha}{2} \left( H_0 \right)^m_i Y_l^m, 
\]

\[
B_i \, dx^i = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( H_1 \right)^m_i Y_l^m \, dp + \left( h_0 \right)^m_i \, Z_l^m \, dx^a, 
\]

\[
\frac{1}{\mu_k} \delta \Pi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \hat{P}_l^m \right) Y_l^m, 
\]

\[
\delta \Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( P_l^m \right) Y_l^m, 
\]

where the notations by Regge–Wheeler and Moncrief have been followed for the different harmonic coefficients.

Since all perturbations are decoupled at linear order, from here on, the \((l, m)\) labels from the harmonic coefficients and the harmonic tensors will be removed. In addition, we define the following shortening for the sum that appears in all decompositions above,

\[
\sum_{l,m} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l}. 
\]

4.2. Effective action

In order to obtain the effective action in terms of the harmonic coefficients, decomposition (21)–(26) is introduced in expression (8). The angular part can be integrated by making use of
the properties of the tensor spherical harmonics as shown in appendix of [23]. In this way, it is easy to obtain the following form for the effective action for the polar part of the linearized perturbations:

\[
\frac{1}{2} \delta^2 S_{\text{polar}} = \int dx^4 \left[ p^{ij} \delta h_{ij,t} + \delta \Pi \delta \Phi_t - C_\delta(H) \right] \\
- B' \delta^2 (H_t) - \frac{\alpha}{2} \delta^2 (H_t) - \frac{\beta'}{2} \delta^2 (H_t) \right]_{\text{polar}} \\
= \sum_{\lambda} \int dt \int d\rho \left[ p_1 h_{1,t} + p_2 H_{2,t} + p_3 K_{3,t} + p_4 G_{4,t} + \rho \eta_{\rho,t} \right] \\
+ \int dt \left\{ F_0 \left[ -\alpha H_0/2 \right] + F_1 \left[ H_1 \right] + F_2 \left[ h_0 \right] \right\} + \ldots, 
\]

(28)

where the dots stand for terms coming from the second perturbation of the background constraints, which do not enter the gauge transformations, and the functionals \( F_0, \ F_1, \ F_2 \) will be defined below. The conjugate momenta are related to the harmonic coefficients given by the expansions (21)–(26) in the following way,

\[
p_1 = \frac{2}{a} \frac{(l+1)}{l} P^*_{h}, 
\]

(29)

\[
p_2 = a^2 P^*_{h}, 
\]

(30)

\[
p_3 = 2a^2 P^*_{k}, 
\]

(31)

\[
p_4 = \lambda a^2 P^*_{g}, 
\]

(32)

\[
p = a^2 \hat{p}^* 
\]

(33)

where the star stands for complex conjugate and

\[ \lambda := \frac{1}{2} \frac{(l+2)!}{(l-2)!} \]

has been defined.

The polar part of the harmonic decomposition of the linearized constraints is given by,

\[
\delta [H] = \sum_{l,m} \mu_{l,m} Y_{l,m} \left\{ H_2 \left[ \Pi_1 (\Pi_1 - \Pi_2) - \frac{l^2 + l + 2}{r^2} - \frac{H}{2} \frac{2}{r^2} \right] - 2H_2 \frac{r'}{r} \\
+ \frac{p_2}{ar^2} (\Pi_1 - \Pi_2) + \frac{1}{2} K \left[ -\Pi_1^2 - \Pi_2^2 + \Phi'^2 \right] \\
- \frac{p_3}{ar^2} \Pi_1 - \frac{2}{r^3} \left( (l-1)(l+2) \right) K + \frac{2}{r^3} \left( r^2 K' \right) + \Phi \Phi' \\
+ \frac{p_4}{ar^2} \Pi_3 + \frac{2 l (l+1)}{r^3} \left( r a^{-1} h_1 \right) - \frac{2}{r^2} G \right\}, \\
\]

\[
\delta \left[ H_\rho \right] = \sum_{l,m} \mu_{l,m} Y_{l,m} \left\{ - \frac{2}{a} \frac{(l-1)^2}{r^2} + \frac{p_1}{r^2} + \frac{2}{ar^2} \frac{r'}{r} + \frac{p}{ar^2} \Phi' - H_2 \Pi_1 - \frac{2}{r^2} \left( r^2 \Pi_1 \right) H_2 
\]

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With these relations at hand, the three generators of polar gauge transformations can be written as

$$F_0[f] = \int dx^3 f \, Y \delta[H], \quad (34)$$

$$F_1[f] = \int dx^3 f \, Y \frac{1}{a} \delta[H_{\rho}], \quad (35)$$

$$F_2[f] = \int dx^3 f \, Z_{a} \frac{1}{r^2} \gamma_{ab} \delta[H_{\phi}] \text{polar}, \quad (36)$$

which act on any smooth arbitrary scalar field $f$. It is possible to calculate the Poisson brackets between different generators,

$$\{ F_0[f_1], F_0[f_2] \} = \int dp \, ar^2 (f_1 f'_2 - f'_1 f_2) \frac{1}{a} \frac{H_{\rho}}{\mu_k}, \quad (37)$$

$$\{ F_0[f_1], F_1[f_2] \} = \int dp \, a f_1 \left( \frac{r^2 f'_2 H}{\mu_k} \right), \quad (38)$$

$$\{ F_0[f_1], F_2[f_2] \} = -l(l+1) \int dp \, a f_1 f'_2 \frac{H}{\mu_k}, \quad (39)$$

$$\{ F_1[f_1], F_1[f_2] \} = \int dp \, ar^2 (f_1 f'_2 - f'_1 f_2) \frac{1}{a} \frac{H_{\rho}}{\mu_k}, \quad (40)$$

$$\{ F_1[f_1], F_2[f_2] \} = 0, \quad (41)$$

$$\{ F_2[f_1], F_2[f_2] \} = l(l+1) \int dp \, a \left( f'_1 f'_2 - f'_1 f_2 \right) \frac{1}{a} \frac{H_{\rho}}{\mu_k}, \quad (42)$$

all vanishing on-shell, which confirms that they are first-class constraints.

### 4.3. Canonical transformations: gauge-invariant variables

Moncrief isolated the Zerilli variable after two canonical transformations on the four pairs $(h_1, p_1), (H_2, p_2), (K, p_3), (G, p_4)$. Here the additional pair $(\phi, p)$ for the scalar field is also present; and the fact that the background is dynamical makes the problem harder. We will instead proceed in five steps, to clarify the role of each step and simplify the computations. In particular we will first eliminate the two gauge degrees of freedom related to the momentum.
constraint (which are rather trivial and very similar to the axial case) and then remove the gauge degree associated with the Hamiltonian constraint, the nontrivial step of this computation.

There are many possible transformations that implement this program, but we would like them to obey certain minimal criteria. First, they should be algebraic transformations so that they do not involve any integration in the process and can be performed explicitly. Second, they should not require dividing by any background object that could vanish, in particular one of the background momenta \( \Pi_\alpha \). And third, in order to obtain a generalization of the Zerilli master variable, all transformations should be well defined in the vacuum limit, that is, when taking vanishing values for the scalar field and its perturbations. The full transformation that will be proposed here completely fulfills the first and third criterion, but it is unclear whether it also satisfies the second one, as will explained below.

The first canonical transformation is motivated by the Gerlach and Sengupta choice of gauge invariants \([10]\),

\[
\begin{align*}
    k_1 &= K + \frac{l(l + 1)}{2} G - 2r \left( a^{-1} h_1 - \frac{r^2}{2} G \right), \\
    k_2 &= H_2 - 2 \left( a^{-1} h_1 - \frac{r^2}{2} G \right), \\
    k_3 &= G, \\
    k_4 &= a^{-1} h_1 - \frac{r^2}{2} G, \\
    k_5 &= \Phi - \left( a^{-1} h_1 - \frac{r^2}{2} G \right) \Phi, \\
\end{align*}
\]

which requires the canonical momenta

\[
\begin{align*}
    \pi_1 &= p_3, \\
    \pi_2 &= p_2, \\
    \pi_3 &= p_4 - \frac{l(l + 1)}{2} p_3 - \frac{1}{2} a \left( r^2 p_1 \right), \\
    \pi_4 &= a p_1 - 2a \left( a^{-1} p_2 \right) + 2r' r p_3 + p' \Phi, \\
    \pi_5 &= p. \\
\end{align*}
\]

In terms of these new variables, the components of the perturbed momentum constraints are written as

\[
\frac{1}{a} \delta \left[ H_{\mu} \right] = \sum_{l,m} \mu_\gamma Y \left\{ \frac{\pi_4}{a r^2} + \frac{\Pi_2}{r^2} \left( r^2 k_1 \right)' - \frac{\Pi_1 k_2'}{r^2} - 2 k_2 \left( r^2 \Pi_1 \right)' - \frac{2}{r^2} \left( r^2 \Pi_1 k_4 \right)' + k_4 \frac{1}{a} \frac{H_{\mu}}{\mu_\gamma} \\
+ \frac{l(l + 1)}{r^2} \Pi_2 \left( k_4 - k_3 r r' \right) + k_4 \Pi_3 \Phi + k_4 \frac{\Pi_3}{r^2} \left( r^2 \right)' + \Pi_3 k_5' \right\},
\]

\[9\]
The explicit form of the perturbation of the Hamiltonian constraint $\delta[H]$ in terms of these variables is not displayed because it is a very lengthy expression and does not contribute in any way to the present discussion.

A second canonical transformation is performed, which converts the momentum constraints into canonical variables. Because of the requirements we want to impose in all our transformations, only the momenta $\pi_4$ and $\pi_3$ can replace the constraints (53) and (54) respectively,

$$\pi_1 = \pi_1 - a r^2 (\Pi_2 k_4)' \tag{55}$$

$$\pi_2 = \pi_2 + \frac{l(l + 1)}{2} a r^2 \Pi_1 k_3 + a r^2 \Pi_1 k_4 - a \left(r^2 \Pi_1\right)' k_4 \tag{56}$$

$$\pi_3 = \frac{1}{2} l(l + 1) a r^2 \left(\frac{\delta[H]_{\text{polar}}}{\mu_8 Z_a}\right) = \pi_3 + ..., \tag{57}$$

$$\pi_4 = r^2 \left(\frac{\delta[H_{\mu}}{\mu_8 Y}\right) = \pi_4 + ..., \tag{58}$$

$$\pi_5 = \pi_5 + \frac{l(l + 1)}{2} a r^2 \Pi_3 k_3 - a \left(r^2 \Pi_3 k_4\right)' \tag{59}$$

The division by the tensor harmonics must be understood just as removing them, as well as the summation symbol, from the above expressions (53)–(54). These last transformations for the momenta do not affect the position variables,

$$\tilde{k}_1 = k_1, \tag{60}$$

$$\tilde{k}_2 = k_2, \tag{61}$$

$$\tilde{k}_3 = k_3, \tag{62}$$

$$\tilde{k}_4 = k_4, \tag{63}$$

$$\tilde{k}_5 = k_5. \tag{64}$$

In terms of these last variables, the perturbative constraints take the following simpler form,

$$\delta[H] = \sum_{l,m} \mu_8 Y \left\{ - \Pi_1 \frac{\tilde{k}_1}{ar^2} + \left(\Pi_1 - \Pi_2\right) \frac{\tilde{k}_2}{ar^2} + \Pi_3 \frac{\tilde{k}_3}{ar^2} - \left(\frac{r^2}{r^2} \tilde{k}_1\right)' \right\} \Phi' + \frac{2}{r^2} \left(\frac{r^3}{r^2} \tilde{k}_1\right)' - \left(\frac{r^2}{r^2} \tilde{k}_2\right)'$$
The gauge freedom contained in the perturbed momentum constraints has been fully isolated. The variables \( \bar{k}_3 \) and \( \bar{k}_4 \) are gauge dependent and non-dynamical because their conjugate momenta \( \bar{\pi}_3 \) and \( \bar{\pi}_4 \) are constrained to vanish. We are left with a system of three degrees of freedom \((\bar{k}_1, \bar{k}_2, \bar{k}_5)\) and the single constraint (65).

Following the same procedure, at this point one should make another canonical transformation and convert the Hamiltonian constraint into one of the variables. Because of the first criterion we want to impose, we cannot convert any of the variables \( \{\bar{k}_1, \bar{k}_2, \bar{k}_5\} \) which appear in (65) into the full constraint. But, because of the second requirement, we cannot do it for any of the momenta \( \{\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_5\} \). Therefore, the idea is to first make a transformation that removes second-order derivatives of \( \bar{k}_1 \) from the constraint (65), so that all first derivatives of the perturbed objects can be absorbed in a single term. We use an arbitrary constant \( \gamma \) to parameterize the transformation,

\[
\begin{align*}
\bar{k}_1' &= \bar{k}_1, \\
\bar{k}_2' &= \bar{k}_2 - \frac{\gamma\bar{k}_1'}{r'} - \gamma\bar{k}_1, \\
\bar{k}_3' &= \bar{k}_3, \\
\bar{k}_4' &= \bar{k}_4, \\
\bar{k}_5' &= \bar{k}_5,
\end{align*}
\]

which will introduce a first derivative of \( \bar{\pi}_2 \) in the Hamiltonian constraint through the transformations of the momenta,

\[
\begin{align*}
\bar{\pi}_1 &= \bar{\pi}_1 + (\gamma - 1)\bar{\pi}_2 - ar\left(\frac{a^{-1}\bar{r}_2}{r'}\right)' , \\
\bar{\pi}_2 &= \bar{\pi}_2, \\
\bar{\pi}_3 &= \bar{\pi}_3, \\
\bar{\pi}_4 &= \bar{\pi}_4, \\
\bar{\pi}_5 &= \bar{\pi}_5.
\end{align*}
\]

In this way, the Hamiltonian constraint can be written as a sum of a full derivative term (which will be later promoted to the polar gauge-invariant geometric master variable) and a linear combination of variables \( \bar{k}_i \) and \( \bar{\pi}_i \) with no derivatives,
\[
\delta[H] = \sum_{l,m} \mu_l \Phi \left\{ \left[ -\Pi_l \frac{\tilde{\Phi}_2}{ar^2} - \frac{1}{r' r} \frac{d}{dr'} \tilde{k}_2 + \frac{1}{r^2} \Phi \tilde{k}_5 - \frac{1}{r'} V'_r \tilde{k}_1 \right] r' + \Pi_l \frac{\tilde{\Phi}_1}{ar^2} + \Pi_l \frac{\tilde{\Phi}_3}{ar^2} + \left( \frac{r (\Pi_l)}{r'} - \Pi_l - \Pi_2 + \gamma \Pi_l \right) \tilde{\Phi}_2 + \frac{r}{r'} V'_r + V'_r \frac{r^2}{2(r')^2} \left( \frac{3(r')^2}{r^2} - \frac{1}{r^2} + \frac{(3) R}{2} \right) \right. \\
- \Pi_3^2 + (1 - \gamma) \Pi_l (\Pi_2 - \Pi_l) - \frac{l(l + 1)}{r^2} (1 + \gamma) + \frac{2}{r^2} (1 - \gamma) \tilde{k}_4 - \Phi' \tilde{k}_5 \\
- \left( V + \frac{3(r')^2}{r^2} \right) \tilde{k}_2 \right\},
\]

(78)

where the background potentials,

\[
V := \frac{1 + l + l^2}{r^2} + \Pi_l (\Pi_2 - \Pi_l) + \frac{(3) R}{2}, \quad V_r := V + (2 \gamma - 3) \frac{(r')^2}{r^2},
\]

(79)

have been defined. As it has been anticipated, this clearly motivates another canonical transformation in which the term that appears inside the full derivative (in the first line of equation (78)) replaces the canonical variable \( \tilde{k}_2 \). Note that using \( \tilde{k}_1 \) instead to replace such term would require dividing by \( V_r \), which is a background object that could vanish, whereas using \( \tilde{k}_3 \) would not provide a well defined vacuum limit. The fourth canonical transformation takes thus the following form:

\[
\tilde{k}_1 = \tilde{k}_1,
\]

(80)

\[
\tilde{k}_2 = -\Pi_1 \frac{\tilde{\Phi}_2}{ar^2} - \frac{1}{r^2} \frac{d}{dr'} \tilde{k}_2 + \frac{1}{r'} \Phi' \tilde{k}_5 - \frac{1}{r} V'_r \tilde{k}_1,
\]

(81)

\[
\tilde{k}_3 = \tilde{k}_3,
\]

(82)

\[
\tilde{k}_4 = \tilde{k}_4,
\]

(83)

\[
\tilde{k}_5 = \tilde{k}_5,
\]

(84)

\[
\tilde{\Phi}_1 = \tilde{\Phi}_1 - \frac{V'_r}{2(r')^2} \tilde{\Phi}_2,
\]

(85)

\[
\tilde{\Phi}_2 = -\frac{r \tilde{\Phi}_2}{2r'},
\]

(86)

\[
\tilde{\Phi}_3 = \tilde{\Phi}_3,
\]

(87)

\[
\tilde{\Phi}_4 = \tilde{\Phi}_4,
\]

(88)

\[
\tilde{\Phi}_5 = \tilde{\Phi}_5 + \frac{\Phi'}{2r'} \tilde{\Phi}_2.
\]

(89)

Now the Hamiltonian constraint does not contain \( \tilde{\Phi}_2 \) and, in addition, it has no explicit dependence on the constant \( \gamma \). But, more importantly, we have achieved what we were looking for: it neither contains derivatives of \( \tilde{k}_1 \).
\[ \delta[H] = \sum_{i,m} \mu_i Y \left\{ Dk_i - \Pi_i \frac{\dot{\bar{k}}_i}{ar^2} + \Pi_i \frac{\bar{k}_5}{ar^2} + \bar{k}_2' + \frac{\bar{k}_2}{2} \left( \frac{V}{r'} + 3r' \right) \right\}, \] (90)

where we have defined the background coefficient

\[ D = \left( \frac{r^2 V}{rr'} \right) + \frac{r^2}{2(r')^2} \left[ V + \left( \frac{r'}{r} \right)^3 \right] \left[ \frac{l(l+1)}{r^2} + \Pi_1 (\Pi_2 - \Pi_1) + (^{3R}) \right] \]

\[ - \left[ \frac{2(l(l+1)}{r^2} + \Pi_2^2 \right]. \] (91)

This fact permits us to perform the final fifth canonical transformation, which converts the Hamiltonian constraint into the first of the variables of the problem,

\[ Q_i = D\bar{k}_i - \Pi_i \frac{\bar{k}_1}{ar^2} + \Pi_i \frac{\bar{k}_5}{ar^2} + \bar{k}_2' + \frac{\bar{k}_2}{2} \left( \frac{V}{r'} + 3r' \right) - \bar{k}_5 \left[ \Phi^r + \frac{\phi'}{2} \left( \frac{V}{r'} + 3r' \right) \right], \] (92)

\[ Z = \bar{k}_2, \] (93)

\[ Q_3 = \bar{k}_3, \] (94)

\[ Q_4 = \bar{k}_4, \] (95)

\[ \phi = \bar{k}_5 + \frac{\Pi_3}{ar^2} \bar{k}_1, \] (96)

\[ P_1 = \frac{\bar{k}_1}{D}, \] (97)

\[ P_2 = \bar{k}_2 + a \left( \frac{\bar{k}_1}{D} \right)' - \frac{\bar{k}_1}{2D} \left( \frac{V}{r'} + 3r' \right), \] (98)

\[ P_3 = \bar{k}_3, \] (99)

\[ P_4 = \bar{k}_4, \] (100)

\[ P_5 = \bar{k}_5 \left[ \Phi^r + \frac{\phi'}{2} \left( \frac{V}{r'} + 3r' \right) \right]. \] (101)

At this point we have succeeded in separating the physical degrees of freedom \((Z, P_2)\) and \((\phi, P_5)\) from the gauge degrees of freedom \((Q_1, P_1), (Q_3, P_3)\) and \((Q_4, P_4)\). However in this last transformation the background object \(D\) appears as denominator and it is not clear to us whether this object can vanish or not. In vacuum, for a Schwarzschild solution, defining \(\Lambda := (l - 1)(l + 2)/2\), we have

\[ D = \frac{1}{1 - 2M/r} \frac{l(l+1)}{r^3} (\Lambda r + 3M), \] (102)

which is always positive. It is reasonable to assume that for spacetimes close enough to Schwarzschild (though possibly dynamical), the variable \(D\) will also be positive. If this was the case we would have succeeded in implementing to completion the procedure while
obeying the three imposed criteria. If not, analyzing the procedure that has been followed, it
seems quite difficult to achieve the construction of gauge-invariant master variables without
dividing by a never vanishing background object for a generic background gauge. Note that,
onece the canonical transformation (43)–(52) is performed, there is no much freedom left in the
procedure if one insists on imposing the three criteria: performing only algebraic
transformations, not dividing by a possibly vanishing background object and having a well
defined vacuum limit. More specifically, the momenta $\pi_3$ and $\pi_4$ are the only variables that
can be used to solve for the constraints (54) and (53) respectively. This leads to the form (65)
of the linearized Hamiltonian constraint. In that expression none of the variables can be used
to solve the constraint algebraically. Thus, next parametrized transformation (68)–(77) is
performed in order to concentrate all derivatives in a unique full derivative. The term inside
this full derivative is then promoted to one of the basic variables. Finally, in expression (90)
the underived variable $\dot{k}_5$ could also be used to solve for that constraint, but this would not obey our third criterion about
having a well defined vacuum limit. In this case the perturbations of the scalar field would be
pure gauge and thus the physical matter degrees of freedom would be encoded in a geometric
pair. In addition we would not get a master variable that could be consider the generalization
of the Zerilli variable.

In any case, there are other routes that one could follow to construct a polar master gauge
invariant. For instance, one could also choose for instance a particular background gauge with
a fixed (nonzero) value of $\Pi_1$ (or $\Pi_3$) and divide by this moment when solving the constraint
for $\pi_1$ (or for $\pi_3$). Another alternative could be to relax the first condition about the algebraic
nature of the transformations.

5. Evolution equations

The variable $Z$ (93) obeys a complicated equation of motion. This section summarizes its
differential structure, and shows that it is indeed a generalization of the Zerilli equation. In
order to simplify the calculations, $(t, \rho = r)$ are chosen as background coordinates. In
addition we will take $\beta = 0$ which, because of the background evolution equation (16),
implies $\Pi_l = 0$. By reversing all canonical transformations, it is straightforward to write the
variable $Z$ in terms of the initial harmonic coefficients (21)–(26). In particular, in this
background gauge the master variable takes the following form:

$$
Z = -\frac{2}{ar}H_2 + \frac{1}{2a} \left(4K_r + 2\Phi_r \varphi - \Pi^2 r^2 G_r \right) + \frac{1}{8ar} [2K + l(l + 1)G]
\times \left\{ 12 - a^2 \left[4 + 4 l(l + 1) + \Pi^2 r^2 \right] - (r\Phi_r)^2 \right\} + \frac{1}{ar^2} \left[ 2 l(l + 1) + \Pi^2 r^2 \right] \dot{h}_1.
$$

(103)

In order to get this expression, it is enough to reverse all transformations presented in the
previous section. The moments $(p_1, p_2, p_3, p)$ can be written in terms of time derivatives of
their corresponding position variable by making use of the linearized evolution equations.
Even so, note that all time derivatives disappear from the expression of the master variable in
the chosen background gauge.

Let us now present the physical evolution equations. The equations of motion for the
master gauge-invariant variables are obtained by direct variation of the action (28),
where the Roman numerals in the different superscripts represent radial derivatives and the dots stand for terms with lower-order radial derivatives. In addition, the subindices of the background-dependent $M$-coefficients denote positions in a matrix. More precisely, $M_{ij}^{(k)}$ would correspond to the slot $(i, j)$ of the matrix that multiplies the $k$th order radial derivative of the column vector $(P_2, Z, P_\phi, \phi)^T$. Before further analyzing this system of equations, let us particularize it to the vacuum case and recover the results obtained in such a scenario by Moncrief [1].

5.1. The vacuum case: the Zerilli equation

For a vacuum spacetime, all background and perturbative fluid variables $\{\Pi_3, \Phi, Q_5, R\}$ disappear from our problem and equations (106)–(107) are empty. In this case, the background gauge conditions we have chosen at the beginning of this section, which implied $\Pi_1 = 0$, also impose a vanishing value for the background momentum $\Pi_2$ due to the constraint (13). In addition, from background equations (18), (19) one gets the explicit form of the metric components, 

\[ \alpha = \frac{1}{a} = \sqrt{1 - \frac{2M}{r}}. \]  

In this way, one can solve equation (105) to write down the gauge-invariant momentum $P_Z$ in terms of the time derivative of its conjugate variable $Z$, 

\[ P_Z = \frac{2\phi A}{l(l + 1)(2\alpha A + 6M)} Z_{,ij}. \]  

For convenience, we define the rescaled variable, 

\[ \chi := -\frac{r^3}{a} \frac{Z}{2\alpha A + 6M}, \]  

which, inverting the canonical transformations that have been performed in the previous section, can be expressed in terms of the initial harmonic coefficients as 

\[ \chi = \frac{(r - 2M)}{3M + Ar} \left\{ rH_2 - r^2 K_r - l(l + 1)h_1 \right\} + rK + \frac{1}{2} l(l + 1) rG. \]  

Finally, introducing relation (109) in equation (104), the Zerilli equation is obtained,

\[ \left( 1 - \frac{2M}{r} \right)^{-1} \left( -\frac{\partial^2 \chi}{\partial t^2} + \frac{\partial^2 \chi}{\partial r^2} \right) - V_Z \chi = 0, \]  

where we have made use of the tortoise coordinates $(t, r^*)$, with $r^* = r + 2M \ln(r/2M - 1)$, and the potential is given by,
Therefore, the gauge-invariant combination \( \chi \) (111) reduces to the Zerilli variable when particularized to vacuum.

5.2. Simplification of the evolution system

The highest radial derivative that appears in the evolution equations (104)–(107) is that of the Z variable. More precisely, in equation (104) it is of sixth order, whereas in equations (106) and (107) it is of fifth order. Nonetheless in equation (105) only radial derivatives of the variable Z up to fourth order appear. Thus, this latter equation can be derived with respect to \( r \) in order to replace those higher-radial derivatives of Z, which appear in other equations, with lower-order derivatives of different variables and a time derivative of Z. In fact, it turns out that it is possible to perform a change of variables so that the highest radial derivative in the equations is of third order only. Let us define the new variables

\[
\begin{align*}
\xi_1 &= B_1 p_2 + B_2 Z_{,r} + B_3 Z_{,r}, \\
\xi_2 &= Z, \\
\xi_3 &= B_4 p_\phi + B_5 Z_{,r}, \\
\xi_4 &= B_6 \phi + B_7 Z_{,r},
\end{align*}
\]

where the \( B_i \) coefficients are given by,

\[
\begin{align*}
B_1 &= \frac{2a^2}{\lambda^2} \left( \frac{r^3}{a} \right)^6 \frac{D}{a^D}, \\
B_2 &= \frac{2a^2}{\lambda^2} \left( \frac{r^4 \Pi_2^3 \Pi_3^2}{a^4 D} \right)^2, \\
B_3 &= -\frac{a^2}{2a^2} \left( \frac{\Pi_3^2}{a^2 D} \right)^2 \Pi_2 \left\{ 4r^2 \Pi_2^3 \Phi_{\rho \rho} + r^3 \Pi_2^2 + l(l+1) \right\} \Pi_2 \left( \Phi_{\rho} \right)^2 + a^3 r^3 \Pi_2^3 \Pi_3^4 \\
&+ \frac{4r^2}{D} \left( D \Pi_2^3 \right) \Pi_2 + r \left[ 3a^2 l(l+1) - 8 \right] \Pi_2 \Pi_3^2 \\
&+ \frac{4}{r} l(l+1) \left[ a^2 l(l+1) - 3 \right] \Pi_2 \right\}, \\
B_4 &= \frac{2a}{\lambda} \left( \frac{r^3 \Pi_2^3}{a^2} \right)^2 \frac{D}{\Pi_2}, \\
B_5 &= \frac{a \rho}{\lambda} \left( \frac{r^3}{a} \right)^2 \Pi_2 \left\{ \left( l^2 + l + 2 \right) a^2 + 2 \right\} \Pi_2 + 2r a^2 D \Pi_3 + 2r \Phi_{\rho \rho} \Pi_2, \\
B_6 &= -B_4,
\end{align*}
\]

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The system of equations (104)–(107) is then rewritten in the following way:

\[
\frac{\partial \xi_i}{\partial t} = \sum_{j=1}^{4} \sum_{k=0}^{3} A_{jk}^i \frac{\partial^2 \xi_j}{\partial r^k},
\]  

(125)

or, by defining the column vector \( \vec{\xi} := (\xi_1, \xi_2, \xi_3, \xi_4)^T \), in matrix notation,

\[
\frac{\partial \vec{\xi}}{\partial t} = \sum_{k=0}^{3} A_k \frac{\partial^2 \vec{\xi}}{\partial r^k}.
\]  

(126)

The first row of the matrix corresponding to third-order radial derivatives is zero, that is, \( A_{13}^{ij} = 0 \), and thus only up to second-order derivatives of \( \xi_i \) appears in the equations. Note that the global differential order of the system of equations has been reduced in three (from up to sixth-order radial derivatives to up to just third order) by the above transformation (114)–(117).

In order to further simplify this set of equations, one could perform another change of variables that takes the matrix coefficients of the higher-order derivatives, in this case \( A_3 \), to its Jordan form. In the axial case such a transformation converted a set of two equations of second-order into a new set of an equation of second-order and another with no radial derivatives [23].

In the present case, all four eigenvalues of the matrix \( A_3 \) are zero and its corresponding Jordan form is,

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(127)

As it is usual, a similarity transformation, implemented by a matrix \( S \), relates the matrix \( A_3 \) with its Jordan form \( J \),

\[
J = S^{-1} A_3 S.
\]  

(128)

With this \( S \) matrix at hand, one can define the normal coordinates \( \vec{\omega} \) as

\[
\vec{\omega} := S^{-1} \vec{\xi}.
\]  

(129)

And the equations of motion for these new variables will take the following form:

\[
\frac{\partial \vec{\omega}}{\partial t} = J \frac{\partial^2 \vec{\omega}}{\partial r^3} + \sum_{k=0}^{2} \Omega_k \frac{\partial^2 \vec{\omega}}{\partial r^k}.
\]  

(130)

It is easy to see that these \( \Omega \) matrices can be written in terms of \( A_k \) matrices in combination with derivatives of the similarity matrix \( S \) in the following way,

\[
\Omega_2 = S^{-1} \left( 3A_3 S_s + A_2 S \right),
\]  

(131)

\[
\Omega_1 = S^{-1} \left( 3A_3 S_{sr} + 2A_2 S_s + A_1 S \right),
\]  

(132)

\[
\Omega_0 = S^{-1} \left( A_3 S_{rr} + A_2 S_{sr} + A_1 S_s + A_0 S - S_{,r} \right).
\]  

(133)
Unfortunately, these matrices are quite involve. In particular, and as opposed to what happened in the axial case, all the components of the matrix $\Omega_2$ are non-vanishing and thus second radial derivatives of all variables appear in all equations of the system (130). Therefore the only advantage of the normal variables $\vec{\omega}$ with respect to the initial variables $\vec{\xi}$ lies in the simplicity of the third-order radial derivative terms, which only appear in two of the variables. Nonetheless, this advantage might not be so relevant if one considers that our final interest is to obtain the perturbed metric in terms of the variables for which we solve. Reconstructing the perturbed metric from the $\vec{\omega}$ variables implies another change of variables more than reconstructing it from $\vec{\xi}$ variables. In addition, $\Omega_k$ matrices are more involve than their $A_k$ counterparts for all $k \leq 2$. The lengthy expressions of $A_k$ and $\Omega_k$ prevents us from providing them here explicitly. Even so, these matrices are available from the author by request.

6. Conclusions

In this paper a generalization of the Zerilli master variable for a specific spherical but dynamical background spacetime has been presented. In order to factorize and remove the angular dependence from the equations of motion, a decomposition on tensor spherical harmonics of the polar part of different perturbative variables has been performed. At linearized level each harmonic coefficient, characterized by $(l, m)$ angular numbers, decouple from the rest due to the symmetry of the background. The perturbative problem has then been formulated on a Hamiltonian framework, which shows very clearly the dynamical role of each object. As it is well known, the second variation of the Einstein–Hilbert action provides an action functional for the perturbative variables. In this way, an effective Hamiltonian can be defined for linearized variables, which is given as a linear combination of a physical Hamiltonian and four constraints, which are the linearized constraints of general relativity. These constraints are the generators of gauge transformations. Thus, in order to construct gauge-invariant master variables, one can perform a canonical transformation so that four of the new variables are equal to the constraints. In this way, for the considered background spacetime, one obtains two pairs of variables that encode the complete physical information of the problem: one pair corresponds to the polar mode of the gravitational wave and the other one to the scalar matter degree of freedom.

In order to be admissible, three conditions have been requested to this canonical transformation, which has been performed as five subsequent transformations to provide a clear view of each step. First, they should be algebraic transformations so that they do not involve any integration and can be performed explicitly. Second, they should not require dividing by any background object that could vanish. And third, they should have a well defined vacuum limit so that we obtain a generalization of the Zerilli variable. This latter in particular implies that the perturbative matter degrees of freedom cannot be used to solve for the perturbative constraints, since such a transformation would not be well defined in the limit that the perturbations of the scalar field vanish. The full transformation that has been proposed here completely fulfills the first and third criteria, but it is unclear whether it also satisfies the second one. In the last (fifth) transformation it turned out to be necessary to divide by the background coefficient $D$ defined in (91). This background coefficient is positive definite in the Schwarzschild case (102), but it is difficult to assert something about it in the general dynamical case.

In principle the master variable that has been found is not unique, since there is apparently much freedom in the canonical transformations that one could perform.
Nonetheless, the three imposed criteria reduce considerably this freedom. In particular note that, once the canonical transformation (43)–(52) is performed, these criteria almost completely single out the subsequent transformations. More specifically, the momenta \( \pi_3 \) and \( \pi_4 \) are the only variables that can be used to solve for the constraints (54) and (53) respectively. This leads to the form (65) of the linearized Hamiltonian constraint. In that expression none of the variables can be used to solve the constraint algebraically. Thus, next parametrized transformation (68)–(77) is performed in order to concentrate all derivatives in a unique full derivative. The term inside this full derivative is then promoted to one of the basic variables. Finally, in expression (90) the variable \( \hat{k}_3 \), which appears with no derivatives, is used to solve for the Hamiltonian constraint. Note that \( \hat{k}_5 \) could also be used to solve for that constraint, but then this transformation would not have a well defined limit when the perturbative matter degrees of freedom vanish and would then violate the third criterion.

Finally, the evolution equations obeyed by the master variables have been obtained. The differential order, in radial derivatives, was initially seven. Nonetheless, it is possible to redefine new master variables so that the highest radial derivative is of third order. Hence contrary to the Zerilli variable, which fulfills a wave equation, these master variables obey equations of higher (radial) order. This could be surprising since the equation of motion of a linear combination of objects, which obey second-order differential equations, is obviously of second-order in terms of the objects themselves. Nonetheless, the fact that these objects are expressed in terms of the combination itself (and its radial derivatives) can increase the differential order of the equation. This is exactly what happens in this case. Unfortunately the obtained equations are quite involved and it is not clear if they could be of practical use. Nevertheless, the fact that the mentioned canonical transformations can be performed algebraically and completely decouple the gauge from the physical degrees of freedom, turns out to be a relevant result in itself, which could pave the way for the construction of a polar master variable for any dynamical spherically symmetric background.

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