Dynamics of a multiplex neural network with delayed couplings

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Abstract  Multiplex networks have drawn much attention since they have been observed in many systems, e.g., brain, transport, and social relationships. In this paper, the non-linear dynamics of a multiplex network with three neural groups and delayed interactions is studied. The stability and bifurcation of the network equilibrium are discussed, and interesting neural activities of the network are explored. Based on the neuron circuit, transfer function circuit, and time delay circuit, a circuit platform of the network is constructed. It is shown that delayed couplings play crucial roles in the network dynamics, e.g., the enhancement and suppression of the stability, the patterns of the synchronization between networks, and the generation of complicated attractors and multi-stability coexistence.

Key words  neural network, time delay, synchronization, coexisting attractor

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1 Introduction

In the past few years, great efforts have been made on the investigations of multiplex networks due to their wide range of applicability in fields such as biology, physics, sociology, epidemiology, and engineering[1–3]. Multiplex networks are termed in disparate ways, e.g., interacting networks, networks of networks, multilevel networks, and hypergraphs[2]. In such networks, the nodes are distributed in individual networks, e.g., layers and loops, each of them accounts for a kind of interaction process existing among the nodes. For example, in the case of a C. elegans neuronal network, a proper way to describe it is a multiplex graph with two sub-networks, one for the chemical synaptic links and one for the gap junction interactions[2]. Moreover, due to the cortical connectivity of mammalian brains, the corticocortical network of
the cat can be constructed by modeling each node (cortical area) with a sub-network of interacting neurons[4–5]. Examples also include air transportation networks, power grids, coupled semiconductor lasers, communication networks, social networks, infectious disease, etc.[1–4,6–9].

The analysis of dynamical behaviors of multiplex networks is one of the focal issues in complex nonlinear systems, and has found important and extensive applications in science and engineering[2,10–16]. For example, experimental evidence shows that chaotic behaviors exist in the brain networks and associate with the information processing, cognitive function, memory storage, and retrieval[10–11]. In multiplex networks, the dynamic processes take place at the same time in interacting sub-networks with different structural and dynamical configurations. The interactions of the sub-networks open up a plethora of fascinating behaviors, e.g., complete/partial synchronization, intra/inter-layer synchronization, cluster synchronization, and explosive synchronization[7,12–13,17–18]. The phenomena of synchronization in networks are of great importance in nature, and have become focal subjects in many fields[7,12–13,19–23]. For instance, the synchronization of neurons plays a crucial role in the context of cognition and learning and the pathogenesis of several neurological diseases, e.g., Parkinson’s disease and essential tremor[7,19,21,24]. It is therefore important and essential to investigate the dynamical behaviors arising from the interactions of separate networks.

Since signal propagation is always non-instantaneous, time delays commonly exist in realistic systems[15,25–29]. In neural networks, for example, the switching speed of action potentials propagating across neuron axons is finite, and the time lapses occurring by both dendritic and synaptic processing are non-negligible[30–31]. Neglecting the time delay in a dynamical system usually leads to false or even wrong results. In recent years, the investigations on the dynamics of multiplex neural networks with time delays have been witnessed[17–18,32–37]. Previous studies were mostly devoted to the dynamical characterization of two-coupled networks since such models are fundamental and easily addressable, e.g., two-layer networks[7,12–13,36,38]. Little attention has been paid to neural networks made up of three or more interconnected networks. However, the multi-coupled structures are ubiquitous in neural systems[2,9,12]. In brain, many interconnected areas (sub-networks) can be seen, which usually consist of neural assemblies and their couplings represent the structural and functional interactions among them[9]. For instance, separate parallel neural loops in the cortex or thalamus operate through the basal ganglia and the interplay of them can lead to the generation of tremor oscillations in Parkinson’s disease and epilepsy[32]. Thus, disregarding the multiplex structure of networks may result in misunderstanding of the properties of neural systems.

Motivated by the above discussion, the purpose of this paper is to study the dynamical behaviors of a multiplex network, as shown in Fig. 1. The system consists of three networks, each of which has an arbitrary number of nodes and couplings between single neurons. Different time delays are introduced into the connections between networks.

Fig. 1 Structure of the multiplex network consisting of three sub-networks 1, 2, and 3 (color online)
The remaining part of this paper is organized as follows. In Section 2, the local and global stability and bifurcation of the trivial equilibrium of the network are analyzed. Case studies of numerical simulations are shown in Section 3. A circuit platform is designed to validate the obtained results in Section 4. Finally, conclusions are made in Section 5.

2 Model and stability analysis

As shown in Fig. 1, the network can be described by

\[
\begin{align*}
\dot{x}_i(t) &= \left\{
\begin{array}{l}
-x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j) + r_1 g(z_i(t - \tau_1)), \quad i = 1, \\
-x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j), \quad 2 \leq i \leq n,
\end{array}
\right. \\
\dot{y}_i(t) &= \left\{
\begin{array}{l}
y_i(t) + \sum_{j=1}^{n} b_{ij} f(y_j) + r_2 g(x_i(t - \tau_2)), \quad i = 1, \\
y_i(t) + \sum_{j=1}^{n} b_{ij} f(y_j), \quad 2 \leq i \leq n,
\end{array}
\right. \\
\dot{z}_i(t) &= \left\{
\begin{array}{l}
z_i(t) + \sum_{j=1}^{n} c_{ij} f(z_j) + r_3 g(y_i(t - \tau_3)), \quad i = 1, \\
z_i(t) + \sum_{j=1}^{n} c_{ij} f(z_j), \quad 2 \leq i \leq n,
\end{array}
\right. \\
\end{align*}
\]

(1)

where \(i, j = 1, 2, 3, \ldots, n\), and \(x_i, y_i, \) and \(z_i\) denote the states of the \(i\)th neuron in Networks 1, 2, and 3, respectively. \(a_{ij}, b_{ij}, \) and \(c_{ij}\) are the connection weights within the networks. \(r_1, r_2, \) and \(r_3\) are the coupling strengths between the first neuron of each sub-network. \(\tau_i\) represents the time delay in the couplings. Without loss of generality, the functions \(f\) within the individual sub-networks and the function \(g\) between sub-networks are absolutely smooth and satisfy \(f(0) = 0\) and \(g(0) = 0\). In this network, the local kinetics of each node is described by the Hopfield neuron\(^{[30]}\).

The linearization of Eq. (1) at the trivial equilibrium of the network can be written as follows:

\[
\dot{u}(t) = -u(t) + L_1 u(t) + r_1 L_2 u(t - \tau_1) + r_2 L_3 u(t - \tau_2) + r_3 L_4 u(t - \tau_3),
\]

(2)

where

\[
u(t) = (x_1, x_2, x_3, \ldots, x_n, \ y_1, y_2, y_3, \ldots, y_n, \ z_1, z_2, z_3, \ldots, z_n)^T, \]

\[
a_{ij} = a_{ij} f'(0), \quad \beta_{ij} = b_{ij} f'(0), \quad \gamma_{ij} = c_{ij} f'(0), \quad k_i = r_i g'(0),
\]

\[
L_1 = \begin{pmatrix} A & O & O \\ O & B & O \\ O & O & C \end{pmatrix}, \quad L_2 = \begin{pmatrix} O & O & K \\ O & O & O \\ O & O & O \end{pmatrix}, \quad L_3 = \begin{pmatrix} O & O & O \\ K & O & O \\ O & O & O \end{pmatrix}, \quad L_4 = \begin{pmatrix} O & O & O \\ O & O & O \\ O & K & O \end{pmatrix},
\]

\[
K = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix},
\]

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and eliminating the harmonic terms, one arrives at roots of the characteristic equation. By separating the real and imaginary parts of \( \Delta \), where 

\[
\begin{align*}
B &= \begin{pmatrix}
\beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\
\beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{n1} & \beta_{n2} & \cdots & \beta_{nn}
\end{pmatrix},
\quad C &= \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\
\gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn}
\end{pmatrix}.
\end{align*}
\]

In the above equations, \( O \) is an \( n \times n \) zero matrix, \( i, j, k, \cdots, n, \) and \( l = 1, 2, 3, \) 

The characteristic equation of the network is 

\[
M(\lambda, \tau) = \begin{pmatrix}
(\lambda + 1)I - A & O & -k_1e^{-\lambda\tau_1}K \\
-k_2e^{-\lambda\tau_2}K & (\lambda + 1)I - B & O \\
O & -k_3e^{-\lambda\tau_3}K & (\lambda + 1)I - C
\end{pmatrix},
\tag{3}
\]

where \( I \) is an \( n \times n \) identity matrix. Thus, the characteristic equation of the network reads 

\[
\Delta(\lambda, \tau) = |(\lambda + 1)I - A| \cdot |(\lambda + 1)I - B| \cdot |(\lambda + 1)I - C|
- k_1e^{-\lambda\tau_1}I \cdot |(\lambda + 1)I_{n-1} - A_1| \cdot |(\lambda + 1)I_{n-1} - B_1| \cdot |(\lambda + 1)I_{n-1} - C_1|
= A(\lambda)B(\lambda)C(\lambda) - ke^{-\lambda\tau}A_1(\lambda)B_1(\lambda)C_1(\lambda)
= P(\lambda) - ke^{-\lambda\tau}Q(\lambda) = 0,
\tag{4}
\]

where \( I_{n-1} \) is an \( (n-1) \times (n-1) \) identity matrix, and 

\[
k = k_1k_2k_3, \quad \tau = \tau_1 + \tau_2 + \tau_3, \quad P(\lambda) = A(\lambda)B(\lambda)C(\lambda), \quad Q(\lambda) = A_1(\lambda)B_1(\lambda)C_1(\lambda),
\]

\[
A(\lambda) = |(\lambda + 1)I - A|, \quad B(\lambda) = |(\lambda + 1)I - B|, \quad C(\lambda) = |(\lambda + 1)I - C|,
\]

\[
A_1(\lambda) = |(\lambda + 1)I_{n-1} - A_1|, \quad B_1(\lambda) = |(\lambda + 1)I_{n-1} - B_1|, \quad C_1(\lambda) = |(\lambda + 1)I_{n-1} - C_1|.
\]

In fact, \( A_1, B_1, \) and \( C_1 \) represent the connection matrices of the three sub-networks without the first neuron, respectively, i.e., 

\[
A_1 = \begin{pmatrix}
\alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\
\alpha_{32} & \alpha_{33} & \cdots & \alpha_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n2} & \alpha_{n3} & \cdots & \alpha_{nn}
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
\beta_{22} & \beta_{23} & \cdots & \beta_{2n} \\
\beta_{32} & \beta_{33} & \cdots & \beta_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{n2} & \beta_{n3} & \cdots & \beta_{nn}
\end{pmatrix}, \quad C_1 = \begin{pmatrix}
\gamma_{22} & \gamma_{23} & \cdots & \gamma_{2n} \\
\gamma_{32} & \gamma_{33} & \cdots & \gamma_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn}
\end{pmatrix}.
\]

The stability of the trivial equilibrium of the network can be determined by the root distribution of the characteristic equation (3). When the network has no coupling time delays, the Routh-Hurwitz criteria can be applied. As the time delays vary, let \( \lambda = \pm i\omega \) (\( \omega > 0 \)) be the roots of the characteristic equation. By separating the real and imaginary parts of \( \Delta(\iota, \tau) = 0 \) and eliminating the harmonic terms, one arrives at 

\[
D(\omega) = P_R^2(\omega) + P_I^2(\omega) - k^2(Q_R^2(\omega) + Q_I^2(\omega)) = 0,
\]

where \( P_R(\omega) = \text{Re}(P(\iota\omega)) \), \( P_I(\omega) = \text{Im}(P(\iota\omega)) \), \( Q_R(\omega) = \text{Re}(Q(\iota\omega)) \), and \( Q_I(\omega) = \text{Im}(Q(\iota\omega)) \). 

Suppose that \( D(\omega) = 0 \) has some positive roots \( \omega_j \). Then, a set of critical time delays can be obtained by \( \tau_{ij} = (\theta_j + 2l\pi)/\omega_j \) (\( l = 0, 1, 2, \cdots \)), where \( \theta_j \in [0, 2\pi) \) yields sets of triangle equations 

\[
\cos \theta_j = \frac{P_R(\omega_j)Q_R(\iota\omega_j) + P_I(\omega_j)Q_I(\iota\omega_j)}{k|Q(\iota\omega_j)|^2}, \quad \sin \theta_j = \frac{P_R(\omega_j)Q_I(\iota\omega_j) - P_I(\omega_j)Q_R(\iota\omega_j)}{k|Q(\iota\omega_j)|^2}.
\]
To determine the variation direction of the real part of Eq. (4) with respect to the time delay, the differentiation of $\lambda$ with respect to $\tau$ yields

$$P'(\lambda)\lambda'(\tau) - kQ'(\lambda)\lambda'(\tau)e^{-\lambda\tau} + kQ(\lambda(\tau) + \tau\lambda'(\tau))e^{-\lambda\tau} = 0.$$ 

Then, one arrives at

$$\text{Re} \left( \frac{d\lambda}{d\tau} \right)_{\lambda=i\omega} = \text{Re} \left( -\frac{i\omega kQ(i\omega)}{P'(i\omega)e^{i\omega\tau} - kQ'(i\omega) + k\tau Q(i\omega)} \right) = 0.5\omega D'(\omega) |S(i\omega)|^2,$$

where $S(i\omega) = P'(i\omega)e^{i\omega\tau} - kQ'(i\omega) + k\tau Q(i\omega)$. It follows that

$$\text{sgn} \left( \frac{d\lambda}{d\tau} \right)_{\lambda=i\omega} = \text{sgn} D'(\omega).$$

Thus, the system undergoes finite stability switches, and must become unstable with an increase in the time delay when $D(\omega) = 0$ has positive roots. On the other hand, it is delay-independently stable or unstable for any given time delay when $D(\omega) = 0$ has no positive root.$^{[25,40]}$

In fact, the stability of the trivial equilibrium of the multiplex network is determined by the characteristic roots of the isolated sub-networks and the product of the coupling strength and the sum of the time delay.

Let $n = 3$, $a_{11} = b_{11} = c_{11} = -1.4$, $a_{12} = b_{12} = c_{12} = 1.3$, $a_{13} = b_{13} = c_{13} = -6$, $a_{21} = b_{21} = c_{21} = 1.1$, $a_{22} = b_{22} = c_{22} = 0$, $a_{23} = b_{23} = c_{23} = 2.6$, $a_{31} = b_{31} = c_{31} = 2.4$, $a_{32} = b_{32} = c_{32} = -2$, $a_{33} = b_{33} = c_{33} = 4$, $r_1 = r_2 = r_3$, $\tau_1 = \tau_2 = \tau_3 = \tau_n$, and $f = g = \tanh$.

Based on the above analysis, the region in the parameter planes indicating the stability of the trivial equilibrium of the triplex network is given in Fig. 2. Along the black and red curves, the characteristic equation of the system has a pair of pure imaginary roots. Moreover, the characteristic equation adds a pair of conjugate roots with positive real parts for each crossing black curves while reduces a pair conjugate roots with positive real parts for each crossing red curves. Besides, the network has delay-independent and delay-dependant stability regions.

Fig. 2  Stability region of the trivial equilibrium of a triplex neural network (color online)

Furthermore, the global asymptotic stability of the trivial equilibrium of the network is discussed as follows. In neural systems, due to the input-output relation of neurons, the activation functions $f$ and $g$ are often assumed to satisfy the normalization, monotonicity, boundedness, and concavity conditions, i.e., $f'(0) = g'(0) = 1$; $f'(v) > 0$ and $g'(v) > 0$ for all $v \in \mathbb{R}$; $-\infty < \lim_{v \to \pm \infty} f(v) < +\infty$ and $-\infty < \lim_{v \to \pm \infty} g(v) < +\infty$; $vf''(v) < 0$ and $vg''(v) < 0$ for all $v \neq 0$. The sigmoid functions, e.g., hyperbolic tangent functions, are widely used in neural networks.
Theorem 1 If \( np + q < 1 \), the trivial equilibrium of the network is globally asymptotically stable for all time delays, where \( p = \max \{|a_{ij}|, |b_{ij}|, |c_{ij}|\} \), and \( q = \max \{|r_1|, |r_2|, |r_3|\} \).

Proof Define the following Lyapunov function:

\[
V(t) = \sum_{i=1}^{n} (x_i^2(t) + y_i^2(t) + z_i^2(t)) + |r_1| \int_{t-\tau_1}^{t} g^2(z_1(s))ds \\
+ |r_2| \int_{t-\tau_2}^{t} g^2(x_1(s))ds + |r_3| \int_{t-\tau_3}^{t} g^2(y_1(s))ds.
\]

Then, one arrives at

\[
\dot{V}(t) = 2 \sum_{i=1}^{n} (x_i(t)\dot{x}_i(t) + y_i(t)\dot{y}_i(t) + z_i(t)\dot{z}_i(t)) + |r_1|(g^2(z_1(t)) - g^2(z_1(t - \tau_1))) \\
+ |r_2|(g^2(x_1(t)) - g^2(x_1(t - \tau_2))) + |r_3|(g^2(y_1(t)) - g^2(y_1(t - \tau_3))) \\
\leq -2 \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2) + na \sum_{i=1}^{n} (x_i^2 + f^2(x_i)) + nb \sum_{i=1}^{n} (y_i^2 + f^2(y_i)) \\
+ nc \sum_{i=1}^{n} (z_i^2 + f^2(z_i)) + |r_1|(z_1^2 + g^2(z_1)) + |r_2|(x_1^2 + g^2(x_1)) + |r_3|(y_1^2 + g^2(y_1)),
\]

where \( a = \max \{|a_{ij}|\} \), \( b = \max \{|b_{ij}|\} \), and \( c = \max \{|c_{ij}|\} \). Since \( vf''(v) < 0 \) and \( vg''(v) < 0 \) for \( v \neq 0 \), one has \( f'(v) \leq f'(0) = 1 \) and \( g'(v) \leq g'(0) = 1 \). In addition, one obtains \( f(v_i(t)) = \varphi_i(t)v_i(t) \) and \( g(v_i(t)) = \phi_i(t)v_i(t) \), where

\[
\varphi_i(t) = \int_{0}^{1} f'(pv_i(t))dp, \quad \phi_i(t) = \int_{0}^{1} g'(pv_i(t))dp.
\]

There exist \( \varphi^* \in (0, 1) \) and \( \phi^* \in (0, 1) \) such that \( \varphi_i(t) \leq \varphi^* \leq 1 \) and \( \phi_i(t) \leq \phi^* \leq 1 \). Hence, one has

\[
\dot{V}(t) \leq -2(1 - np - q) \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2).
\]

Then, \( \dot{V}(t) < 0 \) holds true for \( np + q < 1 \) when \( v \neq 0 \). This completes the proof.

3 Case studies

In this section, the activation functions of neurons are chosen as \( f = g = \tanh \), which is a typical sigmoid function and has been widely used in neural networks.

Case 1 \( n = 3 \), \( a_{11} = b_{11} = c_{11} = -1.4 \), \( a_{12} = b_{12} = c_{12} = 1.3 \), \( a_{13} = b_{13} = c_{13} = -6 \), \( a_{21} = b_{21} = c_{21} = 1.1 \), \( a_{22} = b_{22} = c_{22} = 0 \), \( a_{23} = b_{23} = c_{23} = 2.6 \), \( a_{31} = b_{31} = c_{31} = 2.4 \), \( a_{32} = b_{32} = c_{32} = -2 \), \( a_{33} = b_{33} = c_{33} = 4 \), \( r_1 = r_2 = r_3 = 0.17 \), and \( \tau_1 = \tau_2 = \tau_3 = \tau_5 \).

The trivial equilibrium of the network free of coupling time delays is locally asymptotically stable according to the Routh-Hurwitz criteria. Solving the polynomial \( D(\omega) = 0 \) gives two positive and simple roots \( \omega_1 = 3.26 \) and \( \omega_2 = 3.17 \). Then, two sets of critical time delays can be obtained as \( \tau_{1,t} = 0.15, 2.08, 4.00, \ldots \), and \( \tau_{2,t} = 1.26, 3.25, 5.23, \ldots \). It is easy to check that \( D'(\omega_1) > 0 \) and \( D'(\omega_2) < 0 \) hold. From \( \text{sgn} \Re \left( \frac{D(\tau)}{\omega} \right)_{\theta = \omega} = \text{sgn} D'(\omega) \), the characteristic equation of the network adds a new pair of conjugate roots with positive real parts for each crossing at \( \tau_{1,t} \), but reduces a pair of conjugate roots with positive real parts for each crossing at \( \tau_{2,t} \). Therefore, the trivial equilibrium of the network is locally asymptotically stable for \( \tau \in [0, \tau_{1,0}) \cup \cdots \cup (\tau_{1,t}, \tau_{1,t+1}) \cup \cdots \cup (\tau_{2,14}, \tau_{1,15}) \), and becomes unstable for \( \tau \in \)
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$(\tau_1,0,\tau_2,0) \cup \cdots \cup (\tau_1,l,\tau_2,l) \cup \cdots \cup (\tau_1,15,+,\infty)$. Figure 3(a) shows that the trivial equilibrium of the network loses its stability and a branch of periodic oscillations arising from the Hopf bifurcation comes into being when $\tau = 0.3 \in (\tau_1,0,\tau_2,0)$. Obviously, the corresponding neurons in sub-networks oscillate with the same waveform and in phase with each other, i.e., $x_i = y_i = z_i$.

Figure 3(b) illustrates that the trivial equilibrium remains stable when $\tau = 1.8 \in (\tau_2,0,\tau_1,1)$. Figure 3(c) gives that the trivial equilibrium becomes unstable again and periodic oscillations occur when $\tau = 2.4 \in (\tau_1,1,\tau_2,1)$. As shown in Fig. 3(c), the neurons in the sub-networks move with the same waveform and different phases. The red, green, and blue lines represent the responses of the first neurons in the three sub-networks. It is shown that the network undergoes the stable rest state, completely synchronous periodic oscillation, stable rest state, and periodic oscillations with different phases as the time delay increases.

Fig. 3

Responses of the multiplex network when $a_{11} = b_{11} = c_{11} = -1.4$, $a_{12} = b_{12} = c_{12} = 1.3$, $a_{13} = b_{13} = c_{13} = -6$, $a_{21} = b_{21} = c_{21} = 1.1$, $a_{22} = b_{22} = c_{22} = 0$, $a_{23} = b_{23} = c_{23} = 2.6$, $a_{31} = b_{31} = c_{31} = 2.4$, $a_{32} = b_{32} = c_{32} = -2$, $a_{33} = b_{33} = c_{33} = 4$, $\tau_1 = \tau_2 = \tau_3 = 0.17$, and $\tau_1 = \tau_2 = \tau_3 = \tau_s$ (color online).

To show the effects of the property of the coupling on the network dynamics, let the parameters within the sub-networks be the same as those in the above case study and the time delay $\tau_s = 0.1$. Figure 4(a) gives the asynchronous periodic oscillations with $x_i = -y_i = z_i$ when $k_1 = -k_2 = -k_3$. As shown in Fig. 4(a), the neurons of Networks 1 and 3 oscillate with the same waveform and in phase with each other, but the neurons of Networks 1 and 2 move with the same waveform and half a period out of phase with each other. Figure 4(b) illustrates that the neurons of Networks 1 and 2 move synchronously, but the neurons of Networks 1 and 3 oscillate out-of-phase. Figure 4(c) shows the asynchronous periodic oscillations with $x_i = -y_i = -z_i$ when $k_1 = k_2 = -k_3$. In this case, the neurons of Networks 2 and 3 move synchronously, but the neurons of Networks 1 and 2 oscillate out-of-phase. As shown in Fig. 4, the red solid, green dashed, and blue dotted curves represent the responses of the first neurons.
Fig. 4  Responses of the network when $\tau_s = 0.1$: (a) asynchronous periodic oscillation with $x_i = -y_i = z_i$; (b) asynchronous periodic oscillation with $x_i = y_i = -z_i$; (c) asynchronous periodic oscillation with $x_i = -y_i = -z_i$ (color online)

in the sub-networks. It is interesting that the network exhibits four patterns of synchronization between networks for the same product of coupling strengths, as depicted in Figs.3(a) and 4. It follows that the property of the coupling can be used to regulate the patterns of synchronization between sub-networks.

Case II $n = 3$, $a_{11} = b_{11} = c_{11} = 1.5$, $a_{12} = b_{12} = c_{12} = 2.9$, $a_{13} = b_{13} = c_{13} = 0.7$, $a_{21} = b_{21} = c_{21} = -2$, $a_{22} = b_{22} = c_{22} = 1.18$, $a_{23} = b_{23} = c_{23} = 0$, $a_{31} = b_{31} = c_{31} = 2.98$, $a_{32} = b_{32} = c_{32} = -10$, $a_{33} = b_{33} = c_{33} = 0.47$, $r_1 = r_2 = r_3 = 0.2$, and $r_1 = r_2 = r_3 = \tau_s$.

Figure 5 shows the coexistence of two period-2 orbits and two period-4 oscillations under different initial conditions when $\tau_s = 0.1$. It is seen that the period-2 orbits on the $x_1-x_2$ plane are colored in blue and red lines, while the period-4 trajectories are colored in purplish red and black lines. The blue, red, purplish red, and black lines correspond to the initial conditions, which are defined as $(x_1(0), x_2(0), x_3(0), y_1(0), y_2(0), y_3(0), z_1(0), z_2(0), z_3(0))$, IC1 (0.5, 0.1, 0.2, 0.7, 0.8, 0.3, 0.6, 0.9, 0.4), IC2 (−0.5, −0.1, −0.2, −0.7, −0.8, −0.3, −0.6, −0.9, −0.4), IC3 (0.5, −0.1, 0.2, 0.7, 0.8, 0.3, 0.6, −0.9, 0.4), and IC4 (−0.5, 0.1, −0.2, −0.7, −0.8, −0.3, −0.6, 0.9, −0.4), respectively. By increasing the time delay $\tau_s = 0.6$, Fig.6 illustrates the coexistence of a new pair of period-2 oscillations and two separated chaotic motions. As shown in Fig. 7, the period-2 responses disappear and a pair of period-4 solutions come into being when $\tau_s = 0.8$. Figure 8 gives the coexistence of two chaotic motions and two period-2 responses for $\tau_s = 1.2$. As shown in Figs.5–8, different types of multiple coexisting attractors are observed when the coupling time delays vary.

Figure 9 gives the bifurcation diagram as a function of the coupling time delay $\tau_s$. The Poincaré section is defined by $\sum = \{(\tau_n, x_1) : (x_2 = 0, \dot{x}_2 > 0)$. When the coupling time delay
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Fig. 5  Phase trajectories of the network when $\tau_s = 0.1$: (a) two period-2 oscillations under the initial conditions IC1 and IC2; (b) two period-4 orbits under the initial conditions IC3 and IC4 (color online)

Fig. 6  Phase trajectories of the network when $\tau_s = 0.6$: (a) two period-2 oscillations under the initial conditions IC1 and IC2; (b) two chaotic motions under the initial conditions IC3 and IC4 (color online)

Fig. 7  Phase trajectories of the network when $\tau_s = 0.8$: (a) two period-4 oscillations under the initial conditions IC1 and IC2; (b) two chaotic motions under the initial conditions IC3 and IC4 (color online)

varies, the multiplex network exhibits interesting and complicated behaviors, e.g., multi-periodic orbits and chaotic attractors, showing that time delay can be used to regulate the dynamic performances of networks, including the generation and transition of different complex oscillations.
Fig. 8  Phase trajectories of the network when $\tau_s = 1.2$: (a) two separated chaotic attractors under the initial conditions IC1 and IC2; (b) two coexisting period-2 orbits under the initial conditions IC3 and IC4 (color online)

Fig. 9  Bifurcation diagrams on the Poincaré section under the initial conditions IC1, IC2, IC3, and IC4 when the coupling time delay varies (color online)

4 Circuit implementation

The circuit is constructed based on Hopfield neuron circuit, transfer function circuit, and time delay circuit. As shown in Fig. 10, the Hopfield neuron circuit unit is achieved by inverting adder circuit, integral circuit, and inverting circuit. Blocks with tanh/−tanh represent the positive/negative hyperbolic tangent function circuit between neurons. The hyperbolic tangent function circuit unit is constructed by crucial bipolar transistors, operational amplifiers, potentiometers, and ±15 V direct current (DC) voltage sources.[41–42] Blocks delay1, delay2, and delay3 are time delay circuits. The time delay circuit unit can be achieved by a resistor-capacitor
(RC) low-pass filter consisting of operational amplifiers, capacitors, and resistors. The circuit state equations can be established as follows:

\[
\begin{align*}
R_{x1}C_{x1}\frac{dX_1}{dt} &= -\frac{R_{x4}}{R_{x7}}X_1 + \frac{R_{x4}}{R_{x8}}f(X_1) + \frac{R_{x4}}{R_{x9}}f(X_2) + \frac{R_{x4}}{R_{x10}}f(X_3) + \frac{R_{x4}}{R_{x11}}f(Z_1(t' - \tau_1')) , \\
R_{x2}C_{x2}\frac{dX_2}{dt} &= -\frac{R_{x5}}{R_{x12}}X_2 + \frac{R_{x5}}{R_{x13}}f(X_1) + \frac{R_{x5}}{R_{x14}}f(X_2) + \frac{R_{x5}}{R_{x15}}f(X_3) , \\
R_{x3}C_{x3}\frac{dX_3}{dt} &= -\frac{R_{x6}}{R_{x16}}X_3 + \frac{R_{x6}}{R_{x17}}f(X_1) + \frac{R_{x6}}{R_{x18}}f(X_2) + \frac{R_{x6}}{R_{x19}}f(X_3) , \\
R_{y1}C_{y1}\frac{dY_1}{dt} &= -\frac{R_{y4}}{R_{y7}}Y_1 + \frac{R_{y4}}{R_{y8}}f(Y_1) + \frac{R_{y4}}{R_{y9}}f(Y_2) + \frac{R_{y4}}{R_{y10}}f(Y_3) + \frac{R_{y4}}{R_{y11}}f(X_1(t' - \tau_2')) , \\
R_{y2}C_{y2}\frac{dY_2}{dt} &= -\frac{R_{y5}}{R_{y12}}Y_2 + \frac{R_{y5}}{R_{y13}}f(Y_1) + \frac{R_{y5}}{R_{y14}}f(Y_2) + \frac{R_{y5}}{R_{y15}}f(Y_3) , \\
R_{y3}C_{y3}\frac{dY_3}{dt} &= -\frac{R_{y6}}{R_{y16}}Y_3 + \frac{R_{y6}}{R_{y17}}f(Y_1) + \frac{R_{y6}}{R_{y18}}f(Y_2) + \frac{R_{y6}}{R_{y19}}f(Y_3) , \\
R_{z1}C_{z1}\frac{dZ_1}{dt} &= -\frac{R_{z4}}{R_{z7}}Z_1 + \frac{R_{z4}}{R_{z8}}f(Z_1) + \frac{R_{z4}}{R_{z9}}f(Z_2) + \frac{R_{z4}}{R_{z10}}f(Z_3) + \frac{R_{z4}}{R_{z11}}f(Y_1(t' - \tau_1')) , \\
R_{z2}C_{z2}\frac{dZ_2}{dt} &= -\frac{R_{z5}}{R_{z12}}Z_2 + \frac{R_{z5}}{R_{z13}}f(Z_1) + \frac{R_{z5}}{R_{z14}}f(Z_2) + \frac{R_{z5}}{R_{z15}}f(Z_3) , \\
R_{z3}C_{z3}\frac{dZ_3}{dt} &= -\frac{R_{z6}}{R_{z16}}Z_3 + \frac{R_{z6}}{R_{z17}}f(Z_1) + \frac{R_{z6}}{R_{z18}}f(Z_2) + \frac{R_{z6}}{R_{z19}}f(Z_3).
\end{align*}
\]
where $X_j$, $Y_j$, and $Z_j$ are the output voltages in the circuit, $f$ is the hyperbolic tangent function, $R_{x1}$, $R_{y1}$, and $R_{z1}$ represent resistors, $C_{x1}$, $C_{y1}$, and $C_{z1}$ stand for capacitors, $\tau_{j}^1$, $\tau_{j}^2$, and $\tau_{j}^3$ denote time delays in the blocks delay1, delay2, and delay3, respectively, $j = 1, 2, 3, i = 1, 2, 3, \ldots, 19$, and

\[
R_{x1} = R_{y1} = R_{z1} = 1 \text{k}\Omega, \quad R_{x2} = R_{y2} = R_{z2} = 1 \text{k}\Omega,
\]
\[
R_{x3} = R_{y3} = R_{z3} = 1 \text{k}\Omega, \quad R_{x4} = R_{y4} = R_{z4} = 10 \text{k}\Omega,
\]
\[
R_{x5} = R_{y5} = R_{z5} = 10 \text{k}\Omega, \quad R_{x6} = R_{y6} = R_{z6} = 10 \text{k}\Omega,
\]
\[
R_{x7} = R_{y7} = R_{z7} = 10 \text{k}\Omega, \quad R_{x8} = R_{y8} = R_{z8} = 6.67 \text{k}\Omega,
\]
\[
R_{x9} = R_{y9} = R_{z9} = 3.45 \text{k}\Omega, \quad R_{x10} = R_{y10} = R_{z10} = 14.29 \text{k}\Omega,
\]
\[
R_{x11} = R_{y11} = R_{z11} = 50 \text{k}\Omega, \quad R_{x12} = R_{y12} = R_{z12} = 10 \text{k}\Omega,
\]
\[
R_{x13} = R_{y13} = R_{z13} = 5 \text{k}\Omega, \quad R_{x14} = R_{y14} = R_{z14} = 8.47 \text{k}\Omega,
\]
\[
R_{x16} = R_{y16} = R_{z16} = 10 \text{k}\Omega, \quad R_{x17} = R_{y17} = R_{z17} = 3.36 \text{k}\Omega,
\]
\[
R_{x18} = R_{y18} = R_{z18} = 1 \text{k}\Omega, \quad R_{x19} = R_{y19} = R_{z19} = 21.27 \text{k}\Omega, \quad R_{y1} = 1 \text{k}\Omega,
\]
\[
C_{x1} = C_{y1} = C_{z1} = C_{x2} = C_{y2} = C_{z2} = C_{x3} = C_{y3} = C_{z3} = 1 \mu\text{F}.
\]

It is easy to check that Eq. (1) is the dimensionless form of the circuit equation (6), where

\[
t = \frac{1}{R_{x1}C_{x1}}t', \quad \tau_j = \frac{1}{R_{x1}C_{x1}}\tau_j'.
\]

Moreover, the connections of $R_{x15}$, $R_{y15}$, and $R_{z15}$ should be removed when $a_{23} = b_{23} = c_{23} = 0$.

Figures 11–14 show the phase portraits of the output voltages in the circuit based on the Multisim electronic circuit simulator. As shown in Figs. 11–14, when the coupling time delay varies, the circuit exhibits different patterns of multiple coexisting attractors, which are consistent with the phenomena given in Figs. 5–8.

![Phase portraits of output voltages in the circuit when $\tau_j'' = 0.1$ ms (color online)](image)

Fig. 11 Phase portraits of output voltages in the circuit when $\tau_j'' = 0.1$ ms (color online)

5 Conclusions

In biological and physiological systems, the interacting neural networks are crucial for the function of the brain and the efficient processing of information. The dynamical behaviors of
Fig. 12  Phase portraits of output voltages in the circuit when $\tau'_s = 0.6$ ms (color online)

Fig. 13  Phase portraits of output voltages in the circuit when $\tau'_s = 0.8$ ms (color online)

Fig. 14  Phase portraits of output voltages in the circuit when $\tau'_s = 1.2$ ms (color online)

multiplex neural systems are revealed through an example of three coupled networks, each of which has an arbitrary number of neurons. In the parameter plane of the product of coupling strength and the sum of time delays, the delay-independent and delay-dependent regions of the network equilibrium are shown. By regarding the sum of the coupling time delay as the pa-
rameter, various dynamical phenomena are observed, e.g., multiple stability switches, different patterns of periodic oscillations, the coexistence of two period-2 and two period-4 responses, the coexistence of two period-2 orbits and two chaotic motions, and the coexisting period-4 and chaotic attractors. Moreover, it is found that the excitatory and inhibitory couplings can induce complete and partial synchronization between sub-networks for the same product of coupling strengths. An electronic circuit is designed, and the phenomena agree with the revealed results. The obtained results in this paper can lead to a broader understanding in the mechanisms of the rhythms and complex evolution patterns of neural systems.

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