Conservation law for distributed entanglement of formation and quantum discord

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We present a direct relation, based upon a monogamic principle, between entanglement of formation (EOF) and quantum discord (QD), showing how they are distributed in an arbitrary tripartite pure system. By extending it to a paradigmatic situation of a bipartite system coupled to an environment, we demonstrate that the EOF and the QD obey a conservation relation. By means of this relation we show that in the deterministic quantum computer with one pure qubit the protocol has the ability to rearrange the EOF and the QD, which implies that quantum computation can be understood on a different basis as a coherent dynamics where quantum correlations are distributed between the qubits of the computer. Furthermore, for a tripartite mixed state we show that the balance between distributed EOF and QD results in a stronger version of the strong subadditivity of entropy.

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I. INTRODUCTION

Quantum discord (QD) is a measure of quantum correlation defined by Ollivier and Zurek almost ten years ago [1] and, yet, a subject of increasing interest today [2]. It is well known that, for a bipartite pure state, the definition of QD coincides with that of the entanglement of formation (EOF). But it has remained an open question how those two quantities would be related for general mixed states. Here, we present this desired relation for arbitrarily mixed states and show that the EOF and the QD obey a monogamic relation. Surprisingly, this necessarily requires an extension of the bipartite mixed system to its tripartite purified version. Nonetheless, we obtain a conservation relation for the distribution of EOF and QD in the system - the sum of all possible bipartite entanglement shared with a particular subsystem, as given by the EOF, cannot be increased without increasing, by the same amount, the sum of all QD shared with this same subsystem. When extended to the case of a tripartite mixed state, this relation results in a new proof of the strong subadditivity of entropy, with stronger bounds depending on the balance between the sum of EOF and the sum of QD shared with a particular subsystem.

As an example of the importance of this conservation relation, we explore the distribution of entanglement in the deterministic quantum computation with one single pure qubit and a collection of $N$ mixed states (DQC1). The algorithm, developed by Knill and Laflamme [3], is able to perform exponentially faster computation of important tasks, [4, 5] when compared with well-known classical algorithms, without any entanglement between the pure qubit and the mixed ones [4]. Arguably, the power of the quantum computer is supposed to be related to QD, rather than entanglement [6]. Here, using the conservation relation, we have shown that even in the supposedly entanglement-free quantum computation there is a certain amount of multipartite entanglement between the qubits and the environment, which is responsible for the non-zero QD (See Fig. 1).

II. CONSERVATION RELATION

Let us first consider an arbitrary system represented by a density matrix $\rho_{ABE}$ with $A$ and $B$ representing two subsystems and $E$ representing the environment. It is important to emphasize that the environment, here, is constituted by the universe minus the subsystems $A$ and $B$, since, in this case, $\rho_{ABE}$ is a pure density matrix. There is an important monogamic relation between the entanglement of formation (EOF) [8] and the classical correlation (CC) [9] between the two subsystems.

FIG. 1. (Color Online) Schematic illustration of the DQC1. $A$ represents the pure qubit, $B$ the maximally mixed state, obtained through maximal entanglement with the environment $E$. From the left to the right, $B$ is initially entangled with $E$ (purple bar). The protocol is then executed and $A$, although not directly entangled with $B$, gets entangled with the pair $BE$ as the QD between $A$ and $B$ increase.
developed by Koashi and Winter [10], that we employ to understand the distribution of entanglement. It is given by

$$E_{AB} + J^x_{AE} = S_A,$$  

where $E_{AB} = E(\rho_{AB})$ is the EOF between $A$ and $B$, $J^x_{AE} = J^x(\rho_{AE})$ is the CC between $A$ and $E$, and $S_A = S(\rho_A)$ is the usual Shannon entropy [11] of $A$. Further, $\rho_{AB} = \text{Tr}_E \{ \rho_{ABE} \}$ and analogously for $\rho_{AE}$ and $\rho_A$. Explicitly, CC reads $J^x_{AE} = \max_p \{ S(\rho_A) - \sum_x p_x S(\rho^x_A) \}$ where the maximum is taken over all positive operator valued measurements $\{ \Pi^x_E \}$ performed on subsystem $E$, with probability of $x$ as an outcome, $p_x = \text{Tr}_A \{ \Pi^x_E \rho_{ABE} \Pi^x_E \}$ and $\rho^x_A = \text{Tr}_E \{ \Pi^x_E \rho_{ABE} \Pi^x_E \}/p_x$. One can easily understand Eq. (1). The entropy $S(\rho_A)$ measures the amount of correlation (classical and/or quantum) between $A$ and the external world. If we divide the external world into two parts, $B$ and $E$, the amount of quantum correlation between $A$ and $B$, plus the amount of classical correlation between $A$ and the complementary part $E$, must be equal to $S_A$. In this sense, Eq. (1) poses constraints on the ability that system $A$ has to share correlations with other systems. For this reason it is called a monogamous relation.

We can show a different aspect of Eq. (1) by adding to both of its sides the mutual information between $A$ and $E$, $I_{AE} = S_A + S_E - S_{AE}$. After some manipulation we obtain

$$E_{AB} = \delta^x_{AE} + S_{A|E},$$  

where $S_{A|E} = S_{AE} - S_E$ is the conditional entropy and $\delta^x_{AE} = I_{AE} - J^x_{AE}$ is the QD between subsystem $A$ and the environment $E$. Eq. (2) tells us that the entanglement between two arbitrary subsystems, $A$ and $B$, is related to the quantum discord between one of the subsystems ($A$) and the environment $E$. It is important to note that, although in Eq. (2) the EOF is written as a function of the QD between $A$ and $E$, it is straightforward to write it as a function of the discord between $B$ and $E$. In that case, $E_{AB} = \delta^x_{BE} + S_{B|E}$. In the same way, we can evaluate the QD between the subsystems $A$ and $B$,

$$\delta^x_{AB} = E_{AE} - S_{A|B},$$  

which gives the quantum discord between $A$ and $B$ as a function of the entanglement between $A$ and $E$. Remarking that, since the global state is pure, $S_{AB} = S_E - S_B$ which is in fact the EOF of the partition $E$ with $AB$, $E_{E(AB)}$, minus the EOF of the partition $B$ with $AE$, $E_{B(AE)}$. Thus Eq. (3) can be rewritten as

$$\delta^x_{AB} = E_{AE} - E_{E(AB)} + E_{B(AE)}.$$  

This result shows that the EOF and QD obey a very special monogamic relation, involving bipartite and tripartite entanglement.

We now derive a very simple but powerful result regarding the distribution of bipartite entanglement. Noting that $S_{A|B} = -S_{A|E}$ since $\rho_{ABE}$ is a pure state and summing Eq. (2) and Eq. (3), we obtain

$$E_{AB} + E_{AE} = \delta^x_{AB} + \delta^x_{AE}.$$  

This important monogamic distribution of EOF and QD can also be viewed as a quantum conservation law:

Given an arbitrary tripartite pure system, the sum of all possible bipartite entanglement shared with a particular subsystem, as given by the EOF, can not be increased without decreasing, by the same amount, the sum of all QD shared with this same subsystem.

### III. UNDERSTANDING THE DISTRIBUTION OF ENTANGLEMENT IN THE DQC1

This last fundamental result has remarkable implications in the way that entanglement can be distributed among many parties. For example, we are now able to analyze the power of the quantum computer “without” entanglement in view of this last statement. In this sense, let us consider the DQC1 protocol, where the power of one pure qubit was firstly revealed. It is well-known that any quantum computation executed over $N$ maximally mixed states does not give rise to exponential speedup when compared with the classical computation. However, Knill and Laflamme [3] demonstrated that if just one single pure qubit is added to this set, the situation changes dramatically [4, 5]. For instance, the DQC1 protocol gives an exponential speedup for the computation of the normalized trace of an unitary operator, $2^{-n}\text{Tr}(U_n)$.

The DQC1 consists of a pure qubit that is represented here by the subsystem $A$ and a completely mixed state of $n$ qubits, $I_n/2^n$ that is represented by $B$. As illustrated in Fig. (1), we observe that initially the subsystem $A$ is pure and has zero entanglement and zero discord with respect to $B$ and $E$. On the other hand, the subsystem $B$ is given by the maximally mixed state. It is important to emphasize here that even a completely mixed state manifests its quantumness by the fact that it is impossible to distinguish the infinitely many ensembles that can realize it. An alternative way to look at this property is to consider it as an entangled state with an external environment which has as many degrees of freedom as necessary to purify the whole system. Thus, we consider here the degree of mixture of $B$ as due to the entanglement between an environment $E$, which does not interact with $B$, and $A$. However, it has interacted with $B$ in the past, being thus responsible for its mixedness. This approach has been fundamental for the understanding of important tasks in quantum information like Schumacher compression, quantum state merging, and entanglement theory [11, 12]. Given this initial situation, we consider a circuit as that exposed in Fig. (1). We suppose that the subsystem $A$ is a qubit in the initial state $|0\rangle$ (an eigenvector of the Pauli matrix $\sigma_z$) and apply a Hadamard quantum gate, followed by a control unitary on the remaining $n$ mixed qubit state. Thus, after this process,
the state of the subsystem $A$ and $B$ is given by
\[
\rho_{AB} = \frac{1}{2^{n+1}} \begin{pmatrix} I_n & U_n^\dagger \\ U_n & I_n \end{pmatrix},
\]

since it gives a separated state with respect to $A$ and $B$ [4]. Expanding the state on the eigenstate basis $\{| u_i \rangle \}$ of the unitary operator $U_n$ with eigenvalues $\exp(i \theta_i)$ and considering the purifying system eigenbasis $\{| e_i \rangle \}$, the joint $ABE$ state can be written as
\[
| \psi_{ABE} \rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_i (| 0 \rangle + e^{i \theta_i} | 1 \rangle) \otimes | u_i \rangle \otimes | e_i \rangle.
\]

Thus the expectation values of $\sigma_x$ and $\sigma_y$ on $A$ provide the normalized trace of $U_n$: $\langle \sigma_x \rangle = \text{Re} \{ \text{Tr}(U_n) \}/2^{n+1}$ and $\langle \sigma_y \rangle = -\text{Im} \{ \text{Tr}(U_n) \}/2^{n+1}$. At the end of the process, just before the measurement that determines $\langle \sigma_x \rangle$ and $\langle \sigma_y \rangle$, we will have finite QD between $A$ and $B$, $\delta_{BA}^\epsilon$, but no entanglement between them, $E_{AB} = 0$ [4]. Using the results given in Eq. [3] and Eq. [5], we meticulously examine the EOF and the QD distribution. For this purpose, we examine how the initial entanglement between $B$ and $E$ is affected during the computation. According to Eq. [3], $\delta_{AB}^\epsilon = E_{AE} - E_{\epsilon}^B(E_{BA})$, but from Eq. [7], $E_{AE} = \delta_{BA}^\epsilon = 0$. Similarly, we see that $E_{AB} = \delta_{AE}^\epsilon = 0$, so there really is no bipartite entanglement between $A$ with $B$ or $E$. But in a similar fashion to Eq. [4] we can write
\[
E_{BE} = E_{E(AB)} + \delta_{BA}^\epsilon - E_{A(BE)}
\]
allowing us to analyze the EOF and the QD distribution in the QDC1. Prior the computation, $E_{BE} = E_{E(A)}$, and $\delta_{BA}^\epsilon = E_{A(BE)} = 0$. However, $\delta_{BA}^\epsilon$ increases after the controlled unitary operation, implying necessarily in a redistribution of the entanglement between the parties. In order to $\delta_{BA}^\epsilon$ to increase some multipartite entanglement $E_{A(BE)}$ must exist. This entanglement is indeed signaled by the mixed state of $A$ alone after the computation. Furthermore, using Eq. [6], it is straightforward to show that for the DQC1 this entanglement unbalance can be measured by the QD between $B$ and $E$, since
\[
\delta_{BE}^\epsilon = E_{E(AB)} - E_{A(BE)},
\]
and finally
\[
\delta_{BA}^\epsilon = E_{BE} - \delta_{BE}^\epsilon.
\]

Nevertheless, it is important to emphasize that the power of the quantum computer does not come only from the entanglement present between $B$ and $E$, or even between $A$ and $BE$. In the DQC1, it is clear that it comes from the protocol ability to redistribute entanglement and quantum discord. This property is intrinsic of the protocol and does not rely on the particularities of the environment. The DQC1 protocol ability to transfer entanglement and its efficiency against classical algorithms for special tasks can be tested in the light of the subsystem $B$ initial entanglement with $E$. Had we started with a non-maximally mixed state for $B$, meaning a non-maximally entanglement with $E$, instead of Eq. [7], one would have ended up with $| \psi_{ABE} \rangle = \sum_i c_i (| 0 \rangle + e^{i \theta_i} | 1 \rangle) \otimes | u_i \rangle \otimes | e_i \rangle$. In this case $\{ \sigma_x \}$ or $\{ \sigma_y \}$ gives $\text{Tr}[\rho_B U_n]$ containing, thus, less information about the trace of $U_n$ when $B$ is initially less entangled with $E$. The worst case is when $B$ is in a definite state $| u_i \rangle$ (no entanglement with $E$), when we have access to only one eigenvalue of $U_n$. Curiously this corresponds to the situation where a maximal entanglement between $A$ and $B$ would be available at the end, which certainly does not contribute to any speedup for this special purpose. Therefore, we suggest that one should look carefully at the redistribution of entanglement during any quantum computation, and its implication for the speedup of certain protocols. In the present situation, we see that this ability for entanglement redistribution is a necessary (but not sufficient) ingredient for efficient quantum computation.

IV. STRONGER BOUNDS ON THE ENTROPY STRONG SUBADDITIVITY

At this point, one could imagine what would be the implications of such a relation when some information is lacking for the description of the global state, i. e., when the tripartite state involving systems $A$, $B$, and $E$ is mixed. In that case Eq. [1] becomes an inequality [10] and, therefore, Eq. [2] turns into
\[
E_{AB} \leq \delta_{AE}^\epsilon + S_{A\mid E}.
\]

Similarly, by changing $B$ for $E$ in the equation above, it now reads $E_{AE} \leq \delta_{BA}^\epsilon + S_{A\mid B}$, which when added to Eq. [11] gives
\[
S_B + S_E + \Delta \leq S_{AB} + S_{AE}
\]
with
\[
\Delta = E_{AB} + E_{AE} - \delta_{AB}^\epsilon - \delta_{AE}^\epsilon
\]
being the balance between the entanglement and the quantum discord in the system. The inequality [12] can be stronger than the strong subadditivity (SS) [13],
\[
S_B + S_E \leq S_{AB} + S_{AE},
\]
depending on $\Delta$. For $\Delta > 0$ it gives a remarkable lower bound for $S_{AB} + S_{AE}$, which is more restrictive than [14] and must be fulfilled by any quantum system. Thus, we can define a more restrictive inequality than the SS, $S_B + S_E + \tilde{\Delta} \leq S_{AB} + S_{AE}$,
\[
\text{with } \tilde{\Delta} = \max \{0, \Delta\}
\]
where $\Delta$ is given by the balance between EOF and QD, Eq. [13]. It is important to emphasize that the SS, despite of being more difficult to prove, is essentially derived through extensions of its
classical counterpart, but correlations play a different role in quantum systems. So, it is not surprising that a more restrictive bound may occur.

To exemplify this let us suppose we have a convex mixed state \( \rho_{ABE} = (1 - \lambda) \frac{I_B}{2} + \lambda \rho \), where \( \rho = |\Psi\rangle \langle \Psi| \) is a normalized pure state \( |\Psi\rangle = \beta |[101] + |011\rangle + \alpha |000\rangle \), with \( 2p^2 + \alpha^2 = 1 \), and \( I \) is the identity operator over the joint Hilbert space of \( A + B + E \). Let us define two quantities

\[
I_1 = S_{AB} + S_{AE} - S_B - S_E \geq 0,
\]

and

\[
I_2 = I_1 - \Delta \geq 0.
\]

In Fig. 2 we plot \( I_1 \) and \( I_2 \) as a function of \( \alpha \), and, in the inset, we plot \( \Delta \) for a fixed \( \lambda \approx 0.9 \). It is easy to see that in this situation \( \Delta \) can be positive or negative. When \( \Delta < 0 \) the inequality given by Eq. (12) is weaker than the SS given by Eq. (14). However, when \( \Delta > 0 \), meaning that the EOF of all bipartitions is larger than their QD, Eq. (16) is stronger than Eq. (17), limiting the lower bound for \( S_{AB} + S_{AE} \). This is a strikingly different bound imposed on the entropies of quantum systems, which is not shared by their classical counterpart. The inequality above recovers the SS only when \( \Delta = 0 \), meaning that the distribution of bipartite entanglement is equal to the amount of distributed quantum discord or smaller than that. It is important to emphasize here the essential role that SS plays in classical and quantum information theories. Many fundamental inequalities, as nonnegativity of entropy and subadditivity, can be derived from that. To the best of our knowledge, the only inequality known to be independent is the one proposed in Ref. [14], which is valid when the SS saturates on some particular subsystems configuration. It is straightforward to show that the inequality in Ref. [14] is independent of (17) as well, when \( \Delta > 0 \), since in this case SS can not be saturated. However something else can be learned from this saturation. Given a quadrupartite quantum system \( \rho_{ABCD} \) such that SS is saturated for the three triples \( ABC, CAB, \) and \( ADB \) then, \( I_{CD} \geq I_{C(AB)} \). Substituting \( I_{CD} \) by \( \delta_{CD} \) and using the monogamic relation, Eq. (16), and the conservation law, Eq. (17), it is straightforward to show that when \( I_{CD} \geq I_{C(AB)} \) we have \( \delta_{CD} \geq 0 \). So, as in Eq. (17), the difference between the EOF and the QD is of fundamental importance.

V. CONCLUSION

To summarize, we have given a monogamic relation between the EOF and the QD. For that, we have derived a general interrelation on how those quantities are distributed in a general tripartite system. We applied this relation to show that in the DQC1 the entanglement present between one of the subsystems and the environment is responsible for the non-zero quantum discord.

Since the maximally mixed state is entangled with the environment, we show that the circuit described by the DQC1 distributes this initial entanglement between the pure qubit and the mixed state. Our results suggest that the protocol ability to redistribute entanglement is a necessary condition for the speedup of the quantum computer. In addition, we have extended the discussion for an arbitrary tripartite mixed system showing the existence of an inequality for the subsystems entropies which is stronger than the usual SS.

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[1] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).

[2] B. P. Lanyon, et al., Phys. Rev. Lett. 101, 200501 (2008).
A. Shabani and D. A. Lidar, Phys. Rev. Lett. 102, 100402 (2009), T. Werlang, et al., Phys. Rev. A 80, 024103 (2009), K. Modi, et al., Phys. Rev. Lett. 104, 080501 (2010).

[3] E. Knill and R. Laflamme, Phys. Rev. Lett. 81, 5672 (1998).
[4] D. Poulin, R. Blume-Kohout, R. Laflamme, and H. Olivierv, Phys. Rev. Lett. 92, 177906 (2004).
[5] P. W. Shor and S. P. Jordan, arXiv:0707.2831 (2007).
[6] A. Datta, A. Shaji, and C. M. Caves, Phys. Rev. Lett. 100, 050502 (2008).
[7] C. E. López, G. Romero, F. Lastra, E. Solano, and J. C. Retamal, Phys. Rev. Lett. 101, 080503 (2008); J. Maziero, T. Werlang, F. F. Fanchini, L. C. Céleri, and R. M. Serra, Phys. Rev. A 81, 022116 (2010).
[8] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[9] L. Henderson and V. Vedral, J. Phys. A 34, 6899 (2001); V. Vedral, Phys. Rev. Lett 90, 050401 (2003).
[10] M. Koashi and A. Winter, Phys. Rev. A 69, 022309 (2004).
[11] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).
[12] B. Schumacher, Phys. Rev. A 51, 2738 (1995). M. Horodecki, J. Oppenheim, and A. Winter, Nature 436, 673 (2005); M. Horodecki, J. Oppenheim, and A. Winter, Nature 436, 673 (2005); M. F. Cornelio, M. C. de Oliveira, F. F. Fanchini arXiv:1007.0228 (2010).
[13] E.H Lieb and M.B. Ruskai, J. Math. Phys. 14, 1938 (1973).
[14] N. Linden and A. Winter, Commun. Math. Phys. 259, 129 (2005).