The renormalization group and quark number fluctuations in the Polyakov loop extended quark-meson model at finite baryon density

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Thermodynamics and the phase structure of the Polyakov loop-extended two flavors chiral quark–meson (PQM) model is explored beyond the mean-field approximation. The analysis of the PQM model is based on the functional renormalization group (FRG) method. We formulate and solve the renormalization group flow equation for the scale-dependent thermodynamic potential in the presence of the gluonic background field at finite temperature and density. We determine the phase diagram of the PQM model in the FRG approach and discuss its modification in comparison with the one obtained under the mean-field approximation. We focus on properties of the net-quark number density fluctuations as well as their higher moments and discuss the influence of non-perturbative effects on their properties near the chiral crossover transition. We show that with an increasing net-quark number density the higher order moments exhibit a peculiar structure near the phase transition. We also consider ratios of different moments of the net-quark number density and discuss their role as probes of deconfinement and chiral phase transitions.

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I. INTRODUCTION

Thermodynamic properties of a strongly interacting matter at nonzero baryon density and high temperature were quantified numerically within Lattice QCD (LQCD) [1–3]. The LQCD results demonstrate that QCD exhibits both dynamical chiral symmetry breaking and confinement at finite temperature and densities. The LQCD equation of state indicates a clear separation between the confined hadronic phase and deconfined phase of quark–gluon plasma. However, since the thermodynamics at large baryon densities and near the chiral limit is still not accessible from the first principle LQCD calculations, many phenomenological models and effective theories have been developed [4–18].

The hadronic properties at low energy as well as the nature of the chiral phase transition at finite temperature and densities have been successfully explored and described in such effective models. The physics of color deconfinement and its relation to the chiral symmetry breaking has been recently studied in terms of effective models. The idea to extend the existing Lagrangians such as the Nambu–Jona–Lasinio or the quark–meson, by introducing coupling of quarks to uniform temporal background gauge fields (Polyakov loops) was an important step forward in these studies [7,13].

It was shown that the Polyakov loop extended Nambu–Jona–Lasinio (PNJL) [4] or quark–meson (PQM) [15] models can reproduce essential properties of the QCD thermodynamics obtained in the LQCD already within the mean-field approximation. However, to correctly account for the critical behavior and scaling properties near the chiral phase transition one needs to go beyond the mean-field approximation and include quantum fluctuations and non-perturbative dynamics. This can be achieved by using the methods based on the functional renormalization group (FRG) [19–25].

Following our previous work [26] we use a truncation of the PQM model which is suitable for the functional renormalization group analysis to describe the thermodynamics beyond the mean-field approximation. The functional renormalization group approach in the PQM model is used to take into account fluctuations of the meson fields, while the Polyakov loop is treated as a background field on the mean-field level. In contrast to the previous work [26] we extend our calculations to the finite chemical potential. We determine the phase diagram and the position of the critical end point (CEP) in the PQM model by exploring the dependencies of the chiral order parameter and the quark number susceptibility on thermal variables. We calculate the moments (cumulants) of the net-quark number density fluctuations \(\langle n_{c}\rangle\) at finite temperature and chemical potential in the presence of mesonic fluctuations. We discuss the influence of non-perturbative effects on properties of the first four \(c_{n}\)-moments near the chiral crossover transition. We show that \(c_{n}\)-cumulants exhibit a peculiar structure and for sufficiently large values of the chemical potential can be negative near to the crossover transition. We calculate the ratios \(c_{3}/c_{1}\) and \(c_{4}/c_{2}\) and discuss their roles as probes of deconfinement and chiral phase transitions. We summarized properties of different susceptibilities near the chiral phase transition at finite net-quark density within the Landau mean-field and the scaling theories.
II. THE POLYAKOV-QUARK-MESON MODEL

The model which is used in this paper to explore the chiral phase transition at finite temperature and density within the FRG approach is the Polyakov loop-extended two flavor quark–meson model. In general, the PQM model, being an effective realisation of the low–energy sector of the QCD, cannot describe confinement phenomena because the local $SU(N_c)$ invariance of the QCD is replaced by the global symmetry. However, it was argued that by connecting the chiral quark–meson (QM) model with the Polyakov loop potential the confining properties of QCD can be approximately accounted for [7, 14, 13].

The Lagrangian of the PQM model reads [15]

$$\mathcal{L} = \bar{q} [i\gamma^\mu D_\mu - g(\sigma + i\gamma_5 \vec{\pi})] q + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} (\partial_\mu \vec{\pi})^2 - U(\sigma, \vec{\pi}) - \mathcal{U}(\ell, \ell^*) .$$

The coupling between the effective gluon field and quarks is implemented through the covariant derivative

$$D_\mu = \partial_\mu - iA_\mu,$$

where $A_\mu = g A^a_\mu \lambda^a / 2$. The spatial components of the gluon field are neglected, i.e. $A_\mu = \delta_\mu_0 A_0$. Moreover, $\mathcal{U}(\ell, \ell^*)$ is the effective potential for the gluon field expressed in terms of the thermal expectation values of the color trace of the Polyakov loop and its conjugate

$$\ell = \frac{1}{N_c} \langle \text{Tr}_c L(\vec{x}) \rangle, \quad \ell^* = \frac{1}{N_c} \langle \text{Tr}_c L^\dagger(\vec{x}) \rangle,$$

with

$$L(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^\beta d\tau A_4(\vec{x}, \tau) \right],$$

where $\mathcal{P}$ stands for the path ordering, $\beta = 1/T$ and $A_4 = i A_0$. In the $O(4)$ representation the meson field is introduced as $\phi = (\sigma, \vec{\pi})$ and the corresponding $SU(2)_L \otimes SU(2)_R$ chiral representation is defined by $\sigma + i \vec{\pi} \gamma_5$.

The purely mesonic potential of the model, $U(\sigma, \vec{\pi})$, is defined as

$$U(\sigma, \vec{\pi}) = \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma,$$

while the effective potential of the gluon field is parametrized in such a way as to preserve the $Z(3)$ invariance:

$$\frac{\mathcal{U}(\ell, \ell^*)}{T^4} = - \frac{b_2(T)}{2} \ell^* \ell - \frac{b_3}{6} (\ell^3 + \ell^* \ell^3) + \frac{b_4}{4} (\ell^* \ell)^2 .$$

The parameters,

$$b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3$$

with $a_0 = 6.75, a_1 = -1.95, a_2 = 2.625, a_3 = -7.44, b_3 = 0.75$ and $b_4 = 7.5$ were chosen to reproduce the equation of state of the pure $SU_c(3)$ lattice gauge theory. Consequently, at $T_0 \simeq 270$ MeV the potential [11] yields the first-order deconfinement phase transition. Several alternative parametrizations of the effective potential of the gluon field were also explored in Refs. [27, 29].

A. The FRG method in the PQM model

In order to formulate a non-perturbative thermodynamics within the PQM model we adopt a method based on the functional renormalization group (FRG). The FRG is based on an infrared regularization with the momentum scale parameter of the full propagator which turns the corresponding effective action into the scale $k$-dependent functional $\Gamma_k$ [19, 22].

In the PQM model the formulation of the FRG flow equation would require implementation of the Polyakov loop as a dynamical field. However, in the current calculation we treat the Polyakov loop as a background field which is introduced selfconsistently on the mean-field level.

Following our previous work [20] we formulate the flow equation for the scale-dependent grand canonical potential for the quark and mesonic subsystems

$$\partial_k \Omega_k(\ell, \ell^*; T, \mu) = \frac{k^4}{12 \pi^2} \left\{ \frac{3}{E_\pi} \left[ 1 + 2 n_B(E_\pi; T) \right] + \frac{1}{E_\sigma} \left[ 1 + 2 n_B(E_\sigma; T) \right] \right. \right.$$

$$\left. - \frac{4 N_c N_f}{E_\pi} \left[ 1 - N(\ell, \ell^*; T, \mu) - \tilde{N}(\ell, \ell^*; T, \mu) \right] \right\} .$$

Here $n_B(E_{\pi,\sigma}; T)$ is the bosonic distribution function

$$n_B(E_{\pi,\sigma}; T) = \frac{1}{\exp(E_{\pi,\sigma}/T) - 1}.$$
FIG. 1: The phase diagrams for the PQM model in the mean-field approximation (left panel) and in the FRG approach (right panel). The shaded regions are defined by 5%-deviations of the temperature derivative of the chiral order parameter, $d\sigma/dT$, from its maximal value. The arrows show the lines corresponding to different values of $\mu/T$. The dashed curves indicate isentropes for $s/n_q = 2, 5, 10, 25, 50, 200$.

with the pion and sigma energies given by

$$E_\pi = \sqrt{k^2 + \Omega_k}, \quad E_\sigma = \sqrt{k^2 + \Omega_k' + 2\rho \Omega_k''},$$

where the primes denote derivatives with respect to $\rho$ and $\overline{\Omega} = \Omega + c\sigma$. The functions $N(\ell, \ell^*; T, \mu)$ and $\tilde{N}(\ell, \ell^*; T, \mu)$ defined by

$$N(\ell, \ell^*; T, \mu) = \frac{1 + 2\ell^* \exp[\beta(E_q - \mu)] + \ell \exp[2\beta(E_q - \mu)]}{1 + 3\ell \exp[2\beta(E_q - \mu)] + 3\ell^* \exp[2\beta(E_q - \mu)] + \exp[3\beta(E_q - \mu)]},$$

$$\tilde{N}(\ell, \ell^*; T, \mu) = N(\ell^*, \ell; T, -\mu)$$

are fermionic distributions modified because of the coupling to gluons. The quark energy is defined by

$$E_q = \sqrt{k^2 + 2g^2 \rho}. \quad (11)$$

The minimum of the thermodynamic potential is determined by the stationarity condition

$$\left. \frac{d\Omega_k}{d\sigma} \right|_{\sigma = \sigma_k} = \left. \frac{d\Omega_k'}{d\sigma} \right|_{\sigma = \sigma_k} - c = 0. \quad (12)$$

The flow equation (8) is solved numerically with the initial cutoff $\Lambda = 1.2$ GeV (see details in Ref. [26]). The initial conditions for the flow are chosen to reproduce the vacuum properties: the physical pion mass $m_\pi = 138$ MeV, the pion decay constant $f_\pi = 93$ MeV, the sigma mass $m_\sigma = 600$ MeV, and the constituent quark mass $m_q = 300$ MeV at the scale $k = 0$. The symmetry breaking term, $c = m_\tau^2 f_\pi$, corresponds to an external field and consequently does not flow. In this work we neglect the flow of the Yukawa coupling $g$, which is not expected to be significant for the present studies (see e.g. Refs. [20, 21]).

By solving the equation (8) one gets thermodynamic potential for quarks and mesonic subsystems, $\Omega_{k\to 0}(\ell, \ell^*; T, \mu)$, as the function of the Polyakov loop variables $\ell$ and $\ell^*$. The full thermodynamic potential $\Omega(\ell, \ell^*; T, \mu)$ in the PQM model which includes quarks, mesons and gluons degrees of freedom is obtained by adding to $\Omega_{k\to 0}(\ell, \ell^*; T, \mu)$ the effective gluon potential $U(\ell, \ell^*)$:

$$\Omega(\ell, \ell^*; T, \mu) = \Omega_{k\to 0}(\ell, \ell^*; T, \mu) + U(\ell, \ell^*). \quad (13)$$

At a given temperature and chemical potential, the Polyakov loop variables, $\ell$ and $\ell^*$, are determined by the stationarity conditions:

$$\frac{\partial}{\partial \ell} \Omega(\ell, \ell^*; T, \mu) = 0, \quad (14)$$

$$\frac{\partial}{\partial \ell^*} \Omega(\ell, \ell^*; T, \mu) = 0. \quad (15)$$

The thermodynamic potential (8) does not contain contributions of statistical modes with momenta larger than the cutoff $\Lambda$. In order to obtain the correct high-temperature behavior of thermodynamic functions we
need to supplement the FRG potential with the contribution of high-momentum states. A simplified model for implementing such states was proposed in Ref. \[32\] where the contributions of the $k > \Lambda$ states to the flow is approximated by that of a non-interacting gas of quarks and gluons. For the PQM model we generalize this procedure by including for $k > \Lambda$ the flow equation of interacting quarks with Polyakov loops \[26\]

In the PQM model the thermodynamical potential derived under the mean-field approximation reads \[15\]:

$$\Omega_{MF} = \mathcal{U}(\ell, \ell^*) + U(\langle \sigma \rangle, \langle \pi \rangle = 0) + \Omega_{qq}(\langle \sigma \rangle, \ell, \ell^*).$$

(17)

Here, the contribution of quarks with dynamical mass $m_q = g\langle \sigma \rangle$ is given by

$$\Omega_{qq}(\langle \sigma \rangle, \ell, \ell^*) = -2N_f T \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{N_c E_q}{T} + \ln g^+(\langle \sigma \rangle, \ell, \ell^*; T, \mu) + \ln g^-(\langle \sigma \rangle, \ell, \ell^*; T, \mu) \right\},$$

(18)

where

$$g^+(\langle \sigma \rangle, \ell, \ell^*; T, \mu) = 1 + 3\ell \exp[-(E_q - \mu)/T] + 3\ell^* \exp[-2(E_q - \mu)/T] + \exp[-3(E_q - \mu)/T],$$

(19)

$$g^-(\langle \sigma \rangle, \ell, \ell^*; T, \mu) = g^+(\langle \sigma \rangle, \ell^*, \ell; T, -\mu);$$

(20)

and $E_q = \sqrt{\ell^2 + m_q^2}$ is the quark quasi-particle energy. The first term in Eq. \[18\] is a divergent vacuum fluctuation contribution which has to be properly regularized. Following our previous studies \[33\] we use the dimensional regularization, which introduces an arbitrary scale $M$. The renormalization procedure allows to compensate the physics dependence on this scale. The finite contribution of the vacuum term to Eq. \[18\] reads \[33\]

$$\Omega_{qq}^{vac} = -\frac{N_c N_f}{8\pi^2} m_q^4 \ln \left( \frac{m_q}{M} \right).$$

(21)

To obtain the complete thermodynamic potential of the PQM model we integrate Eq. \[16\] from $k = \infty$ to $k = \Lambda$ where we switch to the PQM flow equation \[10\]. Divergent terms in the high-momentum flow equation \[10\] are independent of mesonic and gluonic fields as well as of temperature and chemical potential. Consequently, such terms can be absorbed to an unobservable constant shift of the thermodynamic potential.

### B. The mean-field approximation

To illustrate the importance of mesonic fluctuations on thermodynamics in the PQM model we will compare the FRG results with those obtained under the mean-field approximation. In the latter, both quantum and thermal fluctuations are neglected and the mesonic fields are replaced by their classical expectation values.
The importance and influence of this contribution on the thermodynamics of chiral models was demonstrated and studied in detail in Refs. [33] and [35].

The equations of motion for the mean fields are obtained by requiring that the thermodynamic potential is stationary with respect to changes of \( \langle \sigma \rangle \), \( \ell \) and \( \ell^* \):

\[
\frac{\partial \Omega_{MF}}{\partial \langle \sigma \rangle} = \frac{\partial \Omega_{MF}}{\partial \ell} = \frac{\partial \Omega_{MF}}{\partial \ell^*} = 0.
\] (22)

The model parameters are fixed to reproduce the same vacuum physics as in the FRG calculation.

### III. THERMODYNAMICS OF THE PQM MODEL

The thermodynamic potential introduced in Eqs. (13) and (17) provides basis to explore critical properties of the PQM model at finite baryon density within the FRG approach and under the mean-field approximation.

To find the potential at finite temperature and chemical potential one needs to solve the FRG flow equation [8]. This equation is solved by numerical methods based on the Taylor series expansion [26, 34]. This method is successful in studying thermodynamics at finite density and temperature [24, 26, 33] in the regime where the system exhibits the crossover or the second-order phase transition. For the solution of the FRG flow equations in the regime of the first-order phase transition, where the thermodynamical potential develops two minimums, different numerical methods are required [52, 56]. In the present work we restrict our considerations only to the parameter range where the PQM model exhibits the crossover or the second-order chiral phase transitions.

A. The phase diagram

The PQM model is expected to belong to the same universality class as QCD and thus should exhibit a generic phase diagram with the critical point at non-vanishing chemical potential. In the chiral limit the phase diagram is identified by divergences of the chiral susceptibility. At finite quark mass the chiral transition is of crossover type. In this case the pseudocritical temperature and chemical potential are located by determining the maximum of the chiral susceptibility or the temperature derivative of the chiral order parameter. The position of the CEP is identified by the properties of the sigma mass. The temperature and chemical potential where the sigma mass vanishes correspond to the position of the CEP. One can equivalently consider the net-quark number fluctuations to identify the critical point which according to \( Z(2) \) universality diverge at the CEP.

Fig. 1 shows the phase diagrams of the PQM model obtained within the mean-field approximation and FRG approach. For the physical pion mass and moderate values of the chemical potential the PQM model exhibits a smooth crossover chiral transition. In Fig. 1 we define the transition region as bands where the temperature derivative of the order parameter exhibits 5%-deviations from its maximal value. At larger chemical potentials the crossover line terminates at the CEP where the transition is of the second order and belongs to the universality class of the three dimensional Ising model.

Comparing the resulting phase diagrams of the PQM model obtained within the mean-field approximation and FRG approach we find a clear shift of the position of the chiral phase boundary to higher temperatures due to mesonic fluctuations. Such shifts were previously found in the QM model within the FRG approach [23, 35].

![FIG. 3: The coefficient \( c_2 \) as a function of temperature for different values of \( \mu/T \) for the PQM model in the mean-field approximation (left panel) and in the FRG approach (right panel).](image_url)
However, in our studies due to the gluonic background, which is explicitly included in our FRG calculations, we also find a significant shift of the CEP to higher temperature.

In the following we focus on observables that are related to the non-vanishing net-quark number density and consider moments of quark number fluctuations. We discuss how such fluctuations depend on the quark chemical potential in the presence of the gluonic background within the FRG approach.

B. Quark number density fluctuations

The fluctuations of conserved charges are observables that provide direct information on critical properties related with the chiral symmetry restoration. Fluctuations related with the baryon number conservation are of particular interest. In an equilibrium medium a divergence of the net-quark number susceptibility is a direct evidence for the existence of the CEP \[10\]. Consequently, any non-monotonic dependence of these fluctuations on the baryon number conservation are of particular interest. In an equilibrium medium a divergence of the net-quark number susceptibility is a direct evidence for the existence of the CEP \[10\].

In a non-equilibrium system, the net-quark number susceptibility was also shown to signal the first-order chiral phase transition due to spinodal decomposition \[10\].

The properties of different moments of the net-quark number fluctuations \[41–43\] which is defined as the ratio

$$ R_{4,2} = \frac{c_4}{c_2}. $$

This key observable is not only sensitive to the chiral but also to the deconfinement transition. At vanishing chemical potential, in the asymptotic regime of high and low temperatures the kurtosis reflects quark content of the baryon-number carrying effective degrees of freedom \[41\] \[43\]. Therefore, at low temperatures in the confined phase, $R_{4,2} \approx N_q^2 = 9$ while for high temperatures one recovers an ideal gas of quarks with $R_{4,2} \sim 1$ \[49\].

The fluctuations of the net-quark number density are observables that are related to the non-vanishing net-quark number density and consider moments of quark number fluctuations. We discuss how such fluctuations depend on the quark chemical potential in the presence of the gluonic background within the FRG approach.

**FIG. 4:** The coefficient $c_3$ as a function of temperature for different values of $\mu/T$ for the PQM model in the mean-field approximation (left panel) and in the FRG approach (right panel).
susceptibility and the kurtosis as signatures of the deconfinement and the chiral phase transition as well as the CEP was discussed \[43\]. The influence and dependence of these fluctuations on the quark mass was also analyzed on the lattice and in effective models \[24, 26, 29, 33, 44\]. However, only little is known about chemical potential dependence of the higher cumulants \(c_n\), particularly with \(n > 2\). Such dependence can be obtained in the PQM model from the thermodynamic potential introduced in Eqs. (13) and (17). In Figs. 2–5 we quantify the first four moments obtained in the PQM model under the mean-field approximation and in the FRG approach for different values of the ratio \(\mu/T\). The lines of the fixed ratio \(\mu/T\) also indicated on the phase diagram in Fig. 1.

Fig. 2 shows the net-quark number density normalized by \(T^3\), \(c_1 = n_q/T^3\), for different values of \(\mu/T\). The cumulant \(c_1\) as well as all \(c_{2n+1}\), for \(n = 1, 2, 3, \ldots\), are vanishing at zero chemical potential, \(\mu = 0\). At finite \(\mu\) and in the chirally broken phase the net-quark number, \(n_q\), is strongly increasing function of \(\mu/T\). In the low temperature phase due to the large dynamical quark mass the \(n_q\) increases approximately as \(\sinh(3\mu/T)\). This is a direct consequence of the "statistical confinement" properties of the PQM model which for small expectation values of Polyakov loops \(l \ll 1\) suppresses the single and double quark modes in the partition function as seen from Eqs. (9) and (19). At high temperatures/chemical potentials \(n_q\) behaves as polynomial in \(\mu/T\). For a fixed \(\mu/T\) and in the pseudo-critical region where the chiral symmetry is partially restored there is a clear change in the temperature dependence of \(n_q\). At the pseudo-critical temperature the density exhibits a kink-like structure which is particularly evident in the mean-field calculations at larger \(\mu/T\). In the FRG calculations and at the corresponding \(\mu/T\) the above kink-like structure in the density \(n_q\) is suppressed. Thus, the meson fluctuations quantified by the FRG method provide smoothing of the net-quark density near the crossover transition.

The appearance of the crossover chiral transition in the PQM model is also transparent when considering the structure of isentropic trajectories. Because of the entropy and baryon-number conservation such trajectories
correspond to contours of the constant entropy density per net-quark number density, $s/n_q$, in the temperature-chemical potential plane. Fig. 1 shows isentropes for different $s/n_q$ in the PQM model obtained under the mean-field approximation and in the FRG approach. There is a clear change of slopes of isentropes along the transition line indicating the change of the equation of state. The qualitative behavior of contours of constant $s/n_q$ obtained in the PQM model is similar to that calculated previously within the QM model \cite{35}. In particular, the isentropes remain smooth when the effect of long-wavelength meson fluctuations is consistently included in the presence of the gluon background. There are also no indications of any focusing towards the CEP in the PQM model.

The influence of the finite chemical potential on $c_2$ (which is proportional to the net-quark number susceptibility $c_2 = \chi_q/T^2$) is shown in Fig. 3 for the mean-field approximation and the FRG approach. At vanishing chemical potential the cumulant $c_2$ increases monotonously with temperature. However, at finite chemical potential, the susceptibility $c_2$ develops a peak structure. The amplitude of this peak increases with the chemical potential towards the CEP where $c_2$ diverges. In the high temperature/chemical potential phase $c_2$ converges to the Stefan Boltzmann limit

$$c_2^{SB} = \frac{N_c N_f}{3} \left[1 + \frac{3}{\pi^2} \left(\frac{\mu}{T}\right)^2\right]. \quad (28)$$

As it is seen in Fig. 3 the peak structure in $c_2$ is more pronounced in the mean-field approximation at $\mu/T = 1$ than in the FRG at $\mu/T = 1.5$. This is in spite of the fact that the location of the CEP in the FRG is closer to the line of $\mu/T = 1.5$ than the corresponding one for the mean-field approximation to $\mu/T = 1$ (see Fig. 1). This shows that the criticality of $c_2$ as a function of the distance to the CEP appears earlier in the mean-field approximation than in the FRG approach. This result is in agreement with the previous studies in the QM model showing that the critical region shrinks due to mesonic fluctuations \cite{29}.

The cumulants $c_1$ and $c_2$ are sensitive to changes in the chemical potential and are influenced by the meson fluctuations and the gluon background. However, both these coefficients are positive for all values of $\mu$ and $T$. At finite chemical potential the positivity of the first two coefficients is not preserved for $c_n$-moments with $n > 2$. Figs. 4 and 5 show the third and the fourth order cumulants of the net-quark number density for different values of $\mu/T$. For a large $\mu/T$ both these susceptibilities exhibit a peculiar structure in the transition region. There is a minimum of $c_3$ which can reach negative value. The amplitudes of this minimum is strongly suppressed by meson fluctuations in FRG approach.

The fourth order cumulant is strictly positive for vanishing chemical potential. However, for higher values of $\mu/T$, $c_4$ becomes negative in the vicinity of the crossover transition. This manifests the broadening of the net-quark number distribution in comparison to the Gaussian one. Large values of $c_4$ in the broken phase infer that the distribution becomes narrower than the Gaussian. The chemical potential independent Stefan-Boltzmann limit $c_4^{SB} = 2N_c N_f/\pi^2$ is reproduced at temperatures $T \gg T_c$.

Comparing the mean-field with the FRG results for both $c_3$ and $c_4$ we conclude that mesonic fluctuations essentially modify properties of different generalized quark susceptibilities. In the transition region $c_3$ and $c_4$ are suppressed in the FRG relative to the mean-field results. Thus, the mean-field approach is only an approximate method to describe static thermodynamics near the chiral phase transition.

We have already indicated that the ratios of different $c_n$ are sensitive to the deconfinement and the chiral phase transitions. Figs. 6 and 7 show the kurtosis $R_{4,2} = c_4/c_2$ calculated as a function of temperature along different paths in the temperature-chemical potential plane quantified by fixed $\mu/T$ and $s/n_q$. In the PQM model the kurtosis drops from $R_{4,2} \simeq 9$ to $R_{4,2} \simeq 1$ in the transi-

![FIG. 7: The temperature dependence of the kurtosis $R_{4,2}$ calculated in the PQM model at fixed values of the entropy density to net-quark density ($s/n_q$) under the mean-field approximation (left panel) and in the FRG approach (right panel).](image)
The height of these peaks depends not only on the pion potential but also on the chemical potential. At low temperatures and at zero chemical potential $c_3/c_1$ vanishes but their ratio is finite. For finite chemical potential the kurtosis becomes negative following the same trends as seen in the fourth order cumulant.

A direct information on quark content of baryon carrying effective degrees of freedom in the low temperature phase is also contained in the $c_3/c_1$ ratio. At low temperature where thermodynamics is dominated by three-quark modes $c_3/c_1 = 9$ for any value of chemical potential. At low temperatures and at zero chemical potential $c_1$ and $c_3$ vanishes but their ratio is finite. At asymptotically large temperatures and for $\mu \to 0$ this ratio diverges. The ratio $c_3/c_1$ similarly as the kurtosis $R_{4,2} = c_4/c_2$ exhibits strong variations in the phase transition region. It develops a peak with height which increases with $\mu/T$ and for sufficiently large $\mu/T$ it develops a deep structure. This is the case for both mean-field and FRG calculations, however variations at corresponding $\mu/T$ in the FRG are strongly suppressed because of meson fluctuations.

C. Scaling properties of generalized susceptibilities at finite chemical potential

The general trends and behavior of different susceptibilities calculated in the last section can be understood considering their scaling properties near the chiral transition. Under the mean-field approximation such scaling can be inferred from the Landau theory of phase transitions, where the singular part of the thermodynamic potential is a polynomial in an order parameter $\sigma$,

$$\Omega^{sing}(T, \mu; \sigma) = \frac{1}{2} a(T, \mu) \sigma^2 + \frac{1}{4} b(T, \mu) \sigma^4 + \frac{1}{6} c \sigma^6 - h \sigma. \quad (29)$$

with $h$ being a symmetry breaking term.

For the chemical potential much smaller than the critical temperature $T_c(\mu = 0)$ of the second-order chiral phase transition the mass term $a(T, \mu)$ is parameterized as

$$a(T, \mu) = A \cdot (T - T_c) + B_2 \mu^2, \quad (30)$$

where both coefficients $A$ and $B_2$ are positive. In general, the effective quartic coupling $b > 0$, is also $T$ and $\mu$ dependent. However, this dependence is irrelevant near the critical line $a(T, \mu) = 0$ and away from the CEP or the tri-critical point (TCP). Therefore in this case we can also neglect the higher order polynomial contribution by setting $c = 0$. In the chiral limit $h = 0$, the critical properties of $c_2$ and $c_4$ are obtained from Eqs. (31) and (32):

$$c_2^{sing} = \frac{AB_2}{bT^2} (T - T_c) \theta(T_c - T), \quad (31)$$

$$c_4^{sing} = \frac{6B_2^2}{b} \theta(T_c - T). \quad (32)$$

Thus, $c_2$ is not differentiable at the critical temperature, while $c_4$ has a discontinuity.

For a finite quark mass i.e. for $h \neq 0$, the second-order transition is turned to the crossover and the sharp structures in $c_2$ and $c_4$ are smoothened. Therefore in the PQM model, the peak structure appearing in $c_4$ is directly linked to the quark mass and closely connected to the partial restoration of the chiral symmetry. From Eq. (32) it is clear, that the kurtosis $R_{4,2}$, driven by the $c_4$ has a discontinuity at $T_c$. 

FIG. 8: The ratio of $c_3$ to $c_1$ as a function of temperature for different values of $\mu/T$ obtained in the PQM model under the mean-field approximation (left panel) and in the FRG approach (right panel).
In the FRG approach the critical behavior of generalized susceptibilities obtained under the mean-field approximation will be modified by the long wave meson fluctuations. Detailed studies in the QM model showed that the FRG method can correctly account for long-range correlations resulting in the O(4) critical behavior of thermodynamic functions [24]. The gluon background is not modifying critical dynamics related with the chiral symmetry. Therefore in the chiral limit the PQM model belongs to the O(4) universality class. The singular part of the thermodynamic potential is controlled by the critical exponents of the three-dimensional O(4)-symmetric spin system. At vanishing chemical potential

$$\Omega^{\text{sing}} \sim (T - T_c)^{2-a},$$

leading to the following scaling of generalized susceptibilities

$$c_{2n}^{\text{sing}} \sim (T - T_c)^{2-n-a},$$

$$c_{2n+1} = 0$$

for \( n = 1, 2, 3, \ldots \)

In the mean-field approach the critical exponent \( a = 0 \). The quantum fluctuations within FRG renormalize the exponent to \( a \approx -0.21 \) expected in the O(4) universality class. Therefore fluctuations lead to weakening of singularities. The finite quark mass further smooths the temperature dependence of \( c_2, c_3 \) and \( c_4 \) as seen in Figs. 3 and 4. The kurtosis follows singular behavior of \( c_4 \) and for \( \mu = 0 \) exhibits a step-like structure at \( T_c \).

For finite chemical potential, but still away from the CEP or the tricritical point (TCP), the coefficient \( a(\mu, T) \) in the Landau potential can be parameterized as

$$a(T, \mu) = A \cdot (T - T_c) + B_1 \cdot (\mu - \mu_c)$$

while the quartic coupling \( b > 0 \) and we still keep \( c = 0 \).

In this case one gets the following expressions for susceptibilities

$$c_1^{\text{sing}} = \frac{B_1 a}{2T^3 b} \theta(-a),$$

$$c_2^{\text{sing}} = \frac{B_2^2}{2b^2 T^2} \theta(-a),$$

$$c_3^{\text{sing}} = c_4^{\text{sing}} = 0.$$  \hspace{1cm} (39)

Thus, \( c_2 \) exhibits a discontinuity while the singular part of \( c_4 \) is vanishing along with \( c_3 \) and higher order cumulants. However, there is the next to leading order contribution to \( a(T, \mu) \) owing to non-vanishing curvature of the transition line in the \((T, \mu)\)-plane. In order to take it into account one needs to add an extra contribution \( B_2 \cdot (\mu - \mu_c)^2 \) to the right-hand side of Eq. (36). In this case the \( c_4^{\text{sing}} \) behaves as in Eq. (32) while the leading contribution to \( c_1^{\text{sing}} \) and \( c_2^{\text{sing}} \) have the structure as in Eq. (37) but with a modified \( a(T, \mu) \). The third order cumulant also develops nonzero values in the broken phase

$$c_3^{\text{sing}} = b \frac{B_2 B_4}{b_4} \theta(-a).$$

As in the case of \( \mu = 0 \), the kurtosis has a step-like behavior but here the curvature of the phase diagram controls whether the kurtosis is an increasing or decreasing function at the phase transition because of two competing step structures in \( c_2^{\text{sing}} \) and \( c_4^{\text{sing}} \).

Including quantum fluctuations as in the FRG approach the above MF scaling is modified to the following O(4) relations

$$\Omega^{\text{sing}} \sim (-\mu)^{2-a} \hspace{1cm} (40)$$

$$c_n^{\text{sing}} \sim (-\mu)^{2-n-a} \hspace{1cm} (41)$$

resulting in divergence of the \( c_3, c_4 \) and all higher order cumulants along the O(4) critical line with the specific heat critical exponent \( \alpha \approx -0.21 \). The kurtosis is clearly divergent at \( T_c = T(\mu_c) \) following the singularity of the \( c_4^{\text{sing}} \) cumulant.

The above scaling properties of the net-quark number fluctuations at finite chemical potential are modified when approaching the TCP in the chiral limit or CEP at finite quark mass. Close to the TCP the effective Landau potential has the structure as in Eq. (29) where the parameter \( a(T, \mu) \) has a linear dependence on the reduced temperature and chemical potential

$$a(T, \mu) = A_a \cdot (T - T_c) + B_a \cdot (\mu - \mu_c).$$

The quartic coupling tends to zero as

$$b(T, \mu) = A_b \cdot (T - T_c) + B_b \cdot (\mu - \mu_c).$$

The six-order coupling \( c > 0 \) as it is required by the stability of the theory. Consequently, within Landau's theory and with the above parametrization of the potential coefficients one gets the following relations for the leading contribution to generalized susceptibilities

$$c_1^{\text{sing}} = \frac{B_a}{2T^3} \sqrt{\frac{-\mu}{c}} \theta(-\mu),$$

$$c_n^{\text{sing}} = \frac{\Gamma(n - \frac{4}{3})}{2T^4} \left[ (b^2 - 4ac) \right]^{\frac{n-4}{3}} \cdot n > 1.$$  \hspace{1cm} (45)

resulting in the divergent kurtosis at the TCP as:

$$R_{4,2}^{\text{sing}} \sim (b^2 - 4ac)^{2/3}.$$  \hspace{1cm} (45)

From Eq. (45) one concludes that the \( c_2^{\text{sing}} \) is inversely proportional to a distance from the TCP along the \( a(T, \mu) = 0 \) line. When approaching the TCP from any other direction which is non-tangential to the critical line, then \( b^2 \ll a \) and \( c_2^{\text{sing}} \) is inversely proportional to the square root of the distance to the TCP [11, 48]. This demonstrates that the critical region being defined by the properties of the \( c_n \) for \( n > 1 \) is elongated along the O(4) critical line.

The thermodynamic properties and critical behavior near TCP are well described by the mean-field theory up to logarithmic corrections, because in this case the upper critical dimension is three.
For the non-zero external field (non-zero quark mass) the three-critical point is turned to the CEP. Following the same mean-field analysis (see also Ref. [48]) we obtain the following leading singular behavior along the linear continuation of the phase-coexistence line to the crossover region and near to the CEP

\[ c_n^{\text{sing}} \sim |v|^{2-\frac{4}{3}n}. \] (46)

For any other directions which are asymptotically not equivalent to the previous one

\[ c_n^{\text{sing}} \sim |u|^{4-n}. \] (47)

where the introduced variables \( v \) and \( u \) are linear combinations of \((T-T_{\text{CEP}})\) and \((\mu-\mu_{\text{CEP}})\). For scaling [46] the kurtosis diverges as \( R_{4,2}^{\text{sing}} \sim |v|^{-3} \), and for scaling [47] the kurtosis \( R_{4,2}^{\text{sing}} \sim |u|^{-2} \).

When going beyond the mean-field approximation by including quantum and thermal fluctuations in the PQM model one expects that along the \( u = 0 \) line the following scaling holds (see also [34]):

\[ c_n^{\text{sing}} \sim |v|^{-[(n-2)(\gamma+\beta)+\gamma]}, \] (48)

and for any other direction

\[ c_n^{\text{sing}} \sim |u|^{-(n-2+\frac{2}{3}\beta)}. \] (49)

Here, the critical exponents \( \gamma \) and \( \beta \) correspond to the three-dimensional spin model belonging to the \( Z(2) \) universality class [50]. The kurtosis is divergent at the CEP, however the strength of the singularity depends on the direction. Approaching the TCP along the \( u = 0 \) line results in

\[ R_{4,2}^{\text{sing}} \sim |v|^{-2(\gamma+\beta)} = |v|^{-2-\gamma+\alpha}, \] (50)

whereas for any other direction

\[ R_{4,2}^{\text{sing}} \sim |u|^{-2}. \] (51)

The properties of the net-quark number density fluctuations and their higher moments obtained in the PQM model at finite chemical potential and the pion mass are qualitatively understood as remnants of the critical structure and scaling behaviors related with the chiral symmetry restoration in the limit of massless quarks.

IV. SUMMARY AND CONCLUSIONS

We have formulated thermodynamics of the Polyakov loop extended quark-meson effective chiral model (PQM), including quantum fluctuations within the functional renormalization group method (FRG). We have solved the flow equation for the scale dependent thermodynamic potential at finite temperature and density in the presence of a background gluonic field.

We have shown that the non-perturbative dynamics introduced by the FRG approach essentially modifies predictions of the model derived under the mean-field approximation. In particular, we have demonstrated quantitative changes of the phase diagram leading to a shift in the position of the critical end point.

We have focused on fluctuations of the net-quark number density and calculated the first four moments near the chiral transition for different values of the chemical potential. We have indicated the role and importance of ratios of different cumulant moments to identify the deconfinement and chiral phase transitions. We have also discussed predictions of the Landau and scaling theories on the critical behavior of the net-quark number density fluctuations and their higher moments in the vicinity of the chiral phase transition.

The extension of the FRG method proposed here to account for the coupling of fermions to the background gluon fields within the quark–meson model is of relevance to understand effectively thermodynamics of the QCD near the chiral phase transition.

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[1] C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann and C. Schmidt, Phys. Rev. D 68, 014507 (2003).
[2] C. R. Allton, M. Doring, S. Ejiri, S.J. Hands, O. Kaczmarek, F. Karsch, E. Laermann and K. Redlich, Phys. Rev. D 71, 054508 (2005)
[3] M. Cheng et al., Phys. Rev. D 77, 014511 (2008).
[4] A. Gocksch and M. Ogilvie, Phys. Rev. D 31, 877 (1985).
[5] M. Buballa, Phys. Rept. 407, 205 (2005).
[6] P. N. Meisinger and M. C. Ogilvie, Phys. Lett. B 379, 163 (1996); P. N. Meisinger, T. R. Miller and M. C. Ogilvie, Phys. Rev. D 65, 034009 (2002).
[7] K. Fukushima, Phys. Lett. B 591, 277 (2004).
[8] F. Sannino, Phys. Rev. D 66, 034013 (2002); A. Mocsy,
F. Sannino and K. Tuominen, Phys. Rev. Lett. 92, 182302 (2004).

[9] C. Ratti, M. A. Thaler and W. Weise, Phys. Rev. D 73, 014019 (2006).

[10] C. Sasaki, B. Friman and K. Redlich, Phys. Rev. D 77, 034024 (2008). Phys. Rev. Lett. 99, 232301 (2007). Phys. Rev. D 75, 074013 (2007).

[11] S. Digal, E. Laermann and H. Satz, Eur. Phys. J. C 18, 583 (2001).

[12] E. Megias, E. Ruiz Arriola and W. Weise, Phys. Rev. D 73, 014019 (2006).

[13] C. Sasaki, B. Friman and K. Redlich, Phys. Rev. D 77, 034024 (2008). Phys. Rev. Lett. 99, 232301 (2007). Phys. Rev. D 75, 074013 (2007).

[14] S. Digal, E. Laermann and H. Satz, Eur. Phys. J. C 18, 583 (2001).

[15] B. J. Schaefer, J. M. Pawlowski and J. Wambach, Phys. Rev. D 76, 074023 (2007).

[16] E. S. Bowman and J. I. Kapusta, Phys. Rev. C 79, 015202 (2009) [arXiv:0810.0042 [nucl-th]].

[17] C. Sasaki and I. Mishustin, arXiv:1005.4811 [hep-ph]. C. Sasaki, arXiv:0910.4375 [hep-ph]. M. Harada, C. Sasaki and S. Takemoto, Phys. Rev. D 81 (2010) 016009.

[18] T. Kahara and K. Tuominen, arXiv:1006.3931 [hep-ph]. L. F. Palhares and E. S. Fraga, arXiv:1006.2357 [hep-ph].

[19] C. Wetterich, Phys. Lett. B 301, 90 (1993).

[20] T. R. Morris, Int. J. Mod. Phys. A 9, 2411 (1994).

[21] U. Ellwanger, Z. Phys. C 62, 503 (1994).

[22] J. Berges, N. Tetradis and C. Wetterich, Phys. Rept. 363, 223 (2002).

[23] B. J. Schaefer and J. Wambach, Phys. Rev. D 75, 085015 (2007).

[24] B. Stokic, B. Friman and K. Redlich, Eur. Phys. J. C 67, 425 (2010).

[25] T. K. Herbst, J. M. Pawlowski and B. J. Schaefer, arXiv:1008.0081 [hep-ph].

[26] B. J. Schaefer, J. M. Pawlowski and K. Redlich, Eur. Phys. J. C 67, 425 (2010).

[27] T. K. Herbst, J. M. Pawlowski and B. J. Schaefer, arXiv:1008.0081 [hep-ph].

[28] V. Skokov, B. Stokic, B. Friman and K. Redlich, Phys. Rev. C 82, 015206 (2010) [arXiv:1004.2665 [hep-ph]].

[29] C. Ratti, S. Roessner and W. Weise, Phys. Lett. B 649, 57 (2007) [arXiv:hep-ph/0701091].

[30] K. Fukushima, Phys. Rev. D 77, 114028 (2008) [Erratum-ibid. D 78, 039902 (2008)] arXiv:0803.3318 [hep-ph].

[31] B. J. Schaefer, M. Wagner and J. Wambach, Phys. Rev. D 81, 074013 (2010) [arXiv:0910.5628 [hep-ph]].

[32] D. U. Jungnickel and C. Wetterich, Phys. Rev. D 53, 5142 (1996).

[33] L. F. Palhares and E. S. Fraga, Phys. Rev. D 78, 025013 (2008) arXiv:0803.0262 [hep-ph]; E. S. Fraga, L. F. Palhares and M. B. Pinto, Phys. Rev. D 79, 065026 (2009) arXiv:0902.1498 [hep-ph].

[34] J. Braun, K. Schwenzel and H. J. Pirner, Phys. Rev. D 70, 085016 (2004).

[35] V. Skokov, B. Friman, E. Nakano, K. Redlich and B. J. Schaefer, arXiv:1005.3166 [hep-ph].

[36] E. F. Litim, Nucl. Phys. B 631, 128 (2002) arXiv:hep-th/0203006.

[37] E. Nakano, B. J. Schaefer, B. Stokic, B. Friman and K. Redlich, arXiv:0907.1344 [hep-ph].

[38] J.A. Adams et al., Mod. Phys. Lett. A 10 (1995) 2367 arXiv:hep-th/9507093.

[39] M. A. Stephanov, PoS LAT2006, 024 (2006) arXiv:hep-lat/0701002.

[40] C. Athanasiou, K. Rajagopal and M. Stephanov, arXiv:1006.4636 [hep-ph].

[41] M. A. Stephanov, Phys. Rev. Lett. 102, 032301 (2009) arXiv:0809.3450 [hep-ph].

[42] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. D 60, 114028 (1999) arXiv:hep-ph/9903292.

[43] S. Ejiri, F. Karsch and K. Redlich, Phys. Lett. B 633, 275 (2006).

[44] F. Karsch, S. Ejiri and K. Redlich, Nucl. Phys. A 774, 619 (2006). S. Ejiri, et al., Nucl. Phys. A 774, 837 (2006).

[45] B. Stokic, B. Friman and K. Redlich, Phys. Lett. B 673 (2009) 192 arXiv:0809.3120 [hep-ph].

[46] W. J. Fu, Y. X. Liu and Y. L. Wu, Phys. Rev. D 81, 014028 (2010) arXiv:0910.5733 [hep-ph].

[47] F. Karsch and K. Redlich, arXiv:1007.2581 [hep-ph].

[48] M. Cheng, C. Jung, F. Karsch, O. Kaczmarek, E. Laermann, R.D. Mawhinney, C. Miao, P. Petreczky, C. Schmidt, W. Soeldner, Phys. Rev. D 79, 074505 (2009).

[49] F. Karsch, PoS CP0407.026 (2007). C. Schmidt, et. al. arXiv:0805.0236 [hep-lat]. F. Karsch, Talk given at the INT Conference ”The QCD Critical Point” (INT-08-2b) (2008).

[50] Y. Hatta and T. Ikeda, Phys. Rev. D 67, 014028 (2003) arXiv:hep-ph/0210284.

[51] More precisely, this number is $6/\pi^2$ due to quantum statistics.

[52] In the Z(2) universality class the $\alpha \approx 0.125$, $\beta = 0.312$ and $\gamma \approx 1.25$. 

[53] L. F. Palhares and E. S. Fraga, Phys. Rev. D 78, 025013 (2008) arXiv:0803.0262 [hep-ph]; E. S. Fraga, L. F. Palhares and M. B. Pinto, Phys. Rev. D 79, 065026 (2009) arXiv:0902.1498 [hep-ph].