We report on duality invariant constraints, which allow a classification of BPS black holes preserving different fractions of supersymmetry. We then relate this analysis to the orbits of the exceptional groups $E_6(6), E_7(7)$, relevant for black holes in five and four dimensions.

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BPS BLACK HOLES, SUPERSYMMETRY AND ORBITS OF EXCEPTIONAL GROUPS

Sergio FERRARA
Theoretical Physics Division, CERN, CH-1211 Geneva 23

Abstract: We report on duality invariant constraints, which allow a classification of BPS black holes preserving different fractions of supersymmetry. We then relate this analysis to the orbits of the exceptional groups $E_6(6), E_7(7)$, relevant for black holes in five and four dimensions.

1 Introduction

Impressive results have recently been obtained in the study of general properties of BPS black holes in supersymmetric theories of gravity. The latter are described by string theory and M-theory [1] whose symmetry properties are encoded in extended supergravity effective field theories.

Of particular interest are extremal black holes in four and five dimensions which correspond to BPS saturated states [2] and whose ADM mass depends, beyond the quantized values of electric and magnetic charges, on the asymptotic value of scalars at infinity. The latter describe the moduli space of the theory. Another physical relevant quantity, which depends only on quantized electric and magnetic charges, is the black hole entropy, which can be defined macroscopically, through the Bekenstein-Hawking area-entropy relation or microscopically, through D-branes techniques [3] by counting of microstates [4]. It has been further realized that the scalar fields, independently of their values at infinity, flow towards the black hole horizon to a fixed value of pure topological nature given by a certain ratio of electric and magnetic charges [5]. These “fixed scalars” correspond to the extrema of the ADM mass in moduli space while the black-hole entropy is the value of the squared ADM mass at this point in $D = 4$ [4] and the power $3/2$ of the ADM mass in $D = 5$. In four dimensional theories with $N > 2$, extremal black-holes preserving one supersymmetry have the further property that all central charge eigenvalues other than the one equal to the BPS mass flow to zero for “fixed scalars”.

The entropy formula turns out to be in all cases a U-duality invariant expression (homogeneous of degree two in $D = 4$ and of degree $3/2$ in $D = 5$) built out of electric and magnetic
charges and as such can be in fact also computed through certain (moduli-independent) topological quantities which only depend on the nature of the U-duality groups and the appropriate representations of electric and magnetic charges. More specifically, in the $N = 8$, $D = 4$ and $D = 5$ theories the entropy was shown to correspond to the unique quartic $E_7$ and cubic $E_6$ invariants built with the 56 and 27 dimensional representations respectively \cite{[5]}, \cite{[6]}. In this report we respectively describe the invariant classification of BPS states preserving different numbers of supersymmetries in $D = 4$ and 5 and then relate this analysis to the theory of BPS orbits of the exceptional groups $E_7(7)$ and $E_6(6)$.

2 BPS Conditions for Enhanced Supersymmetry

In this section we will describe U-duality invariant constraints on the multiplets of quantized charges in the case of BPS black holes whose background preserves more than one supersymmetry \cite{[9]}. We will still restrict our analysis to four and five dimensional cases for which three possible cases exist \textit{i.e.} solutions preserving 1/8, 1/4 and 1/2 of the original supersymmetry (32 charges).

The invariants may only be non zero on solutions preserving 1/8 supersymmetry. In dimensions $6 \leq D \leq 9$ black holes may only preserve 1/4 or 1/2 supersymmetry, and no associated invariants exist in these cases.

The description which follows also make contact with the D-brane microscopic calculation, as it will appear obvious from the formulae given below. We will first consider the five dimensional case.

In this case, BPS states preserving 1/4 of supersymmetry correspond to the invariant constraint $I_3(27) = 0$ where $I_3$ is the $E_6$ cubic invariant \cite{[9]}. This corresponds to the $E_6$ invariant statement that the $\mathbf{27}$ is a null vector with respect to the cubic norm. As we will show in a moment, when this condition is fulfilled it may be shown that two of the central charge eigenvalues are equal in modulus. The generic configuration has 26 independent charges.

Black holes corresponding to 1/2 BPS states correspond to null vectors which are critical, namely

$$\partial I(27) = 0$$

(1)

In this case the three central charge eigenvalues are equal in modulus and a generic charge vector has 17 independent components.

To prove the above statements, it is useful to compute the cubic invariant in the normal frame, given by:

$$I_3(27) = Tr(Z\Omega)^3$$

$$= 6(e_1 + e_2)(e_1 + e_3)(e_2 + e_3)$$

$$= 6s_1s_2s_3$$

(2)
where:

\[ e_1 = \frac{1}{2}(s_1 + s_2 - s_3) \]
\[ e_2 = \frac{1}{2}(s_1 - s_2 + s_3) \]
\[ e_3 = \frac{1}{2}(-s_1 + s_2 + s_3) \]

are the eigenvalues of the traceless antisymmetric $8 \times 8$ matrix. We then see that if $s_1 = 0$ then $|e_1| = |e_2|$, and if $s_1 = s_2 = 0$ then $|e_1| = |e_2| = |e_3|$. To count the independent charges we must add to the eigenvalues the angles given by $USp(8)$ rotations. The subgroup of $USp(8)$ leaving two eigenvalues invariant is $USp(2)^4$, which is twelve dimensional. The subgroup of $USp(8)$ leaving invariant one eigenvalue is $USp(4) \times USp(4)$, which is twenty dimensional. The angles are therefore $36 - 12 = 24$ in the first case, and $36 - 20 = 16$ in the second case. This gives rise to configurations with 26 and 17 charges respectively, as promised.

Taking the case of Type II on $T^5$ we can choose $s_1$ to correspond to a solitonic five-brane charge, $s_2$ to a fundamental string winding charge along some direction and $s_3$ to Kaluza-Klein momentum along the same direction.

The basis chosen in the above example is $S$-dual to the D-brane basis usually chosen for describing black holes in Type IIB on $T^5$. All other bases are related by U-duality to this particular choice. We also observe that the above analysis relates the cubic invariant to the picture of intersecting branes since a three-charge $1/8$ BPS configuration with non-vanishing entropy can be thought as obtained by intersecting three single charge $1/2$ BPS configurations [18], [19], [20].

By using the $S$–$T$-duality decomposition we see that the cubic invariant reduces to $I_3(27) = 10 - 210 - 21_4 + 16_116_110_{-2}$. The 16 correspond to D-brane charges, the 10 correspond to the 5 KK directions and winding of wrapped fundamental strings, the 1 correspond to the N-S five-brane charge.

We see that to have a non-vanishing area we need a configuration with three non-vanishing N-S charges or two D-brane charges and one N-S charge.

Unlike the 4-$D$ case, it is impossible to have a non-vanishing entropy for a configuration only carrying D-brane charges.

We now turn to the four dimensional case.

In this case the situation is more subtle because the condition for the 56 to be a null vector (with respect to the quartic norm) is not sufficient to enhance the supersymmetry. This can be easily seen by going to the normal frame where it can be shown that for a null vector there are not, in general, coinciding eigenvalues. The condition for $1/4$ supersymmetry is that the gradient of the quartic invariant vanish.

The invariant condition for $1/2$ supersymmetry is that the second derivative projected into the adjoint representation of $E_7$ vanish. This means that, in the symmetric quadratic
polynomials of second derivatives, only terms in the 1463 of $E_7$ are non-zero. Indeed, it can be shown, going to the normal frame for the 56 written as a skew $8 \times 8$ matrix, that the above conditions imply two and four eigenvalues being equal respectively.

The independent charges of $1/4$ and $1/2$ preserving supersymmetry are 45 and 28 respectively.

To prove the latter assertion, it is sufficient to see that the two charges normal-form matrix is left invariant by $USp(4) \times USp(4)$, while the one charge matrix is left invariant by $USp(8)$ so the $SU(8)$ angles are $63 - 20 = 43$ and $63 - 36 = 27$ respectively.

The generic $1/8$ supersymmetry preserving configuration of the 56 of $E_7$ with non-vanishing entropy has five independent parameters in the normal frame and $51 = 63 - 12$ $SU(8)$ angles. This is because the compact little group of the normal frame is $SU(2)^4$. The five parameters describe the four eigenvalues and an overall phase of an $8 \times 8$ skew diagonal matrix.

If we allow the phase to vanish, the 56 quartic norm just simplifies as in the five dimensional case:

$$I_4(56) = s_1s_2s_3s_4 = (e_1 + e_2 + e_3 + e_4)(e_1 + e_2 - e_3 - e_4) \times (e_1 - e_2 - e_3 + e_4)(e_1 - e_2 + e_3 - e_4)$$

where $e_i$ ($i = 1, \ldots, 4$) are the four eigenvalues.

$1/4$ BPS states correspond to $s_3 = s_4 = 0$ while $1/2$ BPS states correspond to $s_2 = s_3 = s_4 = 0$.

An example of this would be a set of four D-branes oriented along 456, 678, 894, 579 (where the order of the three numbers indicates the orientation of the brane). Note that in choosing the basis the sign of the D-3-brane charges is important; here they are chosen such that taken together with positive coefficients they form a BPS object. The first two possibilities ($I_4 \neq 0$ and $I_4 = 0, \frac{\partial I_4}{\partial q^i} \neq 0$) preserve $1/8$ of the supersymmetries, the third ($\frac{\partial I_4}{\partial q^i} = 0, \frac{\partial^2 I_4}{\partial q^i \partial q^j} \, |_{Adj \, E_7} \neq 0$) $1/4$ and the last ($\frac{\partial^2 I_4}{\partial q^i \partial q^j} \, |_{Adj \, E_7} = 0$) $1/2$.

It is interesting that there are two types of $1/8$ BPS solutions. In the supergravity description, the difference between them is that the first case has non-zero horizon area. If $I_4 < 0$ the solution is not BPS. This case corresponds, for example, to changing the sign of one of the three-brane charges discussed above. By U-duality transformations we can relate this to configurations of branes at angles such as in [10].

Going from four to five dimensions it is natural to decompose the $E_7 \to E_6 \times O(1, 1)$ where $E_6$ is the duality group in five dimensions and $O(1, 1)$ is the extra T-duality that appears when we compactify from five to four dimensions. According to this decomposition, the representation breaks as: $56 \to 27_1 + 1_{-3} + 27'_{-1} + 1_3$ and the quartic invariant becomes:

$$56^4 = (27_1)^31_{-3} + (27'_{-1})^31_3 + 1_31_31_{-3}1_{-3} + 27_127_127'_{-1}27'_{-1} + 27_127'_{-1}1_31_{-3}$$

The $27$ comes from point-like charges in five dimensions and the $27'$ comes from string-like charges.
Decomposing the U-duality group into T- and S-duality groups, $E_7 \rightarrow SL(2, \mathbb{R}) \times O(6, 6)$, we find $56 \rightarrow (2, 12) + (1, 32)$ where the first term corresponds to N-S charges and the second term to D-brane charges. Under this decomposition the quartic invariant (1) becomes $56^4 \rightarrow 32^4 + (12.12')^2 + 32^2.12.12'$. This means that we can have configurations with a non-zero area that carry only D-brane charges, or only N-S charges or both D-brane and N-S charges.

It is remarkable that $E_{7(7)}$-duality gives additional restrictions on the BPS states other than the ones merely implied by the supersymmetry algebra. The analysis of double extremal black holes implies that $I_4$ be semi-definite positive for BPS states. From this fact it follows that configurations preserving 1/4 of supersymmetry must have eigenvalues equal in pairs, while configurations with three coinciding eigenvalues are not BPS.

To see this, it is sufficient to write the quartic invariant in the normal frame basis. A generic skew diagonal $8 \times 8$ matrix depends on four complex eigenvalues $z_i$. These eight real parameters can be understood using the decomposition [21]:

$$56 \rightarrow (8_v, 2, 1, 1) + (8_s, 1, 2, 1) + (8_c, 1, 1, 2) + (1, 2, 2, 2) \quad (6)$$

under

$$E_{7(7)} \rightarrow O(4, 4) \times SL(2, \mathbb{R})^3 \quad (7)$$

Here $O(4, 4)$ is the little group of the normal form and the $(2, 2, 2)$ are the four complex skew-diagonal elements. We can further use $U(1)^3 \subset SL(2, \mathbb{R})^3$ to further remove three relative phases so we get the five parameters $z_i = \rho_i e^{i\phi/4} (i = 1, \ldots, 4)$.

The quartic invariant, which is also the unique $SL(2, \mathbb{R})^3$ invariant built with the $(2, 2, 2)$, becomes [4]:

$$I_4 = \sum_i |z_i|^4 - 2 \sum_{i<j} |z_i|^2 |z_j|^2 + 4(z_1z_2z_3z_4 + \overline{z}_1\overline{z}_2\overline{z}_3\overline{z}_4)$$

$$= (\rho_1 + \rho_2 + \rho_3 + \rho_4)(\rho_1 + \rho_2 - \rho_3 - \rho_4) \times (\rho_1 - \rho_2 + \rho_3 + \rho_4)(\rho_1 - \rho_2 - \rho_3 + \rho_4)$$

$$+ 8\rho_1\rho_2\rho_3\rho_4 (\cos\phi - 1) \quad (8)$$

The last term is semi-definite negative. The first term, for $\rho_1 = \rho_2 = \rho$ becomes:

$$- [4\rho^2 - (\rho_3 + \rho_4)^2](\rho_3 - \rho_4)^2 \quad (9)$$

which is negative unless $\rho_3 = \rho_4$. So 1/4 BPS states must have

$$\rho_1 = \rho_2 > \rho_3 = \rho_4, \quad \cos\phi = 1 \quad (10)$$

For $\rho_1 = \rho_2 = \rho_3 = \rho$, the first term in $I_4$ becomes:

$$- (3\rho + \rho_4)(\rho - \rho_4)^3 \quad (11)$$

so we must also have $\rho_4 = \rho$, $\cos\phi = 1$ which is the 1/2 BPS condition.

An interesting case, where $I_4$ is negative, corresponds to a configuration carrying electric and magnetic charges under the same gauge group, for example a 0-brane plus 6-brane configuration.
which is dual to a K–K-monopole plus K–K-momentum \[11\], \[12\]. This case corresponds to 
\[z_i = \rho e^{i\phi/4}\]
and the phase is 
\[\tan \phi/4 = e/g\]
where \(e\) is the electric charge and \(g\) is the magnetic charge. Using \(I_4\) we find that 
\(I_4 < 0\) unless the solution is purely electric or purely magnetic. In \[13\] it was suggested that \(0 + 6\) does not form a supersymmetric state. Actually, it was shown in \[14\] that a \(0 + 6\) configuration can be T-dualized into a non-BPS configuration of four intersecting D-3-branes. Of course, \(I_4\) is negative for both configurations. Notice that even though these two charges are Dirac dual (and U-dual) they are not S-dual in the sense of filling out an \(SL(2, \mathbb{Z})\) multiplet. In fact, the K–K-monopole forms an \(SL(2, \mathbb{Z})\) multiplet with a fundamental string winding charge under S-duality \[15\].

3 Duality Orbits for BPS States Preserving Different Numbers of Supersymmetries

In this last section we give an invariant classification of BPS black holes preserving different numbers of supersymmetries in terms of orbits of the 27 and the 56 fundamental representations of the duality groups \(E_6(6)\) and \(E_7(7)\) respectively \[16\], \[17\].

In five dimensions the generic orbits preserving 1/8 supersymmetry correspond to the 26 dimensional orbits \(E_6(6)/F_4(4)\) so we may think the generic 27 vector of \(E_6\) parametrized by a point in this orbit and its cubic norm (which actually equals the square of the black-hole entropy).

The light-like orbit, preserving 1/4 supersymmetry, is the 26 dimensional coset \(E_6(6)/O(5, 4) \oplus T_{16}\) where \(\oplus\) denotes the semidirect product.

The critical orbit, preserving maximal 1/2 supersymmetry (this correspond to \(\frac{\partial I_4}{\partial q_i} \neq 0\) correspond to the 17 dimensional space

\[
\frac{E_6(6)}{O(5, 5) \oplus T_{16}}
\]

In the four dimensional case, we have two inequivalent 55 dimensional orbits corresponding to the cosets \(E_7(7)/E_6(2)\) and \(E_7(7)/E_6(6)\) depending on whether \(I_4(56) > 0\) or \(I_4(56) < 0\). The first case corresponds to 1/8 BPS states with non-vanishing entropy, while the latter corresponds to non BPS states.

There is an additional 55 dimensional light-like orbit \((I_4 = 0)\) preserving 1/8 supersymmetry given by \(\frac{E_7(7)}{F_4(4) \oplus T_{26}}\).

The critical light-like orbit, preserving 1/4 supersymmetry, is the 45 dimensional coset \(E_7(7)/O(6, 5) \oplus (T_{32} \oplus T_1)\)

The critical orbit corresponding to maximal 1/2 supersymmetry is described by the 28
dimensional quotient space

\[
\frac{E_{7(7)}}{E_{6(6)} \otimes T_{27}}
\]  

(13)

We actually see that the counting of parameters in terms of invariant orbits reproduces the counting previously made in terms of normal frame parameters and angles. The above analysis makes a close parallel between BPS states preserving different numbers of supersymmetries with time-like, space-like and light-like vectors in Minkowski space.

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