Topical Review

Discrete time Toda systems

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Abstract

In this paper, we discuss several concepts of the modern theory of discrete integrable systems, including:

- Time discretization based on the notion of Bäcklund transformation.
- Symplectic realizations of multi-Hamiltonian structures.
- Interrelations between discrete 1D systems and lattice 2D systems.
- Multi-dimensional consistency as integrability of discrete systems.
- Interrelations between integrable systems of quad-equations and integrable systems of Laplace type.
- Pluri-Lagrangian structure as integrability of discrete variational systems.

All these concepts are illustrated by the discrete time Toda lattices and their relativistic analogs.

Keywords: discrete integrable systems, multi-dimensional consistency, pluri-Lagrangian structure, discrete Laplace type equations, discrete time Toda lattice

1. Introduction

The one-dimensional lattice with exponential interaction of nearest neighbors, discovered by Toda,

$$\tilde{x}_k = e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}}, \quad (1.1)$$

and its relativistic generalization, discovered by Ruijsenaars,

$$\tilde{x}_k = (1 + \alpha \tilde{x}_{k+1})(1 + \alpha \tilde{x}_k) \frac{e^{\tilde{x}_{k+1} - \tilde{x}_k}}{1 + \alpha^2 e^{\tilde{x}_{k+1} - \tilde{x}_k}} - (1 + \alpha \tilde{x}_k)(1 + \alpha \tilde{x}_{k-1}) \frac{e^{\tilde{x}_k - \tilde{x}_{k-1}}}{1 + \alpha^2 e^{\tilde{x}_k - \tilde{x}_{k-1}}}, \quad (1.2)$$

belong to the most celebrated integrable models. They enjoy a great amount of generalizations and applications in various branches of mathematics and physics. This paper reviews a variety of generalized Toda lattices (TLs) and relativistic TLs, along with their integrable discretizations. This gives us an opportunity to touch upon some of the most important recent developments in the field.
developments of the theory of discrete integrable systems, including the multi-dimensional consistency and pluri-Lagrangian structure. The paper is organized as follows.

In section 2 we quickly review the main integrability attributes of the TL in the Flaschka–Manakov variables. It is one of the basic systems amenable to the Adler–Kostant–Symes scheme, which is presented in section 3. A recipe for integrable discretization of the systems within the AKS scheme is formulated in section 4. It is applied to the TL in the Flaschka–Manakov variables in section 5. Then in section 6 these results are applied to a symplectic realization of the linear Poisson brackets for the Flaschka–Manakov variables, which leads to the most classical exponential TL (1.1) and its time discretization. A variety of relatives of this system, which appear through symplectic realization of different Poisson brackets for the Flaschka–Manakov variables, together with their time discretizations, are treated in section 7. After that, a similar work is done for the relativistic TL: in section 8, the main integrability attributes in the Flaschka–Manakov variables are reviewed, the discretization in these variables is performed in section 9, and various symplectic realizations are presented in sections 10 and 11. An interesting phenomenon is investigated in section 12: it turns out that explicit discretizations of the TL belong to the relativistic Toda hierarchy, the discrete time step playing the role of the inverse speed of light. In section 13, we address an important conceptual twist: time discretizations of 1D evolutionary equations are re-interpreted as lattice 2D systems. In the second half of the paper, we deal with recent conceptual breakthroughs in the theory of discrete integrable systems. In section 14, we discuss the relation of discrete Laplace type equations to quad-equations, and the notion of multi-dimensional consistency of quad-equations as their integrability. This development allows one to derive, in an algorithmic way, zero curvature representations for discrete relativistic Toda type systems, as demonstrated in section 15. Then, we turn to the pluri-Lagrangian theory, which describes integrability features of variational systems. The 1D pluri-Lagrangian theory is formulated in section 16 and is illustrated by the discrete time exponential TL in section 17. The 2D pluri-Lagrangian theory is formulated in section 18 and is illustrated by the discrete time relativistic TL in section 19.

Bibliographic references are given at the end of each section; they are kept to a necessary minimum and therefore are by no means exhaustive. We hope, however, that they will enable an interested reader to get oriented in the relevant literature.

2. TL in Flaschka variables: equations of motion, Lax representation and triHamiltonian structure

Equations of motion of the TL in Flaschka–Manakov variables:

\[ \dot{b}_k = a_k - a_{k-1}, \quad \dot{a}_k = a_k (b_{k+1} - b_k), \quad 1 \leq k \leq N, \]

with one of two types of boundary conditions: open-end \((a_0 = a_N = 0)\), or periodic (all subscripts are taken \((\text{mod} \ N)\), so that \(a_0 \equiv a_N, b_{N+1} \equiv b_1\)).

The Lax representation of the TL which we mainly use in this paper is:

\[ \dot{T} = [T, A_+] = -[T, A_-] \]

with

\[ T(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k+1} + \sum_{k=1}^{N} b_k E_{k} + \lambda \sum_{k=1}^{N} E_{k+1 \cdot k}, \]
\[ A_+ (a, b, \lambda) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad A_- (a, b, \lambda) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1}. \]  

(2.4)

Here and below \( E_{ik} \) stands for the matrix whose only nonzero entry is on the intersection of the \( i \)th row and the \( k \)th column and is equal to 1. Naturally, we set in the periodic case \( E_{N+1,N} = E_{1,N}, E_{N,N+1} = E_{N,1} \). In the open-end case we set \( E_{N+1,N} = E_{N,N+1} = 0 \) and always put \( \lambda = 1 \).

In the notation which will be explained in the next section, we have: \( A_+ = \pi_+ (T) \), \( A_- = \pi_- (T) \), so that (2.2) takes the form
\[
\dot{T} = [T, \pi_+(T)] = -[T, \pi_-(T)],
\]

(2.5)

which is a (prototypical) example of the systems eligible to the AKS (Adler–Kostant–Symes) scheme. Spectral invariants of the matrix \( T \) are integrals of motion of TL; there are \( N \) functionally independent ones.

The phase space of TL can be equipped with three local Poisson brackets which are preserved by this flow which is therefore Hamiltonian with respect to any of these Poisson structures. These brackets are compatible in the sense that their linear combinations are invariant Poisson brackets, as well.

### 2.1. Linear bracket

\[
\{b_k, a_k\} = -a_k, \quad \{a_k, b_{k+1}\} = -a_k.
\]

(2.6)

The Hamilton function of TL in this bracket is
\[
H_2 (a, b) = \frac{1}{2} \text{tr}(T^2) = \frac{1}{2} \sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k.
\]

(2.7)

### 2.2. Quadratic bracket

\[
\begin{align*}
\{b_k, a_k\}_2 &= -b_k a_k, & \{a_k, b_{k+1}\}_2 &= -a_k b_{k+1}, \\
\{b_k, b_{k+1}\}_2 &= -a_k, & \{a_k, a_{k+1}\}_2 &= -a_k a_{k+1}.
\end{align*}
\]

(2.8)

The Hamilton function of TL in this bracket is
\[
H_1 (a, b) = \text{tr}(T) = \sum_{k=1}^{N} b_k.
\]

(2.9)

### 2.3. Cubic bracket

\[
\begin{align*}
\{b_k, a_k\}_3 &= -a_k (b_k^2 + a_k), & \{a_k, b_{k+1}\}_3 &= -a_k (b_{k+1}^2 + a_k), \\
\{b_k, b_{k+1}\}_3 &= -a_k (b_k^2 + b_{k+1}), & \{a_k, a_{k+1}\}_3 &= -2a_k a_{k+1} b_{k+1}, \\
\{b_k, a_{k+1}\}_3 &= -a_k a_{k+1}, & \{a_k, b_{k+2}\}_3 &= -a_k a_{k+1}.
\end{align*}
\]

(2.10)
The Hamilton function of TL in this bracket is
\[ H_0(a, b) = \text{tr}(\log T) = \log(\det T). \] (2.11)
All spectral invariants of the Lax matrix \( T \) (including \( H_0, H_1, H_2 \)) are in involution with respect to any of these three brackets.

2.4. Bibliographical remarks
An integrable lattice with an exponential interaction of nearest neighbors was discovered by Toda [T67]. Since then it became one of the most popular and important integrable models in general. The best general reference remains Toda’s monograph [T89], where also the story of the discovery of this system is given from the first hand. The change of variables \((x, p) \mapsto (a, b)\) was introduced in [F74, M74], along with the Lax representation. Concerning the tri-Hamiltonian structure of the TL: the linear bracket was known from the very beginning, since it is a simple consequence of the Flaschka–Manakov change of variables; the quadratic and the cubic ones appeared in [A79] and in [Ku85], respectively.

3. Adler–Kostant–Symes scheme
Let \( g \) be a Lie algebra of some associative algebra, equipped with a non-degenerate bi-invariant scalar product, which allows us to identify \( g^* \) with \( g \). Let \( g \), as a linear space, be a direct sum of its two subspaces \( g_{\pm} \) which are also Lie subalgebras:
\[ g = g_+ \oplus g_-, \quad [g_+, g_+] \subset g_+, \quad [g_-, g_-] \subset g_- . \] (3.1)
Let \( \pi_+, \pi_- \) denote the projections from \( g \) to the corresponding subspaces, so that for any \( T \in g \) we have:
\[ T = \pi_+(T) + \pi_-(T), \quad \pi_{\pm}(T) \in g_{\pm}. \] (3.2)
Then the AKS scheme deals with explicit solutions and Hamiltonian structures of the flows
\[ \dot{T} = [T, \pi_+(f(T))] = -[T, \pi_-(f(T))], \] (3.3)
with \( \text{Ad} \)-covariant functions \( f : g \to g \).

This general setting is sufficient to ensure that equations (3.3) possess several remarkable properties. First of all, different flows of the type (3.3) commute. Second, they admit an explicit solution in terms of a factorization problem in a Lie group. Both these properties have a purely kinematical nature and do not depend on the Hamiltonian theory (though the latter provides a deeper insight into the situation).

Let \( G \) be a Lie group with the Lie algebra \( g \), and let \( G_+ \) and \( G_- \) be its two subgroups having \( g_+ \) and \( g_- \), respectively, as Lie algebras. Then in a certain neighborhood \( V \) of the group unit \( I \) following factorization is uniquely defined, so that for any \( g \in V \subset G \) we have:
\[ g = \Pi_+(g) \Pi_-(g), \quad \Pi_{\pm}(g) \in G_{\pm}. \] (3.4)
In what follows we suppose, for the sake of notational simplicity, that \( G \) is a matrix group, and write the adjoint action of the group on \( g \) as a conjugation by the corresponding matrices. Correspondingly, we shall call \( \text{Ad} \)-covariant functions \( f : g \to g \) also ‘conjugation covariant’.
This notation has an additional advantage of being applicable also to functions \( F : g \to G \). Namely, we shall call such a function conjugation covariant, if \( F(\text{Ad} g \cdot T) = g F(T) g^{-1} \).
Theorem 3.1. Let \( f : g \to g \) be a conjugation covariant function. Then the solution of the differential equation (3.3) with the initial condition \( T(0) = T_0 \) is given, at least for sufficiently small \( t \), by
\[
T(t) = \Pi_+^{-1} \left( e^{tf(T_0)} \right) \cdot T_0 \cdot \Pi_+ \left( e^{tf(T_0)} \right) = \Pi_- \left( e^{tf(T_0)} \right) \cdot T_0 \cdot \Pi_-^{-1} \left( e^{tf(T_0)} \right). \tag{3.5}
\]

Definition 3.2. Consider the hierarchy of flows (3.3) on \( g \). For an arbitrary conjugation covariant function \( F : g \to G \) define the \( \text{Bäcklund transformation} \) \( B_T F : g \to g \) of this hierarchy as
\[
\tilde{T} = B_T F(T) = \Pi_+^{-1} (F(T)) \cdot T \cdot \Pi_+ (F(T)) = \Pi_- (F(T)) \cdot T \cdot \Pi_-^{-1} (F(T)). \tag{3.6}
\]

One of the most important properties of Bäcklund transformations, implying also other ones, is contained in the following theorem.

Theorem 3.3. For two arbitrary conjugation covariant functions \( F_1, F_2 : g \to G \),
\[
B_T F_2 \circ B_T F_1 = B_T F_2 F_1, \tag{3.7}
\]
so that the Bäcklund transformations \( B_T F_2, B_T F_1 \) commute.

Corollary 3.4.

(a) Any two flows of the type (3.3) commute.
(b) An arbitrary Bäcklund transformation commutes with an arbitrary flow of the hierarchy (3.3). In other words, any Bäcklund transformation maps solutions of (3.3) onto solutions.

Indeed, according to theorem 3.1 any flow governed by a differential equation (3.3) consists of Bäcklund transformations \( B_T F \) with \( F(T) = e^{f(T)} \).

As another important consequence of theorem 3.3 we have the following statement.

Theorem 3.5. Let \( F : g \to G \) be a conjugation covariant function. Consider the formula (3.6) for the Bäcklund transformation \( B_T F \) as the difference equation
\[
\tilde{T} = \Pi_+^{-1} (F(T)) \cdot T \cdot \Pi_+ (F(T)) = \Pi_- (F(T)) \cdot T \cdot \Pi_-^{-1} (F(T)). \tag{3.8}
\]

for \( T = T(n) \), \( \tilde{T} = T(n + 1) \), with the initial condition \( T(0) = T_0 \). Then the solution of this difference equation is given by
\[
T(n) = \Pi_+^{-1} (F^n(T_0)) \cdot T_0 \cdot \Pi_+ (F^n(T_0)) = \Pi_- (F^n(T_0)) \cdot T_0 \cdot \Pi_-^{-1} (F^n(T_0)). \tag{3.9}
\]

Proof. By induction from theorem 3.3,
\[
B_T F_n \circ \ldots \circ B_T F_2 \circ B_T F_1 = B_T F_n F_{n-1} F_{n-2} \ldots F_2 F_1.
\]
In particular, for \( F_1 = F_2 = \ldots = F_n = F \) we obtain:
\[
(B_T F)^n = B_T F^n,
\]
which is the statement of the theorem.

This theorem gives a discrete-time counterpart of theorem 3.1. Comparing the formulas (3.9), (3.5), we see that the map (3.8) is interpolated by the flow (3.3) with the time step \( h \), if
\[ e^{\hbar(T)} = F(T) \iff f(T) = h^{-1} \log(F(T)). \]

For the flow TL, the main ingredients of the AKS construction are as follows.

3.1. Open-end case

For the open-end case we set \( g = \text{gl}(N) \), the algebra of \( N \times N \) matrices with the usual matrix product, the Lie bracket \([L,M] = LM - ML\), and the non-degenerate bi-invariant scalar product \( \langle L, M \rangle = \text{tr}(LM) \) which allows to identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \). We have a splitting (3.1), where \( \mathfrak{g}_+ \) consists of lower triangular matrices, while \( \mathfrak{g}_- \) consists of strictly upper triangular matrices. The Lie group \( G \) corresponding to the Lie algebra \( \mathfrak{g} \) is \( \text{GL}(N) \), the group of \( N \times N \) nondegenerate matrices. The subgroups \( G_+ \), \( G_- \) corresponding to the Lie algebras \( \mathfrak{g}_+ \), \( \mathfrak{g}_- \) consist of non-degenerate lower triangular matrices and of upper triangular matrices with unit diagonal, respectively. The \( \Pi, \Pi_- \) factorization is well known in the linear algebra under the name of the \( LU \) factorization.

3.2. Periodic case

In the periodic case we set \( g \) to be a certain twisted loop algebra over \( \text{gl}(N) \):

\[ g = \{ T(\lambda) \in \text{gl}(N)[\lambda, \lambda^{-1}] : \Omega T(\lambda)\Omega^{-1} = T(\omega \lambda) \} , \]

where \( \Omega = \text{diag}(1, \omega, \ldots, \omega^{N-1}) \), \( \omega = \exp(2\pi i/N) \). The nondegenerate bi-invariant scalar product is chosen as \( \langle L(\lambda), M(\lambda) \rangle = \text{tr}(L(\lambda)M(\lambda))_0 \) the subscript 0 denoting the free term of the formal Laurent series. Again, we have a splitting (3.1), where

\[ g_+ = \{ T(\lambda) \in \text{gl}(N)[\lambda] : \Omega T(\lambda)\Omega^{-1} = L(\omega \lambda) \} , \]
\[ g_- = \{ T(\lambda) \in \text{gl}(N)[\lambda^{-1}] : \Omega T(\lambda)\Omega^{-1} = T(\omega \lambda) \text{ and } T(\infty) = 0 \} .\]

The group \( G \) corresponding to the Lie algebra \( g \) is a twisted loop group,

\[ G = \{ g : \mathbb{C}P^1 \setminus \{0, \infty\} \to \text{GL}(N) : g \text{ regular, } \Omega g(\lambda)\Omega^{-1} = g(\omega \lambda) \} . \]

Its subgroups \( G_+ \) and \( G_- \) corresponding to the Lie algebras \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) are singled out by the following conditions:

\[ G_+ = \{ g : \mathbb{C}P^1 \setminus \{\infty\} \to \text{GL}(N) : g \text{ regular, } \Omega g(\lambda)\Omega^{-1} = g(\omega \lambda) \} , \]
\[ G_- = \{ g : \mathbb{C}P^1 \setminus \{0\} \to \text{GL}(N) : g \text{ regular, } \Omega g(\lambda)\Omega^{-1} = g(\omega \lambda) \text{ and } g(\infty) = I \} . \]

We call the corresponding \( \Pi_+, \Pi_- \) factorization the generalized \( LU \) factorization. It is uniquely defined in a certain neighborhood of the unit element of \( G \). As opposed to the open-end case, finding the generalized \( LU \) factorization is a problem of the Riemann–Hilbert type which is solved in terms of algebraic geometry rather than in terms of linear algebra.

In both cases, open-end and periodic, one has \( f(T) = T \).

3.3. Bibliographical remarks

The foundational references for the AKS-scheme are [A79, Ko79, Sy80, Sy82].
4. Recipe for integrable discretization

The results of the previous section suggest the following prescription.

Recipe. For an integrable system allowing a Lax representation of the form (3.3), an integrable discretization is given by the difference equation (3.6), with some conjugation covariant function $F: g \rightarrow G$ such that $F(T) = I + hf(T) + o(h)$ (Bäcklund transformation close to identity).

Of course, this prescription is only practical if the corresponding factors $\Pi_{\pm}(F(T))$ admit more or less explicit expressions, allowing to write down the corresponding difference equations in a more or less closed form. The choice of $F(T)$ in this transcendental problem. Miraculously, the simplest possible choice $F(T) = I + hf(T)$ works perfectly well for a vast set of examples (when it makes sense, i.e. when $I + hf(T) \in G$), including the TL.

Let us stress once more the advantages of this approach to the problem of integrable discretization.

- The discretizations obtained in this way share the Lax matrix and the integrals of motion with their underlying continuous time systems.
- Suppose that the hierarchy of continuous time systems (3.3) is Hamiltonian with respect to some Poisson bracket. Then our discretizations have the Poisson property with respect to the same bracket.
- The initial value problem for our discrete time equations can be solved in terms of the same factorization in a Lie group as the initial value problem for the continuous time system.
- Interpolating Hamiltonians are granted by-products of this approach.

4.1. Bibliographical remarks

This recipe for integrable discretization was clearly formulated for the first time in [S95, S96], and was put at the basis of a monographic study [S03]. However, a viewpoint according to which Bäcklund transformations lie at the basis of discretization was already pushed forward in [L81]. This is also well established in discrete differential geometry, see [B99, BS08, DSM00].

5. Discretization of the TL in the Flaschka–Manakov variables

We now turn to the problem of finding an integrable time discretization for the flow TL. To this purpose we apply the recipe of section 4 with

$$F(T) = I + hT,$$

i.e. we take as a discretization of the flow TL the map described by the discrete time Lax equation

$$\tilde{T} = \Pi_{+}^{-1}(I + hT) \cdot T \cdot \Pi_{+}(I + hT) = \Pi_{-}(I + hT) \cdot T \cdot \Pi_{-}^{-1}(I + hT).$$

Thus, the main problem is to determine the factors

$$A_{+} = \Pi_{+}(I + hT), \quad A_{-} = \Pi_{-}(I + hT). \quad (5.1)$$
Lemma 5.1. For the matrix $T = T(a, b, \lambda)$ of the form (2.3), the factors (5.1) are of the form

\[ A_+ = \sum_{k=1}^N \beta_k E_{kk} + h\lambda \sum_{k=1}^N E_{k+1,k}, \]

\[ A_- = I + h\lambda^{-1} \sum_{k=1}^N \alpha_k E_{kk+1}. \]

Here, the coefficients $\beta_k = \beta_k(a, b)$ are defined by the relations

\[ \beta_k = 1 + h b_k - \frac{h^2 a_k}{\beta_{k-1}}. \]

and

\[ \alpha_k = \frac{a_k}{\beta_k}. \]

Indeed, it is easy to realize that the factors $A_{\pm}$ must be of the form (5.2), (5.3). The factorization $A_+ A_- = I + hT$ is equivalent to the system

\[ \beta_k + h^2 \alpha_{k-1} = 1 + h b_k, \quad \beta_k \alpha_k = a_k, \]

which, in turn, is equivalent to (5.4), (5.5). In the open-end case, due to $a_0 = 0$, relation (5.4) is uniquely solvable, and leads to explicit expressions in terms of finite continued fractions:

\[ \beta_1 = 1 + h b_1; \quad \beta_2 = 1 + h b_2 = \frac{h^2 a_1}{1 + h b_1}; \quad \cdots ; \]

\[ \beta_N = 1 + h b_N - \frac{h^2 a_{N-1}}{1 + h b_{N-1} - \frac{h^2 a_{N-2}}{1 + h b_{N-2} - \cdots - \frac{h^2 a_1}{1 + h b_1}}}. \]

In the periodic case $\beta_k$ may be expressed as analogous infinite $N$-periodic continued fractions and are, therefore, double-valued functions of $(a, b)$. However, in the limit $h \to 0$ one branch of $\beta_k$ can be singled out by the asymptotics

\[ \beta_k(a, b) = 1 + h b_k + O(h^2). \]

Theorem 5.2. The discrete time Lax equation

\[ \tilde{T} = A_+^{-1} T A_- = A_+ T A_-^{-1} \quad \text{with} \quad A_+ = \Pi_+ (I + hT), \quad A_- = \Pi_- (I + hT) \]

is equivalent to the map $(a, b) \mapsto (\tilde{a}, \tilde{b})$ described by the following equations:

\[ \tilde{b}_k = b_k + h \left( \frac{a_k}{\beta_k} - \frac{a_{k-1}}{\beta_{k-1}} \right), \quad \tilde{a}_k = a_k \frac{\beta_{k+1}}{\beta_k}, \]

where the functions $\beta_k$ are defined by the recurrent relation (5.4).
The map (5.9) will be of a fundamental interest to us in this paper. We denote it by $dTL(h)$; it is a genuine map in the open-end case, while in the periodic case it is a double-valued map (a correspondence). The following statements automatically follow from our construction.

- The map $dTL(h)$ is Poisson with respect to any invariant Poisson bracket of the flow TL.
- The map $dTL(h)$ commutes with all flows of the TL hierarchy (3.3).
- The maps $dTL(h)$ with different $h$ commute among themselves. (However, the notion of commutativity of double-valued maps is non-trivial, see section 17.)
- The map $dTL(h)$ is interpolated by the flow (3.3) with $f(T) = h^{-1} \log (I + h T)$.

For most of the time, the parameter $h$ will be suppressed from the notation.

### 5.1. Bibliographical remarks

Bäcklund–Darboux transformation for the TL in Flaschka–Manakov variables, which essentially coincides with the map given in theorem 5.2, was given for the first time in [MS79]. However, as observed in [S95], it is not different from the $qd$ algorithm well known in the numerical analysis for a long time [Rut57]. Moreover, the latter reference contains also the equations of motion of the TL (under the name of a ‘continuous analogue of the $qd$ algorithm’).

### 6. Symplectic realization of the linear bracket: exponential TL

A great variety of canonical Hamiltonian systems and their equivalent Lagrangian systems arise from TL upon parametrizing various invariant Poisson brackets via the canonical symplectic brackets. A map from $\mathbb{R}^{2N}(x, p)$ equipped with the canonical symplectic structure to a Poisson manifold $(P, \{\cdot, \cdot\})$ is called a symplectic realization of the bracket $\{\cdot, \cdot\}$, if it is Poisson. In particular, the most classical (exponential) form of TL (the original discovery by Toda) appears this way via the Flaschka–Manakov map,

$$a_k = e^{\alpha_k - \alpha_{k-1}}, \quad b_k = p_k.$$  \hspace{1cm} (6.1)

We consider this map for two types of the boundary conditions. For the periodic case we assume that $x_0 = x_N, x_{N+1} = x_1$, while for the open-end case we set $x_0 = \infty, x_{N+1} = -\infty$, which corresponds to $a_N = 0$.

**Proposition 6.1.** The map (6.1) is a symplectic realization of the linear Poisson bracket $\{\cdot, \cdot\}_1$, see (2.6), on the phase space of TL.

**Theorem 6.2.** Pull-back of the flow TL under parametrization (6.1) is a canonical Hamiltonian system with the Hamilton and the Lagrange functions

$$H(x, p) = \frac{1}{2} \sum_{k=1}^N p_k^2 + \sum_{k=1}^N e^{\alpha_k - \alpha_{k-1}}.$$  \hspace{1cm} (6.2)

$$L(x, \dot{x}) = \frac{1}{2} \sum_{k=1}^N \dot{x}_k^2 - \sum_{k=1}^N e^{\alpha_k - \alpha_{k-1}}.$$  \hspace{1cm} (6.3)

The corresponding Newtonian equations of motion:

$$\ddot{x}_k = e^{\alpha_k - \alpha_{k-1}} - e^{\alpha_{k-2}}.$$  \hspace{1cm} (6.4)
Proof. Hamilton function \((6.2)\) is obtained by substituting \((6.1)\) into \((2.7)\). Lagrange function \((6.3)\) is obtained from 
\[ H(x, p) \]
via the Legendre transformation.

For the map \(dTL\), the machinery of Hamiltonian flows is no more available, but it may be successfully replaced by a direct analysis of equations of motion. From now on, we use the undertilde symbols to denote inverse maps (if \(\phi : y \mapsto \tilde{y}\), then \(\phi^{-1} : \tilde{y} \mapsto y\)).

**Theorem 6.3.** Pull-back of the map \(dTL\) under parametrization \((6.1)\) is a symplectic map 
\[(x, p) \mapsto (\tilde{x}, \tilde{p})\]
with the following equations of motion:
\[
\begin{align*}
\tilde{p}_k &= \frac{1}{h} \left( e^{\tilde{x}_k - x_k} - 1 \right) + h e^{x_k - \tilde{x}_k - 1}, \\
\tilde{p}_k &= \frac{1}{h} \left( e^{\tilde{x}_k - x_k} - 1 \right) + h e^{x_k - \tilde{x}_k}.
\end{align*}
\]

The corresponding Newtonian equations of motion:
\[
\begin{align*}
e^{\tilde{x}_k - x_k} - e^{x_k - \tilde{x}_k} &= h^2 \left( e^{x_{k+1} - x_k} - e^{x_k - \tilde{x}_{k-1}} \right).
\end{align*}
\]

**Proof.** Under parametrization \((6.1)\) equations of motion \((5.9)\) of \(dTL\) together with recurrent relation \((5.4)\) for the auxiliary quantities \(\beta_k\) take the following form:
\[
\begin{align*}
\tilde{e}^{\tilde{x}_k - x_k} &= e^{\tilde{x}_k + 1 - x_k}, \\
\tilde{p}_k &= p_k + \frac{h e^{\tilde{x}_k + 1 - x_k}}{\beta_k} = \frac{h e^{x_k - \tilde{x}_k - 1}}{\beta_k}, \\
\beta_k &= 1 + h p_k - \frac{h^2 e^{x_k - \tilde{x}_k - 1}}{\beta_k}.
\end{align*}
\]
Equation \((6.7)\) implies that the quantity \(\beta_k / e^{\tilde{x}_k - x_k}\) is constant, i.e. does not depend on \(k\). Choosing this constant is equivalent to choosing one representative among all possible pull–backs of the map \(dTL\). We set this constant equal to one, so that
\[
\beta_k = e^{\tilde{x}_k - x_k}.
\]
Substituting this into \((6.9)\), we obtain the first equation of motion in \((6.5)\). Finally, the second equation of motion in \((6.5)\) follows from \((6.8)\) by using the previously obtained expressions. Newtonian equations of motion \((6.6)\) follow directly from comparing both equation in \((6.5)\).

We will say that the Hamiltonian system with the Hamilton function \((6.2)\) and the map \((6.5)\) are symplectic realizations of the flow \(TL\) and of the map \(dTL\), respectively. The latter map can be named discrete exponential \(TL\).

**6.1. Bibliographical remarks**

The discrete time \(TL\) \((6.6)\), in the Lagrangian form \((6.5)\), was found in [WT75, TW75] as a Bäcklund transformation for the TL. The Lagrangian function appeared as a generating function of the Bäcklund transformation, which demonstrated also the symplectic nature of this
transformation. Notice that in the infinite lattice situation this Bäcklund transformation is, generically, no longer isospectral; rather, it adds one soliton to the solution.

7. The variety of symplectic realizations of TL and dTL

In this section, we will provide the reader with a list of different symplectic realizations of invariant Poisson brackets on the phase space of TL, as well as of the flow TL and of the map dTL.

7.1. General form of equations of motion

All symplectic realizations of TL share the following general form:

\[ \ddot{x}_k = r(\dot{x}_k)\left(f(x_{k+1} - x_k) - f(x_k - x_{k-1})\right). \]  (7.1)

System (7.1) is Lagrangian, with the Lagrange function

\[ L(x, \dot{x}) = \sum_{k=1}^{N} K(\dot{x}_k) - \sum_{k=1}^{N} U(x_{k+1} - x_k), \]  (7.2)

where \( K''(v) = 1/r(v) \) and \( U'(u) = f(u) \). By the Legendre transformation, one easily finds the Hamilton function \( H(x, p) \).

All symplectic realizations of dTL(h) have the following general structure:

\[ \psi(\tilde{x}_k - \tilde{x}_k; h) = \psi(x_k - \tilde{x}_k; h) - \phi(x_k - \tilde{x}_{k-1}; h). \]  (7.3)

These Newtonian equations of motion admit a Lagrangian formulation. They are identified as discrete time Euler–Lagrange equations

\[ \frac{\partial}{\partial \tilde{x}_k} \left( \Lambda(x, \tilde{x}; h) + L(x, \tilde{x}; h) \right) = 0 \]  (7.4)

for the discrete time Lagrange function \( \Lambda: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \),

\[ \Lambda(x, \tilde{x}; h) = \sum_{k=1}^{N} \Psi(\tilde{x}_k - x_k; h) - \sum_{k=1}^{N} \Phi(x_{k+1} - \tilde{x}_k; h), \]  (7.5)

where \( \Psi(\xi; h) = \psi(\xi; h) \), \( \Phi(\xi; h) = \phi(\xi; h) \). The discrete time Euler–Lagrange equations generate a symplectic map \( F: T^* \mathbb{R}^N \to T^* \mathbb{R}^N \), \( F(x, p) = (\tilde{x}, \tilde{p}) \) by the formulas

\[
\begin{align*}
    p_k &= -\partial \Lambda(x, \tilde{x}; h)/\partial \tilde{x}_k = \psi(\tilde{x}_k - x_k; h) + \phi(x_k - \tilde{x}_{k-1}; h), \\
    \tilde{p}_k &= \partial \Lambda(x, \tilde{x}; h)/\partial x_k = \psi(x_k - \tilde{x}_k; h) + \phi(x_{k+1} - \tilde{x}_k; h).
\end{align*}
\]  (7.6)

In order that equations (7.6) define a map \((x, p) \mapsto (\tilde{x}, \tilde{p})\), the first of these equations should be solvable for \( \tilde{x} \) (at least locally), i.e. the matrix of the mixed partial derivatives of the Lagrange function \( \Lambda \) should be non-degenerate, \( \det(\partial^2 \Lambda/\partial x_k \partial \tilde{x}_j) \neq 0 \).

Function \( \Lambda(x, \tilde{x}; h) \) from (7.5) is a difference approximation to \( L(x, \dot{x}) \) from (7.2) as \( h \to 0 \).

More precisely, assuming that \( \tilde{x} = x + h\dot{x} + O(h^2) \), we find:

\[ h^{-1} \Lambda(x, \tilde{x}; h) = L(x, \dot{x}) + O(h), \]

provided \( h^{-1} \Psi(hv; h) = K(v) + O(h) \) and \( h^{-1} \Phi(au; h) = U(u) + O(h) \).
7.2. Realization of the linear bracket: exponential TL

Symplectic realization of the bracket \{·, ·\}_1:
\[ a_k = e^{x_k+\epsilon p_k}, \quad b_k = p_k. \]

\text{TL :} \quad r(v) = 1, \quad f(u) = e^u,
\text{dTL}(h) : \quad \psi(v; h) = \frac{1}{h}(e^v - 1), \quad \phi(u; h) = he^u.

7.3. Realization of the linear bracket: dual TL

Another symplectic realization of the bracket \{·, ·\}_1:
\[ a_k = e^{p_k}, \quad b_k = x_k - x_{k-1}. \] (7.7)

\text{TL :} \quad r(v) = v, \quad f(u) = u,
\text{dTL}(h) : \quad \psi(v; h) = \log \frac{v}{h}, \quad \phi(u; h) = \log(1 + hu).

7.4. Realization of the quadratic bracket: modified exponential TL

Symplectic realization of the bracket \{·, ·\}_2:
\[ a_k = e^{x_k+\epsilon p_k}, \quad b_k = e^{p_k + e^{x_k-1}} \] (7.8)

\text{TL :} \quad r(v) = v, \quad f(u) = e^u,
\text{dTL}(h) : \quad \psi(v; h) = \log \frac{e^v - 1}{h}, \quad \phi(u; h) = \log(1 + he^u).

7.5. Realization of the linear-quadratic bracket: modified exponential TL with parameter

Symplectic realization of the linear combination \{·, ·\}_1 + \epsilon \{·, ·\}_2 of the brackets (2.6) and (2.8):
\[ a_k = e^{x_k+\epsilon (p_k+p_1)}, \quad b_k = e^{-\epsilon}(e^{p_k} - 1) + e^{u-x_{k-1}}. \] (7.9)

\text{TL :} \quad r(v) = 1 + \epsilon v, \quad f(u) = e^u,
\text{dTL}(h) : \quad \psi(v; h) = \frac{1}{h} \log \left(1 + \frac{\epsilon}{2}(e^v - 1)\right), \quad \phi(u; h) = \frac{1}{h} \log(1 + h e^u).

Particular case \( \epsilon = h \):
\text{dTL}(h) : \quad \psi(v; h) = \frac{v}{h}, \quad \phi(u; h) = \frac{1}{h} \log(1 + h^2 e^u).

7.6. Realization of the cubic-quadratic bracket: multiplicative hyperbolic TL

Symplectic realization of the linear combination \(-\{·, ·\}_3 - 4\beta \{·, ·\}_2\) of the brackets (2.10) and (2.8):
\begin{align}
\begin{split}
    a_k &= \frac{\beta (\coth(x_k-x_{k-1})+1)(\coth(x_{k+1}-x_k)+1)}{\sinh(v_k)}, \\
    b_k &= -\beta (\coth(\beta p_k)+1)(\coth(x_k-x_{k-1}+1) \\
    &\quad -\beta (\coth(\beta p_{k-1})+1)(\coth(x_k-x_{k-1})-1)).
\end{split}
\end{align} \tag{7.10}

\begin{align}
    TL: \quad r(v) &= -(v^2 - \beta^2), \\
    f(u) &= \coth(u),
\end{align}

\begin{align}
    dTL(h): \quad \psi(v; h) &= \frac{1}{2\beta} \log \frac{\sinh(v + \beta h_0)}{\sinh(v - \beta h_0)}, \\
    \phi(u; h) &= \frac{1}{2\beta} \log \frac{\sinh(u + \beta h_0)}{\sinh(u - \beta h_0)},
\end{align}

where \( h_0 = -(1/4\beta) \log(1 - 4\beta h) = h + \mathcal{O}(\beta h^2) \).

7.7 Realization of the cubic bracket. I: multiplicative rational TL

A symplectic realization of the bracket \(-\{\cdot, \cdot\}_3\), see (2.10):

\begin{align}
    a_k &= \frac{1}{(x_k-x_{k-1})(x_{k+1}-x_k) \sinh^2(p_k)}, \\
    b_k &= -\frac{\coth(p_{k-1}) + \coth(p_k)}{x_k-x_{k-1}}
\end{align} \tag{7.11}

\begin{align}
    TL: \quad r(v) &= -(v^2 - 1), \\
    f(u) &= \frac{1}{v},
\end{align}

\begin{align}
    dTL(h): \quad \psi(v; h) &= \frac{1}{2} \log \frac{v + h}{v - h}, \\
    \phi(u; h) &= \frac{1}{2} \log \frac{u + h}{u - h}.
\end{align}

7.8 Realization of the cubic bracket. II: additive rational TL

Another symplectic realization of the bracket \(-\{\cdot, \cdot\}_3\), see (2.10):

\begin{align}
    a_k &= \frac{1}{(x_k-x_{k-1})(x_{k+1}-x_k) p_k^2}, \\
    b_k &= -\frac{1}{x_k-x_{k-1}} \left( \frac{1}{p_{k-1}} + \frac{1}{p_k} \right)
\end{align} \tag{7.12}

\begin{align}
    TL: \quad r(v) &= -v^2, \\
    f(u) &= \frac{1}{u},
\end{align}

\begin{align}
    dTL(h): \quad \psi(v; h) &= \frac{h}{v}, \\
    \phi(u; h) &= \frac{h}{u}.
\end{align}

7.9 Bibliographical remarks

Integrable systems of the form (7.1) were classified in [Y89]. Yamilov’s list coincides with the list of the present section. The fact that all items of this list are various symplectic realizations of the flow TL, was observed in [S97b, S03]. The latter references contain also discretizations of all items of the Yamilov’s list, as well as the fact that they all are various symplectic realizations of the map dTL(h).

It should be mentioned that, if one generalizes the ansatz (7.1) by allowing functions \( f \) to depend on \( x_k, x_{k+1} \) not necessarily through the differences \( x_{k+1} - x_k \), then Yamilov’s list contains one further system, the so called elliptic TL,

\[ \bar{x}_k = (\bar{x}_k^2 - 1) \left( \zeta(x_{k+1} + x_k) - \zeta(x_{k+1} - x_k) + \zeta(x_k + x_{k-1}) + \zeta(x_k - x_{k-1}) - 2\zeta(2x_k) \right). \]
Here \( \zeta(u) \) is the Weierstrass zeta-function. This system was independently found in [Kr00]. Its discretization,

\[
\phi(x_k, \tilde{x}_k; h) + \phi(x_k, \tilde{x}_k; h) - \phi(x_k, \tilde{x}_{k+1}; h) - \phi(x_k, \tilde{x}_{k-1}; h) = 0,
\]

where

\[
\phi(x_0, x_1; \alpha) = \frac{1}{2} \log \frac{\sigma(x_0 + x_1 + \alpha)\sigma(x_0 - x_1 + \alpha)}{\sigma(x_0 + x_1 - \alpha)\sigma(x_0 - x_1 - \alpha)},
\]

(7.13)

was found in [A00, AS04]. The elliptic TL and its discrete time counterpart admit a hyperbolic degeneration (\( \zeta(u) \to \coth(u) \), \( \sigma(u) \to \sinh(u) \)) and a rational degeneration (\( \zeta(u) \to 1/u \), \( \sigma(u) \to u \)).

8. Relativistic TL in Flaschka–Manakov variables: equations of motion, Lax representation and tri-Hamiltonian structure

There exists a very remarkable generalization of TL, called relativistic TL. It is a one-parameter perturbation of TL, and in a certain physical interpretation this (small) parameter \( \alpha \) has the meaning of the inverse speed of light. In all our considerations, a special attention is payed to an immediate and transparent limit \( \alpha \to 0 \). Actually, there are two simplest flows which are perturbations of TL, which we will call the first and the ‘negative first’ flows of the RTL hierarchy. The first flow, denoted hereafter RTL\(_+\)(\( \alpha \)), reads:

\[
\begin{aligned}
\dot{b}_k &= (1 + \alpha b_k)(a_k - a_{k-1}), \\
\dot{a}_k &= a_k(b_{k+1} - b_k + \alpha a_{k+1} - \alpha a_{k-1}), \quad 1 \leq k \leq N.
\end{aligned}
\]

(8.1)

The negative first flow, denoted hereafter RTL\(_-\)(\( \alpha \)), reads:

\[
\begin{aligned}
\dot{b}_k &= \frac{a_k}{1 + \alpha b_{k+1}} - \frac{a_{k-1}}{1 + \alpha b_{k-1}}, \\
\dot{a}_k &= a_k \left( \frac{b_{k+1}}{1 + \alpha b_{k+1}} - \frac{b_k}{1 + \alpha b_k} \right), \quad 1 \leq k \leq N.
\end{aligned}
\]

(8.2)

All conventions about boundary conditions (open-end or periodic) remain valid for the relativistic case.

Remarkably, both flows RTL\(_\pm\)(\( \alpha \)) admit Lax representations falling into the AKS scheme. We define:

\[
L(a, b, \lambda) = \sum_{k=1}^{N} (1 + \alpha b_k)E_{kk} + \alpha \lambda \sum_{k=1}^{N} E_{k+1,k},
\]

(8.3)

\[
U(a, b, \lambda) = I - \alpha \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1},
\]

(8.4)

and

\[
T_1(a, b, \lambda) = L(a, b, \lambda)U^{-1}(a, b, \lambda), \quad T_2(a, b, \lambda) = U^{-1}(a, b, \lambda)L(a, b, \lambda).
\]

(8.5)

For both matrices \( T_{1,2}(a, b, \lambda) \) we have an asymptotic formula:

\[
T_{1,2}(a, b, \lambda) = I + \alpha T(a, b, \lambda) + O(\alpha^2),
\]

(8.6)

where \( T(a, b, \lambda) \) is the Lax matrix of the TL hierarchy.
**Proposition 8.1.** Equations of motion (8.1) of the flow $\text{RTL}_+(\alpha)$ are equivalent to the ‘Lax triads’:

\[
\dot{L} = LB_2 - B_1 L, \quad \dot{U} = UB_2 - B_1 U,
\]

which also imply usual Lax equations for the matrices $T_{1,2}(a, b, \lambda)$:

\[
\dot{T}_i = [T_i, B_i], \quad i = 1, 2.
\]

Here the auxiliary matrices

\[
B_1(a, b, \lambda) = \sum_{k=1}^{N} (b_k + \alpha a_{k-1})E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k},
\]

\[
B_2(a, b, \lambda) = \sum_{k=1}^{N} (b_k + \alpha a_k)E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}
\]

admit the following expressions:

\[
B_i = \pi_+((T_i - I)/\alpha), \quad i = 1, 2.
\]

**Proposition 8.2.** Equations of motion (8.2) of the flow $\text{RTL}_-(\alpha)$ are equivalent to the ‘Lax triads’:

\[
\dot{L} = C_1 L - LC_2, \quad \dot{U} = C_1 U - UC_2,
\]

which also imply usual Lax equations for the matrices $T_{1,2}(a, b, \lambda)$:

\[
\dot{T}_i = [C_i, T_i], \quad i = 1, 2.
\]

Here the auxiliary matrices

\[
C_1(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k}{1 + \alpha b_{k+1}} E_{k,k+1},
\]

\[
C_2(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k}{1 + \alpha b_k} E_{k,k+1}
\]

admit the following expressions:

\[
C_i = \pi_-(((I - T_i^{-1})/\alpha), \quad i = 1, 2.
\]

The Lax representations of the RTL flows lives in the same algebra $g$ as the Lax representation of TL; moreover, Lax representation (8.8) of $\text{RTL}_+(\alpha)$ is an $O(\alpha)$-perturbation of the $\pi_+$ version of the Lax equation (2.5) for TL, while Lax representation (8.13) of $\text{RTL}_-(\alpha)$ is an $O(\alpha)$-perturbation of the $\pi_-$ version of the Lax equation (2.5) for TL.

The flows $\text{RTL}_\pm(\alpha)$, like their non-relativistic counterpart TL, are tri-Hamiltonian. That is, the phase space of RTL can be equipped with three local Poisson brackets which are preserved by these flows which is therefore Hamiltonian with respect to any of these Poisson structures.
These brackets are compatible in the sense that their linear combinations are invariant Poisson brackets, as well.

8.1. Linear bracket

\[
\{b_k, a_k\}_1 = -a_k, \quad \{a_k, b_{k+1}\}_1 = -a_k, \\
\{b_k, b_{k+1}\}_1 = \alpha a_k.
\] (8.17)

The Hamilton functions of RTL\(_{\pm}(\alpha)\) are \(H_2(a, b)\) and \(-\alpha^{-1}H_0(a, b)\), respectively, where

\[
H_2 = \sum_{k=1}^{N} \left( \frac{1}{2} b_k^2 + a_k \right) + \alpha \sum_{k=1}^{N} (b_k + b_{k+1}) a_k + \alpha^2 \sum_{k=1}^{N} \left( \frac{1}{2} a_k^2 + a_k a_{k+1} \right),
\] (8.18)

\[
H_0 = -\alpha^{-1} \sum_{k=1}^{N} \log(1 + \alpha b_k).
\] (8.19)

Notice that the Hamilton function \(-\alpha^{-1}H_0(a, b)\) is singular in \(\alpha\), but it becomes regular upon adding the Casimir function \(\alpha^{-1}H_1(a, b)\), where

\[
H_1 = \sum_{k=1}^{N} b_k + \alpha \sum_{k=1}^{N} a_k.
\] (8.20)

Indeed:

\[-\alpha^{-1}H_0 + \alpha^{-1}H_1 = \sum_{k=1}^{N} \left( \frac{1}{2} b_k^2 + a_k \right) + O(\alpha).
\]

8.2. Quadratic bracket

\[
\{b_k, a_k\}_2 = -b_k a_k, \quad \{a_k, b_{k+1}\}_2 = -a_k b_{k+1}, \\
\{b_k, b_{k+1}\}_2 = -a_k, \quad \{a_k, a_{k+1}\}_2 = -a_k a_{k+1}.
\] (8.21)

Amazingly, this bracket coincides with the invariant Poisson bracket (2.8) of TL. The Hamilton functions of the flows RTL\(_{\pm}(\alpha)\) are \(H_1(a, b)\) and \(-\alpha H_0(a, b)\), respectively.

8.3. Cubic bracket

\[
\{b_k, a_k\}_3 = -a_k (b_k^2 + a_k) - \alpha b_k a_k^2, \\
\{a_k, b_{k+1}\}_3 = -a_k (b_k^2 + a_k) - \alpha a_k^2 b_{k+1}, \\
\{b_k, b_{k+1}\}_3 = -a_k (b_k + b_{k+1}) - \alpha b_k a_k b_{k+1}, \\
\{a_k, a_{k+1}\}_3 = -2a_k b_{k+1} a_{k+1} - \alpha a_k a_{k+1} (a_k + a_{k+1}), \\
\{b_k, a_{k+1}\}_3 = -a_k a_{k+1} - \alpha b_k a_k a_{k+1}, \\
\{a_k, b_{k+2}\}_3 = -a_k a_{k+1} - \alpha a_k a_{k+1} b_{k+2}, \\
\{a_k, a_{k+2}\}_3 = -\alpha a_k a_{k+1} a_{k+2}.
\] (8.22)

The Hamilton functions of the flows RTL\(_{\pm}(\alpha)\) in this bracket are \(\frac{1}{2} C(a, b)\) and \(\frac{1}{2} C(a, b) - \alpha H_0(a, b)\), where
\[ C(a, b) = \sum_{k=1}^{N} \log(a_k). \] (8.23)

8.4. Bibliographical remarks

Relativistic TL was introduced in [Rui90], as the Newtonian equations (1.2). Early references, concerning inverse scattering and finite gap solutions, Lax representations and tri-Hamiltonian structure include: [BR88, BR89a, BR89b, OFZR89, ZTOF91].

The Lax representation in terms of \((L, U) \in g \otimes g\) was given in [S91], where it was pointed out that the corresponding Lax matrices \(T_{1,2}\) build an orbit of a Lie–Poisson group. This was further developed in [FM97, FG00], and generalized for all simple Lie groups in [HKKR90].

9. Discretization of the relativistic TL in the Flaschka–Manakov variables

To find integrable discretization of the flows RTL\(_{\pm}(\alpha)\), we apply the recipe of section 4, i.e. we consider the maps

\[ \tilde{T} = \Pi_{+}^{-1}(F(T)) \cdot T \cdot \Pi_{+}(F(T)) = \Pi_{-}(F(T)) \cdot T \cdot \Pi_{-}^{-1}(F(T)) \]

with \(T = T_1\) or \(T_2\), and with

\[ F(T) = I + \frac{\hbar}{\alpha} (T - I) \text{ for RTL}_+(\alpha), \quad F(T) = I + \frac{\hbar}{\alpha} (I - T^{-1}) \text{ for RTL}_-(\alpha). \]

As a matter of fact, it is much more convenient to work with the discrete time Lax triads, which take the form

\[ \tilde{L} = \Pi_{+}^{-1}(F(T_1)) \cdot L \cdot \Pi_{+}(F(T_2)) = \Pi_{-}(F(T_1)) \cdot L \cdot \Pi_{-}^{-1}(F(T_2)), \]

\[ \tilde{U} = \Pi_{+}^{-1}(F(T_1)) \cdot U \cdot \Pi_{+}(F(T_2)) = \Pi_{-}(F(T_1)) \cdot U \cdot \Pi_{-}^{-1}(F(T_2)). \]

It turns out that for the flow RTL\(_{\pm}\)\((\alpha)\) the version with the \(\Pi_+\) factors is more suitable, while for the RTL\(_-\)\((\alpha)\) flow the version with the \(\Pi_-\) factors is preferable.

Lemma 9.1. The factors \(B_{1,2} = \Pi_{\mp} \left( I + \frac{\hbar}{\alpha} (T_{1,2} - I) \right)\) are of the form

\[ B_1(a, b, \lambda) = \sum_{k=1}^{N} a_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}, \] (9.1)

\[ B_2(a, b, \lambda) = \sum_{k=1}^{N} b_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}, \] (9.2)

where coefficients \(a_k = a_k(a, b)\) are defined by the recurrent relations

\[ a_k = 1 + h b_k + \frac{h(\alpha - h)a_{k-1}}{a_{k-1}}, \] (9.3)

and coefficients \(b_k = b_k(a, b)\) are defined by either of the relations

\[ b_k = b_k(a, b). \]
which are equivalent by virtue of (9.3).

Remark. As in the case of the discretization of TL, in the open–end case due to $a_0 = 0$ one can solve (9.3) explicitly in terms of finite continued fractions:

$$a_1 = 1 + hb_1; \ a_2 = 1 + hb_2 + \frac{h(\alpha - h)a_1}{1 + hb_1}; \ \ldots ;$$

$$a_N = 1 + hb_N + \frac{h(\alpha - h)a_{N-1}}{1 + hb_{N-1} + \ldots + \frac{h(\alpha - h)a_1}{1 + hb_1}}.$$  

In the periodic case $a_k$ can be expressed as infinite $N$-periodic continued fractions, and therefore they are double-valued functions, with one of the branches satisfying $a_k = 1 + h(b_k + \alpha a_k - 1) + O(h^2)$ as $h \to 0$.

Theorem 9.2. Consider the discrete time Lax triads

$$\tilde{L} = B_1^{-1}L B_2, \quad \tilde{U} = B_1^{-1}UB_2,$$  

with $B_{1,2} = \Pi \left( I + \frac{h}{\alpha}(T_{1,2} - I) \right)$. They serve as a Lax representation of the map $(a, b) \mapsto (\tilde{a}, \tilde{b})$ given by

$$1 + \alpha \tilde{b}_k = (1 + \alpha b_k) \frac{b_k}{a_k}, \quad \tilde{a}_k = a_k \frac{b_{k+1}}{a_k},$$  

where functions $a_k$, $b_k$ are defined by (9.3), (9.4). Map (9.6) will be denoted by $\text{dRTL}_+ (\alpha, h)$.

Proof. Matrix equations (9.5), written entry-wise, are equivalent to equations (9.6) in conjunction with

$$h(1 + \alpha \tilde{b}_k) + \alpha a_{k+1} = h(1 + \alpha b_{k+1}) + \alpha b_k, \quad a_k - h\alpha \tilde{a}_k - b_k - h\alpha a_k.$$  

A direct computation shows that, given (9.6), equations (9.7) follow from (9.3), (9.4). This finishes the proof. For later reference, we mention the following formula:

$$h(\tilde{b}_k + \alpha \tilde{a}_{k-1}) - h(b_k + \alpha a_k) = a_k - a_{k+1}.$$  

It follows immediately by eliminating $b_k$ between equations (9.7). □

Next, we turn to discretization of the flow $\text{RTL}_- (\alpha)$.

Lemma 9.3. The factors $C_{1,2} = \Pi \left( I + \frac{h}{\alpha}(I - T^{-1}_{1,2}) \right)$ are of the form

$$C_1(a, b, \lambda) = I + h\lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1},$$  

with $c_k = -\lambda^{-1} h\lambda^{-1}(I - T^{-1}_{1,2})^k$. The factors $C_2(a, b, \lambda) = I + h\lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}$ satisfy

$$\lambda C_1(a, b, \lambda) C_1(a, b, \lambda) = \lambda C_1(a, b, \lambda) C_1(a, b, \lambda).$$

Theorem 9.4. Consider the discrete time Lax triads

$$\tilde{L} = C_1^{-1}L C_2, \quad \tilde{U} = C_1^{-1}UC_2,$$  

with $C_1, C_2$ as above. They serve as a Lax representation of the map $(a, b) \mapsto (\tilde{a}, \tilde{b})$ given by

$$1 + \alpha \tilde{b}_k = (1 + \alpha b_k) \frac{b_k}{a_k}, \quad \tilde{a}_k = a_k \frac{b_{k+1}}{a_k},$$  

where functions $a_k$, $b_k$ are defined by (9.3), (9.4). Map (9.6) will be denoted by $\text{dRTL}_- (\alpha, h)$.
\[ C_2(a, b, \lambda) = I + h\lambda^{-1} \sum_{k=1}^{N} \partial_k E_{k+1}, \quad (9.10) \]

where coefficients \( \partial_k = \partial_k(a, b) \) are defined by the recurrent relations

\[ \partial_k = \frac{a_k}{1 + (\alpha + h)(b_k - h\partial_{k-1})}, \quad (9.11) \]

and coefficients \( \epsilon_k = \epsilon_k(a, b) \) are given by either of two expressions

\[ \epsilon_k = \partial_k \frac{1 + \alpha(b_k - h\partial_{k-1})}{1 + \alpha(b_{k+1} - h\partial_k)} = \partial_{k+1} \frac{\alpha a_k + h\partial_k}{\alpha a_{k+1} + h\partial_{k+1}}, \quad (9.12) \]

which are equivalent by virtue of (9.11).

**Remark.** In the open-end case we have the following finite continued fractions expressions for \( \partial_1 \):

\[
\begin{align*}
\partial_1 &= \frac{a_1}{1 + (\alpha + h)b_1}; \\
\partial_2 &= \frac{a_2}{1 + (\alpha + h)b_2 - \frac{h(\alpha + h)a_1}{1 + (\alpha + h)b_1}}; \\
\partial_{N-1} &= \frac{a_{N-1}}{1 + (\alpha + h)b_{N-1} - \frac{h(\alpha + h)a_{N-2}}{1 + (\alpha + h)b_{N-2}} - \ldots - \frac{h(\alpha + h)a_1}{1 + (\alpha + h)b_1}}.
\end{align*}
\]

In the periodic case these continued fractions are replaced by \( N \)-periodic ones, which represent double-valued functions, with one of the branches satisfying \( \partial_k = a_k/(1 + \alpha b_k) + O(h) \) as \( h \to 0 \).

**Theorem 9.4.** Consider the discrete time Lax triads

\[ \tilde{L} = C_1 L C_2^{-1}, \quad \tilde{U} = C_1 U C_2^{-1}, \quad (9.13) \]

with \( C_{1,2} = \Pi - \left( I + \frac{h}{\alpha}(I - T_{1,2}^{-1}) \right) \). They serve as a Lax representation of the map \((a, b) \mapsto (\tilde{a}, \tilde{b})\) given by

\[ 1 + \alpha \tilde{b}_k = (1 + \alpha b_{k+1}) \frac{\epsilon_k}{\partial_k}, \quad \tilde{a}_k = a_{k+1} \frac{\epsilon_k}{\partial_{k+1}}, \quad (9.14) \]

where functions \( \partial_k, \epsilon_k \) are defined by (9.11), (9.12). Map (9.14) will be denoted \( \text{dRTL}_-(\alpha, h) \).

**Proof.** Matrix equations (9.13), written entry-wise, are equivalent to a system of scalar equations consisting of (9.14) and of

\[ \tilde{b}_k + h\partial_{k-1} = b_k + h\epsilon_k, \quad \alpha \tilde{a}_k - h\partial_k = \alpha a_k - h\epsilon_k, \quad (9.15) \]

Given (9.14), equations (9.15) are consequences of (9.11), (9.12). For the later reference, we observe the following formula:

\[ (\tilde{b}_k + \alpha \tilde{a}_{k-1}) - (b_k + \alpha a_{k-1}) = h\epsilon_k - h\epsilon_{k-1}, \quad (9.16) \]
which follows by eliminating $\delta_k$ between equations (9.15).

By construction, both maps $d\text{RTL}_\pm(\alpha, \hbar)$ are Poisson with respect to each one of compatible Poisson brackets (8.17), (8.21), (8.22), and hence with respect to their arbitrary linear combination. Moreover, these maps share integrals of motion with the flows $\text{RTL}_\pm(\alpha)$, commute with those flows and with one another, and are interpolated by certain flows of the $\text{RTL}(\alpha)$ hierarchy.

9.1. Bibliographical remarks

Discretization of the relativistic TL was performed in [S96] as one of the first applications of our general recipe of integrable discretization.

10. Symplectic realization of the linear bracket: additive exponential relativistic TL

**Proposition 10.1.** The map

$$b_k = p_k - \alpha e^{x_k-\delta_k-1}, \quad a_k = e^{x_k+\delta_k}$$

is a symplectic realization of the linear bracket $\{\cdot, \cdot\}$, see (8.17), on the phase space of $\text{RTL}_\pm(\alpha)$.

**Theorem 10.2.** Pull-back of the flow $\text{RTL}_+ (\alpha)$ under parametrization (10.1) is a Hamiltonian system with the Hamilton and Lagrange functions

$$H(x, p) = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} (1 + \alpha p_k) e^{x_k-\delta_k},$$

$$L(x, \dot{x}) = \frac{1}{2} \sum_{k=1}^{N} \dot{x}_k^2 - \sum_{k=1}^{N} (1 + \alpha \dot{x}_k) e^{x_k-\delta_k} + \frac{\alpha^2}{2} \sum_{k=1}^{N} e^{2(x_k-\delta_k-1)}. \quad (10.3)$$

The corresponding Newtonian equations of motion read:

$$\ddot{x}_k = (1 + \alpha \dot{x}_{k+1}) e^{x_k-\delta_k} - (1 + \alpha \dot{x}_{k-1}) e^{x_{k+1}-\delta_k} - \alpha^2 e^{2(x_k-\delta_k-1)} + \alpha^2 e^{2(x_k-\delta_k+1)}. \quad (10.4)$$

**Proof.** Hamilton function (10.2) is obtained from (8.18) upon substitution (10.1). Straightforward computations lead to Lagrange function (10.3) and to its Euler–Lagrange equations (10.4). \qed

**Theorem 10.3.** Pull-back of the map $d\text{RTL}_+(\alpha, \hbar)$ under parametrization (10.1) is the following symplectic map:

$$\begin{cases}
  p_k = \frac{1}{\hbar} (e^{\tilde{x}_k-\delta_k} - 1) - \frac{(\alpha - \hbar)}{1 - \hbar \alpha} e^{\tilde{x}_k-\delta_k-1} + \alpha e^{x_{k-1}-\delta_k} - \alpha e^{x_{k+1}-\delta_k}, \\
  \tilde{p}_k = \frac{1}{\hbar} (e^{\tilde{x}_k-\delta_k} - 1) - \frac{(\alpha - \hbar)}{1 - \hbar \alpha} e^{\tilde{x}_k+\delta_k} + \alpha e^{x_{k-1}-\delta_k} - \alpha e^{x_{k+1}-\delta_k}. \quad (10.5)
\end{cases}$$

The corresponding Newtonian equations of motion:
\[ e^{\tilde{x}_{\mu}} - e^{x_{\mu}} = h \alpha e^{x_{\mu+1} - x_{\mu}} - h \alpha e^{x_{\mu} - 1} - \frac{h(\alpha - h) e^{\tilde{x}_{\mu+1} - x_{\mu}}}{1 - h \alpha e^{\tilde{x}_{\mu+1} - x_{\mu}}} + \frac{h(\alpha - h) e^{\tilde{x}_{\mu} - 1}}{1 - h \alpha e^{\tilde{x}_{\mu} - x_{\mu}}}. \] (10.6)

**Proof.** The second equation of motion in (9.6) together with equation (9.4) yield:

\[ \tilde{\alpha}_k = \alpha_k \frac{a_{k+1} + h \alpha a_{k+1}}{a_k + h \alpha a_k}. \]

In the parametrization \( a_k = e^{x_{\mu+1} - x_{\mu}} \) this is equivalent to the following quantity being constant, i.e. not depending on \( k \):

\[ e^{\tilde{x}_{\mu}} / (a_k + h \alpha a_k) = \text{const}. \]

Choosing this constant to be equal to 1, we get:

\[ a_k + h \alpha a_k = e^{\tilde{x}_{\mu}}, \] (10.7)

hence

\[ a_k = e^{\tilde{x}_{\mu}} \left( 1 - h \alpha e^{x_{\mu+1} - x_{\mu}} \right). \] (10.8)

and (from (9.4)):

\[ b_k = e^{\tilde{x}_{\mu}} \left( 1 - h \alpha e^{x_{\mu} - 1} \right). \] (10.9)

Further, (10.7), (10.8) allow us to derive from the recurrent relation (9.3):

\[ a_k - h b_k = 1 + \frac{h(\alpha - h) a_{k-1}}{a_k - 1} = 1 + \frac{h(\alpha - h) e^{\tilde{x}_{\mu} - 1}}{1 - h \alpha e^{x_{\mu} - 1}}, \] (10.10)

which, taking into account formulas (10.8) and \( p_k = b_k + \alpha e^{x_{\mu} - 1} \), leads to the first formula (for \( p_k \)) in (10.5). To obtain the second formula (for \( \tilde{p}_k \)) in (10.5), we make use of (9.8). This completes derivation of the Lagrangian equations of motion (10.5). The Newtonian ones follow immediately. □

**Theorem 10.4.** Pull-back of the flow \( \text{RTL} - (\alpha) \) under parametrization (10.1) is a Hamiltonian system with the Hamilton and the Lagrange functions

\[ H(x, p) = \alpha^{-1} \sum_{k=1}^{N} p_k - \alpha^{-2} \sum_{k=1}^{N} \log \left( 1 + \alpha p_k - \alpha^2 e^{x_{\mu} - 1} \right), \] (10.11)

\[ L(x, \dot{x}) = -\alpha^{-2} \sum_{k=1}^{N} \left( \alpha \dot{x}_k + \log(1 - \alpha \dot{x}_k) \right) - \sum_{k=1}^{N} (1 - \alpha \dot{x}_k) e^{x_{\mu} - 1}. \] (10.12)

The corresponding Newtonian equations of motion:

\[ \ddot{x}_k = \left( 1 - \alpha \dot{x}_k \right)^2 \left( (1 - \alpha \dot{x}_{k+1} e^{x_{\mu} - 1} - (1 - \alpha \dot{x}_{k-1}) e^{x_{\mu} - 1} \right). \] (10.13)
Theorem 10.5. Pull-back of the map \( dRTL_{-}(\alpha, h) \) under parametrization (10.1) is the following symplectic map:

\[
\begin{aligned}
\tilde{p}_k &= \frac{e^{\alpha x_k - 1} - 1}{1 - \alpha h^{-1}(e^{\alpha x_k - 1} - 1)} + (\alpha + h) e^{\alpha x_k - 1}, \\
\tilde{\alpha}_k &= \frac{e^{\alpha x_k - 1} - 1}{1 - \alpha h^{-1}(e^{\alpha x_k - 1} - 1)} + (\alpha + h) e^{\alpha x_k - 1} \tilde{\alpha}_k + \alpha e^{\alpha x_k - 1}.
\end{aligned}
\]  
(10.14)

The corresponding Newtonian equations of motion:

\[
\begin{aligned}
\frac{h^{-1}(e^{\alpha x_k} - 1)}{1 - \alpha h^{-1}(e^{\alpha x_k} - 1)} - \frac{h^{-1}(e^{\alpha x_k} - 1)}{1 - \alpha h^{-1}(e^{\alpha x_k} - 1)} &= (\alpha + h)(e^{\alpha x_k} e^{\alpha x_k} - e^{\alpha x_k} - 1) - \alpha(e^{\alpha x_k} - e^{\alpha x_k} - 1).
\end{aligned}
\]  
(10.15)

Proof. The second equation of motion in (9.14), together with the second expression for \( \epsilon_k \) in (9.12), reads:

\[
\tilde{\alpha}_k = a_k + \frac{\alpha a_k + h \delta_k}{\alpha^2 \delta_k + h \delta_k + 1}.
\]

Choosing this constant equal to \( \alpha + h \), we obtain:

\[
h \delta_k = (\alpha + h)(e^{\alpha x_k - 1} - \alpha e^{\alpha x_k - 1} - 1).
\]  
(10.16)

The recurrent relation (9.11) implies:

\[
1 + (\alpha + h)(b_k - h \delta_k) = \frac{a_k}{\delta_k} = \frac{e^{\alpha x_k}}{1 - \alpha h^{-1}(e^{\alpha x_k} - 1)},
\]

which is equivalent to

\[
1 + \alpha(b_k - h \delta_k) = \frac{1}{1 - \alpha h^{-1}(e^{\alpha x_k} - 1)}.
\]  
(10.17)

To compute \( \epsilon_k \), we substitute (10.16), (10.17) into (9.12), and arrive at:

\[
h \epsilon_k = (\alpha + h)(e^{\alpha x_k - 1} - \alpha e^{\alpha x_k - 1} - 1).
\]  
(10.18)

Formula (10.17), together with (10.16) and \( p_k = b_k + \alpha e^{\alpha x_k - 1} \), immediately implies the first formula (for \( p_k \)) in (10.14). To derive the second formula in (10.14) (for \( \tilde{p}_k \)), we make use of (9.16). Now Newtonian equations (10.15) follow readily. \( \square \)
10.1. Bibliographical remarks

Results of this section are from [S97a, S03].

11. The variety of symplectic realizations of $\text{RTL}_\pm(\alpha)$ and $\text{dRTL}_\pm(\alpha, h)$

11.1. General form of equations of motion

Results of the previous section illustrate a long list of Newtonian equations with continuous and discrete time, which can be obtained as symplectic realizations of the flows $\text{RTL}_\pm(\alpha)$ and of the maps $\text{dRTL}_\pm(\alpha, h)$. All the continuous time systems share the following general form of Newtonian equations of motion:

$$\ddot{x}_k = r(x_k) \left( \dot{x}_{k+1} g(x_{k+1} - x_k) - \dot{x}_{k-1} g(x_k - x_{k-1}) + f(x_{k+1} - x_k) - f(x_k - x_{k-1}) \right). \quad (11.1)$$

Systems (11.1) are Lagrangian, with the Lagrange function

$$L(x, \dot{x}) = \sum_{k=1}^{N} K(x_k) - \sum_{k=1}^{N} \dot{x}_k Q(x_{k+1} - x_k) - \sum_{k=1}^{N} U(x_{k+1} - x_k), \quad (11.2)$$

where $K"(v) = 1/r(v)$, $Q'(u) = g(u)$, and $U'(u) = f(u)$.

For all symplectic realizations of $\text{dRTL}_\pm(\alpha, h)$, Newtonian equations of motion have the following general structure:

$$\psi(x_k - x_{k+1}) - \psi(x_k - x_{k-1}) = 0,$$

$$\phi(x_k - x_{k+1}) - \phi(x_k - x_{k-1}) = 0.$$

(11.3)

There are two natural ways to realize these equations as discrete Euler–Lagrange equations.

The first one refers to the discrete Lagrange function

$$\Lambda_+(x, \tilde{x}, \tilde{p}, p) = \sum_{k=1}^{N} \Phi_+(x_k, \tilde{x}_k) - \sum_{k=1}^{N} \Phi_0(x_{k+1} - x_k), \quad (11.4)$$

where $\Phi(x) = \psi(x)$, $\Phi_0(x) = \psi_0(x)$. Corresponding symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ is given by

$$\begin{cases} p_k = \psi(\tilde{x}_k - x_k) + \phi(x_k - x_{k-1}) + \psi_0(x_{k+1} - x_k) - \psi_0(x_k - x_{k-1}), \\ \tilde{p}_k = \psi(\tilde{x}_k - x_k) + \phi(x_{k+1} - \tilde{x}_k). \end{cases} \quad (11.5)$$

This is the general formula for symplectic realizations of $\text{dRTL}_+(\alpha, h)$. Corresponding Newtonian equations coincide with (11.3).

The second possibility is

$$\Lambda_-(x, \tilde{x}, \tilde{p}, p) = \sum_{k=1}^{N} \Phi_-(x_k, \tilde{x}_k) - \sum_{k=1}^{N} \Phi_0(x_{k+1} - \tilde{x}_k), \quad (11.6)$$

which generates map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ given by

$$\begin{cases} p_k = \psi(\tilde{x}_k - x_k) + \phi(x_k - \tilde{x}_{k-1}), \\ \tilde{p}_k = \psi(\tilde{x}_k - x_k) + \phi(x_{k+1} - x_k) - \psi_0(x_{k+1} - \tilde{x}_k) + \psi_0(x_k - \tilde{x}_{k-1}). \end{cases} \quad (11.7)$$

This is the general formula for symplectic realizations of $\text{dRTL}_-(\alpha, h)$. Corresponding Newtonian equations coincide with (11.3), as well.
Note that in all our examples the non-relativistic limit \( \alpha = 0 \) corresponds to \( g(u) = 0 \), resp. \( \psi_0(u) = 0 \), which leads to symplectic realizations of TL, resp. \( dTL(h) \).

### 11.2. Realization of the linear bracket: additive exponential relativistic TL

Here we reproduce the results of the previous section. A symplectic realization of the bracket \( \{ \cdot, \cdot \}_1 \):

\[
b_k = p_k - \alpha e^\alpha u_1, \quad a_k = e^{\alpha + \alpha^{-1} u_1}.
\]

**RTL\(_+\)(\(\alpha\))**: \( r(v) = 1, \quad f(u) = e^\alpha - \alpha^2 e^\alpha, \quad g(u) = \alpha e^\alpha, \)

**dRTL\(_+\)(\(\alpha, h\))**: \( \psi(v) = \frac{1}{\alpha} (e^\alpha - 1), \quad \phi(u) = \frac{(h - \alpha) e^\alpha}{1 - \alpha e^\alpha}, \quad \psi_0(u) = \alpha e^\alpha, \)

**RTL\(_-\)(\(\alpha\))**: \( r(v) = (1 - \alpha) e^\alpha, \quad f(u) = e^\alpha, \quad g(u) = -\alpha e^\alpha, \)

**dRTL\(_-\)(\(\alpha, h\))**: \( \psi(v) = \frac{h^{-1} (e^\alpha - 1)}{1 - \alpha e^\alpha}, \quad \phi(u) = (h + \alpha) e^\alpha, \quad \psi_0(u) = -\alpha e^\alpha. \)

### 11.3. Realization of the linear-quadratic bracket: Ruijsenaars TL

Relativistic deformation of the exponential TL discovered by Ruijsenaars comes from the following symplectic realization of the bracket \( \{ \cdot, \cdot \}_1 + \alpha \{ \cdot, \cdot \}_2 \), see (8.17), (8.21):

\[
b_k = \frac{1}{\alpha} \left( e^{\alpha u_k} - 1 \right), \quad a_k = e^{\alpha + \alpha^{-1} u_1} \phi(u),
\]

**RTL\(_+\)(\(\alpha\))**: \( r(v) = 1 + \alpha v, \quad f(u) = \frac{e^\alpha}{1 + \alpha^2 e^\alpha}, \quad g(u) = \frac{\alpha e^\alpha}{1 + \alpha^2 e^\alpha}, \)

**dRTL\(_+\)(\(\alpha, h\))**: \( \psi(v) = \frac{1}{\alpha} \log \left( 1 + \alpha (e^\alpha - 1) \right), \quad \phi(u) = -\frac{1}{\alpha} \log \left( 1 - \alpha (h + \alpha) e^\alpha \right), \)

\( \psi_0(u) = \frac{1}{\alpha} \log (1 + \alpha^2 e^\alpha). \)

The ingredients of Newtonian equations for RTL\(_-\)(\(\alpha\)), dRTL\(_-\)(\(\alpha, h\)) are obtained from those for RTL\(_+\)(\(\alpha\)), dRTL\(_+\)(\(\alpha, h\)) by changing \( \alpha \) to \(-\alpha\).

### 11.4. Realization of the linear bracket: dual relativistic TL

An alternative symplectic realization of of the linear bracket \( \{ \cdot, \cdot \}_1 \), see (8.17):

\[
b_k = x_k - x_{k-1} - \alpha e^{\alpha x_{k-1}}, \quad a_k = e^x.
\]

**RTL\(_+\)(\(\alpha\))**: \( r(v) = v, \quad f(u) = u, \quad g(u) = \frac{\alpha}{1 + \alpha u}, \)

**dRTL\(_+\)(\(\alpha, h\))**: \( \psi(v) = \log \frac{v}{u}, \quad \phi(u) = \log \frac{1 + hu}{1 + \alpha u}, \quad \psi_0(u) = \log (1 + hu), \)

**RTL\(_-\)(\(\alpha\))**: \( r(v) = v(1 + \alpha v), \quad f(u) = \frac{u}{1 + \alpha u}, \quad g(u) = -\frac{\alpha}{1 + \alpha u}, \)

**dRTL\(_-\)(\(\alpha, h\))**: \( \psi(v) = \log \frac{h^{-1} v}{1 + h^{-1} \alpha (h + \alpha) v}, \quad \phi(u) = \log (1 + (h + \alpha) u), \)

\( \psi_0(u) = -\log (1 + hu). \)
11.5. Realization of the quadratic bracket: modified relativistic TL

A symplectic realization of the bracket \{·, ·\}2, see (8.21):

\[ b_k = e^{\alpha_k} + e^{\alpha_k - u_k}, \quad a_k = e^{h u_k + \epsilon_k}. \]  \hfill (11.11)

\[
\begin{align*}
\text{RTL}_+(\alpha) & : \quad r(v) = v, \quad f(u) = e^u, \quad g(u) = \frac{\alpha e^u}{1 + \alpha e^u}, \\
\text{dRTL}_+(\alpha, h) & : \quad \psi(v) = \log \frac{e^u - 1}{h}, \quad \phi(u) = \log \frac{1 + h e^u}{1 + \alpha e^u}, \quad \psi_0(u) = \log(1 + \alpha e^u), \\
\text{RTL}_-(\alpha) & : \quad r(v) = v(1 - \alpha v), \quad f(u) = \frac{e^u}{1 + \alpha e^u}, \quad g(u) = -\frac{\alpha e^u}{1 + \alpha e^u}, \\
\text{dRTL}_-(\alpha, h) & : \quad \psi(v) = \log \frac{h^{-1}(e^u - 1)}{1 - h^{-1}(e^u - 1)}, \quad \phi(u) = \log (1 + (h + \alpha)e^u), \\
& \quad \quad \psi_0(u) = -\log(1 + \alpha e^u).
\end{align*}
\]

11.6. Realization of the linear-quadratic bracket II: general exponential relativistic TL

A symplectic realization of the bracket \{·, ·\}_1 + \epsilon \{·, ·\}_2, see (8.17), (8.21):

\[ b_k = \frac{1}{\epsilon}(e^{\alpha_k} - 1) + (\epsilon - \alpha)e^{\alpha_k - u_k}, \quad a_k = e^{h u_k + \epsilon_k}. \]  \hfill (11.12)

The corresponding one-parameter family of symplectic realizations interpolates between the additive exponential relativistic TLs for \( \epsilon = 0 \) and the Ruijsenaars TLs for \( \epsilon = \alpha \). This family is the most general relativistic TL with exponential interactions.

\[
\begin{align*}
\text{RTL}_+(\alpha) & : \quad r(v) = 1 + \epsilon v, \quad f(u) = \frac{e^u + \alpha(\epsilon - \alpha)e^u}{1 + \epsilon \alpha e^u}, \quad g(u) = \frac{\alpha e^u}{1 + \epsilon \alpha e^u}, \\
\text{dRTL}_+(\alpha, h) & : \quad \psi(v) = \frac{1}{\epsilon} \log \left( 1 + \frac{h}{\epsilon}(e^u - 1) \right), \quad \phi(u) = \frac{1}{\epsilon} \log \frac{1 + h(\epsilon - \alpha)e^u}{1 + \alpha(\epsilon - h)e^u}, \\
& \quad \quad \psi_0(u) = \frac{1}{\epsilon} \log(1 + \epsilon \alpha e^u), \\
\text{RTL}_-(\alpha) & : \quad r(v) = (1 - \alpha v)(1 + (\epsilon - \alpha)v), \quad f(u) = \frac{e^u}{1 + \epsilon \alpha e^u}, \quad g(u) = -\frac{\alpha e^u}{1 + \epsilon \alpha e^u}, \\
\text{dRTL}_-(\alpha, h) & : \quad \psi(v) = \frac{1}{\epsilon} \log \frac{1 + (\epsilon - \alpha)h^{-1}(e^u - 1)}{1 - h^{-1}(e^u - 1)}, \quad \phi(u) = \frac{1}{\epsilon} \log \left( 1 + \epsilon(h + \alpha)e^u \right), \\
& \quad \quad \psi_0(u) = -\frac{1}{\epsilon} \log(1 + \epsilon \alpha e^u).
\end{align*}
\]

11.7. Realization of the cubic-quadratic bracket: multiplicative hyperbolic relativistic TL

In this paragraph, we restrict ourselves to the ‘first’ flow of the hierarchy (and its discretization), since the resulting equations for the ‘negative first’ one are essentially the same (obtained by the change of \( \alpha \) to \( -\alpha \)). We assume that \( \beta \) is a small parameter and use the following notation:

\[ \alpha_0 = -\frac{1}{4\beta} \log(1 - 4\beta \alpha) = \alpha + O(\beta \alpha^2), \quad h_0 = -\frac{1}{4\beta} \log(1 - 4h \beta) = h + O(\beta h^2), \quad \epsilon = \frac{1}{4\beta}. \]
A symplectic realization of the bracket $-[\cdot, \cdot]_3 - 4\beta[\cdot, \cdot]_2$, see (8.21), (8.22), is given by

$$b_k = u_k + v_{k-1}, \quad a_k = u_k v_k,$$

where

$$u_k = \frac{y_k (1 + \epsilon z_{k-1})}{1 - \epsilon \alpha y_{k-1} z_{k-1}}, \quad v_k = \frac{z_k (1 + \epsilon y_k)}{1 - \epsilon \alpha y_{k-1} z_{k-1}},$$

and

$$y_k = -2\beta(\coth(\beta p_k + \beta \alpha_0) + 1), \quad z_k = 2\beta(\coth(x_{k+1} - x_k - \beta \alpha_0) - 1).$$

(11.13)

\[
RTL_+(\alpha) : \quad r(v) = -(v^2 - \beta^2), \quad f(u) = \frac{\sinh(2u)}{\sinh^2(u) - \sinh^2(\beta \alpha_0)}, \quad g(u) = -\frac{1}{2\beta} \cdot \frac{\sinh(2\beta \alpha_0)}{\sinh^2(u) - \sinh^2(\beta \alpha_0)},
\]

\[
dRTL_+(\alpha, h) : \quad \psi(v) = \frac{1}{2\beta} \log \frac{\sinh(v + \beta \alpha_0)}{\sinh(v - \beta \alpha_0)}, \quad \phi(u) = \frac{1}{2\beta} \log \frac{\sinh(u + \beta \alpha_0 - \beta \alpha_0)}{\sinh(u - \beta \alpha_0 + \beta \alpha_0)},
\]

\[
\psi_0(u) = \frac{1}{2\beta} \log \frac{\sinh(u + \beta \alpha_0)}{\sinh(u - \beta \alpha_0)}.
\]

11.8. Realization of the cubic bracket. I: multiplicative rational relativistic TL

A symplectic realization of the bracket $-[\cdot, \cdot]_3$, see (8.22):

$$a_k = \frac{1}{\sinh^2(p_k) (x_k - x_{k-1} + \alpha \coth(p_{k-1})) (x_{k+1} - x_k + \alpha \coth(p_k))},$$

$$b_k = -\frac{\coth(p_{k-1}) + \coth(p_k)}{x_k - x_{k-1} + \alpha \coth(p_{k-1})}.$$  

(11.14)

\[
RTL_+(\alpha) : \quad r(v) = -(v^2 - 1), \quad f(u) = \frac{u}{u^2 - \alpha^2}, \quad g(u) = -\frac{\alpha}{u^2 - \alpha^2},
\]

\[
dRTL_+(\alpha, h) : \quad \psi(v) = \frac{1}{2} \log \frac{v + h}{v - h}, \quad \phi(u) = \frac{1}{2} \log \frac{u + h - \alpha}{u - h + \alpha}, \quad \psi_0(u) = \frac{1}{2} \log \frac{u + \alpha}{u - \alpha}.
\]

11.9. Realization of the cubic bracket. II: additive rational relativistic TL

Another symplectic realization of the bracket $-[\cdot, \cdot]_3$, see (8.22):

$$a_k = \frac{1}{p_k^2 (x_k - x_{k-1} + \frac{\alpha}{p_{k-1}}) (x_{k+1} - x_k + \frac{\alpha}{p_k})}, \quad b_k = -\frac{1}{x_k - x_{k-1} + \frac{\alpha}{p_{k-1}}} \left( \frac{1}{p_{k-1}} + \frac{1}{p_k} \right)$$

(11.15)

\[
RTL_+(\alpha) : \quad r(v) = -v^2, \quad f(u) = \frac{1}{u}, \quad g(u) = -\frac{\alpha}{u^2},
\]

\[
dRTL_+(\alpha, h) : \quad \psi(v) = \frac{h}{v}, \quad \phi(u) = \frac{h - \alpha}{u}, \quad \psi_0(u) = \frac{\alpha}{u}.
\]
11.10. Bibliographic remarks

In the paper [Rui90] Ruijsenaars introduced ‘relativistic TL’ as the system (1.2). Discretization of this system was performed in [S96]. In [S97a, S03] further systems of relativistic Toda type, both in continuous and in discrete time, were found and identified as symplectic realizations of \( \text{RTL}_\pm (\alpha) \), resp. of \( \text{dRTL}_\pm (\alpha, h) \). The latter result provides a complete understanding for all these systems, including the full set of integrals of motion, Lax representations etc.

In [ASh97a, ASh97b], a classification of ‘integrable’ systems of the type (11.1) was achieved. The notion of ‘integrability’ there is based on the requirement that the form of equations of motion is preserved under a sort of Legendre transformation. This is, \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability. However, the resulting list coincides with the list of symplectic realizations of \( \text{RTL}_\pm (\alpha) \) given in the present section. Thus, \( \text{à posteriori} \), all these systems are integrable also in the classical sense. In [A99], a similar approach was used to perform a classification of ‘integrable’ discrete time systems of the type (11.3). Again, the resulting list coincides with the list of symplectic realizations of \( \text{dRTL}_\pm (\alpha) \) given in the present section.

In the course of classification in [ASh97a, ASh97b, A99], the authors discovered also systems of a more general form, where the functions \( f, g, \psi, \phi \) and \( \psi_0 \) not necessarily depend on the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments. These more general systems are: the relativistic elliptic TL, TL. Indeed, as \( \text{à priori} \), unrelated to the classical notion of Liouville–Arnold integrability, the difference of the arguments.

12. Explicit discretizations of the TL

An important particular case of the map \( \text{dRTL}_\pm (\alpha, h) \) appears when the discrete time step \( h \) coincides with the relativistic parameter \( \alpha \). Continued fractions for \( a_k \), defined by (9.3), obviously degenerate to explicit expressions \( a_k = 1 + h b_k \). The reason for this is clear, since in this case \( B_1,2 = \Pi_+ (T_{1,2}) \), so that, in particular, \( B_1 = \Pi_+ (L U^{-1}) = L \).

\[
1 + \tilde{h} b_k = 1 + \frac{1 + h b_k + h^2 a_k}{1 + h b_{k-1} + h^2 a_{k-1}}, \quad \tilde{a}_k = a_k - \frac{1 + h b_{k+1} + h^2 a_{k+1}}{1 + h b_k + h^2 a_k}.
\]

Thus, the map \( \text{dRTL}_\pm (h, h) \) is an explicit rational map. Moreover, it is birational: to found the inverse map, it is sufficient to observe that \( 1 + \tilde{h} b_k + h^2 a_{k-1} = 1 + h b_k + h^2 a_k \). Of course, the map \( \text{dRTL}_\pm (h, h) \) belongs to the hierarchy RTL(\( h \)); however, it serves as a discretization of TL. Indeed, as \( h \to 0 \), we have:

\[
\tilde{b}_k = b_k + h(a_k - a_{k-1}) + O(h^2), \quad \tilde{a}_k = a_k + h a_k (b_{k+1} - b_k) + O(h^3).
\]

A similar simplification takes place for the map \( \text{dRTL}_\pm (\alpha, h) \) if \( h = -\alpha \). Continued fractions for \( a_k \), defined by (9.11), obviously degenerate to explicit expressions \( a_k = a_k \). The reason is that in this case \( C_{1,2} = \Pi_- (T_{1,2}^{-1}) \), so that, in particular, \( C_2 = \Pi_- (L^{-1} U) = U \).

Formulas (9.14) turn into
This map $dRT_+(-h, h)$ is an explicit rational map which belongs to the hierarchy $RTL(-h)$ and serves as a discretization of $TL$. We now consider various symplectic realizations of these explicit maps. Naturally, they are explicit discretizations of the corresponding symplectic realizations of the flow $TL$. We restrict ourselves to the map $dRTL_+(h, h)$. Newtonian equations of motion have the following general structure:

$$
\psi(x_{k+1} - x_k; h) - \psi(x_k - x_{k-1}; h) = \psi_0(x_{k+1} - x_k; h) - \psi_0(x_k - x_{k-1}; h). \quad (12.3)
$$

They serve as Euler–Lagrange equations for the discrete time Lagrange function

$$
\Lambda(x, \tilde{x}; h) = \sum_{k=1}^N \left( \psi_0(x_{k+1} - x_k; h) - \psi_0(x_k - x_{k-1}; h) \right). \quad (12.4)
$$

The corresponding symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ is given by

$$
\begin{align*}
    p_k &= \psi(\tilde{x}_k - x_k; h) + \psi_0(x_k - x_{k-1}; h) - \psi_0(x_{k+1} - x_k; h), \\
    \tilde{p}_k &= \psi(\tilde{x}_k - x_k; h).
\end{align*} \quad (12.5)
$$

(a) For the pull-back of the map $dRTL_+(h, h)$ under parametrization (11.8) with $\alpha = h$:

$$
\psi(v; h) = \frac{1}{h} (e^v - 1), \quad \psi_0(u; h) = \alpha e^u. \quad (12.6)
$$

(b) For the pull-back of the map $dRTL_+(h, h)$ under parametrization (11.9) with $\alpha = h$:

$$
\psi(v; h) = \frac{v}{h}, \quad \psi_0(u; h) = \frac{1}{h} \log(1 + h^2 e^u). \quad (12.7)
$$

(c) For the pull-back of the map $dRTL_+(h, h)$ under parametrization (11.10) with $\alpha = h$:

$$
\psi(v; h) = \log\left(\frac{v}{h}\right), \quad \psi_0(u; h) = \log(1 + hu). \quad (12.8)
$$

(d) For the pull-back of the map $dRTL_+(h, h)$ under parametrization (11.11) with $\alpha = h$:

$$
\psi(v; h) = \log\left(\frac{e^v - 1}{h}\right), \quad \psi_0(u; h) = \log(1 + h e^u). \quad (12.9)
$$

(e) For the pull-back of the map $dRTL_+(h, h)$ under parametrization (11.12) with $\alpha = h$:

$$
\psi(v; h) = \frac{1}{2\beta} \log\frac{\sinh(v + \beta h_0)}{\sinh(v - \beta h_0)} \quad \psi_0(u; h) = \frac{1}{2\beta} \log\frac{\sinh(u + \beta h_0)}{\sinh(u - \beta h_0)}. \quad (12.10)
$$

(f) For the pull-back of the map $dRTL_+(h, h)$ under parametrization (11.13) with $\alpha = h$:

$$
\psi(v; h) = \frac{1}{2} \log\left(\frac{v + h}{v - h}\right) \quad \psi_0(u; h) = \frac{1}{2} \log\left(\frac{u + h}{u - h}\right). \quad (12.11)
$$

(g) For the pull-back of the map $dRTL_+(h, h)$ under parametrization (11.14) with $\alpha = h$:

$$
\psi(v; h) = \frac{h}{v}, \quad \psi_0(u; h) = \frac{h}{u}. \quad (12.12)
$$
12.1. Bibliographical remarks

The fact that explicit discretization of TL belongs to the relativistic Toda hierarchy was pointed out in [S90a, S90b]. It was used there to introduce Lie-algebraic generalizations of relativistic Toda systems.

13. Discrete 2D equations of Laplace type

We elaborate now on the common features of our discrete time systems.

13.1. Implicit discretizations of TLs

In equation (7.3), we assume that
\[ x_k = x_k(t) = x_{k,n}, \quad \tilde{x}_k = x_k(t + h) = x_{k,n+1}, \quad \bar{x}_k = x_k(t - h) = x_{k,n-1} \]
for \( t = nh \) with \( n \in \mathbb{Z} \). Thus, equation (7.3) can be interpreted as equation
\[ \psi(x_{k,n+1} - x_{k,n}; h) - \psi(x_{k,n} - x_{k,n-1}; h) = \phi(x_{k+1,n} - x_{k,n}; h) - \phi(x_{k,n} - x_{k-1,n+1}; h) \]
for \( x : \mathbb{Z}^2 \to \mathbb{R} \). This is a 2D lattice equation. If we visualize such an equation by connecting all pairs of vertices which appear in its individual terms, then we end up with a stencil shown on figure 1(a).

Introduce the graph \( \Gamma \) with the set of vertices \( V(\Gamma) = \mathbb{Z}^2 \) and with the set of (directed) edges \( E(\Gamma) \) connecting nearest neighbors in the south-to-north direction and in the south-east-to-north-west direction. Clearly, this graph is combinatorially nothing but the regular square lattice, but drawn in \( \mathbb{R}^2 \) in a non-standard way. Equation (13.2) lives on vertex stars of \( \Gamma \).

Moreover, equation (13.2) admits a 2D Lagrangian formulation.

Definition 13.1. Let \( \Gamma \) be a graph with the set of vertices \( V(\Gamma) = \mathbb{Z}^2 \) and with the set of (directed) edges \( E = E(\Gamma) \) of the graph \( \Gamma \) and depending on the values \( x(v_1), x(v_2) \) of a function \( x : V(\Gamma) \to \mathbb{R} \) at the endpoints of \( e \). Introduce the action functional \( S[x] \) for any function \( x : V(\Gamma) \to \mathbb{R} \), as a (formal) sum of Lagrangians over all edges of \( \Gamma \),
\[ S[x] = \sum_{e=(v_1,v_2)\in E(\Gamma)} L(e). \]

A discrete Laplace type system on the graph \( \Gamma \) consists of Euler–Lagrange equations for critical points \( x : V(\Gamma) \to \mathbb{R} \) of the action \( S \), that is, of equations
\[ \frac{\partial S}{\partial x(v)} = 0 \quad \forall v \in V(\Gamma). \]

Clearly, for a given vertex \( v \in V(\Gamma) \), the expression \( \partial S/\partial x(v) \) involves the derivatives of discrete Lagrangians for all edges incident to \( v \), i.e. for the vertex star of \( v \).

Equation (13.2) is Euler–Lagrange equation for the action functional on \( \mathbb{Z}^2 \),
\[ S[x] = \sum_{(k,n)\in \mathbb{Z}^2} \left( \Psi(x_{k,n+1} - x_{n,k}; h) - \Phi(x_{k+1,n} - x_{k,n}; h) \right), \]
where edge Lagrangians are given by
13.2. From 2D to 1D: discrete time evolution

Formulation of definition 13.1 is covariant in the sense that there is no a priori privileged evolution direction on a general graph \( \Gamma \) with \( V(\Gamma) = \mathbb{Z}^2 \). The time evolution arises if we consider a specific Cauchy problem on a graph. This can be formalized as follows.

**Definition 13.2.** Let the vertices of a graph \( \Gamma \) be represented as a disjoint union \( V(\Gamma) = \bigsqcup_{n \in \mathbb{Z}} V_n \) in such a way that any two \( V_n \) and \( V_m \) are in a one-to-one correspondence. A space-time splitting (slicing) of \( \Gamma \) is a sequence of subgraphs \( \{ \Gamma_n \}_{n \in \mathbb{Z}} \) (Cauchy slices) such that:

- \( V(\Gamma_n) = V_n \cup V_{n+1} \),
- the sets \( E(\Gamma_n) \) are disjoint,
- \( E(\Gamma) = \bigsqcup_{n \in \mathbb{Z}} E(\Gamma_n) \).

The discrete time Lagrange function corresponding to the slicing \( \{ \Gamma_n \} \) is obtained by summing up discrete Lagrangians over the edges of one slice \( \Gamma^n \):

\[
\Lambda(x, \tilde{x}) = \sum_{e \in E(\Gamma^n)} \mathcal{L}(e). 
\]  

Let us comment on the notation used on the left-hand side of (13.6). For any fixed \( n \), we denote the space of functions on \( V^n \) by \( \mathcal{X} = \{ x : V^n \to \mathbb{R} \} \) (this space does not depend on \( n \) because any two \( V^n \), \( V^m \) are in a bijection). We take \( \mathcal{X} \) as the configuration space of our discrete time system. Functions on \( V(\Gamma^n) \) are naturally identified with pairs of functions \((x, \tilde{x})\), where \( x = x(n) \) is a function on \( V^n \) and \( \tilde{x} = x(n+1) \) is a function on \( V^{n+1} \). Thus, the space of functions on \( V(\Gamma^n) \) is identified with \( \mathcal{X} \times \mathcal{X} \). We have:

\[
S = \sum_{n \in \mathbb{Z}} \Lambda(x(n), x(n+1)).
\]
There follows from definition 13.2 that each edge of \( \Gamma \) incident to a vertex \( v \in V^n \) belongs to exactly one of the slices \( E(\Gamma^n) \) and \( E(\Gamma^{n-1}) \). Therefore, for any vertex \( v \in V^n \), the part of the action \( S[x] \) containing edges incident to \( v \) is
\[
\sum_{e \in E(\Gamma) \cap E(\Gamma^{n-1})} \mathcal{L}(e) = \Lambda(x(n), x(n+1)) + \Lambda(x(n-1), x(n)) = \Lambda(x, \tilde{x}) + \Lambda(x, x).
\]

It follows that discrete Laplace type equations (13.4) coincide with the discrete time Euler–Lagrange equations (7.4). Equations (7.6) produce out of this system of second order difference equations a symplectic map on \( T^*\mathcal{X} \).

For discrete Toda systems of the type (7.3) with open-end or periodic boundary conditions, \( V(\Gamma) = \{1, 2, \ldots, N\} \times \mathbb{Z} \), and \( V^n = \{1, 2, \ldots, N\} \times \{n\} \). The edges of \( \Gamma \) connect nearest neighbors in the south-to-north direction and in the south-east-to-north-west direction. The relevant slicing is shown on figure 1(b). The space \( \mathcal{X} = \mathbb{R}^N \) consists of finite sequences \( x = (x_k)_{k=1}^N \). Equations (7.6) define a symplectic map on \( T^*\mathbb{R}^N \).

13.3. Explicit discretizations of TLs

We now give a similar consideration of symplectic realizations of the explicit rational map dRTL_{+}(h, h), listed in section 12. Under identification (13.1), equation (12.3) becomes a 2D lattice equation
\[
\psi(x_{k,n+1} - x_{k,n}; h) - \psi(x_{k,n} - x_{k,n-1}; h) = \psi_0(x_{k+1,n} - x_{k,n}; h) - \psi_0(x_{k,n} - x_{k-1,n}; h)
\]  
(13.7)

for a function \( x : \mathbb{Z}^2 \rightarrow \mathbb{R} \). The stencil supporting this equation is shown on figure 2(a).

Equation (12.3) is a discrete Laplace type system on the regular square lattice, according to definition 13.1, because it is Euler–Lagrange equation for the action functional on \( \mathbb{Z}^2 \),
\[
S[x] = \sum_{(k,n) \in \mathbb{Z}^2} \left( \psi(x_{k,n+1} - x_{k,n}; h) - \psi_0(x_{k+1,n} - x_{k,n}; h) \right),
\]
(13.8)

where \( \Psi'(\xi; h) = \psi(\xi; h) \), \( \psi_0'(\xi; h) = \psi_0(\xi; h) \). Thus, discrete Lagrangians are given by
\[
\mathcal{L}(e) = \begin{cases} 
\psi(x_{k,n+1} - x_{k,n}; h) & \text{for } e = ((k,n), (k,n+1)), \\
-\psi_0(x_{k+1,n} - x_{k,n}; h) & \text{for } e = ((k,n), (k+1,n)).
\end{cases}
\]

The space-time splitting of the regular square lattice leading to the discrete time Lagrange function (12.4) is shown on figure 2(b).

13.4. Discretizations of relativistic TLs

Under the usual identification (13.1), equation (11.3) becomes a 2D lattice equation
\[
\psi(x_{k,n+1} - x_{k,n}) - \psi(x_{k,n} - x_{k,n-1}) = \phi(x_{k+1,n+1} - x_{k,n}) - \phi(x_{k,n} - x_{k-1,n+1}) + \psi_0(x_{k+1,n} - x_{k,n}) - \psi_0(x_{k,n} - x_{k-1,n})
\]
(13.9)

for a function \( x : \mathbb{Z}^2 \rightarrow \mathbb{R} \). If we visualize this equation by connecting all pairs of vertices which appear in its individual terms, then we arrive at the stencil shown on figure 3.

Consider the graph \( \Gamma \) with the set of vertices \( V(\Gamma) = \mathbb{Z}^2 \), and with the set of edges \( E(\Gamma) \) connecting nearest neighbors in the south-to-north direction, in the west-to-east direction, and in the south-east-to-north-west direction. Clearly, this graph is combinatorially nothing but the regular triangular lattice. Equation (13.9) lives on vertex stars of \( \Gamma \). Moreover, equation (11.3)
is a discrete Laplace type system on the regular triangular lattice, according to definition 13.1, because it is Euler–Lagrange equation for the action functional on $\mathbb{Z}^2$,

$$S[x] = \sum_{(k,n) \in \mathbb{Z}^2} \left( \Psi(x_{k,n+1} - x_{k,n}) - \Psi_0(x_{k+1,n} - x_{k,n}) - \Phi(x_{k+1,n} - x_{k,n+1}) \right),$$

(13.10)

where $\Psi'(\xi) = \psi'(\xi)$, $\Psi_0'(\xi) = \psi_0'(\xi)$, $\Phi'(\xi) = \phi'(\xi)$. Thus, discrete Lagrangians are given by

$$L(e) = \begin{cases} 
\Psi(x_{k,n+1} - x_{k,n}) & \text{for } e = ((k,n), (k,n+1)), \\
-\Psi_0(x_{k+1,n} - x_{k,n}) & \text{for } e = ((k,n), (k+1,n)), \\
-\Phi(x_{k,n+1} - x_{k+1,n}) & \text{for } e = ((k+1,n), (k,n+1)).
\end{cases}$$

There are two natural ways to introduce a space-time splitting of the regular triangular lattice $\Gamma$. A slice of the first slicing is shown on figure 4(a). Summing up discrete Lagrangians over the edges of one slice $\Gamma^n$, we come to the discrete time Lagrange function (11.4) for maps $dRTL^+/(\alpha, h)$. A slice of an alternative slicing is shown on figure 4(b). It supports discrete time Lagrange function (11.6) for maps $dRTL^-/(\alpha, h)$.

13.5. Bibliographical remarks

Relation between lattice 2D systems and initial value problems for discrete 1D systems was discussed in the literature on many occasions, see [PNC90, QCPN91]. Lagrangian aspects of this relation were considered in [CNP91]. The fact that different initial value problems for one and the same 2D lattice system lead to different 1D systems was instrumental in [SR99] for a construction of a novel 1D integrable system, the relativistic Volterra lattice.

Some of the 2D Laplace type equations appearing as time discretizations of TLs enjoy applications in different areas of mathematics. See, for instance, [BH03] for an application of the discrete additive rational relativistic TL to integrable circle patterns with the combinatorics of the regular hexagonal lattice and with prescribed intersection angles.
14. Integrable discrete Laplace type equations and integrable quad-equations

14.1. Discrete Laplace type equations

We can slightly generalize definition of discrete Laplace type system, by removing the requirement that they come from a variational principle.

**Definition 14.1.** Let $\Gamma$ be a graph, with the set of vertices $V(\Gamma)$ and the set of (directed) edges $E(\Gamma)$. A discrete Laplace type system on $\Gamma$ for a function $x : V(\Gamma) \rightarrow \mathbb{R}$ consists of equations

$$\sum_{e_i = (v_0,v_1) \in \text{star}(v_0)} \phi(x(v_0),x(v_1); e_i) = 0. \quad (14.1)$$

There is one equation for every vertex $v_0 \in V(\Gamma)$; the summation is extended over $\text{star}(v_0)$, the set of edges $e_i \in E(\Gamma)$ adjacent to $v_0$. The functions $\phi = \phi(x_0,x_1; e_i)$ are possibly depending on $e_i$, often through parameters $\alpha : E(\Gamma) \rightarrow \mathbb{R}$, assigned to the edges of $\Gamma$.

We will be mainly working with planar graphs $\Gamma$, that is, with graphs coming from a (strongly regular) polytopal cell decomposition of an oriented surface. In this case, one can introduce the dual graph (cell decomposition) $\Gamma^*$. Each edge $e \in E(\Gamma)$ separates two faces of $\Gamma$, which in turn correspond to two vertices of $\Gamma^*$. A path between these two vertices is then declared to be the edge $e^* \in E(\Gamma^*)$ dual to $e$. If one assigns a direction to an edge $e \in E(\Gamma)$, then it will be assumed that the dual edge $e^* \in E(\Gamma^*)$ is also directed, in a way consistent with the orientation of the underlying surface, namely so that the pair $(e,e^*)$ is positively oriented at its crossing point. This orientation convention implies that $e^{**} = -e$. Finally, the faces of $\Gamma^*$ are in a one-to-one correspondence with the vertices of $\Gamma$: if $v_0 \in V(\Gamma)$, and $v_1, \ldots, v_n \in V(\Gamma)$ are its neighbors connected with $v_0$ by the edges $e_1 = (v_0,v_1), \ldots, e_n = (v_0,v_n) \in E(\Gamma)$, then the face of $\Gamma^*$ dual to $v_0$ is bounded by the dual edges $e_1^* = (y_1,y_2), \ldots, e_n^* = (y_n,y_1)$; see figure 5(b).

14.2. Integrability of discrete Laplace type equations

We will say that a discrete Laplace type system on $\Gamma$ is integrable if it possesses a discrete zero curvature representation. That means the existence of a collection of matrices $L(e^*; \lambda) \in G[\lambda]$ from some loop group $G[\lambda]$, associated to directed edges $e^* \in \hat{E}(\Gamma^*)$ of the dual graph $\Gamma^*$, such that:
The matrix \( L(\varepsilon^*; \lambda) = L(x(v), x(w), \alpha; \lambda) \) depends on the fields \( x(v) \) and \( x(w) \) at the vertices of the edge \( e = (v, w) \in E(\Gamma) \), dual to the edge \( e^* \in E(\Gamma^*) \), as well as on the parameter \( \alpha = \alpha(e) \);

- for any directed edge \( e^* = (y_1, y_2) \), if \( -e^* = (y_2, y_1) \), then
  \[
  L(-e^*, \lambda) = (L(e^*, \lambda))^{-1};
  \]

- for any closed path of directed edges \( e_1^* = (y_1, y_2), e_2^* = (y_2, y_3), \ldots, e_n^* = (y_n, y_1) \), we have
  \[
  L(e_1^*, \lambda) \cdots L(e_n^*, \lambda) = I. \tag{14.3}
  \]

The matrix \( L(e^*; \lambda) \) is interpreted as a transition matrix along the edge \( e^* \in E(\Gamma^*) \), that is, a transition across the edge \( e \in E(\Gamma) \). Under conditions (14.2), (14.3) one can define a wave function \( \Psi : V(\Gamma^*) \to G[\lambda] \) on the vertices of the dual graph \( \Gamma^* \), by the following requirement:

- for any directed edge \( e^* = (y_1, y_2) \), the values of the wave functions at its ends must be connected via
  \[
  \Psi(y_2, \lambda) = L(e^*, \lambda) \Psi(y_1, \lambda). \tag{14.4}
  \]

For an arbitrary graph, the analytical consequences of the zero curvature representation for a given Laplace type system are not clear. However, in the case of regular graphs, like the square lattice or the regular triangular lattice, such a representation may be used to determine conserved quantities for suitably defined Cauchy problems, as well as to apply powerful analytical methods for finding concrete solutions.

### 14.3. Quad-graphs and quad-equations

Although one can consider 2D integrable systems on very different kinds of planar graphs, there is one kind, namely quad-graphs, supporting the most fundamental integrable systems.

**Definition 14.2.** A quad-graph is a planar graph with all quadrilateral faces.

Quad-graphs are privileged because from an arbitrary planar graph \( \Gamma \) one can produce a certain quad-graph \( D \), called the double of \( \Gamma \). The double \( D \) is a quad-graph, constructed from \( \Gamma \) and its dual \( \Gamma^* \) as follows. The set of vertices of the double \( D \) is \( V(D) = V(\Gamma) \sqcup V(\Gamma^*) \). Each pair of dual edges, say \( e = (v_0, v_1) \in E(\Gamma) \) and \( e^* = (y_1, y_2) \in E(\Gamma^*) \), defines a quadrilateral
These quadrilaterals constitute the faces of a quad-graph $D$; see figure 6. Let us stress that edges of $D$ belong neither to $E(\Gamma)$ nor to $E(\Gamma^*)$.

Quad-graphs $D$ coming as doubles are bipartite: the set $V(D)$ may be decomposed into two complementary halves, $V(D) = V(\Gamma) \sqcup V(\Gamma^*)$ (‘black’ and ‘white’ vertices), such that the ends of each edge from $E(D)$ are of different colors. Equivalently, any closed loop consisting of edges of $D$ has an even length.

The construction of the double can be reversed. Start with a bipartite quad-graph $D$. For instance, any quad-graph embedded in a plane or in an open disc is automatically bipartite. Any bipartite quad-graph produces two dual planar graphs $\Gamma$ and $\Gamma^*$, with $V(\Gamma)$ containing all the ‘black’ vertices of $D$ and $V(\Gamma^*)$ containing all the ‘white’ ones, and edges of $\Gamma$ (resp. of $\Gamma^*$) connecting ‘black’ (resp. ‘white’) vertices along the diagonals of each face of $D$. The decomposition of $V(D)$ into $V(\Gamma)$ and $V(\Gamma^*)$ is unique, up to interchanging the roles of $\Gamma$ and $\Gamma^*$.

A privileged role played by the quad-graphs is reflected in the privileged role played in the theory of discrete integrable systems by the so called quad-equations supported by quad-graphs.

**Definition 14.3.** For a given bipartite quad-graph $D$, the system of **quad-equations** for a function $x : V(D) \to \mathbb{R}$ consists of equations of the type

$$Q(x_0, y_1, x_1, y_2) = 0.$$  \hspace{1cm} (14.5)

There is one equation for every face $(x_0, y_1, x_1, y_2)$ of $D$, see figure 7(a). The function $Q$ is supposed to be multi-affine, i.e. a polynomial of degree $\leq 1$ in each argument, so that equation (14.5) is uniquely solvable for any of its arguments. Usually, the function $Q = Q(x_0, y_1, x_1, y_2; \alpha, \beta)$ additionally depends on some parameters assigned to the edges of the quadrilaterals, $\alpha : E(D) \to \mathbb{C}$, opposite edges carrying equal parameters: $\alpha = \alpha(x_0, y_1) = \alpha(y_2, x_1)$ and $\beta = \alpha(x_0, y_2) = \alpha(y_1, x_1)$.

The geometric relation of a given planar graph $\Gamma$ to its double $D$ leads to a relation of discrete Laplace type systems on $\Gamma$ to systems of quad-equations on $D$. The latter relation is based on an intriguing property of quad-equations to have the so called three-leg form.

**Definition 14.4.**

(a) A quad-equation (14.5) possesses a **three-leg form** centered at the vertex $x_0$ if it is equivalent to the equation
ψ(\(x_0, y_1\)) − ψ(\(x_0, y_2\)) = φ(\(x_0, x_1\)) \tag{14.6}

with some functions \(ψ, φ\). The terms on the left-hand side correspond to the ‘short’ legs \((x_0, y_1), (x_0, y_2) \in E(D)\), while the right-hand side corresponds to the ‘long’ leg \((x_0, x_1) \in E(Γ)\), see figure 7(b).

(b) A system of quad-equations (14.5) on the faces of a bipartite quad-graph \(D\) has the legs matching property, if for any edge \((x, y) \in E(D)\), the short leg functions \(ψ(x, y)\) for both quadrilaterals sharing this edge, coincide.

For a system of quad-equations with the leg matching property, consider the following discrete Laplace type equations on the ‘black’ graph \(Γ\), constructed from the ‘long’ legs functions:

\[
\sum_{x_t \in V(Γ), x_t \sim x_0} φ(x_0, x_t) = 0 \quad \forall x_0 \in V(Γ).
\tag{14.7}
\]

**Theorem 14.5.**

(a) The restriction of any solution \(x : V(D) \to \mathbb{R}\) of the system of quad-equations (14.5) to the ‘black’ vertices \(V(Γ)\) satisfies discrete Laplace type equations (14.7).

(b) If the graph \(Γ\) is embedded into a simply connected surface, then, conversely, given a solution \(x : V(Γ) \to \mathbb{R}\) of the Laplace type equations (14.7), there exists a one-parameter family of extensions \(x : V(D) \to \mathbb{R}\) satisfying quad-equations (14.5) on the double \(D\). Such an extension is uniquely determined by the value at one arbitrary vertex of \(V(Γ^*)\).

**Proof.** This follows by summation of quad-equations over the vertex star of \(D\) adjacent to the ‘black’ vertex \(x_0 \in V(Γ)\) (see figure 6), due to the telescoping effect.

14.4. Multi-dimensional consistency of quad-equations

The by now widely accepted notion of integrability of quad-equations is that of multi-dimensional consistency. Consider a system on \(\mathbb{Z}^m\) consisting of (possibly different)
quad-equations \( Q(x, x_i, x_j, x_k) = 0 \) on all affine two-dimensional sublattices \( n_0 + \mathbb{Z}e_i + \mathbb{Z}e_j \). Here \( x \) stands for \( x(n) \) at a generic point \( n \in \mathbb{Z}^m \), and \( x_i = x(n + e_i) \), \( x_{ij} = x(n + e_i + e_j) \), where \( e_i \) is the unit vector of the \( i \)th coordinate direction. Such a system is called multi-dimensionally consistent, if it has a solution whose restrictions on all two-dimensional sublattices are generic solutions of corresponding equations. It turns out that the multi-dimensional consistency of quad-equations follows from 3D consistency, and the latter boils down to a local property for one elementary 3D cube.

**Definition 14.6.** Consider a six-tuple of (a priori different) quad-equations assigned to the faces of a 3D cube:

\[
\begin{align*}
A(x, x_1, x_2, x_{12}) &= 0, & \quad \bar{A}(x_3, x_{13}, x_{23}, x_{123}) &= 0, \\
B(x, x_2, x_3, x_{23}) &= 0, & \quad \bar{B}(x_1, x_{12}, x_{13}, x_{123}) &= 0, \\
C(x, x_1, x_3, x_{13}) &= 0, & \quad \bar{C}(x_2, x_{12}, x_{23}, x_{123}) &= 0.
\end{align*}
\]  

(14.8)

Such a six-tuple is called 3D consistent if, for arbitrary initial data \( x, x_1, x_2, x_3 \), and for \( x_{12}, x_{13}, x_{23} \) determined by using \( A = 0, B = 0, C = 0 \), the three values for \( x_{123} \) determined by using \( \bar{A} = 0, \bar{B} = 0, \text{ or } \bar{C} = 0 \), coincide. See figure 8.

This notion is relevant for systems of quad-equations on quad-graphs, since, under certain mild conditions, a quad-graph can be realized as a quad-surface in a lattice of a sufficiently high dimension. An example relevant to the discrete relativistic Toda type systems will be given in the next section.

The property of 3D consistency is relevant for integrability, since it implies the more traditional attributes thereof, like the discrete zero curvature representation and the existence of permutable Bäcklund transformations.

### 14.5. Bibliographical remarks

Integrability of discrete Laplace type equations on arbitrary planar graphs was discussed in [A01].

The notion of 3D consistency of quad-equations which can be put into the basis of the integrability theory, was clearly formulated in [NW01]. A conceptual breakthrough has been made in [BS02] and [N02], where it was shown that 3D consistency allows one to derive in an
algorithmic way such basic integrability attributes as discrete zero curvature representations and Bäcklund transformations for quad-equations. In [ABS03], this property has been put into the basis of a classification of integrable quad-equations which provided a finite list of such equations known nowadays as the ‘ABS list’.

A relation between integrable quad-equations and discrete Laplace type equations based on the three-leg forms in [BS02]. The existence of the three-leg forms was established for all equations of the ABS list in [ABS03], and was proved for all quad-equations with multi-affine functions $Q$ by V. Adler, see Exercise 6.16 in [BS08].

15. Discrete relativistic Toda systems from quad-equations on the dual kagome lattice

The non-symmetric discrete relativistic Toda type systems live on the regular triangular lattice $T$ and cannot be directly generalized to arbitrary graphs. Therefore, we introduce now the specific notation tailored for the regular triangular lattice. The double of $T$ is the quad-graph $K$ known as the dual kagome lattice (drawn on figure 9 in dashed lines). The latter graph has vertices of two kinds, black vertices of valence 6 and white vertices of valence 3, and edges of three different directions.

Correspondingly, it can be realized as a quad-surface in $\mathbb{Z}^3$, so that the three edge directions are realized as coordinate directions of $\mathbb{Z}^3$. In this realization, black vertices of $K$, that is, vertices of $T$, are the points $(i_1, i_2, i_3) \in \mathbb{Z}^3$ lying in the plane $i_1 + i_2 + i_3 = 0$, while white vertices of $K$ are the points of $\mathbb{Z}^3$ lying in the planes $i_1 + i_2 + i_3 = 1$ (vertices $V$) and $i_1 + i_2 + i_3 = -1$ (vertices $U$). See figure 10.

Theorem 15.1. Each discrete relativistic Toda type system is a restriction to the triangular lattice $T$ of a certain 3D consistent system of quad-equations on the dual kagome lattice $K$ considered as a quad-surface in $\mathbb{Z}^3$.

Proof. The corresponding systems of quad-equations are constructed case by case. The quadrilateral faces of the dual kagome lattice are of three different types (corresponding to three different directions of coordinate planes of $\mathbb{Z}^3$). We will denote them by type I, II, and III. The systems are specified by giving the quad-equations explicitly for each type of quadrilateral faces separately in notation of figure 11. One has to: a) check the 3D consistency of the quad-equations, and b) find the three-leg forms, centered at $x_k$, of quad-equations for all
six quadrilaterals around $x_k$, and then check that adding these three-leg forms results in the corresponding discrete Toda equation. All this is a matter of direct computations.

We only give details for the additive exponential relativistic TL (10.6). For this system, we have the following 3D consistent system of quad-equations:

\begin{align*}
    h(XY + UV) + YV + h^2XU - (1 - h\lambda)XV &= 0, \quad (I) \\
    \alpha(XY + UV) + XV + \alpha^2YU - (1 - \alpha\lambda)XU &= 0, \quad (II) \\
    (h - \alpha)(XY + UV) + (1 - \alpha\lambda)YU - (1 - h\lambda)YV \\
    + h^2(1 - \alpha\lambda)XV - \alpha^2(1 - h\lambda)XU &= 0. \quad (III)
\end{align*}

Three-leg forms of equation (I) centered at $X$ and at $Y$ read:

\begin{align*}
    \frac{Y}{X} + \frac{hU}{X} - \frac{(1 - h\lambda)V}{V + hX} &= 0, \quad \text{resp.} \quad \frac{Y}{X} + \frac{hY}{V} - \frac{(1 - h\lambda)Y}{Y + hU} = 0.
\end{align*}

In notation of figures 11(b) and (e), we obtain:

\begin{align*}
    e^{\tilde{x}_k - \alpha} + \frac{h\tilde{U}_{k+1}}{e^{\tilde{U}_k}} - \frac{(1 - h\lambda)\tilde{V}_k}{\tilde{V}_k + he^{\tilde{U}_k}} &= 0, \quad (N) \\
    e^{\alpha - \alpha} + \frac{he^{\alpha}}{V_k} - \frac{(1 - h\lambda)e^{\alpha}}{e^{\beta} + hU_{k+1}} &= 0. \quad (S)
\end{align*}

For equation (II) we find the following three-leg forms (in notation of figures 11(c) and (d)):

\begin{align*}
    \alpha e^{\alpha - \alpha} + \frac{\tilde{U}_{k+1}}{e^{\alpha}} - \frac{(1 - \alpha\lambda)\tilde{V}_{k+1}}{e^{\alpha} + \alpha V_{k+1}} &= 0. \quad (E)
\end{align*}
Figure 10. Embedding of the triangular lattice and the dual kagome lattice into $\mathbb{Z}^3$.

Figure 11. Notation for single quadrilaterals of the dual kagome lattice around the vertex $x_k$. 
\[ \alpha e^{x_k-x_{k-1}} + \frac{e^{x_k}}{V_k} - \frac{(1-\alpha \lambda)e^{x_k}}{U_k + \alpha e^{x_k}} = 0. \quad \text{(W)} \]

For equation (III) we find the following three-leg forms (in notation of figures 11(a) and (f)):

\[ \frac{(\alpha - \hbar)e^{x_k-x_{k-1}}}{1 - \hbar \alpha e^{x_k-x_{k-1}}} - \frac{(1-\alpha \lambda)e^{x_k}}{U_k + \alpha e^{x_k}} + \frac{(1 - \hbar \lambda)\tilde{V}_k}{V_k + \hbar e^{x_k}} = 0, \quad \text{(NW)} \]

\[ \frac{(\alpha - \hbar)e^{x_{k+1}-x_k}}{1 - \hbar \alpha e^{x_{k+1}-x_k}} - \frac{(1-\alpha \lambda)V_{k+1}}{e^{x_k} + \alpha V_{k+1}} + \frac{(1 - \hbar \lambda)U_{k+1}}{e^{x_k} + \hbar U_{k+1}} = 0, \quad \text{(SE)} \]

Adding/subtracting these six equations, we see that all contributions of the short legs (depending on \( \lambda \)) cancel away, and we are left with the combination of the long legs expressed as (10.6).

\[ \square \]

15.1. Zero curvature representations

The construction of discrete Laplace type systems on graphs from systems of quad-equations allows one to find, in an algorithmic way, discrete zero curvature representations for the former. Indeed, each quad-equation can be viewed as a M"obius transformation of the field at one white vertex of a quad into the field at the other white vertex, with the coefficients dependent on the fields at the both black vertices. The \( PSL(2, \mathbb{R}) \) matrices representing these M"obius transformations play then the role of transition matrices across the edges connecting the black vertices. The property (14.3) is satisfied automatically, by construction.

Specializing this construction to the case of the regular triangular lattice, we denote by \( L_k \) the transition matrix from \( V_k \) to \( V_{k+1} \), and by \( M_k \) the transition matrix from \( V_k \) to \( \tilde{V}_k \) (see figure 9 for notations). Then the discrete zero curvature representation reads:

\[ \tilde{L}_k M_k = M_{k+1} L_k, \quad \text{(15.1)} \]

both parts representing the transition from \( V_k \) to \( \tilde{V}_{k+1} \) along two different paths. Discrete zero curvature representation depending on a spectral parameter \( \lambda \) is one of the central integrability attributes. In particular, it implies that the monodromy matrix

\[ T_N(x, p, \lambda) = L_N(x, p, \lambda) \cdots L_2(x, p, \lambda) L_1(x, p, \lambda) \]

remains isospectral under the discrete time evolution (at least in the case of periodic boundary conditions):

\[ \tilde{T}_N M_1 = M_1 T_N, \]

so that its spectral invariants are integrals of motion of the system.

It is clear (see figure 9) that \( L_k \) is the product of two matrices, the first corresponding to the transition from \( V_k \) to \( U_{k+1} \) across the edge \([x_k, x_{k+1}]\) (equation (S) on figure 11(e)), and the second corresponding to the transition from \( U_{k+1} \) to \( V_{k+2} \) across the edge \([x_{k+1}, x_{k+2}]\) (equation (SE) on figure 11(f)). Similarly, \( M_k \) can be represented as the product of two matrices, the first corresponding to the transition from \( V_k \) to \( \tilde{U}_k \) across the edge \([x_k, x_{k-1}]\) (equation (W) on figure 11(d)), and the second corresponding to the transition from \( \tilde{U}_k \) to \( \tilde{V}_k \) across the edge \([x_k, \tilde{x}_{k-1}]\) (equation (NW) on figure 11(a)). The matrices \( L_k, M_k \) for a given discrete relativistic...
Toda type equation can be computed in a straightforward way, as soon as the generating system of quad-equations mentioned in theorem 15.1 is known.

**Theorem 15.2.** For all discrete relativistic Toda type systems, the transition matrix $L_k$ is local when expressed in terms of canonically conjugate variables, $L_k = L(x_k, p_k; \lambda)$, and does not depend on the time discretization parameter $h$.

**Proof.** This is obtained via direct computations on the case-by-case basis. We illustrate the claims of the theorem with the case (10.6). For this case, we first eliminate $U_{k+1}$ between equations (S) and (SE). Upon taking into account the second equation of motion in (10.5), downshifted in discrete time, we find:

$$
p_k + \lambda + \frac{e^{x_k}}{V_k} + \frac{(1 - \alpha \lambda)V_{k+1}}{e^{x_k} + \alpha V_{k+1}} = 0.
$$

This can be put as $V_{k+1} = L_k[V_k]$ with

$$
L_k = L(x_k, p_k; \lambda) = \begin{pmatrix} p_k + \lambda & e^{x_k} \\ - (1 + \alpha p_k) e^{-x_k} & -\alpha \end{pmatrix}.
$$

Analogously, we eliminate $\tilde{U}_k$ between equations (W) and (NW). Taking into account the second equation of motion in (10.5) with $k \rightarrow k - 1$, we find:

$$
he^{-\tilde{x}_{k-1}}(1 + \alpha \tilde{p}_{k-1}) + \frac{1}{V_k} - \frac{1 - h \lambda}{V_k + he^{x_k}} = 0.
$$

This can be put as $\tilde{V}_k = M_k[V_k]$ with

$$
M_k = M(x_k, \tilde{x}_{k-1}, \tilde{p}_{k-1}; \lambda) = \begin{pmatrix} 1 - h \lambda - h^2(1 + \alpha \tilde{p}_{k-1}) e^{x_{k-1}} & -he^{x_k} \\ h(1 + \alpha \tilde{p}_{k-1}) e^{-x_k} & 1 \end{pmatrix}.
$$

The fact that $L_k$ does not depend on $h$ means that the corresponding symplectic maps $(x, p) \mapsto (\tilde{x}, \tilde{p})$ belong to the same integrable hierarchies as their respective continuous time Hamiltonian counterparts. This confirms once again that these maps serve as Bäcklund transformations for the respective Hamiltonian flows, the Bäcklund parameter being the time step $h$. □

15.2. Bibliographical remarks

Our presentation here follows [BolS10], where one can also find details for other discrete relativistic Toda type systems. For the elliptic Toda systems mentioned in the bibliographical remarks to sections 7 and 11, see [AS04]. These discrete Laplace type systems are obtained by the same procedure as discussed in this section from the master integrable equation Q4 of the ABS list (and its hyperbolic and rational degenerations Q3$f_{\pm 1}$ and Q2). For these equations, all leg functions (the ‘short’ and the ‘long’ ones) are given by (7.13).
16. General theory of discrete one-dimensional pluri-Lagrangian systems

Recall that symplectic maps describing the discrete time TL with different step sizes commute. We now turn to the theory which gives a deep insight into the nature of commuting symplectic maps.

**Definition 16.1 (One-dimensional pluri-Lagrangian problem).** Let $L$ be a discrete 1-form on $\mathbb{Z}^m$ (a function of directed edges $\sigma$ of $\mathbb{Z}^m$ with $L(-\sigma) = -L(\sigma)$), depending on a function $x: \mathbb{Z}^m \to X$, where $X$ is some vector space. It is supposed that $L(\sigma)$ depends on the values of the field $x$ at the endpoints of the edge $\sigma$.

- To an arbitrary discrete curve $\Sigma$ in $\mathbb{Z}^m$ (a concatenation of a sequence of directed edges in $\mathbb{Z}^m$ such that the endpoint of any edge is the beginning of the next one), there corresponds the action functional
  \[ S_\Sigma = \sum_{\sigma \in \Sigma} L(\sigma) \]
  (it depends only on the fields at vertices of $\Sigma$).

- We say that the field $x: V(\Sigma) \to X$ is a critical point of $S_\Sigma$, if at any interior point $n$ of the curve $\Sigma$, we have
  \[ \frac{\partial S_\Sigma}{\partial x(n)} = 0. \]

- We say that the field $x: \mathbb{Z}^m \to X$ solves the pluri-Lagrangian problem for the Lagrangian 1-form $L$ if, for any discrete curve $\Sigma$ in $\mathbb{Z}^m$, the restriction $x|_{V(\Sigma)}$ is a critical point of the corresponding action $S_\Sigma$.

We use the following notations: $x(n)$ at a generic point $n \in \mathbb{Z}^m$, and
\[ x_i = x(n + e_i), \quad x_{-i} = x(n - e_i), \quad i = 1, \ldots, m, \quad (16.1) \]
where $e_i$ is the unit vector of the $i$th coordinate direction, then we assume that
\[ L(\sigma_i) = L(n, n + e_i) = L_i(x, x_i) \quad \Leftrightarrow \quad L(-\sigma_i) = L(n + e_i, n) = -L_i(x, x_i). \]
Here $L_i: X \times X \to \mathbb{R}$ are local Lagrangians corresponding to the edges of the $i$th coordinate direction.

Any interior point of any discrete curve $\Sigma$ in $\mathbb{Z}^m$ is of one of the four types shown on figure 12.

The pieces of discrete curves as on figures 12(b)–(d) will be called 2D corners. Observe that a straight piece of a discrete curve, as on figure 12(a), is a sum of 2D corners, as on figures 12(b) and (c). The whole variety of Euler–Lagrange equations for a pluri-Lagrangian system with $d = 1$ reduces to the following three types of 2D corner equations:
\[ \frac{\partial L_i(x, x_i)}{\partial x} - \frac{\partial L_i(x, x_i)}{\partial x} = 0, \quad (16.2) \]
\[ \frac{\partial L_i(x_{-i}, x)}{\partial x} + \frac{\partial L_i(x, x_i)}{\partial x} = 0, \quad (16.3) \]
\[
\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} - \frac{\partial \Lambda_i(x_{-j}, x)}{\partial x} = 0.
\]

(16.4)

In particular, the standard single-time discrete Euler–Lagrange equation,

\[
\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} + \frac{\partial \Lambda_i(x_i, x_{-i})}{\partial x} = 0,
\]

corresponding to a straight piece of a discrete curve as on figure 12(a) is a consequence of equations (16.2) and (16.3), corresponding to 2D corners as on on figures 12(b) and (c).

To discuss consistency of the system of 2D corner equations, it will be more convenient to re-write them with appropriate shifts, as

\[
\frac{\partial \Lambda_i(x_i, x_{-i})}{\partial x} = \frac{\partial \Lambda_i(x_{-j}, x)}{\partial x} = 0,
\]

(Ei)

\[
\frac{\partial \Lambda_j(x_{-j}, x)}{\partial x} + \frac{\partial \Lambda_j(x_i, x_{-i})}{\partial x} = 0,
\]

(Ej)

\[
\frac{\partial \Lambda_i(x_i, x_{-i})}{\partial x_{ij}} - \frac{\partial \Lambda_j(x_{-j}, x)}{\partial x_{ij}} = 0.
\]

(Eij)

In this form, 2D corner equations (E)–(Eij) correspond to the four vertices of an elementary square $\sigma_{ij}$ of the lattice, as on figure 13(a). Consistency of the system of 2D corner equations (E)–(Eij) should be understood as follows: start with the fields $x$, $x_i$, $x_j$ satisfying equation (E). Then each of equations (Ei), (Ej) can be solved for $x_{ij}$. Thus, we obtain two alternative values for the latter field. Consistency takes place if these values coincide identically (with respect to the initial data), and, moreover, if the resulting field $x_{ij}$ satisfies equation (Eij).

In other words:

**Definition 16.2.** The system of 2D corner equations (E)–(Eij) is called consistent, if it has the minimal possible rank 2, i.e. if exactly two of these four equations are independent.

Observe that 2D corner equations (E)–(Eij) can be put as

![Figure 12. Four types of vertices of a discrete curve. Case (a): two edges of one coordinate direction meet at $n$. Case (b): a negatively directed edge followed by a positively directed edge. Case (c): two equally (positively or negatively) directed edges of two different coordinate directions meet at $n$. Case (d): a positively directed edge followed by a negatively directed edge. Values of the field $x$ at the points of the curve are indicated.](image-url)
\[ \partial S_{ij} / \partial x = 0, \quad \partial S_{ij} / \partial x_i = 0, \quad \partial S_{ij} / \partial x_j = 0, \quad \partial S_{ij} / \partial x_{ij} = 0, \quad (16.5) \]

where \( S_{ij} \) is the action along the boundary of an oriented elementary square \( \sigma_{ij} \) (this action can be identified with the discrete exterior derivative \( d\mathcal{L} \) evaluated at \( \sigma_{ij} \)),

\[ S_{ij} = d\mathcal{L}(\sigma_{ij}) = \Delta_i \mathcal{L}(\sigma_j) - \Delta_j \mathcal{L}(\sigma_i) = \Lambda_i(x, x_i) + \Lambda_j(x, x_j) - \Lambda_i(x, x_j) - \Lambda_j(x, x_i). \]

Here and in what follows, \( \Delta_i = T_i - I \) is the difference operator, \( T_i \) being the shift operator in the \( i \)th coordinate, so that, e.g., \( T_ix = T_ix(n) = x(n + e_i) = x_i \) and \( T_ix = T_ix(n + e_j) = x(n + e_i + e_j) = x_{ij} \).

The main feature of our definition is that the ‘almost closedness’ of the 1-form \( \mathcal{L} \) on solutions of the system of 2D corner equations is, so to say, built-in from the outset.

**Theorem 16.3.** If the system of 2D corner equations (18.2) is consistent, then, for any pair of the coordinate directions \( i, j \), the action \( S_{ij} \) over the boundary of an elementary square of these coordinate directions is constant on solutions:

\[ S_{ij}^{\ell} = \ell_{ij} = \text{const} \quad (\text{mod } \partial S_{ij} / \partial x = 0, \ldots, \partial S_{ij} / \partial x_{ij} = 0). \]

In particular, if all these constants \( \ell_{ij} \) vanish, then the discrete 1-form \( \mathcal{L} \) is closed on solutions of the Euler–Lagrange equations, so that the critical value of the action functional \( S_{\Sigma} \) does not depend on the choice of the curve \( \Sigma \) connecting two given points in \( \mathbb{Z}^m \).

Consistency of the system of 2D corner equations (16.2)–(16.4) is equivalent to existence of a function \( p : \mathbb{Z}^m \to \mathcal{X} \) satisfying all the relations

\[ p = - \frac{\partial \Lambda_i(x, x_i)}{\partial x}, \quad i = 1, \ldots, m, \quad (16.6) \]

\[ p = \frac{\partial \Lambda_i(x, x_i)}{\partial x}, \quad i = 1, \ldots, m. \quad (16.7) \]

Suppose that all the equations (16.6) can be solved for \( x_i \) in terms of \( x, p \), so that equations
\[ p = -\frac{\partial \Lambda(x, x_i)}{\partial x}, \quad p_i = \frac{\partial \Lambda(x, x_i)}{\partial x_i}. \]  
\[ (16.8) \]

define symplectic maps \( F_i : (x, p) \mapsto (x_i, p_i) \).

**Theorem 16.4.** For a consistent one-dimensional pluri-Lagrangian system, maps \( F_i \) commute:

\[ F_i \circ F_j = F_j \circ F_i, \]

\[ (16.9) \]

see figure 13(b). Conversely, for a given system of \( m \) commuting symplectic maps \( F_i \) admitting Lagrangians (generating functions) \( \Lambda_i \), the 1-form \( \mathcal{L} \) on \( \mathbb{Z}^m \) defined by \( \mathcal{L}(\sigma_i) = \Lambda_i(x, x_i) \), generates a consistent one-dimensional pluri-Lagrangian system.

We are mainly interested in one-parameter families of commuting symplectic maps \( F_i : (x, p) \mapsto (x_i, p_i) \), depending on the parameter \( \lambda \), and admitting a generating function \( \Lambda(x, x_i; \lambda) \). To avoid double indices, we will denote its action by a tilde:

\[ F_i : p = -\frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial x}, \quad \tilde{p} = \frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial \tilde{x}}. \]

\[ (16.10) \]

When considering a second such map, say \( F_j \), corresponding to another parameter value \( \mu \), we will denote its action by a hat:

\[ F_j : p = -\frac{\partial \Lambda(x, \hat{x}; \mu)}{\partial x}, \quad \hat{p} = \frac{\partial \Lambda(x, \hat{x}; \mu)}{\partial \hat{x}}. \]

\[ (16.11) \]

We assume that (16.9) is satisfied for any two parameter values \( \lambda, \mu \). Corner equations \((E)\)–\((E_{ij})\) in these new notations take the form

\[ \frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial x} = \frac{\partial \Lambda(x, \tilde{x}; \mu)}{\partial x} = 0, \]

\[ (E) \]

\[ \frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial \tilde{x}} + \frac{\partial \Lambda(\tilde{x}, \hat{x}; \mu)}{\partial \hat{x}} = 0, \]

\[ (E_i) \]

\[ \frac{\partial \Lambda(x, \tilde{x}; \mu)}{\partial \tilde{x}} + \frac{\partial \Lambda(\tilde{x}, \hat{x}; \lambda)}{\partial \hat{x}} = 0, \]

\[ (E_j) \]

\[ \frac{\partial \Lambda(\hat{x}, \tilde{x}; \lambda)}{\partial \hat{x}} = \frac{\partial \Lambda(\hat{x}, \tilde{x}; \mu)}{\partial \hat{x}} = 0. \]

\[ (E_{ij}) \]

**Theorem 16.5.** For a consistent system of corner equations \((E)\)–\((E_{ij})\), the discrete multi-time Lagrangian 1-form is closed on solutions, that is, \( \ell(\lambda, \mu) = 0 \), if and only if \( \partial \Lambda(x, \tilde{x}; \lambda)/\partial \lambda \) is a common integral of motion for all \( F_j \).

**Proof.** According to theorem 16.3,

\[ \Lambda(x, \tilde{x}; \lambda) + \Lambda(x, \tilde{x}; \mu) - \Lambda(x, \hat{x}; \mu) - \Lambda(\hat{x}, \tilde{x}; \lambda) = \ell(\lambda, \mu) \]

\[ (16.12) \]

is constant on solutions of corner equations \((E)\)–\((E_{ij})\). This constant is obviously skew-symmetric: \( \ell(\mu, \lambda) = -\ell(\lambda, \mu) \). Then \( \ell(\lambda, \mu) = 0 \) is equivalent to \( \partial \ell/\partial \lambda = 0 \). Differentiating
equation (16.12) with respect to \( \lambda \) and taking into account that the terms containing \( \partial \tilde{x}/\partial \lambda \) etc, appearing due to the chain rule, vanish by virtue of corner equations \((E)\)–\((E_0)\), we arrive at
\[
\frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial \lambda} = \frac{\partial \Lambda(\hat{x}, \tilde{x}; \lambda)}{\partial \lambda} = 0.
\]
This is equivalent to \( \partial \Lambda(x, \tilde{x}; \lambda)/\partial \lambda \) being an integral of motion for \( F_j \).

16.1. Bibliographical remarks

The first example of the pluri-Lagrangian structure for discrete 1D systems was given in [YLN11]. A general theory was developed in [S13], on which this section is based. The result of theorem 16.5 is a re-formulation of the mysterious ‘spectrality property’ of Bäcklund transformations discovered by Kuznetsov and Sklyanin [KS98]. Spectrality was originally understood as the property of \( \partial \Lambda(x, \tilde{x}; \lambda)/\partial \lambda \) to be a spectral invariant of the Lax matrix for the system at hand. Our re-formulation avoids an a priori knowledge of the Lax matrix. We remark that the problem of completeness of the set of the integrals encoded in this quantity requires for a separate study in both approaches.

17. Commutativity of Bäcklund transformations for exponential TL

Here we illustrate the main constructions by the example of Bäcklund transformations for the exponential TL (6.4). The maps \( dTL(\lambda) = F_i : T^* \mathbb{R}^N \rightarrow T^* \mathbb{R}^N \) are given by equations (6.5):
\[
F_i : \begin{cases} 
   p_k = \frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{\tilde{x}_k - \tilde{x}_k}, \\
   \tilde{p}_k = \frac{1}{\mu} (e^{\tilde{x}_k - x_k} - 1) + \mu e^{\tilde{x}_k - \tilde{x}_k}.
\end{cases}
\]
The corresponding Lagrangian is given by
\[
\Lambda(x, \tilde{x}; \lambda) = \frac{1}{\lambda} \sum_{k=1}^{N} (e^{\tilde{x}_k - x_k} - 1 - (\tilde{x}_k - x_k)) - \lambda \sum_{k=1}^{N} e^{\tilde{x}_k - \tilde{x}_k},
\]
and the standard single-time Euler–Lagrange equations coincide with (6.6) with \( h = \lambda \).

As discussed in the previous section, commutativity of the maps \( F_i, F_j \) (in the open-end case, when they are well-defined, i.e. single-valued) is equivalent to consistency of the system of corner equations:
\[
\begin{align*}
   \frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{\tilde{x}_k - \tilde{x}_k} &= \frac{1}{\mu} (e^{\tilde{x}_k - x_k} - 1) + \mu e^{\tilde{x}_k - \tilde{x}_k}, \\
   \frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{\tilde{x}_k - \tilde{x}_k} &= \frac{1}{\mu} (e^{\tilde{x}_k - x_k} - 1) + \mu e^{\tilde{x}_k - \tilde{x}_k}, \\
   \frac{1}{\mu} (e^{\tilde{x}_k - x_k} - 1) + \mu e^{\tilde{x}_k - \tilde{x}_k} &= \frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{\tilde{x}_k - \tilde{x}_k}, \\
   \frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{\tilde{x}_k - \tilde{x}_k} &= \frac{1}{\mu} (e^{\tilde{x}_k - x_k} - 1) + \mu e^{\tilde{x}_k - \tilde{x}_k}.
\end{align*}
\]
We have to clarify the meaning of the both notions (commutativity of $F_i$, $F_j$ and consistency of corner equations) in the periodic case. To do this, we prove the following statement.

**Theorem 17.1.** Suppose that the fields $x$, $\hat{x}$, $\tilde{x}$ satisfy corner equations (E). Define the fields $\hat{\hat{x}}$, $\tilde{\hat{x}}$ by any of the following two formulas, which are equivalent by virtue of (E):

\[
\frac{1}{\lambda} \left( e^{\hat{\hat{x}}_{k-1}} - 1 \right) - \frac{1}{\mu} \left( e^{\hat{\hat{x}}_{k}} - 1 \right) + \lambda e^{x_{k+1} - \hat{x}_{k}} - \mu e^{x_{k+1} - \tilde{x}_{k}} = 0,
\]
\[
\frac{1}{\lambda} \left( e^{\tilde{x}_{k+1} - \hat{x}_{k}} - 1 \right) - \frac{1}{\mu} \left( e^{\tilde{x}_{k+1} - \hat{x}_{k+1} - 1} \right) + \lambda e^{x_{k} - \tilde{x}_{k+1}} - \mu e^{x_{k} - \tilde{x}_{k+1}} = 0,
\]

called superposition formulas. Then corner equations $(E_i)$, $(E_j)$, $(E_{ij})$ are satisfied, as well.

**Proof.** First of all, we show that equations (S1) and (S2) are indeed equivalent by virtue of (E). For this, we re-write these equations in algebraically equivalent forms:

\[
\frac{\lambda - \mu}{e^{x_{k+1} - \hat{x}_{k+1}} - \lambda \mu} = \lambda e^{x_{k} - \hat{x}_{k+1}} - \mu e^{x_{k} - \tilde{x}_{k+1}},
\]

(17.3)

and

\[
(\lambda - \mu) e^{x_{k+1} - \tilde{x}_{k+1}} = \frac{1}{\mu} \left( e^{\tilde{x}_{k+1} - \hat{x}_{k+1} + 1} - 1 \right) - \frac{1}{\lambda} \left( e^{\tilde{x}_{k+1} - \hat{x}_{k+1} + 1} - 1 \right),
\]

(17.4)

respectively. The left-hand sides of the latter two equations are equal. Thus, their difference coincides with (E).

Second, we show that equations (S1) and (S2) yield $(E_i)$. (For $(E_j)$ everything is absolutely analogous.) For this aim, we re-write these equations in still other algebraically equivalent forms. Namely, (S1) is equivalent to

\[
e^{\hat{\hat{x}}_{k-1}} = \lambda \mu e^{x_{k+1} - \hat{x}_{k}} + \frac{\mu - \lambda}{\mu e^{x_{k} - \hat{x}_{k}} - \lambda},
\]

(17.5)

while (S2) with $k$ replaced by $k + 1$ is equivalent to

\[
\lambda \mu e^{x_{k+1} - \tilde{x}_{k+1}} = e^{\tilde{x}_{k+1}} + \frac{\lambda - \mu}{\mu - \lambda e^{x_{k} - \tilde{x}_{k+1}}}.
\]

(17.6)

An obvious linear combination of these expressions leads to

\[
\frac{1}{\mu} e^{\tilde{x}_{k-1}} + \mu e^{\tilde{x}_{k+1} - \tilde{x}_{k-1}} = \lambda e^{x_{k+1} - \tilde{x}_{k}} + \frac{1}{\lambda} e^{x_{k} + 1} + \frac{\lambda - \mu}{\lambda \mu},
\]

which is nothing but $(E_i)$.

Third, we observe that the sum of equations $(E)$, (S1) and (S2) is nothing but the corner equation $(E_{ij})$.

**Remark.** Observe that equations (S1) and (S2) are quad-equations with respect to

\[
\left( e^{x_{k+1}}, e^{\hat{x}_{k}}, e^{\tilde{x}_{k}}, e^{\hat{\hat{x}}_{k}} \right), \quad \text{resp.} \quad \left( e^{x_{k+1}}, e^{\hat{x}_{k+1}}, e^{\tilde{x}_{k+1}}, e^{\hat{\hat{x}}_{k+1}} \right).
\]
i.e. they can be formulated as vanishing of multi-affine polynomials of the four specified variables. Equations (17.3) and (17.4) are then interpreted as the three-leg forms of the quad-equations, centered at \( x_{k+1} \). Similarly, equations (17.5) and (17.6) are the three-leg forms of the quad-equations, centered at \( \tilde{x}_k \).

Theorem 17.1 allows us to achieve an exhaustive understanding of consistency and commutativity for double-valued Bäcklund transformations. First, suppose that we are given the fields \( x, \tilde{x}, \hat{x} \) satisfying corner equation \((E)\). Each of corner equations \((E_i), (E_j)\) produces two values for \( \tilde{x} \). Then consistency is reflected in the following fact: one of the values for \( \tilde{x} \) obtained from \((E_i)\) coincides with one of the values for \( \tilde{x} \) obtained from \((E_j)\), see figure 14(a). Indeed, this common value is nothing but \( \tilde{x} \) obtained from the superposition formulas \((S_1), (S_2)\), as in theorem 17.1.

The ‘loose ends’ on figure 14(a) are best explained by considering the (double-valued) maps \( F_i, F_j \), i.e. by working with the variables \( (x, p) \) rather than with the variables \( x \) alone. Indeed, each of the compositions \( F_i \circ F_j \) and \( F_j \circ F_i \) is four-valued. It follows from theorem 17.1 that their branches must pairwise coincide, as shown on figure 14(b). Indeed, theorem 17.1 delivers four possible values for \( (\tilde{x}, \tilde{x}, \hat{x}) \) satisfying all corner equations \((E)\)–\((E_{ij})\), namely one \( \tilde{x} \) for each of the four possible combinations of \( (x, \tilde{x}, \hat{x}) \).

**Theorem 17.2.** The quantity

\[
P(x, \tilde{x}; \lambda) = \prod_{k=1}^{N} e^{\tilde{x}_k - x_k} \quad (17.7)
\]

is a common integral of motion for all maps \( F_j \). Equivalently, the discrete multi-time Lagrangian 1-form \( \mathcal{L} \) is closed on any solution of the corner equations \((E)\)–\((E_{ij})\).

**Proof.** First of all, we show that the closure relation \( d\mathcal{L} = \ell(\lambda, \mu) = 0 \) is equivalent to

\[
\sum_{k=1}^{N} (\tilde{x}_k - \tilde{x}_k - \tilde{x}_k + x_k) = 0 \iff \prod_{k=1}^{N} e^{\tilde{x}_k - x_k} = 1. \quad (17.8)
\]

This can be done in two different ways. On one hand, combining \((S_1), (S_2)\) with \((E)\), we arrive at the two formulas

\[
\begin{align*}
\frac{1}{\lambda} e^{\tilde{x}_k - y_{k+1}} - \frac{1}{\mu} e^{\tilde{x}_k - y_{k+1}} - \frac{1}{\lambda} e^{\tilde{x}_k - \tilde{x}_k} + \frac{1}{\mu} e^{\tilde{x}_k - \tilde{x}_k} &= 0, \\
\lambda e^{\tilde{x}_k - y_{k+1}} - \mu e^{\tilde{x}_k - y_{k+1}} - \lambda e^{\tilde{x}_k - \tilde{x}_k} + \mu e^{\tilde{x}_k - \tilde{x}_k} &= 0.
\end{align*}
\quad (17.9)
\]

By virtue of these formulas, most of the terms on the left-hand side of \((16.12)\) with the Lagrange function \((17.2)\) cancel, leaving us with

\[
\ell(\lambda, \mu) = \left( \frac{1}{\lambda} - \frac{1}{\mu} \right) \sum_{k=1}^{N} (\tilde{x}_k - \tilde{x}_k - \tilde{x}_k + x_k).
\]

Alternatively, we can refer to theorem 16.5 stating that \( \ell(\lambda, \mu) = 0 \) is equivalent to \( \partial \Lambda(x, \tilde{x}; \lambda) / \partial \lambda \) being an integral of motion for \( F_j \). One easily computes:
\[ \partial \Lambda(x, \tilde{x}; \lambda) = - \frac{1}{\lambda} \sum_{k=1}^{N} p_k + \frac{1}{\lambda^2} \sum_{k=1}^{N} (\tilde{x}_k - x_k), \]

the first sum on the right-hand side being an obvious integral of motion.

Now the desired result (17.8) can be derived from the following form of the superposition formula:

\[ e^{\tilde{x}_k x - \tilde{x}_k x_i + 1} = \frac{\lambda e^{\tilde{x}_k x_i + 1} - \mu e^{\tilde{x}_k x_i + 1}}{\lambda e^{\tilde{x}_k x} - \mu e^{\tilde{x}_k x}}. \]  

(17.11)

which is in fact equivalent to either of equations (17.9), (17.10). In the periodic case (17.8) follows directly by multiplying equations (17.11) for \(1 \leq k \leq N\), in the open-end case equation (17.11) holds true for \(1 \leq k \leq N-1\) and has to be supplemented by the boundary counterparts

\[ e^{\tilde{x}_k x} = \frac{\lambda e^{\tilde{x}_k x} - \mu e^{\tilde{x}_k x}}{\lambda - \mu}, \quad \tilde{e}^{\tilde{x}_k x} = \frac{\lambda - \mu}{\lambda e^{\tilde{x}_k x} - \mu e^{\tilde{x}_k x}}. \]  

(17.12)

which are equivalent to (E) for \(k = 1\), resp. to \((E_0)\) for \(k = N\).

The conserved quantity (17.7), expressed through \((x, p)\), can be given a beautiful expression in terms of matrices which turn out to be transition matrices of the zero curvature representation for \(F_i\) (but the latter notion is not necessary for establishing the result).

**Theorem 17.3.** Set

\[ L_k(x, p; \lambda) = \begin{pmatrix} 1 + \lambda p_k & -\lambda^2 e^{x_k x_i - 1} \\ 1 & 0 \end{pmatrix}, \]
and
\[ T_N(x,p;\lambda) = L_N(x,p;\lambda) \cdots L_2(x,p;\lambda)L_1(x,p;\lambda). \]

Then in the periodic case conserved quantity (17.7) is an eigenvalue of \( T_N(x,p;\lambda) \), while in the open-end case it is equal to \( \text{tr} T_N(x,p;\lambda) \).

**Proof.** We use the following notation for the action of matrices from \( GL(2, \mathbb{C}) \) on \( \mathbb{C} \) by Möbius transformations:
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} [z] = \frac{az + b}{cz + d}.
\]

With this notation, we can re-write the first equation in (17.1) as
\[ e^{\tilde{x}_k - x} - x_k = (1 + \lambda p_k) - \lambda e^{x_k - x_{k-1}} = L_k(x,p;\lambda) [e^{\tilde{x}_k - x_{k-1}}]. \]

This is equivalent to saying that
\[ L_k(x,p;\lambda) \begin{pmatrix} \gamma_k - 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} \gamma_k \\ 1 \end{pmatrix}, \quad \text{where} \quad \gamma_k = e^{\tilde{x}_k - x}. \]

The proportionality coefficient is easily determined by comparing the second components of these vectors:
\[ L_k(x,p;\lambda) \begin{pmatrix} \gamma_k - 1 \\ 1 \end{pmatrix} = \gamma_k - 1 \begin{pmatrix} \gamma_k \\ 1 \end{pmatrix}. \quad (17.13) \]

Now in the periodic case we see that \( (\gamma_N, 1)^T \) is an eigenvector of \( T_N(x,p;\lambda) \) with the eigenvalue \( \prod_{k=1}^N \gamma_k \). In the open-end case, equation (17.13) holds true for \( 2 \leq k \leq N \), and has to be supplemented by the following two relations:
\[ L_1(x,p;\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + \lambda p_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \gamma_N \\ 1 \end{pmatrix} = \gamma_N. \]

As a consequence,
\[ (1 \quad 0) T_N(x,p;\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \prod_{k=1}^N \gamma_k. \]

Thus, the (11)-entry of \( T_N(x,p;\lambda) \) equals \( \prod_{k=1}^N \gamma_k \). It coincides with \( \text{tr} T_N(x,p;\lambda) \), since the (22)-entry of this matrix vanishes. \( \square \)

17.1. Bibliographical remarks

Presentation here is based on [BPS13]. One finds there similar results for all discrete Toda type systems given in section 7.
18. General theory of discrete two-dimensional pluri-Lagrangian systems

Multi-dimensional consistency is a fundamental integrability concept for quad-equations. In the present section, we address the question about an analog of this property for discrete variational Laplace type systems.

Definition 18.1 (Two-dimensional pluri-Lagrangian problem). Let $L$ be a discrete 2-form on $\mathbb{Z}^m$ (a function of oriented elementary squares $\sigma_{ij} = (n,n + e_i,n + e_j)$ such that $L(\sigma_{ij}) = -L(\sigma_{ji})$), depending on a function $x : \mathbb{Z}^m \to \mathcal{X}$, where $\mathcal{X}$ is some vector space. It is supposed that $L(\sigma_{ij})$ depends on the values $x, x_i, x_j, x_{ij}$ of the field at the four vertices of the elementary square $\sigma_{ij}$.

- To an arbitrary quad-surface $\Sigma$ in $\mathbb{Z}^m$ (an oriented surface in $\mathbb{R}^m$ composed of elementary squares of $\mathbb{Z}^m$), there corresponds the action functional
  $$S_\Sigma = \sum_{\sigma \in \Sigma} L(\sigma)$$
  (it depends only on the fields at vertices of $\Sigma$).
- We say that the field $x : V(\Sigma) \to \mathcal{X}$ is a critical point of $S_\Sigma$, if at any interior point $n \in V(\Sigma)$, we have
  $$\frac{\partial S_\Sigma}{\partial x(n)} = 0.$$
- We say that the field $x : \mathbb{Z}^m \to \mathcal{X}$ solves the pluri-Lagrangian problem for the Lagrangian 2-form $L$ if, for any quad-surface $\Sigma$ in $\mathbb{Z}^m$, the restriction $x|_{V(\Sigma)}$ is a critical point of the corresponding action $S_\Sigma$.

One can show that the vertex star of any interior vertex of an oriented quad-surface $\Sigma$ in $\mathbb{Z}^m$ can be represented as a sum of (oriented) 3D corners in $\mathbb{Z}^{m+1}_m$, see figure 15. Here, a 3D corner is a quad-surface consisting of three elementary squares adjacent to a vertex of valence 3. As a consequence, the action over any vertex star can be represented as a sum of actions for several 3D corners. Thus, Euler–Lagrange equation for any interior vertex $n$ of $\Sigma$ can be represented as a sum of several Euler–Lagrange equations for 3D corners.

This justifies the following fundamental definition:

Definition 18.2. The system of 3D corner equations for a given discrete 2-form $L$ consists of discrete Euler–Lagrange equations for all possible 3D corners in $\mathbb{Z}^m$. If the action over an oriented elementary cube $\sigma_{ijk}$ of the coordinate directions $i,j,k$ (which can be identified with the discrete exterior derivative $dL$ evaluated at $\sigma_{ijk}$) is denoted by
  $$S_{ijk}^j = dL(\sigma_{ijk}) = \Delta_k L(\sigma_{ij}) + \Delta_i L(\sigma_{jk}) + \Delta_j L(\sigma_{ki}),$$
  then the system of 3D corner equations consists of the eight equations
  $$\frac{\partial S_{ijk}^j}{\partial x} = 0, \quad \frac{\partial S_{ijk}^j}{\partial x_i} = 0, \quad \frac{\partial S_{ijk}^j}{\partial x_j} = 0, \quad \frac{\partial S_{ijk}^j}{\partial x_k} = 0,$$
  $$\frac{\partial S_{ijk}^j}{\partial x_{ij}} = 0, \quad \frac{\partial S_{ijk}^j}{\partial x_{jk}} = 0, \quad \frac{\partial S_{ijk}^j}{\partial x_{ik}} = 0, \quad \frac{\partial S_{ijk}^j}{\partial x_{ijk}} = 0,$$  
  (18.2)
  for each triple $i,j,k$. 

Thus, the system of 3D corner equations encompasses all possible discrete Euler–Lagrange equations for all possible quad-surfaces $\Sigma$. In other words, solutions of a two-dimensional pluri-Lagrangian problem as introduced in definition 18.1 are precisely solutions of the corresponding system of 3D corner equations.

Of course, in order that the above definition be meaningful, the system of 3D corner equations has to be consistent:

**Definition 18.3.** The system (18.2) is called consistent, if it has the minimal possible rank $2$, i.e. if exactly two of these equations are independent.

Like in the one-dimensional case, the ‘almost closedness’ of the 2-form $L$ on solutions of the system of 3D corner equations is built-in from the outset.

**Theorem 18.4.** If the system of 3D corner equations (18.2) is consistent, then, for any triple of the coordinate directions $i, j, k$, the action $S^{ijk}$ over an elementary cube of these coordinate directions is constant on solutions:

$$S^{ijk}(x, \ldots, x_{ijk}) = \varepsilon^{ijk} = \text{const} \mod \left( \partial S^{ijk}/\partial x = 0, \ldots, \partial S^{ijk}/\partial x_{ijk} = 0 \right).$$

The most interesting case is, of course, when all $\varepsilon^{ijk} = 0$. Then $dL = 0$, that is, the discrete 2-form $L$ is closed on solutions of the system of 3D corner equations, so that the critical value of the action $S_\Sigma$ does not change under perturbations of the quad-surface $\Sigma$ in $\mathbb{Z}^m$ fixing its boundary.

18.1. Case of three-point 2-forms

We formulated the system of 3D corner equations for a generic 2-form $L$. We now specialize the theory for an important particular ansatz for the discrete 2-form, namely we consider the so called three-point 2-form:

$$L(\sigma_i) = \Psi_i(x_i - x) - \Psi_j(x_j - x) - \Phi_{ij}(x_j - x_i), \quad (18.3)$$

where the Lagrangians $\Psi_i$ and $\Phi_{ij}$ only depend on the differences of the fields at the end points, and the diagonal Lagrangians are skew-symmetric in the sense that $\Phi_{ij}(\xi) = -\Phi_{ji}(-\xi)$. Thus, one considers as a main building block in (13.10) the discrete 2-form rather than edge dependent Lagrangians. This seemingly minor change of viewpoint turns out to be very important conceptually.

For a 3-point 2-form, expression (18.1) specializes to
\[ S^{\dot{i}k} = \Psi_i(x_{ik} - x_i) + \Psi_j(x_{jk} - x_j) + \Psi_k(x_{ik} - x_k) - \Psi_i(x_{ij} - x_i) - \Psi_j(x_{jk} - x_j) - \Psi_k(x_{ik} - x_k) - \Phi_{ij}(x_{ik} - x_{jk}) - \Phi_{jk}(x_{jk} - x_{ik}) + \Phi_{ij}(x_{ij} - x_j) + \Phi_{jk}(x_{jk} - x_k). \] (18.4)

Thus, \( S^{\dot{i}k} \) depends on neither \( x \) nor \( x_{ijk} \), and its domain of definition is better visualized as an octahedron shown in figure 16(a).

Accordingly, the system of corner equations consists of six equations per elementary 3D cube, which we denote by \((\xi_i), \Phi(\xi), (\xi_j), (\xi_k), \Phi(\xi_j), \xi_0)\). To write them down, we set
\[ \psi_i(\xi) = \Psi_i(\xi), \quad \phi_{ij}(\xi) = \Phi_{ij}(\xi). \] (18.5)

In particular, we have: \( \phi_i(\xi) = \phi_i(-\xi_i) \). In terms of these functions, corner equations read:
\[ \psi_i(x_{ij} - x_i) + \phi_{ij}(x_{ij} - x_i) = \psi_k(x_{ik} - x_k) + \phi_{ik}(x_{ik} - x_k), \] (\( \xi_i \))
\[ \psi_i(x_{ij} - x_i) + \phi_{ij}(x_{ij} - x_i) = \psi_i(x_{ij} - x_j) + \phi_{ij}(x_{ij} - x_{jk}). \] (\( \xi_j \))

They can be characterized as 5-point 4-leg equations, see figures 16(b) and (c). The consistency of the system of 3D corner equations is defined literally as in definition 18.3.

3D corner equations for three-point 2-forms are elementary building blocks for discrete Laplace type equations on the regular triangular lattice, like all symplectic realizations of the discrete time relativistic TLs. This is illustrated on figure 17.

Now we turn to the case of one-parameter families of 3D corner equations, where we can obtain results generalizing to 3D the spectrality property of theorem 16.5. We fix the following framework. Suppose that one of the coordinate directions plays a special role (we denote this direction by ‘0’). Assume that all other coordinate directions (denoted by \( i, j, \) etc) correspond to certain instances of a parameter (denoted, respectively, by \( \lambda, \mu, \) etc). Thus,
\[ \Psi_i(\xi) = \Psi_i(\xi; \lambda), \quad \Phi_{ij}(\xi) = \Phi_{ij}(\xi; \lambda, \mu), \quad \Phi_{0i}(\xi) = \Phi_{0i}(\xi; \lambda), \] (18.6)
where \( \Phi(\xi; \lambda, \mu) = -\Phi(-\xi; \mu, \lambda) \). Moreover, we will denote the shifts in the coordinate directions \( i, j \) by tilde and by hat, respectively. The indices for the coordinate direction ‘0’ will be denoted by \( k \), and their shift will not be abbreviated. In this specific context, we can re-write expression (18.4) for \( dL \) as follows:
\[ dL = S^{\dot{i}0} = \Psi(\tilde{x}_{i+1} - x_{i+1}; \lambda) - \Psi(\tilde{x}_{i+1} - x_{i+1}; \mu) - \Psi(\tilde{x}_i - x_i; \lambda) + \Psi(\tilde{x}_i - x_i; \mu) - \Psi_0(\tilde{x}_{i+1} - x_{i+1}; \lambda, \mu) + \Psi_0(\tilde{x}_i - x_i; \lambda, \mu) + \Phi_0(\tilde{x}_{i+1} - x_{i+1}; \lambda, \mu) - \Phi_0(\tilde{x}_i - x_i; \lambda, \mu) + \Phi_0(\tilde{x}_{i+1} - x_{i+1}; \lambda) - \Phi_0(\tilde{x}_i - x_i; \lambda). \] (18.7)

**Theorem 18.5.** A three-point discrete 2-form \( L \) with the discrete edge Lagrangians (18.6) is closed on solutions of the system of 3D corner equations if and only if the latter system admits the conservation law
\[ \Delta_0 P_{i0} = \Delta_0 P_{j0}, \] (18.8)
with the densities
\[ P_{i0} = \frac{\partial \Psi(x_0 - x_0; \lambda)}{\partial \lambda} - \frac{\partial \Phi_0(x_{i0} - \tilde{x}_0; \lambda)}{\partial \lambda}, \] (18.9)
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\[ P_{ij} = \frac{\partial \Psi(\tilde{x}_k - x_k; \lambda)}{\partial \lambda} - \frac{\partial \Phi(\hat{x}_k - \tilde{x}_k; \lambda, \mu)}{\partial \lambda}. \]  

(18.10)

**Proof.** According to theorem 18.4, quantity (18.7) is constant on solutions of the system of 3D corner equations: \( d\mathcal{L} = \ell(\lambda, \mu). \) This constant is obviously skew-symmetric: \( \ell(\mu, \lambda) = -\ell(\lambda, \mu). \) Then \( \ell(\lambda, \mu) = 0 \) is equivalent to \( \partial \ell / \partial \lambda = 0. \) Differentiating equation (18.7) with respect to \( \lambda \) and taking into account that the terms containing \( \partial \tilde{x}_k / \partial \lambda \) etc., appearing due to the chain rule, vanish by virtue of the corresponding 3D corner equations, we arrive at

\[
\frac{\partial \Psi(\tilde{x}_{k+1} - x_{k+1}; \lambda)}{\partial \lambda} - \frac{\partial \Psi(\hat{x}_k - \tilde{x}_k; \lambda)}{\partial \lambda} - \frac{\partial \Phi_0(x_{k+1} - \tilde{x}_k; \lambda)}{\partial \lambda} + \frac{\partial \Phi_0(\tilde{x}_{k+1} - \hat{x}_k; \lambda)}{\partial \lambda} - \frac{\partial \Phi(\hat{x}_{k+1} - \tilde{x}_{k+1}; \lambda, \mu)}{\partial \lambda} + \frac{\partial \Phi(\hat{x}_k - \tilde{x}_k; \lambda, \mu)}{\partial \lambda} = 0.
\]

**Figure 16.** (a) Octahedron supporting \( d\mathcal{L} \) for a three-point discrete 2-form \( \mathcal{L}; \) (b) Stencil supporting 3D corner equation (\( \mathcal{E} \)); (c) Stencil supporting corner equation (\( \mathcal{E}_i \)).

**Figure 17.** Sum of four 3D corner equations (matched at the vertex \( x \), one of the equations being void) results in a planar seven-point Laplace type equation on the regular triangular lattice.
This is equivalent to formula (18.8).

18.2. Bibliographical remarks

Discrete three-point 2-forms as in (18.3) were introduced in [LN09] as an ingenious device to generalize the action (13.10) from $\mathbb{Z}^2$ to an arbitrary quad-surface in a multi-dimensional lattice. For several equations from the ABS list, it was shown in [LN09] that solutions of quad-equations deliver critical points for the action functional over an arbitrary quad-surface in $\mathbb{Z}^m$, and that the critical value of action is invariant under local flips of the quad-surface. This paper pioneered the pluri-Lagrangian theory. In [BS10], a conceptual proof of these facts has been given for all quad-equations of the ABS list. A decisive step, shifting the focus from quad-equations to 3D corner equations as main objects of interest, was made in [BPS14, BPS15]. Our presentation follows that paper. Further developments of the pluri-Lagrangian theory include [LNQ09, LN10, BS14].

19. 3D corner equations for relativistic Toda type systems

Like in the previous section, we now consider the situation where one of the coordinate directions (which we denote as the 0th one) plays a distinguished role. We will use the index $k$ for this coordinate direction only. It will enumerate the sites of relativistic Toda chains. Accordingly, we will only consider surfaces in $\mathbb{Z}^m$ which contain, along with any point, the whole line through this point parallel to the 0th coordinate axis. One can call such surfaces cylindrical. The set of values of $x$ along such a line, $x = \{x_k : k \in \mathbb{Z}\}$, or, upon a finite-dimensional reduction, $x = \{x_k : 1 \leq k \leq N\}$, is an element of the configuration space $X$ of the relativistic TL. For other coordinate directions (denoted by $i, j$), we will use tilde, resp. hat to denote the corresponding shifts. The shift in the 0th coordinate direction will not be abbreviated.

**Definition 19.1.** Consider a pluri-Lagrangian system with a three-point 2-form (18.3). The maps $F_i : T^*X \rightarrow T^*X$, $(x, p) \mapsto (\tilde{x}, \tilde{p})$ are defined as the symplectic maps with the generating functions

$$
\Lambda_i(x, \tilde{x}) = \sum_{n=1}^{N} \Psi_i(\tilde{x}_n - x_k) - \sum_{n=1}^{N} \Psi_0(x_{k+1} - x_k) - \sum_{n=1}^{N} \Phi_0(x_{k+1} - \tilde{x}_k),
$$

(19.1)

thus equations of motion for $F_i$ read:

$$
F_i : \begin{cases}
    p_k = -\frac{\partial \Lambda_i}{\partial x_k} = \psi_i(\tilde{x}_k - x_k) + \phi_0(x_k - \tilde{x}_{k-1}) - \psi_0(x_{k+1} - x_k) + \psi_0(x_k - x_{k-1}), \\
    \tilde{p}_k = \frac{\partial \Lambda_i}{\partial \tilde{x}_k} = \psi_i(\tilde{x}_k - x_k) + \phi_0(x_{k+1} - \tilde{x}_k).
\end{cases}
$$

(19.2)

with the corresponding Euler–Lagrange equations

$$
\psi_i(\tilde{x}_k - x_k) - \psi_i(x_k - x_k) = \psi_0(x_{k+1} - x_k) - \psi_0(x_k - x_{k-1}) + \phi_0(x_{k+1} - x_k) - \phi_0(x_k - \tilde{x}_{k-1}).
$$

(19.3)
The map $F_i$ corresponds to the edges $(x, \tilde{x}) = (x, x_i)$ of the $i$th coordinate direction, to which the strip supporting $\Lambda_i$ projects along the 0th coordinate axis. See the identifications of variables on figure 4(a).

**Theorem 19.2.** If the system of 3D corner equations corresponding to a three-point 2-form (18.3) is consistent, then the maps $F_i, F_j$ commute.

**Proof.** According to theorem 16.4, commutativity of $F_i$ and $F_j$ is equivalent to the consistency of the system of 2D corner equations for the corresponding one-dimensional pluri-Lagrangian system with the multi-time $\mathbb{Z}^2$. This system reads:

$$
\psi_j(\tilde{x}_k - x_k) + \phi_0(x_k - \tilde{x}_{k-1}) = \psi_j(x_k - \tilde{x}_k) + \phi_0(x_k - \tilde{x}_{k-1}), \quad (E)
$$

$$
\psi_j(\tilde{x}_k - x_k) + \phi_0(x_{k+1} - \tilde{x}_k) = \psi_j(\tilde{x}_k - x_k) + \phi_0(x_{k+1} - \tilde{x}_k) - \psi_0(\tilde{x}_{k+1} - \tilde{x}_k), \quad (E_i)
$$

$$
\psi_j(\tilde{x}_k - x_k) + \phi_0(x_{k+1} - \tilde{x}_k) = \psi_j(\tilde{x}_k - x_k) + \phi_0(x_{k+1} - \tilde{x}_k) - \psi_0(\tilde{x}_{k+1} - \tilde{x}_k), \quad (E_j)
$$

$$
\psi_j(\tilde{x}_k - x_k) + \phi_0(\tilde{x}_{k+1} - \tilde{x}_k) = \psi_j(\tilde{x}_k - x_k) + \phi_0(\tilde{x}_{k+1} - \tilde{x}_k). \quad (E_y)
$$

A visualization of the 2D corner equations embedded in $\mathbb{Z}^3$ is given in figure 18. Discrete curves in the multi-time plane $\mathbb{Z}^2$ (including the simplest such curves, the 2D corners themselves) are in a one-to-one correspondence with cylindrical surfaces in $\mathbb{Z}^3$, via the projection along the 0th coordinate direction of $\mathbb{Z}^3$.

Consistency of system $(E)-(E_y)$ is proved with the help of the following statement.

**Theorem 19.3.** Suppose that the system of 3D corner equations corresponding to a three-point 2-form (18.3) is consistent. Let the fields $x, \tilde{x},$ and $\tilde{x}$ satisfy 2D corner equations $(E)$. Define the fields $\hat{x}$ by any of the following four formulas, which are equivalent by virtue of $(E)$:

$$
\psi_j(\hat{x}_k - \tilde{x}_k) + \phi_j(\hat{x}_k - \hat{x}_k) = \psi_j(\tilde{x}_{k+1} - \tilde{x}_k) + \phi_0(x_{k+1} - \tilde{x}_k), \quad (S1a)
$$

$$
\psi_j(\hat{x}_k - \tilde{x}_k) + \phi_j(\hat{x}_k - \hat{x}_k) = \psi_0(x_{k+1} - \hat{x}_k) + \phi_0(x_{k+1} - \hat{x}_k), \quad (S1b)
$$

$$
\psi_j(\tilde{x}_{k+1} - x_{k+1}) + \phi_j(\tilde{x}_{k+1} - \tilde{x}_{k+1}) = \psi_0(\tilde{x}_{k+1} - \tilde{x}_k) + \phi_0(\tilde{x}_{k+1} - \tilde{x}_k), \quad (S2a)
$$

$$
\psi_j(\tilde{x}_{k+1} - x_{k+1}) + \phi_j(\tilde{x}_{k+1} - \tilde{x}_{k+1}) = \psi_0(\tilde{x}_{k+1} - \tilde{x}_k) + \phi_0(\tilde{x}_{k+1} - \tilde{x}_k), \quad (S2b)
$$

called superposition formulae (note that each one of these formulas is local with respect to $\tilde{x}$).

Then the 2D corner equations $(E_i), (E_j),$ and $(E_y)$ are satisfied, as well.

**Proof.** One easily checks that the two corner equations $(E_i), (E_j),$ and $(S1a)-(S2b)$ build nothing but the system of six 3D corner equations ($(\tilde{E}_i)$), $(\tilde{E}_y)$ within one elementary cube of $\mathbb{Z}^3$, see figures 18(a), (d) and 19(a)–(d). Due to consist-
ency of the latter system, as formulated in theorem 19.3, if equation (E) and one of equations (S\textsubscript{1}a)–(S\textsubscript{2}b) hold, then equation (E\textsubscript{ij}) and the remaining three of equations (S\textsubscript{1}a)–(S\textsubscript{2}b) are satisfied, as well. Furthermore, equation (E\textsubscript{i}) is the difference of (S\textsubscript{1}a) and (S\textsubscript{2}a)\textsubscript{k→k−1}, while equation (E\textsubscript{j}) is the difference of (S\textsubscript{1}b) and (S\textsubscript{2}b)\textsubscript{k→k−1}. This completes the proof. □

Theorem 19.3 provides a proof of theorem 19.2 in the case of open-end boundary conditions, where the maps $F_i$ are well-defined. At the same time, it provides us with an exhaustive understanding of commutativity also in the case of periodic boundary conditions, where the maps $F_i$ are double-valued. In this case, each of the compositions $F_i \circ F_j$ and $F_j \circ F_i$ applied to a point $(x, p)$ produces four different branches for $(\tilde{x}, \tilde{p})$. Commutativity is reflected in the following fact: each of the branches of $F_i \circ F_j$ coincides with one of the branches of $F_j \circ F_i$. Indeed, theorem 19.3 delivers four possible values for $(x, \tilde{x}, \hat{x})$ satisfying all 2D corner equations (E)–(E\textsubscript{ij}), namely one $\hat{x}$ for each of the four possible combinations of $(x, \tilde{x}, \hat{x})$. See figure 14. □

### 19.1. Example: Bäcklund transformations for the additive exponential relativistic TL

We consider system (10.6) which is a discretization of (and a Bäcklund transformation for) system (10.4). The corresponding maps $F_i : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ are given by

$$
F_i : \begin{cases}
\rho_k = \frac{1}{2} \left( e^{x_{k+1} - x_k} - 1 \right) + \frac{(\lambda - \alpha) e^{x_k - x_{k-1}}}{1 - \lambda e^{x_k - x_{k+1}}} - \alpha e^{x_k - x_{k-1}} + \alpha e^{x_{k+1} - x_k}, \\
\tilde{\rho}_k = \frac{1}{2} \left( e^{x_{k+1} - x_k} - 1 \right) + \frac{(\lambda - \alpha) e^{x_k - x_{k+1}}}{1 - \lambda e^{x_k - x_{k+1}}}
\end{cases}
$$

(19.4)
Thus,

\[ \psi_i(\xi) = \psi(\xi; \lambda) = \frac{1}{\lambda} (e^\xi - 1), \quad \psi_0(\xi) = \alpha e^\xi, \quad \phi_\alpha(\xi) = \phi_\alpha(\xi; \lambda) = \frac{(\lambda - \alpha)e^\xi}{1 - \lambda \alpha e^\xi}. \]  

(19.5)

One can show that, in order to obtain a consistent system of 3D corner equations, these leg functions have to be supplemented by

\[ \phi_\alpha(\xi) = \phi(\xi; \lambda, \mu) = \frac{e^\xi - 1}{\lambda e^\xi - \mu}. \]  

(19.6)

The corresponding 2D corner equations are given by:

\[ \frac{1}{\lambda} (e^{\xi - \bar{x}_i} - 1) + \frac{(\lambda - \alpha)e^{\xi - \bar{x}_i}}{1 - \lambda \alpha e^{\xi - \bar{x}_i}} = \frac{1}{\mu} (e^{\xi - \bar{x}_i} - 1) + \frac{(\mu - \alpha)e^{\xi - \bar{x}_i}}{1 - \mu \alpha e^{\xi - \bar{x}_i}}, \]  

(E)

\[ \frac{1}{\lambda} (e^{\xi - \bar{x}_i} - 1) + \frac{(\lambda - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \lambda \alpha e^{\xi + \bar{x}_i - \bar{x}_i}} = \frac{1}{\mu} (e^{\xi - \bar{x}_i} - 1) + \frac{(\mu - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \mu \alpha e^{\xi + \bar{x}_i - \bar{x}_i}} + \alpha e^{\xi - \bar{x}_i} - \alpha e^{\xi + \bar{x}_i - \bar{x}_i}, \]  

(E_1)

\[ \frac{1}{\mu} (e^{\xi - \bar{x}_i} - 1) + \frac{(\mu - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \mu \alpha e^{\xi + \bar{x}_i - \bar{x}_i}} = \frac{1}{\lambda} (e^{\xi - \bar{x}_i} - 1) + \frac{(\lambda - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \lambda \alpha e^{\xi + \bar{x}_i - \bar{x}_i}} + \alpha e^{\xi - \bar{x}_i} - \alpha e^{\xi + \bar{x}_i - \bar{x}_i}, \]  

(E_2)

\[ \frac{1}{\lambda} (e^{\xi - \bar{x}_i} - 1) + \frac{(\lambda - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \lambda \alpha e^{\xi + \bar{x}_i - \bar{x}_i}} = \frac{1}{\mu} (e^{\xi - \bar{x}_i} - 1) + \frac{(\mu - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \mu \alpha e^{\xi + \bar{x}_i - \bar{x}_i}}, \]  

(E_3)

while the superposition formulas are given by:

\[ \frac{1}{\mu} (e^{\xi - \bar{x}_i} - 1) + \frac{e^{\xi} - e^{\xi}}{\lambda e^{\xi} - \mu e^{\xi}} = \alpha e^{\xi + \bar{x}_i - \bar{x}_i} + \frac{(\lambda - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \lambda \alpha e^{\xi + \bar{x}_i - \bar{x}_i}}, \]  

(S1a)

\[ \frac{1}{\lambda} (e^{\xi - \bar{x}_i} - 1) + \frac{e^{\xi} - e^{\xi}}{\mu e^{\xi} - \lambda e^{\xi}} = \alpha e^{\xi + \bar{x}_i - \bar{x}_i} + \frac{(\mu - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \mu \alpha e^{\xi + \bar{x}_i - \bar{x}_i}}, \]  

(S1b)

\[ \frac{1}{\lambda} (e^{\xi + \bar{x}_i - \bar{x}_i} - 1) + \frac{e^{\xi + 1} - e^{\xi + 1}}{\mu e^{\xi + 1} - \lambda e^{\xi + 1}} = \alpha e^{\xi + \bar{x}_i - \bar{x}_i} + \frac{(\mu - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \mu \alpha e^{\xi + \bar{x}_i - \bar{x}_i}}, \]  

(S2a)

\[ \frac{1}{\lambda} (e^{\xi + \bar{x}_i - \bar{x}_i} - 1) + \frac{e^{\xi + 1} - e^{\xi + 1}}{\lambda e^{\xi + 1} - \mu e^{\xi + 1}} = \alpha e^{\xi + \bar{x}_i - \bar{x}_i} + \frac{(\lambda - \alpha)e^{\xi + \bar{x}_i - \bar{x}_i}}{1 - \lambda \alpha e^{\xi + \bar{x}_i - \bar{x}_i}}, \]  

(S2b)

Of course, in order to apply our general results, one has to prove consistency of the system of 3D corner equations consisting of \( (E_k)_{k \rightarrow k+1}, \) \( (E_0) \), and \( (S1a)-(S2b) \). For this, one shows by direct computations that any two of these six equations are equivalent by virtue of the following octahedron equation:

\[ \frac{1}{\lambda} e^{\xi + \bar{x}_i - \bar{x}_i} + \frac{1}{\mu} e^{\xi - \bar{x}_i} + \frac{1}{\lambda} e^{\xi - \bar{x}_i} + \frac{1}{\mu} e^{\xi - \bar{x}_i} + \alpha e^{\xi + \bar{x}_i - \bar{x}_i} - \alpha e^{\xi + \bar{x}_i - \bar{x}_i} = 0. \]  

(19.7)
Indeed, octahedron relation (19.7) is an immediate consequence of \((E_k \rightarrow k + 1)\), \((S_1 a)\) and \((S_1 b)\) (or, alternatively, of \((E_{ij})\), \((S_2 a)\) and \((S_2 b)\)). On the other hand, eliminating from any of the corner equations one of the variables by means of the (multi-affine) octahedron equation, we obtain another corner equation.

\[\square\]

**Theorem 19.4.** The system of 3D corner equations consisting of \((E_k \rightarrow k + 1)\), \((E_0)\), and \((S_1 a) - (S_2 b)\) admits the conservation law (18.8). Therefore, the discrete Lagrangian 2-form \(\mathcal{L}\) is closed on any solution of this system.

**Proof.** We apply theorem 18.5. To do this, we compute from (19.5), (19.6):

\[
\Psi(\xi; \lambda) = \frac{1}{\lambda} (e^\xi - 1 - \xi) \Rightarrow \frac{\partial \Psi(\xi; \lambda)}{\partial \lambda} = -\frac{1}{\lambda^2} (e^\xi - 1 - \xi)
\]

\[
\Phi(\xi; \lambda, \mu) = \frac{1}{\mu} + \frac{\mu - \lambda}{\lambda \mu} \log(\lambda e^\xi - \mu) \Rightarrow \frac{\partial \Phi(\xi; \lambda, \mu)}{\partial \lambda} = -\frac{1}{\lambda \mu} + \frac{1}{\lambda} \frac{e^\xi - 1}{\lambda^2} - \frac{1}{\lambda^2} \log(\lambda e^\xi - \mu).
\]

\[
\Phi_0(\xi; \lambda) = -\frac{\lambda - \alpha}{\lambda \alpha} \log(1 - \lambda \alpha e^\xi) \Rightarrow \frac{\partial \Phi_0(\xi; \lambda)}{\partial \lambda} = \frac{1}{\lambda} \frac{\lambda - \alpha}{1 - \lambda \alpha e^\xi} - \frac{1}{\lambda^2} \log(1 - \lambda \alpha e^\xi).
\]

As a consequence, we can write conservation law (18.8) as

\[
\Delta_j (R_{i0} - \frac{1}{\lambda} S_{i0}) = \Delta_0 (R_{ij} - \frac{1}{\lambda} S_{ij}), \quad (19.8)
\]

where

\[
R_{i0} = \frac{1}{\lambda} (e^{\tilde{x}_{i+k}} - 1) + \frac{\lambda - \alpha}{1 - \lambda \alpha e^\xi} e^{\tilde{x}_{i+k}}, \quad R_{ij} = \frac{1}{\lambda} (e^{\tilde{x}_{i+k}} - 1) + \frac{e^{\tilde{x}_{i}}}{{\lambda} e^{x_{i+k}}} - \frac{e^{x_{i+k}}}{{\mu} e^{x_{i+k}}}.
\]
and
\[ S_\theta = (\bar{x}_k - x_k) + \log \left( 1 - \lambda e^{x_k - \bar{x}_k} \right), \quad S_\theta = -x_k + \log \left( \lambda e^{\bar{x}_k} - e^{x_k} \right). \]

We show that the two conservation laws
\[ \Delta_j R_i = \Delta_0 R_j \quad \text{and} \quad \Delta_j S_i = \Delta_0 S_j \]
are satisfied separately. Indeed, \( \Delta_j R_i = \Delta_0 R_j \) is an immediate consequence of (E\_ij), (S1\_a) and (S2\_a), while (the exponentiated form of) \( \Delta_j S_i = \Delta_0 S_j \) reads
\[ e^{\bar{x}_k - x_k + \lambda x_k + 1} - e^{x_k - \bar{x}_k} = \frac{\lambda e^{x_k - \bar{x}_k} - \mu e^{x_k - \bar{x}_k}}{\lambda e^{\bar{x}_k} - \mu e^{x_k}}, \]
and is equivalent to (19.7) (as a straightforward clearing of denominators shows).

As a corollary, \( \sum_{k=1}^{N} S_\theta \) is a common conserved quantity for all \( F_j \). Its exponentiated form is
\[ P(x, \bar{x}; \lambda) = \prod_{k=1}^{N} \gamma_k, \quad \text{where} \quad \gamma_k = e^{x_k - \bar{x}_k} \left( 1 - \lambda e^{x_k - \bar{x}_k} \right). \tag{19.9} \]

Like in the non-relativistic case, one can find a nice expression of this quantity through \( (x, p) \).

**Theorem 19.5.** Set
\[ L_k(x, p; \lambda) = \begin{pmatrix} 1 + \lambda p_k - \lambda e^{x_k - \bar{x}_k} & -\lambda(x - \alpha)e^{x_k - \bar{x}_k} \\ 0 & 1 \end{pmatrix}, \]
and
\[ T_N(x, p; \lambda) = L_N(x, p; \lambda) \cdots L_2(x, p; \lambda) L_1(x, p; \lambda). \]

Then in the periodic case the quantity \( P(x, \bar{x}; \lambda) \) is an eigenvalue of \( T_N(x, p; \lambda) \), while in the open-end case it is equal to \( \text{tr} T_N(x, p; \lambda) \).

**Proof.** The first equation in (19.4) can be put as
\[ e^{x_k - \bar{x}_k} \left( 1 - \lambda e^{x_k - \bar{x}_k} \right) = \left( 1 + \lambda p_k - \lambda e^{x_k - \bar{x}_k} \right) - \frac{\lambda(x - \alpha)e^{x_k - \bar{x}_k}}{e^{x_k - \bar{x}_k} - \left( 1 - \lambda e^{x_k - \bar{x}_k} \right)}. \]

This is equivalent to saying that
\[ L_k(x, p; \lambda) \begin{pmatrix} \gamma_k^{-1} \\ 1 \end{pmatrix} \sim \begin{pmatrix} \gamma_k^0 \\ 1 \end{pmatrix}. \]

From this point, the proof is literally the same as for theorem 17.3.

19.2. Bibliographical remarks

Our presentation here follows [BPS15]. Similar results are obtained there for all relativistic TLs listed in section 11.
20. Conclusions

The TL model, whose 50th anniversary we are currently celebrating, continues to demonstrate its unique and central status in the theory of integrable systems. All relevant structures in the theory are most transparently illustrated on the example of the TL. This is true for the most classical ones, like multi-soliton solutions, Bäcklund transformations, Lax representations in loop algebras, inverse spectral method, multi-Hamiltonian structures and recursion operators, AKS scheme based on factorizations in loop groups, algebro-geometric integration methods. This continues to be true for the most recent findings. Every significant development in the theory of integrable systems is being necessarily probed on this celebrated model. We illustrated this with the AKS-based time discretization, with the concept of multi-dimensional consistency as integrability of discrete systems, and with the pluri-Lagrangian theory of continuous and discrete systems, discussed in detail in the present review. The author is convinced that years to come will surprise us with further fundamental structural insights in the theory of integrable systems and with their further applications in mathematics and natural science, and that these developments will continue to be stimulated by questions about the TL.

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