What can be learnt from pinching a glass?

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It is now well established that glasses feature quasilocalized nonphononic excitations, which follow a universal $\omega^4$ density of states in the limit of low frequencies $\omega$. All glass-specific properties, such as the dependence on the preparation protocol or composition, are encapsulated in the nonuniversal prefactor of the universal $\omega^4$ law. The prefactor, however, is a composite quantity that incorporates information both about the number of quasilocalized nonphononic excitations and their characteristic stiffness, in an apparently inseparable manner. We show that by pinching a glass, i.e. by probing its response to force dipoles, one can disentangle and independently extract the two fundamental pieces of physical information. This analysis reveals that the number of quasilocalized nonphononic excitations follows a Boltzmann-like distribution in terms of the parent temperature from which the glass is quenched. The latter, sometimes termed the fictive (or effective) temperature, plays important roles in non-equilibrium thermodynamic approaches to the relaxation, flow and deformation of glasses. The analysis also shows that the characteristic stiffness of quasilocalized nonphononic excitations can be related to their characteristic size, a long sought-for length scale. These results show that important physical information, which is relevant for various key questions in glass physics, can be obtained through pinching a glass.

Understanding the micromechanical, statistical and thermodynamic properties of soft nonphononic excitations in structural glasses remains one of the outstanding challenges in glass physics, despite decades of intensive research \cite{1–15}. Soft nonphononic excitations are believed to give rise to a broad range of glassy phenomena, many of which are still poorly understood; some noteworthy examples include the universal thermodynamic and transport properties of glasses at temperatures of 10K and lower \cite{1, 6, 16–18}, the low-temperature yielding transition in which a mechanically-loaded brittle glass fails via the formation of highly localized bands of plastic strain \cite{19, 20}, and anomalous, non-Rayleigh wave attenuation rates \cite{21–23}.

Computational studies have been invaluable in advancing our knowledge about the statistical and mechanical properties of soft glassy excitations, and in revealing the essential roles that these excitations play in various glassy phenomena. Dating back to the early 1990’s, Schober and Laird were the first to reveal the existence of low-frequency quasilocalized vibrational modes in a model computer glass \cite{24, 25}. Soon later, Schober and coworkers showed that relaxation events deep in the glassy state exhibit patterns that resemble quasilocalized modes \cite{26}, suggesting a link between soft glassy structures and dynamics. In an important subsequent work \cite{27}, this link was further strengthened by showing that relaxational dynamics in supercooled liquids strongly correlates with quasilocalized low-frequency vibrational modes measured in underlying inherent states. Some years later, it was shown that plastic activity in model structural glasses and soft-sphere packings is governed by nonphononic low-frequency modes \cite{28–30}.

It was, however, only recently that the universal statistical and structural properties of soft quasilocalized modes in glasses were revealed, first in a Heisenberg spin glass in a random field \cite{31}, and later in model structural glasses \cite{11, 12, 32–34}. It is now well accepted that the density of nonphononic quasilocalized modes of frequency $\omega$ grows from zero (i.e. without a gap) as $\omega^4$, independently of microscopic details \cite{11}, preparation protocol \cite{13}, or spatial dimension \cite{12}. Importantly, as shown in \cite{34} and demonstrated again in this work, the $\omega^4$ distribution of quasilocalized modes persists even in inherent states that underlie very deeply supercooled states, i.e. in stable computer glasses whose stability is comparable to conventional laboratory glasses. Furthermore, soft quasilocalized modes have been shown to generically feature a disordered core of linear size of a few particle spacings, decorated by long-range Eshelby-like displacement fields whose amplitude decays as $r^{-d-4}$ at a distance $r$ from the core, in $d$ spatial dimensions \cite{11, 12}.

The key challenge in revealing the statistical, structural and energetic properties of soft quasilocalized modes — to be termed hereafter QLMs — in computer investigations lies in the abundance of spatially-extended low-frequency phonons in structural glasses \cite{32, 35}. These phononic excitations hybridize with quasilocalized excitations, as pointed out decades ago by Schober and Oligschleger \cite{36}. These hybridization processes hinder the accessibility of crucial information regarding characteristic length and frequency scales of QLMs, and regarding their prevalence.

While promising attempts to overcome the aforementioned hybridization issues have been put forward \cite{32, 36–38}, a complete statistical-mechanical picture of QLMs is still lacking. In particular, recent work has revealed that annealing processes affect QLMs in three ways: firstly, the number of QLMs appears to decrease...
In this work, we investigate the effect of very deep supercooling/annealing on the statistical, structural and energetic properties of QLMs in a model computer glass (see Methods section for details). First, we explain why information regarding the number density of QLMs cannot be obtained from the universal vibrational density of states (vDOS) of QLMs alone. Instead, we show that the vDOS grants access to a composite physical observable, which encodes information regarding both the characteristic frequency scale of QLMs, \( \omega_g \), and their number density, \( N \). Then, following recent suggestions [13, 41], we use the average response of the glass to a local pinch — more formally, we use the bulk-average response of a glass to force dipoles — as a measure of \( \omega_g \). This, in turn, allows us to quantitatively disentangle the processes of annealing-induced stiffening of QLMs from their annealing-induced depletion.

Remarkably, this analysis reveals that \( N \) follows an equilibrium-like Boltzmann relation \( N \propto \exp \left( -\frac{E_{\text{QLM}}}{k_B T_p} \right) \), with \( T_p \) denoting the parent temperature from which glassy states are quenched, \( k_B \) is the Boltzmann constant, and \( E_{\text{QLM}} \) is the energetic cost of creating a QLM. That is, our results indicate that QLMs behave as “quasiparticles” whose number density is determined by equilibrium statistical thermodynamics at the parent equilibrium temperature \( T_p \), and that this number density is preserved when the glass goes out-of-equilibrium during a quick quench to a temperature much smaller than \( T_p \). The QLMs thus appear to correspond to configurational degrees of freedom that carry memory of the equilibrium state at \( T_p \), deep into the non-equilibrium glassy state, and in this sense \( T_p \) has a clear thermodynamic interpretation as a non-equilibrium temperature. This physical picture has been for quite some time the cornerstone of the non-equilibrium thermodynamic shear-transformation-zones (STZs) theory of the glassy deformation [42–44], where \( T_p \) is termed a fictive/effective/configurational temperature, once QLMs are identified with STZs, i.e. with glassy “flow defects” [45].

Furthermore, we show that \( \omega_g \) can be used to define a length that appears to match the independently determined core size of QLMs, argued to mark the crossover between the disorder-dominated elastic response of glasses at the mesoscale, and the continuum-like elastic response at the macroscale [46]. Taken together, these results show that important physical information, which is relevant for various key questions concerning the formation, relaxation and flow of glasses, can be obtained through pinching a glass.

FIG. 1. A graphical representation of the population of QLMs in poorly annealed (upper panel) and deeply annealed (bottom panel) 2D computer glasses. Each blob represents a QLM, its size is proportional to our estimation of the mode’s core size \( \xi_{\text{QLM}} \), and the color code represents the mode’s frequency, decreasing from bright to dark; upper (bottom) panel color code range is [0.18,0.42] ([0.54,0.74]), expressed in terms of \( c_\infty/a_0 \), with \( c_\infty \) being the high-\( T_p \) shear wave speed, and \( a_0 \) is the interparticle distance. The typical distance between QLMs, \( \xi_s \), is also marked. Details of the calculation can be found in the SI. Note that the deeply annealed case shown in the bottom panel might be representative of laboratory molecular or metallic glasses.

upon deeper annealing, i.e. they are depleted, as first pointed out in [13, 39]. Secondly, the core size of QLMs, \( \xi_{\text{QLM}} \), was shown to decrease with deeper annealing [11, 40]. Lastly, in [11, 13] it was shown that the characteristic frequencies of QLMs also increase upon deeper annealing, i.e. they stiffen, in addition to their depletion. These three effects, and other concepts discussed below, are graphically illustrated in Fig. 1.
THE QLMS DEPLETION VERSUS STIFFENING CONUNDRUM

It is now established that the vDOS of QLMs, $D(\omega)$, follows a universal law [11, 12, 32–34]

$$D(\omega) = A_4 \omega^4,$$

(1)

at asymptotically low frequencies, $\omega \rightarrow 0$. The prefactor $A_4$ is a non-universal quantity that encodes information about a particular glassy state, most notably its composition (constituent elements, interaction potential etc.) and its preparation protocol [13, 34, 39]. The ultimate goal of this work is to explore the physical information encapsulated in $A_4$ and its dependence on the glass preparation protocol.

![CDF plot](image)

FIG. 2. Cumulative density of states CDF $\equiv \int_0^{\omega^*} D(\omega')d\omega'$ for various parent temperatures $T_p$ (see values in the legend). Inset: the prefactors $A_4$ vs. $T_p$, see text for discussion.

In Fig. 2, we plot the cumulative vDOS, measured in 10000 glassy samples of $N = 2000$ particles, rapidly quenched from parent equilibrium temperatures $T_p$ (as appears in the figure legend) to zero temperature. The system size is chosen so as to avoid hybridization with phonons at the lowest frequencies, as explained in [11]. The figure shows, in agreement with [34], that the $\omega^4$ scaling persists all the way down to the deepest supercooled states accessible to us, $T_p = 1/3$ (the units used to report $T_p$ are defined below). The inset shows that the prefactor $A_4$ varies by nearly 3 orders of magnitude in the simulated $T_p$ range. The huge variability of $A_4$ with the preparation protocol, here quantified by the parent equilibrium temperature $T_p$, indicates dramatic changes in the resulting glassy states, despite the fact that all of them follow the universal $\omega^4$ law.

What physics is encapsulated in $A_4$? To start addressing this question, let us first consider the dimensions of $A_4$. When $D(\omega)$ is integrated over the frequency range in which (1) is valid, one obtains the total number of QLMs per unit volume. Consequently, $A_4$ has the dimensions of a number density over a frequency to the fifth power. In sharp contrast to crystalline solids, where the total number of phononic excitations equals the number of degrees of freedom, in glassy solids there is no constraint whatsoever on the fraction of QLMs out of the total number of vibrational modes. Consequently, we expect to have $A_4 = N \omega_g^{-5}$, where $N$ is proportional to the $a priori$ unknown number density of QLMs, and $\omega_g$ is their characteristic frequency scale [11, 13], which quantifies their stiffness (or alternatively their softness). Therefore, (1) should be rewritten as

$$D(\omega) = N \omega_g^{-5} \omega^4,$$

(2)

which evidently does not allow one to distinguish between changes in the number density of QLMs (e.g. a decrease, i.e. depletion) and in their characteristic frequency (e.g. an increase, i.e. stiffening). How to disentangle the $N$ and $\omega_g$ dependence of $A_4$, and the possible depletion and stiffening of QLMs associated with them, is the question we address next.

ESTIMATING QLMS FREQUENCY SCALE BY PINCHING A GLASS

It is clear from the previous discussion and from (2) that the $T_p$ dependence of $A_4 = N \omega_g^{-5}$, presented in the inset of Fig. 2, cannot be readily used to extract the $T_p$ dependence of $N$ and $\omega_g$ separately. Consequently, one needs additional physical input in order to disentangle the two quantities. Here we follow the suggestion put forward in [13] that the characteristic frequency $\omega_g$ of QLMs can be probed through pinching a glass. Formally, by pinching we mean applying a force dipole $d^{(ij)}$ to a pair of interacting particles $i, j$ in a glassy sample. The displacement response to $d^{(ij)}$, which was shown to resemble the spatial pattern of QLMs [13], can be associated with a frequency $\omega_g^{(ij)}$ (see additional details in the SI). By averaging $\omega_g^{(ij)}$ over many interacting pairs $i, j$ in a glassy sample, one obtains a characteristic frequency scale $\omega_g$. Our suggestion that $\omega_g$ represents the characteristic frequency of QLMs, was discussed in length and tested in [13] under various circumstances; the remainder of the paper is devoted to exploring the implications of this suggestion.

In Fig. 3a we plot the characteristic frequency $\omega_g$ vs. the parent temperature $T_p$, where $\omega_g$ is estimated by the pinching procedure just described. It is observed that $\omega_g$ varies by nearly a factor of 2 at low parent temperatures $T_p$ and reaches a plateau at higher $T_p$. We further find that the sample-to-sample mean athermal shear modulus, $G$, shown in the inset, also plateaus at the same $T_p$ as $\omega_g$ does. Consequently, in what follows we conveniently express temperatures in terms of the onset temperature $T_{onset}$ of the high-$T_p$ plateaus of $G$ and $\omega_g$.

We conclude that, in the $T_p$ range considered here, QLMs appear to stiffen by a factor of approximately 2 with decreasing $T_p$. Interestingly, in [34] it was reported that the boson peak frequency $\omega_{BP}$ varies by approxi-
mately a factor of 2 over a similar range of $T_p$, suggesting that $\omega_{\text{BP}}$ and $\omega_g$ might be related. In [41], a similar proposition was put forward in the context of the unjamming transition [47–49], where it was argued that the renowned ‘unjamming’ frequency scale $\omega_s$ [2, 47] can be extracted by considering the frequencies associated with the responses to a local pinch. However, since $\omega_s$ and $\omega_{\text{BP}}$ may differ [8], it is not currently clear which of these frequencies is better represented by $\omega_g$.

The stiffening of QLMs by a factor of approximately 2 accounts for an approximate 30-fold variation of $A_4$, due to the $\omega_g^{-5}$ dependence in (2). The remaining variation is attributed to the number density of QLMs, $N = A_4 \omega_g^5$, plotted in Fig. 3b. The result indicates that QLMs are depleted by slightly less than 2 orders of magnitude in the simulated $T_p$ range. The strong depletion of QLMs upon deeper supercooling has dramatic consequences for the properties of the resulting glassy states. For example, brittle failure [50, 51] and reduced fracture toughness [52–54] are claimed to be a consequence of this depletion. It is interesting to note that the range of variability observed in Fig. 3b appears to be consistent with a very recent study [55] of the depletion of tunneling two-level systems in stable computer glasses, strengthening the suggestion that a subset of the QLMs are associated with tunneling two-level systems [1, 32, 56].

The results presented in Fig. 3 demonstrate that pinching a glass may offer a procedure to separate the depletion and stiffening processes that take place with progressive supercooling. Next, we aim at exploring the physical implications of disentangling $N$ and $\omega_g$.

### A THERMODYNAMIC SIGNATURE OF THE QLMs

QLMs correspond to compact zones (though they also have long-range elastic manifestations), which are embedded inside a glass, and characterized by particularly soft structures. It is tempting then to think of them as quasiparticles that feature well defined properties (e.g., formation energy). If true, one may hypothesize that QLMs can be created and annihilated by thermodynamic fluctuations and follow an equilibrium distribution at the parent equilibrium temperatures $T_p$. Moreover, their equilibrium thermodynamic nature might be manifested in non-equilibrium glassy states as they become frozen in during the rapid quench upon glass formation.

![Graph showing characteristic frequency $\omega_0$ of quasilocalized modes vs. $T_p$.](image)

**FIG. 3.** (left) The characteristic frequency $\omega_0$ of quasilocalized modes, estimated by the pinching procedure discussed in the text, plotted vs. the parent temperature $T_p$. Inset: the sample-to-sample mean athermal shear modulus, $G$, plotted against $T_p$. (right) $N$ is proportional to the number density of quasilocalized modes, and is plotted here against $T_p$.

![Graph showing density of QLMs vs. $1/T_p$.](image)

**FIG. 4.** The density of QLMs, plotted against $1/T_p$, revealing that it is controlled by a Boltzmann-like factor $e^{-E_{\text{QLM}}/k_B T_p}$, with the parent temperature playing the role of the equilibrium temperature. We find $E_{\text{QLM}} \approx 3.4$, expressed in terms of $k_B T_{\text{core}}$.

As we have now at hand the number density $N$ as a function of $T_p$ (cf. Fig. 3b), we can start testing these ideas. To this aim, we plot in Fig. 4 $N$ vs. $T_p^{-1}$ on a semilogarithmic scale; the outcome reveals a key result: the number density of QLMs follows a Boltzmann-like distribution, with the parent temperature $T_p$ playing the role of the equilibrium temperature, namely

$$N \propto \exp \left( -\frac{E_{\text{QLM}}}{k_B T_p} \right).$$

A corollary of this observation is that QLMs feature a well defined formation energy, $E_{\text{QLM}} \approx 3.4$ (in units of $k_B T_{\text{core}}$). A related observation was made in [57] for reheated stable glasses [58].

The results in (3) and Fig. 4 indicate that QLMs might indeed correspond to a subset of configurational degrees of freedom that equilibrate at the parent temperature $T_p$ and that carry memory of their equilibrium distribution when the glass goes out-of-equilibrium during a quench to lower temperatures. This physical picture strongly resembles the idea of a fictive/effective/configurational temperature, which was quite extensively used in models of the relaxation, flow and deformation of glasses [42–44, 59–62]. This connection is further strengthened in light of available evidence indicating that the cores of
deformation-coupled QLMs are the loci of irreversible plastic events that occur once a glass is driven by external forces [63–65].

The Boltzmann-like relation in (3), when interpreted in terms of STZs, is a cornerstone of the non-equilibrium thermodynamic STZ theory of the glassy deformation [42–44], where \( T_p \) is treated as a thermodynamic temperature that characterizes configurational degrees of freedom and that differ from the bath temperature. The strong depletion of STZs with decreasing \( T_p \), as predicted by the Boltzmann-like relation, was shown to give rise to a ductile-to-brittle transition in the fracture toughness of glasses [52, 53]. This prediction was recently supported by experiments on the toughness of Bulk Metallic Glasses (BMGs), where \( T_p \) was carefully controlled and varied [54].

It is natural to define a length scale corresponding to the typical distance between QLMs as \( \xi_s \sim N^{-1/d_s} \), once their number density \( N \) is at hand. Such a “site length” \( \xi_s \) was introduced in [11], where it was related to the sample-to-sample average minimal QLM frequency \( \langle \omega_{\text{min}} \rangle \) according to \( \langle \omega_{\text{min}} \rangle \sim \omega_g (L/\xi_s)^{d_s/5} \). The latter implies that the lowest QLM frequency is selected among \( (L/\xi_s)^d \times N \) possible candidates, which is directly related to the extreme value statistics of \( \omega_{\text{min}} \) [11]. The site length \( \xi_s \) is expected to control finite-size effects in studies of athermal plasticity in stable glasses, as discussed in detail in [66]. Similar definitions of a site length were proposed in [66, 67]; an important message here is that the disentangling of the stiffening effect from the prefactor \( A_4 \) is imperative for the purpose of obtaining a consistent definition of a length scale in such a setting.

A GLASSY LENGTH SCALE REVEALED BY PINCHING A GLASS

What additional physics can pinching a glass reveal? Up to now we explored the physics of the QLMs number density \( N \); we now turn to the other contribution to \( A_4 \), i.e. to the frequency scale \( \omega_g \) that characterizes the typical stiffness of QLMs. \( \omega_g \) was shown to undergo stiffening with decreasing \( T_p \) (cf. Fig. 3a); is this stiffening related to other properties of QLMs that vary with \( T_p \)? An interesting possibility we explore here is whether it might be related to a glassy length scale that is associated with QLMs.

To that aim we construct a length scale \( \xi_g \) out of \( \omega_g \) according to

\[
\xi_g \equiv c_s/\omega_g ,
\]

where \( c_s \) is the shear wave-speed. This length is similar in spirit to the “boson peak” length \( \xi_{BP} \equiv c_s/\omega_{BP} \) [68]. The physical rationale behind our constructed length \( \xi_g \) is that the emerging length scale is expected to mark a crossover in the elastic response of a glass to a local pinch, as discussed below. In Fig. 5 we plot \( \xi_g \) vs. the parent temperature \( T_p \); we find that \( \xi_g \) decreases upon deeper annealing by approximately 40%, a manifestation of the modest stiffening of the macroscopic shear modulus compared to that of QLM (recall that \( c_s \) is proportional to the square root of the shear modulus). This decreasing length is of unique character amongst the plethora of glassy lengthscales previously put forward in the context of the glass transition, most of which are increasing functions of decreasing temperature or parent temperature [69–74].

In order to shed light on the physical meaning of \( \xi_g \), we consider also (i) the crossover length \( \xi_c \), as observed in the displacement response to local pinches, between an atomistic-disorder-dominated response at distances \( r \lesssim \xi_c \) from the perturbation, to the expected continuum behavior seen at \( r > \xi_c \), and (ii) the core size of QLMs, \( \xi_{\text{QLM}} \), which is known to decrease upon annealing [34, 39, 40], as is also illustrated graphically in Fig. 1. In Fig. 5 we directly compare between \( \xi_g \) and our measurements of \( \xi_c \) and \( \xi_{\text{QLM}} \) (see SI for details). These three lengthscales feature very similar relative variations with \( T_p \), strongly supporting their equivalence. Consequently, we propose that \( \xi_g \) provides a measure of the core size of QLMs, and in light of the suggested relation between the latter and STZs, also of the size of STZs.

Additional insight may be gained by invoking the relation — established in [41] — between \( \omega_g \) and the characteristic frequency \( \omega_c \) that emerges near the unjamming transition [47–49]. Indeed, in the unjamming scenario the length \( \xi_g \) (often denoted \( \ell_c \)) was shown to diverge upon approaching the unjamming point [46], and to mark the crossover between disorder-dominated responses near a local perturbation, and the continuum-like response observed in the far field, away from the perturbation. The same length was shown in [75] to characterize the core size of QLMs near the unjamming point of harmonic-sphere packings. In light of the results shown in Fig. 5, we hypothesize that the fundamental
crossover length — below which responses to local perturbations are microstructural/disorder-dominated, and above which responses to local perturbations follow the expected continuum-like behavior — is, in fact, $\xi_g$, which, in turn, we show to agree well with the size of QLMs.

**SUMMARY AND OUTLOOK**

In this work we have employed a computer glass model, which can be deeply annealed [76], to quantitatively study the variation of the properties of QLMs with the depth of annealing. Most notably, we calculated the variation of the number density, characteristic frequency and core size of QLMs with the parent temperature from which the glass is formed. This has been achieved by assuming that the characteristic frequency scale of QLMs can be estimated through the bulk-average response of a glass to a local pinch. This frequency scale, in turn, allowed us to disentangle the apparently inseparable effects of the depletion and stiffening of QLMs, which are both encoded in the prefactor of the universal $\omega^4$ vibrational density of states of QLMs.

We found that the number density of QLMs follows a Boltzmann-like factor, with the parent temperature — from which equilibrium configurations were vitrified — playing the role of the equilibrium temperature. Consequently, the parent temperature may be regarded as a non-equilibrium temperature that characterizes QLMs deep inside the glassy state. Furthermore, our analysis reveals that both the core size of QLMS, and the mesoscopic length scale that marks the crossover between atomistic-disorder-dominated responses near local perturbations, and continuum like responses far away from local perturbations, can be estimated using the characteristic frequency of QLMs — obtained by pinching the glass —, and the speed of shear waves.

Our results may have important implications for various basic problems in glass physics. We mention a few of them here; first, the Boltzmann-like distribution of the number density of QLMs may play a major role in theories of the relaxation, flow and deformation of glasses, and may support some existing approaches. Second, together with other available observations [34, 41, 75], our results may suggest that the boson peak frequency could be robustly probed by pinching glassy samples, instead of the more involved analysis required otherwise [7, 34]. Finally, the variation of the energy scale proportional to $\omega_g^2$ with annealing temperature appears to match very well the variation of activation barriers required to rationalize fragility measurements in laboratory glasses (compare Fig. 3a with Fig. 8 of [77]). If valid, our results appear to support elasticity-based theories of the glass transition [78–80], and indicate that QLMs play important roles in relaxation processes in deeply supercooled liquids [27]. We hope that these interesting investigation directions will be pursued in the near future.

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**Appendix A: Models and methods**

We employed a computer glass forming model in three dimensions, simulated using the swap Monte-Carlo method, explained e.g. in [76]. The model consists of soft repulsive spheres interacting via a $\propto r^{-10}$ pairwise potential (with $r$ denoting the distance between the centers of a pair of particles), enclosed in a fixed-volume box with periodic boundary conditions. The particles’ sizes are drawn from a distribution designed such that crystallization is avoided [76]. A comprehensive description of the model, and of all parameter choices, can be found in [81], including an important discussion about how we handled large sample-to-sample realization fluctuations of particle sizes that can arise in small system sizes due to the breadth of the employed particle size distribution. Ensembles of 10000 (10000) glassy samples were made for systems of $N = 2000$ ($N = 8000$) particles by instantaneously-quenching (to zero temperature) independent configurations equilibrated at various parent temperatures $T_p$. All data except for those shown in Fig. 5 were calculated using the smaller glasses. Lengths are expressed in terms of $a_0 \equiv (V/N)^{1/d}$ where $V$ is the system’s volume. All particles share the same mass $m$, which we set as our microscopic unit of mass. Frequencies are expressed in terms of $c_\infty/a_0$, where $c_\infty \equiv \sqrt{G_\infty/\rho}$ is the high-$T_p$ shear wave-speed, with $G_\infty$ denoting the high-$T_p$ sample-to-sample mean athermal shear modulus, and $\rho \equiv mN/V$ denotes the mass density. $T_p$ is expressed in terms of the crossover temperature $T_{\text{onset}}$, above which the sample-to-sample mean athermal shear modulus saturates to a high-temperature plateau, as shown in the inset of Fig. 3a and in [81]. In our model we find $G_\infty a_0^3/k_B T_{\text{onset}} \approx 17$.

[1] U. Buchenau, Y. M. Galperin, V. L. Gurevich, and H. R. Schober, Anharmonic potentials and vibrational localization in glasses, Phys. Rev. B 43, 5039 (1991).
[44] M. L. Falk and J. S. Langer, Deformation and failure of amorphous, solidlike materials, Annu. Rev. Condens. Matter Phys. 2, 353 (2011).
[45] M. L. Falk and J. S. Langer, Dynamics of viscoplastic deformation in amorphous solids, Phys. Rev. E 57, 7192 (1998).
[46] E. Lerner, E. DeGiuli, G. During, and M. Wyatt, Breakdown of continuum elasticity in amorphous solids, Soft Matter 10, 5085 (2014).
[47] C. S. O’Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Jamming at zero temperature and zero applied stress: The epitome of disorder, Phys. Rev. E 68, 011306 (2003).
[48] A. J. Liu and S. R. Nagel, The jamming transition and the marginally jammed solid, Annu. Rev. Condens. Matter Phys. 1, 347 (2010).
[49] M. van Hecke, Jamming of soft particles: geometry, mechanics, scaling and isostaticity, J. Phys.: Condens. Matter 22, 033101 (2010).
[50] M. Popovi´ c, T. W. J. de Geus, and M. Wyatt, Elastoplastic description of sudden failure in amorphous materials during quasistatic loading, Phys. Rev. E 98, 040901 (2018).
[51] M. Ozawa, L. Berthier, G. Biroli, A. Rosso, and G. Tarjus, Random critical point separates brittle and ductile yielding transitions in amorphous materials, Proc. Natl. Acad. Sci. U.S.A. 115, 6656 (2018).
[52] C. H. Rycroft and E. Bouchbinder, Fracture toughness of metallic glasses: Annealing-induced embrittlement, Phys. Rev. Lett. 109, 194301 (2012).
[53] M. Vasoya, C. H. Rycroft, and E. Bouchbinder, Notch fracture toughness of glasses: Dependence on rate, age, and geometry, Phys. Rev. Applied 6, 024008 (2016).
[54] J. Ketkaew, W. Chen, H. Wang, A. Datye, M. Fan, G. Pereira, U. D. Schwarz, Z. Liu, R. Yamada, W. Dmowski, M. D. Shattuck, C. S. O’Hern, T. Egami, E. Bouchbinder, and J. Schroers, Mechanical glass transition revealed by the fracture toughness of metallic glasses, Nat. Commun. 9, 3271 (2018).
[55] D. Khomenko, C. Scalliet, L. Berthier, D. R. Reichman, and F. Zamponi, Depletion of two-level systems in ultrastable computer-generated glasses, arXiv preprint arXiv:1910.11168 (2019).
[56] W. Ji, M. Popovi´ c, T. W. J. de Geus, E. Lerner, and M. Wyatt, Theory for the density of interacting quasilocalized modes in amorphous solids, Phys. Rev. E 99, 023003 (2019).
[57] W. Jie, T. De Geus, E. Agoritsas, M. Popovic, and M. Wyatt, in preparation (private communication).
[58] G. Kapteijns, W. Ji, C. Brito, M. Wyatt, and E. Lerner, Fast generation of ultrastable computer glasses by mini- mization of an augmented potential energy, Phys. Rev. E 99, 012106 (2019).
[59] A. Q. Tool, Relation between inelastic deformability and thermal expansion of glass in its annealing range*, J. Am. Ceram. Soc. 29, 240 (1946).
[60] O. S. Narayanaswamy, A model of structural relaxation in glass, J. Am. Ceram. Soc. 54, 491 (1971).
[61] C. A. Angell, K. L. Ngai, G. B. McKenna, P. F. McMillan, and S. W. Martin, Relaxation in glassforming liquids and amorphous solids, J. Appl. Phys. 88, 3113 (2000).
[62] J. C. Mauro, R. J. Loucks, and P. K. Gupta, Fictive temperature and the glassy state, J. Am. Ceram. Soc. 92, 75 (2009).
[63] C. Maloney and A. Lemaître, Universal breakdown of elasticity at the onset of material failure, Phys. Rev. Lett. 93, 195501 (2004).
[64] E. Lerner, Micromechanics of nonlinear plastic modes, Phys. Rev. E 93, 053004 (2016).
[65] Z. Schwartzman-Nowik, E. Lerner, and E. Bouchbinder, Anisotropic structural predictor in glassy materials, Phys. Rev. E 99, 060601 (2019).
[66] E. Lerner, I. Procaccia, C. Rainone, and M. Singh, Protocol dependence of plasticity in ultrastable amorphous solids, Phys. Rev. E 98, 063001 (2018).
[67] S. Karmakar, E. Lerner, and I. Procaccia, Direct estimate of the static length-scale accompanying the glass transition, Physica A 391, 1001 (2012).
[68] L. Hong, V. N. Novikov, and A. P. Sokolov, Dynamic heterogeneities, boson peak, and activation volume in glassforming liquids, Phys. Rev. E 83, 061508 (2011).
[69] J.-P. Bouchaud and G. Biroli, On the adam-gibbs-kirkpatrick-thirumalai-wolynes scenario for the viscosity increase in glasses, J. Chem. Phys. 121, 7347 (2004).
[70] A. Montanari and G. Semerjian, Rigorous inequalities between length and time scales in glassy systems, J. Stat. Phys. 125, 23 (2006).
[71] G. Biroli, J.-P. Bouchaud, A. Cavagna, T. S. Grigera, and P. Verrocchio, Thermodynamic signature of growing amorphous order in glass-forming liquids, Nature Physics 4, 771 EP (2008).
[72] G. M. Hocky, T. E. Markland, and D. R. Reichman, Growing point-to-set length scale correlates with growing relaxation times in model supercooled liquids, Phys. Rev. Lett. 108, 225506 (2012).
[73] C. P. Royall and S. R. Williams, The role of local structure in dynamical arrest, Phys. Rep. 560, 1 (2015).
[74] S. Karmakar, C. Dasgupta, and S. Sastry, Growing length scales and their relation to timescales in glass-forming liquids, Annu. Rev. Condens. Matter Phys. 5, 255 (2014).
[75] M. Shimada, H. Mizuno, M. Wyatt, and A. Ikeda, Spatial structure of quasilocalized vibrations in nearly jammed amorphous solids, Phys. Rev. E 98, 060901 (2018).
[76] A. Ninarello, L. Berthier, and D. Coslovich, Models and algorithms for the next generation of glass transition studies, Phys. Rev. X 7, 021039 (2017).
[77] G. Tarjus, D. Kivelson, S. Mossa, and C. Alba-Simionesco, Disentangling density and temperature effects in the viscous slowing down of glassforming liquids, J. Chem. Phys. 120, 6135 (2004).
[78] J. C. Dyre, T. Christensen, and N. B. Olsen, Elastic models for the non-arrhenius viscosity of glass-forming liquids, J. Non-Cryst. Solids 352, 4635 (2006).
[79] C. Brito, E. Lerner, and M. Wyatt, Theory for swap acceleration near the glass and jamming transitions for continuously polydisperse particles, Phys. Rev. X 8, 031050 (2018).
[80] M. Wyatt and M. E. Cates, Does a growing static length scale control the glass transition?, Phys. Rev. Lett. 119, 195501 (2017).
[81] E. Lerner, Mechanical properties of simple computer glasses, J. Non-Cryst. Solids 522, 119570 (2019).
Supporting Information for: “What can be learnt from pinching a glass?”

In this Supporting Information we provide information about (i) how the bulk-average frequency of the response to a local pinch of the glass, denoted $\omega_g$ in the manuscript, was calculated, (ii) how the modes in Fig. 1 of the main text were calculated, and their size estimated, and (iii) how we estimated the crossover length $\xi_{crossover}$ and the QLMS core size $\xi_{QLMS}$, both appearing in Fig. 5 of the main text.

We recall that lengths are expressed in terms of $a_0 \equiv V/N$ where $V$ is the system’s volume, and $N$ denotes the number of particles. All particles share the same mass $m$, which we set as our microscopic unit of mass. Frequencies are expressed in terms of $c_\infty/a_0$, where $c_\infty \equiv \sqrt{G_\infty/\rho}$ is the high-$T_p$ shear wave-speed, with $G_\infty$ denoting the high-$T_p$ plateau of sample-to-sample mean athermal shear modulus of inherent states (see inset of Fig. 3a of main text), and $\rho \equiv mN/V$ denotes the mass density. Temperatures are expressed in terms of $\xi\equiv\infty^2/\rho$.

We next define the conditional average

$$\rho \equiv \langle \omega_g^{ij} \rangle_{f_{ij}/(p a_0^2) < 10^{-2}} ,$$

where $p$ is the glass pressure (recall that in our computer glass particles interact via a purely repulsive pairwise interaction), and the triangular brackets denote an average taken over all interacting pairs $i,j$ for which the dimensionless pairwise force $f_{ij}/(p a_0^2) < 10^{-2}$.

The reason we chose to only consider weak forces in the estimation of $\omega_g$ can be understood by scatter-plotting $\omega_g^{ij}$ vs. the dimensionless pairwise force $f_{ij}/(p a_0^2)$, calculated in an ensemble of glassy solid quenched from $T_p = 5/9$. Results for other parent temperatures have similar forms. At strong forces the stiffnesses associated with responses to local pinches are substantially higher compared to those associated with weak forces. We find a saturation of the statistics of $\omega_g^{ij}$ at $f_{ij}/(p a_0^2) < 10^{-2}$, marked by the vertical line. (b) & (c) Comparison between the mean frequencies $\omega_g$ calculated with and without filtering by the pairwise forces; the two means differ by $\approx 40\%$ consistently throughout the sampled temperature range.

**FIG. S1.** (a) Scatter plot of $\omega_g^{ij}$ vs. the dimensionless pairwise force $f_{ij}/(p a_0^2)$, calculated in an ensemble of glassy solid quenched from $T_p = 5/9$. Results for other parent temperatures have similar forms. At strong forces the stiffnesses associated with responses to local pinches are substantially higher compared to those associated with weak forces. We find a saturation of the statistics of $\omega_g^{ij}$ at $f_{ij}/(p a_0^2) < 10^{-2}$, marked by the vertical line. (b) & (c) Comparison between the mean frequencies $\omega_g$ calculated with and without filtering by the pairwise forces; the two means differ by $\approx 40\%$ consistently throughout the sampled temperature range.

The reason we chose to only consider weak forces in the estimation of $\omega_g$ can be understood by scatter-plotting $\omega_g^{ij}$ vs. $f_{ij}/(p a_0^2)$, as seen in Fig. S1a. We can clearly see that two families of frequencies are generated by pinching pairs between which strong or weak forces are found. In particular, strongly-interacting pairs tend to generate much stiffer responses (note the logarithmic y-axis). Since these responses are supposed to represent soft, quasi-localized modes, we opt for filtering the responses according to the dimensionless forces $f_{ij}/(p a_0^2)$. Below the chosen threshold $f_{ij}/(p a_0^2) < 10^{-2}$, that can be clearly read off the scatter plot Fig. S1 (vertical yellow line), the statistics of $\omega_g^{ij}$ appears to saturate.

In Fig. S1b,c we examine the effect of filtering interactions by their force on the $T_p$ dependence of $\omega_g$. We see that the relative variation of the two mean frequencies is very similar throughout the sampled parent-temperature range.
S-2. Calculation of soft modes in 2D

In this section we describe how the modes shown in Fig. 1 of the main text were calculated. We employed the two-dimensional version of the same computer glass model used for our study; details about the model can be found in [S3]. Ensembles of glassy samples were quenched from equilibrium parent temperatures of \( T_p = 7/9 \) (expressed in terms of the onset temperature \( T_{\text{onset}} \) as described above) and \( T_p = 17/90 \). We followed the framework put forward in [S4], and calculated solutions\( \pi \) to the equation

\[
\mathcal{H} \cdot \pi = \frac{\pi \cdot \mathcal{H} \cdot \pi}{\partial^4 U / \partial x \partial x \partial x \partial x} \cdot \pi \pi \pi \pi, \tag{S6}
\]

where triple and quadruple contractions are denoted as \( \cdot \) and \( :: \), respectively, and particle indices were suppressed for simplicity. Solutions \( \pi \) to Eq. (S6) were coined ‘quartic modes’ \([S2]\); they resemble soft quasilocalized vibrations which resemble low-frequency quasilocalized vibrations seen below or in between phonon bands \([S4]\), i.e. in the absence of hybridizations with phonons. Solutions to Eq. (S6) were calculated by employing a standard nonlinear conjugate gradient minimization algorithm to find local minima of the cost function \([S4]\)

\[
\mathcal{G}(z) = \frac{(z \cdot \mathcal{H} \cdot z)^2}{\partial^4 U / \partial x \partial x \partial x \partial x} \cdot z z z z, \tag{S7}
\]

where \( z \) represents a displacement field in the \( N \times d \) dimensional configuration space of the glass. It is straightforward to show (see further details in [S4]) that minima of \( \mathcal{G} \) correspond to solutions of Eq. (S6). Initial conditions for the minimization of \( \mathcal{G} \) were obtained by calculating the linear displacement response to a dipolar force, as given by Eq. (S2), for every pair of interacting particles in the glass. In Fig. 1 of the main text, we only show modes \( \pi \) for which \( \omega_\pi = \sqrt{\mathcal{H} \cdot \pi / \pi} < \omega_g / 3 \), where \( \omega_g \) was calculated as described in the previous section. The area of the disordered core of each of the calculated modes was estimated as \( N e \), with \( e \) denoting the participation ratio of the modes, defined for a normalized mode \( \mathbf{z} \) as

\[
e = (N \sum_i (\hat{z}_i \cdot \mathbf{z}_i)^2)^{-1}. \]

The participation ratio is a proxy for the degree of localization of a mode; in particular, for a localized mode one expects \( e \sim O(1/N) \), whereas a spatially-extended mode would give \( e \sim O(1) \). In Fig. 1 of the main text, the area of each blob that represents a soft mode is proportional to \( N e \), and its color represents its frequency \( \omega_\pi \), with dark (bright) colors representing softer (stiffer) modes.

S-3. Estimation of the crossover length

In this section we describe how we measured the crossover length \( \xi_{co} \) between disorder-dominated responses near a local perturbation to continuum-

Eshelby-like algebraic decays away from a local perturbation. The crossover lengths \( \xi_{co} \) extracted from the following analysis are shown in Fig. 5 of the main text.

![Fig. S2](https://example.com/fig_s2.png)

**Fig. S2.** (a) Decay functions \( c(r) \) of the response to local pinches, see text for precise definition. The different curves correspond to measurements performed on glassy samples quenched from \( T_p = 13/9, 10/9, 8/9, 7/9, 13/18, 2/3, 11/18, 5/9, 1/2, 4/9, 7/18 \), from warm to cold colors. (b) Rescaling \( c(r) \) by \( r^{-6} \) allows to robustly identify the crossover length \( \xi_{co} \) between disorder-dominated to continuum-like scaling. (c) & (d) The crossover lengths \( \xi_{co} \) and the factors \( A_1 \) used to collapse the curves in panel (b), plotted against the parent temperature \( T_p \).

In order to estimate the crossover length, we follow the measurement scheme of \([S5]\); this amounts to calculating the response to local dipoles via Eq. (S2), still following the dimensionless-force filtering scheme discussed above. The fields are then normalized, namely for every pair \( ij \) considered, we calculate \( \tilde{u}^{(ij)} = u^{(ij)} / |u^{(ij)}| \). Then, for each interaction \( kl \neq ij \), we compute the square of the projection of the normalized response \( \tilde{u}^{(ij)} \) onto the normalized dipole vector \( d^{(kl)} = d^{(kl)} / |d^{(kl)}| \), i.e. we calculate

\[
C_{ij,kl} \equiv (\tilde{u}^{(ij)} \cdot d^{(kl)})^2. \tag{S8}
\]

\( C_{ij,kl} \) generally depends on the distance \( r_{ij,kl} \) between the interactions \( ij \) and \( kl \), and on their relative orientation.

For each normalized response field \( \tilde{u}^{(ij)} \), we bin \( C_{ij,kl} \) — calculated for all pairs \( kl \neq ij \) — over the distances \( r_{ij,kl} \), and calculate the median of \( C_{ij,kl} \) over all pairs \( kl \) located at similar distances \( r \) away from the excited dipole \( d^{(ij)} \); the average over the excited dipole \( ij \), and over glassy samples, denoted below by \( \langle \cdot \rangle_{ij} \), defines the decay function \( c(r) \), namely

\[
c(r) \equiv \langle \text{median}_{r_{ij,kl} \leq r} (C_{ij,kl}) \rangle_{ij}. \tag{S9}
\]

The decay functions \( c(r) \) are plotted in Fig. S2a. Continuum elasticity would predict that \( c(r) \sim r^{-2d} \) \([S5]\). We
therefore plot in Fig. S2b the rescaled decay functions \( r^6 c(r)/A_1 \) against the rescaled distance \( r/\xi_{\text{QLM}} \) with \( \xi_{\text{QLM}}(T_p) \) denoting the parent-temperature dependent crossover lengths, chosen to collapse the data, as are the constants \( A_1(T_p) \) reported in Fig. S2d. The crossover lengths \( \xi_{\text{QLM}}(T_p) \) are plotted against the parent temperature \( T_p \) in Fig. S2c.

**S-4. Estimation of QLMs core length**

In stable glasses, it becomes difficult to sample many QLMs using a harmonic analysis due to their stiffening and depletion, discussed in length in the main text. As a result of these processes, characteristic frequencies of the softest QLMs tend to overlap with the lowest phonon frequencies, leading to hybridizations of phonons and QLMs, and obscuring a clear picture of QLMs properties and statistics, as demonstrated in Fig. S3.

In order to reveal the properties of QLMs for glasses quenched from all parent temperatures, including in stable glasses, we opt for calculating ‘quartic modes’ as representatives of QLMs, since the former are known to be indifferent to the presence of phonons with comparable frequencies (they show no hybridizations with phonons, as shown in [S4] and in the left panel of Fig. S3). At the same time, quartic modes feature frequencies that are in excellent agreement with QLMs’ frequencies in the absence of hybridizations [S4], as can also be seen in Fig. S3.

We first generated, for each of our glassy samples of \( N=8000 \) particles, a quartic mode as discussed in length the main text. As a result of these processes, characteristic frequencies of the softest QLMs tend to overlap with the lowest phonon frequencies, leading to hybridizations of phonons and QLMs, and obscuring a clear picture of QLMs properties and statistics, as demonstrated in Fig. S3.

In order to reveal the properties of QLMs for glasses quenched from all parent temperatures, including in stable glasses, we opt for calculating ‘quartic modes’ as representatives of QLMs, since the former are known to be indifferent to the presence of phonons with comparable frequencies (they show no hybridizations with phonons, as shown in [S4] and in the left panel of Fig. S3). At the same time, quartic modes feature frequencies that are in excellent agreement with QLMs’ frequencies in the absence of hybridizations [S4], as can also be seen in Fig. S3.

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We first generated, for each of our glassy samples of \( N=8000 \) particles, a quartic mode as discussed in length in Sect. S-0S-2. In this case, however, the initial conditions for finding quartic modes were chosen to be the linear displacement responses to the forces that arise due to imposing simple and pure shear [S6] in all possible Cartesian planes (i.e. \( \hat{x} - \hat{y}, \hat{x} - \hat{z}, \ldots \)). Each such linear response is then used as the initial condition for the minimization of the cost function \( G \), c.f. Eq. (S7). An ensemble of QLMs is constructed by only keeping and considering the QLM \( \pi \) with the smallest frequency \( \omega_\pi = \sqrt{\pi \cdot H \cdot \pi}/\sqrt{m} \) amongst all those calculated for each individual sample, leaving us with 1000 soft QLMs per parent temperature \( T_p \).

In order to demonstrate the utility of quartic modes for the assessment of the core size of QLMs, we scatter-plot in Fig. S3 the participation ratio of both harmonic modes (obtained by a partial diagonalization of the Hessian of the potential energy), and quartic modes (obtained as described above). We show that, at the very lowest frequencies, each harmonic mode overlaps with a quartic mode that our calculation produces, demonstrating that our calculation captures well the QLM away from regimes of strong hybridizations with phonons. These data show that harmonic and quartic mode share very similar localization properties and frequencies, as also discussed in length in [S4], which motivates employing quartic modes as faithful representitives of QLM.

**FIG. S3.** Scatter plot of the participation ratio \( c — \) that quantifies the degree of localization of a mode — vs. frequency, calculated for harmonic (black symbols) and quartic (red symbols) modes in glassy samples of \( N=8000 \) particles quenched from \( T_p = 7/18 \) (left) and \( T_p = 13/9 \) (right). In the analyzed stable glassy samples \( (T_p = 7/18, \text{left panel}) \) phonons and QLMs dwell at similar frequencies, leading to their hybridizations. Quartic modes appear to be entirely indifferent to the presence of these phonons (see left panel).

**FIG. S4.** (a) Decay functions \( c(r) \) of QLMs calculated as explained in this SI. The different curves correspond to measurements performed on glassy samples quenched from the same parent temperatures \( T_p \), as spelled out in the caption of Fig. S2. (b) Rescaling \( c(r) \) by \( r^{-6} \) allows to robustly identify the QLMs linear core size \( \xi_{\text{QLM}} \). (c) & (d) The QLM core length \( \xi_{\text{QLM}} \) and the factors \( A_2 \) used to collapse the curves in panel (b), plotted against the parent temperature \( T_p \).

In order to estimate the linear size of the cores of QLM, each calculated QLM \( \pi \) as described above was normalized \( \bar{\pi} = \pi/|\pi| \); we then identified the pair \( ij \) of interacting particles that maximizes the difference squared \( |\bar{\pi}_i - \bar{\pi}_j|^2 \), and consider this pair as the center of the QLM’s core. We calculated the spatial decay \( c(r) \) of QLMs similarly to the procedure explained in the previous Section for analyzing the spatial decay of the response to a local pinch, with the only differences being
that here that $r$ represents the distance from the aforementioned pair $ij$, and

$$C_{ij,kt} \equiv (\hat{\pi} \cdot \hat{d}^{(kt)})^2,$$  \hspace{1cm} (S10)

$$c(r) \equiv \langle \text{median}_{r_{ij,kt} \leq r} \left( C_{ij,kt} \right) \rangle_{\text{QLMs}}, \hspace{1cm} (S11)$$

where the average is taken over all calculated QLMs.

The results of this calculation are shown in Fig. S4, see figure caption for further details. The lengths $\xi_{\text{QLM}}$ extracted from our analysis are shown in Fig. S4c, and used in Fig. 5 of the main text.

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[S1] E. Lerner, Mechanical properties of simple computer glasses, J. Non-Cryst. Solids 522, 119570 (2019).

[S2] E. Lerner and E. Bouchbinder, A characteristic energy scale in glasses, J. Chem. Phys. 148, 214502 (2018).

[S3] A. Moriel, G. Kapteijns, C. Rainone, J. Zylberg, E. Lerner, and E. Bouchbinder, Wave attenuation in glasses: Rayleigh and generalized-rayleigh scattering scaling, J. Chem. Phys. 151, 104503 (2019).

[S4] L. Gartner and E. Lerner, Nonlinear modes disentangle glassy and Goldstone modes in structural glasses, SciPost Phys. 1, 016 (2016).

[S5] E. Lerner, E. DeGiuli, G. During, and M. Wyart, Breakdown of continuum elasticity in amorphous solids, Soft Matter 10, 5085 (2014).

[S6] C. E. Maloney and A. Lemaître, Amorphous systems in athermal, quasistatic shear, Phys. Rev. E 74, 016118 (2006).