THE TWO-DISTANCE SETS IN DIMENSION FOUR

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1. Introduction and Main Results

Let \( d \geq 1 \) be an integer, and let \( \mathbb{R}^d \) denote the \( d \)-dimensional Euclidean space equipped with the standard inner product \((\,,\,)\) and norm induced metric \( \mu \). Following the terminology of \([10]\), a Euclidean representation of a simple graph \( \Gamma \) on \( n \geq 1 \) vertices is an embedding \( f \) (with real parameters \( \alpha_2 > \alpha_1 > 0 \)) of the vertex set of \( \Gamma \) into \( \mathbb{R}^d \) such that for different vertices \( u \neq v \) we have \( \mu(f(u), f(v)) = \alpha_1 \) if and only if \( \{u, v\} \) is an edge of \( \Gamma \), and \( \mu(f(u), f(v)) = \alpha_2 \) otherwise. The smallest \( d \) for which such a representation exists is denoted by \( \dim_2 \Gamma \). If \( \Gamma \) is neither complete, nor empty, then its image under \( f \) is called an \( n \)-element 2-distance set \([2], [4], [6], [7], [11], [10]\). The representation, as well as the 2-distance set is called spherical, if the image of \( f \) lies on the \((d-1)\)-sphere of radius 1 in \( \mathbb{R}^d \) \([8], [9]\). A spherical representation is called J-spherical \([10]\ Definition 4.1), if \( \alpha_1 = \sqrt{2} \). Graphs on \( n \geq d + 2 \) vertices having a J-spherical representation in \( \mathbb{R}^d \) are in a certain sense extremal \([13]\). We remark that other authors relax the condition \( \alpha_2 > \alpha_1 \) thus essentially identifying the same 2-distance set with a graph \( \Gamma \) and its complement \( \overline{\Gamma} \) \([12], [14]\).

Motivated by a recent problem posed in \([10], Section 4.3\), we continue the computer-aided generation and classification of 2-distance sets in Euclidean spaces \([15]\), a program initiated originally in \([7]\). In particular, we describe the 2-distance sets in \( \mathbb{R}^4 \), that is, we determine all simple graphs \( \Gamma \) with \( \dim_2 \Gamma = 4 \). Since all such graphs are known on at most 6 vertices \([4], [10]\), and it is known that there are no such graphs on more than 10 vertices \([7]\), the aim of this note is to close this gap by classifying the graphs in the remaining cases. The main result is the following.

Theorem 1. The number of \( n \)-element 2-distance sets in \( \mathbb{R}^4 \) for \( n \in \{7, 8, 9\} \) is 33, 20, and 5 up to isometry.

The proof is in part computational, and easily follows from the theory developed earlier in \([15]\), which we briefly outline here for completeness as follows. Assume that \( \Gamma \) is a graph with vertices \( v_1, \ldots, v_n \). Let \( a \) and \( b \) be indeterminates, and associate to \( \Gamma \) a “candidate Gram matrix” \( G(a, b) := aA(\Gamma) + bA(\overline{\Gamma}) + I \), where \( A(\Gamma) \) is the graph adjacency matrix, and \( I \) is the identity matrix of order \( n \). Now let \( f \) be a spherical representation of \( \Gamma \) (with parameters \( \alpha_2 > \alpha_1 > 0 \) as usual) in \( \mathbb{R}^d \). Then the Gram matrix of the representation can be written as \( \langle (f(v_i), f(v_j))_{i,j=1}^n = G(1 - \alpha_1^2/2, 1 - \alpha_2^2/2) \). This correspondence allows us to construct a representation based on solely \( A(\Gamma) \) by exploiting that the rank of \( G(1 - \alpha_1^2/2, 1 - \alpha_2^2/2) \) is at most \( d \). Indeed, if we are given a candidate Gram matrix \( G(a,b) \), then those values \( a^*, b^* \in \mathbb{C} \) for which \( G(a^*, b^*) \) has a certain rank can be found by considering the set of \((d+1) \times (d+1)\) minors of \( G(a,b) \) which should all be vanishing. The arising system of polynomial equations can be analyzed by a standard Gröbner basis computation \([11], [3]\), as detailed in \([15]\). In particular, if no common solutions are found, then the candidate Gram matrix (as well as both \( \Gamma \) and its complement) should be discarded as it cannot correspond to a spherical-2-distance set in \( \mathbb{R}^d \). On the other hand, if some solutions are found, then the candidate Gram matrix survives the test, and one should further ascertain that \( G(a^*, b^*) \) is a positive semidefinite matrix. This can be done by investigating the signs of the coefficients of its characteristic polynomial \([4] Corollary 7.2.4\).

The general case (i.e., when \( f \) is not necessarily spherical) is analogous, but slightly more technical as the image of \( f \) should be translated to the origin first, and then the Gram matrix of this shifted set (which is sometimes called Menger’s matrix) should be considered \([7] Section 7.1\), \([15] Section 4\). In particular, we have

\[
(f(v_i) - f(v_n), f(v_j) - f(v_n)) = (G(\alpha_1^2, \alpha_2^2),_{i,n} + G(\alpha_1^2, \alpha_2^2),_{j,n} - G(\alpha_1^2, \alpha_2^2),_{i,j} + I_{ij})/2, \quad i, j \in \{1, \ldots, n-1\}.
\]

The right hand side describes the entries of a positive semidefinite matrix of rank at most \( d \), which depends on \( A(\Gamma) \) only. This rank condition can be treated in a similar way as discussed previously. In Table \([1]\) we summarize the number of surviving candidate Gram matrices found by a simple backtrack search, and the number of corresponding 2-distance sets. The entry marked by an asterisk indicates that 6 out of the 42 cases are actually the maximum 2-distance sets in \( \mathbb{R}^3 \), see \([3] Section 10\). The proof of Theorem \([1]\) can be obtained by setting \( d = 4 \) and then analyzing one by one the surviving candidate Gram matrices and the corresponding graphs on \( n \in \{7, 8, 9\} \) vertices.

It is known that the maximum cardinality of a 2-distance set in \( \mathbb{R}^4 \) is exactly 10, and the unique configuration realizing this corresponds to the triangular graph \( T(5) \), see \([7]\). We have verified this result independently. Indeed,

June 21, 2018, preprint. This research was supported in part by the Academy of Finland, Grant #289002.

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our computer program identified a single $10 \times 10$ candidate Gram matrix, which cannot be extended any further, and whose spherical representation is shown in Table 2. While the subgraphs of $T(5)$ are obviously spherical 2-distance sets embedded in $\mathbb{R}^4$, there are several additional examples as follows.

**Proposition 2.** The number of 9-point 2-distance sets in $\mathbb{R}^4$ is 5, out of which 2 are spherical.

**Proof.** Our computer program generated 4 candidate Gram matrices in the general case, and 2 in the spherical case, see Table 1. The two spherical cases correspond to the 9-vertex subgraph of $T(5)$, and to the Paley-graph, see Table 2. The remaining two candidate Gram matrices correspond to three nonspherical 2-distance sets, see Table 6.

**Proposition 3.** The number of 8-point 2-distance sets in $\mathbb{R}^4$ is 20, out of which 7 are spherical.

**Proof.** Our computer program generated 13 candidate Gram matrices in the general case, and 6 in the spherical case, see Table 1. In the spherical case five out of the six candidate Gram matrices yielded one spherical 2-distance set, while one resulted in two, see Table 2. The remaining 7 candidate Gram matrices correspond to two nonspherical 2-distance sets each, except for the one which is self-complementary. See Table 6 for some details.

| $n$ | $G(n, b)$ | $G(n, 0)$ | $(a^*, b^*)$ | Remark |
|-----|-----------|-----------|--------------|--------|
| 10  | aaaaaaaa  | $\Gamma_{10, A}$ | $(1, 6, -2/3)$ | $T(5)$, dim$_2(\Gamma_{10, A}) = 5$ |
| 9   | aaaaaaaa  | $\Gamma_B$ | $(1, 6, -2/3)$ | $\Gamma_B \sim T_{10, A} \setminus \{\ast\}$, dim$_2(\Gamma_B) = 5$ |
| 9   | aaaaaaaa  | $\Gamma_{D2}$ | $(1, 4, -1/2)$ | Paley, self-complementary |
| 8   | aaaaaaaa  | $\Gamma_A$ | $(0, -1)$ | 16-cell, spherical, dim$_2(\Gamma_A) = 7$ |
| 8   | aaaaaaaa  | $\Gamma_{K4}$ | $(1 \pm 4\sqrt{2}/9, 1/4, 1/2)$ | |
| 7   | aaaaaaaa  | $\Gamma_{K3}$ | $(1, 6, -2/3)$ | dim$_2(\Gamma_{K3}) = 5$ |
| 7   | aaaaaaaa  | $\Gamma_{K1}$ | $(1/3, 3/5)$ | dim$_2(\Gamma_{K1}) = 6$ |
| 6   | aaaaaaaa  | $\Gamma_{E}$ | $(1, 6, -2/3)$ | $\Gamma_{K3} \sim T_{10, A} \setminus \{\ast\}$, dim$_2(\Gamma_{E}) = 5$ |
| 5   | aaaaaaaa  | $\Gamma_{F}$ | $(1, 4, -1/2)$ | $\Gamma_{D2} \sim T_{10, A} \setminus \{\ast\}$, self-complementary |

Table 2. Spherical 2-distance sets on $n \in \{8, 9, 10\}$ points in $\mathbb{R}^4$.

In the tables the vectorization (i.e., row-wise concatenation) of the lower triangular part of a graph adjacency matrix of order $n$ is denoted by a string of letters $a$ and $b$ of length $n(n - 1)/2$, where letter $a$ indicates adjacent vertices.

**Proposition 4.** The number of 7-point 2-distance sets in $\mathbb{R}^4$ is 33, out of which 23 are spherical.

**Proof.** Our computer program generated 22 candidate Gram matrices in the general case, and 17 in the spherical case, see Table 1. In the spherical case there was a single matrix which did not correspond to any 2-distance sets as it turned out to be indefinite. All the other candidate Gram matrices yielded at least one spherical 2-distance set, see Table 3. The remaining 5 candidate Gram matrices correspond to two nonspherical 2-distance sets each, see Table 6.

| $n$ | $G(n, b)$ | $G(n, 0)$ | $(a^*, b^*)$ | Remark |
|-----|-----------|-----------|--------------|--------|
| 7   | aaaaaaaa  | $\Gamma_{12, A}$ | $(0, -1)$ | J-spherical, dim$_2(\Gamma_{12, A}) = 6$ |
| 7   | aaaaaaaa  | $\Gamma_{12, B}$ | $(1 \pm 4\sqrt{2}/9, 1/4, 1/2)$ | |
| 7   | aaaaaaaa  | $\Gamma_{12, C}$ | $(1 \pm 4\sqrt{2}/9, 1/4, 1/2)$ | dim$_2(\Gamma_{12, C}) = 5$ |
| 7   | aaaaaaaa  | $\Gamma_{12, D}$ | $(1, 6, -2/3)$ | dim$_2(\Gamma_{12, D}) = 5$ |
| 7   | aaaaaaaa  | $\Gamma_{16, D}$ | $(1, 4, -1/2)$ | dim$_2(\Gamma_{16, D}) = 5$ |
| 7   | aaaaaaaa  | $\Gamma_{16, E}$ | $(1/3, 3/5)$ | dim$_2(\Gamma_{16, E}) = 6$ |
| 6   | aaaaaaaa  | $\Gamma_{20, K}$ | $(-1 - \sqrt{7}/8, 3(-1 + \sqrt{7})/8)$ | dim$_2(\Gamma_{20, K}) = 5$ |
| 6   | aaaaaaaa  | $\Gamma_{20, L}$ | $(1, 6, -2/3)$ | dim$_2(\Gamma_{20, L}) = 5$ |
| 5   | aaaaaaaa  | $\Gamma_{20, M}$ | $(1 \pm 4\sqrt{2}/9, 1/4, 1/2)$ | dim$_2(\Gamma_{20, M}) = 5$ |
| 5   | aaaaaaaa  | $\Gamma_{20, N}$ | $(-5/12, 7/24)$ | dim$_2(\Gamma_{20, N}) = 5$ |
| 5   | aaaaaaaa  | $\Gamma_{24, D}$ | $(-1 \pm 3)/8, (1 \mp 3)/8, (1 \mp 3)/8$ | $8n^2 + 32n + 10n - 1 = 0, |n^2| \leq 1$ |

Table 3. Spherical 2-distance sets on $n = 7$ points in $\mathbb{R}^4$.

**Proposition 5** (cf. [4, p. 494], [10 Section 4.3]). The number of 6-point 2-distance sets in $\mathbb{R}^4$ is 145. The number of 6-point spherical 2-distance sets in $\mathbb{R}^4$ is 42, out of which 6 are in fact the maximum 2-distance sets in $\mathbb{R}^3$.
Proof. It is known, see [3, 10], that a graph on 6 vertices can be represented in \( \mathbb{R}^4 \) unless it is a disjoint union of cliques. Since the total number of simple graphs on 6 vertices is 156, out of which 11 are disjoint union of cliques, we find that 145 graphs can be represented in \( \mathbb{R}^4 \). Our computer program generated 30 candidate Gram matrices in the spherical case, see Table 1. There were two indefinite matrices, and the remaining 28 resulted in at least one spherical 2-distance set each. Amongst these, we found the 6 maximum 2-distance sets in \( \mathbb{R}^3 \), denoted by \( \Gamma_{6K}, \Gamma_{6Q}, \Gamma_{6O}, \Gamma_{6B}, \Gamma_{6R}, \Gamma_{6Y} \), see Table 4.

Finally, there are 7 graphs \( \Gamma \) on 5 vertices for which \( \dim_2 \Gamma = 4 \). One particular spherical representation is given of these in Table 5. The number of corresponding nonsymmetric 2-distance sets in \( \mathbb{R}^4 \) in these cases is infinite.

**Table 4.** Spherical 2-distance sets on \( n = 6 \) points in \( \mathbb{R}^4 \)

| \( n \) | \( G(n, 6) \) | \( G(1, 6) \) | \( (a^*, b^*) \) | Remark |
|---|---|---|---|---|
| 6 | \( \Gamma_{6A} \) | \( (0, 1) \) | J-spherical, \( \dim_2(\Gamma_{6A}) = 5 \) |
| 6 | \( \Gamma_{6B} \) | \( (1 \pm \sqrt{5})/4, 1/2 \) | J-spherical, \( \dim_2(\Gamma_{6B}) = 5 \) |
| 6 | \( \Gamma_{6C} \) | \( (1/6, 1/3, 3) \) | J-spherical, \( \dim_2(\Gamma_{6C}) = 4 \) |
| 6 | \( \Gamma_{6D} \) | \( (1/6, 2/3) \) | J-spherical, \( \dim_2(\Gamma_{6D}) = 4 \) |
| 6 | \( \Gamma_{6E} \) | \( (1 \pm \sqrt{5})/4, 1/2 \) | J-spherical, \( \dim_2(\Gamma_{6E}) = 4 \) |
| 6 | \( \Gamma_{6F} \) | \( (1/1, 1/2) \) | J-spherical, \( \dim_2(\Gamma_{6F}) = 4 \) |
| 6 | \( \Gamma_{6G} \) | \( (1, 1/3) \) | J-spherical, \( \dim_2(\Gamma_{6G}) = 4 \) |

**Table 5.** Spherical 2-distance sets on \( n = 5 \) points in \( \mathbb{R}^4 \)

| \( n \) | \( G(n, 5) \) | \( G(1, 5) \) | \( (a^*, b^*) \) | Remark |
|---|---|---|---|---|
| 5 | \( \Gamma_{5A} \) | \( a^* = -1/4 \) | regular 5-cell, \( \dim_2(\Gamma_{5A}) = 4 \) | 1-distance set |
| 5 | \( \Gamma_{5B} \) | \( (1 \pm \sqrt{5})/6, 0 \) | J-spherical, \( \dim_2(\Gamma_{5B}) = 3 \) |
| 5 | \( \Gamma_{5C} \) | \( (0, -1/2) \) | J-spherical, \( \dim_2(\Gamma_{5C}) = 3 \) |
| 5 | \( \Gamma_{5D} \) | \( (0, -1/2) \) | J-spherical, \( \dim_2(\Gamma_{5D}) = 3 \) |
| 5 | \( \Gamma_{5E} \) | \( (1 \pm \sqrt{5})/6, 0 \) | J-spherical, \( \dim_2(\Gamma_{5E}) = 3 \) |

**Table 6.** General (nonsymmetric) 2-distance sets on \( n \in \{7, 8, 9\} \) points in \( \mathbb{R}^4 \)

| \( n \) | \( G(n, 6) \) | \( G(1, 6) \) | \( (a^*, b^*) \) | Remark |
|---|---|---|---|---|
| 7 | \( \Gamma_{7A} \) | \( (1, 3 \pm \sqrt{5})/2 \) | J-spherical, \( \dim_2(\Gamma_{7A}) = 3 \) |
| 7 | \( \Gamma_{7B} \) | \( (1, 3 \pm \sqrt{5})/2 \) | J-spherical, \( \dim_2(\Gamma_{7B}) = 3 \) |
| 7 | \( \Gamma_{7C} \) | \( (1, 3 \pm \sqrt{5})/2 \) | J-spherical, \( \dim_2(\Gamma_{7C}) = 3 \) |
| 7 | \( \Gamma_{7D} \) | \( (1, 3 \pm \sqrt{5})/2 \) | J-spherical, \( \dim_2(\Gamma_{7D}) = 3 \) |

Corollary 6. The number of graphs \( \Gamma \) for which \( \dim_2 \Gamma = 4 \) is 211.

**Proof.** This follows from earlier results in [4, 7], and Theorem 11 the number of such graphs on \( n \in \{5, 6, 7, 8, 9, 10\} \) vertices is 7, 145, 33, 20, 5, and 1, respectively, and there are no such graphs on \( n < 5 \) or \( n > 10 \) vertices.

We conclude this manuscript with the following remark: the classification of the maximum 3-distance sets in \( \mathbb{R}^4 \) has recently been carried out in [15], and therefore data on the (not necessarily largest) candidate Gram matrices is readily available for that case too (see [15, Table 5 and 7]). However, the individual analysis and ultimately the presentation of those tens of thousands of matrices would require considerably more efforts.

Finally, we introduce \( \Gamma \) on 6 vertices for which \( \dim_2 \Gamma = 4 \). One particular spherical representation is given of these in Table 5. The number of corresponding nonsymmetric 2-distance sets in \( \mathbb{R}^4 \) in these cases is infinite.
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