Textile D-forms and $D_{4d}$

Katherine A. Seaton

Department of Mathematics and Statistics, La Trobe University, Bundoora, VIC, Australia

**ABSTRACT**
D-forms were originally created from inflexible materials and have subsequently been considered as abstract mathematical objects. This paper describes a textile instance of a D-form, with ornamentation of the constituent surfaces as the highlighted feature. A set of 11 biscornu has been fashioned to provide a 3D sampler of the axial point group $D_{4d}$ and its subgroups, using hitomezashi. Thus, this paper provides a link between the D-form literature and that of complete symmetry samplers in the fibre arts.

**ARTICLE HISTORY**
Received 16 March 2021
Accepted 5 October 2021

**KEYWORDS**
D-form; textiles; fibre arts; biscornu; hitomezashi; axial point group

**AMS SUBJECT CLASSIFICATIONS**
20D05; 52A15; 97M80

**CONTACT** Katherine A. Seaton k.seaton@latrobe.edu.au Department of Mathematics and Statistics, La Trobe University, Bundoora, VIC 3086, Australia

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1. Introduction

A *D-form* is a 3-dimensional domain created when two isoperimetric developable sheets are joined together by seaming their boundary curves. The same two constituent sheets can give rise to many possible D-forms, by changing the point on each boundary at which the seaming process begins (Wills, 2006). First dreamt into concrete form by the designer Tony Wills (literally dreamt, and literally concrete Wills, 2006, n.d.) some 20 years ago, the idea has subsequently been placed on a firmer mathematical footing (Demaine & O’Rourke, 2007; Demaine & Price, 2010; Pottmann & Wallner, 2001; Sharp, 2005).

Imposing the condition on the constituent surfaces that they each have a smooth and convex boundary curve (Demaine & O’Rourke, 2007), the chief mathematical results are that such a D-form is the convex hull of its seam curve, and that it has no creases (apart from the seam itself). The examples of D-forms in the design literature include objects to which these results do not apply; indeed, the earliest D-forms used squares and shapes with undulating edges (Sharp, 2005, 2009; Wills, 2006).

Of course, mathematicians work with ideal surfaces and curves; the material from which their D-forms are created does not stretch or collapse and their seams do not pucker. In the real world, stiff paper, card or plastic (like overhead projector transparencies) work well as the developable sheets (Bardos, n.d.; Knoll et al., 2008; Sharp, 2009). Wills has made D-forms from plywood or metal both as artistic forms and in professional practice for street architecture (Wills, 2006, n.d.). To design a large final product from scratch, computational tools can be used to match edge length (Orduño et al., 2016), and to predict the outcome (Bourke, n.d.; Gönen et al., 2007), which is otherwise generally hard to anticipate. There is, however, much delight in observing how the 3D object is formed progressively as the edges are brought together. An oft-quoted statement by Wills is to the effect that there is no ugly D-form (Sharp, 2009). In a recent paper, the authors ask whether we perhaps find this beauty because the human senses (unconsciously) appreciate their minimal convex nature (Bohr et al., 2018).

Both Sharp (2005) and Wills (2006) advise D-forms *not* be made of fabric, as it may shear or deform, though concede that this might be an interesting line to pursue. In fact, in a commonplace context which predates Wills’ dream by at least a century, it is precisely the deformation of fabric over a spherical core, which creates a baseball from two non-convex leather ‘flats’, joined with curved seam that pitchers use to their advantage (Thompson, 1998). Recently, specified hierarchies from within the sphericon family of 3D objects were shown to be D-forms, formed by seaming non-convex shapes (Seaton, 2017). The knitted and crocheted surfaces of the objects illustrating that paper were stabilised by felting or lining, or were made from a stiff plastic thread and lightly filled.

In this paper, another extant textile D-form is explored. The D-form cushions called *biscornu* are introduced in Section 2.1, and then in Section 2.2 customary decoration (embroidery) of their surfaces is used to illustrate their underlying symmetry group, that of the square anti-prism.¹

2. Biscornu

2.1. Biscornu as textile D-forms

Biscornu are ornamental cushions, made from two offset squares of the same size (Textile Research Centre Leiden, n.d.). They take their name from a French adjective meaning
Figure 1. An undecorated biscornu, made from two squares of Aida cloth, side length 6 cm.

irregular or more literally multi-horned (c.f. unicorn or tricorn), and are usually made from patch-work fabric, or from even-weave fabric embroidered using a counted-thread technique such as blackwork or cross-stitch, possibly further decorated with beads. Their popularity with crafters was extremely high for about a decade beginning around 2005. They are frequently used as pincushions, or as hanging decorations. Their origin is obscure, and they do not appear to have been connected to any handcraft tradition.

A biscornu is a D-form in the original sense of Wills (2006), since the boundary of a square is not smooth. The two squares are seamed together with each vertex of one square coinciding with the midpoint of an edge of the other, and vice versa. As can be seen in Figure 1 the seam does not lie flat but takes on a zigzag formation, like the crown formation of the molecule cyclo-octasulfur (the most commonly occurring form of sulfur). Strikingly, the emergence of biscornu can be dated to the first decade after Wills proposed D-forms, though there seems to be no evidence that this is not purely coincidence.

As is the case for cyclo-octasulphur, the symmetry group of the biscornu is that of the square anti-prism which it resembles: \( D_{4d} \). Smaller square regions that have as their vertices the midpoints of the original edges play the role of the square faces of the anti-prism, and the corners of the original squares bend over, without forming a sharp folded edge, to provide the triangles between the squares (again, see Figure 1). Unsupported, the fabric will not hold this shape, and the cushion is filled in some way. The inherent irregularity is often heightened by pulling the two squares together with a button sewn at the centre of each (n.d.), but this is not essential and does distort them from being true D-forms. Over-stuffing also needs to be avoided, lest they look more like balls and less like quirky anti-prisms with gently rounded edges.

2.2. An axial point group sampler

In two dimensions, samplers of rosette, frieze and wallpaper groups have been fashioned using various counted-thread techniques. (For a survey of complete symmetry samplers, realisable whilst respecting the constraints of the craft being used, see Goldstine, 2017, July 27–31). We now present images of 11 biscornu designed and constructed to showcase the three-dimensional symmetry group \( D_{4d} \) and each of the possible subgroups, including the trivial subgroup; collectively they comprise a single symmetry sampler of this axial point group. The traditional Japanese sashiko stitching form hitomezashi (Briscoe, 2004; Hayes & Seaton, 2020) has been chosen as the embroidery form used. This set of biscornu was displayed as the artwork (Ir)regularity at the 2021 Bridges conference (Seaton, 2021), the name playing on both the depicted symmetries and the meaning of the word biscornu.
Figure 2. (a) The two squares comprising a biscornu are indicated as seen from above before seaming, and the eight vertices are labelled; the square labelled with Greek letters lies under the square whose vertices are numbered. (b) The rotoreflection operation has been applied to (a), the axis of the rotation being perpendicular to the plane of the page at the position shown by the black dot.

The symmetry group of biscornu, the dihedral group $D_{4d}$, has order 16. One of the group generators is a rotoreflection (of order 8): rotation through $\pi/4$ about an axis through the centre of both squares, together with reflection across a plane halfway between these centres and perpendicular to the rotation axis. The effect of this operation is shown schematically in Figure 2.

The other symmetry group generator, of order 2, is a rotation through $\pi$; it is about a line that lies in the reflection plane mentioned above and that makes angle $\pi/8$ with the symmetry axes of the two squares, as shown in Figure 3(a). This operation, a non-axial rotation, changes the labelling order of the vertices from anti-clockwise to clockwise, as shown in Figure 3(b).

The unadorned biscornu of Figure 1 has the full $D_{4d}$ symmetry, but so too does the foremost object in Figure 4. Both squares comprising this biscornu have been decorated with the same Fibonacci snowflake (Ramírez et al., 2014) which has $D_4$ symmetry. The trivial subgroup has been depicted (the object slightly behind and to the left in Figure 4) by using an aleatoric design; that is, the running stitches have been placed according to the toss of a coin, resulting in no particular symmetry on either square.

There are nine distinct non-trivial subgroups, three each of order 8, 4 and 2.
The first subgroup of order 8 is denoted $S_8$, where this is Schoenflies notation referring to eight-fold rotoreflection, not to be confused with the symmetric group on eight elements, or indeed, the chemical symbol for octasulfur. The two squares comprising the relevant biscornu are shown in Figure 5. The centre of the design is the traditional *yamagata*
Figure 5. The two squares which comprise a bicornu symmetric under the order 8 rotoreflection.

Figure 6. Each square has $D_4$ symmetry, with four reflection axes, two running with the fabric grid and two diagonally. When the squares are offset and seamed together, the diagonal axes of one line up with the ‘square’ reflection axes of the second.

(mountain form). The running stitch borders have opposite chirality on the two squares; under the rotoreflection one maps to the other.

The two constituent squares have been decorated with different patterns of the same symmetry to create a bicornu to depict $D_4$ (see Figures 6 and 7). On one square, *yamagata* extends over the whole surface. On the other, the pattern comprises *jūjizashi* (ten-cross, the character for the number ten resembling a cross) and little squares.

The final order 8 subgroup depicted is $C_{4v}$, shown in Figure 8. The well-kerb (*igetazashi*) design has been surrounded by running stitches of the same chirality on each square. This object is unchanged by rotation about the central axis through $\frac{\pi}{2}$ (order 4) and by the non-axial rotation (order 2).

To illustrate the first of the order 4 subgroups, $C_4$, as shown in Figure 9 one square bears a design with that symmetry (a variant of *kakinohanazashi* persimmon flower stitch, surrounded by wide chiral bands of running stitch) while the other has further symmetry ($D_4$).
Figure 7. When the two squares from Figure 6 are seamed together, the overall object has four reflections and four-fold rotation about the axis. This object represents the subgroup $D_4$.

Figure 8. A biscornu with $C_{4v}$ symmetry. That the running stitches map correctly when the non-axial rotation of Figure 3 is applied can be verified in this photo which highlights an edge, not a face.

Figure 9. A square with the desired symmetry (left) is to be combined with a square that has additional symmetry, to create a biscornu to illustrate $C_4$.

This second square features a pleasing design of squares, crosses and a persimmon flower; in fact it is essentially the reverse fabric (or dual Hayes & Seaton, 2020) of the Fibonacci snowflake in Figure 4.

A similar approach has been used to design a biscornu that has $D_2$ symmetry. As shown in Figure 10 a close-packed arrangement of $jūjizashi$ has been paired with a square that has additional symmetry in its design. When this second square is offset, its two diagonal reflection axes line up with those of the other square, and being symmetric under rotation through $\frac{\pi}{4}$, it is also symmetric under the rotation through $\frac{\pi}{2}$.
In this case, combining a square with the desired symmetry (left) with a square that has additional symmetry, a bicornu to illustrate $D_2$ is created.

Two identical designs which use the mountain form stitch and which combine to give a bicornu with $C_{2v}$ symmetry.

The final subgroup of order four $C_{2v}$ is generated by rotation through $\pi$ about the principal axis and the non-axial rotation, both of order 2. The squares used to create a bicornu with this symmetry bear identical designs (with $D_2$ symmetry) as shown in Figure 11. The designs do not have diagonal symmetry axes, so that when one square is offset the resultant bicornu would have only $C_2$ symmetry were it not that the squares map to each other under the non-axial rotation.

There are two differently generated $C_2$ subgroups of $D_{4d}$. One is generated by the non-axial rotation (only) and is shown in Figure 12. The other is generated by rotation through $\pi$ about the principal axis. The constituent squares of this second $C_2$ subgroup bicornu are shown in Figure 13. The final subgroup, also of order 2, is $C_s$ and corresponds to an object with a single bilateral symmetry. To create such a bicornu, a square with a step design dan tsunagi, that has a single diagonal symmetry axis, has been combined with a design having a single symmetry axis parallel to one pair of sides. The squares are shown in Figure 14.
**Figure 12.** Since the identical squares do not have diagonal symmetry, the resultant biscornu has the non-axial rotation as its only symmetry. The stitch design is called *hirayama michi* (mountain passes).

**Figure 13.** Both the running stitch frame and the central design in the square on the left are symmetric only under rotation through $\pi$. The by-now familiar approach of using a second square with more symmetry, in this case a design of two kinds of persimmon, has been used to yield a biscornu with bilateral symmetry.

**Figure 14.** The *yamagata* form in the square on the left could have been extended; instead, a persimmon has been stitched like a sun above the mountain. The second square can be joined to the first in only one of two ways (not four) in order to yield a $C_4$ biscornu.
3. Conclusion

This paper provides a link between the D-form literature and that of complete symmetry samplers in the fibre arts. While intended to be artistically interesting in their own right, these decorated D-forms with their symmetric designs could be useful as manipulables in the teaching of algebra and its applications. I can attest that devising stitch patterns that have the desired symmetries certainly improves one’s understanding of three-dimensional geometry, as well as being creatively satisfying.

Note

1. There is another set of objects called 15-sided biscornu, made from 15 squares which can be joined together in 3 different configurations; we are not discussing these.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Katherine A. Seaton http://orcid.org/0000-0002-7354-8814

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