A2 Macdonald polynomials:  
a separation of variables

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Abstract

In this paper we construct a discrete linear operator \( K \) which transforms \( A_2 \) Macdonald polynomials into the product of two basic \( \phi_2 \) hypergeometric series with known arguments. The action of the operator \( K \) on power sums in two variables can be reduced to a generalization of one particular case of the Bailey’s summation formula for a very-well-poised \( \psi_6 \) series. We also propose the conjecture for a transformation of \( \psi_6 \) series with different arguments.

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1. Introduction

In the recent paper [1] V.B. Kuznetsov and E.K. Sklyanin proposed a new integral representation for $A_2$ Jack polynomials [2] in terms of $\,_3F_2$ hypergeometric functions and constructed the integral operator $M$ and its inverse $M^{-1}$ which separates variables in $A_2$ Jack polynomials. Also they formulated several conjectures about a structure of separated functions for general $A_n$ case. In fact, their method of a separation of variables originates (see recent review [3]) from the Inverse Scattering Method [4] and $L$-operator formalism applied to Calogero-Sutherland system [5, 6]. It is known that in this case the corresponding classical $r$-matrix depends on dynamical variables [7–9] and does not satisfy the usual classical Yang-Baxter equation. As a result we don’t know a consistent quantization procedure for such kind of integrable systems. Nevertheless, using the classical $L$-operator authors of ref. [1] guessed a quantum version of the equations on the integral operator $M$ which permits a separation of variables for $A_2$ Jack polynomials.

It is well known that the quantum Calogero-Sutherland system has a discrete $q$-finite analog [10]. Corresponding eigenfunctions are called Macdonald $A_n$ polynomials which generalize Jack $A_n$ polynomials for finite $q$. In this paper we will show that a separation variable procedure should work for Macdonald polynomials as well. We will formulate some results concerning general $A_n$ Macdonald polynomials and construct explicitly for $A_2$ case a discrete operator $K$ which apparently separates variables in $A_2$ Macdonald polynomials (we believe that this is true, but a rigorous proof can be reduced to summations of some hypergeometric $q$-series). The paper is organized as follows.

In Section 2 we remind some well known facts about Calogero-Sutherland system. Section 3 contains a brief review of Macdonald symmetric functions related to the root system $A_n$. In Section 4 we give definitions related to basic hypergeometric series and two theorems which show a connection between the spectrum of $A_{n-1}$ Macdonald operators and $\,_{n}\phi_{n-1}$ series. In Section 5 we construct explicitly for $A_2$ case a discrete operator $K$ which separates variables in Macdonald polynomials and formulate the conjecture for the action of this operator on power sums in two variables. Section 6 contains a discussion of some unsolved problems. In Appendix A we write out a set of difference equations on the kernel of the operator $K$ which appear from separated equations of Section 4 for $A_2$ case. At last, Appendix B contains
the calculation of the normalization factor for the kernel of the operator $K$
and several examples which support the conjecture from Section 5.

2. Calogero-Sutherland model

In this section we remind some known results related to Calogero-Sutherland
hamiltonian.

Consider $n$ particles on a circle interacting with a long range potential
[5, 6]. We denote coordinates of the particles by $x_i, i = 1, \ldots, n, 0 \leq x_i \leq L$.
Then a total momentum of the system and hamiltonian which describe a
dynamics of the particles are given by:

\[ P = \sum_{i=1}^{n} \frac{1}{i} \frac{d}{dx_i}, \quad (2.1) \]
\[ H = -\sum_{i=1}^{n} \frac{1}{2} \frac{d^2}{dx_i^2} + g(g-1)\frac{\pi^2}{L^2} \sum_{i<j} \frac{1}{\sin^2\left(\frac{\pi}{L}(x_i - x_j)\right)}. \quad (2.2) \]

As usual set $\theta_j = 2\pi \frac{x_j}{L}$ and $z_j = \exp(i\theta_j)$. Then eigenfunctions of the
equation $H\Psi = E\Psi$ have the following structure

\[ \Psi(\theta) = \Delta^g(\theta) J(\theta), \quad (2.3) \]
\[ \Delta(\theta) = \prod_{i<j} \sin\left(\frac{\theta_i - \theta_j}{2}\right) \quad (2.4) \]

and $J$ is a symmetric Laurent polynomial in $z_j$.

It is naturally to introduce the effective Hamiltonian

\[ \tilde{H} = \Delta^{-g} H \Delta^g, \quad (2.5) \]

which has polynomial eigenfunctions $J$ and can be rewritten in the following
form

\[ \tilde{H} = \sum_{j=1}^{n} (z_j \frac{\partial}{\partial z_j})^2 + g \sum_{i\neq j} \frac{z_i + z_j}{z_i - z_j} (z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j}). \quad (2.6) \]

Now we list some well known facts about operator $\tilde{H}$. 
1. $\tilde{H}$ can be included in a mutually commuting family of differential operators $\tilde{H}_k$, $k = 1, \ldots, n$ in variables $z_j$ [11].

2. Simultaneous polynomial eigenfunctions $J$ of the differential operators $\tilde{H}_k$ are known as Jack polynomials.

3. This system has a discrete $q$-analogue [10]. Corresponding commuting discrete operators are called Macdonald operators. Their simultaneous eigenfunctions are Macdonald polynomials which generalize Jack polynomials for finite $q$.

3. Macdonald symmetric functions

In this section we give a short list of definitions concerning Macdonald polynomials [10] related to $A_{n-1}$ root system. There are several ways to define Macdonald symmetric functions. Keeping in mind a connection with Calogero-Sutherland model we define Macdonald polynomials $P_{\lambda}(x; q, t)$ as simultaneous eigenfunctions of a commuting family of operators $D_r^n$, $r = 0, 1, \ldots, n$ acting in the ring of symmetric functions in $n$ variables.

As usual let $\Lambda_n(q, t)$ be the ring of symmetric functions in $n$ variables $(x_1, \ldots, x_n)$ over field $Q(q, t)$ of rational functions in two independent indeterminates $q$ and $t$.

Define shift operators $T_{x_i}$ by

$$(T_{x_i} f)(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$$

(3.1)

for any polynomial $f(x_1, \ldots, x_n)$.

Now let us introduce a family of mutually commuting operators $D_r^n$, $r = 0, 1, \ldots, n$ as follows

$$D_r^n = \sum_I t^{r(r-1)/2} \prod_{i \in I} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} T_{x_i},$$

(3.2)

where the summation is over all $r$-element subsets $I$ of $(1, 2, \ldots, n)$.

Define a partition $\lambda$ of the weight $|\lambda|$ as a sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \ldots$ such that $|\lambda| = \sum \lambda_i < \infty$. The nonzero’s $\lambda_i$ are
called the parts of $\lambda$ and the number of parts is the length $l(\lambda)$ of the partition $\lambda$. A natural partial ordering for two partitions $\lambda, \mu$ can be defined as follows

$$\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu| \text{ and } \sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i \text{ for all } r \geq 1. \quad (3.3)$$

One of the bases in $\Lambda_n, Q(q,t)$ is given by monomial symmetric functions $m_\lambda$, $l(\lambda) \leq n$:

$$m_\lambda = \sum_{\alpha} x^{\alpha(\lambda)}, \quad (3.4)$$

where $x^\lambda = x_1^{\lambda_1}x_2^{\lambda_2}\ldots$ and the summation is over all distinct permutations $\alpha$ of nonzero parts of the partition $\lambda$.

Further for each partition $\lambda$, $l(\lambda) \leq n$ let coefficients $d_{i_n}^\lambda(\lambda)$ are defined by the following expansion

$$\prod_{i=1}^{n}(1 + Xq^\lambda_i t^{n-i}) = \sum_{i=0}^{n} X^i d_{i_n}^\lambda(\lambda). \quad (3.5)$$

**Theorem 3.1** [10] For each partition $\lambda$, $l(\lambda) \leq n$ there exists a unique symmetric function $P_\lambda(x; q, t) \in \Lambda_n, Q(q,t)$ which satisfies two following conditions:

1. $P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda \mu} m_\mu, \quad u_{\lambda \mu} \in Q(q,t), \quad (3.6)$
2. $D_n^r P_\lambda = d_{i_n}^\lambda(\lambda) P_\lambda. \quad (3.7)$

One can also define the scalar product $\langle \cdot, \cdot \rangle_{q,t}$ [10] such that

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0, \quad \text{if } \lambda \neq \mu, \quad (3.8)$$

but we will not use it in this paper.

### 4. Spectrum of Macdonald operators and basic hypergeometric series

In this section we remind a definition of the basic hypergeometric series $r_{+1} \phi^r_r$ and show that coefficients $d_{i_n}^\lambda(\lambda)$ (see (3.5)) naturally appear from separated equations for the function $n \phi_{n-1}$. 
First of all introduce the following notations (see, for example, [12, 13]):

\[(a; q)\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad (4.1)\]

\[(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad n \in \mathbb{Z}, \quad (4.2)\]

\[(a_1, a_2, \ldots, a_r; q)_n = \prod_{i=1}^{r}(a_i; q)_n, \quad (4.3)\]

where \(|q| < 1, a \in \mathbb{C}, r \geq 0\).

As usual define the basic hypergeometric function \(r+1\phi_r\) as

\[r+1\phi_r(a_1, \ldots, a_{r+1}; b_1, \ldots, b_r; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n(a_2; q)_n \cdots (a_{r+1}; q)_n}{(q; q)_n(b_1; q)_n \cdots (b_r; q)_n} z^n. \quad (4.4)\]

Sometimes we will use a short notation \(r+1\phi_r([a]; [b]; q, z)\).

This function satisfies the difference equation

\[\left[(1 - T_z) \prod_{i=1}^{r}(1 - q^{-1}b_i T_z) - z \prod_{q=1}^{r+1}(1 - a_i T_z)\right] r+1\phi_r([a]; [b]; q, z) = 0. \quad (4.5)\]

Now for each partition \(\lambda, l(\lambda) \leq n\) define two sequences \([a] = (a_1, \ldots, a_n), [b] = (b_1, \ldots, b_{n-1})\)

\[a_i = q^{1+\lambda_n-\lambda_i} q^{i-1-n}, \quad i = 1, \ldots, n, \quad b_j = ta_j, \quad j = 1, \ldots, n-1 \quad (4.6)\]

and introduce functions \(\varphi_\lambda(q, t; z), \psi_\lambda(q, t; z)\):

\[\varphi_\lambda(q, t; z) = z^{\lambda_n} \phi_{n-1}([a]; [b]; q, z) \quad (4.7)\]

\[\psi_\lambda(q, t; z) = \frac{(z, q)_\infty}{(qt^{-n}, q)_\infty} \varphi_\lambda(q, t; z). \quad (4.8)\]

Then

**Theorem 4.1** The function \(\psi_\lambda(q, t; z)\) is a polynomial in \(z\).
Proof: (4.6) we note that
\[
\frac{(a_{i+1};q)_l}{(b_i; q)_l} = \frac{(b_i q^i; q)_{\lambda_i-\lambda_{i+1}}}{(b_i; q)_{\lambda_i-\lambda_{i+1}}}, \quad i = 1, \ldots, n - 1.
\] (4.9)

Then
\[
\psi_{\lambda}(q,t; z) = z^{\lambda_n} \frac{(z, q)_\infty}{(qz^{t-n}, q)_\infty} \sum_{l=0}^{\infty} \frac{(q^{1+\lambda_n-\lambda_1}t^{-n}; q)_l}{(q; q)_l} \prod_{i=1}^{n-1} \left[ \frac{(b_i q^i; q)_{\lambda_i-\lambda_{i+1}}}{(b_i; q)_{\lambda_i-\lambda_{i+1}}} \right] z^l = \]
\[
= z^{\lambda_n} \frac{(z, q)_\infty}{(qz^{t-n}, q)_\infty} \sum_{l=0}^{\lambda_1-\lambda_n} \rho_k(t; q) \frac{(q^{1+\lambda_n-\lambda_1+k}t^{-n}z; q)_\infty}{(q^k z; q)_\infty} = \]
\[
= z^{\lambda_n} \sum_{k=0}^{\lambda_1-\lambda_n} \rho_k(t; q) \frac{(z; q)_k}{(qz^{t-n}; q)_{\lambda_n-\lambda_1+k}} = \sum_{k=\lambda_n}^{\lambda_1} \tilde{\rho}_k(t; q) z^k. \quad Q.E.D.
\]

In fact, this theorem naturally generalizes Theorem 3 from ref. [1].

And now we formulate the main result of this section:

**Theorem 4.2** The function \(\varphi_{\lambda}(q,t; z)\) satisfies the following difference equation
\[
\left[ \sum_{i=0}^{n} (1 - z(qt^{-1})^i) d_{n-i}^i(\lambda)(-T_z)^i \right] \varphi_{\lambda}(q,t; z) = 0, \quad (4.10)
\]
where coefficients \(d_{n}^{i}(\lambda)\) coincide with eigenvalues of Macdonald operators \(D_{n}^{i}\) for the eigenfunction \(P_{\lambda}(x; q,t)\).

Proof: Direct calculation. Q.E.D.

We will call equation (4.10) as the separated equation for \(A_{n-1}\) Macdonald polynomials. This equation strongly supports (and generalizes) the conjecture of ref. [1] concerning a structure of separated functions for \(A_{n-1}\) Jack polynomials. In the next section we will construct explicitly a discrete linear operator \(K\) which provides a separation of variables for \(A_2\) Macdonald polynomials.

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5. Separation of variables for $A_2$ Macdonald polynomials

Hereafter we will consider only the case $n = 3$, which corresponds to $A_2$ Macdonald polynomials.

Using (3.2) we can write out explicitly Macdonald operators for $n = 3$:

\begin{align*}
D^0_3 &= 1, \\
D^1_3 &= \frac{(tx_1 - x_2)(tx_1 - x_3)}{(x_1 - x_2)(x_1 - x_3)}T_{x_1} + (x_1 \to x_2 \to x_3), \\
D^2_3 &= t\frac{(tx_1 - x_3)(tx_2 - x_3)}{(x_1 - x_3)(x_2 - x_3)}T_{x_1}T_{x_2} + (x_1 \to x_2 \to x_3), \\
D^3_3 &= t^3T_{x_1}T_{x_2}T_{x_3}.
\end{align*}

(5.1)

First let us make the following changing of variables:

\begin{align*}
x_1 &= z_1 v, \quad x_2 = z_2 v, \quad x_3 = v, \\
T_{x_1} &= T_{z_1}, \quad T_{x_2} = T_{z_2}, \quad T_{x_3} = T_{z_1}^{-1}T_{z_2}^{-1}T_{v}.
\end{align*}

(5.2)

Then using a homogeneity of $A_2$ Macdonald polynomials $P_\lambda(x_1, x_2, x_3; q, t)$ with respect to $x_i$ we have

\[ P_\lambda(x_1, x_2, x_3; q, t) = v^{\lambda_1 + \lambda_2 + \lambda_3}p_\lambda(z_1, z_2; q, t), \]

(5.4)

where $p_\lambda(z_1, z_2; q, t)$ is a nontrivial function of two new variables $z_1$ and $z_2$.

Now let us suppose an existence of a linear discrete operator $K$

\[ K : p_\lambda(z_1, z_2) \to (K \circ p_\lambda)(z_1, z_2) \]

(5.5)

such that after applying of the $K$ to each $p_\lambda(z_1, z_2)$ a result will satisfy to separated equation (4.10) (at $n = 3$) in each variable $z_i$. In fact, the action of this linear operator can be represented in the form of the summation of $p_\lambda$ with some discrete kernel $K$. 

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Then using a linearity of the operator $K$ let us rewrite separated equations (4.10) as
\[
\left( \sum_{l=0}^3 (1 - z_i(q t^{-1})^l)(-T_{z_i})^l \right) (K \circ d_3^{3-l}(\lambda) \ p_\lambda)(z_1, z_2) = 0. \tag{5.6}
\]
\[
\begin{array}{c}
\uparrow \\
D_3^{3-l} \leftarrow \downarrow \\
\end{array}
\]
Now we can replace eigenvalues $d_3^{3-l}(\lambda)$ of Macdonald polynomials by corresponding Macdonald operators and using a conjugation with respect to the convolution $K \circ p_\lambda$ to move the action of Macdonald operators on the kernel of the operator $K$. In fact, it will correspond to shifts of summation variables on $+1$ or $-1$. After that we can try to satisfy a set of local difference equations on the kernel $K$. And if we solve these equations, probably this linear operator $K$ will separate variables in $A_2$ Macdonald polynomials.

But if summations are not infinite in both directions, some problems with boundary terms can appear under shifts of summation variables. However, it appears that all summations can be chosen from $-\infty$ to $+\infty$ and for each fixed Macdonald polynomial $p_\lambda$ we can choose parameter $t$ in such a way that all series will be convergent. So we will not care about boundary terms anymore.

To write out explicitly the action of the linear operator $K$ on polynomials $p_\lambda(z_1, z_2)$ let us make the following substitution:
\[
z_1 = \sigma \xi, \quad z_2 = \sigma \xi^{-1} \tag{5.7}
\]
\[
T_{z_1} = T_{\sigma}^{1/2} T_{\xi}^{1/2}, \quad T_{z_2} = T_{\sigma}^{1/2} T_{\xi}^{-1/2}. \tag{5.8}
\]
It turns out that the action of the operator $K$ on the variable $\sigma$ is very simple:
\[
K \circ \sigma^l = t^{3l/2} \sigma^l, \tag{5.9}
\]
but the action of $K$ on the variable $\xi$ is quite nontrivial.

So taking into account formulas (5.7-5.9) we can represent the action of $K$ in the form of the following single sum:
\[
K : p_\lambda \rightarrow \sum_{l=-\infty}^{+\infty} K(z_1, z_2; y_1, y_2) \ p_\lambda(y_1, y_2), \tag{5.10}
\]
where
\[ z_1 = t^{3/2}\sigma\xi, \quad z_2 = t^{3/2}\sigma\xi^{-1}, \]  
and \( s \in [0, 1/2] \). We will explain a necessity to introduce the parameter \( s \) later.

Further we will use one of the following equivalent forms for the kernel \( K \):
\[ K(z_1, z_2; y_1, y_2) = K(z; y) = K(\sigma, \xi; \eta) = K(z_1, z_2; s, l), \]  
where arguments satisfy (5.11-5.12).

Note that due to (5.8) we must summarize, in fact, over integer and half-integer powers of \( q \). But we can split the summation in (5.10) into two sums: with the kernel \( K(z_1, z_2; s, l) \) and with the kernel \( K(z_1, z_2; s + 1/2, l) \) and summarize over \( l \) with \( \eta = q^{s+l} \).

Substituting (5.10) into separated equation (5.6) and using explicit form (5.1) of the Macdonald operators one can obtain four linear difference equations on the kernel \( K \). These equations are given in Appendix A. It turns out that they are a consequence of much more simple relations:

\[ T_z T_{y_1} K(z; y) = \frac{t^2(y_1 - 1)(y_2 - qy_1)(z_1 - ty_2)}{(qy_1 - t)(y_2 - y_1)(qz_1 - t^2y_2)} K(z; y), \]  
\[ T_z T_{y_2} K(z; y) = \frac{t^2(y_2 - 1)(y_1 - qy_2)(z_1 - ty_1)}{(qy_2 - t)(y_1 - y_2)(qz_1 - t^2y_1)} K(z; y). \]

Let us make several comments about equations (5.14-5.15). If we consider the limit \( q \to 1 \), then it can be shown that these equations will reduce to differential equations on the kernel of the integral operator which separates variables for \( A_2 \) Jack polynomials [3]. In fact, E.K. Sklyanin showed that one can extract these equations (their classical version) from classical \( L \)-operator related to Calogero-Sutherland system. We obtained (5.14-5.15) explicitly from the separated equations (5.6) for \( A_2 \) Macdonald polynomials. It would be interesting to understand: how to obtain equations (5.14-5.15) from some quantum \( q \)-finite version of the classical \( L \)-operator related to 3-particle Calogero-Sutherland system. For \( n > 3 \) it is rather difficult to work with a generalization of difference equations (A.1-A.2) (see Appendix
A) because of their very complicated structure. But it is likely that equations (5.14-5.15) should have a simple multiplicative generalization.

Now it is not difficult to solve equations (5.14-5.15) with respect to the kernel $K$. The answer is

$$K(\sigma, \xi; \eta) = g(\sigma, \xi; s) \eta(1 - \eta^2) \left( \frac{q \eta}{\sigma t}, q \frac{\eta \xi}{t \sqrt{t}}, q \frac{\eta}{\xi \sqrt{t}}; q \right)_\infty,$$

where

$$g(q^{1/2} \sigma, q^{1/2} \xi, s) = q^{1/2} t^{3/2}.$$  

Let us note that if we set the parameter $s$ in (5.12) as $s = 0$ then the sum in RHS of (5.10) will be equal zero (it can be checked explicitly using (5.16)). So the function $g(\sigma, \xi, s)$ must have a pole at $s = 0$. We will see that this is really the case.

It is enough to calculate the action of the operator $K$ on the power sums $z_1^n + z_2^n$.

Define the following function

$$\Phi(s, \sigma, \xi, t; q, n) = \sum_{l=-\infty}^{\infty} \frac{q^{s+l}(1 - q^{2(s+l)})}{t^{3(s+l)}} \times$$

$$\times (q^{n(s+l)} + q^{-n(s+l)}) \left( \frac{q \sigma}{t}, q \frac{\sigma}{t \sqrt{t}}, q \frac{\xi}{\sqrt{t}}; q \right)_\infty,$$

where $s \in [0, 1]$, $n = 0, 1, 2, 3, \ldots$

It is easy to see that all series in (5.18) are convergent provided that $|q^{1-n}/t^3| < 1$. So for any fixed $n$ we can always choose parameter $t$ in such a way that sums in (5.18) will be convergent and after analytically continue a result for any complex $t$.

Then we have

$$K \circ (z_1^n + z_2^n) = g(\sigma, \xi, s)t^{3n/2} \sigma^n \Phi(s, \sigma, \xi, t; q, n) + \Phi(s + \frac{1}{2}, \sigma, \xi, t; q, n))$$

where $z_1$, $z_2$ satisfy (5.7).
It is naturally to choose the function \( g(\sigma, \xi, s) \) as follows

\[
g(\sigma, \xi, s) = \frac{2}{\Phi(s, \sigma, \xi, t; q, 0) + \Phi(s + 1/2, \sigma, \xi, t; q, 0)}
\] (5.20)

and as a consequence

\[
K \circ 1 = 1.
\] (5.21)

Using expression (B.2) (see Appendix B) for the function \( \Phi(s, \sigma, \xi, t; q, 0) \) it is not difficult to show that the function \( g(\sigma, \xi, s) \) defined by (5.20) satisfies recurrence relations (5.17).

Now let us formulate the following

**Conjecture:**

\[
\Phi(s, \sigma, \xi, t; q, n) = \Phi(s, \sigma, \xi, t; q, 0) P(\sigma, \xi, t; q, n),
\] (5.22)

where \( P(\sigma, \xi, t; q, n) \) is a Laurent polynomial in \( \sigma, \xi \).

Three first nontrivial examples of polynomials \( P(\sigma, \xi, t; q, n) \) are given in Appendix B.

From (5.19-5.22) it is easy to see that the action of the operator \( K \) on power sums does not depend on the parameter \( s \).

So we immediately come to the following statement:

*If the conjecture (5.22) is valid for the arbitrary \( n \), then*

\[
K \circ p_\lambda(z_1, z_2) = c_\lambda(q, t) \psi_\lambda(q, t; t^{3/2} z_1) \psi_\lambda(q, t; t^{3/2} z_2),
\] (5.23)

where \( z_1, z_2 \) satisfy (5.7), the action of the operator \( K \) is given by formulas (5.10, 5.16, 5.20) and the function \( \psi_\lambda(q, t; z) \) is defined by (4.8).

Using **MATHEMATICA** program we checked that factorization formula (5.23) works for all partitions with \(|\lambda| \leq 4\).

Note that the separated equations (5.6) do not contain any information about the normalization multiplier \( c_\lambda(q, t) \) in (5.23).

### 6. Discussion

In this paper we constructed the discrete operator \( K \) which should transform \( A_2 \) Macdonald polynomials into the product of two \( 3\phi_2 \) functions. It will
be a theorem if conjecture (5.22) is valid for the arbitrary integer \( n \). In fact, formula (5.22) connects two very-well-poised \( {}_6\psi_6 \) series with different arguments. We failed to find relation (5.22) in any literature related to basic hypergeometric series. So the proof of (5.22) could be quite interesting from a purely mathematical point of view.

The next problem is to calculate the inverse operator \( K^{-1} \). We know the answer in the limit \( q \to 1 \) (see [1]) when \( K \) is reduced to the integral operator. But the problem of the calculation \( K^{-1} \) in a discrete case can be much more difficult. Possibly the best way to do this is to write a set of difference equations on the inverse operator \( K^{-1} \) and to solve them. Then we will obtain a representation of \( A_2 \) Macdonald polynomials in terms of basic \( {}_3\phi_2 \) hypergeometric series.

But definitely a normalization of this discrete representation will differ from the standard one (3.6). So a separate problem is to calculate the corresponding normalization factor.

And, of course, it would be interesting to generalize these results for \( A_{n-1} \) case. Separated equations (4.10) show that it can be done. But here we come to another problem: what is an underlying algebraic structure for finite \( q \) which permits a separation of variables for Macdonald polynomials? For the classical case of Calogero-Sutherland system the answer is known (see [3]). So the answer can give a new insight on the whole algebraic construction and help to write a discrete representation for \( A_n \) Macdonald polynomials. Possibly it can give a lot of new interesting identities in the theory of \( q \)-series. We hope to consider these problems in a separate paper.

7. Acknowledgments

I would like to thank Prof. G. Andrews who explained me that the expression (5.18) at \( n = 0 \) can be represented in the form of \( {}_6\psi_6 \) series and is summable by Bailey’s formula.

A. Appendix

In this appendix we write out difference equations on the kernel of the linear operator \( K \) which appear from separated equations (5.6).
Using explicit formulas for Macdonald operators (5.1) and separated equations (5.6) we obtain four difference equations on the kernel $K$:

\[
\left\{ \left( t - z_i \right) + \left( 1 - z_i \frac{q^2}{t^3} \right) \left( 1 - y_1 \right) \frac{1 - y_2}{(1 - ty_1)(1 - ty_2)} T z_i T y_1 T y_2 - \left( 1 - z_i \frac{q^2}{t^2} \right) T z_i \times 
\right. \\
\left. \times \left( \frac{ty_1 - qy_2}{(1 - ty_2)} T y_2 + \frac{ty_2 - qy_1}{(1 - ty_1)} T y_1 \right) \right\} K(z; y) = 0, \tag{A.1}
\]

\[
\left\{ \left( 1 - z_i \right) t + \left( 1 - z_i \frac{q^2}{t^2} \right) \frac{1 - qy_1/t}{(1 - qy_2/t)} T z_i T y_1 T y_2 - \left( 1 - z_i \frac{q^2}{t^2} \right) T z_i \times 
\right. \\
\left. \times \left( \frac{ty_1 - qy_2}{(1 - ty_2)} T y_2 + \frac{ty_2 - qy_1}{(1 - ty_1)} T y_1 \right) \right\} K(z; y) = 0, \tag{A.2}
\]

where $i = 1, 2$.

Combining equations (A.1-A.2) one can obtain a nonlinear difference equation in one variable, but its solution should satisfy some additional compatibility conditions. In this way we obtained difference equations (5.14-5.15) which satisfy all necessary conditions.

After some tedious calculations one can check that the solution (5.16) satisfies all equations (A.1-A.2) as well.

**B. Appendix**

In this appendix we calculate the normalization multiplier for the kernel $K$ and give three first nontrivial examples which support the conjecture (5.22).

Using usual definitions for $r\psi_r$ series (see [12, 13]) we can rewrite (5.18) as a sum of two $6\psi_6$ series:

\[
\Phi(s, \sigma, \xi, t; q, n) = \frac{q^s \left( 1 - q^{2s} \right)}{t^{3s}} \left( \frac{q^\sigma q^s}{t\sigma}, \frac{q}{t\sigma}, \frac{q}{q^s}, \frac{q\xi}{q^s}, \frac{q^\xi q^s}{t\xi} \right)_\infty \times \\
\left( \frac{t}{\sigma q^s}, t\sigma q^s, \frac{t}{\xi q^s}, \frac{\sqrt{t} q^s}{\xi} \right)_\infty
\]

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Now we will calculate explicitly the function $\Phi(s, \sigma, \xi, t; q, 0)$ using Bailey’s summation formula [14] for the very-well-poised series $\psi_6$. One can check that arguments of $\psi_6$ in (B.1) satisfy all necessary constraints (at $n = 0$) and so $\Phi(s, \sigma, \xi, t; q, 0)$ is summable. In fact, at $n = 0$ we have a particular case of Bailey’s formula (which contains 5 independent parameters) with 4 independent parameters.

So we have (see [12, 14])

$$
\Phi(s, \sigma, \xi, t; q, 0) = \frac{2q^s(1 - q^{2s})}{t^{3s}(q/t^3)_\infty} \times 
\frac{(q \sigma \xi / t^{3/2}, q \sigma / t^{3/2}, q \sigma \xi / t^{3/2}, q/q \sigma \xi / t^{3/2})_\infty}{(t/q^s, t \sigma q^s, \xi \sqrt{t} q^s, \frac{\sqrt{t}}{\xi} q^s, \frac{\sqrt{t}}{\xi} q^s, \frac{\sqrt{t}}{\xi} q^s, \frac{\sqrt{t}}{\xi} q^s, q, q)}.
$$

(B.2)

Now we give three first nontrivial examples of polynomials $P(\sigma, \xi, t; q, n)$ (see (5.22)):

$$
P(\sigma, \xi, t; q, 1) = \frac{t(\sigma + \sigma^{-1}) + (t + 1)\sqrt{t}(\xi + \xi^{-1})}{2(1 + t + t^2)},
$$

(B.3)

$$
P(\sigma, \xi, t; q, 2) = \frac{t(t(qt - 1)(\sigma^2 + \sigma^{-2}) + (1 + t)(qt^2 - 1)(\xi^2 + \xi^{-2}))}{2(qt^3 - 1)(1 + t + t^2)} + 
\frac{(1 + q)(t^2 - 1)(t\sqrt{t}(\sigma + \sigma^{-1})(\xi + \xi^{-1}) - (1 + t)(1 - t + t^2))}{2(qt^3 - 1)(1 + t + t^2)},
$$

(B.4)

$$
P(\sigma, \xi, t; q, 2) = \frac{(1 + q + q^2)(t^2 - 1)\sqrt{t}}{2(1 + t + t^2)(qt^3 - 1)(qt^3 - 1)} \times
$$

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\[
\sqrt{t}(\sigma + \sigma^{-1})(1 + (1 + q)(t^3 - t) - qt^4 + t(qt^2 - 1)(\xi^2 + \xi^{-2})) + \\
+ (\xi + \xi^{-1})(1 + (1 + q)(t^3 - t^2) - qt^5 + t^2(qt - 1)(\sigma^2 + \sigma^{-2})) + \\
\frac{t^3(qt - 1)(q^2 t - 1)(\sigma^3 + \sigma^{-3}) + (1 + t)(q^2 t^2 - 1)(qt^2 - 1)t^{3/2}(\xi^3 + \xi^{-3})}{2(1 + t + t^2)(q^2 t^3 - 1)(qt^3 - 1)}.
\]

Apparently the function \( P(\sigma, \xi, t; q, n) \) can be written in the form of truncated basic hypergeometric series, but up to now we failed to find this representation for the arbitrary \( n \).

Also one can ask: is it possible to generalize formula (5.22) for the general 5-parametric case? If the answer is yes, then Bailey’s summation formula for \( \psi_6 \) will be the first representative (the case \( n = 0 \)) in a sequence of transformations for very-well-poised \( \psi_6 \) series with different arguments.

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