Excitation theory for space-dispersive active media waveguides

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Abstract.
A unified electrodynamic approach to the guided-wave excitation theory is generalized to the waveguiding structures containing a hypothetical space-dispersive medium with drifting charge carriers possessing simultaneously elastic, piezoelectric and magnetic properties. Substantial features of our electrodynamic approach are: (i) the allowance for medium losses and (ii) the separation of potential fields peculiar to the slow quasi-static waves which propagate in such active media independently of the fast electromagnetic waves of curl nature. It is shown that the orthogonal complementary fields appearing inside the external source region are just associated with a contribution of the potential fields inherent in exciting sources. Taking account of medium losses converts the usual orthogonality relation into a novel form called the quasi-orthogonality relation. Development of the mode quasi-orthogonality relation and the equations of mode excitation is based on the generalized reciprocity relation (the extended Lorentz lemma) specially proved for this purpose with allowing for specific properties of the space-dispersive active media and separating the potential fields. The excitation equations turn out to be the same in form whatever waveguide filling, including both the time-dispersive (bianisotropic) and space-dispersive media. Specific properties of such media are reflected in a particular form of the normalizing coefficients for waveguide eigenmodes. It is found that the separation of potential fields reveals the fine structure of interaction between the exciting sources and mode eigenfields: in addition to the exciting currents (bulk and surface) interacting with the curl fields, the exciting charges (bulk and surface) and the double charge (surface dipole) layers appear to interact with the quasi-static potentials and the displacement currents, respectively.

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1. Introduction

Modern progress of material science and technology opens new potential possibilities in synthesizing complex and composite media with unique electromagnetic properties at microwaves and in optics. This requires revising some problems of guided-wave electrodynamics, in particular, the theory of waveguide excitation by external sources. The guided-wave excitation theory for passive media with isotropic, anisotropic and bianisotropic properties was developed in [1]. Electrodynamic processes in such passive media are fully described by Maxwell’s equations and local constitutive relations, the most general form of which is inherent in bianisotropic media (BAM) with frequency-dependent constitutive parameters. Although magneto-electric phenomena in such media, by their microscopic nature, are brought about by non-locality of short-range polarization responses on electromagnetic actions [2–4], their macroscopic manifestations are actually similar to those for real time-dispersive media. Indeed, the short-range character of these phenomena enables one to use the plane wave representation with wavenumber $k = \omega/c$ so that in the first-order approximation all the constitutive tensor parameters of such a medium become solely local and frequency-dependent (see Ref. [2]).

Unlike passive media, a true active medium requires, in addition to Maxwell’s equations, an appropriate equation of motion for its electrodynamic description. Among active media we restrict our consideration to three kinds:

(i) piezo-dielectrics with elastic properties providing the technological basis for acoustic-wave electronics [5–7],
(ii) dielectrics with ferrimagnetic properties (magnetized ferrites) forming the technological basis for spin-wave electronics [8–10],
(iii) nondegenerate plasmas with drifting charge carriers (in particular, semiconductors with negative differential mobility of hot electrons) constituting the technological basis for plasma-wave electronics [11–14].

For generality, we shall investigate a hypothetical space-dispersive medium possessing simultaneously elastic, piezoelectric, ferrimagnetic and nondegenerate plasma properties in order to provide specific relations for any kind of the complex composite medium as a special case of the general situation developed below.

Space-dispersive properties of such a waveguiding medium are related to non-local effects caused by specific interactions between adjacent particles of the active media, such as elastic interactions in piezo-dielectrics, exchange interactions in ferrites and carrier diffusion effects in nondegenerate plasmas. Neglecting the non-local effects enables the equations of medium motion to be converted into the constitutive relations with frequency-dependent parameters. In other words, such a medium possesses properties of the usual time-dispersive active medium (TDAM) and in this case its electrodynamic properties fully conform to the ordinary anisotropic media examined in [1]. Our subsequent analysis will be based on results obtained there, among them a novel notion of the mode quasi-orthogonality for lossy waveguides (see section 3 of [1]).

The objective in writing the paper is to generalize the guided-wave excitation theory developed previously for the waveguiding structures with time-dispersive bianisotropic media [1] to waveguides filled with a generalized (hypothetical) space-dispersive medium which contains all the piezoelectric, ferrimagnetic and plasma phenomena. Section 2 includes input information about the modal field expansions with separating potential fields to give a physical insight into the nature of the orthogonal complementary fields obtained previously [1]. Since the electrodynamic
description of space-dispersive phenomena requires, in addition to Maxwell’s equation, special equations of medium motion, section 3 is devoted to consideration of the appropriate equations for three kinds of SDAM. Section 4 is fundamental and begins with an examination of the generalized reciprocity relation specially derived in the Appendix. This relation serves as a basis for obtaining the mode quasi-orthogonality relation and the equations of mode excitation taking into account specific contributions from the generalized hypothetical medium. Section 5 contains the general discussion of physical features in the mathematical description concerning the excitation of lossless and lossy systems valid for both BAM (see [1]) and SDAM waveguides. Mathematical notation here is the same as applied in [1], in particular, $A$ stands for scalars, $\mathbf{A}$ for vectors, $\mathbf{A}$ for dyadics and $\mathbf{A}$ for tensors of rank more than two.

2. Modal field expansions with separating potential fields

The fundamental result of the previous examination [1], which was obtained for passive media (including BAM as the most general case), is the incompleteness of eigenmode basis for any waveguiding structure inside the excitation source region. This manifests itself in the fact that the modal field expansions $E_a = \sum_k A_k E_k$ and $H_a = \sum_k A_k H_k$ must be supplemented with the orthogonal complementary fields $E_b$ and $H_b$ which are related to the longitudinal components of the external currents $J_e$ and $J_m$, i.e. the complete electromagnetic fields inside the source region are represented in the following form (cf. equations (4.8) and (4.9) in [1])

$$E(r_t, z) = E_a(r_t, z) + E_b(r_t, z) = \sum_k A_k(z) E_k(r_t, z) + E_k(r_t, z)$$

$$H(r_t, z) = H_a(r_t, z) + H_b(r_t, z) = \sum_k A_k(z) H_k(r_t, z) + H_k(r_t, z)$$

where an unknown longitudinal dependence of the excitation amplitudes $A_k(z)$ is due to exciting external sources as well as the complementary fields $E_b$ and $H_b$.

In [1] the complementary fields received a mathematical substantiation as the orthogonal complement to Hilbert space spanned by the eigenfield basis $\{E_k, H_k\}$, but their physical nature is still not properly understood. The present examination of space-dispersive media enables us to furnish an explanation of these fields as a part of the potential fields generated by external sources.

The basic electrodynamic property of active media is associated with their ability to support propagating the special kind of slow waves (whose propagation velocity is much less than velocity of light characteristic of a medium under consideration) such as the surface acoustic waves (SAW) in elastic piezo-dielectrics [5–7], the magnetostatic spin waves (MSW) in magnetized ferrites [8–10] and the space-charge waves (SCW) in semiconductors with negative differential mobility of electrons [12–16]. These slow waves, being of multimodal character for composite (multilayered) structures, constitute the so-called quasi-static part of the total electromagnetic spectrum, whose principal feature is related to the predominance of a relevant potential field (electric for SAW and SCW or magnetic for MSW) over its curl counterpart.

Representation of the total fields $(\mathbf{E}, \mathbf{H})$ as a sum of their curl $(\mathbf{E}_c, \mathbf{H}_c)$ and potential $(\mathbf{E}_p = -\nabla \varphi, \mathbf{H}_p = -\nabla \psi)$ parts is realizable on the basis of Helmholtz’s
decomposition theorem [17]. In this theorem, the fields of every $k$th eigenmode can be represented as

$$E_k = E_{ck} + E_{pk} = E_{ck} - \nabla \varphi_k \quad H_k = H_{ck} + H_{pk} = H_{ck} - \nabla \psi_k$$  \hspace{1cm} (2.3)$$

where $\nabla \cdot E_{ck} = 0$ and $\nabla \cdot H_{ck} = 0$.

Therefore, instead of one set of the total eigenfields $\{E_k, H_k\}$ forming the basis of Hilbert space, we now have two sets involving the curl eigenfields $\{E_{ck}, H_{ck}\}$ and the quasi-static eigenpotentials $\{\varphi_k, \psi_k\}$. This extends dimensionality of Hilbert space and allows us to anticipate the possibility of expanding the complementary fields $E_b$ and $H_b$ in terms of the scalar potential basis $\{\varphi_k, \psi_k\}$.

Let us apply the vector curl-field basis $\{E_{ck}, H_{ck}\}$ to expand the desired curl fields

$$E_c(r_t, z) = \sum_k A_k(z) E_{ck}(r_t, z) \quad H_c(r_t, z) = \sum_k A_k(z) H_{ck}(r_t, z)$$  \hspace{1cm} (2.4)$$

and the scalar potential basis $\{\varphi_k, \psi_k\}$ to expand the desired quasi-static potentials

$$\varphi(r_t, z) = \sum_k A_k(z) \varphi_k(r_t, z) \quad \psi(r_t, z) = \sum_k A_k(z) \psi_k(r_t, z).$$  \hspace{1cm} (2.5)$$

In this case the complete fields inside the source region can be written, allowing for relations (2.3), in the following form

$$E = E_c - \nabla \varphi = \sum_k A_k(E_{ck} - \nabla \varphi_k) - z_0 \sum_k \frac{dA_k}{dz} \varphi_k$$

$$= \sum_k A_k E_k - z_0 \sum_k \frac{dA_k}{dz} \varphi_k$$  \hspace{1cm} (2.6)$$

$$H = H_c - \nabla \psi = \sum_k A_k (H_{ck} - \nabla \psi_k) - z_0 \sum_k \frac{dA_k}{dz} \psi_k$$

$$= \sum_k A_k H_k - z_0 \sum_k \frac{dA_k}{dz} \psi_k.$$  \hspace{1cm} (2.7)$$

From comparison of equations (2.6) and (2.7) with (2.1) and (2.2) it follows that

$$E_b = -z_0 \sum_k \frac{dA_k}{dz} \varphi_k \quad H_b = -z_0 \sum_k \frac{dA_k}{dz} \psi_k.$$  \hspace{1cm} (2.8)$$

Formulae (2.8) give the expected expansions of complementary fields in terms of the quasi-static potentials of eigenmodes in a specific form involving the derivatives $dA_k/dz$ in place of the amplitudes $A_k$ as expansion coefficients, which vanish outside the source region where $A_k(z) = \text{const}$. As evidently follows from (2.6) and (2.7), the complementary fields (2.8) are in fact a part of the total potential fields associated with external sources, which was formerly unexpandable, whereas the other part is included in the modal expansions.

All the above results lead us to the important conclusion that the complemented Hilbert space spanned by two sets of base functions, consisting of the curl eigenfields $\{E_{ck}, H_{ck}\}$ and the quasi-static eigenpotentials $\{\varphi_k, \psi_k\}$, is closed with respect to any function corresponding to arbitrary external sources because the desired
representations for the curl fields (2.4) and for the quasi-static potentials (2.5) do not contain any orthogonal complements. So, if we entirely exclude the potential fields \( E_p \) and \( H_p \) from our analysis, by using instead their scalar potentials \( \varphi \) and \( \psi \), together with the curl fields \( E_c \) and \( H_c \), the appropriate sets of the mode quantities \{ \varphi_k, \psi_k \} and \{ E_{ck}, H_{ck} \} will constitute a complete basis that produces the modal expansions (2.4) and (2.5) with no orthogonal complements. In consequence, the latter fact provides the disappearance of the effective surface currents \( J_{s,ef} \) and \( J_{m,ef} \) created by the complementary fields and given by formula (4.32) in [1].

In the subsequent examination, we shall consider that the modal expansions (2.4) and (2.5) involve the transverse distributions of physical quantities for all the modes of SDAM waveguide, which are taken to be known from a preliminary solution to the appropriate source-free boundary-value problem. Hence, the basic task is to find the mode excitation amplitudes \( A_k(z) \) inside the external source region.

3. Constitutive relations and equations of motion for SDAM

As was mentioned above, we consider the generalized (hypothetical) medium possessing simultaneously piezoelectrically-elastic, ferrimagnetic and plasma properties and restrict our consideration to the macroscopic model of SDAM, i.e. the medium under examination is regarded as a continuum characterized by pertinent phenomenological parameters. Such is the case in the long-wavelength approximation, when an excitation wavelength is much greater than typical medium dimensions such as interatomic distances for solids or Debye length for plasmas. For hot electrons in non-degenerate semiconductor plasmas, this holds true for the hydrodynamic, quasi-hydrodynamic and local-field approximations [14].

3.1. Piezoelectrically-elastic properties of a medium

The stressed state of an elastic medium is specified by two second-rank tensors: the stress tensor \( \bar{T} \) and the strain tensor \( \bar{S} \). The components \( S_{ij} \) are related to components \( u_i \) of the vector \( \mathbf{u} \) of medium particle displacement by the known relation [5–7]

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right).
\]  

(3.1)

The components \( T_{ij} \) of stress tensor enter into the dynamic equation for the elastic medium written in the ordinary form of Newton’s equation [5–7]

\[
\rho_m \frac{d\mathbf{U}}{dt} = \frac{\partial T_{ij}}{\partial r_j} \quad \text{or} \quad \rho_m \frac{d\mathbf{U}}{dt} = \nabla \cdot \mathbf{T}
\]  

(3.2)

where \( \rho_m \) is the mass density and \( \mathbf{U} = \partial \mathbf{u} / \partial t \equiv \dot{\mathbf{u}} \) is the medium particle velocity. The left part of equation (3.2) involves the total derivative with respect to time \( d\mathbf{U} / dt = \partial \mathbf{U} / \partial t + \mathbf{U} \cdot \nabla \mathbf{U} \) and the right part expresses the dynamic force \( \mathbf{F} = \nabla \cdot \mathbf{T} \) exerted on a unit mass element (still without allowing for dissipative effects).

If the elastic dielectric medium possesses piezoelectricity, both the strain \( \bar{S} \) and the electric field \( \mathbf{E} \) evoke an appearance of the electric polarization \( \mathbf{P} \) and the elastic stress \( \mathbf{T} \). In this case the constitutive relations can be written in one of the conventional forms (ignoring magnetostrictive effects) [5–7]

\[
P_k = \varepsilon_{kij} S_{ij} + \varepsilon_0 \chi^e E_i \quad \text{or} \quad \mathbf{P} = \varepsilon \cdot \mathbf{S} + \varepsilon_0 \chi^e \cdot \mathbf{E}
\]  

(3.3)

\[
T_{ij} = c_{ijkl} S_{kl} - \varepsilon_{kij} E_k \quad \text{or} \quad \mathbf{T} = \varepsilon^e \cdot \mathbf{S} - \varepsilon \cdot \mathbf{E}.
\]  

(3.4)
In view of the relation \( \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \) (3.3) can be rewritten as
\[
D_k = \epsilon_{kij} S_{ij} + \epsilon_{ik} E_i \quad \text{or} \quad D = \tilde{\epsilon} : \tilde{S} + \epsilon_s \cdot \mathbf{E}.
\] (3.5)

In (3.3) – (3.5), the quantities \( \tilde{\chi}^s \) and \( \tilde{\epsilon}^s = \epsilon_0 (\mathbf{I} + \tilde{\chi}^s) \) are the second-rank susceptibility and permittivity tensors, \( \tilde{\epsilon} \) is the third-rank piezoelectric stress tensor and \( \tilde{\epsilon}^e \) is the fourth-rank elastic stiffness tensor (for brevity sake, the superscripts \( S \) and \( E \) will be dropped below).

Acoustic-wave propagation losses in solids are caused by two dissipative effects which can be introduced into the dynamic equation (3.2) phenomenologically by means of the following quantities:

(i) the internal friction stress \( \mathbf{T}^{fr} \) associated with the existence of viscous properties of an elastic medium, which should be added to the stress tensor \( \mathbf{T} \) (to yield the total stress tensor \( \mathbf{T}^\Sigma = \mathbf{T} + \mathbf{T}^{fr} \)) in the form analogous to that used for an isotropic medium [7]
\[
T^{fr}_{ij} = \eta_{ijkl} \frac{\partial S_{kl}}{\partial t} \quad \text{or} \quad \mathbf{T}^{fr} = \tilde{\eta} : \dot{\mathbf{S}}
\] (3.6)
where \( \tilde{\eta} \) is the viscosity tensor considered as phenomenologically given;

(ii) the dynamic friction force \( \mathbf{F}^{fr} \) exerted by imperfections of a crystal lattice on the motion of acoustic phonons, which should be added to the total dynamic force \( \mathbf{F}^\Sigma = \nabla \cdot \mathbf{T}^\Sigma \) in the form of a relaxation term
\[
F^{fr}_i = -\tau^{-1}_{ij} \rho_m U_i \quad \text{or} \quad \mathbf{F}^{fr} = -\tilde{\tau}^{-1} \cdot \rho_m \mathbf{U}
\] (3.7)
where \( \tilde{\tau}^{-1} \) is the inverse relaxation time tensor regarded as phenomenologically given.

Allowing for relations (3.6) and (3.7), equations (3.1) and (3.2) are rewritten in the following form
\[
\frac{\partial S_{ij}}{\partial t} = -\frac{1}{2} \left( \frac{\partial U_i}{\partial r_j} + \frac{\partial U_j}{\partial r_i} \right)
\] (3.8)
\[
\rho_m \frac{dU}{dt} = \nabla \cdot \mathbf{T}^\Sigma + \mathbf{F}^{fr}.
\] (3.9)

In the case of pure harmonic processes (with time dependence in the form of \( \exp(i \omega t) \)), the above equations can be linearized in small-signal quantities (marked by subscript 1 unlike their static values marked by subscript 0) so that \( \rho_m = \rho_{m0} + \rho_{m1} \), \( U = U_1 \), etc with \( |\rho_{m1}| \ll |\rho_{m0}| \).

3.2. Ferrimagnetic properties of a medium

Macroscopic dynamics of a ferrimagnetic medium uniformly magnetized by an external static field \( \mathbf{H}_e^0 \) to the saturation magnetization \( \mathbf{M}_0 \) is described by the equation of motion written for the total magnetization vector \( \mathbf{M} \) in the following form \([8 – 10]\)
\[
\frac{\partial \mathbf{M}}{\partial t} = -\gamma \mu_0 (\mathbf{M} \times \mathbf{H}_{eff}) + \mathbf{R}
\] (3.10)
where \( \gamma = |e|/m_0 \) is the gyromagnetic ratio for magnetism of spin nature.

The effective magnetic field \( \mathbf{H}_{eff} \) takes into consideration all the torque-producing contributions caused, in addition to \( \mathbf{H}_e^0 \), by: (a) the Maxwellian field \( \mathbf{H} \) (satisfying
Maxwell’s equations), (b) the crystal anisotropy field $H_c = -\mathbf{N}_c \cdot \mathbf{M}$ (due to magnetocrystalline anisotropy of a ferrite material), (c) the demagnetizing field $H_d = -\mathbf{N}_d \cdot \mathbf{M}$ (due to shape anisotropy of a ferrite sample), (d) the exchange field $H_{ex} = \lambda_{ex} \nabla^2 \mathbf{M}$ (due to nonuniform exchange interaction of precessing spins), namely [8–10]

$$H_{eff} = H_0^e + H - \mathbf{N} \cdot \mathbf{M} + \lambda_{ex} \nabla^2 \mathbf{M}.$$ (3.11)

Here the net anisotropy tensor $\mathbf{N} = \mathbf{N}_c + \mathbf{N}_d$ allowing for both the magnetocrystalline anisotropy of a medium and the demagnetization anisotropy of a ferrite sample is assumed to be known, as well as the exchange constant $\lambda_{ex}$.

The relaxation term $R$ taking account of magnetic losses in ferrites is written in different forms, among them more convenient for us is the Gilbert form [8–10]

$$R = \alpha \left( \frac{M}{M_0} \times \frac{\partial M}{\partial t} \right)$$ (3.12)

with the damping parameter $\alpha$ considered as phenomenologically given or found from the resonance line half-width measurements as $\alpha = \Delta H/H_0$ [10].

The Maxwellian field $H$, by its sense, is always a signal quantity, i.e. $H = H_1$, unlike the total magnetization $\mathbf{M}$ which is represented by separating a small-signal magnetization $\mathbf{M}_1$ in the form $\mathbf{M}(r, t) = \mathbf{M}_0 + \mathbf{M}_1(r, t)$ where $|\mathbf{M}_1| << |\mathbf{M}_0|$. Then the effective magnetic field (3.11) takes the following form

$$H_{eff} = H_0 + H_1 - \mathbf{N} \cdot \mathbf{M}_1 + \lambda_{ex} \nabla^2 \mathbf{M}_1$$ (3.13)

where $H_0 = H_0^e - \mathbf{N} \cdot \mathbf{M}_0$ is the static field inside a ferrite sample.

### 3.3. Drifting charge carriers in a medium

For the hydrodynamic description of nondegenerate plasmas with drifting streams of mobile charge carriers, it is more suitable and even necessary to apply, instead of the widely-used Eulerian description, a less known polarization description [13, 14, 18, 19].

As known [11–14], in the hydrodynamic model of non-degenerate plasmas the drifting charge carriers (say, electrons) are represented as a charged fluid flow characterized by such macroscopic quantities as the mean electron density $n$ (or the charge density $\rho = en$), the mean electron velocity $\mathbf{v}$ (or the current density $\mathbf{J} = \rho \mathbf{v}$) and the electron temperature $T$ (or the electron pressure $p = nk_B T$). Microscopic processes of scattering and thermal chaotic motion (or diffusion) of carriers are described in this model by such phenomenological parameters as the momentum relaxation time (or the mean time of free path) $\tau$ and the thermal velocity $v_T = (k_B T/m)^{1/2}$ (or the diffuson constant $D = v_T^2 \tau$). When intercarrier (electron-electron) collisions are rather frequent, there is local thermal equilibrium inside the carrier ensemble with the electron temperature $T$ exceeding a lattice temperature $T_0$ for high electric fields.

In approximation of the local thermal equilibrium, the hydrodynamic force equation has the following form [11–14]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{\nabla (nk_B T)}{mn} - \frac{\mathbf{v}}{\tau}$$ (3.14)

where $m$ is the effective electronic mass different from the mass $m_0$ of a free electron.
Equation (3.14) holds true for the case of uniform stationary heating of electrons when the electron viscosity, heat flow and thermal perturbations in an electron ensemble are negligibly small, so that there is no contribution from the so-called thermoforce and \( \nabla p = m v_T^2 \nabla n \) [14]. Such a situation takes place when \( \tau_e << \tau_M \), where \( \tau_e \) is the energy relaxation time determining the rate of electron temperature perturbations and \( \tau_M = e/\sigma = e m^2 / e^2 n \tau \) is the Maxwellian relaxation time determining the time scale of ac changes in the electric field and charge distribution. This condition means that the temperature keeps pace with signal perturbations in the electric field providing a local relationship between \( T \) and \( E \) [14]. The latter allows the momentum relaxation time \( \tau \) to be considered as a function of the electric field magnitude \( E \), which is given phenomenologically or found from measuring the field dependence of mobility \( \mu(E) = (e/m) \tau(E) \) and diffusion constant \( D(E) = v_T^2 \tau(E) \).

Strictly speaking, for plasmas placed in a magnetic field \( \mathbf{B} \) the electron heating is produced by an electric field known as the effective heating field [14]

\[
E_h = \sqrt{\frac{E^2 + (b \cdot E)^2}{1 + b^2}}
\]

where the quantity \( b = \mu B \) takes into account an influence of magnetic fields on the heating effect so that the quantities \( \tau \), \( \mu \) and \( D \) now depend on \( E_h \).

For a small-signal situation when \( E = E_0 + E_1 \), \( B = B_0 + B_1 \), \( E_h = E_{h0} + E_{h1} \) and all ac values (marked by subscript 1) are assumed to be much smaller than their dc counterparts (marked by subscript 0), we have

\[
\tau(E_h) = \tau(E_{h0}) + \left( \frac{d \ln \tau}{d \ln E_h} \right)_{E_{h0}} E_{h1} \equiv \tau_0 + \tau_1
\]

with \( \tau_0 \equiv \tau(E_{h0}) \) and

\[
\frac{\tau_1}{\tau_0} = \left( \frac{d \ln \tau}{d \ln E_h} \right)_{E_{h0}} \frac{E_{h1}}{E_{h0}} = (\kappa_0 - 1) \frac{E_{h1}}{E_{h0}} = (\kappa_0 - 1) \frac{F_0}{E_0} \cdot \frac{E_1 + v_0 \times B_1}{E_0} \quad (3.15)
\]

Here we have denoted [14]

\[
F_0 = \frac{(1 + b_0^2) \left[ E_0 + (b_0 \cdot E_0) b_0 \right]}{(1 + \kappa_0 b_0^2) + \left[ (1 + b_0^2) + (1 - \kappa_0) \right] (b_0 \cdot E_0)^2 / E_0^2} \quad (3.16)
\]

where \( b_0 = \mu_e B_0 \) and the anisotropy coefficient \( \kappa_0 = \mu_d / \mu_e (\leq 1 \), with inequality being true for hot electrons so that \( \kappa_0 < 0 \) for negative differential mobility) is defined as a ratio of the differential \( (\mu_d) \) and static \( (\mu_e) \) mobilities which are equal to [14]

\[
\mu_d = \frac{d \mu(E) E}{d E} \bigg|_{E=E_{h0}} \quad \text{and} \quad \mu_e = \mu(E_{h0}) = \frac{e}{m} \tau(E_{h0}) = \frac{e}{m} \tau_0 .
\]

Thus, the force equation (3.14) describes dynamics of the field-charge perturbations in hot electron gases characterized by the field dependence of momentum relaxation time \( \tau(E) \) given phenomenologically, which is known as the local field approximation [14].

The polarization \( P \) description holds an intermediate position between the well-known Lagrangian \( L \) and Eulerian \( E \) descriptions. To analyze the small-signal processes in drifting carrier streams it is customary to consider two states of the
stream – unperturbed (without a signal) and perturbed (with a signal). For the two states in the \(L\)-description the motion of a particular group of charges (located inside a physically infinitesimal volume called the liquid particle) is described by a time dependence of two radius-vectors \(r_0(t)\) and \(r(t)\) appropriate to the unperturbed and perturbed stream. The fundamental dynamic variable of the polarization description is defined as a difference in these radius-vectors for two positions of the same liquid particle caused by signal action [14, 18, 19]:

\[
r_1(r_0, t) = r(t) - r_0(t)
\]

(3.17)

which is considered as a function of the unperturbed radius-vector \(r_0\) called the electron displacement vector. Hence, after introducing the displacement vector (3.17) the “trajectory” description of electron motion (typical for the \(L\)-variables) is replaced by the “field” description (typical for the \(E\)-variables). Now we deal with a field of the electron displacement \(r_1(r_0, t)\) which is completely identical to the field of the lattice particle displacement \(u(r_0, t)\) applied in elasticity theory (see section 3.1).

According to the Eulerian and polarization descriptions, the total instantaneous velocity \(v(r, t)\) of a particular group of charges satisfying the dynamic equation (3.14) is represented as [14, 18, 19]

\[
v(r, t) = v_0(r) + u_1(r, t, t) = v_0(r_0) + v_1(r_0, t)
\]

(3.18)

where the static velocity \(v_0\) is taken at two positions (perturbed, \(r_0\), for the \(E\)-description and unperturbed, \(r_0\), for the \(P\)-description) of the same group of charges. The small-signal Eulerian \((u_1)\) and polarization \((v_1)\) velocities are defined by relation (3.18) at different space points but at the same point between them there is the following relation: \(v_1 = u_1 + (r_1 \cdot \nabla)v_0\).

The polarization velocity \(v_1\) adheres to the equation of motion in the \(P\)-variables which is obtained from equation (3.14) in the following form [14, 19]

\[
\frac{\partial v_1}{\partial t} + (v_0 \cdot \nabla)v_1 =
\]

\[
= \frac{e}{m} \left( E_1 + (r_1 \cdot \nabla)E_0 + v_1 \times B_0 + v_0 \times B_1 + v_0 \times (r_1 \cdot \nabla)B_0 \right)
\]

\[+ \frac{v_1^2}{\rho_0} \left( \rho_0 \nabla(\nabla \cdot r_1) + \nabla r_1 \cdot \nabla \rho_0 \right) - \frac{v_1}{\tau_0} + \frac{v_0}{\tau_0} \frac{\tau_1 + (r_1 \cdot \nabla)\tau_0}{\tau_0} \]

(3.19)

where the relaxation times \(\tau_0\) and \(\tau_1\) are given by equation (3.13). This equation allows for spatial nonuniformity in dc quantities \(E_0\), \(B_0\), \(\rho_0\) and \(\tau_0\) caused, for instance, by nonuniform doping of a semiconductor.

Between the polarization variables \(v_1\) and \(r_1\) there is the following relation [14, 18, 19]

\[
v_1 = \frac{\partial r_1}{\partial t} + (v_0 \cdot \nabla)r_1
\]

(3.20)

which proves to play a role of the continuity equation

\[
\frac{\partial \rho_1}{\partial t} + \nabla \cdot J_1 = 0.
\]

(3.21)
Eulerian densities of charge \( \rho_1 \) and current \( J_1 = \rho_1 v_0 + \rho_0 u_1 \) are expressed in the \( P \)-variables in terms of the electronic polarization vector \( p_1 = \rho_0 r_1 \) by the following relations \([14, 18, 19]\):

\[
\rho_1 = - \nabla \cdot p_1
\]

\[
J_1 = \frac{\partial p_1}{\partial t} + \nabla \times (p_1 \times v_0).
\]

It is obvious that expressions \((3.22)\) and \((3.23)\) satisfy the continuity equation \((3.21)\) identically. The charge and current densities introduced by these formulae are fully the same as those produced by the motion of an actual dielectric with the polarization vector \( p_1 \) moving at velocity \( v_0 \) \([20]\). It is this fact that has given the name to the polarization description of mobile charges in vacuo and plasmas.

3.4. Electromagnetic properties of a medium

Electromagnetic fields are described by the usual Maxwell equations written for small-signal quantities as

\[
\nabla \times E_1 = - \frac{\partial B_1}{\partial t} \quad \nabla \times H_1 = \frac{\partial D_1}{\partial t} + J_1
\]

\[
\nabla \cdot D_1 = \rho_1 \quad \nabla \cdot B_1 = 0
\]

where the densities of charge \( \rho_1 \) and current \( J_1 \) are represented in \( P \)-variables by \((3.22)\) and \((3.23)\). The induction vectors are associated with the polarization and magnetization vectors by the known relations \([20]\)

\[
D_1 = \varepsilon_0 E_1 + P_1 \quad \text{and} \quad B_1 = \mu_0 (H_1 + M_1).
\]

For the generalized medium under examination, possessing also piezoelectric and magnetic properties, its polarization \( P_1 \) and magnetization \( M_1 \) adhere, respectively, to the constitutive relation \((3.3)\) and to the equation of motion \((3.10)\) written in a linearized form.

The polarization description of plasmas provides a convenient way to make an artificial replacement of plasma by an equivalent magneto-dielectric medium without mobile charges. Indeed, it is known that a real electric dipole \( p_1 \) moving at velocity \( v_0 \) is perceived, by a fixed observer, as two immovable dipoles: electric \( p_1 \) and magnetic \( m_1 = p_1 \times v_0 \) \([20]\). Hence, the polarization description operating with the electronic polarization vector \( p_1 \) allows any charged medium with mobile carriers to be represented as an equivalent polarized (with \( p_1 \)) and magnetized (with \( m_1 \)) medium with no mobile charges. In this case Maxwell’s equations \((3.24)\) by inserting expressions \((3.22)\) and \((3.23)\) can be written in the following form

\[
\nabla \times E_1 = - \frac{\partial B_1}{\partial t} \quad \nabla \times H_1^p = \frac{\partial D_1^p}{\partial t}
\]

\[
\nabla \cdot D_1^p = 0 \quad \nabla \cdot B_1 = 0
\]

where the fundamental vectors \( E_1 \) and \( B_1 \) remain unchanged while the vectors \( H_1 \) and \( D_1 \) are replaced with their equivalent polarization counterparts \( H_1^p \) and \( D_1^p \). The
latter are introduced so that expressions (3.25) would hold their form with the use of these new vectors, namely

\[ D_p^1 = \varepsilon_0 E_1 + P_p^1 \quad \text{and} \quad B_1 = \mu_0 (H_1^p + M_p^1) \]  

(3.27)

where the total polarization and magnetization vectors defined as

\[ P_p^1 = P_1 + p_1 \quad \text{and} \quad M_p^1 = M_1 + m_1 \]  

(3.28)

take into account the contributions from both a crystal lattice \((P_1 \text{ and } M_1)\) and an electron ensemble \((p_1 \text{ and } m_1) \equiv p_1 \times v_0\). Thus, the new equivalent field vectors entering into Maxwell’s equations (3.26) in the polarization description are equal to

\[ D_p^1 = D_1 + p_1 \quad \text{and} \quad H_p^1 = H_1 + v_0 \times p_1. \]  

(3.29)

Therefore, using Maxwell’s equations of the “dielectric” form (3.26) enables one to consider all media including plasmas, as pure dielectric and describe them by applying the equivalent field vectors (3.29), which, in addition to the lattice polarization \((P_1 \text{ and } M_1)\), allow for the electronic polarization \(p_1\) due to mobile charge carriers. In this case, the components of all field vectors, as follows from equations (3.26), are continuous on the boundaries of drifting carrier streams. Such a feature makes it more preferable to use the equivalent “dielectric” form of Maxwell’s equations (without \(J_1\) and \(\rho_1\)) instead of the usual form (3.24) (with \(J_1\) and \(\rho_1\)). The point is that the latter generate the equivalent surface charge \(\rho_{eq}^s = n \cdot p_1\) and surface current \(J_{eq}^s = (n \cdot p_1)v_0\) on the boundaries of drifting carrier streams [14, 15, 21], which ensures discontinuity in the appropriate field components.

4. Orthogonality and quasi-orthogonality of modes and equations of mode excitation

4.1. The generalized reciprocity relation (the extended Lorentz lemma)

Up to this point, we have considered the source-free region of a waveguiding structure with SDAM whose electromagnetic properties are described by the usual Maxwell equations (3.24) or their equivalent “dielectric” form (3.26). The latter form, which allows for the electronic polarization of drifting charge carriers, along with the lattice polarization and magnetization, is more suitable for subsequent examinations. Inside the region of exciting bulk sources \((J_{e1}^b, J_{m1}^b, \rho_{e1}^b, \rho_{m1}^b)\) its curl equations are written for pure harmonic processes in the form

\[ \nabla \times E_1 = -i\omega B_1 - J_{b1}^m \quad \text{and} \quad \nabla \times H_1^p = i\omega D_1^p + J_{b1}^e \]  

(4.1)

where the electric and magnetic sources obey the continuity equations

\[ i\omega \rho_{b1}^e + \nabla \cdot J_{b1}^m = 0 \quad \text{and} \quad i\omega \rho_{b1}^m + \nabla \cdot J_{b1}^e = 0. \]  

(4.2)

As follows from (3.27) – (3.29), in the equivalent magneto-dielectric medium characterized by the total (lattice and electronic) polarization \(P_p^1 = P_1 + p_1\) and magnetization \(M_p^1 = M_1 + m_1\), the field-intensity vectors \(E_1\) and \(H_1^p = H_1 + v_0 \times p_1\) produce the flux-density vectors \(D_1^p\) and \(B_1\):

\[ D_1^p = \varepsilon_0 E_1 + P_p^1 = \varepsilon_0 E_1 + P_1 + p_1 \]  

(4.3)

\[ B_1 = \mu_0 (H_1^p + M_p^1) = \mu_0 (H_1 + M_1) \]  

(4.4)
where the vectors $P_1$, $M_1$ and $p_1$ reflect physical properties of the medium discussed in section 3.

A basis for deriving the equations of mode excitation is the reciprocity relation in complex conjugate form (often called the Lorentz lemma) extended to the generalized medium under consideration with piezoelectrically-elastic, ferrimagnetic and plasma properties.

To derive the conjugate reciprocity relation it is necessary, in addition to the given system of equations (marked with subscript 1), to consider another system (marked with subscript 2 having also small-signal meaning) whose dynamic equations are all taken in complex conjugate form (see (5.25) and (5.26) of [1]). A conventional procedure applied to these equations gives relation (5.27) (see section 5.2.1 in [1]), which can be rewritten with the aid of equations (4.3) and (4.4) in the following form

$$\nabla \cdot \left( E_1 \times H_2^p + E_2^* \times H_1^p \right) = -i \omega \left( P_1 \cdot E_2^* - P_2^* \cdot E_1 \right)$$

$$- \omega \mu_0 \left( M_1 \cdot H_2^p - M_2^* \cdot H_1 \right) - i \omega \left( p_1 \cdot E_2^p - p_2^* \cdot E_1^p \right)$$

$$- \left( J_{b1}^e \cdot E_2^* + J_{b2}^e \cdot E_1 \right) - \left( J_{b1}^m \cdot H_2^p + J_{b2}^m \cdot H_1^p \right)$$

(4.5)

where $E_{1,2}^p = E_{1,2} + v_0 \times B_{1,2}$ is the electric field measured relative to an observer moving with the nonrelativistic velocity $v_0$.

The first three terms on the right of equation (4.5) are calculated in the Appendix by the help of the appropriate equations of medium motion for elastic piezo-dielectrics, magnetized ferrites and drifting charge carrier streams (see section 3). Substitution of equations (A.10), (A.14) and (A.22) into (4.5) finally gives the desired reciprocity relation (cf. equation (5.28) in [1])

$$\nabla \cdot S_{12} + q_{12} = \tau^{(b)}_{12}.$$ (4.6)

Unlike equation (5.28) of [1], here the total power quantity

$$S_{12} = S_{12}^{EM} + S_{12}^{PM}$$ (4.7)

in addition to the usual electromagnetic contribution (with the polarization modification $H_{1,2} \rightarrow H_{1,2}^p$ [13, 21, 22])

$$S_{12}^{EM} = \left( E_1 \times H_2^p + E_2^* \times H_1^p \right)$$ (4.8)

contains also the contribution from non-electromagnetic fields in polarized media

$$S_{12}^{PM} = - \left( \vec{T}_{1}^e \cdot U_2^p + \vec{T}_{2}^e \cdot U_1 \right)$$

$$+ \left( \bar{V}_{1}^m \cdot J_{2}^m + \bar{V}_{2}^m \cdot J_{1}^m \right) + \left( \bar{V}_{1}^e \cdot J_{2}^e + \bar{V}_{2}^e \cdot J_{1}^e \right).$$ (4.9)

Here we have introduced new quantities:

(i) for ferrimagnetic media – a vector $J_1^m$ of the effective magnetization current density and a tensor $V_1^m$ of the effective magnetic (exchange) potential equal to

$$J_1^m = i \omega \mu_0 M_1 \quad V_1^m = - \lambda_{ex} \nabla M_1$$ (4.10)
(ii) for plasma media - a vector $J_1^p$ of the electronic polarization current density and a tensor $V_1^{ek} = V_1^{ek} + V_1^{th}I$ of the effective electronic potential involving the electrokinetic potential tensor $V_1^{ek}$ and the thermal (diffusion) potential $V_1^{th}$ equal to

$$J_1^p = i\omega p_1 \quad V_1^{ek} = \frac{m}{e} v_0 v_1^p \quad V_1^{th} = \frac{m}{e} \frac{v^2}{\rho_0} p_1$$

(4.11)

where $v_1^p = v_1 - (e/2m)(r_1 \times B_0)$ is the resulting small-signal velocity of electrons in the polarization description allowing for their rotation in the static magnetic field $B_0$ with the Larmor angular velocity $\omega_L = -(e/2m)B_0$ [21, 22].

The relaxation processes in the generalized medium are taken into account by the sum of three contributions:

$$q_{12} = 2 \left( \omega^2 \bar{S}^* \cdot \dot{\bar{S}} + \bar{S}_1 + \rho_{m0} U_2^* \cdot \tau^{-1} \cdot U_1 \right) + 2 \nu_{m0} \left( \frac{\omega}{\omega_{m}} \right)^2 \left( M_1 \cdot M_2^* \right)$$

$$+ \frac{1}{\mu_0} \left( (v_1 - \vartheta_1 v_0) \cdot J_2^* + (v_2^* - \vartheta_2^* v_0) \cdot J_1^* \right)$$

(4.12)

where we have introduced (i) for ferrimagnetic media the magnetic relaxation frequency $\nu_m = \alpha \omega_{m} = \alpha_\gamma \mu_0 M_0$ and (ii) for plasma media the quantity

$$\vartheta_1 \equiv \frac{\tau_1 + r_1 \cdot \nabla \tau_0}{\tau_0} = \left( \kappa_0 - 1 \right) \frac{E_0}{E_0} \cdot \frac{E_1 + v_0 \times B_1}{E_0}$$

with the last expression obtained from (3.13) by ignoring spatial static non-uniformities (so that $\nabla \tau_0 = 0$) and assuming that the vector $B_0$ is longitudinal (so that $b_0 \cdot E_0 = b_0 E_0$ and $E_0 = E_0$, as a result of (3.16)) [14].

The interaction of the electromagnetic fields with the bulk external currents is taken into account by the following term in the right-hand side of (4.4):

$$r_{12}^{(b)} = - \left( J_{b_1}^e \cdot H_2^{p*} + J_{b_2}^e \cdot H_1^{p*} \right) - \left( J_{b_1}^{m*} \cdot H_2^{p*} + J_{b_2}^{m*} \cdot H_1^{p} \right).$$

(4.13)

Representation of the electromagnetic field vectors $(E_{1,2}^e, H_{1,2}^{p*})$ as the sum of their curl $(E_{1,2}^{c1,2}, H_{1,2}^{c1,2})$ and potential $(-\nabla \varphi_{1,2}, -\nabla \psi_{1,2})$ parts, as in (2.3), gives

$$\nabla \cdot \left( E_1^e \times H_2^{p*} + H_2^e \times H_1^{p*} \right) = \nabla \cdot \left[ \left( E_{c1}^{e} \times H_{c2}^{p*} + E_{c2}^{e} \times H_{c1}^{p*} \right) \right]$$

$$+ \left( \varphi_1 (i\omega D_{2}^{p*}) + \varphi_2^* (i\omega D_{1}^{p*}) \right) + \left( \psi_1 (i\omega B_{2}) + \psi_2^* (i\omega B_{1}) \right)$$

$$+ \left( J_{b1}^e \cdot \nabla \varphi_2^* + J_{b2}^e \cdot \nabla \varphi_1 \right) + \left( J_{b1}^{m*} \cdot \nabla \psi_2 + J_{b2}^{m*} \cdot \nabla \psi_1 \right)$$

$$- \left( i\omega \rho_{b1}^e \varphi_2 + i\omega \rho_{b2}^{m*} \varphi_1 \right) - \left( i\omega \rho_{b1}^{m*} \psi_2 + i\omega \rho_{b2}^e \varphi_1 \right)$$

(4.14)

where we have used the vector identity $\nabla \cdot (A \times \nabla \phi) = \nabla \cdot (\phi \nabla \times A)$ and equations (4.1) and (4.2).

The use of (4.14) in the reciprocity relation (4.6) leaves expressions (4.9) and (4.12) for $S_{12}^{EM}$ and $q_{12}$ unchanged, but modifies $S_{12}^{EM}$ and $r_{12}^{(b)}$, which are now equal to

$$S_{12}^{EM} = \left( E_{c1} \times H_{c2}^{p*} + E_{c2}^{e} \times H_{c1}^{p*} \right)$$
4.2. Quasi-orthogonality and orthogonality relations for SDAM waveguides

To this end, the differential form (4.6) of the reciprocity relation should be converted into curl and potential parts) provides a basis for deriving the mode quasi-fields or with equations (4.7), (4.9), (4.12), (4.15) and (4.16) for the fields separated from (4.6) (with equations (4.7) – (4.9), (4.12) and (4.13) for the total electromagnetic fields or with equations (4.7), (4.9), (4.12), (4.15) and (4.16) for the fields separated into curl and potential parts) provides a basis for deriving the mode quasi-orthogonality relations and the equations of mode excitation in SDAM waveguides.

Let subscripts 1 and 2 in the reciprocity relation (4.6) correspond to two eigenmodes with numbers \( k \) and \( l \) for a waveguiding structure with SDAM in the absence of sources, i.e. \( r_{12}^{(b)} = 0 \). Then integrating equation (4.6) over the total cross section \( S \) of the waveguide and applying the integral relation (2.25) in [1] give

\[
dP_{kl}(z) + Q_{kl}(z) = 0
\]

where the complex cross-power (or self-power for \( k = l \)) flow transferred jointly by the \( k \)th and \( l \)th modes is

\[
P_{kl}(z) = P_{kl}^{EM} + P_{kl}^{PM} = \frac{1}{4} \int_S \left( S_{kl}^{EM*} + S_{kl}^{PM*} \right) \cdot z_0 \, dS
\]

and the complex cross-power (or self-power for \( k = l \)) loss dissipated jointly by the \( k \)th and \( l \)th modes is

\[
Q_{kl}(z) = \frac{1}{4} \int_S q_{kl}^*(\mathbf{r}_z) \, dS = \frac{1}{4} M_{kl} A_k^* A_l e^{-\frac{1}{2}(\gamma_k^2 + \gamma_l^2)z} = \frac{1}{4} M_{kl} a_k^* a_l.
\]

The quantities \( S_{kl}^{EM} \), \( S_{kl}^{PM} \) and \( q_{kl} \) in these formulas are obtained from (4.18) or (4.13), (4.9) and (4.15) by replacing subscripts 1 and 2 with \( k \) and \( l \), respectively. The quantities \( P_{kl} \) and \( Q_{kl} \) defined by (4.18) and (4.19) determine:

(i) the total power flow (cf (2.34) and (2.46) of [1])

\[
P(z) = \sum_k \sum_l P_{kl}(z) = \frac{1}{4} \sum_k N_k |a_k(z)|^2 + \frac{1}{2} \Re \sum_k \sum_{l>k} N_{kl} a_k^*(z) a_l(z)
\]

Formula (4.13) involves three contributions to the transferred power from the electromagnetic curl fields, quasi-electrostatic potential fields and quasi-magnetostatic potential fields. Formula (4.16) reflects the interactions of the curl fields with the external bulk currents and the potential fields with the external bulk charges (surface exciting sources will be considered later).

The generalized reciprocity relation (or the extended Lorentz lemma) in the form (4.6) (with equations (4.7) – (4.9), (4.12) and (4.13) for the total electromagnetic fields or with equations (4.7), (4.9), (4.12), (4.15) and (4.16) for the fields separated into curl and potential parts) provides a basis for deriving the mode quasi-orthogonality relations and the equations of mode excitation in SDAM waveguides. To this end, the differential form (4.6) of the reciprocity relation should be converted into an integral form.

\[
\begin{align*}
&+ \left( \varphi_1^* (i \omega \mathbf{D}_2^p) + \varphi_2^* (i \omega \mathbf{D}_1^p) \right) \left( \psi_1^* (i \omega \mathbf{B}_2) + \psi_2^* (i \omega \mathbf{B}_1) \right) \\
&+ \left( \left( J_{k1}^c \cdot \mathbf{E}_c^* + J_{k2}^c \cdot \mathbf{E}_c \right) - \left( J_{l1}^m \cdot \mathbf{H}_c^p + J_{l2}^m \cdot \mathbf{H}_c^p \right) \right)
\end{align*}
\]

(4.15)

\[
r_{12}^{(b)} = - \left( J_{k1}^c \cdot \mathbf{E}_c^* + J_{k2}^c \cdot \mathbf{E}_c \right) - \left( J_{l1}^m \cdot \mathbf{H}_c^p + J_{l2}^m \cdot \mathbf{H}_c^p \right)
\]

(4.16)
(ii) the total power loss (cf. equations (2.35) and (2.47) of paper [1])

\[ Q(z) = \sum_k \sum_l Q_{kl}(z) = \frac{1}{4} \sum_k M_k |a_k(z)|^2 + \frac{1}{2} \text{Re} \sum_k \sum_{l>k} M_{kl} a_k^*(z)a_l(z) \]  

(4.21)

where the mode amplitude \( a_k \) is related to the excitation amplitude \( A_k \) by formula

\[ a_k = A_k \exp(-\gamma_k z) \]

for every \( k \)th mode specified by the propagation constant \( \gamma_k = \alpha_k + i\beta_k \) and the set of cross-section eigenfunctions (marked with hat over them) \{ \( \hat{E}_k(r_1), \hat{H}_k(r_1), \) etc \} such that \( \hat{E}_k(r_1, z) = \hat{E}_k(r_1) \exp(-\gamma_k z), \hat{H}_k(r_1, z) = \hat{H}_k(r_1) \exp(-\gamma_k z), \) etc.

In (4.18) – (4.21), following formulae (2.36) and (2.37) of [1], we have introduced the normalizing \( (N_{kl}) \) and dissipative \( (M_{kl}) \) coefficients constructed of the cross section eigenfield vectors. Expression for the normalizing coefficients \( (N_{kl} = N_{kl}^{EM} + N_{kl}^{PM}) \) is given by formula (4.53) or (4.57), while the dissipative coefficients are equal to

\[ M_{kl} = 2 \int_S \left[ \left( \omega^2 (\hat{S}_k^* \cdot \hat{\eta} \cdot \hat{S}_l) + \rho_m (\hat{U}_k^* \cdot \hat{\tau}^{-1} \cdot \hat{U}_l) \right) + \nu_m \mu_0 \left( \frac{\omega}{\omega_m} \right)^2 (M_k^* \cdot M_l) \right. \]

\[ + \frac{1}{2\mu_e} \left( \left( \hat{w}_k^* - \hat{\vartheta}_k \nu_0 \right) \cdot \hat{J}_l^* + \left( \hat{w}_l - \hat{\vartheta}_l \nu_0 \right) \cdot \hat{J}_k^* \right) \bigg] \text{d}S. \]  

(4.22)

Formula (4.17) is in fact the required relation of mode quasi-orthogonality which reflects an independent transmission and dissipation of power by any one of mode pairs \((k, l)\) in a lossy system (see section 3.1 of [1]). The expression of \( P_{kl} \) and \( Q_{kl} \) in terms of \( N_{kl} \) and \( M_{kl} \) is written in (4.18) and (4.19), owing to formulae (2.41) and (2.42) in [1]. Substitution of (4.18) and (4.19) in equation (4.17) yields the desired form of the general quasi-orthogonality relation

\[ (\gamma_k^* + \gamma_l) P_{kl} = Q_{kl} \quad \text{or} \quad (\gamma_k^* + \gamma_l) N_{kl} = M_{kl} \]  

(4.23)

which turns into the usual orthogonality relation for lossless waveguiding structures as a special case with \( Q_{kl} = M_{kl} = 0 \).

Since relations (4.23) are completely coincident with the similar relations (3.6) and (3.7) of [1], the reasoning given there in section 3, concerning the quasi-orthogonality of eigenmodes in lossy waveguides (when \( M_{kl} \neq 0 \)) and the orthogonality of active and reactive eigenmodes in lossless waveguides (when \( M_{kl} = 0 \)), remains true for the waveguiding structures with SDAM.

4.3. Equations of mode excitation for SDAM waveguides

To derive the equations of mode excitation, the fields in the reciprocity relation (4.6) marked by subscript 1 (which will be dropped for exciting sources) are assumed to be the desired fields excited by the bulk and surface sources, whereas those marked by subscript 2 are the known fields of the \( k \)th mode without sources. Any one of the physical quantities \( \Phi \) (e.g. the components of electromagnetic fields, potentials, polarizations, etc) is written for the \( k \)th mode as

\[ \Phi_k(r_1, z) = \hat{\Phi}_k(r_1) e^{-\gamma_k z} \]  

(4.24)

while the same quantity in the source region is represented in the complete form

\[ \Phi_l(r_1, z) = \Phi_a(r_1, z) + \Phi_b(r_1, z) = \sum_l A_l(z) \Phi_l(r_1, z) + \Phi_b(r_1, z) \]
\[
\sum_l A_l(z) \hat{\phi}_l(r_1) e^{-\gamma l z} + \phi_b(r_1, z) \equiv \sum_l a_l(z) \hat{\phi}_l(r_1) + \phi_b(r_1, z)
\]  
(4.25)

involving the modal expansion \( \phi_a \) and the orthogonal complement \( \phi_b \). The latter is true for all physical quantities except the curl fields and quasi-static potentials for which, according to equations (2.4) and (2.5), \( \mathbf{E}_{cb} = \mathbf{H}_{cb}^0 = 0 \) and \( \varphi_b = \psi_b = 0 \).

The separation of potential fields causes the surface boundary conditions (formerly written for the total fields in the form of relations (4.6) and (4.7) in [1]) to be reformulated. Inside the source region, besides the external bulk currents \( \mathbf{J}_s^{e} \) and \( \mathbf{J}_s^{m} \) which results in discontinuity in the normal component of the curl magnetic and electric fields \( \mathbf{n}_s \times \mathbf{H}_1^{cl} \) and \( \mathbf{n}_s \times \mathbf{E}_1^{cl} \), respectively;

(a) the current sheet with the electric and magnetic surface current densities \( \mathbf{n}_s \times \mathbf{E}_1^{cl} \) and \( \mathbf{n}_s \times \mathbf{H}_1^{cl} \), respectively;

(b) the charge sheet with the electric and magnetic surface charge densities \( \rho_s^e \) and \( \rho_s^m \) which results in discontinuity in the normal component of the electric and magnetic inductions \( \mathbf{n}_s \cdot \mathbf{D}_1^{p+} \) and \( \mathbf{n}_s \cdot \mathbf{B}_1^{p+} \), respectively;

(c) the dipole (double charge) sheet with the electric and magnetic surface current densities \( \eta_s^e \) and \( \eta_s^m \) which results in discontinuity in the quasi-static electric and magnetic potentials \( \varphi_1 \) and \( \psi_1 \), respectively.

The corresponding boundary conditions written in pairs for the electric and magnetic sources have the following form [14, 20]:

(a) for the current sheets
\[
\mathbf{n}_s^+ \times \mathbf{E}_1^{cl} + \mathbf{n}_s^- \times \mathbf{E}_1^{cl} = - \mathbf{J}_s^m
\]  
(4.26)
\[
\mathbf{n}_s^+ \times \mathbf{H}_1^{cl} + \mathbf{n}_s^- \times \mathbf{H}_1^{cl} = \mathbf{J}_s^e
\]  
(4.27)

(b) for the charge sheets
\[
\mathbf{n}_s^+ \cdot \mathbf{D}_1^{p+} + \mathbf{n}_s^- \cdot \mathbf{D}_1^{p+} = \rho_s^e
\]  
(4.28)
\[
\mathbf{n}_s^+ \cdot \mathbf{B}_1^{p+} + \mathbf{n}_s^- \cdot \mathbf{B}_1^{p+} = \rho_s^m
\]  
(4.29)

(c) for the dipole sheets
\[
\mathbf{n}_s^+ \varphi_1^+ + \mathbf{n}_s^- \varphi_1^- = \frac{1}{\varepsilon_0} \eta_s^e
\]  
(4.30)
\[
\mathbf{n}_s^+ \psi_1^+ + \mathbf{n}_s^- \psi_1^- = \frac{1}{\mu_0} \eta_s^m
\]  
(4.31)

Here, all the quantities with superscripts \( \pm \) are values taken at points of the contour \( L_s \) lying on its different sides marked by the inward unit normal vector \( \mathbf{n}_s^\pm \).

For deriving the equations of mode excitation it is necessary to integrate the reciprocity relation (4.1) (with replacing subscripts 2 by \( k \)) over the cross section \( S \) of a waveguide by using relation (2.25) in [1], with the result depending on whether the potential fields are separated.

Without separating the potential fields (when \( \mathbf{E}_1 = \mathbf{E}_a + \mathbf{E}_b \) and \( \mathbf{H}_1^p = \mathbf{H}_a^p + \mathbf{H}_b^p \) in accordance with equation (4.25)), the substitution of equation (1.8) (with changing subscripts \( 2 \rightarrow k \)) into the integral relation (2.25) of [1] yields
\[
\int_S \nabla \cdot \mathbf{S}_{ik}^{EM} \, dS = \frac{\partial}{\partial z} \int_S \left( \mathbf{E}_a \times \mathbf{H}_k^{p+} + \mathbf{E}_k^+ \times \mathbf{H}_a^p \right) \cdot \mathbf{z}_0 \, dS
\]
\[ + \int_{L_b} n_b \cdot (E_b \times H_k^{ps} + E_k^s \times H_b^p) \, dl - \int_{L_s} \left[ n_s^+ \cdot (E_1 \times H_k^{ps} + E_k^s \times H_1^p) + n_s^- \cdot (E_1 \times H_k^{ps} + E_k^s \times H_1^p) \right] \, dl = \frac{\partial}{\partial z} \int_S S_{ik}^{EM} \cdot z_0 \, dS \]

\[ + \int_{L_s} \left( J_s^e \cdot E_k^s + J_s^m \cdot H_k^{ps} \right) \, dl + \int_{L_b} \left( J_s^{ef} \cdot E_k^s + J_s^{ef} \cdot H_k^{ps} \right) \, dl \]  

(4.32)

where

\[ S_{ik}^{EM} = E_1 \times H_k^{ps} + E_k^s \times H_1^p. \]  

(4.33)

Here we have used: (i) the boundary conditions (4.6) and (4.7) of paper [1] with the real surface currents \( J_s^e \) and \( J_s^m \) given on the contour \( L_s \) with the unit normal vectors \( n_s^+ \), (ii) the effective surface currents \( J_s^{ef} = -n_b \times H_b^p \) and \( J_s^{ef} = n_b \times E_b \) defined on the boundary contour \( L_b \) of the bulk source area \( S_b \) with the outward unit normal vector \( n_b \) (see (4.24) – (4.28) in [1]).

With separating the potential fields (when \( E_{cb} = H_{cb}^p = 0 \) and \( \varphi_b = \psi_b = 0 \) in accordance with equations (2.4) and (2.5)), the substitution of (4.13) (with changing subscripts 2 \( \rightarrow \) k) into the integral relation (2.25) of [1] yields

\[ \int_S \nabla \cdot S_{ik}^{EM} \, dS = \frac{\partial}{\partial z} \int_S \left( E_{c1} \times H_{ck}^{ps} + E_{ck} \times H_{c1}^{ps} \right) \]  

\[ + \left( \varphi_1 (i\omega D_1^p)^* + \varphi_k (i\omega D_k^p)^* \right) + \left( \psi_1 (i\omega B_1)^* + \psi_k (i\omega B_1)^* \right) \) \cdot z_0 \, dS \]

\[ - \int_{L_s} \left[ n_s^+ \cdot \left( E_{c1} \times H_{ck}^{ps} + E_{ck} \times H_{c1}^{ps} \right) + n_s^- \cdot \left( E_{c1} \times H_{ck}^{ps} + E_{ck} \times H_{c1}^{ps} \right) \right] \, dl \]

\[ - \int_{L_s} \left[ n_s^+ \cdot \left( \varphi_1 (i\omega D_1^p)^* + \varphi_k (i\omega D_k^p)^* \right) + n_s^- \cdot \left( \varphi_1 (i\omega D_1^p)^* + \varphi_k (i\omega D_k^p)^* \right) \right] \, dl \]

\[ - \int_{L_s} \left[ n_s^+ \cdot \left( \psi_1 (i\omega B_1)^* + \psi_k (i\omega B_1)^* \right) + n_s^- \cdot \left( \psi_1 (i\omega B_1)^* + \psi_k (i\omega B_1)^* \right) \right] \, dl. \]

The use of the boundary conditions (4.26) – (4.31) for rearranging line integrals in the last formula gives

\[ \int_S \nabla \cdot S_{ik}^{EM} \, dS = \frac{\partial}{\partial z} \int_S S_{ik}^{EM} \cdot z_0 \, dS + \int_L \left( J_s^e \cdot E_{ck}^s + J_s^m \cdot H_{ck}^{ps} \right) \, dl \]

\[ - \int_{L_s} \left[ (i\omega \rho_k^e \varphi_k^e + i\omega \rho_k^m \psi_k^m) + \left( \frac{n_k^e}{\epsilon_0} \cdot (i\omega D_k^p)^* + \frac{n_k^m}{\mu_0} \cdot (i\omega B_k)^* \right) \right] \, dl \]  

(4.34)

where

\[ S_{ik}^{EM} = \left( E_{c1} \times H_{ck}^{ps} + E_{ck} \times H_{c1}^{ps} \right) + \left( \varphi_1 (i\omega D_1^p)^* + \varphi_k (i\omega D_k^p)^* \right) \]

\[ + \left( \psi_1 (i\omega B_1)^* + \psi_k (i\omega B_1)^* \right) \]  

(4.35)
It should be noted that, as distinct from equation \((4.32)\), expression \((4.34)\) does not contain the effective surface sources like \(J_{e,ef}\) since the curl fields \((2.4)\) and the quasi-static potentials \((2.5)\) have no orthogonal complements.

Substitution of equation \((4.9)\) (with changing subscripts \(2 \rightarrow k\)) into the integral relation \((2.25)\) of [1] yields

\[
\int_S \nabla \cdot S_{1k}^{PM} \, dS = \frac{\partial}{\partial z} \int_S S_{1k}^{PM} \cdot z_0 \, dS \tag{4.36}
\]

where

\[
S_{1k}^{PM} = - \left( T_1^e \cdot U_k^e + \bar{T}_k^{Es} \cdot U_1 \right) + \left( \bar{V}_1^m \cdot J_k^{m*} + \bar{V}_k^{m*} \cdot J_1^m \right) + \left( \bar{V}_1^e \cdot J_k^e + \bar{V}_k^{e*} \cdot J_1^e \right) \tag{4.37}
\]

Integrating the reciprocity relation \((4.6)\) (with changing subscript \(2 \rightarrow k\)) over the cross section \(S\) of a waveguiding structure and using \((4.32)\), \((4.33)\) or \((4.34)\), \((4.35)\) along with \((4.36)\) and \((4.37)\) give the following result:

\[
\frac{dP_{1k}(z)}{dz} + Q_{1k}(z) = R_{1k}(z) \tag{4.38}
\]

where

\[
P_{1k} = P_{1k}^{EM} + P_{1k}^{PM} = \int_S \left( S_{1k}^{EM} + S_{1k}^{PM} \right) \cdot z_0 \, dS \tag{4.39}
\]

\[
Q_{1k} = \int_S q_{1k} \, dS \tag{4.40}
\]

\[
R_{1k} = R_{1k}^{(b)} + R_{1k}^{(s)} = \int_{S_b} r_{1k}^{(b)} \, dS + \int_{L_s} r_{1k}^{(s)} \, dl \tag{4.41}
\]

The loss term \(Q_{1k}\) has the universal form determined by formula \((4.12)\) (with changing subscripts \(2 \rightarrow k\)) for \(q_{1k}\). In contrast, the expressions for \(P_{1k}\) and \(R_{1k}\) are obtained different depending on whether the potential fields are separated.

Without separating the potential fields, the power term \((4.39)\) involves \(S_{1k}^{EM}\) and \(S_{1k}^{PM}\) given by \((4.33)\) and \((4.37)\), respectively. The bulk excitation term determined by \((4.13)\) (with \(J_{b2}^e = 0\) and changing subscripts \(2 \rightarrow k\)) is equal to

\[
R_{1k}^{(b)} = \int_{S_b} r_{1k}^{(b)} \, dS = - \int_{S_b} \left( J_k^e \cdot E_k^e + J_k^m \cdot H_k^{p*} \right) dS \tag{4.42}
\]

and the surface excitation term follows from equation \((4.32)\) in the form

\[
R_{1k}^{(s)} = \int_{L_s + L_b} r_{1k}^{(s)} \, dl = - \int_{L_s} \left( J_k^e \cdot E_k^e + J_k^m \cdot H_k^{p*} \right) dl - \int_{L_b} \left( J_{e,ef}^m \cdot E_k^e + J_{e,ef}^m \cdot H_k^{p*} \right) dl. \tag{4.43}
\]

With separating the potential fields, the quantity \(S_{1k}^{EM}\) and \(S_{1k}^{PM}\) determining the power term \((4.39)\) are given by \((4.35)\) and \((4.37)\), respectively. The bulk excitation
term determined by equations (4.16) (with $J_{k2}^v = \rho_{k2}^v = 0$ and changing subscripts $2 \rightarrow k$) is equal to

$$R_{1k}^{(b)} = \int_{S_{1k}} r_{1k}^{(b)} \, dS = \int_{S_{1k}} \left[ - \left( J_{s}^r \cdot E_{z}^{*} + J_{m}^{B} \cdot H_{z}^{*} \right) \right. + \left( \omega \rho_{p}^{s} \varphi_{k}^{s} + \omega \rho_{m}^{m} \psi_{k}^{m} \right) \right] dS$$

and the surface excitation term, according to equation (4.34), take the form

$$R_{1k}^{(s)} = \int_{L_{1k}} \frac{r_{1k}^{(s)}}{dS} = \int_{L_{1k}} \left[ - \left( J_{s}^r \cdot E_{z}^{*} + J_{m}^{B} \cdot H_{z}^{*} \right) \right. + \left( \frac{\eta_{p}}{\epsilon_{0}} \cdot (\omega D_{k}^{p})^{*} + \frac{\eta_{m}}{\mu_{0}} \cdot (\omega B_{k}^{p})^{*} \right) \right] dS$$

The quadratic (power) quantities $P_{1k}$ and $Q_{1k}$ in the form of (4.38) and (4.40) are constructed of the linear quantities: (i) $\Phi_{k}$ of the form (4.24) and (ii) $\Phi_{l} = \Phi_{a} + \Phi_{b}$ of the form (4.25) involving the modal expansion $\Phi_{a} = \sum_{l} A_{l} \Phi_{l}$ and the orthogonal complement $\Phi_{b}$ (the latter is absent for the curl fields and quasi-static potentials). On this basis, we can rewrite (4.39) and (4.40) as

$$P_{1k} = P_{ak} + P_{bk} \quad P_{ak} = P_{ak}^{EM} + P_{ak}^{PM} = \int_{S} \left( S_{ak}^{EM} + S_{ak}^{PM} \right) \cdot z_{0} \, dS$$

$$Q_{1k} = Q_{ak} + Q_{bk} \quad Q_{ak} = \int_{S} q_{ak} \, dS.$$

Here the subscripts $a$ and $b$ correspond to using the modal expansions $\Phi_{a}$ and the orthogonal complements $\Phi_{b}$ to construct the appropriate quadratic quantities; therewith $P_{bk} = P_{bk}^{EM} + P_{bk}^{PM} = P_{bk}^{PM}$ since always $P_{bk}^{EM} = 0$ by virtue of the fact that without separating potential fields $S_{bk}^{EM} \cdot z_{0} = 0$ because of $E_{b} = z_{0} E_{b}$ and $H_{b}^{p} = z_{0} H_{b}^{p}$, whereas with separating them $S_{bk}^{EM} \equiv 0$ because of (2.4) and (2.5).

The basic property of the orthogonal complement to be (quasi-)orthogonal in power sense with respect to the fields of every $k$th eigenmode, assures the following quasi-orthogonality relation for lossy systems:

$$\frac{dP_{bk}(z)}{dz} + Q_{bk}(z) = 0$$

which is written in agreement with the mode quasi-orthogonality relation (4.17). Hence, with allowing for (4.46) – (4.48) formula (4.38) takes the form

$$\frac{dP_{ak}(z)}{dz} + Q_{ak}(z) = R_{1k}(z)$$

involving solely the modal expansions with no orthogonal complements.

Substitution of the modal expansions into expressions (4.46) and (4.47) for $P_{ak}$ and $Q_{ak}$ allows us to reveal their longitudinal dependence (without an explicit writing of transverse coordinates):

$$P_{ak}(z) = P_{ak}^{EM}(z) + P_{ak}^{PM}(z) \equiv \int_{S} \left( S_{ak}^{EM}(z) + S_{ak}^{PM}(z) \right) \cdot z_{0} \, dS$$
\[ Q_{ak}(z) = \int_S q_{ak}(z) \, dS = \sum_l A_l(z) \int_S q_{lk} \, dS = \sum_l A_l(z) \int_S q_{lk}^* \, dS = \left( \sum_l M_{kl} \, A_l(z) \, e^{-\gamma_i z} \right) \, e^{-\gamma_i z}. \] (4.51)

By inserting equations (4.50) and (4.51) in relation (4.43) and representing the exciting integrals \(R_{1k}\) in the form

\[ R_{1k} = R^{(b)}_{1k} + R^{(s)}_{1k} = \left( R^{(b)}_k + R^{(s)}_k \right) e^{-\gamma_i z} \equiv R_k e^{-\gamma_i z} \]

involving the wave factor \(\exp(-\gamma_i z)\) explicitly, we obtain

\[ \sum_l \left\{ N_{kl} \frac{dA_l(z)}{dz} - \left[ (\gamma_i^k + \gamma_i l) N_{kl} - M_{kl} \right] A_l \right\} \, e^{-\gamma_i z} = R_k \equiv R^{(b)}_k + R^{(s)}_k. \] (4.52)

The square bracket in equation (4.52) vanishes owing to the quasi-orthogonality relation (4.23) and the required equations of mode excitation take the following form:

(i) for the excitation amplitudes \(A_l(z)\)

\[ \sum_l N_{kl} \frac{dA_l(z)}{dz} \, e^{-\gamma_i z} = R^{(b)}_k(z) + R^{(s)}_k(z), \quad k = 1, 2, \ldots \] (4.53)

(ii) for the mode amplitudes \(a_l(z) = A_l(z) \, e^{-\gamma_i z}\)

\[ \sum_l N_{kl} \left( \frac{dA_l(z)}{dz} + \gamma_i a_l(z) \right) = R^{(b)}_k(z) + R^{(s)}_k(z), \quad k = 1, 2, \ldots \] (4.54)

These equations have the general structure applicable for both cases of separating and not separating the potential fields. The only distinction between them consists in different forms of the normalizing coefficients \(N_{kl} = N_{kl}^{EM} + N_{kl}^{PM}\) (electromagnetic and polarized-medium) and the exciting integrals \(R_k = R^{(b)}_k + R^{(s)}_k\) (bulk and surface) which depend on whether the potential fields are separated:

(a) **without separating the potential fields** (when equations (4.42) and (4.43) are valid)

\[ N_{kl} = N_{kl}^{EM} + N_{kl}^{PM} = \int_S \left( \dot{E}_k^* \times \hat{H}_l^p + \dot{E}_l \times \hat{H}_k^{P*} \right) \cdot z_0 \, dS + \int_S \left[ -\left( \hat{T}_k^{E*} \cdot \hat{U}_l + \hat{T}_l^E \cdot \hat{U}_k^* \right) \right] \cdot z_0 \, dS \]

\[ + \left( \dot{V}_k^{m*} \cdot \dot{j}_l^m + \dot{V}_l^m \cdot \dot{j}_k^{m*} \right) + \left( \dot{V}_k^{c*} \cdot \dot{j}_l^c + \dot{V}_l^c \cdot \dot{j}_k^{c*} \right) \cdot z_0 \, dS \] (4.55)

\[ R_k = R^{(b)}_k + R^{(s)}_k = -\int_{S_h} \left( J_b^e \cdot \dot{E}_k^* + J_b^m \cdot \dot{H}_k^{P*} \right) \, dS - \int_{L_s} \left( J_s^e \cdot \dot{E}_k^* + J_s^m \cdot \dot{H}_k^{P*} \right) \, dl \]

\[ - \int_{L_h} \left( J_{s,ef}^e \cdot \dot{E}_k^* + J_{s,ef}^m \cdot \dot{H}_k^{P*} \right) \, dl \] (4.56)
involving necessarily the effective surface currents $\mathbf{J}_{k,\text{ef}}^e$ and $\mathbf{J}_{k,\text{ef}}^m$ which were developed in [1] (see formulae (4.24 – 4.32)).

(b) with separating the potential fields (when equations (4.44) and (4.45) are valid)

\[
N_{kl} = N_{kl}^{\text{EM}} + N_{kl}^{\text{PM}} = \int_S \left[ (\tilde{\mathbf{E}}_{ck}^* \cdot \tilde{\mathbf{H}}_{cl}^p + \tilde{\mathbf{E}}_{cl}^* \cdot \tilde{\mathbf{H}}_{ck}^p) + (\psi_k^* (i \omega \tilde{\mathbf{D}}_l^p) + \phi_l (i \omega \tilde{\mathbf{B}}_k^p)^*) \right]
\]

\[
+ \left[ (\psi_k^* (i \omega \tilde{\mathbf{B}}_l) + \phi_l (i \omega \tilde{\mathbf{D}}_k^p)\right] \cdot z_0 dS + \int_S \left[ -\left( \tilde{T}_{k}^{c*} \cdot \tilde{U}_l + \tilde{T}_{l}^{c} \cdot \tilde{U}_k \right) \right]
\]

\[
+ \left( \tilde{\mathbf{V}}_{k}^{m*} \cdot \tilde{\mathbf{J}}_{l}^{m} + \tilde{\mathbf{V}}_{l}^{m} \cdot \tilde{\mathbf{J}}_{k}^{m*} \right) + \left( \tilde{\mathbf{V}}_{k}^{\psi c*} \cdot \tilde{\mathbf{J}}_{l}^{\psi c} + \tilde{\mathbf{V}}_{l}^{\psi c} \cdot \tilde{\mathbf{J}}_{k}^{\psi c*} \right) \cdot z_0 dS \tag{4.57}
\]

\[
R_k = R_k^{(b)} + R_k^{(s)} = \int_{S_b} \left[ -\left( J_{b}^{e} \cdot \tilde{\mathbf{E}}_{ck}^* + J_{b}^{m} \cdot \tilde{\mathbf{H}}_{ck}^p \right) + (\omega \mu_0 \psi_k^* + \omega \mu_0 \tilde{\mathbf{B}}_k^p) \right] dS
\]

\[
+ \int_{L_k} \left[ -\left( J_{e}^{e} \cdot \tilde{\mathbf{E}}_{ck}^* + J_{m}^{m} \cdot \tilde{\mathbf{H}}_{ck}^p \right) + (\omega \mu_0 \psi_k^* + \omega \mu_0 \tilde{\mathbf{B}}_k^p) \right]
\]

\[
+ \left( \eta_0^e \cdot (i \omega \tilde{\mathbf{D}}_k^p) + \eta_0^m \cdot (i \omega \tilde{\mathbf{B}}_k^p) \right) \right] d\ell \tag{4.58}
\]

involving no effective surface currents, unlike the previous case.

All the quantities $N_{kl}$ and $R_k$ involve the cross-section eigenfunctions (marked by the hat sign above them) and the $z$-dependence of the exciting integrals $R_k(z)$ is due to that of the external bulk and surface sources.

The separation of potential fields has revealed a fine structure of the interaction between the external sources and the mode eigenfields (curl and potential), which is demonstrated by relation (4.58): the currents (electric $\mathbf{J}_{b,\text{ef}}^{e}$ and magnetic $\mathbf{J}_{b,\text{ef}}^{m}$) interact with the curl fields (electric $\tilde{\mathbf{E}}_{ck}$ and magnetic $\tilde{\mathbf{H}}_{ck}^p$), whereas the charges (electric $\rho_{b,\text{ef}}^{e}$ and magnetic $\rho_{b,\text{ef}}^{m}$) interact with the quasi-static potentials (electric $\phi_k$ and magnetic $\psi_k$). Also, there is the interaction of the displacement currents (electric $i \omega \tilde{\mathbf{D}}_k^p$ and magnetic $i \omega \tilde{\mathbf{B}}_k^p$) with the double charge (dipole) layers (electric $\eta_k^e$ and magnetic $\eta_k^m$), if any. Usually, the latter do not exist in real physical situations, but can be introduced by the equivalence principle as equivalent surface sources.

It should be mentioned that the generalized theory of guided-wave interaction, allowing for the potential fields, much like the theory elaborated here, was first developed by the author [22]. Both results are in good agreement only for lossless waveguiding structures, although the problem of the orthogonal complements went unnoticed then. Allowance for losses was made on the basis of the bi-orthogonality relation (instead of the quasi-orthogonality relation, as it is done here) which was obtained by introducing a subsidiary boundary-value problem (called the associated problem) to change artificially the sign of a loss parameter. In so doing, the eigenmode norms for lossy waveguides have no power meaning and the results of mode excitation pose serious difficulties in physical interpretation.
5. General conclusions on the excitation theory for BAM and SDAM waveguides

The final equations (4.53) or (4.54) of the waveguide excitation theory developed are completely identical, by their structure, with equations (5.47) and (5.48) obtained in [1] for the waveguides with bianisotropic media. They constitute an infinite set of coupled differential equations of the first order in the desired modal amplitudes $A_k$ or $a_k$ excited by a given distribution of the external sources (bulk and surface) which enter into the exciting integrals $R^{(b)}_k$ and $R^{(s)}_k$ defined by formulae (4.56) and (4.58). This set of excitation equations can be rewritten in the matrix-operator form

$$\tilde{N} \cdot Z(z) = R(z)$$

where the matrix-operator $\tilde{N}$ formed from the normalizing coefficients is hermitian ($N_{kl} = N_{lk}^*$) and the column-vector $R$ has $R_k$ as components. The column-vector $Z$ is composed from the elements $Z_k = \frac{d}{dz}a_k + \gamma_k a_k = \frac{d}{dz}A_k \exp(-\gamma_k z)$.

Generally, the coupled equations developed hold true for dissipative systems since the coupling coefficient $N_{kl}$ defined by equation (4.55) or (4.57) determines, according to (4.18), the cross-power flow $P_{kl}$ for any one of mode pairs, which is the case for lossy waveguides. It is evident that these equations remain true for nondissipative systems as a special case.

For lossless waveguiding structures, the general quasi-orthogonality relation (4.23) turns into equation (3.10) of paper [1] to produce there the orthonormalization relations (3.18) and (3.24) for the active and reactive modes, respectively. They can be written jointly in the combined form

$$N_{kl} = \begin{cases} N_k \delta_{kl} & \text{– for active modes} \\ N_k \delta_{\tilde{k}\tilde{l}} & \text{– for reactive modes} \end{cases}$$

where the norms $N_k$ possess such a property that for an active mode it is real-valued, whereas for a reactive mode it is complex-valued and $N_k = N_{\tilde{k}}^*$ (see section 3.2 in [1]). Substitution of equation (5.2) into (5.1) yields

$$Z_k = \begin{cases} R_k/N_k & \text{– for active modes} \\ R_{\tilde{k}}/N_{\tilde{k}} & \text{– for reactive modes} \end{cases}$$

As follows from the orthogonality relations (5.2), every eigenmode of a lossless waveguide is orthogonal to all the other modes, except for the only one in combination with fields of which it forms its own norm: the active $k$-mode is non-orthogonal to itself but the reactive $k$-mode is non-orthogonal to its own twin-conjugate $\tilde{k}$-mode for which $\gamma_k + \gamma_{\tilde{k}}^* = 0$. The fields of such a mode, non-orthogonal to the $k$-mode, enter into the exciting integrals and determine the power of interaction with external sources, so as to supply the given $k$-mode independently of the others (see section 3.2 in [1]).

This is true only if the exciting sources are fixed, which lies outside the context of self-consistent treatment usually applied in practice.

The independent excitation of every mode by given sources described by (5.3) is inherent only in lossless waveguides. For a lossy waveguide this is not the case. Even for the fixed exciting sources there is a dissipative coupling among modes expressed by non-zero off-diagonal elements $N_{kl}$ of the normalizing coefficient matrix $\tilde{N}$. Such a dissipative coupling implies that, unlike lossless waveguides, the given external sources
excite the total set of eigenmodes as a whole rather than every mode as a single. It is very important to realize that this mode coupling of dissipative character is apparent since it exists only inside sources. Outside them even the presence of nonzero off-diagonal coefficients \( N_{kl} \) leaves all the eigenmodes uncoupled. Indeed, for the source-free region the right-hand side of equation (5.1) vanishes \((R_k = 0)\) and, if \( N_{kl} \neq 0 \), from here it follows that \( Z_k = 0 \) or \( A_k(z) = \text{constant} \).

Hence, the eigenmodes of a lossy waveguide, being linearly independent solutions to the appropriate boundary-value problem, remain uncoupled outside the source region. Under these conditions, any eigenmode has the constant value of amplitude that was gained from sources at the exit boundary of their existence region and propagates along the waveguide without coupling to other modes. However, the picture of power transfer is more involved. Every \( k \)-mode, besides the self-power flow \( P_k \equiv P_{kk} \), also carries the cross-power flows \( P_{kl} \) in pairs, together with the other \( l \)-modes, which were also excited inside the source region and outside have constant amplitudes. According to the quasi-orthogonality relation in the power form \((4.17)\), any one of mode pairs has a certain value of the cross-power flow \( P_{kl} \), as well as the cross-power loss \( Q_{kl} \). This fact is a physical manifestation of the power non-orthogonality (called the quasi-orthogonality) among eigenmodes outside the source region. It is the existence of \( P_{kl} \) proportional to the normalizing coefficients \( N_{kl} \) that results in the apparent coupling of dissipative character among modes inside the source region, which is described by equations \((4.53)\) or \((4.54)\) with the exciting sources on the right considered as fixed.

At first glance, it may seem that the infinite set of coupled equations like \((5.1)\) obtained for lossy systems reduces to certain mathematical difficulties which are absent for lossless systems described by the uncoupled equations like \((5.3)\). However, the difference between them disappears in the self-consistent formulation of a wave problem. In this case, the external sources appearing in \( R_k \) themselves can be represented as series expansions in terms of eigenmodes of another waveguide, which makes these sources exciting for the waveguide under consideration. Then, the uncoupled equations for single modes, for example \((5.3)\), turn into an infinite set of coupled equations (see Refs. \([13, 22, 23]\) where this technique is described). Solving such coupled equations for lossless systems differs little in computational complexity from an analogous solution for lossy systems. In both cases, to obtain the total solution of coupled equations, allowing for interaction between all modes, is unrealizable in practice. Usually, the major effect of interaction is determined by coupling among finite number of modes. Strongly interacting modes can be separated by the coupled-mode technique \([24]\), which enables an approximate solution to be obtained with a precision sufficient for practical applications.

Thus, we have developed a unified treatment of the electrodynamic theory of guided-wave excitation by external sources, applied to any waveguide with composite and multilayered structures involving complex media, with both bi-anisotropic and space-dispersive properties.

Appendix A. Derivation of the generalized reciprocity relation for space-dispersive active media

Appendix A.1. Contribution from piezoelectrically-elastic properties of a medium

Let us rewrite equations \((3.4) - (3.9)\) for the first system with subscript 1:

\[
T_{1,ij} = c_{ijkl} S_{1,kl} - e_{kij} E_{1,k} 
\]

\[(A.1)\]
\[ D_{1,k} = \epsilon_{kj} S_{1,ij} + \epsilon_{ik} E_{1,i} \quad (A.2) \]

\[ i\omega \rho_m U_{1,i} = \frac{\partial T_{1,ij}^e}{\partial r_j} - \tau_{ij}^{-1} \rho_m U_{1,j} \quad (A.3) \]

\[ T_{1,ij}^e = T_{1,ij} + T_{1,ij}^{fr}, \quad T_{1,ij}^{fr} = i\omega \eta_{ijkl} S_{1,kl} \quad (A.4) \]

\[ i\omega S_{1,ij} = \frac{1}{2} \left( \frac{\partial U_{1,i}}{\partial r_j} + \frac{\partial U_{1,j}}{\partial r_i} \right). \quad (A.5) \]

For the second system, the similar equations are obtained from equations (A.1) through (A.3) by taking complex conjugation and replacing subscripts 1 with 2 (both have the small-signal meaning). The equations obtained in such a way are numbered as (A.1') through (A.5'), but will not be written explicitly for the sake of brevity.

In order to calculate the term
\[ i\omega \left( P_1 \cdot E_2^* - P_2^* \cdot E_1 \right) = i\omega \left( D_1 \cdot E_2^* - D_2^* \cdot E_1 \right) \]
in the right-hand side of equation (A.3), it is necessary to: multiply (A.1) and (A.1') by \(-i\omega S_{2,ij}^*\) and \(i\omega S_{1,ij}\), respectively; multiply (A.2) and (A.2') by \(-i\omega E_{2,k}^*\) and \(i\omega E_{1,k}\), respectively; and to add all the results. Then
\[ i\omega \left( P_1 \cdot E_2^* - P_2^* \cdot E_1 \right) = i\omega \left( T_{1,ij} S_{2,ij}^* - T_{2,ij}^* S_{1,ij} \right) \quad (A.6) \]

where the relations \(\epsilon_{ik} = \epsilon_{ki}\) and \(c_{ijkl} = c_{klij}\) have been applied.

Multiplying (A.5) by \(T_{2,ij}^*\) and (A.5') by \(T_{1,ij}\) gives the following
\[ i\omega \left( T_{1,ij} S_{2,ij}^* - T_{2,ij}^* S_{1,ij} \right) = -\left( T_{1,ij} \left( \frac{\partial U_{2,i}}{\partial r_j} \right) + T_{2,ij}^* \left( \frac{\partial U_{1,i}}{\partial r_j} \right) \right) \]
\[ = -\nabla \cdot \left( T_{1} \cdot U_{2}^* + T_{2}^* \cdot U_{1} \right) + \left( U_{1,i} \left( \frac{\partial U_{2,i}}{\partial r_j} \right) + U_{2,i}^* \left( \frac{\partial U_{1,i}}{\partial r_j} \right) \right) \]
\[ = -\nabla \cdot \left( T_{1}^e \cdot U_{2}^* + T_{2}^e \cdot U_{1} \right) + \left( U_{1,i} \left( \frac{\partial U_{2,i}}{\partial r_j} \right) + U_{2,i}^* \left( \frac{\partial U_{1,i}}{\partial r_j} \right) \right) \]
\[ + \left( T_{1,ij}^fr \left( \frac{\partial U_{2,i}}{\partial r_j} \right) + T_{2,ij}^fr \left( \frac{\partial U_{1,i}}{\partial r_j} \right) \right). \quad (A.7) \]

Multiplying (A.3) by \(U_{2,i}^*\) and (A.3') by \(U_{1,i}\) gives the following
\[ \left( U_{1,i} \left( \frac{\partial U_{2,i}}{\partial r_j} \right) + U_{2,i}^* \left( \frac{\partial U_{1,i}}{\partial r_j} \right) \right) = \rho_m \tau_{ij}^{-1} \left( U_{1,i} U_{2,j}^* + U_{2,i}^* U_{1,j} \right) \]
\[ = 2 \rho_m \tau_{ij}^{-1} U_{1,i} U_{2,j} = 2 \rho_m U_{2}^* \cdot \tau^{-1} \cdot U_{1} \quad (A.8) \]

where the last equality is obtained by using the relation \(\tau_{ij}^{-1} = \tau_{ji}^{-1}\).
To calculate the last term in the right-hand side of \( \text{[A.7]} \), it is necessary to multiply \( \text{[A.4]} \) and \( \text{[A.4']} \) by \( \partial U_{2,i}^*/\partial r_j \) and \( \partial U_{1,i}/\partial r_j \), respectively, and to add the results. Then the use of \( \text{[A.5]} \) and \( \text{[A.5']} \) yields

\[
\left( T^{fr}_{1,ij} \frac{\partial U_{2,i}^*}{\partial r_j} + T^{fr}_{2,ij} \frac{\partial U_{1,i}}{\partial r_j} \right) = i \omega \eta_{ijkl} \left( S_{1,kl} \frac{\partial U_{2,i}^*}{\partial r_j} - S_{2,kl}^* \frac{\partial U_{1,i}}{\partial r_j} \right)
\]

\[
= \left( \frac{\eta_{ijkl}}{2} \right) \left( S_{1,kl} \left( \frac{\partial U_{2,i}^*}{\partial r_j} + \frac{\partial U_{2,i}}{\partial r_j} \right) - S_{2,kl}^* \left( \frac{\partial U_{1,i}}{\partial r_j} + \frac{\partial U_{1,i}}{\partial r_j} \right) \right)
\]

\[
= \omega^2 \eta_{ijkl} \left( S_{1,ij} S_{2,kl}^* + S_{2,ij}^* S_{1,kl} \right) = 2 \omega^2 \eta_{ijkl} S_{1,ij} S_{2,kl}^* = 2 \omega^2 \bar{S}_2 : \bar{\eta} : \bar{S}_1 \quad \text{(A.9)}
\]

where the relations \( \eta_{ijkl} = \eta_{klji} \) and \( \eta_{ijkl} = \eta_{jikl} \) have been applied.

By collecting formulae \( \text{[A.6]} - \text{[A.9]} \) we finally obtain

\[
i \omega \left( P_1 \cdot E_2^* - P_2^* \cdot E_1 \right) = - \nabla \cdot \left( \bar{T}_1^c \cdot U_2^* + \bar{T}_2^c \cdot U_1 \right)
\]

\[
+ 2 \omega^2 \bar{S}_2 : \bar{\eta} : \bar{S}_1 + 2 \rho_m U_2^* \cdot \bar{\tau}^{-1} \cdot U_1.
\]

(A.10)

Appendix A.2. Contribution from ferrimagnetic properties of a medium

For ferrimagnetic magnetic media it is necessary to calculate the term

\[
i \omega \mu_0 \left( M_1 \cdot H_2 - M_2^* \cdot H_1 \right) = i \omega \left( B_1 \cdot H_2 - B_2^* \cdot H_1 \right)
\]

on the right-hand side of \( \text{[A.5]} \). By using \( \text{[B.11]} - \text{[B.13]} \) we write the linearized equation of motion for the first system marked by subscript 1:

\[
i \omega M_1 = - \gamma \mu_0 \left[ M_1 \times H_0 + M_0 \times H_1 - M_0 \times (\bar{N} \cdot M_1) \right]
\]

\[
+ \lambda_{ex} M_0 \times \nabla^2 M_1 + \nu_m \frac{\omega}{\omega_m} M_0 \times M_1 \quad \text{(A.11)}
\]

where \( \nu_m = \alpha \omega_m = \alpha \gamma \mu_0 M_0 \) is the magnetic relaxation frequency.

The similar equation for the second system marked by subscript 2 in place of 1 is obtained from equation (A.11) by taking complex conjugation and numbered as \( \text{(A.11')} \) (without its explicit writing for brevity).

Vector-multiplying equation (A.11) by \( M_2^* \) and taking account of the equalities \( M_{1,2} \cdot H_0 = M_{1,2} \cdot M_0 = 0 \) valid for small signals we obtain

\[
i \omega \left( M_1 \times M_2^* \right) = \gamma \mu_0 \left[ M_0 \left( M_2^* \cdot H_1 \right) - H_0 \left( M_2^* \cdot M_1 \right) - M_0 \left( M_2^* \cdot \bar{N} \cdot M_1 \right) \right]
\]

\[
+ \lambda_{ex} M_0 \left( M_2^* \cdot \nabla^2 M_1 \right) - i \nu_m \frac{\omega}{\omega_m} M_0 \left( M_2^* \cdot M_1 \right) \quad \text{(A.12)}
\]

The similar equation obtained in a such way from equation \( \text{(A.11')} \) is

\[
i \omega \left( M_2^* \times M_1 \right) = - \gamma \mu_0 \left[ M_0 \left( M_1 \cdot H_2^* \right) - H_0 \left( M_1 \cdot M_2^* \right) - M_0 \left( M_1 \cdot \bar{N} \cdot M_2^* \right) \right]
\]

\[
+ \lambda_{ex} M_0 \left( M_1 \cdot \nabla^2 M_2^* \right) - i \nu_m \frac{\omega}{\omega_m} M_0 \left( M_1 \cdot M_2^* \right) \quad \text{(A.13)}
\]
By adding these equations with using the vector-dyadic identity \( \nabla \cdot (\nabla A \cdot B) = B \cdot \nabla A + (\nabla \times A) \cdot (\nabla \times B) + \nabla A \cdot \nabla B \) and allowing for the symmetry of tensor \( \mathcal{N} \) we finally obtain

\[
i \omega \mu_0 \left( M_1 \cdot H_2^* - M_2^* \cdot H_1 \right) =
\]

\[
= \nabla \cdot \left( \bar{V}_m^1 \cdot J_2^* + \bar{V}_m^2 \cdot J_1^* \right) + 2 \mu_0 \nu_m \left( \frac{\omega}{\omega_0} \right)^2 \left( M_1 \cdot M_2^* \right) \tag{A.14}
\]

where the magnetization current \( J_{1,2} = i \omega \mu_0 M_{1,2} \) and the effective magnetic (exchange) potential \( \bar{V}_m^1 = -\lambda_{\text{ex}} \nabla M_{1,2} \) of a ferrimagnetic medium have been used.

**Appendix A.3. Contribution of drifting charge carriers in a medium**

For plasmas with drifting charge carriers, it is necessary to calculate the term

\[
i \omega \left( p_1 \cdot E_2^e - p_2^* \cdot E_1^e \right) \equiv i \omega \left( p_1 \cdot (E_2^s + v_0 \times B_2^s) - p_2^* \cdot (E_1^s + v_0 \times B_1^s) \right)
\]

on the right-hand side of (4.5). To this end, we apply (3.19) and (3.20) for the first system with subscript 1 written in the form

\[
i \omega v_1 + (v_0 \cdot \nabla) v_1 =
\]

\[
= \frac{e}{m} \left\{ E_1 + (r_1 \cdot \nabla) E_0 + v_1 \times B_0 + v_0 \times B_1 + v_0 \times (r_1 \cdot \nabla) B_0 \right\}
\]

\[
+ \frac{v_0^2}{\rho_0} \left[ \rho_0 \nabla (\nabla \cdot r_1) + \nabla r_1 \cdot \nabla \rho_0 \right] - \frac{v_1}{\tau_0} + \frac{v_0}{\tau_0} \frac{\tau_1 + (r_1 \cdot \nabla) \tau_0}{\tau_0} \tag{A.15}
\]

\[
v_1 = i \omega r_1 + (v_0 \cdot \nabla) r_1 \tag{A.16}
\]

and the similar equations for the second system with subscript 2 in place of 1 obtained from (A.15) and (A.16) by taking complex conjugation and numbered as (A.15') and (A.16') (without explicit writing for brevity).

Multiplying (A.15) and (A.15') by \(-i \omega (m/e) \rho_0 r_2^* \) and \(i \omega (m/e) \rho_0 r_1 \), respectively, and adding the results with a combination of terms give

\[
i \omega \left( p_1 \cdot E_2^e - p_2^* \cdot E_1^e \right) = -i \omega \frac{m}{e} \rho_0 \left\{ i \omega \left( r_2^* \cdot v_1 + r_1 \cdot v_2^* \right) \right. \]

\[
+ \left[ r_2^* \cdot (v_0 \cdot \nabla) v_1 - r_1 \cdot (v_0 \cdot \nabla) v_2^* \right] \right\} + i \omega \rho_0 \left[ r_2^* \cdot (r_1 \cdot \nabla) E_0 - r_1 \cdot (r_2^* \cdot \nabla) E_0 \right] \]

\[
+ i \omega \rho_0 \left\{ r_2^* \cdot (v_0 \times B_0) - r_1 \cdot (v_2^* \times B_0) \right\} + \left[ r_2^* \cdot (v_0 \times (r_1 \cdot \nabla) B_0) \right] - r_1 \cdot (v_0 \times (r_2^* \cdot \nabla) B_0) \right\} + i \omega \frac{m}{e} \rho_0 \left\{ \rho_0 r_2^* \cdot \nabla (\nabla \cdot r_1) - \rho_0 r_1 \cdot \nabla (\nabla \cdot r_2^*) \right\} + \]

\[
+ \left[ r_2^* \cdot (\nabla r_1 \cdot \nabla \rho_0) - r_1 \cdot (\nabla r_2^* \cdot \nabla \rho_0) \right] \right\} \]
\[
+ i \omega \rho_0 \frac{m}{\tau_0} e \left[ r_1 \cdot (v_2^* - v_0 \frac{\tau_2^*}{\tau_0} + r_2^* \cdot \nabla \tau_0) - r_2^* \cdot (v_1 - v_0 \frac{\tau_1 + r_1 \cdot \nabla \tau_0}{\tau_0}) \right]. \tag{A.17}
\]

Let us transform the right-hand side of (A.17), with the last terms already in the desired form.

The first term is transformed by using \( \nabla \cdot J_0 = 0 \), and the result of multiplying (A.16) and (A.16') by \(-i \omega (m/e) \rho_0 v_2^* + i \omega (m/e) \rho_0 v_1 \), respectively, into the following form

\[
- i \omega \rho_0 \frac{m}{e} \left\{ i \omega \left[ r_2^* \cdot (v_1 + r_1 \cdot v_2^*) + r_2^* \cdot (v_0 \cdot \nabla) v_1 - r_1 \cdot (v_0 \cdot \nabla) v_2^* \right] \right\}
\]

\[
= \nabla \cdot \left[ i \omega \frac{m}{e} v_0 \left( p_1 \cdot v_2^* - p_2^* \cdot v_1 \right) \right]. \tag{A.18}
\]

The second term vanishes after using the following vector-dyadic identity

\[
(A \times B) \cdot (\nabla \times C) = B \cdot (A \cdot \nabla C) - A \cdot (B \cdot \nabla C) \tag{A.19}
\]

with \( A = r_1 \), \( B = r_2^* \) and \( C = E_0 \), since \( \nabla \times E_0 = 0 \).

The third term is rearranged by employing (A.16), (A.16'), \( \nabla \cdot B_0 = 0 \), \( \nabla \cdot J_0 = 0 \), and the identity

\[
A \times (B \cdot \nabla C) - B \times (A \cdot \nabla C) = \nabla C \cdot (B \times A) - (\nabla \cdot C)(B \times A)
\]

with \( A = r_1 \), \( B = r_2^* \) and \( C = B_0 \). Therefore,

\[
i \omega \rho_0 \left\{ \left[ r_2^* \cdot (v_1 \times B_0) - r_1 \cdot (v_2^* \times B_0) \right] + \left[ r_2^* \cdot (v_0 \times (r_1 \cdot \nabla) B_0) \right] - r_1 \cdot (v_0 \times (r_2^* \cdot \nabla) B_0) \right\}
\]

\[
= - \nabla \cdot \left[ i \omega v_0 \left( p_1 \cdot (r_2^* \times B_0)/2 - p_2^* \cdot (r_1 \times B_0)/2 \right) \right]. \tag{A.20}
\]

The fourth term uses identity (A.19) with \( A = r_1 \), \( B = r_2^* \), \( C = \nabla \rho_0 \) and the fact that \( \nabla \times \nabla \rho_0 = 0 \), to be transformed into the following form

\[
i \omega v_0^2 \frac{m}{e} \left\{ \left[ \rho_0 r_2^* \cdot \nabla (\nabla \cdot r_1) - \rho_0 r_1 \cdot \nabla (\nabla \cdot r_2^*) \right] + \left[ r_2^* \cdot (\nabla r_1 \cdot \nabla \rho_0) \right] - r_1 \cdot (\nabla r_2^* \cdot \nabla \rho_0) \right\}
\]

\[
= \nabla \cdot \left[ i \omega \frac{m}{e} \rho_0 v_0^2 \left( p_2 \nabla \cdot p_1 - p_1 \nabla \cdot p_2 \right) \right] = \nabla \cdot \left( V_{1,2}^{th} J_2^{*} + V_{2}^{th} J_1^{*} \right) \tag{A.21}
\]

where the thermal (diffusion) potential \( V_{1,2}^{th} = (k_B T/e)(\rho_{1,2}/\rho_0) \) has been used.
Substituting \((A.18) - (A.21)\) into \((A.17)\), we finally obtain
\[
i\omega \left( p_1 \cdot E'_2 - p^*_2 \cdot E'_1 \right) = \nabla \cdot \left( \tilde{V}^c_1 \cdot J^*_2 + \tilde{V}^c_2 \cdot J^*_1 \right)
\]
\[\quad + \frac{1}{\mu_e} \left[ \left( v_1 - v_0 \frac{\tau_1 + r_1 \cdot \nabla \tau_0}{\tau_0} \right) \cdot J^*_2 \right] \tag{A.22}
\]
where the electronic polarization current \(J^e_1,2 = i\omega p^e_1,2\) and the effective electronic potential \(\tilde{V}^e_1,2 = \tilde{V}^e_{1,2} + V^\text{therm}_{1,2}\) have been introduced.

References

[1] Barybin A A 1998 Modal expansions and orthogonal complements in the theory of complex media waveguide excitation by external sources for isotropic, anisotropic and bianisotropic media *Progress In Electromagnetics Research* vol 19 ed J A Kong (Cambridge MA: EMW) pp 241–300

[2] Landau L D and Lifshitz E M 1960 *Electrodynamics of Continuous Media* (Reading MA: Addison-Wesley)

[3] Post E J 1962 *Formal Structure of Electromagnetics* (Amsterdam: North-Holland)

[4] O’Dell T H 1970 *The Electrodynamics of Magneto-Electric Media* (Amsterdam: North-Holland)

[5] Auld B A 1973 *Acoustic Fields and Waves in Solids* (New York: Wiley)

[6] Dieulesaint E and Royer D 1980 *Elastic Waves in Solids* (New York: Wiley)

[7] Kino G S 1987 *Acoustic Waves: Devices, Imaging and Analog Signal Processing* (Englewood Cliffs NJ: Prentice-Hall)

[8] Sodha M S and Srivastava N C 1981 *Microwave Propagation in Ferrimagnetics* (New York: Plenum)

[9] Soohoo R F 1985 *Microwave Magnetics* (New York: Harper and Row)

[10] Stancil D D 1993 *Theory of Magnetostatic Waves* (New York: Springer)

[11] Korn G A and Korn T M 1961 *Mathematical Handbook for Scientists and Engineers* (New York: McGraw-Hill)

[12] Bobroff D L 1959 Independent space variables for small-signal electron beam analyses *IRE Trans. Electron Devices* **6** 68–79

[13] Barybin A A 1977 Electrodynamical concepts of wave interactions in thin-film semiconductor structures Pt I *Advances in Electronics and Electron Physics* vol 44 ed L Marton (New York: Academic) pp 99–139; Pt II *ibid* vol 45 pp 1–38

[14] Barybin A A 1986 *Waves in Thin-Film Semiconductor Structures with Hot Electrons* (Moscow: Nauka) (in Russian)

[15] Barybin A A 1975 Boundary conditions on carrier stream surfaces in nondegenerate semiconductor plasmas *J. Appl. Phys.* **46** 1684–96

[16] Barybin A A 1975 Quasistatic solution of normal mode problem in semiconductor films without magnetic field *J. Appl. Phys.* **46** 1697–706

[17] Korn G A and Korn T M 1961 *Mathematical Handbook for Scientists and Engineers* (New York: McGraw-Hill)

[18] Barybin A A 1978 Mathematical treatment of polarization description of nondegenerate semiconductor plasmas *Int. J. Electron* **44** 481–97

[19] Barybin A A 1975 On the generalized theory of normal mode excitation in electromagnetic and polarized medium waveguides by external sources *J. Appl. Phys.* **46** 1707–20

[20] Louisell W H 1960 *Coupled Mode and Parametric Electronics* (New York: Wiley)