ON THE GAP PROPERTY OF A LINEARIZED NLS OPERATOR

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ABSTRACT. We consider general non-radial linearization about the ground state to the cubic nonlinear Schrödinger equation in dimension three. We introduce a new compare-and-conquer approach and rigorously prove that the interval $(0, 1]$ does not contain any eigenvalue of $L_+$ or $L_-$. The method can be adapted to many other spectral problems.

1. Introduction

In this note we consider the nonlinear Schrödinger equation for $\psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$:

$$i\partial_t \psi + \Delta \psi + |\psi|^2 \psi = 0. \quad (1.1)$$

Plugging in the standing wave ansatz $\psi = e^{it} \phi(x)$, we obtain

$$\Delta \phi - \phi + |\phi|^2 \phi = 0. \quad (1.2)$$

Denote by $Q$ the positive radial ground state. We have $Q(x) = y(r) (r = |x|)$, where $y$ solves the nonlinear ODE

$$-y''(r) - \frac{2}{r} y'(r) + y(r) - y^3(r) = 0. \quad (1.3)$$

Consider $\phi = Q + \eta$ with $\eta = \eta_1 + i\eta_2$. Clearly

$$\Delta \phi - \phi + |\phi|^2 \phi = \Delta \eta - \eta + (Q^2 + 2Q\eta_1)(Q + \eta_1 + i\eta_2) - Q^3 + O(|\eta|^2),$$

where $L_+ = -\Delta + 1 - 3Q^2$, $L_- = -\Delta + 1 - Q^2$.

It is known that the essential spectrum of $L_+$ and $L_-$ is $[1, \infty)$. $L_+$ has a unique negative bound state. If $f \perp \Delta Q$, then (below $\langle, \rangle$ denotes the usual $L^2$-inner product for real-valued functions)

$$\langle L_+ f , f \rangle \gtrsim \left( \int_{\mathbb{R}^3} f Q dx \right)^2. \quad (1.5)$$

The kernel of $L_+$ is $\text{span}\{\partial_j Q\}_{j=1}^3$. The kernel of $L_-$ is $\text{span}\{Q\}$. On the other hand it has been long accepted wisdom that $L_+$ and $L_-$ has no eigenvalue in $(0, 1]$, known as the gap property. This gap property plays an important role in the construction of stable manifolds for orbitally unstable NLS (cf. [1] and [3]).

It was numerically verified by Demanet and Schlag in [2] using the Birman-Schwinger method for NLS with nonlinearities $|\psi|^2/\psi$, $\beta_2 < \beta_1 \leq 1$, $\beta_1 \approx 0.913958905$. In recent [4], Costin, Huang and Schlag rigorously proved the gap property under radial assumptions. The main achievements in [3] are two:

1. A remarkably accurate approximate ground state $\tilde{Q}$ which differs from the true ground state by $O(10^{-4})$. More precisely, the point-wise error is at most $7 \cdot 10^{-5} = 7/10^5 e^{-r}$.

2. A robust Wronskian strategy connecting two Jost quasi-solutions: one emanating from $r = 0$, and the other (decaying) solution from $r = \infty$.

The decisive step is to check $\inf_{\lambda \in [0, 1]} |W(\lambda)| > 0$ for $L_+$ and $\inf_{\lambda \in [0, 1]} |W(\lambda)/\lambda| > 0$ for $L_-$, where $\lambda$ is the spectral parameter. This very involved computation was executed in [2] to prove the gap property for the radial case.

The purpose of this note is to give a rigorous proof of this gap property for the full non-radial case. We shall develop a new compare-and-conquer approach which offers an interesting (and perhaps simpler) alternative to the Wronskian strategy developed in [2].
Theorem 1.1. The operator $L_+$ and $L_-$ does not have any $(L^2)$ eigenvalue in $(0, 1]$. For eigenvalue \( \lambda = 0 \), the kernel of $L_+$ is $\text{span}\{\partial_1 Q\}_{j=1}^3$, and the kernel of $L_-$ is $\text{span}\{Q\}$.

Stronger statements can be inferred from our proof but we shall not dwell on this issue here.

Remark 1.1. As expected the spectral analysis requires some nontrivial information of the ground state $Q$. In order to minimize technicality at several places we adopt the approximate solution $Q$ in [2] (which is remarkably close to $Q$ within $10^{-4}$) to extract some powerful point-wise estimates. It is possible to build other high-precision approximations of $Q$ with controlled error estimates. However we shall not dwell on this issue here.

We now explain the main steps of the proof. Consider first the operator $L_+$ and the equation $L_+ u = \lambda u$. The task is to show for $\lambda \in (0, 1]$ the above equation admits no solution in $L^2(\mathbb{R}^3)$. To do this we argue by contradiction and assume that there is an $L^2$ solution for some $\lambda \in (0, 1]$. By standard elliptic theory, it follows that $u \in H^m(\mathbb{R}^3)$ for all $m \geq 1$. In particular $u$ admits a rapidly convergent spherical harmonic expansion

$$u = \sum_{l=0}^{\infty} \sum_{|m| \leq l} R_{ml}(r)Y_l^m(\theta, \phi),$$  

(1.6)

where $Y_l^m(\theta, \phi)$ are $L^2(S^2)$-normalized spherical harmonics and $R_{ml}(r) = \int_{S^2} u(x) Y_l^m(\theta, \phi) d\sigma$.

Remark 1.2. Since $u$ is smooth, by using the Taylor expansion $u(x) = \sum_{|\alpha| \leq k_0} C_\alpha x^\alpha + O(|x|^{k_0+1})$ and the formula for $R_{ml}$, one can infer that $R_{ml}(r)$ has a regular local expansion when $r \to 0+$. This simple yet important observation will be used when we classify the corresponding solutions having regular behavior when $r \to 0+$.

By using the spherical harmonics expansion, we are led to the following set of equations arranged to the ascending order of degree of the spherical harmonics:

$$l = 0 : \quad ( - \partial_{rr} - \frac{2}{r} \partial_r + 1 - \lambda - 3Q^2 ) R_0 = 0;$$

(1.7)

$$l = 1 : \quad ( - \partial_{rr} - \frac{2}{r} \partial_r + \frac{2}{r^2} + 1 - \lambda - 3Q^2 ) R_1 = 0;$$

(1.8)

$$l \geq 2 : \quad ( - \partial_{rr} - \frac{2}{r} \partial_r + \frac{l(l+1)}{r^2} + 1 - \lambda - 3Q^2 ) R_l = 0.$$  

(1.9)

Here $R_0$, $R_1$ and $R_l$ are functions of $r$ only. The main requirements on $R_j$ are two: 1) $R_j \in L^2([0, \infty), rdr)$; 2) $R_j$ has a regular local expansion when $r \to 0+$.

We discuss several cases.

The case $l \geq 2$. By using the point-wise inequality $\frac{6-10\epsilon - 10}{r^2} \geq 3Q^2(r)$, $\forall r > 0$ (see Lemma 5.1), we rule out any nontrivial solution to (1.9) in $L^2(rdr)$.

The case $l = 0$. Denote $\epsilon = 1 - \lambda$, $t = r$ and $F_t(t) = tR_0(t)$. It suffices to consider

$$\begin{cases} F''_t = (\epsilon - 3Q^2)F_t, \quad t > 0; \\ F_t(0) = 0, \quad F'_t(0) = -1. \end{cases}$$  

(1.10)

By a comparison argument (see Lemma 6.1), we show that $F_t$ must change sign at least once, and the first positive zero $t_\epsilon$ of $F_t$ satisfies

$$t_\epsilon \geq t_0 > 0,$$  

(1.11)

where $t_0$ is the first positive zero of $F_0$. We then focus on analyzing the behavior of the solution after its first positive zero. For this it is enough to study the time-shifted equation

$$\begin{cases} \bar{F''}_{t_\epsilon} = (\epsilon - 3Q^2(t + t_\epsilon))\bar{F}_{t_\epsilon}, \quad t > 0; \\ \bar{F}_{t_\epsilon}(0) = 0, \quad \bar{F}'_{t_\epsilon}(0) = 1. \end{cases}$$  

(1.12)

\[\text{1We shall slightly abuse the notation and regard } Q(x) = Q(|x|) = Q(r) \text{ when there is no obvious confusion.}\]
Introduce $q$ solving
\begin{align}
\begin{cases}
q'' = -3Q^2(t + t_0)q, & t > 0; \\
q(0) = 0, & q'(0) = 1.
\end{cases}
\end{align}
(1.13)

We show via comparison arguments (see Theorem 6.1) that $q(t)$ is positive for $t > 0$, and $q(t)/t$ stays bounded from below by a positive constant for $t \in [1, \infty)$. Thanks to another comparison argument, we deduce $\tilde{F}_l(t) \geq q(t)$ for all $t > 0$. This yields the desired conclusion for $l = 0$. Quite interestingly, in some sense we are able to reduce the original $\lambda$-dependent problem to the study of $\lambda = 1$ case.

The case $l = 1$. This is the most involved case since for $\lambda = 0$, $R = -Q'(r)$ solves the equation
\begin{equation}
(-\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{r^2} + 1 - 3Q^2)R = 0.
\end{equation}
(1.14)

If one adopts the Wronskian strategy in this case then one must deal with the degeneracy of $W(\lambda)$ as $\lambda \to 0$. In our compare-and-conquer approach, we first use a local analysis together with suitable normalization to deduce that
\begin{equation}
\text{const} \cdot R_4(r) = r + (1 - \lambda - 3Q^2(0))r^3 + \mathcal{O}(r^4), \quad r \to 0+.
\end{equation}
(1.15)

Denote $t = r$ and $F_\lambda(t) = \text{const} \cdot tR_4(t)$. Then $F_\lambda$ solves
\begin{equation}
F''_\lambda = (1 - \lambda - \frac{2}{t^2} - 3Q^2(t))F_\lambda, \quad 0 < t < \infty;
\end{equation}
(1.16)
and $F_\lambda(t) = t^2 + (1 - \lambda - 3Q^2(0))t^4 + \mathcal{O}(t^5)$, as $t \to 0+$. By a comparison argument (see Proposition 5.1), we show that $F_\lambda$ must change its sign and the first positive zero $t_0$ of $F_\lambda$ satisfies $t_0 \geq 0.2$. It then suffices for us to study the solution after $t \geq t_0$. In Proposition 4.1 we show via a further comparison argument that the corresponding solution must grow in time.

The above concludes the analysis for the operator $L_+$. For $L_-$ the analysis is similar and slightly simpler. The governing equations are
\begin{align}
\begin{cases}
l = 0: & (-\partial_{rr} - \frac{2}{r}\partial_r + 1 - \lambda - Q^2)R_0 = 0; \\
l \geq 1: & (-\partial_{rr} - \frac{2}{r}\partial_r + \frac{l(l + 1)}{r^2} + 1 - \lambda - Q^2)R_l = 0.
\end{cases}
\end{align}
(1.17)
(1.18)

By Lemma 5.1 we have $\frac{2 - \frac{1}{10^{20}}}{r^2} > Q^2(t)$ for all $t > 0$. Thus the equation (1.18) does not admit any nontrivial solution in $L^2(rdr)$. For (1.17) we show in Theorem 7.1 that it does not admit any nontrivial $L^2(rdr)$ solution for $\lambda \in (0, 1]$. The overall strategy is similar to the $L_+$ case.

The rest of this note is organized as follows. In Section 2 we recall some basic ODE Sturm-Liouville type comparison lemma. In Section 3–6 we prove our main result for the operator $L_+$. The last section collects the needed modifications for the operator $L_-$.

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2. Recap of Sturm

We record the following standard Sturm type comparison lemma. We include a simple proof for the sake of completeness.

Lemma 2.1 (Sturm comparison). Let $0 < l_0 < \infty$. Suppose $G = G(t), g = g(t): [0, l_0] \to \mathbb{R}$ are Lipschitz functions satisfying
\begin{equation}
G(t) \geq g(t), \quad \forall 0 \leq t \leq l_0.
\end{equation}
(2.1)

Assume $F, f$ are $C^2$ functions satisfying
\begin{equation}
\begin{cases}
F'' = GF, & 0 < t < l_0; \\
f'' = gf, & 0 < t < l_0; \\
F(0) = f(0) \geq 0, & F'(0) \geq f'(0),
\end{cases}
\end{equation}
(2.2)

\footnote{One possible fix is to work with $W(\lambda)/\lambda$.}
and \( f(t) > 0 \) for all \( 0 < t < l_0 \). Then

\[
F(t) \geq f(t) > 0, \quad \forall 0 \leq t \leq l_0. \tag{2.3}
\]

**Remark 2.1.** More generally, the same conclusion holds if \( f(t) > 0 \) for all \( 0 \leq t < l_0 \) and

\[
\left( \frac{f'}{f} - \frac{F'}{F} \right)_{t=0} \leq 0, \quad 0 < f(0) \leq F(0). \tag{2.4}
\]

**Proof.** We sketch the (standard) argument. First of all it is enough to prove the theorem under the assumption that \( F(t) > 0 \) for all \( 0 < t < l_0 \). Once this is proved, the general case follows by a simple bootstrapping argument. Also one may assume \( F(0) = f(0) > 0 \). The case \( F(0) = f(0) = 0 \) can be treated by a limiting argument.

Denote \( R = R(t) = \frac{F'(t)}{F(t)^2} \), \( r = r(t) = \frac{f'(t)}{f(t)^2} \). Clearly \( (R - r)_{t=0} \geq 0 \). Then

\[
(R - r)' = \frac{F''F - (F')^2}{F^2} - \frac{f''f - (f')^2}{f^2} \tag{2.5}
\]

\[
= G - g - R^2 + r^2 \tag{2.6}
\]

\[
\geq -(R + r)(R - r). \tag{2.7}
\]

Integrating in time then yields that \( R - r \geq 0 \) for all \( t \). Thus

\[
R - r = \left( \log \frac{F(t)}{f(t)} \right)' \geq 0. \tag{2.8}
\]

Thus \( F(t) \geq f(t) \) for all \( 0 \leq t \leq l_0 \). \( \Box \)

**Remark 2.2.** There exists a natural correspondence of our linearized equation to the usual Bessel function, at least near \( r = \infty \). To see this consider the equation

\[
\frac{d^2}{dt^2} F_1 + (3Q^2 - e^2)F_1 = 0. \tag{2.9}
\]

Near \( r = \infty \) one can regard \( Q(t) \sim t^{-1}e^{-t} \). Dropping the \( t^{-2} \) factor, we arrive at the model

\[
\frac{d^2}{dt^2} F = (e^2 - k^2 e^{-2t})F. \tag{2.10}
\]

Make a change of variable \( x = e^{-t} \). Clearly

\[
\frac{d}{dt} F = -u' \cdot e^{-t}, \quad (\text{here we write } F(t) = u(x) = u(e^{-t})), \tag{2.11}
\]

\[
\frac{d^2}{dt^2} F = u'' e^{-2t} + u' e^{-t} = x^2 u'' + xu'. \tag{2.12}
\]

Thus we obtain

\[
x^2 u'' + xu' = (e^2 - k^2 x^2)u. \tag{2.13}
\]

By another change of variable, we arrive at the usual Bessel equation:

\[
x^2 u'' + xu' = (e^2 - x^2)u. \tag{2.14}
\]

3. When \( 0 < \lambda \leq 1 \) solution must change sign

**Lemma 3.1.** Suppose \( F \) is a smooth function solving the linear equation

\[
F'' = \frac{2}{l^2} - 3Q^2(t))F, \quad 1 \leq t < \infty. \tag{3.1}
\]

Then for some constants \( c_1, c_2 \) we have

\[
F(t) = c_1(t^2 + \eta_1(t)) + c_2(\frac{1}{t} + \eta_2(t)), \tag{3.2}
\]

where \( \eta_i(t) \) are smooth functions satisfying

\[
\sup_{1 \leq t < \infty} |e^t \eta_1(t)| + |e^t \eta_2(t)| < \infty. \tag{3.3}
\]
Proof. It suffices for us to exhibit two independent solutions. We consider $\eta_1$ solving the integral equation
\[ \eta_1(t) = \int_1^\infty (s-t) \left( -3Q^2(s)s^2 + \frac{2}{s^2} - 3Q^2(s) \right) \eta_1(s) \, ds, \quad t \geq T_1. \] (3.4)
By taking $T_1$ sufficiently large, one can obtain a contraction in the norm $|e^t \eta_1(t)|_{L^\infty([T_1, \infty))}$. Clearly the function $\Theta_1(t) = t^2 + \eta_1(t)$ solves the original ODE on $(T_1, \infty)$. Solving it backward in time and noting that it is a linear equation, we obtain a smooth solution $\Theta_1(t)$ defined on $[1, \infty)$.

Analogously we can find $\eta_2$ solving
\[ \eta_2(t) = \int_1^\infty (s-t) \left( -3Q^2(s)s^2 + \frac{2}{s^2} - 3Q^2(s) \right) \eta_2(s) \, ds, \quad t \geq T_2. \] (3.5)
The second solution $\Theta_2(t) = \frac{1}{t} + \eta_2(t)$ on $[1, \infty)$ is also easily obtained.

To check the independence of the two solutions one can examine the Wronskian. It is clearly nonzero for large $t$ and hence nonzero for all $t$.

**Proposition 3.1.** Suppose $0 < \lambda \leq 1$ and $F_\lambda = F_\lambda(t)$ solves
\[ F_\lambda''(t) = (1 - \lambda + \frac{2}{t^2} - 3Q^2(t))F_\lambda, \quad 0 < t < \infty. \] (3.6)
To fix the normalization we fix $F_\lambda(t)$ such that
\[ F_\lambda(t) = t^2 + (1 - \lambda - 3Q^2(0))t^4 + O(t^5), \quad \text{as } t \to 0+. \] (3.7)
Then $F_\lambda$ must change its sign at least once on $(0, \infty)$. Moreover the first positive zero $t_0$ of $F_\lambda$ satisfies $t_0 \geq 0.2$.

**Proof.** We first show that $F_\lambda$ must change sign on $(0, \infty)$. Assume that $F_\lambda$ stays positive (note that $F_\lambda$ cannot touch the x-axis on $(0, \infty)$ by uniqueness). Clearly for $t = 0+$, we have
\[ \log F_\lambda = 2\log t + (1 - \lambda - 3Q^2(0))t^2 + O(t^4); \] (3.8)
\[ \frac{F_\lambda'(t)}{F_\lambda(t)} = \frac{2}{t} + 2(1 - \lambda - 3Q^2(0)t + O(t^3)). \] (3.9)
In particular it is not difficult to check that for $t_1 > 0$ sufficiently small, we have
\[ \frac{F_\lambda'(t_1)}{F_\lambda(t_1)} < \frac{\beta'(t)}{\beta(t)}, \quad t = t_1; \] (3.10)
\[ F_\lambda(t_1) < \beta(t_1), \] (3.11)
where $\beta(t) = -c_1tQ'(t)$, and $c_1 > 0$ is sufficiently large. Note that
\[ \beta'' = (1 + \frac{2}{t^2} - 3Q^2(t))\beta. \]
Comparing $\beta$ with $F_\lambda$ on $[t_1, \infty)$ and using the assumption that $F_\lambda$ is positive, we obtain
\[ 0 < F_\lambda(t) < \beta(t), \quad \forall t_1 \leq t < \infty. \] (3.12)
First we discuss the case $\lambda = 1$. By Lemma 3.1 the solution must decay as $t^{-1}$ as $t \to \infty$. But then it clearly contradicts to the upper bound $\beta(t)$ which decays as $O(e^{-t})$.

The case $0 < \lambda < 1$ is similar. One can also obtain a contradiction. Thus $F_\lambda$ must change sign on $(0, \infty)$.

The estimate of $t_0 \geq 0.2$ follows from Lemma 4.2.

## 4. AFTER THE FIRST POSITIVE ZERO

**Lemma 4.1.** We have
\[ 0 < Q(t) \leq 2.714 \frac{1}{t} e^{-\frac{1}{t}}, \quad \forall t \geq 2.5; \] (4.1)
\[ 3Q(t)^2 \leq e^{-2t}, \quad \forall t \geq 5. \] (4.2)
Proof. By Lemma 2.4 in [2], we have
\[
\frac{187}{69} e^{-t} < \tilde{Q}(t) < \frac{350}{129} e^{-t}, \quad \forall t \geq 2.5.
\] (4.3)

Note that \(\frac{350}{129} \approx 2.7101\), and
\[
|Q(t) - \tilde{Q}(t)| \leq 7 \cdot 10^{-5} \cdot \frac{e^{-t}}{1 + t}, \quad \forall t \geq 0.
\] (4.4)

The desired bound for \(t \geq 2.5\) clearly holds.

The proof for \(t \geq 5\) then follows from a similar simple computation. □

**Lemma 4.2.** We have
\[
\frac{2}{t^2} \geq 3Q^2(t), \quad \text{if } 0 < t \leq 0.2 \text{ or } t \geq 1.5.
\] (4.5)

Proof. For \(0 < t \leq 0.2\), thanks to the explicit expression of \(\tilde{Q}(t)\), one can check that
\[
\frac{2}{t^2} - 3(\tilde{Q}(t))^2 > 4.9.
\] (4.6)

Denote \(\eta = \tilde{Q} - Q\) and recall that \(\|\eta\|_\infty < 7 \times 10^{-5}\). Since \(\|\tilde{Q}\|_\infty < 4.4\), we have
\[
|3(\tilde{Q} + \eta)^2 - 3(\tilde{Q})^2| \leq 3Q^2 + 6|\eta||\tilde{Q}| < 0.1.
\] (4.7)

Thus the desired estimate holds for \(0 < t \leq 0.2\).

It is not difficult to verify for \(0 < t \leq 0.2\),
\[
\frac{2}{t^2} - 3(\tilde{Q}(t))^2 > 0.5.
\] (4.8)

Thus the desired estimate holds for \(t \geq 2.5\).

We only need to consider the regime \(1.5 \leq t \leq 2.5\). One can check that for \(1.5 \leq t \leq 2.5\),
\[
\frac{2}{t^2} - 3(\tilde{Q}(t))^2 > 0.29.
\] (4.9)

The desired upper then holds for \(Q\) thanks to (4.7). □

**Proposition 4.1.** Consider
\[
\begin{cases}
G'' = \left(\frac{2}{(t + t_0)^2} - 3Q^2(t + t_0)\right)G, & t > 0; \\
G(0) = 0, G'(0) = 1.
\end{cases}
\] (4.10)

Assume \(t_0 \geq 0.2\). Then \(G(t) > 0\) for all \(t > 0\), and
\[
G(t) > ct^{c/2}, \quad t \geq 2.5,
\] (4.11)

where \(c > 0\), \(c_2 > 0\) are constants.

Proof. Observe that for \(t_0 \geq 1.5\) we have
\[
\frac{2}{(t + t_0)^2} - 3Q^2(t + t_0) \geq 0 \quad \text{for all } t.
\]
In this case the solution obviously grows in time.

Thus it is enough to consider the case \(t_0 \in [0.2, 1.5]\).

By (4.8), we have
\[
\frac{2}{(t + t_0)^2} - 3Q^2(t + t_0) \geq \frac{0.5}{(t + t_0)^2}, \quad t \geq 2.5.
\] (4.12)

Consider the auxiliary system
\[
\begin{cases}
G''_1 = \frac{0.5}{(t + t_0)^2}G_1, & t > 2.5; \\
G_1(2.5) > 0, \quad G'_1(2.5) > 0.
\end{cases}
\] (4.13)

It is not difficult to prove that for some constants \(\alpha_1 > 0\), \(\alpha_2 > 0\), we have \(G_1(t) \geq \alpha_1 t^{\alpha_2}\), for all \(t \geq 2.5\).

It remains for us to check that for \(t \in (0, 2.5]\), \(t_0 \in [0.2, 1.5]\), it holds that
\[
G(t) > 0, \quad \forall 0 < t \leq 2.5;
\] (4.14)
\[
G'(2.5) > 0.
\] (4.15)

Both statements can be verified rather easily numerically (and rigorously).
5. The case \( l \geq 2 \)

**Lemma 5.1.** We have
\[
\frac{6 - 10^{-20}}{t^2} > 3Q^2(t), \quad \forall 0 < t < \infty.
\]

**(Proof.** By Lemma 4.2, we only need to check the regime \( 0.2 \leq t \leq 1.5 \). In this case we have
\[
\frac{6}{t^2} - 3(Q(t))^2 > 2.18.
\]

The desired estimate then follows from (4.7).

Now we consider the equation
\[
\left(-\partial^{tt} + \frac{2}{t} \partial_t + 1 - \lambda - \frac{(l+1)}{t^2} - 3Q^2(t)\right)f = 0,
\]
where \( \lambda \in [0, 1], l \geq 2 \).

Clearly for \( l \geq 2 \), we have the point-wise bound
\[
\frac{l(l+1) - 10^{-20}}{t^2} > 3Q^2(t), \quad \forall t > 0.
\]

It follows that the above system cannot admit any nontrivial \( L^2 \) solution.

6. The case \( l = 0 \)

We consider the equation
\[
\left(-\partial^{tt} + \frac{2}{t} \partial_t + 1 - \lambda - 3Q^2(t)\right)f = 0.
\]

Denote \( F_\epsilon(t) = tf(t) \) and \( \epsilon = 1 - \lambda \in [0, 1] \). It suffices to study the equation
\[
\begin{align*}
F''_\epsilon &= (\epsilon - 3Q^2)F_\epsilon, \quad t > 0; \\
F_\epsilon(0) &= 0, \quad F'_\epsilon(0) = -1.
\end{align*}
\]

We chose the normalization \( F'_\epsilon(0) = -1 \) since \( F_\epsilon \) will change sign at least once. This is proved in the following lemma.

**Lemma 6.1.** Let \( \epsilon \in [0, 1] \). Then \( F_\epsilon \) must change its sign at least once. The first positive zero \( t_\epsilon \) of \( F_\epsilon \) satisfies
\[
t_\epsilon \geq t_0 > 0,
\]
where \( t_0 \) is the first positive zero of \( F_0 \).

**(Proof.** We first show that \( F_\epsilon \) must change its sign. Assume that \( F_\epsilon \) is negative for all \( 0 < t < \infty \). Denote \( G_\epsilon = -F_\epsilon \). Consider
\[
\begin{align*}
G'' &= (1 + \epsilon_0 - 3Q^2)G, \\
G(0) &= 0, \quad G'(0) = 1.
\end{align*}
\]

Here \( \lambda = -\epsilon_0 < 0 \) corresponds to the negative eigenvalue and \( G \) is the corresponding eigen-function which is positive on \((0, \infty)\). Observe that
\[
\begin{align*}
G'' &= (\epsilon - 3Q^2)G_\epsilon, \\
G_\epsilon(0) &= 0, \quad G'_\epsilon(0) = 1.
\end{align*}
\]

Since we assume \( G_\epsilon > 0 \) on \((0, \infty)\), it follows by using comparison that
\[
0 < G_\epsilon(t) \leq G(t), \quad \forall 0 < t < \infty.
\]

Note that \( G(t) \) decays as \( e^{-\sqrt{1 + \lambda} t} \) as \( t \to \infty \). This clearly contradicts the decay of \( G_\epsilon \). Thus we arrive at a contradiction. It follows that \( F_\epsilon \) must change sign at least once on \((0, \infty)\).

The proof of \( t_\epsilon \geq t_0 \) follows by comparing \( F_\epsilon \) with \( F_0 \). \( \Box \)
We now consider
\[
\begin{cases}
F'' = (\epsilon - 3Q^2(t + t_\epsilon)) F, & t > 0; \\
F(0) = 0, & F'(0) = 1.
\end{cases}
\] (6.7)

Note that
\[
\epsilon - 3Q^2(t + t_\epsilon) \geq -3Q^2(t + t_0).
\] (6.8)

We only need to examine the \(\epsilon\)-independent system
\[
\begin{cases}
q'' = -3Q^2(t + t_0) q, & t > 0; \\
q(0) = 0, & q'(0) = 1.
\end{cases}
\] (6.9)

**Theorem 6.1.** We have \(q(t) > 0\) for all \(0 < t < \infty\). Furthermore \(\min_{t \geq 1} \frac{1}{t} q(t) \geq c_0 > 0\) for some constant \(c_0\).

**Proof.** Firstly we observe that it suffices to consider the system
\[
\begin{cases}
F'' = -3Q^2 F, & t > 0; \\
F(0) = 0, & F'(0) = -1.
\end{cases}
\] (6.10)

We only need to show that \(F(t)\) stays positive for \(t > t_0\) and \(F\) remains bounded below for \(t \geq 1 + t_0\).

Step 1: the regime \(0 \leq t \leq 5\). In this step we use rigorous numerics to compute \(F\) to high precision thanks to the explicit form of \(Q\). We obtain
\[
|F(5) - 0.47| < 0.01, \quad |F'(5) - 0.03| < 0.01.
\] (6.11)

Step 2: the regime \(t \geq 5\). By Lemma 6.1, we have
\[
3Q^2(t) \leq e^{-2t}, \quad \forall t \geq 5.
\] (6.12)

Consider the system
\[
\begin{cases}
G'' = -e^{-2t} G, & t \geq 5; \\
G(5) = F(5), & G'(5) = F'(5).
\end{cases}
\] (6.13)

Clearly if \(G\) stays positive, then \(F(t) \geq G(t)\) for all \(t \geq 5\) by using comparison.

We now focus on analyzing \(G\). One can solve the \(G\)-equation explicitly and obtain
\[
G(t) = \alpha_1 J_0(e^{-t}) + \alpha_2 Y_0(e^{-t}), \quad t \geq 5,
\] (6.14)
where \(\alpha_1 > 0, \alpha_2 < 0\). For example if we take \(G(5) = F(5) = 0.48, G'(5) = F'(5) = 0.03\), then
\[
\alpha_1 = 0.326585, \quad \alpha_2 = -0.0486773.
\] (6.15)

More generally if \(|F(5) - 0.47| < 0.01, |F'(5) - 0.03| < 0.01\), then \(\alpha_1 > 0, \alpha_2 < 0\). On the other hand, \(J_0(e^{-t}) > 0 \text{ for } t \geq 5\) and \(J_0(e^{-t}) \to J_0(0) = 1 \text{ as } t \to \infty\). We have \(Y_0(e^{-t}) \to 0 \text{ for } t \geq 5\) and \(Y_0(e^{-t})/t \to -\frac{2}{3} \text{ as } t \to \infty\). It follows that \(G(t) > 0 \text{ for all } t \geq 5\) and \(G(t) \to \infty \text{ as } t \to \infty\). \(\square\)

**Proof of Theorem 6.1** This follows from our analysis for \(l = 0, l = 1\) and \(l \geq 2\) in previous sections. \(\square\)

7. **The operator \(L_\lambda\)**

The proof for \(L_\lambda\) is similar. Thus we only sketch the needed modifications. It suffices to examine the equation
\[
(-\frac{1}{r^2} - \frac{2}{r} \frac{\partial}{\partial r} + 1 - \lambda - Q^2) R_0 = 0
\] (7.1)

Note that for \(\lambda = 0\), \(R_0(r) = Q(r)\) is a solution to the above equation.

Denote \(t = r, H_\epsilon(t) = tR_0(t)\) and \(\epsilon = 1 - \lambda \in [0, 1]\). It suffices to study the equation
\[
\begin{cases}
H'' + (\epsilon - Q^2) H, & t > 0; \\
H(0) = 0, & H'(0) = -1.
\end{cases}
\] (7.2)
**Lemma 7.1.** Let \( \epsilon \in [0, 1) \). Then \( H_\epsilon \) must change its sign at least once. The first zero \( \tau_\epsilon \) of \( H_\epsilon \) satisfies
\[
\tau_\epsilon \geq \tau_0 > 0,
\]
where \( \tau_0 \) is the first zero of \( H_0 \).

**Proof.** The proof is similar to Lemma 6.1. One can use the comparison function \( H_1(r) = \text{const} \cdot rQ(r) \) to deduce that \( F_\epsilon \) for \( \epsilon \in [0, 1) \) must change sign.

Similar to the argument in Section 6, we only need to examine the system
\[
\begin{cases}
p'' = -Q^2(t + \tau_0)p, & t > 0; \\
p(0) = 0, \quad p'(0) = 1.
\end{cases}
\]

**Theorem 7.1.** We have \( p(t) > 0 \) for all \( 0 < t < \infty \). Furthermore \( \min_{t \geq 1} p(t) \geq c_1 > 0 \) for some constant \( c_1 \).

**Proof.** The proof is similar to Theorem 6.1.

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