SOME REMARKS ON THE RADIUS OF SPATIAL ANALYTICITY FOR THE EULER EQUATIONS

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Abstract. We consider the Euler equations on $\mathbb{T}^d$ with analytic data and prove lower bounds for the radius of spatial analyticity $\epsilon(t)$ of the solution using a new method based on inductive estimates in standard Sobolev spaces. Our results are consistent with similar previous results proved by Kukavica and Vicol, but give a more precise dependence of $\epsilon(t)$ on the radius of analyticity of the initial datum.

1. Introduction

Consider, on the $d$-dimensional torus $\mathbb{T}^d$, the Euler equations

\begin{equation}
\frac{\partial u}{\partial t} + P(u \cdot \nabla x u) = 0, \quad \text{div } u = 0,
\end{equation}

where $P$ is the Leray projection in $L^2$ on the subspace of divergence free vector fields. It is well known that the corresponding Cauchy problem is locally well posed in $H^k$ if $k > d/2 + 1$, see e.g. [14, Chapter 17, Section 2]. The analyticity of the solution in the space variables, for analytic initial data is also an important issue, investigated in [1, 2, 3, 4, 10, 11, 12], and one is specially interested in lower bounds for the radius of analyticity $\epsilon(t)$ as $t$ grows.

To be precise, if $f$ is an analytic function on the torus, its radius of analyticity is the supremum of the constants $\epsilon > 0$ such that $\|\partial^\alpha f\|_{L^\infty} \leq C\epsilon^{-|\alpha|}\alpha!$ for some constant $C > 0$. Notice that we can also replace the $L^\infty$ norm with a Sobolev norm $H^k$, $k \geq 0$.

Concerning the Euler equations on the torus, a recent result by Kukavica and Vicol [10] states that for the radius of analyticity $\epsilon(t)$ of any analytic solution $u(t)$ we have the lower bound

\begin{equation}
\epsilon(t) \geq C(1 + t)^{-2} \exp \left( - C_0 \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds \right)
\end{equation}

for a constant $C_0 > 0$ depending on the dimension and $C > 0$ depending on the norm of the initial datum in some infinite order Sobolev space.

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The same authors in [11] obtained a better lower bound for $\varepsilon(t)$ for the Euler equations in a half space replacing $(1+t)^{-2}$ by $(1+t)^{-1}$ in (1.2). Now, one suspects that the same bound should hold also on $T^d$ and on $R^d$. In this paper we give a new and more elementary proof of the results above on $T^d$, which yields some improvements concerning the dependence on the initial data. In fact, it is natural to expect that the constant $C$ in (1.2) should be comparable with the inverse of the radius of analyticity of the initial datum. The dependence found in [10, 11] is due to the fact that the proof given there relies on the energy method in infinite order Gevrey-Sobolev spaces (cf. also [5, 6, 8, 9, 12]). In our recent paper [7] we developed a method for the estimate of the radius of analyticity for semilinear symmetrizable hyperbolic systems based on inductive estimates in standard Sobolev spaces. The purpose of this note is to adapt this method to the Euler equations on $T^d$ and to prove that

\begin{equation}
\varepsilon(t) \geq C(1+t)^{-1} \exp \left( -C_0 \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds \right)
\end{equation}

(as suggested in [11]) with a neat dependence of the constant $C$ on the initial datum.

Namely, we have the following result.

**Theorem 1.1.** Let $k > d/2 + 1$ be fixed. There exists constants $C_0, C_1 > 0$, depending only on $k$ and $d$, such that the following is true.

Let $u_0$ be analytic in $T^d$, $\text{div} \, u_0 = 0$, satisfying

\begin{equation}
\|\partial^\alpha u_0\|_{H^k} \leq BA|\alpha|^{-1}|\alpha|!/(|\alpha| + 1)^2, \quad \alpha \in \mathbb{N}^d,
\end{equation}

for some $B \geq \frac{9}{4}\|u_0\|_{H^{2k+1}}$, $A \geq 1$.

Let $u(t, x)$ by the corresponding $H^k$ maximal solution of the Euler equations (1.1), with $u(0, \cdot) = u_0$. Then $u(t, \cdot)$ is analytic with radius of analyticity

\begin{equation}
\varepsilon(t) \geq A^{-1}(1 + C_1 Bt)^{-1} \exp \left( -C_0 \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds \right).
\end{equation}

The proof is different and more elementary than the one in [10, 11], and it is in part inspired by the arguments in [1]. It proceeds by estimating by induction the growth of the spatial derivatives of $u$ in finite order Sobolev spaces. Moreover, the same argument can be readily repeated replacing $T^d$ by $R^d$. Although in the case of the Euler equations it provides only minor improvements to the results in [10, 11], our method seems to be adaptable also to other types of quasilinear evolution equations and conservation laws. We shall treat these applications in a future paper.

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1We use the sequence $|\alpha|!/(|\alpha| + 1)^2$ in place of $|\alpha|!$ in (1.4) just for technical reasons; this does not change the radius of analyticity.
2. Notation and preliminary results

In the following we use the notation $X \lesssim Y$ if $X \leq CY$ for some constant $C$ depending only on the dimension $d$ and on the index $k$ in Theorem 1.1.

Moreover, as in [1] page 196 we consider the sequence

\[
M_n = \frac{n!}{(n+1)^2}, \quad n \geq 0,
\]

so that

\[
\sum_{\beta < \alpha} \binom{\alpha}{\beta} M_{|\alpha-\beta|} M_{|\beta|+1} \leq C|\alpha| M_{|\alpha|},
\]

for some constant $C > 0$.

We also recall from [14, Chapter 13, Proposition 3.6], for future reference, the following estimates

\[
\| \partial^\alpha u \cdot \partial^\beta v \|_{L^2} \leq C(\|u\|_{L^\infty} \|v\|_{H^m} + \|u\|_{H^m} \|v\|_{L^\infty}), \quad |\alpha| + |\beta| = m
\]

for a constant $C > 0$ depending on $m$ and on the dimension $d$.

We will use the following form of the Gronwall inequality.

**Lemma 2.1.** Let $f(t) \geq 0$, $g(t) \geq 0$, $h(t) \geq 0$ be continuous functions on $[0, T]$ and $C \geq 0$, such that

\[
f(t) \leq C + \int_0^t h(s)f(s) \, ds + \int_0^t g(s) \, ds, \quad t \in [0, T].
\]

Then, with $H(t) := \int_0^t h(s) \, ds$, we have

\[
f(t) \leq e^{H(t)} \left[ C + \int_0^t e^{-H(s)} g(s) \, ds \right], \quad t \in [0, T].
\]

**Proof.** The result can be obtained for example by applying Gronwall lemma in [13, Lemma 2.1.3], and integrating by parts. \qed

Finally we recall from [14, Chapter 17, Section 2 and Exercise 1, page 485] that if $u_0$ is a smooth vector field (and $\text{div} u_0 = 0$), then the maximal $H^k$-solution $u(t)$ of the Euler equations, with $u(0) = u_0$, $k > d/2 + 1$, is smooth as well and moreover the following estimates hold for its Sobolev norms: if $s \geq k > d/2 + 1$,

\[
\|u(t)\|_{H^s} \leq \|u(0)\|_{H^s} \exp \left( C_0 \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds \right)
\]

for some constant $C_0 > 0$ depending on the dimension and on $s$. Indeed, in the sequel we will use these estimates for some fixed $s$ (depending only on $d$).
3. Proof of the main result (Theorem 1.1)

As observed in the previous section, we already know that the solution \( u \) is smooth, since \( u_0 \) is. Now, it is sufficient to prove that for \(|\alpha| = N \geq 2\) we have

\[
(3.1) \quad \frac{\|\partial^\alpha u(t)\|_k}{M_{|\alpha|}} \leq 2BA^{N-1} \exp \left( C_0(N - 1) \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds \right) \left( 1 + C_1Bt \right)^{N-2},
\]

where the sequence \( M_{|\alpha|} \) is defined by (2.1) and \( C_0, C_1 \) are positive constants depending only on \( k \) and \( d \).

We also set

\[
(3.2) \quad \mathcal{E}_N[u(t)] = \sup_{|\alpha|=N} \frac{\|\partial^\alpha u(t)\|_k}{M_{|\alpha|}}.
\]

We proceed by induction on \( N \). The result is true for \( N = 2 \) by (2.4) with \( s = k + 2 \leq 2k + 1 \) and by the assumption \( B \geq \frac{9}{4} \|u_0\|_{H^{2k+1}}, A \geq 1 \). Hence, let \( N \geq 3 \) and assume (3.1) holds for multi-indices \( \alpha \) of length \( 2 \leq |\alpha| \leq N - 1 \) and prove it for \( |\alpha| = N \).

For \( |\alpha| = N, |\gamma| \leq k \) we estimate \( \|\partial^\alpha + \gamma u\|_{L^2} \) starting from the following formula, which is well-known (see e.g. [14, pag. 477]):

\[
(3.3) \quad \frac{d}{dt}\|\partial^{\alpha+\gamma} u\|_{L^2}^2 = -2([\partial^{\alpha+\gamma}, L]u, \partial^{\alpha+\gamma} u)_{L^2},
\]

with \( Lw = L_\alpha w := u \cdot \nabla w \). Now, we have

\[
(3.4) \quad [\partial^{\alpha+\gamma}, L]u = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} \partial^{\alpha-\beta+\gamma-\delta} u \cdot \nabla \partial^{\beta+\delta} u.
\]

We estimate the \( L^2 \) norm of each term, considering first the sum

\[
\sum_{\beta \in A_{\alpha}} \binom{\alpha}{\beta} \sum_{\delta \leq \gamma \atop |\beta|+|\delta| \leq |\alpha|+|\gamma|} \binom{\gamma}{\delta} \|\partial^{\alpha-\beta+\gamma-\delta} u \cdot \nabla \partial^{\beta+\delta} u\|_{L^2},
\]

where

\[
A_{\alpha} := \{ \beta : \beta \leq \alpha, 0 \neq |\beta| \leq |\alpha| - 2 \}.
\]

Using (2.3) and the fact that \( k > d/2 + 1 \) we see that for \( \beta \in A_{\alpha} \) we have

\[
\|\partial^{\alpha-\beta+\gamma-\delta} u \cdot \nabla \partial^{\beta+\delta} u\|_{L^2} \lesssim \|\partial^{\alpha-\beta} u\|_k \|\nabla \partial^{\beta} u\|_k.
\]
By the inductive hypothesis (3.1) (note that $2 \leq |\alpha - \beta| \leq |\alpha| - 1 = N - 1$, $2 \leq |\beta| + 1 \leq N - 1$) we obtain

$$
\sum_{\beta \in A_{\gamma, \delta, \alpha}} \binom{\alpha}{\beta} \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} \|\partial^{\alpha-\beta+\gamma-\delta} u \cdot \nabla \partial^{\beta+\delta} u\|_{L^2}
$$

$$
\lesssim \sum_{\beta < \alpha} \binom{\alpha}{\beta} M_{|\alpha-\beta|} M_{|\beta|+1} B^2 A^{N-1} \exp \left(C_0(N-1) \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds\right) (1 + C_1 Bt)^{N-3}
$$

$$
\lesssim N M_N B^2 A^{N-1} \exp \left(C_0(N-1) \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds\right) (1 + C_1 Bt)^{N-3},
$$

where we used (2.2).

It remains to estimate the $L^2$ norms of the terms in (3.4) when $\beta \leq \alpha$, $\delta \leq \gamma$, and $|\beta| + |\delta| < |\alpha| + |\gamma|$, but the conditions

$$
0 \neq |\beta| \leq |\alpha| - 2
$$

fail.

Consider first the terms where the highest order derivatives fall on a single factor, namely $|\beta| + |\delta| = |\alpha| + |\gamma| - 1$ or $|\beta| + |\delta| = 0$. We distinguish three cases: for the terms with $\beta = \alpha$ and $|\delta| = |\gamma| - 1$ we have

$$
\binom{\alpha}{\alpha} \|\partial^{\gamma-\delta} u \cdot \nabla \partial^{\alpha+\delta} u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \|\partial^{\alpha} u\|_{H^{|\beta|+1}} \leq \|\nabla u\|_{L^\infty} \|\partial^{\alpha} u\|_{H^k}
$$

whereas for those with $|\beta| = |\alpha| - 1$, $\delta = \gamma$ we use

$$
\binom{\alpha}{\beta} \|\partial^{\alpha-\beta} u \cdot \nabla \partial^{\beta+\gamma} u\|_{L^2} \leq |\alpha| \|\nabla u\|_{L^\infty} \sum_{j; \alpha_j \geq 1} \|\partial^{\alpha-e_j} u\|_{H^{k+1}}
$$

$$
\lesssim |\alpha| \|\nabla u\|_{L^\infty} \sum_{j; \alpha_j \geq 1, 1 \leq k \leq d} \|\partial^{\alpha-e_j} u\|_{H^k} \lesssim N M_N \|\nabla u\|_{L^\infty} \mathcal{E}_N[u],
$$

where $\mathcal{E}_N[u]$ is defined in (3.2).

Finally, for the terms with $|\beta| + |\delta| = 0$ we have

$$
\binom{\alpha}{0} \|\partial^{\alpha+\gamma} u \cdot \nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \|\partial^{\alpha} u\|_{H^k}.
$$

We now consider the terms with $0 \neq |\beta| + |\delta| \leq |\alpha| + |\gamma| - 2$ but $\beta = \alpha$ or $|\beta| = |\alpha| - 1$ or $\beta = 0$. 


If $\beta = \alpha$ then $|\delta| \leq |\gamma| - 2 \leq k - 2$ and we can write

$$
\left(\frac{\alpha}{\beta}\right) \left\| \partial^{\gamma-\delta} u \cdot \nabla \partial^{\alpha+\delta} u \right\|_{L^2} \leq \left\| \partial^{\gamma-\delta} u \right\|_{L^\infty} \left\| \nabla \partial^{\alpha+\delta} u \right\|_{L^2} \\
\lesssim \left\| \partial^{\gamma-\delta} u \right\|_{H^k} \left\| \partial^{\alpha} u \right\|_{H^{k-1}} \lesssim \left\| \partial^{\gamma-\delta} u \right\|_{H^k} \sup_{j: \alpha_j \geq 1} \left\| \partial^{\alpha-e_j} u \right\|_{H^k} \\
\lesssim M_{N-1} \left\| u(0) \right\|_{H^{2k}} \exp \left( C_0 \int_0^t \left\| \nabla u(s) \right\|_{L^\infty} \, ds \right) \\
\times BA^{N-2} \exp \left( C_0(N-2) \int_0^t \left\| \nabla u(s) \right\|_{L^\infty} \, ds \right) (1 + C_1 Bt)^{N-3}
$$

where we used (2.4) (with $s = |\gamma - \delta| + k \leq 2k$), the inductive hypothesis (3.1), and the fact that $B > \left\| u_0 \right\|_{H^{2k+1}}$.

If $|\beta| = |\alpha| - 1$ then $|\delta| \leq k - 1$ and we have

$$
\left(\frac{\alpha}{\beta}\right) \left\| \partial^{\gamma-\delta} u \cdot \nabla \partial^{\beta+\delta} u \right\|_{L^2} \lesssim |\alpha| \cdot \left\| \partial^{\gamma-\delta} u \right\|_{L^\infty} \left\| \nabla \partial^{\beta+\delta} u \right\|_{L^2} \lesssim |\alpha| \cdot \left\| \partial^{\gamma-\delta} u \right\|_{H^k} \left\| \partial^{\beta} u \right\|_{H^k} \\
\lesssim NM_{N-1} \left\| u(0) \right\|_{H^{2k}} \exp \left( C_0 \int_0^t \left\| \nabla u(s) \right\|_{L^\infty} \, ds \right) \\
\times BA^{N-2} \exp \left( C_0(N-2) \int_0^t \left\| \nabla u(s) \right\|_{L^\infty} \, ds \right) (1 + C_1 Bt)^{N-3}
$$

Finally, if $\beta = 0$ then $\delta \neq 0$ and we have

$$
\left(\frac{\alpha}{0}\right) \left\| \partial^{\alpha+\gamma-\delta} u \cdot \nabla \partial^{\beta} u \right\|_{L^2} \lesssim \left\| \partial^{\alpha} u \right\|_{H^{k-1}} \left\| \nabla \partial^{\beta} u \right\|_{H^k} \\
\lesssim \sup_{j: \alpha_j \geq 1} \left\| \partial^{\alpha-e_j} u \right\|_{H^k} \left\| \nabla \partial^{\beta} u \right\|_{H^k} \\
\lesssim M_{N-1} B^2 A^{N-2} \exp \left( C_0(N-1) \int_0^t \left\| \nabla u(s) \right\|_{L^\infty} \, ds \right) \\
\times (1 + C_1 Bt)^{N-3}
$$

Summing up, we have

$$
\left\| [\partial^{\alpha+\gamma}, L] u \right\|_{L^2} \lesssim NM_N \left\| \nabla u \right\|_{L^\infty} E_N[u] + NM_N B^2 A^{N-1} \exp \left( C_0(N-1) \int_0^t \left\| \nabla u(s) \right\|_{L^\infty} \, ds \right) (1 + C_1 Bt)^{N-3}.
$$
By applying the Cauchy-Schwarz inequality in $L^2$ in (3.3) and summing over $|\gamma| \leq k$ we then obtain

\[
\frac{d}{dt} \| \partial^\alpha u(t) \|_{H^k} \leq C(N - 1)\| \nabla u(t) \|_{L^\infty} \mathcal{E}_N[u(t)] + CNA^{N-1}B^2 \exp \left( C_0(N - 1) \int_0^t \| \nabla u(s) \|_{L^\infty} \, ds \right) (1 + C_1 Bt)^{N-3}
\]

for a constant $C > 0$ depending only on the dimension $d$ and $k$.

Now we integrate from 0 to $t$ and take the supremum on $|\alpha| = N$. We obtain

\[
\mathcal{E}_N[u(t)] \leq \int_0^t C(N - 1)\| \nabla u(s) \|_{L^\infty} \mathcal{E}_N[u(s)] \, ds + \mathcal{E}_N[u(0)] + \int_0^t CNA^{N-1}B^2 \exp \left( C_0(N - 1) \int_0^s \| \nabla u(\tau) \|_{L^\infty} \, d\tau \right) (1 + C_1 Bs)^{N-3} \, ds.
\]

We can take $C_0 \geq C$, so that Gronwall inequality (Lemma 2.1) gives

\[
\mathcal{E}_N[u(t)] \leq \exp \left( C_0(N - 1) \int_0^t \| \nabla u(s) \|_{L^\infty} \, ds \right) \times \left[ \mathcal{E}_N[u(0)] + CNA^{N-1}B^2 \int_0^t (1 + C_1 Bs)^{N-3} \, ds \right] \leq \exp \left( C_0(N - 1) \int_0^t \| \nabla u(s) \|_{L^\infty} \, ds \right) \times \left[ \mathcal{E}_N[u(0)] + \frac{C}{C_1} \frac{N}{N - 2} BA^{N-1}(1 + C_1 Bt)^{N-2} \right]
\]

Now, we have $\mathcal{E}_N[u(0)] \leq BA^{N-1}$ by the assumption (1.4). If we choose $C_1 = 3C$, so that $3C/C_1 = 1$, since $A \geq 1$, $N \geq 3$ we have

\[
\mathcal{E}_N[u(0)] + \frac{C}{C_1} \frac{N}{N - 2} BA^{N-1}(1 + C_1 Bt)^{N-2} \leq [BA^{N-1} + \frac{3C}{C_1} BA^{N-1}](1 + C_1 Bt)^{N-2} \leq 2BA^{N-1}(1 + C_1 Bt)^{N-2},
\]

and we obtain exactly (3.1) for $|\alpha| = N$. The theorem is proved.

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We use $N/(N - 1) \leq 3/2$ if $N \geq 3$, and also $\frac{d}{dt} \| \partial^\alpha u(t) \|_{H^k} = 2\| \partial^\alpha u(t) \|_{H^{k+1}} \frac{d}{dt} \| \partial^\alpha u(t) \|_{H^k}$, which in fact holds for $\| \partial^\alpha u(t) \|_{H^k} \neq 0$ but a standard argument, see e.g. [13] pag. 47-48, shows that the results below still hold when $\| \partial^\alpha u(t) \|_{H^k} = 0$. 

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