Hybrid numerical model of the plasma flow dynamics in open magnetic systems

V.A. Vshivkov, M.A. Boronina, E.A. Genrikh, G.I. Dudnikova, L.V. Vshivkova, A.M. Sudakov

The Institute of Computational Mathematics and Mathematical Geophysics
SB RAS, Novosibirsk
E-mail: vsh@ssd.sscc.ru

Abstract. The report provides a brief overview of the numerical models used to solve plasma physics problems. The necessity of using hybrid models is shown, and a hybrid model that can be used in computer modeling of plasma flows in open magnetic systems is considered in more detail. The features of using hybrid models in two-dimensional cylindrical coordinates are given. To solve the equations of particle motion, the Boris algorithm is considered and its improvement is proposed based on the analytical solution of equations at a time step. We present a number of results of numerical simulation of the interaction of plasma flows obtained on the basis of a hybrid model. The calculations were performed in relation to the conditions of laboratory experiments with laser plasma at the KI-1 facility of the ILF SB RAS and the conditions of the diamagnetic regime at the CAT facility of the INP SB RAS.

1. Introduction
The numerical models used to solve plasma physics problems can be divided into three groups: hydrodynamic, kinetic, and hybrid. The system of equations of the two-fluid MHD model is obtained from kinetic equations when the local distribution function is slightly different from the Maxwell one. The system of simple plasma transport equations contains the equations of continuity, motion and conservation of thermal energy for electrons and ions [1]

\[
\begin{align*}
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e V_e) &= 0; \\
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i V_i) &= 0; \\
m_e n_e \frac{dV_e}{dt} &= -\nabla p_e - \frac{\partial \pi_{e\beta}}{\partial x_\beta} - en_e \left( E + \frac{1}{c} V_e \times H \right) + R; \\
m_i n_i \frac{dV_i}{dt} &= -\nabla p_i - \frac{\partial \pi_{i\beta}}{\partial x_\beta} + Z e n_i \left( E + \frac{1}{c} V_i \times H \right) - R; \\
3 \frac{\partial T_e}{\partial t} + p_e \nabla \cdot V_e &= -\nabla \cdot q_e - \pi_{e\beta} \frac{\partial V_e}{\partial x_\beta} + Q_e; \\
3 \frac{\partial T_i}{\partial t} + p_i \nabla \cdot V_i &= -\nabla \cdot q_i - \pi_{i\beta} \frac{\partial V_i}{\partial x_\beta} + Q_i,
\end{align*}
\]

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where

\[ p_e = n_e T_e; \quad p_i = n_i T_i; \]

\[ \frac{d_e}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_e \nabla); \quad \frac{d_i}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_i \nabla). \]

This system of equations is supplemented by the Maxwell system of equations for electromagnetic fields.

When modeling a rarefied plasma, the distribution functions are nonequilibrium, so it is necessary to solve the kinetic equations. To study the dynamics of a collisionless plasma, the mathematical model consisting of the Vlasov equations for each plasma component (electrons and ions of different elements) and Maxwell equations for electromagnetic fields is best suited [2]

\[ \frac{\partial f_\alpha}{\partial t} + \mathbf{v} \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H} \right) \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0, \]

\[ \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \sum_\alpha q_\alpha \int f_\alpha \mathbf{v} d\mathbf{v} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \]

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}. \quad (2) \]

\[ \nabla \cdot \mathbf{E} = 4\pi \rho = 4\pi \sum_\alpha q_\alpha \int f_\alpha d\mathbf{v}, \]

\[ \nabla \cdot \mathbf{H} = 0. \]

Here \( \alpha \) is the sort of particles (ions or electrons), \( q_\alpha \) is the particle charge.

For the numerical solution of problems using this model, the method of particles in cells is most effective due to its universality for a wide range of physical parameters [3, 4]. There are a number of 2- and 3-dimensional electromagnetic kinetic PIC codes (KARAT, VORPAL, OOPIC, MANDOR, etc.), which are used, in particular, to solve the problems of the interaction of a laser pulse or electron beam with a plasma. However, the application of the particle method in three-dimensional problems requires large computational resources - memory and computer speed. This is due to the fact that in the particle in cells method in the solution area a grid is introduced (the grid step determines the accuracy of the solution), in each cell of which a sufficiently large number of model particles (up to 1000 particles) is placed. But an even greater limitation on the effectiveness of the numerical model is imposed by the different scales of physical phenomena determined by the behavior of light (electrons) and heavy (ions) plasma particles. In particular, the time step in the numerical model is selected from the conditions of accuracy and stability of motion of the lightest particles modeling the electronic component of the plasma. If the effects under study are determined by the movement of ionic components, then a significant limitation when using the model will be associated with a long counting time.

To reduce the requirements for the speed and memory of a computer, in comparison with completely kinetic models, combined (hybrid) models are used [5-7]. Their feature is that one of the plasma components (ions or electrons) is described kinetically, and the other is considered in the hydrodynamic approximation. Hydrodynamic equations for various plasma components can be obtained from the corresponding kinetic equations, provided that the particle distribution function is equilibrium or close to this state. For example, for an electronic component, these conditions can be a sufficiently large magnetic field or a high plasma density. For each specific problem, an analysis of the possibility of using the hydrodynamic approximation for any plasma component is necessary. In the course of calculations, it is necessary to ensure that the plasma parameters do not go beyond the applicability of the hydrodynamic approximation.
The simplest hybrid electrostatic model can be considered under conditions when the electron temperature is much higher than the ion temperature \( T_e \gg T_i \) [8, 9]. Such conditions are often realized in laboratory experiments and solar wind plasma. In this model, only ionic component motion is considered, and the electron density is described by the Boltzmann distribution \( n(r, t) = n_0 \exp(\varphi(r, t)/T_e) \), where \( \varphi(r, t) \) is the electric field potential, \( n_0 \) is undisturbed plasma density. The system of equations has the form

\[
\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial r} - e \frac{\partial \varphi}{\partial r} \frac{\partial f}{\partial u} = 0, \\
\]

\[
\Delta \varphi = 4\pi e \left[ n_0 \exp\left(\frac{e\varphi}{T_e}\right) - \int_{-\infty}^{\infty} f du \right],
\]

where \( f = f(r, u, t) \) is the ion distribution function, \( u \) is the ion velocity. In this formulation in [6, 7], the problems of the propagation of large-amplitude waves and the decay of an arbitrary discontinuity in the density of ions in a nonisothermal plasma were solved.

One of the first works in which the hybrid model is described is article [5]. It presents hybrid numerical algorithms for modeling low-frequency \( (\omega \ll \omega_{pe}, \omega_{ce}) \) electromagnetic and electrostatic phenomena in a magnetized plasma. Maxwell’s equations are solved within a small transverse bias current (Darwin model) and quasineutrality. Numerical models treat electrons as a massless liquid, and ions as particles. The numerical stability of the algorithms is studied analytically and confirmed by computer experiments.

2. Hybrid model

In this paper, we consider the following hybrid model. In it, the ionic components are described using kinetic equations, and the hydrodynamic type equations are used for the electronic component. In this model, it is assumed that the plasma is quasineutral and bias currents \( \partial E / \partial t \) are neglected.

Equations for ionic components

\[
\frac{\partial f_\alpha}{\partial t} + v \frac{\partial f_\alpha}{\partial r} + \frac{F_\alpha}{m_\alpha} \frac{\partial f_\alpha}{\partial v} = 0, \\
F_\alpha = q_\alpha \left( E + \frac{1}{c} v \times H \right) + R_\alpha.
\]

where the index \( \alpha \) denotes the sort of ions.

Hydrodynamic equations for electrons

\[
m_e \left( \frac{\partial V_e}{\partial t} + (V_e \cdot \nabla)V_e \right) = -e \left( E + \frac{1}{c} V_e \times H \right) - \frac{\nabla p_e}{n_e} + R_e, \\
n_e \left( \frac{\partial T_e}{\partial t} + (V_e \cdot \nabla)T_e \right) + (\gamma - 1)p_e \nabla \cdot V_e = (\gamma - 1)(Q_e - \nabla \cdot q_e).
\]

The quasineutrality condition has the form \( n_e = \sum_\alpha Z_\alpha n_\alpha = n \), where \( n_\alpha = \int f_\alpha(r, v, t) \, dv \). The friction force is calculated by the formulas

\[
R_e = \frac{e}{\sigma} j, \quad R_\alpha = -\frac{Z_\alpha e}{\sigma} j, \quad R_e + \sum_\alpha R_\alpha = 0.
\]

Electron heating \( Q_e \) and heat flux due to thermal conductivity \( q_e \) are calculated by the formulas

\[
Q_e = \frac{j^2}{\sigma}, \quad q_e = -\kappa \nabla T_e,
\]

where \( \sigma \) is the plasma conductivity, \( \kappa \) is the thermal conductivity coefficient, \( p_e = n_e T_e \) is the electron pressure [1].
A system of Maxwell equations without bias currents is added to these equations

\[
\nabla \times \mathbf{H} = \frac{4\pi}{c} j,
\]
\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t},
\]
\[
\nabla \cdot \mathbf{H} = 0,
\]
\[
\nabla \cdot \mathbf{E} = 0.
\]

In the absence of energy flows across the boundaries of the solution domain, the law of conservation of total energy is satisfied for this system

\[
\sum_{\alpha} \frac{m_{\alpha}}{2} \int f_{\alpha}(r,v,t) \ v^2 dv dr + \int \left( n_e \frac{m_e V_e^2}{2} + \frac{H^2}{8\pi} + n_e T_e \frac{1}{\gamma - 1} \right) dr = \text{const}.
\]

The kinetic equation for the ion component is solved using the particle-in-cell method, in which the ion component of the plasma is replaced by a set of model particles. The equations of motion for model particle with number \( j \) are obtained from the characteristics of the kinetic equation

\[
\frac{dr_j}{dt} = v_j,
\]
\[
\frac{dv_j}{dt} = \frac{q_j}{m_j} \left( E + \frac{1}{c} v_j \times H \right) + \frac{1}{m_j} R_{\alpha}.
\]

and the hydrodynamic equations for electrons are solved on a grid, in the same way as the Maxwell equations. This model (4-7) is used to solve the problem of the interaction of an ion beam with a background plasma in the magnetic field of an open plasma trap.

Open plasma traps have a cylindrical geometry, and cylindrical coordinates \((r, \varphi, z)\) can be entered into them. Assuming that all functions are independent of the angular coordinate \( \varphi \), we pass from the three-dimensional problem to the two-dimensional one, where all functions depend on only two spatial variables \((r, z)\). The transition to two-dimensional coordinates significantly saves computer memory and the number of arithmetic operations. However, one cannot take into account the possibility of the development of a number of angular instabilities and the design features of the trap.

From the point of view of implementing the particle-in-cell method, the following features appear in the algorithm:

1) Model particles have different masses (charges) depending on the distance to the axis.
2) The cell volume is proportional to the distance to the axis.
3) Features of setting boundary conditions on the axis for charge density and current appear.
4) In the algorithm for calculating the coordinates and velocities of particles, it is necessary to use the method proposed in [10], which consists in a local transition to Cartesian coordinates, the calculation of new coordinates and particle velocities, and the reverse transition to cylindrical ones.

3. Particle pusher
In the particle-in-cell method, Boris’s algorithm is usually used to calculate new velocities [10]

\[
\frac{v_1 - v^n_{1/2}}{\tau/2} = \frac{q}{m} E^n,
\]
\[
\frac{v_2 - v_1}{\tau} = \frac{q}{mc} \frac{v_2 + v_1}{2} \times H^n,
\]
\[
\frac{\mathbf{v} - \mathbf{v}_2}{\tau / 2} = \frac{q}{m} \mathbf{E}.
\]

At each time step, the coordinate calculation algorithm is divided into three stages. In the first and third, the particle velocity changes only under the influence of an electric field, and in the second - only under the influence of a magnetic field. The circuit has a second order of approximation in time. Despite the formally implicit form, the system of three equations for the three velocity components is easily solved analytically. The advantage of the second stage is that when it is used, the velocity modulus does not change, which corresponds to the law of motion of a charged particle in a magnetic field. Moreover, kinetic energy is retained even if the magnetic field or time step is very large.

Let us consider separately the second stage of the method (motion in a magnetic field). If the magnetic field is directed along the \( z \) axis, \( \mathbf{H} = (0, 0, H_z) \), and the velocity vector \( \mathbf{v}_1 = (1, 0, 0) \), then the vector \( \mathbf{v}_2 \) will lie in the \((x, y)\) plane and have the following coordinates

\[
\mathbf{v}_2 = \left( \frac{4 - \omega^2 \tau^2}{4 + \omega^2 \tau^2} - \frac{4 \omega \tau}{4 + \omega^2 \tau^2}, 0 \right), \quad \omega = \frac{q H_z}{m c}.
\]

(9)

The equation of the second stage in a differential form can be solved analytically. This solution with the same initial data has the form \((t = \tau)\)

\[
\mathbf{v}_2 = (\cos \omega \tau, -\sin \omega \tau, 0).
\]

(10)

Figure 1 shows the solutions of the second stage of the Boris scheme using difference schemes and analytical formulas. For small \( \tau H_z (\omega \tau < 0.5) \), the solutions differ little from each other, and for large, the character of these curves changes. The exact solution corresponds to cyclotron rotation of the ion in a magnetic field, while in the numerical solution the sign of the velocity vector is reversed. It should be noted that for any values of \( \tau H_z \) the length of the velocity vector remains equal to 1, and the scheme is stable. When a particle enters a region with high magnetic field values, then its motion is considered incorrect.

Since when calculating the velocities during the time step \( \tau \), the electric and magnetic fields remain constant, it is possible to include electric fields in the analytical solution. Suppose again that the magnetic field is directed along the \( z \) axis, \( \mathbf{H} = (0, 0, H_z) \), and the velocity and electric field vectors lie in the \((x, y)\) plane, \( \mathbf{v}_1 = (v_x^1, v_y^1, 0), \mathbf{E} = (E_x, E_y, 0) \). Then the solution will look like

\[
v_x^2 = \left( v_x^1 - \frac{c E_y}{H_z} \right) \cos \omega \tau + \left( v_y^1 + \frac{c E_x}{H_z} \right) \sin \omega \tau + \frac{c E_y}{H_z},
\]

(11)
\[ v_{y}^2 = \left( v_{y}^1 + \frac{cE_x}{H_z} \right) \cos \omega \tau - \left( v_{x}^1 - \frac{cE_y}{H_x} \right) \sin \omega \tau - \frac{cE_x}{H_z}. \]

In formulas (11), the velocity separation into two parts is clearly visible: the drift velocity in crossed fields and the Larmor rotation velocity. In the three-dimensional case, for an arbitrary direction of all vectors, the algorithm for calculating the speed at step \( \tau \) has the following form:

1) First, the projections of the velocity and electric field vectors on the direction of the magnetic field are calculated

\[ v_{\parallel}^1 = (v \cdot \mathbf{H}) \frac{H}{H^2}, \quad E_{\parallel} = (E \cdot \mathbf{H}) \frac{H}{H^2}. \]

2) Subtracting from the velocity and electric field vectors the parts parallel to the magnetic field, we obtain the parts of these vectors perpendicular to the magnetic field

\[ v_{\perp}^1 = v_{\parallel}^1, \quad E_{\perp} = E - E_{\parallel}. \]

3) The drift velocity is calculated

\[ v_{dr} = \frac{c}{H^2} E_{\perp} \times H. \]

4) Next, using the analytical formulas, we find new velocity components perpendicular to the magnetic field

\[ v_{\perp x}^2 = (v_{\perp x}^1 - v_{dr,x}) \cos \omega \tau + \frac{1}{H} (v_{1y}^1 H_x - v_{1z}^1 H_y + cE_x) \sin \omega \tau + v_{dr,x}, \]
\[ v_{\perp y}^2 = (v_{\perp y}^1 - v_{dr,y}) \cos \omega \tau + \frac{1}{H} (v_{1z}^1 H_x - v_{1x}^1 H_z + cE_y) \sin \omega \tau + v_{dr,y}, \]
\[ v_{\perp z}^2 = (v_{\perp z}^1 - v_{dr,z}) \cos \omega \tau + \frac{1}{H} (v_{1x}^1 H_y - v_{1y}^1 H_x + cE_z) \sin \omega \tau + v_{dr,z}, \]

where

\[ \omega = \frac{q}{mc} H = \frac{q}{mc} \sqrt{H_x^2 + H_y^2 + H_z^2}. \]

5) The addition to the speed due to the longitudinal electric field and the new longitudinal speed are calculated

\[ v_{\parallel}^2 = v_{\parallel}^1 + \tau \frac{q}{m} E_{\parallel}. \]

6) Adding new longitudinal and lateral speeds, we obtain the full speeds

\[ v_2 = v_{\parallel}^2 + v_{\perp}^2. \]

An important problem in modeling open traps is the motion of model particles in the region of magnetic plugs. It is known that when particles move along an alternating magnetic field, they retain a magnetic moment associated with kinetic energy transverse to the magnetic field [11]. Since at each time step the particle moves in a constant field, it may not retain the magnetic moment and, therefore, may not be reflected from magnetic plugs. To verify this assumption, a test calculation of the motion of a group of particles in the vicinity of a magnetic plug with the same initial velocity was performed. Figure 2 shows the trajectories of three model particles moving in the direction of the \( z \) axis.
As can be seen from the figure, a particle located closer to the axis, flew to the border, and the other two were reflected from the border. Thus, the algorithm used allows one to correctly describe the plasma motion in open magnetic traps.

4. Calculation results

Let us present the results of numerical simulation of the interaction of plasma flows obtained on the basis of a hybrid model. The calculations were performed in relation to the conditions of laboratory experiments with laser plasma at the KI-1 stand [12] and the conditions of the diamagnetic regime at the SAT facility [13].

1. The case of a constant magnetic field. In this formulation, the mechanisms of collisionless interaction of plasma flows and the nature of perturbations generated by a dense plasma piston cloud were studied [2, 6, 14]. Figure 3 shows the magnetic field lines at successive times that characterize the change in the magnetic structure during expansion of a dense plasma cloud in a low-density magnetized background plasma. This process is accompanied by the formation of a magnetic cavity, the size of which depends on the kinetic energy of the cloud and the magnetic field strength.

The evolution of the background plasma density is shown on Figure 4. From this figure it is seen that the formation of a region of reduced density (density cavity) is accompanied by the generation of disturbances that propagate in the form of shock waves. Their structure depends on the angle of propagation with respect to the direction of the magnetic field and on the Alfven-Mach number of the plasma cloud [2].
2. Inhomogeneous magnetic field. In this formulation, the problem of ion beam injection into the magnetic system of an open plasma trap is solved [13]. The initial configuration of the magnetic field and the direction of the incoming beam are shown on Figure 5.

Figure 5. The configuration of the magnetic system of the open trap

Features of this task compared to the previous one are:

1) in the presence of large gradients of the magnetic field of the mirror cell, which requires a small time step.

2) the need for continuous injection of an ion beam that has a directed velocity and a Maxwell velocity distribution (temperature).

3) the need to determine the pressure of the ionic component of the plasma, for the calculation of which an algorithm was used based on the determination of temperature as the dispersion of the particle distribution function.

Here are some simulation results that characterize the possibility of heating and confining the plasma in the diamagnetic mode of an open plasma trap. Presented on Figure 6 the distribution of the beam ions and the background plasma on the (r, z) plane at successive times gives a clear idea of the dynamics of plasma flows in the beam-plasma system under consideration. The movement of the ion beam is accompanied by the formation of a cavity of the density of the background plasma and the displacement of the magnetic field (Figure 7).
Figure 6. Distribution of beam ions (a, c) and background plasma (b, d) on the (r, z) plane at successive times t = 20 (a, b) and t = 35 (c, d).

In Figure 7 (upper) shows the distribution of the magnetic field pressure at a time corresponding to the formation of its quasistationary structure. The pressure of the magnetic field inside the cavity is 10% of its initial value. At the boundary of the cavity, layers of increased magnetic field pressure and ion pressure (Figure 7, lower) are formed, which have comparable values. The results obtained correspond to the available theoretical estimates of plasma dynamics in the diamagnetic regime of the mirror cell and can be used in planning experiments on heating and plasma confinement using the CAT facility (INP SB RAS) [13].

In all the figures presented in the work, physical variables are presented in a dimensionless form. The spatial dimensions are normalized to the dispersion length $L = c/\omega_{i0}$, the time to $t_0 = \omega_{hi}^{-1}$. For the conditions of experiments on CAT facility [13], $B_0 = 0.2$ T, $n_0 = 10^{12} cm^{-3}$, $L = 22$ cm, $t_0 = 5 \cdot 10^{-8}$ s.

Figure 7. Distribution of magnetic field pressure (upper) and ion pressure (t = 15).
5. Conclusion

To solve the problem of plasma dynamics in open magnetic systems, a hybrid model has been created that takes into account the basic laws of physical processes and anomalous dissipation mechanisms. A modification of the Boris algorithm for solving the equations of particle motion is proposed. Two examples of using a hybrid model in cylindrical coordinates are considered. In both cases, the movement of the ion beam is accompanied by the formation of a cavity of the density of the background plasma and the displacement of the magnetic field. The above results give a clear idea of the dynamics of plasma flows in the beam-plasma systems under consideration.

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References

[1] Braginsky S I 1963 Reviews of Plasma Physics vol 1 pp 183-272 (in Russian)
[2] Berezin Yu A, Dudnikova G I, Liseykina T V and Fedoruk M P 2017 Modeling of unsteady plasma processes (Novosibirsk: Novosibirsk State University Press) p 359 (in Russian)
[3] Berezin Yu A and Vshivkov V A 1980 Particle method in the dynamics of rarefied plasma (Novosibirsk: Nauka) p 94 (in Russian)
[4] Birdsall Ch K and Langdon A B 1985 Plasma Physics via Computer Simulation (McGraw-Hill Book Company) p 479
[5] Byers J A, Cohen B I, Condit W C and Hanson J D 1978 J Comput Phys 27 (3) pp 363-96
[6] Berezin Yu A, Dudnikova G I, Fedoruk M P and Vshivkov V A 1998 J. Comp. Math.: Fluid Dynamics 10 pp 117-26
[7] Vshivkova L and Dudnikova G 2017 Lecture Notes in Computer Science vol 10187 LNCS pp 737-43
[8] Malkov M A, Sagdeev R Z, Dudnikova G I, Liseykina T V, Diamond P H, Papadopoulos K, Liu C-S and Su J-J 2016 Physics of Plasmas 23 (4) doi.org/10.1063/1.4945649
[9] Berezin Yu A and Vshivkov V A 1977 Numerical methods in plasma physics (Moskow Nauka) pp 150-53 (in Russian)
[10] Boris J P 1970 Fourth Conference on Numerical Simulation of Plasmas (Washington) pp 3-67
[11] Krall N A and Trivelpiece A W 1973 Principles of plasma physics (McGraw-Hill Book Company) p 686
[12] Zakharov Yu P, Ponomarenko A G, Tishchenko V N, Antonov V M, Melekhov A V, Posukh V G, Prokopov P A and Terekhin V A 2016 Quantum Electronics 46 pp 399-405
[13] Bagryansky P A et al 2016 AIP Conf. Proc. 1771 030015 https://doi.org/10.1063/1.4964171
[14] Dudnikova G I, Orishich A M, Ponomarenko A G, Vshivkov V A and Zakharov Yu P 1990 Plasma Astrophysics ESA SP-311 p 301