In [KLP14], Kapovich, Leeb and Porti gave several new characterizations of Anosov representations $\Gamma \to G$, including one where geodesics in the word hyperbolic group $\Gamma$ map to "Morse quasigeodesics" in the associated symmetric space $G/K$. In analogy with the negative curvature setting, they prove a local-to-global principle for Morse quasigeodesics and describe an algorithm which can verify the Anosov property of a given representation in finite time. However, some parts of their proof involve non-constructive compactness and limiting arguments, so their theorem does not explicitly quantify the size of the local neighborhoods one needs to examine to guarantee global Morse behavior. In this paper, we supplement their work with estimates in the symmetric space to obtain the first explicit criteria for their local-to-global principle. This makes their algorithm for verifying the Anosov property effective. As an application, we demonstrate how to compute explicit perturbation neighborhoods of Anosov representations with two examples.

1 Introduction

Anosov representations were introduced by Labourie and defined in general by Guichard and Wienhard [Lab06; GW12]. An Anosov representation is a homomorphism from a word hyperbolic group $\Gamma$ to a semisimple Lie group $G$ satisfying a strong dynamical condition. These representations have come to be widely studied as an interesting source of infinite covolume discrete subgroups of higher rank semisimple Lie groups, see the surveys [Kas18; KL18a]. This paper is concerned with certifying the Anosov property of a given representation. For some well-studied examples of Anosov representations, such as Hitchin representations and maximal representations of surface groups, the Anosov property can be certified via coarse topological invariants [BIW03]. However, in the most general setting, deciding whether a given representation is Anosov is difficult. Building on the work of Kapovich, Leeb and Porti in [KLP14], we give here the first explicit, finite criteria that certify the Anosov property for a general representation.

One important property of Anosov representations is stability: any sufficiently small perturbation of an Anosov representation remains Anosov. It can happen that a connected component of the representation space consists entirely of Anosov representations, such as the Hitchin component, or the components consisting of maximal representations of surface groups, see also [Wie18; GW18]. In these cases, the Anosov condition is closed: every deformation of such a representation remains Anosov. However the Anosov condition is not closed in general. For instance, given an Anosov representation of a free group, or the representations of surface groups studied by Barbot in [Bar10], it is unclear how large to expect Anosov neighborhoods to be. As an application of our main result, we demonstrate how to construct explicit perturbation neighborhoods of a given Anosov representation with two examples, see Theorem 1.2 and Theorem 1.3.

Anosov representations have come to be viewed as the appropriate generalization to higher rank semisimple Lie groups of convex cocompact actions on rank 1 symmetric spaces. Indeed, when $G$ has real rank 1, a representation of a finitely generated group is Anosov if and only if it has finite kernel and the image is convex cocompact, i.e. acts cocompactly on a nonempty convex subset of the associated negatively curved symmetric space. A finitely generated group of isometries of a negatively curved symmetric space is convex cocompact if and only if it is undistorted, i.e. any orbit map is a quasi-isometric embedding. By the Morse Lemma in hyperbolic geometry, geodesics in $\Gamma$ then map within uniformly bounded neighborhoods of geodesics in the symmetric space. Moreover, the Morse Lemma implies a local-to-global principle for quasigeodesics, allowing one to establish finite criteria for a finitely generated group to be undistorted. One
can then exhaust the group by balls in the Cayley graph and if any such ball passes a finite check then the group is undistorted. This is a semidecidable algorithm to verify undistortion: if the group is undistorted, this algorithm will eventually terminate and certify so; otherwise, it will run on forever.

The naive generalization of convex cocompactness to higher rank turns out to be too restrictive. For example, the work of Kleiner and Leeb and independently Quint implies that a Zariski dense, discrete subgroup of a higher rank simple Lie group which acts cocompactly on a convex subset of the associated symmetric space is a uniform lattice \[ \text{KL00, Qu10}. \] On the other hand, the naive generalization of undistortion to higher rank turns out to be too loose: in his thesis, Guichard described an example of an undistorted subgroup in \( \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) which is unstable, in the sense that representations arbitrarily close to the inclusion fail to have discrete image [Gui04], see also [GGKW17]. In [KLP14], Kapovich, Leeb and Porti describe an example of a discrete undistorted subgroup of \( \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) which is finitely generated but not finitely presentable, using work Baumslag and Roseblade [BR84]. The Anosov property strikes a balance between these two naive generalizations to give a large class of representations that still exhibit good behavior. We will be concerned with a newer characterization that directly strengthens the undistortion condition.

In [KLP14], Kapovich, Leeb and Porti gave several new characterizations of Anosov representations generalizing some of the many characterizations of convex cocompact subgroups. We will use their characterization, called Morse actions, that strengthens the undistortion condition by requiring geodesics in \( \Gamma \) to map to \textit{Morse quasigeodesics}, described below. They prove a suitable generalization of the local-to-global principle for Morse quasigeodesics in higher rank symmetric spaces, see Theorem 1.1 below. They then show the Anosov property is semi-decidable by describing an algorithm which can certify the Anosov property of a given representation of a word hyperbolic group in finite time. However, some parts of their proof involve non-constructive compactness and limiting arguments, so their theorem does not explicitly quantify the size of the local neighborhoods one needs to examine to guarantee global Morse behavior. In order to implement their algorithm, one needs a quantified version of the local-to-global principle as we give here.

Roughly speaking, a quasigeodesic is \textit{Morse} if every finite consecutive subsequence is uniformly close to a \textit{diamond}, which plays the role of a geodesic segment in rank 1. These diamonds are intersections of Weyl cones, see Sections 3.7 and 5.4 and may also be characterized as unions of Finsler geodesic segments, see [KL18b, KL18c]. An infinite Morse quasiray stays within a uniformly bounded neighborhood of a Weyl cone, which plays the role of a geodesic ray in rank 1, and a bi-infinite Morse quasigeodesic stays within a uniformly bounded neighborhood of a parallel set, which plays the role of a geodesic line in rank 1, see Section 6.3. The precise definition of Morse quasigeodesic is given in Section 5.

The main result of this paper is a quantified version of the following theorem due to Kapovich, Leeb and Porti.

**Theorem 1.1** ([KLP14, Theorem 7.18]). For any \( \alpha_{\text{new}} < \alpha_0, D, c_1, c_2, c_3, c_4 \), there exists a scale \( L \) so that every \( L \)-local \((\alpha_0, \tau_{mod}, D)\)-Morse \((c_1, c_2, c_3, c_4)\)-quasigeodesic in \( X \) is an \((\alpha_{\text{new}}, \tau_{mod}, D)\)-Morse \((c'_1, c'_2, c'_3, c'_4)\)-quasigeodesic.

We reprove Theorem 1.1 and obtain the first explicit estimate of \( L \). This appears in Theorem 5.6 which depends on Theorem 5.3 and Theorem 5.4. The theorem statements involve several auxiliary parameters and inequalities, so they are too cumbersome to give here. In order to apply our quantified version of the local-to-global principle and obtain an explicit scale \( L \), on must produce auxiliary parameters satisfying these inequalities; this process is tedious but easy, as we discuss in Section 6. Versions of Theorems 5.1 and 5.6 without explicit conditions are also proved in [KLP14].

As a demonstration of our techniques, we compute explicit perturbation neighborhoods of two Anosov representations into \( \text{SL}(3, \mathbb{R}) \). To quantify the distance between linear representations we use the Frobenius norm on the generators: for a matrix \( A \), let \( |A|_F^2 = \text{trace}(A^T A) \). In both cases we control the orbit map at a basepoint; the Frobenius norm is closely related to distances to that basepoint, see Section 6.3. The first example is a neighborhood of Anosov representations of a free group.

**Theorem 1.2.** Let \( \Gamma_1 \) be the subgroup of \( \text{SL}(3, \mathbb{R}) \) generated by

\[
g = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad h = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix},
\]

2
with \( \tanh t = 0.75 \). If \( \Gamma'_1 \) is generated by \( g', h' \) where \( \max \{ |g - g'|_{Fr}, |h - h'|_{Fr} \} \leq 1.94 \times 10^{-15.598} \), then \( \Gamma'_1 \) is Anosov.

The second example is a neighborhood of Anosov representations of a closed surface group. Let \( \Gamma_2 \) be the subgroup of \( SL(3, \mathbb{R}) \) generated by

\[
S = \left\{ \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \right\}
\]

for \( \log \lambda = \cosh^{-1}(\cot \theta) \). This group is isomorphic to the fundamental group of a closed surface of genus 2, see Section 6.3. In the statement of Theorem 1.3 we control the perturbed representation on a larger generating set \( S' = \{ \gamma \in \Gamma_2 \mid \sqrt{6} |\log \gamma|_{Fr} \leq 9.5 \} \). The finite set \( S' \) contains the standard generating set \( S \) and consists of the elements of \( \Gamma_2 \) which move a basepoint \( p \) in the symmetric space associated to \( SL(3, \mathbb{R}) \) by a distance of at most 9.5. This basepoint is the point stabilized by \( SO(3) \). Using a larger generating set allows us to perturb the initial representation farther.

**Theorem 1.3.** If \( \rho: \Gamma_2 \to SL(3, \mathbb{R}) \) is a representation satisfying \( |\rho(s) - s|_{Fr} \leq 10^{-3.6 \times 10^6} \) for all \( s \in S' \), then \( \rho \) is Anosov.

We briefly sketch the proof of Theorems 1.2 and 1.3. Let \( \Gamma \) denote either \( \Gamma_1 \) or \( \Gamma_2 \). In either case the group \( \Gamma \) acts cocompactly on a closed convex subset of a copy of the hyperbolic plane embedded totally geodesically in the symmetric space associated to \( SL(3, \mathbb{R}) \). We find explicit quasiisometry constants and by the classical Morse Lemma, there exists \( R > 0 \) such that the orbit of any geodesic in \( \Gamma \) is within \( R \) of a geodesic. We slightly relax the Morse quasiisometric parameters of \( \Gamma \) and apply the local-to-global principle Theorem 5.6. This provides a lower bound on \( k \) such that any \( 2k \)-local Morse quasiisometric is a global Morse quasiisometric. We control the perturbation of words of length \( k \) in terms of the perturbation of the generators, completing the proof.

We emphasize that our approach is completely general, in the following sense. Let \( \rho: \Gamma \to G \) be any Anosov representation such that the orbit map at \( p \in X \) has known Morse quasiisometric parameters with respect to a finite symmetric generating set \( S \) for \( \Gamma \). We may then easily produce explicit parameters \( k, \epsilon \) such that: if any other representation \( \rho': \Gamma \to G \) satisfies \( d(\rho(\gamma)p, \rho'(\gamma)p) \leq \epsilon \) for all \( \gamma \in \Gamma \) of word length at most \( k \), then \( \rho' \) is Anosov. Moreover, for linear groups we explicitly bound \( d(\rho(\gamma)p, \rho'(\gamma)p) \) in terms of the word length of \( \gamma \), the Frobenius norms \( |\rho(s)|_{Fr} \) and \( |\rho(s) - \rho'(s)|_{Fr} \), so we obtain a condition on \( \rho' \) just in terms of the generators.

The bulk of the paper is devoted to a proof of Theorem 1.3. We supply a number of estimates in Section 4 related to the geometry of \( X \). An important tool is the \( \zeta \)-angle \( \zeta' \), a \( Stab_{\rho}(p) \)-invariant metric on \( \text{Flag}(\tau_{mod}) \) introduced by Kapovich, Leeb and Porti in [KLP14]. We show that if the \( \zeta \)-angle at \( p \) is close to \( \pi \) then the distance from \( p \) to the corresponding parallel set is small, and vice versa, see Corollary 4.14 and Lemma 4.16. This implies that midpoints of long regular segments have short projections to nearby parallel sets, see Corollary 4.11 and that if regular vectors make a small angle at \( p \), then their corresponding simplices form a small \( \zeta \)-angle, see Lemma 4.10. Several of the estimates use the generalized Iwasawa decomposition of \( G \), see Section 3.3 and properties of Killing vector fields on \( X \). The proof of Lemma 4.16 is particularly involved, because the embedded flag manifolds in the unit tangent sphere at \( p \) are not locally convex. We extend the distance function from a regular point to the relevant flag manifold in Section 3.9 and obtain some control on its Hessian, recorded in Corollary 4.13. This is the primary ingredient in Lemma 4.16 which implies an upper bound on the \( \zeta \)-angles in terms of the regularity parameter \( \alpha_0 \). Many of these estimates appear in [KLP14] without explicit constants because the proofs there use non-constructive compactness or limiting arguments. For example, an abstract proof that the two metrics on the embedded flag manifold are comparable is easy because all metrics on a compact space are comparable. However, producing an explicit local Lipschitz constant comparing the metrics is nontrivial.

Theorem 5.1 guarantees that a sequence \( (x_n) \) with sufficiently spaced points forming \( \zeta \)-angles sufficiently close to \( \pi \) is a Morse quasigeodesic. It is a quantified version of Theorem 7.2 in [KLP14] and shares the same outline. One first shows that the property of “moving away” from a simplex propagates along the sequence, see Section 5.1. This implies that we can extract a simplex \( \tau_- \) that the sequence \( (x_n) \) moves away from (respectively towards) as \( n \) increases (respectively decreases), and a simplex \( \tau_+ \) that the sequence \((x_n)\)
moves away from (respectively towards) as \( n \) decreases (respectively increases). One then verifies that the simplices \( \tau_- \), \( \tau_+ \) are opposite and that the projections to the parallel set \( P(\tau_-, \tau_+) \) define suitable diamonds, making \( (x_n) \) a Morse quasigeodesic.

Theorem 5.4 is a quantified version of Proposition 7.16 in [KLP14]. It states that sufficiently spaced points on Morse quasigeodesics have straight and spaced midpoint sequences. A crucial ingredient is Corollary 4.11 which allows us to force the midpoints to be arbitrarily close to the parallel sets in terms of the Morse and spacing parameters. This guarantees that they appear in nested Weyl cones, and makes the \( \zeta \)-angles arbitrarily straight.

 Armed with the Theorem 5.1 and Theorem 5.4, the proof of Theorem 5.6 is similar to the proof of Theorem 1.1 given in [KLP14]. We start with an \( L \)-local Morse quasigeodesic where \( L \) is large enough to satisfy several explicit inequalities. We then replace our Morse quasigeodesic with a coarsification and take the midpoint sequence. Our assumptions together with Theorem 5.4 shows that this coarse midpoint sequence is sufficiently straight and spaced, see Section 5.3. An application of Theorem 5.4 shows that the midpoint sequence is a Morse quasigeodesic, and since it is a coarse approximation of the original sequence, the original sequence is also a Morse quasigeodesic, completing the proof.

The usual proof of the local-to-global principle in hyperbolic geometry depends on the classical Morse Lemma. A higher rank version of the Morse Lemma was proved by Kapovich, Leeb and Porti in [KLP18]. In particular they prove that the orbit map \( \Gamma \to X \) of a finitely generated group is a coarsely uniformly regular quasiisometric embedding if and only if \( \Gamma \) is word hyperbolic and the orbit map is a Morse quasiisometric embedding. It would be interesting to quantify their higher rank Morse Lemma by producing an explicit Morse parameter for (coarsely) uniformly regular quasiisometric embeddings, but we do not do this here. In the special case of the symmetric space associated to \( \text{SL}(d,\mathbb{R}) \), another proof of the higher rank Morse Lemma appears in [BPS19]. There, Bochi, Potrie and Sambarino give yet another characterization of Anosov representations in terms of cone-types and dominated splittings.

The organization of the paper is as follows. In Section 2 we fix some notation we use throughout the paper. In Section 3 we review some background of symmetric spaces. Much of this section is classical and may be skipped by experts on symmetric spaces, but we point the reader to our definition of regularity in Definition 3.10 and the definition of \( \zeta \)-angle in Definition 3.17. The notion of regularity here is slightly different, but equivalent to, that in [KLP14], see Proposition 3.15. The bulk of the work is in Section 4 where we give several estimates related to the geometry of symmetric spaces. In Section 5 we supplement the proof of the local-to-global principle in [KLP14] with our estimates from Section 4, reproving Theorem 1.1 with explicit bounds. Together with some standard geometric group theory, elementary hyperbolic geometry, and linear algebra in Section 6, this allows us to prove Theorems 1.2 and 1.3.

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2 Notation

We establish our notational conventions in this paper. When possible, we have tried to keep notation consistent with \cite{KLP14, KLP17, Ebe96}.

1. $X = G/K$ will denote a symmetric space of noncompact type. $G$ is assumed to be the connected component of the isometry group of $X$, and $K$ is a maximal compact subgroup of $G$, see Section 3.

2. We let $p, q, r, c$ denote points or curves in $X$. We let $g, h, u, a$ denote elements or curves in $G$.

3. The Lie algebra of $G$ is denoted $\mathfrak{g}$. The Lie algebra of $K$ is denoted $\mathfrak{k}$. When a point $p$ is given, $K$ is the stabilizer of $p$ in $G$. Usually $U, V, W, X, Y, Z$ will denote elements of $\mathfrak{g}$.

4. The orbit map $\text{orb}_p: G \to X$, given by $\text{orb}_p(g) = gp$, has differential $\text{ev}_p: \mathfrak{g} \to T_p X$ at the identity, see Section 3.

5. The Cartan decomposition induced by $p \in X$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. It corresponds to a Cartan involution $\theta_p: \mathfrak{g} \to \mathfrak{g}$, see Section 3.1.

6. The Killing form on $\mathfrak{g}$ is denoted $B$. Each point $p \in X$ induces an inner product $B_p$ on $\mathfrak{g}$ defined by $B_p(X, Y) = -B(\theta_p X, Y)$, see Section 3.1.

7. We assume that the Riemmanian metric $\langle \cdot, \cdot \rangle$ on $X$ is the one induced by the Killing form, see Equation 2.
8. The sectional curvature $\kappa$ of $X$ has image $[-\kappa^2_0, 0]$, see Section 3.3.

9. A maximal abelian subspace of $\mathfrak{p}$ will be denoted $\mathfrak{a}$. The associated restricted roots are denoted $\Lambda \subset \mathfrak{a}^\vee$. A choice of simple roots is denoted $\Delta$, see Section 3.2.

10. Each maximal abelian subspace $\mathfrak{a}$ has an action by the Weyl group and decomposition into Euclidean Weyl chambers denoted $V$, see Section 3.4.

11. There is a vector-valued distance function $\tilde{d} : X \times X \rightarrow V_{mod}$ with image the model Euclidean Weyl chamber, see Equation 6. In [KLP14, KLP17] this map is denoted $\Delta$, and they let $\Delta$ denote the model Euclidean Weyl chamber we call $V_{mod}$. In this paper, $\Delta$ denotes a choice of simple roots.

12. A spherical Weyl chamber $\sigma$ corresponds to a set of simple roots $\Delta$. For a face $\tau$ of $\sigma$ we have

$$\Delta_\tau = \{ \alpha \in \Delta \mid \alpha(\tau) = 0 \}, \quad \Delta_\tau^+ = \{ \alpha \in \Delta \mid \alpha(\text{int } \tau) > 0 \},$$

see Equations 10. We have

$$\tau = \sigma \cap \bigcap_{\alpha \in \Delta_\tau} \ker \alpha, \quad \text{int}_\tau \sigma = \{ X \in \sigma \mid \forall \alpha \in \Delta_\tau^+, \alpha(X) > 0 \}, \quad \partial_\tau \sigma = \sigma \cap \bigcup_{\alpha \in \Delta_\tau^+} \ker \alpha.$$

13. The visual boundary of $X$ is denoted $\partial X$, see Section 3.7. We let $\tau, \sigma$ denote a spherical simplex/chamber in $\mathfrak{a}$ or an ideal simplex/chamber in $\partial X$.

14. There is a type projection $\theta : \partial X \rightarrow \sigma_{mod}$ with image the model ideal Weyl chamber, see Section 3.7.

15. A face of $\sigma_{mod}$ is called a model simplex and denoted $\tau_{mod}$. There is a decomposition $\sigma_{mod} = \text{int}_{\tau_{mod}} \sigma_{mod} \cup \partial_{\tau_{mod}} \sigma_{mod}$, see Section 3.8.

16. We define $(\alpha_0, \tau)$-regular and $(\alpha_0, \tau)$-spanning vectors and geodesics in Section 3.10. We extend that definition to ideal points in Section 3.9.

17. We write $\text{Flag}(\tau_{mod})$ for the set of ideal simplices in $\partial X$ of type $\tau_{mod}$, see Section 3.7.

18. We define Weyl cones $V(x, \alpha_0, \tau), V(x, \text{oist}(\tau))$ and Weyl sectors $V(x, \tau)$ in Section 3.8.

19. We describe the generalized Iwasawa decomposition $G = N_r A_r K$ in Section 3.9.

20. A parallel set is denoted $P(\tau_-, \tau_+)$ for $\tau_-, \tau_+ \in \text{Flag}(\tau_{mod})$. A horocycle is denoted $H(p, \tau)$, see Section 3.10. A diamond is denoted $\Diamond(p, q)$ or $\Diamond_{\alpha_0}(p, q)$, see Section 5.1.

21. For $p \in X$ and $x, y \in X \setminus \{ p \}$, $\angle_p(x, y)$ denotes the Riemannian angle at $p$ between $x$ and $y$. For $\eta, \eta' \in \partial X$, $\angle_{\text{Tits}}(\eta, \eta')$ denotes their Tits angle. If $px$ and $py$ are $\tau_{mod}$-regular and $\tau, \tau' \in \text{Flag}(\tau_{mod})$ then $\angle^\Delta_p(\tau, \tau')$, $\angle_p^{\Delta}(\tau, y)$, $\angle_p(\zeta(\tau), \zeta(py))$ denote the $\zeta$-angles, see Section 3.11.

22. A $(c_1, c_2, c_3, c_4)$-quasigeodesic is a sequence $(x_n)$ (possibly finite, infinite, or biinfinite) in $X$ such that

$$\frac{1}{c_1}|N| - c_2 \leq d(x_n, x_{n+N}) \leq |N|c_3 + c_4.$$

A quasigeodesic is $(\alpha_0, \tau_{mod}, D)$-Morse if for all $x_n, x_m$ there exists a diamond $\Diamond_{\alpha_0}(p, q)$ such that $d(p, x_n), d(q, x_m) \leq D$ and for all $n \leq i \leq m, d(x_i, \Diamond) \leq D$, see Section 5.
3 Background on symmetric spaces

We begin with some background on the structure of symmetric spaces of noncompact type. Experts on symmetric spaces can skip this section, but should note that we assume the metric is induced by the Killing form (see Equation 2), quantify the regularity of geodesics in Definition 3.10, and define the $\zeta$-angle in Definition 3.17. For detailed references on symmetric spaces see [Ebe96; Hel01; Hel79].

A symmetric space is a connected Riemannian manifold $X$ such that for each point $p \in X$, there exists a geodesic symmetry $S_p : X \to X$, an isometry fixing $p$ whose differential at $p$ is $(dS_p)_p = -id_{T_p X}$. A symmetric space is necessarily complete with transitive isometry group. If $X$ is nonpositively curved, it is simply connected. Simply connected Riemannian manifolds admit a de Rham decomposition into metric factors. If $X$ is a nonpositively curved symmetric space with no Euclidean de Rham factors, we say $X$ is a symmetric space of noncompact type. Throughout the paper, $X$ refers to any fixed symmetric space of noncompact type.

The isometry group of $X$ is a semisimple Lie group, and we let $G$ be the identity component of the isometry group. For each point $p \in X$, the stabilizer $K = G_p = \{ g \in G \mid gp = p \}$ is a maximal compact subgroup of $G$. Hence $X$ is diffeomorphic to $G/K$ by the orbit-stabilizer theorem for Lie groups and homogeneous spaces. We write $g$ for the Lie algebra of left invariant vector fields on $G$.

A Killing vector field on a Riemannian manifold is vector field whose induced flow is by isometries. There is a natural linear isomorphism from $g$ to the space of Killing vector fields on $X$ by defining for $X \in g$ the vector field $X^*$ given by

$$X^*_p := \left. \frac{d}{dt} e^{tX} p \right|_{t=0}. \quad (1)$$

The Lie bracket of two Killing vector fields is again a Killing vector field, but the map $X \mapsto X^*$ is a Lie algebra anti-homomorphism: $[X,Y]^* = -[X^*,Y^*]$.

3.1 Cartan decomposition

Each point $p \in X$ induces a Cartan decomposition in the following way. The geodesic symmetry $S_p : X \to X$ induces an involution of $G$ by

$$g \mapsto S_p \circ g \circ S_p.$$ 

The differential is a Lie algebra involution $\vartheta_p : g \to g$, so we may write

$$g = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k} = \{ X \in g \mid \vartheta_p X = X \}$ and $\mathfrak{p} = \{ X \in g \mid \vartheta_p X = -X \}$. Since $\vartheta_p$ preserves brackets, we have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}.$$ 

We denote the orbit map $g \mapsto gp$ by $\text{orb}_p : G \to X$. The differential $(d\text{orb}_p)_1 : g \to T_p X$ has kernel precisely $\mathfrak{k}$. Moreover, $\mathfrak{k}$ is the Lie algebra of $K = G_p$. The restriction $(d\text{orb}_p)_1 : \mathfrak{p} \to T_p X$ is a vector space isomorphism. For any $X \in \mathfrak{g}$, $(d\text{orb}_p)_1 X = ev_p X := X^*_p$, see Equation 1, so we use the less cumbersome notation $ev_p = (d\text{orb}_p)_1 : g \to T_p X$ throughout the paper.

Let $B$ denote the Killing form on $g$ and let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric on $X$. We will assume that for all $X, Y \in \mathfrak{p}$,

$$B(X, Y) = \langle ev_p X, ev_p Y \rangle_p, \quad (2)$$

i.e. that the Riemannian metric on $X$ is induced by the Killing form. The other $G$-invariant Riemannian metrics on $X$ only differ from this one by scaling by a global constant on each de Rham factor of $X$.

Under the identification of $p$ with $T_p X$, the Riemannian exponential map $p \mapsto X$ is given by $X \mapsto e^X p$. In particular, the constant speed geodesics at $p$ are given by $c(t) = e^{tX} p$ for $X \in \mathfrak{p}$.

The point $p \in X$ induces an inner product $B_p$ on $g$ defined by

$$B_p(X, Y) := -B(\vartheta_p X, Y). \quad (3)$$

On $\mathfrak{p}$, $B_p$ is just the restriction of the Killing form $B$, and we have required that the identification of $(\mathfrak{p}, B)$ with $(T_p X, \langle \cdot, \cdot \rangle)$ is an isometry. On $\mathfrak{k}$, $B_p$ is the negative of the restriction of $B$ to $\mathfrak{k}$. Since $\mathfrak{k}$ and $\mathfrak{p}$ are $B$-orthogonal, it follows that $B_p$ is an inner product on $g$. For each $X \in \mathfrak{p}$, $\text{ad} X$ is symmetric with respect to $B_p$ on $g$, and likewise for each $Y \in \mathfrak{k}$, $\text{ad} Y$ is skew-symmetric.
3.2 Restricted root space decomposition

Let \( a \) be a maximal abelian subspace of \( \mathfrak{p} \). Via the adjoint action, \( a \) is a commuting vector space of diagonalizable linear transformations on \( \mathfrak{g} \). Therefore \( \mathfrak{g} \) admits a common diagonalization called the restricted root space decomposition. For each \( \alpha \in a^* \), define

\[
\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \forall A \in a, \text{ad} A(X) = \alpha(A)X \}.
\]

We obtain a collection of roots \( \Lambda = \{ \alpha \in a^* \mid \mathfrak{g}_\alpha \neq 0 \} \) corresponding to the nonzero root spaces. The restricted root space decomposition is then

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha.
\]

For each root \( \alpha \in \Lambda \), define the coroot \( H_\alpha \in a \) by \( \alpha(A) = B(H_\alpha, A) \) for all \( A \in a \). This induces an inner product, also denoted \( B \), on \( a^* \) by defining \( B(\alpha, \beta) := B(H_\alpha, H_\beta) \).

Proposition 3.1 ([Ebe96, Proposition 2.9.3]). \( \Lambda \) is a root system\(^2\) in \((a^*, B)\). That is,

1. The span of \( \Lambda \) is \( a^* \);
2. If \( \alpha \in \Lambda \) and a scalar multiple \( \lambda \alpha \in \Lambda \), then \( \lambda \in \{ \pm \frac{1}{2}, \pm 1, \pm 2 \} \);
3. For each \( \alpha \in \Lambda \), reflection in \( \alpha^\perp \) permutes \( \Lambda \);
4. If \( \alpha, \beta \in \Lambda \), then \( 2 \frac{B(\alpha, \beta)}{B(\alpha, \alpha)} \) is an integer.

In addition, the restricted root space decomposition is \( B_\mathfrak{p} \)-orthogonal.

The Cartan involution restricts to an isomorphism \( \vartheta_\mathfrak{p} : \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha} \) for each \( \alpha \in \Lambda \cup \{0\} \). Thus we have

\[
\mathfrak{p}_\alpha := \mathfrak{p} \cap \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = (\text{id} - \vartheta_\mathfrak{p})\mathfrak{g}_\alpha = (\text{id} - \vartheta_\mathfrak{p})\mathfrak{g}_{-\alpha}.
\]

and

\[
\mathfrak{k}_\alpha := \mathfrak{k} \cap \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = (\text{id} + \vartheta_\mathfrak{p})\mathfrak{g}_\alpha = (\text{id} + \vartheta_\mathfrak{p})\mathfrak{g}_{-\alpha}.
\]

Note that \( \mathfrak{p}_\alpha = \mathfrak{p}_{-\alpha} \) and likewise \( \mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha} \), so for \( \Lambda_+ \) a set of positive roots, we have the decomposition

\[
\mathfrak{g} = a \oplus \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Lambda_+} \mathfrak{p}_\alpha \oplus \bigoplus_{\alpha \in \Lambda_+} \mathfrak{k}_\alpha
\]

which is both \( B_\mathfrak{p} \) orthogonal and \( B \) orthogonal. Some authors use \( \mathfrak{m} = \mathfrak{k}_0 \).

3.3 Curvature and copies of \( \mathbb{H}^2 \)

The curvature tensor \( R \) of \( \mathbb{X} \) may be defined using its Levi-Civita connection \( \nabla \) by

\[
R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]},
\]

for vector fields \( u, v \) on \( \mathbb{X} \). In a symmetric space there is a particularly nice formula for the curvature tensor. Our convention is that the sectional curvature spanned by orthonormal unit vectors \( u, v \) is

\[
\kappa(u \wedge v) = \langle R(u, v)v, u \rangle.
\]

\(^1\)Any finite-dimensional vector space \( a \) of commuting diagonalizable linear transformations admits a “common eigenspace decomposition.” The “eigenvalues” are called roots and are naturally a subset of \( a^* \) and the “eigenspaces” are called root spaces.

\(^2\)Note that this definition of root system is slightly different from the definition that appears in the study of, say, complex semisimple Lie algebras. There, one assumes that the only multiple of a root \( \alpha \) which appears in \( \Lambda \) is \( \pm \alpha \). This assumption does not hold for restricted roots of symmetric spaces, for example it fails in the symmetric space associated to \( SU(2,1) \).
isometric to the hyperbolic plane of curvature $Y$ having lower curvature bound $X$ algebra. Let $X \in \mathfrak{a}, Y \in \mathfrak{p}$ and assume $X, Y$ are orthogonal unit vectors. For any $Y \in \mathfrak{p}$, we may write $Y = \sum_{\alpha \in \Lambda \cup \{0\}} Y_\alpha$ where each $Y_\alpha \in \mathfrak{p}_\alpha$, and recall that this decomposition is $B$-orthogonal, so we have the lower curvature bound

$$\kappa(X^*_p \wedge Y^*_p) = B([-[[X,Y],Y],X]) = B([X,Y],[X,Y]) = -B([X,[X,Y]],Y) = -\sum_{\alpha \in \Lambda_+} B(\alpha(X)^2 Y_\alpha, Y) = -\sum_{\alpha, \beta \in \Lambda_+} \alpha(X)^2 B(Y_\alpha, Y_\beta) \geq -\kappa_0^2$$

where $\kappa_0$ is defined to be the maximum of $\{\alpha(X) \mid \alpha \in \Lambda, X \in \mathfrak{a}, |X| = 1\}$. In general, we have $\kappa_0 \leq 1$, as we now explain. Since $\alpha(X)$ is maximized in the direction of the coroot $H_\alpha$, we have

$$\kappa_0 = \alpha\left(\frac{H_\alpha}{|H_\alpha|}\right) = |H_\alpha|$$

for some $\alpha$. By [Pet06, p. 214.5], we have for $A, A' \in \mathfrak{a}$ that $B(A, A') = \sum_{\beta \in \Lambda} (\dim \mathfrak{g}_\beta) \beta(A) \beta(A')$, so

$$1 = B\left(\frac{H_\alpha}{|H_\alpha|}, \frac{H_\alpha}{|H_\alpha|}\right) = \sum_{\beta \in \Lambda} (\dim \mathfrak{g}_\beta) \beta\left(\frac{H_\alpha}{|H_\alpha|}\right)^2 \geq \alpha\left(\frac{H_\alpha}{|H_\alpha|}\right)^2 = \kappa_0^2.$$

In particular, under this normalization where the symmetric space inherits its metric from the Killing form, the sectional curvature is always bounded between $0$ and $-\frac{1}{2}$.

**Example 3.3.** In $\mathfrak{sl}(d, \mathbb{R})$, each root $\alpha$ has $|H_\alpha| = \frac{1}{\sqrt{d}}$, so we have $\kappa_0 = \frac{1}{\sqrt{d}}$ and the associated symmetric space has lower curvature bound $-\frac{\sqrt{d}}{2}$.

In Section 6 we will need to know the curvature of copies of the hyperbolic plane in $X$. These correspond to copies of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{g}$. Let $\alpha \in \Lambda$ and $X_\alpha \in \mathfrak{g}_\alpha$ such that $B_p(X_\alpha, X_\alpha) = \frac{2}{|H_\alpha|}$. Set $\tau_\alpha := \frac{H_\alpha}{|H_\alpha|} H_\alpha$ so that $\alpha(\tau_\alpha) = 2$. Set $Y_\alpha := -\partial_p X_\alpha \in \mathfrak{g}_{-\alpha}$. Then

$$[\tau_\alpha, X_\alpha] = 2X_\alpha, \quad [\tau_\alpha, Y_\alpha] = -2Y_\alpha, \quad \text{and} \quad [X_\alpha, Y_\alpha] = \tau_\alpha,$$

where the last equality follows from considering $B([X_\alpha, Y_\alpha], A)$ for $A \in \mathfrak{a} = \mathbb{R} H_\alpha \oplus \ker \alpha$. Then $\partial_p X_\alpha + \partial_p Y_\alpha = -\partial_p X_\alpha + \partial_p X_\alpha = -2Y_\alpha$, so $X_\alpha + Y_\alpha \in \mathfrak{p}$ and $|X_\alpha + Y_\alpha|^2 = |X_\alpha|^2_B + |Y_\alpha|^2_B = \frac{4}{|H_\alpha|^2}$. So $\frac{H_\alpha}{|H_\alpha|} (X_\alpha + Y_\alpha)$ and $\frac{H_\alpha}{|H_\alpha|}$ are orthonormal unit vectors in $\mathfrak{p}$, and

$$\kappa\left(\frac{|H_\alpha|}{2} (X_\alpha + Y_\alpha) \wedge \frac{H_\alpha}{|H_\alpha|}\right) = -\alpha\left(\frac{H_\alpha}{|H_\alpha|}\right)^2 \frac{|H_\alpha|}{2} (X_\alpha + Y_\alpha) \right) = -\frac{|H_\alpha|^4}{|H_\alpha|^2} \frac{4}{|H_\alpha|^2} = -\frac{4}{|H_\alpha|^2}$$

by the formula above.

**Example 3.4.** In the symmetric space associated to $\mathfrak{sl}(d, \mathbb{R})$, the root spaces $\mathfrak{g}_\alpha$ are one-dimensional, so the subalgebra $\mathfrak{sl}(2, \mathbb{R})_\alpha$ spanned by $X_\alpha, Y_\alpha, \tau_\alpha$ is uniquely determined by $\alpha$ and we denote it by $\mathfrak{sl}(2, \mathbb{R})_\alpha$. The image of $\mathbb{R} H_\alpha \oplus \mathfrak{p}_\alpha$ under the Riemannian exponential map at $p$ is a totally geodesic submanifold $\mathbb{H}^d_\alpha$ isometric to the hyperbolic plane of curvature $-\frac{1}{d}$.

### 3.4 Weyl chambers and the Weyl group

In this section we describe Weyl faces as subsets of maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}$. In Section 3.7 we will define Weyl faces as subsets of the visual boundary $\partial X$, and explain how the definitions relate.

Let $\Lambda$ be the roots of a restricted root space decomposition of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. For each $\alpha \in \Lambda \subset \mathfrak{a}^*$, the kernel of $\alpha$ is called a **wall**, and a component $C$ of the complement of the union of
the walls is called an open Euclidean Weyl chamber; \( C \) is open in \( \mathfrak{a} \). A vector \( X \in \mathfrak{a} \) is called regular if it lies in an open Euclidean Weyl chamber and singular otherwise. The closure \( V \) of an open Euclidean Weyl chamber is a closed Euclidean Weyl chamber; \( V \) is closed in \( \mathfrak{p} \).

For a closed Weyl chamber \( V \) there is an associated set of positive roots
\[
\Lambda_+ := \{ \alpha \in \Lambda \mid \forall v \in V, \alpha(v) \geq 0 \}
\]
and simple roots \( \Delta \), i.e. those which cannot be written as a sum of two elements of \( \Lambda_+ \), see [Ebe96, p. 2.9.6].

We may define
\[
N_K(a) := \{ k \in K \mid \text{Ad}(k)(a) = a \}, \quad Z_K(a) := \{ k \in K \mid \forall A \in a, \text{Ad}(k)(A) = A \}.
\]
Since the adjoint action preserves the Killing form, \( N_K(a) \) acts by isometries on \( \mathfrak{a} \) with kernel \( Z_K(a) \). We call the image of this action the Weyl group. For each reflection \( r_\alpha \) in a wall, it is possible to find a \( k \in K \) whose action on \( a \) agrees with \( r_\alpha \) [Ebe96, p. 2.9.7]. It is well-known that the Weyl group acts simply transitively on the set of Weyl chambers, which implies it is generated by the reflections in the walls of a chosen Weyl chamber. It is convenient for us to show this fact in Proposition 3.6, since the same techniques provide Corollary 3.13.

(a) The walls of a maximal flat in \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \).
(b) The walls of a maximal flat in \( \text{SL}(4, \mathbb{R})/\text{SO}(4) \).

The Riemannian exponential map identifies maximal abelian subspaces in \( \mathfrak{p} \) isometrically with maximal flats through \( p \). So we can also refer to open/closed Euclidean Weyl chambers in \( X \) as the images of those in some \( \mathfrak{a} \) under this identification. For every \( X \in \mathfrak{p} \), there exists a maximal abelian subspace \( \mathfrak{a} \) containing \( X \), and in \( \mathfrak{a} \), there exists some closed Euclidean Weyl chamber \( V \) containing \( X \).

3.5 A Morse function on flag manifolds

In this subsection, we show that the vector-valued distance function \( \tilde{d} \) on \( X \) (denoted \( d_\Delta \) in [KLP14, KLP17], see Definition 3) is well-defined, and give part of a proof of Theorem 3.8, an important part of the structure theory of symmetric spaces. Along the way we prove the \( \tilde{d} \)-triangle inequality [KLP14, KLP17, KLM09, Par], and provide an estimate on the Hessian of flag manifolds with a particular Riemannian metric, see Proposition 3.6 and Corollary 3.13.

We will use the following proposition. For \( A \in \mathfrak{p} \), let \( \mathfrak{c}_A \) be the intersection of all maximal abelian subspaces containing \( A \).

**Proposition 3.5** ([Ebe96, p. 2.20.18]). Let \( p \in X \) with Cartan decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) and let \( k \in K \) and \( A \in \mathfrak{p} \). If \( \text{Ad}(k)(A) = A \) then for all \( E \in \mathfrak{c}_A \) we have \( \text{Ad}(k)(E) = E \).

Note that there is a typo in Eberlein: the word “maximal” is omitted in the definition of \( \mathfrak{c}_A \). The proof of Proposition 3.5 relies on passing to the compact real form of \( \mathfrak{g}^\mathbb{C} \).

In this section, a **flag manifold** is the orbit of a vector \( Z \in \mathfrak{p} \) under the adjoint action of \( K = \text{Stab}_G(p) \). The following proposition is essentially a standard part of the theory of symmetric spaces, however we will need to extract a specific estimate, recorded in Corollary 3.13 in order to prove Lemma 4.10.
Proposition 3.6 (Cf. [Hel01, Lemma 6.3 p211] and [Ebe85, Proposition 24]). Let $X, Z \in \mathfrak{p}$ be unit vectors. Define

$$f : K \to \mathbb{R}, \quad f(k) := B(X, \text{Ad}(k)Z).$$

1. If $k$ is a critical point for $f$, then $\text{Ad}(k)Z$ commutes with $X$.

2. If $k$ is a local maximum for $f$, then $\text{Ad}(k)Z$ lies in a common closed Weyl chamber with $X$.

3. If $X$ is regular then the function $B(X, \cdot) : \text{Ad}(K)Z \to \mathbb{R}$ is Morse and has a unique local maximum.

4. If $X$ is regular then the distance function $d(X, \cdot) : \text{Ad}(K)Z \to \mathbb{R}$ has a unique local minimum.

Note that $f$ is the composition of the orbit map $K \to \text{Ad}(K)Z$ with the map $B(X, \cdot) : \text{Ad}(K)Z \to \mathbb{R}$.

Proof. 1. Let $Y \in \mathfrak{k}$, viewed as a left-invariant vector field on $K$. If $k$ is a critical point for $f$, then

$$0 = df_k(Y) = \frac{d}{dt} f(ke^{tY}) \bigg|_{t=0} = \frac{d}{dt} B(X, \text{Ad}(ke^{tY})Z) \bigg|_{t=0} = B(X, \text{Ad}(k)(ad(Y)(Z))) = B(X, [Y, Z])[Z'] = B([Z', X], Y')$$

where we write $Y' = \text{Ad}(k)Y$ and $Z' = \text{Ad}(k)Z$. Since $Y'$ is an arbitrary element of $\mathfrak{k}$, $[X, Z'] \in \mathfrak{k}$, and $B$ is negative definite on $\mathfrak{k}$, we can conclude that $[X, Z'] = 0$, which is the claim.

2. At a critical point $k$ for $f$, the Hessian of $f$ at $k$ is a symmetric bilinear form on $T_kK$ determined by

$$\text{Hess}(f)(v, v)_k = (f \circ c)'(0)$$

for any curve $c$ with $c(0) = k$ and $c'(0) = v$. Let $Y \in \mathfrak{k}$, the left-invariant vector fields on $K$, and choose $c(t) = ke^{tY}$. To compute the Hessian of $f$ we only need to compute

$$\frac{d^2}{dt^2} f(ke^{tY}) \bigg|_{t=0} = \frac{d}{dt} B(X, \text{Ad}(ke^{tY})(ad(Y)(Z))) \bigg|_{t=0} = B(X, \text{Ad}(k)([Y, [Y, Z]])) = B(X, Y', Y') = B(TY', Y')$$

where we write $T = \text{ad}(Z') \circ \text{ad}(X)$ as a linear transformation on $\mathfrak{k}$. At a critical point $X$ and $Z'$ commute by part 1, and we can choose a maximal abelian subspace $\mathfrak{a}$ containing both of them, and then consider the corresponding restricted root space decomposition. For $Y_\alpha \in \mathfrak{t}_\alpha$,

$$TY_\alpha = \alpha(Z')\alpha(X)Y_\alpha$$

so the transformation $T$ has the eigenvalue $\alpha(Z')\alpha(X)$ on its eigenspace $\mathfrak{t}_\alpha$ and acts as 0 on $\mathfrak{t}_0$. Since we assumed $k$ is a local maximum for $f$, we have

$$0 \geq \frac{d^2}{dt^2} f(ke^{tY}) \bigg|_{t=0} = B(TY', Y')$$

for all $Y \in \mathfrak{k}$, so for each $\alpha \in \Lambda, \quad \alpha(Z')\alpha(X) \geq 0$, and therefore $X$ and $Z'$ lie in a common closed Weyl chamber.

3. We may assume that $Z$ is a critical point of $f$ by precomposing $f$ with a left translation of $K$. The differential $(\text{d orb})_1 : \mathfrak{k} \to T_k \text{Ad}(K)Z$ is given by $-\text{ad}Z$ and has kernel $\mathfrak{k}_Z = Z_\mathfrak{k}(Z) = \{W \in \mathfrak{k} | [W, Z] = 0\}$ with orthogonal complement $\mathfrak{k}^Z = \bigoplus_{\alpha \in \Lambda, \alpha(Z) > 0} \mathfrak{t}_\alpha$. Then $k$ is a critical point for $f$ if and only if $Z(k) = \text{Ad}(k)Z$ is a critical point for $B(X, \cdot)$. The Hessians satisfy

$$\text{Hess}(B(X, \cdot))((\text{d orb})_kU, (\text{d orb})_kV)_{\text{Ad}(k)Z} = \text{Hess}(f)(U, V)_{k},$$

so by the calculation above the critical points are nondegenerate, occur at $\text{Ad}(k)Z$ when $[\text{Ad}(k)Z, X] = 0$, and have index the number of positive signs in the collection $\alpha(X)\alpha(\text{Ad}(k)Z)$, (weighted by dim $\mathfrak{t}_\alpha$) as $\alpha$ ranges over the roots with $\alpha(Z) > 0$. These can only be nonnegative when $\text{Ad}(k)Z$ lies in the closed Weyl chamber containing $X$.  

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For uniqueness, observe that any two maximizers $Z', Z''$ lie in the closed Weyl chamber containing $X$, and suppose $\text{Ad}(k)(Z') = Z''$. The adjoint action takes walls to walls so $\text{Ad}(k)$ preserves the facet spanned by $Z', Z''$ and hence fixes its soul (i.e. its center of mass) \cite[p65]{Ebe96}. By Proposition 3.5, $\text{Ad}(k)$ fixes each point of the face, and in particular $Z' = Z''$.

4. Since $(p, B)$ is a Euclidean space, 

$$d_p(X, Y)^2 = B(X - Y, X - Y) = B(X, X) + B(Y, Y) - 2B(X, Y)$$

so if $X, Y$ are unit vectors in $p$

$$d_p(X, Y)^2 = 2(1 - B(X, Y))$$

and the distance function $d_p(X, \cdot)$ is minimized when $B(X, \cdot)$ is maximized. Then by part 3, the distance function is uniquely minimized at the unique $\text{Ad}(k)Z$ in the closed Weyl chamber containing $X$.

The next two results are part of the standard theory of symmetric spaces. Since we have already proven Proposition 3.6 it is convenient to give the proofs.

**Corollary 3.7.** [\cite{Ebe96}, Section 2.12] Every $K$-orbit in the unit sphere $S(p)$ intersects each closed spherical Weyl chamber exactly once.

**Proof.** Let $X$ be a regular vector in a chosen Weyl chamber. The $K$-orbit of a unit vector $Z$ is compact and therefore the function $d_p(X, \cdot)$ has a global minimum on $\text{Ad}(K)Z$. But that function has a unique local minimum which must lie in the chosen closed Weyl chamber.

For a point $p \in X$, maximal abelian subspace $a \subset p$ and closed Euclidean Weyl chamber $V \subset a$, we call $(p, a, V)$ a point-chamber triple.

**Theorem 3.8.** [\cite{Ebe96}, Section 2.12] For any two point-chamber triples $(p, a, V), (p', a', V')$ there exists an isometry $g \in G$ taking $(p, a, V)$ to $(p', a', V')$. If $g$ stabilizes $(p, a, V)$, then it acts trivially on it.

**Proof.** The group $G$ acts transitively on $X$, so we may assume that $p' = p$ and then show that an element of $K = \text{Stab}_G(p)$ takes $(a, V)$ to $(a', V')$. Choose any regular unit vectors $X \in V$, $Z \in V'$. Then Proposition 3.6 implies there is an element $k \in K$ such that $\text{Ad}(k)Z$ is in the same open Weyl chamber as $X$. Regular vectors lie in unique Weyl chambers in unique maximal abelian subspaces, so $\text{Ad}(k)a' = a$ and $\text{Ad}(k)V' = V$.

If $g$ fixes $p$ and stabilizes $(a, V)$, then it acts trivially on $V$ by Corollary 3.7. 

\[\blacksquare\]
The above isometry is not necessarily unique. For example, consider hyperbolic space \( \mathbb{H}^n, n \geq 3 \). There a Euclidean Weyl chamber is just a geodesic ray, which has infinite pointwise stabilizer. However the action on \( V \) is unique.

As a corollary, we may define the vector-valued distance function
\[
d̃: \mathbb{X} \times \mathbb{X} \to (\mathbb{X} \times \mathbb{X})/G =: V_{\text{mod}}
\]
to have range a model closed Euclidean Weyl chamber. One could think of \( V_{\text{mod}} \) as some preferred Euclidean Weyl chamber, but it is better to think of it as an abstract Euclidean cone with no preferred basepoint, flat or Weyl chamber. There is an “opposition involution” \( \iota: V_{\text{mod}} \to V_{\text{mod}} \) induced by any geodesic symmetry \( S_p \). On a model pointed flat \( a_{\text{mod}} \), the composition of \( - \text{id} \) with the longest element of the Weyl group restricts to \( \iota \) on the model positive chamber \( V_{\text{mod}} \). Note that \( \iota d(p, q) = \bar{d}(q, p) \).

The triangle inequality implies that for any \( p, p', q, q' \) in a metric space,
\[
|d(p, q) - d(p', q')| \leq d(p, p') + d(q, q').
\]
The next result is the “vector-valued triangle inequality” for symmetric spaces.

**Corollary 3.9** (The \( \bar{d} \)-triangle inequality \[KLP14, KLM09, Par\]). For points \( p, p', q, q' \) in \( \mathbb{X} \),
\[
|\bar{d}(p, q) - \bar{d}(p', q')| \leq d(p, p') + d(q, q').
\]

**Proof.** In a moment we will use the proposition to prove that for any \( p, q, q' \) in \( \mathbb{X} \),
\[
|\bar{d}(p, q) - \bar{d}(p', q')| \leq d(q, q'),
\]
from which the general inequality follows easily:
\[
|\bar{d}(p, q) - \bar{d}(p', q')| = \left| \bar{d}(p, q) - \bar{d}(p, q') + \bar{d}(p, q') - \bar{d}(p', q') \right|
\leq |\bar{d}(p, q) - \bar{d}(p, q')| + |\bar{d}(q', p) - \bar{d}(q', p')| \leq d(q, q') + d(p, p').
\]

To prove \( \bar{d} \) let \( X, Z \in \mathfrak{p} \) such that \( e^X p = q \) and \( e^Z p = q' \). Choose a closed Weyl chamber \( V \) containing \( X \) and the unique \( Z' \) in the \( K \)-orbit of \( Z \) in that Weyl chamber. The map \( \bar{d}(p, e^X p): V \to V_{\text{mod}} \) is an isometry. Note that \( k \mapsto B(X, \text{Ad}(k)Z) \) is maximized when \( k \mapsto B(X, \text{Ad}(k)Z)/|X||Z| \) is maximized, so by Proposition 8.6
\[
|\bar{d}(p, q) - \bar{d}(p, q')|^2 = |X - Z'|^2 = |X|^2 + |Z'|^2 - 2\langle X, Z' \rangle
\leq |X|^2 + |Z|^2 - 2\langle X, Z \rangle = d_p(X, Z)^2 \leq d(q, q')^2
\]
since the Riemannian exponential map is distance non-decreasing by the nonpositive curvature of \( \mathbb{X} \).

### 3.6 Regularity in maximal abelian subspaces

A spherical Weyl chamber is the intersection of a Euclidean Weyl chamber with the unit sphere \( S \) in \( \mathfrak{a} \). A spherical Weyl chamber \( \sigma \) is a spherical simplex, and each of its faces \( \tau \) is called a Weyl face. Each Euclidean (resp. spherical) Weyl face is the intersection of walls of a (resp. as well as \( S \)). The interior of a face \( \text{int}(\tau) \) is obtained by removing its proper faces; the interiors of faces are called open simplices. The unit sphere \( S \) is a disjoint union of the open simplices. If \( \tau \) is the smallest simplex containing a unit vector \( X \) in its interior, we say that \( \tau \) is spanned by \( X \) and \( X \) is \( \tau \)-spanning.

We will quantify the regularity of tangent vectors using a parameter \( \alpha_0 > 0 \). We will show in Proposition 9.15 that our definition of regularity is equivalent to the definition in \[KLP14\]. A similar definition appears in \[KLP18, Definition (2.6)\].

**Definition 3.10** (Regularity). Let \( p \in \mathbb{X} \) and \( \mathbb{X} \) be a closed spherical Weyl chamber and let \( \tau \) be a face of \( \sigma \). Consider the corresponding maximal abelian subspace \( \mathfrak{a} \) in \( \mathfrak{p} \) and set of simple roots \( \Delta \). We define
\[
\Delta_{\tau} = \{ \alpha \in \Delta \mid \alpha(\tau) = 0 \}, \quad \Delta_{\tau}^+ = \{ \alpha \in \Delta \mid \alpha(\text{int} \tau) > 0 \}
\]
A vector \( X \in \mathfrak{a} \) is called \( (\alpha_0, \tau) \)-regular if for each \( \alpha \in \Delta_{\tau}^+, \alpha(X) \geq \alpha_0 |X| \). A geodesic \( c \) at \( p \) is called \( (\alpha_0, \tau) \)-regular if \( c'(0) = dpX \) for an \( (\alpha_0, \tau) \)-regular vector \( X \in \mathfrak{a} \).
It is immediate from the definition that $X$ is $(\alpha_0, \sigma)$-regular for some $\alpha_0 > 0$ and $\sigma$ if and only if $X$ is regular. We define

$$\Lambda_\tau := \{ \alpha \in \Lambda | \alpha(\tau) = 0 \}, \quad \Lambda^+ \tau := \{ \alpha \in \Lambda | \alpha({\text{int}} \, \tau) > 0 \} \quad (7)$$

Observe that $X$ is $(\alpha_0, \tau)$-regular if and only if for each root $\alpha \in \Lambda^+ \tau$ we have $\alpha(X) \geq \alpha_0$.

**Remark 3.11.** The signed distance from a vector $A \in a$ to the wall ker $\alpha$ is $\alpha(A)/|\alpha| \geq \alpha(A)/\kappa_0$.

**Definition 3.12.** A unit vector $X$ is $(\alpha_0, \tau)$-spanning if it is $\tau$-spanning and $(\alpha_0, \tau)$-regular.

We may now record a mild extension of Proposition 3.6 which will appear in Lemma 4.16.

**Corollary 3.13.** Suppose $X \in p$ is an $(\alpha_0, \tau)$-regular unit vector and $Z \in p$ is a $(\zeta_0, \tau)$-spanning unit vector. Then $Z$ is the unique maximum of $B(X, \cdot) : \text{Ad}(K)Z \rightarrow \mathbb{R}$, and for all $U, V \in T_Z \text{Ad}(K)Z$,

$$|\text{Hess}(B(X, \cdot))(U, V)_Z| \geq \frac{\alpha_0 \zeta_0}{2} |B_p(U, V)|.$$

**Proof.** The proof of Proposition 3.6 goes through in this setting, requiring only the following observation: if $X$ is $\tau$-regular and lies in a spherical Weyl chamber $\sigma$, then $\tau$ is a face of $\sigma$. 

3.7 The visual boundary $\partial X$

We say two unit speed geodesic rays $c_1, c_2$ are asymptotic if there exists a constant $D > 0$ such that

$$d(c_1(t), c_2(t)) \leq D$$

for all $t \geq 0$. The asymptote relation is an equivalence relation on unit-speed geodesic rays and the set of asymptote classes is called the visual boundary of $X$ and denoted by $\partial X$. There is a natural topology on $\partial X$ called the cone topology, where for each point $p \in X$ the map $S(T_p X) \rightarrow \partial X$ (which takes a unit tangent vector to the geodesic ray with that derivative) is a homeomorphism. In fact the cone topology extends to $\overline{X} := X \cup \partial X$, yielding a space homeomorphic to a unit ball of the same dimension as $X$.

**Lemma 3.14.** If $c_1$ and $c_2$ are asymptotic geodesic rays then for all $t \geq 0$,

$$d(c_1(t), c_2(t)) \leq d(c_1(0), c_2(0)).$$

**Proof.** The left hand side is convex Ebe96 and bounded above, hence (weakly) decreasing. 

We have a natural action of $G$ on $\partial X$: $g[c] = [g \circ c]$. For $\eta \in \partial X$, we denote the stabilizer

$$G_\eta := \{ g \in G | \text{g}\eta = \eta \}$$

and call $G_\eta$ the parabolic subgroup fixing $\eta$. (Note that in GW12 and GGKW17, $G$ itself is a parabolic subgroup, but in this paper a parabolic subgroup is automatically a proper subgroup.) When $\eta$ is regular $G_\eta$ is a minimal parabolic subgroup of $G$ (sometimes called a Borel subgroup).

Let $\eta, \eta'$ be ideal points in $\partial X$, represented by the geodesics $c(t) = e^{tX}p$ and $c'(t) = e^{tY}q$. Then since $G$ is transitive on point-chamber triples, we can find $g \in G$ such that $gg = p$ and $\text{Ad}(g)Y$ lies in a (closed) Euclidean Weyl chamber in common with $X$. In particular, every $G$ orbit in $\partial X$ intersects every spherical Weyl chamber exactly once.

Each unit sphere $S(p)$ has the structure of a simplical complex compatible with the action of $G$. By Theorem 3.8, this simplicial structure passes to $\partial X$, which is in fact a thick spherical building whose apartments are the ideal boundaries of maximal flats. In [KLP14, KLP17] the spherical building structure on $\partial X$ is used to describe the regularity of geodesic rays. We have used the restricted roots to define regularity and will show the notions are equivalent in Proposition 3.15. When we need to distinguish between simplices in $S(p)$ and simplices in $\partial X$ we call the former spherical and the latter ideal.

An ideal Weyl chamber $\sigma \subset \partial X$ is the image of a spherical Weyl chamber under the identification $S(p) \rightarrow \partial X$. The collection of ideal Weyl chambers is the flag manifold Flag($\sigma_{mod}$). Compared to a spherical simplex, an ideal simplex lacks the data of a basepoint $p \in X$. 

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Define the type map to be \( \theta: \partial X \to \partial X / G =: \sigma_{\text{mod}} \)
with range the model ideal Weyl chamber. The opposition involution \( \iota: V_{\text{mod}} \to V_{\text{mod}} \) induces an opposition involution \( \iota: \sigma_{\text{mod}} \to \sigma_{\text{mod}} \), see the discussion after Equation 4 in the previous subsection. The faces of \( \sigma_{\text{mod}} \) are called model simplices. For a model simplex \( \tau_{\text{mod}} \subset \sigma_{\text{mod}} \), we define the flag manifold \( \text{Flag}(\tau_{\text{mod}}) \) to be the set of simplices \( \tau \) in \( \partial X \) such that \( \theta(\tau) = \tau_{\text{mod}} \). If ideal points \( \eta, \eta' \) span the same simplex \( \tau \), then they correspond to the same parabolic subgroup, so we define \( G_\tau := G_\eta \). A model simplex corresponds to the conjugacy class of a parabolic subgroup of \( G \).

### 3.8 Regularity for ideal points

![Diagram showing three cases of \( (\alpha_0, \tau_{\text{mod}}) \)-regularity for various choices of \( \tau_{\text{mod}} \).](image)

Figure 3: \( (\alpha_0, \tau_{\text{mod}}) \)-regularity for various choices of \( \tau_{\text{mod}} \)

Theorem 3.8 implies that “model roots” are well-defined: if \( g \in G \) takes the point-chamber triple \( (p, a, V) \) to \( (p', a', V') \) and takes the simplex \( \tau \subset \partial V \) to \( \tau' \subset \partial V' \), it also takes \( \Delta_\tau \) to \( \Delta'_\tau \) and \( \Delta_\tau^+ \) to \( \Delta'_\tau^+ \), where \( \Delta \) is the simple roots in \( a^* \) corresponding to \( V \) and \( \Delta' \) is the simple roots in \( a' \) corresponding to \( V' \).

An ideal point \( \eta \in \partial X \) is called \( (\alpha_0, \tau) \)-regular if every geodesic in its asymptote class is \( (\alpha_0, \tau) \)-regular. As soon as one representative of an ideal point is \( (\alpha_0, \tau) \)-regular, every representative is. A vector, geodesic, or ideal point is \( (\alpha_0, \tau_{\text{mod}}) \)-regular if it is \( (\alpha_0, \tau) \)-regular for some simplex \( \tau \) of type \( \tau_{\text{mod}} \).

The open star of a simplex \( \tau \), denoted \( \text{ost}(\tau) \), is the union of open simplices \( \nu \) whose closures intersect \( \tau \). Equivalently, it is the collection of \( \tau \)-regular points in \( \partial X \). For a model simplex, \( \text{int}_{\tau_{\text{mod}}}(\sigma_{\text{mod}}) \) is the collection of \( \tau_{\text{mod}} \)-regular ideal points in \( \sigma_{\text{mod}} \). Equivalently, it is \( \sigma_{\text{mod}} \setminus \bigcup_{\alpha \in \Delta_\tau^+} \ker \alpha \). We have

\[
\tau = \sigma \cap \bigcap_{\alpha \in \Delta_\tau^+} \ker \alpha, \quad \text{int}_\tau \sigma = \{ \eta \in \sigma | \forall \alpha \in \Delta_\tau^+, \alpha(\eta) > 0 \}, \quad \partial_\tau \sigma = \sigma \cap \bigcup_{\alpha \in \Delta_\tau^+} \ker \alpha.
\]

There is a decomposition \( \sigma_{\text{mod}} = \text{int}_{\tau_{\text{mod}}}(\sigma_{\text{mod}}) \cup \partial_{\tau_{\text{mod}}}(\sigma_{\text{mod}}) \).

We call the set of \( (\alpha_0, \tau) \)-regular points the “\( \alpha_0 \)-star of \( \tau \).” We define the closed cone on the \( \alpha_0 \)-star of \( \tau \)

\[
V(p, \text{st}(\tau), \alpha_0) := \{ c_{px}(t) | t \in [0, \infty), x \text{ is } (\alpha_0, \tau) \text{-regular} \}
\]

the cone on the open star of \( \tau \)

\[
V(p, \text{ost}(\tau)) := \{ c_{px}(t) | t \in [0, \infty), x \text{ is } \tau \text{-regular} \}
\]

and the Euclidean Weyl sector

\[
V(p, \tau) := \{ c_{px}(t) | t \in [0, \infty), x \text{ is } \tau \text{-spanning} \}.
\]

\[3\text{In } [KLP14] \text{ the notation ost}(\tau_{\text{mod}}) \text{ was used for what is called int}_{\tau_{\text{mod}}}(\sigma_{\text{mod}}) \text{ here and in } [KLP17].\]
By Lemma 3.14, the Hausdorff distance between $V(p, st(\tau), \alpha_0)$ and $V(q, st(\tau), \alpha_0)$ is bounded above by $d(p, q)$, and the same holds for the open cones $V(p, ost(\tau))$ and $V(q, ost(\tau))$ and for the Weyl sectors $V(p, \tau), V(q, \tau)$.

We now describe the notion of regularity used in [KLP14, KLP17] and show it is equivalent to our definition. We always work with respect to a fixed type $\tau_{mod}$. A subset $\Theta \subset \sigma_{mod}$ is called $\tau_{mod}$-Weyl convex if its symmetrization $W_{\tau_{mod}} \Theta \subset a_{mod}$ is a convex subset of the model apartment $a_{mod}$. Here we think of the Weyl group $W$ as acting on the visual boundary of a model flat with distinguished Weyl chamber $\sigma_{mod}$ and $W_{\tau_{mod}}$ is the subgroup of $W$ stabilizing the simplex $\tau_{mod}$. One then quantifies $\tau_{mod}$-regular ideal points by fixing an auxiliary compact $\tau_{mod}$-Weyl convex subset $\Theta$ of $int(\sigma_{mod}) \subset \sigma_{mod}$.

An ideal point $\eta$ is $\Theta$-regular if $\theta(\eta) \in \Theta$. It is easy to see that the notions of $\Theta$-regularity and $(\alpha_0, \tau_{mod})$-regularity are equivalent.

**Proposition 3.15.** Let $\Delta_{\tau_{mod}} \subset \Delta$ be the model simple roots corresponding to a simplex $\tau_{mod} \subset \sigma_{mod}$. Then

1. If $\Theta$ is a compact subset of $int(\tau_{mod}) \setminus \bigcup_{\alpha \in \Delta_{\tau_{mod}}^+} \ker \alpha$, the quantity $\min(\alpha(\zeta) | \alpha \in \Delta_{\tau_{mod}}^+ \cap \Theta)$ exists and is positive.

2. Every $(\alpha_0, \tau_{mod})$-regular ideal point is $\Theta$-regular for $\Theta = \{\xi \in \sigma_{mod} | \forall \alpha \in \Delta_{\tau_{mod}}^+, \alpha(\zeta) \geq \alpha_0\}$.

**Proof.** We first prove 1. Since $\Theta$ is a compact subset of $\sigma_{mod} \setminus \bigcup_{\alpha \in \Delta_{\tau_{mod}}^+} \ker \alpha$, the quantity $\min(\alpha(\zeta) | \alpha \in \Delta_{\tau_{mod}}^+ \cap \Theta)$ exists and is positive.

We now prove 2. The subset $\Theta = \{\xi \in \sigma_{mod} | \forall \alpha \in \Delta_{\tau_{mod}}^+, \alpha(\zeta) \geq \alpha_0\}$ has symmetrization $W_{\tau_{mod}} \Theta = \{\xi \in a_{mod} | \forall \alpha \in \Delta_{\tau_{mod}}^+, \alpha(\xi) \geq \alpha_0\}$ which is an intersection of finitely many half-spaces together with the unit sphere, so it is compact and convex. Furthermore $\Theta = \sigma_{mod} \cap W_{\tau_{mod}} \Theta$ is a compact subset of $int(\tau_{mod}) \cap \sigma_{mod}$.

### 3.9 Generalized Iwasawa decomposition

Let $p$ be a point in $X, \tau \in \text{Flag}(\tau_{mod})$ and let $X \in \mathfrak{p}$ be $\tau$-spanning. Choose a Cartan subspace $X \in \mathfrak{a} \subset \mathfrak{p}$, with restricted roots $\Lambda$ and a choice of simple roots $\Delta$ associated to $\sigma \supset \tau$. Recalling the notation in Definition 3.10, we define

1. $\mathfrak{a}_\tau = Z(X) \cap \mathfrak{p} = \{Y \in \mathfrak{p} | [X, Y] = 0\}$ and $A_\tau = \exp(\mathfrak{a}_\tau)$. Note that $\mathfrak{a}_\tau$ and $A_\tau$ depend on $p$.

2. The (nilpotent) horocyclic subalgebra $\mathfrak{n}_\tau = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and the (unipotent) horocyclic subgroup $N_\tau = \exp(\mathfrak{n}_\tau)$.

3. The generalized Iwasawa decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_\tau \oplus \mathfrak{n}_\tau$.

4. The generalized Iwasawa decomposition of $G$ is $G = KA_\tau N_\tau = N_\tau A_\tau K$. The indicated decomposition is unique.

Note that our notation differs from [KLP17], where $N_\tau$ denotes the full horocyclic subgroup at $\tau$ and $A_\tau$ is the group of translations of the flat factor of the parallel set defined by $p$ and $\tau$, see Section 3.10. In our notation, $N_\tau$ is the unipotent radical of the parabolic subgroup $G_\tau$, see [Ebe96, p. 2.17].

### 3.10 Antipodal simplices, parallel sets and horocycles

We say a pair of points $\xi, \eta$ in $\partial X$ are antipodal if there exists a geodesic $c$ with $c(-\infty) = \xi$ and $c(+\infty) = \eta$. Equivalently, $\xi, \eta$ are antipodal if there exists a geodesic symmetry $S_p$ taking $\xi$ to $\eta$.

A pair of simplices $\tau_{\pm}$ are antipodal if there exists some $p \in X$ such that $S_p \tau_- = \tau_+$, or equivalently if there exists a geodesic $c$ with $c(-\infty) \in \text{int}(\tau_-)$ and $c(+\infty) \in \text{int}(\tau_+)$. If a model simplex $\tau_{mod}$ is $\tau$-invariant then every simplex $\tau$ of type $\tau_{mod}$ has the same type as any of its antipodes.

For antipodal simplices $\tau_{\pm}$, the parallel set $P(\tau_-, \tau_+)$ is the union of (images of) geodesics $c$ with $c(-\infty) \in \tau_-$ and $c(+\infty) \in \tau_+$. Given one such geodesic $c$, we may also define $P(\tau_-, \tau_+) = P(c)$ to be the union of geodesics parallel to $c$, or equivalently to be the union of maximal flats containing $c$. Antipodal $\tau_{mod}$-regular
points $\xi, \eta$ lie in the boundary of a unique parallel set $P = P(\tau(\xi), \tau(\eta))$. We say in this case that $P$ joins $\xi$ and $\eta$. The parallel set joining a pair of antipodal Weyl chambers is a maximal flat.

The horocycle centered at $\tau \in \text{Flag}(\tau_{\text{mod}})$ through $p \in X$ is denoted $H(p, \tau)$ and is defined to be the orbit $N_\tau \cdot p$. For any $p \in X$ and $\hat{\tau}$ antipodal to $\tau$, the horocycle $H(p, \tau)$ intersects the parallel set $P(\hat{\tau}, \tau)$ in exactly one point. A horocycle is the union of basepoints of strongly asymptotic Weyl sectors/ geodesic rays \cite{KLP14, KLP17}.

Let $\mathcal{H}_\tau$ be the foliation of $X$ by horocycles centered at $\tau$. Let $\text{Opp}(\tau)$ be the set of simplices antipodal to $\tau$, sometimes called the open Schubert stratum. $\text{Opp}(\tau)$ is the unique open and dense orbit of $G_\tau$ in $\text{Flag}(\tau_{\text{mod}})$. Let $\mathcal{P}_\tau$ be the foliation of $X$ by parallel sets $P(\hat{\tau}, \tau)$ with $\hat{\tau} \in \text{Opp}(\tau)$.

Any choice of $\tau \in \text{Flag}(\tau_{\text{mod}})$ induces a topological product structure on the symmetric space

$$X = \text{Opp}(\tau) \times \mathcal{H}_\tau.$$ 

Any further choice of $\hat{\tau} \in \text{Opp}(\tau)$ allows the identifications $N_\tau \to \text{Opp}(\tau)$ and $P(\hat{\tau}, \tau) \to \mathcal{H}_\tau$, given by the maps $n \to n\hat{\tau}$ and $p \to H(p, \tau)$ respectively. Then the symmetric space can be written as

$$X = N_\tau \times P(\hat{\tau}, \tau).$$

In the next example we explain why the Siegel upper half space model for the symmetric space associated to $\text{Sp}(\mathbb{R}^{2n}, \omega)$ is just a special case of this description. Nothing in the paper depends on this example, but the reader might find it a helpful discussion of the concepts in this section.

**Example 3.16.** Let $G = \text{Sp}(\mathbb{R}^{2n}, \omega)$ be the group of symplectic automorphisms of $\mathbb{R}^{2n}$. The space of Lagrangians is one flag manifold of the associated symmetric space $\mathbb{Y}$, corresponding to a vertex $\tau_{\text{Lag}}$ of the model chamber $\sigma_{\text{mod}}$.

A choice of opposite simplices in $\text{Flag}(\tau_{\text{Lag}})$ corresponds to a choice of transverse Lagrangians $\mathbb{R}^{2n} = V \oplus W$. Any other Lagrangian $U$ transverse to $V$ may be written as the graph of a unique symmetric linear map $\phi: W \to V$ where by symmetric we mean $\omega(w, \phi w') + \omega(\phi w, w') = 0$ for all $w, w' \in W$. The decomposition $\mathbb{R}^{2n} = V \oplus W$ allows us to decompose any $g \in \text{Sp}(\mathbb{R}^{2n}, \omega)$ into linear maps $A: V \to V, B: W \to V, C: V \to W, D: W \to W$. If $g$ preserves the decomposition, then $B$ and $C$ are zero and $D$ determines $A$ since $\omega(gw, v) = \omega(w, g^{-1}v)$ holds for all $v \in V, w \in W$.

If $\tau_V \in \text{Flag}(\tau_{\text{Lag}})$ corresponds to $V \in \text{Lag}(\mathbb{R}^{2n}, \omega)$ then $N_{\tau_V}$ via the decomposition above is the set of symplectic automorphisms with $A = \text{id}_V, C = 0, D = \text{id}_W$. Then $B: W \to V$ must be symmetric in the sense above; such linear maps can be identified with the space of real symmetric $n \times n$ matrices $\text{Sym}^2(\mathbb{R}^n)$.

Each point in $\mathbb{Y}$ corresponds to an inner product on $\mathbb{R}^{2n}$ compatible with $\omega$. Restricting the inner product to $W$ yields a map from $\mathbb{Y}$ to the symmetric space associated to $\text{GL}(W)$; this space can be identified with the space of real symmetric positive definite $n \times n$ matrices $\text{PSym}^2(\mathbb{R}^n)$.

Via the decomposition $\mathbb{R}^{2n} = V \oplus W$ we may write any symplectic automorphism as a block matrix and the action on $\mathbb{Y}$ becomes

$$\text{Sp}(\mathbb{R}^{2n}, \omega) \times (\text{Sym}^2(\mathbb{R}^n) + i\text{PSym}^2(\mathbb{R}^n)) \to \text{Sym}^2(\mathbb{R}^n) + i\text{PSym}^2(\mathbb{R}^n), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$ 

This description is known as the Siegel upper half space model of the symmetric space associated to $\text{Sp}(\mathbb{R}^{2n}, \omega)$.

### 3.11 The $\zeta$-angle and Tits angle

We follow \cite{KLP14} in defining the $\zeta$-angle between two simplices at a point $p \in X$. When this angle is close enough to $\pi$, the chambers must be antipodal and $p$ is close to the maximal flat joining the chambers. To make this definition, we first fix the auxiliary data of a $(\zeta_0, \tau_{\text{mod}})$-spanning $i$-invariant ideal point $\zeta = \zeta_{\text{mod}} \in \text{int}(\tau_{\text{mod}})$.

**Definition 3.17** ($\zeta$-angle, cf. \cite{KLP14} Definitions 2.3 and 2.4).

1. For a simplex $\tau \in \text{Flag}(\tau_{\text{mod}})$ let $\zeta(\tau)$ denote the unique point in $\text{int}(\tau)$ of type $\zeta$.

2. For a $\tau_{\text{mod}}$-regular ideal point $\xi \in \partial X$, let $\zeta(\xi) = \zeta(\tau(\xi))$ where $\tau(\xi)$ is the simplex spanned by $\xi$. 

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3. Let \( p \in X \), let \( \tau, \tau' \) be Weyl chambers in \( \partial X \) and let \( x, y \in X \) with \( px \) and \( py \) \( \tau_{\text{mod}} \)-regular. The \( \zeta \)-angle is given by

\[
\angle^\zeta_p(\tau, \tau') := \angle_p(\zeta(\tau), \zeta(\tau')) ,
\]
\[
\angle^\zeta_p(\tau, y) := \angle_p(\zeta(\tau), \zeta(py)) ,
\]
\[
\angle^\zeta_p(x, y) := \angle_p(\zeta(px), \zeta(py)) .
\]

Note there is a typo in the definition of \( \zeta \)-angle in [KLP14, Definition 7.5].

For \( \xi, \eta \in \partial X \), the **Tits angle** is

\[
\angle_{\text{Tits}}(\xi, \eta) := \sup_{p \in X} \angle_p(\xi, \eta) .
\]

Ideal points \( \xi, \eta \) are antipodal if and only if their Tits angle is \( \pi \). For \( p \in X \), \( \xi, \eta \in \partial X \), the equality \( \angle_p(\xi, \eta) = \angle_{\text{Tits}}(\xi, \eta) \) holds if and only if there is a maximal flat \( F \) containing \( p \) with \( \xi, \eta \in \partial F \) and moreover for any \( \xi, \eta \in \partial X \), there exists some maximal flat \( F \) with \( \xi, \eta \in \partial F \) [Ebe96].

For simplices \( \tau, \tau' \) in \( \text{Flag}(\tau) \), we may define

\[
\angle_{\text{Tits}}^\zeta(\tau, \tau') := \angle_{\text{Tits}}(\zeta(\tau), \zeta(\tau')) .
\]

There are only finitely many possible Tits angles between ideal points of fixed type. Therefore, there exists a bound \( \varepsilon(\zeta_{\text{mod}}) \) such that if \( \angle_{\text{Tits}}^\zeta(\tau, \tau') > \pi - \varepsilon(\zeta_{\text{mod}}) \) then \( \tau \) and \( \tau' \) are antipodal, as observed in [KLP14, Remark 2.42]. By Remark 3.11, we have

\[
\sin\left(\frac{1}{2} \varepsilon(\zeta_{\text{mod}})\right) = \min_{\alpha \in X^+_{\text{mod}}} \frac{\alpha(\zeta_{\text{mod}})}{\alpha} \geq \frac{\zeta_0}{\kappa_0} .
\]

By the definition of Tits angle, the same holds if the \( \zeta \)-angle at any point is strictly within \( \varepsilon(\zeta_{\text{mod}}) \) of \( \pi \): the inequality

\[
\angle_{\text{Tits}}^\zeta(\tau, \tau') \geq \angle_{\text{Tits}}^\zeta(\tau, \tau') \geq \pi - \varepsilon(\zeta_{\text{mod}})
\]

implies that \( \tau \) and \( \tau' \) are antipodal. Since \( \zeta_0 \leq \kappa_0 < 2\kappa_0 \) we have

\[
\sin\left(\frac{1}{2} \frac{\zeta_0^2}{\kappa_0}\right) \leq \sin\left(\frac{1}{2} \frac{\zeta_0^2}{\kappa_0}\right) \leq \sin\left(\frac{1}{2} \varepsilon(\zeta_{\text{mod}})\right) ,
\]

and we obtain the estimate \( \frac{\zeta_0^2}{\kappa_0} < \varepsilon(\zeta_{\text{mod}}) \). We record this observation in the following lemma.

**Lemma 3.18** (Cf. [KLP14, Remark 2.42]). If the inequality \( \angle_p(\tau_-, \tau_+) > \frac{\zeta_0^2}{\kappa_0} \) holds for some \( p \in X \) then \( \tau_- \) is antipodal to \( \tau_+ \). In other words, \( \frac{\zeta_0^2}{\kappa_0} < \varepsilon(\zeta_{\text{mod}}) \).
4 Estimates

In this section we prove several estimates used in Section 5 to quantify the higher rank Morse Lemma, cf. [KLP14, Section 7]. We first relate the Riemannian metric on $\mathcal{X}$ to algebraic data on $\mathfrak{g}$. We then control the regularity of perturbations of long, regular geodesic segments via the vector-valued triangle inequality Corollary 5.9. We go on to quantify the exponential decay of the distance of uniformly regular strongly asymptotic cones in Lemma 4.11. We show that if a long regular segment has endpoints at bounded distance from a Weyl cone, its midpoint gets arbitrarily close to the Weyl cone in Corollary 4.11.

Using the auxiliary data of a fixed type $\zeta_{\text{mod}}$ in the interior of $\tau_{\text{mod}}$, we measure the angle between simplices at any point in $\mathcal{X}$, see Section 3.11. We assume throughout the rest of the paper that $\zeta = \zeta_{\text{mod}}$ is $(\zeta_0, \tau_{\text{mod}})$-spanning and $\iota$-invariant, see Definition 3.12 and Section 3.7. It is shown in [KLP14] that these “$\zeta_{\text{mod}}$-angles” control the distance of the given point to the parallel set joining the simplices, and vice versa. We give precise estimates quantifying this control in Lemma 4.14 and Lemma 4.15. Finally, in Lemma 4.16 we give an explicit local Lipschitz constant for the projection of $(\alpha_0, \tau_{\text{mod}})$-regular vectors to simplices in Flag($\tau_{\text{mod}}$).

4.1 Useful properties of the inner product $B_p$ on $\mathfrak{g}$

We remind the reader that our convention is that the Riemannian metric on $\mathcal{X}$ is the one induced by the Killing form, see Equation 2. Recall that each point $p \in \mathcal{X}$ induces an inner product $B_p$ on $\mathfrak{g}$ and the evaluation map $\text{ev}_p : \mathfrak{g} \to T_p \mathcal{X}$, see Section 3.1. We first relate the inner product $B_p$, the Killing form $B$ on $\mathfrak{g}$, and the Riemannian metric $\langle \cdot, \cdot \rangle$ at $p$.

**Lemma 4.1.** For any $X, Y \in \mathfrak{g}$ and $p \in \mathcal{X}$,

$$2(\text{ev}_p X, \text{ev}_p Y) = B(X, Y) + B_p(X, Y).$$

In particular, any $U$ in $\mathfrak{n}_p$ or $\mathfrak{g}_\alpha$ is ad-nilpotent, so $B(U, U) = 0$ and $|U|_{B_p} = \sqrt{2} |\text{ev}_p U|$, see Section 3.9.

Recall that $\vartheta_p$ is a Lie algebra automorphism so $\vartheta_p^2 = [\vartheta_p^2 X, \vartheta_p^2 Y] = B(\vartheta_p^2 X, \vartheta_p^2 Y) = B(X, Y)$.

**Proof.** The kernel of $\text{ev}_p$ is the +1-eigenspace for $\vartheta_p$, so for any $X \in \mathfrak{g}$, $2\text{ev}_p X = \text{ev}_p (X - \vartheta_p X)$ and

$$4(\text{ev}_p X, \text{ev}_p Y)_p = \langle \text{ev}_p X - \vartheta_p X, \text{ev}_p Y - \vartheta_p Y \rangle_p = B(X - \vartheta_p X, Y - \vartheta_p Y)$$

$$= B(X, Y) + B(\vartheta_p X, \vartheta_p Y) - B(\vartheta_p X, Y - \vartheta_p Y) - B(\vartheta_p Y, X - \vartheta_p Y) = 2B(X, Y) + 2B_p(X, Y).$$

Next we show that the transpose on $\text{End} \mathfrak{g}$ with respect to $B_p$ restricts to $-\vartheta_p$ on the image of the adjoint representation.

**Lemma 4.2.** For $X, Y, Z \in \mathfrak{g}$, $B_p(\text{ad} X(Y), Z) = B_p(Y, \text{ad}(-\vartheta_p X)(Z))$.

**Proof.** We have

$$B_p(\text{ad} X(Y), Z) = -B(\vartheta_p \text{ad} X(Y), Z) = -B(\text{ad}(-\vartheta_p X)(\vartheta_p Y), Z)$$

$$= -B(\vartheta_p Y, \text{ad}(-\vartheta_p X)(Z)) = B_p(Y, \text{ad}(-\vartheta_p X)(Z))$$

where we have used that $\text{ad} \vartheta_p X$ is skew-symmetric relative to $B$.

Third, we bound $B(\text{ad} X(Y), Z)$ by the product of the $B_p$-norms of $X, Y$ and $Z$ and bound the operator norm of $\text{ad} X$ by $|X|_{B_p}$ along the way.

**Lemma 4.3.** Let $X, Y, Z \in \mathfrak{g}$ and let $p \in \mathcal{X}$ induce the inner product $B_p$ on $\mathfrak{g}$. Consider the operator norm $|\cdot|_{\text{op}}$ and Frobenius norm $|\cdot|_{\text{Fr}}$ on $\text{End} \mathfrak{g}$ induced by $B_p$. Then

1. $|\text{ad} X|_{\text{op}} \leq |\text{ad} Y|_{\text{Fr}} = |Y|_{B_p}$,
2. $B(X, \text{ad} Y(Z)) \leq |X|_{B_p} |Y|_{B_p} |Z|_{B_p}$, and
3. For $Y \in \mathfrak{p}$, $|[Y, X]|_{B_p} \leq \kappa_0 |Y|_{B_p} |X|_{B_p}$.
Proof. Recall that the operator norm of a linear transformation is the largest singular value, while the Frobenius norm is the square root of the sum of the singular values squared. Therefore
\[ |\text{ad} X|_{op}^2 \leq |\text{ad} X|_{F}^2 = \text{trace}_g(\text{ad}(-\vartheta_p X) \circ \text{ad} X) = B_p(X, X) \]
by Lemma 4.2 proving the first claim. Using this, we have
\[ B(X, \text{ad} Y(Z)) = -B_p(\vartheta_p X, \text{ad} Y(Z)) \leq |\vartheta_p X|_{B_p} |\text{ad} Y(Z)|_{B_p} \leq |X|_{B_p} |\text{ad} Y|_{op} |Z|_{B_p} \leq |X|_{B_p} |Y|_{B_p} |Z|_{B_p}. \]
If \( Y \in \mathfrak{p} \), we may choose a maximal abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \) containing \( Y \) and decompose \( X = \sum_{\alpha \in \Lambda \cup \{0\}} X_\alpha \) according to the associated restricted root space decomposition, which is \( B_p \)-orthogonal. Therefore
\[ ||Y, X||_{B_p}^2 = \left| \sum_{\alpha \in \Lambda} \alpha(Y) X_\alpha \right|_{B_p}^2 = \sum_{\alpha \in \Lambda} \alpha(Y)^2 |X_\alpha|_{B_p}^2 \leq \kappa_0^2 |Y|_{B_p}^2 |X|_{B_p}^2 \]
where \( \kappa_0 \) is the maximum of \( \{ \alpha(A) \mid \alpha \in \Lambda, A \in \mathfrak{a}, |A| = 1 \} \) see Section 3.3.

Fourth, we need to compare the norms induced by \( p, q \in X \) in terms of \( d(p, q) \).

Lemma 4.4. Let \( p, q \in X \), \( g \in G \) and \( X \in \mathfrak{g} \). Then

1. \( \vartheta_{gp} \circ \text{Ad}(g) = \text{Ad}(g) \circ \vartheta_p \).
2. \( |X|_{B_p} = |\text{Ad}(g)X|_{B_{gp}} \).
3. \( |X|_{B_p} \leq e^{\kappa_0 d(p, q)} |X|_{B_q} \).

Proof. The point stabilizer \( G_{gp} \) is \( gG_p g^{-1} \) and it follows that \( \text{Ad}(g) \) takes \( \vartheta_p \) to \( \vartheta_{gp} \). This, together with the \( \text{Ad} \) invariance of the Killing form implies 2. For the last point, choose a maximal flat \( F \) containing \( p \) and \( q \) and let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus \mathfrak{g}_\alpha \) be the restricted root space decomposition corresponding to \( p \) and \( F \). Then we may choose \( A \in \mathfrak{a} \), the maximal abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \) corresponding to \( F \), such that \( e^A p = q \), and then
\[ |X|_{B_p} = |e^{\text{ad} A} X|_{B_q} = \left| \sum_{\alpha \in \Lambda \cup \{0\}} e^{\alpha(A)} X_\alpha \right|_{B_q} \leq e^{\kappa_0 d(p, q)} |X|_{B_q} \]
using the restricted root space decomposition of \( X \) and the fact that the restricted root space decomposition is \( B_q \)-orthogonal.

4.2 Perturbations of long, regular segments

We will need to control the regularity of bounded perturbations of long regular geodesic segments. The following Lemma is an explicit version of Lemma 3.6 in [KLP18]. This assertion also appears in the proof of Lemma 7.10 in [KLP14].

Lemma 4.5. Suppose \( xy \) is an \((\alpha_0, \tau_{\text{mod}})\)-regular geodesic segment with \( d(x, y) \geq l \) and let \( x', y' \) be points in \( X \) satisfying \( d(x, x') \leq \delta_x \) and \( d(y, y') \leq \delta_y \). If
\[ \alpha_0 - \frac{(\delta_x + \delta_y)(\alpha_0 + \kappa_0)}{l - \delta_x - \delta_y} \geq \alpha'_0 \]
then \( x'y' \) is \((\alpha'_0, \tau_{\text{mod}})\)-regular.

Proof. We apply Corollary 3.9 the triangle inequality for \( \tilde{d} \)-distances:
\[ |\tilde{d}(x, y) - \tilde{d}(x', y')| \leq d(x, x') + d(y, y') \leq \delta_x + \delta_y. \]
Similarly, $|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y') \leq \delta_x + \delta_y$, so $d(x',y') \geq l - \delta_x + \delta_y$ and

$$\frac{d(x,y)}{d(x', y')} \geq 1 - \frac{\delta_x + \delta_y}{l} \geq 1 - \frac{\delta_x + \delta_y}{l - \delta_x - \delta_y}.$$

For any $\alpha \in \Delta^+_\text{mod}$,

$$\frac{\alpha(d(x',y'))}{d(x', y')} \geq \frac{\alpha_0d(x,y) - \delta_x \kappa_0 - \delta_y \kappa_0}{d(x', y')} \geq \alpha_0 \left(1 - \frac{\delta_x + \delta_y}{l} - \frac{\delta_x + \delta_y}{l - \delta_x - \delta_y}\right) = \alpha_0 - \frac{(\delta_x + \delta_y)(\alpha_0 + \kappa_0)}{l - \delta_x - \delta_y} \geq \alpha_0'.$$

We will often apply this lemma in the case $\delta_x = \delta_y = D$.

### 4.3 Angle comparison to Euclidean space

When $p, q, r$ are points in $\mathbb{X}$ such that $d(p,q)$ is much larger then $d(q,r)$, we provide an upper bound for the Riemannian angle $\angle p(q,r)$ by comparing to Euclidean space. The following estimate is surely not new, but we could not find a direct reference so we give a proof.

**Lemma 4.6.** Let $p, q, r$ be non-collinear points in $\mathbb{X}$. Then

$$\sin \angle p(q,r) \leq \frac{d(q,r)}{d(p,q)}.$$

The convenience of this estimate is that the third possible distance $d(p, r)$ does not appear.

**Proof.** Let $X,Y \in \mathfrak{p}$ such that $e^Xp = q$ and $e^Yp = r$. Then $|X| = d(p, q)$ and $d(X,Y) \leq d(q,r)$ and we may assume that $d(p,q) > d(q,r)$. In Euclidean space, the comparison holds: among vectors $Y'$ with $d(X,Y') \leq d(X,Y)$, the largest angle occurs for a vector $Y'$ forming a right triangle with $X$ as hypotenuse. Then

$$\sin \angle(X,Y) \leq \sin \angle(X,Y') = \frac{d(X,Y')}{|X|} \leq \frac{d(q,r)}{d(p,q)}.$$

### 4.4 Projecting curves in $G$ to $\mathbb{X}$

In this section we prepare to estimate the length of curves in $\mathbb{X}$ which are images of curves in $G$ under the orbit map. We begin by comparing the speeds of two such curves related by right-translation. We apply this result in the next section to Lemma 4.8 for a curve in $X$ and in the following section to Lemma 4.9 for a curve in the subgroup $N_\tau$.

**Lemma 4.7.** Let $g: \mathbb{R} \to G$ be a curve in $G$, let $h \in G$ and let $p \in \mathbb{X}$. Write $q_h(s) = g(s)hp$. If $\dot{g}(s) = (dl_{g(s)})_X s$ then

$$|q_h(s)| = |ev_p \Ad(h^{-1})X_s|.$$

**Proof.** The curve $q_h(s) = g(s)hp$ has the same speed as $c_h(s) = h^{-1}g(s)hp$ since $h^{-1}$ is an isometry. For $x \in G$, we denote by $\text{conj}_x: G \to G$ the conjugation map $\text{conj}_x(y) = x y x^{-1}$. Writing

$$c_h(s) = p \circ \text{conj}_{h^{-1}} \circ g(s)$$

and differentiating with respect to $s$ we have

$$\dot{c}_h(s) = (d \text{orb}_p)_{h^{-1}} g_h \circ (d \text{conj}_{h^{-1}})_{g(s)} \circ \dot{g}(s).$$

For any $a, b \in G$ and $X \in T_1G$ we have

$$(d \text{conj}_a)_b(dl_b)_1X = (dl_a)_{ba^{-1}}(d^{-1}r_a^{-1})_b(dl_b)_1X = (dl_a)_{ba^{-1}}(dl_a)_{a^{-1}}(dl_a)_{a^{-1}}(dl_a)_{a^{-1}}X = dl_{aba^{-1}} \Ad(a)X.$$
We also have \((d \text{ orb}_p)_a (dl_a)_1 = da_p (d \text{ orb}_p)_1\), so if \(\dot{g}(s) = dl_{g(s)} X_s\), then
\[
\dot{c}_i(s) = (d \text{ orb}_p)_{h^{-1}gh} \circ (dl_{h^{-1}gh})_1 \text{Ad}(h^{-1})X_s = (dh^{-1}gh)_p (d \text{ orb}_p)_1 \text{Ad}(h^{-1})X_s.
\]
This implies
\[
|\dot{q}_a(s)| = |\dot{c}_a(s)| = |(d \text{ orb}_p)_1 \text{Ad}(h^{-1})X_s| = |ev_p \text{Ad}(h^{-1})X_s|
\]
and completes the proof.

4.5 Weyl cones forming small angles

In this subsection, we show that if \(q \in V(p, \text{st}(\tau), \alpha_0)\) and \(r \in V(p, \text{st}(\tau'), \alpha_0)\) with \(d(p, q)\) much larger than \(d(q, r)\), the midpoint of \(pq\) is close to \(V(p, \text{st}(\tau'), \alpha_0)\).

![Figure 5: Weyl cones forming a small angle](image)

Lemma 4.8. Let \(p, q, r \in X\) with \(pq\) an \((\alpha_0, \tau)\)-regular geodesic ray with \(d(p, q) \geq 2l\) and \(d(q, r) \leq D\). If \(m = \text{mid}(p, q)\), \(K = \text{Stab}_G(p)\) and
\[
\alpha_0 - \frac{D(\kappa_0 + \alpha_0)}{2l - D} \geq \alpha'_0 > 0
\]
and
\[
\frac{1}{2}(e^{2\kappa_0 D} - 1)[\sinh(\alpha'_0(2l - D))]^{-2} \leq 3e^{2\kappa_0 D}
\]
then there exists \(k \in K\) such that \(km \in V(p, \text{st}(\tau(pr)), \alpha_0)\) and \(d(m, km)\) is at most \(2De^{\kappa_0 D - \alpha_0}l\).

The first inequality guarantees that \(pr\) is \(\tau_{\text{mod}}\)-regular so that \(\tau(pr)\) is well-defined. The second requirement looks strange and involves an arbitrary choice, but is extremely mild and serves our purposes well. (When we apply this Lemma, we will have a bounded \(D\) and a large \(l\).) Compared to other variations of Lemma 4.8 we could present here, the given version has a less cumbersome upper bound in the conclusion of the Lemma.

Proof. We may assume that \(d(p, q) = 2l\) and \(d(q, r) = D\). Let \(c: [0, D] \to X\) be the unit-speed geodesic from \(q\) to \(r\). We have \(l\) large enough that Lemma 4.8 implies that each ray \(pc(t)\) is \((\alpha'_0, \tau_{\text{mod}})\)-regular and defines a simplex \(\tau_t := \tau(pc(t))\). We may decompose
\[
\dot{c}(t) = N_{c(t)} + T_{c(t)}
\]
so that \(T_{c(t)}\) is tangent to \(V_t := V(p, \text{ost}(\tau_t))\) and \(N_{c(t)}\) is normal to \(V_t\). There is a unique \(X_t \in T^*_x\) such that \(ev_{c(t)} X_t = N_{c(t)}\), and we extend each \(X_t\) to a right-invariant vector field on \(K\). We may view this
time-dependent vector field as vector field supported on a compact neighborhood of \([0, D] \times K\), so it defines a flow and in particular a curve \(k: [0, D] \to K\) with \(k(0) = 1\) and \(k(t) = (X_t)k(t) = (dr_k(t))_1 X_t\).

It is convenient to set \(X_t = \text{Ad}(k(t))Y_t\) and work with the time-dependent vector field \(Y_t \in \mathfrak{t}^\tau\). We have \(\dot{k}(t) = (dr_k(t))_1 Y_t\), so we may view \(Y_t\) as the left-invariant vector field agreeing with \(X_t\) along \(k(t)\).

We may now write \(c(t) = k(t)v(t)\) where \(v(t) \in V(p, \omega(t), \alpha_0)\). Since \(T_{c(t)} = dk(t)v(t)\) we have \(|v| \leq |\dot{c}|\), so

\[
d(k(t)v(0), k(t)v(t)) = d(v(0), v(t)) \leq t \leq D.
\]

Setting \(q(t) = k(t)q\), we have \(|\dot{q}(t)| = |\text{ev}_q Y_t|\) by Lemma 4.7 and by Lemma 4.4.3 we have

\[
2|\text{ev}_q Y_t|^2 - |Y_t|^2_B = |Y_t|^2_{B_q} \leq e^{2\kappa_0 t}|Y_t|^2_{B_{\text{ev}t(t)}} = e^{2\kappa_0 t} \left(2|\text{ev}_v(t) Y_t|^2 - |Y_t|^2_B\right)
\]

where \(|Y_t|^2_B = B(Y_t, Y_t)\) is nonpositive.

For large \(l\), the evaluation of \(Y_t\) at \(v\) bounds the Killing form norm of \(Y_t\): We choose a maximal flat containing \(p\) and \(v = e^A p\) and, suppressing \(t\), write \(Y_t = \sum_{\alpha \in \Lambda^+_p} Y_{\alpha} + Y_{-\alpha} \) with \(Y_{\alpha} \in \mathfrak{g}_{\alpha}\) and compute

\[
|\text{ev}_v Y_t|^2 = |\text{ev}_p \text{Ad}(e^{-A}) Y_t|^2
= \sum_{\alpha \in \Lambda^+_p} \left| \left(e^{\alpha(A)} - e^{-\alpha(A)}\right) \text{ev}_p Y_{\alpha}\right|^2
= \frac{1}{2} \sum_{\alpha \in \Lambda^+_p} \left(e^{\alpha(A)} - e^{-\alpha(A)}\right)^2 |Y_{\alpha}|^2_{B_p}
\geq \frac{1}{2} \sum_{\alpha \in \Lambda^+_p} |2 \sinh(\alpha_0'(2l - t))|^2 |Y_{\alpha}|^2_{B_p}
= \frac{1}{2} \sum_{\alpha \in \Lambda^+_p} |2 \sinh(\alpha_0'(2l - t))|^2 \sum_{\alpha \in \Lambda^+_p} |Y_{\alpha}|^2_{B_p}
= \frac{1}{4} \left|2 \sinh(\alpha_0'(2l - t))\right|^2 (-|Y|^2_B).
\]

This bound \(-|\text{sinh}(\alpha_0'(2l - t))|^2 |Y_t|^2_B \leq |\text{ev}_v(t) Y_t|^2\) together with (8) implies

\[
2|\text{ev}_v Y_t|^2 \leq e^{2\kappa_0 t} |\text{ev}_v(t) Y_t|^2 - (e^{2\kappa_0 D t} - 1)|Y_t|^2_B \leq 2|\text{ev}_v(t) Y_t|^2 \left[e^{2\kappa_0 t} + \frac{1}{2}(e^{2\kappa_0 t} - 1)|\text{sinh}(\alpha_0'(2l - t))|^2\right].
\]

We now write \(m(t) = k(t)m\) where \(m = \text{mid}(p, q) = e^{W} p\) for \(W \in \mathfrak{p}\). For \(t \geq 0\), using \(\alpha(W) \geq \alpha_0 > 0\) for all \(\alpha \in \Lambda^+_p\) and Lemma 4.7 we have

\[
|m(t)|^2 = |\text{ev}_p \text{Ad}(e^{-lw}) Y_t|^2
= \frac{1}{2} \sum_{\alpha \in \Lambda^+_p} \left(e^{\alpha(W)} - e^{-\alpha(W)}\right)^2 |Y_{\alpha}|^2_{B_p}
\leq \frac{1}{2} \sum_{\alpha \in \Lambda^+_p} \left[\left(e^{2\alpha(W)} - e^{-2\alpha(W)}\right) e^{\alpha(W)}\right] |Y_{\alpha}|^2_{B_p}
\leq \frac{1}{2} \sum_{\alpha \in \Lambda^+_p} \left(e^{2\alpha(W)} - e^{-2\alpha(W)}\right)^2 e^{-2\alpha W} |Y_{\alpha}|^2_{B_p}
= e^{-2\alpha W} |\dot{q}(t)|^2
\leq e^{-2\alpha W} \left[e^{2\kappa_0 t} + \frac{1}{2}(e^{2\kappa_0 t} - 1)|\text{sinh}(\alpha_0'(2l - t))|^2\right]
\]

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The length of $m$ is then
\[
\int_0^D |\dot{m}(t)| \, dt \leq \int_0^D e^{-\alpha_0 t} \sqrt{e^{2\kappa_0 t} + \frac{1}{2} (e^{2\kappa_0 t} - 1) [\sinh(\alpha_0'(2l - t))]^{-2}} \, dt
\]
and $k(D)$ is the desired isometry. 

It is possible to give a slightly stronger upper bound in Lemma 4.8, but the improvement would be inconsequential when we apply this Lemma in Section 5 while making the already cumbersome statements even harder to read.

### 4.6 Strongly asymptotic geodesics and Weyl cones

The next estimate says that a point far along an $(\alpha_0, \tau)$-regular geodesic ray gets arbitrarily close to any given parallel set $P(\hat{r}, \tau)$. The following Lemma is a quantified version of Lemma 2.39 in [KLP14].

**Figure 6:** Strongly asymptotic geodesics get close at an exponential rate

**Lemma 4.9.** Let $q, r \in \mathbb{X}$ with $qr$ an $(\alpha_0, \tau)$-regular geodesic ray with ideal endpoint $\eta \in \text{ost}(\tau)$ and $d(q, r) \geq l$. Let $P = P(\hat{r}, \tau)$ be a parallel set with $d(q, P) \leq D$, and let $p \in P$ be the unique point on the horocycle $H(q, \tau)$. Then

\[
d(r, p\eta) \leq De^{\kappa_0 D - \alpha_0 l}.
\]

Proof. We may assume that $d(q, P) = D$ and $d(q, r) = l$. By abuse of notation let $q: [0, D] \to \mathbb{X}$ be the unit speed geodesic segment from $q$ to its nearest point $\bar{q} \in P$. Let $G = N_r A_r K$ be the generalized Iwassawa decomposition associated to $p$ and $\tau$, see Section 3.9. Since $N_r \times A_r \to M, (u, a) \mapsto uap$ is a diffeomorphism, we may write $q(s) = u(s)a(s)p$ for unique curves $u: [0, D] \to N_r$ and $a: [0, D] \to A_r$. Note that $u(D) = 1 = a(0)$, since horocycles at $\tau$ meet parallel sets $P(\hat{r}, \tau)$ in exactly one point.

Writing $c_t(s) = C(s, t) = u(s)a(s)t$ we have $q(s) = C(s, s) = c_s(s)$ so

\[
\dot{q}(s_0) = \left. \frac{\partial C}{\partial s} \right|_{s_0, s_0} + \left. \frac{\partial C}{\partial t} \right|_{s_0, s_0} = \dot{c}_s(s_0) + \left. \frac{\partial C}{\partial t} \right|_{s_0, s_0}
\]

and these vectors are orthogonal, so each has norm bounded by 1. The curve $t \mapsto a(t)p$ has speed bounded by 1 since

\[
\left. \frac{\partial C}{\partial t} \right|_{s_0, t_0} = du(s_0) \frac{d}{dt} a(t)p \bigg|_{t=t_0} ,
\]

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so \(d(p, a(t)p) \leq t \leq D\). We write \(\dot{u}(s) = dl_{u(s)}U_s\) and use Lemmas 4.1, 4.4 and 4.7 to obtain
\[
|\dot{c}_0(s)| = |ev_pU_s| = \frac{1}{\sqrt{2}} |U_s|_{B_p} \leq \frac{1}{\sqrt{2}} e^{\kappa d(p, a(s)p)} |U_s|_{B_{a(s)p}} = e^{\kappa d(p, a(s)p)} |\dot{c}_s(s)| \leq e^{\kappa a}. 
\]

We next need to push this horocyclic curve towards \(\tau\) and check that the length shrinks by at least \(e^{-\alpha t}\). Let \(X \in p\) be the unit vector so that \(gq(t) = u(0)e^{tX}p\). By abuse of notation define the curve \(r_1(s) = u(s)e^{tX}p\) from \(gq\) to \(pq\) and note that \(r_1(0) = u(0)e^{tX}p = r\). We’ve shown that the speed of \(r_0 = c_0\) is at most \(e^{\kappa a}\), and we may conclude after we show that
\[
|r_1'(s)| \leq e^{-\alpha t}|r_0'(s)| 
\]
in the next paragraph.

Define curves \(U_\alpha(s) \in g_\alpha\) by \(\dot{u}(s) = (dl_{u(s)})_1 \sum_{\alpha \in \Lambda^+} U_\alpha(s)\) and using Lemma 4.7 write
\[
|r_1'(s)|_{T_{\kappa(t)}X} = \left| ev_p \text{Ad}(e^{-tX}) \sum_{\alpha \in \Lambda^+} U_\alpha(s) \right|_{T_p X} 
\]
\[
= \left| ev_p \sum_{\alpha \in \Lambda^+} e^{-ta(X)} U_\alpha(s) \right|_{T_p X} 
\]
\[
= \frac{1}{\sqrt{2}} \left| \sum_{\alpha \in \Lambda^+} e^{-ta(X)} U_\alpha(s) \right|_{B_p} 
\]
\[
\leq \frac{1}{\sqrt{2}} e^{-\alpha t} \left| \sum_{\alpha \in \Lambda^+} U_\alpha(s) \right|_{B_p} 
\]
\[
= e^{-\alpha t}|r_0'(s)|_{T_{\kappa(t)}X} 
\]
Integrating this inequality bounds the length of \(r_1\) by \(De^{\kappa a D - \alpha t}\) and completes the proof. 

It is possible to give the slightly stronger upper bound \(\kappa_0^{-1}(e^{\kappa a D} - 1)e^{-\alpha t}\) in Lemma 4.9 but the improvement would be inconsequential when we apply this Lemma in Section 5 while making the already cumbersome statements even harder to read.

The following Lemma is a quantified version of Lemma 2.40 in [KLP14].

**Lemma 4.10.** Let \(p, q, x \in \mathbb{X}\) with \(pq\) an \((\alpha_0, \tau)\)-regular geodesic segment and \(d(p, q) \geq l\) and \(d(p, x) \leq D\). If
\[
\alpha_0 - \frac{D(\alpha_0 + \kappa_0)}{l - D} \geq \alpha_0' 
\]
then
\[
d(q, V(x, \text{st}(\tau), \alpha_0')) \leq De^{\kappa a D - \alpha_0 t}. 
\]

**Proof.** Let \(\eta \in \text{ost}(\tau)\) such that \(pq(+\infty) = \eta\). Let \(y\) be the unique point in the intersection \(P(S_X \tau, \tau) \cap H(p, \tau)\). By Lemma 4.9 we have a point \(q'\) on the image of \(y\eta\) such that \(d(q, q') \leq De^{\kappa a D - \alpha_0 t}\) and we may assume that \(\bar{d}(y, q') = \bar{d}(p, q)\).

Choose chambers \(\sigma, \sigma'\) containing \(\tau\) so that \(yq' \in V(y, \sigma)\) and \(xq' \in V(x, \sigma')\). Then there is a unique (restricted) isometry \(g: V(y, \sigma) \rightarrow V(x, \sigma')\) by Theorem 5.1 and
\[
d(gq', q') = |\bar{d}(x, gq') - \bar{d}(x, q')| = |\bar{d}(y, q') - \bar{d}(x, q')| \leq d(x, y) \leq D. 
\]

The geodesic segment from \(x\) to \(gq'\) has length at least \(l\) and is \((\alpha_0, \tau)\)-regular, so the proof of Lemma 4.9 implies that \(xq'\) is \((\alpha_0', \tau)\)-regular. 

\[\square\]
4.7 Projecting midpoints to Weyl cones

We combine the previous Lemmas 4.8, 4.9 and 4.10 to show that a long regular geodesic segment in a bounded neighborhood of a Weyl cone has its midpoint arbitrarily close to the Weyl cone.

**Corollary 4.11.** Let \( p, q, x \in X \) with \( pq \) an \((\alpha_0, \tau_{\text{mod}})\)-regular geodesic segment with midpoint \( m \), let \( \tau \in \text{Flag}(\tau_{\text{mod}}) \) and let \( V = V(x, \text{ost}(\tau)) \). Assume that \( d(p, x) \leq D, d(q, V) \leq D \) and \( d(p, q) \geq 2l \). If

\[
\alpha_0 - \frac{D(\alpha_0 + \kappa_0)}{l - D} \geq \alpha'_0 > 0
\]

and

\[
\frac{1}{2}(e^{2\kappa_0 D} - 1)[\sinh(\alpha'_0(2l - D))]^{-2} \leq 3e^{2\kappa_0 D}
\]

then

\[d(m, V(x, \text{st}(\tau), \alpha'_0)) \leq 3D e^{\kappa_0 D - \alpha_0 l}\]

**Proof.** By Lemma 4.8 there exists \( m' \in V(p, \text{st}(\tau), \alpha_0) \) with

\[d(m, m') \leq 2D e^{\kappa_0 D - \alpha_0 l}\]

and \( d(p, m') \geq l \). By Lemma 4.10

\[d(m', V(x, \text{st}(\tau), \alpha'_0)) \leq D e^{\kappa_0 D - \alpha_0 l}\]

and the result follows by the triangle inequality. \(\square\)

4.8 Simplex displacement after a short flow

Recall that we have fixed a model type \( \zeta = \zeta_{\text{mod}} \) spanning \( \tau_{\text{mod}} \), see Definition 3.12 and Section 3.8.

**Lemma 4.12.** Let \( p \in X, \zeta = \zeta_{\text{mod}} \in \tau_{\text{mod}}, \) and \( \tau \in \text{Flag}(\tau_{\text{mod}}) \). For any \( X \in \mathfrak{p} \),

\[
\sin \frac{1}{2} \angle_{\mathfrak{p}}^\tau(\tau, e^X \tau) \leq \frac{\kappa_0}{2} |X|_{B_\nu}.
\]

**Proof.** Denote by \( f_\tau \) the Busemann function associated to the ray from \( p \) to \( \zeta(\tau) \) and write \( \text{grad} f_\tau \) for its gradient. Then

\[\angle_{\mathfrak{p}}^\tau(\tau, e^X \tau) = \angle_{\mathfrak{p}}(\text{grad} f_\tau, \text{grad} f_{e^X \tau})\]
and
\[ \sin \frac{1}{2} \angle_p (\text{grad} f_\tau, \text{grad} f_{e \cdot \tau}) = \frac{1}{2} d_{p, X}(\text{grad} f_\tau, \text{grad} f_{e \cdot \tau}). \]

Let \( Z \in p \) be the unit vector so that \( \text{ev}_p Z = (\text{grad} f_{\tau}). \) Decompose \( X = K + Y \) according to the generalized Iwasawa decomposition \( g = K + A^+ + n_0 \), so that flowing by \( Y \) fixes \( \tau \) and therefore commutes with \( \text{grad} f_\tau \), and flowing by \( K \) fixes \( p \), see Section 3.9. We may write \( X = A + \sum_{\alpha \in A^+} (-X_\alpha + \partial_p X_\alpha) \) and \( K = \sum_{\alpha \in A^+} (X_\alpha + \partial_p X_\alpha) \) so \( |K|_{B_p} \leq |X|_{B_p} \). At \( p \) we have

\[
\frac{d}{dt} \left( \text{grad} f_{e \cdot \tau} \right)_{p} \bigg|_{t=0} = \frac{d}{dt} \left( (e^{tX}) \ast (\text{grad} f_\tau) \right)_{p} \bigg|_{t=0} = (\mathcal{L}_{-X} \ast \text{grad} f_\tau)_{p} = [-X^*, \text{grad} f_\tau]_{p} = [(-X + Y)^*, \text{grad} f_\tau]_{p} = [-K^*, \text{grad} f_\tau]_{p} = (\mathcal{L}_{-K} \ast \text{grad} f_\tau)_{p}
\]

where we used Lemma 4.3 in the second inequality. Finally we obtain

\[
|\text{grad} f_\tau - \text{grad} f_{e \cdot \tau}|_{T_p X} \leq \int_0^1 \left| \frac{d}{dt} \text{grad} f_{e \cdot \tau} \bigg|_{T_p X} \right| dt \leq \kappa_0 |X|_{B_p}
\]

which completes the proof.

4.9 The distance to a parallel set bounds the \( \zeta \)-angle

**Corollary 4.13.** Let \( p, q \) be points \( \mathbb{X} \) and \( \tau, \tau' \in \text{Flag}(\tau_{mod}) \). If \( d(p, q) \leq \frac{2}{\kappa_0} \) then

\[
|\angle_p^{\zeta}(\tau, \tau') - \angle_q^{\zeta}(\tau, \tau')| \leq 4 \sin^{-1} \left( \frac{\kappa_0}{2} d(p, q) \right).
\]

**Proof.** Write \( q = e^{-X} p \) for \( X \in p \). We use that \( \zeta \)-angles are \( G \)-invariant, the triangle inequality for quadruples in \( \text{Flag}(\tau_{mod}) \), \( \angle_p^{\zeta} \) and the simplex displacement estimate given by Lemma 4.12

\[
|\angle_p^{\zeta}(\tau, \tau') - \angle_q^{\zeta}(\tau, \tau')| = |\angle_p^{\zeta}(\tau, \tau') - \angle_p^{\zeta}(e^{X} \tau, e^{X} \tau')| \leq \angle_p^{\zeta}(\tau, e^{X} \tau) + \angle_p^{\zeta}(\tau', e^{X} \tau') \leq 4 \sin^{-1} \left( \frac{\kappa_0}{2} |X|_{B_p} \right).
\]

Since \( |X|_{B_p} = d(p, q) \) we are done.

We will often apply Corollary 4.13 in the following form. This result is a quantified version of Lemma 2.43(i) in [KLP14].
**Corollary 4.14.** Let $\tau_+, \tau_-$ be antipodal simplices in $\text{Flag}(\tau_{mod})$ and let $P = P(\tau_-, \tau_+)$ be the parallel set joining them. Let $p$ be any point in $\mathbb{X}$ such that $d(p, P) \leq \frac{2\kappa_0}{\delta}$. Then

$$\angle_p^\zeta(\tau_-, \tau_+) \geq \pi - 4\sin^{-1}\left(\frac{\kappa_0}{2} d(p, P)\right).$$

**Proof.** Since $\angle_p^\zeta(\tau_-, \tau_+) = \pi$ for any $q \in P$, and in particular the projection of $p$ to $P$, the assertion follows immediately from Corollary 4.13. 

### 4.10 The $\zeta$-angle bounds the distance to the parallel set

We continue to work with a fixed $(\zeta_0, \tau_{mod})$-spanning type $\zeta = \zeta_{mod}$ and now assume that $\zeta$ is $\nu$-invariant, see the discussion after Theorem 3.5. The next lemma complements Corollary 4.13 when the $\zeta$-angle at $q \in \mathbb{X}$ between simplices $\tau_\pm \in \text{Flag}(\tau_{mod})$ is near $\pi$, the point $q$ is near the parallel set $P(\tau_-, \tau_+)$. In the proof we use the fact that a vector field $X$ is Killing (if and) only if for all vector fields $V, W$ on $\mathbb{X}$, we have

$$X(V, W) = \langle [X, V], W \rangle + \langle V, [X, W] \rangle,$$

see [O’N83, p. 9.25]. The following result is a quantified version of Lemma 2.43.(ii) in [KLP14].

**Lemma 4.15.** Let $\tau_\pm \in \text{Flag}(\tau_{mod})$ be antipodal simplices and let $q \in \mathbb{X}$. If $\delta \leq \frac{\zeta_0^2}{\kappa_0}$ and $\angle_q^\zeta(\tau_-, \tau_+) \geq \pi - \delta$ then $d(q, P(\tau_-, \tau_+)) \leq \delta/\zeta_0$.

![Figure 8: The $\zeta$-angle at $q$ bounds the distance to $P$](image)

**Proof.** Write $\zeta_\pm$ for the unique ideal points $\tau_\pm$ of type $\zeta$, and choose Busemann functions $f_\pm$ at $\zeta_\pm$. For all $p \in \mathbb{X}$ we have $\cos \angle_p^\zeta(\tau_-, \tau_+) = \cos \angle_p(\zeta_-, \zeta_+) = \langle \text{grad} f_-, \text{grad} f_+ \rangle_p$. Let $\tilde{q} \in P = P(\tau_-, \tau_+)$ be the nearest point on $P$ to $q$, and let $X \in \mathbb{p}_{\tilde{q}}$ such that $c(t) = e^{tX} \tilde{q}$ is the unit-speed geodesic from $\tilde{q}$ to $q$. Either $\angle_q(\zeta_-, \tilde{q}) \geq \frac{\pi}{2} - \frac{\delta}{\kappa_0}$ or $\angle_q(\tilde{q}, \zeta_+) \geq \frac{\pi}{2} - \frac{\delta}{\kappa_0}$, so without loss of generality we may assume the second inequality holds. Let $f: (-\infty, \infty) \to [-1, 1]$ be defined by $f(s) = \langle -X^*, \text{grad} f_+ \rangle_{c(s)}$ and note that $f(s) = \cos \angle_{c(s)}(\tilde{q}, \zeta_+)$ for all $s > 0$. We first show that $f'(s) \geq 0$ for all $s$, so $f$ is (weakly) monotonic.

If $X = A + \sum_{\alpha \in \Lambda^+} -X_\alpha + \vartheta X_\alpha$ in the decomposition associated to $c(s)$ and a chamber containing $\tau_+$,
then for \( K = \sum_{\alpha \in \Lambda^+} X_\alpha + \partial X_\alpha \) and \( Z \in p_{\varepsilon(s)} \) pointing to \( \zeta_+ \) we have

\[
f'(s) = X^* \langle -X^*, \nabla f \rangle_{c(s)} = \langle -X^*, [X^*, \nabla f] \rangle_{c(s)} = \langle -X^*, [K, Z]^* \rangle_{c(s)} = B(-X, -[K, Z]_g) = B(A + \sum_{\beta \in \Lambda^+} -X_\beta + \partial X_\beta, \sum_{\alpha \in \Lambda^+} \alpha(Z)(X_\alpha - \partial X_\alpha)) = \sum_{\alpha \in \Lambda^+} \alpha(Z)B(X_\alpha - \partial X_\alpha, X_\alpha - \partial X_\alpha) \geq \zeta_0 \sum_{\alpha \in \Lambda^+} |X_\alpha + \partial X_\alpha|^2_B.
\]

This shows \( f'(s) \geq 0 \) for all \( s \). Moreover, since \( X^* \) is orthogonal to \( P(\bar{q}, \tau) \) at \( s = 0 \), we have \( |X^*_q|^2 = \sum_{\alpha \in \Lambda^+} |X_\alpha + \partial X_\alpha|^2 \), so \( f'(0) \geq \zeta_0 \). We also have \( |f''(s)| \leq 1 \) since

\[
f''(s) = X^* X^* \langle \nabla f \rangle_{c(s)} = \langle X^*, [X^*, [X^*, \nabla f]] \rangle_{c(s)}
\]

has norm bounded by \( |X|^2_B |K|_{B_p} |Z|_{B_p} \leq \alpha_0^3 \) using Lemma 4.3.

Since \( f'(0) \geq \zeta_0 \) and \( |f''(s)| \leq 1 \), we have \( f(s) \geq s \zeta_0 - \frac{1}{2} \nu_0 s^2 \). Since \( f \) is monotonic, if \( s \geq \frac{\zeta_0}{\nu_0} \) then \( f(s) \geq \frac{\zeta_0}{\nu_0} \geq \frac{\zeta_0^2}{2\nu_0} \). On the other hand, if \( s \leq \frac{\zeta_0}{\nu_0} \) we have \( f(s) \geq \zeta_0 s - \frac{1}{2} \nu_0 s^2 \geq \frac{1}{2} \zeta_0 s \). This implies

\[
\frac{1}{2} \zeta_0 d(p, q) \leq f(d(p, q)) = \cos \zeta_0^*(\bar{q}, \tau_+) \leq \cos \left( \frac{\pi}{2} - \frac{\delta}{2} \right) = \sin \left( \frac{\delta}{2} \right) \leq \frac{\delta}{2}
\]

unless \( d(p, q) > \frac{\zeta_0}{\nu_0} \), which yields \( \frac{\zeta_0^2}{\nu_0} < \delta \) and contradicts our assumption.

If we combine Lemma 4.15 with Lemma 3.18 we see that the stronger bound \( \delta \leq \frac{\zeta_0^2}{\nu_0} \) implies the simplices \( \tau_-, \tau_+ \) are antipodal.

### 4.11 Projecting regular vectors to flag manifolds

We continue to work with a fixed type \( \zeta = \zeta_{mod} \) which is \((\zeta_0, \tau_{mod})\)-spanning. For a \( \tau_{mod}\)-regular \( X \in p \), define \( \zeta(X) \) to be the unique vector in a common closed Weyl chamber as \( X \) of type \( \zeta \). Note that \( \zeta(X) \) is the unique maximizer for \( B(X, \cdot) : \text{Ad}(K)Z \rightarrow \mathbb{R} \) where \( Z \in p \) is any vector of type \( \zeta \) by Corollary 3.12.

This map \( \zeta \) from \( \tau_{mod}\)-regular elements of \( p \) to \( \text{Ad}(K)Z \) is a smooth fiber bundle. In the next lemma we show that nearby \( \tau_{mod}\)-regular points project to nearby points on \( \text{Ad}(K)Z \) in the metric induced by viewing \( \text{Ad}(K)Z \) as a Riemannian submanifold of \( p \). Note that one expects a local Lipschitz constant proportional to \( \frac{1}{\alpha_0} \) by considering vectors near the walls \( \ker \alpha \) for \( \alpha \in \Delta_{x_0}^+ \).

**Lemma 4.16.** Let \( X, X' \) be \((\alpha_0, \tau)\)-regular unit vectors in \( p \) with \( \alpha_0 (X, X') \leq \alpha_0 \). Write \( Z = \zeta(X) \) and \( Z' = \zeta(X') \). Then the Riemannian distance on \( \text{Ad}(K)Z \) from \( Z \) to \( Z' \) is bounded by the distance in \( p \) from \( X \) to \( X' \):

\[
d_{\text{Ad}(K)Z}(Z, Z') \leq \frac{2}{\alpha_0 \zeta_0} d_p(X, X').
\]

**Proof.** Let \( t \rightarrow X_t \) be a unit-speed line segment from \( X \) to \( X' \) in \( p \). Let \( \{X_t^{dimp}\}_{i=1} \) be linear coordinates on \( p \), and we may assume that the derivative of \( t \rightarrow X_t \) is \( \frac{d}{dt} \). Since \( d_p(X, X') \leq \alpha_0 \) each \( X_t \) is \((\alpha_0, \tau_{mod})\)-regular. Write \( Z_t = \zeta(X_t) \) and note that \( t \rightarrow Z_t \) is a smooth curve on \( \text{Ad}(K)Z \). To prove the claim we will show that \( \frac{d Z_t}{dt} \leq \frac{2}{\alpha_0 \zeta_0} \), where we restrict the inner product on \( p \) to a Riemannian metric on \( \text{Ad}(K)Z \).
Restricting the domain of $B$, we write $B \colon p \times \text{Ad}(K)Z \to \mathbb{R}$. Near $(X_0, Z_0) = (X_{t_0}, Z_{t_0})$, we have coordinates $\{Z_j\}_{j=1}^{\dim \text{Ad}(K)Z}$ on $\text{Ad}(K)Z$. We may assume that $Z_t$ is an immersion at $Z_0$ because the set $\{t \mid \frac{dZ_0}{dt} = 0\}$ does not contribute to the arclength of $Z_t$ and furthermore up to a change of coordinates we may assume that $\frac{dZ_t}{dt} = \frac{\partial}{\partial x^1}$. On this coordinate patch $U$, we obtain the function $B_j \colon p \times U \to \mathbb{R}$ defined by $B_j(X'', Z'') := dB(X'', Z'')(\frac{\partial}{\partial x^1})$. Along the curve $t \mapsto (X_t, Z_t)$, the function $B_j$ is identically 0 (where defined) since $Z_t$ maximizes $B(X_t, \cdot)$ on $\text{Ad}(K)Z$. Differentiating $B_j(X_t, Z_t) = 0$ in $t$, we obtain

$$0 = dB_j(X_t, Z_t) \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial Z^1} \right) = \frac{\partial B_j}{\partial x^1} + \frac{\partial B_j}{\partial Z^1}.$$ 

Observe that

$$\frac{\partial B_j}{\partial Z^1}(X_t, Z_t) = \text{Hess}(B) \left( \frac{\partial}{\partial Z^1}, \frac{\partial}{\partial Z^1} \right)_{(X_t, Z_t)} = \text{Hess}(B(X_t, \cdot)) \left( \frac{\partial}{\partial Z^1}, \frac{\partial}{\partial Z^1} \right)_{Z_t},$$

so by Corollary 3.13 we have

$$\left| \frac{\partial B_j}{\partial Z^1} \right| \geq \frac{a_0\xi_0}{2} \left| \left( \frac{\partial}{\partial Z^1}, \frac{\partial}{\partial Z^1} \right) \right|.$$ 

In particular, along $(X_t, Z_t)$ and setting $j = 1$, we have

$$\frac{a_0\xi_0}{2} \left| \frac{\partial}{\partial Z^1} \right|^2 \leq \left| \frac{\partial B_1}{\partial x^1} \right| = \left| B_1 \left( \frac{\partial}{\partial x^1}, Z_t \right) \right| \leq \left| \frac{\partial}{\partial Z^1} \right|$$

since $\frac{\partial}{\partial x^1}$ is a unit vector. We obtain for all $t$

$$\left| \frac{\partial}{\partial Z^1} \right| \leq \frac{2}{a_0\xi_0}$$

and the claim is proven. □

5 Quantified local-to-global principle

In this section we augment the theorems of [KLP14, Section 7] with quantitative estimates. We obtain a precise version of the local-to-global principle which allows us to perturb known Anosov representations by a definite amount, producing new Anosov representations in Section 6.

In rank one, local quasigeodesics of sufficiently good quality are global quasigeodesics, as a consequence of the Morse lemma. The Morse Lemma fails in the Euclidean plane, hence in higher rank, so we must use Morse quasigeodesics as defined in [KLP14]. The strategy here, as in [KLP14], is to show that local Morse quasigeodesics of sufficiently good quality have straight and spaced midpoint sequences which are then globally Morse quasigeodesics. We first give an explicit local criteria for a sequence to be a Morse quasigeodesic.

5.1 Sufficiently straight and spaced sequences are Morse quasigeodesics

We recall some definitions from [KLP14]. A sequence of points $(x_n)$ in $\mathbb{X}$ is $(\alpha_0, \tau_{mod}, \epsilon)$-straight if each geodesic segment $x_n x_{n+1}$ is $(\alpha_0, \tau_{mod})$-regular and if

$$\angle_{x_n}^\tau (x_{n-1}, x_{n+1}) \geq \pi - \epsilon$$

for all $n$. The sequence is $s$-spaced if $d(x_n, x_{n+1}) \geq s$ for all $n$. We say a sequence $(x_n)$ moves $\epsilon$-away from a simplex $\tau$ if for all $n$

$$\angle_{x_n}^\tau (\tau, x_{n+1}) \geq \pi - \epsilon.$$

In this paper we are only interested in discrete sequences of points in $\mathbb{X}$. For us, a $(c_1, c_2, c_3, c_4)$-quasigeodesic is a sequence $(x_n)$ (possibly finite, infinite, or biinfinite) such that

$$\frac{1}{c_1} |N| - c_2 \leq d(x_n, x_{n+N}) \leq |N| c_3 + c_4.$$
A sequence \((x_n)\) is \((c_1, c_2)\)-coarsely spaced (or lower-quasigeodesic) if
\[
\frac{1}{c_1}|N| - c_2 \leq d(x_n, x_{n+N}).
\]
Likewise \((x_n)\) is \((c_3, c_4)\)-coarsely Lipschitz (or upper-quasigeodesic) if
\[
d(x_n, x_{n+N}) \leq |N|c_3 + c_4.
\]

For an \((\alpha_0, \tau_{\text{mod}})\)-regular segment \(pq\), the \((\alpha_0, \tau_{\text{mod}})\)-diamond is the intersection
\[
\diamond_{\alpha_0}(p, q) := V(p, \text{st}(\tau(pq)), \alpha_0) \cap V(q, \text{st}(\tau(qp)), \alpha_0).
\]

Figure 9: The \((\alpha_0, \tau_{\text{mod}})\)-diamond with endpoints \(p\) and \(q\)

A quasigeodesic is \((\alpha_0, \tau_{\text{mod}}, D)\)-Morse if for all \(x_n, x_m\) there exists a diamond \(\diamond_{\alpha_0}(p, q)\) such that
\[
d(p, x_n), d(q, x_m) \leq D
\]
and for all \(n \leq i \leq m\), \(d(x_i, \diamond) \leq D\). In hyperbolic space, quasi-geodesics are automatically Morse by the Morse lemma. In higher rank symmetric spaces of noncompact type, the following theorem allows us to construct Morse quasigeodesics from sufficiently straight and spaced sequences.

There are a few variations of the precise definition of Morse quasi-geodesic in the literature. The definition of Morse quasi-geodesic here is the same as that given in \cite{KLP17, Definition 5.50}, except that we keep track of more constants in the definition of quasigeodesic. This is the same as \cite{KLP14, Definition 7.14} except that we work with sequences rather than paths. Likewise \cite{KL18c, Definition 6.13} defines paths to be Morse quasigeodesics when they satisfy a similar and equivalent, but not identical, property as the one we have given here (the constants will be different). We warn the reader that the notion of Morse quasigeodesic here is not equivalent to the notion of \(1\)-dimensional Morse quasiflat given in \cite{HKS19, HKS20}. Rather, a maximal flat in a symmetric space is an example of a Morse quasiflat.

Define the constant
\[
c_0 := \sum_{\alpha \in \Lambda^+_{\tau_{\text{mod}}}} \dim \mathfrak{g}_\alpha,
\]
equal to the codimension of any parallel set of type \(\tau_{\text{mod}}\). The inequality \(c_0 \geq 1\) always holds. Theorem 5.1 is a quantified version of Theorem 7.2 in \cite{KLP14}.

**Theorem 5.1.** Fix \(\alpha_{\text{new}} < \alpha_0\), \(\delta\) and assume \(\epsilon\) is small and \(s\) is large. Precisely, we assume that:

1. \(5\epsilon \leq \frac{\epsilon_0^2}{\gamma_0}\). This allows us to apply Lemma 3.18 and the angle-to-distance estimate in Lemma 4.13;

2. \(\frac{\epsilon \kappa_0}{\gamma_0} e^{2\alpha_{\text{new}}\epsilon_{\gamma_0} / \epsilon_{\alpha_{0}s}} \leq \sin \left(\frac{\epsilon}{4}\right)\)

so that we may apply the distance-to-angle estimate Lemma 4.14.

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3. \[ \frac{5\epsilon}{\zeta_0} \leq \delta \]
   to control the distance from the sequence to the parallel set;

4. \[ \alpha_0 - \frac{2\delta(\alpha_0 + \kappa_0)}{s - 2\delta} \geq \alpha_{\text{new}} \]
   so that certain projections are \((\alpha_{\text{new}}, \tau_{\text{mod}})\)-regular by Lemma 4.5.

5. \[ 2\epsilon + \sin^{-1} \left( \frac{4\delta}{\alpha_0 \zeta_0 s} \right) < \epsilon(\zeta) \]
   so that certain simplices are antipodal, see Section 3.11.

Then every \((\alpha_0, \tau_{\text{mod}}, \epsilon)\)-straight \(s\)-spaced sequence \((x_n)\) in \(X\) is \(\delta\)-close to a parallel set \(P(\tau_-, \tau_+)\) such that
\[ x_{n \pm m} \in V(x_n, \text{st}(\tau_{\pm}), \alpha_{\text{new}}) \]
for all \(n\) and \(m \geq 1\). It follows that the sequence is coarsely spaced:
\[ d(x_n, x_{n \pm m}) \geq 2\alpha_{\text{new}} \zeta_0 \alpha_0 (s - 2\delta)m - 2\delta, \]
and if \((x_n)\) is coarsely Lipschitz it is then a \((\alpha_{\text{new}}, \tau_{\text{mod}}, \delta)\)-Morse quasigeodesic.

Our proof closely follows [KLP14, Section 7], who prove the same theorem without the explicit assumptions 1 through 5 and without the explicit estimates we obtained in Section 4. Note that the resulting sequence will always be \(\frac{\zeta_0}{\kappa_0}\)-close to the parallel set, even if \(\delta\) is chosen larger than that quantity.

**Proof.** Step 1: Propagation cf. [KLP14, Lemma 7.6]. We show that for sufficiently straight and spaced sequences, the property of moving away from a simplex propagates along the sequence.

![Figure 10: Propagation](image)

Assume that for some simplex \(\tau\) in Flag(\(\tau_{\text{mod}}\)) we have \(\angle_{x_0}^\zeta(\tau, x_1) \geq \pi - 2\epsilon\). Since \(2\epsilon < \epsilon(\zeta)\) by assumption 1, Lemma 3.18 implies that the simplex \(\tau_01\) containing \(x_0x_1(+\infty)\) is antipodal to \(\tau\) and together they define a parallel set \(P = P(\tau, \tau_01)\). By assumption 1 and our angle-to-distance estimate Lemma 4.15 we have \(d(x_0, P) \leq \frac{2\epsilon}{\zeta_0}\). By Lemma 4.9 the geodesic ray from \(x_0\) through \(x_1\) gets arbitrarily close to \(P\) and in particular
\[ d(x_1, P) \leq \frac{2\epsilon}{\zeta_0}e^{2\kappa_0 \epsilon/\zeta_0 - \alpha_0 s} \]
and by assumption 2 and the distance-to-angle estimate Corollary 4.14 we have
\[ \angle_{x_1}^\zeta(\tau, \tau_01) \geq \pi - 4 \sin^{-1} \left( \frac{\epsilon \kappa_0}{\zeta_0} e^{2\kappa_0 \epsilon/\zeta_0 - \alpha_0 s} \right) \geq \pi - \epsilon \]
which then implies that \( \angle_{x_n}^\zeta (\tau, x_0) = \pi - \angle_{x_{n+1}}^\zeta (\tau, x_n) \leq \epsilon \). Straightness and an application of the triangle inequality for \((S(T_x, X), \angle_{x_n})\) implies \( \angle_{x_1}^\zeta (\tau, x_2) \geq \pi - 2\epsilon \). By induction we have that \( \angle_{x_n}^\zeta (\tau, x_{n+1}) \geq \pi - 2\epsilon \) for all \( n \geq 1 \).

Step 2: Extraction cf. [KLP14, Lemma 7.7]. We extract antipodal simplices that the sequence moves away/towards. It follows that the sequence stays near the corresponding parallel set.

For each \( n \) define the compact subsets \( C_n^\pm \subset \text{Flag}(\text{mod}) \)
\[
C_n^\pm := \{ \tau \pm | \angle_{x_n}^\zeta (\tau, x_{n+1}) \geq \pi - 2\epsilon \}.
\]
Each of these is nonempty since \( \angle_{x_n}^\zeta (x_{n+1}, x_n) = \pi \) implies \( \tau(x_{n+1}, x_n) \in C_n^\pm \). By step 1, \( C_n^- \subset C_n^+ \) so there exists \( \tau_- \in \bigcap_n C_n^- \). Similarly, there exists some \( \tau_+ \in \bigcap_n C_n^+ \). Straightness and the triangle inequality imply \( \angle_{x_n}^\zeta (\tau_-, \tau_+) \geq \pi - 5\epsilon \), so \( \tau_\pm \) are antipodal by assumption \( \dagger \) and Lemma \( \text{3.18} \) and define the parallel set \( P = P(\tau_-, \tau_+) \). Since \( 5\epsilon \leq \frac{c_0^2}{2^2} \) by assumption \( \dagger \) the angle-to-distance estimate Lemma \( \text{4.16} \) implies
\[
d(x_n, P) \leq \frac{5\epsilon}{c_0} \leq \delta
\]
with the last inequality from assumption \( \ddagger \).

Step 3: Morseness cf. [KLP14, Lemma 7.9, Lemma 7.10, Corollary 7.13]. We verify that the sequence is a Morse quasi-geodesic. We have already shown the angles are straight enough to guarantee that the projection to \( P \) is bounded. We show that projected rays land in nested cones; it follows that projecting further to the \( \zeta \)-ray yields a monotonic sequence which makes progress bounded away from zero.

![Figure 11: The projection \( \tau_{n+1} \) lands in the Weyl cone \( V(\tau_n, \text{st}(\tau_+), \alpha_{\text{new}}) \)](image)

By assumption \( \dagger \) and Lemma \( \text{4.3} \) we have that the projections \((\tau_n) \) to \( P \) are \( (\alpha_{\text{new}}, \tau_{\text{mod}}) \)-regular. Let \( \xi \) be the ideal point corresponding to the ray \( \tau_n, \tau_{n+1} \). Since the rays \( x_n \xi \) and \( \tau_n \xi \) are asymptotic, their Hausdorff distance is at most \( d(x_n, \tau_n) \leq \delta \), so \( x_{n+1} \) is at most \( 2\delta \) from \( x_n \xi \). Then
\[
\angle_{\text{Tits}}^\zeta (\tau_-, \xi) \geq \angle_{x_n}^\zeta (\tau_-, \xi) \geq \angle_{x_n}^\zeta (\tau_-, x_{n+1}) - \angle_{x_n}^\zeta (x_{n+1}, \xi) \geq \pi - 2\epsilon - \angle_{x_n}^\zeta (x_{n+1}, \xi).
\]
By Lemma \( \text{4.16} \) and Lemma \( \text{4.6} \) we may guarantee that
\[
\sin \angle_{x_n}^\zeta (x_{n+1}, \xi) \leq \frac{2}{\alpha_{\text{new}} \delta} \frac{2\delta}{\delta_0} \leq s
\]
so by assumption \( \ddagger \) this Tits angle is within \( \epsilon(\zeta) \) of \( \pi \), so \( \angle(\tau_-) \) is antipodal to \( \angle(\xi) \), but the only simplex in \( \partial P \) antipodal to \( \tau_- \) is \( \tau_+ \), so \( \tau(\xi) = \tau_+ \) and
\[
\angle_{\tau_n}^\zeta (\tau_-, \tau_n) = \angle_{\tau_n}^\zeta (\tau_-, \xi) = \pi.
\]

\( \ddagger \)The simplices are unique when the sequence is biinfinite, see [KLP14, pp. 7.19, 5.15], but this theorem also applies when the sequence is finite or a Morse quasiray.
We know that $\overline{x}_n\overline{x}_{n+1}$ is $(\alpha_{\text{new}}, \tau_{\text{mod}})$-regular and $\angle_{\overline{x}_n}(\tau_-, \zeta) = \pi$ and these two properties are equivalent to $\overline{x}_{n+1} \in V(\overline{x}_n, \text{st}(\tau_+), \alpha_{\text{new}})$. Using the convexity of Weyl cones and induction, we get that for all $n$ and all $m \geq 1$

$$\overline{x}_{n \pm m} \in V(\overline{x}_n, \text{st}(\tau_\pm), \alpha_{\text{new}}).$$

![Diagram](https://example.com/diagram.png)

Figure 12: Sufficiently straight and spaced sequences have monotonic projections to a geodesic ray

Finally, we want to show the sequence is coarsely spaced. The bound

$$d(x_n, x_{n+m}) \geq 2\alpha_{\text{new}}\zeta_0c_0(s - 2\delta)m - 2\delta$$

will follow from

$$d(\overline{x}_n, \overline{x}_{n+m}) \geq 2\alpha_{\text{new}}\zeta_0c_0(s - 2\delta)m.$$  

Indeed, the sequence $(\overline{x}_n)$ in $P$ is $(s - 2\delta)$-spaced and has a monotonic projection $(\overline{x}_n)$ to the geodesic line $\overline{x}_n\zeta(\tau_+)$ for any $n$ by the nestedness of Weyl cones. By [Ebe96, p. 2.14.5],

$$B(\zeta, d(\overline{x}_n, \overline{x}_{n+1})) = \sum_{\alpha \in \Lambda} \alpha(\zeta)\alpha(d(\overline{x}_n, \overline{x}_{n+1})) \text{ dim } g_\alpha \geq 2\alpha_{\text{new}}\zeta_0c_0d(\overline{x}_n, \overline{x}_{n+1}) \sum_{\alpha \in \Lambda^*} \text{ dim } g_\alpha = 2\alpha_{\text{new}}\zeta_0c_0d(\overline{x}_n, \overline{x}_{n+1}).$$

It follows that the projection $\overline{x}_{n+1}$ lies at least $2\alpha_{\text{new}}\zeta_0c_0(s - 2\delta)$ along the ray $\overline{x}_n\zeta$. \qed

In the final step of the proof we used the regularity of the projections to obtain the linear lower-quasigeodesic constant. When the angular radius of $\sigma_{\text{mod}}$ with respect to $\zeta$ is strictly less than $\pi/2$, the linear lower-quasigeodesic bound can be chosen independent of the regularity. By [KL18b, Lemma 5.8], this happens exactly when $\zeta$ is not contained in a factor of a nontrivial spherical join decomposition of $\sigma_{\text{mod}}$. 

**Remark 5.2.** To provide suitable auxiliary parameters to apply Theorem 5.1 we may first choose $\epsilon$ small enough to satisfy assumptions 1 and 2 and then choose $s$ large enough to satisfy assumptions 3 and 4. When we apply Theorem 5.1 in Section 6 we will choose $\delta = \frac{\zeta_0}{c_0}$ and $\epsilon = \frac{\zeta_0}{c_0}$ and then find a large enough $s$ to satisfy the conditions of Theorem 5.1.

### 5.2 Morse quasigeodesics have straight and spaced midpoints

In this section we show that Morse quasigeodesics of sufficiently good quality have straight and spaced midpoint sequences.

**Definition 5.3** (Cf. [KLP14, Definition 7.14]). For points $p, q$ in $X$ we let mid$(p, q)$ denote the midpoint of the geodesic segment $pq$. A sequence $(p_n)_{n=0}^{n=t_{\text{max}}}$ in $X$ satisfies the $(\alpha_0, \tau_{\text{mod}}, \epsilon, s, k)$-quadruple condition if for all $t_1, t_2, t_3, t_4 \in [t_0, t_{\text{max}}]$ and $t_2 - t_1, t_3 - t_2, t_4 - t_3 \geq k$ the triple of midpoints

$$(\text{mid}(p_1, p_2), \text{mid}(p_2, p_3), \text{mid}(p_3, p_4))$$

is $(\alpha_0, \tau_{\text{mod}}, \epsilon)$-straight and $s$-spaced. (Here $p(t_i) = p_i$.)
Our next theorem says that sufficiently spaced points on Morse quasigeodesics have straight and spaced midpoint sequences. In an effort to make Theorem 5.4 readable, we have given up some control over the required spacing. For example, we use only one auxiliary parameter and the crude estimate \( \sin^{-1}(x) \leq \frac{x}{2} \) for \( 0 \leq x \leq 1 \) (this follows from the fact that \( \sin^{-1} \) is convex). The following result is a quantified version of Proposition 7.16 in \cite{KLP14}.

**Theorem 5.4.** Assume \( k \) is large enough in terms of \( \alpha_{\text{new}} < \alpha_0, D, \epsilon, c_1, c_2 \) and \( s \). To make this precise, we use auxiliary constants \( l, \delta, \alpha_{\text{aux}} \) and make the following assumptions.

1. Let \( k \) be large enough in terms of the quasigeodesic parameters so that if \( |N| \geq k \) then \( d(x_n, x_{n+N}) \geq 2l \). Precisely, let \( k \geq c_1(2l + c_2) \). Our requirements on \( k \) will manifest as requirements on \( l \);

\[
1 \leq 6 \sinh(\alpha_{\text{aux}}(2l - D))^2, \quad 3Dc_0^{\kappa D - \alpha_0l} \leq \delta
\]

so that midpoints are \( \delta \)-close to diamonds by Lemma 4.11.

2. We assume \( \frac{2\alpha_{\text{aux}}}{\kappa_0}(l - \delta - D) \geq s \) to ensure that the midpoints are appropriately spaced.

3. We use an auxiliary parameter \( \alpha_{\text{aux}} \) such that \( \alpha_{\text{new}} < \alpha_{\text{aux}} < \alpha_0 \),

\[
\frac{\alpha_0\delta + 3\alpha_0D + 2\kappa_0D}{l} \leq \alpha_0 - \alpha_{\text{aux}}, \quad \text{and} \quad \frac{2\kappa_0\delta(\alpha_{\text{aux}} + \kappa_0)}{2\alpha_{\text{aux}}(l - \delta - D) - 2\kappa_0\delta} \leq \alpha_{\text{aux}} - \alpha_{\text{new}}.
\]

so that certain perturbations of regular segments are regular by 4.3.

4. We assume

\[
\frac{2}{\alpha_{\text{aux}}\zeta_0}D + \frac{2}{\alpha_{\text{new}}\zeta_0}2\alpha_{\text{aux}}(l - \delta - D) - \delta\kappa_0 + \frac{2}{\alpha_{\text{aux}}\zeta_0}\frac{\delta}{l - D} + \frac{2}{\alpha_{\text{new}}\zeta_0}\frac{\delta}{l - \delta} + 2\kappa_0\delta \leq \frac{\epsilon}{\pi}
\]

5. We assume

\[
\frac{2}{\alpha_{\text{aux}}\zeta_0}D + \frac{2}{\alpha_{\text{new}}\zeta_0}2\alpha_{\text{aux}}(l - \delta - D) - \delta\kappa_0 + \frac{2}{\alpha_{\text{aux}}\zeta_0}\frac{\delta}{l - D} + \frac{2}{\alpha_{\text{new}}\zeta_0}\frac{\delta}{l - \delta} + 2\kappa_0\delta \leq \frac{\epsilon}{\pi}
\]

so that midpoint sequence is straight.

Then every \((\alpha_0, \tau_{\text{mod}}, D)\)-Morse \((c_1,c_2)\)-lower-quasigeodesic satisfies the \((\alpha_{\text{new}}, \tau_{\text{mod}}, \epsilon, s, k')\)-quadruple condition for every \( k' \geq k \).

Note that in assumption 5 we have in particular assumed \( 2\pi\kappa_0\delta < \epsilon \), so the \( \delta \) which appears in the proof is quite small. Our proof follows \cite{KLP14} Proposition 7.16 closely.

**Proof.** Let \((q_n)_{n=t_0}^{n=t_{\text{max}}} \) be an \((\alpha_0, \tau_{\text{mod}}, D)\)-Morse quasigeodesic and let \( t_1, t_2, t_3, t_4 \in [t_0, t_{\text{max}}] \cap \mathbb{Z} \) such that \( t_2 - t_1, t_3 - t_2, t_4 - t_3 \geq k \). We abbreviate \( p_i := q_{t_i} \) and \( m_i := \text{mid}(p_i, p_{i+1}) \). We have \( d(p_i, p_{i+1}) \geq 2l \), \( d(m_i, p_i) \geq l \) and \( d(m_i, p_{i+1}) \geq l \).

To show that the midpoint sequence is \((\alpha_{\text{new}}, \tau_{\text{mod}}, \epsilon)\)-straight it suffices to show that the segment \( m_2m_1 \) is \((\alpha_{\text{new}}, \tau_{\text{mod}})\)-regular and that \( \angle_{m_2}(p_2, m_1) \leq \epsilon/2 \) under our assumptions on \( k \).

![Figure 13: The projections satisfy \( P_2 \in V(m_1, \text{st}(\tau_+), \alpha_{\text{aux}}) \) and \( m_2 \in V(P_2, \text{st}(\tau_+), \alpha_{\text{aux}}) \)](image)

By the Morse property there exists a diamond \( \Diamond_{\alpha_0}(x_1, x_3) \) such that \( d(x_1, p_1), d(x_3, p_3) \leq D \) and \( p_2 \) is in the \( D \)-neighborhood of \( \Diamond_{\alpha_0}(x_1, x_3) \). The diamond spans a unique parallel set \( P = P(\tau_-, \tau_+) \). We denote by \( P_1 \) and \( P_2 \) the projections of \( p_i \) and \( m_i \) to \( P \).
We first observe that $m_1$ is $\delta$-close to $P$ by the midpoint projection estimate Lemma 4.11, we have $d(p_1, x_1) \leq D$, $d(p_2, V(x_1, \text{ost}(\tau(x_1, x_3)))) \leq d(p_2, \Diamond_{\alpha_0}(x_1, x_3)) \leq D$ and $p_1p_2$ is $(\alpha_0, \tau_{\text{mod}})$-regular with $d(p_1, p_2) \geq 2l$ and $l$ large enough by assumption 2 and assumption 4.

$$d(m_1, P) \leq 3D\epsilon^{\kappa_0D - \alpha_0} \leq \delta.$$ 

Next we look at the directions of the segments $\overline{m_2m_1}$ and $\overline{m_2p_2}$ and show that they have the same $\tau$-direction. We have

$$d(\overline{m_2}, V(\overline{p_1}, \text{st}(\tau_+), \alpha_0)) \leq d(p_2, \Diamond_{\alpha_0}(x_1, x_3)) + d(\Diamond_{\alpha_0}(x_1, x_3), \overline{p_1}) \leq 2D$$

since projecting to a closed convex subset is distance-non-increasing. If $c_1$ is the geodesic from $p_1$ through $p_2$, the function $t \mapsto d(c_1(t), V(\overline{p_1}, \text{st}(\tau_+), \alpha_0))$ is convex, which implies $\overline{m_1}$ is $2D$-close to $V(\overline{p_1}, \text{st}(\tau_+), \alpha_0)$. We have $d(\overline{m_1}, \overline{p_1}) \geq l - \delta - D$, so by using the point in $V(\overline{p_1}, \text{st}(\tau_+), \alpha_0)$ within $2D$ of $\overline{m_1}$ and Lemma 4.5 in the presence of assumption 4, we obtain that $\overline{m_1} \in V(\overline{p_1}, \text{st}(\tau_+), \alpha_{\text{aux}})$. Similar arguments show that $\overline{m_1} \in V(\overline{p_2}, \text{st}(\tau_-), \alpha_{\text{aux}})$, or equivalently (by using the geodesic symmetry at mid($\overline{p_1}, \overline{p_2}$)) that $\overline{p_2} \in V(\overline{m_1}, \text{st}(\tau_+), \alpha_{\text{aux}})$. By the nestedness of Weyl cones, $\overline{p_1} \in V(\overline{p_2}, \text{st}(\tau_-), \alpha_{\text{aux}})$ and $\overline{p_2} \in V(\overline{p_1}, \text{st}(\tau_+), \alpha_{\text{aux}})$. Similarly, $\overline{m_2} \in V(\overline{p_2}, \text{st}(\tau_-), \alpha_{\text{aux}})$ and $\overline{p_2} \in V(\overline{m_2}, \text{st}(\tau_-), \alpha_{\text{aux}})$. The convexity of Weyl cones implies that also $\overline{m_1} \in V(\overline{m_2}, \text{st}(\tau_-), \alpha_{\text{aux}})$. In particular, $\angle_{\tau_+}(\overline{m_2}, \overline{m_1}) = 0$.

It is convenient to show that the midpoint sequence is appropriately spaced at this point in the proof, so that we can use the resulting estimate to control the regularity parameters $\alpha_{\text{aux}}$ and $\alpha_{\text{new}}$ and the straightness parameter $\epsilon$. The inclusions $\overline{m_1} \in V(\overline{p_2}, \text{st}(\tau_-), \alpha_{\text{aux}})$ and $\overline{m_2} \in V(\overline{p_2}, \text{st}(\tau_-), \alpha_{\text{aux}})$ imply that $d(\overline{m_1}, \overline{m_2}) \geq \frac{\alpha_{\text{aux}}}{\kappa_0} (d(\overline{m_1}, \overline{p_2}) + d(\overline{p_2}, \overline{m_1}))$. Therefore by assumption 4, the midpoint sequence is appropriately spaced:

$$d(m_1, m_2) \geq d(\overline{m_1}, \overline{m_2}) \geq \frac{\alpha_{\text{aux}}}{\kappa_0} (d(\overline{m_1}, \overline{p_2}) + d(\overline{p_2}, \overline{m_1})) \geq \frac{2\alpha_{\text{aux}}}{\kappa_0} (l - \delta - D) \geq s.$$ 

Using the previous estimate, Lemma 4.5 and assumption 4, we see that $m_2m_1$ and $m_2\overline{m_1}$ are $(\alpha_{\text{aux}}, \tau_{\text{mod}})$-regular and $m_2\overline{p_2}$ is $(\alpha_{\text{aux}}, \tau_{\text{mod}})$-regular.

We may now demonstrate the bound $\angle_{m_2}(p_2, m_1) \leq \epsilon/2$. We have

$$\angle_{m_2}(p_2, m_1) = \angle_{m_2}(p_2, m_1) - \angle_{m_2}(p_2, \overline{m_1})$$

$$\leq \angle_{m_2}(p_2, m_1) - \angle_{m_2}(p_2, \overline{m_1})$$

$$+ \angle_{m_2}(p_2, \overline{m_1}) - \angle_{m_2}(\tau(\overline{m_2}p_2), \tau(\overline{m_2}\overline{m_1}))$$

$$+ \angle_{m_2}(\tau(\overline{m_2}p_2), \tau(\overline{m_2}\overline{m_1})) - \angle_{m_2}(\tau(\overline{m_2}, \overline{m_1}))$$

$$= \angle_{m_2}(p_2, \overline{m_1})$$

By the triangle inequality for quadruples (on the metric space $(\text{Flag}(\tau_{\text{mod}}), \angle_{m_2})$) we have

$$\angle_{m_2}(p_2, \overline{m_1}) \leq \angle_{m_2}(p_2, \overline{m_1}) + \angle_{m_2}(\alpha_0, \overline{m_1}) = 2 \sin^{-1}\left(\frac{1}{2}d_p(Z_3, Z_4)\right)$$

where $Z_1, Z_2, Z_3, Z_4$ are the unit vectors at $m_2$ in the directions $\zeta(m_2p_2), \zeta(m_2p_2), \zeta(m_2m_1), \zeta(m_2\overline{m_1})$ respectively. Let $X_1, X_2, X_3, X_4$ be the unit vectors at $m_2$ which in the directions $p_2, \overline{p_2}, m_1, \overline{m_1}$ respectively. Then by Lemma 4.14 and the angle comparison to Euclidean space Lemma 4.10 we have

$$d(Z_1, Z_2) \leq \frac{2}{\alpha_{\text{aux}}\delta} d(X_1, X_2) = \frac{4}{\alpha_{\text{aux}}\delta} \sin\frac{1}{2}\angle_{m_2}(p_2, \overline{m_1}) \leq \frac{2}{\alpha_{\text{aux}}\delta} \frac{D}{l}.$$ 

Similarly,

$$d(Z_3, Z_4) \leq \frac{2}{\alpha_{\text{aux}}\delta} d(X_3, X_4) = \frac{4}{\alpha_{\text{aux}}\delta} \sin\frac{1}{2}\angle_{m_2}(m_1, \overline{m_1}) \leq \frac{2}{\alpha_{\text{aux}}\delta} \frac{\kappa_0\delta}{2\alpha_{\text{aux}}(l - \delta - D) - \delta\kappa_0}.$$ 

Again by the triangle inequality on $(\text{Flag}(\tau_{\text{mod}}), \angle_{m_2}),$

$$\angle_{m_2}(p_2, \overline{m_1}) \leq \angle_{m_2}(\tau(\overline{m_2}p_2), \tau(\overline{m_2}\overline{m_1})) \leq \angle_{m_2}(p_2, \tau(\overline{m_2}p_2)) + \angle_{m_2}(\overline{m_1}, \tau(\overline{m_2}\overline{m_1})).$$
Asymptotic geodesic rays are bounded by the distance of their tips, so if we let \( c_2 \) be the geodesic ray from \( m_2 \) to \( m_0 \mathbb{P}_2^{(+\infty)} \) we may use Lemma 4.16 to obtain
\[
\sin \frac{1}{2} \angle_{m_2}^c(p_2, \tau(m_2 \mathbb{P}_2)) \leq \frac{1}{\alpha_{aux} \zeta_0} \frac{d(p_2, \text{im}c_2)}{d(m_2, p_2)} \leq \frac{1}{\alpha_{aux} \zeta_0} \frac{\delta}{l - D}.
\]

Similarly by considering the geodesic ray \( c_3 \) from \( m_2 \) through \( m_1 \),
\[
\sin \frac{1}{2} \angle_{m_2}^c(m_1, \tau(m_2 m_1)) \leq \frac{1}{\alpha_{new} \zeta_0} \frac{d(m_1, \text{im}c_3)}{d(m_2, m_1)} \leq \frac{1}{\alpha_{new} \zeta_0} \frac{\delta}{l - \delta}.
\]

Write \( \tau = \tau(m_2 \mathbb{P}_2) \) and \( \tau' = \tau(m_2 m_1) \). By the distance-to-angle estimate Corollary 4.13
\[
\left| \angle_{m_2}^c(m_2 \mathbb{P}_2, \tau(m_2 m_1)) - \angle_{m_2}^c(p_2, m_1) \right| = \left| \angle_{m_2}^c(\tau, \tau') - \angle_{m_2}^c(\alpha, \alpha) \right| \leq 4 \sin^{-1} \left( \frac{\kappa_0 \delta}{2 m_2} \right) \leq 4 \sin^{-1} \left( \frac{\kappa_0 \delta}{2} \right)
\]

Combining these estimates with the fact that \( \sin^{-1}(x) \leq \frac{\pi}{2} x \) for \( 0 \leq x \leq 1 \) yields
\[
\angle_{m_2}^c(p_2, m_1) \leq \frac{\pi}{2} \left[ \frac{2}{\alpha_{aux} \zeta_0} \frac{D}{l} + \frac{2}{\alpha_{new} \zeta_0} \frac{\kappa_0 \delta}{2 \alpha_{aux} (l - D) - \delta} + \frac{2}{\alpha_{aux} \zeta_0} \frac{\delta}{l - D} + \frac{2}{\alpha_{new} \zeta_0} \frac{\delta}{l - \delta} + 2 \kappa_0 \delta \right] \leq \frac{\epsilon}{2}
\]
by assumption I. For similar reasons \( \angle_{m_2}^c(p_3, m_3) \leq \frac{\pi}{2} \), so \( \angle_{m_2}^c(m_1, m_3) \geq \pi - \epsilon \) as desired. We have already shown that \( m_2 m_1 \) is \((\alpha_{new}, \tau_{mod})\)-regular and \( s \)-spaced. For similar reasons the same holds for \( m_2 m_3 \). This concludes the proof.

**Remark 5.5.** To provide suitable auxiliary parameters to apply Theorem 5.4 we may first choose any \( \delta < \frac{\epsilon}{2 \pi \alpha_0} \) and any \( \alpha_{aux} < \alpha_{aux} < \alpha_0 \). Then we may choose \( l \) large enough to satisfy assumptions 2 through 5, which provides a suitable \( k \) via assumption I. When we apply Theorem 5.4 in Section 6 we set \( \delta = \frac{\epsilon}{20 \pi \alpha_0} \) and \( \alpha_{aux} = 0.8 \alpha_0 + 0.2 \alpha_{new} \).

### 5.3 Local-to-global principle for Morse quasigeodesics

An \( L\)-local \((\alpha_0, \tau_{mod}, D)\)-Morse \((c_1, c_2, c_3, c_4)\)-quasigeodesic is a sequence \((x_n)_{n=t_0}^{n=t_{max}} \in X \) such that for \( t_0 \leq t_1 \leq t_2 \leq t_{max} \) with \( t_2 - t_1 \leq L \), the subsequence \((x_n)_{n=t_1}^{n=t_2} \) is an \((\alpha_0, \tau_{mod}, D)\)-Morse \((c_1, c_2, c_3, c_4)\)-quasigeodesic.

We now come to the main result of the paper. The following result is a quantified local-to-global principle for Morse quasigeodesics. Theorem 7.18 says that for any fixed quality of Morse quasigeodesic, there exists a large enough scale so that a local Morse quasigeodesic of that scale and quality is a global Morse quasigeodesic. It is a quantified version of Theorem 7.18 in [KLP14], stated as Theorem 1.1 in the introduction. We will apply Theorem 5.1 and Theorem 5.3. While these theorems have cumbersome statements, finding auxiliary parameters which satisfy the required inequalities is easy, as we discussed in Remark 5.2 and Remark 5.3 and as we demonstrate in the next section.

**Theorem 5.6.** For any \( \alpha_{new} < \alpha_0, D, c_1, c_2, c_3, c_4, \) there exists a scale \( L \) so that every \( L\)-local \((\alpha_0, \tau_{mod}, D)\)-Morse \((c_1, c_2, c_3, c_4)\)-quasigeodesic in \( X \) is an \((\alpha_{new}, \tau_{mod}, D')\)-Morse \((c_1', c_2', c_3', c_4')\)-quasigeodesic. Precisely, \( L = 3k \) is large enough if auxiliary parameters \( \alpha_{aux}, k, \delta, s, \epsilon \) satisfy:

1. \( \epsilon \) is small enough and \( s \) is large enough to satisfy the conditions of Theorem 5.4 for \( \alpha_{new} < \alpha_{aux}, \delta \),
2. \( k \) is large enough in terms of \( \alpha_{aux} < \alpha_0, D, \epsilon, c_1, c_2 \) and \( s \) to satisfy the conditions of Theorem 5.4,

and the sequence has global Morse parameters

1. \( D' = c_3 k + \frac{3}{4} c_4 + \delta \),
2. \( c_1' = 2 \alpha_{new} \zeta_0 c_0(s - 2 \delta) k^{-1} \),
3. \( c_2' = 2 \alpha_{new} \zeta_0 c_0(s - 2 \delta) + 2 \delta + 2 c_3 k + 3 c_4 \),

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4. $c'_3 = c_3 + \frac{c_4}{2}$.
5. $c'_4 = c_4$.

**Proof.** Let $(x_n)_{n=+\infty}^1$ be an $L$-local $(\alpha_0, \tau_{mod}, D)$-Morse $(c_1, c_2, c_3, c_4)$-quasigeodesic. By Theorem 5.4 and assumption 2, each subsequence $(x_n)_{n=+\infty}^1$ satisfies the $(c_{aux}, \tau_{mod}, \epsilon, s, k)$-quadruple condition. In particular, the coarse midpoint sequence $m_n = \text{mid}(x_{nk}, x_{nk+k})$ is $t_{aux}$-$\tau_{mod}$-straight and $s$-spaced. By Theorem 5.3 and assumption 1, the midpoint sequence $(m_n)$ is an $(c_{aux}, \tau_{mod}, \delta)$-Morse $((2\alpha_{aux}\zeta_0 c_0 (s - 2\delta))^{-1}, 2\delta)$-lower quasigeodesic. We now use the midpoint sequence as a coarse approximation of the original sequence to show that $(x_n)$ is a global Morse quasigeodesic.

The subsequences $x_{nk}, x_{nk+1}, \ldots, x_{nk+k}, x_{nk+k}$ are $(c_3, c_4)$-upper-quasigeodesics (because $L \geq k$), so they lie in uniform neighborhoods of each $m_n$: if $|t - nk| \leq \frac{k}{2}$ then

$$d(m_n, x_t) \leq d(n, x_{nk}) + d(x_{nk}, x_t) \leq \frac{d(x_{nk}, x_{nk+k})}{2} + d(x_{nk}, x_t) \leq \frac{c_3}{2}k + \frac{c_4}{2} + c_4 = c_3k + \frac{3}{2}c_4.$$ 

In particular, $(x_{\tau})$ is $(\alpha_{aux}, \tau_{mod}, D')$-Morse for $D' = c_3k + \frac{3}{2}c_4 + \delta$. The midpoint sequence is coarsely spaced:

$$d(m_n, m_{n+N}) \geq 2\alpha_{aux}\zeta_0 c_0 (s - 2\delta)|N| - 2\delta,$$

so the original sequence is also coarsely spaced:

$$d(x_t, x_{t'}) \geq d(m_n, m_{n'}) - d(m_n, x_t) - d(m_n, x_{t'}) \geq 2\alpha_{aux}\zeta_0 c_0 (s - 2\delta)|n - n'| - 2\delta - 2c_3k - 3c_4 \geq 2\alpha_{aux}\zeta_0 c_0 (s - 2\delta)k^{-1}|k - t'|- 2\alpha_{aux}\zeta_0 c_0 (s - 2\delta) - 2\delta - 2c_3k - 3c_4.$$ 

Finally, if a sequence is $(c_3, c_4)$-coarsely Lipschitz on intervals of length $L$, it then satisfies $d(x_n, x_{n+N}) \leq |N|(c_3 + \frac{c_4}{2}) + c_4$ and is $(c_3 + \frac{c_4}{2}, c_4)$-coarsely Lipschitz. \hfill \Box

### 6 Applications of the local-to-global principle

In this section we give two applications of the main result, Theorem 5.4. We describe two explicit neighborhoods of Anosov representations in SL(3, R), one for free groups and another for closed surface groups. Each of them is constructed by perturbing a group acting cocompactly on a convex subset of a totally geodesic hyperbolic plane in the associated symmetric space.

We will need some further estimates in order to quantify these neighborhoods. First we recall a standard proof of the Milnor-Schwarz Lemma so that we may use the explicit quasi-isometry constants it produces. We then give a version of the classical Morse Lemma that will be used in Section 6.3. In Section 6.1.3 we use elementary linear algebra to control the perturbations of long words in a linear group that results from perturbing the generators. We also relate the Frobenius norm on $d \times d$ matrices to the distance in the symmetric space associated to SL(d, R). In the final two sections, we apply the local-to-global principle Theorem 5.4 to describe explicit neighborhoods of Anosov representations.

As one might expect, straightforward applications of Theorem 5.4 as we have done here will yield only very small perturbations. This is partially explained by the following geometric difficulty. The Morse condition implies that the image of each geodesic in the Cayley graph fellow-travels a unique parallel set. After perturbing the representation, one expects the image of the geodesic to fellow-travel a new parallel set. For geodesics through the identity, our techniques merely bound the distance from the perturbed geodesic to its previous parallel set, so for it to fellow-travel for a long time, the perturbation has to be extremely small. If we could identify the new parallel set it fellow-travels and bound the distance to that parallel set, we expect that the perturbation bounds would improve significantly.

#### 6.1 Preliminary estimates

##### 6.1.1 The Milnor-Schwarz Lemma

In this subsection we state and prove a standard result in geometric group theory called the Milnor-Schwarz Lemma. It is a source of concrete quasiisometry parameters for nice enough actions of finitely generated...
groups, such as those we consider in Sections 6.2 and 6.3. The proof given here is taken directly from Sisto’s lecture notes [Sis14].

Lemma 6.1 (Milnor–Švarc Lemma). Let $G$ be a group acting properly discontinuously, cocompactly and by isometries on a proper geodesic space $X$. Choose any $p \in X$. Then the group $G$ has a finite generating set $S$ so that the orbit map at $p$ is a quasi-isometry for $G$ with the word metric induced by $S$. In fact,

$$\text{wl}(g) \leq d(p, gp) + 1, \quad \text{and} \quad d(p, gp) \leq \max_{s \in S} \{d(p, sp)\} \text{wl}(g).$$

Proof. Since the action is cocompact, there exists a constant $R$ so that the $G$-translates of $B_R(p)$ cover $X$. Let $S := \{g \in G \mid d(p, gp) \leq 2R + 1\}$. Since $X$ is proper, the closed ball of radius $R + \frac{1}{2}$ centered at $p$ is compact, and since the action is properly discontinuous, $S = \{g \in G \mid B_{R+\frac{1}{2}}(p) \cap B_{R+\frac{1}{2}}(gp)\}$ is finite. Now let $g \in G$. Choose a minimal geodesic from $p$ to $gp$, and subdivide it with points $p_i$ so that $p = p_0, p_1, p_2, \ldots, p_{n-1}, p_n = gp$ occur monotonically and for $i = 0, 1, 2, \ldots, n - 2$, we have $d(p_i, p_{i+1}) = 1$ and $d(p_{n-1}, p_n) \leq 1$. For each $1 \leq i \leq n - 1$ choose $g_i \in G$ so that $d(g_i, p_i) \leq R$ and set $g_0 = \text{id}$ and $g_n = g$. Then for all $0 \leq i \leq n - 1$, we have

$$d(g_ip, g_{i+1}p) \leq d(g_ip, p_i) + d(p_i, p_{i+1}) + d(p_{i+1}, g_{i+1}p) \leq 2R + 1,$$

which implies that there exists $s_{i+1} \in S$ so that $g_{i+1} = g_is_{i+1}$. For all $1 \leq i \leq n$ it follows that $g_i = s_1s_2s_3\cdots s_i$. Therefore $g$ can be written as a product of $n$ elements of $S$, with $n - 1 \leq d(p, gp)$. It follows that $S$ is a finite generating set for $G$ and the word length of $g$ with respect to $S$ is bounded above by $d(p, gp) + 1$.

We have shown that $S$ is a finite generating set for $G$. Write $g = g_1 \cdots g_n$ with $g_i \in S$. Then

$$d(p, g_1g_2g_3\cdots g_np) \leq d(p, g_1\cdots g_{n-1}p) + d(g_1\cdots g_{n-1}p, g_1\cdots g_{n-1}g_np)$$

$$= d(p, g_1\cdots g_{n-1}p) + d(p, g_np)$$

$$\leq d(p, g_1p) + \cdots + d(p, g_np)$$

$$\leq \max_{s \in S} \{d(p, sp)\} n,$$

so the orbit map at $p$ is $\max_{s \in S} \{d(p, sp)\}$-Lipschitz with respect to the generating set $S$. Note that by the definition of $S$, $\max_{s \in S} \{d(p, sp)\} \leq 2R + 1$. \hfill $\square$

The previous lemma provides quasi-isometry constants in terms of only the constant $R$ so that the image of an $R$-ball covers the quotient. In return we give up control over the generating set. In particular, when we apply Lemma 6.1 to an action of a closed surface group on the hyperbolic plane in Section 6.3 we will give quasiisometry parameters with a nonstandard generating set for the Cayley graph. We will need to control the Frobenious norm of the matrices in our generating set by using Lemma 6.7.

6.1.2 The classical Morse Lemma

In Section 6.3 we will use the following version of the classical Morse Lemma to provide Morse quasiisometry parameters for the orbit map of a surface group acting on a copy of the hyperbolic plane. The following proof is adapted from Bridson-Haefliger [BH99].

Theorem 6.2 (Classical Morse Lemma, Cf. [BH99] Theorem III.H.1.7). Let $D_0$ be an upper bound for

$$\{D \mid D - 1 \leq \delta \log_2(2D + 2M^2l + 6DMl + aM)\}$$

and set $R = D_0 + lM_0 + lM^2 + \frac{\alpha}{2}$. Then:

If $(y_i)_{i=0}^{i=N}$ is a sequence in a $\delta$-hyperbolic geodesic space $\mathcal{Y}$ with

$$d(y_i, y_j) \leq M|j - i| \quad \text{and} \quad |j - i| \leq ld(y_i, y_j) + a$$

then for all $0 \leq n \leq N$, the distance from $y_n$ to a geodesic segment from $y_0$ to $y_N$ is bounded above by $R$. 39
Proof. Let $c: [0, N] \to Y$ be the piecewise geodesic curve with $c(i) = y_i$. Let $D$ be minimal so that the closed $D$-neighborhood of $\text{im } c$ covers the geodesic from $p = y_0$ to $q = y_N$. Choose a point $x_0$ on $pq$ realizing $D$, and choose $y, z$ on $pq$ at distance $2D$ from $x_0$ so that $y, x_0, z$ occurs in order (if $x_0$ is too close to $p$, use $p$ for $y$, and likewise for $z$). Choose $y'$ on $\text{im } c$ within $D$ of $y$, and choose $z'$ similarly. Choose $i, j$ so that $y'$ is on $y_i y_{i+1}$ and $z'$ is on $y_{j-1} y_j$. If $c(t) = y'$ and $c(t') = z'$ then the length of $c$ restricted to the $[t, t']$ is at most

$$\text{length}(c|_{[t, t']}) \leq \text{length}(c|[i, j]) \leq M|j - i| \leq M[ld(y_i, y_j) + a].$$

Also,

$$d(y_i, y_j) \leq d(y_i, y') + d(y', y) + d(y, z) + d(z, z') + d(z', y_j) \leq 2M + 6D$$

and it follows that the curve $c'$ formed by following a geodesic segment from $y$ to $y'$ then along $c$ to $z'$ then along a geodesic segment to $z$ has length at most $2D + M[2M + 6D + a]$. Proposition III.1.6.1 in [BH99] bounds $D$ in terms of the length of $c'$ and $\delta$. In particular

$$D - 1 \leq \delta|\log_2(2D + 2M^2l + 6MDl + aM)|$$

which implies an upper bound $D_0$ on $D$.

Now suppose that $(y_n)_{n=a'}^{b'}$ is a maximal (consecutive) subsequence outside the $D_0$-neighborhood of $pq$. There exist $s, s'$ such that $0 \leq s \leq a'$ and $b' \leq s' \leq N$ within $D_0$ of the same point on $pq$, so $d(c(s), c(s')) \leq 2D_0$. As before, by choosing $m, n$ so that $c(s)$ lies on $y_my_{m+1}$ and $c(s')$ lies on $y_n y_{n+1}$ we have that

$$\text{length}(c|[s, s']) \leq \text{length}(c|[m, n]) \leq M|m - n| \leq M|ld(y_m, y_n) + a)$$

and

$$d(y_m, y_n) \leq d(y_m, c(s)) + d(c(s), c(s')) + d(c(s'), y_n) \leq 2M + 2D_0$$

so we obtain

$$\text{length } c|[s, s'] \leq M[2D_0 + 2M] + a].$$

It follows that $R = D_0 + M[2D_0 + M] + a]$ is an upper bound for the distance from any $y_n$ to $pq$.  

### 6.1.3 Matrix Estimates

In this section we establish a few elementary estimates related to the symmetric space associated to $\text{SL}(d, \mathbb{R})$. We will control perturbations of long words in a generating set in terms of the Frobenious norm of the generators. As noted above, we use a non-standard generating set for the closed surface group, so we also prepare to control the Frobenious norm of the generators in that case. In Sections 6.2 and 6.3 we combine these estimates with the local-to-global principle Theorem 6.0 to guarantee that the Morse subgroups under consideration remain Morse after certain explicit perturbations.

In the rest of the paper, we identify the symmetric space associated to $\text{SL}(d, \mathbb{R})$ with the space of real, symmetric, positive-definite matrices of determinant 1. We remind the reader that we take the Riemannian metric to be induced by the Killing form, so at the identity matrix, the Riemannian metric is $2d$ times the Frobenious inner product $\langle X, Y \rangle_{Fr} = \text{trace}(X^T Y)$.

**Lemma 6.3.** Let $|\cdot|$ be any submultiplicative norm on $d \times d$ matrices. Let $w = g_1g_2\cdots g_{k-1}g_k$ be a product of $k$ matrices, and let $w' = (g_1 + \epsilon_1)(g_2 + \epsilon_2)\cdots(g_{k-1} + \epsilon_{k-1})(g_k + \epsilon_k)$ be a product of perturbed matrices. Suppose that for all $1 \leq i \leq k$, $|g_i| \leq A$ and $|\epsilon_i| \leq \epsilon$. If $k \geq 3$ and $k^{-\frac{1}{2}} \leq 1$ then $|w' - w| \leq 2kA^{k-1}\epsilon$. 

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Proof. We have
\[
|w' - w| = \left| \prod_{i=1}^{k} (g_i + \epsilon_i) - \prod_{i=1}^{k} g_i \right|
\]
\[
= \left| \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq k} g_1 g_2 \cdots g_{i_1} - 1 \epsilon_i g_{i_1 + 1} \cdots g_{i_j} - 1 \epsilon_i g_{i_j + 1} \cdots g_k \right|
\]
\[
\leq kA^{k-1} \epsilon + \left( \binom{k}{2} \right) A^{k-2} \epsilon^2 + \cdots + \left( \binom{k}{j} \right) A^{k-j} \epsilon^j + \cdots + \epsilon^k
\]
\[
= A^k \left[ \left( 1 + \frac{\epsilon}{A} \right)^k - 1 \right]
\]
\[
\leq 2kA^{k-1} \epsilon
\]
where the last line follows from the Taylor approximation \((1 + \frac{\epsilon}{A})^k - 1 \leq k\frac{\epsilon}{A} + \frac{k(k-1)}{2} \left( \frac{\epsilon}{A} \right)^2\), valid when \(\frac{\epsilon}{A} \leq 1\).

We next relate the Riemannian distance in \(X\) to the \(B_p\)-norm on the space of matrices. Recall that when \(p\) is the identity matrix, \(B_p\) is \(2d\) times the Frobenius inner product. We let \(B_p\) be defined on all of \(\mathfrak{gl}(d, \mathbb{R})\) as \(2d\) times the Frobenius inner product.

**Lemma 6.4.** Let \(g \in \text{SL}(d, \mathbb{R})\) and \(p \in X\) be the identity matrix. Then
\[
d_X(gp, p) \leq \sqrt{d(d-1)} |g - 1|_{B_p}.
\]

**Proof.** \(K = \text{SO}(d)\) acts on \((\mathfrak{gl}(d, \mathbb{R}), B_p)\) by isometries on the left and the right, so \(|g - 1|_{B_p} = |e^A - 1|_{B_p}\) where
\[
A = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_d
\end{pmatrix}
\]
is the Cartan projection of \(g\). That is, \(A\) is the unique diagonal matrix with \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d\) and \(\lambda_1 + \cdots + \lambda_d = 0\) such that \(g = k e^{A} k'\) for some \(k, k' \in \text{SO}(d)\). We have \(|A|_{B_p} = d(gp, p)\). Since \(-\lambda_d \leq (d-1)\lambda_1\) and \(\lambda_1^2 \leq (e^{\lambda_1} - 1)^2\),
\[
d(gp, p)^2 = |A|_{B_p}^2 = 2d \sum_{i=1}^{d} \lambda_i^2 \leq 2d^2(d-1)^2 \lambda_1^2 \leq 2d^2(d-1)^2 \sum_{i=1}^{d} (e^{\lambda_i} - 1)^2 = d(d-1)^2 |e^A - 1|_{B_p}^2 = d(d-1)^2 |g - 1|_{B_p}^2.
\]

In the following corollary, we consider a pair of linear representations that map the generating set to nearby generators. We apply a long word to the basepoint using each representation, and bound the resulting distance.

**Corollary 6.5.** Let \(\Gamma\) be a group with symmetric generating set \(S = \{\gamma_1, \ldots, \gamma_n\}\) and let \(\rho\) and \(\rho'\) be two representations of \(\Gamma\) into \(\text{SL}(d, \mathbb{R})\). Assume that
1. For \(i \in \{1, \ldots, n\}\), \(|\rho(\gamma_i)|_{F_r} \leq A\) and \(|\rho(\gamma_i) - \rho'(\gamma_i)|_{F_r} \leq \epsilon\); and
2. \(k \geq 3\) and \(\frac{k - 1}{A} \leq 1\).

Then for any \(\gamma \in \Gamma\) with \(d_S(\gamma, 1) \leq k\), it holds that \(d_S(\rho'(\gamma)p, \rho(\gamma)p) \leq \sqrt{d(d-1)kA^{2k-1}} \epsilon\).
Proof. Let \( g = \rho(\gamma) \) and \( g' = \rho'(\gamma) \) for \( d_S(\gamma, 1) \leq k \). Since the Frobenius norm is submultiplicative we have 
\[
|g^{-1}|_{F_r} \leq A^k
\]
and moreover because of the assumptions, Lemma 6.3 applies and we obtain 
\[
|g - g'|_{F_r} \leq 2kA^{k-1}\epsilon.
\]
We see that
\[
|g^{-1}g' - 1|_{F_r} = |g^{-1}(g' - g)|_{F_r} \leq |g^{-1}|_{F_r}|g' - g|_{F_r} \leq A^k|g' - g|_{F_r} \leq 2kA^{k-1}\epsilon.
\]
Then by applying Lemma 6.4 to \( g^{-1}g' \) we obtain
\[
d(g'p, gp) = d(g^{-1}g'p, p) \leq \sqrt{d(d - 1)} |g^{-1}g' - 1|_{B_p} \leq \sqrt{8d(d - 1)kA^{2k-1}\epsilon}.
\]

In the next lemma we give a precise, quantitative version of the following statement: If a representation \( \rho \) induces a Morse quasiisometric embedding, then its perturbation \( \rho' \) induces a local Morse quasiisometric embedding.

**Lemma 6.6.** Let \( \rho, \rho' : \Gamma \to \text{SL}(d, \mathbb{R}) \) be representations and let \( S \) be a symmetric generating set for \( \Gamma \). If \( d(\rho(\gamma)p, \rho'(\gamma)p) \leq \epsilon \) for all \( d_S(\gamma, 1) \leq k \) and if the orbit map of \( \rho \) at \( p \) is an \((\alpha_0, \tau_{\text{mod}}, D)\)-Morse \((c_1, c_2, c_3, c_4)\)-quasiisometric embedding then the orbit map of \( \rho' \) at \( p \) is a \(2k\)-local \((\alpha_0, \tau_{\text{mod}}, D + \epsilon)\)-Morse \((c_1, c_2 + \epsilon, c_3, c_4 + \epsilon)\)-quasiisometric embedding.

**Proof.** If \( d(\rho(\gamma)p, \rho'(\gamma)p) \leq \epsilon \) for all \( d_S(\gamma, 1) \leq k \), then for every geodesic \((\gamma_n)_{n=-k}^k\) in \( \Gamma \) of length \( 2k \),
\[
d(\rho'(\gamma_n)p, \rho'(\gamma_0)p) = d(\rho'(\gamma^{-1}_0)p, \rho'(\gamma_n)p, p)
\]
is within \( \epsilon \) of \( d(\rho(\gamma_n)p, \rho(\gamma_0)p) \). Additionally, if \( (\rho(\gamma_n)p) \) is within \( D \) of \( \delta(q, r) \), then \( (\rho'(\gamma_0)p) \) is within \( D + \epsilon \) of \( \delta(\rho'(\gamma_0)p, \rho'(\gamma_0)\rho(\gamma_0)^{-1}r) \). In particular, if \( \rho \) induces an \((\alpha_0, \tau_{\text{mod}}, D)\)-Morse \((c_1, c_2, c_3, c_4)\)-quasiisometric embedding then \( \rho' \) induces a \(2k\)-local \((\alpha_0, \tau_{\text{mod}}, D + \epsilon)\)-Morse \((c_1, c_2 + \epsilon, c_3, c_4 + \epsilon)\)-quasiisometric embedding.

When we apply the Milnor-Schwarz lemma we use the generating set \( S = \{ s \in \Gamma \mid d(p, sp) \leq 2R + 1 \} \), and when we apply Corollary 6.5 we need to bound the size of the generating set. The following Lemma helps us do just that.

**Lemma 6.7.** Let \( p \) be the identity matrix in \( \mathbb{X}_d \) and let \( g \in \text{SL}(d, \mathbb{R}) \) such that \( d(p, gp) \leq 2R + 1 \). Let \( |\cdot|_{F_r} \) denote the Frobenius norm. Then
\[
|g|_{F_r} \leq \exp\left(\frac{2R + 1}{\sqrt{2d}}\right).
\]

**Proof.** Combine
\[
|g|^2_{F_r} = |gg^T|_{F_r} = \exp \log gg^T \leq \exp |\log gg^T|_{F_r}
\]
and
\[
\sqrt{\frac{d}{2}} |\log gg^T|_{F_r} = \frac{1}{2} |\log gg^T|_{B_p} = \left| \log \sqrt{gg^T} \right|_{B_p} = d(p, gp) \leq 2R + 1
\]
to obtain
\[
|g|_{F_r} \leq \exp \frac{1}{2} |\log gg^T|_{F_r} \leq \exp \left(\frac{2R + 1}{\sqrt{2d}}\right).
\]

### 6.2 An explicit neighborhood of Anosov free groups

In this section we obtain an explicit non-empty neighborhood of Anosov free groups. Let \( \Gamma_1 \) be the subgroup of \( \text{SL}(3, \mathbb{R}) \) generated by
\[
g = \begin{bmatrix} e^t & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-t} \end{bmatrix}, \quad h = \begin{bmatrix} \cosh t & 0 & \sinh t \\
0 & 1 & 0 \\
\sinh t & 0 & \cosh t \end{bmatrix}.
\]

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As in Section 6.1.3 we identify the associated symmetric space with the space of real, symmetric, positive-definite matrices of determinant 1. Let \( p \in \mathbb{R} \) be the identity matrix. \( \Gamma_1 \) is a subgroup of a copy of \( \text{SL}(2, \mathbb{R}) \) preserving a copy of \( \mathbb{H}^2 \) containing \( p \) of curvature \(-\frac{1}{3}\), see Section 3.3. We will directly estimate the Morse quasi-isometry parameters of the orbit map at \( p \) on \( \Gamma_1 \).

The points \( p, gp, hp \) form an isosceles right triangle:

\[
d(p, gp) = \left\| \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t \end{bmatrix} \right\|_{B_p} = 2\sqrt{3}t = \left\| \begin{bmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{bmatrix} \right\|_{B_p} = d(p, hp).
\]

Figure 14: The Dirichlet domain \( C_p \) in the projective model for \( \mathbb{H}^2 \)

Write \( T = \tanh(t) \). If \( \sqrt{2}T > 1 \), then \( \Gamma_1 \) acts cocompactly on a closed convex subset \( C \) of \( \mathbb{H}^2 \), with a Dirichlet domain \( C_p \). The domain \( C_p \) is an octagon with geodesic boundary and neighbors \( gC_p, g^{-1}C_p, hC_p, h^{-1}C_p \) in \( C \). Since \( C \) is convex, the minimum distance between any pair of neighbors is bounded below by the length of an arc in \( C_p \) joining non-adjacent edges. This has lower bound

\[
c_1^{-1} = \sqrt{3} \min \left\{ t, \frac{1}{2} \log \left( \frac{T^2 + \sqrt{2T^2 - 1}}{T^2 - \sqrt{2T^2 - 1}} \right), \frac{1}{2} \log \left( \frac{1 + 2T\sqrt{1 - T^2}}{1 - 2T\sqrt{1 - T^2}} \right) \right\}.
\]

We also set \( c_3 = 2\sqrt{3}t \). The orbit map is a \((c_1, 0, c_3, 0)\) quasi-isometry. Set \( R = \sqrt{3} \tanh^{-1} (\sqrt{T^{-2} - 2 + 2T^2}) \). Then \( C \) is within the \( R \)-neighborhood of \( \Gamma_1 \cdot p \) and the diameter of \( C_p \) is \( 2R \). The orbit map is \( R \)-Morse.

We are now in position to prove Theorem 1.2.

**Theorem 1.2.** Let \( \Gamma_1 \) be the subgroup of \( \text{SL}(3, \mathbb{R}) \) generated by

\[
g = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad h = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix},
\]

with \( \tanh t = 0.75 \). If \( \Gamma_1' \) is generated by \( g', h' \) where \( \max\{|g - g'|_{F_T}, |h - h'|_{F_T}\} \leq 1.94 \times 10^{-15.598} \), then \( \Gamma_1' \) is Anosov.
Before proceeding to the proof, we discuss how to choose suitable parameters in the application of Theorem 5.6. There are a number of auxiliary parameters appearing in Theorems 5.1 and 5.3. We will choose these auxiliary parameters in the same way in Section 6.3. Because of the large number of auxiliary parameters, it is not clear how to obtain optimal estimates, even when treating Theorems 5.1, 5.4, and 5.6 as black boxes. The choices we make here are simply the result of selecting auxiliary parameters in a few different ways and choosing the best result (smallest k) we achieved. We used a Mathematica notebook to verify the system of inequalities for each theorem.

First we choose auxiliary parameters $\delta = \frac{\epsilon}{\kappa_0}$ and $\alpha_{aux} := 0.5\alpha_0 + 0.5\alpha_{new}$. We apply Theorem 5.1 with $\alpha_{aux} < \alpha_0$ and $\delta = \frac{\epsilon}{\kappa_0}$ by setting $\epsilon = \frac{\epsilon}{5\kappa_0}$ and then choosing $s$ large enough to satisfy the assumptions of the theorem. In Theorem 5.4, for any choice of auxiliary parameters $\delta_{aux} < \frac{\epsilon}{2\kappa_0}$ and any $\alpha_{aux} < \alpha'_{aux} < \alpha_0$, there is a large enough auxiliary parameter $l$ to satisfy the assumptions. We select $\delta_{aux} := 0.1 \frac{\epsilon}{2\kappa_0} = 0.1 \frac{\epsilon}{10\kappa_0}$ and $\alpha'_{aux} := 0.8\alpha_0 + 0.2\alpha_{aux}$.

Proof of Theorem 1.2. As discussed earlier in this section, the orbit map of $\Gamma_1$ is a $(\zeta_0, \sigma_{mod}, 3.18)$-Morse embedding $((1.28)^{-1}, 0, 3.38, 0)$-quasigeodesic embedding. We relax the parameters, asking the perturbation to induce a $34,240$-local $(\zeta_0, \sigma_{mod}, 3.28)$-Morse $(1, 0.1, 3.38, 0.1)$-quasiisometric embedding. By Theorem 5.6, such an orbit map is a global orbit map is a $(0.95\alpha_0, \sigma_{mod}, 38, 577)$-Morse embedding $(98, 77, 270, 3.38, 1)$-quasiisometric embedding.

If $g', h' \in \text{SL}(3, \mathbb{R})$ satisfy $|g - g'|_{F_{r}}$, $|h - h'|_{F_{r}} \leq 1.94 \times 10^{-15}, 598$, then for $d_{1}(w, 1) \leq k = 17, 120$ we have $d(\rho(w)p, \rho'(w)p) \leq 0.1$ by Corollary 6.3, so $\rho'$ also induces a $34,240$-local $(\zeta_0, \sigma_{mod}, 3.28)$-Morse $(1, 0.1, 3.38, 0.1)$-quasiisometric embedding and therefore its orbit map at $p$ is a (global) Morse quasiisometric embedding. In particular, $g', h'$ generate an Anosov subgroup of $\text{SL}(3, \mathbb{R})$ and our proof of Theorem 1.2 is complete.

6.3 An explicit neighborhood of Anosov surface groups

Let $\Gamma_2$ be the subgroup of $\text{SL}(3, \mathbb{R})$ generated by

$$S = \left\{ \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \right\},$$

for $\log \lambda = \cosh^{-1}(\cot \frac{\pi}{3})$. This group acts cocompactly on a complete, totally geodesic submanifold of $\mathbb{X}$ of constant curvature $-\frac{1}{3}$, see Section 3.3, with quotient a closed surface of genus 2. A fundamental domain for this action is given by a regular octagon in $\mathbb{H}^2$ with center $p$, the identity matrix in $\mathbb{X}$. This octagon decomposes into 16 triangles with vertices at the center, the vertices of the octagon, and the midpoints of the edges. These triangles are isosceles with angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$. By the hyperbolic law of cosines (for curvature $-\frac{1}{3}$),

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh \left( \frac{1}{\sqrt{3}} \right),$$

we see that the distance from the center $p$ to the vertex is $R = \sqrt{3 \cosh^{-1} \left( \cot^2 \frac{\pi}{8} \right)}$. The $\Gamma_2$ translates of $B_R(p)$ cover $\mathbb{H}^2$, so by the Milnor-Schwarz Lemma the orbit map orb$p$: $\Gamma_2 \to \mathbb{H}^2$ is a $(1, 1, 2R + 1, 0)$-quasi-isometric embedding. One checks that $2R + 1 \leq 9.5$. Here we use the symmetric generating set $S' = \{ \gamma \in \Gamma_2 \mid d(p, \gamma p) \leq 9.5 \}$. Note that the $S'$ here agrees with the one in the introduction because $d(p, \gamma p) = \sqrt{6} |\log \gamma|_{F_{r}}$. Every geodesic in this copy of $\mathbb{H}^2$ is $(\frac{\alpha}{2\sqrt{3}}, \sigma_{mod})$-regular in $\mathbb{X}$. Representations of this form were studied by Barbot in [Bar10].

We may now prove

Theorem 1.3. If $\rho$: $\Gamma_2 \to \text{SL}(3, \mathbb{R})$ is a representation satisfying $|\rho(s) - s|_{F_{r}} \leq 10^{-3.6 \times 10^6}$ for all $s \in S'$, then $\rho$ is Anosov.

Proof. From the classical Morse Lemma (Theorem 6.2), we get a Morse constant of $D = 163$. Thus the orbit map at $p$ is a $(\frac{\alpha}{2\sqrt{3}}, \sigma_{mod}, 163)$-Morse $(1, 1, 9.5, 0)$-quasiisometric embedding.
We relax the additive parameters by 10 and ask a perturbation to be a $(2.2 \times 10^6)$-local $(\frac{1}{2\sqrt{3}}, \sigma_{mod}, 173)$-Morse $(1, 11, 9.5, 10)$-quasiisometric embedding. By Theorem 5.6 such an orbit map is a global $(\frac{1}{4\sqrt{3}}, \sigma_{mod}, 6.8 \times 10^6)$-Morse $(54, 108, 1.4 \times 10^7, 9.5, 0)$-quasiisometric embedding.

If $\rho: \Gamma_2 \to \text{SL}(3, \mathbb{R})$ is another representation such that $|\rho(s) - s|_{F_E} \leq 10^{-3.6 \times 10^8}$ then for $d_{S'}(w, 1) \leq k = 1.1 \times 10^6$ we have $d_S(\rho(w)p, wp) \leq 10$ by Corollary 6.5, so $\rho$ also induces a $(2.2 \times 10^6)$-local $(\frac{1}{2\sqrt{3}}, \sigma_{mod}, 173)$-Morse $(1, 11, 9.5, 10)$-quasiisometric embedding and therefore a global $(\frac{1}{4\sqrt{3}}, \sigma_{mod}, 6.8 \times 10^6)$-Morse $(54, 108, 1.4 \times 10^7, 9.5, 0)$-quasiisometric embedding. In particular, $\rho$ is Anosov and our proof of Theorem 1.3 is complete. □
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