Toric Sylvester forms and applications in elimination theory

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Abstract

In this paper, we investigate the structure of the saturation of ideals generated by square systems of sparse homogeneous polynomials over a toric variety \(X\) with respect to the irrelevant ideal of \(X\). As our main results, we establish a duality property and make it explicit by introducing toric Sylvester forms, under a certain positivity assumption on \(X\). In particular, we prove that toric Sylvester forms yield bases of some graded components of \(I^{\text{sat}}/I\), where \(I\) denotes an ideal generated by \(n + 1\) generic forms, \(n\) is the dimension of \(X\) and \(I^{\text{sat}}\) the saturation of \(I\) with respect to the irrelevant ideal of the Cox ring of \(X\). Then, to illustrate the relevance of toric Sylvester forms we provide three consequences in elimination theory: (1) we introduce a new family of elimination matrices that can be used to solve sparse polynomial systems by means of linear algebra methods, including overdetermined polynomial systems; (2) by incorporating toric Sylvester forms to the classical Koszul complex associated to a polynomial system, we obtain new expressions of the sparse resultant as a determinant of a complex; (3) we prove a new formula for computing toric residues of the product of two forms.

Keywords: sparse polynomial systems, toric geometry, sparse resultants, algebraic geometry.

1 Introduction

The elimination of variables from a system of homogeneous polynomials is deeply connected to the saturation of ideals with respect to a certain geometrically irrelevant ideal. Thus, the search and study of universal generators of the saturation of an ideal generated by generic homogeneous polynomials is an important topic in elimination theory. Actually, in the classical literature of the previous century such universal generators were called inertia forms by Hurwitz, Mertens, Van der Waerden and many others, including Zariski; see the references in [Jou91; Jou97] and [Zar37]. As examples, Jacobian determinants and resultants associated to a square polynomial system are important inertia forms.

To be more specific, consider the ideal \(I = (F_0, \ldots, F_n)\) where \(F_i\) is the generic homogeneous polynomial of degree \(d_i\) in the (canonically) graded polynomial ring \(C = \mathbb{A}[x_0, \ldots, x_n]\), with \(\mathbb{A}\) standing for the universal ring of coefficients of the \(F_i\)’s. The saturation of the ideal \(I\) with respect to the ideal \(m = (x_0, \ldots, x_n)\), which we denote by \(I^{\text{sat}} = I : m^{\infty}\), is the ideal of inertia forms. In this context, the ideal \(m\) is the (geometrically) irrelevant ideal of the projective space of dimension \(n\) which is associated to \(C\). The elements in \(I\) being trivially inertia forms, \(I^{\text{sat}}/I\) is the natural quotient to study. It turns out that the Jacobian determinant of the \(F_i\)’s is a generator of the graded component of \(I^{\text{sat}}/I\) in degree \(\delta := d_0 + \cdots + d_n - (n + 1)\) and their resultant is a generator of \(I^{\text{sat}}/I\) in degree 0. In order to unravel the structure of \(I^{\text{sat}}/I\) in degrees smaller than \(\delta\), the formalism of Sylvester forms has been introduced and studied by Jouanolou in [Jou97]. His ideas were based on the fact that for each \(\mu = (\mu_0, \ldots, \mu_n) \in \mathbb{N}^{n+1}\) such that \(|\mu| := \sum \mu_i < \min d_i\), it is possible to find a decomposition of each polynomial \(F_i\) of the form

\[
F_i = \sum_j x_j^{\mu_j+1} F_{i,j},
\]

and hence to consider the determinant \(\det(F_{i,j})_{0 \leq i, j \leq n}\). This latter is called a Sylvester form of the \(F_i\)’s and denoted by \(\text{Sylv}_\mu\). Independently of the decomposition, the class of \(\text{Sylv}_\mu\), which is denoted by \(\text{sylv}_\mu\), gives a

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nonzero element in \((I^{\text{sat}}/I)_{\delta - |\mu|}\) and additionally, \((I^{\text{sat}}/I)_{\delta - |\mu|}\) is a free \(A\)-module which can be generated by Sylvester forms of degree \(\delta - |\mu|\). This result is a consequence of a duality property between Sylvester forms and monomials; namely, for all \(\nu < \min d_i\) we have an isomorphism of \(A\)-modules

\[
(I^{\text{sat}}/I)_{\delta - \nu} \simeq \text{Hom}_A(C_\nu, A).
\]

It is made explicit by the equalities

\[
x^{\mu'} \, \text{sylv}_\mu = \begin{cases} 
\text{sylv}_0 & \mu = \mu' \\
0 & \mu \neq \mu'
\end{cases}
\]

where \(\text{sylv}_\mu\) is a generator of \((I^{\text{sat}}/I)_{\delta}\), which is equal, up to a nonzero multiplicative constant, to the class of the Jacobian determinant of the \(F_i\)'s; see [Jou97, §3.10].

The definition and main properties of Sylvester forms have been recently extended to the case of \(n + 1\) generic multi-homogeneous polynomials, i.e. of polynomials defining hypersurfaces over a product of projective spaces of total dimension \(n\); see [BCN22]. In this paper, we develop the theory of Sylvester forms in the general setting of homogeneous polynomials in the coordinate ring of a projective toric variety \(X_\Sigma\). In addition, to illustrate the importance of these forms in elimination theory, we also provide applications to the construction of elimination matrices for overdetermined polynomial systems, and to the computation of toric resultants and toric residues. As far as we know, these applications provide new results in the context of multi-homogeneous polynomial systems as well.

Let \(k\) be an algebraically closed field and \(X_\Sigma\) be a \(n\)-dimensional projective toric variety over \(k\) given by a complete fan \(\Sigma\) in a lattice \(N\). Let \(R\) be the homogeneous coordinate ring of \(X_\Sigma\) over \(k\), also known as the Cox ring of \(X_\Sigma\), which is graded using the combinatorics of \(\Sigma\); see Section 2 or [Cox95] for more details. Choosing a \(n\)-dimensional cone \(\sigma \in \Sigma(n)\), the variables of \(R\) can be written as \(x_1, \ldots, x_n\), which are associated to \(\sigma\), and \(z_1, \ldots, z_r\) for the remaining ones. With these notations, a homogeneous polynomial in \(X_\Sigma\) of degree \(\alpha \in \text{Cl}(X_\Sigma)\), the class group of \(X_\Sigma\), is an element in the graded component \(R_\alpha\) of \(R\) in degree \(\alpha\); it is a \(k\)-linear combination of monomials \(x^{\mu} := x_1^{\mu_1} \cdots x_n^{\mu_n} z_1^{\nu_1} \cdots z_r^{\nu_r}\) of degree \(\alpha\). Now, the generic homogeneous polynomial of degree \(\alpha\) is the polynomial of the form \(\sum_{\sigma \in \Sigma} c_{\mu, \sigma} x^{\mu}\) where the coefficients \(c_{\mu, \sigma}\) are seen as variables. Therefore, being given \(n + 1\) degrees \(\alpha_0, \ldots, \alpha_n\), the associated generic homogeneous polynomial system over \(X_\Sigma\) corresponds to the \(n + 1\) homogeneous polynomials

\[
F_i = \sum_{x^\mu \in R_{\alpha_i}} c_{i, \mu} x^{\mu} \in C = A \otimes_k R = A[x_1, \ldots, x_n, z_1, \ldots, z_r], i = 0, \ldots, n,
\]

where \(A\) is the universal ring of coefficients over \(k\), i.e. \(A = k[\sigma, x^{\mu} : R_{\alpha_i}, i = 0, \ldots, n]\). We define the ideals \(I = (F_0, \ldots, F_n)\) and \(b = (\tilde{x}^{\sigma} : \tilde{x}^{\sigma} = \prod_{\sigma \in \Sigma} x_{\sigma}, \sigma \in \Sigma(n))\), where the latter is the irrelevant ideal of \(X_\Sigma\). The saturation of \(I\) is thus the ideal of \(C\) defined as \(I^{\text{sat}} = (I : b^\infty)\).

As the first main result of this paper, we prove the following duality property, which is a generalization of [BCN22, Theorem A] to the case of a projective toric variety; see Theorem 3.1. We set \(\delta = \alpha_0 + \cdots + \alpha_n - \text{K}_X \in \text{Cl}(X_\Sigma)\), where \(\text{K}_X\) is the anticanonical class of \(X_\Sigma\).

**Theorem.** Let \(X_\Sigma\) be a projective toric variety and let \(\nu \in \text{Cl}(X_\Sigma)\). There exists a region \(\Gamma \subset \text{Cl}(X_\Sigma)\) such that if \(\delta - \nu \not\in \Gamma\) then

\[
(I^{\text{sat}}/I)_{\delta - \nu} \simeq \text{Hom}_A((C/I)_{\nu}, A).
\]

In cases where \((C/I)_{\nu} = C_{\nu}\), the above duality implies that \((I^{\text{sat}}/I)_{\delta - \nu}\) is a free \(A\)-module (see Corollary 3.1) and a natural question is to find explicit bases. This is what we provide as the second main contribution of this paper, by introducing toric Sylvester forms. We first prove that for suitable \(\nu\) and \(x^{\mu} \in R_{\nu}\), each generic homogeneous polynomial \(F_i\) can be decomposed into \(n + 1\) generic homogeneous polynomials \((F_{ij})_{0 \leq j \leq n}\), similarly to (1.1). The existence of such decompositions requires a certain property on \(X_\Sigma\) that we introduce and call positivity property; see Theorem 2.1. Then, from these decompositions we define toric Sylvester forms as the determinants \(\text{sylv}_\mu := \det(F_{ij}) \in I^{\text{sat}}_{\delta - \nu}\) and show that their classes in \((I^{\text{sat}}/I)_{\delta - \nu}\), denoted by \(\text{sylv}_\mu\), are independent of the choice of decompositions. Finally, we get the following explicit duality property; see Theorem 4.1.

**Theorem.** Let \(X_\Sigma\) be a projective toric variety. Then, under suitable conditions on \(\nu \in \text{Cl}(X_\Sigma)\), for any pair \(x^{\mu}, x^{\mu'} \in R_{\nu}\), we have

\[
x^{\mu} \, \text{sylv}_{\mu'} = \begin{cases} 
\text{sylv}_0 & \mu = \mu' \\
0 & \text{otherwise}
\end{cases}
\]

where \(\text{sylv}_0\) is a generator of \((I^{\text{sat}}/I)_{\delta}\). Therefore, \(\{\text{sylv}_\mu\}_{x^{\mu} \in R_{\nu}}\) yields an \(A\)-basis of \((I^{\text{sat}}/I)_{\delta - \nu}\).
In the rest of the paper, we provide three applications to illustrate the practical interest of toric Sylvester forms in elimination theory. The first application deals with elimination matrices. An important question in elimination theory is the study of matrices $M$ with entries in $A$ such that i) their rank drops when specialized in $k$ to a system having solutions in $X_\Sigma$ and ii) when specialized in $k$ to a system having finitely many solutions, their corank coincides with this number of solutions. The first property can be related to resultant theory whilst the second one has practical implications in solving 0-dimensional polynomial systems; see [CDS97; GZK94] for i) and [BT22; EM99] for ii). In this paper, we introduce a new family of elimination matrices by adding to a classical Macaulay-block matrix in some degree $\alpha \in \text{Cl}(X_\Sigma)$, a block-matrix built from the toric Sylvester forms of degree $\alpha$; see Definition 5.1. We call these matrices hybrid elimination matrices and prove their main properties in Theorem 5.1.

Compared with the more classical Macaulay matrices, this new family yields more compact matrices that can still be used for solving 0-dimensional polynomial systems. In addition, we also prove that the construction of hybrid elimination matrices can be extended to polynomial systems with more than $n + 1$ polynomials, providing this way new matrices that can be used to solve overdetermined 0-dimensional polynomial systems; see Theorem 5.3.

Our second application concerns the computation of sparse resultants. A classical result in elimination theory is that the sparse resultant can be computed as the determinant of certain graded components of the Koszul complex built from the considered polynomial system; see [GZK94]. Generalizing a construction of Cattani, Dickenstein and Sturmfels in [CDS97, §2] using the so-called toric Jacobian, we modify the usual Koszul complex by incorporating the Sylvester forms in its last differential and prove that the determinant of some suitable graded parts of this new complex is equal to the sparse resultant, up to a nonzero multiplicative constant in $k$; see Theorem 6.1. This result yields new compact formulas for computing the sparse resultant as a determinant of a complex.

Our third application deals with the computation of toric residues. The toric residue of the generic polynomial system (1.2) was defined by Cox in [Cox96]. It is a map that sends any polynomial in $(C/I)_\delta$ to the fraction field $K(A)$ of $A$. The computation of this residue map by means of determinants has been an active research topic with many contributions, including [Jou97; DK05; CCD97]. In this paper, using toric Sylvester forms we construct matrices whose determinants are used to compute the residue of a product of two forms $PQ$, where $P \in C_\nu$, $Q \in C_{\delta-\nu}$ and $\nu \in \text{Cl}(X_\Sigma)$. This formula can be seen as an extension of a similar formula proved by Jouanolou in the case $X_\Sigma = \mathbb{P}^n$ [Jou97, Proposition 3.10.27]. It yields more compact matrices in comparison with the formula proved by D’Andrea and Khetan in [DK05, Theorem 5.1] for computing the toric residue of a form of degree $\delta$.

The paper is organised as follows. In Section 2, we present all the tools of toric geometry that are needed in the rest of the paper. In particular, we prove the existence of decompositions of forms in a projective toric varieties $X_\Sigma$ satisfying to a certain positivity property. In Section 3, we show that the claimed duality property holds outside a region $\Gamma \subset \text{Cl}(X_\Sigma)$ which depends on the supports of the local cohomology modules of the Cox ring. In Section 4, we define Sylvester forms and show that they give an $A$-basis of $(I^{\text{tor}}/I)^{\delta-\nu}$ for certain degrees $\nu \in \text{Cl}(X_\Sigma)$. In Section 5, we introduce hybrid elimination matrices when $X_\Sigma$ is a smooth projective toric variety; we prove that, under mild conditions, the computation of the local cohomology modules is not necessary and we extend these results to overdetermined systems. In Section 6, we prove that the determinant of certain graded parts of a modified Koszul complex in a region $\Gamma_{\text{Res}} \subset \text{Cl}(X_\Sigma)$ is equal to the sparse resultant, up to a nonzero multiplicative constant in $k$. Finally, in Section 7 we prove a new formula for computing the toric residue of a product of two forms.

2 Preliminaries on toric geometry

In this section, we set our notation and briefly review some material we will use from toric geometry; we refer the unfamiliar reader to the book by Cox, Little and Schenck [CLS12] which offers a detailed treatment of this topic. At the end of the section, we also prove a decomposition result that we will use in order to introduce toric Sylvester forms later on.

Toric varieties. Let $k$ be an algebraically closed field and let $M$ be a lattice of rank $n$. We denote by $N = \text{Hom}(M, \mathbb{Z})$ the dual of $M$, by $T_N = N \otimes k^\times$ the algebraic torus associated to $N$ and we set $M_\mathbb{R} = M \otimes \mathbb{R}$, which are two vector spaces over the real numbers. Let $A = \{m_1, \ldots, m_s\} \subset M$ be a finite set of lattice points and consider its convex hull $\Delta = \text{conv}(A) \subset M_\mathbb{R}$. The projective toric variety $X_\Delta$ can be defined as the algebraic closure of the image of the map

$$\Phi_A : T_N \to \mathbb{P}^{k-1} : t := (t_1, \ldots, t_n) \mapsto (t^{m_1} : \cdots : t^{m_s}).$$

This variety is called toric because the group action of $T_N$ on itself extends to $X_\Delta$ with good geometric properties. Another definition of $X_\Delta$, more intrinsic, can be stated from the normal fan $\Sigma \subset N$ of $\Delta$ so that this variety is also denoted by $X_\Sigma$; see [Cox95].
Example 2.1. If $\Delta$ is a product of simplices of the form $\Delta_{n_j} = \{ t \in \mathbb{R}^{n_j} : t_k \geq 0, \sum_{k=0}^{n_j} t_k \leq 1 \}$ for $j = 1, \ldots, s$, then $X_\Sigma = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$.

The geometric properties of $X_\Sigma$ are deeply connected with the combinatorial properties of the fan $\Sigma$. Thus, $X_\Sigma$ is a smooth variety if and only if $\Sigma$ is smooth, which means that the minimal generators of all cones $\sigma \in \Sigma$ are part of a basis of $N$. Similarly, $X_\Sigma$ is a complete variety if and only if $\Sigma$ is complete, i.e. if $N_\Sigma = \cup_{\sigma \in \Sigma} \sigma$; see [CLS12, Theorem 3.1.10].

We denote by $\Sigma(r)$ the set of $r$-dimensional cones of $\Sigma$, which are also called rays when $r = 1$. We assume that the generators of the rays $u_\rho \in N$ for $\rho \in \Sigma(1)$ are primitive and span the vector space $N_\Sigma$; by [CLS12, Corollary 3.3.10], this condition is equivalent to the toric variety $X_\Sigma$ having no torus factors. Moreover, as $\Delta$ is a bounded polytope, its normal fan $\Sigma$ is complete and its cones are strongly convex. Under these assumptions, $\Sigma(1)$ contains at least $n + 1$ rays.

Let $\text{Cl}(X_\Sigma)$ be the class group of $X_\Sigma$. There is a short exact sequence

$$0 \to M \xrightarrow{\mathcal{P}} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\pi} \text{Cl}(X_\Sigma) \to 0,$$

(2.1) where $M$ is an $n \times (n + r)$ matrix whose rows are the generators of the rays in $\Sigma(1)$ and $\pi$ is chosen accordingly to be a cokernel matrix; see [CLS12, Theorem 4.1.3]. A Cartier divisor $D$ is nef if and only if $D$ is generated by global sections [CLS12, Theorem 6.3.12], and ample if and only if the normal fan of its polytope is $\Sigma$ [CLS12, Proposition 7.2.3]. We notice that if $X_\Sigma$ is a complete projective toric variety then there exists at least one ample divisor; see [Har77, Chapter 2, Theorem 7.10]. Because of that, from now on we assume that $X_\Sigma$ is projective.

The Cox ring and a positivity property. The homogeneous coordinate ring of a toric variety $X_\Sigma$, also known as the Cox ring, is the ring $R = k[x_\rho, \rho \in \Sigma(1)]$ which is $\text{Cl}(X_\Sigma)$-graded by the map $\pi$ in (2.1): $R = \oplus_{\alpha \in \text{Cl}(X_\Sigma)} R_\alpha$, with $R_\alpha = H^0(X_\Sigma, \mathcal{O}_\Sigma(D))$ where $D$ is a torus-invariant Weil divisor such that $[D] = \alpha$ and $\mathcal{O}_\Sigma$ is the structural sheaf of $X_\Sigma$; see [Cox95]. We will use the following notation for the variables of the Cox ring: being given $\sigma \in \Sigma(n)$ a maximal smooth cone, we will denote by $x_1, \ldots, x_r$ the variables associated to the rays $\rho \in \sigma(1)$ and by $z_1, \ldots, z_s$ the remaining variables. According to the choice of $\sigma$, one can always write a matrix of the map $\pi$ in (2.1) under the form

$$\pi = (\mathcal{P} \quad 1d_n),$$

(2.2) $\mathcal{P}$ being a block matrix $(\mathcal{P}_{i,j})_{1 \leq i \leq r, 1 \leq j \leq n}$, whose rows correspond to the relations between $u_\rho$ for $\rho \notin \sigma$ and the basis given by $\sigma$.

Definition 2.1. We say that $X_\Sigma$ has the positivity property with respect to $\sigma$ if there exists a smooth cone $\sigma \in \Sigma(n)$ such that a matrix of the map $\pi$ can be written as in (2.2) with the additional condition that $\mathcal{P}_{i,j} \geq 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n$.

The above positivity property can be understood as saying that the vector $-u_\rho$ belongs to $\sigma$ for all $\rho \notin \sigma(1)$, as each row of $\pi$ corresponds to the identity $u_\rho + \sum_{\rho_\sigma \in \sigma(1)} \mathcal{P}_{i,j} u_{\rho_\sigma} = 0$. A first observation is that not all smooth toric varieties satisfy this property, as shown in the following example.

Example 2.2. Let $\Sigma$ be the complete smooth fan in $\mathbb{R}^2$ with the following rays:

$$\rho_1 = (1, 0), \rho_2 = (0, 1), \rho_3 = (-1, 1), \rho_4 = (-1, 0), \rho_5 = (-1, -1), \rho_6 = (0, -1).$$

It is straightforward to check that for every maximal $\sigma \in \Sigma(2)$, there is $\rho \notin \sigma(1)$ such that $-u_\rho \notin \sigma$.

On the other hand, most of the toric varieties that are of interest for our applications do satisfy the positivity property.

Lemma 2.1. Any product of toric varieties having the positivity property with respect to some $\sigma$ has the positivity property with respect to some other $\sigma'$.

Proof. The cones of the fans in these varieties must be a union of cones of each factor. If one considers $\sigma'$ to be the union of all the cones with respect to which the factors have the positivity property, it must have this property as well.

Example 2.3. The projective space $\mathbb{P}^n$ satisfies the positivity property as the map $\pi$ can be written as $\pi = (1 \cdots 1)$. Therefore, any product of projective spaces has the positivity property by Lemma 2.1. Another classical family of smooth toric varieties are Hirzebruch surfaces $\mathcal{H}_r \subset \mathbb{R}^2$: for each $r \in \mathbb{Z}_{>0}$, it is the variety corresponding to the fan $\Sigma_r$ with rays

$$\rho_1 = (1, 0), \rho_2 = (0, 1), \rho_3 = (-1, r), \rho_4 = (0, -1).$$

Hirzebruch surfaces are smooth and have the positivity property with respect to $\sigma = (\rho_1, \rho_2)$ as $\pi$ can be written as

$$\pi = \begin{pmatrix} 1 & r & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$
**Generic sparse homogeneous polynomial systems.** Let $\Delta_0, \ldots, \Delta_n$ be rational polytopes in $\mathbb{R}^k$. Let $\Sigma$ be the normal fan of the Minkowski sum $\Delta = \sum_{i=0}^n \Delta_i$ and $X_\Sigma$ be the corresponding projective toric variety. Suppose that $X_\Sigma$ has the positivity property with respect to some cone $\sigma \in \Sigma(n)$. The polytopes $\Delta_i$ can be seen as elements $a_i = (a_{i,j}) \in \mathbb{Z}^{\Sigma(1)}$ using the following facet presentation:

$$\Delta_i = \{ m \in \mathbb{Z}^k : (m, a_{i,j}) \geq -a_{i,j}, \rho_j \in \Sigma(1) \}, \quad i = 0, \ldots, n. \tag{2.3}$$

These presentations relate each of the polytopes $\Delta_i$ to Weil divisors that can be written as $\sum_j a_{i,j}D_j$ where $D_j$ is the torus invariant divisor associated with the ray $\rho_j$. Using (2.1), we see that two polytopes that map to the same class in $\text{Cl}(X_\Sigma)$ are translates of each other. For each class $\alpha \in \text{Cl}(X_\Sigma)$, we choose this presentation so that the vertex associated to the cone $\sigma$ is $0 \in M$. In particular, this implies that $a_{i,j} = 0$ for $\rho_j \in (1).

Let $\alpha_0, \ldots, \alpha_n$ be nef classes in $\text{Cl}(X_\Sigma)$ associated to $\Delta_0, \ldots, \Delta_n$, and $R_{\alpha_0}, \ldots, R_{\alpha_n}$ be the corresponding graded components in the Cox ring, respectively. These graded components are finite $k$-vector spaces and have a monomial basis given by $x^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n} z_1^{\mu_{n+1}} \cdots z_r^{\mu_r} \in R$. Let $A = k[c_{i,m} : x^\mu \in R_{\alpha_i}, i = 0, \ldots, n]$ and $C = A[x_1, \ldots, x_n, z_1, \ldots, z_r]$. A generic homogeneous sparse polynomial system of degree $\alpha_0, \ldots, \alpha_n$ is the system obtained from the polynomials

$$F_i = \sum_{x^\mu \in R_{\alpha_i}} c_{i,m} x^\mu \in C = A[x_1, \ldots, x_n, z_1, \ldots, z_r], \quad i = 0, \ldots, n. \tag{2.4}$$

The ring $C$ can be interpreted as the Cox ring of the toric variety $X_\Sigma \times_k \text{Spec}(A)$ over generic coefficients and its graded components are given by $C_{\alpha} = R_{\alpha} \otimes_k A$.

If the system is dehomogenized by setting $z_1 = \cdots = z_r = 1$, the Newton polytopes $F_i$ is $\Delta_i$ for $i = 0, \ldots, n$. Conversely, the polynomials $F_0, \ldots, F_n$ can be defined as the homogenization of the system of polynomials $F_0 = \cdots = F_n = 0$ with supports in the subsets $A_i = \Delta_i \cap M$ for $i = 0, \ldots, n$:

$$\tilde{F}_i = \sum_{m \in A_i} c_{i,m} x^m \in \tilde{C} = A[x_1, \ldots, x_n] \to F_i = \sum_{m \in A_i} c_{i,m} x^{m+a_i} \in C = A[x_1, \ldots, x_n, z_1, \ldots, z_r]$$

where $F$ and $a_i$ appear in (2.1) and (2.3), respectively. By homogenizing the monomials associated to the lattice points in $A_i$, we can choose a monomial basis of $R_{\alpha_i}$ using $\mu = Fm + a_i$; see [BT22, Section 2.2] for more details about homogenization and dehomogenization of sparse polynomial systems.

**Torsion and local cohomology.** From the fan $\Sigma$ of a toric variety $X_\Sigma$, the irrelevant ideal $b$ of its homogeneous coordinate ring $C = A[x_p, p \in \Sigma(1)]$ is defined as

$$b = (x^\sigma \text{ such that } \sigma \in \Sigma(n)), \quad \text{where } x^\sigma = \prod_{\rho \notin \sigma(1)} x_p.$$ 

The $b$-torsion of a graded $C$-module $S$ is classically defined as

$$\Gamma_b(S) = \{ a \in S : b^k a = 0, k \in \mathbb{N} \}$$

and the local cohomology modules $H^j_b(S)$ are then the derived functors of $S \to \Gamma_b(S)$. When the module $S$ is a quotient ring $B = C/I$ for $I = (F_0, \ldots, F_n)$ the ideal defined by (2.4), the 0-th local cohomology is $H^0_b(B) = I^\text{sat}/I$ where $I^\text{sat}$ denotes the saturation of the ideal $I$ with respect to the irrelevant ideal of $C$, i.e. $I^\text{sat} := (I : b^\infty) = \{ p \in C : \exists k \in \mathbb{Z}, b^k p \subset I \}$.

Local cohomology modules are strongly related to sheaf cohomology modules. More precisely, let $S$ be a finitely generated $\text{Cl}(X_\Sigma)$-graded $R$-module with associated coherent sheaf $S$ in $X_\Sigma$ and $\alpha \in \text{Cl}(X_\Sigma)$. If $p \geq 0$, then

$$H^p_b(S)_\alpha \simeq H^{p-1}(X_\Sigma, S(\alpha)). \tag{2.5}$$

Furthermore, the following exact sequence holds:

$$0 \to H^p_b(S)_\alpha \to S_\alpha \to H^0(X_\Sigma, S(\alpha)) \to H^{p-1}_b(S)_\alpha \to 0;$$

see [CLS12, Theorem 9.5.7] for proofs. If $S = R$, then $R_\alpha = H^0(X_\Sigma, \mathcal{O}_\Sigma(\alpha))$ and therefore

$$H^p_b(R) = H^p_b(S)_\alpha = 0.$$ 

which implies that $H^p_b(C) = 0$ for $i = 0, 1$. 

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Notation 2.1. For the sake of simplicity in the notation, for any Cartier divisor $D$ and any integer $p \geq 0$, we will write $H^p(X_S, \alpha)$ in place of $H^p(X_S, \mathcal{O}_S(D))$, where $\alpha = [D] \in \text{Cl}(X_S)$. The theorems of Batyrev-Borisov [BB11, Theorem 2.5] and Demazure [Dem70, Corollary 1] are the main tools that we will rely on in order to analyze the vanishing of sheaf cohomology modules of toric varieties. We recall that, if $\alpha \in \text{Cl}(X_S)$ is a nef class, then
- $H^p(X_S, \alpha) \simeq 0$ for $p > 0$ (Demazure),
- $H^p(X_S, -\alpha) \simeq 0$ if $p \neq \dim \Delta_\alpha$ (Batyrev-Borisov),
- $H^p(X_S, -\alpha) \simeq \oplus_{m \in \text{RelSim}(\Delta_\alpha)} \mathcal{O}_X(-m)$ if $p = \dim \Delta_\alpha$ (Batyrev-Borisov).

Another important result we will use is the toric version of Serre duality: for any $\alpha \in \text{Cl}(X_S)$ and any integer $p \geq 0$, $H^p(X_S, \alpha) \cong H^{n-p}(X_S, -K_X - \alpha)^{\vee}$, where $K_X$ the anticanonical class in $\text{Cl}(X_S)$; see [CLS12, Theorem 9.2.10] for a proof.

Hilbert functions and the Grothendieck-Serre formula. Let $X_S$ be a smooth projective toric variety, $R$ be its Cox ring and $S$ a graded $R$-module. The Hilbert function of $S$ is defined as
$$\text{HF}(S, -) : \text{Cl}(X_S) \rightarrow \mathbb{Z}_{\geq 0} \quad \alpha \mapsto \text{HF}(S, \alpha) := \dim_k(S_\alpha).$$

For $\alpha \gg 0$ (component-wise), this function becomes a (multivariate) polynomial which is called the Hilbert polynomial and is denoted by $\text{HP}(S, \alpha)$; see [MS03, Lemma 2.8].

Remark 2.1. If $S = R/J$ with $J$ an ideal defining a 0-dimensional subscheme in $X_S$, then the Hilbert polynomial of $S$ is a constant which is equal to the number of points counted with multiplicity.

An important relation between the Hilbert function, the Hilbert polynomial and local cohomology modules is given by the Grothendieck-Serre formula; see [MS03, Proposition 2.14]: for any $\alpha \in \text{Cl}(X_S)$,
$$\text{HF}(S, \alpha) = \text{HP}(S, \alpha) + \sum_{i=0}^{d} (-1)^{i} \dim_k H^{i}_{\alpha}(S)_{\alpha}. \quad (2.7)$$

Existence of some decompositions. To conclude this preliminary section, we prove the existence of certain decompositions of the polynomials $F_i$, $i = 0, \ldots, n$, that we will use in Section 4 in order to define toric Sylvester forms. For sake of clarity, we state the following theorem over $k$ and with polynomials in $R$, but this result extends naturally to generic homogeneous polynomials, as defined in (2.4).

Let $J$ be an ideal of $R$ generated by homogeneous polynomials $f_0, \ldots, f_n$ of degree $\alpha_0, \ldots, \alpha_n$, respectively.

Theorem 2.1. Let $X_S$ be a projective toric variety over $k$ with the positivity property with respect to $\sigma$. Let $\nu \in \text{Cl}(X_S)$ be a nef class and $\Delta_\nu$ be the corresponding polytope, written as in (2.3), satisfying $0 \leq \nu_j < \min_i \alpha_{i,j}$ for $\rho_1 \not\in \sigma(1)$. Then, the following two properties hold:
- $R_\nu = (R/J)_\nu$.
- For every $x^\mu \in R_\nu$ and $f_i \in R_{\alpha_i}$ and $i = 0, \ldots, n$, there exists a decomposition of the form
  $$f_i = x^{\mu_{n+1}} f_{i,0} + x^{\mu_{n+2}} f_{i,1} + \cdots + x^{\mu_{n+r}} f_{i,r},$$

  where $f^\mu_{i,j}$, $i, j = 0, \ldots, n$, are homogeneous polynomials in $R$.

Proof. The graded quotient map $R_\nu \rightarrow (R/J)_\nu$ is surjective. Using the degree constraint, its kernel must be zero, giving the first property. On the other hand, we have to prove that for $x^\mu = x^{\mu_1} \cdots x^{\mu_n} \in R_{\nu}$ (recall: $\mu = \nu_m + \nu_f$ for $m \in A_0$), every monomial $x^{\nu_m + a_i} \in R_\nu$ that is not divided by $x^{\mu_{n+1}} \cdots x^{\mu_{n+r}}$ is divided by some of the $x^{\mu_{n+1}} \cdots x^{\mu_{n+1}}$. Using the toric homogenization, the fact that the first does not happen implies that:
$$\langle u_{n+j}, m \rangle + a_{i,j} \leq \mu_{n+j} \quad j = 1, \ldots, r.$$ Considering any of these $j$ and using that $\nu_j = \mu_{n+j} + \sum_{k=1}^{n} \mathcal{P}_{k,j} \mu_k$, through the map $\pi$ defined in Definition 2.1, we get:
$$\langle u_{n+j}, m \rangle + \min_{a_{i,j}} \leq \mu_{n+j} \quad \Rightarrow \quad \langle u_{n+j}, m \rangle + \nu_j < \mu_{n+j} \quad \Rightarrow \quad \langle u_{n+j}, m \rangle + \sum_{k=1}^{n} \mathcal{P}_{k,j} \mu_k < 0 \quad \Rightarrow \quad \sum_{k=1}^{n} \mathcal{P}_{k,j} (\mu_k - \langle u_k, m \rangle) < 0$$

As $X_S$ has the positivity property, then there must be $k \in \{1, \ldots, n\}$ such that $\mu_k - \langle u_k, m \rangle < 0$. This implies that $x^{\mu_{n+1}}$ divides $x^{\nu_m + a_i}$ as the coefficient of $x_k$ in this monomial is $F_{u_k} + a_{i,k} = \langle u_k, m \rangle$. □
This theorem extends trivially to a decomposition of the generic homogeneous sparse polynomials $F_i$ over $X_\Sigma \times_k A$, where $F_i^{\nu}$ are homogeneous polynomials in $C$.

**Corollary 2.1.** If $\Delta_i$ is $n$-dimensional for all $i = 0, \ldots, n$, then the previous theorem holds for $\nu = 0$.

**Proof.** If $\Delta_i$ is $n$-dimensional, then $a_{i,j} > 0$ for $j > n$. Therefore, $0 < \min a_{i,j}$ for $p_j \notin \sigma(1)$, implying the result. \hfill $\square$

3 A duality theorem

In this section, some graded components of the saturation of the ideal generated by $n+1$ generic homogeneous sparse polynomials over a projective toric variety $X_\Sigma$ of dimension $n$, are analyzed via a duality property. We assume that $X_\Sigma$ has the positivity property with respect to $\sigma \in \Sigma(n)$ and we take again the notation of (2.4): $F_0, \ldots, F_n$ are the generic homogeneous polynomials of degree $\alpha_0, \ldots, \alpha_n$, respectively; they are of the form

$$F_i = \sum_{\mu \in C_i} c_{i,\mu} x^\mu \in C = A[x_1, \ldots, x_n, z_1, \ldots, z_r].$$

(3.1)

As a preliminary result, we show that $F_0, \ldots, F_n$ form a regular sequence outside $V(b) \subset \text{Spec}(C)$.

**Lemma 3.1.** The homogeneous generic polynomials $F_0, \ldots, F_n$ define a regular sequence in the localization ring $C_\sigma := C_{\sigma,\bullet}$ for any $\sigma \in \Sigma(n)$.

**Proof.** We claim that $F_0$ is a nonzero divisor in $C$. This follows as a corollary of Dedekind-Mertens Lemma [BJ14, Corollary 2.8], which says that $F$ is a nonzero divisor in $A[x_1, \ldots, x_n]$ if its content ideal is a nonzero divisor in $A$. The content ideal is generated by the coefficients $c_{0,\mu}$ for $x^\mu \in C_0$ and they are all nonzero divisors. Therefore, $F_0$ is a nonzero divisor also in $C_\sigma$ for all $\sigma \in \Sigma(n)$.

Now, as $\Sigma$ always refines the normal fan of $\Delta_i$, we can always find a vertex $a_\sigma \in A_1$ corresponding to the maximal cone $\sigma \in \Sigma(n)$. Let $c_{i,\sigma}$ be the coefficient associated to this vertex. Then, similarly to [BCN22, Lemma 3.2], for any $t \in \{1, \ldots, n-1\}$ there is an isomorphism of algebras

$$B_t^\sigma = (A[x_1, \ldots, x_n, z_1, \ldots, z_r]/(F_0, \ldots, F_t))_\sigma \overset{\sim}{\rightarrow} (A^t_\sigma[x_1, \ldots, x_n, z_1, \ldots, z_r]_\sigma$$

where $A_\sigma^t = k[c_{i,\mu}]$, $c_{i,\mu} \neq c_{i,\sigma}$, $0 \leq i \leq t$, and which maps $c_{i,\sigma} x^{\alpha_{i,\sigma}}$ to $F_1 - c_{i,\sigma} x^{\alpha_{i,\sigma}}$ and which leaves invariant the other variables and coefficients. Applying again Dedekind-Mertens Lemma as above, we deduce that $F_{t+1}$ is a nonzero divisor in $(A^t_\sigma[x_1, \ldots, x_n, z_1, \ldots, z_r]_\sigma$, and therefore in $B_t^\sigma$.

Next, we consider the two canonical spectral sequences associated with the Čech-Koszul double complex $C^\bullet_\bullet(K_\bullet(F))$, where $K_\bullet(F)$ denotes the Koszul complex of the sequence of homogeneous polynomials $F_0, \ldots, F_n$ in $C$. The terms of this Koszul complex are graded free $C$-modules and we denote their homology modules by $H_\mu$ for simplicity in the notation. If we start taking homologies horizontally, the second page is:

$$
\begin{array}{cccccccc}
H^0_b(H_{n+1}) & H^0_b(H_n) & H^0_b(H_{n-1}) & \cdots & H^0_b(H_0) & \equiv I^{\text{nat}}/I \\
0 & 0 & 0 & \cdots & H^0_b(H_0) \\
0 & 0 & 0 & \cdots & H^0_b(H_0) \\
0 & 0 & 0 & \cdots & H^0_b(H_0) \\
0 & 0 & 0 & \cdots & H^0_b(H_0) \\
0 & 0 & 0 & \cdots & H^0_b(H_0) \\
\end{array}
$$

The vanishing of the local cohomology modules $H^i_b(H_j)$ for $i > 0$ and $j > 0$ follows from the fact that the $F_i$’s form a regular sequence after localization by a generator of $b$ by Lemma 3.1. In addition, we deduce that $H_\mu$ are geometrically supported on $V(b)$ for all $p > 0$ by a classical property of Koszul complexes, and hence that $H^0_b(H_p) = H_\mu$ for all $p > 0$.
On the other hand, if we start taking homologies vertically, we obtain the following first page:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H^{n-1}_b(C(-\sum \alpha_i)) & H^{n-1}_b(\oplus_k C(-\sum_{j\neq k} \alpha_j)) & H^{n-1}_b(\oplus_{k,k'} C(-\sum_{j\neq k,k'} \alpha_j)) & \cdots & H^{n-1}_b(C) \\
H^n_0(C(-\sum \alpha_i)) & H^n_0(\oplus_k C(-\sum_{j\neq k} \alpha_j)) & H^n_0(\oplus_{k,k'} C(-\sum_{j\neq k,k'} \alpha_j)) & \cdots & H^n_0(C) \\
H^{n+1}_b(C(-\sum \alpha_i)) & H^{n+1}_b(\oplus_k C(-\sum_{j\neq k} \alpha_j)) & H^{n+1}_b(\oplus_{k,k'} C(-\sum_{j\neq k,k'} \alpha_j)) & \cdots & H^{n+1}_b(C)
\end{array}
\]

using that \(K_j(F) = \oplus_{|J|=J} C(-\sum_{i\in J} \alpha_i)\) for \(J \subset \{0, \ldots, n\}\). We notice that the vanishing of the two first rows follows from Remark 2.6, and also that the vanishing of \(H^n_0(C)\) for all \(p > n+1\) is a consequence of Grothendieck’s vanishing Theorem [Gro75, Theorem 3.6.5].

**Notation 3.1.** The support \(\text{Supp} S\) of a graded module \(S\) is the subset of \(\nu \in \text{Cl}(X_{\Sigma})\) such that \(S_\nu \neq 0\). We denote by \(\Gamma_1\) the support of the modules on the main diagonal, expect on the last row, and by \(\Gamma_0\) the support of the modules in the diagonal under \(\Gamma_1\), expect on the last row again, i.e.

\[
\Gamma_i = \text{Supp}(\oplus_{\delta=\alpha}^\nu H^\delta_0(K_{\nu+i-1}(F))) \quad i = 0, 1
\]

In addition, we define \(\Gamma_{\text{Res}}\) to be the support of all the cohomology modules that are appearing above the diagonal in the first page of the second spectral sequence, i.e. \(\Gamma_{\text{Res}} = \bigcup_{i<j} \text{Supp} H^\delta_{\nu+i}(K_i(F))\). Moreover, from now on, we denote by \(\delta\) the Cartier divisor class \(\alpha_0 + \cdots + \alpha_n - K_X\) where \(K_X\) denotes the anticanonical divisor of \(X_{\Sigma}\).

The comparison of the two above spectral sequences leads to the following duality.

**Theorem 3.1.** Let \(X_{\Sigma}\) be a projective toric variety and let \(\nu \in \text{Cl}(X_{\Sigma})\). If \(\delta - \nu \notin \Gamma_0 \cup \Gamma_1\) then

\[
(I^{\text{sat}}/I)_{\delta-\nu} \simeq \text{Hom}_A((C/I)_{\nu}, A).
\]

**Proof.** From the comparison of the two spectral sequences associated to the double complex \(C^i_0(K_\bullet(F))\), for all \(\delta - \nu \notin \Gamma_0 \cup \Gamma_1\) we get an isomorphism

\[
(I^{\text{sat}}/I)_{\delta-\nu} \simeq \text{Ker} \left( H^{n+1}_b(C(-\sum i \alpha_i)) \to H^{n+1}_b(\oplus_{i \neq j} C(-\sum_i \alpha_i)) \right)_{\delta-\nu}.
\]

Moreover, using toric Serre duality and the relation between sheaf and local cohomology modules, we obtain

\[
H^{n+1}_b(C(-\sum i \alpha_i))_{\delta-\nu} \simeq H^n_b(X_{\Sigma}, -\nu - K_X) \simeq H^0_b(X_{\Sigma}, \nu)^\vee \simeq \text{Hom}_A(C_\nu, A).
\]

By the same argument, we also have \(H^{n+1}(\oplus_{i \neq j} C(-\sum_i \alpha_i))_{\delta-\nu} \simeq \text{Hom}_A(I_\nu, A)\). Using the first isomorphism, we get the duality property.

**Theorem 3.1** holds if \(\delta - \nu \notin \text{Supp} H^{n+1}_b(\oplus_{i \neq j} C(-\sum_i \alpha_i))\), which is a priori not contained in \(\Gamma_0\) or \(\Gamma_1\), but if it does not belong to this support, then we get the following important consequence.

**Corollary 3.1.** Under the assumption of Theorem 3.1, if in addition \(\delta - \nu \notin \text{Supp} H^{n+1}_b(\oplus_{i \neq j} C(-\sum_i \alpha_i))\) then

\[
(I^{\text{sat}}/I)_{\delta-\nu} \simeq \text{Hom}_A(C_\nu, A).
\]

In particular, \((I^{\text{sat}}/I)_{\delta-\nu}\) is a free \(A\)-module whose rank is equal to \(\text{rank}_A C_\nu\).

We notice that the additional condition in the above corollary happens precisely when \(\nu\) does not reach the generating degrees of \(I\), i.e. \(I_\nu = 0\), equivalently \(C_\nu = (C/I)_\nu\) as in (2.1). We also notice that the case \(\nu = 0\), which corresponds to the isomorphism \((I^{\text{sat}}/I)_\delta \simeq A\), was already known; see [BC93; CCD07].
4 Toric Sylvester forms

By Corollary 3.1, some graded components of $I^{\text{sat}}/I$ are free $A$-modules and hence a natural question is to provide explicit $A$-bases for them. This is precisely the goal of this section. We will first describe the graded component $(I^{\text{sat}}/I)_\delta$, which essentially follows from [CCD97], and then introduce Sylvester forms to deal with the other cases. In what follows, we take again the notation of Section 3, in particular $X_\Sigma$ is assumed to be a projective toric variety that has the positivity property with respect to a cone $\sigma$.

Following [CCD97], we find a nonzero element of $(I^{\text{sat}}/I)_\delta \simeq A$ as follows. Using Corollary 2.1, if $\Delta_0, \ldots, \Delta_n$ are $n$-dimensional, one can decompose each polynomial as

$$F_i = x_1 F_{i,1} + \cdots + x_n F_{i,n} + z_1 \cdots z_r F_{i,n+1}.$$  

(4.1)

and consider the determinant

$$\text{Sylv}_0 = \det (F_{i,j})_{0 \leq i, j \leq n}. $$

This homogeneous polynomial is called the toric jacobian; we will denote its class modulo $I$ by $\text{Sylv}_0$.

**Proposition 4.1.** The element $\text{Sylv}_0$ belongs to $(I^{\text{sat}}/I)_\delta$. Moreover, $\text{Sylv}_0$ is independent on the choices of decompositions (4.1) and the choice of $\sigma$ (as long as $X_\Sigma$ has the positivity property with respect to $\sigma$). In addition, $\text{Sylv}_0$ is a generator of $(I^{\text{sat}}/I)_\delta$, which is a free $A$-module of rank 1.

**Proof.** The fact that $\text{Sylv}_0 \in I^{\text{sat}}$ follows from $x_i \text{Sylv}_0 \in I$ for $i = 1, \ldots, n$ and $z_1 \cdots z_r \text{Sylv}_0 \in I$. The $A$-module $(I^{\text{sat}}/I)_\delta$ is free of rank 1 by Theorem 3.1 and the fact that $C_0 \simeq A$. The fact that $\text{Sylv}_0$ is nonzero is a consequence of [CCD97, Theorem 0.2] and the independence of the choice of the decomposition (4.1) is a consequence of the classical Wiebe’s lemma; see [Jou95, Proposition 3.8.1.6].

The fact that $\text{Sylv}_0 \notin I$ and the independence from $\sigma$ is proved in [CCD97, Theorem 0.2] using toric residues, namely that $\text{Residue}_F(\text{Sylv}_0) = 1$, generically. In loc. cit. the hypothesis that the $\alpha_i$’s are $\mathbb{Q}$-ample, for $i = 0, \ldots, n$, is used in order to derive the decomposition (4.1). In our context, we already derived such decomposition in Theorem 2.1 so, as claimed in [CCD97, Remark 2.12, iv], the same property holds in this case.

In order to prove that $\text{Sylv}_0$ has degree $\delta$, we find the degree of each entry $(i,j)$ of the matrix defined by the $F_{i,j}$. In (4.1), we divided a set of monomials of degree $\alpha_i$, by a monomial of degree

$$\begin{cases} 
\pi(e_j) & \text{if the monomial is } x_j 	ext{ for } j = 1, \ldots, n, \\
\pi(\sum_{k=n+1}^r e_j) & \text{if the monomial is } z_1 \cdots z_r,
\end{cases}$$

where $\{e_j\}_{j=1}^{n+r}$ is the canonical basis of $\mathbb{Z}^{n+r}$. On the other hand, the anticanonical class $K_X$ coincides with the degree of the monomial $x_1 \cdots x_n z_1 \cdots z_r$, which is equal to $\pi(\sum_{j=1}^n e_j)$. Therefore, the degree of each of the summands that constitutes the determinant is equal to:

$$\sum_{i=0}^n (\alpha_i - \pi(\sigma_{\sigma((i)}))) = \left( \sum_{i=0}^n \alpha_i - K_X \right) = \delta,$$

where $\sigma \in S_n$ is the element of the symmetric group corresponding to such summand. □

We notice that using Batyrev-Borisov theorem, the Sylvester form $\text{Sylv}_0$ is in correspondence with the unique lattice point in the interior of the polytope $\Delta_\Sigma$ associated to the anticanonical divisor $K_X$.

$$(I^{\text{sat}}/I)_\delta \simeq H^n_{\mathbb{Z}}(C(- \sum_{\alpha_i}))_\delta \simeq H^n(X_\Sigma, -K_X) \simeq \oplus_{m \in \text{Relint}(\Delta_\Sigma)} A^{X}.$$ 

We have proved that the toric Jacobian $\text{Sylv}_0$ yields an $A$-basis of $(I^{\text{sat}}/I)_\delta \simeq A$. Our next step is to construct an $A$-basis of $(I^{\text{sat}}/I)_{\delta - \nu}$ when this latter is a free $A$-module.

**Definition 4.1.** Let $X_\Sigma$ be a projective toric variety with the positivity property with respect to $\sigma$. Let $\nu \in \text{Cl}(X_\Sigma)$ be a nef class and $\Delta_\nu$ be the corresponding polytope, written as in (2.3), satisfying $0 \leq \nu_j < \min_{i,j} \alpha_{i,j}$ for $\rho_j \notin \sigma(1)$. Then, according to Theorem 2.1, being given $x^\mu \in R_\nu$, each polynomial $F_i$ can be decomposed as

$$F_i = x_1^{\mu_{1}+1} \cdots x_n^{\mu_{n}+1} F_{i,0} + x_1^{\mu_{1}+1} F_{i,1} + \cdots + x_n^{\mu_{n}+1} F_{i,n}$$

(4.2)

and the toric Sylvester form $\text{Sylv}_\mu$ is then defined as the determinant

$$\text{Sylv}_\mu = \det(F_{i,j})_{0 \leq i, j \leq n}.$$ 

The class of $\text{Sylv}_\mu$, modulo $I$ is denoted by $\text{Sylv}_\mu$. 


Theorem 4.1. Let \( X_\Sigma \) be a projective toric variety. Then, for any \( \nu \in \text{Cl}(X_\Sigma) \) satisfying the hypotheses of Theorem 2.1 and any pair \( x^\mu, x^\mu' \in R_\nu \):

\[
x^\mu' \text{sylv}_\mu = \begin{cases} \text{sylv}_0 & \mu = \mu' \\ 0 & \text{otherwise.} \end{cases}
\]

The element \( \text{sylv}_\mu \) belongs to \((I^{\text{sat}})_{\delta - \nu}\). Its class \( \text{sylv}_\mu \in (I^{\text{sat}}/I)_{\delta - \nu} \) is a nonzero element which is independent of the choices of decompositions and of \( \sigma \) (as long as \( X_\Sigma \) has the positivity property with respect to \( \sigma \)). Therefore, \( \{\text{sylv}_\mu\}_{\nu \in C_\nu} \) gives a basis of \((I^{\text{sat}}/I)_{\delta - \nu}\).

Proof. First, the fact that \( \text{sylv}_\mu \) has degree \( \delta - \nu \) follows by using the same reasoning as the one at the end of Proposition 4.1. Now, from the decompositions (4.2) and the invariance of the determinant by column operations, we get

\[
x_j^{\mu_j+1} \text{sylv}_\mu = \begin{vmatrix} \cdots & x_j^{\mu_j+1}F_{0,j} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & x_j^{\mu_j+1}F_{0,j} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} \cdots & F_0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & F_n & \cdots \end{vmatrix} \in I,
\]

and the same holds for the monomial \( z_1^{\mu_1+1} \cdots z_r^{\mu_r+1} \). This proves that \( \text{sylv}_\mu \in (I^{\text{sat}})_{\delta - \nu} \). Suppose that for \( x^\mu \neq x^\mu' \in R_\nu \), there exists \( j \in \{1, \ldots, n\} \) such that \( \mu_j' > \mu_j \) and \( x_j^{\mu_j+1} \) divides \( x^\mu' \). Then,

\[
x^\mu' \text{sylv}_\mu = \frac{x^\mu'}{x_j^{\mu_j+1}} x_j^{\mu_j+1} \text{sylv}_\mu \in I \implies x^\mu' \text{sylv}_\mu = 0 \in (I^{\text{sat}}/I)_{\delta - \nu}.
\]

If this does not happen, then \( \mu_j' \leq \mu_j \) for all \( j \in \{1, \ldots, n\} \). Using the positivity property, this implies that \( \sum_{j=1}^n P_j \mu_j' \leq \sum_{j=1}^n P_j \mu_j \) for \( k = 1, \ldots, r \), but if it was an equality, then:

\[
\nu_k = \mu_k + \sum_{j=1}^n P_j \mu_j' = \mu_k + \sum_{j=1}^n P_j \mu_j \quad k = 1, \ldots, r
\]

where we used that \( x^\mu \) and \( x^\mu' \) have the same degree \( \nu \). This would imply that \( x^\mu = x^\mu' \), a contradiction. Otherwise, \( \mu_k' > \mu_k \) for all \( k = 1, \ldots, r \), implying:

\[
x^\mu' \text{sylv}_\mu = \frac{x^\mu'}{x_1^{\mu_1+1} \cdots x_r^{\mu_r+1} z_1^{\mu_1+1} \cdots z_r^{\mu_r+1}} \text{sylv}_\mu \in I \implies x^\mu' \text{sylv}_\mu = 0 \in (I^{\text{sat}}/I)_{\delta - \nu}.
\]

On the other hand, we have

\[
x^\mu \text{sylv}_\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} \sum_{j=1}^n z_1^{\mu_1+1} \cdots z_r^{\mu_r+1} \text{det}(F_{i,j}) = \text{det}(x^\mu F_{i,j})
\]

but at the same time, the decomposition

\[
F_i = x_1^{\mu_1} F_{i,1} + x_2^{\mu_2} F_{i,2} + \cdots + z_1^{\mu_1+1} \cdots z_r^{\mu_r+1} F_{i,n+1}
\]

gives the Sylvester form \( \text{sylv}_0 \), implying the equality and that \( \text{sylv}_\mu \notin I \). The fact that they form a basis follows from the duality in Theorem 3.1 and that \( \text{sylv}_0 \) is a basis of \((I^{\text{sat}}/I)_{\delta} \).

\[
\square
\]

Remark 4.1. We notice that over the field \( k \), the duality between Sylvester forms and monomials in Theorem 4.1 can also be deduced from the global transformation law in [CCD97, Theorem 0.2]. The approach we developed above allows us to work over the ring \( A \), which is the universal ring of coefficients over \( k \).

5 Application to toric elimination matrices

Our main motivation for studying the structure of the saturation of an ideal generated by generic sparse polynomials and for introducing Sylvester forms as generalizations of the toric Jacobian is for applications in elimination theory, in particular for solving sparse polynomial systems. Thus, in this section we introduce and study a new family of matrices whose construction uses toric Sylvester forms. It yields compact elimination matrices that can be used for solving \( 0 \)-dimensional sparse polynomial systems via linear algebra methods; we refer the reader to [EM99; BT22; Tel20] for a thorough exposition of such solving methods that we will not discuss in this paper.

In what follows, we will consider a smooth projective toric variety \( X_\Sigma \) and a generic sparse polynomial system defined by homogeneous polynomials \( F_0, \ldots, F_n \) as defined in (2.4). We need to assume additionally that \( X_\Sigma \) is
smooth because we will use the Grothendieck-Serre formula. However, we notice that this assumption is not strongly restrictive as $X_\Sigma$ could be replaced by one of its desingularization variety. Some desingularization procedures are described in [CLS12, Chapters 10, 11]; if this variety has the positivity property with respect to some $\sigma$, then our approach can be applied by considering the same polytopes as nef divisors in this desingularization.

The elimination matrices we will consider are universal with respect to the coefficients of the $F_i$’s, so we introduce the following notation to state and study rigorously their properties under specialization of these coefficients. Recall that $I$ denotes the ideal in $C$ generated by $F_0, \ldots, F_n$.

**Notation 5.1.** Any specialization (i.e. ring morphism) $\rho : A \to k$ induces a surjective map $C \to R$ where $R = k[x_\rho : \rho \in \Sigma(1)]$ (this map leaves invariant the variables $x_\rho$). For all $i = 0, \ldots, n$ we define $f_i = \rho(F_i) \in R$ and we denote by $I(f)$ the homogeneous ideal $(f_0, \ldots, f_n)$ of $R$ and set $B(f) = R/I(f)$. Moreover, we also set $B^{\sat} = C/I^{\sat}$, $B(f)^{\sat} = R/I(f)^{\sat}$ and $B^{\sat} = C/I^{\sat}(f)$ (observe that $I(f)^{\sat}$ and $I^{\sat}(f)$ are in general not the same ideals). Finally, for any matrix $M$ with coefficients in $A$, we denote by $M(f)$ its specialization by $\rho : A \to k$. We also consider $\Pic(X_\Sigma)$ instead of $\Cl(X_\Sigma)$ as all Weil divisors are Cartier in a smooth variety; see [CLS12, Proposition 4.2.6].

### 5.1 Hybrid elimination matrices

We begin by describing precisely what we mean by an elimination matrix $M$ associated to the polynomials $F_0, \ldots, F_n$. It as a matrix whose columns are filled with coefficients of some homogeneous forms that are of the same degree and that belong to the saturated ideal $I^{\sat} \subset C$. Thus, its entries are polynomials in $A$. Moreover, it is required that for any specialization map $\rho : A \to k$ the following two properties hold:

i) The corank of $M(f)$ is equal to zero if and only if $f_0 = \cdots = f_n = 0$ has no solution in $X_\Sigma$.

ii) If the number of solutions of $f_0 = \cdots = f_n = 0$ is finite in $X_\Sigma$ and equals $\kappa$, then the corank of $M(f)$ is $\kappa$.

Some comments are in order about this definition. The first property yields a certificate of existence of a common root of the $f_i$’s. We notice that this is equivalent to the vanishing of their sparse resultant; we will come back to this topic in the next section. The second property is mainly required for solving 0-dimensional polynomial systems by means of linear algebra techniques based on eigen-computations, as in this approach the common roots of the $f_i$’s are extracted from the cokernel of $M(f)$. We also emphasize that requiring the columns of $M$ to be filled with homogeneous forms in $I^{\sat}$ is also mainly targeted for solving 0-dimensional polynomial systems. Indeed, this guarantees that each common root to the $f_i$’s corresponds to a sub-vector space of the cokernel of $M(f)$, and then the second property implies that the total dimension of these spaces is the total number of roots, counted with multiplicity. From a more geometric point of view, homogeneous forms in $I^{\sat}$ specialize to hypersurfaces in $X_\Sigma$ that vanish scheme-theoretically at the roots of the $f_i$’s in $X_\Sigma$; see [Cox95, Corollary 3.8]. For this reason, in the elimination theory literature these forms are sometimes called inertia forms of the polynomial $F_0, \ldots, F_n$.

A very classical family of elimination matrices is obtained by filling columns with all the multiples of the $F_i$’s of a certain degree. These matrices are usually called Macaulay-type matrices and are widely used for solving 0-dimensional polynomial systems; see [BT22]. To be more precise, these matrices, that we denote by $M_\alpha$, are presentation matrices of the $A$-module $B\alpha$, i.e. are the matrices of the maps

\[
\left( \oplus_{i=0}^n C(-\alpha_i) \right)_\alpha \to C_\alpha,
\]

\[
(G_0, \ldots, G_n) \mapsto \sum_{i=0}^n G_i F_i.
\]

Of course, some conditions on $\alpha \in \Pic(X_\Sigma)$ are required in order to guarantee that $M_\alpha$ is an elimination matrix; see [EM99] or [Tel20, Chapter 5] for good reviews on the properties of Macaulay-type matrices. In our setting, we see them as a particular case of the hybrid construction we now describe.

Applying results we proved in the previous sections, we introduce a new family of elimination matrices by using toric Sylvester forms. We recall that Sylvester forms belong to $I^{\sat}$ by Theorem 4.1.

**Definition 5.1.** Let $\alpha$ be such that $(I^{\sat}/I)_\alpha \cong \oplus_{\mu} A$ is a free $A$-module; e.g. $\alpha \notin \Gamma_0 \cup \Gamma_1$ and $I_{\delta - \alpha} = 0$ as in Corollary 3.1. Consider the map

\[
\left( \oplus_{i=0}^n C(-\alpha_i) \right)_\alpha \oplus (I^{\sat}/I)_\alpha \to C_\alpha,
\]

\[
(G_0, \ldots, G_n, l_\mu) \mapsto \sum_{i=0}^n G_i F_i + \sum_{x_\mu \in C_{\delta - \alpha}} l_\mu \Sylv_{x_\mu},
\]

where $l_\mu \in A$. Its matrix is called a **hybrid elimination matrix** and will be denoted by $M_\alpha$. 
The matrices $\mathbb{H}_\alpha$ are called hybrid because they are composed of two blocks, one from the classical Macaulay-type matrices and another one built from toric Sylvester forms. In particular, we notice that $M_\alpha = \mathbb{H}_\alpha$ if $(I^{sat}/I)\alpha = 0$, so that the family of matrices $\mathbb{H}_\alpha$ can be seen as an extension of the family of Macaulay-type matrices $M_\alpha$; from now on we will use the notation $\mathbb{H}_\alpha$ instead of $M_\alpha$. Showing that these matrices are elimination matrices is the purpose of the next section.

5.2 Main properties

In this section, we first prove that the matrices $\mathbb{H}_\alpha$ introduced in Definition 5.1 are elimination matrices. Then, we give an illustrative example and also provide another criterion to construct the matrices $\mathbb{H}_\alpha$ without relying on the computation of the supports $\Gamma_0$ and $\Gamma_1$.

First, suppose given a specialization map (see Notation 5.1) and a degree $\alpha$. From the results of Section 3 and Section 4, and also Definition 5.1, we deduce that the image of the matrix $\mathbb{H}_\alpha(f)$ is $I^{sat}(f)\alpha$, so that its corank is $HF(B^{sat}(f), \alpha)$. Therefore, a natural question is to compare this Hilbert function of $B^{sat}(f)$ with the one of $(f)\alpha$ in degrees for which hybrid matrices $\mathbb{H}$ are defined; see Definition 5.1.

**Lemma 5.1.** Let $\alpha \notin \Gamma_0 \cup \Gamma_1 \subset \text{Pic}(X_\Sigma)$ and suppose given specialized polynomials $f_0, \ldots, f_n$ defining a 0-dimensional subscheme in $X_\Sigma$, possibly empty, of $\kappa$ points, counted with multiplicity. Then,

$$HF(B^{sat}(f), \alpha) = \text{HF}(B^{sat}(f), \alpha) = \kappa.$$

**Proof.** This proof goes along the same lines as [BCN22, Lemma 2.7]. First, one observes that $I(f) \subset I^{sat}(f) \subset I(f)^{sat}$ so that $B^{sat}(f)$, $B^{sat}(f)$ and $B(f)$ have the same Hilbert polynomial, which is the constant $\kappa$ by our assumption.

Now, $H_i^\alpha(B^{sat}(f)) = 0$ for $i = 0$ and for all $i > 1$ since $V(I(f))$ is finite. Applying Grothendieck-Serre formula, it follows that $HF(B^{sat}(f), \alpha) = \kappa$ for all $\alpha$ such that $H_i^\alpha(B^{sat}(f)) = 0$. Analyzing the two spectral sequences associated to the Čech-Koszul complex of $f_0, \ldots, f_n$, we get that the above vanishing holds for all $\alpha \notin \Gamma_0 \cup \Gamma_1$.

Similarly, Grothendieck-Serre formula and the finiteness of $V(I(f))$ imply that $HF(B^{sat}(f), \alpha) = \kappa$ for all $\alpha$ such that $H_i^\alpha(B^{sat}(f)) = 0$. By [Ch13, Proposition 6.3], the vanishing of these modules can be derived from the similar vanishing conditions $H_i^\alpha(B^{sat})_\alpha = H_i^1(B^{sat})_\alpha = 0$. These latter conditions hold for all $\alpha \notin \Gamma_0 \cup \Gamma_1$, which concludes the proof.

**Remark 5.1.** As a consequence of the above lemma, we notice that, under its assumptions, the canonical map from $I^{sat}_\alpha$ to $I(f)^{sat}_\alpha$, which is induced by a specialization $\rho$, is surjective, i.e. generators of $I(f)^{sat}_\alpha$ can be computed by means of universal formulas.

**Theorem 5.1.** Let $\alpha \notin \Gamma_0 \cup \Gamma_1 \subset \text{Pic}(X_\Sigma)$. Then, $\mathbb{H}_\alpha$ is an elimination matrix, that is:

- $\text{corank}(\mathbb{H}_\alpha(f)) = 0$ if and only if $V(I(f))$ is empty in $X_\Sigma$,
- If $V(I(f))$ is a finite subscheme of degree $\kappa$ in $X_\Sigma$, then $\text{corank}(\mathbb{H}_\alpha(f)) = \kappa$.

**Proof.** The proof of the second item follows directly from Lemma 5.1.

For the first item, if $V(I(f))$ is empty, equivalently $B^{sat}(f) = 0$, then $HF(B^{sat}(f), \alpha) = 0$ by Lemma 5.1. If $V(I(f)) \neq \emptyset$, then the $f_i$’s have a common solution, say the point $p \in X_\Sigma$ (over $k$) with defining ideal $I_p$ (radical and maximal in $R$). Therefore, since $I^{sat}(f) \subset I(f)^{sat} \subset I_p$ and $HF(R/I_p, \beta) = 1$ for all $\beta \in \text{Pic}(X_\Sigma)$ by the maximality of $I_p$, we deduce that $HF(R/I^{sat}(f), \alpha) \neq 0$, for any $\alpha$.

We emphasize that the above results include the classical Macaulay-type matrices we mentioned in the previous section.

**Example 5.1.** Let $M = \mathbb{Z}^2$ and $X_\Sigma$ be the Hirzebruch surface $H_1$ described in Example 2.3. Consider the following polytope presentations:

$$\Delta_i = \{ m \in \mathbb{Z}^2 : \langle m, (1, 1) \rangle \geq 0, \langle m, (0, 1) \rangle \geq 0, \langle m, (-1, -1) \rangle \geq 2, \langle m, (0, -1) \rangle \geq 1 \}, i = 0, 1, 2.$$

$H_1$ has the positivity property with respect to $\sigma = \langle (1, 0), (0, 1) \rangle$. The class in $\text{Pic}(H_1) = \mathbb{Z}^2$ corresponding to these polytopes is $\sigma = (2, 1)$ and we write the corresponding generic sparse homogeneous polynomials as:

$$F_0 = a_0z_1z_2 + a_2x_1z_2 + a_2x_1z_2 + a_4x_2z_1 + a_4x_1x_2 \quad \text{resp.} \quad F_1, F_2 \text{ with coefficients } b_i, c_i \quad i = 0, \ldots, 4.$$
We obtain the diagram in the Figure 1 for the supports \( \Gamma_0, \Gamma_1, \Gamma_{\text{Res}} \) (also \( \Gamma \), which will be defined in Section 6) and we deduce that some elimination matrices \( H_\alpha \) are obtained for \( \alpha \in \{4, 2, 3, 2, (3, 1), (2, 1)\} \). In the cases \( \alpha = (4, 2) \) and \( \alpha = (3, 2) \), we get two Macaulay-type matrices \( (M_\nu) \); see Example 6.1 for more details on their relation with sparse resultant theory. The two other cases give matrices:

- \( \alpha = (3, 1) \). This matrix corresponds to \( \alpha = \delta \) and in this case, we are introducing a Sylvester form. This form is \( \text{Sylv}_0 \) and can be computed, as before, by a determinant that we write as:

\[
\text{det} \begin{pmatrix} a_1 z_1 z_2 + a_2 x_1 z_2 + a_4 x_2 \\ b_1 z_1 z_2 + b_2 x_1 z_2 + b_4 x_2 \\ c_1 z_1 z_2 + c_2 x_1 z_2 + c_4 x_2 \end{pmatrix} = [130] z_1^3 z_2 + [230] x_1 z_1^2 z_2 + [430] x_2 z_1^2,
\]

where \([ijk]\) denotes \( \text{det} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} \). Therefore, the elimination matrix \( H_\alpha \) is of the form:

\[
H_{(3,1)} = \begin{pmatrix} a_0 & 0 & b_0 & 0 & c_0 & 0 & [130] \\ a_1 & a_0 & b_1 & b_0 & c_1 & c_0 & [230] \\ a_2 & a_1 & b_2 & b_1 & c_2 & c_1 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & c_2 & 0 \\ a_3 & 0 & b_3 & 0 & c_3 & 0 & [430] \\ a_4 & a_3 & b_4 & b_3 & c_4 & c_3 & 0 \\ 0 & a_4 & 0 & b_4 & 0 & c_4 & 0 \end{pmatrix}.
\]

This type of matrices for \( \alpha = \delta \) were already known from [CDS97] as the \( \Delta_i \)'s are all equal and ample in \( H_1 \). However, we notice that the block of Sylvester forms is more sparse than if we used the one appearing in that article.

- For \( \alpha = (2, 1) \), we obtain the following matrix \( H_\alpha \) which is built from two different Sylvester forms:

\[
H_{(2,1)} = \begin{pmatrix} a_0 & b_0 & c_0 & [013] & [023] \\ a_1 & b_1 & c_1 & [023] + [014] & [024] + [123] \\ a_2 & b_2 & c_2 & [024] & [124] \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_4 & b_4 & c_4 & 0 & 0 \end{pmatrix}
\]

where the Sylvester forms correspond to the monomial basis \( \{z_1, x_1\} \) in \( C_\nu \) for \( \nu = (1, 0) \). As far as we know, this kind of matrices are new.

**Example 5.2.** Let’s change the system of Example 5.1 by setting \( \alpha_2 = (1, 1) \). This implies that the corresponding generic sparse homogeneous polynomial is:

\[
F_2 = c_0 z_1 z_2 + c_1 x_1 z_2 + c_4 x_2.
\]

Notice that, in this case, the Newton polytopes \( \Delta_i \)'s are not scaled copies of a fixed ample class and \( \alpha_2 \) is not even ample in \( H_1 \). Now, \( \delta = (2, 1) \) and the corresponding Sylvester form would be:

\[
\text{det} \begin{pmatrix} a_1 z_1 z_2 + a_2 x_1 z_2 + a_4 x_2 \\ b_1 z_1 z_2 + b_2 x_1 z_2 + b_4 x_2 \\ c_1 z_1 + c_0 \end{pmatrix} = [130] z_1^2 z_2 + [230] x_1 z_1 z_2 + [430] x_2 z_1,
\]
where \( [ijk] := \det \begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix} \), with the convention that \( c_i = 0 \) if this coefficient does not appear in \( F_2 \). The corresponding elimination matrix is

\[
\mathbb{H}_{(2,1)} = \begin{pmatrix} a_0 & b_0 & c_0 & 0 & [013] \\ a_1 & b_1 & c_1 & 0 & [023] + [014] \\ a_2 & b_2 & 0 & c_1 & 0 \\ a_3 & b_3 & c_3 & 0 & [024] \\ a_4 & b_4 & 0 & c_3 & 0 \end{pmatrix}.
\]

This example shows that we generalized the results of [CCD97] for \( \alpha = \delta \) with the only restriction that the \( \Delta_i \) must be \( n \)-dimensional.

As illustrated in the above Example 5.1, the construction of the elimination matrices \( \mathbb{H}_\alpha \) requires the computation of the support of the local cohomology module \( H^*_\alpha(R) \) over the toric variety \( X_\Sigma \); see [Alt+20] for some interesting results, specially when \( \Sigma \) splits or the rank of \( \text{Pic}(X_\Sigma) \) is 2 or 3, and also [EMS00; Bot11]. However, none of these references covers the case of a general smooth projective variety. The next result provides settings for which these computations can be avoided.

**Notation 5.2.** We use the same notation as in Section 3 and the beginning of this section, writing \( \alpha_i \in \text{Pic}(X_\Sigma) \) for the classes associated to the homogeneous polynomial system, \( K_X \) for the anticanonical divisor, \( \delta = \alpha_0 + \cdots + \alpha_n - K_X \) and \( \nu, \alpha \in \text{Pic}(X_\Sigma) \) as elements in the class group; \( K_j(F) \) for the modules involved in the Koszul complex and \( J \subset \{0, \ldots, n\} \) for a subset of indices.

**Theorem 5.2.** Assume that all the polytopes \( \Delta_i \) are \( n \)-dimensional and that \( \alpha \in \text{Pic}(X_\Sigma) \) satisfies one of the two following properties:

i) \( \alpha = \delta + \nu \) with \( \nu \in \text{Pic}(X_\Sigma) \) a nef class,

ii) \( \alpha = \delta - \nu \) for \( \nu \in \text{Pic}(X_\Sigma) \) a nef class satisfying the hypotheses of Theorem 2.1, and \( \alpha_i - \nu \) is nef for all \( i = 0, \ldots, n \).

Then, \( \mathbb{H}_\alpha \) is an elimination matrix. Moreover, it is purely of Macaulay-type if and only if \( \alpha \) satisfies i).

**Proof.** We begin with case i). If \( \alpha = \delta + \nu \), we have

\[
H^i_\delta(K_j(F))_{\delta + \nu} \simeq H^i_\delta(\oplus_{|J| = j} C(- \sum_{i \in J} \alpha_i))_{\delta + \nu} \simeq \oplus_{|J| = j} H^i_\delta(C)_{\delta + \nu - \sum_{i \in J} \alpha_i},
\]

where we use that the local cohomology functors are exact. Using (2.5), for \( i \geq 2 \) and \( J \subset \{0, \ldots, n\} \), we get:

\[
H^i_\delta(C)_{\delta + \nu - \sum_{i \in J} \alpha_i} \simeq H^{i-1}(X_\Sigma, \sum_{i \in J} \alpha_i - K_X + \nu) \simeq H^{n-i+1}(X_\Sigma, - \sum_{i \notin J} \alpha_i - \nu),
\]

which is zero by Batyrev-Borisov Theorem. The vanishing of \( H^i_\delta(C) \) for \( i = 0, 1 \) follows from (2.6). From here, we check that \( H^0_\delta(B)_{\alpha} \simeq (P^{\text{rat}}/I)_{\alpha} = 0 \) and \( H^1_\delta(B)_{\alpha} = 0 \), as expected.

Now, we turn to the case ii). We have:

\[
H^i_\delta(K_j(F))_{\delta - \nu} \simeq H^i_\delta(\oplus_{|J| = j} C(- \sum_{i \in J} \alpha_i))_{\delta - \nu} \simeq \oplus_{|J| = j} H^i_\delta(C)_{\delta - \nu - \sum_{i \in J} \alpha_i}.
\]

For \( i > 1 \), using Serre duality, we have:

\[
H^i_\delta(C)_{\delta - \nu - \sum_{j \notin J} \alpha_j} \simeq H^{i-1}(X_\Sigma, \sum_{j \notin J} \alpha_j - K_X - \nu) \simeq H^{n-i+1}(X_\Sigma, \nu - \sum_{j \notin J} \alpha_j).
\]

As we supposed that \( \alpha_i - \nu \) is nef, if \( J \neq \{0, \ldots, n\} \), \( \sum_{j \notin J} \alpha_j - \nu \) is also nef and we can apply Batyrev-Borisov Theorem. Moreover, if \( J = \{0, \ldots, n\} \), \( H^{n-i+1}(X_\Sigma, \nu) \) vanishes by Demazure Theorem, unless \( i = n + 1 \). In such case, we have \( H^{n+1}_\delta(K_{n+1})_{\delta - \nu} \simeq C_\nu \). The claimed result follows using the fact that \( \delta - \nu \notin \Gamma_0 \cup \Gamma_1 \) and \( (P^{\text{rat}}/I)_{\delta - \nu} \) has a basis of toric Sylvester forms.

**Corollary 5.1.** If \( \Delta_i \) are \( n \)-dimensional, \( \mathbb{H}_\delta \) is an elimination matrix.

**Proof.** If the \( \alpha_i \) are full-dimensional, \( \nu = 0 \) is nef and satisfies \( 0 \leq \nu_j < \min \alpha_{ij} \), and the \( \alpha_i \) are nef so the previous result holds for \( \delta \).

**Example 5.3.** Referring to Example 5.1, we see that many of the interesting matrices appear using this setting. The matrix of Macaulay-type with \( \alpha = (4,2) \) corresponds to the case i), while the matrix with \( \alpha = (3,2) \) does not belong to either of the two cases as \( \nu = (0,1) \) is not a nef divisor; similarly to the Dixon elimination matrices in [BCN22, Example 5.6]. The matrices with \( \alpha = (3,1), (2,1) \) correspond to case ii) as \( \nu = (0,0), (1,0) \) are nef divisors and so are \( \alpha_i - \nu = (2,1), (1,1) \) for \( i = 0, 1, 2 \).
5.3 Overdetermined sparse polynomial systems

Overdetermined polynomial systems, i.e. systems defined by \( r + 1 \) equations with \( r > n \), appear very often in applications. In this section, we extend the construction of hybrid elimination matrices to this setting. For that purpose, the key property we will prove is that all Sylvester forms built from subsets of \( n + 1 \) polynomial equations in an over-determined system give a generating set of suitable graded components of inertia forms. This is the content of Theorem 5.4.

**Notation 5.3.** Let \( X_\Sigma \) be a smooth projective toric variety. In this subsection \( F_0, \ldots, F_r \) are generic homogeneous sparse polynomials corresponding to the nef classes \( \alpha_0, \ldots, \alpha_r \) and \( I \) is their corresponding ideal, \( B = C/I \) their quotient ring. For each subset \( T \subset \{0, \ldots, r\} \) of cardinality \( n + 1 \), we set \( I_T = (F_i : i \in T) \), \( B_T = C/I_T \) and \( \delta_T = \sum_{\alpha \in T} \alpha - K_X \). We denote by \( \text{Sylv}_{\mu,T} \) the Sylvester forms that can be formed from \( \{F_i\}_{i \in T} \); see Section 4. We also denote by \( K_{\mu}(F) \) the Koszul complex of \( F_0, \ldots, F_r \) and by \( K_{T,\bullet}(F) \) the Koszul complex built from the generators of \( I_T \).

**Theorem 5.3.** Using the previous notation, suppose that there is a subset \( S \subset \{0, \ldots, r\} \) of cardinality \( n + 1 \) and a nef class \( \nu \in \text{Pic}(X_\Sigma) \) satisfying the hypotheses of Theorem 2.1 such that

\[
\forall i \in S \quad j \notin S \quad \alpha_i - \alpha_j \text{ nef and } \forall i \in S \quad \alpha_i - \nu \text{ is nef.}
\]

Then, the set of Sylvester forms

\[
\{\text{sylv}_{\mu,T} : T \subset \{0, \ldots, r\} \text{ such that } |T| = n + 1 \text{ and } x^\nu \in C_{\delta_T - \delta_{S^\nu}}\}
\]

yields a generating set of the \( A \)-module \( (I^\mu/I)_{\delta_{S^\nu}}. \)

**Proof.** This proof can be seen an generalization of a similar result that is proved in the case \( X_\Sigma = \mathbb{P}^n \) in [BCP23, Chapter 3, Proposition 3.23].

First, we use Serre duality and Batyrev-Borisov Theorem in order to compute the local cohomology modules \( H^i_b(K_j(F))_{\delta_{S^\nu}} \), for \( i, j = 0, \ldots, n + 1 \), similarly to Theorem 5.2. Namely, for \( i \geq 2 \)

\[
H^i_b(K_j(F))_{\delta_{S^\nu}} \cong \oplus_{|T|=j} H^i_b(C(- \sum_{k \in T} \alpha_k))_{\delta_{S^\nu}} \cong H^{n+1-i}(X_\Sigma; \sum_{k \in T} \alpha_k - \sum_{k \in S} \alpha_{k'}) + \nu).
\]

The elements in \( S \cap T \) cancel each other, and the rest of elements \( k' \in S \) can be either (i) paired up with \( \alpha_k \) for \( k \in T \) satisfying that \( \alpha_i - \alpha_j \) is nef, (ii) be paired up with \( \nu \) satisfying that \( \alpha_{k'} - \nu \) is nef or, (iii) they are nef themselves. Therefore, the previous cohomology module is of the form \( H^{n+1-i}(X_\Sigma, -\alpha) \) with \( \alpha \) a sum of nef divisors, and we can apply Batyrev-Borisov Theorem to deduce:

\[
H^i_b(K_j(F))_{\delta_{S^\nu}} \cong \begin{cases} \oplus_{|T|=n+1} C_{\nu} & \text{, } \sum_{j \in T} \alpha_j - \sum_{i \in S} \alpha_i + \nu \text{ } i, j = n + 1 \\ 0 & \text{otherwise.} \end{cases}
\]

As a consequence, from the comparison of the two spectral sequences that are considered in Theorem 3.1, we obtain the following transgression map, which is an isomorphism of graded modules:

\[
\tau : H_{n+1}(K_\bullet(F), H^0_b(C))_{\delta_{S^\nu}} \cong H^0_b(B)_{\delta_{S^\nu}}.
\]

For any \( T \subset \{0, \ldots, r\} \), let \( \tau_T \) be the corresponding transgression map for \( K_{T,\bullet}(F) \) and \( B_T \). For each of these Koszul complexes, we have a canonical morphism of complexes \( K_{T,\bullet}(F) \to K_\bullet(F) \) that all together induce the morphism of complexes:

\[
L_\bullet(F) = \bigoplus_{|T|=n+1} K_{T,\bullet}(F) \to K_\bullet(F).
\]

It follows that there is a commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{|T|=n+1} H_{n+1}(K_{T,\bullet}(F), H^0_b(C))_{\delta_{S^\nu}} & \cong H_{n+1}(K_\bullet(F), H^0_b(C))_{\delta_{S^\nu}} \\
\oplus \tau_T & \cong H^0_b(B)_{\delta_{S^\nu}} & \cong H^0_b(B)_{\delta_{S^\nu}}
\end{array}
\]

As the two vertical arrows are isomorphisms, in order to show that the bottom arrow is surjective, it is enough to show that the top arrow is surjective. For that purpose, observe that \( L_{n+1} = K_{n+1} \) by construction and also

\[
\bigoplus_{|T|=n+1} H_{n+1}(K_{T,\bullet}(F), H^0_b(C))_{\delta_{S^\nu}} = \ker(H^0_b(C)_{\delta_{S^\nu}}) \to H^0_b(L_{n+1}(F))_{\delta_{S^\nu}}.
\]
However, by the same argument as before \( H_b^{n+1}(L_n(F)) \delta_{S-\nu} = 0 \), so
\[
\oplus_{|T|=n+1} H_b^{n+1}(K_{T,\bullet}(F), H_b^{n+1}(C)) \delta_{S-\nu} \simeq \oplus_{|T|=n+1} \sum_{\alpha} \delta_{\alpha-\sum \delta_{S-\nu} \alpha+\nu'.}
\]
which is generated by the Sylvester forms at each of these degrees. On the other hand,
\[
H_b^{n+1}(K_{\bullet}(F), H_b^{n+1}(C)) \delta_{S-\nu} \simeq \ker(H_b^{n+1}(K_{n+1}) \delta_{S-\nu} \rightarrow H_b^{n+1}(K_n) \delta_{S-\nu})/ \text{Im}(H_b^{n+1}(K_{n+2}) \delta_{S-\nu} \rightarrow H_b^{n+1}(K_{n+1}) \delta_{S-\nu}).
\]
As above, \( H_b^{n+1}(K_n) \delta_{S-\nu} = 0 \) and:
\[
H_b^{n+1}(K_{\bullet}(F), H_b^{n+1}(C)) \delta_{S-\nu} \simeq H_b^{n+1}(K_{n+1}) \delta_{S-\nu}/ \text{Im}(H_b^{n+1}(K_{n+2}) \delta_{S-\nu} \rightarrow H_b^{n+1}(K_{n+1}) \delta_{S-\nu}).
\]
This implies that the top map in the diagram (5.4) is surjective, as we wanted to prove. It follows that the basis of Sylvester forms of \( \oplus_{|T|=n+1} H_b^{n}(B_T) \delta_{S-\nu} \) is a set of generators of \( H_b^{n}(B) \delta_{S-\nu} = (I^{\text{sat}}/I) \delta_{S-\nu} \).

**Definition 5.2.** Let \( \nu \in \text{Pic}(X_{\Sigma}) \) satisfying the assumptions of Theorem 5.3. We will denote by \( H_{\alpha} \) the matrix of the following map:
\[
(\oplus_{i=0}^n C(-\alpha_i)) \rightarrow \bigoplus_{T \subseteq \{0, \ldots, r\}} (I^{\text{sat}}/I_T) \rightarrow C_{\alpha}(5.5)
\]

\[
(G_0, \ldots, G_n, l_{\mu,T}) \rightarrow \sum_{i=0}^n G_i F_i + \sum_{T \subseteq \{0, \ldots, r\}} \sum_{x^\nu \in C_{S-\alpha}} l_{\mu,T} \text{Sylv}_{\mu,T}
\]

where \( l_{\mu,T} \in A. \)

**Theorem 5.4.** If \( \alpha = \delta_{S-\nu} \) for \( S \subseteq \{0, \ldots, r\} \) and \( \nu \in \text{Pic}(X_{\Sigma}) \) a nef class satisfying the hypotheses of Theorem 5.3, then \( H_{\alpha} \) is an elimination matrix.

**Proof.** The proof follows the same lines as the one given in Theorem 5.1 for \( r = n. \)

**Example 5.4.** Taking again the notation of Example 2.3, we add another polynomial with degree \( \alpha_3 = (2,1) \) in \( H_1 \) and write it in homogeneous coordinates as
\[
F_3 = d_0 z_1^2 z_2 + d_1 x_1 z_1 z_2 + d_2 x_2^2 z_2 + d_3 x_2 z_1 + d_4 x_1 z_2.
\]
Following Theorem 5.4, the matrix \( H_{\alpha} \) for \( \alpha = \delta_{S} \) is
\[
\begin{pmatrix}
 a_0 & 0 & b_0 & 0 & c_0 & 0 & d_0 & 0 & [130]_{abc} & [130]_{abd} & [130]_{acd} & [130]_{bcd} \\
 a_1 & a_0 & b_1 & b_0 & c_0 & d_1 & d_0 & [230]_{abc} & [230]_{abd} & [230]_{acd} & [230]_{bcd} \\
 a_2 & a_1 & b_2 & b_1 & c_1 & d_2 & d_1 & 0 & 0 & 0 & 0 \\
 0 & a_2 & 0 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 \\
 a_3 & 0 & b_3 & 0 & c_3 & d_3 & 0 & [430]_{abc} & [430]_{abd} & [430]_{acd} & [430]_{bcd} \\
 a_4 & a_3 & b_4 & b_3 & c_4 & c_3 & d_4 & d_3 & 0 & 0 & 0 \\
 0 & a_4 & 0 & b_4 & 0 & c_4 & d_4 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where \([ijk]_{abc} = a_i b_j c_k\), and \([ijk]_{abd}, [ijk]_{acd}, [ijk]_{bcd}\) defined accordingly. It is an elimination matrix for the overdetermined polynomial system defined by \( F_0, F_1, F_2 \) and \( F_3. \)

### 6 Sylvester forms and sparse resultants

Resultants are central tools in elimination theory and there is a huge literature on various methods to compute them. A classical result is that the sparse resultant can be computed as the determinant of certain graded components of the Koszul complex built from the considered polynomial system; see for instance [GKZ94; DD00; WZ92; Ben+21]. In this section, we show that Sylvester forms can be incorporated in the usual Koszul complex and obtain this way new expressions for the sparse resultant as the determinant of a complex. This extends results in [CDS97, §2] by providing more compact formulas.

In what follows, we assume that \( X_{\Sigma} \) is a smooth projective toric variety and we take again the notation of Section 3. Thus, \( F_0, \ldots, F_n \) are generic homogeneous sparse polynomial system as defined by (2.4). Let \( A_i \) be
the supports of these polynomial systems for \( i = 0, \ldots, n \), respectively, which can be seen as the lattice points in \( \Delta_i \subset M_i \); see Section 2.

The space of coefficients of the \( F_i \)'s has a natural structure of multi-projective space, as the equations \( F_i = 0 \) are not modified after multiplication by a nonzero scalar. We write it as \( \prod_{i=0}^{n} \mathbb{P}^{A_i} \), where \( \mathbb{P}^{A_i} \) denotes the projective space associated to the coefficients of the polynomial \( F_i \). Let \( Z(F) = \{ x \times (\ldots, c_{\mu}, \ldots) \in X_0 \times \prod_{i=0}^{n} \mathbb{P}^{A_i} : F_0 = \cdots = F_n = 0 \} \) be the incidence variety of \( F_0, \ldots, F_n \). Consider the canonical projection onto the second factor

\[
\pi : X_0 \times \prod_{i=0}^{n} \mathbb{P}^{A_i} \to \prod_{i=0}^{n} \mathbb{P}^{A_i}.
\]

The image of \( Z(F) \) via \( \pi \) is an irreducible hypersurface and the sparse resultant, denoted as \( \text{Res}_A \), is a primitive polynomial in the universal ring of coefficients \( A \) defining the direct image \( \pi_*(Z(F)) \), up to a sign in \( k \), if it is of codimension one, otherwise the sparse resultant is set to 1.

The Koszul complex \( K_{\bullet}(F) \) of the sequence of polynomials \( F_0, \ldots, F_n \) is of the form

\[
K_{\bullet}(F) : K_{n+1} = C(-\sum \alpha_i) \to \ldots \to K_2 = \oplus_{k,k'} C(-\alpha_k - \alpha_{k'}) \to K_1 = \oplus_k C(-\alpha_k) \to C.
\]

It is a graded complex of free graded \( A \)-modules. We already used it several times previously. Gelfand, Kapranov and Zelevinsky proved that the determinant of some of its graded components is equal to the sparse resultant, up to a nonzero scalar in \( k \). More precisely, for all \( \alpha \notin \Gamma_{\text{Res}} \subset \text{Pic}(X_0) \) the strand \( K_{\bullet}(F)_\alpha \) is an acyclic complex of free \( A \)-modules and \( H_0(K_{\bullet}(F)_\alpha) = B_\alpha \). Moreover, if in addition \( (I_{\text{sat}}/I)_\alpha = 0 \) then \( \det(K_{\bullet}(F)_\alpha) \) equals the sparse resultant \( \text{Res}_A \) up to a nonzero scalar; see [GKZ94, Chapter 3, Theorem 4.2] for proofs. We notice that the map on the far right of the complex \( K_{\bullet}(F)_\alpha \) is nothing but the Macaulay-type map (5.1), whose matrix is an elimination matrix of the form \( M_\alpha \). In what follows, we will extend this approach by using Sylvester forms.

In order to incorporate Sylvester forms in the above construction we proceed as follows: we consider a graded strand \( K_{\bullet}(F)_\alpha \) of the Koszul complex, such that \( (I_{\text{sat}}/I)_\alpha \) is a nonzero free \( A \)-module, and we define a new complex, denoted \( K_{\text{sat}}(F)_\alpha \), by adding Sylvester forms to the map on the far right, i.e.

\[
K_{\text{sat}}(F)_\alpha = C(-\sum \alpha_i)_\alpha \xrightarrow{d_0} \ldots \xrightarrow{\oplus_{k,k'} C(-\alpha_k - \alpha_{k'})_\alpha} \oplus_k C(-\alpha_k)_\alpha \oplus (I_{\text{sat}}/I)_\alpha \to C,
\]

which is a graded complex of free \( A \)-modules, that we call the saturated Koszul complex. From its definition, it satisfies \( H_0(K_{\text{sat}}(F)_\alpha) = (B_{\text{sat}})_\alpha \). Moreover, the map on the far right is precisely the map we used to define hybrid elimination matrices \( \Pi_\alpha \). Notice also that if \( (I_{\text{sat}}/I)_\alpha = 0 \), then we recover the usual Koszul complex.

The following result generalizes [GKZ94, Chapter 3, Theorem 4.2], as well as [CD97, Theorem 2.2] where a formula for the sparse resultant as the determinant of a complex incorporating the toric Jacobian, which only applies for \( \alpha = \delta \) and the polytopes \( \Delta_i \), being scaled copies of a given polytope.

**Theorem 6.1.** Let \( \alpha \notin \Gamma_{\text{Res}} \), then \( K_{\text{sat}}(F)_\alpha \) is an acyclic complex of free \( A \)-modules. If \( \alpha = \delta - \nu \) as in Theorem 5.2 ii), then \( \det(K_{\text{sat}}(F)_\alpha) \) equals \( \text{Res}_A \) up to a nonzero multiplicative constant in \( k \).

**Proof.** The acyclicity of \( K_{\text{sat}}(F)_\alpha \) follows from the same property on the Koszul complex \( K_{\bullet}(F)_\alpha \) (observe that the image of \( d_1 \) does not map to \( (I_{\text{sat}}/I)_\alpha \) and the fact that \( (I_{\text{sat}}/I)_\alpha \) is a free \( A \)-module.

The acyclicity property, together with the fact that \( H_0(K_{\text{sat}}(F)_\alpha) = (B_{\text{sat}})_\alpha \), imply that \( \det(K_{\text{sat}}(F)_\alpha) \) and \( \text{Res}_A \) are two polynomials in \( A \) vanishing in the same specializations in \( k \). As a consequence of the Projective Nullstellensatz, we only have to compare their degrees in order to check that they are the same polynomial, up to multiplication by a nonzero constant in \( k \); see [Har77]. As proved in [GKZ94, Section 3, Theorem 14], the determinant of a complex of vector spaces \( V_\bullet : V_{n+1} \to \ldots \to V_1 \to V_0 \) satisfies the equality

\[
\det(V_\bullet) = \bigotimes_i \bigwedge V_i^{(-1)^i}.
\]

This result implies that the degree of the determinant can be calculated as an alternate sum. We know that \( \det(K_{\text{sat}}(F)_\alpha) = \text{Res}_A \) (up to multiplication by a nonzero constant) if \( (I_{\text{sat}}/I)_\alpha = 0 \) and that for \( \alpha \gg 0 \) (component-wise), we have \( \text{HF}(C, \alpha) = \text{HP}(C, \alpha) \). Therefore, for \( \alpha \gg 0 \),

\[
\deg(\text{Res}_A) = \deg \det(K_{\text{sat}}(F)_\alpha) = \sum_{J \subset \{0, \ldots, n\}} (-1)^{|J|} \text{HF}(C, \alpha - \sum_{j \in J} \alpha_i) = \sum_{J \subset \{0, \ldots, n\}} (-1)^{|J|} \text{HP}(C, \alpha - \sum_{j \in J} \alpha_i).
\]

This alternate sum yields a polynomial whose degree coincides with the degree of the resultant. Therefore, for \( \alpha = \delta - \nu \) as in the statement, we have \( (I_{\text{sat}}/I)_\delta = \text{Hom}_A(C_\nu, A) \neq 0 \) and we can check that the difference
of degrees between the previous alternate sum and the degree of the resultant is compensated by \((I^{\ast l}/I)_{\delta - \nu}\) as follows:
\[
\deg \det(K_{\bullet}(F)_{\delta - \nu}) - \deg(\Res_{\Delta}) = \sum_{J \subseteq \{0, \ldots, n\}} (-1)^{|J|} \HF(C, \delta - \nu - \sum_{j \in J} \alpha_j) - \HF(C, \delta - \nu - \sum_{j \in J} \alpha_j).
\]

Using Grothendieck-Serre formula (2.7), we get that this coincides with the quantity
\[
\sum_{J \subseteq \{0, \ldots, n\}} (-1)^{|J|} \prod_{i=0}^{n+1} \dim_k H^i_k(C)_{\delta - \nu - \sum_{j \in J} \alpha_j}.
\]

Applying Theorem 5.2 ii), all the elements in this sum vanish except the term \(H^{n+1}_k(C)_{-K_X - \nu}\), which is counted with the sign \((-1)^{2(n+1)} = 1)\). In particular, using Serre duality, \(H^{n+1}_k(C)_{-K_X - \nu} \cong C_{\nu}\) which is dual to \((I^{\ast l}/I)_{\delta - \nu}\), so this difference is compensated in the complex \(K^{\ast l}_{\bullet}(F)_{\delta - \nu}\), which concludes the proof.

In particular, if the polytopes \(\Delta_i\) are \(n\)-dimensional for \(i = 0, \ldots, n\), we can consider \(\nu = 0\) and we recover the results in [CDS97], with the slight improvement that we do not need the \(\alpha_i\) to be of the form \(k_i \beta\) for \(k_i > 0\) and \(\beta\) an ample class.

**Remark 6.1.** The relation between \((I^{\ast l}/I)_{\delta - \nu}\) and \(\HF(C, \nu)\) in the previous result can be seen as an instance of multivariate Ehrhart reciprocity; see [Bec02, Theorem 2]. This result shows that if \(\HF(C, \nu)\) is the multivariate Hilbert polynomial in \(X_\Delta\) corresponding to the number of lattice points in \(\Delta_\nu\) and \(\HF^\ast(C, \nu)\) is another Hilbert polynomial associated to the number of lattice points in the interior of \(\Delta_\nu\), then \(\HF(C, \nu) = (-1)^n \HF^\ast(C, \nu)\).

From the above result, we can also identify cases where the matrices \(\mathbb{H}_\alpha\) are square matrices, and therefore their determinant (in the usual sense of the determinant of a matrix) is equal to the sparse resultant, up to a nonzero multiplicative constant.

**Corollary 6.1.** Let \(\Gamma = \text{Supp} \oplus k_{-\nu}C(-\alpha_k - \alpha_{k'})\). For \(\alpha \notin \Gamma\), we have \(\deg(\mathbb{H}_\alpha) = \Res_{\Delta}\), up to a nonzero multiplicative constant.

**Proof.** If \(\alpha \notin \Gamma\), then the complex \(K_{\bullet}(F)_{\alpha}\) has only two terms and therefore \(\det(K_{\bullet}(F)_{\alpha}) = \det(\mathbb{H}_\alpha)\).

**Remark 6.2.** Computing the determinant of a complex can be done using some techniques such as Cayley determinants; see [GKZ94, Appendix A], but it is not very practical. However, Theorem 6.1 yields new expressions of the sparse resultant as a ratio of two determinants if \(\alpha \notin \text{Supp} \oplus k_{1, \nu}C(-\alpha_k - \alpha_l - \alpha_m);\) see [CDS97, Corollary 2.4] for a combinatorial characterization of such case.

For Macaulay-type formulas of the form \(M_{\alpha}\), the Canny-Emiris formula gives a possible way to choose a nonzero minor; see [CE93] for the formula and [DJS22] for a proof and the non-vanishing of the minor using tropical deformations. It is an open problem to see whether the conditions on the proof of the Canny-Emiris formula [DJS22] coincide with the Cayley determinant for such choice of a minor. In the case of hybrid elimination matrices \(\mathbb{H}_\alpha\), the Canny-Emiris formula has only been explored in for \(n = 2\) and \(\alpha = \delta\); see [DE01].

**Example 6.1.** Let’s consider the four matrices provided in Example 5.1, which correspond to \(\alpha \in \{(4, 2), (3, 2), (3, 1), (2, 1)\}\). The last three are square matrices while the first one is not. We have drawn the region \(\Gamma\) in brown in Figure 1, in order to indicate the elements that provide a square matrix, as well as \(\Gamma_{\text{Res}}\), in green, for the acyclicity of the complex. For the Macaulay-type matrices, we can combinatorially describe a maximal minor of \(\mathbb{M}_{4,2}\) using the Canny-Emiris formula; see [CE93; DJS22]. The matrix \(\mathbb{H}_{(3,2)}\) is square,
\[
\mathbb{H}_{(3,2)} = \begin{pmatrix}
  a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
  a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
  a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
  0 & a_2 & a_1 & 0 & b_2 & b_1 & 0 & c_2 & c_1 \\
  0 & 0 & a_2 & 0 & 0 & b_2 & 0 & 0 & c_2 \\
  a_3 & 0 & 0 & b_3 & 0 & 0 & c_3 & 0 & 0 \\
  a_4 & a_3 & 0 & b_4 & b_3 & 0 & c_4 & c_3 & 0 \\
  0 & a_4 & a_3 & 0 & b_4 & b_3 & 0 & c_4 & c_3 \\
  0 & 0 & a_4 & 0 & 0 & b_4 & 0 & 0 & c_4
\end{pmatrix},
\]

and it might be obtained using a greedy approach to the same formula (see [CP93; CE22]), but as far as we know, there was no certificate of its existence as a resultant formula until now. The hybrid matrices for \(\alpha = (3, 1), (2, 1)\) are square, but if they weren’t, a procedure for choosing a minor is known for \(n = 2\) and \(\alpha = \delta = (3, 1)\); see [DE01].
7 Toric residue of the product of two forms

Another topic for which Sylvester forms are of interest is the computation of toric residues. These objects were initially introduced by Cox as a way to relate the residue of a family of \( n + 1 \) forms to the integral of a certain form in a toric variety \( \Sigma_X \); see [Cox96]. Being given \( F_0, \ldots, F_n \) generic homogeneous polynomials as in (2.4), and denoting by \( K(A) \) the quotient field of the universal ring of coefficients \( A \), Cox proved the existence of a residue map

\[
\text{Residue}_F : B_0 \rightarrow K(A)
\]

(recall that \( I = (F_0, \ldots, F_n) \) and \( B = C/I \)) which has the following property: for any specialization \( \rho : A \rightarrow k \) (see Notation 5.1) such that the specialized system \( f_0 = \cdots = f_n = 0 \) has no solution in \( X_\Sigma \), the residue map \( \text{Residue}_F : (R/I(f))_\delta \rightarrow k \) is an isomorphism. Cox defined residue maps through trace maps of —Cech cohomology, but it can be characterized through the fact that, if there is no solution in \( X_\Sigma \), \( \rho(\text{sylv}_0) \) is sent to \( \pm 1 \in k \), so generically \( \text{Residue}_F(\text{sylv}_0) = \pm 1 \), as we used in Proposition 4.1. Many authors contributed formulas based on elimination matrices and resultants to compute residues [KS05; DK05; CCD97; CDS97] and also used them in other applications such as polynomial interpolation [Sop07] or mirror symmetry [BM02]. In particular, in [DK05] an explicit formula for computing the toric residue of a form of degree \( \delta \) as a quotient of two determinants “à la Macaulay” is proved.

If a form \( G \) of degree \( \delta \) can be written as a product \( G = PQ \), a natural question is to ask whether one can take advantage of this factorization in the computation of the residue of \( G = PQ \) with respect to the polynomial system \( F_0, \ldots, F_n \). In the case \( \Sigma_X = \mathbb{P}^n \), Jouanolou proved that this is indeed possible by exploiting the duality between the degrees \( \delta - \nu \) and \( \nu \) of \( P \) and \( Q \), respectively; see [Jou97, Proposition 3.10.27]. In what follows, we generalize Jouanolou’s formula to a general smooth projective toric variety \( \Sigma_X \). For that purpose, we will rely on toric Sylvester forms and the elimination matrices \( \mathbb{H}_{\delta,\nu} \) we introduced in Section 5.1. The new formulas we obtain can be seen as an extension of the rational formula “à la Macaulay” proved in [DK05, Corollary 3.4].

Remark 7.1. We notice that in [Jou97], Jouanolou defines the residue as a map onto \( A \), and not in \( K(A) \), by multiplying with \( \text{Res}_A \) in the image; see also [CDS97, Theorem 1.4] for a proof that the product of the residue and the resultant lies in \( A \).

Let \( \mathbb{H}_{\delta,\nu} \) be an elimination matrix that satisfies the assumptions of Theorem 5.2 ii), and let \( \mathbb{H}_{\delta,\nu} \) be a maximal minor of \( \mathbb{H}_{\delta,\nu} \) which contains the entire block built with Sylvester forms. Now, being given two generic forms \( P \in C_\nu \) and \( Q \in C_{\delta,\nu} \), we consider the matrix

\[
\Theta_{\delta,\nu} = \begin{pmatrix}
\mathbb{H}_{\delta,\nu} & \mathbf{q} \\
0 & \mathbf{p}_c^T
\end{pmatrix}
\]

(7.1)

where \( \mathbf{p} \), respectively \( \mathbf{q} \), stands for the vector of coefficients of \( P \), respectively \( Q \). Recall that by construction of the matrix \( \mathbb{H}_{\delta,\nu} \), the matrix \( \mathbb{H}_{\delta,\nu} \) is built as the join of a Macaulay-type block-matrix and another column-block matrix built from Sylvester forms. Thus, the row \( \mathbf{p}_c^T \) is aligned with the second column-block, built from Sylvester forms, of \( \mathbb{H}_{\delta,\nu} \); see Example 5.4 for an illustration.

As a first result, one proves that the residue of the product of two monomials can be computed as a quotient of determinants. In what follows, we denote by \( \mathbb{H}_{\alpha,\beta} \) the submatrix of \( \mathbb{H}_{\delta,\nu} \) that is obtained by deleting the column corresponding to the monomial \( x^\alpha \in C_\nu \) and the row corresponding to the monomial \( x^\beta \in C_{\delta,\nu} \).

Lemma 7.1. Let \( F_0, \ldots, F_n \) be a system of homogeneous polynomials in \( C \) as in (2.4), then for any pair of monomials \( x^\alpha \in C_\nu \) and \( x^\beta \in C_{\delta,\nu} \),

\[
\text{Residue}_F(x^{\alpha+\beta}) = (-1)^{\alpha+\beta} \frac{\det(\mathbb{H}_{\alpha,\beta})}{\det(\mathbb{H}_{\delta,\nu})}.
\]

Proof. Let \( H^\beta \) be the matrix obtained by multiplying the row of \( \det(\mathbb{H}_{\delta,\nu}) \) corresponding to \( x^\beta \) by the monomial \( x^\beta \) itself. Then, by expanding the determinant along this row, one gets:

\[
x^\alpha x^\beta \det(\mathbb{H}_{\delta,\nu}) = x^\alpha \det(H^\beta) = x^\alpha \left( \sum_{\alpha' \in C_\nu} c_{\alpha',\beta} \text{sylv}_{\alpha'} \right) = \sum x^\alpha F_i + c_{\alpha,\beta} \text{sylv}_0.
\]

Taking residues at both sides, we deduce that

\[
\text{Residue}_F(x^{\alpha+\beta}) \det(\mathbb{H}_{\delta,\nu}) = (-1)^{\alpha+\beta} c_{\alpha,\beta}.
\]

Finally, from the expansion of the determinant \( \det(H^\beta) \), one sees immediately that \( c_{\alpha,\beta} = \det(\mathbb{H}_{\alpha,\beta}) \).

We are now ready to prove the claimed formula for the residue of the product of two forms.
Theorem 7.1. Let $F_1, \ldots, F_n$ be a system of homogeneous polynomials in $C$ as in (2.4), then for any pair of forms $P \in C_\nu$ and $Q \in C_{\delta-\nu}$, $$\text{Residue}_F(PQ) = \frac{\det(\Theta_{\delta-\nu})}{\det(\Theta_{\delta-\nu})}.$$  

Proof. Write $P = \sum_{a^x \in C_\nu} p_a x^a$ and $Q = \sum_{b^x \in C_{\delta-\nu}} q_b x^b$. Then, by linearity of residues, we have:

$$\text{Residue}_F(PQ) = \sum_{a^x \in C_\nu, b^x \in C_{\delta-\nu}} p_a q_b \text{Residue}_F(x^{a+b}) = \frac{\sum_{a, b} (-1)^{a+b} p_a q_b \det(\Theta_{a, b})}{\det(\Theta_{\delta-\nu})}.$$  

The numerator is precisely the expansion of the determinant $\det(\Theta_{\delta-\nu})$ of the matrix defined in (7.1), with respect to the last row and column.  

Example 7.1. In Example 5.1, the elimination matrix $\mathbb{H}_\alpha$ for $\alpha = (2, 1)$ is square, therefore we take

$$\mathbb{H}_{(2, 1)} = \mathbb{H}_{(2, 1)} = \begin{pmatrix}
 a_0 & b_0 & c_0 & [013] & [023] \\
 a_1 & b_1 & c_1 & [023] + [014] & [024] + [123] \\
 a_2 & b_2 & c_2 & [024] & [124] \\
 a_3 & b_3 & c_3 & 0 & 0 & q_3 \\
 a_4 & b_4 & c_4 & 0 & 0 & q_4 \\
 0 & 0 & 0 & p_0 & p_1 & 0
\end{pmatrix}.$$  

Let $P = p_0 z_1 + p_1 x_1$ and $Q = q_0 z_1^3 z_2 + q_1 z_1 z_2 x_1 + q_2 z_2 x_1^2 + q_3 z_1 x_2 + q_4 x_1 x_2$ be homogeneous forms in $C_{(1,0)}$ and $C_{(2,1)}$, respectively, then

$$\Theta_{(2, 1)} = \begin{pmatrix}
 a_0 & b_0 & c_0 & [013] & [023] & q_0 \\
 a_1 & b_1 & c_1 & [023] + [014] & [024] + [123] & q_1 \\
 a_2 & b_2 & c_2 & [024] & [124] & q_2 \\
 a_3 & b_3 & c_3 & 0 & 0 & q_3 \\
 a_4 & b_4 & c_4 & 0 & 0 & q_4 \\
 0 & 0 & 0 & p_0 & p_1 & 0
\end{pmatrix}.$$  

and applying Theorem 7.1 we deduce that $\text{Residue}_F(PQ) = \frac{\det(\Theta_{(2, 1)})}{\det(\mathbb{H}_{(2, 1)})}$. For the sake of comparison, let us examine the formula we obtain by developing the product of $P$ and $Q$. In this case, we apply Theorem 7.1 with $\delta = (3, 1)$ and $\nu = 0$, so we have to consider the matrix $\Theta_{(3, 1)}$ which is of the form:

$$\Theta_{(3, 1)} = \begin{pmatrix}
 a_0 & b_0 & c_0 & 0 & 0 & [130] & p_0 q_0 \\
 a_1 & a_0 & b_1 & b_0 & c_1 & c_0 & [230] + p_0 q_1 + p_1 q_0 \\
 a_2 & a_1 & b_2 & b_1 & c_2 & c_1 & 0 & p_0 q_2 + p_1 q_1 \\
 0 & a_2 & 0 & b_2 & 0 & c_2 & 0 & p_1 q_2 \\
 a_3 & 0 & b_3 & 0 & c_3 & 0 & [430] & p_0 q_3 \\
 a_4 & a_3 & b_4 & b_3 & c_4 & c_3 & 0 & p_0 q_4 + p_1 q_3 \\
 0 & a_4 & 0 & b_4 & 0 & c_4 & 0 & p_1 q_4 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$  

since

$$PQ = p_0 q_0 z_1^3 z_2 + (p_0 q_1 + p_1 q_0) z_1^2 z_2 x_1 + (p_0 q_2 + p_1 q_1) z_1 z_2 x_1^2 + p_0 q_2 z_2 x_1^2 +$$

$$+ (p_0 q_4 + p_1 q_3) z_1 x_1 x_2 + p_1 q_2 z_2 x_1^3 + p_1 q_4 x_1^2 x_2.$$  

The expansion of the determinant of $\Theta_{(3, 1)}$ with respect to the last row leads to the same formula as in [DK05, Corollary 3.4].

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