On Algebraic Semigroups and Monoids

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Abstract We present some fundamental results on (possibly nonlinear) algebraic semigroups and monoids. These include a version of Chevalley’s structure theorem for irreducible algebraic monoids, and the description of all algebraic semigroup structures on curves and complete varieties.

Key words: algebraic semigroup, algebraic monoid, algebraic group.

MSC classes: 14L10, 14L30, 20M32, 20M99.
1 Introduction

Algebraic semigroups are defined in very simple terms: they are algebraic varieties endowed with a composition law which is associative and a morphism of varieties. So far, their study has focused on the class of linear algebraic semigroups, that is, of closed subvarieties of the space of \( n \times n \) matrices that are stable under matrix multiplication; note that for an algebraic semigroup, being linear is equivalent to being affine. The theory has been especially developed by Putcha and Renner for linear algebraic monoids, i.e., those having a neutral element (see the books [23, 26]).

In addition, there has been recent progress on the structure of (possibly nonlinear) algebraic monoids: by work of Rittatore, the invertible elements of any irreducible algebraic monoid \( M \) form an algebraic group \( G(M) \), open in \( M \) (see [28, Thm. 1]). Moreover, \( M \) is linear if and only if so is \( G(M) \) (see [29, Thm. 5]). Also, the structure of normal irreducible algebraic monoids reduces to the linear case, as shown by Rittatore and the author: any such monoid is a homogeneous fiber bundle over an abelian variety, with fiber a normal irreducible linear algebraic monoid (see [5, Thm. 4.1], and [27] for further developments). This was extended by the author to all irreducible monoids in characteristic 0 (see [3, Thm. 3.2.1]).

In this article, we obtain some fundamental results on algebraic semigroups and monoids, that include the above structure theorems in slightly more general versions. We also describe all algebraic semigroup structures on abelian varieties, irreducible curves and complete irreducible varieties. The latter result is motivated by a remarkable theorem of Mumford and Ramanujam: if a complete irreducible variety \( X \) has a (possibly nonassociative) composition law \( \mu \) with a neutral element, then \( X \) is an abelian variety with group law \( \mu \) (see [19, Chap. II, §4, Appendix]).

As in [23, 26], we work over an algebraically closed field of arbitrary characteristic (most of our results extend to a perfect field without much difficulty; this is carried out in Subsection 3.5). But we have to resort to somewhat more advanced methods of algebraic geometry, as the varieties under consideration are not necessarily affine. For example, to show that every algebraic semigroup has an idempotent, we use an argument of reduction to a finite field, while the corresponding statement for affine algebraic semigroups follows from linear algebra. Also, we occasionally use some semigroup and monoid schemes (these are briefly considered in [9, Chap. II]), but we did not endeavour to study them systematically.

This text is organized as follows. Section 2 presents general results on idempotents of algebraic semigroups and on invertible elements of algebraic monoids. Both topics are fairly interwoven: for example, the fact that every algebraic monoid having no nontrivial idempotent is a group (whose proof is again more involved than in the linear case) implies a version of the Rees structure theorem for simple algebraic semigroups. In Section 3, we show that
the Albanese morphism of an irreducible algebraic monoid $M$ is a homogeneous fibration with fiber an affine monoid scheme. This generalization of the main result of [3] is obtained via a new approach, based on the consideration of the universal homomorphism from $M$ to an algebraic group. In Section 4, we describe all semigroup structures on certain classes of varieties. We begin with the easy case of abelian varieties; as an unexpected consequence, we show that all the maximal submonoids of a given irreducible algebraic semigroup have the same Albanese variety. Then we show that every irreducible semigroup of dimension 1 is either an algebraic group or an affine monomial curve; this generalizes a result of Putcha in the affine case (see [21, Thm. 2.13] and [22, Thm. 2.9]). We also describe all complete irreducible semigroups, via another variant of the Rees structure theorem. Next, we obtain two general rigidity results; one of them implies (in loose words) that the automorphisms of a complete variety are open and closed in the endomorphisms. This has applications to complete algebraic semigroups, and yields another approach to the above theorem of Mumford and Ramanujam. Finally, we determine all families of semigroup laws on a given complete irreducible variety.

This article makes only the first steps in the study of (possibly nonlinear) algebraic semigroups and monoids, which presents many open questions. From the viewpoint of algebraic geometry, it is an attractive problem to describe all algebraic semigroup structures on a given variety. Our classes of examples suggest that the associativity condition imposes strong restrictions which might make this problem tractable: for instance, the composition laws on the affine line are of course all the polynomial functions in two variables, but those that are associative are obtained from the maps $(x, y) \mapsto 0, x, y, x + y$ or $xy$ by a change of coordinate. From the viewpoint of semigroup theory, it is natural to investigate the structure of an algebraic semigroup in terms of its idempotents and the associated (algebraic) subgroups. Here a recent result of Renner and the author (see [6]) asserting that every algebraic semigroup $S$ is strongly $\pi$-regular (i.e., for any $x \in S$, some power $x^m$ belongs to a subgroup) opens the door to further developments.

**Notation and conventions.** Throughout this article, we fix an algebraically closed field $k$. A variety is a reduced, separated scheme of finite type over $k$; in particular, varieties need not be irreducible. By a point of a variety $X$, we mean a closed (or equivalently, $k$-rational) point; we may identify $X$ to its set of points equipped with the Zariski topology and with the structure sheaf $\mathcal{O}_X$. Morphisms of varieties are understood to be $k$-morphisms.

The textbook [13] will be our standard reference for algebraic geometry, and [10] for commutative algebra. We will also use the books [33] and [19] for some basic results on linear algebraic groups, resp. abelian varieties.
2 Algebraic semigroups and monoids

2.1 Basic definitions and examples

Definition 1. An (abstract) semigroup is a set $S$ equipped with an associative composition law $\mu : S \times S \to S$. When $S$ is a variety and $\mu$ is a morphism, we say that $(S, \mu)$ is an algebraic semigroup.

A neutral (resp. zero) element of a semigroup $(S, \mu)$ is an element $x_o \in S$ such that $\mu(x, x_o) = \mu(x_o, x) = x$ for all $x \in S$ (resp. $\mu(x, x_o) = \mu(x_o, x) = x_o$ for all $x \in S$).

An abstract (resp. algebraic) semigroup $(S, \mu)$ equipped with a neutral element $x_o$ is called an abstract (resp. algebraic) monoid.

An algebraic group is a group $G$ equipped with the structure of a variety, such that the group law $\mu$ and the inverse map $\iota : G \to G$, $g \mapsto g^{-1}$ are morphisms.

Clearly, a neutral element $x_o$ of a semigroup $S$ is unique if it exists; we then denote $x_o$ by $1_S$, or just by 1 if this yields no confusion. Likewise, a zero element is unique if it exists, and we then denote it by $0_S$ or 0. Also, we simply denote the semigroup law $\mu$ by $(x, y) \mapsto xy$.

Definition 2. A left ideal of a semigroup $(S, \mu)$ is a subset $I$ of $S$ such that $xy \in I$ for any $x \in S$ and $y \in I$. Right ideals are defined similarly; a two-sided ideal is of course a left and right ideal.

Definition 3. Given two semigroups $S$ and $S'$, a homomorphism of semigroups is a map $\varphi : S \to S'$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. When $S$ and $S'$ are monoids, we say that $\varphi$ is a homomorphism of monoids if in addition $\varphi(1_S) = 1_{S'}$.

A homomorphism of algebraic semigroups is a homomorphism of semigroups which is also a morphism of varieties. Homomorphisms of algebraic monoids, resp. of algebraic groups, are defined similarly.

Definition 4. An idempotent of a semigroup $S$ is an element $e \in S$ such that $e^2 = e$. We denote by $E(S)$ the set of idempotents.

Idempotents yield much insight in the structure of semigroups; this is illustrated by the following:

Remark 1. (i) Let $\varphi : S \to S'$ be a homomorphism of semigroups. Then $\varphi$ sends $E(S)$ to $E(S')$; moreover, the fiber of $\varphi$ at an arbitrary point $x' \in S'$ is a subsemigroup of $S$ if and only if $x' \in E(S')$.

(ii) Let $S$ be a semigroup, and $M \subseteq S$ a submonoid with neutral element $e$. Then $M$ is contained in the subset

$$\{x \in S \mid ex = xe = x\} = \{exe \mid x \in S\} =: eSe,$$
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which is the largest submonoid of $S$ with neutral element $e$. This defines a bijective correspondence between idempotents and maximal submonoids of $S$.

(iii) Let $S$ be a semigroup, and $e \in E(S)$. Then the subset

$$Se := \{xe \mid x \in S\} = \{x \in S \mid xe = x\}$$

is a left ideal of $S$, and the map

$$\varphi : S \rightarrow Se, \quad x \mapsto xe$$

is a retraction (i.e., $\varphi(x) = x$ for all $x \in Se$). The fiber of $\varphi$ at $e$,

$$Se := \{x \in S \mid xe = e\},$$

is a subsemigroup of $S$. Moreover, the restriction

$$\psi := \varphi|_{eS} : eS \rightarrow eS \cap Se = eSe$$

is a retraction of semigroups, that is, $\psi(x) = x$ for all $x \in eSe$ and $\psi$ is a homomorphism (indeed, $xey = xye$ for all $x \in S$ and $y \in eS$). The fiber of $\psi$ at $e$,

$$eSe := \{x \in S \mid ex = x \text{ and } xe = e\},$$

is a subsemigroup of $S$ with law $(x, y) \mapsto y$ (since $xy = xey = ey = y$ for all $x, y \in eSe$).

When $S$ is an algebraic semigroup, $E(S)$ is a closed subvariety. Moreover, $Se$, $Se$, $eSe$ and $eSe$ are closed in $S$ as well, and $\varphi$ (resp. $\psi$) is a retraction of varieties (resp. of algebraic semigroups). In particular, every maximal abstract submonoid of $S$ is closed.

Similar assertions hold for the right ideal $eS$ and the subsemigroups

$$eS := \{x \in S \mid ex = e\}, \quad eSe := \{x \in S \mid xe = x \text{ and } ex = e\}.$$

An abstract semigroup may have no idempotent; for example, the set of positive integers equipped with the addition. Yet we have:

**Proposition 1.** Any algebraic semigroup has an idempotent.

**Proof.** We use a classical argument of reduction to a finite field. Consider first the case where $k$ is the algebraic closure of a prime field $\mathbb{F}_p$. Then $S$ and $\mu$ are defined over some finite subfield $\mathbb{F}_q$ of $k$, where $q$ is a power of $p$. Thus, for any $x \in S$, the powers $x^n$, where $n$ runs over the positive integers, form a finite subsemigroup of $S$. We claim that some $x^n$ is idempotent. Indeed, we have $x^a = x^b$ for some integers $a > b > 0$. Thus, $x^b = x^{b-a+b} = x^{b+m(a-b)}$ for all $m > 0$. In particular, $x^b = x^{b(a-b+1)}$. Multiplying by $x^{b(a-b-1)}$, we obtain $x^{b(a-b)} = x^{2b(a-b)}$, i.e., $x^{b(a-b)}$ is idempotent.
Next, consider the case of an arbitrary field \( k \). Choose \( x \in S \); then \( S, \mu \) and \( x \) are defined over some finitely generated subring \( R \) of \( k \). In the language of schemes, we have a scheme \( S_R \) together with morphisms

\[
\varphi : S_R \to \text{Spec}(R), \quad \mu_R : S_R \times_{\text{Spec}(R)} S_R \to S_R
\]

and with a section \( x_R : \text{Spec}(R) \to S_R \) of \( \varphi \), such that the \( k \)-scheme \( S_R \times_{\text{Spec}(R)} \text{Spec}(k) \) is isomorphic to \( S \); moreover, this isomorphism identifies \( \mu_R \times_{\text{Spec}(R)} \text{Spec}(k) \) to \( \mu \), and \( x_R \times_{\text{Spec}(R)} \text{Spec}(k) \) to \( x \). Also, \( S_R \) is a semigroup scheme over \( \text{Spec}(R) \) (that is, \( \mu_R \) is associative in an obvious sense), and the morphism \( \varphi \) is of finite type. Denote by \( E(\text{Spec}(R)) \) the subscheme of idempotents of \( S_R \), i.e., \( E(\text{Spec}(R)) \) is the preimage of the diagonal in \( S_R \times S_R \) under the morphism \( S_R \to S_R \times S_R \), \( s \mapsto (\mu_R(s, s), s) \). Then \( E(\text{Spec}(R)) \) is a closed subscheme of \( S_R \); let \( \psi : E(\text{Spec}(R)) \to \text{Spec}(R) \) be the restriction of \( \varphi \).

We claim that the image of \( \psi \) contains all closed points of \( \text{Spec}(R) \). Indeed, consider a maximal ideal \( m \) of \( R \); then \( R/m \) is a finite field. By the first step, the semigroup \( S_R \times_{\text{Spec}(R)} \text{Spec}(R/m) \) (obtained from \( S_R \) by reduction mod \( m \)) contains an idempotent; this yields the claim.

Since \( R \) is noetherian and the morphism \( \psi \) is of finite type, its image is constructible (see e.g. [13, Exer. II.3.19]). In view of the claim, it follows that this image contains the generic point of \( \text{Spec}(R) \) (see e.g. [loc. cit., Exer. II.3.18]), i.e., \( E(\text{Spec}(R)) \) has (not necessarily closed) points over the fraction field of \( R \). Hence \( E(S) \) has (closed) points over the larger algebraically closed field \( k \).

Combining the above proposition with Remark[1](i), we obtain:

**Corollary 1.** Let \( f : S \to S' \) be a surjective homomorphism of algebraic semigroups. Then \( f(E(S)) = E(S') \).

We now present several classes of (algebraic) semigroups:

**Example 1.** (i) Any set \( X \) has two semigroup laws \( \mu_l, \mu_r \) given by \( \mu_l(x, y) := x \) (resp. \( \mu_r(x, y) := y \)) for all \( x, y \in X \). For both laws, every element is idempotent and \( X \) has no proper two-sided ideal.

Also, every point \( x \in X \) defines a semigroup law \( \mu_x \) by \( \mu_x(y, z) := x \) for all \( y, z \in X \). Then \( x \) is the zero element; it is the unique idempotent, and the unique proper two-sided ideal as well.

The maps \( \mu_l, \mu_r, \mu_x \) (\( x \in X \)) will be called the trivial semigroup laws on \( X \). When \( X \) is a variety, these maps are algebraic semigroup laws. Note that every morphism of varieties \( f : X \to Y \) yields a homomorphism of algebraic semigroups \((X, \mu_r) \to (Y, \mu_r)\), and likewise for \( \mu_l \). Also, \( f \) yields a homomorphism \((X, \mu_x) \to (Y, \mu_y)\), where \( y := f(x) \).
(ii) Let $X$ be a set, $Y \subseteq X$ a subset, $\rho : X \to Y$ a retraction, and $\nu$ a semigroup law on $Y$. Then the map

$$\mu : X \times X \to X, \quad (x_1, x_2) \mapsto \nu(\rho(x_1), \rho(x_2))$$

is easily seen to be a semigroup law on $X$. Moreover, $\rho$ is a retraction of semigroups, and $E(X) = E(Y)$. If in addition $X$ is a variety, $Y$ is a closed subvariety and $\rho, \nu$ are morphisms, then $(X, \mu)$ is an algebraic semigroup.

When $Y$ consists of a single point $x$, we recover the semigroup law $\mu_x$ of the preceding example.

(iii) Given two semigroups $(S, \mu)$ and $(S', \mu')$, we may define a composition law $\nu$ on the disjoint union $S \sqcup S'$ by

$$\nu(x, y) = \begin{cases} 
\mu(x, y) & \text{if } x, y \in S, \\
y & \text{if } x \in S \text{ and } y \in S', \\
x & \text{if } x \in S' \text{ and } y \in S, \\
\mu'(x, y) & \text{if } x, y \in S'.
\end{cases}$$

One readily checks that $(S \sqcup S', \nu)$ is a semigroup; moreover, $(S, \mu)$ is a subsemigroup and $(S', \mu')$ is a two-sided ideal. Also, note that $E(S \sqcup S') = E(S) \cup E(S')$.

When $S$ (resp. $S'$) has a zero element $0_S$ (resp. $0_{S'}$), consider the set $S \sqcup_0 S'$ obtained from $S \sqcup S'$ by identifying $0_S$ and $0_{S'}$. One checks that $S \sqcup_0 S'$ has a unique semigroup law $\nu_0$ such that the natural map $S \sqcup S' \to S \sqcup_0 S'$ is a homomorphism; moreover, the image of $0_S$ is the zero element. Here again, $S$ is a subsemigroup of $S \sqcup_0 S'$, and $S'$ is a two-sided ideal; we have $E(S \sqcup_0 S') = E(S) \cup_0 E(S')$.

If in addition $(S, \mu)$ and $(S', \mu')$ are algebraic semigroups, then so are $(S \sqcup S', \nu)$ and $(S \sqcup_0 S', \nu_0)$. This construction still makes sense when (say) $S'$ is a scheme of finite type over $k$, equipped with a closed point $0 = 0_{S'}$ and with the associated trivial semigroup law $\mu_0$. Taking for $S'$ the spectrum of a local ring of finite dimension as a $k$-vector space, and for $0$ the unique closed point of $S'$, we obtain many examples of nonreduced semigroup schemes (having a fat point at their zero element).

(iv) Any finite semigroup is algebraic. In the opposite direction, the (finite) set of connected components of an algebraic semigroup $(S, \mu)$ has a natural structure of semigroup. Indeed, if $C_1, C_2$ are connected components of $S$, then $\mu(C_1, C_2)$ is contained in a unique connected component, say, $\nu(C_1, C_2)$. The resulting composition law $\nu$ on the set of connected components, $\pi_o(S)$, is clearly associative, and the canonical map $f : S \to \pi_o(S)$ is a homomorphism of algebraic semigroups. In fact, $f$ is the universal homomorphism from $S$ to a finite semigroup.

Next, we present examples of algebraic monoids and of algebraic groups:

Example 2. (i) Consider the set $M_n$ of $n \times n$ matrices with coefficients in $k$, where $n$ is a positive integer. We may view $M_n$ as an affine space of dimension
n^2; this is an irreducible algebraic monoid relative to matrix multiplication, the neutral element being of course the identity matrix.

The subspaces $D_n$ of diagonal matrices, and $T_n$ of upper triangular matrices, are closed irreducible submonoids of $M_n$. Note that $D_n$ is isomorphic to the affine $n$-space $\mathbb{A}^n$ equipped with pointwise multiplication.

An example of a closed reducible submonoid of $M_n$ consists of those matrices having at most one nonzero entry in each row and each column. This submonoid, that we denote by $R_n$, is the closure in $M_n$ of the group of monomial matrices (those having exactly one nonzero entry in each row and each column). Note that $R_n = D_n S_n$, where $S_n$ denotes the symmetric group on $n$ letters, viewed as the group of permutation matrices. Thus, the irreducible components of $R_n$ are parametrized by $S_n$. Each such component contains the zero matrix; in particular, $R_n$ is connected.

(ii) A linear algebraic monoid is a closed submonoid $M$ of some matrix monoid $M_n$. Then the variety $M$ is affine; conversely, every affine algebraic monoid is linear (see [9, Thm. II.2.3.3]). It follows that every affine algebraic semigroup is linear as well, see [23, Cor. 3.16].

(iii) Let $A$ be a $k$-algebra (not necessarily associative, or commutative, or unital) and denote by $\text{End}(A)$ the set of algebra endomorphisms of $A$. Then $\text{End}(A)$, equipped with the composition of endomorphisms, is an (abstract) monoid with zero. Its idempotents are exactly the retractions of $A$ to subalgebras. Given such a retraction $e : A \to B$, we have

$$e \text{End}(A) \cong \text{Hom}(A, B), \quad \text{End}(A)e \cong \text{Hom}(B, A), \quad e \text{End}(A)e \cong \text{End}(B),$$

where $\text{Hom}$ denotes the set of algebra homomorphisms. Also, $\text{End}(A)_e$ (resp. $\text{End}(A)_e^-$) consists of those $\varphi \in \text{End}(A)$ such that $\varphi(x) = x$ for all $x \in B$ (resp. $\varphi(x) - x \in I$ for all $x \in A$, where $I$ denotes the kernel of $e$).

If $A$ is finite-dimensional as a $k$-vector space, then $\text{End}(A)$ is a linear algebraic monoid; indeed, it identifies to a closed submonoid of $M_n$, where $n := \dim(A)$.

(iv) Examples of algebraic groups include:
- the additive group $\mathbb{G}_a$, i.e., the affine line equipped with the addition,
- the multiplicative group $\mathbb{G}_m$, i.e., the affine line minus the origin, equipped with the multiplication,
- the elliptic curves, i.e., the complete nonsingular irreducible curves of genus 1, equipped with a base point; then there is a unique algebraic group structure for which this point is the neutral element, see e.g. [13, Chap. II, §4]).

In fact, these examples yield all the connected algebraic groups of dimension 1, see [15, Prop. 10.7.1].

(v) A complete connected algebraic group is called an abelian variety; elliptic curves are examples of such algebraic groups. It is known that every abelian variety $A$ is a commutative group and a projective variety; moreover, the group law on $A$ is uniquely determined by the structure of variety and the neutral element (see [19, Chap. II]).
2.2 The unit group of an algebraic monoid

In this section, we obtain some fundamental results on the group of invertible elements of an algebraic monoid. We shall need the following observation:

**Proposition 2.** Let \((M, \mu)\) be an algebraic monoid. Then \(M\) has a unique irreducible component containing 1: the neutral component \(M^0\). Moreover, \(M^0 X = X M^0 = X\) for any irreducible component \(X\) of \(M\); in particular, \(M^0\) is a closed submonoid of \(M\).

**Proof.** Let \(X, Y\) be irreducible components of \(M\). Then \(XY\) is the image of the restriction of \(\mu\) to \(X \times Y\), and hence is a constructible subset of \(M\); moreover, its closure \(\overline{XY}\) is an irreducible subvariety of \(M\). If \(1 \in X\), then \(Y \subseteq XY \subseteq \overline{XY}\). Since \(Y\) is an irreducible component, we must have \(Y = XY = X Y\); likewise, one obtains that \(YX = Y\). In particular, \(XX = X\), i.e., \(X\) is a closed submonoid. If in addition \(1 \in Y\), then also \(XY = YX = Y\), hence \(Y = X\). This yields our assertions. \(\Box\)

**Remark 2.** Any algebraic group \(G\) is a nonsingular variety, and hence every connected component of \(G\) is irreducible. Moreover, the neutral component \(G^0\) is a closed normal subgroup, and the quotient group \(G/G^0\) parametrizes the components of \(G\).

In contrast, there exist connected reducible algebraic monoids: for example, the monoid \(R_n\) of Example 2 (i). Also, algebraic monoids are generally singular; for example, the zero locus of \(z^2 - xy\) in \(A^3\) equipped with pointwise multiplication.

On a more advanced level, note that any group scheme is reduced in characteristic 0 (see e.g. [9, Thm. II.6.1.1]). In contrast, there always exist nonreduced monoid schemes. For example, one may stick an arbitrary fat point at the origin of the multiplicative monoid \((A^1, \times)\), by the construction of Example 1 (iii).

**Definition 5.** Let \(M\) be a monoid and let \(x, y \in M\). Then \(y\) is a left (resp. right) inverse of \(x\) if \(yx = 1\) (resp. \(xy = 1\)). We say that \(x\) is invertible (also called a unit) if it has a left and a right inverse.

With the above notation, one readily checks that the left and right inverses of any unit \(x \in M\) are equal. Moreover, if \(x' \in M\) is another unit with inverse \(y'\), then \(xy'\) is a unit with inverse \(x'y\). Thus, the invertible elements of \(M\) form a subgroup: the unit group, that we denote by \(G(M)\).

The following result on unit groups of algebraic monoids is due to Rittatore in the irreducible case (see [28, Thm. 1]). The proof presented here follows similar arguments.

**Theorem 1.** Let \(M\) be an algebraic monoid. Then \(G(M)\) is an algebraic group, open in \(M\). In particular, \(G(M)\) consists of nonsingular points of \(M\).
Proof. \begin{align*} G := \{(x, y) \in M \times M^{\text{op}} \mid xy = yx = 1\}, \end{align*}
where $M^{\text{op}}$ denotes the opposite monoid to $M$, i.e., the variety $M$ equipped with the composition law $(x, y) \mapsto yx$. One readily checks that $G$ (viewed as a closed subvariety of $M \times M^{\text{op}}$) is a submonoid; moreover, every $(x, y) \in G$ has inverse $(y, x)$. Thus, $G$ is a closed algebraic subgroup of $M \times M^{\text{op}}$.

The first projection $p_1 : M \times M^{\text{op}} \to M$ restricts to a homomorphism of monoids $\pi : G \to M$ with image being the unit group $G(M)$. In fact, $G$ acts on $M$ by left multiplication: $(x, y) \cdot z := xz$, and $\pi$ is the orbit map $(x, y) \mapsto (x, y) \cdot 1$; in particular, $G(M)$ is the $G$-orbit of 1. The isotropy subgroup of 1 in $G$ is clearly trivial as a set. We claim that this also holds as a scheme; in other words, the isotropy Lie algebra of 1 is trivial as well.

To check this, recall that the Lie algebra $\text{Lie}(G)$ is the Zariski tangent space $T_{(1,1)}(G)$ and hence is contained in $T_{(1,1)}(M \times M^{\text{op}}) \cong T_1(M) \times T_1(M)$. Since the differential at $(1, 1)$ of the monoid law $\mu : M \times M \to M$ is the map
\begin{align*}
T_1(M) \times T_1(M) & \to T_1(M), \quad (x, y) \mapsto x + y,
\end{align*}
we have
\begin{align*}
T_{(1,1)}(G) & \subseteq \{(x, y) \in T_1(M) \times T_1(M) \mid x + y = 0\}.
\end{align*}
Thus, the first projection $\text{Lie}(G) \to T_1(M)$ is injective; but this projection is the differential of $\pi$ at $(1, 1)$. This proves our claim.

By that claim, $\pi$ is a locally closed immersion. Thus, $G(M)$ is a locally closed subvariety of $M$, and $\pi$ induces an isomorphism of groups $G \cong G(M)$. So $G(M)$ is an algebraic group.

It remains to show that $G(M)$ is open in $M$; it suffices to check that $G(M)$ contains an open subset $U$ of $M$ (then the translates $gU$, where $g \in G(M)$, form a covering of $G(M)$ by open subsets of $M$). For this, we may replace $M$ with its neutral component $M^{\text{op}}$ (Proposition \[2\]) and hence assume that $M$ is irreducible. Note that

\begin{align*}
G(M) = \{x \in M \mid xy = z x = 1 \text{ for some } y, z \in M\}
\end{align*}

(then $y = zxy = z$). In other words,
\begin{align*}
G(M) = p_1(\mu^{-1}(1)) \cap p_2(\mu^{-1}(1)),
\end{align*}
where $p_1, p_2 : M \times M \to M$ denote the projections. Also, the set-theoretic fiber at 1 of the restriction $p_1 : \mu^{-1}(1) \to M$ consists of the single point 1. By a classical result on the dimension of fibers of a morphism (see \[13\] Exer. II.3.22), it follows that every irreducible component $C$ of $\mu^{-1}(1)$ containing 1 satisfies $\dim(C) = \dim(M)$, and that the restriction $p_1 : C \to M$ is dominant. Thus, $p_1(C)$ contains a dense open subset of $M$. Likewise, $p_2(C)$ contains a dense open subset of $M$, and hence so does $G(M)$.
Note that the unit group of a linear algebraic monoid is linear, see [23, Cor. 3.26]. Further properties of the unit group are gathered in the following:

**Proposition 3.** Let $M$ be an algebraic monoid, and $G$ its unit group.

(i) If $x \in M$ has a left (resp. right) inverse, then $x \in G$.
(ii) $M \setminus G$ is the largest proper two-sided ideal of $M$.
(iii) If $1$ is the unique idempotent of $M$, then $M = G$.

**Proof.** (i) Assume that $x$ has a left inverse $y$. Then the left multiplication $M \to M$, $z \mapsto zx$ is an injective endomorphism of the variety $M$. By [1, Thm. C] (see also [2]), this endomorphism is surjective, and hence there exists $z \in M$ such that $xz = 1$. Then $y = yxz = z$, i.e., $x \in G$. The case where $x$ has a right inverse is handled similarly.

(ii) Clearly, any proper two-sided ideal of $M$ is contained in $M \setminus G$. We show that the latter is a two-sided ideal: let $x \in M \setminus G$ and $y \in M$. If $xy \in G$, then $y(xy)^{-1}$ is a right inverse of $x$. By (i), it follows that $x \in G$, a contradiction. Thus, $(M \setminus G)M \subseteq M \setminus G$. Likewise, $M(M \setminus G) \subseteq M \setminus G$.

(iii) By Theorem 1, $M \setminus G$ is closed in $M$; also, $M \setminus G$ is a subsemigroup of $M$ by (ii). Thus, if $M \neq G$ then $M \setminus G$ contains an idempotent, in view of Proposition 1. □

### 2.3 The kernel of an algebraic semigroup

In this subsection, we show that every algebraic semigroup has a smallest two-sided ideal (called its kernel) and we describe the structure of that ideal, thereby generalizing some of the known results about the kernel of a linear algebraic semigroup (see [23, 14]).

First, recall that the idempotents of any (abstract) semigroup $S$ are in bijective correspondence with the maximal submonoids of $S$, via $e \mapsto eSe$, and hence with the maximal subgroups of $S$, via $e \mapsto G(eSe)$. Thus, when $S$ is an algebraic semigroup, its maximal subgroups are all locally closed in view of Theorem 1. They are also pairwise disjoint: if $e \in E(S)$ and $x \in G(eSe)$, then there exists $y \in eSe$ such that $xy = yx = e$. Thus, $xSx \supseteq xySyx = eSe$. But $xSx \subseteq eSeSeSe \subseteq eSe$, and hence $xSx = eSe$. So $xSx$ is a closed submonoid of $S$ with neutral element $e$.

Next, we recall the classical definition of a partial order on the set of idempotents of any (abstract) semigroup:

**Definition 6.** Let $S$ be a semigroup and let $e, f \in E(S)$. Then $e \leq f$ if we have $e = ef = fe$.

Note that $e \leq f$ if and only if $e \in fSf$; this is also equivalent to the condition that $eSe \subseteq fSf$. Thus, $\leq$ is indeed a partial order on $E(S)$ (this fact may of course be checked directly). Also, note that $\leq$ is preserved by
every homomorphism of semigroups. For an algebraic semigroup, the partial order \( \leq \) satisfies additional finiteness properties:

**Proposition 4.** Let \( S \) be an algebraic semigroup.

(i) Every subset of \( E(S) \) has a minimal element with respect to the partial order \( \leq \), and also a maximal element.
(ii) \( e \in E(S) \) is minimal among all idempotents if and only if \( eSe \) is a group.
(iii) If \( S \) is commutative, then \( E(S) \) is finite and has a smallest element.

**Proof.** (i) Note that the \( eSe \), where \( e \in E(S) \), form a family of closed subsets of the noetherian topological space \( S \); hence any subfamily has a minimal element. For the existence of maximal elements, consider the family \( S \times SfS := \{(x,xe) \in S \times S | xe = ye \} \) of closed subsets of \( S \times S \). Let \( f \in E(S) \) such that \( e \leq f \). Then \( S \times SfS \subseteq S \times SfS \), since the equality \( xf = yf \) implies that \( xe = xfe = yfe = ye \). Moreover, if \( S \times SfS = S \times SfS \), then \( (x,xe) \in S \times SfS \) for all \( x \in S \), i.e., \( xf = xef \). Hence \( xf = xe \), and \( f = f^2 = fe = e \). Thus, a minimal \( S \times SfS \) (for \( e \) in a given subset of \( E(S) \)) yields a maximal \( e \).

(ii) Let \( e \) be a minimal idempotent of \( S \). Then \( e \) is the unique idempotent of the algebraic monoid \( eSe \). By Proposition 3 (iii), it follows that \( eSe \) is a group. The converse is immediate (and holds for any abstract semigroup).

(iii) Let \( e,f \) be idempotents. Then \( ef = fe \) is also idempotent, and \( ef = e(ef)e = f(ef)f \) so that \( ef \leq e \) and \( ef \leq f \). Thus, any two minimal idempotents are equal, i.e., \( E(S) \) has a smallest element.

To show that \( E(S) \) is finite, we may replace \( S \) with its closed subsemigroup \( E(S) \), and hence assume that every element of \( S \) is idempotent. Then every connected component of \( S \) is a closed subsemigroup in view of Example 1 (iv). So we may further assume that \( S \) is connected. Let \( x \in S \); then \( xS \) is a connected commutative algebraic monoid with neutral element \( x \), and consists of idempotents. Thus, \( G(xS) = \{x\} \). By Theorem 1 it follows that \( x \) is an isolated point of \( xS \). Hence \( xS = \{x\} \), i.e., \( xy = x \) for all \( y \in S \). Since \( S \) is commutative, we must have \( S = \{x\} \).

As a consequence of the above proposition, every algebraic semigroup admits minimal idempotents. These are of special interest, as shown by the following:

**Proposition 5.** Let \( S \) be an algebraic semigroup, \( e \in S \) a minimal idempotent, and \( eSe \) the associated closed subgroup of \( S \).

(i) The map
\[
\rho = \rho_e : S \rightarrow S, \quad s \mapsto s(ese)^{-1}s
\]
is a retraction of varieties of \( S \) to \( SeS \). In particular, \( SeS \) is a closed two-sided ideal of \( S \).
(ii) The map\[\varphi : eSe \times eSe \times eSe \rightarrow S, \quad (x, g, y) \mapsto xgy\]
yields an isomorphism of varieties to its image, $SeS$.

(iii) Via the above isomorphism, the semigroup law on $SeS$ is identified to that on $eSe \times eSe \times eSe$, given by
\[(x, g, y)(x', g', y') = (x, g\pi(y, x')g', y'),\]
where $\pi : eSe \times eSe \rightarrow eSe$ denotes the map $(y, z) \mapsto yz$. This identifies the idempotents of $SeS$ to the triples $(x, \pi(y, x)^{-1}, y)$, where $x \in eSe$ and $y \in eSe$. In particular, $E(SeS) \cong eSe \times eSe$ as a variety.

(iv) The semigroup $SeS$ has no proper two-sided ideal.

(v) $SeS$ is the smallest two-sided ideal of $S$; in particular, it does not depend on the minimal idempotent $e$.

(vi) The minimal idempotents of $S$ are exactly the idempotents of $SeS$.

Proof. (i) Clearly, $\rho$ is a morphism; also, since $s(ese)^{-1}s \in SeSeS$ for all $s \in S$, the image of $\rho$ is contained in $SeS$. Let $z \in SeS$ and write $z = set$, where $s, t \in S$; then
\[\rho(z) = set(ese)^{-1}set = set(ete)^{-1}(ese)^{-1}set = set = z.\]
This yields the assertions.

(ii) Clearly, $\varphi$ takes values in $SeS$. Moreover, for any $s, t$ as above, we have
\[set = se(ese)^{-1}esete(ete)^{-1}t = xgy,\]
where $x := se(ese)^{-1}$, $g := esete$ and $y := (ete)^{-1}et$. Furthermore, $x \in eSe$, $g \in eSe$ and $y \in eSe$. In particular, the image of $\varphi$ is the whole $SeS$. Also, the map
\[\psi : SeS \rightarrow eSe \times eSe \times eSe, \quad set \mapsto (x, g, y)\]
(where $x, g, y$ are defined as above) is a morphism of varieties and satisfies $\varphi \circ \psi = \text{id}$. Thus, it suffices to check that $\psi \circ \varphi = \text{id}$. Let $x \in eSe$, $g \in eSe$, $y \in eSe$ and put $s := xgy$. Then $se = xg$ and $es = gy$. Hence $g = es$, $x = se(ese)^{-1}$ and $y = (ese)^{-1}es$, which yields the desired assertion.

(iii) For the first assertion, just write $(xgy)(x'g'y') = x(g(yx')g')y'$, and note that $xy' \in eSe \subseteq eSe$. The assertions on idempotents follow readily.

(iv) Let $z \in SeS$ and write $z = \varphi(x, g, y)$. Then the subset $SzS$ of $SeS$ is identified with that of $eSe \times eSe \times eSe$ consisting of the triples $(x_1, g_1\pi(y_1, x)g\pi(y, x_2)g_2, y_2)$, where $x, x_2 \in eSe$, $g_1, g_2 \in eSe$ and $y, y_1, y_2 \in eSe$. It follows that $SzS = SeS$; in particular, $SzS$ contains $z$. Hence $SeS$ is the smallest two-sided ideal containing $z$.

(v) Let $I$ be a two-sided ideal of $S$. Then $SeI$ is a two-sided ideal of $S$ contained in $SeS$; hence $SeI = SeS$ by (iv). But $SeI \subseteq I$; this yields our assertions.
If \( f \in E(S) \) is minimal, then \( SfS = SeS \) by (v). Thus, \( f \in SeS \).

For the converse, let \( f \in E(SeS) \). Then \( SfS = SeS \) by (iv), and hence \( fSf = fSfS = f(SeS)f \). Identifying \( f \) to a triple \( (x, \pi(y, x)^{-1}, y) \), one checks as in the proof of (iv) that \( f(SeS)f \) is identified to the set of triples \( (x, g, y) \), where \( g \in eSe \). But \((x, \pi(y, x)^{-1}, y)\) is the unique idempotent of this set. Thus, \( f \) is the unique idempotent of \( fSf \), i.e., \( f \) is minimal.

In view of these results, we may set up the following:

**Definition 7.** The kernel of an algebraic semigroup \( S \) is the smallest two-sided ideal of \( S \), denoted by \( \ker(S) \).

**Remark 3.** (i) As a consequence of Proposition 5, we see that any algebraic semigroup having no proper closed two-sided ideal is simple, i.e., has no proper two-sided ideal at all. Moreover, any simple algebraic semigroup \( S \), equipped with an idempotent \( e \), is isomorphic (as a variety) to the product \( X \times G \times Y \), where \( X := eSe \) and \( Y := eSe \) are varieties, and \( G := eSe \) is an algebraic group. This identifies \( e \) to a point of the form \((x_o, 1, y_o)\), where \( x_o \in X \) and \( y_o \in Y \); moreover, the semigroup law of \( S \) is identified to that as in Proposition 3 (iii), where \( \pi : Y \times X \to G \) is a morphism such that \( \pi(x_o, y) = \pi(x, y_o) = 1 \) for all \( x \in X \) and \( y \in Y \).

Conversely, any tuple \((X, Y, G, \pi, x_o, y_o)\) satisfying the above conditions defines an algebraic semigroup law on \( S := X \times G \times Y \) such that \( e := (x_o, 1, y_o) \) is idempotent and \( eSe = X \times \{(1, y_o)\} \), \( eSe = \{x_o\} \times G \times \{y_o\} \), \( eSe = \{(x_o, 1)\} \times Y \).

This description of algebraic semigroups having no proper closed two-sided ideal is a variant of the classical Rees structure theorem for those (abstract) semigroups that are completely simple, that is, simple and having a minimal idempotent (see e.g. [23, Thm. 1.9]).

(ii) By analogous arguments, one shows that every algebraic semigroup \( S \) contains minimal left ideals, and these are exactly the subsets \( Sf \), where \( f \) is a minimal idempotent. In particular, the minimal left ideals are all closed. Also, given a minimal idempotent \( e \), these ideals are exactly the subsets \( X \times G \times \{y\} \) of \( \ker(S) \), where \( X := eSe \), \( G := eSe \) and \( y \in Y := eSe \), as above. Similar assertions hold of course for the minimal right ideals; it follows that the intersections of minimal left and minimal right ideals are exactly the subsets \( \{x\} \times G \times \{y\} \), where \( x \in X \) and \( y \in Y \).

### 2.4 Unit dense algebraic monoids

In this subsection, we introduce and study the class of unit dense algebraic monoids. These include the irreducible algebraic monoids, and will play an important role in their structure.
Let $M$ be an algebraic monoid, and $G(M)$ its unit group. Then the algebraic group $G(M) \times G(M)$ acts on $M$ via left and right multiplication: $(g,h) \cdot x := gxh^{-1}$. Moreover, the orbit of 1 under this action is just $G(M)$, and the isotropy subgroup scheme of 1 equals the diagonal, $\text{diag}(G(M)) := \{(g,g) \mid g \in G(M)\}$.

**Definition 8.** An algebraic monoid $M$ is *unit dense* if $G(M)$ is dense in $M$.

For instance, every irreducible algebraic monoid is unit dense. An example of a reducible unit dense algebraic monoid consists of $n \times n$ matrices having at most one nonzero entry in each row and each column (Example 2 (i)).

Any unit dense monoid may be viewed as an equivariant embedding of its unit group, in the sense of the following:

**Definition 9.** Let $G$ be an algebraic group. An *equivariant embedding* of $G$ is a variety $X$ equipped with an action of $G \times G$ and with a point $x \in X$ such that the orbit $(G \times G) \cdot x$ is dense in $X$, and the isotropy subgroup scheme $(G \times G)_x$ is the diagonal, $\text{diag}(G)$.

Note that the law of a unit dense monoid is uniquely determined by its structure of equivariant embedding, since that structure yields the law of the unit group. Also, given an affine algebraic group $G$, every affine equivariant embedding of $G$ has a unique structure of algebraic monoid such that $G$ is the unit group, by [28, Prop. 1]. Conversely, every unit dense algebraic monoid with unit group $G$ is affine by Theorem 2 below. For an arbitrary connected algebraic group $G$, the equivariant embeddings of $G$ that admit a monoid structure are characterized in Theorem 4 below.

**Proposition 6.** Let $M$ be an algebraic monoid, and $G$ its unit group. Then the unit group of the neutral component $M^o$ is the neutral component $G^o$ of $G$.

If $M$ is unit dense, then its irreducible components are exactly the subsets $gM^o$, where $g \in G$; they are indexed by $G/G^o$, the group of components of $G$.

**Proof.** Note that $G(M^o)$ is contained in $G$, and open in $M^o$ by Theorem 1. Hence $G(M^o)$ contains an open neighborhood of 1 in $M^o$, or equivalently in $G$. Using the group structure, it follows that $G(M^o)$ is open in $G$; also, $G(M^o)$ is irreducible since so is $M^o$. But the algebraic group $G$ contains a unique open irreducible subgroup: its neutral component. Thus, $G(M^o) = G^o$.

Clearly, $gM^o$ is an irreducible component of $M$ for any $g \in G$, and this component depends only on the coset $gG^o$. If $M$ is unit dense, then any irreducible component $X$ of $M$ contains a unit, say $g$. Since $g^{-1}X$ is an irreducible component containing 1, it follows that $X = gM^o$. If $X = hM^o$ for some $h \in G$, then $g^{-1}h \in G \cap M^o$. Thus, $g^{-1}hG^o$ is an open subset of $M^o$, and hence meets $G^o$; so $g^{-1}h \in G^o$, i.e., $gG^o = hG^o$. □
Proposition 7. Let $M$ be a unit dense algebraic monoid, and $G$ its unit group. Then the kernel, $\ker(M)$, is the unique closed orbit of $G \times G$ acting by left and right multiplication. Moreover, $\ker(M) = GeG$ for any minimal idempotent $e$ of $M$.

Proof. We may choose a closed $G \times G$-orbit $Y$ in $M$. Then

$$MYM = \overline{GYG} \subseteq \overline{GYG} = Y = Y.$$ 

Thus, $Y$ is a two-sided ideal of $M$. Moreover, if $Z$ is another two-sided ideal, then $Z$ is stable by $G \times G$, and meets $Y$ since $YZ \subseteq Y \cap Z$. Thus, $Z$ contains $Y$; this shows that $Y = \ker(M)$. In particular, $Y$ is the unique closed $G \times G$-orbit; this proves the first assertion. The second one follows from Proposition 5.

Proposition 8. Let $M$ be a unit dense algebraic monoid with unit group $G$. Then the following conditions are equivalent for any $x \in M$:

(i) The orbit $Gx$ (for the $G$-action by left multiplication) is closed in $M$.
(ii) $Gx = Mx$.
(iii) $x \in \ker(M)$.

Moreover, all closed $G$-orbits in $M$ are equivariantly isomorphic; in other words, the isotropy subgroup schemes $G_x$, where $x \in \ker(M)$, are all conjugate. Also, each closed orbit contains a minimal idempotent. For any such idempotent $e$, the algebraic group $eMe$ equals $eGe$.

Proof. (i)$\Rightarrow$(ii) Since $Gx$ is closed in $M$, we have $Mx = \overline{Gx} \subseteq \overline{Gx} = Gx$ and hence $Mx = Gx$.

(ii)$\Rightarrow$(iii) We have $Gx = Mx \supset \ker(M)x$ and the latter subset is stable under left multiplication by $G$. Hence $Gx = \ker(M)x$ is contained in $\ker(M)$.

(iii)$\Rightarrow$(i) Let $e$ be a minimal idempotent of $M$. Since $\ker(M) = GeG$, the $G$-orbits in $\ker(M)$ are exactly the orbits $Geg$, where $g \in G$. Since the right multiplication by $g$ is an automorphism of the variety $M$ commuting with left multiplications, these orbits are all isomorphic as $G$-varieties. In particular, they all have the same dimension; hence they are closed in $\ker(M)$, and thus in $M$. Also, the orbit $Geg$ contains $g^{-1}eg$, which is a minimal idempotent since the map $M \to M, x \mapsto g^{-1}xg$ is an automorphism of algebraic monoids. Finally, we have $Ge = Me$ by (i); likewise, $eG = eM$ and hence $eGe = eMe$.

Note that the closed orbits for the left $G$-action are exactly the minimal left ideals (considered in Remark 3(ii) in the setting of algebraic semigroups).

2.5 The normalization of an algebraic semigroup

In this subsection, we begin by recalling some background results on the normalization of an arbitrary variety (see e.g. [10, §4.2, §11.2]). Then we
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discuss the normalization of algebraic semigroups and monoids; as in the previous subsection, this construction will play an important role in their structure.

A variety $X$ is normal at a point $x$ if the local ring $O_{X,x}$ is integrally closed in its total quotient ring; $X$ is normal if it is so at any point. The normal points of a variety form a dense open subset, which contains the nonsingular points. The irreducible components of a normal variety are pairwise disjoint, and each of them is normal.

An arbitrary variety $X$ has a normalization, i.e., a normal variety $\tilde{X}$ together with a finite surjective morphism $\eta: \tilde{X} \to X$ which satisfies the following universal property: for any normal variety $Y$ and any morphism $\varphi: Y \to X$ which is dominant (i.e., the image of $\varphi$ is dense in $X$), there exists a unique morphism $\tilde{\varphi}: Y \to \tilde{X}$ such that $\varphi = \eta \circ \tilde{\varphi}$. Then $\tilde{X}$ is uniquely determined up to unique isomorphism, and $\eta$ is an isomorphism above the open subset of normal points of $X$; in particular, $\eta$ is birational (i.e., an isomorphism over a dense open subset of $X$).

**Proposition 9.** Let $(S, \mu)$ be an algebraic semigroup and let $\eta: \tilde{S} \to S$ be the normalization.

(i) If the morphism $\mu: S \times S \to S$ is dominant, then $\tilde{S}$ has a unique algebraic semigroup law $\tilde{\mu}$ such that $\eta$ is a homomorphism. Moreover, $\eta(E(\tilde{S})) = E(S)$.

(ii) If $S$ is an algebraic monoid (so that $\mu$ is surjective), then $\tilde{S}$ is an algebraic monoid as well, with neutral element the unique preimage of $1_S$ under $\eta$.

Moreover, $\eta$ induces an isomorphism $G(\tilde{S}) \cong G(S)$.

**Proof.** (i) By the assumption on $\mu$, the morphism $\mu \circ (\eta \times \eta): \tilde{S} \times \tilde{S} \to \tilde{S}$ is dominant. Since $\tilde{S} \times \tilde{S}$ is normal, there exists a unique morphism $\tilde{\mu}: \tilde{S} \times \tilde{S} \to \tilde{S}$ such that $\eta \circ \tilde{\mu} = \mu \circ (\eta \times \eta)$. Then $\tilde{\mu}$ is associative, since it coincides with $\mu$ on the dense open subset of normal points; moreover, $\eta$ is a homomorphism by construction. The assertion on idempotents is a consequence of Corollary 1.

(ii) The neutral element $1_S$ is a nonsingular point of $S$ by Theorem 1; thus, it has a unique preimage $1_{\tilde{S}}$ under $\eta$. Moreover, we have for any $\tilde{x} \in \tilde{S}$:

$$\eta(\tilde{\mu}(\tilde{x}, 1_{\tilde{S}})) = \mu(\eta(\tilde{x}), \eta(1_{\tilde{S}})) = \eta(\tilde{x}) = \eta(\tilde{\mu}(1_{\tilde{S}}, \tilde{x})).$$

Thus, $\tilde{\mu}(\tilde{x}, 1_{\tilde{S}}) = \tilde{\mu}(1_{\tilde{S}}, \tilde{x}) = \tilde{x}$ for all $\tilde{x}$ such that $\eta(\tilde{x})$ is a normal point of $S$. By density of these points, it follows that $1_{\tilde{S}}$ is the neutral element of $(\tilde{S}, \tilde{\mu})$. Finally, the assertion on unit groups follows from the inclusion $G(\tilde{S}) \subseteq \eta^{-1}(G(S))$ and from the fact that $\eta$ is an isomorphism above the nonsingular locus of $S$. 

**Remark 4.** (i) For an arbitrary algebraic semigroup $S$, there may exist several algebraic semigroup laws on the normalization $\tilde{S}$ that lift $\mu$. For example, let $x \in S$ and consider the trivial semigroup law $\mu_x$ of Example 1. Then $\mu_{\tilde{x}}$
lifts $\mu$ for any $\tilde{x} \in \tilde{X}$ such that $\eta(\tilde{x}) = x$. In general, such a point $\tilde{x}$ is not unique, e.g., when $S$ is a plane curve and $x$ an ordinary multiple point.

(ii) With the above notation, there may also exist no algebraic semigroup law on $\tilde{S}$ that lifts $\mu$. To construct examples of such algebraic semigroups, consider a normal irreducible affine variety $X$ and a complete nonsingular irreducible curve $C$, and choose a finite surjective morphism $\varphi : C \to \mathbb{P}^1$. Let $Y := C \setminus \{\varphi^{-1}(\infty)\}$; then $Y$ is an affine nonsingular irreducible curve equipped with a finite surjective morphism $\varphi : Y \to \mathbb{A}^1$. Choose a point $x_o \in X$ and let $\gamma : Y \to X \times Y, y \mapsto (x_o, y)$; then $\gamma$ is a section of the second projection $p_2 : X \times Y \to Y$. By [11, Thm. 5.1], there exists a unique irreducible variety $S$ that sits in a co-cartesian diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & \mathbb{A}^1 \\
\downarrow \gamma & & \downarrow \mathrm{id} \\
X \times Y & \xrightarrow{\eta} & S.
\end{array}
\]

Then $\iota$ is a closed immersson, and $\eta$ is a finite morphism that restricts to an isomorphism $(X \setminus \{x_o\}) \times Y \cong S \setminus \iota(\mathbb{A}^1)$ and to the (given) morphism $\{x_o\} \times Y \to \mathbb{A}^1$, $(x_o, y) \mapsto \varphi(y)$. In particular, $\eta$ is the normalization; $S$ is obtained by “pinching $X \times Y$ along $\{x_o\} \times Y$ via $\varphi$”. Since the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & \mathbb{A}^1 \\
\downarrow \gamma & & \downarrow \mathrm{id} \\
X \times Y & \xrightarrow{\varphi \circ p_2} & \mathbb{A}^1
\end{array}
\]

commutes, it yields a unique morphism $\rho : S \to \mathbb{A}^1$ such that $\rho \circ \iota = \mathrm{id}$ and $\rho \circ \eta = \varphi \circ p_2$. The retraction $\rho$ defines in turn an algebraic semigroup law $\mu$ on $S$ by $\mu(s, s') := \iota(\rho(s)\rho(s'))$ as in Example II (ii).

We claim that $\mu$ does not lift to any algebraic semigroup law on $X \times Y$, if the curve $C$ is nonrational. Indeed, any such lift $\tilde{\mu}$ satisfies

\[
\eta(\tilde{\mu}((x, y), (x', y')) = \mu(\eta(x, y), \eta(x', y'))
\]
\[
= \iota(\rho(\eta(x, y)\rho(\eta(x', y'))) = \iota(\varphi(y)\varphi(y'))
\]

for any $x, x' \in X$ and any $y, y' \in Y$. As a consequence, $\tilde{\mu}((x, y), (x', y'))$ only depends on $(y, y')$, and this yields an algebraic semigroup law on $Y$ such that $\varphi$ is a homomorphism. But such a law does not exist, as follows e.g. from Theorem 5 below.
3 Irreducible algebraic monoids

3.1 Algebraic monoids with affine unit group

The aim of this subsection is to prove the following result, due to Rittatore for irreducible algebraic monoids (see [29, Thm. 5]). The proof presented here follows his argument closely, except for an intermediate step (Proposition 10).

**Theorem 2.** Let $M$ be a unit dense algebraic monoid, and $G$ its unit group. If $G$ is affine, then so is $M$.

**Proof.** Let $\eta : \tilde{M} \to M$ denote the normalization. Then $\tilde{M}$ is an algebraic monoid with unit group isomorphic to $G$, by Proposition 9. Moreover, $G$ is dense in $\tilde{M}$ since it is so in $M$. If $\tilde{M}$ is affine, then $M$ is affine by a result of Chevalley: the image of an affine variety by a finite morphism is affine (see [13, Exer. II.4.2]). Thus, we may assume that $M$ is normal. Then $M$ is the disjoint union of its irreducible components, and each of them is isomorphic (as a variety) to the neutral component $M^o$ (Proposition 9). So we may assume in addition that $M$ is irreducible.

By Proposition 7, the connected algebraic group $G \times G$ acts on $M$ with a unique closed orbit. In view of a result of Sumihiro (see [34]), it follows that $M$ is quasiprojective; in other words, there exists a locally closed immersion $\iota : M \to \mathbb{P}^n$ for some positive integer $n$. (We may further assume that $\iota$ is equivariant for some action of $G$ on $\mathbb{P}^n$; we will not need that fact in this proof). Then the pull-back $L := \iota^*O_{\mathbb{P}^n}(1)$ is an ample line bundle on $M$. The associated principal $\mathbb{G}_m$-bundle $\pi : X \to M$ (where $X$ is the complement of the zero section in $L$) is the pull-back to $M$ of the standard principal $\mathbb{G}_m$-bundle $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Thus, $X$ is a locally closed subvariety of $\mathbb{A}^{n+1}$, and hence is quasi-affine.

By Proposition 10 below (a version of [29, Thm. 4]), $X$ has a structure of a normal irreducible algebraic monoid such that $\pi$ is a homomorphism. Since that monoid is quasi-affine, it is in fact affine by a result of Renner (see [25, Thm. 4.4]). Moreover, the map $\pi : X \to M$ is the categorical quotient by the action of $\mathbb{G}_m$; hence $M$ is affine. 

**Proposition 10.** Let $M$ be a normal irreducible algebraic monoid, and assume that its unit group $G$ is affine. Let $\varphi : L \to M$ be a line bundle, and $\pi : X \to M$ the associated principal $\mathbb{G}_m$-bundle. Then $X$ has a structure of a normal irreducible algebraic monoid such that $\pi$ is a homomorphism.

**Proof.** By [10, Lem. 4.3], the preimage $Y := \pi^{-1}(G)$ has a structure of algebraic group such that the restriction of $\pi$ is a homomorphism; we then have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow Y \longrightarrow G \longrightarrow 1,$$
where $G_m$ is contained in the center of $Y$. Thus, the group law $\mu_G : G \times G \to G$ sits in a cartesian square

$$
\begin{array}{ccc}
Y \times_{G_m} Y & \xrightarrow{\mu_Y} & Y \\
\pi \times \pi & \downarrow & \pi \\
G \times G & \xrightarrow{\mu_G} & G,
\end{array}
$$

where $Y \times_{G_m} Y$ denotes the quotient of $Y \times Y$ by the action of $G_m$ via $t \cdot (y,z) = (ty,t^{-1}z)$, and $\mu_Y$ stands for the group law on $Y$. Via the correspondence between principal $G_m$-bundles and line bundles, this translates into a cartesian square

$$
\begin{array}{ccc}
p_1^*(L|_G) \otimes p_2^*(L|_G) & \xrightarrow{\cong} & L|_G \\
\varphi \times \varphi & \downarrow & \varphi \\
p_1^* \mu_G(L|_G) & \mu_G(L|_G) & \xrightarrow{\cong} & L|_G,
\end{array}
$$

where $p_1, p_2 : G \times G \to G$ denote the projections. In other words, we have an isomorphism

$$
p_1^*(L|_G) \otimes p_2^*(L|_G) \cong \mu_G(L|_G)
$$

of line bundles over $G \times G$.

Over $M \times M$, this yields an isomorphism

$$
p_1^*(L) \otimes p_2^*(L) \cong \mu^*(L) \otimes \mathcal{O}_{M \times M}(D),
$$

where $D$ is a Cartier divisor with support in $(M \times M) \setminus (G \times G)$. Since $G$ is affine, the irreducible components $E_1, \ldots, E_n$ of $M \setminus G$ are divisors of $M$. Thus, the irreducible components of $(M \times M) \setminus (G \times G)$ are exactly the divisors $E_i \times M$ and $M \times E_j$, where $i,j = 1, \ldots, n$. Hence

$$
D = p_1^*(D_1) + p_2^*(D_2)
$$

for some Weil divisors $D_1, D_2$ with support in $M \setminus G$. In particular, the pullback of $D$ to $M \times G$ is $D_1 \times G$. Since $D$ is Cartier, so is $D_1$; likewise, $D_2$ is Cartier. We thus obtain an isomorphism

$$
p_1^*(L) \otimes p_2^*(L) \cong \mu^*(L) \otimes p_1^*(\mathcal{O}_M(D_1)) \otimes p_2^*(\mathcal{O}_M(D_2))
$$

of line bundles over $M \times M$. We now pull back this isomorphism to $M \times \{1\}$. Note that $\mu^*(L)|_{M \times \{1\}} = L = p_1^*(L)|_{M \times \{1\}}$; also, $p_1^*(\mathcal{O}_M(D_1))|_{M \times \{1\}} = \mathcal{O}_M(D_1)$, and both $p_2^*(L)|_{M \times \{1\}}, p_2^*(\mathcal{O}_M(D_2))|_{M \times \{1\}}$ are trivial. Thus, $\mathcal{O}_M(D_1)$ is trivial; one shows similarly that $\mathcal{O}_M(D_2)$ is trivial. Hence we have in fact an isomorphism

$$
p_1^*(L) \otimes p_2^*(L) \cong \mu^*(L).
$$
As above, this translates into a cartesian square
\[
\begin{array}{ccc}
X \times_{S_m} X & \longrightarrow & X \\
\pi \times \pi & \downarrow & \downarrow \pi \\
M \times M & \longrightarrow & M.
\end{array}
\]
In turn, this yields a morphism \(\nu : X \times X \rightarrow X\) which lifts \(\mu : M \times M \rightarrow M\) and extends the group law \(Y \times Y \rightarrow Y\). It follows readily that \(\nu\) is associative and has \(1_Y\) as a neutral element.

A noteworthy consequence of Theorem 2 is the following sufficient condition for an algebraic monoid to be affine, which slightly generalizes [5, Cor. 3.3]:

**Corollary 2.** Let \(M\) be a unit dense algebraic monoid having a zero element. Then \(M\) is affine.

**Proof.** Consider the action of the unit group \(G\) on \(M\) via left multiplication. This action is faithful, and fixes the zero element. It follows that \(G\) is affine (see e.g. [7, Prop. 2.1.6]). Hence \(M\) is affine by Theorem 2. \(\square\)

### 3.2 Induction of algebraic monoids

In this subsection, we show that any unit dense algebraic monoid has a universal homomorphism to an algebraic group, and we study the fibers of this homomorphism.

**Proposition 11.** Let \(M\) be a unit dense algebraic monoid, and \(G\) its unit group.

(i) There exists a homomorphism of algebraic monoids \(\varphi : M \rightarrow \mathcal{G}(M)\), where \(\mathcal{G}(M)\) is an algebraic group, such that every homomorphism of algebraic monoids \(\psi : M \rightarrow \mathcal{G}\), where \(\mathcal{G}\) is an algebraic group, factors uniquely as \(\varphi\) followed by a homomorphism of algebraic groups \(\mathcal{G}(M) \rightarrow \mathcal{G}\).

(ii) We have \(\mathcal{G}(M) = \varphi(M) = \varphi(G) = G/H\), where \(H\) denotes the smallest normal subgroup scheme of \(G\) containing the isotropy subgroup scheme \(G_x\) for some \(x \in \ker(M)\).

**Proof.** We show both assertions simultaneously. Let \(\psi : M \rightarrow \mathcal{G}\) be a homomorphism as in the statement. Then \(\psi|_G\) is a homomorphism of algebraic groups, and hence its image is a closed subgroup of \(\mathcal{G}\). Since \(M\) is unit dense, it follows that \(\psi(M) = \psi(G)\). Let \(K\) be the scheme-theoretic kernel of \(\psi|_G\). Then \(K\) is a normal subgroup scheme of \(G\), and \(\psi\) induces an isomorphism from \(G/K\) to \(\psi(G)\); we may thus view \(\psi\) as a \(G\)-equivariant homomorphism
$M \to G/K$. In particular, for any $x \in M$, the map $g \mapsto \psi(g \cdot x)$ yields a morphism $G \to G/K$ which is equivariant under the action of $G$ by left multiplication, and invariant under the action of the isotropy subgroup scheme $G_x$ by right multiplication. Thus, $K$ contains $G_x$; hence $K$ contains $H$, and $\psi|_G$ factors as the quotient homomorphism $\gamma : G \to G/H$ followed by the canonical homomorphism $\pi : G/H \to G/K$.

Next, choose $x \in \text{ker}(M)$; then $Gx = Mx$ by Proposition 8. Thus, the morphism $M \to Mx$, $y \mapsto yx$ may be viewed as a morphism $M \to Gx \cong G/G_x$. Composing with the morphism $G/G_x \to G/H$ induced by the inclusion of $G_x$ in $H$, we obtain a morphism $\varphi : M \to G/H$. Clearly, $\varphi$ is $G$-equivariant, and $\varphi(1)$ is the neutral element of $G/H$. Thus, the restriction $\varphi|_G$ is the quotient homomorphism $\gamma$. By density, $\varphi$ is a homomorphism of monoids, and $\psi = \pi \circ \varphi$. So $\varphi$ is the desired homomorphism. \hfill \Box

Remark 5. (i) As a consequence of the above proposition, the smallest subgroup scheme of $G$ containing $G_x$ is independent of the choice of $x \in \text{ker}(M)$. This also follows from the fact that the subgroup schemes $G_x$, where $x \in \text{ker}(M)$, are all conjugate in $G$ (Proposition 8). By that proposition, we may take for $x$ any minimal idempotent of $M$.

(ii) As another consequence, any irreducible semigroup $S$ has a universal homomorphism to an algebraic group (in the sense of the above proposition). Indeed, choose an idempotent $e$ in $S$, and consider a homomorphism of semigroups $\psi : S \to G$, where $G$ is an algebraic group. Then $\psi(x) = \psi(exe)$ for all $x \in S$; moreover, $eSe$ is an irreducible monoid with neutral element $e$. Thus, there exists a unique homomorphism $\pi : G(eSe) \to G$ such that $\psi(x) = \pi(\phi(exe))$ for all $x \in S$, where $\phi : eSe \to G(eSe)$ denotes the universal homomorphism. Then we must have $\pi(\phi(eye)) = \pi(\phi(exe)\phi(e))$ for all $x, y \in S$. Let $H$ denote the smallest normal subgroup scheme of $G(eSe)$ containing the image of the morphism

$$S \times S \longrightarrow G(eSe), \quad (x, y) \longmapsto \phi(exe)\phi(exe)^{-1}\phi(eye)^{-1},$$

and let $\varphi : S \to G(eSe)/H$ denote the homomorphism that sends every $x$ to the image of $exe$. Then $\pi$ factors as $\varphi$ followed by a unique homomorphism of algebraic groups $G(eSe)/H \to G$, i.e., $\varphi$ is the desired homomorphism. Note that $\varphi : S \to G(S)$ is surjective by construction; in particular, $G$ is connected.

Proposition 12. Keep the notation and assumptions of Proposition 8.

(i) If $H$ is an algebraic group (e.g., if char($k$) = 0), then the scheme-theoretic fibers of $\varphi$ are reduced.

(ii) If $M$ is normal, then $H$ is a connected algebraic group; moreover, the scheme-theoretic fibers of $\varphi$ are reduced and irreducible.

Proof. (i) Denote by $\gamma : G \to G/H$ the quotient homomorphism and form the cartesian square
Since $\gamma$ and $\varphi$ are equivariant for the actions of $G$ by left multiplication, $X$ is equipped with a $G$-action such that $\gamma'$ and $\varphi'$ are equivariant. Denote by $N$ the (scheme-theoretic) fiber of $\varphi'$ at the neutral element $1_G$. Then the morphism

$$G \times N \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

is an isomorphism with inverse given by $x \mapsto (\varphi'(x), \varphi'(x)^{-1} \cdot x)$. Moreover, the fiber of $\varphi'$ at every $g \in G$ is $g \cdot N \cong N$. If $H$ is an algebraic group (i.e., if $H$ is smooth; this holds when char$(k) = 0$), then the morphism $\gamma$ is smooth as well; hence so is $\gamma'$. It follows that $X$ is reduced. But $X \cong G \times N$ and hence $N$ is reduced. If in addition $H$ is connected, then the fibers of $\gamma$ are irreducible; hence the same holds for $\gamma'$, and $X$ is irreducible. As above, it follows that $N$ is irreducible.

(ii) Consider the reduced neutral component $H^\text{red}_\text{o} \subseteq H$; then $H^\text{red}_\text{o}$ is a closed normal subgroup of $G$. Moreover, the natural map $\delta : G/H^\text{red}_\text{o} \rightarrow G/H$ is a finite morphism and sits in a commutative square

$$G \xrightarrow{\iota} M \xrightarrow{\epsilon} G/H$$

where $\iota$ denotes the inclusion. Let $\Gamma \subseteq M \times G/H^\text{red}_\text{o}$ be the closure of the graph of $\epsilon$. Then the projection $p_1 : \Gamma \rightarrow M$ is a finite morphism, and an isomorphism over the dense open subset $G$ of $M$. Since $M$ is normal, it follows that $p_1$ is an isomorphism, i.e., $\epsilon$ extends to a morphism $\psi : M \rightarrow G/H^\text{red}_\text{o}$. As $\epsilon$ is a homomorphism of algebraic groups, $\psi$ must be a homomorphism of monoids. Thus, $\psi$ factors through $\varphi$, and hence $H^\text{red}_\text{o} = H$. In other words, $H$ is a connected algebraic group.

But in general, the scheme-theoretic fibers of the homomorphism $\varphi : M \rightarrow G(M)$ are reducible; also, these fibers are nonreduced in prime characteristics, as shown by the following:

**Example 3.** Consider the monoid $A^3$ equipped with pointwise multiplication, and the locally closed subset

$$M := \{(x, y, z) \mid z^n = xy^n \text{ and } x \neq 0\},$$

where $n$ is a positive integer. Then $M$ is an irreducible commutative algebraic monoid with unit group
isomorphic to \( \mathbb{G}_m^2 \) via the projection \((x, y, z) \mapsto (y, z)\). Moreover, \( \ker(M) = Me \), where \( e := (1, 0, 0) \) is the unique minimal idempotent. Since \( M \) is commutative, the isotropy subgroup scheme \( H \) of Proposition 11 is just \( Ge \); the latter is the scheme-theoretic kernel of the homomorphism \( x : G \to \mathbb{G}_m \).

Thus,

\[
H \cong \{(y, z) \in \mathbb{G}_m^2 \mid y^n = z^n\} \cong \mathbb{G}_m \times \mu_n,
\]

where \( \mu_n \) denotes the subgroup scheme of \( n \)th roots of unity. The universal homomorphism \( \varphi : M \to G/H \) is identified to \( x : M \to \mathbb{G}_m \), and this identifies the fiber of \( \varphi \) at 1 to the submonoid scheme \((y^n = z^n)\) of \((k^2, \times)\). The latter scheme is reducible when \( n \geq 2 \), and nonreduced when \( n \) is a multiple of \( \text{char}(k) \).

We keep the notation and assumptions of Proposition 11 and denote by \( N \) the scheme-theoretic fiber of \( \varphi \) at 1. Assume in addition that \( H \) is an algebraic group (this holds e.g. if \( M \) is normal or if \( \text{char}(k) = 0 \)). Then \( N \) is reduced by Proposition 12, also, \( N \) is a closed submonoid of \( M \), containing \( H \) and stable under the action of \( G \) on \( M \) by conjugation (via \( g \cdot x := gxg^{-1} \)). Moreover, the map \( \pi : G \times N \to M, \ (g, y) \mapsto gy \)

is a homomorphism of algebraic monoids, where \( G \times N \) is equipped with the composition law

\[
(g_1, y_1)(g_2, y_2) := (g_1g_2, g_2^{-1}y_1g_2y_2)
\]

with unit \((1_G, 1_N)\) (this defines the semi-direct product of \( G \) with \( N \)). Finally, \( \pi \) is the quotient morphism for the action of \( H \) on \( G \times N \) via

\[
h \cdot (g, y) := (gh^{-1}, hy).
\]

In other words, the morphism \( \varphi : M \to G/H \) identifies \( M \) to the fiber bundle \( G \times^H N \to G/H \) associated to the principal \( H \)-bundle \( G \to G/H \) and to the variety \( N \) on which \( H \) acts by left multiplication. We say that the algebraic monoid \( M \) is induced from \( N \).

If we no longer assume that \( H \) is an algebraic group, then \( N \) is just a submonoid scheme of \( M \), and the above properties hold in the setting of monoid schemes. We now obtain slightly weaker versions of these properties in the setting of algebraic monoids.

**Proposition 13.** Let \( M \) be a unit dense algebraic monoid, \( G \) its unit group, \( \varphi : M \to G/H \) the universal homomorphism to an algebraic group, and \( N \) the scheme-theoretic fiber of \( \varphi \) at 1. Denote by \( H_{\text{red}} \) (resp. \( N_{\text{red}} \)) the reduced scheme of \( H \) (resp. \( N \)).
(i) $H_{\text{red}}$ is a closed normal subgroup of $G$ and $N_{\text{red}}$ is a closed submonoid of $M$, stable under the action of $G$ by conjugation.

(ii) $G \times \mathcal{H}_{\text{red}} N_{\text{red}}$ is an algebraic monoid, and the natural map

$$\psi : G \times \mathcal{H}_{\text{red}} N_{\text{red}} \longrightarrow M$$

is a finite bijective homomorphism of algebraic monoids.

(iii) $N_{\text{red}}$ is unit dense and its unit group is $H_{\text{red}}$.

(iv) $\psi$ is birational.

Proof. (i) The assertion on $H_{\text{red}}$ is well-known. That on $N_{\text{red}}$ follows readily from the fact that $N$ is a closed submonoid scheme of $M$, stable under the $G$-action by conjugation.

(ii) The natural map $G/H_{\text{red}} \longrightarrow G/H$ is a purely inseparable homomorphism of algebraic groups, and hence is finite and bijective. Also, $G \times \mathcal{H}_{\text{red}} N$ is the fibered product of $M = G \times H$ and $G/H_{\text{red}}$ over $G/H$. Thus, $G \times \mathcal{H}_{\text{red}} N$ is a monoid scheme; moreover, the natural morphism $G \times \mathcal{H}_{\text{red}} N \to M$ is finite, and bijective on closed points. As $G \times \mathcal{H}_{\text{red}} N_{\text{red}} = (G \times \mathcal{H}_{\text{red}} N)_{\text{red}}$, this yields our assertions.

(iii) Since $M$ is unit dense with unit group $G$ and $\psi$ is a homeomorphism, we see that $G \times \mathcal{H}_{\text{red}} N_{\text{red}}$ is unit dense with unit group $G$ as well. It follows that $G \times N_{\text{red}}$ is unit dense with unit group $G \times \mathcal{H}_{\text{red}}$. Thus, $H_{\text{red}}$ is the unit group of $N_{\text{red}}$ and is dense there.

(iv) Just note that $\psi$ restricts to the natural isomorphism $G \times \mathcal{H}_{\text{red}} H_{\text{red}} \cong G$; moreover, $G \times \mathcal{H}_{\text{red}} H_{\text{red}}$ is a dense open subset of $G \times \mathcal{H}_{\text{red}} N_{\text{red}}$. $\square$

Example 4. Assume that $\text{char}(k) = p > 0$. Consider the monoid

$$M := \{ (x, y, z) \in k^3 \mid z^p = xy^p \text{ and } x \neq 0 \}$$

relative to pointwise multiplication, as in Example 3. Recall from this example that $G \cong \mathbb{G}_m^2$ and that the universal homomorphism $\varphi : M \to G/H$ is just $x : M \to \mathbb{G}_m$, with scheme-theoretic fiber $N$ at 1 being the submonoid scheme $(z^p = y^p)$ of $(k^2, \times)$. It follows that $N_{\text{red}} \cong (k^1, \times)$, $H_{\text{red}} \cong \mathbb{G}_m$ and $G \times \mathcal{H}_{\text{red}} N_{\text{red}} \cong \mathbb{G}_m \times k^1$, where the right-hand side is equipped with pointwise multiplication. One checks that $\psi : G \times \mathcal{H}_{\text{red}} N_{\text{red}} \to M$ is identified with the map $(t, u) \mapsto (t^p, t, u)$.

Returning to the general setting, we now relate the idempotents and kernel of $M$ with those of $N_{\text{red}}$:

**Proposition 14.** Keep the notation and assumptions of Proposition 13.

(i) $E(M) = E(N_{\text{red}})$.

(ii) The assignment $I \mapsto I \cap N_{\text{red}}$ defines a bijection between the two-sided ideals of $M$ and those two-sided ideals of $N_{\text{red}}$ that are stable under conjugation by $G$. The inverse bijection is given by $J \mapsto GJ$. 


(iii) We have \( \ker(M) \cap N_{\text{red}} = \ker(N_{\text{red}}) \) and \( \ker(M) = G \ker(N_{\text{red}}) \).

Proof. (i) Clearly, \( E(N_{\text{red}}) \subseteq E(M) \). Moreover, if \( e \in E(M) \) then \( \varphi(e) = 1_{G/H} \) and hence \( e \in N \), i.e., \( e \in N_{\text{red}} \).

(ii) Consider a two-sided ideal \( I \) of \( M \). Then \( J := I \cap N_{\text{red}} \) is a two-sided ideal of \( N_{\text{red}} \), stable under conjugation by \( G \) (since so are \( I \) and \( N_{\text{red}} \)). Moreover, \( I = GJ \), since \( M = GN_{\text{red}} \) and \( I = GI \).

Conversely, let \( J \) be a two-sided ideal of \( N_{\text{red}} \), stable under conjugation by \( G \). Then \( I := GJ \) is closed in \( M \) and satisfies \( I \cap N_{\text{red}} = J \), as follows easily from the fact that \( \psi : G \times H_{\text{red}} N_{\text{red}} \to M \) is a homeomorphism. Moreover, \( GIG = GJG = GJ = I \) by stability of \( J \) under conjugation. Since \( M \) is unit dense and \( I \) is closed in \( M \), it follows that \( MIM = I \); in other words, \( I \) is a two-sided ideal.

(iii) Since \( N_{\text{red}} \) is stable under conjugation by \( G \), so is \( \ker(N_{\text{red}}) \). In view of (ii), it follows that \( \ker(M) \cap N_{\text{red}} = \ker(N_{\text{red}}) \). Together with Proposition 6 this yields \( \ker(M) = G \ker(N_{\text{red}})G = G \ker(N_{\text{red}}) \). \( \square \)

Remark 6. (i) If \( N \) is reduced, then any homomorphism of algebraic monoids from \( N \) to an algebraic group is trivial.

Indeed, let \( \kappa : N \to G \) be such a homomorphism. We may assume that \( \kappa \) is the universal morphism \( N \to H/K \) of Proposition 11 where \( K \) is a normal subgroup scheme of \( H \). Then the \( G \)-action on \( N \) by conjugation yields a \( G \)-action on \( H/K \), compatible with the conjugation action on \( H \); thus, \( K \) is a normal subgroup scheme of \( G \). Moreover, the \( H \)-equivariant morphism \( \kappa \) induces a \( G \)-equivariant morphism

\[
\psi : M = G \times H N \longrightarrow G \times H H/K \cong G/K, \quad (g, y) H \longmapsto (g, \kappa(y)) H.
\]

Also, \( \psi(1) \) is the neutral element of \( G/K \), since \( \kappa(1) \) is the neutral element of \( H/K \). It follows that \( \psi(xy) = \psi(x)\psi(y) \) for all \( x \in G \) and \( y \in M \), and hence for all \( x, y \in M \). By Proposition 11 \( \psi \) factors through \( \varphi \), and hence \( K = H \).

We do not know if the analogous statement holds for \( N_{\text{red}} \) when \( N \) is nonreduced.

(ii) Let \( M \) be a unit dense algebraic monoid, and \( M^o \) its neutral component. Then \( M^o \) is stable under the action of the unit group \( G \) by conjugation on \( M \); also, \( M = GM^o \) by Proposition 6. Thus, \( G \times G^o M^o \) is an algebraic monoid, the disjoint union of the irreducible components of \( M \). Moreover, the map

\[
\varphi : G \times G^o M^o \longrightarrow M, \quad (g, x) G^o \longmapsto gx
\]

is a homomorphism of algebraic monoids, which is readily seen to be finite and birational. Hence \( \varphi \) is an isomorphism whenever \( M \) is normal.

For an arbitrary (unit dense) \( M \), it follows that \( E(M) = E(M^o) \). Indeed, \( \varphi \) is surjective, and hence restricts to a surjection \( E(G \times G^o M^o) \to E(M) \) by Corollary 1. Also, \( E(G \times G^o M^o) = E(M^o) \), since the unique idempotent of \( G/G^o \) is the coset of \( 1_G \).
3.3 Structure of irreducible algebraic monoids

We begin this subsection by presenting some classical results on the structure of an arbitrary connected algebraic group $G$. By Chevalley’s structure theorem, $G$ has a largest connected affine normal subgroup $G_{\text{aff}}$; moreover, the quotient group $G/G_{\text{aff}}$ is an abelian variety. In other words, $G$ sits in a unique exact sequence of connected algebraic groups

$$1 \rightarrow G_{\text{aff}} \rightarrow G \xrightarrow{\alpha} A \rightarrow 1,$$

where $G_{\text{aff}}$ is linear, and $A := G/G_{\text{aff}}$ is an abelian variety. This exact sequence is generally nonsplit; yet $G$ has a smallest closed subgroup $H$ such that $\alpha|_H$ is surjective. Moreover, $H$ is connected, contained in the center of $G$, and satisfies $O(H) = k$. In fact, $H$ is the largest closed subgroup of $G$ satisfying the latter property, which defines the class of anti-affine algebraic groups; we denote $H$ by $G_{\text{ant}}$. Finally, we have the Rosenlicht decomposition: $G = G_{\text{aff}}G_{\text{ant}}$, and $(G_{\text{ant}})_{\text{aff}}$ is the connected neutral component of the scheme-theoretic intersection $G_{\text{aff}} \cap G_{\text{ant}}$. In other words, we have an isomorphism of algebraic groups

$$G \cong (G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}},$$

and the quotient group scheme $(G_{\text{aff}} \cap G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}}$ is finite. We refer to [7] for an exposition of these results and of further developments.

We shall obtain a similar structure result for an arbitrary irreducible algebraic monoid; then the unit group is a connected algebraic group by Theorem 1. Our starting point is the following:

Proposition 15. Let $M$ be an irreducible algebraic monoid, $G$ its unit group, $\varphi : M \rightarrow G(M) = G/H$ the universal homomorphism to an algebraic group, and $N$ the scheme-theoretic fiber of $\varphi$ at 1. Then $H$ and $N$ are affine.

Proof. Recall from Proposition 11 that $H$ is the normal subgroup scheme of $G$ generated by $G_x$, where $x$ is an arbitrary point of $\ker(M)$. Since $G_x$ is the isotropy subgroup scheme of a point for a faithful action of $G$ (the action on $M$ by left multiplication), it follows that $G_x$ is affine (see e.g. [7, Cor. 2.1.9]). The image of $G_x$ under the homomorphism $\alpha : G \rightarrow A$ is affine (as the image of an affine group scheme by a homomorphism of group schemes) and proper (as a subgroup scheme of the abelian variety $G/G_{\text{aff}}$), hence finite. But $\alpha(G_x) = \alpha(H)$ by the definition of $H$ and the commutativity of $A$; hence $\alpha(H)$ is finite. Also, the kernel of the homomorphism $\alpha|_H$ is a subgroup scheme of $G_{\text{aff}}$, and hence is affine. Thus, the reduced scheme $H_{\text{red}}$ is an extension of a finite group by an affine algebraic group, and hence is affine. Thus, so is $N_{\text{red}}$ in view of Theorem 2 and of Proposition 11. It follows that $N$ is affine, by [13, Exer. III.3.1]).
Remark 7. If \( \text{char}(k) = 0 \), then \( N \) is reduced and any homomorphism from \( N \) to an algebraic group is trivial by Remark 6 (i). If in addition \( N \) is irreducible (e.g., if \( M \) is normal), then \( \ker(N) \) is generated by the minimal idempotents. Indeed, the unit group \( H \) of \( N \) is generated by the conjugates of the isotropy group \( H_e \) for some idempotent \( e \in \ker(N) \), by Proposition 11 (ii). So the assertion follows from \([24, \text{Thm. 2.1}]\).

A noteworthy consequence of Proposition 15 is the following:

Corollary 3. Any irreducible algebraic monoid is quasiprojective.

Proof. With the notation of the above proposition, the morphism \( \varphi \) is affine, since \( M = G \times^H N \) where \( N \) is affine. Moreover, \( G/H \) is quasiprojective since so is any algebraic group (see e.g. \([7, \text{Prop. 3.1.1}]\)). Thus, \( M \) is quasiprojective as well. \( \square \)

Another consequence is a version of Chevalley’s structure theorem for an irreducible algebraic monoid; it generalizes \([5, \text{Thm. 1.1}]\), where the monoid is assumed to be normal.

Theorem 3. Let \( M \) be an irreducible algebraic monoid, \( G \) its unit group, and \( M_{\text{aff}} \) the closure of \( G_{\text{aff}} \) in \( M \).

(i) \( M_{\text{aff}} \) is an irreducible affine algebraic monoid with unit group \( G_{\text{aff}} \).
(ii) The action of \( G_{\text{aff}} \) on \( M_{\text{aff}} \) extends to an action of \( G = G_{\text{aff}}G_{\text{ant}} \), where \( G_{\text{ant}} \) acts trivially.
(iii) The natural map \( G_{\text{ant}} \times^{G_{\text{ant}} \cap G_{\text{aff}}} M_{\text{aff}} \to G \times^{G_{\text{aff}}} M_{\text{aff}} \) is an isomorphism of irreducible algebraic monoids. Moreover, the natural map

\[
\kappa : G \times^{G_{\text{aff}}} M_{\text{aff}} \to M
\]

is a finite birational homomorphism of algebraic monoids.
(iv) \( E(M) = E(M_{\text{aff}}) \) and \( \ker(M) = G \ker(M_{\text{aff}}) \).
(v) \( M \) is normal if and only if \( M_{\text{aff}} \) is normal and \( \kappa \) is an isomorphism. Then the assignment \( I \mapsto I \cap M_{\text{aff}} \) defines a bijection between the two-sided ideals of \( M \) and those of \( M_{\text{aff}} \); the inverse bijection is given by \( J \mapsto GJ \).

In particular, \( \ker(M) \cap M_{\text{aff}} = \ker(M_{\text{aff}}) \).

Proof. (i) Clearly, \( M_{\text{aff}} \) is an irreducible submonoid of \( M \), and \( G(M_{\text{aff}}) \) contains \( G_{\text{aff}} \) as an open subgroup. Since \( G(M_{\text{aff}}) \) is connected, it follows that \( G(M_{\text{aff}}) = G_{\text{aff}} \). Hence \( M_{\text{aff}} \) is affine by Theorem 2.

(ii) follows readily from the Rosenlicht decomposition: since \( G_{\text{aff}} \cap G_{\text{ant}} \) is contained in the center of \( G_{\text{aff}} \), its action on \( M_{\text{aff}} \) by conjugation is trivial. Thus, the \( G_{\text{aff}} \)-action by conjugation on \( M_{\text{aff}} \) extends to an action of \( G \cong (G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{aff}} \cap G_{\text{ant}}) \), where \( G_{\text{ant}} \) acts trivially.

(iii) The first assertion follows from the Rosenlicht decomposition again, since that decomposition yields an isomorphism \( G \cong G_{\text{ant}} \times^{G_{\text{ant}} \cap G_{\text{aff}}} G_{\text{aff}} \) of principal \( G_{\text{aff}} \)-bundles over \( G/G_{\text{aff}} \cong G_{\text{ant}}/(G_{\text{ant}} \cap G_{\text{aff}}) \). For the second
assertion, note first that \( \kappa \) restricts to the natural isomorphism \( G \times \text{G}_{\text{aff}} \rightarrow G \), and hence is a birational homomorphism of algebraic monoids. To show that \( \kappa \) is finite, we use the isomorphism \( M \cong G \times H_N \) of Subsection 3.2. Here \( H \) and \( N \) are affine by Proposition 15; also, the natural map \( G \times H_{\text{red}} \rightarrow M \) is finite and bijective by Proposition 13. It follows that the analogous map

\[
\gamma : G \times H_{\text{red}} N_{\text{red}} \rightarrow M
\]

is finite and surjective. But \( H_{\text{red}} \subseteq \text{G}_{\text{aff}} \) since \( H \) is affine. Thus,

\[
G \times H_{\text{red}} N_{\text{red}} \cong G \times \text{G}_{\text{aff}} (G \times H_{\text{red}} N_{\text{red}}).
\]

Moreover, \( N_{\text{red}} \subseteq M_{\text{aff}} \), since \( N_{\text{red}} \) is the closure in \( M \) of \( H_{\text{red}} \subseteq \text{G}_{\text{aff}} \); hence \( G \text{aff} N_{\text{red}} \subseteq M_{\text{aff}} \). So \( \gamma \) factors as the natural map

\[
\beta : G \times \text{G}_{\text{aff}} (G \times H_{\text{red}} N_{\text{red}}) \rightarrow G \times \text{G}_{\text{aff}} M_{\text{aff}}
\]

(induced from the map \( \delta : G \times H_{\text{red}} N_{\text{red}} \rightarrow M_{\text{aff}} \), followed by \( \kappa \). Now \( \delta \) is the restriction of \( \gamma \) to a closed subvariety, and hence is finite; thus, its image \( G \text{aff} N_{\text{red}} \) is closed in \( M_{\text{aff}} \). But \( M_{\text{aff}} = \text{G}_{\text{aff}} \), and hence \( \delta \) is surjective. Hence \( \beta \) is finite and surjective. Since \( \gamma = \kappa \circ \beta \), it follows that \( \kappa \) is finite and surjective as well.

(iv) Let \( e \in E(M) \). By Corollary 1, we may lift \( e \) to some idempotent \( f \) of \( G \times \text{G}_{\text{aff}} M_{\text{aff}} \). Then the image of \( f \) in \( G/G_{\text{aff}} \) is the neutral element, and hence \( f \in M_{\text{aff}} \) so that \( e \in E(M_{\text{aff}}) \). The converse is obvious.

Next, choose \( e \) minimal. Then \( \ker(M) = G e G \) by Proposition 4 and hence \( \ker(M) = G(G_{\text{aff}} e G_{\text{aff}}) \) in view of the Rosenlicht decomposition. But \( G_{\text{aff}} e G_{\text{aff}} = \ker(M_{\text{aff}}) \) since \( e \) is a minimal idempotent of \( G_{\text{aff}} \).

(v) Assume that \( M \) is normal. By (iii) and Zariski’s Main Theorem, it follows that \( \kappa \) is an isomorphism. In particular, \( G \times \text{G}_{\text{aff}} M_{\text{aff}} \) is normal. Since the natural morphism \( G \times M_{\text{aff}} \rightarrow G \times \text{G}_{\text{aff}} M_{\text{aff}} \) is smooth, it follows that \( G \times M_{\text{aff}} \) is normal (e.g., by Serre’s criterion); hence so is \( M_{\text{aff}} \). The converse is straightforward. This proves the first assertion.

The second assertion is proved by the argument of Proposition 13 (ii); note that any two-sided ideal of \( M_{\text{aff}} \) is stable under conjugation by \( G \), in view of (ii) above.

\( \square \)

Example 5. Let \( n \) be a positive integer, \( \mu_n \) the group scheme of \( n \)th roots of unity, and \( A \) an abelian variety containing \( \mu_n \) as a subgroup scheme (any ordinary elliptic curve will do). As in Example 3 let \( N \) be the submonoid scheme \((z^n = y^n)\) of \((\mathbb{A}^2, \times)\), and \( H \) the unit subgroup scheme of \( N \); then \( H \cong \mu_n \times \mathbb{G}_m \). Next, let \( G := A \times \mathbb{G}_m \); this is a connected commutative algebraic group containing \( H \) as a subgroup scheme. Finally, let

\[
M := G \times H N = A \times \mu_n N.
\]
Then one checks that \( M \) is an irreducible algebraic monoid with unit group \( G \). Clearly, \( G_{\text{aff}} = G_m \) and \( A(G) = A \); also, one checks that \( M_{\text{aff}} = (\mathbb A^1, \times) \) and hence \( G \times G_{\text{aff}} M_{\text{aff}} \cong A \times \mathbb A^1 \). The morphism \( \kappa : G \times G_{\text{aff}} M_{\text{aff}} \to M \) sends the closed subscheme \( \mu_n \times \{0\} \) to 0, and restricts to an isomorphism over the complement. So \( M \) is obtained from \( A \times \mathbb A^1 \) by pinching \( \mu_n \times \{0\} \) to a point.

In view of Theorem 3, we may transfer information from affine algebraic monoids (about which much is known, see [23, 26]) to general ones. For example, the minimal idempotents of any irreducible algebraic monoid are all conjugate under the unit group, since this holds in the affine case by [23, Prop. 6.1, Cor. 6.8]. Another noteworthy corollary is the following relation between the partial order on idempotents and limits of one-parameter subgroups:

**Corollary 4.** Let \((S, \mu)\) be an irreducible algebraic semigroup, and \( e, f \in E(S) \). Then \( e \leq f \) if and only if there exists a homomorphism of algebraic semigroups \( \lambda : (\mathbb A^1, \times) \to (S, \mu) \) such that \( \lambda(0) = e \) and \( \lambda(1) = f \).

**Proof.** The “if” implication is obvious (and holds in every algebraic semigroup). For the converse, assume that \( e \leq f \). Then \( e \in f S f \) and the latter is an irreducible algebraic monoid. Thus, we may assume that \( S \) itself is an irreducible algebraic monoid, and \( f \) is the neutral element. In view of Theorem 3 we may further assume that \( S \) is affine. Then the assertion follows from [22, Thm. 2.9, Thm. 2.10]. \( \square \)

### 3.4 The Albanese morphism

By [30, Sec. 4], every irreducible variety \( X \) admits a universal morphism to an abelian variety: the *Albanese morphism*,

\[
\alpha : X \longrightarrow A(X).
\]

The group \( A(X) \) is generated by the differences \( \alpha(x) - \alpha(y) \), where \( x, y \in X \). Also, \( X \) admits a universal rational map to an abelian variety: the *Albanese rational map*,

\[
\alpha_{\text{rat}} : X - \longrightarrow A(X)_{\text{rat}}.
\]

The maps \( \alpha \) and \( \alpha_{\text{rat}} \) are uniquely determined up to translations and isomorphisms of the algebraic group \( A(X) \). Moreover, there exists a unique morphism

\[
\beta : A(X)_{\text{rat}} \longrightarrow A(X)
\]

such that \( \alpha = \beta \circ \alpha_{\text{rat}} \). The morphism \( \beta \) is always surjective; when \( X \) is nonsingular, it is an isomorphism. For an arbitrary \( X \), we have \( A(X)_{\text{rat}} = A(U) \), where \( U \subseteq X \) denotes the nonsingular locus; in particular, \( \alpha_{\text{rat}} \) is defined at any nonsingular point of \( X \).
When \(X\) is equipped with a base point \(x\), we may assume that \(\alpha(x)\) is the origin of \(A(X)\). If \(X\) is nonsingular at \(x\), then we may further assume that \(\alpha_{\text{rat}}(x)\) is the origin of \(A(X)_{\text{rat}}\). Then \(\alpha\) and \(\alpha_{\text{rat}}\) are unique up to isomorphisms of algebraic groups.

Next, observe that the Albanese morphism of a connected linear algebraic group \(G\) is constant: indeed, \(G\) is generated by rational curves, and any morphism from such a curve to an abelian variety is constant. For a connected algebraic group \(G\) (not necessarily linear), it follows that \(\alpha = \alpha_{\text{rat}}\) is the quotient homomorphism by the largest connected affine subgroup \(G_{\text{aff}}\). This determines the Albanese rational map of an irreducible algebraic monoid \(M\), which is just the Albanese morphism of its unit group. Some properties of the Albanese morphism of \(M\) are gathered in the following:

**Proposition 16.** Let \(M\) be an irreducible algebraic monoid with unit group \(G\). Then the map \(\alpha : M \to A(M)\) is a homomorphism of algebraic monoids, and an affine morphism. Moreover, the map \(\beta : A(M)_{\text{rat}} = A(G) \to A(M)\) is an isogeny. If \(M\) is normal, then \(\beta\) is an isomorphism.

**Proof.** The monoid law \(\mu : M \times M \to M, (1_M, 1_M) \mapsto 1_M\) induces a morphism of varieties \(A(\mu) : A(M \times M) \to A(M), 0 \mapsto 0\). Since \(A(M \times M) = A(M) \times A(M)\), it follows that \(A(\mu)\) is a homomorphism; hence so is \(\alpha\). In particular, \(\alpha\) factors through the universal homomorphism \(\varphi : M \to G/H\) of Proposition 11. Hence \(A(M) = A(G/H) = G/G_{\text{aff}} H\), where \(G_{\text{aff}} H\) is a normal subgroup scheme of \(G\) such that the quotient \(G_{\text{aff}} H/G_{\text{aff}} \cong H/(H \cap G_{\text{aff}})\) is finite. Write \(M = G \times^H N\) as in Proposition 13, then

\[
M \cong G \times^{G_{\text{aff}} H} (G_{\text{aff}} H \times^H N)
\]

and this identifies \(\alpha\) with the natural map to \(G/G_{\text{aff}} H\), with fiber \(G_{\text{aff}} H \times^H N\). But that fiber is affine, since so are \(N\) and \(G_{\text{aff}} H/H \cong G_{\text{aff}}/(G_{\text{aff}} \cap H)\). It follows that the morphism \(\alpha\) is affine. Also, \(\beta\) is identified with the natural homomorphism \(G/G_{\text{aff}} \to G/G_{\text{aff}} H\); hence the kernel of \(\beta\) is isomorphic to \(G_{\text{aff}} H/H\), a finite group scheme.

If \(M\) is normal, then \(M \cong G \times^{G_{\text{aff}}} M_{\text{aff}}\) by Theorem 8. Thus, the natural map \(M \to G/G_{\text{aff}}\) is the Albanese morphism. \(\square\)

Consider for instance the monoid \(M\) constructed in Example 5. Then \(A(M) \cong A/\mu_n\) and \(A(G) \cong A\); this identifies \(\beta\) to the quotient morphism \(A \to A/\mu_n\).

Returning to our general setting, recall that every irreducible algebraic monoid may be viewed as an equivariant embedding of its unit group. For an arbitrary equivariant embedding \(X\) of a connected algebraic group \(G\), we may again identify \(A(X)_{\text{rat}}\) with \(A(G)\); when \(X\) is normal, we still have \(A(X) = A(X)_{\text{rat}}\) as a consequence of [4] Thm. 3. But the morphism \(\alpha\) is generally nonaffine, and the finiteness of \(\beta\) is an open question in this setting.

We now characterize algebraic monoids among equivariant embeddings:
Theorem 4. Let $X$ be an equivariant embedding of a connected algebraic group $G$. Then $X$ has a structure of algebraic monoid with unit group $G$ if and only if the Albanese morphism $\alpha : X \to A(X)$ is affine.

Proof. In view of Proposition 16, it suffices to show that $X$ is an algebraic monoid if $\alpha$ is affine. Note that $\alpha$ is $G \times G$-equivariant for the given action of $G \times G$ on $X$, and a transitive action on $A(X)$. It follows that $A(X) \cong (G \times G)/(K \times K)_{\text{diag}}(G)$ for a unique normal subgroup scheme $K$ of $G$; then $A(X) \cong G/K$ equivariantly for the left (or right) action of $G$. Moreover, $\alpha$ is a fiber bundle of the form

$$G \times G \times (K \times K)_{\text{diag}}(G) \to (G \times G)/(K \times K)_{\text{diag}}(G),$$

where $Y$ is a scheme equipped with an action of $(K \times K)_{\text{diag}}(G)$; for the left (or right) $G$-action, this yields the fiber bundle $G \times K \to G/K$. Since $\alpha$ is affine, so is $Y$. Also, $Y$ meets the open orbit $G \cong (G \times G)/_{\text{diag}}(G)$ along a dense open subscheme isomorphic to $K$, where $K \times K$ acts by left and right multiplication, and $\text{diag}(G)$ by conjugation. Thus, the group scheme $K$ is quasi-affine, and hence is affine.

We now show that the group law $\mu_K : K \times K \to K$ extends to a morphism $\mu_Y : Y \times Y \to Y$, by following the argument of [28, Prop. 1]. The left action $K \times Y \to Y$ and the right action $Y \times K \to Y$ restrict both to $\mu_K$ on $K \times K$, and hence yield a morphism $(K \times Y) \cup (Y \times K) \to Y$. Since $Y$ is affine, it suffices to show the equality

$$\mathcal{O}((K \times Y) \cup (Y \times K)) = \mathcal{O}(Y \times Y).$$

But $\mathcal{O}(Y \times Y) = \mathcal{O}(Y) \otimes \mathcal{O}(Y) \subseteq \mathcal{O}(K) \otimes \mathcal{O}(K) = \mathcal{O}(K \times K)$, since $K$ is dense in $Y$. Moreover,

$$\mathcal{O}((K \times Y) \cup (Y \times K)) = (\mathcal{O}(K) \otimes \mathcal{O}(Y)) \cap (\mathcal{O}(Y) \otimes \mathcal{O}(K)),$$

where the intersection is considered in $\mathcal{O}(K) \otimes \mathcal{O}(K)$. Now for any vector space $V$ and subspace $W$, we easily obtain the equality $(W \otimes V) \cap (V \otimes W) = W \otimes W$ as subspaces of $V \otimes V$. When applied to $\mathcal{O}(Y) \subseteq \mathcal{O}(K)$, this yields the desired equality.

Since $\mu_Y$ is associative on the dense subscheme $K$, it is associative everywhere; likewise, $\mu_Y$ admits $1_K$ as a neutral element. Thus, $\mu_Y$ is an algebraic monoid law on $Y$. We may now form the induced monoid $G \times^K Y$ as in Subsection 3.2 to get the desired structure on $X$. \qed
3.5 Algebraic semigroups and monoids over perfect fields

In this subsection, we extend most of the above results to the setting of algebraic semigroups and monoids defined over a perfect field. We use the terminology and results of [33], especially Chapter 11 which discusses basic rationality results on varieties.

Let $F$ be a subfield of the algebraically closed field $k$. We assume that $F$ is perfect, i.e., every algebraic extension of $F$ is separable; we denote by $\bar{F}$ the algebraic closure of $F$ in $k$, and by $\Gamma$ the Galois group of $\bar{F}$ over $F$.

We say that an algebraic semigroup $(S, \mu)$ (over $k$) is defined over $F$, or an algebraic $F$-semigroup, if $S$ is an $F$-variety and the morphism $\mu$ is defined over $F$. Then the set of $\bar{F}$-points, $S(\bar{F})$, is a subsemigroup of $S$ equipped with an action of $\Gamma$ by semigroup automorphisms, and the fixed point subset $S(\bar{F})^\Gamma$ is the semigroup of $F$-points, $S(F)$.

Note that an algebraic $F$-semigroup may well have no $F$-point; for example, an $F$-variety without $F$-point equipped with the trivial semigroup law $\mu_l$ or $\mu_r$. But this is the only obstruction to the existence of $F$-idempotents, as shown by the following:

**Proposition 17.** Let $(S, \mu)$ be an algebraic $F$-semigroup.

(i) $E(S)$ and $\ker(S)$ (viewed as closed subsets of $S$) are defined over $F$.
(ii) If $S$ is commutative, then its smallest idempotent is defined over $F$.
(iii) If $S$ has an $F$-point, then it has an idempotent $F$-point.

**Proof.** (i) Clearly, $E(S)$ and $\ker(S)$ are defined over $\bar{F}$ and their sets of $\bar{F}$-points are stable under the action of $\Gamma$ on $S(\bar{F})$. Thus, $E(S)$ and $\ker(S)$ are defined over $F$ by [33] Prop. 11.2.8(i).

(ii) is proved similarly.

(iii) Let $x \in S(F)$ and denote by $(x)$ the closure in $S$ of the set $\{x^n, n \geq 1\}$. Then $(x)$ is a closed commutative subsemigroup of $S$, defined over $F$ by [33] Lem. 11.2.4. In view of (ii), $(x)$ contains an idempotent defined over $F$. □

We do not know if any algebraic $F$-semigroup $S$ has a minimal idempotent defined over $F$. This holds if $S$ is irreducible, as we will see in Proposition 19.

First, we record two rationality results on algebraic monoids:

**Proposition 18.** Let $(M, \mu, 1_M)$ be an algebraic monoid with unit group $G$ and neutral component $M^o$. If $M$ and $\mu$ are defined over $F$, then so are $1_M$, $G$ and $M^o$. Moreover, the inverse map $\iota : G \to G$ is defined over $F$.

**Proof.** Observe that $1_M$ is the unique point $x \in M$ such that $xy = yx = y$ for all $y \in M(\bar{F})$ (since $M(\bar{F})$ is dense in $M$). It follows that $1_M \in M(\bar{F})$; also, $1_M$ is $\Gamma$-invariant by uniqueness. Thus, $1_M \in M(F)$. 
The assertion on $G$ follows from \cite[Prop. 11.2.8(ii)]{33}. It implies that $G^o$ is defined over $F$ by [loc. cit., Prop. 12.1.1]. Since $M^o$ is the closure of $G^o$ in $M$, it is also defined over $F$ in view of [loc. cit., Prop. 11.2.8(i)].

It remains to show that $\iota$ is defined over $F$; equivalently, its graph is an $F$-subvariety of $G \times G$. But this graph equals
\[
\{(x, y) \in G \times G \mid xy = 1\} = \mu_G^{-1}(1_M),
\]
where $\mu_G : G \times G \to G$ denotes the restriction of $\mu$, and $\mu_G^{-1}(1_M)$ stands for the set-theoretic fiber. Moreover, this fiber is defined over $F$ in view of \cite[Cor. 11.2.14]{33}.

**Proposition 19.** Let $(M, \mu, 1_M)$ be an irreducible algebraic monoid with unit group $G$. If $(M, \mu)$ is defined over $F$, then the universal homomorphism to an algebraic group, $\varphi : M \to G(M)$, is defined over $F$ as well. Moreover, $G_{\text{aff}}$ and $M_{\text{aff}}$ are defined over $F$.

**Proof.** The first assertion follows from the uniqueness of $\varphi$ by a standard argument of Galois descent, see \cite[Chap. V, §4]{31}. The (well-known) assertion on $G_{\text{aff}}$ is proved similarly; it implies the assertion on $M_{\text{aff}}$ by \cite[Prop. 11.2.8(i)]{33}.

Returning to algebraic semigroups, we obtain the promised:

**Proposition 20.** Let $(S, \mu)$ be an irreducible algebraic $F$-semigroup. If $S$ has an $F$-point, then some minimal idempotent of $S$ is defined over $F$.

**Proof.** By Proposition \ref{17}, we may choose $e \in E(S(F))$. Then $eSe$ is a closed irreducible submonoid of $S$, and is defined over $S$ in view of \cite[Prop. 11.2.8(i)]{33} again. Moreover, any minimal idempotent of $eSe$ is a minimal idempotent of $S$. So we may assume that $S$ is an irreducible monoid, $M$. In view of Theorem \ref{3} and Proposition \ref{19} we may further assume that $M$ is affine. Then the unit group of $M$ contains a maximal torus $T$ defined over $F$, by Proposition \ref{18} and \cite[Thm. 13.3.6, Rem. 13.3.7]{33}. The closure $\bar{T}$ of $T$ in $M$ is defined over $F$, and meets $\ker(M)$ in view of \cite[Cor. 6.10]{23}. So the (set-theoretic) intersection $N := \bar{T} \cap \ker(M)$ is a commutative algebraic semigroup, defined over $F$ by \cite[Thm. 11.2.13]{33}. Now applying Proposition \ref{17} to $N$ yields the desired idempotent.

**Remark 8.** The above observations leave open all the rationality questions for an algebraic semigroup $S$ over a field $F$, not necessarily perfect. In fact, $S$ has an idempotent $F$-point if it has an $F$-point, as follows from the main result of \cite{6}. But some results do not extend to this setting; for example, the kernel of an algebraic $F$-monoid may not be defined over $F$, as shown by a variant of the standard example of a linear algebraic $F$-group whose unipotent radical is not defined over $F$ (see \cite[Exp. XVII, 6.4.a)]{32} or \cite[12.1.6]{33}; specifically, replace the multiplicative group $\mathbb{G}_m$ with the monoid $(\mathbb{A}^1, \times)$.
in the construction of this example). Also, note that Chevalley’s structure theorem fails over any imperfect field $F$ (see [32, Exp. XVII, App. III, Cor.], and [35] for recent developments). Thus, $G_{\text{aff}}$ may not be defined over $F$ with the notation and assumptions of Proposition 19. Yet the Albanese morphism still exists for any $F$-variety equipped with an $F$-point (see [36, App. A]) and hence for any algebraic $F$-semigroup equipped with an $F$-idempotent.

4 Algebraic semigroup structures on certain varieties

4.1 Abelian varieties

In this subsection, we begin by describing all the algebraic semigroup laws on an abelian variety. Then we apply the result to the study of the Albanese morphism of an irreducible algebraic semigroup.

Proposition 21. Let $A$ be an abelian variety with group law denoted additively, $\mu$ an algebraic semigroup law on $A$, and $e$ an idempotent of $(A, \mu)$; choose $e$ as the neutral element of $(A, +)$.

(i) There exists a unique decomposition of algebraic semigroups

$$(A, \mu) = (A_0, \mu_0) \times (A_l, \mu_l) \times (A_r, \mu_r) \times (B, +)$$

where $A_0$, $A_l$, $A_r$ and $B$ are abelian varieties, and $\mu_0$ (resp. $\mu_l$, $\mu_r$) the trivial semigroup law on $A_0$ (resp. $A_l$, $A_r$) defined in Example 1 (i).

(ii) The corresponding projection $\varphi : A \to B$ is the universal homomorphism of $(A, \mu)$ to an algebraic group. Moreover, we have $E(S) = \{e\} \times A_l \times A_r \times \{e\}$ and $\ker(S) = \{e\} \times A_l \times A_r \times B$.

Proof. (i) By [19, Chap. II, §4, Cor. 1], the morphism $\mu : A \times A \to A$ satisfies

$$\mu(x, y) = \varphi(x) + \psi(y) + x_0,$$

where $x_0 \in A$ and $\varphi$, $\psi$ are endomorphisms of the algebraic group $A$. Since $\mu(e, e) = e$ and $\varphi(e) = \psi(e) = e$, we have $x_0 = e$, i.e., $\mu(x, y) = \varphi(x) + \psi(y)$. Now the associativity of $\mu$ is equivalent to the equality

$$\varphi \circ \varphi(x) + \varphi \circ \psi(y) + \psi(z) = \varphi(x) + \psi \circ \varphi(y) + \psi \circ \psi(z),$$

that is, to the equalities

$$\varphi \circ \varphi = \varphi, \quad \varphi \circ \psi = \psi \circ \varphi, \quad \psi \circ \psi = \psi.$$

This easily yields the desired decomposition, where $A_0 := \text{Ker}(\varphi) \cap \text{Ker}(\psi)$, $A_l := \text{Im}(\varphi) \cap \text{Ker}(\psi)$, $A_r := \text{Ker}(\varphi) \cap \text{Im}(\psi)$, and $B := \text{Im}(\varphi) \cap \text{Im}(\psi)$.
so that \( \varphi \) (resp. \( \psi \)) is the projection of \( A \) to \( A_l \times B \) (resp. \( A_r \times B \)). The uniqueness of this decomposition follows from that of \( \varphi \) and \( \psi \).

(ii) Let \( \gamma : (A, \mu) \to \mathcal{G} \) be a homomorphism to an algebraic group. Then the image of \( \gamma \) is a complete irreducible variety, and hence generates an abelian subvariety of \( \mathcal{G} \). Thus, we may assume that \( \mathcal{G} \) is an abelian variety, with group law also denoted additively. As above, we have \( \gamma(x) = \pi(x) + x'_0 \), where \( \pi : A \to \mathcal{G} \) is a homomorphism of algebraic groups and \( x'_0 \in \mathcal{G} \). Since \( \gamma(e) \) is idempotent, we obtain \( x'_0 = 0 \), i.e., \( \gamma : (A, +) \to \mathcal{G} \) is also a homomorphism. It follows readily that \( \gamma \) sends \( A_0 \times A_l \times A_r \times \{e\} \) to 0. So \( \gamma \) factors as \( \varphi \) followed by a unique homomorphism \( \gamma' : B \to \mathcal{G} \). This proves the assertion on \( \varphi \); those on \( E(S) \) and \( \ker(S) \) are easily checked.

**Proposition 22.** Let \((S, \mu)\) be an irreducible algebraic semigroup, \( e \in E(S) \), and \( \alpha : S \to A(S) \) the Albanese morphism; assume that \( \alpha(e) = 0 \).

(i) There exists a unique algebraic semigroup law \( A(\mu) \) on \( A(S) \) such that \( \alpha \) is a homomorphism.

(ii) Let \( \varphi : (A(S), A(\mu)) \to B(S) \) be the universal homomorphism to an algebraic group. Then the map \( eS_e \to B(S), x \mapsto \varphi(\alpha(x)) \) is the Albanese morphism of \( eS_e \).

**Proof.** (i) follows from the functorial properties of the Albanese morphism (see [30, Sec. 2]) by arguing as in the beginning of the proof of Proposition 16.

(ii) Consider the inclusions \( eS_e \subseteq eS \subseteq S \). Each of them admits a retraction, \( x \mapsto xe \) (resp. \( x \mapsto ex \)). Thus, the corresponding morphisms of Albanese varieties, \( A(eS_e) \to A(eS) \to A(S) \), also admit retractions, and hence are closed immersions. So we may identify \( A(eS_e) \) with the subgroup of \( A(S) \) generated by the differences \( \alpha(exe) - \alpha(eye) \), where \( x, y \in S \). But \( \alpha(exe) = A(\mu)(\alpha(e), A(\mu)(\alpha(x), \alpha(e))) \) and \( \alpha(e) \) is of course an idempotent of \((A(S), A(\mu))\). Hence \( \alpha(e) = (e, a_l, a_r, e) \) in the decomposition of Proposition 21. Using that decomposition, we obtain \( \alpha(exe) = (e, a_l, a_r, b(x)) \), where \( b(x) \) denotes the projection of \( \alpha(x) \) to \( B(S) \). As a consequence, \( \alpha(exe) - \alpha(eye) = (e, e, e, b(x) - b(y)) \); this yields the desired identification of \( A(eS_e) \) to \( B(S) \).

Combined with Proposition 16, the above result yields:

**Corollary 5.** Let \( S \) be an irreducible algebraic semigroup.

(i) All the maximal submonoids of \( S \) have the same Albanese variety, and all the maximal subgroups have isogenous Albanese varieties.

(ii) The irreducible monoid \( eS_e \) is affine for all \( e \in E(S) \) if \( eS_e \) is affine for some \( e \in E(S) \).

**Remark 9.** (i) With the notation and assumptions of Proposition 22, the morphism \( \varphi \circ \alpha : S \to B(S) \) is the universal homomorphism to an abelian variety.
Also, recall from Remark 5 that there exists a universal homomorphism to an algebraic group, \( \psi : S \rightarrow G(\mathcal{S}) \), and that \( G(\mathcal{S}) \) is connected. It follows that \( B(\mathcal{S}) \) is the Albanese variety of \( G(\mathcal{S}) \).

(ii) Consider an irreducible algebraic semigroup \((S, \mu)\) and its rational Albanese map \( \alpha_{\text{rat}} : S \rightarrow A(S)_{\text{rat}} \). If the image of \( \mu : S \times S \rightarrow S \) meets the domain of definition of \( \alpha_{\text{rat}} \), then there exists a unique algebraic semigroup structure \( A(\mu) \) on \( A(S)_{\text{rat}} \) such that \( \alpha_{\text{rat}} \) is a ‘rational homomorphism’, i.e., \( \alpha_{\text{rat}}(\mu(x, y)) = A(\mu)(\alpha_{\text{rat}}(x), \alpha_{\text{rat}}(y)) \) whenever \( \alpha_{\text{rat}} \) is defined at \( x, y \in S \) and at \( \mu(x, y) \) (as can be checked by the argument of Proposition 22). But this does not hold for an arbitrary \((S, \mu)\); for example, if \( S \subseteq \mathbb{A}^3 \) is the affine cone over an elliptic curve \( E \subseteq \mathbb{P}^2 \) and if \( \mu = \mu_0 \). Here 0, the origin of \( \mathbb{A}^3 \), is the unique singular point of \( S \), and \( \alpha_{\text{rat}} \) is the natural map \( S \setminus \{0\} \rightarrow E \).

4.2 Irreducible curves

In this subsection, we classify the irreducible algebraic semigroups of dimension 1; those having a nontrivial law (as defined in Example 1 (i)) turn out to be algebraic monoids.

Such semigroups include of course the connected algebraic groups of dimension 1, presented in Example 2 (iv). We now construct further examples: let \((a_1, \ldots, a_n)\) be a strictly increasing sequence of positive integers having no nontrivial common divisor, and consider the map

\[ \varphi : \mathbb{A}^1 \rightarrow \mathbb{A}^n, \quad x \mapsto (x^{a_1}, \ldots, x^{a_n}). \]

Then \( \varphi \) is a homomorphism of algebraic monoids, where \( \mathbb{A}^1 \) and \( \mathbb{A}^n \) are equipped with pointwise multiplication. Also, one checks that the morphism \( \varphi \) is finite; hence its image is a closed submonoid of \( \mathbb{A}^n \), containing the origin as its zero element. We denote this monoid by \( M(a_1, \ldots, a_n) \), and call it an affine monomial curve; it only depends on the abstract submonoid of \((\mathbb{Z}, +)\) generated by \( a_1, \ldots, a_n \). One may check that \( \varphi \) restricts to an isomorphism \( \mathbb{A}^1 \setminus \{0\} \cong M(a_1, \ldots, a_n) \setminus \{0\} \); also, \( M(a_1, \ldots, a_n) \) is singular at the origin unless \( \varphi \) is an isomorphism, i.e., unless \( a_1 = 1 \).

**Theorem 5.** Let \( S \) be an irreducible curve, and \( \mu \) a nontrivial algebraic semigroup structure on \( S \). Then \((S, \mu)\) is either an algebraic group or an affine monomial curve.

**Proof.** As the arguments are somewhat long and indirect, we divide them into four steps.

**Step 1:** we show that every idempotent of \( S \) is either a neutral or a zero element.
Let \( e \in E(S) \). Since \( Se \) is a closed irreducible subvariety of \( S \), it is either the whole \( S \) or a single point; in the latter case, \( Se = \{ e \} \). Thus, one of the following cases occurs:

(i) \( Se = eS = S \). Then any \( x \in S \) satisfies \( xe = ex = x \), i.e., \( e \) is the neutral element.

(ii) \( Se = \{ e \} \) and \( eS = S \). Then for any \( x, y \in S \), we have \( xe = e \) and \( ey = y \).

Thus, \( xy = xey = ey = y \). So \( \mu = \mu_e \) in the notation of Example 1 (i), a contradiction since \( \mu \) is assumed to be nontrivial.

(iii) \( eS = \{ e \} \) and \( Se = S \). This case is excluded similarly.

(iv) \( Se = eS = \{ e \} \). Then \( e \) is the zero element of \( S \).

**Step 2:** we show that if \( S \) is complete, then it is an elliptic curve.

For this, we first reduce to the case where \( S \) has a zero element. Otherwise, \( S \) has a neutral element by Step 1. Hence \( S \) is a monoid with unit group \( G \) being \( \mathbb{G}_a \), \( \mathbb{G}_m \) or an elliptic curve, in view of the classification of connected algebraic groups of dimension 1. In the latter case, \( G \) is complete and hence \( G = S \). On the other hand, if \( G = \mathbb{G}_a \) or \( \mathbb{G}_m \), then \( S \setminus G \) is a nonempty closed subsemigroup of \( S \) in view of Proposition 3. Hence \( S \setminus G \) contains an idempotent, which must be the zero element of \( S \) by Step 1. This yields the desired reduction.

The semigroup law \( \mu : S \times S \to S \) sends \( S \times \{ 0 \} \) to the point 0. By the rigidity lemma (see e.g. [19, Chap. II, §4]), it follows that \( \mu(x, y) = \varphi(y) \) for some morphism \( \varphi : S \to S \). The associativity of \( \mu \) yields

\[
\varphi(z) = (xy)z = x(yz) = \varphi(yz) = \varphi(\varphi(z))
\]

for all \( x, y, z \in S \); hence \( \varphi \) is a retraction to its image. Since \( S \) is an irreducible curve, either \( \varphi = \text{id} \) or the image of \( \varphi \) consists of a single point \( x \). In the former case, \( \mu = \mu_e \), whereas \( \mu = \mu_x \) in the latter case. Thus, the law \( \mu \) is trivial, a contradiction.

**Step 3:** we show that if \( S \) is an affine monoid, then it is isomorphic to \( \mathbb{G}_a \), \( \mathbb{G}_m \) or an affine monomial curve.

We may view \( S \) as an equivariant embedding of its unit group \( G \), and that group is either \( \mathbb{G}_a \) or \( \mathbb{G}_m \). Since \( \mathbb{G}_a \cong \mathbb{A}^1 \) as a variety, any affine equivariant embedding of \( \mathbb{G}_a \) is \( \mathbb{G}_a \) itself. So we may assume that \( G = \mathbb{G}_m \). Then the coordinate ring \( \mathcal{O}(S) \) is a subalgebra of \( \mathcal{O}(\mathbb{G}_m) = k[x, x^{-1}] \), stable under the natural action of \( \mathbb{G}_m \). It follows that \( \mathcal{O}(S) \) has a basis consisting of Laurent monomials, and hence that

\[
\mathcal{O}(S) = \bigoplus_{n \in \mathcal{M}} x^n,
\]

where \( \mathcal{M} \) is a submonoid of \((\mathbb{Z}, +)\). Moreover, since \( \mathbb{G}_m \) is open in \( S \), the fraction field of \( \mathcal{O}(S) \) is the field of rational functions \( k(x) \); it follows that \( \mathcal{M} \) generates the group \( \mathbb{Z} \). Thus, either \( \mathcal{M} = \mathbb{Z} \) or \( \mathcal{M} \) is generated by finitely
many integers, all of the same sign and having no nontrivial common divisor.
In the former case, \( S = \mathbb{G}_m \); in the latter case, \( S \) is an affine monomial curve.

**Step 4:** in view of Step 2, we may assume that the irreducible curve \( S \) is noncomplete, and hence is affine. Then it suffices to show that \( S \) has a nonzero idempotent: then \( S \) is an algebraic monoid by Step 1, and we conclude by Step 3. We may further assume that \( S \) is nonsingular: indeed, by the nontriviality assumption, the semigroup law \( \mu : S \times S \to S \) is dominant. Using Proposition [9], it follows that the normalization \( \tilde{S} \) (an irreducible nonsingular curve) has a compatible algebraic semigroup structure; then the image of a nonzero idempotent of \( \tilde{S} \) is a nonzero idempotent of \( S \).

So we assume that \( S \) is an affine irreducible nonsingular semigroup of dimension 1, having a zero element 0, and show that \( S \) has a neutral element. We use the “right regular representation” of \( S \), i.e., its action on the coordinate ring \( \mathcal{O}(S) \) by right multiplication; specifically, an arbitrary point \( x \in S \) acts on \( \mathcal{O}(S) \) by sending a regular function \( f \) on \( S \) to the regular function \( x \cdot f : y \mapsto f(yx) \). This yields a map

\[
\varphi : S \longrightarrow \text{End}(\mathcal{O}(S)), \quad x \longmapsto x \cdot f
\]

which is readily seen to be a homomorphism of abstract semigroups. Moreover, the action of \( S \) on \( \mathcal{O}(S) \) stabilizes the maximal ideal \( m \) of 0, and all its powers \( m^n \). This defines compatible homomorphisms of abstract semigroups

\[
\varphi_n : S \longrightarrow \text{End}(m/m^n) \quad (n \geq 1).
\]

Since \( S \) is a nonsingular curve, we have compatible isomorphisms of \( k \)-algebras

\[
m/m^n \cong k[t]/t^n k[t],
\]

where \( t \) denotes a generator of the maximal ideal \( m\mathcal{O}_{S,0} \) of the local ring \( \mathcal{O}_{S,0} \); the right-hand side is the algebra of truncated polynomials at the order \( n \). Thus, an endomorphism \( \gamma \) of \( m/m^n \) is uniquely determined by \( \gamma(\bar{t}) \), where \( \bar{t} \) denotes the image of \( t \) mod \( t^n k[t] \). Moreover, the assignment \( \gamma \mapsto \gamma(\bar{t}) \) yields compatible isomorphisms of abstract semigroups

\[
\text{End}(m/m^n) \xrightarrow{\cong} tk[t]/t^n k[t],
\]

where the semigroup law on the right-hand side is the composition of truncated polynomials. Thus, we obtain compatible homomorphisms of abstract semigroups

\[
\psi_n : S \longrightarrow tk[t]/t^n k[t].
\]

Clearly, the right-hand side is an algebraic semigroup. Moreover, \( \psi_n \) is a morphism: indeed, for any \( f \in \mathcal{O}(S) \), we have \( f(yx) = \sum_{i \in \mathcal{I}} f_i(x)g_i(y) \) for some finite collection of functions \( f_i, g_i \in \mathcal{O}(S) \) (since the semigroup law is a
morphism). In other words, \( x \cdot f = \sum_{i \in I} f_i(x)g_i \). Thus, the matrix coefficients of the action of \( x \) in \( \mathcal{O}(S)/m^n \), and hence in \( m/m^n \), are regular functions of \( x \).

We claim that there exists \( n \geq 1 \) such that \( \psi_n \neq 0 \). Otherwise, we have \( \varphi_n(x) = 0 \) for all \( n \geq 1 \) and all \( x \in S \). Since \( \bigcap_{n} m^n = \{0\} \), it follows that \( \varphi(x) \) sends \( m \) to 0. But \( \mathcal{O}(S) = k \oplus m \), where the line \( k \) of constant functions is fixed pointwise by \( \varphi(x) \). Hence \( \varphi(x) = \varphi(0) \) for all \( x \), i.e., \( f(yx) = f(0) \) for all \( f \in \mathcal{O}(S) \) and all \( x, y \in S \). Thus, \( yx = 0 \), i.e., \( \mu = \mu_0 \); a contradiction.

Now let \( n \) be the smallest integer such that \( \psi_n \neq 0 \). Then \( \psi_n \) sends \( S \) to the quotient \( t^{n-1}k[t]/t^n k[t] \), i.e., to the semigroup of endomorphisms of the algebra \( k[t]/t^n k[t] \) given by \( \bar{t} \mapsto ct^{n-1} \), where \( c \in k \). If \( n \geq 3 \), then the composition of any two such endomorphisms is 0, and hence \( \psi_n(xy) = 0 \) for all \( x, y \in S \). Thus, \( xy \) belongs to the fiber of \( \psi_n \) at 0, a finite set containing 0. Since \( S \) is irreducible, it follows that \( xy = 0 \), i.e., \( \mu = \mu_0 \); a contradiction. Thus, \( \psi_n(xy) \) is nonconstant, and we obtain a nonconstant morphism \( \psi = \psi_2 : S \to A^1 \), where the semigroup law on \( A^1 \) is the multiplication. The image of \( \psi \) contains 0 and a nonempty open subset \( U \) of the unit group \( \mathbb{G}_m \). Then \( UU = \mathbb{G}_m \) and hence \( \psi \) is surjective. By Proposition 1, it follows that there exists an idempotent \( e \in S \) such that \( \psi(e) = 1 \). Then \( e \) is the desired nonzero idempotent.

Remark 10. One may also deduce the above theorem from the description of algebraic semigroup structures on abelian varieties (Proposition 21), when the irreducible curve \( S \) is assumed to be nonsingular and nonrational. Then the Albanese morphism of \( S \) is a locally closed embedding in its Jacobian variety \( A \). It follows that \( A \) has no trivial summand \( A_0 \), \( A_1 \) or \( A_r \) (otherwise, the projection to that summand is constant since \( \mu \) is nontrivial; as the differences of points of \( S \) generate the group \( A \), this yields a contradiction). In other words, the inclusion of \( S \) into \( A \) is a homomorphism for a suitable choice of the origin of \( A \). This implies that \( S = A \), and we conclude that \( S \) is an elliptic curve equipped with its group law.

### 4.3 Complete irreducible varieties

In this subsection, we obtain a description of all complete irreducible algebraic semigroups, analogous to that of the kernels of algebraic semigroups presented in Proposition 5.

**Theorem 6.** There is a bijective correspondence between the following objects:

- the triples \((S, \mu, e)\), where \( S \) is a complete irreducible variety, \( \mu \) an algebraic semigroup structure on \( S \), and \( e \) an idempotent of \((S, \mu)\),
- the tuples \((X, Y, G, \iota, \rho, x_0, y_0)\), where \( X \) (resp. \( Y \)) is a complete irreducible variety equipped with a base point \( x_0 \) (resp. \( y_0 \)), \( G \) is an abelian variety,
\[ \iota : X \times G \times Y \to S \text{ is a closed immersion, and } \rho : S \to X \times G \times Y \text{ a retraction of } \iota. \]

This correspondence assigns to any such tuple, the algebraic semigroup structure \( \nu \) on \( X \times G \times Y \) defined by
\[ \nu((x, g, y), (x', g', y')) := (x, gg', y') \]
and then the algebraic semigroup structure \( \mu \) on \( S \) defined by
\[ \mu(s, s') := \iota(\nu(\rho(s), \rho(s'))). \]
The idempotent is \( e := \iota(x_0, 1_G, y_0) \). Moreover, \( \iota \) and \( \rho \) are homomorphisms of algebraic semigroups.

The inverse correspondence will be constructed at the end of the proof. We begin that proof with three preliminary results.

**Lemma 1.** Let \( \varphi : X \to Y \) be a morphism of varieties, where \( X \) is complete and irreducible; assume that \( \varphi \) has a section (for example, \( \varphi \) is a retraction of \( X \) to a subvariety \( Y \)). Then \( Y \) is complete and irreducible as well. Moreover, the map \( \varphi^* : O_Y \to \varphi_* (O_X) \) is an isomorphism; in particular, the fibers of \( \varphi \) are connected.

**Proof.** Note that \( \varphi \) is surjective, since it admits a section. This readily yields the first assertion.

Next, consider the Stein factorization of \( \varphi \) as the composition
\[ X \xrightarrow{\varphi'} X' \xrightarrow{\psi} Y, \]
where \( \varphi' \) is the natural morphism to the Spec of the sheaf of \( O_Y \)-algebras \( \varphi_* (O_X) \), and \( \psi \) is finite (see [13, Cor. III.11.5]). Then \( \varphi' \) is surjective, and hence \( X' \) is a complete irreducible variety. Also, given a section \( \sigma \) of \( \varphi \), the map \( \varphi' \circ \sigma \) is a section of \( \psi \). In view of the irreducibility of \( X' \) and the finiteness of \( \psi \), it follows that \( \psi \) is an isomorphism; this yields the second assertion. \( \square \)

**Lemma 2.** Let \( S \) be a complete irreducible algebraic semigroup, and \( e \) an idempotent of \( S \). Then \( xy = xey \) for all \( x, y \in S \).

**Proof.** Recall that the map \( \varphi : S \to eS, x \mapsto ex \) is a retraction. Thus, its fibers are connected by Lemma 1. Let \( F \) be a (set-theoretic) fiber. Then the morphism \( \mu : S \times S \to S, (x, y) \mapsto xy \) sends \( \{e\} \times F \) to a point. By the rigidity lemma (see e.g. [19, Chap. II, §4]), \( \mu(\{x\} \times F) \) consists of a single point for any \( x \in S \). Thus, the map \( y \mapsto xy \) is constant on the fibers of \( \varphi \). Since \( \varphi(y) = \varphi(ey) \) for all \( y \in S \), this yields the statement. \( \square \)

**Lemma 3.** Keep the assumptions of the above lemma.
The closed submonoid $eSe$ of $S$ is an abelian variety.

(ii) The map $\varphi : S \to eSe$, $x \mapsto exe$ is a retraction of algebraic semigroups.

(iii) The above map $\varphi$ is the universal homomorphism to an algebraic group.

Proof. (i) By Proposition 3 (iii), it suffices to show that $e$ is the unique idempotent of $eSe$. But if $f \in E(eSe)$, then $xy = xfy$ for all $x, y \in S$, by Lemma 2. Taking $x = y = e$ yields $e = efe = f$.

(ii) By Lemma 2 again, we have $exye = exye = (exe)(eye)$ for all $x, y \in S$.

(iii) Let $G$ be an algebraic group and let $\psi : S \to G$ be a homomorphism of algebraic semigroups. Then $\psi(e) = 1$ and hence $\psi(x) = \psi(exe)$ for all $x \in S$. Thus, $\psi$ factors uniquely as the homomorphism $\varphi$ followed by some homomorphism of algebraic groups $eSe \to G$. □

Remark 11. By Lemma 3, every idempotent $e$ of a complete irreducible algebraic semigroup $(S, \mu)$ is minimal. Moreover, by Lemma 2, the image of the morphism $\mu$ is exactly the kernel of $S$; this is a simple algebraic semigroup in view of Proposition 5. One may thus deduce part of Theorem 6 from the structure of simple algebraic semigroups presented in Remark 3 (i). Yet we will provide a direct, self-contained proof by adapting the arguments of Proposition 5.

Proof of Theorem 6.

One readily checks that the map $\nu$ (resp. $\mu$) as in the statement yields an algebraic semigroup structure on $X \times G \times Y$ (resp. on $S$); compare with Example 1 (ii).

Conversely, given $(S, \mu, e)$ as in the statement, consider

$$X := eSe, \quad G := eSe, \quad Y := eSe$$

with the notation of Remark 1 (ii). Then $G$ is an abelian variety by Lemma 8. Let $\iota : X \times G \times Y \to S$ denote the multiplication map: $\iota(x, g, y) = xgy$.

Finally, define a map $\rho : S \to S \times G \times S$ by

$$\rho(s) = (s(ese)^{-1}, ese, (ese)^{-1}s).$$

Then $s(ese)^{-1} \in X$, since $es(ese)^{-1} = ese(ese)^{-1} = e$ and $s(ese)^{-1}e = s(ese)^{-1}$. Likewise, $(ese)^{-1}s \in Y$. So the image of $\rho$ is contained in $X \times G \times Y$.

We claim that $\rho \circ \iota$ is the identity of $X \times G \times Y$. Indeed, $(\rho \circ \iota)(x, g, y) = \rho(xgy)$. Moreover, $exgy = g$ so that

$$\rho(xgy) = (xgyg^{-1}, g, g^{-1}gy).$$

Now $xgyg^{-1} = xgyeg^{-1} = xgyg^{-1}x = xe = x$ and likewise, $g^{-1}xgy = y$. This proves the claim.

By that claim, $\iota$ is a closed immersion, and $\rho$ a retraction of $\iota$. Also, we have for any $x, x' \in X$, $g, g' \in G$ and $y, y' \in Y$: 
\[
x g y x' g' y' = x g y e x' g' y' = x g e x' g' y' = x g e y'.
\]

In other words, \( \iota \) is a homomorphism of algebraic semigroups, where \( X \times G \times Y \) is given the semigroup structure \( \nu \) as in the statement.

We next claim that \( \rho \) is a homomorphism of algebraic semigroups as well. Indeed,
\[
\rho(ss') = (ss'(ess'e)^{-1}, ess'e, (ess'e)^{-1}ss')
\]
and hence, using Lemma 3
\[
\rho(ss') = (ss'(es'e)^{-1}(ese)^{-1}, eses'e, (es'e)^{-1}(ese)^{-1}ss').
\]
Moreover,
\[
ss'(es'e)^{-1}(ese)^{-1} = ses'e(es'e)^{-1}(ese)^{-1} = se(ese)^{-1} = s(ese)^{-1}
\]
by Lemma 2 and likewise \((es'e)^{-1}(ese)^{-1}ss' = (es'e)^{-1}s'\). Thus,
\[
\rho(ss') = (s(ese)^{-1}, eses'e, (es'e)^{-1}s') = \nu(\rho(s), \rho(s'))
\]
as required.

Finally, we claim that \( ss' = \iota(\nu(\rho(s), \rho(s')))) \). Indeed, the right-hand side equals
\[
s(ese)^{-1}eses'e(es'e)^{-1}s' = ses' = ss'
\]
in view of Lemma 2 again.

Remark 12. (i) The description of algebraic semigroup laws on a given abelian variety \( A \) (Proposition 21) may of course be deduced from Theorem 3 with the notation of that theorem, the inclusion \( \iota \) and retraction \( \rho \) yield a decomposition \( A \cong A_0 \times A_1 \times A_r \times B \), where \( A_1 := X, A_r := Y, B := G \) and \( A_0 \) denotes the fiber of \( \rho \) at 0. Yet the original proof of Proposition 21 is simpler and more direct.

(ii) As a direct consequence of Theorem 4, every algebraic semigroup law on a complete irreducible curve is either trivial or the group law of an elliptic curve. This yields an alternative proof of part of the classification of irreducible algebraic semigroups of dimension 1 (Theorem 3); but in fact, both arguments make a similar use of the rigidity lemma.

(iii) As another consequence of Theorem 4, for any complete irreducible algebraic semigroup \((S, \mu)\), the closed subset \( E(S) \) of idempotents is an irreducible subsemigroup. Indeed, choosing \( e \in E(S) \), we have with the notation of that theorem
\[
E(S) = \iota(X \times \{1_G\} \times Y) \cong X \times Y.
\]
Moreover, \( \mu(\iota(x, 1_G, y), \iota(x', 1_G, y')) = \iota(x, 1_G, y') \) with an obvious notation.

(iv) In fact, some of the ingredients of Theorem 4 only depend of \((S, \mu)\), but not of the choice of \( e \in E(S) \). Specifically, note first that the projections \( \varphi : E(S) \to X, \psi : E(S) \to Y \) are independent of \( e \). Indeed, let \( f \in E(S) \)
and write \( f = \iota(x, 1_G, y) \). Then \( f E(S) = \iota((x, 1_G)) \times Y \) and hence \( f E(S) \) is the fiber of \( \varphi \) at \( f \); likewise, the fiber of \( \psi \) at \( f \) is \( E(S)f \).

As seen in Remark 11, \( \ker(S) = SeS \) is isomorphic to \( X \times G \times Y \) via \( \iota \). The resulting projection \( \gamma : \ker(S) \to G \) is the universal homomorphism to an algebraic group by Lemma 8, and hence is also independent of \( e \); its fiber at \( 1_G \) is \( E(S) \). In particular, the algebraic group \( G = G(S) \) is independent of \( e \). Note however that \( G = e Se \), viewed as a subgroup of \( S \), does depend on the choice of the idempotent \( e \). Indeed, \( e Se = \iota((x_o, y_o)) \) with the notation of Theorem 6, while \( f Sf = \iota((x) \times G \times \{y\}) \) for \( f \) as above.

The map \( \rho : S \to X \times G \times Y \) satisfies \( \iota \circ \rho = \rho_e \), where \( \rho_e : S \to \ker(S) \) denotes the retraction \( s \mapsto s(e s e)^{-1} s \) of Proposition 8. We check that \( \rho_e \) is independent of \( e \) (this also follows from Proposition 23 below). Let \( f \in E(S), s \in S \), and write \( f = \iota(x, 1_G, y), \rho(s) = (x_s, g_s, y_s) \). Then \( f s f = \iota(x, g_s, y) \) and hence \( (f s f)^{-1} = \iota(x, g_s^{-1}, y) \). Thus,

\[
\rho_f(s) = s(f s f)^{-1} s = \iota(x_s, g_s, y_s) = \rho_e(s).
\]

Finally, consider the action of the abelian variety \( G \) on \( \ker(S) \cong X \times G \times Y \) via translation on the second factor:

\[
g' \cdot \iota(x, g, y) := \iota(x, gg', y).
\]

We check that this action lifts to an action of \( G \) on \( S \) such that \( \rho : S \to \ker(S) \) is equivariant. For any \( s, s' \in S \), define

\[
s' \cdot s := s(e s e)^{-1} s' s.
\]

Then we have

\[
s' \cdot s = s(e s e)^{-1} es'e es = s(e s e)^{-1} e s'es(e s e)^{-1} s.
\]

It follows that \( s' \cdot s = es'e \cdot s = es'e \cdot \rho(s) \). Moreover, \( s' \cdot S = \ker(S) \) and the endomorphism \( s \mapsto s' \cdot s \) of \( \ker(S) \) is just the translation by \( es'e \in G \) on \( G = e Se \); we have

\[
s' \cdot s_1 s_2 = s_1 s' s_2
\]

for all \( s_1, s_2 \in S \). Also, one may check as above that \( s' \cdot s \) is independent of the choice of \( e \).

(v) Theorem 5 extends readily to those irreducible algebraic semigroups that are defined over a perfect subfield \( F \) of \( k \), and that have an \( F \)-point; indeed, this implies the existence of an idempotent \( F \)-point by Proposition 17.

Likewise, the results of Subsections 4.1 and 4.2 extend readily to the setting of perfect fields. In view of Theorem 6, every nontrivial algebraic semigroup law \( \mu \) on an irreducible curve \( S \) is commutative; by Proposition 17 again, it follows that \( S \) has an idempotent \( F \)-point whenever \( S \) and \( \mu \) are defined over \( F \).
4.4 Rigidity

In this subsection, we obtain two rigidity results (both possibly known, but for which we could not locate adequate references) and we apply them to the study of endomorphisms of complete varieties.

Our first result is a scheme-theoretic version of a classical rigidity lemma for irreducible varieties (see [8, Lem. 1.15]; further versions can be found in [20, Prop. 6.1]).

**Lemma 4.** Let \( f : X \to Y \) and \( g : X \to Z \) be morphisms of schemes of finite type over \( k \), satisfying the following assumptions:

(i) \( f \) is proper and the map \( f^\# : \mathcal{O}_Y \to f_* (\mathcal{O}_X) \) is an isomorphism.

(ii) There exists a \( k \)-rational point \( y_0 \in Y \) such that \( g \) maps the scheme-theoretic fiber \( f^{-1}(y_0) \) to a single point.

(iii) \( f \) has a section, \( s \).

(iv) \( X \) is irreducible.

Then \( g \) factors through \( f \); specifically, \( g = h \circ f \), where \( h := g \circ s \).

**Proof.** We first treat the case where \( y_0 \) is the unique closed point of \( Y \). We claim that \( X \) is the unique open neighborhood of \( f^{-1}(y_0) \). Indeed, given such a neighborhood \( U \) with complement \( F := X \setminus U \), the image \( f(F) \) is closed, since \( f \) is proper. If \( F \) is nonempty, then \( f(F) \) contains \( y_0 \), a contradiction.

Let \( z_o \in Z \) be the point \( g(f^{-1}(y_0)) \), and choose an open affine neighborhood \( W \) of \( z_o \) in \( Z \). Then \( g^{-1}(W) = X \) by the claim together with (iii); thus, we may assume that \( Z = W \) is affine. Then \( g \) is uniquely determined by the homomorphism of algebras \( g^\# : \mathcal{O}(Z) \to \mathcal{O}(X) \). But the analogous map \( f^\# : \mathcal{O}(Y) \to \mathcal{O}(X) \) is an isomorphism in view of (iv). Thus, there exists a morphism \( h' : Y \to Z \) such that \( g = h' \circ f \). Then \( h = g \circ s = h' \circ f \circ s = h' \); this completes the proof in that case (note that the assumptions (i) and (ii) suffice to conclude that \( g \) factors through \( f \)).

Next, we treat the general case. The scheme \( Y' := \text{Spec}(\mathcal{O}_{Y,y_0}) \) has a unique closed point \( y'_0 \) and comes with a flat morphism \( \psi : Y' \to Y \), \( y'_0 \mapsto y_0 \). Moreover, \( X' := X \times_Y Y' \) is equipped with morphisms \( f' : X' \to Y' \), \( g' = g \circ p_X : X' \to Z \) that satisfy (i) (since taking the direct image commutes with flat base extension, see [13, Prop. III.9.3]) and (ii). Also, note that \( f' \) has a section \( s' \) given by the morphism \( s \circ \psi \times \text{id} : Y' \to X \times Y' \). By the preceding step, we thus have \( g' = h' \circ f' \), where \( h' := g' \circ s' \). It follows that there exists an open neighborhood \( V \) of \( y_0 \) in \( Y \) such that \( g = h \circ f \) over \( f^{-1}(V) \).

We now consider the largest subscheme \( W \) of \( X \) over which \( g = h \circ f \), i.e., \( W \) is the preimage of the diagonal in \( Z \times Z \) under the morphism \( g \times (h \circ f) \). Then \( W \) is closed in \( X \) and contains \( f^{-1}(V) \). Since \( X \) is irreducible, it follows that \( W = X \).

**Remark 13.** The assertion of Lemma 4 still holds under the assumptions (ii), (iii) and the following (weaker but more technical) versions of (i), (iv):
(i) If $f$ is proper, and for any irreducible component $Y'$ of $Y$, the scheme-theoretic preimage $X' := f^{-1}(Y')$ is an irreducible component of $X$. Moreover, the map $f^{\#} : \mathcal{O}_{Y'} \to f_*(\mathcal{O}_{X'})$ is an isomorphism, where $f' : X' \to Y'$ denotes the restriction of $f$.

(iv) $X$ is connected.

Indeed, let $Y_o$ be an irreducible component of $Y$ containing $y_o$; then $X_o := f^{-1}(Y_o)$ is an irreducible component of $X$. Moreover, the restrictions $f_o : X_o \to Y_o$, $g_o : X_o \to Z$, and $s_o : Y_o \to X_o$ satisfy the assumptions of Lemma 4. By that lemma, it follows that $g_o = g_o \circ s_o \circ f_o$, i.e., $g = h \circ f$ on $X_o$. In particular, $g$ maps the scheme-theoretic fiber of $f$ at any point of $Y_o$ to a single point.

Next, let $Y_1$ be an irreducible component of $Y$ intersecting $Y_o$. Then again, $X_1 := f^{-1}(Y_1)$ is an irreducible component of $X$; moreover, the restrictions $f_1 : X_1 \to Y_1$, $g_1 : X_1 \to Z$, and $s_1 : Y_1 \to X_1$ satisfy the assumptions of the above lemma, for any point $y_1$ of $Y_o \cap Y_1$. Thus, $g = h \circ f$ on $X_o \cup X_1$. Iterating this argument completes the proof in view of the connectedness of $X$.

As a first application of the above lemma and remark, we present a rigidity result for retractions; further applications will be obtained in the next subsection.

**Proposition 23.** Let $X$ be a complete irreducible variety, and $\varphi$ a retraction of $X$ to a subvariety $Y$. Let $T$ be a connected scheme of finite type over $k$, equipped with a $k$-rational point $t_o$, and let $\Phi : X \times T \to X$ be a morphism such that the morphism $\Phi_{t_o} : X \to X$, $x \mapsto \Phi(x, t_o)$ equals $\varphi$.

(i) There exists a unique morphism $\Psi : Y \times T \to X$ such that $\Phi(x, t) = \Psi(\varphi(x), t)$ on $X \times T$.

(ii) If $\Phi$ is a family of retractions to $Y$ (i.e., $\Phi(y, t) = y$ on $Y \times T$), then $\Phi$ is constant (i.e., $\Phi(x, t) = \varphi(x)$ on $X \times T$).

**Proof.** Consider the morphisms

\[
\begin{align*}
f : X \times T &\to Y \times T, \quad (x, t) \mapsto (\varphi(x), t), \\
g : X \times T &\to X \times T, \quad (x, t) \mapsto (\Phi(x, t), t).
\end{align*}
\]

Then the assumption (i)' of Remark 13 holds, since $\varphi_*(\mathcal{O}_X) = \mathcal{O}_Y$ in view of Lemma 1. Also, the assumption (ii) of Lemma 4 holds for any point $(y, t_o)$, where $y \in Y$, and the assumption (iii) of that lemma holds with $s$ being the inclusion of $Y \times T_o$ in $X \times T_o$. Finally, the assumption (iv)' of Remark 13 is satisfied, since $T$ is connected. By that remark, we thus have $g = g \circ s \circ f$ on $X \times T$. Hence there exists a unique morphism $\Psi : Y \times T \to X$ such that $\Phi(x, t) = \Psi(\varphi(x), t)$ on $X \times T$, namely, $\Psi(y, t) := \Phi(y, t)$. If $\Phi$ is a family of retractions, then we get that $\Phi(x, t) = \varphi(x)$ on $X \times T$. \qed
Remark 14. The preceding result has a nice interpretation when \( X \) is projective. Then there exists a quasiprojective \( k \)-scheme, \( \text{End}(X) \), which represents the endomorphism functor of \( X \), i.e., for any noetherian \( k \)-scheme \( T \), the set of \( T \)-points \( \text{End}(X)(T) \) is naturally identified with the set of endomorphisms of \( X \times T \) over \( T \); equivalently,

\[
\text{End}(X)(T) = \text{Hom}(X \times T, X).
\]

Moreover, each connected component of \( \text{End}(X) \) is of finite type. These results hold, more generally, for the similarly defined functor \( \text{Hom}(X,Y) \) of morphisms from a projective scheme \( X \) to another projective scheme \( Y \) (see [12, p. 21]). The composition of morphisms yields a morphism of \( \text{Hom} \) functors, and hence of \( \text{Hom} \) schemes by Yoneda's lemma. In particular, \( \text{End}(X) \) is a monoid scheme; its idempotent \( k \)-points are exactly the retractions with source \( X \).

Returning to the setting of an irreducible projective variety \( X \) together with a retraction \( \varphi : X \to Y \), we may identify \( \varphi \) with the idempotent endomorphism \( e \) of \( X \) with image \( Y \). Now Proposition 23 yields that the connected component of \( e \) in \( \text{End}(X) \) is isomorphic to the connected component of the inclusion \( Y \to X \) in \( \text{Hom}(Y,X) \), by assigning to any \( \phi \in \text{Hom}(Y,X)(T) = \text{Hom}(Y \times T, X) \), the composition \( \psi \circ (\varphi \times \text{id}) \in \text{Hom}(X \times T, X) \). Moreover, this isomorphism identifies the connected component of \( e \) in

\[
\text{End}(E)_e := \{ \Phi \in \text{End}(X) \mid \Phi \circ e = e \}
\]
to the (reduced) point \( e \).

Next, we obtain our second rigidity result:

**Lemma 5.** Let \( X \) be a complete variety, \( T \) a connected scheme of finite type over \( k \), and

\[
\Phi : X \times T \to X \times T, \quad (x,t) \mapsto (\varphi(x,t),t)
\]

an endomorphism of \( X \times T \) over \( T \). Assume that \( T \) has a point \( t_o \) such that \( \Phi_{t_o} : X \to X, x \mapsto \varphi(x,t_o) \) is an automorphism. Then \( \Phi \) is an automorphism.

**Proof.** Note that \( \Phi \) is proper, as the composition of the closed immersion \( X \times T \to X \times X \times T, (x,t) \mapsto (x,\varphi(x,t),t) \) and of the projection \( X \times X \times T \to X \times T, (x,y,t) \mapsto (y,t) \).

We now show that the fibers of \( \Phi \) are finite. Assuming the contrary, we may find a complete irreducible curve \( C \subseteq X \) and a point \( t_1 \in T \) such that \( \varphi : X \times T \to X \) sends \( C \times \{t_1\} \) to a point. By the rigidity lemma, it follows that the restriction of \( \varphi \) to \( C \times T \) factors through the projection \( C \times T \to T \). Taking \( t = t_o \), we get a contradiction.

The morphism \( \Phi \) is finite, since it is proper and its fibers are finite; it is also surjective, since for any \( t \in T \), the map \( \Phi_t : X \to X, x \mapsto \varphi(x,t) \) is a finite endomorphism of \( X \) and hence is surjective.
We now claim that $\Phi$ restricts to an automorphism of $X \times V$, for some open neighborhood $V$ of $t_0$ in $T$. This claim is proved in \[17\] Lem. \[1.10.1\]; we recall the argument for completeness. Since $\Phi$ is proper, the sheaf $\Phi_* (\mathcal{O}_{X \times T})$ is coherent; it is also flat over $T$, since $\Phi$ lifts the identity of $T$. Moreover, the map $\Phi^* : \mathcal{O}_{X \times T} \to \Phi_* (\mathcal{O}_{X \times T})$ induces an isomorphism $\Phi^* : \mathcal{O}_X \to (\Phi_{t_0})_* (\mathcal{O}_X)$. In view of a version of Nakayama’s lemma (see \[17\] Prop. \[I.7.4.1\]), it follows that $\Phi^*$ is an isomorphism over a neighborhood of $t_0$. This yields the claim.

By that claim, the points $t \in T$ such $\Phi_t$ is an isomorphism form an open subset of $T$. Since $T$ is connected, it suffices to show that this subset is closed.

For this, we may assume that $T$ is an irreducible curve; replacing $T$ with its normalization, we may further assume that $T$ is nonsingular. By shrinking $T$, we may finally assume that it has a point $s$ such that $\varphi_t$ is an automorphism for all $t \in T \setminus \{s\}$; we have to show that $\varphi_s$ is an automorphism as well.

If $X$ is normal, then so is $X \times T$; moreover, the above endomorphism $\Phi$ is finite and birational, and hence an automorphism. Thus, every $\varphi_t$ is an automorphism.

For an arbitrary $X$, consider the normalization $\eta : \tilde{X} \to X$. Then $\Phi$ lifts to an endomorphism $\tilde{\Phi} : \tilde{X} \times T \to \tilde{X} \times T$, which is an automorphism by the above step. In particular, $\varphi_s$ lifts to an automorphism $\tilde{\varphi}_s$ of $\tilde{X}$. We have a commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}_s} & \tilde{X} \\
\downarrow \eta & & \downarrow \eta \\
X & \xrightarrow{\varphi_s} & X
\end{array}
$$

and hence a commutative diagram of morphisms of sheaves

$$
\begin{array}{ccc}
\mathcal{O}_X & \longrightarrow & (\varphi_s)_* (\mathcal{O}_X) \\
\downarrow \eta_* (\mathcal{O}_X) & & \downarrow \eta_* (\tilde{\varphi}_s)_* (\mathcal{O}_{\tilde{X}}).
\end{array}
$$

Moreover, the bottom horizontal arrow in the latter diagram is the identity (as $(\tilde{\varphi}_s)_* (\mathcal{O}_{\tilde{X}}) = \mathcal{O}_{\tilde{X}}$), and the other maps are all injective. Thus, $\mathcal{O}_X \subseteq (\varphi_s)_* (\mathcal{O}_X) \subseteq \eta_* (\mathcal{O}_X)$, and hence the iterates $(\varphi_s^n)_* (\mathcal{O}_X)$ form an increasing sequence of subsheaves of $\eta_* (\mathcal{O}_X)$. As the latter sheaf is coherent, we get

$$(\varphi_s^n)_* (\mathcal{O}_X) = (\varphi_s^{n+1})_* (\mathcal{O}_X) \quad (n \gg 0).$$

Since $\varphi_s$ is finite and surjective, it follows that $\mathcal{O}_X = (\varphi_s)_* (\mathcal{O}_X)$ and hence that $\varphi_s$ is an isomorphism.

A noteworthy consequence of Lemma 5 is the following:

**Corollary 6.** (i) Let $M$ be a complete algebraic monoid. Then $G(M)$ is a union of connected components of $M$. In particular, if $M$ is connected then it is an abelian variety.
(ii) Let $S$ be a complete algebraic semigroup and let $e, f$ be distinct idempotents such that $e \leq f$. Then $e$ and $f$ belong to distinct connected components of $S$. In particular, if $S$ is connected then every idempotent is minimal.

Proof. (i) Let $T$ be a connected component of $M$ containing a unit $t_o$. Applying Lemma 5 to the morphism $M \times T \to M \times T$, $(x, t) \mapsto (xt, t)$, we see that the map $x \mapsto xt$ is an isomorphism for any $t \in T$. Likewise, the map $x \mapsto tx$ is an isomorphism as well. Thus, $t$ has a left and a right inverse in $M$, and hence is a unit. So $T$ is contained in $G(M)$.

Alternatively, we may deduce the statement from Theorem 5: indeed, $G(M)$ contains no subgroup isomorphic to $G_a$ or $G_m$, since the latter do not occur as unit groups of complete irreducible monoids. By Chevalley’s structure theorem, it follows that the reduced neutral component of $G(M)$ is an abelian variety. Thus, $G(M)$ is complete, and hence closed in $M$. But $G(M)$ is open in $M$, hence the assertion follows.

(ii) Assume that $e$ and $f$ belong to the same connected component $T$ of $S$. Then $T$ is a closed subsemigroup, and hence we may assume that $S$ is connected. Now $fSf$ is a complete connected algebraic monoid, and hence an abelian variety. It follows that $e = f$, a contradiction. $\square$

Remark 15. (i) Like for Proposition 23, the statement of Lemma 5 has a nice interpretation when $X$ is projective. Then its functor of automorphisms is represented by an open subscheme $\text{Aut}(X)$ of $\text{End}(X)$ (see [12, p. 21]); in fact, $\text{Aut}(X)$ is the unit group scheme of the monoid scheme $\text{End}(X)$. Now Lemma 5 implies that $\text{Aut}(X)$ is also closed in $\text{End}(X)$. In other words, $\text{Aut}(X)$ is a union of connected components of $\text{End}(X)$.

For an arbitrary complete variety $X$, the automorphism functor defined as above is still represented by a group scheme $\text{Aut}(X)$; moreover, each connected component of $\text{Aut}(X)$ is of finite type (see [18, Thm. 3.7] for these results). We do not know if $\text{End}(X)$ is representable in this generality; yet the above interpretation of Lemma 5 still makes sense in terms of functors.

(ii) Let $X$ and $T$ be complete varieties, where $T$ is irreducible, and let $\mu : X \times T \to X$ be a morphism such that $\mu(x, t_o) = x$ for some $t_o \in T$ and all $x \in X$. Then by Lemma 5 the map $\mu_t : x \mapsto \mu(x, t)$ is an automorphism for any $t \in T$. This yields a morphism of schemes

$$\varphi : T \to \text{Aut}(X), \quad t \mapsto \mu_t$$

such that $\varphi(t_o)$ is the identity. Hence $\varphi$ sends $T$ to the neutral component $\text{Aut}^\circ(X)$. Consider the subgroup $G$ of $\text{Aut}^\circ(X)$ generated by the image of $T$; then $G$ is closed and connected by [9] Prop. II.5.4.6, and hence is an abelian variety. In loose words, the morphism $\mu$ arises from an action of an abelian variety on $X$.

(iii) Let $X$ be a complete irreducible variety, and $\mu : X \times X \to X$ a morphism such that $\mu(x, x_o) = \mu(x_o, x) = x$ for some $x_o \in X$ and all $x \in X$. Then the above morphism $\varphi : X \to \text{Aut}^\circ(X)$ satisfies $\varphi(x)(x_o) = x$, and
hence is a closed immersion; we thus identify $X$ to its image in $\text{Aut}^o(X)$. As seen above, $X$ generates an abelian subvariety $G$ of $\text{Aut}^o(X)$. The natural action of $G$ on $X$ is transitive, since the orbit $Gx_o$ contains $Xx_o = X$. Thus, $X$ itself is an abelian variety on which $G$ acts by translations. Moreover, since $Gx_o = Xx_o$, we have $G = XGx_o$, where $Gx_o$ denotes the isotropy subgroup scheme of $x_o$. As $G$ is commutative and acts faithfully and transitively on $X$, this isotropy subgroup scheme is trivial, i.e., $G = X$. In conclusion, $X$ is an abelian variety with group law $\mu$ and neutral element $x_o$. This result is due to Mumford and Ramanujam, see [19, Chap. II, §4, Appendix].

4.5 Families of semigroup laws

**Definition 10.** Let $S$ be a variety, and $T$ a $k$-scheme. A family of semigroup laws on $S$ parameterized by $T$ is a morphism $\mu : S \times S \times T \to S$ such that the associativity condition

$$\mu(s, \mu(s', s'', t), t) = \mu(\mu(s, s', t), s'', t)$$

holds on $S \times S \times S \times T$.

Such a family yields a structure of semigroup scheme on $S \times T$ over $T$: to any scheme $T'$ equipped with a morphism $\theta : T' \to T$, one associates the (abstract) semigroup consisting of all morphisms $\sigma : T' \to S$, equipped with the law $\mu_\theta$ defined by

$$\mu_\theta(\sigma, \sigma') = \mu(\sigma, \sigma', \theta).$$

In particular, any $k$-rational point $t_o$ of $T$ yields an algebraic semigroup structure on $S$,

$$\mu_{t_o} : S \times S \to S, \quad (s, s') \mapsto \mu(s, s', t_o).$$

This sets up a bijective correspondence between families of semigroup laws on $S$ parameterized by $T$, and structures of $T$-semigroup scheme on $S \times T$.

For example, every algebraic semigroup law $S \times S \to S, \ (s, s') \mapsto ss'$ defines a family of semigroup laws on $S \times S$ parameterized by $S$, via

$$\mu : S \times S \times S \to S, \quad (s, s', t) \mapsto sts'.$$

If $S$ is irreducible and complete, and $e \in S$ is idempotent, then $\mu_e(s, s') = ss'$ in view of Lemma 2. More generally, for any $t \in S$, we have $\mu_t(s, s') = t \cdot ss'$, with the notation of Remark 12 (iv). In other words, the family $\mu$ arises from the action of the abelian variety $G = eS e$ on $S$, defined in that remark.

We now generalize this construction to obtain all families of semigroup structures on a complete irreducible variety, under a mild assumption on the parameter scheme.
Theorem 7. Let $S$ be a complete irreducible variety, $T$ a connected scheme of finite type over $k$, and $\mu : S \times S \times T \to S$ a family of semigroup laws. Choose a $k$-point $t_o \in T$ and denote by $\ker(S)$ the kernel of $(S, \mu_{t_o})$, by $\rho : S \to \ker(S)$ the associated retraction, and by $G$ the associated abelian variety; recall that $G$ acts on $\ker(S)$ by translations.

Then there exist unique morphisms $\varphi : \ker(S) \times T \to S$ and $\gamma : T \to G$ such that

$$\mu(s, s', t) = \varphi(\mu_{t_o}(s, s'), t)$$

on $S \times S' \times T$, and that the composition $\rho \circ \varphi : \ker(S) \times T \to \ker(S)$ is the translation $(s, t) \mapsto \gamma(t) \cdot s$.

Conversely, given $\varphi : \ker(S) \times T \to S$ such that there exists $\gamma : T \to G$ satisfying the preceding condition, the assignment $(s, s', t) \mapsto \varphi(\mu_{t_o}(s, s'), t)$ yields an algebraic semigroup law over $T$. Moreover, $\varphi(s, t_o) = s$ on $\ker(S)$, and $\gamma(t_o) = 1_G$.

Proof. Denote for simplicity $\mu_{t_o}(s, s')$ by $ss'$. We begin by showing that there exists a unique morphism $\varphi : \ker(S) \times T \to S$ such that $\mu(s, s', t) = \varphi(ss', t)$. For this, we apply Lemma 4 and the subsequent Remark 13 to the morphisms

$$\mu_{t_o} \times \id : S \times S \times T \to \ker(S) \times T, \quad \mu \times \id : S \times S \times T \to S \times T.$$

To check the corresponding assumptions, note first that $\mu_{t_o}$ has a section $\sigma : \ker(S) \longrightarrow S \times S, \quad s \longmapsto (s, s(e se)^{-2} s)$, where $e$ denotes a fixed idempotent of $(S, \mu_{t_o})$. (Indeed, let $\iota : X \times G \times Y \to S$ be the associated closed immersion with image $\ker(S)$. Then

$$\sigma(\iota(x, g, y)) = (\iota(x, g, y), \iota(x, 1_G, y))$$

as an easy consequence of Theorem 6.) Thus, $\mu_{t_o}(\sigma(\iota(x, g, y))) = \iota(x, g, y)$.) By Lemma 1 it follows that the map $\mu_{t_o}^{\#} : \mathcal{O}_{\ker(S)} \to (\mu_{t_o})^{\#}(\mathcal{O}_{S \times S})$ is an isomorphism. Thus, $\mu_{t_o}$ satisfies the assumption (i)' of Remark 13 hence so does $\mu_{t_o} \times \id$. Also, the assumption (ii) of Lemma 1 holds for any point $(s, t_o)$ with $s \in \ker(S)$, and the assumption (iii) of that lemma holds as well, since $\sigma \times \id$ is a section of $\mu_{t_o} \times \id$. Finally, $S \times S \times T$ is connected, i.e., the assumption (iv)' of Remark 13 is satisfied. Hence that remark yields the desired morphism $\varphi$.

In particular, $ss' = \mu(s, s', t_o) = \varphi(ss', t_o)$ for all $s, s' \in S$. Since the image of $\mu_{t_o}$ equals $\ker(S)$, it follows that $\varphi(s, t_o) = s$ for all $s \in \ker(S)$.

Next, consider the morphism

$$\Psi := (\rho \circ \varphi) \times \id : \ker(S, \mu_{t_o}) \times T \longrightarrow \ker(S, \mu_{t_o}) \times T.$$

Then $\Psi_{t_o}$ is the identity by the preceding step; thus, $\Psi$ is an automorphism in view of Lemma 5. In other words, $\Psi$ arises from a morphism...
\[ \pi : T \rightarrow \text{Aut}(\ker(S)), \quad t_o \mapsto \text{id}. \]

Since \( T \) is connected, the image of \( \pi \) is contained in \( \text{Aut}^o(\ker(S)) \). We identify \( \ker(S) \) with \( X \times G \times Y \) via \( \iota \). Then the natural map

\[ \text{Aut}^o(X) \times \text{Aut}^o(G) \times \text{Aut}^o(Y) \rightarrow \text{Aut}^o(\ker(S)) \]

is an isomorphism by [7, Cor. 4.2.7]. Moreover, \( \text{Aut}^o(G) \cong G \) via the action of \( G \) on itself by translations, see e.g. [loc. cit., Prop. 4.3.2]. Thus, we have

\[ \Psi(x, g, y, t) = (\alpha(x, t), g + \gamma(t), \beta(y, t), t) \]

for uniquely determined morphisms \( \alpha : X \times T \rightarrow X \), \( \beta : Y \times T \rightarrow Y \) and \( \gamma : T \rightarrow G \) such that \( \alpha \times \text{id} \) is an automorphism of \( X \times T \) over \( T \), and likewise for \( \beta \times \text{id} \).

We now use the assumption that \( \mu \) is associative. This is equivalent to the condition that

\[ \varphi(s\varphi(s's'', t), t) = \varphi(\varphi(s's', t)s'', t) \]

on \( S \times S \times X \). Let \( \psi := \rho \circ \varphi \), then

\[ \psi(s\psi(s's'', t), t) = \psi(\psi(s's', t)s'', t) \]

on \( \ker(S) \times \ker(S) \times \ker(S) \times T \), since \( ss' = \rho(s)\rho(s') = \rho(s's') \) on \( S \times S \). In view of the equalities \( \psi(x, g, y) = (\alpha(x, t), g + \gamma(t), \beta(y, t)) \) and \( (x, g, y)(x', g', y') = (x, gg', y') \), the above associativity condition for \( \psi \) yields that \( \alpha(x, t) = \alpha(\alpha(x, t), t) \) on \( X \times T \), and \( \beta(y, t) = \beta(\beta(y, t), t) \) on \( Y \times T \). As \( \alpha \times \text{id} \) and \( \beta \times \text{id} \) are automorphisms, it follows that \( \alpha(x, t) = x \) and \( \beta(y, t) = y \). Thus,

\[ \psi(x, g, y, t) = (x, g + \gamma(t), y), \]

that is, \( \rho \circ \varphi \) is the translation by \( \gamma \).

For the converse, let \( \varphi, \gamma \) be as in the statement. Then the morphism

\[ \mu : S \times S \times T \rightarrow S \times T, \quad (s, s', t) \mapsto \varphi(ss', t) \]

satisfies the associativity condition, since

\[ \varphi(s\varphi(s's'', t), t) = \varphi(\rho(s)\rho(\varphi(s's'', t)), t) = \varphi(\gamma(t) \cdot \rho(s)\rho(s')\rho(s''), t) \]

and the right-hand side is clearly associative. Moreover, as already checked, \( \varphi(s, t_o) = s \) on \( \ker(S) \); it follows that

\[ \gamma(t_o) \cdot s = (\rho \circ \varphi)(s, t_o) = \rho(s) = s \]

for all \( s \in \ker(S) \). Thus, \( \gamma(t_o) = 1_G \). \( \square \)

**Remark 16.** (i) With the notation and assumptions of Theorem 7, one can easily obtain further results on the semigroup scheme structure of \( S \times T \) over
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For example, one may check that the idempotent sections of the projection $S \times T \to T$ are exactly the morphisms

$$T \to S, \quad t \mapsto \varphi(\gamma(t)^{-2} \cdot \varepsilon(t)),$$

where $\varepsilon : T \to E(S, \mu_{t_0})$ is a morphism. In particular, any such semigroup scheme has an idempotent section.

(ii) Consider the functor of composition laws on a variety $S$, i.e., the contravariant functor from schemes to sets given by $T \mapsto \text{Hom}(S \times S \times T, S)$; then the families of algebraic semigroup laws yield a closed subfunctor (defined by the associativity condition). When $S$ is projective, the former functor is represented by a quasiprojective $k$-scheme,

$$\text{CL}(S) := \text{Hom}(S \times S, S);$$

moreover, each connected component of $\text{CL}(S)$ is of finite type over $k$ (as mentioned in Remark [14]). Thus, the latter subfunctor is represented by a closed subscheme,

$$\text{SL}(S) \subseteq \text{CL}(S).$$

In particular, $\text{SL}(S)$ is quasi-projective, and its connected components are of finite type.

By Theorem [7], the connected component of $\mu_{t_0}$ in $\text{SL}(S)$ is identified with the closed subscheme of $\text{Hom}(\text{ker}(S), S) \times G$ consisting of those pairs $(\varphi, \gamma)$ such that $\rho \circ \varphi$ is the translation by $\gamma$. Via the assignment $(\varphi, \gamma) \mapsto (\gamma^{-1} \cdot \varphi, \gamma)$ (where $\gamma^{-1} \cdot \varphi$ is defined as in Remark [12] (iv)), the above component of $\text{SL}(S)$ is identified with the closed subscheme of $\text{Hom}(\text{ker}(S), S) \times G$ consisting of those pairs $(\sigma, \gamma)$ such that $\rho \circ \sigma = \text{id}$, that is, $\sigma$ is a section of $\rho$. This identifies the universal semigroup law on the above component, with the morphism

$$(s, s') \mapsto \gamma \cdot \sigma(\mu_{t_0}(s, s')).$$

Note that the scheme of sections of $\rho$ is isomorphic to an open subscheme of the Hilbert scheme, $\text{Hilb}(S)$, by assigning to every section its image (see [12, p. 21]). This open subscheme is generally nonreduced, as shown by a classical example where $S$ is a ruled surface over an elliptic curve $C$. Specifically, $S$ is obtained as the projective completion of a nontrivial principal $\mathbb{G}_a$-bundle over $C$, and $\rho : S \to C$ is the ruling; then the section at infinity of $\rho$ yields a fat point of $\text{Hilb}(S)$, as follows from obstruction theory (see e.g. [17, Sec. I.2]). As a consequence, the scheme $\text{SL}(S)$ is generally nonreduced as well.

(iii) The families of semigroup laws on further classes of varieties are worth investigating. Following the approach of deformation theory, one may consider those families of semigroup laws $\mu$ on a prescribed variety $S$ that are parameterized by the spectrum of a local artinian $k$-algebra $R$ with residue field $k$, and that have a prescribed law $\mu_{t_0}$ at the closed point. Then the
first-order deformations (i.e., those parameterized by $\text{Spec}(k[t]/(t^2))$) form a $k$-vector space which may well be infinite-dimensional; this already happens when $S$ is the affine line, and $\mu_t$, the multiplication.

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References

1. Ax, J.: The elementary theory of finite fields. Ann. Math. 88, 239–271 (1968)
2. Borel, A.: Injective endomorphisms of algebraic varieties. Arch. Math. 20, No. 5, 531–537 (1969)
3. Brion, M.: Local structure of algebraic monoids. Mosc. Math. J. 8, No. 4, 647–666 (2008)
4. Brion, M.: Some basic results on actions of nonaffine algebraic groups. In: Campbell, H.E.A. et al. (eds.) Symmetry and Spaces, pp. 1-20. Birkhäuser, Basel (2010)
5. Brion, M., Rittatore, A.: The structure of normal algebraic monoids. Semigroup Forum 74, No. 3, 410–422 (2007)
6. Brion, M., Rittatore, A.: Algebraic semigroups are strongly $\pi$-regular. Available on the arXiv: [link](http://arxiv.org/pdf/1209.2042.pdf)
7. Brion, M., Samuel, P., Uma, V.: Lectures on the structure of algebraic groups and geometric applications. Hindustan Book Agency, Chennai (2013)
8. Debarre, O.: Higher-dimensional Algebraic Geometry. Springer-Verlag, New York (2001)
9. Demazure, M., Gabriel, P.: Groupes algébriques. Masson, Paris (1970)
10. Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. Springer-Verlag, New York (1995)
11. Ferrand, D.: Conducteur, descente et pincement. Bull. Soc. Math. France 131, No. 4, 553–585 (2003)
12. Grothendieck, A.: Techniques de construction et théorèmes d’existence en géométrie algébrique IV : les schémas de Hilbert. In: Séminaire Bourbaki 6, pp. 249–276. Soc. Math. France, Paris (1995)
13. Hartshorne, R.: Algebraic Geometry. Springer-Verlag, New York (1977)
14. Huang, W.: The kernel of a linear algebraic semigroup. Forum Math. 17, No. 5, 851–869 (2005)
15. Kempf, G.: Algebraic Varieties. Cambridge University Press, Cambridge (1993)
16. Knop, F., Kraft, H., Vust, T.: The Picard group of a $G$-variety. In: Kraft, H. et al. (eds.) Algebraische Transformationsgruppen und Invariantentheorie, pp. 77-87. Birkhäuser, Basel (1989)
17. Kollár, J.: Rational curves on algebraic varieties. Springer-Verlag, Berlin (1995)
18. Matsumura, H., Oort, F.: Representability of group functors, and automorphisms of algebraic schemes. Invent. Math. 4, No. 1, 1–25 (1967)
19. Mumford, D.: Abelian Varieties. Second edition. With appendices by C. P. Ramanujam and Yuri Manin. Oxford University Press, Oxford (1974)
20. Mumford, D., Fogarty, J., Kirwan, F.: Geometric Invariant Theory. Third edition. Springer-Verlag, Berlin, 1994
21. Putcha, M.S.: On linear algebraic semigroups. Trans. Amer. Math. Soc. 259, No. 2, 457–469 (1980)
22. Putcha, M.S.: On linear algebraic semigroups. III. Internat. J. Math. Math. Sci. 4, No. 4 667–690 (1981); corrigendum, ibid., 5, No. 1, 205–207 (1982)
23. Putcha, M.S.: Linear Algebraic Monoids. Cambridge University Press, Cambridge (1988)
24. Putcha, M.S.: Products of idempotents in algebraic monoids. J. Aust. Math. Soc. 80, No. 2, 193–203 (2006)
25. Renner, L.E.: Quasi-affine algebraic monoids. Semigroup Forum 30, No. 2, 167–176 (1984)
26. Renner, L.E.: Linear Algebraic Monoids. Springer-Verlag, New York (2005)
27. Renner, L.E., Rittatore, A.: The ring of regular functions of an algebraic monoid. Trans. Am. Math. Soc. 363, No. 12, 6671–6683 (2011)
28. Rittatore, A.: Algebraic monoids and group embeddings. Transform. Groups 3, No. 4, 375–396 (1998)
29. Rittatore, A.: Algebraic monoids with affine unit group are affine. Transform. Groups 12, No. 3, 601–605 (2007)
30. Serre, J.-P.: Morphismes universels et variété d’Albanese. In: Exposés de séminaires (1950–1999). Soc. Math. France, Paris (2001)
31. Serre, J.-P.: Groupes algébriques et corps de classes. Hermann, Paris (1959)
32. Séminaire de Géométrie Algébrique du Bois-Marie 1962/64 (SGA3), Schémas en Groupes II. Springer-Verlag, New York (1970)
33. Springer, T.A.: Linear Algebraic Groups. Second edition. Birkhäuser, Boston (1998)
34. Sumihiro, H.: Equivariant completion. J. Math. Kyoto Univ. 14, No. 1, 1–28 (1974)
35. Totaro, B.: Pseudo-abelian varieties. Ann. Sci. Éc. Norm. Sup. (4) 46, No. 5 (2013)
36. Wittenberg, O.: On Albanese torsors and the elementary obstruction. Math. Ann. 340, No. 4, 805–838 (2008)