Nonextensive statistical effects in nuclear physics problems

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Abstract. Recent progresses in statistical mechanics indicate the Tsallis nonextensive thermostatistics as the natural generalization of the standard classical and quantum statistics, when memory effects and long range forces are not negligible. In this framework, weakly nonextensive statistical deviations can strongly reduce the puzzling discrepancies between experimental data and theoretical predictions for solar neutrinos and for pion transverse-momentum correlations in Pb-Pb high-energy nuclear collisions.

1 Introduction

Nuclear and, in general, many-particle systems are often in regimes where concepts of statistical physics come into play. Long-range forces, memory effects, sizeable correlations and fluctuations of the observables provide demanding testing grounds of the equilibrium and non-equilibrium statistical mechanics theory.

In many physical applications, the kinetic description of the system is developed in the weak-coupling limit defined by the condition \( \tau_b \ll (\tau_\lambda, \tau_u) \), where \( \tau_b, \tau_\lambda \) and \( \tau_u \) are the duration of a binary collision, the mean free time between subsequent collisions and the characteristic time of the mean-field potential, respectively [1]. In this case, the many-body collision process is well described independent binary collisions in the framework of the standard Boltzmann equation and in terms of the \( \delta(t-t') \) function for the fluctuating stochastic Langevin force (Markovian approximation).

If the mean-field characteristic time becomes comparable to the duration of a binary collision (\( \tau_b \geq \tau_u \)), the independent binary collision approximation is no longer correct and the Markovian description breaks down: memory effects become important.

Tsallis [2] has recently advanced a generalization of the conventional Boltzmann-Gibbs thermostatistics that overcomes the inability of the conventional statistical mechanics to tackle those many physical problems with long-range interactions, long-range microscopic memory, or fractal space-time constraints. There exist already many applications: astrophysical self-gravitating systems [3], the solar neutrino problem [4], distribution of peculiar velocities of galaxy clusters [5], cosmology [6], many-body theory, dynamical linear response theory and variational methods [7].
In this paper, after a brief review of the Tsallis thermostatistics in Sec. II, we discuss (Sec. III) how a weakly non-ideal stellar plasma, such as the solar interior, can produce an equilibrium velocity distribution that deviates from the standard one and how this effect can be relevant to the solar neutrino problem. In Sec. IV, we consider the interpretation of the pion transverse-momentum NA49 experimental data in high-energy Pb-Pb collisions in the framework of the nonextensive thermostatistics. Although the physics of the two problems is very different, they are both characterized by memory effects and long-range forces and, consequently, show nonextensive statistical behaviors.

2 Nonextensive statistical mechanics

The principal features of the Tsallis generalized thermostatistics is based upon the following two postulates [2].

– Given $p_k$ the probability in any $W$ different microstates $k$, the entropy of a system is defined as

$$S_q = \frac{1}{q-1} \sum_{k=1}^{W} p_k (1 - p_k^{q-1}), \quad (1)$$

where $q$ is a fixed real parameter. The generalized entropy has the usual properties of positivity, equiprobability, concavity, irreversibility and, in the limit $q \to 1$, is equal to the conventional Boltzmann-Gibbs entropy $S = -\sum_k p_k \log p_k$.

– The mean value of an observable $O$, whose value in the microstate $k$ is $O_k$, is defined as

$$\langle O \rangle_q = \sum_{k=1}^{W} p_k^q O_k. \quad (2)$$

The $S_q$ entropy is nonextensive. If $A$ e $B$ are two independent systems $A$ e $B$, i.e., the probability of composite system $A + B$ factorizes into $p_{A+B}(u_A,u_B) = p_A(u_A)p_B(u_B)$, the global entropy is not the sum of the entropies of the subsystems, but

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B). \quad (3)$$

The single particle distribution function is obtained by the usual procedure of maximizing the Tsallis entropy with the constraints that the average internal energy and the average number of particles remain constant:

$$f(v) = \left[ 1 - (1 - q)\frac{mv^2}{2kT} \right]^{1/(1-q)}. \quad (4)$$

When the entropic parameter $q$ is smaller than 1 the distribution has an upper cut-off: $mv^2/2 \leq kT/(1 - q)$ (the tail is depleted). The distribution correctly
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reduces to the exponential Maxwell-Boltzmann distribution in the limit \( q \to 1 \). When the parameter \( q \) is greater than 1, there is no cut-off and the (power-law) decay is slower than exponential (the tail is enhanced).

These classical distribution can be generalized to the quantum case under some not too restrictive approximation (see Ref.[8] for details) resulting in the following analytical expression for the mean occupational number:

\[
\langle n \rangle_q = \frac{1}{[1 + (q - 1)\beta(E - \mu)]^{1/(q-1)} + 1}, \tag{5}
\]

where \( \beta = 1/kT \), and the + (−) sign applies to fermions (bosons). In the limit \( q \to 1 \) (extensive statistics), we recover the conventional Fermi-Dirac and Bose-Einstein distribution.

3 Non-ideal stellar plasma and solar neutrino fluxes

3.1 Plasma parameter and solar interior

The stellar core is usually described as an ideal plasma in the Debye-Hückel mean-field approximation. However, density and temperature conditions in stellar plasmas, such as the core of the Sun, of brown dwarfs or of Jupiter, suggest that the microscopic diffusion of electrons and ions could be nonstandard and that the Debye-Hückel approximation is not sufficiently accurate.

The appropriate theoretical approach and the effective interactions used to describe a plasma can be deduced from its plasma parameter \( \Gamma \)

\[
\Gamma = \frac{(Ze)^2}{akT}, \tag{6}
\]

where \( a = n^{-1/3} \) is of the order of the interparticle average distance (\( n \) is the average density). The plasma parameter is a measure of the ratio of the mean (Coulomb) potential energy and the mean kinetic (thermal) energy. On the basis of its value, we can distinguish three different regimes.

– \( \Gamma \ll 1 \). The plasma is described by the Debye-Hückel mean-field theory as a dilute weakly-interacting gas. The screening Debye length

\[
R_D = \sqrt{\frac{kT}{4\pi e^2 \sum_i Z_i^2 n_i}}, \tag{7}
\]

is much greater than the average interparticle distance \( a \), hence there is a large number of particles in the Debye sphere \((N_D \equiv (4\pi/3)R_D^3)\). Collective degrees of freedom are present (plasma waves), but they are weakly coupled to the individual degrees of freedom (ions and electrons) and, therefore, do not affect their distribution. Binary collisions through screened forces produce the standard velocity distribution.
– $\Gamma \approx 0.1$. The mean Coulomb energy potential is not much smaller than the thermal kinetic energy and the screening length $R_D \approx a$. It is not possible to clearly separate individual and collective degrees of freedom. The presence of at least two different scales of energies of the same rough size produces deviations from the standard statistics which describe the system in terms of a single scale, $kT$.

– $\Gamma > 1$. This is a high-density/low-temperature plasma; the Coulomb interaction and quantum effects start to dominate and determine the structure of the system.

In the solar interior the plasma parameter is $\Gamma_\odot \approx 0.1$; therefore, the solar core is a weakly non-ideal plasma where the Debye-Hückel conditions are only approximately verified.

The reaction time necessary to build up screening after a hard collision can be estimated from the inverse solar plasma frequency $t_{pl} = \omega_{pl}^{-1} = \sqrt{m/(4\pi ne^2)} \approx 10^{-17}$ sec, and it is comparable to the collision time $t_{coll} = (\sigma v n)^{-1} \approx 10^{-17}$ sec. Therefore, several collisions are likely necessary before the particle loses memory of the initial state and the scattering process can not be considered Markovian. In addition to many-body collisional effects, electric microfields are present and the distribution and the fluctuations of these microfields must be carefully considered, since they modify the usual Boltzmann kinetics.

At the light of the above considerations, we conclude that density and temperature conditions of the solar core suggest that non-Markovian memory effects and long-range forces are present. Because the solar core lies in an intermediate region of the plasma parameter, the corrections to the standard Maxwell-Boltzmann distribution should be small and affect mainly the more energetic particles. However we will see in next subsection that this small deviation strongly affects the nuclear reaction rates and, consequently, the determination of the solar neutrino fluxes.

### 3.2 Thermonuclear reaction rate and solar neutrino fluxes

The nuclear reaction rates in the stellar interiors play a crucial rôle in the understanding of the structure and evolution of stars.

The reaction rate per particle pair is defined as

$$\langle \nu \sigma \rangle = \int_0^\infty f(v) \sigma v \, dv,$$

where the particles distribution function $f(v)$ is a local function of the temperature.

The kinetic energy of the ions in the solar core is essentially the thermal energy $kT_\odot \approx 1.36$ keV which is far below the Coulomb barrier, therefore only a small number of particles in the high-energy tail of the distribution has a chance of reacting: this high-energy tail plays a crucial rôle for the reaction rates, which becomes very sensitive to small changes of the distribution.
For small deviations from the standard extensive equilibrium statistics, we can write the generalized Tsallis distribution (4) as follows [4]

\[ f(E) \sim \langle kT \rangle^{-3/2} e^{-E/kT - \delta(E/kT)^2}, \]

where \( \delta = (1 - q)/2. \)

The presence of even a tiny deviation from MB in the solar core produces large changes of the subbarrier nuclear reaction rates and, consequently, of the predicted neutrino fluxes. Following the general homology relationships for the variations of physical inputs, see for instance Ref. [9], we can estimate the effect of the non-Maxwellian distribution on the fluxes [4]:

\[ R_j = \frac{\Phi_j}{\Phi_j^{(0)}} = e^{-\delta_j \beta_j}, \]

for the fluxes \( j = ^7\text{Be}, ^8\text{B}, ^{13}\text{N} \) and \(^{15}\text{O}, \) while we use the solar luminosity constraint to determine the pp flux, \( R_{pp} = 1 + 0.087 \times (1 - R_{Be}) + 0.010 \times (1 - R_{N}) + 0.009 \times (1 - R_{O}), \) and keep fixed the ratio \( \xi \equiv \Phi_{pep}/\Phi_{pp} = 2.36 \times 10^{-3}. \)

The power indices \( \beta_j, \) that appear in Eq. (10), depend on the nuclear reaction considered and their values have been taken from Ref. [9]. In principle, there could be a different parameter \( \delta_j \) for each reaction and it should be possible to calculated them from the specific interactions in the solar plasma core. For the purpose of estimating the possible effects of this mechanism on the solar neutrino fluxes, we used two simple models where \( \delta_j \) were used as free parameters.

The first model uses the same \( \delta \) for all the reactions; the best fit to the experimental data give \( \delta = 0.005 \) with a corresponding \( \chi^2 = 35. \) The second model fits two different \( \delta \)’s, one (\( \delta_{(17)} \)) for the \( p^+7\text{Be} \) reaction and the other (\( \delta_{(34)} \)) for the \( ^3\text{He}^+4\text{He} \) reaction. The best result gives \( \chi^2 = 20 \) with \( \delta_{(17)} = -0.018 \) and \( \delta_{(34)} = 0.030. \) The negative value of \( \delta_{(17)} \) means that the corresponding distribution has an enhanced tail and that the \( p^+7\text{Be} \) reaction rate increases.

In spite of the fact that the values of \( \chi^2 \) are much smaller than the ones in the SSM (\( \chi^2_{SSM} > 74 \)), they are still large: this mechanism cannot solve the solar neutrino problem. However, even if \( \delta \) is small, it has non-trivial consequences on the neutrino fluxes: the boron (beryllium) flux can change of as much as 50% (30%).

### 4 Fluctuations and correlations in high-energy nuclear collisions

The recent event-by-event analysis of central Pb+Pb collisions at 158 A GeV (NA49 coll.) has spurred great interest because of the strong suppression of the pion transverse-momentum fluctuations in Pb+Pb respect to \( p+p \) collisions. A reduction of the fluctuations appears reasonable because correlations should be washed out by meson-baryon and meson-meson rescatterings; however, the size of this suppression is not understood.
Gaźdżicki and Mrówczyński [10] have introduced a definition of the event-by-event transverse-momentum fluctuations that is independent of the particle multiplicity:

\[ \Phi_{p_\perp} = \sqrt{\frac{\langle Z_{p_\perp}^2 \rangle}{\langle N \rangle}} - \sqrt{\frac{z_{p_\perp}^2}{\langle N \rangle}} , \]  

where \( z_{p_\perp} = p_\perp - \overline{p_\perp} \) is a single-particle measure of the deviation of \( p_\perp \) from its average value, and where \( Z_{p_\perp} = \sum_{i=1}^{N} (p_{\perp i} - \overline{p_\perp}) \) is the corresponding total contribution (\( N \) is the number of the particles in the event).

In a very recent paper, Mrówczyński [11] has calculated this correlation measure \( \Phi_{p_\perp} \) for a pion gas in global equilibrium within the standard extensive thermodynamics. His result (\( \Phi_{p_\perp} = 6.5 \text{ MeV at } T = 120 \text{ MeV} \)) is sensibly greater than the experimental value \( \Phi_{p_\perp} = 0.7 \pm 0.5 \text{ MeV} \), which has been measured in the central Pb+Pb collision by the NA49 collaboration [12]).

What is the origin of such a large discrepancy between theoretical results and experimental data? Can the conditions of density and temperature at the early stage collisions modify the correlations between the produced particles?

The quoted theoretical result [11] has been obtained within the framework of the conventional quantum statistical mechanics. Therefore, pions, which are sufficiently light to show quantum degeneracy, follow the standard Bose-Einstein distribution. However, it is common opinion that, because of the extreme conditions of density and temperature in ultrarelativistic heavy ion collisions, memory effects and long-range color interactions affect the thermalization process [13, 14, 15]. In fact, if the quark-gluon plasma forms at the early stage of the collision, strong chromoelectric fields appear within the parton gas. Recent investigations, both for for QED and QCD, have shown that these strong fields are connected to the presence of non-Markovian process in the kinetic equation [16].

The foregoing considerations suggest to calculate the equilibrium correlation measure in the framework of the nonextensive statistics.

On the basis of the generalized thermodynamics relations [2], it can be shown that

\[ \overline{z_{p_\perp}^2} = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \left[ p_\perp - \overline{p_\perp} \right]^2 \langle n \rangle_q , \]  

where \( \langle n \rangle_q \) is the mean occupation number of Eq.(5) and

\[ \frac{\langle Z_{p_\perp}^2 \rangle}{\langle N \rangle} = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} \left[ p_\perp - \overline{p_\perp} \right]^2 \langle \Delta n^2 \rangle_q , \]  

where

\[ \overline{p_\perp} = \frac{1}{\rho} \int \frac{d^3p}{(2\pi)^3} p_\perp \langle n \rangle_q , \quad \rho = \int \frac{d^3p}{(2\pi)^3} \langle n \rangle_q \]  

and

\[ \langle \Delta n^2 \rangle_q = \frac{1}{\beta} \frac{\partial \langle n \rangle_q}{\partial \mu} = \frac{\langle n \rangle_q}{1 + (q-1)\beta(E - \mu)} (1 + \langle n \rangle_q) , \]
are the particle fluctuations in nonextensive \((q \neq 1)\) statistics. For \(q = 1\), we recover the well known fluctuations expression for fermions \((-\)) and bosons \((+\)). From Eq. (15), we see that the expression for the generalized fluctuations has the same structure of the standard one modified by the factor \(1/[1+(1-q)\beta(E-\mu)]\).

For the boson case a value \(q > 1\) implies a very strong suppression of \(\Phi_{p_{\perp}}\). If we consider a pion gas and fix the freeze-out temperature to \(T = 120\) MeV (obtained from the analysis of single particle spectra), we reproduce the experimental value \((\Phi_{p_{\perp}} = 0.7 \pm 0.5\) MeV) using \(q = 1.015\). Hence, a small deviation from the standard statistics \((q - 1 = 0.015)\) is sufficient for eliminating the puzzling discrepancy between theoretical calculations and experimental data.

Finally, we observe that fluctuations are the more strongly modified the larger the mass of the detected particle for a given value of the deformation parameter \(q\). Hence, this model predicts stronger nonextensive effects for correlations of heavier mesons and baryons: future measurements of \(\Phi_{p_{\perp}}\) for these particles should find an even larger reduction when plasma is formed. New data and investigations are necessary to gain a deeper understanding of the high-energy heavy-ion observables.

5 Conclusion

We have considered the relevance of the generalized nonextensive statistics to the solar neutrino problem and to the interpretation of the pion transverse-momentum correlations in Pb-Pb high energy collisions experimental data. Although the two problems involve rather different physics, both appear to show memory effects and long range forces. Discrepancies between experimental data and theoretical models can be strongly reduced when weakly-nonextensive \((|q - 1| < 0.02)\) statistical effects are considered.

In the solar neutrino problem a small deviation from the Maxwell-Boltzmann distribution produces strong modifications of the thermonuclear reaction rates and, consequently, modifies neutrino fluxes by amounts comparable to those that constitute the solar neutrino problem without affecting bulk properties such as the sound speed or hydrostatic equilibrium.

In high-energy ion collisions, nonextensive statistics predicts a reduction of the correlation measure \(\Phi_{p_{\perp}}\) in Pb-Pb collisions \((\Phi_{p_{\perp}} = 0.7 \pm 0.5\) MeV) respect to p-p collisions \((\Phi_{p_{\perp}} = 4.2 \pm 0.5\) MeV) due to plasma effects. This reduction has been seen in the experiments and could be interpreted as a signature of the transition to a quark-gluon-plasma phase driven by the extreme conditions of density and temperature in the early stage of the collision.

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