DIFFERENTIAL EQUATIONS FOR SEPTIC THETA FUNCTIONS

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Abstract. We demonstrate that quotients of septic theta functions appearing in S. Ramanujan’s Notebooks and in F. Klein’s work satisfy a new coupled system of nonlinear differential equations with interesting symmetric form. This differential system bears a close resemblance to an analogous system for quintic theta functions. The proof extends a technique used by Ramanujan to prove the classical differential system for normalized Eisenstein series on the full modular group. In the course of our work, we show that Klein’s quartic relation induces new symmetric representations for low weight Eisenstein series in terms of weight one modular forms of level seven.

1. Introduction

Let \(|q| < 1\), and

\[ a(q) = -\sum_{n=-\infty}^{\infty} (-1)^n q^{(14n+5)^2/56}, \quad b(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(14n+3)^2/56}, \]
\[ c(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(14n+1)^2/56}. \]

These septic theta functions have an interesting provenance. They first appeared in the 1879 work of Felix Klein [13] and were studied independently by Ramanujan [19, p. 300]. With entirely different motivations, Klein and Ramanujan derived identities between these theta functions equivalent to Klein’s eponymous quartic relation

\[ a^3(q)b(q) + b^3(q)c(q) + c^3(q)a(q) = 0. \] (1.1)

The septic theta functions \(a(q), b(q),\) and \(c(q)\) also appear in connection with Ramanujan’s seventh order mock theta functions [22]. We augment the work of Klein and Ramanujan by formulating a new coupled system of differential equations for these theta functions. Our approach is based on Ramanujan’s famous proof of the coupled differential system for the normalized Eisenstein series on the full modular group [21]

\[ q \frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}, \quad q \frac{dE_4}{dq} = \frac{E_2 E_4 - E_6}{3}, \quad q \frac{dE_6}{dq} = \frac{E_2 E_6 - E_4^2}{2}, \] (1.2)

where the Eisenstein series \(E_k = E_k(q)\) are defined by

\[ E_{2k}(q) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n}, \] (1.3)

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and where $\zeta$ is the analytic continuation of the Riemann $\zeta$-function. Ramanujan proved by formulating two identities involving the classical Weierstrass zeta function

$$\zeta(\theta | q) = \frac{1}{2} \cot \frac{\theta}{2} + \frac{\theta}{12} - 2\theta \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} + 2 \sum_{n=1}^{\infty} \frac{q^n \sin n\theta}{1 - q^n}. \quad (1.4)$$

We follow Ramanujan’s lead to derive a new differential system from elementary properties of elliptic functions. Our work culminates in a curiously symmetric coupled differential system for the quotients

$$x(q) = q^{7/8}(q^7; q^7)^3 \frac{b(q)}{c(q)}$$
$$y(q) = -q^{7/8}(q^7; q^7)^3 \frac{a(q)}{b(q)}$$
$$z(q) = q^{7/8}(q^7; q^7)^3 \frac{c(q)}{a(q)}$$

where here and throughout the paper, we employ the notation

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \lim_{n \to \infty} \prod_{k=0}^{n-1} (1 - aq^k).$$

**Theorem 1.1.** Let $P(q) = E_2(q^7)$, where $E_2(q) = 1 - 24 \sum_{n=1}^{\infty} (\sum_{d|n} d) q^n$. Then

$$q \frac{d}{dq} x = \frac{x}{12} \left(5y^2 + 5z^2 - 7x^2 - 20yz - 52xy + 7P\right), \quad (1.7)$$
$$q \frac{d}{dq} y = \frac{y}{12} \left(5z^2 + 5x^2 - 7y^2 + 20xz - 52yz + 7P\right), \quad (1.8)$$
$$q \frac{d}{dq} z = \frac{z}{12} \left(5x^2 + 5y^2 - 7z^2 - 20xy + 52xz + 7P\right), \quad (1.9)$$

$$q \frac{d}{dq} P(q) = \frac{7}{12} \left(p^2 - x^4 + 4x^3y + 12xy^3 - y^4 - 12x^3z + 4y^3z - 4xz^3 + 12yz^3 - z^4\right).$$

The form and symmetry present in this differential system is also exhibited by a recently derived coupled system satisfied by quintic theta functions $\mathbf{8}$, defined by

$$A(q) = q^{1/5} (q; q)_{\infty}^{-3/5} \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2 - 3n)/2}, \quad B(q) = (q; q)_{\infty}^{-3/5} \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2 - n)/2}.$$  

**Theorem 1.2.** Let $Q(q) = E_2(q^5)$, where $E_2(q) = 1 - 24 \sum_{n=1}^{\infty} (\sum_{d|n} d) q^n$. Then

$$q \frac{d}{dq} A = \frac{1}{60} A \left(7B^{10} - 5A^{10} - 66A^5B^5 + 5Q\right), \quad (1.10)$$
$$q \frac{d}{dq} B = \frac{1}{60} B \left(7A^{10} - 5B^{10} + 66A^5B^5 + 5Q\right), \quad (1.11)$$

$$q \frac{d}{dq} Q = \frac{5}{12} \left(Q^2 - B^{20} + 12B^{15}A^5 - 14B^{10}A^{10} - 12B^5A^{15} - A^{20}\right). \quad (1.12)$$
Our formulas for Eisenstein series in Theorem 4.5 will demonstrate the equivalence of the fourth differential equation of Theorem 1.1 and the first equation of Ramanujan’s differential system (1.2). To accomplish this, we rely on septic parameterizations for Eisenstein series from [3].

Each differential system appearing here is analogous to corresponding nonlinear coupled systems for modular forms of lower level. In particular, Theorems 1.2 and 1.3 are analogous to several coupled systems for the cubic theta functions [11, 17] and [9, 18], respectively. Most of the coupled systems discussed so far are subsumed (see [12, §1]) by a more general system for the parameters [10]

\[ e_\alpha(q) = 1 + 4 \tan(\pi \alpha) \sum_{n=1}^{\infty} \frac{\sin(2n\pi \alpha)q^n}{1 - q^n}, \quad P_\alpha(q) = 1 - 8 \sin^2(\pi \alpha) \sum_{n=1}^{\infty} \frac{\cos(2n\pi \alpha)q^n}{1 - q^n}, \]

\[ Q_\alpha(q) = 1 - 8 \tan(\pi \alpha) \sin^2(\pi \alpha) \sum_{n=1}^{\infty} \frac{\sin(2n\pi \alpha)q^n}{1 - q^n}. \] (1.17)

**Theorem 1.3.** Let \( e_\alpha(q) \), \( P_\alpha(q) \), and \( Q_\alpha(q) \) be defined as in (1.17). Then for \( \alpha \neq 1/2 \),

\[ \frac{d}{dq} e_\alpha = \frac{\csc^2(\pi \alpha)}{4} (e_\alpha P_\alpha - Q_\alpha), \] (1.18)

\[ \frac{d}{dq} P_\alpha = \frac{\csc^2(\pi \alpha)}{4} P_\alpha^2 - \frac{1}{2} \cot^2(\pi \alpha) e_\alpha Q_\alpha + \frac{1}{2} \cot(\pi \alpha) \cot(2\pi \alpha) e_{1-2\alpha} Q_\alpha, \] (1.19)

\[ \frac{d}{dq} Q_\alpha = \frac{1}{4} Q_\alpha P_\alpha \csc^2(\pi \alpha) + \frac{1}{2} P_{1-2\alpha} Q_\alpha \csc^2(2\pi \alpha) - \frac{1}{2} e_{1-2\alpha} Q_\alpha \cot^2(2\pi \alpha) \]

\[ + \frac{3}{2} e_\alpha e_{1-2\alpha} Q_\alpha \cot(2\pi \alpha) - e_\alpha^2 Q_\alpha \cot^2(\pi \alpha). \] (1.20)

In Section 2, we formulate relevant elliptic function identities in terms of these parameters. In Section 3, we write the quotients \( x(q) \), \( y(q) \), and \( z(q) \) as linear combinations of \( e_{1/7}(q) \), \( e_{2/7}(q) \), and \( e_{3/7}(q) \). Section 4 culminates in a proof of Theorem 1.1 and introduces new a set of symmetric parameterizations for Eisenstein series of weight four and six in terms of the septic parameters (1.5) – (1.6). These formulas constitute a septic reprise of symmetric quintic parameterizations for Eisenstein series from [3].
2. Elliptic modular preliminaries

The purpose of this section is to introduce results from the theory of elliptic modular functions necessary for our further work. A critical component in our proof of Theorem 1.1 is Lemma 2.1 where Lambert series representations are derived for polynomials of degree two in the parameters $e_\alpha(q)$. These rather unconventional parameters are customarily expressed in terms of the logarithmic derivative of the Jacobi theta function

$$\theta_1(z \mid q) = -i q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz}$$

(2.1)
given by the equivalent representations

$$\frac{\theta_1'(z \mid q)}{\theta_1(z \mid q)} = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2nz$$

(2.2)

$$= i - 2i \sum_{n=1}^{\infty} \frac{q^n e^{2iz}}{1 - q^n e^{2iz}} + 2i \sum_{n=0}^{\infty} \frac{q^n e^{-2iz}}{1 - q^n e^{-2iz}}.$$  (2.3)

We will also require the familiar identity for the Jacobi theta function

$$\left(\frac{\theta_1'(y \mid q)}{\theta_1(y \mid q)}\right)'(x \mid q) - \left(\frac{\theta_1'(x \mid q)}{\theta_1(x \mid q)}\right)'(y \mid q) = \frac{\theta_1'(0 \mid q)^2 \theta_1(x - y \mid q) \theta_1(x + y \mid q)}{\theta_1^2(x \mid q) \theta_1^2(y \mid q)}.$$  (2.4)

In order to relate the parameters $e_\alpha(q)$ to the quotients of theta functions appearing in (1.5)–(1.6), we will logarithmically differentiate the infinite product representations

$$x(q) = \frac{q(q^7; q_7^2)^2/(q^2; q_7^2)^2}{(q^3; q_7^2)^2/(q^4; q_7^2)^2}$$,

$$y(q) = \frac{q(q^7; q_7^2)^2(q^6; q_7^2)^2}{(q^2; q_7^2)^2(q^5; q_7^2)^2(q^4; q_7^2)^2},$$

$$z(q) = \frac{(q^7; q_7^2)^2(q^4; q_7^2)(q^4; q_7^2)(q^7; q_7^2)^2}{(q; q_7^2)^2(q^5; q_7^2)^2(q^6; q_7^2)^2}.$$  (2.5)

These product formulations are consequences of the Jacobi triple product formula

$$\theta_1(z \mid q) = -i q^{1/8} e^{iz}(q; q_\infty)(qe^{2iz}; q_\infty)(e^{-2iz}; q_\infty).$$  (2.7)

By differentiating (2.7) at the origin, we obtain

$$\theta_1'(q) := \lim_{z \to 0} \frac{\theta_1(z \mid q)}{z} = 2q^{1/8}(q; q_\infty)^3.$$  (2.8)

We may also apply (2.7) to derive the subsequently useful product representations

$$\theta_1(\pi \tau \mid q^7) = i q^{3/8} (q; q_\infty^7)(q^6; q_\infty^7)(q^7; q_\infty^7),$$

(2.9)

$$\theta_1(2\pi \tau \mid q^7) = i q^{-1/8} (q^2; q_\infty^7)(q^5; q_\infty^7)(q^7; q_\infty^7),$$

(2.10)

$$\theta_1(3\pi \tau \mid q^7) = i q^{-3/8} (q^3; q_\infty^7)(q^4; q_\infty^7)(q^7; q_\infty^7).$$

(2.11)

Our proof of Theorem 1.1 will employ a number of classical elliptic function identities for the Weierstrass $\zeta$-function defined by (1.3) and the Weierstrass $\wp$-function determined by $(d/dz)\zeta(z \mid q) = -\wp(z \mid q)$. In particular, we will make use of an identity
prominent in Ramanujan’s proof of the differential system for Eisenstein series (1.2). Ramanujan used elementary trigonometric identities to prove [7, p. 135] (cf. [1])

\[
\left( \zeta(z \mid q) - \frac{z E_2(q)}{12} \right)^2 = \varphi(z \mid q) - \frac{1}{6} + 4 \sum_{n=1}^{\infty} \frac{q^{2n} \cos(nz)}{(1 - q^{2n})^2}. \tag{2.12}
\]

Identity (2.12) played a key role in Ramanujan’s proof [21] of (1.2) and facilitates the main results of the present paper by inducing Lambert series expansion for \( e^{2\alpha}(q) \).

Corresponding Lambert expansions for \( e^{\alpha}(q) e^{1-2\alpha}(q) \) will depend on the Frobenius-Stickelberger pseudo-addition formula [6], [23, p. 459] for the Weierstrass \( \zeta \)-function

\[
\{ \zeta(a) + \zeta(b) + \zeta(c) \}^2 = \zeta'(a) + \zeta'(b) + \zeta'(c), \quad a + b + c = 0. \tag{2.13}
\]

The next lemma translates (2.12)–(2.13) into forms involving the series from (1.17).

**Lemma 2.1.** Let \( e_{\alpha} \) and \( P_{\alpha} \) be defined by (1.17), and let \( E_2(q) \) denote the normalized Eisenstein series of weight 2. Then

\[
e_{\alpha}^2(q) = 1 + \frac{16}{\cot^2(\pi \alpha)} \sum_{n=1}^{\infty} \frac{q^n \cos(2\pi \alpha n)}{(1 - q^n)^2} + \frac{8}{\cot^2(\pi \alpha)} \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \tag{2.14}
\]

\[
- \frac{8}{\cot^2(\pi \alpha)} \sum_{n=1}^{\infty} \frac{q^n n \cos(2n\pi \alpha)}{1 - q^n},
\]

\[
\left( \cot(\pi(1 - 2\alpha))e_{1-2\alpha}(q) + 2 \cot(\pi \alpha) e_{\alpha}(q) \right)^2 \tag{2.15}
\]

\[
= \csc^2((1 - 2\alpha)\pi) P_{1-2\alpha}(q) + 2 \csc^2(\pi \alpha) P_{\alpha}(q) - E_2(q).
\]

**Proof.** Recast (2.12) in the form

\[
\left( \frac{1}{4} \cot \theta + \sum_{n=1}^{\infty} \frac{q^n \sin(2n\theta)}{1 - q^n} \right)^2 \tag{2.16}
\]

\[
= \left( \frac{1}{4} \cot \theta \right)^2 + \sum_{n=1}^{\infty} \frac{q^n \cos(2n\theta)}{(1 - q^n)^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} (1 - \cos(2n\theta)).
\]

Equation (2.14) follows from setting \( \theta = 2\pi \alpha \) in (2.16). Identity (2.13) takes the form

\[
\left( \frac{1}{2} \cot \theta - \cot \frac{\theta}{2} + \sum_{n=1}^{\infty} \frac{q^n (2 \sin(2n\theta) - 4 \sin(n\theta))}{1 - q^n} \right)^2 \tag{2.17}
\]

\[
= \frac{1}{4} \csc^2 \theta + \frac{1}{2} \csc^2 \frac{\theta}{2} - \sum_{n=1}^{\infty} \frac{nq^n (2 \cos(2n\theta) + 4 \cos(n\theta))}{1 - q^n}.
\]

Equation (2.15) may be obtained by setting \( \theta = 2\pi \alpha \) in (2.17). □
Lemma 2.2. Let \( e_\alpha(q) \) be defined by (1.17). Then

\[
e_\alpha^2(q) = 1 + \sum_{n=1}^{\infty} \frac{\delta_\alpha(n) n q^n}{1 - q^n} + \sum_{n=1}^{\infty} \frac{\lambda_\alpha(n) q^n}{(1 - q^n)^2},
\]

(2.18)

\[
e_\alpha(q)e_{1-2\alpha}(q) = 1 + \sum_{n=1}^{\infty} \frac{\kappa_\alpha(n) n q^n}{1 - q^n} + \sum_{n=1}^{\infty} \frac{\mu_\alpha(n) q^n}{(1 - q^n)^2},
\]

(2.19)

where

\[
\delta_\alpha(q) = 16 \tan^2(\pi \alpha) \sin^2(\pi n \alpha), \quad \lambda_\alpha(n) = 16 \tan^2(\pi \alpha) \cos(2\pi n \alpha),
\]

\[
\kappa_\alpha(n) = 8 \tan(\pi \alpha) \tan(2\pi \alpha) \sin^2(\pi n \alpha), \quad \mu_\alpha(n) = 4 \tan(\pi \alpha) \tan(2\pi \alpha) \left(4 \cos(2\pi n \alpha) + \cos(4\pi n \alpha)\right).
\]

Proof. Equation (2.18) follows from (2.14) and elementary trigonometric identities. To prove (2.19), expand the left side of (2.15) and subtract the squared terms from both sides to obtain

\[
e_{1-2\alpha}(q)e_\alpha(q) = \frac{\tan(\pi \alpha) \tan(2\pi \alpha)}{4} \left(E_2(q) + \cot^2((1 - 2\alpha)\pi)e_{1-2\alpha}^2(q)
\right.

\[+ 4 \cot^2(\pi \alpha)e_\alpha^2(q) - \csc^2((1 - 2\alpha)\pi)P_{1-2\alpha}(q) - 2 \csc^2(\pi \alpha)P_\alpha(q)\right).\]

Next, apply (2.18) on the right side of (2.20) to the terms

\[\cot^2((1 - 2\alpha)\pi)e_{1-2\alpha}^2(q) \quad \text{and} \quad 4 \cot^2(\pi \alpha)e_\alpha^2(q)\]

Thus, (2.20) may be expressed in the form

\[
e_{1-2\alpha}(q)e_\alpha(q) = \sum_{n=1}^{\infty} \frac{8 \tan(2\pi \alpha) \tan(\pi \alpha) \sin^2(\pi n \alpha) n q^n}{1 - q^n}
\]

(2.21)

\[+ \sum_{n=1}^{\infty} \frac{4 \tan(2\pi \alpha) \tan(\pi \alpha) \left(4 \cos(2\pi n \alpha) + \cos(4\pi n \alpha)\right) q^n}{(1 - q^n)^2}.
\]

Comparing the right side of (2.21) with (2.19), we arrive at the claimed identity. \( \square \)

Lemma 2.3. For any sequence \( \{a_n\}_{n=1}^{\infty} \) periodic modulo seven, such that the the series are absolutely convergent, we have

\[
\sum_{n=1}^{\infty} \frac{a_n q^n}{(1 - q^n)^2} = \sum_{n=1}^{\infty} \frac{a_n}{1 - q^{7n}} \sum_{m=1}^{7} a_m q^{nm}.
\]

(2.22)

Proof. To prove (2.22), express the sum on the left as the derivative of a geometric series and invert the order of summation to yield

\[
\sum_{n=1}^{\infty} \frac{a_n q^n}{(1 - q^n)^2} = \sum_{n=1}^{\infty} a_n q^n \frac{d}{dq} \left( \frac{1}{1 - q^n} \right) q^{n(k-1)} = \sum_{m=1}^{7} a_m \sum_{k=1}^{\infty} k q^{mk} \sum_{n=0}^{\infty} q^{7nk}.
\]

(2.23)

By expanding the innermost sum of (2.23) as a geometric series, we obtain (2.22). \( \square \)
3. Elliptic interpolation of septic theta functions

We now apply the results of the Section 2 to study the functions $x(q), y(q),$ and $z(q)$. Our goal in the next Lemma is to obtain representations $x(q), y(q),$ and $z(q)$ as linear combinations of logarithmic derivatives of theta functions denoted $\theta_{1/7}(q), \theta_{2/7}(q), \theta_{3/7}(q)$.

Lemma 3.1.

\[ x(q) = \alpha_1 \theta_{1/7}(q) + \alpha_2 \theta_{2/7}(q) + \alpha_3 \theta_{3/7}(q), \]
\[ y(q) = \beta_1 \theta_{1/7}(q) + \beta_2 \theta_{2/7}(q) + \beta_3 \theta_{3/7}(q), \]
\[ z(q) = \gamma_1 \theta_{1/7}(q) + \gamma_2 \theta_{2/7}(q) + \gamma_3 \theta_{3/7}(q), \]

where

\[ \alpha_1 = \frac{1}{14} \left( 1 - 3 \cos \left( \frac{3\pi}{14} \right) \csc \left( \frac{\pi}{7} \right) \right), \quad \alpha_2 = \frac{1}{14} \left( 1 + 6 \sin \left( \frac{3\pi}{14} \right) \right), \]
\[ \alpha_3 = \frac{1}{14} \left( 1 - 6 \sin \left( \frac{\pi}{14} \right) \right), \]
\[ \beta_1 = \frac{1}{56} \left( 4 - \csc \left( \frac{\pi}{14} \right) \left( 2 + \csc \left( \frac{3\pi}{14} \right) \right) \right), \]
\[ \beta_2 = \frac{1}{14} \left( 1 + 4 \sin \left( \frac{3\pi}{14} \right) - 2 \cos \left( \frac{\pi}{7} \right) \right), \]
\[ \beta_3 = \frac{1}{28} \left( 2 - 4 \sin \left( \frac{\pi}{14} \right) + \csc \left( \frac{3\pi}{14} \right) \right), \]
\[ \gamma_1 = \frac{1}{28} \left( 4 + \csc \left( \frac{\pi}{14} \right) \left( 3 + \csc \left( \frac{3\pi}{14} \right) \right) \right), \]
\[ \gamma_2 = \frac{1}{7} \left( 1 - 5 \sin \left( \frac{3\pi}{14} \right) + 3 \cos \left( \frac{\pi}{7} \right) \right), \]
\[ \gamma_3 = \frac{1}{28} \left( 4 + 8 \sin \left( \frac{\pi}{14} \right) - 3 \csc \left( \frac{3\pi}{14} \right) \right). \]

Proof. In order to obtain relevant Lambert series expansions for the series from (1.5)–(1.6), we employ three theta function identities derived by Z.-G. Liu [15, pp. 67-68]

\[ q^{-1} \theta_1'(q^7) \theta_1(2\pi \tau | q^7) = -2i + \frac{\theta_1'}{\theta_1}(2\pi \tau | q) - 2 \frac{\theta_1'}{\theta_1}(3\pi \tau | q^7), \]
\[ q^{-1/2} \theta_1'(q^7) \theta_1(\pi \tau | q^7) = \frac{\theta_1'}{\theta_1}(\pi \tau | q^7) - 2 \frac{\theta_1'}{\theta_1}(2\pi \tau | q^7) + \frac{\theta_1'}{\theta_1}(3\pi \tau | q^7), \]
\[ q^{1/2} \theta_1'(q^7) \theta_1(3\pi \tau | q^7) = 2i + \frac{\theta_1'}{\theta_1}(\pi \tau | q^7) + \frac{\theta_1'}{\theta_1}(2\pi \tau | q^7) + \frac{\theta_1'}{\theta_1}(3\pi \tau | q^7). \]

These may be reformulated through the use of (2.3), (2.8), and (2.9)–(2.11) as

\[ x(q) = \sum_{n=1}^{\infty} a_n q^n, \quad y(q) = \sum_{n=1}^{\infty} b_n q^n, \quad z(q) = 1 + \sum_{n=1}^{\infty} c_n q^n, \]
where \(a_n, b_n, c_n\) are periodic sequences modulo seven defined by
\[
\{a_n\}_{n=0}^7 = \{0, 1, -1, -2, 2, 1, -1\}, \quad \{b_n\}_{n=0}^7 = \{0, 1, -2, 1, -1, 2, -1\}, \quad \{c_n\}_{n=0}^7 = \{0, 2, 1, 1, -1, -1, -2\}.
\]
The periodic odd sequence \(a_n\) has the standard discrete Fourier representation \[24\]
\[
f(n) = \sum_{m=1}^{3} \ell_m \sin \left(\frac{2\pi m x}{7}\right), \quad \text{where} \quad \ell_m = \frac{2}{7} \sum_{k=0}^{6} a_k \sin \left(\frac{2\pi mk}{7}\right). \tag{3.16}
\]
Thus, from (3.1) and (3.15), and referring to the definition of \(e\) where
\[
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\]
the resultant identities to derive, from (2.3), (2.8), and (2.9)-(2.11),

The simplified form for each of these constants is given in Lemma 3.1. Each triple of constants \(\alpha, \beta, \gamma\) may be similarly constructed as corresponding multiples of the finite Fourier coefficients for the sequences \(b_n\) and \(c_n\), respectively. \(\square\)

To derive parameterizations for Eisenstein series necessary to prove Theorem 1.1 we require a parameterization for the Hecke Eisenstein series of weight one twisted by the septic Jacobi symbol. This is an immediate consequence of the formulas on line 3.15.

**Lemma 3.2.** Let \(\left(\frac{n}{7}\right)\) denote the Jacobi symbol modulo seven. Then
\[
1 + 2 \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n}{1 - q^n} = x(q) - y(q) + z(q). \tag{3.17}
\]

Our derivation of further parameterizations for Eisenstein series depend fundamentally on Klein’s quartic identity (1.1). We therefore give an elementary proof of (1.1). From (1.5)-(1.6), we may rephrase equation (1.1) in terms of the quadratic (3.18).

**Lemma 3.3.**
\[
x(q)y(q) - x(q)z(q) + y(q)z(q) = 0. \tag{3.18}
\]

*Proof. Right replacement of \(q\) by \(q^7\) in (2.1), and make the respective substitutions
\[
(x, y) = (\pi \tau, 2\pi \tau), \quad (\pi \tau, 3\pi \tau), \quad (2\pi \tau, 3\pi \tau) \tag{3.19}
\]
in the resultant identities to derive, from (2.3), (2.8), and (2.9)-(2.11),
\[
D_1(q) - D_2(q) = y(q)z(q), \quad D_1(q) - D_3(q) = x(q)z(q), \tag{3.20}
\]
\[
D_2(q) - D_3(q) = x(q)y(q), \tag{3.21}
\]
where \(D_1(q), D_2(q),\) and \(D_3(q)\) take the form
\[
D_1(q) = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} + \sum_{n=1}^{\infty} \frac{nq^{6n}}{1 - q^{7n}}, \quad D_2(q) = \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{7n}} + \sum_{n=1}^{\infty} \frac{nq^{5n}}{1 - q^{7n}}, \tag{3.22}
\]
\[
D_3(q) = \sum_{n=1}^{\infty} \frac{nq^{3n}}{1 - q^{7n}} + \sum_{n=1}^{\infty} \frac{nq^{4n}}{1 - q^{7n}}. \tag{3.23}
\]

Identity (3.18) follows immediately from (3.20)-(3.21). \(\square\)
4. A proof of the septic system

We now present a proof of Theorem 1.1 through a sequence of elementary lemmas. We first show that the logarithmic derivatives \( x(q), y(q), \) and \( z(q) \) coincide with the relevant quadratics on the right side of (1.7)–(1.9). To prove the final equation of Theorem 1.1 we derive a parameterization for \( E_4(q^7) \) in terms of \( x(q), y(q), \) and \( z(q) \).

**Lemma 4.1.** Let \( x = x(q), y = y(q), \) and \( z = z(q) \) be defined by (1.5)–(1.6). Then

\[
  \frac{d}{dq} \log x = \frac{5y^2 + 5z^2 - 7x^2 - 20yz - 52xy + 7p}{12} = 1 + \sum_{n=1}^{\infty} f(n) \frac{qn^n}{1 - q^n}, \tag{4.1}
\]

\[
  \frac{d}{dq} \log y = \frac{5x^2 + 5z^2 - 7y^2 + 20xz - 52yz + 7p}{12} = 1 + \sum_{n=1}^{\infty} g(n) \frac{qn^n}{1 - q^n}, \tag{4.2}
\]

\[
  \frac{d}{dq} \log z = \frac{5x^2 + 5y^2 - 7z^2 - 20xy + 52xz + 7p}{12} = 1 + \sum_{n=1}^{\infty} h(n) \frac{qn^n}{1 - q^n}, \tag{4.3}
\]

where \( f, g \) and \( h \) are periodic arithmetic functions modulo seven defined by

\[
  \{f(n)\}_{n=0}^6 = \{-2, 0, -12, 2, -1, 0\}, \quad \{g(n)\}_{n=0}^6 = \{-2, -1, 0, 0, 2, -1\}, \tag{4.4}
\]

\[
  \{h(n)\}_{n=0}^6 = \{-2, 2, 0, -1, -1, 0, 2\}. \tag{4.5}
\]

**Proof.** The claimed equality between the extreme sides of (4.1) follows immediately from the Jacobi triple product representation (2.5) for \( x(q) \). To prove the rightmost equality of (4.1), we begin by applying Lemma 3.1 to write the middle expression of (4.1) as a polynomial of degree two in \( e_1/7(q), e_2/7(q), \) and \( e_3/7(q) \). With the constants \( \alpha_i, \beta_i, \gamma_i, i = 1, 2, 3 \), defined as in Lemma 3.1, we find

\[
  5y^2 + 5z^2 - 52xy - 7x^2 - 20yz = (5\beta_1^2 + 5\gamma_1^2 - 52\alpha_1\beta_1 - 7\alpha_1^2 - 20\beta_1\gamma_1)e_{1/7}^2
\]

\[+ (5\beta_2^2 + 5\gamma_2^2 - 52\alpha_2\beta_2 - 7\alpha_2^2 - 20\beta_2\gamma_2)e_{2/7}^2
\]

\[+ (5\beta_3^2 + 5\gamma_3^2 - 52\alpha_3\beta_3 - 7\alpha_3^2 - 20\beta_3\gamma_3)e_{3/7}^2
\]

\[+ (10\beta_1\beta_2 + 10\gamma_1\gamma_2 - 52\alpha_1\beta_1 - 52\alpha_2\beta_2 - 14\alpha_1\alpha_2 - 20\beta_1\gamma_1 - 20\beta_2\gamma_2)e_{1/7}e_{2/7}
\]

\[+ (10\beta_2\beta_3 + 10\gamma_2\gamma_3 - 52\alpha_3\beta_2 - 52\alpha_2\beta_3 - 14\alpha_2\alpha_3 - 20\beta_2\gamma_3 - 20\beta_3\gamma_2)e_{2/7}e_{3/7}
\]

\[+ (10\beta_1\beta_3 + 10\gamma_1\gamma_3 - 52\alpha_3\beta_1 - 52\alpha_1\beta_3 - 14\alpha_1\alpha_3 - 20\beta_1\gamma_3 - 20\beta_3\gamma_1)e_{1/7}e_{3/7}
\]

\[:= \Phi_1 e_{1/7}^2 + \Phi_2 e_{2/7}^2 + \Phi_3 e_{3/7}^2 + \Phi_4 e_{1/7}e_{2/7} + \Phi_5 e_{1/7}e_{3/7} + \Phi_6 e_{2/7}e_{3/7} \tag{4.7}
\]

\[
  = \sum_{k=1}^{6} \Phi_{k} + \sum_{n=1}^{\infty} \frac{a_n q^n}{1 - q^n} + \sum_{n=1}^{\infty} \frac{b_n q^n}{(1 - q^n)^2}, \tag{4.8}
\]

where \( a_n \) and \( b_n \) are sequences periodic modulo seven defined through Lemma 2.2 by

\[
  a_n = \sum_{k=1}^{3} \Phi_k \delta_{k/7}(n) + \sum_{r=1}^{6} \Phi_r \kappa_{(k-3)/7}(n), \quad b_n = \sum_{k=1}^{3} \Phi_k \lambda_{k/7}(n) + \sum_{r=1}^{6} \Phi_r \mu_{(k-3)/7}(n). \tag{4.9}
\]
By employing exact precision arithmetic in Mathematica 9.0 yields
\[ \{a_n\}_{n=0}^6 = \{0, 37, 25, 61, 61, 25, 37\}, \quad \{b_n\}_{n=0}^6 = \{222, -37, -37, -37, -37, -37, -37\}, \]
\[ \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 = 5. \quad (4.11) \]

After applying Lemma 2.3, the rightmost series of (4.10) takes the form
\[ \sum_{n=1}^{\infty} \frac{b_n q^n}{(1 - q^n)^2} = \sum_{n=1}^{\infty} \frac{n}{1 - q^n} \sum_{m=1}^{7} (-37) q^{mn} + 259 \sum_{n=1}^{\infty} \frac{nq^{7n}}{1 - q^{7n}} = -37 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}. \quad (4.12) \]

Therefore, by applying the calculations on lines (4.6)–(4.11), we may write
\[ \frac{5y^2 + 5z^2 - 7x^2 - 20yz - 52xy + 7P}{12} = 1 + \sum_{n=1}^{\infty} \frac{(a_n - 37)nq^n}{12(1 - q^n)} - 2 \sum_{n=1}^{\infty} \frac{7nq^{7n}}{1 - q^{7n}}. \quad (4.13) \]

The final equality of (4.13) demonstrates the truth of the first claim of Lemma 4.1. Proofs of the latter two identities of the lemma are similar. We omit the details. \[\square\]

Our proof of Lemma 4.1 addresses the first three equations (1.7)–(1.9) of Theorem 1.1. It remains for us to prove the last equation of Theorem 1.1. In the next sequence of lemmas we will show that this equation is a reparameterization of Ramanujan’s original differential equation for weight two Eisenstein series given by (1.2)
\[ q \frac{dE_2}{dq} = \frac{E_2^3 - E_4}{12}. \quad (4.14) \]

To exhibit the equivalence of (4.14) and the last equation of Theorem 1.1, it suffices to derive a corresponding parameterization for the Eisenstein series of weight 4 and argument \( q^7 \); namely,
\[ E_4(q^7) = x^4 - 4x^3y + 12x^3z - 12xy^3 + 4xz^3 + y^4 - 4y^3z - 12yz^3 + z^4. \quad (4.15) \]

The validity of Equation (4.15) will be addressed following our proof of Theorem 4.5. We construct relevant parameterizations for Eisenstein series in terms of septic parameters from Klein’s relation (3.18) as well as a representation appearing in the next lemma for the Hauptmodul on \( \Gamma_0(7) \) as a rational function of \( x, y, z \).

**Lemma 4.2.** Let \( x = x(q) \), \( y = y(q) \), and \( z = z(q) \) be defined by (1.5) (1.6). Then
\[ \frac{(q; q)_\infty^4}{q(q^7; q^7)_\infty^4} = \frac{z^2 - xz - y^2 - 6yz}{yz}. \quad (4.16) \]

**Proof.** From [5, p. 88] (cf. [14, p. 838]) we have
\[ j_7 := \frac{(q; q)_\infty^4}{q(q^7; q^7)_\infty^4} = \frac{a^5b^5 + b^5c^5 + c^5a^5 - 5a^2b^2c^2}{a^2b^2c^2}, \quad (4.17) \]
where \( a, b \) and \( c \) are defined by (1.14)–(1.15). Klein’s quartic relation (1.1) implies
\[ ab^5 + bc^5 + ca^5 - 5a^2b^2c^2 = ab^5 + ca^5 + 6bc^5 + \frac{5b^4c^3}{a}. \quad (4.18) \]
Hence, by (4.17), and (4.18),
\[
\frac{(q; q)_\infty^4}{q(q^7; q^7)_\infty^4} = \frac{a^2b^5 + ca^6 + 6abc^5 + 5b^4c^3}{a^3b^2c^2} = \frac{z^2 - xz - y^2 - 6yz}{yz}, \tag{4.19}
\]
where last equality of (4.19) follows from (1.5)–(1.6).

Lemma 4.3. Let \(x = x(q), y = y(q), \) and \(z = z(q)\) be defined by (1.5)(1.6). Then
\[
q^2\left(\frac{q^7; q^7}{q; q}\right)_\infty^7 = xyz, \quad q(q; q)_\infty^3\left(\frac{q^7; q^7}{q; q}\right)_\infty^3 = x \left( z^2 - y^2 + 6xy - 7xz \right), \tag{4.20}
\]
\[
\frac{q(q; q)_\infty^7}{(q^7; q^7)_\infty} = x^3 - 32x^2y + 13xy^2 - y^3 + 45x^2z - 13xz^2 + z^3. \tag{4.21}
\]

Proof. The leftmost equation on line (4.20) follows from the Jacobi triple product representations (2.5) for \(x, y, \) and \(z\). To derive, the second equation of (4.20), multiply (4.16) by the left equation of (4.20) to derive
\[
q(q; q)_\infty^3\left(\frac{q^7; q^7}{q; q}\right)_\infty^3 = x \left( z^2 - y^2 + 6xy - 7xz \right) - 6x(xy - xz + yz). \tag{4.22}
\]
Klein’s quartic relation (3.18) may be applied to (4.22) to arrive at the rightmost equation of (4.20). Equation (4.21) may be derived by multiplying the second equation of (4.20) by equation (4.16) and similarly applying Klein’s quartic relation. \(\square\)

We now employ formulas for Eisenstein series equivalent to those appearing in Ramanujan’s Lost Notebook [20, p. 53] (see also [2] [16]) to construct relevant representations for Eisenstein series. The following representations were formulated by S. Cooper and P. C. Toh [4, p. 176] from Ramanujan’s septic representations for Eisenstein series.

Lemma 4.4. Let
\[
\sigma = 1 + 2\sum_{n=1}^\infty \left(\frac{N}{7}\right) \frac{q^n}{1 - q^n}, \quad Z = \frac{(q; q)_\infty^7}{(q^7; q^7)_\infty}, \quad X = \frac{q(q^7; q^7)_\infty^4}{(q; q)_\infty^4}.
\]

Then
\[
E_4(q) = \sigma(1 + 245X + 2401X^2), \tag{4.23}
E_4(q^7) = \sigma(1 + 5X + X^2), \tag{4.24}
E_6(q) = Z^2(1 - 490X - 21609X^2 - 235298X^3 - 823543X^4), \tag{4.25}
E_6(q^7) = Z^2(1 + 14X + 63X^2 + 70X^3 - 7X^4). \tag{4.26}
\]

The parameterizations from Lemma 4.4 may be transcribed in equivalent form as polynomials in \(x(q), y(q), \) and \(z(q)\). These beautiful formulas comprise the last ingredient needed for our proof of Theorem 1.1. Their intentional symmetry is one of the infinitely many equivalent septic formulations made possible by Klein’s relation (3.18).
Theorem 4.5.

\[
E_4(q) = x^4 - 116x^3y + 116xy^3 + y^4 - 116x^3z + 848xyz^2 + 848x^2yz \\
- 848xy^2z - 116y^3z + 116xz^3 + 116yz^3 + z^4,
\]

\[
E_4(q^7) = x^4 + 4x^3y - 4xy^3 + y^4 + 4x^3z + 8xyz + 8x^2yz - 8xy^2z \\
+ 4y^2z - 4xz^3 - 4yz^3 + z^4,
\]

\[
E_6(q) = x^6 + 258x^5y - 5904x^4y^2 - 5904x^2y^4 - 258xy^5 + y^6 + 258x^5z \\
+ 7310x^3y^2z + 7310x^2y^3z + 258y^5z - 5904x^4z^2 + 7310x^3yz^2 \\
- 8751x^2y^2z^2 - 7310x^2yz^3 - 7310xyz^4 - 258y^5z - 258yz^5 + z^6,
\]

\[
E_6(q^7) = x^6 + 6x^5y + 18x^4y^2 + 18x^2y^4 - 6xy^5 + y^6 + 6x^5z + 2x^3y^2z \\
+ 2x^2y^3z + 6y^5z + 18x^4z^2 + 2x^3yz^2 - 57x^2y^2z^2 - 2x^3z^2 \\
+ 18y^4z^2 - 2x^2yz^3 - 2xy^2z^3 + 18x^2z^4 + 18y^2z^4 - 6x^5 \\
- 6yz^5 + z^6.
\]

Proof. To prove (4.28), apply (4.24) and the formulas from Lemmas 4.2-4.3 to derive

\[
Z\sigma(1 + 5X + X^2)
\]

\[
- (x^4 + 4x^3y + 4x^3z + 8x^2yz - 4xy^2z - 8xyz^2 - 4xz^3 + y^4 + 4y^3z - 4y^3z^3 + z^4)
\]

\[
= (xy - xz + yz)\frac{-42x^4z^2 - 79x^3y^2z + 258x^3yz^2 + 52x^3z^3 - 37x^2y^4}{(zx - z^2 + y^2 + 6yz)^2}
\]

\[
+ (xy - xz + yz)\frac{-184x^2y^3z + 3x^2y^2z^2 - 6x^2yz^3 + 30x^2z^4 + 45xy^5}{(zx - z^2 + y^2 + 6yz)^2}
\]

\[
+ (xy - xz + yz)\frac{+292xy^4z + 90xy^3z^2 - 449xy^2z^3 + 334xyz^4 - 48xz^5}{(zx - z^2 + y^2 + 6yz)^2}
\]

\[
+ (xy - xz + yz)\frac{-10y^6 - 84y^5z - 133y^4z^2 + 96y^3z^3 + 121y^2z^4 - 70yz^5 + 8z^6}{(zx - z^2 + y^2 + 6yz)^2}.
\]

Klein’s relation (3.18) implies that the difference of the expression on line (4.32) and
the line immediately following is zero. Since the other relations of Theorem 4.5 are
similarly derived, we omit the details. \hfill \square

To derive the last equation of Theorem 1.1 and complete the proof of Theorem 1.1,

note that by (3.18), the difference of the right sides of (4.28) and (4.15) equals

\[
8(xy - xz + yz)(x^2 + y^2 + z^2) = 0.
\]

Parameterizations appearing in Theorem 4.5 are distinguished from equivalent represen-
tations in [2, 4, 16, 20] by the apparent coefficient symmetry. These representations
are analogous to balanced quintic parameterizations from [3]. Symmetric septic repre-
sentations for more general Eisenstein series will be explored in a subsequent paper.
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