The Erdős-Szekeres problem and an induced Ramsey question

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Abstract

Motivated by the Erdős-Szekeres convex polytope conjecture in \( \mathbb{R}^d \), we initiate the study of the following induced Ramsey problem for hypergraphs. Given integers \( n > k \geq 5 \), what is the minimum integer \( g_k(n) \) such that any \( k \)-uniform hypergraph on \( g_k(n) \) vertices with the property that any set of \( k + 1 \) vertices induces 0, 2, or 4 edges, contains an independent set of size \( n \). Our main result shows that \( g_k(n) > 2^{cn^{k-4}} \), where \( c = c(k) \).

1 Introduction

Given a finite point set \( P \) in \( d \)-dimensional Euclidean space \( \mathbb{R}^d \), we say that \( P \) is in \textit{general position} if no \( d + 1 \) members lie on a common hyperplane. Let \( ES_d(n) \) denote the minimum integer \( N \), such that any set of \( N \) points in \( \mathbb{R}^d \) in general position contains \( n \) members in \textit{convex position}, that is, \( n \) points that form the vertex set of a convex polytope. In their classic 1935 paper, Erdős and Szekeres [1] proved that in the plane, \( ES_2(n) \leq 4^n \). In 1960, they [2] showed that \( ES_2(n) \geq 2^{n-2}+1 \) and conjectured this to be sharp for every integer \( n \geq 3 \). Their conjecture has been verified for \( n \leq 6 \) [1, 8], and determining the exact value of \( ES_2(n) \) for \( n \geq 7 \) is one of the longest-standing open problems in Ramsey theory/discrete geometry. Recently [9], the second author asymptotically verified the Erdős-Szekeres conjecture by showing that \( ES_2(n) = 2^{\Theta(n^{1/(d-1)})} \).

In higher dimensions, \( d \geq 3 \), much less is known about \( ES_d(n) \). In [3], Károlyi showed that projections into lower-dimensional spaces can be used to bound these functions, since most generic projections preserve general position, and the preimage of a set in convex position must itself be in convex position. Hence, \( ES_d(n) \leq ES_2(n) = 2^{n+o(n)} \). However, the best known lower bound for \( ES_d(n) \) is only on the order of \( 2^{n^{1/(d-1)}} \), due to Károlyi and Valtr [4]. An old conjecture of Füredi (see Chapter 3 in [5]) says that this lower bound is essentially the truth.

Conjecture 1.1. For \( d \geq 3 \), \( ES_d(n) = 2^{\Theta(n^{1/(d-1)})} \).

It was observed by Motzkin [6] that any set of \( d + 3 \) points in \( \mathbb{R}^d \) in general position contains either 0, 2, or 4 \((d+2)\)-tuples not in convex position. By defining a hypergraph \( H \) whose vertices are \( N \)

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points in $\mathbb{R}^d$ in general position, and edges are $(d+2)$-tuples not in convex position, then every set of $k+1$ vertices induces 0, 2, or 4 edges. Moreover, by Carathéodory’s theorem (see Theorem 1.2.3 in [8]), an independent set in $H$ would correspond to a set of points in convex position. This leads us to the following combinatorial parameter.

Let $g_k(n)$ be the minimum integer $N$ such that any $k$-uniform hypergraph on $N$ vertices with the property that every set of $k+1$ vertices induces 0, 2, or 4 edges, contains an independent set of size $n$. For $k \geq 5$, the geometric construction of Károlyi and Valtr [4] mentioned earlier implies that

$$g_k(n) \geq ES_{k-2}(n) \geq 2^{cn^{1/(k-3)}},$$

where $c = c(k)$. One might be tempted to prove Conjecture 1.1 by establishing a similar upper bound for $g_k(n)$. However, our main result shows that this is not possible.

**Theorem 1.2.** For each $n \geq k \geq 5$ there exists $c = c(k) > 0$ such that $g_k(n) > 2^{cn^{k-4}}$.

In the other direction, we can bound $g_k(n)$ from above as follows. For $n \geq k \geq 5$ and $t \leq k$, let $h_k(t, n)$ be the minimum integer $N$ such that any $k$-uniform hypergraph on $N$ vertices with the property that any set of $k+1$ vertices induces at most $t$ edges, contains an independent set of size $n$. In [7], the authors proved the following.

**Theorem 1.3 ([7]).** For $k \geq 5$ and $t \leq k$, there is a positive constant $c' = c'(k, t)$ such that

$$h_k(t, n) \leq \text{twr}(c'n^{k-t} \log n),$$

where $\text{twr}$ is defined recursively as $\text{twr}_1(x) = x$ and $\text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}$.

Hence, we have the following corollary.

**Corollary 1.4.** For $k \geq 5$, there is a constant $c' = c'(k)$ such that

$$g_k(n) \leq h_k(4, n) \leq 2^{2c'n^{k-4} \log n}.$$

It is an interesting open problem to improve either the upper or lower bounds for $g_k(n)$.

**Problem 1.5.** Determine the tower growth rate for $g_k(n)$.

Actually, this Ramsey function can be generalized further as follows: for every $S \subset \{0, 1, \ldots, k\}$, define $g_k(n, S)$ to be the minimum integer $N$ such that any $N$-vertex $k$-uniform hypergraph with the property that every set of $k+1$ vertices induces $s$ edges for some $s \in S$, contains an independent set of size $n$. General results for $g_k(n, S)$ may shed light on classical Ramsey problems, but it appears difficult to determine even the tower height for any nontrivial cases.
2 Proof of Theorem 1.2

Let \( k \geq 5 \) and \( N = 2^{c_k k - 4} \) where \( c = c_k > 0 \) is sufficiently small to be chosen later. We are to produce a \( k \)-uniform hypergraph \( H \) on \( N \) vertices with \( \alpha(H) \leq n \) and every \( k + 1 \) vertices of \( H \) span 0, 2, or 4 edges. Let \( \phi : \binom{[N]}{k-3} \to \binom{[k-1]}{2} \) be a random \( \binom{[k-1]}{2} \)-coloring, where each color appears on each \( (k-3) \)-tuple independently with probability \( 1/\binom{[k-1]}{2} \). For \( f = (v_1, \ldots, v_{k-1}) \in \binom{[N]}{k-1} \), where \( v_1 < v_2 < \cdots < v_{k-1} \), define the function \( \chi_f : \binom{[k-1]}{2} \to \binom{[k-1]}{2} \) as follows: for all \( \{i, j\} \in \binom{[k-1]}{2} \), let

\[
\chi_f(f \setminus \{v_i, v_j\}) = \{i, j\}.
\]

We define the \( (k-1) \)-uniform hypergraph \( G \), whose vertex set is \( [N] \), such that

\[
G = G_\phi := \left\{ f \in \binom{[N]}{k-1} : \phi(f \setminus \{u, v\}) = \chi_f(f \setminus \{u, v\}) \text{ for all } \{u, v\} \in \binom{f}{2} \right\}.
\]

For example, if \( k = 4 \) (which is excluded for the theorem but we allow it to illustrate this construction) then \( \phi : [N] \to \{12, 13, 23\} \) and for \( f = (v_1, v_2, v_3) \), where \( v_1 < v_2 < v_3 \), we have \( f \in G \) iff \( \phi(v_1) = 23, \phi(v_2) = 13, \text{ and } \phi(v_3) = 12 \).

Finally, we define the \( k \)-uniform hypergraph \( H \), whose vertex set is \( [N] \), such that

\[
H = H_\phi := \left\{ e \in \binom{[N]}{k} : |G[e]| \text{ is odd} \right\}.
\]

Claim 2.1. \(|H[S]| \) is even for every \( S \in \binom{[N]}{k+1} \).

Proof. Let \( S \in \binom{[N]}{k+1} \) and suppose for contradiction that \(|H[S]| \) is odd. Then

\[
2|G[S]| = \sum_{f \in G[S]} 2 = \sum_{f \in G[S]} \sum_{e \in \binom{f}{2}} 1 = \sum_{e \in \binom{f}{2}} |G[e]| = \sum_{e \notin H[S]} |G[e]| + \sum_{e \in H[S]} |G[e]|.
\]

The first sum on the RHS above is even by definition of \( H \) and the second sum is odd by definition of \( H \) and the assumption that \(|H[S]| \) is odd. This contradiction completes the proof.

Claim 2.2. \(|G[e]| \leq 2 \) for every \( e \in \binom{[N]}{k} \).

Proof. For sake of contradiction, suppose that for \( e = (v_1, \ldots, v_k) \), where \( v_1 < \cdots < v_k \), we have \(|G[e]| \geq 3 \). Let \( e_p = e \setminus \{v_p\} \) for \( p \in [k] \) and suppose that \( e_i, e_j, e_l \in G \) with \( i < j < l \). In what follows, we will find a set \( S \) of size \( k-3 \), where \( S \subseteq e_i \) and \( S \subseteq e_l \), such that \( \chi_{e_i}(S) \neq \chi_{e_l}(S) \). This will give us our contradiction since \( e_i, e_l \in G \) implies that \( \chi_{e_i}(S) = \phi(S) = \chi_{e_l}(S) \).

Let \( Y = e \setminus \{v_i, v_j, v_l\} \) and \( Y' = Y \setminus \{\min Y\} \). Let us first assume that \( i > 1 \) so that \( \min Y = v_1 \). In this case,

\[
\chi_{e_i}(Y' \cup \{v_j\}) = \{1, l-1\},
\]

since we obtain \( Y' \cup \{v_j\} \) from \( e_i \) by removing \( \min Y \) and \( v_l \) which are the first and \((l-1)\)st elements of \( e_i \). Similarly,

\[
\chi_{e_l}(Y' \cup \{v_j\}) = \{1, i\},
\]
since we obtain $Y' \cup \{v_j\}$ from $e_l$ by removing $\min Y$ and $v_l$ which are the first and $i$th elements of $e_i$. Because $l > i + 1$, we conclude that $\chi_{e_i}(Y' \cup \{v_j\}) \neq \chi_{e_l}(Y' \cup \{v_j\})$ as desired.

Next, we assume that $i = 1$ and $\min Y = v_q$ where $q > 1$. In this case,

$$\chi_{e_l}(Y' \cup \{v_j\}) = \{q - 1, l - 1\},$$

since we obtain $Y' \cup \{v_j\}$ from $e_l$ by removing $v_q$ and $v_l$ which are the $(q - 1)$st and $(l - 1)$st elements of $e_l$. Similarly,

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{1, q'\} \quad \text{where} \quad q' = q \text{ if } q < l \text{ and } q' = q - 1 \text{ if } q > l,$$

since we obtain $Y' \cup \{v_j\}$ from $e_l$ by removing $v_i = v_1$ and $v_q$ which are the first and $q'$th elements of $e_i$. If $q = 2$, then we immediately obtain $\chi_{e_i}(Y' \cup \{v_j\}) \neq \chi_{e_l}(Y' \cup \{v_j\})$ as desired. On the other hand, if $q = 2$, then $q' = q - 2$ as well and $l \geq 4$, so $l - 1 \neq q'$ and again

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{q - 1, l - 1\} \neq \{1, q'\} = \chi_{e_l}(Y' \cup \{v_j\}).$$

This completes the proof of the claim. \hfill \Box

Let $T_3$ be the $(k - 1)$-uniform hypergraph with vertex set $S$ with $|S| = k + 1$ and three edges $e_1, e_2, e_3$ such that there are three pairwise disjoint pairs $p_1, p_2, p_3 \in \binom{S}{2}$ with $p_i = \{v_i, v'_i\}$ and $e_i = S \setminus p_i$ for $i \in \{1, 2, 3\}$.

Claim 2.3. $T_3 \not\subset G$.

Proof. Suppose for a contradiction that there is a subset $S \subset [N]$ of size $k + 1$ such that $T_3 \subset G[S]$. Using the notation above, assume without loss of generality that $v_1 = \min \cup_i p_i$ and $v_2 = \min(p_2 \cup p_3)$. Let $Y = S \setminus (p_1 \cup p_3)$ and note that $Y \in \binom{e_1 \cup e_3}{k - 3}$. Let $Y_1 \subset Y$ be the set of elements in $Y$ that are smaller than $v_1$, so we have the ordering

$$Y_1 < v_1 < v_2 < \{v_3, v'_3\}.$$

Now, $\chi_{e_1}(Y)$ is the pair of positions of $v_2$ and $v'_3$ in $e_1$. Both of these positions are at least $|Y_1| + 2$ as $Y_1 \cup \{v_2\}$ lies before $p_3$. On the other hand, the smallest element of $\chi_{e_3}(Y)$ is $|Y_1| + 1$ which is the position of $v_1$ in $e_3$. This shows that $\chi_{e_1}(Y) \neq \chi_{e_3}(Y)$, which is a contradiction as both must be equal to $\phi(Y)$ as $e_1, e_3 \subset G$. \hfill \Box

We now show that every $(k + 1)$-set $S \subset [N]$ spans 0, 2 or 4 edges of $H$. Let $G'$ be the graph with vertex set $S$ and edge set $\{S \setminus f : f \in G[S]\}$. So there is a 1-1 correspondence between $G[S]$ and $G'$ via the map $f \rightarrow S \setminus f$. If $G'$ has a vertex $x$ of degree at least three, then $|G[S \setminus \{x\}]| \geq 3$ which contradicts Claim 2.2. Therefore $G'$ consists of disjoint paths and cycles. Next, observe that Claim 2.3 implies that $G'$ does not contain a matching of size three, for the complementary sets of this matching yield a copy of $T_3 \subset G$. This immediately implies that $k = 5$, for otherwise we obtain a 3-matching in $G'$. Moreover, the only way to avoid a 3-matching when $k = 5$ is for $G'$ to consist of two components each of which contains a two edge path so we may assume that $G'$ is of this form, with paths $abc, uvw$. If both $uvw$ and $abc$ are triangles, then $|H[S]| = 0$ as any 5-set $A$ in $S$ contains precisely two edges of $G'$ from $A$ to $S \setminus A$ which yields
Let us now argue that \( \alpha(H) \leq n \), which is a straight-forward application of the probabilistic method. Indeed, we will show that this happens with positive probability and conclude that an \( H \) with this property exists. For a given \( k \)-set, the probability that it is an edge of \( H \) is \( p < 1 \), where \( p \) depends only on \( k \). Consequently, the probability that \( H \) has an independent set of size \( n \) is at most

\[
\binom{N}{n} (1 - p)^{c'n^{k-3}}
\]

for some \( c' > 0 \). Note that the exponent \( k - 3 \) above is obtained by taking a partial Steiner \((n, k, k - 3)\) system \( S \) within a potential independent set of size \( n \) and observing that we have independence within the edges of \( S \). A short calculation shows that this probability is less than 1 as long as \( c \) is sufficiently small. This completes the proof of Theorem 1.2. \[\square\]

References

[1] P. Erdős, G. Szekeres, A combinatorial problem in geometry, *Compos. Math.* 2 (1935), 463–470.

[2] P. Erdős, G. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 3–4 (1960-61), 53–62.

[3] G. Károlyi, Ramsey-remainder for convex sets and the Erdős-Szekeres theorem, *Discrete Applied Mathematics* 109 (2001), 163–175.

[4] G. Károlyi, P. Valtr, Point configurations in \( d \)-space without large subsets in convex position, *Disc. Comp. Geom.* 30 (2003), 277–286.

[5] J. Matoušek, *Lectures in Discrete Geometry*, Springer, 2002.

[6] T. Motzkin, Cooperative classes of finite sets in one and more dimensions, *Journal of Combinatorial Theory* 3 (1967), 244–251.

[7] D. Mubayi, A. Suk, The Erdos-Hajnal hypergraph Ramsey problem, submitted.

[8] G. Szekeres, L. Peters, Computer solution to the 17-point Erdős-Szekeres problem, *ANZIAM Journal* 48 (2006), 151–164.

[9] A. Suk, On the Erdős-Szekeres convex polygon problem, *Journal of the American Mathematical Society* 30 (2017), 1047–1053.