AN IMPROVED STABILITY METHOD FOR LINEAR SYSTEMS WITH FAST-VARYING DELAYS

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Abstract: Stability of linear systems with uncertain bounded time-varying delays is studied under assumption that the nominal delay values are not equal to zero. An input-output approach to stability of such systems is known to be based on the bound of the $L^2$-norm of a certain integral operator. There exists a bound on this operator in two cases: in the case where the delay derivative is not greater than 1 and in the case without any constraints on the delay derivative. In the present note we fill the gap between the two cases by deriving a tight operator bound which is an increasing and continuous function of the delay derivative upper bound $d \geq 1$. For $d \to \infty$ the new bound corresponds to the second case and improves the existing bound. As a result, delay-derivative-dependent frequency-domain and time-domain stability criteria are derived for systems with the delay derivative greater than 1.

Keywords: time-varying delay, stability, input-output approach, $L^2$-norm.

1. INTRODUCTION

Two main approaches have been applied to stability analysis of linear systems with uncertain time-varying delay: a direct Lyapunov approach and an input-output approach (see e.g. Gu et al., 2003), which reduces the stability analysis of the uncertain system to the analysis of the class of systems with the same nominal part but with additional inputs and outputs. In the existing literature the uncertain time-varying delay has been divided into two types: the *slowly-varying* delay (with delay derivative less than $d < 1$) and the *fast-varying delay* (without any constraints on the delay derivative) (see e.g. Kolmanovskii & Myshkis, 1999; Niculescu, 2001). Recently a third type of *moderately varying delay* has been revealed in Fridman & Shaked (2005), where the delay derivative is not greater than 1 (almost for all $t$). This has been obtained by applying the input-output approach to stability. It is known Gu et al. (2003), Kao & Lincoln (2004) that the latter approach to systems with time-varying bounded delays is based on the bound of the $L^2$-norm of a certain integral operator.

In the present paper we fill the gap between the case of the delay derivative not greater than 1 and the fast-varying delay by deriving a new integral operator bound. This bound is an increasing and continuous function of the delay derivative bound $d \geq 1$. In the limit case (where $d \to \infty$) which corresponds to the fast-varying delay, the new bound improves the existing one. As a result, improved frequency-domain and time-domain sta-
bility criteria are derived for systems with the delay derivative greater than 1.

**Notation:** Throughout the paper the superscript \( 'T' \) stands for matrix transposition, \( \mathbb{R}^n \) denotes the \( n \) dimensional Euclidean space with vector norm \( \| \cdot \| \), \( \mathbb{R}^{n \times m} \) is the set of all \( n \times m \) real matrices, and the notation \( P > 0 \), for \( P \in \mathbb{R}^{n \times n} \) means that \( P \) is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by *\( A \), \( L_2 \) is the space of square integrable functions \( v : [0, \infty) \rightarrow C^n \) with the norm \( \| v \|_{L_2} = \left( \int_0^\infty \| v(t) \|^2 dt \right)^{1/2} \), \( \| A \| \) denotes the Euclidean norm of a \( n \times n \) (real or complex) matrix \( A \), which is equal to the maximum singular value of \( A \). For a transfer function matrix of a stable system \( G(s) \), \( s \in \mathbb{C} \)

\[
\| G \|_\infty = \sup_{-\infty < w < \infty} \| G(iw) \|, \quad i = \sqrt{-1}.
\]

## 2. PROBLEM FORMULATION

We consider the following linear system with uncertain time-varying delay \( \tau(t) \) :

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)),
\]

where \( x(t) \in \mathbb{R}^n \) is the system state, \( A_i \), \( i = 0, 1 \) are constant matrices.

The uncertain delay \( \tau(t) \) has a form

\[
\tau(t) = h + \eta(t), \quad |\eta(t)| \leq \mu \leq h,
\]

where \( h \) is a known nominal delay value and \( \mu \) is a known upper bound on the delay uncertainty. In the existing literature Kolmanovskii & Myshkis (1999), Niculescu (2001), Gu et al. (2003) the following types of uncertain time-varying delays are usually considered:

**Case A (slowly-varying delay):** \( \tau(t) \) is a differentiable almost everywhere function, satisfying

\[
\dot{\tau}(t) = \dot{\eta}(t) \leq d = 1 + p,
\]

where \( -1 \leq p < 0 \);

**Case B (fast-varying delay):** \( \tau(t) \) is a measurable (e.g. piecewise-continuous) function.

Recently a moderately-varying delay with \( \dot{\tau}(t) \leq d = 1 \) was introduced in Fridman & Shaked (2005). In the present note we enlarge the latter class of delays as follows:

**Case C (moderately-varying delay):** \( \tau(t) \) is a differentiable almost everywhere function, satisfying (3) with \( p \geq 0 \).

In the present note we will improve the stability results in cases B and C by applying input-output approach and by deriving new inequalities. The results are easily generalized to the case of any finite number of the delays.

We represent (1) in the form:

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - h) - A_1 \int_{-h}^{-h-\eta} \dot{x}(t + s)ds.
\]

Following Fridman & Shaked (2005) we introduce the following auxiliary system:

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - h) + \mu A_1 u(t),
\]

with the feedback

\[
u(t) = -\frac{1}{\mu} \sqrt{F(p)} \int_{-h-\eta}^{-h} y(t + s)ds,
\]

where \( F : [-1, \infty] \rightarrow R^+ \) is a scalar function which will be shortly defined and \( p \) is given by (3). The results for the delay of case B correspond to \( p = \infty \), i.e. to \( F(\infty) \) in the input-output model (5), (6). Substitution of (6) in (5) readily leads to (4).

We are looking for \( F(p) \) which satisfies the following inequality

\[
\| u \|_{L_2}^2 \leq \| y \|_{L_2}^2, \quad \forall \ y \in L_2[0, \infty), \ y|_{[-\infty, 0]} \equiv 0,
\]

where \( u \) is given by (6). This is equivalent to the fact that \( \mu \sqrt{F(p)} \) is an upper bound on the \( L_2 \)-norm of the integral operator \( \Delta : L_2[0, \infty) \rightarrow L_2[0, \infty) \)

\[
z(t) = \Delta y(t) = \int_{-h-\eta}^{-h} y(t + s)ds, \quad y|_{[-\infty, 0]} \equiv 0,
\]

i.e. that

\[
\| z \|_{L_2}^2 \leq \mu^2 F(p) \| y \|_{L_2}^2, \quad \forall \ y \in L_2[0, \infty), \ y|_{[-\infty, 0]} \equiv 0.
\]

Our objective is to find \( F(p) \) (as small as possible) such that (7) (or equivalently (9)) holds.

For \(-1 \leq p < 0 \) (case A) it was established in Gu et al. (2003) that \( F(p) \) can be chosen to be 1. For \( p \geq 0 \) the following was found in Fridman & Shaked (2005): \( F(0) = 1 \) and \( F(p) = 2 \) for \( p \in (0, \infty] \).

We note that the value 1 of \( F(p) \) for \(-1 \leq p \leq 0 \) can not be improved (i.e. chosen to be less than 1). Indeed, taking constant delay \( \eta(t) \equiv \mu \), which satisfies the condition of case A for any \(-1 \leq p \leq 0 \), we consider the functions \( y_\theta(t) = 1 \) as \( 0 \leq t \leq \theta \), and \( y_\theta(t) = 0 \) as \( t > \theta \). Using formula (6) with \( F(p) = 1 \) we immediately obtain

\[
\| y_\theta \|_{L_2}^2 = \theta^2, \quad \| u \|_{L_2}^2 = (\theta - \mu)^2 + \frac{2}{3} \mu^2,
\]

and hence \( \| u \|_{L_2}/\| y_\theta \|_{L_2} \rightarrow 1 \) as \( \theta \rightarrow \infty \).

In the present paper we will improve the values of \( F(p) \) for \( p > 0 \) by showing that \( F(p) \) can be chosen as a continuous increasing function of \( p \geq 0 \) satisfying \( F(0) = 1 \) (as in Fridman &
Shaked (2005)), but $F(p) < F(\infty) = 1.75$ for $p > 0$. The improved values of $F(p)$ will readily lead to improved stability criteria.

3. MAIN RESULTS

3.1 New Bounds

Proofs of the Lemmas of this section are given in the Appendix.

**Lemma 1.** Consider case C. For all $y(t) \in L_2[0,\infty)$ and such that $y(t) = 0 \forall t \leq 0$ and for $u(t)$ given by (6) inequality (7) holds with $F$ given by

$$F(p) = \begin{cases} \frac{4p}{2p + 1}, & \text{if } 0 \leq p < 1, \\ \frac{p+1}{4p-1}, & \text{if } p \geq 1. \end{cases}$$

(10)

As it was mentioned above, $F$ is increasing continuous function satisfying for $p > 0$ the following inequality: $1 = F(0) < F(p) < \lim_{p \to \infty} F(p) = 7/4$.

**Lemma 2.** Consider case B. For all $y(t) \in L_2[0,\infty)$ and such that $y(t) = 0 \forall t \leq 0$ and for $u(t)$ given by (6) inequality (7) holds with $F(\infty) := 7/4$.

**Remark 1.** The value $7/4 = 1.75$ for $F(\infty)$ in Lemma 2 is not far from an optimal one. The following example shows that it cannot be less than 1.5. Namely, define scalar functions $y(t)$ and $\eta(t)$ by

$$y(t) = \begin{cases} t, & \text{if } 0 \leq t \leq \mu, \\ \mu - t, & \text{if } \mu \leq t \leq 2\mu, \\ 0, & \text{if } y(2\mu - y) < 0, \end{cases}$$

$$\eta(t) = \begin{cases} -\mu, & \text{if } t \leq \mu, \\ \mu, & \text{if } t > \mu. \end{cases}$$

Setting in (6) $F(\infty) = 3/2$ we have $u(t) = -\frac{1}{\mu \sqrt{3/2}} z(t)$, where

$$z(t + h) = \int_{t-\eta(t)}^{t} y(s) ds$$

3.2 A Tight Frequency-Domain Stability Criterion

We assume

**A1** Given the nominal value of the delay $h > 0$, the nominal system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h),$$

(11)

is asymptotically stable.

The auxiliary system (5) can be written as $y = Gu$ with the transfer matrix

$$G(s) = \sqrt{F(p)} s I (s I - A_0 - A_1 e^{-hs})^{-1} A_1,$$

(12)

By the small gain theorem (see e.g. Gu et al. (2003) the system (1) is input-output stable (and thus asymptotically stable, since the nominal system is time-invariant) if $\|G\|_\infty < 1$. A stronger result may be obtained by scaling $G$.

**Theorem 1.** Consider (1) with delay given by (2). Under A1 the system is asymptotically stable if there exists non-singular matrix $X$ such that

$$\|G_X\|_\infty < 1, \quad G_X(s) = X G(s) X^{-1},$$

(13)

where $G$ is given by (12) with $F(p)$ of (10) and where $p \in [0,\infty)$ corresponds to case C, while $F(\infty) = 7/4$ corresponds to case B.

**Remark 2.** From Theorem 1 it follows that under A1 (1) is asymptotically stable if

$$\mu < \frac{1}{\sqrt{F(p)}} k = \frac{1}{\|s I (s I - A_0 - A_1 e^{-hs})^{-1} A_1\|_\infty}.$$ 

By Fridman & Shaked (2005) $F(p) = 2, p > 0$ and thus (1) with $\tau(t) \leq 1 + p, p > 0$ or with $\tau(t)$ of case B) is asymptotically stable for $\tau(t) \in [h - \mu, h + \mu]$, where $\mu < 0.7071 k$. By the new bounds of Lemma 2 and Lemma 1 we obtain a wider stability intervals:

$$p = 0.1, \tau(t) \leq 1.1, F(p) = 1.0909, \mu < 0.9574 k;$$

$$p = 0.5, \tau(t) \leq 1.5, F(p) = 1.3333, \mu < 0.8660 k;$$

$$p = 1, \tau(t) \leq 2, F(p) = 1.5, \mu < 0.8165 k;$$

$$p = \infty, \text{ case } B, F(p) = 1.75, \mu < 0.7559 k.$$ 

(14)

3.3 On Improved Stability Criteria in the Time-Domain

By applying the time-domain results of Fridman & Shaked (2005) via descriptor model transformation with the corresponding simple Lyapunov-Krasovskii functional we obtain:

**Theorem 2.** System (1) is asymptotically stable for all delays of (2), if there exist $n \times n$ matrices
In (Fridman, 2004) the maximum value of \( \mu \), for which the system is asymptotically stable, was derived. The existing bound in case B is also improved. In the past, case C was treated as case B, which was restrictive. The new bounds lead to improved stability criteria and gives tools for further improvements.

5. APPENDIX

Proof of Lemma 2. Denote by \( \varphi : \mathbb{R}^2 \rightarrow \{0, 1\} \) the characteristic function of the domain \( D \) in the positive quadrant, bounded by the line \( s = t - h \) and by the graph of the function \( s = t - h - \eta(t) \), i.e.,

\[
\varphi(t, s) = \begin{cases} 1, & \text{if } (t - h - s)(t - h - \eta(t) - s) \leq 0, \\ 0, & \text{if } (t - h - s)(t - h - \eta(t) - s) > 0 \end{cases}
\]

is feasible. Here \( \mathcal{F}(p) \) is given by (10) in case C and \( \mathcal{F}(\infty) = 7/4 \) in case B.

LMI (15) is convex in \( \mathcal{F}(p) \) and thus the smaller values of \( \mathcal{F}(p) \) lead to a less restrictive condition. The time-domain criteria give sufficient conditions for the frequency domain Theorem 1.

Example (Kharitonov & Niculescu, 2003): Consider the system

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x(t - \tau(t)), \quad \tau(t) = 1 + \eta(t), \quad |\eta(t)| \leq \mu, \quad \tau(t) \leq d.
\]

In (Fridman, 2004) the maximum value of \( \mu \), for which the system is asymptotically stable, was found to be \( \mu = 0.271 \) for all \( d \geq 1 \). The latter result was less restrictive than the one by (Kharitonov & Niculescu, 2003). By the time domain criterion of (Fridman & Shaked, 2005) for \( d = 1 \) the corresponding value of \( \mu \) is greater (\( \mu = 0.384 \)), while for \( d > 1 \) the result is the same (\( \mu = 0.271 \)). Theorem 2 of the present paper leads to a wider stability interval for \( d > 1 \) (see Table 1).

Note that the results by Kao & Rantzer (2005) do not improve the existing results by descriptor approach and do not treat the case of \( h - \mu > 0 \).

4. CONCLUSIONS

Linear systems with bounded time-varying delays are analyzed under the assumption that the nominal delay values are not equal to zero. Two cases of delay are considered: case B (without any constraints on the delay derivative) and case C (where the delay derivative is not greater than \( d \geq 1 \)).

An input-output approach to stability of such systems is known to be based on the bound of the \( L_2 \)-norm of a certain integral operator. In the present paper for the first time a tight \( d - \text{dependent} \) bound is derived. The existing bound in case B is also improved. In the past, case C was treated as case B, which was restrictive. The new bounds lead to improved stability criteria and gives tools for further improvements.

Table 1. Maximum value of \( \mu \)

| \( d = 1 \) | \( d = 1.1 \) | \( d = 1.5 \) | \( d = 2 \) | fast delay |
|---|---|---|---|---|
| 0.384 | 0.367 | 0.331 | 0.313 | 0.289 |

0 < \( P_1, P_2, P_3 \), \( S > 0 \), \( Y_1, Y_2, T, R, R_0 \) such that the following Linear Matrix Inequality (LMI)

\[
\begin{bmatrix}
\Gamma_n & \begin{bmatrix} \mu P_1^T A_1 & 0 \\ \mu P_2^T A_1 & \mathcal{F}(p) R_n \end{bmatrix} \\
0 & -
\end{bmatrix} < \begin{bmatrix}
0 & 0 \\
-\mu R_n & -\mathcal{F}(p) R_n
\end{bmatrix},
\]

where

\[
\Psi_n = \begin{bmatrix}
P^T & 0 \\
0 & \begin{bmatrix} 0 & \frac{I}{\mu} \\ \frac{I}{\mu} & 0 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} T & 0 \\
0 & T \end{bmatrix} h \begin{bmatrix} 0 & T \\
0 & -T \end{bmatrix},
\]

is feasible. Here \( \mathcal{F}(p) \) is given by (10) in case C and \( \mathcal{F}(\infty) = 7/4 \) in case B.

By the Riesz-Thorin interpolation theorem (see, for example, Okikiolu (1971), Theorem 5.1.3),

\[
\|K\|_{L_2} \leq \sqrt{\|K\|_{L_1} \cdot \|K\|_{L_\infty}}.
\]

Since \( k(s_1, s_2) \geq 0 \) and \( k(s_1, s_2) = k(s_2, s_1) \), by the well-known formulas for the \( L_1 \) and \( L_\infty \)-norms, we have

\[
\|K\|_{L_1} = \|K\|_{L_\infty} \leq \sup_{s \in [0, \infty)} K(s),
\]

where \( K(s) = K(s) ds_1 \), and hence we decide that

\[
\|K\|_{L_2} \leq \sup_{s \in [0, \infty)} K(s).
\]

We shall show that \( K(t) \leq 7/4 \mu^2 \) for all \( t \in [0, \infty) \).

Without loss of generality assume that \( \eta(t) > 0 \). Geometrically, \( K(t) \) is the area of the part \( D(t) \) of the domain \( D \) cut out by the strip \( t - h \geq s_2 \geq t - h - \eta(t) \) (double shaded region in Figure 1).

Thus, \( D(t) \) lies inside the parallelogram

\[
\Pi(t) := \{(s_1, s_2) \in \mathbb{R}^2 : t - h \leq s_2 \leq t - h - \eta(t), \ s_1 - h - \mu \leq s_2 \leq s_1 - h + \mu \}
\]

(see Figure 2). Divide \( \Pi(t) \) by the vertical lines \( s_1 = t_4 \) and \( s_1 = t_5 \) into two trapezes of total
Without loss of generality we assume that the domain above the value of $(s_1)$ (shaded region in Figure 3(b)).

For $0 < p < 1$, the area $\mu^2 - (\mu - \eta(t))^2$, and a square (see Figure 2). The intersection of $D(t)$ with a vertical line $s_1 = \sigma$, where $t_4 \leq \sigma \leq t_5$, is contained either in the segment $[\tau - h, t - h]$, or in the segment $[\tau - h, t - h]$, and hence the area of $D(t) \cap \{ t - \eta(t) \leq s_1 \leq t \}$ does not exceed

$$\int_{t - \eta(t)}^{t} \max \{(\tau - (t - \eta(t)), t - \tau) d\tau = \frac{3}{4} \eta(t)^2 \}.$$ 

So we derive the required bound (9) from the evident inequality

$$\mu^2 - (\mu - \eta(t))^2 + \frac{3}{4} \eta(t)^2 \leq \frac{7}{4} \mu^2,$$

which geometrically means that the maximal area domain $D(t)$ looks as the shaded region in Figure 2.

**Proof of Lemma 1.** We interpret the function $F(p)$ geometrically as follows:

- For $p \geq 1$, $F(p) \mu^2$ is the area of the domain

$D_p := \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq \mu, t - 2\mu \leq s \leq t, (t - \mu - s)(2s + 2pt - (3p + 1)\mu) \geq 0 \}$

(shaded region in Figure 3(a)),

- For $0 \leq p < 1$, $F(p) \mu^2$ is the area of the domain

$D_p = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq \mu, t - 2\mu \leq s \leq t, (t - \mu - s)(s + pt - 2p\mu) \leq 0 \}$

(shaded region in Figure 3(b)).

As in the proof of Lemma 2, we estimate from above the value of $K(s_2)$. i.e., the area of the domain $D(t)$.

Without loss of generality we assume that $\eta$ is a smooth function, whose zero locus is locally finite and consists of only simple roots. Fix $t > 0$. Assume that $\eta(t) > 0$, which, in particular, means that the line $s_1 = t$ crosses the domain $D(t)$ along the segment $t_1 \leq s_2 \leq t_2$ (cf. Figure 2). Put $N = \#(\eta^{-1}(0) \cap (t - \mu - \eta(t), t + \mu))$ (i.e., the number of zeroes in the interval $(t_3, t_6)$ in Figure 2).

Consider few possibilities.

**Step 1.** Suppose that $N = 0$. Then $D(t)$ is contained in the part of the parallelogram $\Pi(t)$, right to the line $s_2 = s_1 - h$ (see Figure 2). Then its area does not exceed $\eta(t)\mu \leq \mu^2 \leq F(p) \mu^2$.

**Step 2.** Suppose that $N = 1$. The zero $\tau = \eta^{-1}(0) \cap (t - \mu - \eta(t), t + \mu)$ corresponds to an intersection point of the graphs of the functions $s_2 = s_1 - h$ and $s_2 = s_1 - h - \eta(s_1)$. This intersection point lies either right to the line $l_1 := \{ s_1 = t \}$ or below the line $l_2 := \{ s_2 = ps_1 + (p - 1)t - h - \eta(t) \}$ (see Figure 3(c)). In the former case, the domain $D(t)$ remains below the line $s_2 = s_1 - h$, that is we have the upper bound from Step 1. In the latter case, the part of domain $D(t)$ in the half-plane $s_1 \geq \tau$ should lie below the line $s_2 = s_1 - h$ and above the line $l_3 = \{ s_2 = ps_1 + (p - 1)\tau - h \}$, and the part of $D(t)$ in the half-plane $s_1 \leq \tau$ should lie below the line $l_3$ and above the line $s_2 = s_1 - h$ (shaded region in Figure 3(c)). It is an elementary geometry exercise to show that the area of the shaded region in Figure 3(c) does not exceed $F(p) \mu^2$.

**Step 3.** We intend to show that the case $N > 1$ reduces to the above considered cases $N = 0$ or 1. For, we need the following auxiliary geometric statement. Consider the parallelogram $\Pi(t)$, some points $x_1, x_2, x_3$ with decreasing coordinates, lying on the line $s_2 = s_1 - h$ below the line $l_2 = \{ s_2 = ps_1 + (p - 1)t - h - \eta(t) \}$ (see Figure 4(a)). Draw the lines $l_4, l_5$ with slope $-p$ through the points $x_1, x_3$, respectively, and the vertical line $l_6$ through $x_2$. Denote by $F(x_2)$ the area of the domain $\delta(x_2)$, lying inside $\Pi(t)$, between the lines $l_4, l_5$, and in the two sectors, bounded by the lines...
to the vertical line through $\delta$. In the former situation, when replacing $\delta$ does not change, $F$ attains its maximum either at $x_2 = x_1$ or at $x_2 = x_3$.

Indeed, when $x_2$ runs along a subsegment of $[x_1, x_3]$ such that the combinatorial type of $\delta(x_2)$ does not change, $F$ is a quadratic polynomial in the abscissa $\tau$ of $x_2$ with the positive second derivative. That is, in this subsegment, $F$ attains its maximum at an endpoint. If such an endpoint differs from $x_1$ and $x_3$, then it corresponds to the situation when

- either $x_2$ belongs to a side of $\Pi(t)$ (see, for example, Figure 4(d));
- or the domain $\delta(x_2)$ intersects with some side of $\Pi(t)$ at a point (see, Figure 4(b,c)).

In the former situation, $\delta(x_2)$ entirely lies right to the vertical line through $x_2$, and then monotonically grows as $x_2$ tends to $x_3$. In the latter situation, when replacing $x_2$ by $x_1$, the domain $\delta(x_2)$ turns into the domain $\delta(x_1)$ by getting rid of $\delta(x_2) \cap \{s_1 \leq \tau\}$ and adding the fragment $\delta_+$ (the trapeze, bounded by the vertical line through $x_2$, dashed line, the upper side of $\Pi(t)$, and the line $s_2 = s_1 - h$ in Figure 4(b)). In the latter situation, when replacing $x_2$ by $x_3$, the domain $\delta(x_2)$ turns into the domain $\delta(x_3)$ by getting rid of $\delta(x_2) \cap \{s_1 \geq \tau\}$ and adding the fragment $\delta_+$ (the trapeze, bounded by the vertical line through $x_2$, dashed line, the lower side of $\Pi(t)$, and the line $s_2 = s_1 - h$ in Figure 4(b)).

One can easy see that the area of $\delta(x_2)$ in the above situation is less than the maximum of the areas of $\delta(x_1)$ and $\delta(x_3)$. The same conclusion one can derive in the situation, presented in Figure 4(c).

**Step 4.** Suppose that $N > 1$. We can take $N$ to be odd, adding if necessary one more zero close to $t - \mu - \eta(t)$. Denote by $x_1, x_2, ..., x_n$ the corresponding intersection points of the graph of $s_2 = s_1 - h - \eta(s_1)$ with the line $s_2 = s_1 - h$, numbered by the decreasing coordinates. Through each point $x_{2i-1}$ we draw a line $l_i'$ with slope $-p$, $1 \leq i \leq (n + 1)/2$, and through each point $x_{2i}$ we draw a vertical line $l_i''$, $1 \leq i \leq n/2$. Thus, $D(t)$ is the union of the regions in $\Pi(t)$, bounded by the introduced lines as follows (marked by dots in Figure 4(e)):

- above the line $l_i'$ and below the line $s_2 = s_1 - h$;
- below the line $l_i'$, right to the line $l_i''$, and above the line $s_2 = s_1 - h$, $1 \leq i \leq n/2$;
- to the line $l_i''$, above the line $l_i''$, and below the line $s_2 = s_1 - h$, $1 \leq i \leq n/2$;
- below the line $l_{(n+1)/2}'$ and above the line $s_2 = s_1 - h$.

Using the statement of Step 3, we can move the point $x_2$ either to the position $x_1$, or $x_3$ and increase the area of $\delta(t)$. On the other hand, each of these limit positions for $x_2$ means that we, in fact, have reduced two zeroes of $\eta$ in the interval $(t - \mu - \eta(t), t + \mu)$. Thus, we inductively come to the case $N = 1$ treated in Step 2.

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