On the Parallelism of Extremal Processes and Subordinators

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Abstract

We study the extremal processes through Feller semigroups theory from which it is possible to observe some parallelism with subordinators. Consequently, we observe that an extremal process possesses concepts analogous to those of Laplace exponent and Lévy measure of a subordinator. This approach also yields a short proof of the invariance principle of the extremal processes.

1 Introduction

The origin of the extremal processes dates back to the works of Dwass [Dwa64] and Lamperti [Lam64]. In the former, some properties of these processes are studied and in the latter, an invariance principle is established. Extremal processes have appeared in some asymptotic results of several kinds of stochastic processes, for example: in connection with fractional Brownian motion [KO99], some subordinators [KL13], [Wat80] and more recently with continuous state branching processes [FM19]. In all these works, the stochastic process that appears in the limit is some transformation of the Watanabe extremal process, see example (4.6) for the definition.

Except for some particular cases ([Wat80] and [KL13]), it seems that the fact that every extremal process is a Feller process has gone unnoticed all this time. So, the main purpose of this note is to fill this theoretical gap and clarify the parallelism between extremal processes and subordinators.

Definition 1.1 (Distribution concentrated on $[0, +\infty)$). A distribution function $F$ concentrated in $[0, +\infty)$ is a mapping from $[0, +\infty)$ to $[0, 1]$ such that

1. It is non-decreasing, right-continuous,
2. $\inf\{x > 0 : F(x) > 0\} = 0$ and $\lim_{x \to +\infty} F(x) = 1$.

Nowadays, the next is the classical definition of the extremal processes provided in [RR73] see also [EKM13] p. 250 or [Res13] p. 179.

Definition 1.2 ($F$-Extremal Processes). For any distribution function $F$ concentrated in $[0, +\infty)$, an $F$-extremal process $(M_t)_{t \geq 0}$ on $\mathbb{R}_+ := [0, +\infty)$ is a stochastic process that satisfies

$$
\mathbb{P}(M_{t_1} \leq x_1, M_{t_2} \leq x_2, \ldots, M_{t_m} \leq x_m) = F^{t_1} \left( \bigwedge_{i=1}^m x_i \right) F^{t_2-t_1} \left( \bigwedge_{i=2}^m x_i \right) \cdots F^{t_m-t_{m-1}} \left( x_m \right),
$$

(1)
where $F^t(x) = (F(x))^t$, $0 \leq t_1 < \cdots < t_m$ and $x_1, \cdots, x_m \in [0, +\infty)$.

We define $Q(x) := -\log(F(x))$ and we will call this function the Dwass exponent of the extremal process. The measure $\pi$ on $(0, \infty)$ defined by $\pi(x, \infty) := Q(x)$ will play an important role in this note, and we will call it the Dwass measure of the extremal process. At this point, this terminology might appear gloomy but it will make sense given the results of the next sections. Roughly speaking, the Dwass exponent and the Dwass measure will play the same role as the Laplace exponent and the Lévy measure of a subordinator.

**Remark 1.3.** An $F$-extremal process can be constructed for every distribution function concentrated on $[0, \infty)$; indeed, it is enough to apply Kolmogorov’s extension theorem.

The Definition 1.2 is motivated by the fact that an $F$-extremal process $(M_t)_{t \geq 0}$ is “max” infinitely indivisible, this is, for every $n \in \mathbb{N}$, there exists $\{(M_t^{(i)})_{t \geq 0}\}_{i=1}^n$ i.i.d. $F^{1/n}$-extremal processes such that

$$(M_t)_{t \geq 0} \overset{d}{=} \left(\bigvee_{i=1}^n M_t^{(i)}\right)_{t \geq 0}.$$  

**Organization.** In Section 2, we define and study some of the properties of the extremal Feller semigroups, and we observe that the classical definition of the extremal process coincides with the stochastic process associated to this Feller semigroup. In particular, we obtain a proof of the invariance principle of the extremal process via Feller semigroup theory. In Section 3, we recall the “Lévy-Itô decomposition” of the extremal processes and observe that these also satisfy a kind of Lévy-Khintchine formula. Then we observe that a subordinator and the extremal processes associated to it have the same Laplace symbol. In section 4, we introduce the concept of extremal functional of a Markov process and prove that the time change of an extremal functional by the inverse of the local time at zero of the Markov process is an extremal process, some examples are gathered.

## 2 Feller Semigroup Construction of Extremal processes

The purpose of this section is to provide a construction of the extremal processes via Feller semigroups, which, from this point of view, it is more natural than the usual definition (definition 1.2) found in the literature.

### 2.1 The Extremal Feller Semigroup

**Definition 2.1** (Extremal Convolution). Let $\mu$ and $\nu$ be two probability measures on $\mathbb{R}_+ := [0, \infty)$. The extremal convolution of $\mu$ and $\nu$ denoted by $\mu \gamma \nu$ is the probability measure on $\mathbb{R}_+$, such that

$$\int_{\mathbb{R}_+} f(z) (\mu \gamma \nu)(dz) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(x \vee y) \mu(dx)\nu(dy),$$

(2)
for every \( f \in C_0(\mathbb{R}_+) \). Here and in what follows, \( C_0(\mathbb{R}_+) \) are the continuous functions vanishing at \(+\infty\), that is, \( \lim_{x \to +\infty} f(x) = 0 \).

**Remark 2.2.** Observe that \( f \mapsto \int_{\mathbb{R}_+} f(x \lor y) \mu(dx)\nu(dy) \) is a positive continuous linear functional on \( C_0(\mathbb{R}_+) \), so that the Riesz-Markov theorem guarantees the existence and uniqueness of the probability measure \( \mu \lor \nu \).

Throughout the sequel \( \mathcal{B}(\mathbb{R}_+) \) will denote the Borel \( \sigma \)-field on \( \mathbb{R}_+ \), \( \mathcal{M}_b(\mathbb{R}_+) \) the class of bounded \( \mathcal{B}(\mathbb{R}_+) \)-measurable functions. If \( \mathcal{C} \) is a family of functions from \( \mathbb{R}_+ \) to \( \mathbb{R} \), \( \sigma(\mathcal{C}) \) will denote the smallest \( \sigma \)-field on \( \mathbb{R}_+ \) for which the functions on \( \mathcal{C} \) are Borel-measurable.

**Lemma 2.3** (Characterization of the Extremal Convolution). For \( \mu, \nu \) and \( \lambda \) probabilities measures on \( \mathcal{B}(\mathbb{R}_+) \), \( \mu \lor \nu = \lambda \) if and only if \( \mu([0,a]) \cdot \nu([0,a]) = \lambda([0,a]) \), for every \( a \geq 0 \).

**Proof.** Define \( \mathcal{H} := \{ f \in \mathcal{M}_b(\mathbb{R}_+) : \int_{\mathbb{R}_+} f(z) \lambda(dz) = \int_{\mathbb{R}_+} f(x \lor y) \mu(dx)\nu(dy) \} \).

\( (\Rightarrow) \) Suppose that \( \mu \lor \nu = \lambda \), this means that \( C_0(\mathbb{R}_+) \subseteq \mathcal{H} \). Since \( C_0(\mathbb{R}_+) \) is closed under pointwise multiplication then, by the functional version of the monotone class theorem (see [RY13, p. 3]), \( \mathcal{H} \) contains all the bounded \( \sigma(C_0(\mathbb{R}_+)) \)-measurable functions. Remember that the Baire \( \sigma \)-algebra \( \sigma(C_0(\mathbb{R}_+)) \) coincides with the Borel sigma algebra \( \mathcal{B}(\mathbb{R}_+) \), this is due to the fact that \( \mathbb{R}_+ \) is locally compact, separable and metrizable. So, in particular, \( \mathcal{H} \) contains the functions \( x \mapsto 1_{[0,a]}(x) \) and this clearly imply that \( \mu([0,a]) \cdot \nu([0,a]) = \lambda([0,a]) \), for every \( a \geq 0 \).

\( (\Leftarrow) \) Suppose that \( \mu([0,a]) \cdot \nu([0,a]) = \lambda([0,a]) \), for every \( a \geq 0 \), this means that \( \mathcal{J} \subseteq \mathcal{H} \), where \( \mathcal{J} := \{ 1_{[0,a]} : a \geq 0 \} \). By the same argument as before, we get that \( \mathcal{H} \) contains all the bounded \( \sigma(\mathcal{J}) \)-measurable functions. Once again is well-known that \( \sigma(\mathcal{J}) = \mathcal{B}(\mathbb{R}_+) \). In particular, \( \mathcal{H} \) contains \( C_0(\mathbb{R}_+) \) so that \( \mu \lor \nu = \lambda \). \( \square \)

**Definition 2.4** (Extremal Convolution Semigroup). A family of measures \( (\mu_t)_{t \geq 0} \) on \( \mathcal{B}(\mathbb{R}_+) \) is an extremal convolution semigroup if

1. \( \mu_t \lor \mu_s = \mu_{s+t} \) for \( s, t \geq 0 \),

2. \( \mu_0 = \delta_0 \) and \( \lim_{t \downarrow 0} \mu_t = \delta_0 \) vaguely.

The next result is analogous to the classical construction of subordinators via Feller semigroups which is done for example in [Ber99], or [BSW13 pp. 32-33] and [RY13 p. 96].

**Proposition 2.5.** For \( (\mu_t)_{t \geq 0} \) a family of probability measures on \( \mathcal{B}(\mathbb{R}_+) \), the following conditions are equivalent:

(a) The transition probabilities functions defined by

\[
P_t(x, A) := \int_{\mathbb{R}_+} 1_A(x \lor y)\mu_t(dy), \quad A \in \mathcal{B}(\mathbb{R}_+), \ t \geq 0,
\]

(3) determine a Feller semigroup on \( C_0(\mathbb{R}_+) \), that is, the family \( (P_t)_{t \geq 0} \) of operators on \( \mathcal{M}_b(\mathbb{R}_+) \) defined by \( P_t f(x) := \int_{\mathbb{R}_+} f(x) P_t(x, dy) = \int_{\mathbb{R}_+} f(x \lor y)\mu_t(dy), x \in \mathbb{R}_+ \), is a Feller semigroup on \( C_0(\mathbb{R}_+) \).
(b) The family \((\mu_t)_{t \geq 0}\) is an extremal convolution semigroup.

Moreover, if conditions (a) and/or (b) are satisfied then \(\mu_t([0,a]) = F^a(t)\) for \(a \geq 0\), \(t \geq 0\), for some distribution function \(F\) concentrated on \(\mathbb{R}_+\). Conversely, if \(\mu_t\) is defined by \(\mu_t([0,a]) = F^a(t)\), for \(a \geq 0\), \(t \geq 0\), then \(\[0, a\] \times \mathbb{R}_+\) determines a Feller semigroup on \(C_0(\mathbb{R}_+)\).

**Proof.** The proof of the equivalence of conditions (a) and (b) is analogous to the case of subordinators. Nevertheless, for the sake of completeness, we write it down.

Observe that an application of the dominated convergence theorem guarantees that \(P_t(C_0(\mathbb{R}_+)) \subseteq C_0(\mathbb{R}_+)\). Indeed, \(x \mapsto x \lor y = (x + y + |x - y|)/2\) is a continuous function and if \(x_n \to x\) as \(n \to \infty\), then

\[
\lim_{n \to \infty} P_tf(x_n) = \int_{\mathbb{R}_+} \lim_{n \to \infty} f(x_n \lor y) \mu_t(dy) = \int_{\mathbb{R}_+} f(x \lor y) \mu_t(dy) = P_tf(x),
\]

Now, if \(f \in C_0(\mathbb{R}_+)\), once again the dominated convergence theorem guarantees

\[
\lim_{x \to +\infty} P_tf(x) = \int_{\mathbb{R}_+} \lim_{x \to +\infty} f(x \lor y) \mu_t(dy) = 0, \quad f \in C_0(\mathbb{R}_+)\]

so that \(P_t(C_0(\mathbb{R}_+)) \subseteq C_0(\mathbb{R}_+)\).

\(a) \Leftrightarrow (b)\) Denote by \(f_x\) the function \(f_x : \mathbb{R}_+ \to \mathbb{R}\) defined by \(f_x(y) := f(x \lor y)\). Observe that

\[
P_{s+t}f(x) = \int_{\mathbb{R}_+} f(x \lor y) \mu_{s+t}(dy) = \int_{\mathbb{R}_+} f_x(y) \mu_{s+t}(dy), \quad (4)
\]

moreover

\[
P_tP_s f(x) = \int_{\mathbb{R}_+} P_s f(x \lor z) \mu_t(dz)
= \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(y \lor z \lor x) \mu_t(dz) \mu_s(dy)
= \int_{\mathbb{R}_+ \times \mathbb{R}_+} f_x(y \lor z) \mu_t(dz) \mu_s(dy). \quad (5)
\]

Then the semigroup property of \((P_t)_{t \geq 0}\) is read as

\[
\int_{\mathbb{R}_+} f_x(y) \mu_{s+t}(dy) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} f_x(y \lor z) \mu_t(dz) \mu_s(dy), \quad x \in \mathbb{R}_+, \ f \in C_0(\mathbb{R}_+), \quad (6)
\]

and this is equivalent to the fact that \(\mu_t \lor \mu_s = \mu_{s+t}\), \(s, t \geq 0\). Remember that \(\lim_{t \to 0} \|P_tf - f\| = 0\) for each \(f \in C_0(\mathbb{R}_+)\) is equivalent to \(\lim_{t \to 0} P_tf(x) = f(x), \ x \in \mathbb{R}_+, \ f \in C_0(\mathbb{R}_+)\), see for example [RY13 p. 89], but this is condition 2 of Definition 2.4. We have verified the equivalence of (a) and (b).

Let us verify the last statements of the proposition. Suppose (b), define \(F_t(a) := \mu_t([0,a])\) for \(a, t \geq 0\), by Lemma 2.3 and the fact that \(\mu_{s+t} = \mu_s \lor \mu_t\), we have

\[
F_s(a)F_t(a) = F_{s+t}(a), \quad (7)
\]
and observe that $t \mapsto F_t(a)$ is decreasing. Then the solutions to the Cauchy equations (7) are given by $F_t(a) = (F_1(a))^t$. On the other hand, from the condition 2 in Definition 2.4 it follows that
\[
\text{lim}_{t \to 0} F_t(a) = \text{lim}_{t \to 0} (F_1(a))^t = \delta_0([0,a]) = 1, \quad \text{for each } a > 0, \quad \text{since } \delta_0(\partial_{\mathbb{R}_+}[0,a]) = \delta_0(\{a\}) = 0.
\]
Then $F_1(a) > 0$ for every $a > 0$. So that $\mu_t([0,a]) = (F_1(a))^t$, $a \geq 0$, $t \geq 0$, and $F_t$ is a distribution function concentrated on $[0,\infty)$.

Conversely, suppose that $\mu_0 = \delta_0$ and that $\mu_t$ is defined by $\mu_t([0,a]) = F^t(a)$, $t > 0$, for $F$ a distribution function concentrated on $[0,\infty)$. Define the distribution function $\tilde{F}$ on $\mathbb{R}$, as $\tilde{F}(x) := 1_{\mathbb{R}_+}(x)F(x)$, then $\tilde{F}^t$ converges weakly on $\mathbb{R}_+$ to $\delta_0$. Indeed $\delta_0((-\infty,a]) = 1_{\{a\geq 0\}}$, so that the points of continuity of $a \mapsto \delta_0((-\infty,a])$ are $(-\infty,0) \cup (0, +\infty)$ and since $0 < \tilde{F}^t(a) \leq 1$ for each $a > 0$ then $\text{lim}_{t \to 0} \tilde{F}^t(a) = 1$, so that $\tilde{F}^t$ converges weakly to $\delta_0$ on $\mathbb{R}$, and this clearly implies condition 2 in Definition 2.4. And condition 1 of Definition 2.4 follows from Lemma 2.3.

**Definition 2.6 (Extremal Feller Semigroup).** A Feller semigroup $(T_t)_{t \geq 0}$ is an extremal Feller semigroup if it has the form of the Feller semigroup in Proposition 2.5, that is, if it can be written as
\[
T_t f(x) = \int_{\mathbb{R}_+} f(x \lor y) F^t(dy), \quad f \in C_0(\mathbb{R}_+),
\]
for a distribution function $F$ concentrated on $\mathbb{R}_+$.

As before, we will call the function $Q$ defined by $Q(x) = -\log(F(x))$ the Dwass exponent of the extremal Feller semigroup and the measure $\pi$ on $(0,\infty)$ defined by $\pi(x,\infty) = Q(x)$ the Dwass measure of the extremal Feller semigroup.

**Remark 2.7.** In the literature, one can find this semigroup written in the equivalent but non compact form $T_t g(x) = g(x)F^t(x) + t \int_{(x,\infty)} g(y)F^{t-1}(y)f(y)dy$, for some very particular distribution functions $F$, such that $F(dy) = f(y)dy$, see for example [RY13, p. 504] or [Dwa64]. But of course, this does not reveal the underlying structure of the extremal processes.

Remember (see [Bas81]) that a Lipschitz semigroup $(T_t)_{t \geq 0}$ with Lipschitz constant $e^{Kt}$ is a Feller semigroup satisfying $\|T_t u\|_{\text{Lip}} \leq e^{Kt}\|u\|_{\text{Lip}}$ for $u \in \text{Lip} (\mathbb{R}_+)$, where
\[
\text{Lip} (\mathbb{R}_+) := \left\{ u \in C(\mathbb{R}_+) : \|u\|_{\text{Lip}} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|} < \infty \right\}.
\]

**Lemma 2.8.** For an extremal Feller semigroup $(T_t)_{t \geq 0}$, we have
\[
(a) \quad T_t f(x) = -\int_{(x,\infty)} f'(y)F^t(y)dy \quad \text{for } f \in C_0^1([0, +\infty)),
\]
\[
(b) \quad (T_t)_{t \geq 0} \text{ is Lipschitz semigroup with Lipschitz constant equal to 0},
\]
\[
(c) \quad T_t (C_c(\mathbb{R}_+)) \subseteq C_c(\mathbb{R}_+).
\]
Proof. (a) Observe that \( \int_{\mathbb{R}_+} f(x \vee y) F^t(dy) = f(x) F^t(x) + \int_{(x, \infty)} f(y) F^t(dy) \), whence the result follows from the integration by parts formula \( \int_{(x, \infty)} f(y) F^t(dy) + \int_{(x, +\infty)} F^t(y) f'(y)dy + f(x) F^t(x) = 0 \).

(b) Since a Lipschitz function is absolutely continuous and its derivative is essentially bounded by the Lipschitz constant, hence by the same argument as before

\[
|T_tf(x) - T_tf(y)| \leq \int_{(x,y)} |f'(z)| F^t(z)dy \leq (y-x) \cdot \|f\|_{\text{Lip}}, \quad f \in \text{Lip}(\mathbb{R}_+),
\]

for \( 0 \leq x < y \).

(c) Just observe that if \( f(x) = 0 \) for all \( x \geq a \), with \( a \in \mathbb{R}_+ \), then \( T_tf(x) = \int_{[0,a]} f(x \vee y) F^t(dy) = 0 \) for all \( x \geq a \). \( \square \)

Consider the operator \( \tau_x : C_0(\mathbb{R}_+) \rightarrow C_0(\mathbb{R}_+) \), defined by \( (\tau_x f)(y) := f(x \vee y) \). The extremal Feller semigroups are the only Feller semigroups that are invariant under this mapping, more precisely.

**Lemma 2.9** (Characterization of an Extremal Feller Semigroup). A Feller semigroup \( (T_t)_{t \geq 0} \) is an extremal Feller semigroup if and only if

\[
T_t(\tau_x f) = \tau_x(T_t f), \quad f \in C_0(\mathbb{R}_+), \; t \geq 0.
\]

**Proof.** \( (\Leftarrow) \) By Riesz-Markov's theorem \( T_tf(x) = \int_{\mathbb{R}_+} f(y) \mu_t(x, dy) \), for some transition probabilities \( \mu_t(x, dy), \; x, t \geq 0 \). By (9), we have

\[
T_tf(x) = \tau_x(T_tf)(0) = T_t(\tau_x f)(0) = \int_{\mathbb{R}_+} f(x \vee y) \mu_t(0, dy), \quad f \in C_0(\mathbb{R}_+), \; t \geq 0,
\]

and, by Proposition 2.5, we are done.

\( (\Rightarrow) \) It is trivial. \( \square \)

**Lemma 2.10.** The measures \( \nu_t(dy) := t^{-1}F^t(dy) \), restricted to \( (0, \infty) \), converge vaguely to \( \pi \) as \( t \to 0 \).

**Proof.** Let \( K \) a compact subset of \( (0, \infty) \), and \( a_K, b_K > 0 \) such that \( K \subseteq (a_K, b_K] \). Observe that

\[
\nu_t(K) \leq \frac{F^t(b_K) - F^t(a_K)}{t} = \frac{e^{-tQ(b_K)} - e^{-tQ(a_K)}}{t} \leq Q(a_K) - Q(b_K), \quad t > 0,
\]

so that \( \{\nu_t\}_{t>0} \) is vaguely relatively compact, see [Bou13, Ch. III, §4, No. 9]. Finally, the only possible limit is \( \pi \) since \( \lim_{t \to 0} \nu_t(a, b] = Q(a) - Q(b) = \pi(a, b] \), for every \( a, b > 0 \). \( \square \)

**Proposition 2.11** (Infinitesimal Generator of an Extremal Feller Semigroup). The infinitesimal generator \( (A, \mathcal{D}(A)) \) of an extremal Feller semigroup \( (\mathcal{E}) \) satisfies \( \mathcal{D}(A) = C_0(\mathbb{R}_+) \), moreover

\[
Af(x) = \int_0^\infty f(x \vee y) - f(x) \mu(dy), \quad f \in C_c((0, +\infty), \quad (10)
\]

where \( \pi \) is the Dwass measure of the extremal process.
Remark 2.12. Once again observe that in the formula (10) it can be appreciated the parallelism between the extremal processes and the subordinators. Indeed, the infinitesimal generator of a driftless subordinator satisfies
\[ Af(x) = \int_{0}^{\infty} f(x + y) - f(x) \Pi(dy), \quad f \in C_{0}^{1}[0, \infty), \]
where \( \Pi \) is the Lévy measure of the subordinator.

Proof. It is enough to compute the infinitesimal generator in a pointwise way, \cite[pg. 43]{Paz12}. We have
\[ \frac{T_{t}f(x) - f(x)}{t} = \int_{0}^{\infty} f(x \vee y) - f(x) \nu_{t}(dy), \quad t > 0. \]
But since \( y \mapsto f(x \vee y) - f(x) \) is an element of \( C_{c}((0, \infty)) \), then the conclusion follows from Lemma 2.10.

The property of being an extremal process is preserved after Bochner’s subordination, as it happens with subordinators. Remember that if \( \psi \) is the Laplace exponent of a subordinator \((S_{t})_{t \geq 0}\), the subordinated semigroup denoted by \((T_{t}^{\psi})_{t \geq 0}\) of the semigroup \((T_{t})_{t \geq 0}\) with respect to \( \psi \) is defined by
\[ T_{t}^{\psi}f := \int_{\mathbb{R}^{+}} T_{s}f \mu_{t}(ds), \quad t \geq 0, \]
where \( \mu_{t}(ds) := \mathbb{P}(S_{t} \in ds) \). For more information about subordinated semigroups see \cite[Ch. 13]{SSV12}.

Proposition 2.13. Let \((T_{t})_{t \geq 0}\) be an extremal Feller semigroup with Dwass exponent \( Q \) and \( \psi \) be the Laplace exponent of a subordinator, then the subordinated semigroup \((T_{t}^{\psi})_{t \geq 0}\) is an extremal Feller semigroup with Dwass exponent \( Q^{\psi} \) given by
\[ Q^{\psi}(x) = \psi(Q(x)). \]

Proof. Clearly, the subordinated semigroup \((T_{t}^{\psi})_{t \geq 0}\) satisfies (9) since \((T_{t})_{t \geq 0}\) does, so that by Lemma 2.9 the Feller semigroup \((T_{t}^{\psi})_{t \geq 0}\) is an extremal Feller semigroup. By the Lemma (2.8), we see that, for \( f \in C_{0}^{1}([0, +\infty)) \)
\[ T_{t}^{\psi}f(x) = \int_{\mathbb{R}^{+}} T_{s}f(x) \mu_{t}(ds) \]
\[ = -\int_{\mathbb{R}^{+}} \int_{(x, \infty)} F^{s}(y) f'(y) dy \mu_{t}(ds) \]
\[ = -\int_{(x, \infty)} f'(y) \int_{\mathbb{R}^{+}} e^{-sQ(y)} \mu_{t}(ds) dy \]
\[ = -\int_{(x, \infty)} f'(y)e^{-t\psi(Q(y))} dy. \]
So that \( Q^{\psi}(y) = \psi(Q(y)). \)
2.2 The Equivalent Definitions of the Extremal Processes

It turns out that the Feller process (started at 0) with Feller semigroup given by (8) coincides with the Definition of an $F$-extremal process, as the next proposition shows.

**Proposition 2.14 (Agreement of the Definitions).** Let $(X_t)_{t \geq 0}$ be a stochastic process (starting at 0) with Feller semigroup given by (8). Then, it satisfies (1), that is

$$
\mathbb{P}_0 (X_{t_1} \leq x_1, \ldots, X_{t_m} \leq x_m) = \prod_{j=1}^{m} F^{t_j - t_{j-1}} \left( \bigwedge_{i=j}^{m} x_i \right),
$$

where $0 = t_0 < t_1 < \cdots < t_m$.

**Proof.** The finite-dimensional distributions are given by

$$
\int_{\mathbb{R}_+} P_{t_1}(0, dy_1) 1_{\{y_1 \leq x_1\}} \cdots \int_{\mathbb{R}_+} P_{t_m-t_{m-1}}(y_{m-1}, dy_m) 1_{\{y_m \leq x_m\}}
$$

$$
= \int_{(\mathbb{R}_+)^{m}} \prod_{i=1}^{m} 1_{\{y_i \leq x_i\}} P_{t_m-t_{m-1}}(y_{m-1}, dy_m) \cdots P_{t_1}(0, dy_1)
$$

$$
= \int_{(\mathbb{R}_+)^{m}} \prod_{i=1}^{m} 1_{\{y_i \leq x_i\}} 1_{\{y_{m-1} \vee y_m \leq x_m\}} \mu_{t_m-t_{m-1}}(dy_m) P_{t_{m-1}-t_{m-2}}(y_{m-2}, dy_{m-1}) \cdots P_{t_1}(0, dy_1)
$$

$$
= \int_{(\mathbb{R}_+)^{m}} \prod_{i=1}^{m} 1_{\{y_i \leq x_i\}} 1_{\{y_{m-1} \vee y_m \leq x_m\}} \mu_{t_m-t_{m-1}}(dy_m) \cdots \mu_{t_1}(dy_1)
$$

$$
= \int_{(\mathbb{R}_+)^{m}} \prod_{i=1}^{m} 1_{\{y_i \leq x_i\}} \mu_{t_m-t_{m-1}}(dy_m) \cdots \mu_{t_1}(dy_1)
$$

$$
= \int_{(\mathbb{R}_+)^{m}} \prod_{i=1}^{m} 1_{\{y_i \leq x_i\}} \mu_{t_m-t_{m-1}}(dy_m) \cdots \mu_{t_1}(dy_1)
$$

$$
= \int_{(\mathbb{R}_+)^{m}} \prod_{i=1}^{m} 1_{\{y_i \leq x_i\}} \mu_{t_m-t_{m-1}}(dy_m) \cdots \mu_{t_1}(dy_1)
$$

$$
= \prod_{j=1}^{m} F^{t_j - t_{j-1}} \left( \bigwedge_{i=j}^{m} x_i \right)
$$

It is usually proved (or at least mentioned) that an extremal process is stochastically continuous, Markovian, and has a cadlag version, see [Res13, Proposition 4.7]. All these properties now follow from the general theory of Feller processes and the fact that every extremal process is a Feller process.

We can now give an alternative definition of the extremal processes which is more convenient in several situations, as we will see in the next sections.
**Definition 2.15 (Extremal Process).** We say that \((X_t)_{t \geq 0}\) is an extremal process if

1. \(X_0 = 0\)

2. \((X_t)_{t \geq 0}\) admits a right-continuous version.

3. \(X_{s+t} \overset{d}{=} X'_s \vee X_t, \ t, s \geq 0,\) where \(X'_s \overset{d}{=} X_s\) and \(X'_s \perp X_t.\) That is, \((X'_t)_{t \geq 0}\) is an independent copy of \((X_t)_{t \geq 0}\) that could be defined in a larger probability space.

**Theorem 2.16.** For an extremal process \((X_t)_{t \geq 0}\), the family of operators \((T_t)_{t \geq 0}\), on \(M_b(\mathbb{R}^+)\), defined by

\[
T_t f(x) = \mathbb{E}[f(x \vee X_t)], \quad t \geq 0,
\]

is an extremal Feller semigroup.

**Proof.** Observe that, for \(t, s \geq 0\)

\[
T_{s+t} f(x) = \mathbb{E}[f(x \vee X_{s+t})] = \mathbb{E}[f(x \vee X_t \vee X'_s)] = \mathbb{E} \left[ \mathbb{E}[f(X_t \vee y)] \big| y = x \vee X'_s \right] = \mathbb{E} \left[ T_t f(x \vee X'_s) \right] = T_s T_t f(x),
\]

so that \((T_t)_{t \geq 0}\) has the semigroup property. Since \((X_t)_{t \geq 0}\) admits a right-continuous version, then

\[
\lim_{t \to 0} T_t f(x) = \lim_{t \to 0} \mathbb{E}[f(x \vee X_t)] = \mathbb{E}[f(x \vee X_0)] = f(x), \text{ for } f \in C_0(\mathbb{R}^+).
\]

The continuity of \(x \mapsto x \vee y\) implies that \(T_t(C_0(\mathbb{R}^+)) \subseteq C_0(\mathbb{R}^+).\) Then \((T_t)_{t \geq 0}\) is a Feller semigroup and clearly satisfies \([9],\) hence is an extremal Feller semigroup. \(\square\)

### 2.3 Invariance Principle of the Extremal Processes

We now will exploit the fact that the extremal processes are Feller processes to obtain a short proof of the invariance principle of the extremal processes, which was first observed in [Lam64]. Another indirect proof appears in [Res13, pp. 211-215] which is based on convergence of point processes and it turns out to be quite long.

Consider the Markov chains \(\{X^{(n)}_k(x); \ k \geq 0\}_{n \in \mathbb{N}}\) defined by

\[
X^{(n)}_k(x) := \frac{\sum_{i=0}^{k} \xi^{(n)}_i(x) - a_n}{b_n} := \frac{(b_n x + a_n) \vee \bigvee_{i=1}^{k} \xi_i - a_n}{b_n},
\]

here \(\{\xi_i\}_{i=1}^{\infty}\) are i.i.d. with distribution function \(F\) and \(b_n > 0.\)

**Theorem 2.17.** The stochastic process defined by \(Y^{(n)}_t := X^{(n)}_{nt}(0), \ t \geq 0,\) converges to the \(G\)-extremal process in the Skorokhod’s space \(J_1\) if \(G_n(x) := F^n(b_n x + a_n) \to G(x)\) as \(n \to \infty.\)
Proof. Let us first observe that \( \mathcal{D} := \text{Lip}(\mathbb{R}_+) \cap C_c(\mathbb{R}_+) \) is a core, it is enough to see that \( T_t(\mathcal{D}) \subseteq \mathcal{D} \) and \( \mathcal{D} \) is dense on \( C_0(\mathbb{R}_+) \), see [Kal02, p. 374]. The fact that \( T_t(\mathcal{D}) \subseteq \mathcal{D} \) follows from Lemma 2.8. Since \( \text{Lip}(\mathbb{R}_+) \cap C_c(\mathbb{R}_+) \) is dense on \( C_c(\mathbb{R}_+) \) (see [Car00]) and \( C_0(\mathbb{R}_+) \) is dense on \( C_0(\mathbb{R}_+) \) then \( \mathcal{D} \) is dense on \( C_0(\mathbb{R}_+) \).

It is enough to apply the well-known result about weak convergence of Feller processes that appears in [Kal02, Theorem 19.28]. Observe that \( \mathbb{P}[X_k^n(x) \leq y] = 1_{\{x \leq y\}} F^k(b_n y + a_n) \). By the same reasoning as in Lemma (2.8), we get that
\[
T_t^{(n)} f(x) := \mathbb{E} \left[ f(X_{nt}^{(n)}(x)) \right] = f(x) F^{nt}(b_n x + a_n) + \int_{(x,\infty)} f(y) F^{nt}(b_n dy + a_n)
\]
for \( f \in \mathcal{D} \). If \( T_t^G f(x) := \int_{\mathbb{R}_+} f(x \lor y) G^t(dy) \) then
\[
\sup_{x \in \mathbb{R}_+} \left| T_t^{(n)} f(x) - T_t^G f(x) \right| \leq \int_{\mathbb{R}_+} \left| G^{nt/n}(y) - G^t(y) \right| \cdot |f'(y)| dy, \quad f \in \mathcal{D},
\]
since \( [nt]/n \to t \) as \( n \to \infty \), and an application of the dominated convergence theorem guarantees that \( T_t^{(n)} f \) converges strongly to \( T_t^G f \).

\[ \square \]

3 Poisson Construction of Extremal Processes

3.1 Lévy-Itô Decomposition and Lévy Khintchine Formula

It is possible to construct the extremal processes via Poisson random measure. The following result is not new, see [Res13, p. 180].

Lemma 3.1 (Lévy-Itô Descomposition). Let \( N = \sum_{i \in \mathbb{I}} \delta_{(s_i, x_i)} \) a Poisson point process on \( \mathbb{R}_+ \times (0, \infty) \), with intensity \( dt \times \mu \) such that \( 0 < \pi(x, \infty) < \infty \) for every \( x > 0 \). Then \( X_t := \bigvee_{s_i \leq t} x_i := \sup_{s_i \leq t} x_i \) is an extremal process with Dwass measure \( \pi \).

Remark 3.2. Once more, observe the parallelism existing with subordinators, indeed if \( \sum_{t_i \leq t} \) is written instead of \( \bigvee_{t_i \leq t} \), a subordinator is obtained as long as \( \pi \) is a Lévy measure.

In particular, for every subordinator, it is possible to associate an extremal process in the following way, which will be useful in the incoming subsection.

Corollary 3.3 (Extremal Process Associated to a Spectrally Positive Lévy Process). Let \( (Y_t)_{t \geq 0} \) be a spectrally positive Lévy Process with Lévy measure II. Define the stochastic process \( (X_t)_{t \geq 0} \) by
\[
X_t := \sup_{0 \leq s \leq t} \Delta Y_s, \quad t \geq 0,
\]
where $\Delta Y_s := Y_s - Y_{s-}$, $s > 0$, with $Y_{s-}$ the left limit of $s \mapsto Y_s$ at $s$. Then $(X_t)_{t \geq 0}$ is an extremal process with Dwass measure $\pi = \Pi$. We will say that $(X_t)_{t \geq 0}$ is the extremal process associated to the spectrally positive Lévy processes $(Y_t)_{t \geq 0}$.

Observe that $(Y_t)_{t \geq 0}$ could have purely discontinuous and strictly increasing sample paths, but the sample paths of $(X_t)_{t \geq 0}$ are piecewise constant.

**Example 3.4** (Cauchy Process $\to$ Watanabe’s Extremal Process). Recall that a (one-sided) Cauchy processes $(Y_t)_{t \geq 0}$ is a Lévy process with Lévy measure given by $\Pi(dx) = 1_{\{x>0\}}/x^2dx$ where the constant $\theta > 0$, its associated extremal process $(X_t)_{t \geq 0}$ has a distribution function given by

$$F(dx) = e^{-\frac{1}{x}}dx.$$  

The extremal process $(X_t)_{t \geq 0}$ is known as Watanabe’s process or canonical extremal process, see [RY13, p. 504] and [KO99]. This Poisson representation of Watanabe’s process was used by Kasahara to give a proof of Kotani’s lemma, see [IH06, p. 90]. Another remarkable feature of the canonical extremal process is that it is a $1$-self similar process, i.e. $\{X_{at}, t \geq 0\} \overset{d}{=} \{aX_t, t \geq 0\}$ for every $a > 0$, just as the Cauchy process.

**Example 3.5** (Stable Processes $\to$ Fréchet Extremal Processes). For $\alpha \in (1,2)$, consider a spectrally positive $\alpha$-stable Lévy process $(Y_t)_{t \geq 0}$, that is, $(Y_t)_{t \geq 0}$ has Lévy measure given by $\Pi(dx) = c1_{\{x>0\}}/x^{1+\alpha}dx$ where the constant $c > 0$, its associated extremal process $(X_t)_{t \geq 0}$ has a distribution function given by

$$F(dx) = e^{-\frac{c}{\alpha}x^{\alpha}}dx.$$  

The extremal process $(X_t)_{t \geq 0}$ is known as the $\alpha$-Fréchet process.

**Corollary 3.6** (Lévy-Khintchine Formula). With the notation of the previous lemma

$$E[1_{\{X_t \leq \lambda\}}] = \exp\left(-t \int_{(0,\infty)} \left(1 - 1_{\{x \leq \lambda\}}\right) \pi(dx)\right).$$

**Remark 3.7.** Once again, observe that if $e^{-\lambda x}$ is written instead of $1_{\{x \leq \lambda\}}$, the Lévy-Khintchine formula for subordinators is obtained.

4 Extremal Functionals of a Markov Process

Consider a Brownian motion $t \mapsto B_t$ and $t \mapsto \tau_t$ the right inverse of its local time at zero. As far as we know, the fact that $t \mapsto \sup_{s \leq \tau_t} B_s$ is an extremal process dates back, at least, to the work of Kasahara [Kas82]. The purpose of this section is to generalize this observation and to put it in a clearer context. In particular, it will be possible to associate with each (nice) Markov process an extremal process and its Dwass exponent can be given in terms of the excursion measure.

The next definition is analogous to that of an additive functional, but we write a $\lor$ where it usually appears a $+$, just as we have been doing all this time.
Definition 4.1 (Extremal Functional). A stochastic process \((E_t)_{t \geq 0}\) is an extremal functional of a Markov processes \(X = (\Omega, F_t, X_t, \theta_t, P_x)\) if it satisfies:

1. \(E_0 = 0\),
2. \((E_t)_{t \geq 0}\) is adapted to \((F_t)_{t \geq 0}\),
3. \(E_{s+t} = E_t \vee (E_s \circ \theta_t)\) for \(s, t \geq 0\).

Lemma 4.2. With the notation of the previous definition:

1. The stochastic process defined by \(E_t := \sup_{s \leq t} f(X_s)\), \((t \geq 0)\),

is an extremal functional.

2. Given an additive functional \((A_t)_{t \geq 0}\) the stochastic process defined by

\[ E_t := \sup_{s \leq t} f(X_s) \cdot \Delta A_s, \quad (t \geq 0), \]

is an extremal functional.

3. If \(M_t^{(a)} := 1_{\{E_t \leq a\}}\) then \((M_t^{(a)})_{t \geq 0}\) is a multiplicative functional.

Proof. To prove 2, note that if \((A_t)_{t \geq 0}\) is an additive functional, then \(\Delta A_{s+t} = (\Delta A_s) \circ \theta_t\), which yields

\[ E_{s+t} = \sup_{u \leq s+t} f(X_u) \cdot \Delta A_u \]

\[ = \left( \sup_{u \leq s} f(X_u) \cdot \Delta A_u \right) \vee \left( \sup_{t<s \leq s+t} f(X_u) \cdot \Delta A_u \right) \]

\[ = \left( \sup_{u \leq s} f(X_u) \cdot \Delta A_u \right) \vee \left( \sup_{u \leq s+t} f(X_{u+t}) \cdot \Delta A_{u+t} \right) \]

\[ = E_t \vee (E_s \circ \theta_t). \]

Part 1, follows in a similar way. To prove 3, note that if \((E_t)_{t \geq 0}\) is an extremal functional then

\[ M_{s+t}^{(a)} = 1_{\{E_{s+t} \leq a\}} = 1_{\{E_t \leq a\} \vee (E_s \circ \theta_t) \leq a} = 1_{\{E_t \leq a\}} \cdot (1_{\{E_s \leq a\} \circ \theta_t}) = M_t^{(a)} \cdot (M_s^{(a)} \circ \theta_t) \]

Proposition 4.3 (Change of Time of an Extremal Functional). Let \(X = (\Omega, F_t, X_t, \theta_t, P_x)\) be a Markov process. Suppose that there exists a family of local times \((L_t)_{t \geq 0}\) at 0, such that \(\lim_{t \to +\infty} L_t = \infty\), and consider \(\tau_t := \inf\{s \geq 0 : L_s > t\}\) the right inverse of the local time. Then for any càdlàg extremal functional \((E_t)_{t \geq 0}\), the stochastic process \((E_{\tau_t})_{t \geq 0}\) is an extremal process under \(P_0\).
Proof. Let us stress that if $S$ and $T$ are two stopping times then $(E_S) \circ \theta_T$ is the map $\omega \mapsto E_{S(\theta_T(\omega))}(\theta_T(\omega))$, whereas $E_{S(\theta_T)}$ is the map $\omega \mapsto E_{S(\omega)}(\theta_T(\omega))$. Now, observe that

$$(E_{\tau_{s+1}})(\omega) = (E_{\tau_{s+1}} - \tau_{s+1})\left(\theta_{\tau_s}(\omega)\right)$$

$$(E_{\tau_{s+1}})(\omega) = (E_{\tau_s})(\omega) \vee (E_{\tau_{s+1}})(\theta_{\tau_s}(\omega))$$

$$(E_{\tau_{s+1}})(\omega) = (E_{\tau_s})(\omega) \vee (E_{\tau_s} \circ \theta_{\tau_s}(\omega))$$

then

$$\mathbb{P}_0 \left[ E_{\tau_s} \leq x, E_{\tau_s} \circ \theta_{\tau_s} \leq y \right] = \mathbb{E}_0 \left[ 1_{\left\{ E_{\tau_s} \leq x \right\}} \cdot 1_{\left\{ E_{\tau_s} \circ \theta_{\tau_s} \leq y \right\}} \right]$$

$$= \mathbb{E}_0 \left[ 1_{\left\{ E_{\tau_s} \leq x \right\}} \mathbb{E}_0 \left[ 1_{\left\{ E_{\tau_s} \leq y \right\}} \mid \mathcal{F}_{\tau_s} \right] \right]$$

$$= \mathbb{E}_0 \left[ 1_{\left\{ E_{\tau_s} \leq x \right\}} \mathbb{E}_{x_{\tau_s}} \left[ 1_{\left\{ E_{\tau_s} \leq y \right\}} \right] \right]$$

$$= \mathbb{P}_0 \left[ E_{\tau_s} \leq x \right] \cdot \mathbb{P}_0 \left[ E_{\tau_s} \leq y \right],$$

that is $E_{\tau_s} \perp E_{\tau_s} \circ \theta_{\tau_s}$ and $E_{\tau_s} \circ \theta_{\tau_s} \overset{d}{=} E_{\tau_s}$, therefore $(E_{\tau_s})_{t \geq 0}$ is an extremal process in the sense of Definition 2.15. \qed

Remark 4.4. Compare this result with the known fact that if $(A_t)_{t \geq 0}$ is an additive functional of a Markov process then $(A_{\tau_t})_{t \geq 0}$ is a subordinator, see for example [Ber96, p. 120-121].

Corollary 4.5. The stochastic process $(S(t))_{t \geq 0}$ is an extremal process, where $S_t := \sup_{s \leq t} X_s$. Moreover, if $n$ denotes the excursion measure associated to the Markov process $(X_t)_{t \geq 0}$, then the Dwass exponent of this extremal process is given by

$$Q(x) = n \left( u : \sup_{t < R} u(t) > x \right),$$

where $R$ is the length of the excursion.

Proof. Consider $\Lambda_x = \{ u : \sup_{t < R} u(t) > x \}$ then

$$\{ S(t) \leq x \} = \{ T_{\Lambda_x} > t \},$$

where $T_{\Lambda_x} := \inf\{ t > 0 : \Delta N_{\Lambda_x}^u > 0 \}$ with $N_{\Lambda_x}^u = \sum_{u \leq s} 1_{\Lambda_x} e(u), s \geq 0$, here $(e_t)_{t \geq 0}$ is the excursion process associated to $(X_t)_{t \geq 0}$. Remember that $s \mapsto N_{\Lambda_x}^u$ is an homogeneous Poisson process with intensity $n(\Lambda_x)$ then

$$\mathbb{P} \left[ S(t) \leq x \right] = e^{-t \cdot n(\Lambda_x)}.$$
The quantity \( n(u : \sup_{t < R} u(t) > x) \) often appears in the theory of Markov excursions. In this context, it acquires an interesting interpretation.

**Example 4.6 (The Extremal Watanabe Process).** In the particular case where \((X_t)_{t \geq 0}\) is a standard Brownian motion, the resulting extremal process is the Watanabe’s process, see \([RY13, p. 504]\) and \([KO99]\).

**Example 4.7.** For \((X_t)_{t \geq 0}\) a spectrally negative Lévy process, with 0 a regular point for itself and characteristic exponent \(\Psi\) satisfying \(\int_{-\infty}^{+\infty} \Re \left( \frac{1}{1+\Psi(y)} \right) \, dy < +\infty\), we have

\[
Q(x) = 1/\lfloor L_{T_x} \rfloor = \pi/\int_{-\infty}^{+\infty} (1 - \cos(xy)) \cdot \Re \left( \frac{1}{\Psi(y)} \right) \, dy.
\]

Here \(T_x\) is the first passage time at \(x\) and \(\Re(z)\) is the real part of \(z \in \mathbb{C}\). See \([Ber96, p. 144]\) for the proof of the last equality.

The following example provides an alternative approach to Proposition 6 from \([FM19]\). There can be found a complete and deep relation between flows of continuous state branching processes and extremal processes.

Let us present the *height process* \((H_t)_{t \geq 0}\) introduced by Le Gall and Le Jan \([LGLJ98]\). \((H_t)_{t \geq 0}\) is obtained as a functional \(H(X)\) of a spectrally positive Lévy process \((X_t)_{t \geq 0}\) with exponent \(\psi\), more precisely, for any \(t > 0\), we define the Lévy process reversed at time \(t\) by \(\tilde{X}_s^{(t)} := X_t - X_{t-s}^-\) for \(0 \leq s \leq t\) (with the convention \(X_0^- = 0\)), and consider its associated supremum \(\tilde{S}_s^{(t)} = \sup_{r \leq s} \tilde{X}_r^{(t)}\). Since \((\tilde{S}_s^{(t)} - \tilde{X}_s^{(t)})_{s \leq t}\) has the same law as the so-called reflected process \((S_s - X_s)_{s \leq t}\) for which 0 is a regular point, then we define \(H_t\) as the (markovian) local time at level zero at time \(t\) of the process \((\tilde{S}_s^{(t)} - \tilde{X}_s^{(t)})\). In order to complete the definition let us specify the normalization of the local time: let \((L_t^{-1})_{t \geq 0}\) be the right inverse of the local time \((L_t)_{t \geq 0}\) at zero of \((S_t - X_t)_{t \geq 0}\), then \((X_t^{-1})_{t \geq 0}\) is a (possibly killed) subordinator, the so-called ladder height process, whose Laplace exponent is given by \(\mathbb{E}[\exp\{-\lambda X_t^{-1}\}] = e^{-c t \psi(\lambda)/\lambda}\), where the positive constant \(c\) depends on the normalization of \((L_t)_{t \geq 0}\). We fix the normalization so that \(c = 1\). Let us recall the Ray-Knight theorem for the height process \((H_t)_{t \geq 0}\). For any \(x, t \geq 0\) the local time \(L_t^x\) of \(H\) at time \(t\) and at level \(x\) can be defined via the approximation (see \([DLG02, Prop. 1.3.3]\))

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{s \leq t} \left| \frac{1}{\varepsilon} \int_0^s 1\{x<H_r,x+s\} \, dr - L^x_s \right| \right] = 0.
\]

Moreover, the following occupation time formula is satisfied

\[
\int_0^t g(H_s) \, ds = \int_0^{+\infty} g(a)L^a_t \, dx
\]

for every nonnegative measurable function \(g\) on \(\mathbb{R}_+\) and any \(t \geq 0\). Next, for any \(t > 0\), set \(\tau_t = \inf\{s \geq 0 : X_s = -t\}\). Then, Theorem 1.4.1 in \([DLG02]\) asserts that \((L^x_{\tau_t})_{x \geq 0}\) is a continuous state branching process started at \(t\).
Proposition 4.8 (Extinction Process of CSBP flows). Let \( ((L^y_{\tau t})_{y \geq 0}; t \geq 0) \) be a flow of continuous state branching processes with branching mechanism \( \psi \). Suppose that extinction always occurs, that is \( \int_1^\infty \frac{du}{\psi(u)} < \infty \). Then the extinction process \( T_x = \inf\{y > 0 : L^y_{\tau t} = 0\} \) is given by in terms of the height process \((H_t)_{t \geq 0}\) by the identity

\[
T_x = \sup_{s \leq \tau_x} H_s, \quad x \geq 0. \tag{14}
\]

Moreover, \((T_x)_{x \geq 0}\) is an extremal process with Dwass exponent \( Q \) given in terms of the branching mechanism \( \psi \) by the identity

\[
\int_0^\infty \frac{du}{Q(u)} = x, \quad x \geq 0. \tag{15}
\]

Proof. Let us verify the identity (14). By the occupation time formula (13) applied for the constant function \( g \equiv 1 \) at time \( \tau_x \), we have

\[
\tau_x = \int_0^\infty L^s_{\tau_x} ds = \int_0^{T_x} L^s_{\tau_x} ds = \int_0^{\tau_x} 1_{\{0 \leq H_s \leq T_x\}} ds,
\]

where the last equality follows by applying one more time the occupation time formula (13). By Theorem 1.4.3. from [DLG02], it follows that \((H_t)_{t \geq 0}\) has continuous sample paths a.s. therefore \( \sup_{s \leq \tau_x} H_s \leq T_x \). By the same reasoning as before, for every \( \epsilon > 0 \), we have

\[
\tau_x < \int_0^{T_x} 1_{\{0 \leq H_s \leq T_x - \epsilon\}},
\]

which implies that \( \sup_{s \leq \tau_x} H_s > T_x - \epsilon \), letting \( \epsilon \) goes to zero, we obtain that \( \sup_{s \leq \tau_x} H_s \geq T_x \) and the identity (14) is established. The fact that \((T_x)_{x \geq 0}\) is an extremal process and the identity (15) follows from Corollary 1.4.2 in [DLG02], Corollary 4.5 and the fact that the height process is regenerative at zero.

Remark 4.9. Even though the height processes are not in general Markov processes, the corollary 4.5 still holds since the height processes are regenerative at zero.

Example 4.10. (Extinction process of BESQ) In the case \( \psi(u) = 2u^2 \), the corresponding flow of continuous state branching processes \((L^y_{\tau t})_{y \geq 0}; t \geq 0)\) coincides with the flow of squared Bessel processes, namely

\[
X_t(x) = x + 2 \int_0^t \sqrt{X_s(x)} dB_s, \quad t \geq 0
\]

where \((B_t)_{t \geq 0}\) is a Brownian motion. In particular, for \( 0 < x < y \)

\[
\mathbb{P}(T_x = T_y) = \mathbb{P}(T_x = T_x \lor T_{y-x}) = \mathbb{P}(T_{y-x} < T_x) = \int_0^\infty F^{y-x}(z) F^x(dz) = \int_0^{+\infty} z \frac{y-x}{z} dz = \frac{x}{y},
\]

15
where $T'_{y-x}$ has the same law as $T_{y-x}$ and is independent of $T_{x}$, the distribution function $F$ is such that $\mathbb{P}(T_{x} \leq z) = F_{x}(z)$.

In [Kat15, Section 1.13], in the setting of Bessel flows, it is defined $T^{x}$ for $z > 0$, as the first time that a Bessel process started at $z$ hits the $\{0\}$. Theorem 1.2 in [Kat15] asserts that, for dimension $\delta \in (3/2, 2)$, $\mathbb{P}(T^{x} = T^{y}) > 0$, and for dimension $\delta \in [1, 3/2]$, the process $(T_{x})_{x \geq 0}$ is strictly increasing, that is $\mathbb{P}[T_{x} < T_{y}] = 1$. Here, we have covered the case $\delta = 0$

Example 4.11. (Extinction Process of a Rough Continuous-State Branching Processes) This example shows that the extinction process of a Rough continuous-state branching process is a Fréchet extremal processes. For $\alpha \in (0, 1)$, let $(X_{t})_{t \geq 0}$ an $(1 + \alpha)$-stable spectrally positive Lévy process, and let $L_{t}^{x}$ its local time at level $x \in \mathbb{R}$ and at time $t \geq 0$. In [Xu21a], it is proved that $(L_{t}^{x})_{x \geq 0}$ is a solution of the Volterra equation

$$L_{t}^{x} = t + \int_{0}^{x} \int_{0}^{\infty} \int_{0}^{L_{t}^{y}} (W(x-y) - W(x-y-w)) \tilde{N}_{x}(dy, dw, dz), \quad x \geq 0$$

where $W(x) = \frac{x^{\alpha}}{\alpha \Gamma(\alpha + 1)}$ and $\tilde{N}_{x}(dy, dw, dz)$ is a compensated Poisson random measure on $(0, \infty) \times \mathbb{R}_{+}^{2}$ with intensity $dy \nu_{x}(dw, dz)$, where $\nu_{x}(dy)$ is the Lévy measure of $(X_{t})_{t \geq 0}$ given by $\nu_{x}(dy) := \frac{\cos(\alpha+1)}{1(1-\alpha)} y^{-\alpha-2} dy, y > 0$. Moreover, in [Xu21a], it is proved that $Z_{t} := \inf\{y > 0 : L_{t}^{y}\} < +\infty$ which by the same reasoning as above is $Z_{t}$ equal to $\sup_{s \leq t} X_{s}$ and, by Example 4.7, this is an Extremal process with Dwass exponent given by

$$Q_{\alpha}(x) = \pi / \int_{-\infty}^{+\infty} (1 - \cos(xy)) \cdot \Re \left( \frac{1}{\Psi(y)} \right) dy = \frac{c \Gamma(1 + \alpha)}{x^{\alpha}}$$
In [Xu21b], it is proved that
\[ P[Z_t \geq x] \sim t\Gamma(1 + \alpha)x^{-\alpha} \quad (x \to \infty), \]
but this follows readily from the fact that \((Z_t)_{t \geq 0}\) is an Extremal process since \(P[Z_t \geq x] = 1 - \exp\{-tQ_\alpha(x)\} = 1 - \exp\{-t\Gamma(1 + \alpha)/x^\alpha\} \). 

Remark 4.12. For \(\alpha \in (0, 1)\), let \((X_t)_{t \geq 0}\) an \((1 + \alpha)\)-stable spectrally positive Lévy process with Lévy measure given by \(\nu_\alpha(dy) := \frac{c(\alpha+1)}{\Gamma(1-\alpha)}y^{-\alpha-2}dy, \ y > 0\). We have
\[ \Psi(y) = \kappa|y|^{1+\alpha}(1 - i\tan(\pi(\alpha+1)/2)\sgn(y)) \]
where \(\kappa = -\Gamma(-1 - \alpha)\cos(\pi(1 + \alpha)/2) \cdot \frac{c(\alpha+1)}{\Gamma(1-\alpha)}\). Therefore
\[ \Re\left(\frac{1}{\Psi(y)}\right) = \left(\frac{\Gamma(1 - \alpha)}{c\cos(\pi(1 + \alpha)/2)\Gamma(-\alpha)\alpha}\right) \cdot |y|^{-1-\alpha}\cos^2(\pi(1 + \alpha)/2) \]
\[ = |y|^{-1-\alpha} \cdot \frac{\cos(\pi(1 + \alpha)/2)}{c}. \]

Using than \(\int_0^{+\infty}(1 - \cos(xy))/y^{1+\alpha}dy = |x|\Gamma(1 - \alpha)\sin(\pi(1 + \alpha)/2)/\alpha\), we obtain, for \(x > 0\),
\[ \int_{-\infty}^{+\infty}(1 - \cos(xy)) \cdot \Re\left(\frac{1}{\Psi(y)}\right)dy = x^\alpha \cdot \frac{-\sin(\pi(1 + \alpha))\Gamma(1 - \alpha)}{c \cdot \alpha} \]
\[ = x^\alpha \cdot \frac{-\Gamma(1 - \alpha)\pi}{c\alpha\Gamma(1 + \alpha)\Gamma(-\alpha)} \]
\[ = \frac{x^\alpha \cdot \pi}{c\Gamma(1 + \alpha)}. \]

Therefore \(Q_\alpha(x) = x^{-\alpha}c\Gamma(1 + \alpha)\).
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