LOCAL EXACT CONTROLLABILITY TO TRAJECTORIES OF THE MAGNETO-MICROPOLAR FLUID EQUATIONS

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Abstract. In this paper we prove the exact controllability to trajectories of the magneto-micropolar fluid equations with distributed controls. We first establish new Carleman inequalities for the associated linearized system which lead to its null controllability. Then, combining the null controllability of the linearized system with an inverse mapping theorem, we deduce the local exact controllability to trajectories of the nonlinear problem.

1. Introduction and statement of main results. Let \( \Omega \) be a bounded connected domain in \( \mathbb{R}^d \), \( d \in \{2, 3\} \), whose boundary \( \partial \Omega \) is regular enough. Let \( T > 0 \) and we will use the notations \( Q = \Omega \times (0, T) \), \( \Sigma = \partial \Omega \times (0, T) \), and we denote by \( n(x) \) the outward unit normal to \( \partial \Omega \) at the point \( x \in \partial \Omega \).

We consider the controllability of the following magneto-micropolar fluid equations:

\[
\begin{align*}
\begin{cases}
y_t - \Delta y + (y \cdot \nabla) y - (B \cdot \nabla) B + \nabla p + \nabla \left( \frac{|B|^2}{2} \right) = \text{curl} \omega + u1_\Omega & \text{in } Q, \\
\omega_t - \Delta \omega - (d-2)\nabla (\nabla \cdot \omega) + (y \cdot \nabla) \omega + \omega = \text{curly} + w1_\Omega & \text{in } Q, \\
B_t - \Delta B + (y \cdot \nabla) B - (B \cdot \nabla) y = P(v1_\Omega) & \text{in } Q, \\
\nabla \cdot y = \nabla \cdot B = 0 & \text{in } Q, \\
y(0) = y^0, \ \omega(0) = \omega^0, \ B(0) = B^0 & \text{in } \Omega,
\end{cases}
\end{align*}
\]

where \( y \) and \( B \) respectively describe the flow velocity vector and the magnetic field vector,

\[
\omega = \begin{cases}
scalar \ angular \ velocity \ if \ d = 2, \\
(\omega_1(x, t), \omega_2(x, t), \omega_3(x, t)) \ angular \ velocity \ vector \ if \ d = 3,
\end{cases}
\]

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\[ p \text{ is a scalar pressure, while } y^0, \omega^0 \text{ and } B^0 \text{ are the given initial velocity, initial angular velocity and initial magnetic field, and } (u, w, v) \text{ stands for control functions acting on a small nonempty open subset } \mathcal{O} \text{ of } \Omega. \]

Here we have used the following notations:

In the case \( d = 2 \), we denote \( \text{curl} a = \partial_{x_2} a_1 - \partial_{x_1} a_2 \) for a vector function \( a = (a_1, a_2) \), and \( \text{curl} b = (\partial_{x_2} b, -\partial_{x_1} b) \) for a scalar function \( b \).

In the case \( d = 3 \), we denote

\[
\text{curl} = (\partial_{x_2} a_3 - \partial_{x_3} a_2, \partial_{x_3} a_1 - \partial_{x_1} a_3, \partial_{x_1} a_2 - \partial_{x_2} a_1)
\]

for a vector function \( a = (a_1, a_2, a_3) \).

In this work, the control function acting on the equations satisfied by the magnetic \( B \) is assumed to have the form

\[
P(v_{1\mathcal{O}}) = v_{1\mathcal{O}} + \nabla \chi, \text{ for some } \chi \in L^2(0, T; H^1(\Omega)). \tag{2}
\]

This form of the control \( v \) has been also considered in recent works on the local exact controllability of the MHD system \([4, 5, 18, 19]\). There is only a recent result on the controllability of MHD system \([3]\) in which the control acting on the magnetic field has support in an arbitrarily small open subset of the spatial domain, i.e., the control has the form \( 1_{\mathcal{O}}P_{\mathcal{O}}(v_{1\mathcal{O}}) \), where \( P_{\mathcal{O}} \) is the classical Helmholtz projector related to \( \mathcal{O} \) (i.e., the orthogonal projection operator from \( L^2(\mathcal{O})^d \) onto the completion of the set \( \{ v \in C_0^\infty(\mathcal{O})^d | \nabla \cdot v = 0 \text{ in } \mathcal{O} \} \) in the norm of \( L^2(\mathcal{O})^d \). However, since the boundary conditions on the magnetic field in our system is different from that in \([3]\), so here we cannot use ideas in \([3]\) to establish our Carleman estimate for the component \( C \) of the adjoint system respectively to the magnetic field. Hence, we are not able to get an estimate of the right-hand side of the component \( C \) having the form \( \int_\mathcal{O} e^{-2\alpha \zeta^3} |P_{\mathcal{O}}C|^2 dx dt \) as in \([3]\). So we only obtain the controllability of \((1)\) with the control function acting on the magneto field has the form \((2)\). The controllability of \((1)\) with the control function acting on the magneto field has the form \( 1_{\mathcal{O}}P_{\mathcal{O}}(v_{1\mathcal{O}}) \) remains an open question.

The magneto-micropolar fluid is a model of fluids in which micro-structures of the fluid and its electronic-magnetic properties are taken into account. In the past years, there have been a number of works devoted to studying mathematical questions related to the magneto-micropolar fluid equations. The existence and uniqueness of weak/strong solutions to \((1)\) were studied in \([8, 14, 25, 27, 28]\). The regularity and blow-up criterion of solutions were studied in \([13, 23, 33, 35]\). Besides, the long-time behavior of solutions was investigated in \([1, 6, 21, 22, 24, 29]\). However, to the best of our knowledge, there is no work on the controllability of the magneto-micropolar fluid equations. This is the motivation of the present paper. Because here we focus on the controllability, we have omitted some physical constants in this model.

It is noticed that the magneto-micropolar fluid equations contain the micropolar equations (when \( B = 0 \), the MHD equations (when \( \omega = 0 \), the Navier-Stokes equations (when \( B = 0 \) and \( \omega = 0 \) as particular cases. The local exact controllability of the Navier-Stokes equations has been studied extensively in many works, see e.g. \([10, 12, 26]\) and references therein. In recent years, the local exact controllability of the MHD system was also studied by a number of authors in \([3, 4, 5, 18, 19]\), and that of the micropolar fluid equation was studied in \([9, 17]\).

To study system \((1)\), we use the following function spaces

\[
H = \{ y \in L^2(\Omega)^d \mid \nabla \cdot y = 0 \text{ and } y \cdot n = 0 \text{ on } \partial \Omega \}.
\]
with the norm
\[ \| y \|_H = \left( \sum_{i=1}^{d} \int_{\Omega} |y_j|^2 \, dx \right)^{1/2}, \]
and
\[ V = \{ y \in H^1_0(\Omega)^d \mid \nabla \cdot y = 0 \text{ in } \Omega \} \]
with the norm
\[ \| y \|_V = \left( \sum_{i=1}^{d} \int_{\Omega} |\nabla y_j|^2 \, dx \right)^{1/2}. \]

The main question considered in this paper is whether \( (1) \) is locally exactly controllable to the trajectories.

Let us fix a regular trajectory \( (y, p, \omega, B) \) of the non-controlled system corresponding to \( (1) \), i.e.,
\[ \begin{cases} 
\eta_t - \Delta \eta - (\eta \cdot \nabla) \eta + (\eta \cdot \nabla) p + \nabla \left( |B|^2 \right) = \text{curl} \omega & \text{in } Q, \\
\omega_t - \Delta \omega - (d-2) \nabla (\nabla \cdot \omega) + (\eta \cdot \nabla) \omega + \omega = \text{curl} \eta & \text{in } Q, \\
B_t - \Delta B + (\eta \cdot \nabla) B - (\eta \cdot \nabla) \eta = 0 & \text{in } Q, \\
\nabla \cdot \eta = \nabla \cdot B = 0 & \text{in } Q, \\
\eta = 0, \omega = 0, B = 0 & \text{on } \Sigma, \\
\eta(0) = \eta^0, \omega(0) = \omega^0, B(0) = B^0 & \text{in } \Omega, \end{cases} \]
for some initial data \( (\eta^0, \omega^0, B^0) \).

We will assume that \( (\eta, \omega, B) \) satisfies
\[ (\eta, \omega, B) \in L^\infty(Q)^5 \text{ if } d = 2, \tag{4} \]
and
\[ (\eta, \omega, B) \in L^\infty(Q)^9 \text{ if } d = 3. \tag{5} \]

As long as the initial conditions are concerned, we will assume that
\[ (y^0, \omega^0, B^0) \in E_0 := \begin{cases} 
H \times L^2(\Omega) \times H & \text{if } d = 2, \\
(H \cap L^4(\Omega))^3 \times L^4(\Omega)^3 \times (H \cap L^4(\Omega)^3) & \text{if } d = 3. \end{cases} \tag{6} \]

We are now ready to formulate the main results in the present paper. First, the result in the case of two dimensions is given in the following theorem.

**Theorem 1.1.** Let \( d = 2 \). Assume that \( (\eta, \omega, B) \) satisfies \( (4) \). Then \( (1) \) is locally exactly controllable to \( (\eta, \omega, B) \) at any time \( T > 0 \), that is, there exists \( \epsilon > 0 \) such that, for any initial data \( (y^0, \omega^0, B^0) \) satisfying \( (6) \) and
\[ \| y^0 - \eta^0 \|_H + \| \omega^0 - \omega^0 \|_{L^2(\Omega)} + \| B^0 - B^0 \|_H < \epsilon, \]
there exist controls \( (u, w, v) \in L^2(O \times (0, T))^5 \) such that the solution \( (y, p, \omega, B) \) of \( (1) \) satisfying
\[ y(\cdot, T) = \eta(\cdot, T), \omega(\cdot, T) = \omega(\cdot, T) \text{ and } B(\cdot, T) = B(\cdot, T) \text{ in } \Omega. \]

The following theorem is the result in the case of three dimensions.
Theorem 1.2. Let $d = 3$. Assume that $(\bar{y}, \bar{p}, \bar{\omega}, \bar{B})$ satisfies (5). Then (1) is locally exactly controllable to $(\bar{y}, \bar{p}, \bar{\omega}, \bar{B})$ at any time $T > 0$, that is, there exists $\varepsilon > 0$ such that, for any initial data $((y^0, \omega^0, B^0))$ satisfying (6) and

$$
\|y^0 - \bar{y}^0\|_{H^\alpha L^4(\Omega)^3} + \|\omega^0 - \bar{\omega}^0\|_{L^4(\Omega)^3} + \|B^0 - \bar{B}^0\|_{H^\alpha L^4(\Omega)^3} < \varepsilon,
$$

there exist controls $(u, w, v) \in L^2(\mathcal{O} \times (0, T))^9$ such that the solution $(y, p, \omega, B)$ of (1) satisfying

$$
y(\cdot, T) = \bar{y}(\cdot, T), \quad \omega(\cdot, T) = \bar{\omega}(\cdot, T) \quad \text{and} \quad B(\cdot, T) = \bar{B}(\cdot, T) \quad \text{in} \quad \Omega.
$$

Remark 1. From the above theorems, by taking $\omega = 0$ and $B = 0$ we recover the local exact controllability result in [26] for Navier-Stokes equations, which improved the previous results in [10] and references therein. Moreover, by taking $B = 0$ only, we improved the previous result on local exact controllability to trajectories of the micropolar fluids in [9] in the sense that a weaker regularity of the given trajectory and initial data is required.

Our strategy is as follows: Let the trajectory $(\bar{y}, \bar{p}, \bar{\omega}, \bar{B})$ be given in (3) satisfying (4) or (5). Firstly, let us introduce the auxiliary nonlinear system:

$$
\begin{align*}
    \bar{y}_t - \Delta \bar{y} + ((\bar{y} + \bar{p}) \cdot \nabla) \bar{y} + (\bar{y} \cdot \nabla) \bar{p} - ((\bar{B} + \bar{B}) \cdot \nabla) \bar{B} \\
    - (\bar{B} \cdot \nabla) \bar{B} + \nabla \bar{p} + \frac{1}{2} \nabla((\bar{B} + \bar{B}) \cdot \nabla) + \frac{1}{2} \nabla(\bar{B} \cdot \nabla) = \text{curl} \bar{\omega} + u1_\mathcal{O} \quad \text{in} \quad Q, \\
    \bar{\omega}_t - \Delta \bar{\omega} - (d - 2) \nabla(\nabla \cdot \bar{\omega}) + ((\bar{y} + \bar{p}) \cdot \nabla) \bar{\omega} \\
    + (\bar{y} \cdot \nabla) \bar{\omega} + \bar{\omega} = \text{curl} \bar{\eta} + w1_\mathcal{O} \quad \text{in} \quad Q, \\
    \bar{B}_t - \Delta \bar{B} + ((\bar{y} + \bar{p}) \cdot \nabla) \bar{B} + (\bar{y} \cdot \nabla) \bar{B} \\
    - ((\bar{B} + \bar{B}) \cdot \nabla) \bar{y} - (\bar{B} \cdot \nabla) \bar{y} = P(v1_\mathcal{O}) \quad \text{in} \quad Q, \\
    \nabla \cdot \bar{y} = \nabla \cdot \bar{B} = 0 \quad \text{in} \quad Q, \\
    \bar{y}(0) = \bar{y}^0, \quad \bar{\omega}(0) = \bar{\omega}^0, \quad \bar{B}(0) = \bar{B}^0 \quad \text{in} \quad \Omega.
\end{align*}
$$

(7)

Setting $(y, p, \omega, B) = (\bar{y} + \bar{p}, \bar{p}, \bar{\omega} + \bar{\omega}, \bar{B} + \bar{B})$, it is seen that to prove the main results, what we have to do is to prove the local null controllability of (7). In other words, we have to show that, for some $\varepsilon > 0$, whenever the initial datum in (7) satisfies

$$
\|((\bar{y}^0, \bar{\omega}^0, \bar{B}^0))\|_{E_0} < \varepsilon,
$$

we can find controls $u, w$ and $v$ such that the associated solution $(\bar{y}, \bar{p}, \bar{\omega}, \bar{B})$ of (7) satisfies

$$
\bar{y}(\cdot, T) = 0, \quad \bar{\omega}(\cdot, T) = 0 \quad \text{and} \quad \bar{B}(\cdot, T) = 0 \quad \text{in} \quad \Omega.
$$

To do this, we will follow the strategy introduced by Fursikov and Imanuvilov [12] in the context of Navier-Stokes equations. Let us consider the linearized system
where \( f, f_1, f_2 \) and \( f_3 \) are functions that decay exponentially to zero as \( t \to T^− \).

We will prove that, under appropriate assumptions for \( f, f_1, f_2 \) and \( f_3 \), these above linear system (8) is null controllable. After that, combining the null controllability of (8) with an inverse mapping theorem, it will lead to the local null exact controllability of (7).

A basic tool for proving the null controllability of (8) is a global Carleman inequality for solutions to the following associated adjoint system

\[
\begin{cases}
\varphi_t - \Delta \varphi - (D^s \varphi) \bar{\gamma} + (D^a C) \bar{B} + \nabla \pi = \text{curl} \varphi + \left( \nabla \psi \right) \bar{\omega} + g_1 & \text{in } Q, \\
\psi_t - \Delta \psi - (d - 2) \nabla (\nabla \cdot \psi) + (\bar{\gamma} \cdot \nabla) \psi + \psi = \text{curl} \varphi + g_2 & \text{in } Q, \\
-C_t - \Delta C + (D^s \varphi) \bar{B} - (D^a C) \bar{\gamma} + \nabla r = g_3 & \text{in } Q, \\
\varphi = 0, \ \psi = 0, \ C = 0 & \text{on } \Sigma, \\
\varphi(T) = \varphi^T, \ \psi(T) = \psi^T, \ C(T) = C^T & \text{in } \Omega.
\end{cases}
\]

(8)

Here we have used the notations \( D^s := \nabla + \imath \nabla \) and \( D^a := \nabla - \imath \nabla \). In (9), the pressure functions are \( \pi, r \).

To obtain the above main results, which particularly improve some recent related results, we have to establish new necessary Carleman inequalities. This is in fact the main contribution of our paper.

Let us explain the method used to construct our Carleman inequality. Firstly, using the Carleman estimate in [20, Theorem 4.1] (see also in [26, Theorem 3.4]) for the Stokes system with suitable \( f \), we get the global integral estimates for the component \( \varphi \) in both cases \( d = 2 \) and \( d = 3 \). Since the magneto field has the homogeneous Dirichlet condition and the equation satisfying the magneto field has an addition pressure, then the global integral estimates for the component \( C \) can be established as same as the estimates for the component \( \varphi \). The global integral estimate for the component \( \psi \) is obtained separately in two cases \( d = 2 \) and \( d = 3 \). In the case \( d = 2 \), we can use the Carleman inequality directly for the heat equation to the component \( \psi \) to get the estimate for \( \psi \). However, in the case \( d = 3 \), we cannot use the Carleman inequality directly for the heat equations to the component \( \psi \) since the equation satisfying by \( \psi \) has the term \( \nabla (\nabla \cdot \psi) \). To overcome this difficulty, we exploit some ideas in [17] by using the Carleman inequality [20, Theorem 2.2] for the nonhomogeneous heat equations with suitable powers of the weight functions. Then, we can establish our new Carleman estimates with slightly weaker requirement of
the regularity of the trajectory as that in the case of micropolar fluid equations [9, Proposition 4].

The paper is organized as follows. In Section 2, we establish new Carleman inequalities for the solutions to the adjoint linearized system. Section 3 is devoted to proving Theorem 1.1 and Theorem 1.2. We first use the new Carleman inequality to prove the null controllability of the linearized system, then the conclusion of the proof of the main results is obtained by combining the null controllability of the linearized system and an inverse mapping theorem. In the Appendix we recall some well-known Carleman inequalities which are used in the proof.

2. Carleman inequalities.

2.1. Statement of Carleman inequalities. In this subsection, we will formulate a suitable Carleman estimate for the adjoint system (9). To do this, we introduce some weight functions. Let $\hat{O} \subset \subset O$ and $\eta^0 \in C^2(\hat{O})$ satisfy

$$\eta^0 > 0 \text{ in } O, \; \eta^0 \equiv 0 \text{ on } \partial O \text{ and } |\nabla \eta^0| > 0 \text{ in } \Omega \setminus \hat{O}. \quad (10)$$

The existence of such a function $\eta^0$ was given in [11, Lemma 1.1]. Let $\ell \in C^\infty([0,T])$ be a function such that

$$\begin{cases}
\ell(t) > 0 & \text{for all } t \in [0,T], \\
\ell(t) = t & \text{for all } t \in [0,T/4], \\
\ell(t) = T - t & \text{for all } t \in [3T/4, T].
\end{cases}$$

We now consider the following weight functions

$$\alpha(x,t) = \frac{e^{\lambda (\|\eta^0\|_\infty + m_2)} - e^{\lambda (\|\eta^0(x)\|_\infty + m_1)}}{\ell(t)^4}, \quad \xi(x,t) = \frac{e^{\lambda (\|\eta^0\|_\infty + m_2)} - e^{\lambda m_1}}{\ell(t)^4},$$

$$\alpha^*(t) = \max_{x \in \Omega} \alpha(x,t) = \alpha|_{\partial \Omega}(t) = \frac{e^{\lambda (\|\eta^0\|_\infty + m_2)} - e^{\lambda m_1}}{\ell(t)^4},$$

$$\xi^*(t) = \min_{x \in \hat{O}} \xi(x,t) = \xi|_{\partial \Omega}(t) = \frac{e^{\lambda m_1}}{\ell(t)^4},$$

where $\lambda \geq 1$ and $m_1, m_2$ are two constants chosen for the moment such that $m_1 \leq m_2$ and $\exists \zeta > 0$ (independent of $\lambda$) such that $\forall \lambda \geq 1,

|\partial_\alpha| \leq \zeta^{5/4}, \text{ and } |\partial^2_\alpha| \leq \zeta^{3/2}.$

For example, we can choose with $m_0 \geq 0,$

$$m_1 = (4 + m_0)\|\eta^0\|_\infty, \quad m_2 = (4 + m_0 + \frac{m_0}{4})\|\eta^0\|_\infty.$$

**Theorem 2.1.** Let $d = 2$. Assume that the trajectory $(\bar{y}, \bar{p}, \bar{z}, \bar{B})$ satisfies (4), $(g_1, g_2, g_3) \in L^2(Q)^5$. Then there exist some positive constants $\hat{C}, \bar{s}_0$ and $\hat{\lambda}_0$, only depending on $\Omega$ and $\partial O$, such that the solution $(\varphi, \psi, C)$ of (9) satisfies

$$s^{-1} \int_Q e^{-2s\alpha} \xi^{-1} (|\psi|^2 + |\Delta \psi|^2) \, dx \, dt + s^3 \lambda^4 \int_Q e^{-2s\alpha} \xi^3 |\psi|^2 \, dx \, dt$$

$$+ s^2 \lambda^2 \int_Q e^{-2s\alpha} \xi |\nabla \psi|^2 \, dx \, dt + s^{-1} \int_Q e^{-2s\alpha} \xi^{-1} (|\nabla \text{curl} \phi|^2 + |\nabla \text{curl} C|^2) \, dx \, dt$$
Step 1. Estimation of global terms $\varphi$ and $C$: Notice that the system for the components $\varphi$ (and $C$) in the adjoint system (9) can be viewed as the Stokes system (43) in the Appendix with $t$ replaced by $T - t$ and $f = (D^s \varphi) \mathbf{g} - (D^s C) \mathbf{B} + \text{curl} \psi + (\nabla \psi) \mathbf{w} + g_1$ (and $f = -(D^s \varphi) \mathbf{B} + (D^s C) \mathbf{g} + g_3$). So, applying Lemma 4.3
in the Appendix to components \( \varphi \) (and \( C \)) in (9) we get some positive constants \( s_0 \geq 1, \lambda_0 \geq 1 \) and \( C > 0 \) such that

\[
\begin{align*}
&\quad s^{-1} \int_Q e^{-2s\alpha}(\|\nabla \varphi\|^2 + |\nabla C|^2)dxdt \\
&+ s\lambda^2 \int_Q e^{-2s\alpha}(\|\nabla C\|^2 + |\nabla C|^2)dxdt \\
&+ \lambda^2 \int_Q e^{-2s\alpha}(\|\nabla C\|^2 + |\nabla C|^2)dxdt + s^2\lambda^4 \int_Q e^{-2s\alpha}\xi^2(\|\varphi\|^2 + |C|^2)dxdt \\
&\leq C \left( (\|\varphi\|^2_{\infty} + |B|_{\infty}^2) \right) \int_Q e^{-2s\alpha}(\|\nabla \varphi\|^2 + |\nabla C|^2)dxdt \\
&+ (1 + \|\varphi\|^2_{\infty}) \int_Q e^{-2s\alpha}\|\nabla \psi\|^2dxdt \\
&+ \int_Q e^{-2s\alpha}(\|\psi\|^2 + |g|^2)dxdt + s^3\lambda^4 \int_{Q \times (0,T)} e^{-2s\alpha}\xi^3(\|\varphi\|^2 + |C|^2)dxdt
\end{align*}
\]

for any \( s \geq s_0 \) and \( \lambda \geq \lambda_0 \), where we have used the fact that \( |\nabla \varphi|^2 \leq C|\nabla \varphi|^2 \) and \( |\nabla \psi|^2 \leq C|\nabla \psi|^2 \).

Therefore, taking \( \lambda \geq \max\{\lambda_0, C(\|\varphi\|_{\infty} + |B|_{\infty})\} \), we have from (14) that

\[
\begin{align*}
&\quad s^{-1} \int_Q e^{-2s\alpha}(\|\nabla \varphi\|^2 + |\nabla C|^2)dxdt \\
&+ s\lambda^2 \int_Q e^{-2s\alpha}(\|\nabla C\|^2 + |\nabla C|^2)dxdt \\
&+ \lambda^2 \int_Q e^{-2s\alpha}(\|\nabla C\|^2 + |\nabla C|^2)dxdt + s^2\lambda^4 \int_Q e^{-2s\alpha}\xi^2(\|\varphi\|^2 + |C|^2)dxdt \\
&\leq C \left( (\|\varphi\|^2_{\infty} + |B|_{\infty}^2) \right) \int_Q e^{-2s\alpha}\|\nabla \psi\|^2dxdt \\
&+ \int_Q e^{-2s\alpha}(\|\psi\|^2 + |g|^2)dxdt + s^3\lambda^4 \int_{Q \times (0,T)} e^{-2s\alpha}\xi^3(\|\varphi\|^2 + |C|^2)dxdt
\end{align*}
\]

\[ (15) \]

**Step 2. Estimation of global term** \( \psi \): We will consider two cases:

**Case** \( d = 2 \). Using the Carleman estimate (40) in the Appendix for \( \psi \) in (9) with \( d = 2 \), we deduce that

\[
\begin{align*}
&\quad s^{-1} \int_Q e^{-2s\alpha}(\|\psi\|^2 + |\Delta \psi|^2)dxdt + s^3\lambda^4 \int_Q e^{-2s\alpha}\xi^3|\psi|^2dxdt \\
&+ s\lambda^2 \int_Q e^{-2s\alpha}\|\nabla \psi\|^2dxdt \leq C \left( s^3\lambda^4 \int_{Q \times (0,T)} e^{-2s\alpha}\xi^3|\psi|^2 \right)
\end{align*}
\]
+ \left( \int_{Q} e^{-2s\alpha}\left( |\psi|^2 + |\text{curl}\varphi|^2 + |g_2|^2 \right) dxdt + \|\varphi\|_{\infty}^2 \int_{Q} e^{-2s\alpha} |\nabla\psi|^2 dxdt \right), \quad (16)

for \( s \geq C(T^3 + T^4) \) and \( \lambda \geq C \).

**Case d = 3.** We apply the divergence operator to the equation satisfied by \( \psi \) in (9) with \( d = 3 \) to deduce that

\[
- \partial_t (\nabla \cdot \psi) - 2\Delta (\nabla \cdot \psi) = \nabla \cdot \left( \psi + (\varphi \cdot \nabla)\psi + g_2 \right). \quad (17)
\]

Thus, we apply the Carleman estimate (42) in the Appendix for the equation (17) with different powers of \( \xi \). More precisely, we apply that Carleman inequality to \( s^{1/2} \xi^{1/2} \nabla \cdot \psi \) and we get that

\[
\int_{Q} e^{-2s\alpha} |\nabla (\nabla \cdot \psi)|^2 dxdt + s^2 \lambda^2 \int_{Q} e^{-2s\alpha} \xi^2 |\nabla \cdot \psi|^2 dxdt
\]

\[
\leq C \left( s^2 \lambda^2 \int_{\hat{\Omega} \times (0,T)} e^{-2s\alpha} \xi^2 |\nabla \cdot \psi|^2 dxdt + s^{1/2} \left\| e^{-s\alpha} \xi^{1/4} \nabla \cdot \psi \right\|_{L^2(\Sigma)}^2 \right)
\]

\[
+ s \int_{Q} e^{-2s\alpha} \xi |\psi|^2 dxdt + s \int_{Q} e^{-2s\alpha} \xi |\nabla \psi|^2 dxdt + s \int_{Q} e^{-2s\alpha} \xi |g_2|^2 dxdt
\]

(18)

for \( s \geq s_0 \) and \( \lambda \geq \lambda_0 \), where \( \hat{\Omega} \subset \subset \hat{\Omega} \subset \subset \Omega \).

On the other hand, since \( \psi \) satisfies the system

\[
\begin{align*}
-\partial_t (\nabla \cdot \psi) - \Delta \psi &= \nabla (\nabla \cdot \psi) - (\varphi \cdot \nabla)\psi - \psi + \text{curl}\varphi + g_2 \quad \text{in } Q, \\
\psi &= 0 \quad \text{on } \Sigma,
\end{align*}
\]

(19)

then using the Carleman (40) in the Appendix for \( \psi \) in (19), we deduce that

\[
\int_{Q} e^{-2s\alpha} \xi^{-1} \left( |\psi_t|^2 + |\Delta \psi|^2 \right) dxdt
\]

\[
+ s^3 \lambda^4 \int_{Q} e^{-2s\alpha} \xi^3 |\psi|^2 dxdt + s \lambda^2 \int_{Q} e^{-2s\alpha} \xi |\nabla \psi|^2 dxdt
\]

\[
\leq C \left( s^3 \lambda^4 \int_{\hat{\Omega} \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 dxdt + \int_{Q} e^{-2s\alpha} \xi \left| \nabla (\nabla \cdot \psi) \right|^2 dxdt \right)
\]

\[
+ \int_{Q} e^{-2s\alpha} \left( |\psi|^2 + |\text{curl}\varphi|^2 + |g_2|^2 \right) dxdt + \|\varphi\|_{\infty}^2 \int_{Q} e^{-2s\alpha} |\nabla \psi|^2 dxdt \right), \quad (20)
\]

for \( s \geq C(T^3 + T^4) \) and \( \lambda \geq C \). Combining (18) and (20) yields the estimate

\[
\int_{Q} e^{-2s\alpha} \xi^{-1} \left( |\psi_t|^2 + |\Delta \psi|^2 \right) dxdt
\]

\[
+ s^3 \lambda^4 \int_{Q} e^{-2s\alpha} \xi^3 |\psi|^2 dxdt + s \lambda^2 \int_{Q} e^{-2s\alpha} \xi |\nabla \psi|^2 dxdt
\]
for any $\varepsilon > 0$. Hence, choosing $\varepsilon$ sufficiently small, one infers from (21) that

$$
\int_{Q} e^{-2s\alpha} |\nabla(\nabla \cdot \psi)|^2 dx dt + s^2 \int_{Q} e^{-2s\alpha} \xi^2 |\nabla \cdot \psi|^2 dx dt
\leq C \left( \int_{Q} e^{-2s\alpha} |\nabla (\nabla \cdot \psi)|^2 dx dt + s^2 \int_{Q} e^{-2s\alpha} \xi^2 |\nabla \cdot \psi|^2 dx dt \right).
$$

We now estimate the trace terms. From the definition of $\| \cdot \|_{H^{1/2}(\Sigma)}$, we have

$$
\int_{Q} e^{-s\alpha} e^{1/4} e^{-s\alpha} |\nabla \cdot \psi|^2 dx dt + s^4 \int_{Q} e^{-2s\alpha} \xi^3 |\nabla \cdot \psi|^2 dx dt + s^2 \int_{Q} e^{-2s\alpha} \xi^2 |\nabla \cdot \psi|^2 dx dt
\leq C \left( \int_{Q} e^{-s\alpha} e^{1/4} e^{-s\alpha} |\nabla \cdot \psi|^2 dx dt + s^4 \int_{Q} e^{-2s\alpha} \xi^3 |\nabla \cdot \psi|^2 dx dt + s^2 \int_{Q} e^{-2s\alpha} \xi^2 |\nabla \cdot \psi|^2 dx dt \right).
$$

(22)

We see that $\sigma_1 \psi$ satisfies

$$
\begin{cases}
-\partial_t (\sigma_1 \psi) - \Delta (\sigma_1 \psi) - \nabla (\nabla \cdot (\sigma_1 \psi)) = -\sigma_1 (\nabla \cdot (\sigma_1 \psi)) - \sigma_1 \psi + \sigma_1 \psi + \sigma_1 \psi + \sigma_1 g_2 & \text{in } Q, \\
\sigma_1 \psi = 0 & \text{on } \Sigma, \\
(\sigma_1 \psi)(T) = 0 & \text{in } \Omega.
\end{cases}
$$

Hence, using a similar classical energy estimate for the heat equation, we get
\[ \| \sigma_1 \psi \|^2_{L^2(0,T;H^1(\Omega)^2)} + \| \sigma_1 \psi \|^2_{H^1(0,T;L^2(\Omega)^2)} \leq C \int_Q \sigma_2^2 (|\text{curl} \varphi|^2 + |g_2|^2) \, dx \, dt \\
+ \| \mathcal{F} \|^2_\infty \int_Q \sigma_1^1 |\nabla \psi|^2 \, dx \, dt + \int_Q (|\sigma'_1|^2 + \sigma_3^2) |\psi|^2 \, dx \, dt. \]  
(24)

Since \( |\sigma'_1| \leq C s^{5/4} (\xi^*)^{11/8} e^{-sa} \), one deduces from (24) and (23) that
\[ s^{1/2} \| e^{-s a} \xi^{1/4} \nabla \cdot \psi \|^2_{H^{1/4} (\Sigma)} \leq C \left( \int_Q e^{-2sa} \xi^{1/2} |\text{curl} \varphi|^2 \, dx \, dt + s^{1/2} \int_Q e^{-2sa} \xi^{1/2} |g_2|^2 \, dx \, dt \right) \\
+ \| \mathcal{F} \|^2_\infty s^{1/2} \int_Q e^{-2sa} \xi^{1/2} |\nabla \psi|^2 \, dx \, dt + s^{5/2} \int_Q e^{-2sa} \xi^{11/4} |\psi|^2 \, dx \, dt. \]

Combining (22) and (24), we get
\[ s^{-1} \int_Q e^{-2sa} \xi^{-1} \left( |\psi|^2 + |\Delta \psi|^2 \right) \, dx \, dt + s^3 \lambda^4 \int_Q e^{-2sa} \xi^3 |\psi|^2 \, dx \, dt \\
+ s \lambda^2 \int_Q e^{-2sa} \xi |\nabla \psi|^2 \, dx \, dt + \int_Q e^{-2sa} |\nabla (\nabla \cdot \psi)|^2 \, dx \, dt \\
+ s^2 \lambda^2 \int_Q e^{-2sa} \xi^2 |\nabla \cdot \psi|^2 \, dx \, dt \leq C \left( s^3 \lambda^4 \int_Q e^{-2sa} \xi^3 |\psi|^2 \, dx \, dt \\
+ s^{1/2} \int_Q e^{-2sa} \xi^{1/2} |\text{curl} \varphi|^2 \, dx \, dt + s \int_Q e^{-2sa} \xi |g_2|^2 \, dx \, dt \right), \]  
(25)

for \( s \geq \max \{s_0, C(T^3 + T^4) \} \) and \( \lambda \geq \max \{\lambda_0, C(1 + \| \mathcal{F} \|_\infty) \} \). Here we have used the fact that \( s^{1/2} \xi^{1/2} \geq C \) for \( s \geq C T^4 \).

Step 3. Conclusion.

Conclusion of Theorem 2.1. Combining (15) and (16) with note that \( |\text{curl} \varphi|^2 \leq C |\nabla \varphi|^2 \), we get (12) for \( s \geq s_0 (T^3 + T^4) \) and \( \lambda \geq \lambda_0 (1 + \| \mathcal{F} \|_\infty + \| \mathcal{G} \|_\infty + \| \mathcal{H} \|_\infty) \), where \( \lambda_0 = \max \{\lambda_0, C \} \) and \( \hat{s}_0 = \max \{s_0, C \} \). This completes the proof of Theorem 2.1.

Conclusion of Theorem 2.2. Combining (15) and (25), we get (13) for \( \lambda \geq \hat{\lambda}_0 (1 + \| \mathcal{F} \|_\infty + \| \mathcal{G} \|_\infty + \| \mathcal{H} \|_\infty) \) and for any \( s \geq \hat{s}_0 (T^3 + T^4) \). This completes the proof of Theorem 2.2.

3. Proof of the main results. In this section, we will give the proof of Theorem 1.2, i.e. the result in the case of three dimensions. The proof of Theorem 1.1 (the result in the case \( d = 2 \)) is very similar to that in the case \( d = 3 \), so it is omitted here.

3.1. Null controllability for the linear system (8). We now prove the null controllability for the system (8) and this will be crucial when proving the local controllability of (1) in the next subsection.
We can rewrite problem (8) as follows

\[
\begin{aligned}
L(\vec{y}, \vec{\omega}, \vec{B}) + (\nabla p, 0, 0) &= (f_1 + u_1, f_2 + w_1, f_3 + P(v_1)) & \text{in } Q,

\nabla \cdot \vec{y} = \nabla \cdot \vec{B} = 0 & \text{in } Q,

\vec{y} = 0, \vec{\omega} = 0, \vec{B} = 0 & \text{on } \Sigma,

\vec{y}(0) = \vec{y}^0, \vec{\omega}(0) = \vec{\omega}^0, \vec{B}(0) = \vec{B}^0 & \text{in } \Omega,
\end{aligned}
\]

where

\[
L(\vec{y}, \vec{\omega}, \vec{B}) = (L_1(\vec{y}, \vec{\omega}, \vec{B}), L_2(\vec{y}, \vec{\omega}), L_3(\vec{y}, \vec{B}))
\]

with

\[
\begin{aligned}
L_1(\vec{y}, \vec{\omega}, \vec{B}) &:= \vec{y}_t - \Delta \vec{y} + (\vec{\omega} \cdot \nabla) \vec{y} + (\vec{y} \cdot \nabla) \vec{\omega} - (\vec{B} \cdot \nabla) \vec{B} - (\vec{\omega} \cdot \nabla) \vec{B} - \text{curl} \vec{\omega},

L_2(\vec{y}, \vec{\omega}) &:= \vec{\omega}_t - \Delta \vec{\omega} - \nabla (\vec{\omega} \cdot \vec{\omega}) + (\vec{\omega} \cdot \nabla) \vec{\omega} + (\vec{y} \cdot \nabla) \vec{\omega} - \text{curl} \vec{y},

L_3(\vec{y}, \vec{B}) &:= \vec{B}_t - \Delta \vec{B} + (\vec{y} \cdot \nabla) \vec{B} - (\vec{\omega} \cdot \nabla) \vec{B} + (\vec{\omega} \cdot \nabla) \vec{\omega} - (\vec{B} \cdot \nabla) \vec{y}.
\end{aligned}
\]

We would like to find the controls \((u, v, \omega)\) such that the solution \((\vec{y}, \vec{\omega}, \vec{B})\) to (26) satisfies

\[
\vec{y}(T) = 0, \vec{\omega}(T) = 0, \vec{B}(T) = 0 \text{ in } \Omega.
\]

We first deduce the Carleman inequality with weight functions that do not vanish at \(t = 0\). More precisely, let us consider the function

\[
\tilde{\ell}(t) = \begin{cases} 
\ell(T/2) & \text{if } 0 \leq t \leq T/2, \\
\ell(t) & \text{if } T/2 \leq t \leq T,
\end{cases}
\]

and we define new weight functions

\[
\begin{aligned}
\beta(x, t) &= \frac{e^{\lambda \|\eta^0\|_\infty + m_2} - e^{\lambda \|\eta^0(x) + m_1\|_\infty}}{\ell(t)^4}, \\
\gamma(x, t) &= \frac{e^{\lambda \|\eta^0(x) + m_1\|_\infty}}{\ell(t)^4}, \\
\beta^*(t) &= \max_{x \in \Omega} \beta(x, t), \quad \gamma^*(t) = \min_{x \in \Omega} \gamma(x, t).
\end{aligned}
\]

We will prove the following lemma.

**Lemma 3.1.** Let \(s\) and \(\lambda\) be like in Theorem 2.2. Then there exists a positive constant \(C_0\) depending on \(T, s\) and \(\lambda\), such that every solution \((\varphi, \psi, C)\) of (9) satisfies

\[
\begin{aligned}
\|\varphi(0)\|_{L^2(\Omega)^3}^2 + \|\psi(0)\|_{L^2(\Omega)^3}^2 + \|C(0)\|_{L^2(\Omega)^3}^2 &+ \iint_Q e^{-2s\beta} \gamma^2 (|\varphi|^2 + |C|^2) dx dt + \iint_Q e^{-2s\beta} \gamma^3 |\psi|^2 dx dt \\
&+ \iint_Q e^{-2s\beta^*} \gamma^* (|\nabla \varphi|^2 + |\nabla \psi|^2 + |\nabla C|^2) dx dt \\
&\leq C \left( \iint_Q e^{-2s\beta} (|g_1|^2 + |g_3|^2) dx dt + \iint_Q e^{-2s\beta} |g_2|^2 dx dt \\
&+ \iint_{\sigma \times (0, T)} e^{-2s\beta} \gamma^3 (|\varphi|^2 + |\psi|^2 + |C|^2) dx dt \right).
\end{aligned}
\]
Proof. The proof of this lemma is similar to those in some recent works on the controllability of the fluid models (see for instance [16]). More precisely, this lemma is a consequence of (13) and energy estimates satisfied by solutions of (9). In what follows, we only give the sketch of the proof.

We introduce a function \( \vartheta \in C^1([0,T]) \) such that
\[
\vartheta \equiv 1 \text{ in } [0,T/2], \quad \vartheta = 0 \text{ in } [3T/4,T].
\]
Then \((\vartheta \varphi, \vartheta \psi, \vartheta C)\) satisfies
\[
\begin{cases}
-(\vartheta \varphi)_t - \Delta (\vartheta \varphi) - (D^s \vartheta \varphi) \vec{y} + (D^d \vartheta C) \vec{B} + \nabla (\vartheta \pi) = \text{curl}(\vartheta \psi) + (\nabla (\vartheta \psi)) \vec{w} + \vartheta g_1 - \vartheta' \varphi & \text{in } Q, \\
-(\vartheta \psi)_t - \Delta (\vartheta \psi) - \nabla (\vartheta \vec{v}) - (\vec{y} \cdot \nabla)(\vartheta \psi) + \vartheta \psi = \text{curl}(\vartheta \varphi) + \vartheta g_2 & \text{in } Q, \\
-(\vartheta C)_t - \Delta (\vartheta C) + (D^s \vartheta \varphi) \vec{B} - (D^d \vartheta C) \vec{B} + \nabla (\vartheta \pi) = \vartheta g_3 - \vartheta' C & \text{in } Q, \\
\nabla \cdot (\vartheta \varphi) = \nabla \cdot (\vartheta C) = 0 & \text{in } Q, \\
\vartheta \varphi = 0, \vartheta \psi = 0, \vartheta C = 0 & \text{on } \Sigma, \\
(\vartheta \varphi)(T) = 0, (\vartheta \psi)(T) = 0, (\vartheta C)(T) = 0 & \text{in } \Omega.
\end{cases}
\]
(29)

Multiplying (29)_1 by \( \vartheta \varphi \), (29)_2 by \( \vartheta \psi \), (29)_3 by \( \vartheta C \), then integrating over \( \Omega \) and using the Cauchy inequality, there exists a positive constant \( C \) depending on \( ||\vec{y}||_{\infty}, ||\varpi||_{\infty}, ||\vec{B}||_{\infty} \) such that
\[
-\frac{d}{dt} \int_{\Omega} (|\vartheta \varphi|^2 + |\vartheta \psi|^2 + |\vartheta C|^2) \, dx + \int_{\Omega} (|\nabla (\vartheta \varphi)|^2 + |\nabla (\vartheta \psi)|^2 + |\nabla (\vartheta C)|^2) \, dx
\leq C \left( \int_{\Omega} (|\vartheta \varphi|^2 + |\vartheta \psi|^2 + |\vartheta C|^2) \, dx + \int_{\Omega} (|\vartheta g_1|^2 + |\vartheta g_2|^2 + |\vartheta g_3|^2) \, dx \\
+ \int_{\Omega} |\vartheta'|^2 (|\varphi|^2 + |\psi|^2 + |C|^2) \, dx \right).
\]
(30)

So, from inequality (30) we get the energy estimate
\[
||\vartheta \varphi||_{L^\infty(0,T;H^9)}^2 + ||\vartheta \varphi||_{L^2(0,T;V)}^2 + ||\vartheta \psi||_{L^2(0,T;L^2(\Omega)^3)}^2 + ||\vartheta \psi||_{L^2(0,T;H^9(\Omega)^3)}^2 + ||\vartheta C||_{L^2(0,T;H^9(\Omega)^3)}^2 + ||\vartheta' \varphi||_{L^2(0,T;V)}^2
\leq C(T) \left( ||\vartheta \varphi||_{L^2(0,T;L^2(\Omega)^3)}^3 + ||\vartheta' \psi||_{L^2(0,T;H^9(\Omega)^3)}^3 + ||\vartheta C||_{L^2(0,T;H^9(\Omega)^3)}^3 + ||\vartheta(g_1, g_2, g_3)||_{L^2(0,T;H^9(\Omega)^3)}^3 \right).
\]

This implies that
\[
||\varphi(0), \psi(0), C(0)||_{L^2(\Omega)^3}^2 + ||\varphi||_{L^2(0,T/2;H^9(\Omega)^3)}^2 + ||\psi||_{L^2(0,T/2;L^2(\Omega)^3)}^2 + ||C||_{L^2(0,T/2;H^9(\Omega)^3)}^2
\leq C(T) \left( ||\varphi(0), \psi(0), C(0)||_{L^2(T/2,3T/4;L^2(\Omega)^3)}^2 + ||\varphi||_{L^2(0,T;H^9(\Omega)^3)}^2 + ||\varphi||_{L^2(0,T;V)}^2 \right).
\]

From the last inequality and the fact that
\[
0 < e^{-2s\beta} \gamma^3, e^{-2s\beta} \gamma^2, e^{-2s\beta} \gamma^* \leq C, \forall t \in [0,T/2]; \quad e^{-2s\beta} \geq C, \forall t \in [0,3T/4],
\]
we have
\[
||\varphi(0)||_{L^2(\Omega)^3}^2 + ||\psi(0)||_{L^2(\Omega)^3}^2 + ||C(0)||_{L^2(\Omega)^3}^2
\]
\[ + \int_0^{T/2} \int_0^\Omega e^{-2s_\beta \gamma^2 (|\varphi|^2 + |C|^2)} dx \, dt + \int_0^T \int_\Omega e^{-2s_\beta \gamma^3 |\psi|^2} dx \, dt \]
\[ + \int_0^{T/2} \int_\Omega e^{-2s_\beta \gamma^* (|\nabla \varphi|^2 + |\nabla \psi|^2 + |\nabla C|^2)} dx \, dt \]
\[ \leq C \left( \int_0^{3T/4} \int_\Omega e^{-2s_\beta \gamma^2 (|\varphi|^2 + |C|^2)} dx \, dt + \int_0^{3T/4} \int_\Omega e^{-2s_\beta \gamma^3 |\psi|^2} dx \, dt \right) \]
\[ + \int_0^{3T/4} \int_\Omega e^{-2s_\beta (|g_1|^2 + |g_3|^2)} dx \, dt + \int_0^{3T/4} \int_\Omega e^{-2s_\beta \gamma |g_2|^2} dx \, dt \right). \] (31)

Note that, since \( \beta = \alpha \) in \( \Omega \times (T/2, T) \), we have
\[ \int_0^T \int_\Omega e^{-2s_\beta \gamma^2 (|\varphi|^2 + |C|^2)} dx \, dt + \int_0^T \int_\Omega e^{-2s_\beta \gamma^3 |\psi|^2} dx \, dt \]
\[ + \int_0^T \int_\Omega e^{-2s_\beta \gamma^* (|\nabla \varphi|^2 + |\nabla \psi|^2 + |\nabla C|^2)} dx \, dt \]
\[ \leq \int\int_Q e^{-2s_\alpha \xi^2 (|\varphi|^2 + |C|^2)} dx \, dt + \int\int_Q e^{-2s_\alpha \xi^3 |\psi|^2} dx \, dt \]
\[ + \int\int_Q e^{-2s_\alpha \xi^* (|\nabla \varphi|^2 + |\nabla \psi|^2 + |\nabla C|^2)} dx \, dt \]
\[ \leq C \left( \int\int_Q e^{-2s_\alpha (|g_1|^2 + |g_3|^2)} dx \, dt + \int\int_Q e^{-2s_\alpha \xi |g_2|^2} dx \, dt \right) \]
\[ + \int\int_{\partial \times (0, T)} e^{-2s_\alpha \xi^3 (|\varphi|^2 + |\psi|^2 + |C|^2)} dx \, dt, \] (32)

for some positive constant \( C \) depending on \( s_0, \lambda_0 \). Here, we have used the Carleman inequality (13) with note that
\[ \int\int_Q e^{-2s_\alpha \xi^* |\nabla \varphi|^2} dx \, dt \leq C \int\int_Q e^{-2s_\alpha \xi^* |\nabla \varphi|^2} dx \, dt \]
since \( \varphi = 0 \) on \( \Sigma \) and \( \nabla \cdot \varphi = 0 \) in \( \Omega \).

Now, since
\[ e^{-2s_\beta}, e^{-2s_\beta \gamma}, e^{-2s_\beta \gamma^3} \geq C > 0 \ \forall t \in [0, T/2], \]
we conclude from (32) that
\[ \int_0^T \int_\Omega e^{-2s_\beta \gamma^2 (|\varphi|^2 + |C|^2)} dx \, dt + \int_0^T \int_\Omega e^{-2s_\beta \gamma^3 |\psi|^2} dx \, dt \]
\[ + \int_0^T \int_\Omega e^{-2s_\beta \gamma^* (|\nabla \varphi|^2 + |\nabla \psi|^2 + |\nabla C|^2)} dx \, dt \]
\[ \leq C \left( \int\int_Q e^{-2s_\beta (|g_1|^2 + |g_3|^2)} dx \, dt + \int\int_Q e^{-2s_\beta \gamma |g_2|^2} dx \, dt \right) \]
\[ + \int\int_{\partial \times (0, T)} e^{-2s_\beta \gamma^3 (|\varphi|^2 + |\psi|^2 + |C|^2)} dx \, dt. \] (33)
Combining (33) and (31) we get (28).

Now, we proceed to define the spaces where (26)-(27) will be solved. The main space will be

\[
E = \left\{ (\tilde{y}, \tilde{p}, \tilde{\omega}, \tilde{B}, u, w, v) : e^{s\beta} \tilde{y}, e^{s\beta} \gamma^{-1/2} \tilde{\omega}, e^{s\beta} \tilde{B} \in L^2(Q)^3, \\
e^{s\beta} \gamma^{-3/2}(u_1, w_1, P(v_1)) \in L^2(Q)^9, \\
e^{s\beta/2}(\gamma^*)^{-1/4} \tilde{y} \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^4(0, T; L^{12}(\Omega)^3), \\
e^{s\beta/2}(\gamma^*)^{-1/4} \tilde{\omega} \in L^2(0, T; H^1(\Omega)^3) \cap L^\infty(0, T; L^2(\Omega)^3) \cap L^4(0, T; L^{12}(\Omega)^3), \\
e^{s\beta/2}(\gamma^*)^{-1/4} \tilde{B} \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^4(0, T; L^{12}(\Omega)^3), \\
e^{s\beta}(\gamma^*)^{-1/2} (L_1(\tilde{y}, \tilde{\omega}, \tilde{B}) + \nabla \tilde{p} - u_1) \in L^2(0, T; W^{1,6}(\Omega)^3), \\
e^{s\beta}(\gamma^*)^{-1/2} (L_2(\tilde{y}, \tilde{\omega}) - w_1) \in L^2(0, T; W^{1,6}(\Omega)^3), \\
e^{s\beta}(\gamma^*)^{-1/2} (L_3(\tilde{y}, \tilde{B}) - P(v_1)) \in L^2(0, T; W^{1,6}(\Omega)^3) \right\}.
\]

Observe that \( E \) is a Banach space with the norm

\[
\| (\tilde{y}, \tilde{p}, \tilde{\omega}, \tilde{B}, u, w, v) \|_E^2 = \|(e^{s\beta} \tilde{y}, e^{s\beta} \gamma^{-1/2} \tilde{\omega}, e^{s\beta} \tilde{B}) \|_{L^2(Q)^3}^2 + \|(e^{s\beta} \gamma^{-3/2}(u_1, w_1, P(v_1)) \|_{L^2(Q)^9}^2 + \|e^{s\beta/2}(\gamma^*)^{-1/4} \tilde{y} \|_{L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^4(0, T; L^{12}(\Omega)^3)}^2 \\
+ \|e^{s\beta/2}(\gamma^*)^{-1/4} \tilde{\omega} \|_{L^2(0, T; H^1(\Omega)^3) \cap L^\infty(0, T; L^2(\Omega)^3) \cap L^4(0, T; L^{12}(\Omega)^3)}^2 \\
+ \|e^{s\beta/2}(\gamma^*)^{-1/4} \tilde{B} \|_{L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^4(0, T; L^{12}(\Omega)^3)}^2 \\
+ \|e^{s\beta}(\gamma^*)^{-1/2} (L_1(\tilde{y}, \tilde{\omega}, \tilde{B}) + \nabla \tilde{p} - u_1) \|_{L^2(0, T; W^{1,6}(\Omega)^3)}^2 \\
+ \|e^{s\beta}(\gamma^*)^{-1/2} (L_2(\tilde{y}, \tilde{\omega}) - w_1) \|_{L^2(0, T; W^{1,6}(\Omega)^3)}^2 \\
+ \|e^{s\beta}(\gamma^*)^{-1/2} (L_3(\tilde{y}, \tilde{B}) - P(v_1)) \|_{L^2(0, T; W^{1,6}(\Omega)^3)}^2.
\]

**Remark 3.** We can see that if \((\tilde{y}, \tilde{p}, \tilde{\omega}, \tilde{B}, u, w, v) \in E\) then \((\tilde{y}, T) = 0, \tilde{\omega}, (., T) = 0, \tilde{B}, (., T) = 0 \in \Omega, \) so \((\tilde{y}, \tilde{p}, \tilde{\omega}, \tilde{B}, u, w, v)\) solves a null controllability problem for system (26) with an appropriate right-hand side \((f_1, f_2, f_3).\)

We will prove the following result.

**Proposition 1.** Assume that \((\tilde{y}, \tilde{p}, \tilde{\omega})\) satisfies (5) and \((\tilde{y}^0, \tilde{\omega}^0, \tilde{B}^0) \in (H \cap L^4(\Omega)^3) \times L^4(\Omega)^3 \times (H \cap L^4(\Omega)^3).\) Furthermore, assume that

\[
e^{s\beta}(\gamma^*)^{-1/2} (f_1, f_2, f_3) \in (L^2(0, T; W^{1,6}(\Omega)^3))^3.
\]

Then, there exist control functions \(u \in L^2(0 \times (0, T))^3, w \in L^2(0 \times (0, T))^3 \) and \(v \in L^2(0 \times (0, T))^3\) such that if \((\tilde{y}, \tilde{\omega}, \tilde{B})\) is the associated solution to (26), one has \((\tilde{y}, \tilde{p}, \tilde{\omega}, \tilde{B}, u, w, v) \in E.\) In particular, \((\tilde{y}, T) = 0, \tilde{\omega} (., T) = 0, \tilde{B} (., T) = 0 \in \Omega.\)

**Proof.** The proof is similar to that of Proposition 2 in [16] (see also Proposition 3 in [10]), so in what follows we only give the sketch of the proof.

Let \(L^*\) be defined by

\[
L^*(\chi, \kappa, \rho) = (L^*_1(\chi, \kappa, \rho), L^*_2(\chi, \kappa, L^*_3(\chi, \rho))
\]
with
\[ L_1^*(\chi, \kappa, \rho) = -\chi_t - \Delta \chi - (D^\rho \chi)_{\bar{\gamma}} + (D^\rho \rho)_{\bar{B}} - \text{curl} \kappa - (\nabla \kappa)_{\bar{\omega}}, \]
\[ L_2^*(\chi, \kappa) = -\kappa_t - \Delta \kappa - \nabla (\nabla \cdot \kappa) - (\bar{\gamma} \cdot \nabla) \kappa + \kappa - \text{curl} \chi, \]
\[ L_3^*(\chi, \rho) = -\rho_t - \Delta \rho + (D^\rho \chi)_{\bar{B}} - (D^\rho \rho)_{\bar{\gamma}}, \]
and let us introduce the space
\[ X_0 = \left\{ (\chi, \sigma, \kappa, \rho, \zeta) \in C^2(\Omega)^3 \times C^1(\Omega) \times C^2(\Omega)^3 \times C^2(\Omega)^3 \times C^1(\Omega)^1 : \nabla \cdot \chi = \nabla \cdot \rho = 0 \text{ in } Q, \chi = 0, \kappa = 0, \rho = 0 \text{ on } \Sigma \right\}. \]

Then, we consider the following variational problem: find \((\hat{x}, \hat{\sigma}, \hat{\kappa}, \hat{\rho}, \hat{\zeta})\) such that
\[ a((\hat{x}, \hat{\sigma}, \hat{\kappa}, \hat{\rho}, \hat{\zeta}), (\chi, \sigma, \kappa, \rho, \zeta)) = \langle G, (\chi, \sigma, \kappa, \rho, \zeta) \rangle, \quad \forall (\chi, \sigma, \kappa, \rho, \zeta) \in X_0, \quad (34) \]
where
\[ a((\hat{x}, \hat{\sigma}, \hat{\kappa}, \hat{\rho}, \hat{\zeta}), (\chi, \sigma, \kappa, \rho, \zeta)) = \int_Q e^{-2s^2\beta} (L_1^*(\hat{x}, \hat{\kappa}, \hat{\rho}) + \nabla \hat{\sigma}) \cdot (L_1^*(\chi, \kappa, \rho) + \nabla \sigma) dxdt + \int_Q e^{-2s^2\beta} \gamma L_2^*(\hat{x}, \hat{\kappa}) \cdot L_2^*(\chi, \kappa) dxdt + \int_Q e^{-2s^2\beta} \left( L_3^*(\hat{x}, \hat{\rho}) + \nabla \hat{\zeta} \right) \cdot \left( L_3^*(\chi, \rho) + \nabla \zeta \right) dxdt + \int_Q e^{-2s^2\beta} \gamma^3 (\hat{\chi}_1 \cdot \chi_1 + \hat{\kappa} \chi_1 + \hat{\rho} \chi_1) dxdt, \]
and
\[ \langle G, (\chi, \sigma, \kappa, \rho, \zeta) \rangle = \int_0^T \langle f_1, \chi \rangle_{H^{-1}(\Omega)^3, H^1_0(\Omega)^3} dt + \int_0^T \langle f_2, \kappa \rangle_{H^{-1}(\Omega)^3, H^1_0(\Omega)^3} dt + \int_0^T \langle f_3, \rho \rangle_{H^{-1}(\Omega)^3, H^1_0(\Omega)^3} dt + \int_\Omega \left( \hat{y}^0 \cdot \chi(0) + \bar{\omega}^0 \cdot \kappa(0) + \tilde{B}^0 \cdot \rho(0) \right) dx. \]

From the Carleman inequality (28) applied to functions of \(X_0\), which implies that \(a(\cdot, \cdot)\) is a scalar product on \(X_0\). Therefore, we can consider the space \(X\), the completion of \(X_0\) with respect to the norm associated to \(a(\cdot, \cdot)\) (denoted by \(\| \cdot \|_X\)). Then \(X\) is a Hilbert space and \(a(\cdot, \cdot)\) is well-defined, continuous and definite positive on \(X\). Furthermore, thanks to (28), we see that the linear form \((\chi, \sigma, \kappa, \rho, \zeta) \mapsto \langle G, (\chi, \sigma, \kappa, \rho, \zeta) \rangle\) is well-defined and bounded on \(X\). Consequently, in view of Lax-Milgram's lemma, there exists a unique solution \((\hat{x}, \hat{\sigma}, \hat{\kappa}, \hat{\rho}, \hat{\zeta})\) of (34).

Let \((\hat{y}, \hat{\omega}, \hat{B})\) and \((\hat{u}, \hat{w}, \hat{v})\) be given by
\[
\begin{align*}
\dot{\hat{y}} &= e^{-2s^2\beta} (L_1^*(\hat{x}, \hat{\kappa}, \hat{\rho}) + \nabla \hat{\sigma}) \quad \text{in } Q, \\
\dot{\hat{\omega}} &= e^{-2s^2\beta} \gamma L_2^*(\hat{x}, \hat{\kappa}) \quad \text{in } Q, \\
\dot{\hat{B}} &= e^{-2s^2\beta} \left( L_3^*(\hat{x}, \hat{\rho}) + \nabla \hat{\zeta} \right) \quad \text{in } Q, \\
(\hat{u}, \hat{w}, \hat{v}) &= -e^{-2s^2\beta} \gamma^3 (\hat{x}_1, \hat{\kappa}_1, \hat{\rho}_1) \quad \text{in } Q.
\end{align*}
\]
Then, it is readily seen that they satisfy
\[
\int_{Q} e^{2s\beta} \left( |\dot{y}|^2 + \gamma^{-1} |\dot{\omega}|^2 + |\dot{B}|^2 \right) \, dx \, dt
\]
\[
+ \int_{Q} e^{2s\beta} \gamma^{-3} \left( |\dot{u}1\sigma|^2 + |\dot{\omega}1\sigma|^2 + |\dot{v}1\sigma|^2 \right) \, dx \, dt
\]  
(36)
\[ = a((\check{\chi}, \check{\sigma}, \check{\kappa}, \check{\rho}), (\check{\chi}, \check{\sigma}, \check{\kappa}, \check{\rho})) < +\infty. \]
Moreover, we can see from (36) that \((\hat{y}, \hat{\omega}, \hat{B}) \in L^2(Q)^9, \hat{u}1\sigma \in L^2(Q)^3, \hat{\omega}1\sigma \in L^2(Q)^3, \hat{\omega}1\sigma \in L^2(Q)^3\). On the other hand, from (34) and (35), we see that \((\hat{y}, \hat{\omega}, \hat{B})\) together with some pressure \(\hat{p}\) is the unique solution of (26) which is defined by the transposition with \(u = \hat{u}, w = \hat{w}, \nu = \hat{v}\).

Finally, we must check that \((\hat{y}, \hat{\rho}, \hat{\omega}, \hat{B}, \hat{\dot{u}}, \hat{\dot{w}}, \hat{\dot{v}})\) belongs to \(E\). We already know that
\[
e^{s\beta}(\hat{y}, \gamma^{-1}\hat{\omega}, \hat{B}) \in L^2(Q)^9, \quad e^{2s\beta}\gamma^{-3}(\hat{u}1\sigma, \hat{\omega}1\sigma, P(\hat{v}1\sigma)) \in L^2(Q)^9, \]
\[
e^{s\beta}(\gamma^{-1/2}(L1(\hat{y}, \hat{\omega}, \hat{B}) + \nabla \hat{p} - \hat{u}1\sigma)) \in L^2(0, T; W^{-1,6}(\Omega)^3), \]
\[
e^{s\beta}(\gamma^{-1/2}(L2(\hat{y}, \hat{\omega}) - \hat{\omega}1\sigma)) \in L^2(0, T; W^{-1,6}(\Omega)^3), \]
\[
e^{s\beta}(\gamma^{-1/2}(L3(\hat{y}, \hat{B}) - P(\hat{v}1\sigma)) \in L^2(0, T; W^{-1,6}(\Omega)^3). \]
Therefore, it remains to check that
\[
e^{s\beta/2}(\gamma^{-1/4}\hat{y}) \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^4(0, T; L^{12}(\Omega)^3), \]
\[
e^{s\beta/2}(\gamma^{-1/4}\hat{\omega}) \in L^2(0, T; H^4(\Omega)^3) \cap L^\infty(0, T; L^2(\Omega)^3) \cap L^4(0, T; L^{12}(\Omega)^3), \]
\[
e^{s\beta/2}(\gamma^{-1/4}\hat{B}) \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^4(0, T; L^{12}(\Omega)^3). \]
To this end, let us set
\[
(y^*, \omega^*, B^*) = e^{s\beta/2}(\gamma^{-1/4}(\hat{y}, \hat{\omega}, \hat{B}), \quad p^* = e^{s\beta/2}\hat{p}, \quad \hat{f}^1, \hat{f}^2, \hat{f}^3) = e^{s\beta/2}(\gamma^{-1/4}(f^1 + \hat{u}1\sigma, f^2 + \hat{\omega}1\sigma, f^3 + P(\hat{v}1\sigma)). \]
Then they satisfy
\[
\begin{aligned}
y_t^* - \Delta y^* + (\bar{y} \cdot \nabla) y^* + (y^* \cdot \nabla) \bar{y} - (B^* \cdot \nabla) B^* - (B^* \cdot \nabla) B + \nabla p^* \\
+ \nabla(B \cdot B^*) - \text{curly} y^* = f^*_1 + (e^{s\beta/2}(\gamma^{-1/4}) t\hat{y}) \quad \text{in } Q,
\end{aligned}
\]
\[
\omega_t^* - \Delta \omega^* - \nabla(\nabla \cdot \omega^*) + (\bar{y} \cdot \nabla) \omega^* + (y^* \cdot \nabla) \omega \\
+ \text{curly} y^* = f^*_2 + (e^{s\beta/2}(\gamma^{-1/4}) t\hat{\omega}) \quad \text{in } Q,
\]
\[
B_t^* - \Delta B^* + (\bar{y} \cdot \nabla) B^* + (y^* \cdot \nabla) \bar{B} - (B^* \cdot \nabla) y^* - (B^* \cdot \nabla) \pi \\
= f^*_3 + (e^{s\beta/2}(\gamma^{-1/4}) t\hat{B}) \quad \text{in } Q,
\]
\[
\nabla \cdot y^* = \nabla \cdot B^* = 0 \quad \text{in } Q, \]
\[
y^*_t = 0, \quad \omega^*_t = 0, \quad B^*_t = 0 \quad \text{on } \Sigma, \]
\[
(y^*(0), \omega^*(0), B^*(0)) = e^{s\beta/2}(\gamma(0)^*)^{-1/4} \left( \tilde{y}^0, \tilde{\omega}^0, \tilde{B}^0 \right) \quad \text{in } \Omega.
\]
We can see that
\[
f^*_1 + (e^{s\beta/2}(\gamma^{-1/4}) t\hat{y}) \in L^2(0, T; H^{-1}(\Omega)^3), \]
\[
f^*_2 + (e^{s\beta/2}(\gamma^{-1/4}) t\hat{\omega}) \in L^2(0, T; H^{-1}(\Omega)^3), \]
\[
f^*_2 + (e^{s\beta/2}(\gamma^{-1/4}) t\hat{B}) \in L^2(0, T; H^{-1}(\Omega)^3). \]
Moreover, \( y^*(0) \in H, \omega^*(0) \in L^2(\Omega)^3 \) and \( B^*(0) \in H \). Therefore, from the well-known result in [28], we know that

\[
\begin{align*}
    y^* &\in L^2(0,T; V) \cap L^\infty(0,T; H) \\
    \omega^* &\in L^2(0,T; H^1(\Omega)^3) \cap L^\infty(0,T; L^2(\Omega)^3) \\
    B^* &\in L^2(0,T; V) \cap L^\infty(0,T; H).
\end{align*}
\]

We now have to prove that \((y^*, \omega^*, B^*) \in (L^4(0,T; L^{12}(\Omega)^3))^3\).

To prove \( y^* \in L^4(0,T; L^{12}(\Omega)^3) \), we follow the arguments in [10]. To do this, let \( b \in L^\frac{4}{3}(0,T; L^{12}(\Omega)^3) \) and we consider the following Stokes system

\[
\begin{cases}
   -z_t - \Delta z + \nabla h = b & \text{in } Q, \\
   \nabla \cdot z = 0 & \text{in } Q, \\
   z = 0 & \text{on } \Sigma, \\
   z(T) = 0 & \text{in } \Omega.
\end{cases}
\]  

(37)

We know (see [10, Lemma 2], the proof uses regularity properties for the Stokes system [15] and some fine interpolation results [30]) that the system (37) has a unique solution \((z, h)\) satisfying

\[
z \in L^2(0,T; W^{1,6/5}_0(\Omega)^3) \cap C([0,T]; L^{4/3}(\Omega)^3),
\]

which depends continuously on \( b \) in these spaces. Then \( y^* \) satisfies

\[
\iint_Q y^* \cdot b \, dx \, dt = \int_\Omega e^{\beta^*(0)/2} \tilde{g}^0 \cdot z(0) dx + \int_0^T \langle F_1^*, z \rangle_{W^{-1,s}(\Omega)^3, W^{1,6/5}_0(\Omega)^3} dt.
\]

Here

\[
F_1^* = f_1^* + (e^{\beta^*/2} (\gamma^*）^{-1/4} ) \tilde{g} - (\tilde{g} \cdot \nabla) y^* - (y^* \cdot \nabla) \tilde{g} + (\tilde{B} \cdot \nabla) B^* + (B^* \cdot \nabla) B - \nabla (B \cdot B^*) + \text{curl} \omega^*,
\]

and \((z, q)\) is the solution to (37) associated to \( b \).

We know that \( z(0) \in L^4(\Omega)^3, \nabla z \in L^2(0,T; L^6(\Omega)^3) \). Remark that \((y^*, \omega^*, B^*) \in L^2(0,T; L^6(\Omega)^3)^3\), all terms of the previous definition make sense by virtue of (38) and the assumption \( \tilde{g}^0 \in L^4(\Omega)^3 \). Therefore,

\[
y^* \in \left( L^2(0,T; W^{1,6/5}_0(\Omega)^3) \cap C([0,T]; L^{4/3}(\Omega)^3) \right)' = L^4(0,T; L^{12}(\Omega)^3).
\]

We remark that, by the same above argument, one obtains

\[
(\omega^*, B^*) \in (L^4(0,T; L^{12}(\Omega)^3))^2.
\]

This ends the proof of Proposition 1. \( \square \)

3.2. Local controllability of the semilinear problem. In this subsection we give the proof of Theorem 1.2 by using similar arguments as in pioneering works [10, 16].

We will use the following inverse mapping theorem (see [2]).

Theorem 3.2. Let \( B_1 \) and \( B_2 \) be two Banach spaces and let \( A: B_1 \to B_2 \) satisfy \( A \in C^1(B_1; B_2) \). Assume that \( b_1 \in B_1, A(b_1) = b_2 \) and that \( A'(b_1): B_1 \to B_2 \) is surjective. Then, there exists \( \varepsilon > 0 \) such that, for every \( b' \in B_2 \) satisfying \( \| b' - b_2 \|_{B_2} < \varepsilon \), there exists a solution of the equation

\[
A(b) = b', \ b \in B_1.
\]
In our setting, we use this theorem with the spaces \( \mathcal{B}_1 = E, \mathcal{B}_2 = X \times Y \), where

\[
X = \left( L^2(e^{s^*}(\gamma^*)^{-1/2}(0,T); W^{-1,s}(\Omega)^3) \right)^3,
\]

and

\[
Y = H \cap L^4(\Omega)^3 \times L^4(\Omega)^3 \times H \cap L^4(\Omega)^3.
\]

Then, we consider the operator

\[
\mathcal{A}(\vec{y}, \vec{p}, \vec{\omega}, \vec{B}, u, w, v) = \left( A_1(\vec{y}, \vec{p}, \vec{\omega}, \vec{B}, u), A_2(\vec{y}, \vec{\omega}, w), A_3(\vec{y}, \vec{B}, v), \vec{y}(0), \vec{\omega}(0), \vec{B}(0) \right)
\]

with

\[
A_1(\vec{y}, \vec{p}, \vec{\omega}, \vec{B}, u) = L_1(\vec{y}, \vec{\omega}, \vec{B}) + (\vec{y} \cdot \nabla)\vec{y} - (\vec{B} \cdot \nabla)\vec{B} + \nabla \vec{p} + \frac{1}{2} \nabla(\vec{B} \cdot \vec{B}) - u1_O,
\]

\[
A_2(\vec{y}, \vec{\omega}, w) = L_2(\vec{y}, \vec{\omega}) + (\vec{y} \cdot \nabla)\vec{\omega} - w1_O,
\]

\[
A_2(\vec{y}, \vec{B}, v) = L_3(\vec{y}, \vec{B}) + (\vec{y} \cdot \nabla)\vec{B} - (\vec{B} \cdot \nabla)\vec{y} - P(v1_O).
\]

To apply Theorem 3.2, we first check that the operator \( \mathcal{A} \) is of class \( C^1(\mathcal{B}_1, \mathcal{B}_2) \). Indeed, all terms arising in the definition of \( \mathcal{A} \) are linear and consequently \( C^1 \), except for \((\vec{y} \cdot \nabla)\vec{y} - (\vec{B} \cdot \nabla)\vec{B} + \frac{1}{2} \nabla(\vec{B} \cdot \vec{B})\), \((\vec{y} \cdot \nabla)\vec{\omega}\), and \((\vec{y} \cdot \nabla)\vec{B} - (\vec{B} \cdot \nabla)\vec{y}\).

However, the operators

\[
((\vec{y}, \vec{p}, \vec{\omega}, \vec{B}, u, w, v), (\vec{y}, \vec{p}, \vec{\omega}, \vec{B}, \vec{u}, \vec{w}, \vec{v})) \mapsto \left( (\vec{y} \cdot \nabla)\vec{y} - (\vec{B} \cdot \nabla)\vec{B} + \frac{1}{2} \nabla(\vec{B} \cdot \vec{B}), (\vec{y} \cdot \nabla)\vec{\omega}, (\vec{y} \cdot \nabla)\vec{B} - (\vec{B} \cdot \nabla)\vec{y} \right)
\]

are continuous from \( \mathcal{B}_1 \times \mathcal{B}_2 \) to \( X \). So it suffices to prove their continuity from \( \mathcal{B}_1 \times \mathcal{B}_2 \) into \( G_1 \).

First, notice that

\[
e^{s^*\gamma^* - 1/2}((\vec{y}, \vec{p}, \vec{\omega}, \vec{B}) \in (L^4(0,T; L^{12}(\Omega)^3))^3 \quad (39)
\]

for any \((\vec{y}, \vec{p}, \vec{\omega}, \vec{B}, u, w, v) \in \mathcal{B}_1\).

The nonlinear term \((\vec{y} \cdot \nabla)\vec{y} - (\vec{B} \cdot \nabla)\vec{B} + \frac{1}{2} \nabla(\vec{B} \cdot \vec{B})\): We have

\[
\left\| e^{s^*\gamma^* - 1/2}((\vec{y} \cdot \nabla)\vec{y} - (\vec{B} \cdot \nabla)\vec{B} + \frac{1}{2} \nabla(\vec{B} \cdot \vec{B})) \right\|_{L^2(0,T; W^{-1,s}(\Omega)^3)} \\
\leq C \left( \left\| e^{s^*\gamma^* - 1/2}((\vec{y} \cdot \nabla)\vec{y}) \right\|_{L^4(0,T; L^{12}(\Omega)^3)} + \left\| e^{s^*\gamma^* - 1/2}(\vec{B} \cdot \nabla)\vec{B} \right\|_{L^2(0,T; L^6(\Omega)^3)} \\
+ \left\| e^{s^*\gamma^* - 1/2}(\vec{B} \cdot \vec{B}) \right\|_{L^2(0,T; L^{6}(\Omega)^3)} \right)
\]

\[
\leq C \left( \left\| e^{s^*\gamma^* - 1/2}((\vec{y} \cdot \nabla)\vec{y}) \right\|_{L^4(0,T; L^{12}(\Omega)^3)} + \left\| e^{s^*\gamma^* - 1/2}(\vec{B} \cdot \nabla)\vec{B} \right\|_{L^2(0,T; L^6(\Omega)^3)} \\
+ \left\| e^{s^*\gamma^* - 1/2}(\vec{B} \cdot \vec{B}) \right\|_{L^2(0,T; L^{6}(\Omega)^3)} \right).
\]

So, it follows from (39) that \((\vec{y} \cdot \nabla)\vec{y} - (\vec{B} \cdot \nabla)\vec{B} + \frac{1}{2} \nabla(\vec{B} \cdot \vec{B})\) belongs to the class of \( C^1 \).

The nonlinear term \((\vec{y} \cdot \nabla)\vec{\omega}\): We have

\[
\left\| e^{s^*\gamma^* - 1/2}(\vec{y} \cdot \nabla)\vec{\omega} \right\|_{L^2(0,T; W^{-1,s}(\Omega)^3)} \\
\leq C \left\| e^{s^*\gamma^* - 1/2}(\vec{y} \cdot \nabla)\vec{\omega} \right\|_{L^2(0,T; L^6(\Omega)^3)} \\
\leq C \left\| e^{s^*\gamma^* - 1/2}(\vec{y} \cdot \nabla)\vec{\omega} \right\|_{L^2(0,T; L^{12}(\Omega)^3)} + \left\| e^{s^*\gamma^* - 1/2}(\vec{y} \cdot \nabla)\vec{\omega} \right\|_{L^2(0,T; L^{12}(\Omega)^3)}.
\]
So, from (39) we have that \((\tilde{y} \cdot \nabla) \tilde{\omega}\) belongs to the class of \(C^1\).

**The nonlinear term:** \((\tilde{y} \cdot \nabla)B - (\tilde{B} \cdot \nabla)\tilde{y}\): We have the same estimates as in term \((\tilde{y} \cdot \nabla)\tilde{y} - (\tilde{B} \cdot \nabla)\tilde{B}\). Hence, this term belongs to the class of \(C^1\).

Therefore, we have proved that \(A \in C^1(B_1, B_2)\) with
\[
A'(0,0,0,0,0,0,\tilde{y}, \tilde{p}, \tilde{\omega}, \tilde{B}, u, w, v) = \left( L(\tilde{y}, \tilde{\omega}, \tilde{B}) + \nabla \tilde{p} - u \gamma_1 - w \gamma_1 - P(\gamma_1), \tilde{\omega}, \tilde{B} \right),
\]
for all \((\tilde{y}, \tilde{p}, \tilde{\omega}, \tilde{B}, u, w, v) \in B_1\).

In view of the null controllability result for the linearized system (8) given in Proposition 1, we can see that \(A'(0,0,0,0,0,0)\) is surjective.

As a consequence, we can apply Theorem 3.2 for \(b_1 = (0,0,0,0,0,0,0,0,0)\) to get the existence of \(\varepsilon > 0\) such that if \(||\tilde{y}(0), \tilde{\omega}(0), \tilde{B}(0)||_Y \leq \varepsilon\), then we can find controls \(u, w, v\) so that the associated solution to (7) satisfies \(\tilde{y}(., T) = 0, \tilde{\omega}(., T) = 0, \tilde{B}(., T) = 0\) in \(\Omega\). This completes the proof of Theorem 1.2.

4. **Appendix: Some well-known Carleman estimates.** With the weight functions \(\alpha\) and \(\xi\) defined in (11), we now recall some well-known Carleman estimates, which have been used in our proofs above.

**Lemma 4.1.** [11] Let \(O\) be a nonempty open subset of \(\Omega\). For all \(q \in L^2(0,T; H^1_0(\Omega)) \cap H^2(\Omega)\), there exists \(C > 0\) depending on \(\Omega\) and \(O\) such that
\[
s^{-1} \int_Q e^{-2s\alpha} \xi^{-1} (|q_t|^2 + |\Delta q|^2) \, dx \, dt + s^3 \lambda^4 \int_Q e^{-2s\alpha} \xi^3 |q|^2 \, dx \, dt + s^2 \lambda^2 \int_Q e^{-2s\alpha} \xi |\nabla q|^2 \, dx \, dt \\
\leq C \left( \int_Q e^{-2s\alpha} |q_t| + \Delta q|^2 \, dx \, dt + s^3 \lambda^4 \int_{\partial \times (0,T)} e^{-2s\alpha} \xi^3 |q|^2 \, dx \, dt \right) \quad (40)
\]
for any \(s \geq C(T^3 + T^4)\) and any \(\lambda \geq C\).

Consider the equation
\[
y_t - \Delta y = F_0 + \sum_{j=1}^3 \partial_j F_j \text{ in } Q, \quad (41)
\]
where \(F_0, F_1, F_2, F_3 \in L^2(Q)\). Then we have the following result.

**Lemma 4.2.** [20, Theorem 2.2] Let \(\tilde{O}\) be a nonempty open subset of \(\Omega\). There exist \(s_0 \geq 1\), \(\lambda_0 \geq 1\) and a constant \(C > 0\) (independent of \(s \geq s_0\) and \(\lambda \geq \lambda_0\)) such that for every \(y \in L^2(0,T; H^1(\Omega)) \cap H^2(0,T; H^{-1}(\Omega))\) satisfying (41), we have for every \(s \geq s_0\) and for every \(\lambda \geq \lambda_0\),
\[
s^{-1} \int_Q e^{-2s\alpha} \xi^{-1} |\nabla y|^2 \, dx \, dt + s \lambda^2 \int_Q e^{-2s\alpha} \xi |y|^2 \, dx \, dt
\]
\[
\frac{s\lambda^2}{e^{2s\alpha}} \int_{\Omega} e^{-2s\alpha} |y|^2 \, dx \, dt + s^{-1/2} \left\| e^{-2s\alpha} \frac{\xi}{\lambda} \right\|_{H^{1/4} (\Sigma)}^2 \\
+ s^{-2} \lambda^{-2} \int_{Q} e^{-2s\alpha} |F_0|^2 \, dx \, dt + \sum_{j=1}^{3} \int_{Q} e^{-2s\alpha} |F_j|^2 \, dx \, dt \right).
\]
(42)

Recall here that
\[
\|y\|_{H^{1/4} (\Sigma)} = \left( \|y\|_{L^2 (0,T;H^{1/2} (\partial \Omega))}^2 + \|y\|_{H^{1/4} (0,T;L^2 (\partial \Omega))}^2 \right)^{1/2}.
\]

Let us now consider the following Stokes system
\[
\begin{cases}
z_t - \Delta z + \nabla q = f & \text{in } Q, \\
\nabla \cdot z = 0 & \text{in } Q, \\
z = 0 & \text{on } \Sigma, \\
z(0) = z_0 & \text{in } \Omega,
\end{cases}
\]
(43)

with \(z_0 \in V\) and \(f \in L^2 (0,T;L^2 (\Omega)^d)\). Then we have the following result for solutions to (43).

**Lemma 4.3.** [20] Let \(\mathcal{O}\) be a nonempty open subset of \(\Omega\). There exist \(s_0 \geq 1, \lambda_0 \geq 1\) and \(C > 0\) such that for \(s \geq s_0\) and \(\lambda \geq \lambda_0\) and for every solution \(z\) to the Stokes system (43), we have

\[
\frac{s^{-1}}{e^{2s\alpha}} \int_{Q} e^{-2s\alpha} |\xi|^2 \, dx \, dt + s^{\lambda^2} \int_{Q} e^{-2s\alpha} |\nabla \xi|^2 \, dx \, dt \\
+ \lambda^2 \int_{Q} e^{-2s\alpha} |\nabla z|^2 \, dx \, dt + s^{2\lambda^4} \int_{Q} |\xi|^2 |z|^2 \, dx \, dt \\
\leq C \left( \int_{Q} e^{-2s\alpha} |f|^2 \, dx \, dt + s^3 \lambda^4 \int_{\mathcal{O} \times (0,T)} e^{-2s\alpha} |z|^2 \, dx \, dt \right).
\]

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