We establish a scaling limit for autonomous stochastic Newton equations, the solutions are often called nonlinear stochastic oscillators, where the nonlinear drift includes a mean field term of McKean type and the driving noise is Gaussian. Uniform convergence in $L^2$ sense is achieved by applying $L^2$-type estimates and the Gronwall Theorem. The approximation is also called Smoluchowski-Kramers limit and is a particular averaging technique studied by Papanicolaou. It reveals an approximation of diffusions with a mean-field contribution in the drift by stochastic nonlinear oscillators with differentiable trajectories.

1 Introduction

In E. Nelson [18] Brownian motion is constructed as a scaling limit of a family of Ornstein-Uhlenbeck position processes which possess differentiable sample paths by construction. The so-called ”Ornstein-Uhlenbeck theory of Brownian motion” constitutes a dynamical theory of Brownian motion. In principle, this result goes back
to [20] and [14] and also Chandrasekhar [8] studied this kind of limits. The second order stochastic Newton equation is represented as a degenerate system of first order Itô equations. For more details see Nelson [18] and the references therein. Degenerate diffusion processes which satisfy stochastic Newton equations driven by Brownian motion have been investigated in many works e.g. [1, 2, 17]. For applications in the setting of diffusions on manifold, see [10]. The broader class of Lévy processes also includes processes with jumps and hence can explain more phenomena. We proved a Smoluchowski-Kramers limit for stochastic Newton equations driven by Lévy noise see [4, 5] in the absence of mean-field terms. In the PhD thesis of Zhang [21] the reader finds references concerning generalizations to an infinite dimensional setting. Also it should be mentioned that the Smoluchowski-Kramers approximation carries over to nonlinear stochastic oscillators driven by Fractional Brownian motion [6, 9].

We would like to emphasize that the mathematical techniques and the mode of convergence in the works mentioned above differ substantially.

There are studies prior to this work considering the differential approximation in the presence of mean field, one of these is [17]. In that paper, a differentiable approximation for the stochastic Liénard equation with mean-field of McKean type is achieved. In the theory of differential games with a large number of players or in financial mathematics in particular for pricing models for markets with a large number of traders, stochastic differential equations with drift of mean-field type appear naturally [11].

Here the mean-field term contains no dissipation. In this paper we pursue a scaling limit as treated in [4, 5, 18] for stochastic Newton equations driven by Brownian motion with a nonlinear drift term ($\beta K$), $\beta > 0$, where $K$ is globally Lipschitz continuous, moreover, it depends only on the position process and a mean-field term in form of the expectation value of the position process.

2 Preliminaries

In this section we recall a representation for Itô processes with linear dissipation and explain the scaling limit we are using.

**Proposition 1.** Consider the equation

$$dY_t = -aY_t dt + f(t) dt + dB_t,$$

where $a > 0$, $B_t$, $t \geq 0$, is standard Brownian motion, and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\int_0^T e^{as}|f(s)|ds < \infty$. Then we know that the solution exists and is given by

$$Y_t = e^{-at}x_0 + \int_0^t e^{-a(t-s)}f(s)ds + \int_0^t e^{-a(t-s)}dB_s,$$

(1)
for some initial value $x_0$. For a proof see for example [19].

A stochastic Newton equation is an Itô equation which is second order in time, hence is given by a system of a first order ordinary differential equation and a stochastic differential equation. The dependent variables are traditionally denoted by $x_t$ and $v_t$, respectively, whence the independent variable $t \geq 0$ is interpreted as time. In physical models $x_t$ describes the position of a particle at time $t > 0$. It is assumed in this paper that the velocity $\frac{dx}{dt} = v$ exists and satisfies the so called Langevin equation with a nonlinear drift depending on the position and its expectation value.

We introduce a scaling in form of a parameter $\gamma > 0$ for the stochastic Newton equation with mean-field having the following form

$$
\begin{align*}
\frac{dx}{dt} &= v_t dt \\
\frac{dv}{dt} &= -\gamma v_t dt + K(x_t - \mathbb{E}[x_t])dt + \sqrt{\gamma} dB_t.
\end{align*}
$$

Deviating from Narita, in our model the nonlinear part of the drift includes the mean field term $\mathbb{E}[x_t]$, moreover, it does not contain the marginal process $v_t$ which is why this term does not explicitly contain any scaling parameter. We emphasize that the deterministic system corresponding to this system apart from the dissipation $\gamma v_t$ does not depend on the scaling parameter. Moreover, for $K(x) = -\frac{\partial V}{\partial x}$ we may define a Hamiltonian $H(x,v) = \frac{1}{2}v^2 + V(x)$, such that $\frac{\partial H}{\partial v} = \frac{\partial H}{\partial v} = v$ and $\frac{\partial H}{\partial v} = -\beta K(x)$.

Averaging properties of such Hamiltonian systems have been studied in [3, 15].

For the solution of (2) we scale the time according to $t' = \frac{1}{\gamma}t$, and we define the scaled process $x^\gamma(t') = x(\gamma t')$ and $v^\gamma(t') = \gamma v(\gamma t')$. Moreover, we introduce a new Brownian motion $\tilde{B}_{t'} = \frac{1}{\sqrt{\gamma}}B(\gamma t')$ for a new Brownian motion $\tilde{B}$. Then $(x^\gamma(t'), v^\gamma(t'))$ satisfy the following stochastic differential equation

$$
\begin{align*}
\frac{dx^\gamma}{dt'} &= v^\gamma_t dt' \\
\frac{dv^\gamma}{dt'} &= -\gamma^2 v^\gamma_t dt' + \gamma^2 K(x^\gamma_t - \mathbb{E}[x^\gamma_t])dt' + \gamma^2 dB_{t'},
\end{align*}
$$

where in a final step the Langevin equation has been multiplied by $\gamma$. For the sake of a short notation we replace $t'$ by $t$, $\tilde{B}_t$ by $B_t$, and set $\gamma^2 = \beta$. Then the two differential equations combine to the initial value problem:

$$
\begin{align*}
\frac{dx^\beta}{dt} &= v^\beta_t dt \\
\frac{dv^\beta}{dt} &= -\beta v^\beta_t dt + \beta K(x^\beta_t - \mu^\beta_t)dt + \beta dB_t,
\end{align*}
$$

with initial state $(x^\beta_0, v^\beta_0) = (x_0, v_0)$, where $\mu^\beta_t = \mathbb{E} \left[ x^\beta_t \right]$ denotes the expectation, the possibly nonlinear function $K$ satisfies a global Lipschitz condition, i.e. $|K(x - y)| \leq \kappa|x - y|$ for $\kappa > 0$, and $B_t$ is standard Brownian motion.
3 Formulation of the main result

The system we finally derived will be investigated in this work. When examining the equation in the variable \( v_t \), apparently, the drift has the time scale \( \beta t \) and \( \beta B_t \) has the faster time scale \( \beta^2 t \). Thus, for \( \beta \) tending to infinity the time is sent to infinity while the Brownian paths performs arbitrarily fast oscillations around the trajectories of the deterministic system given by the drift.

Let us formulate our main result. In a first step we introduce the Itô process in \( \mathbb{R} \) solving the stochastic differential equation

\[
dy_t = K(y_t - \mathbb{E}[y_t]) + dB_t,
\]

with \( y(0) = x_0 \) and \( K \) as in (3).

**Theorem 1.** Let \((x^\beta_t, v^\beta_t)\) be the solution of (3) and let \( \Phi = (x_0, v_0) \) be any two-dimensional random vector independent of the Brownian motion \( B_t, \ t \geq 0 \), such that

\[
\mathbb{E} \left[ |\Phi|^2 \right] < M < \infty. \tag{6}
\]

Then

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^\beta_t - y_t|^2 \right] \to 0 \quad \text{as} \quad \beta \to \infty,
\]

for every \( T < \infty \), where \( y(t) \) is the solution of (5).

The idea of the proof is to use Gronwall lemma repeatedly to show that \( \mu_t \) and \( \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^\beta_t| \right] \) are uniformly bounded on \([0, T]\). Before we proceed let us make the following important remarks.

**Remark 1.** As stated in the introduction the function \( K \) is supposed to be globally Lipschitz continuous throughout the paper. Moreover, assume all conditions stated in Theorem 1 hold. Then by Arnold [7], and for more recent results see [12], the differential equations (3) and (5) have a pathwise unique solution \((x^\beta(t), v^\beta(t))\) with initial states \((x_0, v_0)\) satisfying the moment estimates

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( |x^\beta_t| + |v^\beta_t| \right)^2 \right] \leq L_\beta,
\]

for every \( T < \infty \) with a constant \( L_\beta > 0 \) depending on \( \beta \) and \( T \).

**Remark 2.** For the same assumptions as in Remark 1 we have for \( y_t \) defined in (5) with \( y(0) = x_0 \):

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |y_t|^2 \right] \leq L,
\]

for every \( T < \infty \) and a constant \( L > 0 \).
Remark 3. The process $y_t$, $t \geq 0$, is defined in the same space as the coordinate processes $x^\beta_t$, $t \geq 0$ in (3). In differential geometrical terms this means that the driving Brownian motion $B_t$, $t \geq 0$, changed from the tangent space $\mathbb{R}$ where the coordinate process $v_t$, $t \geq 0$, is defined to the manifold which trivially is $\mathbb{R}$. Moreover, the nonlinear vector field $K(x_t - \mathbb{E}[x_t])$ changes from the cotangent space to the tangent space.

Using Proposition 1 the integral equation corresponding to the first equation of (3) becomes

$$x^\beta_t = x_0 + v_0 \int_0^t e^{-\beta s} ds + \beta \int_0^t \int_s^t e^{-\beta(s-u)} K(x^\beta_u - \mu^\beta_s) duds + \beta \int_0^t \int_s^t e^{-\beta(s-u)} dB_udu.$$  

(7)

We change the order in the double integral to obtain

$$x^\beta_t = x_0 + I_0^\beta(t) + \int_0^t e^{\beta s} K(x^\beta_s - \mu_s^\beta) ds + \beta \int_0^t \int_s^t e^{-\beta u} dB_udu,$$

where $x_0 = x^\beta(0)$ and $I_0^\beta(t) = I^\beta(0, t) = \frac{v_0}{\beta}(1 - e^{-\beta t})$. Partial integration reveals

$$x^\beta_t = x_0 + I_0^\beta(t) + I_1^\beta(t) + \int_0^t (1 - e^{-\beta(t-s)}) K(x^\beta_s - \mu_s^\beta) ds + B_t,$$

(8)

where $I_1^\beta(t) = -e^{-\beta t} \int_0^t e^{\beta s} dB_s$. For the sake of a short notation we sometimes drop the time parameter, i.e. $I_1^\beta := I_1^\beta(\cdot)$.

4 Auxiliary estimates

We evaluate the upper bound for the expectation value of each $I_i^\beta(t)$, $i = 0, 1$, given above. For the deterministic integral $I_0^\beta$ the $k^{th}$ absolute moment, $k \in \mathbb{N}$, is trivially estimated

$$\mathbb{E}\left[|I_0^\beta|^k\right] = \frac{1}{\beta^k} (1 - e^{-\beta t})^k \leq \frac{1}{\beta^k}. $$  

(9)

Moreover, for $I_1^\beta(t)$ we use Itô isometry to have

$$\mathbb{E}\left[|I_1^\beta(t)|^2\right] \leq \mathbb{E}\left[e^{-2\beta t} \left(\int_0^t e^{\beta u} dB_u\right)^2\right] = \frac{1}{2\beta} (1 - e^{-2\beta t}) \leq \frac{1}{2\beta}. $$

(10)
Using the Lipschitz condition on $K$ we estimate $|x^\beta_t|$ in (8) by:

$$
|x^\beta_t| \leq |x_0| + \frac{1}{\beta} + |I^\beta_t| + |B_t| + \kappa \int_0^t |x^\beta_s - \mu^\beta_s| ds.
$$  (11)

By monotonicity of integration there holds $|\mu_t| \leq \mathbb{E} \left[ |x^\beta_t| \right]$. Taking the expectation and inserting (9) and (10) reveals

$$
\mathbb{E} \left[ |x^\beta_t| \right] \leq \mathbb{E} \left[ |x_0| \right] + \frac{1}{\beta} + \mathbb{E} \left[ |I^\beta_t| \right] + \mathbb{E} \left[ |B_t| \right] + 2\kappa \int_0^t \mathbb{E} \left[ |x^\beta_s| \right] ds
\leq M^\frac{\beta}{2} + \frac{1}{\sqrt{2\beta}} + \sqrt{t} + 2\kappa \int_0^t \mathbb{E} \left[ |x^\beta_s| \right] ds,
$$

where we used the fact that $\mathbb{E} \left[ |x_0|^2 \right] \leq M$ and $\mathbb{E} \left[ |B_t| \right] \leq \sqrt{t}$. Next we consider the moment of the supremum of the square of the solution $x^\beta(t)$, for which we derive a bound uniform in the parameter $\beta$ and $t \in [0, T]$.

**Lemma 1.** For the same assumption as in Theorem 1, let $(x^\beta_t, v^\beta_t)$ be the solution of (3) with initial state $\Phi$. Then we have

$$
\sup_{\beta > 1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^\beta(t)|^2 \right] \leq H(T)
$$

for arbitrary fixed $T < \infty$, where $H(T)$ is a positive constant independent of $\beta$ as $\beta$ tends to infinity.

**Proof of Lemma 1**

Let $T < \infty$ be arbitrary but fixed. Consider any $t \in [0, T]$. We return to equation (11), square both sides, apply Jensen’s inequality before taking the supremum over the interval $[0, t]$, and estimate:

$$
\sup_{0 \leq u \leq t} |x^\beta_t|^2 \leq 5 \left( |x_0|^2 + \frac{1}{\beta^2} \sup_{0 \leq u \leq t} |I^\beta_t|^2 + \sup_{0 \leq u \leq t} |B_u|^2 + 2\kappa^2 T \int_0^t |x^\beta_s|^2 + \mathbb{E} \left[ |x^\beta_s|^2 \right] ds \right)
$$

where we have used that we have positive integrands and the estimate $|\mu_t|^2 \leq \mathbb{E} \left[ |x^\beta_t|^2 \right]$.

Since $B_t$ is a martingale, Doob’s inequality reveals $\mathbb{E} \left[ \sup_{0 \leq u \leq t} |B_u|^2 \right] \leq 4 \mathbb{E} \left[ |B_t|^2 \right]$, respectively, by inserting (7) and the recent estimates into

$$
\sup_{0 \leq u \leq t} |I^\beta_1(u)|^2 \leq 2 \sup_{0 \leq u \leq t} |B_u|^2 + 2 \sup_{0 \leq u \leq t} |B_u|^2 t \sup_{0 \leq u \leq t} (1 - e^{-2\beta u}).
$$
we obtain that for all $0 \leq t \leq T$:
\[
\mathbb{E} \left[ \sup_{0 \leq u \leq t} |x_u^\beta|^2 \right] \leq 5 \left( \mathbb{E} \left[ |x_0^\beta|^2 \right] + \frac{1}{\beta^2} + 9T + 8T^2 + 4\kappa^2 T \int_0^T \mathbb{E} \left[ \sup_{0 \leq u \leq s} |x_u^\beta|^2 \right] ds \right).
\]
(13)

Let us now turn to the core of the proof Lemma 1. By the assumption on the initial values $x_0, v_0$ we have
\[
\mathbb{E} \left[ |x_0|^2 \right] < M < \infty.
\]
For $\beta > 1$ and hence for $\beta \to \infty$ on a given interval $[0, T]$ we define the constant
\[
H_0(T) := 5M + 5 + 45T + 40T^2
\]
which combines all additive constants in equation (13) and rewrite
\[
\mathbb{E} \left[ \sup_{0 \leq u \leq t} |x^\beta(t)|^2 \right] \leq \left[ H_0(T) + 20\kappa^2 T \int_0^T \mathbb{E} \left[ \sup_{0 \leq u \leq s} |x_u^\beta|^2 \right] ds \right].
\]

Gronwall’s lemma then reveals:
\[
\mathbb{E} \left[ \sup_{0 \leq u \leq t} |x^\beta(t)|^2 \right] \leq H(T),
\]
(14)

with $H(T) = H_0(T)e^{\theta T^2}$ and $\theta := 20\kappa^2$ where $\kappa$ is the Lipschitz constant of $K$. The bound holds uniformly for $t \in [0, T]$ and for $\beta$ sufficiently large, e.g. $\beta > 1$, and hence for $\beta \to \infty$, which completes the proof.

Next we give the proof of the main result of this paper.

5 Proof of Theorem 1

Under the same assumptions as in Theorem 1. Let $(x_t^\beta, v_t^\beta)$ and $y_t$ be the solutions of (3) and (5) with initial states $(x_0^\beta, v_0^\beta) = \Phi = (x_0, v_0)$ and $y_0 = x_0$, respectively. Combining (5) and (8) we have
\[
x_t^\beta - y_t = \int_0^t K(x_s^\beta - \mu_s^\beta)ds - \int_0^t K(y_s - \mu_s)ds + I_0^\beta(t) + I_1^\beta(t) + I_2^\beta(t),
\]
where $I_i^\beta(t), i = 0, 1,$ are as before, $I_2(t) = -e^{-\beta t} \int_0^t e^{\beta s} K(x_s^\beta - \mu_s^\beta)ds,$ and $\mu_s = \mathbb{E}[y_s]$. We exploit that $K$ is Lipschitz continuous with constant $\kappa$ and rearrange the arguments in the norm, to get
\[
|x_t^\beta - y_t| \leq |I_0^\beta| + |I_1^\beta| + |I_2^\beta| + \int_0^t |K(x_s^\beta - \mu_s^\beta) - K(y_s - \mu_s)|ds
\]
\[
\leq \kappa \int_0^t |x_s^\beta - y_s|ds + \kappa \int_0^t |\mu_s^\beta - \mu_s|ds + \sum_{i=0}^2 |I_i^\beta(t)|.
\]
By inserting the definition of $\mu^\beta_s$ and $\mu_s$ and by using Jensen’s inequality, in particular that $|\mu^\beta_s - \mu_s|^2 \leq \mathbb{E}\left[|x^\beta_s - y_s|^2\right]$, we estimate further

$$
|x^\beta_t - y_t|^2 \leq 5 \left( \kappa^2 t \int_0^t |x^\beta_s - y_s|^2 ds + \kappa^2 t \int_0^t \mathbb{E}\left[|x^\beta_s - y_s|^2\right] ds + \sum_{i=0}^2 |I^\beta_i(t)|^2 \right).
$$

Taking first the supremum and then the expectation value yields for $0 \leq t \leq T < \infty$, that

$$
\mathbb{E}\left[\sup_{0 \leq u \leq t} |x^\beta_u - y_u|^2\right] \leq D \int_0^t \mathbb{E}\left[\sup_{0 \leq u \leq s} |x^\beta_u - y_u|^2\right] ds + 5 \sum_{i=0}^2 \mathbb{E}\left[\sup_{0 \leq u \leq t} |I^\beta_i(u)|^2\right]
$$

with $D := 10\kappa^2 T$. Direct calculation together with the previous bounds (9) and (10) reveals that the last term on the right hand side is uniformly bounded. To this end we proceed in the same way as in (12) and (13), in particular we use that $K$ is Lipschitz continuous as well as Jensen’s inequality to find

$$
|I^\beta_2(t)|^2 \leq te^{-2\beta t} \int_0^t e^{2\beta s}(K(x^\beta_s - \mu^\beta_s))^2 ds \leq \kappa^2 te^{-2\beta t} \int_0^t e^{2\beta s}|x^\beta_s - \mu_s|^2 ds
$$

$$
\leq 2\kappa^2 te^{-2\beta t} \int_0^t e^{2\beta s}|x^\beta_s|^2 ds.
$$

Since the integrand is positive taking the supremum reveals the following estimate:

$$
\sup_{0 \leq u \leq t} |I^\beta_2(u)|^2 \leq 2\kappa^2 t \sup_{0 \leq u \leq s} |x^\beta_s|^2 \frac{1}{e^{2\beta T}} (1 - e^{-2\beta T}) \leq \kappa^2 T \sup_{0 \leq u \leq t} |x^\beta_u|^2 \frac{1}{e^{2\beta T}}.
$$

Taking expectation and inserting the bound (14) we obtain:

$$
\mathbb{E}\left[\sup_{0 \leq u \leq t} |I^\beta_2(u)|^2\right] \leq \frac{\kappa^2}{\beta} T H(T) e^{\beta T^2}.
$$

(15)

For $\beta$ sufficiently large, e.g. $\beta \geq 1$, we get the following bound

$$
\sum_{i=0}^2 \mathbb{E}\left[\sup_{0 \leq u \leq t} |I^\beta_i(u)|^2\right] \leq \frac{1}{\beta} \Lambda(T)
$$

(16)

with $\Lambda(T) := 5 \left( \kappa^2 TH(T)e^{\beta T^2} + \frac{3}{2} \right)$ where we combined the bounds (9), (10), and (15). By introducing the bound (16) into (5) we arrive at the inequality

$$
\mathbb{E}\left[\sup_{0 \leq u \leq t} |x^\beta_u - y_u|^2\right] \leq \frac{1}{\beta} \Lambda(T) + D \int_0^t \mathbb{E}\left[\sup_{0 \leq u \leq s} |x^\beta_u - y_u|^2\right] ds
$$
which allows to apply Gronwall’s lemma a last time, to give

\[ \mathbb{E} \left[ \sup_{0 \leq u \leq t} |x_u^\beta - y_u|^2 \right] \leq \frac{1}{\beta} \Lambda(T) e^{DT^2} \]

with constant \( D \) as defined in (5) which holds uniformly on the given time interval \([0, T]\) and does not depend on \( \beta \) for \( \beta \) sufficiently large, while \( \frac{1}{\beta} \) tends to zero as \( \beta \) tends to infinity. This concludes the proof of Theorem 1.

6 Section

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