Fermionic superfluid properties in a one-dimensional optical lattice

Theja N. De Silva
Department of Physics, Applied Physics and Astronomy,
The State University of New York at Binghamton, Binghamton, New York 13902, USA.

We discuss various superfluid properties of a two-component Fermi system in the presence of a tight one-dimensional periodic potential in a three-dimensional system. We use a zero temperature mean field theory and derive analytical expressions for the Josephson current, the sound velocity and the center of mass oscillations in the BCS-Bose Einstein condensation crossover region.

1. INTRODUCTION

Recent experiments with trapped alkaline Fermi gases have shown promising directions toward understanding the many body physics in various disciplines. These systems have become a focus of attention due to their easy controllability. For example, the controllability of interaction strength using a magnetically tuned Feshbach resonance and of the spatial structure using counter propagating laser beams allow one to study a wide range of quantum mechanical phenomena. One such phenomena is a dissipationless flow at low temperatures. The focus of this paper is the analytical study of dissipationless transport properties of a stack of weakly-coupled two-dimensional (2D) Fermi systems. Such a system can be created by applying a relatively strong one-dimensional (1D) optical lattice potential to an ordinary three-dimensional system. One natural analogy of this system is the weakly coupled copper-oxide planes of high temperature superconductors where Bose condensed paired electrons flow dissipationlessly in the crystal lattice.

In this paper, we derive analytical expressions for the energy and the density in the limit of weakly coupled layers using a zero temperature mean field theory. Using the energy and the density expressions together with gap equation, we then derive analytical expressions for various dynamical quantities in the Bardeen, Cooper and Schrieffer-Bose Einstein condensation (BCS-BEC) crossover region. A similar discussion of tunneling dynamics using a path integral approach can be found in ref. [2]. A hydrodynamic approach in the weakly interacting BCS limit and a Bogoliubov-de Gennes approach in the unitarity limit can be found in ref. [3] and ref. [4] respectively. First, we calculate the superfluid oscillation (Bloch oscillation) amplitudes due to the linear force on atoms in the presence of Bloch bands and find that the amplitudes are larger for weak periodic potentials and for weak attractive interactions. Second, we calculate the sound velocity along the direction of 1D lattice potential and find that the sound velocity strongly depends on the lattice height but weakly depends on the interaction for tight periodic potentials. Finally, we calculate the dipole oscillation frequencies in the BCS-BEC crossover region in the presence of a harmonic external trapping potential.

In the absence of a periodic potential, dipole oscillations are undamped and the oscillation frequencies are independent of the two-body interaction (generalized Kohn’s theorem [5]). However, in the presence of a 1D periodic potential, we find that the dipole oscillations are damped and strongly depend on the two-body interactions. Such damping for a 1D Bose gas and a non-interacting Fermi gas have already been observed [6].

The paper is organized as follows. In section II, we present the theoretical formulation. In section III, we present the results and discussion followed by the conclusion in section IV.

II. FORMALISM

We consider an interacting two-component Fermi atomic gas trapped in one dimensional optical lattice created by standing laser waves. The 1D optical potential modulated along z-axis has the form \( V = \frac{sE_R}{2}\sin^2(2\pi z/\lambda) \). Here \( \lambda \) is the wavelength of the of the laser beam, \( E_R = \frac{h^2}{2M}(2\pi/\lambda)^2/(2M) \) is the recoil energy and \( s \) is a dimensionless parameter which accounts the intensity of the laser beam. The periodicity of the 1D optical lattice along z-direction is \( d = \lambda/2 \). When the laser power is large, the atomic system forms a stack of weakly coupled 2D planes. We take the model Hamiltonian of the system as \( H = \sum_{\sigma} H_j \), where \( j \) is the layer index.

\[
H_j = \int d^2r \left\{ \sum_{\sigma} \psi^{\dagger}_{j\sigma}(r) \left[ -\frac{\nabla_{2D}^2}{2M} + V_{\text{ho}}(j) - \mu_\sigma \right] \psi_{j\sigma}(r) + \right. \\
\left. + i \sum_{\sigma} \left[ \psi^{\dagger}_{j\sigma}(r) \psi_{j+1\sigma}(r) + \text{hc} \right] \\
+ U_{2D} \psi^{\dagger}_{j1}(r) \psi^{\dagger}_{j1}(r) \psi_{j1}(r) \psi_{j1}(r) \right\}
\]

where \( V_{\text{ho}} = 1/2M \omega_z^2 r^2 + 1/2M \omega_y^2 \sigma^2 j^2 \) is the external trapping potential with \( r^2 = x^2 + y^2 \), \( \nabla_{2D} \) is the 2D gradient operator and \( U_{2D} \) is the 2D interaction strength. The operator \( \psi^{\dagger}_{j\sigma}(r) \) creates a fermion of mass \( M \) in in \( j \)th plane with spin \( \sigma = \uparrow, \downarrow \) at position \( r = (x, y) \). We consider a tight 1D lattice where interlayer tunneling en-
energ\(y t\) is small and consider the following expression proposed in ref. [3] and ref. [3] for the weakly coupled layers. In terms of dimensionless parameters, the tunneling energy for a 1D lattice proposed in ref. [3] is \(t_a/E_R = \pi s(\pi^2/4 - 1) \exp[-\pi^2 \sqrt{\pi s}/4]\). This contrasts with the isotropic three-dimensional (3D) lattice tunneling parameter \(t_b/E_R = (2s^{3/4}/\sqrt{\pi}) \exp[-2\sqrt{s}]/\sqrt{\pi}\) given in ref. [8]. Both of these are derived using a harmonic approximation around the minima of the optical lattice potential. Notice that these ratios can be varied by changing the laser intensity. As depth of the optical potential increases, the atomic wave function becomes more and more localized and the tunneling amplitude decreases. For comparison, these tunneling amplitudes are shown in FIG. (1) as a function of the laser intensity. For asymptotically deep lattices and any separable potential, the tunneling energy should be independent of the laser intensity.

FIG. 1: Tunneling energies as function of laser intensity. Gray line: Tunneling energy \(t_a\) proposed in ref. [3] for 3D optical potentials. Black line: Tunneling energy \(t_b\) proposed in ref. [3] for 1D optical potentials. The dimensionless parameter \(s\) accounts the laser intensity.

The Fourier transform of the Hamiltonian gives

\[
H = \sum_{k,\sigma} (\epsilon_k - \mu_\sigma) a_{m\sigma}^\dagger(k) a_{m\sigma}(k) + t \sum_{k,\sigma} \left[ a_{m+1\sigma}^\dagger(k) a_{m\sigma}(k) + h.c \right] + \sum_{km} \left[ \Delta_m a_{m\uparrow}^\dagger(k) a_{m\downarrow}(-k) + h.c \right] - \sum_{m} \left| \Delta_m \right|^2 / U_{2D}
\]

where \(\epsilon_k = \hbar^2 k^2/(2M)\) with \(k^2 = k_x^2 + k_y^2\). Notice that we have absorbed the external trapping potential into the chemical potential. One can treat this local chemical potential with local density approximation. In this paper, we take the external trapping potential into account when we calculate the dipole oscillations, otherwise we treat the system as homogenous. In order to take into account superfluid flow along \(z\)-direction, we take superfluid order parameter in the form \(\Delta_m = \Delta \exp[im\phi]\). The periodicity along \(z\)-direction allows us to write the Fermi operators,

\[
a_{m\uparrow}(k) = \sum_{k_z} \exp[i k_z m + i m\phi/2] c_{\uparrow}(k)
\]

\[
a_{m\downarrow}(k) = \sum_{k_z} \exp[-i k_z m + i m\phi/2] c_{\downarrow}(k)
\]

Transforming the Hamiltonian given in Eq. (2) using above transformation followed by the usual Bogoliubov transformation, the Hamiltonian per plane can be expressed as

\[
H/N = \sum_{k,\sigma} \left[ \sqrt{\epsilon_k - \mu}^2 + \Delta^2 - 2t \cos(k_z d) \sin(\phi/2) \right] (4)
\]

where \(N\) is the number of planes and \(\epsilon_k = \epsilon_k + 2t \cos(k_z d) \cos(\phi/2)\). Notice that the quasi particle energy depends on the phase of the superfluid order parameter. In the normal phase, the phase factor \(\phi\) can have multiple values of \(\pi\) [10]. The grand potential of the system \(\Omega = \ln[Z_G]\) with \(Z_G = Tr \exp[-\beta H]\) at zero temperature is

\[
\Omega = \sum_{k_z} \int \frac{d^2k}{(2\pi)^2} \left[ \epsilon_k - \mu - \sqrt{(\epsilon_k - \mu)^2 + \Delta^2} \right] - \frac{\Delta^2}{U_{2D}} (5)
\]

The 2D contact interaction \(U_{2D}\) is related to the bound state energy \(E_B\) as

\[
\frac{1}{U_{2D}} = - \int \frac{d^2k}{(2\pi)^2} \frac{1}{\hbar^2 k^2/M + E_B} (6)
\]

The bound state energy \(E_B = (C \hbar^2 \omega_L/\pi) \exp(\sqrt{2\pi \hbar^2 L/a_s})\), where \(a_s\) is the 3D s-wave scattering length and \(C \approx 0.915\). Notice that the bound state energy depends not only on the scattering length but also on the lattice potential. Performing the momentum integrals, the grand potential is

\[
\Omega = \frac{m}{2\pi \hbar^2} \left\{ -\frac{\mu^2}{2} - \frac{\Delta^2}{4} - \frac{\mu^2}{2} \sqrt{\mu^2 + \Delta^2} \right\} - \left(1 + \frac{3\Delta^2 \mu}{2(\mu^2 + \Delta^2)^{3/2}}\right) \cos^2(\phi/2) t^2 (7)
\]
Josephson current and Bloch oscillations

For the solid-state electronic systems, the size of the Cooper pair is usually larger than the lattice spacing and one can treat the system as homogeneous when calculating physical quantities such as the supercurrent. However, the periodicity of the cold atomic system is adjustable. As a result one can expect new phenomena on optical lattices with Fermi superfluid. For example, oscillating fermionic superfluidity which is absent in the bulk solid can be observed with the periodic potentials. These oscillations may be used as a probe of BCS-BEC crossover and of the normal to superfluid transitions. We calculate the particle supercurrent along z-direction using

\[ j(\phi) = \frac{m}{2\pi\hbar} \frac{\sin \phi}{h} \left\{ \frac{1}{2} + \frac{3\Delta^2 \mu + 2\mu^3}{4(\mu^2 + \Delta^2)^{3/2}} \right\} t^2 \]

In early quantum theory of electrical conductivity, Bloch and Zener \[11\] predicted that when a static electric field is applied to a crystalline electronic system, instead of uniform motion one would naively expect, electron in the crystal should oscillate. However these so-called Bloch oscillations have never been observed in the bulk crystals. The reason is that the period of Bloch oscillations is much larger than the electron scattering rate. As the cold atomic superfluid in optical lattices are clean and controllable, these systems have shown to be well suited for the observation of Bloch oscillations. In fact, the oscillations are already observed in one dimensional optical potentials \[12\].

We use superfluid velocity along the z-direction to get some understanding about the amplitude of the Bloch oscillations. Consider an atom is in the Bloch state \[|l, k_0\rangle\], where \(l\) is the discrete band index and \(k_0\) is the continuous quasimomentum. If the atom feels a constant force \(F\) (in optical lattices, one can mimic a constant force on atoms by introducing a tunable frequency difference between two counter propagating laser beams), the above Bloch state evolve to another state \(|l, k(\tau)\rangle\), where \(k(\tau) = k_0 + F\tau/h\) is the quasi momentum at time \(\tau\). We assumed that the force is weak enough and the 1D lattice is tight enough not to induce the inter band transitions. Therefore setting \(\phi = F\tau/h\) in Eq. \[11\], we can calculate current due to the induced Bloch oscillations. Then we calculate the superfluid velocity using

\[ \nu_{SF}(F\tau/h) = j(F\tau/h)/n. \]

III. RESULTS AND DISCUSSION

All our derivations for physical quantities will be calculated from Eq. \[4\], Eq. \[5\] and Eq. \[4\]. For example, the solutions of Eq. \[8\] for various parameters are shown in FIG. \[2\]. For the case of decoupled layers (in the limit \(t = 0\)), Eq. \[8\] gives \(\Delta = \sqrt{E_B(E_B + 2\mu)}\). This results the grand potential \(\Omega = -[m/(8\pi\hbar^2)](E_B + 2\mu)^2\) and the number density \(n = m/(2\pi\hbar^2)(E_B + 2\mu)\).

In the following subsections we calculate the Josephson current, the sound velocity and the dipole oscillation frequencies. First we calculate the oscillatory superfluid velocity along the direction of optical lattice through the Josephson current in the absence of harmonic potential. Next, we consider the propagation of sound along z-direction and calculate the sound velocity in a homogeneous system. Later we include the external harmonic potential to investigate the dipole oscillations of the atomic cloud.
of the phonon dispersion, propagation of the sound along z-direction. The form than the lattice periodicity.

superfluidity is satisfied if the Cooper pair size is smaller than the Fermi velocity.

These Bloch oscillations, superfluid velocity should not be observed in each plane. In order to observe these Bloch oscillations, superfluid velocity should not exceed the critical velocity. This Landau criterion for superfluidity is satisfied if the Cooper pair size is smaller than the lattice periodicity.

**Propagation of sound**

We consider a homogenous system and the propagation of the sound along z-direction. The form of the phonon dispersion $\omega = c_z q$ gives the sound velocity along z-direction $c_z = \sqrt{(n/\bar{M})(\partial \mu/\partial n)}$, where the effective mass of the atoms $\bar{M}$ is given by $\bar{M}^{-1} = \langle \partial^2 E_i/\partial k^2 \rangle$, where $E_i = 2t[1 - \cos(k_\perp d)]$. Using $\langle \partial^2 E_i/\partial k^2 \rangle = 2t^2/\hbar^2(\cos(k_\perp d/\hbar)) = (2t^2/\hbar^2)(1/n) \sum_k n_k \cos(k_\perp d/\hbar)$ with atom density $n = \sum_k n_k$ and summing over the momentum states,

$$
\frac{1}{\bar{M}} = \frac{2t^2}{\hbar^2} \frac{\cos(\phi/2)t^2}{\sqrt{\Delta^2 + \mu^2}} \left\{ \frac{3\Delta^2 \mu + 2\Delta^2 \sqrt{\Delta^2 + \mu^2} \times \cos^3(\phi/2)t^4 + O(t^6)}{2(\Delta^2 + \mu^2)^{3/2}(\mu + \sqrt{\Delta^2 + \mu^2})} \right\}
$$

Notice that the presence of periodic lattice is taken into account through both the effective mass and the compressibility $\partial \mu/\partial n$. And the ratio $c_\perp^2/c_z^2 = \bar{M}$, where $c_\perp$ is the sound velocity in transverse direction. Calculating the compressibility using the number equation (Eq. (9)), the sound velocity is

$$
c_z = \sqrt{\frac{2t^2}{\hbar^2} \cos(\phi/2)} \left\{ t + \frac{3\Delta^2 \mu \cos(\phi/2)t^3}{4(\Delta^2 + \mu^2)^2(\mu + \sqrt{\Delta^2 + \mu^2})} \right\} + O(t^5)
$$

The sound velocity as a function of $s$ for $\phi = 0$ is given in FIG. 4. As seen in Eq. (13), the sound velocity is independent of $\Delta$ and $\mu$ for the lowest order in tunneling $t$. As a result, the sound velocity is almost same in the entire BCS-BEC crossover region. In the absence of a periodic potential, the sound velocity is smaller than the Fermi velocity. Therefore the sound cannot propagate in a degenerate non-interacting Fermi
Dipole oscillations

The Bloch Oscillations discussed in the previous subsection are due to the atoms oscillations along the Bloch band under a linear force, and the dipole oscillations or the center of mass oscillations are due to the confinement in the external trapping potential. Even though the theoretical difference is clear, the probing of these two types of oscillations may be very complicated. One possible way of distinguishing these two types of oscillations is from observation of the damping rate. If the momentum distribution is narrower than the first Brillouin zone, then the damping rate of the Bloch oscillation will be zero.

The frequency of the dipole oscillations can be derived from the sum rule approach. Defining the dipole operator $D = \sum_j z_j$, the energy weight moment $m_1 = (1/2)\langle[D,[H,D]]\rangle$ and the inverse-energy weight moment $m_{-1} = N_0/(2M\omega_z^2)$ can be calculated. Here $H$ is the Hamiltonian, $[A,B] = AB - BA$ is the usual commutator, $N_0$ is the total number of atoms, and $\langle..\rangle$ is average over the ground states. Then the upper bound of the dipole oscillation frequency $\omega_d$ is calculated from the expression [14],

$$\hbar\omega_d = \sqrt{\frac{m_1}{m_{-1}}}$$

This gives the frequency of the center of mass oscillation along the 1D lattice $\omega_d = \omega_z \sqrt{M/M}$. Notice in the absence of optical lattice $\omega_d = \omega_z$ is independent of the interaction.

The dipole oscillation frequency as a function of $s$ is given in FIG. (5) for $\phi = 0$. In the entire BCS-BEC region, the dipole oscillations show significant reduction in the presence of 1D lattice. This reduction is stronger in the BEC regime where $\Delta$ is larger. As evidence from the FIG. (5), the oscillation frequency shows a weak chemical potential (hence the total number) dependence.

IV. CONCLUSIONS

We study some of the dynamical properties of a two-component Fermi gas trapped in a parabolic plus 1D periodic potential. We use a zero temperature mean field theory in the limit of tight 1D confinement. We find that the Bloch oscillations become weaker as the interaction strength changes from weakly attractive limit to weakly repulsive limit through the unitarity regime. Furthermore, we find that the sound velocity is independent of the interaction for very tight 1D confinements. Finally, we find that the dipole oscillation frequencies strongly depend on both the interaction and the 1D periodic potential.

V. ACKNOWLEDGEMENTS

This work was supported by Binghamton University. We are very grateful to Kaden Hazzard for useful discussions on the dimensional dependence of the tunneling energies.

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