UNIFIED FORMALISM FOR THE GENERALIZED 
kth-ORDER HAMILTON-JACOBI PROBLEM

LEONARDO COLOMBO∗
MANUEL DE LEÓN†

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM).
C/ Nicolás Cabrera 15. Campus Cantoblanco UAM. 28049 Madrid. Spain

PEDRO DANIEL PRIETO-MARTÍNEZ‡
NARCISO ROMÁN-ROY§

Departamento de Matemática Aplicada IV.
Universitat Politècnica de Catalunya-Barcelona Tech.
Edificio C-3, Campus Norte UPC. C/ Jordi Girona 1. 08034 Barcelona. Spain

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Abstract

The geometric formulation of the Hamilton-Jacobi theory enables us to generalize it to systems of higher-order ordinary differential equations. In this work we introduce the unified Lagrangian-Hamiltonian formalism for the geometric Hamilton-Jacobi theory on higher-order autonomous dynamical systems described by regular Lagrangian functions.

Key words: Hamilton-Jacobi equation, Higher-order systems, Skinner-Rusk formalism.

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1 Introduction

The geometric formulation of the Hamilton-Jacobi theory given in [2] and [4] enables us to generalize it to systems of higher-order ordinary differential equations. This generalization has been done recently for the Lagrangian and Hamiltonian formalism of higher-order autonomous mechanical systems described by regular Lagrangian functions [3]. The aim of this work is to give a unified Lagrangian-Hamiltonian version of this theory for these kinds of systems, using the unified framework introduced by Skinner and Rusk [8]. The advantage of this formulation is that it compresses the Lagrangian and Hamiltonian Hamilton–Jacobi problems into a single formalism which allows to recover both of them in a simple way, and it is specially interesting when dealing with singular systems.

All the manifolds are real, second countable and $C^\infty$. The maps and the structures are assumed to be $C^\infty$. Sum over repeated indices is understood.

2 Higher-order tangent bundles

Let $Q$ be a $n$-dimensional manifold, and $k \in \mathbb{Z}^+$. The $k$th-order tangent bundle of $Q$ is the $(k+1)n$-dimensional manifold $T^kQ$ made of the $k$-jets of the bundle $\pi: \mathbb{R} \times Q \to \mathbb{R}$ with fixed source point $t = 0 \in \mathbb{R}$; that is, $T^0Q = J^0_0\pi$.

We have the following natural projections (for $r \leq k$):

$$
\rho^k_r : T^kQ \longrightarrow T^rQ \quad ; \quad \beta^k : T^kQ \longrightarrow Q \quad ; \quad j^k_0 \phi \longmapsto \beta^k(\phi(0))
$$

where $j^k_0 \phi$ denotes a point in $T^kQ$; that is, the equivalence class of a curve $\phi: I \subset \mathbb{R} \to Q$ by the $k$-jet equivalence relation. Notice that $\rho^k_0 = \beta^k$, where $T^0Q$ is canonically identified with $Q$, and $\rho^k_k = \text{Id}_{T^kQ}$. Observe also that $\rho^l_s \circ \rho^r_t = \rho^r_s$, for $0 \leq s \leq l \leq r \leq k$.

If $\phi: \mathbb{R} \to Q$ is a curve in $Q$, the canonical lifting of $\phi$ to $T^kQ$ is the curve $j^k\phi: \mathbb{R} \to T^kQ$ defined as the $k$-jet lifting of $\phi$ restricted to $T^kQ \hookrightarrow J^k\pi$ (see [3]).
3 The Hamilton-Jacobi problem in the Skinner-Rusk formalism

Let \( Q \) be a \( n \)-dimensional smooth manifold modeling the configuration space of a \( k \)-th order autonomous dynamical system with \( n \) degrees of freedom, and let \( \mathcal{L} \in C^\infty(T^k Q) \) be a Lagrangian function for this system, which is assumed to be regular. In the Lagrangian-Hamiltonian formalism, we consider the bundle \( W = T^{2k-1} Q \times_{T^k Q} T^\ast(T^{k-1} Q) \) with canonical projections \( \text{pr}_1: W \to T^{2k-1} Q \) and \( \text{pr}_2: W \to T^\ast(T^{k-1} Q) \). It is clear from the definition that the bundle \( W \) fibers over \( T^{k-1} Q \). Let \( p: W \to T^{k-1} Q \) be the canonical projection. Obviously, we have \( p = \rho_{k-1} \circ \text{pr}_1 = \pi_{T^{k-1} Q} \circ \text{pr}_2 \). Hence, we have the following commutative diagram

\[
\begin{array}{ccc}
T^{2k-1} Q & \xrightarrow{\rho_{k-1}^2} & T^k Q \\
\downarrow & & \downarrow \\
W & \xrightarrow{\pi_{T^{k-1} Q}} & T^\ast(T^{k-1} Q) \\
\end{array}
\]

We consider in \( W \) the presymplectic form \( \Omega = \text{pr}_1^\ast \omega_{k-1} \in \Omega^2(W) \), where \( \omega_{k-1} \in \Omega^2(T^\ast(T^{k-1} Q)) \) is the canonical symplectic form. In addition, from the Lagrangian function \( \mathcal{L} \), and using the canonical coupling function \( \mathcal{C} \in C^\infty(W) \), we construct a Hamiltonian function \( H \in C^\infty(W) \) as \( H = \mathcal{L} - \mathcal{L} \). Thus, the dynamical equation for the system is

\[
i(X_{\mathcal{L}H})\Omega = dH , \quad X_{\mathcal{L}H} \in \mathfrak{X}(W).
\]  

(1)

Following the constraint algorithm in [5], a solution to the equation (1) exists on the points of a submanifold \( j_0: \mathcal{W}_o \hookrightarrow W \) which can be identified with the graph of the Legendre-Ostrogradsky map \( \mathcal{F}\mathcal{L}: T^{2k-1} Q \to T^\ast(T^{k-1} Q) \) associated to \( \mathcal{L} \). If the Lagrangian function is regular, then there exists a unique vector field \( X_{\mathcal{L}H} \) solution to (1) and tangent to \( \mathcal{W}_o \) (see [6]).

3.1 The generalized Hamilton-Jacobi problem

We first state the generalized version of the Hamilton-Jacobi problem. Following the same patterns as in [2], [3] and [4] (see also an approach to the problem for higher-order field theories in [9]), the natural definition for the generalized Hamilton-Jacobi problem in the Skinner-Rusk setting [7], [8] is the following.

**Definition 1** The generalized \( k \)-th order Lagrangian-Hamiltonian Hamilton-Jacobi problem (or generalized \( k \)-th order unified Hamilton-Jacobi problem) consists in finding a section \( s \in \Gamma(p) \) and a vector field \( X \in \mathfrak{X}(T^{k-1} Q) \) such that the following conditions are satisfied:

1. The submanifold \( \text{Im}(s) \hookrightarrow W \) is contained in \( \mathcal{W}_o \).
2. If \( \gamma: \mathbb{R} \to T^{k-1} Q \) is an integral curve of \( X \), then \( s \circ \gamma: \mathbb{R} \to \mathcal{W} \) is an integral curve of \( X_{\mathcal{L}H} \), that is,

\[
X \circ \gamma = \dot{s} \implies X_{\mathcal{L}H} \circ (s \circ \gamma) = \dot{s} \circ \gamma.
\]

(2)

It is clear that the vector field \( X \in \mathfrak{X}(T^{k-1} Q) \) cannot be chosen independently from the section \( s \in \Gamma(p) \). Indeed, following the same pattern as in [2] we can prove:
Proposition 1 The pair \((s, X) \in \Gamma(p) \times \mathcal{X}(T^{k-1}Q)\) satisfies the two conditions in Definition 7 if, and only if, \(X_{LH}\) and \(X\) are \(s\)-related.

Corollary 1 If \(s \in \Gamma(p)\) and \(X \in \mathcal{X}(T^{k-1}Q)\) satisfy the two conditions in Definition 7 then \(X = T_p \circ X_{LH} \circ s\).

That is, the vector field \(X \in \mathcal{X}(T^{k-1}Q)\) is completely determined by the section \(s \in \Gamma(p)\), and it is called the vector field associated to \(s\). Therefore, the search of a pair \((s, X) \in \Gamma(p) \times \mathcal{X}(T^{k-1}Q)\) satisfying the two conditions in Definition 7 is equivalent to the search of a section \(s \in \Gamma(p)\) such that the pair \((s, T_p \circ X_{LH} \circ s)\) satisfies the same condition. Thus, we can restate the problem as follows.

Proposition 2 The generalized \(k\)-th-order unified Hamilton-Jacobi problem for \(X_{LH}\) is equivalent to finding a section \(s \in \Gamma(p)\) satisfying the following conditions:

1. The submanifold \(\text{Im}(s) \hookrightarrow \mathcal{W}\) is contained in \(\mathcal{W}_0\).

2. If \(\gamma : \mathbb{R} \rightarrow T^{k-1}Q\) is an integral curve of \(T_p \circ X_{LH} \circ s \in \mathcal{X}(T^{k-1}Q)\), then \(s \circ \gamma : \mathbb{R} \rightarrow \mathcal{W}\) is an integral curve of \(X_{LH}\), that is
\[
T_p \circ X_{LH} \circ s \circ \gamma = \dot{s} \implies X_{LH} \circ (s \circ \gamma) = \dot{s} \circ \gamma.
\]

Proposition 3 The following assertions on a section \(s \in \Gamma(p)\) are equivalent.

1. \(s\) is a solution to the generalized \(k\)-th-order unified Hamilton-Jacobi problem.

2. The submanifold \(\text{Im}(s) \hookrightarrow \mathcal{W}\) is invariant under the flow of the vector field \(X_{LH}\) solution to equation (1) (that is, \(X_{LH}\) is tangent to the submanifold \(\text{Im}(s)\)).

3. The section \(s\) satisfies the dynamical equation \(i(X)(s^*\Omega) = d(s^*H)\), where \(X = T_p \circ X_{LH} \circ s\) is the vector field associated to \(s\).

(Proof) The proof is analogous to that of Proposition 6 and Theorem 2 in [2].

Coordinate expression. Let \((q_0^A, \ldots, q_{2k-1}^A, p_0^A, \ldots, p_{k-1}^A)\) be a set of local coordinates in \(Q\), with \(1 \leq A \leq n\), and \((q_0^A, \ldots, q_{2k-1}^A, p_0^A, \ldots, p_{k-1}^A)\) the induced local coordinates in \(\mathcal{W}\) (see [7] for details). Then, local coordinates in \(\mathcal{W}\) adapted to the p-bundle structure are \((q_i^A, q_j^A, p_i^A)\), where \(0 \leq i \leq k - 1\), \(k \leq j \leq 2k - 1\). Hence, a section \(s \in \Gamma(p)\) is given locally by \(s(q_i^A) = (q_i^A, s_j^A, \alpha_A^i)\), where \(s_j^A, \alpha_A^i\) are local functions in \(T^{k-1}Q\).

From Proposition 3 an equivalent condition for a section \(s \in \Gamma(p)\) to be a solution of the generalized \(k\)-th-order unified Hamilton-Jacobi problem is that the dynamical vector field \(X_{LH}\) is tangent to the submanifold \(\text{Im}(s) \hookrightarrow \mathcal{W}\), which is defined locally by the constraints \(q_j^A - s_j^A = 0\) and \(p_i^A - \alpha_A^i = 0\). From [7], the vector field \(X_{LH}\) solution to equation (1) is given locally by
\[
X_{LH} = \sum_{l=0}^{2k-2} q_{l+1}^A \frac{\partial}{\partial q_i^A} + F^A \frac{\partial}{\partial q_{2k-1}^A} + \frac{\partial L}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + \left( \frac{\partial L}{\partial q_l^A} - p_i^A \right) \frac{\partial}{\partial p_i^A},
\]
where \( F^A \) are the functions solution to the following system of \( n \) equations

\[
(-1)^k (F^B - d_T(q_{2k-1}^B)) \frac{\partial^2 L}{\partial q_k^B \partial q_k^B} + \sum_{i=0}^{k} (-1)^i d_T \left( \frac{\partial L}{\partial q_i^A} \right) = 0.
\]

Hence, requiring \( X_{LH}(q_j^A - s_j^A) = 0 \) and \( X_{LH}(p_{2}^j - \alpha_j^A) = 0 \) we obtain the following system of \( 2kn \) partial differential equations on \( \text{Im}(s) \)

\[
s_{j+1}^A - q_{i+1}^B \frac{\partial s_i^A}{\partial q_i^B} - s_k^B \frac{\partial s_{2k-1}^A}{\partial q_{2k-1}^B} = 0 ; \quad F^A - q_{i+1}^B \frac{\partial s_{2k-1}^A}{\partial q_{2k-1}^B} - s_k^B \frac{\partial s_{2k-1}^A}{\partial q_{2k-1}^B} = 0
\]

\[
\frac{\partial L}{\partial q_i^A} - q_{i+1}^B \frac{\partial \alpha_i^A}{\partial q_i^B} - s_k^B \frac{\partial \alpha_{2k-1}^A}{\partial q_{2k-1}^B} = 0 ; \quad \frac{\partial L}{\partial q_i^A} - \alpha_{i+1}^A - q_{i+1}^B \frac{\partial \alpha_{i+1}^A}{\partial q_i^B} - s_k^B \frac{\partial \alpha_{2k-1}^A}{\partial q_{2k-1}^B} = 0
\]

This is a system of \( 2kn \) partial differential equations with \( 2kn \) unknown function \( s_j^A, \alpha_j^B \). Hence, a section \( s \in \Gamma(p) \) is a solution to the generalized \( k \)-order Lagrangian-Hamiltonian Hamilton-Jacobi problem if, and only if, its component functions satisfy the local equations \([3]\).

### 3.2 The Hamilton-Jacobi problem

In general, to solve the generalized \( k \)-order Hamilton-Jacobi problem is a difficult task since we must find \( kn \)-dimensional submanifolds of \( \mathcal{W} \) contained in the submanifold \( \mathcal{W}_o \) and invariant by the dynamical vector field \( X_{LH} \). Hence, it is convenient to consider a less general problem and require some additional conditions to the section \( s \in \Gamma(p) \) \([1, 2]\).

**Definition 2** The \( k \)-order Lagrangian-Hamiltonian Hamilton-Jacobi problem consists in finding sections \( s \in \Gamma(p) \) solution to the generalized \( k \)-order unified Hamilton-Jacobi problem such that \( s^* \Omega = 0 \). Such a section is called to the \( k \)-order Lagrangian-Hamiltonian Hamilton-Jacobi problem.

From the definition of \( \Omega \in \Omega^2(\mathcal{W}) \) we have

\[
s^* \Omega = s^* (pr_2^* \omega_{k-1}) = (pr_2 \circ s)^* \omega_{k-1}.
\]

Hence, \( s^* \Omega = 0 \) if, and only if, \( (pr_2 \circ s)^* \omega_{k-1} = 0 \). As \( \Gamma(\pi_{T^k-1}Q) = \Omega^1(T^{k-1}Q) \), the section \( pr_2 \circ s \in \Gamma(\pi_{T^k-1}Q) \) is a 1-form in \( T^{k-1}Q \), and from the properties of the tautological form \( \theta_{k-1} \) of the cotangent bundle \( T^*(T^{k-1}Q) \) we have

\[
(pr_2 \circ s)^* \omega_{k-1} = (pr_2 \circ s)^*(-d \theta_{k-1}) = -d((pr_2 \circ s)^* \theta_{k-1}) = -d(pr_2 \circ s).
\]

Hence, the condition \( s^* \Omega = 0 \) is equivalent to \( pr_2 \circ s \in \Omega^1(T^{k-1}Q) \) being a closed 1-form. Therefore, the Hamilton-Jacobi problem can be reformulated as follows.

**Proposition 4** The \( k \)-order unified Hamilton-Jacobi problem is equivalent to finding sections \( s \in \Gamma(p) \) solution to the generalized \( k \)-order unified Hamilton-Jacobi problem such that \( pr_2 \circ s \) is a closed 1-form in \( T^{k-1}Q \).

Taking into account the new assumption \( s^* \Omega = 0 \) in Definition \([3]\), a consequence of Proposition \([3]\) is the following result.
Proposition 5. The following assertions on a section \( s \in \Gamma(p) \) satisfying \( s^*\Omega = 0 \) are equivalent:

1. \( s \) is a solution to the \( k \)th-order unified Hamilton-Jacobi problem.
2. \( d(s^*H) = 0 \).
3. \( \text{Im}(s) \) is an isotropic submanifold of \( \mathcal{W} \) invariant by \( X_{LH} \).
4. The integral curves of \( X_{LH} \) with initial conditions in \( \text{Im}(s) \) project onto the integral curves of \( X = T_p \circ X_{LH} \circ s \).

Coordinate expression. From [7], the Hamiltonian function in \( \mathcal{W} \) has coordinate expression

\[
H = q^A_i + p^i_A - L(q^A_0, \ldots, q^A_k),
\]

Thus, its differential is given locally by

\[
dH = -\frac{\partial L}{\partial q^A_0} dq^A_0 + \left( p^i_A - \frac{\partial L}{\partial q^A_{i+1}} \right) dq^A_{i+1} + q^A_{i+1} dp^i_A.
\]

Hence, the condition \( d(s^*H) = 0 \) in Proposition 5 holds if, and only if, the following \( kn \) partial differential equations are satisfied

\[
q^{B}_{i+1} \frac{\partial \alpha^i_A}{\partial q^B_0} + s^B_i \frac{\partial \mu^{k-1}}{\partial q^B_0} + \alpha^{k-1} \frac{\partial s^B_k}{\partial q^A_{0}} - \left( \frac{\partial L}{\partial q^A_0} + \frac{\partial L}{\partial q^B_{k}} \frac{\partial s^B_k}{\partial q^A_{0}} \right) = 0,
\]

where \( 1 \leq l \leq k - 1 \).

Equivalently, we can require the 1-form \( \text{pr} \circ s \in \Omega^1(T^{k-1}Q) \) to be closed, that is, \( d(\text{pr} \circ s) = 0 \).

Locally, this condition reads

\[
\frac{\partial \alpha^i_A}{\partial q^B_j} = \frac{\partial \alpha^j_B}{\partial q^A_i} = 0, \text{ with } A \neq B \text{ or } i \neq j.
\]

Therefore, a section \( s \in \Gamma(p) \) is a solution to the \( k \)th-order Lagrangian-Hamiltonian Hamilton-Jacobi problem if, and only if, the local functions \( s^A_j, \alpha^i_A \) satisfy the system of partial differential equations given by (3) and (4), or, equivalently (3) and (5). Observe that the system of partial differential equations may not be \( C^\infty(U) \)-linearly independent.

3.3 Relation with the Lagrangian and Hamiltonian formalisms

Finally, we state the relation between the solutions of the Hamilton-Jacobi problem in the unified formalism and the solutions of the problem in the Lagrangian and Hamiltonian settings given in [3].

Theorem 1. Let \( \mathcal{L} \in C^\infty(T^kQ) \) be a hyperregular Lagrangian function.

1. If \( s \in \Gamma(p) \) is a solution to the (generalized) \( k \)th-order Lagrangian-Hamiltonian Hamilton-Jacobi problem, then the sections \( s_{\mathcal{L}} = \text{pr}_1 \circ s \in \Gamma(p_{2k-1}) \) and \( \alpha = \text{pr}_2 \circ s \in \Omega^1(T^{k-1}Q) \) are solutions to the (generalized) \( k \)th-order Lagrangian and Hamiltonian Hamilton-Jacobi problems, respectively.
2. If \( s_L \in \Gamma(\rho_{k-1}^2) \) is a solution to the (generalized) \( k \)-th order Lagrangian Hamilton-Jacobi problem, then \( s = j_o \circ \text{pr}_1^{-1} \circ s_L \in \Gamma(p) \) is a solution to the (generalized) \( k \)-th order Lagrangian-Hamiltonian Hamilton-Jacobi problem.

If \( \alpha \in \Omega^1(T^{k-1}Q) \) is a solution to the (generalized) \( k \)-th order Hamiltonian Hamilton-Jacobi problem, then \( s = j_o \circ \text{pr}_2^{-1} \circ \alpha \in \Gamma(p) \) is a solution to the (generalized) \( k \)-th order Lagrangian-Hamiltonian Hamilton-Jacobi problem.

(Proof) The proof of the first item follows the same patterns that the proof of Theorem 1 in [3]. For the second item, the key point is to take into account that the maps \( \text{pr}_1: W \to T^{2k-1}Q \) and \( \text{pr}_2: W \to T^*(T^{k-1}Q) \) are diffeomorphisms, and that the dynamical vector field \( X_{LH} \in \mathfrak{X}(W) \) solution to equation (1) is tangent to \( W_o \), and therefore is \( j_o \)-related to a vector field \( X_o \in \mathfrak{X}(W_o) \) for which it is possible to state an equivalent Hamilton-Jacobi problem.

3.4 An example: A (homogeneous) deformed elastic cylindrical beam with fixed ends

Consider a deformed elastic cylindrical beam with both ends fixed (see [7] and references therein). The problem is to determine its shape; that is, the width of every section transversal to the axis. This gives rise to a 1-dimensional second-order dynamical system, which is autonomous if we require the beam to be homogeneous. Let \( Q \) be the 1-dimensional smooth manifold modeling the configuration space of the system with local coordinate \((q_0)\). Then, in the natural coordinates of \( T^2Q \), the Lagrangian function for this system is

\[
\mathcal{L}(q_0, q_1, q_2) = \frac{1}{2} \mu q_2^2 + \rho q_0,
\]

where \( \mu, \rho \in \mathbb{R} \) are constants, and \( \mu \neq 0 \). This a regular Lagrangian function because the Hessian matrix

\[
\left( \frac{\partial^2 \mathcal{L}}{\partial q_2 \partial q_2} \right) = \mu,
\]

has maximum rank equal to 1 when \( \mu \neq 0 \).

In the induced natural coordinates \((q_0, q_1, q_2, q_3, p^0, p^1)\) of \( W \), the coordinate expressions of the presymplectic form \( \Omega = \text{pr}_2^* \omega_1 \in \Omega^2(W) \) and the Hamiltonian function \( H = \mathcal{C} - \mathcal{L} \in C^\infty(W) \) are

\[
\Omega = dq_0 \wedge dp^0 + dq_1 \wedge dp^1 ; \quad H = q_1 p^0 + q_2 p^1 - \frac{1}{2} \mu q_2^2 - \rho q_0.
\]

Thus, the semispray of type 1 \( X_{LH} \in \mathfrak{X}(W) \) solution to the dynamical equation (1) and tangent to the submanifold \( W_o = \text{graph}(FL) \hookrightarrow W \) has the following coordinate expression

\[
X_{LH} = q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \frac{\rho}{\mu} \frac{\partial}{\partial q_3} + \rho \frac{\partial}{\partial p^0} - p^0 \frac{\partial}{\partial p^1}.
\]

In the following we state the equations for the (generalized) Lagrangian-Hamiltonian Hamilton-Jacobi problem for this dynamical system.

In the generalized Lagrangian-Hamiltonian Hamilton-Jacobi problem we look for sections \( s \in \Gamma(p) \), given locally by \( s(q_0, q_1) = (q_0, q_1, s_2, s_3, \alpha^0, \alpha^1) \), such that the submanifold \( \text{Im}(s) \hookrightarrow W \) is invariant under the flow of \( X_{LH} \in \mathfrak{X}(W) \). Since the constraints defining locally \( \text{Im}(s) \) are
\[ q_2 - s_2 = 0, \quad q_3 - s_3 = 0, \quad p^0 - \alpha^0 = 0, \quad p^1 - \alpha^1 = 0, \]
then the equations for the section \( s \) are
\[
\begin{align*}
  s_3 - q_1 & \frac{\partial s_2}{\partial q_0} - s_2 \frac{\partial s_2}{\partial q_1} = 0 ; & \frac{\rho}{\mu} - q_1 \frac{\partial s_3}{\partial q_0} - s_2 \frac{\partial s_3}{\partial q_1} = 0, \\
  \rho - q_1 & \frac{\partial \alpha^0}{\partial q_0} - s_2 \frac{\partial \alpha^0}{\partial q_1} = 0 ; & -\alpha^0 - q_1 \frac{\partial \alpha^1}{\partial q_0} - s_2 \frac{\partial \alpha^1}{\partial q_1} = 0.
\end{align*}
\]

For the Lagrangian-Hamiltonian Hamilton-Jacobi problem, we require in addition the section \( s \in \Gamma(\rho^W) \) to satisfy \( s^* \Omega = 0 \) or, equivalently, the form \( \text{pr}_2 \circ s \in \Omega^1(TQ) \) to be closed. In coordinates, if \( s = (q_0, q_1, s_2, s_3, \alpha^0, \alpha^1) \), then the 1-form \( \text{pr}_2 \circ s \) is given by
\[
\begin{align*}
  \text{pr}_2 \circ s & = \alpha^0 dq_0 + \alpha^1 dq_1.
\end{align*}
\]
Hence, a section \( s \in \Gamma(p) \) solution to the unified Hamilton-Jacobi problem for this system must satisfy the following system of 5 partial differential equations
\[
\begin{align*}
  s_3 - q_1 & \frac{\partial s_2}{\partial q_0} - s_2 \frac{\partial s_2}{\partial q_1} = 0 ; & \frac{\rho}{\mu} - q_1 \frac{\partial s_3}{\partial q_0} - s_2 \frac{\partial s_3}{\partial q_1} = 0, \\
  \rho - q_1 & \frac{\partial \alpha^0}{\partial q_0} - s_2 \frac{\partial \alpha^0}{\partial q_1} = 0 ; & -\alpha^0 - q_1 \frac{\partial \alpha^1}{\partial q_0} - s_2 \frac{\partial \alpha^1}{\partial q_1} = 0.
\end{align*}
\]
Observe that, when the condition \( \text{Im}(s) \subseteq W_o = \text{graph} F^L \) is required, these equations project to the Lagrangian or Hamiltonian equations for the Hamilton-Jacobi problem [3].

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