IKT-approach to MHD turbulence

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Abstract

An open issue in turbulence theory is related to the determination of the exact evolution equation for the probability density associated to the relevant (stochastic) fluid fields. Such an equation in the usual approaches to turbulence reproduces, at most in an approximate sense, the correct fluid equations. In this paper we present a statistical model which applies to an incompressible, resistive and quasi-neutral magnetofluid. The approach is based on the formulation of an inverse kinetic theory (IKT) for the full set of MHD equations appropriate for an incompressible, viscous, quasi-neutral, isentropic, isothermal and resistive magnetofluid. Basic feature of the new approach is that it relies on first principles - including in particular the exact validity of the fluid equations - and thus permits the determination of the correct evolution equation for the probability density. Specific application of the theory here considered concerns the case of statistically homogeneous and stationary MHD turbulence.

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I. INTRODUCTION

The goal of this paper is to investigate a well-known theoretical issue of fluid dynamics. This is concerned with the rigorous formulation of the so-called turbulence problem, i.e., the description of the dynamics of a classical fluid in the presence of stochastic sources. We refer here, in particular, to the statistical description of turbulence for incompressible fluids based on its phase-space approach, namely achieved by representing the turbulent fluid means of a suitable phase-space probability density function (pdf). In such a context, an open issue is related to the determination of the exact evolution equation for the probability density associated to the relevant (stochastic) fluid fields in an incompressible turbulent fluid. It is well known, in fact, that in customary approaches to turbulence (see for example Monin and Yaglom [1975 and Pope, 2000]) this equation reproduces only in an approximate sense the correct fluid equations. In this paper we present a statistical theory which applies to an incompressible, resistive and quasi-neutral magnetofluid.

The purpose of this paper is to propose a solution based on the formulation of an inverse kinetic theory (IKT) for an incompressible, viscous, quasi-neutral, isentropic and resistive magnetofluid. Basic feature of the theory is that it permits the determination of the exact evolution equation for the probability density.

A. Deterministic and stochastic descriptions of fluids

In fluid dynamics the state of the fluid is assumed to be prescribed by an appropriate set of suitably smooth functions \( \{Z\} \equiv \{Z_i, i = 1, n\} \) denoted as fluid fields. These are required to be real functions which are at least continuous in all points of a closed set \( \Omega \times I \), closure of the extended configuration domain \( \Omega \times I \). In the remainder we shall require that:

1. \( \Omega \) (configuration domain) is a bounded subset of the Euclidean space \( E^3 \) on \( R^3 \);

2. \( I \) is identified, when appropriate, either with a bounded time interval, \( i.e., I=[t_0, t_1] \subseteq R \), or with the same time axis \( (R) \);

3. in the open set \( \Omega \times I \) the functions \( \{Z\} \), are assumed to be solutions of an appropriate closed set of PDE’s, denoted as fluid equations;
4. by assumption, these equations together with appropriate initial and boundary conditions are required to define a well-posed problem with unique strong solution defined everywhere in $\Omega \times I$.

Depending on the definition of initial conditions (A), boundary conditions (B), possible volume forces (C), as well as, more generally, also the functional form assumed for the fluid equations (D), two types of descriptions of the fluid, deterministic and stochastic, can in principle be distinguished, in which the fluid fields are treated respectively as deterministic or stochastic functions (see definitions in Appendix). The two descriptions are obtained, respectively, when all conditions (A), (B), (C) and (D) are deterministic, or at least one of them [A, B, C or D] is prescribed in a stochastic sense (see below). For definiteness, in the following (except for Sec.7) we shall formally rule out case D, imposing that the functional form of the fluid equations remains the same in both cases.

B. Two types of turbulence - Homogeneous and stationary turbulence

The stochastic description of fluids is usually denoted as turbulence\(^\text{4}\). In this respect, however, two different types of phenomena should be distinguished, \textit{i.e.},

- \textit{physical turbulence}

- \textit{numerical turbulence}.

The first one arises due to "physical effects", namely the specific definition adopted for the fluid equations and for the related initial and boundary conditions (these definitions may be considered, in a sense, as physical characteristics of a fluid, \textit{i.e.}, as related to the properties of the fluid). Thus, physical turbulence may appear only as a result of the \textit{stochastic "sources"} indicated above. The first two (A and B) are due to possible stochastic initial and boundary conditions. In fact, the initial and/or boundary state of the fluid may not be known deterministically, as implied by measurement errors, or by the action of some type of initial and/or boundary volume/surface stochastic forcing. Finally (case D), it is obvious that the fluid equations for the stochastic fluids fields (\textit{stochastic fluid equations}) may, in principle, differ in other ways (besides C), from the corresponding deterministic...
fluid equations. Their precise prescription is therefore essential for the definition of specific
turbulence models.

Instead, numerical turbulence occurs only in numerical simulations and is due specifically to approximations involved in the numerical solution method (for the fluid equations). Hence, it is necessarily related to the specific algorithms adopted for the construction of the numerical solutions.

Both types of turbulence may produce, in principle, ”real” effects to be observed in numerical experiments. However, it should be stressed that, in practice, the distinction between the two phenomena may be difficult, or even impossible, to ascertain in numerical simulations. Nevertheless, from theoretical perspective it is obvious that the distinction between the two phenomena should be made.

This paper deals with exact properties and implications of the fluid equations (and the related initial and boundary conditions), disregarding completely issue related to the possible choice of the approximate solution methods adopted for their numerical solution. Hence, we will face only the first type of phenomena. For this reason in the following the term ”turbulence” will be meant, in a proper sense, only in the physical sense here specified.

Turbulence is therefore characterized by the presence of stochastic fluid fields $Z_i \in \{Z\}$, for $i = 1, n$ (with $\{Z\}$ defining the state of the fluid), which are assumed to depend on suitable hidden variables $\alpha = \{\alpha_i, i = 1, k\} \in V_\alpha \subseteq \mathbb{R}^k$, with $k \geq 1$, which are assumed independent of $(r,t)$. Hidden variables by assumption cannot be known deterministically (in other words, in the context of classical mechanics, they are not ”observables”), and hence are necessarily stochastic. This means that by assumption: 1) the hidden variables $\alpha$ are characterized by a probability density $g$, denoted as reduced stochastic probability density function (rs-pdf) defined on $V_\alpha$, with $V_\alpha$ a non-empty subset of $\subseteq \mathbb{R}^k$ and $k \geq 1$; 2) that the fluid fields $\{Z\}$ will generally be considered as stochastic functions (see Appendix).

Critical issues in turbulence theory are in principle related both to the definition of the hidden variables $\alpha$ as well as the determination of the corresponding rs-pdf $g$. Both depend necessarily on the specific assumptions concerning the source of stochasticity. Therefore, it is obvious that for both of them the definition is non-unique. In particular, the precise definition of $g$ may depend on the specific source (of stochasticity). This is obvious - for example - when its origin is either provided by boundary conditions (i.e., surface forces) or respectively by volume forces: both sources, in fact, can be prescribed in principle in
arbitrary ways. It follows that, in general, $g$ should be considered of the form

$$g = g(r,t,\alpha),$$

(1)

with $(r,t) \in \Omega \times I$. This corresponds to the so-called (statistically) nonhomogenous and nonstationary turbulence. Particular solutions are, however, provided by the requirements that there results identically either $g = g(t,\alpha)$ (homogeneous turbulence), or $g = g(r,\alpha)$ (stationary turbulence).

In subsequent sections (Sec.2-6) we intend to present a mathematical formulation of turbulence in magnetofluids which pertains specifically to the validity of both assumptions, i.e.,

$$g = g(\alpha).$$

(2)

This defines the so-called homogenous and stationary turbulence. Possible generalizations of the theory to nonhomogeneous and nonstationary turbulence are discussed in Sec.7. For this purpose, all the fluid fields $\{Z\}$ will be generally assumed stochastic (i.e., non-deterministic), leaving unspecified the definition of the hidden variables and - apart for the previous requirement - the form of the rs-pdf.

C. Turbulence in magnetofluids

The phenomenon of turbulence in magnetofluids (MHD turbulence) is nowadays playing a major role in plasma physics and fluid dynamics research. We refer here in particular to incompressible magneto- and conducting fluids for which the fluid description is appropriate (for such systems typically a statistical description based on the microscopic dynamics is not possible). The widespread picture of MHD-turbulence phenomena occurring in these fluids consists of an ensemble of finite-amplitude waves with random phase. Examples of turbulence models of this type are well known. For example, Iroshnikov 3 and Kraichnan 6 independently assumed that MHD turbulence occurs as a result of nonlinear interactions between Alfven wave packets 7, 8. However, there is an increasing evidence that this picture is an oversimplification. In particular, MHD turbulence may include fluctuations whose phase-coherence characteristics are incompatible with wave-like properties. The latter are the so-called coherent structures, like clumps, holes, current filaments, shocks, magnetic islands, vortices, convective cells, zonal flows, streamers, etc. 9, 10.
In fluid turbulence the signature of the presence of coherent structures is provided by the existence of non-Gaussian features in the probability density. This is usually identified with the velocity-difference probability density function, traditionally adopted for the description of homogeneous turbulence. In the past several statistical models have been proposed for its determination, which include the mapping-closure method, the test-function method and the field-theoretical approach \[11, 12, 13\]. Nevertheless, despite the progress achieved in modelling key features of the basic phenomenology, still missing is a consistent, theory-based, statistical description of MHD turbulence. Clearly, such formulation - if achievable at all - should rely exclusively on a rigorous, deductive formulation of the turbulence-modified fluid equations following from the fluid equations.

D. Goals of the investigation

Based on a recently proposed inverse kinetic theory (IKT) for classical and quantum fluids (Ellero and Tessarotto, 2004-2007 \[14, 15, 16, 17\]), and in particular its formulation for MHD equations \[18, 19\], a phase-space statistical model of turbulence is proposed. The present approach, which represents a generalization of the one recently developed for incompressible neutral fluids \[3, 20\], allows to determine exactly the stochastic evolution equation of the local position-velocity joint probability density function (local pdf). The local pdf, rather than the velocity-increment pdf traditionally adopted in turbulence theory, is in fact found to be meaningful for the dynamics of the fluid. Indeed, we intend to prove that the stochastic fluid fields which characterize the turbulent magnetofluid are uniquely determined by means of suitable velocity-moments of the local pdf. Nevertheless, as previously pointed out \[3, 17\], IKT’s are intrinsically non-unique in character, in particular because they may be also formulated in such a way to apply only to a suitable subset of the fluid equations. In this respect two types of phase-space approaches can in principle be developed, namely either:

- a Complete IKT, yielding the full set of stochastic fluid equations which advance in time the stochastic fluid fields \(\{Z\}\);
- a Reduced IKT, which applies only to a suitable subset of fluid equations, to be identified with the set fluid equations advancing in time only \(\{\langle Z \rangle\}\).
In particular the Complete IKT can be non-uniquely prescribed (see THM’s 1 and 2) so that:

1. the time-evolution of the stochastic fluid fields \( \{Z\} \) describing the state of a turbulent magnetofluid is uniquely determined by means of a suitable phase-space classical dynamical system (the MHD-dynamical system);

2. the MHD-dynamical system is uniquely associated to an appropriate inverse kinetic equation (IKE) which advances in time an appropriate phase-space probability density \( f \) (pdf);

3. the inverse kinetic equation (IKE) is such that its velocity moments coincide with the complete set of stochastic(MHD) fluid equations. In particular, the IKE’s which advance in time the stochastic-averaged pdf \( \langle f \rangle \) as well its stochastic fluctuation \( \delta f = f - \langle f \rangle \), determine respectively the fluid equations for stochastic-averaged fluid fields \( \{\langle Z \rangle\} \) and the stochastic fluctuations \( \{\delta Z\} = \{Z - \langle Z\rangle\} \);

4. under appropriate initial and smoothness conditions, the strict positivity of \( f \) is assured for arbitrary strictly positive initial pdf’s;

5. under the same assumptions for \( \langle f \rangle \), exhibits an irreversible time-evolution (see the Corollary of THM.2)

Instead, as previously pointed out \([3, 19]\), the Reduced IKT (see THM.3) can be realized so that:

1. the time evolution of the stochastic averaged fluid fields is similarly uniquely determined by means of a suitable dynamical system and an appropriate inverse kinetic equation advancing in time the stochastic-averaged pdf \( \langle f \rangle \);

2. the strict positivity of \( \langle f \rangle \) is assured;

3. the inverse kinetic equation for \( \langle f \rangle \) is Markovian.

Key feature of present theory is that - unlike several other approaches to be found in the literature (for a review see for example, Monin and Yaglom, 1975 \([1]\) and Pope, 2000 \([2]\)) - the relationship between fluid fields and the pdf is exact (and not just only approximate). In
particular, the evolution of the stochastic-averaged fluid fields is determined via a suitable nonlinear transformation (here generated by an appropriate dynamical system). This is achieved by the construction of a Vlasov-type kinetic equation which advances in time the local pdf and - as a consequence - also the stochastic fluid fields of the magnetofluid.

II. MOTIVATIONS: IKT APPROACH FOR MAGNETOFLUIDS

A fundamental aspect of fluid dynamics is the construction of phase-space approaches for realistic fluids. Indeed, phase-space techniques are well known both in classical and quantum fluid dynamics. In fact, generally the fluid equations represent a mixture of hyperbolic and elliptic PDE’s, which are extremely hard to study both analytically and numerically. This has motivated in the past efforts to replace them with other equations, possibly simpler to solve or mathematically more elegant. In this connection a particular viewpoint - which applies in principle both to classical and quantum fluids - is represented by the class of so-called inverse problems, involving the search of a so-called inverse kinetic theory (IKT) able to yield the complete set of fluid equations for the fluid fields, via a suitable correspondence principle. As a consequence the fluid equations are identified with appropriate moment equations constructed in terms of the relevant kinetic equation. This raises the interesting question whether the theory can be generalized to arbitrary classical magnetofluids. The issue is relevant at least for the following reasons: a) the proliferation of numerical algorithms in MHD which reproduce the behavior of incompressible fluids only in an asymptotic sense; b) the possible verification of conjectures involving the validity of appropriate equations of state for the fluid pressure; c) the ongoing quest for efficient numerical solution methods for the MHD equations, with particular reference to Lagrangian methods. Finally, another important motivation is the possibility of achieving an exact solution to this problem, based on inverse kinetic theory (IKT) (see Tessarotto et al., 2008 [18]). In fact an aspect of fluid dynamics is represented by the class of so-called inverse problems, involving the search of IKT’s able to yield identically a prescribed set of fluid equations. A few examples of IKT’s which hold both for classical and quantum fluids have been recently investigated (Tessarotto et al., 2000-2007 [14, 15, 16, 17, 21]).

In particular, among the possible IKT’s, special have IKT’s which pertain to incompressible fluids described in terms of strong solutions of the corresponding fluid equations. In
such a case IKT’s can be defined in such a way that the kinetic equation which advances in time the kinetic distribution function \textit{generates a suitable classical dynamical system}. Phase-space approaches of this type have been already achieved for several types of fluids (Ellero and Tessarotto, 2000-2008 [15, 16, 17, 21]). In particular, the same approach can be extended in principle also to incompressible magnetofluids. Without loss of generality, in the following we consider the case of the MHD equations describing an incompressible, viscous, quasi-neutral, isentropic, isothermal and resistive magnetofluid. Starting point is the assumption that there exists a suitable phase-space classical dynamical system, to be denoted as \textit{MHD-dynamical system},

\[ \mathbf{x}_o \to \mathbf{x}(t) = T_{t,t_o} \mathbf{x}_o, \]  

defined in such a way that \textit{its evolution operator} \( T_{t,t_o} \) \textit{uniquely advances in time the fluid fields} \[ 17 \]. The dynamical system (and the operator \( T_{t,t_o} \)) is assumed to be generated by a suitably smooth vector field \( \mathbf{X}(\mathbf{x}, t; Z) \), i.e., as a solution of the initial-value problem

\[
\begin{cases}
\frac{d}{dt} \mathbf{x} = \mathbf{X}(\mathbf{x}, t; Z), \\
\mathbf{x}(t_o) = \mathbf{x}_o,
\end{cases}
\]

where \( \mathbf{X}(\mathbf{x}, t; Z) \) is required to be - in an appropriate way - functionally dependent on the set of fluid fields \( \{Z\} \) which characterize the fluid. Here \( \mathbf{x} = (\mathbf{r}_1, \mathbf{v}_1) \in \Gamma \) and respectively \( \mathbf{r}_1 \) and \( \mathbf{v}_1 \) denote an appropriate "state-vector" and the corresponding "configuration" and "velocity" vectors. In particular, it is assumed that \( \mathbf{x} \) spans a phase-space of dimension \( 2n \) \( (\Gamma \subseteq \mathbb{R}^{2n}) \), where by assumption \( n \geq 3 \). Therefore, introducing the corresponding phase-space probability density function (pdf) \( f(\mathbf{x}, t; Z) \), in \( \Gamma \) it fulfills necessarily the integral Liouville equation

\[ J(t; Z) f(\mathbf{x}(t), t; Z) = f(\mathbf{x}_o, t_o, Z), \]

to be viewed in the following as a \textit{Lagrangian inverse kinetic equation} (Lagrangian IKE), with \( J(t; Z) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right| \) denoting the Jacobian of the map \( 3 \). In particular, let us assume for definiteness that \( 3 \) is a diffeomorphism at least of class \( C^2(\Gamma \times I \times I) \), with \( I \subset \mathbb{R} \) denoting an appropriate finite time interval. Then, if the initial pdf \( f(\mathbf{x}_o, t_o, Z) \) is suitably smooth it follows that the pdf \( f(\mathbf{x}, t; Z) \) satisfies also the differential Liouville equation

\[ L(Z) f(\mathbf{x}, t; Z) = 0, \]
\( L(Z) \) denoting the corresponding streaming operator

\[
L(Z) \cdot \equiv \frac{\partial}{\partial t} \cdot + \frac{\partial}{\partial x} \cdot \{X(x,t;Z)\}.
\]  

(7)

This equation [and the equivalent Lagrangian equation (5)] will be considered in the following as an \textit{Eulerian inverse kinetic equation} (Eulerian IKE) for a suitable set of fluid equations. In other words, the arbitrariness of the dynamical system \( i.e., of X(x,t;Z) \), will be used to seek a representation of the vector field \( X(x,t;Z) \) such that that suitable the velocity moments of \( f(x,t;Z) \) can be identified with the relevant fluid fields characterizing a prescribed fluid.

Let us consider, for definiteness, the case of an incompressible, viscous, quasi-neutral, isentropic, isothermal and resistive magnetofluid subject to the condition of isentropic flow. By assumption, the relevant fluids

\[
\{Z\} \equiv \{\rho = \rho_o > 0, V, p \geq 0, B, S_T\},
\]

(8)

\( i.e., \) respectively the mass density, the fluid velocity, the fluid pressure, the magnetic field and the thermodynamic entropy, are assumed to be defined in the whole existence domain \( \overline{\Omega} \times I \). In particular if \( \Omega \) denotes an open connected subset of \( R^3 \), its closure \( \overline{\Omega} \) by definition is the set where the mass density is a constant \( \rho = \rho_o > 0 \). We shall assume that the fluid fields are continuous in \( \overline{\Omega} \times I \), satisfy a suitable set of \textit{deterministic MHD equations} in the open set \( \Omega \times I \) and, moreover, fulfill appropriate initial and (Dirichlet) boundary conditions respectively at \( t = t_o \) and on the boundary of the configuration domain \( \delta \Omega \).

These requirements are represented respectively by the equations

\[
\begin{cases}
\rho = \rho_o, \\
\nabla \cdot V = 0, \\
\nabla \cdot B = 0 \\
NV = 0, \\
V \cdot NV = 0, \\
N_B B = 0, \\
\frac{\partial}{\partial t} S_T = 0, \\
Z(r,t_o) = Z_o(r), \\
Z(r,t)|_{\delta \Omega} = Z_w(r,t)|_{\delta \Omega}.
\end{cases}
\]

(9)
Here the notation is standard. Thus, $N$ and $N_B$ are the nonlinear Navier-Stokes and Faraday-Neumann differential operator

$$N \mathbf{V} = \frac{\partial}{\partial t} \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{1}{\rho_o} [\nabla p - \mathbf{f}] - \nu \nabla^2 \mathbf{V} \tag{10}$$

and

$$N_B \mathbf{B} = \frac{\partial}{\partial t} \mathbf{B} + \mathbf{V} \cdot \nabla \mathbf{B} - \mathbf{h} - \frac{c}{4\pi \sigma} \nabla^2 \mathbf{B}. \tag{11}$$

Here the vector fields $\mathbf{f}$ and $\mathbf{h}$, to be denoted as *volume forces*, read respectively

$$\mathbf{f} = \rho_o \mathbf{g} + \frac{1}{4\pi} \mathbf{B} \cdot \nabla \mathbf{B} - \nabla \left( \frac{B^2}{8\pi} \right), \tag{12}$$

$$\mathbf{h} = - \mathbf{B} \cdot \nabla \mathbf{V}. \tag{13}$$

where the mass density $\rho_o$, the kinematic viscosity $\nu$ and the conductivity $\sigma$ are all real positive constants to be considered non-stochastic. Furthermore, in the first equation, $\rho_o \mathbf{g}$ denotes the gravitational force density and the remaining terms the Lorentz force density. In addition, the first three equation in (9) denote respectively the so-called *incompressibility, isochoricity and divergence-free* (for $\mathbf{B}$) *conditions*. The remaining ones include, instead, the *Navier-Stokes equation*, the energy equation, the *Faraday-Neumann equation* for $\mathbf{B}$ and the isentropic entropy condition (with $S_T$ denoting the thermodynamic entropy). Finally, the last two equations denote respectively the initial and Dirichlet boundary conditions. Thus, by taking the divergence of the N-S equation, it follows the Poisson equation for the fluid pressure $p$ which reads

$$\nabla^2 p = - \rho_o \nabla \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) + \nabla \cdot \mathbf{f}, \tag{14}$$

with $p$ to be assumed non negative and bounded in $\bar{\Omega} \times \bar{T}$. Finally, in the following and consistent with Eqs.(9), the thermodynamic entropy $S_T$ will be assumed as a constant in the whole extended configuration domain $\bar{\Omega} \times I$.

### III. IKT FOR THE DETERMINISTIC MHD EQUATIONS

A prerequisite for the subsequent formulation of a turbulence theory based on the phase-space approach here adopted [3] (see subsequent Sections 4-6) is the development of a Complete IKT for the deterministic fluid equations [see Eqs.(9)]. As recently pointed out in Ref.[18], this goal can be achieved by introducing suitable generalization of the of the theory
earlier developed for the incompressible Navier-Stokes equations \[15, 16, 17, 21\]. For definiteness, let us introduce the notations \( \mathbf{r}_1 = (\mathbf{r}, s) \), \( \mathbf{v}_1 = (\mathbf{v}, y) \) and \( \mathbf{X} = \{\mathbf{v}_1, F_1(x, t; f, Z)\} \), where the vectors \( \mathbf{r} \) and \( \mathbf{v}_1 \) span, respectively, the whole configuration domain of the fluid \( \overline{\Omega} \) and the 3-dimensional velocity space \( \mathbb{R}^3 \). Moreover \( s \in \Omega, y \in \mathbb{R}^3 \) are two additional real vector variables, with \( s \) denoting in particular an ignorable configuration-space vector [both for the fluid fields and the kinetic distribution function \( f(x, t; Z) \)]. The streaming operator \( L(Z) \) in this case reads

\[
L(Z) \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{v}} \cdot \{F(x, t; f, Z)\} + \frac{\partial}{\partial y} \cdot \{Y(x, t; Z)\},
\]

where \( \{Z\} \) are the fluid fields and the vector field

\[
F_1(x, t; f, Z) = \{F(x, t; f, Z), Y(x, t; f, Z)\}
\]

can be interpreted as an effective mean field force acting on a particle state \( \mathbf{x} = (\mathbf{r}, s, \mathbf{v}, y) \). In the following we intend to prove that, at least in a suitable finite time-interval \( I \), the fluid fields \( Z(\mathbf{r}, t) \) can be uniquely identified with the velocity moments \( \int d\mathbf{v}_1 G(x, t)f(x, t; Z) \), where \( f(x, t; Z) \) is properly defined kinetic distribution function and \( G(x, t) \) appropriate weight functions. More precisely, there results:

\[
\begin{align*}
1 &= \int d\mathbf{v} dy f(x, t; Z), \\
\mathbf{V}(r,t) &= \int d\mathbf{v} dy \mathbf{v} f(x, t; Z), \\
p_1(r,t) &= \rho_o \int d\mathbf{v} dy \frac{u^2}{3} f(x, t; Z), \\
\mathbf{B}(r,t) &= \int d\mathbf{v} dy \mathbf{y} f(x, t; Z), \\
S_T(t) &= S(f(t)),
\end{align*}
\]

(correspondence principle). In particular \( \mathbf{V}, p \) and \( \mathbf{B} \) are identified respectively with the moments \( G(x, t) = \mathbf{v}, \rho_o u^2/3, \mathbf{y} \), where \( \mathbf{u} = \mathbf{v} - \mathbf{V} \). In addition, if the same distribution function \( f(t) \equiv f(x, t; Z) \) is strictly positive in the whole set \( \Gamma \times I \), \( S(f(t)) \) is the statistical Boltzmann-Shannon entropy functional

\[
S(f(t)) = -\int_{\Gamma} d\mathbf{x} f(x, t; Z) \ln f(x, t; Z).
\]

Consistent with Eqs.\((19)\) we shall impose that \( S(f(t)) \) exists for all \( t \in I \), and that there results identically (\( \forall t \in I \)) the conservation law

\[
S(f(t)) = \text{const},
\]
(constant H-theorem). To reach the proof, let us first show that, by suitable definition of the "force" fields \( \mathbf{F}(x, t) \) and \( \mathbf{Y}(x, t) \), a particular solution (which defines a kinetic equilibrium \[35\]) of the IKE \[6\] is delivered by the Maxwellian distribution:

\[
f_M(x, t; Z) = \frac{\rho}{\pi^2 v_{\text{th},p} v_{\text{th},T}} \exp \left\{-X^2 - Y^2\right\}.
\]

Here we denote

\[
X^2 = \frac{u^2}{v_{\text{th},p}^2},
\]

\[
Y^2 = \frac{w^2}{v_{\text{th},T}^2},
\]

where \( u = v - V \), \( w = y - B \), \( v_{\text{th}}^2 = 2\tilde{p}_1/\rho_o \) and \( v_{\text{th},T} \) is an 'a priori' arbitrary strictly positive constant. Furthermore

\[
p_1(r, t) = p_0(t) + p(r, t)
\]

is the kinetic pressure. In these definitions, \( p_0(t) \) (to be denoted as pseudo-pressure) is an arbitrary strictly positive and suitably smooth function defined in \( I \), while the mass \( m > 0 \) is an arbitrary real constant. The following theorem can immediately be proven:

**Theorem 1 - Complete IKT for the deterministic MHD equations**  
Let us assume that in the existence domain \( \Omega \times I \) the fluid fields \[36\] are strong solutions of the MHD fluid equations \[9\]. Furthermore, let us identify the vector fields \( \mathbf{F}(x, t; f, Z) \) and \( \mathbf{Y}(x, t; f, Z) \) with

\[
\mathbf{F}(x, t; f_M, Z) = \mathbf{F}_0 + \mathbf{F}_1,
\]

\[
\mathbf{Y}(x, t; f_M, Z) = \mathbf{u} \cdot \nabla \mathbf{B} - h - \frac{c}{4\pi\sigma} \nabla^2 \mathbf{B} + \mathbf{w} \cdot \nabla \mathbf{B},
\]

where \( \mathbf{F}_0 \) and \( \mathbf{F}_1 \) read respectively

\[
\mathbf{F}_0(x, t; f_M, Z) = \frac{1}{\rho_o} \mathbf{f} + \frac{1}{2} \mathbf{u} \cdot \nabla \mathbf{V} + \frac{1}{2} \nabla \mathbf{V} \cdot \mathbf{u} + \nu \nabla^2 \mathbf{V},
\]

\[
\mathbf{F}_1(x, t; f_M, Z) = \frac{\mathbf{u}}{2p_1} \frac{D}{Dt} p_1 + \frac{v_{\text{th}}^2}{2} \nabla \ln p_1 \left\{ \frac{\mathbf{u}^2}{v_{\text{th}}^2} - \frac{3}{2} \right\},
\]

while

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla
\]
and \( \frac{D}{Dt} p_1 \) is uniquely prescribed consistent with the energy equation, namely

\[
\frac{D}{Dt} p_1 = \frac{\partial}{\partial t} p_1 - \rho_o \left[ \frac{\partial}{\partial t} V^2/2 + V \cdot \nabla V^2/2 - \frac{1}{\rho_o} V \cdot f - \nu V \cdot \nabla^2 V \right].
\]  

(29)

Finally, let us assume that the initial kinetic probability density \( f(t_o) \equiv f(x,t_o,Z) \) belongs to the functional class \( \{ f(t_o) \} \), defined so that \( f(t_o) \) fulfils at time \( t = t_o \) the moment equations \( \{ T \} \), while the initial fluid fields \( Z(r,t_o) = Z_o(r) \) are suitably prescribed.

It follows that:

1) the local Maxwellian distribution \( \{ 20 \} \) is a particular solution of IKE [Eq.(6)];
2) this is a solution of the same equation if an only if the fluid fields \( \{ Z \} \) satisfy the fluid equations \( \{ 9 \} \);
3) the velocity-moment equations obtained by taking the weighted velocity integrals of Eq.(6) with the weights

\[
G(x,t) = \rho_o, v, \rho_o u^2/3, y, w^2/3
\]

(30)
deliver identically the same fluid equations \( \{ 9 \} \);
4) the initial kinetic probability density \( f(t_o) \equiv f(x,t_o,Z) \) is assumed to satisfy the principle of entropy maximization (PEM) \[34\], which requires that

\[
\delta S(f(t_o)) = 0;
\]  

(31)

5) \( \forall t, t_o \in I \), the statistical entropy \( S(f(t)) \) satisfies the constraint

\[
S(f(t_o)) = S(f(t))
\]

(constant-H theorem).

PROOF

First, we notice that if the fluid equations \( \{ 9 \} \) are satisfied identically in \( \Omega \times I \), the proof that \( \{ 20 \} \) is a particular solution of the IKE [Eq.(6)] follows by direct differentiation. The converse implication, i.e., the proof that if \( \{ 20 \} \) is a solution of Eq.(6) then the fluid equations \( \{ 9 \} \) are satisfied identically in \( \Omega \times I \), follows by evaluating in particular the \( (v_1-) \) velocity-space moments of Eq.(6) for the weights \( \{ 30 \} \) \( G = \rho_o (v - V)^2/3 \) and \( w^2/3 \). These moments deliver respectively the two isochoricity conditions

\[
\nabla \cdot V = 0,
\]

(33)

\[
\nabla \cdot B = 0.
\]

(34)
The proof of Proposition 3) follows by noting that the extremal solution of the variational principle (31) coincides necessarily with the Maxwellian distribution $f_M(x, t_o; Z)$ [see Eq. (20)]. Finally, to satisfy Proposition 4) it is sufficient to notice that the pseudo-pressure $p_o(t)$ [which enters the definition of the kinetic pressure (23)] can always be defined in such a way to satisfy, at least in a finite time interval $I$, the constraint (32). Q.E.D.

In analogy to the case of isothermal fluids the present theorem can be generalized in a straightforward way to a suitably smooth non-Maxwellian initial distribution function [16].

IV. STOCHASTIC MHD FLUID EQUATIONS

Let us now construct the stochastic MHD fluid equations appropriate in the case of homogeneous and stationary turbulence. In principle this requires the adoption of an appropriate turbulence (or stochastic) model. This concerns the introduction of suitable hypotheses, i.e.,

1. the definition of an appropriate set of hidden variables $\alpha$;
2. the definition of the associated rs-pdf $g$;
3. the assumption that at least some of the fluid fields, together with their initial and boundary values on $\delta\Omega$, and/or the volume forces $f$ and $h$ depend in a suitable way from the hidden variables $\alpha$;
4. the prescription of the form of the stochastic fluid equations.

It should be stressed that in principle the definition of the $\alpha$’s - and consequently of $g$ and of the dependencies on $\alpha$ to be suitably prescribed in the fluid equations - remains completely arbitrary. Regarding item 4), in particular, it is obvious that ‘a priori’ also the stochastic fluid equations might differ from the deterministic equations (9). Thus, for example, the condition of incompressibly of the fluid $\rho = \rho_o$ might be violated in a turbulent fluid due to the possible presence of stochastic mass density fluctuations (see Sec.7).

For definiteness, let us assume that the fluid fields are stochastic, i.e., they depend smoothly on the hidden variables $\alpha$, while the stochastic fluid equations retain the same functional form of the corresponding deterministic equations (9). This implies that the fluid remains, in particular, incompressible, viscous, quasi-neutral, isentropic, isothermal and resistive magnetofluid, which - in particular - implies necessarily $\delta\rho \equiv 0$ and $\delta S_T \equiv 0$. Hence
the meaningful fluid fields are only represented by the subset

$$\{Z(\alpha)\} \equiv \{\mathbf{V}(r,t,\alpha), p(r,t,\alpha) \geq 0, \mathbf{B}(r,t,\alpha)\}. \quad (35)$$

Together with their initial and boundary values on $\partial \Omega$ (defined respectively by the vector fields $Z_o(r,\alpha)$ and $Z(r,t,\alpha)|_{\partial \Omega} \equiv Z_w(r,t,\alpha)|_{\partial \Omega}$), and the volume forces $f(r,t,\alpha), h(r,t,\alpha)$, we shall require that they all admit the stochastic decompositions:

$$Z(r,t;\alpha) = \langle Z(r,t,\alpha) \rangle + \delta Z(r,t,\alpha), \quad (36)$$
$$Z_o(r;\alpha) = \langle Z_o(r,\alpha) \rangle + \delta Z_o(r,\alpha), \quad (37)$$
$$Z_w(r,t,\alpha) = \langle Z_w(r,t,\alpha) \rangle + \delta Z_w(r,t,\alpha), \quad (38)$$
$$f(r,t,\alpha) = \langle f(r,t,\alpha) \rangle + \delta f(r,t,\alpha). \quad (39)$$

Here the stochastic-averaging operator $\langle \rangle$ is defined according to Eq.(77), while $\langle Z \rangle, \langle Z_o \rangle, \langle Z_w \rangle, \langle f \rangle$ and $\delta Z, \delta Z_o, \delta Z_w$ and $\delta f$ are denoted respectively as stochastic averages and stochastic fluctuations.

In the remainder the precise definition of the operator $\langle \rangle$ [i.e., the specification of $g(\alpha)$ and of the hidden variables $\alpha$] is not needed. The only requirement to be introduced is that - consistent with the assumption (2) - it commutes with all the differential operators appearing in the MHD equations, in particular, $\frac{\partial}{\partial t}$, $\nabla$ and $\nabla^2$. In such a case the fluids equations for the averaged fields $\{\langle Z(\alpha) \rangle\}$ are immediately found to be

$$\begin{cases} 
\langle \rho \rangle = \rho_o, \\
\nabla \cdot \langle \mathbf{V} \rangle = 0, \\
\nabla \cdot \langle \mathbf{B} \rangle = 0, \\
\langle N \rangle \langle \mathbf{V} \rangle + \langle \delta N \delta \mathbf{V} \rangle = 0, \\
\langle \mathbf{V} \rangle \cdot [\langle N \rangle \langle \mathbf{V} \rangle + \langle \delta N \delta \mathbf{V} \rangle] = 0, \\
\langle N_B \rangle \langle \mathbf{B} \rangle + \langle \delta N_B \delta \mathbf{B} \rangle = 0, \\
\frac{\partial}{\partial t} \langle \mathbf{S} \rangle = 0, \\
\langle Z(r,t_o;\alpha) \rangle = \langle Z_o(r;\alpha) \rangle, \\
\langle Z(r,t;\alpha) \rangle|_{\delta \Omega} = \langle Z_w(r,t;\alpha) \rangle|_{\delta \Omega}, 
\end{cases} \quad (40)$$
which are denoted as the \textit{stochastic-averaged MHD equations} for \{⟨Z⟩\}. Analogous equations hold for the stochastic fluctuations \{δZ(α)\}:

\[
\begin{align*}
\nabla \cdot \delta \mathbf{V} &= 0, \\
\nabla \cdot \delta \mathbf{B} &= 0, \\
⟨N⟩ \delta \mathbf{V} + δN \delta \mathbf{V} - ⟨δNδ\mathbf{V}⟩ &= 0, \\
⟨\mathbf{V}⟩ \cdot [⟨N⟩ \delta \mathbf{V} + δN \delta \mathbf{V} - ⟨δNδ\mathbf{V}⟩] &= 0, \\
⟨N_B⟩ δ \mathbf{B} + δN_B δ \mathbf{B} - ⟨δN_B δ \mathbf{B}⟩ &= 0, \\
δZ(r, t_o; α) &= δZ_o(r; α), \\
δZ(r, t; α)|_{δΩ} &= δZ_w(r, t; α)|_{δΩ}
\end{align*}
\]

\textit{(stochastic-fluctuating MHD equations)}. Here the notation is standard \cite{18, 19}. Finally, \langle N \rangle, δN and respectively \langle N_B \rangle, δN_B are the nonlinear operators

\[
\langle N \rangle \langle \mathbf{V} \rangle = \frac{∂}{∂t} \langle \mathbf{V} \rangle + \langle \mathbf{V} \rangle \cdot \nabla \langle \mathbf{V} \rangle + \frac{1}{\rho_o} [\nabla ⟨p⟩ - ⟨f⟩] - ν∇^2 \langle \mathbf{V} \rangle ,
\]

\hspace{1cm}(42)

\[
δN δ \mathbf{V} = δ \mathbf{V} \cdot \nabla δ \mathbf{V} + \frac{1}{\rho_o} [∇δp - δf] - ν∇^2 δ \mathbf{V},
\]

\hspace{1cm}(43)

and

\[
\langle N_B \rangle \langle \mathbf{B} \rangle = \frac{∂}{∂t} \langle \mathbf{B} \rangle + \langle \mathbf{V} \rangle \cdot \nabla \langle \mathbf{B} \rangle - \langle \mathbf{h} \rangle - \frac{c}{4 π \sigma} ∇^2 \langle \mathbf{B} \rangle ,
\]

\hspace{1cm}(44)

\[
δN_B δ \mathbf{B} = δ \mathbf{V} \cdot \nabla δ \mathbf{B} - δ \mathbf{h} - \frac{c}{4 π \sigma} ∇^2 δ \mathbf{B}.
\]

\hspace{1cm}(45)

V. COMPLETE IKT FORMULATION FOR THE STOCHASTIC MHD EQUATIONS

As indicated above (see also related discussion in Ref. \cite{3}) in principle the problem of the formulation of a stochastic IKT can be posed either for the full set of fluid equations or only for the stochastic-averaged equations (40) (see also Sec.6).

In the first case let us assume, for greater generality, that the pdf can depend on the hidden variables both implicitly, via the fluid fields, and possibly also explicitly, i.e., it is stochastic local pdf of the form \(f(x,t; Z(α), α)\). The formulation of the IKT in this case follows directly from THM.1 by imposing that \(f(x,t; Z(α), α)\) satisfies the stochastic Eulerian IKE

\[
L(Z(α))f(x,t; Z(α), α) = 0.
\]

\hspace{1cm}(46)
Here, the streaming operator $L(Z(\alpha))$ is again defined by Eq. (15), while by assumption the vector fields $F$ and $Y$ depend on $\alpha$ only implicitly via the fluid fields $\{Z(\alpha)\}$. Next, let us introduce for the pdf and the streaming operator the stochastic decompositions

$$f(t; \alpha) \equiv f(x,t; Z(\alpha), \alpha) = \langle f(x,t; Z(\alpha), \alpha) \rangle + \delta f(x,t; Z(\alpha), \alpha), \quad (47)$$

$$L(Z(\alpha)) = \langle L(Z(\alpha)) \rangle + \delta L(Z(\alpha)), \quad (48)$$

where $\langle \rangle$ is again the stochastic-averaging operator (77). Requiring that $\langle \rangle$ commutes also with the differential operators $\partial/\partial v$ and $\partial/\partial y$, Eq. (76) yields the two stochastic kinetic equations advancing in time respectively $\langle f \rangle \equiv \langle f(x,t; Z(\alpha), \alpha) \rangle$ and $\delta f \equiv \delta f(x,t; Z(\alpha), \alpha)$, i.e.

$$\langle L(Z) \rangle \langle f \rangle + \langle \delta L(Z) \delta f \rangle = 0, \quad (49)$$

$$\langle L(Z) \rangle \delta f + \delta L(Z) \delta f - \langle \delta L(Z) \delta f \rangle = 0. \quad (50)$$

Thanks to THM.1, Eq. (46) [or the equivalent Eqs. (49) and (50)] provides, in principle, the exact (and unique) evolution of the stochastic pdf $f(x,t; Z(\alpha), \alpha)$. Furthermore, thanks to the correspondence principle [given by Eqs. (17)], also of stochastic fluid fields $\{Z\}$ are uniquely determined [3].

In this Section we intend to formulate a Complete IKT, yielding the full set of stochastic fluid equations advancing in time the stochastic fluid fields $\{Z\}$, here represented by the stochastic MHD problem defined above. The basic result has the flavor of:

**Theorem 2 - Complete IKT for the stochastic MHD equations** Requiring that the fluid fields $\{Z(\alpha)\} \equiv \{V, p \geq 0, B\}$ are stochastic [in the sense the of the definition given above], and the correspondence principle

$$\begin{align*}
1 &= \int dv dy f(x,t; Z(\alpha), \alpha), \\
V(r,t; \alpha) &= \int dv dy v f(x,t; Z(\alpha), \alpha), \\
p_1(r,t) &= \rho_0 \int dv dy \frac{1}{2} u^2 f(x,t; Z(\alpha), \alpha), \\
B(r,t; \alpha) &= \int dv dy y f(x,t; Z(\alpha), \alpha), \\
S_T(t; \alpha) &= S(f(t; \alpha)),
\end{align*} \quad (51)$$

holds identically in $\Gamma \times I$ (where $\Gamma$ is the closure of $\Gamma = \Omega \times \mathbb{R}^3$) then provided the mean-field force

$$F_1(x,t; f, Z(\alpha), \alpha) = \{F(x,t; f, Z(\alpha), \alpha), Y(x,t; f, Z(\alpha), \alpha)\} \quad (52)$$
is still defined by Eqs. (24) - (26) (with fluid fields to be considered stochastic functions) it follows that:

1) the local distribution $f_M(x, t; Z(\alpha))$, i.e., the Maxwellian distribution (20) with stochastic fluid fields [or, equivalent, the contributions $\langle f_M(x, t; Z(\alpha)) \rangle$ and $\delta f_M(x, t; Z(\alpha))$] is a particular solution of the inverse kinetic equation (46) [respectively of Eqs. (49)-(50)] if only if the fluid fields $\{\langle Z(\alpha) \rangle\}$ and $\{\delta Z(\alpha)\}$ satisfy respectively the stochastic fluid equations (40) and (41);

2) $f_M(x, t; Z(\alpha))$ maximizes the Boltzmann-Shannon entropy $S(f(t; \alpha))$ in the functional class $\{f(x, t; \langle Z(\alpha) \rangle, \alpha)\}$ defined so that the pdf satisfies solely the constraints (51), where the stochastic-averaged fluid fields $\{Z(\alpha)\}$ are considered prescribed;

3) the velocity-moment equations obtained by taking the weighted velocity integrals of Eq. (46) [or, equivalent, Eqs. (49)-(49)] with the weights (30) deliver identically the same stochastic fluid equations (40) and (41).

**PROOF**

The proof is an immediate consequence of THM.1 and follows by invoking the stochastic decompositions (47) and (48), together with the assumption (2) for the rs-pdf $g$, which assures the equivalence of the two stochastic kinetic equations (49)-(49) with Eq. (6). In particular, it is immediate by direct evaluation to verify that stochastic fluid equations (40) and (41) are recovered from Eqs. (49)-(49) by taking their moments (30). Q.E.D.

Main consequences of THM.2 are that:

1. the inverse kinetic equation, either represented in the Eulerian or Lagrangian forms. In particular, the Eulerian equation is provided by Eq. (46) [or by the equivalent Eqs. (49) and (50)], while the Lagrangian equation, in view of Eq. (5), is again of the form (5), namely

$$J(t; Z(\alpha), \alpha) f(x(t, \alpha), t; Z(\alpha), \alpha) = f(x_o, t_o, Z(\alpha), \alpha)$$

where $x_o \rightarrow x(t, \alpha)$ is the *stochastic MHD-dynamical system* determined by the (stochastic) phase-space Lagrangian equation (4) with vector field $X(x, t; Z(\alpha))$, $J(t; Z(\alpha), \alpha) = \left| \frac{\partial x(t)}{\partial x_o} \right|$ denoting the corresponding Jacobian.

2. the two phase-space representations, provided respectively by the Eulerian and Lagrangian IKE’s, are equivalent. In fact, by construction, both equations satisfy iden-
tically the stochastic MHD equations, including the appropriate initial and boundary conditions imposed on the fluid fields;

3. the time-evolution of the stochastic pdf $f$ is, in turn, uniquely determined by the stochastic dynamical system defined above [see, in fact, the Lagrangian inverse kinetic equation Eq.(53)].

4. The stochastic-averaged pdf $\langle f \rangle$ can be proven to be strictly positive. This result is important to assure both that $\langle f \rangle$ is truly a probability density and that its time evolution is (possibly) irreversible. This result can be shown to be a consequence of THM.1.

VI. MARKOVIAN REDUCED IKT FOR THE STOCHASTIC MHD EQUATIONS

In the traditional approach to turbulence theory main emphasis is related to the construction of the statistical equation advancing in time the stochastic-averaged pdf, rather than on the complete pdf. An example is provided by stochastic models - based on Markovian Fokker-Planck models of small-scale fluid turbulence recently investigated in the literature by several authors (including: Naert et al., 1997 [22]; Friedrich and Peinke et al., 1999 [23]; Luck et al., 1999 [24]; Cleve et al., 2000 [25]; Ragwitz and Kantz, 2001 [26]; Renner et al., 2001, 2002 [27, 28]; Hosokawa, 2002 [29]).

An interesting issue is whether the possible Markovian character of the statistical equation advancing in time the stochastic-averaged pdf, in the present case to be identified with the local pdf $\langle f \rangle$, can be established based on an inverse kinetic theory approach.

As previously pointed out [3, 20], the stochastic IKE provided by Eq.(46), or the equivalent equations (49) and (50), are formally similar to the Vlasov equation arising in the kinetic theory of quasi-linear and strong turbulence for Vlasov-Poisson plasmas [30, 31, 32] and related renormalized kinetic theory [33], which are known to lead generally to a non-Markovian kinetic equation for $\langle f \rangle$ alone. Nevertheless, the stochastic-averaged kinetic equation [i.e., Eq. (49)] is known to be amenable, under suitable assumptions, to an approximate Fokker-Planck kinetic equation advancing in time $\langle f \rangle$ alone. This is achieved by formally constructing a perturbative solution of the equation (50) for the stochastic perturbation $\delta f$. To obtain a convergent perturbative theory, however, this usually requires the
adoption of a suitable renormalization scheme in order to obtain a consistent kinetic equation for \( \langle f \rangle \). This raises the interesting question whether in the framework of IKT there exist possible alternatives based on the construction of exact Markovian IKT formulations.

A solution to this problem is provided by the construction of an IKT only for the stochastic-averaged equations \([40]\), i.e., to be achieved by means of a Reduced IKT \([3]\).

Here we intend to show that this leads to an inverse kinetic equation of the form \((5)\),

\[
L(\langle Z(\alpha) \rangle) f(x, t; \langle Z(\alpha) \rangle) = 0, \tag{54}
\]

which is manifestly Markovian \([3, 19]\) and satisfies also a constant H-theorem (i.e., it conserves the Boltzmann-Shannon entropy). Here \( f \equiv f(x, t; \langle Z \rangle) \) is the Eulerian local pdf for the stochastic INSE problem, which advances in time the stochastic-averaged fluid fields \( \{\langle Z(\alpha) \rangle\} \) and \( L(\langle Z(\alpha) \rangle) \) is the corresponding streaming operator, to be defined in terms of the vector field

\[
F_1(x, t; f, \langle Z(\alpha) \rangle) = \{F(x, t; f, \langle Z(\alpha) \rangle), Y(x, t; f, \langle Z(\alpha) \rangle)\}, \tag{55}
\]

with \( F_1 \) to be interpreted again as a mean field force acting on a particle with state \( x \). Besides the requirement of validity of the fluid equations \([40, 16]\), let us impose that vector \( X \) has the form \( X = \{v, y, F, Y\} \), where \( F(x, t; f, \langle Z(\alpha) \rangle) \) and \( Y(x, t; f, \langle Z(\alpha) \rangle) \) are both assumed generally functionally dependent on the local pdf. Again, by appropriate choice of the mean field forces \( F, Y \), the moment equations can be proven to satisfy identically the fluid equations \([\text{Eqs. } (40)]\), as well the appropriate initial and boundary conditions. Requiring that \( f \) is a strictly positive let us require that there results identically in \( \Gamma \times I \):

\[
\begin{aligned}
1 &= \int dv dy f(x, t; \langle Z \rangle), \\
\langle V(r, t; \alpha) \rangle &= \int dv dy \dot{v} f(x, t; \langle Z \rangle), \\
\bar{p}_1(r, t) &= \rho_0 \int dv dy \frac{1}{2} (u)^2 f(x, t; \langle Z \rangle) \\
\langle B(r, t; \alpha) \rangle &= \int dv dy \dot{y} f(x, t; \langle Z \rangle), \\
\langle S_T(t; \alpha) \rangle &= S(f(x, t; \langle Z \rangle)).
\end{aligned} \tag{56}
\]

Here \( S(f(x, t; \langle Z \rangle, \alpha)) \) is the (Shannon) entropy integral

\[
S(f(x, t; \langle Z \rangle, \alpha)) = -\int_{\Gamma} dx f(x, t; \langle Z(\alpha) \rangle) \ln f(x, t; \langle Z(\alpha) \rangle), \tag{57}
\]

\( u = v - V(r, t; \alpha) \) is the relative velocity, while

\[
\bar{p}_1(r, t) = P_0(t) + \langle p(r, t; \alpha) \rangle \tag{58}
\]
is the kinetic pressure and $P_0(t)$ a smooth real function (pseudo-pressure) to be suitably defined (see below). Again the form of the local pdf $f \equiv f(x; t; \langle Z \rangle)$ can be chosen in such a way to satisfy the principle of entropy maximization (PEM) [34], i.e., imposing, at the initial time $t = t_o$, the variational equation $\delta S(f) = 0$, with $\delta^2 S(f) < 0$, while requiring that $f$ belongs to the functional class $\{ f(x; t; \langle Z(\alpha) \rangle) \}$, manifestly prescribed by the constraints [56]. Hence, PEM implies yields necessarily (for $t = t_o$) that the initial pdf must be of the form

$$f_M(x; t; \langle Z \rangle) = \frac{1}{\pi^2 v_{th}^2} \exp \left\{ -X^2 - Y^2 \right\}. \quad (59)$$

Here, in difference to Eqs.(21) and (22), $X^2$ and $Y^2$ now denote

$$X^2 = \langle u \rangle^2 / v_{th}^2, \quad (60)$$

$$Y^2 = \langle w \rangle^2 / v_{th,T}^2, \quad (61)$$

where $u = v - V$, $w = y - B$, $v_{th}^2 = 2 \tilde{\rho}_1 / \rho_o$ and $v_{th,T}^2$ is an ‘a priori’ arbitrary strictly positive constant. Eq.[54] implies the construction of a suitable classical dynamical system, defined by a phase-space map $x_o \to x(t) = T_{t,t_o} x_o$, where $T_{t,t_o}$ is the evolution operator [16] generated by the stochastic phase-space Lagrangian equation (1) with vector field $X(x; t, \langle Z(\alpha) \rangle)$ and Jacobian $J(x(t), t, \langle Z(\alpha) \rangle)$.

Therefore, the Eulerian IKE (54) can also be represented in the Lagrangian form [equivalent to Eq.(54)]

$$J(x(t), t, \langle Z(\alpha) \rangle) f(x(t), t; \langle Z(\alpha) \rangle) = f(x_o, t_o; \langle Z_o(\alpha) \rangle) \equiv f_o(x_o; \langle Z_o(\alpha) \rangle), \quad (62)$$

(Lagrangian IKE), Here, $f_o(x_o; \langle Z_o(\alpha) \rangle)$ is a suitably smooth initial pdf and

$$J(x(t), t; \langle Z(\alpha) \rangle) = \left| \frac{\partial x(t)}{\partial x_o} \right| \quad (63)$$

is the Jacobian of the map $x_o \to x(t)$. Then it is immediate to prove that the stochastic MHD problem admits a Reduced IKT. For the sake of simplicity here we present the result which holds in the case of a Maxwellian (local) pdf of the type [59].

**Theorem 3** - Markovian Reduced IKT for the stochastic-averaged MHD equations
Let us assume that: A\(_1\) the stochastic MHD problem \([40]\) admits a smooth strong solution in \(\Gamma \times I\), with I a finite time interval; A\(_2\) the mean-field force

\[
\mathbf{F}_1(\mathbf{x}, t, f, \langle Z(\alpha) \rangle) = \{\mathbf{F}(\mathbf{x}, t; f, \langle Z(\alpha) \rangle), \mathbf{Y}(\mathbf{x}, t; f, \langle Z(\alpha) \rangle)\}
\]  

(64)

is defined by

\[
\mathbf{F}(\mathbf{x}, t; f_M, \langle Z(\alpha) \rangle) = \mathbf{F}_0 + \mathbf{F}_1,
\]

(65)

\[
\mathbf{Y}(\mathbf{x}, t; f_M, \langle Z(\alpha) \rangle) = \langle \mathbf{u} \cdot \nabla \mathbf{B} \rangle - \langle \mathbf{h} \rangle - \frac{c}{4\pi\sigma} \nabla^2 \langle \mathbf{B} \rangle + \langle \mathbf{w} \nabla \cdot \mathbf{B} \rangle,
\]

(66)

where \(\mathbf{F}_0\) and \(\mathbf{F}_1\) read respectively

\[
\mathbf{F}_0(\mathbf{x}, t; f_M, \langle Z(\alpha) \rangle) = \frac{1}{\rho_o} \langle f \rangle + \frac{1}{2} \langle \mathbf{u} \cdot \nabla \mathbf{V} \rangle + \frac{1}{2} \langle \nabla \mathbf{V} \cdot \mathbf{u} \rangle + \nu \nabla^2 \langle \mathbf{V} \rangle,
\]

(67)

\[
\mathbf{F}_1(\mathbf{x}, t; f_M, \langle Z(\alpha) \rangle) = \frac{1}{2} \langle \mathbf{u} \rangle \frac{1}{\rho_1} + \frac{v_{th}^2}{2} \nabla \ln \rho_1 \left\{ \frac{\langle u \rangle^2}{2} - \frac{3}{2} \right\},
\]

(68)

while

\[
A \equiv \frac{\partial}{\partial t} (P_0(t) + \langle p \rangle) + \langle \mathbf{V} \rangle \cdot \nabla (P_0(t) + \langle p \rangle)
\]

(69)

is prescribed by the requirement of validity of the stochastic-averaged energy equation \([19]\); A\(_3\) \(f(\mathbf{x}, t; \langle Z(\alpha) \rangle)\) satisfies suitable initial and boundary condition consistent with the initial-boundary value problem \([40]\) (see Ref. \([14]\)); A\(_4\) the initial local pdf, \(f(\mathbf{x}, t_0; \langle Z_0(\alpha) \rangle)\), is suitably smooth and strictly positive. It follows that:

1) the velocity-moment equations of IKE [54] evaluated for the weight functions \(G(\mathbf{x}, t) = v, \frac{1}{3} u^2\) coincide with the stochastic INSE equations;

2) if \(f(t_0, \langle Z(\alpha) \rangle)\) is strictly positive so it is \(f(t, \langle Z(\alpha) \rangle)\) in the whole domain \(\Gamma \times I\);

3) the Maxwellian local pdf \([59]\) is a particular solution of IKE if and only if the fluid fields \(\{\langle Z \rangle\}\) satisfy the stochastic-averaged MHD problem \([47]\);

4) the pseudo-pressure \(P_o(t)\) can be uniquely determined in the time interval \(I\) in such a way that, denoting \(f(t, \langle Z(\alpha) \rangle) \equiv f(\mathbf{x}, t; \langle Z \rangle)\), in the same time interval \(I\) there results identically

\[
\frac{\partial}{\partial t} S(f(t, \langle Z(\alpha) \rangle)) = 0
\]

(70)

(constant H-theorem).

PROOF
The proof is immediate for proposition 1), while 2) follows invoking the Lagrangian IKE (62) and by noting that by construction \( J(\mathbf{x}(t), t; \langle Z(\alpha) \rangle) > 0 \) in the whole domain \( \Gamma \times I \).

To prove 3) let us assume that a strong solution of the stochastic INSE problem exists. In such a case it is immediate to prove that \( f_M(\mathbf{x}(t), t; \langle Z \rangle) \) is a particular solution of the inverse kinetic equation (54). In fact, substituting (59) in the inverse kinetic equation (54) it follows:

\[
L f_M(\mathbf{x}, t; \langle Z(\alpha) \rangle) = \left\{ \frac{\partial}{\partial t} \langle V \rangle + \mathbf{v} \cdot \nabla \langle V \rangle \right\} \cdot \frac{\langle u \rangle \rho_o}{\hat{p}_1} f_M + \\
+ \left\{ \frac{\partial}{\partial t} \langle B \rangle + \mathbf{v} \cdot \nabla \langle B \rangle \right\} \cdot \frac{2 \langle w \rangle}{\nu_{th,T}^2} f_M \\
+ \left\{ \frac{\partial}{\partial t} \ln \hat{p}_1 + \mathbf{v} \cdot \nabla \ln \hat{p}_1 \right\} \left\{ \frac{\langle u \rangle^2}{\nu_{th}^2} - \frac{3}{2} \right\} f_M - \\
+ \left\{ -\mathbf{F} \cdot \frac{\langle u \rangle \rho_o}{\hat{p}_1} - \mathbf{Y} \cdot \frac{2 \langle w \rangle}{\nu_{th,T}^2} + \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F} + \frac{\partial}{\partial \mathbf{y}} \cdot \mathbf{Y} \right\} f_M,
\]

which manifestly implies Eqs.(40). Instead, if we assume that in \( \Gamma \times I \), \( f \equiv f_M(\mathbf{x}, t; \langle Z(\alpha) \rangle) \) is a particular solution of the inverse kinetic equation (IKE) (54), which fulfills identically the constraint equation (70), thanks to proposition 1) it follows that the fluid fields \( \langle V \rangle, \langle p \rangle \) are solutions of the INSE equations. Finally to prove proposition 4) and 5), let us invoke IKE to evaluate the entropy production rate, which reads

\[
\frac{\partial}{\partial t} S(f(\mathbf{x}, t; \langle Z(\alpha) \rangle)) = -\int_{\Gamma} d\mathbf{x} f(\mathbf{x}, t; \langle Z \rangle) \left[ \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{x}, t; f) + \frac{\partial}{\partial \mathbf{y}} \cdot \{ \mathbf{Y}(\mathbf{x}, t; f) \} \right].
\]

Hence, in a bounded time interval \( I \) it always possible to satisfy the constraint placed by the constant H-theorem, requiring

\[
\frac{\partial}{\partial t} S(f(\mathbf{x}, t; \langle Z(\alpha) \rangle)) = -\frac{3}{2} \int_{\Omega} d^3 \mathbf{r} \frac{1}{P_0(t) + \langle p \rangle} \left[ \frac{\partial}{\partial t} (P_0(t) + \langle p \rangle) + \langle V \rangle \cdot \nabla (P_0(t) + \langle p \rangle) \right] = 0.
\]

This delivers an ordinary differential equation for the pseudo-pressure \( P_0(t) \). Assuming that the fluid fields \( \langle V \rangle, \langle p \rangle \) are suitably smooth, this equation can always be fulfilled at least in the case in which the domain of existence \( I \) is a finite time interval. Q.E.D.

Let us briefly analyze the consequences of the theorem, with particular reference to the condition of strict positivity of the pdf [proposition 2)] and the constant H-theorem [Eq.(73), see proposition 5)]. It is well-known that H-theorem assures, for arbitrary (and suitably
smooth) initial conditions of the pdf, also the strict positivity of the kinetic distribution function. Nevertheless, in view of the validity of proposition 2), the requirement of its validity is, in principle, unnecessary. In fact, the H-theorem is only necessary because of the requirements posed by the condition of isentropic flow [see Eqs.(40)] and the correspondence principle [defined by Eqs.(56)]. However, there is (another) important physical consequence of the theorem. In fact, if the requirement is posed that the initial pdf $f(t_0, \langle Z(\alpha) \rangle)$ satisfies PEM, i.e., it maximizes $S(f(t_0, \langle Z(\alpha) \rangle))$, the constant H-theorem assures that in the domain of existence $I$, $f(t, \langle Z(\alpha) \rangle)$ must necessarily maximize $S(f(t, \langle Z(\alpha) \rangle))$. Therefore, if for all $t \in I$, $f(t, \langle Z(\alpha) \rangle)$ is assumed to belong to the same functional class specified above $\{f(t, \langle Z(\alpha) \rangle)\}$, then necessarily $f(t, \langle Z(\alpha) \rangle)$ is a kinetic equilibrium, to be identified with the local Maxwellian distribution (59). Similar conclusions apply manifestly to THM.2 too, provided the functional class is suitably defined (see THM2.) and the kinetic equilibrium is identified with (20).

Another interesting issue is provided by the comparison between the two stochastic IKT-approaches discussed above, i.e., the Complete and the Reduced IKT’s here developed. Key differences between them are manifestly provided both by the form of the inverse kinetic equations and the choice of the initial conditions for the corresponding pdf. This is due to the different definition of the mean field force in the two cases, given respectively by Eqs.(52),(64) and moreover by the different requirements posed in the two case by PEM (the principle of entropy maximization; Janys, 1957 [34]). In fact, the initial pdf $f(t_0, \langle Z(\alpha) \rangle)$, which in both cases is an extremal of the Boltzmann-Shannon entropy, namely is such that $\delta S(f(t_0, \langle Z(\alpha) \rangle)) = 0$, belongs to different functional classes $\{f(t_0, \langle Z(\alpha) \rangle)\}$. The precise definition of $\{f(t_0, \langle Z(\alpha) \rangle)\}$ in the two cases is actually important since it determines uniquely the initial condition [see related discussion in Refs [3] and [20]]. Provided either the total initial fluid fields $Z(r,t_0;\alpha) = Z_o(r;\alpha)$ or only the stochastic averaged ones $\langle Z(r,t_0;\alpha) \rangle = \langle Z_o(r;\alpha) \rangle$ are prescribed.

In both cases [see THM’s 1, 2 and 3] the initial kinetic probability density is provided by a Maxwellian-type kinetic equilibrium distribution. This is found as the ”most-likely” initial kinetic probability density at $t = t_o$, in agreement with PEM. In particular, in the first case the distribution $f_M(x,t;Z)$ [see Eq.(20)] follows by requiring that $\{f(t_o, \alpha)\}$ is prescribed imposing the constraints defined by Eqs.(17) - to be evaluated at $t = t_o$ - while in the second $f_M(x,t;\langle Z \rangle)$ [see Eq.(59)] follows from the constraints (56). This means that in the first
In the first case the initial fluid fields the compete set of fluid fields \( \{Z\}_{t=t_0} \) must be considered as prescribed, while in the second only the stochastic-averaged fluid fields \( \{\langle Z \rangle\}_{t=t_0} \) is initially known (so that in this case the initial value of the stochastic fluctuations \( \{\delta Z\}_{t=t_0} \) remains in this case arbitrary).

VII. GENERALIZATIONS TO NONHOMOGENEOUS AND NONSTATIONARY TURBULENCE

Turbulence models depend both on the definitions of the hidden variables \( \alpha \) and of the related rs-pdf \( g \) (see also previous section). This involves specifically the case in which \( g \) is taken of the form \( g(r,t;\alpha) \), which corresponds to the assumption of nonhomogeneous and nonstationary turbulence. It is obvious that, unless specific additional assumptions are introduced, such models are non-unique, due to the arbitrariness of their possible choices [both for \( \alpha \) and \( g \)]. Their unique determination, however, may in principle be achieved either based on suitable 'ad hoc' mathematical models or the phenomenology of (magneto-)fluids, based either on first principles (i.e., implied by the underlying microscopic molecular dynamics and statistical mechanics), or the observation of real fluids or obtained from numerical experiments. Nevertheless, despite this arbitrariness, in the particular case in which the functional form of the fluid equations [i.e., Eqs. (9)] remains unchanged, the generalization of the present theory is still possible. This requires solely suitable smoothness assumptions to be satisfied by the rs-pdf. On the contrary, a similar conclusion is not obvious if the form of the fluid equations [Eqs. (9)] is modified. In fact, in such a case, i.e., for arbitrary turbulence-modified fluid equations, the existence of an IKT-approach 'a priori' cannot generally be assurred.

VIII. CONCLUDING REMARKS

A statistical model of MHD turbulence has been pointed out utilizing the inverse kinetic theory (IKT) recently developed for classical and quantum fluids [14, 15, 16, 17, 21]. Basic feature of the theory is that the IKT has been constructed in such a way to satisfy exactly the stochastic MHD equations describing an incompressible, viscous, quasi-neutral, isentropic, isothermal and resistive magnetofluid. As consequence an inverse kinetic equation has been
determined which uniquely advances in time the local pdf.

The main result of the paper is represented by THM.2 and 3, yielding IKT-approaches to the stochastic MHD equations and particularly by the inverse kinetic equations \([i.e., \text{Eqs.}(49),(50) \text{ and } (54)]\), which advance in time respectively the stochastic or the stochastic-averaged local pdf. The first two equations \([i.e., \text{Eqs.}(49) \text{ and } (50)]\), determine uniquely the evolution of the stochastic fluid fields, \(i.e., \text{both the stochastic averages } \{\langle Z(\alpha) \rangle \} \text{ and the corresponding stochastic fluctuations } \{\delta Z(\alpha)\}. \) Instead, Eq.\((54)\) determines only \(\{\langle Z(\alpha) \rangle \} \) \(i.e., \text{provided } \{\delta Z(\alpha)\} \text{ is known}. \) In both cases the inverse kinetic equations have the property that their velocity-moments (to be defined in terms of appropriate weighted velocity-space integrals) satisfy identically a closure condition. In fact, there exists, by construction, a subset of the moment equations which is closed. Such a set is defined so that it coincides with the appropriate stochastic fluid equations \([i.e., \text{either Eqs.}(40) \text{ or } (41)]\).

The theory displays several interesting new features, with respect to customary phase-space statistical approaches to turbulence (see for example [1, 2]). In particular:

1. it is based on the introduction of a local position-velocity joint probability density function (local pdf), rather than the usual velocity-difference pdf (which typically affords only an approximate description);

2. Eulerian and Lagrangian phase-space formulations are manifestly equivalent;

3. a remarkable, and in a sense, surprising feature of the present theory is that the equation advancing in time the stochastic-averaged pdf is can also be set in the form of a Markovian kinetic equation [9]. This feature is actually "built in" the IKT approach, thanks to the definition here adopted for the time evolution operator \(T_{t,t_0}\), which advances in time the local pdf (and hence the same fluid fields).

In our view the IKT-theory here presented is a useful setting for the investigation of theoretical and mathematical aspects of turbulence phenomena which may potentially occur in a variety of fluids (both incompressible and compressible) and in particular magnetofluids. Several interesting applications and generalizations of the theory are possible, which include - for example - the investigation of turbulence in fluid mixtures and in thermo-magnetofluids. The proper treatment of such phenomena requires a consistent description of phase-space dynamics, to be based on the IKT approach developed in this paper. We remark, in this
connection, that in the present approach both the specification of the hidden variables $\alpha$ and of the reduced stochastic probability density ($g$) remains in principle completely arbitrary. This feature is potentially important since it permits, in principle, the systematic treatment of all the possible different sources of stochasticity here pointed out, a problem which remains still fundamentally unsolved to date. Finally, the extension of the theory to nonhomogeneous and nonstationary turbulence has been pointed out (Sec.7). Related developments will be discussed in greater detail elsewhere.

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IX. APPENDIX: STOCHASTIC AND DETERMINISTIC FUNCTIONS

A possible convenient definition of stochastic function $A$ (to be identified with the generic fluid field $Z$) is the following one:

Definition - Stochastic function

A function $A$ is denoted as stochastic on the set $\Omega \times T \times V_\alpha$ if:

1) $A \equiv A(r, t, \alpha)$ is assumed to be a real function defined on the set $\Omega \times T \times V_\alpha$ which depends on the real hidden-variables (to be denoted as stochastic parameters) $\alpha \in V_\alpha \subseteq \mathbb{R}^k$;

2) the $\alpha$'s are characterized by a probability density $g$, denoted as reduced stochastic probability density function (rs-pdf) defined on $V_\alpha$, with $V_\alpha$ a non-empty subset of $\subseteq \mathbb{R}^k$ and $k \geq 1$. Hence, $g$ is necessarily non-negative ($g \geq 0$) [or even strictly positive ($g > 0$)] and such that

$$\int_{V_\alpha} d\alpha g = 1,$$

$(75)$

$d\alpha$ denoting the canonical measure on $V_\alpha$;
3) introducing the stochastic averaging operator

\[ \langle \cdot \rangle \equiv \int_{V_{\alpha}} d\alpha g \cdot, \]  

(76)

where the integration is performed at constant \((r, t) \in \overline{\Omega} \times \overline{T}\), the stochastic function \(A(r, t, \alpha)\) is assumed to admit everywhere in \(\overline{\Omega} \times \overline{T}\) the stochastic average

\[ \langle A \rangle \equiv \int_{V_{\alpha}} d\alpha g A(r, t, \alpha); \]  

(77)

In contrast, a deterministic function can be defined as:

**Definition - Deterministic function**

A function \(A(r, t, \alpha)\) is denoted as deterministic if there results identically on the set \(\overline{\Omega} \times V_{\alpha} \times \overline{T}\)

\[ A = \langle A \rangle. \]  

(78)

This requires that that either \(A\) is independent of the parameters \(\alpha\) or the probability density \(g\) is a \(k\)-dimensional Dirac delta, *i.e.*, it is of the form \(g = \delta^{(k)}(\alpha - \alpha_o) \equiv \prod_{i=1}^{k} \delta(\alpha_i - \alpha_{oi})\), with \(\alpha_o\) a prescribed element of \(V_{\alpha}\).

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