ON A GENERALIZATION OF BAER THEOREM

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Abstract. R. Baer has proved that if the factor-group \( G/\zeta_n(G) \) of a group \( G \) by the member \( \zeta_n(G) \) of its upper central series is finite (here \( n \) is a positive integer) then the member \( \gamma_{n+1}(G) \) of the lower central series of \( G \) is also finite. In particular, in this case, the nilpotent residual of \( G \) is finite. This theorem admits the following simple generalization that has been published recently by M. de Falco, F. de Giovanni, C. Musella and Ya. P. Sysak: "If the factor-group \( G/Z \) of a group \( G \) modulo its upper hypercenter \( Z \) is finite then \( G \) has a finite normal subgroup \( L \) such that \( G/L \) is hypercentral". In the current article we offer a new simpler very short proof of this theorem and specify it substantially. In fact, we prove that if \( |G/Z| = t \) then \( |L| \leq t^k \), where \( k = \frac{1}{2} (\log_t p + 1) \), and \( p \) is the least prime divisor of \( t \).

1. Introduction

One of the important long-standing results in the Theory of Groups is a classical theorem due to I. Schur [8], which establishes a connection between the factor-group \( G/\zeta(G) \) of a group \( G \) modulo its center \( \zeta(G) \) and the derived subgroup \( [G,G] \) of \( G \). It follows from Schur's theorem [8] that if \( G/\zeta(G) \) is finite then \( [G,G] \) is also finite. A natural question related to this result appears here, namely the question regarding the relationship between the orders \( |G/\zeta(G)| \) and \( |[G,G]| \). J. Wiegold in the paper [9] obtained the following answer to this question. Let \( G \) be a group such that \( |G/\zeta(G)| = t \) is finite. J. Wiegold proved that there exists a function \( w \) such that \( |[G,G]| \leq w(t) \). He also was able to obtain for this function the value \( w(t) = t^m \) where \( m = \frac{1}{2} (\log_t p - 1) \) and \( p \) is the least prime divisor of \( t \). Later on, J. Wiegold was able to show that this boundary value may be attained if and only if \( t = p^n \) for some prime \( p \) [10]. When \( t \) has more than one prime divisor, the picture becomes more complicated.

Various generalizations of Schur's theorem can be found in the mathematical literature. One of the most interesting approaches would be studying the properties of the following question: study properties of the factor-group \( G/\zeta(G) \) such that the derived subgroup \( [G,G] \) satisfies the same property. A class of groups \( \mathcal{X} \) is said to be a Schur class if for every group \( G \) such that \( G/\zeta(G) \in \mathcal{X} \) the derived subgroup \( [G,G] \) also belongs to \( \mathcal{X} \). Schur's classes were introduced in the paper [3]. Besides of the obvious examples of the classes of finite and of locally finite groups, the class of polycyclic–by–finite groups and the class of Chernikov groups are also Schur's
classes (see, for example, [7] Theorem 3.9]). In this paper [3] other Schur’s classes were found as well.

In the paper [11] R. Baer generalized Schur’s theorem in a different direction. We recall that the upper central series of a group $G$ is the ascending series

$$\langle 1 \rangle = \zeta_0(G) \leq \zeta_1(G) \leq \cdots \leq \zeta_\alpha(G) \leq \cdots \leq \zeta_\lambda(G) = \zeta_\infty(G)$$

given by $\zeta_1(G) = \gamma(G)$ is the center of $G$, and recursively $\zeta_\alpha(G)/\zeta_{\alpha+1}(G) = \zeta(G/\zeta_{\alpha}(G))$ for all ordinals $\alpha$ and $\zeta_\lambda(G) = \bigcup_{\mu<\lambda} \zeta_\mu(G)$ for every limit ordinal $\lambda$. The last term $\zeta_\infty(G)$ of this series is called the upper hypercenter of $G$. $G$ itself is called hypercentral if $\zeta_\infty(G) = G$. In general, the length of the upper central series of $G$ is denoted by $zI(G)$. On the other hand, the lower central series of $G$ is the descending series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_\alpha(G) \geq \gamma_\gamma(G) \geq \cdots$$

given by $\gamma_2(G) = [G, G]$, and recursively $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$ for all ordinals $\alpha$ and $\gamma_\lambda(G) = \bigcap_{\mu<\lambda} \gamma_\mu(G)$ for every limit ordinal $\lambda$.

R. Baer proved that if for some positive integer $n$ the factor-group $G/\zeta_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite too (11). In particular, in this case the nilpotent residual of $G$ (that is, the intersection of all normal subgroups $N$ of $G$ such that $G/N$ is nilpotent) is finite. Very recently, in the paper [2], M. de Falco, F. de Giovanni, C. Musella and Ya. P. Sysak obtained the following generalization of this result:

**Theorem A.** Let $G$ be a group and let $Z$ be the upper hypercenter of $G$. If $G/Z$ is finite, then $G$ has a finite normal subgroup $L$ such that $G/L$ is hypercentral.

In Section 2 we provide an elementary proof of this result, which is considerably shorter than the original one.

Just as in the theorem of Schur, the question on finding a relationship between the factor-group $G/\zeta_\infty(G)$ and the hypercentral residual of $G$ (the intersection of all normal subgroups $N$ of $G$ such that $G/N$ is hypercentral) appears to be very natural. More specifically, is there a function (depending on the order of $G/\zeta_\infty(G)$) that bounds the order of the hypercentral residual of $G$? In this note we show that Theorem A can be significantly improved. We prove that the order of the hypercentral residual of $G$ is bounded by a function of the order of $G/\zeta_\infty(G)$ and moreover we are able to give an explicit form of this function. Thus the main result of the current note is the following

**Theorem B.** Let $G$ be a group and let $Z$ be the upper hypercenter of $G$. Suppose that $G/Z$ is finite and put $|G/Z| = t$. Then $G$ has a finite normal subgroup $L$ such that $G/L$ is hypercentral. Moreover, $|L| \leq t^k$, where $k = \frac{1}{2} (\log_p t + 1)$ and $p$ is the least prime divisor of $t$.

2. **A short proof of Theorem A**

The proof makes use of an auxiliary result by N. S. Hekster [5, Lemma 2.4].

**Lemma 2.1** (HN1986). Let $G$ be a group, $K$ a subgroup of $G$, and suppose that $G = K\zeta_n(G)$ for some positive integer $n$. Then the following properties holds.

1. $\gamma_{n+1}(G) = \gamma_{n+1}(K)$.
2. $\zeta_n(K) = K \cap \zeta_n(G)$.
(3) \( \gamma_{n+1}(G) \cap \zeta_n(G) = \gamma_{n+1}(K) \cap \zeta_n(K) \).

Proof of Theorem A. We note that if \( zl(G) \) is finite, the result follows from Baer’s theorem \([1]\). Therefore we may suppose that \( zl(G) \) is infinite. Let \( K \) be a finitely generated subgroup with the property \( G = ZK \). We have that \( K \) is nilpotent–by–finite (see \([7\), Proposition 3.19\] for example). Since \( G/Z \) is not nilpotent, neither is \( K \). Set \( r = zl(K) \) and let \( C \) be the upper hypercenter of \( K \). We claim that \( C = C \cap Z \). For, otherwise \( CZ/Z \not\in \langle 1 \rangle \), which means that the upper hypercenter of \( G/Z \) is not identity, a contradiction. Then \( C = C \cap Z \) as claimed. By Baer’s theorem \([1]\), \( \gamma_{r+1}(K) \) is finite. It follows that the nilpotent residual \( L \) of \( K \) is finite.

We now consider the local system \( L \) consisting of all finitely generated subgroups of \( G \) that contains \( K \). Pick \( V \in L \) and let \( CV \) be the upper hypercenter of \( V \). Clearly we have \( G = ZV \) and then \( CV = V \cap Z \). Since \( V \leq KZ \) and \( K \leq V \), we have \( V = K(V \cap Z) = KCV \). Put \( n = zl(V) \). Since \( V = KCV \), we have that \( \gamma_{n+1}(V) = \gamma_{n+1}(K) \) by Lemma \([22]\). In particular, \( \gamma_{n+1}(K) \) is normal in \( V \). Since \( L \) is a characteristic subgroup of \( \gamma_{n+1}(K) \), \( L \) is normal in \( V \). Since this holds for each \( V \in L \), \( L \) is normal in \( G = \bigcup_{V \in L} V \). We have

\[
G/ZL \cong (G/L)(ZL/L) = (KZ/L)/(ZL/L) = (K/L)(ZL/L)/(ZL/L) \cong
\]

\[
\cong (K/L)/((K/L) \cap (ZL/L)).
\]

Since \( K/L \) is nilpotent, so is \( G/ZL \). Since the hypercenter of \( G/L \) includes \( ZL/L \), \( G/L \) has to be hypercentral. \(\Box\)

3. Proof of Theorem B

Let \( G \) be a group, \( R \) a ring and \( A \) an \( RG \)–module. We construct the upper \( RG \)–central series of \( A \) as the ascending chain of submodules

\[
\{0\} = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \cdots \lambda,
\]

where \( A_1 = \zeta_{RG}(A) = \{ a \in A \mid a(g - 1) = 0 \} \), \( A_{\alpha+1}/A_\alpha = \zeta_{RG}(A/A_\alpha) \) for every ordinal \( \alpha < \lambda \) and \( \zeta_{RG}(A/A_\lambda) = \{0\} \). The last term \( A_\lambda \) of this series is called the upper \( RG \)–hypercenter of \( A \) and will denoted by \( \zeta_{RG}^\infty(A) \). If \( A = A_\lambda \), then \( A \) is said to be \( RG \)–hypercentral. Moreover, if \( \lambda \) is finite, then \( A \) is said to be \( RG \)–nilpotent.

Let \( B \leq C \) be \( RG \)–submodules of \( A \). The factor \( C/B \) is called \( G \)–eccentric if \( C_G(C/B) \neq G \). An \( RG \)–submodule \( C \) of \( A \) is said to be \( RG \)–hypereccentric if it has an ascending series of \( RG \)–submodules

\[
\{0\} = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\lambda = C
\]

such that every factor \( C_{\alpha+1}/C_\alpha \) is a \( G \)–eccentric simple \( RG \)–module.

It is said that the \( RG \)–module \( A \) has the \( Z \)–decomposition if we can express

\[
A = \zeta_{RG}^\infty(A) \bigoplus E_{RG}^\infty(A),
\]

where \( E_{RG}^\infty(A) \) is the maximal \( RG \)–hypereccentric \( RG \)–submodule of \( A \) (D. I. Zaicev \([11]\)). We note that, if \( A \) has the \( Z \)–decomposition, then \( E_{RG}^\infty(A) \) includes every \( RG \)–hypereccentric \( RG \)–submodule and, in particular, it is unique. Indeed, put \( E = E_{RG}^\infty(A) \) and let \( B \) be a \( RG \)–hypereccentric \( RG \)–submodule of \( A \). If \( (B + E)/E \) is non-zero, then it has a non-zero simple \( RG \)–submodule \( U/E \), say. Since \( (B + E)/E \cong B/(B \cap E) \), \( U/E \) is \( RG \)–isomorphic to some simple \( RG \)–factor of \( B \) and then \( G \neq C_G(U/E) \). But \( (B + E)/E \leq A/E \cong \zeta_{RG}^\infty(A) \) and then
Lemma 3.1. Let $G$ be a finite nilpotent group and let $A$ be a $ZG$–module. Suppose that the additive group of $A$ is periodic. Then $A$ has the $Z$–decomposition.

Proof. Since $G$ is finite, $A$ has a local family $\mathcal{L}$ consisting of finite $ZG$–submodules. If $B \in \mathcal{L}$, applying the results of [11], $B$ has the $Z$–decomposition. Pick now $C \in \mathcal{L}$ such that $B \leq C$. Then we have

$$B = E_{ZG}(B) \bigoplus E_{ZG}(C), C = E_{ZG}(C) \bigoplus E_{ZG}(C).$$

Clearly $E_{ZG}(B) \leq E_{ZG}(C)$ and, since $E_{ZG}(C)$ includes every $ZG$–hypereccentric $ZG$–submodule, $E_{ZG}(B) \leq E_{ZG}(C)$. It follows that

$$E_{ZG}(A) = \bigcup_{B \in \mathcal{L}} E_{ZG}(B), E_{ZG}(A) = \bigcup_{B \in \mathcal{L}} E_{ZG}(B).$$

Therefore $A = E_{ZG}(A) \bigoplus E_{ZG}(A)$. □

Lemma 3.2. Let $G$ be a finite group and $Z$ a $G$–invariant subgroup of the hypercenter of $G$. Put $|G/Z| = t$. Then there exists a function $f_1$ such that the nilpotent residual of $G$ has order at most $f_1(t)$.

Proof. The subgroup $Z$ has a series of $G$–invariant subgroups

$$(1) = Z_0 \leq Z_1 \leq \cdots \leq Z_n = Z,$$

whose factors $Z_{j+1}/Z_j$ are $G$–central. Applying a result due to L. A. Kaloujnine [6], the factor-group $G/C_G(Z)$ is nilpotent. Put $C = C_G(Z)$ so that $Z \leq C_G(C)$. In particular, $|G/C_G(C)| \leq t$. Clearly $C \cap Z \leq \zeta(C)$ and so $C/(Z \cap C) \cong CZ/Z$ is a finite group of order at most $t$. By Wiegold’s theorem [8], the derived subgroup $D = [C, C]$ has order at most $w(t)$. We note that $D$ is $G$–invariant and $C/D$ is abelian. By the facts proved above, the factor-group $(G/D)/C_{G/D}(C/D)$ is nilpotent. By Lemma [8] the $ZG$–module $C/D$ has the $Z$–decomposition, that is $C/D = E_{ZG}(C/D) \bigoplus E_{ZG}(C/D)$. Clearly, $(C \cap Z)/D \leq E_{ZG}(C/D)$ and then $L/D = E_{ZG}(C/D)$ has order at most $t$. Hence $(C/D)/(L/D)$ is $ZG$–hypercentral. In other words, the hypercenter of $G/L$ contains $C/L$. Since $G/C$ is nilpotent so is $G/L$. Finally, $|L| = |D|/|L/D| \leq tw(t) = tm^t = tm^t$, where $m = \frac{1}{2}(\log_t p - 1)$ and $p$ is the least prime divisor of $t$, so that $m + 1 = \frac{1}{2}(\log_t p - 1) + 1 = \frac{1}{2}(\log_t p + 1)$. Therefore, it suffices to put $f_1(t) = t^k$, where $k = \frac{1}{2}(\log_t p + 1)$ and $p$ is the least prime divisor of $t$. □

If $G$ is a group, we denote by $\text{Tor}(G)$ the maximal periodic normal subgroup of $G$. $\text{Tor}(G)$ is a characteristic subgroup of $G$ and, if $G$ is locally nilpotent, $G/\text{Tor}(G)$ is torsion-free.

Lemma 3.3. Let $G$ be a finitely generated group and $Z$ a $G$–invariant subgroup of the hypercenter of $G$. Suppose that $|G/Z| = t$ is finite. Then $G$ has a finite normal subgroup $L$ such that $G/L$ is nilpotent. Moreover, $|L| \leq f_1(t)$.

Proof. Since $G/Z$ is finite, $Z$ is finitely generated. It follows that $Z$ is nilpotent. Moreover, $zL(G)$ is finite. By Baer’s theorem [1], $G$ has a finite normal subgroup $F$ such that $G/F$ is nilpotent. Being finitely generated, $G/F$ has a finite periodic part $\text{Tor}(G/F) = K/F$. As we remarked above, the factor-group $(G/F)/(K/F) \cong$
G/K = B is torsion-free and nilpotent. We have that the subgroup Z is nilpotent and T = Tor(G) is finite. Then Z has a torsion-free normal subgroup U such that the orders of the elements of Z/U are the divisors of some positive integer k (see [4] Proposition 2) for example). Put V = Z^k so that V ≤ U and V is also torsion-free. By construction, V is G–invariant and G/V is periodic. Being finitely generated nilpotent–by–finite, C = G/V is finite. By Lemma 3.2 the nilpotent residual D of C has order at most f_1(t).

Clearly V ∩ K = {1}. Applying Remak’s theorem, we obtain an embedding G ≤ G/V × G/K = C × B = H. Since B is torsion-free nilpotent, the nilpotent residual of H is exactly D. It follows that G/(G ∩ D) ∼= GD/D ≤ H/D is nilpotent. This shows that G ∩ D includes the nilpotent residual L of G. In particular, L is finite and moreover |L| ≤ |G| ≤ |D| ≤ f_1(t).

We are now in a position to show the main result of this paper

Proof of Theorem B. Since G/Z is finite, there exists a finitely generated subgroup K such that G = KZ. We pick the family Σ of all finitely generated subgroups of G that contains K. Clearly G is FC–hypercentral and then every finitely generated subgroup of G is nilpotent–by–finite (see [7] Proposition 3.19) for example). If U ∈ Σ, then the hypercenter of U includes a U–invariant subgroup U ∩ Z = Z_U such that U/Z_U is nilpotent and has order at most t. By Lemma 3.3 U has a finite normal subgroup H_U such that U/H_U is nilpotent and |H_U| ≤ f_1(t). Being finite-by–nilpotent, the nilpotent residual L_U of U is finite and L_U has order at most f_1(t).

Pick Y ∈ Σ such that |L_Y| is maximal and let Σ_1 be the family of all finitely generated subgroups of G that contains Y. Pick U ∈ Σ_1. Then Y ≤ U. The factor-group U/L_U is nilpotent and, since Y/(Y ∩ L_U) ∼= YL_U/L_U ≤ U/L_U, Y/(Y ∩ L_U) is nilpotent. It follows that L_Y ≤ Y ∩ L_U and then L_Y ≤ L_U. But |L_Y| is maximal, so that L_U = L_Y. In particular, L_Y is normal in U for every U ∈ Σ_1. Then L_Y is normal in U ∈ Σ_1 U = G and U/L_Y is nilpotent. Thus G/L_Y has a local family of nilpotent subgroups, that is G/L_Y is locally nilpotent. Then (G/L_Y)/(ZL_Y/L_Y) is nilpotent since it is finite. It follows that G/L_Y is hypercentral, because the upper hypercenter of G/L_Y includes ZL_Y/L_Y.

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