Adjoint Representations of the Symmetric Group

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Abstract

We study the restriction to the symmetric group, $S_n$, of the adjoint representation of $GL_n(\mathbb{C})$. We determine the irreducible constituents of the space of symmetric as well as the space of skew-symmetric $n \times n$ matrices as $S_n$-modules.

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To the memory of our beloved mentor and a good friend, Jeff Remmel.

1 Introduction

In [3], the first author and Jeff Remmel introduced the notion of “loop-augmented rooted forest” and explained its combinatorial representation theoretic role for the conjugation action of the symmetric group on certain subsets of the partial transformation semigroup. In this note, we present an application of this development in a basic Lie theory context.

Let $G$ be a Lie group and let $\mathfrak{g}$ denote the Lie algebra of $G$. The conjugation action $G \times G \to G$, $(g, h) \mapsto ghg^{-1}$, $g, h \in G$ leads to a linear representation of $G$ on its tangent space at the identity element,

$$\text{Ad} : G \to \text{Aut}(\mathfrak{g})$$

$$g \mapsto \text{Ad}_g.$$  \hfill (1.1)

The representation (1.1), which is called the adjoint representation, has a fundamental place in the structure theory of Lie groups. It has a concrete description when $G$ is a closed subgroup of $GL_n(\mathbb{C})$, the general linear group of $n \times n$ matrices. In this case, the Lie algebra
$\mathfrak{g}$ of $G$ is a Lie subalgebra of the $n \times n$ matrices, and the adjoint representation (1.1) is given by

$$\text{Ad}_g(X) = gXg^{-1},$$

for $g \in G, \ X \in \mathfrak{g}$.

Let $S_n$ denote the group of permutations of the set $\{1, \ldots, n\}$. We view $S_n$ as a subgroup of $\text{GL}_n(\mathbb{C})$ by identifying its elements with $n \times n$ 0/1 matrices with at most one 1 in each row and each column. The basic representation theoretic question that we address here is the following:

What are the irreducible constituents of the $S_n$-representation that is obtained from (1.1) by restriction?

Surprisingly, even though the adjoint representation is at the heart of Lie theory, to the best of our knowledge the answer to our question is missing from the literature, at least, it is not presented in the way that we are answering it. To state our theorem we set up the notation.

It is well known that the finite dimensional irreducible representations over $\mathbb{Q}$ of $S_n$ are indexed by the integer partitions of $n$. The Frobenius character map, $V \mapsto F_V$ is an assignment of symmetric functions to the finite dimensional representations of $S_n$. (We will explain this in more detail in the sequel.) In particular, if $V$ is the irreducible representation determined by an integer partition $\lambda$, then $F_V$ is a Schur symmetric function, denoted by $s_{\lambda}$. Furthermore, if $V = \bigoplus V_i$ is a decomposition of $V$ into $S_n$-submodules, then $F_V = \sum F_{V_i}$. Our first main result is as follows.

**Theorem 1.2.** Let $n$ be an integer such that $n \geq 2$. The Frobenius character of the adjoint representation of $S_n$ on $\text{Mat}_{n}(\mathbb{C})$ is given by

$$F_{\text{Mat}_{n}(\mathbb{C})} = \begin{cases} 2s_2 + 2s_{1,1} & \text{if } n = 2; \\ 2s_3 + 3s_{2,1} + s_{1,1,1} & \text{if } n = 3; \\ 2s_n + 3s_{n-1,1} + s_{n-2,2} + s_{n-2,1,1} & \text{if } n \geq 4. \end{cases} \tag{1.3}$$

The space of symmetric $n \times n$-matrices, which we denote by $\text{Sym}_n(\mathbb{C})$, is closed under the adjoint action of the orthogonal group, $O_n(\mathbb{C}) := \{ g \in \text{GL}_n(\mathbb{C}) : gg^\top = id \}$. However, $\text{Sym}_n(\mathbb{C})$ is not closed under the adjoint action of $\text{GL}_n(\mathbb{C})$. Nevertheless, since $S_n$ is a subgroup of $O_n(\mathbb{C})$, we see that the representation

$$\text{Ad} : S_n \to \text{Aut}(\text{Sym}_n(\mathbb{C})) \tag{1.4}$$

is defined. Moreover, since there is a direct sum decomposition

$$\text{Mat}_n(\mathbb{C}) = \text{Sym}_n(\mathbb{C}) \oplus \text{Skew}_n(\mathbb{C}),$$
where $\text{Skew}_n(\mathbb{C})$ is the space of $n \times n$ skew-symmetric matrices, we have the complementary adjoint representation

$$\text{Ad} : S_n \to \text{Aut}(\text{Skew}_n(\mathbb{C})) \quad (1.5)$$

as well.

Our second main result is the following

**Theorem 1.6.** Let $n$ be an integer such that $n \geq 2$. The Frobenius character of the adjoint representation $(1.4)$ of $S_n$ on $\text{Sym}_n(\mathbb{C})$ is given by

$$F_{\text{Sym}_n(\mathbb{C})} = \begin{cases} 2s_2 + s_{1,1} & \text{if } n = 2; \\ 2s_3 + 3s_{2,1} & \text{if } n = 3; \\ 2s_n + 2s_{n-1,1} + s_{n-2,2} & \text{if } n \geq 4. \end{cases} \quad (1.7)$$

**Corollary 1.8.** Let $n$ be an integer such that $n \geq 2$. The Frobenius character of the adjoint representation $(1.5)$ of $S_n$ on $\text{Skew}_n(\mathbb{C})$ is given by

$$F_{\text{Skew}_n(\mathbb{C})} = \begin{cases} s_{1,1} & \text{if } n = 2; \\ s_{1,1} & \text{if } n = 3; \\ s_{n-1,1} + s_{n-2,1} & \text{if } n \geq 4. \end{cases} \quad (1.9)$$

## 2 Preliminaries

### 2.1 Symmetric functions and plethysm.

The *$k$-th power-sum symmetric function*, denoted by $p_k$, is the sum of $k$-th powers of the variables. For an integer partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_l)$, we define $p_{\lambda}$ to be equal to $\prod_i p_{\lambda_i}$. The *$k$-th complete symmetric function*, $h_k$, is defined to be the sum of all monomials $x_1^{a_1} \cdots x_r^{a_r}$ with $\sum a_i = k$ and $h_{\lambda}$ is defined to be equal to $\prod_{i=1}^{l} h_{\lambda_i}$. The *Schur function associated with* $\lambda$, denoted by $s_{\lambda}$, is the symmetric function defined by the determinant $\det(h_{\lambda_i+j-i})_{i,j=1}^{l}$. In particular, we have, for all $k \geq 1$, that $s_{(k)} = h_k$. For easing the notation, we denote $s_{(k)}$ by $s_k$.

The type of a conjugacy class $\sigma$ in $S_n$ is the partition $\lambda$ whose parts correspond to the lengths of the cycles that appear in the cycle decomposition of an element $x \in \sigma$. It is well known that the type is independent of the element $x$, furthermore, it uniquely determines the conjugacy class.

The power sum and Schur symmetric functions will play special roles in our computations via the *Frobenius character map*

$$F : \text{class functions on } S_n \to \text{symmetric functions}$$

$$\delta_{\sigma} \mapsto \frac{1}{n!} p_{\lambda},$$

where $S_n$ is the symmetric group of degree $n$. The Frobenius map $F$ is a homomorphism from the ring of class functions on $S_n$ to the ring of symmetric functions. It is defined by $F(\delta_{\sigma}) = \frac{1}{n!} p_{\lambda}$, where $\lambda$ is the type of $\sigma$. The Frobenius map is a fundamental tool in the representation theory of symmetric groups and has applications in algebraic combinatorics and algebraic geometry.
where $\sigma \subset S_n$ is a conjugacy class of type $\lambda$ and $\delta_\sigma$ is the indicator function
\[
\delta_\sigma(x) = \begin{cases} 
1 & \text{if } x \in \sigma; \\
0 & \text{otherwise}. 
\end{cases}
\]

It turns out that if $\chi_\lambda$ is the irreducible character of $S_n$ indexed by the partition $\lambda$, then $F(\chi_\lambda) = s_\lambda$. In the sequel, we will not distinguish between representations of $S_n$ and their corresponding characters. In particular, we will often write the Frobenius character of an orbit to mean the image under $F$ of the character of the representation of $S_n$ that is defined by the action on the orbit. If $V$ is an $S_n$-representation, then we will denote its Frobenius character by $F_V$.

For two irreducible characters $\chi_\mu$ and $\chi_\lambda$ indexed by partitions $\lambda$ and $\mu$, the Frobenius character image of the “plethysm” $\chi_\lambda[\chi_\mu]$ is the plethystic substitution $s_\lambda[s_\mu]$ of the corresponding Schur functions. Roughly speaking, the plethysm of the Schur function $s_\lambda$ with $s_\mu$ is the symmetric function obtained from $s_\lambda$ by substituting the monomials of $s_\mu$ for the variables of $s_\lambda$. In the notation of \cite{5}; the plethysm of symmetric functions is the unique map $[\cdot] : \Lambda \times \Lambda \to \Lambda$ satisfying the following three axioms:

P1. For all $m, n \geq 1$, $p_m[p_n] = p_{mn}$.

P2. For all $m \geq 1$, the map $g \mapsto p_m[g]$, $g \in \Lambda$ defines a $Q$-algebra homomorphism on $\Lambda$.

P3. For all $g \in \Lambda$, the map $h \mapsto h[g]$, $h \in \Lambda$ defines a $Q$-algebra homomorphism on $\Lambda$.

Although the problem of computing the plethysm of two (arbitrary) symmetric functions is very difficult, there are some useful formulas for Schur functions:

\[
s_\lambda[g + h] = \sum_{\mu, \nu} c_{\mu, \nu}^\lambda(s_\mu[g])(s_\nu[h]), \tag{2.1}
\]

and

\[
s_\lambda[gh] = \sum_{\mu, \nu} \gamma_{\mu, \nu}^\lambda(s_\mu[g])(s_\nu[h]). \tag{2.2}
\]

Here, $g$ and $h$ are arbitrary symmetric functions, $c_{\mu, \nu}^\lambda$ is a scalar, and $\gamma_{\mu, \nu}^\lambda$ is $\frac{1}{n!}\langle \chi_\lambda, \chi_\mu \chi_\nu \rangle$, where the pairing stands for the standard Hall inner product on characters.

In (2.1) the summation is over all pairs of partitions $\mu, \nu \subset \lambda$, and the summation in (2.2) is over all pairs of partitions $\mu, \nu$ such that $|\mu| = |\nu| = |\lambda|$. In the special case when $\lambda = (n)$, or $(1^n)$ we have

\[
s_{(n)}[gh] = \sum_{\lambda \vdash n} (s_\lambda[g])(s_\lambda[h]), \tag{2.3}
\]

\[
s_{(1^n)}[gh] = \sum_{\lambda \vdash n} (s_\lambda[g])(s_\lambda[h]), \tag{2.4}
\]

where $\lambda'$ denotes the conjugate of $\lambda$. 
2.2 Background on partial transformations.

Let \( n \) denote a positive integer, \([n]\) and \([n]\) denote the sets \(\{1, \ldots, n\}\) and \(\{n\} \cup \{0\}\), respectively. We will use the following basic notation for our semigroups:

- \( \mathcal{F}_{\text{Full}}_n \): the full transformation semigroup on \([n]\);
- \( \mathcal{P}_n \): the semigroup of partial transformations on \([n]\);
- \( \mathcal{C}_n \): the set of nilpotent partial transformations on \([n]\).

A partial transformation on \([n]\) is a function \( f : A \rightarrow [n] \), where \( A \) is a nonempty subset of \([n]\). A full transformation on \([n]\) is a function \( g : [n] \rightarrow [n] \). We note that there is an “extension by 0” morphism from partial transformations on \([n]\) into full transformations on \([n]\),

\[
\varphi_0 : \mathcal{P}_n \rightarrow \mathcal{F}_{\text{Full}}_n
\]

\[
f \mapsto \varphi_0(f)
\]

which is defined by

\[
\varphi_0(f)(i) = \begin{cases} 
  f(i) & \text{if } i \text{ is in the domain of } f; \\
  0 & \text{otherwise.}
\end{cases}
\]

The map \( \varphi \) is an injective semigroup homomorphism. Since \( \mathcal{F}_{\text{Full}}_n \) contains a zero transformation it makes sense to talk about nilpotent partial transformations in \( \mathcal{P}_n \). In particular, \( \mathcal{C}_n \) is well defined although its definition requires the embedding of \( \mathcal{P}_n \) into \( \mathcal{F}_{\text{Full}}_n \). The unit group of \( \mathcal{P}_n \) is equal to \( S_n \), therefore, its conjugation action of \( \mathcal{P}_n \) makes sense. Furthermore, the set of nilpotent partial transformations is stable under this action of \( S_n \).

The combinatorial significance of \( \mathcal{C}_n \) stems from the fact that there is a bijection between \( \mathcal{C}_n \) and the set of labeled rooted forests on \( n \) vertices. (Hence, \( |\mathcal{C}_n| = (n+1)^{n-1} \).) The conjugation action of \( S_n \) on \( \mathcal{C}_n \) translates into the permutation action of \( S_n \) on the labels. This idea, which we started to use in [1], extends in a rather natural way to all directed graphs that correspond to the elements of \( \mathcal{P}_n \). Indeed, let \( \tau \) be a partial transformation from \( \mathcal{P}_n \). We view \( \tau \) as an \( n \times n \) 0/1 matrix with at most one 1 in each column, so, it defines a directed, labeled graph on \( n \) vertices; there is a directed edge from the \( i \)-th vertex to the \( j \)-th vertex if the \((i, j)\)-th entry of the matrix is 1. The underlying graph of this labeled directed graph depends only on the \( S_n \)-conjugacy class of \( \tau \). To explain the consequences of this identification, next, we will re-focus on the elements of \( \mathcal{C}_n \) and on the corresponding labeled rooted forests. What we are going to state for \( \mathcal{C}_n \) extends to all partial transformations and to the corresponding labeled directed graphs.

A pair \((\tau, \phi)\), where \( \tau \) is a rooted forest on \( n \) vertices and \( \phi \) is a bijective map from \([n]\) onto the vertex set of \( \tau \) is called a labeled rooted forest. As a convention, when we talk about labeled rooted forests, we will omit writing the corresponding labeling function despite the fact that the action of \( S_n \) does not change the underlying forest but the labeling function only. In particular, when we write \( S_n \cdot \tau \) we actually mean the orbit

\[
S_n \cdot (\tau, \phi) = \{ (\tau, \phi') : \phi' = \sigma \cdot \phi, \ \sigma \in S_n \}.
\]
The right hand side of (2.5) is an $S_n$-set, hence it defines a representation of $S_n$. More generally, to any partial transformation $\tau$ in $P_n$, we associate the representation corresponding to the orbit $S_n \cdot \tau$. We refer to the resulting representation by the odun of $\tau$, and denote it by $o(\tau)$.

The odun depends only on $\tau$, not on the labels, so, we write $\text{Stab}_{S_n}(\tau)$ to denote the stabilizer subgroup of the pair $(\tau, \phi)$. As an $S_n$-module, the vector space of functions on the right cosets, that is $\mathbb{C}[S_n/\text{Stab}_{S_n}(\tau)]$ is isomorphic to the odun of $\tau$. As a representation of $S_n$, this is equivalent to the induced representation $\text{Ind}^{S_n}_{\text{Stab}_{S_n}(\tau)} 1$. Next, following [1], we present an example to demonstrate the computation of the Frobenius character of $\text{Ind}^{S_n}_{\text{Stab}_{S_n}(\tau)} 1$.

**Example 2.6.** Let $\tau$ be the rooted forest depicted in Figure 2.1 and let $\tau_i$, $i = 1, 2, 3$ denote its connected components (from left to right in the figure). Let $F_o(\tau)$ and $F_o(\tau_i)$, $i = 1, 2, 3$ denote the corresponding Frobenius characters. Since $\tau_1 \neq \tau_2 = \tau_3$, we have

$$F_o(\tau) = F_o(\tau_1) \cdot s_2[F_o(\tau_2)].$$  

(2.7)

Here, $s_2$ is the Schur function $s_{(2)}$, the bracket stands for the plethysm of symmetric functions and the dot stands for ordinary multiplication. More generally, if a connected component $\tau'$, which of course is a rooted tree appears $k$-times in a forest $\tau$, then $F_o(\tau)$ has $s_k[F_o(\tau')]$ as a factor. Now we proceed to explain the computation of the Frobenius character of a rooted tree. As an example we use $F_o(\tau_1)$ of Figure 2.1. The combinatorial rule that we obtained in [1] is simple; it is the removal of the root from the tree. The effect on the Frobenius character of this simple rule is as follows: Let $\tau'_1$ denote the rooted forest that we obtain from $\tau_1$ by removing the root. Then $F_o(\tau_1) = s_1 \cdot F_o(\tau'_1)$. Thus, by the repeated application of this rule and the previous factorization rule, we obtain $F_o(\tau_1) = s_1 \cdot F_o(\tau'_1) = s_1 \cdot s_1 \cdot s_4[s_1]$. It follows from the definition of plethysm that $s_k[s_1] = s_k$ for any nonnegative integer $k$.

Therefore,

$$F_o(\tau_1) = s_1^2 \cdot s_4. \quad (2.8)$$

We compute $F_o(\tau_2)$ by the same method;

$$F_o(\tau_2) = s_1^5 \cdot s_2. \quad (2.9)$$

By putting (2.7)–(2.9) together, we arrive at the following satisfactory expression for the Frobenius character of $\tau$:

$$F_o(\tau) = F_o(\tau_1) \cdot s_2[F_o(\tau_2)] = s_1^2 \cdot s_4 \cdot s_2[s_1^5 \cdot s_2]. \quad (2.10)$$
Note that the expansion of $s_2[s_5^5 \cdot s_2]$ in the Schur basis is computable by a recursive method by applying (2.3) and using Thrall’s formula [7] (see [6, Chapter I, Section 8, Example 9]). However, the resulting expression is rather large, so, we omit writing it here.

A loop-augmented forest is a rooted forest such that there is at most one loop at each of its roots. See, for example, Figure 2.2, where we depict a loop-augmented forest on 22 vertices and four loops. It is a well known variation of the Cayley’s theorem that the number of labeled forests on $n$ vertices with $k$ roots is equal to $\binom{n-1}{k-1} n^{n-k}$. See [4], Theorem D, pg 70. It follows that the number of loop-augmented forests on $n$ vertices with $k$ roots is $2^k \binom{n-1}{k-1} n^{n-k}$. It follows from generating function manipulations that the number of loop-augmented forests on $n$ vertices for $n \geq 2$ is $2n^{n-3}$.

Next, we will explain how to interpret loop-augmented forests in terms of partial functions. Let $\sigma$ be a loop-augmented forest. Then some of the roots of $\sigma$ have loops. There is still a partial function for $\sigma$, as defined in the previous paragraph for a rooted forest. The loops in this case correspond to the “fixed points” of the associated function. Indeed, for the loop at the $i$-th vertex we have a 1 at the $(i, i)$-th entry of the corresponding matrix representation of the partial transformation. Recall that the permutation action of $\mathcal{S}_n$ on the labels translates to the conjugation action on the incidence matrix. By an appropriate relabeling of the vertices, the incidence matrix of a loop-augmented rooted forest can be brought to an upper-triangular form. Clearly, the conjugates of a nilpotent (respectively unipotent) matrix are still nilpotent (respectively unipotent). Let $U$ be an arbitrary upper triangular matrix. Then $U$ is equal to a sum of the form $D+N$, where $D$ is a diagonal matrix and $N$ is a nilpotent matrix. If $\sigma$ is a permutation matrix of the same size as $U$, then we conclude from the equalities $\sigma \cdot U = \sigma U \sigma^{-1} = \sigma D \sigma^{-1} + \sigma N \sigma^{-1}$ that the conjugation action on loop-augmented forests is equivalent to the simultaneous conjugation action on labeled rooted forests and the representation of $\mathcal{S}_n$ on diagonal matrices.

The following two theorems from [3] describe, respectively, the stabilizer of a labeled loop-augmented rooted forest and the character of the corresponding odun.

**Theorem 2.11.** Let $f$ be a partial transformation of the form $f = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$, where $\sigma \in \mathcal{S}_k$ is a permutation and $\tau \in \mathcal{C}_{n-k}$ is a nilpotent partial transformation. In this case, the stabilizer subgroup in $\mathcal{S}_n$ of $f$ has the following decomposition:

$$
\text{Stab}_{\mathcal{S}_n}(f) = Z(\sigma) \times \text{Stab}_{\mathcal{S}_{n-k}}(\tau),
$$

Figure 2.2: A loop-augmented forest.
where $Z(\sigma)$ is the centralizer of $\sigma$ in $S_k$ and $\text{Stab}_{S_{n-k}}(\tau)$ is the stabilizer subgroup of $\tau$ in $S_{n-k}$.

Theorem 2.12. Let $f$ be a partial function representing a loop-augmented forest on $n$ vertices. Then $f$ is similar to a block diagonal matrix of the form $egin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$ for some $\sigma \in S_k$ and $\tau \in C_{n-k}$. Furthermore, if $\nu$, which is a partition of $k$, is the conjugacy type of $\sigma$ in $S_k$ and if the underlying rooted forest of the nilpotent partial transformation $\tau$ has $\lambda_1$ copies of the rooted tree $\tau_1$, $\lambda_2$ copies of the rooted forest $\tau_2$ and so on, then the character of $o(f)$ is given by

$$\chi_{o(f)} = \chi^\nu \cdot \chi_{o(\tau)} = \chi^\nu \cdot (\chi^{(\lambda_1)}[\chi_{o(\tau_1)}]) \cdot (\chi^{(\lambda_2)}[\chi_{o(\tau_2)}]) \cdots (\chi^{(\lambda_r)}[\chi_{o(\tau_r)}]).$$

3 Proof of Theorem 1.2

Let $i$ and $j$ be two integers from $[n]$. We denote by $E_{i,j}$ the $n \times n$ 0/1 matrix with 1 at its $(i,j)$-th entry and 0’s elsewhere. As a vector space, $\text{Mat}_n(\mathbb{C})$ is spanned by $E_{i,j}$’s. In fact, \{\{E_{i,j} : i, j \in [n]\}\} constitute a basis,

$$\text{Mat}_n(\mathbb{C}) = \bigoplus_{i,j \in [n]} \mathbb{C}E_{i,j}. \quad (3.1)$$

It is clear that $E_{i,j}$’s are actually partial transformation matrices. Equivalently, we will view $E_{i,j}$’s as labeled loop-augmented rooted forests, essentially in two types. The adjoint representation of $S_n$ on $\text{Mat}_n(\mathbb{C})$ is completely determined by the action of $S_n$ on these two types of labeled loop-augmented rooted forests;

1. $i, j \in [n]$ and $i \neq j$. In this case, the labeled loop-augmented rooted forest corresponding to $E_{i,j}$ is as in Figure 3.1.

   ![Figure 3.1: The basis element $E_{i,j}$ as a labeled loop-augmented rooted forest.](image)

By [1, Corollary 6.2], the Frobenius character of the $S_n$-module structure on the orbit $S_n \cdot E_{i,j}$ is given by

$$F_{o(E_{i,j})} = s_1 \cdot s_1 \cdot s_{n-2}[s_1] = s_1^2 s_{n-2}.$$

By applying the Pierri rule (twice), we find that

$$F_{o(E_{i,j})} = \begin{cases} s_2 + s_{1,1} & \text{if } n = 2; \\ s_3 + 2s_{2,1} + s_{1,1,1} & \text{if } n = 3; \\ s_n + 2s_{n-1,1} + s_{n-2,2} + s_{n-2,1,1} & \text{if } n \geq 4. \end{cases} \quad (3.2)$$
2. \(i, j \in [n]\) and \(i = j\). In this case, the labeled loop-augmented rooted forest corresponding to \(E_{i,i}\) is as in Figure 3.2.

By Theorem 2.12, the Frobenius character of the \(S_n\)-module structure on the orbit \(S_n \cdot E_{i,i}\) is given by

\[
F_{o(E_{i,i})} = s_1 \cdot s_{n-1}[s_1] = s_1 s_{n-1}.
\]

By Pierri rule, we see that

\[
F_{o(E_{i,j})} = s_n + s_{n-1,1}. \tag{3.3}
\]

Note that \(E_{k,l} \in S_n \cdot E_{i,j}\) for all \(k, l \in [n]\) with \(k \neq l\). Indeed, \(S_n\) acts on \(E_{i,j}\) by permuting the labels on the vertices. Note also that \(E_{k,k} \in S_n \cdot E_{i,i}\) for all \(k \in [n]\). Therefore, in the light of direct sum (3.1), by combining (3.2) and (3.3), we see that the Frobenius character of the adjoint representation of \(S_n\) on \(\text{Mat}_n(\mathbb{C})\) is as we claimed in Theorem 1.2. This finishes the proof of our first main result.

4 Proofs of Theorem 1.6 and Corollary 1.8

First, we have some remarks about the vector space basis for \(\text{Sym}_n(\mathbb{C})\). For \(i, j \in [n]\) with \(i \neq j\), we set

\[
F_{i,j} := E_{i,j} + E_{j,i}.
\]

A vector space basis for \(\text{Sym}_n(\mathbb{C})\) is given by the union

\[
\{E_{i,i} : i = 1, \ldots, n\} \cup \{F_{i,j} : i, j \in [n], i \neq j\}. \tag{4.1}
\]

Notice that the matrices \(F_{i,j}'s\) are partial transformation matrices as well. However, this time, the directed graph corresponding to \(F_{i,j}\) is not a forest. See Figure 4.1.

Figure 3.2: The basis element \(E_{i,i}\) as a labeled loop-augmented rooted forest.

Figure 4.1: The basis element \(F_{i,j}\) as a directed graph.
Lemma 4.2. Let $i$ and $j$ be two elements from $[n]$ such that $i \neq j$. The orbit of the adjoint action of $S_n$ on the matrix $F_{i,j}$ is the same as the permutation action of $S_n$ on the set of all labelings of the vertices of the directed graph in Figure 4.2. In particular, $F_{1,2} \in S_n \cdot F_{i,j}$.

Proof. Since $i$ and $j$ are two different but otherwise arbitrary elements from $[n]$, it suffices to prove our second claim only. Also, without loss of generality we will assume that $i < j$.

Now, we apply the adjoint action of the transposition $(1, j)$ to $F_{i,j}$;

$$\text{Ad}_{(1,j)}(F_{i,j}) = F_{1,j}.$$ 

Next, we apply the adjoint action of the transposition $(2, j)$ to $F_{1,j}$;

$$\text{Ad}_{(2,j)}(F_{1,j}) = F_{1,2}.$$ 

Therefore, $\text{Ad}_{(2,j)(1, j)}(F_{i,j}) = F_{1,2}$. This finishes the proof. \hfill \Box

In the notation of [2], any element from (4.1) is a partial involution. Following our arguments from [3] for the idempotents of $P_n$, next, we will compute the stabilizer subgroup of the partial involution $F_{i,j}$. Let us mention in passing that for all $i$ in $[n]$, the matrix $E_{i,i}$ is already an idempotent, hence, we know its stabilizer subgroup.

Lemma 4.3. The odun of $F_{i,j}$, $o(F_{i,j})$ is equal to the $S_n$-module

$$o(F_{i,j}) = \oplus_{i,j \in [n], \ i \neq j} C F_{i,j}.$$ 

The stabilizer subgroup of $F_{i,j}$ in $S_n$ is isomorphic to the parabolic subgroup $S_2 \times S_{n-2}$.

Proof. As we already mentioned before, the adjoint (conjugation) action of $S_n$ on partial transformation matrices amounts to the permutation action of $S_n$ on the labels of the associated graph (Lemma 4.2). Our first claim readily follows from this argument. Now, without loss of generality, we assume that $F_{i,j} = F_{1,2}$. In other words,

$$F_{i,j} = F_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0_{n-2} \end{bmatrix}. \quad (4.4)$$ 

Clearly, the stabilizer subgroup in $S_n$ of the matrix (4.4) consists of matrices of the form

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix},$$

where $\sigma_1 \in S_2$ and $\sigma_2 \in S_{n-2}$. \hfill \Box
Proof of Theorem 1.6. It follows from Lemma 4.3 that the representation of $\mathcal{S}_n$ on the orbit $o(F_{i,j}) \cong \mathcal{S}_n \cdot F_{1,2}$ is isomorphic to the left multiplication action of $\mathcal{S}_n$ on the right coset space $\mathbb{C}[\mathcal{S}_n/\mathcal{S}_2 \times \mathcal{S}_{n-2}]$. It follows from definitions that this representation is isomorphic to
\begin{equation}
\mathbb{C}[\mathcal{S}_n/\mathcal{S}_2 \times \mathcal{S}_{n-2}] \cong \text{Ind}_{\mathcal{S}_2 \times \mathcal{S}_{n-2}}^{\mathcal{S}_n} 1. \tag{4.5}
\end{equation}
In particular, the Frobenius character of (4.5) is given by $F_{o(F_{i,j})} = s_2s_{n-2}$. By the Pierri rule, we have
\begin{equation}
F_{o(F_{i,j})} = \begin{cases}
s_2 & \text{if } n = 2; \\
s_3 + s_{2,1} & \text{if } n = 3; \\
s_n + s_{n-1,1} + s_{n-2,2} & \text{if } n \geq 4.
\end{cases} \tag{4.6}
\end{equation}
The rest of the proof follows from combining (4.6) with the formula (3.3). \hfill \Box

Proof of Corollary 1.8. The Frobenius character of $\text{Mat}_n(\mathbb{C})$ is the sum of the Frobenius characters of $\text{Sym}_n(\mathbb{C})$ and $\text{Skew}_n(\mathbb{C})$. The proof now is a consequence of Theorems 1.2 and 1.6. \hfill \Box

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