On the Non-Uniqueness of Solutions to the Average Cost HJB for Controlled Diffusions with Near-Monotone Costs

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Abstract

We present a theorem for verification of optimality of controlled diffusions under the average cost criterion with near-monotone running cost, without invoking any blanket stability assumptions. The implications of this result to the policy iteration algorithm are also discussed.

Index Terms

controlled diffusions, near-monotone costs, Hamilton–Jacobi–Bellman equation, policy iteration

I. INTRODUCTION

The theory of ergodic control of diffusions under near-monotone costs, in the absence of blanket stability assumptions, lacks a satisfactory result for the verification of optimality. This has to do with the non-uniqueness of solutions to the associated Hamilton–Jacobi–Bellman equation (HJB). The results available assert the uniqueness (up to a constant) of a value function which is bounded below that solves the HJB provided the optimal value of the average cost, which appears in the equation as a parameter, is selected. However the optimal value of the average cost is unknown and this leads to a circularity. In an effort to fill this gap we present a verification theorem which to the best of our knowledge is new.

We are concerned with controlled diffusion processes $X = \{X_t, \ t \geq 0\}$ taking values in the $d$-dimensional Euclidean space $\mathbb{R}^d$, and governed by the Itô stochastic differential equation

$$dX_t = b(X_t, U_t) \, dt + \sigma(X_t) \, dW_t.$$  \hspace{1cm} (1)

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All random processes in (1) live in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $W$ is a $d$-dimensional standard Wiener process independent of the initial condition $X_0$. The control process $U$ takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t, \omega) \in [0, \infty) \times \Omega$. Moreover, it is non-anticipative: for $s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s \triangleq$ the completion of $\sigma\{X_0, U_r, W_r, \ r \leq s\}$ relative to $(\mathcal{F}, \mathbb{P})$.

Such a process $U$ is called an admissible control, and we let $\mathfrak{U}$ denote the set of all admissible controls. We impose fairly standard assumptions on the drift $b$ and the diffusion matrix $\sigma$ to guarantee existence and uniqueness of solutions to (1), namely that the diffusion is locally non-degenerate and that $b$ and $\sigma$ have at most affine growth, are continuous and locally Lipschitz in $x$ uniformly in $u \in \mathbb{U}$. For the precise statements of these assumptions see Section II-A.

Let $c:\mathbb{R}^d \times \mathbb{U} \to \mathbb{R}$ be a nonnegative continuous function which is referred to as the running cost. We assume that the cost function $c:\mathbb{R}^d \times \mathbb{U} \to \mathbb{R}_+$ is continuous and locally Lipschitz in its first argument uniformly in $u \in \mathbb{U}$. As well known, the ergodic control problem, in its almost sure (or pathwise) formulation, seeks to a.s. minimize over all admissible $U \in \mathfrak{U}$ the quantity

$$
\limsup_{t \to \infty} \frac{1}{t} \int_0^t c(X_s, U_s) \, ds.
$$

(2)

A weaker, average formulation seeks to minimize

$$
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}^U[c(X_s, U_s)] \, ds.
$$

(3)

We let $\varrho^*$ denote the infimum of (3) over all admissible controls. We assume that $\varrho^* < \infty$.

An fairly general class of running cost functions arising in practice for which the ergodic control problem is well behaved are the near-monotone ones. Let $M^* \in \mathbb{R}_+ \cup \{\infty\}$ be defined by

$$
M^* \triangleq \liminf_{|x| \to \infty} \min_{u \in \mathbb{U}} c(x, u).
$$

(4)

The running cost function $c$ is called near-monotone if $\varrho^* < M^*$. Note that ‘norm-like’ functions $c$ are always near-monotone. The advantage of this class of problems is that no blanket stability (ergodicity) assumption is imposed. Indeed, models of controlled diffusions enjoying a uniform geometric ergodicity do not arise often in applications. What we frequently encounter in practice is a running cost which has a structure that penalizes unstable behavior and thus renders all stationary optimal controls stable. Such is the case for quadratic costs typically used in linear control models. Throughout this paper we assume that the running cost is near-monotone.
Solutions to the ergodic control problem can be constructed via the HJB equation

$$
\sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2 V}{\partial x_i \partial x_j}(x) + H(x, \nabla V) = \rho ,
$$

(5)

with $\rho = \rho^*$ where $a = [a^{ij}]$ is the symmetric matrix $\frac{1}{2} \sigma \sigma^T$ and

$$
H(x, p) \triangleq \min_u \left[ b(x, u) \cdot p + r(x, u) \right] .
$$

(6)

The real-valued function $V$ is bounded below in $\mathbb{R}^d$ and lives in $C^2(\mathbb{R}^d)$, the space of twice-continuous differentiable functions on $\mathbb{R}^d$. The resulting characterization is that a stationary Markov control $v^*$ is optimal for the ergodic control problem if and only if it is an a.e. measurable selector from the minimizer of (5), i.e., if and only if it satisfies

$$
b(x, v^*(x)) \cdot \nabla V(x) + r(x, v^*(x)) = H\left(x, \nabla V(x)\right) \text{ a.e. in } \mathbb{R}^d.
$$

A. Non-Uniqueness of solutions to the HJB

Obtaining solutions to (5) is further complicated by the fact that $\rho^*$ is unknown. Even though there exists a unique (up to a constant) solution $V \in C^2(\mathbb{R}^d)$ which is bounded below when $\rho = \rho^*$, the HJB equation admits in general many solutions for $\rho \neq \rho^*$ [1]. We next review the example in [2, Section 3.8.1]. Consider the one-dimensional controlled diffusion $dX_t = U_t \, dt + dW_t$, with $U = [-1, 1]$ and running cost $c(x) = 1 - e^{-|x|}$. If we define

$$
\xi_\rho \triangleq \log \frac{3}{2} + \log(1 - \rho) , \quad \rho \in \left[\frac{1}{3}, 1\right)
$$

and

$$
V_\rho(x) \triangleq 2 \int_{-\infty}^{x} e^{2|y-\xi_\rho|} dy \int_{-\infty}^{y} e^{-2|z-\xi_\rho|}(\rho - c(z)) \, dz , \quad x \in \mathbb{R} ,
$$

then direct computation shows that

$$
\frac{1}{2} V''_\rho(x) - |V'_\rho(x)| + c(x) = \rho \quad \forall \rho \in \left[\frac{1}{3}, 1\right) ,
$$

and so the pair $(V_\rho, \rho)$ satisfies the HJB for any $\rho \in \left[\frac{1}{3}, 1\right)$.

With this example in mind, the question we pose is the following: given a solution pair $(V, \rho)$ of the HJB how does one verify if a control obtained from the minimizer is indeed optimal, or equivalently whether $\rho = \rho^*$? As far as we know, the existing theory lacks a satisfactory verification theorem.
B. A verification theorem

We start by comparing the Markov control obtained from the minimizer of the HJB for the example in Section I-A to the value of $\varrho$. A stationary Markov control corresponding to the solutions of this HJB is $w_\varrho(x) = -\text{sign}(x - \xi_\varrho)$. The controlled process under $w_\varrho$ has invariant probability density $\psi_\varrho(x) = e^{-2|x-\xi_\varrho|}$. A simple computation shows that

$$\int_{-\infty}^{\infty} c(x)\psi_\varrho(x) \, dx = \varrho - \frac{2}{3}(1 - \varrho)(3\varrho - 1) \quad (7)$$

for all $\varrho \in \left[\frac{1}{3}, 1\right]$. Thus if $\varrho > \frac{1}{3}$, then $\varrho$ is not the average cost for the controlled process under $w_\varrho$. This motivates the following definition.

**Definition 1.1:** A solution pair $(V, \varrho) \in C^2(\mathbb{R}^d) \times \mathbb{R}_+$ of the HJB equation (5) with $\varrho \in [0, M^*)$ is said to be compatible if $V$ is bounded below in $\mathbb{R}^d$, and for some measurable selector $v : \mathbb{R}^d \to U$ from the minimizer of (5) the associated invariant probability measure $\mu_v$ of the diffusion controlled by $v$ satisfies

$$\beta(v) \triangleq \int_{\mathbb{R}^d} c(x, v(x)) \, \mu_v(dx) = \varrho. \quad (8)$$

**Remark 1.1:** Since $V$ in Definition 1.1 is bounded below it follows from the Foster–Lyapunov stability criteria that every measurable selector from the minimizer of the HJB is a Markov control under which the diffusion is positive recurrent and hence it admits an invariant probability measure (see (12) in Section II-C).

The main result of this paper is the following.

**Theorem 1.1:** Provided $c$ is near-monotone and bounded in $\mathbb{R}^d \times \mathbb{U}$ then there exists a unique compatible solution pair $(V, \varrho) \in C^2(\mathbb{R}^d) \times \mathbb{R}_+$ of the HJB equation (5), with $V$ satisfying $V(0) = 0$. Moreover $\varrho = \varrho^*$ and any measurable selector $v : \mathbb{R}^d \to \mathbb{U}$ from the minimizer of (5) is an optimal stationary Markov control.

Theorem 1.1 offers a satisfactory verification theorem. First note that by Theorem 1.1 if a solution pair $(V, \varrho)$ is compatible then every measurable selector $v$ from the minimizer of the HJB satisfies (8). Therefore it suffices to verify (8) for any such $v$. Going back to the example in Section I-A it is clear from (7) that the pair $(V_\varrho, \varrho)$ is compatible for $\varrho = \frac{1}{3}$. This suffices to assert that $\varrho^* = \frac{1}{3}$.

The organization of the paper is as follows. In Section II we introduce the notation used in the paper, we provide a precise statement concerning the assumptions on the model data, and we review some basic definitions and results for controlled diffusions. Section III is devoted to the proof of the main result. In Section IV we discuss some implications of the results for the policy iteration algorithm. Concluding remarks are in Section V.
II. Notation, Assumptions and Some Basic Definitions

The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $| \cdot |$. The set of non-negative real numbers is denoted by $\mathbb{R}_+$. $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{I}$ denotes the indicator function. We denote by $\tau(A)$ the first exit time of the process $\{X_t\}$ from the set $A \subset \mathbb{R}^d$, defined by

$$
\tau(A) \triangleq \inf \{ t > 0 : X_t \not\in A \}.
$$

The closure, the boundary and the complement of a set $A \subset \mathbb{R}^d$ are denoted by $\overline{A}$, $\partial A$ and $A^c$, respectively. The open ball of radius $R$ in $\mathbb{R}^d$, centered at the origin, is denoted by $B_R$, and we let $\tau_R \triangleq \tau(B_R)$, and $\bar{\tau}_R \triangleq \tau(B_R^c)$.

The term domain in $\mathbb{R}^d$ refers to a nonempty, connected open subset of the Euclidean space $\mathbb{R}^d$. For a domain $D \subset \mathbb{R}^d$, the space $C^k(D)$ ($C^\infty(D)$) refers to the class of all real-valued functions on $D$ whose partial derivatives up to order $k$ (of any order) exist and are continuous, and $C_b(D)$ denotes the set of all bounded continuous real-valued functions on $D$. Also the space $L^p(D)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes) of measurable functions $f$ satisfying $\int_D |f(x)|^p \, dx < \infty$, and $L^\infty(D)$ is the Banach space of functions that are essentially bounded in $D$. The standard Sobolev space of functions on $D$ whose generalized derivatives up to order $k$ are in $L^p(D)$, equipped with its natural norm, is denoted by $W^{k,p}(D)$, $k \geq 0$, $p \geq 1$.

In general if $\mathcal{X}$ is a space of real-valued functions on $Q$, $\mathcal{X}_{loc}$ consists of all functions $f$ such that $f \varphi \in \mathcal{X}$ for every $\varphi \in C^\infty_c(Q)$, the space of smooth functions on $Q$ with compact support. In this manner we obtain for example the space $W^{2,p}_{loc}(Q)$.

We adopt the notation $\partial_i \triangleq \frac{\partial}{\partial x_i}$ and $\partial_{ij} \triangleq \frac{\partial^2}{\partial x_i \partial x_j}$ for $i, j \in \mathbb{N}$. We often use the standard summation rule that repeated subscripts and superscripts are summed from 1 through $d$. For example,

$$
a^{ij} \partial_{ij} \varphi + b^i \partial_i \varphi \triangleq \sum_{i,j=1}^d a^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^d b^i \frac{\partial \varphi}{\partial x_i}.
$$

A. Assumptions on the Data

The drift $b = [b^1, \ldots, b^d]^T : \mathbb{R}^d \times U \mapsto \mathbb{R}^d$ the diffusion matrix $\sigma = [\sigma^{ij}] : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$ and the running cost $c : \mathbb{R}^d \times U \mapsto \mathbb{R}_+$ are continuous and satisfy the following growth, local Lipschitz and local non-degeneracy properties: For each $R > 0$ there exists a constant $\kappa_R$ such that for all $x, y \in B_R$ and
$u \in U$ it holds that
\[
|b(x, u) - b(y, u)| + \|\sigma(x) - \sigma(y)\| \leq \kappa_R|x - y|,
\]
\[
|c(x, u) - c(y, u)| \leq \kappa_R|x - y|,
\]
\[
\det[a(x)] \geq \kappa_R^{-1},
\]
and
\[
|b(x, u)|^2 + \|\sigma(x)\|^2 \leq \kappa_1(1 + |x|^2),\quad \forall (x, u) \in \mathbb{R}^d \times U,
\]
where $\|\sigma\|^2 \triangleq \text{trace}(\sigma\sigma^T)$ and ‘det’ denotes the determinant.

**B. Controlled extended generator**

In integral form, (1) is written as
\[
X_t = X_0 + \int_0^t b(X_s, U_s) \, ds + \int_0^t \sigma(X_s) \, dW_s.
\]
(9)
The second term on the right hand side of (9) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (1), if it is $\mathcal{F}_t$-adapted, continuous in $t$, defined for all $\omega \in \Omega$ and $t \in [0, \infty)$, and satisfies (9) for all $t \in [0, \infty)$ at once a.s.

With $u \in U$ treated as a parameter, we define the family of operators $L^u : C^2(\mathbb{R}^d) \mapsto C(\mathbb{R}^d)$ by
\[
L^u f(x) = a^{ij}(x) \partial_{ij} f(x) + b^i(x, u) \partial_i f(x),\quad u \in U.
\]

We refer to $L^u$ as the **controlled extended generator** of the diffusion. The HJB equation in (5) then takes the form
\[
\min_{u \in U} [L^u V(x) + c(x, u)] = \varrho,\quad x \in \mathbb{R}^d.
\]

Of fundamental importance in the study of functionals of $X$ is Itô’s formula. For $f \in C^2(\mathbb{R}^d)$ and with $L^u$ as defined in (II-B),
\[
f(X_t) = f(X_0) + \int_0^t L^u f(X_s) \, ds + M_t,\quad \text{a.s.,}
\]
(10)
where
\[
M_t \triangleq \int_0^t \langle \nabla f(X_s), \sigma(X_s) \, dW_s \rangle
\]
is a local martingale. In this paper we also use Krylov’s extension of the Itô formula \cite[p. 122]{Krylov} which extends (10) to functions $f$ in the Sobolev space $W_{\text{loc}}^{2,p}(\mathbb{R}^d)$, for $p > d$. 

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C. Markov controls

Recall that a control is called stationary Markov if \( U_t = v(X_t) \) for a measurable map \( v : \mathbb{R}^d \to \mathbb{U} \). Correspondingly, the equation

\[
X_t = x_0 + \int_0^t b(X_s, v(X_s)) \, ds + \int_0^t \sigma(X_s) \, dW_s
\]

(11)
is said to have a strong solution if given a Wiener process \((W_t, \mathcal{F}_t)\) on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), there exists a process \(X\) on \((\Omega, \mathcal{F}, \mathbb{P})\), with \(X_0 = x_0 \in \mathbb{R}^d\), which is continuous, \(\mathcal{F}_t\)-adapted, and satisfies (11) for all \(t\) at once, a.s. A strong solution is called unique, if any two such solutions \(X\) and \(X'\) agree \(\mathbb{P}\)-a.s., when viewed as elements of \(C([0, \infty), \mathbb{R}^d)\). It is well known that under our assumptions on the data, for any stationary Markov control \(v\), (11) has a unique strong solution [4].

Let \(U_{SM}\) denote the set of stationary Markov controls. Under \(v \in U_{SM}\), the process \(X\) is strong Markov, and we denote its transition function by \(P^v(t, x, \cdot)\). It also follows from the work of [5] that under \(v \in U_{SM}\), the transition probabilities of \(X\) have densities which are locally Hölder continuous. Thus \(L^v\) defined by

\[
L^v f(x) = a^{ij}(x) \partial_{ij} f(x) + b^i(x, v(x)) \partial_i f(x)
\]

for \(v \in U_{SM}\) and \(f \in C^2(\mathbb{R}^d)\), is the generator of a strongly-continuous semigroup on \(C_b(\mathbb{R}^d)\), which is strong Feller. We let \(\mathbb{P}^v_x\) denote the probability measure and \(\mathbb{E}^v_x\) the expectation operator on the canonical space of the process under the control \(v \in U_{SM}\), conditioned on the process \(X\) starting from \(x \in \mathbb{R}^d\) at \(t = 0\).

Recall that control \(v \in U_{SM}\) is called stable if the associated diffusion is positive recurrent. We denote the set of such controls by \(U_{SSM}\), and let \(\mu_v\) denote the unique invariant probability measure on \(\mathbb{R}^d\) for the diffusion under the control \(v \in U_{SSM}\). It is well known that \(v \in U_{SSM}\) if and only if there exists an inf-compact function \(V \in C^2(\mathbb{R}^d)\), a bounded domain \(D \subset \mathbb{R}^d\), and a constant \(\varepsilon > 0\) satisfying

\[
L^v V(x) \leq -\varepsilon \quad \forall x \in D^c.
\]

(12)

III. Proof of the main result

The ergodic control problem for near-monotone cost functions is characterized by Theorem 3.1 below which combines Theorems 3.4.7, 3.6.6 and 3.6.10, and Lemmas 3.6.8 and 3.6.9 in [2].

We need the following definition: For \(v \in U_{SSM}\), \(\varrho > 0\) and \(r > 0\) define

\[
\Psi^v_r(x; \varrho) \triangleq \mathbb{E}^v_x \left[ \int_0^{\tau_r} (c(X_t, v(X_t)) - \varrho) \, dt \right], \quad x \in B^c_r,
\]

(13)
where as defined in Section II, \( \bar{\tau}_r \) stands for \( \tau(B_c^r) \). Note that \( \Psi_r^v(x; \varrho) \) is always finite if \( \beta(v) < \infty \), with \( \beta \) as defined in (8).

**Theorem 3.1:** There exists a unique solution \( V^* \in C^2(\mathbb{R}^d) \) to the HJB equation

\[
\min_{u \in U} \left[ L^u V^*(x) + c(x, u) \right] = \varrho^*, \quad x \in \mathbb{R}^d.
\]

that is bounded below in \( \mathbb{R}^d \) and satisfies \( V^*(0) = 0 \). Also, a control \( v^* \in \mathcal{U}_{SM} \) is optimal with respect to the criteria (2) and (3) if and only if it satisfies

\[
b(x, v(x)) \cdot \nabla V^*(x) + r(x, v(x)) = H(x, \nabla V^*(x)) \quad \text{a.e. in } \mathbb{R}^d.
\]

Moreover, we have

\[
V^*(x) = \limsup_{r \downarrow 0} \inf_{v \in \mathcal{U}_{SM}} \Psi_r^v(x; \varrho^*)
\]

\[
= \Psi_r^v(x; \varrho^*) + \mathbb{E}^v_x \left[ V^*(X_{\bar{\tau}_r}) \right]
\]

for all \( x \in \mathbb{R}^d \) and \( r > 0 \).

It follows by (12) and the near-monotone hypothesis that the optimal control \( v^* \) in Theorem 3.1 is stable.

We need the following lemma.

**Lemma 3.2:** Let \( (V, \varrho) \in C^2(\mathbb{R}^d) \times \mathbb{R}_+ \) be a compatible solution pair to (5) and \( v : \mathbb{R}^d \to \mathcal{U} \) a measurable selector from the minimizer of (6). Then

\[
V(x) = \Psi_r^v(x; \varrho) + \mathbb{E}^v_x \left[ V(X_{\bar{\tau}_r}) \right], \quad \forall r > 0, \quad \forall x \in B_r^c.
\] (14)

**Proof:** By Dynkin’s formula for any \( R > r > 0 \) we have

\[
V(x) = \mathbb{E}^v_x \left[ \int_0^{\bar{\tau}_r \wedge \tau_R} (c(X_t, v(X_t)) - \varrho) \, dt 
+ V(X_{\tau_r}) \mathbb{I}\{\bar{\tau}_r < \tau_R\} + V(X_{\tau_R}) \mathbb{I}\{\bar{\tau}_r \geq \tau_R\} \right]
\] (15)

Since \( V \) is bounded below

\[
\liminf_{R \to \infty} \mathbb{E}^v_x \left[ V(X_{\tau_R}) \mathbb{I}\{\bar{\tau}_r \geq \tau_R\} \right] \geq 0 \quad \forall x \in \mathbb{R}^d.
\] (16)

Applying Fatou’s lemma to (15) and using (16) we obtain

\[
V(x) \geq \Psi_r^v(x; \varrho) + \mathbb{E}^v_x \left[ V(X_{\tau_r}) \right], \quad \forall r > 0, \quad \forall x \in B_r^c.
\] (17)
From (17) we obtain that
\[ V(x) \geq V(0) + \lim_{r \downarrow 0} \Psi_r^v(x, \varrho) \quad \forall x \in \mathbb{R}^d. \] (18)

Since \( v \in \mathcal{U}_{SSM} \), then by Lemma 3.7.8 (ii) in [2] the function \( \Psi_0^v \) defined by
\[ \Psi_0^v(x) \triangleq \lim_{r \downarrow 0} \Psi_r^v(x, \beta(v)) \quad \forall x \in \mathbb{R}^d. \] (19)
lives in \( W_{loc}^{2,p}(\mathbb{R}^d) \), for any \( p > d \), and satisfies
\[ L^v \Psi_0^v(x) + c(x, v(x)) = \beta(v) \quad \text{on} \quad \mathbb{R}^d. \] (20)

Since \( \varrho = \beta(v) \), from (18)–(19) we obtain \( V - \Psi_0^v \geq V(0) \), and also by (20) we have \( L^v(V - \Psi_0^v) = 0 \).
Therefore by the strong maximum principle we obtain \( V - V(0) = \Psi_0^v \). Also by (3.7.50) in [2]
\[ \Psi_0^v(x) = \Psi_r^v(x; \beta(v)) + E_x^v[\Psi_0^v(X_{\tau_r})], \quad \forall r > 0, \quad \forall x \in B_r^c, \]
from which (14) follows since \( \Psi_0^v = V - V(0) \).

Remark 3.1: It follows by Lemma 3.2 and [2, Corollary 3.7.3] that if \( (V, \varrho) \in C^2(\mathbb{R}^d) \times \mathbb{R}_+ \) is a compatible solution pair of (5) then
\[ \frac{1}{t} \mathbb{E}_x^v[V(X_t)] \xrightarrow{t \to \infty} 0. \] (21)
The converse also holds. Therefore (21) can be used in the place of (8) to verify optimality of a solution to the HJB.

We continue with the proof of the main result.

Proof of Theorem 1.1: Let \((\hat{V}, \hat{\varrho})\) be a compatible solution pair to (5) and \( \hat{v} : \mathbb{R}^d \to \mathbb{U} \) an associated measurable selector from the minimizer of (6). For each \( R > 0 \) define
\[ b_R(x, u) \triangleq \begin{cases} b(x, u) & \text{if} \ |x| < R, \\ b(x, \hat{v}(x)) & \text{if} \ |x| \geq R, \end{cases} \]
\[ c_R(x, u) \triangleq \begin{cases} c(x, u) & \text{if} \ |x| < R, \\ c(x, \hat{v}(x)) & \text{if} \ |x| \geq R. \end{cases} \]
Consider the following family of diffusions, parameterized by \( R > 0 \), given by
\[ dX_t = b_R(X_t, U_t) \, dt + \sigma(X_t) \, dW_t, \] (22)
with associated running costs \( c_R(x, u) \). For each \( \alpha \in (0, 1] \) the discounted optimal cost \( V_\alpha^R \) defined by
\[ V_\alpha^R(x) \triangleq \inf_{U \in \mathcal{U}} \mathbb{E}_x^U \left[ \int_0^\infty e^{-\alpha s} c_R(X_s, U_s) \, ds \right] \]
relative to the controlled diffusion in (22) lives in \( \mathcal{W}^{2,p}_{\text{loc}}(\mathbb{R}^d) \), for any \( p > d \), and satisfies
\[
a^{ij}(x) \partial_{ij} V^R_\alpha(x) + H_R(x, \nabla V^R_\alpha) = \alpha V^R_\alpha(x) \quad \text{a.e. in } \mathbb{R}^d,
\]
with
\[
H_R(x, p) \triangleq \min_{u \in U} \left[ b^*_R(x, u) p + c_R(x, u) \right].
\]
(23)

Note that \( V^R_\alpha \) may not live in \( C^2(\mathbb{R}^d) \), since \( b_R \) and \( c_R \) are not necessarily continuous in \( x \) for \( |x| > \mathbb{R} \). Nevertheless, the compactness of the embedding \( \mathcal{W}^{2,p}(B_R) \hookrightarrow C^{1,r}(\overline{B_R}) \), \( r < 1 - \frac{d}{p} \), for \( p > d \), implies that \( \nabla V^R_\alpha \) is Hölder continuous in \( \overline{B_R} \). This has two implications:

1) There exists a measurable selector from the minimizer in the definition of the Hamiltonian \( H_R \).

2) The restriction of \( V^R_\alpha \) to \( B_R \) is in \( C^2(B_R) \).

Fix \( R > 0 \). The running cost \( c_R \) is clearly near monotone for the diffusion in (22). Therefore we may apply the standard theory in [2, Section 3.6.2] to assert that \( V^R_\alpha(x) \) converges uniformly on compact sets in \( \mathbb{R}^d \) to some \( V^R_\xi \) as \( \alpha \downarrow 0 \), and that the pair \( (V^R_\alpha, \xi) \) satisfies
\[
a^{ij}(x) \partial_{ij} V^R_\alpha(x) + H_R(x, \nabla V^R_\alpha) = \xi \quad \text{a.e. in } \mathbb{R}^d.
\]

It is also the case that \( V^R \) is bounded below and admits the following stochastic representation: for any measurable selector \( v_R \) from the minimizer in (23) we have
\[
V^R(x) = \Psi^R(x; \xi) + E^R_x[V^R(X_\tau_R)], \quad \forall x \in B^c_r.
\]
(24)

Also \( \xi = \beta(v_R) \). It is also clear that \( \xi \leq \hat{\xi} \) for all \( R \geq 0 \). This is because \( \alpha V^R_\alpha(0)|_{R=0} \rightarrow \hat{\xi} \) as \( \alpha \downarrow 0 \), and \( V^R_\alpha(0) \) is non-increasing in \( R \). Since \( v_R \) agrees with \( \hat{v} \) on \( B^c_R \) and \( \xi \leq \hat{\xi} \), we obtain
\[
\Psi^R(x; \xi) = \Psi^\hat{v}_R(x; \xi) \geq \Psi^\hat{v}_R(x; \hat{\xi}),
\]
which together with Lemma 3.2 and (24) implies that
\[
\hat{V}(x) \leq V^R(x) + \max_{\partial B^c_R} \hat{V} - \min_{\partial B^c_R} V^R \quad \forall x \in B^c_R.
\]
(25)

Therefore since
\[
\lim_{t \to \infty} \frac{1}{t} E^R_x[V^R(X_t)] = 0
\]
by [2 Corollary 3.7.3], the bound in (25) shows that the same applies to \( \hat{V} \). Applying Dynkin’s formula to
\[
L^v R \hat{V}(x) + c(x, v_R(x)) \geq \hat{\xi}
\]
and using the just established fact that \( \frac{1}{t} \mathbb{E}^{v_R}_{x} [V(X_t)] \xrightarrow{t \to \infty} 0 \) and the definition \( q_R = \beta(v_R) \) we obtain \( q_R \geq \hat{\theta} \). Therefore it must be the case that

\[
\hat{\theta} = q_R \quad \forall R > 0.
\]  

(26)

Define the function \( F_R : [0, 1] \to \mathbb{R}_+ \) by

\[
F_R(\alpha) = \begin{cases} 
\alpha V^R_{\alpha}(0) & \text{for } \alpha \in (0, 1] \\
q_R & \text{for } \alpha = 0.
\end{cases}
\]

It is a simple matter to verify that \( \alpha \mapsto V^R_{\alpha}(0) \) is continuous on \( (0, 1] \). Therefore \( F_R \) is continuous on \( [0, 1] \) for each fixed \( R > 0 \). It is also evident that \( R \mapsto F_R \) is non-increasing. Therefore by Dini’s theorem \( F_R \) converges uniformly on \( [0, 1] \) to some non-negative function \( F_\infty \) as \( R \to \infty \), and as a result \( F_\infty \) is continuous on \( [0, 1] \). It is a standard matter to show that for any \( \alpha \in (0, 1] \) the function \( V^R_{\alpha} \) converges uniformly on compact sets of \( \mathbb{R}^d \) as \( R \to \infty \) to some \( V^\infty_{\alpha} \in W^{2,p}_{\text{loc}}(\mathbb{R}^d) \), for any \( p > d \), and that the limit is a solution of

\[
\min_{u \in U} \left[ L^u V^\infty_{\alpha}(x) + c(x, u) \right] = \alpha V^\infty_{\alpha}(x), \quad x \in \mathbb{R}^d.
\]  

(27)

By elliptic regularity \( V^\infty_{\alpha} \in C^2(\mathbb{R}^d) \). Since \( c \) is bounded, (27) has a unique nonnegative solution in \( C^2(\mathbb{R}^d) \) which admits the stochastic representation

\[
V^\infty_{\alpha}(x) \triangleq \inf_{U \in \mathcal{U}} \mathbb{E}^U_x \left[ \int_0^\infty e^{-\alpha s} c(X_s, U_s) \, ds \right].
\]

It is well known that \( \alpha V^\infty_{\alpha}(0) \to q^* \) as \( \alpha \downarrow 0 \) [2, Theorem 3.6.6]. Since \( F_R \) converges uniformly on \( [0, 1] \), we have

\[
\lim_{R \to \infty} q_R = \lim_{\alpha \downarrow 0} \lim_{R \to \infty} F_R(\alpha) \\
= \lim_{\alpha \downarrow 0} F_\infty(\alpha) \\
= \lim_{\alpha \downarrow 0} \alpha V^\infty_{\alpha}(0) \\
= q^*. \]  

(28)

By (26) and (28) we obtain \( \hat{\theta} = q^* \). \( \square \)

IV. A REMARK ON THE POLICY ITERATION ALGORITHM

A good part of the difficulty in obtaining a solution to the HJB equation lies in the fact that the optimal cost \( q^* \) is not known. The policy iteration (PIA) provides an iterative procedure for obtaining the HJB equation via iterations of linear equations.
Recall the definitions of $M^*$, $\beta$, $\Psi^v_0$ and $\Psi^v_r$ in (4), (8), (13) and (19), respectively. Under the near-monotone hypothesis, if $v \in \mathcal{U}_{SSM}$ and $\beta(v) < M^*$ then $\Psi^v_0$ is the unique solution $V$ of the Poisson equation

$$L^v V(x) + c(x, v(x)) = \beta(v), \quad x \in \mathbb{R}^d$$

in $\mathcal{W}^{2,p}_{loc}(\mathbb{R}^d)$, $p > d$, which is bounded below and satisfies $V(0) = 0$. Note also that (12) implies that any control $v$ satisfying $\varrho_v < M^*$ is stable.

We write the PIA in following form:

**Algorithm 4.1 (Policy Iteration):**

1) Initialization. Set $k = 0$ and select any $v_0 \in \mathcal{U}_{SM}$ such that $\beta(v_0) < M^*$. Set $V_0 = \Psi^v_0$ and $\varrho_0 = \beta(v_0)$.
2) Policy improvement. Select an arbitrary $v_{k+1} \in \mathcal{U}_{SM}$ which satisfies

$$v_{k+1}(x) \in \text{Arg min}_{u \in U} \left\{ b^i(x, u) \partial_i V_k(x) + c(x, u) \right\}, \quad x \in \mathbb{R}^d.$$

3) Value determination. Let $V_{k+1} = \Psi^{v_{k+1}}_0$ and $\varrho_{k+1} = \beta(v_{k+1})$. If $\varrho_{k+1} = \varrho_k$ stop.

It is well known and straightforward to show that, provided $c$ is near-monotone, then over any iteration of the PIA $\varrho_k$ is a non-increasing sequence. Also for some positive numbers $\alpha_k$ and $\gamma_k$, $k \geq 0$, such that $\alpha_k \downarrow 1$ and $\gamma_k \downarrow 0$ as $k \to \infty$ it holds that (see [6, Theorem 4.4])

$$\alpha_k V_{k+1}(x) + \gamma_{k+1} \leq \alpha_k V_k + \gamma_k \quad \forall k \in \mathbb{N}.$$

The near monotone hypothesis along with the fact that $\varrho_k$ is non-increasing imply that the density of the invariant probability measure $\mu_{v_k}$ is locally bounded away from zero uniformly in $k \in \mathbb{N}$. Using the above properties one can show that $V_k$ converges uniformly on compact sets of $\mathbb{R}^d$ to some $\hat{V} \in C^2(\mathbb{R}^d)$ which together with the constant $\hat{\varrho} = \lim_{k \to \infty} \varrho_k$ form a solution pair for the HJB equation (5) (see [7] Lemma 2 and Corollary 1).

Observe that at every iteration the PIA returns a pair of the form $(V_k, \varrho_k) = (\Psi^{v_k}_0, \beta(v_k))$. Hence a pair $(V, \varrho) \in C^2(\mathbb{R}^d) \times \mathbb{R}_+$ is an equilibrium of the PIA if and only if it is a compatible solution pair to the HJB equation. Therefore, if the running cost is bounded, then by Theorem 1.1 the only equilibrium of the PIA is the optimal pair $(V^*, \varrho^*)$ in Theorem 3.1. However this does not imply that the PIA always converges to the optimal pair $(V^*, \varrho^*)$. Because is does not preclude the possibility that the iterates $(V_k, \varrho_k)$ may have a limit point $(\hat{V}, \hat{\varrho})$ which is not an equilibrium of the PIA. Observe that the map $v \mapsto \mu_v$ from $\mathcal{U}_{SSM}$ under the topology of Markov controls (see [2] Section 2.4) to the set of invariant probability measures under the Prohorov topology is not in general continuous. As a result if
\{v_k\} \subset \mathcal{U}_{SSM} \text{ is a sequence which converges under the topology of Markov controls but } \{\mu_{v_k}\} \text{ is not tight, we may obtain}
\lim_{k \to \infty} \beta(v_k) > \beta\left(\lim_{k \to \infty} v_k\right).

It is interesting to note that if we allow a transfinite number of iterations then, provided the running cost is bounded, convergence to the optimal value can be obtained. We give a brief description of this transfinite recursion in the next paragraph. For a more sophisticated use of transfinite iterations in dynamic programming we refer the reader to [8].

The recursion on the ordinals is defined as follows: We denote the algorithm as \((V_{k+1}, \varrho_{k+1}) = \mathcal{T}(V_k, \varrho_k)\). Note that \(\mathcal{T}\) is not really a map since the measurable selector from the minimizer at each step is not unique, but we don’t delve into the formalism of inductive definability because the recursion is quite intuitive, and also because it is straightforward to demonstrate that it terminates at a countable ordinal.

Let \(\omega_1\) denote the first uncountable ordinal. Let \(V_0 = \Psi_0\) and \(\varrho_0 = \beta(v_0)\), and for every ordinal \(\xi < \omega_1\) define \(\{(V_\xi, \varrho_\xi), \xi < \omega_1\}\) by
\[(V_\xi, \varrho_\xi) = \mathcal{T}\left(\lim_{\eta < \xi} V_\eta, \lim_{\eta < \xi} \varrho_\eta\right).
\tag{29}

If \(\xi\) is a limit ordinal, then since Algorithm [4,1] (which is defined on \(\mathbb{N}\)) converges, it follows that
\[\left(\hat{V}(\xi), \hat{\varrho}(\xi)\right) \triangleq \left(\lim_{\eta < \xi} V_\eta, \lim_{\eta < \xi} \varrho_\eta\right)\]
is a solution of the HJB. Suppose \(\left(\hat{V}(\xi), \hat{\varrho}(\xi)\right)\) is not a compatible pair, otherwise the recursion terminates. Then by (29) we obtain \(V_\xi = \hat{V}(\xi)\) and \(\varrho_\xi < \hat{\varrho}(\xi)\) which imply that \((V_\xi, \varrho_\xi)\) does not solve the HJB. Therefore we must have \(\varrho_{\xi+1} < \varrho_\xi\). Set \(\delta_\xi = \varrho_{\xi+1} - \varrho_\xi\). Since only a countable number of the \(\delta_\xi\) can be positive it follows that there exists \(\xi^* < \omega_1\) such that \(\delta_{\xi^*} = 0\). Therefore the recursion terminates at a countable ordinal.

V. CONCLUDING REMARKS

Theorem [1.1] fills a gap in the theory of ergodic control of diffusions under near-monotone costs, albeit under the assumption of bounded running cost. This assumption was only used to assert that (27) has a unique non-negative solution. Therefore whenever this can be established for a particular problem the hypothesis of bounded running costs can be waived.

There are also some standard situations when the problem can be mapped to an equivalent problem with bounded costs. Suppose that \(c\) satisfies
\[
\sup_{x \in \mathbb{R}^d, u, u' \in \mathcal{U}} \frac{c(x, u')}{c(x, u)} < \infty.
\]
A particular case when this happens is of course when the running cost does not depend on the control. We leave it to the reader to verify that if we define
\[ g(x) \triangleq 1 + \min_{u \in U} c(x, u), \quad \bar{\sigma} \triangleq \frac{\sigma}{g}, \quad \text{and} \quad \tilde{b} \triangleq \frac{b}{g}, \]
then the controlled diffusion with data \( \tilde{b}, \bar{\sigma} \) and running cost
\[ \tilde{c} \triangleq \frac{g^*}{\min_{\sigma \in \sigma^*}} \frac{1 + c - \bar{\sigma}^*}{g} \]
is an equivalent optimal control problem which satisfies the assumptions of Theorem 1.1 and hence the conclusions of this theorem apply to the original problem.

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