Pure Spinors for General Backgrounds†

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Abstract

We show the equivalence of the different types of pure spinor constraints geometrically derived from the Free Differential Algebras of $\mathcal{N} = 2$ d=10 supergravities. Firstly, we compute the general solutions of these constraints, using both a $G_2$ and an SO(8) covariant decomposition of the 10d chiral spinors. Secondly, we verify that the number of independent degrees of freedom is equal to that implied by the Poincaré pure spinor constraints so-far used for superstrings, namely twenty two. Thirdly, we show the equivalence between the FDA type IIA/B constraints among each other and with the Poincaré ones.

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1 Introduction

In order to have a constructive derivation of the pure spinor sigma model on general supergravity backgrounds, we decided in [1] to derive it from the rheonomic approach to supergravity. The latter is based on a Free Differential Algebra (FDA) and the pure spinor formulation is based on the BRST extension of that FDA. The closure of the BRST algebra leads to pure spinor constraints which look different from those used on [2, 3, 4]. Therefore, it is crucial to prove the equivalence of the new constraints with the old ones. A similar analysis was performed in a previous paper [5] and recently discussed also in a conference proceedings [6].

We discuss the pure spinor constraints as derived from FDA in the type IIA and type IIB backgrounds. Let us name the solutions of such equations the FDA pure spinors. On the other hand let us denote Poincaré pure spinors those which solve the constraints so-far
used for type II superstrings [2, 3, 4] which read as follows

\[ \bar{\lambda}_1 \Gamma^a \lambda_1 = 0, \quad \bar{\lambda}_2 \Gamma^a \lambda_2 = 0. \]  

(1.1)

We use the notation \( \lambda_A \) (with \( A = 1, 2 \)) for the pure spinors and we distinguish between type IIA and IIB by choosing the chirality of \( \lambda_2 \).

Although the choice of the Poincaré constraints (1.1) is feasible they imply unconventional superspace constraints for supergravity. Therefore it becomes quite difficult to construct explicitly the corresponding pure spinor sigma given a solution of the supergravity field equations. On the other hand, the new pure spinor constraints derived in [1] from the FDA structure are those naturally adapted to a generic background and allow the immediate writing of the corresponding pure spinor string action on any supergravity on-shell configuration.

Obviously, we need to show that these new constraints lead to the correct amount of independent degrees of freedom to cancel the conformal central charge.

The three situations, Poincaré, type IIA and IIB are summarized in the table 1. In paper [1], we deduced the pure spinor constraints from the closure of the FDA algebra in its extended ghost-form by suppressing all the other ghosts except those of supersymmetry. It is important to clarify how the constraints in table [1] have to be understood. The latter are too strong for a 10d target-space vielbein \( V^a \) and therefore we have to project them on the 2d string worldsheet by embedding it into the target-space. Explicitly the vielbeins \( V^a \) must be replaced by their pull-back onto the worldsheet, namely:

\[ V^a \mapsto \Pi^a_+ e^+ + \Pi^a_- e^- \]  

(1.2)

where \( e^\pm \) denote the worldsheet zweibein.

In this way, we are able to prove that the number of independent degrees of freedom is the same in all cases.

The proof of the equivalence is done in two ways. First, we find the solutions of the pure spinor constraints using an SO(8) and a \( G_2 \) decomposition, respectively. Both solutions

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\(^1\)The adopted name Poincaré refers to the fact that these constraints can be deduced by exploiting two copies of the Poincaré Lie superalgebra in \( d=10 \), a left-handed and a right-handed one in the type IIA case and two independent left-handed ones in the type IIB case. It is important to stress the word Lie superalgebra as opposed to the word FDA. Indeed the different constraints which are naturally adapted to a generic supergravity background follow from the complete algebraic structure underlying supergravity which is the FDA extension of the superPoincaré Lie algebra and which mixes the two spinor chiralities because of the Ramond-Ramond \( p \)-forms
Table 1: We list the pure spinor constraints and the RR or NSNS fields to which they are associated. We denote by $\lambda_A$ with $A = 1, 2$ the pure spinor either in type IIA and type IIB. In the former case $A = 1, 2$ are of opposite chirality, while for the latter they have the same chirality. $T$ stands for the torsion constraint. The vielbein $V^a$ is the pull-back on the worldsheet of the target space vielbein.

| Poincaré | type IIA | type IIB |
|----------|----------|----------|
| $T: \sum_A \bar{\Lambda}_A \Gamma^a \lambda_A = 0$ | $T: \sum_A \bar{\Lambda}_A \Gamma^a \lambda_A = 0$ | $T: \sum_A \bar{\Lambda}_A \Gamma^a \lambda_A = 0$ |
| $\sum_A (-)^A \bar{\Lambda}_A \Gamma^a \lambda_A = 0$ | $B_{[2]}: \sum_A (-)^A \bar{\Lambda}_A \Gamma^a \lambda_A V^a = 0$ | $B_{[2]}: \sum_A (-)^A \bar{\Lambda}_A \Gamma^a \lambda_A V^a = 0$ |
| $C_{[3]}: \bar{\Lambda}_1 \Gamma^{[a_1 b_1]} \lambda_2 V_{a_2} V_{b_2} = 0$ | $C_{[2]}: \bar{\Lambda}_1 \Gamma^a \lambda_2 V^a = 0$ | $C_{[2]}: \bar{\Lambda}_1 \Gamma^a \lambda_2 V^a = 0$ |
| $C_{[1]}: \bar{\Lambda}_1 \lambda_2 = 0$ | | |

are quite interesting: the former provides a solution in terms of an infinite number of fields (this solution is very similar to the one proposed in [7]); the latter shares several similarities, since it preserves only the $G_2$ invariance, but it can be expressed in terms of a finite number of fields. We provide a complete solution on a single patch and one has to extend the solution to the entire space as usual. Then, we prove the equivalence between type IIA and type IIB pure spinors, by showing that there exists a simple map between the two set of constraints which is written in terms of Dirac matrices. Finally we show the equivalence of the solutions for the type IIB and the Poincaré case by mapping one solution in the other.

To complete the proof we observe that in the type IIA case, all constraints can be cast into a single Lorentz tensor of anti-de Sitter group $SO(2, 10)$ which can be written as follows

$$\Lambda \Gamma_{[\Sigma \Xi]} \Lambda = 0, \quad \Sigma, \Xi = 1, \ldots, 12 \quad (1.3)$$

and therefore one can skew-diagonalize the constraint by an $SO(2, 10)$ rotation which can be expressed in terms of a rotation of $Spin(2, 10)$ on the spinors $\Lambda$. Notice that also the Poincaré pure spinor constraints can be rotated in the same way and therefore, if the matrices have the same rank (i.e. the same number of non-zero skew-eigenvalues) then we can map the Poincaré constraints into the type IIA ones. Indeed, it is easy to show that the rank of the pure spinor matrix is four in all cases so that there are two independent sets of skew-eigenvalues. Thus, by a $Spin(2, 10)$ transformation, we can rotate one type of constraints into the others.
The paper is organized as follows: sec. 2 provides an extract of the derivation of the pure spinor constraints from the FDA of supergravity. In sec. 3 we discuss the solution of Poincaré pure spinor constraints by means of a $G_2$ decomposition. In sec. 4, we compute the number of independent components using either a $G_2$ or an $SO(8)$ decomposition. In sec. 5, we derive the equations of pure spinors for the type IIA case in the $G_2$ decomposition and finally in sec. 6 we prove the equivalence.

## 2 Pure Spinor Constraints from FDA

Using the same argument used in paper [1] for the type IIA case, the pure spinor constraints implied by the constrained ghost extension of the type IIB FDA curvatures are as follows:

\[ i\lambda \Gamma^a \lambda \approx 0 \quad (2.1) \]

\[ \Lambda^\eta_+ \lambda \Gamma^a \lambda V^a + \Lambda^\eta_- \lambda \Gamma^a \lambda V^a \approx 0, \quad \eta = 1, 2 \quad (2.2) \]

\[ \lambda \Gamma_{abc} \lambda V^a \wedge V^b \wedge V^c \approx 0 \quad (2.3) \]

where the $2 \times 2$ complex matrix $\Lambda$ denotes the coset representative of $SU(1,1)/U(1)$, by $V^a$ we denote the 10d vielbein and $\lambda$ is a Weyl commuting spinor. Eq. (2.1) follows from the BRST variation of the diffeomorphism ghosts $\xi^a$, when they are set to zero, eq. (2.2) follows from the BRST variation of the 2-form ghosts $a_{[2-i,i]}$ when they are set to zero and by the same token eq. (2.3) follows from the BRST variation of the 4–form ghosts. Only the first of these equations is background independent. On the contrary the second and the third depend also on the vielbein, namely on the background. The constraint (2.3) has no projection on the string-worldvolume and hence no relevance. The constraint (2.2) instead, written in real notation looks like follows:

\[ d^{A|BC} \xi_B \Gamma^B \lambda_C V^a = 0 \quad (2.4) \]

where $A, B, C = 1, 2$ so that the constraints to be solved are:

\[ 0 = \xi_A \Gamma^A \lambda V^a, \quad 0 = d^{A|BC} \xi_B \Gamma^B \lambda_C V^a \quad (2.5) \]

where the $d^{A|BC}$ tensor denotes the Clebsh-Gordon coefficients for the decomposition of two doublets of $SO(2)$ into one doublet. So that, $d^{1|BC} \propto (\sigma^3)^{BC}$ and $d^{2|BC} \propto (\sigma^1)^{BC}$.

The structure of the pure spinor constraints in the presence of non trivial backgrounds were also obtained in [2] where the couplings with RR fields are essential for D-brane actions.
3 Poincaré Pure Spinors

The Poincaré constraints used by Berkovits are rather different from ours. They are encoded in the following background independent equations

$$\sum_{A=1,2} \bar{\lambda}_A \Gamma^\mu \lambda_A = 0, \quad \sum_{A=1,2} (-)^A \bar{\lambda}_A \Gamma^\mu \lambda_A = 0$$

(3.1)

Hence in this case we have to solve only the constraint:

$$\bar{\lambda} \Gamma^\mu \lambda = 0$$

(3.2)

where $\lambda$ is a chiral spinor $\Gamma_{11} \lambda = \lambda$. We use the gamma matrix basis well adapted to the 1-brane case which we describe in the appendix and we solve the $d = 10$ chirality condition by posing:

$$\lambda = \phi_+ \otimes \zeta_+ + \phi_- \otimes \zeta_-$$

(3.3)

where $\phi_{\pm}$ are 2-component SO(1, 1) chiral spinors and $\zeta_{\pm}$ are 16-components SO(8) spinors also chiral:

$$\gamma_3 \phi_{\pm} = \pm \phi_{\pm} \ ; \ T_9 \zeta_{\pm} = \pm \zeta_{\pm}$$

(3.4)

In the chosen basis we have:

$$\phi_+ = \begin{pmatrix} \varphi_+ \\ 0 \end{pmatrix} \ ; \ \phi_- = \begin{pmatrix} 0 \\ \varphi_- \end{pmatrix}$$

$$\zeta_+ = \begin{pmatrix} 0 \\ \omega_+ \end{pmatrix} \ ; \ \zeta_- = \begin{pmatrix} \omega_- \\ 0 \end{pmatrix}$$

(3.5)

where $\varphi_{\pm}$ are just complex numbers while $\omega_{\pm}$ are 8-components complex SO(7) spinors.

In view of the charge conjugation matrix (6.16) the pure spinor constraint (3.2) reduces to:

$$0 = \lambda^T C \Gamma^\mu \lambda = \begin{cases} \phi_+^T \epsilon \gamma_i \phi_+ \zeta_+^T \zeta_+ + \phi_-^T \epsilon \gamma_i \phi_- \zeta_-^T \zeta_- = 0 \ (i = 0, 1) \\ \phi_+^T \epsilon \phi_- \zeta_+^T T_I \zeta_- = 0 \ (I = 1, \ldots, 8) \end{cases}$$

(3.6)

which further reduces to:

$$0 = \varphi_+^2 \zeta_+^T \zeta_+ - \varphi_-^2 \zeta_-^T \zeta_-$$

$$0 = -\varphi_+^2 \zeta_+^T \zeta_+ - \varphi_-^2 \zeta_-^T \zeta_-$$

$$0 = 2 \varphi_+ \varphi_- \zeta_+^T T_I \zeta_-$$

(3.7)
Since $\varphi_\pm$ are 1-component objects, namely complex numbers, eq.s (3.6) yield:

$$\begin{align*}
\zeta_+^T \zeta_+ &= 0 \\
\zeta_+^T T_I \zeta_- &= 0
\end{align*} \Rightarrow \begin{cases} 
\omega_+^T \omega_+ &= 0 \\
\omega_+^T \tau^\alpha \omega_- &= 0 \\
\omega_+^T \omega_- &= 0
\end{cases} \quad (3.8)
$$

The constraints in eq.(3.8) can be solved in various ways. We have four branches of a singular solution depending only on 7 complex parameters and a regular solution depending on 11 complex parameters.

**The singular solution with 7-parameters** Let $\omega = \{\omega_1, \ldots, \omega_8\}$ be an 8-component complex spinor fulfilling the equation

$$\omega^T \omega = 0 \quad (3.9)$$

then eq.s(3.8) can be solved by setting either:

1) $\omega_+ = \omega$ ; $\omega_- = 0$ or

2) $\omega_+ = 0$ ; $\omega_- = \omega$ or

3) $\omega_+ = \omega$ ; $\omega_- = \omega$ or

4) $\omega_+ = \omega$ ; $\omega_- = -\omega$

**The regular solution with 11 parameters** Let $\varpi^\alpha$ and $\chi^\alpha$ be two 7-components complex vectors satisfying the constraints:

$$\begin{align*}
\varpi \cdot \varpi &\equiv \varpi^\alpha \varpi^\alpha = 0 \\
\varpi \cdot \chi &\equiv \varpi^\alpha \chi^\alpha = 0
\end{align*} \quad (3.10)$$

We can solve eq.s(3.8) setting either

1) $$
\begin{align*}
\frac{\omega_+^\alpha}{\omega_+} &= \frac{\varpi^\alpha}{\varpi^\alpha} \quad ; \quad \omega_+^8 &= 0 \\
\frac{\omega_-^\alpha}{\omega_-} &= \frac{a^{\alpha \beta \gamma}}{\varpi^\beta \chi^\gamma} \quad ; \quad \omega_-^8 &= 0
\end{align*} \quad (3.11)
$$

or

2) $$
\begin{align*}
\omega_+^\alpha &= a^{\alpha \beta \gamma} \varpi^\beta \chi^\gamma \quad ; \quad \omega_+^8 &= 0 \\
\omega_-^\alpha &= \varpi^\alpha \quad ; \quad \omega_-^8 &= 0
\end{align*} \quad (3.12)$$
In this way we have 11 eleven parameters in each of the two pure spinors $\lambda_1$ and $\lambda_2$. Indeed $\pi^\alpha$ counts for 6 because its norm is zero and $\pi^\alpha$ counts for 5 because it is orthogonal to a vector of vanishing norm and because it is defined up to a gauge transformation $\chi^\alpha \mapsto \chi^\alpha + x \pi^\alpha$ where $x$ is the gauge parameter. This makes the correct counting 22.

Together with the ghost fields $\lambda$, one has to consider their conjugate momenta $w$. We recall the quadratic part of the action for free pure spinors (here we consider only the left-moving sector for simplicity)

$$S = \int w_\alpha \bar{\partial} \lambda^\alpha = \int w^T \bar{\partial} \lambda. \quad (3.13)$$

where we have also neglected the coupling with the homomorphic form $\Omega$ (see for example [9, 10, 11]) since it does not enter in the present discussion. We have also used the matrix notation $w^T$ to denote the spinor $w_\alpha$.

We observe that if $\lambda^\alpha$ satisfies the pure spinor constraints, which are first class constraints (since their Poisson brackets vanish), then there are gauge symmetries generated by them. If we denote by $q = \oint \Lambda^m \bar{\Gamma}^m \lambda$ the charge associated to that gauge symmetry and $\Lambda^m$ a set of gauge parameters, then we have the gauge transformations

$$\delta w_\alpha = 2 \Lambda^m (C \Gamma^m \lambda)_\alpha. \quad (3.14)$$

Now, in order to use the decomposition (3.3), we insert it in the above equation and we use the fact that the spinors $\phi_\pm$ have only one non-zero component. Hence, their value can be reabsorbed into $\zeta_\pm$ and they can be set equal to unit versors $\phi_+ = (1, 0)$ and $\phi_- = (0, 1)$, yielding

$$S = \int (\phi_+ \otimes \bar{\partial} \zeta_+ + \phi_- \otimes \bar{\partial} \zeta_-) . \quad (3.15)$$

Since the spinors $\phi_\pm$ are orthogonal to each other, we can decompose the conjugates $w_\alpha$ as follows

$$w = \phi_+ \otimes w^- + \phi_- \otimes w^+ \quad (3.16)$$

where $w^\pm$ are 8-dimensional spinors and the action becomes

$$S = \int (w^- \bar{\partial} \omega_+ + w^+ \bar{\partial} \omega_-) . \quad (3.17)$$

Here we have plugged the definitions (3.3). Hence from the pure spinor constraints written in terms of $\omega_\pm$ (see eq. (3.3)), it is straightforward to get

$$\delta w^\pm = \Lambda^\pm w^\pm + \tilde{\Lambda} w^\mp + \Lambda^\alpha \pi^\alpha w^\mp, \quad (3.18)$$

7
where $\Lambda^\pm, \Lambda$ and $\Lambda_\alpha$ are 10 gauge parameters obtained by decomposing the vector $\Lambda_m$ in representations $(2,0) + (0,0) + (0,7) + (0,1)$ of $\text{SO}(1,1) \otimes \text{SO}(7)$ as in (3.14). The pure spinor constraints are not irreducible and therefore the gauge symmetries are not all independent. It is easy to see that one can use the three gauge parameters $\Lambda^\pm, \Lambda$ to set some components of $w^\pm$ to zero, but the gauge transformation of the 8th-component (because of the ansatz (3.12)) works as follows

$$\delta w^+_8 = \Lambda_\alpha \varpi^\alpha, \quad \delta w^-_8 = \Lambda_\alpha \alpha^\beta \chi^\beta \chi^\gamma.$$ (3.19)

This implies that the two components of $w^\pm$ can be set to zero by using the gauge parameters $\Lambda_\alpha$. The independent gauge parameters is the space complementary to that spanned by the solutions of

$$\Lambda_\alpha \varpi^\alpha = 0, \quad \Lambda_\alpha \alpha^\beta \chi^\beta \chi^\gamma = 0.$$ (3.20)

It is easy to count the gauge parameters by observing that the two constraints imply that there are 5 free parameters, defined up to some the gauge symmetries $\Lambda_\alpha \rightarrow \Lambda_\alpha + x \varpi^\alpha + y \chi^\alpha + z a_\alpha \beta \gamma \chi^\beta \chi^\gamma$ with $x, y, z$ the corresponding gauge parameters. This yields the wanted gauge parameter counting: seven parameters $\Lambda_\alpha$ minus two constraints (3.20) minus the three gauge symmetries makes two. Therefore, the number of non-vanishing $w$ can be fixed to eleven.

4 PS for IIB backgrounds

4.1 Solution with $G_2$ decomposition

Let us now compute the solution of the pure spinor constraints in the case of type IIB backgrounds and and let us show that there is a 22-parameter solution also for them, although differently constructed. Also in this case we use a well adapted basis of gamma matrices and we search for a $G_2$-covariant parametrization of the solution.

Let us then consider eq.s (2.5) and let us treat them as before by setting the following tensor product parametrization:

$$\lambda_A = \phi_+ \otimes \zeta_A^+ + \phi_- \otimes \zeta_A^-$$ (4.1)

where:

$$\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\zeta_A^+ = \begin{pmatrix} 0 \\ \omega_A^+ \end{pmatrix}; \quad \zeta_A^- = \begin{pmatrix} \omega_A^- \\ 0 \end{pmatrix}$$ (4.2)
In writing eq.s 4.2, we have observed that the unique component of \( \phi_\pm \) can always be reabsorbed in the normalization of \( \omega_\pm^\pm \) and hence set to one, as already noted.

Using a well adapted basis where, defining, \( V^a = \Pi^a_i \epsilon_i \) we have:

\[
\Pi^j_i = \delta^j_i \quad \text{and zero otherwise} \tag{4.3}
\]

the constraints to be solved reduce to the following ones:

\[
\phi^T_+ \epsilon \gamma_i \phi_+ \zeta^+_A \cdot \zeta^-_A + \phi^T_- \epsilon \gamma_i \phi_- \zeta^-_A \cdot \zeta^-_A = 0 \tag{4.4}
\]

\[
d^{ABC} \left( \phi^T_+ \epsilon \gamma_i \phi_+ \zeta^+_B \cdot \zeta^+_C + \phi^T_- \epsilon \gamma_i \phi_- \zeta^-_B \cdot \zeta^-_C \right) = 0
\]

\[
\phi^T_+ \epsilon \phi_- \zeta^+_A \cdot T^I \zeta^- = 0.
\]

Elaborating eq.s 4.4 a little further we reduce them to the following ones in terms of 8-component SO(7)-spinors:

\[
\omega^\pm_1 \cdot \omega^\pm_1 = 0 \tag{4.5}
\]

\[
\omega^\pm_2 \cdot \omega^\pm_2 = 0
\]

\[
\omega^\pm_1 \cdot \omega^\pm_2 = 0
\]

\[
\omega^+_1 \cdot \omega^-_1 + \omega^+_2 \cdot \omega^-_2 = 0
\]

\[
\omega^+_1 \cdot \tau^\alpha \omega^-_1 + \omega^+_2 \cdot \tau^\alpha \omega^-_2 = 0
\]

It is now easy to present the 22-parameter solutions of the above constraints. Let

\[
\omega^\alpha ; \quad \pi^{\alpha} ; \quad \xi^{\alpha} ; \quad \chi^{\alpha} \tag{4.6}
\]

be a set of four 7-component vectors (fundamental representations of \( G_2 \)) subject to the following constraints:

\[
\omega \cdot \omega = 0 \tag{4.7}
\]

\[
\pi \cdot \pi = 0 \tag{4.8}
\]

\[
a^{\alpha \beta \gamma} \chi_\alpha \pi_\beta \omega_\gamma = 0 \tag{4.9}
\]

\[
a^{\alpha \beta \gamma} \xi_\alpha \pi_\beta \omega_\gamma = 0 \tag{4.10}
\]

the solutions of the constraints (4.3,4.4,4.1) is given by the following positions:

\[
\omega^+_1 = (\omega^\alpha, 0)
\]

\[
\omega^-_2 = (\pi^\alpha, 0)
\]

\[
\omega^-_1 = (a^{\alpha \beta \gamma} \chi_\beta \omega_\gamma, \chi \cdot \omega)
\]

\[
\omega^+_2 = (a^{\alpha \beta \gamma} \xi_\beta \pi_\gamma, \xi \cdot \pi) \tag{4.11}
\]
It is easy to count the number of parameters and verify that it amounts to 22 independent ones. Indeed \( \varpi \) and \( \pi \) count 6 each being of vanishing norm, while \( \chi \) and \( \xi \) count 5 each because they are subject to the constraints (4.9, 4.10) and defined up to a gauge transformation. Hence we have a 22-parameter solution also for the FDA pure spinor constraints for type IIB.

The intersection of the this solution with the Poincaré solution is obtained by imposing the extra condition:

\[
\varpi \cdot \pi = 0 \tag{4.12}
\]

that reduces the space to a 21-parameter one and does not yield the correct counting.

Again, we want to check that the conjugate momenta have the correct degrees of freedom also in the FDA case. For that we introduce the two sets of 8-dimensional spinors \( w^\pm_A \) with \( A = 1, 2 \) and we derive their gauge transformations from the reduced equations (4.5) to get

\[
\delta w^\pm_1 = 2\Lambda^\pm_1 \omega^\pm_1 + \hat{\Lambda}^\pm \omega^\mp_1 + \Lambda_\alpha \tau^\alpha \omega^\mp_1,
\]

\[
\delta w^\pm_2 = 2\Lambda^\pm_2 \omega^\pm_2 + \hat{\Lambda}^\pm \omega^\mp_2 + \Lambda_\alpha \tau^\alpha \omega^\mp_2,
\]

where \( \Lambda^\pm_A, \hat{\Lambda}^\pm, \hat{\Lambda} \) and \( \Lambda_\alpha \) are the gauge parameters. As before, we can use \( \Lambda^-_A \) to fix the two 8th-components of \( w^\pm_A \) to zero and we can use the remaining parameters to fix other two components of \( w^\pm_A \) by a combination of \( \Lambda_\alpha \) and \( \hat{\Lambda} \). We are left with \( \hat{\Lambda}^\pm \) and \( \Lambda^+_A \) two set other fours to zero. Finally, we can use other twos of \( \Lambda_\alpha \) to set the components of \( w^\pm_A \) to zero. However, of the original 7 components of \( \Lambda_\alpha \), 4 components are irrelevant and among the other three components there are no residual gauge symmetries. Indeed, by setting to zero the gauge transformation of the 8th components of \( w^\pm_{8A} \), there is no residual gauge symmetry and we have exactly 3 gauge parameter \( \Lambda_\alpha \). So, the total counting is again 22.

Summarizing also in the FDA case out of the fourteen gauge parameters corresponding to the fourteen pure spinor constraints, only ten are irreducibles and can be used to gauge away ten components of the 32 \( w^\pm_A \).

### 4.2 Solution with SO(8) symmetry

It is convenient to solve the FDA constraints also in a SO(8) basis. For that we use the solution for a single pure spinor in an SO(8) basis of the constraints

\[
\omega^\pm_2 \cdot \omega^\mp_2 = 0, \quad \omega^+_2 \sigma^I \omega^-_2 = 0 \tag{4.15}
\]
where $\sigma^I$ are the Pauli matrices in eight dimensions and $\omega_2^\pm$ are $8_c$ and $8_s$ representation of spin(8), respectively. The index $I$ instead runs over $1, \ldots, 8$ in the $8_v$ representation.

To solve (4.15) one can make the ansatz \[ \omega^-_2 = \eta_I \sigma^I \omega^+_2 \] with $\eta_I$ an eight dimensional vector in the $8_v$ representation. This ansatz solves the equation $\omega^-_2 \cdot \omega^-_2 = 0$ and the third equation in (4.15) if $\omega^+_2 \cdot \omega^+_2 = 0$. This implies that there are 7 independent components for $\omega^+_2$ and $\omega^-_2$ is expressed in terms of $\omega^+_2$ and in terms of $\eta_I$. However, the latter are defined up to an infinite number of gauge degrees of freedom (since we can shift $\eta_I$ with $\eta_I + v \sigma^I \omega^+_2$ where $v \in 8_s$ and again the latter is defined up to gauge degrees of freedom. This procedure iterates up to infinity) which effectively makes the counting of independent components of $\omega^-_2$ equal to 4. Explicitly, this can be done by breaking SO(8) to SU(4) and suppressing one of the two four into which the $8_v$ breaks up. So, this sums up to 11 components for $\omega^\pm_2$.

Next, we make the ansatz

\[ \omega^\pm_1 = \alpha^\pm \omega^\pm_2 + \alpha_I^\pm \sigma^I \omega^\mp_2. \]  \hspace{1cm} (4.16)

where $\omega^\pm_2$ solves equation (4.15). Here $\alpha^\pm$ and $\alpha_I^\pm$ are $2 + 2 \times 8$ independent degrees of freedom which parametrize the solution for $\omega^\pm_1$ in terms of $\omega^\pm_2$. Notice that the amount of parameters exceeds the wanted independent components of $\omega^\pm_1$. This means that we have to reduce the number of them by imposing some relations. Indeed, by inserting the ansatz (4.16) into (4.5), and using (4.15), we get that $\alpha^\pm$ are free independent parameters, but $\alpha_I^\pm$ are constrained by

\[ \alpha^+_I \alpha^-_J - \alpha^+_J \alpha^-_I = 0. \]  \hspace{1cm} (4.17)

To derive (4.17) one has to use the commutation relations and the symmetry properties of the products of the Pauli matrices and using the fact that $\omega^\pm_2$ are bosonic quantities. The most general solution of (4.17) has indeed $8 + 1$ parameters. Then, the total independent parameters which describe the solution for $\omega^\pm_1$ are effectively 11, which is the correct counting. So, the solution is asymmetric, but it takes into account the SO(8) symmetry.

In order to compare this solution of the type FDA constraints with a symmetric solution of the Poincaré ones we can observe that the asymmetric solutions for $\omega^\pm_1$ is written in terms of $\omega^\pm_2$. On a patch where $\alpha^+ \neq 0$ we can solve $\omega^+_2$ in terms of $\omega^+_1$ yielding

\[ \omega^+_2 = \frac{1}{\alpha^+}(\omega^+_1 + \alpha^+_I \sigma^I \omega^-_2). \]  \hspace{1cm} (4.18)

Inserting this result into $\omega^-_1$, one gets

\[ \omega^-_1 = \alpha^- \omega^-_2 + \frac{1}{\alpha^+} \alpha^+_I \sigma^I \omega^+_1 + \frac{1}{\alpha^+} \alpha^-_J \alpha^+_I \sigma^I \sigma^J \omega^-_2 \]  \hspace{1cm} (4.19)
\[
\omega_1^- = \alpha^- \omega_2^- + \frac{1}{\alpha^+} \alpha_j^+ \sigma^I \omega_1^+ + \frac{1}{2 \alpha^+} \left( \alpha_j^- \alpha_j^+ \sigma^I \sigma^I + \alpha_j^- \alpha_j^+ \sigma^I \sigma^J \right) \omega_2^- 
\]

So, finally if we choose to have \( \alpha^+ \alpha^- = \alpha^- \alpha_j^+ \delta^{IJ} \) (notice that this equation is again a cone) we get \( \omega_1^- = \alpha_j^- \sigma^I \omega_1^+ \) which has the form of the solution for \( \omega_2^- \) and it is symmetric. Notice that we have chosen \( \alpha^- \) to put the solution in the wanted form and this reduces the amount of independent dof to 21. For a generic solution of Poincaré pure spinor, one has to uplift the constraints on \( \alpha \)'s in order to satisfy the symmetric constraints. Also in SO(8) we find that the intersection space of solutions has 21 parameters.

So, we have found a map between the solution for type FDA type IIB pure spinors to the symmetric solution of the Poincaré constraints.

## 5 FDA PS for IIA backgrounds

Let us now compute the solution of the pure spinor constraints in the case of type IIA backgrounds and let us show that there is a 22-parameter solution also for them, although differently constructed. Also in this case we use a well adapted basis of gamma matrices and we search for a \( G_2 \) invariant parametrization of the solution. We show that reducing the pure spinor constraints to the \( G_2 \) basis, one gets the same equations as in (4.5).

As displayed in table 11 we have the following PS constraints:

\[
0 = \sum \lambda_A \Gamma_2 \lambda_A, \quad \quad 0 = \sum (-)^I \lambda_A \Gamma_2 \lambda_A V^{\mu}, \quad \quad (5.1)
\]

\[
0 = \lambda_1 \Gamma_{ab} \lambda_2 V^a V^b, \quad \quad 0 = \lambda_1 \lambda_2 = 0. \quad \quad (5.2)
\]

The first two constrains come from the torsion and from the \( B_{[2]} \) form, whilst the other two are coming from the variation of the RR fields \( C_{[1]} \) and \( C_{[3]} \).

As above, we write the PS \( \lambda_I \) by decomposing them using the same basis and we get the two structures

\[
\lambda_1 = \phi_+ \otimes \zeta_1^+ + \phi_- \otimes \zeta_1^- \quad \quad (5.3)
\]
\[
\lambda_2 = \phi_+ \otimes \zeta_2^- + \phi_- \otimes \zeta_2^+ \quad \quad (5.4)
\]
and inserting them into (5.1)-(5.2) we get the following equations

\begin{align*}
0 &= \phi^T_+ \epsilon \phi_+ (\zeta^+_1 \zeta^+_1 \pm \zeta^-_2 \zeta^-_2) + \phi^T_- \epsilon \phi_- (\zeta^-_1 \zeta^-_1 \pm \zeta^+_2 \zeta^+_2), \\
0 &= \phi^T_+ \epsilon \phi_+ (\zeta^+_1 T^t \zeta^-_1 + \zeta^-_2 T^t \zeta^+_2), \\
0 &= \phi^T_+ \epsilon \phi_+ (\zeta^+_1 \zeta^+_2 - \zeta^-_1 \zeta^-_2), \\
0 &= \phi^T_+ \epsilon \phi_+ (\zeta^+_1 \zeta^+_2 + \zeta^-_1 \zeta^-_2),
\end{align*}

which completely reduce to eq.s (4.5). This shows that reducing the equations to the present $G_2$ basis, one can verify the T-duality of the FDA PS constraints. It follows that the solutions are also isomorphic even though the set of constraints are different. We conclude that all three sets of constraints are equivalent even though the solutions differ (but with the same number of parameters).

### 5.1 From IIB to IIA FDA PS

In order to map the pure spinor constraints of type IIA to those of type IIB, we consider the following map

$$
\lambda_2 \rightarrow \left( \alpha \Gamma^+ + \beta \Gamma^- \right) \lambda_2.
$$

(5.6)

Inserting this map into the two constraints $\lambda_1 \lambda_2 = 0$ and $\lambda_1 \Gamma^+ \lambda_2 = 0$ using an adapted basis, we get

\begin{align*}
\alpha \lambda_1 \Gamma^+ \lambda_2 + \beta \lambda_1 \Gamma^- \lambda_2 &= 0, \\
\alpha \lambda_1 \Gamma^+ \lambda_2 &= 0
\end{align*}

(5.7)  

and if $\alpha, \beta \neq 0$, they imply the pure spinor constraints of the type IIB FDA (in an adapted basis). Next equations, we study the constraint coming from the torsion and from the NS-NS 2-form. They read, in an adapted basis, as follows $\lambda_1 \Gamma^\pm \lambda_1 = 0$ and $\lambda_2 \Gamma^\pm \lambda_2 = 0$. Recalling that $(\Gamma^\pm)^T = \Gamma^\mp$ we get that using the map (5.6), these constraints are mapped into each other. The remaining constraints are easily shown to be equivalent. This completes the proof that the FDA type IIA and type IIB constraints are equivalent.

### 6 Overall Equivalence

In the previous sections we have shown that Poincaré and FDA constraints have solutions with the same number of parameters which can be mapped into each other. This suggests
that the very constraint equations are equivalent in the sense that they can be mapped one into the others. This is precisely what we are proving in this section.

To this end, we observe that the FDA type IIA constraints can be organized as follows

\[ \Lambda \Gamma_{\Sigma,\Xi} \Lambda F^{\Sigma,\Xi}_{\Omega,\Lambda} = 0 \]  

(6.1)

where \( \Lambda \) is the spinor obtained by combining the two chiral spinors \( \lambda_A \) and \( \Gamma_{\Sigma,\Xi} \) are Dirac matrices of SO(2,10). The indices \( \Sigma, \Xi, \Omega, \Delta \) run over 1, \ldots, 12. Explicitly we have

\[
\begin{pmatrix}
0 & \Lambda \Gamma_{1,2} \Lambda & \ldots & \Lambda \Gamma_{1,11} \Lambda & \Lambda \Gamma_{1,12} \Lambda \\
-\Lambda \Gamma_{1,2} \Lambda & 0 & \ldots & \Lambda \Gamma_{2,11} \Lambda & \Lambda \Gamma_{2,12} \Lambda \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
-\Lambda \Gamma_{1,11} \Lambda & -\Lambda \Gamma_{2,11} \Lambda & \ldots & 0 & \Lambda \Gamma_{11} \Lambda \\
-\Lambda \Gamma_{1,12} \Lambda & -\Lambda \Gamma_{2,12} \Lambda & \ldots & -\Lambda \Gamma_{11} \Lambda & 0
\end{pmatrix}
\]  

(6.2)

where for simplicity we have used the indices from 1 to 10 for the vectors in 10d and \( \Gamma_{\Sigma,12} = \Gamma_{\Sigma} \) and \( \Gamma_{12,12} = 0 \). The matrix (6.2) is antisymmetric and therefore it can be skew-diagonalized by an SO(2,10) rotation. By the invariant theory, an antisymmetric matrix has only the skew-eigenvalues as invariants and therefore if two matrices have the same number of eigenvalues (same rank) they are equivalent. Once we have established the rank and found the rotation \( R \) of SO(2,10), we can find the corresponding rotation on the spinors \( \Lambda \) as a rotation of Spin(32). After recalling this fact, it is straightforward to verify that both \( F^{\Sigma,\Xi}_{\Omega,\Lambda} \) representing the FDA and the Poincaré case have only two non-vanishing skew-eigenvalues. Hence the same rank. Indeed, the set of constraints (6.1) are viewed as 12d covariant constraints and therefore the rotations of them can be achieved by a Spin(32) rotation on the spinors \( \Lambda \). This completes the proof.

**Remarks and Conclusions**

- This analysis implies that the canonical form of supergravity as formulated in the FDA approach and that corresponding to the unconventional superspace constraints derived from the pure spinor formulation given in [12] are related by “superconformal” transformation of SO(2,10). This also confirms the results of our work [1].

- As already pointed out, the Poincaré constraints are background independent and the solution it does not depend upon the point on the base manifold. In our case, the FDA constraints are soldered on the base manifold and therefore, one has to choose
an adapted basis to solve. However, since the spirit of the entire constructions is to avoid solving them, actually there is no practical difference. On the other hand, the advantage of the FDA approach to pure spinor is the conventional framework for the supergravity and therefore yields an explicit recipe for the construction of the pure spinor sigma model on any supergravity background.

- In the case of heterotic sigma model, the FDA pure spinor constraints coincide with the Poincaré ones as been noticed by [13].

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Appendix A: $D = 1 + 9$ basis of gamma matrices well adapted to $10 = 1 + 1 \oplus 8$

In the discussion of the BRST invariant string action and in order to prove relevant Fierz identities we need to use a different basis of gamma matrices, well adapted to the subalgebra:

$$\text{SO}(1, 1) \oplus \text{SO}(8) \subset \text{SO}(1, 9)$$

We obtain a $32 \times 32$ realization of the SO(1, 9) Clifford algebra by writing:

$$\Gamma = \begin{cases} \Gamma_i = \gamma_i \otimes T_9 & ; \; i = 0, 1, \\ \Gamma_{1+\Lambda} = 1 \otimes T_I & ; \; I = 1, 2, \ldots, 8 \end{cases}$$

where $\gamma_i$ are $2 \times 2$ gamma matrices for the SO(1, 1) Clifford algebra, namely:

$$\{\gamma_i, \gamma_j\} = 2 \eta_{ij} = \text{diag}\{+, -\}$$

while $T_I$ are $16 \times 16$ gamma matrices for the SO(8) Clifford algebra with negative metric:

$$\{T_I, T_J\} = -2 \delta_{I,J}$$

As an explicit representation of the $d = 2$ gamma matrices we can take the following ones in terms of Pauli matrices:

$$\gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
On the other hand the SO(8) Clifford algebra with negative metric admits a representation in terms of completely real and antisymmetric matrices. We adopt the following one:

\[
T_I = \begin{cases} 
T_\alpha = \sigma_1 \otimes \tau_\alpha & ; \alpha = 1, 2, \ldots, 7 \\
T_8 = i \sigma_2 \otimes 1_{8 \times 8} & 
\end{cases} \tag{6.8}
\]

where \(\tau_\alpha\) denotes the \(8 \times 8\) completely antisymmetric realization of the SO(7) Clifford algebra with negative metric:

\[
\{\tau_\alpha, \tau_\beta\} = -2 \delta_{\alpha\beta} ; \quad \tau_\alpha = -(\tau_\alpha)^T
\tag{6.9}
\]

given by:

\[
(\tau_\alpha)_{\beta\gamma} = a_{\alpha\beta\gamma} ; \quad (\tau_\alpha)_{8\beta} = -(\tau_\alpha)_{8\beta} = \delta_{\alpha\beta}
\tag{6.10}
\]

where the completely antisymmetric tensor \(a_{\alpha\beta\gamma}\) encodes the structure constants of the octonion algebra or, equivalently corresponds to the components of the unique G\(_2\) invariant 3–form. Explicitly the tensor \(a_{\alpha\beta\gamma}\) is defined by its seven non vanishing components:

\[
\begin{align*}
a_{123} &= -1 ; & a_{136} &= -1 \\
a_{145} &= -1 ; & a_{235} &= -1 \\
a_{246} &= 1 ; & a_{347} &= -1 \\
a_{567} &= -1 ; & \text{all other vanish}
\end{align*}
\tag{6.11}
\]

The tensor \(a_{\alpha\beta\gamma}\) satisfies the following identity:

\[
a_{\alpha\beta\gamma} a_{\delta\eta\gamma} = \delta_{\alpha\delta} \delta_{\beta\eta} - \delta_{\alpha\eta} \delta_{\beta\delta} - \tilde{a}_{\alpha\beta\delta\eta}
\tag{6.12}
\]

where the complete antisymmetric 4-index tensor \(\tilde{a}_{\alpha\beta\delta\eta}\) is the dual of \(a_{\alpha\beta\gamma}\). Its non vanishing components are the following ones:

\[
\begin{align*}
\tilde{a}_{1234} &= -1 ; & \tilde{a}_{1357} &= 1 \\
\tilde{a}_{1256} &= -1 ; & \tilde{a}_{1467} &= -1 \\
\tilde{a}_{2367} &= -1 ; & \tilde{a}_{2457} &= -1 \\
\tilde{a}_{3456} &= -1 ; & \text{all other vanish}
\end{align*}
\tag{6.13}
\]

Finally the \(16 \times 16\) matrix \(T_9\) which anticommutes with all the \(T_A\) has, in this basis, the following structure:

\[
T_9 = -\sigma_3 \otimes 1_{8 \times 8}
\tag{6.14}
\]

The charge conjugation matrix, with respect to which we have:

\[
C \Gamma_\sharp C^{-1} = -\Gamma_\sharp^T
\tag{6.15}
\]
is given by:

\[ C = \varepsilon \otimes 1_{16 \times 16} ; \quad (\varepsilon \equiv i \sigma_2) \]  

(6.16)

In the paper, we use also the notation \( \sigma^I \) for the \( 8 \times 8 \) Dirac matrices for the blockdiagonal pieces of \( T_I \). Notice that \( \sigma^1 = \pm 1_{8 \times 8} \) depending on the chirality.
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