Experimental assessment of the speed of light perturbation in free-fall absolute gravimeters

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Abstract

Precision absolute gravity measurements are growing in importance, especially in the context of the new definition of the kilogram. For the case of free fall absolute gravimeters with a Michelson-type interferometer tracking the position of a free falling body, one of the effects that needs to be taken into account is the ‘speed of light perturbation’ due to the finite speed of propagation of light. This effect has been extensively discussed in the past, and there is at present a disagreement between different studies. In this work, we present the analysis of new data and confirm the result expected from the theoretical analysis applied nowadays in free-fall gravimeters. We also review the standard derivations of this effect (by using phase shift or Doppler effect arguments) and show their equivalence.

Keywords: speed of light perturbation, absolute gravimeters, gravimetry, watt balance, SI

(Some figures may appear in colour only in the online journal)

1. Introduction

The accurate determination of local acceleration due to gravity, \( g \), is important in many different scientific areas like geodesy or geophysics. In metrology \( g \) plays a crucial role in the so-called watt balance experiment, aiming for a new definition of the kilogram [1].

The most widely used technique for the precision measurement of gravitational acceleration is to track the position of a free-falling body by means of a Michelson interferometer. Nowadays, the relative uncertainty of these kinds of instruments is a few parts in \( 10^9 \) [2]. This is smaller than the perturbation coming from the finite speed of propagation of light in the set-up. Thus, the latter (known as ‘speed of light correction’) should be taken into account to properly identify local gravitational acceleration with current data. The ‘speed of light correction’ has been computed in several works in the past [3–11]. A systematic derivation and comparison of the previous different results (with their limitations) has been presented recently [12, 13].

These results were questioned by the authors of [15], who proposed a reconsideration of the theoretical derivation that deviates from [12] (and previous works [3–11]). The controversy was deepened after the claim in [16] of the first measurement of the ‘speed of light perturbation’ that apparently confirmed the calculations in [15]. The results and methods of [15, 16] were questioned in a series of works [17, 18] (see also [19]) as well as more recently in [20].

The above controversy, which also has a direct impact on watt balance experiments [21, 22], calls for an independent theoretical analysis and experimental study of the ‘speed of light perturbation’. This is the idea behind the present work.
We present here the analysis of new sets of data from different gravimeters and find experimental agreement with the classical result given in [3–12]. We also review the derivation of the ‘speed of light perturbation’ by two different methods and show their mutual agreement and equivalence to the results in [3–12].

The paper is organized as follows: in section 2 we review the derivation of the ‘speed of light perturbation’ in Michelson interferometers with two different approaches and show that both calculations coincide. The results of section 2 are applied for the two different cases in section 3: the constant velocity and free falling moving mirror. The latter is the relevant one for gravimetry. Finally, in section 4, we perform a data analysis with an extensive uncertainty evaluation and present a measurement of the effect of the ‘speed of light perturbation’. We summarize our results in the conclusion. Further descriptions of the data can be found in annex A.

2. Michelson interferometer with a moving and a reference mirror

As mentioned in the introduction, it seems worth carefully reviewing the derivation of the ‘speed of light perturbation’ in free fall gravimeters with a Michelson interferometer. We will present two different derivations; the first based on time delay and the second on Doppler shift, and then confirm their equivalence. A large part of our analysis can be found in earlier references, e.g. [3–12, 17].

2.1. Displacement measurements with Michelson interferometers

The working principle of displacement measuring systems based on interferometry is schematically represented in figure 1. It is described using a 2D reference system with coordinates y and z.

The light produced by a laser beam source, which we describe here by an electric field $\overline{E}_b(t, y, z)$, is sent into a beam splitter. The incoming beam at the splitting point 0 is characterized by the electric field

$$\overline{E}_b(0, t) = \overline{E}_{b,0} e^{-iomega t},$$

where $omega$ corresponds to the frequency of the laser beam and where we assume that the polarization of the beam $\overline{E}_{b,0}$ is constant in time. We have identified the origin of the coordinates with the splitting point 0. We define the y-axis to coincide with the direction of propagation of the incoming beam. After splitting, the measuring beam, $\overline{E}_{mb}$, propagates in a direction perpendicular to $y$ and that we identify with the $z$ direction. This beam is reflected by a moving mirror with trajectory $z_{mm}(t)$ (the $z$ coordinate grows in the direction of propagation of the reflected beam, downwards in figure 1) and generates an electric field at the origin, $\overline{E}_{mb}(0, t)$. The rest of the original laser beam (the reference beam, $\overline{E}_{rb}$) propagates in the y direction and is reflected from a fixed reference mirror placed at distance $L$ and produces the field $\overline{E}_{rb}(t, 0)$ at the origin. The

![Figure 1](image)

$6$ This is also valid for fields with a time-varying frequency $omega$ such that $domega/dt \ll omega T_i$. This condition will hold in our experimental setup.
where the previous expression defines $\Delta t(t)$ after introducing the reflection time

$$t_r \equiv t - \Delta t(t).$$

We have made explicit that the variable $\Delta t$ depends on time. Assuming that the reflection is instantaneous, the relation between the reflected and the incident waveform is given by a phase shift with a constant $\phi_r$, close to $\pi$ for an ideal mirror. Finally, the incident wave was originated after splitting at time: $t = 2\Delta t$. Following the previous logic, one writes

$$\mathcal{E}_\text{mb}(t, 0) = \mathcal{E}_\text{mb}(t - \Delta t, -c\Delta t) = e^{i\omega t} \mathcal{E}_\text{mb}(t - 2\Delta t),$$

(5)

where $\mathcal{E}_\text{mb}$ is the electric field of the incident beam (the beam after the original laser beam splits).

A similar expression holds for the reference beam, but this time with a fixed distance $L$ and with a different incident beam

$$\mathcal{E}_\text{ib}(t, 0) = e^{i\omega t} \mathcal{E}_\text{ib}(t - 2L/c, 0).$$

(6)

Since both incident beams come from the splitting of the original signal, they share the same phase with the incoming beam $\mathcal{E}_\text{mb}$ in (1) except for an irrelevant constant.

The previous formulae (5) and (6) are very generic. In the case of an optical interferometer, one can recall from (1) that we are dealing with monochromatic electromagnetic waves. The two beams produced after splitting will be waves with the same frequency. Applying the previous formulae we then find

$$\mathcal{E}_\text{mb}(t, 0) = e^{i\omega t} \mathcal{E}_\text{mb}(t - 2L/c, 0),$$

(7)

The intensity is given by

$$I(t) = I_0 \cdot \cos^2 \left[ \omega \cdot (\Delta t(t) - L/c) \right] = I_0 \cdot \cos^2 (\pi \omega z(t)/\lambda),$$

(8)

where $I_0$ is a constant irrelevant for the analysis of the trajectory. The relevant point is that the intensity function depends only on the variable $\Delta t$ that corresponds to the time needed for the light to travel from the moving mirror to the origin, where interference occurs. The intensity function is independent of any Doppler shifted frequency. It can also be deduced from equation (8) that interference fringes occur every time the function $\Delta t(t)$ changes by $\lambda/2$, where $\lambda = 2\pi c/\omega$. As mentioned above, one also realizes that by timing and counting the occurrence of the interference fringes, the position of the moving prism at the time when reflection happens, as seen by the detector, can be determined as a function of time. We also want to recall that the previous formula is just based on the constant speed of light $c$ and is not testing any other aspect of special relativity [14].

2.3. Derivation of the intensity signal by considering the Doppler shift

The previous derivation can be found in many classical textbooks, e.g. [23]. Still, another method of computing the same observable is based on considerations involving the Doppler shift. Since part of the discrepancy in the works [12] and [15] came about from using this second method, it seems relevant to derive the previous result again and show that both derivations agree. For this, let us consider the electric field of the beam reflected from the mirror as a function of time. The wave has a time varying frequency $\tilde{\omega}_\text{mb}$ such that the electric field reads

$$\mathcal{E}_\text{mb}(t, 0) = e^{i\omega t} \mathcal{E}_\text{mb}(t, 0).$$

(9)

To compute the time dependent frequency, one recalls that this is generated by the absorption and emission of the wave by the moving mirror, which happens at the reflection time $t_r$ defined in (2). This yields the double Doppler shift of the frequency

$$\tilde{\omega}_\text{mb} = \left( \frac{1 + v(t_r)c}{1 - v(t_r)c} \right) \omega,$$

(10)

where $v(t_r)$ is the velocity of the moving mirror at the time of reflection. To show the equivalence between equations (9) and (7), it is important to realize that the velocity in the previous formula corresponds to the variation of the position with respect to the reflection time

$$v(t_r) = \frac{dz(t_r)}{dt} = -c \left( \frac{dt}{dt_r} - 1 \right).$$

(11)

where in the second equality we used (4). This gives

$$\int_0^{t_r} d\tau \tilde{\omega}_\text{mb} = \omega (2t_r - t) = \omega (2z(t_r)/c),$$

and expressions (9) and (7) coincide for any trajectory $z(t)$. This confirms the equivalence between both derivations [17].

It seems interesting to understand why the double Doppler shift did not appear in the derivation of the previous section. For this, let us reconsider the intermediate steps in equation (5). Indeed, after reflection by the moving mirror, the frequency of the measuring beam that we observe at time $t$ changed to the value $\tilde{\omega}_\text{mb}$. Then equation (5) implies

$$\mathcal{E}_\text{mb}(t, 0) e^{-i\tilde{\omega}_\text{mb}(t - z(t)/c)} = e^{i\omega t} \mathcal{E}_\text{mb}(t, 0) e^{-i\omega t + i\tilde{\omega}_\text{mb}(t - z(t)/c)}.$$

From the previous expression we can find the value of $\mathcal{E}_\text{mb}(t, 0)$. Then we find for the field at time $t$ and at the origin,

$$\mathcal{E}_\text{mb}(t, 0) = e^{i\omega t} \mathcal{E}_\text{mb}(t, 0),$$

(10)

$$\mathcal{E}_\text{mb}(t, 0) = e^{i\omega t} \mathcal{E}_\text{mb}(t, 0) e^{-i\omega t + i\tilde{\omega}_\text{mb}(t - z(t)/c)} = e^{i\omega t} \mathcal{E}_\text{mb}(t, 0) e^{-i\omega t + i\tilde{\omega}_\text{mb}(t - z(t)/c)}.$$
where in the last equality we used \( z_{mm}(t_1)/c = t_1 - t \). Thus, the final expression simply depends on the phase change along the light ray (as expected) and is strictly identical to equation (7).

3. Two cases: constant velocity and free falling mirrors

In this section, we will apply the general formula (8) to two special cases of motion of the moving mirror: constant velocity and free fall in the gravity field of the Earth. The latter is of immediate relevance to the experimental analysis in the subsequent section.

3.1. Constant velocity

Let us first consider the case of a mirror moving along the z axes from an initial position \( z_0 \) at a constant velocity \( v_0 \).

\[
z_{mm}(t) = z_0 + v_0 \cdot t.
\]

From the definition of \( \Delta t \), equation (3), we find,

\[
\Delta t = \frac{z_0 + v_0 \cdot t}{v_0 - c}.
\]

This eventually translates into the intensity of the form (see (8))

\[
I = I_0 \cdot \cos^2 \left[ \phi \cdot \left( \frac{1}{1 - v_0/c} (z_0 + v_0 \cdot t) + \frac{L_f}{c} \right) \right].
\]

As we know from the general treatment, the trajectory dependent term corresponds to the time of reflection of the wave

\[
z_{mm}(t - \Delta t) = \frac{1}{1 - v_0/c} (z_0 + v_0 \cdot t).
\]

In this last equation we see the correction factor \( I/(1 - v_0/c) \), which corresponds to the double Doppler shift also discussed in [15].

3.2. Constant acceleration

Let us now turn to a freely falling mirror in the gravitational field of the Earth. For our analysis we will consider the approximation in which the latter produces an acceleration characterized by a constant term and a gradient term

\[
\frac{d^2 z_{mm}(t)}{dt^2} = g + \Gamma \cdot z_{mm}(t). \tag{11}
\]

Inclusion of the vertical gravity gradient is necessary in order to achieve the relative precision of a few parts in \( 10^9 \) in the measurements of \( g \). Indeed, the measured values of \( \Gamma \) are in the range of \( 3 \mu \text{Gal cm}^{-1} \) (1 \( \mu \text{Gal} = 10^{-9} \text{ m s}^{-2} \)) with an associated relative uncertainty of \( 3\% \) \( (k = 2) \). In the previous formula \( g \) is the acceleration at the origin of the coordinates.

In this theoretical treatment we omit other sources of disturbances, such as self-attraction, as they essentially depend on the properties of the experimental apparatus. Their discussion is postponed until the next section.

After integration equation (11) and expanding in \( \Gamma t^2 \) one finds

\[
z_{mm}(t) = z_0 (1 + \Gamma t^2/2) + v_0 t (1 + \Gamma t^2/6) + g t^2/2 \left( 1 + \Gamma t^2/12 \right). \tag{12}
\]

Even if the previous equation yields a quartic equation for \( \Delta t(t) \), by noticing that \( \Delta t(t) \) is of order \( z_{mm}(t)/c \) one can reduce the calculation of the first ‘speed of light’ perturbation to the solution of a quadratic equation. Neglecting the contributions of \( \mathcal{O}(c^{-3}) \) and \( \mathcal{O}(\Gamma c^{-2}) \), one finds

\[
\Delta t = \frac{- [z_0 + v_0 t + (g + \Gamma z_0) t^2/2 + 1/6 \Gamma v_0 z_0 t^3 + 1/24 \Gamma g t^4]/c}{(v_0^2 + g z_0) t + 3/2 g v_0 t^2 + g^2 t^3/2)/c^2}.
\]

Finally, this generates a time dependent intensity (8) characterized by the function

\[
z_{mm}(t) = z_0 (t - \Delta t) = z_0 + v_0 t + g t^2/2 + (v_0 z_0 + (\frac{g}{2} + g z_0) t + 3/2 g v_0 t^2 + g^2 t^3/2)/c + \Delta z^G(t), \tag{13}
\]

where the piece depending on the gravitational gradient reads

\[
\Delta z^G(t) \equiv \Gamma t^2/2 (z_0 + v_0 t/3 + 1/12 \Gamma g t^2).
\]

In equation (13) we see how the perturbation due to the finite value of \( c \) enters into the final formula. Notice that it does not affect the contribution from the gradient at the order of interest.

For gravimetry, one needs the connection between the time variations of the intensity and the value of the local gravitational acceleration \( g \). This follows simply from taking the second derivative of the variable inside the cosine in (8). At the desired accuracy one gets

\[
\frac{d^2 z_{mm}(t - \Delta t)}{dt^2} = g + 3 \left( \frac{g \cdot v_0 + g^2 \cdot t}{c} \right) + \Delta g^G(t) \tag{14}
\]

\[
= g + g_{\text{bias}}(t) + \Delta g^G(t).
\]

In the previous expression we have denoted by \( \Delta g^G(t) \) the contribution coming from the gradient and introduced the quantity related to the speed of light by

\[
g_{\text{bias}}(t) \equiv 3 \cdot \left( \frac{g \cdot v_0 + g^2 \cdot t}{c} \right). \tag{15}
\]

This coincides with the standard result revised in [12].

Our aim is to verify formula (15) for the speed of light perturbation experimentally. To be open to possible experimental surprises, we will treat the multiplication factor on the right side of (15) as a free parameter in the analysis of the data and replace (15) by the function

\[
g_{\text{bias}}(a_c, t) \equiv a_c \left( \frac{g \cdot v_0 + g^2 \cdot t}{c} \right). \tag{16}
\]

The result derived in [15] corresponds to \( a_c = 2 \). We believe that this is due to an error in the analysis of [15] where it was
not taken into account that the Doppler change in frequency (which occurs at reflection) and interference at the beam splitter happen at different times.

4. Experimental study

The purpose of our study is to determine experimentally the proportionality factor $a_c$ defined in (16). We used the data acquired by different free fall absolute gravimeters at different sites. The working principles of this kind of device have been widely described elsewhere [2]. In these instruments, the trajectory of a corner cube, free falling in the Earth’s gravity field, is measured with a Michelson type interferometer.

To illustrate the measuring principle of a free fall absolute gravimeter through the time dependence of the intensity, equation (8), we show an expected interference pattern in figure 2. From (8) and (13) one deduces that if there were no speed-of-light perturbation, interference fringes would occur each time the mirror moves by a distance $\lambda/2$, hence the shrinking of the spacing between the fringes for accelerated trajectories. Note, however, that the ‘speed of light perturbation’ in (13) modifies this relation between the position of the fringes and the trajectory of the mirror.

By measuring the times at which the intensity (8) goes through the mean value, $t_i$ (in the case illustrated by figure 2 the mean value is shifted to zero), we can associate this to a zero of $I(t)$ and determine the set of points $z_i$ (up to an irrelevant constant). These position–time pairs ($z_i; t_i$) allow the function $z_{\text{mean}}(t_i)$ appearing in (8) to be reconstructed.

4.1. Experimental description

To determine the proportionality factor $a_c$ in (16), we use the same method as Rothleitner in [16]. This method is based on ignoring in (13) the contribution coming from the speed of light perturbation and fitting the data by the formula obtained in the limit $c \to \infty$. This leads to the following model function

$$z_{\text{mean}}(t) = z_{\text{mean}}(t - \Delta t) = (z_0 + v_0 t + g t^2/2) + \Delta z^2(t), \quad (17)$$

to which the data of each drop was fitted using the least squares method.

Since the gravity gradient is known from other experiments (cf. below) the fit produces experimental values for $z_{\text{mean}}$, $v_0$ and $g$. The value of $g$ measured this way, that we call $g_{\text{meas}}$, will produce a value varying as a function of the total drop time $T$ and the initial velocity $v_0$. The study of this variation allows one to determine $a_c$.

The applied procedure is illustrated by figure 3. In our experiment, typically a few 1000 position–time pairs ($z_i; t_i$) were acquired for a single drop. By removing the fringes at the end of the drop (last fringe removal) the obtained values for $g_{\text{meas}}(T, v_0)$ will all have the same initial velocity $v_0$ but their total drop times $T$ will be different. By removing fringes at the beginning of the drop (first fringe removal) the obtained values $g_{\text{meas}}(T, v_0)$ will have different drop times $T$ as well as different initial velocities $v_0$.

The number of drops processed per set (a set is given by the number of drops acquired at a given date at a given station) varies between 2000 and 5000 with a typical drop interval of 10 s. To single out the speed of light perturbation and exclude any influence from time dependent gravity variations, two different approaches are possible. In the first one, the variation of $g_{\text{meas}}(T, v_0)$, as a function of the drop time $T$ and the initial velocity $v_0$, should be estimated for each single drop of the set. Because all geophysical perturbations can be considered as constant during a drop time, no geophysical corrections have to be applied here. The mean variation from all drops of the set can be used to evaluate the proportionality factor.

In the second approach, $g_{\text{meas}}(T, v_0)$ is estimated for each drop time for all drops of the set. Because of the time interval between the drops, these values have to be corrected for all classical geophysical perturbations. The mean values of the
obtained $\Delta g_{\text{mean}}(T_v)$ can then be used to estimate the proportionality factor. Independently of the processing approach, the data has to be corrected for the self-attraction [24] and transferred to the same height. Because both methods described above yield to the same result, we decided to keep the second approach since it is more representative of the conventional processing algorithm used in absolute gravimetry.

To focus on the speed of light perturbation in the measured value of gravitational acceleration, we subtracted the minimum value of the set from $g_{\text{mean}}(T_v)$ to obtain $\Delta g_{\text{mean}}(T_v)$. These values were then least square fitted to

$$\Delta g_{\text{mean}}(T_v) = a_c \cdot \Delta g_{\text{theo}}(g, T_v),$$

(18)

with $\Delta g_{\text{theo}}(g, T_v) = (g \cdot v_0 + 0.5 \cdot g^2 \cdot T \cdot (1 + \eta_2/5))/c$ and where

$$\eta_2 \equiv 5 \cdot \lambda_b \cdot (\lambda_b^2 - 3)/\left(7 \cdot (3 \cdot \lambda_b^2 - 5)\right),$$

with $\lambda_b \equiv 1/(1 + 2 \cdot v_0/g(T))$. The right hand side of equation (18) is obtained by ponderation of equation (16) with a suitable weighting function which depends on the measuring scheme of the graviometer and takes into account that the data acquired by the instrument is equally spaced in distance [11, 12]. Independently of whether the fringes were removed from the beginning or from the end, for each $g_{\text{mean}}$, a pair ($\Delta g_{\text{theo}}; \Delta g_{\text{theo}}$) was estimated. These pairs were then used to estimate the proportionality factor $a_c$.

The impact of the speed of light perturbation on the estimation of $g$ has been summarized in [11, 15]. Considering a total drop time $T = 0.2$ s and an initial velocity $v_0 = 0.2$ m s$^{-1}$, the correction due to the finite speed of light can vary from 11 μGal for $a_c = 3$ to 7 μGal for $a_c = 2$.

### 4.2. Experimental results

In this chapter, the results obtained by analyzing data from three different instruments that have been setup at eight different stations are presented.

**Instruments:**

- Gravimeter FG5X-209 from the Federal Institute of Metrology METAS.
- Gravimeter FG5X-311 from Micro-g Lacoste.
- Gravimeter FG5X-216 from the University of Luxembourg.

**Stations:**

- WANA: Absolute reference station at METAS.
- Zimm: Station of the Swiss geodetic gravity reference network.
- AI, A3, B3, B5, C3, C4: Reference stations at the underground laboratory in Walferdange, Luxembourg.

#### 4.2.1. Uncertainty evaluation

The evaluation of the uncertainty associated with $a_c$ is made in two steps. First the uncertainties related to $\Delta g_{\text{theo}}$ and $\Delta g_{\text{mean}}$ are estimated. Then the uncertainty for $a_c$, obtained by the least square in (18) is calculated.

#### 4.2.1.1. Uncertainty evaluation for $\Delta g_{\text{theo}}$ and $\Delta g_{\text{mean}}$

The uncertainty associated with $\Delta g_{\text{theo}}$ is estimated by applying the law of propagation as described in [29]. The results of this evaluation are summarized in table 1.

The result given in table 1 clearly shows that even with a very conservative evaluation, the uncertainty associated with $\Delta g_{\text{theo}}(g, T_v)$ stays very small.

For the evaluation of the uncertainty associated with $\Delta g_{\text{mean}}$ it is important to recall that in the present experiment, we are interested in changes of acceleration as a function of the total drop time $T$ and $v_0$ and not in the absolute value. This ‘differential’ mode eliminates almost all systematic errors. Nevertheless, there are a number of error sources that are dependent of the total drop time $T$ and $v_0$. To estimate their influence we divided them into two groups. The first group contains the sources that we could not include in our model. These are the laser beam diffraction correction, the frequency dependent phase shift in the electronics, the corner cube rotation, the residual air friction and the residual ground vibrations. The contribution of the second group, formed by the self-attraction and the gradient is discussed in a separate paragraph below.

The impact of the first group, without ground vibrations, has been conservatively evaluated to be of order 0.5 μGal, in agreement with [11, 16, 26]. Regarding the vibrations, in the same manner as in [16], we estimated their influence by a spectral analysis of the residual function, where the residual function corresponds to the difference between the model given by equation (17) and the measurements. Figure 4 shows the average amplitude spectrum from the residuals of more than 2500 drops acquired with instrument 209 on station WANA in the METAS watt balance laboratory.

The spectrum signature shown in figure 4, is very similar to that obtained with the other instruments analyzed in the context of the present study, independently of the site where the measurements were made. This indicates that the oscillations are probably generated by the instruments themselves. The error in $g$ induced by instrumental oscillations has been discussed in [27] and more recently in [28]. The oscillation related error is dependent on the amplitude, the frequency, the phase and the total drop time $T$. To evaluate this error, the average amplitude and phase spectrums were estimated for each set. From these spectrums, the error induced by the four main amplitudes in the frequency range between 0 Hz to 150 Hz were estimated by applying the formulas given in [27, 28]. The oscillation dependent error obtained for the different sets of data acquired on station WANA by instrument 209 are shown in figure 5. The mean error at the lowest drop time is around 0.3 μGal with a standard deviation of 0.5 μGal and tends to zero at longer drop times.

| $x_i$ / unit | $\delta(x_i)$ / unit | $u(x_i)$ / unit |
|-------------|----------------------|-----------------|
| $v_0$       | 0.20 m s$^{-2}$       | 0.001 m s$^{-2}$| 0.003 μGal    |
| $g$         | 9.81 m s$^{-2}$       | 0.001 m s$^{-2}$| 0.001 μGal    |
| $T$         | 0.15 s               | 0.001 s         | 0.032 μGal    |
| $\Delta g_{\text{theo}}$ | | 0.032 μGal |
Another straight-forward and indicative evaluation of the influence of instrumental oscillations is to convert the amplitude of the main harmonics into acceleration. The estimate obtained by this method is of the order of 0.9 μGal. This value is in agreement with the estimate made previously as well as with the estimate given by Rothleitner in [16]. To underpin the different estimations made above, the uncertainty resulting from the sum of all the sources of the first group was determined by evaluating \( g_{\text{meas}} \) as described in 4.1, but including in the model the perturbation due to the finite speed of light (i.e. using the correct formula (13), in which case the theoretical expectation is simply \( g \)). The standard deviation of \( g_{\text{meas}} \) was estimated at 1 μGal (\( k = 1 \)) which is in agreement with our estimations as well as with those given in [16]. The uncertainty contributions to \( \Delta g_{\text{meas}} \) are finally summarized in table 2.

4.2.12. Uncertainty evaluation for \( a_c \)  The method used for the proportionality factor estimate for one set is the Levenberg–Marquardt least square orthogonal distance method in which the uncertainties \( u_{g_{\text{meas}}} \) and \( u_{g_{\text{theo}}} \) associated with \( \Delta g_{\text{meas}} \) and \( \Delta g_{\text{theo}} \) respectively are included. The software IGOR Pro\(^{10} \) was used to perform the fit. The uncertainty, \( u_{a_{\text{c fit}}} \), associated with \( a_c \) given by the fitting procedure has been estimated at 0.3 (\( k = 1 \)).

As mentioned above, the influence of self-attraction and the gradient are also dependent on the total drop time. To estimate the contribution of these two factors to \( a_c \), a Monte-Carlo simulation was performed. For that, a synthetic data set was generated, taking into account the models of self-attraction [24] and the gravity gradient [25]. The first and second order parameters of the gravity gradient were varied with a normal distribution and a standard deviation of 5% from the effective value. The self-attraction function was varied around its nominal value with a normal distribution and a standard deviation of 3%. With 2000 data sets, the uncertainty \( u_{a_{\text{c mont}}} \) in \( a_c \) due to gradient and self-attraction was estimated at 0.3 (\( k = 1 \)). For this estimation we considered the total drop time, which leads to a conservative evaluation. The uncertainty associated with the proportionality factor \( a_c \) estimated for one set is summarized in table 3.

4.2.2. Results

4.2.2.1. Determination of the proportionality factor \( a_c \)  For the experimental determination of the proportionality factor \( a_c \) we used the data sets acquired by the three gravimeters, 209, 311

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Table 2. The uncertainty budget associated with \( \Delta g_{\text{meas}} \), \( x_i \) represents the parameters and \( u(x_i) \) represents the contribution of the parameter \( x_i \) to the uncertainty \( u_{g_{\text{meas}}} \).

| \( x_i \)               | \( u(x_i) \)/μGal |
|------------------------|-------------------|
| Laser beam diffraction | 0.50              |
| Electronic phase shift |                   |
| Corner cube rotation   |                   |
| Residual air friction  |                   |
| Standard deviation of the mean | 0.30 |
| Vibrations             | 1.00              |
| \( u_{g_{\text{meas}}} \) | 1.16              |

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Figure 4. The average amplitude spectrum obtained from more than 2500 drops acquired at station WANA in the watt balance laboratory of METAS with instrument 209.

Figure 5. Error due to the instrument oscillations estimated for the sets acquired on station WANA by the instrument 209. Each dashed curve corresponds to the error estimated for one set. The continuous curve represents the mean value of all individual curves with its associated uncertainty. For a total drop time \( T = 0.13 \) s the mean error is estimated at around 0.3 μGal with a standard deviation of 0.5 μGal. The mean error tends to zero with increasing drop time.

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\(^{10}\) IGOR Pro, WaveMetrics Inc., Version 6.21, ODRPACK95.
and 316 at WANA, Zimm and Walferdange. As an example, figure 6 shows $\Delta g_{\text{meas}}$ as a function of $\Delta g_{\text{theo}}$ for the data sets acquired by instrument 209 at WANA in the watt balance laboratory from METAS.

At WANA, the mean value of the proportionality factors $a_c$ was estimated at $a_c^{\text{WANA}} = 3.4$, with an uncertainty of 0.43 ($k = 1$). The contributions to the uncertainty are summarized in table 4.

In the same manner as described above, the proportionality factor $a_{c, \text{site}}$ has been evaluated for three different instruments at different sites. The results are presented in figure 7 and summarized in table 5 (more details of the measurements are given in annex A).

Table 5 and figure 7 show that the proportionality factors $a_{c, \text{site}}$, obtained with the datasets of three different instruments at different sites are in complete agreement. Based on these five results the mean value of the proportionality factor is estimated at $a_c^{\text{site}} = 3.2 \pm 0.44$ ($k = 1$) where the contributions to the uncertainty are summarized in table 6.

4.2.2.2. Comparison at B3. The evaluation of the proportionality factor $a_c^{B3}$ presented by Rothleitner [16] has been made with a dataset acquired by instrument 216 at station B3 in the underground laboratory in Walferdange. In the context of the present study, the dataset used by Rothleitner has been reprocessed with our procedure, together with three other datasets acquired at B3 by instrument 209 and 216. The obtained results are presented in figure 8 and summarized in table 7.

Figure 8 shows that the first estimate of $a_c^{B3}$ by 216 at B3, denoted by the symbol ▲, present a significant offset to the three other estimates. This disagreement is confirmed by the calculation of the compatibility index $E_a$ defined by

$$E_a = \frac{|x_i - x_{\text{ref}}|}{\sqrt{U^2(x_i) + U^2(x_{\text{ref}})}},$$

where $x_i$ is the $i$th estimation, $x_{\text{ref}}$ the estimated reference value and $U$ the respective uncertainties. An $E_a$ factor larger than 1 indicates that the two values are incompatible, as their difference cannot be covered by their uncertainties. This means that either one of the two values is corrupted or that the declared uncertainties are too small.

In our case, $x_i$ is the estimated $a_c$, and $x_{\text{ref}}$ the weighted mean value of the $a_c$.

The $E_a$ factors in table 7 indicate that the first estimation of $a_c^{B3}$ ($B3/216$, 04.10.12) is incompatible with the reference value. After withdrawing the incompatible value we get a weighted mean value of $a_c^{B3} = 3.1 \pm 0.9$ ($k = 2$). This result is in agreement with the result obtained in the previous paragraph.

5. Conclusion

In this work, we have theoretically and experimentally established the value of the proportionality factor used in speed of light corrections in the position measurements of free falling
be characterized by the time dependent pattern (8), where $z_{mm}(t)$ is given by (13). In (13) one clearly sees the perturbation related to the finiteness of $c$. This coincides with previous results in the literature, see [12] and references therein.

For the experimental confirmation of the $1/c$ perturbation we analyzed 28 datasets, which correspond to more than 50000 drops, from three different instruments on eight different sites. The results of this analysis agree with the theoretical expectations. They also display coherent instrument behavior, independent of the site or their respective configuration setup. We parameterized the effect (and possible deviations) by the parameter $\alpha_c$ in (16). Our theoretical result is $\alpha_c = 3$ while the experimental analysis yields $\tilde{\alpha}_c^{\text{site}} = 3.2 \pm 0.9$ ($k = 2$) from the combined results of sites Zimm, WANA and Walferdange, while from site B3 in Walferdange one obtains $\tilde{\alpha}_c^{\text{B3}} = 3.1 \pm 0.9$ ($k = 2$).

Previous to this work, the perturbation due to the finite speed of light was experimentally assessed by Rothleitner et al [16]. They obtained a proportionality factor of $\alpha_c = 2$, which coincides with their theoretical expectation derived in [15]. We have reprocessed the data used by Rothleitner in [16] with our own software and have showed that the result obtained with this data is not in agreement with the three other results obtained with data acquired at the same site by the same instrument and a second one. Even if we could not identify any clear error in their experimental analysis, we think that our results (with more data and from different stations) hint at some possible anomaly in their dataset. Concerning the derivation in [15] we think that their analysis does not properly take into account the fact that the interference and reflection times do not coincide.

The precise determination of the Earth’s gravitational field is a key element in the definition of the kilogram through a watt balance experiment [31, 32]. Our results confirm the traditional theoretical treatment of one of the most important corrections for achieving the desired relative uncertainty of a few parts in $10^9$, thus paving the way for the feasibility and reliability of the method.

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Annex A. Description of the datasets

A.1. FG5X 209 at station WANA

In the context of the watt balance experiment [22], METAS has built a dedicated laboratory in which five absolute gravity

| Table 5. The proportionality factors $\alpha_c^{\text{site}}$ and their respective uncertainty estimated by three different instruments at different sites. |
|--------------------------------------|--------|--------|
| Station/Instrument                  | $\alpha_c^{\text{site}}$ | $u_{\alpha_c^{\text{site}}}$ |
| Zimm/209                            | 3.3    | 0.43   |
| WANA/209                            | 3.4    | 0.43   |
| Wall/209                            | 3.2    | 0.43   |
| Wall/311                            | 3.1    | 0.43   |
| Wall/216                            | 3.1    | 0.43   |

| Table 6. The uncertainty budget associated with $\alpha_c^{\text{site}}$ estimated at three different sites. $x_i$ represents the parameters and $u(x_i)$ represents the contribution of the parameter $x_i$ to the uncertainty $u_{\alpha_c^{\text{site}}}$. |
|--------------------------------------|--------|--------|
| $x_i$                                | $u(x_i)$ | $\alpha_c$ |
| Reproducibility (standard deviation of the mean of the $\alpha_c^{\text{site}}$) | 0.10    | 0.43    |
| $u_{\alpha_c^{\text{site}}}$         | 0.44    |         |

| Table 7. The proportionality factor $\alpha_c^{B3}$ estimated by instruments 216 and 209 in the underground laboratory in Walferdange on station B3. |
|--------------------------------------|--------|--------|
| Station/Instrument                  | $\alpha_c^{B3}$ | $u_{\alpha_c^{B3}}$ | $E_w$ |
| B3/216 (04.10.12)                   | 1.6    | 0.9    | 1.3   |
| B3/216 (27.05.14)                   | 3.1    | 0.9    | 0.4   |
| B3/209 (06.11.13)                   | 3.1    | 0.9    | 0.4   |
| B3/209 (06.11.13)                   | 3.2    | 0.9    | 0.5   |
| MeanWeighted                         | 2.8    | 0.4    |       |

mirrors by Michelson interferometers. This is particularly relevant for determining the Earth’s gravitational acceleration with a relative uncertainty of a few parts in $10^9$. Given the past controversy in the value and origin of this correction we devoted the first two sections to giving a thorough review of the effect by two independent methods. Our final theoretical result is that the interference pattern in the intensity of the combined beams measured at the detector of figure 1 will
The proportionality factor evaluated at station WANA by instrument 209. The value has been estimated at 3.4 with a standard deviation of the mean of 0.08.

| Date          | Gravi | Station | \(a_c\) |
|---------------|-------|---------|---------|
| 25.02.2013    | 209   | WANA    | 3.1     |
| 27.02.2013    | 209   | WANA    | 3.2     |
| 29.04.2013    | 209   | WANA    | 3.2     |
| 25.06.2013    | 209   | WANA    | 3.3     |
| 02.07.2013    | 209   | WANA    | 2.9     |
| 03.07.2013    | 209   | WANA    | 3.5     |
| 29.07.2013    | 209   | WANA    | 3.4     |
| 04.08.2013    | 209   | WANA    | 3.5     |
| 28.08.2013    | 209   | WANA    | 3.6     |
| 28.10.2013    | 209   | WANA    | 3.1     |
| 04.11.2013    | 209   | WANA    | 3.3     |
| 13.11.2013    | 209   | WANA    | 3.8     |
| 18.11.2013    | 209   | WANA    | 3.8     |
| \(a_c\) mean |       |         | 3.4     |

The proportionality factor evaluated at station Zimm by instrument 209. The value has been estimated at 3.3 with a standard deviation in the mean of 0.11.

| Date          | Gravi | Station | \(a_c\) |
|---------------|-------|---------|---------|
| 12.03.2013    | 209   | Zimm    | 3.1     |
| 13.03.2013    | 209   | Zimm    | 3.5     |
| 14.03.2013    | 209   | Zimm    | 3.4     |
| \(a_c\) mean |       |         | 3.3     |

The proportionality factor evaluated at stations A, B, and C in the underground laboratory in Walferdange with instrument 209. The value has been estimated at 3.2 with a standard deviation in the mean of 0.11.

| Date          | Gravi | Station | \(a_c\) |
|---------------|-------|---------|---------|
| 05.11.2013    | 209   | A3      | 3.7     |
| 06.11.2013    | 209   | B3      | 3.1     |
| 06.11.2013    | 209   | B3      | 3.2     |
| 06.11.2013    | 209   | B3      | 3.3     |
| 07.11.2013    | 209   | C3      | 2.9     |
| 07.11.2013    | 209   | C3      | 3.0     |
| \(a_c\) mean |       |         | 3.2     |

The proportionality factor evaluated at stations A, B, C, and D in the underground laboratory in Walferdange with instrument 209. The value has been estimated at 3.1 with a standard deviation in the mean of 0.16.

| Date          | Gravi | Station | \(a_c\) |
|---------------|-------|---------|---------|
| 07.11.2013    | 311   | A1      | 2.8     |
| 04.11.2013    | 311   | B5      | 3.5     |
| 05.11.2013    | 311   | B5      | 3.0     |
| 06.11.2013    | 311   | C4      | 3.2     |
| \(a_c\) mean |       |         | 3.1     |

A.2. FG5X 209 at station Zimm

The gravity station Zimm is a reference station in the Swiss geodetic gravity reference network. The value of \(g\) has been measured at the station once a year for about 10 years. The standard deviation at this station is less than 2 \(\mu\)Gal. The results obtained are summarized in table A2.

A.3. FG5X 209 in the underground laboratory in Walferdange

The absolute gravity stations of the underground laboratory in Walferdange have been used for more than 10 years for conducting regional and international comparisons [25]. During the last key comparison [30] the value of \(g\) was measured at stations A3, B3 and C3. The results obtained at the different stations in that laboratory are given in tables A3 and A4.

### References

[1] Eichenberger A, Genevès G and Gournay P 2009 Determination of the Planck constant by means of a watt balance *Eur. Phys. J. 172B* 363

[2] Niebauer T M, Sasagawa G S, Faller J E and Klopping F 1995 A new generation of absolute gravimeters *Metrologia* 32 159–80

[3] Hammond J A and Faller J E 1971 Results of absolute gravity determinations at a number of different sites *J. Geophys. Res.* 76 7850–2

[4] Arnautov G P, Gik L D, Kalish Y N and Stus Y F 1972 *Investigation of Systematic Errors of Gravity Acceleration Measurements by the Free-Fall Method Measurements of the Absolute Value of Gravity Acceleration* (Novosibirsk: Institute of Automation and Electrometry) pp 32–57 (Engl. transl.)

[5] Arnautov G P, Gik L D, Kalish E N, Koronkevitch V P, Malyshev I S, Nesterikhin Y E, Stus Y F and Tarasov G G 1974 High-precision laser gravimeter *Appl. Opt.* 13 310–3

[6] Arnautov G P et al 1979 Measurement of the absolute acceleration due to gravity using a laser ballistic gravimeter *Sov. J. Quantum Electron.* 9 333

[7] Zumberge M 1981 A portable apparatus for absolute measurements of the Earth’s gravity PhD Thesis University of Colorado

[8] Arnautov G P, Boulanger Y D, Kalish E N, Koronkevitch V P, Stus Y F and Tarasuk V G 1983 ‘Gabl’, an absolute free-fall laser gravimeter *Metrologia* 19 49–55

[9] Kuroda K and Mio N 1991 Correction to interferometric measurements of absolute gravity arising from the finite speed of light *Metrologia* 28 75–8

[10] Hanada H 1996 Development of absolute gravimeter with a rotating vacuum pipe and study of gravity variation *Publ. Natl Astron. Obs. Japan* 4 75–134

[11] Robertsson L 2005 Absolute gravimetry in a shifted Legendre basis *Metrologia* 42 458–63

[12] Nagornyi V D et al 2011 Correction due to the finite speed of light in absolute gravimeters *Metrologia* 48 101

[13] Nagornyi V D et al 2011 Correction due to the finite speed of light in absolute gravimeters (Corrigendum) *Metrologia* 48 231

[14] Nagornyi V D, Zanimonskij Y M and Zanimonskij Y Y 2012 Can corner-cube absolute gravimeters sense the effects of special relativity? (arXiv:1205.5084v1 [physics.ins-det])

[15] Rothleitner Ch and Francis O 2011 Second-order Doppler-shift correction in free-fall absolute gravimeters *Metrologia* 48 187–95
[16] Rothleitner Ch, Niebauer T M and Francis O 2014 Measurement of the speed-of-light perturbation of free-fall absolute gravimeters Metrologia 51 1.9
[17] Nagornyi V D et al 2011 Relativity, Doppler shifts and retarded times in deriving the correction for the finite speed of light: a comment on ‘Second-order Doppler-shift corrections in free-fall absolute gravimeters’ Metrologia 48 437
[18] Nagornyi V D 2014 Comment on ‘Measurement of the speed-of-light perturbation of free-fall absolute gravimeters’ Metrologia 51 563
[19] Rothleitner Ch and Francis O 2011 Reply to ‘Comment on second-order Doppler-shift corrections in free-fall absolute gravimeters’ Metrologia 48 442
[20] Shao C-G, Tan Yu-J, Li J and Hu Z-K 2015 The speed of light perturbation in absolute gravimeters from the viewpoint of ‘relativistic geometry’ Metrologia 52 324
[21] Stock M 2013 Watt balance experiments for the determination of the Planck constant and the redefinition of the kilogram Metrologia 50 R1–16
[22] Baumann H et al 2013 Design of the new METAS watt balance experiment Mark II Metrologia 50 235
[23] Hariharan P 2003 Optical Interferometry 2nd edn (Amsterdam: Elsevier)
[24] Niebauer T M, Billson R, Schiel A, van Westrum D and Klopping F 2013 The self-attraction correction for the FG5X absolute gravity meter Metrologia 50 1–8
[25] Francis O et al 2012 Final report of the regional key comparison EURAMET.M.G-K1: European Comparison of Absolute Gravimeters ECAG-2011 Metrologia 49 07014
[26] Robertsson L 2007 On the diffraction correction in absolute gravimetry Metrologia 44 35
[27] Murata I 1978 A transportable apparatus for absolute measurement of gravity Bull. Earthq. Res. Inst. 53 49–130
[28] Svitlov S 2012 Frequency-domain analysis of absolute gravimeters Metrologia 49 706–26
[29] JCGM 2008 Guide to the Expression of Uncertainty in Measurement (www.bipm.org)
[30] Francis O et al 2015 CCM.G-K2 key comparison Metrologia 52 07009
[31] Merlet S et al 2008 Micro-gravity investigations for the LNE watt balance project Metrologia 45 265–74
[32] Baumann H, Klingelé E E, Eichenberger A, Richard P and Jeckelmann B 2009 Evaluation of the local value of the earth gravity field in the context of the new definition of the kilogram Metrologia 46 178–86