Onsets and Outflow Distributions in Abelian and Stochastic BTW Models

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We study the onset of bulk avalanches and boundary outflow in the Bak Tang Wiesenfeld (BTW) model and in the stochastic BTW models by computer simulation. We also study the dependency of these two onset times on system sizes. We observe that these two onset times follow simple power-law dependency on system sizes. We estimate these power-law exponents both for the BTW model and for the stochastic BTW models. We observe the evolution of the density of the system and estimate self-organized critical (SOC) state density both for BTW and stochastic BTW models. We study the boundary distribution for BTW and stochastic BTW models and show that boundary distribution does not follow a power-law distribution. In this paper, all the investigations are done for one-dimensional, two-dimensional, and three-dimensional cases.

I. INTRODUCTION

BTW model was first introduced by Bak Tang Wiesenfeld (BTW) (1988) [1] in terms of a two-dimensional cellular automata model. The model well-explained self-organized critical systems and introduced computer simulations to analyze such systems. One can often see self-organized critical systems in nature. The example given by Bak et. al. [1] was sandpiles. Like sandpiles other granular systems, earthquake, landslide also shows self-organise criticality, and there are many more examples. The steady-state dynamics of the model show a power-law distribution in probability for the occurrence of the avalanches or relaxation clusters of a certain size. Extensive work has been done so far to investigate the properties of the model in steady-state (Majumdar, Satya N and Dhar (1991) [2]; Priezzhev, VB and Ktitarev (1996) [3]; L. Pietronero, A. Vespignani (1994) [4]; Lübeck, S and Usadel, KD (1997) [5]; Manna, SS (1991) [6]). The model has been solved exactly using the cumulative properties of the particle addition operators (Majumdar, Satya N and Dhar (1991) [2]). The entropy, height probability, height correlation, all these properties of the critical state has been observed and analytically calculated in (Majumdar, Satya N and Dhar (1991) [2]; Priezzhev, VB and Ktitarev (1996) [3]). However, the critical exponents have not been calculated analytically. This drives an extensive numerical study toward the estimation of various exponents of the system (Manna, SS (1991) [6]; Ben-Hur, Asa and Biham, Ofer (1996) [7]; Priezzhev, VB and Ktitarev, DV (1996) [8]-Lübeck, S and Usadel, KD (1997) [5]). The critical exponents of size and lifetime distribution have been calculated for avalanches starting at the boundaries of the system (Ivashkevich, EV and Ktitarev, DV and Priezzhev, VB (1994) [9]). Using the renormalization scheme, avalanche exponents have been calculated in (L. Pietronero, A. Vespignani (1994) [4]). Some previous attempts are there to calculate the onset of avalanche dependence on system size (Bhowal, Ajanta (1998) [10]) using the abelian BTW model, but according our knowledge no such extensive study of calculating boundary distributions and onsets and their behaviours for 1D, 2D and 3D systems for abelian and stochastic BTW models have done yet.

In this paper, we investigated the onset of bulk (first toppling in the system) and boundary avalanches (escaping the 1st particle through the boundary) for 1-dimensional (1D), 2-dimensional (2D), and 3-dimensional (3D) systems for the BTW model, and see their dependence with system size. We have done the same analysis for the stochastic BTW model. For 1, 2, and 3 dimensions the stochastic model have been defined and analyzed. All these studies are basically in the subcritical region of the time evolution of the BTW model. We also have studied the evolution of system weight, in terms of the average number of points in the system on a time slice. We have studied the outflow weight on each time slice and investigated the outflow distribution for all 1.2- and 3-dimension systems with abelian and stochastic BTW model.

II. MODEL AND SIMULATION

The model is based on cellular automata. The actual model described in BTW paper [1] was a two-dimensional (2D) cellular automation, where random points were dropped on a two-dimensional lattice. If the number of points on any lattice site reaches a threshold value (for 2D systems it is 4, which is the coordination number in 2D) then that point avalanches and distributes 4 points among its nearest neighbors. So, in 2D, the model is $a(i,j) = a(i,j) + 1$ (randomly), If $a(i,j) \geq 4$ then that site relaxes as

- $a(i \pm 1, j) = a(i \pm 1, j) + 1$
- $a(i, j \pm 1) = a(i, j \pm 1) + 1$
- $a(i, j) = a(i, j) - 4$

In 2D stochastic BTW model two mutually orthogonal directions are chosen randomly. When a lattice point avalanche, it avalanche by distributing four points equally to the neighbours sitting on those randomly chosen directions. So, $a(i,j) = a(i,j) + 1$ (randomly), If $a(i,j) \geq 4$ then a random number generates which ran-
domly choose a set of two coordinates- positive $x$ and positive $y$ ($x+$, $y+$) or negative $x$ and negative $y$ ($x-$, $y-$).

The lattice site then relaxes 4 points by distributing them in any of the randomly chosen set of coordinates. As any randomly chosen set of coordinates has two coordinates, each will receive 2 points. So,

either $a(i+1, j) = a(i+1, j) + 2$, $a(i, j+1) = a(i, j+1) + 2$

or, $a(i-1, j) = a(i-1, j) + 2$, $a(i, j-1) = a(i, j-1) + 2$

will happen, and for both the cases, $a(i, j) = a(i, j) - 4$

The 1D extension of the model is done by taking care of the coordination number (number of nearest neighbour) in that space.

The lattice site then relaxes 4 points by distributing them randomly (or, $a(i, j) = a(i, j) - 4$)

In this case, there are two possibilities- (1) randomly choosing a set of three orthogonal directions $x+$, $y+$, $z+$ or $x-$, $y-$, $z-$ then distributing 6 points uniformly in any of the chosen set of direction (as there are 3 members in a set, each will get 2

The 1D and 3 dimension (3D) extension of the model is equal to 2 (the coordination number in 3D) then that point

relaxes (avalanches) as

$$a(i \pm 1, j) = a(i \pm 1, j) + 1,$$

$$a(i, j \pm 1) = a(i, j \pm 1) + 1,$$

$$a(i, j, k \pm 1) = a(i, j, k \pm 1) + 1$$

and $a(i, j, k) = a(i, j, k) - 6$.

This is basically equal distribution of points among the nearest neighbours. In case of 3D stochastic model there are two possibilities- (1) randomly choosing a set of three orthogonal directions $x+$, $y+$, $z+$ or $x-$, $y-$, $z-$ then distributing 6 points uniformly in any of the chosen set of direction (as there are 3 members in a set, each will get 2
DENSITY OF POINTS VS TIME FOR 2D SYSTEM, \( L = 100 \)

**FIG. 5.** Density of lattice point versus time plot for 2D abelian BTW model (labeled as normal model) (purple, saturation density is 2.10 ± 0.01) and for 2D stochastic BTW model (green, saturation density is 1.93 ± 0.01)

DENSITY OF POINTS VS TIME FOR 3D SYSTEM, \( L = 30 \)

**FIG. 6.** Density of lattice point versus time plot for 3D abelian BTW model (labeled as normal BTW model) (purple, saturation density is 3.10 ± 0.01), for 3D stochastic BTW model-1 (green, saturation density is 2.93 ± 0.01) and for 3D stochastic BTW model-2 (light blue, saturation density is 2.76 ± 0.01)

DENSITY OF POINTS VS TIME FOR 1D SYSTEM, \( L = 10 \)

**FIG. 7.** Boundary distribution (frequency of outflow size versus outflow size) plot for 1D abelian BTW system for system size \( L = 10 \)

DENSITY OF POINTS VS TIME FOR 2D SYSTEM, \( L = 10 \)

**FIG. 8.** Boundary distribution (frequency of outflow size versus outflow size) plot for 2D abelian (green, labeled as normal model) and stochastic BTW (purple) system for system size \( L = 10 \)

points), this named as- 3D stochastic BTW model-1.

(2) randomly choosing a set of two mutually orthogonal directions \( x^+, y^-, y^+, z^- \) or \( z^+, x^- \) then distribute 6 points among any of the chosen set of directions (as there are two member in a set, each will get 3 points) this we named as- 3D stochastic BTW model-2.

The model gives a power law distribution of the frequency of bulk avalanches. That is shown in the bak et. al. [1] for 2D and 3D abelian BTW model. We confirmed that for stochastic models the power law distribution of bulk avalanches sustained.

The onset of bulk avalanches is the time slice in which the first toppling happens inside the bulk of the system. The onset of boundary avalanches is the time slice in which first point flow out (escapes) from the system through the boundary.

For a fixed system size and in the same system environment, every time the onset time will be different, as it is probabilistic to happen a toppling anywhere in the system. So, we considered an ensemble of systems and took the ensemble average of onset time. For both onset of bulk avalanche and onset of boundary outflow this ensemble averaging is done. All the onset time specified in this paper are average onset time.

The system weight at any time slice is the total number of points in the system. The average system weight is the average number of point (height) at any lattice site. The boundary weight at a time slice is the number of points escape through the boundary at that time slice.

For any dimension, the system size is defined by its length (L), as we always consider equal length for each dimensions of a given system (square for 2D, cube for 3D).
III. NUMERICAL RESULTS AND ANALYSIS

In FIG. 1 we have plotted the total number of points on the lattice and total number of boundary outflow (points escaped from the lattice) with respect to time. This plot is done for 2D system with $L = 50$ (length of the side of the square lattice) but the behavior is exactly same for 1D and 3D systems for any size ($L$). FIG. 1 shows that, in small time regime the total number of points on the lattice increases with time, this is obvious as we are randomly adding points to the system. So, the total number of points in the system primarily increases. At this small time regime the boundary outflow is zero, as any disturbance in the bulk would take some time to reach the boundary. After some time the number of points on the lattice get saturated (SOC reached). In this regime the boundary outflow get linear with time. So, at this region any externally added point will ultimately escape through the boundary (in most of the cases, but there exists some time slices in this region, for which no boundary outflow happens).

The density of points on a lattice can be defined as the total number of lattice points divided by the size of the lattice (length $L$ for 1D system, Area $L \times L$ for 2D system, Volume $L \times L \times L$ for 3D system). For a fixed dimension and for a fixed model the density of lattice points saturates at a definite value and does not depend on the lattice size. This saturation density is the self-organized critical (SOC) density. In FIG. 2 it is shown that for 1D the SOC density is $0.99 \pm 0.01$ and it does not depend on system size, For 2D the SOC density is $2.10 \pm 0.01$ and for 3D the SOC density is $3.10 \pm 0.01$ FIG. 3. If any abelian BTW system poses these densities then that system can be thought of as a self-organized critical (SOC) system. All this SOC densities are for abelian BTW model. However for stochastic BTW model the SOC density is different for a fixed dimension and fixed sized system.

The saturation density (SOC density) of the lattice...
FIG. 13. Onset of boundary outflow versus system size plot for 1D system. For abelian BTW model (purple) the exponent $b = 0.983 \pm 0.001$ and for stochastic BTW model (green), the exponent $b = 0.483 \pm 0.001$

FIG. 14. Onset of boundary outflow versus system size plot for 2D system. For abelian BTW model (purple) and for stochastic BTW model the exponent is same, which is $b = 1.76 \pm 0.01$

is different for stochastic BTW model. The density of lattice for abelian BTW model and for stochastic BTW model are shown in the FIG. 4 (for 1D system), FIG. 5 (for 2D) and FIG. 6 (for 3D). For 1D Stochastic BTW model the saturation value is $0.50 \pm 0.01$. For 2D stochastic BTW model it is $1.93 \pm 0.01$. As for 3D system we have two stochastic BTW models, for stochastic BTW model-1 the saturation density is $2.93 \pm 0.01$ and for stochastic BTW model-2 the saturation density is $2.76 \pm 0.01$. All these densities implies SOC density for the corresponding systems.

FIG. 7 shows the frequency of boundary outflow versus boundary outflow size plot for 2D abelian and stochastic BTW models. FIG. 9 shows frequency of boundary outflow versus boundary outflow size plot for 3D abelian and stochastic BTW models. FIG. 7, FIG. 8 and FIG. 9 shows that unlike bulk avalanches frequency versus bulk avalanches size (shown in Bak et. al. [1]), boundary avalanche frequency versus boundary avalanche size does not follow any well behaved power law distribution for both abelian and stochastic BTW model.

FIG. 10, FIG. 11 and FIG. 12 are showing average internal (bulk) onset time versus system size plot for 1D, 2D and 3D systems respectively. Plots are showing power law dependency of bulk onset time on system sizes. So, bulk onset time is proportional to $L^b$, where $L$ is the system size and $b$ is the exponent. Exponents $b$ are positive for all three dimension systems. Stochastic BTW model will give same onset time as abelian BTW model, as onset time does not depend on the toppling rule, it only depends on the first toppling in the system. All the exponents are tabulated in TABLE 1. Error bars are showing statistical errors (standard deviation on both side of the data point) in bulk onset times. As system size is a defined quantity, it has no error.

FIG. 13, FIG. 14 and FIG. 15 are showing onset of boundary outflow versus system size plot for 1D, 2D and 3D systems respectively. Plots are showing power law dependency of onset of boundary outflow on system size. So, onset of boundary outflow is proportional to $L^b$, where $L$ is the system size and $b$ is the exponent. Exponents $b$ are positive for all three dimension systems for abelian and stochastic BTW models. For 1D system the exponents are different for abelian BTW model and stochastic BTW model (FIG. 13). For 2D and 3D systems abelian BTW model and stochastic BTW model only has two directions, and in a single time slice only one point can escape from a boundary. FIG. 8 shows frequency of boundary outflow versus boundary outflow size plot for 2D abelian and stochastic BTW models. FIG. 9 shows frequency of boundary outflow versus boundary outflow size plot for 3D abelian and stochastic BTW models. FIG. 7, FIG. 8 and FIG. 9 shows that unlike bulk avalanches frequency versus bulk avalanches size (shown in Bak et. al. [1]), boundary avalanche frequency versus boundary avalanche size does not follow any well behaved power law distribution for both abelian and stochastic BTW model.
has same exponents. All the exponents are tabulated in TABLE I. Error bars are showing statistical errors (standard deviation on both side of the data point) in onset of boundary avalanche times. Here also, as system size is a defined quantity, it has no error.

### IV. SUMMARY

In this paper we studied how critical density (saturation density) depends on the model. For a fixed model the critical density of the system does not depend on the system size, we have shown this for 1D (FIG. 2) and 3D (FIG. 3) systems. However the critical density depends on the model. Critical densities for abelian BTW model and for stochastic BTW model is calculated for 1D (FIG. 4), 2D (FIG. 5) and 3D (FIG. 6) systems. If a system poses these densities then that system will behave as SOC system.

We have plotted the boundary outflow distribution (frequency of outflow versus outflow size) for both abelian BTW model and stochastic BTW model for 1D (FIG. 7), 2D (FIG. 8) and 3D (FIG. 9) systems. We have shown that unlike bulk toppling distribution (bulk toppling frequency versus bulk toppling size) boundary outflow does not follow any power law distribution for both abelian BTW and stochastic BTW models.

The onset of bulk avalanches follows power law distribution with system size ($L^b$). The exponent $b$ is positive for all the cases. The onset of bulk avalanches does not depend on the chosen model, as any rule of toppling specified by the model will work after the first toppling, and onset is the first toppling. For 1D (FIG. 10), 2D (FIG. 11) and 3D (FIG. 12) systems these power law exponents has been calculated and tabulated in TABLE I.

The onset of boundary outflow also follows power law distribution with system size ($L^b$). Here also all the exponents ($b$) are positive. These exponents does depend on the model for 1D only (FIG. 13). However for 2D and 3D systems these exponents does not depend on the model. We get same exponent for abelian BTW model and for stochastic BTW model for 2D (FIG. 14) and 3D (FIG. 15) systems.

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