Planar coincidences for $N$-fold symmetry

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Abstract

The coincidence problem for planar patterns with $N$-fold symmetry is considered. For the $N$-fold symmetric module with $N < 46$, all isometries of the plane are classified that result in coincidences of finite index. This is done by reformulating the problem in terms of algebraic number fields and using prime factorization. The more complicated case $N \geq 46$ is briefly discussed and $N = 46$ is described explicitly.

The results of the coincidence problem also solve the problem of colour lattices in two dimensions and its natural generalization to colour modules.

1 Introduction

The concept of coincidence site lattices (CSLs) arises in the crystallography of grain and twin boundaries [1]. Different domains of a crystal do have a relationship: There is a sublattice common to both domains across a boundary, and this is the CSL. This can be seen as the intersection of a perfect lattice with a rotated copy of it where the set of points common to both forms a sublattice of finite index, the CSL. Up to now, CSLs have been investigated only for special cases, for example for cubic or hexagonal crystals [2]. With the advent of quasicrystals infinitely many new cases arise: quasicrystals also have grain boundaries, and one should know the coincidence site quasilattices [3, 4]. In a rather different context, multiple coincidences of families of 1D quasicrystals have been applied in constructing quasicrystals with arbitrary symmetry (in higher dimensions) [5, 6]. An application of these results was made by Rivier and Lawrence [7] to crystalline grain boundaries, which themselves turn out to be quasicrystalline. This is an important example of the relevance of a coincidence quasilattice. The experimental evidence was provided indirectly by Sass, Tan and Balluffi in the 1970s [8], but beautifully by the observations of growth of
quasicrystalline grain at the grain boundary between two crystals by Cassada, Shiflet and Poon [9] and by Sidhom and Portier [10]. Gratias and Thalal [11], on the other hand, used quasicrystal concepts in a different context to embed the two crystal grains adjacent to a grain boundary in a higher dimensional perfect lattice. So an extension of the CSL analysis to more general discrete structures is desirable.

In this paper we give a unified treatment of the coincidence problem for planar structures with general \( N \)-fold rotation symmetry, extending previous [3, 13] and parallel [12] work and putting it in a more general setting. This is what is needed for quasicrystalline \( T \)-phases which are quasiperiodic in a plane and periodically stacked in the third dimension. Icosahedral symmetry in 3D requires different methods and will be described separately [14]. Common to both is the necessity of an attack in two stages: not only do we have to find the coincidence isometries (the universal part of the problem), but also the specific modifications of the atomic surfaces (also called windows or acceptance domains) that are needed to describe the set of coinciding points.

In order to describe this scenario, we start with the coincidence problem of the square lattice \( \mathbb{Z}^2 \). The set of coincidence transformations for \( \mathbb{Z}^2 \) forms a group, the generators of which can be given explicitly through their connection with Gaussian integers. Simultaneously, the so-called \( \Sigma \)-factor or coincidence index can be calculated for an arbitrary CSL isometry. Though this is not new, the approach we use here can be generalized to quasiperiodic planar patterns with \( N \)-fold symmetry. The description of this more general case and the tools necessary to tackle it is the main aim of this article.

In two dimensions the classification of CSLs is the same as the classification of colour lattices with rotational symmetry ([15], Section 5.8). In that setting the \( \Sigma \)-factor or coincidence index is the number of colours and the different coloured sublattices are the different cosets of the CSL in the original lattice. This is because, as long as the symmetry group consists of 2D rotations, all members of the symmetry group commute with the CSL rotation, thus ensuring that the CSL is invariant under the symmetry group. For indecomposable groups in higher dimensions no non-trivial orthogonal transformation commutes with all symmetries so there is no longer this equivalence. The only non-trivial rotation groups of 2D lattices are \( C_3 \), \( C_4 \) and \( C_6 \). The prime numbers \( p \) for which there exist \( p \)-colour lattices with these symmetries are listed in [13], p. 76, and coincide with the sets of primes in the denominators of the Dirichlet series given at the end of Section 4 for the cases \( n = 3 \) and \( n = 4 \). (Note that \( N = 2n \) for \( n \) odd and \( N = n \) otherwise as will be explained later.) For non-lattices the solution of the CSL problem in 2D can be regarded as a classification of colour modules in the plane. An \( r \)-colour \( n \)-module is a pair of \( n \)-modules (\( \mathcal{M}, \mathcal{M}_1 \)) such that \( \mathcal{M}_1 \) has index \( r \) in \( \mathcal{M} \) and is invariant under the symmetry group of \( \mathcal{M} \), while no other coset of \( \mathcal{M}_1 \) has this property (see Section 3 for a definition of \( n \)-modules). The colour of a point in \( \mathcal{M} \) is then determined by its coset mod \( \mathcal{M}_1 \). In this light, the results of Sections 3 and 6 can be interpreted as finding, for each \( n \), the numbers \( r \) for which there are \( r \)-colour \( n \)-modules and what these \( n \)-modules
are.

The paper is organized as follows. In Section 2, we review the coincidence problem for the square lattice and formulate it in terms of Gaussian integers. This enables us to describe the group structure and the coincidence indices explicitly and to introduce the concepts needed for the generalization in Section 3. There, the main structure is derived with the aid of the algebraic number theory of cyclotomic fields, followed by various explicitly worked out cases in Section 4. They include 8-, 10- and 12-fold symmetry, the most important cases for quasicrystalline $T$-phases, and thus cover all cases linked to quadratic irrationalities \[16\]. In Section 5 we then show, in an illustrative way, how to use the method for the eightfold symmetric Ammann–Beenker rhombus pattern and the tenfold symmetric Tübingen triangle tiling. We give an explicit formula for the necessary correction of the coincidence index. In Section 6 we discuss certain details to be dealt with for $N \geq 46$, where the variety of modules rapidly increases, though this does not affect the generality of our findings. The case $N = 46$ ($n = 23$) is presented in some detail. This is followed by some concluding remarks, while the two appendices cover further examples (Appendix A) and proofs of technical results used in Sections 3 and 4 (Appendix B).

## 2 The square lattice: a warm-up exercise

Let us consider the CSL problem for the square lattice $\mathbb{Z}^2$. We focus on pure rotations first and deal with the easy extension to reflections later. Consider therefore a rotation (i.e., an element of the group $SO(2) = SO(2, \mathbb{R})$) and ask for the condition that it maps some lattice point to another one. Clearly, rotations through multiples of $\pi/2$ do this. They form the cyclic group $C_4$ — an index 2 subgroup of $D_4$, the point group of $\mathbb{Z}^2$.

But there are more cases, as can already be seen from the growing number of lattice points on expanding circles, summarized in the coefficients of the theta-function of the lattice, cf. \[17\],

\[
\Theta_{\mathbb{Z}^2}(x) = \sum_{q \in \mathbb{Z}^2} x^{||q||^2} = (\vartheta_3(x))^2
\]

\[
= 1 + \sum_{M=1}^{\infty} r(M)x^M
\]

\[
= 1 + 4x + 4x^2 + 4x^4 + 8x^5 + \ldots
\]

Here, $\vartheta_3(x) = \sum_{q \in \mathbb{Z}^2} x^{||q||^2} = 1 + 2x + 2x^4 + 2x^9 + \ldots$ is Jacobi’s theta-function and $r(M)$ denotes the number of integral solutions of the equation $a^2 + b^2 = M$, see \[18\] for details on $r(M)$. This number is only slowly increasing but is unbounded, so there is an infinite number of rotations that map one lattice point to another.

As is obvious (cf. \[19\] and references therein), the set of coincidence rotations (or CSL rotations) consists of all rotations $R$ through angles $\varphi$ with $\sin(\varphi) = a/m$.
and \( \cos(\varphi) = \frac{b}{m} \) rational, and hence is identical with the group \( SO(2, \mathbb{Q}) \). This requires integral solutions of the Diophantine equation

\[
a^2 + b^2 = m^2, \tag{2}
\]

where we need consider only the primitive solutions, i.e., \( \gcd(a, b) = 1 \). They are, of course, given by the primitive Pythagorean triples \([18]\). For a primitive solution, the set of coinciding points forms a sublattice of \( \mathbb{Z}^2 \) of index \( m \), whence \( 1/m \) is the fraction of lattice points coinciding. We call \( m \) the coincidence index of \( R \), denoted by \( \Sigma_{\mathbb{Z}^2}(R) \), or \( \Sigma(R) \) for short. This index is often called the \( \Sigma \)-factor \([2, 3, 4]\).

The number of CSL rotations with given index

Without determining the rotations explicitly we can calculate their possible indices and the number of different rotations with each index as follows.

The number of primitive solutions of Eq. (2) can be derived from the well-known formula (cf. \([18]\))

\[
r(M) = 4(d_1(M) - d_3(M)), \tag{3}
\]

(where \( d_k(M) \) counts the number of divisors of \( M \) of the form \( 4\ell + k \)) for the total number of integer solutions of

\[
a^2 + b^2 = M. \tag{4}
\]

If we write \( M = 2^z M_1 M_3 \), where \( M_1 \) and \( M_3 \) are maximal divisors of \( M \) composed of primes congruent to 1 or 3 \((\text{mod } 4)\), respectively, then Eq. (4) can be equivalently expressed as

\[
r(M) = \begin{cases} 4d(M_1), & \text{if } M_3 \text{ is a square}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}
\]

where \( d(M_1) \) counts all the divisors of \( M_1 \). When (as in our case) \( M \) is a square, the first alternative in Eq. (5) occurs. The number of primitive solutions, \( r^*(m^2) \), of Eq. (2) can now be derived from the “input-output” principle (cf. \([18]\), Thm 260) as

\[
r^*(m^2) = r(m^2) - \sum_p r\left(\left(\frac{m}{p}\right)^2\right) + \sum_{p,p'} r\left(\left(\frac{m}{pp'}\right)^2\right) - \sum_{p,p',p''} r\left(\left(\frac{m}{pp'p''}\right)^2\right) + \cdots, \tag{6}
\]

where \( p \) runs through all prime factors of \( m \), \( pp' \) through all pairs of distinct prime factors of \( m \), and so on. After substituting Eq. (5) in the right hand side of Eq. (6) and then counting the contributions of the factors of \( m \) one at a time, it can be seen that

\[
r^*(m^2) = \begin{cases} 4d^*(m), & \text{if } m \text{ has prime factors } \equiv 1 \ (4) \text{ only}, \\ 0, & \text{otherwise}, \end{cases} \tag{7}
\]

where \( d^*(m) \) counts the squarefree divisors of \( m \). We note that the number of CSL’s (as distinct from CSL rotations) of index \( m \) in the square lattice is a quarter this
number, since each is itself a square lattice stabilized by the rotation group of the square (of order 4). (Note however that not every sublattice with square symmetry is a CSL.)

So far we have:

**Theorem 1** The coincidence indices of the square lattice are precisely the numbers \( m \) with prime factors \( \equiv 1 \pmod{4} \) only. The number of coincidence rotations \( \hat{f}(m) \) with a given index \( m \) is

\[
\hat{f}(m) = 4d^*(m)
\]

and the number of CSL’s with index \( m \) is

\[
f(m) = d^*(m).
\]

**CSL rotations and Gaussian integers**

We have settled the question of what numbers occur as coincidence indices of CSL rotations of \( \mathbb{Z}^2 \) and how many rotations there are with each index, but there is still more to be said.

We have seen that the set of CSL rotations forms a group \((SO(2, \mathbb{Q}), \text{in fact})\). Let us introduce the notation

\[
SOC(\mathbb{Z}^2) := \{ R \in SO(2) \mid \Sigma(R) < \infty \}
\]

for it. We shall investigate its structure and derive independent generators.

The most transparent proof of Eq. (3) (that given in [18]) depends on factorization in the ring of Gaussian integers. By making direct use of this idea we not only find independent generators for \( SOC(\mathbb{Z}^2) \) but also have a method that readily generalizes to other lattices and modules.

To this end, we consider the lattice \( \mathbb{Z}^2 \) as the ring \( \mathbb{Z}[i] \) of Gaussian integers, i.e., with \( i = \sqrt{-1} \),

\[
\mathbb{Z}[i] = \{ a + ib \mid a, b \in \mathbb{Z} \}
\]

(11)

together with the (number theoretic) norm

\[
\text{norm}(a + ib) = (a + ib)(a - ib) = |a + ib|^2.
\]

(12)

The ring \( \mathbb{Z}[i] \) consists of all algebraic integers in \( \mathbb{Q}(i) = \{ a + ib \mid a, b \in \mathbb{Q} \} \), which is both a quadratic and a cyclotomic field. The coincidence rotation problem is then equivalent to finding all numbers of norm 1 in \( \mathbb{Q}(i) \) because rotation through an angle means multiplication with the corresponding complex number on the unit circle and a coincidence can only happen if this complex number is in \( \mathbb{Q}(i) \).

Any such number can uniquely be written (up to units) as the quotient of two Gaussian integers,

\[
e^{i\varphi} = \frac{a + ib}{c + id}
\]

(13)
with coprime Gaussian integers \( \alpha, \beta \) of identical norm, \( \text{norm}(a+ib) = \text{norm}(c+id) = \ell \), say. Now, we can profit from unique factorization in \( \mathbb{Z}[i] \) because any integer \( \alpha \in \mathbb{Z}[i] \) divides its norm:
\[
\alpha \mid \text{norm}(\alpha) \, .
\]
(14)

Let \( \ell = p_1^{\nu_1} \cdots p_r^{\nu_r} \) be the (unique) factorization of \( \ell \) of \( (13) \) into “ordinary” primes of \( \mathbb{Z} \), called rational primes from now on. If any rational prime \( p_j \) stayed prime in \( \mathbb{Z}[i] \) (i.e., did not split into two Gaussian integers), which happens if \( p_j \equiv 3 \pmod{4} \), it would appear both in the numerator and the denominator of \( (13) \) which is inconsistent with coprimality. Thus such a rational prime cannot divide \( \ell \).

A similar argument applies to the prime 2 which, although it splits as \( 2 = i(1-i)^2 \), also has only one Gaussian prime factor up to units. The remaining primes are \( \equiv 1 \pmod{4} \) and split as \( p = \omega_p \overline{\omega}_p \) into two Gaussian integers. One of them appears in the numerator of \( (13) \), the other in the denominator, if \( p \mid \ell \). Of course, the actual choice of \( \omega_p \) is only unique up to units and up to taking the complex conjugate which reflects the point symmetry of the square lattice! One convenient choice for uniqueness (which we will now take) is a rotation angle in the interval \((0, \pi/4)\).

This, in fact, solves the above problem constructively: any CSL rotation can be written in the form
\[
\exp(i\varphi) = \varepsilon \cdot \prod_{p \equiv 1 \pmod{4}} \left( \frac{\omega_p}{\overline{\omega}_p} \right)^{n_p}
\]
(15)
where \( n_p \in \mathbb{Z} \), \( \varepsilon \) is a unit in \( \mathbb{Z}[i] \) and \( P \) denotes the set of rational primes. Since the group of units in \( \mathbb{Z}[i] \) is nothing but \( C_4 \), we find
\[
\text{SOC}(\mathbb{Z}^2) \simeq C_4 \times \mathbb{Z}^{(\aleph_0)}
\]
(16)
and the generators are \( i \) (for \( C_4 \)) and \( \omega_p/\overline{\omega}_p \) for rational primes \( p \equiv 1 \pmod{4} \). By \( \mathbb{Z}^{(\aleph_0)} \) we mean, as usual, the infinite Abelian group that consists of all finite integer linear combinations of the (countably many) generators. The coincidence index \( m \) is obviously 1 for the units in \( C_4 \) and \( p = \text{norm}(\omega_p) \) for the other generators because this counts the number of residue classes of the CSL in \( \mathbb{Z}^2 \). If the CSL rotation \( R \) is factorized as in Eq. \( (15) \), we thus find
\[
\Sigma(R) = \prod_{p \equiv 1 \pmod{4}} p^{|n_p|} \, .
\]
(17)

This solves the rotation part in principle, one can now work along the primes \( p \equiv 1 \pmod{4} \) to write down the generators explicitly, e.g.,
\[
\frac{4 + 3i}{5}, \frac{12 + 5i}{13}, \frac{15 + 8i}{17}, \frac{21 + 20i}{29}, \frac{35 + 12i}{37}, \frac{40 + 9i}{41}, \text{etc} \, ,
\]
where the number on the unit circle is shown in a form with denominator \( p \) and rotation angle in \((0, \pi/4)\). All other CSL rotations are obtained by combinations, and one can regain the formula of Theorem 1 for the number of them with index \( m \). Since \( d^*(m) \) is a multiplicative function (i.e., \( d^*(m_1 m_2) = d^*(m_1) d^*(m_2) \)) for coprime
\[ m_1, m_2 \) and \( d^r(p^r) = 2 \) for a prime power \( p^r \) \((r \geq 1)\), we obtain for \( f(m) = d^r(m) \) the Dirichlet series generating function \[ \Phi(s) = \sum_{m=1}^{\infty} \frac{d^r(m)}{m^s} = \prod_{p \equiv 1 (4)} \left( 1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots \right) = \prod_{p \equiv 1 (4)} \frac{1 + p^{-s}}{1 - p^{-s}} \] (18)

and the Dirichlet series generating function for \( \hat{f}(m) \) is \( 4\Phi(s) \).

Finally, the full group of CSL isometries, \( OC(\mathbb{Z}^2) \), is the semidirect product of the rotation part \( SOC(\mathbb{Z}^2) \) (normal subgroup) with the group \( \mathbb{Z}_2 \) generated by complex conjugation (= reflection in the \( x \)-axis):

\[ OC(\mathbb{Z}^2) = SOC(\mathbb{Z}^2) \times_s \mathbb{Z}_2 . \] (19)

Here conjugation of a rotation through an angle \( \varphi \) by complex conjugation results in the inverse rotation through \( -\varphi \). Let us give a brief justification of Eq. (19). Since \( O(2) = SO(2) \times_s \mathbb{Z}_2 \) (semidirect product) with the \( \mathbb{Z}_2 \) of Eq. (19), any planar isometry \( T \) with \( \det(T) = -1 \) can uniquely be written as the product

\[ T = R(\varphi) \cdot T_x \] (20)

of a rotation through \( \varphi \) with \( T_x \), the reflection in the \( x \)-axis. But \( T_x \) leaves \( \mathbb{Z}^2 \) invariant, so \( T \) is a coincidence isometry if and only if \( R(\varphi) \) is a coincidence rotation.

The calculation of coincidence indices is also simple in this case. The coincidence index for the reflection \( T_x \) is 1. For an arbitrary element of \( OC(\mathbb{Z}^2) \), we either meet a rotation (where we know the result already) or use the factorization (20) again. Then, the coincidence index is identical with that of its rotation part, so Eq. (20) is all that is needed. This solves the coincidence problem for the square lattice completely and we have

**Theorem 2** The group of coincidence isometries of the square lattice \( \mathbb{Z}^2 \) is

\[ OC(\mathbb{Z}^2) \simeq O(2, \mathbb{Q}) \simeq (C_4 \times \mathbb{Z}(\mathbb{R}_0)) \times_s \mathbb{Z}_2 . \] (21)

This group is fully characterized by Eqs. (17), (18) and (19), and the coincidence index of an element (20) is given by Eqs. (17) and (18).

### 3 More generality: the unique factorization case

As briefly explained in the introduction, the corresponding programme for a locally finite tiling \( \mathcal{T} \) with \( N \)-fold symmetry (or rather for its set of vertex sites) consists of two steps, the first being the solution of the coincidence problem for the limit translation module \( \mathcal{M}(\mathcal{T}) \) of \( \mathcal{T} \) (see [20] for details about this concept). For the
moment, we consider only tilings with the property that the set of vertex sites of $T$ is a subset of $\mathcal{M}(T)$, a condition we shall come back to in Section 5. Furthermore, we assume $\mathcal{M}(T)$ to be what is termed an “$N$-lattice” in [21] but which we shall call an “$n$-module” (where $N = 2n$ for $n$ odd and $N = n$ otherwise) in line with the mathematical practice of reserving the word “lattice” for discrete subgroups. The principal $n$-module (the “standard $N$-lattice” of [21]) is the additive subgroup of $\mathbb{R}^2$, generated by the vectors of the regular $n$-star,

$$(\cos(2\pi k/n), \sin(2\pi k/n)), \ k = 0, \ldots, n - 1.$$ \hfill (22)

The other modules are the non-trivial subgroups of the principal module that are invariant under rotation about the origin through $2\pi/n$. Modules that differ only in scale and orientation are regarded as equivalent.

Because all modules are invariant under rotation through $\pi$ (since if $x$ is in the module then so is $-x$), an $n$-module with $n$ odd is invariant not only under rotation through $2\pi/n$ but also through $\pi/n$. So $n$-modules and $N$-modules are the same. In view of this we shall assume throughout that $n$ is either odd or divisible by 4, though this necessitates bearing in mind that for odd $n$ an $n$-module has $2n$-fold symmetry. The opposite convention is used in [21], but the one used here is more convenient for expressing results about cyclotomic fields that we shall need later because it gives $n$ the parity of the discriminant of the corresponding field.

The first stage of our analysis, occupying all but Section 5, is to investigate coincidence rotations for modules and their associated coincidence site modules, which we designate CSMs.

**Symmetric modules and cyclotomic fields**

Viewed as complex numbers, the vectors (22) are $\xi^k$, where $\xi$ is a primitive $n$th root of 1, and the modules are subsets of the cyclotomic field $K = \mathbb{Q}(\xi)$. The principal module is precisely the ring of integers $\mathcal{O}_K$ of $K$, since it is known that \(\{1, \xi, \xi^2, \ldots, \xi^{\phi(n)-1}\}\) is a basis for the integers of $K$, where $\phi(n)$, the Euler totient function of $n$, is the degree of $K$ over $\mathbb{Q}$, cf. Chapter 9 of [23]. The other modules are the ideals of $\mathcal{O}_K$ (to be defined later), modules being equivalent precisely when they belong to the same ideal class (defined in Section 6).

In this section, at the expense of discussing only 29 modules (see [25], [21]), we restrict attention to values of $n$ for which all $n$-modules are equivalent. Because of the connection with algebraic number theory we call this the “class number 1” case and use the designation “CN1” to indicate results that are special to this case. (The reason behind this terminology is explained in Section 6. Briefly, it is the case when the $n$th cyclotomic field has class number 1.) The class number 1 assumption simplifies the treatment in two ways:

1) it is enough to solve the coincidence problem for the principal module $\mathcal{O}_K$ only, since all others are equivalent to it; and
2) in the class number 1 case each integer in $\mathcal{O}_K$ has a factorization into irreducible integers that is unique apart from multiplying the factors by units. (Because of the unique factorization these irreducible integers can safely be called primes in the class number 1 case.)

Though a convenience, the restriction to class number 1 is by no means essential: with only minor modifications our method applies to any 2D module, as outlined in Section 6.

As in the previous section, a coincidence rotation that takes $\beta$ to $\alpha$, say, $(\alpha, \beta \in \mathcal{O}_K)$ can be represented by the point $\gamma = \alpha/\beta$ on the unit circle. So the CSM problem amounts to finding the structure of the set of numbers $\gamma$ in $K$ with

$$|\gamma| = 1$$

(a subgroup of the multiplicative group of $K$).

The CSM associated with $\gamma$ is $\mathcal{O}_K \cap \gamma\mathcal{O}_K = \text{num}(\gamma)\mathcal{O}_K$, where $\text{num}(\gamma)$, the numerator of $\gamma$, is given by

$$\text{num}(\gamma) = \gcd(\nu \in \mathcal{O}_K \mid \nu/\gamma \in \mathcal{O}_K),$$

and is unique up to multiplication by a unit. In particular, $\text{num}(\gamma) \mid \alpha$. The index of this module in the original module $\mathcal{O}_K$ is $\text{norm}(\text{num}(\gamma))$, the absolute norm of $\text{num}(\gamma)$, (22 4.4 and Cor. 2.96). (Since units have norm 1 this is independent of the particular numerator chosen. All conjugates of the field $K$ are complex, so norms of numbers in $K$ are products of pairs of complex conjugates and hence positive.)

Eq. (23) can be reformulated as an algebraic condition with the aid of the maximal real subfield $L$ of $K$:

$$L := \mathbb{Q}(\xi + \xi^{-1}) = \mathbb{Q}(\cos \frac{2\pi}{n}).$$

It is known that when $K$ has unique factorization $L$ does too (see p. 231 of [27]). As an extension of $L$, $K$ has degree 2 and the set of conjugates over $L$ of a number $\gamma \in K$ is just the complex conjugate pair $\{\gamma, 1/\gamma\}$. Consequently, the relative norm of $\gamma$ over $L$, $\text{norm}_{K/L}(\gamma)$, is given by

$$\text{norm}_{K/L}(\gamma) = |\gamma|^2.$$  

In this notation, the absolute norm of $\gamma$ is $\text{norm}(\gamma) = \text{norm}_{K/\mathbb{Q}}(\gamma)$ and we have the relation

$$\text{norm}_{K/\mathbb{Q}}(\gamma) = \text{norm}_{L/\mathbb{Q}}(\text{norm}_{K/L}(\gamma)).$$

Relative norms of integers in $K$ are integers in $L$ and norms of units are units. As in the previous section (where $L = \mathbb{Q}$), $\alpha \mid \text{norm}_{K/L}(\alpha) = \alpha\overline{\alpha}$ for every integer $\alpha$ of $K$, so the only possible prime factors of $\alpha$ in $K$ are those that divide $\text{norm}(\alpha)$.
Cyclotomic numbers on the unit circle

When a planar module $\mathcal{M}$ intersects a rotated or reflected copy of itself in a submodule of finite index, the isometry (rotation or reflection) is again called a coincidence isometry. The set of coincidence isometries of $\mathcal{M}$ is denoted by $OC(\mathcal{M})$. It is again a group, with $SOC(\mathcal{M})$ being its subgroup of rotations. (These concepts can be put in a much more general setting. Some slight extensions of them are already required for the examples in Appendix A, for example.)

In view of Eq. (26) and our representation of $SOC(\mathcal{O}_K)$ as the elements of $K$ on the unit circle, we have

$$SOC(\mathcal{O}_K) \simeq \{ \gamma \in K \mid \text{norm}_{K/L}(\gamma) = 1 \}.$$  \hfill (28)

To analyze the right hand side further we need some facts about the arithmetic of $K$ and $L$. First, the units $\varepsilon$ of $K$ with $|\varepsilon| = 1$ are precisely the powers of $\xi$, though in general there are also infinitely many units not on the unit circle. (This follows, e.g., from [28], Lemma 1.6, and the last sentence of the remark following it.) Second, if a prime $\varrho$ of $L$ has two non-associated prime factors in $K$ (i.e., their ratio is not a unit) then they can be taken as complex conjugates, $\omega$ and $\overline{\omega}$. This is because $\omega | \varrho$ implies $\overline{\omega} | \varrho$, so, if $\omega$ and $\overline{\omega}$ are not associates, $\omega \overline{\omega}$ is an integer in $\mathcal{O}_L$ dividing $\varrho$, hence is an associate of $\varrho$. (Here $\mathcal{O}_L$ is the ring of integers of $L$, of course.) Conversely, if $\varrho$ is divisible by just the prime $\omega$ in $K$ and no other (up to units), then, as $\overline{\omega}$ also divides $\varrho$, $\omega/\overline{\omega}$ must be a unit. Thus a prime $\omega \in \mathcal{O}_K$ divides a prime $\varrho \in \mathcal{O}_L$ with distinct factors if and only if $\overline{\omega}$ is not an associate of $\omega$. By Eq. (26), $\text{norm}_{K/L}(\omega) = \text{norm}_{K/L}(\overline{\omega})$.

Now suppose that $\gamma \in K$ satisfies $\text{norm}_{K/L}(\gamma) = 1$ and write $\gamma = \alpha/\beta$, where $\alpha$, $\beta$ are integers of $\mathcal{O}_K$ with no common factor. Then

$$\text{norm}_{K/L}(\alpha) = \text{norm}_{K/L}(\beta) = \nu \in \mathcal{O}_L$$ \hfill (29)

and every prime factor of $\nu$ must factorize into two non-associated primes of $K$, one of which divides $\alpha$ only and the other $\beta$ only. Since any such pair can be chosen to be complex conjugates, $\gamma$ can be written as

$$\gamma = \varepsilon \prod_k \left( \frac{\omega_k}{\overline{\omega}_k} \right)^{n_k},$$ \hfill (30)

with $\varepsilon$ a unit of $K$ and the $n_k$’s in $\mathbb{Z}$. Taking absolute values in (30) shows that $|\varepsilon| = 1$, whence $\varepsilon$ is a root of unity. Different values of the $n_k$’s give $\gamma$’s with different prime factorizations, which are therefore not associates, and different roots of unity $\varepsilon$ give different $\gamma$’s within each set of associates. So in this more general situation we again have explicit presentations of $SOC(\mathcal{O}_K)$ and $OC(\mathcal{O}_K)$ almost identical to those for $SOC(\mathbb{Z}^2)$ and $OC(\mathbb{Z}^2)$ in the previous section. These are, for $SOC$,

$$SOC(\mathcal{O}_K) \simeq \langle \xi \rangle \times_{(\omega,\overline{\omega}) \in \Omega} \left( \frac{\omega}{\overline{\omega}} \right)^{n_0} \simeq C_N \times \mathbb{Z}^{(n_0)},$$ \hfill (31)
where Ω is the set of complex conjugate pairs of non-associated primes in K (and N = lcm(n, 2) as usual) and, for OC,

\[ OC(O_K) = SOC(O_K) \times_s \langle \overline{\cdot} \rangle, \]  

where \( \overline{\cdot} \) is complex conjugation and its action on \( SOC(O_K) \) is clear.

The coincidence index of the typical rotation (30) of \( SOC(O_K) \) is the absolute norm of its numerator:

\[ \prod_k (\text{norm}_{K/Q}(\omega_k))^{|n_k|}. \]  

(33)

For each prime pair \( \{\omega, \overline{\omega}\} \in \Omega \), the common value norm(\( \omega \)) = norm(\( \overline{\omega} \)) is a rational prime power \( p^d \). We call these prime powers the basic indices of \( O_K \) and the primes \( p \) themselves the complex splitting primes for \( K \) (because, in their factorization over \( K \), they contain at least one complex conjugate pair of distinct primes). Then (31), (32) and (33) show that:

**Proposition 1** (CN1) An integer \( m \in \mathbb{N} \) is a coincidence index if and only if it is a product of basic indices.

To find out what basic indices there are and count how many members of the group \( SOC(O_K) \) have given index we need to determine how each rational prime \( p \) factorizes in the fields \( L \) and \( K \). It will turn out that the basic indices are powers of distinct primes and that whether a power of \( p \) is a basic index (and what this power is) depends only on the residue class of \( p \mod n \).

**Factorization of primes in algebraic number fields**

Before considering \( K \) and \( L \) specifically we describe how primes factorize in a general algebraic number field extension \( F(\alpha) \supset F \) of degree \( D \). (We shall use the standard notation \( F(\alpha)/F \) to denote such an extension. A detailed account of the material in this section can be found in Chapter 2 of [23].)

Let \( f(x) = 0 \) be the minimal equation satisfied by \( \alpha \) with coefficients in \( F \) (so the degree of \( f(x) \) is \( D \)) and let \( O \) and \( O' \) be the rings of integers of \( F \) and \( F(\alpha) \).

An *ideal* of \( O \) is a subset \( a \) of \( O \) (non-empty and \( \neq \{0\} \)) such that \( \alpha + \beta \in a, \forall \alpha, \beta \in a, \) and \( \lambda \alpha \in a, \forall \lambda \in O, \alpha \in a. \) We use the notation \( (\alpha, \beta, \ldots)_O \) to denote the smallest ideal containing \( \alpha, \beta, \ldots \) (where these are numbers in \( O \)). Ideals have a natural multiplication, defined by \( ab = (\alpha\beta | \alpha \in a, \beta \in b)_O \) and \( O \) itself is the multiplicative identity. There is an infinite set of *prime ideals* in \( O \) and every ideal can be uniquely factorized into prime ideals.

Every ideal \( a \) in \( O \) extends to an ideal \( (a)_O' \) in \( O' \), but \( O' \) also has other ideals not of this form. For an ideal \( a' \) of \( O' \) the *relative norm*, norm\(_{F(\alpha)/F}(a')\), is defined as \( \text{norm}_{F(\alpha)/F}(\alpha) | \alpha \in a' \) — an ideal of \( O \). Norms are completely multiplicative (i.e., \( \text{norm}(ab) = \text{norm}(a)\text{norm}(b) \)). Let \( p \) be a prime ideal in \( O \). Then \( (p)_O' \) factorizes into prime ideals in \( O' \) as

\[ (p)_{O'} = p_1^{e_1} \cdots p_g^{e_g}, \]  

(34)
and for each \( k = 1, \ldots, g \)

\[
norm_{F(\alpha)/F}(p_k) = p^{d_k},
\]

(35)

where \( d_k \) is the residue class degree of \( p_k \). Taking norms of both sides of Eq. (34) shows that

\[
d_1e_1 + \cdots + d_ge_g = D,
\]

(36)

For the special case of normal field extensions (i.e., extensions where \( F(\alpha) \) contains not only \( \alpha \) itself but also all other roots of \( f(x) = 0 \)) we have

\[
e_1 = \cdots = e_g \quad \text{and} \quad d_1 = \cdots = d_g.
\]

So in this case

\[
(p)_{O'} = (p_1 \cdots p_g)^e,
\]

(37)

where each \( p_k \) has the same degree \( d \) and without ambiguity we can define \( \deg_{F(\alpha)}(p) := d \) and \( e_{F(\alpha)}(p) := e \). Also \( e_{F(\alpha)}(p) > 1 \) only for the finitely many primes \( p \) that divide the discriminant of the extension \( F(\alpha)/F \). (Such primes are called ramified.)

Another special case is field extensions where \( \alpha \) can be chosen so that \( O' = O[\alpha] \) (\( O' \) has a simple integral basis over \( O \)). In this case the factorization of a prime \( p \) of \( O \) into prime ideals of \( O' \) mimics the factorization of \( f(x) \) into irreducible factors over the finite residue class field \( O/p \). So if \( p \) factorizes as in Eq. (34) then

\[
f(x) \equiv f_1(x)^{e_1} \cdots f_g(x)^{e_g} \pmod{p}
\]

(38)

where each \( f_k \) is irreducible of degree \( d_k \) and distinct \( f_k \)'s correspond to distinct primes \( p_k \). This provides a simple way of calculating the degrees and multiplicities of the prime factors of \( p \).

The three extensions we have to deal with — \( K/Q \), \( L/Q \) and \( K/L \) — are all normal and have simple integral bases, so all the above results apply to them. Also relative degrees are multiplicative: if \( p \) is a rational prime having a prime factor \( \varrho \) in \( O_L \) which in turn has a prime factor \( \omega \) in \( O_K \) then

\[
\deg_{K/Q}(\omega) = \deg_{K/L}(\omega) \cdot \deg_{L/Q}(\varrho).
\]

(39)

In particular, \( \deg_{K/L}(\omega) \) is the same for all prime factors \( \omega \) in \( O_K \) of the same rational prime \( p \).

The primes of \( O_K \) in the non-associated pairs \( \{\omega, \overline{\omega}\} \) are precisely the unramified primes of relative degree 1 over \( O_L \). In view of (33) and the normality of \( K \) and \( L \), \( \Omega \) is the set of all pairs of distinct prime factors \( \{\omega, \overline{\omega}\} \) in \( O_K \) that divide rational primes \( p \) with

\[
\deg_{L}(p) = \deg_{K}(p) \quad (= d, \ \text{say})
\]

(40)

and, for any such \( \omega \), the absolute norm (cf. (27) above) is

\[
norm(\omega) = p^d.
\]

(41)

We have:

**Proposition 2** (CN1) The complex splitting primes for \( K \) are the rational primes that satisfy (40) and the basic indices of \( O_K \) are the powers \( p^d \) of these primes.
How to calculate the CSL group and its coincidence indices

Getting more explicit information about $SOC(O_{\mathcal{K}})$ and its coincidence indices comes down to finding $\deg_L(p)$ and $\deg_K(p)$ for rational primes $p$. The following facts are sufficient to do this; we state them here and justify them in Appendix B.

To reiterate our notation: $K = \mathbb{Q}(\xi)$ and $L = \mathbb{Q}(\xi + \xi^{-1})$, where $\xi$ is a primitive $n$th root of 1, and $p$ is any rational prime.

**Fact 1** If $p \nmid n$ then $\deg_K(p)$ is the smallest $d \in \mathbb{N}$ such that $n$ divides $p^d - 1$.

**Fact 2** If $p \nmid n$ then $\deg_L(p)$ is the smallest $d \in \mathbb{N}$ such that $n$ divides at least one of $p^d + 1$ or $p^d - 1$.

**Fact 3** (a) If $n = p^r$, for some $r$, then $p$ is not a complex splitting prime and $\deg_K(p) = 1$.

(b) More generally, if $n = p^r n_1$ with $p \nmid n_1$ then $p$ is a complex splitting prime in $K$ if and only if it is a complex splitting prime in $K_1$ (the cyclotomic field of $n_1$th roots of unity). Moreover, $\deg_K(p) = \deg_{K_1}(p)$.

Although these facts alone clearly enable us to identify the complex splitting primes and calculate their degrees and multiplicities, it is nevertheless worth listing some general consequences of them.

**Remark 1** These facts show that whether a prime $p$ is a complex splitting prime of $n$ and what its degree $d$ is depend only on the residue class of $p \mod n$.

**Remark 2** Since, for a prime $\omega$ in $O_{\mathcal{K}}$ dividing $p$, $\text{norm}(\omega) = p^d$, where $d = \deg_K(p)$, Fact 1 has the well-known consequence that $\text{norm}(\omega) \equiv 1 \mod n$ for every prime $\omega$ in $O_{\mathcal{K}}$ with $\omega \nmid n$. In particular, every coincidence index $m$ with $\gcd(m, n) = 1$ satisfies $m \equiv 1 \mod n$.

**Remark 3** When $p \nmid n$ is not a complex splitting prime, $\deg_K(p) = 2\deg_L(p)$, so $\deg_K(p)$ is even. Facts 1, 2, and 3(a) show that, conversely, if $n$ is an odd prime power then no prime $p$ with $d = \deg_K(p)$ even is a complex splitting prime. This is because if $n \mid p^d - 1 = (p^{d/2} - 1)(p^{d/2} + 1)$ but $n \nmid p^{d/2} - 1$ then, since $\gcd(p^{d/2} - 1, p^{d/2} + 1) = 2$ (or 1 if $p = 2$), $n \mid p^{d/2} + 1$, so $\deg_L(p) = d/2$.

So, for an odd prime power, $p$ is a complex splitting prime if and only if $\deg_K(p)$ is odd and $p \nmid n$, and it is unnecessary to compute degrees over $L$ in this case.

**Remark 4** By Fact 1 the unramified primes with $\deg_K(p) = 1$ (i.e., the primes that *split completely* in $K$) are precisely those $\equiv 1 \mod n$. So these primes are always complex splitting primes.

**Remark 5** Facts 1 and 2 show that, for primes $p \equiv -1 \mod n$, $\deg_K(p) = 2$ and $\deg_L(p) = 1$, so these primes are never complex splitting primes. Consequently, for every $n$, the proportion of integers that are coincidence indices is 0.

In the next section we apply these facts and remarks to calculate coincidence indices of specific modules.
The number of coincidences with given index

Let \( \hat{f}(m) = N \cdot f(m) \) be the number of elements of \( SOC(\mathcal{O}_K) \) with index \( m \). The computational convenience of representing \( \hat{f}(m) \) this way arises from the fact that \( f(m) \) is more fundamental: it is a multiplicative function of \( m \) and, as for the square lattice, it counts the CSMs with index \( m \), since the rotation group of each module is the group of roots of unity in \( K \) and has order \( N \). In the general case, \( f(m) \) cannot be described as simply as in Eq. (9), but its Dirichlet series generating function does have a very simple expression in terms of the \( \zeta \)-functions of the fields \( K \) and \( L \). Also, for quite sizeable individual values of the index, the number of coincidence isometries can be calculated from (33) and knowledge of the identity, degrees and exponents of the complex splitting primes (or, equivalently, from the generating function).

From the decomposition (31) of \( SOC(\mathcal{O}_K) \) and the function (33) it can be seen that \( f(m) \) is multiplicative, i.e., \( \gcd(m_1, m_2) = 1 \) implies \( f(m_1 m_2) = f(m_1) f(m_2) \). This makes its Dirichlet series

\[
\sum_{m=1}^{\infty} \frac{f(m)}{m^s}
\]  

(42)

a convenient tool for studying \( f \): it can be expressed as an “Euler product”

\[
\prod_p \left( \sum_{r=1}^{\infty} \frac{f(p^r)}{p^{rs}} \right),
\]  

(43)

with one Euler factor for each prime \( p \), and the individual Euler factors are straightforward to compute. (The series we obtain will all be absolutely convergent in the right half-plane \( \Re(s) > 1 \) and extendable to meromorphic functions on the whole plane. For using the series formally to calculate individual values of \( f \) these analytic properties are irrelevant, but they play an essential rôle in calculating the asymptotic average value of \( f \).)

Suppose the rational prime \( p \) is divisible by the pairs \( \{\omega_1, \overline{\omega}_1\}, \ldots, \{\omega_{g/2}, \overline{\omega}_{g/2}\} \) of non-associated primes in \( K \) and that each \( \omega_j \) has \( \text{norm}_{K/Q}(\omega_j) = p^d \). Then \( f(p^k) \) is the coefficient of \( p^{-ks} \) in

\[
\left( \cdots + \frac{1}{p^{2ds}} + \frac{1}{p^{ds}} + 1 + \frac{1}{p^{ds}} + \frac{1}{p^{2ds}} + \cdots \right)^{g/2},
\]  

(44)

the product of \( g/2 \) two-way infinite sums (one for each pair \( \{\omega_j, \overline{\omega}_j\} \) each having one term for each value of the corresponding \( n_k \) in (33). (The symmetry of the sums arises from the fact that the index depends only on \( |n_k| \), of course.) On summing the series this becomes

\[
\left( \frac{1 + p^{-ds}}{1 - p^{-ds}} \right)^{g/2}.
\]  

(45)

Since \( f(m) \) is multiplicative, for a general \( m \) it is the coefficient of \( m^{-s} \) in

\[
\prod_{C' \not\parallel m} \left( \frac{1 + p^{-ds}}{1 - p^{-ds}} \right)^{g/2},
\]  

(46)
where \( \mathcal{C} \) is the set of complex splitting primes for \( K \), and the values of \( d \) and \( g \) are those appropriate to each individual prime \( p \). This, in turn, is the coefficient of \( m^{-s} \) in the infinite product

\[
\Phi_K(s) = \prod_{p \in \mathcal{C}} \left( \frac{1 + p^{-ds}}{1 - p^{-ds}} \right)^{g/2}.
\]  

(47)

To express this more simply we introduce the Dedekind \( \zeta \)-functions of number fields \([22, 28]\). The \( \zeta \)-function of a general algebraic number field \( F \) is the Dirichlet series generating function for the number of ideals \( a \) of \( \mathcal{O} \) with \( \text{norm}(a) = m \), hence is given by

\[
\zeta_F(s) = \sum_a \frac{1}{\text{norm}(a)^s} = \prod_p \left( 1 - \frac{1}{\text{norm}(p)^s} \right)^{-1},
\]

(48)

where \( a \) runs through all ideals of \( F \) and \( p \) through all prime ideals. When \( F \) is normal we can collect together prime ideals \( p \) dividing the same rational prime \( p \) to put the product on the right in the form

\[
\prod_p \left( 1 - \frac{1}{p^{ds}} \right)^{-g},
\]

(49)

where, for each rational prime \( p \), \( d = \text{deg}_{F/Q}(p) \) and \( g \) is the number of prime ideals of \( F \) dividing it.

A particular case of this is

\[
\zeta_Q(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1},
\]

(50)

which is the Riemann \( \zeta \)-function \( \zeta(s) \) itself.

The following table compares the Euler factors of \( \zeta_K(s) \) and \( \zeta_L(2s) \) for each rational prime \( p \), there being three cases to consider. (It follows from Lemma 3 of \([24]\) and Prop. 2.15(b) of \([28]\) that the third case, \( e_K(p) \neq e_L(p) \), occurs for at most one prime \( p \): the prime, if any, a power of which is equal to \( n \).)

| \( p \) | Field | Degree | Distinct prime factors | Euler factor |
|--------|-------|--------|------------------------|--------------|
| Complex splitting | \( K \) | \( d \) | \( g \) | \( (1 - p^{-ds})^{-g} \) |
| | \( L \) | \( d \) | \( g/2 \) | \( (1 - p^{-2ds})^{-g/2} \) |
| Not complex splitting and \( e_K(p) = e_L(p) \) | \( K \) | \( d \) | \( g \) | \( (1 - p^{-ds})^{-g} \) |
| | \( L \) | \( d/2 \) | \( g \) | \( (1 - p^{-ds})^{-g} \) |
| \( e_K(p) \neq e_L(p) \) | \( K \) | 1 | 1 | \( (1 - p^{-s})^{-1} \) |
| | \( L \) | 1 | 1 | \( (1 - p^{-2s})^{-1} \) |
On taking the quotients of the Euler factors arising from $K$ and $L$ and comparing with Eq. (47), we see that

$$\frac{\zeta_K(s)}{\zeta_L(2s)} = \Phi_K(s) \left(1 + \frac{1}{p^s}\right)^*,$$

the star indicating that the second factor on the right is present only if $n$ is a power of a prime $p$. So

$$CN1 \Rightarrow \Phi_K(s) = \begin{cases} (1 + p^{-s})^{-1}\zeta_K(s)/\zeta_L(2s), & \text{if } n \text{ is a power of a prime } p, \\ \zeta_K(s)/\zeta_L(2s), & \text{if not.} \end{cases} \quad (52)$$

We summarize this in

**Theorem 3** Let $n$ be one of the 29 numbers for which the cyclotomic field $K$ of $n$th roots of unity has class number 1. Then the group of coincidence rotations of an $n$-fold symmetric module is the direct product of its finite rotation symmetry group $C_N$ and countably many infinite cyclic groups, as in (31), and the full group of coincidence isometries is the extension of this by a reflection symmetry. The coincidence index of such an isometry is given by (33) and (30). The Dirichlet series generating function for

$$f(m) = \begin{cases} \text{number of CSMs of index } m, \\ \frac{1}{N} \times \text{number of coincidence rotations of index } m \end{cases} \quad (53)$$

is given by (52).

A principal use of a Dirichlet series is to find asymptotic formulæ for sum functions of its coefficients by means of residue calculus. In the present instance this technique shows, for example, that

$$\text{Number of CSMs of index } < X = \sum_{m<X} f(m) \sim X \cdot \{\text{residue of } \Phi_K(s) \text{ at } s = 1\}. \quad (55)$$

In view of Eq. (52), this residue can be computed from known formulæ for the residues of $\zeta$-functions at 1 and values of $\zeta$-functions at 2. The value of the residue can be regarded as the “average number of CSMs” with a given arbitrarily chosen positive integer as index.

### 4 Examples: $N = 6, 4, 10, 14, 8, \text{ and } 12$

After the general derivation of the previous section, let us present some examples explicitly. We select those relevant to known crystals and quasicrystals. For each example we list:
(a) The fields $K$ and $L$ and the degree $[K : \mathbb{Q}]$ of $K$ over $\mathbb{Q}$.

(b) A table giving, for each residue class mod $n$ containing primes $p$, $\deg_K(p)$ (and, if necessary, $\deg_L(p)$ too). In the bottom line of the table (where $\deg_K(p)$ is given) the degrees of complex splitting primes are underlined. With each table is a comment describing which facts and remarks from the previous section were used to compute it.

(c) A list of the types of basic indices, using the notation that $p_b^{(a)}$ represents the $b$th powers of all primes congruent to $a$ mod $n$.

(d) The Dirichlet series generating function of $f(m)$, given as a ratio of $\zeta$-functions, as an Euler product and expanded explicitly as far as the 12th nonzero term. (The same notation as in (c) is used for the primes in the Euler product.)

(e) An explicit formula for $f(m)$ in the style of Eq. (9). In these formulæ $e_p$ denotes the largest exponent $e$ for which $p^e | m$.

(f) The average value of $f(m)$, as defined above.

The smallest coincidence indices can be read off as the denominators (with $s = 1$) of the Dirichlet series, with the corresponding values of $f(m)$ as the numerators. All values of $f(m)$ for $m > 1$ are even, reflecting the geometrical fact that the reverse of a coincidence rotation is also a coincidence rotation.

$n = 3$, the triangular (or hexagonal) lattice

$K = \mathbb{Q}(\sqrt{-3})$, $L = \mathbb{Q}$, $[K : \mathbb{Q}] = 2$.

| $p \pmod{3}$ | 1 | 2 | 3 |
|--------------|---|---|---|
| $\deg_K(p)$  | 1 | 2 | 1 |

Computed using Facts 1 and 3(a) and Remark 3.

Basic indices: $p_{(1)}$

Dirichlet series:

$$\left(1 + \frac{1}{3^s}\right)^{-1} \frac{\zeta_K(s)}{\zeta(2s)} = \prod \frac{1 + p_{(1)}^{-s}}{1 - p_{(1)}^{-s}}$$

$$= 1 + \frac{2}{7^s} + \frac{2}{13^s} + \frac{2}{19^s} + \frac{2}{31^s} + \frac{2}{37^s} + \frac{2}{43^s} + \frac{2}{61^s} + \frac{2}{67^s} + \frac{2}{73^s} + \frac{2}{79^s} + \cdots$$

Number of CSLs with index $m$:

$$f(m) = \begin{cases} 2, & \text{if } m \text{ is a product of basic indices}, \\ 0, & \text{otherwise}. \end{cases}$$

Average number of CSLs:

$$\frac{\sqrt{3}}{2\pi} \simeq 0.276$$
\( n = 4 \), the square lattice
\( K = \mathbb{Q}(i), L = \mathbb{Q}, [K : \mathbb{Q}] = 2. \)

\[
\begin{array}{c|ccc}
 p \pmod{4} & 1 & 2 & 3 \\
\hline
\deg_L(p) & 1 & 1 & 1 \\
\deg_K(p) & 1 & 1 & 2 \\
\end{array}
\]

Computed from Facts 1, 2 and 3(a).

Basic indices: \( p_{(1)} \)

Dirichlet series:
\[
\left( 1 + \frac{1}{2^s} \right)^{-1} \frac{\zeta_K(s)}{\zeta(2s)} = \prod \frac{1 + p_{(1)}^{-s}}{1 - p_{(1)}^{-s}} \\
= 1 + \frac{2}{3^s} + \frac{2}{7^s} + \frac{2}{13^s} + \frac{2}{17^s} + \frac{2}{23^s} + \frac{2}{41^s} + \frac{2}{61^s} + \frac{2}{65^s} + \frac{2}{73^s} + \cdots
\]

Number of CSLs with index \( m \):
\[
f(m) = \begin{cases} 
2, & \text{if } m \text{ is a product of basic indices,} \\
0, & \text{otherwise.}
\end{cases}
\]

Average number of CSLs:
\[
\frac{1}{\pi} \simeq 0.318
\]

\( n = 5 \), the 10-fold module
\( K = \mathbb{Q}(e^{2\pi i/5}), L = \mathbb{Q}(\sqrt{5}), [K : \mathbb{Q}] = 4. \)

\[
\begin{array}{c|ccccc}
p \pmod{5} & 1 & 2 & 3 & 4 & 5 \\
\hline
\deg_K(p) & 1 & 4 & 4 & 2 & 1 \\
\end{array}
\]

Computed using Facts 1 and 3(a) and Remark 3.

Basic indices: \( p_{(1)} \)

Dirichlet series:
\[
\left( 1 + \frac{1}{5^s} \right)^{-1} \frac{\zeta_K(s)}{\zeta(2s)} = \prod \left( \frac{1 + p_{(1)}^{-s}}{1 - p_{(1)}^{-s}} \right)^2 \\
= 1 + \frac{4}{11^s} + \frac{4}{13^s} + \frac{4}{17^s} + \frac{4}{19^s} + \frac{4}{31^s} + \frac{8}{43^s} + \frac{4}{47^s} + \frac{4}{53^s} + \frac{4}{59^s} + \frac{4}{61^s} + \cdots
\]

Number of CSMs with index \( m \):
\[
f(m) = \begin{cases} 
4c_p, & \text{if } m \text{ is a product of basic indices,} \\
0, & \text{otherwise.}
\end{cases}
\]

Average number of CSMs:
\[
\frac{5 \log \tau}{\pi^2} \simeq 0.244
\]
\( n = 7 \), the 14-fold module

\[ K = \mathbb{Q}(e^{2\pi i/7}), \quad L = \mathbb{Q}(\cos(2\pi/7)), \quad [K : \mathbb{Q}] = 6. \]

\[
\begin{array}{c|cccccccc}
 p \mod 7 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 \text{deg}_K(p) & 1 & 3 & 6 & 3 & 6 & 2 & 1 \\
\end{array}
\]

Computed using Facts 1 and 3(a) and Remark 3.

Basic indices: \( p_{(1)}, p_{(2)}^3, p_{(4)}^3 \)

Dirichlet series:
\[
\left( 1 + \frac{1}{7^s} \right)^{-1} \frac{\zeta_K(s)}{\zeta_L(2s)} = \prod \left( \frac{1 + p_{(1)}^{-s}}{1 - p_{(1)}^{-s}} \right)^3 \frac{(1 + p_{(2)}^{-3s})(1 + p_{(4)}^{-3s})}{(1 - p_{(2)}^{-3s})(1 - p_{(4)}^{-3s})}
\]
\[
= 1 + \frac{2}{7^s} + \frac{4}{11^s} + \frac{2}{25^s} + \frac{4}{41^s} + \frac{2}{73^s} + \frac{4}{89^s} + \frac{4}{97^s} + \frac{2}{121^s} + \frac{4}{137^s} + \cdots
\]

Number of CSMs with index \( m \):
\[
f(m) = \begin{cases} 
\prod_{\substack{p \mid m \\
p \equiv 1 \pmod{7}}} (4e_p^2 + 2) \prod_{\substack{p \mid m \\
p \not\equiv 1 \pmod{7}}} 2, & \text{if } m \text{ is a product of basic indices,} \\
0, & \text{otherwise.}
\end{cases}
\]

Average number of CSMs:
\[
\frac{21\sqrt{7}R}{16\pi^3} \approx 0.235,
\]

where \( R \) (the regulator of \( K \)) is given by
\[
\frac{R}{4} = \log^2(2\cos \frac{2\pi}{7}) - \log(2\cos \frac{\pi}{7}) \log(2\cos \frac{3\pi}{7}) \approx 0.525.
\]

\( n = 8 \), the 8-fold module

\[ K = \mathbb{Q}(e^{\pi i/4}), \quad L = \mathbb{Q}(\sqrt{2}), \quad [K : \mathbb{Q}] = 4. \]

\[
\begin{array}{c|cccc}
 p \mod 8 & 1 & 2 & 3 & 5 \\
 \text{deg}_L(p) & 1 & 1 & 2 & 2 \\
 \text{deg}_K(p) & 1 & 1 & 2 & 2 \\
\end{array}
\]

Computed from Facts 1 and 3(a).

Basic indices: \( p_{(1)}, p_{(3)}^2, p_{(5)}^2 \)

Dirichlet series:
\[
\left( 1 + \frac{1}{2^s} \right)^{-1} \frac{\zeta_K(s)}{\zeta_L(2s)} = \prod \left( \frac{1 + p_{(1)}^{-s}}{1 - p_{(1)}^{-s}} \right)^2 \frac{(1 + p_{(3)}^{-2s})(1 + p_{(5)}^{-2s})}{(1 - p_{(3)}^{-2s})(1 - p_{(5)}^{-2s})}
\]
\[
= 1 + \frac{2}{9^s} + \frac{4}{11^s} + \frac{2}{25^s} + \frac{4}{41^s} + \frac{2}{73^s} + \frac{4}{89^s} + \frac{4}{97^s} + \frac{4}{109^s} + \frac{2}{121^s} + \frac{4}{137^s} + \cdots
\]
Number of CSMs with index \( m \):

\[
f(m) = \begin{cases} 
\prod_{p \mid m} 4e_p \prod_{p \nmid m} 2, & \text{if } m \text{ is a product of basic indices} \\
0, & \text{otherwise.}
\end{cases}
\]

Average number of CSMs:

\[
\frac{2\sqrt{2}\log(1 + \sqrt{2})}{\pi^2} \simeq 0.253
\]

\( n = 12 \), the 12-fold module

\( K = \mathbb{Q}(e^{\pi i/6}), \ L = \mathbb{Q}(\sqrt{3}), [K : \mathbb{Q}] = 4. \)

| \( p \pmod{12} \) | 1 | 2 | 3 | 5 | 7 | 11 \\
|------------------|---|---|---|---|---|-----|
| \( \deg_L(p) \)  | 1 | 1 | 1 | 2 | 2 | 1   \\
| \( \deg_K(p) \)  | 1 | 2 | 2 | 2 | 2 | 2   |

Computed from Facts 1 and 2 and the cases \( n = 3 \) and \( n = 4 \) using Fact 3(b).

Basic indices: \( p_{(1)}, p_{(5)}^2, p_{(7)}^2 \)

Dirichlet series:

\[
\frac{\zeta_K(s)}{\zeta_L(2s)} = \prod \left( \frac{1 + p_{(1)}^{-s}}{1 - p_{(1)}^{-s}} \right)^2 \frac{(1 + p_{(5)}^{-2s})(1 + p_{(7)}^{-2s})}{(1 - p_{(5)}^{-2s})(1 - p_{(7)}^{-2s})}
\]

\[
= 1 + \frac{4}{13^x} + \frac{2}{25^x} + \frac{4}{37^x} + \frac{2}{49^x} + \frac{4}{61^x} + \frac{4}{73^x} + \frac{4}{97^x} + \frac{4}{109^x} + \frac{4}{157^x} + \frac{8}{181^x} + \cdots
\]

Number of CSMs with index \( m \):

\[
f(m) = \begin{cases} 
\prod_{p \mid m} 4e_p \prod_{p \nmid m} 2, & \text{if } m \text{ is a product of basic indices} \\
0, & \text{otherwise.}
\end{cases}
\]

Average number of CSMs:

\[
\frac{\sqrt{3}\log(2 + \sqrt{3})}{\pi^2} \simeq 0.231
\]

5 Application to 2D quasicrystals

The reader might like to see at least one or two examples where we apply the above results to planar quasicrystals. For simplicity, we consider the eightfold symmetric Ammann–Beenker tiling and the decagonal Tübingen triangle tiling [29] here, while the slightly more complicated rhombic Penrose tiling is discussed in Appendix A.
The Ammann–Beenker tiling

Consider the eightfold symmetric Ammann–Beenker tiling of Fig. 1 and, in particular, the coincidence problem of its vertex points for rotations around the symmetry centre. The underlying module is the standard eightfold module of rank 4, usually obtained as projection of the hypercubic lattice $\mathbb{Z}^4$ to a suitably chosen 2D plane. This plane, and its perpendicular complement, are eigenspaces of an eightfold rotation.

The set of vertex sites of this tiling is just the subset of module points whose corresponding points in $\mathbb{Z}^4$ perpendicularly project into a certain regular octagonal window. It is clear then that a coincidence of vertex sites implies one in the module, but also the converse is true due to the way the tiling sites are distributed over the module.

A coincidence rotation can be lifted to 4-space whence it also affects the window. In fact, a coincidence point must have perpendicular projections both in the original and in the rotated window! But this results in a slight modification of the fraction $21$.
of coinciding points which has to be corrected by an acceptance factor $A$. This is nothing but the area ratio of the intersection of the rotated windows with the original window, see Fig. 2. For a coincidence rotation through $\phi$, it turns out to be

$$A = 1 - (1 - \frac{1}{\sqrt{2}}) \sin(\hat{\psi}) \sin(\frac{\pi}{4} - \hat{\psi}) ,$$

(56)

where $\hat{\psi} \in [0, \pi/4]$ via

$$\hat{\psi} = \psi - \left[ \frac{4\psi}{\pi} \right] \cdot \frac{\pi}{4},$$

(57)

and $\psi$, the rotation angle in perpendicular or internal space, is related to the angle $\varphi = 2 \arctan(a + b\sqrt{2})$ through an algebraic conjugation:

$$\psi = 2 \arctan(a - b\sqrt{2}) .$$

(58)

The acceptance factor (56) is 1 for symmetry rotations and smaller otherwise, the minimum value being $A_{\text{min}} \simeq 0.957$ at $\pi/8$. The set of coinciding points almost looks like an Ammann–Beenker pattern again, but some points are missing: the quantity $1 - A$ is the frequency of such failures which were observed in [3]. With a more complicated window, star-shaped say, the acceptance factor would also become more complicated: with some choices of window it can even be zero for certain angles. But we will not go into further details here.

The Tübingen triangle tiling

Let us now consider the coincidence problem for the vertices of the decagonal triangular tiling of Fig. 3. All vertex sites belong to the standard tenfold module which
can be obtained by projection of the root lattice $A_4$ to a suitably chosen plane $[29]$. For simplicity, we consider the cartwheel tiling (which is singular) because it has full $D_{10}$ symmetry in the sense that a $D_{10}$ operation produces mismatches of density zero in the plane (along worms). We thus have coincidence fraction 1 in this case. Also, all other coincidences of the tenfold module are realized. As in the previous example, one has to correct the coincidence fraction, this time by rotating a decagon (the window of the vertex sites) and intersecting it with the original one. Let us give the correction formula in slightly more generality. If the window were a regular $n$-gon, the analogue of Eq. (56) would read

$$A = 1 - \left( \frac{\sin(\alpha/2)}{\sin(\alpha)} \right)^2 \sin(\psi) \sin(\alpha - \psi), \quad (59)$$

where $\alpha = 2\pi/n$, $\psi = \psi - \left[ \frac{n\psi}{2\pi} \right] \cdot \frac{2\pi}{n}$, and $\psi$ is related to $\varphi$ via an algebraic conjugation.

In the present instance, this relation is that $\tan(\varphi/2)$ can be expressed in the form

$$\tan \frac{\varphi}{2} = (a + b\tau) \sin \frac{2\pi}{5} \quad (a, b \in \mathbb{Q}) \quad (60)$$

and then

$$\tan \frac{\psi}{2} = (a + b\tau') \sin \frac{4\pi}{5}, \quad (61)$$

where $\tau' = -1/\tau$ is the conjugate of $\tau$ in $\mathbb{Q}(\tau)$.

Twelvefold symmetric tilings

As well as eight- and tenfold symmetries, twelvefold symmetry is of practical interest. Here the calculation of $\psi$ is very similar to the eightfold case: given the angle

$$\phi = 2 \arctan(a + b\sqrt{3}) \quad (62)$$

in tiling space one obtains the angle

$$\psi = 2 \arctan(a - b\sqrt{3}) \quad (63)$$

in internal space, which can be used with Eq. (59).

6 Beyond unique factorization

In Section 3, we restricted ourselves to the “class number 1” case, where there is essentially only one $n$-module. We now show how our method can be adapted to other cases, too. The smallest value of $n$ to which Section 3 does not apply is 23 ($N = 46$), mentioned in $[21]$. Here, the cyclotomic field has class number 3, so there are 3 distinct modules with 46-fold symmetry. (The number of modules increases rapidly with $n \quad [26][21].$)
Figure 3: Central patch of the cartwheel version of the tenfold symmetric Tübingen triangle tiling.
6.0.1 Ideals and ideal classes

Let $F$ be algebraic number field with ring of integers $\mathcal{O}$. The set of ideals of $\mathcal{O}$ can be extended to form a group by admitting fractional ideals of the form

$$ab^{-1} = \{\gamma | \gamma \beta \in a \ \forall \beta \in b\}, \quad (64)$$

where $a$ and $b$ are ideals as defined in Section 3. (A fractional ideal need not be a subset of $\mathcal{O}$.) The identity element of the group of fractional ideals is $\mathcal{O}$. A principal ideal is a fractional ideal of the form

$$(\gamma)\mathcal{O} = \{\gamma \alpha | \alpha \in \mathcal{O}\}, \quad (65)$$

generated by the single number $\gamma$. When unique factorization into irreducible integers fails in $F$ then some ideals must necessarily be non-principal. Two fractional ideals $a$ and $b$ are equivalent if $b = \gamma a$ for some $\gamma \in F$. The equivalence classes, called ideal classes, form a quotient group of the group of fractional ideals called the ideal class group, $H = H(F)$, which turns out to be finite. Its order is called the class number, $h(F)$. The identity element of $H$ is the class of principal ideals.

The ideal classes inherit complex conjugation from $F$: each ideal class $C \in H$ has a complex conjugate class $\overline{C}$. Complex conjugation is an automorphism of $H$ of order 2.

6.0.2 Ideals as modules

Our definition of $n$-modules makes them ideals in the ring of integers of the $n$th cyclotomic field (and with any broader definition an $n$-module would certainly be equivalent to one of these). Multiplication by a complex number $\gamma$ is equivalent to a combined rotation and scale change in the plane, so equivalent ideals certainly correspond to equivalent modules. Conversely, equivalent modules can be transformed into each other by multiplication by a complex number $\gamma$, and if both modules are subsets of an algebraic number field $K$ then $\gamma$ is in $K$ and the corresponding ideals are equivalent.

So the set of $n$-modules up to equivalence corresponds to the class group of the $n$th cyclotomic field.

6.0.3 Coincidence rotations in the general case

With class number $> 1$, $n$-modules are no longer all equivalent. So, for comprehensiveness, we need to consider not just $OC(\mathcal{O}_K)$ but also $OC(c)$ for an arbitrary ideal $c$ of $\mathcal{O}_K$.

There are two problems to be overcome in extending our method to the general $n$-module:

(1) How to classify which of the products on the right of Eq. (30) give rise to numbers $\gamma$ with $|\gamma| = 1$ (when some $\omega_k$’s are non-principal ideals) and
(2) how to choose a representative of the reflection coset of $OC$, for modules not invariant under complex conjugation, and how to calculate coincidence indices of reflections from it.

In this subsection we address the first of these.

For the fractional ideal $a\overline{a}^{-1}$ to give rise to a number $\gamma \in K$ with $|\gamma| = 1$ two conditions are necessary (and the conjunction of these conditions is also sufficient). They are

(A) $a\overline{a}^{-1}$ is principal, and

(B) for every $\delta$ such that $(\delta)_C = a\overline{a}^{-1}$, $\delta\overline{\epsilon} = \epsilon\overline{\epsilon}$ for some unit $\epsilon$ of $K$.

Condition (B) arises because $\gamma = \epsilon\delta$ in Eq. (30) gives $\delta\overline{\epsilon} = \epsilon^{-1}\overline{\epsilon}^{-1}$. Condition (A) is tantamount to saying that the ideals $a$ and $\overline{a}$ are equivalent, in other words that $a$ belongs to a class in $H_1$, the subgroup of $H$ consisting of classes $C$ with $\overline{C} = C$. It is easily checked that Condition (B) also depends only on the class of $a$ and is preserved under multiplication and inversion of classes. For Condition (B) to be applicable at all $a$ must belong to a class in $H_1$. Consequently Condition (B) is equivalent to $a$ belonging to a class in a certain subgroup $H_2$ of $H_1$.

When Condition (B) is satisfied the numbers $\gamma = \xi\delta\overline{\epsilon}^{-1}$, where $\xi$ runs through the $N$ roots of 1 in $K$, satisfy $|\gamma| = 1$. In this case $\text{num}(\gamma)$ is the ideal $a$ and can still be defined exactly as in Eq. (24), provided that “gcd” is interpreted as meaning “the ideal generated by”. Again $c \cap \gamma c = \text{num}(\gamma)c$ and the coincidence index associated with the rotation $\gamma$ is $\text{norm}(\text{num}(\gamma))$ (independent of the ideal $c$). In the general case, when the $\omega$'s may be non-principal ideals, a member of the product group on the right of Eq. (31) is a pair (root of unity, fractional ideal of the form $a\overline{a}^{-1}$) and the argument of Section 3 shows that elements of $\text{SOC}(c)$ correspond precisely to those pairs with the class of $a$ in $H_2$. Such pairs form a subgroup of finite index in the full product group. One can choose a set of generators for this subgroup in much the same way as one chooses a basis for a lattice of finite index in a given lattice, and as in that case there is an infinite number of such bases and no canonical choice.

Although $\text{SOC}(c)$ has independent generators as a group, the set of coincidence indices in general no longer has independent generators as a semigroup.

6.0.4 Coincidence reflections in the general case

Our second problem was how to calculate indices of coincidence reflections for a module class in which no module is invariant under complex conjugation. Choose, for simplicity, a prime ideal $p$ in the class (which is possible since every ideal class is known to contain infinitely many prime ideals). Then $p \cap \overline{p} = p\overline{p}$ has index $\text{norm}(p)$ in $p$. Every coincidence reflection of $p$ has the form $\rho = \gamma\overline{\epsilon}$ for some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$. Being a coincidence reflection on $p$, $\rho(\alpha) = \beta$ for some $\alpha, \beta \in p$. Hence
\[ \gamma = \beta/\alpha \in K. \] The index of \( \rho \) is the index of \( p \cap \gamma p \) in \( p \) which is
\[
\begin{cases} 
\text{norm(num(\gamma))norm}(p), & \text{if } p \not| \text{num}(\gamma), \\
\text{norm(num(\gamma))}/\text{norm}(p), & \text{if } p | \text{num}(\gamma).
\end{cases}
\]

We note that there is a reflection of index 1 if and only if the class of \( p \) is in \( H_2 \) (when we can choose \( \gamma \) to be a generator of the fractional ideal \( p/\overline{p} \)) and that in that case \( (66) \) agrees with our previous way of calculating the index. When the class of \( p \) is not in \( H_2 \) the smallest reflection index is got by taking \( (\gamma) = p a/\overline{p} a \), where \( a \) is the ideal of minimal norm such that the class of \( pa \) is in \( H_2 \). Of course, \( OC(p) \) is \( OC(p) \) conjugated by reflection in the \( x \)-axis (corresponding isometries having the same index). We note that this is consistent with \( (66) \); just replace \( p \) and \( \gamma \) by their complex conjugates.

**Theorem 4** The group of coincidence rotations of a general \( n \)-fold symmetric module is the direct product of its finite rotation symmetry group \( C_n \) and countably many infinite cyclic groups which can be effectively computed and depend only on \( n \). The index of any coincidence rotation so presented can be calculated explicitly. Any such module is equivalent to some prime ideal in the cyclotomic field of \( n \)th roots of unity, and in this form complex conjugation represents the coset of coincidence reflections whose indices can be computed from \( (66) \) (they depend not only on \( n \) but on the individual module). Such a module need not have exact reflection symmetry.

The following table list some statistics for the first few cyclotomic fields with \( h > 1 \). We follow Washington\(^{28} \) in listing fields with their degree, \( \phi(n) \), as the primary order and \( n \) as the secondary order. For each field we give \( n, N, H, H_1, H_2 \), the smallest rotation index and the smallest reflection index of the non-principal modules (for the principal module it is always 1). In brackets after each index we give the number of different rotations or reflections with that index. For all fields on our list \( H_2 \) is the trivial subgroup consisting only of the identity element \( E \) of \( H \). Also complex conjugation acts on the class group as multiplicative inversion for all these fields.

| \( n \) | \( N \) | Degree | \( H \) | \( H_1 \) | \( H_2 \) | Min. rotation index | Min. reflection index of non-principal modules |
|---|---|---|---|---|---|---|---|
| 23 | 46 | 22 | \( C_3 \) | \( \{E\} \) | \( \{E\} \) | 599 \((22)\) | 47 \((11)\) |
| 39 | 78 | 24 | \( C_2 \) | \( C_2 \) | \( \{E\} \) | 157 \((24)\) | 13 \((2)\) |
| 52 | 52 | 24 | \( C_3 \) | \( \{E\} \) | \( \{E\} \) | 313 \((24)\) | 13 \((1)\) |
| 56 | 56 | 24 | \( C_2 \) | \( C_2 \) | \( \{E\} \) | 64 \((2)\) | 8 \((2)\) |
| 72 | 72 | 24 | \( C_3 \) | \( \{E\} \) | \( \{E\} \) | 729 \((2)\) | 9 \((1)\) |
| 29 | 58 | 28 | \( C_2^3 \) | \( C_2^3 \) | \( \{E\} \) | 493 \((28)\) | 59 \((4)\) |
| 31 | 62 | 30 | \( C_9 \) | \( \{E\} \) | \( \{E\} \) | 595 \((30)\) | \( \{32 \((1)\) : \text{order 9}\) |
| & & | & & | & & | & & | \( 125 \((5)\) : \text{order 3}\) |

The two sets of figures in the last entry are due to the fact that non-principal modules with different orders in the class group of \( Q(\sqrt{\cos 2\pi/31}) \) have different minimum reflection indices.
6.0.5 Another example: $N = 46$

To illustrate the results of the previous subsection we treat in detail the case $n = 23$ (with 46-fold symmetry). For this $n$, the class group $H$ of $K$ is $H = \{E, C, C^2\}$, where $C^3 = E$ and $C = C^2$. Hence $H_1 = \{E\}$ and therefore $H_2 = \{E\}$ too.

The methods of Section 3 show that the complex splitting primes are precisely those that are quadratic residues mod 23 and for these $\deg(p) = 1$ or 11 according to whether $p \equiv 1 \pmod{23}$ or not. The prime ideals $p$ of $O_K$ that divide a given rational prime $p$ are either all principal or all non-principal (because the Galois group Gal($K/\mathbb{Q}$) permutes them transitively) and in the non-principal case fall into complex conjugate pairs of ideals, one from each of the classes $C$ and $C^2$. We partition the set of pairs $\Omega$ into the sets $\Omega_1, \Omega_2$ as follows:

$$\Omega_1 = \{\{\omega_1, \overline{\omega}_1\}, \{\omega_2, \overline{\omega}_2\}, \ldots\} \quad (67)$$

$$\Omega_2 = \{\{p_1, \overline{p}_1\}, \{p_2, \overline{p}_2\}, \ldots\}, \quad (68)$$

where the $\omega_i$'s are numbers (corresponding to principal ideals) and where in $\Omega_2$ we have chosen $p_i \in C$, $\overline{p}_i \in C^2$ for each $i$. Finding all numbers of $K$ on the unit circle is equivalent to finding all principal ideals with $K/L$-norm equal to $\mathcal{O}_L$. (The numbers $\gamma$ are then the sets of associates of the generators of these ideals.) These ideals are precisely those of the form

$$\prod_l \left(\frac{\omega_l}{\overline{\omega}_l}\right)^{m_l} \prod_k \left(\frac{p_k}{\overline{p}_k}\right)^{n_k} \quad (69)$$

with $\sum n_k$ divisible by 3 (since each $p_k \overline{p}_k^{-1}$ belongs to the class $C^2$ of order 3). This group of ideals has each $\omega_l/\overline{\omega}_l$ as an independent generator of the first factor, and a set of independent generators of the second factor can be chosen as follows:

$$(p_1/\overline{p}_1)^3, \quad p_1 p_2/\overline{p}_1 \overline{p}_2, \quad \overline{p}_2 p_3/p_2 \overline{p}_3, \ldots \quad (70)$$

Although this exhibits $SOC(c)$ as having independent generators as a group, the set of coincidence indices no longer has independent generators as a semigroup. Instead of basic coincidence indices one has the prime powers

$$p \quad (p \equiv 1 \pmod{23}) \quad \text{and} \quad p^{11} \quad (p \equiv 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 \pmod{23})$$

which can be partitioned into two classes $P_1$ and $P_2$ (corresponding to $\Omega_1$ and $\Omega_2$) according to whether or not the prime ideals dividing $p$ are principal. As examples:

$$599, 691, 829, 59^{11}, 101^{11} \in P_1$$

and

$$47, 139, 277, 461, 967, 2^{11}, 3^{11}, 13^{11}, 29^{11}, 31^{11}, 41^{11}, 71^{11}, 73^{11} \in P_2.$$ 

These examples were computed using the observation (derived from the last paragraph of Chapter 1 of [28]) that $p$ factorizes into principal primes if and only if it
factorizes into principal primes in \( \mathbb{Q}(\sqrt{-23}) \). A necessary and sufficient condition for this is the solubility of the Diophantine equation \( 6x^2 + xy + y^2 = p \).

The general product of these numbers has the form
\[
m = p_1^{a_1} \cdots p_r^{a_r} (p_{r+1}^{11})^{a_{r+1}} \cdots (p_s^{11})^{a_s} \times \{P_1\text{-factors}\},
\]
where \( p_1, \ldots, p_s^{11} \) are in \( P_2 \) with \( p_1, \ldots, p_r \equiv 1 \pmod{23} \) and \( p_{r+1}, \ldots, p_s \not\equiv 1 \pmod{23} \). Now, for \( k = r + 1 \ldots s \) define
\[
\epsilon_k = \begin{cases} 
0 & \text{if } 3 \mid a_k, \\
1 & \text{if not.}
\end{cases}
\]
Then \( m \) is a coincidence index if and only if
\[
a_1 + \cdots + a_r + \epsilon_{r+1} + \cdots + \epsilon_s \neq 1.
\]

[The reason for this is that in choosing a principal ideal giving index \( m \) we can arrange that \( \sum n_k \) is divisible by 3 in (69) by changing the sign of some \( n_k \)'s provided at least two \( n_k \)'s are not divisible by 3. Primes of degree 11 are divisible by only one pair of primes in \( K \), but primes of degree 1 are divisible by 11 such pairs, so for these we can easily arrange that no \( n_k \) is divisible by 3.]

Consequently the first three rotation coincidence indices are 1, 599, 691, the smallest not composed entirely of primes \( \equiv 1 \pmod{23} \) is \( 2^{11}47 = 96256 \) and the smallest with no prime factors \( \equiv 1 \pmod{23} \) is \( 2^{11}3^{11} = 362797056 \).

The Dirichlet series generating function of \( f(m) \) can be found much as before, except that the contribution from non-principal ideals with \( K/L \)-norm equal to \( O_L \) must be omitted. This can be done using the three characters of the class group: we form three Dirichlet series (Hecke \( L \)-series), one for each character, by multiplying each norm in the series by the value of the character on its ideal. The required generating function is then the average of these three series.

For the principal character (identically equal to 1) the corresponding Dirichlet series is exactly as in Eq. (52), namely
\[
\left(1 + \frac{1}{23^s}\right)^{-1} \frac{\zeta_K(s)}{\zeta_L(2s)}.
\]

For a non-principal character \( \chi \) the Euler factor for a prime \( p \) occurring in \( P_1 \) is exactly as in (44) and (45). For a prime \( p \) occurring in \( P_2 \), however, the Euler factor is
\[
\left( \cdots + \frac{\eta}{p^{2ds}} + \frac{\eta^2}{p^{4ds}} + 1 + \frac{\eta^2}{p^{6ds}} + \frac{\eta^2}{p^{8ds}} + \cdots \right)^{g/2} = \left( \frac{1 - 2p^{-ds}}{1 - p^{-ds}} \right)^{g/2},
\]
where \( \eta^3 = 1 \). Since this does not depend on which primitive cube root of unity \( \eta \) is, the Dirichlet series formed with the characters \( \chi \) and \( \overline{\chi} \) are the same and we have
\[
\sum_{m=1}^\infty \frac{f(m) \chi(m)}{m^s} = \frac{1}{3} \left(1 + \frac{1}{23^s}\right)^{-1} \frac{\zeta_K(s)}{\zeta_L(2s)} \left\{ 1 + 2 \prod_{p \in P_2} \left( \frac{1 - 2p^{-s}}{1 + p^{-s}} \right)^{11} \prod_{q \in P_2} \left( \frac{1 - 2q^{-s}}{1 + q^{-s}} \right) \right\}.
\]
where the first product is over the primes \( p \) in \( P_2 \) (which are \( \equiv 1 \mod 23 \)) and the second is over the 11th powers \( q \) in \( P_2 \).

In line with earlier examples we give the first 12 nonzero terms:

\[
1 + \frac{22}{599^s} + \frac{22}{691^s} + \frac{22}{829^s} + \frac{110}{1151^s} + \frac{22}{2209^s} + \frac{22}{2347^s} + \frac{22}{2393^s} + \frac{22}{3037^s} + \frac{22}{3313^s} + \frac{22}{3359^s} + \frac{22}{4463^s} + \cdots
\]

Note that this applies to all three modules.

For the principal module the reflection indices are the same as the rotation indices. For the two non-principal modules, however, the first three reflection indices are 47, 139, 277.

7 Concluding remarks

Let us summarize our results. We have solved the coincidence problem for planar patterns with \( N \)-fold symmetry by number theoretic methods. The first stage consisted of the analysis of lattices and modules in the plane where the coincidence indices are integers.

For various cases of interest we have given the solution explicitly, in particular describing the set of possible coincidence indices and the number of coincidence isometries with given index. The method is described in sufficient detail to allow other examples along these lines to be worked out. This is relatively easy for \( N < 46 \), but the complication increases astronomically for larger \( N \) as foreshadowed even in the example \( N = 46 \), where the class number is only 3.

The second stage was the explicit investigation of discrete structures associated with a given module. Here, in the non-periodic case, the calculation of the coincidence ratio requires a non-integral correction factor. We have demonstrated its calculation in several examples.

Furthermore, the approach via algebraic number fields automatically yields sets of independent generators for the CSM group and therefore an explicit description of it. The group structure is interesting in itself because we deal here with infinite discrete groups that are countably generated and the structure of such groups is not at all obvious.

An obvious next step is to extend the investigation to 3D examples. This is not only an interesting extension of the technique, but may have concrete realizations. There are two cases to consider: first the T-phases, i.e. quasicrystals which have a unique quasiperiodic plane and are periodic in the third direction. The CSMs for rotations around the unique axis are the ones treated in this paper. CSMs around other axes occur only when special relations hold between the lattice constants in the plane and perpendicular to the plane, a result familiar from the hexagonal case \[2\]. There are also near-coincidences with small misfits between the two grains, but it is beyond our scope to deal with these. The second case is the icosahedral one, the only remaining non-crystalline symmetry in 3D. Here we do not have such a powerful tool as the complex numbers and the structure of the CSM groups is more
complicated, even the rotation part being non-Abelian in general. Some results are reported in [13] and will be described more fully in [14].

Appendix A: other rotation centres

In the main text, we have analyzed the standard situation of coincidence rotations around lattice (or module) points. Here, we will briefly comment on rotations around other centres in the lattice case and on situations with more than one translation class of points.

n=4: the square lattice revisited

Another obvious rotation problem is that around the centre of a Delaunay cell of $\mathbb{Z}^2$, $(\frac{1}{2}, \frac{1}{2})$ say. This point represents the only class of deep holes of $\mathbb{Z}^2$, cf. [17], and has the entire point group $D_4$ of $\mathbb{Z}^2$ as site symmetry. It is obvious that the coincidence problem is equivalent to that of the point set $\Gamma$ defined by

$$\Gamma = \{a + ib \mid a, b \in \mathbb{Z}, a + b \text{ odd}\}, \quad (A1)$$

which is obtained from $\mathbb{Z}^2 - (\frac{1}{2}, \frac{1}{2})$ via rotation through $\pi/4$ and dilation by $\sqrt{2}$.

Observe that (A1) can be rewritten as

$$\Gamma = \{\alpha \in \mathbb{Z}[i] \mid \alpha \not\equiv 0 (1+i)\} \quad (A2)$$

which solves the problem: as was shown in Section 2, the coincidence rotations of $\mathbb{Z}^2$ can be factorized, the generators being $e^{i\phi} = i$ (rotation through $\pi/2$) or of the form $e^{i\phi} = \omega_p/\overline{\omega}_p$ with $N(\omega_p) = p \equiv 1 \ (4)$, hence $\omega_p \not\equiv 0 (1+i)$. The former still is a symmetry of $\Gamma$ (index 1), and we also get the latter because both numerator and denominator are in $\Gamma$. Also, the reflection in the $x$-axis remains a coincidence operation of index 1. Summarizing:

$$OC(\Gamma) = OC(\mathbb{Z}^2), \quad (A3)$$

and the coincidence indices are unchanged.

n=3: the hexagonal packing

Consider the Voronoi complex of the triangular lattice — it is a packing made from regular hexagons — and let $H$ be its vertex set. Let us consider rotations around the centre of a hexagon which is a point of maximal site symmetry $D_6$. If we rotate the complex through $\pi/6$ and dilate by $\sqrt{3}$, then $H$ can be characterized as

$$H = \{\alpha \in \mathbb{Z}[\varrho] \mid \alpha \not\equiv 0 (1+\varrho)\} \quad (A4)$$

where $\varrho = \frac{1+i\sqrt{3}}{2}$. Since $N(1+\varrho) = 3$ and 3 is not a complex splitting prime in $\mathbb{Q}(\varrho)$, we find again all rotations and reflections which we had already for the triangular lattice:

$$OC(H) = OC(A_2) \ , \quad (A5)$$

and also the indices remain unchanged.
n=3: coincidence definition revisited

Slightly different is the situation if we keep the entire set of lattice points, but rotate around the centre of a Delaunay cell: the latter is a triangle and its centre has only $D_3$ site symmetry. We rotate again through $\pi/6$ and dilate by $\sqrt{3}$ which gives the point set

$$ G = \{ \alpha \in \mathbb{Z}[\rho] \mid \alpha \equiv 1 \pmod{1+\rho} \}.$$

(A6)

Here, a rotation through $\pi/3$ would change the congruence class of $G$ from 1 to $-1$, so it is no longer a coincidence rotation. This reduces the torsion part of $OC$ from $C_6$ to $C_3$ in agreement with the reduced site symmetry while all other generators remain unchanged. In particular, the reflection in the $x$-axis leaves $G$ invariant and the index formula applies for all remaining elements.

One might also consider possible variants of the coincidence concept here: a rotation through $\pi/3$ alone does not produce a coincidence for the set $G$, while the same rotation followed by a suitable translation can give a coincidence of index 1. The latter might be more important when the connection to grain boundary growth is considered. Indeed, especially in view of applications to nonperiodic discrete point sets, one might define (with obvious meaning)

$$ \inf_{t \in \mathbb{R}^2} [P : P \cap (RP + t)] \quad \text{(A7)}$$

to be the coincidence index of an isometry $R$ acting on a point set $P$. This gets rid of the dependence of the index on the rotation centre and comes closer to the idea of optimal fitting of grain fragments.

n=5: the rhombic Penrose tiling

A complication here is that the vertex sites of the rhombic Penrose tiling $T$ fall into 4 different translation classes with respect to the uniquely defined limit translation module $\mathcal{M}(T)$, compare [29, 20]. We identify $\mathcal{M}(T)$ with the projection of the 4D root lattice $A_4$ into tiling space for definiteness. Then each point class has its own window of pentagonal shape. The windows come in two different sizes (related by a factor of $\tau = (1 + \sqrt{5})/2$) and in pairs related by rotation through $\pi$, compare [29]. The vertices of the rhombi are not points of the module $\mathcal{M}(T)$ (which also means that none of them is a “standard” rotation centre).

Let us now consider the coincidence problem of the set of vertex sites with all translation classes identified. To be explicit, we take the rhombic version of the cartwheel pattern where the rotation centre is not a rhombus vertex but coincides with the centre of a regular decagon filled with rhombi. This point is a representative of the fifth translation class, so far absent. The cartwheel tiling has $D_{10}$ symmetry in the sense that any $D_{10}$-operation either maps the tiling upon itself (thus, in particular, the set of vertex sites) or produces at most a mismatch of density zero (along the well-known worms). All these operations thus have coincidence ratio 1. The corresponding rotation in window space maps windows to windows, because
they appear in $D_{10}$-orbits around the origin. More than this, it maps translation classes of windows to translation classes of windows.

For other coincidence isometries, we first observe that the integral span of all vertex points is again a planar module of rank 4, in our explicit case the projection of the weight lattice $A_4^*$, the dual of $A_4$, into tiling space. This module is equivalent to $M(\mathcal{T})$ and possesses therefore the same coincidence isometries, namely those described in Section 3. Consequently, we find all these also as coincidence isometries of the rhombic cartwheel tiling. The coincidence ratio must now be corrected in a similar way to that of the Ammann–Beenker tiling in Section 4, but the window system requires a slightly more complicated calculation which we will not present here.

Even more complicated would be the coincidence analysis for rotations around vertex points, in particular with various point classes distinguished. The methods needed are in principle those described for $n = 3$ above, but details will not be given here.

n=12: a square-triangle tiling

Quasiperiodic square-triangle tilings are attractive for a number of reasons. We mention them because they can have 12-fold symmetry in the sense of mismatches of at most density zero under $D_{12}$-operations, see [30] for an example. There, all vertex points are in one translation class, so no problem occurs and we find all coincidence isometries of Section 3. But for the correction factor due to window overlaps one encounters a new type of complication: the window is fractally shaped and consequently we see no way of calculating this factor. It is left as an exercise for fractal readers.

Appendix B: proofs

Here we give the promised references and proofs of Facts 1–3 in Section 3.

Fact 1 This is proved in [24] (Lemma 4) or [28] (Theorem 2.13) for example, but we sketch the proof here as it leads on naturally to the proof of Fact 2, which is less commonly found in the literature.

Let $P$ be a prime factor of $p$ in $K$. Since $p \nmid n$ the $n$th roots of 1 in $K$ are distinct mod $P$. (The most straightforward way to see this is from the identity

$$n = \prod_{k=1}^{n-1} (1 - \xi^k), \quad (B1)$$

got by putting $x = 1$ in $(x^n - 1)/(x - 1)$, and noting that every difference of roots of unity is an associate of $1 - \xi^k$ for some $k$.) The residue class field $\mathbb{F}_P = \mathbb{Z}[\xi]/P$ is a finite field generated over $\mathbb{F}_p$ by the residue class $\xi^*$ of $\xi$, and since distinct roots of unity are distinct mod $P$, the order of $\xi^*$ in $\mathbb{F}_P$ is $n$. Every finite extension of $\mathbb{F}_p$ is
normal with cyclic Galois group generated by the Frobenius automorphism \( x \mapsto x^p \), whose order is the degree \( d \) of the extension. Consequently the degree \([\mathbb{F}_P : \mathbb{F}_p]\) is the smallest \( d \) with \( \xi^{*p^d} = \xi^* \); that is, the smallest \( d \) with \( n \mid (p^d - 1) \). This establishes Fact 1, since \([\mathbb{F}_P : \mathbb{F}_p]\) is the degree of the minimal polynomial satisfied by \( \xi \) mod \( p \) and hence is \( \deg_K(p) \).

**Fact 2** Analogously to the above proof, \( \deg_L(p) \) is the degree \( d' \) of the residue class field extension \( \mathbb{F}_p/\mathbb{F}_p' \), where \( p \) is the prime of \( L \) divisible by \( P \) and \( \mathbb{F}_p \) is the residue class field \( \mathbb{Z}[\xi + \xi^{-1}]/P \). Clearly \([\mathbb{F}_P : \mathbb{F}_p]\) \( \leq [K : L] = 2 \), so the \( d \) of Fact 1 is either \( d' \) or \( 2d' \). If \( d \) is odd then \( d' = d \), the order of \( p \) mod \( n \), and no power of \( p \) is congruent to \(-1\) mod \( n \).

To treat the case of even \( d \) we first note that if \( \xi^{*k_1} + \xi^{*-k_1} = \xi^{*k_2} + \xi^{*-k_2} \) in \( \mathbb{F}_P \) (where \( 0 \leq k_1, k_2 < n \)) then either \( \xi^{*k_1} = \xi^{*k_2} \) or \( \xi^{*k_1} = \xi^{*-k_2} \). This is because \( \xi^{*k_j}, \xi^{*-k_j} \) are the two roots in \( x \) of

\[
x^2 - (\xi^{*k_j} + \xi^{*-k_j})x + 1 = 0, \quad (j = 1, 2)
\]

and when the equations are the same the roots must match in some order. Now \( d' = d/2 \) if and only if \( \mathbb{F}_p \) is the unique subfield of index 2 in \( \mathbb{F}_P \), this being the fixed field of the element \( x \mapsto x^{p^{d/2}} \) of order 2 in the Galois group of \( \mathbb{F}_P/\mathbb{F}_p \). So \( d' = d/2 \) if and only if

\[
\xi^* + \xi^{-1} = (\xi^* + \xi^{-1})p^{d/2} = \xi^{*p^{d/2}} + \xi^{-p^{d/2}},
\]

which requires \( \xi^{-1} = \xi^{p^{d/2}} \) (equivalent to \( n \mid p^{d/2} + 1 \)), since \( \xi^* \neq \xi^{p^{d/2}} \). The exponent \( d/2 \) here is plainly minimal, since \( n \mid p^{a} + 1 \Rightarrow n \mid p^{2a} - 1 \).

**Fact 3** Part (a) is a result of the fact that when \( n = p^r \) then \( p \) is totally ramified in \( K \) (that is, \( p \) is the \( \phi(p^r) \)th power of a degree 1 prime of \( K \)), see for example \[24\], Lemma 3. As a consequence \( \deg_K(p) = 1 \) and \( p \) is not a complex splitting prime because it has only one prime factor in \( K \).

For part (b) we refer to the Hasse diagram of field inclusions in Figure 4. Here \( K_1 \) and \( K_2 \) are the cyclotomic fields of \( n_1 \)th and \( p^r \)th roots of unity and \( L_1 \) and \( L_2 \) their maximal real subfields. Then \( K = K_1K_2 \), the compositum of \( K_1 \) and \( K_2 \), and, since \( p \nmid n_1, K_1 \cap K_2 = \mathbb{Q} \) (see \[22\] Thm. 9.52 or \[28\] Prop. 2.4). Let \( p = p_K \) be a prime of \( K \) dividing \( p \). For an arbitrary subfield \( F \) of \( K \) we denote by \( p_F \) the prime ideal of \( F \) that is divisible by \( p \).

Because \( p \) is unramified in \( L_1 \) and \( K_1 \) but totally ramified in \( K_2 \) it follows that \( p_{L_1} \) and \( p_{K_1} \) are totally ramified in \( K_2L_1 \) and \( K \) and, in particular,

\[
\deg_{K_2L_1/L_1}(p_{K_2L_1}) = \deg_{K/K_1}(p_K) = 1.
\]

Consequently

\[
de_{K/K_2L_1}(p) = \deg_{K_2L_1/L_1}(p_{K_2L_1}).
\]

Now look at the fields \( L_1L_2, K_2L_1, L \) and \( K \). Since \( p_{L_2} \) ramifies in \( K_2 \) but \( p \) is unramified in \( L_1, p_{L_1L_2} \) ramifies in \( K_2L_1 \) and hence in \( K \). By Prop. 2.15(b) of \[28\],
Figure 4: Hasse diagram.
\( p_L \) is unramified in \( K \), and hence \( p_{L_1} \) ramifies in \( L \). We now have

\[
\deg_{L/L_1 L_2}(p_L) = \deg_{K_2 L_1/L_1 L_2}(p_{K_2 L_1}) = 1
\]

whence

\[
\deg_{K/L}(p) = \deg_{K_2/L_1}(p).
\]

Since \( p_L \) is unramified in \( K \) and \( p_{L_1} \) is unramified in \( K_1 \), Eqs. (B5) and (B7) imply that \( p_L \) factors into two primes of \( K \) if and only if \( p_{L_1} \) factors into two primes of \( K_1 \). Finally, \( \deg_K(p) = \deg_{K_1}(p) \) is an immediate consequence of the fact that \( p_{K_1} \) is totally ramified in \( K \).

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