Thick branes and Gauss-Bonnet self-interactions

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Abstract

Thick branes obtained from a bulk action containing Gauss-Bonnet self-interactions are analyzed in light of the localization properties of the various modes of the geometry. The entangled system describing the localization of the tensor, vector and scalar fluctuation is decoupled in terms of variables invariant for infinitesimal coordinate transformations. The dynamics of the various zero modes is discussed and solved in general terms. Provided the four-dimensional Planck mass is finite and provided the geometry is everywhere regular, it is shown that the vector and scalar zero modes are not localized. The tensor zero mode is localized leading to four-dimensional gravity. The general formalism is illustrated through specific analytical examples.

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I. INTRODUCTION

If space-time has more than four dimensions, the Einstein-Hilbert term is not the only geometrical action leading to equations of motion involving (at most) second order derivatives of the metric [1]. In dimensions larger than four, the usual Einstein-Hilbert action can indeed be supplemented with higher order curvature corrections without generating, in the equations of motion, terms containing more than two derivatives of the metric with respect to the space-time coordinates [2]. If this is the case, the quadratic part of the action can be written in terms of the Euler-Gauss-Bonnet combination:

\[
R_{EGB}^2 = R^{ABCD}R_{ABCD} - 4R^{AB}R_{AB} + R^2.
\] (1.1)

In four dimensions the EGB is a topological term and it coincides with the Euler invariant: its contribution to the equations of motion can be rearranged in a four-divergence which does not contribute to the classical equations of motion. In more than four dimensions the EGB combination leads to a ghost-free theory and it appears in different higher dimensional contexts. In string theory the EGB indeed appears in the first string tension correction to the (tree-level) effective action [4–7]. In supergravity the EGB is required in order to supersymmetrize the Lorentz-Chern-Simons term.

Higher dimensional gravity theories have been investigated in connection with possible alternatives to Kaluza-Klein compactification [8,10–13]. For a more complete account of the various perspectives of the problem see the recent review [14].

In this context the gravity part of the action is usually taken to be the Einstein-Hilbert term. Recently, various investigations took into account the possible contribution of higher derivatives terms in the action and mainly in the five-dimensional case [15–18]. Various physical frameworks can be invoked in order to include higher order curvature corrections, whose motivation may range from back-reaction effects [19] to some interesting connections with string models [15]. For the rôle of quadratic counter-terms in the context of the AdS/CFT correspondence see for instance [22]. Quadratic corrections may also play a
role in dimensions larger than five. In \cite{20} seven-dimensional warped solutions have been
discussed in the case when hedgehogs configurations are present together with quadratic
self-interactions. In \cite{21} the simultaneous presence of dilaton field and quadratic corrections
has been investigated in a more string theoretical perspective and mainly in six-dimensions.

Suppose now that a smooth domain-wall solution is used in order to localize fields of
various spin as in \cite{8,9}. It is then interesting to investigate how the localization of the metric
fluctuations (coupled to the fluctuations of the wall) is affected by the addition of EGB
self-interactions. Smooth domain wall configurations leading to a fully regular geometry
can be constructed \cite{23–28}. The warp factors interpolates in a regular way between two $AdS_5$
geometries. If the gravity action is selected to be the Einstein-Hilbert term, the
scalar and vector zero modes of the geometry are not localized on the wall \cite{29} while the
tensor zero mode is localized, leading, ultimately, to ordinary four-dimensional gravity. Once
quadratic self-interactions are included in the picture, important effects on the localization of
gravity \cite{30} have been discussed and it has been also argued that singularity free domain-wall
solutions are possible if the quadratic part of the action is expressed in terms of Gauss-Bonnet
combination \cite{31}.

The aim of the present investigation is to study the fluctuations of scalar walls when
Gauss-Bonnet self-interactions are included. Most of this analysis can be developed in
general terms, i.e. without specifying the explicit solution describing the domain-wall. The
fluctuations of the metric can be studied in terms of fully gauge-invariant quantities so that
the obtained results will be independent on the specific coordinate system. Technically this
is made possible by generalizing the Bardeen formalism \cite{32} to the case of non-compact extra
dimensions whose dynamics is described in terms of a specific quadratic theory of gravity of
the type of the one recalled in Eq. (1.1). The application of the present formalism to the
context of large (but compact) extra-dimensions \cite{33–35} will not be directly examined here.

The plan of the present investigation is then the following. In Section II the basic
equations describing the background dynamics will be presented. In Section III the fully
gauge-invariant approach to linearized fluctuations will be discussed and applied to the prob-
lem at hand. Section IV deals the localization of the tensor fluctuations whereas in Section V the localization of the vector and scalar fluctuations will be analyzed. Section VI contains some specific studies of the localization of metric fluctuations in the case of analytical and singularity-free thick brane solutions induced by Gauss-Bonnet self-interactions. Some concluding remarks are presented in Section VII. Various technical results are collected in the Appendix.

II. BRANE SOURCES WITH QUADRATIC GRAVITY IN THE BULK

Consider a bulk action containing EGB self-interactions together with a brane source describing the spontaneous breaking of $D$-dimensional Poincaré invariance:

$$S = \int d^Dx \sqrt{|G|} \left[ -\frac{R}{2\kappa} - \alpha' R_{\text{EGB}}^2 + \frac{1}{2} G^{AB} \partial_A \varphi \partial_B \varphi - V(\varphi) \right],$$

(2.1)

where $\kappa = 8\pi G_D$ is related to the $D$-dimensional Planck mass. The sign of the coupling appearing in front of the EGB combination has been taken in order to match the sign obtained from the low-energy string effective action. Dimensionally, $[\alpha'] = L^{D-4}$ where $L$ is a generic length scale. The potential $V(\varphi)$ will be assumed to be symmetric for $\varphi \rightarrow -\varphi$. For sake of simplicity, one can think of the case where the potential is

$$V(\varphi) = V_0 \left[ c_1 \left( \frac{\varphi}{\varphi_0} \right)^4 + c_2 \left( \frac{\varphi}{\varphi_0} \right)^2 + c_3 \right],$$

(2.2)

where $c_1$, $c_2$ and $c_3$ are numerical coefficients of order one (no hierarchy is assumed among them). The equations of motion derived from the action of Eq. (2.1)

$$R^B_A - \frac{1}{2} \delta^B_A R = \kappa T^B_A - 2\alpha' \kappa Q^B_A,$$

(2.3)

$$G^{AB} \nabla_A \nabla_B \varphi + \frac{\partial V}{\partial \varphi} = 0,$$

(2.4)

are written in terms of the brane energy-momentum tensor

$$T^B_A = \partial_A \varphi \partial^B \varphi - \delta^B_A \left[ \frac{1}{2} G^{MN} \partial_M \varphi \partial_N \varphi - V(\varphi) \right],$$

(2.5)

and of the Lanczos tensor
\[ Q_A^B = \frac{1}{2} \delta_A^B R_{EGB}^2 - 2 R A^B + 4 R A R C R B^C + 4 R C D R A C B D - R A C D E R B C D E, \]  
\[ \text{accounting for the contribution of quadratic corrections to the equations of motion.} \]

In the contracted form of Eqs. (2.3)–(2.6)
\[ R_B^A = \kappa \tau_B^A - \epsilon \left[ \delta_B^A - \frac{1}{d} R_{EGB}^2 - 2 R R_A^B + 4 R A R C R B^C + 4 R C D R A C B D - 2 R A C D E R B C D E \right], \]
we set, for notational convenience, \( \epsilon = 2\alpha'\kappa \) and \( d = D - 2 \). In Eq. (2.7)
\[ \tau_A^B = T_A^B - \frac{T}{d} \delta_A^B, \]
and \( T \equiv T_A^A \) is the trace of the brane energy-momentum tensor. The \( D \)-dimensional metric will be taken in the form
\[ ds^2 = a^2(w) [dt^2 - dx_1^2 - \ldots - dx_d^2 - dw^2], \]
where the ellipses stand for the \( d \) spatial coordinates on the brane and while \( w \) is the bulk coordinate. \footnote{The Latin (uppercase) indices run over the whole \( D \)-dimensional space-time whereas the Greek indices run over the \( (d + 1) \)-dimensional subspace.}

Defining \( H = a'/a \), in the metric of Eq. (2.9) the EGB combination can be written as
\[ R_{EGB}^2 = \frac{d(d+1)}{a^4} \left[ (d-1)(d-2) H^4 + 4(d-1) H' H^2 \right], \]
where the prime denotes the derivation with respect to the bulk coordinate \( w \). The non-vanishing components of the Lanczos tensor are
\[ Q_{\mu \nu} = \frac{1}{2a^2} \left[ d (d-1)(d-2)(d-3) H^4 + 4 d (d-1)(d-2) H' H^2 \right] \eta_{\mu \nu}, \]
\[ Q_{w w} = -\frac{1}{2a^2} d (d+1)(d-1)(d-2) H^4, \]
where \( \eta_{\mu \nu} \) is the Minkowski metric in \( (d + 1) \)-dimensions.
Using Eqs. (2.10)–(2.12) the explicit form of Eqs. (2.3)–(2.4) can be obtained for a metric of the type of the one reported in Eq. (2.9):

\[ dH' + \frac{d(d - 1)}{2} H^2 = -\kappa \left[ \frac{\varphi'^2}{2} + Va^2 \right] + \frac{\epsilon}{a^2} \left[ \frac{d(d - 1)(d - 2)(d - 3)}{2} \mathcal{H}^4 + 2d(d - 1)(d - 2)\mathcal{H}^2 \mathcal{H}' \right], \]

\[ \frac{d(d + 1)}{2} H^2 = \kappa \left[ \frac{\varphi'^2}{2} - Va^2 \right] + \frac{\epsilon}{2a^2} d(d + 1)(d - 1)(d - 2)\mathcal{H}^4, \]

\[ \varphi'' + dH\varphi' - \frac{\partial V}{\partial \varphi} a^2 = 0. \]  

By combining Eqs. (2.13)–(2.17) we obtain

\[ \varphi'^2 = \frac{d}{\kappa} (\mathcal{H}^2 - \mathcal{H}') q(w), \]

\[ V + \frac{d}{2\kappa a^2} \left\{ (\mathcal{H}' + a\mathcal{H}^2) - \frac{\epsilon}{a^2} (d - 1)(d - 2) [(d - 1)\mathcal{H}^4 + 2\mathcal{H}^2 \mathcal{H}'] \right\} = 0, \]

where

\[ q(w) = \left[ 1 - \frac{2\epsilon}{a^2} (d - 1)(d - 2) \mathcal{H}^2 \right] \]

has been defined since it naturally appears in the analysis of the metric fluctuations.

The relation among the reduced Planck mass and the higher dimensional Planck mass is modified with respect to the case when quadratic curvature corrections are absent. In the interesting case when \( d = 3 \)

\[ M_P^2 \simeq M^3 \int_{0}^{\infty} a^3(w) \left[ 1 + \frac{4\epsilon}{a^2} (\mathcal{H}^2 + 2\mathcal{H}') \right] dw. \]

In the case when \( \epsilon = 0 \), Eq. (2.19) turns into the the well know relation connecting the four-dimensional Planck mass to the five-dimensional one.

### III. GAUGE-INVARIENT THEORY OF LINEARIZED FLUCTUATIONS

When EGB self-interactions are included in the action the theory of linearized fluctuations is more complicated than in the case where the bulk action only contains the Einstein-Hilbert term. The system of equations describing the fluctuations can be written, formally, as
\[ \delta R_A^B = \kappa \delta \tau_A^B - 2\alpha' \kappa \left[ \delta Q_A^B - \frac{1}{d} \delta Q \delta_A^B \right], \quad (3.1) \]

with

\[ \delta \tau_A^B = \partial_A \varphi \partial^B \chi + \partial_A \chi \partial^B \varphi - \frac{2}{d} \delta_A^B \frac{\partial V}{\partial \varphi} \chi, \quad (3.2) \]

and where \( \chi \) denotes the fluctuation of \( \varphi \). The fluctuations of the Lanczos tensor (and of its trace) appearing in Eq. (3.1) can be written as

\[ \delta Q_A^B = \left[ \frac{1}{2} \delta R^2_{EGB} \delta_A^B - 2R \delta R_A^B - 2\delta R^B A + 4\delta R_{AC} R^B_C + 4 \right] \delta R^C_B \]

\[ + \, 4\delta R_{CD} \overline{R}^{CBD}_A - 4R_{ACDE} \overline{R}^{BCDE} - 2R_{ACDE} \delta R^{BCDE} \], \quad (3.3) \]

\[ \delta Q = \left( \frac{d-2}{2} \right) \delta R^2_{EGB} \equiv (d-2) \left[ R \delta R - 2R_{MN} \delta R^{MN} - 2\delta R_{MN} \overline{R}^{MN} \right], \quad (3.4) \]

\[ + \, 1 \delta R_{MNAB} R^{MNAB} + \frac{1}{2} \delta R_{MNAB} \overline{R}^{MNAB} \], \quad (3.5) \]

\[ \delta R = R_{MN} \delta G^{MN} + \overline{G}_{MN} \delta R^{MN}, \quad (3.6) \]

where the symbol \( \delta \) indicates the fluctuations of the various tensors to first order in the amplitude of the metric fluctuations

\[ G_A^B(x^\mu, w) = G_A^B(w) + \delta G_A^B(x^\mu, w), \quad (3.7) \]

and where the over-line reminds that the corresponding quantities are evaluated on the background. Eqs. (3.1)–(3.6) should be supplemented with

\[ \delta G^{AB}(\partial_A \partial_B \varphi - \Gamma^C_{AB} \partial_C \varphi) + \overline{G}^{AB}(\partial_A \partial_B \chi - \Gamma^C_{AB} \partial_C \chi - \delta \Gamma^C_{AB} \partial_C \varphi) + \frac{\partial V}{\partial \varphi} \chi = 0, \quad (3.8) \]

which is the perturbed counterpart of Eq. (2.4).

The gauge-invariant theory of linearized fluctuations \[29,32\] can be generalized to the case when the gravity action is not in the Einstein-Hilbert form. Even if the metric fluctuations are not, by themselves, invariant under infinitesimal coordinate transformations, fully gauge-invariant equations can be obtained using a two-step procedure which will be now illustrated.

Eqs. (3.1)–(3.8) should be written in general terms without choosing a specific coordinate system but keeping all the \((d+2)(d+3)/3\) (15 for \(d = 3\)) degrees of freedom of the
metric. Then the variation can be written in fully gauge-invariant terms by selecting the appropriate variables invariant under infinitesimal coordinate transformations. This procedure is strongly reminiscent of what is done in the context of the Bardeen formalism [32] and of its generalization to the case of compact Kaluza-Klein dimensions [33,37].

The second step will be to write down, explicitly, Eqs. (3.1)–(3.8) in terms of the gauge-invariant fluctuations and to decouple the system. In order to complete this second step the background equations (2.13)–(2.17) will be used into the perturbed equations together with the constraints arising from the off-diagonal components of the perturbed equations.

A. Fully gauge invariant approach

Without assuming any specific gauge the fluctuations of the Ricci tensor can be obtained, after a tedious calculation:

\[
\delta R^w = \frac{(d+1)}{a^2} \left\{ \psi'' + H(\xi' + \psi') + 2H'\xi \\
- \partial_\alpha \partial^\alpha [(C - E')' + H(C - E') - \xi] \right\},
\]

\[
\delta R^\mu = \frac{1}{a^2} \left\{ d\partial_\mu (H\xi' + \psi') + \frac{1}{2} \partial_\alpha \partial^\alpha (D_\mu - f_\mu') \right\},
\]

\[
\delta R^\nu = \frac{1}{a^2} \left\{ h^\nu'' + dHh^\nu' - \partial_\alpha \partial^\nu h^\nu_{\mu} \\
+ \delta^\nu_\mu [\psi'' + (2d + 1)H\psi' - \partial_\alpha \partial^\alpha \psi + H\xi' + 2(H' + dH^2)\xi - H\partial_\alpha \partial^\alpha (C - E')] \\
+ \partial_\mu \partial^\nu [(E' - C)' + dH(E' - C) + \xi - (d - 1)p] \\
+ [(\partial_\mu f^\nu)'' + dH(\partial_\mu f^\nu)'] - [(\partial_\nu D^\mu)' + dH(\partial_\nu D^\mu)] \right\},
\]

where the various functions appearing in the fluctuations come from the perturbed form of the metric which has been taken as

\[
\delta G_{AB} = a^2(w) \left( 2h_{\mu\nu} + (\partial_\mu f_\nu + \partial_\nu f_\mu) + 2\eta_{\mu\nu} \psi + 2\partial_\alpha \partial_\beta E_{\mu} D_\nu + \partial_\mu C \right).
\]

In Eq. (3.12) \(h_{\mu\nu}\) is a divergence-less and trace-less rank-two tensor in the \((d+1)\)-dimensional Poincaré invariant space-time. The vectors \(f_\mu\) and \(D_\mu\) are both divergence-less. The scalar
fluctuations are parametrized by the four independent functions $\xi$, $\psi$, $C$ and $E$. The number of independent functions parameterizing the fluctuations of the metric is then $(d+2)(d+3)/2$.

Under an infinitesimal coordinate transformation of the form

$$x^A \rightarrow \tilde{x}^A = x^A + \epsilon^A,$$  \hspace{1cm} (3.13)

the fluctuations of the metric transform according to the usual expression involving the Lie derivative in the direction of the vector $\epsilon^A$

$$\delta \tilde{G}_{AB} = \delta G_{AB} - \nabla_A \epsilon_B - \nabla_B \epsilon_A,$$  \hspace{1cm} (3.14)

where $\epsilon_A = a^2(w)(\epsilon_\mu, -\epsilon_w)$ and where the gauge functions can be written as

$$\epsilon_\mu = \partial_\mu \epsilon + \zeta_\mu$$  \hspace{1cm} (3.15)

with $\partial_\mu \zeta^\mu = 0$.

As defined in Eq. (3.12), the tensor modes of the metric fluctuations, i.e. $h_{\mu\nu}$ are automatically invariant under the transformation (3.13), whereas the vectors and the scalars are not gauge-invariant. This is the source of the lack of gauge-invariance of Eqs. (3.9)–(3.11). However, since there are two scalar gauge functions [i.e. $\epsilon$ and $\epsilon_w$], and one vector gauge function [i.e. $\zeta_\mu$], two gauge-invariant scalar fluctuations and one gauge-invariant vector fluctuation can be constructed. The gauge-invariant scalars are

$$\Psi = \psi - \mathcal{H}(E' - C),$$

$$\Xi = \xi - \frac{1}{a}[a(C - E')]'$$

The gauge-invariant vector is

$$V_\mu = D_\mu - f'_\mu.$$  \hspace{1cm} (3.18)

The invariance of Eqs. (3.16)–(3.18) with respect to infinitesimal coordinate transformations can be verified by using the explicit form of Eq. (3.14), namely:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu},$$  \hspace{1cm} (3.19)
for the tensors and
\[
\tilde{f}_\mu = f_\mu - \zeta_\mu, \quad (3.20)
\]
\[
\tilde{D}_\mu = D_\mu - \zeta'_\mu. \quad (3.21)
\]
for the vectors. Since according to Eq. (3.14) the scalars transform as
\[
\tilde{E} = E - \epsilon, \quad (3.22)
\]
\[
\tilde{\psi} = \psi - \mathcal{H}\epsilon_w, \quad (3.23)
\]
\[
\tilde{C} = C - \epsilon' + \epsilon_w, \quad (3.24)
\]
\[
\tilde{\xi} = \xi + \mathcal{H}\epsilon_w + \epsilon'_w. \quad (3.25)
\]
the invariance of Eqs. (3.16)–(3.17) is immediately verified. Sometimes, in cosmological applications, the gauge-invariant fluctuations are called Bardeen potentials.

Using Eqs. (3.16)–(3.18) into Eqs. (3.9)–(3.11) the fluctuations of the Ricci tensor can be written as the sum of a fully gauge-invariant part and of a second term which will vanish using the equations of the background:

\[
\delta R^w_w = \delta^{(g)} R^w_w - [\mathcal{R}^w_w]'(C - E'), \quad (3.26)
\]
\[
\delta R^w_\mu = \delta^{(g)} R^w_\mu - \mathcal{R}^w_\mu\partial_\mu(C - E') + \mathcal{R}^\nu_\mu\partial_\nu(C - E'), \quad (3.27)
\]
\[
\delta R^\nu_\mu = \delta^{(g)} R^\nu_\mu - [\mathcal{R}^\nu_\mu]'(C - E'), \quad (3.28)
\]
where \(\delta^{(g)}\) denotes a variation which preserves gauge-invariance and where

\[
\delta^{(g)} R^w_w = \frac{1}{a^2} \left\{ (d + 1)[\Psi'' + \mathcal{H}(\Psi' + \Xi') + 2\mathcal{H}'\Xi] + \partial_\alpha \partial^\alpha \Xi \right\}, \quad (3.29)
\]
\[
\delta^{(g)} R^w_\mu = \frac{1}{a^2} \left\{ d\partial_\mu[\mathcal{H}\Xi + \Psi'] + \frac{1}{2} \partial_\alpha \partial^\alpha V_\mu \right\}, \quad (3.30)
\]
\[
\delta^{(g)} R^\nu_\mu = \frac{1}{a^2} \left\{ \delta^\nu_\mu[\Psi'' + (2d + 1)\mathcal{H}\Psi' - \partial_\alpha \partial^\alpha \Psi + \mathcal{H}\Xi' + 2(\mathcal{H}' + d\mathcal{H}^2)\Xi]
+ \partial_\mu \partial^\nu[\Xi - (d - 1)\Psi] - [(\partial_\mu V^\nu)'] + d\mathcal{H}(\partial_\mu V^\nu)] + h^\nu_\mu'' + d\mathcal{H} h^\nu_\mu' - \partial_\alpha \partial^\alpha h^\nu_\mu \right\}. \quad (3.31)
\]
The same procedure outlined in the case of the Ricci tensors should be repeated for all the tensors appearing in Eqs. (3.1)–(3.8). For sake of simplicity, in view of the rather
heavy algebraic expressions, the tensor, vector and scalar modes will be separately dis-
cussed. Moreover, the relevant technical results will be collected in the various sections of
the Appendix.

Under infinitesimal coordinate transformations also the scalar field fluctuation $\chi$ changes
as
$$\tilde{\chi} = \chi - \varphi' \epsilon_w,$$  \hspace{1cm} (3.32)
and the corresponding gauge-invariant variable is
$$X = \chi - \varphi'(E' - C).$$  \hspace{1cm} (3.33)
Also for the fluctuations of the brane source the same procedure outlined for the Ricci tensors
should be performed so that fully gauge-invariant fluctuations can be obtained.

IV. LOCALIZATION OF TENSOR MODES

Taking the contracted form of Einstein equations into account, Eqs. (3.1)–(3.8) per-
turbed to first order in the amplitude of (gauge-invariant) tensor fluctuations $h_\nu^{\mu}$ can be
written as
\begin{equation}
\begin{align}
\left[1 - 2\epsilon R\right] \delta R_\nu^{\mu} + 2\epsilon \left\{2[\delta R_{\alpha\beta} R_\mu^{\alpha\beta} + R_{\alpha\beta} \delta R_\mu^{\alpha\beta} + R_{ww} \delta R_\mu^{ww}] \right. \\
- \left[\delta R_{\mu CDE} R^{CDE}_{\nu} + R_{\mu CDE} \delta R^{CDE}_{\nu}\right] + 2[\delta R_{\mu\alpha} R^{\alpha}_{\nu} + R_{\mu\alpha} \delta R^{\alpha}_{\nu}] \right\} = 0.
\end{align}
\end{equation}
(4.1)
where the symmetries of the background (i.e. $R_{\mu\nu} = 0$, etc.) have been used, when possible,
in order to simplify the contractions. Recalling now the explicit form of the background
tensors reported in Eqs. (A.3)–(A.6) and their perturbed version reported in Eqs. (B.1)–
(B.3) of Appendix B the following equation can be obtained
\begin{equation}
\begin{align}
\left\{1 - \frac{2\epsilon}{a^2} H^2(d - 1)(d - 2)\right\} h_\nu^{\mu} \\
+ \left\{dH - \frac{2\epsilon H}{a^2}(d - 1)(d - 2)[2H' + (d - 2)H^2] h_\nu^{\mu}\right\} h_\nu^{\mu} \\
- \left\{1 - \frac{2\epsilon}{a^2}(d - 2) \left[2H' + (d - 3)H^2\right]\right\} \partial_{\alpha} \partial_{\nu} h_\mu^{\nu} = 0.
\end{align}
\end{equation}
(4.2)
The different contributions appearing in Eq. (4.1) and leading to Eq. (4.2) are listed in Eqs. (B.4)–(B.6) of Appendix B. Using now the explicit definition of \( q(w) \) reported in Eq. (2.18), Eq. (4.2) becomes

\[
qh''_{\mu} + (dHq + q')h'_{\mu} - \left[ q + \frac{q'}{(d-1)\mathcal{H}} \right] \partial_{\alpha}\partial^{\alpha}h_{\mu} = 0, \tag{4.3}
\]

where the relation

\[
q' = \frac{4\epsilon}{a^2}(d-1)(d-2)\mathcal{H}(\mathcal{H}^2 - \mathcal{H}'). \tag{4.4}
\]

By eliminating first derivative, Eq. (4.3) can be finally put in a Schrödinger-like form, namely

\[
\mu'' - \frac{(\sqrt{s})'}{\sqrt{s}}\mu - \frac{r}{s}\partial_{\alpha}\partial^{\alpha}\mu = 0. \tag{4.5}
\]

where, for a generic tensor polarization,

\[
\mu(w, x^\mu) = \sqrt{s(w)}h(x^\mu, w), \tag{4.6}
\]

and where

\[
s(w) = a^d\left[ 1 - 2\epsilon(d-1)(d-2)\frac{\mathcal{H}^2}{a^2} \right] \equiv a^d q,
\]

\[
r(w) = a^d\left\{ 1 - 2\frac{\epsilon(d-2)}{a^2}\left[ 2\mathcal{H}' + (d-3)\mathcal{H}^2 \right] \right\} \equiv a^d\left( q + \frac{q'}{(d-1)\mathcal{H}} \right). \tag{4.7}
\]

Not surprisingly the same equation can be directly obtained by perturbing the action (2.11) to second order in the amplitude of the tensor modes with the result that, up to total derivatives,

\[
\delta^{(2)}S = \frac{1}{2} \int d^{d+2}x \left[ -s(w)h'^2 + r(w)\partial_{\alpha}h\partial^{\alpha}h \right], \tag{4.8}
\]

where \( h \) represents, as usual, a generic tensor polarization.

By using the Euler-Lagrange equations, Eq. (4.2) can be obtained. In terms of \( \mu \) the action of Eq. (4.8) can be written as

\[
\delta^{(2)}S = \frac{1}{2} \int d^{d+2}x \left[ -\mu'^2 - \frac{(\sqrt{s})''}{\sqrt{s}}\mu^2 + \frac{r}{s}\partial_{\alpha}\mu\partial^{\alpha}\mu \right]. \tag{4.9}
\]
By taking the functional derivative of the above action with respect to $\mu$ the related equations of motion is, as expected, Eq. (4.5).

The obtained result has been derived without specifying the background geometry but only assuming the form of the metric, i.e. Eq. (2.9). Equations (4.2)–(4.5) hold then in general for the theory described by the action (2.1). From Eq. (4.5), the equation for the mass eigenstates is

$$-\frac{d^2 \mu_m}{dw^2} + V(w)\mu_m = m^2 r s \mu_m,$$

where

$$V(w) = \mathcal{L}^2 - \mathcal{L}', \quad \mathcal{L} = -\frac{(\sqrt{s})'}{\sqrt{s}}. \quad (4.11)$$

In a context of supersymmetric quantum mechanics $\mathcal{L}$ is the superpotential \[38\]. In terms of the superpotential Eq. (4.10) can be written in terms of two first order differential operators.

From this form of the equation for the mass eigenstates, it follows that if the lowest mass eigenstate is normalizable the spectrum does not contain tachyonic modes.

The lowest mass eigenstate related to Eq. (4.2) is given by

$$\mu_0(w) \simeq \sqrt{s(w)} = a^{d/2} \sqrt{1 - \frac{2\epsilon}{a^2} (d - 2)(d - 1)}, \quad (4.12)$$

and the corresponding normalization integral can now be written, assuming the background is invariant under $w \to -w$ symmetry

$$2 \int_0^\infty a^d(q(w)) \, dw = 2 \int_0^\infty a^d(w) \left[ 1 - \frac{2\epsilon}{a^2} (d - 2)(d - 1) \right] \, dw, \quad (4.13)$$

where the first equality follows from the definition of Eq. (2.18). Concerning eq. (4.13) few remarks are in order. Consider for sake of concreteness the case $d = 3$. It will now be shown that if the four-dimensional Planck mass is finite and if the background geometry is regular everywhere, then the tensor zero mode will always be localized.

From Eq. (2.19) the four-dimensional Planck mass can be written as

$$M_P^2 = M^3 \int_0^\infty a^3 \left[ q + \frac{8\epsilon}{a^2} (\mathcal{H}' + \mathcal{H}^2) \right] \, dw. \quad (4.14)$$
Using now the fact that \( a(H^2 + H') = a'' \), Eq. (4.14) can be further modified

\[
M_P^2 = M^3 \left\{ \int_0^\infty a^3 q \, dw \right\} + 8\epsilon M^3 [a']^3_0 \tag{4.15}
\]

The second term in Eq. (4.13) should be finite if the geometry is regular everywhere. The first term is exactly proportional to the integral appearing in the normalization condition of the tensor zero mode of Eq. (4.13). Hence, if the four-dimensional Planck mass is finite, the tensor zero mode is localized.

Notice that, if \( q(w) \) would diverge at some value of \( w \), then a singularity should be expected in the background. Suppose, in fact, that \( q(w) \) diverges at some \( w \). If this is the case, from Eq. (2.16) it is easy to show that in order to have \( \varphi' \) regular, \( (H' - H^2) \) should go to zero. But \( H' \sim H^2 \) implies that \( H \sim (w_0 - w)^{-1} \), which produces, in its turn, a singularity in the curvature invariants (A.3)–(A.5). Thus, if \( q(w) \) at some value of \( w \) either \( \varphi'(w) \) diverges or the curvature invariant diverge, or both.

V. LOCALIZATION OF VECTOR AND SCALAR MODES

A. The vector modes

The evolution of the gauge-invariant vector fluctuation \( V_{\mu} \) can be obtained, after some algebra, from the \((\mu, \nu)\) and \((\mu, w)\) component of Eq. (3.1). The detailed results for the fluctuations of the Riemann and Lanczos tensor perturbed in the amplitude of the gauge-invariant vector modes of the geometry are reported in Appendix C. From the \((\mu, \nu)\) and \((\mu, w)\) components of Eq. (3.1) the resulting equations are, respectively,

\[
q(w)[\partial^\nu V_\mu]' + \left\{ dH - \frac{2\epsilon}{a^2}(d-1)(d-2)[2HH' + (d-2)H^2] \right\} V_\mu = 0, \tag{5.1}
\]

\[
-\partial_\alpha \partial^\alpha V_\mu = 0. \tag{5.2}
\]

Eq. (5.1) is obtained from Eqs. (C.4)–(C.6) whereas Eq. (5.2) can be derived from Eqs. (C.7)–(C.9). Both sets of equations are collected in Appendix C. Eq. (5.2) implies that \( V_\mu \) is massless whereas Eq. (5.1) allows to determine the evolution of the zero mode. Recalling
the definition of the background function $q(w)$ given in Eq. (2.18), Eq. (5.2) can be written as

$$\mathcal{V}_\mu' + \left( \frac{d}{2} \mathcal{H} + \frac{q'}{2q} \right) \mathcal{V}_\mu = 0,$$

(5.3)

where $\mathcal{V}_\mu = a^{d/2} \sqrt{q} V_\mu$ is the canonical normal mode of the vector action perturbed to second order in the amplitude of the amplitude of the metric fluctuations, namely

$$\delta^{(2)} S_V = \int d^{d+1} w \frac{1}{2} \left[ \eta^{\alpha\beta} \partial_\alpha \mathcal{V}_\mu \partial_\beta \mathcal{V}_\mu \right].$$

(5.4)

For each vector polarization, the evolution of the zero mode is given by:

$$V_0 \simeq \frac{1}{a^{d/2} \sqrt{q}}.$$  

(5.5)

The corresponding normalization integral will then be

$$2 \int_0^\infty \frac{dw}{a^d(w)q(w)}.$$  

(5.6)

By comparing Eq. (5.6) with Eq. (4.13) it can be immediately appreciated that the integrands appearing in the two expressions are one the inverse of the other. If the tensor modes are localized, then the vector modes will not be localized.

**B. The scalar modes**

Using the definition of $q(w)$ the evolution equations for the gauge-invariant scalar fluctuations of the geometry can be written from the explicit expressions of Eqs. (3.1)–(3.8). The details are reported in Appendix C. For the $(w,w)$ component the results is

$$q \partial_\alpha \partial^\alpha \Xi + (d + 1)q \Psi'' + (d + 1) \mathcal{H} \left[ 1 - \frac{2\epsilon}{a^2} (d - 1)(d - 2) (2 \mathcal{H}' - \mathcal{H}^2) \right] + (d + 1) \mathcal{H} q \Xi' + 2(d + 1) \Xi \left\{ \mathcal{H}' + \frac{\varphi^2}{2} - \frac{2\epsilon}{a^2} (d - 1)(d - 2) \left[ 2 \mathcal{H}' \mathcal{H}^2 - \mathcal{H}^4 \right] \right\} + \varphi' X' + \frac{1}{d \varphi} \frac{\partial V}{\partial \varphi} a^2 X = 0,$$

(5.7)

\(^2\)For sake of simplicity, in the following part and in the related Appendix D natural gravitational units $2\kappa = 1$ will be used.
whereas the \((\mu, \nu)\) component leads to

\[
q\Psi'' + \mathcal{H}\Psi' \left\{ (2d + 1) - \frac{2\epsilon}{a^2} (d - 1)(d - 2) [2\mathcal{H}' + (2d - 1)\mathcal{H}^2] \right\} + q\mathcal{H}\Xi' - q\partial_\alpha \partial^\alpha \Psi + 2\Xi \left\{ (\mathcal{H}' + d\mathcal{H}^2)^2 - \frac{2\epsilon}{a^2} (d - 1)(d - 2) [(d - 1)\mathcal{H}^4 + 2\mathcal{H}'\mathcal{H}^2] \right\} + \frac{1}{d} \frac{\partial V}{\partial \phi} a^2 X = 0, \tag{5.8}
\]

if \(\mu = \nu\), and to

\[
q\Xi - (d - 1)\Psi \left\{ 1 - \frac{2\epsilon}{a^2} (d - 2) [2\mathcal{H}' + (d - 3)\mathcal{H}^2] \right\} = 0. \tag{5.9}
\]

if \(\mu \neq \nu\). Finally, the \((\mu, w)\) component of Eq. (3.1) produces the constraint

\[
dq \left( \Psi' + \mathcal{H}\Xi \right) + \frac{1}{2} \phi' X = 0. \tag{5.10}
\]

The perturbed scalar field equation becomes

\[
X'' + d\mathcal{H}X' - \partial_\alpha \partial^\alpha X - \frac{\partial^2 V}{\partial \phi^2} a^2 X + \phi' [(d + 1)\Psi' + \Xi'] + 2(\phi'' + d\mathcal{H}\phi')\Xi = 0. \tag{5.11}
\]

Subtracting now Eq. (5.8) from Eq. (5.7)

\[
q\partial_\alpha \partial^\alpha \left( \Psi + \Xi \right) + dq \left[ \Psi'' + \mathcal{H} \left( \Xi' - \Psi' \right) \right] + dq' \left( \Psi' + \mathcal{H}\Xi \right) + X'\phi' = 0. \tag{5.12}
\]

Using now Eq. (5.9) in order to eliminate \(\Xi\) from the constraint, Eq. (5.10) can be written as

\[
X = -\frac{2qd}{\phi'} \left\{ \Psi' + \Psi \left[ (d - 1)\mathcal{H} + \frac{q'}{q} \right] \right\}. \tag{5.13}
\]

Inserting Eqs. (5.9)–(5.13) into Eq. (5.12) a decoupled equation for the Bardeen potential is obtained

\[
\Psi'' - \left[ 1 + \frac{q'}{\mathcal{H}q} \right] \partial_\alpha \partial^\alpha \Psi + \Psi' \left[ d\mathcal{H} + \frac{2q'}{q} - 2\phi'' \right] + \Psi \left[ \frac{q''}{q} + \frac{q'\mathcal{H}'}{q} \mathcal{H} + (d - 1)\mathcal{H} q' \right] - 2(d - 1)\mathcal{H} \frac{\phi''}{\phi'} - 2\frac{\phi''}{\phi'} \frac{q'}{q} + 2(d - 1)\mathcal{H}' = 0. \tag{5.14}
\]

Rescaling now \(\Psi\) according to

\[
\Phi = \frac{a^{d/2} q}{\phi'} \Psi, \tag{5.15}
\]
the following equation can be obtained from Eq. (5.13)

$$\Phi'' - \left(\frac{1}{z}\right)'' \Phi - \left(1 + \frac{q'}{Hq}\right) \partial_\alpha \partial^\alpha \Phi = 0,$$

where

$$z(w) = \frac{a^{d/2} \varphi'}{H}.$$  \hspace{1cm} (5.17)

In order to get to eq. (5.16) the following background relation [obtained by deriving Eq. (2.16)] has been used

$$\frac{\mathcal{H}''}{\mathcal{H}} = 2 \mathcal{H}' - \left(2 \varphi'' - \frac{q'}{q}\right) \left[\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right].$$  \hspace{1cm} (5.18)

By virtue of the constraint (5.10) the equation obeyed by $\Phi$ is also obeyed by the appropriately rescaled $\Xi$ variable.

The normalization condition for the lowest mass eigenstate related to Eq. (5.16) can be now written as

$$2 \int_0^\infty dw \left|\frac{\varphi'}{\mathcal{H}}\right|^2 \equiv \int_0^\infty dw \frac{\mathcal{H}^2}{a^d \varphi'^2}. $$  \hspace{1cm} (5.19)

It will now be shown that this integral cannot be convergent. Consider the case $d = 3$.

The integrand appearing in Eq. (5.19) can be rearranged by using Eq. (2.16) which should always hold since it is a background relation. Therefore,

$$\int_0^\infty dw \frac{\mathcal{H}^2}{a^d \varphi'^2} = 6 \int_0^\infty dw \frac{\mathcal{H}^2}{a^d q} \left(\frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}'}\right).$$  \hspace{1cm} (5.20)

The expression appearing in the denominator is $a^{-3} q^{-1}$. This is exactly the inverse of the integrand arising in the definition of the Planck mass. Now, in Eqs. (4.14) and (4.15) it has been shown that in order to have a finite four-dimensional Planck mass $a^3 q$ should be convergent everywhere and, in particular, at infinity. Therefore $a^{-3} q^{-1}$ will be strongly divergent. In the same limit, if the curvature invariant are regular at infinity, $\mathcal{H}^2/(\mathcal{H}^2 - \mathcal{H}')$ cannot converge fast enough to make the whole integral convergent.
VI. PHYSICAL EXAMPLES

In the previous sections various conditions on the localization of the zero modes of the geometry have been obtained without specifying the background solution. It is interesting to apply them to some specific case of (analytical) thick brane solution with EGB corrections.

In order to solve Eqs. (2.13)–(2.15) a useful approach is to fix the geometry. Then, from Eq. (2.16), the relation \( \varphi(w) \) can be obtained after one integration. Then, by inverting this relation, \( w = w(\varphi) \) can be inserted in Eq. (2.17) and \( V(\varphi) \) determined. Following this approach it can be obtained, for instance,

\[
a(x) = \frac{a_0}{\sqrt{x^2 + 1}},
\]

(6.1)

\[
\varphi(x) = \varphi_0 \frac{x}{\sqrt{x^2 + 1}} + \varphi_1,
\]

(6.2)

\[
V(\varphi) = V_0 \left[ \left( \frac{d+3}{2} \right) \left( \frac{\varphi - \varphi_1}{\varphi_0} \right)^4 - (d+3) \left( \frac{\varphi - \varphi_1}{\varphi_0} \right)^2 + 1 \right],
\]

(6.3)

where

\[
a_0 = \sqrt{2 \epsilon (d-1) (d-2)} b,
\]

(6.4)

\[
\varphi_0 = \sqrt{\frac{d}{\kappa}}, \quad V_0 = \frac{d}{4 \kappa \epsilon (d-1) (d-2)},
\]

(6.5)

and \( \varphi_1 \) is an integration constant. In Eqs. (6.1)–(6.16), \( x = bw \) is the bulk coordinate rescaled through the brane thickness. In this dimension-less coordinate the brane core is located for \( |x| \leq 1 \). Far from the core, i.e. \( |x| \gg 1 \), the limit of the solution is \( AdS_{d+2} \) space, in fact

\[
\lim_{|x| \to \infty} a(x) = \frac{A}{w}
\]

(6.6)

where \( A \) is the “radius” of \( AdS \) space. For instance, in the \( d = 3 \) case, \( A = 2\sqrt{2\alpha' \kappa} \) can be identified with the radius of the \( AdS_5 \) space. Similar solutions have been discussed in [31].

Over these solutions the curvature invariants and the EGB combination (appearing in the quadratic part of the action) are all regular for any value of the bulk coordinate. In fact,
Eqs. (A.3)–(A.5) in the specific background provided by Eqs. (5.1)–(5.4), the explicit form of the curvature invariants can be obtained:

\[ R^2 = \frac{(d+1)^2 [(d+2) x^2 - 2]^2}{4 (d-2)^2 (d-1)^2 c^2 (x^2 + 1)^2}, \]  
\[ R^{AB} R_{AB} = \frac{(d+1) \left[ d^2 x^4 + 2 (x^2 - 1)^2 + d (3 x^2 - 1) (x^2 - 1) \right]}{4 (d-2)^2 (d-1)^2 c^2 (x^2 + 1)^2}, \]  
\[ R^{ABCD} R_{ABCD} = \frac{(d+1) \left[ 2 - 4 x^2 + (2 + d) x^4 \right]}{2 (d-2)^2 (d-1)^2 c^2 (x^2 + 1)^2}. \]  

All the curvature invariants go to constant for large \(|x|\) and they have two minima and a maximum around the origin.

Let us now focus our attention to the case \(d = 3\). In this case the integral appearing in the definition of the Planck mass [see Eq. (2.19)] is finite and the integrand always convergent, as it can be directly checked using Eq. (6.1) into Eq. (2.19).

As argued in the Section IV the tensor zero mode will also be normalizable. In fact, for the background of Eqs. (6.1)–(2.17) the normalization condition of eq. (4.13) reads

\[ 2 \int_0^\infty \frac{dw}{a^3(w)q(w)} \equiv 16 b^3 c^{3/2} \int_0^\infty \frac{dx}{(1 + x^2)^{5/2}}, \]  

which is clearly convergent. Consider now the vector modes discussed in Section V. On the basis of the results of Eq. (5.6) the vector zero mode is not normalizable. In fact, from Eq. (5.6) the normalization integral can be reduced to

\[ \int_0^\infty (1 + x^2)^{5/2} dx \]  

which is not convergent at infinity.

Consider finally the scalars discussed in Section V. From the general condition derived in Eq. (5.19), the normalization integral reduces to

\[ 2 \int_0^\infty \frac{dw}{|z(w)|^2} = \frac{1}{4 b^3 c^{3/2} \phi_0^2} \int_0^\infty x^2 (1 + x^2)^{5/2}, \]  

which is divergent at infinity.
VII. CONCLUDING REMARKS

A gauge-invariant framework for the analysis of the fluctuations of the various modes of a warped geometry has been developed in the case when the gravity action contains quadratic curvature corrections parametrized in the Gauss-Bonnet form.

General conditions for the localization of the tensor, vector and scalar fluctuations of the metric have been derived. If physical domain-wall solutions are described using a bulk action containing the Gauss-Bonnet combination the evolution equations of the fluctuations can be reduced to a set of decoupled second order (Schrödinger-like) differential equations.

The lowest mass eigenstate of the tensor fluctuations is always localized provided the four-dimensional Planck mass is finite. On the contrary, under the same assumption, the scalar and vector zero modes are not localized. The only other assumptions used in the analysis are that the geometry is fully regular and that the background is symmetric for $w \to -w$. Specific examples of thick domain-wall solutions induced by Gauss-Bonnet self-interactions have been also studied providing, in this way, concrete examples of the general features illustrated in the analysis of the fluctuations.

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APPENDIX A: EXPLICIT FORM OF QUADRATIC INVARIANTS

Useful background relations will be now collected. From the explicit form of the Riemann tensors
\[ R_{\mu w}^{\nu} = \mathcal{H}' \eta_{\mu \nu}, \]
\[ R_{\alpha \nu \beta}^{\mu} = [\delta_{\nu}^{\mu} \eta_{\alpha \beta} - \delta_{\beta}^{\mu} \eta_{\alpha \nu}] \mathcal{H}^2, \] (A.1)
the corresponding explicit form of the Ricci tensors
\[ R_{\mu \nu} = (\mathcal{H}' + d \mathcal{H}^2) \eta_{\mu \nu}, \]
\[ R_{w w} = -(d + 1) \mathcal{H}', \] (A.2)
leads to the explicit expression of the curvature invariants
\[ R^2 = \frac{(d + 1)^2}{a^4} \left[ 4 \mathcal{H}'^2 + d^2 \mathcal{H}^4 + 4d \mathcal{H}' \mathcal{H}^2 \right], \] (A.3)
\[ \overline{R}_{AB}^{A} \overline{R}_{AB}^{B} = \frac{(d + 1)}{a^4} \left[ (d + 2) \mathcal{H}'^2 + d^2 \mathcal{H}^4 + 2d \mathcal{H}' \mathcal{H}^2 \right], \] (A.4)
\[ \overline{R}_{ABCD}^{A} \overline{R}_{ABCD}^{B} = \frac{(d + 1)}{a^4} \left[ 4 \mathcal{H}'^2 + 2d \mathcal{H}' \right], \] (A.5)
whereas the explicit form of the EGB combination is
\[ R^2_{\text{EGB}} = \frac{d(d + 1)}{a^4} \left[ (d - 1)(d - 2) \mathcal{H}^4 + 4(d - 1) \mathcal{H}' \mathcal{H}^2 \right]. \] (A.6)

The covariant components of the Lanczos tensor are instead:
\[ \overline{Q}_{\mu \nu} = \frac{d(d - 1)(d - 2)}{2 a^2} \{ (d - 3) \mathcal{H}^4 + 2 \mathcal{H}'^2 \} \eta_{\mu \nu}, \] (A.7)
\[ \overline{Q}_{w w} = - \frac{d(d + 1)(d - 1)(d - 2)}{2a^2} \mathcal{H}^4. \] (A.8)

APPENDIX B: THE TENSOR PROBLEM

In the present Appendix the perturbed form of the various quantities entering Eqs. (3.1)–(3.8) will be reported in the case of the tensor modes of the geometry. The only non-vanishing fluctuations of the Riemann and Ricci tensors can be written, in this case, as
Inserting the terms reported in Eqs. (B.4)–(B.6) into Eq. (4.1) the explicit form of the vector perturbations of the metric will now be reported. In particular, for the amplitude of the vector fluctuations of the metric can be obtained. Using Eqs. (C.1)–(C.3), the explicit form of Eq. (3.1) perturbed to first order in the amplitude of the vector modes of the geometry can be obtained and it is given in Eq. (4.2).

**APPENDIX C: THE VECTOR PROBLEM**

The gauge-invariant fluctuations of the Riemann tensors to first order in the amplitude of the vector perturbations of the metric will now be reported.

\[
\delta^{(g)} R^\nu_{\mu
u w w} = - (h'' + \mathcal{H} h''),
\]

\[
\delta^{(g)} R^\alpha_{\mu
u} = \partial_\beta [ - \partial^\alpha h_{\mu\nu} + \partial_\mu h^\alpha_{\nu} + \partial_\nu h^\alpha_{\mu} - \partial_\nu [ - \partial^\alpha h_{\mu\beta \nu} + \partial_\beta h^\alpha_{\mu} ] + \mathcal{H} [ \eta_{\mu\nu} h^\alpha_{\beta} - \eta_{\mu\beta} h^\alpha_{\nu} ] + \mathcal{H} [ \delta^\alpha_{\beta} (h''_{\mu\nu} + 2\mathcal{H} h_{\mu\nu}) - \delta^\alpha_{\beta} (h''_{\mu\beta} + 2\mathcal{H} h_{\mu\beta}) ],
\]

\[
\delta^{(g)} R^\nu_{\mu} = \frac{1}{a^2} (h''_{\mu\nu} + \partial_\nu h''_{\mu} - \partial_\alpha \partial^\alpha h''_{\mu}).
\]

Hence the different contributions appearing in Eq. (4.1) can be written in an explicit form and they are

\[
[1 - 2\varepsilon \overline{R}] \delta^{(g)} R^\nu_{\mu} = \frac{1}{a^2} \left( 1 - \frac{2\varepsilon}{a^2} \left[ 2\mathcal{H}' (d - 1) + d(d - 3) \mathcal{H}^2 \right] \right) (h''_{\mu\nu} + \partial_\nu h''_{\mu} - \partial_\alpha \partial^\alpha h''_{\mu}),
\]

\[
4\varepsilon [ \delta^{(g)} R_{\alpha\beta} \overline{R}^{\alpha\beta} + \overline{R}_{\alpha\beta} \delta^{(g)} R_{\mu}^{\alpha\beta} + 2\overline{R}_{\mu \nu w w} \delta^{(g)} R_{\mu \nu} - \delta^{(g)} R_{\mu \nu \nu w w} - \delta^{(g)} R_{\mu \nu \nu w w} \overline{R}^{\nu w w} ] = \frac{4\varepsilon}{a^4} \left( h''_{\mu\nu} [(d + 1) \mathcal{H}' - \mathcal{H}^2] + h''_{\mu\nu} [2d \mathcal{H} \mathcal{H}' + d(d - 2) \mathcal{H}^2] - [\mathcal{H}' + (d - 1) \mathcal{H}^2] \partial_\alpha \partial^\alpha h''_{\mu} \right)
\]

\[
2\varepsilon [ 2\delta^{(g)} R_{\alpha\beta} \overline{R}^{\alpha\beta} + 2\overline{R}_{\alpha\beta} \delta^{(g)} R_{\mu}^{\alpha\beta} + 2\overline{R}_{\mu \nu w w} \delta^{(g)} R_{\mu \nu} - \delta^{(g)} R_{\mu \nu \nu w w} - \delta^{(g)} R_{\mu \nu \nu w w} \overline{R}^{\nu w w} ] = - \frac{8\varepsilon}{a^4} \left( \mathcal{H}' h''_{\mu\nu} + [\mathcal{H} \mathcal{H}' + (d - 1) \mathcal{H}^2] h''_{\mu\nu} - \mathcal{H}^2 \partial_\alpha \partial^\alpha h''_{\mu} \right)
\]

Inserting the terms reported in Eqs. (B.4)–(B.6) into Eq. (4.1) the explicit form of the evolution of the tensor modes of the geometry can be obtained and it is given in Eq. (4.2).
For the \((\mu, w)\) component we have

\[
[1 - 2\varepsilon R] \delta^{(g)} R^w_{\mu w} = \frac{2}{a^2} \partial_{\alpha} \partial^{\alpha} V_{\mu} \left[ 1 - \frac{2\varepsilon}{a^2} [2(d + 1) \mathcal{H}' + d(d + 1) \mathcal{H}^2] \mathcal{H}^2 \right], \quad (C.7)
\]

\[
4\varepsilon [\delta^{(g)} R^{w\alpha}_{\mu} R^{\alpha w} + \delta^{(g)} R^{aw}_{\mu}] = \frac{4\varepsilon}{a^4} \partial_{\alpha} \partial^{\alpha} V_{\mu} [d \mathcal{H}^2 + (d + 2) \mathcal{H}], \quad (C.8)
\]

\[
2\varepsilon [2\delta^{(g)} R_{CD} R^{wD}_{\mu} + 2\delta^{(g)} R_{\mu} - \delta^{(g)} R_{\mu CDE} R^{CDE}] = \frac{2\varepsilon}{a^4} [(d - 1) \mathcal{H}^2 - \mathcal{H}'] \partial_{\alpha} \partial^{\alpha} V_{\mu}. \quad (C.9)
\]

Using Eqs. (C.4)–(C.6), Eq. (5.1) can be derived. Similarly, Eqs. (C.7)–(C.9) give Eq. (5.2).

### APPENDIX D: THE SCALAR PROBLEM

The gauge-invariant fluctuations of the Riemann tensors to first order in the scalar fluctuations of the metric

\[
\delta^{(g)} R^w_{\mu w} = \eta_{\mu w} \left[ \Psi'' + \mathcal{H}(\Psi' + \Xi') + 2\mathcal{H}'(\Psi + \Xi) \right], \quad (D.1)
\]

\[
\delta^{(g)} R^\mu_{\alpha \beta w} = [\delta^\beta_{\mu} \partial_{\alpha} \Psi' - \eta_{\alpha \beta} \partial^{\nu} \Psi'] - \mathcal{H} [\partial^{\nu} \Xi \eta_{\alpha \beta} - \delta^\mu_{\beta} \partial_{\alpha} \Xi], \quad (D.2)
\]

\[
\delta^{(g)} R^\alpha_{\mu \beta \nu} = \delta^\alpha_{\nu} \partial_{\mu} \partial_{\beta} \Psi - \delta^\alpha_{\beta} \partial_{\mu} \partial_{\nu} \Psi - \eta_{\mu \nu} \partial_{\beta} \partial^{\alpha} \Psi + \eta_{\mu \beta} \partial_{\nu} \partial^{\alpha} \Psi
+ [2\mathcal{H} \Psi' + 2\mathcal{H}^2 (\Psi + \Xi)] \eta_{\mu \nu} \delta^\alpha_{\beta} - \eta_{\alpha \beta} \delta^\nu_{\nu}], \quad (D.3)
\]

\(^3\)Notice that, in the following equations, \(\partial_{\mu} V_{\nu} = \frac{1}{2}(\partial_{\mu} V_{\nu} + \partial_{\nu} V_{\mu})\)
allow to compute the various components of the perturbed form of Eqs. (3.1). The \((\mu, w)\)
component of Eq. (3.1) can be obtained from the following expression

\[
\delta^{(gi)} R^{w}_{\mu} - \frac{1}{2} \partial_{\mu} \chi \partial^{w} \varphi = \frac{1}{a^2} \left[ \partial_{\mu} \left( d\Psi' + d\mathcal{H} \Xi - \frac{1}{2} \varphi' \chi \right) \right],
\]  

(D.4)

\[
\epsilon \left[ -2 R^{(gi)} R^{w}_{\mu} + 4 \delta^{(gi)} R^{\mu w} \right] = \frac{2 \epsilon}{a^2} [2 \mathcal{H}' - 2 d^2 \mathcal{H}^2] \partial_{\mu} [d\Psi' + d\mathcal{H} \Xi],
\]  

(D.5)

\[
4 \epsilon \left[ \delta^{(gi)} R^{w}_{\alpha \beta} \right] = \frac{4 \epsilon}{a^2} \left[ H' - H^2 \right] \partial_{\mu} \left[ d\Psi' + d\mathcal{H} \Xi \right].
\]  

(D.6)

The EGB combination appearing in Eq. (3.1) does not contribute to the off-diagonal components of the perturbed equations but it does contribute to the diagonal components. From the fluctuation of the EGB combination

\[
\mathcal{K} = \epsilon \delta^{(gi)} R_{EGB}^2 \equiv \frac{4 \epsilon}{a^2} \left\{ (d - 1) \mathcal{H}^2 \partial_{\alpha} \partial^{\alpha} \Xi - (d - 1) [\mathcal{H}' + (d - 2) \mathcal{H}^2] \partial_{\alpha} \partial^{\alpha} \Psi \right. \\
+ (d - 1) (d + 1) \mathcal{H}^2 \varphi' + (d - 1) (d + 1) \mathcal{H} \partial^{\alpha} \Xi' + (d - 1) (d + 1) \mathcal{H} \varphi' \Xi' + (d - 1) (d + 1) \mathcal{H} \varphi' \Xi \right\},
\]  

(D.8)

the \((w, w)\) and \((\mu, \nu)\) components of Eq. (3.1) can be obtained. In fact, defining

\[
\mathcal{M}_{A}^{B} = \delta^{(gi)} R_{A}^{B} - \frac{1}{2} \partial_{A} \varphi \partial^{B} X + \frac{1}{d} \frac{\partial V}{\partial \varphi} X \delta_{A}^{B},
\]

\[
\mathcal{N}_{A}^{B} = -2 \epsilon \left[ R^{(gi)} R_{A}^{B} + R^{(gi)} \right],
\]

\[
\mathcal{O}_{A}^{B} = 4 \epsilon \left\{ \delta^{(gi)} R^{(gi)} + R^{(gi)} \right\},
\]

\[
\mathcal{S}_{A}^{B} = 4 \epsilon \left\{ \delta^{(gi)} R^{(gi)} + R^{(gi)} \right\},
\]

\[
\mathcal{T}_{A}^{B} = -4 \epsilon \left\{ R^{(gi)} \right\},
\]  

(D.9)

the equation of the fluctuations is given, according to Eq. (3.1), by:

\[
\mathcal{M}_{A}^{B} \mathcal{R}^{B}_{A} + \mathcal{N}_{A}^{B} \mathcal{O}_{A}^{B} + \mathcal{S}_{A}^{B} \mathcal{T}_{A}^{B} = 0
\]  

(D.10)

Hence, the \((w, w)\) equation can be derived from the following explicit expressions
Using Eqs. (D.11)–(D.15) together with (D.8) into Eq. (D.10) Eq. (5.7) is recovered.  

The \((\mu, \nu)\) component can be similarly obtained from

\[
\mathcal{M}^w = \frac{1}{a^2} \left\{ \partial_\alpha \partial^\alpha \Xi + (d + 1)[\Psi'' + \mathcal{H}(\Psi' + \Xi')] + 2(d + 1)\mathcal{H}' \phi^2] \Xi + \phi' X' + \frac{1}{d} \frac{\partial V}{\partial \phi} a^2 X \right\}, \tag{D.11}
\]

\[
\mathcal{N}^w = \frac{-2e}{a^4} \left\{ (d + 1)(4\mathcal{H}' + d\mathcal{H}^2) \partial_\alpha \partial^\alpha \Xi - 2d(d + 1)\mathcal{H}' \partial_\alpha \partial^\alpha \Psi + 4(d + 1)^2[2\mathcal{H}'^2 + \mathcal{H}^2 \mathcal{H}'] \Xi + (d + 1)^2[4\mathcal{H}' + d\mathcal{H}^2](\Psi'' + \mathcal{H}\Xi') \right\} + (d + 1)^2[2(d + 2)\mathcal{H}' + d\mathcal{H}^2] \mathcal{H} \Psi' \tag{D.12}
\]

\[
\mathcal{O}^w = \frac{4e}{a^4} (d + 1)\mathcal{H}' \left\{ 2\partial_\alpha \partial^\alpha \Xi + 2(d + 1)[\Psi'' + \mathcal{H}(\Psi' + \Xi')] + 4(d + 1)\mathcal{H}' \Xi \right\}, \tag{D.13}
\]

\[
\mathcal{S}^w = \frac{4e}{a^4} \left\{ (d + 1)[d\mathcal{H}^2 + 2\mathcal{H}'] \Psi'' + (d + 1)[2(d + 1)\mathcal{H}' + d\mathcal{H}^2] \mathcal{H} \Psi' - 2d\mathcal{H}' \partial_\alpha \partial^\alpha \Psi + 4(d + 1)\mathcal{H}'(\mathcal{H}' + d\mathcal{H}^2) \Xi + (d + 1)[2\mathcal{H}' + d\mathcal{H}'] \mathcal{H} \Xi' (2\mathcal{H}' + d\mathcal{H}^2) \partial_\alpha \partial^\alpha \Xi \right\} \tag{D.14}
\]

\[
\mathcal{T}^w = \frac{-8e}{a^4} \mathcal{H}' \left\{ (d + 1)[\Psi'' + \mathcal{H}(\Psi' + \Xi')] + 2\mathcal{H}' \Xi + \partial_\alpha \partial^\alpha \Xi + \partial^\alpha \partial^\alpha \Xi \right\}. \tag{D.15}
\]

Using Eqs. (D.11)–(D.15) together with (D.8) into Eq. (D.10) Eq. (5.7) is recovered.
\[-4\mathcal{H}^2 \partial_\alpha \partial^\alpha \Psi + 8d\mathcal{H}^3 \Psi'] + \partial_\mu \partial^\nu \left[ 4\mathcal{H}' H - 4(d - 1)\mathcal{H}^2 \Psi \right]. \]

(D.20)

Again using Eqs. (D.16)–(D.20) together with Eq. (D.8) into Eq. (D.10), Eqs. (5.8) and (5.9) (reported in the text) are recovered.
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