Finite Size Scaling of the Spin stiffness of the Antiferromagnetic $S = \frac{1}{2}$ XXZ chain

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November 7, 2018

Abstract. We study the finite size scaling of the spin stiffness for the one-dimensional $s = \frac{1}{2}$ quantum antiferromagnet as a function of the anisotropy parameter $\Delta$. Previous Bethe ansatz results allow a determination of the stiffness in the thermodynamic limit. The Bethe ansatz equations for finite systems are solvable even in the presence of twisted boundary conditions, a fact we exploit to determine the stiffness exactly for finite systems allowing for a complete determination of the finite size corrections. Relating the stiffness to thermodynamic quantities we calculate the temperature dependence of the susceptibility and its finite size corrections at $T = 0$. A Luttinger liquid approach is used to study the finite size corrections using renormalization group techniques and the results are compared to the numerically exact results obtained using the Bethe ansatz equations. Both irrelevant and marginally irrelevant cases are considered.

PACS. 7 5.10.-b, 75.10.Jm, 75.40.Mg

1 Introduction

The general XXZ Hamiltonian on a ring of size $L$ is given by

$$H = H_0 + H_\ell(\Delta) = \frac{J}{2} \sum_{i=1}^L (S_i^z S_{i+1}^z + h\hbar c) + J\Delta \sum_{i=1}^L S_i^+ S_{i+1}^-,$$

(1)

Here $H_0$ is the free part and $H_\ell(\Delta)$ is the interacting part. It is well known that this model is solvable when periodic boundary conditions are applied [5]. However, the same model can also be solved under more general boundary conditions, in particular under the so called twisted boundary conditions [6,7,8] defined by:

$$S_{L+1}^z = S_1^z, \quad S_{L+1}^+ = S_1^+ e^{+i\varphi},$$

(2)

where $\varphi$ is the twist angle. The application of such boundary conditions is equivalent to considering a system threaded by a magnetic flux of strength $\varphi(h\hbar c/e)$ [9].

This fact was exploited by Shastry and Sutherland [6,7,8] to calculate transport properties of the system. Notably, the total ground state energy as a function of $\varphi$, $E_0(\varphi)$, can be calculated and hence also the spin current and spin stiffness. In the thermodynamic limit they showed that the spin stiffness is given by:

$$\rho = \frac{\pi}{4 \mu(\pi - \mu)}.$$

(3)

Hence, for the antiferromagnetic Heisenberg model, $\Delta = 1$, $\rho/J = 1/4$. In the context of mesoscopic physics it is quite interesting to study transport properties for finite systems where coherence effects are important [5] and notably the finite size dependence of the current, susceptibility and stiffness are important for a complete understanding of the experimental results. The stiffness, $\rho$, of the system is closely related to the dc conductivity of the system and a non-zero $\rho$ implies quasi long-ranged correlations with power-law decay. In the present paper we show that it is possible to calculate numerically exactly the spin stiffness, $\rho(L)$, for a finite system. $\rho(L)$ can then be used to calculate the susceptibility for finite systems and finite temperatures. Using a Luttinger liquid approach it is possible to understand quite completely the structure of the finite-size corrections and we compare these perturbative results to the numerical ones.

We consider exclusively the antiferromagnetic (AF) case with $J = 1$ and we shall mainly be concerned with the regime where the anisotropy parameter, $\Delta$, lies between $\Delta = 0$ (XY) and $\Delta = 1$ (XXX). It is well known that in the regime $\Delta \in [0,1]$ this model display gapless excitations with power law correlations and off-diagonal long-range order characterized by a non-zero stiffness, $\rho$, in close analogy to the superfluid order parameter in the two-dimensional classical $XY$ model. For $\Delta > 1$ the model Eq. (1) enters a phase with Ising like AF order and a non-zero gap. Following the above remarks, this transition, occurring at $\Delta = 1$, can be viewed as a metal-insulator transition [10]. We concentrate on the region $\Delta \in [0,1]$ with emphasis on the finite-size corrections at $\Delta = 1$. We first briefly review the finite-size scaling of the stiffness in section 2 then we discuss the simple free case, $\Delta = 0$, (XY).
2 Scaling of Stiffness

Suppose a twist of size $\varphi$ is applied at the boundary of an otherwise uniform system. It is natural to expect that this will give rise to a uniform phase gradient

$$\nabla \theta = \varphi / (aL),$$

throughout the system with $L$ sites and lattice spacing $a$. We can now define the stiffness with respect to the resulting change in the ground-state energy density in the following way:

$$\delta e_0 = \frac{1}{\hbar} \rho (\nabla \theta)^2,$$

where $\delta e_0$ is the change in the ground-state energy density when the twist $\varphi$ is applied. It then follows that the stiffness is given by the following expression:

$$\rho (L) = \frac{(aL)^2 \partial^2 e_0 (\varphi)}{\hbar} \mid_{\varphi=0}. \tag{6}$$

Hence, $\rho$ has dimension of inverse (length)$^d$. In the vicinity of a quantum critical point or line we can invoke hyperscaling and two-scale factor universality (see reference for a discussion and references) to argue that

$$\rho L^{d-2} \xi = C,$$ \tag{7}

where $C$ is a universal constant. Applying standard finite size scaling theory we then expect the stiffness to obey the following finite size scaling ansatz:

$$\rho (L) = L^{-d-z+2} \rho (L^{1/\nu} \delta),$$ \tag{8}

where $\delta$ is the distance to the quantum critical point. In a phase with long-range order, the stiffness should diverge and in the absence of long-range order we expect $\rho$ to vanish exponentially with the system size. Hence, Eq. (8) describes the finite-size corrections close to a critical point. In the present case of the one-dimensional Heisenberg chain we expect to have $z = 1$ and consequently $-d - z + 2 = 0$. Since the twist is applied in the $XY$ component of the spins we expect the resulting stiffness to be non-zero and universal in the critical region between $\Delta \in [-1, 1]$. Eq. (8) then tells us that the leading finite-size corrections in this region are absent and in the thermodynamic limit we expect $\rho$ to jump discontinuously at $\Delta = -1, 1$, as noted in Ref. 3. It is important to note that the above finite-size scaling analysis do not take into account corrections to scaling which, as we shall see later on, are especially important close to $\Delta = 1$.

3 Free case, $\Delta = 0$

At $\Delta = 0$, $H = H_0$ and the Hamiltonian becomes equivalent to a system of free fermions that can be diagonalized explicitly in $k$-space through the use of the Jordan-Wigner transformation $S_\uparrow = 1/2 - n_j$, and $S_\downarrow = \psi_j e^{i\pi \sum l \neq j n_l}$. The $\psi_j$ satisfies fermionic commutation relations, $\{\psi_j^\dagger, \psi_j\} = \delta_{ij}$, and $n_j = \psi_j^\dagger \psi_j$ is the number of fermions (spin down) at the j-site. We can now write $H_0$ in $k$-space

$$H_0 = -J \sum_k \cos(k) \psi_k \psi_k^\dagger. \tag{9}$$

The antiferromagnetic ground state is in the $S_{tot}^z = 0$ sector, which corresponds to half-filling ($n_{tot} = L/2$). The effect of the twist $\varphi$ is a shift in the momentum space: $k \rightarrow k + \varphi$, this can also be seen explicitly from the Bethe ansatz equation (see section 4). Therefore, we obtain a rather simple expression for the ground state energy per site versus $\varphi$:

$$\frac{e_0 (L, \varphi)}{J} = 1 \cos \left( \frac{\varphi}{L} \right). \tag{10}$$

Using Eq. (10), the finite size corrections for the stiffness are then:

$$\frac{\hbar}{Ja^2} \rho (L) = \frac{1}{L \sin (\pi / L)} \sim \frac{1}{\pi} + \frac{\pi}{6L^2} + O \left( \frac{1}{L^4} \right). \tag{11}$$

The corrections of order $O \left( \frac{1}{L^2} \right)$ can be understood in terms of conformal field theory. We also have the universal corrections to the free energy which are given by $e_0 (L) = e_0^\infty - \frac{\pi u}{\sin (\pi / L)} + O \left( \frac{1}{L^2} \right)$. Here, $u$ is the velocity of excitations ($u = 1$ in the free case) and $c$ is the conformal anomaly number; $c = 1$ for the $s = 1/2$ XXZ chain.

4 Renormalization group analysis

4.1 Bosonization of the XXZ chain

In this section, we introduce the bosonization of interacting fermionic systems like the XXZ chain (using Jordan-Wigner transformation) or the Hubbard model, which are Luttinger liquids. In a few words, it consists in linearizing the dispersion relation near the Fermi points and studying the low energy excitations close to those points. Moreover, for commensurate fillings of order $n$, umklapp scattering ($n$ left electrons become right or vice-versa) is possible and gives rise to an additional Sine-Gordon term in the Hamiltonian. In the continuum limit and in the $S_{tot}^z = 0$ sector (corresponding to half-filling), the XXZ chain is governed by a Hamiltonian expressed in terms of conjugate boson fields $\Phi(x)$ and $\Pi(x)$ as follows:

$$H = \int \frac{dx}{2} \left\{ \frac{uK}{\pi} [\pi \Pi (x)]^2 + \frac{\pi u}{K} [\partial_x \Phi (x)]^2 \right\}$$

$$+ \frac{J\Delta}{2(\pi a)^2} \int dx \cos (4\Phi (x)). \tag{12}$$
shown in Fig. 1 where distinct behaviors are observed: \[ \text{[16]} \]. The renormalization flow diagram for this model is equivalent to considering an infinite chain at \( 0 \), but due to the invariance of the model it should be charge stiffness, given by \( \rho = \frac{uK}{\pi} \), and the susceptibility, \( \chi = \frac{K}{u\pi} \).

The susceptibility can then be obtained from \( u \) and \( \rho \) using the hydrodynamic relation \[ \chi = \frac{\rho}{u^2} \].

In the thermodynamic limit, we will note \( u^* \) and \( K^* \) for the Luttinger liquid parameters. By using the thermodynamic relations from above and comparing to exact results obtained from Bethe ansatz, we can express these parameters versus \( \Delta = \cos(\mu) \): \[ \frac{u^*(\Delta)}{\pi} = \frac{\pi \sin(\mu)}{2} \] \[ K^*(\Delta) = \frac{\pi}{2(\pi - \mu)} \].

### 4.2 RG equations

Now, we have to study the Sine-Gordon term in Eq. \[ \text{[12]} \] \((g \int dx \cos(4\Phi(x)))\) as a perturbation and examine the effect of a renormalization under a change of length scale. We then obtain:

\[ \frac{dg}{dl} = (2 - 4K)g + O(g^2) \]

\[ \frac{dK}{dl} = -Ag^2 \],

where \( g = J\Delta/(2(\pi a)^2) \) is the coupling, \( l = \ln(L) \) and \( A \) is a constant. Those equations are identical to the Kosterlitz-Thouless renormalization group analysis \[ \text{[13]} \] used in the classical 2d XY model and which gives a description of the superfluid transition. There is a powerful analogy between the superfluid density which vanishes at the transition and the spin stiffness in the quantum XXZ chain. Here, the renormalization is done versus the chain length \( L \) at \( T = 0 \), but due to the invariance of the model it should be equivalent to considering an infinite chain at \( T = u/L \) \[ \text{[14]} \]. The renormalization flow diagram for this model is shown in Fig. 1 where distinct behaviors are observed:

- For \( 1/2 < K \leq 1 \) (\( \Delta \in [0,1] \)), the perturbation \( g \) remains irrelevant and vanishes rapidly with the size \( L \), and \( K \) goes to \( K^* \).
- At the isotropic point \( K = 1/2 \) (\( \Delta = 1 \)), the perturbation is marginally irrelevant implying a logarithmically decreasing \( g \) and \( K \rightarrow K^* \).
- For \( K < 1/2 \) (\( \Delta > 1 \)), the umklapp term, \( g \), increases with the system size because the perturbation is relevant and \( K \) vanishes. A gap opens up in the spectrum and the stiffness falls to zero.

![Figure 1. RG flow of the Sine-Gordon model.](image)

### 5 Finite size scaling

The integration of the RG equations \[ \text{[15]} \] from \( L_0 \) to \( L \) gives us the finite-size scaling of \( K(L) \) and \( g(L) \). The corrections to \( g \) have already been evaluated by Cardy \[ \text{[17]} \] and by Lukyanov \[ \text{[18]} \]. They are very important to calculate the corrections to the ground-state energy \[ \text{[19]} \]. At the isotropic point, the coupling \( g \) is marginally irrelevant and the integration of Eq. \[ \text{[17]} \] up to \( O(g^2) \) gives rise to logarithmic corrections to the ground-state energy \[ \text{[19]} \] :

\[ e_0(L) = e_0^\infty - \frac{\pi u}{6L^2}\left[c + 8\pi^3bg(L)^3\right], \]

where \( c = 1 \) and \( b = 4/\sqrt{3} \) is determined by a three-point function \[ \text{[14]} \]. Here, the corrections of \( K \) will imply corrections to the stiffness and susceptibility in two cases: when the perturbation is irrelevant and marginally irrelevant.

### 5.1 Irrelevant perturbations: power-law corrections

In the anisotropic antiferromagnetic regime \( (1/2 < K^* \leq 1) \), we can integrate Eq. \[ \text{[17]} \] from \( L_0 \) to \( L \), assuming the
integration with $L \gg L_0$. We then obtain the finite size corrections to $K$:

$$K(L) = K^* + \frac{(K(L_0) - K^*)(1 - 2K^*)}{1 - K^* - K(L_0)} \left( \frac{L}{L_0} \right)^{8(K^* - 1/2)} + O \left( \frac{L}{L_0} \right)^{-16(K^* - 1/2)}. \tag{19}$$

We see that the exponent of $1/L$ lies between $0^+$ and 4. In the free case ($K = 1$), we find an exponent 4 which seems to contradict the finite-size corrections found in Eq. (14). However, it must be noted that terms in $1/L^2$, the so-called 'analytical' corrections, are expected to appear in all quantities since they are related to the conformal symmetry of the fixed-point Hamiltonian [1,14]. We can also emphasize that these analytical corrections dominate when $K \geq 3/4$ (i.e., $\Delta \leq 1/2$) and are sub-dominant otherwise. This will be shown precisely with the numerical Bethe ansatz calculation.

From Eq. (19), we expect that to the lowest order, the finite size corrections for the stiffness and the susceptibility are:

If $1/2 < \Delta < 1$:

$$\rho(L) - \frac{uK^*}{\pi} \sim \chi(L) - \frac{K^*}{u\pi} \sim \left( \frac{1}{L} \right)^{8(K^* - 1/2)},$$

If $0 \leq \Delta \leq 1/2$:

$$\rho(L) - \frac{uK^*}{\pi} \sim \chi(L) - \frac{K^*}{u\pi} \sim \left( \frac{1}{L} \right)^2. \tag{20}$$

5.2 Marginal irrelevant perturbations: logarithmic corrections

At the isotropic point, we can integrate Eq. (17) at the lowest order, taking initial conditions at $L_0$. We find that $K$ is decreasing logarithmically:

$$K(L) = \frac{1}{2} + \frac{K(L_0) - \pi}{1 + 4(K(L_0) - \pi/4) \ln(L/L_0)}. \tag{21}$$

Such behavior is in agreement with the logarithmic corrections obtained by Loss et al. [20]. At first order in $1/\ln(L/L_0)$, the finite size scaling correction for the stiffness and the susceptibility are:

$$\frac{\rho(L)}{J} \sim \frac{1}{4} + \frac{\rho(L_0)/J - 1}{1 + 8(\rho(L_0)/J - 1/4) \ln(L/L_0)}, \tag{22}$$

$$J\chi(L) \sim \frac{1}{\pi^2} + \frac{J\chi(L_0) - 1/\pi^2}{1 + 2\pi^2(J\chi(L_0) - 1/\pi^2) \ln(L/L_0)}. \tag{23}$$

As noted previously, these results give the temperature dependence of these quantities by taking $T = u/L$ and we obtain results in agreement with previous work by Eggert et al. [11]. These predictions will be compared to numerically exact results in section 6.

6 Bethe Ansatz Equations

The solution of the one-dimensional antiferromagnetic Heisenberg model due to Bethe [1] can be extended to quite general boundary conditions. Hamer, Quispel and Batchelor [3] exploited this fact to calculate the ground state energy in the thermodynamic limit for the Hamiltonian with twisted boundary conditions, Eq. (3), as a function of the applied twist $\varphi$. Using this result the stiffness of the system can be calculated $\rho$ using the relation Eq. (3). If twisted boundary conditions are imposed the expression for the ground state energy per site remains unchanged and are the same as for the uniform case:

$$\frac{e_0(L, \varphi)}{J} = \frac{\Delta}{4} - \frac{1}{L} \sum_{l=1}^{L/2} \cos(k_l(\varphi)). \tag{24}$$

The change in the boundary conditions enters in the equations determining the $L/2$ quasi-momenta, $k_l$, only in the following manner:

$$k_l = \frac{1}{L}[2\pi I_l + \varphi - \sum_{n \neq l} \Theta_{l,n}], \tag{25}$$

where $\Theta_{l,n}$ is given by:

$$\Theta_{l,n} = 2 \arctan \left[ \frac{\Delta \sin(k_l - k_n)}{\cos(k_l + k_n)/2 - \Delta \cos(k_l - k_n)/2} \right]. \tag{26}$$

For the ground state, the set of integers $I_l$ is given by $I_l = l - (L/2 + 1)/2$, $l \in [1, L/2]$. Using the above equations it is possible to numerically determine the quasi-momenta, $k_l$, even for relatively large systems and for non-zero $\varphi$.

Using Eq. (3) and Eq. (24), we can now formally write an equation for $\rho(L)$:

$$\frac{\rho(L)}{J} = \frac{1}{L} \sum_{l=1}^{L/2} \partial^2 k_l/\partial \varphi^2 \sin(k_l) + \left[ \partial k_l / \partial \varphi \right]^2 \cos(k_l) \big|_{\varphi=0}. \tag{27}$$

Hence it is possible to rather easily calculate $\rho(L)$ if the derivatives $\partial^2 k_l / \partial \varphi^2$ and $\partial k_l / \partial \varphi$ can be calculated. For $\partial k_l / \partial \varphi$ we obtain the following expression:

$$\frac{\partial k_l}{\partial \varphi} = \frac{1}{L} \sum_{n=1}^{L/2} \left[ \frac{\partial \Theta(k_l, k_n)}{\partial k_l} \frac{\partial k_l}{\partial k_n} - \frac{\partial \Theta(k_l, k_n)}{\partial k_n} \frac{\partial k_l}{\partial k_n} \right]. \tag{28}$$

Since the derivatives $\partial \Theta(k_l, k_n) / \partial k_l$ do not depend on $\partial k_l / \partial \varphi$, but only on the previously determined quasi-momenta $k_l$, this is a simple matrix equation from which $\partial k_l / \partial \varphi$ can be determined using standard linear algebra routines. An equivalent expression exists for $\partial^2 k_l / \partial \varphi^2$ which also reduces to a linear algebra problem once $\partial k_l / \partial \varphi$ is known. Hence, $\rho(L)$ can be determined numerically exactly once the $k_l$’s have been obtained.
7 Numerical results

We begin by a discussion of our results for the spin stiffness, \( \rho(L) \), for finite rings of size \( L \). In Fig. 2 our results for the spin stiffness are shown as a function of the anisotropy \( \Delta \) for different ring sizes, \( L \). As noted in section 3.2 we expect the leading finite size correction to be absent in the gapless regime \( \Delta \in [0, 1] \). This is clearly the case. Only when \( \Delta > 1 \) do these corrections become important, the system opens up a gap and the correlation length, \( \xi \), becomes finite. In this regime, an explicit expression for \( \xi \) has been obtained by Baxter [21] and due to this finite correlation length we expect \( \rho(L) \) to vanish in an exponential manner with the system size, \( \rho(L) \sim e^{-L/\xi} \). For \( \Delta < 1 \), the finite size corrections are expected to have the form \( (21) \). Finally, at \( \Delta = 1 \), we expect logarithmic corrections of the form \( (22) \). These logarithmic corrections are non-negligible and remain sizable for systems of macroscopic size. Hence, the discontinuous jump in \( \rho \) at \( \Delta = 1 \) is difficult to observe.

7.1 Power-law corrections

We now turn to a discussion of our results for the finite size corrections in the critical region \( \Delta < 1 \). Following our results from the previous sections, we expect \( \rho(L) \) to be independent of \( L \) to leading order and the corrections to scaling to acquire a power-law dependence with an exponent depending on \( K^* \).

Writing \( \rho(L) = \rho + O(L^{-\alpha}) \), with \( \rho \) given by Eq. (15), we can determine the exponent \( \alpha \) from the numerically exact results obtained from the Bethe ansatz. We have determined this exponent in the region \( \Delta < 1 \), and our results are shown in Fig. 3. The solid line indicates the renormalization group result, Eq. (19). As previously explained we expect on general grounds always to have corrections to scaling of the form \( 1/L^2 \), as explicitly shown in Eq. (11). Corrections of such a form dominate when \( 0 < \Delta < 1/2 \). Accordingly, the corrections to scaling coming from the marginally irrelevant coupling are dominant when \( 1/2 < \Delta < 1 \).

7.2 Logarithmic corrections

Finally we discuss our results for the isotropic point. At \( \Delta = 1 \), the perturbation to the fixed point Hamiltonian is marginally irrelevant, giving rise to finite size corrections of logarithmic form for the spin stiffness, Eq. (22), and the susceptibility, Eq. (24). In order to compare our numerical results to the RG results we use our results for the largest possible system size as the reference point, \( L_0 \), and perform the comparison for \( L < L_0 \). Taking \( L_0 = 10000 \), we show in Fig. 4 our results for stiffness as a function of system size, \( L \). The solid line indicates the RG result for the stiffness, Eq. (22), and the numerical results from the Bethe ansatz solution are shown as circles. Numerical calculations have been performed for system sizes up to 10000 sites. A good agreement between the RG results and the numerical results is evident even down to rather small system sizes. From Eq. (15) we see that the susceptibility follows a similar form.

Using the relationship \( L \leftrightarrow u/T \), we can obtain information about the temperature dependence of the susceptibility in the thermodynamic limit from our results.
obtained at $T = 0$ for finite systems. The temperature dependence of the Drude weight has recently been calculated using the thermodynamic Bethe ansatz [12] and is currently a topic of discussion [23]. Previously, the low temperature behavior of the susceptibility has also been studied in detail by Eggert et al. [16] and Lukyanov [18]. In analogy with the work of Eggert et al. [16], we obtain from Eq. (28) the following expression for the low temperature dependence of the susceptibility in the thermodynamic limit:

$$J\chi(T) \approx \frac{1}{\pi^2} + \frac{J\chi(T_0) - 1/\pi^2}{1 + 2\pi^2(J\chi(T_0) - 1/\pi^2)\ln(T_0/T)}. \quad (29)$$

As previously done, we use our largest system size to define an equivalent temperature, $T_0$, and use the RG expression Eq. (29) for higher temperatures (smaller system sizes). Taking $T_0 = u/L_0 = \pi J/(2 * L_0) \sim 1.57 \times 10^{-4}$ ($L_0 = 10000$), we show in Fig. 5 the RG result, Eq. (29) (solid line) along with the exact numerical results from the Bethe ansatz (circles) and the result of Ref. [18]. As already shown by Eggert et al. [16] for the isotropic case, the agreement between the RG result and the Bethe ansatz results is excellent at low temperatures (large system sizes). However, the result of Lukyanov [18] seems to work better over the complete temperature range. The logarithmic dependence of the susceptibility at $\Delta = 1$ is clearly visible and the derivative with respect to temperature of $\chi(T)$ diverges as $T \to 0$ and $\chi(T) \to 1/\pi^2$.

As pointed out in the previous sections, we expect a power-law dependence of $\chi(T)$ with the temperature for $\Delta < 1$ with an exponent of 2 for $\Delta < 1/2$ and a non-trivial exponent in the regime $1/2 < \Delta < 1$. In the inset of Fig. 5, $\chi(T)$ is shown for several different values of $\Delta < 1$. For $\Delta > 1$ the system opens up a gap and the susceptibility decreases exponentially at low temperatures.

### 7.3 Coupling term $g$

From our results for the susceptibility (or equivalently the stiffness) it is possible to obtain an estimate of the flow of the Sine-Gordon coupling term $g(L)$ with the system size $L$. Previous studies have obtained the same flow from the ground state energy [12]. At the isotropic point ($\Delta = 1$), we have seen that the perturbation term in Eq. (12), $g \int dx \cos(4\Phi(x))$, is marginally irrelevant. Hence, we expect a logarithmic behavior of $g(L)$. From the results of Eggert et al. [16] for the susceptibility in the $k = 1$ WZW non-linear $\sigma$ model [24], the coupling $g$ can be expressed in term of $\chi(L)$ obtained numerically from the solution of the Bethe ansatz equations as follows:

$$g^{BAE}_\chi(L) = \frac{\pi\sqrt{3}}{2}(J\chi(L) - \frac{1}{\pi^2}). \quad (30)$$

As mentioned, it is also possible to estimate $g(L)$ from the finite size corrections to the ground state energy [12], Eq. (13). In this case one finds:

$$g^{BAE}_{GS}(L) = \left[\frac{12L^2/\pi^2(e_0^\infty - e_0(L)) - 1}{32\pi^3/\sqrt{3}}\right]^{1/3}, \quad (31)$$

where $e_0(L)$ again is determined from the numerical solution of the Bethe ansatz equations. Finally, we can compare these two estimates to the RG result for $g(L)$ which...
In this equation, we use $\chi$ to determine the bare coupling $g_0$. We can also compare our results to the general expression, including higher order corrections, derived by Lukyanov [18] for $g(L)$ which is, in our notation:

$$g^{-1} + \frac{\sqrt{3}}{8 \pi} \ln(g) = \frac{\sqrt{3}}{8 \pi} \ln(\frac{2 \sqrt{2}}{3 \sqrt{3}} e^{\gamma + 0.25} \times L),$$

where $\gamma$ is the Euler constant. In Fig. 6 we show results for $g_{BAE}^{GS}(L)$ (solid diamonds) and $g_{BAE}^{RG}(L)$ (o). The solid line indicates the RG result, Eq. (32) and the dashed line indicates the result of Lukyanov [18]. The largest system size, $L_0 = 10000$, has again been used to determine the bare parameters in Fig. 6. As evident from the results shown in Fig. 6 there is an excellent agreement between the numerical estimates for $g(L)$ obtained from the ground state energy $\epsilon_0(L)$ and the susceptibility, $\chi(L)$ and the RG result, Eq. (32). The result of Lukyanov [18] agrees very well with the numerical results over the entire range of $L$. $\Delta = 1$ where logarithmic corrections are dominant, our results are in good agreement with previously known results. From the finite size dependence of the spin stiffness and susceptibility we are able to obtain precise results for the low temperature behavior of the susceptibility in the thermodynamic limit. The described effects should be observable in mesoscopic systems where a detailed understanding of the finite size behavior is crucial.

We thank P. Azaria for useful discussions.

8 Conclusion

The finite size corrections to the spin stiffness and subsequently the susceptibility have been studied in detail for the $S = 1/2$ antiferromagnetic spin chain as a function of the anisotropy parameter $\Delta$. At the isotropic point,

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