THE INDEX OF REDUCIBILITY OF PARAMETER IDEALS
IN LOW DIMENSION

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Abstract. In this paper we present results concerning the following question: If \( M \) is a finitely-generated module with finite local cohomologies over a Noetherian local ring \((A, m)\), does there exist an integer \( \ell \) such that every parameter ideal for \( M \) contained in \( m^\ell \) has the same index of reducibility? We show that the answer is yes if \( \dim M = 1 \) or if \( \dim M = 2 \) and depth \( M > 0 \).

This research is closely related to work of Goto-Suzuki and Goto-Sakurai; Goto-Sakurai have supplied an answer of yes in case \( M \) is Buchsbaum.

1. Introduction

Let \( A \) be a \( d \)-dimensional Noetherian local ring with maximal ideal \( m \) and residue field \( k = A/m \), and let \( M \) be a finitely generated \( A \)-module. Recall that a submodule of \( M \) is called irreducible if it cannot be written as the intersection of two larger submodules. It is well known that every submodule \( N \) of \( M \) can be expressed as an irredundant intersection of irreducible submodules, and that the number of irreducible submodules appearing in such an expression depends only on \( N \) and not on the expression [ShV, p. 92-3].

For an ideal \( I \) of \( A \), we say that \( I \) is cofinite on \( M \) if the module \( M/IM \) has finite length. For an ideal \( I \) of \( A \) which is cofinite on \( M \), the index of reducibility of \( I \) on \( M \) is defined as the number of submodules appearing in an irredundant expression of \( IM \) as an intersection of irreducible submodules of \( M \). We denote the index of reducibility of \( I \) on \( M \) by \( N_A(I; M) \).

The smallest number of generators of an ideal which is cofinite on \( M \) is the dimension of \( M \), and a cofinite ideal having this minimal number of generators is called a parameter ideal for \( M \). Our interest in the index of reducibility of parameter ideals stems from the relationship with the Cohen-Macaulay and Gorenstein properties. In 1956, D. G. Northcott proved that in a Cohen-Macaulay local ring, the index of reducibility of any parameter ideal depends only on the ring [N, Theorem 3]. This result extends to modules, and the common index of reducibility of parameter ideals for a Cohen-Macaulay module \( M \) is called the (Cohen-Macaulay) type of \( M \). We denote the type of \( M \) by \( t_A(M) \).

As a partial converse, Northcott along with D. Rees proved in [NR1] that if every parameter ideal of \( A \) is irreducible, then \( A \) is Cohen-Macaulay. This provides an attractive characterization of a Gorenstein local ring as a local ring in which every parameter ideal is irreducible.

One might suspect that the Cohen-Macaulay property of a local ring is characterized by the constant index of reducibility of parameter ideals. However, in 1964,
M. Narita in [EN] gave an example of a Noetherian local ring in which the index of reducibility of each parameter ideal is two, and yet the ring is not Cohen-Macaulay.

In 1984, S. Goto and N. Suzuki generalized the example of Endo-Narita, as well as undertaking a study of the supremum of the index of reducibility of parameter ideals for $M$. We refer to this supremum as the Goto-Suzuki type (GS-type) of $M$, and denote it by $\mathfrak{r}_A(M)$. In the case where $M$ is Cohen-Macaulay, the GS-type $\mathfrak{r}_A(M)$ is equal to the type $r_A(M)$. However, Goto-Suzuki provide examples where the GS-type of a Noetherian local ring is infinity.

We introduce some terminology in order to state one of the main results of Goto-Suzuki. We denote the $i$th local cohomology module of $M$ with respect to $m$ by $H^i_m(M)$. We say that $M$ has finite local cohomologies if the modules $H^i_m(M)$ have finite length for $i \neq d$. We use $\lambda_A(M)$ to denote the length of $M$, and we set $\mathfrak{S}(M) = \lambda_A(\text{Hom}_A(k, M))$, the socle dimension of $M$. We let $\mu_A(M)$ denote the minimal number of generators of $M$, and let $E$ denote the injective hull of the residue field $k$. The main result of Goto-Suzuki concerning the GS-type of a module having finite local cohomologies is the following:

**Theorem 1.1** (Goto-Suzuki). Let $M$ be a finitely generated $d$-dimensional $A$-module with finite local cohomologies. Then we have the following inequalities:

\[
\sum_{i=0}^{d} \binom{d}{i} \mathfrak{S}(H^i_m(M)) \leq \mathfrak{r}_A(M) \leq \sum_{i=0}^{d-1} \binom{d}{i} \lambda_A(H^i_m(M)) + \mu_A(K),
\]

where $\hat{A}$ is the $m$-adic completion of $A$ and $K = \text{Hom}_A(H^d_m(M), E)$ is the canonical module of the completion of $M$.

**Proof.** See [GSu, Theorems 2.1 and 2.3].

In 1994, Kawasaki used this result to determine conditions under which a module having finite GS-type is Cohen-Macaulay. The main result of Kawasaki in [K] is that if $A$ is the homomorphic image of a Cohen-Macaulay ring, then $M$ is Cohen-Macaulay if and only if $\mathfrak{r}_A(M)$ is finite, $M_p$ is Cohen-Macaulay for all primes in the support of $M$ with $\text{dim } M_p < r_A(M)$, and all the associated primes of $M$ have the same dimension.

In two recent papers [GS1, GS2], Goto with H. Sakurai has returned to the study of the index of reducibility of parameter ideals in order to investigate when the equality $I^2 = QI$ holds for a parameter ideal $Q$ in $A$, where $I = (Q :_A m)$. According to earlier research of A. Corso, C. Huneke, C. Polini, and W. Vasconcelos [CHP, CP, CPV], this equality holds for all parameter ideals $Q$ in case $A$ is a Cohen-Macaulay ring which is not regular. Goto-Sakurai generalize this and say that if $A$ is a Buchsbaum ring whose multiplicity is greater than 1, then the equality $I^2 = QI$ holds for any parameter ideal whose index of reducibility is the GS-type of $A$. Thus, if $A$ is a Buchsbaum ring whose multiplicity is greater than 1 and if $A$ has constant index of reducibility of parameter ideals, then the equality $I^2 = QI$ holds for all parameter ideals $Q$.

Most pertinent to our discussion is Corollary 3.13 of Goto-Sakurai in [GS1], which states that if $A$ is a Buchsbaum ring of positive dimension, then there is an integer $\ell$ such that the index of reducibility of $Q$ is independent of $Q$ and equals
τ_A(Q) for all parameter ideals Q ⊆ m^ℓ. In view of this, it is natural to ask the following question:

**Question 1.2.** Suppose (A, m) is a Noetherian local ring having finite local cohomologies. Is there an integer ℓ such that the index of reducibility of any parameter ideal contained in m^ℓ is the same?

We note that if A is a ring satisfying the hypothesis of the question, and if the answer to the question is yes for A, then the common index of reducibility of parameter ideals in high powers of the maximal ideal is equal to the lower bound of Goto-Suzuki: \( \sum_{i=0}^{d} \binom{d}{i} \mathcal{S} (H^i_m(A)) \). This is because implicit in the proof of the lower bound [GS] Theorem 2.3] we find that given any system of parameters \( x_1, \ldots, x_d \) for A, there are integers \( n_i \) such that the parameter ideal \( (x_1^{n_1}, \ldots, x_d^{n_d})A \) has index of reducibility \( \sum_{i=0}^{d} \binom{d}{i} \mathcal{S} (H^i_m(A)) \).

The main result of the current paper is the following:

**Theorem 1.3.** Let (A, m) be a Noetherian local ring and let M be a finitely-generated A-module of dimension \( d \leq 2 \). Suppose either M has dimension 1, or M has finite local cohomologies and depth at least one. Then there exists an integer ℓ such that for every parameter ideal q for M contained in m^ℓ, the index of reducibility of q on M is independent of q and is given by

\[
N_A(q; M) = \sum_{i=0}^{d} \binom{d}{i} \mathcal{S} (H^i_m(M)).
\]

**Proof.** See Theorem 2.3 and 3.3. □

As a corollary of this result we prove that a Noetherian local ring A of dimension at most 2 is Gorenstein if and only if A has finite local cohomologies, and inside every power of the maximal ideal of A there exists an irreducible parameter ideal. A Noetherian local ring having the property that every power of its maximal ideal contains an irreducible parameter ideal is called an approximately Gorenstein ring, or is said to have small cofinite irreducibles (SCI). The author’s original motivation for studying the index of reducibility of parameter ideals comes from questions that arose while studying M. Hochster’s paper [H] exposing the relationship between modules having SCI and the condition that cyclic purity implies purity.

We present an example of a complete Noetherian local ring of dimension \( d \) and depth \( d - 1 \) (\( d > 1 \)), such that \( H^{d-1}_m(A) \) is not finitely generated, and such that in every power of the maximal ideal there is a parameter ideal with index of reducibility 2 and a parameter ideal with index of reducibility 3. This example is obtained as an idealization, so we present several basic results relating the index of reducibility with an idealization. Using a result of C. Lech [L], we are able to obtain such an example among Noetherian local domains.

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2. BACKGROUND AND DIMENSION ONE

We begin with some terminology.
Definitions and Notation 2.1. Let $A$ denote a Noetherian local ring with maximal ideal $m$ and residue field $k = A/m$, let $I$ denote an ideal of $A$, and let $M$ denote a finitely generated $A$-module.

(1) The socle of $M$ is defined to be $S(M) = (0:_M m)$. Note that $S(M) \cong \text{Hom}_A(k, M)$. The socle is naturally a vector space over $k$, and we denote its dimension by $\mathfrak{S}(M)$.

(2) Suppose $M$ has depth $t$. The type of $M$ is defined by
\[
\text{r}_A(M) = \dim_k \text{Ext}_A^1(k, M).
\]
Note that if $I$ is cofinite on $M$, then $N_A(I; M) = \text{r}_A(M/IM)$, and if $M$ has depth 0, then $\mathfrak{S}(M) = \text{r}_A(M)$. See [BH, p. 13] for more information on the type of a module.

(3) Given an $A$-algebra $B$, we define $M_B = M \otimes_A B$. Then $(A/I)_B \cong B/IB$, and if $B$ is flat over $A$, then for any submodule $N$ of $M$, we have $M_B/N_B \cong (M/N)_B$.

(4) We say that an ideal $J$ of $A$ is a reduction of $I$ if $J \subseteq I$ and there is some integer $n$ such that $I^{n+1} = JI^n$. We say that $J$ is a minimal reduction of $I$ if there is no reduction of $I$ properly contained in $J$. The reduction number of $I$ with respect to $J$ is defined as
\[
\text{red}_J(I) = \min\{n : I^{n+1} = JI^n\}.
\]
We define the reduction number of $J$ to be
\[
\text{red}(I) = \min\{\text{red}_J(I) : J \text{ is a minimal reduction of } I\}.
\]

We must reduce to the case that the residue field is infinite in order to obtain a principal reduction of the maximal ideal. Thus our first result is a basic lemma concerning the behavior of the index of reducibility under a flat, local change of base. We omit the proof of Lemma 2.2 but cite [Na, Theorem 19.1], [BH, Proposition 1.2.16] and [M, Theorem 7.4] for relevant related results.

Lemma 2.2. Suppose $A$ and $B$ are Noetherian local rings with maximal ideals $m$ and $n$, respectively, let $M$ be a finitely generated $A$-module, and suppose $B$ is a flat $A$-algebra for which $n = mB$. Let $I$ be an ideal of $A$ and let $N$ be a submodule of $M$.

(1) If $M$ has finite length, then $M_B$ has finite length, and $\lambda_B(M_B) = \lambda_A(M)$.

(2) If $I$ is an ideal of $A$ which is cofinite on $M$, then $IB$ is cofinite on $M_B$.

(3) If $I$ is an ideal of $A$ which is cofinite on $M$, then
\[
N_B(IB; M_B) = N_A(I; M).
\]

(4) $(N :_M I)_B \cong (N_B :_{M_B} IB)$.

(5) $H^n_B(M_B) = H^0_B(M_M)$. 

Now we are prepared to prove that parameter ideals in high powers of a one dimensional Noetherian local ring all have the same index of reducibility.

Theorem 2.3. Suppose $A$ is a Noetherian local ring with maximal ideal $m$, and let $M$ be a finitely generated $A$-module of dimension 1. Set $W = H^2_m(M)$. There is an integer $t$ such that the index of reducibility of any parameter ideal $q \subseteq m^t$ is independent of $q$ and is given by
\[
N_A(q; M) = \mathfrak{S}(M) + \text{r}_A(M/W).
\]
When \( M = A \) and \( k \) is infinite, the integer \( \ell \) may be taken to be
\[
\ell = \max\{c, d\} + 1
\]
where \( c = \text{red}(m) \) and \( d \) is the smallest integer with \( m^d \cap W = 0 \).

**Proof.** First we show that it suffices to prove the theorem in the case where the residue field \( k \) is infinite. Suppose \( B \) is any local flat \( A \)-algebra whose maximal ideal \( n \) is extended from \( A \) and whose residue field is infinite (for instance, \( B = A[x]/m_A[x] \) for an indeterminate \( x \)). Then according to Theorem A.11 on p. 415 of \( [BH] \), the \( B \)-module \( M_B \) has dimension 1. Therefore we may apply the theorem to \( M_B \); let \( \ell \) be the integer guaranteed by the theorem. Set \( V = H^0_B(M_B) \); then according to Part 1 of Lemma 2.2, we have \( V = W_B \).

Suppose \( q \) is a parameter ideal for \( M \) that is contained in \( m^\ell \). Then \( qB \) is a parameter ideal for \( M_B \) which is contained in \( n^\ell \), so we have
\[
N_B(qB; M_B) = S(M_B) + r_B(M_B/V) \tag{2.1}
\]
According to Part 3 of Lemma 2.2, the left side of equality \( \text{(2.1)} \) equals \( N_A(q; M) \), so it remains to see that \( S(M_B) = S(M) \) and \( r_B(M_B/V) = r_A(M/W) \). For the first equality, if \( W = 0 \) then \( V = 0 \) and we have nothing to show. Otherwise, \( M \) has depth 0 and thus according to Proposition 1.2.16 part (a) from \( [BH] \), so does \( M_B \), so by part (b) of the same Proposition, we have
\[
S(M_B) = r_B(M_B) = r_A(M) = S(M) \tag{2.2}
\]

To see the equality \( r_B(M_B/V) = r_A(M/W) \), we first note that
\[
M_B/V = M_B/W_B \cong (M/W)_B
\]
Hence, by another application of Proposition 1.2.16 part (b) from \( [BH] \), we have
\[
r_B(M_B/V) = r_B((M/W)_B) = r_A(M/W) \tag{2.2}
\]
Thus we see that we may assume the residue field \( k \) is infinite.

Our second task is to see that we may replace the ring \( A \) by the ring \( C = A/I \), where \( I \) is the annihilator of \( M \). Let \( q \) denote an ideal of \( A \) which is a parameter ideal for \( M \). The \( A \)-module \( M \) becomes an \( C \)-module in a natural way, and the submodule structure of \( M \) remains unchanged. The maximal ideal of \( A \) extends to that of \( C \), the residue field of \( C \) is \( k \), and \( qC \) is an ideal of \( C \) which is a parameter ideal for \( M \). If \( N \) is any finitely generated \( A \)-module annihilated by \( I \), then as sets we have \( \text{Hom}_A(k, N) = \text{Hom}_C(k, N) \), and we easily check that this equality is actually an isomorphism of \( A \)-modules. Thus \( S(N) \) does not depend on whether we view \( N \) as an \( A \)-module or a \( C \)-module. Since the numbers \( N_A(q; M) \), \( S(M) \), and \( r_A(M/W) \) are all calculated as socle dimensions of quotients of \( M \), they do not change when we view \( M \) as an \( C \)-module. Thus we replace \( A \) by \( C \) and assume that \( A \) has Krull dimension one.

If \( W = 0 \), then \( M \) is Cohen-Macaulay, \( S(M) = 0 \), and for a parameter ideal \( q = aA \) we have
\[
N_A(q; M) = S(M/aM) = \dim_k \text{Ext}^1_A(k, M) = r_A(M) \tag{2.2}
\]
Thus the proof is complete in this case.

Now suppose \( W \neq 0 \). Since \( W \) has finite length and
\[
\bigcap_{i=1}^{\infty}(m^n M \cap W) = 0,
\]
there is some integer $d$ with $\mathfrak{m}^d M \cap W = 0$.

Since the residue field is infinite, $\mathfrak{m}$ has a principal reduction; i.e., there is an element $x \in \mathfrak{m}$ and an integer $c \geq 1$ such that $\mathfrak{m}^{c+1} = x\mathfrak{m}^c$ [BH Corollary 4.6.10, p. 191]. Set $\ell = \max\{c, d\} + 1$. Then we have arrived at a situation where any parameter $a$ for $M$ which is in $\mathfrak{m}^\ell$ is of the form $a = xy$ with $y \in \mathfrak{m}^d$. Furthermore, since we have assumed $M$ is faithful, the parameters for $M$ are just the parameters for $A$ [E Proposition 10.8, p. 237]. Since the parameters for the one-dimensional ring $A$ are those elements not in any minimal prime ideal of $A$, we see that if $a = xy$ is a parameter for $M$, then so are $x$ and $y$.

Since $aM \cap W = 0$, we have $(W + aM)/aM \cong W$, so, as in the proof of [GS1 Proposition (2.4)], we see that the top row in the following commutative diagram is exact:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & W & \longrightarrow & \frac{M}{aM} & \longrightarrow & \frac{M}{aM+W} & \longrightarrow & 0 \\
& & f & & \downarrow g & & \\
& & & \frac{M}{xM+W} & & \\
\end{array}
\]

The maps $f$ and $g$ are each given by multiplication by $y$. The map $f$ is well-defined since $y(xM + W) = aM + yW = aM$, and $g$ is just the composition $g = \beta f$.

To see that $g$ is injective, view $g$ as multiplication by $y$ as follows:

\[
\frac{M/W}{x(M/W)} \xrightarrow{y = y} \frac{M/W}{a(M/W)}
\]

Since $M/W$ is a Cohen-Macaulay module and $y$ is a parameter for $M$, and thus for $M/W$, we have that $y$ is regular on $M/W$. Hence

\[
(xy(M/W) :_{M/W} y) = x(M/W),
\]

so that $g$ is injective.

Information on the dimension of the socles is obtained by applying the functor $\text{Hom}_A(k, -)$; what results is the following exact commutative diagram, where the maps induced by $f$ and $g$ are still injective:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_A(k, W) & \longrightarrow & \text{Hom}_A(k, \frac{M}{aM}) & \longrightarrow & \text{Hom}_A(k, \frac{M}{aM+W}) \\
& & & \downarrow \text{Hom}_A(k,f) & & \downarrow \text{Hom}_A(k,g) & \\
& & & \text{Hom}_A(k, \frac{M}{xM+W}) & & \\
\end{array}
\]

Here we use $\beta^*$ to denote $\text{Hom}_A(k, \beta)$.

The important point is that since $x$ and $a$ are still parameters for the Cohen-Macaulay module $M/W$, we have

\[
\dim_k \text{Hom}_A \left( k, \frac{M}{aM+ W} \right) = \dim_k \text{Hom}_A \left( k, \frac{M/W}{a(M/W)} \right) = N_A(aA; M/W) = r_A(M/W),
\]
and similarly
\[ \dim_k \text{Hom}_A \left( k, \frac{M}{aM + W} \right) = r_A \left( M/W \right). \]
The map \( \text{Hom}_A (k, g) \) is an injection of \( k \)-vector spaces, each of dimension \( r_A (M/W) \), hence this map is an isomorphism. From the surjectivity of this map, it follows that \( \beta^* \) is surjective, so that the top row of Diagram 2.4 is exact. Since the length of the middle module is \( N_A (aA; M) \) and the left-hand module is the socle of \( W \), which is the socle of \( M \), we complete the proof in dimension one using the additivity of length.

\[ \square \]

3. Dimension 2

We thank the referee for pointing out the fascinating technique in this section, and for pointing out the recent work of Goto-Sakurai [GS1, GS2]. Goto-Sakurai successfully apply this technique to the case of Buchsbaum local rings of arbitrary dimension [GS1 Corollary 3.13].

Suppose \( q = (x_1, \ldots, x_d)A \) is an ideal generated by a system of parameters \( x_1, \ldots, x_d \) for a finitely-generated module \( M \) of dimension \( d \) over a Noetherian local ring \((A, m)\). We may form a direct system of modules by setting \( M_i = M/(x_1, \ldots, x_i)M \) and defining maps from \( M_i \to M_{i+1} \) given by multiplication by \( x_{i+1} \). It is known that the direct limit of this system is \( H_m^d (M) \) [BH Theorem 3.5.6]. From this point of view, we see that there is a canonical homomorphism from \( M/qM \) to \( H_m^d (M) \).

For an ideal \( I \) of a Noetherian local ring \((A, m)\) and a finitely-generated module \( M \) we define \( U (I; M) \) to be the unmixed component of the submodule \( IM \) of \( M \); that is, \( U (I; M) \) is the intersection of the primary components of the submodule \( IM \) whose associated primes have maximal dimension, equal to \( \dim M/IM \).

In the course of our proof for dimension 2 we will need to mention several generalizations of the notion of a regular sequence. These definitions can be found in the appendix of [SY1], which is a good source for information concerning modules having finite local cohomologies.

**Definition 3.1.** Suppose \((A, m)\) is a Noetherian local ring, \( M \) is a finitely-generated \( A \)-module of dimension \( d > 0 \), and let \( a \) be an \( m \)-primary ideal. A system of elements \( x_1, \ldots, x_r \) is called an \( a \)-weak \( M \)-sequence if
\[
((x_1, \ldots, x_i-1)M :M x_i) \subseteq ((x_1, \ldots, x_{i-1})M :M a)
\]
for all \( i = 1, \ldots, r \).

Let \( x_1, \ldots, x_d \) be a system of parameters for \( M \) and let \( q = (x_1, \ldots, x_d)A \). We say that \( x_1, \ldots, x_d \) is a standard system of parameters of \( M \) if \( x_1^{n_1}, \ldots, x_d^{n_d} \) is a \( q \)-weak \( M \)-sequence for all \( n_1, \ldots, n_d \geq 1 \).

We say that \( a \) is a standard ideal with respect to \( M \) if every system of parameters of \( M \) contained in \( a \) is a standard system of parameters of \( M \).

We begin by isolating a general statement concerning the index of reducibility of parameter ideals in the case where \( M \) has finite local cohomologies.

**Proposition 3.2.** Let \((A, m, k)\) be a Noetherian local ring and let \( M \) be a finitely-generated \( d \)-dimensional \( A \)-module with \( d > 0 \) such that \( M \) has finite local cohomologies. There exists an integer \( \ell \) such that for every parameter ideal \( q = (x_1, \ldots, x_d) \)
Theorem 3.3. Suppose \( \mathfrak{a} \) and use the surjectivity on the socles to obtain the result.

\[
(3.7) \quad 0 \rightarrow (3.6)
\]

thus by Equation 3.4 we see that

\[
(3.8) \quad U (x_1, \ldots, x_d) \subseteq \mathfrak{a}
\]

Set \( U = U ((x_1, \ldots, x_d) A; M) \). Since \( U_i = ((x_1, \ldots, x_d) M : M x_i) \), we have that

\[
(3.1) \quad N^A (q; M) = \mathfrak{a} \left( \sum_{i=1}^d U_i + x_i M \right) + \mathfrak{a} (H^d_m (M)),
\]

where \( U_i = U ((x_1, \ldots, x_d) A; M) \).

Proof. Since \( M \) has finite local cohomologies, we have a standard ideal \( \mathfrak{a} \) for \( M \) \cite[Corollary 18, p. 264]{SV}; hence every system of parameters of \( M \) contained in \( \mathfrak{a} \) is an \( \mathfrak{a} \)-weak \( M \)-sequence \cite[Theorem 20, p. 264]{SV}. Thus, given any system of parameters \( x_1, \ldots, x_d \) of \( M \) contained in \( \mathfrak{a} \) and any integer \( n \geq 1 \), we have by \cite[Lemma 23]{SV}

\[
(\mathfrak{a}^{n+1} : M) = (x_1, \ldots, x_d) + \sum_{i=1}^d ((x_1, \ldots, x_i) M : M a).
\]

We note that this right hand side is equal to

\[
(3.3) \quad \sum_{i=1}^d ((x_1, \ldots, x_d) M : M x_i) + x_i M).
\]

Set \( U_i = U ((x_1, \ldots, x_d) A; M) \). Since \( U_i = ((x_1, \ldots, x_d) M : M x_i) \), we have that

\[
(3.4) \quad ((x_1^{n+1}, \ldots, x_d^{n+1}) M : M x_1 \cdots x_d^n) = \sum_{i=1}^d (U_i + x_i M).
\]

An important component of this proof is that according to \cite[Lemma 3.12]{GS1}, we may choose \( \ell \) large enough so that for any ideal \( q \) generated by a system of parameters for \( M \), if \( q \subseteq m^\ell \) then the canonical map \( \phi : M / q M \rightarrow H^d_m (M) \) is surjective on the socles; that is, \( \text{Hom}_A (k, \phi) \) is surjective. We also require \( \ell \) to be large enough so that \( m^\ell \subseteq \mathfrak{a} \).

Let \( q = (x_1, \ldots, x_d) \) be a parameter ideal of \( M \) contained in \( m^\ell \) and set \( U_i = U ((x_1, \ldots, x_d) A; M) \). Let \( K \) denote the kernel of the canonical map \( \phi \) from \( M / q M \) to \( H^d_m (M) \). According to the definition of the direct limit, we have

\[
(3.5) \quad K = \bigcup_{n \geq 1} \frac{(x_1^{n+1}, \ldots, x_d^{n+1}) M : M x_1 \cdots x_d^n}{q M},
\]

thus by Equation 3.4 we see that

\[
(3.6) \quad K = \sum_{i=1}^d U_i + x_i M q M.
\]

Now all that is left is to apply the socle functor \( \text{Hom}_A (k, -) \) to the exact sequence

\[
(3.7) \quad 0 \rightarrow K \rightarrow M / q M \rightarrow H^d_m (M)
\]

and use the surjectivity on the socles to obtain the result. \( \square \)

**Theorem 3.3.** Suppose \( (A, \mathfrak{m}) \) is a Noetherian local ring and let \( M \) be a finitely-generated \( A \)-module of dimension 2 such that \( M \) has positive depth, and \( H^1_m (M) \)
is finitely generated. Then there exists an integer $\ell$ such that for every parameter ideal $q$ of $M$, if $q \subseteq m^\ell$ then the index of reducibility of $q$ on $M$ is given by
\begin{equation}
N_A(q; M) = 2 \cdot \mathfrak{S} \left( H^1_m(M) \right) + \mathfrak{S} \left( H^2_m(M) \right).
\end{equation}

In particular, parameter ideals for $M$ in large powers of $m$ all have the same index of reducibility.

Proof. We begin by obtaining an integer $\ell$ from Proposition 3.2. As in the proof of that proposition, we may assume that $\ell$ is large enough so that every system of parameters of $M$ contained in $m^\ell$ is a standard system of parameters of $M$.

Let $a, b$ be a system of parameters of $M$ contained in $m^\ell$ and set $q = (a, b)_A$. Then we have
\begin{equation}
N_A(q; M) = \mathfrak{S} \left( \frac{U_a + bM}{qM} + \frac{U_b + aM}{qM} \right) + \mathfrak{S} \left( H^2_m(M) \right),
\end{equation}
where $U_a = U(aA; M) = (aM :_M b)$, and similarly for $U_b$.

Since $M$ has positive depth, $a$ and $b$ are regular elements on $M$. According to Theorem and Definition 17, p. 261, $a$ and $b$ both kill $H^1_m(M)$. Thus from the long exact sequence for local cohomology obtained from the short exact sequence
\begin{equation}
0 \longrightarrow M \longrightarrow M/aM \longrightarrow M/aM \longrightarrow 0
\end{equation}
we see that $H^0_m(M/aM) \cong H^1_m(M)$. Since $M/aM$ has dimension 1, $H^0_m(M/aM)$ is just $U(aA; M)/aM = (aM :_M b)/aM$. Furthermore, we have a surjective homomorphism $(aM :_M b) \to ((aM :_M b) + bM)/bM$ whose kernel is $(aM :_M b) \cap bM$. Using the definition of a standard system of parameters, we see that this last expression is just $aM$. Thus we have seen that $H^3_m(M) \cong (aM :_M b) + bM)/bM$. The same holds if we interchange $a$ and $b$.

At this point all that remains is to see that the sum
\begin{equation}
\frac{U_a + bM}{qM} + \frac{U_b + aM}{qM}
\end{equation}
is direct. To this end, we note that
\begin{equation}
(U_a + bM) \cap (U_b + aM) = (U_a \cap (U_b + aM)) + bM = (U_a \cap U_b) + aM + bM
\end{equation}
Using the fact that $a$ and $b$ are regular on $M$ and form a standard system of parameters of $M$, we see that
\begin{equation}
U_a \cap U_b = (aM :_M b) \cap (bM :_M a) = aM \cap bM.
\end{equation}
Thus
\begin{equation}
(U_a + bM) \cap (U_b + aM) = qM,
\end{equation}
the sum is direct, and our proof is complete. \hfill \Box

Corollary 3.4. Let $(A, m)$ be a Noetherian local ring of dimension at most 2. Then $A$ is Gorenstein if and only if $A$ has finite local cohomologies and every power of the maximal ideal contains an irreducible parameter ideal.

Proof. If $A$ is Gorenstein, then all the local cohomology modules other than $H^2_m(A)$ are zero, and all the parameter ideals are irreducible.

For the other direction, note that it suffices to show that $A$ is Cohen-Macaulay, since a Cohen-Macaulay local ring with an irreducible parameter ideal is precisely a
Gorenstein local ring. The theorem is trivial in dimension zero: a zero dimensional Noetherian local ring is Gorenstein if and only if the zero ideal is irreducible. When the dimension of $A$ is one, the result essentially goes back to Northcott-Rees [NR1, Lemma 7]: If $A$ has SCI, then $A$ has positive depth.

In dimension 2, using the fact that the depth is positive, we know from Theorem 3.2 that all parameter ideals in a high power of the maximal ideal have the same index of reducibility, namely $2 \cdot \mathcal{S}(H^1_m(A)) + \mathcal{S}(H^2_m(A))$. According to our hypothesis, this integer must be 1. Since $H^2_m(A)$ is a nonzero Artinian module, it has a nonzero socle. Thus is has a cyclic socle, and $H^1_m(A)$ is an Artinian module with zero socle. Thus $H^1_m(A)$ is zero, so that $A$ is Cohen-Macaulay. □

**Remark 3.5.** The technique in this section can be used to give a different proof in the case of dimension 1.

### 4. An Example

We begin with a lemma concerning idealizations.

**Definition 4.1.** Given a ring $R$ and an $R$-module $M$ we define the ring $R \ltimes M$ to be the symmetric algebra of $M$ modulo the square of the positive piece. Thus $R \ltimes M$ is a graded ring whose degree 0 piece is $R$, whose degree 1 piece is $M$, and whose components of degree greater than 1 are zero. This ring is called the idealization of $R$ with $M$, or the trivial extension of $R$ by $M$.

For more information on idealization, we refer the reader to [Na, p. 18] or [BH, Exercise 3.3.22].

**Lemma 4.2.**

1. Let $R$ be a ring and let $M$ be an $R$-module. If $I$ is an ideal of $R$, then
   \[ \frac{R \ltimes M}{I(R \ltimes M)} \cong \frac{R}{I} \ltimes \frac{M}{IM}. \]

2. Let $(R, m, k)$ be a Noetherian local ring and let $M$ be an $R$-module. Set $A = R \ltimes M$. Then
   \[ S(A) = S(R) \cap \text{ann}_R(M) + S(M). \]
   In particular, we have
   \[ \mathcal{S}(A) = \mathcal{S}(M) + \dim_k(S(R) \cap \text{ann}_R(M)). \]

3. Let $(R, m, k)$ be a Noetherian local ring and let $M$ be an $R$-module. Set $A = R \ltimes M$. If $q$ is an irreducible $m$-primary ideal of $R$ which does not contain $\text{ann}_R(M)$, then $qA$ is a parameter ideal of $A$ for which
   \[ N_A(qA; A) = N_R(q; M) + 1. \]

**Proof.**

1. As $R$-modules, we have the isomorphism, and we immediately see that this homomorphism respects multiplication.

2. We regard $A$ as $A = R + Mt$ with $t$ an indeterminate and $t^2 = 0$. The maximal ideal of $A$ is $m + Mt$. An element $r + mt$ is in $S(A)$ if and only if
   \[ 0 = (r + mt)(m + Mt) = rm + (rMt + mmt), \]
   which happens precisely when $r \in S(R) \cap \text{ann}_R(M)$ and $m \in S(M)$, as desired. The statement about dimensions now follows from the fact that the $k$-vector space structure of the socle is induced through its graded $R$-module structure.
(3) We have
\[ N_A(q; A) = \mathfrak{S}(A/qA) = \mathfrak{S}\left(\frac{R}{q} \times \frac{M}{qM}\right) = \mathfrak{S}(M/qM) + \dim_k S(R/q) \cap \text{ann}_{R/q}(M/qM) = N_R(q; M) + \dim_k S(R/q) \cap \text{ann}_{R/q}(M/qM) \]

Now since \( q \) is an irreducible \( m \)-primary ideal, the socle of \( R/q \) is simple and essential, and is thus contained in every nonzero submodule of \( R/q \). Hence all that remains is to note that \( \text{ann}_{R/q}(M/qM) \) is not zero, since it contains \( (q + \text{ann}_{R}(M))/q \).

\[ \square \]

We thank the referee for suggesting the following example in dimension 2, and for pointing out the interesting paper of C. Lech [1].

**Example 4.3.** Let \( k \) be a field and let \( d > 1 \) be an integer. Put \( R = k[[x, y, z_3, \ldots, z_d]] \), the formal power series ring over \( k \) in \( d \) variables. Let \( M = R/(x^2 R) \) and set \( A = R \times M \). Let \( w \) be a new variable, then \( A \cong T/(x^2, w^2)T \), where \( T = k[[x, y, z_3, \ldots, z_d, w]] \). Thus \( A \) is a complete Noetherian local ring of dimension \( d \) and depth \( d - 1 \).

For each integer \( n \geq 3 \) we define two parameter ideals of \( R \):
\[ q = (x^n, y^n, z_3^n, \ldots, z_d^n)R \quad \text{and} \quad q' = ((x + y)^n, xy^{n-1}, z_3^n, \ldots, z_d^n)R. \]
Let \( Q = qA \) and \( Q' = q'A; \) these are parameter ideals of \( A \) contained in \( m^n \). We show that \( N_A(Q; A) = 2 \) while \( N_A(Q'; A) = 3 \).

Neither \( Q \) nor \( Q' \) contain \( \text{ann}_R(M) = x^2R \), and since \( R \) is regular (hence Gorenstein), each of \( q \) and \( q' \) are irreducible. Hence from Part 3 of Lemma [1.2] we see that to decide the index of reducibility of \( Q \) and \( Q' \), we just need to consider
\[ N_R(q; M) = \dim_k R/(q + x^2 R), \]
and similarly for \( q' \).

We have
\[ q + x^2 R = (x^2, y^n, z_3^n, \ldots, z_d^n)R \quad \text{and} \quad q' + x^2 R = (x^2, xy^{n-1}, y^n, z_3^n, \ldots, z_d^n)R. \]
The first of these ideals is a parameter ideal for the Gorenstein ring \( R \), and is thus irreducible. For the second we have
\[ (x^2, xy^{n-1}, y^n, z_3^n, \ldots, z_d^n)R = (x^2, y^{n-1}, z_3^n, \ldots, z_d^n)R \cap (x, y^n, z_3^n, \ldots, z_d^n)R. \]
Hence \( N_A(Q; A) = 1 + N_R(q; M) = 2 \) while \( N_A(Q'; A) = 1 + N_R(q'; M) = 3 \).

**Remark 4.4.** We would like to point out that a bad example such as this may be obtained among local Noetherian domains. The referee kindly brought our attention to a paper of C. Lech [1] which proves that a complete Noetherian local ring \( A \) is the completion of a Noetherian local domain if and only if the prime ring of \( A \) is a domain that acts without torsion on \( A \), and the maximal ideal of \( A \) is not associated to 0. (Here the prime ring of a ring is the subring generated by the multiplicative identity.)
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