Explicit Rational Solution of the KZ Equation (example)

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Abstract

We investigate the Knizhnik-Zamolodchikov linear differential system. The coefficients of this system are rational functions. We have proved that the solution of the KZ system is rational when $k$ is equal to two and $n$ is equal to three (see [5]). In this paper, we construct the corresponding solution in the explicit form.

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Introduction

We will consider the system of the form:

\[
\frac{dW}{dz} = -2A(z)W, \tag{0.1}
\]

where \(A(z)\) and \(W(z)\) are 3×3 matrices, \(z_1 \neq z_2\). We suppose that \(A(z)\) has the form

\[
A(z) = \frac{P_1}{z - z_1} + \frac{P_2}{z - z_2}. \tag{0.2}
\]

Here:

\[
P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.3}
\]

\[
P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{0.4}
\]

The matrices \(P_1\) and \(P_2\) are connected with the matrix representation of the symmetric group. System (0.1) is a special case of the Knizhnik-Zamolodchikov [1], [2]. We have proved that the solution of system (0.1) is rational [5]. In this paper, we construct the corresponding solution in the explicit form. We consider the case when \(S_3\) and use the method of L. Sakhnovich [3].

1 Main Notions, The Coefficients of the solution in the neighborhood of \(z = \infty\)

The solution \(W(z)\) of system (0.1) has the form [5]:

\[
W(z) = \frac{L_1}{(z - z_1)^2} + \frac{L_2}{(z - z_1)} + \frac{L_3}{(z - z_2)^2} + \frac{L_4}{z - z_2} + z^2G_{-2} + zG_{-1} + G_0. \tag{1.1}
\]

In a neighborhood of \(z = \infty\) the solution \(W(z)\) can be represented in the form

\[
W(z) = \sum_{k=-2}^{\infty} z^{-k}G_k, \tag{1.2}
\]
where the coefficients $G_k$ are defined by the relations (see [3]).

\[
[(q + 1)I_3 - 2T]G_{q+1} = 2 \sum_{r+s=q} T_r G_s, \quad r \geq 0
\]  

(1.3)

and

\[
T_r = z_1^{r+1}P_1 + z_2^{r+1}P_2, \quad T = P_1 + P_2.
\]  

(1.4)

The eigenvalues of $T$ are

\[
\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1.
\]  

(1.5)

The corresponding eigenvectors have the forms:

\[
\ell_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \ell_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \ell_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.
\]  

(1.6)

First, we will begin with finding all of the coefficients from $G_{-2}$ to $G_{-4}$. The eigenvalues of matrix $2T$ are twice the eigenvalues of the matrix $T$. Thus we get:

\[
\mu_1 = 4, \quad \mu_2 = 2, \quad \mu_3 = -2.
\]  

(1.7)

The eigenvectors remain unchanged.

The smallest eigenvalue of $2T$ is equal to $(-2)$. That is why we begin with the coefficient $G_{-2}$. From equation (1.3) we can say that

\[
(-2I_3 - 2T)G_{-2} = 0.
\]  

(1.8)

Using equation (1.6) and (1.8) we conclude that

\[
G_{-2} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}
\]  

(1.9)

When the coefficient is $G_{-1}$, equation (1.3) takes the form

\[
(-I_3 - 2T)G_{-1} = 2T_0 G_{-2}
\]  

(1.10)

From the last relation we find that

\[
G_{-1} = 2 \begin{bmatrix} -z_1 - z_2 \\ z_2 \\ z_1 \end{bmatrix}.
\]  

(1.11)
When \( q + 1 = 0 \), we get the relation:

\[
-2TG_0 = 2(T_0G_{-1} + T_1G_{-2}).
\]  

(1.12)

From this we find that

\[
G_0 = \begin{bmatrix}
-z_1^2 + 4z_1z_2 - z_2^2 \\
z_1(z_1 - 2z_2) \\
z_2(-2z_1 + z_2)
\end{bmatrix}.
\]  

(1.13)

When \( q + 1 = 1 \) we obtain:

\[
(I_3 - 2T)G_1 = 2(T_0G_0 + T_1G_{-1} + T_2G_{-2}).
\]  

(1.14)

Now we have

\[
G_1 = 2 \begin{bmatrix}
0 \\
(z_1 - z_2)^3 \\
-(z_1 - z_2)^3
\end{bmatrix}.
\]  

(1.15)

When \( q + 1 = 2 \):

\[
(2I_3 - 2T)G_2 = 2(T_0G_1 + T_1G_0 + T_2G_{-1} + T_3G_{-2}).
\]  

(1.16)

**Remark 1.1**  
When \( q + 1 = -1, 0, 1 \) the matrices \((q + 1)I_3 - 2T\) are invertible. That is why \( G_{-1}, G_0, \) and \( G_1 \) are correctly defined by formulas (1.11), (1.13), and (1.15). The situation changes when \( q + 1 = 2 \) because 2 is an eigenvalue of the matrix \( 2T \). In this case, the matrix \( 2I_3 - 2T \) is not invertible. The right-hand side of equation (1.16) has the form:

\[
\begin{bmatrix}
4(z_1 - z_2)^4 \\
-2(z_1 - z_2)^4 \\
-2(z_1 - z_2)^4
\end{bmatrix}.
\]  

(1.17)

The eigenvalues of \((2I_3 - 2T)\) are

\[
\mu_1 = -2, \quad \mu_2 = 0, \quad \mu_3 = 4.
\]  

(1.18)

The right side of (1.16) is the linear combination of the vectors \( \ell_1 \) and \( \ell_3 \). From relations (1.6), (1.16), and (1.18) we obtain

\[
G_2 = \begin{bmatrix}
(z_1 - z_2)^4 \\
-\frac{1}{2}(z_1 - z_2)^4 \\
-\frac{1}{2}(z_1 - z_2)^4
\end{bmatrix}.
\]  

(1.19)
When \( q + 1 = 3 \) we obtain:

\[
(3I_3 - 2T)G_3 = 2(T_0G_2 + T_1G_1 + T_2G_0 + T_3G_{-1} + T_4G_{-2}). \tag{1.20}
\]

Using our previous results we find that

\[
G_3 = \begin{bmatrix}
\frac{3}{5}(z_1 - z_2)^4(z_1 + z_2) \\
\frac{1}{5}(z_1 - z_2)^3(6z_1^2 - 25z_1z_2 + 9z_2^2) \\
-\frac{1}{5}(z_1 - z_2)^3(9z_1^2 - 25z_1z_2 + 6z_2^2)
\end{bmatrix}. \tag{1.21}
\]

When \( q + 1 = 4 \) we use the formula

\[
(4I_3 - 2T)G_4 = 2(T_0G_3 + T_1G_2 + T_2G_1 + T_3G_0 + T_4G_{-1} + T_5G_{-2}). \tag{1.22}
\]

The right side of (1.22) has the form

\[
\begin{bmatrix}
\frac{9}{10}(z_1 - z_2)^4(3z_1^2 - 4z_1z_2 + 3z_2^2) \\
\frac{1}{10}(z_1 - z_2)^3(15z_1^3 - 29z_1^2z_2 - 50z_1z_2^2 + 24z_2^3) \\
-\frac{1}{10}(z_1 - z_2)^3(24z_1^3 - 50z_1^2z_2 - 29z_1z_2^2 + 15z_2^3)
\end{bmatrix}. \tag{1.23}
\]

The case when \( q + 1 = 4 \) is similar to the case when \( q + 1 = 2 \) (see Remark 1.1). The eigenvalues of \((4I_3 - 2T)\) are

\[
\mu_1 = 0, \quad \mu_2 = 2, \quad \mu_3 = 6. \tag{1.24}
\]

The right side of (1.22) is the linear combination of the vectors \( \ell_2 \) and \( \ell_3 \).

From relations (1.22), (1.23), and (1.24) we obtain

\[
G_4 = \begin{bmatrix}
\frac{3}{10}(z_1 - z_2)^4(3z_1^2 - 4z_1z_2 + 3z_2^2) \\
\frac{1}{10}(z_1 - z_2)^3(15z_1^3 - 29z_1^2z_2 - 50z_1z_2^2 + 24z_2^3) \\
-\frac{1}{10}(z_1 - z_2)^3(24z_1^3 - 50z_1^2z_2 - 29z_1z_2^2 + 15z_2^3)
\end{bmatrix}. \tag{1.25}
\]

From (1.1) we wind up with the following system:

\[
L_2 + L_4 = G_1 \tag{1.26}
\]
\[
L_1 + L_2z_1 + L_3 + L_4z_2 = G_2 \tag{1.27}
\]
\[
2L_1z_1 + L_2z_1^2 + 2L_3z_2 + L_4z_2^2 = G_3 \tag{1.28}
\]
\[
3L_1z_1^2 + L_2z_1^3 + 3L_3z_2^2 + L_4z_2^3 = G_4 \tag{1.29}
\]
System (1.26)-(1.29) can be written in the matrix form:

\[ SX = Y, \quad (1.30) \]

where

\[
S = \begin{bmatrix}
0 & I_3 & 0 & I_3 \\
I_3 & z_1 & I_3 & z_2 \\
2z_1I_3 & z_1^2I_3 & 2z_2I_3 & z_2^2I_3 \\
3z_1^2I_3 & z_1^3I_3 & 3z_2^2I_3 & z_2^3I_3
\end{bmatrix},
\quad (1.31)
\]

\[ X = \text{col}[L_1, L_2, L_3, L_4], \quad (1.32) \]

\[ Y = \text{col}[G_1, G_2, G_3, G_4]. \quad (1.33) \]

In equation (1.30) the matrices \( S \) and \( Y \) are known, but the matrix \( X \) is unknown.

From relation (1.30) we get that

\[ X = S^{-1}Y \quad (1.34) \]

where

\[
S^{-1} = \begin{bmatrix}
-\frac{z_1z_2^2}{(z_1-z_2)^2}I_3 & \frac{z_2(2z_1+z_2)}{(z_1-z_2)^3}I_3 & -\frac{z_1+2z_2}{(z_1-z_2)^2}I_3 & \frac{1}{(z_1-z_2)^3}I_3 \\
\frac{z_2^2}{(z_1-z_2)^2}I_3 & -\frac{6z_1z_2}{(z_1-z_2)^3}I_3 & \frac{3z_1+z_2}{(z_1-z_2)^2}I_3 & \frac{2}{(z_1-z_2)^3}I_3 \\
-\frac{z_2^2}{(z_1-z_2)^2}I_3 & \frac{z_1(z_1+z_2)}{(z_1-z_2)^3}I_3 & -\frac{z_1+2z_2}{(z_1-z_2)^2}I_3 & \frac{1}{(z_1-z_2)^3}I_3 \\
\frac{z_2^3}{(z_1-z_2)^2}I_3 & \frac{6z_1z_2}{(z_1-z_2)^3}I_3 & \frac{3z_1+z_2}{(z_1-z_2)^2}I_3 & \frac{2}{(z_1-z_2)^3}I_3
\end{bmatrix},
\quad (1.35)
\]

Thus, we find that

\[
L_1 = \begin{bmatrix}
\frac{1}{10}(3z_1 - 7z_2)(z_1 - z_2)^3 \\
\frac{1}{10}(3z_1 - 7z_2)(z_1 - z_2)^3 \\
-\frac{1}{5}(3z_1 - 7z_2)(z_1 - z_2)^3
\end{bmatrix},
\quad (1.36)
\]

\[
L_2 = \begin{bmatrix}
0 \\
\frac{1}{5}(3z_1 - 7z_2)(z_1 - z_2)^2 \\
-\frac{1}{5}(3z_1 - 7z_2)(z_1 - z_2)^2
\end{bmatrix},
\quad (1.37)
\]

\[
L_3 = \begin{bmatrix}
\frac{1}{10}(7z_1 - 3z_2)(z_1 - z_2)^3 \\
-\frac{1}{5}(7z_1 - 3z_2)(z_1 - z_2)^3 \\
\frac{1}{10}(7z_1 - 3z_2)(z_1 - z_2)^3
\end{bmatrix},
\quad (1.38)
\]
and
\[
L_4 = \begin{bmatrix}
0 \\
\frac{1}{5}(7z_1 - 3z_2)(z_1 - z_2)^3 \\
-\frac{1}{5}(7z_1 - 3z_2)(z_1 - z_2)^3
\end{bmatrix}.
\] (1.39)

This way we have proved the following statement:

**Proposition 1** System (0.1) has the following solution:

\[
W_1(z) = \frac{L_1}{(z - z_1)^2} + \frac{L_2}{(z - z_1)} + \frac{L_3}{(z - z_2)^2} + \frac{L_4}{(z - z_2)} + z^2G_{-2} + zG_{-1} + G_0.
\] (1.40)

The matrices $G_k$ and $L_k$ are defined by the relations (1.9), (1.1), (1.13), and (1.36) - (1.39).

To find the next solution to the system (0.1) we consider the case

\[
g_k = 0 \quad \text{when} \quad k < 2
\] (1.41)

In this case, we have

\[
g_2 = \text{col}[0, 1, -1].
\] (1.42)

From relation (1.3) we get:

\[
(3I_3 - 2T)g_3 = T_0g_2.
\] (1.43)

From this we find that

\[
g_3 = \frac{1}{5} \begin{bmatrix}
z_1 - z_2 \\
2z_1 + 3z_2 \\
-3z_1 - 2z_2
\end{bmatrix}.
\] (1.44)

In order to find $g_4$ we will use equation (1.3) again;

\[
(4I_3 - 2T)g_4 = T_0g_3 + T_1g_2.
\] (1.45)

The right side of the equation (1.45) has the form:

\[
\frac{1}{5} \begin{bmatrix}
7(z_1 - z_2)(z_1 + z_2) \\
\frac{1}{5}z_1^2 + z_1z_2 + \frac{8}{5}z_2^2 \\
-8z_1^2 - z_1z_2 - z_2^2
\end{bmatrix} = \begin{bmatrix}
\phi \\
-\frac{1}{5}\phi \\
-\frac{1}{5}\phi
\end{bmatrix} + \begin{bmatrix}
0 \\
\psi + \frac{1}{5}\phi \\
-\psi - \frac{1}{5}\phi
\end{bmatrix},
\] (1.46)

where

\[
\phi = \frac{7}{5}(z_1 - z_2)(z_1 + z_2), \quad \psi = \frac{1}{5}z_1^2 + z_1z_2 + 8z_2^2
\] (1.47)
Analogously (1.34) we can write:

\[ X = S^{-1}Y, \]  

(1.48)

where

\[ X = \text{col}[M_1, M_2, M_3, M_4], \]  

(1.49)

\[ Y = \text{col}[0, g_2, g_3, g_4]. \]  

(1.50)

Thus, we can say that

\[ M_1 = \begin{bmatrix} \frac{5z_1+3z_2}{10(z_1-z_2)} & \frac{z_1z_2+9(-2z_1^2+2z_1z_2+z_2^2)}{30(z_1-z_2)^2} - \frac{z_1z_2-3z_1(z_1-4z_2)}{30(z_1-z_2)^2} \end{bmatrix}, \]  

(1.51)

\[ M_2 = \begin{bmatrix} \frac{-4(z_1+z_2)}{5(z_1-z_2)^2} & -\frac{24z_1^2+46z_1z_2-12z_2^2}{15(z_1-z_2)^3} -\frac{12z_1^2+46z_1z_2-24z_2^2}{15(z_1-z_2)^3} \end{bmatrix}, \]  

(1.52)

\[ M_3 = \begin{bmatrix} \frac{-3z_1+5z_2}{10(z_1-z_2)} & \frac{z_1z_2+3(4z_1-z_2)z_2}{30(z_1-z_2)^2} - \frac{z_1z_2+9(z_1^2+2z_1z_2-2z_2^2)}{30(z_1-z_2)^2} \end{bmatrix}, \]  

(1.53)

and

\[ M_4 = \begin{bmatrix} \frac{4(z_1+z_2)}{5(z_1-z_2)^2} & -\frac{24z_1^2+46z_1z_2-12z_2^2}{15(z_1-z_2)^3} -\frac{12z_1^2+46z_1z_2-24z_2^2}{15(z_1-z_2)^3} \end{bmatrix}. \]  

(1.54)

This way we have proved the following statement:

**Proposition 2** System (0.1) has the following solution:

\[ W_2(z) = \frac{M_1}{(z-z_1)^2} + \frac{M_2}{(z-z_1)} + \frac{M_3}{(z-z_2)^2} + \frac{M_4}{(z-z_2)}. \]  

(1.55)

The matrices \( M_k \) are defined by the relations (1.51) - (1.54).
The next solution of system (0.1) has the form

\[
W_3(z) = \frac{N_1}{(z - z_1)^2} + \frac{N_2}{(z - z_1)} + \frac{N_3}{(z - z_2)^2} + \frac{N_4}{(z - z_2)}. \tag{1.56}
\]

In order to find \( N_1, N_2, N_3 \), and \( N_4 \) we consider the case when

\[
G_k = 0 \quad \text{when} \quad k < 4 \quad \text{and} \quad G_4 = \ell_1. \tag{1.57}
\]

From relations (1.34) and (1.35) we have

\[
N_1 = N_3 = \frac{1}{(z_1 - Z_2)^2} \ell_1 \tag{1.58}
\]

\[
N_2 = -N_4 = \frac{2}{(-z_1 + z_2)^3}. \tag{1.59}
\]

The main theorem follows from Propositions 1 - 3.

**Theorem 1**

*The general solution of system (0.1) has the form:*

\[
W_z = \alpha_1 W_1(z) + \alpha_2 W_2(z) + \alpha_3 W_3(z), \tag{1.60}
\]

*where \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) are arbitrary constants.*

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