Quantization of the non-projectable 2+1D Hořava theory: The second-class constraints

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Abstract

We present the quantization of the 2+1 dimensional nonprojectable Hořava theory. The central point of the approach is that this is a theory with second-class constraints, the quantization procedure must take account of them. We consider all the terms in the Lagrangian that are compatible with the foliation-preserving-diffeomorphisms symmetry, up to the $z = 2$ order which is the minimal order indicated by power-counting renormalizability. The measure of the path integral must be adapted to the second-class constraints, and this has consequences in the quantum dynamics of the theory. Since this measure is defined in terms of Poisson brackets between the second-class constraints, we develop all the Hamiltonian formulation of the theory with the full Lagrangian. We found the soundly result that the lapse function (and the metric) acquires a totally regular form. The quantization requires the incorporation of a Lagrange multiplier for a second-class constraint and fermionic ghosts associated to the measure of the second-class constraints. It turns out that these unphysical variables have related propagators. These results signal that the second-class-constraint sector play a crucial role for the consistent renormalization of the theory.
1 Introduction

Hořava theory [1, 2] is a geometrical field theory that may be used to study quantum gravity since it is power-counting renormalizable and unitary. As a field theory, it has some open questions that deserve deep analysis. For example, the consistent quantization of its nonprojectable version is still pending. This is a rather nondirect program since the nonprojectable theory has second-class constraints, any scheme of quantization must take account of them. Once a consistent framework for such a quantized theory has been established, an important application for it is to prove (or disprove) its renormalizability. On the contrary, the projectable version has not second-class constraints, hence its quantization can be achieved with the standard techniques of gauge field theories. Indeed, the renormalizability of the projectable version has been proven [3]. Moreover, it is a theory with asymptotic freedom in 2+1 dimensions [4]. Quantum corrections to the 2+1 projectable theory has been studied in Ref. [5].

The geometrical framework introduced in the Hořava theory is to represent the gravitating space as a foliation of spacelike hypersurfaces along a given direction of time. The foliation is considered as absolute, that is, it cannot be changed by a symmetry transformation. The theory is formulated in terms of the Arnowitt-Deser-Misner (ADM) variables: the spatial metric $g_{ij}$, the lapse function $N$ and the shift vector $N_i$, which are natural for such a foliation [6]. The gauge symmetry of the theory is given by the group of all the foliation-preserving diffeomorphisms (FDiff) acting on these variables. We emphasize that the presence of this gauge symmetry does not guarantee that the standard quantization procedures for gauge theories –Faddeev-Popov [7], BRST, background-field method, etc., are sufficient to perform the quantization of this theory, due to the presence of the second-class constraints. The second-class constraints are not associated to gauge symmetries. Besides the quantization of the gauge sector, one must find a way to incorporate the second class constraints as restrictions on the phase space, hence making a consistent quantization.

The FDiff gauge symmetry leads to a theory that includes higher order terms in spatial derivatives. As a consequence, it is expected that the renormalizability of the theory is improved with respect to general relativity, since the behavior of propagators is improved in the ultraviolet. Simultaneously, unitarity can be safe since no higher time derivatives are generated. The theory has two versions, the projectable version where the lapse function $N$ is a function only of time and the nonprojectable version where $N$ may depend on the time and the space. The projectable condition is preserved by the FDiff symmetry group, hence the projectable case constitutes an independent formulation. The nonprojectable version has field equations closer to the Einstein equations, and its 3 + 1 formulation has more chance of surviving the observational tests than the projectable case. In the nonprojectable version, a fundamental extension of the Lagrangian was proposed in Ref. [8] by including terms depending on the FDiff-covariant vector $a_i = \partial_i \ln N$. 

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Computations in quantum gravity are typically difficult. In the case of the non-projectable $3 + 1$ Hořava gravity, the Lagrangian includes a number of the order of $10^2$ different terms that are compatible with the FDiff gauge symmetry. However, when the dimensionality of the space is reduced to $2 + 1$, the number of independent terms in the Lagrangian reduces drastically and the theory is still interesting for doing quantum gravity. Indeed, an outstanding feature of the $2 + 1$ Hořava theory is that it propagates a physical, scalar, degree of freedom, unlike $2 + 1$ general relativity which is a topological theory. Thus, $2 + 1$ Hořava theory is a three-dimensional model with a particle carrying the local gravitational interaction, hence, in principle, perturbative quantization based on Feynman propagators and Feynman graphs can be used. An exception occurs when the theory is formulated at the critical point, in which case a physical mode is suppressed. This critical point, whose definition depends on the dimensionality of the theory, has been called the kinetic-conformal point [9]. In the $2 + 1$ dimensional case the theory becomes topological at the critical point, since the only propagating mode disappears. This case deserves a separate study, here we only consider the noncritical formulation.

Our objective in this paper is to perform a detailed analysis of the quantization of the $2 + 1$ nonprojectable Hořava theory. As we have commented, an essential feature of the nonprojectable theory is the presence of second-class constraints. This fact forces us to consider the quantization in rather different approaches to ones used in general relativity, projectable Hořava theory and gauge theories in general with only first-class constraints. As it is well known, there are two main routes to manage theories with second-class constraints, namely, the path integral quantization with the appropriate measure for the second-class constraints and the Dirac brackets in the operator formalism. We focus on the path integral quantization, since it is more adaptable to a gravitational field theory as the Hořava theory. Since the measure corresponding to the second-class constraints is defined in terms of the canonical variables and their Poisson brackets [10], our approach is based on the Hamiltonian formulation of the theory.

We consider the full $2 + 1$ theory. Hence we consider in the Lagrangian all the inequivalent terms that are compatible with the FDiff symmetry, up to the minimal order in spatial derivatives required by power-counting renormalizability, which is $z = 2$ in the $2 + 1$ theory [1]. This yields a Lagrangian with terms of second, third and fourth order in spatial derivatives. We combine the perturbative approach, where the constraints can be solved and one can obtain the propagator of the theory, with formal nonperturbative and nonreduced approaches.

We comment that, in spite of being a nontopological theory, the $2 + 1$ Hořava theory still shares some features with $2 + 1$ general relativity. A fundamental issue with consequences in the local quantization is the definition of asymptotic flatness, which in particular is relevant for the Hamiltonian formulation of the theory. In $2 + 1$ general relativity the definition of asymptotic flatness is not based on having a fixed metric at infinity, unlike the $3 + 1$ case. The definition lies on the existence of the exact solution corresponding to the gravitational field of a massive point particle.
This solution is a locally flat cone with a deficit angle that depends on the mass of the particle \([11]\). An asymptotically flat configuration is then a configuration that approaches this solution for large enough distances. As a consequence, the dominant mode in the expansion is not fixed functionally, as we have commented. In a previous paper \([12]\), we studied the analogous situation in the 2+1 nonprojectable Hořava theory, finding that the same solution for the massive point particle is valid in the Hořava theory. Thus, we proposed the same definition of asymptotic flatness as in 2+1 general relativity \([13]\) for the three-dimensional Hořava theory. More consequences on the value of the energy and the role of the higher order terms were considered in that reference.

2 The nonprojectable Hořava theory in 2 spatial dimensions

The starting point is the definition of a foliation formed by two-dimensional spatial slices, the foliation being defined along a direction of time. This setting is considered as absolute, it can not be changed by a symmetry transformation. Thus, the underlying gauge symmetry group is given by the diffeomorphisms that preserve the foliation, FDiff. Under a FDiff transformation, the coordinates \((t, \vec{x})\) transform as

\[
\delta t = f(t), \quad \delta x^i = \zeta^i(t, \vec{x}).
\]  (2.1)

The Hořava theory is formulated in the Arnowitt-Deser-Misner (ADM) variables, which are the spatial metric \(g_{ij}\), lapse function \(N\) and shift function \(N_i\). Under a FDiff transformation, the ADM variables transform as

\[
\delta N = \zeta^k \partial_k N + f \dot{N} + \dot{f} N, \quad \partial N_i = \zeta^k \partial_k N_i + N_k \partial_i \zeta^k + \dot{\zeta}^i g_{ij} + f \dot{N}_i + \dot{f} N_i, \quad \delta g_{ij} = \zeta^k \partial_k g_{ij} + 2 g_{k(i} \partial_{j)} \zeta^k + f \dot{g}_{ij}. \quad (2.2)
\]

With the FDiff gauge symmetry one may define the Hořava theory in the projectable and the nonprojectable formulations. In the projectable version the lapse function is a function only on time, \(N(t)\). This condition is preserved by the FDiff symmetry, as it can be directly deduced from (2.2). We will focus in the nonprojectable version and the minimum grade of anisotropy to ensure the renormalization of the Hořava theory.

The action of the nonprojectable Hořava theory in 2+1 dimensions is

\[
S = \int dt d^2x \sqrt{\bar{g}N} \left( G^{ijkl} K_{ij} K_{kl} - V \right),
\]  (2.5)

where the extrinsic curvature is defined by

\[
K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - 2 \nabla_{(i} N_{j)} \right).
\]  (2.6)
and the hypermatrix $G^{ijkl}$ is a four-index metric

$$G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl}. \quad (2.7)$$

The dimensionless parameter $\lambda$ that appears inside of the kinetic term play role fundamental in the theory. The matrix $G^{ijkl}$ has inverse if only if $\lambda \neq 1/2$, a condition that we assume throughout this paper. On the contrary, when this parameter acquires the critical value $\lambda = 1/d$, where $d$ is the spatial dimension, the kinetic term acquires conformal invariance, although the whole theory is not conformally invariant due to the potential that breaks this symmetry. The critical theory can become conformal if the potential is conformal, like a Cotton-square term and the potential studied in [14]. At the critical point the extra scalar mode is eliminated due to the raising of two second-class constraints [15]. Recently it has been focused the quantization of the Hořava theory at the kinetic-conformal point in $3 + 1$ dimensions [9].

The potential $\mathcal{V}$ must be invariant under FDiff. It must be formed by invariants written in terms of the spatial metric and the acceleration vector [8]

$$a_k = \partial_k \ln N. \quad (2.8)$$

The full potential in $2 + 1$ dimensions, considering all the terms up to $z = 2$, is given by [16]

$$\mathcal{V} = -\beta R - \alpha a^2 + \alpha_1 R^2 + \alpha_2 a^4 + \alpha_3 R a^2 + \alpha_4 a^2 \nabla_k a^k + \alpha_5 R \nabla_k a^k + \alpha_6 \nabla^l a^k \nabla_l a_k + \alpha_7 (\nabla_k a^k)^2 \quad (2.9)$$

where $\beta, \alpha$, and the $\alpha_{1,\ldots,7}$ are independent coupling constants.

## 3 Canonical formulation

### 3.1 Hamiltonian and constraints

Our principal aim is to address the path integral quantization of the theory. To this end we perform the canonical formulation. The nonreduced phase space is spanned by the conjugate pairs $(g_{ij}, \pi^{ij})$ and $(N, P_N)$. The first (primary) constraint that arises in the formulation is the vanishing of the momentum conjugate to the lapse function,

$$P_N = 0, \quad (3.1)$$

since the Lagrangian in (2.5) does not depend on the time derivative of $N$. Given the Lagrangian in (2.5), the conjugated momentum tensor of the spatial metric has the form

$$\frac{\pi^{ij}}{\sqrt{g}} = G^{ijkl} K_{kl}. \quad (3.2)$$
The time derivative of the metric $\dot{g}_{ij}$ can be completely solved from this expression if only if the coupling constant is $\lambda \neq 1/2$. This is the mathematical reason behind the critical case $\lambda = 1/2$.

As happens in the canonical formulation of general relativity [4], the Legendre transformation automatically incorporates the momentum constraint

$$\mathcal{H}^i = -2\nabla_j \pi^{ij}, \quad (3.3)$$

which generates spatial coordinate transformations only in $(g_{ij}, \pi^{ij})$. Consequently, the shift function $N_i$ is regarded as a Lagrange multiplier. We are interested in the full generator of the spatial diffeomorphisms, therefore we must include the generator of the spatial coordinate transformations in $(N, P_N)$ [17], hence we redefine (3.3) by

$$\mathcal{H}^i = -2\nabla_j \pi^{ij} + P_N \partial_i N. \quad (3.4)$$

Unlike general relativity, the bulk part of the Hamiltonian does not arise as a sum of the primary constraints, it arises instead as

$$H = \int d^2x \left( \frac{N}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} + \frac{\lambda}{1-2\lambda} \pi^2 \right) + N \sqrt{g} V + N_i \mathcal{H}^i + \sigma P_N \right), \quad (3.5)$$

where the function $\sigma$ and $N_i$ are the Lagrange multipliers of the primary constraints.

Before we proceed further, we parenthetically comment on the definition of asymptotic flatness in the 2+1 gravitational theories. The exact solution of a particle at rest in general relativity is a flat cone with a deficit angle that depends of the mass of the particle [11]. This motivates the definition of asymptotically flat condition in general relativity [13]. In the Hořava theory in 2 + 1 dimensions we found [12] that the same solution is valid, hence we introduce the same definition of asymptotic flatness in 2 + 1 Hořava theory, under which the canonical variables behave asymptotically as

$$g_{ij} = r^{-\mu} (\delta_{ij} + \mathcal{O}(r^{-1})), \quad \pi^{ij} \sim \mathcal{O}(r^{-2}), \quad N = 1 + \mathcal{O}(r^{-1}), \quad (3.6)$$

where $\mu$ is an arbitrary constant. The asymptotically flatness conditions impose restrictions on the differentiability of the Hamiltonian (3.5), hence we must add a counterterm to the Hamiltonian (3.5) for its differentiability

$$\mathcal{E} = +2\pi \beta \mu. \quad (3.7)$$

This energy term is the same that general relativity to except by the presence of the constant $\beta$.

Now we move to the time preservation of the primary constraints. The preservation of the $P_N = 0$ generates a secondary constraint, the Hamiltonian constraint

$$\mathcal{H} = \frac{1}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} + \frac{\lambda}{1-2\lambda} \pi^2 \right) + \sqrt{g} \left( V - \frac{1}{N} B(N) \right), \quad (3.8)$$
where the term $B$ stands for the total divergences,

\begin{align}
    B &\equiv -2\alpha \nabla_k (Na^k) + 4\alpha_2 \nabla_k (Na^2 a^k) + 2\alpha_3 \nabla_k (NR a^k) - \alpha_4 (\nabla^2 (Na^2 a^k) \\
    & - 2\nabla_l (\nabla_k (a^k) Na^l)) - \alpha_5 \nabla^2 (NR) - 2\alpha_6 \nabla^k \nabla^l (N\nabla_l a_k) - 2\alpha_7 \nabla^2 (N\nabla_l a_l^i).
\end{align}

(3.9)

Note that the integral of the Hamiltonian constraint (3.8) has the form

\[
\int d^2 x N H = \int d^2 x N \left( \frac{\pi ij \pi ij}{\sqrt{g}} + \frac{\lambda}{1 - 2\lambda} \frac{\pi^2}{\sqrt{g}} + \sqrt{g} \lambda \right) - \int d^2 x \sqrt{g} B.
\]

(3.10)

According to asymptotically flat conditions, the last integral in (3.10) is zero; hence, it does not contribute to the background energy.

The next step is the preservation of the Hamiltonian constraint. The following bracket is useful to this end,

\[
\{ H(y), \int d^2 x \pi \} =
\int d^2 x \left( \mathcal{H} \pi \eta + 2 \left[ f_1 \nabla_k \eta - f_1 \nabla_k \pi \right] \\
- \pi^{kl} \nabla_l f_1 - f_3 \pi^{kl} a_l + \tau_{ij} \pi^{ij} + \tau^i \pi^j + \nabla^i \pi^j - \tau^i \pi^j + \tau \nabla^i \pi^j \\
+ \frac{\partial_l (N f_3)}{N} \left( \frac{1}{2} \tau g^{kl} - \tau^{kl} \right) + (2\alpha_4 - 4\alpha_7) \pi^{ij} a^i a^j \\
+ \frac{\alpha_5}{2} \tau R + 2\alpha_7 \pi^{ij} a^i a^j \right) + 2\alpha_6 \left[ \frac{1}{2} \pi^{ij} \left( \nabla_i (N \nabla^j a^k) - N M^{ij k} \right) \right] \left( \nabla_k \eta - \eta \partial_k N \right)
\]

\[
+ 2N \left[ 
- \alpha_3 - \alpha_6 \right] \left( \pi_{ij} \nabla^i \nabla_j \left( Na^k \partial_k \eta' \right) - \nabla^2 \left( Na^k \partial_k \eta' \right) \right) - \alpha_5 \left( \pi_{ij} \nabla^i \nabla_j \left( \nabla^k \partial_k \eta' \right) \right) \\
- \pi^{ij} \nabla_i \partial_k \eta' \left( N \partial_k \eta' \right) - 2\alpha_6 \left( - \frac{1}{2} \tau_{ij} \nabla_l \left( Na^l \partial_k \eta' \right) \right) \\
+ \tau_{ij} \nabla^i \left( \partial_j \left( N \partial_k \eta' \right) \right) - \frac{1}{2} \tau \nabla_l \left( \partial_j \left( \nabla^k \partial_k \eta' \right) \right) \right) \\
+ \tau_{ij} \nabla^i \left( \partial_j \left( \nabla^k \partial_k \eta' \right) \right) \\
- \frac{1}{2} \nabla \left( Na^l \partial^k \partial_k \eta' \right) \right) - \sqrt{g} \frac{\sigma}{N} \left( B \eta' - 2\nabla^k (\eta a_k f_2) + \nabla^2 (\eta f_3) + 2\alpha_6 \nabla^k \nabla^l (\eta \nabla_l a_k) \\
+ 2N f_3 a^k \nabla_k \eta' + f_3 N \nabla^2 \eta' - \nabla^k (2N f_2 \nabla_k \eta') - 8\alpha_3 \nabla_l (N a^l \partial^k \nabla_k \eta') \\
+ 2\alpha_4 \nabla^2 (Na^k \nabla_k \eta') - 2\alpha_4 \nabla_k (Na^k \nabla^2 \eta') + 2\alpha_7 \nabla^2 (N \nabla^2 \eta') + 2\alpha_6 N \nabla_l a_k \nabla^k \nabla^l \eta' \\
+ 2\alpha_6 \nabla_k \nabla_l (N \nabla^k \nabla^l \eta') \right),
\]

(3.11)

where $\eta$ is arbitrary function and we define $\eta' = \frac{\eta}{N}$. The symbols introduced above
are

\[
\begin{align*}
\tau_{kl} &= \pi_{kl} + \frac{\lambda}{1 - 2\lambda}\pi_{kl}, \quad \tau = g^{kl}\tau_{kl}, \quad (3.12) \\
f_1 &= -\beta + 2\alpha_1 R + \alpha_3 a^2 + \alpha_5 \nabla_{k} a^k, \quad (3.13) \\
f_2 &= -\alpha + \alpha_3 R + 4\alpha_4 \nabla_{k} a^k + 2\alpha_2 a^2, \quad (3.14) \\
f_3 &= \alpha_4 a^2 + \alpha_5 R + 2\alpha_7 \nabla_{k} a^k, \quad (3.15) \\
f_4 &= -\alpha + 2\alpha_2 a^2 + \alpha_4 (a^2 + \nabla_{i} a^i) + \alpha_5 R + \frac{1}{2}\alpha_6 R, \quad (3.16) \\
\mathcal{Z}^{(ij)k} &= \frac{1}{2} f_3 g^{ij} a^k - \alpha_6 \left( a^i \nabla^k a^j + a^i \nabla^j a^k - a^k \nabla^i a^j \right), \quad (3.17) \\
\mathcal{M}^{ij} &= a^i \nabla^j a^k + a^i \nabla_{i} a^k + g^{ki} \nabla_{j} a^i. \quad (3.18)
\end{align*}
\]

To obtain the desired time preservation of the Hamiltonian constraint, we put \( \eta = \delta^2 \) into the bracket (3.11). This yields an elliptic partial differential equation for the Langrange multiplier \( \sigma \),

\[
0 = -\sqrt{g} \left( \frac{2}{N} (\alpha_6 + \alpha_7) \nabla^i \sigma + 4\alpha^k (\alpha_6 + \alpha_7) \nabla_k \nabla^2 \sigma' + 2 \left( f_3 - f_2 - \frac{1}{N} \alpha_4 \nabla_k \left( N a^k \right) \right. \right. \\
\left. + \frac{2}{N} \alpha_7 \nabla^{i} \nabla^{j} \nabla^{k} \nabla^{l} \right) \nabla^2 \sigma' + \left( -8\alpha_2 a^i \nabla^i (Na^l) + \alpha_6 \left( 4 \nabla_l a_k + 2 \frac{1}{N} \nabla_k \nabla_l N \right) \right) \nabla^2 \sigma' + \left( -2\alpha_3 - \alpha_6 \right) \nabla_k \left( Na^k \left( \nabla^i \nabla^j \left( N \tau_{ij} \right) - \nabla^2 \left( N \tau \right) \right) \right) \\
\left. - 2\alpha_3 \partial_k \partial^k \left( N \left( \nabla^i \nabla^j \left( N \tau_{ij} \right) - \nabla^2 \left( N \tau \right) \right) \right) - 4\alpha_4 \nabla_k \left( Na^k a^i \nabla^j \left( N \tau_{ij} \right) \right) \\
+ 4\alpha_6 \nabla^i \left( -\frac{1}{2} Na^i \nabla_l \left( N \tau_{ij} \right) + N \nabla^l \partial_i \left( Na^l \tau_{ij} \right) - \frac{1}{2} N \nabla_l \left( a^k \nabla_k \left( N \tau \right) \right) \right) \\
+ 2\alpha_7 \partial_k \partial^k \left( Na^i \nabla^j \left( N \tau_{ij} \right) - \frac{1}{2} N \nabla^l a^k \right) - \nabla_l \left( N P \right) - Na_k P - \nabla_k \left( \mathcal{H} \nabla^k \right) \right),
\]

where

\[
P = 2 \left( f_1 \nabla^i a^{kl} + (1 + \omega) (\pi \nabla^k f_1 - f_1 \nabla^k \pi) - \pi^{kl} \nabla^i f_1 - f_3 \tau^{kl} a_l + \tau_{ij} \mathcal{Z}^{(ij)k} \right. \\
\left. - \tau a^k f_4 + 2\tau^{kl} a_l (f_2 - \frac{1}{2} \alpha_6 R) + (4\alpha_2 - 2\alpha_4) \tau_{ij} a^i a^j a^k + \frac{\partial_l (N f_3)}{N} \left( \frac{1}{2} \tau g^{kl} - \tau^{kl} \right) \right. \\
\left. + (2\alpha_4 - 4\alpha_7) \tau_{ij} \nabla^i a^j a^k - \partial^k (\alpha_4 \tau_{ij} a^i a^j + \frac{\alpha_5}{2} \tau R + 2\alpha_7 \tau_{ij} \nabla^i a^j) - \alpha_4 a^k \nabla^l a^i \right) \\
\left. + 2\alpha_6 \left( \frac{1}{2} \tau \nabla_l (N \nabla^l a^k) - N \mathcal{M}^{(ij)k}_{\tau_{ij}} \right) \right).
\]

(3.20)
Dirac’s procedure ends at this step; the preservation of the Hamiltonian constraint has generated a elliptic partial differential equation, therefore there are no more constraints.

Summarizing, we have found two primary constraints, (3.1) and (3.4), one secondary constraint (3.8), and a elliptic partial differential equation for the Langrange multiplier $\sigma$. We have shown that the boundary terms do not contribute to the background energy because of the asymptotic flat conditions, therefore to have a differentiable Hamiltonian we just need to add the energy term (3.7).

The differentiable Hamiltonian with all the constraints incorporated has the form

$$H = \int d^3x \left( \frac{N}{\sqrt{g}} \left( \pi^i \pi^i + \frac{\lambda}{1 - 2\lambda} \pi^2 \right) + N \sqrt{g} \mathcal{V} + N_i \mathcal{H}^i + \sigma P_N + A \mathcal{H} \right) + \mathcal{E},$$

where $A$ is a Lagrange multiplier (we assume that it decays asymptotically fast enough). Once all constraints have been incorporated to the Hamiltonian with its respectively Lagrange multiplier, the equations of motion of the canonical variables in the Hamiltonian formalism are

$$\dot{N} = N^k \nabla_k N + \sigma,$$  

$$\dot{g}_{ij} = \frac{2}{\sqrt{g}} (N + A) \left( \pi_{ij} + \frac{\lambda}{1 - 2\lambda} g_{ij} \pi \right) + 2 \nabla_{(i} N_{j)},$$

(3.22)  

(3.23)
\[
\dot{\pi}^{ij} = -(N + A) \left( -\frac{1}{2} g^{ij} \left( \pi^{kl} \pi_{kl} + \frac{\lambda}{1 - 2\lambda} \pi^2 \right) + \frac{2}{\sqrt{g}} \left( \pi^{ik} \pi^j_k + \frac{\lambda}{1 - 2\lambda} \pi^{ij} \right) \right) + \frac{1}{2} g^{ij} \sqrt{g} \nabla^2 + \sqrt{g} \left( -\frac{1}{2} f_1 g^{ij} R - f_2 a^i a^j - f_3 \nabla^i a^j - 2\alpha_6 \nabla^i a^k \nabla^j a_k \right) \right)
\]
3.2 Algebra of the constraints

In this section we show the algebra of the constraints. The Poisson brackets that involve the momentum constraint are

\[
\left\{ \int d^2 x \varepsilon^k \mathcal{H}_k, \int d^2 y \eta^i \mathcal{H}_i \right\} = \int d^2 x \mathcal{H}_l \mathcal{L}_7 \eta^l, \tag{3.25}
\]

\[
\left\{ \int d^2 x \varepsilon^k \mathcal{H}_k, \int d^2 y \eta^i \mathcal{H} \right\} = \int d^2 x \mathcal{H} \mathcal{L}_7 \eta, \tag{3.26}
\]

\[
\left\{ \int d^2 x \varepsilon^k \mathcal{H}_k, \int d^2 y \eta^i P_N \right\} = \int d^2 x P_N \mathcal{L}_7 \eta. \tag{3.27}
\]

This confirms that the momentum constraint (3.34) is a first class constraint. It is the generator of the gauge symmetry associated to the spatial transformations.

The Poisson bracket of \( P_N \) with itself is zero. The Hamiltonian constraint does not commute with itself neither with \( P_N \), therefore the constraints \( \mathcal{H} \) and \( P_N \) are second class constraints. The Poisson brackets between them are

\[
\left\{ \int d^2 x \varepsilon^k \mathcal{H}, \int d^2 y \eta^i P_N \right\} = \int d^2 x \sqrt{g} \eta^i \left( B^i - 2 \nabla^k (\epsilon a_k f_2) + \nabla^2 (\epsilon f_3) + 2 \alpha_6 \nabla^k \nabla^i (\epsilon \nabla_1 \alpha) + 2 \alpha_3 \mathcal{H}_l (N a^k \nabla_1 \epsilon) + 2 \alpha_4 \nabla^2 (N a^k \nabla_1 \epsilon) - 2 \alpha_4 \nabla_1 (N a^k \nabla^2 \epsilon) + 2 \alpha_7 \nabla^2 (N \nabla^2 \epsilon) + 2 \alpha_6 N \nabla_1 \alpha \nabla^k \nabla^i \epsilon + 2 \alpha_6 \nabla_1 \nabla_1 (N \nabla^k \nabla^i \epsilon) \right), \tag{3.28}
\]

\[
\left\{ \int d^2 x \varepsilon^k \mathcal{H}, \int d^2 y \eta^i \mathcal{H} \right\} = 2 \int d^2 x \left( f_1 \nabla^i \pi^k l + \frac{1 - \lambda}{1 - 2 \lambda} (\pi \nabla^k f_1 - f_1 \nabla^k \pi) - \pi^k l \nabla^i f_1 - f_3 \tau^k l a_1 + \tau^k l j \zeta^i j - \tau^k a_1 f_1 + 2 \tau^k l a_1 \left( f_2 - \frac{1}{2} \alpha_6 R \right) + (4 \alpha_2 - 4 \alpha_4) \tau^k l a_i a^i a^k + \frac{\partial l (N f_3)}{N} \left( \frac{1}{2} \tau^k l \nabla^k - \tau^k l \right) + \frac{2 \alpha_4 - 4 \alpha_7}{2} \tau^j l \nabla^k a^k \nabla^k \eta^i - \partial^k (\alpha_4 \tau^k l a^i a^j + \frac{1}{2} \alpha_5 \tau R)
\right) \tag{3.29}
\]

where \( \epsilon, \eta \) are arbitrary functions and \( (\cdot)' = (\cdot)/N \).

The canonical variables are \( \{ g_{ij}, \pi^i j, N, P_N \} \) (8 variables). There are four functional degrees of freedom eliminated by the constraints. The gauge symmetry of the
spatial diffeomorphisms gives two gauge degrees of freedom. Therefore, among the
original eight degrees of freedom in the nonreduced phase space, six are unphysical,
leaving two propagating degrees of freedom. This represents even scalar degree of
freedom in the theory. This mode renders the theory nontopological.

4 Quantization in the reduced phase space

4.1 Linearized theory

We start with the perturbations around the configuration that corresponds to the
“Minkowski” solution,

\[ g_{ij} = \delta_{ij}, \quad N = 1, \quad N_i = 0. \] (4.1)

We also consider \( \mu = 0 \). The perturbations are parameterized according to

\[ g_{ij} = \delta_{ij} + h_{ij}, \quad N = 1 + n, \quad A = a, \quad \pi^i = p^i, \quad N^i = n^i, \quad P_N = p_N. \] (4.2)

The perturbations decay as \( h_{ij} \sim O(r^{-1}) \), \( p^i \sim O(r^{-2}) \) and \( n \sim O(r^{-1}) \). These
linearized variables transform under FDiff as

\[ \delta n = \dot{f}, \]
\[ \delta n^i = \dot{\zeta}^i, \] (4.3)
\[ \delta h_{ij} = 2 \partial^i \partial_j h_{ij}, \] (4.4)

where the functions \( f, \zeta \) are infinitesimal FDiff parameters.

The linearized Hamiltonian to second order evaluated in the phase space is

\[
H \approx \int d^2 x \left( \frac{\lambda}{1 - 2\lambda} p^2 - \beta \left( - \partial^2 h + \partial_i \partial_j h^{ij} + h^{ij} \partial^2 h_{ij} - 2h^{ij} \partial_i \partial_j h_{ij} \right) + h^{ij} \partial_i \partial_j h_{ij} + \frac{1}{2} \partial^i \partial^j \partial_i \partial_j h_{ij} - \partial^i \partial^k \partial_i \partial_j h^{ij} + \frac{1}{2} \partial^i \partial^k \partial_i \partial_j h_{ij} \right) \\
\quad + \left( n + \frac{1}{2} h \right) \left( - \partial^2 h + \partial_i \partial_j h^{ij} \right) - \alpha \partial_k n \partial^k n + \alpha_1 \left( - \partial^2 h + \partial_i \partial_j h^{ij} \right)^2 \\
\quad - \alpha_5 \partial^2 n \left( - \partial^2 h + \partial_i \partial_j h^{ij} \right) + \alpha_6 \partial^i \partial^j n \partial_i \partial_j n + \alpha_7 \partial^2 n \partial^2 n, \]
\] (4.5)

and the equation of motion up to first order are

\[
\dot{N} \approx \sigma, \] (4.6)
\[
\dot{g}_{ij} \approx 2 \left( p_{ij} + \frac{\lambda}{1 - 2\lambda} \delta_{ij} p \right) + 2 \partial_i (n_j), \] (4.7)
\[
\dot{\pi}^{ij} \approx \beta \partial^i \partial^j (n + a) - 2 \partial^i \partial^j \left( 2\alpha_1 \left( - \partial^2 h + \partial_k \partial_l h^{kl} \right) + \alpha_5 \partial^2 n \right) - \beta \delta^{ij} \partial^2 (n + a) \\
\quad + 2 \delta^{ij} \partial^2 \left( 2\alpha_1 \left( - \partial^2 h + \partial_k \partial_l h^{kl} \right) + \alpha_5 \partial^2 n \right). \] (4.8)
We introduce the orthogonal tranverse and longitudinal decomposition

\[ h_{ij} = \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) h^T + \partial_i h_j, \tag{4.9} \]

We impose the transverse gauge

\[ \partial_i h_{ij} = 0, \tag{4.10} \]

under which all the longitudinal sector of the metric is eliminated, \( h^L_{ij} = 0 \). The linearized momentum constraint eliminates the longitudinal sector of \( p^{ij} \), since

\[ \partial_i p^{ij} = 0, \tag{4.11} \]

hence \( p^L_i = 0 \). The Hamiltonian constraint to first order is given by

\[ \beta \partial^2 h^T - \alpha_5 \partial^4 h^T + 2 \alpha_2 \partial^2 n + 2 (\alpha_6 + \alpha_7) \partial^4 n = 0, \tag{4.12} \]

we may solve for \( n \),

\[ n = \frac{1}{2} \left( \frac{-\beta + \alpha_5 \partial^2}{\alpha + (\alpha_6 + \alpha_7) \partial^2} \right) h^T \]

\[ = \mathcal{P} h^T. \tag{4.13} \]

Therefore the momentum and Hamiltonian constraints and the transverse gauge fix the variables \( h^L_i, p^L_i \) and \( n \), leaving the transverse sector the pair \( \{ h^T, p^T \} \) and the Legendre multiplier \( n_i \) active. The longitudinal sector of the linearized evolution equation of the metric \( g_{ij} \) [17] yields an equation for \( n_i \),

\[ \partial^2 n_i + \partial^i \partial_j n_j = -\frac{2 \lambda}{1 - 2 \lambda} \partial_i p^T, \tag{4.14} \]

whose solution is

\[ n_i = -\frac{\lambda}{1 - 2 \lambda} \partial_i \left( \frac{1}{\partial^2} p^T \right). \tag{4.15} \]

The traces of the linearized eqs. [17] and [18] lead automatically to their transverse sector. The trace of these equations, after using eq. [13] lead to

\[ \dot{h}^T = 2 \left( \frac{1 - \lambda}{1 - 2 \lambda} \right) p^T, \tag{4.16} \]

\[ \dot{p}^T = -\beta \partial^2 (n + a) - 4 \alpha_1 \partial^4 h^T + 2 \alpha_5 \partial^4 n, \tag{4.17} \]

theses equations imply

\[ \ddot{h}^T = 2 \left( \frac{1 - \lambda}{1 - 2 \lambda} \right) \left( -\beta \mathcal{P} \partial^2 h^T - \beta \partial^2 a + [2 \alpha_5 \mathcal{P} - 4 \alpha_1] \partial^4 h^T \right). \tag{4.18} \]

This represent the propagating equation of the scalar mode \( \{ h^T, p^T \} \) of the complete nonprojetable Hořava theory in 2 + 1 dimensions [16].
4.2 The reduced Hamiltonian and the propagator of the physical mode

We show the quadratic Hamiltonian (4.5) in the transverse gauge evaluated on the phase space, the reduced Hamiltonian,

\[ H_{\text{RED}} = \int d^2x \left( p^T M p^T + h^T \bar{M} h^T \right), \]  

(4.19)

where

\[ M = \frac{1 - \lambda}{1 - 2\lambda}, \]  

(4.20)

\[ \bar{M} = (\beta + \beta P + \alpha P^2) \partial^2 + (\alpha_1 + \alpha_5 P + [\alpha_6 + \alpha_7] P^2) \partial^4. \]  

(4.21)

We look for the propagator of the independent physical mode in the transverse gauge for the full Hořava theory in 2 + 1 dimensions. The path integral in the reduced phase space has the form

\[ Z_0 = \int Dp^T Dp^T \exp \left[ i \int dt d^2x \left( p^T \dot{h}^T - H_{\text{RED}} \right) \right], \]  

(4.22)

where \( H_{\text{RED}} \) is reduced Hamiltonian density of the equation (4.19). After of Gaussian integration in \( p^T \) we obtain the path integral in the noncanonical form

\[ Z_0 = \int Dh^T \exp \left[ i \int dt d^2x \left( \frac{1}{4M} \dot{h}^T \dot{h}^T - H_{\text{RED}} \right) \right], \]  

(4.23)

we can get the propagator of the physical mode

\[ < h_{ij} h_{kl} > = \frac{\theta_{ij} \theta_{kl}}{\omega^2/4M + (\beta + \beta P + \alpha P^2) \vec{k}^2 + (\alpha_1 + \alpha_5 P + [\alpha_6 + \alpha_7] P^2) \vec{k}^4}, \]  

(4.24)

where

\[ \theta_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}. \]  

(4.25)

5 The path integral in the nonreduced phase space

5.1 Nonperturbative formalism

In the Hamiltonian formulation of a field theory, the recipe for the measure of the gauge sector was provided by Faddeev [18]. In this formulation, the gauge symmetries have first-class constraints associated. The measure is then given by the brackets between the first-class constraints and the chosen gauge-fixing conditions, which, by definition, must be nonzero brackets. In our case the first class constraints
are the components of the momentum constraints $\mathcal{H}^i$. Let us denote by $\chi^i = 0$ the associated gauge-fixing conditions. The measure of the gauge sector is given by

$$\det\{\mathcal{H}^k, \chi^l\}. \quad (5.1)$$

Regarding the second-class constraints, let us introduce a common notation for them, namely, $\theta_1 = \mathcal{H}$ and $\theta_2 = P_N$. The measure corresponding to the second-class constraints is defined by Senjanovic \cite{10} as

$$\sqrt{\det\{\theta_p, \theta_q\}}. \quad (5.2)$$

The path integral in terms of the nonreduced canonical variables has then the form

$$Z_0 = \int DV \delta(\mathcal{H}^i) \delta(\chi^i) \delta(\theta_1) \delta(\theta_2) e^{iS_{can}}, \quad (5.3)$$

where the measure and the action are given respectively by

$$DV = Dg_{ij} D\pi^{ij} DNP_N \times \det\{\mathcal{H}^k, \chi^l\} \sqrt{\det\{\theta_p, \theta_q\}}, \quad (5.4)$$

$$S_{can} = \int dt d^2 x \left( P_N \dot{N} + \pi^{ij} \dot{g}_{ij} - \frac{N}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} + \frac{\lambda}{1 - 2\lambda} \pi^2 \right) - N \sqrt{g} \right). \quad (5.5)$$

### 5.1.1 The measure of the gauge sector

Since we have the momentum constraint explicitly, Eq. (3.4), we can make explicit computations on the measure of the gauge sector, taking a quite general gauge-fixing condition $\chi^i$. This must be a condition that fixes the freedom of choosing spatial coordinates. For simplicity, let us consider that this condition only involves the spatial metric, $\chi^i = \chi^i(h)$, but otherwise arbitrary. Using the expression (3.4), we obtain the bracket

$$\{\mathcal{H}^k(x), \chi_l(y)\} = 2 \int d^2 z \nabla_{(i} \frac{\delta \mathcal{H}^k(h)}{\delta h_{ij)} (\delta \chi_l(h))} = -2 \delta^{kj} \nabla_j \left( \frac{\delta \chi_l}{\delta h_{ij}} \right). \quad (5.6)$$

A important question one can pose here is whether this approach for dealing with the gauge fixing and the first-class constraints is equivalent to the usual approach (Faddeev-Popov) for incorporating the gauge-fixing condition in the path integral quantization of gauge theories, typically formulated in terms of covariant Lagrangians. To answer this question, we evaluate the Faddeev-Popov determinant of the standard approach. We need the gauge-transformed field, where the gauge symmetry is an infinitesimal spatial diffeomorphism, then

$$h^{\xi}_{ij} = h_{ij} + 2 \nabla (\xi_j). \quad (5.7)$$
The Faddeev-Popov factor becomes
\[
\frac{\delta \chi_l(h^\epsilon(x))}{\delta \zeta_k(y)} = \int d^2 z \frac{\delta \chi_i(h^\epsilon(x))}{\delta h_{ij}^\epsilon(z)} \frac{\delta h_{ij}^\epsilon(z)}{\delta \zeta_k(y)} = 2 \int d^2 z \nabla_i (\delta_j^k \delta^2) \frac{\delta \chi_i(h^\epsilon)}{\delta h_{ij}^\epsilon}.
\] (5.8)

This shows that, at least for a general gauge-fixing condition that depends on the spatial metric, \(\chi^i(h)\), the measure in the Hamiltonian formulation (5.6) and the measure in the Fadeev-Popov approach coincide.

If we consider the transverse gauge condition, \(\chi_j = \partial_i h_{ij} = 0\), the bracket of the measure becomes
\[
\{H^k, \chi_l\} = \delta^{i j} \partial^2 \delta^2 + \partial^i \partial^j \delta^2 + (\delta^k \Gamma^i_{ij} + \Gamma^j_k) \partial^j \delta^2.
\] (5.9)

In the path integral, the determinant of the gauge sector is incorporated to the action by means of ghosts fields,
\[
\det \{H^k, \chi_l\} = \int D\bar{\eta} D\eta \exp \left( i \int dt dx \bar{\eta} \{H^k, \chi_l\} \eta \right).
\] (5.10)

### 5.1.2 The second-class-constraint measure

There is an important simplification because the matrix of brackets between the second-class constraints acquires a triangular form,
\[
\{\theta_p, \theta_q\} = \begin{pmatrix}
\{H, H\} & \{H, P_N\} \\
\{P_N, H\} & 0
\end{pmatrix},
\] (5.11)

then the determinant of the matrix Poisson bracket is a quadratic form,
\[
\sqrt{\det \{\theta_p, \theta_q\}} = \det \{H, P_N\}.
\] (5.12)

The bracket between the Hamiltonian constraint and \(P_N\) is show in (3.28), setting \(\eta = \epsilon = \delta^2\)
\[
\{H, P_N\} = \frac{\sqrt{g}}{N} \left( B_{ij} + 2 \nabla^k (a_k f_2) + \nabla^2 (f_3) + 2 \alpha_6 \nabla^k \nabla^l (\nabla_i a_k)ight.
+ 2N f_2 a_k \nabla_k \left( \frac{\cdot}{N} \right) + f_3 N \nabla^2 \left( \frac{\cdot}{N} \right) - \nabla^k \left( 2N f_2 \nabla_k \left( \frac{\cdot}{N} \right) \right)
- 8\alpha_2 \nabla_i \left( N a_l a_k \nabla_k \left( \frac{\cdot}{N} \right) \right)
+ 2\alpha_4 \nabla^2 \left( N a_k \nabla_k \left( \frac{\cdot}{N} \right) \right)
- 2\alpha_4 \nabla_k \left( N a_k \nabla^2 \left( \frac{\cdot}{N} \right) \right) + 2\alpha_6 \nabla_k \nabla_l \left( N \nabla^k \nabla^l \left( \frac{\cdot}{N} \right) \right)
+ 2\alpha_6 N \nabla_i a_k \nabla^k \nabla^l \left( \frac{\cdot}{N} \right) + 2\alpha_6 \nabla_i \nabla_l \left( N \nabla^k \nabla^l \left( \frac{\cdot}{N} \right) \right)
\right) \delta.
\] (5.13)

We may promote the measure (5.12) to the action by means of ghosts fields. We define the fermionic fields \(\bar{\eta}\) and its conjugate \(\bar{\eta}\), such that
\[
\det \{H, P_N\} = \int D\bar{\eta} D\eta \exp \left( i \int dt dx^2 \bar{\eta} \{H, P_N\} \eta \right).
\] (5.14)
5.2 Perturbations in the path integral: linearized theory

We study the quadratic quantum action. It takes the form

\[
S_{\text{can}} \approx \int dt \, d^2 x \left( p_n \dot{n} + p^{ij} \dot{h}_{ij} - \left( p^{ij} p_{ij} + \frac{\lambda}{1 - 2\lambda} \dot{p}^2 \right) - \alpha_6 \partial^i \partial^j n \partial_i \partial_j n - \alpha_7 \partial^2 n \partial^2 n 
- \alpha_1 \left( -\partial^2 h + \partial_i \partial_j h^{ij} \right)^2 - \alpha_5 \left( -\partial^2 h + \partial_i \partial_j h^{ij} \right) \partial^2 n \right). 
\]

(5.15)

The ghosts are of first order, therefore the determinant of second-class constraint must be considered at zero order,

\[
\det \{ \mathcal{H}, P_N \}_0 = \int \mathcal{D} \bar{\eta} \mathcal{D} \eta \exp \left( i \int dt \, d^2 x \bar{\eta} \left( 2(\alpha_6 + \alpha_7) \partial^4 \eta \right) \right). 
\]

(5.16)

To further advance in the computations, we adopt a partial reduction in the phase space. This means that we eliminate explicitly all the variables belonging to the gauge sector, but leaving the second-class constraints unsolved. This allows us to find the propagators of all the sector that is not associated to the gauge symmetry. To eliminate the gauge sector easily, we impose the transverse gauge, \( \chi_j = \partial_i h_{ij} = 0 \). This implies that the longitudinal sector of the metric is eliminated. Moreover, the linearized momentum constraint eliminates its longitudinal sector, so \( h_{i}^L = p_{i}^L = 0 \). The resulting path integral is

\[
Z_0 = \int \mathcal{D} h^T \mathcal{D} p^T \mathcal{D} n \mathcal{D} p_N \mathcal{D} \bar{\eta} \mathcal{D} \eta \delta(\mathcal{H}) \delta(p_N) \exp \left( i \int dt \, d^2 x \left( p_N \dot{n} - M \left( p^T - \frac{1}{2M} \dot{h}^T \right)^2 
+ \frac{1}{4M} \dot{h}^T \dot{h}^T - \left( h^T (\alpha_1 \Delta^2) h^T + n((\alpha_6 + \alpha_7) \Delta^2) n - h^T (\alpha_5 \Delta^2) n 
+ \bar{\eta} \left( 2(\alpha_6 + \alpha_7) \partial^4 \eta \right) \right) \right), 
\]

(5.17)

where we have used that, under integration,

\[
\delta(h_{i}^L) \delta(p_{i}^L) \delta(\chi_j) \det \{ \mathcal{H}, \chi_j \} = 1. 
\]

(5.18)

The variables \( p_n \) and \( p^T \) are not associated to the gauge symmetries, but them can be integrated directly (\( p_n \) is trivial and the integration on \( p^T \) is Gaussian). Finally, we may promote the delta \( \delta(\mathcal{H}) \) to the Lagrangian by using a Lagrange multiplier, which we denote by \( a \), such that the path integral includes the integration in \( a \). In the perturbative approach, \( a \) is considered as a variable of linear order, hence, in
the quadratic action we need the expression of the Hamiltonian constraint \( \mathcal{H} \) up to linear order. The linear-order expression for \( \mathcal{H} \) can be taken from Eq. \((1.12)\),

\[
\mathcal{H} = \beta \partial^2 h^T - \alpha_5 \partial^4 h^T + 2\alpha \partial^2 n + 2(\alpha_6 + \alpha_7)\partial^4 n. \tag{5.19}
\]

We are interested in the case UV behavior, we discard all the lower order terms, keeping only the \( z = 2 \) terms. After these steps, we arrive at the path integral

\[
\mathcal{Z}_0 = \int \mathcal{D}h^T \mathcal{D}n \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp \left[ i \int dt d^2 x \left( \frac{1}{4M} h^T \dot{h}^T - \alpha_1 \partial^4 h^T - (\alpha_6 + \alpha_7)n \partial^4 n + \alpha_5 h^T \partial^4 n + a \left( -\alpha_5 \partial^4 h^T + 2(\alpha_6 + \alpha_7)\partial^4 n \right) + 2(\alpha_6 + \alpha_7)\bar{\eta} \partial^4 \bar{\eta} \right) \right]. \tag{5.20}
\]

Our notation for the propagators is as follows: they are obtained from

\[
\begin{pmatrix}
  h^T & n & a & \bar{\eta} & \eta
\end{pmatrix} \begin{pmatrix}
  \mathcal{M}_1 & 0 & 0 \\
  0 & \mathcal{M}_2 & 0 \\
  \bar{\eta} & \eta
\end{pmatrix} \begin{pmatrix}
  h^T \\
  n \\
  a \\
  \bar{\eta} \\
  \eta
\end{pmatrix}.
\]

Therefore, to get the propagators we must invert the matrix inside this product, where

\[
\mathcal{M}_1 = \begin{pmatrix}
  \frac{\omega^2}{4M} + \alpha_1 k^4 & -\alpha_5 k^4/2 & \alpha_5 k^4/2 \\
  -\alpha_5 k^4/2 & (\alpha_6 + \alpha_7) k^4 & -(\alpha_6 + \alpha_7) k^4 \\
  \alpha_5 k^4/2 & -(\alpha_6 + \alpha_7) k^4 & 0
\end{pmatrix},
\]

\[
\mathcal{M}_2 = \begin{pmatrix}
  0 & -(\alpha_6 + \alpha_7) k^4 \\
  (\alpha_6 + \alpha_7) k^4 & 0
\end{pmatrix}.
\]

Thus, we obtain the propagators

\[
<h^T h^T> = 4M \left( \omega^2 + M \left( 4\alpha_1 - \frac{\alpha_5^2}{\alpha_6 + \alpha_7} \right) k^4 \right)^{-1}, \tag{5.24}
\]

\[
nn = \frac{\alpha_5^2 M}{(\alpha_6 + \alpha_7)^2} \left( \omega^2 + M \left( 4\alpha_1 - \frac{\alpha_5^2}{\alpha_6 + \alpha_7} \right) k^4 \right)^{-1}, \tag{5.25}
\]

\[
<h^T n> = \frac{2\alpha_5 M}{\alpha_6 + \alpha_7} \left( \omega^2 + M \left( 4\alpha_1 - \frac{\alpha_5^2}{\alpha_6 + \alpha_7} \right) k^4 \right)^{-1}, \tag{5.26}
\]

\[
<h^T a> = 0 \tag{5.27}
\]

\[
<na> = -\frac{1}{(\alpha_6 + \alpha_7)k^4}, \tag{5.28}
\]

\[
<aa> = -\frac{1}{(\alpha_6 + \alpha_7)k^4}, \tag{5.29}
\]

\[
<\bar{\eta}\eta> = \frac{1}{(\alpha_6 + \alpha_7)k^4}. \tag{5.30}
\]
We observe that the propagators of $h^T$ and $n$ get a regular form. That is, any nonregular part of the propagator of the lapse function, which is the obstruction to apply the criterium of renormalizability found in Ref. [3], has disappeared, once the second-class constraint are taken into account. Secondly, there is an evident similarity between the propagator of the Lagrange multiplier $a$ and the one of the ghosts $\bar{\eta}, \eta$. We consider that this is again an evidence of the importance of the second-class constraint sector, since these ghosts fields are the ones incorporating the measure of the second-class constraint, a measure that must be included by consistency [10]. It seems that it compensates the presence of the unphysical Lagrange multiplier $a$.

5.3 Interactions

The previous result on the propagator of the ghosts fields introduced by the measure of the second-class constraints suggest to further explore the quantum formulation of this theory. To this end we expect that the next order in perturbations, which yields nontrivial interactions, may be useful. We list the third-order version of the measure of the second-class constraints and the potential

$$\det\{\mathcal{H}, P_N\} \approx \int \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp \left( i \int dt d^2x \bar{\eta} \left( -2(\alpha_6 + \alpha_7) \left( -\partial^4 + \partial^4 n + n \partial^4 
+ 2\partial_k n \partial^2 \partial^k \right) + (4\alpha_4 - 2\alpha_6 - 8\alpha_7) \partial_i \partial_j n \partial^i \partial^j + (-3\alpha_4 - 2\alpha_6 + 2\alpha_7) \partial^2 n \partial^2 
+ 2(-\alpha_3 + \alpha_5) \left( -\partial_k \partial^2 h + \partial_k \partial_i \partial_j h^{ij} \right) \partial^k 
+ (-\alpha_3 + 2\alpha_5) \left( -\partial^2 h + \partial_i \partial_j h^{ij} \right) \partial^2 
+ (\alpha_6 + \alpha_7) \left( h \partial^4 - 2\partial^2 \left( \delta^{ij} \chi_{ijk} \partial^k \right) \right) + \alpha_6 \left( -\partial_k \partial^2 h \partial^k + \partial_k \partial_i \partial_j h^{ij} \partial^k \partial^2 - \partial^2 h \partial^2 \right) 
+ \partial_i \partial_j h^{ij} \partial^2 - 2\delta^{ij} \chi_{ijk} \partial^k \partial^2 \right) + 2\alpha_7 \left( -\partial^2 h \partial_i \partial_j - 2\partial^2 h \partial^i k \partial_i \partial_j \partial k - 2h^{ij} \partial^2 \partial_i \partial_j \right) 
- \partial_i h^{ij} \partial_j \partial^2 + \frac{1}{2} \partial^2 h \partial^k \partial^2 \right) \right) \right),$$

(5.31)

where

$$\chi_{ijk} = \frac{1}{2} \left( \partial_i h_{jk} + \partial_j h_{ik} - \partial_k h_{ij} \right).$$

(5.32)
The potential up to third order takes the form

\[
N\sqrt{\mathcal{g}} V \approx \alpha_6 \partial^i \partial^j \partial^k \partial^l n \partial_i \partial_j \partial^k \partial^l n + \alpha_7 \partial^2 n \partial^2 n + \alpha_1 \left(-\partial^2 h + \partial_i \partial_j h^{ij}\right)^2 \\
+ \alpha_5 \left(-\partial^2 h + \partial_i \partial_j h^{ij}\right) \partial^2 n + \alpha_4 \partial^2 n \partial^k n \partial_k n - \alpha_6 \left(\partial h n \partial^i \partial^j n \partial_i \partial_j n \right) \\
+ n \partial_k \partial^i n \partial^k \partial_i n - \alpha_7 \left(2 \partial h n \partial^2 n + n \partial^2 n \partial^2 n \right) + \alpha_1 n \left(-\partial^2 h + \partial_i \partial_j h^{ij}\right)^2 \\
+ \alpha_3 \left(-\partial^2 h + \partial_i \partial_j h^{ij}\right) \partial^k n \partial_k n + \alpha_5 \left(-\partial^2 h + \partial_i \partial_j h^{ij}\right) \left(\frac{1}{2} k \partial^2 n \right) \\
- h^{kl} \partial_k \partial_i n - \partial^k n \partial_k n - \partial^k n \gamma^l p \chi_{lp} + \left(h^{ij} \partial^2 h_{ij} - 2 h^{ij} \partial_i \partial_j h^{ij} \right) \\
+ h^{ij} \partial_i \partial_j h + \frac{1}{2} \partial h^{ij} \partial^i h_{ij} - \partial h^{lk} \partial_i h^{i_k} + \frac{1}{2} \partial h^{lk} \partial_k h \right) \partial^2 n \right) \\n+ \alpha_6 \left(-2 h^{ij} \partial^k \partial_j n \partial_k \partial_i n + \frac{1}{2} \partial h^{ij} \partial^i n \partial_i \partial_j n - 2 \chi_{ijk} \partial^k \partial^i n \partial^j n \right) \\
+ \alpha_7 \left(\frac{1}{2} \partial h^{ij} \partial^2 n \partial^2 n - 2 h^{ij} \partial_i \partial_j n \partial^2 n - 2 \partial^2 n \partial^i n \delta^j k \chi_{jki} \right) \\
+ \alpha_1 \left(\frac{1}{2} k \left(-\partial^2 h + \partial_i \partial_j h^{ij}\right)^2 + 2 \left(-\partial^2 h + \partial_k \partial_i h^{kl}\right) \left(h^{ij} \partial^2 h_{ij} \right. \\
- 2 h^{ij} \partial_i \partial_j h + h^{ij} \partial_i \partial_j h + \frac{1}{2} \partial h^{ij} \partial^i h_{ij} - \partial h^{ij} \partial h^{ij} \right) + \frac{1}{2} \partial h^{ij} \partial h_{ij} \right) \right).
\]

6 Conclusions

We have considered the consistent quantization of the 2+1 nonprojectable Hořava theory, considering all the terms in the Lagrangian that are covariant under the FDiff gauge symmetry. Our central focus has been in the presence of the second-class constraints, which requires that the quantization must be addressed in a different way to the pure gauge theories, that is, gauge theories without second-class constraints. We highlight that the measure for the second-class constraints is known, at least in the Hamiltonian formalism [10].

We have performed the full Hamiltonian analysis of the theory, finding all the constraints explicitly, classifying the constraints between first and second class. As expected, the momentum constraint is the only first-class constraint. It is associated to the symmetry of arbitrary spatial diffeomorphisms over each leaf of the foliation, which is the only gauge symmetry of the theory in the strict sense. The set of constraints is complete in the sense that their preservations lead to elliptic differential equations for Lagrange multipliers. An application of this analysis is the characterization of the physical propagating mode of the theory. We have shown this by means of a perturbative analysis.
The Hamiltonian formulation of the theory has enabled us to obtain the measure of the second-class constraints explicitly. A central result is that the matrix of Poisson brackets of the second-class constraints acquires a quadratic form. This simplify the square root of the measure, such that it can be incorporated to the quantum Lagrangian by means of fermionic ghosts. We have extracted further consequences of the measure using perturbative analysis. We have found the rather surprising result that, when the Hamiltonian constraint and the measure are considered in the quantization procedure, the lapse function acquires a completely regular form. There are no nonregular terms in the propagator of the lapse function, nor in the one of the scalar mode of the spatial metric. The would-be nonregular part was previously found in Ref. [3] as an obstruction to achieve the renormalization of the nonprojectable theory by means of the technique of regular propagators. On the basis of this, we consider that all the second-class-constraint sector, which means that we must incorporate these constraint to the quantization procedure, together with their associated measure, plays a fundamental role in the consistent quantization of the theory, and even in the proof of its renormalization. More analysis in this direction is required.

Further computations on the quantization of the theory with interactions are also required. In particular they may shed light on the renormalization, since explicit amplitudes can be computed. With this aim, we have presented the quantum theory, that is, the measure and the potential, at cubic order in perturbations. We expect that this work can be extended with the computations of the (scalar) graviton scattering and to explore its renormalization.

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References

[1] P. Hořava, “Quantum Gravity at a Lifshitz Point”, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].

[2] P. Hořava, “Membranes at Quantum Criticality”, JHEP 0903, 020 (2009) [arXiv:0812.4287 [hep-th]].

[3] A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov and C. F. Steinwachs, “Renormalization of Hořava gravity”, Phys. Rev. D 93, 064022 (2016) [arXiv:1512.02250 [hep-th]].
[4] A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov and C. F. Steinwachs, “Hořava Gravity is Asymptotically Free in 2 + 1 Dimensions”, Phys. Rev. Lett. 119, 211301 (2017) [arXiv:1706.06809 [hep-th]].

[5] T. Griffin, K. T. Grosvenor, C. M. Melby-Thompson and Z. Yan, “Quantization of Hořava gravity in 2+1 dimensions”, JHEP 1706, 004 (2017) [arXiv:1701.08173 [hep-th]].

[6] R. L. Arnowitt, S. Deser and C. W. Misner, “The Dynamics of general relativity”, Gen. Rel. Grav. 40, 1997 (2008) [arXiv:gr-qc/0405109 [gr-qc]].

[7] L. D. Faddeev and V. N. Popov, “Feynman Diagrams for the Yang-Mills Field”, Phys. Lett. 25B, 29 (1967).

[8] D. Blas, O. Pujolas and S. Sibiryakov, “Consistent Extension of Hořava Gravity”, Phys. Rev. Lett. 104, 181302 (2010) [arXiv:0909.3525 [hep-th]].

[9] J. Bellorín and A. Restuccia, “Quantization of the Hořava theory at the kinetic-conformal point”, Phys. Rev. D 94, 064041 (2016) [arXiv:1606.02606 [hep-th]].

[10] P. Senjanovic, “Path Integral Quantization of Field Theories with Second Class Constraints”, Annals Phys. 100, 227 (1976) Erratum: [Annals Phys. 209, 248 (1991)].

[11] S. Deser, R. Jackiw and G. t Hooft, “Three-Dimensional Einstein Gravity: Dynamics of Flat Space”, Ann. Phys. (N.Y.) 152, 220 (1984).

[12] J. Bellorín and B. Droguett, “Point-particle solution and the asymptotic flatness in 2+1D Hořava gravity”, Phys. Rev. D 100, 064021 (2019) [arXiv:1905.02836 [gr-qc]].

[13] A. Ashtekar and M. Varadarajan, “A Striking property of the gravitational Hamiltonian”, Phys. Rev. D 50, 4944 (1994) [gr-qc/9406040].

[14] J. Bellorín and B. Droguett, “Dynamics of the anisotropic conformal Hořava theory versus its kinetic-conformal formulation”, Phys. Rev. D 98, 086008 (2018) [arXiv:1807.01293 [hep-th]].

[15] J. Bellorin, A. Restuccia and A. Sotomayor, “Consistent Hořava gravity without extra modes and equivalent to general relativity at the linearized level”, Phys. Rev. D 87, no.8, 084020 (2013) [arXiv:1302.1357 [hep-th]].

[16] T. P. Sotiriou, M. Visser and S. Weinfurtner, Lower-dimensional Hořava-Lifshitz gravity, Phys. Rev. D 83, 124021 (2011) [arXiv:1103.3013 [hep-th]].

[17] W. Donnelly and T. Jacobson, “Hamiltonian structure of Hořava gravity”, Phys. Rev. D 84, 104019 (2011) [arXiv:1106.2131 [hep-th]].
[18] L. D. Faddeev, “Feynman integral for singular Lagrangians”, Theor. Math. Phys. 1 (1969) 1 [Teor. Mat. Fiz. 1 (1969) 3].