SO(N) invariant Wess-Zumino action and its quantization

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Abstract

A consistent quantization procedure of anomalous chiral models is discussed. It is based on the modification of the classical action by adding Wess-Zumino terms. The $SO(3)$ invariant WZ action for the $SO(3)$ model is constructed. Quantization of the corresponding modified theory is considered in details.

1 Introduction

In this paper we consider a possibility of consistent canonical quantization of anomalous gauge theories based on the modification of the classical action by adding the Wess-Zumino (WZ) term. It is known that the straightforward quantization of anomalous models leads to inconsistent theory breaking either unitarity or Lorentz covariance or renormalizability [1, 2, 3]. Usually the absence of anomalies is considered as a criterion

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of choosing physically acceptable model. However there exists another point of view according to which the appearance of anomalies leads to increasing of the number of degrees of freedom: some excitations which can be eliminated in the classical theory by gauge transformation become physical. This point of view was adopted by Polyakov [4] in the relativistic string model, by Jackiw and Rajaraman [5] in the chiral Schwinger model and by Faddeev and Shatashvili [6] in nonabelian anomalous chiral models. However at present it is not yet clear whether this point of view may lead to a consistent theory. A problem arises already in the process of quantizing anomalous models. As was pointed out by Faddeev [7] in anomalous models some classical first class constraints transform into the second class constraints. For example, if one quantizes the theory in the temporal gauge \( A_0 = 0 \) one should select the physical subspace by imposing the condition \( \varphi|\psi > = 0 \) where \( \varphi \) is a Gauss law. In nonanomalous models \( \varphi \) form a set of first class constraints generating the gauge transformations. However in the presence of anomaly the algebra of constraints is modified. In the commutation relations the Schwinger term arises which has a meaning of a 2-cocycle on the gauge group [7, 8, 9]. That means the conditions \( \varphi|\psi > = 0 \) are inconsistent and the quantization procedure should be revised.

The origin of appearance of anomalies is impossibility to introduce for anomalous models a gauge invariant intermediate regularization which is necessary to give a precise mathematical meaning to the quantum theory. Any regularization changes the type of constraints transforming some of the first class constraints to the second class ones. It suggests that one possibility to perform a consistent quantization is to introduce a Lagrangian regularization on the classical level and apply the canonical quantization to the regularized theory. An alternative way is to introduce a regularization in the framework of BRST quantization scheme [10]. The first procedure was successively applied to the abelian chiral models and to the string model in refs. [1, 2, 3]. As there is no gauge invariant regularization, the resulting theory is not gauge invariant and depends crucially on the particular regularization chosen. For example in the two dimensional chiral Schwinger model some regularizations lead to the consistent theory with the positive definite Hamiltonian while the others generate non physical ghost states. Although the resulting theory in this case is not gauge invariant the invariance may be easily restored by introducing a new field \( g \) with values in the gauge group. In the case of chiral gauge theories which we shall consider below it can be done by making a gauge transformation \( A_\mu \rightarrow A_\mu^g; \psi \rightarrow \psi^g \) and considering \( g \) as a new variable with the transformation law \( g \rightarrow h^{-1}g \). Taking into account that the classical action is gauge invariant one sees that the dependence on the chiral field \( g \) enters only via regularizing fields. Integrating over all the fields except for \( A_\mu \) and \( g \) one gets an effective action for these fields. The part of this action which depends on \( g \) appears due to the gauge noninvariance of the regularization and is a 1-cocycle on the gauge group. When the regularization parameter goes to infinity this part reduces to the WZ action [14]. This action may be calculated also by the algebraic and geometric methods [4, 8, 13, 10]. Its gauge variation gives the anomaly which is therefore an infinitesimal 1-cocycle. It suggests another possibility of quantizing anomalous theories. One can modify the classical gauge invariant action by adding the WZ term and quantize the modified action [8]. The modified classical action is not gauge invariant however one can quantize it by imposing some gauge condition and then prove that the quantum observables do not depend on the choice of gauge condition.

The form of the WZ action depends on the regularization, the difference being a trivial 1-cocycle. Although the difference is a trivial 1-cocycle it may change drastically
the physical content of the theory, as we have already seen in the chiral Schwinger model. From the point of view of physical applications it would be of interest to analyze the freedom in choosing the modified action related to nonuniqueness of the WZ action. Below we shall show that in the case of $SU(N)$ chiral gauge models one can choose a regularization preserving the gauge invariance with respect to $SO(N)$ subgroup and calculate the corresponding WZ action, the anomaly and the 2-cocycle appearing in the commutator of the Gauss law [17]. We consider in details the canonical quantization of the $SU(3)$ model. In this case the symplectic form of the WZ action is degenerate and the quantization procedure which has been used for nondegenerate case by Faddeev and Shatashvili [3] should be modified. Due to degeneracy of the symplectic form secondary constraints appear which together with the primary constraints form a set of secondary class ones. We construct a path integral representation for the generating functional in the temporal gauge and prove the gauge invariance of physical observables [18].

The paper is organised as follows. In the second part we derive the expression for the $SO(N)$ invariant WZ action. We present also an alternative expression for the WZ action in terms of the chiral fields with values in the coset space $SU(N)/SO(N)$. Using this action we calculate the anomaly. In the third section we calculate the infinitesimal 2-cocycle appearing in the anomalous constraints commutator and establish that it vanishes on the $so(N)$ subalgebra. The fourth section is devoted to the canonical quantization of the WZ actions with degenerate symplectic forms. We consider as the simplest example the two dimensional chiral $SU(2)$ gauge model. In the two dimensional case there is a family of WZ actions parametrized by one parameter $a$ and the choice of $a = 0$ corresponds to the WZ action with the degenerate symplectic form. As was mentioned by Shatashvili [19] this case differs from the others and requires a special analysis. In the fifth section we discuss the four dimensional model with the $SO(3)$ invariant WZ action. We point out a parametrization of the coset space $SU(N)/SO(N)$ reducing the WZ action to a pure four dimensional one and then perform canonical quantization. As was expected we find new physical degrees of freedom but contrary to the standard WZ action we get not four but two new degrees of freedom.

2 The $SO(N)$ invariant WZ action

We consider the model described by the classical action

$$S = \int d^4x \left[ -\frac{1}{4}(F_{\mu\nu}^a)^2 + i\bar{\psi}\gamma^\mu(\partial_\mu + A_\mu)\psi \right], \quad (2.1)$$

Where $A_\mu$ is an $SU(N)$ Yang-Mills field and $\psi \equiv \frac{1}{2}(1 + \gamma_5)\psi$ is a chiral fermion in the fundamental representation of the $SU(N)$ generated by the antihermitian matrices $\lambda^a$

$$\text{tr} \lambda^a \lambda^b = -\frac{1}{2}\delta_{ab}; \quad [\lambda^a, \lambda^b] = f^{abc} \lambda^c, \quad (2.2)$$

The action (2.1) is invariant with respect to the gauge transformations,

$$A_\mu \rightarrow A_\mu^g = g^{-1}A_\mu g + g^{-1}\partial_\mu g$$
$$\psi \rightarrow \psi^g = g^{-1}\psi, \quad g \in SU(N). \quad (2.3)$$
Due to the gauge invariance of the action (2.1) in the classical theory one can impose any admissible gauge condition. However it is well known that because of the quantum anomaly in quantum theory the equivalence of different gauges is lost which leads to the inconsistency of the model.

The existence of the quantum anomaly is related to the absence of a $SU(N)$ invariant regularization of the action (2.1). However it is known that $SU(N)$ group has nonanomalous subgroups, the $SO(N)$ subgroup being a maximal one. It suggests that there exists a regularization preserving the invariance with respect to the $SO(N)$ subgroup. Indeed, such a regularization can be described by the following Lagrangian

$$
\mathcal{L} = i\bar{\psi}\gamma^\mu (\partial_\mu + A_\mu)\psi + \sum_{r=1}^{2K-1} \left[ i\bar{\psi}_r\gamma^\mu (\partial_\mu + A_\mu)\psi_r - M_r\psi_r^T C\psi_r - M_r\bar{\psi}_r C\bar{\psi}_r^T \right] \\
+i \sum_{r=1}^{2K} \left[ (-1)^r \bar{\phi}_r\gamma^\mu (\partial_\mu + A_\mu)\phi_r - \sum_{r,s=1}^{2K} (M_{rs}\phi_r^T C\phi_s - M_{rs}\bar{\phi}_r C\bar{\phi}_s^T) \right] 
$$

(2.4)

Here $\psi_r$ are the anticommuting Pauli–Villars spinors and $\phi_r$ are the commuting ones. $M_{rs}$ is an antisymmetric matrix. The standard Pauli–Villars conditions are assumed. The matrix $C$ is the charge conjugation matrix. The only terms, which are not invariant under the gauge transformation (2.3) of all fields, are the mass terms for the Pauli–Villars fields. The mass terms transform as follows

$$
M_r\bar{\psi}_r C\psi_r^T \rightarrow M_r\bar{\psi}_r Cgg^T\psi_r^T. 
$$

(2.5)

One sees that for $g \in SO(N)$, $gg^T = 1$ this mass term is invariant, and therefore the regularization preserves the $SO(N)$ gauge invariance.

It follows that the anomaly calculated with the help of this regularization vanishes on the $so(N)$ subalgebra. The standard anomaly [2] does not possess this property. Anomaly can be defined as a gauge variation of the functional called the WZ action given by the following equation,

$$
e^{i\alpha_1(A,g)} = \frac{\det(\gamma^\mu(\partial_\mu + A^g_\mu))}{\det(\gamma^\mu(\partial_\mu + A_\mu))}. 
$$

(2.6)

The value of this determinants depend on the particular regularization used. Below we shall calculate the WZ action corresponding to the regularization (2.4) assuming that necessary counterterms are introduced. It follows directly from eq.(2.6) that the WZ action satisfies the condition

$$
\alpha_1(A, g_1) + \alpha_1(A^{g_1}, g_2) = \alpha_1(A, g_1 g_2) \quad (mod \, 2\pi). 
$$

(2.7)

This equation is a definition of a 1-cocycle on the gauge group (see Appendix A). Different regularizations lead to the WZ actions which differ by a trivial 1-cocycle. We call cocycle to be trivial if it can be presented as a finite gauge variation of a local functional $\alpha_0(A)$ (0-cochain),

$$
\alpha_1^{triv}(A, g) = \alpha_0(A^g) - \alpha_0(A) 
$$

(2.8)

It is worthwhile to note that in anomalous theories the WZ action is not a trivial 1-cocycle.

In the case of regularization (2.4) the WZ action $\alpha_1^{ort}$ possesses the $SO(N)$ invariance

$$
\alpha_1^{ort}(A, gh) = \alpha_1^{ort}(A, g), 
$$

(2.9)
where \( h \in SO(N) \).

Let us stress that in eq.(2.9) the field \( A \) is not transformed. Eq.(2.9) is a direct consequence of the invariance of the gauge transformed mass term (2.3) under the transformation
\[
g \rightarrow gh, \quad h \in SO(N).
\] (2.10)

Eq.(2.9) expresses the hidden symmetry of the Wess–Zumino action in our case. Hidden symmetries of this type in connection with models on homogeneous spaces were discussed in refs.[20, 21, 22]. It follows from eqs.(2.7,2.9) that the Wess–Zumino action vanishes if the chiral field \( g \) belongs to the orthogonal subgroup \( SO(N) \)
\[
\alpha_1^{\text{ort}}(A, h) = 0, \quad h \in SO(N)
\] (2.11)
The geometric origin of the existence of such Wess–Zumino action is the triviality of the cohomology group \( H^5(SO(N)) \).

To calculate the \( SO(N) \) invariant WZ action we shall use the fact that as was discussed above any WZ action can be presented in the form
\[
\alpha_1^{\text{ort}}(A, g) = \alpha_1(A, g) + \alpha_0(A^g) - \alpha_0(A)
\] (2.12)
Here \( \alpha_1(A, g) \) is the ”standard” Wess–Zumino action
\[
\alpha_1(A, g) = \int d^4x [d^{-1} \kappa(g)] - \frac{i}{48\pi^2} \epsilon^{\mu
u \lambda \sigma} \text{tr} [(A_\mu \partial_\nu A_\lambda + \partial_\mu A_\nu A_\lambda + A_\mu A_\nu A_\lambda)g_\sigma - \frac{1}{2} A_\mu g_\nu A_\lambda g_\sigma - A_\mu g_\nu g_\lambda g_\sigma]
\] (2.13)
and we use the notations
\[
\int d^4x d^{-1} \kappa(g) \equiv -\frac{i}{240\pi^2} \int_{M_5} d^5x \epsilon^{pqrst} \text{tr} (g_p g_q g_r g_s g_t)
\] (2.14)
\[
g_\mu = \partial_\mu g g^{-1}.
\] (2.15)
In eq.(2.14) the integration goes over a five-dimensional manifold whose boundary is the usual four-dimensional space.

The functional \( \alpha_0(A^g) - \alpha_0(A) \) is a trivial local 1-cocycle which can be determined from eq.(2.11). The explicit form of \( \alpha_1(A, g) \) (eq.(2.13)) dictates the following ansatz for \( \alpha_0(A) \):
\[
\alpha_0(A) = -\frac{i}{48\pi^2} \int d^4x \epsilon^{\mu \nu \lambda \sigma} \text{tr} (a_1 A_\mu A_\nu A_\lambda A_\sigma^T + a_2 A_\mu A_\nu A_\lambda A_\sigma^T + a_3 A_\mu A_\nu A_\lambda A_\sigma^T + a_4 A_\mu A_\nu A_\lambda A_\sigma^T + a_5 A_\mu A_\nu A_\lambda A_\sigma^T + a_6 A_\mu A_\nu A_\lambda A_\sigma^T + a_7 A_\mu A_\nu A_\lambda A_\sigma^T + a_8 A_\mu A_\nu A_\lambda A_\sigma^T + a_9 A_\mu A_\nu A_\lambda A_\sigma^T + a_{10} A_\mu A_\nu A_\lambda A_\sigma^T + a_{11} A_\mu A_\nu A_\lambda A_\sigma^T + a_{12} A_\mu A_\nu A_\lambda A_\sigma^T)
\] (2.16)
where \( A_\mu^T \) is a transposed matrix \( A_\mu \). We choose this ansatz because it is the most general local functional which is metric independent and is invariant under global \( SO(N) \) transformations.

Let us stress that to satisfy eq.(2.11) it is necessary to introduce the terms depending not only on \( A_\mu \) but also on \( A_\mu^T \). Eq.(2.11) determines uniquely the coefficients \( a_i, b_i \). As a result:
\[
\alpha_0(A) = -\frac{i}{48\pi^2} \int d^4x \epsilon^{\mu \nu \lambda \sigma} \text{tr} (A_\mu A_\nu A_\lambda A_\sigma^T - \frac{1}{4} A_\mu A_\nu A_\lambda A_\sigma^T + \partial_\mu A_\nu A_\lambda A_\sigma^T + A_\mu \partial_\nu A_\lambda A_\sigma^T)
\] (2.17)
Obviously one can add also any trivial local $SO(N)$ invariant 1-cocycle. The corresponding infinitesimal 1-cocycle (anomaly) is calculated in a standard way

$$
\int d^4x \, \epsilon^a(x) \Lambda^a_{\text{ort}}(A) = \alpha^\text{ort}_1(A^h, h^{-1}g) - \alpha^\text{ort}_1(A, g) \tag{2.18}
$$

where $h = 1 + \epsilon^a \lambda^a$.

It looks as follows

$$
\Lambda^a_{\text{ort}}(A) = \frac{i}{48\pi^2} \epsilon^{\mu
u\lambda\sigma} \text{tr} \left[ (\lambda^a + \lambda^a T)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\nu A_\lambda A_\sigma - A_\nu A_\lambda A_\sigma - \frac{1}{2} A_\nu A_\lambda A_\sigma T - \frac{1}{2} \partial_\nu A_\lambda A_\sigma T - A_\nu A_\lambda A_\sigma T - \frac{1}{2} A_\nu A_\lambda A_\sigma T - A_\nu A_\lambda A_\sigma T + \frac{1}{2} A_\mu A_\nu A_\lambda A_\sigma + \frac{1}{2} A_\mu A_\nu A_\lambda A_\sigma T + \frac{1}{2} A_\mu A_\nu A_\lambda A_\sigma T \right] \tag{2.19}
$$

One sees that on the subgroup $SO(N)$ ($\lambda^a = -\lambda^a T$) this anomaly vanishes. The Wess–Zumino consistency condition \cite{14} (infinitesimal version of (2.7)) is obviously satisfied because our anomaly differs from the standard one by the trivial 1-cocycle.

The additional $SO(N)$ invariance of the Wess–Zumino action $\alpha^\text{ort}_1(A, g)$ means that it depends in fact not on all the elements of $SU(N)$ but only on the elements of the homogeneous space $SU(N)/SO(N)$. One can introduce coordinates on this homogeneous space and express the Wess–Zumino action in terms of these coordinates.

The natural coordinates are symmetric and unitary matrices

$$
s = gg^T \tag{2.20}
$$

This choice is suggested by the form of the mass term in the regularized Lagrangian \eqref{2.15}. As follows from eq.\eqref{2.5} after the gauge transformation it depends only on the combination $gg^T$. The gauge group transforms the coordinates $s$ in the following manner

$$
s \rightarrow g^{-1} s g^{-1} T. \tag{2.21}
$$

In terms of these coordinates the Wess–Zumino action looks as follows:

$$
\alpha^\text{ort}_1 = \int d^4x \left[ \frac{1}{2} d^{-1} \kappa(s) - \frac{i}{48\pi^2} \epsilon^{\mu\nu\lambda\sigma} \text{tr} \left[ \left( \partial_\mu A_\nu - A_\nu A_\lambda \right) + \partial_\nu A_\lambda A_\sigma - \frac{1}{2} A_\nu A_\lambda A_\sigma - \frac{1}{2} A_\nu A_\lambda A_\sigma T - \frac{1}{2} A_\nu A_\lambda A_\sigma T - A_\nu A_\lambda A_\sigma T + \frac{1}{2} A_\mu A_\nu A_\lambda A_\sigma + \frac{1}{2} A_\mu A_\nu A_\lambda A_\sigma T + \frac{1}{2} A_\mu A_\nu A_\lambda A_\sigma T \right] \right] \tag{2.22}
$$

where $s_\mu = \partial_\mu s^{-1}$.

The derivation is straightforward but some comments are in order. Using the equality

$$
g^T_\mu = s^{-1} (s_\mu - g_\mu) s \tag{2.23}
$$
we express \( q_\mu^T \) in terms of \( g_\mu \) and \( s_\mu \) and then comparing the terms of a given order in \( A_\mu \) and applying again eq. (2.23) we find the expression (2.24). This action may be used for the construction of the symplectic form defining the integration measure in the path integral. It is worthwhile to emphasize that contrary to the standard case the action (2.22) depends not only on the chiral current \( \partial_\mu s s^{-1} \), belonging to the Lie algebra of the group, but also on the coordinates of the homogeneous space \( SU(N)/SO(N) \). It may be of importance for analyzing possible stationary points of the effective action.

3 Anomalous constraints commutator

In this section we shall calculate the 2-cocycle associated to the Wess–Zumino action (2.12). This 2-cocycle appears as the Schwinger term in the constraints commutator and can be calculated either by direct summation of the Feynman diagrams [23, 24] or by using the path integral representation for the commutator [25]. We use the second approach. According to the Bjorken-Johnson-Low (BJL) formula the matrix element of the equal time commutator may be expressed in terms of the expectation value of \( T \)-product as follows:

\[
\lim_{q_0 \to \infty} q_0 \int dt' e^{iq_0(t'-t)} \langle \bar{\psi} | A(x,t') B(y,t) | \psi \rangle = i \langle \bar{\psi} | [A(x,t), B(y,t)] | \psi \rangle
\]

(3.1)

For the expectation value of \( T \)-product one can write the representation in terms of the path integral

\[
\langle \bar{\psi} | A(x,t') B(y,t) | \psi \rangle = \int d\mu e^{iS} A(x,t') B(y,t)
\]

(3.2)

Here it is understood that the integration goes over the fields satisfying the boundary conditions corresponding to the initial and final states \( | \psi \rangle \) and \( \langle \bar{\psi} | \). Following the approach of [25] we can consider the chiral \( SU(N) \) Yang–Mills model in the Hamiltonian gauge \( A_0 = 0 \). In this gauge the \( S \)-matrix element can be written as the path integral

\[
\langle \alpha | \beta \rangle = \int d\mu \delta(A_0) e^{iS},
\]

(3.3)

where in the first order formalism

\[
S = \int d^4x \left[ E_\iota^a \dot{A}_\iota^a - \frac{1}{2} (E_\iota^a)^2 - \frac{1}{4} (F_\iota^{ai})^2 + A_0^a G^a + i\bar{\psi} \gamma_0 \partial_0 \psi - i\bar{\psi} \gamma_i (\partial_i - A_i) \psi \right]
\]

(3.4)

In the nonanomalous case the constraints \( G^a = (\nabla_i E_i)^a \) form a Lie algebra

\[
[G^a(x), G^b(y)] = i f^{abc} G^c(y) \delta(x - y)
\]

(3.5)

However as was shown in refs. [25, 23, 24] in the anomalous theory this relation is violated and the Schwinger term arises.

To calculate this Schwinger term we make the gauge transformation of the variables in the integral (3.3). The transformed integral may be written in the form

\[
\langle \bar{\psi} | \psi \rangle = \int d\mu \delta(A_0) e^{iS} \exp \left\{ -i \int d^4x g_0^a G^a(x) + i\alpha_1^{ort}(A, g) \big|_{A_0 = -g_0} \right\}
\]

(3.6)
Here the 1-cocycle arises due to the noninvariance of the regularization in accordance with eq. (2.17).

Using the representation for the chiral field \( g = e^a \) and taking into account that the integral (3.6) does not depend on \( g \) we can put equal to zero variation of this integral over \( u \). To the second order in \( u \) one has:

\[
\frac{1}{2} \int d^4x d^4y \langle \bar{\phi} T \tilde{G}^a(x) \tilde{G}^b(y) | \phi \rangle \partial_0 u^a(x) \partial_0 u^b(y) + \\
\frac{i}{2} \int d^4x f^{abc} u^a(x) \partial_0 u^b(y) \langle \bar{\phi} T \tilde{G}^a(x) | \phi \rangle + \\
\frac{1}{48\pi^2} \int d^4x \langle \bar{\phi} | \exp \{ \epsilon_{ijk} \partial_i A_j ( \partial_k u(x), \partial_0 u(x)) \} | \phi \rangle + ... = 0. \tag{3.7}
\]

Here we introduced the notation:

\[
\tilde{G}^a(x) = G^a(x) - \frac{i}{48\pi^2} \epsilon_{ijk} \text{tr} \left[ (\lambda^a + \lambda^a T) (A_i \partial_j A_k + \partial_i A_j A_k + A_i A_j A_k) - \lambda^a \{ \partial_i A_j, A_k^T \} \right] - \lambda^a \{ \partial_i A_j, A_k^T \}
\tag{3.8}
\]

And ... denotes the terms which do not contribute in the BJL limit.

In the process of derivation of eqs. (3.7, 3.8) we used the explicit form of \( \alpha^\text{ort}_1 \) (2.12, 2.17) up to the second order in \( u \),

\[
\alpha^\text{ort}_1(-g_0, A_i, g_0) = \frac{i}{48\pi^2} \text{tr} \int dt \int \{ (AdA + dAA + A^3 - AA^T A + (T) - \\
- \{ dA, A^T \} (\partial_0 u + \frac{1}{2}[u, \partial_0 u]) + + (2dA + A^2)(du \partial_0 u + \partial_0 u du) + \\
- 2A \partial_0 u Adu - \partial_0 A[A, du + \frac{1}{2}[u, du]] + + \{(dg_0 + [A, g_0])(g[A^g, A^T g]g^{-1} - \\
- [A, A^T] - \partial_0 A(g[A^g, A^T g]g^{-1} - [A, A^T]) \}. \tag{3.9}
\]

where (\( T \)) stands for transposition of the first four terms. The coefficients of \( \partial_0 A \) can be combined with \( E \) leading to the shift of \( E_i^a \),

\[
E_i^a \rightarrow E_i^a + \frac{i}{48\pi^2} \epsilon_{ijk} \text{tr} \lambda^a \{ A_j, g_k \} + g \{ A_j^g, A_k^{Tg} \} g^{-1} - \{ A_j, A_k^T \} \tag{3.10}
\]

Combining the coefficients of \( g_0 \) with \( G^a(x) \) we get

\[
\int d\mu \exp\{ i \int d^4x (E_i^a \dot{A}_i^a - \tilde{H}(E_i, A_i, \partial_0 A_j) - g_0^a \tilde{G}^a(x)) + \\
+ \frac{i}{48\pi^2} \int dt \int dA (du \partial_0 u + \partial_0 u du)] \tag{3.11}
\]

Eq. (3.7) follows directly from eq. (3.11).

To get the expression for the commutator of \( \tilde{G} \) we apply to eq. (3.7) the operator:

\[
\lim_{(p_0 - q_0) \rightarrow +\infty} \frac{p_0 - q_0}{p_0 q_0} \int dx_0 dy_0 \varepsilon^{p_0 x_0 + i q_0 y_0} \frac{\delta}{\delta u^a(x)} \frac{\delta}{\delta u^b(y)} \tag{3.12}
\]

Taking the limit we get the result

\[
[\tilde{G}^a(x), \tilde{G}^b(y)] = i f^{abc} \tilde{G}^c(y) \delta(x - y) - \frac{1}{24\pi^2} \epsilon_{ijk} \text{tr} (\partial_i A_j \{ \lambda^a, \lambda^b \}) \partial_k^c \delta(x - y). \tag{3.13}
\]
Let us note that the commutator of $\tilde{G}$ coincides with the analogous commutator obtained in ref.\[25\] with the different Wess–Zumino action. However the definition of $\tilde{G}$ in our case is different. If one comes back to the $G$ one gets

$$[G^a(x), G^b(y)] = if^{abc} G^c(y) \delta(x - y) + a_{2,ort}^{ab}(A; x, y)$$ (3.14)

Here $a_{2,ort}^{ab}$ is the ultralocal 2-cocycle

$$a_{2,ort}^{ab}(A; x, y) = -\frac{1}{48\pi^2} \epsilon_{ijk} \text{tr} \left( [\lambda^a + \lambda^a T, \lambda^b + \lambda^b T] \times \right.$$

$$\times (A_i \partial_j A_k + \partial_i A_j A_k - A_i A_j A_k) + A_i A_j T A_k + A_i T \partial_j A_k + \partial_i T A_j A_k) +$$

$$+ (\lambda^a + \lambda^a T)(\partial_i A_j - \partial_i T A_j - A_i T A_j)(\lambda^b + \lambda^b T) A_k -$$

$$- (\lambda^b + \lambda^b T)(\partial_i A_j - \partial_i T A_j - A_i T A_j)(\lambda^a + \lambda^a T) A_k)$$ (3.15)

This cocycle differs from the one obtained in ref.\[25\]–\[24\] by trivial 2-cocycle. It vanishes if at least one of the constraints $G^a$ corresponds to the subgroup $SO(N)$. We note that the addition of any trivial 1-cocycle having topological nature does not change the commutator of modified constraints $\tilde{G}$.

Eq. (3.14) shows that the constraints commutator does not vanish on the constraint surface and therefore the quantization in the temporal gauge is inconsistent. To avoid this problem we shall add following the approach of Faddeev and Shatashvili the WZ action \[ to the classical action (2.1). As the gauge variation of the WZ action compensates the anomaly, one can hope that the quantization of the modified action will lead to a consistent theory. We continue to use the temporal gauge $A_0 = 0$ in spite of the fact that the modified classical action is not gauge invariant. We shall show that when the quantum corrections are taken into account the gauge invariance is restored.

The WZ action is the first order action for the chiral fields and to quantize it one needs to find the symplectic form. If this form is nondegenerate the quantization is performed in a standard way and has been done in ref. \[3\]. However in the case of $SO(3)$ invariant WZ action considered above the symplectic form is degenerate and the quantization requires more careful analysis. To illustrate the main ideas we consider firstly more simple case of two dimensional $SU(2)$ theory with a degenerate symplectic form.

### 4 Quantization of two dimensional $SU(2)$ model

The two dimensional chiral $SU(2)$ Yang-Mills theory is described by the action

$$S_{YM} = \int d^2 x \left( -\frac{1}{4e^2} (F^a_{\mu\nu})^2 + i\bar{\psi} \gamma^\mu (\partial_\mu + A_\mu) \psi \right),$$ (4.1)

where $\gamma^0 = \sigma^1$, $\gamma^1 = \sigma^2$ and $\psi \equiv \frac{(1 + i \gamma_5)}{2} \psi$, $\gamma_5 \equiv i \sigma^1 \sigma^2 = -\sigma^3$. The algebra of two dimensional $\gamma$-matrices allows to rewrite this action in the form

$$S_{YM} = \int d^2 x \left( -\frac{1}{4e^2} (F^a_{\mu\nu})^2 + i\bar{\psi}^+(\partial_+ + A_+) \psi \right),$$ (4.2)

where $\partial_+ = \partial_0 + \partial_1$; \quad $A_+ = A_0 + A_1$.

On the classical level this action possesses the usual gauge invariance, however as is well-known quantum corrections violate this invariance. To restore the gauge invariance
one can following Faddeev and Shatashvili [6] add to the action (4.1) the corresponding Wess-Zumino action, which in our case looks as follows:

\[ S_{WZ} = \frac{1}{12\pi} \int_{M^+} d^3x \epsilon^{ijk} \text{tr} g_i g_j g_k + \frac{1}{4\pi} \int d^2x \epsilon^{\mu\nu} \text{tr} (g_\mu A_\nu + \frac{a}{2} (A_\mu + g_\mu)^2 - \frac{a}{2} (A_\mu)^2) \]  

Here \( g_i = \partial_i g g^{-1} \), \( g \in SU(2) \), \( \epsilon^{ijk} \) and \( \epsilon^{\mu\nu} \) are antisymmetric tensors, \( M^+ \) is a three-dimensional manifold whose boundary is the usual two-dimensional space and \( a \) is an arbitrary parameter depending on the regularization used to calculate the WZ action.

If \( a \) is different from zero the action is nondegenerate and this case was considered by Shatashvili [19]. In particular when \( a = -1 \) the model is exactly soluble [27]. The case \( a = 0 \) is exceptional. In this case the WZ action does not depend on the space-time metrics and its symplectic form is degenerate (this is true for any gauge group). Below we carry out the Hamiltonian analysis of this case.

Following the strategy discussed in the Introduction we impose some gauge condition (here we shall use the temporal gauge \( A_0 = 0 \)), apply the canonical formalism and construct the path integral representation for the generating functional. Then we prove the gauge invariance of the integration measure justifying thus the possibility of imposing gauge condition before the quantization.

The first problem in applying the canonical quantization is the three dimensional term in the Wess-Zumino action (4.3). It is known that this term depends only on the values of the chiral field \( g \) on the two-dimensional boundary (more exactly by \( \text{mod } 2\pi \)) and therefore one can choose such a parametrization of the field \( g \) in which this term can be written explicitly as a two-dimensional one. We use the parametrization of the \( SU(2) \) group by the fields \( \phi^A \), satisfying the following condition:

\[ \text{tr} (g_A g_B g_C) = 6\pi \epsilon_{ABC}, \]  

where \( g_A = \frac{\partial g}{\partial \phi^A} g^{-1} = \partial_A g g^{-1} \) is a right-invariant vector field on the \( SU(2) \) group.

In terms of the fields \( \phi^A \) any right-invariant current \( g_i \) can be expressed by the following formula:

\[ g_i = \frac{\partial g}{\partial x^i} g^{-1} = g_A \partial_i \phi^A \]  

Due to the condition (4.4) the Haar measure \( dgg^{-1} \) on the \( SU(2) \) group is proportional to \( d\phi^1 d\phi^2 d\phi^3 \).

Using the parametrization by the fields \( \phi^A \) and imposing the light-cone gauge one can rewrite the sum of (4.2) and (4.3) as follows:

\[ S = \int d^2x \left( \frac{1}{2e^2} (\partial_0 A^a)^2 + \frac{1}{2} \epsilon_{ABC} \epsilon^{\mu\nu} \phi^A \partial_\mu \phi^B \partial_\nu \phi^C + \frac{1}{4\pi} \text{tr} (g_A A) \partial_0 \phi^A + i\psi^+ \partial_0 \psi + i\psi^+ \partial_1 \psi + i\psi^+ A \psi \right) \]  

Here \( A = A_1 \).

Introducing the canonically-conjugated momenta for the fields \( A \) and \( \phi^A \) one can present the action (4.6) in an equivalent form:

\[ S = \int d^2x \left( E_a \partial_0 A^a + p_A \partial_0 \phi^A - \frac{1}{2} (E_0)^2 \right) + \]
\[ + \lambda^A (p_A + \epsilon_{ABC} \phi^B \partial_1 \phi^C - \frac{1}{4\pi} \text{tr} (g_A A) + \]
\[ + i\psi^+ \partial_0 \psi + i\psi^+ \partial_1 \psi + i\psi^+ A\psi) \quad (4.7) \]

From (4.7) one can conclude that
\[ H = \frac{1}{2} (E_a)^2 - i\psi^+ \partial_1 \psi - i\psi^+ A\psi \quad (4.8) \]
is the Hamiltonian and
\[ C_A = p_A + \epsilon_{ABC} \phi^B \partial_1 \phi^C - \frac{1}{4\pi} \text{tr} (g_A A) \quad (4.9) \]
are the primary constraints of the model.

The next step in the canonical quantization is the calculation of secondary constraints. The simplest way seems to be to find all null-vectors of the matrix of the Poisson brackets of the primary constraints. Then for every null-vector \( \epsilon_a \) one can form a linear combination of the primary constraints \( C_\alpha = C_A \epsilon_A \), which commutes with all primary constraints on the constraints surface. The secondary constraints are then given by the Poisson brackets of \( C_\alpha \) and the Hamiltonian \( H \).

In our case the matrix of the Poisson brackets the primary constraints is equal to:
\[ \Omega_{AB}(x^1, y^1) = \{ C_A(x^1), C_B(y^1) \} = \Omega_{AB}(x^1) \delta(x^1 - y^1) \]
\[ = \frac{1}{4\pi} \text{tr} ([g_A, g_B](g_1(x^1) + A(x^1))) \delta(x^1 - y^1) \quad (4.10) \]

This matrix is ultralocal and in fact coincides with the symplectic form for the Wess-Zumino action. There is only one null-vector of \( \Omega_{AB} \) (in every space point) equal to
\[ \epsilon^A(x^1) = \frac{1}{4\pi} \epsilon^{ABC} \text{tr} ([g_B, g_C](g_1(x^1) + A(x^1))) = \epsilon^{ABC} \Omega_{BC}(x^1) \quad (4.11) \]
Calculating the Poisson bracket of the constraint \( \tilde{C}(x^1) = C_A(x^1) \epsilon^A(x^1) \) and \( H \) one gets up to the primary constraints the secondary constraint:
\[ C(x^1) = 4\pi \{ H, \tilde{C}(x^1) \} = \text{tr} (E(x^1)(g_1(x^1) + A(x^1))) \quad (4.12) \]
In this equation we omitted the term proportional to \( C_A \). The primary constraints \( C_A(x^1) \) and the secondary constraint \( C(x^1) \) form a set of second-class constraints and the matrix of the Poisson brackets of the constraints is equal to:
\[ M(x^1, y^1) = \begin{pmatrix} \Omega_{AB}(x^1, y^1) & v_A(x^1, y^1) \\ -v_B(y^1, x^1) & 0 \end{pmatrix} \quad (4.13) \]
where
\[ v_A(x^1, y^1) = \{ C_A(x^1), C(y^1) \} \]
\[ = \text{tr} (g_A(\partial_1 E - [g_1, E] - \frac{1}{4\pi} (g_1 + A)) \delta(x^1 - y^1) \]
\[ - \text{tr} (g_A E(x^1)) \partial_1^2 \delta(x^1 - y^1) \quad (4.14) \]
It is not difficult to show that the determinant of the matrix $M$ is equal to
\[
\det M = (\det \epsilon^{ABC} \Omega_{AB} v_C)^2 = (\det e^A v_A)^2
\]
and
\[
e^A v_A(x^1, y^1) = \text{tr} ((g_1 + A)(\nabla_1 E - \frac{1}{4\pi}(g_1 + A))) \delta(x^1 - y^1)
\]
up to the secondary constraint $C(x^1)$. Now one can write the expression for the generating functional of the model:
\[
Z = \int DA_0 D\phi Dp D\psi (\det M)^{\frac{1}{2}} \delta(C) \delta(C_A) \exp \left\{ i \int d^2 x \left( E_a \partial_0 A^a + p_A \partial_0 \phi^A - \frac{1}{2} (E_a)^2 + i\psi^+ \partial_0 \psi + i\psi^+ \partial_1 \psi + i\psi^+ A \psi \right) \right\}
\]
Integrating over $p_A$ and introducing the integration over $A_0$ in the path integral one gets
\[
Z = \int DA_0 D\phi Dp D\psi (\det M)^{\frac{1}{2}} \delta(C) \delta(C_A) \exp \left\{ i(S_{YM} + S_{WZ}) \right\}
\]
It is obvious from eqs.(4.12, 4.16) that the integration measure in eq.(4.18) is gauge-invariant apart from the gauge-fixing condition and the fermion measure. Therefore one can easily show that the modified Gauss-law constraints form the $SU(2)$ gauge algebra:
\[
[G_a(x^1), G_b(y^1)] = i\epsilon_{abc} G_c(x^1) \delta(x^1 - y^1)
\]
where
\[
G(x^1) = \nabla_1 E(x^1) - \frac{1}{4\pi} g_1(x^1) + j_0(x^1)
\]
Indeed in our case the gauge variation of the WZ action exactly compensates anomaly arising due to noninvariance of the fermionic measure and all other factors are gauge invariant. Therefore one can repeat all the arguments given in the proceeding section to show that the Gauss law has the form (4.19).

Due to this fact one can select the physical subspace imposing the condition $G_a |\Psi >= 0$ on the state vectors. The number of the physical degrees of freedom can be now easily calculated. All vector fields are unphysical due to the Gauss-law constraints and there is only one physical degree of freedom for three chiral fields $\phi_a$ due to the four second-class constraints.

This result is in accordance with the intuitive expectations. In the classical theory we had the system with three first class constraints eliminating all the bosonic degrees of freedom. Due to the quantum anomaly the first class constraints transform to the second class ones and one secondary constraint arises. Thus we have a system of four second class constraints eliminating two degrees of freedom. One degree survives as a physical excitation.

5 Quantization of the four dimensional SU(3) model

Now we are ready to quantize the four dimensional chiral $SU(3)$ Yang-Mills theory. The complete action of the model is described by the sum of the Yang-Mills action and the
$SO(3)$ invariant WZ action,
\[ S = \int d^4x \left(-\frac{1}{4e^2} (F_{\mu\nu}^a)^2 + i\bar{\psi} \gamma^\mu (\partial_\mu + A_\mu) \psi + \alpha_1^{\text{ort}}(A, s) \right) \] (5.1)

where $\alpha_1^{\text{ort}}$ is given by (2.22).

To apply the canonical formalism to the model one needs, as was mentioned in the Introduction, to reduce the five-dimensional term in the Wess-Zumino action to a four-dimensional one. To do it one can use the fact that any symmetric unitary matrix can be represented in the following form:
\[ s = \omega D\omega^T \] (5.2)

where $\omega$ is an orthogonal matrix $\omega\omega^T = 1$ and $D$ is a diagonal unitary matrix.

Using this representation and and the 1-cocycle condition
\[ \alpha_1^{\text{ort}}(A^h, h^{-1}sh^{-1}T) = \alpha_1^{\text{ort}}(A, s) - \alpha_1^{\text{ort}}(A, hh^T) \quad (\text{mod } 2\pi) \] (5.3)
one can show the validity of the following equation:
\[ \alpha_1^{\text{ort}}(A, \omega D\omega^T) = \alpha_1^{\text{ort}}(A^\omega, D) \] (5.4)

The five-dimensional term is equal to zero for any diagonal matrix and therefore the parametrization (5.2) solves the problem of reducing the Wess-Zumino action to a four-dimensional form.

Let us now represent the matrix $D$ in the form $D = e^{u^a T_a}$, where matrices $T_a$ belong to the Cartan subalgebra of the $su(N)$ algebra, and use an arbitrary parametrization of the $SO(N)$ group by fields $\phi^A$. Then introducing the canonically conjugated momenta for the fields $A_i$, $\phi^A$ and $u^a$ and imposing the temporal gauge $A_0 = 0$ one can rewrite the action (5.1) as follows:
\[ S = \int d^4x (\Pi_i^a \partial_0 A_i^a + p_A \partial_0 \phi^A + \pi_\alpha \partial_0 u^\alpha - \frac{1}{2}(\Pi_i^a - \Delta E_i^a)^2 - \frac{1}{4}(F_{ij}^a)^2 + \lambda^\alpha C_\alpha + \lambda^A C_A + \mathcal{L}_\psi) \] (5.5)

\[ \Delta E_i^a = -\frac{i}{48\pi^2} \epsilon^{ijk} \text{tr} (T_a \omega (\{A_j^\omega, u_k\} - \frac{1}{2}(DA_j^{\omega,T} D^{-1}, u_k) + \{A_j^\omega, DA_k^{\omega,T} D^{-1}\}) \omega^{-1}) \]
\[ A_i^\omega = \omega^{-1} A_i^\omega + \omega^{-1} \partial_\omega; \quad u_i = \partial_\omega u = \partial_\omega DD^{-1} \] (5.6)

and $C_p = (C_\alpha, C_A)$ are the primary constraints of the model
\[ C_\alpha = \pi^\alpha - \frac{i}{48\pi^2} \epsilon^{ijk} \text{tr} T_a (\{\partial_\alpha A_j^\omega, A_k^\omega\} + A_i^\omega A_j^\alpha A_k^\omega - \frac{1}{2}(\partial_\alpha A_j^\omega, DA_k^{\omega,T} D^{-1}) - A_i^\omega DA_j^{\omega,T} D^{-1} A_k^\omega - A_i^\omega u_j A_k^\omega) \] (5.7)
\[ C_A = p_A + \frac{i}{48\pi^2} \epsilon^{ijk} \text{tr} T_a (\{\partial_0 A_j^\omega, u_k\} - \frac{1}{2} D(\partial_0 A_j^{\omega,T}, u_k) D^{-1} + A_i^\omega u_j A_k^\omega - D^{-1} A_i^\omega u_j A_k^\omega D - DA_i^{\omega,T} D^{-1} A_j^\omega u_k - u_k A_i^\omega DA_j^{\omega,T} D^{-1} + \frac{1}{2}(A_i^\omega, DA_j^{\omega,T} D^{-1}, u_k) - u_i A_j^\alpha u_k + \{\partial_\omega A_j^\omega, DA_k^{\omega,T} D^{-1} - A_k^{\omega,T}\} + D^{-1}(A_i^\omega, \partial_0 A_j^\omega) D - A_i^\omega, \partial_0 A_j^\omega \} + D^{-1} A_i^\omega A_j^\alpha A_k^\omega D - A_i^\omega A_j^\omega A_k^\omega D - D^{-1} A_i^\omega DA_j^{\omega,T} D^{-1} A_k^\omega \] (5.8)
As was mentioned above, the matrix of the Poisson brackets of the primary constraints coincides with the symplectic form and is equal to:

\[
\Omega_{pq}(x, y) = \{C_p(x), C_q(y)\} = \Omega_{pq}(x) \delta(x - y)
\]

\[
\Omega_{pq}(x) = \frac{i}{96\pi^2} \epsilon_{ijk} \text{tr}([s_p, s_q] (\frac{1}{2} \{\tilde{A}_i, \tilde{F}_{jk}\} - \tilde{A}_i \tilde{A}_k \tilde{A}_k) + s_p (\frac{1}{2} \tilde{F}_{ij} - \tilde{A}_i \tilde{A}_j) s_q \tilde{A}_k - s_q (\frac{1}{2} \tilde{F}_{ij} - \tilde{A}_i \tilde{A}_j) s_p \tilde{A}_k) \quad (5.9)
\]

where

\[
s_\alpha = \frac{\partial s}{\partial u^\alpha} s^{-1} = \omega \lambda_{\alpha} \omega^{-1}; \quad s_A = \frac{\partial s}{\partial \phi^A} s^{-1} = \omega (\omega_A - D\omega_A D^{-1}) \omega^{-1};
\]

\[
\omega_A = \omega^{-1} \frac{\partial \omega}{\partial \phi^A} \quad (5.10)
\]

and

\[
\tilde{F}_{ij} = F_{ij} - s F_{ij}^T s^{-1}; \quad \tilde{A}_i = A_i + s A_i^T s^{-1} + s_i \quad (5.11)
\]

In parametrization we use the symplectic matrix is an antisymmetric 5 \times 5 matrix of the following type

\[
\Omega = \left( \begin{array}{cc} \{C_\alpha, C_\beta\} & \{C_\alpha, C_A\} \\ \{C_B, C_\beta\} & \{C_A, C_B\} \end{array} \right) \quad (5.12)
\]

where \{C_\alpha, C_\beta\} is nondegenerate block. For any antisymmetric matrix with nondegenerate block \(A\)

\[
\begin{pmatrix} A & f \\ -f^T & B \end{pmatrix} \quad (5.13)
\]

the equation for null vectors

\[
\begin{pmatrix} \alpha_r \\ \beta_r \end{pmatrix} \quad (5.14)
\]

can be reduced to the following equation for the component \(\beta_r\)

\[
(B + f^T A^{-1} f) \beta_r = 0 \quad (5.15)
\]

Substituting to this equation the explicit form of \(A, B\) and \(f\) from eq. (5.12) we see that for the case of a general position the only null vector is equal to

\[
e^p(x) = \epsilon^{pqrst} \Omega_{qr}(x) \Omega_{st}(x) \quad (5.16)
\]

As before the secondary constraint is given by the Poisson bracket of the constraint \(\tilde{C}(x) = C_p(x) e^p(x)\) and the Hamiltonian \(H = \int d^3 x (\frac{1}{2} (\Pi_a - \Delta E_a)^2 + \frac{1}{4} (F_{ij}^a)^2)\):

\[
\tilde{C}(x) = \{C_p(x) e^p(x), H\} \sim \epsilon^{pqrst} R_p \Omega_{qr}(x) \Omega_{st}(y) \quad (5.17)
\]

where

\[
R_p = \epsilon^{ijk} \text{tr} s_p (\{E_i, F_{jk} - \frac{1}{2} s F_{jk}^T s^{-1} - \tilde{A}_j \tilde{A}_k\} + \tilde{A}_i E_{ij} \tilde{A}_k) \quad (5.18)
\]
This formula follows from the equation

\[ \{ C_p(x), E_i^a(y) \} \sim \epsilon^{ijk} \text{tr} s_p(\{ \lambda^a, F_{jk} - \frac{1}{2}sF^T_{jk}s^{-1} - \tilde{A}_j\tilde{A}_k \} - \tilde{A}_j\lambda^a\tilde{A}_k)\delta(x - y) \] (5.19)

As it is shown in the Appendix B the secondary constraint transforms under the gauge transformation as follows:

\[ C(x) \rightarrow \det \left( \frac{\partial \phi^p}{\partial \tilde{\phi}^q} \right) \Omega_{pq}(x, y) v_p(x, y) - v_q(y, x) v(x, y) \] (5.20)

where \( \phi^p \) are the coordinates of the point \( s \) on the coset space \( SU(3)/SO(3) \) (\( u^\lambda \) and \( \phi^A \) in our case) and \( \tilde{\phi}^p \) are the coordinates of the gauge-transformed point \( g^{-1}sg^{-1}\top \).

In other words the function \( \tilde{\phi}(\phi) \) defines the change of the field \( \phi^p \) under the gauge transformation. The five primary constraints \( C_p(x) \) and the secondary constraint \( C(x) \) form a set of second-class constraints with the following matrix of the Poisson brackets of the constraints:

\[ M(x, y) = \left( \begin{array}{cc} \Omega_{pq}(x, y) & v_p(x, y) \\ -v_q(y, x) & v(x, y) \end{array} \right) \] (5.21)

where

\[ v_p(x, y) = \{ C_p(x), C(y) \}, \quad v(x, y) = \{ C(x), C(y) \} \] (5.22)

The matrix \( M(x, y) \) has the following gauge transformation law (see Appendix B):

\[ M(x, y) \rightarrow \left( \begin{array}{cc} \frac{\partial \phi^p}{\partial \tilde{\phi}^r}(x) & 0 \\ 0 & \det \left( \frac{\partial \phi^q}{\partial \tilde{\phi}^q}(x) \right) \end{array} \right) \left( \begin{array}{cc} \Omega_{pq}(x, y) & v_p(x, y) \\ -v_q(y, x) & v(x, y) \end{array} \right) \times \left( \begin{array}{cc} \frac{\partial \phi^q}{\partial \tilde{\phi}^r}(y) & 0 \\ 0 & \det \left( \frac{\partial \phi^p}{\partial \tilde{\phi}^p}(y) \right) \end{array} \right) \] (5.23)

Due to eq.(5.23) \( (\det M)^{\frac{1}{2}} \) transforms as follows:

\[ (\det M)^{\frac{1}{2}} \rightarrow \left( \det \frac{\partial \phi}{\partial \tilde{\phi}} \right)^2 (\det M)^{\frac{1}{2}} \] (5.24)

Now we can prove the gauge invariance of the integration measure in the path integral for the generating functional:

\[ Z = \int DA_p DED\phi D\psi D\bar{\psi}(\det M)^{\frac{1}{2}}\delta(C)\delta(A_0)\exp \{ i(S_{YM} + S_{WZ}) \} \] (5.25)

Taking into account eqs.(5.19) and (5.24) and the transformation law \( D\phi \rightarrow \det \left( \frac{\partial \phi}{\partial \tilde{\phi}} \right) D\phi \) we see that the measure \( D\phi(\det M)^{\frac{1}{2}}\delta(C) \) is gauge invariant. Hence in the complete analogy with the discussion at the end of the preceding section we justified the possibility of imposing the gauge condition before the quantization and of selecting the physical subspace by the Gauss-law constraints.

The number of the physical degrees of freedom can be now easily calculated. Due to the Gauss-law constraints there are \( 2 \times 8 \) vector degrees of freedom (8 is the dimension...
of $SU(3))$ and due to the six second-class constraints there are two bosonic degrees of freedom. Let us remind that in the case of the standard Wess-Zumino action one would get four bosonic degrees of freedom and thus these models differ crucially from each other in spite of the fact that the difference between these Wess-Zumino actions is a local trivial 1-cocycle. Let us finally note that one could use such a parametrization of the coset space $SU(3)/SO(3)$ that the invariant measure is proportional to $d\phi^1...d\phi^5$. In this case the secondary constraint and det $M$ are gauge-invariant.

6 Discussion

In this paper we showed that if one modifies the classical action of anomalous model by adding the corresponding WZ action the theory can be consistently quantized and new degrees of freedom appear. The number of new degrees of freedom depends on the particular choice of the WZ action. In the case of four dimensional $SU(N)$ gauge models the minimal number of new degrees of freedom arises if one uses the $SO(N)$ invariant WZ action described above. At present we have no reliable calculation scheme that makes difficult more detailed analysis of the model. One could try to develop some perturbation expansion in terms of vector fields as for $A_\mu = 0$ the WZ action is reparametrization invariant and describes the exactly soluble model. Unfortunately the point $A_\mu = 0$ is a singular one and this procedure fails. In the case of two dimensional model the perturbative expansion in the coupling constant can be developed in the light-cone gauge. However to get really interesting results one needs some nonperturbative approach which at present is not known.

Finally we mention that the path integral representation for generating functionals in models considered above can be written in an alternative form as the path integrals of the exponent of the Lagrangian action. As was shown in the paper [28] in the path integral for systems with secondary constraints one can make a canonical transformation eliminating the secondary constraint and thus to rewrite it as the integral of the exponent of the Lagrangian action. The price one pays for it is the appearances of the new local measure in the path integral. In this approach the symmetry properties of the action in particular gauge invariance are manifest but one needs to study the new local measure. In perturbation theory this measure can be done trivial but in a general case the problem remains open.

The analysis of the models described above was simplified due to the fact that in these cases the number of chiral fields was equal to $d + 1$ where $d$ is space-time dimension. However it can be generalized to arbitrary $SU(N)$ chiral model.

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A APPENDIX

In this Appendix we shall give some information about the cocycles and show how the $SO(N)$ invariant WZ action can be obtained in the framework of descent procedure.
Let us consider a group $G$ acting on the space $M$, i.e. $g : m \to mg$, $g \in G$, $m \in M$ and real functions $\alpha_0(m)$, $\alpha_1(m; g)$, ..., $\alpha_n(m; g_1, ..., g_n)$, ... form the sequence of functions depending on the point $m$ and on the ordered set $g_1, ..., g_n$. The operator $\delta$ which acts from the space with number $n$ to the space with number $n + 1$ according to the following rule

$$
(\delta\alpha_n)(m; g_1, ..., g_{n+1}) = \alpha_n(mg_1; g_2, ..., g_{n+1}) - \alpha_n(m; g_1g_2, ..., g_{n+1}) + ...
+ (-1)^n \alpha_n(m; g_1, g_2, ..., g_n)
$$

is a coboundary operator ($\delta^2 = 0$) A function $\alpha_n$ which satisfies the equation $\delta\alpha_n = 0$ is called $n$-cocycle. A cocycle which can be presented in the form $\alpha_n = \delta\beta$ is called coboundary or trivial cocycle. We use the term trivial cocycle in a more narrow sense to denote a cocycle which is a coboundary of a local functional. In our case the elements of $M$ are the Yang-Mills fields and $G$ is the group of gauge transformations. In this paper we presented a construction of 1-cocycle on the gauge group (WZ action) and infinitesimal 2-cocycle (Schwinger term in Gauss law commutator), now we shall show how these cocycles can be constructed with the help of descent procedure.

We shall use the language of external forms. In this language the Yang-Mills field is described by matrix valued 1-form $A = A_\mu dx^\mu$. The Yang-Mills field strength is described by the 2-form $F = dA + A^2$, the wedge product is assumed.

Led us consider the Minkovski space-time as embedded into some higher dimensional space. In this space we can write the closed gauge invariant 6-form

$$
\omega_{-1} = \frac{1}{48\pi^2} \text{tr} F^3
$$

This form is locally exact

$$
\omega_{-1} = d\omega_0
$$

where

$$
\omega_0 = \frac{1}{48\pi^2} \text{tr} (F^2 A - FA^3 + \frac{2}{3} A^5) + d\omega' \pmod{Z}
$$

where $\omega'$ is an arbitrary 4-form. The 5-form $\omega_0$ being integrated over a five dimensional manifold gives five dimensional Chern-Simons action. Action of the coboundary operator $\delta$ which in this case coincides with gauge variation on $\omega_0$ gives a closed 5-form

$$
d\delta \omega_0 = \delta d\omega_0 = \delta \omega_{-1} = 0
$$

the explicit form of $\delta \omega_0$ is given by the equation

$$
\delta \omega_0 = \frac{1}{48\pi^2} \text{tr} (dgg^{-1})^5 + \frac{i}{48\pi^2} d(\text{tr}[ (AdA + dAA + A^3) dgg^{-1} - \frac{1}{2} Adgg^{-1} Adgg^{-1} - A(dgg^{-1})^3] + \delta \omega'(A))
$$

From this explicit expression follows that the $\delta \omega_0$ is locally exact. Let us introduce the notation $d^{-1} \kappa$

$$
\int_{M_4} d^{-1} \kappa = \frac{i}{240\pi^2} \int_{D_5} \kappa
$$

where $D_5$ is a five dimensional disc with boundary $M_4$, the usual Minkovski space. Then locally

$$
\delta \omega_0 = d\omega_1
$$
where $\omega_1$ is defined up to a global exact form and a coboundary. Integrating this form over the four dimensional space we get

$$\frac{1}{2\pi} \alpha_1(A, g) = \int_{M_4} \omega_1(A, g) = \int_{D_5} d\omega_1(A, g) = \int_{D_5} \delta \omega_0(A, g)$$  \hspace{1cm} \text{(A.9)}$$

The functional $\alpha_1$ is defined up to the coboundary $\delta \alpha_0$

$$\alpha_0(A) = \int_{M_4} \omega'(A)$$  \hspace{1cm} \text{(A.10)}$$

From eq. (A.9) it follows that

$$\delta \alpha_1 = 0 \pmod{2\pi}$$  \hspace{1cm} \text{(A.11)}$$

i.e. $\alpha_1$ is 1-cocycle.

If to choose $\omega'$ to be zero \[7\] one gets the standard WZ action which breaks the $SO(N)$ invariance. The corresponding $\omega_0$ is not $SO(N)$ invariant for $d\omega' = 0$. Choosing $\omega'$ in such a way to restore the $SO(N)$ gauge invariance of $\omega_0$ we shall get by the descent procedure the $SO(N)$ invariant WZ action presented above. Using $\delta$ and $d^{-1}$ to continue the descent procedure we can calculate 2-cocycle. It is also defined up to a coboundary and using this freedom one can get the Schwinger term vanishing on $SO(N)$ subgroup.

**B APPENDIX**

In this appendix we prove the gauge invariance of the integration measure

$$DA_i DE_i D\phi (det M)^{1/2} \delta(C)$$

where $C$ is the secondary constraint

$$C(x) = \{C_p(x) e^p(x), H\} \sim \epsilon^{pqrs} R_p Q_{qr}(x) Q_{st}(y)$$

Where

$$R_p = tr s_p f;$$

$$\Omega_{pq} = \frac{i}{96\pi^2} \epsilon^{ijk} \text{tr} ([s_p, s_q] f_i + s_p f_i s_q \tilde{A}_k - s_q f_i s_p \tilde{A}_k)$$

Explicit forms of $f$, $f_1$ and $f_i$ are given in the section 5.

$M$ is the matrix of Pisson brackets of the constraints

$$M(x, y) = \begin{pmatrix}
\Omega_{pq}(x, y) & v_p(x, y) \\
-v_q(y, x) & v(x, y)
\end{pmatrix}$$

where

$$v_p(x, y) = \{C_p(x), C(y)\}, \quad v(x, y) = \{C(x), C(y)\}$$

Firstly we prove that the gauge transformation of $C(x)$ has a form

$$C(x) \rightarrow \det \left( \frac{\partial \phi^p}{\partial \phi^q} \right) C(x)$$

(B.5)
It follows from the explicit expressions for \( f, f_1 \) and \( f_i \) and \( \tilde{A}_i \) that under the gauge transformation they change as follows

\[
f_\ast \to g^{-1} f_\ast g
\]

The law for \( s_p(\phi) \) follows from the definition of \( s_p \)

\[
\tilde{s}_p(\phi) = \frac{\partial s(\phi)}{\partial \phi^p} s^{-1}(\phi) = g^{-1} \frac{\partial s(\phi)}{\partial \phi^p} s^{-1}(\phi) g = g^{-1} s_q(\phi) \frac{\partial \phi^q}{\partial \phi^p}
\]

From eqs. (B.2, B.3, B.7) we get the transformation law for \( C(x) \)

\[
C(x) \to e^{\rho q r s} \frac{\partial \phi^p}{\partial \phi^q} \cdots \frac{\partial \phi^t}{\partial \phi^p} R_{\rho q r s} \Omega_{\rho q r s} = det \left( \frac{\partial \phi}{\partial \phi} \right) C(x)
\]

To get the transformation law for \( det M \) we use the following equations

\[
v_p(x, y) \to \frac{\partial \phi^q}{\partial \phi^p}(x)v_q(x, y) det \left( \frac{\partial \phi}{\partial \phi} \right)(y)
\]

\[
v(x, y) \to det \left( \frac{\partial \phi}{\partial \phi} \right)(x)v(x, y) det \left( \frac{\partial \phi}{\partial \phi} \right)(y)
\]

which are valid on the constraint surface \( C = 0 \). These equations lead to the following transformation of the matrix \( M \)

\[
M(x, y) \to \begin{pmatrix}
\frac{\partial \phi^q}{\partial \phi^p}(x) & 0 \\
0 & det \left( \frac{\partial \phi^q}{\partial \phi^p}(x) \right)
\end{pmatrix} \begin{pmatrix}
\Omega_{pq}(x, y) & v_p(x, y) \\
v_q(x, y) & -v_q(y, x)
\end{pmatrix}
\times
\begin{pmatrix}
\frac{\partial \phi^t}{\partial \phi^s}(y) & 0 \\
0 & det \left( \frac{\partial \phi^t}{\partial \phi^s}(y) \right)
\end{pmatrix}
\]

and the \( det M \) transforms as follows

\[
(det M)^{1/2} \to \left( det \frac{\partial \phi}{\partial \phi} \right)^2 (det M)^{1/2}
\]

Let us firstly prove eq. (B.10). The constraint \( C(y) \) has the following structure

\[
C(y) = tr E(y) B(y)
\]

where \( B(y) \) depends on \( s(y), \tilde{A}_i(y) \) and \( s_p(y) \). Using the Leibnitz rule one can write the Poisson bracket of \( C_p(x) \) and \( C(y) \) in the form

\[
v_p(x, y) = tr \{ C_p(x), E_i(y) \} B_i(y) + tr E_i(y) \{ C_p(x), B(y) \}
\]
To prove eqs. (B.17–B.19) we note that the integration measure (B.1) is gauge invariant. Using eqs. (B.8) and (B.16) we see that this term is in agreement with eq. (B.10). To find the transformation law of the second term we need the relations

\[ \{ C_p(x), s(y) \} \rightarrow \frac{\partial \phi^n}{\partial \phi^p}(x)g^{-1}(y)\{ C_q(x), s(y) \} g^{-1,T}(y) \]  
(B.17)

\[ \{ C_p(x), \bar{A}_i \} \rightarrow \frac{\partial \phi^{p_1}}{\partial \phi^p}(x)\frac{\partial \phi^{q_1}}{\partial \phi^n}(y)g^{-1}(y) \times \]

\[ \times \{ C_{p_1}(x), s_{q_1}(y) \} g^{-1,T}(y) - g^{-1}s_{q_1}g\frac{\partial \phi^{p_1}}{\partial \phi^p} \frac{\partial \phi^{q_1}}{\partial \phi^n} \delta(x - y) \]  
(B.19)

To prove eqs. (B.17, B.19) we note that \( C_p \) can be presented in the form

\[ C_p(x) = \pi_p + \varphi_p(s, A) \]  
(B.20)

where \( \pi_p \) is the canonical momentum for \( \phi^p \). The second term in eq. (B.20) depends only on \( s \) and \( A_i \) and therefore commutes with \( s \) and \( A_i \). The Poisson bracket of \( \pi_p \) with any functional \( W \) of \( \phi^p \) is equal to

\[ \{ \pi_p(x), W(\phi) \} = -\frac{\delta}{\delta \phi^p(x)}W(\phi) \]  
(B.21)

If the last term in eq. (B.19) were absent one would get for \( v_p \) the transformation law (B.10). This term leads to an additional contribution in the transformation law for \( v_p(x) \). However one can show that this contribution vanishes on the constraint surface \( C = 0 \). To prove it let us note that the constraint \( C \) can be represented as follows

\[ C(x) = \text{tr} \left( \epsilon^{pqrst}(s_p \otimes s_q \otimes s_r \otimes s_s \otimes s_t) V(x) \right) \]  
(B.22)

where \( V(x) \) does not depend on \( s_p \). Thus the additional term is equal to

\[ \Delta = \text{tr} \left[ \epsilon^{uqrst}(s_{p_{1qrst}} - s_{q_{p1rst}} + s_{q_{r1pst}} - s_{q_{ps1t}} + s_{p_{1qrst}}) \frac{\partial^2 \phi^{p_1}}{\partial \phi^n \partial \phi^u} V(x) \right] \delta(x - y) \]  
(B.23)

where \( s_{pqrst} = s_p \otimes s_q \otimes s_r \otimes s_s \otimes s_t \). The combination in the curly brackets is proportional to

\[ \epsilon^{uqrst}(s_p_{1qrst} - \ldots) \sim \delta_{p_1 u} \epsilon^{uqrst} s_{uqrst} \]  
(B.24)

It is obvious from eq. (B.24) that \( \Delta \) is proportional to \( C \). Therefore we proved the validity of eq. (B.10). Eq. (B.11) can be derived in a similar way. Taking into account that \( DA \) and \( DE \) are invariant and \( D\phi \) transforms as follows

\[ D\phi \rightarrow \text{det} \left( \frac{\partial \phi}{\partial \phi} \right) D\tilde{\phi} \]  
(B.25)

we see that the integration measure (B.1) is gauge invariant.
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