MORE SMOOTHLY REAL COMPACT SPACES

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Abstract. A topological space $X$ is called $\mathcal{A}$-real compact if every algebra homomorphism from $\mathcal{A}$ to the reals is an evaluation at some point of $X$, where $\mathcal{A}$ is an algebra of continuous functions. Our main interest lies on algebras of smooth functions. Arias-de-Reyna has shown that any separable Banach space is smoothly real compact. Here we generalize this result to a huge class of locally convex spaces including arbitrary products of separable Fréchet spaces.

In [KMS] the notion of real compactness was generalized by defining a topological space $X$ to be $\mathcal{A}$-real-compact if every algebra homomorphism $\alpha: \mathcal{A} \to \mathbb{R}$ is just the evaluation at some point $a \in X$, where $\mathcal{A}$ is some subalgebra of $C(X, \mathbb{R})$. In case $\mathcal{A}$ equals the algebra $C(X, \mathbb{R})$ of all continuous functions this condition reduces to the usual real-compactness. Our main interest lies on algebras $\mathcal{A}$ of smooth functions. In particular we showed in [KMS] that every space admitting $\mathcal{A}$-partitions of unity is $\mathcal{A}$-real-compact. Furthermore any product of the real line $\mathbb{R}$ is $C^\infty$-real-compact. A question we could not solve was whether $\ell^1$ is $C^\infty$-real-compact, despite the fact that there are no smooth bump functions. [AdR] had already shown that this is true not only for $\ell^1$, but for any separable Banach space.

The aim of this paper is to generalize this result of [AdR] to a huge class of locally convex spaces, including arbitrary products of separable Fréchet spaces.

Convention. All subalgebras $\mathcal{A} \subseteq C(X, \mathbb{R})$ are assumed to be real algebras with unit and with the additional property that for any $f \in \mathcal{A}$ with $f(x) \neq 0$ for all $x \in X$ the function $1/f$ lies also in $\mathcal{A}$.

1. Lemma. Let $\mathcal{A} \subseteq C(X, \mathbb{R})$ be a finitely generated subalgebra of continuous functions on a topological space $X$. Then $X$ is $\mathcal{A}$-real-compact.

Proof. Let $\alpha: \mathcal{A} \to \mathbb{R}$ be an algebra homomorphism. We first show that for any finite set $\mathcal{T} \subseteq \mathcal{A}$ there exists a point $x \in X$ with $f(x) = \alpha(f)$ for all $f \in \mathcal{T}$.

For $f \in \mathcal{A}$ let $Z(f) := \{x \in X : f(x) = \alpha(f)\}$. Then $Z(f) = Z(f - \alpha(f)1)$, since $\alpha(f - \alpha(f)1) = 0$. Hence we may assume that all $f \in \mathcal{T}$ are even contained in $\ker \alpha = \{f : \alpha(f) = 0\}$. Then $\bigcap_{f \in \mathcal{T}} Z(f) = Z(\sum_{f \in \mathcal{T}} f^2)$. The
sets \( Z(f) \) are not empty, since otherwise \( f \in \ker \alpha \) and \( f(x) \neq 0 \) for all \( x \), so \( 1/f \in \mathcal{A} \) and hence \( 1 = f/f \in \ker \alpha \), a contradiction to \( \alpha(1) = 1 \).

Now the lemma is valid, whether the condition "finitely generated" is meant in the sense of an ordinary algebra or even as an algebra with the additional assumption on non-vanishing functions, since then any \( f \in \mathcal{A} \) can be written as a rational function in the elements of \( \mathcal{F} \). Thus \( \alpha \) applied to such a rational function is just the rational function in the corresponding elements of \( \alpha(\mathcal{F}) = \mathcal{F}(x) \) and is thus the value of the rational function at \( x \). \( \Box \)

2. **Corollary.** Any algebra-homomorphism \( \alpha: \mathcal{A} \rightarrow \mathbb{R} \) is monotone.

**Proof.** Let \( f_1 \leq f_2 \). By Lemma 1 there exists an \( x \in X \) such that \( \alpha(f_i) = f_i(x) \) for \( i = 1, 2 \). Thus \( \alpha(f_1) = f_1(x) \leq f_2(x) = \alpha(f_2) \). \( \Box \)

3. **Corollary.** Any algebra-homomorphism \( \alpha: \mathcal{A} \rightarrow \mathbb{R} \) is bounded, for every convenient algebra structure on \( \mathcal{A} \).

By a convenient algebra structure we mean a convenient vector space structure for which the multiplication \( \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \) is a bilinear bornological mapping. A convenient vector space is a separated locally convex vector space that is Mackey complete, see [FK].

**Proof.** Suppose that \( f_n \) is a bounded sequence, but \( |\alpha(f_n)| \) is unbounded. Replacing \( f_n \) by \( f_n^2 \) we may assume that \( f_n \geq 0 \) and hence also \( \alpha(f_n) \geq 0 \). Choosing a subsequence we may even assume that \( \alpha(f_n) \geq 2^n \). Now consider \( \sum_n f_n/2^n \). This series converges in the sense of Mackey, and since the bornology on \( \mathcal{A} \) is complete, the limit is an element \( f \in \mathcal{A} \). Applying \( \alpha \) yields

\[
\alpha(f) = \alpha \left( \sum_{n=0}^{N} \frac{1}{2^n} f_n + \sum_{n>N} \frac{1}{2^n} f_n \right) = \sum_{n=0}^{N} \frac{1}{2^n} \alpha(f_n) + \alpha \left( \sum_{n>N} \frac{1}{2^n} f_n \right)
\]

\[
\geq \sum_{n=0}^{N} \frac{1}{2^n} \alpha(f_n) + 0 = \sum_{n=0}^{N} \frac{1}{2^n} \alpha(f_n),
\]

where we applied to the function \( \sum_{n>N} \frac{1}{2^n} f_n \geq 0 \) that \( \alpha \) is monotone. Thus the series \( \sum_{n=0}^{N} \alpha(f_n)/2^n \) is bounded and increasing, hence converges, but its summands are bounded by 1 from below. This is a contradiction. \( \Box \)

4. **Definition.** We recall that a mapping \( f: E \rightarrow F \) between convenient vector spaces is called smooth (\( C^\infty \) for short) if the composite \( f \circ c: \mathbb{R} \rightarrow F \) is smooth for every smooth curve \( c: \mathbb{R} \rightarrow E \). It can be shown that under these assumptions derivatives \( f^{(p)}: E \rightarrow L^p(E, F) \) exist. See [FK].

A mapping is called \( C^\infty_c \) if in addition all derivatives considered as mappings \( d^p f: E \times E^p \rightarrow F \) are continuous.

Now we generalize Lemma 5 and Proposition 7 of [AdR] to arbitrary convenient vector spaces.

5. **Definition.** Let \( \mathcal{A} \subseteq C(X, \mathbb{R}) \) be a set of continuous functions on \( X \). We say that a space \( X \) admits large carriers of class \( \mathcal{A} \) if for every neighborhood \( U \) of a point \( p \in X \) there exists a function \( f \in \mathcal{A} \) with \( f(p) = 0 \) and \( f(x) \neq 0 \) for all \( x \notin U \).
Every \( \mathcal{A} \)-regular space \( X \) admits large \( \mathcal{A} \)-carriers, where \( X \) is called \( \mathcal{A} \)-regular if for every neighborhood \( U \) of a point \( p \in X \) there exists a function \( f \in \mathcal{A} \) with \( f(p) > 0 \) and \( f(x) = 0 \) for \( x \notin U \). The existence of large \( \mathcal{A} \)-carriers follows by using the modified function \( \bar{f} := f(a) - f \).

In [AdR, Proof of Theorem 8] it is proved that every separable Banach space admits large \( C^\infty_c \)-carriers. The carrying functions can even be chosen as polynomials as shown in Lemma 7 below.

6. Lemma. Let \( E \) be a convenient vector space, \( \{x'_n : n \in \mathbb{N}\} \subset E' \) be bounded, \( (\lambda_n) \in \ell^1(\mathbb{N}) \). Then the series \( (x, y) \mapsto \sum_{n=1}^{\infty} \lambda_n x_n'(x) x_n'(y) \) converges to a continuous symmetric bilinear function on \( E \times E \).

\textbf{Proof.} Clearly the function converges pointwise. Since the sequence \( \{x'_n\} \) is bounded, it is equicontinuous, hence bounded on some neighborhood \( U \) of 0, so there exists a constant \( M \in \mathbb{R} \) such that \( |x'_n(U)| \leq M \) for all \( n \in \mathbb{N} \). For \( x, y \in U \) we have \( |\sum_{n=1}^{\infty} \lambda_n x_n'(x) x_n'(y)| \leq \sum_{n=1}^{\infty} |\lambda_n| M^2 \), which suffices for continuity of a bilinear function. \( \square \)

7. Lemma. Let \( E \) be a Banach space that is separable or whose dual is separable for the topology of pointwise convergence. Then \( E \) admits large carriers for continuous polynomials of degree 2.

\textbf{Proof.} If \( E \) is separable there exists a dense sequence \( (x_n) \) in \( E \). By the Hahn-Banach theorem [J, 7.2.4] there exist \( x'_n \in E' \) with \( x'_n(x_n) = |x_n| \) and \( |x'_n| \leq 1 \).

Claim. \( \sup_n |x'_n(x)| = |x| \).

Since \( |x'_n| \leq 1 \) we have \( (\leq) \). For the converse direction let \( \delta > 0 \) be given. By denseness there exists an \( n \in \mathbb{N} \) such that \( |x_n - x| < \frac{\delta}{2} \). So we have

\[
|x| \leq |x_n| + |x - x_n| < |x'_n(x_n)| + \frac{\delta}{2} \\
\leq |x'_n(x)| + |x'_n(x - x_n)| + \frac{\delta}{2} < |x'_n(x)| + \delta.
\]

If the dual \( E' \) is separable for the topology of pointwise convergence, then let \( x'_n \) be a sequence that is weakly dense in the unit ball of \( E' \). Then \( |x| = \sup_n |x'_n(x)| \).

In both cases the continuous polynomials of Lemma 6

\[
x \mapsto \sum_{n=1}^{\infty} x'_n(x - a)^2 / n^2
\]

vanish exactly at \( a \). \( \square \)

8. Lemma. Let \( \alpha : \mathcal{A} \rightarrow \mathbb{R} \) be an algebra homomorphism and assume that some subset \( \mathcal{A}_0 \subset \mathcal{A} \) exists and a point \( a \in X \) such that \( \alpha(f_0) = f_0(a) \) for all \( f_0 \in \mathcal{A}_0 \) and such that \( X \) admits large carriers of class \( \mathcal{A}_0 \).

Then \( \alpha(f) = f(a) \) for all \( f \in \mathcal{A} \).

\textbf{Proof.} Let \( f \in \mathcal{A} \) be arbitrary. Since \( X \) admits large \( \mathcal{A}_0 \)-carriers, there exists for every neighborhood \( U \) of \( a \) a function \( f_U \in \mathcal{A}_0 \) with \( f_U(a) = 0 \) and \( f_U(x) \neq 0 \) for all \( x \in U \). By Lemma 1 there exists a point \( a_U \) such that \( \alpha(f) = f(a_U) \) and \( \alpha(f_U) = f_U(a_U) \). Since \( f_U \in \mathcal{A}_0 \), we have \( f_U(a_U) = \ldots \)
\[ \alpha(f_U) = f_U(a) = 0, \text{ hence } a_U \in U. \] Thus the net \( a_U \) converges to \( a \) and consequently \( f(a) = f(\lim a_U) = \lim f(a_U) \) since \( f \) is continuous. \( \square \)

Now we generalize Proposition 2 and Lemma 3 of [BBL]. For every convenient vector space \( E \), let a subalgebra \( \mathcal{A}(E) \) of \( C(E, \mathbb{R}) \) be given, such that for every \( f \in L(E, F) \) the image of \( f^* \) on \( \mathcal{A}(F) \) lies in \( \mathcal{A}(E) \). Examples are \( C_c^{\infty}, \mathcal{C}^{\infty} \cap \mathcal{C}, \mathcal{C}^o := C_c^{\infty} \cap \mathcal{C}^o, \mathcal{C}^o \cap \mathcal{C} \), where \( \mathcal{C}^o \) denotes the algebra of real analytic functions in the sense of [KM] and suitable algebras of functions of finite differentiability like \( \text{Lip}^m \) (see [FK]) or \( C_c^m \).

9. Theorem. Let \( E_i \) be \( \mathcal{A} \)-real-compact spaces that admit large carriers of class \( \mathcal{A} \). Then any closed subspace of the product of the spaces \( E_i \) and, in particular, every projective limit of these spaces, has the same properties.

Proof. First we show that this is true for the product \( E \). We use Lemma 8 with \( \mathcal{A}(E) \) for \( \mathcal{A} \) and the vector space generated by \( \bigcup_i \{ f \circ \text{pr}_i : f \in \mathcal{A}(E_i) \} \) for \( \mathcal{A}_0 \), where \( \text{pr}_i : E = \prod E_i \to E_j \) denotes the canonical projection. Let the finite sum \( f = \sum_i f_i \circ \text{pr}_i \) be an element of \( \mathcal{A}_0 \). Since \( \alpha \circ \text{pr}_i{}^* : \mathcal{A}(E_i) \to \mathcal{A}(E) \to \mathbb{R} \) is an algebra homomorphism, there exists a point \( a_i \in E_i \) such that \( \alpha(f_i \circ \text{pr}_i) = \alpha(\alpha \circ \text{pr}_i{}^*)(f_i) = f_i(a_i) \). Let \( a \) be the point in \( E \) with coordinates \( a_i \). Then

\[ \alpha(f) = \alpha\left( \sum_i f_i \circ \text{pr}_i \right) = \sum_i \alpha(f_i \circ \text{pr}_i) = \sum_i f_i(a_i) = \sum_i (f_i \circ \text{pr}_i)(a) = f(a). \]

Now let \( U \) be a neighborhood of \( a \) in \( E \). Since we consider the product topology on \( E \), we may assume that \( a \in \prod U_i \subset U \), where \( U_i \) are neighborhoods of \( a_i \) in \( E_i \) and are equal to \( E_i \) except for \( i \) in some finite subset \( F \) of the index set. Now choose \( f_i \in \mathcal{A}(E_i) \) with \( f_i(a_i) = 0 \) and \( f_i(x) \neq 0 \) for all \( x \notin U_i \). Consider \( f = \sum_{i \in F}(f_i \circ \text{pr}_i)^2 \in \mathcal{A}_0 \). Then \( f(a) = \sum_{i \in F} f_i(a_i)^2 = 0 \). Furthermore \( x \notin U \) implies that \( x_i \notin U_i \) for some \( i \), which turns out to be in \( F \), and hence \( f(x) \geq f_i(x_i)^2 > 0 \). So we may apply Lemma 8 to conclude that \( \alpha(f) = f(a) \) for all \( f \in \mathcal{A}(E) \).

Now we prove the result for a closed subspace \( F \subset E \). Again we want to apply Lemma 8, this time with \( \mathcal{A}(F) \) for \( \mathcal{A} \) and \( \{ f|_F : f \in \mathcal{A}(E) \} \) for \( \mathcal{A}_0 \). Since \( \alpha \circ \text{incl}^* : \mathcal{A}(E) \to \mathcal{A}(F) \to \mathbb{R} \) is an algebra homomorphism there exists an \( a \in E \) with \( \alpha(f|_F) = f(a) \) for all \( f \in \mathcal{A}(E) \) with \( a \in F \). Now let \( U \) be a neighborhood of \( a \) in \( E \); then there exists an \( f_U \in \mathcal{A}(E) \) with \( f_U(a) = 0 \) and \( f_U(x) \neq 0 \) for all \( x \notin U \). By Lemma 1 there exists a point \( a_U \in F \) such that \( f_U(a_U) = \alpha(f_U|_F) = f_U(a) = 0 \). Hence \( a_U \) is in \( U \), and thus is a net in \( F \) that converges to \( a \). In particular, \( a \in F \) since \( F \) is closed in \( E \). If \( V \) is a neighborhood of \( a \) in \( F \) then there exists a neighborhood \( U \) of \( a \) in \( E \) with \( U \cap F \subset V \) and hence an \( f \in \mathcal{A}_0 \) with \( f(a) = 0 \) and \( f(x) \neq 0 \) for all \( x \notin U \). So again Lemma 8 applies. \( \square \)

10. Remark. Theorem 9 shows that a closed subspace of a product of certain \( \mathcal{A} \)-real-compact spaces is again \( \mathcal{A} \)-real-compact. Of course the natural question arises of whether the result remains true for arbitrary \( \mathcal{A} \)-real-compact spaces.

The question is even open—whether the product of two \( \mathcal{A} \)-real-compact spaces is \( \mathcal{A} \)-real-compact, or whether a closed subspace of an \( \mathcal{A} \)-real-compact space is \( \mathcal{A} \)-real-compact.
space is \( \mathcal{A} \)-real-compact, or whether a projective limit of a projective system of \( \mathcal{A} \)-real-compact spaces is \( \mathcal{A} \)-real-compact.

11. **Corollary.** Let \( E \) be a separable Fréchet space (e.g., a Fréchet-Montel space); then every algebra homomorphism on \( C^\infty(E, \mathbb{R}) \) or on \( C^\infty_c(E, \mathbb{R}) \) is a point evaluation. The same is true for any product of separable Fréchet spaces.

**Proof.** Any Fréchet space has a countable Basis \( \mathcal{U} \) of absolutely convex 0-neighborhoods, and since it is complete, it is a closed subspace of the product \( \prod_{u \in \mathcal{U}} E(u) \). The \( E(u) \) are the normed spaces formed by \( E \) modulo the kernel of the Minkowski functional generated by \( U \). As quotients of \( E \) the spaces \( E(u) \) are separable if \( E \) is such. So the completion \( \widehat{E(u)} \) is a separable Banach space, and hence by [AdR, Theorem 8] \( \widehat{E(u)} \) is \( C^\infty_c \)-real-compact and admits large \( C^\infty_c \)-carriers. By Theorem 9 the same is true for the given Fréchet space. So the result is true for \( C^\infty_c(E, \mathbb{R}) \). Since \( E \) is metrizable this algebra coincides with \( C^\infty(E, \mathbb{R}) \), see [K, 82].

Now for a product \( E \) of metrizable spaces the two algebras \( C^\infty(E, \mathbb{R}) \) and \( C^\infty_c(E, \mathbb{R}) \) again coincide. This can be seen as follows. For every countable subset \( A \) of the index set, the corresponding product is separable and metrizable, hence \( C^\infty \)-real-compact. Thus there exists a point \( x_A \) in this countable product such that \( \alpha(f) = f(x_A) \) for all \( f \) that factor over the projection to that countable subproduct. Since for \( A_1 \subset A_2 \) the projection of \( x_{A_1} \) to the product over \( A_1 \) is just \( x_{A_1} \) (use the coordinate projections composed with functions on the factors for \( f \)), there is a point \( x \) in the product whose projection to the subproduct with index set \( A \) is just \( x_A \). Every Mackey continuous function and, in particular, every \( C^\infty \)-function, depends only on countably many coordinates, thus factors over the projection to some subproduct with countable index set \( A \), hence \( \alpha(f) = f(x_A) = f(x) \). This can be shown by the same proof as for a product of factors \( \mathbb{R} \) in [FK, Theorem 6.2.9] since the result of [M, 1952] is valid for a product of separable metrizable spaces. □

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