Research Article
New Properties on Degenerate Bell Polynomials

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The aim of this paper is to study the degenerate Bell numbers and polynomials which are degenerate versions of the Bell numbers and polynomials. We derive some new identities and properties of those numbers and polynomials that are associated with the degenerate Stirling numbers of both kinds.

1. Introduction

The Bell number Bel_n counts the number of partitions of a set with n elements into disjoint nonempty subsets. The Bell polynomials Bel_n(x), also called Touchard or exponential polynomials, are natural extensions of Bell numbers. The partial and complete Bell polynomials, which are multivariate generalizations of the Bell polynomials, have diverse applications not only in mathematics but also in physics and engineering as well (see [1]).

For instance, the following formula, due to Faà di Bruno formula:

\[
\frac{d^n}{dt^n} f \circ g(t) = \sum_{k=0}^{n} \binom{n}{k} \left( g'(t) \right)^k B_{n,k} \left( g(t) \right)
\]

(1)

gives an explicit formula for higher derivatives of composite functions. Here, the partial Bell polynomials B_{n,k} (x_1, x_2, \ldots, x_{n-k+1}) are defined by

\[
B_{n,k} (x_1, x_2, \ldots, x_{n-k+1}) = \frac{n!}{\prod_{i=1}^{n-k+1} i! \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)} \quad \text{for} \quad (n \geq k \geq 0),
\]

(2)

where the sum runs over all nonnegative integers i_1, i_2, \ldots, i_{n-k+1}, satisfying i_1 + i_2 + \cdots + i_{n-k+1} = k and i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n (see [1], p. 133). Then, the complete Bell polynomials are given by B_n (x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k} (x_1, x_2, \ldots, x_{n-k+1}) x^k, \quad (n \geq 1), \quad \text{and} \quad B_n (x|1, 1, \ldots, 1) = \sum_{k=1}^{n} B_{n,k} (1, 1, \ldots, 1) x^k = \text{Bel}_n(x), \quad (n \geq 1).

As a degenerate version of those Bell polynomials and numbers, the degenerate Bell polynomials Bel_{n,k} (x) and numbers Bel_{n,1} (see (17)) are introduced and studied under the different names of the partially degenerate Bell polynomials and numbers in [2]. Some interesting identities for them were obtained in connection with Stirling numbers of the first and second kinds [2]. We hope that we will be able to find many interesting applications of these polynomials and numbers in near future.
In [3], Carlitz initiated the exploration of degenerate Bernoulli and Euler polynomials, which are degenerate versions of the ordinary Bernoulli and Euler polynomials. Along the same line as Carlitz’s pioneering work, intensive studies have been done for degenerate versions of quite a few special polynomials and numbers (see [2–10] and the references therein). It is worthwhile to mention that these studies of degenerate versions have been done not only for some special numbers and polynomials but also for transcendental functions like gamma functions (see [8]). The studies have been carried out by various means like combinatorial methods, generating functions, differential equations, umbral calculus techniques, $p$-adic analysis, and probability theory.

The aim of this paper is to further investigate the degenerate Bell polynomials and numbers by means of generating functions. In more detail, we derive several properties and identities of those numbers and polynomials which include recurrence relations for degenerate Bell polynomials (see Theorems 1, 3, 4, and 8), and expressions for them that can be derived from repeated applications of certain operators to the exponential functions (see Theorem 2, Proposition 1), the derivatives of them (Corollary 1), the antiderivatives of them (see Theorem 6), and some identities involving them (see Theorems 5, 9). For the rest of this section, we recall some necessary facts that are needed throughout this paper.

For any $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$e^x_\lambda(t) = \sum_{k=0}^{\infty} \frac{(x)_k \lambda^k}{k!},$$

(3)

(see [9]), where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda) \cdots \big(x-(n-1)\lambda\big), \quad (n \geq 1).$$

(4)

When $x = 1$, we see use the notation $e_t(\lambda) = e^1_\lambda(t)$.

In [3], Carlitz introduced the degenerate Bernoulli numbers given by

$$\frac{t}{e^t(\lambda) - 1} = \sum_{n=0}^{\infty} B_{n,\lambda} \frac{t^n}{n!}.$$  

(5)

Note that $\lim_{\lambda \to 0} B_{n,\lambda} = B_n$, where $B_n$ are the ordinary Bernoulli numbers given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$  

(6)

(see [1–14]).

From (5), we deduce that

$$\beta_{0,\lambda} = 1, \quad \beta_{1,\lambda} = \frac{1}{2} - \frac{1}{2} \beta_{2,\lambda} = \frac{1}{6} \lambda^2 - \frac{1}{6},$$

$$\beta_{3,\lambda} = \frac{1}{4} \lambda^3 - \frac{1}{4} \lambda, \quad \beta_{4,\lambda} = \frac{19}{30} \lambda^4 + \frac{2}{3} \lambda^2 - \frac{1}{30},$$

$$\beta_{5,\lambda} = \frac{9}{4} \lambda^5 - \frac{5}{2} \lambda^3 + \frac{1}{4} \lambda, \quad \beta_{6,\lambda} = \frac{863}{84} \lambda^6 + 12 \lambda^4 - \frac{7}{4} \lambda^2 + \frac{1}{42},$$

$$\beta_{7,\lambda} = \frac{1375}{24} \lambda^7 - 70 \lambda^5 + \frac{105}{8} \lambda^3 - \frac{5}{12} \lambda,$$

$$\beta_{8,\lambda} = \frac{33953}{90} \lambda^8 + 480 \lambda^6 - \frac{1624}{15} \lambda^4 + \frac{50}{9} \lambda^2 - \frac{1}{30}.$$  

(8)

from which we compute the first few values of $\beta_{n,\lambda}$ as follows:

$$\beta_{0,\lambda} = 1, \quad \beta_{1,\lambda} = \frac{1}{2} - \frac{1}{2} \beta_{2,\lambda} = \frac{1}{6} \lambda^2 + \frac{1}{6}, \quad \beta_{3,\lambda} = \frac{1}{4} \lambda^3 - \frac{1}{4} \lambda, \quad \beta_{4,\lambda} = \frac{19}{30} \lambda^4 + \frac{2}{3} \lambda^2 - \frac{1}{30},$$

$$\beta_{5,\lambda} = \frac{9}{4} \lambda^5 - \frac{5}{2} \lambda^3 + \frac{1}{4} \lambda, \quad \beta_{6,\lambda} = \frac{863}{84} \lambda^6 + 12 \lambda^4 - \frac{7}{4} \lambda^2 + \frac{1}{42},$$

$$\beta_{7,\lambda} = \frac{1375}{24} \lambda^7 - 70 \lambda^5 + \frac{105}{8} \lambda^3 - \frac{5}{12} \lambda,$$

$$\beta_{8,\lambda} = \frac{33953}{90} \lambda^8 + 480 \lambda^6 - \frac{1624}{15} \lambda^4 + \frac{50}{9} \lambda^2 - \frac{1}{30}.$$  

(8)

It is well known that the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n, k) x^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{x^n}{n!}$$

(9)

(see [14]), where $(x)_0 = 1$, $(x)_n = x(x-1) \cdots (x-n+1)$, $(n \geq 1)$.

As the inversion formula of (9), the Stirling numbers of the second kind are given by

$$x^n = \sum_{n=k}^{\infty} S_2(n, k) x^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{x^n}{n!}$$

(10)

(see [14]).

The degenerate Stirling numbers of the first kind are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n, k) (x)_{k,\lambda} = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{x^n}{n!}$$

(11)

(see [5]), and the degenerate Stirling numbers of the second kind are given by

$$(x)_{n,\lambda} = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) (x)_{k,\lambda} = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{x^n}{n!}$$

(12)

(see [5, 7]).

Here, $\log_t (1 + t)$ is the degenerate logarithm given by (18).

We also recall the degenerate absolute Stirling numbers of the first kind that are defined by

$$\langle x \rangle_n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_\lambda \langle x \rangle_{k,\lambda}$$

(13)

(see [10]), where
\[ \langle x \rangle_0 = 1, \quad \langle x \rangle_n = x(x+1) \cdots (x+n-1), \quad (n \geq 1), \]
\[ \langle x \rangle_{0,l} = 1, \quad \langle x \rangle_{n,l} = x(x+\lambda)(x+2\lambda) \cdots (x+(n-1)\lambda), \quad (n \geq 1). \]  

(14)

It is well known that the Bell polynomials are defined by

\[ e^x(e^{t-1}) = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!} \]  

(15)

(see [12, 13]).

When \( x = 1 \), \( \text{Bel}_n = \text{Bel}_n(1) \) are called the Bell numbers.

From (12), we note that

\[ \text{Bel}_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k, \]  

(16)

(see [12, 13]).

In [2], the degenerate Bell polynomials are defined by

\[ e^{x(t\lambda)} = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!} \]  

(17)

Note that \( \lim_{\lambda \longrightarrow 0} \text{Bel}_{n,\lambda}(x) = \text{Bel}_n(x). \) For \( x = 1 \), \( \text{Bel}_{n,\lambda} = \text{Bel}_{n,\lambda}(1) \) are called the degenerate Bell numbers.

\[ \begin{align*}
\text{Bel}_{0,\lambda}(x) &= 1, \quad \text{Bel}_{1,\lambda}(x) = x, \quad \text{Bel}_{2,\lambda}(x) = (x)\lambda + (x^2 + x), \\
\text{Bel}_{3,\lambda}(x) &= (2x)\lambda^2 + (-3x^2 + 3x)\lambda + (x^3 + 3x^2 + x), \\
\text{Bel}_{4,\lambda}(x) &= (-6x)\lambda^3 + (11x^2 + 12x)\lambda^2 + (-6x^3 - 18x^2 - 7x)\lambda + (x^4 + 6x^3 + 7x^2 + x), \\
\text{Bel}_{5,\lambda}(x) &= (24x)\lambda^4 + (-50x^2 - 60x)\lambda^3 + (35x^3 + 110x^2 + 50x)\lambda^2 + (-10x^4 - 60x^3 - 75x^2 - 15x)\lambda \\
&\quad + (x^5 + 10x^4 + 25x^3 + 15x^2 + x), \\
\text{Bel}_{6,\lambda}(x) &= (-120x)\lambda^5 + (274x^2 + 274x)\lambda^4 + (-225x^3 - 675x^2 - 225x)\lambda^3 \\
&\quad + (85x^4 + 510x^3 + 595x^2 + 85x)\lambda^2 + (-15x^5 - 150x^4 - 375x^3 - 225x^2 - 15x)\lambda \\
&\quad + (x^6 + 15x^5 + 65x^4 + 90x^3 + 31x^2 + x), \\
\text{Bel}_{7,\lambda}(x) &= (720x)\lambda^6 + (-1764x^2 - 1764x)\lambda^5 + (1624x^3 + 4872x^2 + 1624x)\lambda^4 \\
&\quad + (-735x^4 - 4410x^3 - 5145x^2 - 735x)\lambda^3 + (175x^5 + 1750x^4 + 4375x^3 + 2625x^2 + 175x)\lambda^2 \\
&\quad + (21x^6 - 315x^5 - 1365x^4 - 1890x^3 - 651x^2 - 21x)\lambda + (x^7 + 21x^6 + 140x^5 + 350x^4 + 301x^3 + 63x^2 + x), \\
\text{Bel}_{8,\lambda}(x) &= (-5040x)\lambda^7 + (13068x^2 + 13068x)\lambda^6 + (-13132x^3 - 39396x^2 - 13132x)\lambda^5 \\
&\quad + (6769x^4 + 40614x^3 + 47383x^2 + 6769x)\lambda^4 + (-1960x^5 - 19600x^4 - 49000x^3 - 29400x^2 - 1960x)\lambda^3 \\
&\quad + (332x^6 + 4830x^5 + 20930x^4 + 28980x^3 + 9982x^2 + 322x)\lambda^2 \\
&\quad + (-28x^7 - 588x^6 - 3920x^5 - 9800x^4 - 8428x^3 - 1764x^2 - 28x)\lambda \\
&\quad + (x^8 + 28x^7 + 266x^6 + 1050x^5 + 1701x^4 + 966x^3 + 127x^2 + x). 
\end{align*} \]  

(22)

The compositional inverse of \( e_1(t) \) is given by \( \log_3(1 + t) \), namely, \( e_1(\log_3(1 + t)) = t = \log_3(1 + t) \), where

\[ \log_3(1 + t) = \frac{1}{\lambda} \left( (1 + t)^\lambda - 1 \right) = \sum_{n=1}^{\infty} \lambda^{n-1} \left( 1 - \frac{t^n}{n!} \right), \]  

(18)

(see [5]).

Note that \( \lim_{\lambda \longrightarrow 0} \log_3(1 + t) = \log(1 + t) \).

From (17), we note that

\[ \text{Bel}_{n,\lambda}(x) = e^{-x} \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} x^k = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k, \]  

(19)

(see [2]).

From (12), we can deduce the recurrence relation given by

\[ S_{2,\lambda}(n+1,k) = S_{2,\lambda}(n,k-1) + (k - n\lambda) S_{2,\lambda}(n,k), \]  

(20)

and the values

\[ S_{2,\lambda}(n,0) = 0, \quad (n \geq 1), \quad S_{2,\lambda}(n,n) = 1, \quad (n \geq 0). \]  

(21)

Now, we compute from (19), (3), and (21) the first few degenerate Bell polynomials as follows:
In Figure 1, we plot the shapes of Bell polynomials $\text{Bel}_k(x)$, $\lambda$. The upper-left graph looks different from the others. However, of course they all go to infinity as $x$ tends to infinity.

2. Some New Properties on Degenerate Bell Polynomials

Let $a$ be a nonzero constant. First, we observe that

\[
\frac{d^n}{dt^n} e^{a t} e^{e^{x} t} = \frac{d^n}{dt^n} \sum_{k=0}^{\infty} \frac{a^k}{k!} e^k (t) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (k)_{n,k} e^{k-n-1} (t)
\]

\[
= \frac{1}{(1+\lambda t)^n} e^{ae^{t} (t) - n \lambda}
\]

Therefore, by (23), we obtain the following lemma.

**Lemma 1.** For $n \geq 0$, the $n$th derivative of $e^{a t} e^{e^{x} t}$ is given by

\[
\frac{d^n}{dt^n} e^{a t} e^{e^{x} t} = \frac{1}{(1+\lambda t)^n} e^{ae^{t} (t) - n \lambda}.
\]  (24)

Let $x = e^{t}$ in (23). Then, we have

\[
\frac{d}{dt} e^{x} = \frac{d}{dx} e^{x} = \frac{1}{1+\lambda t} \frac{d}{dx} e^{x}, \quad n \geq 0.
\]  (25)

By Lemma 1 and (25), we get

\[
\left(x^{1-n} \frac{d}{dx}\right)^n e^{x} = x^{-n} \sum_{k=0}^{\infty} (k)_{n,k} \frac{k!}{k!} e^{ae^{t} (t) - n \lambda}, \quad n \geq 0.
\]  (26)

Let

\[
S_{n,k} = \sum_{k=0}^{\infty} (k)_{n,k} \frac{k!}{k!} e^{ae^{t} (t) - n \lambda}, \quad n = 0, 1, 2, \ldots
\]  (27)

Then, we note from (19) that we have

\[
e^{\text{Bel}_{n,k} (a x)} = S_{n,k}.
\]  (28)

The generating function of $S_{n,k}$ is given by

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\[ e^{c_\lambda(t)} = \sum_{n=0}^{\infty} S_{n,\lambda} \frac{t^n}{n!} \]  \hspace{1cm} (29)

Indeed, this can be seen from the following:

\[ \sum_{n=0}^{\infty} S_{n,\lambda} \frac{t^n}{n!} = e^{\sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(1) \frac{t^n}{n!}} = e^{e^{c_\lambda(t)} - 1} = e^{c_\lambda(t)}. \]  \hspace{1cm} (30)

Taking the derivative with respect to \( t \) on both sides of (30), we have

\[ \sum_{n=0}^{\infty} \frac{n}{n!} S_{n,\lambda} \frac{t^n}{m!} = \sum_{n=0}^{\infty} \lambda \_t \sum_{m=0}^{\infty} S_{m,\lambda} \frac{t^m}{m!} \]  \hspace{1cm} (31)

\[ = \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \left( \begin{array}{c} n \\ m \end{array} \right) \lambda \_t \sum_{m=0}^{\infty} S_{m,\lambda} \frac{t^m}{m!} \]  \hspace{1cm} = \sum_{m=0}^{\infty} \left( \begin{array}{c} n \\ m \end{array} \right) \lambda \_t S_{m,\lambda} \frac{t^m}{m!} \]  \hspace{1cm} = \sum_{m=0}^{\infty} \left( \begin{array}{c} n \\ m \end{array} \right) \lambda \_t S_{m,\lambda} \frac{t^m}{m!}. \]  \hspace{1cm} (32)

Thus, by comparing the coefficients on both sides of (31) and from (28), we obtain the following theorem.

**Theorem 1.** For \( n \geq 0 \), the following recurrence relation holds:

\[ \text{Bel}_{n+1,\lambda} = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \sum_{n=m+1,\lambda}. \]  \hspace{1cm} (33)

Assume that the following identity holds:

\[ \left( x^{1-\lambda} \frac{d}{dx} \right)^n e^x = \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} x^{k-n}. \]  \hspace{1cm} (34)

Then, we have

\[ \left( x^{1-\lambda} \frac{d}{dx} \right)^{n+1} e^x = x^{1-\lambda} \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} x^{k-n}. \]  \hspace{1cm} (35)

\[ = x^{1-\lambda} \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} (k-n) x^{k-n-1}. \]  \hspace{1cm} (36)

\[ = \sum_{k=0}^{\infty} \frac{(k)_{n+1,\lambda}}{k!} x^{k-(n+1)} \]  \hspace{1cm} (37)

From the first equality in (35) and (27), we see that we have

\[ S_{n,\lambda} = \left( x^{1-\lambda} \frac{d}{dx} \right)^n e^x \bigg|_{x=1}. \]  \hspace{1cm} (38)

Clearly, \( S_{0,\lambda} = S_{1,\lambda} = e \). We can check that

\[ \left( x^{1-\lambda} \frac{d}{dx} \right)^2 e^x = (1 - \lambda)x^{1-2\lambda} e^x + x^{2-2\lambda} e^x, \]  \hspace{1cm} (39)

\[ x^{1-\lambda} \frac{d}{dx} \right)^3 e^x = x^{1-3\lambda} e^x (1 - \lambda)x_{l_2,\lambda} + (1 - \lambda)x_{l_3,\lambda}. \]  \hspace{1cm} (40)

Therefore, by (40) and Theorem 2, we obtain the following theorem.

**Theorem 2.** For \( n \geq 0 \), the following relations hold true:

\[ \left( x^{1-\lambda} \frac{d}{dx} \right)^n e^x = \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} x^{k-n} = x^{-n} \text{Bel}_{n,\lambda}(x) e^x. \]  \hspace{1cm} (41)

From (17), we note that
\[
\sum_{n=0}^{\infty} \frac{d}{dx} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!} = \frac{d}{dx} e^x (e_\lambda(t) - 1) e^x (e_\lambda(t) - 1)
\]

\[
= \left( \sum_{m=0}^{\infty} \frac{(1)_m}{m!} \right) \frac{d}{dx} \sum_{m=0}^{\infty} \text{Bel}_{m,\lambda}(x) \frac{t^n}{m!}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{n}{m} \left[ \text{Bel}_{m,\lambda}(x) \left( \lambda^n - \lambda^{n-m} \right) \right] \frac{t^n}{n!}
\]

(42)

Thus, by comparing the coefficients on both sides of (42), we get

\[
\frac{d}{dx} \text{Bel}_{n,\lambda}(x) = \text{Bel}'_{n,\lambda}(x) = \sum_{m=0}^{n-1} \binom{n}{m} \text{Bel}_{m,\lambda}(x) (1 - \lambda)^{n-m,\lambda},
\]

(43)

Taking the derivative with respect to \( t \) on both sides of (17), we have

\[
\frac{d}{dt} e^x (e_\lambda(t) - 1) = \sum_{n=0}^{\infty} \frac{d}{dx} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!}
\]

(44)

On the other hand,

\[
\frac{d}{dt} e^x (e_\lambda(t) - 1) = x e_\lambda(t) (e_\lambda(t) - 1)
\]

\[
= x \sum_{m=0}^{\infty} \frac{(1 - \lambda)_m}{m!} \sum_{n=0}^{\infty} \text{Bel}_{m,\lambda}(x) \frac{t^n}{m!}
\]

\[
= x \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \text{Bel}_{m,\lambda}(x) (1 - \lambda)^{n-m,\lambda} \frac{t^n}{n!}
\]

(45)

Therefore, by (44) and (45), we obtain the following theorem.

**Theorem 4.** For \( n \geq 0 \), the following recurrence relation is valid:

\[
\text{Bel}_{n+1,\lambda}(x) = x \sum_{m=0}^{n} \binom{n}{m} \text{Bel}_{m,\lambda}(x) (1 - \lambda)^{n-m,\lambda}.
\]

(46)

**Remark 1.** Theorems 3 and 4 and (43) give us the following:

\[
\text{Bel}_{n+1,\lambda}(x) = x \sum_{m=0}^{n} \binom{n}{m} \text{Bel}_{m,\lambda}(x) (1 - \lambda)^{n-m,\lambda} - n\lambda \text{Bel}_{n,\lambda}(x)
\]

(47)

This implies that the following identity must hold true:

\[
n \text{Bel}_{n,\lambda}(x) = x \sum_{m=0}^{n} \binom{n}{m} (n-m) \text{Bel}_{m,\lambda}(x) (1 - \lambda)^{n-m,\lambda},
\]

the validity of which follows from Theorem 4.

From Theorem 3, we note that

\[
x \text{Bel}'_{n,\lambda}(x) = x \frac{d}{dx} \text{Bel}_{n,\lambda}(x) = \text{Bel}_{n+1,\lambda}(x) - x \text{Bel}_{n,\lambda}(x)
\]

\[
+ n\lambda \text{Bel}_{n,\lambda}(x)
\]

\[
= \text{Bel}_{n+1,\lambda}(x) - (x - n\lambda) \text{Bel}_{n,\lambda}(x)
\]

(49)

Therefore, by (49), we obtain the following corollary.

**Corollary 1.** For \( n \geq 1 \), we have the following identity:

\[
x \frac{d}{dx} \text{Bel}_{n,\lambda}(x) = x \sum_{m=0}^{n-1} \binom{n}{m} \text{Bel}_{m,\lambda}(x) (1 - \lambda)^{n-m,\lambda} + n\lambda \text{Bel}_{n,\lambda}(x).
\]

(50)

We observe that
\[ x^{-\lambda} \frac{d}{dx} \left( x^{-n} \text{Bel}_{n,\lambda} (x)e^x \right) = x^{-\lambda} \frac{d}{dx} \left( x^{-n} \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} x^k \right) \]
\[ = \sum_{k=0}^{\infty} \frac{(k)_{n+1,\lambda}}{k!} x^{k-(n+1)\lambda} \]
\[ = x^{-(n+1)\lambda} \left( \sum_{k=0}^{\infty} \frac{(k)_{n+1,\lambda}}{k!} x^k \right) e^x \]
\[ = x^{-(n+1)\lambda} \text{Bel}_{n+1,\lambda} (x)e^x, \quad (n \geq 0). \quad (51) \]

Thus, by (51), we get
\[ x^{-\lambda} \frac{d}{dx} \left( x^{-n} \text{Bel}_{n,\lambda} (x)e^x \right) = x^{-(n+1)\lambda} \text{Bel}_{n+1,\lambda} (x)e^x, \quad (n \geq 0). \quad (52) \]

From (17), we have
\[ \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda} (x+y) \frac{t^n}{n!} = e^{(x+y)(\epsilon_1(t)-1)} = e^{\epsilon_1(t)-1} \cdot e^y (\epsilon_1(t)-1) \]
\[ = \sum_{l=0}^{\infty} \text{Bel}_{l,\lambda} (x) \frac{t^l}{l!} \sum_{m=0}^{\infty} \text{Bel}_{m,\lambda} (x) \frac{t^m}{m!} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} \text{Bel}_{l,\lambda} (x) \text{Bel}_{n-l,\lambda} (y) \right) \frac{t^n}{n!}. \quad (53) \]

Therefore, by comparing the coefficients on both sides of (53), we obtain the following theorem.

**Theorem 5.** For \( n \geq 0 \), the following binomial identity holds:
\[ \text{Bel}_{n,\lambda} (x+y) = \sum_{l=0}^{n} \binom{n}{l} \text{Bel}_{l,\lambda} (x) \text{Bel}_{n-l,\lambda} (y). \quad (54) \]

From (17), we note that
\[ \sum_{n=0}^{\infty} \int_{0}^{x} \text{Bel}_{n,\lambda} (y) dy \frac{t^n}{n!} = \int_{0}^{x} e^{\epsilon_1(t)-1} dy \quad (55) \]

On the other hand, we also have
\[ \int_{0}^{x} e^{\epsilon_1(t)-1} dt = \frac{1}{\epsilon_1(t) - 1} \left[ e^{\epsilon_1(t)-1} \right]_{0}^{x} \]
\[ = \frac{1}{\epsilon_1(t) - 1} \left( e^{\epsilon_1(t)-1} - 1 \right) = \frac{1}{\epsilon_1(t) - 1} \sum_{k=1}^{\infty} \text{Bel}_{k,\lambda} (x) \frac{t^k}{k!} \]
\[ = t \sum_{k=0}^{\infty} \frac{\text{Bel}_{k+1,\lambda} (x)}{k + 1} \frac{t^k}{k!} = \sum_{l=0}^{\infty} \beta_{l,\lambda} \int_{0}^{x} \text{Bel}_{k+1,\lambda} (y) \frac{t^k}{k!} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{n}{k} \text{Bel}_{k+1,\lambda} (x) \beta_{n-k,\lambda} \right) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \sum_{k=0}^{n} \frac{n+1}{k+1} \text{Bel}_{k+1,\lambda} (x) \beta_{n-k,\lambda} \right) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \sum_{k=1}^{n+1} \text{Bel}_{k+1,\lambda} (x) \beta_{n-k,\lambda} \right) \frac{t^n}{n!}. \quad (56) \]
Therefore, by (55) and (56), we obtain the following theorem.

**Theorem 6.** For \( n \geq 0 \), the antiderivative of \( \text{Bel}_{n, \lambda}(x) \) is given by

\[
\int_0^x \text{Bel}_{n, \lambda}(x) \, dx = \frac{1}{n + 1} \sum_{k=0}^{n+1} \binom{n+1}{k} \beta_{n+1-k, \lambda} \text{Bel}_{k, \lambda}(x),
\]

(57)

where \( \beta_{n, \lambda} \) are Carlitz’s degenerate Bernoulli numbers given by\( (t/(e^{t} - 1)) = \sum_{n=0}^{\infty} \beta_{n, \lambda} \left( t^n/n! \right) \).

For \( k \geq 0 \), by (12), we get

\[
\sum_{n=k}^{\infty} S_{2, \lambda}(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( e^{t} - 1 \right)^{k} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left( \frac{t^n}{n!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left( j \right)_{n, \lambda} \right) \frac{t^n}{n!}
\]

(58)

By comparing the coefficients on both sides of (58), we have

\[
\frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left( j \right)_{n, \lambda} = \begin{cases} S_{2, \lambda}(n, k), & \text{if } n \geq k, \\ 0, & \text{if } 0 \leq n \leq k - 1. \end{cases}
\]

(59)

Let \( D = (d/dx) \), and let \( y = x^p \). As \( x^{1-\lambda}D = p y^{1-(\lambda/p)}(d/dy) \), we have

\[
(x^{1-\lambda}D)^n e^{ax^p} = \left( p y^{1-(\lambda/p)} \frac{d}{dy} \right)^n e^{ap} = p^n y^{-(nk/p)} \text{Bel}_{n, \lambda/p}(ay)e^{ap} = p^n x^{-nk} \text{Bel}_{n, \lambda/p}(ax^p)e^{ax^p}.
\]

Thus, we have

\[
(x^{1-\lambda}D)^n e^{ax^p} = p^n x^{-nk} \text{Bel}_{n, \lambda/p}(ax^p)e^{ax^p}, \quad (n \geq 0).
\]

(60)

Therefore, by (60), we obtain the following proposition.

**Proposition 1.** For \( n \geq 0 \), we have the following operational formula:

\[
x^{nk} (x^{1-\lambda}D)^n e^{ax^p} = p^n \text{Bel}_{n, \lambda/p}(ax^p)e^{ax^p},
\]

(62)

where \( D = (d/dx) \).

From (12), we note that

\[
\sum_{k=0}^{n+1} S_{2, \lambda}(n+1, k) (x)_k = (x)_{n+1, \lambda} = (x - n\lambda) (x)_{n, \lambda}
\]

(63)

\[
= (x - n\lambda) \sum_{k=0}^{n} S_{2, \lambda}(n, k) (x)_k = \sum_{k=0}^{n} S_{2, \lambda}(n, k) (x - k + n\lambda) (x)_k
\]

\[
= \sum_{k=0}^{n} S_{2, \lambda}(n, k) (x)_k + \sum_{k=0}^{n} S_{2, \lambda}(n, k) (k - n\lambda) (x)_k
\]

\[
= \sum_{k=0}^{n+1} S_{2, \lambda}(n, k-1) (x)_k + \sum_{k=0}^{n} S_{2, \lambda}(n, k) (k - n\lambda) (x)_k
\]

\[
= \sum_{k=0}^{n+1} \left( S_{2, \lambda}(n, k-1) + S_{2, \lambda}(n, k) (k - n\lambda) \right) (x)_k.
\]
By (63), we get
\[ S_{2,\lambda}(n+1, k) = S_{2,\lambda}(n, k-1) + (k-n\lambda)S_{2,\lambda}(n, k), \tag{64} \]
where \(0 \leq k \leq n+1\).

We prove the next theorem by induction on \(n\).

**Theorem 7.** Assume that \(f\) is an infinitely differentiable function. Then, for \(n \geq 0\), the following operational formula holds:

\[
(x^{\lambda-1}D)^n f(x) = \sum_{k=0}^{n} S_{2,\lambda}(n, k)x^{k-n\lambda}D^k f(x) \tag{65}
\]

where \(D = (d/dx)\).

**Proof.** The statement is obviously true for \(n = 0\). Assume that it is true for \(n\), \((n \geq 0)\).

Let \(f(x) = e^x\). Then, we have
\[
x^n(x^{\lambda-1}D)^n e^x = \left( \sum_{k=0}^{n} S_{2,\lambda}(n, k)x^k \right) e^x = Bel_{n,\lambda}(x)e^x. \tag{67}
\]

Observe that, for any \(a\), we have
\[
(x^{\lambda-1}D)^n x^a = (\alpha)_{n,\lambda} x^{a-n\lambda}. \tag{68}
\]

By the Leibniz rule, we get
\[
(x^{\lambda-1}D)^n(fg) = \sum_{l=0}^{n} \binom{n}{l} \left[ (x^{\lambda-1}D)^{n-l} f \right] \left[ (x^{\lambda-1}D)^l g \right]. \tag{69}
\]

From Theorem 2, we note that
\[
(x^{\lambda-1}D)^n(-m\lambda Bel_{m,\lambda}(x)) = \sum_{j=0}^{m} S_{2,\lambda}(m, j) \left[ (x^{\lambda-1}D)^{n-k} x^{-m\lambda} \right] \]
\[
= \sum_{j=0}^{m} S_{2,\lambda}(m, j)(j-m\lambda)_{n-k,\lambda} x^{-m\lambda-(n-k)\lambda} = \sum_{j=0}^{m} S_{2,\lambda}(m, j)(j-m\lambda)_{n-k,\lambda} x^{-m\lambda-(n-k)\lambda} \tag{72}
\]
\[
= \sum_{j=0}^{m} S_{2,\lambda}(m, j) \frac{(j)_m k\lambda}{(j)_m} x^{-m\lambda-(n-k)\lambda}. \]
By (71) and (72), we get
\[ x^{-(m+n)\lambda} e^x \text{Bel}_{m,n,\lambda}(x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^{-k\lambda} \text{Bel}_{k,\lambda}(x) e^x \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) S_{2,\lambda}(m, j) \frac{(j)_{m+n-k,\lambda}}{(j)_{m,\lambda}} \]
\[ = x^{-(m+n)\lambda} e^x \sum_{k=0}^{n} \sum_{j=0}^{m} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} m \\ j \end{array} \right) S_{2,\lambda}(m, j) \frac{(j)_{m+n-k,\lambda}}{(j)_{m,\lambda}} x^j. \]
(73)

Therefore, by comparing the coefficients on both sides of (73), we obtain the following theorem.

**Theorem 8.** For \( m, n \geq 0 \), we have the following expression:
\[ \text{Bel}_{m,n,\lambda}(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} m \\ j \end{array} \right) S_{2,\lambda}(m, j) \frac{(j)_{m+n-k,\lambda}}{(j)_{m,\lambda}} x^j. \]
(74)

Taking \( x = 1 \) in (74), we have
\[ \text{Bel}_{m,n,\lambda} = \sum_{k=0}^{n} \sum_{j=0}^{m} \left( \begin{array}{c} n \\ k \end{array} \right) S_{2,\lambda}(m, j) \frac{(j)_{m+n-k,\lambda}}{(j)_{m,\lambda}} \]
(75)

From (19) and (68), we note that
\[ (x^{1-\lambda}D)^n \text{Bel}_{m,\lambda}(x) = (x^{1-\lambda}D)^n \sum_{k=0}^{m} S_{2,\lambda}(m, k)x^k \]
\[ = \sum_{k=0}^{m} S_{2,\lambda}(m, k)(k)_{n,\lambda} x^{k-n}. \]
(76)

On the other hand, by Leibniz rule (69) and Theorem 2, we get
\[ (x^{1-\lambda}D)^n \text{Bel}_{m,\lambda}(x) = (x^{1-\lambda}D)^n \sum_{k=0}^{m} S_{2,\lambda}(m, k)x^k \]
\[ = \sum_{k=0}^{m} S_{2,\lambda}(m, k)(k)_{n,\lambda} x^{k-n}. \]

By (68) and (69) and Theorem 2, we easily get
\[ (x^{1-\lambda}D)^{n-k}(x^{m\lambda} e^{-x}) = \sum_{j=0}^{n-k} \left( \begin{array}{c} n-k \\ j \end{array} \right) (x^{1-\lambda}D)^{n-k-j}(x^{m\lambda} e^{-x}) \]
\[ = \sum_{j=0}^{n-k} \left( \begin{array}{c} n-k \\ j \end{array} \right) (m\lambda)_{j,\lambda} x^{m\lambda-j\lambda} e^{-x} \text{Bel}_{n-k-j,\lambda}(-x)x^{-(n-k-j)\lambda} \]
(78)
From (77) and (78), we have

$$
(x^{1-\lambda})^\theta \text{Bel}_{m,\lambda}(x) \\
= \sum_{k=0}^{n} \binom{n}{k} x^{m-\lambda k} \text{Bel}_{m+k,\lambda}(x)e^x \sum_{j=0}^{n-k} \binom{n-k}{j} (m\lambda)_j x^{m+n-k-j} \text{Bel}_{n-k-j,\lambda}(-x)e^{-x} \\
= \sum_{k=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \text{Bel}_{m+k,\lambda}(x) \text{Bel}_{n-k-j,\lambda}(-x) (m\lambda)_j x^{m+n-k-j}.
$$

(79)

Therefore, by (76) and (79), we obtain the following theorem.

**Theorem 9.** For $m, n \geq 0$, the following identity holds true.

$$
\sum_{k=0}^{n} S_{n,k}(m, k)(n)_k x^k \\
= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \text{Bel}_{m+k,\lambda}(x) \text{Bel}_{n-k-j,\lambda}(-x) (m\lambda)_j x^{m+n-k-j}.
$$

(80)

By (11) and (13), we easily get

$$
(-1)^{n-k} S_{n,k}(n, k) = \binom{n}{k}, \quad (0 \leq k \leq n).
$$

(81)

Indeed,

$$
\sum_{n=0}^{\infty} \langle x \rangle_n t^n = \left( \frac{1}{1-t} \right)^x = e^x (\log x (1-t)) \\
= \sum_{k=0}^{\infty} (-x)_{k+1} (\log x (1-t))^k \\
= \sum_{k=0}^{\infty} (-1)^k \langle x \rangle_{k+1} \sum_{n=k}^{\infty} S_{n,k}(n, k) \frac{(-t)^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-1)^{n-k} S_{n,k}(n, k) \langle x \rangle_{k+1} \right) \frac{t^n}{n!}.
$$

(82)

Therefore, by (82), we get

$$
\langle x \rangle_n = \sum_{k=0}^{n} (-1)^{n-k} S_{n,k}(n, k) \langle x \rangle_{k+1}.
$$

(83)

Replacing $t$ by $\log x (1+t)$ in (17), we get

$$
e^x = \sum_{k=0}^{\infty} \text{Bel}_{k,\lambda}(x) \frac{1}{k!} (\log x (1+t))^k \\
= \sum_{k=0}^{\infty} \text{Bel}_{k,\lambda}(x) \sum_{n=k}^{\infty} S_{n,k}(n, k) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \text{Bel}_{k,\lambda}(x) S_{n,k}(n, k) \right) \frac{t^n}{n!}.
$$

(84)

Thus, from (81) and (84), we get

$$
x^n = \sum_{k=0}^{n} \text{Bel}_{k,\lambda}(x) (-1)^{n-k} \binom{n}{k}.
$$

(85)

From (13), we note that

$$
\sum_{k=0}^{n+1} \left[ \binom{n+1}{k} \langle x \rangle_{k+1} = \langle x \rangle_{n+1} = \langle x \rangle_n + \sum_{k=0}^{n} \binom{n}{k} \langle x \rangle_{k+1} \right. \\
= \sum_{k=0}^{n} \sum_{k=0}^{n} \binom{n}{k} \langle x \rangle_{k+1} + \sum_{k=0}^{n} \binom{n}{k} \langle x \rangle_{k+1} \right. \\
= \sum_{k=0}^{n} \binom{n}{k} \langle x \rangle_{k+1} + \sum_{k=0}^{n} \binom{n}{k} \langle x \rangle_{k+1} \\
= \sum_{k=0}^{n} \left[ \binom{n}{k} + (n-k) \langle x \rangle_{k+1} \right.
$$

(86)

Thus, by comparing the coefficients on both sides of (86), we get

$$
\left[ \binom{n+1}{k} \right]_{k+1} = \left[ \binom{n}{k} \right]_{k+1} + (n-k) \left[ \binom{n}{k} \right]_{k+1}, \quad (0 \leq k \leq n+1).
$$

(87)

### 3. Conclusion

Here, we studied by means of generating functions the degenerate Bell polynomials which are degenerate versions of the Bell polynomials. In more detail, we derived recurrence relations for degenerate Bell polynomials (see Theorems 1, 3, 4, and 8), and expressions for them that can be derived from repeated applications of certain operators to the exponential functions (see Theorem 2 and Proposition 1), the derivatives of them (Corollary 1), the antiderivatives of them (see Theorem 6), and some identities involving them (see Theorems 5 and 9).

As one of our future projects, we would like to continue to study degenerate versions of certain special polynomials and numbers and their applications to physics, science, and...
engineering as well as to mathematics. An earlier version of this paper has been presented as preprint in [15].

**Data Availability**

No data were used to support this study.

**Disclosure**

An earlier version of the paper has been presented as preprint in the following link: https://arxiv.org/abs/2108.06260.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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