Injectives in Residuated Algebras

HECTOR FREYTES, BUENOS AIRES *

Abstract

Injectives in several classes of structures associated with logic are characterized. Among the classes considered are residuated lattices, MTL-algebras, IMTL-algebras, BL-algebras, NM-algebras and bounded hoops.

Keywords: Injectives, absolute retracts, residuated structures, hoops, BL-algebras.

Mathematics Subject Classification: 06F05, 08B30, 03G10, 03G25

Introduction

Residuated structures, rooted in the work of Dedekind on the ideal theory of rings, arise in many fields of mathematics, and are particularly common among algebras associated with logical systems. They are structures \langle A, \odot, \rightarrow, \leq \rangle such that \( A \) is a nonempty set, \( \leq \) is a partial order on \( A \) and \( \odot \) and \( \rightarrow \) are binary operations such that the following relation holds for each \( a, b, c \) in \( A \):

\[ a \odot b \leq c \iff a \leq b \rightarrow c. \]

Important examples of residuated structures related to logic are Boolean algebras (corresponding to classical logic), Heyting algebras (corresponding to intuitionism), residuated lattices (corresponding to logics without contraction rule [18]), BL-algebras (corresponding to Hájek’s basic fuzzy logic [14]), MV-algebras (corresponding to Łukasiewicz many-valued logic [8]). All these examples, with the exception of residuated lattices are hoops [4], i. e., they satisfy the equation \( x \odot (x \rightarrow y) = y \odot (y \rightarrow x) \).

*During the preparation of this paper the author was supported by a Fellowship from the FOMEC Program. The author expresses his gratitude to Roberto Cignoli, for his advice during the preparation of this paper, and to the Referee for her/his many suggestions to improve the presentation of this paper.
The aim of this paper is to investigate injectives and absolute retracts in classes of residuated lattices and bounded hoops. In §2 and §3 we also present some results on injectives in more general varieties.

The paper is structured as follows. In §1 we recall some basic definitions and properties. In §2 we show that under some mild hypothesis on a variety $V$ of algebras, the existence of nontrivial injectives is equivalent to the existence of a self-injective maximum simple algebra. In §3 we use ultrapowers to obtain lattice properties of the injectives in varieties of ordered algebras. The results of §2 and §3 are applied in §4, §7 and in §14 to the study of injectives in varieties of residuated lattices, prelinear residuated lattices and bounded hoops, respectively. In the remaining sections we consider injectives in several subvarieties of residuated lattices which appear in the literature. The results obtained are summarized in Table 1.

## 1 Basic Notion

We recall from [1] and [5] some basic notion of injectives and universal algebra. Let $\mathcal{A}$ be a class of algebras. For all algebras $A, B$ in $\mathcal{A}$, $[A, B]_\mathcal{A}$ will denote the set of all homomorphism $g : A \rightarrow B$. An algebra $A$ in $\mathcal{A}$ is injective iff for every monomorphism $f \in [B, C]_\mathcal{A}$ and every $g \in [B, A]_\mathcal{A}$ there exists $h \in [C, A]_\mathcal{A}$ such that $hf = g$; $A$ is self-injective iff every homomorphism from a subalgebra of $A$ into $A$, extends to an endomorphism of $A$. An algebra $B$ is a retract of an algebra $A$ iff there exists $g \in [B, A]_\mathcal{A}$ and $f \in [A, B]_\mathcal{A}$ such that $fg = 1_B$. It is well known that a retract of an injective object is injective. An algebra $B$ is called an absolute retract in $\mathcal{A}$ iff it is a retract of each of its extensions in $\mathcal{A}$. For each algebra $A$, we denote by $Con(A)$, the congruence lattice of $A$, the diagonal congruence is denoted by $\Delta$ and the largest congruence $A^2$ is denoted by $\nabla$. A congruence $\theta_M$ is said to be maximal iff $\theta_M \neq \nabla$ and there is no congruence $\theta$ such that $\theta_M \subset \theta \subset \nabla$. An algebra $I$ is simple iff $Con(I) = \{\Delta, \nabla\}$. A nontrivial algebra $T$ is said to be minimal in $\mathcal{A}$ iff for each nontrivial algebra $A$ in $\mathcal{A}$, there exists a monomorphism $f : T \rightarrow A$. A simple algebra $I_M$ is said to be maximum simple iff for each simple algebra $I$, $I$ can be embedded in $I_M$. A simple algebra is hereditarily simple iff all its subalgebras are simple. An algebra $A$ is semisimple iff it is a subdirect product of simple algebras. An algebra $A$ is rigid iff the identity homomorphism is the only automorphism. An algebra $A$ has the congruence extension property (CEP) iff for each subalgebra $B$ and $\theta \in Con(B)$ there is a $\phi \in Con(A)$
such that $\theta = \phi \cap A^2$. A variety $\mathcal{V}$ satisfies CEP iff every algebra in $\mathcal{V}$ has the CEP. It is clear that if $\mathcal{V}$ satisfies CEP then every simple algebra is hereditarily simple.

## 2 Injectives and simple algebras

**Definition 2.1** Let $\mathcal{V}$ be a variety. Two constant terms $0, 1$ of the language of $\mathcal{V}$ are called *distinguished constants* iff $A \models 0 \neq 1$ for each nontrivial algebra $A$ in $\mathcal{V}$.

**Lemma 2.2** Let $\mathcal{A}$ be a variety with distinguished constants $0, 1$ and let $A$ be a nontrivial algebra in $\mathcal{A}$. Then $A$ has maximal congruences, and for each simple algebra $I \in \mathcal{A}$, all homomorphisms $f : I \to A$ are monomorphisms.

**Proof:** Since for each homomorphism $f : A \to B$ such that $B$ is a nontrivial algebra, $f(0) \neq f(1)$ then for each $\theta \in \text{Con}(A) \setminus \{A^2\}$, $(1, 0) \notin \theta$. Thus a standard application of Zorn lemma shows that $\text{Con}(A) \setminus \{A^2\}$ has maximal elements. The second claim follows from the simplicity of $I$ and $f(0) \neq f(1)$. □

**Theorem 2.3** Let $\mathcal{A}$ be a variety with distinguished constants $0, 1$ having a minimal algebra. If $\mathcal{A}$ has nontrivial injectives, then there exists a maximum simple algebra $I$.

**Proof:** Let $A$ be a nontrivial injective in $\mathcal{A}$. By Lemma 2.2 there is a maximal congruence $\theta$ of $A$. Let $I = A/\theta$ and $p : A \to I$ be the canonical projection. Since $\mathcal{A}$ has a minimal algebra it is clear that for each simple algebra $J$, there exists a monomorphism $h : J \to A$. Then the composition $ph$ is a monomorphism from $J$ into $I$. Thus $I$ is a maximum simple algebra. □

We want to establish a kind of the converse of the above theorem.

**Theorem 2.4** Let $\mathcal{A}$ be a variety satisfying CEP, with distinguished constants $0, 1$. If $I$ is a self-injective maximum simple algebra in $\mathcal{A}$ then $I$ is injective.

**Proof:** For each monomorphism $g : A \to B$ we consider the following diagram in $\mathcal{A}$:
By CEP, $I$ is hereditarily simple. Hence $f(A)$ is simple and $Ker(f)$ is a maximal congruence of $A$ such that $(0, 1) \notin Ker(f)$. Further $Ker(f)$ can be extended to a maximal congruence $\theta$ in $B$. It is clear that $(0, 1) \notin \theta$ and $\theta \cap A^2 = Ker(f)$. Thus if we consider the canonical projection $p : B \to B/\theta$, then there exists a monomorphism $g' : f(A) \to B/\theta$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{f} & I \\
g & | & \\
B \\
\end{array}$$

Since $I$ is maximum simple, $B/\theta$ is isomorphic to a subalgebra of $I$. Therefore, since that $I$ is self-injective, there exists a monomorphism $\varphi : B/\theta \to I$ such that $\varphi g' = 1_{f(A)}$. Thus $(\varphi p)g = f$ and $I$ is injective. \hfill \Box

**Lemma 2.5** If $A$ is a rigid simple injective algebra in a variety, then all the subalgebras of $A$ are rigid. \hfill \Box

## 3 Injectives, ultrapowers and lattice properties

We recall from [3] some basic notions on ordered sets that will play an important role in what follows. An ordered set $L$ is called **bounded** provided it has a smallest element 0 and a greatest element 1. The **decreasing segment** $[a]$ of $L$ is defined as the set $\{ x \in L : x \leq a \}$. The increasing segment $[a]$ is defined dualy. A subset $X$ of $L$ is called **down directed** **(upper directed)** iff for all $a, b \in X$, there exists $x \in X$ such that $x \leq a$ and $x \leq b$ ($a \leq x$ and $b \leq x$).

**Lemma 3.1** Let $L$ be a lattice and $X$ be a down (upper) directed subset of $L$ such that $X$ does not have a minimum (maximum) element. If $F$ is the filter in $\mathcal{P}(X)$ generated by the decreasing (increasing) segments of $X$, then there exists a nonprincipal ultrafilter $U$ such that $F \subseteq U$. 

4
Proof: Let \((a],(b]\) be decreasing segments of \(X\). Since \(X\) is a down directed subset, there exists \(x \in X\) such that \(x \leq a\) and \(x \leq b\), whence \(x \in (a] \cap (b]\) and \(\mathcal{F}\) is a proper filter of \(\mathcal{P}(X)\). By the ultrafilter theorem there exists an ultrafilter \(\mathcal{U}\) such that \(\mathcal{F} \subseteq \mathcal{U}\). Suppose that \(\mathcal{U}\) is the principal filter generated by \((c]\). Since \(X\) does not have a minimum element, there exists \(x \in X\) such that \(x < c\). Thus \((x] \in \mathcal{U}\) and it is a proper subset of \((c]\), a contradiction. Hence \(\mathcal{U}\) is not a principal filter. By duality, we can establish the same result when \(X\) is an upper directed set.

**Definition 3.2** A variety \(\mathcal{V}\) of algebras has lattice-terms iff there are terms of the language of \(\mathcal{V}\) defining on each \(A \in \mathcal{V}\) operations \(\lor, \land\), such that \(\langle A, \lor, \land \rangle\) is a lattice. \(\mathcal{V}\) has bounded lattice-terms if, moreover, there are two constant terms \(0, 1\) of the language of \(\mathcal{V}\) defining on each \(A \in \mathcal{V}\) a bounded lattice \(\langle A, \lor, \land, 0, 1 \rangle\). The order in \(A\), denoted by \(L(A)\), is called the natural order of \(A\).

Observe that each subvariety of a variety with (bounded) lattice-terms is also a variety with (bounded) lattice-terms.

Let \(\mathcal{V}\) be a variety with lattice-terms and \(A \in \mathcal{V}\). \(A^X/\mathcal{U}\) will always denote the ultrapower corresponding to a down (upper) directed set \(X\) of \(A\) with respect to the natural order, without smallest (greatest) element and a nonprincipal ultrafilter \(\mathcal{U}\) of \(\mathcal{P}(X)\), containing the filter generated by the decreasing (increasing) segments of \(X\). For each \(f \in A^X\), \([f]\) will denote the \(\mathcal{U}\)-equivalence class of \(f\). Thus \([1_X]\) is the \(\mathcal{U}\)-equivalence class of the canonical injection \(X \hookrightarrow A\) and for each \(a \in A\), \([a]\) is the \(\mathcal{U}\)-equivalence class of the constant function \(a\) in \(A^X\). It is well known that \(i_A(a) = [a]\) defines a monomorphism \(A \rightarrow A^X/\mathcal{U}\) (see [6, Corollary 4.1.13]).

**Theorem 3.3** Let \(\mathcal{V}\) be a variety with lattice-terms. If there exists an absolute retract \(A\) in \(\mathcal{V}\), then each down directed subset \(X \subseteq A\) has an infimum, denoted by \(\bigwedge X\). Moreover if \(P(x)\) is a first-order positive formula (see [6]) of the language of \(\mathcal{V}\) such that each \(a \in X\) satisfies \(P(x)\), then \(\bigwedge X\) also satisfies \(P(x)\).

**Proof:** Let \(X\) be a down directed subset of the absolute retract \(A\). Suppose that \(X\) does not admit a minimum element and consider an ultrapower \(A^X/\mathcal{U}\). Since \(A\) is an absolute retract there exists a homomorphism \(\varphi\) such that the following diagram is commutative:
We first prove that $\varphi([1_X])$ is a lower bound of $X$. Let $a \in X$. Then $[1_X] \leq [a]$ since $\{x \in X : 1_X(x) \leq a(x)\} = \{x \in X : x \leq a\} \in U$. Thus $\varphi([1_X]) \leq \varphi([a]) = a$ and $\varphi([1_X])$ is a lower bound of $X$. We proceed now to prove that $\varphi([1_X])$ is the greatest lower bound of $X$. In fact, if $b \in A$ is a lower bound of $X$ then for each $x \in X$ we have $b \leq x$. Thus $[b] \leq [1_X]$ since $\{x \in X : b(x) \leq 1_X(x)\} = \{x \in X : b \leq x\} = X \in U$. Now we have $b = \varphi([b]) \leq \varphi([1_X])$. This proves that $\varphi([1_X]) = \bigwedge X$. If each $a \in X$ satisfies the first order formula $P(x)$ then $[1_X]$ satisfies $P(x)$ and, since $P(x)$ is a positive formula, it follows from ([6, Theorem 3.2.4] ) that $\varphi([1_X])$ satisfies $P(x)$.

In the same way, we can establish the dual version of the above theorem. Recalling that a lattice is complete iff there exists the infimum $\bigwedge X$ (supremum $\bigvee X$), for each down directed (upper directed) subset $X$, we have the following corollary:

**Corollary 3.4** Let $\mathcal{V}$ be a variety with lattice-terms. If $A$ is an absolute retract in $\mathcal{V}$, then $L(A)$ is a complete lattice. $\square$

## 4 Residuated Lattices and Semisimplicity

**Definition 4.1** A *residuated lattice* [18] or *commutative integral residuated 0,1-lattice* [17], is an algebra $\langle A, \wedge, \vee, \circ, \to, 0, 1 \rangle$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following axioms:

1. $\langle A, \circ, 1 \rangle$ is an abelian monoid,
2. $L(A) = \langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice,
3. $(x \circ y) \to z = x \to (y \to z),$
4. $((x \to y) \circ x) \wedge y = (x \to y) \circ x,$
5. $(x \wedge y) \to y = 1.$
A is called an involutive residuated lattice or Girard monoid [15] if it also satisfies the equation:

6. \((x \to 0) \to 0 = x\).

A is called distributive if satisfies 1. – 5. as well as:

7. \(x \land (y \lor z) = (x \land y) \lor (x \land z)\).

The variety of residuated lattices is denoted by \(RL\), and the subvariety of Girad monoids is noted by \(GM\). Following the notation used in [17], the variety of residuated lattices that satisfy the distributive law is denoted by \(DRL\), and \(DGM\) will denote the variety of distributive Girad monoids. It is clear that 0,1 are distinguished constant terms in \(RL\). Moreover, \(\{0, 1\}\) is a subalgebra of each nontrivial \(A \in RL\), which is a boolean algebra. Hence \(\{0, 1\}\) with its natural boolean algebra structure is the minimal algebra in each nontrivial subvariety of \(RL\). Thus the variety \(BA\) of boolean algebras is contained in all nontrivial varieties of residuated lattices. On each residuated lattice \(A\) we can define a unary operation \(\neg\) by \(\neg x = x \to 0\). We also define for all \(a \in A\), \(a^1 = a\) and \(a^{n+1} = a^n \circ a\). An element \(a\) in \(A\) is called idempotent iff \(a^2 = a\), and it is called nilpotent iff there exists a natural number \(n\) such that \(a^n = 0\). The minimum \(n\) such that \(a^n = 0\) is called nilpotence order of \(a\). An element \(a\) in \(A\) is called dense iff \(\neg a = 0\) and it is called a unity iff for all natural numbers \(n\), \(\neg(a^n)\) is nilpotent. The set of dense elements of \(A\) will be denoted by \(Ds(A)\). We recall now some well-known facts about implicative filters and congruences on residuated lattices. Let \(A\) be a residuated lattice and \(F \subseteq A\). Then \(F\) is an implicative filter iff it satisfies the following conditions:

1. \(1 \in F\),
2. if \(x \in F\) and \(x \to y \in F\) then \(y \in F\).

It is easy to verify that a nonempty subset \(F\) of a residuated lattice \(A\) is an implicative filter iff for all \(a, b \in A\):

- If \(a \in F\) and \(a \leq b\) then \(b \in F\),
- if \(a, b \in F\) then \(a \circ b \in F\).
Note that an implicative filter $F$ is proper iff $0$ does not belong to $F$. The intersection of any family of implicative filters of $A$ is again an implicative filter of $A$. We denote by $\langle X \rangle$ the implicative filter generated by $X \subseteq A$, i.e., the intersection of all implicative filters of $A$ containing $X$. We abbreviate this as $\langle a \rangle$ when $X = \{a\}$ and it is easy to verify that $\langle X \rangle = \{x \in A : \exists w_1 \cdots w_n \in X \text{ such that } x \geq w_1, \circ \cdots, \circ w_n\}$. For any implicative filter $F$ of $A$, $\theta_F = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in F\}$ is a congruence on $A$. Moreover $F = \{x \in A : (x, 1) \in \theta_F\}$. Conversely, if $\theta \in \text{Con}(A)$ then $F_\theta = \{x \in A : (x, 1) \in \theta\}$ is an implicative filter and $(x, y) \in \theta$ iff $(x \rightarrow y, 1) \in \theta$ and $(y \rightarrow x, 1) \in \theta$. Thus the correspondence $F \rightarrow \theta_F$ is a bijection from the set of implicative filters of $A$ onto the set $\text{Con}(A)$. If $F$ is an implicative filter of $A$, we shall write $A/F$ instead of $A/\theta_F$, and for each $x \in A$ we shall write $x/\theta_F$ for the equivalence class of $x$.

**Proposition 4.2** If $A$ is a subvariety of $\mathcal{RL}$, then $A$ satisfies CEP.

**Proof:** This follows from the same argument used in ([4, Theorem 1.8]). \(\square\)

If $A$ is a residuated lattice then we define

$$\text{Rad}(A) = \bigcap \{F : F \text{ is a maximal implicative filter in } A\}.$$ 

It is clear that $A$ is semisimple iff $\text{Rad}(A) = \{1\}$. If $A$ is a subvariety of $\mathcal{RL}$, we denote by $\text{Sem}(A)$ the subclass of $A$ whose elements are the semisimple algebras of $A$. Thus we have $\text{Sem}(A) = \{A/\text{Rad}(A) : A \in A\}$.

**Proposition 4.3** Let $A$ be a residuated lattice. Then:

1. $A$ is simple iff for each $a < 1$, $a$ is nilpotent.
2. $\text{Rad}(A) = \{a \in A : a \text{ is unity}\}$.
3. $\text{Ds}(A)$ is an implicative filter in $A$ and $\text{Ds}(A) \subseteq \text{Rad}(A)$.

**Proof:** 1) Trivial. 2) See ([15, Lemma 4.6]). 3) Follows immediately from 2. \(\square\)

If $\text{Rad}(A)$ has a least element $a$, i.e., $\text{Rad}(A) = [a]$, then $a$ is called the principal unity of $A$. It is clear that a principal unity is an idempotent element and that it generates the radical.
Lemma 4.4 Let $A$ be a residuated lattice having a principal unity $a$. If $x \in \text{Rad}(A)$, then $x \rightarrow \neg a = \neg a$.

Proof: $x \rightarrow \neg a = \neg(x \circ a) = \neg a$ since $a$ is the minimum unity.  \hfill \square

Proposition 4.5 Let $A$ be a linearly ordered residuated lattice. Then:

1. $a$ is a unity in $A$ iff $a$ is not a nilpotent element.

2. If $a$ is a unity in $A$, then $\neg a < a$.

Proof: 1) If $a < 1$ and there exists a natural number $n$ such that $a^n = 0$, then $\neg(a^n) = 1$ and $a$ is not a unity. Conversely, suppose $a$ is not a unity. Since $A$ is linearly ordered, we must have $a^n \leq \neg(a^n) < \neg(a^n)$. Hence $a^{2n} = 0$ and $a$ is nilpotent, which is a contradiction. 2) Is an obvious consequence of 1).  \hfill \square

Corollary 4.6 Let $A$ be a residuated lattice such that there exists an embedding $f : A \rightarrow \prod_{i \in I} L_i$, with $L_i$ a linearly ordered residuated lattice for each $i \in I$. Then $a$ is a unity in $A$ iff for each $i \in I$, $a_i = \pi_i f(a)$ is a unity in $L_i$, where $\pi_i$ is the $i$th-projection onto $L_i$.

Proof: If $a$ is a unity in $A$ then $a_i = \pi_i f(a)$ is a unity in $L_i$, because homomorphisms preserve unities. Conversely, suppose that $a$ is not a unity. Therefore there is an $n$ such that $\neg(a^n)$ is not nilpotent, and hence $\neg(a^n) \not\leq \neg(a^n)$. Since $f$ is an embedding and since $L_i$ is linearly ordered for each $i \in I$, there exists $j \in I$ such that $\neg(a^n_j) \nleq \neg(a^n_j)$, and by Proposition 4.5 $a_j$ is not a unity in $L_j$.  \hfill \square

Proposition 4.7 Let $A$ be a subvariety of $\mathcal{RL}$. Then $\text{Sem}(A)$ is a reflective subcategory, and the reflector $[1]$ preserves monomorphism.

Proof: If $A \in A$, for each $x \in A$, $[x]$ will denote the $\text{Rad}(A)$-congruence class of $x$. We define $S(A) = A/\text{Rad}(A)$, and for each $f \in [A, A']_A$, we let $S(f)$ be defined by $S(f)([x]) = [f(x)]$ for each $x \in A$. Since homomorphisms preserve unity, we obtain a well defined function $S(f) : A/\text{Rad}(A) \rightarrow A'/\text{Rad}(A')$. It is easy to check that $S$ is a functor from $A$ to $\text{Sem}(A)$. To show that $S$ is a reflector, note first that if $p_A : A \rightarrow A/\text{Rad}(A)$ is the canonical projection, then the following diagram is commutative:
Suppose that $B \in \mathcal{S}(A)$ and $f \in [A, B]_A$. Since $\text{Rad}(B) = \{1\}$, the mapping $[x] \mapsto f(x)$ defines a homomorphism $g : A/\text{Rad}(A) \rightarrow B$ that makes the following diagram commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\text{Rad}(A) & \xrightarrow{S(f)} & A/\text{Rad}(A') \\
\end{array}
$$

and it is obvious that $g$ is the only homomorphism in $[A/\text{Rad}(A), B]_{\text{Sem}(A)}$ making the triangle commutative. Therefore we have proved that $\mathcal{S}$ is a reflector. We proceed to prove that $\mathcal{S}$ preserves monomorphisms. Let $f \in [A, B]_A$ be a monomorphism and suppose that $(S(f))(x) = (S(f))(y)$, i.e., $[f(x)] = [f(y)]$. Then for each number $n$ there exists a number $m$ such that $0 = (((f(x) \rightarrow f(y))^n)^m = f(((x \rightarrow y)^n)^m)$. Since $f$ is a monomorphism then $((x \rightarrow y)^n)^m = 0$ and $x \rightarrow y \in \text{Rad}(A)$. Interchanging $x$ and $y$, we obtain $[x] = [y]$ and $S(f)$ is a monomorphism. \(\square\)

**Corollary 4.8** Let $\mathcal{A}$ be a subvariety of $\mathcal{RL}$. If $\mathcal{A}$ is injective in $\text{Sem}(\mathcal{A})$ then $\mathcal{A}$ is injective in $\mathcal{A}$.

**Proof:** It is well known that if $\mathcal{D}$ is a reflective subcategory of $\mathcal{A}$ such that the reflector preserves monomorphisms then an injective object in $\mathcal{D}$ is also injective in $\mathcal{A}$ [1, I.18]. Then this theorem follows from Propositions 4.7. \(\square\)

We will say that a variety $\mathcal{A}$ is **radical–dense** provided that $\mathcal{A}$ is a subvariety of $\mathcal{RL}$ and $\text{Rad}(A) = Ds(A)$ for each $A$ in $\mathcal{A}$. An example of a radical-dense variety is the variety $\mathcal{H}$ of Heyting algebras (i.e., $\mathcal{RL}$ plus the equation $x \odot y = x \land y$).

**Theorem 4.9** Let $\mathcal{A}$ be a radical-dense variety. If $\mathcal{A}$ is a non-semisimple absolute retract in $\mathcal{A}$, then $\mathcal{A}$ has a principal unity $\epsilon$ and $\{0, \epsilon, 1\}$ is a subalgebra of $\mathcal{A}$ isomorphic to the three element Heyting algebra $H_3$. 

Proof: Let $A$ be a non-semisimple absolute retract. Unities are characterized by the first order positive formula $\neg x = 0$ because $\text{Rad}(A) = Ds(A)$. Since $Ds(A)$ is a down-directed set, by Theorem 3.3 there exists a minimum dense element $\epsilon$. It is clear that $\epsilon$ is the principal unity and since $\epsilon < 1$, \{0, \epsilon, 1\} is a subalgebra of $A$, which coincides with the three element Heyting algebra $H_3$. □

Definition 4.10 Let $\mathcal{A}$ be a radical-dense variety. An algebra $T \in \mathcal{A}$ is called a test$_d$-algebra iff there are $\epsilon, t \in \text{Rad}(T)$ such that $\epsilon$ is an idempotent element, $t < \epsilon$ and $\epsilon \rightarrow t \leq \epsilon$.

An important example of a test$_d$-algebra is the totally ordered four element Heyting algebra $H_4 = \{0 < b < a < 1\}$ whose operations are given as follows:

$$x \odot y = x \land y,$$

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y. \end{cases}$$

Theorem 4.11 Let $\mathcal{A}$ be a radical-dense variety. If $\mathcal{A}$ has a nontrivial injective and contains a test$_d$-algebra $T$, then all injectives in $\mathcal{A}$ are semisimple.

Proof: Suppose that there exists a non-semisimple injective $A$ in $\mathcal{A}$. Then by Lemma 4.9, there is a monomorphism $\alpha : H_3 \rightarrow A$ such that $\alpha(a)$ is the principal unity in $A$. Let $i : H_3 \rightarrow T$ be the monomorphism such that $i(a) = \epsilon$. Since $A$ is injective, there exists a homomorphism $\varphi : T \rightarrow A$ such that the following diagram commutes

$$\begin{array}{ccc}
H_3 & \xrightarrow{\alpha} & A \\
i \downarrow & \equiv & \downarrow \varphi \\
T & \xrightarrow{i} & A
\end{array}$$

Since $\alpha(a)$ is the principal unity in $A$ and $t \leq \epsilon$, then, by commutativity, $\varphi(\epsilon) = \varphi(t) = \alpha(a)$. Thus $\varphi(\epsilon \rightarrow t) = 1$, which is a contradiction since by hypothesis $\varphi(\epsilon \rightarrow t) \leq \varphi(\epsilon) = \alpha(a) < 1$. Hence $\mathcal{A}$ has only semisimple injectives. □
5 Injectives in $\mathcal{RL}$, $\mathcal{GM}$, $\mathcal{DRL}$ and $\mathcal{DG}$

**Proposition 5.1** Let $A$ be a residuated lattice. Then the set $A^\circ = \{(a,b) \in A \times A : a \leq b\}$ equipped with the operations

\[(a_1,b_1) \wedge (a_2,b_2) := (a_1 \wedge a_2, b_1 \wedge b_2),\]
\[(a_1,b_1) \vee (a_2,b_2) := (a_1 \vee a_2, b_1 \vee b_2),\]
\[(a_1,b_1) \odot (a_2,b_2) := (a_1 \odot a_2, (a_1 \odot b_2) \lor (a_2 \odot b_1)),\]
\[(a_1,b_1) \rightarrow (a_2,b_2) := ((a_1 \rightarrow a_2) \land (b_1 \rightarrow b_2), a_1 \rightarrow b_2).\]

is a residuated lattice, and the following properties hold:

1. The map $i : A \to A^\circ$ defined by $i(a) = (a,a)$ is a monomorphism.
2. $\neg (a,b) = (\neg b, \neg a)$ and $\neg (0,1) = (0,1)$.
3. $A$ is a Girard monoid iff $A^\circ$ is a Girard monoid.
4. $A$ is distributive iff $A^\circ$ is distributive.

**Proof:** See [15, IV Lemma 3.2.1].

**Definition 5.2** We say that a subvariety $\mathcal{A}$ of $\mathcal{RL}$ is $\odot$-closed iff for all $A \in \mathcal{A}$, $A^\circ \in \mathcal{A}$.

**Theorem 5.3** If a subvariety $\mathcal{A}$ of $\mathcal{RL}$ is $\odot$-closed, then $\mathcal{A}$ has only trivial absolute retracts.

**Proof:** Suppose that there exists a non-trivial absolute retract $A$ in $\mathcal{A}$. Then by Proposition 5.1 there exists an epimorphism $f : A^\circ \to A$ such that the following diagram is commutative

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow{i} & \equiv & \downarrow{f} \\
A^\circ & \end{array}
\]

Thus there exists $a \in A$ such that $f(0,1) = a = f(a,a)$. Since $(0,1)$ is a fixed point of the negation in $A^\circ$ it follows that $0 < a < 1$. We have $f(a,1) = 1$. Indeed, $(0,1) \rightarrow (a,a) = ((0 \rightarrow a) \land (1 \rightarrow a), 0 \rightarrow a) = (a,1)$. Thus $f(a,1) = f(0,1 \rightarrow (a,a)) = f(0,1) \rightarrow f(a,a) = a \rightarrow a = 1$. In view
of this we have \( 1 = f(a, 1) \odot f(a, 1) = f((a, 1) \odot (a, 1)) = f(a \odot a, (a \odot 1) \lor (a \odot 1)) = f((a \odot a, a)) \leq f((a, a)) = a \), which is a contradiction since \( a < 1 \).

Hence \( \mathcal{A} \) has only trivial absolute retracts.

\[ \square \]

**Corollary 5.4** \( \mathcal{RL}, \mathcal{GM}, \mathcal{DRL} \) and \( \mathcal{DG} \mathcal{M} \) have only trivial absolute retracts and injectives.

\[ \square \]

## 6 Injectives in SRL-algebras

**Definition 6.1** A SRL-algebra is a residuated lattice satisfying the equation:

\[ (S) \quad x \land \neg x = 0 \]

The variety of SRL-algebras is denoted by \( \text{SRL} \).

**Proposition 6.2** If \( A \) is a SRL-algebra, then \( 0 \) is the only nilpotent in \( A \).

**Proof:** Suppose that there exists a nilpotent element \( x \) in \( A \) such that \( 0 < x \), having nilpotence order equal to \( n \). By the residuation property we have \( x^{n-1} \leq \neg x \). Thus \( x^{n-1} = x \land x^{n-1} \leq x \land \neg x = 0 \), which is a contradiction since \( x \) has nilpotence order equal to \( n \).

\[ \square \]

**Corollary 6.3** Let \( A \) be a subvariety of \( \text{SRL} \). Then the two-element boolean algebra is the maximum simple algebra in \( A \) and \( \text{Sem}(A) = BA \).

**Proof:** Follows from Propositions 6.2 and 4.3.

\[ \square \]

**Corollary 6.4** If \( A \) is a subvariety of \( \text{SRL} \) then \( A \) is a radical-dense variety.

**Proof:** Let \( A \) be an algebra in \( A \) and let \( a \) be a unity. Thus \( \neg a \) is nilpotent and hence \( \neg a = 0 \).

\[ \square \]

**Corollary 6.5** If \( A \) is a subvariety of \( \text{SRL} \), then all complete boolean algebras are injectives in \( A \).

**Proof:** By Corollary 6.3 the two-element boolean algebra is the maximum simple algebra in \( A \). Since it is self-injective, by Theorem 2.4 it is injective. Since complete boolean algebras are the retracts of powers of the two-element boolean algebra, the result is proved.

\[ \square \]

As an application of this theorem we prove the following results:
Corollary 6.6 In SRL and H, the only injectives are complete boolean algebras.

Proof: Follows from Corollary 6.5 and Theorem 4.11 because the testₐ-algebra $Hₐ$ belongs to both varieties. □

Remark 6.7 The fact that injective Heyting algebras are exactly complete boolean algebras was proved in [2] by different arguments.

7 MTL-algebras and absolute retracts

Definition 7.1 An MTL-algebra [12] is a residuated lattice satisfying the pre-linearity equation

\[(Pl) \quad (x \rightarrow y) \vee (y \rightarrow x) = 1\]

The variety of MTL-algebras is denoted by $\mathcal{MTL}$.

Proposition 7.2 Let $A$ be a residuated lattice. Then the following conditions are equivalent:

1. $A \in \mathcal{MTL}$.

2. $A$ is a subdirect product of linearly ordered residuated lattices.

Proof: [15, Theorem 4.8 p. 76]. □

Corollary 7.3 $\mathcal{MTL}$ is subvariety of $\mathcal{DRL}$. □

Corollary 7.4 Let $A$ be a MTL-algebra.

1. If $A$ is simple, then $A$ is linearly ordered.

2. If $e$ is a unity in $A$, then $\neg e < e$.

Proof: 1) Is an immediate consequence of Proposition 7.2. 2) If we consider that the $i$th-coordinate $\pi_i f(e)$ of $e$ in the subdirect product $f : A \rightarrow \prod_{i \in I} L_i$ is a unity, for each $i \in I$, then by Proposition 4.5, $-\pi_i f(e) < \pi_i f(e)$. Thus $-e < e$. □
To obtain the analog of Theorem 4.9 for varieties of MTL-algebras, we cannot use directly Theorem 3.3, because the property of being a unity is not a first order property. We need to adapt the proof of Theorem 4.9 to this case:

**Theorem 7.5** Let $A$ be a subvariety of $MTL$. If $A$ is an absolute retract in $A$ then $A$ has a principal unity $e$ in $A$.

**Proof:** By Proposition 7.2 we can consider a subdirect embedding $f : A \to \prod_{i \in I} L_i$ such that $L_i$ is linearly ordered. We define a family $H(L_i)$ in $A$ as follows: for each $i \in I$

(a) if there exists $e_i = \min\{u \in L_i : u \text{ is unity}\}$ then $H(L_i) = L_i$,

(b) otherwise, $X = \{u \in L_i : u \text{ is unity}\}$ is a down-directed set without least element. Then by Proposition 3.3 we can consider an ultra-product $L_i^{X/\mu}$ of the kind considered after Definition 3.2. We define $H(L_i) = L_i^{X/\mu}$. It is clear that $H(L_i)$ is a linearly ordered $A$-algebra. If we take the class $e_i = [1_X]$ then $e_i$ is a unity in $H(L_i)$ since for every natural number $n$, $0 < e_i^n$ iff $\{x \in X : 0 < (1_X(x))^n\} \in \mathcal{U}$ and $\{x \in X : 0 < (1_X(x))^n = x^n\} = X \in \mathcal{U}$.

We can take the canonical embedding $j_i : L_i \to H(L_i)$ and then for each $i \in I$ we can consider $e_i$ as a unity lower bound of $L_i$ in $H(L_i)$. By Corollary 4.6, $(e_i)_{i \in I}$ is a unity in $\prod_{i \in I} H(L_i)$. Let $j : \prod_{i \in I} L_i \to \prod_{i \in I} H(L_i)$ be the monomorphism defined by $j((x_i)_{i \in I}) = (j_i(x_i))_{i \in I}$. Since $A$ is an absolute retract there exists an epimorphism $\varphi : \prod_{i \in I} H(L_i) \to A$ such that the following diagram commutes:

$$
\begin{array}{cccc}
A & \xrightarrow{f} & \prod_{i \in I} L_i & \xrightarrow{j} & \prod_{i \in I} H(L_i) \\
& & \equiv & & \downarrow \varphi \\
& & 1_A & & A \\
\end{array}
$$

Let $e = \varphi((e_i)_{i \in I})$. It is clear that $e$ is a unity in $A$ since $\varphi$ is an homomorphism. If $u$ is a unity in $A$ then $(e_i)_{i \in I} \leq jf(u)$ and by commutativity of the above diagram, $e = \varphi((e_i)_{i \in I}) \leq \varphi jf(u) = u$. Thus $e = \min\{u \in A : u \text{ is unity}\}$ resulting in $\text{Rad}(A) = [e]$. □
8  Injectives in WNM-algebras and $\mathcal{MTL}$

Definition 8.1 A $\textit{WNM-algebra}$ (weak nilpotent minimum) [12] is an MTL-algebra satisfying the equation

\[(W) \quad \neg(x \circ y) \lor ((x \land y) \rightarrow (x \circ y)) = 1.\]

The variety of WNM-algebras is noted by $\textit{WNM}$.

Theorem 8.2 The following conditions are equivalent:

1. $I$ is a simple WNM-algebra.

2. $I$ has a coatom $u$ and its operations are given by

\[
x \circ y = \begin{cases} 
0, & \text{if } x, y < 1 \\
x, & \text{if } y = 1 \\
y, & \text{if } x = 1
\end{cases}
\]

\[
x \rightarrow y = \begin{cases} 
1, & \text{if } x \leq y \\
y, & \text{if } x = 1 \\
u, & \text{if } y < x < 1.
\end{cases}
\]

Proof: $\Rightarrow$. For $\text{Card}(I) = 2$ this result is trivial. If $\text{Card}(I) > 2$ then we only need to prove the following steps:

a) If $x, y < 1$ in $I$ then $x \circ y = 0$: Since $I$ is simple, equation (W) implies that $x^2 = 0$ for each $x \in I \setminus \{1\}$. Hence if $x \leq y < 1$, then $x \circ y \leq y \circ y = 0$.

b) $I$ has a coatom: Let $0 < x < 1$. We have that $\neg x < 1$ and, since $I$ is simple, we also have $\neg \neg x < 1$. Then by a) it follows that $\neg x \leq \neg \neg x \leq \neg \neg x = \neg x$, i.e., $\neg x = \neg \neg x$. If $0 < x, y < 1$, again by a) we have $\neg x \circ \neg y = 0$. Thus $\neg x \leq \neg \neg y = \neg y$. By interchanging $x$ and $y$ we obtain the equality $\neg x = \neg y$. Now it is clear that if $0 < x < 1$, then $u = \neg x$ is the coatom in $I$.

c) If $y < x < 1$ then $x \rightarrow y = u$: Since $x \rightarrow y = \bigvee\{t \in I : t \circ x \leq y\}$, this supremum cannot be 1 because $y < x$. Thus, in view of item a), $x \rightarrow y$ is the coatom $u$.

$\Leftarrow$ Immediate. \hfill \square
Example 8.3 We can build simple WNM-algebras having arbitrary cardinality if we consider an ordinal $\gamma = \text{Suc}(\text{Suc}(\alpha))$ with the structure given by Proposition 8.2, taking $\text{Suc}(\alpha)$ as coatom. These algebras will be called ordinal algebras.

Proposition 8.4 WNM and MTL have only trivial injectives.

Proof: Follows from Proposition 2.3 since these varieties contain all ordinal algebras.

9 Injectives in SMTL-algebras

Definition 9.1 An SMTL-algebra [13] is a MTL-algebra satisfying equation $(S)$. The variety of SMTL-algebras is denoted by $\text{SMTL}$.

Proposition 9.2 The only injectives in $\text{SMTL}$ are complete boolean algebras.

Proof: Follows from Corollary 6.5 and Theorem 4.11 since the test$_d$-algebra $H_4$ belongs to $\text{SMTL}$.

10 Injectives in $\Pi SMTL$-algebras

Definition 10.1 A $\Pi SMTL$-algebra [12] is a SMTL-algebra satisfying the equation:

$$(\Pi) \quad (\neg\neg z \odot ((x \odot z) \rightarrow (y \odot z))) \rightarrow (x \rightarrow y) = 1.$$ 

The variety of $\Pi SMTL$-algebras is denoted by $\Pi SMTL$.

Proposition 10.2 Let $A$ be an $\Pi SMTL$-algebra. Then 1 is the only idempotent dense element in $A$.

Proof: By equation II it is easy to prove that, for each dense element $\epsilon$, if $\epsilon \odot x = \epsilon \odot y$ then $x = y$. Thus if $\epsilon$ is an idempotent dense then $\epsilon \odot 1 = \epsilon \odot \epsilon$ and $\epsilon = 1$.

Theorem 10.3 Let $A$ be a subvariety of $\Pi SMTL$. Then the injectives in $A$ are exactly the complete boolean algebras.

Proof: Follows from Corollary 6.5, Theorem 4.9 and Proposition 10.2.
11 Injectives in BL, MV, PL, and in Linear Heyting algebras

Definition 11.1 A BL-algebra [14] is an MTL-algebra satisfying the equation

\[(B)\quad x \odot (x \rightarrow y) = x \land y\]

We denote by \(\mathcal{BL}\) the variety of BL-algebras. Important subvarieties of \(\mathcal{BL}\) are the variety \(\mathcal{MV}\) of multi-valued logic algebras (MV-algebras for short), characterized by the equation \(\neg\neg x = x\) [8, 14], the variety \(\mathcal{PL}\) of product logic algebras (PL-algebras for short), characterized by the equations \((\Pi)\) plus \((S)\) [14, 9], and the variety \(\mathcal{HL}\) of linear Heyting algebras, characterized by the equation \(x \odot y = x \land y\) (also known as Gödel algebras [14]).

Remark 11.2 It is well known that \(\mathcal{MV}\) is generated by the MV-algebra \(R_{[0,1]} = ([0,1], \odot, \rightarrow, \land, \lor, 0, 1)\) such that \([0,1]\) is the real unit segment, \(\land, \lor\) are the natural meet and join on \([0,1]\) and \(\odot\) and \(\rightarrow\) are defined as follows: \(x \odot y := \max(0, x+y-1)\), \(x \rightarrow y := \min(1, 1-x+y)\). \(R_{[0,1]}\) is the maximum simple algebra in \(\mathcal{MV}\) (see [8, Theorem 3.5.1]). Moreover \(R_{[0,1]}\) is a rigid algebra (see [8, Corollary 7.2.6]), hence self-injective. Injective MV-algebras were characterized in [16, Corollary 2.11]) as the retracts of powers of \(R_{[0,1]}\).

Proposition 11.3 If \(A\) is a subvariety of \(\mathcal{PL}\), then the only injectives of \(A\) are the complete boolean algebras.

Proof: Follows from Theorem 10.3 since \(\mathcal{PL}\) is a subvariety of \(\Pi SMTL\). \(\square\)

Proposition 11.4 The only injectives in \(\mathcal{HL}\) are the complete boolean algebras.

Proof: Follows from Corollary 6.5 and Theorem 4.11 since the algebra \(test_d H_4\) lies in \(SMTL\). \(\square\)

Proposition 11.5 \(\mathcal{BL}\) is a radical-dense variety.

Proof: See [10, Theorem 1.7 and Remark 1.9]. \(\square\)

Proposition 11.6 Injectives in \(\mathcal{BL}\) are exactly the retracts of powers of the MV-algebra \(R_{[0,1]}\).

Proof: By Remark 11.2 and Propositions 11.5 and 2.4, retracts of a power of the \(R_{[0,1]}\) are injectives in \(\mathcal{BL}\). Thus by Theorem 4.11, they are the only possible injectives since \(H_4\) lies in \(\mathcal{BL}\). \(\square\)
12 Injectives in IMTL-algebras

Definition 12.1 An involutive MTL-algebra (or IMTL-algebra) [12] is a MTL-algebra satisfying the equation

\[(I) \quad \neg\neg x = x.\]

The variety of IMTL-algebras is noted by \(\text{IMTL}\).

An interesting IMTL-algebra, whose role is analogous to \(H_3\) in the radical-dense varieties, is the four element chain \(I_4\) defined as follows:

\[
\begin{array}{c|cccc}
\circ & 1 & a & b & 0 \\
1 & 1 & a & b & 0 \\
a & a & a & 0 & 0 \\
b & b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\quad \rightarrow
\begin{array}{c|cccc}
\neg & 1 & a & b & 0 \\
1 & 1 & a & b & 0 \\
a & 1 & 1 & b & b \\
b & 1 & 1 & 1 & a \\
0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Theorem 12.2 Let \(\mathcal{A}\) be a subvariety of \(\text{IMTL}\). If \(\mathcal{A}\) is a non-semisimple absolute retract in \(\mathcal{A}\), then \(\mathcal{A}\) has a principal unity \(\epsilon\) and \(\{0, \neg\epsilon, \epsilon, 1\}\) is a subalgebra of \(\mathcal{A}\) which is isomorphic to \(I_4\).

Proof: Follows from Theorem 7.5.

Definition 12.3 Let \(\mathcal{A}\) be a subvariety of \(\text{IMTL}\). An algebra \(\mathcal{T}\) is called test\(_{I}\)-algebra iff, it has a subalgebra \(\{0, \neg\epsilon, \epsilon, 1\}\) isomorphic to \(I_4\) and there exists \(t \in \text{Rad}(\mathcal{T})\) such that \(t < \epsilon\).

Theorem 12.4 Let \(\mathcal{A}\) be a subvariety of \(\text{IMTL}\). If \(\mathcal{A}\) has a nontrivial injective and contains a test\(_{I}\)-algebra, then injectives are semisimple.

Proof: Let \(\mathcal{T}\) be a test\(_{I}\)-algebra and \(t \in \text{Rad}(\mathcal{T})\) such that \(t < \epsilon\). We can consider a subdirect embedding \(f: \mathcal{T} \rightarrow \prod_{j \in J} H_j\) such that \(L_j\) is linearly ordered. Let \(x_j = \pi_j f(x)\) for each \(x \in \mathcal{T}\) and \(\pi_j\) the \(j\)th-projection. Since \(t < \epsilon\), exists \(s \in J\) such that \(\neg\epsilon_s < \neg t_s < t_s < \epsilon_s\) and by Corollary 4.6, \(t_s\) and \(\epsilon_s\) are unities in the chain \(H_s\) with \(\epsilon_s\) idempotent. Note that \(H_s\) is also a test\(_{I}\)-algebra. To see that \(\epsilon_s \rightarrow t_s \leq \epsilon\), observe first that \(0 < \epsilon_s \neg t_s\) since, if \(\epsilon_s \neg t_s = 0\) then \(\epsilon_s \leq \neg t_s = t_s\) which is a contradiction. Consequently,
Thus we can conclude that $\epsilon_s \to t_s = -(\epsilon_s \odot t_s) \leq -\epsilon = \epsilon_s$. Suppose that there exists a non-semisimple injective $A$ in $\mathcal{A}$. Then by Theorem 12.2, let $\alpha : I_4 \to A$ be a monomorphism such that $\alpha(a)$ is the principal unity in $A$. Let $i : I_4 \to H_s$ be the monomorphism such that $i(a) = \epsilon_s$. Since $A$ is injective, there exists a homomorphism $\varphi : H_s \to A$ such that the following diagram commutes:

$$
\begin{array}{ccc}
I_4 & \xrightarrow{\alpha} & A \\
\downarrow{i} & \equiv & \downarrow{\varphi} \\
H_s & & \\
\end{array}
$$

Since $\alpha(a)$ is the principal unity in $A$ and $t_s \leq \epsilon_s$ then, by commutativity, $\varphi(\epsilon_s) = \varphi(t_s) = \alpha(a)$. Thus $\varphi(\epsilon_s \to t_s) = 1$, which is a contradiction since $\varphi(\epsilon_s \to t_s) \leq \varphi(\epsilon_s) = \alpha(a) < 1$. Hence $A$ has only semisimple injectives. □

**Proposition 12.5** $\mathcal{IMTL}$ has only trivial injectives.

**Proof:** Suppose that there exists nontrivial injectives in $\mathcal{IMTL}$. By Theorem 2.3 there is a simple maximum algebra $I$ in $\mathcal{IMTL}$. We consider the six elements $IMTL$ chain $I_6$ defined as follows:

| $\odot$ | 1 | $a_1$ | $t$ | $a_2$ | $a_3$ | 0 |
|--------|---|--------|----|--------|--------|---|
| 1      | 1 | $a_1$  | $t$ | $a_2$  | $a_3$  | 0 |
| $a_1$  | $a_1$| $a_2$ | $a_3$| 0      | 0      | $a_1$|
| $t$    | $t$ | $a_3$ | 0    | 0      | 0      | $t$ |
| $a_2$  | $a_2$| $a_3$ | 0    | 0      | 0      | $a_2$|
| $a_3$  | $a_3$| 0     | 0    | 0      | 0      | $a_3$|
| 0      | 0  | 0     | 0    | 0      | 0      | 0  |

Since $I$ is simple maximum we can consider $I_6$ and $R_{[0,1]}$ as subalgebras of $I$. In view of this and using the nilpotence order we have that $1/2 < t < 3/4$ since $I$ is a chain. Therefore we can consider $u = \bigvee_{R_{[0,1]}} \{x \in R_{[0,1]} : x < t\}$ and $v = \bigwedge_{R_{[0,1]}} \{x \in R_{[0,1]} : x > t\}$ and it is clear that $u, v \in R_{[0,1]}$ since $R_{[0,1]}$ is a complete algebra. Thus $u < t < v$. This contradicts the fact that the order of $R_{[0,1]}$ is dense. Consequently $\mathcal{IMTL}$ has only trivial injectives. □
13 Injectives in NM-algebras

**Definition 13.1** A nilpotent minimum algebra (or *NM-algebra*) [12] is an IMTL-algebra satisfying the equation $(W)$.

The variety of NM-algebras is noted by $\mathcal{NM}$. As an example we consider $N_{[0,1]} = \langle [0,1], \odot, \rightarrow, \wedge, \vee, 0, 1 \rangle$ such that $[0,1]$ is the real unit segment, $\wedge, \vee$ are the natural meet and join on $[0,1]$ and $\odot$ and $\rightarrow$ are defined as follows:

$$x \odot y = \begin{cases} x \wedge y, & \text{if } 1 < x + y \\ 0, & \text{otherwise}, \end{cases}$$

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ \max(y, 1-x), & \text{otherwise}. \end{cases}$$

Note that $\{0, \frac{1}{2}, 1\}$ is the universe of a subalgebra of $N_{[0,1]}$, that we denote by $L_3$. The subvariety of $\mathcal{NM}$ generated by $L_3$ coincides with the variety $L_3$ of three-valued Lukasiewicz algebras (see [19, 7]).

**Proposition 13.2** $L_3$ is the maximum simple algebra in $\mathcal{NM}$, and it is self-injective.

**Proof:** Let $I$ be a simple algebra such that $\text{Card}(I) > 2$. By Theorem 8.2 $I$ has a coatom $u$ satisfying $\neg x = u$ for each $0 < x < 1$. Thus $x = \neg \neg x = \neg u = u$ for each $0 < x < 1$. Consequently $\text{Card}(I) = 3$ and $I = L_3$. □

**Corollary 13.3** $\text{Sem}(\mathcal{NM}) = L_3$. □

**Proposition 13.4** Injectives in $\mathcal{NM}$ coincide with complete Post algebras of order 3.

**Proof:** By Proposition 8.2, Theorem 2.4 and Theorem 12.4 injectives in $\mathcal{NM}$ are semisimple since $N_{[0,1]}$ is an algebra $\text{Test}_I$. Thus by Proposition 13.3 and [19], [7, Theorem 3.7], complete Post algebras of order 3 are the injectives in $\mathcal{NM}$. □

14 Injective bounded hoops

**Definition 14.1** A hoop [4] is an algebra $\langle A, \odot, \rightarrow, 1 \rangle$ of type $(2,2,0)$ satisfying the following axioms:
1. \((A, \odot, 1)\) is an abelian monoid,
2. \(x \rightarrow x = 1\),
3. \((x \rightarrow y) \odot x = (y \rightarrow x) \odot y\),
4. \(x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z\).

The variety of hoops is noted \(\mathcal{HO}\). Every hoop is a meet semilattice, where the meet operation is given by \(x \land y = x \odot (x \rightarrow y)\). Let \(A\) be a hoop. If \(A\) has smallest element 0, we can define an unary operation \(\neg\) by \(\neg x = x \rightarrow 0\). A subset \(F\) of \(A\) is a filter iff \(1 \in F\) and \(F\) is closed under \(\odot\). As in residuated lattices, filters and congruences can be identified [4].

**Definition 14.2** A Wajsberg hoop [4] is a hoop that satisfies the following equation

\[(T) \quad (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.\]

Each Wajsberg hoop is a lattice, in which the join operation is given by \(x \lor y = (x \rightarrow y) \rightarrow y\).

**Definition 14.3** A bounded hoop is an algebra \(\langle A, \odot, \rightarrow, 0, 1 \rangle\) of type \(\langle 2, 2, 0, 0 \rangle\) such that:

1. \(\langle A, \odot, \rightarrow, 1 \rangle\) is a hoop
2. \(0 \rightarrow x = 1\).

The variety of bounded hoop is noted by \(\mathcal{BH}_0\). Observe that since 0 is in the clone of hoop operation, we require that for each morphism \(f\), \(f(0) = 0\). In the same way as in the case of residuated lattices, for each bounded hoop \(A\), we can consider \(Ds(A)\) the set of dense elements of \(A\), and this is an implicative filter of \(A\).

**Proposition 14.4** A bounded simple hoop is a simple MV-algebra.

**Proof:** Let \(I\) be a simple hoop. Then by [4, Corollary 2.3] it is a totally ordered Wajsberg hoop. If 0 is the smallest element in \(I\) then by the equation \((T), \neg \neg x = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow x) \rightarrow x = 1 \rightarrow x = x\). Hence it is an MV-algebra. Since the MV-congruences are in correspondence with implicative filters, \(I\) is a simple MV-algebra. \(\square\)
| Variety | Equations | Injectives |
|---------|-----------|------------|
| RL      | RL + x ∧ (y ∨ z) = (x ∧ y) ∨ (x ∧ z) | Trivial |
| DRL     | RL + ¬¬x = x | Trivial |
| GM      | GM + x ∧ (y ∨ z) = (x ∧ y) ∨ (x ∧ z) | Trivial |
| DGM     | GM + x ∧ (y ∨ z) = (x ∧ y) ∨ (x ∧ z) | Trivial |
| TL      | RL + (x → y) ∨ (y → x) = 1 | Trivial |
| WNL     | WNL + ¬(x ⊗ y) ∨ ((x ⊗ y) → (x ⊗ y)) = 1 | Trivial |
| LTL     | LTL + ¬¬x = x | Trivial |
| BL      | BL + x ∧ y = x ⊗ (x → y) | Retracts of powers of $R_{[0,1]}$ |
| MV      | VL + ¬¬x = x | Retracts of powers of $R_{[0,1]}$ |
| BHL      | BHL + ¬¬x = x | Retracts of powers of $R_{[0,1]}$ |
| SRL     | SRL + x ∧ ¬x = 0 | Complete boolean algebras |
| SMTL    | SMTL + x ∧ ¬x = 0 | Complete boolean algebras |
| NMSMTL  | NMSMTL + ¬¬x ⊗ ((x ⊗ z) → (y ⊗ z)) ≤ (x → y) | Complete boolean algebras |
| PL      | PL + x ∧ y = x ⊗ (x → y) | Complete boolean algebras |
| NM      | NM + ¬¬x = x | Complete boolean algebras |

Table 1: Injectives in Varieties of Residuated Algebras

**Proposition 14.5** Let $I, J$ be simple hoops with smallest elements $0_I, 0_J$ respectively. If $\varphi : I \to J$ is a hoop homomorphism then $\varphi$ is also an MV-homomorphism, i.e., $\varphi(0_I) = 0_J$.

**Proof:** Suppose that $\varphi(0_I) = a$. Since $J$ is simple, there exists a natural number $n$ such that $a^n = 0_J$. Thus we have, $\varphi(0_I) = \varphi(0_I^n) = (\varphi(0_I))^n = a^n = 0_J$. □

The following two results are obtained in the same way as Theorems 4.9 and 4.11 respectively.

**Theorem 14.6** Let $A$ be a subvariety of $\mathcal{BH}_0$. If $A$ is a non-semisimple absolute retract in $A$, then $D_s(A)$ has a least element $e$, i.e., $D_s(A) = \{e\}$ and $\{0, e, 1\}$ is a subalgebra of $A$ isomorphic to the three element Heyting algebra $H_3$. □

**Theorem 14.7** Let $A$ be a subvariety of $\mathcal{BH}_0$. If $A$ has a nontrivial injectives and contains the Heyting algebra $H_4$ then injectives are semisimple. □

**Corollary 14.8** Injectives in $\mathcal{BH}_0$ are exactly the retracts of powers of the MV-algebra $R_{[0,1]}$. 23
Proof: By Proposition 14.4, semisimple bounded hoops are MV-algebras. Therefore $R_{[0,1]}$ is the maximum simple algebra and it is self injective by Proposition 14.5. Thus by Theorem 2.4 retracts of powers of the MV-algebra $R_{[0,1]}$ are injectives in $\mathcal{BH}_0$. By Theorem 14.7 they are the only injectives, because $H_4$ lies in $\mathcal{BH}_0$. □

References

[1] R. Balbes and Ph. Dwinger, **Distributive Lattices**, University of Missouri Press, Columbia, 1974.

[2] R. Balbes, A. Horn, **Injective and Projective Heyting algebras**, Trans. Amer. Math. Soc. **148** (1970), 549–559.

[3] G. Birkhoff, **Lattice Theory**, 3rd Ed., Amer. Math. Soc., Providence, Rh. I., 1967.

[4] W.J. Blok, I.M.A. Ferreirim, **On the structure of hoops**, Algebra Univers. **43** (2000), 233–257.

[5] S. Burris, H. P. Sankappanavar, **A Course in Universal Algebra**, Graduate Text in Mathematics, Vol. 78. Springer-Verlag, New York Heidelberg Berlin, 1981.

[6] C. C. Chang, H. J. Keisler, **Model theory**, North-Holland, Amsterdam-London-New York-Tokio, 1994.

[7] R. Cignoli, **Representation of Lukasiewicz and Post algebras by continuous functions**, Colloq. Math., **24** (1972), 128–138.

[8] R. Cignoli, M. I. D’Ottaviano and D. Mundici, **Algebraic foundations of many-valued reasoning**, Kluwer, Dordrecht-Boston-London, 2000.

[9] R. Cignoli and A. Torrens, **An algebraic analysis of product logic**, Mult. Valued Log., **5** (2000), 45-65.

[10] R. Cignoli and A. Torrens, **Hájek basic fuzzy logic and Lukasiewicz infinite-valued logic**, Arch. Math. Logic, **42** (2003), 361–370.

[11] R. Cignoli and A. Torrens, **Free algebras in varieties of BL-algebras with a boolean retract**, Algebra Univers., **48** (2002), 55–79.
[12] F. Esteva and L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems, 124 (2001), 271–288.

[13] F. Esteva, L. Godo, P. Hájek and F. Montagna: Hoops and fuzzy logic, to appear.

[14] P. Hájek, Metamathematics of fuzzy logic, Kluwer, Dordrecht-Boston-London, 1998.

[15] U. Höhle, Commutative, residuated l-monoids. In: Non-classical Logics and their applications to Fuzzy Subset, a Handbook on the Mathematical Foundations of Fuzzy Set Theory, U. Höhle, E. P. Klement, (Editors). Kluwer, Dordrecht, 1995.

[16] D. Gluschankof, Prime deductive systems and injective objects in the algebras of Lukasiewicz infinite-valued calculi, Algebra Universal. 29 (1992), 354–377.

[17] P. Jipsen and C. Tsinakis, A Survey of Residuated Lattices, Ordered Algebraic Structures, Proceedings of the Gainesville Conference. Edited by Jorge Martinez, Kluwer Academic Publishers, Dordrecht, Boston, London 2001

[18] T. Kowalski and H. Ono, Residuated Lattices: An algebraic glimpse at logics without contraction, Preliminary report, 2000.

[19] L. Monteiro, Sur les algèbres de Lukasiewicz injectives, Proc. Japan Acad, 41 (1965), 578–581.

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria
1428 Buenos Aires - Argentina
e-mail: hfreytes@dm.uba.ar