A NEW APPLICATION
OF THE REPRODUCING KERNEL METHOD

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ABSTRACT. We give a new implementation of the reproducing kernel method to investigate difference equations in this paper. We obtain the solutions in terms of convergent series. The method of obtaining the approximate solution in form of an algorithm is presented. We demonstrate some experiments to prove the accuracy of the technique.

1. Introduction. Difference equations depend to differential equations as discrete mathematics depends to continuous mathematics. Difference equations come to exist naturally as discretised analogues of differential equations, and they also seem in their own right, e.g., in the recurrence formulae for special functions and orthogonal polynomials [5].

We investigate difference equations by reproducing kernel method in this work. A reproducing kernel Hilbert space approach in meshless collocation method has been investigated in [6]. Piecewise reproducing kernel method for linear impulsive delay differential equations with piecewise constant arguments has been worked in [14]. Numerical solution of integro-differential equations of high-order Fredholm has been found by the simplified reproducing kernel method in [16]. Application of reproducing kernel Hilbert space method for solving second-order fuzzy Volterra integro-differential equations has been given in [10]. Reproducing kernel method for solving nonlinear fractional fredholm-integrodifferential Equation has been searched in [12]. For more details see [3, 9, 2, 11, 15, 4].

We prepare the paper as: Section 2 introduces several reproducing kernel spaces for differential equations. Section 3 is devoted to some reproducing kernel functions for difference equations. The acting in \((V,\langle \cdot , \cdot \rangle_3)\) and a linear operator are given in Section 4. Section 5 presents the main results. Some examples are shown in Section 6. Some conclusions are presented in the final section.

2. Reproducing kernel Hilbert spaces for differential equations.

Definition 2.1. We describe \(G_2^1[0, 1]\) as:
\[
G_2^1[0, 1] = \{ u \in AC[0, 1] : u' \in L^2[0, 1] \}.
\]

The inner product and the norm in \(G_2^1[0, 1]\) are defined by
\[
\langle u, g \rangle_{G_2^1} = u(0)g(0) + \int_0^1 u'(x)g'(x)dx, \quad u, g \in G_2^1[0, 1],
\]

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Lemma 2.2. \( G_2^1[0,1] \) is a reproducing kernel space. We obtain the reproducing kernel function \( A_y \) of this space as [7, page 17]:

\[
A_y(x) = \begin{cases} 
1 + x, & x \leq y, \\
1 + y, & x > y.
\end{cases}
\]

Definition 2.3. We describe the space \( H_2^2[0,1] \) as:

\[
H_2^2[0,1] = \{ u \in AC[0,1] : u', u'' \in AC[0,1], u^{(3)} \in L^2[0,1] \}.
\]

We have the inner product and the norm as:

\[
\langle u, g \rangle_{H_2^2} = u(0)g(0) + \int_0^1 u''(x)g''(x)dx, \quad u, g \in H_2^2[0,1],
\]

and

\[
\|u\|_{H_2^2} = \sqrt{\langle u, u \rangle_{H_2^2}}, \quad u \in H_2^2[0,1].
\]

Lemma 2.4. \( H_2^2[0,1] \) is a reproducing kernel space. We obtain the reproducing kernel function \( T_y \) of this space as [7, page 17]:

\[
T_y(x) = \begin{cases} 
1 + xy + \frac{x^2}{2} - \frac{x^3}{6}, & x \leq y, \\
1 + xy + \frac{x^2}{2} - \frac{x^3}{6}, & x > y.
\end{cases}
\]

Definition 2.5. We describe the space \( W_2^3[0,1] \) by

\[
W_2^3[0,1] = \{ u \in AC[0,1] : u', u'' \in AC[0,1], u^{(3)} \in L^2[0,1] \}.
\]

The inner product and the norm in \( W_2^3[0,1] \) are given as:

\[
\langle u, g \rangle_{W_2^3} = \sum_{i=0}^2 u^{(i)}(0)g^{(i)}(0) + \int_0^1 u^{(3)}(x)g^{(3)}(x)dx, \quad u, g \in W_2^3[0,1],
\]

and

\[
\|u\|_{W_2^3} = \sqrt{\langle u, u \rangle_{W_2^3}}, \quad u \in W_2^3[0,1].
\]

Lemma 2.6. The space \( W_2^3[0,1] \) is a reproducing kernel space, and its reproducing kernel function \( R_y \) is given by [7, page 17]

\[
R_y(x) = \begin{cases} 
1 + xy + \frac{x^2}{4} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{120}, & x \leq y, \\
1 + xy + \frac{x^2}{4} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{120}, & x > y.
\end{cases}
\]

3. Reproducing kernel functions for difference equations.

Definition 3.1. We describe the space \( V \) as:

\[
V = \{ u : [0, N] \cap \mathbb{Z} \rightarrow \mathbb{R} \}.
\]

The inner product and the norm in \( \langle V, \langle \cdot, \cdot \rangle_1 \rangle \) are presented as:

\[
\langle u, g \rangle_1 = u(0)g(0) + \sum_{x=0}^{N-1} \Delta u(x) \Delta g(x), \quad u, g \in V, \quad N \in \mathbb{N},
\]
\[ \|u\|_1 = \sqrt{\langle u, u \rangle_1}, \quad u \in V. \]  

**Lemma 3.2.** \((V, \langle \cdot, \cdot \rangle_1)\) is the reproducing kernel space. We obtain the reproducing kernel function \(B_y\) of this space as [1]:

\[ B_y(x) = \begin{cases} 
1 + y, & y \leq x, \\
1 + x, & y > x.
\end{cases} \]

**Proof.** Note that

\[ \Delta B_y(x) = \begin{cases} 
0, & y \leq x, \\
1, & y > x.
\end{cases} \]

Let \(u \in V\) and \(y \in [0, N] Z\). Then by Definition 3.1 we have

\[
\langle u, B_y \rangle_1 = u(0)B_y(0) + \sum_{x=0}^{N-1} \Delta u(x) \Delta B_y(x)
\]

\[
= u(0) + \sum_{x=0}^{y-1} \Delta u(x)
\]

\[
= u(0) + u(y) - u(0)
\]

\[
= u(y).
\]

\[ \square \]

**Definition 3.3.** The inner product and the norm in \((V, \langle \cdot, \cdot \rangle_2)\) are given by [1]:

\[
\langle u, g \rangle_2 = u(0)g(0) + \Delta u(0)\Delta g(0) + \sum_{x=0}^{N-2} \Delta^2 u(x)\Delta^2 g(x), \quad u, g \in V, \quad N \in \mathbb{N}, \quad N \geq 1,
\]

and

\[ \|u\|_2 = \sqrt{\langle u, u \rangle_2}, \quad u \in V, \]

for any real \(x\)

\[
x^2 = x(x-1), \quad x^3 = x(x-1)(x-2).
\]

**Lemma 3.4.** \((V, \langle \cdot, \cdot \rangle_2)\) is a reproducing kernel space. We obtain the reproducing kernel function \(M_y\) of this space as [1]:

\[ M_y(x) = \begin{cases} 
1 + xy + \frac{y^2(x-1)}{2} - \frac{y^3}{6}, & y \leq x, \\
1 + yx + \frac{x^2(y-1)}{2} - \frac{x^3}{6}, & y > x.
\end{cases} \]

**Proof.** First note that

\[ \Delta M_y(x) = \begin{cases} 
y + \frac{y^2}{2}, & y \leq x, \\
y + x(y-1) - \frac{x^2}{2}, & y > x,
\end{cases} \]

\[ \Delta^2 M_y(x) = \begin{cases} 
0, & y \leq x, \\
y - 1 - x, & y > x.
\end{cases} \]
Let $u \in V$ and $y \in [0, N - 1] \mathbb{Z}$. Then by Definition 3.3, we have

$$
\langle u, M_y \rangle_2 = u(0)M_y(0) + \Delta u(0)\Delta M_y(0) + \sum_{x=0}^{N-2} \Delta^2 u(x)\Delta^2 M_y(x)
$$

$$
= u(0) + y\Delta u(0) - \sum_{x=0}^{y-1} (x - y + 1)\Delta^2 u(x)
$$

$$
= u(0) + y\Delta u(0) - \sum_{x=0}^{y-1} \Delta (x - y)\Delta u(x) + \sum_{x=0}^{y-1} \Delta u(x)
$$

$$
= u(0) + y\Delta u(0) - 0\Delta u(y) - y\Delta u(0) + u(y) - u(0)
$$

$$
= u(y).
$$

\[\square\]

**Definition 3.5.** We describe the inner product and the norm in $(V, \langle \cdot, \cdot \rangle_3)$ as:

$$
\langle u, g \rangle_3 = \sum_{i=0}^{2} \Delta^i u(0)\Delta^i g(0) + \sum_{x=0}^{N-3} \Delta^3 u(x)\Delta^3 g(x), \quad u, g \in V, \quad N \in \mathbb{N}, \quad N \geq 2
$$

and

$$
\|u\|_3 = \sqrt{\langle u, u \rangle_3}, \quad u \in V,
$$

for any real $x$

$$
x^4 = x(x-1)(x-2)(x-3), \quad x^5 = x(x-1)(x-2)(x-3)(x-4).
$$

**Lemma 3.6.** $(V, \langle \cdot, \cdot \rangle_3)$ is a reproducing kernel space. We obtain the reproducing kernel function $F_y$ of this space as [1]:

$$
F_y(x) = \begin{cases} 
1 + xy + \frac{x^2}{4} + \frac{x^3}{12} - \frac{y^4}{24} + \frac{y^5}{120}, & y \leq x, \\
1 + xy + \frac{x^2}{4} + \frac{x^3}{12}y - \frac{x^4}{24} + \frac{x^5}{120}, & y > x.
\end{cases}
$$

**Proof.** Note that

$$
\Delta F_y(x) = \begin{cases} 
y + \frac{y^2}{2} + \frac{(x-1)y^3}{6} - \frac{y^4}{24}, & y \leq x, \\
y + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^4}{24} + \frac{x^5}{24}, & y > x,
\end{cases}
$$

$$
\Delta^2 F_y(x) = \begin{cases} 
\frac{y^2}{2} + \frac{y^3}{6}, & y \leq x, \\
\frac{y^2}{2} + \frac{x(y-1)(y-2)}{2} - \frac{x^2}{2} + \frac{x^3}{6}, & y > x,
\end{cases}
$$

$$
\Delta^3 F_y(x) = \begin{cases} 
0, & y \leq x, \\
\frac{(y-1)(y-2)}{2} - x(y - 2) + \frac{x^2}{2}, & y > x.
\end{cases}
$$
Let $u \in E$ and $y \in [0, N - 2]_Z$. Then by Definition 3.5 we have
\[
\langle u, F_y \rangle_3 = u(0)F_y(0) + \Delta u(0)\Delta F_y(0) + \Delta^2 u(0)\Delta^2 F_y(0) + \sum_{x=0}^{N-3} \Delta^3 u(x)\Delta^3 F_y(x)
\]
\[
= u(0) + y\Delta u(0) + \frac{y(y - 1)}{2} \Delta^2 u(0)
+ \sum_{x=0}^{y-1} \left( \frac{(y - 1)(y - 2)}{2} - (y - 2)x + \frac{x(x - 1)}{2} \right) \Delta^3 u(x)
= u(0) + y\Delta u(0) + \frac{y(y - 1)}{2} \Delta^2 u(0)
+ \sum_{x=0}^{y-1} \Delta \left( \left( \frac{(y - 1)(y - 2)}{2} - (y - 2)(x - 1) + \frac{(x - 1)(x - 2)}{2} \right) \Delta^2 u(x) \right)
- \sum_{x=0}^{y-1} (x - y + 1) \Delta^2 u(x)
= u(0) + y\Delta u(0) + \frac{y(y - 1)}{2} \Delta^2 u(0)
+ \left( \frac{(y - 1)(y - 2)}{2} - (y - 2)(y - 1) + \frac{(y - 1)(y - 2)}{2} \right) \Delta^2 u(y)
- \left( \frac{y(y - 1)}{2} - (y - 2)(0 - 1) + \frac{(0 - 1)(0 - 2)}{2} \right) \Delta^2 u(0)
- \sum_{x=0}^{y-1} \Delta ((x - y)\Delta u(x)) + \sum_{x=0}^{y-1} \Delta u(x)
= u(0) + y\Delta u(0) - (y - y)\Delta u(y) + (0 - y)\Delta u(0) + u(y) - u(0)
= u(y).
\]

4. **Solutions in** $(V, \langle \cdot, \cdot \rangle_3)$. The solutions of the following problem is considered in the reproducing kernel space $(V, \langle \cdot, \cdot \rangle_3)$.
\[\Delta^2 u(x) + \Delta u(x) - u(x) = 0.\]  
(3)
We describe the linear operator $L : (V, \langle \cdot, \cdot \rangle_3) \to (V, \langle \cdot, \cdot \rangle_1)$ as:
\[Lv = \Delta^2 v + \Delta v - v.\]  
(4)
We have the following problem.
\[\Delta^2 u(x) + \Delta u(x) - u(x) = 0,\]  
(5)
with the boundary conditions
\[u(0) = u(1) = 1.\]  
(6)
This problem changes to the following problem by homogenizing the boundary conditions.

\[
\begin{cases}
Lv = 1, \\
v(0) = v(1) = 0,
\end{cases}
\]

where

\[v(x) = u(x) - 1.\]

**Lemma 4.1.** \(L\) is a bounded operator.

**Proof.** We should prove

\[\|Lu\|_1^2 \leq M \|u\|_3^2,\]

where \(M\) is a positive constant. By (1) and (2) we get

\[\|Lu\|_1^2 = \langle Lu, Lu \rangle_1 = [Lu(0)]^2 + \sum_{x=0}^{N-1} [\Delta Lu(x)]^2.\]  

(8)

We have

\[u(x) = \langle u(\cdot), F_x(\cdot) \rangle_3\]

and

\[Lu(x) = \langle u(\cdot), LF_x(\cdot) \rangle_3,\]

by reproducing property. So, we get

\[|Lu| \leq \|u\|_3 \|LF_x\|_3 = M_1 \|u\|_3,\]

thus

\[[Lu(0)]^2 \leq M_3^2 \|u\|_3^2.\]

Since

\[\Delta Lu(x) = \langle u(\cdot), \Delta F_x(\cdot) \rangle_3,\]

we get

\[|\Delta Lu| \leq \|u\|_3 \|\Delta F_x\|_3 = M_2 \|u\|_3,\]

so we have

\[[\Delta Lu]^2 \leq M_2^2 \|u\|_3^2,\]

that is,

\[\|Lu\|_1^2 = [Lu(0)]^2 + \sum_{x=0}^{N-1} [\Delta Lu(x)]^2 \leq \left(M_3^2 + NM_2^2\right) \|u\|_3^2,\]

where \(M = M_3^2 + NM_2^2\) is a positive constant.

\[\square\]

In similar way one can show that \(L : (V, \langle \cdot, \cdot \rangle_2) \to (V, \langle \cdot, \cdot \rangle_1)\) is a bounded linear operator.
5. The main results. From Eq. (4) it is clear that $L : (V, \langle \cdot , \cdot \rangle_3) \rightarrow (V, \langle \cdot , \cdot \rangle_1)$ is a bounded linear operator. Put $\varphi_i = B_{x_i}$ and $\psi_i = L^* \varphi_i$, where $L^*$ is conjugate operator of $L$. The orthonormal system $\{ \hat{\Psi}_i \}_{i=1}^N$ of $(V, \langle \cdot , \cdot \rangle_3)$ can be derived from Gram-Schmidt orthogonalization process of $\{ \psi_i \}_{i=1}^N$,

$$\hat{\psi}_i = \sum_{k=1}^i \beta_{ik} \varphi_k, \quad (\beta_{ii} > 0, \ i = 1, 2, \ldots). \quad (9)$$

**Theorem 5.1.** For Eq. (4) $\{ \Psi_i \}_{i=1}^N$ is a complete system in $(V, \langle \cdot , \cdot \rangle_3)$, and $\Psi_i = LF_{x_i}(x)$.

**Proof.** We have

$$\Psi_i = L^* \varphi_i = \langle L^* \varphi_i, F_x \rangle_3$$
$$= \langle \varphi_i, LF_x \rangle_1$$
$$= \langle LF_x, B_{x_i} \rangle_1$$
$$= LF_x(x_i)$$
$$= LF_{x_i}(x).$$

Clearly $\Psi_i \in (V, \langle \cdot , \cdot \rangle_3)$. For each fixed $u \in (V, \langle \cdot , \cdot \rangle_3)$, if

$$\langle u, \Psi_i \rangle_3 = 0, \ i = 1, 2, \ldots,$$

then

$$0 = \langle u, \Psi_i \rangle_3$$
$$= \langle u, L^* \varphi_i \rangle_3$$
$$= \langle Lu, \varphi_i \rangle_1$$
$$= \langle Lu, B_{x_i} \rangle_1$$
$$= Lu(x_i), \ i = 1, 2, \ldots.$$ 

Hence, $Lu = 0$. From the existence of $L^{-1}$, it follows that $u = 0$. □

**Theorem 5.2.** If $u$ is the exact solution of (7), then

$$u = \sum_{i=1}^N \sum_{k=1}^i \beta_{ik} \hat{\Psi}_i. \quad (10)$$

**Proof.** By Theorem (5.1) $\{ \Psi_i \}_{i=1}^N$ is a complete system in $(V, \langle \cdot , \cdot \rangle_3)$. Thus

$$u = \sum_{i=1}^N \langle u, \hat{\Psi}_i \rangle_3 \hat{\Psi}_i$$
$$= \sum_{i=1}^N \sum_{k=1}^i \beta_{ik} \langle u, \Psi_k \rangle_3 \hat{\Psi}_i$$
$$= \sum_{i=1}^N \sum_{k=1}^i \beta_{ik} \langle u, L^* \varphi_k \rangle_3 \hat{\Psi}_i$$
$$= \sum_{i=1}^N \sum_{k=1}^i \beta_{ik} \langle Lu, \varphi_k \rangle_1 \hat{\Psi}_i$$
\[ u_n = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} \widehat{\Psi}_i, \quad n \leq N. \]  

The approximate solution \( u_n \) can be found as:

\[ u_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \widehat{\Psi}_i, \quad n \leq N. \]  

(11)

Obviously

\[ \|u_n - u\|_3 \to 0, \quad n \to N. \]

\[ \|u_n - u\|_3 \to 0, \quad n \to N. \]

\[ \|u_n - u\|_3 \to 0, \quad n \to N. \]

**Theorem 5.3.** If \( u \in (V, \langle \cdot, \cdot \rangle_3) \), then

\[ \|u_n - u\|_3 \to 0, \quad n \to N. \]

A sequence \( \|u_n - u\|_3 \) is monotonically decreasing in \( n \).

**Proof.** From (10) and (11), it follows that

\[ \|u_n - u\|_3 = \left\| \sum_{i=n+1}^{N} \sum_{k=1}^{i} \beta_{ik} \widehat{\Psi}_i \right\|_3. \]

Thus

\[ \|u_n - u\|_3 \to 0, \quad n \to N. \]

In addition

\[ \|u_n - u\|_3^2 = \left\| \sum_{i=n+1}^{N} \sum_{k=1}^{i} \beta_{ik} \widehat{\Psi}_i \right\|_3^2. \]

Clearly, \( \|u_n - u\|_3 \) is monotonically decreasing in \( n \).

6. **Numerical results.**

**Example 1.** We take into consideration the first-order difference equation

\[ \Delta u(x) - u(x) = 1, \]  

(12)

with the boundary conditions

\[ u(0) = 0, \quad u(1) = 1. \]  

(13)

The exact solution of (12)–(13) is presented by [13]

\[ u(x) = 2^x - 1. \]

We show our results in Table 1.
Example 2. We research the Beverton–Holt difference equation
\[ u(x + 1) = \frac{vK(x)u(x)}{K(x) + (v - 1)u(x)}. \] (14)
The exact solution of (14) is presented by [8]
\[ u(x) = \frac{1}{(\frac{1}{2})^x \frac{1}{u(0)} + \sum_{i=0}^{x-1} (\frac{1}{2})^{x-i} \frac{1}{K(i)}}. \]
Taking \( v = 2 \) and using the above method we obtain Table 2.

Example 3. We investigate the logistic difference equation
\[ u(x + 1) = ru(x) \left(1 - \frac{u(x)K(x)}{K(x)} \right). \] (15)
Exact solution for different value of \( x \) is found by recursive formulae. Taking \( r = \frac{1}{2} \) and using the above method we obtain Table 3.

Example 4. We take into consideration the Airy difference equation
\[ u(x + 1) = (x + 2)u(x) - u(x - 1). \] (16)
with the boundary conditions
\[ u(0) = 0, \quad u(1) = 1. \] (17)
Exact solution for different value of \( x \) is found by recursive formulae. After homogenizing the boundary conditions and using the above method we obtain Table 4.

Example 5. Let us consider the second-order difference equation
\[ \Delta^2 u(x) + \Delta u(x) - u(x) = 0, \] (18)
with the boundary conditions
\[ u(0) = u(1) = 1. \] (19)
The exact solution of (18)–(19) is given as [13]
\[ u(x) = \frac{\sqrt{5}}{5} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^x - \left( \frac{1 - \sqrt{5}}{2} \right)^x \right). \]
We obtain Table 5 by reproducing kernel method.

Example 6. Let us consider the second-order difference equation
\[ \Delta^2 u(x) + u(x + 1) = 0, \] (20)
with the boundary conditions
\[ u(0) = 0, \quad u(1) = 1. \] (21)
Exact solution for different value of \( x \) is found by recursive formulae. We get Table 6 after homogenizing the boundary conditions.
| $x$ | ES                       | AS                       | AE          | RE       | CPU time(s) |
|-----|--------------------------|--------------------------|-------------|----------|-------------|
| 50  | $1.125899907 \times 10^{15}$ | $1.125899907 \times 10^{15}$ | 0.0         | 0.0      | 0.078       |
| 100 | $1.2676506 \times 10^{10}$  | $1.2676506 \times 10^{10}$  | 0.0         | 0.0      | 0.032       |
| 150 | $1.427247693 \times 10^{45}$ | $1.42721 \times 10^{45}$         | 0.0         | 0.0000264095715 | 0.094         |
| 200 | $1.606938044 \times 10^{90}$ | $1.606938044 \times 10^{90}$ | 0.0         | 0.0      | 0.031       |
| 250 | $1.809251394 \times 10^{75}$ | $1.809251394 \times 10^{75}$ | 0.0         | 0.0      | 0.063       |
| 300 | $2.037035976 \times 10^{90}$ | $2.037035976 \times 10^{90}$ | 0.0         | 0.0      | 0.078       |

Table 1. Numerical results for Example 1.

| $x$ | ES                    | AS                    | AE          | RE       | CPU time(s) |
|-----|-----------------------|-----------------------|-------------|----------|-------------|
| 10  | 39.88315482           | 39.88315374           | $1.08 \times 10^{-6}$ | 2.707910156 $\times 10^{-8}$ | 0.344       |
| 20  | 19.999998093          | 19.99998111           | $1.8 \times 10^{-7}$ | 9.000008582 $\times 10^{-9}$ | 0.421       |
| 30  | 39.999999989          | 39.99999960           | $2.9 \times 10^{-7}$ | 7.25000002 $\times 10^{-9}$ | 0.344       |
| 40  | 20.000000000          | 19.99999629           | $3.8 \times 10^{-6}$ | $1.9 \times 10^{-7}$    | 0.390       |
| 50  | 40.000000000          | 39.99999340           | $6.6 \times 10^{-6}$ | $1.65 \times 10^{-7}$   | 0.437       |
| 100 | 20.000000000          | 19.99999843           | $1.57 \times 10^{-6}$ | $7.85 \times 10^{-8}$   | 0.405       |

Table 2. Numerical results for Example 2.

| $x$ | ES                    | AS                    | AE          | RE       | CPU          |
|-----|-----------------------|-----------------------|-------------|----------|--------------|
| 10  | 0.004173544379        | 0.004173091           | 4.53379 $\times 10^{-7}$ | 0.000108631647 | 0.046     |
| 20  | 0.000004074670078     | 0.000003549           | 5.25670078 $\times 10^{-7}$ | 0.1290092371 | 0.078      |
| 30  | 3.979168669 $\times 10^{-9}$ | 6.56 $\times 10^{-7}$ | 6.520208313 $\times 10^{-7}$ | 163.8585558 | 0.063     |
| 40  | 3.885060902 $\times 10^{-12}$ | 1.66 $\times 10^{-7}$ | 1.659661141 $\times 10^{-7}$ | 4.2717.47067 | 0.031     |
| 50  | 3.794830960 $\times 10^{-15}$ | 3.68 $\times 10^{-7}$ | 3.679999962 $\times 10^{-7}$ | 9.697401546 $\times 10^{-7}$ | 0.032     |
| 60  | 3.705889610 $\times 10^{-18}$ | 9.72 $\times 10^{-7}$ | 9.720000000 $\times 10^{-7}$ | 2.622852007 $\times 10^{-11}$ | 0.078     |
| 70  | 3.705889610 $\times 10^{-18}$ | 6.891 $\times 10^{-7}$ | 6.891000000 $\times 10^{-7}$ | 1.859472549 $\times 10^{-11}$ | 0.078     |
| 80  | 3.53421174 $\times 10^{-24}$ | 1.675 $\times 10^{-7}$ | 1.675 $\times 10^{-7}$ | 4.739387799 $\times 10^{-16}$ | 0.078     |
| 90  | 3.451378652 $\times 10^{-27}$ | 7.3 $\times 10^{-8}$ | 7.3 $\times 10^{-8}$ | 2.115096817 $\times 10^{-19}$ | 0.047     |
| 100 | 3.370486965 $\times 10^{-30}$ | 0.00000104 | 0.000001048 | 3.109342985 $\times 10^{-21}$ | 0.031     |

Table 3. Numerical results for Example 3.
7. **Conclusion.** In this paper, we investigated the reproducing kernel method for investigating the difference equations. For illustration purposes, we considered six examples which were selected to show the computational accuracy. It may be concluded that, the reproducing kernel method is very powerful and efficient in finding approximate solution for wide classes of problem. The approximate solution obtained by the present method is uniformly convergent. As seen in the tables the results were obtained in a very short time. Furthermore, the obtained results are very accurate. Clearly, the series solution methodology can be applied to much more complicated nonlinear difference equations.

| x  | ES          | AS          | AE | RE | CPU time(s) |
|----|-------------|-------------|----|----|-------------|
| 10 | 1.543083500 × 10^7 | 1.543083500 × 10^7 | 0.0 | 0.0 | 0.063       |
| 20 | 1.890687562 × 10^19 | 1.890687562 × 10^19 | 0.0 | 0.0 | 0.140       |
| 30 | 2.996465452 × 10^33 | 2.996465452 × 10^33 | 0.0 | 0.0 | 0.094       |
| 40 | 1.209470191 × 10^49 | 1.209470191 × 10^49 | 0.0 | 0.0 | 0.031       |
| 50 | 1.240894842 × 10^64 | 1.240894842 × 10^64 | 0.0 | 0.0 | 0.016       |
| 60 | 4.047603113 × 10^81 | 4.047603113 × 10^81 | 0.0 | 0.0 | 0.125       |
| 70 | 6.76635157 × 10^99  | 6.76635157 × 10^99  | 0.0 | 0.0 | 0.046       |
| 80 | 4.604129154 × 10^118 | 4.604129154 × 10^118 | 0.0 | 0.0 | 0.078       |
| 90 | 1.072315597 × 10^138 | 1.072315597 × 10^138 | 0.0 | 0.0 | 0.109       |
| 100| 7.467889258 × 10^157 | 7.467889258 × 10^157 | 0.0 | 0.0 | 0.047       |

**Table 4. Numerical results for Example 4.**

| x  | ES          | AS          | AE | RE | CPU time(s) |
|----|-------------|-------------|----|----|-------------|
| 50 | 1.258626873 × 10^10 | 1.258626873 × 10^10 | 0.0 | 0.0 | 2.465       |
| 100| 3.542248316 × 10^20 | 3.542248316 × 10^20 | 0.0 | 0.0 | 2.386       |
| 150| 9.969215980 × 10^30 | 9.969215980 × 10^30 | 0.0 | 0.0 | 2.512       |
| 200| 2.805711470 × 10^41 | 2.805711470 × 10^41 | 0.0 | 0.0 | 2.247       |
| 250| 7.896324908 × 10^51 | 7.896324908 × 10^51 | 0.0 | 0.0 | 2.340       |
| 300| 2.222322136 × 10^62 | 2.222322136 × 10^62 | 0.0 | 0.0 | 2.168       |
| 350| 6.254448414 × 10^72 | 6.254448414 × 10^72 | 0.0 | 0.0 | 2.247       |
| 400| 1.760236480 × 10^83 | 1.760236480 × 10^83 | 0.0 | 0.0 | 2.262       |

**Table 5. Numerical results for Example 5.**
| \( x \) | ES | AS | AE | CPU time(s) |
|------|-----|-----|----|------------|
| 10   | -1  | -0.9999995 | 5 \times 10^{-7} | 0.094 |
| 20   | 1   | 0.999992593 | 9.7407 \times 10^{-5} | 0.109 |
| 30   | 0   | 0.000006701 | 6.701 \times 10^{-7} | 0.110 |
| 40   | -1  | -0.999991867 | 8.133 \times 10^{-6} | 0.062 |
| 50   | 1   | 0.999997854 | 2.146 \times 10^{-6} | 0.063 |
| 60   | 0   | 0.000008068 | 8.068 \times 10^{-6} | 0.15 |
| 70   | -1  | -0.999983411 | 1.6589 \times 10^{-5} | 0.062 |
| 80   | 1   | 0.999969194 | 3.0806 \times 10^{-5} | 0.062 |
| 90   | 0   | 0.000008144 | 8.144 \times 10^{-6} | 0.78 |
| 100  | -1  | -0.999917201 | 8.2799 \times 10^{-5} | 0.047 |

Table 6. Numerical results for Example 6.

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