Recursive Polynomial Remainder Sequence  
and the Nested Subresultants

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Abstract. We give two new expressions of subresultants, nested subresultant and reduced nested subresultant, for the recursive polynomial remainder sequence (PRS) which has been introduced by the author. The reduced nested subresultant reduces the size of the subresultant matrix drastically compared with the recursive subresultant proposed by the authors before, hence it is much more useful for investigation of the recursive PRS. Finally, we discuss usage of the reduced nested subresultant in approximate algebraic computation, which motivates the present work.

1 Introduction

The polynomial remainder sequence (PRS) is one of fundamental tools in computer algebra. Although the Euclidean algorithm (see Knuth [1] for example) for calculating PRS is simple, coefficient growth in PRS makes the Euclidean algorithm often very inefficient. To overcome this problem, the mechanism of coefficient growth has been extensively studied through the theory of subresultants; see Collins [2], Brown and Traub [3], Loos [4], etc. By the theory of subresultant, we can remove extraneous factors of the elements of PRS systematically.

In our earlier research [5], we have introduced a variation of PRS called “recursive PRS,” and its subresultant called “recursive subresultant.” The recursive PRS is a result of repeated calculation of PRS for the GCD and its derivative of the original PRS, until the element becomes a constant. Then, the coefficients of the elements in the recursive PRS depend on the coefficients of the initial polynomials. By the recursive subresultants, we have given an expression of the coefficients of the elements in the recursive PRS in certain determinants of coefficients of the initial polynomials. However, as the recursion depth of the recursive PRS has increased, the recursive subresultant matrices have become so large that use of them have often become impractical [6].

In this paper, we give two other expressions of subresultants for the recursive PRS, called “nested subresultant” and “reduced nested subresultant.” The nested subresultant is a subresultant with expression of “nested” determinants, used to show the relationship between the recursive and the reduced nested subresultants. The reduced nested subresultant has the same form as the result of
Gaussian elimination with the Sylvester’s identity on the nested subresultant, hence it reduces the size of the subresultant matrix drastically compared with the recursive subresultant. Therefore, it is much more useful than the recursive subresultant for investigation of the recursive PRS.

This paper is organized as follows. In Sect. 2, we review the concept of the recursive PRS and the recursive subresultant. In Sect. 3, we define the nested subresultant and show its equivalence to the recursive subresultant. In Sect. 4, we define the reduced nested subresultant and show that it is a reduced expression of the nested subresultant. In Sect. 5, we discuss briefly usage of the reduced nested subresultant in approximate algebraic computation.

2 Recursive PRS and Recursive Subresultants

Let $R$ be an integral domain and $K$ be its quotient field, and polynomials $F$ and $G$ be in $R[x]$. When we calculate PRS for $F$ and $G$ which have a nontrivial GCD, we usually stop the calculation with the GCD. However, it is sometimes useful to continue the calculation by calculating the PRS for the GCD and its derivative; this is used for square-free decompositions. We call such a PRS a “recursive PRS.”

To make this paper self-contained, we briefly review the definitions and the properties of the recursive PRS and the recursive subresultant, with necessary definitions of subresultants (for detailed discussions, see Terui [5]). In this paper, we follow definitions and notations by von zur Gathen and Lücking [7].

2.1 Recursive PRS

Definition 1 (Polynomial Remainder Sequence (PRS)). Let $F$ and $G$ be polynomials in $R[x]$ of degree $m$ and $n$ ($m > n$), respectively. A sequence $(P_1, \ldots, P_l)$ of nonzero polynomials is called a polynomial remainder sequence (PRS) for $F$ and $G$, abbreviated to prs($F,G$), if it satisfies $P_1 = F$, $P_2 = G$, $\alpha_i P_{i-2} = q_{i-1} P_{i-1} + \beta_i P_i$, for $i = 3, \ldots, l$, where $\alpha_3, \ldots, \alpha_l, \beta_3, \ldots, \beta_l$ are elements of $R$ and $\deg(P_{i-1}) > \deg(P_i)$. A sequence $((\alpha_3, \beta_3), \ldots, (\alpha_l, \beta_l))$ is called a division rule for prs($F,G$). If $P_1$ is a constant, then the PRS is called complete.

Definition 2 (Recursive PRS). Let $F$ and $G$ be the same as in Definition 1. Then, a sequence $(P_1^{(1)}, \ldots, P_{l_1}^{(1)}, P_1^{(2)}, \ldots, P_{l_2}^{(2)}, \ldots, P_1^{(t)}, \ldots, P_{l_t}^{(t)})$ of nonzero polynomials is called a recursive polynomial remainder sequence (recursive PRS) for $F$ and $G$, abbreviated to rprs($F,G$), if it satisfies

\[
P_1^{(1)} = F, \quad P_2^{(1)} = G, \quad P_{i_1}^{(1)} = \gamma_1 \cdot \gcd(P_1^{(1)}, P_2^{(1)}) \quad \text{with } \gamma_1 \in R,
\]

\[
(P_1^{(1)}, P_2^{(1)}, \ldots, P_{l_1}^{(1)}) = \text{prs}(P_1^{(1)}, P_2^{(1)}),
\]

\[
P_1^{(k)} = P_{l_{k-1}}^{(k-1)}, \quad P_2^{(k)} = \frac{d}{dx} P_{l_{k-1}}^{(k-1)}, \quad P_{l_{k}}^{(k)} = \gamma_k \cdot \gcd(P_1^{(k)}, P_2^{(k)}) \quad \text{with } \gamma_k \in R, \quad (1)
\]

\[
(P_1^{(k)}, P_2^{(k)}, \ldots, P_{l_k}^{(k)}) = \text{prs}(P_1^{(k)}, P_2^{(k)}),
\]
for \( k = 2, \ldots, t \). If \( \alpha_i^{(k)}, \beta_i^{(k)} \in R \) satisfy \( \alpha_i^{(k)} P_{i-2}^{(k)} = q_{i-1}^{(k)} P_{i-1}^{(k)} + \beta_i^{(k)} P_i^{(k)} \) for \( k = 1, \ldots, t \) and \( i = 3, \ldots, l_k \), then a sequence \( (\alpha_3^{(1)}, \beta_3^{(1)}), \ldots, (\alpha_{l_t}^{(t)}, \beta_{l_t}^{(t)}) \) is called a division rule for \( \text{rprs}(F, G) \). Furthermore, if \( P_{l_t}^{(t)} \) is a constant, then the recursive PRS is called complete. \( \square \)

In this paper, we use the following notations. Let \( c_i^{(k)} = \text{lc}(P_i^{(k)}) \), \( n_i^{(k)} = \deg(P_i^{(k)}) \), \( f_0 = m \) and \( j_k = n_i^{(k)} \) for \( k = 1, \ldots, t \) and \( i = 1, \ldots, l_k \), and let \( d_i^{(k)} = n_i^{(k)} - n_{i+1}^{(k)} \) for \( k = 1, \ldots, t \) and \( i = 1, \ldots, l_k - 1 \).

2.2 Recursive Subresultants

We construct “recursive subresultant matrix” whose determinants represent the elements of the recursive PRS by the coefficients of the initial polynomials.

Let \( F \) and \( G \) be polynomials in \( R[x] \) such that

\[
F(x) = f_mx^m + \cdots + f_0x^0, \quad G(x) = g_nx^n + \cdots + g_0x^0, \tag{2}
\]

with \( m \geq n > 0 \). For a square matrix \( M \), we denote its determinant by \( |M| \).

**Definition 3 (Sylvester Matrix and Subresultant Matrix).** Let \( F \) and \( G \) be as in (2). The Sylvester matrix of \( F \) and \( G \), denoted by \( \text{N}(F, G) \) in (3), is an \((m + n) \times (m + n)\) matrix constructed from the coefficients of \( F \) and \( G \). For \( j < n \), the \( j \)-th subresultant matrix of \( F \) and \( G \), denoted by \( \text{N}^{(j)}(F, G) \) in (3), is an \((m + n - j) \times (m + n - 2j)\) sub-matrix of \( \text{N}(F, G) \) obtained by taking the left \( n - j \) columns of coefficients of \( F \) and the left \( m - j \) columns of coefficients of \( G \).

\[
\text{N}(F, G) = \begin{pmatrix}
f_m & g_n & \cdots & \cdots & \cdots \\
f_{m-1} & f_m & g_{n-1} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
f_1 & f_2 & \cdots & f_m & g_0 \\
0 & f_1 & \cdots & f_m & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
f_0 & \cdots & \cdots & \cdots & f_m \\
\end{pmatrix}, \quad \text{N}^{(j)}(F, G) = \begin{pmatrix}
f_m & g_n & \cdots & \cdots & \cdots \\
f_{m-1} & f_m & g_{n-1} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
f_1 & f_2 & \cdots & f_m & g_{n-j} \\
0 & f_1 & \cdots & f_m & g_{n-2j} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
f_0 & \cdots & \cdots & \cdots & f_m \\
\end{pmatrix}. \tag{3}
\]

**Definition 4 (Recursive Subresultant Matrix).** Let \( F \) and \( G \) be defined as in (2), and let \( (P_1^{(1)}, \ldots, P_t^{(1)}, \ldots, P_1^{(t)}, \ldots, P_t^{(t)}) \) be complete recursive PRS for \( F \) and \( G \) as in Definition 2. Then, for each tuple of numbers \( (k, j) \) with \( k = 1, \ldots, t \) and \( j = j_k - 2, \ldots, 0 \), define matrix \( \text{N}^{(k,j)} = \hat{\text{N}}^{(k,j)}(F, G) \) recursively as follows.

1. For \( k = 1 \), let \( \hat{\text{N}}^{(1,j)}(F, G) = \text{N}^{(j)}(F, G) \).
2. For \( k > 1 \), let \( \hat{\text{N}}^{(k,j)}(F, G) \) consist of the upper block and the lower block, defined as follows:
   (a) The upper block is partitioned into \((j_{k-1} - j_k - 1) \times (j_{k-1} - j_k - 1)\) blocks with diagonal blocks filled with \( \hat{\text{N}}_{U}^{(k-1,j_k-1)} \), where \( \hat{\text{N}}_{U}^{(k-1,j_k-1)} \) is a sub-matrix of \( \hat{\text{N}}^{(k-1,j_{k-1})}(F, G) \) obtained by deleting the bottom \( j_k + 1 \) rows.
(b) Let $\bar{N}^{(k-1,j_k-1)}_L$ be a sub-matrix of $\bar{N}^{(k-1,j_k-1)}_L$ obtained by taking the bottom $j_k-1$ rows, and let $\bar{N}^{(k-1,j_k-1)}_R$ be a sub-matrix of $\bar{N}^{(k-1,j_k-1)}_L$ by multiplying the $(j_k-1 + \tau)$-th rows by $\tau$ for $\tau = j_k-1, \ldots, 1$, then by deleting the bottom row. Then, the lower block consists of $j_k-1 - j$ blocks of $\bar{N}^{(k-1,j_k-1)}_L$ such that the leftmost block is placed at the top row of the container block and the right-side block is placed down by 1 row from the left-side block, then followed by $j_k-1 - j$ blocks of $\bar{N}^{(k-1,j_k-1)}_L$ placed by the same manner as $\bar{N}^{(k-1,j_k-1)}_L$.

Readers can find the structures of $\bar{N}^{(k,j)}(F,G)$ in the figures in Terui [5]. Then, $\bar{N}^{(k,j)}(F,G)$ is called the $(k,j)$-th recursive subresultant matrix of $F$ and $G$.

**Proposition 1.** For $k = 1, \ldots, t$ and $j < j_{k-1} - 1$, the numbers of rows and columns of $\bar{N}^{(k,j)}(F,G)$, the $(k,j)$-th recursive subresultant matrix of $F$ and $G$ are $(m + n - 2j_1)\left\{\prod_{l=2}^{k-1}(2j_{l-1} - 2j_l - 1)\right\}(2j_{k-1} - 2j - 1) + j$, $(m + n - 2j_1)\left\{\prod_{l=2}^{k-1}(2j_{l-1} - 2j_l - 1)\right\}(2j_{k-1} - 2j - 1)$, respectively.

**Definition 5 (Recursive Subresultant).** Let $F$ and $G$ be defined as in (2), and let $(P_1^{(1)}, \ldots, P_l^{(1)}, \ldots, P_i^{(t)}, \ldots, P_t^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition 4. For $j = j_{k-1} - 2, \ldots, 0$ and $\tau = j, \ldots, 0$, let $\bar{N}^{(k,j)}_\tau(F,G)$ be a sub-matrix of the $(k,j)$-th recursive subresultant matrix $\bar{N}^{(k,j)}(F,G)$ obtained by taking the top $(m + n - 2j_1)\left\{\prod_{l=2}^{k-1}(2j_{l-1} - 2j_l - 1)\right\}(2j_{k-1} - 2j - 1) - 1$ rows and the $((m + n - 2j_1)\left\{\prod_{l=2}^{k-1}(2j_{l-1} - 2j_l - 1)\right\}(2j_{k-1} - 2j - 1) + j - \tau)$-th row (note that $\bar{N}^{(k,j)}_\tau$ is a square matrix). Then, the polynomial $\bar{S}_{k,j}(F,G) = |\bar{N}^{(k,j)}_0| x^j + \cdots + |\bar{N}^{(k,j)}_0| x^0$ is called the $(k,j)$-th recursive subresultant of $F$ and $G$.

**3 Nested Subresultants**

Although the recursive subresultant can represent the coefficients of the elements in recursive PRS, the size of the recursive subresultant matrix becomes larger rapidly as the recursion depth of the recursive PRS becomes deeper, hence making use of the recursive subresultant matrix become more inefficient.

To overcome this problem, we introduce other representations for the subresultant which is equivalent to the recursive subresultant up to a constant, and more efficient to calculate. The nested subresultant matrix is a subresultant matrix whose elements are again determinants of certain subresultant matrices (or even the nested subresultant matrices), and the nested subresultant is a subresultant whose coefficients are determinants of the nested subresultant matrices.

In this paper, the nested subresultant is mainly used to show the relationship between the recursive subresultant and the reduced nested subresultant, which is defined in the next section.
Definition 6 (Nested Subresultant Matrix). Let $F$ and $G$ be defined as in (2), and let $(P_1^{(1)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition (2). Then, for each tuple of numbers $(k, j)$ with $k = 1, \ldots, t$ and $j = j_k - 1 - 2, \ldots, 0$, define matrix $\tilde{N}^{(k,j)}(F, G)$ recursively as follows.

1. For $k = 1$, let $\tilde{N}^{(1,j)}(F, G) = N^{(j)}(F, G)$.

2. For $k > 1$ and $\tau = 0, \ldots, j_k - 1$, let $\tilde{N}^{(k-1,j_k-1)}(\tau)$ be a sub-matrix of $N^{(k-1,j_k-1)}(\tau)$ by taking the top $(n_1^{(k-1)} + n_2^{(k-1)} - 2j_k - 1)$ rows and the $(n_1^{(k-1)} + n_2^{(k-1)} - j_k - \tau)$-th row (note that $N^{(k-1,j_k-1)}$ is a square matrix). Now, let

$$\tilde{N}^{(k,j)}(F, G) = N^{(j)}\left(\tilde{S}_{k-1,j_k-1}(F, G), \frac{d}{dx}\tilde{S}_{k-1,j_k-1}(F, G)\right),$$

where $\tilde{S}_{k-1,j_k-1}(F, G)$ is defined by Definition (2). Then, $\tilde{N}^{(k,j)}(F, G)$ is called the $(k, j)$-th nested subresultant matrix of $F$ and $G$.

Definition 7 (Nested Subresultant). Let $F$ and $G$ be defined as in (2), and let $(P_1^{(1)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition (2). For $j = j_k - 2, \ldots, 0$ and $\tau = j, \ldots, 0$, let $\tilde{N}^{(k,j)}(F, G)$ be a sub-matrix of the $(k, j)$-th nested recursive subresultant matrix $N^{(k,j)}(F, G)$ obtained by taking the top $n_1^{(k)} + n_2^{(k)} - 2j - 1$ rows and the $(n_1^{(k)} + n_2^{(k)} - j - \tau)$-th row (note that $N^{(k,j)}$ is a square matrix). Then, the polynomial $\tilde{S}_{k,j}(F, G) = |\tilde{N}_0^{(k,j)}| x^0 + \cdots + |\tilde{N}_j^{(k,j)}| x^j$ is called the $(k, j)$-th nested subresultant of $F$ and $G$.

We show that the nested subresultant is equal to the recursive subresultant up to a sign.

Theorem 1. Let $F$ and $G$ be defined as in (2), and let $(P_1^{(1)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition (2). For $k = 2, \ldots, t$ and $j = j_k - 1 - 2, \ldots, 0$, define $u_{k,j}$, $b_{k,j}$, $r_{k,j}$, and $R_k$ as follows: let

$$u_{k,j} = (m + n - 2j_1) \left\{ \prod_{l=1}^{k-1} (2j_l - 2j_l - 1) \right\} (2j_k - 1 - 2j_1 - 1)$$

with $u_k = u_{k,j_k}$ and $u_1 = m + n - 2j_1$, $b_{k,j} = 2j_k - 2j_1 - 1$ with $b_k = b_{k,j_k}$ and $b_1 = 1$, $r_{k,j} = (-1)^{u_k-1} (1 + 2 + \cdots + (b_{k,j} - 1))$ with $r_k = r_{k,j_k}$ and $r_1 = 1$ for $j < n$, and $R_k = (R_k-1)^b_{k,j} r_k$ with $R_0 = R_1 = 1$. Then, we have

$$\tilde{S}_{k,j}(F, G) = (R_k-1)^{b_{k,j}} r_k \tilde{S}_{k,j}(F, G).$$

To prove Theorem 1 we prove the following lemma.

Lemma 1. For $k = 1, \ldots, t$, $j = j_k - 1 - 2, \ldots, 0$ and $\tau = j, \ldots, 0$, we have

$$|\tilde{N}^{(k,j)}(F, G)| = (R_k-1)^{b_{k,j}} r_k |\tilde{N}^{(k,j)}(F, G)|.$$

Proof. By induction on $k$. For $k = 1$, it is obvious by the definitions of the recursive and the nested subresultants. Assume that the lemma is valid for $1, \ldots, k-1$. 


Then, for \( \tau = j_{k-1}, \ldots , 0 \), we have \( \tilde{N}_{\tau}^{(k-1,j_{k-1})} = R_{k-1}[\tilde{N}_{\tau}^{(k-1,j_{k-1})}] \). For an element in recursive PRS \( P_i^{(k)}(x) \), expressed as \( P_i^{(k)}(x) = a_{i,n_1}^{(k)}x^{n_1^{(k)}} + \cdots + a_{i,0}^{(k)}x^0 \), denote the coefficient vector for \( P_i^{(k)}(x) \) by \( \bar{p}_i^{(k)} = (a_{i,n_1}^{(k)}, \ldots , a_{i,0}^{(k)}) \). Then, there exist certain eliminations and exchanges on columns which transform \( \tilde{N}_{\tau}^{(k-1,j_{k-1})} \) to \( \tilde{M}_{\tau}^{(k-1,j_{k-1})} \) by an example.

To illustrate the idea of reduction of the nested subresultant matrix with the Sylvester’s identity to “flat” representation, or a representation without nested determinants; the nested subresultant matrix has “nested” representation of subresultant matrices, which makes practical use difficult. However, in some cases, by Gaussian elimination of the matrix with the Sylvester’s identity after some precomputations, we can reduce the representation of the nested subresultant matrix to “flat” representation, or a representation without nested determinants; this is the reduced nested subresultant (matrix). As we will see, the size of the reduced nested subresultant matrix becomes much smaller than that of the recursive subresultant matrix.

First, we show the Sylvester’s identity (see also Bariess [8]), then explain the idea of reduction of the nested subresultant matrix with the Sylvester’s identity by an example.

4 Reduced Nested Subresultants

The nested subresultant matrix has “nested” representation of subresultant matrices, which makes practical use difficult. However, in some cases, by Gaussian elimination of the matrix with the Sylvester’s identity after some precomputations, we can reduce the representation of the nested subresultant matrix to “flat” representation, or a representation without nested determinants; this is the reduced nested subresultant (matrix). As we will see, the size of the reduced nested subresultant matrix becomes much smaller than that of the recursive subresultant matrix.

First, we show the Sylvester’s identity (see also Bariess [8]), then explain the idea of reduction of the nested subresultant matrix with the Sylvester’s identity by an example.
Lemma 2 (The Sylvester’s Identity). Let $A = (a_{ij})$ be $n \times n$ matrix, and, for $k = 1, \ldots, n-1$, $i = k+1, \ldots, n$ and $j = k+1, \ldots, n$, let $a^{(k)}_{i,j} = \begin{bmatrix} a_{11} & \cdots & a_{1k} & a_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{kj} \\ a_{11} & \cdots & a_{ik} & a_{ij} \end{bmatrix}$. Then, we have $|A| \left( a^{(k-1)}_{kk} \right)^{n-k-1} = \begin{bmatrix} a_{k+1,k+1} & \cdots & a_{k+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,k+1} & \cdots & a_{n,n} \end{bmatrix}$. □

Example 1. Let $F(x)$ and $G(x)$ be defined as

$$F(x) = a_6 x^6 + a_5 x^5 + \cdots + a_0, \quad G(x) = b_5 x^5 + b_4 x^4 + \cdots + b_0,$$

with $a_6 \neq 0$ and $b_5 \neq 0$. We assume that vectors of coefficients $(a_6, a_5)$ and $(b_5, b_4)$ are linearly independent as vectors over $K$, and that $\text{prs}(F,G) = (P^{(1)}_1 = F, P^{(1)}_2 = G, P^{(1)}_3 = \gcd(F,G))$ with $\deg(P^{(1)}_3) = 4$. Consider the $(2,2)$-th nested subresultant; its matrix is defined as

$$\tilde{N}^{(2,2)} = \begin{pmatrix} A_4 & 4A_4 \\ A_3 & 3A_3 \quad 4A_4 \\ A_2 & 2A_2 \quad 3A_3 \\ A_1 & A_2 \quad 2A_2 \quad 3A_3 \quad A_1 \end{pmatrix}, \quad A_j = \begin{bmatrix} a_6 & b_5 \\ a_5 & b_4 & b_5 \\ a_4 & b_3 & b_4 & b_5 \\ a_3 & b_2 & b_3 & b_4 & b_5 \end{bmatrix}, \quad (10)$$

for $j \leq 4$ with $b_j = 0$ for $j < 0$. Now, let us calculate the leading coefficient of $\tilde{S}_{2,2}(F,G)$ as

$$|\tilde{N}^{(2,2)}_2| = \begin{vmatrix} A_4 & 4A_4 \\ A_3 & 3A_3 \quad 4A_4 \\ A_2 & 2A_2 \quad 3A_3 \\ A_1 \end{vmatrix} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 & b_5 \\ a_4 & b_3 & b_4 & b_5 \\ a_3 & b_2 & b_3 & b_4 & b_5 \end{vmatrix} = |H| = |(H_{p,q})|,$$

(11)

To apply the Sylvester’s identity on $H$, we make the $(3,1)$ and the $(3,2)$ elements in $H_{p,2}$ and $H_{p,3}$ ($p = 1, 2, 3$) equal to those elements in $H_{p,1}$, respectively, by adding the first and the second rows, multiplied by certain numbers, to the third row. For example, in $H_{1,2}$, calculate $x_{12}$ and $y_{12}$ by solving a system of linear equations

$$\begin{cases} a_6 x_{12} + a_5 y_{12} = -4a_4 + a_4 = -3a_4 \\ b_5 x_{12} + b_4 y_{12} = -4b_3 + b_3 = -3b_3 \end{cases}, \quad (12)$$

(Note that $\mathbf{12}$ has a solution in $K$ by assumption), then add the first row multiplied by $x_{12}$ and the second row multiplied by $y_{12}$, respectively, to the
third row. Then, we have \( H_{1,2} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \\ a_4 & h_{12} \end{vmatrix} \) with \( h_{12} = 4b_4 + y_{12}b_5 \). Doing similar calculations for the other \( H_{p,q} \), we calculate \( h_{p,q} \) for \( H_{p,q} \) similarly as in the above. Finally, by the Sylvester’s identity, we have

\[
\begin{vmatrix} \hat{N}_2^{(2,2)} \end{vmatrix} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \\ a_4 & h_{12} \end{vmatrix} \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \\ a_3 & b_3 \end{vmatrix} \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \\ a_2 & b_1 \end{vmatrix} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \end{vmatrix} \begin{vmatrix} \hat{N}_2^{(2,2)} \end{vmatrix},
\]

(13)

note that we have derived \( \hat{N}_2^{(2,2)} \) as a reduced form of \( \tilde{N}_2^{(2,2)} \).

**Definition 8 (Reduced Nested Subresultant Matrix).** Let \( F \) and \( G \) be defined as in Definition 2, and let \( F_i^{(t)}, \ldots, P_1^{(t)} \) be complete recursive PRS for \( F \) and \( G \) as in Definition 4. Then, for each tuple of numbers \((k, j)\) with \( k = 1, \ldots, t \) and \( j = j_{k-1} - 2, \ldots, 0 \), define matrix \( \hat{N}^{(k,j)}(F,G) \) recursively as follows.

1. For \( k = 1 \), let \( \hat{N}^{(1,j)}(F,G) = \hat{N}^{(j)}(F,G) \).
2. For \( k > 1 \), let \( \hat{N}^{(k-1,j_{k-1})}_U(F,G) \) be a sub-matrix of \( \hat{N}^{(k-1,j_{k-1})}(F,G) \) by deleting the bottom \( j_{k-1} + 1 \) rows, and \( \hat{N}^{(k-1,j_{k-1})}_L(F,G) \) be a sub-matrix of \( \hat{N}^{(k-1,j_{k-1})}(F,G) \) by taking the bottom \( j_{k-1} + 1 \) rows, respectively. For \( \tau = j_{k-1}, \ldots, 0 \) let \( \hat{N}^{(k-1,j_{k-1})}_U(F,G) \) be a sub-matrix of \( \hat{N}^{(k-1,j_{k-1})}(F,G) \) by putting \( \hat{N}^{(k-1,j_{k-1})}_U(F,G) \) on the top and the \( (j_{k-1} - \tau + 1) \)-th row of \( \hat{N}^{(k-1,j_{k-1})}_L(F,G) \) in the bottom row. Let \( \hat{A}^{(k-1)} = |\hat{N}^{(k-1,j_{k-1})}| \) and construct a matrix \( H \) as

\[
H = (H_{p,q}) = \hat{N}^{(j)} \left( \hat{A}^{(k-1)}(x), \frac{d}{dx} \hat{A}^{(k-1)}(x) \right),
\]

(14)

where \( \hat{A}^{(k-1)}(x) = \hat{A}^{(k-1)}_0 x_{j_{k-1}} + \cdots + \hat{A}^{(k-1)}_0 x^0 \). Since \( \hat{N}^{(k-1,j_{k-1})}_U \) and a row vector in the bottom, we express \( \hat{N}^{(k-1,j_{k-1})}_U = (U^{(k)})^{(k)} \), where \( U^{(k)} \) is a square matrix and \( v^{(k)} \) is a column vector, and the row vector by \( (b_{p,q})^{(k)} \) and \( (g_{p,q})^{(k)} \), where \( b_{p,q}^{(k)} \) is a row vector and \( g_{p,q}^{(k)} \) is a number, respectively, such that

\[
H_{p,q} = \begin{vmatrix} U^{(k)} \\ b_{p,q}^{(k)} \\ g_{p,q}^{(k)} \end{vmatrix},
\]

(15)

with \( b_{p,q}^{(k)} = 0 \) and \( g_{p,q}^{(k)} = 0 \) for \( H_{p,q} = 0 \). Furthermore, we assume that \( U^{(k)} \) is not singular. Then, for \( p = 1, \ldots, n_1^{(k)} + n_2^{(k)} - j \) and \( q = 2, \ldots, n_1^{(k)} + n_2^{(k)} - j \), calculate a row vector \( x^{(k)}_{p,q} \) as a solution of the equation \( x^{(k)}_{p,q} U^{(k)} = b_{p,1}^{(k)} \), and
define \( h_{p,q}^{(k)} \) as \( h_{p,q}^{(k)} = x_{p,q}^{(k)} v^{(k)} \). Finally, define \( \hat{N}^{(k,j)}(F,G) \) as

\[
\hat{N}^{(k,j)}(F,G) = \begin{pmatrix}
U^{(k)} & v^{(k)} & \cdots & v^{(k)} \\
\hat{b}_{1,1}^{(k)} & g_{1,1}^{(k)} & h_{1,1,j}^{(k)} & \cdots & h_{1,j,k}^{(k)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{b}_{t,1}^{(k)} & g_{t,1}^{(k)} & h_{t,1,j}^{(k)} & \cdots & h_{t,j,k}^{(k)} \\
\end{pmatrix},
\]

(16)

By induction on \( k \), . . . , \( k \) Proposition 2. Then, \( \hat{N}^{(k,j)}(F,G) \) is called the \((k,j)\)-th reduced nested subresultant matrix of \( F \) and \( G \).

Proposition 2. For \( k = 1, \ldots, t \) and \( j < j_{k-1} - 1 \), the numbers of rows and columns of the \((k,j)\)-th reduced nested subresultant matrix \( \hat{N}^{(k,j)}(F,G) \) are \((m+n+2(k-1)-2j)\) and \((m+n-(2k-1)-2j)\) respectively.

Proof. By induction on \( k \). It is obvious for \( k = 1 \). Assume that the proposition is valid for \( 1, \ldots, k-1 \). Then, the numbers of rows and columns of matrix \( U^{(k)} \) in (10) are equal to \((m+n+2(k-1)-2j_{k-1}-1)\) respectively. Therefore, by (16) and (17), we prove the proposition for \( k \).

Note that, as Proposition 2 shows, the size of the reduced nested subresultant matrix, which is at most the sum of the degree of the initial polynomials, is much smaller than that of the recursive subresultant matrix (see Proposition 1).

Definition 9 (Reduced Nested Subresultant). Let \( F \) and \( G \) be defined as in (2), and let \( (P_1^{(1)}, \ldots, P_t^{(1)}, \ldots, P_1^{(t)} \ldots, P_t^{(t)}) \) be complete recursive PRS for \( F \) and \( G \) as defined in (3). For \( j = j_{k-1} - 2, \ldots, 0 \) and \( \tau = j, \ldots, 0 \), let \( \hat{N}^{(k,j)}_\tau = \hat{N}^{(k,j)}_\tau(F,G) \) be a sub-matrix of the \((k,j)\)-th reduced nested subresultant matrix \( \hat{N}^{(k,j)}(F,G) \) obtained by the top \( m+n-2(k-1)-2j-1 \) rows and the \((m+n-2(k-1)-2j+\tau)\)-th row (note that \( \hat{N}^{(k,j)}_\tau(F,G) \) is a square matrix). Then, the polynomial \( \hat{S}^{(k,j)}_\tau(F,G) = |\hat{N}^{(k,j)}_\tau(F,G)|x^j + \cdots + |\hat{N}^{(k,j)}_0(F,G)|x^0 \) is called the \((k,j)\)-th reduced nested subresultant of \( F \) and \( G \).

Now, we derive the relationship between the nested and the reduced nested subresultants.

Theorem 2. Let \( F \) and \( G \) be defined as in (2), and let \( (P_1^{(1)}, \ldots, P_t^{(1)}, \ldots, P_1^{(t)} \ldots, P_t^{(t)}) \) be complete recursive PRS for \( F \) and \( G \) as defined in (3). For \( k = 2, \ldots, t \), \( j = j_{k-1} - 2, \ldots, 0 \) and \( k,j \) as defined in (10), define \( \hat{B}_{k,j} \) and \( \hat{R}_k \) as \( \hat{B}_{k,j} = |U^{(k)}|^{j_{k-1}} \) with \( \hat{B}_k = \hat{B}_{k,j} \) and \( \hat{B}_1 = \hat{B}_2 = 1 \), and \( \hat{R}_k = (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{j_{k-1}} \) with \( \hat{R}_1 = \hat{R}_2 = 1 \), respectively. Then, we have

\[
\hat{S}^{(k,j)}(F,G) = (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{j_{k-1}} \hat{B}_{k,j} \cdot \hat{S}^{(k,j)}(F,G).
\]

(18)
To prove Theorem 2, we prove the following lemma.

**Lemma 3.** For \( k = 1, \ldots, t, \ j = j_{k-1} - 2, \ldots, 0 \) and \( \tau = j, \ldots, 0 \), we have
\[
|\tilde{N}^{(k,j)}_\tau(F, G)| = (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{j_{k-1}, \hat{B}_{k,j}} |\tilde{N}^{(k,j)}_\tau(F, G)|.
\]

**Proof.** By induction on \( k \). For \( k = 1 \), it is obvious from the definitions of the nested and the reduced nested subresultants. Assume that the lemma is valid for \( 1, \ldots, k-1 \). Then, for \( \tau = j_{k-1}, \ldots, 0 \), we have
\[
|\tilde{N}^{(k-1,j_{k-1})}_\tau(F, G)| = (\hat{R}_{k-2} \cdot \hat{B}_{k-2})^{j_{k-1},j_{k-1}} \hat{B}_{k-1,j_{k-1}} |\tilde{N}^{(k-1,j_{k-1})}_\tau(F, G)|
= (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{j_{k-1},j_{k-1}} |\tilde{N}^{(k-1,j_{k-1})}_\tau(F, G)|.
\]
Let \( \tilde{A}^{(k-1)}_l = |\tilde{N}^{(k-1,j_{k-1})}_\tau| \) and \( \hat{A}^{(k-1)}_l = |\tilde{N}^{(k-1,j_{k-1})}_\tau| \). Then, by the definition of the \((k,j)\)-th nested subresultant, we have
\[
|\tilde{N}^{(k,j)}_\tau| = |\hat{A}^{(k-1)}_{j_{k-1}} \hat{A}^{(k-1)}_{j_{k-1}}| \hat{A}^{(k-1)}_{j_{k-1}} |\tilde{N}^{(k,j)}_\tau| = (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{j_{k-1}, \hat{B}_{k,j}} |H'|,
\]
where \( \hat{A}^{(k-1)}_l = 0 \) for \( l < 0 \) and \( H' = (H'_{p,q}) \) is defined as \( H'_{p,q} \) with \( \tilde{A}^{(k-1)}_l \) replaced by \( \hat{A}^{(k-1)}_l \) (note that \( \tilde{N}^{(k,j)}_\tau \)) and \( H' \) are square matrices of order \( J_{k,j} \). Then, by Definition 8, we can express \( H'_{p,q} = |U^{(k)}_{p,q} | b^{(k)}_{p,q} | \) with \( b^{(k)}_{p,q} = 0 \) and \( g^{(k)}_{p,q} = 0 \) for \( H'_{p,q} = 0 \). Note that, for \( p = 1, \ldots, J_{k,j} \), we have \( b^{(k)}_{p,q} = b^{(k)}_{p,q} \) and \( g^{(k)}_{p,q} = g^{(k)}_{p,q} \) for \( p = 1, \ldots, J_{k,j} \), and \( b^{(k)}_{p,q} = b^{(k)}_{p,q} \) and \( g^{(k)}_{p,q} = g^{(k)}_{p,q} \) where \( b^{(k)}_{p,q} \) and \( g^{(k)}_{p,q} \) are defined in (15), respectively. Furthermore, by the definition of \( \tilde{N}^{(k)}_\tau \) in Definition 8, we have
\[
|H'| = \begin{vmatrix}
U^{(k)}_{1} & v^{(k)}_{1} & \cdots & v^{(k)}_{1, J_{k,j}} \\
\tilde{b}^{(k)}_{1, J_{k,j} - 1, 1} & h^{(k)}_{1, J_{k,j} - 1, 1} & \cdots & h^{(k)}_{1, J_{k,j} - 1, J_{k,j}} \\
\tilde{b}^{(k)}_{j_{k,j} - 1, 1} & h^{(k)}_{j_{k,j} - 1, 1} & \cdots & h^{(k)}_{j_{k,j} - 1, J_{k,j}} \\
\tilde{b}^{(k)}_{J_{k,j} - 1, 1} & h^{(k)}_{J_{k,j} - 1, 1} & \cdots & h^{(k)}_{J_{k,j} - 1, J_{k,j}} \\
\end{vmatrix}.
\]
By Lemma 2, we have
\[
|H'| = |U^{(k)}_{1} | h^{(k)}_{1, J_{k,j} - 1} |\tilde{N}^{(k,j)}_\tau(F, G)| = \hat{B}_{k,j} |\tilde{N}^{(k,j)}_\tau(F, G)|,
\]
hence, by putting (24) into (22), we prove the lemma. □

Remark 1. We can estimate arithmetic computing time for the \((k, j)\)-th reduced nested resultant matrix \(\hat{N}^{(k,j)}\) in (16), as follows. The computing time for the elements \(h_{p,q}\) is dominated by the time for Gaussian elimination of \(U^{(k)}\). Since the order of \(U^{(k)}\) is equal to \(m + n - 2(k - 2) - 2j_{k-1}\) (see Proposition 2), it is bounded by \(O((m + n - 2(k - 2) - 2j_{k-1})^3)\), or \(O((m + n)^3)\) (see Golub and van Loan [9] for example). We can calculate \(\hat{N}^{(k,j)}(F,G)\) for \(j < j_{k-1} - 2\) by \(\hat{N}^{(k,0)}(F,G)\), hence the total computing time for \(\hat{N}^{(k,j)}\) for the entire recursive PRS \((k = 1, \ldots, t)\) is bounded by \(O(t(m + n)^3)\) (see also for the conclusion). □

5 Conclusion and Motivation

In this paper, we have given two new expressions of subresultants for the recursive PRS, the nested subresultant and the reduced nested subresultant. We have shown that the reduced nested subresultant matrix reduces the size of the matrix drastically to at most the sum of the degree of the initial polynomials compared with the recursive subresultant matrix. We have also shown that we can calculate the reduced nested subresultant matrix by solving certain systems of linear equations of order at most the sum of the degree of the initial polynomials.

A main limitation of the reduced nested subresultant in this paper is that we cannot calculate its matrix in the case the matrix \(U^{(k)}\) in (15) is singular. We need to develop a method to calculate the reduced nested subresultant matrix in the case such that \(U^{(k)}\) is singular in general.

From a point of view of computational complexity, the algorithm for the reduced nested subresultant matrix has a cubic complexity bound in terms of the degree of the input polynomials (see Remark 1). However, subresultant algorithms which have a quadratic complexity bound in terms of the degree of the input polynomials have been proposed (10, 11); the algorithms exploit the structure of the Sylvester matrix to increase their efficiency with controlling the size of coefficients well. Although, in this paper, we have primarily focused our attention into reducing the structure of the nested subresultant matrix to “flat” representation, development of more efficient algorithm such as exploiting the structure of the Sylvester matrix would be the next problem. Furthermore, the reduced nested subresultant may involve fractions which may be unusual for subresultants, hence more detailed analysis of computational efficiency including comparison with (ordinary and recursive) subresultants would also be necessary.

We expect that the reduced nested subresultants can be used for including approximate algebraic computation, especially for the square-free decomposition of approximate univariate polynomials with approximate GCD computations based on Singular Value Decomposition (SVD) of subresultant matrices (12, 13), which motivates the present work. We can calculate approximate square-free decomposition of the given polynomial \(P(x)\) by several methods including calculation of the approximate GCDs of \(P(x), \ldots, P^{(n)}(x)\) (by \(P^{(n)}(x)\) we denote the \(n\)-th derivative of \(P(x)\)) or those of the recursive PRS for \(P(x)\) and
$P'(x)$; as for these methods, we have to find the representation of the subresultant matrices for $P(x), \ldots, P^{(n)}(x)$, or that for the recursive PRS for $P(x)$ and $P'(x)$, respectively. While several algorithms based on different representation of subresultant matrices have been proposed ([14] [15]) for the former approach, we expect that our reduced nested subresultant matrix can be used for the latter approach. To make use of the reduced nested subresultant matrix, we need to reveal the relationship between the structure of the subresultant matrices and their singular values; this is the problem on which we are working now.

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