A passive CPHASE gate via cross-Kerr nonlinearities

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A fundamental and open question is whether cross-Kerr nonlinearities can be used to construct a controlled-phase (CPHASE) gate. Here we propose a gate constructed from a discrete set of atom-mediated cross-Kerr interaction sites with counter-propagating photons. We show that the average gate fidelity \( F \) between a CPHASE and our proposed gate increases as the number of interaction sites increases and the spectral width of the photon decreases, e.g. with 12 sites we find \( F > 99\% \).

Photons are attractive in quantum information processing as flying qubits and as a quantum computing platform. To realize the full benefits of quantum photonic applications, a nonlinearity or photon-photon interaction is usually required. However, photons only interact in contrived situations [1], thus most interactions between photons are effective, i.e. mediated by matter. For optical quantum computing, in a dual rail encoding, a natural entangling gate is the controlled-phase (CPHASE) gate [2, 3]. Unfortunately, the photon-photon interactions required for a CPHASE gate are hard to engineer. Thus, much of the progress in the field of optical quantum computing has focused on the KLM scheme [4] or measurement-based quantum computing [5–7], which circumvent these issues by use of nondeterministic measurement-induced nonlinearities.

Cross-Kerr interactions have been suggested as a route to a deterministic Fredkin gate by Milburn [8] and a CPHASE gate by Chuang and Yamamoto [9]. These proposals have received less attention than linear-optical schemes due to two obstacles. First, bulk cross-Kerr nonlinearities have historically been very small [10]. However, experiments in cavity-QED [11], circuit-QED [12], and ensemble systems [13], have already demonstrated large cross phase shifts of order one radian per photon.

Second, single-mode analyses fail to account for multimode effects that preclude a high-fidelity CPHASE gate, as pointed out by Shapiro [14] and Gea-Banacloche [15]. In principle, a CPHASE gate could be implemented by a frequency-local interaction, i.e., with a Hamiltonian proportional to \( a^\dagger(\omega)a(\omega)b^\dagger(\omega)b(\omega) \). However, physically-realistic cross-Kerr effects are spatially localized, e.g., \( a^\dagger(x)a(x)b^\dagger(x)b(x) \), since they must be mediated by atoms. This creates a tension between the spectral width of the quanta and the response time of the Kerr medium. If two temporally broad (spectrally narrow) photons impinge on the medium, they are likely to both be absorbed by the atoms, but not at the same interaction site, so no interaction occurs. When temporally narrow (spectrally broad) photons impinge on the medium the atoms cannot absorb the photons before they leave the interaction site, and again no interaction occurs. Shapiro [14] arrives at similar conclusions, via a phenomenological model of the cross-Kerr interaction, that includes a fidelity-degrading phase-noise [16] term. In an intermediate regime, a more fundamental problem with spatially-local interactions is that they generate spectral entanglement [15], e.g. when different frequencies gather different cross-phase shifts, or there is frequency mixing. As a consequence of these arguments, it has become folklore that the multi-mode nature of photons is a fundamental obstacle for constructing a CPHASE gate from Kerr nonlinearities, even in absence of other imperfections.

Here we provide a counter-example to this claim, by constructing a high-fidelity CPHASE gate using photons that counter-propagate through \( N \) atom-mediated cross-Kerr interaction sites. In particular, as \( N \) increases and the spectral width of the photons decreases, our proposal tends to a perfect CPHASE gate. Furthermore, since we do not rely on any phenomenology, our results unambiguously show that the multimode nature of the field is not a fundamental obstacle to quantum computation.

There are other proposals for CPHASE gates based on atom-mediated interactions, see Refs. [17–21]. Our proposal was motivated by Ref. [22], where a CPHASE gate was built by a random walk of counter-propagating qubit waves. Counter-propagating photonic wave packets, with interactions mediated by Rydberg atoms or atomic vourps, were investigated in Refs. [23–26]. Our work improves on previous proposals in two ways. First, our construction requires no active elements, such as error correction, control pulses, switches, or memories. Second, high fidelities (\( F > 99\% \)) are obtainable with relatively few interaction sites (\( N = 12 \)).

Our main goal is to construct a gate that entangles two qubits encoded in dual-rail states (see e.g. [2]) or, equivalently, enact the two-mode transformation:

\[
\begin{align}
|0\rangle_a \otimes |0\rangle_b &\rightarrow |0\rangle_a \otimes |0\rangle_b \quad (1a) \\
|0\rangle_a \otimes |1\zeta\rangle_b &\rightarrow |0\rangle_a \otimes |1\zeta\rangle_b \quad (1b) \\
|1\zeta\rangle_a \otimes |0\rangle_b &\rightarrow |1\zeta\rangle_a \otimes |0\rangle_b \quad (1c) \\
|1\zeta\rangle_a \otimes |1\zeta\rangle_b &\rightarrow e^{i\phi} |1\zeta\rangle_a \otimes |1\zeta\rangle_b , \quad (1d)
\end{align}
\]

where \( a \) and \( b \) are photonic modes, \( |0\rangle \) indicates a multimode vacuum, \( |1\zeta\rangle = \int d\omega \, \xi(\omega) a^\dagger(\omega) \, |0\rangle \) is a single photon in the wave packet \( \xi(\omega) \), and \( [a(\omega), a^\dagger(\omega')] = 0 \).

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actions are of the form $\delta(\omega - \omega')$. Any nontrivial phase ($0 < \phi < 2\pi$) in Eq. (1d) enables quantum computation, but we are interested in the case $\phi = \pi$, which corresponds to the CPHASE gate.

To characterize the action of a medium on multimode light, we use the $S$-matrix from scattering theory. The $S$-matrix is a unitary matrix connecting asymptotic input and output field states i.e. $|\omega_{\text{out}}\rangle = S |\omega_{\text{in}}\rangle$, while capturing the relevant effects of the medium. The ideal $S$-matrices corresponding to Eqs. (1a) to (1d) would be $S_{\text{id},1}(\omega_k; \nu_k) = \delta(\omega_k - \nu_k)$, for single-photon states, and $S_{\text{id},2}(\omega_a, \omega_b; \nu_a, \nu_b) = e^{i\phi} S_{\text{id},1}(\omega_a; \nu_a) S_{\text{id},1}(\omega_b; \nu_b)$ for two-photon states, where input (output) frequencies are denoted by $\nu_k$ ($\omega_k$), for $k = \{a, b\}$. Typically, however, the actual $S$-matrices for matter-mediated interactions are of the form $S_{\text{act},1}(\omega_k; \nu_k) = e^{i\phi_k} \delta(\omega_k - \nu_k)$ and $S_{\text{act},2}(\omega_a, \omega_b; \nu_a, \nu_b) = S_{\text{act},1}(\omega_a; \nu_a) S_{\text{act},1}(\omega_b; \nu_b) + C \delta(\omega_a + \omega_b - \nu_a - \nu_b)$, where the coefficient $C$ depends on all frequencies and the parameters of the interaction mediators [27, 28]. The phase $e^{i\phi_k}$ in $S_{\text{act},1}$ leads to a deformation of the single-photon wave packets, while the function $\delta(\omega_a + \omega_b - \nu_a - \nu_b)$ in $S_{\text{act},2}$, which arises from energy conservation [27], is usually identified as the source of spectral entanglement.

One important choice we make is to ignore single-photon deformation, which is enforced by mapping all $S$-matrices as $S \rightarrow S_{\text{id},1}^\dagger(\omega_a; \nu_a) S_{\text{id},1}^\dagger(\omega_b; \nu_b) S$. Most previous proposals do not do this (e.g. [15]), which accounts for part of the discrepancy in the maximum fidelities obtained. Single-photon deformation could have two negative effects for our proposal. First, it might disrupt linear-optical steps of the computation. This can be avoided by ensuring all photons are deformed equally at each computational time step [29]. Second, our results are obtained for specific input wave packet shapes, so single-photon effects could significantly degrade the fidelity of subsequent gates; later, we show that this is not the case for a few rounds of deformation. It is then possible to use measurement-based quantum computing, where each photon experiences at most two CPHASE gates [30], or teleportation-based error correction [31]. Finally, it is also possible to physically undo this deformation if necessary, as proposed e.g. in Ref. [19].

Ideally, we would like to show that the $S$-matrix for our proposal approaches $S_{\text{id},2}$ in some limit. However, it is sufficient for this to hold only for the particular states that we are considering. Thus, to gauge the quality of our operation, we use the average gate fidelity [32]

$$F(\phi) := \int d\psi \langle \psi | S_{\text{id},1}(\phi)^\dagger S_{\text{act}} | \psi \rangle | \langle \psi | S_{\text{id},1}(\phi)^\dagger S_{\text{act}} | \psi \rangle$$

(2)

where the integration is taken over the Haar measure of the joint Hilbert space (for further details, see Appendix A). For our gate to be useful for quantum computation, it suffices that $F = 1 - \epsilon$, where $\epsilon$ is some constant threshold [33].

Single- and two-site gate fidelities. We begin by examining $F$ for wave packets scattering from a single site, as well as two sites in a co- and counter-propagating arrangement. The discrete Kerr interaction we consider is depicted in Fig. 1 (a). The unit cell we repeat in Fig. 1 (b), call it $G = (L, H)$, can be described using “LH” [34–36] parameters from input-output theory, where $L$ is a vector of operators that couple the field to the system and $H$ is the system Hamiltonian. The LH parameters for our unit cell are

$$G = \left[ \frac{\sqrt{7}}{2} (A_\downarrow - A_\uparrow), \Delta/2 (I - A_\uparrow) + \frac{\Delta}{2} (I - B_\uparrow) + \chi(I - A_\uparrow)(I - B_\downarrow) \right],$$

(3)

where $A_\downarrow$ and $A_\uparrow$ are the atomic lowering and Pauli $Z$ operators for atom $A$, and likewise for $B_\downarrow$ and $B_\uparrow$. We cascaded $N$ unit cells with co- and counter-propagating fields and computed the corresponding S-matrices, as detailed in Ref. [28] and which we reproduce in Appendix B. The final ingredient needed to calculate $F$ is the wave packet shape, which we choose to be Gaussian with detuning $\omega_0$ (i.e. carrier frequency $\omega_c = \Delta + \omega_0$) and bandwidth $\sigma$, i.e.

$$\xi(\omega) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp \left[ - \frac{(\omega - \omega_c)^2}{4\sigma^2} \right].$$

(4)

In Fig. 2, column 1, we display the plots for parameters ($\omega_0, \gamma, \chi$) which maximize the fidelity of counter-propagating wave packets. Relative to the single site, we observe a clear increase of the maximum obtainable fidelity when the photons are counter-propagating, and a
decrease when they are co-propagating, as illustrated in the top row. In the limit of large $\chi$ and $\omega_0 = 0$, the phase shift is always either 0 or $\pi$, corresponding to the identity or CPHASE gate respectively (see the second row). We observe that counter-propagating wave packets tend to perform better than co-propagating for a large region of the parameter space, but there are exceptions.

In Fig. 2, column 2, we display a parameter regime where co-propagating photons obtain their maximum fidelity and outperform the other two cases. The explanation for this is the following. In this regime, the co-propagating case seems to suffer more spectral entanglement, but also acquire a larger phase shift, than the other two (see the dashed lines), and the tradeoff between these effects leads to a higher fidelity with the CPHASE gate. However, these effects are linked in such a way that this peak fidelity and the maximum phase are still much inferior to the best obtained by the single-site and counter-propagating cases in other parameter regimes. Nonetheless, this suggests it is possible to use a perturbative approach to construct long weakly-coupled atom chains where the rate at which phases and spectral entanglement accumulate are more benign (e.g. [19]). It is also interesting that the peak of the fidelity in the co-propagating case happens for larger $\sigma$ than for the counter-propagating case in all three columns, which could lead to a CPHASE gate for spectrally broader photons.

In Fig. 2, column 3, we display parameters that maximize the single-site fidelity. As we generically expect, the counter-propagating wave packets outperform the single-site and co-propagating ones both in fidelities and phase shifts. This happens even when $\omega_0 \neq 0$, indicating that our conclusions are somewhat robust with respect to being off-resonance.

$N$-site gate fidelities. We now investigate the average fidelity of our proposal to the CPHASE gate as we increase the number of interaction sites. Based on observations from the two-site case, we restrict our analysis to counter-propagating photons, working on-resonance ($\omega_0 = 0$), and take $\chi \rightarrow \infty$ limit, since this yields the most promising results. We also take $\gamma = 1$, since choosing other values effectively rescales $\sigma$ when working on-resonance. Thus, the average gate fidelity $F$ is a function of the number of interaction sites $N$ and the photon bandwidth $\sigma$: $F(\sigma, N)$.

In Fig. 3(a) we plot the average gate fidelity, as a function of $\sigma$, for increasing $N$. Notice that, as the number of interaction sites increases, the maximum average fidelity increases, indicating that the resulting operation is sequentially closer to a CPHASE gate. Also notice that, besides attaining higher maximum values, the fidelity curve is also becoming broader (albeit in logarithmic scale). This means that, as the number of sites is increased, the proposed CPHASE gate becomes more broadband, or robust with respect to the spectral bandwidth of the photon. The highest value for the fidelity in Fig. 3(a) is 0.996, when $N = 20$.

In Fig. 3(b), we investigate the maximum of the average fidelity $F_{\text{max}}$ and its corresponding value of $\sigma_{\text{max}}$ as a function of $N$. We see that $1 - F_{\text{max}}$ is monotonically decreasing and $\sigma_{\text{max}}$ slowly tends towards the plane-wave limit. For the observed behaviour of Fig. 3(a-b) we predict that, in order to obtain $F_{\text{max}} = 0.999$, we would need $N \simeq 50$ and $\sigma \simeq 0.014s^{-1}$. Fig. 3(b) also shows that, for
FIG. 3. (Color online). (a) Average gate fidelity between our proposal and the CPHASE as a function of frequency bandwidth \( \sigma \) for increasing number of interaction sites \( N \). (b) We take the maximum of the gate fidelity in (a), and plot the infidelity \( 1 - F \) and the corresponding maximizing \( \sigma_{\text{max}} \) as functions of the number of interaction sites \( N \). Small red dots correspond to \( 1 - F \) for photons that have undergone two rounds of single-photon deformation. The dashed lines correspond to the fits \( 1 - F(\sigma_{\text{max}}, N) = 0.537N^{-1.61} \) and \( \sigma_{\text{max}} = 0.350N^{-0.81} \), where we fit to \( N \in [4, 20] \).

\( N > 5 \), the fidelity is not significantly affected by using single-photon wave packets that have suffered one or two rounds of deformation. Another feature apparent in Fig. 3(a) is that, for fixed \( \sigma \), the advantage gained from increasing \( N \) eventually saturates. This is explored further in Fig. 4. An intuitive explanation is: interpret \( 1/\gamma \) as the typical timescale before an excited atom re-emits a photon, then \( t_{\text{mw}} \approx N/\gamma \) is the time that each wave packet remains inside the medium. Thus, if the wave packets have temporal width of \( t_w \approx 1/\sigma \), when \( N \) is roughly \( \gamma/\sigma \) the chain becomes “long enough” to contain the entire wave packets, and the interaction saturates.

Our results show that, to obtain higher-fidelity gates, one has to move to smaller values of \( \sigma \) together with longer atomic chains. In fact, in Ref. [28], we and Gea-Banacloche have shown that, in the limit where \( \sigma \rightarrow 0 \) and \( N \rightarrow \infty \), the \( S \)-matrix for the \( N \)-site case tends to the ideal one (modulo single-photon deformation)

\[
S_{\text{act}, 1}(\omega_1; \nu_1; \nu_2) = -S_{\text{act}, 1}(\omega_1; \nu_2)S_{\text{act}, 1}(\omega_1; \nu_1).
\]

This is independent of the specific wave packet shape, further motivating our choice to ignore single-photon deformation. The results presented here are more relevant for implementations, as one only needs to increase \( N \) and decrease \( \sigma \) until the fidelity surpasses the threshold necessary for fault-tolerant computation.

**Discussion.** Our goal was to determine if it possible to build a passive CPHASE gate using cross-Kerr interactions, and we have shown that it is. Importantly, our results do not contradict those of Ref. [14], which uses a phenomenological model of a cross-Kerr medium. Using that model, a CPHASE gate might indeed be unachievable. However, our results are based on a fully multimode treatment of the field and a fully microscopic treatment of the interaction mediators. Thus we believe they do provide a counter-example against the stronger claim, which is frequently propagated the literature, that the multimode nature of the field is a fundamental physical obstacle to implementing a CPHASE gate. Furthermore, our proposal enjoys two advantages over prior proposals. First, our gate is passive, i.e., it does not require active error correction, such as the principal mode projection technique used in Ref. [19]. Also, our proposal requires fewer interaction sites to achieve a fixed fidelity, e.g. in Ref. [19] the authors estimate they need \( 10^6 \) interaction sites for a 95% fidelity with a CPHASE gate, whereas our proposal achieves that value with 5 sites. Admittedly, our proposal is a proof-of-principle result that will be challenging to construct in practice. Further, we hope our construction will inspire others to devise simpler and less resource-intensive proposals. There are plenty of avenues to explore, e.g. placing the atomic interaction sites inside cavities to gain a cavity enhancement [37], or varying the atom parameters along the chain [38] while simultaneously varying the input photon wave packet shape. Our analysis did not include any additional imperfections, and we leave as future work to adapt our model to include other effects such as losses, emission into non-guided modes, coupling to a thermal bath, etc. In Appendix C we propose an update to a set of rules, initially laid out by Gea-Banacloche [15], that must be satisfied by any theoretical proposal for a realistic CPHASE gate, based on conclusions drawn from this work and [28].
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Appendix A: Average gate fidelity

In this paper, we gauge the quality of our operation using the standard average gate fidelity [32]:

$$F_1(\phi) := \int d\psi \langle \psi | S_{\text{id}}(\phi) S_{\text{act}} | \psi \rangle \langle \psi | S_{\text{act}}^\dagger S_{\text{id}}(\phi) | \psi \rangle$$  \hspace{1cm} (A1)

where the integration is over the two-qubit Haar measure, and $S_{\text{id}}$ and $S_{\text{act}}$ include the removal of the single-photon deformation. Equation (A1) is the same as Eq. (2) in the main text. In order to carry out the averaging explicitly, one parameterizes $|\psi\rangle$ as [39]

$$|\psi\rangle = e^{i\chi} \cos \theta_0 |00\rangle + e^{i\chi_1} \sin \theta_0 \cos \theta_1 |01\rangle + e^{i\chi_2} \sin \theta_0 \sin \theta_1 \cos \theta_2 |10\rangle + e^{i\chi_3} \sin \theta_0 \sin \theta_1 \sin \theta_2 |11\rangle,$$

where $0 \leq \chi_i < 2\pi$ and $0 \leq \theta_i < \pi/2$. In this parameterization, the Haar measure in the integration is

$$d\psi = 48(2\pi)^{-4} (\sin \theta_0)^5 \cos \theta_0 (\sin \theta_1)^3 \cos \theta_1 (\sin \theta_2)^5 \cos \theta_2 d\theta_0 d\theta_1 d\theta_2 d\chi_0 d\chi_1 d\chi_2 d\chi_3.$$ 

Since our analysis removes all single-photon deformation, we can leverage the fact that the only state affected by the interaction is $|\xi_1\rangle_a \otimes |\xi_1\rangle_b$ and that our gate conserves photon number to write

$$F_1(\phi) = \frac{1}{10} \left(6 + 3\text{Re}(e^{i\phi} F) + |F|^2\right).$$  \hspace{1cm} (A2)

Here $F$ is the overlap between the single- and two-photon wave packets:

$$F = \int [\xi_1^{\text{out}}(\nu_a, \nu_b)]^* \xi_2^{\text{out}}(\nu_a, \nu_b) d\nu_a d\nu_b,$$  \hspace{1cm} (A3)

where

$$\xi_i^{\text{out}}(\nu_a, \nu_b) = \int \xi_i^{\text{in}}(\omega_a) \xi_i^{\text{in}}(\omega_b) |\nu_a \nu_b\rangle S_i |\omega_a \omega_b\rangle d\omega_a d\omega_b$$  \hspace{1cm} (A4)

for $i = 1, 2$ are the propagated two-photon wave packets, respectively, according to the S-matrices computed in Ref. [28].

Often, $F$ is used as the main figure of merit, since it relates the transformed two-photon wave packet to two copies of a single-photon wave packet, thus directly measuring undesired effects such as spectral entanglement. However, we believe that the average fidelity, for our purposes, is a more transparent and unambiguous figure of merit, because it determines how well one approximates a desired gate in the computational state space. Although the average gate fidelity is not the figure of merit that appears in the threshold theorem directly, these quantities can be related [33].

We should also point out that Shapiro, in Ref. [14], considers a slightly different figure of merit, where the average in Eq. (A1) is done over all product two-qubit states. Using that definition, the state $|\psi\rangle$ can be parameterized as

$$|\psi\rangle = (\cos \theta_0 |0\rangle + e^{i\chi_0} \sin \theta_0 |1\rangle)(\cos \theta_1 |0\rangle + e^{i\chi_1} \sin \theta_1 |1\rangle),$$

with the Haar measure given by

$$d\psi = 4(2\pi)^{-2} \sin \theta_0 \cos \theta_1 d\theta_0 \sin \theta_1 \cos \theta_1 d\theta_1 d\chi_0 d\chi_1,$$

and our Eq. (A2) would become

$$F_2(\phi) = \frac{1}{18} \left(11 + 5\text{Re}(e^{i\phi} F) + 2|F|^2\right).$$  \hspace{1cm} (A5)

We have found that both definitions lead to essentially the same results. This is illustrated in Fig. 5, where we see that $F_1 < F_2$ because $F_2$ is less conservative. As $F$ approaches $-1$ (which would correspond to a perfect CPHASE gate) the difference between $F_1(\pi)$ and $F_2(\pi)$ becomes negligible. We use Eq. (A2) because of its connections to the threshold theorem, and because it can capture other undesired effects such as e.g. entanglement breaking.
FIG. 5. (Color online). Comparison between average fidelities using a Haar average over full two-qubit Hilbert space \((F_1)\) and over the two single-qubit Hilbert spaces \((F_2)\). Here \(\chi = 5\), \(\gamma = 4.5\), and \(\omega_0 = 1.1\).

Appendix B: S-matrices

For completeness, in this Appendix we write the S-matrices computed in Ref. [28]. We begin by defining the shorthands

\[
\Gamma(\omega) := \frac{\gamma}{2} + i(\Delta - \omega),
\]

\[
\Gamma_i(\omega) := \frac{\gamma_i}{2} + i(\Delta_i - \omega_i),
\]

which will be used when we have a single and multiple sites, respectively. Although we computed some of these S-matrices with different parameters \((\gamma_i, \Delta_i, \chi_i)\) for each site, in the numerical work reported in the main text we assume them to be equal at all sites for simplicity.

a. Single site

For a single interaction site, the S-matrices for co- and counter-propagating photons are equivalent. In this case, we have the single-photon S-matrix

\[
S_{\text{act},1}(\omega_k; \nu_k) = -\frac{\Gamma(\omega_k)}{\Gamma(\omega_k)} \delta(\omega_k - \nu_k).
\]

for \(k = a, b\). The two-photon S-matrix is

\[
S_{\text{act},2}(\omega_a, \omega_b; \nu_a, \nu_b) = S_{\text{act},1}(\omega_a; \nu_a)S_{\text{act},1}(\omega_b; \nu_b) - i \frac{\chi^2}{\pi} \left(1 + \frac{2i\chi}{\Gamma(\omega_a) + \Gamma(\omega_b)}\right)^{-1} \delta(\omega_a + \omega_b - \nu_a - \nu_b) \frac{\Gamma(\nu_b)\Gamma(\nu_a)\Gamma(\omega_b)\Gamma(\omega_a)}{\Gamma(\nu_b)\Gamma(\nu_a)\Gamma(\omega_b)\Gamma(\omega_a)}.
\]

b. Two sites, co-propagating photons

For two interaction sites in the co-propagating arrangement, we have the single-photon S-matrix

\[
S_{\text{act},1}(\omega_k; \nu_k) = \frac{\Gamma(\omega_k)\Gamma_1(\omega_k)}{\Gamma(\omega_k)\Gamma(\omega_k)} \delta(\omega_k - \nu_k)
\]
for $k = a, b$. The two-photon S-matrix is

$$S_{\text{act,2}}(\omega_a, \omega_b; \nu_a, \nu_b) = S_{\text{act,1}}(\omega_a; \nu_a)S_{\text{act,1}}(\omega_b; \nu_b) - \frac{\delta(\omega_a + \omega_b - \nu_a - \nu_b)}{\pi} \times \left[ \frac{\Gamma_2^*(\omega_a)\Gamma_2^*(\omega_b)}{\Gamma_2(\omega_a)\Gamma_2(\omega_b)} \left( 1 + \frac{2i\chi_1}{\Gamma_1(\omega_a) + \Gamma_1(\omega_b)} \right)^{-1} \frac{\chi_1\gamma_1^2}{\Gamma_1(\nu_b)\Gamma_1(\nu_a)\Gamma_1(\omega_a)\Gamma_1(\omega_b)} \right. \\
+ \left. \frac{\Gamma_1^*(\omega_a)\Gamma_1^*(\omega_b)}{\Gamma_1(\omega_a)\Gamma_1(\omega_b)} \left( 1 + \frac{2i\chi_2}{\Gamma_2(\omega_a) + \Gamma_2(\omega_b)} \right)^{-1} \frac{\chi_2\gamma_2^2}{\Gamma_2(\nu_b)\Gamma_2(\nu_a)\Gamma_2(\omega_a)\Gamma_2(\omega_b)} \right].$$

For two interaction sites in the counter-propagating arrangement, the single-photon S-matrix is obviously the same as for the co-propagating arrangement. The two-photon S-matrix, however, is

$$S_{\text{act,2}}(\omega_a, \omega_b; \nu_a, \nu_b) = S_{\text{act,1}}(\omega_a; \nu_a)S_{\text{act,1}}(\omega_b; \nu_b) - \frac{\delta(\omega_a + \omega_b - \nu_a - \nu_b)}{\pi} \times \left[ \frac{\Gamma_2^*(\omega_a)\Gamma_2^*(\omega_b)}{\Gamma_2(\omega_a)\Gamma_2(\omega_b)} \left( 1 + \frac{2i\chi_1}{\Gamma_1(\omega_a) + \Gamma_1(\omega_b)} \right)^{-1} \frac{\chi_1\gamma_1^2}{\Gamma_1(\nu_b)\Gamma_1(\nu_a)\Gamma_1(\omega_a)\Gamma_1(\omega_b)} \right. \\
+ \left. \frac{\Gamma_1^*(\omega_a)\Gamma_1^*(\omega_b)}{\Gamma_1(\omega_a)\Gamma_1(\omega_b)} \left( 1 + \frac{2i\chi_2}{\Gamma_2(\omega_a) + \Gamma_2(\omega_b)} \right)^{-1} \frac{\chi_2\gamma_2^2}{\Gamma_2(\nu_b)\Gamma_2(\nu_a)\Gamma_2(\omega_a)\Gamma_2(\omega_b)} \right].$$

c. Two sites, counter-propagating photons

For two interaction sites in the counter-propagating arrangement, the single-photon S-matrix is obviously the same as for the co-propagating arrangement. The two-photon S-matrix, however, is

$$S_{\text{act,2}}(\omega_a, \omega_b; \nu_a, \nu_b) = S_{\text{act,1}}(\omega_a; \nu_a)S_{\text{act,1}}(\omega_b; \nu_b) - \frac{\delta(\omega_a + \omega_b - \nu_a - \nu_b)}{\pi} \times \left[ \frac{\Gamma_2^*(\omega_a)\Gamma_2^*(\omega_b)}{\Gamma_2(\omega_a)\Gamma_2(\omega_b)} \left( 1 + \frac{2i\chi_1}{\Gamma_1(\omega_a) + \Gamma_1(\omega_b)} \right)^{-1} \frac{\chi_1\gamma_1^2}{\Gamma_1(\nu_b)\Gamma_1(\nu_a)\Gamma_1(\omega_a)\Gamma_1(\omega_b)} \right. \\
+ \left. \frac{\Gamma_1^*(\omega_a)\Gamma_1^*(\omega_b)}{\Gamma_1(\omega_a)\Gamma_1(\omega_b)} \left( 1 + \frac{2i\chi_2}{\Gamma_2(\omega_a) + \Gamma_2(\omega_b)} \right)^{-1} \frac{\chi_2\gamma_2^2}{\Gamma_2(\nu_b)\Gamma_2(\nu_a)\Gamma_2(\omega_a)\Gamma_2(\omega_b)} \right].$$

d. N sites, counter-propagating photons

For the N-site case, we only computed the S-matrices in the counter-propagating arrangement, and under the assumption of translation invariance, i.e. $\gamma_i = \gamma$, $\Delta_i = \Delta$, and $\chi_i = \chi$. From this, we obtain the single-photon S-matrix

$$S_{\text{act,1}}(\omega_k; \nu_k) = \left( \frac{\Gamma_1^*(\omega_k)}{\Gamma(\omega_k)} \right)^N \delta(\omega_k - \nu_K),$$

for $k = a, b$, and the two-photon S-matrix is

$$S_{\text{act,2}}(\omega_a, \omega_b; \nu_a, \nu_b) = S_{\text{act,1}}(\omega_a; \nu_a)S_{\text{act,1}}(\omega_b; \nu_b) - \frac{i\chi_2\gamma_2^2}{\pi} \left( \frac{\Gamma_1^*(\omega_a)\Gamma_1^*(\omega_b)}{\Gamma(\omega_a)\Gamma(\omega_b)} \right) \left[ \sum_{j=1}^{N-1} \left( \frac{\Gamma_1^*(\omega_b)\Gamma_1^*(\omega_a)}{\Gamma(\omega_b)\Gamma(\omega_a)} \right) \left( \frac{\Gamma_1^*(\omega_b)\Gamma_1^*(\omega_a)}{\Gamma(\omega_b)\Gamma(\omega_a)} \right)^{-1} \delta(\omega_a + \omega_b - \nu_a - \nu_b) \right].$$

Appendix C: Problem statement and rules for passive CPHASE gate via cross-Kerr interactions

In this section we update Gea-Banacloche’s [15] suggested requirements for proposals of CPHASE gates. Paraphrasing, Gea-Banacloche’s [15] requirements were: (GB1) clearly local and physically realizable Hamiltonians; (GB2) localized wave packets using quantized multimode fields; (GB3) report gate fidelities computed; and (GB4) a realistic estimate of any residual losses or decoherence mechanisms.

We are particularly interested in determining if it is possible to construct a passive CPHASE gate using cross-Kerr interactions with the photons entering the device synchronously. By passive, we mean there should be no control pulses applied to the medium (atoms) unless they are static, and no active error correction.

We suggest the following requirements for “in principle” theory proposals
P1: Microscopic model.

Ideally, the medium should be constructed from a microscopic model using clearly local, passive, and physically realizable Hamiltonians. If the model is a hybrid model, e.g., light coupling to a spin wave, one must actually model the coupling efficiency of this interface. This is comparable to (GB1).

One point which is often overlooked is that results obtained using particular phenomenological models cannot be extended as “physical principles”, since a different medium could be constructed, out of different microscopic components, for which that result does not apply.

P2: Multimode analysis.

The analysis should be done with a quantized multimode single photons, e.g., \( |1_\xi\rangle = \int d\nu \xi(\nu)a^\dagger(\nu)|0\rangle \) where \([a(\omega), a^\dagger(\nu)] = \delta(\omega - \nu)\). If possible report the relevant S-matrices or the output wave functions. This is an extension to (GB2).

P3: Reflection must be explicitly dealt with.

Either the proposal should use circulators or isolators to remove reflection altogether, or it should be shown theoretically that reflection can be suppressed enough to obtain a high-fidelity operation. For example, in proposals using continuous mediums, like atomic vapours, reflection is not usually modelled.

P4: Quality of the operation.

Standard figures-of-merit are the average gate fidelity and the diamond norm between the proposal and the CPHASE gate, restricted to the encoded state space. This is comparable to (GB3).

P5: Imperfections.

After making a convincing case for a particular theory proposal, one should then take imperfections into account. Examples include loss out of guided modes and coupling atoms (Kerr medium) to a thermal reservoir. However, it is important to keep in mind that the final word in the feasibility of any proposal is that of experiments. This means that the final proposal should model all (and only) realistic sources of noise for a particular implementation, and that results derived from this procedure should not be generalized to other experimental implementations. Finally, at this point it might be worthwhile to determine the fault tolerance threshold for the proposed imperfect gate. These points are made in (GB4).

We now also point out a few things that, in our view, might not be necessary.

P6: Perfect operation.

It is not necessary that the operation is perfect, or that it can be made arbitrarily good. The threshold theorem of fault-tolerant quantum computing guarantees that there exist some constant threshold above which error correction can be used to perform arbitrarily-long computations (this is why the average gate fidelity or the diamond norm should be used, see point P4 above). The exact value of the threshold is dependent on the noise model which should include, among other effects, the imperfections from point P5 above.

P7: Single-photon distortion.

It is preferable to obtain a high-fidelity operation without deformation of single-photon wave packets. But, as we argued in the main text, that might not be strictly necessary since there are a number of ways to work around it.