On Decay of K-theory

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Closed string tachyon condensation resolves the singularities of nonsupersymmetric orbifolds, however the resolved space typically has fewer D-brane charges than that of the orbifold. The description of the tachyon condensation process via a gauged linear sigma model enables one to track the topology as one passes from the sigma model’s “orbifold phase” to its resolved, “geometric phase,” and thus to follow how the D-brane charges disappear from the effective spacetime dynamics. As a mathematical consequence, our results point the way to a formulation of a “quantum McKay correspondence” for the resolution of toric orbifold singularities.

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1. Introduction and summary

The study of tachyon condensation on unstable localized defects such as D-branes, NS5-branes, and orbifolds has yielded a number of insights into the structure of string theory. For example, open string tachyon condensation provides one route to the topological classification of D-branes via K-theory. The decays of localized defects via closed string tachyon condensation have exhibited striking analogies to the open string case [1,2,3,4]. There is, however, one notable difference. In open string tachyon condensation, the charges of D-branes remain invariant. However, in the typical closed string tachyon condensation described in the above references, it seems that D-brane charge “disappears.” The present paper is an attempt to understand in more precise terms how it disappears, and where it goes.

Our central tool for answering the question will be the world sheet renormalization group (RG). The worldsheet RG has proven to be a reliable tool in analyzing the possible decays and their endpoints under tachyon condensation for both open and closed strings (see [4] for a review). Nonconformal backgrounds in the worldsheet field theory provide a way of continuing off-shell; RG flows interpolate between classical solutions, and thus provide information about the effective action and the topology of the configuration space. Tachyon condensation corresponds to adding a relevant operator to the worldsheet Lagrangian describing the background in which perturbative strings propagate; the endpoint of tachyon condensation in this context is the IR fixed point of the worldsheet renormalization group flow.

Typically, it is difficult to follow the renormalization group trajectory of a generic perturbation of the UV fixed point all the way to its far IR limit; nonperturbative information is required. Such information is provided by the chiral ring (the BPS states) of $\mathcal{N} = 2$ extended worldsheet supersymmetric theories [3]; the renormalization of such states is under good control and enables one to understand the structure of flows preserving the $\mathcal{N} = 2$ structure.

In fact, in $\mathcal{N} = 2$ conformal field theories, there are two rings, due to the independent left- and right-moving supersymmetry algebras: The chiral ring, consisting of operators that are left-chiral and right-chiral; and the twisted chiral ring, whose operators are left-chiral and right-anti-chiral. One can preserve one or the other but not both along RG flows.
Orbifolds $\mathbb{C}^d/\Gamma$, where $\Gamma \subset U(d)$, provide a large class of examples. In the present work, we will mostly consider the abelian orbifolds $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ defined by the discrete group action on $\mathbb{C}^2$

$$(X,Y) \rightarrow (\omega X, \omega^{p} Y) \quad (1.1)$$

where $\omega = \exp[2\pi i/n]$. Note that for $p = n - 1$, the rotation is in $SU(2)$ rather than $U(2)$ so that spacetime supersymmetry is preserved; these are the well-known $A_{n-1}$ ALE orbifolds.

The orbifold twisted chiral ring is built out of the $\mathbb{Z}_n$ twist operators $T_\kappa$, $\kappa = 1, \ldots, n-1$ for each separate complex plane:

$$T_\kappa = T_{\kappa/n}^{(X)} T_{\kappa p/n}^{(Y)} \quad (1.2)$$

where $\{\xi\}$ denotes the fractional part of $\xi$, $0 \leq \{\xi\} < 1$. We concentrate here on the GSO projection corresponding to type 0 strings, for which all of these operators are present in the string spectrum. The operators (1.2) carry the $U(1)_X \times U(1)_Y$ $R$-charges

$$(\kappa/n, \{\kappa p/n\}) \quad (1.3)$$

corresponding to charges under axial rotations of the $X$ and $Y$ planes, respectively (for a more detailed discussion, see [2,4]). A plot of these charges for the twisted chiral ring of the $n(p) = 10(3)$ orbifold is shown in figure 1.

Figure 1. $R$-charge vectors $\frac{1}{n}v_\kappa = (\kappa/n, \{\kappa p/n\})$ of twist fields $T_\kappa$, $\kappa = 1, \ldots, 9$, for $n(p) = 10(3)$, together with the (untwisted sector) twisted chiral fields $V_X, V_Y$ representing the volume forms on the $X, Y$ complex planes. The solid blue dots indicate the generators of the chiral ring.

\footnote{1 The bulk tachyon will be fine-tuned to zero. We will comment on the type II theory in section 8.}
The BPS property determines the total (left plus right) conformal scaling dimensions
\[ \Delta_\kappa = \frac{\kappa}{n} + \left\{ \frac{\kappa p}{n} \right\} \]
of the twist operators \( T_\kappa \). Thus the chiral operators below the diagonal line in figure 1 are relevant operators corresponding to closed string tachyons; those on the line generate marginal deformations, and those above it yield irrelevant perturbations. Since the twist operators are the only relevant deformations that we will be considering, we will henceforth abuse language and drop the modifier ‘twisted’ when referring to the twisted chiral ring, and simply call it the chiral ring. Renormalization group flows generated by a single relevant scaling operator were considered in [1,2,3,4]. Below, in section 6, we will present a picture of the generic RG flow.

For the supersymmetric \( C^2/Z_n \) orbifolds, it is well-known that the generic deformation of the orbifold CFT by the \( n - 1 \) marginal twist fields resolves the orbifold singularity, yielding a nonsingular \( A_{n-1} \) ALE manifold. The algebro-geometric procedure for resolving the singularity consists of excising the singular point at the origin by blowing it up into a chain of \( n - 1 \) \( \mathbb{P}^1 \)'s intersecting in a pattern specified by the \( A_{n-1} \) Dynkin diagram. For the general non-supersymmetric \( C^2/Z_{n(p)} \) orbifolds, there is a similar resolution of the singularity known as the Hirzebruch-Jung (HJ) or minimal resolution [6,7,8]. This resolution consists of excising the orbifold point and inserting a chain of \( r \) \( \mathbb{P}^1 \)'s, where \( r \) is the number of terms in the continued fraction expansion of \( n/p \).

\[
\frac{n}{p} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\ddots - \frac{1}{a_r}}}} := [a_1, \ldots, a_r],
\]
for integers \( a_\alpha \geq 2 \). This resolution describes the geometry of the orbifold theory perturbed in a generic way by the twist operators that generate the chiral ring. The resolution is called “minimal” because there are other resolutions of the singularity with more \( \mathbb{P}^1 \)'s in the resolution chain. These are associated to continued fractions (1.5) having some of the \( a_\alpha = 1 \). More details of these resolutions will be given below in sections four and five.

Let us now consider D-branes in these models. An orbifold conformal field theory admits a canonical set of ‘fractional’ D-branes [9,10]. For any representation of \( Z_n \) there is a corresponding fractional brane. These branes carry charges that couple to corresponding RR gauge fields. The fractional brane charges generate the entire lattice of possible D-brane charges. Mathematically, these objects generate the equivariant K-theory of the orbifold [11,12]. For the supersymmetric \( C^2/Z_{n(n-1)} \) orbifolds, this agrees nicely with the
compact K-theory of the resolved $A_{n-1}$ ALE space: There are $(n-1)$ classes corresponding to line bundles on the resolving spheres, together with the D0 brane. This pleasant correspondence, whose physical realization is so natural, is part of the story of the McKay correspondence (For a sampling of references in the math literature, see [13,14,15,16,17] and in the physics literature, see [18,19,20,21,22,23,24,25,26]).

When we consider the orbifolds $\mathbb{C}^2/\mathbb{Z}_n(p)$, things are not so simple. For general $\mathbb{Z}_n(p)$, it will still be the case that the equivariant K-theory of the orbifold is the representation ring of $\mathbb{Z}_n$, so that the lattice of orbifold D-brane charges is isomorphic to $\mathbb{Z}^n$. That is, there are still $n$ distinct kinds of compactly supported D-brane at the orbifold point. However, in general the K-theory lattice of D-brane charges of the smooth Hirzebruch-Jung space has a rank smaller than $n$. For $p \neq n-1$ one has $r < n-1$ in (1.5), so that there are $r < n-1$ $\mathbb{P}^1$’s needed to smooth the orbifold singularity, and thus, taking into account the D0 brane, $r + 1 < n$ generators of the compactly supported K-theory lattice of the resolved space. Simply put, there are not enough cycles in the resolved space to wrap D-branes on to account for all the independent D-brane charges of the orbifold one started with. Where did the extra D-brane charges go? One of the main goals of this paper is to provide an answer to this question.

In order to answer this question it is very convenient to introduce a gauged linear sigma model (GLSM) [27], for which the UV fixed point is the orbifold conformal field theory. The GLSM construction also contains couplings to twisted sector tachyons which resolve the orbifold singularity, realized as Fayet-Iliopoulos parameters of its abelian gauge dynamics. Some aspects of this type of GLSM were studied in [3].

The GLSM consists of a $U(1)^r$ gauge group coupled to charged chiral matter fields. It has both Coulomb and Higgs branches of its configuration space. In the IR, the Higgs branch can be interpreted as a nonlinear sigma model whose target space is a resolution of $\mathbb{C}^2/\mathbb{Z}_n(p)$ such as the Hirzebruch-Jung resolution. Of course, along the RG flow one has a massive 2D quantum field theory, and the data of the closed string geometry are undergoing RG flow. However, at a fixed RG scale one can speak of the D-branes in the

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2 There is a perfect pairing of the $K$-theory with the compact $K$-theory. The $K$-theory has a natural basis of $n$ distinct canonical line bundles (including the trivial bundle) on the smooth space corresponding to space-filling D4-branes with magnetic monopoles threading the various $\mathbb{P}^1$’s of the resolution.

3 Naturally associated to the description of the singularity and its resolution in toric geometry, c.f. [7].
massive Higgs branch theory. In the IR where the target space of the Higgs branch is a smooth manifold, these D-branes have a geometrical interpretation as branes wrapping nontrivial cycles of the smooth Hirzebruch-Jung manifold; they are therefore interpreted as K-theory classes of the resolved space. These D-brane charges are the $r + 1$ “obvious” charges.

We will also find $n - r - 1$ independent additional D-branes living on the Coulomb branch of the configuration space, one for each of the distinct massive vacua on this branch. In section 7.1 below we will show that these objects supply the extra charges needed to account fully for all the D-brane charge present in the orbifold. This is the resolution of our puzzle.

There are three interesting byproducts of this result.

The first byproduct is an improved understanding of the endpoint of the generic RG flow associated to closed string tachyon condensation of the $\mathbb{F}^2/\mathbb{Z}_n(p)$ orbifold, extending the results of \cite{1,2,3,4}. The general picture is one of several ALE spaces separating from one another along the flow. The precise pattern of ALE spaces is encoded in the minimal continued fraction expansion (1.3). To each consecutive subsequence $[a_\alpha, ..., a_{\alpha + \ell}]$ of (1.3) having all $\ell$ of the $a$’s equal to two, there is an $A_\ell$ ALE space comprising one component of the IR limit of the geometry. The various ALE components are separated by an infinite distance as one flows to the IR. This picture is explained in section 6 below.

The second byproduct is a suggestion for the form of the effective action of the RR gauge fields that couple to the corresponding decoupling D-branes; this effective action appears to take the same form as one finds in the analogous open string examples:

$$S_{\text{eff}}^{RR} = \int d^6x f(T) [F_{RR}^2 + \ldots],$$

where $f(T) \rightarrow 0$ as the tachyon condenses. The RG approach thus appears to support the idea that the decoupling of effective fields and charges under tachyon condensation takes a universal form for both open and closed string degrees of freedom.

The third byproduct is an application to mathematics, where our construction suggests a generalization of the McKay correspondence mentioned above. A generalization of the McKay correspondence to orbifolds of the type $\mathbb{F}^2/\mathbb{Z}_n(p)$ has been discussed in the mathematical literature \cite{8,28,29,30}. These authors compared the equivariant K-theory of the orbifold with the compactly supported K-theory of the minimal resolution. In physical terms, they were able to characterize “special representations” of the quantum $\mathbb{Z}_n$
symmetry of the orbifold corresponding to D-branes on the Hirzebruch-Jung resolution in such a way as to map the K-theory of the former onto the latter (the isomorphism even extends to the derived categories). In section 5 we show that the GLSM point of view makes the construction of these “special representations” very natural. The K-theory of the minimal resolution is generated by tautological sheaves – line bundles associated to a principal $U(1)^r$ bundle over the resolution via the unitary irreps $\rho_i$ of $U(1)^r$. We show that at the orbifold point the gauge group $U(1)^r$ is spontaneously broken to a discrete $\mathbb{Z}_n$ subgroup, and that when restricted to $\mathbb{Z}_n$, the representations $\rho_i$ are precisely the “special representations” of [8,28,29,30]. We expect that our construction will apply to all orbifolds that admit a toric resolution, thus allowing a construction of such “special representations” in a large class of examples. More interestingly, the resolution of our paradox suggests that, by adjoining the charges of the D-branes of the massive vacua on the Coulomb branch of the GLSM to the K-theory of the resolved space, there should be a “quantum McKay correspondence” that holds for a large class of orbifold singularities that admit toric resolutions.

2. The Gauged Linear Sigma Model

The orbifold spacetimes we consider are toric varieties, that is, they can be described as quotients by a $U(1)^r$ action. A simple way of generating such a quotient is to employ a gauged linear sigma model (GLSM). In this section we will recall some of the standard facts which will be important in what follows. All of the results can be found in [27,31,32,33]. For brief summaries, see [34,35].

Complex target space geometry and worldsheet supersymmetry imply $\mathcal{N} = 2$ worldsheet supersymmetry. Therefore, consider $r$ abelian $\mathcal{N} = 2$ gauge fields $V_\alpha$, $\alpha = 1, ..., r$ coupled to $r + d$ $\mathcal{N} = 2$ chiral matter fields $X_i$ with charges $Q_{\alpha i}$. The field strengths of the gauge fields are contained in twisted chiral superfields $\Sigma = \frac{1}{2}(\overline{D}, D^*)$. The classical Lagrangian is

$$\mathcal{L} = \int d^4\theta \left( X_i e^{2Q_{\alpha i}} V_\alpha X_i - \frac{1}{2e_\alpha^2} \Sigma_\alpha \Sigma_\alpha \right) - \frac{1}{2} \left( \int d^2\tilde{\theta} t_\alpha \Sigma_\alpha + c.c. \right), \quad (2.1)$$

4 Here the bar denotes worldsheet complex conjugation, and the star denotes complex conjugation in field space. We will usually follow the conventions of Hori and Vafa [33].
where repeated indices are summed and

$$t_\alpha = \zeta_\alpha - i\theta_\alpha$$  \hspace{1cm} (2.2)$$

combines the Fayet-Iliopoulos (FI) parameter $\zeta$ and theta angle $\theta$ for the $\alpha$th gauge field; $d^2\theta$ is the twisted chiral superspace measure.

In order to define the quantum theory we must renormalize the theory. Accordingly we introduce a momentum cutoff $\Lambda$ and fix a renormalization scale $\mu$. The 1-loop renormalization of the FI parameters is

$$t_\alpha,\text{eff}(\mu) = t_\alpha,\text{bare} + \sum_{i=1}^{r+d} Q_{\alpha i} \log \frac{\mu}{\Lambda}$$  \hspace{1cm} (2.3)$$

where $t_\alpha,\text{bare}$ are bare parameters defined at the momentum cutoff scale $\Lambda$. Note that the theory also has dimension one couplings $e_\alpha$. The renormalized theory is defined by taking $\Lambda \to +\infty$ holding $t_\alpha,\text{eff}(\mu), \mu$ and $e_\alpha$ fixed. The scale dependence of couplings depends crucially on the sign of the beta function, which is governed by:

$$b_\alpha := \sum_i Q_{\alpha i}$$  \hspace{1cm} (2.4)$$

Note that this requires that we take $t_{\alpha,\text{bare}} \to -\infty$ if $b_\alpha < 0$ and $t_{\alpha,\text{bare}} \to +\infty$ if $b_\alpha > 0$.

Our general strategy will be to use the GLSM to define a model which, at a high energy scale, say $\mu \sim \Lambda$, is close to the $\mathbb{C}^2/\mathbb{Z}_n$ orbifold CFT fixed point, and whose RG flow is “close” to that of the orbifold CFT perturbed by relevant operators from the chiral ring, see figure 2. We are then interested in the low energy behavior of the theory, that is, in the IR limit of the RG flow of such theories. This should be a good approximation to the RG flow of the perturbed orbifold theory. Therefore, we now turn to a discussion of the low energy physics of the GLSM.
Figure 2. Schematic description of the renormalization group trajectories for the couplings \(\exp[-t\alpha(\mu)/b_\alpha], 1/e_\alpha\). The RG fixed point at the origin is the orbifold CFT. The flow out of the fixed point along the horizontal axis is the RG flow of the orbifold perturbed by relevant chiral operators. This flow is the limit (indicated by the dashed blue line) of flows defined by choosing a fiducial scale \(\mu^*\) and sending \(e\) at that scale to \(\infty\).

To determine the low energy behavior we must examine the potential energy for the fields. In the classical theory the potential energy takes the form:

\[
U_{\text{classical}} = \sum_{\alpha=1}^{r} \frac{e_\alpha^2}{2} (M_\alpha(X) - \zeta_\alpha)^2 + \sum_{\alpha,\beta=1}^{r} \bar{\sigma}_\alpha \sigma_\beta \sum_{i=1}^{r+d} Q_{\alpha i} Q_{\beta i} |X_i|^2 \tag{2.5}
\]

In the second term \(\sigma_\alpha = \Sigma_\alpha |\tilde{\theta}|=0\). The first term comes from solving the equation of motion for the auxiliary field \(D_\alpha\) in the vector multiplet, where we have defined

\[
M_\alpha(X) := \sum_i Q_{\alpha i} |X_i|^2. \tag{2.6}
\]

The classical ground states are easily determined: Both terms in (2.5) are positive semidefinite and there are in general two branches of solutions. The first, the Higgs branch, has \(\sigma_\alpha = 0\) and nonvanishing values of \(X_i\). In general, nonvanishing values of \(X_i\) transform nontrivially under \(U(1)^r\), that is, these classical VEV’s break the \(U(1)^r\) gauge symmetry, hence the name “Higgs branch.” The Higgs branch is described by solving:

\[
\sum_i Q_{\alpha i} |X_i|^2 = \zeta_\alpha. \tag{2.7}
\]

Let us denote the solution set to (2.7) by \(S_{\zeta} \subset \Phi^{r+d}\). Taking into account gauge invariance, we see that the classical Higgs branch of vacua is the set \(S_{\zeta}/U(1)^r\).
Mathematically, one can view the functions $M_\alpha(X)$ as Hamiltonian functions on a phase space whose symplectic form is the Kähler form $\Omega = \frac{i}{2} dX_i \wedge d\bar{X}_i$. Then the conditions (2.7) (called the moment map equations) fix a level set of $M_\alpha$, and the quotient by the $U(1)^r$ torus action generated by the $M_\alpha$ results in a Hamiltonian reduction of the phase space to $\mathcal{S}_\zeta/U(1)^r$. In this setting, the reduction is known as a Kähler quotient.

If the solution space $\mathcal{S}_\zeta$ admits $X_i = 0$ for some $i$ then there can be another branch of classical vacua, the Coulomb branch, where some subgroup of $U(1)^r$ is unbroken, and some $\sigma_\alpha$ can take (continuous) nonzero expectation values. However, for generic values of $\zeta_\alpha$ such branches are absent.

Now let us turn to the quantum mechanical theory renormalized as in (2.3). There are still Higgs and Coulomb branches, but there are also several important modifications of the above description of the space of vacua.

The Higgs branch equations (2.7) are modified by setting $M_\alpha(X) = \zeta_{\alpha,\text{eff}}(\mu)$. The IR physics is determined by the behavior of $\mathcal{S}_{\zeta(\mu)}/U(1)^r$ for $\mu \to 0$. As we will see in sections 3,4,5 the nature of the model depends very strongly on the sign of $b_\alpha$. In the case where the space $\mathcal{S}_{\tilde{\zeta}(\mu)}/U(1)^r$ is a smooth manifold with $c_1(TX) < 0$ the low energy dynamics is that of a nonlinear sigma model with target space $\mathcal{S}_{\bar{\zeta}(\mu)}/U(1)^r$.

Since the sigma model metric is expected to be renormalized, the Kähler quotient metric will not be precisely that of the renormalized theory at a finite scale $\mu$. Nevertheless, we will use the Kähler quotient metric as a qualitative guide to the geometry.

The most dramatic modification of the configuration space of the quantum theory takes place on the Coulomb branch. We will assume that $b_\alpha \neq 0$ in what follows. In the classical theory the Coulomb branch is either absent, or is a continuous manifold. In the quantum theory, the Coulomb branch is a discrete set of vacua, which are massive if the kinetic term for $\sigma_\alpha$ is nonsingular. These vacua are essential in the quantum McKay correspondence, so let us recall how they arise.

The Coulomb branch vacua arise (for $\sum Q_{\alpha i} \neq 0$) when the $\sigma_\alpha$ gain vacuum expectation values, giving mass to all the $X_i$. The $X_i$ fields can then be integrated out; doing so results in an effective twisted superpotential:

$$\tilde{W}_{\text{eff}} = -\sum_{\alpha=1}^{r} \Sigma_\alpha \left( t_{\alpha,\text{eff}}(\mu) + \sum_{i=1}^{r+d} Q_{\alpha i} \log \left( \frac{1}{e^{\mu} \sum_{\beta=1}^{r} Q_{\beta i} \Sigma_\beta} \right) \right)$$

(2.8)

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5 See a related discussion in [32] where the RG flow on the Higgs branch is described as a flow from a LG model to a nonlinear model with $c_1(TX) < 0$ or a flow from a nonlinear model with $c_1(TX) > 0$ to a LG model, depending on the sign of $b_\alpha$. 

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Here – and here only – is the transcendental number 2.71..., not the gauge coupling) which neatly summarizes the running of the couplings. By standard holomorphy arguments, this formula is exact. The bosonic effective potential of the twisted scalars $\sigma_\alpha$ deduced from $\tilde{W}_{\text{eff}}$ is:

$$U(\sigma) = \sum_{\alpha=1}^{r} \frac{e_{\alpha}^2}{2} t_{\alpha,\text{eff}}(\mu) + \sum_{i=1}^{r+d} Q_{\alpha i} \log \left( \frac{1}{\mu} \sum_{\beta=1}^{r} Q_{\beta i} \sigma_\beta \right)^2$$  \hspace{1cm} (2.9)

Integrating out the $X_i$ fields is justified at momentum scales below the scale set by their masses; this is determined by the VEV’s of the $\sigma_\alpha$, namely

$$\langle \sigma_\alpha \rangle \sim \mu \exp \left[ -t_{\alpha,\text{eff}}(\mu) / b_\alpha \right].$$  \hspace{1cm} (2.10)

Because of the effects of pair creation on the vacuum, we must minimize over the branches of $\theta_\alpha \rightarrow \theta_\alpha + 2\pi n_\alpha$. This renders the potential (2.9) single-valued in $\sigma_\alpha$. Generically, this potential has several local minima, which are the vacua of the Coulomb branch. The kinetic terms of the $\sigma$ fields likewise are expected to receive renormalization. These effects can be very important, for example, in 5-brane physics, but we believe that they are not important in the examples studied in this paper. It would be good to clarify this point.

Finally, we note that in order to reproduce the physics of the orbifold CFT in the UV, one must consider the limit $e_{\alpha}^2 \rightarrow \infty$ of the renormalized theory. The renormalization group trajectories take the schematic form of figure 2 above, and only approach the RG trajectory of the perturbed orbifold in this limit. We should make sure that $e_{\alpha} \rightarrow \infty$ does not violate any of the key assumptions made above; in particular, one should check the self-consistency of the effective action (2.9) that leads to the Coulomb branch vacua. The limit $e_{\alpha}^2 \rightarrow \infty$ leaves fixed the scales (2.10) set by the VEV’s of the $\sigma_\alpha$. The dimensionless scalars $\hat{\sigma}_\alpha = \sigma_\alpha / e_\alpha$ with canonical kinetic terms have masses $\tilde{U}'' \sim e_{\alpha}^4 / \langle \sigma_\alpha \rangle^2$. Thus the $\sigma_\alpha$ fluctuations about the Coulomb branch minima (2.10) become very heavy in this limit and decouple. The end result is that the Coulomb branch vacua are well-separated from the Higgs branch vacuum by a large potential barrier. This is the essential fact we will need.

3. Warmup: $\mathbb{C}^2 / \mathbb{Z}_{a(1)}$

Perhaps the simplest class of examples of the phenomenon of “disappearing” topology are the orbifolds $\mathbb{C}^2 / \mathbb{Z}_{a(1)}$. In this section, we will review how this space and its resolution
are described as different “phases” of the GLSM, and give the prototype of the resolution of the puzzle stated in the introduction.

Consider the $U(1)$ Kähler quotient on $(X_0, X_1, X_2)$ with charges $(1, -a, 1)$ where $a$ is a positive integer. There is a single D-term equation (2.7)

$$\mu := |X_0|^2 - a|X_1|^2 + |X_2|^2 = \zeta$$

which has a solution set $S_\zeta \subset \mathbb{C}^3$. We want to describe the geometry of the quotient $S_\zeta/U(1)$ which arises when we fix the action of the gauge symmetry on $S_\zeta$. This depends strongly on the sign of $\zeta$; for $\zeta < 0$ one finds the $\mathbb{Z}_a(1)$ orbifold singularity at the origin, and for $\zeta > 0$ one finds a smooth resolution of the space.

### 3.1. Description of the quotient space: $\zeta < 0$

For $\zeta < 0$ the variable $X_1$ is necessarily nonzero and can be used to fix a gauge (up to a discrete quotient):

$$(X_0, X_1, X_2) = (e^{i\theta} \xi_1, e^{-ia\theta}|X_1|, e^{i\theta} \xi_2)$$

where $\xi_1, \xi_2 \in \mathbb{C}^2$. Choosing this positive root means that $(\xi_1, \xi_2)$ are gauge invariant up to multiplication by $\omega \in \mathbb{Z}_a$,

$$(\xi_1, \xi_2) \sim (\omega \xi_1, \omega \xi_2), \quad \omega \in \mathbb{Z}_a.$$  

Plugging (3.2) into the GLSM action (2.1), as $\zeta \to -\infty$ the D-term potential freezes the value of $|X_1| \to \infty$, and $\xi_{1,2}$ become free fields. The residual discrete $\mathbb{Z}_a$ gauge action results in the orbifold space $\mathbb{C}^2/\mathbb{Z}_a(1)$.

For finite $\zeta < 0$, the level set $\mu = \zeta$ is simply the subspace

$$|X_1| = + \frac{1}{\sqrt{a}} \sqrt{\rho_1^2 + \rho_2^2} - \zeta$$

where we introduce the magnitude and phase

$$\xi_i := \rho_i e^{i\gamma_i} \quad i = 1, 2.$$  

Note that since $\zeta < 0$ the argument of the square root in (3.4) is always positive. Thus, restricting to fixed value of $|X_1|$, the quotient space is a Lens space $L(a, 1) = S^3/\mathbb{Z}_a(1)$. Roughly speaking, as a space, $S$ is a $U(1)$ bundle over the orbifold $\mathbb{C}^2/\mathbb{Z}_a(1)$. This is not completely accurate because at $\xi_1 = \xi_2 = 0$ the orbifold group acts nontrivially on the fiber.

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6 In complex dimension $d = 2$, it will prove convenient to shift the range of the index on the chiral field by one: $i = 0, \ldots, r + 1$ rather than $1, \ldots, r + 2$. We will do so in the remainder of the discussion.
\section*{3.2. Description of the quotient for $\zeta > 0$}

For $\zeta > 0$, the choice (3.4) still solves the D-term equations (3.1), provided we satisfy the inequality $\rho_1^2 + \rho_2^2 \geq \zeta$; however, the locus $\xi_1 = \xi_2 = 0$ – which resulted in a singular $U(1)$ quotient for $\zeta < 0$ – is lifted from the effective field space for $\zeta > 0$. If we consider the minimal 3-sphere

$$|\xi_1|^2 + |\xi_2|^2 = \zeta$$

(3.6)

in $S_\zeta$ then $X_1 = 0$ and the $U(1)$ group is still completely unbroken by $X_1$. Therefore in the quotient $S_\zeta/U(1)$ this subset projects to $\mathbb{P}^1$. In other words,

$$\{X_1 = 0\} \cap S_\zeta \rightarrow \mathbb{P}^1,$$  

(3.7)

which is to be contrasted with the $U(1)$ quotient of the subset

$$\{X_1 = \epsilon\} \cap S_\zeta;$$

(3.8)

the fiber of this latter space when projected to the quotient is $\mathbb{Z}_a$, so these subspaces project to Lens spaces $L(a, 1)$ inside the quotient. Thus, the singular set of the $U(1)$ quotient for $\zeta < 0$ has been replaced by a nonsingular $\mathbb{P}^1$ for $\zeta > 0$. That is, the singularity has been resolved.

A special case is $a = 1$, where the ‘orbifold’ phase $\zeta < 0$ is nonsingular, and the $\zeta \rightarrow -\infty$ space is $\mathbb{C}^2/\mathbb{Z}_1 \equiv \mathbb{C}^2$. The regime $\zeta > 0$ is however still nontrivial, and describes the blowup of $\mathbb{C}^2$ at a point. Note in particular that the topology of the blown up space is different.

An alternative way to see the geometry for $\zeta > 0$ uses the complex geometry of the quotient space. For $\zeta > 0$, either $X_0 \neq 0$ or $X_2 \neq 0$ for every point on the solution set $S$. Therefore, we divide the solution set into two patches, $X_2 \neq 0$ and $X_0 \neq 0$, and introduce gauge invariant holomorphic coordinates

$$z_+ = X_0/X_2$$

$$p_+ = X_1X_2^a$$

(3.9)

on the patch $X_2 \neq 0$, and

$$z_- = X_2/X_0$$

$$p_- = X_1X_0^a$$

(3.10)
on the patch $X_0 \neq 0$. We now recognize that $z_{\pm}$ are coordinates on $\mathbb{P}^1$, and $p_- = p_+ z_{+}^a$, so that $p_{\pm}$ are fiber coordinates on the complex line bundle $\pi : \mathcal{O}(-a) \to \mathbb{P}^1$.

Incidentally, the relationship between the holomorphic coordinates $(z, p)$ and the coordinates $\xi_i$ defined above is the following. On the patch $X_2 \neq 0$ the coordinates $(z_+, p_+)$ relate to the coordinates $(\xi_1, \xi_2)$ as follows:

$$z_+ = \xi_1 / \xi_2$$

$$p_+ = \xi_2^a \sqrt{(|\xi_1|^2 + |\xi_2|^2 - \zeta) / a}$$

(3.11)

### 3.3. Homology and K-theory

The homology and cohomology groups are

$$H_j(\mathcal{O}(-a)) = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z} & j = 2 \\ 0 & j = 4 \end{cases}$$

(3.12)

$$H^j(\mathcal{O}(-a)) = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z} & j = 2 \\ 0 & j = 4 \end{cases}$$

Poincare duality says that $H^k(X) \cong H_{n-k}(X, \partial X) \cong H_{n-k, \text{cpt}}(X)$ and $H^k_{\text{cpt}}(X) = H^k(X, \partial X) \cong H_{n-k}(X)$. The intersection form on $H_2$ is simply $-a$.

The K-theory is isomorphic to the cohomology (as $\mathbb{Z}$-modules, not as rings), so

$$K^0(X) = \mathbb{Z} \oplus \mathbb{Z}$$

(3.13)

$$K^0_{\text{cpt}}(X) = \mathbb{Z} \oplus \mathbb{Z}.$$  

Now let us compare this with the equivariant K-theory of Kähler quotient in the $\zeta < 0$ phase:

$$K_G(\mathfrak{q}^2) \cong R(G)$$

(3.14)

since $\mathfrak{q}^2$ is equivariantly contractible ($R(G)$ is the representation ring of $G$). For us,

$$R(\mathbb{Z}_a) = \mathbb{Z}[x] / (x^a = 1)$$

(3.15)

as a $\mathbb{Z}$-module this is $\mathbb{Z}^a$, and this seems to have little to do with the K-theory of the Hirzebruch-Jung resolution space $\mathcal{O}(-a)$. This is an example of our basic paradox.
3.4. The canonical line bundle and connection

In accounting for D-brane charges it is useful to have a clear idea of a basis for the K-theory. There is a canonical line bundle \( \mathcal{R} \to \mathcal{O}(-a) \) which is the complex line bundle associated to the principal \( U(1) \) bundle \( \mathcal{S}_\zeta \to \mathcal{O}(-a) \) by the fundamental representation. This bundle carries a canonical connection defined by the 1-form on \( \mathbb{C}^3 \):

\[
\Theta = \frac{i}{2N} \left[ \sum_i Q_i (\bar{X}_i dX_i - X_i d\bar{X}_i) \right].
\]

(3.16)

where

\[
N = \sum_i Q_i^2 |X_i|^2
\]

(3.17)

Restricted to a gauge orbit (3.2) this connection gives \( \Theta = d\theta \); moreover, the Lie derivative along \( d/d\theta \) is zero, and so \( \Theta \) is indeed a connection. As shown in [36,37], this gauge field can be incorporated into an \( \mathcal{N} = 2 \) supersymmetric boundary interaction in the GLSM. The 2-form \( d\Theta|_{\mathcal{S}_\zeta} \) is a basic form for \( \zeta > 0 \), i.e., we can write it as \( \pi^*(F) \). Restricting \( F \) to \( p = 0 \), we find \( F \) integrates to \( 2\pi \) on this sphere, so that the Chern-class of the line bundle is +1. The bundle \( \mathcal{R} \) together with the trivial bundle generate the first line of (3.13).

3.5. Resolution of the puzzle for \( \mathbb{C}^2/\mathbb{Z}_{a(1)} \)

In this simple case of a single \( U(1) \) gauge field, the minima of the effective potential (2.9) lie at

\[
\sigma_1^{(t)} = \Lambda c \exp \left[ \frac{t_{1,\text{bare}} + 2\pi i \ell}{a - 2} \right] = \mu c \exp \left[ \frac{t_{1,\text{eff}}(\mu) + 2\pi i \ell}{a - 2} \right], \quad \ell = 1, \ldots, a - 2.
\]

(3.18)

where \( c = \exp[-a \log(-a)/(a - 2)] \), and we are assuming \( a \neq 2 \). For \( a > 2 \), we see that there are \( a - 2 \) supersymmetric vacua at large \( \sigma \) for large positive \( t_{1,\text{eff}}(\mu) \). For such values of \( t_{1,\text{eff}}(\mu) \) the picture of Coulomb branch vacua will be accurate. D-branes in such massive vacua are localized at the extrema (3.18) of the effective superpotential, \( \mathbb{L} \) thus there are \( a - 2 \) independent D-branes associated to the Coulomb branch vacua; these, together with the two localized brane charges of the Higgs branch \( \mathcal{O}(-a) \) geometry, account for the full rank \( a \) of the equivariant K-theory lattice of D-brane charges of the orbifold.

\footnote{This is true in the topologically twisted theory simply because in the topological theory every primitive idempotent of the Frobenius algebra leads to a single D-brane \[38\]. An explicit construction of the \( \mathcal{N} = 2 \) Landau-Ginzburg boundary states is described in [36,37,39].}
Of course, for \( a = 2 \) one has the spacetime supersymmetric \( A_1 \) ALE space as the resolution of the orbifold. The coupling \( t_1 \) is marginal, and there is no Coulomb branch of the GLSM configuration space. Of course, we weren’t looking for one, since the two localized brane charges of the geometrical resolution of the orbifold account for all of the D-brane charges found in the orbifold.

Finally, for \( a = 1 \) one has the blowup of \( \mathbb{C}^2 \) discussed in subsection 2 above. In this case, the RG flow runs in the opposite direction: In the flow to the IR, i.e. as \( \mu \to 0 \), the \( \mathbb{P}^1 \) of the resolved space blows down to a point and disappears, so that \( \mathbb{C}^2 / \mathbb{Z}_1 = \mathbb{C}^2 \) is the IR limit of the flow; the UV fixed point is then the sigma model with the \( \mathbb{P}^1 \) infinitely blown up. Thus again the UV fixed point theory has one more D-brane charge than the IR geometry (associated to the \( \mathbb{P}^1 \)). This “missing” charge is again found on the Coulomb branch in the IR theory; it is just that according to (2.3), \( \text{Re} (t_1, \text{eff} (\mu)) \to -\infty \) rather than \( +\infty \) along the flow, compatible with the VEV for \( \sigma_1 \) given by (3.18), increasing as \( \mu \to 0 \).

In the next few sections, we will generalize this result to arbitrary nonsupersymmetric \( \mathbb{C}^2 / \mathbb{Z}_{n(p)} \) orbifolds. We will see that, apart from the additional complication of multiple \( U(1) \) gauge fields, corresponding to multiple curves in the resolution of the orbifold singularity, the basic structure is much the same as we have just encountered in the rank one case. In particular, a careful analysis of the Coulomb and Higgs branches of the GLSM will account for all of the D-brane charges of the orbifold CFT.

4. Generalized Cartan matrices and continued fractions

The general orbifold \( \mathbb{C}^2 / \mathbb{Z}_{n(p)} \) can be realized as a quotient of an \((r + 2)\)-dimensional space by a \( U(1)^r \) action generalizing the construction of the previous section. This allows one to resolve the singularity in a similar fashion. The resolution involves a sequence of \( r \) blowups, and should therefore be realized as a phase in a \( U(1)^r \) GLSM. The algebraic geometry of the resolution is for instance explained in [6,8]. The sequence of blowups produces a chain of \( \mathbb{P}^1 \)'s; the north pole of the \( \alpha^{th} \) \( \mathbb{P}^1 \) intersects the south pole of the \((\alpha + 1)^{st} \) \( \mathbb{P}^1 \), and the self-intersection numbers are \(-a_\alpha\), \( \alpha = 1, \ldots, r \). Among the toric

---

8 This example supports the idea of the Coulomb branch vacua decoupling from the Higgs branch, rather than forming a throat as in fivebrane physics. The formation of such a throat here would imply the existence of a stable object in string theory on flat space carrying no conserved charges.
resolutions of the orbifold there is a “minimal resolution” of smallest $r$. This minimal resolution has all $a_\alpha \geq 2$.

The blown up space for any such resolution (not just the minimal one) is covered by $r + 1$ coordinate patches $\mathbb{C}^2$. On overlaps, the coordinates $u$, $v$ of successive patches are related via

$$v_{\alpha + 1} = u_{\alpha}^{-1}$$
$$u_{\alpha + 1} = v_{\alpha} u_{\alpha}^{a_\alpha} \quad .$$

(4.1)

Note the appearance of the transition functions for the patches (3.9), (3.10) covering $\mathcal{O}(-a_\alpha)$, so that indeed the curves have the advertised self-intersection numbers. Note also that the coordinate on the normal bundle of the $\alpha^{th}$ $\mathbb{P}^1$ is the projective coordinate on the $(\alpha + 1)^{st}$ $\mathbb{P}^1$, etc; one sees directly that the intersection of the $\alpha^{th}$ and $(\alpha + 1)^{st}$ spheres is the point $u_\alpha = u_{\alpha + 1} = 0$.

We would now like to use this data specifying a resolution of the singularity to define a gauged linear sigma model. The fact that each $\mathbb{P}^1$ of the resolution has the structure of $\mathcal{O}(-a_\alpha)$ leads us to define the $r \times (r + 2)$ charge matrix:

$$Q_{\alpha i} = -a_\alpha \delta_{\alpha i} + \delta_{\alpha + 1, i} + \delta_{\alpha - 1, i} \quad .$$

(4.2)

where $0 \leq i \leq r + 1$ and $1 \leq \alpha \leq r$. We also denote by $C_{\alpha\beta}$ the $r \times r$ square matrix $-Q_{\alpha\beta}$ for $1 \leq \beta \leq r$. This is a symmetric matrix which we refer to as a generalized Cartan matrix.

In the remainder of this subsection we will gather some mathematical facts about the singularity resolution that will be of use to us in the sequel; we also indicate the physical interpretation of some of these mathematical identities.

The ordered set of positive integers $a_\alpha$ appearing in the continued fraction expansion (1.3) can be used to generate two related sets of integers $q_i$ and $p_i$, $i = 0, \ldots, r + 1$ via the recursion relations:

$$p_{j - 1}/p_j = [a_j, a_{j + 1}, \ldots, a_r]$$
$$q_{j + 1}/q_j = [a_j, \ldots, a_1] \quad , \quad 1 \leq j \leq r \quad (4.3)$$

(where the fractions are in lowest terms). In addition we define $p_{r + 1} = 0$ and $q_0 = 0$. In particular (4.3) for $j = 1$ gives $n = p_0$ and $p = p_1$. \footnote{Using the recursion relations below one can show that $n = \det[C_{\alpha\beta}]$ while $p$ is the determinant of the first minor.}
The sequences of integers $p_j$, and $q_j$ will be very useful in what follows. The first notable fact is that, for the minimal resolution having all $a_\alpha \geq 2$, the vectors

$$\frac{1}{n} v_j = \frac{1}{n} (q_j, p_j) , \quad j = 0, \ldots, r + 1$$

(4.4)

are the $U(1)_x \times U(1)_y$ $R$-charges of a set of chiral operators $\{ T_{q_\alpha} \}$, $\alpha = 1, \ldots, r$, which generate the chiral ring of the orbifold $\mathbb{C}^2/\mathbb{Z}_n(p)$.

To see this, note that the vectors $v_i = (q_i, p_i)$ satisfy a repackaged form of the recursion relation:

$$a_i v_i = v_{i-1} + v_{i+1} \quad 1 \leq i \leq r .$$

(4.5)

with the boundary conditions $q_0 = 0$, $q_1 = 1$ for the $q_i$, and $p_{r+1} = 0$, $p_r = 1$ for the $p_i$. It thus follows that

$$v_j = q_j v_1 + B_j v_0 ;$$

(4.6)

here $B_\alpha$ are integers satisfying the recursion relation $a_j B_j = B_{j+1} + B_{j-1}$, $B_0 = 1$, $B_1 = 0$. A solution of these recursion relations with the required initial conditions is $nB_j = p_j - p_1 q_j$, and hence

$$(q_j, p_j) = q_j v_1 + (0, nB_j)$$

(4.7)

Now, for the minimal resolution when all the $a_\alpha > 1$, the sequences $q_i, p_i$ satisfy

$$q_0 = 0 < q_1 = 1 < \ldots < q_{r+1} = n$$

$$p_0 = n > p_1 = p > \ldots > p_r = 1 > p_{r+1} = 0 ,$$

(4.8)

and hence the vectors $v_i$ lie in the fundamental domain. However, from (1.3) we see that the $R$-charge of the chiral field in the $\kappa$ twisted sector is just

$$\frac{\kappa}{n} v_1 \mod 1 ,$$

(4.9)

and so we identify (4.7) with the $R$-charges of the $\kappa = q_\alpha$ twisted sector.

Finally, to see that these are the generators of the chiral ring we proceed as follows. First note that for all resolutions (minimal or not) it is true that

$$\frac{p_i}{q_i} > \frac{p_{i+1}}{q_{i+1}} ,$$

(4.10)

in other words, the slopes of successive $R$-charge vectors is always decreasing. This is because, by (1.3), the $\alpha^{th}$ vector lies between its neighbors. Now consider the parallelogram
spanned by \( v_{i-1}, v_{i+1} \) and draw the line through \( v_{i-1} + v_{i+1} \). From this figure it is clear that if \( a_i \geq 2 \) then \( v_{i-1}, v_i, v_{i+1} \) form part of the boundary of a convex region in the \((q, p)\) plane (known as the *Newton boundary*). On the other hand, if \( a_i = 1 \) \( v_{i-1}, v_i, v_{i+1} \) certainly do not form a convex boundary. For the minimal resolution we can write all the R-charge vectors in the fundamental domain as positive integral combinations of the generating set of vectors \( v_i \). Assuming the OPE coefficients are generically nonzero, we conclude that these twist fields are a set of generators of the chiral ring.

Equation (4.5) implies that one can write relations on the orbifold chiral ring,

\[
(T_{q_\alpha})^{\alpha} = T_{q_{\alpha-1}} T_{q_{\alpha+1}} , \quad \alpha = 1, \ldots, r
\]  

(4.11)

(where we have defined \( T_0 = V_Y \) and \( T_{r+1} = V_X \), the untwisted sector “volume form” chiral operators of the \( X, Y \) planes of \( \mathbb{C}^2 \)). These ring relations simply encode the additivity of twist quantum numbers in the \( X, Y \) planes of \( \mathbb{C}^2 \).

It follows from (4.5) that \( v_i \times v_{i+1} = v_{i-1} \times v_i \), and hence:

\[
q_i p_{i-1} - q_{i-1} p_i = n , \quad 1 \leq i \leq r + 1.
\]  

(4.12)

More generally, one can show that, for \( i > j + 1 \),

\[
q_i p_j - p_i q_j = n' n , \quad 0 \leq i \leq r + 1
\]  

(4.13)

where (for \( i > j \)) the continued fraction \([a_{j+1}, \ldots, a_{i-1}] = n'/p' \) determines \( n' \). We will make extensive use of these identities below.

There is also a nice formula for the inverse of the generalized Cartan matrix \( C_{\alpha\beta} \) in terms of the \((q_i, p_i)\)\(^{10}\)

\[
(C^{-1})^{\alpha\beta} = \begin{cases} \frac{1}{n} q_\alpha p_\beta & 1 \leq \alpha \leq \beta \leq r \\ \frac{1}{n} p_\alpha q_\beta & 1 \leq \beta \leq \alpha \leq r \end{cases}.
\]  

(4.14)

One easily proves this claim using (4.5) and (4.12).

Finally, we come to an important identity on continued fractions. Let us define \([x, y] = x - 1/y \) for any pair of *real* numbers \( x, y \), and then define multiple continued fractions via \([x, y, z] := [x, [y, z]]\)\(^{11}\) A simple computation shows that

\[
[x + 1, 1, y + 1] = [x, y] ;
\]  

(4.15)

---

\(^{10}\) This equation elegantly generalizes the standard formula for the inverse Cartan matrix of the \( A_r \) Dynkin diagram defined by \( a_\alpha = 2 \), where \( q_i = i \) and \( p_i = (r + 1 - i) \), \( 0 \leq i \leq r + 1 \).

\(^{11}\) Warning: The ordering of the brackets matters.
this is why the continued fraction expansion of $n/p$ is only unique if all the $a_\alpha > 1$.

Returning to the resolution of $\Phi^2/\mathbb{Z}_{n(p)}$, the minimal resolution of the singularity is defined by the criterion that all the $a_\alpha > 1$. As we have mentioned, there are ‘non-minimal’ resolutions of the singularity obtained by blowing up the point of intersection of the $k^{th}$ and $(k+1)^{st}$ $\mathbb{P}^1$’s in the resolution chain. Since the space was nonsingular before this operation, one is blowing up a point on what is locally $\mathbb{C}^2$, and this results in a curve of self-intersection $-1$. The effect on the continued fraction expansion is

$$\frac{n}{p} = [a_1, \ldots, a_k, a_{k+1}, \ldots, a_r] \rightarrow \frac{n}{p} = [a_1, \ldots, a_k + 1, a_{k+1} + 1, \ldots, a_r]. \quad (4.16)$$

This expanded sequence may be used to define a charge matrix (4.2) and hence a GLSM with $U(1)^{r+1}$ gauge group. One may readily check that the original sequences of integers $p_\beta, q_\beta, \beta = 1, \ldots, r$ is unaltered, and that a new pair $p_* = p_k + p_{k+1}$ and $q_* = q_k + q_{k+1}$ is added. More precisely, the sequence of integers on the RHS of (4.16) defines a set of vectors $\hat{v}_I, I = 0, \ldots, r + 2$ related to the original charge vectors by

$$\hat{v}_i = v_i \quad i = 0, \ldots, k$$
$$\hat{v}_{k+1} = v_k + v_{k+1}$$
$$\hat{v}_i = v_{i-1} \quad i = k + 2, \ldots, r + 2. \quad (4.17)$$

Associated to $\hat{v}_{k+1}$ is a dependent chiral field $\mathcal{T}_{\hat{q}_{k+1}} = \mathcal{T}_{\hat{q}_k} \mathcal{T}_{\hat{q}_{k+2}}$ in the chiral ring of the orbifold twist fields.\(^{12}\) The blowing up procedure can of course be repeated any number of times.

5. Geometry of $U(1)^r$ quotients

In this section we will discuss the geometry of the solution of the D-term equations (2.7) for the charge matrix $Q_{\alpha i}$ defined in (4.2). We will show that if $\zeta_\alpha < 0$ for all $\alpha$ then this space is simply a $U(1)^r$ “bundle” over the orbifold $\Phi^2/\mathbb{Z}_{n(p)}$. (The quotation marks refer to the fact that the fibration degenerates over the origin, because $\mathbb{Z}_n$ fixes the origin.) When some of the $\zeta_\alpha > 0$ there is a topology change and we get a partial resolution of the singularity. If all $\zeta_\alpha > 0$ then we have a $U(1)^r$ bundle over a toric resolution of the

\(^{12}\) When it exists; it may happen that the candidate operator $\mathcal{T}_{\hat{q}_{k+1}}$ lies outside of unitarity bounds on the chiral ring of the $\mathcal{N} = 2$ orbifold CFT.
singularity. When all the $a_\alpha > 1$ this is the Hirzebruch-Jung, or minimal resolution of the orbifold singularity associated to the continued fraction $n/p = [a_1, \ldots, a_r]$.

One clear way to understand the geometry of the quotient space is to make a change of basis on the generators of the $U(1)^r$ gauge group so as to diagonalize the $U(1)^r$ action on the $X_\alpha$, $\alpha = 1, \ldots, r$. One thus defines

$$\phi_\beta = C_{\beta\alpha} \theta_\alpha \quad (5.1)$$

so that the gauge rotation acts as

$$X_\beta \to e^{i\phi_\beta} X_\beta \quad (5.2)$$

with no sum, for $1 \leq \beta \leq r$.

This change of basis of the gauge group generators leads to a corresponding change in the D-term equations. We define

$$R_{\alpha i} = C_{\alpha \beta}^{-1} Q_{\beta i} = \frac{p_\alpha}{n} \delta_{i,0} + \frac{q_\alpha}{n} \delta_{i,r+1} - \delta_{\alpha,i} \quad (5.3)$$

where in the second equation we have used (4.14). Accordingly the D-term equations become

$$n R_{\beta i} |X_i|^2 = p_\beta |X_0|^2 - n |X_\beta|^2 + q_\beta |X_{r+1}|^2 = n \zeta'_\beta \quad (5.4)$$

where we define

$$\zeta'_\alpha := C_{\alpha \beta}^{-1} \zeta_\beta \quad (5.5)$$

Let us denote the solution set of (5.4) by $S_{\zeta'}$. We are interested in the geometry of the quotient space $S_{\zeta'}/U(1)^r$.

The different phases of the linear sigma model are again controlled by the signs of the $\zeta_\alpha$ (not the $\zeta'_\alpha$, since as we shall see it is the former that are the physical sizes of the $\mathbb{P}^1$’s in the resolution chain). Nevertheless, as we discuss in the next section, discussions of the renormalization group flow are most naturally expressed in terms of $\zeta'_\alpha$ since their $\beta$ function is directly related to the $R$-charges of the perturbations.
5.1. Description of the quotient when all $\zeta_\alpha < 0$

Consider first the orbifold phase of large negative $\zeta_\alpha$. Since the matrix elements of $C^{-1}$ are all positive, when all the $\zeta_\alpha$ are negative the $\zeta'_\alpha$ are also negative. The D-term constraints then force $|X_\beta| > 0$. One may then fix the gauge analogous to (3.2) by choosing $X_\beta > 0$. That is, we can write the general element on the solution set in the form:

$$(X_0, X_\beta, X_{r+1}) = \left(e^{i\vec{p} \cdot \vec{\phi}/n} \xi_1, e^{-i\phi_\beta |X_\beta|}, e^{i\vec{q} \cdot \vec{\phi}/n} \xi_2\right)$$

(5.6)

(where e.g. $\vec{p} \cdot \vec{\phi} = \sum_{\alpha} p_\alpha \phi_\alpha$, and $\beta = 1, \ldots, r$), again up to a discrete quotient. The unbroken discrete gauge symmetry is in general a subgroup of $\mathbb{Z}_n$ for a given $U(1)$, acting as

$$(\xi_1, \xi_2) \sim (\omega^{p_\beta} \xi_1, \omega^{q_\beta} \xi_2)$$

(5.7)

(it may happen that the greatest common divisor of $n$, $p_\beta$, and $q_\beta$ is greater than one). Because of the identity (4.13), the $(q_\beta)$th power of the $\alpha$th group action (5.7) is identical to the $(q_\alpha)$th power of the $\beta$th group action (5.7); the various group actions are simply different elements of the same $\mathbb{Z}_n$. Our canonical choice is to consider the $\mathbb{Z}_n$ to be generated by (5.7) for $\beta = 1$, with $p_1 = p$ and $q_1 = 1$. We conclude that if all $\zeta_\beta < 0$ then fixing the gauges $|X_\beta| > 0$ leaves unbroken a single $\mathbb{Z}_{n(p)}$ gauge symmetry out of the $U(1)^r$ and hence the quotient is $\mathbb{Q}^2/\mathbb{Z}_{n(p)}$.

5.2. Description in the region $\zeta_\alpha > 0$

It is trivial to solve the D-term equations in the form (5.4). Let us write the solution as

$$|X_\alpha|^2 = \frac{p_\alpha}{n} \rho_1^2 + \frac{q_\alpha}{n} \rho_2^2 - \zeta'_\alpha$$

(5.8)

where again $\rho_i = |\xi_i|$, $i = 1, 2$. In the region $\mathcal{D}$ where

$$p_\alpha |X_0|^2 + q_\alpha |X_{r+1}|^2 > n \zeta'_\alpha$$

(5.9)

for all $\alpha$, we can fix the gauge and parametrize the solution of the level set $\mu_\alpha = \zeta_\alpha$ by

$$(X_0, X_1, \ldots, X_{r+1}) = (e^{i\theta_1} \xi_1, e^{-i\phi_1} |X_1|, \ldots, e^{-i\phi_r} |X_r|, e^{i\theta_r} \xi_2)$$

(5.10)

where $\phi_\alpha = C_{\alpha\beta} \theta_\beta$. 21
Figure 3.  (a) When all $\zeta_\alpha > 0$ in the minimal Hirzebruch-Jung resolution, the convex region $D$ in the $\rho_1^2-\rho_2^2$ plane allowed by the $D$-term constraints $|X_\alpha| > 0$ is indicated by the shaded region. The $\alpha^{th}$ segment is the longitudinal direction of the minimal size curve $C_\alpha$ of the resolved space.  

(b) Blowing down, say, the curve $C_\alpha$ is achieved by adjusting the associated FI parameter $\zeta_\alpha$ to negative values, so that the constraint (5.9) is redundant; the line segment $|X_\alpha| = 0$ shrinks away.

When we use the data of the minimal resolution, the region (5.9) describes a convex region $D$ in the $(|X_0|^2,|X_{r+1}|^2)$ plane which is indicated in figure 3. The region is convex due to the property (4.10), which says that the normals to the constraint boundaries have monotonically decreasing slope. The boundaries of $D$ are line segments where $|X_\alpha| = 0$. On this line segment we cannot fix the gauge freedom as in (5.10). In particular, the $U(1)$ gauge freedom associated with the angle $\phi_\alpha$ cannot be fixed.

The line segment $|X_\alpha| = 0$ is most usefully described as the equation

$$|X_{\alpha-1}|^2 + |X_{\alpha+1}|^2 = \zeta_\alpha$$  \hspace{1cm} (5.11)

with $0 \leq |X_{\alpha\pm 1}|^2 \leq \zeta_\alpha$. On the interior of this segment all $|X_\beta|^2 > 0$ for $\beta \neq \alpha$, by convexity. Accordingly, we may partially fix the gauge by choosing $X_i > 0$ for $i = 0, \ldots, \alpha - 2$ and $i = \alpha + 2, \ldots, r + 1$. Using (4.14) and (4.13) it is easy to show that $\phi_1, \ldots, \phi_{\alpha-2}, \phi_{\alpha+2}, \ldots, \phi_{r+1}$ are completely fixed, while the single remaining $U(1)$ gauge freedom is described by

$$\phi_{\alpha - 1} = \phi_{\alpha + 1} = -\frac{1}{a_\alpha} \phi_\alpha$$  \hspace{1cm} (5.12)
Therefore, the quotient of (5.11) by this remaining gauge freedom is simply the standard Hopf fibration over \( \mathbb{P}^1 \). Thus, by including the boundary \( X_\alpha = 0 \) into the Kähler quotient we are including a sphere, which we denote as \( C_\alpha \). This sphere is the zero-section of the normal bundle with \( X_\alpha \) as holomorphic normal coordinate. From (5.12) we learn that the transition function of the normal bundle over \( C_\alpha \) is \( e^{-ia_\alpha \phi} \) where \( \phi \) is the azimuthal coordinate on \( \mathbb{P}^1 \), and hence the self-intersection number of \( C_\alpha \) is \( -a_\alpha \). The endpoints of the interval \( X_{\alpha \pm 1} = 0 \) describe the intersection with the spheres \( C_{\alpha \pm 1} \). In this way one verifies again the fact that the intersection form of the minimal resolution is

\[
C_\alpha \cdot C_\beta = -C_{\alpha \beta}.
\] (5.13)

The above description of \( C_\alpha \) makes it straightforward to compute the volume of the curve \( C_\alpha \) in the Kähler quotient metric. To do this we further fix the gauge by requiring that \( X_{\alpha - 1} > 0 \). Setting \( z = X_{\alpha + 1}/X_{\alpha - 1} \), a stereographic coordinate for \( \mathbb{P}^1 \), we have

\[
X_{\alpha - 1} = \frac{\sqrt{\zeta_\alpha}}{1 + |z|^2}.
\] (5.14)

Now, restricting the Kähler form

\[
\Omega = \frac{i}{2} \sum_{i=0}^{r+1} dX_i \wedge d\bar{X}_i
\] (5.15)

to this gauge slice we get

\[
\iota^* (\Omega) = \frac{1}{2} d(|z|^2 X_{\alpha - 1}^2) \wedge d\theta
\] (5.16)

where \( \theta \) is the phase of \( z \). It follows that

\[
\text{vol}(C_\alpha) = \pi \zeta_\alpha.
\] (5.17)

One must use caution when interpreting (5.17) in the quantum theory. First of all the kinetic terms in the sigma model limit takes the metric of the target space away from the metric induced by the Kähler quotient. Moreover, the Kähler quotient metric itself is only an approximation to the the renormalized spacetime in certain regions of spacetime. Indeed, the literal Kähler quotient metric is not ALE; rather, it has a fairly intricate structure at infinity. However, we should take the continuum limit \( \Lambda \to +\infty, \zeta_{\text{bare}} \to -\infty \), while working at finite values of the chiral fields \( X_i \). In this regime we expect the induced metric to be a reliable qualitative guide to the geometry.
5.3. Homology, Cohomology, and K-theory for the toric resolutions

Let $\mathcal{X}$ be a toric resolution of $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ corresponding to $n/p = [a_1, \ldots, a_r]$. We need not assume it is the minimal resolution.

Since the homology and cohomology groups are homotopy invariants, we can compute them from the deformation retract of $\mathcal{X}$ to the chain of spheres. It follows that:

$$H_j(\mathcal{X}) = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}^r & j = 2 \\ 0 & j = 4 \end{cases}$$

$$H^j(\mathcal{X}) = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}^r & j = 2 \\ 0 & j = 4 \end{cases}$$

As we have seen in the previous section, the intersection form on $H_2$ is simply $-C_{\alpha\beta}$.

Now, Poincare duality for the smooth space $\mathcal{X}$ says that $H^k(\mathcal{X}) \cong H_{n-k}(\mathcal{X}, \partial \mathcal{X}) \cong H_{n-k,cpt}(\mathcal{X})$ and $H^k_{cpt}(\mathcal{X}) = H^k(\mathcal{X}, \partial \mathcal{X}) \cong H_{n-k}(\mathcal{X})$. It follows that

$$H_{j,cpt}(\mathcal{X}) = \begin{cases} 0 & j = 0 \\ \mathbb{Z}^r & j = 2 \\ \mathbb{Z} & j = 4 \end{cases}$$

$$H^j_{cpt}(\mathcal{X}) = \begin{cases} 0 & j = 0 \\ \mathbb{Z}^r & j = 2 \\ \mathbb{Z} & j = 4 \end{cases}$$

Evidently, there is no torsion in the cohomology, and, by the Atiyah-Hirzebruch spectral sequence, no torsion in the K-theory.

It follows that

$$K^0(\mathcal{X}) = \mathbb{Z} \oplus \mathbb{Z}^r$$

$$K^0_{cpt}(\mathcal{X}) = \mathbb{Z} \oplus \mathbb{Z}^r$$

while $K^1 = 0$.

There is a natural basis for $K^0(\mathcal{X})$ provided by the tautological line bundles. First, we take $\mathcal{O}$, the trivial complex line bundle. This corresponds to a single $Dp$-brane wrapping

\footnote{A nice description of the K-theory of toric varieties is given in \cite{40}. The author states that his results are only guaranteed for compact toric varieties. It would be nice to know if they apply to the noncompact case.}
the space $\mathcal{X}$ with trivial Chan-Paton bundle and zero connection. Next, we construct tautological line bundles $\mathcal{R}_\alpha$ corresponding to $Dp$-branes filling $\mathcal{X}$ with magnetic monopoles in the different exceptional divisors $\mathcal{C}_\alpha$. These are constructed as follows:

In the geometrical phase we have a $G = U(1)^r$ principal bundle $\mathcal{S}_\zeta \to \mathcal{X}$. Let us denote a generic element $g \in G$ by

$$g = (e^{i\theta_1}, \ldots, e^{i\theta_r}) := (g_1, \ldots, g_r)$$

and acting on the chiral fields as

$$X_i \to \prod_\alpha g_\alpha \cdot X_i .$$

We can define a natural collection of line bundles $\mathcal{R}_\alpha$ as the associated bundle to the representation $\rho_\alpha (g) = e^{i\theta_\alpha}$. These are

$$\mathcal{R}_\alpha := (\mathcal{S}_\zeta \times \mathbb{C})/U(1)^r$$

where the $\alpha^{th}$ $U(1)^r$ action is

$$g \cdot (X, v) = (X \cdot g, e^{-i\theta_\alpha} v)$$

There is a canonical pairing of $K(\mathcal{X}) \otimes K_{cpt}(\mathcal{X}) \to \mathbb{Z}$ given by the index theorem. Under the Chern isomorphism this is the same as the pairing $H^*(\mathcal{X}) \otimes H^*_{cpt}(\mathcal{X}) \to \mathbb{Z}$. Thus, $\mathcal{O}$ is dual to a D0-brane supported at a point on $\mathcal{X}$, and there is a nondegenerate pairing between 4-branes with 2-brane charge $\mathcal{R}_\alpha$ and 2-branes wrapping $\mathcal{C}_\alpha$. These physical statements have precise mathematical analogs, and indeed a basis for the compactly supported derived category of $\mathcal{X}$ (and hence of the compactly supported $K$-theory) is described in [28,29,8].

Now, if we compare this with the equivariant K-theory appropriate to the orbifold then

$$K_\Gamma(\mathbb{C}^2) = \mathbb{Z} \oplus \mathbb{Z}^{n-1}$$

where the first summand corresponds to the regular representation and hence to a D0 brane, while the second summand corresponds to the fractional branes. Comparing this with the compactly supported $K$-theory of $\mathcal{X}$ we see that for the non-supersymmetric case when $r < n - 1$ we have a mismatch of the K-theory of the orbifold and of its smooth resolution, and thus a general statement of the problem of what happens to the extra $n - r - 1$ topological charges, and the D-branes they couple to. The resolution of this puzzle will be a generalization of the one we found for the rank one case in section 3. However, before we discuss it, let us first turn to a more detailed examination of the tautological bundles $\mathcal{R}_\alpha$. 

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5.4. Canonical line bundles and special representations of $\mathbb{Z}_n$

In the mathematics literature [28, 29, 8] there is a statement of a generalized McKay correspondence under which the tautological bundles $R_\alpha$ on the minimal resolution correspond to so-called “special representations” of $\mathbb{Z}_n$. In this section we will give a simple physical interpretation of this concept.

As we have seen, when all the $\zeta_\alpha > 0$ there is a set of tautological line bundles associated with the unitary irreps $\rho_i$ of $G = U(1)^r$. In the phase where $\zeta_\alpha < 0$, the gauge group $U(1)^r$ is spontaneously broken to $\Gamma = \mathbb{Z}_n$ by the nonzero VEV’s of the chiral scalar fields $X_\alpha$. We may identify this group $\Gamma$ with the orbifold group of the CFT. The representations $\rho$ of $G$ may be restricted to the subgroup $\Gamma$, and as such are unitary irreps of $\Gamma$. These are the “special representations” of the mathematical literature.\[14\] We will now explain this construction in more detail.

Consider the tautological bundles $R_\alpha$ in the geometrical phase, $\zeta_\alpha > 0$ for all $\alpha = 1, \ldots, r$. If we “continue” in $\zeta$ to the region with all $\zeta_\alpha < 0$ what happens to the quotient space (5.23)?\[15\]

As we have seen in sec. 5.1, in the orbifold phase the gauge group is broken to a $\Gamma = \mathbb{Z}_n$ subgroup of $G$. Let us describe this subgroup more precisely. The $X_\beta$ for $1 \leq \beta \leq r$ have nonzero vev’s, so by (5.10) one has

$$\phi_\beta = 2\pi m_\beta$$

(5.25)

for integers $m_\beta$; these phases are related to those of $U(1)^r$ in equation (5.21), via

$$\theta_\alpha = C^{-1}_{\alpha\beta} \phi_\beta = \frac{1}{n} \left( \sum_{\alpha < \beta} q_{\alpha\beta} \phi_\beta + \sum_{\alpha \geq \beta} p_{\alpha\beta} \phi_\beta \right)$$

(5.26)

Due to equations (4.13), (5.25), one can replace $q_{\alpha\beta}$ by $p_{\alpha\beta}$ in the first sum, modulo $2\pi \mathbb{Z}$. We then have that the unbroken subgroup is the set of elements

$$\left( \exp[i\theta_1], \ldots, \exp[i\theta_r] \right) = \left( \exp[2\pi i \frac{p_1}{n} \hat{q} \cdot \hat{m}], \exp[2\pi i \frac{p_2}{n} \hat{q} \cdot \hat{m}], \ldots, \exp[2\pi i \frac{p_r}{n} \hat{q} \cdot \hat{m}] \right)$$

(5.27)

\[14\] A similar construction has been used by P. Mayr in [21].

\[15\] By “continue” we mean that one can consider the total space $S$ of the family of manifolds $S_\zeta$ over the base space of all $\zeta$. There is a $U(1)^r$ action on $S$ and we can form the associated line bundle over the quotient. We choose a single representation of $U(1)^r$ and compare the same line bundle restricted to the fiber over $\zeta > 0$ and $\zeta < 0$.  

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as \( \vec{m} \) ranges over all elements of \( \mathbb{Z}^r \).

Now, since \( q_1 = 1 \), the quantity \( \vec{q} \cdot \vec{m} \) takes on all integral values. Therefore, the unbroken subgroup is precisely the \( \mathbb{Z}_n \) subgroup described by integer powers of

\[
\hat{g} := \left( \exp[2\pi i \frac{p_1}{n}], \exp[2\pi i \frac{p_2}{n}], \ldots, \exp[2\pi i \frac{p_r}{n}] \right) . \tag{5.28}
\]

Let us now consider the “evolution” of the line bundle \( R_{\alpha} \) associated to the gauge group \( U(1)^r \) by the representation

\[
\rho_{\alpha}(g) = e^{i\theta_{\alpha}} \tag{5.29}
\]

as we proceed form \( \zeta_{\alpha} > 0 \) to \( \zeta_{\alpha} < 0 \). Evidently, this becomes a line bundle associated to the unbroken gauge group \( \Gamma \subset U(1)^r \) (generated by (5.28)) in the orbifold phase. By equation (5.28) that representation is the \( p_{\alpha}^{th} \) power of “the” fundamental representation of \( \mathbb{Z}_n \).

The reason for the quotes is that there is some ambiguity in this statement since one must choose a generator of the dual group \( \widehat{\mathbb{Z}}_n \cong \mathbb{Z}_n \) in order to speak of the \( p_{\alpha}^{th} \) power of a generator of \( \widehat{\mathbb{Z}}_n \). Since \( p_r = 1 \), we are implicitly choosing the generator \( \rho_f \) that takes

\[
\hat{g} \rightarrow \exp[2\pi i/n] , \tag{5.30}
\]

and then the special representations are

\[
\rho_{\alpha} = (\rho_f)^{p_{\alpha}} \tag{5.31}
\]

which is the main result of this subsection.\(^{16}\)

In summary, the natural line bundles \( R_{\alpha} \) of the resolved Hirzebruch-Jung space “analytically continue” to the \( r \) “special representations” of \( \mathbb{Z}_n \) in the quiver picture of the orbifold. We think this gives a nice picture of the “generalized McKay correspondence,” advocated in [28,29,8]. Having understood this, we also see that the “special” representations just defined are not so special. By additional blowups of the sort described at the end of section 4, one introduces additional R-charge vectors \( v_* = (q_*, p_*) \) associated to the resolution of the singularity. If the \( p_* \) do not belong to the set \( \{p_1, \ldots, p_r\} \) of the minimal resolution, we can account for more of the \( \mathbb{Z}_n \) representations associated to fractional branes of the orbifold. In fact, this is only part of the story, as we will see shortly.

\(^{16}\) One could however make other choices. For example, one could consider the resolution chain in the reverse order, via \( n/p' = [a_r, a_{r-1}, \ldots, a_1] \). Geometrically this is the same space, and one readily sees that \( p' = q_r \) of the original sequence \( [a_1, \ldots, a_r] \). By (4.13), \( p'p = p_rq_1 = 1 \) mod \( n \), and so this choice of generator of \( \mathbb{Z}_{n(p)} \) is simply the \( p' \) power of \( \hat{g} \). The result of reversing the sequence of \( a_{\alpha} \) is to interchange the \( p \)'s and \( q \)'s, so that the \( \mathbb{Z}_n \) representations associated to the line bundles become \( \rho_f^{p_{\alpha}} \).
5.5. Connections on the canonical line bundles $\mathcal{R}_\alpha$

In order to write boundary interactions in the GLSM we need an actual connection on the line bundles $\mathcal{R}_\alpha$. In this section we will write formulae for one natural set of connections, $\Theta_\alpha$. We hope these will prove useful in future studies of the fate of D-branes in the geometrical phase using the methods of [36,37,11,33,42,13,14].

A general principle gives us a natural $G$-connection on the principal $G$ bundle $S_\zeta$. Whenever there is a $G$-invariant metric on a principal $G$-bundle $P$, there is a natural connection: The horizontal subspaces are the orthogonal complements to the $G$-orbits. In terms of connection 1-forms the canonical vector fields define a map $B : g \rightarrow TP$. The metrics allow us to define $B^\dagger : TP \rightarrow g$, that is, a Lie-algebra-valued 1-form. The connection form is

$$\Theta = \frac{1}{B^\dagger B} B^\dagger$$

(5.32)

Applying this principle to the present example we get the following: Define

$$\tilde{\Theta}_\alpha = \frac{i}{2} \sum_i Q_{\alpha i} (\bar{X}_i dX_i - X_i d\bar{X}_i)$$

$$N_{\alpha\beta} = \sum_i Q_{\alpha i} Q_{\beta i} |X_i|^2$$

(5.33)

then

$$\Theta_\alpha = N_{\alpha\beta}^{-1} \tilde{\Theta}_\beta$$

(5.34)

is the natural connection on the line bundle $\mathcal{R}_\alpha$.

The connection (5.34) arises naturally as a boundary interaction in the gauged linear sigma model [36,37]. The generalization of the treatment of [36] section 6 to multiple gauge fields is

$$L_{bdy} = \frac{i}{2} n_{\alpha} Q_{\alpha i} (\bar{X}_i DX_i - X_i D\bar{X}_i) - n_\alpha v_\alpha.$$  

(5.35)

Here $v_\alpha$ are the $U(1)^r$ gauge fields (the supersymmetrization of this interaction is discussed for instance in [37]). The covariant derivative $DX_i = \partial X_i + Q_{\alpha i} v_\alpha X_i$ leads to

$$L_{bdy} = \frac{i}{2} n_{\alpha} Q_{\alpha i} (\bar{X}_i \partial X_i - X_i \partial \bar{X}_i) + \frac{1}{2} v_\beta [n_\alpha N_{\alpha\beta} - n_\beta],$$

(5.36)

17 Note that $N_{\alpha\beta}$ is just the mass matrix for the $\sigma_\alpha$ fields in (2.3).
and the constraint imposed by \( v_\alpha \) on the boundary gives \( n_\alpha / \zeta_\alpha = N^{-1}_{\alpha \beta} n_\beta \). Plugging back into \( L_{bdy} \) gives
\[
L_{bdy} = \frac{i}{2} n_\alpha (N^{-1})_{\alpha \beta} Q_{\beta i} (\bar{X}_i \partial X_i - X_i \partial \bar{X}_i)
\] (5.37)
which is the gauge field (5.34). The integers \( n_\alpha \) are the induced \( D(p - 2) \) charges carried by a \( Dp \) brane parallel to the Hirzebruch-Jung space.

The curvatures \( \pi^*(F_\alpha) = d\Theta_\alpha \) generate \( H^2(\mathcal{X}) \), and moreover are dual to the the two-cycles \( C_\alpha \) of the resolved space:
\[
\int_{C_\beta} \frac{F_\alpha}{2\pi} = \delta_{\alpha \beta}.
\] (5.38)
Let us pause to demonstrate this result. Describe the curve \( C_\beta \) by the region \( X_\beta = 0 \) where we fix the gauge (away from the poles) so that all other \( X \)'s are positive, except for \( X_{\beta + 1} = z X_{\beta - 1} \). It is then straightforward to see that
\[
X_{\beta - 1} = \sqrt{\frac{\zeta_\beta}{1 + |z|^2}},
\] (5.39)
and all the other \( X_\gamma \) may be solved for in terms of \( z \). Now we integrate the (1,1)-form \( d\Theta_\alpha \) over this region in \( X \)-space; after some algebra one finds
\[
\int_{C_\beta} F_\alpha = \int_0^\infty \frac{\partial}{\partial \rho^2} \left( \rho^2 X_{\beta - 1}^2 (N^{-1})_{\alpha,\gamma} Q_{\gamma,\beta + 1} \right) d(\rho^2) \wedge d\theta
\] (5.40)
where \( z = \rho e^{i\theta} \). Evaluating the total derivative, there is no contribution at \( \rho^2 = 0 \), and the contribution at \( \rho^2 = \infty \) is just
\[
2\pi \zeta_\beta \left( (N^{-1})_{\alpha,\gamma} Q_{\gamma,\beta + 1} \right) \bigg|_{X_\beta = 0 \cap X_{\beta - 1} = 0}.
\] (5.41)
In appendix A we show that
\[
(N^{-1})_{\alpha \gamma} Q_{\gamma,\beta + 1} = |X_{\beta + 1}|^{-2} \delta_{\alpha \gamma} = \frac{1}{\zeta_\beta} \delta_{\alpha \beta}
\] (5.42)
at the point \( X_\beta = X_{\beta - 1} = 0 \), and so indeed \( \int_{C_\beta} F_\alpha = 2\pi \delta_{\alpha \beta} \) as claimed.

Since the boundary interaction (5.35) has an \( \mathcal{N} = 2 \) supersymmetric completion, it follows that the curvatures \( F_\alpha \) are type (1,1). We do not know if they are Hermitian-Yang-Mills. If we introduce a boundary into the GLSM then there is simultaneous boundary RG flow along with the bulk RG flow of the localized tachyons. We expect that \( F_\alpha \) will
become non-normalizable for those curves that expand out to infinite radius while the \( F_{\alpha} \) corresponding to the \(-2\) curves will flow to the Hermitian-Yang-Mills connections of Kronheimer and Nakajima [46].

The reader should be warned that the naive D-brane charge formula need not apply in this case since the closed string background is not on-shell. Indeed, the integral

\[
\int_X \frac{F_\alpha}{2\pi} \wedge \frac{F_\beta}{2\pi}
\]

may be carried out explicitly, and is not \(-C_{\alpha\beta}^{-1}\). There is a nontrivial contribution from the Chern-Simons term at infinity. It would be very interesting to know if there is a generalization of the standard Chern-Simons coupling to D-branes appropriate to the examples we are discussing.

### 6. RG flow of the Higgs branch

The analysis of the previous section explains precisely which RR charges associated to fractional D-branes are “lost” when the orbifold singularity is resolved by tachyon condensation to its Hirzebruch-Jung minimal resolution. The Hirzebruch-Jung space is the Higgs branch of the GLSM configuration space. In the next section we will show that these “missing” RR charges are recovered by a careful analysis of the Coulomb branch of the GLSM, generalizing the rank one case analyzed in section 3. Actually, it will turn out that there is an interesting interplay between the Higgs and Coulomb branches of the GLSM that takes place under RG flow, where topological charge can pass from the Higgs branch to the Coulomb branch along the flow to the IR. We begin the story with a description of the RG flows of the Higgs branch.

Consider a generic (i.e. not necessarily minimal) resolution of \( \mathbb{C}^2/\mathbb{Z}_{n(p)} \) given by the GLSM of the previous section. The geometry flows with RG scale according to the flow of the FI parameters \( \zeta_\alpha(\mu) = \text{Re}(t_{\alpha,\text{eff}}(\mu)) \) given in (2.3). To say that a certain collection of curves is blown up one should specify the worldsheet scale at which the target space geometry is being considered. Individual curves of the resolution may blow up or down along the flow, or remain of fixed size.

It turns out that the discussion of the renormalization group flow is much clearer once one diagonalizes the \( U(1)^r \) gauge group action as in the previous section (see eq. (5.2)) and accordingly uses the FI parameters \( \zeta'_\alpha \). These satisfy the RG flow

\[
\zeta'_{\alpha,\text{eff}}(\mu) = \zeta'_{\alpha,\text{bare}} - (1 - \Delta_\alpha) \log(\frac{\mu}{\Lambda})
\]

\[
\Delta_\alpha = \frac{p_\alpha}{n} + \frac{q_\alpha}{n}.
\]
We recognize that $1 - \Delta_\alpha$ are the scale dimensions of the couplings $\lambda_\alpha$ to the twist operators $T_{q_\alpha}$ in the chiral ring of the orbifold CFT (1.2). As discussed in section 4, $\frac{1}{n}(q_\alpha, p_\alpha)$ are the R-charges of this operator.

A convenient way to picture the resolution is given in figure 3a. The D-term constraints imply the inequalities (5.9), whose intersection is a convex region $D$ in the $\rho_1^2 - \rho_2^2$ plane (recall that $\rho_{1,2}$ are the magnitudes of the gauge invariant coordinates $\xi_{1,2}$ on the target manifold). The boundary $\partial D$ has two semi-infinite segments corresponding to $\rho_{1,2} = 0$, and finite line segments corresponding to the curves blown up in the resolution. Roughly speaking, each curve is a circle bundle (of the azimuthal direction) over the corresponding line segment (the longitudinal direction). The finite boundary segments of $\partial D$ are along the straight lines $|X_\alpha|^2 = 0$

$$\frac{p_\alpha}{n} \rho_1^2 + \frac{q_\alpha}{n} \rho_2^2 = \zeta'_\alpha(\mu) \quad (6.2)$$

in the $\rho_1^2 - \rho_2^2$ plane, where the running FI parameter is given by (6.1). The $\alpha^{th}$ boundary thus moves with a speed $\frac{\partial}{\partial \log \mu} \zeta'_\alpha(\mu)$ along the flow, in the direction $(p_\alpha, q_\alpha)$.

To see whether a given segment of the boundary corresponding to the $\alpha^{th}$ $\mathbb{P}^1$ is shrinking or growing along the flow, one needs to determine whether the endpoints of that segment are moving toward or away from one another. The intersection of the $\alpha^{th}$ and the $\beta^{th}$ boundaries is at the point $P_{\alpha \beta} := (\rho_1^2, \rho_2^2)|_{X_\alpha = X_\beta = 0} = \frac{n}{p_\alpha q_\beta - p_\alpha p_\beta}(q_\beta \zeta'_\beta(\mu) - q_\alpha \zeta'_\alpha(\mu), p_\alpha \zeta'_\alpha(\mu) - p_\alpha \zeta'_\beta(\mu)) \quad (6.3)$

The velocity of this point along the RG flow is

$$u_{\alpha\beta} = \frac{\partial}{\partial \log \mu} P_{\alpha\beta} = \frac{n}{q_\alpha p_\beta - p_\alpha q_\beta}(q_\beta(1 - \Delta_\alpha) - q_\alpha(1 - \Delta_\beta), p_\alpha(1 - \Delta_\beta) - p_\beta(1 - \Delta_\alpha)) \quad (6.4)$$

In particular,

$$u_{\alpha, -1} = (q_{\alpha - 1} - q_\alpha + 1, p_\alpha - p_{\alpha - 1} + 1) \quad (6.5)$$

One way to determine whether the two endpoints of an interval are separating or approaching is to consider the relative slopes of their corresponding velocity vectors. One finds that the condition that two endpoints $P_{\alpha - 1, \alpha}$ and $P_{\alpha, \alpha + 1}$ are separating, so that $C_\alpha$ is growing in size along the RG flow, is that

$$u_{\alpha, -1} \times u_{\alpha + 1, \alpha} = n(a_\alpha - 2)(1 - \Delta_\alpha) > 0 \quad (6.6)$$
where we have used (4.12) and (4.13).

We see from this result that the curves of the minimal resolution, once blown up, stay blown up; (6.6) implies that in the minimal resolution, \(-2\) curves remain of fixed size (their boundary segments remain of fixed length), and curves with self-intersection \(-3\) and below grow in size. This is illustrated in figure 4 for the example of \(n(p) = 10(3)\), whose minimal resolution corresponds to the continued fraction \(10/3 = [4, 2, 2]\). The general picture of the RG flow can thus be summarized as a splitting of the continued fraction expansion

\[
[\cdots, 2, \ldots, 2, \cdots] \rightarrow [2, \ldots, 2] \oplus [2, \ldots, 2] \oplus \cdots \quad (6.7)
\]

where all \(a_\alpha > 2\) in between the subsequences of \(a_\alpha = 2\). Thus the IR limit is a collection of ALE spaces \(A_{\ell_1} \oplus A_{\ell_2} \oplus \cdots\).

**Figure 4.** Generic RG trajectory for \(n(p) = 10(3)\), for which \(10/3 = [4, 2, 2]\) specifies the minimal resolution. The \(-4\) curve blows up to infinite size along the flow, while the two \(-2\) curves remain of fixed size.

Of course, if we send an FI parameter \(\zeta' \rightarrow -\infty\), its associated wall moves down and to the left toward infinity and stays there under RG flow, and imposes no constraint. For instance, if we turn on only a single relevant perturbation \(\Sigma_\alpha'\) starting from the orbifold fixed point (it need not even be one associated to a curve in the minimal resolution), RG
flow will make a single curve that grows in size. The convex region $\mathcal{D}$ will be bounded by the walls $\rho_1 = 0$, $\rho_2 = 0$, and the curve $|X_\alpha| = 0$, see figure 5. The endpoints of the $|X_\alpha| = 0$ segment will move up the $\rho_{1,2}$ axes along the flow to the IR. Since the remaining curves of the resolution remain blown down, there will be “daughter” singularities at the north and south poles of $C_\alpha$. This gives a picture of the formation and separation of such daughter singularities, previously analyzed in [1,2,3,4].

Figure 5. The special RG flow for the minimal resolution of $n(p) = 10(3)$, in which only the coupling $\zeta'_2$ is allowed to flow out of the orbifold point. Two daughter singularities, $n'(p') = 4(1)$ and $n''(p'') = 2(1)$, sit at the poles of $C_2$ and separate to infinite distance along the flow.

6.1. Non-minimal resolutions of the singularity: RG flow prunes the Higgs branch

In sections 4 and 5.4, we discussed the effect of an additional blowup of the resolved Hirzebruch-Jung singularity

$$\frac{n}{p} = [a_1, \ldots, a_k, a_{k+1}, \ldots, a_r] \longrightarrow \frac{n}{p} = [a_1, \ldots, a_k + 1, 1, a_{k+1} + 1, \ldots, a_r] \ . \quad (6.8)$$

The space with the extra blowup can be realized as a $U(1)^{r+1}$ GLSM with charge matrix (4.2) determined by the blown up sequence. The extra R-charge vector $v_*$ is $v_* = (q_*, p_*) = v_k + v_{k+1}$. There is an extra canonical line bundle associated to the additional $U(1)$ gauge group in the GLSM via the construction of section 5, which continues from the geometrical phase to the orbifold phase as the line bundle associated to the $\mathbb{Z}_n$ representation $(\rho_f)^{p_*}$. 33
Thus, by additional blowups one might think that we can keep track of more than the ‘special’ $\mathbb{Z}_n$ representations of the minimal resolution of the singularity. Alas, the condition (6.6) shows that all such additional curves are blown down under RG flow to the IR, see figure 6.

**Figure 6.** (a) By fine tuning the FI parameters, in some range of worldsheet scales one can arrange that an extra curve is blown up in the singularity resolution. (b) RG flow blows down the extra curve.

Equation (6.6) says that if an $a_\alpha = 1$ curve corresponds to a relevant operator $1 - \Delta_\alpha > 0$, then the line segment that defines the minimal size curve $C_\alpha$ shrinks away, because the flow of the neighboring walls outcompetes the flow of the $\alpha^{th}$ wall (6.2); and if the corresponding operator is irrelevant, the boundary in figure 6 is moving to the lower left and disappears from the geometrical region $\rho_i^2 > 0$ altogether. One can also easily check that for multiple blowups, the flow blows down in succession all the additional curves beyond those of the minimal resolution, until in the far IR of the RG flow one is left with the minimal resolution.\(^{18}\)

18 One might have worried that blowing up twice, e.g. $[\ldots, a_k, a_{k+1}, \ldots] \rightarrow [\ldots, a_k + 2, 1, 2, a_{k+1} + 1, \ldots]$, turns a $-1$ curve into a $-2$ curve which is then stable under RG flow. However, the movement of the walls in figure 6 is controlled by the R-charge vectors, which haven’t changed. One has added an extra segment to the boundary in figure 6a, but the walls formed by the $k^{th}$ and $(k + 1)^{st}$ D-term constraints are still closing in.
Thus, one may choose an RG trajectory for which, in some range of worldsheet scales \( \mu \), additional curves beyond the minimal resolution are blown up. In that range of scales, these curves are part of the resolved geometry \( \mathcal{X} \) and therefore of the Higgs branch of the GLSM configuration space. Eventually, however, these curves are blown down in a generic RG flow; in a sense, the scaling operators \( \mathcal{T}_s \) which couple to the extra curves are ‘not as relevant’ as those which blow up the curves of the minimal resolution, and thus can’t compete with them in the long run.

The precise sense in which these operators are ‘not as relevant’ is shown in figure 7. As discussed in section 4, the generators of the chiral ring are in one-to-one correspondence with the curves of the minimal resolution, and their R-charges define the polygonal boundary (the Newton boundary) of a convex region in the space of R-charges. In other words, the line passing through any pair of points \( \mathbf{v}_\alpha, \mathbf{v}_{\alpha+1} \) has the R-charges of all other operators lying above it. In this sense, the operators corresponding to the curves of the minimal resolution are the ‘most relevant’.

![Figure 7. The Newton boundary of chiral operators for \( n(p) = 10(3) \), is the solid line bounding the shaded polygon. The generators of the chiral ring lie on the Newton boundary, and all other chiral operators lie above it.](image)

7. The Coulomb branch of the GLSM

The results of the previous section show that all line bundles but those associated to the “special” representations of \( \mathbb{Z}_n \) disappear from the purview of the Higgs branch of the GLSM under RG flow. However, the associated topological charges do not simply disappear from the full GLSM configuration space. Each RR gauge field of the orbifold
CFT is built from a vertex operator that creates one of the supersymmetric ground states of the orbifold; if we can follow the ground states, we can follow the topological charge.

The simplest way to characterize the RR ground states not present in the Higgs branch is through the effective twisted chiral superpotential (2.8). To analyze them, it is convenient to introduce (following [33]) a set of twisted chiral fields $Y_i$ which dualize (à la Buscher [47,48]) the phases of the $X_i$. The basic feature of the duality transformation that we will use is its effect on the twisted superpotential

$$\tilde{W} = \sum_{\alpha=1}^{r} \Sigma_\alpha \left( \sum_{i=0}^{r+1} Q_{\alpha i} nY_i - t_\alpha(\mu) \right) + \mu \sum_i \lambda_i e^{-nY_i}$$

(7.1)

where

$$t_\alpha(\mu) := t_{\alpha, \text{bare}} + \sum_{i=0}^{r+1} Q_{\alpha i} \log \left( \frac{\mu}{\Lambda} \right).$$

(7.2)

Eliminating the $Y_i$ by their equation of motion gives back (2.8). Instead, we will eliminate the $\Sigma_\alpha$ and $Y_\alpha$, $\alpha = 1, ..., r$, by their equations of motion to get (in terms of $u_0 = (\mu \lambda_0)^{1/n} \exp[-Y_0]$ and $u_{r+1} = (\mu \lambda_{r+1})^{1/n} \exp[-Y_{r+1}]$)

$$\tilde{W} = u_0^n + u_{r+1}^n + \sum_{\alpha=1}^{r} \lambda'_\alpha u_0^{p_\alpha} u_{r+1}^{q_\alpha} ,$$

(7.3)

where

$$\lambda'_\alpha = \lambda_\alpha \Lambda^{1-\Delta_\alpha} e^{t'_{\alpha, \text{bare}}} = \lambda_\alpha \mu^{1-\Delta_\alpha} e^{t'_{\alpha, \text{eff}}(\mu)} .$$

(7.4)

Note that the $\alpha^{th}$ monomial in the sum in (7.3) is just

$$\Sigma'_\alpha := (\lambda'_\alpha)^{-1} C_{\alpha \beta} \Sigma_\beta = u_0^{p_\alpha} u_{r+1}^{q_\alpha}$$

(7.5)

(we can also extend this to define $\Sigma'_0 = u_0^n$, $\Sigma'_{r+1} = u_{r+1}^n$). The scaling dimensions of these operators thus identifies them as $\Sigma'_\alpha \propto T_{q_\alpha}$.

One should be careful in the use of the ‘mirror transformation’ [34,33]. The mirror transformation amounts to T-duality on the phase of the $X_j$; however, in the geometrical phase, the minimal volume cycles $C_\alpha$ are at $X_\alpha = 0$ where this T-duality is ill-defined. It is therefore unclear to what extent the effective superpotential (7.3) will accurately capture the properties of the ‘geometrical’ supersymmetric ground states associated to the homology of the resolved Hirzebruch-Jung space of the Higgs branch. We do however expect it to describe correctly the vacua of the Coulomb branch which are supported away
from the origin, and it is only for this purpose that we will employ it. One could carry out the whole analysis of massive vacua in terms of the $\Sigma'_\alpha$ without introducing the auxiliary fields $Y_i$; it is merely for convenience that we introduce them.

Note that the ‘mirror’ $\mathbb{Z}_n$ transformation

$$(u_0, u_{r+1}) \sim (\omega u_0, \omega^{-p} u_{r+1}) \quad (7.6)$$

leaves the effective superpotential (7.3) invariant – it fixes all the $\Sigma'_\alpha$. Indeed it is a gauge symmetry remnant of the duality transformation and therefore we should quotient the LG model by its action. Thus massive vacua of (7.3) come in orbits of length $n$ in $u$-space. In general complex dimension $d$, the dualized theory must be orbifolded by $(\mathbb{Z}_n)^{d-1}$.

7.1. Counting the vacua

The effect of turning on the FI couplings $\lambda_\alpha$ is to move a subset of the critical points of (7.3) out to large $|u|$ for large $|\lambda|$. A simple way to see this is to rescale

$$u_0 \to zu_0$$
$$u_{r+1} \to zu_{r+1}$$
$$\lambda'_\alpha \to z^{n-p_\alpha-q_\alpha} \lambda'_\alpha \quad (7.7)$$

which homogeneously rescales the twisted superpotential (7.3); thus, extrema of the potential scale to large $u_0, u_{r+1}$ at large $\lambda'$. On the other hand, the ‘geometrical region’ of the configuration space – the Higgs branch – remains at $|u| \sim 0$ as the FI parameters are made large. One can understand this latter property in terms of the original variables $\Sigma_\alpha, X_i$; equation (5.9) says that the geometrical region (the Hirzebruch-Jung space obtained by solving the D-term equations) is far from the origin in the variables $X_0, X_{r+1}$. The component potential term $Q_{\alpha i}^2 |\sigma_\alpha|^2 |X_i|^2$ in (2.5) then forces $\sigma_\alpha \sim 0$ and thus $u_0, u_{r+1} \sim 0$ by (7.3). On the other hand, vacua with $u_0, u_{r+1}$ large have $|X| \sim 0$. Thus for large $\zeta$ the vacua of the Higgs branch and the vacua of the Coulomb branch are well separated, with a potential barrier between them.

19 The $U(1)^d$ $R$-charges of the twist operators are of the form $\frac{1}{n} v_j = j w \, (\text{mod } \mathbb{Z})$ in each component), for $j = 1, \ldots, n - 1$. There are $d - 1$ independent vectors orthogonal to $w$, whose rational components are in $\frac{1}{n} \mathbb{Z}$ and define a $\mathbb{Z}^{d-1}$ action shifting the $Y$’s, which fixes the $\Sigma_\alpha$. This transformation therefore represents a redundancy of the description under which the $Y$’s should be identified. Equation (7.6) represents the special case $d = 2$.

20 See the last paragraph of section 2 for a more accurate description.
This flow to infinite separation is the standard mechanism by which RR ground states decouple in $\mathcal{N} = 2$ supersymmetric field theories; here the decoupling has an interpretation of decoupling certain K-theory charges from the geometrical spacetime as one perturbs away from the orbifold geometry.

Let us now count how many vacua decouple to large $|u|$ in this manner. In fact, it turns out to be easier to count how many remain behind (at $u_0 = u_{r+1} = 0$) when all the $\lambda_\alpha$ are nonzero and generic. We can do this by turning on a small additional perturbation

$$\delta\tilde{W} = \epsilon u_0$$

in the Landau-Ginsburg potential (7.3), and counting the number of independent solutions that are near the origin (i.e. that smoothly approach $u = 0$ as $\epsilon \to 0$) as opposed to those which are $O(1)$ for $\epsilon \to 0$.

There are several types of scaling solutions near the origin when we do this. As noted at the end of the previous section, the chiral ring generators define a Newton boundary of monomials. The line passing through two adjacent vectors $v_\alpha, v_{\alpha+1}$ has all monomials invariant under the mirror $\mathbb{Z}_n$ (7.6) lying on or above it. For example, in figure 7, the Newton boundary is the polygon with vertices specified by the $R$-charge vectors for $V_Y, T_1 \equiv \Sigma'_1, T_4 \equiv \Sigma'_2, T_7 \equiv \Sigma'_3,$ and $V_X$.

Thus we can define a scaling $u_0 \sim \epsilon^{\mu}, u_{r+1} \sim \epsilon^\nu$ such that $\Sigma'_\alpha \sim \Sigma'_{\alpha+1}$ in their scaling, with all other $\Sigma'_\beta$ scaling as the same or higher powers of $\epsilon$. This means that when analyzing critical points of the potential that lie stably near the origin and scale in this way, we may ignore all the other terms in the potential and focus on the perturbing term plus these two monomials:

$$\tilde{W}^{(\alpha)}_{\text{eff}} = \epsilon u_0 + \chi'_\alpha u_0^q u_{r+1}^a + \chi'_{\alpha+1} u_0^{p_{\alpha+1}} u_{r+1}^{q_{\alpha+1}}.$$ (7.9)

Dropping all constant coefficients, the variational equations are of the form

$$\begin{align*}
\epsilon &= u_0^{p_\alpha-1} u_{r+1}^{q_\alpha} + u_0^{p_{\alpha+1}-1} u_{r+1}^{q_{\alpha+1}} \\
0 &= u_0^{p_\alpha} u_{r+1}^{q_\alpha-1} + u_0^{p_{\alpha+1}} u_{r+1}^{q_{\alpha+1}-1}.
\end{align*}$$ (7.10)

Note that for $\alpha = 0$ the second equation does not involve $u_{r+1}$ because $q_0 = 0$ (so the first term is actually absent) and $q_1 = 1$. Then the first equation is solved only by letting $u_{r+1} \to \infty$ (as one sees by introducing an infinitesimal $\epsilon'$ on the LHS of the second
equation) and so we are not counting solutions that are stably near the origin for small \( \epsilon \). Therefore we only consider these equations for \( \alpha = 1, \ldots, r \).

We solve these equations as follows: Without loss of generality we can take \( \epsilon \) real and positive; then we set

\[
    u_0 = e^{2\pi i \varphi_0} \epsilon^\mu, \quad u_{r+1} = e^{2\pi i \varphi_{r+1}} \epsilon^\nu \tag{7.11}
\]

with \( \mu = \frac{q_{\alpha+1} - q_\alpha}{n - q_{\alpha+1} + q_\alpha}, \quad \nu = \frac{p_\alpha - p_{\alpha+1}}{n - q_{\alpha+1} + q_\alpha} \). Plugging into (7.10), one has a solution for every \( \varphi_0, \varphi_{r+1} \mod 1 \) such that:

\[
    \varphi_0 (p_\alpha - 1) + \varphi_{r+1} q_\alpha = \varphi_0 (p_{\alpha+1} - 1) + \varphi_{r+1} q_{\alpha+1} = 0 \mod 1 \tag{7.12}
\]

One can check that for such solutions the implicit assumptions that \( u_0, u_{r+1} \) are nonzero and stably near zero are both satisfied.

Now, let us count these solutions. We may interpret (7.12) as the defining equations for the lattice \( \mathbf{L}^* \) in \( \mathbb{R}^2 \) dual to the lattice \( \mathbf{L} \) spanned by \( w_1 = (p_\alpha - 1, q_\alpha) \) and \( w_2 = (p_{\alpha+1} - 1, q_{\alpha+1}) \). We are only interested in the number of vectors in \( \mathbf{L}^*/\mathbb{Z}^2 \) since \( \varphi_0, \varphi_{r+1} \) are only defined modulo 1. The number of such vectors is the volume of the unit cell of \( \mathbf{L} \), and hence the \( \alpha \)-th scaling solution has \( w_2 \times w_1 = n + q_\alpha - q_{\alpha+1} \) zeroes near the origin, for a total of

\[
    \sum_{\alpha=1}^r (n + q_\alpha - q_{\alpha+1}) = rn + q_1 - q_{r+1} = (r - 1)n + 1 \tag{7.13}
\]

critical points stably at the origin for small \( \epsilon \).

But this is exactly what we were looking to find! There are \( (n - 1)^2 \) critical points of (7.13) under variation of \( u_0, u_{r+1} \); if \((r - 1)n + 1\) are left at the origin under the generic perturbation by the \( \lambda_\alpha \), then there should be \((n - r - 1)n\) critical points away from the origin even at \( \epsilon = 0 \). These are arranged in \((n - r - 1)n\) orbits of length \( n \) under the mirror \( \mathbb{Z}_n \) (7.4). Since we must quotient by this action we conclude there are \( n - r - 1 \) massive vacua under the perturbations of the superpotential which blow up the exceptional curves of the minimal resolution \( \mathbb{P} \). This corresponds precisely to the number of nontrivial fractional

\[\text{nondegenerate, but expect this to be so generically. It does not immediately follow from nondegeneracy that the vacuum is massive. In order to conclude this we need to know about the kinetic terms for the } u \text{ fields. We are assuming that they are well approximated by standard kinetic terms.}\]
branes we are expecting to lose given the K-theory of the resolved Hirzebruch-Jung space described above.

One might ask if the UV limit of the D-branes of the Coulomb branch transform under the quantum $\mathbb{Z}_n$ symmetry in representations that are complementary to the special representations. This is indeed true. We saw in the previous section that the additional curves of a non-minimal resolution of the singularity were in general associated to additional $\mathbb{Z}_n$ representations. We also saw that the additional curves were blown down along RG flow to the infrared. What happens is that the RR ground states associated to these representations pass onto the Coulomb branch of the configuration space.

In terms of the mirror LG picture, even though the term $u_0^{a_0} u_{r+1}^{a_r+1}$ in the potential is above the convex hull (the Newton boundary) set by the monomials $u_0^{a_0} u_{r+1}^{a_r+1}$ of the minimal resolution, by suitably making the coupling $\lambda'_r$ large enough the former term will be just as important as the latter terms. This is the reflection in the LG picture of the fine tuning of the FI parameters that adds an extra segment to the boundary of the region $\mathcal{D}$, as in figure 6a. Then, in the counting of vacua of the Coulomb branch, one should split up the perturbations controlled by (7.9) for $\alpha = k$ into two parts, one for the pair $u_0^{a_0} u_{r+1}^{a_r+1}$, $u_0^{a_0} u_{r+1}^{a_r+1}$ and another for the pair $u_0^{a_0} u_{r+1}^{a_r+1}$, $u_0^{a_0} u_{r+1}^{a_r+1}$. The sum (7.13) will have an extra term, and so one will find one more vacuum (more precisely, an orbit of (7.6) of length $n$ of vacua) near the origin.

In other words, by suitable fine tuning the RR vacuum corresponding to the extra $-1$ curve remains in the region of small $|u|$ where it has a geometrical interpretation on the Higgs branch (and the LG picture of the Coulomb branch is not actually reliable); the associated RR gauge field doesn’t decouple from spacetime dynamics, at least for some range of RG scales where the extra curve has a size much larger than the string scale. Eventually, the curve shrinks away along the RG flow and joins the other massive LG vacua on their exodus from the Higgs branch of the configuration space, since the critical points of the LG potential (7.3) are generically controlled by the most rapidly growing terms – namely those on the Newton boundary.

7.2. The ‘optimal’ resolution

It would then appear that one can treat any of the vacua, massive or massless, associated to any particular perturbation of the chiral ring of the orbifold CFT, and follow how it joins or leaves the Higgs branch of the GLSM configuration space. The ideal starting point would employ a $U(1)^{n-1}$ GLSM rather than $U(1)^r$, and let fields decouple as one
prescribes. One now also has the degrees of freedom to express the gauge field for any nontrivial line bundle $R_j$ present at the orbifold fixed point, $j = 1, \ldots, n - 1$, in terms of GLSM fields via (5.34). The logical starting point uses the D-term equations written in the diagonalized basis (5.4), where one has a direct relation to scaling operators $T_j$ and the associated $\mathbb{Z}_n$ representations $\rho_j$. Naively, it would appear that one can fine tune so that any representation appears in the Higgs branch of the configuration space by blowing up non-minimally.

An exception to this prescription arises when the R-charge vector of the corresponding orbifold twist operator is not primitive, i.e. is a power of another twist operator; then the non-primitive twist operator does not correspond to a distinct $\mathbb{P}^1$ in the resolution. Nevertheless, these may also be put into a canonical form. To see this, start with the D-terms put in the diagonalized form (5.4). Whenever $(p_\alpha, q_\alpha) = k(p_\beta, q_\beta)$ for some $\beta$, we can rewrite the $\alpha$ th constraint by taking a linear combination with the $\beta$ th constraint as

$$k|X_\beta|^2 - |X_\alpha|^2 = -k\zeta'_\beta + \zeta'_\alpha \equiv \hat{\zeta}'_\alpha . \quad (7.14)$$

When $\hat{\zeta}'_\alpha < 0$, $|X_\alpha|$ is forced to be nonzero, and we can gauge fix the $U(1)$ action associated to the above D-term constraint via fixing the phase of $X_\alpha$. There is no residual gauge action and the remainder of the theory is unaffected. If on the other hand $\hat{\zeta}'_\alpha > 0$, we have $|X_\beta|$ forced to be nonzero, and the $U(1)$ action may be gauge fixed by fixing the phase of $X_\beta$. As in section 3, this leaves a residual $\mathbb{Z}_k$ symmetry which acts on the normal bundle to the $\beta$ th $\mathbb{P}_1$ in the resolution chain; it does not yield an independent $\mathbb{P}^1$ of the resolved space. A special case of this is the $\mathbb{Z}_n(1)$ orbifold, where blowing up to infinity via the chiral operator $W^k$ (where $W$ generates the chiral ring) leads to the daughter space $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_k)$ [4].

The general setup is thus indeed to start with a $U(1)^{n-1}$ GLSM, in the diagonal basis where the $U(1)$ charges are related to the R-charges of the twist fields, so that the D-term constraints are

$$R_{\kappa i}|X_i|^2 \equiv \frac{p_\kappa}{n}|Y_0|^2 + \frac{q_\kappa}{n}|Y_n|^2 - |Y_\kappa|^2 = \zeta'_\kappa , \quad (7.15)$$

with the R-charge vectors given by the full set of orbifold R-charges (1.3). There is a unique ‘optimal’ resolution containing all primitive vectors in the set of R-charge vectors of the orbifold twist fields with a generalized Dynkin diagram of rank $s$. Divide the index set $\{1, \ldots, n - 1\}$ into the subset $\{\alpha_1, \ldots, \alpha_s\}$ associated to this optimal resolution, and the complement $\{\xi_1, \ldots, \xi_{n-1-s}\}$. Denote by $\hat{B}_{\kappa \mu}$ the embedding of its generalized Cartan
matrix acting nontrivially only on the subset \(\{\alpha_i\}\) within the full index space; and denote by \(\hat{M}_{\kappa\mu}\) the lower triangular transformation which has the effect of turning \((7.13)\) into \((7.14)\) for all the \(\{\xi_j\}\) that correspond to non-primitive \(R\)-charge vectors. A canonical charge matrix for the \(U(1)^{n-1}\) GLSM is thus \(\hat{Q} = (\hat{B} + \hat{M})\hat{R}\). We may define the geometrical FI parameters controlling the sizes of cycles as \(\hat{\zeta}_\kappa = \hat{Q}_{\kappa\mu}\hat{\zeta}_\mu\)' as usual. By construction, the \(R\)-charges of the twisted perturbations \((q_j, p_j), j = 1, \ldots, n-1\), will coincide with those of the orbifold \((1.3)\).

Because we now have a \(U(1)\) gauge field corresponding to each representation of \(\mathbb{Z}_{n(p)}\), one can follow what happens to each representation as we move around the parameter space of the twist fields. The optimal resolution just defined keeps them all around if we tune the \(\zeta_k'\) to be large and positive in the appropriate range – all representations of \(\mathbb{Z}_{n(p)}\) can be found as canonical line bundles on the resolved space. RG flow results in a subset of the curves of the optimal resolution being blown down. Various curves leave the resolution chain, and the line bundles become associated to D-branes of the massive vacua in the Coulomb branch that decouple.

Let us illustrate this procedure via the example of \(n(p) = 10(3)\), for which the chiral ring \(R\)-charges are depicted in figure 1. The minimal resolution is associated to the continued fraction \(10/3 = [4, 2, 2]\) with corresponding generators \(\{T_1, T_4, T_7\}\). The optimal resolution involving resolution vectors associated to twist fields is given by the continued fraction \(10/3 = [6, 1, 3, 1, 4, 2]\), with corresponding chiral ring elements (in sequence) \(\{T_1, T_6, T_9, T_4, T_7\}\)\(^{22}\). Left out are the generators \(T_2 = (T_1)^2, T_3 = (T_1)^3, \) and \(T_8 = (T_4)^2\).

The combination of the generalized Cartan matrix embedded as \(\hat{B}\) in the \(U(1)^9\), as well as the lower triangular matrix \(\hat{M}\) acting on the twists that are non-primitive, yields the charge matrix:

\[
\hat{Q} = (\hat{B} + \hat{M}) \cdot \hat{R} = \begin{pmatrix}
1 & -6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}
\]  

\((7.16)\)

\(^{22}\) Note that the generators/R-charge vectors in the resolution chain are in order of decreasing \(p_\alpha/q_\alpha\).
Here $\hat{R}_{\alpha i}$ is the diagonalized matrix from (7.15). Thus we see that rows 1,6,5,9,4,7 indeed form the standard charge matrix of the blowup. The additional structure from rows 2,3,8 will have the same consequence as in the discussion after equation (7.14); in other words, no effect in the regime of large negative FI parameter, while for large positive FI parameter they will induce some extra orbifolding of the normal bundle to a blown up curve.

### 7.3. Deformed chiral ring relations

As an aside, we note that another use of the dual Landau-Ginsburg potential (7.3) is to derive the deformation of the chiral ring that occurs when one perturbs away from the orbifold point $\lambda_{\alpha} = 0$ in the parameter space. Note that, due to equation (4.12), the equations for a critical point can be written

$$n \frac{(\Sigma'_{\alpha})^{q_{\alpha}+1}}{(\Sigma'_{\alpha+1})^{q_{\alpha}}} + \sum_{\beta} p_{\beta} \Sigma'_{\beta} = 0 \quad (7.17)$$

for all $\alpha = 1, \ldots, r$. Therefore

$$\frac{(\Sigma'_{\alpha})^{q_{\alpha}+1}}{(\Sigma'_{\alpha+1})^{q_{\alpha}}} = \frac{(\Sigma'_{\alpha-1})^{q_{\alpha}}}{(\Sigma'_{\alpha})^{q_{\alpha-1}}} \quad (7.18)$$

The consequence of this relation is (again due to equation (4.12), and the fact that $v_{\alpha} = (q_{\alpha}, p_{\alpha})$ obey the resolution vector relations $a_{\alpha} v_{\alpha} = v_{\alpha+1} + v_{\alpha-1}$)

$$(\Sigma'_{\alpha})^{a_{\alpha}} = (\Sigma'_{\alpha-1})(\Sigma'_{\alpha+1}) \quad , \quad \alpha = 2, \ldots, r - 1. \quad (7.19)$$

This relation only holds for $\alpha = 1, r$ if we use a substitution of $\Sigma'_{0}$ or $\Sigma'_{r+1}$, and then we would need a ring relation for them. Since $\Sigma'_{0}, \Sigma'_{r+1}$ correspond to nonnormalizable modes one should use the relations on the $v_{\alpha}$ (and the fact that $q_{1} = 1$ and $p_{r} = 1$), to rewrite the equations of motion for $u_{0}, u_{r+1}$ as

$$0 = n(\Sigma'_{1})^{a_{1}} + \sum_{\alpha=1}^{r} p_{\alpha} \lambda'_{\alpha} \Sigma'_{\alpha} \Sigma'_{2} \quad (7.20)$$

$$0 = n(\Sigma'_{r})^{a_{r}} + \sum_{\alpha=1}^{r} q_{\alpha} \lambda'_{\alpha} \Sigma'_{\alpha} \Sigma'_{r-1} \quad .$$

This pair of equations, together with (7.13), are some (but not all) of the deformed chiral ring relations.

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23 We are assuming that the $u'$s are not near zero so that we can freely multiply these relations by $\Sigma'_{\alpha}$. 

43
8. The spacetime effective action

The decoupling from the Higgs branch of D-branes and RR fields witnessed above should be reflected in the structure of the spacetime effective action. We will propose such an effective action in this section. But first we will discuss the effect of the GSO projection on the spectrum and dynamics of the $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ orbifolds.

8.1. GSO projections and RR fields

The chiral ring of the orbifold contains the set of BPS protected twist operators (1.2), which are holomorphic under the natural choice of complex structure for the $\mathbb{C}^2$ coordinates $X, Y$. However, there is another ring

$$\Sigma_{j/n}^{(X)}(\Sigma_{1-(jp/n)}^{(Y)})^* \quad (8.1)$$

which is BPS under a different linear combination $G_x^* + G_y^*$ of the supersymmetry currents of the component theories, the one natural to the opposite complex structure obtained via $Y \rightarrow Y^*$. We can call the ring of these latter operators the $(c_x, a_Y)$ ring, and the ring of operators (1.2) the $(c_x, c_Y)$ ring.

The type 0 theory that we have been discussing in fact contains both the $(c_x, c_Y)$ and $(c_x, a_Y)$ rings. The type II GSO projection demands invariance under

$$H_1 \rightarrow H_1 + p\pi \quad , \quad H_2 \rightarrow H_2 - \pi \quad (8.2)$$

where $H_i$ are the bosonized worldsheet fermions; this keeps some of each ring, namely the $(c_x, c_Y)$ states with $[jp/n] \in 2\mathbb{Z} + 1$ and the $(c_x, a_Y)$ states with $[jp/n] \in 2\mathbb{Z}$ (here $[\xi] = \xi - \{\xi\}$ denotes the integer part of $\xi$). It is only for the supersymmetric orbifold that the entire $(c_x, a_Y)$ ring is projected out and the entire $(c_x, c_Y)$ ring is preserved by the GSO projection.

The fact that some of the generators of the $(c_x, c_Y)$ ring are projected out means that one is obstructed in the type II theory from fully resolving the singularity using Kähler deformations alone. It is for this reason that we have focussed our attention on the type 0 theory.\textsuperscript{24}

\textsuperscript{24} A. Adams has suggested to us that one might be able to resolve the singularity fully in the type II theory by employing the $(c_x, a_Y)$ operators, which deform the algebraic equations embedding the singularity in $\mathbb{C}^{\ell+2}$ (c.f. [4]). However, since these operators are BPS under a different choice of complex structure, they are not protected from renormalization within the same scheme that protects the $(c_x, c_Y)$ operators. Thus it is difficult to follow their effect under the finite deformation required to obtain a geometrical picture of the target space.
Spectral flow in the $\mathcal{N} = 2$ U(1) R-charge generates a RR ground state, and associated RR gauge field, for every allowed element of each ring. Thus in the type II theory, there are only $n - 1$ RR bispinor fields, some associated to the $(c_x, c_y)$ ring and some to the $(c_x, a_y)$ ring. In the type 0 theory there are twice as many RR gauge fields coming from the twisted sectors as in type II, because one keeps both parities of spinor in forming the RR bispinors.

8.2. The spacetime effective action

We now come to the question of the effective action for the RR gauge fields coupling to the disappearing fractional branes. The disappearance at the IR fixed point of the topological charge that they couple to suggests that these fields decouple from the effective spacetime dynamics. There are two canonical mechanisms for this decoupling: Spontaneously symmetry breaking by the Higgs mechanism, giving the gauge fields a large mass; or some sort of confinement mechanism (sometimes called ‘classical confinement’ due to the fact that we are in tree level string theory).

The Higgs mechanism would require a condensation of D-branes, which are the only objects charged under the RR-gauge symmetry. However, in the massive LG vacua that we have seen are associated to the decoupling RR ground states, D-branes only get heavier as we move away from the UV orbifold fixed point. This Higgs mechanism is ruled out.

The fact that the decoupling generators of the K-theory lattice are associated to massive vacua of the GLSM is rather reminiscent of a similar situation in open string RG flows [49], where the classical confinement mechanism occurs. Consider the open string tachyon in the $Dp$-$\bar{D}p$ system. In the wordsheet RG approach to tachyon condensation, the tachyon condensate to a $D(p - 2)$ brane appears as a boundary mass term. Under the flow, all boundary operators are flowing to the identity operator in the infrared; there are no physical excitations of the brane system away from the massive minimum of the effective potential. One is thus led to conjecture [50] a form of the effective action for the open string degrees of freedom

$$S_{\text{eff}}^{\text{open}} = \int d^{p+1}x \; f(T)[F_{\text{open}}^2 + \ldots],$$

where $f(T) \to 0$ as the tachyon condenses. This proposal has been verified in simple examples [51,52].

The picture of the RR gauge fields which decouple under the closed string tachyon perturbations considered here is quite similar. The RR gauge fields associated to the
massive vacua which are decoupling are such that all their excitations are flowing to the identity operator in the IR of the flow. The RG flows of the GLSM suggest a conjecture analogous to (8.3) for the RR gauge fields coupling to the disappearing K-theory charges:

\[ S_{\text{eff}}^{RR} = \int d^6x \sum_{i=1}^{n-1} f_i(T) [(F_{RR}^{(i)})^2 + \ldots] , \]  

In perturbation theory \( f_i(T) \sim 1 + \mathcal{O}(T) \) (for more precise formulae, see [2]). However, nonperturbatively, it must be that \( f_i(T) \to 0 \) as the tachyon condenses for those gauge fields whose charges disappear.\(^{25}\)

The mechanism of decoupling of gauge charge, and excitations that couple to it on localized unstable objects, would thus appear to be rather universal in perturbative string theory.

9. Discussion

What general lessons can we draw from the above considerations? Perhaps the most important one is that K-theory might continue to play an interesting role in closed string tachyon decay. One general viewpoint on the relation of K-theory and D-branes is that K-theory is an invariant of boundary RG flow, for a fixed bulk CFT. In general, there is no particular reason to think that K-theory of spacetime should be an invariant of bulk RG flow. However, \( \mathcal{N} = 2 \) worldsheet supersymmetry gives additional structure that allows one to relate D-branes and K-theory in a way that is preserved under bulk RG flow. Furthermore, the \( \mathcal{N} = 2 \) preserving localized tachyon perturbations lead to a controlled family of closed string RG flows. In this case one might expect to be able to determine the “fate” of the K-theory charges. This is what we have accomplished in the present paper.

In essence, the topology of the target space is defined by specifying precisely the UV fixed point theory on the worldsheet that describes the unstable, tachyonic vacuum of the closed string theory. One might be able to regard this as a manifestation of UV/IR duality (the global structure of spacetime is related to short-distance structure on the worldsheet).

\(^{25}\) The effective action of this closed string theory is actually an integral over 10 dimensions. However, the couplings to the RR fields associated with the twisted sectors are weighted by wavefunctions which fall off roughly as \( \sim e^{-r/\ell_{\text{string}}} \) in the directions transversal to the orbifold. Thus, at low energies and long distances, the effective action is an integral over the orbifold fixed point locus.
For example, one could have considered the Hirzebruch-Jung space with an extra million non-minimal blowups to be the UV theory one starts with; this will have a K-theory lattice whose rank is increased by a million over that of the minimal resolution, then at some crossover scale the extra curves will blow down and the extra structure decouples, leaving behind the minimal resolution and its K-theory lattice of rank $r + 1$. Or one can consider the orbifold fixed point which also flows to the same IR theory, which will in general have a K-theory lattice smaller than the above in rank, but still larger than that of the IR Hirzebruch-Jung space. Different UV theories having different K-theory can flow to the same Higgs branch geometry in the IR. The full UV theory keeps track of all the topology, some of which moves to non-geometrical branches of the configuration space along the flow to the IR. The Higgs branch of the IR limit contains only the topology of the $r$ curves of the minimal resolution.

It is interesting to contrast our discussion of D-branes in the massive vacua with the paper of Hori, Iqbal, and Vafa [36]. These authors studied D-branes on compact toric manifolds with $c_1(X) > 0$. Such sigma models are good UV fixed points for the massive D-branes. Under RG flow to the IR these manifolds shrink to zero size, and the IR description of the theory is a Landau-Ginzburg orbifold. The latter is more appropriately described by the Coulomb branch vacua. Thus, one should speak of either the Higgs branch, or the Coulomb branch, and the mirror correspondence of D-branes discussed in [36] is a correspondence between the IR and UV description of the “same” branes. In the examples studied in this paper the Hirzebruch-Jung manifold has $c_1(X) < 0$, and appears in the IR, not the UV region of the theory. Hence, in the flow to the IR one is forced to discuss both the Higgs and the Coulomb branches.

The phenomenon of open string tachyon condensation has been the focus of some very interesting investigations in string field theory in the past few years. A corresponding theory for closed string tachyon condensation is glaringly absent. The spacetime picture we have advocated for the disappearance of RR U(1) gauge fields under localized closed string tachyon condensation suggests a natural set of conjectures to which one might try to apply the techniques of closed string field theory.

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Appendix A. Some properties of the matrix \( \mathcal{N} \)

A.1. Proof of (5.42)

We wish to show that

\[
\mathcal{N}^{-1} \mathcal{Q} = |X_{\beta+1}|^{-2} \delta_{\alpha\beta} \tag{A.1}
\]
on the surface \( X_{\beta} = X_{\beta-1} = 0 \). To do this, first note that \( \mathcal{N} \) is block diagonal on this surface, namely \( \mathcal{N} = 0 \) for \( \alpha < \beta, \gamma \geq \beta \). This implies that \( \mathcal{N}^{-1} \mathcal{Q} = 0 \) for \( \alpha < \beta \). We can thus concentrate on the lower right block of entries for \( \alpha, \gamma \geq \beta \). Call this block \( \mathcal{N}_{\alpha\gamma} \), and the corresponding block of \( \mathcal{Q} \) we will call \( \mathcal{Q}_{\alpha\gamma} \).

It is not hard to show that

\[
\mathcal{N}_{\alpha\gamma} = L \cdot D \cdot U \tag{A.2}
\]

where

\[
L_{\alpha\gamma} = \mathcal{Q}_{\alpha\gamma+1}, \quad U_{\alpha\gamma} = \mathcal{Q}_{\alpha+1,\gamma}, \quad D_{\alpha\gamma} = diag(|X_{\beta+1}|^2, \ldots, |X_{r+1}|^2) \tag{A.3}
\]

are lower triangular, upper triangular, and diagonal matrices, respectively. Furthermore,

\[
(L^{-1} \cdot Q)_{\alpha\gamma} = \delta_{\alpha\gamma-1} \tag{A.4}
\]

for \( \alpha \geq \beta, \gamma \geq \beta + 1 \). Then we have

\[
(\mathcal{N}^{-1} \cdot Q) = U^{-1} \cdot D^{-1} \cdot L^{-1} \cdot Q \tag{A.5}
\]

and the RHS is manifestly upper triangular in the relevant block. This means that

\[
(\mathcal{N}^{-1} \cdot Q)_{\beta,\beta+1} = |X_{\beta+1}|^{-2} \quad (\mathcal{N}^{-1} \cdot Q)_{\alpha,\beta+1} = 0 \quad \alpha > \beta \tag{A.6}
\]

which is what we were to show.
A.2. An explicit inverse for \( N \)

It is possible to give an explicit inverse for the matrix \( N \). While it is not used in the text, this formula is slightly nontrivial, and might prove useful in future investigations. So we give it here.

In order to invert \( N \) consider the matrix

\[
T = D^{-1} C^{-1} N C^{-1} D^{-1}
\]

where

\[
D_{\alpha\beta} = \delta_{\alpha\beta} |X_\alpha|.
\]  

(A.7)

This matrix is of the form

\[
T = 1 + v_1 v_1^T + v_2 v_2^T
\]

where

\[
(v_1)_\alpha = \frac{|X_0| p_\alpha}{|X_\alpha| n},
\]

\[
(v_2)_\alpha = \frac{|X_{r+1}| q_\alpha}{|X_\alpha| n}.
\]

(A.9)

The inverse of a matrix of the form (A.8) is

\[
T^{-1} = 1 - \frac{1}{\Delta} \left[ (1 + v_2^2) v_1 v_1^T - (1 + v_2^2) v_2 v_2^T + (v_1 \cdot v_2) (v_1 v_1^T + v_2 v_2^T) \right]
\]

(A.10)

where it is convenient to introduce \( \Delta := 1 + v_1^2 + v_2^2 + v_1^2 v_2^2 - (v_1 \cdot v_2)^2 \). Applying (A.10) to our case we find:

\[
N^{-1}_{\alpha\beta} = \sum_\gamma \frac{C^{-1}_{\alpha\gamma} C^{-1}_{\beta\gamma}}{|X_\gamma|^2} \left[ (1 + v_2^2)|X_0|^2 U_\alpha U_\beta + (1 + v_1^2)|X_{r+1}|^2 V_\alpha V_\beta \right.
\]

\[
- \frac{|X_0|^2 |X_{r+1}|^2}{n^2} \left( \sum_\gamma \frac{p_\gamma q_\gamma}{|X_\gamma|^2} \right) (U_\alpha V_\beta + V_\alpha U_\beta) \left] \right.
\]

(A.11)

where we need to introduce vectors:

\[
U_\alpha := \sum_\gamma \frac{C^{-1}_{\alpha\gamma} p_\gamma}{|X_\gamma|^2}
\]

(V.12)

\[
V_\alpha := \sum_\gamma \frac{C^{-1}_{\alpha\gamma} q_\gamma}{|X_\gamma|^2}.
\]

Using this formula it is possible to give a completely explicit formula for the Kähler quotient metric on \( S_\zeta/U(1)^r \).
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