1. Introduction

The group $SL(n,\mathbb{R})$ acts on $\mathbb{R}^n \setminus \{0\}$ transitively by $g \cdot y = gy$ for $g \in SL(n,\mathbb{R})$ and $y \in \mathbb{R}^n$. Given a curve $\gamma$ in $\mathbb{R}^n \setminus \{0\}$, if $\det(\gamma, \gamma_s, \ldots, \gamma_s^{(n-1)})$ is positive then there is an orientation preserving parameter $x$ unique up to translation (in fact, $\frac{dx}{ds} = \det(\gamma, \gamma_s, \ldots, \gamma_s^{(n-1)})^{\frac{1}{n(n-1)}}$) such that

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) = 1,$$

(1.1)

where $\gamma_x^{(i)} = \frac{d^i \gamma}{dx^i}$. Take $x$ derivative of (1.1) to get

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-2)}, \gamma_x^{(n)}) = 0.$$

This implies that

$$\gamma_x^{(n)} = u_1 \gamma + u_2 \gamma_x + \cdots + u_{n-1} \gamma_x^{(n-2)},$$

for some smooth functions $u_1, \ldots, u_{n-1}$. This parameter $x$ is called the central affine arc-length parameter, $g = (\gamma, \ldots, \gamma_x^{(n-1)})$ the central affine moving frame, and $u_i$ the $i$-th central affine curvature of $\gamma$ for $1 \leq i \leq n-1$ (cf. [3, 7]). Note that

$$u_i = \det(\gamma, \gamma_x, \ldots, \gamma_x^{(i-2)}, \gamma_x^{(n-2)}, \gamma_x^{(i)}, \ldots, \gamma_x^{(n-1)}),$$

and

$$g_x = g(b + u),$$

(1.2)

where $b = \sum_{i=1}^{n-1} e_{i+1,i}$ and $u = \sum_{i=1}^{n-1} u_i e_{in}$.

Let $I = S^1$ or $\mathbb{R}$,

$$\mathcal{M}_n(I) = \{ \gamma : I \rightarrow \mathbb{R}^n \setminus \{0\} \mid \det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) = 1 \},$$

$$V_n = \oplus_{i=1}^{n-1} \mathbb{R} e_{in} \subset sl(n, \mathbb{R}),$$

and $\Psi : \mathcal{M}_n(I) \rightarrow C^\infty(\mathbb{R}, V_n)$ be the central affine curvature map defined by

$$\Psi(\gamma) = u = \sum_{i=1}^{n-1} u_i e_{in},$$

where $u_1, \ldots, u_{n-1}$ are the central affine curvatures along $\gamma$. It follows from the existence and uniqueness for ordinary differential equations that we have the following.

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(i) Given a smooth function \( u = \sum_{i=1}^{n-1} u_ie_{in} \in C^\infty(\mathbb{R}, V_n) \) and \( c_0 \in SL(n, \mathbb{R}) \), there exists a unique smooth \( g : \mathbb{R} \to SL(n, \mathbb{R}) \) satisfying (1.2) and \( g(0) = c_0 \). Moreover, let \( \gamma = gp_0 \), then \( u = \Psi(\gamma) \) and \( g \) is the central affine moving frame for \( \gamma \), where \( p_0 = (1, 0, \ldots, 0)^T \).

(ii) \( \Psi(\gamma_1) = \Psi(\gamma_2) \) if and only if there exists a constant \( c_0 \in SL(n, \mathbb{R}) \) such that \( \gamma_2 = c_0\gamma_1 \).

Hence \( \Psi \) induces a bijection from the orbit space \( \mathcal{M}_n(\mathbb{R}) \to SL(n, \mathbb{R}) \) and \( \{ u_1, \ldots, u_n-1 \} \) forms a complete set of local differential invariants for curves in \( \mathbb{R}^n \setminus \{0\} \) under the group \( SL(n, \mathbb{R}) \). For example, curves with zero central affine curvatures in \( \mathbb{R}^n \setminus \{0\} \) are of the form

\[
\gamma(x) = c \left( 1, x, \frac{x^2}{2}, \ldots, \frac{x^{n-1}}{(n-1)!} \right)^T,
\]

for some \( c \in SL(n, \mathbb{R}) \).

Let \( y \in C^\infty(\mathbb{R}, \mathbb{R}^n) \). We say \( \eta \) is a differential polynomial in \( y \) if \( \eta \) is a polynomial in \( y \) and its \( x \)-derivatives.

Note that \( \xi(\gamma) = \xi_0\gamma + \xi_1\gamma_x + \ldots + \xi_{n-1}\gamma_x^{(n-1)} \) is tangent to \( \mathcal{M}_n(\mathbb{R}) \) at \( \gamma \) if and only if

\[
\sum_{i=0}^{n-1} \det(\gamma, \ldots, \gamma_x^{(i-1)}, \xi_0(i), \xi_x(i+1), \ldots, \gamma_x^{(n-1)}) = 0. \tag{1.3}
\]

A central affine curve flow is an evolution equation on \( \mathcal{M}_n(\mathbb{R}) \) of the form

\[
\gamma_t = \xi(\gamma) = \xi_0(u)\gamma + \xi_1(u)\gamma_x + \ldots + \xi_{n-1}(u)\gamma_x^{(n-1)}, \tag{1.4}
\]

where \( \xi \) is tangent to \( \mathcal{M}_n(\mathbb{R}) \) at \( \gamma \) and \( \xi_0, \ldots, \xi_{n-1} \) are differential polynomials in the central affine curvatures \( u_1(\cdot, t), \ldots, u_{n-1}(\cdot, t) \) of \( \gamma(\cdot, t) \). In particular, \( \xi(\gamma) \) must satisfy (1.3).

It is easy to see that a central affine curve flow is invariant under the action of \( SL(n, \mathbb{R}) \) on \( \mathbb{R}^n \setminus \{0\} \) and translations in the \((x, t)\)-plane. In other words, if \( \gamma(x, t) \) is a solution of (1.4), then so is

\[
\gamma(x, t) = c\gamma(x + r_1, t + r_2),
\]

where \( c \in SL(n, \mathbb{R}) \) and \( r_1, r_2 \in \mathbb{R} \) are constants.

Let \( n \geq 3 \). A direct computation implies that \( X(\gamma) = y_1(\gamma)\gamma + \gamma_{xx} \) is tangent to \( \mathcal{M}_n(\mathbb{R}) \) at \( \gamma \) if and only if \( y_1(\gamma) = -\frac{2}{n}u_{n-1} \). In other words,

\[
\gamma_t = -\frac{2}{n}u_{n-1}\gamma + \gamma_{xx}, \tag{1.5}
\]

is one of the simplest central affine curve flows on \( \mathcal{M}_n(\mathbb{R}) \) for \( n \geq 3 \), where \( u_{n-1}(\cdot, t) \) is the \((n-1)\)-th central affine curvature for \( \gamma(\cdot, t) \). This curve flow turns out to be integrable. In fact, if \( \gamma \) is a solution of the central affine curve flow (1.5), then its central affine curvature \( u(\cdot, t) = \Psi(\gamma(\cdot, t)) \) is a solution of the second flow in the \( A_{n-1}^{(1)} \)-KdV hierarchy. The \( A_{n-1}^{(1)} \)-KdV hierarchy was constructed by Drinfeld and Sokolov in [4] and they showed that it is equivalent to the Gelfand-Dicky (GD) hierarchy. Although the curve flow
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(1.5) takes a very simple form for all \( n \), the formula for the second flow (2.12) in the \( A_{n-1}^{(1)} \)-KdV hierarchy is rather complicated. So the central affine curve flow (1.5) can be viewed as a simple and natural geometric interpretation of the second flow in the \( A_{n-1}^{(1)} \)-KdV hierarchy.

Next we state some of our main results:

(a) We construct a sequence of commuting higher order central affine curve flows on \( M_n(\mathbb{R}) \) such that the second flow is (1.5).

(b) We prove that the affine curvature map \( \Psi \) gives a one to one correspondence between solutions of the \( j \)-th central affine curve flow modulo the action of \( SL(n, \mathbb{R}) \) and solutions of the \( j \)-th flow in the \( A_{n-1}^{(1)} \)-KdV hierarchy.

(c) For \( n \neq 3 \), the third central affine curve flow on \( M_n(\mathbb{R}) \) is

\[
\gamma_t = \left( -\frac{3}{n} u_{n-2} + \frac{3(n-3)}{2n} (u_{n-1})_x \right) \gamma - \frac{3}{n} u_{n-1} \gamma_x + \gamma_{xxx}. 
\]  

(1.6)

Note that when \( n = 2 \), we have \( q = u_1 \) is the central affine curvature for \( \gamma \) and \( \gamma_{xx} = q \gamma \). Then (1.6) becomes

\[
\gamma_t = \frac{1}{4} q_x \gamma - \frac{1}{2} q \gamma_x, 
\]  

(1.7)

which is the central affine curve flow on \( \mathbb{R}^2 \setminus 0 \) first studied in [7]. It was proved in [7] that if \( \gamma \) is a solution of (1.7) then \( q \) is a solution of the KdV. If \( \gamma \) is a solution of (1.6) then the central affine curvatures give rise to a solution of the third flow of the \( A_{n-1}^{(1)} \)-KdV hierarchy. Hence (1.6) is the natural generalization of Pinkall’s 2-d central affine curve flow (1.7) in \( n \)-dimension.

(d) We use the solution of the Cauchy problem of the second flow of the \( A_{n-1}^{(1)} \)-hierarchy to solve the Cauchy problems for (1.5) with periodic initial data and also with initial data whose central affine curvatures are rapidly decaying.

(e) We obtain a bi-Hamiltonian structure and a sequence of Poisson structures \( \{\cdot,\cdot\}_j \) on \( M_n(S^1) \) for the central affine curve flow hierarchy. We prove that the bi-Hamiltonian structure arises naturally from the Poisson structures on \( M_n(S^1) \) of certain co-adjoint orbits.

(f) Since the \( \mathbb{R}^{n-1} \)-action on \( M_n(S^1) \) generated by the first \( (n - 1) \) central affine curve flows commutes with the \( SL(n, \mathbb{R}) \)-action, the direct product \( SL(n, \mathbb{R}) \times \mathbb{R}^{n-1} \) acts on \( M_n(S^1) \). We show that the bi-Hamiltonian structure on \( M_n(S^1) \) induces weak symplectic forms on the orbits space \( M_n(S^1)/SL(n, \mathbb{R}) \) and \( M_n(S^1)/SL(n, \mathbb{R}) \times \mathbb{R}^{n-1} \) respectively.

We note that results (a), (b) and one Poisson structure were obtained in [3] for (1.5) when \( n = 3 \).

In a forthcoming paper [12], we construct Bäcklund transformations for (2.12) and for the central affine curve flow (1.5). Note that \( u = 0 \) is a solution
of (2.12) and the corresponding solutions of (1.5) are the $SL(n, \mathbb{R})$-orbit of the first column $\gamma_0(x, t)$ of $\exp(bx + b^2 t)$. For example,

$$
\gamma_0(x, t) = (1, x, \frac{1}{2} x^2 + t), \\
\gamma_0(x, t) = (1, x, \frac{1}{2} x^2 + t, \frac{1}{3!} x^3 + xt)
$$

are solutions of (1.5) on $M_3(\mathbb{R})$ and $M_4(\mathbb{R})$ respectively with zero central affine curvatures. In [12], we apply BTs to $\gamma_0$ to construct infinitely many families of explicit soliton and rational solutions of (2.12) and (1.5).

This paper is organized as follows: In section 2, we give a brief review of the $A_{n-1}^{(1)}$-KdV hierarchy. In section 3, we prove results (a)-(d). We review the bi-Hamiltonian structure of the $A_{n-1}^{(1)}$-KdV hierarchy and compute the kernels of the bi-Hamiltonian structures in section 4. We write down the formula for the bi-Hamiltonian structure of (1.5) and prove results (e), (f) in the last section.

2. THE $A_{n-1}^{(1)}$-KdV HIERARCHY

Drinfeld and Sokolov constructed a hierarchy of KdV type for each affine Kac-Moody algebra and showed that the $A_{n-1}^{(1)}$-KdV hierarchy is equivalent to the GD$_n$-hierarchy. In this section, we give a brief review of the construction of the $A_{n-1}^{(1)}$-KdV hierarchy (cf. [3]).

We first set up some notations. Let

$$
B_+ = \{ y = (y_{ij}) \in sl(n, \mathbb{R}) \mid y_{ij} = 0, i > j \}, \\
N_+ = \{ y = (y_{ij}) \in sl(n, \mathbb{R}) \mid y_{ij} = 0, i \geq j \}, \\
T_n = \{ y \in gl(n, \mathbb{R}) \mid y_{ij} = 0, i \neq j \},
$$

denote the subalgebras of upper triangular, strictly upper triangular matrices in $sl(n, \mathbb{R})$ and diagonal matrices in $gl(n, \mathbb{R})$ respectively, and $N_+$ the corresponding Lie subgroups of $N_+$.

Let

$$
L(sl(n, \mathbb{R})) = \{ \xi(\lambda) = \sum_j \xi_j \lambda^j \mid \xi_j \in sl(n, \mathbb{R}) \}.
$$

For $\xi \in L(sl(n, \mathbb{R}))$, we use the following notation:

$$
\xi_+ = \sum_{j \geq 0} \xi_j \lambda^j, \quad \xi_- = \sum_{j < 0} \xi_j \lambda^j.
$$

Let

$$
J = e_{1,n} \lambda + b, \quad b = \sum_{i=1}^{n-1} e_{i+1,i}.
$$
A direct computation shows that
\[
 J^n = \lambda I_n, \\
 J^i = (b^i)^{n-i} \lambda + b^i, \quad J^{-i} = (b^i)^i + b^{n-i} \lambda^{-1}, \quad 1 \leq i \leq n - 1. \tag{2.1}
\]

Given \( u \in C^\infty(\mathbb{R}, \mathcal{B}_+) \), a direct computation (cf. [10]) shows that there exists a unique \( Y(u, \lambda) \in \mathcal{L}(sl(n, \mathbb{C})) \) satisfying
\[
\begin{cases}
    [\partial_x + J + u, Y(u, \lambda)] = 0, \\
    Y(u, \lambda)^n = \lambda I_n.
\end{cases} \tag{2.2}
\]
Moreover, the coefficients of the power series expansion of \( Y(u, \lambda) \) in \( \lambda \) can be computed from (2.2) and they are polynomial differentials of \( u \).

Given \( j \not\equiv 0 \pmod{n} \), let \( Y(u, \lambda) \) denote the solution of (2.2), and write
\[
(Y(u, \lambda))^j = \sum_{-\infty}^{[\frac{j}{n}] + 1} Y_{j,i}(u) \lambda^i. \tag{2.3}
\]

It was known (cf. [4], [10]) that if \( u \in C^\infty(\mathbb{R}, V_n) \) then there is a unique \( \zeta_j(u) \in \mathcal{N}_+ \) such that
\[
\begin{align*}
    [\partial_x + b + u, Y_{j,0}(u) - \zeta_j(u)] &\in V_n = \bigoplus_{i=1}^{n-1} \mathbb{R} e_{in} \tag{2.4}
\end{align*}
\]
and entries of \( \zeta_j(u) \) are differential polynomials in \( u \). The \( A_{n-1}^{(1)} \)-KdV hierarchy constructed by Drinfeld-Sokolov in [4] is the sequence of the following flows on \( C^\infty(\mathbb{R}, V_n) \) with \( j \geq 0 \) and \( j \not\equiv 0 \pmod{n} \):
\[
u_t = [\partial_x + b + u, Y_{j,0}(u) - \zeta_j(u)]. \tag{2.5}
\]
Set
\[
Z_j(u, \lambda) = (Y(u, \lambda)^j)^+ - \zeta_j(u) = \sum_{0 \leq i \leq [\frac{j}{n}] + 1} Z_{j,i}(u) \lambda^i. \tag{2.6}
\]
So we have
\[
Z_{j,i}(u) = \begin{cases}
    Y_{j,i}(u), & i > 0, \\
    Y_{j,0}(u) - \zeta_j(u), & i = 0.
\end{cases}
\]

**Proposition 2.1.** Given \( u \in C^\infty(\mathbb{R}, V_n) \), then the following statements are equivalent
\[
\begin{align*}
    (i) \quad &u \text{ is a solution of (2.5), i.e., } u_t = [\partial + b + u, Z_{j,0}(u)], \\
    (ii) \quad &[\partial_x + J + u, \partial_t + Z_j(u, \lambda)] = 0, \tag{2.7}
\end{align*}
\]
for all parameter \( \lambda \in \mathbb{C} \) (i.e. (2.7) is the Lax pair of (2.5)),
\[
(iii) \quad [\partial_x + b + u, \partial_t + Z_{j,0}(u)] = 0, \text{ which is the Lax pair (2.7) with parameter } \lambda = 0,
\]
Example 2.3. [The $A^{(1)}_n$-KdV hierarchy]

The $A^{(1)}_n$-KdV hierarchy is the KdV hierarchy: Let $u = \left( \begin{array}{cc} 0 & q \\ 0 & 0 \end{array} \right)$, $Y(u, \lambda)$ be the solution of (2.2) for $u$. Then it can be checked that

\[
E^{-1}E_x = J + u,
E^{-1}E_t = Z_j(u, \lambda),
E(x, t, \lambda) = E(x, t, \lambda).
\]

(2.8)

We call a solution $E$ of (2.8) a frame of the solution $u$ of the $j$-th flow (2.5) of the $A^{(1)}_{n-1}$-KdV hierarchy. Note the third condition of (2.8) implies that $E(x, t, \lambda) \in GL(n, \mathbb{R})$ for $\lambda \in \mathbb{R}$. It follows from $d(\ln \det(E)) = \text{tr}(E^{-1}dE)$ and $\text{tr}(J + u) = \text{tr}(Z_j(u, \lambda)) = 0$ that we have

**Corollary 2.2.** If $E(x, t, \lambda)$ is a frame of a solution $u$ of the $j$-th flow in the $A^{(1)}_{n-1}$-KdV hierarchy, then $\det(E(x, t, \lambda))$ is independent of $x, t$.

Example 2.4. [The second flow in the $A^{(1)}_{n-1}$-KdV hierarchy]

Given $u = \sum_{i=1}^{n-1} u_i e_i$, after a straightforward but lengthy computation, we get

\[
Z_2(u, \lambda) = (Y(u, \lambda)^2)_+ - \zeta_2(u) = (e_{1,n-1} + e_{2,n})\lambda + Z_{2,0}(u),
\]

(2.9)

\[
Z_{2,0}(u) = Y_{2,0}(u) - \zeta_2(u) = \sum_{i=1}^{n-2} e_{i+2,i} + \sum_{i \leq j} q_{i,j} e_{i,j},
\]

(2.10)

where

\[
\begin{align*}
q_{i,j} &= -\frac{2}{n} (j-1) u_{n-1}^{j-i}, & & j \leq n-2, \ i \leq j, \\
q_{1,n-1} &= u_1 - \frac{2}{n} u_{n-1}^{n-2}, \\
q_{n-1,1} &= -\frac{2}{n} (n-1) u_{n-1}^{n-2} + u_1, & & 2 \leq i \leq n-2, \\
q_{n-1,n-1} &= \frac{n-2}{n} u_{n-1}, \\
q_{1,n} &= (u_1)_x - \frac{2}{n} u_{n-1}^{n-1}, \\
q_{i,n} &= u_{i-1} + (u_i)_x - \frac{2}{n} (n-1) u_{n-1}^{n-1}, & & 2 \leq i \leq n-1, \\
q_{n,n} &= \frac{n-2}{n} u_{n-1}. 
\end{align*}
\]

(2.11)
So the second flow in the $A_{n-1}^{(1)}$-KdV hierarchy is

\[
\begin{cases}
(u_1)_t = (q_{1,n})_x + u_1 q_{n,n} - \sum_{j=1}^{n-1} u_j q_{1,j}, \\
(u_2)_t = (q_{2,n})_x + q_{1,n} + u_2 q_{n,n} - \sum_{j=2}^{n-1} u_j q_{2,j}, \\
(u_i)_t = (q_{i,n})_x + q_{i-1,n} + u_i q_{n,n} - \sum_{j=i}^{n-1} q_{j,j} u_j - u_{j-2}, \quad 3 \leq i \leq n-1,
\end{cases}
\]

where $q_{i,j}$ is given by (2.11). In particular, when $n = 3$, we get

\[
Z_2(u, \lambda) = (e_{12} + e_{23})\lambda + \begin{pmatrix} -\frac{2}{3}u_2 & u_1 - \frac{2}{3}(u_2)_x & (u_1)_x - \frac{2}{3}(u_2)_{xx} \\ 0 & \frac{1}{3}u_2 & u_1 - \frac{1}{3}(u_2)_x \\ 1 & 0 & \frac{1}{3}u_2 \end{pmatrix},
\]

and the second flow in the $A_{2}^{(1)}$-KdV hierarchy is

\[
\begin{cases}
(u_1)_t = (u_1)_{xx} - \frac{2}{3}(u_2)_{xxx} + \frac{2}{3}u_2(u_2)_x, \\
(u_2)_t = -(u_2)_{xx} + 2(u_1)_x.
\end{cases}
\]

3. CENTRAL AFFINE CURVE FLOWS AND THE $A_{n-1}^{(1)}$-KDV HIERARCHY

Let $g = (\gamma, \gamma_x, \ldots, \gamma_{2}^{(n-1)})$ be the central affine moving frame along $\gamma \in \mathcal{M}_{n}(\mathbb{R})$. It follows from (2.10) and (2.11) that the first column of $Z_{2,0}(u)$ is $(-\frac{2}{3}u_{n-1}, 0, 1, 0, \ldots, 0)^t$. So the central affine curve flow (1.5) can be written as $\gamma_t = gZ_{2,0}(u)p_0$, where

\[p_0 = (1, 0, \ldots, 0)^t.\]

This motivates us to consider for every positive $j$, $j \not\equiv 0 \pmod{n}$, the following curve flow

\[\gamma_t = gZ_{j,0}(u)p_0.\]

In this section, we will prove the following results:

1. $\xi(\gamma) = gZ_{j,0}(u)p_0$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma$. So (3.1) is a flow equation on $\mathcal{M}_n(\mathbb{R})$, which will be called the $j$-th central affine curve flow. The proof of $\xi(\gamma)$ is in $T\mathcal{M}_n(\mathbb{R})_{\gamma}$ for small $n$ and $j$ is not difficult because we have explicit formulas for $Z_{j,0}(u)$. But the proof of this fact for general $n, j$ is more complicated.

2. The affine curvature map $\Psi$ gives a one to one correspondence between the space of solutions of the $j$-th central affine curve flow (3.1) on $\mathbb{R}^n \setminus \{0\}$ modulo $SL(n, \mathbb{R})$ and the space of solutions of the $j$-th flow,

\[u_t = [\partial_x + b + u, Z_{j,0}(u)],\]

in the $A_{n-1}^{(1)}$-KdV hierarchy.

3. We use solution of the Cauchy problem for the $j$-th flow in the $A_{n-1}^{(1)}$-KdV hierarchy to solve the Cauchy problem for (3.1) with initial data that is periodic or has rapidly decaying central affine curvatures.
(4) Given $u \in C^\infty(\mathbb{R}, V_n)$ and $v = \sum_{i=1}^{n-1} v_i e_{ni} \in C^\infty(\mathbb{R}, V'_n)$, we show that there is a unique $C = (C_{ij})$ in $C^\infty(\mathbb{R}, \mathfrak{sl}(n, \mathbb{R}))$ such that $[\partial_x + b + u, C] \in V_n$ and $C_{ni} = v_i$ for $1 \leq i \leq n - 1$. Let

$$P_u : C^\infty(\mathbb{R}, V'_n) \to C^\infty(\mathbb{R}, \mathfrak{sl}(n, \mathbb{R}))$$

denote the map defined by $P_u(v) = C$. We prove that $P_u$ is a linear differential operator in $v$ with differential polynomials of $u$ as coefficients and give an algorithm to compute it. We will use this operator to write down the formula for the Poisson structures of the $A_{n-1}^{(1)}$-KdV hierarchy and central affine curve flows in sections 4 and 5.

When $n = 3$, results (1) and (2) were also obtained in [3].

First we use (1.3) and a direct computation to get a necessary and sufficient condition for $\xi$ being tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma$.

**Proposition 3.1.** Let $\gamma \in \mathcal{M}_n(\mathbb{R})$, and $u = \Psi(\gamma)$ the central affine curvature of $\gamma$. Then there is a differential polynomial $\phi_0$ in $u$ and $\xi_1, \ldots, \xi_{n-1}$ such that $\xi = \sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ if and only if

$$\xi_0 = \phi_0(\xi_1, \ldots, \xi_{n-1}, u).$$

**Example 3.2.** We use (1.3) to compute $\phi_0$ for small $n$ and obtain that $\xi = \sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma$ if and only if

- $\xi_0 = -\frac{1}{2} \xi_1'$, for $n = 2$,
- $\xi_0 = -\frac{1}{3}(\xi_2'' + 3 \xi_1' + 2u_2 \xi_2)$, for $n = 3$,
- $\xi_0 = -\frac{1}{4}(\xi_3''' + 4 \xi_2' + 6 \xi_1' + 3u_3 \xi_3 + 5u_3 \xi_3' + 3u_2 \xi_3 + 2u_3 \xi_2)$, for $n = 4$.

For $n = 5$, we have

$$\xi_0 = -\frac{1}{5}(\xi_4^{(4)} + 5 \xi_3'(3) + 10 \xi_2'' + 10 \xi_1' + 6u_3 \xi_4 + 9u_3 \xi_4' + 4(u_4 \xi_4)'' + 3(u_4 \xi_4)' + 2u_4 \xi_4'' + 3(u_4 \xi_4)' + 4u_4 \xi_4' + 4u_2 \xi_4 + 3u_3 \xi_3 + 2u_4 \xi_2 + 2u_3 \xi_3).$$

Here we use $y'$ to denote $y_x$. For general $n$, we have

$$\xi_0 = -\frac{1}{n}(\xi_{n-1})_y^{(n-1)} + \ldots.$$ 

We want to prove that (3.4) is a flow on $\mathcal{M}_n(\mathbb{R})$, i.e., show that $\xi(\gamma) = gZ_{j,0}(u)p_0$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma$. We do not have formulas explicit enough for $\phi_0$ and for $Z_{j,0}(u)$ to check directly. But $Z_{j,0}(u)$ satisfies the condition that $[\partial_x + b + u, Z_{j,0}(u)]$ lies in $V_n$. The following theorem shows that this condition is enough to prove that $gZ_{j,0}(u)p_0$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma$.
Theorem 3.3. Let \( u \in C^\infty(\mathbb{R}, V_n) \), and \( g \in C^\infty(\mathbb{R}, SL(n, \mathbb{R})) \) satisfying \( g^{-1}g_x = b + u \). Suppose \( C = (C_{ij}) \in C^\infty(\mathbb{R}, sl(n, \mathbb{R})) \) satisfies \([\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n)\). Then

1. \( gC = (\eta_1, \eta_2, \ldots, \eta_n) \) for some \( \eta \in C^\infty(\mathbb{R}, \mathbb{R}^{n \times 1}) \),
2. let \( \gamma \) denote the first column of \( g \), then \( \gamma \in M_n(\mathbb{R}) \) and \( \xi(\gamma) = (3.3) \) is tangent to \( M_n(\mathbb{R}) \) at \( \gamma \).

Proof. Note that \( g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) \) and \( g^{-1}g_x = b + u \). Let

\[
\rho = [\partial_x + b + u, C] = C_x + [b + u, C].
\]

Then \( C_x = \rho - [b + u, C] \). It follows that

\[
(gC)_x = g_xC + gC_x = gC(b + u) + g\rho. \tag{3.3}
\]

Let \( \eta_j \) denote the \( j \)-th column of \( gC \). Since the first \((n - 1)\) columns of \( \rho \) are zero, we use (3.3) to get

\[
(\eta_1)_x = \eta_2, \quad (\eta_2)_x = \eta_3, \quad \ldots, \quad (\eta_{n-1})_x = \eta_n. \tag{3.4}
\]

This proves (i).

Since \( g \in C^\infty(\mathbb{R}, SL(n, \mathbb{R})) \) and \( g^{-1}g_x = b + u \), the first column \( \gamma \) of \( g \) lies in \( M_n(\mathbb{R}) \). To prove the rest of (ii), it is equivalent to prove \( \xi(\gamma) \) satisfies (i.3). Let \( C_i \) denote the \( i \)-th column of \( C \). Then \( \xi(\gamma) = gC_1 = \eta_1 \). It follows from (3.4) that we have

\[
(\xi(\gamma))^{(i-1)}(\gamma_x)_x = (\eta_1)_x^{(i-1)} = \eta_i = gC_i, \quad 2 \leq i \leq n.
\]

So \( \det(\gamma, \ldots, \gamma_x^{(i-2)}, \xi(\gamma^{(i-1)}), \gamma_x^{(i)}, \ldots, \gamma_x^{(n-1)}) = C_{ii} \). But \( \text{tr}(C) = 0 \). So \( \xi(\gamma) \) satisfies (i.3). This proves (ii). \( \square \)

Corollary 3.4. Let \( \gamma \in M_n(\mathbb{R}) \), \( g \) the central affine moving frame along \( \gamma \), and \( u = g^{-1}g_x - b \) the central affine curvature of \( \gamma \). Let \( Z_{\gamma 0}(u) \) be as defined in (2.6). Then

1. \( gZ_{\gamma 0}(u) = (\eta_1, \eta_x, \ldots, \eta_x^{(n-1)}) \) for some \( \eta \in C^\infty(\mathbb{R}, \mathbb{R}^{n \times 1}) \),
2. \( \eta \) is tangent to \( M_n(\mathbb{R}) \) at \( \gamma \),
3. (3.1) is a central affine curve flow on \( M_n(\mathbb{R}) \).

Corollary 3.5. Let \( g, \nu, C = (C_{ij}) \) be as in Theorem 3.3. Then

(i) entries of \( C \) are differential polynomials in \( u, C_{21}, \ldots, C_{n1} \),
(ii) for \( 2 \leq i \leq n - 1 \), we have

\[
C_{ni} = C_{n-i+1,1} + (i - 1)(C_{n-i+2,1})_x + \phi_i(C_{n1}, \ldots, C_{n-i+3,1}). \tag{3.5}
\]

for some linear differential operators \( \phi_i \) with differential polynomials in \( u \) as coefficients.

Proof. By Theorem 3.3, \( \sum_{i=1}^n C_{i1} \gamma_x^{(i-1)} \) is tangent to \( M_n(\mathbb{R}) \) at \( \gamma = gp_0 \), where \( p_0 = (1, 0, \ldots, 0)^t \). Proposition 3.1 implies that there exist a differential polynomial \( \phi_0 \) such that \( C_{11} = \phi_0(C_{21}, \ldots, C_{n1}) \). By Theorem 3.3 (i),
we have
\[
\left(\sum_{i=1}^{n} C_{ii} \gamma_x^{(i-1)}(i-1)\right) = \left(\sum_{i=1}^{n} C_{ij} \gamma_x^{(i-1)}\right).
\]

(3.6)

Compare coefficient of \(\gamma_x^{(n-1)}\) of (3.6) to get (3.5). Compare coefficient of \(\gamma_x^{(i-1)}\) of (3.6) for \(2 \leq i \leq n - 1\) to see that \(C_{ij}\)'s are differential polynomial of \(C_{21}, \ldots, C_{n1}\).

\[\square\]

**Corollary 3.6.** Let \(\gamma, u, g, C = (C_{ij})\) be as in Theorem 3.3 and \(C = (C_{ij})\). Then entries of \(C\) are differential polynomial of \(C_{n1}, \ldots, C_{n,n-1}\) and \(u\).

In particular, we get the following.

**Corollary 3.7.** Let \(u \in C^\infty(\mathbb{R}, V_n)\), and \(Z_{j,0}(u)\) defined by (2.6). Suppose \(C = (C_{ij}) \in C^\infty(\mathbb{R}, sl(n, \mathbb{R}))\) satisfying \([\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n)\) and \(C_{ni}\) is equal to the \(ni\)-th entry of \(Z_{j,0}(u)\) for all \(1 \leq i \leq n - 1\). Then \(C = Z_{j,0}(u)\).

In the next two examples we compute the \(C = (C_{ij})\) that satisfies the assumption of Theorem 3.3 in terms of \(C_{21}, \ldots, C_{n1}\) and also in terms of \(C_{n1}, \ldots, C_{n,n-1}\) explicitly for \(n = 3\) and 4.

**Example 3.8.** For \(n = 3\), let \(\gamma \in M_3(\mathbb{R})\), \(g = (\gamma, \gamma_x, \gamma_x^{(2)})\), and \(u = u_1 e_{13} + u_2 e_{23} = g^{-1} g_x - b\), where \(b = e_{21} + e_{32}\). Then \(\gamma_x^{(3)} = u_1 \gamma + u_2 \gamma_x\). Let \(C = (C_{ij}) \in C^\infty(\mathbb{R}, sl(3, \mathbb{R}))\) satisfying \([\partial_x + b + u, C] \in V_3\) and \(C_{ii} = y_i - 1\) for \(1 \leq i \leq 3\). Then there is \(\delta \gamma = y_0 \gamma + y_1 \gamma_x + y_2 \gamma_{xx} \in TM_n(\mathbb{R})\gamma\) such that \(C = g^{-1} \delta g\), where \(\delta g = (\delta \gamma, (\delta \gamma)_x, \ldots, (\delta \gamma)_x^{(n-1)})\). We first write \(C = g^{-1} \delta g\) in terms of the 21-th and 31-th entries:

\[
g^{-1} \delta g = \begin{pmatrix}
y_0 & y_0' + u_1 y_2 & a_{13} \\
y_1 & y_0' + y_1' + u_2 y_2 & a_{23} \\
y_2 & y_1' + y_2' & a_{33}
\end{pmatrix},
\]

where

\[
y_0 = -\frac{1}{3} y_2'' - y_1' - \frac{2}{3} u_2 y_2,
\]

\[
a_{13} = y_0'' + u_1 y_1 + 2 u_1 y_1' + y_1' y_2,
\]

\[
a_{23} = 2 y_0' + y_1'' + u_2 y_1 + 2 u_2 y_2' + u_1 y_2 + u_2 y_1,
\]

\[
a_{33} = y_0 + 2 y_1' + y_2' + u_2 y_2 = \frac{2}{3} y_2'' + y_1' + \frac{1}{3} u_2 y_2.
\]

Next we write \(g^{-1} \delta g\) in terms of the 31-th and 32-th entries:

\[
g^{-1} \delta g = \begin{pmatrix}
-v_2' + \frac{2}{3} v_1'' - \frac{2}{3} u_2 v_1 & * & * \\
v_2 - v_1' & v_2' - \frac{1}{3} v_1'' & * \\
v_1 & v_2 & v_2 - \frac{1}{3} v_1'' + \frac{1}{3} u_2 v_1
\end{pmatrix}.
\]

(3.7)
Example 3.9. For $n = 4$, let $\gamma \in \mathcal{M}_4(\mathbb{R})$, $g = (\gamma, \gamma_x, \ldots, \gamma^{(4)}_x)$, $u = g^{-1} g_x - b = \sum_{i=1}^3 u_i e_{4i}$, and $\delta \gamma = \sum_{i=0}^3 y_i (\gamma_x)^{(i)}$. Then $\gamma_x^{(4)} = u_1 \gamma + u_2 \gamma_x + u_3 \gamma_{xx}$ and

$$g^{-1} \delta g = 
\begin{pmatrix}
y_0 & * & * & * 
y_1 & * & * & * 
y_2 & * & * & * 
y_3 & y'_2 + y''_2 + 2y'_3 + y_1 + u_3 y_3 & \cdot \cdot \cdot 
\end{pmatrix},
$$

where

$$y_0 = -\frac{1}{4} (y_3^{(3)} + 4y''_2 + 6y'_1 + 3u'_3 y_3 + 5u_3 y'_3 + 3u_2 y_3 + 2u_3 y_2),$$

$$a_{14} = y_3^{(3)} + 3y''_2 + 3y'_1 + y_0 + 2u'_3 y_3 + 3u_3 y'_3 + u_2 y_3 + u_3 y_2$$

$$= \frac{3}{4} y_3^{(3)} + 2y''_2 + \frac{3}{2} y'_1 + \frac{5}{4} y''_3 y_3 + \frac{7}{4} u_3 y'_3 + \frac{1}{4} (u_2 y_3 + 2u_3 y_2).$$

We can also write $g^{-1} \delta g$ in terms of $v = v_1 e_{41} + v_2 e_{42} + v_3 e_{43}$:

$$g^{-1} \delta g = 
\begin{pmatrix}
a_{11} & * & * & * 
v_1 & * & * & * 
v_2 & v_1' & v_2 & v_3 & a_{14} 
\end{pmatrix},
$$

where

$$a_{11} = -\frac{1}{4} (3v_1^{(3)} - 8v''_2 + 6v'_1 - 3(u_3 v_1)' + 3u_2 v_1 + 2u_3 v_2),$$

$$a_{14} = \frac{1}{4} v_1^{(3)} - v''_2 + \frac{3}{2} v'_1 - \frac{1}{4} (u_3 v_1)' + \frac{1}{4} u_2 v_1 + \frac{1}{2} u_3 v_2.$$

We use Corollary 3.6 to define a differential operator $P_u$:

Definition 3.10. Fix $u \in C^\infty(\mathbb{R}, V_n)$, let

$$P_u : C^\infty(\mathbb{R}, V^t_n) \to C^\infty(\mathbb{R}, sl(n, \mathbb{R}))$$

denote the map defined by $P_u(v) = C$, where $C$ is the unique $sl(n, \mathbb{R})$-valued map satisfies $\pi_0(C) = v$ and $[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n)$, where $\pi_0$ is the canonical projection defined by

$$\pi_0 : sl(n, \mathbb{C}) \to V^t_n; \quad \pi_0(y) = \sum_{i=1}^{n-1} y_{ni} e_{ni}, \quad \text{for} \quad y = (y_{ij}).$$

Corollary 3.11. Let $u \in C^\infty(\mathbb{R}, V_n)$, $P_u$ the operator defined by Definition 3.10 and $v \in C^\infty(\mathbb{R}, V^t_n)$. Then

(i) the entries of $P_u(v)$ are differential polynomials in $u$ and $v$,

(ii) if $g$ is the central affine moving frame along $\gamma \in \mathcal{M}_n(\mathbb{R})$ and $u = g^{-1} g_x - b$, then there exists $\delta \gamma$ tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma$ such that $P_u(v) = g^{-1} \delta g$, where $\delta g = (\delta \gamma, (\delta \gamma)_x, \ldots, (\delta \gamma)^{(n-1)}_x)$,

(iii) $\xi(\gamma) = g P_u(v) p_0$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma$, where $p_0 = (1, 0, \ldots, 0)^t$.

Corollary 3.12. Let $g$ be the central affine moving frame along $\gamma \in \mathcal{M}_n(\mathbb{R})$, $u = g^{-1} g_x - b$ the central affine curvature, $\delta \gamma \in T \mathcal{M}_n(\mathbb{R}) \gamma$, and $\delta g = (\delta \gamma, \ldots, (\delta \gamma)^{(n-1)}_x)$. Then

$$g^{-1} \delta g = P_u(\pi_0(g^{-1} \delta g)),\quad \text{where} \quad \pi_0 : sl(n, \mathbb{C}) \to V^t_n.$$
where \( \pi_0 \) is the canonical projection from \( sl(n, \mathbb{R}) \) to \( V_n^1 \).

**Proof.** Let \( C = g^{-1}\delta g \). Take variation of \( g^{-1}g_x = b + u \) to get

\[
-C(b + u) + g^{-1}(\delta g)_x = \delta u.
\]

This implies that

\[
C_x = -g^{-1}g_xg^{-1}\delta g + g^{-1}(\delta g)_x = [C, b + u] - \delta u.
\]

So \([\partial_x + b + u, C] = \delta u \in V_n\). By definition of \( P_u \), we get \( C = P_u(\pi_0(C)) \). \( \square \)

Next we use \( P_u \) to rewrite the \( j \)-th flow in the \( A_{n-1}^{(1)} \)-KdV hierarchy.

**Corollary 3.13.** Let \( \pi_0 : sl(n, \mathbb{R}) \to V_n^1 \) be the canonical projection, \( Y(u, \lambda) \) the solution of \((2.2)\), and \( Y_{j,0}(u), Z_{j,0}(u, \lambda) = Y_{j,0}(u, \lambda) - \zeta_j(u) \) as in \((2.6)\). Then

(i) \( Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u))) = P_u(\pi_0(Y_{j,0}(u))) \),

(ii) the \( j \)-th flow \((3.2)\) of the \( A_{n-1}^{(1)} \)-KdV hierarchy on \( C^\infty(\mathbb{R}, V_n) \) can be written as

\[
\gamma_j = [\partial_x + b + u, P_u(\pi_0(Y_{j,0}(u)))].
\]

**Proof.** Recall that \( Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u) \) and \( \zeta_j(u) \in C^\infty(\mathbb{R}, \mathcal{N}_+) \) is the unique one that makes \([\partial_x + b + u, Z_{j,0}(u)] \in V_n\). Since \( \zeta_j \in \mathcal{N}_+, \pi_0(Z_{j,0}(u)) = \pi_0(Y_{j,0}(u)) \). By definition of \( P_u \), we have \( Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u))) \). This proves the corollary. \( \square \)

**Example 3.14.** [Higher order central affine curve flows]

We obtain the first few terms of the power series of \( Y(u, \lambda) \) in \( \lambda \) by solving \((2.2)\). Then we can use \( P_u \) to compute the fourth and the fifth central affine curve flows on \( \mathcal{M}_3(\mathbb{R}) \) as follows:

\[
\gamma_t = -\frac{1}{9}(2u'' - 3u' - 2u_2^2)\gamma + \frac{1}{3}(u_2' - u_1)\gamma_x - \frac{u_2}{3}\gamma_{xx},
\]

\[
\gamma_t = \frac{1}{9}(-u''_1 + u_1u_2)\gamma - \frac{1}{9}(u''_2 - 3u'_1 + u_2^2)\gamma_x + \frac{1}{3}(u'_2 - 2u_1)\gamma_{xx}.
\]

For \( n \neq 3 \), the third central affine curve flow on \( \mathcal{M}_n(\mathbb{R}) \) is the flow \((1.6)\) given in the introduction:

\[
\gamma_t = \left(-\frac{3}{n}u_{n-2} + \frac{3(n-3)}{2n}(u_{n-1})_x\right) \gamma - \frac{3}{n}u_{n-1}\gamma_x + \gamma_{xx}.
\]

As explained in the introduction that when \( n = 2 \), this is the Pinkall’s central affine curve flow on \( \mathbb{R}^2 \setminus 0 \),

\[
\gamma_t = \frac{1}{4}q_x\gamma - \frac{1}{2}q\gamma_x.
\]

So \((1.6)\) is a natural analogue of Pinkall’s flow in \( n \)-dimension \( (n \neq 3) \).
Theorem 3.15. Assume \( j \neq 0 \mod n \). Let \( \gamma \) be a solution of (3.1) on \( \mathcal{M}_n(\mathbb{R}) \), and \( \Psi \) the central affine curvature map. Then \( u(\cdot, t) = \Psi(\gamma(\cdot, t)) = \sum_{i=1}^{n-1} u_i(\cdot, t)e_{in} \) is a solution of the \( j \)-th flow (3.2) in the \( A_{n-1}^{(1)} \)-KdV hierarchy.

Proof. Let \( g(\cdot, t) = (g_1, g_2, \ldots, g_{n-1})(\cdot, t) \) be the central affine moving frame for \( \gamma(\cdot, t) \in \mathcal{M}_n(\mathbb{R}) \). Then \( g^{-1}g_x = b + u \). Next we compute \( g_t \). Let \( \eta_i \) denote the \( i \)-th column of \( gZ_j(0)u \). Since \( [\partial_x + b + u, Z_j(0)] \in V_n \), Theorem 3.3 implies that \( \eta_i = (\eta_i^i)^{(i-1)} \). We have \( \gamma_t = \eta_i \). So a direct computation implies that \( (\gamma_i^i)^{(i-1)} = (\gamma_t^i)^{(i-1)} = \eta_i^i \) for \( 2 \leq i \leq n \). This proves that \( g_t = gZ_j(0)u \). It follows from Proposition 2.1 that \( u \) is a solution of (3.2).

The converse is also true.

Theorem 3.16. Let \( u = \sum_{i=1}^{n-1} u_i e_{in} \) be a solution of the \( j \)-th flow (3.2) in the \( A_{n-1}^{(1)} \)-KdV hierarchy, and \( c_0 \in SL(n, \mathbb{R}) \) a constant. Then

1. there exists a unique \( g : \mathbb{R}^2 \to SL(n, \mathbb{R}) \) satisfies
   \[
   \begin{cases}
   g^{-1}g_x = b + u, \\
   g^{-1}g_t = Z_j(0)u, \\
   g(0, 0) = c_0.
   \end{cases}
   \]

2. \( \gamma(x, t) := g(x, t)p_0 \) is a solution of the \( j \)-th central affine curve flow (3.1) with \( u(\cdot, t) = \Psi(\gamma(\cdot, t)) \), where \( p_0 = (1, 0, \ldots, 0)^t \) and \( \Psi \) is the central affine curvature map.

Proof. Statement (i) follows from Proposition 2.1.

Note that \( g_x = g(\gamma_1) \) implies \( g = (\gamma, \gamma_2, \ldots, \gamma_{n-1}) \) and \( \gamma_x = u_1 \gamma + \ldots + u_{n-1} \gamma_{n-2} \). So \( \gamma(\cdot, t) \in \mathcal{M}_n(\mathbb{R}) \) and \( u_1, \ldots, u_{n-1} \) are the central affine curvatures of \( \gamma(\cdot, t) \). Since \( g_t = gZ_j(0)u \), we get \( \gamma_t = gtp_0 = gZ_j(0)u)p_0 \), which is the \( j \)-th central affine curve flow.

Corollary 3.17. Let \( \Psi \) be the central affine curvature map, and \( \gamma_1, \gamma_2 \) solutions of (1.5) on \( \mathcal{M}_n(\mathbb{R}) \). Then

1. \( \Psi(\gamma_1(\cdot, t)) = \Psi(\gamma_2(\cdot, t)) \) if and only if there is a constant \( c_0 \) in \( SL(n, \mathbb{R}) \) such that \( \gamma_2 = c_0 \gamma_1 \).
2. \( \Psi \) induces a bijection between the space of solutions of (1.5) modulo \( SL(n, \mathbb{R}) \) and the space of solutions of (2.12).

Next we discuss the Cauchy problem for the \( j \)-th central affine curve flow (3.1). The Cauchy problem for the \( A_{n-1}^{(1)} \)-KdV hierarchy (3.8) is solved for an open dense subset of rapidly decaying smooth initial data (cf. [2]). As a consequence, we get the solution for the Cauchy problem for the curve flow (3.1):
Theorem 3.18. [Cauchy problem with rapidly decaying affine curvatures]
Let \( j \geq 1 \) and \( j \neq 0 \mod n \). Given \( \gamma_0 \in \mathcal{M}_n(\mathbb{R}) \) with rapidly decaying central affine curvatures \( u^0_i, \ldots, u^0_{n-1} \), let \( g_0 \) be the central affine moving frame along \( \gamma_0 \). Suppose \( u = \sum_{i=1}^{n-1} u_i e_{in} \) is the solution of the \( j \)-th flow (3.2) in the \( A^{(1)}_{n-1} \)-KdV hierarchy with \( u(x,0) = \sum_{i=1}^{n-1} u^0_i(x) e_{in} \). Let \( g(x,t) : \mathbb{R}^2 \to SL(n,\mathbb{R}) \) be the solution of (3.9) with initial data \( c_0 = g_0(0) \). Then \( \gamma = gp_0 \) is the solution of the \( j \)-th central affine curve flow (3.1) with \( \gamma(x,0) = \gamma_0(x) \). Moreover, the central affine curvatures of \( \gamma(\cdot,t) \) are also rapidly decaying.

Finally, we use the solution of Cauchy problem of the second flow of the \( A^{(1)}_{n-1} \)-KdV hierarchy with periodic initial data to solve the Cauchy problem for the curve flow (1.5) with periodic initial data. By Theorem 3.16 we only need to solve the period problem of (3.9). In fact, we have the following.

Theorem 3.19. [Cauchy Problem with periodic initial data]
Let \( \gamma_0 \in \mathcal{M}_n(S^1) \), and \( u^0_1, \ldots, u^0_{n-1} \) the central affine curvatures of \( \gamma_0 \). Suppose \( u = \sum_{i=1}^{n-1} u_i e_{in} \) is the solution of the periodic Cauchy problem of (3.2) with initial data \( u(x,0) = \sum_{i=1}^{n-1} u^0_i(x) e_{in} \). Let \( g : \mathbb{R}^2 \to SL(n,\mathbb{R}) \) be the solution of (3.9) with initial data \( c_0 = g_0(0) \), where \( g_0 \) is the central affine frame along \( \gamma_0 \). Then \( \gamma = gp_0 \) is a solution of (3.1) with initial data \( \gamma(x,0) = \gamma_0(x) \). Moreover, \( \gamma(x,t) \) is periodic in \( x \) and \( \{u_i(\cdot,t), 1 \leq i \leq n-1\} \) are the central affine curvatures for \( \gamma(\cdot,t) \).

Proof. Note that both \( g_0 \) and \( g(\cdot,0) \) satisfy the same ordinary differential equations, \( g^{-1}g_x = b + u(x,0) \), and have the same initial data. So the uniqueness of ordinary differential equations implies that \( g(x,0) = g_0(x) \). It follows from Theorem 3.16 that \( \gamma(x,t) = g(x,t)p_0 \) is a solution of the curve flow (3.1). Moreover, \( \gamma(x,0) = g(x,0)p_0 = \gamma_0(x) \). It remains to prove that \( \gamma \) is periodic in \( x \).

Since \( \gamma_0 \) is periodic with period \( 2\pi \), \( g_0 \) and \( u_0 \) are periodic in \( x \) with period \( 2\pi \). Since \( u(x,t) \) is periodic in \( x \), so is \( Z_{j,0}(u) \). It suffices to prove
\[
y(t) = g(2\pi,t) - g(0,t)
\]
is identically zero. To do this, we calculate
\[
y_t = g_t(2\pi,t) - g_t(0,t) \\
= (gZ_{j,0}(u))(2\pi,t) - (gZ_{j,0}(u))(0,t) = (g(2\pi,t) - g(0,t))Z_{j,0}(u(0,t)) \\
= y(t)Z_{j,0}(u(0,t)).
\]
Since \( g_0 \) is periodic in \( x \) with period \( 2\pi \), \( y(0) = g(2\pi,0) - g(0,0) = 0 \). Note that \( Z_{j,0}(u(0,t)) \) is given and 0 is the solution of \( y_t = yZ_{j,0}(u(0,t)) \) with the same initial condition \( y(0) = 0 \). So it follows from the uniqueness of ordinary differential equations that \( y \) is identically zero. 

\[\square\]
4. Bi-Hamiltonian structure for the $A_{n-1}^{(1)}$-KdV hierarchy

We first review the bi-Hamiltonian structure for the $A_{n-1}^{(1)}$-KdV hierarchy given in [4]. Then we write down the formulas for these two Poisson structures in terms of the operator $P_u$ defined in Definition 3.10 and compute the kernel. The Hamiltonian theory of the $A_{n-1}^{(1)}$-KdV hierarchy works for either the phase space $S(\mathbb{R}, V_n)$ of rapidly decaying smooth maps or the phase space $C^\infty(S^1, V_n)$. To simplify the presentation, we will only discuss the Hamiltonian theory for the $A_{n-1}^{(1)}$-KdV hierarchy on $C^\infty(S^1, V_n)$.

The group $C^\infty(S^1, N_+)$ acts on $C^\infty(S^1, B_+)$ by gauge transformation

$$g \ast q = g(b + q)g^{-1} - g_x g^{-1} - b,$$

or equivalently,

$$g \ast (\partial_x + b + q) = g(\partial_x + b + q)g^{-1} = \partial_x + b + g \ast q.$$

It was proved in [4] (also in [10]) that $C^\infty(S^1, V_n)$ is a cross section of the gauge action $C^\infty(S^1, N_+)$ on $C^\infty(S^1, B_+)$, i.e., given $q \in C^\infty(S^1, B_+)$ there exist unique $\Delta \in C^\infty(S^1, N_+)$ and $u \in C^\infty(S^1, V_n)$ such that $\Delta \ast q = u$. In other words, each gauge orbit of $C^\infty(S^1, N_+)$ meets $C^\infty(S^1, V_n)$ exactly once. So $C^\infty(S^1, V_n)$ is isomorphic to the orbit space $C^\infty(S^1, B_+)/C^\infty(S^1, N_+)$ and the ring of (differential) functionals on $C^\infty(S^1, V_n)$ is isomorphic to the ring of $C^\infty(S^1, N_+)$-invariant functionals on $C^\infty(S^1, B_+)$.

The two Poisson structures in [4] are defined for $C^\infty(S^1, N_+)$ invariant functionals on $C^\infty(S^1, W_n)$. So they are Poisson structures on $C^\infty(S^1, V_n)$.

Given a functional $F : C^\infty(S^1, V_n) \to \mathbb{R}$, let $\tilde{F} : C^\infty(S^1, B_+) \to \mathbb{R}$ be the functional defined by $\tilde{F}(q) = F(u)$ if $u \in C^\infty(S^1, V_n)$ lies in the same $C^\infty(S^1, N_+)$-orbit as $q$. In other words, $\tilde{F}$ is the unique $C^\infty(S^1, N_+)$-invariant functional satisfying $\tilde{F}(u) = F(u)$ for all $u \in C^\infty(S^1, V_n)$.

Let $\langle , \rangle$ be the bi-linear form on $C^\infty(S^1, sl(n, \mathbb{R}))$ defined by

$$\langle y_1, y_2 \rangle = \int \text{tr}(y_1(x)y_2(x))dx.$$ 

The gradient of $\tilde{F}(q)$ at $q \in C^\infty(S^1, B_+)$ is the unique element in $C^\infty(S^1, B_-)$ defined by

$$d\tilde{F}_q(y) = \langle \nabla \tilde{F}(q), y \rangle$$

for all $y \in C^\infty(S^1, B_+)$. The gradient $\nabla F(u)$ at $u \in C^\infty(S^1, V_n)$ is the unique element in $C^\infty(S^1, V_n^*)$ satisfying

$$dF_u(v) = \langle \nabla F(u), v \rangle$$

for $v \in C^\infty(S^1, V_n)$. 

The two Poisson structures on $C^\infty(S^1,V_n)$ given in [4] are defined as follows:

\begin{align}
\{F_1,F_2\}_1(u) &= \langle [e_1n, \nabla F_1(u)], \nabla F_2(u) \rangle, \quad (4.1) \\
\{F_1,F_2\}_2(u) &= \langle [\partial x + b + u, \nabla F_1(u)], \nabla F_2(u) \rangle. \quad (4.2)
\end{align}

where $\hat{\nabla}$ is the $(S^1,N_\pm)$-invariant functionals defined by $F_i$ for $i = 1,2$.

The following theorem relates $\nabla \hat{F}(u)$ and $\nabla F(u)$ for $u \in C^\infty(S^1,V_n)$.

**Theorem 4.1.** Let $F$ be a functional on $C^\infty(S^1,V_n)$, and $\hat{F}$ the functional on $C^\infty(S^1,B_+)$ invariant under $C^\infty(S^1,N_+)$ defined by $F$. Then

$$\nabla \hat{F}(u) = P_u(\nabla F(u)),$$

where $u \in C^\infty(S^1,V_n)$ and $P_u$ is the operator defined Definition 3.10.

**Proof.** Note that the infinitesimal vector field $\tilde{\xi}$ defined by $\xi$ in $C^\infty(S^1,N_-)$ for the gauge action is

$$\tilde{\xi}(q) = -[\partial x + b + q, \xi],$$

where $q \in C^\infty(S^1,B_+)$. By assumption, $\hat{F}(f \ast q) = \hat{F}(q)$ for all $q \in C^\infty(S^1,B_+)$ and $f \in C^\infty(S^1,N_+)$. Take the variation in $f$ to get $d\hat{F}_q(\tilde{\xi}(q)) = 0$. But

$$d\hat{F}_q(\tilde{\xi}(q)) = \langle \nabla \hat{F}(q), \tilde{\xi}(q) \rangle = -\langle \nabla \hat{F}(q), [\partial_x + b + q, \xi] \rangle = \langle [\partial_x + b + q, \nabla \hat{F}(q)], \xi \rangle$$

for all $\xi \in C^\infty(S^1,N_+)$. So

$$[\partial x + b + u, \nabla \hat{F}(q)] \in C^\infty(S^1,B_-). \quad (4.3)$$

To prove $\nabla \hat{F}(u) = P_u(\nabla F(u))$ for $u \in C^\infty(S^1,V_n)$ is equivalent to prove

$$d\hat{F}_u(y) = \langle P_u(\nabla F(u)), y \rangle \quad (4.4)$$

for all $y \in C^\infty(S^1,B_+)$. We first prove (4.4) for $y \in C^\infty(S^1,V_n)$. Given $u,v \in C^\infty(S^1,V_n)$, we have

$$dF_u(v) = \langle \nabla F(u), v \rangle = d\hat{F}_u(v) = \langle \nabla \hat{F}(u), v \rangle.$$

So $\langle \nabla F(u) - \nabla \hat{F}(u), v \rangle = 0$ for all $v \in C^\infty(S^1,V_n)$. This implies that

$$\pi_0(\nabla \hat{F}(u)) = \nabla F(u),$$

where $\pi_0 : sl(n,\mathbb{R}) \rightarrow V^*_n$ is the canonical projection. By definition of $P_u$, we have $\pi_0(P_u(\nabla F(u))) = \nabla F(u)$. So we obtain $d\hat{F}_u(v) = dF_u(v) = \langle P_u(\nabla F(u)), v \rangle$, i.e., (4.4) is true for $y \in C^\infty(S^1,V_n)$.

Since $C^\infty(S^1,V_n)$ is a cross section of the gauge action of $C^\infty(S^1,N_+)$ on $C^\infty(S^1,B_+)$, the tangent space of $C^\infty(S^1,B_+)$ at $u \in C^\infty(S^1,V_n)$ can be written as a direct sum of $C^\infty(S^1,V_n)$ and the tangent space of the
Then we have $C^\infty(S^1, N_+)$-orbit at $u$. Since $\tilde{F}$ is invariant under $C^\infty(S^1, N_+)$, we have $d\tilde{F}_u(\xi(u)) = 0$ for all $\xi \in C^\infty(S^1, N_+)$. So 

\[
\langle P_u(\nabla F(u)), \xi(u) \rangle = \langle P_u(\nabla F(u)), -[\partial_x + b + u, \xi] \rangle
\]

\[
= \langle [\partial_x + b + u, P_u(\nabla F(u))], \xi \rangle.
\]

By definition of $P_u$, we have $[\partial_x + b + u, P_u(\nabla F(u))] \in C^\infty(S^1, V_n)$. Since $\xi \in C^\infty(S^1, N_+)$, we conclude that $\langle P_u(\nabla F(u)), \xi(u) \rangle = 0$. This proves $d\tilde{F}_u(\xi(u)) = \langle P_u(\nabla F(u)), \xi(u) \rangle = 0$. So \ref{F2} is true for $y$ in the tangent space of $C^\infty(S^1, N_+)$-orbit at $u$. This completes the proof. \hfill $\Box$

So we can write down these Poisson structures on $C^\infty(S^1, V_n)$ in terms of $\nabla F_1, \nabla F_2$ and $P_u$ as follows:

\[
\{F_1, F_2\}_1(u) = \langle [e_{1n}, P_u(\nabla F_1(u))], P_u(\nabla F_2(u)) \rangle,
\]

\[
\{F_1, F_2\}_2(u) = \langle [\partial_x + b + u, P_u(\nabla F_1(u))], P_u(\nabla F_2(u)) \rangle,
\]

where $P_u : C^\infty(S^1, V'_n) \rightarrow C^\infty(S^1, sl(n, \mathbb{R}))$ is defined in Definition \ref{C}. Let 

\[
(J_i)_u : C^\infty(S^1, V'_n) \rightarrow C^\infty(S^1, V_n)
\]

be the Poisson operator corresponding to $\{ , \}_i$ at $u$ for $i = 1, 2$, i.e., $(J_i)_u$ is defined by 

\[
\{F_1, F_2\}_i(u) = \langle (J_i)_u(\nabla F_1(u), \nabla F_2(u)).
\]

Then the Hamiltonian equation for a functional $H : C^\infty(S^1, V_n) \rightarrow \mathbb{R}$ with respect to $\{ , \}_i$ is 

\[
u_t = (J_i)_u(\nabla H(u)).
\]

First we compute the formula for the Poisson operator $J_1$.

**Proposition 4.2.** The Poisson operator $(J_1)_u : C^\infty(S^1, V'_n) \rightarrow C^\infty(S^1, V_n)$ is of the form $(J_1)_u(\xi) = \sum_{i=1}^{n-1} (L_i)_u(\xi)e_{in}$ with 

\[
(L_i)_u(\xi) = n(\xi_{n-i}) + k_i(\xi_1, \ldots, \xi_{n-i-1}),
\]

where $\xi = \sum_{i=1}^{n-1} \xi_{e_{in}}$ and $k_i$'s are linear differential operators with differential polynomials of $u$ as coefficients.

**Proof.** By Corollary \ref{C11}, entries of $P_u(v)$ are differential polynomials of $u$ and $v$. So we can use integration by part to compute $(J_1)_u$. We proceed as follows: Let $u = \sum_{i=1}^{n-1} u_i e_{in}$, and 

\[
\xi = \sum_{i=1}^{n-1} \xi_{e_{ni}} := \nabla F_1(u), \quad \eta = \sum_{i=1}^{n} \eta_i e_{ni} := \nabla F_2(u),
\]

\[
\begin{align*}
C &= (C_{ij}) = P_u(\xi), \\
D &= (D_{ij}) = P_u(\eta).
\end{align*}
\]

Then we have 

\[
\{F_1, F_2\}_1(u) = \langle [e_{1n}, C], D \rangle = \int \sum_{i=1}^{n} C_{ni} D_{i1} - C_{i1} D_{ni} dx.
\]
Let $g : \mathbb{R} \rightarrow SL(n, \mathbb{R})$ be a solution of $g^{-1}g_u = b + u$, $\gamma$ the first column of $g$. Then $\gamma \in M_n(\mathbb{R})$, $g$ is the central affine moving frame along $\gamma$, and $u = \Psi(\gamma)$. By Corollary 3.1, there is a $\delta g$ tangent to $M_n(\mathbb{R})$ at $\gamma$ such that $C = g^{-1}\delta g$, where $\delta g = (\delta \gamma, \ldots, (\delta \gamma)^{(n-1)})$. Since $\delta \gamma = \sum_{i=1}^{n} C_{1i} \gamma_{(i-1)}$ is tangent to $M_n(\mathbb{R})$, it follows from Proposition 3.1 that

$$C_{11} = \phi_0(C_{21}, \ldots, C_{n1}) = f(\xi_1, \ldots, \xi_{n-1}).$$

By definition, $C_{ni} = \xi_i$ and $D_{ni} = \eta_i$. By Corollary 3.6 there is a differential polynomial $f_n$ such that

$$C_{nn} = f_n(\xi_1, \ldots, \xi_{n-1}), \quad D_{nn} = f_n(\eta_1, \ldots, \eta_{n-1}).$$

We then use integration by part to write down the Poisson operator $(J_1)_u$. To get $(L_j)_u(\xi)$, we only need to calculate the terms involving $\eta_j = D_{nj}$ in $\sum_{i=1}^{n} C_{ni} D_{ni} - C_{ni} D_{i1}$. We use (3.3) to compute these terms as follows:

$$D_{nj} C_{j1} + C_{nn} (D_{nn} - D_{11}) - \sum_{i=1}^{n-1-j} C_{ni} D_{i1}
= \eta_j C_{j1} + \xi_1 \left( \sum_{i=1}^{n-1} D'_{i+1, i} \right) - C_{n,n+1-j} D_{n+1-j,1} - \sum_{i=1}^{n-j} C_{ni} D_{i1}
= \eta_j (\xi_{n+1-j} - (n-j)\xi_{n-j} + \varphi_{n-j}(u, \xi_1, \ldots, \xi_{n-j-1})) + \xi_1 \sum_{i=1}^{n-1} D'_{i+1, i}
- \xi_{n+1-j}(\eta_j - (j-1)\eta_{j-1} + \varphi_{j-1}(u, \eta_1, \ldots, \eta_{j-2})) - \sum_{i=1}^{n-j} \xi_i D_{i1}.$$

Notice that $\xi_1 \sum_{i=1}^{n-1} D'_{i+1, i}$ only depends on $\xi_1, \eta_1, \ldots, \eta_{n-1}$, and $\xi_i D_{i1}$ is a differential polynomial in $u$, $\xi_i$ and $\eta_1, \ldots, \eta_{n+1-i}$ for each $i$. Therefore, to consider the term $\xi_{n-j}$ in the the coefficient of $\eta_j$ in $\sum_{i=1}^{n-j} \xi_i D_{i1}$, we only need to calculate $\xi_{n-j} D_{n-j,1}$. Again, use $D_{n-j,1} = \eta_j - j\eta_j + \varphi_{j}(u, \eta_1, \ldots, \eta_{j-1})$ and integration by parts to see that the coefficients of $\eta_j$ is $-n\xi_{n-j} + \text{a differential operator depending on } u, \xi_1, \ldots, \xi_{n-j-1}$. □

**Corollary 4.3.** The dimension of the kernel of $(J_1)_u$ is $n - 1$.

**Proof.** If $(J_1)_u(\xi) = 0$ with $\xi = \sum_{i=1}^{n-1} \xi_i e_{ni}$, then Proposition 4.2 implies that $(L_{n-1})_u(\xi) = n(\xi_1)_x = 0$. So $\xi_1 = c_1$ a constant. Use $(L_{n-2})_u(\xi) = n(\xi_2)_x + k_{n-2}(\xi_1) = 0$ to see that $\xi_2 = -\frac{1}{n} k_{n-2}(c_1)x + c_2$ for some constant $c_2$. The corollary follows from induction. □

Next we compute the formula for the second Poisson operator $J_2$.

**Proposition 4.4.** The Poisson operator $(J_2)_u : C^\infty(S^1, V^*_u) \rightarrow C^\infty(S^1, V_n)$ is

$$(J_2)_u(v) = [\partial_x + b + u, P_u(v)]. \quad (4.5)$$
Proof. By definition of $P_\eta$, we have $[\partial_x + b + u, P_\eta(\nabla F_1(u))] \in C^\infty(S^1, V_n)$. So we have

$\langle [\partial_x + b + u, P_\eta(\nabla F_1(u))], P_\eta(\nabla F_2(u)) \rangle = \langle [\partial_x + b + u, P_\eta(\nabla F_1(u))], \nabla F_2(u) \rangle$.

Hence the second Poisson structure can be written as

$\{F_1, F_2\}_2(u) = \langle [\partial_x + b + u, P_\eta(\nabla F_1(u))], \nabla F_2(u) \rangle$.

(4.6)

which proves that $(J_2)_u(v) = [\partial_x + b + u, P_\eta(v)]$ is the second Poisson operator.

In the following examples, we compute explicit formulas for $J_1$ and $J_2$ when $n = 2, 3, 4$.

Example 4.5. For $n = 2$, write $u = qe_{21}$, $\nabla F_1(u) = \xi e_{21}$, and $\nabla F_2(u) = \eta e_{21}$, then

$P_\eta(\nabla F_1(u)) = \begin{pmatrix} -\frac{1}{2}\xi & -\frac{1}{2}\xi_x + q\xi \\ \xi & \frac{1}{2}\xi_x \end{pmatrix}$

and

$P_\eta(\nabla F_2(u)) = \begin{pmatrix} -\frac{1}{2}\eta_x & -\frac{1}{2}\eta_x + q\eta \\ \eta & \frac{1}{2}\eta_x \end{pmatrix}$.

So we get

$\{F_1, F_2\}_1(u) = 2\int \xi' \eta dx$,

$\{F_1, F_2\}_2(u) = -\int (\frac{1}{2}e^{(3)} \xi - 2q\xi_x - q_x\xi) \eta dx$,

and the corresponding Poisson operators are

$(J_1)_u(e_{21}) = 2\xi_x e_{12}$,

$(J_2)_u(e_{21}) = (-\frac{1}{2}e^{(3)} + 2q\xi_x + q_x\xi) e_{12}$.

This is the bi-Hamiltonian structure for the KdV hierarchy.

Example 4.6. For $n = 3$, write $\xi = \nabla F_1(u) = \xi_1 e_{31} + \xi_2 e_{32}$, $\eta = \nabla F_2(u) = \eta_1 e_{31} + \eta_2 e_{32}$, $C = P_\eta(\xi) = (C_{ij})$, and $D = P_\eta(\eta) = (D_{ij})$. Use (3.7) and integration by part to compute directly to get

$\{F_1, F_2\}_1(u) = \int \sum_{i=1}^3 C_{3i}D_{i1} - C_{i1}D_{3i} dx = 3 \int (\xi_1' \eta_2 + \xi_2' \eta_1) dx$.

Hence

$(J_1)_u(\xi_1 e_{31} + \xi_2 e_{32}) = 3(\xi_1' e_{13} + \xi_2' e_{23})$.

The formula of this Poisson operator was also obtained in [3].

The formula of the second Poisson operator is more complicated. Use (4.5) and a direct computation to see that

$(J_2)_u(\xi) = (C_{13} + u_1(C_{33} - C_{11}) - u_2 C_{12}) e_{13}$

$+ (C_{23} + C_{13} + u_2(C_{33} - C_{22}) - u_1 C_{21}) e_{23}$. 

This gives
\[(J_2)_u(\xi_1e_{31} + \xi_2e_{32}) = (A_1)_u(\xi)e_{13} + (A_2)_u(\xi)e_{23},\]
where
\[(A_1)_u(\xi) = \frac{2}{3}\xi_1^{(5)} - \xi_2^{(4)} - \frac{2}{3}(u_2\xi_1)^{(3)} - \frac{2}{3}u_2\xi_1^{(3)} + u_1''\xi_1 + 2u_1\xi_1'
+ 3u_1\xi_2' + u_2\xi_2'' + \frac{2}{3}u_2(u_2\xi_1)'\]
\[(A_2)_u(\xi) = \xi_1^{(4)} - 2\xi_2^{(3)} - (u_2\xi_1)'' + 2u_2\xi_2' + u_2\xi_2 + u_1\xi_1' + 2u_1\xi_1.\]

**Example 4.7.** For \(n = 4\), let \(\xi = \nabla F_1(u) = \xi_1e_{41} + \xi_2e_{42} + \xi_3e_{43}\), \(\eta = \nabla F_2(u) = \eta_1e_{41} + \eta_2e_{42} + \eta_3e_{43}\), \(P_u(\xi) = (C_{ij})\) and \(P_u(\eta) = (D_{ij})\). Then the first Poisson structure is
\[
\{F_1, F_2\}_1(u) = \int \sum_{j=1}^{4}(C_{4j}D_{j1} - C_{1j}D_{4j})dx
= 4\int (\xi_3' - \frac{1}{2}\xi_2'' + \frac{1}{2}\xi_1^{(3)} - \frac{1}{4}(2u_3\xi_1' + u_3\xi_1))\eta_1 + (\xi_2' + \frac{1}{4}\xi_1'')\eta_2 + \xi_1\eta_3 dx.
\]
Therefore,
\[\quad (J_1)_u(\xi) = \begin{pmatrix} 4\xi_3' - 2\xi_2'' + 2\xi_1'' - 2u_3\xi_1' - u_3\xi_1 \\ 4\xi_2' + 2\xi_1'' \\ 4\xi_1'' \end{pmatrix}.
\]
The formula for \((J_2)_u\) can be computed in a similar way as in the case \(n = 3\), but it is very long and complicated, so we do not include here.

Next we review the construction of conservation laws of the \(A^{(1)}_{n-1}\)-KdV hierarchy.

**Theorem 4.8.** ([1]) Given \(u \in C^\infty(\mathbb{R}, V_n)\), there exists a unique
\[T = \sum_{i=0}^{\infty} T_i \lambda^{-i}\]
such that \(T_0 \in C^\infty(\mathbb{R}, N_+)\), the first column of \(T\) is \((1, 0, \cdots, 0)^t\), and
\[T(\partial_x + J + u)T^{-1} = \partial_x + J + \sum_{i=0}^{\infty} f_i(u)J^{-i}, \quad f_i(u) \in C^\infty(S^1, \mathbb{R}). \quad (4.7)\]
Moreover, these \(f_i\)’s are differential polynomials of \(u\).

Equation (4.7) gives
\[TJT^{-1} + TuT^{-1} - TxT^{-1} = J + \sum_{i=0}^{\infty} f_i(u)J^{-i},\]
or equivalently,
\[TJ + Tu - Tx = JT + \sum_{i=0}^{\infty} f_i(u)J^{-i}T. \quad (4.8)\]
To compute $f_i(u)$’s we need to use the following proposition proved in [4].

**Proposition 4.9.** Let $T_n$ denote the subalgebra of all diagonal matrices in $gl(n, \mathbb{R})$. Then given $\xi = \sum_{i<k} \xi_i \lambda^i \in \mathcal{L}(sl(n, \mathbb{R}))$, there exist unique $h_i \in T_n$ such that $\xi = \sum_{i<k} h_i \lambda^i$.

Although $J$ and diagonal matrices do not commute, they satisfy the following relation: For $h = \text{diag}(h_1, \ldots, h_n) \in T_n$,

$$Jh = h^\sigma J, \quad \text{where } h^\sigma = \text{diag}(h_n, h_1, \ldots, h_{n-1}).$$

By Proposition 4.9, we can write $T = \sum_{i \geq 0} h_i J^{-i}$ with $h_i \in C^\infty(\mathbb{R}, T_n)$. Use (4.9), i.e., $Jh = h^\sigma J$, to write both sides of (4.8) as power series in $J$ with $C^\infty(\mathbb{R}, T_n)$ coefficients on the left. Then $f_i(u)$’s can be computed by equating the coefficients of $J^j$’s.

**Theorem 4.10.** ([4]) Let $f_i(u)$ be as in Theorem 4.8 and $H_i$ the functional on $C^\infty(S^1, V_n)$ defined by

$$H_i(u) = n \oint f_i(u) dx.$$ 

Then $\nabla H_j(u) = \pi_0(Y_{j,0}(u))$, where $\pi_0 : sl(n, \mathbb{R}) \to V_n^\dagger$ is the canonical projection and $Y_{j,0}(u)$ is defined by (2.3). Moreover, the $j$-th flow (3.1) in the $A_n^{(1)}$-KdV hierarchy is the Hamiltonian equation of $H_j$ with respect to $\{.,\}$, and is also the Hamiltonian equation of $H_{j+n}$ with respect to $\{.,\}_1$. In particular, the $j$-th flow can be written as

$$u_j = [\partial_x + b + u, P_u(\nabla H_j(u))] = (J_2)_u(\nabla H_j(u)) = (J_1)_u(\nabla H_{n+j}(u)).$$

**Example 4.11.** For $n = 3$, let $u = u_1 e_{13} + u_2 e_{23}$. Then we have

$$f_1(u) = \frac{1}{3} u_2, \quad f_2(u) = \frac{1}{3} (u_1 - u_2'),$$

$$f_3(u) = \frac{1}{9} (2u_2'' - 3u_1'), \quad f_4(u) = \frac{1}{9} (2u_2'' - u_2'' - u_1 u_2),$$

$$f_5(u) = \frac{1}{81} (3u_2''' - 9u_1''' - u_2''' - 9u_1^2 + 9u_2 u_1' + 18u_2' u_1 + 3u_2 u_1'').$$

So the conservation laws are

$$H_1(u) = \oint u_2 dx, \quad H_2(u) = \oint u_1 dx, \quad H_3(u) = 0,$$

$$H_4(u) = -\frac{1}{3} \oint u_1 u_2 dx,$$

$$H_5(u) = -\frac{1}{27} \oint (u_2^3 + 9u_1^2 - 9u_1 u_2 + 3u_2 u_1')^2 dx.$$ 

Note that $f_i(u)$ is a total derivative when $i \equiv 0 \pmod{3}$ and

$$\nabla H_1(u) = e_{32}, \quad \nabla H_2(u) = e_{31}, \quad \nabla H_4(u) = -\frac{1}{3} u_2 e_{31} - \frac{1}{3} u_1 e_{32},$$

$$\nabla H_5(u) = -\frac{1}{3} (u_2' - 2u_1) e_{31} + \frac{1}{9} (2u_2'' - 3u_1' - u_2') e_{32}. $$
Since \( P_u(\nabla H(u)) = g^{-1}\delta g \) with \( \pi_0(g^{-1}\delta g) = \nabla H(u) \), where \( \pi_0 \) is the canonical projection defined in Definition 3.10, use Example 3.8 to see that

\[
P_u(\nabla H_1(u)) = \begin{pmatrix} 0 & 0 & u_1 \\ 1 & 0 & u_2 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
Z_{2,0}(u) = P_u(\nabla H_2(u)) = \begin{pmatrix} -\frac{2}{3}u_2 & u_1 - \frac{2}{3}u'_2 & u'_1 - \frac{2}{3}u''_2 \\ 0 & \frac{1}{3}u_2 & u_1 - \frac{1}{3}u'_2 \\ 1 & 0 & \frac{1}{3}u'_2 \end{pmatrix},
\]

\[
Z_{4,0}(u) = P_u(\nabla H_4(u)) = \begin{pmatrix} -\frac{2u'_2 + 3u'_2 + 2u_2}{u_2} & * & * \\ \frac{u'_2 - u_1}{3} & \frac{u''_2 - u_2}{9} & 0 \\ -\frac{u_2}{3} & -\frac{u_1}{3} & \frac{u''_2 - 3u'_2 - u^2}{9} \end{pmatrix},
\]

\[
Z_{5,0}(u) = P_u(\nabla H_5(u)) = \begin{pmatrix} \frac{4u_1u_2 - u'_1}{9} & * & * \\ \frac{3u'_2 - u''_2 - u^2}{u'_2 - 2u_1} & \frac{2u''_2 - u''_2 + u_2u'_2 - 2u_1u_2}{9} & 0 \\ \frac{3u''_2 - 3u'_2 - u^2}{9} & \frac{u''_2 - u_1 - u_2u'_2 - 2u_1u_2}{9} & * \end{pmatrix}.
\]

**Example 4.12.** Let \( n = 4 \) and \( u = u_1e_{14} + u_2e_{24} + u_3e_{34} \). We use Theorem 4.8 to compute the first several conservation laws and obtain

\[
f_1(u) = \frac{1}{4}u_3, \quad f_2(u) = \frac{1}{4}(u_2 - \frac{3}{2}(u - 3)x), \quad f_3(u) = \frac{1}{4}(u_1 - \frac{3}{2}(u_2)x + \frac{5}{4}(u_3)^{(3)}x + \frac{1}{8}u_3^2),
\]

\[
H_1(u) = \oint u_3 dx, \quad H_2(u) = \oint u_2 dx, \quad H_3(u) = \oint u_1 + \frac{1}{8}u_3^2 dx.
\]

For general \( n \), we have

\[
H_1(u) = \oint u_{n-1} dx, \quad H_2(u) = \oint u_{n-2} dx, \quad H_3(u) = \oint u_{n-3} + \frac{n-3}{2n}u^2_{n-1} dx.
\]

For \( 4 \leq i \leq n-1 \), we have \( H_i(u) = n \oint f_i(u) dx \), where

\[
f_i(u) = \frac{1}{n}u_{n-i} + h_i(u_{n-1}, u_{n-2}, \cdots, u_{n-i+1})
\]

for some differential polynomial \( h_i \).

The following Theorem shows that the Hamiltonian vector field of \( H_j \) is zero w.r.t. \( \{\cdot, \cdot\}_1 \) for \( 1 \leq j \leq n-1 \).

**Theorem 4.13.** Let \( H_j \) be the functional in Theorem 4.10 for the \( A^{(1)}_{n-1} \)-KdV hierarchy. Then \( (J_1)_u(\nabla H_j(u)) = 0, 1 \leq j \leq n-1 \).
By Corollary 3.13, there is a \( \delta \) such that

\[
\left[ \partial_x + J + u, Y(u, \lambda)^j \right] = 0 \quad \text{for} \quad 1 \leq j \leq n - 1.
\]

Since \( \left[ \partial_x + b + u, (b^j)^{n-j} \right] = [Y_{j,0}(u), e_{1n}] \). Recall that \( Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u) \) for some unique \( \zeta_j(u) \in C^\infty(S^1, \mathcal{N}_+) \). But \( [\eta_j(u), e_{1n}] = 0 \). So we get

\[
\left[ Z_{j,0}(u), e_{1n} \right] = [Y_{j,0}(u), e_{1n}] = [\partial_x + b + u, (b^j)^{n-j}].
\]

By Corollary 3.13, \( Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u))) \). Corollary 3.11 implies that there is a \( \delta_j \gamma \) such that \( Z_{j,0}(u) = g^{-1}\delta_1 g \). Let \( \xi_1 = \pi_0(Z_{j,0}(u)) \), \( \xi_2 \in C^\infty(S^1, V_n^j) \). Then we have

\[
\langle [Z_{j,0}(u), e_{1n}], P_u(\xi_2) \rangle = \langle [\partial_x + b + u, (b^j)^{n-j}], P_u(\xi_2) \rangle
\]

which is zero because \( [\partial_x + b + u, P_u(\xi_2)] \in V_n \) by definition of \( P_u \) and \( (b^j)^{n-j} \) is in \( \mathcal{N}_+ \). This proves that \( \langle (J_1)_u(\pi_0(Z_{j,0}(u))), \xi_2 \rangle = 0 \) for all \( \xi_2 \in V_n^j \). Hence \( \pi_0(Z_{j,0}(u)) \) lies in the kernel of \( (J_1)_u \). By Theorem 4.10, \( \nabla H_j(u) = \pi_0(Z_{j,0}(u)) \) for \( 1 \leq j \leq n - 1 \). So we get \( (J_1)_u(\nabla H_j(u)) = 0, 1 \leq j \leq n - 1 \). \( \square \)

As a consequence of Corollary 4.3 and Theorem 4.13, we obtain the following.

**Corollary 4.14.** The kernel \( (J_1)_u \) is spanned by \( \nabla H_1(u), \ldots, \nabla H_{n-1}(u) \), where \( H_i \)’s are the Hamiltonians given in Theorem 4.10.

Next we derive some properties of the central affine curvature map \( \Psi \) and use them to compute the kernel of \( (J_2)_u \).

**Proposition 4.15.** Let \( \Psi : \mathcal{M}_n(\mathbb{R}) \to C^\infty(\mathbb{R}, V_n) \) be the affine curvature map. Then:

1. \( \Psi(\gamma_1) = \Psi(\gamma_2) \) if and only if there exists a constant \( c_0 \in SL(n, \mathbb{R}) \) such that \( \gamma_2 = c_0 \gamma_1 \).
2. \( \Psi : \mathcal{M}_n(\mathbb{R}) \to C^\infty(\mathbb{R}, V_n) \) is onto.

**Proof.** Statement (1) follows from the uniqueness of ordinary differential equations, and statement (2) follows from the existence of ordinary differential equations. \( \square \)

**Proposition 4.16.** Let \( \Psi : \mathcal{M}_n(\mathbb{R}) \to C^\infty(\mathbb{R}, V_n) \) be the affine curvature map and \( u = \Psi(\gamma) = \sum_{i=1}^{n-1} u_ie_{im} \). Then

\[
d\Psi_\gamma(\delta \gamma) = [\partial_x + b + u, g^{-1}\delta g] = (J_2)_u(\pi_0(g^{-1}\delta g)),
\]
where \( g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) \) is the central affine frame for \( \gamma \) and \( \delta g = (\delta \gamma, (\delta \gamma)_x, \ldots, (\delta \gamma)_x^{(n-1)}) \).

Proof. Take variation of \( g^{-1}g_x = b + u \) to get
\[
\delta u = -(g^{-1}\delta g)(b + u) + g^{-1}(\delta g)_x.
\]
Set \( \eta = g^{-1}\delta g \) and compute directly \( \eta_x \) to get \( \eta_x = -[b + u, \eta] + \delta u \). But \( \Psi(\gamma) = g^{-1}g_x \), where \( g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) \). Hence \( d\Psi(\delta \gamma) = \delta u = [\partial_x + b + u, g^{-1}\delta g] \).

Corollary 4.17. Let \( \Psi : \mathcal{M}_n(S^1) \to C^\infty(S^1, V_n) \) be the affine curvature map, \( \Psi(\gamma) = u \), and \( g \) the central affine moving frame along \( \gamma \in \mathcal{M}_n(S^1) \). Then
\begin{enumerate}
  \item \( \text{Ker}(d\Psi(\gamma)) = \{c_0 \gamma | c_0 \in \text{sl}(n, \mathbb{C})\} \),
  \item \( \Psi^{-1}(\Psi(\gamma)) \) is the \( SL(n, \mathbb{R}) \)-orbit at \( \gamma \).
\end{enumerate}

Corollary 4.18. Let \( u \in C^\infty(S^1, V_n) \), and \( g : S^1 \to GL(n, \mathbb{R}) \) such that \( g^{-1}g_x = b + u \). Let \( v \in C^\infty(S^1, V_n^1) \). If \((J_2)_u(v) = 0\), then there is a constant \( c_0 \in \text{sl}(n, \mathbb{R}) \) such that \( v = \pi_0(g^{-1}c_0g) \), where \( \pi_0 : \text{sl}(n, \mathbb{R}) \to V^1_n \) is the canonical projection.

Proof. Let \( \gamma \) denote the first column of \( g \). The equation \( g_x = g(b + u) \) implies that \( g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) \). By Corollary 3.11 there exist \( \delta \gamma \in T\mathcal{M}_n(\mathbb{R})_\gamma \) such that \( P_u(v) = g^{-1}\delta g \), where \( \delta g = (\delta \gamma, \ldots, (\delta \gamma)_x^{(n-1)}) \). So
\[
(J_2)_u(v) = [\partial_x + b + u, g^{-1}\delta g] = d\Psi(\delta \gamma) = 0.
\]

By Corollary 4.17 there exists content \( c_0 \in \text{sl}(n, \mathbb{R}) \) such that \( \delta \gamma = c_0 \gamma \). Therefore
\[
\delta g = (c_0 \gamma, \ldots, (c_0 \gamma)_x^{(n-1)}) = c_0(\gamma, \ldots, \gamma_x^{(n-1)}) = c_0g.
\]
This proves that \( g^{-1}\delta g = g^{-1}c_0g \). Since \( v = \pi_0(P_u(v)) \), \( v = \pi_0(g^{-1}c_0g) \).

It is known that \{, \}_1 and \{, \}_2 are compatible (cf. [4]), i.e. \( c_1\{, \}_1 + c_2\{, \}_2 \) is a Poisson structure on \( C^\infty(S^1, V_n) \) for all real constants \( c_1, c_2 \). It is standard in the literature (cf. [6], [11]) that we can use these two compatible Poisson structures to generate a sequence of Poisson structures:
\[
\{F_1, F_2\}_j(u) = ((J_j)_u(\nabla F_1(u)), \nabla F_2(u)),
\]
where
\[
J_j = J_2(J_1^{-1}J_2)^{j-2}.
\]

Corollary 4.19. Let \( 1 \leq j \leq n - 1 \), and \( k \geq 0 \). Then the \( (nk + j) \)-th flow in the \( A_{n-1}^{(1)} \text{-KdV hierarchy} \) is
\[
u_{nk+j} = (J_{k+2})_u(\nabla H_{j}(u)) = J_2(J_1^{-1}J_2)^{k}(\nabla H_{j}(u)),
\]
where \( H_j \) is the functional on \( C^\infty(S^1, V_n) \) defined in Theorem 4.10.
Proposition 5.1. Let $H$ be a functional on $C^\infty(S^1, V_n)$, $\delta \gamma$ the Hamiltonian vector field for $\hat{H} = H \circ \Psi$ with respect to $\{ \cdot, \}_{j}^\wedge$. Then
\[ \partial_x + b + u, g^{-1}\delta g = (J_j)_u(\nabla H(u)), \]
where $g$ is the central affine moving frame along $\gamma$, $u = \Psi(\gamma)$, and $\delta g = (\delta_\gamma, \ldots, (\delta_\gamma)_{x}^{(n-1)})$.

Proof. Since $\{ \cdot, \}_{j}^\wedge$ is the pull back of $\{ \cdot, \}_{j}$, we have
\[ d\Psi(\delta_\gamma) = (J_j)_u(\nabla H(u)). \]
By Proposition 4.16, $d\Psi(\delta_\gamma) = [\partial_x + b + u, g^{-1}\delta g]$. □

The following three corollaries follow from Theorem 4.10, $\Psi^*\{ \cdot, \}_{j} = \{ \cdot, \}_{j}^\wedge$, and Corollary 4.19.
Corollary 5.2. Let $H_j$ be functionals on $C^\infty(S^1, V_n)$ defined in Theorem 4.10 and $\hat{H}_j = H_j \circ \Psi$. Then the $j$-th central affine curve flow (3.1) is the Hamiltonian equation for $\hat{H}_j$ and $\hat{H}_{n+j}$ with respect to $\{,\}_2^\wedge$ and $\{,\}_1^\wedge$ respectively.

Corollary 5.3. Let $H_j$ and $\hat{H}_j$ be as in Corollary 5.2. Then the central affine curve flow (1.5) is the Hamiltonian equation for $\hat{H}_2(\gamma) = \oint u_{n-2} dx$ and $\hat{H}_{n+2}$ with respect to $\{,\}_2^\wedge$ and $\{,\}_1^\wedge$ respectively, where $u = \Psi(\gamma) = \sum_{i=1}^{n-1} u_i e_i$.

Corollary 5.4. Let $1 \leq j \leq n - 1$ and $k \geq 0$. Then the $(nk+j)$-th central affine curve flow on $M_n(S^1)$ is

$$\gamma_{nk+j} = g Z_{nk+j,0}(u)p_0 = g(P_u((J_{1}^{-1} J_{2})^k_u(\nabla H_j(u))))p_0,$$

where $p_0 = (1,0,\ldots,0)^t$, $g$ the central affine moving frame along $\gamma$, and $u = \Psi(\gamma)$.

Proposition 5.5. Let $F_1, F_2$ be functionals on $C^\infty(S^1, V_n)$, and $\delta_i \gamma$ the Hamiltonian vector field for $\tilde{F}_1 = F_1 \circ \Psi$ for $i = 1, 2$. Then

$$\{ \tilde{F}_1, \tilde{F}_2 \}_j^\wedge(\gamma) = -g^{-1}\delta_1 g,(J_{2}^{-1} J_{2})^k_u(\pi_0(\delta_2 g)), \quad \text{where } g \text{ is the central affine moving frame along } \gamma, \ u = \Psi(\gamma), \ \text{and } \delta_i g = (\delta_{i} \gamma, (\delta_i \gamma)_x, \ldots, (\delta_i \gamma)^{(n-1)}).

Proof. By Proposition 4.16, $d\Psi(\delta \gamma) = (J_{2})_u(\pi_0(\delta g))$. Since $d\Psi(\delta \gamma) = (J_{2})_u(\nabla F_2(u))$ for $i = 1, 2$, we get $\nabla F_2(u) = (J_{2}^{-1} J_{2})_u(\pi_0(\delta_2 g))$. So we have $(J_{2})_u(\nabla F_2(u)) = J_{2}(\pi_0(\delta_2 g))$. Compute directly to get

$$\{ \tilde{F}_1, \tilde{F}_2 \}_j^\wedge(\gamma) = \{ F_1, F_2 \}_j(u) = \langle (J_{2})_u(\nabla F_1(u)), \nabla F_2(u) \rangle = \langle J_{2}(\pi_0(\delta_2 g), (J_{2}^{-1} J_{2})_u(\pi_0(\delta_2 g) \rangle$$

$$= -\langle \pi_0(\delta_2 g), (J_{2}^{-1} J_{2})_u(\pi_0(\delta_2 g) \rangle$$

Let $(M, w)$ be a symplectic manifold, and $X_f$ the Hamiltonian vector field of $f$. Then $w(X_{f_1}, X_{f_2}) = \{ f_1, f_2 \}$. Motivated by this formula and Proposition 5.3 we define

$$\langle \tilde{w}_j \rangle_{\gamma}(\delta_1 \gamma, \delta_2 \gamma) = -g^{-1}\delta_1 g,(J_{2}^{-1} J_{2})_u(\pi_0(\delta_2 g)), \quad (5.1)$$
Theorem 5.7. So the orbit symplectic structure is obi-linear form the co-Adjoint orbit of infinitesimal vector field is
\[ \langle \xi, \eta \rangle = \langle [\partial_x + b + u, g^{-1}\delta_1 g], g^{-1}\delta_2 g \rangle, \]

\[ \langle \tilde{\omega}_2 \rangle \gamma(\delta_1 \gamma, \delta_2 \gamma) = -\langle g^{-1}\delta_1 g, (J_2)_u(\pi_0(g^{-1}\delta_2 g) \rangle = -\langle g^{-1}\delta_1 g, [\partial_x + b + u, g^{-1}\delta_2 g] \rangle = \langle [\partial_x + b + u, g^{-1}\delta_1 g], g^{-1}\delta_2 g \rangle, \]

\[ \langle \tilde{\omega}_3 \rangle \gamma(\delta_1 \gamma, \delta_2 \gamma) = -\langle g^{-1}\delta_1 g, (J_1)_u(\pi_0(g^{-1}\delta_2 g) \rangle = -\langle \pi_0(g^{-1}\delta_1 g), (J_1)_u(\pi_0(g^{-1}\delta_2 g) \rangle = \langle (J_1)_u(\pi_0(g^{-1}\delta_1 g), \pi_0(g^{-1}\delta_2 g) \rangle = \langle [e_{1n}, g^{-1}\delta_1 g], g^{-1}\delta_2 g \rangle. \]

So we obtain

**Proposition 5.6.** Let $\tilde{\omega}_j$ be the 2-form defined by (5.1). Then

\[ (\tilde{\omega}_2) \gamma(\delta_1 \gamma, \delta_2 \gamma) = \langle [\partial_x + b + u, g^{-1}\delta_1 g], g^{-1}\delta_2 g \rangle, \]

\[ (\tilde{\omega}_3) \gamma(\delta_1 \gamma, \delta_2 \gamma) = \langle [e_{1n}, g^{-1}\delta_1 g], g^{-1}\delta_2 g \rangle. \]

In the following we want to prove that $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are the pull back of certain co-Adjoint orbit symplectic forms. Let $M$ be the co-Adjoint orbit of $G$ on the dual $G^*$ of the Lie algebra $G$ at $\ell_0 \in G^*$. The orbit symplectic form on $M$ is defined by

\[ (\tau_\ell(x), \tilde{\eta}(x)) = \tau(\xi, \eta), \]

where $\ell \in M$, and $\tilde{\xi}$ and $\tilde{\eta}$ are infinitesimal vector fields corresponding to the co-Adjoint action generated by $\xi, \eta \in G$.

We identify the Adjoint orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ on $C^\infty(\mathbb{R}, sl(n, \mathbb{R}))$ as the co-Adjoint orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ via the following non-degenerate bi-linear form

\[ \langle \xi, \eta \rangle = \int_0^1 \text{tr}(\xi(x)\eta(x))dx. \]

**Theorem 5.7.** Let $\mathcal{O}_1$ denote the Adjoint orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ on $C^\infty(S^1, sl(n, \mathbb{R}))$ at the constant loop $e_{1n}$, $\tau_1$ the orbit symplectic form on $\mathcal{O}_1$, and $\mathfrak{f}_1$ the map from $M_n(S^1)$ to $\mathcal{O}_1$ defined by $\mathfrak{f}_1(\gamma) = ge_{1n}g^{-1}$, where $g$ is the central affine moving frame along $\gamma$. Then $\mathfrak{f}_1^* \tau_1 = \tilde{\omega}_3$.

**Proof.** Given $\xi \in C^\infty(S^1, sl(n, \mathbb{R}))$, a direct computation implies that the infinitesimal vector field is

\[ \tilde{\xi}(ge_{1n}g^{-1}) = [\xi, ge_{1n}g^{-1}]. \]

So the orbit symplectic structure is

\[ (\tau_1)_{ge_{1n}g^{-1}}([\xi, ge_{1n}g^{-1}], [\eta, ge_{1n}g^{-1}]) = \langle ge_{1n}g^{-1}, [\xi, \eta] \rangle. \]

The differential of $\mathfrak{f}_1$ at $\gamma$ is

\[ d(\mathfrak{f}_1)_\gamma(\delta \gamma) = [\delta gg^{-1}, ge_{1n}g^{-1}]. \]
induce weak symplectic forms on the orbit spaces $M$ still have the Hamiltonian theory (cf. [1]). Below we show that at dimension, a weak symplectic form need not be symplectic, but we can dimension, then a weak symplectic form is symplectic. When which is equal to ($\hat{w}_3$, $\delta_1 \gamma$, $\delta_2 \gamma$).

Let $\mathbb{R}\partial_x + C^\infty(S^1, sl(n, \mathbb{R}))$ denote the Lie algebra with bracket defined by

$$[r_1\partial_x + u, r_2\partial_x + v] = r_1 v_x - r_2 u_x + [u, v], \quad r_1, r_2 \in \mathbb{R}.$$ 

It is known (cf. [8], [9]) that the dual of the central extension of the loop algebra $C^\infty(S^1, sl(n, \mathbb{R}))$ defined by the 2-cocycle

$$\rho(\xi, \eta) = \oint \text{tr}(\xi(x)\eta(x))dx$$

can be identified as the Lie algebra $\mathbb{R}\partial_x + C^\infty(S^1, sl(n, \mathbb{R}))$. The co-Adjoint action corresponds to the gauge action,

$$g \cdot (\partial_x + u) = g(\partial_x + u)g^{-1} = \partial_x + gug^{-1} - g_xg^{-1}.$$

**Theorem 5.8.** Let $O_2$ denote the gauge orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ at $\partial_x$, $\tau_2$ the orbit symplectic form on $O_2$, and $\xi_2 : \mathcal{M}_n(S^1) \to O_2$ the map defined by $\xi_2(\gamma) = g^{-1}g_x$, where $g$ is the central affine moving frame along $\gamma$. Then $\xi_2(\tau_2) = \hat{w}_2$.

**Proof.** The infinitesimal vector field on $O_2$ given by the gauge action for $\xi \in C^\infty(S^1, sl(n, \mathbb{R}))$ is

$$\dot{\xi}(\partial_x + v) = -[\partial_x + v, \xi].$$

Note that

$$\xi_2(\gamma) = \partial_x + g^{-1}g_x = \partial_x + b + u = \partial_x + b + \Psi(\gamma).$$

By Proposition 4.16

$$d(\xi_2, \gamma) = d\Psi_\gamma(\delta \gamma) = [\partial_x + b + u, g^{-1}\delta g].$$

Then

$$\xi_2(\gamma) = (\tau_2)_{\partial_x + b + u}(d\xi_2(\delta \gamma), d\xi_2(\delta \gamma))$$

$$= (\tau_2)_{\partial_x + b + u}(\partial_x + b + u, g^{-1}\delta_1 g), [\partial_x + b + u, g^{-1}\delta_2 g])$$

$$= ([\partial_x + b + u, g^{-1}\delta_1 g], g^{-1}\delta_2 g),$$

which is equal to ($\hat{w}_2$, $\delta_1 \gamma$, $\delta_2 \gamma$). \qed

Recall that a weak symplectic form on $M$ is a closed 2-form on $M$ such that $w_x(v_1, v_2) = 0$ for all $v_2 \in T_{\partial_x}$. It implies that $v_1 = 0$. If $M$ is of finite dimension, then a weak symplectic form is symplectic. When $M$ is of infinite dimension, a weak symplectic form need not to be symplectic, but we can still have the Hamiltonian theory (cf. [4]). Below we show that $\hat{w}_2$ and $\hat{w}_3$ induce weak symplectic forms on the orbit spaces $\mathcal{M}_n(S^1)/SL(n, \mathbb{R})$ and $\mathcal{M}_n(S^1)/(SL(n, \mathbb{R}) \times \mathbb{R}^{n-1})$ respectively.
Theorem 5.9. The 2-form \( \tilde{w}_2 \) induces a weak symplectic form on the orbit space \( \mathcal{M}_n(S^1)/\text{SL}(n, \mathbb{R}) \).

Proof. By Theorem 5.8, \( \tilde{w}_2 \) is a closed 2-form. It follows from (5.2) that 
\[
(\tilde{w}_2)(\gamma_1, \gamma_2) = 0 \quad \text{for all} \quad \gamma_1, \gamma_2.
\]
This implies that \( \tilde{w}_2 \) is a weak symplectic form. This finishes the proof.\( \square \)

We consider the \( \mathbb{R}^{n-1} \)-action on \( \mathcal{M}_n(S^1) \) generated by the first \( (n-1) \) central affine curve flows. Since the central affine curve flows commute with the \( \text{SL}(n, \mathbb{R}) \)-action, the product group \( \text{SL}(n, \mathbb{R}) \times \mathbb{R}^{n-1} \) acts on \( \mathcal{M}_n(S^1) \).

Theorem 5.10. The 2-form \( \tilde{w}_3 \) induces a weak symplectic form on the orbit space \( \mathcal{M}_n(S^1)/(\text{SL}(n, \mathbb{R}) \times \mathbb{R}^{n-1}) \).

Proof. By Theorem 5.8, \( \tilde{w}_3 \) is a closed 2-form. The formula of \( \tilde{w}_3 \) implies that 
\[
(\tilde{w}_3)(\gamma_1, \gamma_2) = 0 \quad \text{for all} \quad \gamma_1, \gamma_2.
\]
This implies that \( \tilde{w}_3 \) is a weak symplectic form. This finishes the proof.\( \square \)

To end this section, we write down \( \tilde{w}_2(X,Y) \) and \( \tilde{w}_3(X,Y) \) in terms of determinants involving \( X, Y \in T\mathcal{M}_n(S^1) \) and derivatives of \( X \) and \( Y \):

Theorem 5.11. Let \( X, Y \) be tangent vectors of \( \mathcal{M}_n(S^1) \) at \( \gamma \). Then

\[
(\tilde{w}_2)_\gamma(X,Y) = -\sum_{i}^{n-1} \int \det(\gamma, \ldots, \gamma^{(i-2)}, X^{(i)}, \gamma^{(i)}), Y^{(i)}) \, dx
\]

\[
+ \sum_{i,j} \int u_j \det(\gamma, \ldots, \gamma^{(i-1)}, X^{(j-1)}, \gamma^{(i)}, Y^{(i-1)}) \, dx,
\]

\[
(\tilde{w}_3)_\gamma(X,Y) = -\sum_{i}^{n-1} \int \det(\gamma, \ldots, \gamma^{(i-2)}, X, \gamma^{(i)}), Y^{(i-1)}) \, dx
\]

\[
- \sum_{i}^{n-1} \int \det(\gamma, \ldots, \gamma^{(i-2)}, X^{(i-1)}, \gamma^{(i)}, Y) \, dx.
\]

Proof. Let \( \gamma, g, u, \delta_1, \delta_2 \) be as in Proposition 5.3.

\[
C = (C_{ij}) = g^{-1}\delta_1 g, \quad D = (D_{ij}) = g^{-1}\delta_2 g,
\]
and $C_i, D_t$ the $i$-th column of $C$ and $D$ respectively. By Corollary 3.5, we can express $C_i$’s as differential polynomials in $C_1$. Similarly, $D_i$’s can be expressed as differential polynomials in $D_1$. Moreover, $C_i = \langle C_{1i}, \ldots, C_{ni} \rangle$ is the coordinate of $\langle \delta_1 \gamma \rangle_x^{(i-1)}$ with respect to the frame $g = \langle \gamma, \gamma_0^{(n-1)} \rangle$, i.e., $\langle \delta_1 \gamma \rangle_x^{(i-1)} = \sum_{k=1}^n C_{ki} \gamma_x^{(k-1)}$.

Recall that if $Y = \sum_{i=1}^n y_i \gamma_x^{(i-1)}$, then

$$Y' = \sum_{i=1}^n (y'_i + y_{i-1} + u_i y_n) \gamma_x^{(i-1)}.$$ 

Write

$$\langle \delta_1 \gamma \rangle_x^{(n)} = \sum_{i=1}^n \xi_i \langle \delta_1 \gamma \rangle_x^{(i-1)},$$

Then $\xi = (\xi_1, \ldots, \xi_n)^t = C_n' + (b + u)C_n$. By Proposition 5.6, we have

$$(\hat{w}_3)_\gamma (\delta_1 \gamma, \delta_2 \gamma) = \oint \sum_{i=1}^n C_{ni} D_{i1} - C_{i1} D_{ni} \, dx,$$

$$(\hat{w}_2)_\gamma (\delta_1 \gamma, \delta_2 \gamma) = \oint \sum_{i=1}^n (C(b + u))_{in} D_{ni} - (C_x + (b + u)C)_{in} D_{ni} \, dx$$

$$= \oint \sum_{i=1}^n \sum_{j=1}^{n-1} C_{ij} u_j D_{ni} - \sum_{i=1}^n \xi_i D_{ni} \, dx.$$ 

This gives an algorithm to compute $(\hat{w}_2)_\gamma$ and $(\hat{w}_3)_\gamma$.

We compute $(\hat{w}_3)_\gamma$ as follows: Let $X = \delta_1 \gamma$, $Y = \delta_2 \gamma$, then

$$(\hat{w}_3)_\gamma(X, Y) = \oint \sum_{i=1}^n C_{ni} D_{i1} - C_{i1} D_{ni} \, dx$$

$= \oint C_{nn} D_{n1} - C_{n1} D_{nn} + \sum_{i=1}^{n-1} C_{ni} D_{i1} - C_{i1} D_{ni} \, dx$$

$= \oint C_{n1} \sum_{i=1}^{n-1} D_{ii} - (\sum_{i=1}^{n-1} C_{ii}) D_{n1} + \sum_{i=1}^{n-1} C_{ni} D_{i1} - C_{i1} D_{ni} \, dx.$

Note that $\det(\gamma, \ldots, \gamma_x^{(n-1)}) = 1$ and the $k$-th column of $C$ and $D$ are the coefficients of $X_x^{(k-1)}$ and $Y_x^{(k-1)}$ written as a linear combination of $\gamma, \ldots, \gamma_x^{(n-1)}$. So we have

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(i-2)}, X_x^{(k-1)}, \gamma_x^{(i)}, \ldots, Y_x^{(i-1)}) = C_{ik} D_{n\ell} - C_{nk} D_{i\ell}. \quad (5.3)$$

Substitute (5.3) into the above formula for $w(\gamma, X, Y)$ to get the formula for $(\hat{w}_3)_\gamma$ as stated in the theorem.
For \((\hat{w}_2)_\gamma\), from

\[
(\hat{w}_2)_\gamma(X, Y) = \oint \sum_{i=1}^{n} \sum_{j=1}^{n-1} C_{ij} u_j D_{ni} - \sum_{i=1}^{n} \xi_i D_{ni} dx,
\]

and \(\operatorname{tr}(C) = \operatorname{tr}(D) = 0\) and \((5.3)\), we get the formula for \((\hat{w}_2)_\gamma\) as stated in the theorem. □

**Example 5.12.** For \(n = 2\), Theorem 5.11 gives

\[
(\hat{w}_2)_\gamma(X, Y) = -\oint \det(X', Y') + u_1 \det(X, Y) dx,
\]

\[
(\hat{w}_3)_\gamma(X, Y) = -2 \oint \det(X, Y) dx.
\]

These are the 2 forms given in [5] and [7] respectively.

Next we compute \(\hat{w}_1\) and see that it is given by an integral-differential operator. Write \(X = -\frac{1}{2} \xi' \gamma + \xi \gamma_x\), \(Y = -\frac{1}{2} \eta' \gamma + \eta \gamma_x\), then

\[
(\hat{w}_1)_\gamma(X, Y) = \oint \xi K_{u_1}(\eta) dx,
\]

where

\[
K_{u_1}(\eta) = -\frac{1}{8} \eta_x^{(5)} + u_1 \eta_x^{(3)} + \frac{3}{2} (u_1)_x \eta_x + (u_1)_{xx} \eta_x + \frac{1}{4} (u_1)_x \eta_x + 2 u_1 \eta_x - 2 u_1 (u_1)_x \eta + \frac{1}{2} \oint (u_1)_x \eta dx
\]

**Example 5.13.** For \(n = 3\), we get

\[
(\hat{w}_3)_\gamma(X, Y) = -3 \oint \det(X, \gamma', Y) dx,
\]

which is the 2 form given in [3]. We also have

\[
(\hat{w}_2)_\gamma(X, Y) = -\oint \det(X'', \gamma', Y') + \det(\gamma, X'', Y') dx
\]

\[
+ \oint u_1 (\det(X, \gamma, Y) + \det(\gamma, X, Y')) dx
\]

\[
+ \oint u_2 (\det(X', \gamma', Y) + \det(\gamma, X', Y')) dx.
\]
Example 5.14. For \( n = 4 \), let \( |v_1, v_2, v_3, v_4| = \det(v_1, v_2, v_3, v_4) \). Then we have

\[
(\hat{w}_3)(X, Y) = -2 \int \det(X, \gamma', \gamma'', Y) + \det(\gamma, \gamma', X', Y') dx,
\]

\[
(\hat{w}_2)(X, Y) = - \int \left| X^{(4)}, \gamma', \gamma'', Y \right| + \left| \gamma, X^{(4)}, \gamma'', Y' \right| + \left| \gamma, \gamma', X^{(4)}, Y'' \right| dx
\]

\[+ \int u_3 \left( \left| X'', \gamma', \gamma'' \right| + \left| \gamma, X'', \gamma'' \right| + \left| \gamma, \gamma', X'', Y'' \right| \right) dx\]

\[+ \int u_2 \left( \left| X', \gamma', \gamma'' \right| + \left| \gamma, X', \gamma'' \right| + \left| \gamma, \gamma', X' \right| \right) dx\]

\[+ \int u_1 \left( \left| X, \gamma', \gamma'' \right| + \left| \gamma, X, \gamma'' \right| + \left| \gamma, \gamma', X \right| \right) dx.\]
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