On the entire self-shrinking solutions to Lagrangian mean curvature flow II

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Abstract
We prove the rigidity of entire graphic Lagrangian self-shrinkers in \((\mathbb{R}^{2n}, g_\tau)\), where \(g_\tau = \sin \tau \delta_0 + \cos \tau g_0\) is a linear combination of the Euclidean metric \(\delta_0\) and the pseudo metric \(g_0 = 2 \sum_i dx_i dy_i\) with \(\tau \in (0, \frac{\pi}{2})\), complementing the previous results for \(\tau = 0\) and \(\tau = \frac{\pi}{2}\); actually we obtain Bernstein theorems for three corresponding nonlinear elliptic equations between the Monge–Ampère equation (\(\tau = 0\)) and the special Lagrangian equation (\(\tau = \frac{\pi}{2}\)). Moreover, we find Bernstein theorem fails when \(\tau \in (-\frac{\pi}{4}, 0)\) and entire graphic spacelike self-shrinker in Minkowski spaces share this non-rigidity property.

Mathematics Subject Classification 53E10 · 53A10

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1 Introduction

Let \( g_\tau = \sin \tau \delta_0 + \cos \tau g_0 \) be the linear combination of the standard Euclidean metric

\[
\delta_0 = \sum_{i=1}^{n} dx_i \otimes dx_i + \sum_{j=1}^{n} dy_j \otimes dy_j
\]

and the pseudo-Euclidean metric

\[
g_0 = \sum_{i=1}^{n} dx_i \otimes dy_i + \sum_{j=1}^{n} dy_j \otimes dx_j
\]
on \( \mathbb{R}^n \times \mathbb{R}^n \) with \( \tau \in [0, \frac{\pi}{2}] \). This family of metrics was first studied in [26], where Warren established a calibration theory for spacelike Lagrangian submanifolds in \((\mathbb{R}^{2n}, g_\tau)\) (for general calibration theory for Lagrangian submanifolds in Riemannian and pseudo-Riemannian manifolds, see Harvey–Lawson [13] and Mealy [20]) and reformulated some celebrated Bernstein type results of Jörgens [17], Calabi [2], Pogorelov [22], Jost–Xin [18], Flanders [12], and Yuan [28] for the extremal Lagrangian submanifolds, whose potentials satisfy

\[
F_\tau (\lambda(D^2 u)) = c \quad \text{on} \quad \mathbb{R}^n,
\]

where

\[
F_\tau (\lambda) = \begin{cases} 
\frac{1}{2} \sum_{i=1}^{n} \ln \lambda_i, & \tau = 0, \\
\sqrt{a^2 + 1} \sum_{i=1}^{n} \ln \frac{\lambda_i + a - b}{\lambda_i + a + b}, & 0 < \tau < \frac{\pi}{4}, \\
-\sqrt{2} \sum_{i=1}^{n} \ln \frac{1}{1 + \lambda_i}, & \tau = \frac{\pi}{4}, \\
\sqrt{a^2 + 1} \sum_{i=1}^{n} \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
\sum_{i=1}^{n} \arctan \lambda_i, & \tau = \frac{\pi}{2},
\end{cases}
\]

\( a = \cot \tau, \quad b = \sqrt{|\cot^2 \tau - 1|}, \quad \lambda(D^2 u) = (\lambda_1, \ldots, \lambda_n) \) are the eigenvalues of \( D^2 u \), and \( c \) is a constant.

It is also interesting to study the parabolic analogue of (1.1), namely

\[
v_t = F_\tau (\lambda(D^2 v))
\]

and its underlying geometric objects. For an admissible solution \( v \) to (1.2) (see Definition 1.1 below), let \( M_t = \{(x, Dv(x, t)) \mid x \in \mathbb{R}^n\} \) be the gradient graph of \( v(\cdot, t) \) and \( X_t \) be the embedding map of \( M_t \). Since \( v \) is admissible, \( M_t \) is spacelike in \((\mathbb{R}^{2n}, g_\tau)\) (see (2.1) below). It is not hard to see that \( M_t \) is Lagrangian with respect to the usual symplectic structure of \( \mathbb{R}^{2n} \). By Proposition 2.1, there exists a family of diffeomorphisms

\[
\psi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]
such that the family of embeddings

\[
\tilde{X}_t := X_t \circ \psi_t = (\psi_t, Dv(\psi_t, t))
\]
is a solution to the Lagrangian mean curvature flow in \((\mathbb{R}^{2n}, g_\tau)\):
\[
\frac{d\tilde{X}_t}{dt} = H,
\]
where \(H\) is the mean curvature vector of \(M_t\) at \(\tilde{X}_t\) in \((\mathbb{R}^{2n}, g_\tau)\).

An important class of solutions to the mean curvature flow are self-shrinking solutions, whose profiles, self-shrinkers, satisfy
\[
H = -\frac{1}{2} X_{\perp},
\]
where \(\perp\) denotes the orthogonal projection into the normal bundle (see [16]). One goal of this paper is to give fine classifications of Lagrangian self-shrinkers with entire potentials in \((\mathbb{R}^{2n}, g_\tau)\).

For such a Lagrangian self-shrinker \(M = \{(x, Du(x)) \mid x \in \mathbb{R}^n\}\), by Proposition 2.4, up to an additive constant, (1.4) is equivalent to the following equation
\[
F_\tau(\lambda(D^2 u)) = \frac{1}{2} \langle x, Du \rangle - u,
\]
where \(\langle \cdot, \cdot \rangle\) denotes the standard inner product on \(\mathbb{R}^n\). Some Bernstein type results for (1.5) have been obtained for \(\tau = \frac{\pi}{2}\) and \(\tau = 0\) as follows.

For \(\tau = \frac{\pi}{2}\), (1.5) is the special Lagrangian type equation
\[
\sum_{i=1}^{n} \arctan \lambda_i (D^2 u) = \frac{1}{2} \langle x, Du \rangle - u.
\]
This equation was first investigated by Chau–Chen–He [3, 4]. By using the existence [4] and uniqueness [6] results for the Lagrangian mean curvature flow, Chau–Chen–He proved that any solution to (1.6) must be a quadratic polynomial if the Hessian of \(u\) is uniformly bounded between \(-1\) and \(1\). The same rigidity for solutions with arbitrarily bounded Hessian was derived from a Liouville type property for ancient solutions to parabolic equations in [21]. The first author and Wang [15] proved the rigidity of convex solutions. Later, Chau–Chen–Yuan [5] dropped the assumption on the Hessian completely. They observed that a natural geometric quantity, the phase \(\phi = \sum_{i=1}^{n} \arctan \lambda_i (D^2 u)\) obeys a second order elliptic equation with an “amplifying force”. Then they constructed a specific barrier function to show the phase is constant via the maximum principle. Finally the homogeneity of the self-similar term on the right-hand side of the equation leads to the quadratic conclusion. This quantity \((\phi = F_\tau(\lambda(D^2 u))\) in general) plays a very important role there and in later studies. Combining the drift Laplacian operator introduced by Colding–Minicozzi [10] and the phase equation in [5], Ding–Xin [11] gave an integral proof of the same result.

For \(\tau = 0\), (1.5) is the Monge–Ampère type equation
\[
\frac{1}{2} \ln \det D^2 u = \frac{1}{2} \langle x, Du \rangle - u.
\]
The first author [14] showed that any admissible solution to (1.7) leads to a self-shrinking solution to the Lagrangian mean curvature flow in the pseudo-Euclidean space \((\mathbb{R}^{2n}, g_0)\).

Several authors proved that entire solutions to (1.7) are quadratic under inversely quadratic decay conditions on \(D^2 u\) by different methods. Huang–Wang [15] used the flow method by carrying out Calabi’s third order derivatives estimate for the parabolic Monge–Ampère equation. Chau–Chen–Yuan [5] constructed a specific barrier function to show the phase is
constant. In [11], combining the drift Laplacian operator and the phase equation in [5], Ding-Xin gave a complete improvement by dropping additional decay assumptions in an integral way. In [25], the third author reproved Ding-Xin’s optimal result via a pointwise approach, which also works for a large class of equations including Hessian quotient type.

The first goal of this paper is to settle the Bernstein problem for the remaining cases of (1.5), i.e., \( \tau = \frac{\pi}{4}, \) \( 0 < \tau < \frac{\pi}{2} \) and \( \frac{\pi}{4} < \tau < \frac{\pi}{2} \), respectively. Our results are the followings:

**Theorem 1.1** Let \( u \) be an entire smooth solution on \( \mathbb{R}^n \) to

\[
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{1 + \lambda_i} = \frac{1}{2} \langle x, Du \rangle - u
\]

that satisfies

\[
D^2 u > -I_n \quad \text{or} \quad D^2 u < -I_n.
\]

Then \( u \) is a quadratic polynomial.

**Theorem 1.2** Let \( u \) be an entire smooth solution on \( \mathbb{R}^n \) to

\[
\frac{\sqrt{a^2 + 1}}{2b} \sum_{i=1}^{n} \ln \frac{\lambda_i + a - b}{\lambda_i + a + b} = \frac{1}{2} \langle x, Du \rangle - u
\]

that satisfies

\[
D^2 u > -(a - b)I_n \quad \text{or} \quad D^2 u < -(a + b)I_n,
\]

where \( a = \cot \tau, \) \( b = \sqrt{\cot^2 \tau - 1}, \) \( 0 < \tau < \frac{\pi}{4}. \) Then \( u \) is a quadratic polynomial.

**Theorem 1.3** Let \( u \) be an entire smooth solution on \( \mathbb{R}^n \) to

\[
\frac{\sqrt{a^2 + 1}}{b} \sum_{i=1}^{n} \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b} = \frac{1}{2} \langle x, Du \rangle - u,
\]

where \( a = \cot \tau, \) \( b = \sqrt{1 - \cot^2 \tau}, \) \( \frac{\pi}{4} < \tau < \frac{\pi}{2}. \) Then \( u \) is a quadratic polynomial.

Our requirements for the Hessian of \( u \) in Theorems 1.1 and 1.2 are just the following admissibility conditions (cf. [1]):

**Definition 1.1** Let \( \Gamma \subset \mathbb{R}^n \) be the open set that has the following properties:

(i) For any \( \lambda \in \Gamma, \)

\[
\frac{\partial F_\tau(\lambda)}{\partial \lambda_i} > 0 \quad i = 1, \ldots, n.
\]

(ii) \( \Gamma \) is symmetric, namely it is invariant under interchange of any two \( \lambda_i. \)

We say a smooth solution \( v \) to (1.2) is admissible if \( \lambda(D^2 v(x, t)) \in \Gamma \) for all \( (x, t) \) in the domain of \( v, \) and a smooth solution \( u \) to (1.5) is admissible if \( \lambda(D^2 u(x)) \in \Gamma \) for all \( x \) in the domain of \( u. \)

Equations (1.2) and (1.5) are parabolic and elliptic respectively for admissible solutions. Furthermore, our rigidity results also hold for solutions on domains of \( \mathbb{R}^n \) that blow up on the boundaries. These can be seen as analogues to the results of Trudinger–Wang (Theorem 2.1 in [23]), Ding–Xin (Theorem 2.3 in [11]) and Li–Xu–Yuan (Theorem 1.1 in [19]).
Theorem 1.4 Let $u$ be an admissible solution to (1.8), (1.9) or (1.10) on a domain $\Omega \subset \mathbb{R}^n$. Assume that $|Du| = \infty$ or $|u| = \infty$ on $\partial \Omega$. Then $\Omega = \mathbb{R}^n$ and $u$ is a quadratic polynomial.

Till now, the angle $\tau$ considered for $g_\tau$ lies in $[0, \frac{\pi}{2}]$. One may ask the rigidity issue for $\tau$ in a broader range. It is obvious that $g_\tau = g_{\tau+2\pi}$ and $g_{\pi-\tau}|_u = g_{\tau-\tau}|_u$, where $g_{\pi-\tau}|_u$ denotes the induced metric on the gradient graph of $u$ from $g_{\pi-\tau}$. When $\tau \in [-\frac{\pi}{2}, -\frac{\pi}{4}]$, $g_\tau$ is negative semi-definite, so $g_\tau|_u$ cannot be a Riemannian metric. Thus the remaining case is $\tau \in (-\frac{\pi}{4}, 0)$. In this case, $g_\tau|_u$ is a Riemannian metric if and only if

$$-(b + a)I_n < D^2 u < (b - a)I_n,$$

where $a = \cot \tau$, $b = \sqrt{\cot^2 \tau - 1}$, and the self-shrinker equation is

$$\frac{\sqrt{a^2 + 1}}{2b} \sum_{i=1}^n \ln \frac{b + a + \lambda_i}{b - a - \lambda_i} = \frac{1}{2} \langle x, Du \rangle - u. \quad (1.11)$$

However, the rigidity phenomenon becomes very different in this situation.

Theorem 1.5 Equation (1.11) admits non-quadratic incomplete entire admissible solutions in each dimension.

The elliptic coefficients of the solution constructed in the proof of Theorem 1.5 grows fast (exponentially) in one direction. Under certain non-singularity conditions (in integral forms) on the elliptic coefficients, we can still draw the quadratic conclusion. For instance, if the induced metric (see (2.1) below) is complete, by Chen-Qiu’s result (see Theorem 2 in [8]), $u$ is quadratic. Here we give another non-singularity condition in terms of the image of gradient map.

Theorem 1.6 Let $u$ be an admissible solution to (1.11) on a domain $\Omega \subset \mathbb{R}^n$. Assume either the map $(b + a)x + Du$ or $(b - a)x - Du$ from $\Omega$ to $\mathbb{R}^n$ is surjective. Then $\Omega = \mathbb{R}^n$ and $u$ is a quadratic polynomial.

Remark 1.7 After taking the trivial transform

$$w(x) = \frac{b}{\sqrt{a^2 + 1}} u \left( \frac{(a^2 + 1)^{\frac{1}{2}}}{b} x \right) + \frac{a}{2b} |x|^2,$$

we may get a more concise equivalent form of (1.11), namely

$$\frac{1}{2} \sum_{i=1}^n \ln \frac{1 + \mu_i}{1 - \mu_i} = \frac{1}{2} \langle x, Dw \rangle - w, \quad (1.12)$$

where $\mu = (\mu_1, \ldots, \mu_n)$ are the eigenvalues of $D^2 w$. After the Lewy-Yu rotation, i.e., the $\pi/4$-rotation (see [28] and (7.9) below), (1.12) becomes the Monge–Ampère type equation (1.7); but the image of the map $x - Dw$, namely the domain for the new potential, needs not be the whole $\mathbb{R}^n$. As a matter of fact, for the counterexample in Theorem 1.5, the domain of the new potential is not the whole $\mathbb{R}^n$. We will also directly prove the constructed solution is incomplete.

There is a similar non-rigidity phenomenon for the equation

$$\text{div} \left( \frac{Df}{\sqrt{1 - |Df|^2}} \right) = \frac{\langle x, Df \rangle - f}{2\sqrt{1 - |Df|^2}}, \quad |Df| < 1. \quad (1.13)$$
If $f$ is an entire solution to (1.13) on $\mathbb{R}^n$, then $M_f = \{(x, f(x)) \mid x \in \mathbb{R}^n\}$ is an entire self-shrinking solution to the mean curvature flow in the Minkowski space $\mathbb{R}_1^{n+1}$. If the induced metric is complete, by Chen–Qiu’s result [8], $f$ must be linear, corresponding to a hyperplane. But without completeness assumption, there are counterexamples.

**Theorem 1.8** Equation (1.13) admits nonlinear incomplete entire smooth solutions in each dimension.

This non-rigidity phenomenon contrasts sharply with the Bernstein properties of maximal hypersurfaces in the Minkowski spaces [9] and graphical self-shrinking hypersurfaces in the Euclidean spaces [19, 24].

The main arguments for the rigidity theorems rely on a priori estimates and barrier constructions for the phase. Some calculations and arguments are inspired by [5]. The construction of the counterexample is based on ODE analysis. Combining the results in previous studies and here, we have a complete answer to the rigidity problem for entire solutions to (1.5). There seems to be an interesting phenomenon: as $\tau$ increases in $(-\frac{\pi}{4}, \frac{\pi}{2}]$, the rigidity of Eq. (1.5) gains; on the contrary, as $\tau$ increases, the rigidity of Eq. (1.1) decreases.

The organization of this paper is as follows. In the next section, we deduce equation (1.5) and show its equivalency to the self-shrinker equation (1.4) up to an additive constant. In Sect. 3, we prove Theorem 1.1 by the argument of Legendre transform and an integral estimate with a suitable test function; as a byproduct of our argument, we give a Liouville type result for a $p$-Laplace type equation. In Sects. 4 and 5, we prove Theorem 1.2 and Theorem 1.3 via a pointwise approach and a transform respectively. The proof of Theorem 1.4 is given in Sect. 6. In Sect. 7, Theorems 1.5 and 1.8 are proved by constructing non-trivial solutions to second order ODEs; Theorem 1.6 is proved by making use of the Lewy–Yuan rotation.

### 2 Lagrangian self-shrinkers in $(\mathbb{R}^{2n}, g_\tau)$

Throughout the following, Einstein’s convention of summation over repeated indices is adopted. We denote, for a smooth function $u$,

$$ u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \quad \ldots \quad (2.1) $$

**Proposition 2.1** If (1.2) admits an admissible solution $v(x, t)$ on $\mathbb{R}^n \times \mathcal{I}$, where $\mathcal{I}$ is a time interval, then there exists a family of diffeomorphisms

$$ \psi_t : \mathbb{R}^n \to \mathbb{R}^n $$

for $t \in \mathcal{I}$, such that

$$ \tilde{X}_t = (\psi_t, Dv(\psi_t, t)) $$

is a solution to the Lagrangian mean curvature flow in $(\mathbb{R}^{2n}, g_\tau)$.

**Proof** Let $e_i = (0, \ldots, 1, \ldots, 0)$ be the $i$-th coordinate vector in $\mathbb{R}^{2n}$, $i = 1, \ldots, 2n$. Denote the inner product of $(\mathbb{R}^{2n}, g_\tau)$ by $(\cdot, \cdot)_\tau$. The tangent vector fields of $M_t = \{(x, Dv(x, t)) \mid x \in \mathbb{R}^n\}$ are spanned by

$$ E_i = e_i + v_{ij}e_{n+j}, \quad i = 1, \ldots, n. $$

The induced metric $g$ on $M_t$ is
\[ g_{ij} = \langle E_i, E_j \rangle_\tau \]
\[ = \langle e_i + v_{ik} e_{n+k}, e_j + v_{jl} e_{n+l} \rangle_\tau \]
\[ = \sin \tau (\delta_{ij} + v_{ik} v_{kj}) + 2 \cos \tau v_{ij}. \]  
(2.1)

Denote \((g_{ij})^{-1}\) by \((g^{ij})\). It is not hard to see that
\[ g_{ij} = \partial F_\tau (\lambda(D^2 v)) \]
\[ \frac{\partial \tau}{\partial v_{ij}}. \]

Let \(\nabla\) denote the Levi–Civita connection of \(g_\tau\). We have \(\nabla_{E_j} E_i = v_{ijk} e_{n+k}\). The mean curvature of \(M_t\) is computed as
\[ H = g^{ij} \left( \nabla_{E_i} E_j \right)_\perp = \left( g^{ij} v_{ijk} e_{n+k} \right)_\perp = \left( \partial_k F_\tau (\lambda(D^2 v)) e_{n+k} \right)_\perp. \]  
(2.2)

By the evolution equation (1.2) of \(v\),
\[ H = (v_{ik} e_{n+k})_\perp = (0, Dv_t)_\perp. \]  
(2.3)

Take a family of diffeomorphisms \(\psi_t : \mathbb{R}^n \times I \to \mathbb{R}^n\) that satisfies
\[ (I_n, D^2 v(\psi_t, t)) \cdot \frac{d\psi_t}{dt} = -(0, Dv_t(\psi_t, t))^\top, \]
where \(\cdot\) denotes matrix product and \(^\top\) denotes the projection to the tangent bundle of \(M_t\). Set \(\tilde{X}_t = X_t \circ \psi_t\). It follows that
\[ \frac{d\tilde{X}_t}{dt} = \left( \frac{d\psi_t}{dt}, Dv_t(\psi_t, t) + D^2 v(\psi_t, t) \cdot \frac{d\psi_t}{dt} \right) \]
\[ = (0, Dv_t(\psi_t, t))_\perp \]
\[ = H(\tilde{X}_t). \]

This completes the proof. \(\square\)

**Definition 2.1** We say \(v(x, t)\) is a self-shrinking solution to (1.2) if \(v(x, t)\) is a solution that satisfies
\[ v(x, t) = -t v \left( \frac{x}{\sqrt{-t}}, -1 \right) \text{ for } t \in (-\infty, 0). \]  
(2.4)

**Proposition 2.2** If \(v(x, t)\) is a self-shrinking solution to (1.2), then \(v(x, -1)\) satisfies (1.5).

**Proof** By (1.2) and (2.4), one can easily check that
\[ F_\tau (\lambda(D^2 v(x, t))) = \frac{\partial v}{\partial t}(x, t) \]
\[ = -v \left( \frac{x}{\sqrt{-t}}, -1 \right) + \frac{1}{2} \left( Dv \left( \frac{x}{\sqrt{-t}}, -1 \right), \frac{x}{\sqrt{-t}} \right). \]

Taking \(t = -1\), we see that \(v(x, -1)\) satisfies (1.5). \(\square\)
Proposition 2.3 Let \( u(x) \) be a smooth solution to (1.5). Define
\[
v(x, t) = -tu \left( \frac{x}{\sqrt{-t}} \right), \quad t \in (-\infty, 0).
\]
Then \( v(x, t) \) is a self-shrinking solution to (1.2).

**Proof** By a simple computation and (1.5), one checks directly that
\[
\frac{\partial v}{\partial t}(x, t) = -u \left( \frac{x}{\sqrt{-t}} \right) + \frac{1}{2} \left( Du \left( \frac{x}{\sqrt{-t}} \right), \frac{x}{\sqrt{-t}} \right),
\]
\[
= F_\tau \left( \lambda \left( D^2 u \left( \frac{x}{\sqrt{-t}} \right) \right) \right)
\]
\[
= F_\tau \left( \lambda \left( D^2 v(x, t) \right) \right).
\]
Thus we obtain the desired result. \( \square \)

Proposition 2.4 Let \( u \) be a smooth function on \( \mathbb{R}^n \). Then \( M = \{(x, Du(x)) | x \in \mathbb{R}^n \} \) is a Lagrangian self-shrinker in \( (\mathbb{R}^{2n}, g_z) \) if and only if \( u \) is an admissible solution to (1.5) up to an additive constant.

**Proof** As (2.2), the mean curvature of \( M \) is
\[
H = (0, DF_\tau(\lambda(D^2 u)))^\perp.
\]
The normal part of the position vector is
\[
X^\perp = (x, Du)^\perp.
\]
It follows that
\[
H + \frac{X^\perp}{2} = \frac{1}{2} (x, D(2F_\tau(\lambda(D^2 u)) + u))^\perp.
\]
Therefore, \( M \) is a Lagrangian self-shrinker if and only if \( (x, D(2F_\tau(\lambda(D^2 u)) + u)) \) is tangential, namely spanned by \( E_i \) (1 \( \leq i \leq n \)) for all \( x \in \mathbb{R}^n \). Comparing the coefficients, we find that this is equivalent to
\[
-x_i u_{ij} = 2\partial_j F_\tau(\lambda(D^2 u)) + u_j \quad \text{for} \quad j = 1, \ldots, n.
\]
Namely
\[
\partial_j F_\tau(\lambda(D^2 u)) = \partial_j \left( \frac{1}{2} \langle x, Du \rangle - u \right) \quad \text{for} \quad j = 1, \ldots, n. \quad (2.5)
\]
Since \( \mathbb{R}^n \) is connected, (2.5) is equivalent to (1.5) up to an additive constant. \( \square \)

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1, and the two key ingredients of the proof are the Legendre transform and an integral estimate. The two cases \( D^2 u > -I_n \) and \( D^2 u < -I_n \) are related by the transform
\[
\tilde{u} = -|x|^2 - u, \quad (3.1)
\]
which preserves (1.8), so we only need to consider the first one.
Let \( w = \frac{1}{2} |x|^2 + u \). Straightforward calculations yield \( \lambda(D^2 w) = \lambda(D^2 u) + 1 \) and
\[
\frac{1}{2} \langle x, Du \rangle - u = \frac{1}{2} \langle x, Dw \rangle - w.
\]
Thus \( w \) is strictly convex and satisfies
\[
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{\mu_i} = \frac{1}{2} \langle x, Dw \rangle - w,
\tag{3.2}
\]
where \( \mu = (\mu_1, \ldots, \mu_n) \) are the eigenvalues of \( D^2 w \). Take the Legendre transform
\[
y = Dw, \quad w^*(y) = \langle x, Dw \rangle - w.
\]
Then \( D w^*(y) = x, D^2 w^*(y) = (D^2 w(x))^{-1} \). It follows that
\[
\frac{1}{2} \langle y, D w^* \rangle - w^* = -\frac{1}{2} \langle x, Dw \rangle + w,
\]
and \( w^* \) is strictly convex. By (3.2), \( w^* \) satisfies the following equation
\[
\sqrt{2} \Delta w^* = \frac{1}{2} \langle y, D w^* \rangle - w^*.
\tag{3.3}
\]

Next we show that

**Claim 3.1** \( Dw(\mathbb{R}^n) = \mathbb{R}^n \), namely the domain of \( w^* \) is the entire \( \mathbb{R}^n \).

**Proof of Claim 3.1** This proof is partially inspired by the proof of Theorem 2.6 in [11]. Since \( w \) is strictly convex, for any fixed \( \theta \in S^{n-1} \), where \( S^{n-1} \) denotes the unit sphere in \( \mathbb{R}^n \), \( \langle \theta, Dw(r\theta) \rangle \) monotonically increases as a function of \( r \) in \( [0, +\infty) \). Suppose that there exists some \( \theta_0 \in S^{n-1} \) and \( \beta_0 > 0 \) such that
\[
\lim_{r \to +\infty} \langle \theta_0, Dw(r\theta_0) \rangle \leq \beta_0 < \infty.
\]
Due to the convexity of \( w \), we have
\[
\frac{1}{2} r \langle \theta_0, Dw(r\theta_0) \rangle - w(r\theta_0) \geq \frac{1}{2} r \langle \theta_0, Dw(r\theta_0) \rangle - [r \langle \theta_0, Dw(r\theta_0) \rangle + w(0)]
\]
\[
= -\frac{1}{2} r \langle \theta_0, Dw(r\theta_0) \rangle - w(0)
\]
\[
\geq -\frac{\beta_0}{2} r - w(0).
\]
Using (3.2), we get \( w(0) > 0 \) and
\[
\sqrt{2} \sum_{i=1}^{n} \frac{1}{\mu_i(r\theta_0)} \leq \frac{\beta_0}{2} r + w(0).
\]
It follows that
\[
\mu_i(r\theta_0) > \frac{1}{\frac{\beta_0}{2} r + 2w(0)}, \quad i = 1, 2, \ldots, n.
\]
Then we have
\[
\langle \theta_0, D w(r \theta_0) - D w(0) \rangle = \int_0^r \langle \theta_0, D^2 w(t \theta_0) \cdot \theta_0 \rangle \, dt \\
\geq \int_0^r \frac{dt}{\beta_0 t + 2w(0)} = \frac{1}{\beta_0} \ln \left(1 + \frac{\beta_0}{2w(0)} r\right).
\] (3.4)

As \( r \to +\infty \), the left-hand side of (3.4) is finite by the assumption, while the right-hand side of (3.4) blows up. Thus we get a contradiction. Hence, for any \( \theta \in S_n^{n-1} \),

\[
\lim_{r \to +\infty} \langle \theta, D w(r \theta) \rangle = +\infty.
\] (3.5)

Due to the convexity of \( w \) and continuity of \( D w \), (3.5) implies that \( w \) has a superlinear growth. It follows that for any \( y \in \mathbb{R}^n \), \( \langle y, x \rangle - w(x) \) attains its global maximum of \( \mathbb{R}^n \) at some finite point \( x_0 \). Since \( y \) is arbitrary, we conclude that \( D w(\mathbb{R}^n) = \mathbb{R}^n \). \qed

Up to this moment, to prove Theorem 1.1, we need only to show that any strictly convex entire solution to (3.3) is a quadratic polynomial.

Set

\[
\phi := \frac{1}{2} \langle y, D w^* \rangle - w^* = \frac{1}{2} y_l w^*_{lj} - w^*.
\] (3.6)

Recall that the repeated indices are summed. Taking derivatives of (3.6) twice, we obtain

\[
\phi_{ij} = \frac{1}{2} y_l w^*_{lj}.
\] (3.7)

A differentiation of (3.3) with respect to \( x_l \) yields

\[
\sqrt{2} \Delta w^*_l = \phi_l.
\] (3.8)

Combining (3.7) and (3.8), we get

\[
\Delta \phi = \frac{\sqrt{2}}{4} y_l \phi_l.
\] (3.9)

Next we show that any positive entire solutions to (3.9) must be constant. Then by (3.7), the strictly convex entire solution \( w^* \) to (3.3) must be a quadratic polynomial, and Theorem 1.1 is proved. In fact, we prove the following more general result:

**Proposition 3.1** Let \( p > 1 \), \( K > 0 \) be constants and \( h \geq 0 \) satisfies

\[
\text{div} (|Dh|^{p-2} Dh) = K x_i h_i |Dh|^{p-2} \quad \text{on} \quad \mathbb{R}^n.
\] (3.10)

Then \( h \) must be constant.

**Proof** Let \( \eta \) be a smooth cut-off function satisfying:

\[
\begin{cases}
\eta \equiv 1 & \text{in } B_R, \\
0 \leq \eta \leq 1 & \text{in } B_{2R}, \\
\eta \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{2R}, \\
|D\eta| \leq \frac{1}{R} & \text{in } \mathbb{R}^n.
\end{cases}
\]

where and in the sequel, \( B_R \) denotes a ball in \( \mathbb{R}^n \) centered at the origin with radius \( R \); and we use “\( \lesssim \)” to drop out some positive constants independent of \( R \) and \( h \).
By the divergence theorem, with \( \rho = - \frac{K}{2} |x|^2 - h \), we compute
\[
\int_{\mathbb{R}^n} D_i (|Dh|^{p-2} h_i) \ e^\rho \eta^p \ dx = - \int_{\mathbb{R}^n} |Dh|^{p-2} h_i D_i (e^\rho \eta^p) \ dx
\]
\[
= - \int_{\mathbb{R}^n} |Dh|^{p-2} h_i (\rho_i \eta^p + p \eta_i \eta^{p-1}) \ e^\rho \ dx
\]
\[
= \int_{\mathbb{R}^n} K x_i h_i |Dh|^{p-2} \ e^\rho \eta^p \ dx + \int_{\mathbb{R}^n} |Dh|^{p-2} \ e^\rho \eta^p \ dx
\]
\[= - p \int_{\mathbb{R}^n} |Dh|^{p-2} h_i \eta_i \eta^{p-1} \ e^\rho \ dx. \quad (3.11)
\]

Using Eqs. (3.10) in (3.11) we get
\[
\int_{\mathbb{R}^n} |Dh|^p \ e^\rho \eta^p \ dx = p \int_{\mathbb{R}^n} |Dh|^{p-2} h_i \eta_i \eta^{p-1} \ e^\rho \ dx
\]
\[\lesssim \frac{1}{R} \int_{\mathbb{R}^n} |Dh|^{p-1} \eta^{p-1} \ e^\rho \ dx
\]
\[\lesssim \varepsilon \int_{\mathbb{R}^n} |Dh|^p \ e^\rho \eta^p \ dx + \frac{1}{R^p} \int_{B_{2R}} \ e^\rho \ dx, \quad (3.12)
\]
where in the last step, we have used the Young’s inequality with exponent pair \((p, \frac{p}{p-1}, p)\). For \( h \geq 0 \) we have
\[
\int_{\mathbb{R}^n} \ e^\rho \ dx < +\infty.
\]

Taking \( \varepsilon \) small and letting \( R \to +\infty \) in (3.12), one must get \( |Dh| \equiv 0 \) in \( \mathbb{R}^n \). The proof of Proposition 3.1 is completed. \( \square \)

4 Proof of Theorem 1.2

We prove Theorem 1.2 by constructing a barrier function. The case \( D^2 u > -(a - b) I_n \) and the case \( D^2 u < -(b + a) I_n \) are related by the transform
\[
\tilde{u} = -a |x|^2 - u,
\]
which also preserves (1.9), so we only need to consider the first one.

**Proof** Take \( w = u + \frac{a-b}{2} |x|^2 \). Then \( w \) is strictly convex and satisfies
\[
\frac{\sqrt{a^2 + 1}}{2b} \sum_{i=1}^n \ln \left( \frac{\mu_i}{\mu_i + 2b} \right) = \frac{1}{2} \langle x, Dw \rangle - w, \quad (4.1)
\]
where \( \mu = (\mu_1, \ldots, \mu_n) \) are the eigenvalues of \( D^2 w \). Define \( F (D^2 w) \) by
\[
F (D^2 w) = \frac{\sqrt{a^2 + 1}}{2b} \sum_{i=1}^n \ln \left( \frac{\mu_i}{\mu_i + 2b} \right),
\]
and the coefficients \( a_{ij} (D^2 w) \) by
\[
a_{ij} (D^2 w) = \frac{\partial F (D^2 w)}{\partial w_{ij}}.
\]
At any point, we can rotate the coordinates to diagonalize \( D^2 w \) and \((a^{ij})\) simultaneously. In the new coordinates, \( D^2 w \) and \((a^{ij})\) are \( \text{diag}\{\mu_1, \ldots, \mu_n\} \) and \( \text{diag}\left\{\frac{\sqrt{a^2+1}}{\mu_1(\mu_1+2b)}, \ldots, \frac{\sqrt{a^2+1}}{\mu_n(\mu_n+2b)}\right\} \) respectively. As \( w \) is convex, \((a^{ij})\) is positive-definite. We have
\[
a^{ij} w_{ij} = \sum_{k=1}^{n} \frac{\sqrt{a^2+1}}{\mu_k + 2b} < \frac{n \sqrt{a^2+1}}{2b}.
\]

Define the phase
\[
\phi = \frac{1}{2} \langle x, Dw \rangle - w. \tag{4.2}
\]

By (4.1), \( \phi < 0 \). Taking derivatives of (4.2) twice, we obtain
\[
\phi_{ij} = \frac{1}{2} x_s w_{ijs}. \tag{4.3}
\]

A differentiation of (4.1) with respect to \( x_s \) yields
\[
a^{ij} w_{ijs} = \phi_s. \tag{4.4}
\]

Combining (4.3) and (4.4), we get
\[
a^{ij} \phi_{ij} - \frac{1}{2} x_s \phi_s = 0.
\]

Thus \( \phi \) satisfies an elliptic equation without zeroth order term. The corresponding elliptic operator is
\[
\mathcal{L} = a^{ij} \partial^2_{ij} - \frac{1}{2} x_s \partial_s.
\]

Define \( \tilde{w}(x) = w(x) - \langle Dw(0), x \rangle \). Obviously, \( \tilde{w} \) is strictly convex and \( D\tilde{w}(0) = 0 \). So \( \tilde{w} \) is proper. And we have
\[
\mathcal{L}\tilde{w} = a^{ij} \tilde{w}_{ij} - \frac{1}{2} x_s \tilde{w}_s
\]
\[
= a^{ij} w_{ij} - \frac{1}{2} \langle x, D\tilde{w} \rangle
\]
\[
< \frac{n \sqrt{a^2+1}}{2b} - \frac{1}{2} \langle x, D\tilde{w} \rangle.
\]

Due to the convexity of \( \tilde{w} \),
\[
\langle x, D\tilde{w}(x) \rangle \geq \tilde{w}(x) - \tilde{w}(0).
\]

Since \( \tilde{w} \) is proper, there exists \( R_0 > 0 \) such that for \( x \in \mathbb{R}^n \setminus B_{R_0} \),
\[
\tilde{w}(x) \geq |\tilde{w}(0)| + \frac{n \sqrt{a^2+1}}{b}.
\]

Hence, \( \mathcal{L}\tilde{w} < 0 \) for \( x \in \mathbb{R}^n \setminus B_{R_0} \).

For any \( \varepsilon > 0 \), take a barrier function \( b_\varepsilon(x) \) defined by
\[
b_\varepsilon(x) = \varepsilon \tilde{w}(x) + \max_{\partial B_{R_0}} \phi.
\]

Clearly we have
\[
\mathcal{L} b_\varepsilon \leq \mathcal{L} \phi \quad \text{for} \quad x \in \mathbb{R}^n \setminus B_{R_0}.
\]
and

\[ b_\varepsilon(x) \geq \phi(x) \text{ on } \partial B_{R_0}. \]

Since \( \phi < 0 \) and \( \tilde{w} \) is proper, we also have

\[ b_\varepsilon(x) > \phi(x) \text{ as } x \to +\infty. \]

The weak maximum principle then implies

\[ \varepsilon \tilde{w}(x) + \max_{\partial B_{R_0}} \phi \geq \phi(x) \text{ for } x \in \mathbb{R}^n \setminus B_{R_0}. \]

Letting \( \varepsilon \to 0 \), we obtain

\[ \max_{\partial B_{R_0}} \phi \geq \phi(x) \text{ for } x \in \mathbb{R}^n \setminus B_{R_0}. \]

So \( \phi \) attains its global maximum in the closure of \( B_{R_0} \). Hence \( \phi \) is a constant by the strong maximum principle. Finally by (4.3), \( w \) must be a quadratic polynomial, and so is \( u \). \( \square \)

5 Proof of Theorem 1.3

We prove Theorem 1.3 by transforming (1.10) into (1.6).

**Proof** By the difference formula for tangent,

\[ \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b} = \arctan \left( \frac{\lambda_i + a}{b} \right) - \frac{\pi}{4}. \]

Take

\[ w(x) = \frac{b}{\sqrt{a^2 + 1}} u \left( \frac{(a^2 + 1)^{\frac{1}{2}}}{b} x \right) + \frac{a}{2b} |x|^2 - \frac{n \pi}{4}. \]  

(5.1)

Then \( w \) satisfies

\[ \sum_{i=1}^{n} \arctan \mu_i = \frac{1}{2} \langle x, Dw \rangle - w, \]

where \( \mu = (\mu_1, \ldots, \mu_n) \) are the eigenvalues of \( D^2 w \). According to Theorem 1.1 in [5] or Theorem 1.2 in [11], \( w \) is a quadratic function, and so is \( u \). This completes the proof. \( \square \)

6 Proof of Theorem 1.4

**Proof** Suppose \( u \) is an admissible solution to (1.5). Fixing any \( \theta \in \mathbb{S}^{n-1} \), consider a function \( q_\theta(r) \) defined on \( (0, +\infty) \) by

\[ q_\theta(r) = u(r\theta) / r^2. \]

By equation (1.5),

\[ q_\theta'(r) = \frac{ru_r(r\theta) - 2u(r\theta)}{r^3} = \frac{2F_1(\lambda(D^2 u(r\theta)))}{r^3}. \]

**Case I** If \( u \) is an admissible solution to (1.8), then \( D^2 u > -I_n \) or \( D^2 u < -I_n \). We first assume \( D^2 u > -I_n \). By (1.8), we have

\[ q_\theta'(r) < 0. \]
Thus, for $r \geq 1$,

$$\frac{u(r\theta)}{r^2} \leq u(\theta).$$

This implies that for $x \in \mathbb{R}^n \setminus B_1$, $u(x) \leq C |x|^2$, where $C = \max_{\partial B_1} u$. Since $D^2 u \succ -I_n$, $u + |x|^2$ is convex. Hence $u$ has at most a quadratic growth. It follows that $|Du|$ has at most a linear growth. For the case $D^2 u \prec -I_n$, we can get the same growth estimates via the transform (3.1). Consequently, if $|Du|$ or $u$ blows up on $\partial \Omega$, $\Omega$ must be $\mathbb{R}^n$.

The proof for Eq. (1.9) is almost the same as above proof for Eq. (1.8).

**Case 2** If $u$ is a solution to (1.10), we have

$$\left| \frac{q'(\theta)}{r^2} \right| < \frac{n\pi \sqrt{a^2 + 1}}{2b} r^{-3}.$$  

Since $r^{-3} \in L^1([1, +\infty))$, $q'(r)$ is integrable. It follows that $u$ has at most a quadratic growth. Therefore, if $u$ blows up on $\partial \Omega$, then $\Omega$ must be the entire $\mathbb{R}^n$. If $|Du|$ blows up on $\partial \Omega$, by transform (5.1) and Theorem 1.1 in [19], we also conclude that $\Omega = \mathbb{R}^n$.

### 7 Proof of Theorems 1.5, 1.6 and 1.8

**Proof of Theorem 1.5** For simplicity, we take the following trivial transform

$$w(x) = \frac{b}{\sqrt{a^2 + 1}} u \left( \frac{(a^2 + 1)^{\frac{1}{2}}}{b} x \right) + \frac{a}{2b} |x|^2. \quad (7.1)$$

Then $w$ satisfies

$$\frac{1}{2} \sum_{i=1}^{n} \ln \frac{1 + \mu_i}{1 - \mu_i} = \frac{1}{2} \langle x, Dw \rangle - w, \quad (7.2)$$

where $\mu = (\mu_1, \ldots, \mu_n)$ are the eigenvalues of $D^2 w$, and

$$-I_n < D^2 w < I_n.$$  

We first construct a nontrivial one dimensional entire admissible solution to (7.2), namely a solution to

$$\frac{1 + w''}{1 - w''} = \exp \left( tw' - 2w \right), \quad (7.3)$$

where $w_1 = w_1(t)$. Then $w(x) = w_1(x_1)$ gives a nontrivial solution for arbitrary dimensions.

Let $\phi = \frac{1}{2} tw' - w_1$. Similar calculations as in the proof of Theorem 1.1 give

$$\phi'' = \frac{2te^{2\phi}}{(1 + e^{2\phi})^2} \phi'. \quad (7.4)$$

This equation has both even and odd symmetries, namely if $\phi(t)$ is a solution to (7.4), then both $\phi(-t)$ and $-\phi(-t)$ are solutions to (7.4). For any $a_0, a_1 \in \mathbb{R}$, by the Cauchy-Kovalevskaya theorem, there exists a unique local solution around $t = 0$ to (7.4) that satisfies $\phi(0) = a_0$, $\phi'(0) = a_1$. If $a_1 = 0$, then $\phi(t) \equiv a_0$, corresponding to a trivial solution. By the even symmetry, we may assume $a_1 > 0$. Next we show this local solution can be extended to the whole $\mathbb{R}$.
Suppose \((-l_-, l_+)\) is the maximal interval where the solution exists. We first show \(l_+ = +\infty\). As \(\phi'(0) = a_1 > 0\), from (7.4) we see \(\phi'\) monotonically increases in \([0, l_+)\), so

\[
\phi(t) \geq a_1 t + a_0 \quad \text{in } [0, l_+).
\]  

(7.5)

It follows that

\[
\phi''(t) \leq 2te^{-2a_1 t - 2a_0} \phi'(t) \quad \text{in } [0, l_+).
\]

Namely

\[
(\ln \phi')' \leq 2te^{-2a_1 t - 2a_0} \quad \text{in } [0, l_+).
\]

Since \(te^{-2a_1 t} \in L^1([0, +\infty))\), \(\phi'(t)\) is bounded in \([0, l_+)\). Then \(\phi(t)\) and \(\phi'(t)\) can be smoothly extended to and across \(l_+\) if \(l_+ \) is finite. Consequently, \(l_+ = +\infty\).

As \(\phi\) satisfies (7.4) in \((-l_-, 0]\) with \(\phi(0) = a_0\) and \(\phi'(0) = a_1\), by the odd symmetry, \(\phi(t) = -\phi(-t)\) satisfies (7.4) in \([0, l_-)\) with \(\phi(0) = -a_0\) and \(\phi'(0) = a_1\). Via a similar analysis, we conclude that \(\phi(t)\) satisfies \(\phi(t) \geq a_1 t - a_0\) in \([0, l_-)\) and it follows that \(l_- = +\infty\). Back to \(\phi(t)\), we have

\[
\phi(t) \leq a_1 t + a_0 \quad \text{in } (-\infty, 0] .
\]  

(7.6)

Estimates (7.5) and (7.6) will also be used in the proof of incompleteness and non-surjectiveness.

From the relation \(\phi = \frac{1}{2} t w_1' - w_1\), we obtain \(w_1(0) = -a_0\) and \(w_1'(0) = -2a_1\). Equation (7.3) reads

\[
w_1'' = \frac{e^{2\phi} - 1}{e^{2\phi} + 1} .
\]

Integrating above equation twice, we get an entire non-quadratic solution to (7.3). Apparently, there is a corresponding non-quadratic entire admissible solution to (1.11).

In the following, we show this counterexample is incomplete, and neither \((b + a)x + Du\) nor \((b - a)x - Du\) is surjective.

**Incompleteness**

Although Chen–Qiu’s result (Theorem 2 in [8]) implies the constructed nonlinear solution is incomplete, we give a direct proof here. We only need to prove \((\mathbb{R}^n, \tilde{g})\) is incomplete, where \(\tilde{g}_{ij} = \delta_{ij} - w_{ij} w_{ij}\), since the transform (7.1) induces an isometry between the metrics induced by the potentials \(w\) and \(u\). By the symmetry of \(\tilde{g}\), the \(x_1\)-axis is a geodesic, whose length is

\[
l = \int_{-\infty}^{+\infty} \sqrt{1 - w_1''(t)} \, dt = 2 \int_{-\infty}^{+\infty} \frac{e^{\phi(t)}}{e^{2\phi(t)} + 1} \, dt.
\]

Due to the estimates (7.5) and (7.6),

\[
\int_{-\infty}^{+\infty} \frac{e^{\phi(t)}}{e^{2\phi(t)} + 1} \, dt < 2e^{a_0} \int_{0}^{+\infty} e^{-a_1 t} \, dt < \infty.
\]

Hence, \(\tilde{g}\) is incomplete at both ends of the \(x_1\)-axis.

**Non-surjectiveness**

By the transform (7.1), it suffices to prove neither \(x + Dw\) nor \(x - Dw\) is surjective. According to the construction of \(w\), we only need to show both the images of \(x_1 + w_1'\) and \(x_1 - w_1'\) are not the whole \(\mathbb{R}\).
Due to $|w''_1| < 1$ and the estimate (7.6), we have

$$\inf_{x_1 \in \mathbb{R}} (x_1 + w'_1(x_1)) = w'_1(0) - \int_{-\infty}^{0} (1 + w''_1(t)) \, dt = a_1 - 2 \int_{-\infty}^{0} \frac{e^{2\varphi}}{e^{2\varphi} + 1} \, dt > a_1 - 2e^{a_0} \int_{0}^{+\infty} e^{-2a_1t} \, dt > -\infty.$$ 

Similarly,

$$\sup_{x_1 \in \mathbb{R}} (x_1 - w'_1(x_1)) = -w'_1(0) + \int_{0}^{+\infty} (1 - w''_1(t)) \, dt = -a_1 + 2 \int_{0}^{+\infty} \frac{1}{e^{2\varphi} + 1} \, dt < -a_1 + 2e^{-a_0} \int_{0}^{+\infty} e^{-2a_1t} \, dt < +\infty.$$ 

Hence, neither $x + Dw$ nor $x - Dw$ is surjective. \hfill \Box

**Proof of Theorem 1.8** Given $\rho \in (0, 1]$, we construct a nonlinear entire solution to the following ODE

$$\frac{\rho^2 \varphi''(t)}{\rho^2 - \varphi'(t)^2} = \frac{1}{2} \left( t \varphi'(t) - \varphi(t) \right)$$

(7.7)

that satisfies $|\varphi'| < \rho$. Then one can easily verify that $f(x) = \varphi(x_1) + \langle \xi, x' \rangle$ is a nonlinear entire admissible solution to (1.13), where $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\xi$ is an arbitrary vector in $\mathbb{R}^{n-1}$ with $|\xi| = \sqrt{1 - \rho^2}$.

For any $a_0, a_1 \in \mathbb{R}$ with $|a_1| < \rho$, by the Cauchy-Kowalevskaya theorem, there exists a unique local solution around $t = 0$ to (7.7) that satisfies $\varphi(0) = a_0$, $\varphi'(0) = a_1$. We assume $a_0 \neq 0$, otherwise $\varphi(t) \equiv a_1 t$, corresponding to a linear solution. Since (7.7) has odd symmetry, without loss of generality we may assume $a_0 < 0$. Then by the equation, $\varphi''(0) = \frac{1}{2\rho^2}(a_1^2 - \rho^2)a_0 \neq 0$, which implies the local solution is nonlinear. Next we show this local solution can be extended to the whole $\mathbb{R}$.

Suppose $(-l_-, l_+)$ is the maximal interval where the solution exists and satisfies $|\varphi'| < \rho$. For any finite $t \in [0, l_+)$, an integration of both sides of (7.7) on $[0, t]$ yields

$$\ln \frac{\rho + \varphi'(t)}{\rho - \varphi'(t)} = \frac{1}{\rho} \int_{0}^{t} (s\varphi'(s) - \varphi(s)) \, ds.$$  

(7.8)

If $l_+$ is finite, then the right-hand side of (7.8) remains bounded as $t \to l_+$, thus the solution can be extended to and across $l_+$. Hence, $l_+ = \infty$; similarly, $l_- = +\infty$.

For the intrinsic incompleteness of $Mf$, due to the even symmetry of $Df$ with respect to $x'$, it suffices to prove

$$\int_{-\infty}^{+\infty} \sqrt{1 - |Df(x_1, 0)|^2} \, dx_1 = \int_{-\infty}^{+\infty} \sqrt{\rho^2 - \varphi'(t)^2} \, dt < \infty.$$ 

Let $\psi(t) = t\varphi'(t) - \varphi(t)$. Then

$$\psi'(t) = t\varphi''(t) = \frac{t}{2} \left( 1 - \frac{\varphi'(t)^2}{\rho^2} \right) \psi(t).$$
Note that $\psi(0) = -a_0 > 0$ and $|\varphi'(t)| < \rho$, so $\psi(t)$ monotonically increases in $[0, +\infty)$. Then from (7.8), we see $\rho - \varphi'(t)$ decays at least exponentially as $t \to +\infty$. Hence, $\int_0^{+\infty} \sqrt{\rho^2 - \varphi'(t)^2} \, dt < \infty$. Symmetrically, $\int_{-\infty}^0 \sqrt{\rho^2 - \varphi'(t)^2} \, dt < \infty$. This proves the intrinsic incompleteness of $M_f$.

\begin{remark}
Any entire one dimensional admissible solution to (1.12) leads to an entire one dimensional admissible solution to (1.13) by the differentiation relation

$$f(x) = \varphi'(x).$$

This leads to another proof based on Theorem 1.5.

In the end, we prove Theorem 1.6 via the Lewy-Yuan rotation.

\begin{proof}[Proof of Theorem 1.6]
The assumption that the map $(b + a)x + Du$ or $(b - a)x - Du$ is surjective corresponds to the map $x + Dw$ or $x - Dw$ is surjective. Since $-I_n < D^2 w < I_n$, both $x + Dw$ and $x - Dw$ map $\Omega$ to $\mathbb{R}^n$ injectively. If in addition $x - Dw$ is surjective, then $x - Dw : \Omega \to \mathbb{R}^n$ is a diffeomorphism. Take the Lewy-Yuan rotation

$$\begin{cases}
\hat{y} = \frac{1}{\sqrt{2}} (x + Dw), \\
\hat{x} = \frac{1}{\sqrt{2}} (x - Dw).
\end{cases}$$

(7.9)

Since $\mathbb{R}^n$ is simply connected, by the Lagrangian condition (preserved under the $\frac{\pi}{4}$-rotation), $\hat{y}$ is the gradient of an entire function on $\mathbb{R}^n$. Denote this function by $\hat{w}$. Actually, up to a constant, $\hat{w}$ can be taken as (see [7, 27])

$$\hat{w} = w(x) - \frac{1}{2} x \cdot Dw(x) + \frac{|x|^2}{4} - \frac{|Dw(x)|^2}{4}.$$  

We can check that

$$D^2 \hat{w}(\hat{x}) = (I_n + D^2 w(x)) (I_n - D^2 w(x))^{-1} > 0,$$

and

$$\frac{1}{2} \hat{x} \cdot D\hat{w}(\hat{x}) - \hat{w}(\hat{x}) = \frac{1}{2} x \cdot Dw(x) - w(x).$$

Thus $\hat{w}$ satisfies

$$\frac{1}{2} \ln \det D^2 \hat{w} = \frac{1}{2} \langle \hat{x}, D\hat{w} \rangle - \hat{w}.$$  

According to Theorem 1.1 in [11], $\hat{w}$ must be a quadratic polynomial, so are $w$ and $u$. It follows that $\Omega = \mathbb{R}^n$.

Apparently, $-w$ is also an entire admissible solution to (7.2). If $x + Dw$ is surjective, then $x - D(-w)$ is surjective. According to the discussion above, $-w$ is a quadratic function. Thus we also draw the conclusion. \hfill \Box

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\end{acknowledgements}
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