THE LALONDE-MCDUFF CONJECTURE FOR NILMANIFOLDS

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Abstract. We prove that any Hamiltonian bundle whose fiber is a nilmanifold c-splits.

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1. Introduction

Let \((M, \omega)\) be a closed symplectic manifold. A fibre bundle \((M, \omega) \to P \to B\) is called Hamiltonian if its structural group can be reduced to the group of Hamiltonian diffeomorphisms. The following conjecture was posed by Lalonde and McDuff in [LM]

Conjecture 1.1. Every Hamiltonian fiber bundle c-splits. This means that there is an additive isomorphism

\[ H^*(P) \cong H^*(B) \otimes H^*(M) \]

where \(H^*\) denotes cohomology with real coefficients.

The c-splitting conjecture holds in many cases, see [H], [K], [LM], but the general case is still not resolved. In particular, an argument of Blanchard, [B] shows that it holds when the cohomology of the fiber satisfies the hard Lefschetz condition. The purpose of this paper is to prove the c-splitting conjecture for fibers which are symplectic nilmanifolds. A nilmanifold is a compact homogeneous space of the form \(N/\Gamma\), where \(N\) is a simply connected nilpotent Lie group and \(\Gamma\) is a discrete co-compact subgroup (i.e. a lattice). Let us recall that these manifolds do not satisfy the hard Lefschetz condition except for tori ([BQ], [M], [TO]).

In this article we prove the following theorem.

Theorem 1.2. Let \((N/\Gamma, \omega)\) be a symplectic nilmanifold and \((N/\Gamma, \omega) \hookrightarrow P \to B\) be a Hamiltonian bundle over a simply connected CW-complex. Then \(H^*(P) \cong H^*(B) \otimes H^*(N/\Gamma)\) as algebras.

This theorem implies the following result.

Theorem 1.3. Let \((N/\Gamma, \omega)\) be a symplectic nilmanifold. Then any Hamiltonian bundle \((N/\Gamma, \omega) \hookrightarrow P \to B\) c-splits.

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Proof. Since $B\text{Ham}(N/\Gamma)$ is simply connected then the universal Hamiltonian bundle with fiber $N/\Gamma$

$$N/\Gamma \to E\text{Ham}(N/\Gamma) \times_{\text{Ham}(N/\Gamma)} N/\Gamma \to B\text{Ham}(N/\Gamma)$$

is c-split by Theorem 1.2. It follows from [LM, Lemma 4.1(i)] that any Hamiltonian bundle with fiber $N/\Gamma$ c-splits. □

We will prove Theorem 1.2 in the next section. The first step is to establish the result for $B = S^2$. In [M2], McDuff proved that every Hamiltonian bundle over $S^2$ c-splits. The proof is difficult and requires hard analytic tools. In the special case of nilmanifold fiber our approach gives a different and easy proof. The main tool used in this paper is the Sullivan model of a fibration. Also, we use the following characterization of Hamiltonian bundles:

Theorem 1.4. [LM Lemma 2.3] If $\pi_1(B) = 0$ then a symplectic bundle $M \hookrightarrow P \to B$ is Hamiltonian if and only if the class $[\omega] \in H^2(M)$ extends to $a \in H^2(P)$.

2. Proof of Theorem 1.2

2.1. Some facts from rational homotopy theory. Basic reference for rational homotopy theory is [FHT]. We also keep terminology and notation close to [FHT]. In this paper all algebras and cohomologies are considered over the field of real numbers.

A free commutative cochain algebra $(\Lambda V, d)$ such that

1. $V$ is a graded vector space and $V = \{V^p\}_{p \geq 1}$,
2. $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence of graded subspaces such that $d = 0$ in $V(0)$ and $d : V(k) \to \Lambda V(k-1)$, $k \geq 1$,
3. $\text{Im} d \subset \Lambda^+ V \cdot \Lambda^+ V$, where $\Lambda^+ V = \bigoplus_{q=1}^{\infty} \Lambda^q V$ and $\Lambda^q V$ is the linear span of the elements $v_1 \wedge \cdots \wedge v_q$, $v_i \in V$,
4. there is a quasi-isomorphism

$$m : (\Lambda V, d) \to A_{PL}(X).$$

is called the minimal Sullivan model for a path connected topological space $X$. Here $A_{PL}(X)$ denotes the cochain algebra of polynomial differential forms on $X$.

If condition (3) is not satisfied then $(\Lambda V, d)$ is called the Sullivan model for $X$. For any path connected topological space $X$ there exists a minimal Sullivan model $(\Lambda V, d)$ and this is uniquely determined up to isomorphism. If $X$ is a finite CW complex then each $V^p$ is finite dimensional.

Let $f : X \to Y$ be a continuous map between path connected topological spaces. A choice of $m : (\Lambda V_Y, d) \to A_{PL}(Y)$ and $m'$ :
(ΛV_X, d') \to A_{PL}(X)$ determines a unique homotopy class of morphisms between Sullivan models of $Y$ and $X$ respectively. A morphism $f^{*} : (ΛV_Y, d) \to (ΛV_X, d')$ such that $m'f^{*}$ and $A_{PL}(f)m$ are homotopic is called a Sullivan representative of $f$. Here $A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X)$ is a morphism induced by $f$.

The following Theorem can be found in [AP]. See also Proposition 15.5 in [FHT].

**Theorem 2.1.** (Grivel-Halperin-Thomas) Let $\pi : E \to B$ be a Serre fibration of path connected spaces and $F = \pi^{-1}(b)$ be the fiber over the base-point $b$. Suppose that:

1. $F$ is path connected,
2. $\pi_1(B)$ acts nilpotently on $H^k(F)$ for all $k \geq 1$,
3. either $B$ or $F$ is of finite type.

Then there exist a commutative diagram of cochain algebra morphisms

\[
\begin{array}{ccc}
A_{PL}(B) & \longrightarrow & A_{PL}(P) \\
\uparrow m_B & & \uparrow m \\
(ΛV_B, d) & \longrightarrow & (ΛV_B \otimes ΛV_F, D) \\
\downarrow m_F & & \downarrow m_F \\
(ΛV_B, d) & \longrightarrow & (ΛV_F, \bar{d})
\end{array}
\]

in which $(ΛV_B, d)$ is minimal Sullivan model for $B$, $(ΛV_F, \bar{d})$ is minimal Sullivan model for $F$, $m$ is a quasi-isomorphism and $Dv = db, b \in ΛV_B$.

\[
Dv - \bar{d}v \in Λ^+V_B \otimes ΛV_F, v \in V_F.
\]

2.2. A minimal Sullivan model for $S^2$. The fundamental class $[S^2] \in H_2(S^2; \mathbb{Z})$ determines a class $\omega \in H^2(A_{PL}(S^2))$ such that $\langle \omega, [S^2] \rangle = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the paring between cohomology and homology. Let $Φ$ be a cocycle representing the cohomology classes $ω$ and $Φ^2 = dΨ$. Then the a minimal Sullivan model for $S^2$ is given by

\[
m : (Λ(a, b), d) \to A_{PL}(S^2)
\]

where $deg(a) = 2$, $deg(b) = 3$, $ma = Φ$, $mb = Ψ$.

2.3. A minimal Sullivan model for a nilmanifold. Let $N/\Gamma$ be a nilmanifold of dimension $n$. In this case, the differential graded algebra $A^{DR}_{N}(N)$ of right-invariant forms on $N$ is a Sullivan minimal model for $N/\Gamma$. In other words, there is a dual basis $\{x_1, x_2, ..., x_n\}$ to the basis $\{X_1, X_2, ..., X_n\}$ of $n$ such that

\[
(Λ^n, \bar{d}) = (Λ(x_1, x_2, ..., x_n), \bar{d})
\]

with $deg(x_k) = 1, k \in \{1, 2, ..., n\}$ is a minimal model of $N/\Gamma$. Here $n$ denotes the Lie algebra of the Lie group $N$. 
We call a form $\alpha$ on $N/\Gamma$ homogeneous if the pullback of $\alpha$ to $N$ is right-invariant. Thus, any cohomology class $[\alpha] \in H^*(G/\Gamma)$ is represented by a homogeneous form $\alpha_h$. Recall ([M1], [IRTU]) that the cohomology class of the symplectic structure is represented by a homogeneous symplectic form. Therefore, it is represented by a degree 2 element of the minimal model which we can write as $\omega = \sum a_{ij}x_i x_j$ and the following conditions are satisfied:

1. $d\omega = 0$
2. for every $X \in \mathfrak{n}$
   $$\omega(X, -) = 0 \Rightarrow X = 0$$

2.4. **Proof of Theorem 1.2.** Let $N/\Gamma$ be a symplectic nilmanifold of dimension $2n$ and

$$N/\Gamma \hookrightarrow P \rightarrow B$$

be a Hamiltonian bundle over a simply connected CW-complex. The assumptions of Theorem 2.1 are satisfied since $N/\Gamma$ is of finite type. Thus we obtain a commutative diagram

$$
\begin{array}{ccc}
A_{PL}(B) & \longrightarrow & A_{PL}(P) & \longrightarrow & A_{PL}(N/\Gamma) \\
\uparrow m_B \cong & & \uparrow m_P \cong & & \uparrow m_{N/\Gamma} \cong \\
(\Lambda V_B, d) & \longrightarrow & (\Lambda V_B \otimes \Lambda n^*, D) & \longrightarrow & (\Lambda n^*, d)
\end{array}
$$

in which $(\Lambda V_B, d)$ is a minimal Sullivan model for $B$, $(\Lambda n^*, d)$ is a minimal Sullivan model for $N/\Gamma$ and

$$Db = db, \ b \in \Lambda V_B$$

$$Dv - \overline{dv} \in \Lambda^+(V_B \otimes \Lambda n^*, v \in n^*).$$

We will show that $Dv = \overline{dv}$, $v \in n^*$. This implies that $H^*(P) = H^*(B) \otimes H^*(N/\Gamma)$

**Case 1.** $B = S^2$. Let $((\Lambda(a, b) \otimes \Lambda(x_1, x_2, ..., x_{2n}), D)$ with $deg(x_k) = 1$, $k \in \{1, 2, ..., 2n\}$ be the Sullivan model for $P$. By degree reasons the differential $D$ is given by

$$D(x_k) = \alpha_k a + \overline{d}x_k$$

where $\alpha_k \in \mathbb{R}$

Let $\omega = \sum a_{ij} x_i x_j$ represent the cohomology class of the symplectic form on $N/\Gamma$ in the minimal model $\Lambda(x_1, x_2, ..., x_{2n})$. Since the bundle is Hamiltonian we have $D(\omega) = 0$ (Theorem 1.4).
Thus
\[0 = D(\omega) = D\left(\sum a_{ij} x_i x_j\right) = \sum a_{ij} (Dx_i x_j - x_i Dx_j) = \sum a_{ij} ((\alpha_i a + \overline{dx}_i) x_j - x_i (\alpha_j a + \overline{dx}_j)) = \sum a_{ij} (\alpha_i a x_j - \alpha_j x_i a) + \overline{d}\omega\]
and we obtain
\[\sum a_{ij} (\alpha_i x_j - \alpha_j x_i) = 0. \tag{1}\]
Take \(X \in \mathfrak{n}\) such that \(x_k(X) = \alpha_k\). Note that
\[\omega(X, -) = \sum a_{ij} (\alpha_i x_j - \alpha_j x_i).\]
It follows from (1) that
\[X = 0, \text{ and thus } \alpha_k = 0, \text{ for all } k.\]
Thus \(D(x_k) = \overline{dx}_k\) and \(H^*(P) = H^*(S^2) \otimes H^*(N/\Gamma)\).

**Case 2.** \(B\) is a simply connected finite CW complex. First, we will choose a minimal Sullivan model for \(B\). Let \(v_1, ..., v_m\) be a basis of \(H_2(B, \mathbb{R})\). Since \(B\) is simply connected the Hurewicz Theorem asserts that the Hurewicz map is an isomorphism \(\pi_2(B) \to H_2(B, \mathbb{Z})\). This extends to an isomorphism \(\pi_2(B) \otimes \mathbb{Z} \to H_2(B, \mathbb{R})\). Thus \(v_i\) can be represented by maps \(f_i : S^2 \to B\). Let \(a_1, ..., a_m\) be the basis for \(H^2(B, \mathbb{R})\) defined by \(\langle a_j, H_* f_i([S^2]) \rangle = \delta_{ij}\), where \(H_* f_i : H_*(S^2) \to H_*(B)\) and \(\langle , \rangle\) denotes the paring between cohomology and homology. Let \(\Phi_j \in A_{PL}(B)\) be the cocycle representing the cohomology class \(a_j\). Since \(V_B^2 \simeq H^2\) then we can choose \(V_B\) and a quasi-isomorphism
\[m : (\Lambda V_B, \overline{d}) \to A_{PL}(B),\]
such that \(ma_j = \Phi_j, \ j \in \{1, \ldots, m\}\). Hence \(f_i^* a_j = \delta_{ij} a\) for any \(f_i^*\).
Here \(f_i^* : (\Lambda V_B, \overline{d}) \to (\Lambda(a, b), d)\) is a Sullivan representative of \(f_i\).

Now, let
\[(\Lambda(V_B) \otimes \Lambda(x_1, x_2, ..., x_{2n}), \overline{D})\]
with \(deg(x_k) = 1, \ k \in \{1, 2, ..., 2n\}\) be the Sullivan model for \(P\). Again, from degree reasons the differential \(\overline{D}\) is given by
\[\overline{D}(x_k) = \sum \alpha_{kj} a_j + \overline{dx}_k,\]
where \(deg(a_j) = 2, \ \alpha_{kj} \in \mathbb{R}\).
Now, consider pullback of the bundle to $S^2$ by the map $f_i$ for each $i \in \{1, 2, ..., m\}$. Then we can write the commutative diagram

\[
\begin{array}{ccc}
(\Lambda(x_1, x_2, ..., x_{2n}), \overline{d}) & \overset{\hat{f}_i}{\longrightarrow} & (\Lambda(x_1, x_2, ..., x_{2n}), \overline{d}) \\
\uparrow & & \uparrow \\
(\Lambda(a, b) \otimes \Lambda(x_1, x_2, ..., x_{2n}), D) & \overset{\hat{f}_i}{\longrightarrow} & (\Lambda(V_B) \otimes \Lambda(x_1, x_2, ..., x_{2n}), \overline{D}) \\
\uparrow & & \uparrow \\
(\Lambda(a, b), d) & \overset{f^*_i}{\longleftarrow} & (\Lambda V_B, \overline{d})
\end{array}
\]

It follows from Case 1 and the commutativity of the diagram

\[
\hat{f}_i^*(\overline{D}(x_1)) = \hat{f}_i^*(\sum \alpha_{1j}a_j + \overline{dx}_1) = \sum \alpha_{1j}f_i^*(a_j) + \overline{dx}_1 = \sum \alpha_{1j}(\delta_{ij}a) + \overline{dx}_1 = \alpha_{1i}a + \overline{dx}_1 = D(\hat{f}_i^*(x_1)) = D(x_1) = \overline{dx}_1
\]

Hence $\alpha_{1i} = 0$, for all $i \in \{1, 2, ..., m\}$ The above argument applied to $x_2, x_3, ..., x_{2n}$ gives $\overline{D}(v) = \overline{dv}$, $v \in \mathfrak{n}^*$.

**Case 3.** $B$ is a simply connected CW-complex. A Sullivan model for $P$ is of the form $(AV_B \otimes \Lambda n^*, D)$, where $AV_B$ is a minimal Sullivan model for $B$. Assume that the condition $Dv = \overline{dv}$, $v \in \mathfrak{n}^*$ is not satisfied. Then we can choose a finite subcomplex $B_1 \subset B$ such that the restriction of the bundle $P$ over $B_1$ has a Sullivan model of the form $(AV_{B_1} \otimes \Lambda n^*, D)$, $V_{B_1} \subset V_B$ and the condition $Dv = \overline{dv}$, $v \in \mathfrak{n}^*$ is not satisfied. However, this contradicts Case 2. □

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