Entropy Current Formalism for Supersymmetric Theories

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\textbf{Abstract}

The recent developments in fluid/gravity correspondence give a new impulse to the study of fluid dynamics of supersymmetric theories. In that respect, the entropy current formalism requires some modifications in order to be adapted to supersymmetric theories and supergravities. We formulate a new entropy current in superspace with the properties: 1) it is conserved off-shell for non dissipative fluids, 2) it is invariant under rigid supersymmetry transformations 3) it is covariantly closed in local supersymmetric theories 4) it reduces to its bosonic expression on space-time.
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1 Introduction

Recent developments in fluid/gravity correspondence [1, 2, 3] motivate a deeper analysis of the fluid dynamics in the context of supersymmetric theories and of supergravity. In the present work, we take a first step toward that extension by analyzing the definition of the entropy current for non dissipative fluids (see for example [4, 5, 6, 7, 8]) and by providing its supersymmetric generalization. The starting point is a convenient formulation of the fluid dynamics in terms of the comoving coordinates of the fluid (see [10, 11]). The Eulerian description in terms of spacetime-dependent quantities is replaced by a new set of comoving coordinates $\phi^I$ (with $I = 1, \ldots, d$, $d$ being the space dimensions) which are spacetime fields. In terms of those, in the case of non-dissipative fluids, one can easily write down a Lagrangian whose field equations are the relativistic generalization of the well-known Navier-Stokes equations. One can also easily define several interesting thermodynamical quantities such as the entropy, the energy density, chemical potentials and so on. This formalism permits also a direct verification of Maxwell equations for thermodynamics. Finally, all techniques of quantum field theory can be used to investigate the quantum properties of fluids (see for example [5]).

Recently a series of interesting papers [12, 6, 13] appeared on the subject by exploring the fluid dynamics from the point of view of comoving coordinates and discussing the role of the entropy current in that context. In particular they claim that the entropy current of a given system must have the following properties: 1) It is dual to a $d$-form in $(d + 1)$-space-time dimensions; 2) It is conserved off-shell. It is easy to show that the expression

$$J^{(1)} = \star (d\phi^1 \wedge \cdots \wedge d\phi^d)$$

where the star symbol is the Hodge-dual star operator in $d + 1$ dimensions, has the correct properties. In addition, it cannot be written as a $d$-exact expression since the comoving coordinates $\phi^I$ are not globally defined. The entropy density can be computed by considering the Hodge dual of $J^{(1)}$. In papers [6, 8, 7], this formalism has been applied to normal fluids as well as to superfluids and, there, all quantities are computed in terms of the comoving coordinates and of one additional degree of freedom $\psi$. A new symmetry has been advocated in order to describe the superfluid in a suitable phase and the spectrum of waves in that fluid have been taken into account. We briefly review that formalism in Section 2 in order to set up the stage for our developments.

The next step is to provide a supersymmetric extension. Since the coordinates $\phi^I$ represent a set of comoving coordinates of the fluid, it is natural to introduce a set of anticommuting coordinates $\theta^a$ for describing the fluid fermionic degrees of freedom (see [11]). Using the analogy with the Green-Schwarz superstring and with the supermembrane we define a supersymmetric 1-form $\Pi^I$ replacing the 1-form $d\phi^I$ of the bosonic theory. In that context, we discuss the generalization of the action for supersymmetric fluids with all symmetries.

Finally, we can provide the supersymmetric extension of the entropy current. We have to recall that the entropy current is associated with volume preserving diffeomorphisms and therefore the supersymmetric extension must play a similar role of volume preserving superdiffeomorphisms. In that case the form is an integral form (see for example [14, 15, 16, 17, 18, 19] and references therein) whose complete expression on the supermanifold

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1 For earlier works on supersymmetric description of fluids see for instance [9]
\[ \mathcal{M}^{(d+1|m)} \text{ is} \]
\[ J^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_d} \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_d} \theta^m \cdot \theta^1 \wedge \delta(d\theta^1) \wedge \cdots \wedge \delta(d\theta^m), \]

\( d \) being the spatial dimensions and \( m \) the dimension of the spinor representation, that is the number of fermionic coordinates. \(^2\) This expression transforms as a Berezinian under superdiffeomorphisms and the Dirac delta of 1-superforms \( d\theta^\alpha \) is a symbol that has the usual properties of distributions as explained in [18]. The Dirac delta-functions of 1-forms \( d\theta^\alpha \) can be understood by assuming that the fermionic 1-forms are indeed commuting quantities and therefore it becomes pivotal to define an integration measure in this space. One way – although it is not the only one – is to use the atomic measure given by the distributional Dirac delta. The main property of that distribution is locality which plays an important role in our construction.

Given the new formula for the entropy current \( J \), we can compute the entropy density \( s \) and we discuss some implications. As an important application, we generalize our construction to supergravity.

The paper is organized as follows:
In Section 2 we review the Lagrangian approach for fluid dynamics and introduce the entropy current.
In Section 3 we set up the stage for the supersymmetric extension of the bosonic theory and, as a warm-up exercise, we discuss the symmetries and equations of motion of a supersymmetric effective theory for fluid dynamics in a 1+1 dimensional model. We then extend the considerations to a general \( d + 1 \) dimensional Lagrangian.
In Section 4 we give a general expression for the entropy current and entropy density in superspace. We will first discuss the properties of the entropy current for a 1+1 dimensional model, then for a general \( d + 1 \) dimensional theory and finally we propose a possible expression of it in \( N = 1 \) supergravity.
The Appendices contain several technical details.

2 Comoving Coordinate Formalism

In this section we shortly review the Lagrangian approach developed in refs. [10, 4, 6, 8], which is based on the use of the comoving coordinates of the fluid as fundamental fields, adopting the same notations as [6]. Their approach will be useful for the extension of the formalism to the supersymmetric case.

From a physical point of view one assumes that the hydrodynamics of a perfect fluid can be formulated as a low energy effective Lagrangian of massless fields which are thought of as the Goldstone bosons of a broken symmetry, namely space translations (broken by the presence of phonons), and is invariant under the symmetry associated with conserved charges. The effective complete Lagrangian would be a derivative expansion in terms of the breaking parameters (mean free path and mean free time). One tries to determine the low energy Lagrangian by symmetry requirements.

Working, for the sake of generality, in \( d + 1 \) space-time dimensions, one introduces \( d \) scalar fields \( \phi^I(x^I, t), I = 1, \ldots, d \) as Lagrangian comoving coordinates of a fluid element

\(^2\) For the sake of clarity, we assume here and in the following that the \( \theta^\alpha \) are Majorana spinors (as it is in four dimensions) or Majorana-Weyl spinors. The case of Dirac or pseudo-Majorana spinors (as it happens e.g. in D=5 supersymmetric theories) can be dealt with in an analogous way.
at the point $x^I$ at time $t$ such that, at equilibrium, the ground state is described by $\Phi^I = x^I$ and requires, in absence of gravitation, the following symmetries:

\[
\delta \phi^I = a^I \quad (a^I = \text{const.}),
\]

\[
\phi^I \to O^I_J \phi^J, \quad (O^I_J \in \text{SO}(d)),
\]

\[
\phi^I \to \xi^I(\phi), \quad \det(\partial \xi^I/\partial \phi^J) = 1.
\]

Furthermore, if there is a conserved charge (particle number, electric charge etc.), then the associated symmetry cannot be described by transformations acting on the fields $\phi^I$, since they are non compact and they cannot describe particle number conservation. Therefore one introduces a new field $\psi(x^I, t)$ which is a phase, that is it transforms under $U(1)$ as follows

\[
\psi \to \psi + c, \quad (c = \text{const}.).
\]

Finally one must take into account that the particle number is comoving with the fluid, giving rise to a (matter) conserved current

\[
\partial_\mu j^\mu = 0,
\]

where

\[
j^\mu = n u^\mu, \quad u^2 = -1,
\]

$n$ being the particle number density and $u^\mu$ the fluid four-velocity defined below. Moreover, if the charge flows with the fluid, charge conservation is obeyed separately by each volume element. This means that the charge conservation is not affected by an arbitrary comoving position-dependent transformation

\[
\psi \to \psi + f(\phi^I)
\]

$f$ being an arbitrary function. This extra symmetry requirement on the Lagrangian is dubbed \textit{chemical-shift symmetry}.

From these premises the authors of [6] construct the low energy Lagrangian respecting the above symmetries. At lowest order the Lagrangian will depend on the first derivatives of the fields through invariants respecting the symmetries \(2.1\)-\(2.4\) and \(2.7\):

\[
\mathcal{L} = \mathcal{L}(\partial \phi^I, \partial \psi).
\]

For this purpose one introduces the following current which respects the symmetries \(2.1\)-\(2.3\):

\[
J^\mu = \frac{1}{d!} \epsilon^{\mu,\nu_1,\ldots,\nu_d} \epsilon_{I_1,\ldots,I_d} \partial_{\nu_1} \phi^{I_1} \cdots \partial_{\nu_d} \phi^{I_d},
\]

and enjoys the important property that its projection along the comoving coordinates does not change:

\[
J^\mu \partial_\mu \phi^I = 0.
\]

Introducing the Hodge-dual star operator in $d+1$ dimensions, this is equivalent to saying that the spatial $d$-form current

\[
J^{(d)} = -* J^{(1)} = \frac{1}{d!} \epsilon_{I_1,\ldots,I_d} d\phi^{I_1} \wedge \ldots \wedge d\phi^{I_d}
\]
where

\[ J^{(1)} = \frac{1}{d!} \epsilon_{\mu_1 \ldots \mu_d} \epsilon_{I_1 \ldots I_d} \partial^{\mu_1} \phi^{I_1} \ldots \partial^{\mu_d} \phi^{I_d} \, dx^\mu = (-1)^d \left( \frac{1}{d!} \epsilon_{I_1 \ldots I_d} d\phi^{I_1} \wedge \cdots \wedge d\phi^{I_d} \right), \]

is closed identically, that is it is locally an exact form. Hence it is natural to define the fluid four-velocity as aligned with \( J^\mu \):

\[ J^\mu = b \, u^\mu \to b = \sqrt{-J^\mu J_\mu} = \sqrt{\det(B^{IJ})}, \]

(2.13)

where \( B^{IJ} \equiv \partial_\mu \phi^I \partial_\mu \phi^J \). From a physical point of view, the property of \( J^\mu \) to be identically closed identifies it as the entropy current of the perfect fluid, so that \( b = s \), \( s \) being the entropy density. Using the entropy current \( J^\mu \) one finds that, by virtue of eq. (2.10), the quantity \( J^\mu \partial_\mu \psi \) is invariant under (2.7).

Summarizing, a low energy action invariant under (2.1)-(2.4) and (2.7), can depend on \( \phi^I \) and \( \psi \) only through \( J^\mu \) and \( J^\mu \partial_\mu \psi \), and, being a Poincaré invariant, it can be written as follows:

\[ S = \int d^4x F(b, y), \]

(2.14)

where \( y \) is

\[ y = u^\mu \partial_\mu \psi = \frac{J^\mu \partial_\mu \psi}{b}. \]

(2.15)

Computing the Noether current associated with the symmetry (2.4) one derives

\[ j^\mu = F_y u^\mu \to F_y \equiv n, \]

(2.16)

which identifies \( n \) as the particle number density. The Noether currents associated with the infinite symmetry (2.7) are

\[ j^\mu_{(f)} = F_y u^\mu f(\phi^f), \]

(2.17)

and these currents are also conserved by virtue of the \( j^\mu \)-conservation.

By coupling (2.14) to worldvolume gravity we can obtain the energy-momentum tensor by taking, as usual, the derivative with respect to a background metric:

\[ T_{\mu\nu} = (y_F - b F_b) u_\mu u_\nu + \eta_{\mu\nu} (F - b F_b). \]

(2.18)

On the other hand, from classical fluid-dynamics, we also have

\[ T_{\mu\nu} = (p + \rho) u_\mu u_\nu + \eta_{\mu\nu} \rho, \]

(2.19)

from which we identify the pressure and density

\[ \rho = y F_y - F \equiv y n - F , \quad p = F - b F_b. \]

(2.20)

From the derivation of the energy-momentum tensor one can easily obtain the entropy density, the temperature, the Maxwell equations and so on (see [6] for a complete review). In particular, it turns out that the quantity \( y \) defined in eq. (2.15) coincides with the chemical potential \( \mu \). To see this, it suffices to compare the first principle

\[ p + \rho = T s + \mu n, \]

(2.21)
with (2.20). Using \( F_y = n \) and \( b = s \), we then find
\[
\frac{\partial F}{\partial s} = -T , \quad y = \mu .
\]
(2.22)

We conclude that the Lagrangian density is a function of \( s \) and \( \mu \)
\[
F = F(s, \mu) .
\]
(2.23)

Let us remark that the Lagrangian used in this setting does not allow at first sight
for the presence of a kinetic term for the dynamical field \( \psi \), namely \( X = \partial_\mu \psi \partial^\mu \psi \). In
fact we could also consider, besides \( X \), further Poincaré invariants of the form \( Z^I = \partial_\mu \psi \partial^\mu \phi^I \). However, it can be proven that the quantities \( X, Z^I \), together with \( B^{IJ} \), are
not independent of \( y \) since the following relation holds:
\[
y^2 = -\partial_\mu \psi \partial^\mu \psi + \partial_\mu \psi \partial^\mu \phi^I B^{-1}_{IJ} \partial_\nu \psi \partial^\nu \phi^J .
\]
(2.24)

Therefore a dependence of the Lagrangian on \( X \) is somewhat implicit in \( y^2 \).

A Lagrangian exclusively depending on \( X \), i.e. of the form \( F(X) \), has been considered,
for instance, in [8], to describe superfluids at \( T = 0 \). The use of the variables \( X, Z^I \), even
though redundant for ordinary fluids, can be useful in order to describe superfluids as
a spontaneously broken phase of a field theory with chemical-shift symmetry invariance.
This idea is elaborated in [20].

3 Supersymmetric Extension. Lagrangian and Equations of Motion

In order to generalize to the supersymmetric case the lagrangian formalism and the
definition of the entropy current reviewed in the previous section let us first set up our
formalism.

A basis of 1-superforms in a general rigid \((d + 1|m)\)-superspace can be given in terms of the supervielbein \( \{ \Pi^a, \Psi^\alpha \} \) by:
\[
\Pi^a = d\phi^a + \frac{i}{2} \bar{\theta} \Gamma^a d\theta , \quad \Psi^\alpha = d\theta^\alpha
\]
(3.1)
where \( a = (0, I), (I = 1, \cdots d) \) and \( \alpha = (1, \cdots m) \) run over the bosonic and fermionic
directions of superspace respectively. Here \( \Gamma^a \) are the Clifford algebra \( \Gamma \)-matrices in
\( d + 1 \)-dimensions, while \( \theta \) and \( d\theta \) denote the matrix form of the Majorana spinors in the
\( m = 2^{d/2} \)-dimensional spinor representation of \( \text{SO}(d, 1) \).\footnote{For Majorana-Weyl spinors, the dimension of the representation is instead \( m = 2^{(d-1)/2} \).} In particular \( \bar{\theta} \equiv \theta^I \Gamma^0 = \theta^T C \),
\( C = (C_{\alpha\beta}) \) being the charge-conjugation matrix. The space-like fields \( \phi^I(x, t) \) can be
taken as the comoving coordinate fields of the bosonic theory, while the spinors \( \theta^\alpha(x, t) \)
are the fermionic coordinates of superspace and we added a time-like bosonic field \( \phi^0 \) to
complete the superspace (see also [11]).

Together with the supersymmetric extension of \( d\phi^I \) we also introduce a 1-form \( \Omega \), represent-
ing the supersymmetric extension of the chemical shift field-strength \( d\psi \):
\[
\Omega = d\psi + i\bar{\tau} d\theta .
\]
(3.2)
τ being a new Majorana spinor.

The tangent vectors
\[ \partial_a = \frac{\partial}{\partial \phi^a}, \quad D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} (\bar{\theta} \Gamma^a)_\alpha \partial_a \]  
(3.3)

are dual to the supervielbein and the related supersymmetry transformations are:
\[ \delta_{\epsilon_D} \phi^a = \frac{i}{2} \epsilon \Gamma^a \theta, \quad \delta_{\epsilon_D} \theta = \epsilon, \]  
(3.4)

while the supervielbein transforms as\(^4\)
\[ \delta_{\epsilon_D} \Pi^a = i \epsilon \Gamma^a d\theta, \quad \delta_{\epsilon_D} \psi = d\epsilon = 0. \]  
(3.8)

We further assume that the phase \( \psi(x) \) and its supersymmetric partner \( \tau(x) \) are invariant under the rigid supersymmetry generated by the Killing vector \( \bar{c} \). We may, however, extend the chemical shift “internal” symmetry to superspace performing the following superdiffeomorphism on \( \psi \):
\[ \delta \psi = f(\phi, \theta) \]  
(3.9)

with \( f(\phi, \theta) \) arbitrary superfield. Assuming \( \delta \tau = Df \), the 1-form \( \Omega \) acquires the following chemical shift transformation:
\[ \delta \Omega = \frac{\partial f}{\partial \phi^I} \Pi^I. \]  
(3.10)

In order to catch the relevant points of the supersymmetric generalization of the bosonic theory in the simplest way, it is convenient to first restrict ourselves to the \( 1 + 1 \) dimensional case, that will be dealt with in the next subsection, postponing its extension to \( (d + 1) \) space-time dimensions to the following subsection.

### 3.1 Supersymmetric Effective Theory in Two Space-Time Dimensions

In 1+1-dimensions we have just one comoving coordinate \( \phi(x, t) \) and the embedding superspace has two bosonic and one fermionic dimensions\(^5\). Therefore eqs (3.1) and (3.2) imply:

\(^4\)Note that the supervielbein \( \{\Pi^a, \psi^\alpha\} \) are left invariant under the Killing vectors transformations \( \bar{Q}_\alpha \) generators of the supersymmetry algebra, namely
\[ \delta_{\epsilon_Q} \phi^a = -\frac{i}{2} \epsilon \gamma^a \theta, \quad \delta_{\epsilon_Q} \theta = \epsilon \]  
(3.5)
\[ \delta_{\epsilon_Q} \Pi = \delta_{\epsilon_Q} d\theta = 0 \]  
(3.6)

where
\[ Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} \Gamma^a (\bar{\theta} \gamma^a)_\alpha \partial_a \rightarrow \{Q_\alpha, D_\beta\} = 0 \]  
(3.7)

are the fermionic generators of the superPoincaré algebra (anti)commuting with the tangent vectors \( D_\alpha \).

\(^5\)Recall that in 1+1-dimensions the spinors are Majorana-Weyl so that they have just one component.
\[ \Pi = d\phi + \theta d\theta, \quad \Psi = d\theta, \quad \Omega = d\psi + \tau d\theta, \quad (3.11) \]

where we have denoted by \( \phi \) the spatial component of \( \phi^a \) in two dimensions. The supersymmetry transformations are generated by

\[ D = \partial_\theta + \theta \partial_\phi, \quad (3.12) \]

which satisfies \( D^2 = -\partial_\phi \). Setting \( \epsilon D \equiv \tilde{\epsilon} \), the supersymmetry transformations are now:

\[ \delta_\xi \phi = \epsilon \theta, \quad \delta_\xi \theta = \epsilon, \quad (3.13) \]

while the spatial component of the bosonic vielbein transforms as

\[ \delta_\xi \Pi = 2 \epsilon d\theta, \quad \delta_\xi d\theta = d\epsilon = 0, \quad (3.14) \]

since the spinor parameter \( \epsilon \) is constant. Furthermore

\[ \delta_\xi \psi = \delta_\xi \tau = 0. \quad (3.15) \]

In terms of these variables, we can build the following quantities

\[ B = -\Pi \wedge \ast \Pi = \Pi_{\mu \nu} \Pi^\mu \Pi^\nu d^2 x = \hat{B} d^2 x, \quad Y = \Omega \wedge \Pi = e^{\mu \nu} \Omega_{\mu \nu} \Pi^\mu d^2 x = \hat{Y} d^2 x, \quad (3.16) \]

where we denoted by the same letter, though with a hat on the top, the corresponding quantity modulo the volume form, that is \( \hat{B} = \Pi_{\mu \nu} \hat{\eta}^{\mu \nu} \Pi^\nu, \hat{Y} = \Omega_{\mu \nu} \hat{e}^{\mu \nu} \Pi^\mu, b = \sqrt{\hat{B}} \).

The variation of the Poincaré-invariant superfields under a generic variation of \( \phi \) and of \( \psi \) is

\[ \delta B = -2 \Pi \wedge \ast d\delta \phi, \quad \delta Y = -\Pi \wedge d\delta \psi + \Omega \wedge d\delta \phi. \quad (3.17) \]

Therefore, if the action is given by

\[ S = \int d^2 x F[b, Y], \quad (3.18) \]

as an integral of a local functional, we get the equations of motion

\[ d \left[ b^{-1} F_\phi \Pi^\ast + F_\phi \Omega \right] = d^\ast J_\phi = 0, \quad (3.19) \]

\[ d \left[ F_Y \Pi \right] = d^\ast j = 0, \quad (3.20) \]

where \( F_b = \partial F / \partial b, F_Y = \partial F / \partial Y \), and we have defined the 1-forms:

\[ J_\phi = b^{-1} F_\phi \Pi + F_Y \ast \Omega; \quad j = F_Y \ast \Pi, \quad (3.21) \]

which are the Noether currents associated with the shift symmetries \( \phi \to \phi + \epsilon', \psi \to \psi + c \). The variations for \( \theta \) and \( \tau \) are

\[ \delta B = -2 \Pi \wedge \ast (d\delta \theta + \theta d\delta \phi), \quad \delta Y = -\Pi \wedge (\delta \tau d\theta + \tau d\delta \phi) + \Omega \wedge (d\delta \theta + \theta d\delta \phi), \quad (3.22) \]
and the corresponding equations of motion are
\[
2^* J_\phi \wedge d\theta + d\tau \wedge ^* j = 0, \\
^* j \wedge d\theta = 0, 
\] (3.23)

We can also compute the supercurrent and the current associated with the chemical shift symmetry. Recalling that the supercurrent is obtained by variation with respect to \(d\theta\), we find
\[
j_S = -2^* J_\phi + \tau ^* j, 
\] (3.24)

which, by the equations of motion, enjoys the property: \(d^* j_S = 0\).

We now show that the action functional is \textit{invariant under supersymmetry}. Indeed, if we restrict the variation of the action (3.18) to the supersymmetry transformations (3.13), (3.14) and (3.15) we find
\[
\delta \tilde{\epsilon} S = 2 \int [^* J_\phi \wedge d(\epsilon \theta)] 
\] (3.25)

so that, by partial integration and use of the equation of motion, it follows
\[
\delta \tilde{\epsilon} S = 0. 
\] (3.26)

This is of course a major result of our approach.

Concerning the chemical shift symmetry, we consider the possibility of constant symmetry, namely \(f = a + \omega \theta\) where \(a\) and \(\omega\) are commuting and anticommuting parameters of constant type, respectively. Therefore, it is easy by Noether method to compute the following two currents
\[
J^{(a)} = j, \quad J^{(\omega)} = j \theta, 
\] (3.27)

which can be obviously cast into a supermultiplet. Notice that \(d^* J^{(a)} = 0\) as follows from the second equation of (3.19), while \(d^* J^{(\omega)} = 0\) follows from the second equation of (3.23).

### 3.2 The General \(d + 1\) Dimensional Case

In this section we generalize the lagrangian given in the previous section in the \(1 + 1\)-dimensional case to superspace with \((d + 1)\)-space-time dimensions and \(m\) fermionic directions.

We generalize equations (3.16) by defining the following quantities:
\[
B^{IJ} = -\Pi^I \wedge ^* \Pi^J = \Pi^I \Pi^I \mu^d d^{d+1} x = \hat{B}^{IJ} d^{d+1} x, \\
Y = \Omega \wedge \Pi^1 \cdots \wedge \Pi^d = \frac{1}{d!} \epsilon^{\mu_1 \cdots \nu_d} \epsilon_{l_1 \cdots l_d} \Omega_{\mu} \Pi_{\nu_1} \cdots \Pi_{\nu_d} d^{d+1} x = \hat{Y} d^{d+1} x, 
\] (3.28)

where we have denoted by the same letter, though with a hat on the top, the corresponding factor multiplying \((d + 1)\)-dimensional volume form. Note in particular that [6]:
\[
\hat{b} = \sqrt{\det \hat{B}^{IJ}}, 
\] (3.29)
and
\[ \dot{b}_{ij} \bigg|_{\theta=0} = s ; \quad \dot{\gamma} \bigg|_{\theta=0} = y. \]  
(3.30)

where \( b = s \) and \( y = \mu \) are the entropy density and the chemical potential as defined in Section 2.

On the basis of the previous discussion the action generalizing (3.18) will be written as the following integral of a local functional
\[ S = \int d^{d+1}x F[b, \dot{Y}]. \]  
(3.31)

The results of the 1 + 1-dimensional case are then easily generalized as follows. The variation of the Poincaré-invariant superfields under a generic variation of \( \phi^I \) and of \( \psi \) is
\[ \delta B^{IJ} = -2\Pi^{I} \wedge \star d\delta\phi^{J}, \]
\[ \delta Y = d\delta\psi \wedge \Pi^{1} \wedge \ldots \wedge \Pi^{d} + \frac{1}{(d-1)!} \epsilon^{I_{1}\ldots I_{d-1}} \Omega_{\mu_{1}\ldots\mu_{d-1}} \wedge \Pi^{I_{1}} \wedge \ldots \wedge \Pi^{I_{d-1}} \wedge d\delta\phi^{I_{d}}, \]
and the following equations of motion are obtained:
\[ d^* J^{(1)} = d \left[ b F_{b} B_{IJ}^{-1} \Pi^{J} + \frac{1}{(d-1)!} \epsilon^{I_{1}\ldots I_{d-1}} F_{Y} \Omega_{\mu_{1}\ldots\mu_{d-1}} \epsilon^{I_{1}\ldots I_{d-1}} \Pi^{I_{1}} \wedge \ldots \wedge \Pi^{I_{d-1}} \wedge d\phi^{I_{d}}, \right] \]
\[ d^* j^{(1)} = d \left[ F_{Y} \Pi^{1} \wedge \ldots \wedge \Pi^{d} \right] = 0, \]
(3.33)
(3.34)

where \( F_{b} = \partial F / \partial b, F_{Y} = \partial F / \partial Y \) and we have introduced the two currents:
\[ J_{I}^{(1)} = \dot{b} F_{b} B_{I}^{-1} \Pi^{I} + \frac{1}{(d-1)!} \epsilon^{I_{1}\ldots I_{d-1}} F_{Y} \Pi^{I_{1}} \wedge \ldots \wedge \Pi^{I_{d-1}} \Omega_{\mu_{1}\ldots\mu_{d-1}} \epsilon^{I_{1}\ldots I_{d-1}} \Pi^{I_{1}} \wedge \ldots \wedge \Pi^{I_{d-1}} \wedge dx_{\mu}, \]
\[ j^{(1)} = -F_{Y} \epsilon^{\mu_{1}\ldots\mu_{d}} dx_{\mu} \Pi^{1} \wedge \ldots \wedge \Pi^{d} = (-)^{d+1} F_{Y} \star J^{(d)}, \]
(3.35)
(3.36)

where
\[ J^{(d)} = \Pi^{1} \wedge \ldots \wedge \Pi^{d} \equiv \frac{1}{d!} \epsilon_{I_{1}\ldots I_{d}} \Omega_{\mu_{1}\ldots\mu_{d}} \Pi^{I_{1}} \wedge \ldots \wedge \Pi^{I_{d}}. \]

It is straightforward to verify that the (3.35) and (3.36) are the Noether currents associated with the constant translational symmetries \( \phi^I \to \phi^I + c^I \) and \( \psi \to \psi + c \). Furthermore, a general variation of the fermionic fields \( \theta \) and \( \tau \) gives
\[ \delta B^{IJ} = -i\Pi^{I} \wedge \star (\delta \bar{\theta} \Gamma^{J} d\theta + \bar{\theta} \Gamma^{J} d\theta), \]
\[ \delta Y = i (\delta \bar{\tau} d\theta + \bar{\tau} d\delta\theta) \wedge J^{(d)} + \frac{i}{2 (d-1)!} \epsilon_{I_{1}\ldots I_{d-1}} \Omega_{\mu_{1}\ldots\mu_{d-1}} \wedge \Pi^{I_{1}} \wedge \ldots \wedge \Pi^{I_{d-1}} \wedge (\delta \bar{\theta} \Gamma^{I_{d}} d\theta + \bar{\theta} \Gamma^{I_{d}} d\delta\theta), \]
(3.38)

so that the corresponding fermionic equations of motion are
\[ J_{I}^{(1)\mu} \Gamma^{\mu} \partial_{\mu} \theta + \eta_{C} J^{\mu} \partial_{\mu} \tau = 0, \]
(3.39)

where \( \eta_{C} \) is the sign appearing in the relation \( \bar{\tau} d\theta = \eta_{C} \bar{\theta} \tau \) and depends on the property of the charge-conjugation matrix in (\( d + 1) \)-dimensions.

Finally the equation of motion obtained by varying \( \tau \) reads:
\[ J^{(d)} \wedge d\theta = 0. \]
(3.40)
3.2.1 The Energy-Momentum Tensor

In equation (3.29) we have seen that the generalization of the entropy density $b \equiv s$
\[ s(x) = s[\partial_\mu \phi^I] = \sqrt{\det \partial_\mu \phi^I \partial^\mu \phi^J} \]  
(3.41)
to a supersymmetric setting is simply obtained by replacing the purely spatial rigid vielbein $d\phi^I$ with its supersymmetric version $\Pi^I(x, \theta) = d\phi^I + \frac{i}{2} \Gamma^I d\theta$, so that the entropy density superfield is given by
\[ \hat{s}(x) = \hat{s} \left[ \partial_\mu \phi^I(x), \partial_\mu \theta^\alpha(x) \right] = \sqrt{\det \Pi^I \Pi^J}. \]  
(3.42)

In an analogous way we generalize the bosonic variable of Section 2, $y = (J^\mu / s) \partial_\mu \psi$, to
\[ \hat{y} = \frac{J^{(1)\mu}}{\hat{s}} \Omega_\mu = \frac{\hat{Y}}{\hat{s}}, \]  
(3.43)
where $J^{(1)\mu}$ is the natural extension of the current $J^{(1)}$ defined in equations (2.11) and (2.12), namely
\[ J^{(1)\mu} = \frac{1}{d!} \epsilon^{\mu \nu_1 \cdots \nu_d} \epsilon_{\nu_1 \cdots \nu_d} \Pi^{I_1} \cdots \Pi^{I_d}. \]  
(3.44)
the extra index zero being added to comply with the notation of the integral forms given in the next section.

The generalization of the energy-momentum tensor to the supersymmetric theory is then obtained by varying the action functional of the superfields, $\mathcal{S}$, given by (3.31):
\[ \mathcal{S} = \int d^{d+1} x \sqrt{-g} F(\hat{s}, \hat{y}), \]  
(3.45)
with respect to a probe metric $g_{\mu \nu}$. It is straightforward to see that we obtain for the energy-momentum tensor formally the same result as in the non-supersymmetric case, the only difference being the substitution of $b, y$ with their supersymmetric counterparts $\hat{s}(x, \theta), \hat{y}(x, \theta)$:
\[ T_{\mu \nu} = \left( \frac{\partial F}{\partial \hat{y}} - \hat{s} \frac{\partial F}{\partial \hat{s}} \right) u_\mu u_\nu + \eta_{\mu \nu} \left( F - \hat{s} \frac{\partial F}{\partial \hat{s}} \right). \]  
(3.46)

It follows that also the relation between the thermodynamic functions and the superfields $\hat{s}(x, \theta), \hat{y}(x, \theta)$ remains formally the same, namely
\[ p = F(\hat{s}, \hat{y}) - \hat{s}(x, \theta) F_s, \]  
(3.47)
\[ \rho = \hat{y}(x, \theta) \hat{n} - F(\hat{s}, \hat{y}), \]  
(3.48)
where $\hat{n}(x, \theta)$ is the superfield generalizing the particle number density. We see that the pressure, the energy density and the particle number density become superfields whose $\theta = 0$ components give the usual field variables $\rho(x), p(x), n(x)$. The same of course happens for all the other thermodynamical variables.

The generalization of the entropy density given in equation (3.42) implies that the current $J^{(d)}$ defined in equation (3.37) should be the natural extension to the supersymmetric case of the bosonic entropy current since its $\theta = 0$ component coincides with equation (2.12). However, it does not enjoy the important property of being closed, so that it raises several problems of interpretation. For this reason we shall introduce in the next section, using the framework of the integral forms, two alternative definitions of entropy current.

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6Here and in the following we will generally call “entropy current” the $d$-form associated (through Hodge-duality) to the entropy-current 1-form.
4 The Supersymmetric Entropy Current

The formalism of the integral forms allows us to generalize the notion of entropy current to a supersymmetry-invariant (modulo a total derivative), closed form defined on a \((d+1|m)\)-superspace. Such generalization is however not unique. We could consider, for instance, the following supersymmetric extension of the purely bosonic entropy current (2.9):

\[
J^{(d|m)} = \frac{1}{d!} \epsilon_{I_1...I_d} \Pi^{I_1} \wedge ... \wedge \Pi^{I_d} \prod_{\alpha=1}^{m} \delta(d\theta^\alpha) = J^{(bos)} \prod_{\alpha=1}^{m} \delta(d\theta^\alpha),
\]

(4.1)

where \(J^{(bos)}\) coincides with the \(J^{(d)}\) entropy current of the purely bosonic theory, equation (2.11), as we are going to show below.

It has the following properties:

1. It is a purely spatial \(d\)-form on space-time;
2. It is conserved off-shell;
3. Its zero-picture part, at \(\theta = 0\), reduces to the bosonic expression in ordinary hydrodynamics;
4. It is invariant under supersymmetry.

The expression (4.1) is a so called integral form. Its distinctive feature is the presence of the distribution \(\delta(d\theta)\) (see Appendix A.2), defined as the Dirac delta-function of the differential \(d\theta\) (recall that \(d\theta^\alpha\) are commuting quantities) and \(m\) is the dimension of the spinor representation in \((d + 1)\)-dimensions.

To prove the above properties, it is mandatory to recall the main properties of integral forms (see Appendix A.2). First of all \(\delta(d\theta^\alpha)\) enjoys the usual equation \(d\theta^\alpha \delta(d\theta^\alpha) = 0\). This justifies the alternative expression of \(J^{(d|m)}\) in terms of \(J^{(bos)}\) given in equation (4.1). In addition, \(\delta(d\theta^\alpha)\) carries no form-degree and therefore, multiplying it by any number of Dirac delta functions \(\delta(d\theta^\alpha)\), \(\delta(d\theta^\alpha_2)\), ... a \(d\)-form remains a \(d\)-form. However, we can assign a new quantum number \(q\), dubbed picture number, which takes into account the number of the Dirac delta-functions \(\delta(d\theta^\alpha)\). Thus a \(p\)-form of picture \(q\) is denoted by \(\omega^{(p|q)}\). Notice that by using the properties of the Dirac delta-functions it is easy to show that \(\delta(d\theta^\alpha) \wedge \delta(d\theta^\beta) = -\delta(d\theta^\beta) \wedge \delta(d\theta^\alpha)\); as \(d\theta^\alpha\) are commuting quantities, we have instead \(d\theta^\alpha \wedge d\theta^\beta = d\theta^\beta \wedge d\theta^\alpha\). Therefore any integral in superspace of a \(p\)-form \((p \leq d + 1)\) with a given picture-number \(q\) cannot have more than one delta-function of a given differential \(d\theta^\alpha\).

One can also consider any number \(n\) of derivatives of a Dirac delta-function \(\delta^{(n)}(d\theta) = \frac{d^n \delta(d\theta)}{d(d\theta)^n}\), each derivative lowering the degree of the form of one unit. Therefore an additional factor \(\delta^{(n)}(d\theta)\) lowers the form degree of \(n\) units. This implies that we can have forms of negative degree. In particular one can show the identity \(d\theta \delta'(d\theta) = -\delta(d\theta)\) and, in general, \((d\theta)^n \delta^{(n)}(d\theta) = (-)^n \delta(d\theta)\). Finally, as far as integration is concerned, the following formula holds (see eq. (A.12)) :

\[
\int \prod_{\alpha=m}^{1} \theta^\alpha \prod_{\beta=1}^{m} \delta(d\theta^\beta) = 1.
\]

(4.2)
For a more detailed discussion see Appendix A2.

The entropy current (4.1) is exactly invariant under rigid supersymmetry as it can be easily proven as follows:

\[
\delta \epsilon J^{(d|m)} = \frac{1}{(d-1)!} \epsilon_{I_1 \ldots I_d} \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_{d-1}} \wedge \frac{i}{2} \epsilon \Gamma^{I_d} d\theta \bigwedge_{\alpha=1}^{m} \delta (d\theta^\alpha) = 0, \tag{4.3}
\]

where we have used the property \(\delta \epsilon \delta (d\theta^\alpha) = \delta' (d\theta^\alpha) d\epsilon^\alpha = 0\), and the fact that the \(d\theta\) in the variation of \(\Pi^I\) is annihilated by the integral forms. This current is also trivially conserved off-shell.

Although the zero-picture part of \(J^{(d|m)}\) yields the right bosonic entropy current, it would be desirable to relate it to the fluid entropy \(S\) in geometrical way which is intrinsic to the supermanifold: Just as the entropy is the integral of \(J^{(bos)}\) over a spatial hypersurface, it would be natural to express, in a supersymmetric context, the same entropy as the integral over a spatial \((d|m)\) hypersurface \((SS\Sigma)^{(d|m)}_t\) at a given time \(t\), of the corresponding supersymmetric current:

\[
S(t) = \int_{(SS\Sigma)^{(d|m)}_t} J^{(d|m)}. \tag{4.4}
\]

A drawback of the definition (4.1) is however that the integral of \(J^{(d|m)}\) over a superhypersurface does not reduce to the integral over a spatial hyper-surface of \(J^{(bos)}\). In fact such an integral would be zero since:

\[
\int_{(SS\Sigma)^{(d|m)}_t} J^{(d|m)} = \int_{(SS\Sigma)^{(d|m)}_t} J^{(bos)} \bigwedge_{\alpha=1}^{m} \delta (d\theta^\alpha) = \int_{(SS\Sigma)^{(d|m)}} \int_{Berezin} J^{(bos)} = 0, \tag{4.5}
\]

where, for the integration of integral forms, we have used the prescription in [18], summarized in Appendix A.2 (see in particular eq. (A.12).) One could still express the entropy in terms of the integral over the whole superspace of a quantity of the form \(J^{(d|m)} \wedge \ast J^{(d|m)}\), by suitably defining the Hodge-star operation in superspace. Such definition is however subtle and we refrain from dealing with it here.

The generalization of the bosonic entropy current to a closed supersymmetry-invariant (modulo a total derivative) current in superspace is however not unique. Instead of (4.1), we could alternatively define a (super) entropy current in a slightly different fashion as follows:

\[
\mathcal{J}^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_d} \Pi^{I_1} \wedge \cdots \wedge \Pi^{I_{d-1}} \wedge (\theta^\alpha)^m \bigwedge_{\alpha=1}^{m} \delta (d\theta^\alpha), \tag{4.6}
\]

where we have used the short-hand notation:

\[
(\theta)^m \equiv \prod_{\alpha=m}^{1} \theta^\alpha. \tag{4.7}
\]

The quantity \(\mathcal{J}^{(d|m)}\) has the following properties:

1. It is a purely spatial \(d\)-form on space-time and a Grassmann density in superspace;
2. By integration on the fermionic volume element it reduces to the bosonic expression \(J^{(bos)}\).
3. It is conserved off-shell;
4. As it transforms as a total differential under the supersymmetry transformations (3.13), (3.14), its associated conserved charge, the entropy, is invariant under supersymmetry.

In particular the second property allows us to consistently write

$$S(t) = \int_{(S^2)^{(d|m)}} \mathcal{J}^{(d|m)} ,$$

which is the natural relation we were looking for.

To prove the above properties, let us first inspect the simpler, 1 + 1-dimensional case, where all the relevant properties of the entropy current are already at work, and then the general $d+1$ dimensional case.

### 4.1 Two-dimensional case

According to the properties of $\mathcal{J}^{(d|m)}$ listed above, in the 1+1-dimensional case the entropy current must satisfy the following requirements:

1. It is a one-form in the bosonic coordinates and a density in the fermionic sector of superspace.
2. By integration on the fermionic volume element it reduces to the bosonic expression $d\phi$.
3. It is conserved off-shell.
4. It transforms as a total differential under the supersymmetry transformations (3.13), (3.14).

The following expression satisfies the requirements:

$$\mathcal{J} = \Pi \wedge \theta \delta (d\theta) \equiv d\phi \wedge \theta \delta (d\theta) ,$$

where $\delta (d\theta)$ is the Dirac delta function of the differential $d\theta$.

With the definitions introduced above, we can write the entropy current in (4.9) as $\mathcal{J}^{(1|1)}$. By construction the superspace integration of $\mathcal{J}^{(1|1)}$ reduces to the bosonic expression $\mathcal{J}^{(bos)} = d\phi$, so that the second property is satisfied. Moreover we note that $\mathcal{J}^{(1|1)} = \Pi \wedge \theta \delta (d\theta) = d\phi \wedge \theta \delta (d\theta) \equiv \mathcal{J}^{(bos)} \wedge \theta \delta (d\theta)$ where we have used $d\theta \wedge \delta (d\theta) = 0$. It follows

$$d\mathcal{J}^{(1|1)} = d(\mathcal{J}^{(bos)} \wedge \theta \delta (d\theta)) = d (d\phi \wedge \theta \delta (d\theta)) = -d\phi \wedge d\theta \delta (d\theta) = 0 .$$

Finally let us show its property under a supersymmetry transformation:

$$\delta_{\epsilon} \mathcal{J}^{(1|1)} \equiv \ell_{\epsilon} \mathcal{J}^{(1|1)} = (t_\epsilon d + d t_\epsilon) (d\phi \wedge \theta \delta (d\theta)) = d (t_\epsilon \delta \mathcal{J}^{(1|1)})$$

where $\ell_{\epsilon}$ is the Lie derivative in superspace. Equation (4.11) proves the requirement 4.

Next we note that in the case of the bosonic entropy current $J = d\phi$, we can construct an infinite number of currents by multiplying it by any function $f(\phi)$, since then $J_f = \cdots$
\( f(\phi)d\phi \) is clearly closed. It is immediate to verify that the same construction can be extended to the entropy supercurrent. Indeed, introducing an arbitrary superfield \( f(\phi, \theta) \) and defining \( J_f^{(1|1)} = f(\phi, \theta)J^{(1|1)} \), by exterior differentiation it immediately follows:

\[
\begin{align*}
\text{d}J_f^{(1|1)} = \text{d} \left( f(\phi, \theta)J^{(1|1)} \right) = df(\phi, \theta) \wedge J^{(1|1)} = 0. 
\end{align*}
\] (4.12)

However, in our supersymmetric setting, we can still construct another infinite set of conserved currents.

Let us introduce the following expression

\[
\eta^{(0|1)} = (\theta \delta(d\theta) + \Pi \wedge \delta'(d\theta)). 
\] (4.13)

\( \eta^{(0|1)} \) is a \((0|1)\)-form since the first term is a pure Dirac delta function (which carries no form degree) and the second term is made of a 1-form, namely \( \Pi \), and a \((-1)\)-form, namely \( \delta'(d\theta) \). Acting on \( \eta^{(0|1)} \) with the differential \( \text{d} \), we have

\[
\begin{align*}
\text{d}\eta^{(0|1)} &= \text{d} \left( \theta \delta(d\theta) + \Pi \wedge \delta'(d\theta) \right) \\
&= \theta \delta'(d\theta) \wedge \Pi - \Pi \wedge \delta'(d\theta) = 0. 
\end{align*}
\] (4.14)

In addition, we can define a new current

\[
\eta^{(-1|1)} = (\theta \delta'(d\theta) + \Pi \wedge \delta''(d\theta)) . 
\] (4.15)

where \( \delta''(d\theta) \) is the second derivative of the Dirac delta function. The quantity \( \eta^{(-1|1)} \) is a \((-1)\) form because the derivatives on Dirac delta functions count as negative form number. Again, \( d\eta^{(-1|1)} = 0 \) using the properties of delta functions. Proceeding in this way we can define an infinite set of currents of the form

\[
\eta^{(-n|1)} = (\theta \delta^{(n)}(d\theta) + \Pi \wedge \delta^{(n+1)}(d\theta)), 
\] (4.16)

satisfying \( d\eta^{(-n|1)} = 0 \).

### 4.2 Supersymmetric Entropy Current in \( d+1 \) dimensions

As for the action and the equations of motion it is straightforward to generalize the entropy current from two to \((d+1)\)-dimensions.

In the same way as for the two-dimensional case, we show that \( J^{(d|m)} \), as introduced in [16], is off-shell closed. Indeed recalling that \( d\theta^\alpha \delta(d\theta^\alpha) = 0 \), so that

\[
J^{(d|m)} = \frac{1}{d!} \epsilon_{I_1...I_d} d\phi^{I_1} \wedge \cdots \wedge d\phi^{I_d} (\theta)^m \sum_{\alpha=1}^{m} \delta(d\theta^\alpha), 
\] (4.17)

we have, neglecting an overall sign

\[
\begin{align*}
\text{d}J^{(d|m)} &= \frac{1}{d!} \epsilon_{I_1...I_d} d\phi^{I_1} \wedge \cdots \wedge d\phi^{I_d} \sum_{\beta=1}^{m} d\theta^\beta \prod_{\sigma \neq \beta} \theta^\sigma \sum_{\alpha=1}^{m} \delta(d\theta^\alpha) = 0 . 
\end{align*}
\] (4.18)

Therefore the entropy current is a closed form. This in particular implies that the entropy \( S \) is conserved. Indeed, from eq. [13], we have
\[
0 = \int_{\partial(\Sigma)^{(d-1)\text{m}}} d\mathcal{J}^{(d\text{m})} = \int_{\partial(S\Sigma)^{(d\text{m})}} \mathcal{J}^{(d\text{m})} = \int_{\partial(M)^{(d)}} \mathcal{J}^{(\text{bos})} = S(t_2) - S(t_1), \tag{4.19}
\]

where \(\partial(S\Sigma)^{(d\text{m})}\) denotes the boundary of \(S\Sigma^{(d+1)\text{m}}\), represented by two space-like hypersurfaces \((S\Sigma)_1^{(d\text{m})}\), \((S\Sigma)_2^{(d\text{m})}\) at the times \(t_1\), \(t_2\). The integration on the Grassmann volume element \(\prod_{\alpha=1}^m \theta^\alpha \wedge_{\beta=1}^m \delta(d\theta^\beta)\) is performed using the property \([4.2]\). Equation \((4.19)\) thus implies the entropy conservation.

Finally, we consider the transformation law of \(\mathcal{J}^{(d\text{m})}\) under supersymmetry. We have

\[
\delta_{\xi'} \mathcal{J}^{(d\text{m})} \equiv \ell_{\xi'} \mathcal{J}^{(d\text{m})} = (\tau_{\xi'} + d\tau_{\xi'}) \left[ d\phi^{I_1} \wedge \cdots \wedge d\phi^{I_d}(\theta)^m \wedge_{\alpha=1}^m \delta(d\theta^\alpha) \right] \tag{4.20}
\]

Integrating eq. \((4.20)\) over a spatial \((d\text{m})\) hypersurface \((S\Sigma)^{(d\text{m})}\) at a given time \(t\) and using \((4.8), (4.20)\):

\[
\delta_{\xi'} S = \int_{(S\Sigma)^{(d\text{m})}} \delta_{\xi'} \mathcal{J}^{(d\text{m})} = \int_{(S\Sigma)^{(d\text{m})}} d \left( \tau_{\xi'} \mathcal{J}^{(d\text{m})} \right) = \int_{\partial(S\Sigma)^{(d-1)\text{m}}} \tau_{\xi'} \mathcal{J}^{(d\text{m})} = 0, \tag{4.21}
\]

the last integral being zero since the boundary \(\partial(S\Sigma)^{(d-1)\text{m}}\) of \((S\Sigma)^{(d\text{m})}\) is located at spatial infinity, where we assume all fields to vanish together with their derivatives.

Actually, as in the bosonic case, there is an infinity of \(d\text{-form currents which are off-shell closed (and therefore their Hodge-dual are conserved). Indeed if we define

\[
\mathcal{J}_f^{(d\text{m})} = f(\phi^I, \theta) \mathcal{J}^{(d\text{m})}, \tag{4.22}
\]

then

\[
d \left[ f(\phi^I, \theta) \mathcal{J}^{(d\text{m})} \right] = \left[ \frac{\partial f}{\partial \phi^I} d\phi^I + \frac{\partial f}{\partial \theta^\alpha} d\theta^\alpha \right] \mathcal{J}^{(d\text{m})} = 0. \tag{4.23}
\]

Finally, we show that we can define an infinite set of fermionic closed \((-n|1)\)-superforms analogous to those defined in the two-dimensional case (see eqs \((4.14), (4.15), (4.16)\)). They are defined as

\[
\hat{J}^\alpha_{\beta_1 \cdots \beta_n} = \mathbb{P}^{\beta_1 \cdots \beta_n} \hat{J}^\alpha_{\beta_1 \cdots \beta_n}, \tag{4.24}
\]

where \(\mathbb{P}\) is the projector onto the irreducible \(n\)-fold symmetric product of the spinorial representation and

\[
\hat{J}^\alpha_{\beta_1 \cdots \beta_n} = \theta^\alpha \partial_{\beta_1} \cdots \partial_{\beta_n} \prod_{\beta} \delta(d\theta^\beta) + \frac{i}{d} \Pi_I (\Gamma^I C^{-1})^{\alpha\gamma} \partial_{\gamma} \partial_{\beta_1} \cdots \partial_{\beta_n} \prod_{\beta} \delta(d\theta^\beta),
\]

\[
\partial_{\beta} \equiv \frac{\partial}{\partial d\theta^\beta}, \tag{4.25}
\]

where \(C\) is the charge-conjugation matrix. The fermionic currents satisfy the conservation equation

\[
d \left( \mathbb{P}^{\beta_1 \cdots \beta_n} \hat{J}^\alpha_{\beta_1 \cdots \beta_n} \right) = 0. \tag{4.26}
\]

The proof is given in Appendix \([A.4]\).
4.3 Generalized Expression for the Entropy Current

In the present section, we consider a generalized form of the entropy current and its relation with the one given above. This is related to the fact that the integral forms can be seen also from a gauge fixing point of view.

In our case, for the 1+1 dimensional case we can consider the following expressions

\[ P = d\Phi + \Theta d\Theta , \quad \delta(d\Theta) , \quad d\Theta . \] (4.27)

where \( \Phi \) and \( \Theta \) are functions of the coordinates \( \phi, \theta \). Therefore the generalised expression becomes

\[ \tilde{J}^{(1|1)} = P \wedge \Theta \delta(d\Theta) . \] (4.28)

It is straightforward to connect it to the original formula (4.9) by expressing (4.27) in terms of the coordinates \( \phi, \theta \). This can be easily done by observing

\[ \tilde{J}^{(1|1)} = \left[ d\phi \left( \partial_\phi \Phi + \Theta \partial_\theta \Theta \right) + d\theta \left( \partial_\theta \Phi + \Theta \partial_\phi \Theta \right) \right] \Theta \delta \left( d\theta \partial_\theta \Theta + d\phi \partial_\phi \Theta \right) \] (4.29)

Using now \( \Theta = \partial_\theta \Theta \equiv f(\phi) \theta ; \quad \partial_\phi \Theta = \partial_\phi f \theta \)

we obtain

\[ \tilde{J}^{(1|1)} = \left[ d\phi \partial_\phi \Phi + d\theta \partial_\theta \Phi \right] \Theta \left[ \delta(d\theta) + d\phi \frac{1}{\partial_\theta \Theta} \partial_\phi \Theta \partial_\phi f \theta \delta'(d\theta) \right] = d\phi \partial_\phi \Phi \wedge \theta \delta(d\theta) . \]

This proves that the relation between the two formulas is simply the determinant of the (bosonic) Jacobian matrix \( J \) which, in the \( d = 1 \) case, is just \( J = \partial_\phi \Phi \).

The generalization of the above derivation to the \( (d+1) \)-dimensional case is straightforward, though more involved. Consider the following super-reparametrization:

\[ \phi^I , \theta^\alpha \rightarrow \Phi^I(\phi^J, \theta^\beta) , \Theta^\alpha(\phi^J, \theta^\beta) , \] (4.30)

and the corresponding generalized entropy current:

\[ \tilde{J}^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_d} P^{I_1} \wedge \ldots \wedge P^{I_d} \left( \prod_{\beta = m}^1 \Theta^\beta \right) \wedge_{\alpha=1}^m \delta(d\Theta^\alpha) , \] (4.31)

where

\[ P^I = d\Phi^I + \frac{i}{2} \tilde{\Theta} \Gamma^I d\Theta . \] (4.32)

\(^7\) This was the original point of view for introducing the PCO (Picture Changing Operator) which are written in terms of integral form in superstring formulation. To make a long story short, we recall that in the case of superstring the gauge symmetry is a local symmetry plus worldsheet diffeomorphisms and therefore its quantization proceeds by fixing those symmetries by a gauge-fixing-BRST methods. In that process, we have to choose a background metric and a background gravitino. For example, one simple choice is to set the gravitino to zero. However, the corresponding ghost – needed to implement the BRST formalism for that gauge symmetry – is a commuting ghost (usually denoted by \( \beta \)) and the functional integral on it yields the Dirac delta function for the gravitino. Obviously, one can choose a different gauge fixing. [23, 24, 25].
Following the derivation in Appendix A.3 one finds:

\[ \tilde{J}^{(d|m)} = \det(\mathbf{J}) \frac{1}{d!} \epsilon_{I_1 \ldots I_d} \Pi^{I_1} \wedge \ldots \wedge \Pi^{I_d} \left( \prod_{\beta=m}^{1} \theta^{\beta} \right) \bigwedge_{\alpha=1}^{m} \delta(d \theta^\alpha) = \det(\mathbf{J}) \mathcal{J}^{(d|m)} , \]  

(4.33)

where \( \mathbf{J} = (J^I_J) \) is the bosonic block of the super-Jacobian at \( \theta^\alpha = 0 \).

The generalized expression (4.28) (or (4.31)) could be the appropriate form in order to study the supersymmetric entropy current in a fluid/gravity context (see for example [11, 13, 22]). In the bosonic case the entropy density is proportional to the black-hole horizon area. Then the supersymmetric version of the area increase theorem would not change and be still expressed as the statement

\[ \frac{\partial}{\partial \lambda} \det \mathbf{J} \geq 0 , \]  

(4.34)

where \( \lambda \) is the additional bosonic coordinate orthogonal to the super surface whose volume element is the (1|1) integral form (4.27).

Let us check how the supersymmetry transformations act on this generalized expression for the entropy current. Upon a supersymmetry transformation, using eq. (4.33) we find (up to an overall sign):

\[ \delta \tilde{\epsilon} \tilde{J}^{(d|m)} = \delta \tilde{\epsilon} \left( \det(\mathbf{J}(\phi)) \right) \mathcal{J}^{(d|m)} + \det(\mathbf{J}(\phi)) \delta \tilde{\epsilon} \tilde{J}^{(d|m)} = \frac{\partial}{\partial \phi^K} \det(\mathbf{J}) \delta \tilde{\epsilon} \phi^K + \right. \\
+ \det(\mathbf{J}) d \left( \iota_{\tilde{\epsilon}} \mathcal{J}^{(d|m)} \right) = \det(\mathbf{J}) d \left( \iota_{\tilde{\epsilon}} \mathcal{J}^{(d|m)} \right) , \]  

(4.35)

where we have used the property that \( \delta \tilde{\epsilon} \phi^K \propto \bar{\theta} \Gamma^K \epsilon \) is annihilated when multiplied by \( \mathcal{J}^{(d|m)} \) due to the presence of the factor \( \prod_{\alpha} \theta^\alpha \) in the latter. From eq. (4.35) we see that, by supersymmetry, \( \tilde{\mathcal{J}}^{(d|m)} \) still transforms by a total derivative only if the relation between \( \Phi^I \) and \( \phi^I \) is a volume preserving diffeomorphism, i.e. if \( \det(\mathbf{J}) = 1 \).

### 4.4 Entropy Current for Supergravity

The generalization of the entropy current of a fluid to the case of local supersymmetry can be given but is somewhat problematic. First of all, it is not interesting from the point of view of the fluid/gravity correspondence, where the holographic entropy corresponding to the horizon-area of the boosted black-hole solution is instead in correspondence with the entropy of the fluid at the boundary. However, we expect an entropy current in supergravity to exist. It should reduce, in the rigid limit, to the supersymmetric entropy current given in the previous sections and, in analogy to what happens in the rigid case, it should be (Lorentz)-covariantly closed and should transform as a total covariant derivative under local supersymmetry.

Let us work, for the sake of simplicity, in the case of \( N = 1, D=4 \) supergravity coupled to a set of chiral multiplets \( (z^i, \chi^i) \) together with their hermitian conjugates \( (\bar{z}^i, \bar{\chi}^i) \) \((i, \bar{i} = 1, \ldots n_c)\). Let \( \Pi^a \) be the (bosonic) vielbein \((a = 0, 1, 2, 3 \) denote anholonomic space-time indices) and \( \Psi^a \) the gravitino (fermionic) one-form in superspace. In terms of a holonomic basis (spanned by \((d\phi^\mu, d\theta^\alpha)\)), the anholonomic basis of 1-forms in superspace is defined by:

\[ \Pi^a = \Pi^a_{\mu} d\phi^\mu + \Pi^a_\alpha d\theta^\alpha \]  

\[ \Psi^a = \Psi^a_{\mu} d\phi^\mu + \Psi^a_\beta d\theta^\beta . \]  

(4.36, 4.37)
We use the notation $a, b, c, \ldots$ for Lorentz vector indices, $\mu, \nu, \ldots$ for curved vector indices, while Greek indices $\alpha, \beta, \gamma, \ldots$ will denote the spinor-component indices of the gravitino 1-form, running from 1 to 4. Like in the rigid supersymmetry case, $I, J, K, \ldots$ will denote 3-dimensional spatial vector indices.

The supersymmetry transformation laws under a superdiffeomorphism $\theta^\alpha \rightarrow \theta^\alpha + \epsilon^\alpha(\phi, \theta)$ are:

$$\delta_\epsilon \Pi^a = i\bar{\epsilon} \Gamma^a \Psi, \quad (4.38)$$

$$\delta_\epsilon \Psi = -\nabla \epsilon + L_a \Gamma^{ab} \Pi_b + [(\text{Re } S) + i\gamma^5(\text{Im } S)] \Gamma^a \epsilon \Pi_a, \quad (4.39)$$

where $L_a = \frac{i}{2} \chi^i \Gamma^{ab} \chi^j g_{ij}$ is a current of spin-$\frac{1}{2}$ left-handed and right-handed chiral fields $\chi^i, \chi^j$ respectively, $g_{ij}$ is the Kaehler metric of the scalar-fields $\sigma$-model and $S(z^i, \bar{z}^i) \equiv W(z)e^{K}$ is the gravitino mass, $W$ being the superpotential and $K(z^i, \bar{z}^i)$ the Kaehler potential.

The most natural extension to supergravity of eq. (4.6), which has the property of reducing to (4.6) in the rigid limit, is given by the following expression:

$$J_{(3|4)} = \frac{1}{3!} \epsilon_{I_1 \ldots I_4} \Pi^{I_1} \wedge \Pi^{I_2} \wedge \Pi^{I_3} \Pi^{I_4} \prod_{\alpha=1}^4 \theta^\alpha \bigwedge_{\beta=1}^4 \delta(d\theta^\beta). \quad (4.40)$$

Here we used as fermionic coordinates the same set used in rigid superspace (“free falling” frame in the fermionic directions of superspace). To make this expression manifestly covariant in superspace, we can equivalently rewrite it as:

$$J_{(3|4)} = \frac{1}{3!} \epsilon_{I_1 \ldots I_4} \Pi^{I_1} \wedge \Pi^{I_2} \wedge \Pi^{I_3} \prod_{\alpha=1}^4 \xi^\alpha \bigwedge_{\beta=1}^4 \delta(\Psi^\beta) \quad (4.41)$$

where we have introduced a set of spinors $\xi^\alpha$ in superspace, defined by $\xi^\alpha(x, \theta) \equiv \Psi_\beta^\alpha \theta^\beta$ (see eq. (4.37)). Note that $J_{(3|4)}$ is a 3-form with picture-number 4. We now show that it is (covariantly) closed. Indeed

$$\nabla J_{(3|4)} = \frac{1}{4} \epsilon_{I_1 I_2 I_3} \bar{\Psi} \wedge \Gamma^{I_4} \Psi \wedge \Pi^{I_1} \wedge \Pi^{I_2} \wedge \Pi^{I_3} \prod_{\alpha=1}^4 \xi^\alpha \bigwedge_{\beta=1}^4 \delta(\Psi^\beta) +$$

$$+ \frac{2}{3} \epsilon_{I_1 I_2 I_3} \Pi^{I_1} \wedge \Pi^{I_2} \wedge \Pi^{I_3} \prod_{\alpha=1}^4 \xi^\alpha \bigwedge_{\beta=1}^4 \delta(\Psi^\beta) \wedge \nabla \Psi^\beta \bigwedge_{\alpha \neq \beta} \delta(\Psi^\alpha) +$$

$$+ \frac{2}{3} \epsilon_{I_1 I_2 I_3} \Pi^{I_1} \wedge \Pi^{I_2} \wedge \Pi^{I_3} \left[ \sum_{\alpha=1}^4 \nabla \xi^\alpha \prod_{\gamma \neq \alpha=4}^1 \xi^\gamma \bigwedge_{\beta=1}^4 \delta(\Psi^\beta) \right] \quad (4.42)$$

where we used the torsion constraint in superspace, discussed in (A.33) in the Appendix, implying $\nabla \Pi^I = \mathcal{D} \Pi^I = d\Pi^I - \omega_a^I \wedge \Pi^a = \frac{i}{2} \bar{\Psi} \Gamma^I \Psi$, where $\mathcal{D}$ denotes Lorentz-covariant derivative.

Because of the presence of the current $\bar{\Psi} \Gamma^I \Psi$, the first line of eq. (4.42) actually vanishes in force of the identity $\Psi^\alpha \delta(\Psi^\alpha) = 0$. As far as the second and third terms are concerned, to directly show that they sum to zero is a bit involved, however they are easily shown to vanish by using, instead of (4.41), the equivalent expression (4.40) for the entropy current, since in this formulation we can make use of the relation $d\theta^\alpha \delta(d\theta^\alpha) = 0$. 

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In conclusion:

$$\nabla J_{(3|4)} = 0.$$ (4.43)

Moreover, it is easy to show that $J_{(3|4)}$ is invariant under supersymmetry transformations, up to a total covariant derivative. To this purpose, we can use the general property that the supersymmetry transformation is actually a Lie derivative in superspace along the fermionic tangent vectors dual to the gravitini $\Psi$. Moreover, if the Lie derivative acts on Lorentz-covariant forms, say $\omega$, we may replace the ordinary differential with Lorentz-covariant differentials:

$$\delta_\epsilon \omega = \ell_\epsilon \omega + \omega(\tau_\epsilon \omega) = \tau_\epsilon \nabla \omega + \nabla (\tau_\epsilon \omega).$$ (4.44)

We then have:

$$\delta_\epsilon J_{(3|4)} = \tau_\epsilon \nabla J_{(3|4)} + \nabla (\tau_\epsilon J_{(3|4)}) = \nabla (\tau_\epsilon J_{(3|4)}).$$ (4.45)

The above definition (4.41) of the entropy current, which is a direct extension of the definition in rigid superspace (4.6), is not completely satisfying since it does not account for the dynamics of the gravitino fields. An alternative definition, which encodes a non trivial dynamics of superspace, is the one generalizing to local supersymmetry the definition (4.1), that is:

$$J^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 I_2 I_3} \prod_{I=1}^{d} \Pi_1^{I_1} \cdots \prod_{I=1}^{m} \delta (\Psi^\alpha) = J^{(bos)} \prod_{I=1}^{m} \delta (\Psi^\alpha).$$ (4.46)

This expression is covariantly closed (a proof will be given in Appendix A.3). Its physical relation to the bosonic entropy current requires further study. We just observe that the above form is related to its rigid counterpart (4.1) through the super-determinant of the supervielbein matrix

$$J^{(d|m)} = \text{sdet}(E_{MN}) \frac{1}{d!} \epsilon_{I_1 I_2 I_3} d\phi^{I_1} \cdots d\phi^{I_d} \prod_{I=1}^{m} \delta (d\theta^\alpha),$$ (4.47)

where

$$E^M_N = \begin{pmatrix} \Pi^I_j & \Pi^I_\alpha \\ \Psi^\beta_j & \Psi^\beta_\alpha \end{pmatrix}.$$ (4.48)

We refer the reader to Appendix A.3 for a proof of the above statement.

5 Conclusions

In this paper we have developed a supersymmetric extension of the non dissipative fluid dynamics using as a starting point the effective theory approach of [6]. By introducing the supersymmetric partners of the comoving coordinate fields and suitably extending the entropy density and chemical potential fields to superfields, we have shown that the action functional is invariant under supersymmetry transformations. In this setting the bosonic entropy density turns out to be the lowest component of the supersymmetric extension. A drawback of this result is that the natural supersymmetric extension of the current density d-form $J^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 I_2 I_3} \prod_{I=1}^{d} \Pi_1^{I_1} \cdots \prod_{I=1}^{m} \delta (d\theta^\alpha)$ is not closed. We have then
developed an alternative approach to the supersymmetric entropy current based on the formalism of the integral forms. In this new setting the new current is identically closed as it happens in the bosonic case and the Berezin integration in superspace reproduces the bosonic entropy formula. Moreover in this new setting an extension of the entropy current to supergravity seems feasible and we have explicitly constructed it in the case of $\mathcal{N}=1$ supergravity coupled to chiral multiplets.

It would be interesting to see what is the relation, if any, between the two approaches and to explicitly work out the physical implications in both cases. This is left to future investigation.

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A Appendix

A.1 Conventions

We generally use $\mu, \nu, \cdots = 0, 1, \cdots, d$ to denote space-time indices and $\alpha, \beta, \cdots = 1, \cdots, m$ to denote spinor indices. We adopt the “mostly plus” signature of the metric and the following definition of the Hodge dual operation in $D = d + 1$ space-time dimensions:

$$
\star (dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}) = \frac{1}{(D-k)!|g|^{\frac{1}{2}}} \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_D} dx_{\mu_{k+1}} \wedge \cdots \wedge dx_{\mu_D}.
$$

(A.1)

In this convention we have:

$$
\star \star \omega^{(p)} = (-1)^{p(D-p)+1} \omega^{(p)}, \quad \omega^{(p)} \wedge \star \eta^{(p)} = -\sqrt{|g|} \frac{p!}{p!} \omega_{\mu_1 \cdots \mu_p} \eta^{\mu_1 \cdots \mu_p} d^Dx,
$$

(A.2)

where we have used, for a generic $p$–form $\omega^{(p)}$, the following representation:

$$
\omega^{(p)} = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}.
$$

(A.3)

We also use the convention:

$$
dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} = \epsilon^{\mu_1 \cdots \mu_D} d^Dx.
$$

(A.4)

A.2 Properties of the Integral Forms

In this section we briefly recall the definition of “integral forms” and their main properties referring mainly to [15] for a detailed exposition.

The problem is that we can build the space $\Omega^k$ of $k$-superforms out of basic 1-superforms $d\theta^a$ and their wedge products, however these products are necessarily commutative, since the $\theta^a$’s are odd variables. Therefore, together with the differential operator $d$, the spaces $\Omega^k$ form a differential complex

$$
0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} \cdots
$$

(A.5)
which is bounded from below, but not from above.

One simple way to define the “integral forms” is to introduce a new sheaf containing, among other objects to be defined, new basic forms $\delta(d\theta)$. We think of $\delta(d\theta)$ as an operator acting formally on the space of superforms as the usual Dirac’s delta measure. We write this as

$$\langle f(d\theta), \delta(d\theta) \rangle = f(0),$$

where $f$ is a superform. Moreover we consider more general objects such as the derivatives $\delta^{(n)}(d\theta)$. Here we have

$$\langle f(d\theta), \delta^{(n)}(d\theta) \rangle = -\langle f'(d\theta), \delta^{(n-1)}(d\theta) \rangle = (-1)^n f^{(n)}(0),$$

like the usual Dirac $\delta$ measure. Moreover we can consider objects such as $g(d\theta)\delta(d\theta)$, which act by first multiplying by $g$ then applying $\delta(d\theta)$ (in analogy with a measure of type $g(x)\delta(x)$), and so on. The formal properties above imply in addition some simple relations:

$$\delta(d\theta) \wedge \delta(d\theta') = -\delta(d\theta') \wedge \delta(d\theta), \quad d\theta \wedge \delta(d\theta) = 0, \quad d\theta \wedge \delta'(d\theta) = -\delta(d\theta). \quad (A.6)$$

The systematic exposition of these rules can be found in [14]. An interesting consequence of this procedure is the existence of “negative degree” forms, which are those which reduce the degree of forms (e.g. $\delta'(d\theta)$ has degree $-1$).

We introduce also the picture number by counting the number of delta functions (and their derivatives) and we denote by $\Omega^r|_s$ the $r$-forms with picture $s$. For example the integral form

$$dx^{[\mu_1} \wedge \ldots \wedge dx^{\mu_l]} \wedge d\theta^{[\alpha_1} \wedge \ldots \wedge d\theta^{[\alpha r]} \wedge \delta(d\theta^{[\alpha_{r+1}]} \wedge \ldots \wedge \delta(d\theta^{[\alpha_{r+l}]})) \quad (A.7)$$

is an $(r+l)$-form with picture number $s$. All indices $\mu_i$ and $\alpha_{r+1}, \ldots, \alpha_{r+s}$ are antisymmetrized while $\alpha_1, \ldots, \alpha_r$ are symmetrized. Indeed, by also adding derivatives of delta forms $\delta^{(n)}(d\theta)$, even negative form-degree can be considered, e.g. a form of the type:

$$\delta^{(n_1)}(d\theta^{|n_1}) \wedge \ldots \wedge \delta^{(n_s)}(d\theta^{|n_s}) \quad (A.8)$$

is a $-(n_1 + \ldots + n_s)$-form with picture $s$. Clearly $\Omega^k|0$ is just the set $\Omega^k$ of superforms, for $k \geq 0$.

We can formally expand the Dirac delta functions in series

$$\delta \left( d\theta^1 + d\theta^2 \right) = \sum_j \frac{(d\theta^2)^j}{j!} \delta^{(j)}(d\theta^1) \quad (A.9)$$

Recall that any usual superform is a polynomial in the $d\theta$, therefore only a finite number of terms really matter in the above sum, when we apply it to a superform. In fact, applying the formulae above, we have for example,

$$\left\langle (d\theta^1)^k, \sum_j \frac{(d\theta^2)^j}{j!} \delta^{(j)}(d\theta^1) \right\rangle = (-1)^k (d\theta^2)^k \quad (A.10)$$

Notice that this is equivalent to the effect of replacing $d\theta^1$ with $-d\theta^2$. We could have also interchanged the role of $\theta^1$ and $\theta^2$ and the result would be to replace $d\theta^2$ with $-d\theta^1$. Both
procedures correspond precisely to the action we expect when we apply the \(\delta(d\theta^1 + d\theta^2)\) Dirac measure.

The integral forms form a new complex as follows

\[
\ldots \xrightarrow{d} \Omega^{(r|q)} \xrightarrow{d} \Omega^{(r+1|q)} \ldots \xrightarrow{d} \Omega^{(p+1|q)} \xrightarrow{d} 0 \quad (A.11)
\]

where \(\Omega^{(p+1|q)}\) is the top “form” \(dx^{[\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1]} \wedge \delta(d\theta^{[\alpha_1}) \wedge \ldots \wedge \delta(d\theta^{\alpha_q]})\) which can be integrated on the supermanifold. As in the usual commuting geometry, there is an isomorphism between the cohomologies \(H^{(0|0)}\) and \(H^{(p+1|q)}\) on a supermanifold of dimension \((p+1|q)\). In addition, one can define two operations acting on the cohomology groups \(H^{(r|s)}\) which change the picture number \(s\) (see [14]).

Given a function \(f(x, \theta)\) on the superspace, we define its integral by the super top-form \(\omega^{(p+1|q)} = f(x, \theta) d^{p+1} x \delta(d\theta^1) \ldots \delta(d\theta^q)\) belonging to \(\Omega^{(p+1|q)}\) as follows

\[
\int_{\mathbb{R}^{(p+1|q)}} \omega^{(p+1|q)} = \frac{1}{q!} \epsilon^{\alpha_1 \ldots \alpha_q} \partial_{\theta^{\alpha_1}} \ldots \partial_{\theta^{\alpha_q}} \int_{\mathbb{R}^{p+1}} f(x, \theta) d^{p+1} x \quad (A.12)
\]

where the last equality is obtained by integrating on the delta functions and selecting the bosonic top form. The remaining integrals are the usual integral of densities and the Berezin integral. It is easy to show that indeed the measure is invariant under general coordinate changes and the density transform as a Berezinian with the superdeterminant (see Appendix [A.3]). Note that in particular we have

\[
\int \theta^m \ldots \theta^1 \wedge_{\beta=1}^m \delta(d\theta^\beta) = 1. \quad (A.13)
\]

### A.3 Transformation Properties of \(\tilde{J}^{(d|m)}\)

In this Appendix we wish to prove eq. (4.33). To this end it is instructive to start from the slightly more general situation of a current having the form:

\[
\mathbb{J}^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_d} P^{I_1} \ldots P^{I_d} \left( \prod_{\beta=m}^1 \Theta^\beta \right) \wedge_{\alpha=1}^m \delta(\Psi^\alpha), \quad (A.14)
\]

where:

\[
P^I = P^I J \, d\phi^J + P^I_\alpha \, d\theta^\alpha, \quad \Psi^\alpha = \Psi^\alpha_I \, d\phi^I + \Psi^\alpha_\beta \, d\theta^\beta, \quad (A.15)
\]

are even and odd 1-forms, respectively, and \(\Theta^\beta(\phi, \theta)\) are odd superfield 0-forms. Expanding \(\Theta^\alpha\) in products of \(\theta\)'s we find:

\[
\Theta^\beta(\phi, \theta) = \Theta^\alpha_{\beta}(\phi) \, \theta^\beta + \Theta^\alpha_{\beta\gamma}(\phi) \, \theta^\beta \theta^\gamma + \ldots \quad (A.16)
\]

We see that only the first term in the above expansion contributes to the \(m\)-fold product of the \(\Theta\)'s, so that:

\[
\prod_{\beta=1}^m \Theta^\beta = \text{det}(\Theta^\alpha_{\beta}(\phi)) \prod_{\beta=1}^m \theta^\beta. \quad (A.17)
\]
As far as the integral-form part of the current is concerned, let us use the following properties:

\[
\bigwedge_{\alpha=1}^{m} \delta(M^\alpha \beta \, d\theta^\beta) = \frac{1}{\text{det}(M^\alpha \beta)} \bigwedge_{\alpha=1}^{m} \delta(d\theta^\alpha), \tag{A.18}
\]

\[
\delta(d\theta^\alpha + F^\alpha_I \, d\phi^I) = \sum_{k=0}^{m} \frac{1}{k!} \delta^{(k)}(d\theta^\alpha) \left(F^\alpha_I \, d\phi^I\right)^k, \tag{A.19}
\]

where \(\delta^{(k)}\) denotes the \(k\)th-order derivative of the delta-function and \(F^\alpha_I \, d\phi^I\) the \(k\)-fold wedge product of the form \(F^\alpha_I \, d\phi^I\).

Using (A.18) and (A.19), we can rewrite (A.14) as follows:

\[
\mathcal{J}^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_d} P^I_{I_1} \wedge \ldots \wedge P^I_{I_d} \frac{\delta^{(0)}}{\text{det}(\Theta^\alpha_\beta)} \frac{\delta^{(0)}}{\text{det}(\Psi^\alpha_\beta)} \left(\Theta^\alpha_\beta\right)^m \bigwedge_{\alpha=1}^{m} \delta(d\theta^\alpha) \left(F^\alpha_I \, d\phi^I\right)^k, \tag{A.20}
\]

where \(F^\alpha_I \equiv \Psi^{-1}_\alpha_\beta \, \Psi^\beta_J\). Next we use the fact that the presence of \(\prod_{\beta=1}^{m} \theta^\beta\) singles out the order-0 terms in \(\theta\) of all the other factors in the above expression. In particular, being \(P^I_{I_\alpha}\) and \(F^\alpha_I\) odd functions of \(\phi, \theta\), they vanish when multiplied with \(\prod_{\beta=1}^{m} \theta^\beta\), so that:

\[
\mathcal{J}^{(d|m)} = \text{det}(P^{(0)I}_I) \frac{\delta^{(0)}}{\text{det}(\Theta^\alpha_\beta)} \frac{\delta^{(0)}}{\text{det}(\Psi^\alpha_\beta)} \left(\Theta^\alpha_\beta\right)^m \bigwedge_{\alpha=1}^{m} \delta(d\theta^\alpha), \tag{A.21}
\]

where \(P^{(0)I}_I(\phi)\) and \(\Theta^\alpha_\beta(\phi)\), \(\Psi^\alpha_\beta(\phi)\) denote the \(\theta^\alpha = 0\) components of the matrices \(P^{(0)I}_I(\phi, \theta)\) and \(\Theta^\alpha_\beta(\phi, \theta)\).

Consider now the particular case in which \(\Psi^\alpha = d\Theta^\alpha\) and \(P^I\) have the form in (4.32). We have:

\[
P^{(0)I}_I = \left. \frac{\partial \Theta^\alpha}{\partial \phi^I} \right|_{\theta=0} = J^I, \quad \Theta^\alpha_\beta = \left. \frac{\partial \Theta^\alpha}{\partial \theta^\beta} \right|_{\phi=0} = \Theta^\alpha_\beta. \tag{A.22}
\]

The current \(\mathcal{J}^{(d|m)}\) coincides with \(\hat{J}^{(d|m)}\) and we find eq. (4.33).

Inspired by the alternative definition of the entropy current in (4.1) and its supergravity version (4.14), let us repeat the above derivation for the following form:

\[
\mathcal{J}^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_d} P^I_{I_1} \wedge \ldots \wedge P^I_{I_d} \bigwedge_{\alpha=1}^{m} \delta(\Psi^\alpha). \tag{A.23}
\]

Using (A.18) and (A.19), we can write:

\[
\mathcal{J}^{(d|m)} = \frac{1}{d!} \epsilon_{I_1 \ldots I_d} P^I_{I_1} \wedge \ldots \wedge P^I_{I_d} \frac{\delta^{(k)}}{\text{det}(\Psi^\alpha_\beta)} \bigwedge_{\alpha=1}^{m} \sum_{k=0}^{m} \frac{1}{k!} \delta^{(k)}(d\theta^\alpha) \left(F^\alpha_I \, d\phi^I\right)^k. \tag{A.24}
\]

Notice, however, that now all the terms in the sum on the right hand side contribute, and each will select a different term in the expansion of the product of the \(P\)’s. A careful derivation yields:

\[
\mathcal{J}^{(d|m)} = \text{sdet}(E^M_N) \frac{1}{d!} \epsilon_{I_1 \ldots I_d} d\phi^{I_1} \wedge \ldots \wedge d\phi^{I_d} \bigwedge_{\alpha=1}^{m} \delta(d\theta^\alpha). \tag{A.25}
\]
where the the indices $M, N$ run over the spatial and spinor ones $I, \alpha$, the matrix $E^M_N$ is defined as:

$$E^M_N = \begin{pmatrix} P^I_j & P^I_\alpha \\ \Psi^I_j & \Psi^I_\alpha \end{pmatrix},$$  \hspace{1cm} (A.26)

and “sdet” denotes the super-determinant:

$$\text{sdet}(E^M_N) = \frac{1}{\text{det}(\Psi^I_\beta)} \text{det} (P^I_j - P^I_\beta \Psi^{-1\beta}_\alpha \Psi^{\alpha}_j).$$ \hspace{1cm} (A.27)

### A.4 Proof of eq. (4.26)

We wish here to show that the form defined in (4.24) is closed. We start computing the exterior derivative of the current

$$\tilde{J}_{\beta_1 \cdots \beta_n}^a \equiv \theta^\alpha \partial_{\beta_1} \cdots \partial_{\beta_n} \prod_\beta \delta(d\theta^\beta) = \frac{i}{d} \prod_I (\Gamma^I C^{-1})^{\alpha \gamma} \partial_{\beta} \cdots \partial_{\beta_n} \prod_\beta \delta(d\theta^\beta),$$  \hspace{1cm} (A.28)

where $C$ is the charge-conjugation matrix. We shall need to use the properties:

$$d\theta^\alpha \partial_{\beta_1} \cdots \partial_{\beta_n} \prod_\beta \delta(d\theta^\beta) = k \delta^\alpha_{(\beta_1} \partial_{\beta_2} \cdots \partial_{\beta_k)} \prod_\beta \delta(d\theta^\beta),$$

$$d\theta^\alpha d\theta^\beta \partial_{\beta_1} \cdots \partial_{\beta_n} \prod_\beta \delta(d\theta^\beta) = k(k - 1) \delta^{(\alpha \gamma)}_{(\beta_1 \beta_2} \partial_{\beta_3} \cdots \partial_{\beta_k) \prod_\beta \delta(d\theta^\beta).$$ \hspace{1cm} (A.29)

Exterior derivation of (A.28) then yields:

$$d\tilde{J}_{\beta_1 \cdots \beta_n}^a = n \delta^\alpha_{(\beta_1} \partial_{\beta_2} \cdots \partial_{\beta_n)} \prod_\beta \delta(d\theta^\beta) + \frac{i}{d} \frac{i}{2} d\theta^T C \Gamma_I d\theta (\Gamma^I C^{-1})^{\alpha \gamma} \partial_{\gamma} \partial_{\beta_1} \cdots \partial_{\beta_n} \prod_\beta \delta(d\theta^\beta) =$$

$$= n \delta^\alpha_{(\beta_1} \partial_{\beta_2} \cdots \partial_{\beta_n)} \prod_\beta \delta(d\theta^\beta) - \frac{n(n + 1)}{2d} (\Gamma^I C^{-1})^{\alpha \gamma} (C \Gamma_I)(\gamma_{\beta_1} \partial_{\beta_2} \cdots \partial_{\beta_n}) \prod_\beta \delta(d\theta^\beta)$$

$$= n \delta^\alpha_{(\beta_1} \partial_{\beta_2} \cdots \partial_{\beta_n)} \prod_\beta \delta(d\theta^\beta) - \frac{n}{2d} (\Gamma^I C^{-1})^{\alpha \gamma} (2(C \Gamma_I)_{\gamma_{\beta_1} \partial_{\beta_2} \cdots \partial_{\beta_n}} + (n - 1)(C \Gamma_I)_{(\beta_1 \beta_2} \partial_{\beta_3} \cdots \partial_{\beta_n)} ) \prod_\beta \delta(d\theta^\beta) =$$

$$= - \frac{n(n - 1)}{2d} (\Gamma^I C^{-1})^{\alpha \gamma} (C \Gamma_I)_{(\beta_1 \beta_2} \partial_{\beta_3} \cdots \partial_{\beta_n)} \partial_\gamma \prod_\beta \delta(d\theta^\beta),$$ \hspace{1cm} (A.30)

the latter term vanishes upon contraction with $\mathbb{P}$.

### A.5 Covariant Closure of $J^{(3|4)}$

In this Appendix we wish to prove that the current $J^{(3|4)}$ is covariantly closed. The supertorsion $T^a$ and the gravitino field-strength $\rho$ 2-forms in $N = 1$ superspace are defined as follows:

$$T^a = dV^a - \omega^a_b \wedge V^b - \frac{i}{2} \bar{\Psi} \Gamma^a \Psi,$$ \hspace{1cm} (A.31)

$$\rho \equiv \nabla \Psi = d\Psi - \frac{1}{4} \Gamma^{ab} \omega_{ab} \Psi,$$ \hspace{1cm} (A.32)
where $\omega^{ab}$ is the spin connection. Their on-shell parametrization in superspace (superspace constraints) is

$$T^\alpha = 0 \quad (A.33)$$

$$\rho = \rho_{ab} V^a \wedge V^b + L_a \Gamma^a \Psi \wedge V^b + \left((\text{Re} \, S) + i\Gamma^5 (\text{Im} \, S)\right) \Gamma^a \Psi \wedge V_a \quad (A.34)$$

where $\rho_{ab} V^a \wedge V^b$ is the supercovariant gravitino field-strength, $L_a = \frac{i}{2} \chi^i \Gamma^{ab} \chi_i g_{ij}$ is a current of spin-$\frac{1}{2}$ left-handed and right-handed chiral fields $\chi^i, \chi_i$ respectively, $g_{ij}$ is the Kaehler metric of the scalar-fields $\sigma$-model and $S(z^i, z^\dagger) \equiv W(z)e^{\frac{K}{2}}$ is the gravitino mass, $W$ being the superpotential and $K(z^i, z^\dagger)$ the Kaehler potential.

We now show that the entropy current defined in (4.46) is (covariantly) closed. Indeed

$$\nabla J^{(3|4)} = \frac{i}{4} \epsilon I_1 I_2 I_3 V^{I_1} \wedge V^{I_2} \wedge V^{I_3} \wedge \left[ \sum_{\beta=1}^{4} \delta (\Psi^\beta) \wedge \nabla \Psi^\beta \wedge \delta (\Psi^\alpha) \right]$$

where we used the constraint (A.33), implying $dV^I - \omega^I_a \wedge V^a = \frac{1}{2} \bar{\Psi} \Gamma^I \Psi$.

Because of the presence of the current $\bar{\Psi} \Gamma^I \Psi$, the first line of eq. (A.35) actually vanishes in force of the identity $\Psi^\alpha \delta (\Psi^\alpha) = 0$. As far as the second term is concerned, we observe that substituting to $\nabla \Psi^\beta = \rho^\beta$ the right hand side of equation (A.34), we get four kinds of contributions: the terms with five vierbeine identically vanish in four dimensions; as far as the contribution of the other three terms is concerned, they contain the products $\Gamma_5 \Gamma^{ab} \Psi \wedge V_a, \Gamma^a \Psi \wedge V_a$ and $\Gamma_5 \Gamma^a \Psi \wedge V_a$. All of them, however, give a vanishing contribution owing to the traceless property of the $\Gamma$-matrix algebra. Take for example the term

$$-i(\text{Im} \, S) \Gamma^a \Gamma^5 \Psi \wedge V_a.$$  

We have:

$$-i(\text{Im} \, S) \epsilon I_1 ... I_3 \nabla V^{I_1} \wedge V^{I_2} \wedge V^{I_3} \wedge \left[ \sum_{\beta=1}^{4} \delta (\Psi^\beta) \wedge (\Gamma^a \Gamma^5)^\gamma_{\beta} \Psi^\gamma \wedge \delta (\Psi^\alpha) \right]$$

$$= i(\text{Im} \, S) \epsilon I_1 ... I_3 \nabla V^{I_1} \wedge V^{I_2} \wedge V^{I_3} \wedge \left[ \sum_{\beta=1}^{4} (\Gamma^a \Gamma^5)^\beta_{\gamma} \wedge \sum_{\alpha \neq \beta}^{4} \delta (\Psi^\alpha) \right] = 0 \quad (A.36)$$

where we used the property $\delta (\Psi^\gamma) \Psi^\gamma = -\delta (\Psi^\gamma)$.

By the same argument one finds that also the terms proportional to $\Gamma^a \Psi$ and $\Gamma^5 \Gamma^{ab} \Psi$ give vanishing contribution since $Tr (\Gamma^5 \Gamma^{ab}) = Tr (\Gamma^a) = 0$.

In conclusion:

$$\nabla J^{(3|4)} = 0 \quad (A.37)$$

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\textsuperscript{8}Note that in this formalism $\rho_{ab} V^a \wedge V^b$ is the supercovariant gravitino field-strength while $\rho_{\mu\nu}$ is the ordinary field-strength $\nabla_{[\mu} \Psi_{\nu]}$.  

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