Casimir effects in monatomically thin insulators polarizable perpendicularly: nonretarded approximation

Gabriel Barton
Department of Physics and Astronomy, University of Sussex, Brighton BN1 9QH, UK
E-mail: g.barton@sussex.ac.uk

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Abstract. A flat monatomically thin insulating sheet is modelled initially as a square lattice and then as an amorphous distribution of harmonic oscillators, polarizable only perpendicularly to the sheet. In an approximation neglecting dissipation and retardation, we calculate the polarizability $X(\omega, \mathbf{k})$ per unit area as a function of the frequency $\omega$ and surface-parallel wave-vector $\mathbf{k}$ of an externally applied electric field. To find the underlying so-called local fields one must first replace the familiar three-dimensional Lorenz–Lorentz accounts of dielectric functions with their well-established and very different two-dimensional analogues. Image fields are given by weighted integrals over $X(0, \mathbf{k})$; the poles of $X(\omega, \mathbf{k})$ identify the normal-mode frequencies $\tilde{\omega}(\mathbf{k})$. The Hamiltonian version of the theory is quantized via the normal modes; from it we determine the van der Waals interaction of the sheet with a nearby atom, and between two dynamically identical parallel sheets.
1. Introduction

1.1. Background and motivation

In theories of Casimir effects, reflecting layers of nominally infinitesimal thickness play at least three different roles. (i) For a spherical shell taken from the start as perfectly reflecting, Boyer (1968) found that the Casimir stress on it is directed outward, a paradoxical conclusion that has occasioned a vast literature. (ii) Regarding many of their electric properties, giant carbon molecules like $C_{60}$ can be modelled quite well as monatomically thin plasma sheets (Barton and Eberlein 1991), with parameters fitted to a single base plane in graphite as discussed by Fetter (1973, 1974). On extending this model to couple such two-dimensional (2D) plasmas to the quantized Maxwell field, one finds that the expression reported by Boyer is a mathematically fascinating but physically far-subdominant part of the energy of spherical plasma shells (Barton 2004a, 2004b, 2004c), as is its analogue for indefinitely extended plane plasma sheets (Barton 2005a, 2005b). (iii) Very different treatments of 2D electron gases via Dirac-like equations underlie the fast-burgeoning theories of graphene layers (see e.g. Katsnelson 2012), which at low frequencies behave in ways totally unlike the 2D plasmas considered under (ii). Latterly, attention has been focussed also on their van der Waals and Casimir interactions with perfect and imperfect metals (references in section 5.3).

Here, however, we shall be concerned with a different generalization of thin-sheet Casimir physics, from 2D plasmas mimicking conductors to 2D insulators, where the central problem is how to take proper account of the local-field effects which dominate their electromagnetic response functions. One prompt for trying to spell out the elements of the theory is a recent paper (Parashar et al 2012, cited as PMSS; see also Milton et al 2013), whose conclusions...
depend in part on an assertion that sheets polarizable only perpendicularly leave electromagnetic fields wholly unaffected. On examining simple oscillator models of monatomically thin sheets it emerges that this is false: our main purpose is to study the polarizability of such sheets, and some of the physical properties that the polarizability governs.

We shall consider only polarization perpendicular to the sheet, because the theory of parallel polarization, though it features more intricate local fields, is not basically different from the theory of plasma sheets: one merely adds a tangential restoring force to the equation of motion of the charged fluid modelling the plasma. Technically speaking, the fields due to parallel polarization can be ascribed equally well to the surface-charge density produced by its surface divergence, just as for a plasma; whereas perpendicular polarization is unique in producing, as we shall see, discontinuities in the fields but no charge densities whatever.

The sheet is taken to be flat and to extend indefinitely in the \((x, y)\) plane; it is modelled as monatomically thin, and its constituent atoms as dynamically identical linear simple-harmonic oscillators perpendicular to the sheet. The oscillator frequency, internal mass, momentum, coordinate and Hamiltonian are \(\Omega, m, p, \zeta\) and \(h_0 = p^2/2m + m\Omega^2\zeta^2/2\). The atomic (i.e. the individual oscillator) polarizability\(^2\) is

\[
\Pi(\omega) = \alpha/(1 - \omega^2/\Omega^2), \quad \alpha = e^2/m\Omega^2,
\]

and each oscillator is coupled to electric fields only in the dipole approximation: \(h_{\text{int}} = -e\zeta E_z\).

Recall that for atoms (unlike oscillators) \(\alpha\) is generally of the order of \((\text{atomic radius})^3\). The scenario where \(\Omega \to \infty\) at fixed finite \(\alpha\), i.e. where \(\Pi(\omega) = \alpha\) for all \(\omega\), we call the nondispersive limit.

Dissipation is disregarded; so is retardation\(^3\), which will be taken into account in a future paper, cited here as II. Conventionally, the nonretarded approximation is identified as the formal limit \(c \to \infty\); here it applies provided all physically pertinent distances are far below \(c/\Omega\), and also far below \(c/\omega\) for all important frequencies \(\omega\). We expect, and shall find, that the model makes sense only subject to certain stability conditions of the kind familiar in systems of coupled oscillators. For instance, the normal modes of just two such dipole–dipole-coupled oscillators a distance \(a\) apart have squared frequencies \(\Omega^2(1 + \alpha/a^3)\), indicating instability unless \(a > \alpha^{1/3}\). As regards the response functions and normal modes of the sheet, our basic reasoning is classical, except insofar as it assumes that stable atoms with given \(\Pi(\omega)\) exist.

We shall start by determining the response of the sheet to an externally applied field

\[
E_{\text{app}}(s, z = 0) = \tilde{2}E_{\text{app}}(k) \exp(-i\omega t + ik \cdot s), \quad s = (x, y);
\]

at this stage \(\omega\) and \(k\) are merely mathematical parameters chosen independently, regardless of how (2) might be realized in practice. Hats denote unit vectors. Generally we shall suppress the factor \(\exp(-i\omega t)\) common to all the fields in question. When \(\omega^2/\Omega^2\) is very close to 1, i.e. near resonance, one must, in the denominator of \(\Pi\), allow for the natural width \(\Gamma = \alpha\Omega^4/3c^3\) of the oscillator by replacing \(\omega \to \omega - i\Gamma/2\). As an evidently relativistic correction this is, in

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1. The conditions for matching the fields across 2D plasmas are the same whether the sheets are flat or curved; but for insulators the generalization from flats to say spherical shells is a problem in its own right, whose solution is by no means simple.
2. We use unrationalized Gaussian units: \(D = \varepsilon E = E + 4\pi P\) and \(\nabla \cdot E = 4\pi \times \text{(charge density)}\).
3. For nondispersive and parallel-polarizable sheets Fosco et al (2012) formulate a relativistic theory, which it would be instructive to compare with ours if it were extended to perpendicular polarizability.
principle, beyond the scope of the nonretarded approximation: it will feature in II, but here we disregard it, excluding near-resonance behaviour\(^4\) from our remit.

For reference we quote the potential \(\psi\) and the field \(\mathbf{e} = -\nabla \psi\) at \((s, z)\) due to a polarized atom modelled as a point dipole \(\mathbf{\hat{z}} q_0\) placed at the origin:

\[
\psi(s, z) = \frac{q_0 z}{\sqrt{s^2 + z^2}^{3/2}}, \quad \mathbf{e}(s, z) = \frac{q_0}{\sqrt{s^2 + z^2}^{3/2}} \{\mathbf{\hat{s}}(3zs) + \mathbf{\hat{z}}(2z^2 - s^2)\}.
\]

The part of \(\mathbf{e}\) proportional to \(\mathbf{\hat{z}} q_0 \delta(s) \delta(z)\) has been dropped: it is ineffective because it cannot contribute to the interaction between the atoms constituting the sheet, and because its source cannot polarize itself.

The calculations will demonstrate that physically reasonable models of minimally thin sheets can indeed be polarized in this way. The contrary assertions by PMSS are invalidated by a hidden inadequacy of their initial Ansatz (2a), which for our scenario and in our units would assign to them a dielectric function \(\varepsilon = 1 + 4\pi \lambda e^2 \mathbf{\hat{z}}^2 \delta(z)\). One can see the inadequacy by taking \(\delta(z) = 0\) outside and \(\delta(z) = 1/d\) inside a sheet of thickness \(d\) (as do PMSS), and calculating the polarization \(\mathbf{\hat{z}} P\) produced by a finite externally applied field \(\mathbf{\hat{z}} E_{\text{app}}\). Then the total perpendicular dipole moment per unit area is \(Q = \mathbf{\hat{z}} P = E_{\text{app}} \lambda e^2 d / (d + 4\pi \lambda e^2)\), whence \(\lim_{d \to 0} Q = 0\). In fact it is well known that the dielectric responses of effectively 2D bodies cannot be described by means of dielectric functions of anything like the kind assigned to three-dimensional (3D) ones. For instance, in 2D the familiar Lorenz–Lorentz (Clausius–Mossotti) theories of 3D local fields must be re-designed root and branch: the more recent history of this and of related problems\(^5\) can be sampled say through the work of Philpott and Sherman (1975), Lee and Bagchi (1980), Christiansen et al (1998) and Ryazanov and Tishchenko (2006).

\[\begin{align*}
1.2. \text{Preview} \\
\end{align*}\]

Section 2 calculates the polarizability \(X(k, \omega)\) of a simple square lattice. The prime object is to introduce 2D local fields and the associated stability criteria in a setting and for a model that are long-established, and unproblematic as long as the atomic diameter is well below the lattice spacing \(a\). The second equation of (7) gives the result in terms of the function \(f(k)\) defined over the first Brillouin zone by (8). The poles of \(X\) identify the normal-mode frequencies \(\tilde{\omega}(k)\) via (11).

Section 3.1 defines our model of an amorphous sheet. It takes the atoms to be distributed at random, except that no two can come closer than a crucial minimum distance \(b\). The polarizability \(X(k, \omega)\) is given by (25) in terms of the strength parameter \(\mu = \pi na/b\), and of the function \(L(k = kb)\) defined and then approximated in (21)–(23). A plausibility argument anticipates a Debye cutoff \(k < k_D\), supported in section 3.4 by a count of normal modes. The squared eigenfrequencies are given by (39). Meanwhile, section 3.2 explicates how, once \(X\) is known, the expressions for potential and field off the sheet can be matched across it. Section 3.3 determines the potential (36) due to the polarization produced by a point charge \(e\) at a distance \(\xi\) from the sheet; remarkably, when \(\xi \gg b\) the charge experiences the same force (38) as it would from a point dipole at the image position \(-\xi\).

\[\begin{align*}
4 \quad \text{Near resonance the local-field effects which will presently be seen to govern the polarizability of the sheet become overwhelming, and can produce very peculiar patterns of the kind explored by Volkov and Kaplan (2010). For the theory of the atomic polarizability in general see Loudon and Barnett (2006) and Berman et al (2006).} \\
5 \quad \text{Some further discussion, and a more leisurely review, are contained in two unpublished papers by the present writer (Barton 2007, 2008 respectively).}
\end{align*}\]
Section 4 uses the polarizability and the normal modes from section 3 to reformulate the theory in canonical terms: the Hamiltonian $H_0$ of the sheet is identified in section 4.1 (which summarizes itself), and quantized in section 4.2. Equation (55) expresses $H_0$ in terms of the familiar normal-mode creation and annihilation operators (54), linked to the primary canonical operators $q(k), p(k)$ through (52) and (53). Applications require $H_0$, plus the operator $\psi(s, z)$ for the potential off the sheet as expressed in terms of the $q(k)$ by (56).

Section 5 illustrates applications. Section 5.1 calculates the van der Waals potential $U(z)$, equation (58), between the sheet and a ground-state atom at a distance $z$. When $z \gg b$ this reduces to the relatively simple form (59), falling proportionally to $1/z^3$. In the nondispersive limit it reduces further, and somewhat surprisingly, to the quasi-classical expression $U_{\text{nd}} = -\mu b [0 \cdot |d|^2] / 4 [1 + 2\mu] z^4$, where $d$ is the $z$-component of the atomic dipole-moment operator, and $[0 \cdot |d|^2] / 0$ its mean-square value in the ground-state. Section 5.2 finds the electrostatic contribution $\beta(\mu, \kappa_D)$ to the cohesive energy of the sheet per unit area: $\beta$ is positive (repulsive) for small $\kappa_D$, but turns negative for the values of $\mu$ and $\kappa_D$ most likely to be of interest. Finally, section 5.3 considers two identical sheets separated by a distance $L = b\lambda$. The normal-mode frequencies $\tilde{\omega}(\kappa, \lambda)$ are given by (68), and the exact van der Waals potential $u(\mu, \kappa_D, \lambda)$ per unit area by (69). For $\lambda \gg 1$ it reduces to $u \simeq -3(\pi/32)\hbar\Omega n^2\alpha^2 / (1 + 2\mu)^{3/2} L^4$, as indicated by (70).

Finally, section 6 summarizes the main results, and outlines some points needing to be kept in mind when considering their implications.

2. Square lattice

We study a square lattice with lattice constant $a$, so that $n \equiv (\text{number of atoms}) / (\text{unit area}) = 1/a^2$. It is acted on by an applied field $\hat{z} \tilde{E}_{\text{app}}(k) \exp(i \mathbf{k} \cdot \mathbf{s})$, producing (dipole moment)/(unit area) = $\hat{z} \tilde{Q}(s)$. Defining polarizability/(unit area) $\equiv \chi'$, one has\(^6\)

$$Q(s) = \hat{z} \tilde{Q}(k) \exp(i \mathbf{k} \cdot \mathbf{s}), \quad \tilde{Q}(k) = \chi'(k) \tilde{E}_{\text{app}}(k).$$

(4)

Our object is to determine $\chi'$.

The local field experienced by an atom at $s$ is $\hat{z} E_{\text{eff}}(s) = \hat{z} [E_{\text{app}}(s) + E_{\text{self}}(s)]$, with

$$E_{\text{self}}(k) = G(k) \tilde{Q}(k)$$

(5)

the Fourier transform of the field due to all the other atoms. Thus

$$\tilde{Q} = \chi' \tilde{E}_{\text{app}} = n \Pi \tilde{E}_{\text{eff}} = n \Pi (\tilde{E}_{\text{app}} + G \tilde{Q}) \quad \Rightarrow \quad \chi' = \frac{n \Pi}{1 - n \Pi G}.$$  

(6)

It must be kept in mind that $E_{\text{self}}(s)$ is defined only on the sheet, and has physical significance only at positions occupied by an atom: it is not the field $E_{\text{pol}}(s, z) = -\nabla \psi_{\text{pol}}(s, z)$ perceived elsewhere as due to the polarization of the sheet. In particular, the next section will spell out that the response functions governing $\psi_{\text{pol}}$ cannot be read directly from $G$ or from its analogues, but must be determined in their own right.

Meanwhile, the central problem is to find $G$. Define

$$\kappa_1 = k_1 a, \quad \kappa_2 = k_2 a, \quad -\pi \leq (\kappa_1, \kappa_2) < \pi, \quad \kappa = (\kappa_1, \kappa_2).$$

\(^6\) Strictly speaking, the dipole approximation is reliable only if $a \gg 2\pi/k$ and $a \gg a_0$, where $a_0 = \sqrt{\hbar/m\Omega}$ is the oscillator range-parameter (or, if the oscillators are place-holders for atoms, the atomic radius).
It proves convenient to introduce a dimensionless function $f(\kappa)$ such that
\[ G = -f(\kappa)/a, \quad X = n\Pi/[1 + (\Pi/a^3) f(\kappa)], \]
where the prime excludes (only) $v_1 = 0 = v_2$.

Define
\[ \zeta(3/2) \equiv \sum_{\nu=1}^{\infty} \frac{1}{\nu^{3/2}} \simeq 2.612, \quad \beta(3/2) \equiv \sum_{\nu=0}^{\infty} (-1)^\nu/(2\nu + 1)^{3/2} \simeq 0.864; \]
then
\[ Z \equiv \sum_{v_1,v_2=-\infty}^{\infty} \frac{1}{(v_1^2 + v_2^2)^{3/2}} = 4\zeta(3/2)\beta(3/2) \simeq 9.034. \]

The mathematics associated with $Z$ and with the function $f$ has a distinguished history dating back at least as far as Hardy (1919)\(^7\); the challenge is to express slowly converging double sums in terms of far more easily manageable ordinary sums. For the physics in a modern context see Ambjorn and Wolfram (1983), and the appendix to Christiansen et al (1998); for mathematical sophistication and extensive references see McPhedran et al (2004, 2007).

For long waves ($\kappa \ll 1$)
\[ f = Z - 2\pi\kappa - C\kappa^2 + \cdots, \quad C \simeq 0.974. \]

Thus at the zone centre $f(0,0) = Z$. At the edge-centres and at the corners, asymptotics supplemented with some numerics yield $f(\pi,0) \simeq -0.935$ and $f(\pi,\pi) \simeq -2.646$. Figure 1(b) from Christiansen et al (1998) indicates that $f$ is negative all along the edges, minimal at the corners and zero on just one closed locus surrounding the centre.

Normal modes have nonzero $Q$ in absence of an applied field, whence they are signalled by the poles of $X$. Thus the dispersion relation for their frequencies $\bar{\omega}$ reads
\[ 1 + \frac{n\Pi}{a} f(\kappa) = 0 : \quad \frac{n\alpha}{a} f(\kappa) = -1 + \frac{\bar{\omega}^2}{\Omega^2}, \quad \frac{\bar{\omega}^2}{\Omega^2} = 1 + \frac{n\alpha}{a} f(\kappa). \]

For long waves, to first order in $ak = \kappa$,
\[ \frac{\bar{\omega}^2}{\Omega^2} = \left[ 1 + \frac{n\alpha Z}{a} \right] - 2\pi n\kappa + O(\kappa^2). \]

Remarkably, the frequency drops with rising wave-number. The minimal value of $f$ at the zone corners shows that
\[ (\bar{\omega}^2/\Omega^2)_{\text{min}} \simeq 1 - \alpha(1.38/a)^3. \]

If $a < 1.38\alpha^{1/3}$, the sheet would be unstable, presumably in the sense of polarizing spontaneously\(^8\); compare this with the instability condition $a < \alpha^{1/3}$ we have noted for just two atoms.

\[^7\] A paper the present writer cannot recover.

\[^8\] The magnitude of such spontaneous polarization is controlled by nonlinearities excluded from our Ansatz (1) for the atomic polarizability. About phase transitions we are emphatically silent.
3. Amorphous sheet

3.1. Generalities

Our model of such a sheet takes the atoms to be distributed over the plane at random, with (mean number)/(unit area) = \( n \), and makes a rudimentary allowance for pair correlations by imposing a minimum interatomic distance \( b \). Accordingly, the field experienced by each atom, called the local field \( E_{\text{loc}} \), is the applied field plus the field \( E_{\text{self}} = \hat{z}E_{\text{self}} \) due to the polarized sheet treated as continuous, but with a circular hole of radius \( b \) centred on the atom in question. (The main reason why hydrodynamic models of plasma sheets are so much easier to handle than insulators is precisely that their electrons are delocalized, whence the effective fields they experience are not rapidly varying, and are indeed the same as the macroscopic fields perceived outside.) As to orders of magnitude, one expects

\[
b \sim \text{lattice spacings} \gtrsim (\text{Bohr radius}), \quad n \sim 1/b^2, \quad \alpha \lesssim b^3. \tag{14}\]

We consider applied fields (2); then

\[
\tilde{E}_{\text{loc}} = \tilde{E}_{\text{app}} + \tilde{E}_{\text{self}}, \tag{15}\]

and the equation of state reads

\[
\tilde{Q} = n\Pi(\tilde{E}_{\text{app}} + \tilde{E}_{\text{self}}) = X(\mathbf{k})\tilde{E}_{\text{app}}, \tag{16}\]

with \( X(\mathbf{k}) \) a polarizability per unit area. Next, on the same pattern as for the lattice, we define a function \( G(\mathbf{k}) \) such that

\[
\tilde{E}_{\text{self}} = G\tilde{Q} = G X(\mathbf{k})\tilde{E}_{\text{app}}, \quad X = n\Pi/(1 - n\Pi G). \tag{17}\]

To determine \( G \), we combine the \( z \)-components of the fields at \((\mathbf{s}, 0)\) due to the dipoles \( \hat{z}\tilde{Q} \exp(\mathbf{i}\mathbf{k} \cdot \mathbf{s})d^2s' \) at \((\mathbf{s}', 0)\):

\[
E_{\text{self}}(\mathbf{s}, z = 0) = G\tilde{Q}(\mathbf{k})\exp(\mathbf{i}\mathbf{k} \cdot \mathbf{s}), \quad G = -\int_b^{\infty} ds' \int_{-\pi}^{\pi} d\phi' \exp(\mathbf{i}ks' \cos \phi')/s'^3. \tag{18}\]

On scaling \( ks' = x \) and redefining

\[
\kappa \equiv kb, \tag{19}\]

we have

\[
G = - (\pi/b) \mathcal{L}(\kappa), \tag{20}\]

\[
\mathcal{L} = 2\kappa \int_x^{\infty} dx J_0(x)/x^2 = \{-2\kappa + 2(\kappa^2 + 1)J_0(\kappa) - 2\kappa J_1(\kappa) - \pi \kappa^2[J_0(\kappa)H_1(\kappa) - J_1(\kappa)H_0(\kappa)]}, \tag{21}\]

where the \( H_i \) are Struve functions. Asymptotically

\[
\mathcal{L}(\kappa \ll 1) = 2 - 2\kappa + \kappa^2/2 + O(\kappa^4), \tag{22}\]

\[
\mathcal{L}(\kappa \gg 1) = \sqrt{\frac{2}{\pi}} \left\{ \cos(\kappa + \pi/4) \left[ \frac{2}{\kappa^{3/2}} - \frac{1185}{64\kappa^{7/2}} + \cdots \right] + \sin(\kappa + \pi/4) \left[ \frac{21}{4\kappa^{5/2}} - \frac{42735}{512\kappa^{9/2}} + \cdots \right] \right\}. \tag{23}\]
Figure 1. Curve A: $\mathcal{L}(\kappa)$, equation (21); along the horizontal axis, read $x = \kappa$. Curve B: $\Phi_1(\kappa_D)$, equation (63); along the horizontal axis, read $x = \kappa_D$.

The function $\mathcal{L}(\kappa)$ is plotted in figure 1: it turns negative at $\kappa_0 \simeq 1.63$, with a first (and absolute) minimum $\mathcal{L}_{\text{min}}(\kappa_m \simeq 2.6) \simeq -0.24$.

It will prove convenient to introduce a dimensionless coupling constant

$$\mu \equiv \pi n\alpha/b; \quad (24)$$

then the polarizability reads

$$X = n\alpha/[1 - \omega^2/\Omega^2 + \mu\mathcal{L}]. \quad (25)$$

Plausibly, $\mu$ attains its maximal value $M$ when $\alpha = b^3/8$ is the static polarizability of a perfectly conducting sphere with radius $b/2$. For orientation, square and hexagonal lattices (labelled sq and hx respectively) with nearest-neighbour distance $b$ have

$$n_{\text{sq}} = 1/b^2, \quad M_{\text{sq}} = \pi/8 = 0.392; \quad n_{\text{hx}} = 2/\sqrt{3}b^2, \quad M_{\text{hx}} = \pi/4\sqrt{3} = 0.453. \quad (26)$$

Roughly speaking, $\mu$ encodes the strength of the response of an element of the sheet to the field it experiences, while $\mathcal{L}$ encodes the effect on this response of variation parallel to the sheet.

As to $\mathcal{G}$, we repeat the caution voiced in the preceding section just below equation (6), and complement it by spelling out the physically pertinent potential $\psi_{\text{pol}}$ off the sheet. We do this from first principles, i.e. directly from Coulomb’s law, in order to dispel any (though misconceived) reservations about admitting delta-singularities \textit{ab initio}. Thus, in virtue of the first equation of (3), and then changing the integration variable to $\sigma = s' - s$,

$$Q(s) = \tilde{Q}(k) \exp(ik \cdot \sigma) \Rightarrow \psi_{\text{pol}}(s, z) = \int d^2s' \tilde{Q}(k) \exp(ik \cdot s') \frac{z}{[(s - s')^2 + z^2]^{3/2}},$$

$$\psi_{\text{pol}}(s, z) = \tilde{Q}(k) \exp(ik \cdot s) \int d^2\sigma \frac{z \exp(ik \cdot s')}{[\sigma^2 + z^2]^{3/2}}$$

$$= \tilde{Q}(k) \exp(ik \cdot s) \int_0^{\infty} d\sigma \frac{z J_0(k\sigma)}{[\sigma^2 + z^2]^{3/2}} = \tilde{Q}(k) \exp(ik \cdot s) 2\pi \frac{z \exp(-k|z|)}{|z|}. \quad (27)$$
Accordingly
\[ \psi_{\text{pol}}(s, z) = \text{sign}(z)2\pi \tilde{Q}(k) \exp(i k \cdot s - k |z|), \]  
(28)

\[ \mathbf{E}_{\text{pol}}(s, z) = -\nabla \psi_{\text{pol}} = -\hat{z} \delta(z)4\pi \tilde{Q}(k) + 2\pi k \tilde{Q}(k) \exp(i k \cdot s - k |z|) \{-\text{sign}(z)i \hat{k} + \hat{z} \}. \]  
(29)

Finally, in view of the minimum separation \( b \), and anticipating the Debye cutoff presently to be motivated for normal modes, we assume that the polarization cannot support Fourier components \( \tilde{Q}(k) \) with arbitrarily large \( k \); in other words we impose the condition
\[ k \leq k_D \Rightarrow \kappa < \kappa_D \equiv k_D b, \]  
(30)

expecting \( \kappa_D \) to be of order unity. Correspondingly, all integrands under \( \int d^2 k \ldots \) are understood to carry a step-function factor \( \theta(k_D - k) \) or \( \theta(\kappa_D - \kappa) \), which will not be shown explicitly. Improvement on these prescriptions would require microscopic study of the insulating sheets in question: meanwhile, the Debye cutoffs are best regarded as part of the definition of our crude but hopefully serviceable model.

### 3.2. Matching conditions

In one sense, the delta-function part of (29) is just a place holder: off the sheet one can simply discard it, while a microscopic description on the sheet requires the full theory featuring \( E_{\text{self}} \) and \( \mathcal{L} \). Nevertheless the equation of state governing the physics everywhere else can be encapsulated into the standard macroscopic conditions for matching potential \( \psi \) and total electric field \( \mathbf{E} = -\nabla \psi \) across the sheet. On defining discontinuities by
\[ \text{disc } [\psi] \equiv \lim_{z \to 0+} \psi(z) - \lim_{z \to 0-} \psi(z), \]  
(31)

and similarly for other fields, the matching conditions read
\[ \text{disc } [E_3] = 0, \quad \text{disc } [\psi] = \text{disc } [\psi_{\text{pol}}] = 4\pi Q \Rightarrow \]  
(32)

\[ \text{disc } [\tilde{E}_3] = \text{disc } [\tilde{E}_{\text{pol}}] = -\text{disc } [\nabla \| \tilde{\psi}_{\text{pol}}] = -4\pi \nabla \| \tilde{Q} = -4\pi i k X \tilde{E}_{\text{app}3}. \]  
(33)

Though (32) follows with perfect generality from Gauss’s law, here we have derived (33) only via the electrostatic representation \( \mathbf{E} = -\nabla \psi \), unwarranted beyond the nonretarded regime. However, in fact it continues to apply subject to Maxwell’s equations, which entail also that \( E_3 \) is the only Maxwell-field component having a \( \delta(z) \) singularity on the surface. The proof is straightforward but tedious, and is given in II.

Finally it may bear repeating that in view of (32) and of \( \text{disc } [\psi] = - \int_{0-}^{0+} dz E_3 \), nonzero polarization \( Q \) automatically entails a singularity \( 4\pi Q(s)\delta(z) \) in \( E_3 \); the connection is explicated text-book fashion say by Barton (1989).

### 3.3. Image potential

To illustrate the uses of the theory in section 3.1, we calculate the image potential and the image force generated by a point charge \( e \) fixed to the right of the sheet at \((0, \zeta)\). (Since this is a zero-frequency scenario, the result remains unaffected by retardation: one need merely envisage the Coulomb gauge, and note that photons do not then couple to \( e \).)
We start from the applied potential
\[ \psi_{\text{app}}(s, z; \zeta) = \frac{e}{[s + (z - \zeta)]^{1/2}} = \int d^2 k \tilde{\psi}_{\text{app}}(k) \exp(ik \cdot s), \tag{34} \]
\[ \tilde{\psi}_{\text{app}}(k) = (e/2\pi) \exp(-k |z - \zeta|)/k. \tag{35} \]
The \( z \)-component of the applied field experienced by the sheet is
\[ E_{\text{app3}}(s) = \int d^2 k \tilde{E}_{\text{app3}}(k) \exp(ik \cdot s), \quad \tilde{E}_{\text{app3}}(k) = (e/2\pi) \exp(-k \zeta). \]
Since it is time-independent one has \( \omega = 0 \), whence
\[ X = n\alpha/[1 + \mu \mathcal{L}(\zeta)] \Rightarrow \tilde{Q}(k) = [n\alpha/(1 + \mu \mathcal{L})](e/2\pi) \exp(-k \zeta); \]
substitution into (28) and integration over the polar angle then lead to
\[ \psi_{\text{pol}}(s, z; \zeta) = \text{sign}(z)2\pi e \int_0^{k_0} dk \left[ \frac{n\alpha}{1 + \mu \mathcal{L}} \right] J_0(ks) \exp[-k(\zeta + |z|)]. \tag{36} \]
The integral is awkward because of the factor \([ ... ]\) and of the finite upper limit. However, if at least one of \( \zeta, s, |z| \) is much larger than \( b \), then \( \psi_{\text{pol}} \) is dominated by contributions from small \( k \), and can be approximated by replacing \( \mathcal{L}(kb) \to \mathcal{L}(0) = 2 \), and extending \( \int_0^{k_0} dk \to \int_0^{\infty} dk \ldots \). This yields
\[ \psi_{\text{pol}}(s, z; \zeta) \approx \text{sign}(z)e^{2\pi n\alpha} \frac{1}{1 + 2\mu} \int_0^{\infty} dkk J_0(ks) \exp[-k(\zeta + |z|)] \]
\[ = \text{sign}(z) \left[ \frac{2\mu be}{1 + 2\mu} \right] \frac{\zeta + |z|}{[s + (\zeta + |z|)]^{3/2}}. \tag{37} \]
Remarkably, on the right of the sheet this is the potential due to a point dipole \( 2\mu be/(1 + 2\mu) \) at the image position, a distance \( \zeta \) behind the sheet; on the left of the sheet, it is the potential due to a point dipole \( -2\mu be/(1 + 2\mu) \) coinciding with the point charge.

The force on the point charge is
\[ f_{\text{pol}} = -e \left( \hat{z} \frac{\partial \psi_{\text{pol}}}{\partial z} \right)(s = 0, z = \zeta) \approx -\hat{z} \left[ \frac{\mu be^2}{1 + 2\mu} \right] \frac{1}{\zeta^3}; \tag{38} \]
the exact expression follows trivially on adapting (36), and the approximation from (37).

It might seem paradoxical that \( \psi_{\text{pol}} \) fails to vanish as \( \zeta \to 0 \), when the charge sits on the sheet, and cannot therefore polarize it in the first place. In fact \( \psi_{\text{pol}}(s, z; 0) \) is zero while \( \lim_{\zeta \to 0 \pm} \psi_{\text{pol}}(s, z; \zeta) \) are not: the approach to the limit is nonuniform, with the zero sitting at the point of discontinuity between peaks of opposite signs to its left and right.

3.4. Normal modes

The normal-mode frequencies \( \tilde{\omega}(k) \) are signalled by the poles of \( X \), whence the dispersion relation reads
\[ \tilde{\omega}^2/\Omega^2 = 1 + \mu \mathcal{L}, \tag{39} \]
allowing the response function (25) to be re-expressed as
\[ X(\omega, \kappa) = n\alpha \Omega^2/[(\tilde{\omega}(\kappa) - \omega)^2]. \tag{40} \]
For long waves, to first order in $\kappa = kb$
\[
\bar{\omega}^2 / \Omega^2 = 1 + 2 \mu \left(1 - kb + O(kb)^2\right) \simeq [1 + 2\mu] - [2\mu] \kappa.
\] (41)

Here too $\bar{\omega} / \Omega$ drops as $k$ rises\(^9\). For comparison, the analogous equation (12) for the square lattice can be written as $(\bar{\omega}^2 / \Omega^2)_{sq} \simeq [1 + (n\alpha/a)Z] - [2\mu] ak$.

To ensure that normal modes do not outnumber the pertinent degrees of freedom of the atoms, one must impose a Debye cutoff
\[ k < k_D, \quad \kappa < \kappa_D, \quad \pi k_D^2 / (2\pi)^2 = n \sim O(1/b^2). \] (42)

For our two simple lattices
\[ \kappa_{Dsq} = 2\sqrt{\pi} = 3.54, \quad \kappa_{Dhx} = \sqrt{8\pi/\sqrt{3}} = 3.81, \] (43)

where the $\kappa_D$ are the values prescribed by (42) for an amorphous sheet, given the surface densities $n$ in (26). By (39) the sheet remains stable if $\mu |L_{min}| < 1$, as is likely since $|L_{min}| \simeq 0.24$ while (26) suggests $\mu \lesssim M \lesssim 1$. Otherwise translation invariance would presumably be broken by a standing wave of spontaneous antiferroelectric polarization\(^10\), with the lowest wave-number for which $1 + \mu L$ vanishes.

Finally, we anticipate retardation corrections to the extent of noting that our surface-bound normal modes are strictly stable only if $\bar{\omega}(k) < ck$; otherwise they decay by emitting radiation with frequency $\bar{\omega}(k)$ and wave-vector $\mathbf{K} \equiv \langle \mathbf{k}, \hat{\mathbf{k}}_3 \rangle$, where $\hat{\mathbf{k}}_3 \equiv \sqrt{\bar{\omega}^2(k)/c^2 - \mathbf{k}^2}$. Maxwellian theory turns these modes into narrow resonances of the TM reflection and transmission amplitudes. Plasma sheets behave differently, because, without a mechanical restoring force, i.e. lacking a finite frequency parameter like $\Omega$, resonances are replaced by threshold singularities at zero frequency.

4. Canonical theory

We continue to disregard relativistic and retardation effects, and thus the quantized Maxwell field, which will be considered in paper II. Meanwhile, the interaction energy $dH_{ext}$ of a surface element with an externally generated applied field, and the electrostatic interaction energy $dH_{es}$ of two surface elements with each other, are
\[ dH_{ext} = -d^2s Q(s) E_{app,z}(s), \quad d^2 H_{es} = d^2s d^2 s' Q(s) Q(s') / |s - s'|^3. \] (44)

4.1. Hamiltonian

The canonical 2D fields are $p(s)$, having dimensions $[MT^{-1}]$, and dimensionless $q(s)$ normed so that $Q(s) = \varphi q(s)$; the coefficient $\varphi$ will be identified presently. Then the Hamiltonian for a single isolated sheet is
\[ H_0 = \int d^2s \left\{ \frac{1}{2m} p^2 + \frac{1}{2}m \Omega^2 q^2 \right\} + \frac{1}{2} \varphi^2 \int \int d^2s d^2 s' \frac{\theta (|s - s'| - b) q(s) q(s')}{|s - s'|^3}, \] (45)

\(^9\) By contrast, $\omega$ rises with $k$ both for the (TM) bound mode on a plasma sheet (readily calculable from Barton 2005a, equations (B6, B5)), and for the (purely relativistic) TE mode on a graphene sheet (Bordag 2012).

\(^10\) Antiferroelectric polarization is the only kind in question here, because two dipoles located sideways—on attract (only) if they are antiparallel. By contrast, plasma sheets are immune to instability.

New Journal of Physics 15 (2013) 063028 (http://www.njp.org/)
where $\theta$ is the Heaviside step-function, and the last term is the Coulomb self-energy of the sheet. We fit $H_0$ to our model from section 3, i.e. to $n$ oscillators per unit area, each with internal displacement $\zeta$, charge $e$, and static polarizability $\alpha = e^2/m\Omega^2$, by equating their internal (nonelectrostatic) potential energies. This shows that $(m\Omega^2 q^2/2 = nm\Omega^2 \zeta^2/2) \Rightarrow (q = \sqrt{n}\zeta)$; then $Q = ne\zeta = \varphi q = \varphi \sqrt{n}\zeta$ yields $\varphi = \sqrt{n}e$, whence

$$\varphi^2 = ne^2 = n\alpha m\Omega^2.$$  \hspace{1cm} (46)

Hamilton’s equations read

$$\dot{q}(s) = \frac{\delta H}{\delta p(s)} = \frac{p(s)}{m},$$ \hspace{1cm} (47)

$$\dot{p}(s) = -\frac{\delta H}{\delta q(s)} = -m\Omega^2 q(s) + \varphi^2 \int d^2s', \theta(|s - s'| - b) \frac{q(s')}{|s - s'|^3}.$$ \hspace{1cm} (48)

Define Fourier transforms\(^\text{11}\)

$$\begin{bmatrix} \hat{p}(s) \\ \hat{q}(s) \end{bmatrix} = \int d^2k \begin{bmatrix} \hat{p}(k) \exp(-ik \cdot s) \\ \hat{q}(k) \exp(ik \cdot s) \end{bmatrix}, \quad \begin{bmatrix} \hat{p}(-k) \\ \hat{q}(-k) \end{bmatrix} = \begin{bmatrix} \hat{p}^*(k) \\ \hat{q}^*(k) \end{bmatrix},$$ \hspace{1cm} (49)

where classically the superscript $+$ indicates the complex and quantally the Hermitian conjugate. It is straightforward to verify that

$$\frac{1}{(2\pi)^2} \hat{H} = \int d^2k \left\{ \frac{\hbar}{2m} \hat{p}^* \hat{p} + \frac{1}{2} m\Omega^2 + \varphi^2 \frac{\pi}{b} \mathcal{L}(kb) \right\} \hat{q}^* \hat{q}.$$ \hspace{1cm} (50)

with $\mathcal{L}$ from (21)--(23). Accordingly, the Fourier representation diagonalizes $H_0$, as by translation invariance it must; and it identifies the normal-mode frequencies $\tilde{\omega}$ through

$$\tilde{\omega}^2(k) = \Omega^2 + \varphi^2 \frac{\pi}{mb} \mathcal{L}(kb) = \Omega^2 \left[ 1 + \mu \mathcal{L}(kb) \right],$$ \hspace{1cm} (51)

tallying with (39).

4.2. Quantization

The theory is quantized by imposing the equal-time commutation rules

$$[q(s), p(s')] = i\hbar \delta(s - s') \Rightarrow [q(k), p(k')] = -\frac{i\hbar}{(2\pi)^2} \delta(k - k').$$ \hspace{1cm} (52)

They are satisfied by introducing annihilation and creation operators in standard fashion:

$$\hat{q}(k) = \frac{1}{2\pi} \sqrt{\frac{\hbar}{2m\omega}}(a(k) + a^\dagger(-k)), \quad \hat{p}(k) = \frac{i}{2\pi} \sqrt{\frac{\hbar m\omega}{2}}(-a(-k) + a^\dagger(k)), \hspace{1cm} (53)$$

$$[a(k), a(k')] = 0, \quad [a(k), a^\dagger(k')] = \delta(k - k').$$ \hspace{1cm} (54)

Then

$$H_0 = \int d^2k \hbar \tilde{\omega}(k) \frac{1}{2} \{a^\dagger(k) a(k) + a(k) a^\dagger(k)\}.$$ \hspace{1cm} (55)

\(^\text{11}\) In principle $q(k), p(k)$ and $a(k)$ below should carry tildes, which it is now safe to drop in order to ease the notation.

New Journal of Physics 15 (2013) 063028 (http://www.njp.org/)
Recall that the potential $\psi$ generated by $Q$ outside the sheet is odd in $z$, whence $E_q = -\nabla \psi$ is odd and $E_z = -\partial \psi / \partial z$ is even:

$$\psi(s, z) = \varphi z \frac{\int d^2 s' q(s')}{(s-s')^2 + z^2} = \text{sign}(z) 2\pi \varphi \int d^2 k q(k) \exp(i k \cdot s - k |z|),$$

and

$$E(s, z) = 2\pi \varphi \int d^2 k q(k) \exp(i k \cdot s - k |z|)(-i \dot{\kappa} \text{sign}(z) + \dot{z}).$$

### 5. Applications

#### 5.1. van der Waals interaction between the sheet and an atom

We consider the van der Waals interaction energy $U(z)$ between the sheet and an isotropic atom in its ground state, fixed at a distance $z$. Write the atomic energy eigenstates and eigenvalues as $|j\rangle$ and $\epsilon_j$, ground state $j = 0$, with $\epsilon_{j0} = \epsilon_j - \epsilon_0$; the atomic dipole-moment operator as $d$; and, by hindsight, define

$$D_{j0}^2 = |\langle j| d_s|0\rangle|^2 + |\langle j| d_{||}|0\rangle|^2 / 2, \quad \sum_j D_{j0}^2 = (2/3) (0|d^2|0).$$

The interaction Hamiltonian is $H_{\text{int}} = -E d$, with $E$ from (57). In view of the first equation of (53) and of (46), and in an obvious shorthand, second-order perturbation theory yields

$$U = -\sum_j \int d^2 k \frac{|\langle j| k| H_{\text{int}}|0\rangle|^2}{[\epsilon_{j0} + \hbar \omega(k)]} = -\pi n \alpha \hbar \Omega^2 \sum_j \int_0^{k_0} dk \frac{k^3 \exp(-2kz) D_{j0}^2}{\omega(k)[\epsilon_{j0} + \hbar \omega(k)]}. \quad (58)$$

The integral is messy, on account of its finite upper limit and of the awkward variation of $\omega$ via $L(kb)$. We evaluate it only for $z \gg b$: then it is dominated by $kz \ll 1$, so that $\omega(k) \simeq \omega(0) = \Omega \sqrt{1 + 2\mu}$ and $\int_0^{k_0} \omega(k) \sim \int_0^{k_0} dk \ldots \simeq \int_0^{\infty} dk \ldots$. Hence

$$U(z \gg b) \simeq \frac{\pi n \alpha \hbar \Omega}{\sqrt{1 + 2\mu}} \int_0^{\infty} dk k^3 \exp(-2kz) \sum_j \frac{D_{j0}^2}{[\epsilon_{j0} + \hbar \Omega \sqrt{1 + 2\mu}]},$$

$$= -\frac{3\pi}{8} \frac{n \alpha \hbar \Omega}{z^4 \sqrt{1 + 2\mu}} \sum_j \frac{D_{j0}^2}{[\epsilon_{j0} + \hbar \Omega \sqrt{1 + 2\mu}]}. \quad (59)$$

This may be contrasted with the long-distance but still nonretarded van der Waals potential of an atom near a plasma sheet, which falls proportionally to $1/z^3$.

In the nondispersive limit $\hbar \omega(k) = \hbar \Omega \sqrt{1 + \mu L(k)} \rightarrow \infty$ one has $[\epsilon_{j0} + \hbar \omega(k)] \rightarrow \hbar \omega(k)$. Then closure over the atomic states reduces (58), paradoxically, to the prima facie classical expression

$$\frac{\hbar \Omega}{\epsilon_{j0}} \gg 1: \quad U \simeq -\frac{\pi n \alpha}{3} \frac{2}{|0|d^2|0\rangle \int_0^{k_0} dk k^3 \exp(-2kz) \frac{[1 + \mu L(kb)]}{[1 + \mu L(kb)]} \rightarrow \frac{\mu b(0)d^2|0\rangle}{4 [1 + 2\mu] z^3}. \quad (60)$$

---

12 The writer is indebted to Claudia Eberlein for spotting an error in the first version of this calculation.
5.2. Cohesive energy

The electrostatic contribution to the ground-state cohesive energy per unit area is

\[
\beta = \int \frac{d^2 k}{(2\pi)^2} \frac{\hbar (\tilde{\omega}(k) - \Omega)}{2} = \frac{\hbar \Omega}{4\pi b^2} F, \\
F(\mu, \kappa_D) = \int_0^{\kappa_D} d\kappa \kappa \left\{ \sqrt{1 + \mu \mathcal{L}(\kappa)} - 1 \right\}:
\]

(61)

the second term within the braces subtracts the energy that the oscillators would have if they were infinitely far apart. For stable systems the radicand is positive. Since \( \mathcal{L} \) can change sign, the sign of \( F \) is a matter for calculation.

For weak coupling we approximate by keeping only the first term in the Taylor series for \( \{ \ldots \} \):

\[
F \simeq F_w = \frac{1}{2} \mu \int_0^{\kappa_D} d\kappa \kappa \mathcal{L}(\kappa) = \mu \Phi(\kappa_D),
\]

(62)

\[
\Phi \equiv \int_0^{\kappa_D} d\kappa \kappa^2 \int_0^\infty dx \frac{J_0(x)}{x^2} = \frac{1}{8} \kappa_D J_1(\kappa_D) + \frac{1}{6} \kappa_D^2 \mathcal{L}(\kappa_D).
\]

(63)

Figure 1 plots \( \Phi \) alongside \( \mathcal{L} \): it is clear from (62) that the extrema of \( \Phi \) coincide with the zeros of \( \mathcal{L} \).

For our two lattices, we take \( \kappa_D \) from (43), and then, for orientation, evaluate \( F \) with \( \mu \to M \) from (26). This gives

\[
\Phi_{sq} = -0.176, \quad F_{wsq} = -0.0692; \quad \Phi_{hx} = -0.238, \quad F_{whx} = -0.108.
\]

(64)

For strong coupling, the integral in the second equation of (61) must be evaluated numerically. For instance,

\[
F_{sq} = -0.0817, \quad F_{hx} = -0.125,
\]

(65)

not all that different from the somewhat cavalier estimates (64).

5.3. van der Waals interaction between two sheets

Call the sheets 1 and 2, a distance \( L \) apart. Call the normal-mode frequencies \( \tilde{\omega}(k, L) \), so that \( \tilde{\omega}(k) = \tilde{\omega}(k, \infty) \). From the second equation of (44), their interaction Hamiltonian may be written as

\[
W = -\int d^2 s Q^{(2)}(s) E_z^{(1)}(s, L)
\]

\[
= -\left(2\pi\right)^2 \varphi^2 \int d^2 k \exp(-kL) \frac{1}{2} \left\{ q^{(1)+}(k)q^{(2)}(k) + q^{(2)+}(k)q^{(1)}(k) \right\}.
\]

(66)

Combining \( W \) with the \( q^*q \) part of \( H_0^{(1)} + H_0^{(2)} \) as given by (50, 51), we see that the \( \tilde{\omega}^2(k, L) \), are the eigenvalues of the potential-energy matrix \( \ldots \) in

\[
[q^{(1)+}, q^{(2)+}] \frac{1}{2} m \begin{bmatrix}
\tilde{\omega}^2(k), & -\left(2\pi \varphi^2 k/m\right) \exp(-kL) \\
\left(-2\pi \varphi^2 k/m\right) \exp(-kL), & \tilde{\omega}^2(k)
\end{bmatrix}
\]

Thus\(^\text{13}\)

\[
\tilde{\omega}_\pm(k, L) = \left[ \tilde{\omega}^2(k) \pm \left(2\pi \varphi^2 k/m\right) \exp(-kL) \right]^{1/2},
\]

(67)

\(^\text{13}\) The methods from section 3 can of course derive (67) without reference to Hamiltonians: one starts by identifying the applied field experienced by each sheet as the polarization field due to the other.
conveniently scaled to
\[ \lambda \equiv L/b : \quad \bar{\omega}_\pm(\kappa, \lambda) / \Omega = [1 + \mu L(\kappa) \pm 2\mu \kappa \exp(-\lambda \kappa)]^{1/2}. \] (68)

The subscripts specify the sign in (68), which is the same as the parity of \( q \) with respect to the median plane.

Because of the minus sign in the radicand of \( \bar{\omega}_-(\kappa, \lambda) \), at small \( \lambda \) the two-sheet system is at greater risk from instability than a single sheet. In fact the limit \( \lambda \to 0 \) of (68) appears to make no physical sense. For instance, one might plausibly expect that at zero separation the two sheets, each with surface number-density \( n \), merge into a single sheet with surface density \( 2n \); but then \( \bar{\omega}_-(\kappa, L = 0; n) \) would reduce to \( \bar{\omega}(\kappa; 2n) \) (as happens for plasma sheets), whereas comparing (68) with (39) one sees that it does not. Nor is it clear whether taking \( \lambda < 1 \) is compatible with the minimum dipole–dipole distance \( b \) assumed for single sheets. In a preliminary exploration it seems premature to pursue such questions at length. Here we merely observe that with the values of \( \mu \) and \( \kappa_D \) for square or for hexagonal lattices, though \( \lambda = 0 \) would admit instability, \( \lambda \geq 1 \) does not; fudge the issue by imposing the condition \( \lambda > 1 \); and continue to assume stability without further comment.

The van der Waals potential energy per unit area is
\[ u = \int \frac{d^2k}{(2\pi)^2} \frac{\hbar}{2} \{ \bar{\omega}_+(k, L) + \bar{\omega}_-(k, L) - 2\bar{\omega}(k) \} \]
\[ = \left( \frac{\hbar \Omega}{4\pi b^2} \right) \int_0^{\kappa_D} d\kappa \kappa \left[ [1 + \mu L(\kappa) + 2\mu \kappa \exp(-\lambda \kappa)]^{1/2} \right. \]
\[ + \left. [1 + \mu L(\kappa) - 2\mu \kappa \exp(-\lambda \kappa)]^{1/2} - 2[1 + \mu L(\kappa)]^{1/2} \right]. \] (69)

This integral too is messy, for much the same reasons as the one in (58), and becomes reasonably transparent only if \( \lambda \gg 1 \). Then it is dominated by small \( \kappa \), whence one can replace \( L(\kappa) \to L(0) = 2 \), expand \( \ldots \) by powers of \( \exp(-\lambda \kappa) \), keep only the first noncancelling terms, and extend \( \int_0^{\kappa_D} d\kappa \ldots \to \int_0^\infty d\kappa \ldots : \)
\[ u (\lambda \gg 1) \simeq -\frac{\hbar \Omega}{4\pi b^2 (1 + 2\mu)^{3/2}} \int_0^\infty d\kappa \kappa^3 \exp(-2\kappa \lambda) = -\frac{3\pi}{32} \frac{\hbar \Omega n^2 \alpha^2}{L^4 [1 + 2\mu]^{3/2}}. \] (70)

This may be compared with \( u \sim 1/L^{5/2} \) for two plasma sheets (Bordag 2006)\(^{14} \), and with \( u \sim 1/L^3 \) for two graphene sheets (Drosdoff et al 2012, Klimchitskaya and Mostepanenko 2013).

6. Summary

To resume, our main results are as follows. Recall that they apply in the van der Waals regime, disregarding retardation from the start.

(i) From monatomically thin sheets polarizable only along their normal, externally applied fields evoke detectable responses of the same kind as they do from macroscopic parallel-sided slabs. (ii) However, their polarizability must be calculated using ideas and methods specific to 2D systems: in particular, to such sheets the familiar 3D concepts of dielectric functions and Lorenz–Lorentz relations do not apply, and provide no sort of handle (cf point (xi) below). (iii) The classic example of the simple-square 2D lattice in the point-dipole approximation is readily

\(^{14} E_{TM} \) in the equation between his (41) and (42).
adapted to a rough model of an amorphous sheet, which yields results in almost closed form. (iv) It is of the essence that the polarizability exhibits spatial dispersion, i.e. that it is a function not only of the frequency $\omega$ (via the underlying atomic polarizability), but also of the wave-vector $\mathbf{k}$ parallel to the sheet. It determines (v) the image field evoked by an external charge; (vi) the frequencies of the normal modes; and, after quantization, (vii) the electrostatic contribution to the cohesive energy; plus (viii) the van der Waals attraction between a sheet and a nearby atom, and between two parallel sheets.

In considering the implications, one needs to exercise some circumspection. (ix) Our object has been to try and clarify some traditional and quite basic questions about the Casimir-type physics of minimally thin surfaces, rather than to derive accurate expressions for realistic systems. Specifically, the dipole approximation, while it minimizes technical complications, is unlikely to prove wholly adequate at the densities in question (except perhaps for some very dilute adsorbed monolayers): at separations of no more than a few atomic units, (a) higher multipoles become competitive, and (b) the wave-functions of electrons in different atoms may overlap non-negligibly. For a critique on such lines, see e.g. Kempa et al (2005).

(x) Real sheets admit parallel as well as perpendicular polarization. The former can be either curl-free or divergence free, making three types of normal mode for given $\mathbf{k}$, and the effects (v)–(viii) listed above are governed jointly by all three.

(xi) As already stressed in section 1.1, our truly 2D sheets have little in common with parallel-sided slabs containing large numbers $N$ of lattice planes, not even with slabs whose overall thickness is small on any macroscopic scale. The central problem for such slabs is to find out how $N$ governs departures from the local field appropriate to bulk material, and thereby position-dependent departures from the Lorenz–Lorentz formula for the dielectric function. This is much the same as the problem of the variation of the local field near the surface of a half-space. It can be resolved only by calculation: surprisingly perhaps, essentially bulk conditions turn out to prevail already a very few ($\lesssim 10$) lattice planes away from the nearest surface. (Clear accounts are given by Mahan and Obermair 1969 and Mahan 1972.) Thus our 2D formulae apply when $N = 1$; macroscopic (continuum) electrostatics should apply at least roughly when $N$ is well above say 10; and numerics are needed in between.

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New Journal of Physics 15 (2013) 063028 (http://www.njp.org/)
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