Optimal replenishment policy for multi-item probabilistic inventory model with all-units discount

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Abstract. In this paper, we develop a multi item inventory model for the retailer by considering all-units discount offered by the supplier. Demands in this model are assumed to be probabilistic following a Gamma distribution with certain values of shape and scale parameters. The aim of the model is to determine the optimal replenishment policy that minimize the total inventory cost. The total inventory cost was composed from the purchasing, ordering, handling or storing and shortage costs. All shortages are handled by backorder. We compare the individual and joint replenishment policies for the model and develop an algorithm to find the optimal replenishment policy. Numerical examples are given to give a better understanding of the model in comparing the individual and joint replenishment policies. We also perform sensitivity analysis by changing values of shape and scale parameters of the demand distribution to analyse the effects on the optimal solution.

1. Introduction

Inventory has become one of important factors in manufacturing, trading or service industries. Inventory can come in the forms of raw materials, work-in-process or finished goods. There are several reasons for a company to maintain some inventories in its storage. They can be used as decoupling inventory, cycle inventory, pipeline inventory or buffer inventory ([9]). Also, there are a lot of idle money attached in the inventory, therefore a good inventory management is important and necessary. There are five factors affecting inventory policy decisions, viz. system structure, items, market characteristics, lead times and costs ([6]). Managing inventory should also consider several costs involved such as purchasing, ordering, holding and shortage costs. Over the last few decades, there a lot of mathematical models dealing with inventory control problems. Basically most of the models deal with how much to order (order size) and when to order to minimize the total inventory cost. Among these models, the basic and simplest model is known as the Economic Order Quantity (EOQ) ([11],[10]). In the EOQ model there are only ordering and holding cost to consider and demands are assumed constant and known, so there are no shortage cost. As a model, the EOQ model is simple and has become a building block in developing other mathematical models that are more realistic. Some authors include factors such as deterioration (for perishable items), discount, multi item, deterministic or probabilistic demand distributions. These factors make the models developed more complex and complicated although they are more realistic compared to the EOQ model.

Regarding the deterioration factor, Chang, et.al. [2] have developed an inventory model by considering limited shelf space and price-dependent demand. An uncertain deterioration rate became a central topic in developing a mathematical model for production system control for one item ([11]).
Recently, Pervin, et.al [8] have incorporated stochastic deterioration for a deterministic inventory control with shortage, time-dependent demand and time-varying holding cost in their model. A comprehensive review on deteriorating item has been conducted by Li et. al [5].

When dealing with multi item, the model becomes more complex since there is also decision on the policy of replenishment, whether individual replenishment or joint replenishment. Zhang and Wang [14] combine the problem of deterioration and multi item in the EOQ setting and develop a simple algorithm to find the optimal solution. Considering product assortment, shelf space and display area in a model to find the optimal replenishment is the focus in Hariga et. al [3]. Zhang and Du [13] proposed a multi item with limited capacity model in the newsboy setting.

In this paper, we propose a mathematical model for a multi item problem with all-units discount. The model is probabilistic in terms of demand distribution where we use a class of Gamma distribution and all shortages are handled with backorder. Gamma distribution is used in the model since it can approximate other distributions such as exponential and normal distribution for certain value of its parameters. Some authors have employed Gamma distributions such as in Namit and Chen [7] that use it in demand during lead time. Tyworth and Ganeshan [12] suggested a simple method to find the optimal solution in a (Q,r) model with Gamma lead-time demand. The objective of our model is to determine the optimal replenishment that minimize the annual total inventory cost. Our model is an extension of the model in Lesmono and Limansyah [4] where we add all-units discount and proposed a new algorithm to find the optimal time between replenishment for joint replenishment policy. This is one of the contribution of this paper. We also provide sensitivity analysis on the optimal replenishment policy by changing the parameters’ values of Gamma distribution.

We organize this paper as follows. In Section 2 we propose our model, replenishment policy and algorithm to find the optimal solution. We provide numerical results in Section 3 along with sensitivity analysis on the optimal solution. Conclusions and further research directions are relegated in Section 4.

2. The model

The basic model was developed by Lesmono and Limansyah [4] by considering an inventory model with Gamma lead-time demand. In this paper we add the all-units discount to the model and develop an algorithm to find the optimal solution. All assumptions and notations are the same as in Lesmono and Limansyah [4]. Adding all-units discount will have impact in the purchasing cost and the algorithm in finding the optimal solution. Basically, the total inventory cost is composed from purchasing cost, ordering cost, holding cost and shortage cost.

Since there are all-units discount from the supplier, then the price per unit item $P_1$ can be modelled as a step function for each price break $U_j$ as follows.

$$P_i = \begin{cases} a_k & \text{for } U_k \leq Q < U_1 \\ a_1 & \text{for } U_1 \leq Q < U_2 \\ \vdots \\ a_j & \text{for } U_j \leq Q < U_{j+1} \end{cases}$$

where $a_k > a_{k+1}, U_k < U_{k+1}, k = 0,1,2, ... j − 1$ for each unit of goods. The annual purchasing cost becomes $P_iD$ where $D$ is the annual demand.

Every time an order is placed, there are some ordering costs that has to be paid by the retailer. If the retailer has to pay an amount of $S$ every time an order is placed, then the annual ordering cost becomes $SD/Q$ where $Q$ is the order quantity.

The formulation for annual holding cost is given by $P_i h \left[ \frac{Q}{2} + R - E(X) \right]$ where $h$ is the fraction of holding cost relative to purchasing cost, $X$ is lead-time demand random variable and $R$ is the reorder
point. Shortage occurs when the number of demand during lead time exceeds the reorder point. The annual shortage cost is given by

\[
\pi \frac{D}{Q} \int_{R}^{x} (x-R) f(x) dx
\]

where \( \pi \) is the shortage cost per unit. Combining the annual purchasing, ordering, holding and shortage costs, we have the annual total inventory cost as follows

\[
TAC(Q, R) = P_i D + \frac{SD}{Q} + P_i h \left[ \frac{Q}{2} + R - E(X) \right] + \pi \frac{D}{Q} \int_{R}^{x} (x-R) f(x) dx
\]

(1)

The necessary conditions to get the minimum total annual inventory cost are:

\[
\frac{\partial TAC}{\partial Q} = 0 \text{ and } \frac{\partial TAC}{\partial R} = 0
\]

Using these two conditions we have

\[
Q = \sqrt{2D \left( S + \pi \int_{R}^{x} (x-R) f(x) dx \right) \frac{P_i h}{P_i h}}
\]

(2)

\[
\int_{R}^{x} f(x) dx = \frac{P_i h Q}{\pi D}.
\]

(3)

For the individual replenishment policy, the algorithm to find the obtain the optimal order quantity and reorder point is as follows.

At each price break given by the supplier:

1. Calculate the value of reorder point \( R \) using (3).
2. Calculate the order quantity \( Q \) using (2).
3. Compare the value of \( Q \) with the price break \( U \). If \( Q \) is in the interval of \( U_j \leq Q < U_{j+1} \), then \( Q \) is valid.
4. When \( Q \) is not valid, then
   (i) For \( Q < U_j \), set \( Q = U_j \).
   (ii) For \( Q > U_{j+1} \), set \( Q = U_{j+1} \).
5. Calculate \( TAC \) for each valid \( Q \) and for all possible \( U \).
6. Choose the value of \( Q \) that gives minimum \( TAC \).

When retailers put an order for all item at the same time, then the joint replenishment policy is employed. In this situation, the decision variable is the joint replenishment time \( T \), or we are dealing with the periodic review model. The time between replenishment is fixed but the order quantity for each item varies during the planning horizon.

Basically, the annual total inventory cost for the joint replenishment policy is the summation over the number of items of the annual total inventory cost in (1) with two adjustments. The first adjustment lies in the ordering cost. When retailers put an order for all item together, they demand cheaper ordering cost compared when items are ordered separately. Let the joint ordering cost is \( S^* \). For the second adjustment, we use the relation \( Q = DT \) in the model for the individual replenishment policy and then find the optimal \( T \). Considering these two adjustments, then our model becomes:

\[
TAC(T) = \frac{S^*}{T} + \sum_{i=1}^{n} \left\{ P_i D_i + P_i h_i \left( \frac{TD_i}{2} + R_i - E(X_i) \right) + \frac{\pi_i}{T} \int_{R_i}^{x} (x_i - R_i) f(x_i) dx_i \right\}
\]

(4)
where $n$ is the total number of items. The optimal time of replenishment for the joint replenishment policy is given by

$$T = \sqrt{\frac{2 \left(S^* + \sum_{i=1}^{n} \pi_i \int_{R_i}^{\infty} (x_i - R_i) f(x_i) dx_i\right)}{\sum_{i=1}^{n} P_i D_i h_i}} \quad (5)$$

From the relation $Q = DT$, we can have an all-units discount scheme for $n$ items as follows.

$$P_1 = \begin{cases} P_{1,1}, & T < a_{1,1} \\ P_{1,2}, & a_{1,1} \leq T < a_{1,2} \\ \vdots & \vdots \\ P_{1,n_1-1}, & a_{1,n_1-2} \leq T < a_{1,n_1-1} \\ P_{1,n_1}, & a_{1,n_1-1} \leq T < a_{1,n_1} \\ P_{1,n_1+1}, & T \geq a_{1,n_1} \\ P_{2,1}, & T < a_{2,1} \\ P_{2,2}, & a_{2,1} \leq T < a_{2,2} \\ \vdots & \vdots \\ P_{2,n_2-1}, & a_{2,n_2-2} \leq T < a_{2,n_2-1} \\ P_{2,n_2}, & a_{2,n_2-1} \leq T < a_{2,n_2} \\ P_{2,n_2+1}, & T \geq a_{2,n_2} \end{cases}$$

$$P_2 = \begin{cases} \vdots \end{cases}$$

$$P_n = \begin{cases} P_{n,1}, & T < a_{n,1} \\ P_{n,2}, & a_{n,1} \leq T < a_{n,2} \\ \vdots & \vdots \\ P_{n,n_{n-1}}, & a_{n,n_{n-2}} \leq T < a_{n,n_{n-1}} \\ P_{n,n_n}, & a_{n,n_{n-1}} \leq T < a_{n,n_n} \\ P_{n,n_{n+1}}, & T \geq a_{n,n_n} \end{cases}$$

To find the optimal solution for the joint replenishment policy, we propose the following algorithm:

1. Sort the value of $T$ based on the all possibilities of price per item.
2. Calculate the value of $T$ for each interval using (5).
3. When the obtained value of $T$ from step 2 is in its interval, then $T$ is valid.
4. Calculate TAC for valid $T$ and $T = a_{ij}$ for every $i$ and $j$.
5. Choose the value of $T$ that minimizes $TAC$.

3. **Numerical Results**

For the numerical experiments, we consider a retailer who sells three different products and gets these products from one supplier. Lead time demands follow Gamma distributions with different values of shape and scale parameters ($\alpha$ and $\beta$) and other data are given in Table 1 below.
We compare two replenishment policy, individual and joint replenishment policy for the data above. Results for individual replenishment policy and joint replenishment policy are given in Table 1. For the joint replenishment policy, the joint ordering cost $S^*$ is $11.

| Table 1. Data | Product 1 | Product 2 | Product 3 |
|---------------|-----------|-----------|-----------|
| Lead time Demand | 700 | 1200 | 550 |
| Ordering Cost ($) | 5 | 4 | 6 |
| Holding cost fraction | 1% | 0.5% | 2% |
| Shortage cost ($) | 2 | 1 | 1.5 |
| Purchase cost ($) | 10 for Q ≤ 220 | 6 for Q ≤ 320 | 12 for Q ≤ 180 |
| | 8 for Q ≥ 221 | 5 for Q ≥ 321 | 10 for Q ≥ 181 |
| $\alpha$ | 4 | 3 | 4 |
| $\beta$ | 3 | 5 | 2 |

We can see from Table 2 that the total annual inventory cost is lower for joint replenishment policy than for individual replenishment policy although the difference is not so significant. The total purchase cost for individual and joint are the same since the optimal order quantity for each product lies in the same interval of the discount scheme. Joint replenishment policy gives lower ordering and holding costs since the optimal order quantity is also smaller. This makes the shortage cost for the joint replenishment policy a bit higher.

| Table 2. Results for individual and joint replenishment policy |
|---------------------------------------------------------------|
| Individual Replenishment | Joint Replenishment |
|----------------------------|---------------------|
| Q                          | Product 1 | Product 2 | Product 3 | Product 1 | Product 2 | Product 3 |
| 302                        | 632       | 183       | 240       | 411       | 189       |
| 26                         | 36        | 18        | 0.342     |
| Total Purchase Cost        | 5600      | 6000      | 5500      | 5600      | 6000      | 5500      |
| Total Ordering Cost        | 11.58     | 7.59      | 18.04     | 32.16     |
| Total Holding Cost         | 13.21     | 8.46      | 20.30     | 10.70     | 5.66      | 20.81     |
| Total Shortage Cost        | 0.51      | 0.31      | 0.26      | 0.65      | 0.47      | 0.25      |
| Total Inventory Cost       | 5625.3    | 6016.36   | 5538.59   | 5611.34   | 6006.13   | 5521.06   |
| TAC                        | 17,180.25 | 17,170.69 |

Table 3 below give results for sensitivity analysis of the parameter $\beta$ of the Gamma distribution for individual replenishment policy ($\alpha$ equals 4, 3 and 4 for product 1, 2 and 3 respectively). We increase and decrease the parameter $\beta$ by one for product 1, 2 and 3. From Table 3 we can see that in general, changing the value of parameter $\beta$ does not substantially affect the total annual inventory cost for both individual and joint replenishment policy. The difference is around $3 for each replenishment policy and around $9 if we compare the individual and joint replenishment policy. The same situation happens when we do sensitivity analysis for parameter $\alpha$ in Table 4. Here the values of $\beta$ are kept fixed of 3, 5 and 2 for product 1, 2 and 3 respectively and we increase and decrease the values of $\alpha$ by one. Here the difference is smaller, only around $1 for each replenishment policy.
Table 3. Sensitivity analysis for $\beta$

| Product | $\beta$ | 2 | 4 | 1 | 4 | 6 | 3 |
|---------|---------|---|---|---|---|---|---|
| Product 1 | 300 | 631 | 183 | 303 | 634 | 184 |
| Product 2 | 17 | 28 | 8 | 35 | 43 | 26 |
| Total Purchase Cost | 5600 | 6000 | 5500 | 5600 | 6000 | 5500 |
| Total Ordering Cost | 11.64 | 7.60 | 18.03 | 11.52 | 7.56 | 17.92 |
| Total Holding Cost | 12.747 | 8.2911 | 19.1 | 13.68 | 0.38 | 0.49 |
| Total Shortage Cost | 0.39 | 0.29 | 0.27 | 0.64 | 0.38 | 0.49 |
| Total Inventory Cost | 5624.77 | 6016.18 | 5537.40 | 5625.83 | 6016.50 | 5539.63 |

Joint Replenishment Policy

| T | 0.3411 |
| Q | 238.77 |
| Total Purchase Cost | 5600 |
| Total Ordering Cost | 32.248 |
| Total Holding Cost | 10.27 |
| Total Shortage Cost | 0.49 |
| Total Inventory Cost | 5610.76 |
| TAC | 17,178.36 |

Table 4. Sensitivity analysis for $\alpha$

| Product | $\alpha$ | 3 | 2 | 3 | 5 | 4 | 5 |
|---------|---------|---|---|---|---|---|---|
| Product 1 | 302 | 632 | 183 | 303 | 634 | 183 |
| Product 2 | 21 | 27 | 14 | 30 | 43 | 20 |
| Total Purchase Cost | 5600 | 6000 | 5500 | 5600 | 6000 | 5500 |
| Total Ordering Cost | 11.57 | 7.59 | 18 | 11.54 | 7.57 | 17.97 |
| Total Holding Cost | 13.06 | 8.33 | 19.94 | 13.33 | 8.51 | 20.36 |
| Total Shortage Cost | 0.53 | 0.32 | 0.34 | 0.594 | 0.37 | 0.39 |
| Total Inventory Cost | 5625.16 | 6016.236 | 5537.40 | 5625.83 | 6016.44 | 5538.72 |
| TAC | 17,179.67 |

Joint Replenishment Policy

| T | 0.3426 |
| Q | 239.82 |
| Total Purchase Cost | 5600 |
| Total Ordering Cost | 32.11 |
| Total Holding Cost | 10.55 |
| Total Shortage Cost | 0.66 |
| Total Inventory Cost | 5611.22 |
| TAC | 17,170.15 |

4. Conclusions and further research

We developed a mathematical model for probabilistic inventory problem with all-units discount where lead time demand is Gamma distributed with different values of shape and scale parameters for each product. Algorithms to find the optimal solution are proposed for individual and joint replenishment policies. According to our numerical experiments with three products, it is found that the joint replenishment policy gives smaller total annual inventory cost compared to the individual replenishment policy. Sensitivity analysis for the shape and scale parameters of the Gamma distribution are also
perform and we found that the difference in the total annual inventory cost is not quite substantial, only around $1 to $3 for individual replenishment policy and around $9 for joint replenishment policy.

In our model, we only incorporate all-units discount offered by supplier and have not yet considered other factors such as deterioration and the possibility for retailers to return unsold or deteriorated goods to supplier at some costs. These factors will be point of interest for further research.

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