Time Homogeneous Diffusion with Drift and Killing to Meet a Given Marginal

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Abstract

In this article, it is proved that for any probability law \( \mu \) over \( \mathbb{R} \), a given drift field \( b : \mathbb{R} \to \mathbb{R} \) and a given measurable killing field \( k : \mathbb{R} \to \mathbb{R}_+ \) which satisfy hypotheses stated in the article and a given terminal time \( t > 0 \), there exists an \( m \), an \( \alpha \in (0, 1] \), an initial condition \( x_0 \in \mathbb{R} \) and a process \( X \) with infinitesimal generator \( \left( \frac{1}{2} \frac{d^2}{dx^2} + \frac{d}{dm} - \frac{dK}{dm} \right) \) where \( k = \frac{dK}{dx} \) such that for any Borel set \( B \in B(\mathbb{R}) \),

\[ P(X_t \in B|X_0 = x_0) = \alpha \mu(B). \]

Firstly, it is shown the problem with drift and without killing can be accommodated, after a simple co-ordinate change, entirely by the proof in [20]. The killing field presents additional problems and the proofs follow the lines of [20] with additional arguments.

1 Introduction

1.1 Results and Method of Proof

Let \( \mu \) be a probability measure over \( \mathbb{R} \) and \( b : \mathbb{R} \to \mathbb{R} \) a given function satisfying Hypothesis 1.1 below. Set

\[ \tilde{b}(x) = \begin{cases} b(x) & x \in \text{supp}(\mu) \\ 0 & x \notin \text{supp}(\mu) \end{cases} \quad (1) \]

Hypothesis 1.1 (Hypothesis on drift \( b \) and measure \( \mu \)). The target probability measure and the drift \( (\mu, b) \) satisfy two conditions. The first is (2) given below:

\[ \int_{-\infty}^{\infty} \left( \int_{0 \wedge x}^{0 \vee x} e^{F(b,y)} dy \right) \mu(dx) < +\infty \quad (2) \]

where

\[ F(b,y) = 4 \sup_{\xi: \max_j (t_{j+1} - t_j) < 1, (0 \wedge y) = t_0 < \ldots < t_n = (0 \vee y)} \left\{ \sum_{i=0}^{n-1} \left( \frac{1}{2} \int_{t_i}^{t_{i+1}} \tilde{b}(x)dx \right)^2 \vee \int_{t_i}^{t_{i+1}} |\tilde{b}(x)|dx \right\}. \quad (3) \]

and \( \tilde{b} \) is defined by (1). Here the maximum is taken over sequences of length \( n \) for all \( n \in \mathbb{N} \).
The second is (4) below: let \( l_-(x) = \sup\{ y \in \text{suppt}(\mu) \cap (-\infty, x) \} \) and let \( l_+(x) = \inf\{ y \in \text{suppt}(\mu) \cap (x, +\infty) \} \) where \( \text{suppt} \) means support. Then
\[
\sup_{x \in \mathbb{R}} \lim_{h \downarrow 0} \int_{l_-(x)-h}^{l_+(x)+h} |\tilde{b}(x)| \, dx < 1. \tag{4}
\]

This article addresses the following problem: suppose that \((\mu, b)\) satisfy (2) and (4) of Hypothesis 1.1.

Let \( k \) denote the killing field and \( K \) the function defined by:
\[
K(x) = \begin{cases} 
\int_{(0,x] \cap \text{suppt}(\mu)} k(y) \, dy & x > 0 \\
- \int_{[x,0) \cap \text{suppt}(\mu)} k(y) \, dy & x \leq 0 
\end{cases}
\]

It is shown that there exists a string measure \( m \), an \( \alpha \in (0, 1) \) and an \( x_0 \in \mathbb{R} \) such that
\[
\frac{1}{2} \frac{d^2}{dmdx} + b \frac{d}{dm} - \frac{dK}{dm} \tag{5}
\]
is the infinitesimal generator of a process \( X \) satisfying \( \mathbb{P}(X_t \in B|X_0 = x_0, X_t \notin \{D\}) = \mu(B) \) for all \( B \in \mathcal{B}(\mathbb{R}) \), \( X_t \in \{D\} \) denotes that the process has been killed by time \( t \). From the definition of the operator, \( \tilde{b} \frac{d}{dm} = b \frac{d}{dm} \) on the domain of definition.

If \( t \) is replaced by an exponential time, \( \alpha \), \( x_0 \) and \( m \) are uniquely determined and an explicit construction is given. If \( t \) is a deterministic time, only existence is given, although the method of proof may indicate how to provide approximations.

**Remark** It is straightforward (and easier) to obtain the existence of a measure \( m \) which gives an \( \alpha > 0 \) and a process with infinitesimal generator
\[
\left( \frac{1}{2} \frac{d^2}{dmdx} + b \frac{d}{dm} \right) - k \tag{6}
\]
for a given drift \( b \) and killing \( k \), which has distribution
\[
\mathbb{P}(X_\tau \in D) = 1 - \alpha \quad \mathbb{P}(X_\tau \in A) = \alpha \mu(A) \quad \forall A \in \mathcal{B}(\mathbb{R}),
\]
where \( \tau \) is the terminal time, \( \mu \) is the prescribed measure and \( D \) denotes the cemetery state; \( \{X_\tau \in D\} \) denotes that the process has been killed before time \( \tau \). As with the case discussed in this article, with similar proofs, there is uniqueness and explicit construction when stopped at an independent geometric / exponential time.

The line of proof is as follows:

1. Discrete time and finite state space are considered; conditions under which a suitable Markov chain with a given distribution when stopped at an independent geometric time are established. The solution, when it exists, is unique and the construction explicit.

2. This is then extended to establish conditions under which there exists a Markov chain with a given distribution when stopped at an independent negative binomial time. This uses the fact that a negative binomial variable is the sum of independent identically distributed geometric
variables and uses a fixed point theorem. For the problem of finding an infinitesimal generator of the form of (5) or (6), substantial modifications of the arguments in [20] are required when killing is introduced.

3. Limits of negative binomial times by reducing the time mesh are taken to obtain a time with Gamma distribution as in [20]. Limits are then taken to obtain a deterministic time. The arguments are along similar lines to those of [20], with some crucial modifications.

4. Finally, arbitrary state space is considered. As in [20], the target measure is approximated by a sequence of atomised measures. The drift is dealt with by a change of co-ordinates and the sequence of atomised measures in the transformed co-ordinates is considered. The killing is dealt with by considering the process without killing, together with the conditional distribution of the killing time. Both of these converge. The problem is to ensure that the diffusion coefficient does not tend to infinity and the probability that the process has been killed does not tend to 1 as the limit is taken.

1.2 Background

The problem of constructing a gap diffusion with a given law with compact support at an independent exponential time has been discussed fully by Cox, Hobson and Obłój in [4] (2011). The problem of constructing a martingale diffusion that has law \( \mu \) at a fixed time \( t \) has been solved by Jiang and Tao in [11] (2001) under certain smoothness assumptions. Recently, Forde in [8] (2011) extended the work of Cox, Hobson and Obłój [4] to provide a process with prescribed joint law for the process at an independent exponential time \( \tau \) and its supremum over the time interval \([0, \tau]\).

For any prescribed measure \( \mu \), the problem of finding a martingale diffusion with given marginal \( \mu \) at a fixed time \( t > 0 \) was solved in [20] (2013). Independently and simultaneously, Ekström, Hobson, Janson and Tysk [7] (2013) found a different proof; in [7], the target distribution is again approximated by atomic measures, but general results from algebraic topology to conclude existence of a limit. In [19] (1972), Monroe constructs a general symmetric stable process with a prescribed marginal at a fixed time, but does not require that the resulting process satisfies a martingale property.

1.3 Motivation

The subject of strong Markov processes generated by Krein- Feller generalised second order differential operators is of great interest in its own right and, more specifically, the inverse problem, of computing a function \( a \) to give a solution \( f \) to the parabolic equation

\[
\frac{\partial f}{\partial s} = a \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} + b \frac{\partial}{\partial x} - k \right) f.
\]

Here \( a \) is understood as \( \frac{1}{m'(x)} \) and the initial condition at \( s = 0 \) is a dirac mass \( f(0, x) = \delta_{x_0}(x) \) for some \( x \in \mathbb{R} \); the end condition \( f(t, x) \) for \( s = t > 0 \) is prescribed.

The operator \( \frac{d^2}{dxdx} \) and its spectral theory were introduced by Krein [16] (1952) and, for a more developed treatment, Kac and Krein [12] (1958); a lucid account of the spectral theory is given by Dym and McKean [6] (1976). The operator, as a generator of a strong Markov process is discussed
in Knight [14] (1981) where it is referred to as a *gap diffusion* and Kotani and Watanabe [15] (1982) where it is referred to as a *generalised diffusion*.

**Mimicking**  Given a semimartingale \( \xi \), it is of interest in applications from Stochastic Control to find a Markov process \( X \) where some of the marginal laws are the same as those of \( \xi \). A Markov process with such a property preserves the features of the original process \( \xi \) that are of importance and the Markov structure facilitates computations. Quantities that only involve the marginal laws of \( \xi \) may be computed using the infinitesimal generator of the mimicking Markov process. An early and important example of mimicking is by Gyöngy [9] (1986), where a process that can be expressed by an Itô integral is mimicked by a process that satisfies a stochastic differential equation. The method used by Gyöngy [9] is to construct the Green measure of the process \( (t, \xi_t) \) with a given killing rate and identify it with the Green measure of the process \( (t, X_t) \) with another killing rate, where \( X_t \) is any solution of an SDE, constructed to give the appropriate marginals. The article by Kurtz and Stockbridge [17] (1998) is important for applications; given a solution of a controlled martingale problem it is shown under general conditions that there exists a solution having Markov controls which has the same cost as the original solution.

Let \( (p_t) \) be a family of marginal densities of a martingale. It is well known that for convex \( \phi : \mathbb{R}^d \to \mathbb{R} \), \( \int \phi(y)p_t(y)dy \geq \int \phi(y)p_s(y)dy \) for all \( t \geq s \). Kellerer [13] (1972) proved the converse, that for any family of probability distributions with this property, there exists a Markov process with these marginal densities and, furthermore, this process is a submartingale. Yor and co-authors [2], [10](2010) and [18](2002) extended this in several ways. For example, in [18], Madan and Yor give three different constructions of Markov process with a specified flow of marginal distributions, the first of which is based on the solution of Azéma and Yor [1](1979) to the Skorohod embedding problem, the second follows the method introduced by Dupire involving continuous martingales [5] (1994) and the last constructs the mimicking process as a time-changed Brownian motion.

**Mathematics of Finance**  In recent years, interest in gap diffusion operators and their associated processes has been strongly renewed by applications to the field of modelling financial markets. This motivation is discussed briefly here.

The general motivating problem within finance is that of automating the pricing and risk management of derivative securities. More specifically, it is the problem of pricing a wide range of European style options given the current market price of the underlying asset and market option quotes of European call options at a range of strikes \( K \) for a maturity \( t \) or, more generally, several maturities. This is a problem of practical importance; from listed option prices, the problem of inferring option prices at non-listed strikes and maturities arises both on exchanges and with over-the-counter transactions.

This problem is addressed by Carr in [3], who develops one such model, known as the Local Variance Gamma model. The problem of determining the appropriate inputs for a model so that the output is consistent with a specified set of market prices is known as calibration. The reader is referred to the introduction of [20].

The addition of drift and killing are both have clear importance for financial modelling; the idea of a bond with fixed, guaranteed interest is ever more obsolescent within the financial world and the price of the numéraire, as well as the asset, are modelled by a stochastic process. The covariation between
the price of the numéraire and the price of the asset changes the drift of the asset price process, hence
the requirement to incorporate a drift $b$. The inclusion of a killing field extends the class of models
available.

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2 Definitions, Infinitesimal Generators and Processes

A definition of the operator $G = \frac{d^2}{dm dx}$ used in [5] may be found in Dym and McKean [6] or Kotani
and Watanabe [15]. The Kotani Watanabe definition is more useful in this setting, because it extends
to strings defined over the whole real line. The domain of the operator, denoted $D(G)$ is the space of
functions $f \in \mathcal{B}(\mathbb{R})$ such that there exists an $m$-measurable function $g$ satisfying
$\int_{x_1}^{x_2} g^2(x) m(dx) < +\infty$ for all $-\infty < x_1 < x_2 < +\infty$ such that

$$f(x) = f(x_0) + (x - x_0)f'_-(x_0) + \int_{x_0}^{x} \int_{x_0}^{y} g(z) m(dz) dy \quad \forall -\infty < x_0 < x < -\infty$$

where $f'_-$ denotes the left derivative, $\int_a^b$ denotes integration over $(a, b]$ and $\int_a^b$ denotes integration
over the closed interval $[a, b]$. The quantity $Gf = \frac{d^2}{dm dx} f$ is defined as $g$. Similarly, $f \in D(b\frac{d}{dm})$ if
$f \in D(G)$ and there exists an $m$-measurable function $h$ satisfying $\int_{x_1}^{x_2} |b(x)h(x)| m(dx) < +\infty$ for all
$-\infty < x_1 < x_2 < -\infty$ such that

$$\int_{x}^{y} b(z)f'_-(z) dz = \int_{x}^{y} b(z)h(z) m(dz) \quad \forall -\infty < x < y < +\infty.$$

The quantity $\frac{df}{dm}$ is defined as $h$.

As remarked earlier, for $z \notin \text{suppt}(m)$ and $f \in D(b\frac{d}{dm})$, $b(z) \frac{d}{dm} f(z) = 0$, so that Expression (5) is the
same whether $\tilde{b}$ or $b$ is used.

Note The definition of the domain of the operator is not discussed further in this article, since
the method of proof does not require it, but it is reasonably straightforward to show that, for any
process $X$ obtained, if $f \in D(G) \cap D(b\frac{d}{dm})$, then for all $t > 0$, $F(t, .) \in D(G) \cap D(b\frac{d}{dm})$ where
$F(t, x) = \mathbb{E}[f(X_t) | X_0 = x]$. It follows from the analysis given that $F$ thus defined satisfies:

$$\frac{\partial}{\partial t} F = \frac{1}{2} \frac{\partial^2}{\partial m \partial x} F + b \frac{\partial}{\partial m} F - \frac{\partial K}{\partial m} F.$$

When $m$ has a well defined density $m' > 0$, let $a = \frac{1}{m'}$, then (5) may be written as:

$$a \left( \frac{1}{2} \frac{d^2}{dx^2} + b \frac{d}{dx} - k \right).$$

When a finite discrete state space $S = \{i_1, \ldots, i_M\}$ is considered, the generator may be written as:
\[
a \left( \frac{1}{2} \Delta + b \nabla - k \right). 
\]

where the definitions of the operators \(\Delta\) and \(\nabla\) are given in Definition 2.1 below.

**Definition 2.1** (Laplacian and Derivative, Discrete state space). Consider a state space

\[ S = \{i_1, \ldots, i_M\}, \quad i_1 < \ldots < i_M. \]

For a function \(f : S \to \mathbb{R}\), the Laplace operator \(\Delta\) is defined as follows:

\[
\begin{align*}
\Delta f(i_1) &= \Delta f(i_M) = 0 \\
\Delta f(i_j) &= \frac{2}{(i_{j+1}-i_j)(i_{j-1}-i_j)} \left( f(i_{j+1}) - f(i_j) + \frac{i_{j+1}-i_{j-1}}{i_{j+1}-i_{j-1}} f(i_{j-1}) \right) \quad j = 2, \ldots, M - 1 \\
\end{align*}
\]

(8)

The derivative operator \(\nabla\) is defined as follows:

\[
\begin{align*}
\nabla f(i_1) &= \nabla f(i_M) = 0 \\
\nabla f(i_j) &= \frac{f(i_{j+1})-f(i_{j-1})}{i_{j+1}-i_{j-1}} \quad 2 \leq j \leq M - 1 \\
\end{align*}
\]

(9)

**Remarks**

1. If the function \(f\) is defined on an interval \((y_0, y_1)\), \(f \in C^2((y_0, y_1))\) (twice differentiable with continuous second derivative) and a sequence \(S_n\) is considered, where \(S_n = \{i_{n,1}, \ldots, i_{n,M_n}\}\), \(i_{n,j} < i_{n,j+1}\), \(i_{n,1} \downarrow y_0\) and \(i_{n,M_n} \uparrow y_1\) and \(\lim_{n \to +\infty} \max_j (i_{n,j+1} - i_{n,j}) = 0\), with \(\Delta_{S_n}\) the operator defined on \(S_n\), then \(\lim_{n \to +\infty} \Delta_{S_n} f = \frac{d^2}{dx^2} f\). Note that the function \(f\) has been defined on \(C^2((y_0, y_1))\). The sense in which convergence is meant is: let \(j_n(x) = \max\{j : i_{n,j} \leq x\}\) then for all \(x \in (y_0, y_1)\),

\[
\lim_{n \to +\infty} \left| \frac{2}{(i_{n,j_n}(x)+1 - i_{n,j_n}(x)) (i_{n,j_n}(x)-i_{n,j_n}(x)-1)} \left( \frac{i_{n,j}(x) - i_{n,j}(x)-1}{i_{n,j}(x)+1 - i_{n,j}(x)-1} f(i_{n,j}(x)+1) \\
- f(i_{n,j}(x)) + \frac{i_{n,j}(x)+1 - i_{n,j}(x)}{i_{n,j}(x)+1 - i_{n,j}(x)-1} f(i_{n,j}(x)-1) \right) - \frac{d^2}{dx^2} f(x) \right| = 0.
\]

2. If the function \(f\) is defined on the whole interval \((y_0, y_1)\) and \(f \in C^1(\mathbb{R})\) (differentiable, continuous first derivative) and a sequence \(S_n\) is considered where \(S_n = \{i_{1,n}, \ldots, i_{M,n}\}\), \(i_{1,n} < i_{j+1,n}\), \(i_{1,n} \downarrow y_0\) and \(i_{M,n} \uparrow y_1\) and \(\nabla_n\) the operator defined on \(S_n\), then \(\lim_{n \to +\infty} \nabla_n f = \frac{d}{dx} f\) in the sense that for \(f \in C^1((y_0, y_1))\) (differentiable with continuous derivative, for all \(x \in (y_0, y_1)\)

\[
\lim_{n \to +\infty} \left| \frac{f(i_{j_n}(x)+1) - f(i_{j_n}(x)-1)}{i_{j_n}(x)+1 - i_{j_n}(x)-1} - \frac{d}{dx} f(x) \right| = 0.
\]

if \(\lim_{n \to +\infty} \max_j (i_{n,j+1} - i_{n,j}) = 0\) and \(j_n(x) = \max\{j : i_{n,j} \leq x\}\).
Notation  Let $S = \{i_1, \ldots, i_M\}$, let $b : S \setminus \{i_1, i_M\} \to \mathbb{R}$ and $k : S \setminus \{i_1, i_M\} \to \mathbb{R}_+$. The following notation will be used: let $b = (b_2, \ldots, b_{M-1})$ where (with slight abuse of notation) $b_j = b(i_j)$ and $k = (k_2, \ldots, k_{M-1})$ where (same notation) $k_j = k(i_j)$. The notation $\tilde{k}_j$ will be used to denote the following:

$$\tilde{k}_j = (i_{j+1} - i_j)(i_j - i_{j-1}) k_j \quad j \in (2, \ldots, M - 1)$$

(10)

and $\tilde{k} = (\tilde{k}_2, \ldots, \tilde{k}_{M-1})$.

For all results with finite state space, the following hypothesis will be required:

**Hypothesis 2.2.** For a discrete, finite state space $S = \{i_1, \ldots, i_M\}$ where $i_1 < \ldots < i_M$, the vector $b$ satisfies the condition:

$$-\frac{1}{i_{j+1} - i_j} < b_j < \frac{1}{i_j - i_{j-1}} \quad j = 2, \ldots, M - 1.$$  (11)

Set

$$\begin{cases}
q_{j,j+1} = \frac{i_{j+1} - i_j}{i_{j+1} - i_{j-1}}(1 + (i_{j+1} - i_j)b_j) & j = 2, \ldots, M - 1 \\
q_{j,j-1} = \frac{i_{j+1} - i_j}{i_{j+1} - i_{j-1}}(1 - (i_j - i_{j-1})b_j) & j = 2, \ldots, M - 1
\end{cases}$$

(12)

Condition (11) is necessary and sufficient to ensure that $q_{j,j+1}$ and $q_{j,j-1}$ are non negative for each $j$. With these definitions of $b$, $k$ and $\tilde{k}$, the following definitions are made for the transitions (in discrete time) and the intensities (in continuous time) of the Markov processes that are of interest.

**Definition 2.3** (Transition Matrix). Let $\Delta = (\lambda_2, \ldots, \lambda_{M-1}) \in \mathbb{R}^{M-2}$ and, for $h < \frac{1}{\max_j \lambda_j(1+k_j)}$, let $P(h)(\tilde{k}, \Delta)$ be the $M + 1 \times M + 1$ matrix defined by:

$$\begin{cases}
\tilde{P}_{j,M+1}(\tilde{k}, \Delta) = h\lambda_j \tilde{k}_j & j = 2, \ldots, M - 1 \\
\tilde{P}_{1,M+1}(\tilde{k}, \Delta) = \tilde{P}_{M,M+1}(\tilde{k}, \Delta) = 0 \\
\tilde{P}_{j+1,M+1}(\tilde{k}, \Delta) = 1 & j = 1, \ldots, M \\
\tilde{P}_{j,M}(\tilde{k}, \Delta) = 1 - \lambda_j(1 + \tilde{k}_j)h & j = 2, \ldots, M - 1 \\
\tilde{P}_{1,j}(\tilde{k}, \Delta) = \tilde{P}_{M,j}(\tilde{k}, \Delta) = 0 \\
\tilde{P}_{j,j+1}(\tilde{k}, \Delta) = h\lambda_j q_{j,j+1} & j = 2, \ldots, M - 1 \\
\tilde{P}_{j,j-1}(\tilde{k}, \Delta) = h\lambda_j q_{j,j-1} & j = 2, \ldots, M - 1 \\
\tilde{P}_{j,k}(\tilde{k}, \Delta) = 0 & |j - k| \geq 2, \ (j, k) \in \{1, \ldots, M\}^2
\end{cases}$$

(13)

Let $P(h)(\tilde{k}, \Delta)$ denote the $M \times M$ matrix defined by $P_{ij}(h)(\tilde{k}, \Delta) = \tilde{P}_{ij}(h)(\tilde{k}, \Delta)$ for $(i, j) \in \{1, \ldots, M\}^2$.

**Definition 2.4** (Intensity Matrix). Let

$$\Theta(\tilde{k}, \Delta) = \lim_{h \to 0} \frac{1}{h} \left( \tilde{P}(h)(\tilde{k}, \Delta) - I \right)$$

(14)
It is straightforward to see that the matrix $\Theta(\mathbf{k}, \Delta)$ satisfies:

$$
\begin{cases}
\Theta_{j,M+1}(\mathbf{k}, \Delta) = \lambda_k j & j = 2, \ldots, M - 1 \\
\Theta_{1,M+1}(\mathbf{k}, \Delta) = \Theta_{M,M+1}(\mathbf{k}, \Delta) = 0 \\
\Theta_{M+1,j}(\mathbf{k}, \Delta) = 0 & j = 1, \ldots, M + 1 \\
\Theta_{j,j}(\mathbf{k}, \Delta) = -\lambda_j (1 + \tilde{\lambda}_j) & j = 2, \ldots, M - 1 \\
\Theta_{12}(\mathbf{k}, \Delta) = \Theta_{M,M-1}(\mathbf{k}, \Delta) = 0 \\
\Theta_{j,j+1}(\mathbf{k}, \Delta) = \lambda_j q_{j,j+1} & j = 2, \ldots, M - 1 \\
\Theta_{j,j-1}(\mathbf{k}, \Delta) = \lambda_j q_{j,j-1} & j = 2, \ldots, M - 1 \\
\Theta_{j,k}(\mathbf{k}, \Delta) = 0 & |j - k| \geq 2, \quad (j, k) \in \{1, \ldots, M\}
\end{cases}
$$

(15)

which is the intensity matrix of a Continuous Time Markov Chain on state space $\{1, \ldots, M + 1\}$.

**Note** The dependence on $\mathbf{k}$ and $\Delta$ for $\tilde{P}^{(h)}(\mathbf{k}, \Delta)$, $P^{(h)}(\mathbf{k}, \Delta)$ and $\Theta(\mathbf{k}, \Delta)$ will be suppressed; these will be written as $\tilde{P}^{(h)}$, $P^{(h)}$ and $\Theta$ respectively.

For $h < \frac{1}{\max_j \lambda_j (1 + k_j)}$, $\tilde{P}^{(h)}$ is the one-step transition matrix for a time homogeneous Markov process $X^{(h)}$, with time step $h$, satisfying

$$
\mathbb{P}(X^{(h)}_{h(t+1)} = i_k | X^{(h)}_{ht} = i_j) = \tilde{P}^{(h)}_{jk}.
$$

As discussed in [20], as $h \to 0$, the process $X^{(h)} \to X$ (convergence in the sense of finite dimensional marginals) to a continuous time Markov chain with intensity matrix $\Theta = \lim_{h \to 0} \frac{1}{h} (\tilde{P}^{(h)} - I)$ from Definition 2.4.

**Lemma 2.5.** Let $S = \{i_1, \ldots, i_M\}$ and let $X$ be a continuous time Markov process on $S \cup \{D\}$ with transition intensity matrix from Definition 2.4 Equation (11) in the sense that

$$
\begin{cases}
\lim_{h \to 0} \frac{1}{h} \mathbb{P}(Y_h = i_k | Y_0 = i_j) = \Theta_{jk} \quad (j, k) \in \{1, \ldots, M + 1\}^2 \quad j \neq k, \\
\lim_{h \to 0} \frac{1}{h} \left( \mathbb{P}(Y_h = i_j | Y_0 = i_j) - 1 \right) = \Theta_{jj} \quad j \in \{1, \ldots, M + 1\}
\end{cases}
$$

Let

$$
\begin{cases}
\lambda_j = \lambda_j (i_{j+1} - i_j) (i_j - i_{j-1}) & j = 2, \ldots, M - 1 \\
a_1 = a_M = 0 \\
a_j = \lambda_j (i_{j+1} - i_j) (i_j - i_{j-1}) & j = 2, \ldots, M - 1
\end{cases}
$$

(16)

and (where the notation is clear) let $\alpha : S \to \mathbb{R}_+$ denote the function defined by $a(i_1) = a(i_M) = 0$, $a(i_j) = a_j$ for $j = 2, \ldots, M - 1$. Then $Y$ has infinitesimal generator

$$
\alpha \left( \frac{1}{2} \Delta + b \nabla - k \right).
$$

**Proof** Recall the definition of $\mathbf{k}$ (Equation (10)). Let $f$ be a function defined on $\{i_1, \ldots, i_M\}$ and let $F(t, i_j) = \mathbb{E}_{ij}[f(Y_t)]$. Then, for $j = 2, \ldots, M - 1$,
\begin{align*}
\frac{\partial}{\partial t} F(t, i_j) &= \lim_{h \to 0} \frac{F(t + h, i_j) - F(t, i_j)}{h} = \lim_{h \to 0} \frac{1}{h} \mathbb{E}_{i_j} [f(Y_{t+h}) - f(Y_t)] \\
&= \lim_{h \to 0} \frac{1}{h} (\mathbb{E}_{i_j} [F(t, Y_h)] - F(t, i_j)) \\
&= \lambda_j \left( q_{j,j+1} F(t, i_{j+1}) - F(t, i_j) + q_{j,j-1} F(t, i_{j-1}) - \tilde{k}_j F(t, i_j) \right) \\
&= \lambda_j \left( \left( \frac{i_j - i_{j-1}}{i_{j+1} - i_j} F(t, i_{j+1}) - F(t, i_j) + \frac{i_{j+1} - i_j}{i_{j+1} - i_{j-1}} F(t, i_{j-1}) \right) \\
&\quad + (i_{j+1} - i_j)(i_j - i_{j-1}) b_j \left( \frac{F(t, i_{j+1}) - F(t, i_{j-1})}{i_{j+1} - i_{j-1}} - \tilde{k}_j F(t, i_j) \right) \right) \\
&= a_j \left( \frac{1}{2} \Delta + b_j \nabla - k_j \right) F(t, i_j)
\end{align*}

For \( j \in \{1, M\} \),

\[ \frac{\partial}{\partial t} F(t, i_j) = 0 \]
as required. \( \square \)

3 Coordinate change to deal with the drift

The addition of the drift \( b \) can be dealt with through a simple change of co-ordinates, described here. The aim is to find a mapping of the process from space \( S \) (the state space of the process) to a space \( R \) such that the transformed process is drift free. For finite state space, \( S = \{i_1, \ldots, i_M\}, i_1 < \ldots < i_M \) the aim is to find a map \( \kappa \) where (with abuse of notation) \( \kappa_j = \kappa(i_j) \) for \( j = 1, \ldots, M \) where \( \kappa_j < \kappa_{j+1} \) for \( j = 1, \ldots, M - 1 \) such that \( q_{j,j-1} \) and \( q_{j,j+1} \), for \( j = 2, \ldots, M - 1 \), defined by (12), satisfy:

\[ q_{j,j+1} = \frac{\kappa_j - \kappa_j - 1}{\kappa_j - 1} \quad q_{j,j-1} = \frac{\kappa_{j+1} - \kappa_j}{\kappa_{j+1} - 1} \]

(17)

Let

\[ \delta_j = i_j - i_{j-1}, \quad j = 2, \ldots, M. \]

Directly from (12) and (17), it follows that \( \kappa_j - \kappa_{j-1} = \epsilon_j \) for \( j = 2, \ldots, M \) where \( \epsilon_2, \ldots, \epsilon_M \) satisfy

\[ \frac{\epsilon_j}{\epsilon_{j+1} + \epsilon_j} = \frac{\delta_j}{\delta_j + \delta_{j+1}} + \frac{\delta_j \delta_{j+1}}{\delta_j + \delta_{j+1}} b_j. \]

It follows that \( \xi = (\epsilon_2, \ldots, \epsilon_M) \) satisfies

\[ \frac{\epsilon_{j+1}}{\epsilon_j} = \frac{\delta_{j+1}}{\delta_j} \frac{1 - \delta_j b_j}{1 + \delta_{j+1} b_j} \quad j = 2, \ldots, M - 1. \]

(18)

Clearly, (18) does not determine \( \kappa \) uniquely; two additional conditions have to be specified. In order to ‘centre’ \( \kappa \) and ensure that, when the discussion is extended in Section 5 for the case of arbitrary measure \( \mu \) on \( R \), when an appropriate sequence of atomised measures \( \mu^{(N)} \) is considered, the processes on \( \kappa^{(N)} \) have suitable convergence properties, the following choice is made.
Let
\[ e_- = \inf \left\{ j : \sum_{i=1}^{j} p_i \geq \alpha \right\}, \quad e_+ = \sup \left\{ j : \sum_{i=j}^{M} p_i \geq \alpha \right\}. \tag{19} \]
where \(0 < \alpha < 0.5\) is a number chosen such that \(e_- < e_+\) (the inequality is strict). This is possible if \(S\) has 3 or more distinct states. Let \(K = (e_+ - e_-) \vee 1\). Then \(\epsilon = (\epsilon_2, \ldots, \epsilon_M)\) defined by
\[ \epsilon_{j+1} = \frac{\delta_{j+1} \prod_{k=1}^{j} \left( \frac{1-\delta_k b_k}{1+\delta_{k+1} b_k} \right)}{\sum_{a=e_-}^{e_+} \delta_{a+1} \prod_{k=1}^{a} \left( \frac{1-\delta_k b_k}{1+\delta_{k+1} b_k} \right)}. \tag{20} \]
satisfies (18) and \(\kappa_{e_+} - \kappa_{e_-} = K\).

Define \(\kappa\) in the following way: let \(e\) denote a median; a number such that \(\sum_{i=1}^{e} p_i \geq \frac{1}{2}\) and \(\sum_{i=e}^{M} p_i \geq \frac{1}{2}\) and set \(\kappa_{e} = 0\). For \(j \neq e\), set
\[ \begin{cases} \kappa_{e+j+1} = \kappa_{e+j} + \epsilon_{j+1} & j = 0, \ldots, M - e - 1 \\ \kappa_{e-j-1} = \kappa_{e-j} - \epsilon_{e-j-1} & j = 0, \ldots, e - 2 \end{cases} \tag{21} \]
where \(\epsilon\) is defined by (20). It is easy to see that \(\kappa\) thus defined, satisfies (17).

4 A Function to Accommodate the Killing Field

The method of proof adopted in this article is to try and rephrase the problem, as much as possible, in the language of [20] and to use as much of the technique from [20] as possible. The previous section introduced a co-ordinate change to deal with the drift \(b\); under the co-ordinate change, the problem with drift, but without killing, reduces to that of [20]. The introduction of killing presents other problems: firstly, even without drift, the initial condition is no longer as clear as it was in [20] when the killing field is non trivial. It cannot be taken as simply the expectation of the target distribution, since the process in the time interval \([0, t]\), conditioned on being alive at time \(t\), is no longer a martingale. Secondly, the process is killed at rate \(\lambda_j \tilde{k}_j\) on site \(i_j\), where \(\underline{\Lambda}\) is the holding intensity vector which is to be computed. This feeds into the equation required to obtain the intensities and there is no longer an explicit expression, even for the process stopped at an independent exponential time, like the formula \(\lambda_j = \frac{1}{p_j} F_j(p)\) that was available in [20]. The function \(G\) described in this section plays the role of \(F\) in [20].

Let \(p = (p_1, \ldots, p_M)\) satisfy \(\min_j p_j > 0\) and \(\sum_{j=1}^{M} p_j = 1\). Let \(S = \{i_1, \ldots, i_M\}, i_1 < \ldots < i_M\) be a finite state space with \(M\) elements; \(\hat{i} = (i_1, \ldots, i_M)\) will be used to denote the elements of the space. Let \(b = (b_2, \ldots, b_{M-1})\) satisfy Hypothesis 2.2 and let \(\kappa = (\kappa_1, \ldots, \kappa_M)\) denote the coordinate change of \(\hat{i}\) defined by Section 3. Let \(k = (k_2, \ldots, k_{M-1}) \in \mathbb{R}^{M-2}_+\) denote the killing field and let \(\tilde{k}\) be defined by Equation (10). Let \(G_j(t, p) : j \in \{1, \ldots, M\}\) be defined as follows:
The following lemma shows that such a function is well defined, which is a necessary step in accommodating the killing field.

**Lemma 4.1.** For a given \( \underline{p} \in \mathbb{S}^M \), there exists \( (\mathcal{G}_1, \ldots, \mathcal{G}_M) \) satisfying Equation (22).

**Proof** Consider \((\alpha_1, \ldots, \alpha_M)\) such that \(\alpha_j > 0\) for all \(j\) and \(\sum_{j=1}^M \alpha_j = 1\). Now consider, for some \(k \in \{2, \ldots, M-1\}\), \(\beta_k > 0\) and, for \(j \neq k\), \(\beta_j = \frac{1-\beta_k}{\alpha_k} \alpha_j\). Then \(\sum_{j=1}^M \beta_j = 1\). Let \(y = \sum_{j=1}^M \kappa_j \alpha_j\) and let \(z = \sum_{j=1}^M \kappa_j \beta_j\), then

\[
z = y \left( 1 - \beta_k \right) + \left( \frac{\beta_k - \alpha_k}{1 - \alpha_k} \right) \kappa_k \Rightarrow y = z + \left( \frac{\beta_k - \alpha_k}{1 - \beta_k} \right) (z - \kappa_k).
\]

so that, for \(z < \kappa_k\), it follows that \(\beta_k < \alpha_k \Rightarrow y < \kappa_k\) and \(\beta_k > \alpha_k \Rightarrow y < z < \kappa_k\). It therefore follows that \(z < \kappa_k \Rightarrow y < \kappa_k\).

Let \(\mathcal{G}_j^+ = \mathcal{G}_{M,j}^+ = \mathcal{G}_{M,j}^- = 0\) and define \(\mathcal{G}_j^+\) and \(\mathcal{G}_j^-\) for \(j = 2, \ldots, M-1\) by

\[
\begin{align*}
\mathcal{G}_j^- &= \frac{1}{tp_j (\kappa_j, z_j)} \sum_{a=1}^{j-1} (\kappa_j - \kappa_a) p_a (1 + t \mathcal{G}_a^- K_a) \quad 2 \leq j \leq M - 1 \\
\mathcal{G}_j^+ &= \frac{1}{tp_j (\kappa_j, z_j)} \sum_{a=j+1}^M (\kappa_a - \kappa_j) p_a (1 + t \mathcal{G}_a^+ K_a) \quad 2 \leq j \leq M - 1
\end{align*}
\]

Then these are well defined and positive. Define \(x_{0,j}\) by:

\[
\begin{align*}
x_{0,1} &= \frac{\sum_{a=1}^M \kappa_a p_a (1 + t \mathcal{G}_a^+ \bar{K}_a)}{\sum_{a=1}^M p_a (1 + t \mathcal{G}_a^+ \bar{K}_a)} \\
x_{0,j} &= \frac{\sum_{a=1}^M \kappa_a p_a (1 + t \mathcal{G}_a^+ \bar{K}_a) + \sum_{a=j+1}^M \kappa_a p_a (1 + t \mathcal{G}_a^- \bar{K}_a)}{\sum_{a=1}^M p_a (1 + t \mathcal{G}_a^+ \bar{K}_a) + \sum_{a=j+1}^M p_a (1 + t \mathcal{G}_a^- \bar{K}_a)} \quad j = 2, \ldots, M \\
x_{0,M+1} &= \frac{\sum_{a=1}^M \kappa_a p_a (1 + t \mathcal{G}_a^+ \bar{K}_a)}{\sum_{a=1}^M p_a (1 + t \mathcal{G}_a^+ \bar{K}_a)}
\end{align*}
\]

Clearly \(\kappa_1 < x_{0,j} < \kappa_M\) for each \(j \in \{1, \ldots, M+1\}\). To prove the lemma, it is necessary and sufficient to show that there is a \(j\) such that \(\kappa_{j-1} < x_{0,j} \leq \kappa_j\). Note that \(x_{0,M} < \kappa_M\). Suppose that \(x_{0,j} < \kappa_{j-1}\). Then, it follows from the argument above that \(x_{0,j-1} < \kappa_{j-1}\). If \(x_{0,j-1} > \kappa_{j-2}\), then existence has been established; otherwise, proceed inductively. Since \(\kappa_1 < x_{0,j} < \kappa_M\) for all \(j\), the result follows.

Let
\[
x_0 = \frac{\sum_{j=1}^{M} \kappa_j p_j (1 + tG_j(t, p)k_j)}{\sum_{j=1}^{M} p_j (1 + tG_j(t, p)k_j)}.
\]  

(24)

Let \( k = (k_1, \ldots, k_M) \) be the killing field, let \( \tilde{k} \) be defined by (10). This will give the initial condition for the process for geometric/exponential stopping times. \( x_0 \) may be considered as the average of \( \kappa \) under the measure \( Q(t, p, G(t, p)) \) where the quantity \( Q \) is defined by (25) below:

\[
Q_j(s, p, \Lambda) = \frac{p_j (1 + s\tilde{k}_j\lambda_j)}{\sum_{i=1}^{M} p_i (1 + s\tilde{k}_i\lambda_i)} \quad j \in \{1, \ldots, M\}
\]

(25)

(the killing field is considered fixed; this quantity will be considered as a function of time variable, the target probability and the intensities when it is used later).

5 Results

This section states the main results of the article, which are given as Theorems 5.1, 5.2, 5.3 and 5.4.

These theorems are stated separately, because each of them is of use in its own right. Firstly, Theorem 5.1 concerns Exponential and Geometric times. In this setting, an explicit solution can be obtained; \( \alpha \) and \( \Lambda \) are determined uniquely and there are equations to produce the explicit values. Theorem 5.2 considers Negative Binomial and Gamma times. Uniqueness is not shown, but the result comes in terms of the solution to an explicit fixed point problem. Theorem 5.3 takes an appropriate limit to obtain the result for deterministic times. While the result of Theorem 5.3 is the objective, the result of Theorem 5.2 which is a step along the way has an interesting interpretation in terms of the Local Variance Gamma Model of Carr [3] (which may be discussed at greater length in future work) and is therefore stated as a theorem in its own right. Theorems 5.1, 5.2 and 5.3 consider discrete state spaces.

Theorem 5.4 considers arbitrary probability measures where the measure and drift satisfy Hypothesis 1.1 are considered.

For Theorems 5.1, 5.2 and 5.3, let \( p = (p_1, \ldots, p_M) \in \mathbb{S}^{M} \), defined by (23). Let \( S = \{i_1, \ldots, i_M\} \), \( i_1 < \ldots < i_M \) be a finite state space with \( M \) elements and let \( b = (b_2, \ldots, b_{M-1}) \) satisfy Hypothesis 2.2 and let \( \tilde{k} \) be defined by (10).

Theorem 5.1. There exists a unique \( \Lambda = (\lambda_2, \ldots, \lambda_{M-1}) \in \mathbb{R}^{M-2}, \alpha \in (0, 1), l \in \{2, \ldots, M\} \) and \( \beta \in (0, 1] \) such that for all

\[
h \in \left(0, \min_{j} \left(\frac{1}{\lambda_j (1 + k_j)}\right)\right] .
\]

\( \tilde{P}^{(h)} \) given by Definition 2.3 is the one step transition matrix (time step length \( h \)) for a Markov chain \( X^{(h)} \) with state space \( S = \{i_1, \ldots, i_M, D\} \) that satisfies

\[
\alpha p_j = (1 - \beta) \mathbb{P}\left(X^{(h)}_{ht} = i_j | X^{(h)}_0 = i_{l-1}\right) + \beta \mathbb{P}\left(X^{(h)}_{ht} = i_j | X^{(h)}_0 = i_{l}\right) \quad j = 1, \ldots, M,
\]
where \( \tau \) is independent of \( X^{(h)} \) and satisfies \( \tau \sim Ge(a) \) with \( a = \frac{t}{t+h} \), so that \( \mathbb{E}[h \tau] = \frac{ah}{1-a} = t \). The constant \( \alpha \) satisfies:

\[
\alpha = \frac{1}{1 + t \sum_{j=1}^{M} p_j \lambda_j k_j} \tag{26}
\]

while \( \beta \) satisfies:

\[
\beta = \frac{x_0 - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} \tag{27}
\]

where \( x_0 \) is defined by (24). The intensity vector \( \lambda \) satisfies:

\[
\lambda_j = G_j, \quad j = 2, \ldots, M - 1 \tag{28}
\]

where \( G \) is defined by (22), existence of such a function given by Lemma 4.1. Taking \( h \to 0 \), there exists a continuous time Markov chain \( Y \) with state space \( S \), where for each \( j = 2, \ldots, M - 1 \), site \( i_j \) has holding intensity \( \lambda_j \) given by the same formula, and \( Y \) satisfies

\[
\alpha p_j = (1 - \beta) \mathbb{P}(Y_T = i_j | Y_0 = i_{l-1}) + \beta \mathbb{P}(Y_T = i_j | Y_0 = i_l), \quad \tau \sim \text{Exp}(1/t) \quad \text{(that is, exponential, with expected value } \mathbb{E}[\tau] = t), \quad \alpha \text{satisfies (26) and } \beta \text{satisfies (27).}
\]

Let \( \lambda \) satisfy (16) with \( \lambda \) given by (28), then the infinitesimal generator of the process \( Y \) is given by:

\[
a \left( \frac{1}{2} \Delta + b \nabla - k \right). \tag{29}
\]

The quantity \( \beta \) is interpreted in the following way: the process \( X^{(h)} \) has initial condition \( X^{(h)}_0 = x_0 \in (i_{l-1}, i_l) \) such that

\[
\mathbb{P}(X^{(h)}_{0+} = i_l | X^{(h)}_0 = x_0) = \beta \quad \mathbb{P}(X^{(h)}_{0+} = i_{l-1} | X^{(h)}_0 = x_0) = (1 - \beta).
\]

Now consider negative binomial times.

**Theorem 5.2.** For any \( r \geq 1 \), there exists an \( l \in \{2, \ldots, M\} \), a vector \( \lambda = (\lambda_2, \ldots, \lambda_{M-1}) \in \mathbb{R}_+^{M-2} \), a \( \beta \in (0, 1] \), and an \( h_0 \in (0, 1) \), such that for all \( h \in (0, h_0) \), there is an \( \alpha \) satisfying

\[
\alpha \in \left[ \left( 1 + \frac{(k\lambda)^* t}{r} \right)^{-r}, 1 \right] \tag{30}
\]

where \( (k\lambda)^* = \max_p \tilde{k}_p \lambda_j \) and \( \tilde{P}^{(h)} \) from Definition 2.5 is the one step transition matrix for a time homogeneous discrete time Markov chain \( X^{(h)} \), time step length \( h \) such that

\[
(1 - \beta) \mathbb{P}(X^{(h)}_{ht} = i_j | X^{(h)}_0 = i_{l-1}) + \beta \mathbb{P}(X^{(h)}_{ht} = i_j | X^{(h)}_0 = i_l) = \alpha p_j \quad j = 1, \ldots, M
\]

where \( \tau \sim NB(r, \frac{1}{t+h}) \), so that \( \mathbb{E}[h \tau] = t \).

By taking the limit \( h \to 0 \), there is a continuous time, time homogeneous Markov chain \( Y \) with transition intensity matrix \( \Theta \) given by (15), Definition 2.4, such that

\[
(1 - \beta) \mathbb{P}(Y_T = i_j | Y = i_{l-1}) + \beta \mathbb{P}(Y_T = i_j | Y_0 = i_l) = \alpha p_j \quad j = 1, \ldots, M
\]
where $\alpha$ satisfies (30) and $T \sim \text{Gamma}(r, \frac{1}{T})$; that is, $T$ is a Gamma time, with density function

$$f_T(x) = \frac{r^r}{\Gamma(r)} x^{r-1} e^{-xr/t} \quad x \geq 0$$

and expected value $\mathbb{E}[T] = t$.

This is extended to deterministic time:

**Theorem 5.3.** For a given $t > 0$, there exists a vector $\underline{\lambda} = (\lambda_2, \ldots, \lambda_{M-1}) \in \mathbb{R}^{M-2}_{+}$, an

$$\alpha \in [\exp \{ -t(k\lambda)^* \}, 1],$$

where $(k\lambda)^* = \max_j \lambda_j k_j$, an $l \in \{2, \ldots, M\}$, a $\beta \in (0, 1]$, such that $\Theta$ (Equation (15) Definition 2.4) is the intensity matrix for a time homogeneous continuous time Markov chain $X$ such that

$$\beta P(X_t = i_j | X_0 = i_t) + (1 - \beta) P(X_t = i_j | X_0 = i_{t-1}) = \alpha p_j \quad j = 1, \ldots, M.$$  

Again, if $a$ satisfies (16), then the infinitesimal generator of $X$ is defined by (29).

Finally, the continuous limit in the space variable can be taken.

**Theorem 5.4.** Let $\mu$ be a probability measure on $\mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ a measurable function satisfying Hypothesis 1.1 and $k : \mathbb{R} \to \mathbb{R}_{+}$ a measurable function. Then there exists a string measure $m$, a point $x_0 \in \mathbb{R}$, an $\alpha \in (0, 1]$ and a function $K$ satisfying

$$\frac{dK}{dx}(x) = \begin{cases} k(x) & x \in \text{suppt}(m) \\ 0 & x \notin \text{suppt}(m) \end{cases}$$

such that $\frac{1}{2} \frac{d^2}{dm dx} + b \frac{d}{dm} - \frac{dK}{dm}$ is the infinitesimal generator of a process $X$ which satisfies

$$P(X_t \leq x | X_0 = x_0) = \alpha \mu((-\infty, x]).$$

The initial condition $X_0 = x_0$ is interpreted as follows: let $S$ denote the support of $\mu$. Let $z_- = \sup \{ y < z | y \in S \}$ and $z_+ = \inf \{ y > z | y \in S \}$. Then there is a $\beta \in (0, 1]$ such that

$$\beta P(X_t \leq x | X_0 = x_0+) + (1 - \beta) P(X_t \leq x | X_0 = x_0-) = \alpha \mu((-\infty, x]).$$

That is, if $x_0 \notin S$, then the process immediately jumps into $S$, taking values $x_0+$ or $x_0-$ with probabilities $\beta$ and $1 - \beta$ respectively:

$$P(X_{0+} = x_0+ | X_0 = x_0) = \beta \quad P(X_{0+} = x_0- | X_0 = x_0) = 1 - \beta.$$

Note: if $m$ has a density $m'$, then the infinitesimal generator may be written as

$$\frac{1}{m'} \left( \frac{1}{2} \frac{d^2}{dx^2} + b \frac{d}{dx} - k \right).$$
6 Proofs of the results when $k \equiv 0$

For Theorems 5.1, 5.2 and 5.3 which consider a finite state space $S = \{i_1, \ldots, i_M\}$, let $\kappa$ be defined by (21). For $\kappa$ so defined, the quantities $q_{i,j+1}$ and $q_{i,j-1}$ from (12) satisfy (17). With $k \equiv 0$, the problem is therefore that of finding a martingale generalised diffusion and is therefore solved in the article [20] with the change of co-ordinates described above.

For Theorem 5.4, the proof also follows similarly to that of [20], with the following alterations. As in [20], let

$$b_N^h = \int_{i_{N,j-1}^-}^{i_{N,j+1}^-} b(x)dx$$

where $\int_{a_+}^b$ means $\int_{(a,b)}$, the integral over the open interval. Note that (11) of Hypothesis 2.2 is satisfied if:

$$-1 < \min_{j \in \{2, \ldots, M_N - 1\}} \frac{i_{N,j+1} - i_{N,j}}{i_{N,j+1} - i_{N,j-1}} \int_{i_{N,j-1}^-}^{i_{N,j+1}^-} b(x)dx \leq \max_{j \in \{2, \ldots, M_N - 1\}} \frac{i_{N,j} - i_{N,j-1}}{i_{N,j+1} - i_{N,j-1}} \int_{i_{N,j-1}^-}^{i_{N,j+1}^-} b(x)dx < 1$$

From (2) and (4) of Hypothesis 2.1, it follows that (11) of Hypothesis 2.2 is clearly satisfied for sufficiently large $N$, say $N > N_0$. For the remainder of the argument, only $N > N_0$ is considered. Using $b_N^h$ defined by (32), let $\Delta_N = (\lambda_{N,2}, \ldots, \lambda_{N,M_N - 1})$ where $\lambda_{N,j}$ denotes the holding intensity for site $i_{N,j}$, $j = 2, \ldots, M_N - 1$ that provide a solution to the marginal distribution problem. Let

$$q_{i,j}^{(N)} = \begin{cases} \lambda_j (i_{N,j+1} - i_{N,j})(i_{N,j} - i_{N,j-1}) & j = 2, \ldots, M_N - 1 \\ 0 & j = 1, M_N \end{cases}$$

and

$$m^{(N)}(\{i_{N,j}\}) = \begin{cases} \frac{(i_{j+1} - i_{j-1})}{a_{i,j}^{(N)}} & j \in \{2, \ldots, M_N - 1\} \\ +\infty & j = 1, M_N \end{cases}$$

The measure $m^{(N)}$ has support $\{i_{N,1}, \ldots, i_{N,M_N}\}$. Then, from the arguments of [20], there is a limiting measure $m$ such that for any $-\infty < x < y < +\infty$ there exists a subsequence $(N_j)_{j \geq 1}$ satisfying


\[
\lim_{j \to +\infty} \sup_{-\infty < x < a < b < +\infty} \left| m^{(N_j)}([a, b]) - m([a, b]) \right| = 0. \tag{38}
\]

Using the notation of Section 3 let \( \delta_{N,j} = i_{N,j} - i_{N,j-1} \) for \( j = 2, \ldots, M_N \), \( \epsilon_{N,j} = \kappa_{N,j} - \kappa_{N,j-1} \) and

\[
e_{N,-} = \inf \left\{ j : \sum_{i=1}^{j} p^{(N)}_i \geq \alpha \right\}, \quad e_{N,+} = \sup \left\{ j : \sum_{i=j}^{M_N} p^{(N)}_i \geq \alpha \right\}. \tag{39}
\]

where \( 0 < \alpha < 0.5 \) is a number chosen such that there exists an \( a \) and \( b \) such that \( a < b \) and \( \alpha > \mu((\infty, a]) \) and \( \alpha > \mu([b, +\infty)) \) and an \( N_0 \) such that \( \inf_{N > N_0} |i_{N,e_{N,+}} - i_{N,e_{N,-}}| > 0 \) (strict inequality). Only \( N > N_0 \) where this condition and (11) are satisfied will be considered. Let \( K_N = i_{N,e_{N,+}} - i_{N,e_{N,-}} \) and

\[
\epsilon_{N,j+1} = K_N \frac{\delta_{N,j+1} \prod_{k=1}^{j} \left( \frac{1-\delta_{N,k} b^{(N)}_k}{1+\delta_{N,k+1} b^{(N)}_k} \right) \sum_{a=e_{N,-}}^{e_{N,+}-1} \delta_{N,a+1} \prod_{k=1}^{a} \left( \frac{1-\delta_{N,k} b^{(N)}_k}{1+\delta_{N,k+1} b^{(N)}_k} \right)}{\prod_{k=1}^{e_{N,+}-1} \delta_{a+1} \prod_{k=1}^{a} \left( \frac{1-\delta_{N,k} b^{(N)}_k}{1+\delta_{N,k+1} b^{(N)}_k} \right)}. \tag{40}
\]

Let \( e_N \) denote a number such that \( \sum_{i=1}^{e_N} p^{(N)}_j \geq \frac{1}{2} \) and \( \sum_{i=e_N}^{M_N} p^{(N)}_j \geq \frac{1}{2} \). Set

\[
\begin{cases}
\kappa_{N,e_N} = 0 \\
\kappa_{N,e_N+j+1} = \kappa_{N,e_N+j} + \epsilon_{N,e_N+j+1}, \quad j = 0, \ldots, M_N - e_N - 1 \\
\kappa_{N,e_N-j} = \kappa_{N,e_N-j} - \epsilon_{N,e_N-j}, \quad j = 0, \ldots, e_N - 2.
\end{cases} \tag{41}
\]

Now note that

\[
\begin{align*}
i_{N,e_{N,+}} \to c_+ &:= \sup \left\{ x \in \text{suppt}(\mu) : \mu((\infty, x)) < 1 - \alpha \right\} \\
i_{N,e_{N,-}} \to c_- &:= \inf \left\{ x \in \text{suppt}(\mu) : \mu((\infty, x]) \geq \alpha \right\}
\end{align*}
\]

so that \( K_N \to c_+ - c_- \), a well defined positive limit and that, by construction, \( \kappa_{N,e_{N,+}} - \kappa_{N,e_{N,-}} = K_N \) for each \( N \).

The function of Hypothesis 1.1 (2) is to ensure that in the new co-ordinates, the process has a well defined expected value. The following lemma demonstrates that the hypothesis is sufficient.

**Lemma 6.1.** With \( b^{(N)}_j \) defined by (35) and \( (\mu, b) \) satisfying (2),

\[
\sup_N \sum_{j=1}^{M_N} |\kappa_{N,j}| p^{(N)}_j < +\infty. \tag{42}
\]

**Proof of lemma 6.1** Let \( e_{N,+} \) and \( e_{N,-} \) denote the indices defined in (39) and let

\[
C_N = K_N \frac{\prod_{k=1}^{e_{N,+}-1} \left( \frac{1-\delta_{k} b^{(N)}_k}{1+\delta_{k+1} b^{(N)}_k} \right) \sum_{a=e_{N,-}}^{e_{N,+}-1} \delta_{a+1} \prod_{k=1}^{a} \left( \frac{1-\delta_{k} b^{(N)}_k}{1+\delta_{k+1} b^{(N)}_k} \right)}{\prod_{k=1}^{e_{N,+}-1} \delta_{a+1} \prod_{k=1}^{a} \left( \frac{1-\delta_{k} b^{(N)}_k}{1+\delta_{k+1} b^{(N)}_k} \right)}. \]

where, as above, \( K_N = i_{N,e_{N,+}} - i_{N,e_{N,-}} = \kappa_{N,e_{N,+}} - \kappa_{N,e_{N,-}} \). Recall the definition of \( \kappa_{N,e} \) given by (16), that \( e_N \) is the index such that \( \kappa_{N,e_N} = 0 \). Also, \( \delta_{N,j} = (i_{N,j} - i_{N,j-1}) \). Recall the definition of \( b^{(N)} \) from (35). Then for \( j > m_N \),
\[ \kappa_{N,j} = C_N \sum_{k=e_N+1}^{j-1} (i_{N,k+1} - i_{N,k}) \prod_{l=e_N}^{k} \left( \frac{1 - \delta_{N,l} b_l^{(N)}}{1 + \delta_{N,l+1} b_l^{(N)}} \right) \]

\[ = C_N \sum_{k=e_N}^{j-1} (i_{N,k+1} - i_{N,k}) \exp \left\{ \sum_{l=e_N}^{k} \ln(1 - \delta_{N,l} b_l^{(N)}) - \ln(1 + \delta_{N,l+1} b_l^{(N)}) \right\} \]  

Similarly, for \( j < e_N \), so that \( \kappa_{N,j} < 0 \),

\[ -\kappa_{N,j} = C_N \sum_{i=j}^{e_N-1} (i_{N,k+1} - i_{N,k}) \exp \left\{ \sum_{l=i}^{e_N-1} \ln(1 - \delta_{N,l} b_l^{(N)}) - \ln(1 + \delta_{N,l+1} b_l^{(N)}) \right\} \]  

Note that

\[ C_N = K_N \left( \sum_{a=e_N-1}^{e_N-1} \delta_{a+1} \prod_{k=a+1}^{e_N-1} \left( \frac{1 + \delta_{k+1} b_k^{(N)}}{1 - \delta_{k} b_k^{(N)}} \right) \right) + \sum_{a=e_N}^{e_N-1} \delta_{a+1} \prod_{k=e_N}^{a} \left( \frac{1 - \delta_{k} b_k^{(N)}}{1 + \delta_{k+1} b_k^{(N)}} \right) \]  

\[ \leq K_N \left( \sum_{a=e_N-1}^{e_N-1} \delta_{a+1} \prod_{k=a+1}^{e_N-1} \left( \frac{1 - \delta_{k+1} b_k^{(N)}}{1 - \delta_{k} b_k^{(N)}} \right) \right) + \sum_{a=e_N}^{e_N-1} \delta_{a+1} \prod_{k=e_N}^{a} \left( \frac{1 - \delta_{k} b_k^{(N)}}{1 + \delta_{k+1} b_k^{(N)}} \right) \]  

\[ \leq \prod_{k=e_N+1}^{e_N} \left( \frac{1 + \delta_{k} b_k^{(N)}}{1 - \delta_{k+1} b_k^{(N)}} \right) \sum_{a=e_N}^{e_N-1} \delta_{a+1} \prod_{k=a+1}^{e_N} \left( \frac{1 - \delta_{k} b_k^{(N)}}{1 - \delta_{k+1} b_k^{(N)}} \right) \]  

\[ \leq \exp \left\{ 2 \int_{i_{N,-1}}^{e_N} |b(x)|dx + 2 \sum_{a=e_N}^{e_N-1} \left( \int_{i_{N,a+1}}^{i_{N,a}} b(x)dx \right)^2 \right\} \]

where the inequalities:

\[ \ln(1 + x) < x \quad -\ln(1 - x) < x + \frac{x^2}{2} \quad x > -1 \]  

have been used. Using \( i_{N,e_N-1} \downarrow c_- \) and \( i_{N,e_N+1} \uparrow c_+ \) together with (2) gives that \( C_N \) is uniformly bounded by a constant \( C < +\infty \). It follows that:

\[ \sum_j |\kappa_{N,j}| p_j^{(N)} \leq C \left( \sum_{j=e_N+1}^{M_N} p_j^{(N)} \sum_{k=e_N}^{j-1} (i_{N,k+1} - i_{N,k}) \right) \]

\[ \times \exp \left\{ 2 \int_{e_N}^{i_{N,k+1}} |b(x)|dx + 2 \sum_{a=e_N}^{k} \left( \int_{i_{N,a+1}}^{i_{N,a}} b(x)dx \right)^2 \right\} \]

\[ + \sum_{j=1}^{e_N-1} p_j^{(N)} \sum_{k=0}^{j} (i_{N,k} - i_{N,k-1}) \]

\[ \times \exp \left\{ 2 \int_{i_{N,k-1}}^{e_N} |b(x)|dx + 2 \sum_{a=k-1}^{e_N} \left( \int_{i_{N,a+1}}^{i_{N,a}} b(x)dx \right)^2 \right\} \]

\[ \leq C \int_{-\infty}^{\infty} \left( \int_{0}^{e_N} e^{F(b,y)}dy \right) \mu(dx) \]

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where $F$ is defined by (3) and the result now follows directly from (2).

**Lemma 6.2.** Let $\kappa^{(N)}$ denote the function

$$
\kappa^{(N)}(x) = \begin{cases}
\kappa_{N,1}, & x < i_{N,1} \\
\kappa_{N,j} + (\kappa_{N,j+1} - \kappa_{N,j}) \frac{x - i_{N,j}}{\delta_{N,j+1}} - i_{N,j}, & x \in [i_{N,j}, i_{N,j+1}) \quad 1 \leq j \leq M_N - 1 \\
\kappa_{N,M_N}, & x \geq i_{N,M_N}
\end{cases}
$$

(46)

There is a non-decreasing map $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ such that for any $-\infty < a < b < +\infty$,

$$
\lim_{N \to +\infty} \sup_{x \in [i_{N,1} \wedge a, i_{N,M_N} \wedge b]} |\kappa^{(N)}(x) - \kappa(x)| = 0
$$

(47)

**Sketch of Proof** Firstly, note that

$$
\frac{d\kappa^{(N)}}{dx} = \frac{\kappa_{N,j+1} - \kappa_{N,j}}{\delta_{N,j+1}} \quad x \in [i_{N,j}, i_{N,j+1}).
$$

(48)

and it is clear from (40) that $\frac{d\kappa^{(N)}}{dx}$ has a well defined limit, which exists by (2). Furthermore, $i_{N,\epsilon N}$ has a well defined limit and $\kappa^{(N)}(i_{N,\epsilon N}) = 0$ for each $N$. The result follows.

Let $X^{(N)}$ denote the process generated by $a^{(N)} \left( \frac{1}{2} \Delta_S + b^{(N)} \nabla S_N \right)$ where $\Delta_S$ and $\nabla S_N$ are the Laplacian and gradient operators respectively defined on $S_N = \{i_{N,1}, \ldots, i_{N,M_N}\}$ (Definition 2.1) and, with $m$ satisfying (38), let $X$ denote the process generated by $\left( \frac{1}{2} \frac{d^2}{da^2} + b \frac{d}{da} \right)$. Let $Y^{(N)} = \kappa^{(N)}(X^{(N)})$ where $\kappa^{(N)}$ is defined by (46) and the mapping $\kappa$ by (47) and $Y = \kappa(X)$. Then $Y^{(N)}$ is a process with state space $\mathcal{R}_N = \{\kappa_{N,1}, \ldots, \kappa_{N,M_N}\}$ where site $\kappa_{N,j}$ has holding intensity $\lambda_j^{(N)}$ for $j = 1, \ldots, M_N$ and, when it jumps from $\kappa_{N,j}$ for $j \in \{2, \ldots, M_N - 1\}$, it jumps to $\kappa_{N,j+1}$ with probability $\frac{\kappa_{N,j+1} - \kappa_{N,j-1}}{\kappa_{N,j+1} - \kappa_{N,j-1}}$ and to $\kappa_{N,j-1}$ with probability $\frac{\kappa_{N,j-1} - \kappa_{N,j}}{\kappa_{N,j+1} - \kappa_{N,j-1}}$. In short, it is a process with infinitesimal generator $\frac{\tilde{a}^{(N)}}{2} \Delta_{\mathcal{R}_N}$, where $\Delta_{\mathcal{R}_N}$ denotes the Laplace operator defined on functions on $\mathcal{R}_N$ (Definition 2.1) and (with reduction in the notation which is clear)

$$
\tilde{a}_j^{(N)} := \tilde{a}^{(N)}(\kappa_{N,j}) = \begin{cases}
\lambda_j^{(N)}(\kappa_{N,j+1} - \kappa_{N,j})(\kappa_{N,j} - \kappa_{N,j-1}) & j = 2, \ldots, M_N - 1 \\
0 & j = 1, M_N.
\end{cases}
$$

Let $\tilde{m}^{(N)}$ denote the measure supported on $\{\kappa_{N,1}, \ldots, \kappa_{N,M_N}\}$ defined by

$$
\begin{align*}
\tilde{m}^{(N)}(\{\kappa_{N,j}\}) &= \frac{(\kappa_{N,j+1} - \kappa_{N,j-1})}{\tilde{a}_j^{(N)}} \\
\tilde{m}^{(N)}(\{\kappa_{N,1}\}) &= \tilde{m}^{(N)}(\{\kappa_{N,M_N}\}) = +\infty
\end{align*}
$$

It follows from the convergence results of (38) and (47) that there is a limit $\tilde{m}$ such that for the convergent subsequence of (38)

$$
\lim_{j \to +\infty} \sup_{-\infty < \kappa(x) < a < b < \kappa(y) < +\infty} \left| \tilde{m}^{(N)}([a, b]) - \tilde{m}([a, b]) \right| = 0.
$$

(49)

As in [20], convergence of processes is based on the following result:
Theorem 6.3 (Characterisation of generalised diffusion). Let \( W \) denote a standard Wiener process starting from 0 and let \( \phi(s,a) \) denote its local time at site \( a \in \mathbb{R} \), at time \( s \geq 0 \). Let \( m \) be a measure on \( \mathbb{R} \). Let

\[
T(z,s) = \int_{\mathbb{R}} \phi(s,a-z)m(da)
\]

and

\[
T^{-1}(z,s) = \inf \left\{ r \mid \int_{\mathbb{R}} \phi(r,a-z)m(da) \geq s \right\}.
\]

Then \( Y(t,z) = z + W(T^{-1}(z,t)) \) is a strong Markov process with infinitesimal generator \( \frac{1}{2} \frac{d}{dx}^2 \).

This result is stated in Kotani-Watanabe [15].

Let \( f_{0}^{(N)} = \sum_{j=1}^{M_{N}} \kappa_{N,j}p_{j}^{(N)} \). It follows from Equation (47) and the definition of \( p_{j}^{(N)} \) (Equation (44)) that there exists an \( f_{0} \) such that \( \lim_{N \to +\infty} |f_{0}^{(N)} - f_{0}| = 0 \). It therefore follows from the convergence of Equation (49), that there is a subsequence such that for all \( \epsilon > 0 \)

\[
\lim_{j \to +\infty} \mathbb{P} \left( \sup_{0 \leq s \leq t} |Y_{s}^{(N)}(f_{0}^{(N)}) - Y_{s}(f_{0})| > \epsilon \right) = 0,
\]

where an initial condition \( y \) for \( Y_{s}^{(N)}(y) \) is interpreted as:

\[
\left\{ \begin{array}{l}
\mathbb{P} \left( Y_{0}^{(N)} = \kappa_{N,I_{N}(y)} | Y_{0}^{(N)} = y \right) = \frac{y - \kappa_{N,I_{N}(y)}}{\kappa_{N,I_{N}(y)} - y} = \beta_{N} \\
\mathbb{P} \left( Y_{0}^{(N)} = \kappa_{N,I_{N}(y)-1} | Y_{0}^{(N)} = y \right) = \frac{\kappa_{N,I_{N}(y)} - y}{\kappa_{N,I_{N}(y)} - y} = 1 - \beta_{N}
\end{array} \right.
\]

and \( I_{N}(y) \) is defined such that \( \kappa_{N,I_{N}(y)-1} < y \leq \kappa_{N,I_{N}(y)} \). Let \( y_{-} = \lim_{N \to +\infty} \kappa_{N,I_{N}(f_{0}^{(N)})-1} \) and let \( y_{+} = \lim_{N \to +\infty} \kappa_{N,I_{N}(f_{0}^{(N)})} \). Then, when \( y_{-} < y_{+} \) where the inequality is strict, the initial condition \( f_{0} \) for process \( Y(f_{0}) \) is interpreted as:

\[
\mathbb{P}(Y_{0} = y_{+} | Y_{0} = f_{0}) = \frac{f_{0} - y_{-}}{y_{+} - y_{-}} = \beta \quad \mathbb{P}(Y_{0} = y_{-} | Y_{0} = f_{0}) = \frac{y_{+} - f_{0}}{y_{+} - y_{-}} = 1 - \beta.
\]

From this,

\[
\lim_{j \to +\infty} \mathbb{P} \left( \sup_{0 \leq s \leq t} |X_{s}^{(N)} - X_{s}| > \epsilon \right) = 0
\]

where \( X^{(N)} \) satisfies:

\[
\mathbb{P}(X_{0}^{(N)} = i_{N,I_{N}(f_{0}^{(N)})}) = \beta_{N} \quad \mathbb{P}(X_{0}^{(N)} = i_{N,I_{N}(f_{0}^{(N)})-1}) = 1 - \beta_{N}
\]

and \( \beta_{N}(i_{N,I_{N}(f_{0}^{(N)})-1},i_{N,I_{N}(f_{0}^{(N)})}) \to (\beta,x_{-},x_{+}) \) and \( X \) satisfies:

\[
\mathbb{P}(X_{0} = x_{+}) = \beta \quad \mathbb{P}(X_{0} = x_{-}) = 1 - \beta.
\]

It follows that

\[
\lim_{j \to +\infty} \sup_{x} \left| \mathbb{P} \left( X_{t}^{(N)} \leq x \right) - \mathbb{P} \left( X_{t} \leq x \right) \right| = 0
\]

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and hence that

\[ P(X_t \leq x) = \mu((-\infty, x]) \]

where \( X \) is a diffusion process with infinitesimal generator \( \left( \frac{1}{2} d^2 \mu \right) + b \frac{d \mu}{dm} \) as required.

\[ \square \]

7 Introducing the Killing Field: Preliminary Results

Attention is now turned to the problem of introducing a killing field, \( k \). The following sections prove the theorems of the article stated in Section 5; this section presents preliminary results and notation.

The transition from finite state space to arbitrary measure on \( \mathbb{R} \) follows the same proof as [20], together with the arguments of Section 6, with only a few additions. Some discussion is necessary for modifying the proofs of [20] so that they can accommodate killing for geometric / exponential times and then to modify the fixed point theorem so that the transition can be made to negative binomial times. Once negative binomial times are accommodated, the limiting arguments to obtain the result for deterministic time is straightforward and the limiting argument to obtain the result for arbitrary state space follows directly from the analysis of [20].

Recall the definitions of \( \tilde{P} \) (Equation (13) Definition 2.3). Let

\[ \tilde{N}_t = \frac{t+h}{h} \left( I - \frac{t}{t+h} \tilde{P}(h) \right) \]

For the problem without drift or killing, this quantity appeared crucially in establishing the result for geometric times in [20], with \( \frac{1}{1-a} = \frac{tah}{h} \) giving \( a = \frac{t}{t+h} \).

The entries of \( \tilde{N} \) may be computed quite easily and are given in (51) below:

\[
\tilde{N}_{t;j,k}(\kappa, \lambda) = \begin{cases} 
1 & k = j = M + 1 \\
0 & j = M + 1, k \neq M + 1 \\
-t\lambda_j \tilde{k}_j & 1 \leq j \leq M, k = M + 1 \\
1 + t\lambda_j (1 + \tilde{k}_j) & k = j, 1 \leq j \leq M \\
-t\lambda_j \left( \frac{\kappa_j - \kappa_{j+1}}{\kappa_{j+1} - \kappa_j} \right) & k = j + 1, 2 \leq j \leq M - 1 \\
-t\lambda_j \left( \frac{\kappa_{j+1} - \kappa_{j-1}}{\kappa_{j+1} - \kappa_j} \right) & k = j - 1, 2 \leq j \leq M - 1 \\
0 & j = 1, k \in \{2, \ldots, M\} \\
0 & j = M, k \in \{1, \ldots, M - 1\} \\
0 & (j, k) \in \{1, \ldots, M\}^2 \ |j - k| \geq 2 
\end{cases}
\]  

(51)

where \( \tilde{k} \) is defined by (10). Note that this is independent of \( h \). Let \( \mathcal{N}_t \) denote the \( M \times M \) matrix such that \( \mathcal{N}_{t;j,k} = \tilde{N}_{t;j,k} \) for \( (j, k) \in \{1, \ldots, M\}^2 \).

Note The notation will be suppressed; \( \tilde{N}_t(\kappa, \lambda) \) and \( \mathcal{N}_t(\kappa, \lambda) \) will be written as \( \tilde{N} \) and \( \mathcal{N} \) respectively. Some particular variables (\( t \) or \( \lambda \)) may be introduced if they are of particular concern for the point under discussion.
It is straightforward to compute that $\sum_{k=1}^{M+1} \tilde{N}_{t,j,k} = 1$, but there does not seem to be a direct method to control the absolute values of the entries of the matrix. Control is therefore obtained by using the inverse. One result used in the sequel is that for integer $p \geq 1$, all the entries of $\tilde{N}^{-p}$ are non-negative, bounded between 0 and 1 and that for each $j$, $\sum_{k=1}^{M+1} (\tilde{N}^{-p})_{jk} = 1$. This follows from the following representation.

**Lemma 7.1.** For integer $p \geq 1$, $\tilde{N}^{-p}$ has representation:

$$ (\tilde{N}^{-p})_{jk} = \mathbb{P}(Z_T = k|Z_0 = j) $$

where $Z$ is a continuous time Markov chain on $\{1,\ldots,M+1\}$, with intensity matrix $\Theta$ (Equation (15), Definition 2.3) and $T \sim \text{Gamma}(p, \frac{t}{p})$, using the parametrisation of a Gamma distribution from (31) (that is the sum of $p$ independent exponential variables, each with intensity parameter $p/t$).

**Proof of lemma 7.1** Let $a = \frac{t}{t+h}$. Then, for $h < \frac{1}{\max_j \lambda_j (1+k_j)}$, where $\tilde{k}$ is defined by (10),

$$ (\tilde{N}^{-p})_{i,j} = \left( \frac{1}{1-a}(I-a\tilde{P}(h)) \right)_{i,j}^{-p} $$

$$ = (1-a)^p \sum_{k=0}^{\infty} \binom{p+k-1}{k} a^k (\tilde{P}(h))^k = (1-a)^p \sum_{k=0}^{\infty} \binom{p+k-1}{k} a^k \mathbb{P}(Z_{kh}^{(h)} = j|Z_0^{(h)} = i) $$

where $Z^{(h)}$ is a Markov chain with state space $\{1,\ldots,M+1\}$ and one-step transition matrix $\tilde{P}(h)$ (where $h$ is the time step length) defined by (13) Definition 2.3. Since

$$ \mathbb{P}(\tau = k) = \binom{p+k-1}{k} a^k (1-a)^p \quad k = 0,1,2,\ldots $$

is the probability mass function of an $\text{NB}(p,a)$ random variable, it follows that

$$ (\tilde{N}^{-p})_{i,j} = \mathbb{P}\left( Z_{h\tau}^{(h)} = j|Z_0^{(h)} = i \right) \quad \tau \sim \text{NB}\left(p, \frac{t}{t+h}\right) $$

so that $E[\tau] = \frac{kp(t/(t+h))}{(k/(t+h))}$ = $pt$. The fact that $h\tau$ converges in distribution to $T \sim \text{Gamma}(p,t)$ as $h \to 0$, $Z^{(h)}$ converges (in the sense of finite dimensional marginals) to the required continuous time Markov chain $Z$ and $Z_{h\tau}^{(h)} \xrightarrow{h \to 0} Z_T$ follows the proof found in [20].

The following precautionary lemma is introduced to deal with a problem that does not arise in [20]; it is necessary to establish that the Fixed Point Theorem (Theorem 9.1, which is the heart of the proof) does not give a process that is dead with probability 1 at the terminal time.

**Lemma 7.2.** For integer $p \geq 1$, there exists a constant $c > 0$ such that for all $j \in \{1,\ldots,M\}$, $(\tilde{N}^{-p})_{j,M+1} < 1-c$. 

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Proof Using the representation of the previous lemma: \((\tilde{N}_t^{-p})_{j,M+1} = \mathbb{P}(Z_T = k|Z_0 = j)\), recall that the embedded discrete time chain has transitions \(p_{j,j+1} = \frac{\alpha_{j,j+1}}{1+k_j}, p_{j,j-1} = \frac{\alpha_{j,j-1}}{1+k_j}, p_{j,j+M} = \frac{k_j}{1+k_j}\). If the process reaches site 1, it remains there; if the process reaches site \(M\) it remains there. It follows that, for \(j \neq M\),

\[
1 - \tilde{N}_{j,M+1} \geq \prod_{i=2}^{M-1} \left( \frac{\alpha_{i,i-1}}{1+k_i} \right) > 0
\]
as required. This is a lower bound on the probability that the process never reaches the cemetery site \(M + 1\).

The following lemma is used in the Fixed Point Theorem, to show that as \(\epsilon \to 0\), the sequence of fixed points for the approximating problems remains bounded.

**Lemma 7.3.** If \(\lambda_j \to +\infty\) then \((\tilde{N}_t^{-1})_j \to 0\) and consequently \((\tilde{N}_t^{-p})_j \to 0\) for any integer \(p \geq 1\) where the notation \(\cdot_j\) denotes the \(j\)th column of the matrix.

**Proof** Let \(\beta_{lj} = (\tilde{N}_t^{-1})_{lj}\). Then \(0 \leq \beta_{lj} \leq 1\). \(\beta\) satisfies the following system:

\[
- q_{l,l-1} \beta_{l-1,j} + \left( 1 + k_l \right) + \frac{1}{t \lambda_l} \beta_{lj} - \beta_{k,k+1} \beta_{k+1,j} = \begin{cases} 0 & k \neq j \\ \frac{1}{\lambda_j} & l = j \end{cases}
\]

where \(q_{l,l-1}, q_{l,l+1}\) for \(l = 2, \ldots, M - 1\) is defined by (12) and, by definition \(q_{l,l-1} \beta_{l-1,j} = 0\) for \(l = 1\) and \(q_{l,l+1} \beta_{l+1,j} = 0\) for \(l = M\).

From the a priori bounds on \(\beta_{lj}\) (namely \(0 \leq \beta_{lj} \leq 1\) and \(\sum_j \beta_{lj} = 1\) for each \(l\)) which follow directly from Lemma 7.4 it follows from (53) that \(\beta_{lj} \to 0\) for all \(l = 1, \ldots, M\).

The following lemma is key to proving Theorem 5.1 since the proof of Theorem 5.1 boils down to solving the system of equations defined by (51) given below.

**Lemma 7.4.** Let \(p\) be a probability measure over \(\{1, \ldots, M\}\). There exists a unique \(\alpha \in (0, 1]\), \(l \in \{2, \ldots, M\}\), \(x_0 \in (\kappa_{l-1}, \kappa_l)\) and \(\lambda \in \mathbb{R}_+^{M-2}\) satisfying (51):

\[
(\tilde{p} \tilde{N}_t)_k = \begin{cases} \frac{x_k - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} & l = k \\ \frac{x_k - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} & k = l - 1 \\ 0 & \text{otherwise} \end{cases}
\]

where \(\tilde{p}_k = \alpha p_k\) for \(k = 1, \ldots, M\) and \(\tilde{p}_{M+1} = 1 - \alpha\). The solution is the following: \(\alpha\) satisfies Equation (26), \(x_0\) satisfies Equation (27) and \(\lambda = (\lambda_2, \ldots, \lambda_{M-1})\) satisfies Equation (28), where \(G\) is defined by Equation (22) and \(Q\) is defined by Equation (29).

**Proof** The equation given by (54) for \(k = M + 1\) is:

\[
(1 - \alpha) \tilde{N}_{t:M+1,M+1} + \alpha \sum_{j=1}^{M} p_j \tilde{N}_{t,j,M+1} = 0,
\]

which is:
\[1 - \alpha = \alpha t \sum_{j=1}^{M} p_j \lambda_j \tilde{k}_j,\]

where \((\tilde{k}_1, \ldots, \tilde{k}_M)\) are defined by [10]. It follows that \(\alpha \in (0, 1]\) is required to satisfy:

\[\alpha = \frac{1}{1 + t \sum_{j=1}^{M} p_j \lambda_j \tilde{k}_j},\]

so that, if there is a solution, then \(\alpha\) is uniquely determined with this value. Let \(x_0 \in \mathbb{R}, \ l \in \{2, \ldots, M\}\) and let \(\tilde{\mathbf{u}}(l, x_0)\) satisfy:

\[v_j(x_0, l) = \begin{cases} \frac{x_0 - \kappa_{l-1}}{\kappa_{l} - \kappa_{l-1}} & j = l \\ \frac{\kappa_l - x_0}{\kappa_{l} - \kappa_{l-1}} & j = l - 1 \\ 0 & j \neq l, l - 1. \end{cases}\]

There are \(M\) equations involving the \(M - 2\) unknowns, \(\lambda_2, \ldots, \lambda_{M-1}\). These equations are:

\[
\begin{align*}
\alpha p_1 - t\lambda_2 \alpha p_2 q_{21} &= v_1 \\
-t\lambda_{j-1} \alpha p_{j-1} q_{j-1, j} + \alpha p_j (1 + t\lambda_j (1 + \tilde{k}_j)) - t\lambda_{j+1} \alpha p_{j+1} q_{j+1, j} &= v_j & j = 2, \ldots, M - 1 \\
-t\lambda_{M-1} q_{M-1, M} \alpha p_{M-1} + \alpha p_M &= v_M
\end{align*}
\]  

Set \(\Lambda_j = t\alpha p_j \lambda_j\) and \(\tilde{p}_j = \alpha p_j (1 + t\lambda_j \tilde{k}_j)\). Since

\[\alpha = \frac{1}{1 + \sum_{j=1}^{M} \frac{p_j \lambda_j \tilde{k}_j}{1 + t\lambda_j \tilde{k}_j}},\]

it follows from the definition of \(Q\) (Equation (25)) that \(\tilde{\mathbf{p}} = Q(t, \tilde{p})\) and \(\sum_{j=1}^{M} \tilde{p}_j = 1\). The system of equations (55) may be written, with these values, as (56):

\[
\begin{align*}
\tilde{p}_1 - \Lambda_2 q_{21} &= v_1 \\
-\Lambda_{j-1} q_{j-1, j} + (\tilde{p}_j + \Lambda_j) - \Lambda_{j+1} q_{j+1, j} &= v_j & j = 2, \ldots, M - 1 \\
-\Lambda_{M-1} q_{M-1, M} + \tilde{p}_M &= v_M
\end{align*}
\]

which is a linear system of \(M\) equations with \(M - 2\) unknowns. To show that it is of rank at most \(M - 2\): summing both left hand side and right hand side give 1 for any choice of \(x_0\).

Also,

\[\sum_{j} \kappa_j v_j = \frac{x_0 - \kappa_{l-1}}{\kappa_{l} - \kappa_{l-1}} + \kappa_{l-1} \frac{\kappa_{l} - x_0}{\kappa_{l} - \kappa_{l-1}} = x_0,\]

It follows that \(x_0\) is required to satisfy

\[x_0 = \alpha \sum_{j=1}^{M} \kappa_j p_j (1 + t\lambda_j \tilde{k}_j) = \frac{\sum_{j=1}^{M} \kappa_j p_j (1 + t\lambda_j \tilde{k}_j)}{\sum_{j=1}^{M} p_j (1 + t\lambda_j \tilde{k}_j)}\]

It follows that if there is a solution, then \(\alpha, l\) and \(x_0\) are uniquely determined with the values given in the statement of the lemma.
Since $\sum_{j=1}^{M} \tilde{p}_j = 1$, it follows that the system of equations given by (56) is that studied in [20]. From [20], it follows that $\Lambda$ satisfies:

$$\Lambda = \mathcal{L}(\tilde{p})$$

where

$$\mathcal{L}_j(\tilde{p}) = \left\{ \begin{array}{ll}
\frac{(\kappa_{j+1} - \kappa_{j-1})}{(\kappa_{j+1} - \kappa_j)(\kappa_j - \kappa_{j-1})} & 2 \leq j \leq l - 1 \\
\frac{(\kappa_{j+1} - \kappa_{j-1})}{(\kappa_{j+1} - \kappa_j)(\kappa_j - \kappa_{j-1})} & l \leq j \leq M - 1 \\
0 & j = 1 \text{ or } M.
\end{array} \right.$$

Therefore any solution satisfies (57):

$$\lambda_j \tilde{k}_j = \frac{1}{tp_j} \mathcal{L}_j(Q(t, \tilde{p})) = \frac{1 + t\lambda_j \tilde{k}_j}{tQ(t, \tilde{p})} \mathcal{L}_j(Q(t, \tilde{p})) = \frac{1}{t}(1 + t\lambda_j \tilde{k}_j) \mathcal{F}_j(Q(t, \tilde{p})) \quad j = 2, \ldots, M - 1 \quad (57)$$

where $\mathcal{F}$ is defined (as in [20]) by (58):

$$\mathcal{F}_j(p) = \left\{ \begin{array}{ll}
\frac{1}{p_j} \mathcal{L}_j(p) & j = 2, \ldots, M - 1 \\
0 & j = 1, M
\end{array} \right. \quad (58)$$

That is, $\Lambda$ is a solution if and only if $\lambda_1 = \lambda_M = 0$ and for $j = 2, \ldots, M - 1$,

$$\lambda_j = \frac{1}{t}(1 + t\lambda_j \tilde{k}_j) \mathcal{F}_j(Q(t, \tilde{p})) = \frac{1}{t}(1 + t\lambda_j \tilde{k}_j) \mathcal{F}_j(Q(t, \tilde{p})) \frac{1}{p_j(1 + t\lambda_j \tilde{k}_j)\alpha} = \frac{1}{tp_j} \frac{(\kappa_{j+1} - \kappa_{j-1})}{(\kappa_{j+1} - \kappa_j)(\kappa_j - \kappa_{j-1})} \times \{ \begin{array}{ll}
\sum_{i=1}^{j-1} (\kappa_j - \kappa_i)p_i(1 + t\lambda_i \tilde{k}_i) & 2 \leq j \leq l - 1 \\
\sum_{i=j+1}^{M} (\kappa_i - \kappa_j)p_i(1 + t\lambda_j \tilde{k}_i) & l \leq j \leq M - 1
\end{array} \right. \quad (59)$$

where $\mathcal{G}$ is defined by (22). The function $\mathcal{G}$ exists by Lemma 4.1. The proof of Lemma 7.4 is complete. \qed

8 Stopping at Independent Geometric or Exponential Time

The purpose of this section is to prove Theorem 5.1.

Proof of Theorem 5.1 This is equivalent to existence and uniqueness of an $l \in \{2, \ldots, M\}$, $\beta \in (0, 1)$, $\alpha \in (0, 1]$ and a $\Lambda \in \mathbb{R}^{M-2}$ such that $\tilde{P}^{(h)}$ (Definition 2.3, Equation (13)) is the transion matrix for a chain $X^{(h)}$ such that $\tilde{p}$ defined as:

$$\tilde{p}_j = \left\{ \begin{array}{ll}
\alpha p_j & j = 1, \ldots, M \\
1 - \alpha & j = M + 1
\end{array} \right. \quad (60)$$

satisfies:
\[ \hat{p}_j = (1 - a) \left( (1 - \beta)(I - a\bar{P}^{(h)})_{l,j}^{-1} + \beta(I - a\bar{P}^{(h)})_{l-1,j}^{-1} \right) \]

where \( X_0^{(h)} = x_0 \) for some \( x_0 \in (i_{l-1}, i_l] \) and \( \beta \in (0, 1] \) is a number such that

\[ \beta = \mathbb{P}(X_{0+}^{(h)} = i_{l-1} | X_0^{(h)} = x_0) \quad (1 - \beta) = \mathbb{P}(X_{0+}^{(h)} = i_l | X_0^{(h)} = x_0), \]

It follows that

\[ \frac{1}{1 - a} \left( \hat{p}(I - a\bar{P}^{(h)}) \right)_k = \frac{t + h}{h} \left( \hat{p}(I - \frac{t}{t + h} \bar{P}^{(h)}) \right)_k = \begin{cases} 1 - \beta & l = k \\ \beta & k = l - 1 \\ 0 & \text{otherwise}, \end{cases} \]

which is equivalent to showing existence of an \( \alpha, l, \beta \) and \( \lambda \) such that

\[ (\hat{p}\mathcal{N})_k = \begin{cases} (1 - \beta) & l = k \\ \beta & k = l - 1 \\ 0 & \text{otherwise}. \end{cases} \]

The result now follows directly from Lemma 7.4 with \( \kappa = (\kappa_1, \ldots, \kappa_M) \) the change of coordinates described in Section 3 and

\[ \begin{cases} \alpha = \frac{1}{1 + t \sum_{j=1}^M p_j \lambda_j k_j} \\ \beta = \frac{\kappa_l - x_0}{\kappa_l - \kappa_{l-1}} \\ \lambda_j = G_j(p, k, \kappa) \end{cases} \quad x_0 = \frac{\sum_{j=1}^M \kappa_j p_j (1 + t \lambda_j k_j)}{\sum_{j=1}^M p_j (1 + t \lambda_j k_j)} \quad (61) \]

where \( G \) is defined by (22) and \( \tilde{k} \) by (10).

The result now follows for \( 0 < h < \frac{1}{\max_j \lambda_j (1 + k_j)} \). The limiting argument to obtain a continuous time process as \( h \to 0 \), which has the prescribed marginal when stopped at an exponential time is given in [20].

The case with drift and killing on a finite state space, where the process is stopped at an independent exponential time, has now been solved for both of the problems.

9 Negative Binomial, Gamma and Deterministic Time

This section is devoted to the proofs of Theorems 5.2 and 5.3. They follow the lines of the proofs in [20], with some additional ideas required to deal with the killing field.

9.1 Proof of Theorem 5.2

This follows by appealing to the fixed point theorem, Theorem 9.1. As before, let \( \tau \sim NB(r, a) \), with \( a = \frac{t}{t+hr} \), so that \( E[\tau] = \frac{ra}{1+a} = t \). Then, with \( \tilde{P} \) defined by Equation (13) Definition 2.3

\[ \frac{1}{1 - a} (I - a\bar{P}^{(h)}) = \frac{t + hr}{hr} \left( I - \frac{t}{t + hr} \bar{P}^{(h)} \right) = \frac{(t/r) + h}{h} \left( I - \frac{(t/r)}{(t/r) + h} \bar{P}^{(h)} \right) = \tilde{N}_{1/r}. \]
If \( \tau \sim NB(r, a) \) with \( a = \frac{t}{r + lr} \), then \( \Lambda \) provides a solution if and only if there is an \( \alpha \in (0, 1) \), an \( l \) and an \( x_0 \in (\kappa_{l-1}, \kappa_l) \) such that

\[
\alpha p_j = \frac{\kappa_l - x_0}{\kappa_l - \kappa_{l-1}} \mathbb{P}(X_{h_\tau}^{(h)} = i_j | X_0^{(h)} = i_{l-1}) + \frac{x_0 - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} \mathbb{P}(X_{h_\tau}^{(h)} = i_j | X_0^{(h)} = i_l) \quad j = 1, \ldots, M
\]

Let \( \hat{p}_j = \alpha p_j \) for \( j = 1, \ldots, M \) and \( \hat{p}_{M+1} = 1 - \alpha \). Then \( (\alpha, x_0, \Lambda) \) provide a solution if and only if

\[
\hat{p}_j = \frac{\kappa_l - x_0}{\kappa_l - \kappa_{l-1}} \sum_{k=0}^{\infty} \mathbb{P}(X_{hk}^{(h)} = i_j | X_0^{(h)} = i_{l-1}) \mathbb{P}(\tau = k)
+ \frac{x_0 - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} \sum_{k=0}^{\infty} \mathbb{P}(X_{hk}^{(h)} = i_j | X_0^{(h)} = i_l) \mathbb{P}(\tau = k)
= (1 - a)^r \left( \frac{\kappa_l - x_0}{\kappa_l - \kappa_{l-1}} \sum_{k=0}^{\infty} \binom{k + r - 1}{k} a^k \left( \frac{\tilde{\tau}(\kappa_{l-1} | \kappa_l - 1)_{l-1, j}}{\kappa_l - \kappa_{l-1}} \right) + \frac{x_0 - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} \sum_{k=0}^{\infty} \binom{k + r - 1}{k} a^k \left( \frac{\tilde{\tau}(\kappa_{l-1} | \kappa_l - 1)_{l, j}}{\kappa_l - \kappa_{l-1}} \right) \right)
= \frac{\kappa_l - x_0}{\kappa_l - \kappa_{l-1}} (I - a \tilde{\tau}(\kappa_{l-1} | \kappa_l - 1)_{l-1, j}) + \frac{x_0 - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} (I - a \tilde{\tau}(\kappa_{l-1} | \kappa_l - 1)_{l, j})
\]

Let \( \mathbf{v} \) be the \( M + 1 \) vector and \( \mathbf{v} \) the \( M \) vector defined by \( v_l = \tilde{v}_l = \frac{x_0 - \kappa_{l-1}}{\kappa_l - \kappa_{l-1}} \), \( v_{l-1} = \tilde{v}_{l-1} = \frac{\kappa_l - x_0}{\kappa_l - \kappa_{l-1}} \) and \( v_j = 0 \) for \( j \neq l - 1, l \). It follows that a solution is provided by any \( \alpha \in (0, 1) \), \( \Lambda \) and \( x_0 \) such that

\[
\hat{p}_j N_{l, r}^{(r-1)} = \mathbf{v}
\]

holds. Let

\[
q = \frac{1}{\sum_{j,k} p_{jk} N_{l, r}^{(r-1)} N_{l, r}^{(r-1)}},
\]

then \( \Lambda \) provides a solution for all \( h \in \left( 0, \frac{1}{\max_j \lambda_j (1 + k_j)} \right) \) where \( \kappa \) is defined by (10), if and only if there is an \( \alpha \in (0, 1] \) such that \( \alpha q N_{l, r}(\Lambda) = \mathbf{v} \). It follows from Lemma 7.4 that \( \Lambda \) is a solution if and only if

\[
\lambda_j = G_j \left( Q \left( \frac{t}{r}, q \right) \right) \quad j = 1, \ldots, M
\]

(63)

where \( G \) is defined by (22) and \( Q \) by (25). Here

\[
\alpha = \frac{1}{1 + \frac{r}{t} \sum_{j=1}^{M} q_j \lambda_j k_j} \quad x_0 = \frac{\sum_{j=1}^{M} j q_j (1 + \frac{r}{t} \lambda_j k_j)}{\sum_{j=1}^{M} q_j (1 + \frac{r}{t} \lambda_j k_j)}.
\]

The existence of a \( \Lambda \) satisfying (63) follows from the fixed point Theorem 9.1. For the bounds on \( \alpha \), let \( \sigma = \inf \{ t : X_t \in D \} \), then

\[
\mathbb{P}(\sigma \geq nh) \geq (1 - (\lambda k)^n)^n
\]

so that, for \( \tau \sim NB \left( r, \frac{1}{t + lr} \right) \) independent of \( \sigma \) and using \( a = \frac{t}{t + lr} \),

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\[ \alpha = \mathbb{P}(\tau < \sigma) = \sum_{n=0}^{\infty} \mathbb{P}(\sigma > n|\tau = n)\mathbb{P}(\tau = n) \]

\[ = \sum_{n=0}^{\infty} \mathbb{P}(\sigma \geq n + 1)\mathbb{P}(\tau = n) \geq \sum_{n=0}^{\infty} (1 - (\lambda k)^n) n \left( \frac{n + r - 1}{r - 1} \right) a^n (1 - a)^{r-1} \]

\[ = \left( 1 + \frac{t(\lambda k)^n}{r} \right)^{-r} \]

as required. These results hold for all \( h > 0 \) and hence in the continuous time limit as \( h \to 0 \). Details of the convergence of finite dimensional marginals are given in [20].

9.2 Proof of Theorem [5.3]

This follows almost directly from the proof of Theorem [5.2] the problem is to show that when the limit is taken, the result is non-trivial. Let \( \Lambda^{(r)} \) denote a solution for the process stopped at an independent Gamma\((r, \frac{1}{r})\) time (parametrisation for the distribution defined by [33]). Let \( \mathbb{P}(X^{(r)}_t \in \{D\}) = 1 - \alpha_r \) (where \( \{D\} \) denotes the ‘cemetery’; \( 1 - \alpha_r \) is the probability that the process has been killed by time \( T_r \)). Note that \( \lambda_1 = \lambda_M = 0 \) (and hence there is no killing of the process once it has reached sites \( i_1 \) or \( i_M \)). This implies that \( \mathbb{P}(X^{(r)}_t \in \{D\}) > \prod_{j=1}^{M-1} \left( \frac{q_{M-1} q_{M-2} \ldots q_{M-j} M-j-1}{1+k_{M-j}} \right) \). This lower bound is constructed by considering that when the process jumps from site \( j \), it jumps to \( j+1 \) or \( j-1 \) or \( D \) with probabilities \( \frac{q_{j+1}}{1+k_j} \) and \( \frac{k_j}{1+k_j} \) respectively. Once it reaches site \( 1 \), it remains there for all time. It follows that \( \inf_r \alpha_r > 0 \).

Now suppose that there is a subsequence \( r_k \) and a \( j \in \{2, \ldots, M-1\} \) such that \( \lambda_j^{(r_k)} \to 0 \) then, in the limit, if the process reaches site \( i_j \), it remains there for all time, so that either \( p_{j+1} = \ldots = p_M = 0 \) (if \( x_0 \leq i_j \)) which is a contradiction, or \( p_1 = \ldots = p_{j-1} = 0 \) (if \( x_0 \geq i_j \)), again a contradiction.

It follows that there are two constants \( 0 < c < C < +\infty \) such that \( c < \inf_j \inf_r \lambda_j^{(r)} \leq \sup_j \sup_r \lambda_j^{(r)} < C \) and hence it follows that there is a limit point \( \Lambda \) of \( \Lambda^{(r)} \) which provides a solution. The lower bound (52) follows by taking the limit as \( r \to +\infty \) in (30).

Note At this point there is a (minor) divergence when one tries to establish existence of \( m \) such that the generator defined by [5] has the required properties. When considering this problem, there are killing rates \( k_1 \) and \( k_M \) on sites \( i_1 \) and \( i_M \) respectively, which are not necessarily 0. But after the process reaches either of these sites, the killing rate is exponential and therefore the process survives with positive probability for any finite time and it is straightforward to obtain an upper bound on the killing probability which is strictly less than 1 when the process is stopped at a Gamma\((r, \frac{1}{r})\) time for fixed \( t > 0 \); the upper bound is independent of \( r \geq 1 \).

9.3 Fixed Point Theorem

For fixed \( \lambda \) let \( \mathbb{N} : \mathbb{R}^M \times \mathbb{R}^{M-2} \to \mathbb{R}^M \) denote the function defined by:

\[ \mathbb{N}(\mathbf{p}, \Lambda) = \frac{1}{\sum_{j,k} p_j N_{jk}(\Lambda)} p_N(\Lambda) = \frac{1}{\sum_j p_j (1 + \frac{1}{r} \lambda_j k_j)} p_N(\Lambda). \]
where \( \tilde{k} \) is defined by (10). Directly from the definition, for any \( p \in \mathbb{R}_+^M \) and \( \Lambda \in \mathbb{R}_+^{M-2} \),
\[
\sum_j h_j(p, \Lambda) = 1. \quad (65)
\]
From the definition, it is also clear that \( h(\alpha p, \Lambda) = h(p, \Lambda) \) for any \( p \in \mathbb{R}_+^M \), \( \alpha \in \mathbb{R}\setminus\{0\} \) and \( \Lambda \in \mathbb{R}_+^{M-2} \).

For \( r \geq 2 \), set
\[
h^{(r)}(p, \Lambda) = h(h^{(r-1)}(p, \Lambda), \Lambda). \quad (66)
\]

**Theorem 9.1** (Fixed Point Theorem). Set
\[
A(\Lambda, p)(j) := G_j \left( \frac{t}{r}, Q \left( \frac{t}{r}, h^{(r-1)}(p, \Lambda), \Lambda \right) \right),
\]
where \( Q \) defined by (25). There exists a solution \( \Lambda \) to the equation
\[
\Lambda = A(\Lambda, p). \quad (68)
\]

**Proof of Theorem 9.1** The proof follows the lines of [20]. As in [20], for \( p \in \mathbb{R}_+^M \), set
\[
C(p, \epsilon) = \sum_{j=1}^M \left( \frac{p_j}{\sum_{k=1}^M (p_k \lor \epsilon)} \lor \epsilon \right). \quad (69)
\]
For any \( p \in \mathbb{R}_+^M \) and \( \epsilon \in (0, 1) \), \( C(p, \epsilon) \leq M \). For \( p \in \mathbb{R}_+^M \) such that \( \sum_{k=1}^M (p_k \lor 0) \geq 1 \), it follows that for any \( \epsilon \in [0, 1) \), \( \sum_{k=1}^M (p_k \lor \epsilon) \geq 1 \) and hence that
\[
M \geq C(p, \epsilon) \geq \sum_{j=1}^M \left( \frac{p_j}{\sum_{k=1}^M (p_k \lor \epsilon)} \lor \frac{\epsilon}{\sum_{k=1}^M (p_k \lor \epsilon)} \right) = 1.
\]
Let \( P^{(\epsilon)} : \mathbb{R}_+^M \to \mathbb{R}_+^M \) denote the function
\[
(P^{(\epsilon)}_{j}(p)) = \frac{1}{C(p, \epsilon)} \left( \frac{p_j}{\sum_{k=1}^M (p_k \lor \epsilon)} \lor \epsilon \right) \quad (70)
\]
where \( C \) is defined by (69) so that \( \sum_{j=1}^M P_{j}^{(\epsilon)}(p) = 1 \). It follows that for any \( p \in \mathbb{R}_+^M \),
\[
\min_j P_{j}^{(\epsilon)}(p) \geq \frac{\epsilon}{M}.
\]
Set
\[
A^{(\epsilon)}(\Lambda, p)(j) = G_j \left( \frac{t}{r}, Q \left( \frac{t}{r}, P^{(\epsilon)}(h^{(r-1)}(p, \Lambda)), \Lambda \right) \right). \quad (71)
\]

**Lemma 9.2.** For each \( \epsilon > 0 \), there exists a \( K(\epsilon) < +\infty \) such that
\[
\sup_{\Lambda \in \mathbb{R}_+^{M-2}} \max_j A^{(\epsilon)}(\Lambda, p) \leq K(\epsilon).
\]
Proof Consider Equation (22). If \( p_j \geq \epsilon \) for all \( j \in \{1, \ldots, M\} \), then \( A^{(e)}(\underline{\lambda}, \underline{p})(j) \leq f_j \) where \( f_j \) satisfies

\[
\begin{cases}
  f_j = a + b \sum_{i=1}^{j-1} f_i & j = 2, \ldots, M \\
f_1 = 0 & \\
a = \frac{1}{\epsilon} \max_j \frac{r(\kappa_{j+1} - \kappa_j - 1)(\kappa_{M} - \kappa_1)}{r(\kappa_{j+1} - \kappa_j)(\kappa_{j} - \kappa_{j-1})} & b = \max_j \overline{k}_j
\end{cases}
\]

where \( \overline{k}_j : j = 1, \ldots, M \) is defined by (10). The solution to this equation is

\[
f_1 = 0 \quad f_j = a(1 + b)^{j-2} \quad j \geq 2.
\]

and hence

\[
0 \leq \min_j G_j \leq \max_j G_j \leq a(1 + b)^{M-2}.
\]

This depends on \( \epsilon \), but it does not depend on \( \underline{\lambda} \).

It is clear, from the construction, that for \( \epsilon > 0 \), \( A^{(e)}(\cdot, \underline{p}) \) is continuous in \( \underline{\lambda} \). Therefore, by the Schauder fixed point theorem, there is a solution \( \underline{\lambda}^{(e)} \) to the equation

\[
\underline{\lambda} = A^{(e)}(\underline{\lambda}, \underline{p}).
\]

Let \( \lambda^{(e)} \) denote a fixed point (solution) and let

\[
\underline{h}_\epsilon = P^{(e)}(h^{(r-1)}(p, \lambda^{(e)})),
\]

where \( P^{(e)} \) is defined by (70), so that

\[
\lambda_j^{(e)} = G_j \left( \frac{t}{r}, Q \left( \frac{t}{r}, \underline{h}_\epsilon \right) \right) \quad j = 1, \ldots, M
\]

where \( G \) is defined by (22) and \( Q \) is defined by (25). It is required to show:

- \( \limsup_{\epsilon \to 0} \max_j \lambda_j^{(e)} < +\infty \)
- \( \liminf_{\epsilon \to 0} \min_j \lambda_j^{(e)} > 0 \)

Showing \( \limsup_{\epsilon \to 0} \max_j \lambda_j^{(e)} < +\infty \) It follows from (73), using the definition of \( G \) (Equation (22)) and the definition of \( Q \) (Equation (62)) that:

\[
\frac{t}{r} \underline{h}_\epsilon j \lambda_j^{(e)} \left( 1 + \frac{t}{r} \overline{k}_j \lambda_j^{(e)} \right) = \frac{(\kappa_{j+1} - \kappa_{j-1})}{(\kappa_{j+1} - \kappa_j)(\kappa_j - \kappa_{j-1})} \times \begin{cases}
\sum_{i=1}^{j-1} (\kappa_j - \kappa_i) h_{\epsilon,i} \left( 1 + \frac{t}{r} \overline{k}_i \lambda_i^{(e)} \right)^2 & 2 \leq j \leq l - 1 \\
\sum_{i=j+1}^{M} (\kappa_i - \kappa_j) h_{\epsilon,i} \left( 1 + \frac{t}{r} \overline{k}_i \lambda_i^{(e)} \right)^2 & l \leq j \leq M - 1
\end{cases}
\]

It follows that
\[ h_{e,j}^{\lambda_j^{(e)}} \rightarrow +\infty. \] (75)

This can be seen inductively from (74): recall that \( h_{e,j} > 0 \) for each \( j \) and \( \sum_{j=1}^M h_{e,j} = 1. \) Since \( \lambda_1^{(e)} = \lambda_M^{(e)} \equiv 0, \) the result is clearly true for \( j = 2 \) and \( M-1 \) and, furthermore, there are uniform bounds on \( \frac{1}{r} h_{e,2} \lambda_2^{(e)} \left( 1 + \frac{1}{r} k_2 \lambda_2^{(e)} \right) \) and \( \frac{1}{r} h_{e,M-1} \lambda_{M-1}^{(e)} \left( 1 + \frac{1}{r} k_{M-1} \lambda_{M-1}^{(e)} \right) \) and hence on \( \frac{1}{r} h_{e,2} \left( 1 + \frac{1}{r} k_2 \lambda_2^{(e)} \right)^2 \) and \( \frac{1}{r} h_{e,M-1} \left( 1 + \frac{1}{r} k_M - 1 \lambda_M - 1^{(e)} \right)^2. \) It follows inductively that there are uniform bounds on \( \frac{1}{r} h_{e,j} \left( 1 + \frac{1}{r} k_j \lambda_j^{(e)} \right)^2 \) which hold for all \( j \) and hence \( (75) \) follows.

Set

\[ K(\Delta, \epsilon) := \sum_{j=1}^M \left( h_j^{(r-1)}(p, \Delta) \lor \epsilon \right) \quad \text{and} \quad K_\epsilon := K(\Delta^{(e)}, \epsilon), \]

then, from (65), it is clear that \( K_\epsilon \geq 1. \) Set

\[ C_\epsilon = C \left( h^{(r-1)}(p, \Delta^{(e)}) \right) \]

where \( C \) is the function defined by (69).

Let

\[ \tilde{\mathcal{N}}^{(e)} = \tilde{\mathcal{N}}(\Delta^{(e)}), \quad \mathcal{N}^{(e)} = \mathcal{N}(\Delta^{(e)}). \]

The first equality below follows from the definition of \( \tilde{\mathcal{N}} \) by (72) and (70). The second equality follows from the definition of \( \tilde{\mathcal{N}} \) by (64) and \( \tilde{\mathcal{N}}^{(r)} \) given by (66) together with the identity: \( \sum_{j=1}^M h_j^{(r-1)} = 1, \) which follows from (66) and (65). Recall that \( p \) and \( \mathcal{N} \) are taken as row vectors.

\[ h_{e,j} = \frac{1}{C_\epsilon} \left( \frac{1}{K_\epsilon} h_j^{(r-1)}(p, \Delta^{(e)}) \lor \epsilon \right) = \frac{1}{C_\epsilon K_\epsilon} \sum_k (p \mathcal{N}^{(r-1)})_k \left( \mathcal{N}^{(r-1)} \right)_j \lor \frac{\epsilon}{C_\epsilon}, \]

where \( (p \mathcal{N}^{(r-1)})_j = \sum_k p_k (\mathcal{N}^{(r-1)})_{k,j}; (\mathcal{N}^{(r-1)})_{k,j} \) being the \( (k, j) \) component of the matrix \( \mathcal{N}^{(r-1)}. \)

Set

\[ \tilde{\mathcal{H}}_{e,j} = \left\{ \begin{array}{ll} \frac{1}{C_\epsilon K_\epsilon} \sum_k (p \mathcal{N}^{(r-1)})_k (\mathcal{N}^{(r-1)})_j & h_{e,j} \geq \frac{\epsilon}{C_\epsilon} \\ \frac{\epsilon}{C_\epsilon} & h_{e,j} < \frac{\epsilon}{C_\epsilon} \end{array} \right. \] (76)

then, since \( \sum_{j=1}^M p_j = 1, \)

\[ h = p \tilde{\mathcal{H}}_e. \] (77)

Define \( p^{(e)} \) as:

\[ p^{(e)} := \frac{1}{\sum_l (p \tilde{\mathcal{H}}_e \mathcal{N}^{(e)-(r-1)})_l} p \tilde{\mathcal{H}}_e \mathcal{N}^{(e)-(r-1)}. \] (78)
By construction, \( \sum_{j=1}^{M} p_{j}^{(e)} = 1 \). Furthermore, it follows from the definition that \( p^{(e)} \) satisfies:

\[
p^{(e)} = \frac{1}{\sum_{i}(b_{i} N^{(e)-(r-1)}_{i})} b_{i} N^{(e)-(r-1)}.
\]

(79)

From the characterisation given by Lemma 7.1 it follows that \( 0 \leq (\hat{N}^{(e)-(r-1)})_{j,k} \leq 1 \) for each each \( (j,k) \in \{1,\ldots,M+1\}^{2} \). Furthermore, \( h_{\epsilon,j} \geq 0 \) for all \( \epsilon > 0 \) and all \( j \in \{1,\ldots,M\} \). From this, it follows that \( p_{j}^{(e)} \geq 0 \) for each \( j \in \{1,\ldots,M\} \).

From (18) and (19), it follows that

\[
h_{\epsilon} = \left( \sum_{i} \left( p H_{\epsilon} N^{(e)-(r-1)}_{i} \right) \right) p^{(e)} N^{(e)r-1}.
\]

Set

\[
S_{\epsilon} = \left\{ \beta | h_{\epsilon,\beta} = \frac{\epsilon}{C_{\epsilon}} \right\}
\]

(80)

Let \( Y \) be a continuous time Markov chain with state space \( \{1,\ldots,M+1\} \) with transition intensity matrix given by Equation (15), Definition 2.4. Let \( T \) denote an independent time with distribution \( T \sim \text{Gamma} (r-1, \frac{1}{\lambda}) \). Let

\[
c(m) = 1 - \mathbb{P}(Y_{T} = M+1 | Y_{0} = m).
\]

It follows from (22) that \( (\hat{N}^{(e)-(r-1)})_{M+1,j} = 0 \) for \( j = 1,\ldots,M \). From this it follows that

\[
\hat{N}^{(e)-(r-1)} = \begin{pmatrix} N^{(e)-(r-1)} & -N^{(e)-(r-1)} v \\ 0 & 1 \end{pmatrix}
\]

where \( v \) is an \( M \)-row vector of \( 0 \)s, and \( u \) is the \( M \)-column vector with \( v_{j} = (\hat{N}^{(e)r-1})_{j,M+1}, j = 1,\ldots,M \).

It follows that for \( m_{2} \in S_{\epsilon} \),

\[
\sum_{k} (N^{(e)-(r-1)})_{m_{1},k} \hat{H}_{k,m_{2}} = \frac{\epsilon}{C_{\epsilon}} \sum_{k=1}^{M} (N^{(e)-(r-1)})_{m_{1},k} = \frac{\epsilon}{C_{\epsilon}} (1 - (N^{(e)-(r-1)})_{m_{1},M+1}) = \frac{\epsilon}{C_{\epsilon}} c(m_{1}).
\]

It follows that:

\[
\left( N^{(e)-(r-1)} \hat{H}_{\epsilon} \right)_{m_{1},m_{2}} = \begin{cases} \frac{1}{c \epsilon k_{2}} \sum_{j} p_{j} (N^{(\epsilon)-(r-1)})_{j,k} I(m_{1},m_{2}) & m_{2} \notin S_{\epsilon} \\ \frac{1}{c \epsilon} c(m_{1}) & m_{2} \in S_{\epsilon} \end{cases}
\]

(81)

where \( I(m_{1},m_{2}) = \begin{cases} 1 & m_{1} = m_{2} \\ 0 & m_{1} \neq m_{2} \end{cases} \). Set

\[
F_{\epsilon} := N^{(e)-(r-1)} \hat{H}_{\epsilon} \quad \text{so that} \quad N^{(e)r-1} F_{\epsilon} = \hat{H}_{\epsilon}.
\]

(82)

It follows (from (78), using (77) in the denominator)

\[
p^{(e)} = \frac{1}{\sum_{j,k} h_{\epsilon,j} (N^{(e)-(r-1)})_{j,k}} p(N^{(e)r-1} F_{\epsilon} N^{(e)-(r-1)}).
\]

(83)
Let $\Lambda_\epsilon$ be the matrix such that

$$
\Lambda_{\epsilon;m_1,m_2} = \begin{cases} 
1 & m_1 = m_2, m_2 \not\in S_\epsilon \\
0 & \text{otherwise}.
\end{cases}
$$

Let $I_\epsilon$ denote the matrix with entries:

$$
I_{\epsilon;m_1,m_2} = \begin{cases} 
c(m_1) & m_2 \in S_\epsilon \\
0 & \text{otherwise}
\end{cases}
$$

Then $I_\epsilon$ has column $(c(1), \ldots, c(M))^t$ for each $m_2 \in S_\epsilon$ and the remaining columns are columns of 0s. Then (81) may be written, using $F_{\epsilon}$ from (82) as:

$$
F_{\epsilon} = \frac{1}{C_\epsilon K_\epsilon K_\epsilon} \sum_k (p \tilde{N}_\epsilon^{-1})_k \Lambda_\epsilon + \frac{\epsilon I_\epsilon}{C_\epsilon},
$$

so that

$$
\mathbb{P}^{(\epsilon)} = \frac{1}{\sum_k (h_k N_\epsilon^{-1})_k} \left( \frac{p \tilde{N}_\epsilon^{-1} \Lambda_\epsilon N_\epsilon^{(\epsilon)-(r-1)}}{C_\epsilon K_\epsilon \sum_k (p \tilde{N}_\epsilon^{-1})_k} + \frac{\epsilon I_\epsilon N_\epsilon^{(\epsilon)-(r-1)}}{C_\epsilon} \right). 
$$

Note that, since $(\tilde{N}_\epsilon^{(\epsilon)-(r-1)})_{jk} = (N_\epsilon^{(\epsilon)-(r-1)})_{jk}$ for $(j, k) \in \{1, \ldots, M\}^2$, and $\sum_{k=1}^{M+1} (\tilde{N}_\epsilon^{(\epsilon)-(r-1)})_{jk} = 1$ for all $j \in \{1, \ldots, M\}$, it follows that

$$
\sum_k (h_k N_\epsilon^{(\epsilon)-(r-1)})_k = 1 - \sum_k h_{\epsilon;k}(\tilde{N}_\epsilon^{(\epsilon)-(r-1)})_{k,M+1} > c_1 > 0
$$

(85)

where $c_1$ does not depend on $\epsilon$, by Lemma 7.2.

**Note** Similarly to the note at the end of Subsection 9.2, this is the other point where an additional argument is required when an infinitesimal generator given by (6) is required, since there is killing at sites $i_1$ and $i_M$ at rates $k_1$ and $k_M$ respectively, which are not necessarily 0. The modification is similar. Consider the proof of Lemma 7.2 after the process eventually reaches state 1 or state M, which it does with positive probability, the killing rate after it hits these sites is bounded; it has rate $k_1$ on site 1 and $k_M$ on site M and with Generator (6) the kill rate does not depend on $\Lambda$. Hence a $c_1 > 0$ may be obtained independent of $\epsilon$ such that (85) holds.

For any invertible matrix $S$, the eigenvalues of $S^{-1} A S$ are the same as the eigenvalues of $A$. It follows that the eigenvalues of $N_\epsilon^{(\epsilon)-(r-1)} \Lambda_\epsilon N_\epsilon^{(\epsilon)-(r-1)}$ are the eigenvalues of $\Lambda_\epsilon$; 0 with multiplicity equal to the number of elements of $S_\epsilon$ and the remaining eigenvalues all 1. Similarly, the eigenvalues of $I_\epsilon$ are bounded independently of $\epsilon$, since each entry of the $M \times M$ matrix lies in $[0, 1]$. It follows that

$$
\lim_{\epsilon \to 0} \frac{\epsilon}{C_\epsilon} p \tilde{N}_\epsilon^{-1} I_\epsilon N_\epsilon^{(\epsilon)-(r-1)} = 0.
$$

It now follows directly that if $K_\epsilon \sum_k (p \tilde{N}_\epsilon^{-1})_k \xrightarrow{\epsilon \to 0} +\infty$, then $p^{(\epsilon)} \xrightarrow{\epsilon \to 0} 0$, contradicting the fact that $p^{(\epsilon)}_j \geq 0$ for each $j$ and $\sum_j p^{(\epsilon)}_j = 1$ for each $\epsilon \in (0, 1)$. 

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Therefore:

\[ 0 \leq \inf_{\epsilon} K_{\epsilon} \left( \sum_{k} (p N^{(\epsilon)r-1})_k \right) \leq \sup_{\epsilon} K_{\epsilon} \left( \sum_{k} (p N^{(\epsilon)r-1})_k \right) < +\infty. \]

From the definition of \( K_{\epsilon} \),

\[ K_{\epsilon} \left( \sum_{k} (p N^{(\epsilon)r-1})_k \right) = \sum_{k=1}^{M} \left( \frac{(p N^{(\epsilon)r-1})_k \vee \left( \sum_{k} (p N^{(\epsilon)r-1})_k \right)}{\epsilon} \right). \]

From the above,

\[ 0 \leq \inf_{\epsilon} \sum_{k} (p N^{(\epsilon)r-1})_k \leq \sup_{\epsilon} \sum_{k} (p N^{(\epsilon)r-1})_k < +\infty \]

and

\[ \sup_{\epsilon} \max_{k} \left( \frac{(p N^{(\epsilon)r-1})_k \vee 0}{\epsilon} \right) < +\infty, \]

from which

\[ \sup_{\epsilon} \max_{k} \left| (p N^{(\epsilon)r-1})_k \right| < +\infty. \]

Set \( \lambda^{(\epsilon)} = \max_{j} \lambda_j^{(\epsilon)} \) and let

\[ N^{* (\epsilon)} = \frac{1}{\lambda^{(\epsilon)}} N^{(\epsilon)} \]

(that is, divide every element by \( \lambda^{(\epsilon)} \)). Then if there is a sequence \( \epsilon_n \to 0 \) such that \( \lambda^{(\epsilon_n)} \to +\infty \), any limit point \( N^{*} \) of \( N^{* (\epsilon_n)} \) satisfies

\[ 0 = p N^{* (r-1)}. \]

It follows from the construction of \( N^{*} \) that the rank \( \rho \) of \( N^{*} \) is the number of components of \( \lambda \) such that \( \lim_{n \to +\infty} \frac{\lambda^{(\epsilon_n)}}{\lambda^{(\epsilon_n)}} > 0 \), where where \( \lambda^{(\epsilon_n)} \) is a sequence that gives the limit point. This is seen as follows: consider the lowest index \( k_1 \) such that \( \lim_{n \to +\infty} \frac{\lambda^{(\epsilon_n)}}{\lambda^{(\epsilon_n)}} > 0 \), then \( N^{*} \) in the limit, column \( k_1 - 1 \) will have exactly one entry; element \( N^{*}_{k_1,k_1-1} \) will be the only non-zero element of column \( k_1 \). Suppose \( k_1 < \ldots < k_\rho \) are the relevant indices, then the columns \((N^{*}_{1,k_1-1}, \ldots, N^{*}_{\rho,k_\rho-1})\) provide an upper triangular matrix, with elements \( N^{*}_{k_j,k_j-1} \neq 0 \) and \( N_{p,k_j-1} = 0 \) for all \( p \geq k_j + 1 \), proving that \( N^{*} \) is of rank \( \rho \).

Therefore \( N^{* (r-1)} \) is of rank \( \rho \) and the non-zero rows of \( N^{* (r-1)} \) are those corresponding to the indices \( k : \lim_{n \to +\infty} \frac{\lambda^{(\epsilon_n)}}{\lambda^{(\epsilon_n)}} > 0 \). Since the space spanned by the \( \rho \) rows is of rank \( \rho \), it follows that \( p_k = 0 \) for each of these \( p_k \), which is a contradiction (since, by hypothesis, \( p_k > 0 \) for each \( k \)). Hence

\[ \sup_{\epsilon} \lambda^{(\epsilon)} < +\infty. \]
Showing \(\inf \min_j \lambda_j^{(e)} > 0\). Now suppose that there is a subsequence \(\lambda_j^{(e_n)} \xrightarrow{n \to +\infty} 0\) for some \(j \in \{2, \ldots, M - 1\}\). As before, \(\lambda^{(e)} = \max_j \lambda_j^{(e)}\). Recall the representation from Lemma 7.1 that
\[
(\hat{N}^{-(r-1)})_{j,k}(\lambda) = \mathbb{P}(Y_T = k | Y_0 = j)
\]
where \(Y\) is a continuous time Markov chain with state space \(\{1, \ldots, M + 1\}\), with intensity matrix given by Equation (15), Definition 2.4 and \(T\) is an independent random variable with distribution \(T \sim \text{Gamma}(r - 1, \frac{\xi}{\lambda})\). Recall that \(\sup_j \max_k \lambda_j^{(e)} < +\infty\) and suppose that \(\lambda_j^{(e_n)} \to 0\)

If \(\lambda_j^{(e_n)} \to 0\) for some \(m_1 \leq j < m_2\) where \(m_1 < m_2\), then, letting \(\tau_j = \inf \{r | X_r = j\}\) and \(\pi_j(dr)\) the probability measure such that \(\mathbb{P}(\tau_j \in A) = \int_A \pi_j(dr)\), then

\[
\mathbb{P}(Y_T^{(e_n)} = m_2 | Y_0^{(e_n)} = m_1) = \int_0^\infty \mathbb{P}(Y_s^{(e_n)} = m_2 | Y_0^{(e_n)} = m_1, T = s)\pi_T(s)ds
\]

\[
\int_0^\infty \int_0^s \mathbb{P}(Y_T^{(e_n)} = j | Y_0^{(e_n)} = m_1)\mathbb{P}(Y_s^{(e_n)} = m_2 | Y_0^{(e_n)} = j)\pi_j(dr)\pi_T(s)ds
\]

so that if \(\lambda_j^{(e_n)} \to 0\), then \(\mathbb{P}(Y_s^{(e_n)} = k | Y_0^{(e_n)} = j) \to 0\) for all \(k\). It follows that

\[
\mathbb{P}(Y_T^{(e_n)} = m_2 | Y_0^{(e_n)} = m_1) = (\mathcal{N}^{-(r-1)})_{m_1, m_2}(\lambda^{(e_n)}) \to 0
\]

for all \((m_1, m_2)\) such that \(m_1 \leq j < m_2\). Similarly, if \(\lambda_j^{(e_n)} \xrightarrow{n \to +\infty} 0\) for \(m_1 \geq j > m_2\), then (86) holds.

It follows that \(\mathcal{N}^{(r-1)}_{kp}(\lambda^{(e_n)}) \to 0\) for all \((k, p)\) such that \(k \leq j < p\) or \(k \geq j > p\).

Furthermore, it follows from (74) that for any sequence with limit point \(\lambda^{(0)}\) such that \(\lambda_j^{(e_n)} \to 0\) for some \(j\), there is an \(l \in \{1, \ldots, M\}\) such that \(j \leq l - 1\), then \(h_k^{(r-1)}(p, \lambda^{(0)}) \leq 0\) for all \(1 \leq k \leq j - 1\) and if \(j \geq l\) then \(h_k^{(r-1)}(p, \lambda^{(0)}) \leq 0\) for all \(j + 1 \leq k \leq M\).

Now recall that
\[
P(p) = p^\mathcal{N}^{(r-1)}(\lambda)\mathcal{N}^{-(r-1)}(\lambda) = \left( \sum_{j,k} p_{j,k} (\mathcal{N}^{(r-1)})_{j,k}(\lambda) \right) h^{(r-1)}(p, \lambda)\mathcal{N}^{-(r-1)}(\lambda)
\]

Recall the definition of \(\lambda^{(e)}\) given by (73) and let \(l\) denote the index from the definition of \(\mathcal{G}\) in (22).

With \(\lambda = \lambda^{(0)}\) and considering the zeroes of \(\mathcal{N}^{-(r-1)}(\lambda^{(0)})\), it follows that if \(j \leq l - 1\), then \(p_1 \leq 0, \ldots, p_{j-1} \leq 0\), which is a contradiction. If \(j \geq l\) then \(p_{j+1} \leq 0, \ldots, p_M \leq 0\), which is a contradiction.

It follows that any limit point \(\lambda\) satisfies the bounds of and consequently that \(h_{0,1}, \ldots, h_{0,M} > 0\) consequently that \(h(p, \lambda) = h_0\) and hence that is satisfied and that \(\lambda\) satisfies equation. The theorem is proved.

\[\square\]

10 Proof of Theorem 5.4

Following the proof of Theorem 5.3, the theorem is already proved for a finite state space \(S = \{i_1, \ldots, i_M\}\); let \(a\) satisfy \(a_j = \lambda_j(i_{j+1} - i_j)(i_j - i_{j-1})\) then, following Lemma 2.5 the continuous
time, time homogeneous Markov process $X$ that satisfies Theorem 5.3 has infinitesimal generator

$$\mathcal{L} = a \left( \frac{1}{2} \Delta + b \nabla - k \right)$$

where the operators $\Delta$ and $\nabla$ are defined by Equations (8) and (9) respectively, where $\mathcal{L}$ means:

$$\mathcal{L}f(i_j) = a_j \left( \frac{1}{2} \Delta + b_j \nabla - k_j \right) f(i_j) \quad j = 1, \ldots, M \quad a_1 = a_M = 0.$$

For a probability distribution $\mu$ over $\mathbb{R}$, the proof follows the same lines as the proof already given for $k \equiv 0$. Set

$$\mathcal{S}_N = \{i_{N,1}, \ldots, i_{N,M_N}\}$$

and let $\mathcal{P}^{(N)}$ be defined by Equation (34). Let $\Delta_N$ denote a solution to the terminal distribution problem for distribution $\mathcal{P}^{(N)}$ over space $\mathcal{S}_N$. With the reduction of notation used throughout, let

$$a_j^{(N)} = a^{(N)}(i_{N,j}) = \lambda_j^{(N)}(i_{N,j+1} - i_{N,j})(i_{N,j} - i_{N,j-1}) \quad j = 2, \ldots, M_N - 1,$$

Let $\mathcal{L}^{(N)}$ be the infinitesimal generator defined by

$$\mathcal{L}^{(N)}f(i_{N,j}) = a^{(N)}(i_{N,j}) \left( \frac{1}{2} \Delta_N + b_j^{(N)} \nabla_N - k_j^{(N)} \right) f(i_{N,j}) \quad j = 2, \ldots, M_N - 1$$

where $\Delta_N$ and $\nabla_N$ are the Laplacian and gradient operators defined on $\mathcal{S}_N$ (Definition 2.1), the approximate drift field $(b_2^{(N)}, \ldots, b_{M_N-1}^{(N)})$ defined by (39) and

$$k_j^{(N)} = \frac{1}{i_{N,j+1} - i_{N,j-1}} \int_{i_{N,j-1}}^{i_{N,j+1}} k(x)dx \quad j = 2, \ldots, M_N - 1$$

where $\int_a^b$ means integration over the interval $(a,b)$. Then $\mathcal{L}^{(N)}$ is the infinitesimal generator of the process $X^{(N)}$ with state space $\mathcal{S}_N \cup \{D\}$, where $D$ denotes a cemetery, such that there is an $l_N$, an $\alpha_N \in (0,1)$ and a $\beta_N \in (0,1)$ such that

$$\begin{cases} \beta_N \mathbb{P} \left( X^{(N)}_t = i_{N,j} | X_0 = i_{N,l_N} \right) + (1 - \beta_N) \mathbb{P} \left( X^{(N)}_t = i_{N,j} | X_0 = i_{N,l_N-1} \right) = \alpha_N p_j^{(N)} \\ j = 1, \ldots, M_N \\ \beta_N \mathbb{P} \left( X^{(N)}_t = D | X_0 = i_{N,l_N} \right) + (1 - \beta_N) \mathbb{P} \left( X^{(N)}_t = D | X_0 = i_{N,l_N-1} \right) = 1 - \alpha_N \end{cases}$$

(89)

The quantity $\beta_N$ may be interpreted in the following way: there is a point $x_{N,0} \in (i_{N,l_N-1}, i_{N,l_N}]$, denoting the initial condition, such that

$$\mathbb{P} \left( X^{(N)}_{0+} = i_{N,l_N} | X^{(N)}_0 = x_{N,0} \right) = \beta_N, \quad \mathbb{P} \left( X^{(N)}_{0+} = i_{N,l_N-1} | X^{(N)}_0 = x_{N,0} \right) = 1 - \beta_N.$$

Let $\tilde{X}^{(N)}$ denote the process with infinitesimal generator $a^{(N)} \left( \frac{1}{2} \Delta_N + b^{(N)} \nabla_N \right)$, then

$$X^{(N)}_t = \begin{cases} \tilde{X}^{(N)}_t & t \leq \tau_N \\ D & t > \tau_N \end{cases}$$
where $\tau_N$ is a random time satisfying

$$\mathbb{P}(\tau_N \geq s|\bar{X}_t^{(N)}) = \exp \left\{ -\int_0^s a^{(N)}(\bar{X}_r^{(N)})k(\bar{X}_r^{(N)})dr \right\} \quad s \geq 0. \tag{90}$$

It follows from Theorem 6.3 that for a sequence $z_j \to z$, there exists a $\bar{X}$ such that

$$\lim_{j \to +\infty} \mathbb{P}\left( \sup_{0 \leq s \leq t} \left| \bar{X}_s^{(N)}(z_{N_j}) - \bar{X}_s(z) \right| > \epsilon \right) = 0. \tag{91}$$

Let $\tau$ denote a random time satisfying

$$\mathbb{P}(\tau \geq s|\bar{X}_t) = \exp \left\{ -\int_0^s \frac{dK}{dm}(\bar{X}_r)dr \right\} \quad s \geq 0. \tag{92}$$

It follows from Equations (90) and (92) that for a sequence $z_{N_j} \to z$ such that $z_{N_j} \in S_{N_j}$ for each $j$,

$$\lim_{j \to +\infty} \mathbb{P} \left( \left\{ \tau^{(N_j)} \leq t \right\} \right) - \mathbb{P} \left( \left\{ \tau \leq t \right\} \right) = 0$$

and

$$\lim_{j \to +\infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P}(\bar{X}_t^{(N_j)}(z_{N_j}) \leq x) \cap \{\tau^{(N_j)} > t\} - \mathbb{P}((\bar{X}_t(z) \leq x) \cap \{\tau > t\}) \right| = 0$$

from which it follows that $X$ is a process with infinitesimal generator $\frac{1}{2} a^2 + b \frac{d}{dm} - \frac{dK}{dm}$ with the required distribution at time $t > 0$.

Finally, it has to be shown that for the sequence of measures $m^{(N)}$ there does not exist a subsequence such that $m^{(N_j)} \to 0$, which would correspond to $\mathbb{P}(X_t^{(N_j)} = D) \xrightarrow{j \to \infty} 1$.

Let $m^{(N)}$ denote the sequence of measures corresponding to the atomised state spaces. For a killing field function $k$, define $k^{(N)}$ as

$$k^{(N)}(x) = \begin{cases} 0 & x \leq i_{N,1} \\ \frac{1}{i_{N,j}-i_{N,j-1}} \int_{i_{N,j-1}}^{i_{N,j}} k(x)dx & i_{N,j-1} < x \leq i_{N,j} \\ 0 & x > i_{N,M_N} \end{cases}$$

Let $\alpha^{(N)}$ solve:

$$\begin{cases} \frac{\partial}{\partial t} \alpha^{(N)} = a^{(N)} \left( \frac{1}{2} \Delta_N + b^{(N)} \nabla_N - k^{(N)} \right) \alpha^{(N)} \\ \alpha^{(N)}(0,.) = 1 \end{cases} \tag{93}$$

so that $\alpha^{(N)}(t,x) > 0$ for all $t < +\infty$, all $N < +\infty$ and all $x \in \mathbb{R}$. Let $l^{(N)} = \log \alpha^{(N)}$ and $l = \lim_{j \to +\infty} l^{(N_j)}$. With the assumption $m^{(N_j)} \to +\infty$, $l$ satisfies

$$\frac{1}{2} l_{xx} + (l_x)^2 + bl_x = k.$$ 

and $f^{(N)}(t;x,y)$ solve

$$\begin{cases} \frac{\partial}{\partial t} f^{(N)} = a^{(N)} \left( \frac{1}{2} \Delta_N + b^{(N)} \nabla_N - k^{(N)} \right) f^{(N)} \\ f^{(N)}(0,x,y) = \delta_y(x) \end{cases}$$
where the operators $\Delta_N$ and $\nabla_N$ are the Laplacian and drift operators (Definition 2.1) defined on the space $\mathcal{S}_N = (i_{N,1}, \ldots, i_{N,M_N})$, applied to the $x$ variable. Let $q(t; x, y) = \frac{f(t; x, y)}{\alpha(t, x)}$, then for each $s > 0$,

$$
\int_B q(s; x, y) dy = \mathbb{P}(X_s \in B | X_0 = x, X_t \notin D) \quad \forall B \in \mathcal{B}(\mathbb{R})
$$

and hence, for some $x_0 \in \mathbb{R}$,

$$
\int_B q(t; x_0, y) dy = \mu(B) \quad \forall B \in \mathcal{B}(\mathbb{R})
$$

and $q$ satisfies:

$$
\begin{align*}
\left\{ \frac{\partial}{\partial s} q^{(N)} &= a^{(N)} \left( \frac{1}{2} \Delta_N + (b^{(N)} + \frac{\sum_{N} \alpha^{(N)}}{\alpha^{(N)}}) \nabla_N \right) q^{(N)} \\
q^{(N)}(0, x, y) &= \delta_y(x)
\end{align*}
$$

where $a^{(N)}$ solves (93). Now let $l_- = \inf \text{supp}(\mu)$ and $l_+ = \sup \text{supp}(\mu)$. Assuming that $m^{(N)}([x, y]) \rightarrow 0$ for all $l_- < x < y < l_+$, it follows that $q^{(N)} \rightarrow q$ which satisfies

$$
\frac{1}{2} \frac{\partial^2}{\partial x^2} q(x, y) + (b + l_x) \frac{\partial}{\partial x} q(x, y) \equiv 0.
$$

From this it follows that $q(x, y) = \lim_{t \rightarrow +\infty} p_t(x, y)$ where $p_t(x, y) = \mathbb{E}_x[\delta_y(Z_t)]$, $Z_t$ is a time homogeneous Markov process with infinitesimal generator $\frac{1}{2} \frac{\partial^2}{\partial x^2} + (b + l_x) \frac{\partial}{\partial x}$. This is clearly irreducible by the hypotheses on $b$. This follows, because $\kappa(Z_t)$ is a standard Wiener process, where $\kappa$ is the function defined by Lemma 6.2 satisfying (47). It follows that $q(x, y) = q(x)$; it does not depend on $y$. Since $\sum_{j=1}^{MN} q^{(N)}(x, i_{N,j}) = 1$ for each $N$, it follows that $q(x) \equiv 0$. This contradicts the fact that $\int_A q(x_0, y) dy = \mu(A)$ for some $x_0 \in \mathbb{R}$. \hfill $\square$

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