Non-equilibrium universe and black hole evaporation

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Abstract
The evaporation of the black holes during the very early universe is studied. Starting from black hole filled universe, the distributions of particle species are calculated and showed, that they differ remarkably from the corresponding equilibrium distribution. This may have great impact to the physics of the very early universe. Also the evolution of the universe during the evaporation has been studied.
It is well known that in the very early universe the particles could not be equi-
librated due to a too short equilibration time \[^{[1, 2, 3]}\]. Indeed, the only possibility to have equilibrium distributions at times \(t \sim 10^{-34}\) s corresponding equilibration tem-
perature \(T_{eq} \sim 10^{15}\) K is to assume, that there is some process producing particles readily to equilibrium distribution. (It should be noted that in this case it is not really question of thermodynamical equilibrium because there are no interactions maintain-
ing it.) Sometimes it is assumed that \(e.g.\) black hole evaporation could be such a process \[^{[3]}\]. When the particles are originated from small black holes near Planck scale which evaporate due to Hawking process \[^{[4]}\], the radiation at any given time is thermal with temperature equal to Hawking temperature.

In the present paper we calculate the distribution of particles originated from black holes. We suppose that the black holes are generated at Planck time through quantum gravitation. After Planck time they begin to decay emitting particles. We show that the resulting particle distributions are not the equilibrium one, but differs crucially from it. This difference may play remarkable role \(e.g.\) in inflation \[^{[5]}\]\, because the effective, ensemble corrected potential \[^{[3]}\] differs now from the one calculated with an equilibrium thermal bath \[^{[7]}\].

In the present study, for simplicity, we neglect the curvature effects, arising at Planck scale due to ambiguous concept of vacuum state \[^{[8]}\]. By assuming isotropy, the comoving observers have same vacuum and hence, whenever the average energy per particle is clearly smaller than Planck mass, the analysis should be applicable. Consistently we omit also all direct couplings of quantum fields to curvature and treat the gravitational field classically.

Let the evolution of the universe be given by the scale parameter \(R(t)\) and suppose that at Planck time no other matter exists in the universe than black holes. The radius of a (non-rotating) black hole with mass \(m_{BH}\) is given by \(r_{BH} = 2m_{BH}/M_{Pl}^2\), where \(M_{Pl}\) is the Planck mass. The black hole surface area is now given by

\[
A_{BH} = \frac{16\pi m_{BH}^2}{M_{Pl}^4}.
\]

(1)

For simplicity, suppose that all black holes have same mass \(m_i\) at initial time and their number density \(n(t_i)\). Because there is no other type of matter than black holes, the universe is at \(t_i\) matter dominated, \(i.e.\) \(R(t) \propto t^{2/3}\). The ratio of the density of the universe to the critical density at that time is now given by \(\Omega(t_i) = 8\pi m_i n(t_i)/(3M_{Pl}^2 H(t_i)^2)\), where the Hubble parameter is \(H(t_i) = \frac{2}{3} t_i^{-1}\). Thus we can parametrize the initial density with density ratio \(\Omega_i\).

Because a black hole radiates at any time thermal radiation with the temperature given by the Hawking temperature \(T_{BH} = M_{Pl}^2/(8\pi m_{BH})\) through its surface, the

\[^{1}\]The existence of small black holes at early times would have also many other interesting consequences, see \[^{[6]}\].
loss of energy in time unit is $A_{BH} h_s T_{BH}^4 \pi^2 / 30$, where $h_s$ is an effective number of particles lighter than black hole mass. So, as well known, the dynamics of a black hole evaporation is governed by the equation
\[ \dot{m}_{BH} = -\frac{h_s}{15 \times 8^3 \pi^2 m_{BH}^2} \cdot \frac{M_{Pl}^4}{m_{BH}^4}, \]
where we have omitted the absorption of radiation back to the black holes. This is well motivated whenever the radiation density $E_{rad}$ is small, i.e. $E_{rad} A_{BH} \ll |\dot{m}_{BH}|$, which is the case. The solution of the equation (2) reads
\[ m_{BH}(t) = m_i \left( 1 - \frac{t - t_i}{\tau} \right)^{1/3}, \]
where the black hole life-time $\tau$ is given by
\[ \tau = \frac{5 \times 8^3 \pi^2 m_i^3}{M_{Pl}^4 h_s}. \]

We express the initial black hole mass in terms of the inverse of $\tau$
\[ m_i = \frac{M_{Pl}}{8} \left( \frac{h_s}{5^2 \delta} \right)^{1/3}, \]
where $\delta \equiv 1/(M_{Pl}\tau)$ has to be a small number $\ll 1$, i.e. the black hole life-time is much longer than Planck time. Using the expression for $\Omega_i$ and (3) we can give the black hole number density
\[ n(t_i) = \frac{4 M_{Pl}}{3 \pi^2} \left( \frac{5}{\pi h_s} \right)^{1/3} \Omega_0 \delta^{1/3}. \]

We are now ready to calculate the distribution of the particles emitted by the black holes. We further simplify the situation by stating that all emitting particles are essentially massless, so that the effective particle number $h_s$ remains constant during the evaporation. Strictly speaking, this is not true, but gives a reasonable approximation of the situation, where no specific model with its particle contents is given. Also, we suppose that the black holes decay before the thermalization starts. This means that no thermalizing processes are effective during the evaporation and hence not taken into account in the calculation. Whether this is a good assumption or not is studied later in this paper.

\footnote{Indeed, one has to assume that $t_i + \tau \gg t_{Pl}$, because otherwise quantum gravitational effects would certainly not be omitted.}

\footnote{The massive case is as well calculable as the massless one, but leads to introducing of numerous new parameters, which is avoided here.}
During a time \([t, t + dt]\) \((t > t_i)\) the energy emitted by a black hole and distributed to \(h_s\) particle species (degrees of freedom) is \(dE = -\dot{m}_{BH}(t)\ dt\). Because at a given time the radiation emitted obeys thermal spectrum with temperature \(T_{BH} (\beta_{BH} = 1/T_{BH})\), the energy per degree of freedom within momentum between \(p\) and \(p + dp\) is given by

\[
dE(p + dp) - dE(p) = \frac{k_+ dE}{\omega(p)} f^+(p, \beta_{BH}(t))d^3p, \quad (7)
\]

where the fraction \(k_+ dE/h_s\) represents the (total) energy per degree of freedom \((k_+\) is 1 for bosons and \(\frac{7}{8}\) for fermions appearing due to different statistics). The other fraction expresses the part of the energy within the given momentum range and the functions \(f^\pm\) are the usual Fermi-Dirac (+) and Bose-Einstein (-) distribution functions. Thus, the particle number density (at the given momentum range) is simply given by

\[
dE(p + dp) - dE(p) = \frac{k_+ dE}{\omega(p)} \frac{f^\pm(\beta_{BH}p)}{\int d^3k \omega(k) f^\pm(\beta_{BH}k)} \quad (8)
\]

Taking into account that the black hole density reads \(n(t) = n(t_i) (R(t_i)/R(t))^3\), one obtains the distribution \(df_{f(b)}\) of a degree of freedom:

\[
df_{f(b)}(p, t) \frac{d^3p}{(2\pi)^3} = n(t) \frac{dE(p + dp) - dE(p)}{\omega(p)} \quad (9)
\]

Combining (8) and (9) we write

\[
df_{f(b)} = -n(t) \frac{k_+ \dot{m}_{BH}(t)}{h_s} \frac{(2\pi)^3 f^\pm(\beta_{BH}p)}{\int d^3k \omega(k) f^\pm(\beta_{BH}k)}. \quad (10)
\]

Eq. (10) represents the distribution generated during a time-interval \([t, t + dt]\). The overall distribution thus if we want to end up to at a given time \(t\) is obtained by summing all contribution from \(t_i\) to \(t\), i.e. all times \(t'\), \(t_i \leq t' \leq t\). These contribution are, however, red-shifted by a factor \(R(t')/R(t)\), hence the final particle distribution at time \(t\) is given by

\[
f_{f(b)}(p; t) = -\int_{t_i}^{t} dt' n(t') \frac{k_+ \dot{m}_{BH}(t')}{h_s} \frac{(2\pi)^3 f^\pm(\beta_{BH}(t')p)}{\int d^3k \omega(k) f^\pm(\beta_{BH}(t')k)}. \quad (11)
\]

The formula above, Eq. (11), describes in principle completely the particle distributions created by black hole evaporation. It can easily be generalized to the case, where the (initial) black holes are not equally massive, but have some distribution. In this case one just have to sum over all contributions emerging due to different black hole masses.

To be able to calculate the distributions in practise, one has to solve the evolution of the scale parameter \(R(t)\). For that purpose we have to determine first the time \(t_{EQ}\).  

when the universe enters the radiation dominated era after the beginning of the black hole evaporation. We have to calculate when the energy density of the black holes, \( \mathcal{E}_{\text{BH}} \), equals the energy density of radiation, \( \mathcal{E}_{\text{rad}} \), they have emitted: \( \mathcal{E}_{\text{BH}}(t_{\text{EQ}}) = \mathcal{E}_{\text{rad}}(t_{\text{EQ}}) \).

Now, the (average) energy density of the black holes at a given time \( t \) is given by
\[
\mathcal{E}_{\text{BH}}(t) = m_{\text{BH}}(t)n(t_i)(R(t_i)/R(t))^3. 
\]
During a matter dominated time \( R(t) \propto t^{2/3} \) and thus
\[
\mathcal{E}_{\text{BH}}(t) = m_{\text{BH}}(t)n(t_i)\left(\frac{t_i}{t}\right)^2, \quad t_i < t < t_{\text{EQ}}. 
\]
On the other hand the radiation energy density at \( t < t_{\text{EQ}} \) is given by
\[
\mathcal{E}_{\text{rad}} = \sum_i g_i \int \frac{d^3k}{(2\pi)^3} f_i(k,t)\omega_i(k) = g_f \int \frac{d^3k}{(2\pi)^3} f_f(k,t)\omega(k) + g_b \int \frac{d^3k}{(2\pi)^3} f_b(k,t)\omega(k),
\]
where we have summed over all relevant bosonic \( (g_b) \) and fermionic \( (g_f) \) degrees of freedom. Eq. (13) can simplified to
\[
\mathcal{E}_{\text{rad}} = -\int_{t_i}^{t} dt' \dot{m}_{\text{BH}}(t') \left(\frac{R(t')}{R(t)}\right)^4 n(t_i) \left(\frac{R(t_i)}{R(t')}\right)^3,
\]
where \( -\dot{m}_{\text{BH}}(t') \left(\frac{R(t')}{R(t)}\right)^4 \) corresponds the energy radiated at \( t' \) by black holes and redshifted until time \( t \), whereas \( n(t_i) \left(\frac{R(t_i)}{R(t')}\right)^3 \) represents the number density of decaying black holes at the time when the radiation was produced. Evidently, Eq. (13) can be obtained directly without using any knowledge about particle distributions. Indeed, the connection between Eqs. (13) and (14) requires the identification \( h_* = g_* \), where \( g_* \) is the usual effective number of degrees of freedom \( g_* = \frac{7}{8}g_f + g_b \). In the rest of the paper we apply this identification.

It should be noted here, that the correct evolution of the cosmic scale parameter at matter dominated era is not really as simple as \( R(t) \propto t^{2/3} \) but more complicated. This is due to the fact that the correct energy density of matter, \( i.e. \) black holes at that time reads
\[
\mathcal{E}_{\text{BH}}(t) = m(t)n(t_i)(R(t_i)/R(t))^3, 
\]
which is not as simple as just dilution of the matter. Applying this equation to the Einstein equation, one obtains
\[
R(t) = \frac{3}{2} \left(\frac{8\pi \mathcal{E}(t_i)}{3M_{\text{Pl}}^2}\right)^{1/2} \frac{6\pi}{7} \left[ 1 - \left(\frac{m(t)}{m_i}\right)^{7/2} \right]^{2/3} 
\]
which, however, is in quite good agreement with the usual matter dominated form. Moreover, the calculation of the scale parameter at the radiation dominated era leads

\(^4\)Here we remind that, for sake of simplicity, we have assumed that all relevant particles are essentially massless, thus \( \omega(k) = k \).
to a complicated integro-differential equation. Therefore we choose here to use the conventional forms of the evolution parameter $R$, which anyway gives good impression of the physics involved. It is now simple matter to take the expression for $\mathcal{E}_{BH}$ and $\mathcal{E}_{rad}$ from Eqs. (12) and (14) (and relation $R(t) \propto t^{2/3}$) and equate them. One obtains a rather simple equation

$$m_{BH}(t_{EQ}) t_{EQ}^{2/3} = - \int_{t_i}^{t_{EQ}} dt \frac{t^{2/3}}{R(t)} \tilde{m}_{BH}(t).$$ \hfill (17)

It is noteworthy, that eq. (17) does not depend on the initial black hole number density $n(t_i)$ but only on initial time and black hole mass. According to the approximated analysis, the radiation dominance begins quite late compared to the black hole life-time. When $t_i/\tau$ varies from 0 to 0.5, the combination $(t_{EQ} - t_i)/\tau$ decreases from 0.928 to 0.908. In any case over 90% of the black hole life-time belongs to the matter dominated era. This is a consequence of the fact that the soft radiation produced at the early phase of the black hole life-time further red-shifts during the expansion of the universe. Using the expression (16) for cosmic scale parameter, one obtains $(t_{EQ} - t_i)/\tau = 0.924$ which is in good agreement with the approximated result. Therefore, it is reasonable to use the simpler equations.

The scale factor is now simply given by equation

$$R(t) = \left[ \theta(t - t_{EQ}) \left( \frac{t}{t_{EQ}} \right)^{1/2} + \theta(t_{EQ} - t) \left( \frac{t}{t_{EQ}} \right)^{2/3} \right] R(t_i) \left( \frac{t_{EQ}}{t_i} \right)^{2/3},$$ \hfill (18)

where $\theta$ is the usual Heaviside-function. Eq. (18) can be inserted to formulas like (14) and (14). In particular we are now interested in the energy density of the universe at the end of the black hole evaporation era: the energy density at that time determines the later evolution of the universe. We obtain the energy density at a given time $t$

$$\mathcal{E}(t) = - \left( \frac{R(t_i)}{R(t_f)} \right)^3 n(t_i) \int_{t_i}^{t} dt' \tilde{m}_{BH}(t') \frac{R(t')}{R(t_f)}$$

$$= - \left( \frac{t_i}{t_{EQ}} \right)^2 \left( \frac{t_{EQ}}{t_f} \right)^{3/2} n(t_i) \int_{t_i}^{t} dt' \tilde{m}_{BH}(t')$$

$$\times \left[ \theta(t' - t_{EQ}) \left( \frac{t'}{t_{EQ}} \right)^{1/2} + \theta(t_{EQ} - t') \left( \frac{t_{EQ}}{t_f} \right)^{1/2} \left( \frac{t'}{t_{EQ}} \right)^{2/3} \right],$$ \hfill (19)

where $t_i < t < t_f$. For times $t > t_f$ the energy density is only red-shifted as usual. We define a pseudo-temperature $\tilde{T}$ to describe the energy density and particle distribution by setting simply

$$\frac{\pi^2}{30} g_* \tilde{T}(t)^4 = \mathcal{E}(t).$$ \hfill (20)

Now, $\tilde{T}$ is by no means a real temperature, simply because there exists no thermal equilibrium. However, it describes the energy density in familiar way and, moreover,
when the thermal equilibrium is finally maintained, $\tilde{T}$ coincides with the true temperature. Therefore, at the time $t_f$ it is enlightening to compare the actual particle distribution to the thermal one at pseudo-temperature $\tilde{T}$. In the Figure 1a we have displayed the pseudo-temperature $\tilde{T}$ at $t_f$ as a function of $t_i/\tau$ ($t_i = t_{Pl}$).

![Figure 1a](image)

![Figure 1b](image)

**Figure 1:** (a) The pseudo-temperature $\tilde{T}$ at $t = t_f$ as a function of the ratio $t_i/\tau$. ($t_i = t_{Pl}, \Omega_i = 1$.) (b) Evolution of the pseudo-temperature $\tilde{T}$ as a function of $(t - t_i)/\tau$. The upper curve corresponds $t_i/\tau = 0.35$ whereas the lower one $t_i/\tau = 0.1$. ($\Omega_i = 1$.)

An interesting feature emerges in the behaviour of $\tilde{T}$ (i.e. energy density). For $t_i/\tau < z_c \equiv 0.334$ the pseudo-temperature has a minimum during the black hole evaporation era. In the Fig. 1b we have plotted the evolution of two pseudo-temperatures, one with $t_i/\tau = 0.1 < z_c$ (lower curve) and another with $t_i/\tau = 0.35 > z_c$ (upper curve). We have scaled the time so, that $\tilde{T}$ is presented as function of $(t - t_i)/\tau$. Note, that the large $t_i/\tau$ values correspond small black holes with evaporation times comparable to $t_i$. Thus, if $t_i \simeq t_{Pl}$, the approximations are not very good and. Practically in all relevant cases the pseudo-temperature has a minimum.

In this stage we have to make a notion about the homogeneity of the matter and radiation in the universe. All calculations done as far, and all calculations to be done, assume that the universe is homogeneous at large scales. We can consider the matter to be homogeneously distributed if there are several black holes within a single horizon volume. On the other hand the radiation can be viewed to be roughly homogeneous, if the distance between the radiation sources, i.e. black holes is shorter than the length propagated by radiation. These two conditions are essentially equivalent leading
to \( n(t) > t^{-3} \). In particular, this requirement at transition time \( t_{\text{EQ}} \) means, that \( \delta < 0.14(\Omega_i)^{3/2} \). So, if \( \Omega_i = 1 \) we obtain \( \delta < 0.14 \), which in terms of black hole mass reads \( m_{\text{BH}} > 0.356(h_*/160)^{1/3} \).

One can also ask, when the last emitted radiation (at \( t = t_f \)) is homogenized. Spatially homogeneous distribution is reached at time \( t_h \) (assuming the radiation domination after \( t_f \)) if \( n(t_f)^{-1/3}R(t_h)/R(t_f) < t_h - t_f \). Performing a numerical calculation it shows up (with \( \Omega_i = 1 \)), that \( (t_h - t_f)/\tau \) varies from 0 to 3 as \( \delta \) increases from 0 to 0.5. Thus the homogenization time is of the same order of magnitude as \( t_f \) itself or even shorter.

Keeping in mind all above, we are now finally ready to calculate the distributions themselves. Applying Eqs. \( (2), (3), (18) \) and formulae for black hole density and temperature to the Eq. \( (11) \), one obtains the particle distributions. The distributions can be written as

\[
    f_{f(b)}(p; t) = \frac{16n_i}{M_{\text{Pl}}^2} \int_{t_i}^t dt' m_{\text{BH}}(t')^2 \left( \frac{R(t_i)}{R(t')} \right)^3 f^\pm \left( \frac{R(t)}{R(t')} \beta_{\text{BH}}(t')p \right).
\]

In the Fig. \( 2a \) we have displayed the distribution at \( t_f \) for bosons multiplied by \( p^2 \) as a function of the momentum \( p \) together with a reference distribution. The reference distribution is the equilibrium distribution with \( T = \tilde{T}(t_f) \). All are given in Plank units with \( \Omega_i = 1 \). The shape of distribution given in Fig. \( 2a \) is generic and it is noteworthy that the black hole originated distribution has more power in large momenta. On the other hand, the IR end of the distribution, \( p \sim 0 \), there is less power. Indeed, the IR behaviour of reads \( f_b \sim \kappa/p \) like for equilibrium (where \( f^{-} \sim T/p \)), but

\[
    \kappa = \frac{2n_i}{\pi M_{\text{Pl}}^2} \frac{R(t_i)}{R(t_f)} \int_{t_i}^{t_f} dt m_{\text{BH}}(t) \left( \frac{R(t_i)}{R(t)} \right)^2 < \tilde{T}(t_f)
\]

for all relevant \( \delta \): the ratio \( \tilde{T}(t_f)/\kappa \) is for all \( \delta < 0.5 \) larger than \( \sim 5.6 \). This may be important, because the IR effects are essential in many physical considerations, in particular when some kind of effective potential and/or interactions are used.

The fermionic distribution function is presented in Fig. \( 2b \). As for bosonic case we have presented for comparison the equilibrium distribution at \( \tilde{T}(t_f) \), too. The shape of the distribution in Fig. \( 2b \) is generic, and again there is more power at large \( p \) in \( f_f \) than in equilibrium distribution. In general, the distributions arising from black hole evaporation are flatter and lower than corresponding equilibrium distributions. This can be understood, because at late times \( t \sim t_f \) the black hole radiates particles with large momenta. The particle density, \( n_{b(f)} \), is thus always lower than at equilibrium with same energy density, \( n_{b(f)}^{\text{EQ}} \). The ratio \( n_{b(f)}/n_{b(f)}^{\text{EQ}} < 0.36 \) for all \( \delta < 0.5 \) and is proportional to \( \delta^{1/6} \) when \( \delta \to 0 \).

Finally the question, how small the parameter \( \delta \) can be, \textit{i.e.} how large the black holes can be. If one supposes that the black holes evaporate before the thermalization
Figure 2: (a) Boson distribution function $p^2 f_b(p; t_f)$ as a function of momentum $p$ (solid line). For reference, also the equilibrium distribution at the temperature $\tilde{T}(t_f)$ has been presented (dashed line). $\delta = 0.1$. (b) Corresponding fermion distribution function $p^2 f_f(p; t_f)$ (solid line) with equilibrium reference curve at $\tilde{T}(t_f)$ (dashed line). $\delta = 0.1$.

takes place due to GUT processes [1], strong interactions [2] or electroweak interactions [3], one has to require that $t_f \ll t_{th} = 1/\Gamma$, where $\Gamma$ is the (average) thermalization rate of the particles involved. If $\Gamma \sim \alpha \tilde{T}$, where inspired by GUT’s $\alpha \sim 3 \times 10^{-2}$, we conclude that $\delta / \tilde{T} \gg \alpha^2$ or $\delta \gg 3 \times 10^{-8}$. This estimate has many uncertainties e.g. because as dimensional parameter the pseudo-temperature $\tilde{T}$ has been used. Nevertheless, this approximation gives a general idea about the time-scales involved. On the other hand, one should address some interest to the question, when the masses are negligible. Concerning the tree-level masses, general remarks are impossible without specifying a particular model. However, in the case of the ensemble masses (“thermal masses”) some remarks can be done. The ensemble mass $m_e \sim g \tilde{T}$, where $g$ is a generic coupling, should be small compared to the black hole temperature $T_{BH}$, i.e. $m_e \ll T_{BH}$. This can be rewritten as $\delta^{1/3} / \tilde{T} \gg g \pi (\hbar / (5\pi^2 1/3) M_{Pl}^{-1}$, where $T_{BH} = T_{BH}(t_i)$ has been used because it corresponds the lowest black hole temperature giving a conservative limit. Now the left hand side of this equation is always larger than 7.9 (at $\delta = 0.5$). Thus (for $h_* = 160$) the ensemble mass can be ignored, if the coupling $g \ll 1.7$!

In the present paper a general study of the particle distributions inspired by the black hole evaporation is presented. It is found that the distribution produced differs clearly from the equilibrium distributions. The distribution has much more power
at large momenta and the boson distribution has slower increase when \( p \to 0^+ \). The conditions of validity of the analysis have been studied and found that under wide range of parameters calculation can be performed with good reliability. Also the evolution of the universe has been studied during the black hole evaporation era. In particular, the time when it makes transition from matter domination to radiation domination and the generic bahaviour of radiation density are calculated. These considerations may have great impact to the calculations of the very early universe. However, more detailed studies are definitely needed, when any particular model is considered.

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