Reconstruction from blind experimental data for an inverse problem for a hyperbolic equation

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Abstract

We consider the problem of the reconstruction of dielectrics from blind backscattered experimental data. The reconstruction is done from time domain data, as opposed to a more conventional case of frequency domain data. Experimental data were collected using a microwave scattering facility which was built at the University of North Carolina at Charlotte. This system sends electromagnetic pulses into the medium and collects the time-resolved backscattered data on a part of a plane. The spatially distributed dielectric constant \( \varepsilon_r(x), x \in \mathbb{R}^3 \) is the unknown coefficient of a wave-like PDE. This coefficient is reconstructed from those data in blind cases. To do this, a globally convergent numerical method is used.

Keywords: coefficient inverse problem (CIP), finite element method, globally convergent numerical method for CIP, experimental backscattered data

(Some figures may appear in colour only in the online journal)
1. Introduction

We consider the problem of the reconstruction of refractive indices or dielectric constants of unknown targets, i.e., blind experiments, placed in a homogeneous medium from time domain backscattering experimental data. This is a coefficient inverse problem (CIP) for the 3D wave-like equation. Experimental data were collected using a microwave scattering facility which was recently built at the University of North Carolina at Charlotte. These data are generated using a single location of the point source. The backscattered signal is measured on a part of a plane, i.e., over a limited range of backscattered angles. In our experiments of this paper, we image targets located in the air. A potential application of this work is in imaging of explosives. We note that explosives can be located in air or homogeneous background [14], e.g. improvised explosive devices (IEDs). The work on real data for the case when targets are buried under the ground is ongoing.

The model under consideration in this work is also applicable to ground-penetrating radar (GPR), see e.g. [10] and the references therein. GPR is used to detect and locate underground objects. For the mathematical theory of GPR, see e.g. [16]. However, in GPR, the transmitter is usually moved along with one or many receivers, i.e., many sources are used unlike in our experiments in which we use only one point source. This problem can also be found in seismology in which the geological structures and/or materials of the earth’s upper layers are determined based on the reflections of seismic waves by the ground layers, see e.g. [7, 17].

Different imaging methods have been reported in the literature for reconstructing geometrical information about targets, such as their shapes, sizes and locations, see e.g. [6, 10, 18, 19]. On the other hand, the target’s material properties (in our model, the refractive indices), which are our main interest, enhance the characterization of targets, but these are much more difficult to compute. The most popular method in GPR and seismology is to linearize the inverse problem using the Born approximation. As to the gradient-like methods, we refer to, e.g., [1, 9, 11, 21] and references therein. Convergence of these methods is guaranteed only if the starting point of iterations is chosen to be sufficiently close to the correct solution. On the other hand, for an inverse problem similar to the one considered in this work, it was shown in section 5.8.4 of [3] that the gradient method failed to work for experimental forward scattered data of [3] in the case when its starting point was knowledge of the background medium.

In this paper, we present a combination of the approximately globally convergent method of [3] with a finite element method (FEM) for the numerical solution of this CIP. Given a certain function computed by the technique of [3], the FEM reconstructs the unknown coefficient in an explicit form. We believe that our results, if not adequate in their raw form, can go on to be used as initial guesses for locally convergent methods in order to obtain even better shape accuracy, see e.g. section 5.9 in [3], where the image obtained by the globally convergent method for forward scattered experimental data was refined via a locally convergent adaptivity technique.

We now indicate main differences between this paper and our related past work processing measured data. Our research group has previously reported the method of [3] applied to experimental data, see chapter 5 of [3, 14] and references cited there. In chapter 5 of [3] and some of our journal publications cited there, the globally convergent method of [3] was applied to 3D forward scattered data. Backscattered data are much harder to work with than the forward scattered data because the backscattered signals are typically significantly weaker and unwanted signals from neighboring structures mixed with the target’s signals, see figure 2(a). In [14], the case of 1D backscattered experimental data was considered. In this paper, the incident wave from a point source was measured at only one backscattering location. Therefore, only a 1D profile of the dielectric constant was reconstructed. In this paper and our
Table 1. Object names.

| Object number | Name of the object                        |
|---------------|------------------------------------------|
| 1             | A piece of oak                           |
| 2             | A piece of pine                          |
| 3             | A metallic sphere                        |
| 4             | A metallic cylinder                      |
| 5             | Blind target                             |
| 6             | Blind target                             |
| 7             | Blind target                             |
| 8             | Doll, air inside, blind target           |
| 9             | Doll, metal inside, blind target         |
| 10            | Doll, sand inside, blind target          |
| 11            | Two metallic blind targets               |

recent preprint [20], this is the first time we test 3D backscattered experimental data for the globally convergent method, coupled with the FEM method, with a new data pre-processing procedure. However, there are a number of significant differences between this paper and [20].

First, in this paper, we describe the combination of the globally convergent algorithm and the FEM method. In contrast, [20] is devoted to a detailed analysis of the data pre-processing procedure. Secondly, we present here a case of two targets being illuminated simultaneously, which is not analyzed in [20], see tables 1, 3 and figure 7. Finally, the image post-processing procedure here (section 7.2.2) is different from the one presented earlier [20]. The latter leads to some differences in resulting images.

We have collected backscattered time-resolved experimental data of electromagnetic waves propagating in a non-attenuating medium. As was pointed out in [3, 14], the main difficulty of working with measured data is the huge mismatch between these data and that produced by idealized computational simulations. Conventional data denoising techniques do not help with measured data because of its richer structure. Therefore, it is unlikely that any numerical method would successfully invert the raw data. To process the data to look somewhat similar to the corresponding simulated data, a heuristic data pre-processing procedure is applied. The pre-processed data are then used as the input for the globally convergent method.

The approximately globally convergent method of [3] relies on the structure of the underlying PDE operator and does not use optimization techniques. Each iterative step consists of solutions of two problems: the Dirichlet boundary value problem for an elliptic PDE and the Cauchy problem for the underlying hyperbolic PDE. ‘Approximate global convergence’ (global convergence in short) means that we use a certain reasonable approximate mathematical model. The approximation is used because one inevitably faces substantial challenges when trying to develop globally convergent numerical methods for multidimensional CIPs for hyperbolic PDEs with a single source. It is rigorously established in the framework of this model that the method of [3] results in obtaining some points in a small neighborhood of the exact coefficient without a priori knowledge of any point in this neighborhood, see theorem 2.9.4 in [3] and theorem 5.1 in [4]. The distance between those points and the exact solution depends on the error in the data, the step size $h$ of a certain discretization of the pseudo-frequency interval and the computational domain $\Omega$ where the inverse problem is solved (see section 4.2 for definition of $h$). The knowledge of the background medium in $\Omega$ is not required by this method. Because of these theorems, convergence analysis is not presented here. A rigorous definition of the approximate global convergence property can be found in section 1.1.2 of [3] and in [4].
use a mild approximation, since it amounts only to the truncation of a certain asymptotic series and it is used only on the first iterative step (see section 4.1). The validity of this approximate model was verified computationally on both synthetic and transmitted experimental data in [3, 4] as well as in the current work in the case of measured backscattered data.

An outline of this paper is as follows. In section 2, we state the forward and inverse problems. In section 3, we describe the experimental data and briefly outline the data preprocessing procedure. In section 4, we briefly outline the method of [3]. In section 5, we describe a version of the FEM which works for our case. In section 6, we describe our algorithm. In section 7, we outline some details of our numerical implementation. Results are presented in section 8 and a summary is in section 9.

2. Statement of forward and inverse problems

Let \( \Omega \subset \mathbb{R}^3 \) be a convex bounded domain with the boundary \( \partial \Omega \in C^3 \). Denote by \( \mathbf{x} = (x, y, z) \in \mathbb{R}^3 \). We model the electromagnetic wave propagation in an isotropic and non-magnetic space \( \mathbb{R}^3 \) with the dimensionless coefficient \( \varepsilon_r(\mathbf{x}) \), which describes the spatially distributed dielectric constant of the medium. We consider the following Cauchy problem for the hyperbolic equation:

\[
\varepsilon_r(\mathbf{x}) u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty),
\]

\[
u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0).
\]

We assume that the coefficient \( \varepsilon_r(\mathbf{x}) \) of equation (2.1) is such that

\[
\varepsilon_r(\mathbf{x}) \in C^\omega(\mathbb{R}^3), \quad \varepsilon_r(\mathbf{x}) \in [1, b], \quad \varepsilon_r(\mathbf{x}) = 1 \text{ for } \mathbf{x} \in \mathbb{R}^3 \setminus \Omega,
\]

where \( b = \text{const.} > 1 \). We \textit{a priori} assume knowledge of the constant \( b \), which amounts to the knowledge of the set of admissible coefficients in (2.3). However, we do not assume that the number \( b - 1 \) is small, i.e., we do not impose small-value assumptions on the unknown coefficient \( \varepsilon_r(\mathbf{x}) \). Below \( C^{k+\alpha} \) are Hölder spaces, where \( k \geq 0 \) is an integer and \( \alpha \in (0, 1) \).

Let \( \Gamma \subset \partial \Omega \) be a part of the boundary \( \partial \Omega \). Later, we will designate \( \Gamma \) as the backscattered side of \( \Omega \) and will explain how we deal with the absence of the data at \( \partial \Omega \setminus \Gamma \).

\textbf{Coefficient Inverse Problem. Suppose that the coefficient } \varepsilon_r(\mathbf{x}) \textbf{ satisfies (2.3). Determine the function } \varepsilon_r(\mathbf{x}) \textbf{ for } \mathbf{x} \in \Omega, \textbf{ assuming that the following function } g(\mathbf{x}, t) \textbf{ is known for a single source position } \mathbf{x}_0 \notin \overline{\Omega}.

\[
u(\mathbf{x}, t) = g(\mathbf{x}, t), \quad \forall (\mathbf{x}, t) \in \Gamma \times (0, \infty).
\]

The function \( g(\mathbf{x}, t) \) in (2.4) models time-dependent measurements of the wave field at the part \( \Gamma \) of the boundary \( \partial \Omega \) of the domain of interest \( \Omega \). We assume below that the source position is fixed and \( \mathbf{x}_0 \notin \overline{\Omega} \). This assumption allows us to simplify the resulting integral–differential equation because \( \delta(\mathbf{x} - \mathbf{x}_0) = 0 \) in \( \overline{\Omega} \). The assumption \( \varepsilon_r(\mathbf{x}) = 1 \) for \( \mathbf{x} \in \mathbb{R}^3 \setminus \Omega \) means that the coefficient \( \varepsilon_r(\mathbf{x}) \) has a known constant value outside of the domain \( \Omega \).

This is a CIP with single measurement data. The uniqueness for such CIPs in the multidimensional case is currently known only if the function \( \delta(\mathbf{x} - \mathbf{x}_0) \) in (2.2) is replaced by a function \( f(\mathbf{x}) \), such that \( \Delta f(\mathbf{x}) \not= 0 \forall \mathbf{x} \in \overline{\Omega} \). A proper example of such a function \( f \) is a narrow Gaussian centered around \( \mathbf{x}_0 \), which approximates the function \( \delta(\mathbf{x} - \mathbf{x}_0) \) in the distribution sense. From the physics standpoint, this Gaussian is equivalent to \( \delta(\mathbf{x} - \mathbf{x}_0) \). That uniqueness theorem can be proved by the method which was originated in [8]. This method is based on Carleman estimates, also see, e.g., sections 1.10, 1.11 of the book [3] about this method. The authors believe that, because of very real significance of the applications of
this theory, it makes sense to develop numerical methods for this CIP without completely addressing the uniqueness question.

The function \( u(x, t) \) in (2.1) represents the voltage of one component of the electric field \( E(x, t) = (E_x, E_y, E_z)(x, t) \). In our computer simulations, the incident field has only one non-zero component \( E_y \). This component propagates along the \( z \)-axis until it reaches the target, where it is scattered. So, we assume that in our experiment \( u(x, t) = E_y(x, t) \).

We now comment on five main discrepancies between our mathematical model (2.1)–(2.3) and the processing and interpretation of measured data. The first discrepancy which causes the main difficulties is the aforementioned huge mismatch between experimental data and computational simulations. The second one is that although we realize that equation (2.1) in the case of the variable coefficient \( \varepsilon_r(x) \) can be derived from Maxwell equations only in the 2D case, we use it to model the full 3D case. The reason we do so is that our current receiver can measure only one of the polarization components of the scattered electric field \( E \). A second separate experiment or a second antenna would be needed to measure both. In addition, if using a more complicated mathematical model than the one of (2.1), for example, one that includes vector scattering and thus depolarization effects on scattering, then one would need to develop a globally convergent inverse method for this case. The latter is a quite time-consuming task with yet unknown outcome. Equation (2.1) was used in chapter 5 of [3] for the case of transmitted experimental data and accurate solutions were obtained. A partial explanation of the latter can be found in [5], where the Maxwell’s system of equations in a non-magnetic and non-conductive medium was solved numerically in time domain. It was shown numerically in section 7.2.2 of [5] that the component of the vector \( E(x, t) = (E_x, E_y, E_z)(x, t) \), which was initially incident upon the medium, dominates two other components. This is true for at least a rather simple medium such as ours. Therefore, the function \( u(x, t) \) in (2.1) represents the voltage of the computed component \( E_y(x, t) \) of the electric field, which is emitted and measured by our antennas.

The third discrepancy is that the condition \( \varepsilon_r(x) \in C(\mathbb{R}^3) \) is violated on the inclusion/background interface in our experiments. The fourth discrepancy is that formally equation (2.1) is invalid for the case when metallic targets are present. On the other hand, it was demonstrated computationally in [14] that one can treat metallic targets as dielectrics with large dielectric constants, which we call \textit{appearing dielectric constant},

\[
\varepsilon_r \text{ (metallic target)} \in (10, 30).
\]  

Modeling metallic targets as integral parts of the unknown coefficient \( \varepsilon_r(x) \) is convenient for the above application to imaging of explosives. Indeed, IEDs usually consist of mixtures of some dielectrics with a number of metallic parts. Such targets are heterogeneous ones and we consider three heterogeneous cases in section 8.2. On the other hand, modeling metallic parts within heterogeneous targets as a separate problem, being part of the target, while still assuming an unknown background medium does result in additional complications of the already difficult problem.

The fifth discrepancy is that we have also used an incident plane wave instead of the point source in our computations. We have discovered that the plane wave case works better in image reconstructions than the point source, while the point source case is more convenient for the convergence analysis in [3, 4]. In addition, since the distance between our measurement plane and the targets is much larger than the wavelength of our signal, it is reasonable to approximate the incident wave as a plane wave.

Thus, our results of section 8.2 demonstrate the well-known fact that computational results are often less pessimistic than the theory, since the theory cannot grasp all nuances of
the reality. In summary, we believe that accurate solutions of the above CIP for experimental data justify our mathematical model.

3. Experimental data

3.1. Data collection

Figure 1(a) is a photograph illustrating the data collection. The data collection is carried out in a regular room, which contains office furniture, computers, etc. Keeping in mind our desired applications (see Introduction), we intentionally did not arrange a special anechoic chamber, which would protect our data from unwanted signals caused by reflections from various objects in the room. Below $x$ and $y$ are horizontal and vertical axis on the measurement plane, respectively, and the $z$-axis is perpendicular to the measurement plane, the positive direction of $z$-axis is in the direction from the target to the measurement plane. We make our coordinates dimensionless as $x' = x/(1 \text{ m})$, where ‘m’ stands for meter. However, we use the same notation for brevity. Hence, below, e.g. 0.05 of length actually means 5 cm.

The two main equipment items in our system are a picosecond pulse generator (figure 1(b)) and a Textronix real time oscilloscope (figure 1(c)). The picosecond pulse generator generates electromagnetic pulses. The duration of each pulse is 300 ps. This pulse goes to the transmitter,
which is a horn antenna (source). The transmitter directs the pulse into the medium which contains targets of interest. The electromagnetic wave is scattered by the targets and the backscattered signals are detected by a polarization-sensitive small probe detector. The detected signal is recorded by the real time oscilloscope. The oscilloscope produces a digitized time-resolved signal with the step size in time increments of 10 ps. The total time of over which measurements are collected for one outgoing pulse is $10^{-8}$ s.

To decrease the measurement noise, the pulse is generated 800 times for each position of the detector, the backscatter signal is also measured 800 times and the resulting signals are averaged. The detector moves in both horizontal and vertical directions covering the square $\{ -0.5 < x, y < 0.5 \}$ on the measurement plane. We have chosen the step size of this movement to be 0.02. Although we can choose any step size, we found that 0.02 provides a good compromise between the precision of measurements and the total time spent on data collection.

The distance between our targets and the measurement plane is approximately 0.8 with about 0.05 deviations, and the wavelength of our signal is about 0.03. Therefore, the distance between the measurement plane and our targets is about 26 wavelengths. This is in the far field.

3.2. Data pre-processing

The main difficulty working with measured data is that there is a mismatch between these data and numerically simulated data. Indeed, figure 2(a) depicts a sample of experimentally measured data for a wooden block at one position of the detector, see figure 1(a) for data collection scheme. In this figure, the direct signal is the signal going directly to the receiver. We use this direct signal as the time reference for data pre-processing. Unwanted signals are due to reflections of the electromagnetic wave from several objects present in the room, such as the stand on which the target is placed. Figure 2(b) presents the numerically simulated data for the same target, see section 7.1 for data simulations. These figures show a huge mismatch
between real and simulated data. Therefore, data pre-processing is necessary. We refer to [20] for details of our data pre-processing procedure. The main steps of this procedure include the following.

1. **Time-zero correction.** The time-zero correction is to shift the measured data in time so that its starting time is the same as when the incident pulse is emitted from the transmitter. This is done using the direct signals from the transmitter to the detector as the time reference.

2. **Extraction of scattered signals.** Apart from the backscattered wave from the targets, our measured data also contain various other signals, e.g. the direct signals from the horn to the detector, the scattered signals from structures inside the room at a similar range, etc. What we need, however, are the scattered signals from just the target volume. To obtain them, we single out the scattered signals originating from the target volume only and remove all unwanted signals.

3. **Data propagation.** After acquiring the scattered signals from the target, the next step of the data pre-processing is to propagate the data closer to the target reconstruction domain. In the current work, a time-reversal method is used for this purpose. The idea of this method is to solve the wave equation backward in time. The data on the measurement plane are used as the Dirichlet boundary condition for the corresponding initial boundary value problem, see section 3.2.1 of [20] for details. This way, we can approximate the scattered wave on a plane which is much closer to the targets than the measurement plane. The distance between that propagated plane and the front surface of a target is usually between 0.02 and 0.06 (compare with the 0.8 distance from the measurement plane). There are two reasons for doing this. The first one is that the method of [3] works with the Laplace transform of function \( u(x, t) \) (section 4) and that Laplace transform decays exponentially in terms of the time delay, which is proportional to the distance from the target to the measurement plane. Hence, the amplitude of the Laplace-transformed measured data on the measurement plane is very small and can be dominated by computational round-off error. The second reason is that this propagation procedure helps to substantially reduce the computational cost since the computational domain for the inverse problem is reduced.

4. **Data calibration.** Finally, since the amplitudes of the experiment’s incident and scattered waves are usually significantly different from simulations, we find it convenient to bring the former to the same level of amplitude as the latter. This is done using a known target referred to as the **calibrating object**.

In this paper, the result of the above data pre-processing is then used as the measured data \( g(x, t) \) on the backscattering boundary \( \Gamma \) of our computational domain \( \Omega \) for the inverse problem.

### 4. The approximately globally convergent method in brief

In this section, we briefly outline the globally convergent method for the reader’s convenience. We refer to chapter 2 of [3] and [4] for more details.

In our inverse algorithm, we use the Laplace transform of the function \( u(x, t) \),

\[
w(x, s) = \int_0^\infty u(x, t) e^{-st} dt, \quad \text{for } s > s_0 = \text{const.} > 0,
\]

where \( s_0 \) is a certain number. We assume that the number \( s_0 \) is sufficiently large, and we call the parameter \( s \) **pseudo frequency**. It follows from theorem 2.7.2 of [3] that \( w(x, s) > 0 \). Hence, we can consider the functions \( v(x, s), q(x, s) \) defined by

\[
v(x, s) = \frac{\ln w(x, s)}{s^2}, \quad q(x, s) = \frac{\partial v(x, s)}{\partial s}.
\]

(4.2)
From (4.2), we obtain
\[ v(x, s) = - \int_s^\tau q(x, \tau) \, d\tau + V(x, \tau), \] (4.3)
where the truncation pseudo frequency \( \tau > s \) is a large number, which is chosen numerically, see section 8 for details. We call \( V(x, \tau) \) the \textit{tail function}, and it is unknown. It can be shown that the functions \( q \) and \( V \) satisfy the following nonlinear integral differential equation (see [3]):
\[ \Delta q - 2s^2 V q \int_s^\tau \nabla q(x, \tau) \, d\tau + 2s \left( \int_s^\tau \nabla q(x, \tau) \, d\tau \right)^2 + 2s^2 \nabla q \nabla V \\
- 4s V \int_s^\tau \nabla q(x, \tau) \, d\tau + 2s (\nabla V)^2 = 0, x \in \Omega, s \in [\xi, \tau]. \] (4.4)
In addition, (2.4) and (4.2) lead to the following Dirichlet boundary condition for the function \( q \):
\[ q(x, s) = \tilde{\psi}(x, s), \forall (x, s) \in \Gamma \times [\xi, \tau], \] (4.5)
\[ \tilde{\psi}(x, s) = \frac{\partial_s (\ln \psi)}{s^2} - \frac{2 \ln \psi}{s^3}. \] (4.6)
Here, \( \psi(x, s) \) is the Laplace transform of the function \( g(x, t) \) in (2.4). We now need to complement the boundary data (4.5) at the backscattering side \( \Gamma \) with the boundary data at the rest of the boundary \( \partial \Omega \). Using computationally simulated data, it was shown numerically in section 6.8.5 of [3] as well as in [4] that it is reasonable to approximate the boundary data on \( \partial \Omega \setminus \Gamma \) by the solution of the forward problem for the homogeneous medium for the case \( \epsilon_s(x) = 1 \). Thus, we use below the following Dirichlet boundary condition for the function \( q(x, s) \):
\[ q(x, s) = \psi(x, s), \forall (x, s) \in \partial \Omega \times [\xi, \tau], \] (4.7)
\[ \psi(x, s) = \begin{cases} \tilde{\psi}(x, s), \forall (x, s) \in \Gamma \times [\xi, \tau], \\
\psi^0(x, s), \forall (x, s) \in (\partial \Omega \setminus \Gamma) \times [\xi, \tau]. \end{cases} \] (4.8)
where the function \( \psi^0(x, s) \) is the function \( \tilde{\psi}(x, s) \) in (4.6) computed for the case \( \epsilon_s(x) \equiv 1 \).

Even though equation (4.4) with the boundary condition (4.7) has two unknown functions \( q \) and \( V \), we can approximate both of them because approximation procedures for them are different, see section 7.1. After these two functions are approximated in \( \Omega \), the coefficient \( \epsilon_s(x) \) can be easily calculated, see section 5.

4.1. \textit{The first approximation for the tail function}

To start the iterations, we need the first approximation \( V_{1,0}(x) \) for the tail function. In this section, we show how to calculate \( V_{1,0}(x) \), see also [3, 4].

Let the function \( \epsilon^*_s(x) \) satisfying (2.3) be the exact solution of our CIP for the exact data \( g^* \) in (2.4). We also denote by \( w^*, v^*, q^*, V^* \) and \( \psi^* \) respectively the corresponding functions \( w, v, q, V \) and \( \psi \) associated with the exact functions \( \epsilon^*_s \) and \( g^* \). From (4.2) and (4.3), we have
\[ V^*(x, \tau) = v^*(x, \tau) = \frac{\ln w^*(x, \tau)}{\tau^2}. \] (4.9)
Following (4.8), we assume that \( \psi^*(x, s) = \psi^0(x, s) \) for \( (x, s) \in (\partial \Omega \setminus \Gamma) \times [\xi, \tau] \). Hence, (4.4) and (4.7) hold for the functions \( q^* \) and \( \psi^* \).
It follows from [3] (section 2.3) that, under some conditions, there exists a function $p^*(x) \in C^{2+\alpha}(\overline{\Omega})$, such that the following asymptotic behaviors hold:

$$V^*(x, \gamma) = \frac{\ln w^*(x, \gamma)}{\gamma^2} = \frac{p^*(x)}{\gamma} + O\left(\frac{1}{\gamma^2}\right), \quad \gamma \to \infty,$$

(4.10)

$$q^*(x, \gamma) = \partial_\gamma V^*(x, \gamma) = -\frac{p^*(x)}{\gamma^2} + O\left(\frac{1}{\gamma^3}\right), \quad \gamma \to \infty.$$

(4.11)

Since $|O(1/\gamma^2)| \ll 1/\gamma$ and $|O(1/\gamma^3)| \ll 1/\gamma^2$ for sufficiently large $\gamma$, then, to obtain our approximate mathematical model, we truncate both asymptotics (4.10) and (4.11) leaving only the first term in each of them. We want to point out that we use this truncation only to obtain the first approximation $V_{1,0}(x)$ for the tail function. However, we do not truncate asymptotics (4.10), (4.11) on following iterations. It can be derived from the estimate (4.17) (below) that this truncation provides a good approximation for the exact coefficient $\varepsilon^*_s(x)$ already on the first iteration of the algorithm of section 6.1 (good approximation within the framework of our approximate mathematical model). Theorem 2.9.4 of [3] and theorem 5.1 of [4] guarantee that, given the so defined function $V_{1,0}(x)$, some subsequent iterations of the algorithm of section 6.1 also provide good approximations for $\varepsilon^*_s(x)$. We have observed computationally that it is better to use several iterations than just one and to develop a stopping criterion, see section 6.2 for the latter. Thus, using this truncation, we obtain the following approximate mathematical model.

**Approximate mathematical model.** We assume that there exists a function $p^*(x) \in C^{2+\alpha}(\overline{\Omega})$ such that the exact tail function $V^*(x, s)$ has the form

$$V^*(x, s) = \frac{p^*(x)}{s} = \frac{\ln w^*(x, s)}{s^2}, \quad \forall s \geq \gamma,$$

(4.12)

and the function $q^*(x, s)$ is given by

$$q^*(x, s) = -\frac{p^*(x)}{s^2}.$$

(4.13)

We note that the functions $q^*$ and $V^*$ satisfy (4.4). Setting in (4.4) $s = \gamma$ and taking into account (4.12) and (4.13), we obtain the following Dirichlet boundary value problem for $p^*$:

$$\Delta p^* = 0 \text{ in } \Omega, \quad p^* \in C^{2+\alpha}(\overline{\Omega}),$$

(4.14)

$$p^*|_{\partial \Omega} = -\gamma^2 \psi^*(x, \gamma).$$

(4.15)

Because of (4.14), (4.15) and (4.12), we set for the first tail

$$V_{1,0}(x) = \frac{p(x)}{\gamma},$$

(4.16)

where $p(x)$ is the solution of the problem (4.14)–(4.15) with $\psi^*$ being replaced by the boundary function $\psi$ defined by (4.8).

Using (4.12)–(4.16) and Schauder theorem [15], we obtain

$$\|V_{1,0}(x) - V^*(x, \gamma)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C\gamma\|\psi^*(x, \gamma) - \psi(x, \gamma)\|_{C^{2+\alpha}(\partial \Omega)},$$

(4.17)

where the number $C = C(\Omega) > 0$ depends only from the domain $\Omega$. Hence, the error in the calculation of $V_{1,0}(x)$ depends only on the error in the boundary data $\psi(x, \gamma)$. On the other hand, since the boundary function $\psi(x, s)$ is generated by the function $g(x, t)$ in (2.4), then the error in $\psi(x, \gamma)$ is generated by the error in the measurements. Estimate (4.17) is one of the elements of the proof of the approximate global convergence theorem for this numerical method, see theorem 2.9.4 in [3] and theorem 5.1 in [4].
4.2. Discretization with respect to the pseudo-frequency

To approximate both functions \( q \) and \( V \) using (4.4) and (4.7), we consider a layer stripping procedure with respect to \( s \). We divide the interval \([s_0, \bar{s}]\) into \( N \) small subintervals with the uniform step size \( h = s_{n-1} - s_n \). Here, \( s = s_N < s_{N-1} < \ldots < s_0 = \bar{s} \). We approximate the function \( q(x, s) \) as a piecewise constant function with respect to \( s \), i.e., we assume that \( q(x, s) = q_n(x) \) for \( s \in [s_n, s_{n-1}) \). Hence, using (4.3), we approximate the function \( v(x, s_n) \) as

\[
v(x, s_n) = -h \sum_{j=0}^{n} q_j(x) + V(x, \bar{s}), q_0(x) := 0.
\]  

(4.18)

To obtain a sequence of Dirichlet boundary value problems for elliptic PDEs for functions \( q_n(x) \), we introduce the \( s \)-dependent Carleman Weight Function \( C_{n, \mu}(s) = \exp[\mu(s - s_{n-1})] \), where \( \mu \gg 1 \) is a large parameter. In our numerical studies, we take \( \mu = 20 \). This function mitigates the influence of the nonlinear term in the resulting integral–differential equations on every pseudo-frequency interval \((s_n, s_{n-1})\).

We next multiply both sides of equation (4.4) by \( C_{n, \mu}(s) \) and integrate with respect to \( s \in (s_n, s_{n-1}) \). We obtain

\[
\Delta q_n - A_{1,n} \left( \frac{1}{h} \sum_{j=0}^{n-1} \nabla q_j - \nabla V_n \right) \nabla q_n
\]

\[
= B_n (\nabla q_n)^2 - A_{2,n} h^2 \left( \sum_{j=0}^{n-1} \nabla q_j \right)^2 + 2 A_{2,n} \nabla V_n \left( h \sum_{j=0}^{n-1} \nabla q_j \right) - A_{2,n} (\nabla V_n)^2,
\]

(4.19)

Here, \( V_n(x) \) is an approximation of the tail function \( V(x) \) which corresponds to function \( q_n(x) \) (section 6.1). Numbers \( A_{1,n}, A_{2,n}, B_n \) are computed explicitly. Furthermore, \( B_n = O(1/\mu), \mu \to \infty \). For this reason, we ignore the nonlinear term in (4.19), thus setting

\[
B_n (\nabla q_n)^2 := 0.
\]  

(4.20)

Note that (4.20) is not a linearization, since (4.19) contains products \( \nabla q_j \nabla q_i \) and also because the tail function \( V_n \) depends nonlinearly on functions \( q_j \), see step 6 in section 6.1.

5. A finite element method for the reconstruction of \( \varepsilon_f(x) \)

In this section, we explain how we compute functions \( \varepsilon_n(x) \) on every pseudo-frequency interval \((s_n, s_{n-1})\) using the FEM. Once the functions \( q_j, j = 1, \ldots, n \) along with the function \( V_n \) in (4.19) are calculated, we compute the function \( v_n(x) \) using the direct analogue of (4.18),

\[
v_n(x) = -h \sum_{j=0}^{n} q_j(x) + V_n(x), x \in \Omega.
\]

Using (4.2), we set

\[
w_n(x) = \exp \left[ s_n^2 v_n(x) \right].
\]  

(5.1)
To compute the function $\varepsilon_{mn}(x)$, we note that it follows from (2.1), (2.2) and (4.1) that the function $w_n(x, s_n)$ satisfies the following problem:

$$\Delta w_n - \frac{c}{\gamma_n} \varepsilon_{mn}(x) w_n = 0 \text{ in } \Omega, \quad (5.2)$$

$$\frac{\partial}{\partial n} w_n \big|_{\partial \Omega} = f_n(x), \quad (5.3)$$

where

$$f_n(x) = \frac{\partial}{\partial n} \exp \left[ \frac{c}{\gamma_n} v_n(x) \right] \text{ for } x \in \partial \Omega. \quad (5.4)$$

To compute the function $\varepsilon_{mn}(x)$ from (5.2), (5.3) and (5.4), we apply a version of the FEM as described below in sections 5.1, 5.2.

### 5.1. Spaces of finite elements

Following [13], we discretize in computations in our bounded domain $\Omega \subset \mathbb{R}^3$ by an unstructured tetrahedral mesh $T$ using non-overlapping tetrahedral elements $K \in \mathbb{R}^3$. The elements $K$ are such that $T = \{ K_1, \ldots, K_m \}$, where $m$ is the total number of elements in $\Omega$, and

$$\Omega = \bigcup_{K \in T} K = K_1 \cup K_2 \ldots \cup K_m.$$ 

We associate with the mesh $T$ the mesh function $h = h(x)$ as a piecewise constant function, such that

$$h(x) = h_K, \forall K \in T,$$

where $h_K$ is the diameter of $K$ which we define as the longest side of $K$. We impose the following shape regularity assumption of the mesh $T$ for every element $K \in T$

$$a_1 \leq h_K \leq r' a_2, \quad a_1, a_2 = \text{const.} > 0, \quad (5.5)$$

where $r'$ is the radius of the maximal sphere contained in the element $K$.

We define the set of polynomials $P_r(K)$ as

$$P_r(K) = \left\{ v : v(x, y, z) = \sum_{0 \leq i+j+l \leq r} c_{ijl} x^i y^j z^l, (x, y, z) \in K, c_{ijl} \in \mathbb{R}, \forall K \in T \right\}. \quad (5.6)$$

We introduce now the finite element space $V_h$ as

$$V_h = \{ v(x) \in H^1(\Omega) : v \in C(\Omega), v|_K \in P_1(K) \forall K \in T \},$$

where $P_1(K)$ denotes the set of linear functions on $K$ defined by (5.6) for $r = 1$. Hence, the finite element space $V_h$ consists of continuous piecewise linear functions in $\Omega$. To approximate functions $\varepsilon_{mn}$, we introduce the space of piecewise constant functions $C_h$,

$$C_h := \{ u \in L_2(\Omega) : u|_K \in P_0(K), \forall K \in T \},$$

where $P_0(K)$ is the piecewise constant function on $K$ defined by (5.6) for $r = 0$.

### 5.2. A finite element method

To compute the function $\varepsilon_{mn}$ from (5.2), we formulate the FEM for problem (5.2)–(5.4) as follows. Find the function $\varepsilon_{mn} \in C_h$ for the known function $w_n \in V_h$, such that

$$(\varepsilon_{mn} w_n, v) = -\frac{1}{s_n^2} (\nabla w_n, \nabla v) + \frac{1}{s_n^2} (f_n, v)_{\partial \Omega}, \forall v \in V_h, \quad (5.7)$$

where $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega)$. 

12
We expand \( w_n \) in terms of the standard continuous piecewise linear functions \( \{ \varphi_i \}_{i=1}^P \) in the space \( V_h \) as

\[
  w_n(x) = \sum_{l=1}^P w_{n,l} \varphi_l(x),
\]

where \( w_{n,l} \) denote the nodal values of the function \( w_n \) at the nodes \( l \) of the elements \( K \) in the mesh \( T \). We can determine \( w_{n,l} \) by knowing the already computed functions \( v_{n,l} \) using the following analogue of (5.1)

\[
  w_n(x) = \exp \left[ \sum_{l=1}^P \varepsilon_n \varphi_l(x) \right], \quad \forall x \in \Omega.
\]

Substitute (5.8) into (5.7) and choose \( v(x) = \varphi_j(x) \). Then we obtain the following linear algebraic system of equations:

\[
  \sum_{l,j=1}^P \varepsilon_{n,l,j}(w_{n,l} \varphi_l, \varphi_j) = -\frac{1}{s_n^2} \sum_{l,j=1}^P w_{n,l} (\nabla \varphi_l, \nabla \varphi_j) + \frac{1}{s_n^2} \sum_{j=1}^P [f_n, \varphi_j],
\]

where \([ \cdot, \cdot \] is the scalar product in \( L^2(\partial \Omega) \). System (5.9) can be rewritten in the matrix form for the unknown vector \( \varepsilon_n = \{ \varepsilon_{n,l,j} \}^P_{l,j=1} \) and known vector \( w_n = \{ w_{n,l} \}^P_{l=1} \) as

\[
  M \varepsilon_n = -\frac{1}{s_n^2} G w_n + \frac{1}{s_n^2} F.
\]

Here \( M \) is the block mass matrix in space, \( G \) is the stiffness matrix corresponding to the term containing \( (\nabla \varphi_l, \nabla \varphi_j) \) in (5.9) and \( F \) is the load vector. At the element \( K \), the matrix entries in (5.10) are explicitly given by

\[
  M_{l,j}^K = (w_{n,l}\varphi_l, \varphi_j)^K, \quad G_{l,j}^K = (\nabla \varphi_l, \nabla \varphi_j)^K, \quad F_{n,j}^K = (f_n, \varphi_j)^K.
\]

To obtain an explicit scheme for the computation of coefficients \( \varepsilon_{n,l,j} \), we approximate the matrix \( M \) by the lumped mass matrix \( M^l \) in space, i.e., the diagonal approximation is obtained by taking the row sum of \( M \) [3]. We obtain

\[
  \varepsilon_{n,l} = -\frac{1}{s_n^2} (M^l)^{-1} G w_n + \frac{1}{s_n^2} (M^l)^{-1} F.
\]

Note that for the case of linear Lagrange elements which are used in our computations in section 8, we have \( M = M^l \). Thus, this procedure does not include approximation errors in this case.

6. The approximately globally convergent algorithm

We present now our algorithm for the numerical solution of equations (4.19) and computing the functions \( \varepsilon_{n,l,j} \) using equation (5.11). In this algorithm, the index \( i \) denotes the number of inner iterations inside every pseudo-frequency interval \( (s_n, s_{n-1}) \) when we update tails.

6.1. The algorithm

Step 0. Set \( q_0 = 0 \). Compute the initial tail function \( V_{1.0}(x, y) \) \( \in C^{2+\alpha}((\Omega)) \) as in (4.16).

Step 1. Here we describe iterations which update tails inside every pseudo-frequency interval \( (s_n, s_{n-1}) \). Let \( n \geq 1, \quad i \geq 1. \) Suppose that functions \( q_j, j = 1, \ldots, n-1 \) are
Inverse Problems 30 (2014) 025002 L Bellina et al

computed. Solve the Dirichlet boundary value problem for the function \( q_{n,i}(x) \) \( \in C^{2\alpha+\delta}(\Omega) \),

\[
\Delta q_{n,i} - A_{ln} \left( \sum_{j=1}^{n-1} \nabla q_j \right) \cdot \nabla q_{n,i} + A_{ln} \nabla q_{n,i} \cdot \nabla V_{n,i-1} = -A_{2n} h^2 \left( \sum_{j=1}^{n-1} \nabla q_j \right)^2 + 2A_{2n} \nabla V_{n,i-1} \cdot \left( h \sum_{j=1}^{n-1} \nabla q_j \right) - A_{2n} (\nabla V_{n,i-1})^2,
\]

\( q_{n,i}(x) = \psi_{n,i}(x), \quad x \in \partial \Omega. \) \hfill (6.1)

Step 2. Compute functions \( v_{n,i}(x) \) and \( w_{n,i}(x) \).

\[
v_{n,i}(x) = -hq_{n,i}(x) - h \sum_{j=0}^{n-1} q_j(x) + V_{n,i}(x),
\]

\[
w_{n,i}(x) = \exp[\varepsilon x^n v_{n,i}(x)].
\]

Step 3. Compute function \( \tau_{r,n,i} \in C_h \) via backwards calculations, using the finite element formulation of equation (5.11) as

\[
\tau_{r,n,i}(x) = -\frac{1}{s_n^2} (M^t)^{-1} G w_{n,i} + \frac{1}{s_n^2} (M^t)^{-1} F.
\]

Since by (2.3), we should have \( \varepsilon r(x) \geq 1 \), \( \forall x \in \mathbb{R}^3 \), and also since we need to extend the function \( \tau_{r,n,i}(x) \) outside of the domain \( \Omega \) by unity, we set

\[
\varepsilon_{r,n,i}(x) = \begin{cases} 
\tau_{r,n,i}(x) & \text{if } \tau_{r,n,i}(x) \geq 1, \\
1 & \text{if either } \tau_{r,n,i}(x) < 1, \quad \text{or } x \in \mathbb{R}^3 \setminus \Omega.
\end{cases}
\] \hfill (6.2)

Step 4. Solve the forward problem (2.1)–(2.2) with \( \varepsilon_{r}(x) := \varepsilon_{r_{n,i}}(x) \) and compute the Laplace transform (4.1) for \( s = s_n \). We obtain the function \( w_{n,i}(x, s_n) \).

Step 5. Update the tail function as

\[
V_{n,i}(x) = \frac{\ln w_{n,i}(x, s_n)}{s_n^2}.
\] \hfill (6.3)

Continue inner iterations with respect to \( i \) until the stopping criterion of step 1 of section 6.2 is met at \( i = m_n \).

Step 6. Set for the pseudo-frequency interval \([s_n, s_{n-1})\)

\[
q_{n}(x) := q_{n,m_n}(x), \quad \varepsilon_{r_{n}}(x) := \varepsilon_{r_{n,m_n}}(x), \quad V_{n+1,0}(x) := \frac{\ln w_{n,m_n}(x, s_n)}{s_n^2} := V_{n}(x),
\] \hfill (6.4)

Step 7. If either the stopping criterion with respect to \( n \) of step 4 of section 6.2 is met, or \( n = N \), then set the resulting function \( \varepsilon_{r_{n}}(x) \) as the solution of our CIP. Otherwise, set \( n := n + 1 \) and go to step 1.
6.2. The stopping criterion

When testing the algorithm of section 6.1 on experimental data, we have developed a reliable stopping criterion for iterations \((n, i)\) in this algorithm. On every pseudo-frequency interval \((s_n, s_{n-1})\), we define ‘first norms’ \(D_{n,0}\) as

\[
D_{n,0} = ||V_{n,0}|r| - \hat{V}_n||_{L_2(\Gamma)}.
\]

In (6.5), the function \(V_{n,0}\) is the computed tail functions at the inner iteration \(i = 0\) as in (6.4). Functions \(\hat{V}_n\) in (6.5) are obtained from the known measured function \(g(x, t)\) in (2.4) as

\[
\hat{V}_n(x) = \frac{\ln W(x, s_n)}{s_n^2}, \quad x \in \Gamma,
\]

where \(W(x, s_n)\), \(x \in \Gamma\) is the Laplace transform of the function \(g(x, t)\) at \(s = s_n\).

We have observed that the computed ‘first norms’ \(D_{n,0}\) always achieve only one minimum at a certain \(n = \pi\), where the number \(\pi\) depends on the specific set of experimental data. Furthermore, in non-blind cases of non-metallic targets, the corresponding values of \(\max_{x} \epsilon_{\pi,0}(x)\) were in good agreement with \textit{a priori} known ones. However, in the cases of non-blind imaging of metallic targets, we have observed that \(5 \leq \max_{x} \epsilon_{\pi,0}(x) \leq 10\). This is in contradiction with (2.5). Therefore, we have developed the following stopping criterion which consists of four steps.

**The Stopping Criterion**

The first step in our criterion is for stopping inner iterations with respect to \(i\) in step 5 of section 6.1. As to steps 2–4, they are for stopping outer iterations with respect to \(n\) (step 7 in section 6.1). First, we define numbers \(D_{n,i}\) as

\[
D_{n,i} = ||V_{n,i}|r| - \hat{V}_n||_{L_2(\Gamma)}.
\]

In (6.7), functions \(V_{n,i}\) are computed tail functions corresponding to \(\epsilon_{\pi,i}\) (step 6 in section 6.1), and functions \(\hat{V}_n = \hat{V}_n(x, s_n)\) are connected with the measured data. They are calculated using (6.6).

- **Step 1.** Iterate with respect to \(i\) and stop iterations at \(i = m_n \geq 1\), such that either \(D_{n,i} \geq D_{n,i-1}\) or \(i = i_{\text{max}}\).

- **Step 2.** For every \(n\), compute ‘final norms’ \(D_{n,m_n}\) as

\[
D_{n,m_n} = ||V_{n+1,0}|r| - \hat{V}_n||_{L_2(\Gamma)}.
\]

In (6.9), functions \(V_{n+1,0}(x)\) are computed as in (6.4).

- **Step 3.** Compute the number \(\overline{N}\) of the pseudo frequency interval, such that the first norms \(D_{n,0}\) in (6.5) achieve their first minima with respect to \(n\) and obtain corresponding \(\epsilon_{\pi,0}(x)\) on this interval. Compute the number \(\overline{M}\) of the pseudo-frequency interval, such that the final norms \(D_{n,m_n}\) in (6.9) achieve their first minima or they are stabilized with respect to \(n\), and obtain the corresponding \(\epsilon_{\pi,0}(x)\) on this interval. Next, compute the number \(\tilde{\epsilon}_r\),

\[
\tilde{\epsilon}_r = \begin{cases} 
\max_{x} \epsilon_{\pi,0}(x), & \text{if } \overline{M} < \overline{N}, \\
\max_{x} \epsilon_{\pi,0}(x), & \text{if } \overline{M} \geq \overline{N}.
\end{cases}
\]

- **Step 4.** If \(\tilde{\epsilon}_r < 5\) or \(\tilde{\epsilon}_r > 10\), then take the final reconstructed value of the refractive index \(n = \sqrt{\tilde{\epsilon}_r}\). As the computed function \(\epsilon_r(x)\), take

\[
\epsilon_{r,\text{comp}}(x) = \begin{cases} 
\epsilon_{\pi,0}(x), & \text{if } \overline{M} < \overline{N}, \\
\epsilon_{\overline{N},0}(x), & \text{if } \overline{M} \geq \overline{N}
\end{cases}
\]
Conditions of our stopping criterion were achieved in all our computations, regardless of computational errors. As to the first condition of (6.8), since functions $\tilde{V}_n$ are obtained from the measured data, then this condition means that, on the part $\partial \Omega_1$ of the boundary $\Omega_1$, the tail function $\tilde{V}_n$, should not deviate from the data more than the previous tail function $\tilde{V}_{n-1}$. On the other hand, the second condition of (6.8) ensures that the number of iterations with respect to $i$ should not be too large. Note that the above stopping criterion might be improved in our future research. Figure 3 displays a typical $n$-dependence of sequences $D_{n,0}$ and $D_{n,m}$.

7. Some details of the numerical implementation

In this section, we present some additional details of our numerical implementation. Because of (2.5), we define in all our tests the upper value of the function $\varepsilon_r(x)$ as $b = 15$, see (2.3). Thus, we set lower and upper bounds for the reconstructed function $\varepsilon_r(x)$ in $\Omega$ as

$$M_{\varepsilon_r} = \{\varepsilon_r(x) : \varepsilon_r(x) \in [1, 15]\}.$$  

As to the lower bound, we ensure it via (6.2). We ensure the upper bound 15 similarly via truncating to 15 those values of $\varepsilon_{r,\text{comp}}(x)$ which exceed this number. To solve Dirichlet boundary value problems (6.1), we use the FEM. We reconstruct refractive indices rather than dielectric constants of materials since they can be directly measured.

To compare our computational results with directly measured refractive indices $n = \sqrt{\varepsilon_r}$ of dielectric targets and effective dielectric constants of metallic targets (see (2.5)), we consider maximal values of computed functions $\varepsilon_{r,\text{comp}}(x)$,

$$\varepsilon_r^{\text{comp}} = \max_{\Omega} \varepsilon_{r,\text{comp}}(x), \quad n^{\text{comp}} = \sqrt{\varepsilon_r^{\text{comp}}},$$  

and stop iterations. However, if $5 \leq \tilde{\varepsilon}_r \leq 10$, then continue iterations and compute the number $\tilde{N} \in (N + 1, N]$ of the pseudo-frequency interval, such that the global minimum with respect to $n$ of final norms $D_{n,m}$ in (6.9) is achieved. Then, similarly with (6.10), compute the number $\varepsilon_{\tilde{N}}$,

$$\varepsilon_{\tilde{N}} = \max_{\Omega_1} \varepsilon_{r,\tilde{N},0}(x)$$  

and take $n = \sqrt{\varepsilon_{\tilde{N}}}$ as the final reconstructed value of the refractive index. Also, take function $\varepsilon_{r,\text{comp}}(x) = \varepsilon_{r,\tilde{N},0}(x)$ as the computed coefficient $\varepsilon_r(x)$ and stop the iterations.

Figure 3. Behavior of norms $D_{n,0}$ (solid curve) and $D_{n,m}$ (dashed curve) for object 1.
see step 4 of section 6.2 for the definition of $\varepsilon_{r,\text{comp}}(x)$. Using experimental data for non-blind targets and comparing reconstruction results with cases of synthetic data, we have found that our algorithm provided accurate results with the following pseudo-frequency interval, which we use in all our computations
\[ s \in [8, 10], \omega = 8, \bar{\omega} = 10, h = 0.05. \]

## 7.1. Computations of the forward problem

As is clear from step 4 of section 6.1, we need to solve the forward problem (2.1), (2.2) at each iterative step of the inner iterations to update the tail via (6.3). Since it is impossible to computationally solve equation (2.1) in the infinite space $\mathbb{R}^3$, we work with a truncated domain. Namely, we choose the domain $G$ as
\[ G = \{ x = (x, y, z) \in (-0.56, 0.56) \times (-0.56, 0.56) \times (-0.16, 0.1) \}. \]

We use the hybrid FEM/FDM method described in [2] and the software package WaveESP [22]. We split $G$ into two subdomains $G_{\text{FEM}} = \Omega$ and $G_{\text{FDM}}$ so that $G = G_{\text{FEM}} \cup G_{\text{FDM}}$. We solve the forward problem in $G$ and the inverse problem via the algorithm of section 6.1 in $\Omega$. The space mesh in $G_{\text{FEM}}$ and in $G_{\text{FDM}}$ consists of tetrahedral and cubes, respectively. Below
\[ G_{\text{FEM}} = \Omega = \{ x = (x, y, z) \in (-0.5, 0.5) \times (-0.5, 0.5) \times (-0.1, 0.04) \}. \] (7.3)

Since by (2.3) $\varepsilon_r(x) = 1$ in $G_{\text{FDM}}$, then it is computationally efficient to use FDM in $G_{\text{FDM}}$ and to use FEM in $G_{\text{FEM}} = \Omega$, as it is done in the hybrid method of [2].

The front and back sides of the rectangular block $G$ are $\{ z = 0.1 \}$ and $\{ z = -0.16 \}$, respectively. The boundary of the domain $G$ is $\partial G = \partial_1 G \cup \partial_2 G \cup \partial_3 G$. Here, $\partial_1 G$ and $\partial_2 G$ are, respectively, front and back sides of the domain $G$, and $\partial_3 G$ is the union of left, right, top and bottom sides of this domain. The front side $\Gamma$ of the rectangular block $\Omega$ where the propagated data $g(x, t)$ in (2.4) are given is
\[ \Gamma = \{ x \in \partial \Omega : z = 0.04 \}. \] (7.4)

Now we describe the forward problem which is used in our computations. To compute tail functions $V_{n, i}$ via steps 4, 5 of the algorithm of section 6.1, we computationally solve the following forward problem in our tests:

\[
\begin{align*}
\varepsilon_r(x)u_{tt} - \Delta u &= 0, & \text{in } G \times (0, T), \\
u(x, 0) &= 0, \quad u_t(x, 0) = 0, & \text{in } G, \\
\partial_n u &= f(t) , & \text{on } \partial_1 G \times (0, t_1], \\
\partial_n u &= -\partial_2 u , & \text{on } \partial_1 G \times (t_1, T), \\
\partial_n u &= -\partial_3 u , & \text{on } \partial_2 G \times (0, T), \\
\partial_n u &= 0, & \text{on } \partial_3 G \times (0, T),
\end{align*}
\] (7.5)

where $f(t)$ is the amplitude of the initialized plane wave,
\[ f(t) = \sin \omega t , \quad 0 \leq t \leq t_1 := \frac{2\pi}{\omega}. \]

We use $\omega = 30$ and $T = 1.2$. We solve problem (7.5) using the explicit scheme with the time step size $\tau = 0.003$, which satisfies the CFL condition.

## 7.2. Two stages

Our reconstruction procedure is done in two stages described in this section.
7.2.1. **First stage.** In the first stage, we follow the algorithm of section 6.1. We have observed that this stage provides accurate locations of targets of interest. It also provides accurate values of refractive indices \( n = \sqrt{\epsilon(r)} \) of dielectric targets and large values of effective dielectric constants \( \epsilon_r \) for metallic targets, see (6.10) and (6.12). However, the algorithm of section 6.1 does not reconstruct well the sizes/shapes of targets. Thus, we need a post-processing procedure for all of our targets for the following three tests.

Next, we determine minimal values of refractive indices \( n \) that this stage provides accurate locations of targets of interest. It also provides accurate their shapes. It is more challenging to compute the sizes of targets in the constants of metallic targets and (3) to image the locations of targets, their sizes and sometimes targets, (2) to reconstruct refractive indices of dielectric targets and appearing dielectric targets, (1) to differentiate between dielectric and metallic targets and (2) to image the locations of targets, their sizes and sometimes appearing dielectric targets, see (6.12).

The goals of our computational studies are (1) to differentiate between dielectric and metallic targets, (2) to reconstruct refractive indices of dielectric targets and appearing dielectric targets, (3) to image the locations of targets, their sizes and sometimes appearing dielectric targets, see (6.12).

Next, we set \( \tilde{\epsilon}_{r,1}(x) = \left\{ \begin{array}{ll} \epsilon_r(x) & \text{if } \epsilon_r(x) > 0.5 \max \epsilon_r(x), \\ 1 & \text{otherwise.} \end{array} \right. \) (7.6)

Next, we determine minimal \( x_{\min}, y_{\min} \) and maximal \( x_{\max}, y_{\max} \) values in \( x \) and \( y \) directions, where the function \( \tilde{\epsilon}_{r,1}(x) > 1 \). Next, we set \( \epsilon_{r,1}(x) := \left\{ \begin{array}{ll} \tilde{\epsilon}_{r,1}(x) & \text{if } x \in [x_{\min}, x_{\max}], y \in [y_{\min}, y_{\max}], \\ 1 & \text{otherwise} \end{array} \right. \)

and proceed with step 5 of the algorithm of section 6.1. In this second stage, we perform the same number of iterations with respect to both indices \( n, i \) as we did for those on the first stage. We are concerned in the second stage only with the size and shapes of targets and we are not concerned with values of \( \epsilon_r^{\text{comp}}, n^{\text{comp}} \). Rather, we take these values from the first stage. Let \( \tilde{\epsilon}_r(x) \) be the function \( \epsilon_r(x) \) obtained at the last iteration of the second stage. After this, we form the image of the target based on the function \( \epsilon_{r,\text{image}}(x) \):

\[
\epsilon_{r,\text{image}}(x) = \left\{ \begin{array}{ll} \tilde{\epsilon}_r(x) & \text{if } \tilde{\epsilon}_r(x) \geq 0.9 \max \tilde{\epsilon}_r(x), \\ 1 & \text{otherwise.} \end{array} \right.
\]

8. **Results**

The goals of our computational studies are (1) to differentiate between dielectric and metallic targets, (2) to reconstruct refractive indices of dielectric targets and appearing dielectric targets, (3) to image the locations of targets, their sizes and sometimes appearing dielectric targets, see (6.12). It is more challenging to compute the sizes of targets in the \( z \)-direction (i.e., depth) than in \( x-, y\)-directions.

8.1. **Three tests**

To see how sensitive the algorithm is to \( x, y \) sizes of the block \( \Omega \) as well as to the mesh step size \( h_\Delta \) in computations of both forward and inverse problems, we run the above numerical procedure for all of our targets for the following three tests.

Test 1. The domain \( \Omega \) for the computation of the CIP is as in (7.3) and the mesh step size is \( h_\Delta = 0.02 \). Recall that the distance between neighboring positions of our detector on the measurement plane \( P_m \) is also 0.02.

Test 2. The domain \( \Omega \) is as in (7.3), but the mesh step size here is \( h_\Delta = 0.01 \).

Test 3. In this test, we shrink the domain \( \Omega \) in \( x-, y\)-directions, while keeping the same mesh size \( h_\Delta = 0.02 \) as in test 1. In this test,

\[
G_{\text{FEM}} = \Omega = \{x = (x, y, z) \in (-0.2, 0.2) \times (-0.2, 0.2) \times (-0.1, 0.04)\},
\]

\[
M_{\text{ef}} = \{\epsilon_r(x) : \epsilon_r(x) \in [1, 15]\}.
\]
Table 2. Computed $n^{\text{comp}}$ and directly measured refractive indices of dielectric targets together with both measurement and computational errors as well as the average error. Note that the average computing errors are at least three times less than the average error of direct measurements.

| Target number | 1 | 2 | 5 | 8 | 10 | Average error |
|---------------|---|---|---|---|----|---------------|
| Blind/non-blind (yes/no) | no | no | yes | yes | yes |                |
| Measured $n$, error | 2.11, 19% | 1.84, 18% | 2.14, 28% | 1.89, 30% | 2.1, 26% | 24% |
| $n^{\text{comp}}$ of test 1, error | 1.92, 10% | 1.8, 2% | 1.83, 17% | 1.86, 2% | 1.92, 9% | 8% |
| $n^{\text{comp}}$ of test 2, error | 2.07, 2% | 2.01, 10% | 2.21, 3% | 1.83, 3% | 2.2, 5% | 4.6% |
| $n^{\text{comp}}$ of test 3, error | 2.017, 5% | 2.013, 9% | 2.03, 5% | 1.97, 4% | 2.02, 4% | 5% |

Table 3. Computed appearing dielectric constants $\varepsilon^{\text{comp}}$ of metallic target number 3,4,6,7,11 as well as of the target number 9 which is a metal covered by a dielectric.

| Target number | 3 | 4 | 6 | 7 | 9 | 11 |
|---------------|---|---|---|---|---|----|
| Blind/non-blind (yes/no) | no | no | yes | yes | yes | yes |
| $\varepsilon^{\text{comp}}$ of test 1 | 14.4 | 15.0 | 15 | 13.6 | 13.6 | 13.1 |
| $\varepsilon^{\text{comp}}$ of test 2 | 15 | 15 | 15 | 14.1 | 14.1 | 15 |
| $\varepsilon^{\text{comp}}$ of test 3 | 15 | 15 | 15 | 14 | 14.06 |    |

8.2. Reconstructions

We have collected experimental data for 11 targets listed in table 1. Five targets were dielectrics, five were metallic and one was metal covered by a dielectric. We had a total seven blind cases: three dielectric and three metallic targets and one unknown target was the above mixture of metal and dielectric. Three out of eleven targets were heterogeneous and all three were unknown prior to data processing and applying the inversion algorithm. Heterogeneous targets we designed to model explosive devices in which explosive materials are masked by dielectrics.

When proceeding with the algorithm of section 6.1, we first assign the Dirichlet boundary condition $\psi(\mathbf{x}, s)$ at $\partial\Omega$ for function $q(\mathbf{x}, s)$ following (4.5), (4.7) and (4.8), in which case $\Gamma$ is as in (8.1). Next, we calculate functions $\psi_n(\mathbf{x})$ as in (4.19). Figure 4 presents typical behavior of functions $\psi_n(\mathbf{x})$ at $\partial\Omega$ for some objects of table 1. To have a better visualization, these figures are zoomed to $0.4 \times 0.4$ square from the $1 \times 1$ square.

Table 2 lists both computed $n^{\text{comp}}$ and directly measured refractive indices $n$ of dielectric targets for tests 1–3, see (7.2) for $n^{\text{comp}}$. This table also shows the measurement error in direct measurements of $n$. These direct measurements were made by the traditional phase shift monitoring method [12]. Table 3 lists calculated effective dielectric constants $\varepsilon^{\text{comp}}$ of the metallic targets. Recall that $\varepsilon_r = n^2$. We see from table 2 that $(n^{\text{comp}})^2 < 4.9$ for all dielectric targets. This is consistent with step 4 of the stopping criterion. On the other hand, in table 3, $\varepsilon^{\text{comp}}_r > 12$ for all metallic targets. Thus, our algorithm can confidently differentiate between dielectric and metallic targets.

One can derive several important observations from table 2. First, in all three tests and for all targets, the computational error is significantly less than the error associated with direct measurements. Thus, the average computational error is significantly less than the average measurement error in all three tests. Second, computed refractive indices are within reasonable error estimates in all cases. The accuracy of all three tests is about the same.

Table 3 provides information about computed appearing dielectric constants $\varepsilon^{\text{comp}}_r$ of metallic targets, see (2.5) and (7.2). Note that in test 3, the first four numbers $\varepsilon^{\text{comp}}_r = 15.$
This coincides with the upper bound in (8.2). On the other hand, $\varepsilon_{\text{comp}} = 14 < 15$ for the target number 9. This is assumed to be because target number 9 is a mixture of metal and dielectric. An important observation, which can be deduced from table 2 is that our algorithm confidently computes large inclusion/background contrasts exceeding 10:1. It is well known that optimization methods of conventional least squares residual functionals usually cannot image large contrasts.

All targets, except for target number 8, 9, 10, were homogeneous materials, i.e., they were comprised of a single substance only. However, target numbers 8–10 were heterogeneous...
structures; see table 1 for descriptions of all targets. Target number 8 was a wooden doll which was empty inside. In the case of target number 9, a piece of a metal was inserted inside that doll. Under these circumstances, only the metal was imaged, because its reflection was much stronger than from the wood. In the case of target number 10, sand was partly inserted inside that same doll.

Figure 5 displays 3D images of some targets for test 1 after the first and the second (post-processing) stages described in section 7.2. Figures 6, 7 display 3D images of targets 8, 9, 10 and 11 for all three tests.

Note that it is difficult to estimate well the size of a target in the \( z \)-direction. Nevertheless, one can observe that rather good shapes and sizes of targets are computed in the case of blocks and cylinders, see figure 5. As to the doll, neither of the tests imaged the shapes of targets 8–10 accurately. Nevertheless, the location of the doll as well as its sizes in \( x \)-, \( y \)-directions are well estimated, see figures 6.

9. Summary

We conducted experiments to measure the backscattered time-resolved data from targets illuminated by a short electromagnetic pulse. We have applied the approximately globally convergent numerical method of [3] to these data. Results for four non-blind and seven blind cases show good accuracy of reconstruction of the refractive indices of the dielectric targets and of the effective dielectric constants of the metallic targets. In the case of dielectrics, the average reconstruction error was at least three times lower than the error of direct measurements. We were confidently able to differentiate between metallic and dielectric targets. In particular, we
Figure 6. Computed images of target numbers 8, 9, 10 (see table 1). Thin lines indicate correct shapes. To have better visualization, we have zoomed in on the images of tests 1, 2 from the domain \( \Omega_1 \) defined by (7.3) to the domain (8.1).

Figure 7. Computed images of target number 11 of table 1. Thin lines indicate correct shapes of two inclusions to be reconstructed. To have a better visualization, we have zoomed in on the images of tests 1, 2 from the domain \( \Omega_1 \) defined by (7.3) to the domain (8.1).
have accurately computed maximal values of refractive indices/dielectric constants of three blind experiments involving heterogeneous targets. These targets were designed to represent simplified models of improvised explosive devices IEDs, which are likely to be heterogeneous.

The locations of targets and their sizes in x-, y-directions are accurately reconstructed. The most difficult cases of sizes in the z-direction (depth) are well reconstructed in some cases. In addition, shapes of some targets are well reconstructed in some cases. We believe that a follow up application of the locally convergent adaptivity technique of [6] might improve the reconstructions of the shapes of targets or making measurements over a wider range of backscattered angles. The adaptivity takes the solution obtained by the approximately globally convergent method as the starting point for the minimization of the Tikhonov functional on a sequence of adaptively refined meshes. A significant refinement via this adaptivity was demonstrated in [6] for the case of transmitted simulated data and in section 5.9 of [3] for the case of transmitted experimental data, see figures 5.13 and 5.16 in [3].

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