Effect of tendon structural control on the appearance of horseshoes chaos on a cantilever beam due to seismic action

Blaise Romeo Nana Nbendjo and Uwe Dorka

Abstract
This article deals with the problem of improving the seismic strength of mechanical structures using a tendon system as actuation device. It consists of a pair of tension cables transmitting a control torque to the structure at the moment arm position. The purpose of the article is to establish the analytical framework consisting of mathematical modeling of cantilever beam under active tendon structural control, analyze the stability, and determine the physical characteristics of tendon system leading to the suppression of horseshoes chaos. The control efficiency is found by analyzing the behavior of the controlled system through the Melnikov methods. We also provide the critical control gain parameters leading to the efficiency of the control process.

Keywords
Cantilever beam, tendon systems, seismic action, Melnikov theory, horseshoes chaos

Introduction
For years, the problem of seismic-induced vibrations in mechanical structures has received broad attentions. It has now reached the stage where active systems have been installed in full-scale structures.1,2 This article demonstrates theoretically a possibility of implementation of active tendon control to a full-scale structure under actual ground motions. One method used to handle this problem is the installation of automatic active control forces in mechanical structures as mentioned by Yao.3 A number of structural concepts have been identified4 which allow rigid body control and four concepts (Base Isolation, Hysteretic Device System, Tendon System, and Pagoda System) have been suggested for seismic control.5 Among those, active control using tendons has been one of the most promising techniques to implement control mechanisms.6–9 These systems consist of a set of prestressed tendons connected to a structure where a servomechanism controls their tensions. The tendon is generally modeled as a simple spring or as a spring and dashpot in parallel. This simplicity makes it attractive and also helpful for retrofitting or strengthening an existing structure.

Figure 1 illustrates the idea of a tendon control system for suppressing cantilever beam vibrations. A pair of actuators at the beam root activates the tendons (i.e. tension cables) to rotate a pair of moment arms attached at a proper position of the structure. Thereby,
the beam motion is actively controlled using the feedback signals from sensing devices located on top of the beam. The tendon control is suitable to this task not only because the hardware is simple to implement but also because a robust collocated control is realized using “non-collocated” sensors and actuators.8,9

In this article, we are interested by the condition for which tendon system could avoid the global bifurcation before and after loss of stability.10,11 Since these conditions can be detected by means of basin of attraction, it is important to obtain the criteria for theoretically quenching chaotic dynamics. This may imply the existence of fractal basin boundaries and the so-called horseshoes structure of chaos. It will be shown that under suitable hypotheses, chaos arises quite naturally in the dynamics of the cantilever beam subjected to seismic action. Chaotic motions of a beam were earlier studied by Holmes and Marsden,12 and a general formalism of a cantilever beam under tendon system control subjected to seismic action is obtained by standard methods is given by

\[
\rho S \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} + \lambda \frac{\partial y}{\partial t} + \Gamma(t) \frac{\partial^2 y}{\partial x^2} - k \int_0^l \frac{\partial y}{\partial x} \, dx \frac{\partial^2 y}{\partial x^2} = z_c \delta(x-l)
\]

where \( y = y(x,t) \) is the lateral deflexion. \( \lambda \) represents the viscous damping, \( k \) represents the nonlinear stiffness, and \( \Gamma(t) \) is the ground acceleration applied to the beam. The nonlinear term expresses the fact that the axial force in the beam increases with lateral deflexion, leading to an increase in restoring forces. \( z_c \) is the control force and \( \delta(x-l) \) materializes the fact that this force acts on top of the beam. The control force given by Zhang et al.7 is as follows

\[
z_c(t) = 4k_c \cos \alpha_c \left[ s_1 y(t-t_c) + s_2 \frac{\partial y}{\partial t} (t-t_c) \right]
\]

where \( k_c \) is the tendon stiffness, \( \alpha_c \) the tendon inclination, \( s_1 \) and \( s_2 \) are the control gain parameters. \( t_c \) and \( t_k \) are time delays for displacement and velocity feedback force in the system, respectively. We notice that \( x \) and \( \dot{x} \) are written as a function of \( t-t_c \) and \( t-t_k \), respectively. In fact, tendons are viscoelastic structures which mean they exhibit both elastic and viscous behaviors. When stretched, the stress–strain curve starts with a very low stiffness region, and then the structure becomes significantly stiffer and behaves reasonably until it begins to fail. The tendon can be usually installed by drilling holes into existing walls or columns and is anchored in both ends. One of these ends may carry advanced control mechanisms, like (nonlinear) springs or shape-memory-alloy devices.

The cantilever beam is fixed at its base (\( x = 0 \)) and free at the top (\( x = l \)), which implies the following boundary conditions

\[
y(0,t) = 0, \quad \frac{\partial y}{\partial x}(0,t) = 0
\]
\[ \gamma(0, t) = \frac{\partial y}{\partial x}(0, t) = \frac{\partial^2 y}{\partial x^2}(l, t) = \frac{\partial^3 y}{\partial x^3}(l, t) = 0 \]  

Carrying out the conventional Galerkin averaging, we obtain a set of \( n \)th second-order ordinary differential equations coupled. Assuming \( n = 1 \), we obtain the following ordinary differential equation as the non-dimensional equation of motion for the first mode of the beam

\[ \ddot{x}(\tau) + 2\xi \dot{x} + \left( 1 + \frac{\Gamma(\tau)}{\Gamma_{cr}} \right)x(\tau) + \beta x^3 = \gamma_1 \dot{x}(\tau - \tau_z) + \gamma_2 x(\tau - \tau_z) \]

with \( \xi = \lambda/2\sqrt{\rho SEI}, \beta = -k_2/ESI_1, \Gamma_{cr} = I_1EI/I_3, f_0 = 4k_2 \cos(q_1l_1/l_1EI), \tau_z = \omega_0t_z, \gamma_1 = f_0s_1, \gamma_2 = f_0\omega_0s_2, \) and \( \omega_0 = \sqrt{EI}/\rho S \).

We remind readers of the following

\[ I_1 = \int_0^l \Phi^I(x)\Phi(x)dx, \quad I_2 = \frac{\int_0^l \left( \int_0^l \Phi^I(x)dx \right)\Phi(x)\Phi^I(x)dx}{\int_0^l \Phi^2(x)dx}, \]

\[ I_3 = \int_0^l \Phi^I(x)\Phi(x)dx, \quad I_4 = \frac{\int_0^l \Phi^2(x)dx}{\int_0^l \Phi^2(x)dx} \]

where \( \Phi(x) = -(\cosh(k_1l) + \cos(k_1l)/\sinh(k_1l) + \sin(k_1l))/[(\sin(k_1x) - \sin(k_1x)) + \cos(k_1x) - \cos(k_1x)] \).

To compute these quantities, we utilized Maple and after some algebraic manipulations, we have obtained

\[ I_1 = 12.36255606/l^4, \quad I_2 = 2.362759229/l^3, \]

\[ I_3 = 0.8582473444/l^2, \quad I_4 = 4.000002152/l \]

The earthquake signal can be modeled as filtered white noise process, while the filter design is based on a prescribed spectrum of ground motion. Here, the Kanai–Tajimi spectral description of the ground motion is used

\[ S(\omega) = S_0 \frac{\omega^4 + 4\zeta_0^2\omega^2}{\omega^2 - \omega_0^2} \]

where \( \omega_0, \zeta_0, \) and \( S_0 \) are parameters which depend on the soil characteristics and seismic intensity. The transient or non-stationary feature of the earthquake is introduced through an amplitude modulating function. An equivalent expression for the evolutionary of earthquake acceleration for elastic–plastic single-degree-of-freedom structures has been presented by Abbas and Manohar, Abbas, and Ndemanou et al. Where at the first step, the ground acceleration is represented as a product of a Fourier series and an enveloping function as follows

\[ \Gamma(\tau) = e(\tau) \left[ \sum_{i=1}^{N} A_i \cos(\omega_i \tau) + B_i \sin(\omega_i \tau) \right] \]

where \( A_i \) and \( B_i \) are 2N unknown constants and \( \omega_i, i = 1, 2, \ldots, N \), are the frequencies embedded in the ground acceleration \( \Gamma \) which are selected such that they span the frequency range \( (\omega_0, \omega_c) \). The function \( e(\tau) \) represents the enveloping function that imparts transient nature to the earthquake acceleration. In this study, the envelope function \( e(\tau) \) is given by

\[ e(t) = A_0[\exp(-\alpha_1 \tau) - \exp(-\alpha_2 \tau)] \]

where \( A_0, \alpha_1, \) and \( \alpha_2 \) are the parameters of the enveloping function. The maximum value of the enveloping is per the above expression is unity. In this study, for numerical purposes, the frequencies presented in the ground acceleration are selected such that they span the frequency range \( (\omega_0 = 0.2 \text{ Hz}, \omega_c = 25 \text{ Hz}) \). During this analysis, \( A_0 = 2.17, \alpha_1 = 0.13, \) and \( \alpha_2 = 0.50 \). These choices represent the earthquake duration to be about 30 s, that is, typical of the magnitude 7.0. As an illustration, the time history of the optimal ground acceleration and associated Fourier amplitude spectrum for the earthquake load for this case is shown in Figure 2. This critical acceleration will be used for numerical simulation.
Effect of tendon system control on the stability of control design and on the appearance of horseshoes chaos

On the stability of the control design

We assume that the tendon system is locked to provide the maximum suppression of oscillations. Then, the amplitude dynamics describes the rate of amplitude change depending on the physical parameters of the controlled system. Taking into account all the component acting on the system, the non-dimensional governing equation describing the physical system under tendon control is given by

\[
\dot{\chi}(\tau) + 2\xi \dot{\chi} + \left( 1 + \frac{A_0}{c_{cr}} \left[ \exp(-\alpha_1 \tau) - \exp(-\alpha_2 \tau) \right] \right) \sum_{i=1}^{N} A_i \cos(\omega_i \tau) + B_i \sin(\omega_i \tau) \right) \chi(\tau) = 0
\]

In the autonomous case, the system’s stability is explored using the Lyapunov concept that examining the fundamental solution \( e^{Sd} \) (\( S \) is the Lyapunov exponent). The characteristic equation of the eigensystem is then given by

\[
S^2 + (2\xi - \gamma_2 e^{-S_1}) S + 1 - \gamma_1 e^{-S_2} = 0
\]

which is known as a quasipolynomial. To study the stability of the controlled system with respect to the control gain parameters \( \gamma_1 \) and \( \gamma_2 \), the method of D-subdivision is used. The stability boundaries are determined by the points that yield either zero root or a pair of pure imaginary roots of the quasipolynomial. After analysis, we reach to the conclusion that the control system will remain stable if the control gain parameters verify these conditions \( \gamma_1 < 2\xi \) and \( \gamma_2 < 1 \). Thus, we come to the fact that the viscoelastic parameters along with time delays have an important effect on the efficiency of the control process. For illustration, we have solved numerically this equation using fourth-order Runge–Kutta algorithm. We have plotted the evolution of the amplitude of vibration as a function of time in Figure 3. It is viewed in Figure 3(a) (\( \gamma_1 = 0.4 \) and \( \gamma_2 = -0.02 \)) that the amplitude decreases as function of time leading to stability, while in Figure 3(b) (\( \gamma_1 = 0.4 \) and \( \gamma_2 = 0.01 \)), the amplitude increases with time leading to instability of the control system. It also appears that as the damping coefficient of the controller increases, the gap between the amplitude of the controlled and uncontrolled systems decreases, meaning that as tendon is damped, the quality of control is destroyed. These findings should be considered for a better implementation of this concept. The analysis of the effect of time delay on the control process gives rise to the conclusion that the stable domain in control space parameters leading to the efficiency of the control is drastically reduced as the feedback delay increases.\(^7,10\)

On the appearance of horseshoes chaos

Holmes and Marsden\(^12\) have studied the buckled beam subjected to linear damping and periodic transverse forcing. They presented a Melnikov-type technique for a class of infinite-dimensional systems and gave a criterion under which the Smale horseshoes chaos appears. In fact, the Melnikov theory has been developed to predict the splitting of homoclinic or heteroclinic orbits under non-autonomous perturbations. In particular, it can be used to establish the existence, or non-existence, of transverse homoclinic or heteroclinic orbits in dynamic systems upon adding small non-autonomous terms to the governing vector field. Transverse homoclinic or heteroclinic orbits, in turn, imply the existence of horseshoes, and therefore of chaotic dynamics. The basin of attraction is generally used as an indicator for the existence of horseshoes chaos. In this work, one would like to know how the control strategy affects the Melnikov criterion or in what range of the control parameters the heteroclinic chaos in our model could be inhibited? To deal with such a question, let us express the dynamic structure as follows

\[
\dot{U} = F(U) + eG(U)
\]

Figure 3. Effect of tendon parameters on the stability of the control system: (a) time history without tendon and (b) time history with tendon.
where $U = (x; y = \dot{x})$ is the state vector, $F = (y; -\chi - \beta \chi^3)$, and $G = (0; 2\xi \ddot{x} + \Gamma(\tau)/\Gamma_c)\chi(\tau) + \gamma_1 \chi(\tau - \tau_1) + \gamma_2 \chi(\tau - \tau_2)$). The unperturbed system has three fixed points. A hyperbolic fixed point and two elliptic fixed points. Thus, we can compute the Melnikov function\textsuperscript{11,12} and obtained the following result

\[ M(\tau_0) = -2\xi J_0 + \gamma_1 J_{\tau_1} + \gamma_2 J_{\tau_2} \]

\[ -\frac{A_0}{\Gamma_c\sqrt{2\beta^2}} \sum_{i=1}^{N} \left( J_{i_l}^A + J_{i_l}^B \right) \cos \omega_i \tau_0 \]

\[ + \sum_{i=1}^{N} \left( J_{i_l}^A - J_{i_l}^B \right) \sin \omega_i \tau_0 \]

(12)

where

\[ J_0 = -\frac{2\sqrt{2}}{3\beta} \]

\[ J_{i_l}^A = 2\pi \sqrt{2} A_i[D_i(\alpha_1) + D_i(\alpha_2)] \]

\[ J_{i_l}^B = 2\pi \sqrt{2} B_i[E_i(\alpha_1) + E_i(\alpha_2)] \]

\[ J_{\tau_1} = \frac{1}{\sqrt{2\beta^2} \sinh^2 \left( \frac{\tau_1}{\sqrt{2}} \right)} \left[ 2\tau_1 - \sqrt{2} \sinh \left( 2\tau_1 \right) \right] \]

\[ J_{\tau_2} = -\frac{2}{\beta \sinh^3 \left( \frac{\tau_2}{\sqrt{2}} \right)} \left[ \tau_2 \cosh \left( \frac{\tau_2}{\sqrt{2}} \right) - \sqrt{2} \sinh \left( \frac{\tau_2}{\sqrt{2}} \right) \right] \]

(13)

with

\[ D_i(\alpha) = \frac{2\alpha \omega_i \cos \left( \frac{\pi \alpha}{2} \right) \sinh \left( \frac{\pi \alpha}{2} \right) + (\alpha^2 - \omega_i^2) \cosh \left( \frac{\pi \alpha}{2} \right) \sin \left( \frac{\pi \alpha}{2} \right)}{\cos \left( \frac{\pi \alpha}{2} \right) - \cosh \left( \pi \omega_i \sqrt{2} \right)} \]

\[ E_i(\alpha) = \frac{-2\alpha \omega_i \sin \left( \frac{\pi \alpha}{2} \right) \cosh \left( \frac{\pi \alpha}{2} \right) + (\alpha^2 - \omega_i^2) \sin \left( \frac{\pi \alpha}{2} \right) \cos \left( \frac{\pi \alpha}{2} \right)}{\cos \left( \frac{\pi \alpha}{2} \right) - \cosh \left( \pi \omega_i \sqrt{2} \right)} \]

From equation (11), we get the condition for the appearance of the Melnikov chaos in the space parameters $(\gamma_1, \gamma_2)$ which is given by the equation

\[ A_0 \sum_{i=1}^{N} \left( J_{i_l}^A + J_{i_l}^B + J_{2l}^B - J_{2l}^A \right) \]

\[ -\Gamma_c \sqrt{2\beta^2} \left( -2\xi J_0 + \gamma_1 J_{\tau_1} + \gamma_2 J_{\tau_2} \right) > 0 \]

(14)

Figure 4 displays the domain in parameter space $(\gamma_1, \gamma_2)$ leading to the occurrence or suppression of horseshoes chaos. We remind readers that this figure is obtained using the following dimensionless parameters $\xi = 0.0001$ and $\beta = -0.095$ (here, we have considered a steel beam with $S = 0.5 \text{ m}^2$, $E = 200 \text{ MPa}$, $\rho = 7850 \text{ kg m}^{-3}$, $\lambda = 192 \text{ N s}^{-1}$, and $L = 2 \text{ m}$). The shaded region represents the parameter space for which horseshoes chaos cannot appear. It is interesting to verify whether the good range of control parameter predicted by analytic explanation is really safe for chaos. To deal with this problem, we have looked for the fractality of the basin of attraction by solving numerically the base equation.

Figure 5 displays the basin of attraction for the control gain parameters taken in regions I and II. These figures confirm the previous investigation, since the boundary is regular for the control parameter taken in the good region and fractal for the other region. Taking into account the delay, Figure 6 shows that the basin where the boundaries were regular becomes fractal because of delays, meaning that the delays can disrupt the control strategy.

**Conclusion**

This article describes the tendon control strategy in the case of a cantilever beam excited by earthquake. The mathematical modeling of the system under control takes into account the fact that, through the tendon, the kinetic and potential energy of the system is kept small, resulting in small displacements and forces. The concept is especially attractive for historical structures because their installation requires minimal intervention and can be easily removed without causing external visual impact. The earthquake model is expanded as a Fourier series, of unknown coefficients, that is modulated by an
enveloping function. It appears after dynamics analysis that forces in the tendons that develop during the earthquake have a stabilizing effect on the structure. Focusing on the occurrence of chaotic dynamics so-called horseshoes chaos, the analysis using Melnikov theory shows the effectiveness of the control strategy presented here in the sense that by taking into consideration a selective viscoelastic parameter of the tendon system, one can completely suppress the appearance of horseshoes chaos in the system. Those predictions are confirmed and complemented by the numerical simulations from which we illustrate the fractal nature of the basins of attraction. It can be viewed that the threshold amplitude of earthquake excitation for the onset of chaos will move upward as the physical parameters are taken in a good region.

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