

Supermanifolds and local functors of points

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Abstract

We study the functor of points and the local functor of points (here called the Weil–Berezin functor) for smooth and holomorphic supermanifolds, providing characterization theorems and fully discussing the representability issues. In the end we examine applications to differential calculus including the transitivity theorems.

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1
1 Introduction

Since the 1970s the foundations of supergeometry have been investigated by several physicists and mathematicians. Most of the treatments (e.g. [BL75, Kost77, Ber87, Lei80, Man88, DM99, Var04]) present supermanifolds as classical manifolds where the structure sheaf is modified so that the sections are allowed to take values in $\mathbb{Z}_2$-graded commutative algebras. In other words, anticommuting coordinates are introduced as odd elements in the structure sheaf and the sheaf itself is assumed to be locally of a certain form, in the case of smooth supermanifolds, $C^\infty(\mathbb{R}^p) \otimes \Lambda_q$, with $\Lambda_q$ denoting the Grassmann algebra in $q$ generators. This approach is very much in the spirit of classical algebraic geometry and it dates back to the seminal works of F. A. Berezin and D. A. Leites [BL75], and B. Kostant [Kost77].

It is nevertheless only later in [Man88, DM99], that the parallelism with classical algebraic geometry is fully worked out and the functorial language starts to be used systematically. In particular the functor of points approach becomes a powerful device allowing, among other things, to give a rigorous meaning to otherwise just formal expressions and to recover some geometric intuition otherwise lost. In this approach, a supermanifold $M$ is fully recovered by the knowledge of its functor of points, $S \mapsto M(S) := \text{Hom}(S, M)$, which assigns to each supermanifold $S$, the set of the $S$-points of $M$, $M(S)$. The crucial result in this context is Yoneda’s lemma; it asserts that morphisms between supermanifolds are in bijective correspondence with natural transformations between the corresponding functors of points (Yoneda’s lemma is actually a more general result valid for generic locally small categories; see, for example, [Mac71]).

Whereas in the sheaf theoretic approach the nilpotent coordinates are introduced enlarging the sheaf of a classical manifold without modifying the underlying topological space, another possible approach to supermanifolds theory consists in introducing new local models for the underlying set itself ([Bat80, Rog80, DeW84, Shv84]). In this setting, supermanifolds are obtained by gluing domains of the form $\Lambda_0^p \times \Lambda_1^q$, where $\Lambda_0$ and $\Lambda_1$ are the even and odd part of some Grassmann algebra $\Lambda$. This idea is actually the original physicists approach to supergeometry and only later its mathematical foun-
M. Batchelor, following B. DeWitt, considers supermanifolds which are locally isomorphic to $(\Lambda_L)^p \times (\Lambda_L)^q$ where $\Lambda_L$ is an arbitrary fixed Grassmann algebra with $L > q$ generators; the topology is non-Hausdorff and the smooth functions over a supermanifold are defined in an ad hoc way in order to obtain the same class of morphisms as in Kostant’s approach. From another point of view, A. S. Shvarts and A. A. Voronov (see [Shv84, Vor84]) consider simultaneously every finite dimensional Grassmann algebra $\Lambda$ and local models that vary functorially with $\Lambda$; the topology is the classical Hausdorff one and morphisms between superdomains are defined as appropriate natural transformations between them. Both approaches turn out to be equivalent to Kostant’s original formulation, as it is proved in [Bat80] and [Vor84]. Other possible frameworks involving a wider class of supermanifold morphisms have been considered in the literature (see, for example, [DeW84] and [Rog80]). In particular in DeWitt’s approach, supermanifolds have a non-Hausdorff topology and are modeled on a Grassmann algebra with countable infinitely many generators. For a detailed review of some of these approaches we refer the reader to [BBHR91, Rog07].

This paper is devoted to understand the approach to supermanifolds theory via the local functor of points, which associates to each supermanifold $M$ the set of its $A$-points for all super Weil algebras $A$. These are finite dimensional commutative superalgebras of the form $A = K \oplus \hat{A}$ with $K = \mathbb{R}$ or $\mathbb{C}$, and $\hat{A}$ a nilpotent ideal. We focus on the categories of smooth and holomorphic supermanifolds, trying, when possible, to keep a unified treatment. In the case of $M$ a smooth manifold, we have $M_A = \text{Hom}_{\text{SAAlg}}(\mathcal{O}(M), A)$ in striking analogy with the functor of points previously described. In fact, when $A$ is a Grassmann algebra, $M_A$ is indeed the set of the $\mathcal{R}^{0/q}$-points of the supermanifold $M$ in the sense specified above, for suitable $q$. As we have defined it, the local functor of points does not determine the supermanifold, differently from what happens for the functor of points, unless we put an extra structure on $M_A$, in other words, unless we carefully define the category of arrival for the functor $A \mapsto M_A$.

Our approach is a slight modification of the one in [Shv84, Vor84], by Shvarts and Voronov, the main difference being that they consider Grassmann algebras instead of super Weil algebras. In this sense our work is mainly providing a new insight into well known results and clarifies the representability issues of the functor of points and the local functor of points.
that are often overlooked in most of the literature, but are nevertheless very important, since they are often the only means to handle geometrically supergeometric objects.

The local functor of points that we examine in our work and we call the \textit{Weil–Berezin functor} has the advantage to bring differential calculus naturally into the picture. Classically the importance of Weil algebras in the study of jet structures over manifolds was first pointed out by A. Weil in [Wei53] (for more details on this topic we refer to [KMS93]). In the supersetting Koszul takes this same perspective in [Kosz83]. By the way, we are going to review some of the differential supergeometric results using the Weil–Berezin functor.

The paper is organized as follows.

In section 2 we review some basic definitions of supergeometry like the definition of superspace, supermanifold and its associated functor of points as it is discussed in J. Bernstein’s notes by P. Deligne and J. W. Morgan [DM99]. We briefly recall the representability problem and we state the representability theorem for supermanifolds.

In section 3 we introduce super Weil algebras with their basic properties. The basic observation is that super Weil algebras are, by definition, local algebras. At a heuristic level it is clear they are well suited to study the local properties of a supermanifold, or, in other words, the properties of the stalks at the various points of a supermanifold. Once we define the functor \( A \mapsto M_A \) from the category of super Weil algebras to the category of sets, it is only natural to look for an analogue of Yoneda’s lemma, in other words for a result that allows to retrieve the supermanifold from its local functor of points. It turns out (see subsection 3.3) that, as it is stated, this result is not true; in order for the local functor of points to be able to characterize the supermanifold, its category of arrival (sets, as we defined it) needs to be suitably specialized by giving to each set \( M_A \) an extra structure.

In section 4, we discuss the modifications we need to introduce in order to obtain a bijective correspondence between supermanifold morphisms and natural transformations between the local functors of points. Following closely what is proved in [Shv84, Vor84], it turns out that it is necessary to endow the set \( M_A \) with the structure of an \( A_0 \)-smooth manifold (see definition 4.1). We call the functor \( A \to M_A \), with \( M_A \) an \( A_0 \)-smooth manifold, the \textit{Weil–Berezin functor of} \( M \). The main result is that, in such a context,
the analogue of Yoneda’s lemma holds (see theorem 4.5), and as a consequence supermanifolds embed in a full and faithful way into the category of Weil–Berezin functors (Shvarts embedding).

In analogy with the classical theory, it is only natural to ask under which conditions a generic local functor is representable. We prove a representability theorem for generic functors from the category of super Weil algebras to the category of $A_0$-smooth manifolds.

We end the section by giving a brief account of the functor of $A$-points originally described by Shvarts, which is a restriction of the Weil–Berezin functor to Grassmann algebras which form a full subcategory of super Weil algebras.

In section 5 we examine some aspects of super differential calculus on supermanifolds in the language of the Weil–Berezin functor. We describe the finite support distributions over the supermanifold $M$ and their relations with the $A$-points of $M$, establishing a connection between our treatment and Kostant’s seminal approach to supergeometry.

We also prove the super version of the Weil transitivity theorem, which is a key tool for the study of the infinitesimal aspects of supermanifolds, and we apply it to define the “tangent functor” of $A \mapsto M_A$.

Acknowledgements. We want to thank prof. G. Cassinelli, prof. M. Duflo, prof. P. Michor, and prof. V. S. Varadarajan for helpful discussions.

2 Basic Definitions

In this section we recall few basic definitions in supergeometry. Our main references are [Kost77, Man88, DM99, Var04].

2.1 Supermanifolds

Let the ground field $K$ be $\mathbb{R}$ or $\mathbb{C}$.

A super vector space is a $\mathbb{Z}_2$-graded vector space, i.e. $V = V_0 \oplus V_1$; the elements in $V_0$ are called even, those in $V_1$ odd. An element $v \neq 0$ in $V_0 \cup V_1$ is said homogeneous and $p(v)$ denotes its parity: $p(v) = 0$ if $v \in V_0$, $p(v) = 1$ if $v \in V_1$. $K^{p|q}$ denotes the super vector space $K^p \oplus K^q$. 
A *superalgebra* \( A \) is an algebra that is also a super vector space, \( A = A_0 \oplus A_1 \), and such that \( A_i A_j \subseteq A_{i+j} \) (mod 2). \( A_0 \) is an algebra, while \( A_1 \) is an \( A_0 \)-module. \( A \) is said to be *commutative* if for any two homogeneous elements \( x \) and \( y \)

\[
xy = (-1)^{p(x)p(y)} yx.
\]

The category of commutative superalgebras is denoted by \( \text{SAlg}_K \) or simply by \( \text{SAlg} \) when no confusion is possible. From now on all superalgebras are assumed to be commutative unless otherwise specified.

**Definition 2.1.** A *superspace* \( S = (|S|, \mathcal{O}_S) \) is a topological space \( |S| \), endowed with a sheaf of superalgebras \( \mathcal{O}_S \) such that the stalk at each point \( x \in |S| \), denoted by \( \mathcal{O}_{S,x} \), is a local superalgebra.

**Definition 2.2.** A *morphism* \( \varphi: S \to T \) of superspaces is a pair \( (|\varphi|, \varphi^*) \), where \( |\varphi|: |S| \to |T| \) is a continuous map of topological spaces and \( \varphi^*: \mathcal{O}_T \to |\varphi|^* \mathcal{O}_S \), called pullback, is such that \( \varphi^*(M_{|\varphi|(x)}) \subseteq M_x \) where \( M_{|\varphi|(x)} \) and \( M_x \) denote the maximal ideals in the stalks \( \mathcal{O}_{T,|\varphi|(x)} \) and \( \mathcal{O}_{S,x} \) respectively.

**Example 2.3 (The smooth local model).** The superspace \( \mathbb{R}^{p|q} \) is the topological space \( \mathbb{R}^p \) endowed with the following sheaf of superalgebras. For any open set \( U \subseteq \mathbb{R}^p \) define

\[
\mathcal{O}_{\mathbb{R}^{p|q}}(U) := \mathcal{C}^\infty_{\mathbb{R}^p}(U) \otimes \Lambda_{\mathbb{R}}(\vartheta_1, \ldots, \vartheta_q)
\]

where \( \Lambda_{\mathbb{R}}(\vartheta_1, \ldots, \vartheta_q) \) is the real exterior algebra (or Grassmann algebra) generated by the \( q \) variables \( \vartheta_1, \ldots, \vartheta_q \) and \( \mathcal{C}^\infty_{\mathbb{R}^p} \) denotes the \( \mathcal{C}^\infty \) sheaf on \( \mathbb{R}^p \).

**Example 2.4 (The holomorphic local model).** The superspace \( \mathbb{C}^{p|q} \) is the topological space \( \mathbb{C}^p \) endowed with the following sheaf of superalgebras. For any open set \( U \subseteq \mathbb{C}^p \) define

\[
\mathcal{O}_{\mathbb{C}^{p|q}}(U) := \mathcal{H}_{\mathbb{C}^p}(U) \otimes \Lambda_{\mathbb{C}}(\vartheta_1, \ldots, \vartheta_q)
\]

where \( \mathcal{H}_{\mathbb{C}^p} \) denotes the holomorphic sheaf on \( \mathbb{C}^p \) and \( \Lambda_{\mathbb{C}}(\vartheta_1, \ldots, \vartheta_q) \) is now the complex exterior algebra.

By a common abuse of notation, \( K^{m|n} \) denotes both the super vector space \( K^m \oplus K^n \) and the superspace defined above.
Definition 2.5. A smooth (resp. holomorphic) supermanifold of dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ which is locally isomorphic to $\mathbb{R}^p|q$ (resp. $\mathbb{C}^p|q$), i.e. for all $x \in |M|$ there exist open sets $x \in V_x \subseteq |M|$ and $U \subseteq \mathbb{R}^p$ (resp. $\mathbb{C}^p$) such that:

$$\mathcal{O}_M|_{V_x} \cong \mathcal{O}_{\mathbb{R}^p|q}|_U \quad \text{(resp. } \mathcal{O}_M|_{V_x} \cong \mathcal{O}_{\mathbb{C}^p|q}|_U).$$

In particular supermanifolds of the form $(U, \mathcal{O}_{K^p|q}|_U)$ are called superdomains. A morphism of supermanifolds is simply a morphism of superspaces. $\text{SMan}_K$ (or simply $\text{SMan}$) denotes the category of supermanifolds.

If $U$ is open in $|M|$, $(U, \mathcal{O}_M|U)$ is also a supermanifold and it is called the open supermanifold associated to $U$. We shall often refer to it just by $U$, whenever no confusion is possible.

In order to avoid duplications and heavy notations, we will simply refer to supermanifolds when the distinction between the smooth and the holomorphic case is immaterial. Moreover if $M$ is a supermanifold, we will denote by $\mathcal{O}(M)$ the superalgebra $\mathcal{O}_M(|M|)$ of global sections on $M$.

Suppose $M$ is a supermanifold and $U$ is an open subset of $|M|$. Let $\mathcal{J}_M(U)$ be the ideal of the nilpotent elements of $\mathcal{O}_M(U)$. $\mathcal{O}_M/\mathcal{J}_M$ defines a sheaf of purely even algebras over $|M|$ locally isomorphic to $\mathcal{C}^\infty(\mathbb{R}^p)$ (resp. $\mathcal{H}(\mathbb{C}^p)$). Therefore $\tilde{M} := (|M|, \mathcal{O}_M/\mathcal{J}_M)$ defines a classical manifold, called the reduced manifold associated to $M$. The projection $s \mapsto \tilde{s} := s + \mathcal{J}_M(U)$, with $s \in \mathcal{O}_M(U)$, is the pullback of the embedding $\tilde{M} \rightarrow M$.

In the following we denote by $\text{ev}_x(s) := \tilde{s}(x)$ the evaluation of $s$ at $x \in U$. It is also possible to check that $|\varphi|^*(\tilde{s}) = \varphi^*(s)$, so that the morphism $|\varphi|$ is automatically smooth (resp. holomorphic). Moreover since the maximal ideal $\mathcal{M}_x$ in the stalk $\mathcal{O}_{M,x}$ is given by the germs of sections whose value at $x$ is zero, we have that the locality condition in the case of supermanifold morphisms is automatically satisfied.

There are several equivalent ways to assign a morphism between two supermanifolds. The following result can be found in [Man88, ch. 4].

Proposition 2.6 (Chart theorem). Let $U$ and $V$ two smooth or holomorphic superdomains, i.e. two open subsupermanifolds of $K^{p|q}$ and $K^{m|n}$ respectively. There is a bijective correspondence between

1. superspace morphisms $U \rightarrow V$;
(2) superalgebra morphisms $\mathcal{O}(V) \to \mathcal{O}(U)$;

(3) the set of pullbacks of a fixed coordinate system on $V$, i.e. $(m|n)$-uples 
\[ (s_1, \ldots, s_m, t_1, \ldots, t_n) \in \mathcal{O}(U)_0^m \times \mathcal{O}(U)_1^n \]

such that $(\tilde{s}_1(x), \ldots, \tilde{s}_m(x)) \in |V|$ for each $x \in |U|$.

Any supermanifold morphism $M \to N$ is then uniquely determined by a collection of local maps, once atlases on $M$ and $N$ have been fixed. A morphism can hence be given by describing it in local coordinates.

In the smooth category a further simplification occurs: we can assign a morphism between supermanifolds by assigning the pullbacks of the global sections (see [Kost77, §2.15]), i.e.
\[
\text{Hom}_{\text{SMan}_R}(M, N) \cong \text{Hom}_{\text{SAlg}_R}(\mathcal{O}(N), \mathcal{O}(M)).
\] (2.1)

The essential point here is that, borrowing some terminology from algebraic geometry, smooth supermanifolds are an “affine” category. By this we mean that the knowledge of the superalgebra of global sections allows us to fully reconstruct the supermanifold obtaining its structure sheaf by a localization procedure (see, for example, [BBHR91]).

The theory of supermanifolds resembles very closely the classical theory. One can, for example, define tangent bundles, vector fields and the differential of a morphism similarly to the classical case. For more details see [Kost77, Lei80, Man88, DM99, Var04].

2.2 The functor of points

Due to the presence of nilpotent elements in the structure sheaf of a supermanifold, supergeometry can also be equivalently and very effectively studied using the language of functor of points, a very used tool in algebraic geometry applications. We briefly review it; the interested reader can consult [DM99, §2.9] or [Var04, §4.5].

Let us first fix some notation we will use throughout the paper. If $A$ and $B$ are two categories, $[A, B]$ denotes the category of functors between $A$ and $B$. Clearly, the morphisms in $[A, B]$ are the natural transformations. Moreover we denote by $A^{\text{op}}$ the opposite category of $A$, so that the category of contravariant functors between $A$ and $B$ is identified with $[A^{\text{op}}, B]$. For more details we refer to [Mac71].
Definition 2.7. Given a supermanifold $M$, we define its functor of points

$$M(\cdot) : \text{SMan}^{\text{op}} \longrightarrow \text{Set}$$

as the functor from the opposite category of supermanifolds to the category of sets defined on the objects as

$$M(S) := \text{Hom}(S, M)$$

and on the morphisms according to

$$M(\varphi) : M(S) \longrightarrow M(T)$$

$$f \longmapsto f \circ \varphi$$

where $\varphi : T \to S$.

The elements in $M(S)$ are also called the $S$-points of $M$.

Given two supermanifolds $M$ and $N$, Yoneda’s lemma establishes a bijective correspondence

$$\text{Hom}_{\text{SMan}}(M, N) \longleftrightarrow \text{Hom}_{[\text{SMan}^{\text{op}}, \text{Set}]}(M(\cdot), N(\cdot))$$

between the morphisms $M \to N$ and the natural transformations $M(\cdot) \to N(\cdot)$ (see [Mac71, ch. 3] or [EH00, ch. 6]). This allows us to view a morphism of supermanifolds as a family of morphisms $M(S) \to N(S)$ depending functorially on the supermanifold $S$. In other words, Yoneda’s lemma provides an immersion

$$\mathcal{Y} : \text{SMan} \longrightarrow [\text{SMan}^{\text{op}}, \text{Set}]$$

of $\text{SMan}$ into $[\text{SMan}^{\text{op}}, \text{Set}]$ that is full and faithful. There are however objects in $[\text{SMan}^{\text{op}}, \text{Set}]$ that do not arise as the functors of points of a supermanifold. We say that a functor $\mathcal{F} \in [\text{SMan}^{\text{op}}, \text{Set}]$ is representable if it is isomorphic to the functor of points of a supermanifold.

Observation 2.8. We first notice that for any supermanifold $M$ the set $M(K^{0|0}) = \text{Hom}_{\text{SMan}}(K^{0|0}, M) \cong |M|$ as sets, since $K^{0|0}$ is just a point. So the functor of points allows us to recover the set of the points of the topological space $|M|$ underlying $M$. The knowledge of the set-theoretical points $M(K^{0|0})$ however is far from enough to reconstruct the supermanifold $M$ and this is because of two distinct reasons:
(1) All the elements of \(M(K^{0|0})\) annihilate the nilpotent part of the sheaf, so they give us no information on the odd part of the structure sheaf.

(2) The functor \(M(\cdot)\) takes values in the category of sets, hence \(M(K^{0|0})\) is just a set and does not contain any information on the differentiable structure of \(M\), even in the classical setting.

The supermanifold \(M\) is then recaptured only from the knowledge of its \(S\)-points, for all the supermanifolds \(S\).

We now want to recall a representability criterion, which allows to single out, among all the functors from the category of supermanifolds to sets, those that are representable, i.e. those that are isomorphic to the functor of points of a supermanifold. In order to do this, we need to generalize the notion of open submanifold and of open cover, to fit this more general functorial setting.

**Definition 2.9.** Let \(U\) and \(F\) be two functors \(\mathbf{SMan}^{\text{op}} \to \mathbf{Set}\). \(U\) is a *subfunctor* of \(F\) if \(U(S) \subseteq F(S)\) for all \(S \in \mathbf{SMan}\) and this inclusion is a natural transformation. We denote it by \(U \subseteq F\).

We say that \(U\) is an *open subfunctor* of \(F\) if for all supermanifolds \(T\) and all natural transformations \(\alpha: T(\cdot) \to F\), \(\alpha^{-1}(U) = V(\cdot)\), where \(V\) is open in \(T\). If \(U\) is also representable we say that \(U\) is an *open supermanifold subfunctor*.

Let \(U_i\) be open supermanifold subfunctors of \(F\). We say that \(\{U_i\}\) is an *open cover* of a functor \(\mathbf{SMan}^{\text{op}} \to \mathbf{Set}\) if for all supermanifolds \(T\) and all natural transformations \(\alpha: T(\cdot) \to F\), \(\alpha^{-1}(U_i) = V_i(\cdot)\) and \(V_i\) cover \(T\).

Any functor \(F: \mathbf{SMan}^{\text{op}} \to \mathbf{Set}\) when restricted to the category of open subsupermanifolds of a given supermanifold \(T\) defines a presheaf over \(|T|\).

**Definition 2.10.** A functor \(F: \mathbf{SMan}^{\text{op}} \to \mathbf{Set}\) is said to be a *sheaf* if it has the sheaf property, that is, if \(\{T_i\}\) is an open cover of a supermanifold \(T\) and we have a family \(\{\alpha_i\}\), \(\alpha_i \in F(T_i)\), such that \(\alpha_{i | T_i \cap T_j} = \alpha_{j | T_i \cap T_j}\); then there exists a unique \(\alpha \in F(T)\) mapping to each \(\alpha_i\) (for more details see [Vis07]).

In particular, when \(F\) is restricted to the open subsupermanifolds of a given supermanifold \(T\), it is a sheaf over \(|T|\).

We are ready to state a representability criterion which gives necessary and sufficient conditions for a functor from \(\mathbf{SMan}^{\text{op}}\) to \(\mathbf{Set}\) to be representable. This is a very formal result and for this reason it holds as it is for
very different categories as smooth and holomorphic supermanifolds and even for superschemes (for more details on this category see [CF07]). A complete description of the classical representability criterion in algebraic geometry can be found in [DG70, ch. 1]. For the super setting see [FLV07].

**Theorem 2.11 (Representability criterion).** A functor \( \mathcal{F}: \text{SMan}^{\text{op}} \rightarrow \text{Set} \) is representable if and only if:

1. \( \mathcal{F} \) is a sheaf, i.e. it has the sheaf property;
2. \( \mathcal{F} \) is covered by open supermanifold subfunctors \( \{ U_i \} \).

### 3 Super Weil algebras and \( A \)-points

In this section we introduce the category \( \text{SWA} \) of super Weil algebras. These are finite dimensional commutative superalgebras with a nilpotent graded ideal of codimension one. Super Weil algebras are the basic ingredient in the definition of the Weil–Berezin functor and the Shvarts embedding. The easiest examples of super Weil algebras are Grassmann algebras. These are the only super Weil algebras that can be interpreted as algebras of global sections of supermanifolds, namely \( K^{0|q} \). Given a supermanifold \( M \), we want to define a functor \( M(\_): \text{SWA} \rightarrow \text{Set} \) assigning to each super Weil algebra \( A \) the set of \( A \)-points \( M_A \). If \( A \) is a Grassmann algebra, the \( A \)-points of \( M \) are identified with the usual \( K^{0|q} \)-points in the functor of points language described in the previous section. Unfortunately this functor is not adequate to fully describe the supermanifold \( M \); as we shall see at the end of this section, the arriving category needs to have an additional structure in order for \( M(\_): \text{SWA} \rightarrow \text{Set} \) to contain the same information as \( M \).

#### 3.1 Super Weil algebras

We now define the category of *super Weil algebras*. The treatment follows closely that contained in [KMS93, § 35] for the classical case.

**Definition 3.1.** We say that \( A \) is a (real or complex) *super Weil algebra* if it is a commutative unital superalgebra over \( K \) and

1. \( \dim A < \infty \),
2. \( A = K \oplus \hat{A} \),
(3) \( \hat{A} = \hat{A}_0 \oplus \hat{A}_1 \) is a graded nilpotent ideal.

The category of super Weil algebras is denoted by \( \text{SWA} \).

We also define the height of \( A \) as the lowest \( r \) such that \( \hat{A}^{r+1} = 0 \) and the width of \( A \) as the dimension of \( \hat{A}/\hat{A}^2 \).

Notice that super Weil algebras are local superalgebras, i.e. they contain a unique maximal graded ideal.

**Remark 3.2.** As a direct consequence of the definition, each super Weil algebra has an associated short exact sequence:

\[
0 \longrightarrow K \overset{j_A}{\longrightarrow} A = K \oplus \hat{A} \overset{pr_A}{\longrightarrow} A/\hat{A} \cong K \longrightarrow 0.
\]

Clearly the sequence splits and each \( a \in A \) can be written uniquely as

\[
a = \bar{a} + \hat{a}
\]

with \( \bar{a} \in K \) and \( \hat{a} \in \hat{A} \).

**Example 3.3 (Dual numbers).** The simplest example of super Weil algebra in the classical setting is \( K(x) = K[x]/\langle x^2 \rangle \) the algebra of dual numbers. Here \( x \) is an even indeterminate which is nilpotent of degree two.

**Example 3.4 (Super dual numbers).** The simplest non trivial example of super Weil algebra in the super setting is \( K(x, \vartheta) = K[x, \vartheta]/\langle x^2, x\vartheta, \vartheta^2 \rangle \) where \( x \) and \( \vartheta \) are respectively even and odd indeterminates.

**Example 3.5 (Grassmann algebras).** The polynomial algebra in \( q \) odd variables \( \Lambda(\vartheta_1, \ldots, \vartheta_q) \) is another example of super Weil algebra. Grassmann algebras are actually a full subcategory of \( \text{SWA} \).

Let

\[
K[k|l] := K[x_1, \ldots, x_k] \otimes \Lambda(\vartheta_1, \ldots, \vartheta_l)
\]

denote the superalgebra of polynomials on \( K \) in \( k \) even and \( l \) odd variables. \( K[k|l] \) is not a super Weil algebra, unless \( k = 0 \), however every finite dimensional graded quotient \( K[k|l]/J \), with \( J \) graded ideal, is a super Weil algebra. Also the converse holds and actually we have a more general result.

**Lemma 3.6.** Let \( k, l \in \mathbb{N} \) and let \( A \) be one of the following three superalgebras:
• the polynomial algebra $K[k|l]$;
• the formal series algebra $K[[k|l]] := K[[x_1, \ldots, x_k]] \otimes \Lambda(\vartheta_1, \ldots, \vartheta_l)$;
• the stalk $\mathcal{O}_{K^{k|l},0}$.

Let $\mathcal{M} \subset A$ denote the maximal ideal of the elements without constant term and let $J \subset A$ be some graded ideal of finite codimension. Then

(1) $\mathcal{M}^r \subseteq J$ for some $r$;
(2) $A/J$ is a super Weil algebra and, if $s$ is its height, $\mathcal{M}^{s+1} \subseteq J$;
(3) every super Weil algebra can be obtained in this way for suitable $k$, $l$ and $J$.

Proof. (1) is clear in the first two cases and in the third if $K = C$, since we have the identification of $\mathcal{O}_{C^{k|l},0}$ with the superalgebra of converging power series. In the smooth case this can be proved following exactly the same pattern used in [KMS93, lemma 8.35.4] for the corresponding classical statement and super Nakayama lemma (see, for example, [Var04, lemma 4.7.1]).

(2) descends from (1) and the fact that $A = K \oplus \mathcal{M}$.

For (3), let $A$ be a super Weil algebra. As $\hat{A}$ is a finite dimensional graded algebra, it is generated by finitely many elements, $x_1, \ldots, x_k$ and $\vartheta_1, \ldots, \vartheta_l$, even and odd, respectively. Since $K[k|l]$ is the free unital commutative superalgebra over $K$ generated by these elements, $A$ is a quotient of it. Moreover the projection $K[k|l] \to A$ preserves the parity so that its kernel $J$ is graded and $A \cong K[k|l]/J$. We obtain the result from (1) and noting that

$$K[k|l]/\mathcal{M}^r \cong K[[k|l]]/\mathcal{M}^r \cong \mathcal{O}_{K^{k|l},0}/\mathcal{M}^r.$$ 

The last identification is a consequence of the super version of Hadamard’s lemma that we shall recall in 3.13 below. \hfill $\square$

### 3.2 $A$-points

**Definition 3.7.** Let $M$ be a supermanifold, $x \in |M|$ and $A$ a super Weil algebra. We define the set of $A$-points near $x$ as

$$M_{A,x} := \text{Hom}_{\text{SAlg}}(\mathcal{O}_{M,x}, A)$$
and the set of $A$-points as
\[ M_A := \bigcup_{x \in |M|} M_{A,x}. \]

If $x_A \in M_{A,x}$, we call $\tilde{x}_A := x$ the base point of $x_A$.

**Observation 3.8.** Notice that, since $O_{M,x}$ is a local algebra, $M_{K,x}$ contains only the evaluation $ev_x$ and hence $M_K$ is identified with the set of topological points of $M$. Moreover, for each $A \in \mathbf{SWA}$ and $x_A \in M_A$, we have that
\[ x_A = ev_{\tilde{x}_A} + L \]
where $\text{Im}(L) \subseteq \mathbb{A}$.

We can consider the functor
\[ M(\cdot) : \mathbf{SWA} \longrightarrow \mathbf{Set} \]
defined on the objects as
\[ A \mapsto M_A \]
and on morphisms as $\rho \mapsto \rho$, with $\rho \in \text{Hom}_{\mathbf{SAlg}}(A,B)$ and
\[ \rho : M_A \longrightarrow M_B, \quad x_A \longmapsto \rho \circ x_A. \]

**Remark 3.9.** Observe that the only local superalgebras which are equal to $O(M)$ for some supermanifold $M$ are those of the form $\Lambda_{K}(\vartheta_1, \ldots, \vartheta_q) = O(K^{0|q})$. In fact as soon as $M$ has a nontrivial even part, the algebra $O(M)$ becomes infinite dimensional. For this reason this functor is quite different from the functor of points borrowed from algebraic geometry and detailed in the previous section.

**Notation 3.10.** Here we introduce a multiindex notation that we will use in the following. Let $\{x_1, \ldots, x_p, \vartheta_1, \ldots, \vartheta_q\}$ be a system of coordinates. If $\nu = (\nu_1, \ldots, \nu_p) \in \mathbb{N}^p$, we define
\[ x^{\nu} := x_1^{\nu_1} x_2^{\nu_2} \cdots x_p^{\nu_p}, \]

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\[ \nu! := \prod_{i} \nu_i!, \quad \text{and} \quad |\nu| := \sum_{i} \nu_i. \]

If \( J = \{ j_1, \ldots, j_r \} \subseteq \{ 1, \ldots, q \}, \)

we define

\[ \vartheta^J := \vartheta_{j_1} \vartheta_{j_2} \cdots \vartheta_{j_r} \]

with \( 1 \leq j_1 < \cdots < j_r \leq q; \) \(|J|\) denotes the cardinality of \( J. \)

In order to understand the structure of \( MA \) we need some preparation.

We start with a well known result that holds for smooth supermanifolds and
for holomorphic superdomains \( U \subseteq \mathbb{C}^{p|q}. \)

**Lemma 3.11 ("Super" Milnor’s exercise).** Denote by \( X \) either

(1) a smooth supermanifold or

(2) an open holomorphic superdomain in \( \mathbb{C}^{p|q}. \)

The superalgebra maps \( \mathcal{O}(X) \to K \) are exactly the evaluations \( \text{ev}_x : s \mapsto \tilde{s}(x) \)

in the points \( x \in |X|. \) In other words there is a bijective correspondence

between \( \text{Hom}_{\text{SAlg}}(\mathcal{O}(X), K) \) and \(|X|\).

**Proof.** This is a simple consequence of the chart theorem 2.6 and eq. (2.1),
considering that \( \mathcal{O}(K^{0|0}) = K \) and the pullback of a morphism \( \varphi : K^{0|0} \to M \)

is the evaluation at \(|\varphi|(K^0). \) \( \square \)

**Remark 3.12.** Notice that the lemma does not hold for a generic holomor-
phic supermanifold. This is due to the fact that there could be not enough
global holomorphic sections to separate the points. In other words, holomor-
phic supermanifolds are not characterized by the superalgebra of their global
sections, exactly as it happens for their classical counterparts.

Let \( \psi \in \text{Hom}_{\text{SAlg}}(\mathcal{O}(X), A) \). Due to the previous lemma, there exists a
unique point of \(|X|\), that we denote by \( \tilde{\psi}, \) such that \( \text{pr}_A \circ \psi = \text{ev}_{\tilde{\psi}}, \)

where \( \text{pr}_A \) is the projection \( A \to K. \) We thus have a map

\[ \text{Hom}_{\text{SAlg}}(\mathcal{O}(X), A) \to \text{Hom}_{\text{SAlg}}(\mathcal{O}(X), K) \cong |X| \]

\[ \psi \mapsto \text{pr}_A \circ \psi = \text{ev}_{\tilde{\psi}}. \]

The other result we need to obtain a first characterization of \( MA, \) gives
some insight on the structure of the stalk at a given point.
Lemma 3.13 (Hadamard’s lemma). Suppose $M$ is a (smooth or holomorphic) supermanifold, $x \in |M|$ and \{ $x_i, \vartheta_j$ \} is a system of coordinates in a neighborhood of $x$. Denote as usual by $\mathcal{M}_x$ the ideal of the germs of sections whose value at $x$ is zero. For each $[s] \in \mathcal{O}_{M,x}$ and $k \in \mathbb{N}$ there exists a polynomial $P$ in $[x_i]$ and $[\vartheta_j]$ such that

$$[s] - P \in \mathcal{M}_x^k.$$ 

Proof. The holomorphic case is trivial since the stalk at $x$ identifies with convergent power series. The proof in the smooth case can be found, for example, in [Leib80, § 2.1.8] or [Var04, ch. 4].

As a consequence we have the following proposition.

Proposition 3.14. (1) Each element $x_A$ of $M_A$ is determined by the images of the germs of a system of local coordinates $[x_i], [\vartheta_j]$ around $\tilde{x}_A$. Conversely, given $x \in |M|$, a system of local coordinates $\{ x_i \}_{i=1}^p$, $\{ \vartheta_j \}_{j=1}^q$ around $x$ and elements $\{ x_i \}_{i=1}^p$, $\{ \theta_j \}_{j=1}^q$, $x_i \in A_0$, $\theta_j \in A_1$, \footnote{The reader should notice the difference between \{ $x_i, \vartheta_j$ \} and \{ $x_i, \theta_j$ \}.} such that $\tilde{x}_i = \tilde{x}_i(x)$, there exists a unique morphism $x_A \in \text{Hom}_{\text{SAlg}}(\mathcal{O}_{M,x}, A)$ such that

\[
\begin{align*}
  x_A([x_i]) &= x_i \\
  x_A([\vartheta_j]) &= \theta_j.
\end{align*}
\] (3.3)

(2) If $U$ is a coordinate chart around $x$, there is a bijective correspondence

$$U_A = \bigsqcup_{x \in U} \text{Hom}_{\text{SAlg}}(\mathcal{O}_{M,x}, A) \longrightarrow \text{Hom}_{\text{SAlg}}(\mathcal{O}_M(U), A).$$

Proof. Let us consider (1). Suppose that $x_A$ is given. We want to show that the images of the germs of local coordinates $x_A([x_i])$, $x_A([\vartheta_j])$ determine $x_A$ completely. This follows noticing that

- the image of a polynomial section under $x_A$ is determined,
- there exists $k \in \mathbb{N}$ such that the kernel of $x_A$ contains $\mathcal{M}_x^k$ (see lemma 3.6)
and using previous lemma. We now come to existence. Suppose the eq. (3.3) are given and let $[s]$ be a germ at $x$. We define $x_A([s])$ through a formal Taylor expansion. More precisely let

$$s = \sum_{J \subseteq \{1, \ldots, q\}} s_J \partial^J$$

be a representative of $[s]$ near $x$, where the $s_J$ are smooth/holomorphic functions in $x_1, \ldots, x_p$. Define

$$x_A(s) = \sum_{\nu \in \mathbb{N}^p} \frac{1}{\nu!} \frac{\partial^\nu s_J}{\partial x^\nu} \bigg|_{(\bar{x}_1, \ldots, \bar{x}_p)} \bar{x}^\nu \theta^J. \quad (3.4)$$

This is the way in which the purely formal expression

$$s(x_A) = s(\bar{x}_1 + \bar{x}_1, \ldots, \bar{x}_p + \bar{x}_p, \theta_1, \ldots, \theta_q)$$

is usually understood. Eq. (3.4) has only a finite number of terms due to the nilpotency of the $\bar{x}_i$ and $\theta_j$. It is clear from eq. (3.4) that $x_A(s)$ does not depend on the chosen representative. Finally $x_A$ so defined is a superalgebra morphism since, for each $[s], [t] \in \mathcal{O}_{M,x}$,

$$x_A(st) = \sum_{\nu \in \mathbb{N}^p} \lambda(K, J) \frac{1}{\nu!} \frac{\partial^\nu (s_K t_J, K)}{\partial x^\nu} \bar{x}^\nu \theta^J$$

$$= \sum_{\nu-\mu \in \mathbb{N}^p, K, J} \lambda(K, J) \frac{1}{\nu!} \left(\begin{array}{c} \nu \\ \mu \end{array}\right) \frac{\partial^\nu s_K}{\partial x^\mu} \frac{\partial^{\nu-\mu} t_J, K}{\partial x^{\nu-\mu}} \bar{x}^\nu \theta^J$$

$$= \sum_{\nu, \mu, J, K} \left[ \frac{1}{\mu!} \frac{\partial^\mu s_K}{\partial x^\mu} \bar{x}^\mu \theta^K \right] \left[ \frac{1}{(\nu-\mu)!} \frac{\partial^{\nu-\mu} t_J, K}{\partial x^{\nu-\mu}} \bar{x}^{\nu-\mu} \theta^J \right]$$

where $\begin{pmatrix} \nu \\ \mu \end{pmatrix} = \prod_i \begin{pmatrix} \nu_i \\ \mu_i \end{pmatrix}$ and $\lambda(K, J)$ is defined to be $\pm 1$ according to $\theta^K \theta^J = \lambda(K, J) \theta^J$.

Let us now consider (2). Define the map $\eta_x : \mathcal{O}_M(U) \to \mathcal{O}_{M,x}$ assigning to each section over $U$ the corresponding germ at $x \in U$, and consider

$$\bigsqcup_{x \in U} \Hom_{SA}(\mathcal{O}_{M,x}, A) \longrightarrow \Hom_{SA}(\mathcal{O}_M(U), A) \quad x_A \longmapsto x_A \circ \eta_{x_A}. $$

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We show that it is invertible. Let \( \psi \in \text{Hom}_{SAlg}(\mathcal{O}_M(U), A) \). If \( \{ x_i, \vartheta_j \} \) is a coordinate system on \( U \), due to (1), the set \( \{ \psi(x_i), \psi(\vartheta_j) \} \) uniquely determines an element in \( \text{Hom}_{SAlg}(\mathcal{O}_M, \psi), A) \). It is easy to show that this is the required inverse. \( \Box \)

In the smooth category the above setting can be somewhat simplified. This is essentially due to eq. (2.1) and the related discussion. Let us see in detail this point, summarized in prop. 3.16.

**Lemma 3.15.** Let \( M \) be a smooth supermanifold. Let \( s \in \mathcal{O}(M) \) and let \( \psi \in \text{Hom}_{SAlg}(\mathcal{O}(M), A) \). Assume that \( s \) is zero when restricted to a certain neighbourhood of \( \tilde{\psi} \) (see eq. (3.2)). Then \( \psi(s) = 0 \).

**Proof.** Suppose \( U \ni \tilde{\psi} \) is such that \( s|_U = 0 \). Let \( t \in \mathcal{O}(M)(U) \) be such that \( \text{supp}(t) \subset U \) and \( t|_V = 1 \), where the closure of \( V \) is contained in \( U \). Then
\[
0 = \psi(st) = \psi(s)\psi(t).
\]
Hence \( \psi(s) = 0 \), since \( \psi(t) \) is invertible being \( \text{ev}_{\tilde{\psi}}(t) = 1 \). \( \Box \)

**Proposition 3.16.** If \( M \) is a smooth supermanifold
\[
M_A \cong \text{Hom}_{SAlg}(\mathcal{O}(M), A)
\]
in a functorial way.

**Proof.** Clearly each \( x_A \in M_A \) can be identified with a superalgebra map \( \mathcal{O}(M) \to A \) by composing it with the natural map \( \eta_x : \mathcal{O}(M) \to \mathcal{O}_{M,x} \). Vice versa let \( \psi \in \text{Hom}_{SAlg}(\mathcal{O}(M), A) \). In the smooth category, given a germ \( [s] \in \mathcal{O}_{M,\tilde{\psi}} \), there exists a global section \( s \in \mathcal{O}(M) \) such that \( \eta_x(s) = [s] \). Since the image of \( s \) under \( \psi \) depends only on the germs of \( s \) at \( \tilde{\psi} \), \( \psi \) determines an element of \( M_{A,x} \). In fact, let \( s' \in [s] \) and let \( U \) be a neighbourhood of \( \tilde{\psi} \) such that \( s'|_U = s|_U \). It is always possible to find a smaller neighbourhood \( V \subset U \) and \( u, v_1, v_2 \in \mathcal{O}(M) \) such that \( s = u + v_1, s' = u + v_2 \) and \( v_1|_V = 0 \).

Then, due to the previous lemma, \( \psi(s) = \psi(u) = \psi(s') \). The functoriality is clear. \( \Box \)

**Observation 3.17.** Let \( U \) be a chart in a supermanifold \( M \) with local co-
ordinates \( \{ x_i, \vartheta_j \} \). By point (2) of prop. 3.14 we have an injective map
\[
U_A \longrightarrow A^p_0 \times A^q_1
\]
\[
x_A \longmapsto (x_1, \ldots, x_p, \theta_1, \ldots, \theta_q) := (x_A(x_1), \ldots, x_A(\vartheta_q)).
\]

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We can think of it heuristically as the assignment of $A$-valued coordinates $\{x_i, \theta_j\}$ on $U_A$. As we are going to see in theorem 4.2 the components of the coordinates $\{x_i, \theta_j\}$, given by $\langle a_k^*, x_i \rangle$, $\langle a_k^*, \theta_j \rangle$ with respect to a basis $\{a_k\}$ of $A$ are indeed the coordinates of a smooth or holomorphic manifold. The base point $\bar{x}_A \in U$ has coordinates $(\bar{x}_1, \ldots, \bar{x}_p)$. In this language, if $\rho: A \to B$ is a super Weil algebra morphism, the corresponding morphism $\rho: M_A \to M_B$ is “locally” given by

$$\rho \times \cdots \times \rho: A_0^p \times A_1^q \longrightarrow B_0^p \times B_1^q.$$ (3.5)

This is well defined since $\rho$ does not change the base point.

If $M = \mathbb{K}^{p|q}$ we can also consider the slightly different identification

$$\mathbb{K}_{A}^{p|q} \longrightarrow (A \otimes \mathbb{K}_{\mathbb{K}_p^{p|q}})_0$$

$$x_A \longmapsto \sum_i x_A(e_i^*) \otimes e_i$$

where $\{e_1, \ldots, e_{p+q}\}$ denotes a homogeneous basis of $\mathbb{K}^{p|q}$ and $\{e_1^*, \ldots, e_{p+q}^*\}$ its dual basis. Here a little care is needed as we already remarked at the beginning of section 2. In the literature the name $\mathbb{K}^{p|q}$ is used for two in principle different objects: it may indicate the super vector space $\mathbb{K}^{p|q} = \mathbb{K}^p \oplus \mathbb{K}^q$ or the superdomain $(\mathbb{K}^{p, C_{\mathbb{K}^p}} \otimes \Lambda)_q$. In the previous equation the first $\mathbb{K}^{p|q}$ is viewed as a superdomain, while the last as a super vector space. Likewise the $\{e_i^*\}$ are interpreted both as vectors and sections of $\mathcal{O}(\mathbb{K}^{p|q})$. As we shall see in subsection 4.1 the functor

$$A \longmapsto (A \otimes \mathbb{K}^{p|q})_0$$

recaptures all the information about the superdomain $\mathbb{K}^{p|q}$, so that the two in principle different ways of looking at $\mathbb{K}^{p|q}$ become then identified. This result will hence establish a quite natural way to identify the two objects. With this identification, the superdomain morphism $\rho: \mathbb{K}^{p|q}_A \to \mathbb{K}^{p|q}_B$ corresponds to the super vector space morphism

$$\rho \otimes 1: (A \otimes \mathbb{K}^{p|q})_0 \longrightarrow (B \otimes \mathbb{K}^{p|q})_0.$$ 

### 3.3 Natural transformations between functors of $A$-points

In the previous subsection we have seen that, somehow mimicking the functor of points approach to supermanifolds, it is possible to associate to each
supermanifold $M$ a functor

$$\mathbf{SWA} \rightarrow \mathbf{Set}$$

$$A \mapsto M_A.$$

Hence we have a functor:

$$\mathcal{B}: \mathbf{SMan} \rightarrow [\mathbf{SWA}, \mathbf{Set}].$$

The natural question about such a functor is whether $\mathcal{B}$ is a full and faithful embedding or not. In this subsection, we show that $\mathcal{B}$ is not full, in other words, there are many more natural transformations between $M(\cdot)$ and $N(\cdot)$ than those coming from morphisms from $M$ to $N$. We will show this by giving a simple example. Then, in prop. 3.21, we will see a characterization of the natural transformation between two superdomains.

Let us start our discussion. We first want to show that the natural transformations $M(\cdot) \rightarrow N(\cdot)$ arising from supermanifold morphisms $M \rightarrow N$ have a very peculiar form. Indeed, a morphism $\varphi: M \rightarrow N$ of supermanifolds induces a natural transformation between the corresponding functors of $A$-points given by

$$\varphi_A: M_A \rightarrow N_A$$

$$x_A \mapsto x_A \circ \varphi^*$$

for all super Weil algebras $A$. Let $M = K^{p|q}$ and $N = K^{m|n}$, and denote respectively by $\{x_i, \theta_j\}$ and $\{x'_i, \theta'_j\}$ two systems of coordinates over them. With these assumptions, $\varphi$ is determined by the pullbacks of the coordinates of $N$, while the $A$-point $\varphi_A(x_A)$ is determined by

$$(x'_1, \ldots, x'_m, \theta'_1, \ldots, \theta'_n) := (x_A \circ \varphi^*(x'_1), \ldots, x_A \circ \varphi^*(\theta'_n)) \in A^m \times A^n.$$

If $(x_1, \ldots, x_p, \theta_1, \ldots, \theta_q)$ denote the images of the coordinates of $M$ under $x_A$ ($x_1 = x_A(x_1)$, etc.) and $\varphi^*(x'_k) = \sum_j s_{k,j} \theta^J \in \mathcal{O}(K^{p|q})_0$, where the $s_{k,j}$ are functions on $K^p$, then we have

$$x'_k = x_A \circ \varphi^*(x'_k) = \sum_{J \subseteq \{1, \ldots, q\}} \frac{1}{\nu!} \frac{\partial^{[\nu]} s_{k,j}}{\partial x^\nu} \bigg|_{(\bar{x}_1, \ldots, \bar{x}_p)} \hat{x}^\nu \theta^J \quad (3.6)$$

and similarly for the odd coordinates (see prop. 3.14). Notice that if we pursue the point of view of observation 3.17, i.e. if we consider $\{x_i, \theta_j\}$
as $A$-valued coordinates of $K^p|q_A$, this equation can be read as a coordinate expression for $\phi_A$.

Not all the natural transformations $M(\cdot) \rightarrow N(\cdot)$ arise in this way. This happens also for purely even manifolds, as we see in the next example.

**Example 3.18.** Let $M$ and $N$ be two smooth manifolds and let $\phi : M \rightarrow N$ be a map (smooth or not). The natural transformation $\alpha(\cdot) : M(\cdot) \rightarrow N(\cdot)$

$$\alpha_A : M \rightarrow N$$

$$x_A \mapsto ev_{\phi(x_A)}$$

is not of the form seen above, even if $\phi$ is assumed to be smooth, while we still have $\phi = \alpha_K$.

We end this subsection with a technical result, essentially due to A. A. Voronov (see [Vor84]), characterizing all possible natural transformations between the functors of $A$-points of two superdomains, hence comprehending also those not arising from supermanifold morphisms.

**Definition 3.19.** Let $U$ be an open subset of $K^p$. We denote by $A_{p|q}(U)$ the unital commutative superalgebra of formal series with $p$ even and $q$ odd generators and coefficients in the algebra $F(U, K)$ of arbitrary functions on $U$, i.e.

$$A_{p|q}(U) := F(U, K)[[X_1, \ldots, X_p, \Theta_1, \ldots, \Theta_q]].$$

An element $F \in A_{p|q}(U)$ is of the form

$$F = \sum_{\nu \in \mathbb{N}^p} f_{\nu,J} X^\nu \Theta^J$$

where $f_{\nu,J} \in F(U, K)$ and $\{X_i\}$ and $\{\Theta_j\}$ are even and odd generators. $A_{p|q}(U)$ is a graded algebra: $F$ is even (resp. odd) if $|J|$ is even (resp. odd) for each term of the sum.

Let us introduce a partial order between super Weil algebras by saying that $A' \leq A$ if and only if $A'$ is a quotient of $A$.

**Lemma 3.20.** The set of super Weil algebras is directed, i.e., if $A_1$ and $A_2$ are super Weil algebras, then there exists $A$ such that $A_1 \leq A$. 

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As above, Proof.

The parity of its image is the same as that of \( F \) we will see that it is determined by an unique \( F \) determines a map \( U \) such that, if \( \alpha \) and \( \beta \) restrictions imposed on the first \( m \) \( F \) \( \] respectively. The set of natural transformations in \( [\text{SWA}, \text{Set}] \) between \( U(\cdot) \) and \( V(\cdot) \) is in bijective correspondence with the set of elements of the form

\[
F = (F_1, \ldots, F_{m+n}) \in (\mathfrak{A}_{p|q}(|U|))^m \times (\mathfrak{A}_{p|q}(|U|))^n
\]

such that, if \( F_k = \sum_{\nu,J} f_{\nu,J}^k x^n \Theta^J \),

\[
(f_{0,0}^1(x), \ldots, f_{0,0}^m(x)) \subseteq |V| \quad \forall x \in |U|.
\]

Proposition 3.21. Let \( U \) and \( V \) be two superdomains in \( K^{p|q} \) and \( K^{m|n} \) respectively. The set of natural transformations in \( [\text{SWA}, \text{Set}] \) between \( U(\cdot) \) and \( V(\cdot) \) is in bijective correspondence with the set of elements of the form

\[
F = (F_1, \ldots, F_{m+n}) \in (\mathfrak{A}_{p|q}(|U|))^m \times (\mathfrak{A}_{p|q}(|U|))^n
\]

such that, if \( F_k = \sum_{\nu,J} f_{\nu,J}^k x^n \Theta^J \),

\[
(f_{0,0}^1(x), \ldots, f_{0,0}^m(x)) \subseteq |V| \quad \forall x \in |U|.
\]

Proof. As above, \( K^{p|q}_A \) is identified with \( A_p^0 \times A_1^q \) and consequently a map \( K^{p|q}_A \rightarrow K^{m|n}_A \) consists of a list of \( m \) maps \( A_p^0 \times A_1^q \rightarrow A_0 \) and \( n \) maps \( A_p^0 \times A_1^q \rightarrow A_1 \). In the same way, \( U_A \) is identified with \( |U| \times \hat{A}_0^p \times \hat{A}_1^q \).

Let \( F = (F_1, \ldots, F_{m+n}) \) be as in the hypothesis. A formal series \( F_k \) determines a map \( |U| \times \hat{A}_0^p \times \hat{A}_1^q \subseteq A_p^0 \times A_1^q \rightarrow A \) in a natural way, defining

\[
F_k(x_1, \ldots, x_p, \theta_1, \ldots, \theta_q) := \sum_{\nu \in \mathbb{N}^p} f_{\nu,J}^k(x_1, \ldots, x_p) \hat{x}^n \Theta^J.
\]

The parity of its image is the same as that of \( F_k \). Then, in view of the restrictions imposed on the first \( m \) \( F_k \) given by eq. (3.7), \( F \) determines a map \( U_A \rightarrow V_A \) and, varying \( A \in \text{SWA} \), a natural transformation \( U(\cdot) \rightarrow V(\cdot) \), as it is easily checked.

Let us now suppose now that \( \alpha(\cdot) : U(\cdot) \rightarrow V(\cdot) \) is a natural transformation. We will see that it is determined by an unique \( F \) in the way just explained.

Let \( A \) be a super Weil algebra of height \( r \) and

\[
x_A = (\bar{x}_1 + \check{x}_1, \ldots, \bar{x}_p + \check{x}_p, \theta_1, \ldots, \theta_q) \in A_p^0 \times A_1^q \cong K^{p|q}_A
\]

with \( \bar{x}_A \in |U| \). Let us consider the super Weil algebra

\[
\hat{A} := (K[z_1, \ldots, z_p] \otimes \Lambda(\zeta_1, \ldots, \zeta_q)) / \mathcal{M}^s
\]

with \( s > r \) (\( \mathcal{M} \) is as usual the maximal ideal of polynomials without constant term) and the \( \hat{A} \)-point

\[
y_{\bar{x}_A} := (\bar{x}_1 + z_1, \ldots, \bar{x}_1 + z_p, \zeta_1, \ldots, \zeta_q) \in \hat{A}_0^p \times \hat{A}_1^q \cong K^{p|q}_A.
\]
A homomorphism between two super Weil algebras is clearly fixed by the images of a set of generators, but this assignment must be compatible with the relations between the generators. The following assignment is possible due to the definition of \( \hat{A} \). If \( \rho_{xA} : \hat{A} \to A \) denotes the map

\[
\begin{cases}
  z_i \mapsto \tilde{x}_i \\
  \zeta_j \mapsto \theta_j,
\end{cases}
\]

then clearly \( \rho_{xA}(y_{\bar{x}_A}) = x_A \).

Let \( (\alpha_{\hat{A}})_k \) with \( 1 \leq k \leq m + n \) be a component of \( \alpha_{\hat{A}} \), and let

\[
(\alpha_{\hat{A}})_k(y_{\bar{x}_A}) = \sum_{\nu,j} a^k_{\nu,j}(\bar{x}_A)\tilde{x}^\nu \zeta^j
\]

with \( a^k_{\nu,j}(\bar{x}_A) \in K \) and \( (a^1_{0,0}(\bar{x}_A), \ldots, a^m_{0,0}(\bar{x}_A)) \in |V| \); the sum is on \(|J|\) even (resp. odd), if \( k \leq m \) (resp. \( k > m \)). Due to the functoriality of \( \alpha(\cdot) \)

\[
(\alpha_A)_k(x_A) = (\alpha_{\hat{A}})_k \circ \rho_{xA}(y_{\bar{x}_A}) = \rho_{xA} \circ (\alpha_{\hat{A}})_k(y_{\bar{x}_A}) = \sum_{\nu,j} a^k_{\nu,j}(\bar{x}_A)\tilde{x}^\nu \theta^j,
\]

so that there exists a non unique \( F \) such that \( F(x_A) = \alpha_A(x_A) \). Moreover \( F(x_{A'}) = \alpha_{A'}(x_{A'}) \) for each \( A' \preceq A \) and \( x_{A'} \in U_{A'} \) (it is sufficient to use the projection \( A \to A' \)).

If \( F' \) is another list of formal series with this property, there exists a super Weil algebra \( A'' \) such that \( F(x_{A''}) \neq F'(x_{A''}) \) for some \( x_{A''} \in U_{A''} \). Indeed if a component \( F_k \) differs in \( f^k_{\nu,j} \), it is sufficient to consider \( A'' := K[p|q]/M^s \) with \( s > \max(|\nu|, q) \).

4. The Weil–Berezin functor and the Shvarts embedding

In the previous section we saw that the functor

\[
B : \text{SMan} \longrightarrow \text{[SWA, Set]}
\]

does not define a full and faithful embedding of \( \text{SMan} \) in \( \text{[SWA, Set]} \). Roughly speaking, the root of such a difficulty can be traced to the fact that the functor \( B(M) : \text{SWA} \to \text{Set} \) looks only to the local structure of the supermanifold
hence it looses all the global information. For the functor of points as we described it in 2.2, we obtain a full and faithfull embedding thanks to the Yoneda’s lemma. If we try to reproduce its proof in this different setting, we see that the main obstacle is that \( 1_M \) can no longer be seen as an \( M \)-point of the supermanifold \( M \) itself. The following heuristic argument gives a hint on how such a difficulty can be overcome.

It is well known (see, for example, [DM99, § 1.7]) that given graded vector spaces \( V = V_0 \oplus V_1 \) and \( W = W_0 \oplus W_1 \), there is a bijective correspondence between graded linear maps \( V \rightarrow W \) and functorial families of \( \Lambda_0 \)-linear maps between \( (\Lambda \otimes V)_0 \) and \( (\Lambda \otimes W)_0 \), for each Grassmann algebra \( \Lambda \). This result goes under the name of even rule principle. Since vector spaces are local models for manifolds, the even rule principle seems to suggest that each \( M_A \) should be endowed with a local structure of \( A_0 \)-module. This vague idea is made precise with the introduction of the category \( A_0 \text{-Man} \) of \( A_0 \)-smooth manifolds. We then prove that each \( M_A \) can be endowed in a canonical way with the structure of \( A_0 \)-manifold. This construction allows to specialize the arrival category of the functor of \( A \)-points associated to a supermanifold \( M \) and to define the Weil–Berezin functor of \( M \) as

\[
M(\cdot) : \text{SWA} \rightarrow A_0 \text{Man}
\]

\[
A \mapsto M_A.
\]

In this way we can define a functor

\[
\mathcal{S} : \text{SMan} \rightarrow [\text{SWA}, A_0 \text{Man}]
\]

\[
M \mapsto M(\cdot)
\]

where \([\text{SWA}, A_0 \text{Man}]\) is an appropriate subcategory of \([\text{SWA}, A_0 \text{Man}]\) that will be specified by definition 4.3. We will call \( \mathcal{S} \) the Shvarts embedding.

As it turns out, this definition of the local functor of points allows to recover the correct natural transformations without any artificial condition. More precisely, lemma 4.7 will show that the natural transformations \( \alpha(\cdot) : M(\cdot) \rightarrow N(\cdot) \) arising from supermanifolds morphisms are exactly those for which \( \alpha_A \) is \( A_0 \)-smooth for each \( A \). \( \mathcal{S} \) is then a full and faithful embedding. Moreover \( \mathcal{S} \) preserves the products. In particular this implies that if \( G \) is a group object in \( \text{SMan} \), i.e. it is a super Lie group, then \( \mathcal{S}(G) \) is a group object too, i.e. it takes values in the category of the \( A_0 \)-smooth Lie group.

Exactly as for the functor of points, the functor \( \mathcal{S} \) is not an equivalence of categories, so that the problem of characterizing the functors in the image
of \( S \) arises naturally (representability problem). A criterion characterizing the representable functors is then given in subsection 4.2.

### 4.1 \( \mathcal{A}_0 \)-smooth structure and its consequences

Preliminary to everything is the following (rather long) definition of \( \mathcal{A}_0 \)-manifold and of the category \( \mathcal{A}_0 \text{Man} \). For a more detailed discussion see for example [Shu99] and references therein.

**Definition 4.1.** Fix an even commutative finite dimensional algebra \( \mathcal{A}_0 \) and let \( L \) be a finite dimensional \( \mathcal{A}_0 \)-module. Let \( M \) be a manifold. An \( L \)-chart on \( M \) is a pair \( (U, h) \) where \( U \) is open in \( M \) and \( h: U \to L \) is a diffeomorphism onto its image. \( M \) is an \( \mathcal{A}_0 \)-manifold if it admits an \( L \)-atlas. By this we mean a family \( \{ (U_i, h_i) \}_{i \in \mathcal{A}} \) where \( \{ U_i \} \) is an open covering of \( M \) and each \( (U_i, h_i) \) is an \( L \)-chart, such that the differentials

\[
d(h_i \circ h_j^{-1})_{h_i(x)}: T_{h_i(x)} L \cong L \longrightarrow L \cong T_{h_j(x)} L
\]

are isomorphisms of \( \mathcal{A}_0 \)-modules for all \( i, j \) and \( x \in U_i \cap U_j \).

If \( M \) and \( N \) are \( \mathcal{A}_0 \)-manifolds, a *morphism* \( \varphi: M \to N \) is a smooth map whose differential is \( \mathcal{A}_0 \)-linear at each point. We also say that such morphism is \( \mathcal{A}_0 \)-*smooth*. We denote by \( \mathcal{A}_0 \text{Man} \) the category of \( \mathcal{A}_0 \)-manifolds.

We define also the category \( \mathcal{A}_0 \text{Man} \) in the following way. The objects of \( \mathcal{A}_0 \text{Man} \) are manifolds over generic finite dimensional commutative algebras. The morphisms in the category are defined as follows. Denote by \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) two commutative finite dimensional algebras, and let \( \rho: \mathcal{A}_0 \to \mathcal{B}_0 \) be an algebra morphism. Suppose \( M \) and \( N \) are \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) manifolds respectively, we say that a morphism \( \varphi: M \to N \) is \( \rho \)-smooth if \( \varphi \) is smooth and

\[
(\text{d}\varphi)_x (a v) = \rho(a)(\text{d}\varphi)_x (v)
\]

for each \( x \in M, v \in T_x(M) \), and \( a \in \mathcal{A}_0 \).

Notice that \( \mathcal{A}_0 \)-linearity always implies \( \mathbb{K} \)-linearity, in particular, in the complex case, \( \mathcal{A}_0 \)-manifolds are holomorphic.

The above definition is motivated by the following theorems. In order to ease the exposition we first give the statements of the results and we postpone their proofs to the last part of this subsection.

**Theorem 4.2.** Let \( M \) be a smooth (resp. holomorphic) supermanifold, and let \( \mathcal{A} \) be a real (resp. complex) super Weil algebra.
(1) $M_A$ can be endowed with a unique $A_0$-manifold structure such that, for each open subsupermanifold $U$ of $M$ and $s \in \mathcal{O}_M(U)$ the map defined by

$$\hat{s}: U_A \to A$$

$$x_A \mapsto x_A(s)$$

is $A_0$-smooth.

(2) If $\varphi: M \to N$ is a supermanifold morphism, then

$$\varphi_A: M_A \to N_A$$

$$x_A \mapsto x_A \circ \varphi^*$$

is an $A_0$-smooth morphism.

(3) If $B$ is another super Weil algebra and $\rho: A \to B$ is an algebra morphism, then

$$\rho_0: M_A \to M_B$$

$$x_A \mapsto \rho \circ x_A$$

is a $\rho|_{A_0}$-smooth map.

The above theorem says that supermanifolds morphisms give rise to morphisms in the $A_0\text{Man}$ category. From this point of view the next definition is quite natural.

**Definition 4.3.** We call $[[\text{SWA}, A_0\text{Man}]]$ the subcategory of $[\text{SWA}, A_0\text{Man}]$ whose objects are the same and whose morphisms $\alpha_{(\cdot)}$ are the natural transformations $\mathcal{F} \to \mathcal{G}$, with $\mathcal{F}, \mathcal{G}: \text{SWA} \to A_0\text{Man}$, such that

$$\alpha_A: \mathcal{F}(A) \to \mathcal{G}(A)$$

is $A_0$-smooth for each $A \in \text{SWA}$.

Theorem 4.2 allows us to give more structure to the arrival category of the functor of $A$-points. More precisely we have the following definition, which is the central definition in our treatment of the local functor of points.

**Definition 4.4.** Let $M$ be a supermanifold. We define the Weil-Berezin functor of $M$ as

$$M_{(\cdot)}: \text{SWA} \to A_0\text{Man}$$

$$A \mapsto M_A.$$  

(4.1)
Moreover we define the Shvarts embedding
\[ S : \text{SMan} \longrightarrow [[\text{SWA}, \mathcal{A}_0\text{Man}]] \]
\[ M \mapsto \iota_{M(\cdot)} \]

We can now state one of the main results in this paper.

**Theorem 4.5.** \( S \) is a full and faithful embedding, i.e. if \( M \) and \( N \) are two supermanifolds, and \( M(\cdot) \) and \( N(\cdot) \) their Weil–Berezin functors, then
\[ \text{Hom}_{\text{SMan}}(M, N) \cong \text{Hom}_{[[\text{SWA}, \mathcal{A}_0\text{Man}]}}(M(\cdot), N(\cdot)). \]

**Observation 4.6.** If we considered the bigger category \([\text{SWA}, \mathcal{A}_0\text{Man}]\) instead of \([[\text{SWA}, \mathcal{A}_0\text{Man}]]\), the above theorem is no longer true. In example 3.18 we examined a natural transformation between functors from \( \text{SWA} \) to \( \text{Set} \), which did not come from a supermanifolds morphism. If, in the same example, \( \varphi \) is chosen to be smooth, we obtain a morphism in \([\text{SWA}, \mathcal{A}_0\text{Man}]\) that is not in \([[\text{SWA}, \mathcal{A}_0\text{Man}]]\). Indeed, it is not difficult to check that if \( \pi_A : A \to A \) is given by \( a \mapsto \tilde{a} \), then \( \alpha_A \) (in the example) is \( \pi_{A_0} \)-linear.

We now examine the proofs of theorems 4.2 and 4.5. First we need to prove theorem 4.5 in the case of two superdomains \( U \) and \( V \) in \( \mathbb{K}^{p|q} \) and \( \mathbb{K}^{m|n} \) respectively (lemma 4.7). As usual, if \( A \) is a super Weil algebra, \( U_A \) and \( V_A \) are identified with \( |U| \times \hat{A}^p_0 \times A^0_1 \) and \( |V| \times \hat{A}^m_0 \times A^1_1 \) (see observation 3.17). Then they have a natural structure of open subsets of \( A_0 \)-modules. The next lemma is due to A. A. Voronov in [Vor84] and it is the local version of theorem 4.5.

**Lemma 4.7.** A natural transformation \( \alpha(\cdot) : U(\cdot) \to V(\cdot) \) comes from a supermanifold morphism \( U \to V \) if and only if \( \alpha_A : U_A \to V_A \) is \( A_0 \)-smooth for each \( A \).

**Proof.** Due to prop. 3.21 we know that \( \alpha(\cdot) \) is determined by \( m \) even and \( n \) odd formal series of the form \( F_k = \sum_{\nu, J} f^{k}_{\nu, J} X^\nu \Theta^J \) with \( f^{k}_{\nu, J} \) arbitrary functions in \( p \) variables satisfying eq. (3.7). Moreover as we have seen in the discussion before example 3.18 a supermanifold morphism \( \varphi : U \to V \) gives rise to a natural transformation \( \varphi_A : U_A \to V_A \) whose components are of the form of eq. (3.6).

Let us suppose that \( \alpha_A \) is \( A_0 \)-smooth. This clearly happens if and only if all its components are \( A_0 \)-smooth and the smoothness request for all \( A \) forces all coefficients \( f^{k}_{\nu, J} \) to be smooth.
Let \((\alpha_A)_k\) be the \(k\)-th component of \(\alpha_A\) and let \(i \in \{1, \ldots, p\}\). We want to study
\[
\omega: A_0 \longrightarrow A_j
\]
\[x_i \longmapsto (\alpha_A)_k(x_1, \ldots, x_i, \ldots, x_p, \theta_1, \ldots, \theta_q),
\]
supposing the other coordinates fixed \((j = 0 \text{ if } 1 \leq k \leq p \text{ or } j = 1 \text{ if } p < k \leq p + q)\). Since \(x_i \in A_0\) commutes with all elements of \(A\),
\[
\omega(x_i) = \sum_{t \geq 0} a_t(\tilde{x}_i) \tilde{x}_i^t
\]
(4.2)
with
\[
a_t(\tilde{x}_i) := \sum_{\nu, J} f^k_{\nu, J}(\tilde{x}_1, \ldots, \tilde{x}_i, \ldots, \tilde{x}_p) \tilde{x}_i^{(\nu - t\delta_i)} \theta^J
\]
(4.3)
\((t\delta_i\) is the element of \(\mathbb{N}^p\) with \(t\) at the \(i\)-th component and 0 elsewhere).

If \(y = \tilde{y} + \hat{y} \in A_0\) and \(\omega\) is \(A_0\)-smooth
\[
\omega(x_i + y) - \omega(x_i) = d\omega_{x_i}(y) + o(y) = (\tilde{y} + \hat{y})d\omega_{x_i}(1_A) + o(y)
\]
(4.4)
(where \(1_A\) is the unit of \(A\)). On the other hand, from eq. (4.2) and defining
\[
a'_t(\tilde{x}_i) := \sum_{\nu, J} \partial_i f^k_{\nu, J}(\tilde{x}_1, \ldots, \tilde{x}_i, \ldots, \tilde{x}_p) \tilde{x}_i^{(\nu - t\delta_i)} \theta^J
\]
(4.5)
\((\partial_i\) denotes the partial derivative respect to the \(i\)-th variable), we have
\[
\omega(x_i + y) - \omega(x_i) = \sum_{t \geq 0} a_t(\tilde{x}_i + \tilde{y}) (\tilde{x}_i + \hat{y})^t - \sum_{t \geq 0} a_t(\tilde{x}_i) \tilde{x}_i^t
\]
\[
= \sum_{t \geq 0} \left( a'_t(\tilde{x}_i) \tilde{y} \tilde{x}_i^t + a_t(\tilde{x}_i) t \tilde{x}_i^{t-1} \hat{y} + o(y) \right)
\]
\[
= \tilde{y} \sum_{t \geq 0} a'_t(\tilde{x}_i) \tilde{x}_i^t + \hat{y} \sum_{t \geq 0} (t + 1) a_{t+1}(\tilde{x}_i) \tilde{x}_i^t + o(y).
\]
(4.6)
Thus, comparing eq. (4.4) and (4.6), we get that the identity
\[
(\tilde{y} + \hat{y})d\omega_{x_i}(1_A) = \tilde{y} \sum_{t \geq 0} a'_t(\tilde{x}_i) \tilde{x}_i^t + \hat{y} \sum_{t \geq 0} (t + 1) a_{t+1}(\tilde{x}_i) \tilde{x}_i^t
\]
must hold and, consequently, also the following relations must be satisfied:

$$\sum_{t \geq 0} a'_t(\tilde{x}_i) \tilde{x}_i^t = \sum_{t \geq 0} (t + 1)a_{t+1}(\tilde{x}_i) \tilde{x}_i^t$$

and then, from eq. (4.3) and (4.5),

$$\sum_{\nu,J} \partial_i f_{\nu,J}^k(\tilde{x}_1, \ldots, \tilde{x}_p) \tilde{x}_i^{\nu} \theta^J = \sum_{\nu,J} (\nu_i + 1)f_{\nu+\delta_i,J}^k(\tilde{x}_1, \ldots, \tilde{x}_p) \tilde{x}_i^{\nu} \theta^J.$$

Let us fix $\nu \in \mathbb{N}^p$ and $J \subseteq \{1, \ldots, q\}$. If $A = K[p,q]/\mathcal{M}^s$ with $s > \max(|\nu| + 1, q)$ ($\mathcal{M}$ is as usual the maximal ideal of polynomials without constant term), we note that necessarily, due to the arbitrariness of $(x_1, \ldots, \theta_q)$,

$$\partial_i f_{\nu,J}^k = (\nu_i + 1)f_{\nu+\delta_i,J}^k$$

and, by recursion, $(\alpha_A)_k$ is of the form of (3.6) with $s_{k,J} = f_{0,J}^k$.

Conversely, let $(\alpha_A)_k$ be of the form of eq. (3.6). By linearity, it is $A_0$-linear if and only if it is $A_0$-linear in each variable. It is $A_0$-linear in the even variables for what has been said above and in the odd variables since it is polynomial in them.

In particular the above discussion shows also that any superdiffeomorphism $U \to U$ gives rise, for each $A$, to an $A_0$-smooth diffeomorphism $U_A \to U_A$ and then each $U_A$ admits a canonical structure of $A_0$-manifold.

We now use the results obtained for superdomains in order to prove theorems 4.2 and 4.5 in the general supermanifold case. We need to recall the following elementary result from ordinary differential geometry.

**Lemma 4.8.** Let $X$ be a set. Suppose a countable covering $\{U_i\}$ and a collection of injective maps $h_i: U_i \to K^n$ are given, satisfying the following conditions:

1. For each $i$ and $j$, $h_i(U_i \cap U_j)$ is open in $K^n$;
2. $h_j \circ h_i^{-1}: h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$ is a diffeomorphism;
3. For each $x$ and $y$ in $X$, $x \neq y$, there exists $V_x \subseteq U_i$ and $V_y \subseteq U_j$ such that $x \in V_x$, $y \in V_y$, $V_x \cap V_y = \emptyset$, $h_i(V_x)$ and $h_j(V_y)$ both open.

Then $X$ admits a unique smooth manifold structure such that $\{(U_i, h_i)\}$ defines an atlas over it.
We leave the proof of this lemma to the reader and we return to the proofs of theorems 4.2 and 4.5.

Proof of theorem 4.2. Let \( \{ (U_i, h_i) \} \) be an atlas over \( M \) and \( p|q \) the dimension of \( M \). Each chart \((U_i, h_i)\) of such an atlas induces a chart \(((U_i)_A, (h_i)_A)\), \((U_i)_A = \bigsqcup_{x \in U_i} M_{A,x}\), over \( M \) given by

\[
(h_i)_A: (U_i)_A \longrightarrow K^{p|q}_A \quad x_A \longmapsto x_A \circ h_i^*.
\]

The coordinate changes are easily checked to be given, with some abuse of notation, by \((h_i \circ h_j^{-1})_A\), which are \( A_0 \)-smooth due to lemma 4.7. The uniqueness of the \( A_0 \)-manifold structure is clear. This proves the first point.

The other two points concern only the local behavior of the considered maps and are clear in view of lemma 4.7 and eq. (3.5). \( \square \)

Proof of theorem 4.5. Lemma 4.7 accounts for the case in which \( M \) and \( N \) are superdomains. For the general case, let us suppose we have

\[
\alpha \in \text{Hom}_{[[\text{SWA}, A_0\text{Man}]]}(M_{(\cdot)}, N_{(\cdot)}).
\]

Fixing a suitable atlas of both supermanifolds, we obtain, in view of lemma 4.7, a family of local morphisms. Such a family will give a morphism \( M \rightarrow N \) if and only if they do not depend on the choice of the coordinates. Let us suppose that \( U \) and \( V \) are open subsupermanifolds of \( M \) and \( N \) respectively, \( U \cong K^{p|q}, V \cong K^{m|n} \), such that \( \alpha_k(\{U\}) \subseteq \{V\} \), and

\[
h_i: U \longrightarrow K^{p|q} \quad k_i: V \longrightarrow K^{m|n} \quad i = 1, 2
\]

are two different choices of coordinates on \( U \) and \( V \) respectively. The natural transformations

\[
(\hat{\phi}_i)_{(\cdot)} := (k_i)_{(\cdot)} \circ (\alpha_{(\cdot)})_{U_i} \circ (h_i^{-1})_{(\cdot)}: K^{p|q}_{(\cdot)} \longrightarrow K^{m|n}_{(\cdot)}
\]

give rise to two morphisms \( \hat{\phi}_i: K^{p|q} \rightarrow K^{m|n} \). If

\[
\varphi_i := k_i^{-1} \circ \hat{\phi}_i \circ h_i: U \longrightarrow V,
\]

we have \( \varphi_1 = \varphi_2 \) since \( (\varphi_i)_{(\cdot)} = (\alpha_{(\cdot)})_{U_i} \) and two morphisms that give rise to the same natural transformation on a superdomain are clearly equal. \( \square \)
We end this subsection with the next proposition stating that the Shvarts embedding preserves products and, in consequence, group objects.

**Proposition 4.9.** For all supermanifolds $M$ and $N$,

$$S(M \times N) \cong S(M) \times S(N).$$

Moreover $S(K^{0|0})$ is a terminal object in the category $[[\text{SWA}, A_0\text{Man}]]$.

**Proof.** The fact that $(M \times N)_A \cong M_A \times N_A$ for all $A$ can be checked easily. Indeed, let $z_A \in (M \times N)_A$ with $z_A = (x, y)$, we have that $\mathcal{O}_x$ and $\mathcal{O}_y$ naturally inject in $\mathcal{O}_{z_A}$. Hence $z_A$ defines, by restriction, two $A_0$-points $x_A \in M_A$ and $y_A \in N_A$. Using prop. 3.14 and rectangular coordinates over $M \times N$ it is easy to check that such a correspondence is injective, and is also a natural transformation. Conversely, if $x_A \in M_{A,x}$ and $y_A \in N_{A,y}$, they define a map $z_A : \mathcal{O}_x \otimes \mathcal{O}_y \to A$ through $z_A(s_1 \otimes s_2) = x_A(s_1) \cdot y_A(s_2)$. Using again prop. 3.14, it is not difficult to check that this requirement uniquely determines an element in $(M \times N)_{A,(x,y)}$ and that this correspondence defines an inverse for the morphism $(M \times N)(\cdot) \to M(\cdot) \times N(\cdot)$ defined above.

Along the same lines it can be proved that, a similar condition for the morphisms holds. Finally $S(K^{0|0})$ is a terminal object, since $K^{0|0}_A = K^0$ for all $A$. \qed

It is easy to check that the stated result is equivalent to the fact that $S$ preserves finite products for arbitrary many objects.

We now consider a super Lie group $G$, i.e. a group object in the category of (smooth or holomorphic) supermanifolds. First we recall briefly the notion of group object.

**Definition 4.10.** A group object in some category with finite products and terminal object $\mathcal{T}$, is an object $G$ with three arrows

$$\mu_G : G \times G \to G \quad i_G : G \to G \quad e_G : \mathcal{T} \to G$$

satisfying the usual commutative diagrams for multiplication, inverse and unit respectively.

**Observation 4.11.** In a locally small category $\mathcal{C}$ we have that, equivalently, a group is an object $G$ whose functor of points $G(\cdot)$ takes value in the category of groups $\text{Grp}$, i.e. $G$ is a group object if there exists a functor $\mathcal{C}^{op} \to \text{Grp}$ that, composed with the forgetful functor $\text{Grp} \to \text{Set}$, equals $G(\cdot)$. 

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Corollary 4.12. If \( G \) is a super Lie group, \( S(G) \) with the arrows \( S(\mu_G), S(i_G) \) and \( S(e_G) \) is a group object in \([\text{SWA}, \mathcal{A}_0\text{Man}]\). This means that the Weil–Berezin functor of \( G \) takes values in the category of \( A_0 \)-smooth Lie groups.

Proof. This is an immediate consequence of prop. 4.9. \( \square \)

For more information on group objects and product preserving functors, see [Vis07].

4.2 Representability of the Weil–Berezin functor

Next definition is the natural generalization of the classical one to the Weil–Berezin functor setting.

Definition 4.13. We say that a functor 

\[ \mathcal{F}: \text{SWA} \rightarrow \mathcal{A}_0\text{Man} \]

is representable if there exists a supermanifold \( M_\mathcal{F} \) such that \( \mathcal{F} \cong (M_\mathcal{F})(\cdot) \) in \([\text{SWA}, \mathcal{A}_0\text{Man}]\).

Notice that we are abusing the category terminology, that considers a functor \( \mathcal{F} \) to be representable if and only if \( \mathcal{F} \) is isomorphic to the Hom functor.

Due to theorem 4.5, if a functor \( \mathcal{F} \) is representable, then the supermanifold \( M_\mathcal{F} \) is unique up to isomorphism. Next example shows that there exists non representable functors.

Example 4.14. Consider the constant functor \( \text{SWA} \rightarrow \mathcal{A}_0\text{Man} \) defined as \( A \mapsto K \) on the objects (\( K \cong A/\hat{A} \) is an \( A \)-module) and \( \rho \mapsto 1_K \) on the morphisms. This functor is not representable in the sense explained above.

In this subsection we look for conditions ensuring the representability for a functor \( \mathcal{F}: \text{SWA} \rightarrow \mathcal{A}_0\text{Man} \).

Since \( \mathcal{F}(K) \) is a manifold, we can consider an open set \( U \subseteq \mathcal{F}(K) \). If \( A \) is a super Weil algebra and \( \underline{\text{pr}}_A := \mathcal{F}(\text{pr}_A) \), where \( \text{pr}_A \) is the projection \( A \rightarrow K \), \( \underline{\text{pr}}_A^{-1}(U) \) is an open \( A_0 \)-submanifold of \( \mathcal{F}(A) \). Moreover, if \( \rho: A \rightarrow B \) is a superalgebra map, since \( \text{pr}_B \circ \rho = \text{pr}_A \), \( \underline{\rho} := \mathcal{F}(\rho) \) can be restricted to

\[ \underline{\rho}_{\underline{\text{pr}}_A^{-1}(U)}: \underline{\text{pr}}_A^{-1}(U) \rightarrow \underline{\text{pr}}_B^{-1}(U). \]
We can hence define the functor
\[
\mathcal{F}_U : \text{SWA} \rightarrow \mathcal{A}_0\text{Man}
\]
\[
A \mapsto \text{pr}^{-1}_A(U)
\]
\[
\rho \mapsto \text{pr}^{-1}_A(U).
\]

**Proposition 4.15 (Representability).** A functor
\[
\mathcal{F} : \text{SWA} \rightarrow \mathcal{A}_0\text{Man}
\]
is representable if and only if there exists an open cover \(\{U_i\}\) of \(\mathcal{F}(K)\) such that \(\mathcal{F}_{U_i} \cong (\tilde{V}_i)(\cdot)\) with \(\tilde{V}_i\) superdomains in a fixed \(K^{pq}\).

**Proof.** The necessity is clear due to the very definition of supermanifold. Let us prove sufficiency. We have to build a supermanifold structure on the topological space \(|\mathcal{F}(K)|\). Let us denote by \((h_i)(\cdot) : \mathcal{F}_{U_i} \rightarrow (\tilde{V}_i)(\cdot)\) the natural isomorphisms in the hypothesis. On each \(U_i\), we can put a supermanifold structure \(\tilde{U}_i\), defining the sheaf \(\mathcal{O}_{\tilde{U}_i} := (h_i^{-1})_K^* \mathcal{O}_{\tilde{V}_i}\). Let \(k_i\) be the isomorphism \(\tilde{U}_i \rightarrow \tilde{V}_i\) and \((k_i)(\cdot)\) the corresponding natural transformation. If \(U_{i,j} := U_i \cap U_j\), consider the natural transformation \((h_{i,j})(\cdot)\) defined by the composition
\[
(k_i^{-1})(\cdot) \circ (h_i)(\cdot) \circ (h_j^{-1})(\cdot) \circ (k_j)(\cdot) : (U_{i,j}, \mathcal{O}_{\tilde{U}_i|U_{i,j}}(\cdot)) \rightarrow (U_{i,j}, \mathcal{O}_{\tilde{U}_i|U_{i,j}}(\cdot))
\]
where in order to avoid heavy notations we didn’t explicitly indicate the appropriate restrictions. Each \((h_{i,j})(\cdot)\) is a natural isomorphism in \([\text{SWA}, \mathcal{A}_0\text{Man}]\) and, due to lemma 4.7, it gives rise to a supermanifold isomorphism
\[
h_{i,j} : (U_{i,j}, \mathcal{O}_{\tilde{U}_i|U_{i,j}}(\cdot)) \rightarrow (U_{i,j}, \mathcal{O}_{\tilde{U}_i|U_{i,j}}(\cdot)).
\]
The \(h_{i,j}\) satisfy the cocycle conditions \(h_{i,i} = 1\) and \(h_{i,j} \circ h_{j,k} = h_{i,k}\) (restricted to \(U_i \cap U_j \cap U_k\)). This follows from the analogous conditions satisfied by \((h_i)(\cdot) : \mathcal{F}_{U_i} \rightarrow (\tilde{V}_i)(\cdot)\) for each \(A \in \text{SWA}\). The supermanifolds \(\tilde{U}_i\) can hence be glued (for more information about the construction of a supermanifold by gluing see for example [DM99, ch. 2] or [Var04, § 4.2]). Denote by \(M_\mathcal{F}\) the manifold thus obtained. Moreover it is clear that \(\mathcal{F}\) is represented by the supermanifold \(M_\mathcal{F}\). Indeed, one can check that the various \((h_i)(\cdot)\) glue together and give a natural isomorphism \(h(\cdot) : \mathcal{F} \rightarrow (M_\mathcal{F})(\cdot)\).

**Remark 4.16.** The supermanifold \(M_\mathcal{F}\) admits a more synthetic characterization. In fact it is easily seen that \(|M_\mathcal{F}| := |\mathcal{F}(K)|\) and
\[
\mathcal{O}_{M_\mathcal{F}}(U) := \text{Hom}_{[\text{SWA}, \mathcal{A}_0\text{Man}]}(\mathcal{F}_U, K^{pq}_{(\cdot)}).
\]
4.3 The functors of $\Lambda$-points

In this subsection we want to give a brief exposition of the original approach of A. S. Shvarts and A. A. Voronov (see [Shv84, Vor84]). In their work they considered only Grassmann algebras instead of all super Weil algebras. There are some advantages in doing so: Grassmann algebras are many fewer, moreover, as we noticed in remark 3.9, they are the sheaf of the super domains $K^{0|q}$ and so the restriction to Grassmann algebras of the local functors of points can be considered as a true restriction of the functor of points. Finally the use of Grassmann algebras is also used by A. S. Shvarts to formalize the language commonly used in physics.

On the other hand the use of super Weil algebras has the advantage that we can perform differential calculus on the Weil–Berezin functor as we shall see in section 5. Indeed prop. 5.3 is valid only for the Weil–Berezin functor approach, since not every point supported distribution can be obtained using only Grassmann algebras. Also theorem 5.5 and its consequences are valid only in this approach, since purely even Weil algebras are considered.

If $M$ is a supermanifold and $\Lambda$ denotes the category of Grassmann algebras, we can consider the two functors

\[
\Lambda \to \text{Set} \\
\Lambda \to \text{A}_{0}\text{Man}
\]

in place of those introduced by eq. (3.1) and eq. (4.1) respectively. As in the case of A-points, with a slight abuse of notation we denote by $M_\Lambda$ the $\Lambda$-points for each of the two different functors. What we have seen previously — except subsection 5.1 — still remains valid in this setting, provided we substitute systematically $\text{SWA}$ with $\Lambda$; in particular theorems 4.2 and 4.5 still hold true. They are based on prop. 3.21 and lemma 4.7 that we state here in their original formulation as it is contained in [Vor84].

**Proposition 4.17.** The set of natural transformations between $\Lambda \mapsto K^{p|q}_\Lambda$ and $\Lambda \mapsto K^{m|n}_\Lambda$ is in bijective correspondence with

\[
(\mathfrak{A}_{p|q}(K^p))^m_0 \times (\mathfrak{A}_{p|q}(K^p))^n_1.
\]

A natural transformation comes from a supermanifold morphism $K^{p|q} \to K^{m|n}$ if and only if it is $\Lambda_0$-smooth for each Grassmann algebra $\Lambda$. 

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Proof. See proofs of prop. 3.21 and lemma 4.7. The only difference is in the first proof. Indeed the algebra (3.8) is not a Grassmann algebra. So, if $A = \Lambda_n = \Lambda(\varepsilon_1, \ldots, \varepsilon_n)$, we have to consider

$$\hat{A} := \Lambda_{2p(n-1)+q} = \Lambda(\eta_i, a, \xi_i, a, \zeta_j)$$

$(1 \leq i \leq p, 1 \leq j \leq q, 1 \leq a \leq n - 1)$. A $\Lambda_n$-point can be written as

$$x_{\Lambda_n} = \left( u_1 + \sum_{a<b} \varepsilon_a \varepsilon_b k_{1,a,b}, \ldots, u_p + \sum_{a<b} \varepsilon_a \varepsilon_b k_{p,a,b}, \kappa_1, \ldots, \kappa_q \right)$$

with $u_i \in K$, $k_{i,a,b} \in (\Lambda_n)_0$ and $\kappa_j \in (\Lambda_n)_1$. Its image under a natural transformation can be obtained taking the image of the $\Lambda_{2p(n-1)+q}$-point

$$y_{\overline{\Lambda}_n} := \left( u_1 + \sum_{a=1}^{n-1} \eta_{1,a} \xi_{1,a}, \ldots, u_p + \sum_{a=1}^{n-1} \eta_{p,a} \xi_{p,a}, \xi_1, \ldots, \xi_q \right)$$

and applying the map $\Lambda_{2p(n-1)+q} \to \Lambda_n$

$$\begin{align*}
\eta_{i,a} &\mapsto \varepsilon_a \\
\xi_{i,a} &\mapsto \sum_{b>a} \varepsilon_b k_{i,a,b} \\
\zeta_j &\mapsto \kappa_j
\end{align*}$$

to each component. The nilpotent part of each even component of $y_{\overline{\Lambda}_n}$ can be viewed as a formal scalar product between $(\eta_{i,1}, \ldots, \eta_{i,n-1})$ and $(\xi_{i,1}, \ldots, \xi_{i,n-1})$. This is stable under formal rotations and the same must be for its image. So $\eta_{i,a}$ and $\xi_{i,a}$ can occur in the image only as a polynomial in $\sum_a \eta_{i,a} \xi_{i,a}$. In other words the image of $y_{\overline{\Lambda}_n}$ (and then of $x_{\Lambda_n}$) is polynomial in the nilpotent part of the coordinates. \hfill \Box

5 Applications to differential calculus

In this section we discuss some aspects of super differential calculus on supermanifolds using the language of the Weil–Berezin functor. In particular we establish a relation between the $A$-points of a supermanifold $M$ and the finite support distributions over it, which play a crucial role in Kostant’s seminal approach to supergeometry.

We also prove the super version of the Weil transitivity theorem, which is a key tool for the study of the infinitesimal aspects of supermanifolds, and we apply it in order to define the “tangent functor” of $A \mapsto M_A$. 

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5.1 Point supported distributions and \(A\)-points

In this subsection we want to introduce and discuss Kostant’s approach (see [Kost77]) using the Weil–Berezin functor formalism.

Let \((|M|, \mathcal{O}_M)\) a supermanifold of dimension \(p|q\) and \(x \in |M|\). As in [Kost77, § 2.11], let us consider the distributions with support at \(x\).

**Definition 5.1.** If \(\mathcal{O}^*_{M,x}\) is the algebraic dual of the stalk at \(x\), the distributions with support at \(x\) of order \(k\) are defined as:

\[
\mathcal{O}^{k*}_{M,x} := \{ v \in \mathcal{O}^*_{M,x} \mid v(M^k + 1) = 0 \}
\]

where \(M_x\) is, as usual, the maximal ideal of the germs of sections that are zero if evaluated at \(x\). Clearly \(\mathcal{O}^{k*}_{M,x} \subseteq \mathcal{O}^{k+1*}_{M,x}\). The distributions with support at \(x\) are given by the union

\[
\mathcal{O}^*_{M,x} := \bigcup_{k=0}^{\infty} \mathcal{O}^{k*}_{M,x}.
\]

**Observation 5.2.** The distributions \(\mathcal{O}^{k*}_{M,x}\) form a super vector space: an even distribution is 0 on an odd germ and vice versa. If \(x_1, \ldots, x_p, \vartheta_1, \ldots, \vartheta_q\) are coordinates in a neighbourhood of \(x\), a distribution of order \(k\) is of the form

\[
v = \sum_{\nu \in \mathbb{N}^p} \sum_{J \subseteq \{1, \ldots, q\}} a_{\nu, J} \text{ev}_x \frac{\partial^{|
u|} \partial J}{\partial x^\nu \partial \vartheta^J}
\]

with \(a_{\nu, J} \in \mathbb{K}\). This is immediate since we have the following isomorphisms:

\[
\mathcal{O}^{k*}_{M,x} \cong (C^{\infty}_{M,x} \otimes \Lambda(\vartheta_1, \ldots, \vartheta_q))^* \cong C^{\infty,*}_{M,x} \otimes \Lambda(\vartheta_1, \ldots, \vartheta_q)^*
\]

and \(C^{\infty,*}_{M,x} = \sum a_{\nu, J} \text{ev}_x \frac{\partial^{|\nu|}}{\partial x^\nu} \) because of the classical theory.

**Proposition 5.3.** Let \(A\) be a super Weil algebra and \(A^*\) its dual. Let

\[
x_A: \mathcal{O}_{M,x} \longrightarrow A
\]

be an \(A\)-point near \(x \in |M|\). If \(\omega \in A^*\), then

\[
\omega \circ x_A \in \mathcal{O}^*_{M,x}.
\]
Moreover each element of \( \mathcal{O}_{M,x}^{k_*} \) can be obtained in this way with

\[
A = \mathcal{O}_{M,x} / \mathcal{M}_{x}^{k+1} \cong K[p|q] / \mathcal{M}_0^{k+1}
\]

(see lemma 3.6).

**Proof.** If \( A \) has height \( k \), since \( x_A(\mathcal{M}_x) \subseteq \hat{A} \), \( \omega \circ x_A \in \mathcal{O}_{M,x}^{k_*} \). If vice versa \( v \in \mathcal{O}_{M,x}^{k_*} \), it factorizes through

\[
\mathcal{O}_{M,x} \xrightarrow{pr} \mathcal{O}_{M,x} / \mathcal{M}_x^{k+1} \xrightarrow{\omega} K
\]

with a suitable \( \omega \). \( \Box \)

In the next observation we relate the finite support distributions, together with their interpretation via the Weil–Berezin functor, with the tangent superspace.

**Observation 5.4.** Let us first recall that the tangent superspace to a supermanifold \( M \) at a point \( x \) is the super vector space consisting of all the derivations of the stalk at \( x \):

\[
T_x(M) := \{ v : \mathcal{O}_{M,x} \rightarrow K \mid v \text{ is a derivation} \}.
\]

As in the classical setting we can recover the tangent space by using the algebra of **super dual numbers**. Let us consider \( A = K(e, \varepsilon) = K[e, \varepsilon]/(e^2, e\varepsilon, \varepsilon^2) \) be the super Weil algebra of super dual numbers (see example 3.4). If \( x_A \in M_{A,x} \) and \( s, t \in \mathcal{O}_{M,x} \), we have

\[
x_A(st) = ev_x(st) + x_e(st)e + x_\varepsilon(st)\varepsilon
\]

with \( x_e, x_\varepsilon : \mathcal{O}_{M,x} \rightarrow K \). On the other hand

\[
x_A(st) = x_A(s)x_A(t)
\]

\[
= ev_x(s)ev_x(t) + (x_e(s)ev_x(t) + ev_x(s)x_e(t))e
\]

\[
+ (x_\varepsilon(s)ev_x(t) + ev_x(s)x_\varepsilon(t))\varepsilon.
\]

Then \( x_e \) (resp. \( x_\varepsilon \)) is a derivation of the stalk that is zero on odd (resp. even) elements and so \( x_e \in T_0(x(M)) \) (resp. \( x_\varepsilon \in T_1(x(M)) \)). The map

\[
T(M) := \bigsqcup_{x \in M} T_x(M) \rightarrow M_{K(e, \varepsilon)}
\]

\[
v_0 + v_1 \mapsto ev_x + v_0e + v_1\varepsilon
\]
(with \(v_i \in T_i(M)\)) is an isomorphism of vector bundles over \(\tilde{M} \cong M_K\), where \(\tilde{M}\) is the classical manifold associated with \(M\), as in subsection 2.1 (see also [KMS93, ch. 8] for an exhaustive exposition in the classical case). The reader should not confuse \(T(M)\), which is the classical bundle obtained by the union of all the tangent superspaces at the different points of \(|M|\), with \(T_M\) which is the super vector bundle of all the derivations of \(\mathcal{O}_M\).

5.2 Transitivity theorem and applications

We now want to give a brief account on how we can perform differential calculus using the language of \(A\)-points. The essential ingredient is the super version of the transitivity theorem that we discuss below.

In the following, when classical smooth (resp. holomorphic) manifolds are considered, \(\mathcal{O}\) denotes the corresponding sheaf of smooth (resp. holomorphic) functions.

**Theorem 5.5 (Weyl transitivity theorem).** Let \(M\) be a smooth (resp. holomorphic) supermanifold, \(A\) a super Weil algebra and \(B_0\) a purely even Weil algebra, both real (resp. complex). Then

\[
(M_A)_B \cong M_{A \otimes B_0}
\]

as \((A_0 \otimes B_0)\)-manifolds.

**Proof.** Let \(\mathcal{O}_{M_A}\) and \(\mathcal{O}^A_{M_A}\) be the sheaves of smooth (resp. holomorphic) maps from the classical manifold \(M_A\) to \(K\) and \(A\) respectively. Clearly \(\mathcal{O}^A_{M_A} \cong A \otimes \mathcal{O}_{M_A}\) through the map \(f \mapsto \sum_i a_i \otimes \langle a_i^*, f \rangle\), where \(\{a_i\}\) is a homogeneous basis of \(A\).

If \(x_A \in M_A\), let

\[
\tau_{x_A} : \mathcal{O}_{M, \tilde{x}_A} \longrightarrow \mathcal{O}^A_{M, x_A} \cong A \otimes \mathcal{O}_{M, x_A}
\]

\[
[s]_{\tilde{x}_A} \longmapsto [\hat{s}]_{x_A}
\]

where, if \(s \in \mathcal{O}_M(U)\) and \(\tilde{x}_A \in U\),

\[
\hat{s} : y_A \longmapsto y_A(s)
\]

for all \(y_A \in M_A\) such that \(\tilde{y}_A \in U\) (it is not difficult to show that this map descends to a map between stalks).
Recalling that
\[ (M_A)_{B_0} := \bigsqcup_{x_A \in M_A} \text{Hom}_{\text{SAlg}}(\mathcal{O}_{M_A,x_A}, B_0) \]
\[ M_{A \otimes B_0} := \bigsqcup_{x \in |M|} \text{Hom}_{\text{SAlg}}(\mathcal{O}_{M,x}, A \otimes B_0), \]
we can define a map
\[ \xi: (M_A)_{B_0} \longrightarrow M_{A \otimes B_0} \]
\[ X \longmapsto \xi(X) \]
setting
\[ \xi(X): [s]_X \longmapsto (1_A \otimes X)\tau_X([s]_X). \]
This definition is well-posed since \( \xi(X) \) is a superalgebra map, as one can easily check.

Fix now a chart \((U,h)\), \(h: U \to K^{|p|q}\), in \(M\) and denote by \((U_A,h_A)\), \((U_{A \otimes B_0}, h_{A \otimes B_0})\) and \((U_A \otimes B_0, h_A \otimes B_0)\) the corresponding charts lifted to \(M_A\), \((M_A)_{B_0}\) and \(M_{A \otimes B_0}\) respectively. If \(\{e_1, \ldots, e_{p+q}\}\) is a homogeneous basis of \(K^{|p|q}\), we have (here, according to observation 3.17, we tacitly use the identification \(K^{|p|q}_A \cong (A \otimes K^{|p|q})_0\))
\[ (h_A)_{B_0} : (U_A)_{B_0} \longrightarrow (A \otimes B_0 \otimes K^{|p|q})_0 \]
\[ X \longmapsto \sum_{i,j} a_i \otimes X(h_A^*(a_i^* \otimes e_j^*)) \otimes e_j \]
and
\[ h_{A \otimes B_0} : U_{A \otimes B_0} \longrightarrow (A \otimes B_0 \otimes K^{|p|q})_0 \]
\[ Y \longmapsto \sum_k Y(h^*(e_k^*)) \otimes e_k. \]
Then, since
\[ \xi(X)(h^*(e_k^*)) = (1 \otimes X)(\hat{h^*}(e_k^*)) = (1 \otimes X)(\sum_i a_i \otimes h_A^*(a_i^* \otimes e_k^*)), \]
we have
\[ h_{A \otimes B_0} \circ \xi \circ (h_A)^{-1} = 1_{(h_A)_{B_0}((U_A)_{B_0})}. \]
This entails in particular that \( \xi \) is a local \((A_0 \otimes B_0)\)-diffeomorphism. The fact that it is a global diffeomorphism follows noticing that it is fibered over the identity, being
\[ \xi(X) \in M_{A \otimes B_0, X}. \]
From now on we assume all supermanifolds to be smooth.

We want to briefly explain some applications of the Weil transitivity theorem to the smooth category. Let $M$ be a smooth supermanifold and let $A$ be a real super Weil algebra. As we have seen in the previous section, $M_A$ has a natural structure of classical smooth manifold and, due to prop. 3.16, we can identify $M_A$ with the space of superalgebra maps $\mathcal{O}(M) \to A$.

**Definition 5.6.** If $x_A \in M_A$, we define the space of $x_A$-linear derivations of $M$ ($x_A$-derivations for short) as the $A$-module

$$\text{Der}_{x_A}(\mathcal{O}(M), A) := \left\{ X \in \text{Hom}(\mathcal{O}(M), A) \mid \forall s, t \in \mathcal{O}(M), \quad X(st) = X(s)x_A(t) + (-1)^{p(X)p(s)}x_A(s)X(t) \right\}.$$  

**Proposition 5.7.** The tangent superspace at $x_A$ in $M_A$ canonically identifies with $\text{Der}_{x_A}(\mathcal{O}(M), A)_0$.

**Proof.** If $R(e)$ is the algebra of dual number (see example 3.3), $(M_A)_{R(e)}$ is isomorphic, as a vector bundle, to the tangent bundle $T(M_A)$, as we have seen in observation 5.4. Due to theorem 5.5, we thus have an isomorphism

$$\xi: T(M_A) \cong (M_A)_{R(e)} \longrightarrow M_A \otimes R(e).$$

On the other hand, it is easy to see that $x_{A \otimes R(e)} \in M_A \otimes R(e)$ can be written as $x_{A \otimes R(e)} = x_A \otimes 1 + v_{x_A} \otimes e$, where $x_A \in M_A$ and $v_{x_A}: \mathcal{O}(M) \to A$ is a parity preserving map satisfying the following rule for all $s, t \in \mathcal{O}(M)$:

$$v_{x_A}(st) = v_{x_A}(s)x_A(t) + x_A(s)v_{x_A}(t).$$

Then each tangent vector on $M_A$ at $x_A$ canonically identifies a even $x_A$-derivation and, vice versa, each such derivation canonically identifies a tangent vector at $x_A$. \qed

We conclude studying more closely the structure of $\text{Der}_{x_A}(\mathcal{O}(M), A)$. The following proposition describes it explicitly.

---

2We recall that if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ are super vector spaces then $\text{Hom}(V, W)$ denotes the super vector space of all linear morphisms between $V$ and $W$ with the gradation $\text{Hom}(V, W)_0 := \text{Hom}(V_0, W_0) \oplus \text{Hom}(V_1, W_1)$, $\text{Hom}(V, W)_1 := \text{Hom}(V_0, W_1) \oplus \text{Hom}(V_1, W_0)$. 

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Let $K$ be a right $A$-module and let $L$ be a left $B$-module for some algebras $A$ and $B$. Suppose moreover that an algebra morphism $\rho: B \to A$ is given. One defines the $\rho$-tensor product $K \otimes_{\rho} L$ as the quotient of the vector space $K \otimes L$ with respect to the equivalence relation
\[ k \otimes b \cdot l \sim k \cdot \rho(b) \otimes l \]
for all $k \in K$, $l \in L$ and $b \in B$.

Moreover, if $M$ is a supermanifold, we denote by $T_M$ the super tangent bundle of $M$, i.e. the sheaf defined by $T_M := \text{Der}(\mathcal{O}_M)$.

**Proposition 5.8.** Let $M$ be a smooth supermanifold and let $x \in |M|$. Denote $T_{M,x}$ the germs of vector fields at $x$. One has the identification of left $A$-modules

\[ \text{Der}_{x_A}(\mathcal{O}(M), A) \cong A \otimes T_{\tilde{x}_A}(M) \cong A \otimes_{x_A} T_{M,\tilde{x}_A}. \]

This result is clearly local so that it is enough to prove it in the case $M$ is a superdomain. Next lemma do this for the first identification. The second descends from eq. (5.1), since $T_{M,\tilde{x}_A} = \mathcal{O}_{M,\tilde{x}_A} \otimes T_{\tilde{x}_A}(M)$.

**Lemma 5.9.** Let $U$ be a superdomain in $\mathbb{R}^{p|q}$ with coordinate system $\{x_i, \vartheta_j\}$, $A$ a super Weil algebra and $x_A \in U_A$. To any list of elements

\[ f = (f_1, \ldots, f_p, F_1, \ldots, F_q) \quad f_i, F_j \in A \]

there corresponds a $x_A$-derivation

\[ X_f: \mathcal{O}(U) \longrightarrow A \]

given by

\[ X_f(s) = \sum_i f_i x_A \left( \frac{\partial s}{\partial x_i} \right) + \sum_j F_j x_A \left( \frac{\partial s}{\partial \vartheta_j} \right). \quad (5.1) \]

$X_f$ is even (resp. odd) if and only if the $f_i$ are even (resp. odd) and the $F_j$ are odd (resp. even). Moreover any $x_A$-derivation is of this form for a uniquely determined $f$.

**Proof.** That $X_f$ is a $x_A$-derivation is clear. That the family $f$ is uniquely determined is also immediate from the fact that they are the value of $X_f$ on the coordinate functions.
Let now $X$ be a generic $x_A$-derivation. Define

$$f_i = X(x_i), \quad F_j = X(\vartheta_j),$$

and

$$X_f = f_i x_A \circ \frac{\partial}{\partial x_i} + F_j x_A \circ \frac{\partial}{\partial \vartheta_j}.$$ 

Let $D = X - X_f$. Clearly $D(x_i) = D(\vartheta_j) = 0$. We now show that this implies $D = 0$. Let $s \in O(U)$. Due to lemma 3.13, for each $x \in U$ and for each integer $k \in \mathbb{N}$ there exists a polynomial $P$ in the coordinates such that $[s]_x - [P]_x \in \mathcal{M}_x^{k+1}$. Due to Leibniz rule $D(s - P) \in \hat{A}^k$ and, since clearly $D(P) = 0$, $D(s)$ is in $\hat{A}^k$ for arbitrary $k$. So we are done. \qed

The previous result gives the following corollary.

**Corollary 5.10.** We have the identification

$$T_{x_A} M_A \cong \left( A \otimes T_{x_A}(M) \right)_0 \cong \left( A \otimes_{x_A} T_{M,x_A} \right)_0.$$

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