Atomic Polarisation Rings and Transparency Tunnels in a Relativistically Rotating Gas

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Electromagnetism and light-matter interaction in rotating systems is a rich area of ongoing research. We study the interaction of light with a gas of two-level atoms confined to a rotating disk. We numerically solve the optical Bloch equations to investigate the how relativistic rotational velocities influence the atoms’ coherence and inversion. The results are used to predict the fluorescence seen by an observer at rest with the optical source in the laboratory frame. Competing physical effects due to time dilation and motion-induced detuning strongly modify solutions to the Bloch equations when the gas’s velocity becomes relativistic. When spontaneous emission is accounted for, a regime is identified where light’s coherence and interaction with the atoms is enhanced through weak Doppler detuning compared to the stationary gas case. Assuming weak, resonant, continuous plane wave pumping in the laboratory frame, this leads to a doughnut shaped fluorescence pattern surrounding a central transparent region as seen by laboratory frame observers. The results support the intuitive, special-relativistic approach of assigning instantaneously comoving frames to understand noninertial motion.

I. INTRODUCTION

The physical theory of relativistic rotation has been in development for over a century since Einstein’s conception of special relativity, and was a source of inspiration for his general theory. The fact that publications on this subject continue to appear in the literature up to the present [1–3] suggests it is still not completely understood. Even the simple idea of a uniformly rotating frame runs into conceptual difficulties as there is a radius beyond which comoving objects exceed the speed of light, the so-called light cylinder [4]. Several authors have attempted to construct “relativistic rotational transforms” to circumvent this problem [5–9].

The observation that light can be affected by the motion of media it travels through predates Maxwell’s equations. Fizeau demonstrated the Fresnel drag of light by moving water in 1851, showing by interference that light’s velocity increases/decreases when propagating with/against the fluid flow [10]. After the advent of relativity and Maxwell’s theory of electromagnetism, Minkowski developed a general framework for studying light in moving media by Lorentz transforming Maxwell’s equations to the comoving frame of the material [11]. This approach leads to Minkowski’s constitutive equations, which relate the material excitation fields $D$ and $H$ to the electric $E$ magnetic $B$ fields in a medium moving at fixed velocity relative to the observer [12, 13]. Complications arise in anisotropic [14] and inhomogeneous [15] media. More recently, Leonhardt and Piwnicki [16] have examined the problem for slowly moving media from a geometrical optics perspective, expanding on Gordon’s original idea [17] that moving dielectrics may act as effective gravitational fields for light.

A key result of special relativity is that Maxwell’s equations are Lorentz invariant, meaning their mathematical structure does not change when transforming between different inertial frames (though the fields themselves may). However, a uniformly rotating frame is noninertial as comoving observers experience a centripetal acceleration. As Lorentz transforms are only well-defined between inertial frames, it is not obvious how to relate spacetime coordinates in the comoving frame to the laboratory frame (which sees the material rotating). Ehrenfest’s paradox [18] exemplifies this problem for a disc with radius $R$ when measured at rest, rotating with angular velocity $\Omega$ relative to a stationary observer. The observer sees the lengths of bodies moving towards them shortened by a factor $\gamma = 1/\sqrt{1 - (\Omega R/c)^2}$, where $c$ is the speed of light, due to length contraction. They would conclude that the circumference of the disc rotating underneath them is $2\pi R' = 2\pi R/\gamma$. This implies a disc of radius $R'$, however the radial axis between the disc rim and the origin is perpendicular to the rotation and so should be $R$ in both the comoving and observer frames. The resolution of this contradiction comes from the impossibility of synchronising clocks around a rotating ring [19] [20].

For the most part, experimental work in this area appears to have been limited to general studies in simple dielectric materials, typically at slow medium velocities. The Wilson-Wilson experiment [21] (repeated and verified by Hertzberg et.al. [22]) showed that rotating a dielectric cylindrical shell, in the presence of a static magnetic field oriented along its axis, induced a electric field between its inner and outer surfaces. The strength of the electric field agreed with the result predicted Einstein and Laub [23] using Minkowskii’s constitutive relations, as applied to the cylinder’s rotational motion rather than inertial translation as they were originally derived for. As noted in ref. [24], applying theory valid for special relativistic inertial motion to non-inertial
rotation was considered naïve and controversial. A series of publications attempted to find a more satisfactory explanation [25–29]. The most recent contribution on this problem by Canovan and Tucker [24] derived a general result for the electric and magnetic fields in the rotating cylinder using differential forms, which in the limit of a non-relativistic rotational velocity recovers the solution found by the experiments [21, 22]. This suggests that when transforming electromagnetic fields to reference frames of non-inertial observers, the method of Lorentz transforming the fields from the inertial frame to another inertial frame, instantaneously comoving with the non-inertial observer, is at least a reasonable approximation. However, Mashhoon’s articles [30–32] demonstrate that measurements by non-inertial observers are necessarily nonlocal and have to account for the field’s history by integrating over the observer’s past worldline. The form of the kernel for this integral has yet to be determined [33], but the nonlocal contribution should vanish in the limits of either weak accelerations or vanishing wavelength (the eikonal/ray optics limit) [32]. From the above discussions it is clear that the physics of electromagnetism in non-inertial, relativistically moving systems is still not completely understood. More concrete thought experiments in this area would help provide more direct tests of the theory developed so far and hopefully possibilities for experimental observations which would provide decisive answers.

In this work, we will focus on the optical Bloch equations for a gas of two-level atoms confined to a disc, rotating at a constant angular frequency \( \Omega \) relative to the source of a coherent, monochromatic light field. To our knowledge such a system has not been studied before, yet the Bloch equations give a straightforward and quantitatively accurate model of light interacting with simple atomic matter. As such they are fairly simple to study in a rotating frame. We first solve the full Bloch equations numerically in a simplified rectilinear problem where the atoms interact with this field. In the comoving frame, all that remains is to specify how the atoms’ speed varies from zero at the inner edge \( y = 0 \) to 99% of the speed of light at the outer edge \( y = y_{\text{max}} \), i.e. \( \Omega y_{\text{max}} = 0.99c \). We take \( y_{\text{max}} = 1 \) in some arbitrary length scale. In this case an inertial frame comoving with the gas can be identified for each position \( y \), which can related to the laboratory frame using standard Lorentz transformations:

\[
\begin{align*}
t' &= \gamma(y) \left( t - v_x(y)x/c^2 \right) \\
x' &= \gamma(y) \left( x - v_x(y)t \right) \\
y' &= y
\end{align*}
\]

where here \( \gamma(y) = 1/\sqrt{1-(\Omega y/c)^2} \). We assume the atoms have a simple electric dipole moment \( \mu \) and no magnetic moment in their rest frame (a good approximation for noble gas atoms with all electronic shells full), do not interact with each other or the channel boundaries and that their configuration does not allow for collective optical excitations. Both the flow and the light are homogeneous in the horizontal coordinate \( x \), considered infinite in extent. The electric and magnetic field vectors from the continuous-wave laboratory frame source are \( E = E_0 \hat{y} \), \( B = (E_0/c)\hat{x} \) given \( E_0 = A \exp(-i\omega_E t) \) with \( A, \omega_E \) constant. Standard Lorentz transformations then give the fields seen in the atoms’ comoving frame :

\[
\begin{align*}
E_{\perp}' &= \gamma(y)(E_{\perp} + v \times B) \\
B_{\perp}' &= \gamma(y) \left( B_{\perp} - \frac{1}{c^2} v \times E \right) \\
E_{\parallel}' &= E_{\parallel} \\
B_{\parallel}' &= B_{\parallel}
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which after simplification gives

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\begin{align*}
E' &= \gamma(y)E_0 \hat{y} = \gamma(y)E \\
B' &= \frac{E_0}{c} \left( \hat{x} - \gamma(y)\frac{\Omega y}{c^2} \hat{z} \right) = B - \gamma(y)\frac{\Omega y}{c^2} E_0 \hat{z}.
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This specifies the electromagnetic field in the frame corotating with the atoms; all that remains is to specify how the atoms interact with this field. In the comoving frame, the atoms are at rest and their interaction with light only depends on their electric dipole moment, so the magnetic field \( B' \) is irrelevant. The electric field orientation is the same in both frames, however its strength is boosted by \( \gamma(y) \) in the comoving frame. Further the optical frequency \( \omega_E' = \omega_E/\gamma(y) \) seen by the atoms will vary with \( y \) due to the (transverse) relativistic Doppler shift.

II. LINEAR FLOW PROBLEM

As a preliminary step we will demonstrate the essential physics in a Cartesian analogue for the rotating disc, a sheet of gas flowing along a channel in the \( x \) direction with a linear velocity gradient \( \Omega \) in the transverse \( y \) direction: \( v_x(y) = \Omega y \hat{x} \). Light propagates orthogonally to this channel in the \( z \) direction. We choose the flow velocity gradient \( \Omega \) and the width of the gas channel such that the atoms’ speed varies from zero at the inner edge \( y = 0 \) to 99% of the speed of light at the outer edge \( y = y_{\text{max}} \), i.e. \( \Omega y_{\text{max}} = 0.99c \). We take \( y_{\text{max}} = 1 \) in some arbitrary length scale. In this case an inertial frame comoving with the gas can be identified for each position \( y \), which can related to the laboratory frame using standard Lorentz transformations:

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E_{\parallel}' &= E_{\parallel} \\
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which after simplification gives

\[
\begin{align*}
E' &= \gamma(y)E_0 \hat{y} = \gamma(y)E \\
B' &= \frac{E_0}{c} \left( \hat{x} - \gamma(y)\frac{\Omega y}{c^2} \hat{z} \right) = B - \gamma(y)\frac{\Omega y}{c^2} E_0 \hat{z}.
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This results in a detuning across the disc, which may be comparable to the transition frequency of the two-level system. Hence the rotating wave approximation is not necessarily valid and more general Bloch equations are required. The two-level system is described by the state \( |\psi(t')\rangle = c_g(t')|g\rangle + c_e(t')|e\rangle \) given that the ground state is \( |g\rangle \) and the excited state is \( |e\rangle \) and \( c_g, c_e \) are the complex amplitudes of each in the atomic state at time \( t' \). This can be expressed in terms of a density matrix

\[
\dot{\rho} = \begin{pmatrix}
\rho_{gg} & \rho_{ge} \\
\rho_{eg} & \rho_{ee}
\end{pmatrix} = \begin{pmatrix}
c_g c_g^* & c_g c_e^* \\
c_e c_g^* & c_e c_e^*
\end{pmatrix}
\]  

(4)

If spontaneous emission is accounted for by an excited state inverse lifetime \( \Gamma \) and the Hamiltonian for the two-level atom interacting with the light field is

\[
\dot{H}(t') = \frac{1}{2} \begin{pmatrix}
g |\mu \cdot E(t')|g \end{pmatrix} \left( \frac{g |\mu \cdot E(t')|e}{\hbar \omega_0} \right)
\]  

(5)

the evolution of the density matrix is given by Liouville’s equation [34]

\[
\frac{d}{dt} \dot{\rho} = i \frac{\hbar}{\omega} [\dot{\rho}, \dot{H}] - \left( -\frac{1}{2} \rho_{gg} \frac{\Gamma}{\hbar} \rho_{ee} - \frac{1}{2} \rho_{ee} \rho_{gg} \right)
\]  

(6)

Introduce the Bloch vector, defined by [35]

\[
b = \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix} = \begin{pmatrix}
2 Re(\rho_{eg}) \\
2 Im(\rho_{eg}) \\
\rho_{ee} - \rho_{gg}
\end{pmatrix}
\]  

(7)

This is related to the inversion (population difference between excited and ground atomic states) \( W = b_3 \) and dimensionless atomic polarisation (coherence) \( P = (b_1 + ib_2) / 2 \). Defining the Rabi frequency by \( \langle g | \mu \cdot E(t') | e \rangle = \hbar \Omega_R(y) \exp(-i \omega_E(y)b t') \) (\( y \)-dependent due to the Lorentz boost of the electric field), the optical Bloch equations for these variables are derived from eq. 6, making use of the constraints \( \rho_{ee} + \rho_{gg} = 1 \) and \( \rho_{eg} = \rho_{ge}^* \).

\[
\begin{align*}
b_1' &= -\omega_0 b_2 + \omega_R \sin(\omega_E(y)b t') b_3 - \frac{\Gamma}{2} b_1 \\
b_2' &= \omega_0 b_1 - \omega_R \cos(\omega_E(y)b t') b_3 - \frac{\Gamma}{2} b_2 \\
b_3' &= -\omega_R \sin(\omega_E(y)b t') b_1 + \omega_R \cos(\omega_E(y)b t') b_2 - \Gamma (b_3 + 1)
\end{align*}
\]  

(8)

given a laboratory frame optical frequency \( \omega_E \) and an atomic transition frequency \( \omega_0 \). Note the time derivative indicated by dots is with respect to \( t' \) as defined above and not \( t \). Hence we solve the Bloch equations in the co-moving frame of atoms at each position \( y \); transforming the results back to the laboratory frame simply means reversing the Lorentz transform in eqs. 1 to express them in terms of laboratory time \( t \). The flow induces a \( y \)-dependent time dilation for the atoms, \( t' = t/\gamma(y) \). Additionally the detuning increases with \( y \), reducing the medium’s response away from the origin.

We solve the Bloch equations 8 by numerically integrating them with a simple Matlab script, replacing the time derivative by a central finite difference approximation. Figure 1 shows the physical quantities \( W, |P| \) associated with the solution \( b(y, t) \) to equations 8 in the laboratory frame, assuming the linear velocity profile from above with \( \Omega = 0.99c/y_{\text{max}} \), no spontaneous emission \( \Gamma = 0 \) and resonant interactions in the laboratory frame \( \omega_E = \omega_0 \). The atoms are initially prepared in a state \( b = (1/\sqrt{2}, -1/\sqrt{2}, 0) \). For larger values of \( y \approx 0.8 \) the Doppler shift is obvious where inversion and polarisation oscillations are weakened, as the strong detuning suppresses light-matter coupling. Close to the channel’s edge \( y = 1 \) time dilation dominates and the system’s evolution is retarded as seen by the laboratory observer, while the diverging Rabi frequency boosts the inversion. Clearly
the result at $y = 0$ corresponds to the stationary case. There is however an intermediate region around $y = 0.25$ where the inversion oscillations are strengthened while the polarisation is periodically driven to zero; here the Rabi frequency is close to its stationary value while the Doppler detuning $\Delta = \omega_0 - \omega_E$ is small relative to $\omega_0$ (see figure 2).

Repeating the simulation with spontaneous emission $\Gamma = \omega_0/50$ yields figure 3. As expected the spontaneous process leads to decoherence of the light-matter interaction overall, but it is reduced in the same region where figure 1 shows enhanced coupling. For the parameters chosen, both the detuning and $\Omega_R$ are small relative to $\omega_0$ so the rotating wave approximation may be valid to some extent. In this case, steady state solutions to the Bloch equations are easily obtained by setting the time derivatives to 0. Given the Doppler detuning, the steady state values of the inversion and polarisation magnitude are respectively [35]

$$W_s(y) = \frac{2\Omega_R^2}{\Gamma^2 + 4\Delta^2 + 2\Omega_R^2} - 1$$

$$|P_s(y) = \frac{\Omega_R\sqrt{\Gamma^2 + 4\Delta^2}}{\Gamma^2 + 4\Delta^2 + 2\Omega_R^2}$$

(9)

Figure 4 compares these solutions to the final inversion and polarisation shown in figure 3. For $y < 0.8$ the agreement is quite strong. Deviations from the steady state as $y \to 1$ arise as the rotating wave approximation breaks down as both the detuning and Rabi frequency begin to diverge. Figure 4 shows the detuning increases slowly while the Rabi frequency is roughly constant for small $y$; this modest range of detuning causes the polarisation peak around $y = 0.3$. Where the detuning is weak, the atomic ensemble approaches the threshold of population inversion ($W = 0$) at which it becomes transparent, limited by the size of $\Gamma$. This transparency is suppressed by the detuning at larger values of $y$, as light-matter coupling becomes less efficient and larger fractions of the population remain in the ground state. Increasing the laboratory frame Rabi frequency up to $\omega_0$ causes the steady state transparency region to broaden and the polarisation peak to shift outwards to larger $y$. In the strong pumping regime the entire channel becomes transparent as the gas sits just below the population inversion threshold for all $y$. The steady state inversion and polarisation solutions for different values of the laboratory frame Rabi frequency (i.e. $\Omega_R(y = 0)$ in the comoving frame) are plotted in figure 5.

### III. SOLID ROTATION PROBLEM

Now consider an equivalent system in polar coordinates $(t', r, \theta)$: a rotational atomic flow $v_\theta = \Omega r \hat{\theta}$ confined to a disc interacting with light with laboratory frame polarisation $\mathbf{E} = E_0 \hat{r}$, $\mathbf{B} = (E_0/c)\hat{\theta}$, as might be found in a coaxial cable TEM mode. At the disc’s edge, the atoms’ tangential velocity reaches 0.99$c$. Again we assume no interactions with the disc’s boundary or between the atoms. The frames comoving with the atoms are accelerated and non-inertial, so are not strictly connected to the laboratory frame by Lorentz transforms, which are only defined between pairs of inertial frames. However, the experimental findings of the Wilson-Wilson experiment [21, 22] and the concurring results of Canovan and Tucker [24] suggest that Lorentz transforming to a local inertial frame, momentarily-comoving with the rotating atoms, yields at least an accurate approximation for the rotating frame electromagnetic fields. We therefore define a series of instantaneously comoving frames, each at a fixed radius $r = R$, by

$$t' = \gamma(R) (t - \Omega R^2/c^2 \theta)$$

$$\theta' = \gamma(R) (\theta - \Omega t)$$

$$R' = R$$

(10)

where $\gamma(R) = 1/\sqrt{1 - (\Omega R/c)^2}$. The fields seen by the atoms in this frame are then obtained by the same transforms (eqs. 2) as used in the cartesian case:

$$\mathbf{E}' = \gamma(R) E_0 \hat{r}$$

$$\mathbf{B}' = \frac{E_0}{c} \hat{\theta} - \frac{\Omega R}{c^2} E_0 \hat{z}$$

(11)
FIG. 4. Lineout of the inversion a) and polarisation magnitude b) data shown in figure 3 at the final time $t_f = 50$, solid blue lines. Dashed red lines indicate the steady state solutions to the optical Bloch equations in the rotating wave approximation.

Within these approximations, we expect the results from the linear flow problem discussed in the previous section to translate directly over to this rotational case. The only significant difference is that the spatial coordinate in the direction of motion now imposes periodic boundary conditions on the external fields and the atoms’ response, in both laboratory and rotating frames. Figure 6 shows images of the $W, |P|$ distributions obtained by visualising the data in figure 3 on a polar $(r, \theta)$ grid, assuming the polar result is completely symmetric in $\theta$. Under the above arguments and approximations, this is equivalent to solving the Bloch equations 8 in the rotating frame, assuming the same parameters and input condition as those used for figure 3. Over time, the atom disc tends to support optical excitations only within a ring centred around $r \approx 0.3$. This corresponds to the same region in the Cartesian problem where the boosted Rabi frequency supports coupling which would otherwise be eliminated by Doppler detuning or decoherence. An interesting consequence of time dilation is that radial oscillations of the polarisation’s phase accumulate and blueshift continuously in laboratory time $t$. Presumably the blueshift continues until the wavelength becomes comparable to the atomic radius, at which point they are no longer resolved by the gas and aliasing sets in. The additional magnetic field component along $\hat{z}$ in the rotating frame leads to the instantaneously-comoving Poynting vector being oriented partially along the rotation direction. However there is no component of the Poynting vector along $\hat{r}$ in either laboratory or rotating frames, therefore these oscillations carry no energy. They are simply a phase variation that builds due to a combination of time dilation and detuning.

![Figure 5](image.png)

FIG. 5. Plots of the steady state inversion a) and polarisation b) based on the analytical solutions in equations 9 for varying size of the laboratory frame Rabi frequency relative to the atomic transition frequency.

Lastly we calculate and visualise the intensity of fluorescent emission seen in the laboratory frame due to the steady-state atomic polarisation solutions shown in figure 5b). The permittivity of noble gases whose interaction with light can be well described by a two-level system is close to that of vacuum, so in the co–rotating atomic frame the fluorescent electric field is $E'_f = \epsilon_0 p_0 |P_s(r)| \hat{r}$ with the orientation fixed by the pump field, $p_0$ the atomic dipole density and $\epsilon_0$ the permittivity of vacuum. This time-varying electric field will have an associated magnetic field $B'_f = (\epsilon_0 p_0 |P_s(r)| / c) \theta$. Using the inverse of the transformations in equations 2 gives the electric and magnetic fields in the laboratory due to the gas’s polarisation:

$$E_f = \gamma(r) \epsilon_0 p_0 |P_s(r)| \hat{r}$$

$$B_f = \frac{\epsilon_0 p_0 |P_s(r)|}{c} \left( \hat{\theta} - \gamma(r) \frac{\Omega r}{c} \hat{z} \right)$$ (12)
for which the Poynting vector is

\[ \mathbf{S}_f = \varepsilon_0^3 c \gamma(r) p_0^2 |P_s(r)|^2 \left( \gamma(r) \frac{\Omega r}{c} \hat{\theta} + \hat{z} \right). \]  

(13)

The observed fluorescent intensity will be the root-mean-square magnitude of this:

\[ I_f = \frac{1}{2} \varepsilon_0^3 c \gamma(r)^2 p_0^2 |P_s(r)|^2. \]  

(14)

The steady-state fluorescent intensity seen by a laboratory frame observer is shown in figure 7 for three different pumping strengths. When the gas is weakly pumped \( \Omega_R(0) = 0.1 \omega_0 \) fluorescence peaks around an inner ring (as may be inferred from the polarisation pattern in figure 6d) as well as the around disc’s edge. The edge emission dominates under strong pumping \( \Omega_R(0) \approx \omega_0 \). This is easily understood as the fluorescent intensity is proportional to \( \gamma^2 \), so it will be strongest where the gas’s velocity approaches the relativistic limit, provided significant polarisation is supported by pumping there.

**IV. DISCUSSION**

The final inversion and polarisation images shown in figure 6 c) and d) respectively allow two observations to be made from the laboratory observer’s perspective. Firstly, Doppler detuning at larger radii leads to only a narrow region of partially excited gas (where \( W > -1 \)) around \( r = 0 \). Laboratory observers perceive a transparent tunnel through the rotating gas, surrounded by absorbing atoms in their ground states. Secondly, the ring of boosted polarisation implies that observers should see a doughnut-like intensity structure in the emitted fluorescence, despite the gas being pumped homogeneously in their reference frame. This appears true when the gas is modestly pumped (\( \Omega_R(0) \approx 0.1 \omega_0 \)) based on the results of figure 7. With weak pumping the emission consists of a central spot, while under strong pumping it is most intense around the disc’s edge where relativistic enhancement effects are greatest. Further from figure 5a) the central transparency region widens as pumping strength increases.

We note that analysing the light-matter interaction in the atomic rest frame greatly simplifies the problem. In the laboratory frame, the atoms acquire significant magnetic moments due to their relativistic motion \[36\] so their interaction with both the electric and magnetic fields would have to be accounted for. Even to first order in \( v/c \), the atomic Hamiltonian would be modified by a Röntgen term \( \propto \mathbf{v} \cdot (\mathbf{B} \times \mu) \), which would shift the energy separation between ground and excited states depending on \( r \) \[37\].

Using instantaneously comoving inertial frames to describe the rotating system neglects the nonlocality discussed by Mashhoon \[31, 32\] which would require the past field’s history, as well as a description of the disc...
being accelerated from rest to the uniform rate $\Omega$, to be accounted for. If the disc is already rotating with angular frequency $\Omega$ when the light source is switched on in the laboratory frame, we may assume this nonlocality has a small, transient effect which is negligible at longer times. The Unruh effect may also be significant here; the atoms experience a constant centripetal acceleration, so from an extension Unruh’s original analysis [38] they see a non-trivial vacuum [39]. A potential consequence would be that atoms may be excited by Unruh radiation, leading to an extra term in eq. 6 describing acceleration-dependent spontaneous absorption. The Unruh effect for detectors with discrete energy levels in uniform circular motion around a cavity has been explored by Levin, Peleg and Pereg [40] and Davies, Dray and Manogue in free space [41]. Both parties conclude that the detector remains unexcited providing its rotational velocity is subluminal, i.e. $\Omega R < c$. The Unruh effect was analysed more generally by Korsbakken and Leinaas [42] for systems undergoing both linear and rotational acceleration. They found that detector excitation is possible with an additional, weaker linear acceleration from negative energy modes (the same which give rise to amplification in the Penrose process [43, 44] and the Zel’dovich effect [45]), but again only when the detector rotates superluminally. Hence we take the Unruh effect to be irrelevant for the two-level atoms considered here.

If linear polarisation $E \parallel \hat{y}$ was used in section 3, the electric field $E'$ would break rotational symmetry. This would result in the Rabi frequency varying with $\theta$ and a much more complicated system, where the atoms would experience periodic driving.

V. CONCLUSIONS

The optical Bloch equations have been solved to predict the response of two-level atoms in relativistic linear and rotational motion with respect to a laboratory frame continuous wave light source and observer. At atomic velocities approaching the speed of light, a combination of time dilation and relativistic detuning of the optical carrier and Rabi frequencies modify the emission seen from the flowing gas significantly compared to when the gas is stationary in the laboratory frame. When the gas is weakly and homogeneously pumped, a laboratory observer will see a ring structure in the rotating gas’s fluorescence emission and perceive the centre of rotation as being transparent. These results may be relevant to studies of radiation from rapidly rotating astronomical bodies. Observation of such effects would prove useful in validating the instantaneously-comoving frame approach to relativistic rotation of Einstein and Laub as a practical approximation over more complicated methods.

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