p-adic boundary laws and Markov chains on trees

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Abstract
In this paper, we consider a potential on general infinite trees with q spin values and nearest-neighbor p-adic interactions given by a stochastic matrix. We show the uniqueness of the associated Markov chain (splitting Gibbs measures) under some sufficient conditions on the stochastic matrix. Moreover, we find a family of stochastic matrices for which there are at least two p-adic Markov chains on an infinite tree (in particular, on a Cayley tree). When the p-adic norm of q is greater (resp. less) than the norm of any element of the stochastic matrix then it is proved that the p-adic Markov chain is bounded (resp. is not bounded). Our method uses a classical boundary law argument carefully adapted from the real case to the p-adic case, by a systematic use of some nice peculiarities of the ultrametric (p-adic) norms.

Keywords Cayley trees · Boundary laws · Gibbs measures · Translation invariant measures · p-adic numbers · p-adic probability measures · p-adic Markov chain · Non-Archimedean probability

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1 Introduction

In this paper, we develop a boundary law argument to study $p$-adic Markov chains on general trees. In the real case Markov chains on trees are particular cases of Gibbs measures corresponding to a Hamiltonian with nearest-neighbor interactions. In the theory of Gibbs measures on trees (see [9, Chapter 12] and [23]) the main problem is to describe the set of limiting Gibbs measures corresponding to a given Hamiltonian. A complete analysis of this set is often a difficult problem, this is even not completely described for the Ising model (see [4–6,25] for some recent results).

Parallel to the real-valued Gibbs measures, the $p$-adic Gibbs measures are studied using the $p$-adic mathematical physics in [3,13,14,24,30]. A $p$-adic distribution is an analogue of ordinary distributions that takes values in a ring of $p$-adic numbers [1,12,13]. Analogically to a measure on a measurable space, a $p$-adic measure is a special case of a $p$-adic distribution. A $p$-adic distribution taking values in a normed space is called a $p$-adic measure if the values on compact open subsets are bounded.

It is known that some $p$-adic models in physics cannot be described using ordinary Kolmogorov’s probability theory [14,16,18,30]. In [15] the $p$-adic probability theory was developed using the theory of non-Archimedean measures [29]. In [7,11,19–21,26] various models of statistical physics in the context of $p$-adic fields are studied.

In probability theory Kolmogorov’s extension theorem (see, e.g., [28, Chapter II, § 3, Theorem 4, page 167]), says that a compatibility condition of a sequence of probability measures ensures that there exists a unique (limit) measure. This theorem is used to introduce (real-valued) Markov chains on trees (see [9, Chapter 12]) by the notion of a boundary law. A $p$-adic analogue of Kolmogorov’s theorem was proved in [8]. Such a $p$-adic Kolmogorov theorem allows us to construct wide classes of stochastic processes and to develop statistical mechanics in the context of $p$-adic theory [17–21].

In the present paper we introduce $p$-adic Markov chains on general infinite trees. Such chains are constructed by $p$-adic boundary laws (for the real case see [9, Chapter 12]). We also discuss the uniqueness and boundedness of the $p$-adic Markov chain. The boundedness of the $p$-adic measure is needed to integrate $p$-adic valued functions [12,13,27], and also to consider conditional expectations [12,17]. Note that $p$-adic measures are also useful in $p$-adic $L$-functions following the works of B. Mazur (see [10,16] for details).

The paper is organized as follows. Section 2 presents definitions and known results. Section 2.2 is devoted to an introduction of $p$-adic Markov chains through boundary laws. Section 3 (resp. Sect. 4) is devoted to finding a sufficient condition of the uniqueness (resp. non-uniqueness) of $p$-adic Markov chain. In Sect. 5, we give some conditions ensuring that the $p$-adic Markov chain is (resp. not) bounded.

2 Preliminaries

2.1 $p$-adic numbers and measures

Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number $p$, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, $m$ is a
positive integer, and $n$ and $m$ are relatively prime with $p$: $(p, n) = 1, (p, m) = 1$. The $p$-adic norm of $x$ is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

This norm is non-Archimedean and satisfies the so-called strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$ 

We will often use the following fact:

If $|x|_p \neq |y|_p$ then $|x + y|_p = \max\{|x|_p, |y|_p\}$. \hspace{1cm} (2.1)

The completion of $\mathbb{Q}$ with respect to the $p$-adic norm defines the $p$-adic field $\mathbb{Q}_p$. Any $p$-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \cdots),$$ \hspace{1cm} (2.2)

where $\gamma(x) \in \mathbb{Z}$ and the integers $x_j$ satisfy: $x_0 > 0, 0 \leq x_j \leq p - 1$ (see [16,27,30]). In this case, $|x|_p = p^{-\gamma(x)}$.

Our analysis will strongly rely on nice properties of the $p$-adic norm, and on the two following classical results in $p$-adic algebra.

**Theorem 1** ([16,30]) The equation $x^2 = a$, $0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + \cdots)$, $0 \leq a_j \leq p - 1$, $a_0 > 0$ has a solution $x \in \mathbb{Q}_p$ if and only if the following conditions are fulfilled:

(i) $\gamma(a)$ is even;

(ii) $a_0$ is a quadratic residue modulo $p$ if $p \neq 2$; $a_1 = a_2 = 0$ if $p = 2$.

The elements of the set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ are called $p$-adic integers.

The following statement is known as Hensel’s lemma [2, Theorem 3.15].

**Theorem 2** Let $F(x) = \sum_{i=0}^{n} c_i x^i$ be a polynomial whose coefficients are $p$-adic integers. Let $F'(x) = \sum_{i=0}^{n} i c_i x^{i-1}$ be the derivative of $F(x)$. Assume there exist $a_0 \in \mathbb{Z}_p$ and $\gamma \in \{0, 1, 2, \ldots\}$ such that

$$F(a_0) \equiv 0 \pmod{p^{2\gamma+1}},$$

$$F'(a_0) \equiv 0 \pmod{p^{\gamma}},$$

$$F'(a_0) \not\equiv 0 \pmod{p^{\gamma+1}}.$$ 

Then, there exists $a \in \mathbb{Z}_p$ such that $F(a) = 0$ and $a \equiv a_0 \pmod{p^{\gamma+1}}$.

Given $a \in \mathbb{Q}_p$ and $r > 0$ put

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$
The $p$-adic logarithm is defined by the series

$$
\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n},
$$

which converges for $x \in B(1, 1)$; the $p$-adic exponential is defined by

$$
\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},
$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

**Lemma 1** ([16]) Let $x \in B(0, p^{-1/(p-1)})$, then

$$
|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p, \quad |\log_p(1 + x)|_p = |x|_p,
$$

$$
\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1 + x)) = 1 + x.
$$

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets of $X$. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a $p$-adic measure if for any $A_1, \ldots, A_n \in \mathcal{B}$ such that $A_i \cap A_j = \emptyset, i \neq j$, the following holds:

$$
\mu \left( \bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu(A_j).
$$

A $p$-adic measure is called a $p$-adic probability measure if $\mu(X) = 1$, see, e.g., [12,29]. Let us warn that due to the different axiomatic and ring of values, some intuitive properties of sets of probability measures (like e.g., some convex properties) are not valid anymore [24].

### 2.2 Tree

A tree is a connected graph without cycles (see [22] for more details). Let $T = (V, L)$ be a tree, where $V$ is the set of vertices and $L$ is the set of edges. Two vertices $x$ and $y$ are called nearest neighbors if there exists an edge $b \in L$ connecting them. We will use the notation $b = \langle x, y \rangle$ for the edge connecting the vertices $x$ and $y$. A collection of nearest neighbor pairs $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_{d-1}, y \rangle$ is called a path from $x$ to $y$. The distance $d(x, y)$ on the tree is the number of edges of the shortest path from $x$ to $y$.

For $z \in V$, we denote

$$
L^z = \{ \langle x, y \rangle \in L : d(z, x) = d(z, y) + 1 \},
$$

$$
\bar{z}L = \{ \langle x, y \rangle \in L : d(z, y) = d(z, x) + 1 \}.
$$
3 $p$-adic Markov chain and boundary laws

We consider a system with nearest neighbor interactions on a tree where the spins assigned to the vertices of the tree take values in the set $\Phi := \{1, 2, \ldots, q\}$. A configuration $\sigma_A$ on $A \subset V$ is then defined as a function $x \in A \mapsto \sigma_A(x) \in \Phi$. The set of all configurations is $\Phi^A$.

By $p$-adic probability vector we mean a vector with $p$-adic valued coordinates summing to 1. A $p$-adic stochastic matrix is a matrix with each row being a $p$-adic probability vector. For each edge $b = \langle x, y \rangle \in L$ we consider a stochastic matrix $P_b = (P_b(i, j))_{i, j=1}^q$. For each $x \in V$ consider a probability vector $\alpha_x = (\alpha_{1, x}, \ldots, \alpha_{q, x})$.

For any edge $b = \langle x, y \rangle \in L$ we assume that

$$\alpha_{i, x} P_b(i, j) = \alpha_{j, y} P_b(j, i), \ \forall i, j \in \Phi. \ (3.1)$$

**Definition 1** A $p$-adic probability distribution (measure) $\mu$ is called a $p$-adic Markov chain with transition matrices $(P_b)_{b \in L}$ and marginal distribution $\alpha_x$ at $x \in V$ if for all finite, connected set $\Lambda \subset V$, and all $\zeta \in \Phi^\Lambda$ and $z \in \Lambda$ the following holds

$$\mu(\sigma_{\Lambda} = \zeta) = \alpha_z(z_{\zeta}) \prod_{\langle x, y \rangle \in \zeta L: \ x, y \in \Lambda} P_{\langle x, y \rangle}(\zeta_x, \zeta_y). \ (3.2)$$

Note that the reversibility condition (3.1) is equivalent to the statement that the expression on the right of (3.2) is independent of the choice of $z \in \Lambda$.

Consider for each edge $b = \langle x, y \rangle \in L$ a matrix $Q_b = (Q_b(i, j))_{i, j=1}^q$. We always assume

$$Q_{\langle x, y \rangle}(i, j) = Q_{\langle y, x \rangle}(j, i), \sum_{j=1}^q Q_{\langle x, y \rangle}(j, i) = 1. \ (3.3)$$

Let $z(x, y) = (z_1(x, y), \ldots, z_q(x, y))$ be a vector in $\mathbb{Q}_p$.

**Definition 2** For $(Q_b)_{b \in L}$ satisfying (3.3), a $p$-adic boundary law$^1$ $\{z(x, y)\}_{\langle x, y \rangle \in L}$ is such that for any $\langle x, y \rangle \in L$, and for all $i \in \Phi$, it holds

$$z_i(x, y) = c(x, y) \prod_{v \in \partial\{x\} \setminus \{y\}} \sum_{j \in \Phi} z_j(v, x) Q_{\langle v, x \rangle}(j, i), \ (3.4)$$

where $c(x, y)$ is an arbitrary constant (not depending on $i \in \Phi$).

$^1$ Compare with real boundary law of [9, Definition (12.10)].

\[\text{Springer}\]
Using (3.3) and proceeding as in the classical case of [9, Formula (12.13), page 243], one directly gets that each boundary law
\[ z = \{ z(x, y) = (z_1(x, y), \ldots, z_q(x, y)) \}_{(x,y) \in L} \]
defines a \( p \)-adic Markov chain \( \mu^z \): for any finite connected set \( \emptyset \neq \Lambda \subset V \) (and \( \bar{\Lambda} = \Lambda \cup \partial \Lambda \)), one has
\[
\mu^z(\sigma_{\bar{\Lambda}} = \varsigma) = \frac{1}{Z_{\bar{\Lambda}}} \prod_{x \in \partial \Lambda} z_{\varsigma(x)}(x, x_{\Lambda}) \prod_{b \in L: \ b \cap \Lambda \neq \emptyset} Q_b(\varsigma_b),
\]
where \( Z_{\bar{\Lambda}} = Z_{\bar{\Lambda}}(z) \) is the normalizing factor, \( x_{\Lambda} \) denotes the unique neighbor of \( x \in \partial \Lambda \) belonging to \( \Lambda \), and \( \varsigma_b = (\varsigma(u), \varsigma(v)) \), for \( b = (u, v) \). We stress that the first condition in (3.3), which is [9, Formula (12.9)], is needed to check that \( \mu^z \) is a well defined \( p \)-adic Markov chain.

4 Criterion for uniqueness of the \( p \)-adic Markov chain

A \( p \)-adic Markov chain can be considered as a particular case of \( p \)-adic Gibbs measure defined through the \( p \)-adic exponential \( \exp_p(x) \), with \( |x|_p < p^{-1/(p-1)} \) [19]. As it was mentioned above, the set of values of a \( p \)-adic norm \( |\cdot|_p \) is \( \{ p^m : m \in \mathbb{Z} \} \), so the condition \( |x|_p < p^{-1/(p-1)} \) is equivalent to the condition \( |x|_p \leq \frac{1}{p} \). Consequently, we shall restrict part of the analysis to quantities belonging to the set:
\[ \mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : |x-1|_p \leq \frac{1}{p} \right\}. \]
The following lemma will also be useful (see [19, Lemma 4.6]).

Lemma 2 If \( a_i \in \mathbb{Q}_p \) for all \( i = 1, \ldots, m \) are such that
\[ |a_i|_p = 1, \ |a_i - 1|_p \leq M, \]
then
\[
\left| \prod_{i=1}^m a_i \right|_p = 1, \ \left| \prod_{i=1}^m a_i - 1 \right|_p \leq M.
\]
Without loss of generality, we set hereafter \( z_q(v, x) \equiv 1 \) (a normalization at \( q \)). Then, the condition (3.4) for the stochastic matrix \( Q_b = (Q_b(i, j))_{i,j=1}^q \) reads
\[
z_i(x, y) = \prod_{v \in \partial(x) \setminus \{y\}} \frac{1 + \sum_{j=1}^{q-1} (z_{j}(v, x) - 1) Q_{(v,x)}(j, i)}{1 + \sum_{j=1}^{q-1} (z_{j}(v, x) - 1) Q_{(v,x)}(j, q)}, \ i = 1, 2, \ldots, q - 1.
\]
Here we have used

$$Q_{(v,x)}(q,i) = 1 - \sum_{j=1}^{q-1} Q_{(v,x)}(j,i), \quad i = 1, 2, \ldots, q.$$  

In this section, we examine the conditions on the parameters $k \geq 1$ and on $Q_b$ for the existence and the uniqueness of the solutions of the equation (4.1).

For the uniqueness, we assume that the matrix $Q_b = (Q_b(i,j))_{i,j=1}^q$ satisfies the following conditions

$$|Q(x,y)(j,i)|_p \leq 1, \quad |Q(x,y)(j,i) - Q(x,y)(j,q)|_p \leq \frac{1}{p}, \quad \forall(x,y), \forall i, j. \quad (4.2)$$

**Theorem 3** Assume each vertex of the tree has degree at least 2 and that the matrix $Q_b = (Q_b(i,j))_{i,j=1}^q$ satisfies (3.3) and (4.2). Then, the equation (4.1) has a unique solution $z(x,y) \equiv (1,1,\ldots,1) \in \mathcal{E}_p^{q-1}$, $(x,y) \in L$.

**Proof** Since

$$\sum_{j=1}^q Q_{(v,x)}(j,i) = 1, \quad \forall(v,x), \forall i,$$

it follows that $z(x,y) \equiv (1,1,\ldots,1)$ is a solution to (4.1).

We show its uniqueness. For $z = (z_1,\ldots,z_{q-1}) \in \mathbb{Q}_p^{q-1}$, we introduce the norm

$$\|z\| = \max_i |z_i|_p.$$  

Let $z(x,y) \in \mathcal{E}_p^{q-1}$, $(x,y) \in L$ be a solution. Denote

$$K_i \equiv K_i(v,x,q) = \frac{1 + \sum_{j=1}^{q-1} (z_j(v,x) - 1) Q_{(v,x)}(j,i)}{1 + \sum_{j=1}^{q-1} (z_j(v,x) - 1) Q_{(v,x)}(j,q)} \quad (4.3)$$

Using (2.1), (4.2) and Lemma 2, we calculate $|K_i|_p$:

$$|K_i|_p = \left| \frac{1 + \sum_{j=1}^{q-1} (z_j(v,x) - 1) Q_{(v,x)}(j,i)}{1 + \sum_{j=1}^{q-1} (z_j(v,x) - 1) Q_{(v,x)}(j,q)} \right|_p = 1.$$  

Let us now estimate $|K_i - 1|_p$ using (4.2):

$$|K_i - 1|_p = \left| \frac{\sum_{j=1}^{q-1} (z_j(v,x) - 1) [Q_{(v,x)}(j,i) - Q_{(v,x)}(j,q)]}{1 + \sum_{j=1}^{q-1} (z_j(v,x) - 1) Q_{(v,x)}(j,q)} \right|_p.$$  

\[\begin{align*}
\frac{q-1}{j=1}\sum_{j=1}^{q-1} \left[ z_j(v, x) - 1 \right] \left[ Q_{(v,x)}(j, i) - Q_{(v, x)}(j, q) \right] \\
\leq \max_{j} \left| z_j(v, x) - 1 \right| p \left| Q_{(v,x)}(j, i) - Q_{(v, x)}(j, q) \right| p \\
\leq \frac{1}{p} \| z(v, x) - 1 \| \leq \frac{1}{p} \| z(\hat{v}, x) - 1 \|, 
\end{align*}\]

where we have used the hypothesis

\[\left| Q_{(v,x)}(j, i) - Q_{(v, x)}(j, q) \right| p \leq \frac{1}{p},\]

and \( \hat{v} \equiv \hat{v}(x, y) \) is defined by

\[\| z(\hat{v}, x) - 1 \| = \max_{v \in \partial [x] \setminus [y]} \| z(v, x) - 1 \|.\]

Thus \( K_i \) satisfies the conditions of Lemma 2, and we have

\[\begin{align*}
\| z_i(x, y) - 1 \| p &= \left| \prod_{v \in \partial [x] \setminus [y]} \frac{1 + \sum_{j=1}^{q-1} (z_j(v, x) - 1) Q_{(v,x)}(j, i)}{1 + \sum_{j=1}^{q-1} (z_j(v, x) - 1) Q_{(v, x)}(j, q)} - 1 \right| p \\
&= \left| \prod_{v \in \partial [x] \setminus [y]} K_i - 1 \right| p \leq \frac{1}{p} \| z(\hat{v}, x) - 1 \|. 
\end{align*}\] (4.4)

Consequently,

\[\| z(x, y) - 1 \| \leq \frac{1}{p} \| z(\hat{v}, x) - 1 \|.\] (4.5)

Since this estimation is true for arbitrary edge \( (x, y) \in L \), one can start from any edge and then iterate the estimation (4.5), to obtain the following

\[\| z(x, y) - 1 \| \leq \frac{1}{p^n} \| z(\hat{v}^{(n)}, \hat{v}^{(n-1)}) - 1 \| \leq \frac{1}{p^n+1}.\] (4.6)

which as \( n \to \infty \) gives \( z(x, y) = 1 \). \( \square \)

Denote by \( \mu^1 \) the \( p \)-adic Markov chain which corresponds to \( z(x, y) \equiv 1, \ldots, 1 \).

**Corollary 1** Under the conditions of Theorem 3, there exists a unique \( p \)-adic Markov chain, which satisfies that for any finite connected set \( \emptyset \neq \Lambda \subset V \) (and \( \bar{\Lambda} = \Lambda \cup \partial \Lambda \)),

\[\mu^1(\sigma_{\bar{\Lambda}} = \varsigma) = \frac{1}{Z_{\bar{\Lambda}}} \prod_{b \in L; b \cap \Lambda \neq \emptyset} Q_b(\varsigma_b),\] (4.7)
where
\[
Z_{\tilde{\Lambda}} = \sum_{\sigma \in \Omega_{\tilde{\Lambda}}} \prod_{b \in L: b \cap \Lambda \neq \emptyset} Q_b(\sigma_b).
\] (4.8)

5 Criterion for non-uniqueness of the \(p\)-adic Markov chains

5.1 On a regular tree

Consider the Cayley tree of order \(k \geq 1\). Suppose the matrix \(Q_b\) in the system of equations (4.1) satisfies the condition
\[
Q_{(v,x)}(1,i) = Q_{(v,x)}(1,q), \quad \forall (v,x), \quad i = 2, \ldots, q - 1.
\] (5.1)

We assume further that \(Q_{(v,x)}(1,1)\) and \(Q_{(v,x)}(1,q)\) are independent on \((v,x)\), that is
\[
\alpha \equiv Q_{(v,x)}(1,1), \quad \beta \equiv Q_{(v,x)}(1,q), \quad \forall (v,x) \in L.
\] (5.2)

**Theorem 4** If (5.1), (5.2) are satisfied, \(\alpha, \beta\) are \(p\)-adic integers, and there exists \(\gamma \in \{0, 1, 2, \ldots\}\) such that
\[
k(\beta - \alpha) + 1 \equiv 0 \pmod{p^{2^\gamma + 1}},
\]
\[
k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2) \equiv 0 \pmod{p^\gamma},
\]
\[
k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2) \neq 0 \pmod{p^{\gamma + 1}},
\] (5.3)

then the equation (4.1) has at least two solutions.

**Proof** We shall prove that the equation (4.1) has two constant (translational-invariant) solution \(z(x,y) \equiv z, \quad \forall (x,y) \in L\). The first solution is already known: \(z(x,y) \equiv (1, \ldots, 1)\). We shall show that the system (4.1) has a solution of the following form
\[
z = \{z(x,y) = (z, 1, 1, \ldots, 1)_{(x,y) \in L}, \quad z \neq 1\}.
\]

Then from (4.1), for the Cayley tree of order \(k \geq 2\), we get
\[
z = \left(\frac{1 - \alpha + \alpha z}{1 - \beta + \beta z}\right)^k.
\] (5.4)

Independently on parameters, this equation has solution \(z = 1\). We are going to find conditions on \(\alpha \neq \beta\) and on \(k\) to have at least one solution \(z \neq 1\).

The equation (5.4) can be written as \(F(z) = 0\) with
\[
F(z) = z(1 - \beta + \beta z)^k - (1 - \alpha + \alpha z)^k.
\]
We are interested in the solution of $G(z) = F(z) \equiv 0$, where

$$G(z) = 1 + \sum_{j=1}^{k} \binom{k}{j} (z\beta^j - \alpha^j)(z - 1)^{j-1}.$$ 

Since $\alpha, \beta$ are $p$-adic integers, $G(z)$ has only $p$-adic integer coefficients. Now we shall check the other conditions of Hensel’s lemma (see Theorem 2). Take $a_0 = 1$. Then, we have

$$G(1) = 1 + k(\beta - \alpha)$$

and

$$G'(1) = k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2).$$

Therefore by (5.3), the conditions of Hensel’s lemma are satisfied for $G(z)$. Hence there exists a $p$-adic integer $a$ such that $G(a) = 0$ and $a \equiv a_0 \pmod{p^\gamma+1}$, i.e., $G(z) = 0$ has a solution $z = a$. Since $a_0 = 1$, we have $a \equiv 1 \pmod{p^\gamma+1}$. Thus $a \in E_p$. This proves the theorem.

**Remark 1** Note that if $p$ divides $k(\beta - \alpha) + 1$ then $p$ does not divide $\beta - \alpha$, therefore $|\beta - \alpha|_p = 1 > \frac{1}{p}$, i.e., the condition (4.2) is not satisfied.

Let us give some examples of parameters satisfying the conditions of Theorem 4:

**Example 1** The case $\gamma = 0$:

(a) Let $k = 1$. Then, the equation $G(z) = 0$ has a unique solution $z = a = \frac{\alpha-1}{\beta}$. The condition (5.3) of Theorem 4 is equivalent to

$$|\beta - \alpha + 1|_p \leq \frac{1}{p}, \ |\beta|_p = 1.$$ 

This implies $|\alpha - 1|_p = 1$, and $|a| = 1, |a - 1| \leq \frac{1}{p}$. Thus, the solution $z = a$, other than the solution $z = 1$, is also in $E_p$.

(b) Take $k = 2, p = 3, \alpha = 2, \beta = 3$. Then $k(\beta - \alpha) + 1 = 3 \equiv 0 \pmod{3}$ and $k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2) = 11 \not\equiv 0 \pmod{3}$. For these parameters the equation (5.4) has three solutions:

$$z_0 = 1, \ z_1 = \frac{7 - \sqrt{13}}{18}, \ z_2 = \frac{7 + \sqrt{13}}{18}.$$ 

Note (see Theorem 1) that $\sqrt{13}$ exists in $Q_3$. Moreover, it can be calculated$^2$:

$$\sqrt{13} = 1 + 2 \cdot 3 + 3^2 + 3^5 + 2 \cdot 3^6 + \cdots.$$ 

Then, we get

$$|z_1|_3 = \left| \frac{7 - \sqrt{13}}{18} \right|_3 = \left| \frac{3^2 + 3^5 + 2 \cdot 3^6 + \cdots}{2 \cdot 3^2} \right|_3 = 1,$$

$^2$ [http://www.numbertheory.org/php/p-adic.html](http://www.numbertheory.org/php/p-adic.html).
\[|z_1 - 1|_3 = \left| \frac{-11 - \sqrt{13}}{18} \right|_3 = \left| \frac{3^3 + 3^5 + 2 \cdot 3^6 + \ldots}{2 \cdot 3^2} \right|_3 = \frac{1}{3}.\]

Hence \(z_1 \in \mathcal{E}_3\), and \(z_1\) plays the role of \(\alpha \in \mathcal{E}_3\) mentioned in the proof of Theorem 4. On the other hand, we have \(z_1z_2 = \frac{1}{9}\). Consequently \(|z_1z_2|_3 = 9\). Since \(|z_1|_3 = 1\), we obtain \(|z_2|_3 = 9\). Thus \(z_2 \notin \mathcal{E}_3\).

**Example 2** The case \(\gamma = 1\): Take \(k = 2, p = 3, \alpha = 6, \beta = 19\). Then

\[
k(\beta - \alpha) + 1 = 27 \equiv 0 \pmod{33},
\]

\[
k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2) = 363 \equiv 0 \pmod{3},
\]

\[
k\beta + \frac{k(k-1)}{2}(\beta^2 - \alpha^2) = 363 \not\equiv 0 \pmod{32}.
\]

In this case, the equation (5.4) has three solutions:

\[z_0 = 1, \quad z_1 = \frac{359 - 39\sqrt{61}}{722}, \quad z_2 = \frac{359 + 39\sqrt{61}}{722}.\]

We have \(|z_1 - 1|_3 = \left| \frac{363 - 39\sqrt{61}}{722} \right|_3 \leq \frac{1}{3}\). Thus \(z_1 \in \mathcal{E}_3\). Similarly, one can see that \(z_2 \notin \mathcal{E}_3\).

As a corollary of Theorem 4, we have the following.

**Theorem 5** If the conditions of Theorem 4 are satisfied then for the matrix \(Q_b\) on the Cayley tree of order \(k \geq 1\), there are at least two \(p\)-adic Markov chains.

**Remark 2** Theorem 4 can be generalized as follows: fix \(m \in \{1, 2, \ldots, q - 1\}\) and assume

\[
Q_{\langle v, x \rangle}(j, i) = Q_{\langle v, x \rangle}(j, q), \quad \forall \langle v, x \rangle, \quad j = 1, \ldots, m; \quad i = m + 1, \ldots, q - 1. \quad (5.5)
\]

Suppose \(\sum_{j=1}^{m} Q_{\langle v, x \rangle}(j, i)\) and \(\sum_{j=1}^{m} Q_{\langle v, x \rangle}(j, q)\) are independent on \(\langle v, x \rangle\), i.e.,

\[
A \equiv \sum_{j=1}^{m} Q_{\langle v, x \rangle}(j, i), \quad B \equiv \sum_{j=1}^{m} Q_{\langle v, x \rangle}(j, q), \quad \forall \langle v, x \rangle \in L, \quad i = 1, \ldots, m. \quad (5.6)
\]

Under the above mentioned conditions one can show that the system (4.1) has a solution of the following form

\[
z = \left\{ \underbrace{z(x, y) = (z, z, \ldots, z, 1, 1, \ldots, 1)}_{m} \right\}_{\langle x, y \rangle \in L}, \quad z \neq 1.
\]
Then from (4.1), for the Cayley tree of order $k \geq 2$, we get

$$z = \left( \frac{1 - A + Az}{1 - B + Bz} \right)^k. \quad (5.7)$$

This equation is identical with (5.4) and it has non-unique solutions when $A$ and $B$ (replacing $\alpha$ and $\beta$) satisfy the conditions mentioned in Theorem 4.

### 5.2 Extension on a non-regular tree

Consider now a general tree $T$, with each vertex having at least two nearest neighbors. Recall that $L$ is the set of all edges of $T$. Such a tree contains a Cayley tree (of some order $k \geq 1$) as a subtree, which we denote by $\Gamma^k$. Let $L_k$ be the set of all edges of $\Gamma^k$, i.e., $L_k \subset L$.

Assume on $\Gamma^k$, the conditions of Theorem 4 are satisfied. Then, we have a boundary law of the form

$$z = \{z(x, y) = (z, 1, 1, \ldots, 1) \} \{x, y\} \in L_k, \quad z \neq 1. \quad (5.8)$$

Let $g(z) = \frac{1 - \alpha + \alpha z}{1 - \beta + \beta z}$. Define on the edges $\langle x, y \rangle$ of the general tree $T$ the following vector-valued function

$$l = \{l(x, y) = (l_1(x, y), 1, \ldots, 1) \}, \quad (5.9)$$

where

$$l_1(x, y) = \begin{cases} 
  z, & \text{if } \langle x, y \rangle \in L_k, \\
  1, & \text{if } \langle x, y \rangle \in L, \ x \in L \setminus L_k, \\
  zg(z), & \text{if } \langle x, y \rangle \in L, \ x \in \Gamma^k, \ y \in L \setminus L_k.
\end{cases} \quad (5.10)$$

and $z$ is defined in (5.8).

For $i = 2, \ldots, q - 1$, we assume

$$Q_{\langle v, x \rangle}(1, i) = Q_{\langle v, x \rangle}(1, q), \quad \text{for } \langle v, x \rangle \text{ with } v \in \Gamma^k, \ x \in L \setminus L_k. \quad (5.11)$$

and show that $l$ defined by (5.9) satisfies the equation (4.1).

For coordinates $l_i(x, y) = 1$, $i = 2, 3, \ldots, q - 1$, from (4.1) we have

$$1 = l_i(x, y) = \prod_{v \in \partial(x) \setminus [y]} \frac{1 + (l_1(v, x) - 1)Q_{\langle v, x \rangle}(1, i)}{1 + (l_1(v, x) - 1)Q_{\langle v, x \rangle}(1, q)}, \quad i = 2, \ldots, q - 1. \quad (5.12)$$

Therefore, by (5.1), (5.10) and (5.11), one can see that the right-hand side of (5.12) is always 1.

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Now we show that \( l_1(x, y) \) also satisfies (4.1). Indeed, we note that \( \partial \{x\} \setminus \{y\} = A_k(x, y) \cup B_k(x, y) \), where \( A_k(x, y) = (\partial \{x\} \setminus \{y\}) \cap L_k \) and \( B_k(x, y) = (\partial \{x\} \setminus \{y\}) \cap (L \setminus L_k) \).

We thus have the following three possible cases:

Case: \( x, y \in \Gamma^k \). In this case, \( l_1(x, y) = z \) and \( A_k(x, y) \) has \( k \) elements. Therefore, the equation (4.1) for \( l_1(x, y) \) is reduced to \( z = (g(z))^k \), which is satisfied by the conditions of Theorem 4.

Case: \( \langle x, y \rangle \in L, \ x \in L \setminus L_k \). Then, \( A_k(x, y) = \emptyset \) and hence the equation (4.1) for \( l_1(x, y) \) is reduced to the identity \( 1 = 1 \).

Case: \( \langle x, y \rangle \in L, \ x \in \Gamma^k, \ y \in L \setminus L_k \). In this case, \( A_k(x, y) \) contains \( k + 1 \) elements, and we have \( l_1(v, x) = z \) for all \( v \in A_k(x, y) \). Thus the equation (4.1) has the form \( l_1(x, y) = (g(z))^{k+1} \). Using \( z = (g(z))^k \), we get \( l_1(x, y) = zg(z) \) as in the definition (5.9). Thus \( l(x, y) \) satisfies the equation (4.1).

Denote by \( \mu^l \) the \( p \)-adic Markov chain corresponding to \( l \) given by (5.9).

We have proved the following theorem.

**Theorem 6** Let \( T \) be a tree containing a Cayley tree \( \Gamma^k \) of order \( k \geq 1 \), as a subtree. Suppose the conditions of Theorem 4 are satisfied on \( \Gamma^k \). If (5.11) is satisfied, then on the tree \( T \) there are at least two \( p \)-adic Markov chains (one is \( \mu^l \) and the other is \( \mu^1 \)).

### 6 Criterion for the (un-)boundedness of the \( p \)-adic Markov chains

Now we are interested in finding out whether a \( p \)-adic Markov chain is bounded.

Let \( \{z(x, y) \in \mathcal{E}_p, \ \langle x, y \rangle \in L\} \) be a boundary law for the matrix \( Q_b = (Q_b(i, j)) \) and \( \mu^z \) be the corresponding \( p \)-adic Markov chain.

**Theorem 7** The following hold

1. If \( \max_{i, j \in \Phi} |Q_b(i, j)|_p \leq |q|_p \) for all \( b \in L \), then the \( p \)-adic Markov chain \( \mu^z \) is bounded;
2. If \( \min_{i} \max_{j} |Q_b(i, j)|_p > |q|_p \) for all \( b \in L \), then the \( p \)-adic Markov chain \( \mu^z \) is not bounded.

**Proof** It suffices to show that for any finite connected set \( \emptyset \neq \Lambda \subset V \) (denote \( \bar{\Lambda} = \Lambda \cup \partial \Lambda \)), and any \( \varsigma \in \Omega_\Lambda \), one has \( |\mu^z(\sigma_{\bar{\Lambda}} = \varsigma)|_p \leq M \), for some \( M > 0 \). Using (3.5), we get

\[
|\mu^z(\sigma_{\bar{\Lambda}} = \varsigma)|_p = \frac{\prod_{x \in \partial \Lambda} z_{\varsigma}(x, x_\Lambda) \prod_{b \in L : b \cap \Lambda \neq \emptyset} Q_b(\varsigma_b)}{\sum_{\varphi_{\bar{\Lambda}}} \prod_{x \in \partial \Lambda} z_{\varphi}(x, x_\Lambda) \prod_{b \in L : b \cap \Lambda \neq \emptyset} Q_b(\varphi_b)}_p. \quad (6.1)
\]

Let us calculate

\[
\mathcal{Z} = \sum_{\varphi_{\bar{\Lambda}}} \prod_{x \in \partial \Lambda} z_{\varphi}(x, x_\Lambda) \prod_{b \in L : b \cap \Lambda \neq \emptyset} Q_b(\varphi_b)_p
\]
\[ \sum_{\phi_A} \prod_{x \in \partial \Lambda} z_{\phi(x)}(x, x_A) - 1 \prod_{b \in L: b \cap \Lambda \neq \emptyset} Q_b(\phi_b) + \sum_{\phi_A} \prod_{b \in L: b \cap \Lambda \neq \emptyset} Q_b(\phi_b) \cdot \]

The set \( \tilde{\Lambda} \) can be decomposed as

\[ \tilde{\Lambda} = \partial \Lambda \cup \partial_{int} \Lambda \cup \partial_{int}(\Lambda \setminus \partial_{int} \Lambda) \cup \cdots \cup \{x_0\}, \]

where \( \partial_{int} A = \{x \in A : \exists y \in V \setminus A, \langle x, y \rangle\} \). Since \( Q_b \) is stochastic for any \( b \in L \) we get

\[ \sum_{\phi_A} \prod_{b \in L: b \cap \Lambda \neq \emptyset} Q_b(\phi_b) = \sum_{\phi_A} \prod_{b \in L: b \subset (\Lambda \setminus \partial_{int} \Lambda) \times (\Lambda \setminus \partial_{int} \Lambda)} Q_b(\phi(x), \phi(y)) \]

\[ = \cdots = \sum_{\phi(x_0) = 1} q = q. \]

(1) Under the conditions of the part 1), we have (note that \( |q|_p \leq 1 \))

\[ Z = \left| \sum_{\phi_A} \left[ \prod_{x \in \partial \Lambda} z_{\phi(x)}(x, x_A) - 1 \right] \prod_{b \in L: b \cap \Lambda \neq \emptyset} Q_b(\phi_b) + q \right|_p = |q|_p. \]

Thus

\[ |\mu^{z}(\sigma_A = \varsigma)|_p = Z^{-1} \left| \prod_{x \in \partial \Lambda} z_{\varsigma(x)}(x, x_A) \prod_{b \in L: b \cap \Lambda \neq \emptyset} Q_b(\xi_b) \right|_p \leq \frac{|q|_{\tilde{\Lambda}}}{|q|_p} \leq 1. \ (6.2) \]

(2) Suppose now the conditions of part 2) are satisfied. For a marginal on the two-site volume, i.e., an edge \( b = \langle x, y \rangle \), corresponding to a boundary law \( z = (z_1(x, y), \ldots, z_q(x, y)) \), when \( \sigma(x) = i \) is fixed we have

\[ \mu^{z}_b(i, \sigma(y)) = \frac{Q_b(i, \sigma(y))z_{\sigma(y)}(x, y)}{\sum_{\phi(y) = 1} Q_b(i, \phi(y))z_{\phi(y)}(x, y)}. \]
Therefore,

\[
|\mu^z_b(i, \sigma(y))|_p = \left| \frac{Q_b(i, \sigma(y)) z_{\sigma(y)}(x, y)}{\sum_{\varphi(y)=1}^q [z_{\varphi(y)}(x, y) - 1] Q_b(i, \varphi(y)) + \sum_{\varphi(y)=1}^q Q_b(i, \varphi(y))} \right|_p = \frac{|Q_b(i, \sigma(y))|_p}{\sum_{\varphi(y)=1}^q [z_{\varphi(y)}(x, y) - 1] Q_b(i, \varphi(y)) + q}.
\]

(6.3)

In order to show that the measure \( \mu^z \) is not bounded, it is enough to show that its marginal measure is not bounded. Let \( \pi = \{x_0, x_1, \ldots\} \) be an arbitrary infinite path in the tree. The marginal measure \( \mu^z_\pi \) has the form

\[
\mu^z_\pi(\omega_n) = \alpha_{\omega_n(x_0)} \prod_{m=0}^{n-1} \mu^z_{\{x_m, x_{m+1}\}}(\omega_n(x_m), \omega_n(x_{m+1})).
\]

(6.4)

Here \( \omega_n : \{x_0, \ldots, x_n\} \to \Phi = \{1, 2, \ldots, q\} \) is a configuration on \( \{x_0, \ldots, x_n\} \) and \( \alpha_i \) is a coordinate of the invariant stochastic vector of the matrix \( \left( \mu^z_{\{x_0, x_1\}}(i, j) \right)_{i,j=1,\ldots,q} \).

To ensure that \( |\alpha_{\omega_n(x_0)}|_p > c \) for some \( c > 0 \). We can choose the value \( i_0 = \omega_n(x_0) \) (of the configuration \( \omega_n \) on the vertex \( x_0 \)) such that

\[
|\alpha_{i_0}|_p = \max_{s \in \Phi} |\alpha_s|_p.
\]

Then, since \( \alpha \) is a probability vector we have

\[
1 = \left| \sum_{s=1}^q \alpha_s \right|_p \leq \max_{s \in \Phi} |\alpha_s|_p = |\alpha_{\omega_n(x_0)}|_p.
\]

Having \( i_0 \), we choose the value \( i_1 = \omega_n(x_1) \) of the configuration \( \omega_n \) to satisfy

\[
|Q_b(i_0, i_1)|_p = \max_j |Q_b(i_0, j)|_p.
\]

By iterating, we define \( i_m = \omega(x_m) \) to have

\[
|Q_b(i_{m-1}, i_m)|_p = \max_j |Q_b(i_{m-1}, j)|_p, \quad m \geq 1.
\]
Then for the above constructed $\omega_n$, by (6.3) we get

$$\left| \mu^z_{(x_m, x_{m+1})}(i_m, i_{m+1}) \right|_p \geq \frac{|Q_b(i_m, i_{m+1})|_p}{\max \left\{ \frac{1}{p} \max_j |Q_b(i_m, j)|_p, |q|_p \right\}} \geq p, \ m = 1, 2, \ldots$$

(6.5)

Here, at the last step we have used the following (which is true by the condition of the part 2) of theorem)

$$|q|_p < |Q_b(i_m, i_{m+1})|_p.$$

Consequently, for such a configuration $\omega_n$, from (6.4) and (6.5), we find that

$$\mu^z_\pi(\omega_n) \geq p^n,$$

i.e., $\mu^z$ is not bounded.

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