ON GAUSSIAN APPROXIMATION FOR M-ESTIMATOR

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ABSTRACT. This study develops a non-asymptotic Gaussian approximation theory for distributions of M-estimators, which are defined as maximizers of empirical criterion functions. In existing mathematical statistics literature, numerous studies have focused on approximating the distributions of the M-estimators for statistical inference. In contrast to the existing approaches, which mainly focus on limiting behaviors, this study employs a non-asymptotic approach, establishes abstract Gaussian approximation results for maximizers of empirical criteria, and proposes a Gaussian multiplier bootstrap approximation method. Our developments can be considered as extensions of the seminal works (Chernozhukov, Chetverikov and Kato (2013, 2014, 2015)) on the approximation theory for distributions of suprema of empirical processes toward their maximizers. Through this work, we shed new lights on the statistical theory of M-estimators. Our theory covers not only regular estimators, such as the least absolute deviations, but also some non-regular cases where it is difficult to derive or to approximate numerically the limiting distributions such as non-Donsker classes and cube root estimators.

1. INTRODUCTION

This study focuses on non-asymptotic approximations for distributions of M-estimators. Let $Z_1,\ldots,Z_n$ be independently and identically distributed (i.i.d.) random variables in a measurable space $(\mathcal{Z},\mathcal{F})$ with a generating measure $P_Z$. Let $\Theta$ be a Banach space equipped with a norm $\| \cdot \|$, and $f_\theta: \mathcal{Z} \to \mathbb{R}$ be a measurable criterion function for $\theta \in \Theta$. The M-estimator $\hat{\theta}$ maximizes the empirical criterion

$$Q_n(\theta) := \frac{1}{n} \sum_{i=1}^{n} f_\theta(Z_i),$$

over the parameter space $\Theta$. We are interested in approximating the distribution of $\hat{\theta}$, i.e., $P(\hat{\theta} \in A)$ for any Borel subset $A \subset \Theta$. In particular, we establish a Gaussian approximation result for the distribution of the M-estimator:

$$\sup_A |P(\hat{\theta} \in A) - P(\hat{\theta}_G \in A)| = O(r_n),$$

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for some positive sequence \( r_n \) decaying to zero, where \( \hat{\theta}_G \) maximizes a Gaussian process \( G(\theta) \) with the same mean and covariance of \( Q_n(\theta) \).

The approximation theory for distributions of M-estimators is one of central topics in mathematical statistics. Its general theory is summarized in [38, 37, 20] for example. Also, there are numerous specific applications [39, 29, 28, 23, 16, 3, 18] and generalizations [19, 34, 33, 7, 21], among many others. In the statistics literature, it is common to employ limiting approximations. In a regular setup where the empirical criterion is sufficiently smooth, the M-estimator is typically asymptotically normal with the \( \sqrt{n} \) convergence rate. Its properties as a distributional approximation is well investigated [24, 25]. For certain non-regular setups or models, the M-estimators yield more intricate limiting distributions, such as the Chernoff distribution [8]. More generically, the argmax continuous mapping theorem gives limiting distributions constructed by Gaussian processes [14, 31]. For more general parameter spaces, Donsker’s theorem takes Brownian bridges as limiting distributions of empirical processes [13]. These asymptotic approximation results have been widely used in statistics to construct inference methods for the parameters.

In contrast to the existing approaches, this study develops a non-asymptotic approximation theory for the distributions of M-estimators, which does not require knowledge of the limiting distribution. Specifically, we derive a tractable random variable \( \hat{\theta}_G \) whose distribution is sufficiently close to that of the target M-estimator \( \hat{\theta} \) for each \( n \). The random variable \( \hat{\theta}_G \) used in the approximation consists of the maximizer of a Gaussian process on the parameter space.

Our theoretical results are summarized as follows. First, we develop an abstract non-asymptotic approximation theorem for the distributions of M-estimators (Theorem 1). We present sufficient conditions for the main theorem and also provide a convergence analysis for M-estimators (Corollaries 1 and 2). Second, we develop a useful multiplier bootstrap method based on the non-asymptotic approximation theorem and show its validity (Theorem 2). A hypothesis testing method based on the bootstrap theorem is also discussed (Proposition 2). Third, we extend our approximation theory to semiparametric M-estimators with nuisance parameters (Theorem 3), as well as M-estimators with non-Donsker parameter spaces (Theorem 4). For the non-Donsker case, we provide a coupling inequality with an additional smoothness assumption. Finally, we illustrate usefulness of our non-asymptotic approximation theory by three examples: a cubic root estimator, least absolute deviation, and minimum volume prediction.

Compared to the conventional asymptotic approach, there are at least three attractive features of our non-asymptotic approximation approach. First, it may work under weaker conditions than those for deriving the limit distributions. For regular cases, it is often the case that the criterion functions are sufficiently smooth. For some non-regular cases, it is not trivial to derive the limiting distributions of the M-estimators, such as criteria defined by non-Donsker classes. Also, the naive nonparametric bootstrap is known to be asymptotically invalid [1, 22, 32]. In contrast, our non-asymptotic approach does not require differentiability of the criterion functions, even for their expectations. Moreover, our non-asymptotic approximation and associated bootstrap theory may
work even for such non-regular cases. Second, based on the non-asymptotic approximation, our bootstrap method is computationally attractive in some non-regular cases. For example, the Chernoff distribution is given by the maxima of a Gaussian process, which requires a lot of samplings to calculate their quantiles, as noted by [17]. Our method can avoid such difficulty by the Gaussian multiplier bootstrap. Third, our method is applicable regardless of whether the limiting distribution is available, e.g., the distribution may not be available for a large-scale parameter space, such as a non-Donsker class. Even in such scenarios, our method can approximate the distribution of the M-estimator.

Our theoretical developments build upon the seminal works by Chernozhukov, Chetverikov and Kato [9, 12, 10, 11], which develop the approximation theory for distributions of suprema of empirical processes. We extend their various techniques to be adapted to investigate argmax values as well. In particular, our own technical contributions are summarized as follows. First, we provide a representation of the distribution of the M-estimator based on a transformation of the distribution of suprema of empirical processes. This representation allows us to apply the Gaussian approximation theory from [11] with a smooth transformation of Gaussian variables. In this process, we also derive a new Stein-type equality to relax existing conditions on moments. Second, we develop a conditional anti-concentration inequality for Gaussian variables that prevents degeneracy of the argmax distribution. This is an extension of the anti-concentration inequality used to study the Gaussian approximation. Since our approximation of argmax uses a difference between two Gaussian processes, we need to develop a conditional version of the conventional anti-concentration inequality. Third, we develop an eigenvalue inequality to ensure that several specific M-estimators satisfy the conditions of our theory. Since our non-asymptotic theory employs relatively different regularity conditions from the conventional asymptotic theories, we provide a detailed evaluation for eigenvalues of certain covariance matrices based on the criterion function.

1.1. Organization. In Section 2, we present an abstract approximation theorem and its application to general M-estimators. In Section 3, we develop a multiplier bootstrap method and hypothesis testing procedure, and show their validity. Section 4 extends our approximation theory to semiparametric M-estimators with nuisance parameters. Section 5 studies the M-estimator whose parameter space is non-Donsker. In Section 6, we illustrate our approximation theory by some examples. Section 7 concludes. Technical proofs are contained in the supplementary material.

1.2. Notation. Let $|M| := \{1, 2, \ldots, M\}$ for $M \in \mathbb{N}$. For a vector $x \in \mathbb{R}^M$, $x_m$ denotes the $m$-th element of $x$ for $m \in [M]$. For a function $f : \Omega \to \mathbb{R}$, $\|f\|_{L^q(P)} := \left(\int_{\Omega} f(x)^q dP(x)\right)^{1/q}$ is the $L^q$-norm with $q \in (1, \infty)$ and a base measure $P$ on $\Omega$. If $P$ is the Lebesgue measure, we just write it as $\|f\|_{L^q}$. $\|f\|_{L^\infty} := \sup_{x \in \Omega} |f(x)|$ is the sup-norm. $\log^c x$ denotes $(\log x)^c$. For two sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$, $a_n \gtrsim b_n$ denotes that there exists a constant $c > 0$ such that $a_n \geq cb_n$ for all $n \geq \pi$ with some finite $\pi \in \mathbb{N}$. $a_n \lessapprox b_n$ denotes its opposite. $a_n \asymp b_n$ denotes both $a_n \gtrsim b_n$ and $a_n \lessapprox b_n$ hold. For an event $E$, $1\{E\}$ denotes the indicator function which is 1 if $E$ is true, and 0 otherwise. For a set
Ω equipped with a distance \( d \) and radius \( \varepsilon > 0 \), \( \mathcal{N}(\varepsilon, \Omega, d) := \inf \{ N \mid \{ \omega_j \}_{j=1}^N \subset \Omega \text{ s.t. } \bigcup_{j=1}^N \{ \omega : d(\omega, \omega_j) \leq \varepsilon \} \supset \Omega \} \) is an \((\varepsilon)\)-covering number, which is the minimum number of \( \varepsilon \)-balls covering \( \Omega \). Similarly, \( \mathcal{D}(\varepsilon, \Omega, d) := \sup \{ N \mid \{ \omega_j \}_{j=1}^N \subset \Omega \text{ s.t. } d(\omega, \omega_j) \geq \varepsilon \} \) is an \((\varepsilon)\)-packing number, which is the maximum number of \( \varepsilon \)-separated points. We refer the set \( \{ \omega_j \}_{j=1}^N \) as an \((\varepsilon)\)-packing set of \( \Omega \) in \( d \).

## 2. Gaussian Approximation for M-estimator

In this section we develop a Gaussian approximation for the distribution of the M-estimator. With i.i.d. observations \( Z_1, \ldots, Z_n \) from \( P_Z \) on \( \mathcal{Z} \) and a criterion function \( f_\theta : \mathcal{Z} \to \mathbb{R} \) for \( \theta \in \Theta \), we consider the empirical criterion \( Q_n(\theta) \) as in (1) and the M-estimator \( \hat{\theta} := \arg \max_{\theta \in \Theta} Q_n(\theta) \). If there are multiple maxima of \( Q_n(\theta) \), we arbitrarily pick one of them.

The aim of this study is to approximate the distribution of \( \hat{\theta} \) for each \( n \). To this end, we use a maximizer of a Gaussian process as an approximator. In particular, we consider a tight Gaussian process \( \{ G(\theta) \}_{\theta \in \Theta} \) with its mean \( \mathbb{E}_G[G(\theta)] = \mathbb{E}_Z[f_\theta(Z)] \) and covariance function \( \text{Cov}_G(G(\theta), G(\theta')) = \text{Cov}_Z(f_\theta(Z), f_{\theta'}(Z)) \) for \( \theta, \theta' \in \Theta \). Existence of this Gaussian process will be shown under the conditions below. In this section, we establish an approximation result for the distribution of \( \hat{\theta} \) by that of

\[
\hat{\theta}_G := \arg \max_{\theta \in \Theta} G(\theta). \tag{2}
\]

By Theorem 1 in [4], a maximizer of a Gaussian process on a separable space is known to be unique almost surely.

### 2.1. Basic Assumption.

Letting

\[
\tilde{\mathcal{M}}_k := \mathbb{E}_Z \left[ \sup_{\theta \in \Theta} |f_\theta(Z) - \mathbb{E}_Z[f_\theta(Z)]|^k \right],
\]

for \( k \in \mathbb{N} \), we impose the following basic conditions.

**Assumption 1 (Basic).** The following conditions hold:

(A1) \( \Theta \) is separable and pointwise measurable; there exists a countable set \( \widetilde{\Theta} \subset \Theta \) such that for any \( \theta \in \Theta \) there exists a sequence \( \{ \tilde{\theta}_i \}_{i \in \mathbb{N}} \subset \widetilde{\Theta} \) satisfying \( \tilde{\theta}_i \to \theta \) as \( i \to \infty \).

(A2) For any \( \theta, \theta' \in \Theta \), it holds \( \mathbb{E}_Z[|f_\theta(Z) - f_{\theta'}(Z)|] \leq c_\ell \| \theta - \theta' \|^q \) with a constant \( c_\ell > 0 \) and \( q \in (0, 1] \).

(A3) There exist constants \( C > 0 \) and \( \alpha \in (0, 2) \) such that

\[
\log \mathcal{N}(\varepsilon, \Theta, \| \cdot \|) \leq C e^{-\alpha},
\]

holds for all \( \varepsilon \in (0, 1) \).

(A4) There exists an envelope function \( F : \mathcal{Z} \to \mathbb{R} \) such that \( F(z) \geq \sup_{\theta \in \Theta} |f_\theta(z)| \) for any \( z \in \mathcal{Z} \), and \( \| F \|_{L^2(P_Z)} \vee \| F \|_{L^\infty(P_Z)} \leq b < \infty \) for some \( b > 0 \), and \( \mathcal{M}_3 \) is finite.
The condition (A1) deals with the nonmeasurability of suprema of uncountable sets (see p.108 in [38]). Based on this assumption, we can characterize $\sup_{\theta \in \Theta} Q_n(\theta)$ by using $\sup_{\theta \in \tilde{\Theta}} Q_n(\theta)$, which is a measurable map from $\mathcal{Z}$ to $\mathbb{R}$. The measurability of the supremum will be used to handle the maximizer of $Q_n(\theta)$.

The condition (A2) is the Lipschitz (or Hölder) continuity of $E_{\mathcal{Z}}[|f_\theta(Z)|]$ in $\theta$. Since the continuity is imposed on the expected value, this condition may hold even when $f_\theta(\cdot)$ is discontinuous in $\theta$. In addition, our condition does not require differentiability in $\theta$, only the continuity. Hence, it is a relatively mild condition in the literature for M-estimators (see [33, 21] for recent examples), which typically requires (twice) differentiability of the expected criterion function. See our illustrations for the least absolute deviation in Section 6.

The condition (A3) is on the metric entropy of $\Theta$. Similar conditions are frequently used in the studies that used the empirical process theory (see [38] for an overview). This condition is satisfied for a compact finite-dimensional parameter space and an infinite-dimensional functional space with sufficient smoothness. In our analysis, the condition (A3) is employed to guarantee the existence of the Gaussian process $G(\theta)$, which is shown by Lemma 8 (see also Lemma 2.2 in [11]) using the notion of pre-Gaussianity in [38]. We will relax this condition in Section 5.

The condition (A4) is on integrability of the envelope function $F$ and the third moment of $f_\theta(Z)$. This moment condition is utilized to apply approximation techniques using Stein’s identity.

2.2. Covariance Assumption. We additionally impose an assumption on positive definiteness of the covariance function $\text{Cov}_Z(f_\theta(Z), f_\theta'(Z))$. For a finite parameter subset $T \subset \Theta$, consider a matrix $\Sigma = \text{Var}_Z((f_\theta(Z))_{\theta \in T}) \in \mathbb{R}^{|T| \times |T|}$ whose $(i, j)$-th element is $\text{Cov}_Z(f_{\theta_i}(Z), f_{\theta_j}(Z))$ with $\theta_i, \theta_j \in T$. We introduce the following terminology.

**Definition 1** (Coherently positive definite covariance). A $|T| \times |T|$ covariance matrix $\Sigma$ is coherently positive definite with $\sigma^2$, if for any $A \subset T$ with $A^c = T \setminus A$ and $\Sigma = \begin{bmatrix} \Sigma_A & \Sigma_{AA^c} \\ \Sigma_{A^cA} & \Sigma_{A^c} \end{bmatrix}$, the diagonal elements of the matrix $\Sigma_A - \Sigma_{AA^c} \Sigma_{A^c}^{-1} \Sigma_{A^cA}$ are no less than $\sigma^2$.

We impose the following assumption on coherent positive definiteness of covariance matrices induced by $f_\theta(Z)$.

**Assumption 2** (Coherent covariance function). For any $\delta \in (0, 1]$, there exists a $\delta$-packing set $T$ of $\Theta$ and constant $c > 0$ such that a covariance matrix $\Sigma = \text{Var}_Z((f_\theta(Z))_{\theta \in T})$ is coherently positive definite with $\sigma^2 \geq c\delta^2\kappa$ for some $\kappa \in (0, 1)$.

This condition can be considered as strengthening of positive definiteness of the covariance function, which avoids degeneracy of the covariance matrices. Based on this condition, we can guarantee that the maximum value of the approximating Gaussian process does not degenerate to a single point [11, 10], and thus its maximizer has sufficient variations. In Section 6, we provide some examples that satisfy this assumption.
To understand the nature of this condition, we provide a sufficient condition for coherent positive definiteness. Let $\lambda_{\min}(\Sigma)$ be the smallest eigenvalue of a matrix $\Sigma$.

**Proposition 1** (Coherent eigenvalue condition). If a symmetric positive definite matrix $\Sigma$ satisfies $\lambda_{\min}(\Sigma) \geq \sigma^2$, then $\Sigma$ is coherently positive definite with $\sigma^2$.

*Proof of Proposition 1.* It follows by Corollary 2.3 (page 50) in [40].

This result suggests that Assumption 2 is closely related to minimum eigenvalues of covariance matrices.

2.3. **Main Theorem.** We now present our main result of this paper, a Gaussian approximation for the distribution of the M-estimator. Let $H(\varepsilon) := \log \mathcal{N}(\varepsilon, \Theta, \| \cdot \|)$, and $J(\varepsilon) := \int_0^\varepsilon \sqrt{1 + H(\delta)} d\delta$, for $\varepsilon \in (0, 1]$ be the metric entropy and its integration. Our abstract approximation theorem is presented as follows.

**Theorem 1** (Gaussian approximation). Suppose that Assumptions 1 and 2 are satisfied. Then, for any Borel subset $A \subset \Theta$, $\varepsilon \in (0, 1]$, and $n \geq n_0$ with an existing $n_0 \in \mathbb{N}$, we have

$$\left| P_Z(\hat{\theta} \in A) - P_G(\hat{\theta}_G \in A) \right| \leq C \left\{ \varepsilon + \frac{H(\varepsilon)}{n^{5/8}} + \frac{\Delta(n, \varepsilon)}{\varepsilon^\kappa} \right\},$$

where $C > 0$ is a constant depending only on $M_1, M_2, \lambda_3, c_\ell, q$, and $b$, and $\Delta(n, \varepsilon)$ is defined as

$$\Delta(n, \varepsilon) = \varepsilon + \frac{J(\varepsilon) + \varepsilon \sqrt{\log(1/\varepsilon)}}{n^{1/2}} + \frac{\sqrt{\log(1/\varepsilon)} + \log(1/\varepsilon)}{n}.$$

This theorem characterizes the approximation error for the distribution on the M-estimator $\hat{\theta}$ under the generating distribution $P_Z$ by using its Gaussian counterpart $\hat{\theta}_G$ under $P_G$. Applications of this theorem for common finite- and infinite-dimensional parameter spaces are presented as follows.

**Corollary 1** (Finite-dimensional M-estimator). Let $\Theta$ be a compact and convex subset of $\mathbb{R}^d$ and $\| \cdot \|$ be the Euclidean norm. Suppose Assumptions 1 and 2 hold. Then

$$\left| P_Z(\hat{\theta} \in A) - P_G(\hat{\theta}_G \in A) \right| = O\left( n^{-5/8} \log n \right).$$

*Proof of Corollary 1.* By the setting of $\Theta \subset \mathbb{R}^d$ and Lemma 7, we have

$$H(\varepsilon) \lesssim \log C_\Theta + \log(1/\varepsilon^d), \quad J(\varepsilon) \lesssim \varepsilon \sqrt{\log(1/\varepsilon)},$$

for $\varepsilon \in (0, 1)$. Combining these bounds with Theorem 1 we obtain

$$\left| P_Z(\hat{\theta} \in A) - P_G(\hat{\theta}_G \in A) \right| \lesssim \varepsilon + \frac{\log(1/\varepsilon)}{n^{5/8}} + \varepsilon^{1-\kappa} \sqrt{\log(1/\varepsilon)}.$$

By setting $\varepsilon = n^{-5/(8-8\kappa)}$, the conclusion follows.
Corollary 2 (Infinite-dimensional M-estimator). Suppose Assumptions \(^1\) and \(^2\) hold with \(\alpha / 2 + \kappa < 1\). Then
\[
\left| \mathbb{P}_Z\left( \hat{\theta} \in A \right) - \mathbb{P}_G\left( \hat{\theta}_G \in A \right) \right| = O\left( n^{-5/8} \sqrt{n^{-\left(1-\alpha/2-\kappa\right)/\alpha}} \sqrt{n^{-\left(5-5\kappa-\alpha\right)/(8-8\kappa)}} \right).
\]

Proof of Corollary 2. By the condition (A3) in Assumption \(^1\) and Lemma \(^7\) we have \(J(\varepsilon) \lesssim \varepsilon^{1-\alpha/2}\). Combining this with Theorem \(^1\) yields
\[
\left| \mathbb{P}_Z\left( \hat{\theta} \in A \right) - \mathbb{P}_G\left( \hat{\theta}_G \in A \right) \right| \lesssim \varepsilon^{1-\alpha/2} + \varepsilon^{-\alpha} + \varepsilon^{1-\alpha-\kappa}.
\]
By setting \(\varepsilon = n^{-5/(8+4\alpha-8\kappa)} \sqrt{n^{-\alpha}} \sqrt{n^{-\left(5-5\kappa-\alpha\right)/(8-8\kappa)}}\), the conclusion follows.

2.4. Sketch of Proof for Theorem \(^1\) The proof of Theorem \(^1\) consists of three main steps: (i) discretization of \(\Theta\), (ii) smoothing and the Stein approximation, and (iii) conditional anti-concentration. In this subsection, we provide a rough overview. The rigorous full proof will be provided in the Appendix.

(i) Discretization of \(\Theta\): We consider a finite subset \(\Theta_M \subset \Theta\) with \(|\Theta_M| = M\) and approximate the distribution of \(\hat{\theta} = \arg\max_{\theta \in \Theta} Q_n(\theta)\) by that of \(\arg\max_{\theta \in \Theta_M} Q_n(\theta)\). Fix \(A \subset \Theta\) and define its complement \(A^c = \Theta \setminus A\). Let \(\Theta_M \subset \Theta\) be the finite subset such that \(\Theta_M\) is an \(\varepsilon\)-cover of \(\Theta\), that is, for any \(\theta \in \Theta\) there exists \(\theta' \in \Theta_M\) such that \(|\theta - \theta'| \leq \varepsilon\). We also define their discretized analogs \(A_M := A \cap \Theta_M\) and \(A_M^c := A^c \cap \Theta_M\). When \(A_M\) and \(A_M^c\) are non-empty, we obtain
\[
\mathbb{P}_Z\left( \hat{\theta} \in A \right) \approx \mathbb{P}_Z\left( \max_{\theta \in A_M} Q_n(\theta) - \max_{\theta \in A_M^c} Q_n(\theta) + n^{-1/2} \Delta(n, \varepsilon) \geq 0 \right) =: P_1.
\]
To obtain this approximation, we evaluate the effect of the discretization for the parameter space \(\sup_{\theta, \theta'}: \|\theta - \theta'\| \leq \varepsilon \{Q_n(\theta) - Q_n(\theta')\}\) by the empirical process technique (Lemma 2.2 in \([11]\)). Similarly, we have
\[
\mathbb{P}_G\left( \hat{\theta}_G \in A \right) = \mathbb{P}_G\left( \max_{\theta \in A} G(\theta) - \max_{\theta \in A^c} G(\theta) \geq 0 \right) \approx \mathbb{P}_G\left( \max_{\theta \in A_M} G(\theta) - \max_{\theta \in A_M^c} G(\theta) + n^{-1/2} \Delta(n, \varepsilon) \geq 0 \right) =: P_2.
\]

(ii) Smoothing and the Stein approximation: In this step, evaluate the difference between the probabilities \(P_1\) and \(P_2\). The key technique is the approximation by Stein’s identity \([11, 9, 15]\). That is, for a smooth function \(h \in C^3(\mathbb{R}^M)\) and i.i.d. random vectors \(X_1, \ldots, X_n \in \mathbb{R}^M\) with \(\bar{X} := n^{-1/2} \sum_{i=1}^n X_i\), we can obtain \(\mathbb{E}_X[h(\bar{X})] \approx \mathbb{E}_W[h(W)]\), where \(W\) is an \(M\)-dimensional Gaussian random vector such that \(\mathbb{E}_W[W] = \mathbb{E}_X[\bar{X}]\) and \(\text{Var}_W(W) = \text{Var}_X(\bar{X})\). We can rewrite the probability \(P_1\) as
\[
P_1 = \mathbb{E}_Z \left\{ 1 \left[ \max_{\theta \in A_M} Q_n(\theta) - \max_{\theta \in A_M^c} Q_n(\theta) + n^{-1/2} \Delta(n, \varepsilon) \geq 0 \right] \right\},
\]
and $P_2$ with a similar form. Here, owing to the discretization step, $h_1$ is a function of $n^{-1/2} \sum_{i=[n]} X_i$ with $X_i = (f_{\theta_m}(Z_i))_{m \in [M]}$ and the same part for $P_2$ is a function of a sum of the Gaussian vector $W_i = (G_{\theta_m})_{m \in [M]}$. Thus, the approximation technique may show $P_1 \approx P_2$.

To apply the technique, we approximate the part $h_1$ by a smooth function of $\bar{X}$. By using the softmax approximation and soft-step function by [9], we can show that

$$|P_1 - P_2| = O\left( \frac{\bar{M}^3 H(n, \varepsilon)}{n^{5/8}} \right).$$

In this step, we develop the local coupling technique (Lemma 3) and reduce some restrictions such as truncation or higher moments in [9].

(iii) Conditional anti-concentration: To finish the proof, we move the term $n^{-1/2} \Delta(n, \varepsilon)$ from the inside of $P_2$ to its outside, that is, we show

$$P_2 \approx \mathbb{P}_G \left( \max_{\theta \in \Lambda_M} G(\theta) - \max_{\theta \in \Lambda_M} G(\theta) \geq 0 \right) + O\left( \frac{1}{n^{1/2} \sigma} \Delta(n, \varepsilon) \right),$$

where $\sigma$ is a lower bound of covariance appearing in Assumption 2. To this aim, the anti-concentration inequality [10, 9] is often utilized to show non-degeneracy of the maximum of the Gaussian vectors. However, the existing inequality does not work in this setting because of the multiple maxima. Therefore we develop the conditional anti-concentration inequality (Lemma 5) to adapt to our setting.

By combining these steps, we get the result in Theorem 1. □

3. Bootstrap Approximation

In this section we discuss a multiplier bootstrap method to approximate the distribution of the Gaussian counterpart $\hat{\theta}_G$. Since computing maximizers of Gaussian processes is costly (as discussed in [17]), the bootstrap approach provides a practical alternative to approximate the distribution of the M-estimator.

Let $e_i \sim \mathcal{N}(0, 1)$ for $i \in [n]$ be standard normal multipliers which are independent of $Z_1, \ldots, Z_n$, and $P_n(\theta) := n^{-1} \sum_{i=1}^n f_\theta(Z_i)$ be the empirical criterion function. Define the multiplier bootstrap process as

$$B_n(\theta) := P_n(\theta) + E_n(\theta),$$

where

$$E_n(\theta) := \frac{1}{n} \sum_{i=1}^n e_i(f_\theta(Z_i) - P_n(\theta)).$$

Then our bootstrap counterpart for $\hat{\theta}$ is defined as

$$\hat{\theta}_B := \arg\max_{\theta \in \Theta} B_n(\theta).$$
Let \( P_{e|Z} \) be the conditional distribution of \( e_1, \ldots, e_n \) given \( Z_1, \ldots, Z_n \). We obtain the following result for the bootstrap approximation.

**Theorem 2** (Bootstrap approximation). Let \( \Theta \) be a compact and convex subset of \( \mathbb{R}^d \) and \( \| \cdot \| \) be the Euclidean norm. Suppose that Assumptions 1 and 2 are satisfied. Then, for any Borel subset \( A \subset \Theta, \varepsilon \in (0, 1] \), and \( n \geq n_0 \) with an existing \( n_0 \in \mathbb{N} \), we have

\[
\left| P_{e|Z} \left( \hat{\theta}_B \in A \right) - P_G \left( \hat{\theta}_G \in A \right) \right| \leq C \left\{ \varepsilon + \frac{H(\varepsilon)}{n^{5/8}} + \frac{\Delta(n, \varepsilon)}{\varepsilon^\kappa} (\sqrt{H(\varepsilon)} + 1) \right\},
\]

where \( C > 0 \) is a constant depending only on \( \sigma, \bar{M}_1, \bar{M}_3, c, \) and \( b \).

This theorem characterizes the approximation error for the distribution of the Gaussian counterpart \( \hat{\theta}_G \) under \( P_G \) by the Gaussian multiplier bootstrap distribution. Owing to this result, we can give analogous corollaries for the finite- and infinite-dimensional parameter spaces.

**Corollary 3** (Bootstrap for finite-dimensional \( \Theta \)). Let \( \Theta \) be a compact and convex subset of \( \mathbb{R}^d \) and \( \| \cdot \| \) be the Euclidean norm. Suppose Assumptions 1 and 2, and the setting in Corollary 1 hold. Then

\[
\left| P_{e|Z} \left( \hat{\theta}_B \in A \right) - P_G \left( \hat{\theta}_G \in A \right) \right| = O \left( n^{-5/8} \log n \right).
\]

**Corollary 4** (Bootstrap for infinite-dimensional \( \Theta \)). Suppose Assumptions 1 and 2 with \( \alpha/2 + \kappa < 1 \), and the setting in Corollary 2 hold. Then

\[
\left| P_{e|Z} \left( \hat{\theta}_B \in A \right) - P_G \left( \hat{\theta}_G \in A \right) \right| = O \left( n^{-5/8} \vee n^{-(1-\alpha/2-\kappa)/\alpha} \vee n^{-(5-5\kappa-\alpha)/(8-8\kappa)} \right).
\]

The proofs of these corollaries are omitted because they are almost same as those of Corollaries 1 and 2.

### 3.1. Illustration: Hypothesis Testing

Theorems 1 and 2 can be used to conduct hypothesis testing on the true parameter value defined as

\[
\theta_0 := \arg \max_{\theta \in \Theta} \mathbb{E}_Z [f_\theta(Z)].
\]

We assume that \( \theta_0 \) is unique, and consider the testing problem for the null hypothesis \( H_0 : \theta_0 = \theta^* \) against the alternative \( H_1 : \theta_0 \neq \theta^* \), where \( \theta^* \in \Theta \) is a hypothetical value. Suppose \( \Theta \) contains the origin \{0\}.

Here we present a testing procedure by sample splitting. We split the observations \{\( Z_1, \ldots, Z_n \)\} into disjoint sets \( Z' \) and \( Z'' \). Pick a significance level \( s \in (0, 1) \). Based on \( Z' \), we compute \( \hat{\theta}_B \) repeatedly and find a set \( \bar{A}_s \) such that

\[
P_{e|Z'} \left( \hat{\theta}_B \in \bar{A}_s \right) = 1 - s.
\]
The set $\bar{A}_s$ satisfying this property is not unique, but a reasonable choice is the one with the minimum volume. Then by using the other observations $Z''$, we compute $\hat{\theta}$ and construct an acceptance region for $H_0$ as

$$\hat{A}_s^* := \{ \theta^* - \hat{\theta} + \theta_s \mid \theta_s \in \bar{A}_s \}.$$ 

The validity of this acceptance region is established as follows.

**Proposition 2** (Hypothesis testing). *Suppose that the setting in Theorem 2 and $\hat{A}_s^* \subset \Theta$ hold. Then under the null hypothesis $H_0 : \theta_0 = \theta^*$ and for any $s \in (0, 1)$, it holds

$$\mathbb{P}(Z \left( \theta_0 \in \hat{A}_s^* \right) \rightarrow 1 - s, \text{ as } n \rightarrow \infty.$$*

The proof is contained in Appendix. By considering different hypothetical values for $\theta^*$, we can also construct a confidence set for $\theta_0$.

4. **M-estimator with Nuisance Parameters**

Our approximation theory can be extended to the case where the criterion function contains nuisance parameters. This extension is particularly useful to allow for semiparametric models. Let $\theta \in \Theta$ be our parameters of interest and $\eta \in \mathcal{H}$ be nuisance parameters. Let $\Theta \times \mathcal{H}$ be a joint parameter space with a norm $\| \cdot \|_{\Theta \times \mathcal{H}} := \| \cdot \|_\Theta + \| \cdot \|_\mathcal{H}$. For $(\theta, \eta) \in \Theta \times \mathcal{H}$, let $f_{\theta, \eta} : \mathcal{Z} \rightarrow \mathbb{R}$ be a criterion function and define its empirical mean

$$S_n(\theta, \eta) := \frac{1}{n} \sum_{i=1}^{n} f_{\theta, \eta}(Z_i).$$

Then we consider the semiparametric M-estimator

$$\hat{\theta}_S := \arg\max_{\theta \in \Theta} \max_{\eta \in \mathcal{H}} S_n(\theta, \eta),$$

where $\eta$ is profiled out. To approximate the distribution of $\hat{\theta}_S$, we introduce a tight Gaussian process $\{ \Gamma(\theta, \eta) \mid (\theta, \eta) \in \Theta \times \mathcal{H} \}$ with its mean $\mathbb{E}_\Gamma[\Gamma(\theta, \eta)] = \mathbb{E}_Z[f_{\theta, \eta}(Z)]$ and covariance function $\text{Cov}_\Gamma(\Gamma(\theta, \eta), \Gamma(\theta', \eta')) = \text{Cov}_Z(f_{\theta, \eta}(Z), f_{\theta', \eta'}(Z))$ for all $(\theta, \eta), (\theta', \eta') \in \Theta \times \mathcal{H}$. Its existence is shown in Lemma 8 in Appendix. The Gaussian counterpart of $\hat{\theta}_S$ is defined as

$$\hat{\theta}_\Gamma := \arg\max_{\theta \in \Theta} \max_{\eta \in \mathcal{H}} \Gamma(\theta, \eta).$$

We note that this setup covers semiparametric M-estimators, where $\Theta$ is finite-dimensional and $\mathcal{H}$ is infinite-dimensional.

We now give some assumptions to characterize approximation errors for the distribution of $\hat{\theta}_S$ by that of $\hat{\theta}_\Gamma$. Most assumptions are analogous to the ones for the no nuisance parameter case in Section 2. Define

$$\tilde{\mathcal{M}}_{S, k} := \mathbb{E}_Z \left[ \sup_{(\theta, \eta) \in \Theta \times \mathcal{H}} |f_{\theta, \eta}(Z) - \mathbb{E}_Z[f_{\theta, \eta}(Z)]|^k \right].$$
We impose the following assumptions.

**Assumption 3** (Basic for nuisance case). The following conditions hold:

(A1) $\Theta \times \mathcal{H}$ is separable and pointwise measurable.
(A2) For any $(\theta, \eta), (\theta', \eta') \in \Theta \times \mathcal{H}$, $\|f_{\theta, \eta} - f_{\theta', \eta'}\|_{L^2(P)} \leq c_\ell^q \|\theta - \theta'\|_{\Theta \times \mathcal{H}}^q$ holds with constants $c_\ell > 0$ and $q \in (0, 1]$.
(A3) There exist constants $C > 0$ and $\alpha \in (0, 2)$ such that
$$\log \mathcal{N}(\mathcal{E}, \Theta \times \mathcal{H}, \|\cdot\|_{\Theta \times \mathcal{H}}) \leq C e^{-\alpha},$$
holds for all $\mathcal{E} \in (0, 1)$.
(A4) There exists an envelope function $F_S : \mathcal{X} \rightarrow \mathbb{R}$ such that $F_S(z) \geq \sup_{(\theta, \eta) \in \Theta \times \mathcal{H}} |f_{\theta, \eta}(z)|$ for any $z \in \mathcal{X}$ and $\|F_S\|_{L^2(P)} \vee \|F_S\|_{L^\infty(P)} \leq b < \infty$ with a constant $b > 0$, and $\mathcal{M}_{S, 3}$ is finite.

**Assumption 4** (Coherent covariance function for nuisance case). For any $\delta > 0$, there exists a $\delta$-packing set $T$ of $\Theta \times \mathcal{H}$ in $\|\cdot\|_{\Theta \times \mathcal{H}}$ and a constant $c > 0$ such that a covariance matrix $\Sigma_A = \text{Var}_Z((f_{\theta, \eta}(Z))_{(\theta, \eta) \in T})$ is coherently positive definite with $\sigma^2 \geq c \delta^{2\kappa}$ for some $\kappa \in (0, 1)$.

Similar comments to Assumptions 1 and 2 in Section 2 apply. Our assumptions cover semiparametric settings, where either $\theta$ or $\eta$ is infinite-dimensional. The case where both are finite-dimensional is covered as a special case. A Gaussian approximation for the semiparametric M-estimator $\hat{\theta}_S$ is established as follows.

**Theorem 3** (Semiparametric M-estimator). Suppose that Assumptions 3 and 4 are satisfied. Then
$$\left| \mathbb{P}_Z(\hat{\theta}_S \in A) - \mathbb{P}_T(\hat{\theta}_T \in A) \right| = O\left( n^{-5/8} \vee n^{-(1-\alpha/2-\kappa)/\alpha} \vee n^{-1/(5-5\kappa-\alpha)/(8-8\kappa)} \right).$$

This result holds for a broad class of semiparametric estimators. For the semiparametric estimator, the Gaussian multiplier bootstrap also provides an analogous approximation result as in Theorem 2. Since the configuration of the proof is nearly identical, we omit it here.

5. **M-estimator with Non-Donsker Class**

Additional interest is to conduct a distributional approximation to the M-estimator when $\Theta$ is not a Donsker class, e.g., a square root of the metric entropy of $\Theta$ is not integrable. In other words, the condition (A3) in Assumption 1 can be violated. One typical example of the non-Donsker class is the set of bounded Lipschitz functions on a $d$-dimensional space, $\mathcal{B}L([0, 1]^d) := \{ f : [0, 1]^d \rightarrow \mathbb{R} \mid \|f\|_{L^\infty} \leq C, |f(x) - f(x')| \leq C \|x - x'\|_2, \forall x, x' \in [0, 1]^d, x \neq x' \}$ with a finite constant $C > 0$. When the input dimension $d$ is no less than 2, its covering number is $\mathcal{N}(\mathcal{E}, \mathcal{B}L([0, 1]^d), \|\cdot\|_{L^2}) = O(\mathcal{E}^{-d})$ (see Section 8 in [13]), so the integral of the covering number diverges and hence Theorem 1 is not applicable.

To handle the non-Donsker parameter space, we consider a sieve approximation. For $K \in \mathbb{N}$, we consider a sieve space $\Theta_K \subset \Theta_{K+1} \subset \cdots \subset \Theta$. A typical example of the sieve space is a (closure
of) linear hull of orthogonal bases, i.e., \( \Theta_K := \{ \theta = \sum_{j=1}^{K} w_j \phi_j | w_j \in \mathbb{R}, j \in [K] \} \) with the basis function \( \phi_j \) such as the trigonometric basis. For the bounded Lipschitz case, we can find \( \Theta_K \) by splines such that \( \sup_{f \in BL([0,1]^d)} \inf_{\theta' \in \Theta_K} \| f - \theta' \|_{L^2} = O(K^{-1/d}) \) (for an overview, see Section 13 in [30]).

We define the argmax of the Gaussian process over the sieve space as

\[
\hat{\theta}_K = \arg\max_{\theta \in \Theta_K} G(\theta).
\]

For this case, we impose the following assumption on the sieve space.

**Assumption 5** (Metric entropy of sieve space). \( \Theta_K \) is a Banach space and there exists \( C > 0 \) which satisfies

\[
\log \mathcal{N}(\epsilon, \Theta_K, \| \cdot \|) \leq C \log(1/\epsilon^K),
\]

for any \( \epsilon \in (0, 1) \) and \( K \in \mathbb{N} \). Further, there exist constants \( c_\lambda, \lambda > 0 \) which satisfy

\[
\tilde{\Delta}(K) := \sup_{\theta \in \Theta} \inf_{\theta' \in \Theta_K} \| f_\theta - f_{\theta'} \|_{L^1(P_n)} \leq c_\lambda K^{-\lambda},
\]

for any \( K \in \mathbb{N} \), where \( P_n = n^{-1} \sum_{i=1}^{n} \delta_{Z_i} \) is the empirical measure.

The first condition naturally holds with the typical sieve space \( \Theta_K = \{ \theta = \sum_{j=1}^{K} w_j \phi_j \} \) since each \( \theta \in \Theta_K \) is characterized by \( K \)-dimensional parameters. The second condition is on an order \( \alpha \) and \( \phi_j \) is a trigonometric basis, then the second condition holds true with \( \lambda = \alpha/d \).

We additionally introduce a smoothness assumption on the distribution of argmax values. For a set \( A \subset \Theta \), we define \( A^\delta := \{ \theta \in \Theta | \inf_{\theta' \in A} \| \theta - \theta' \| \leq \delta \} \) with \( \delta > 0 \).

**Assumption 6** (Smooth maximum). There exist \( \overline{\delta}, C > 0 \) which satisfy

\[
\mathbb{P}_G \left( \max_{\theta \in A} G(\theta) - \max_{\theta \in A^\delta} G(\theta) + \delta \geq 0 \right) - \mathbb{P}_G \left( \max_{\theta \in A^\delta} G(\theta) - \max_{\theta \in (A^\delta)^c} G(\theta) \geq 0 \right) \leq C \delta,
\]

for any \( \delta \in (0, \overline{\delta}) \) and Borel subset \( A \subset \Theta \).

This assumption requires that when the set \( A \) for calculating the maximum value varies slightly by \( \delta \), the value only changes by about \( O(\delta) \). This assumption is satisfied if, for example, a mean function of \( G(\theta) \) is sufficiently smooth.

Under these assumptions, we obtain the following coupling inequality:

**Theorem 4.** Suppose the conditions (A1) (A2) (A4) in Assumption[7] and Assumptions[2][5] and[6] hold. Further, assume that \( \Theta \) is a pre-Gaussian class, that is, there exists a Gaussian process \( G \) on \( \Theta \), and set \( K = n^{1/3} \). Then, with probability at least \( 1 - C'(n^{-1/8} + n^{-\lambda/3}) \log n \), we obtain

\[
| \hat{\theta} - \hat{\theta}_K | \leq C'(n^{-1/8} + n^{-\lambda/3}) \log n,
\]
for some constant $C' > 0$.

This coupling result is slightly weaker than the one used for showing consistency of the distribution function in the sense of the Kolmogorov distance like Theorem 1. However, it can be strengthened if the supremum of the empirical process can be bounded as in Lemma 2.3 in [12].

6. Examples

We provide several examples to illustrate usefulness of our non-asymptotic approximation theorems. In this section, we give only an overview of the results, and technical details are presented in Appendix.

6.1. Cube Root Estimator. As a non-trivial example, we consider a relative frequency estimator, which is a prototype of the mode estimator and converges at the cube root rate (see [19]). The criterion function is

$$f_\theta(z) = 1\{\theta - 1 \leq z \leq \theta + 1\}, \quad (3)$$

with the parameter space $\Theta = [0, 1]$. For simplicity, assume that $Z_i$ follows the uniform distribution on $[0, 1]$, and consider the M-estimator for this criterion.

In this setup, it is easy to verify Assumption 1. the conditions (A1), (A3), and (A4) are trivially satisfied. The condition (A2) also holds with $c_\ell = 1$ and $q = 1$. On the other hand, it is not trivial to verify Assumption 2. To this aim, we consider equally-spaced parameter grids $\{\theta_j\}_j \subset \Theta$, where $\theta_j = -1 + \delta(j - 1/2)$ for $j = 1, 2, \ldots, \lfloor 2/\delta \rfloor$. Note that $\|\theta_j - \theta_{j'}\| \geq \delta > 0$ for $j \neq j'$. We have $\mathbb{E}[f_{\theta_j}(Z)] = 1/2$ and $\text{Cov}_Z(f_{\theta_j}(Z), f_{\theta_{j'}}(Z)) = 7/4 - \delta|j - j'|$. Let $p = \lfloor 2/\delta \rfloor - 1$ and consider the covariance matrix $\Sigma = \text{Cov}_Z(f_{\theta_1}(Z), \ldots, f_{\theta_{p+1}}(Z))$, which is written as

$$\Sigma = \begin{pmatrix}
    c & c - \delta & \ldots & c - (p - 1)\delta & c - p\delta \\
    c - \delta & c & \ldots & c - (p - 2)\delta & c - (p - 1)\delta \\
    c - 2\delta & c - \delta & \ddots & c - (p - 2)\delta & c - (p - 1)\delta \\
    \vdots & \ddots & \ddots & \ddots & c - (p - 1)\delta \\
    c - p\delta & c - (p - 1)\delta & \ldots & c - \delta & c \\
\end{pmatrix}, \quad (4)$$

and $c = 7/4$. To study the coherency of $\Sigma$, the following result is useful.

**Proposition 3.** For each $\delta > 0$, $\Sigma$ defined in (4) is coherently positive definite with $c\delta$ for some $c > 0$.

This result shows that the criterion (3) satisfies Assumption 2 with $\kappa = 1/2$, and we obtain the following corollary.

**Corollary 5.** In the setting of this subsection, the criterion function (3) satisfies Assumptions 7 and 2 with $\kappa = 1/2$.

All details including the proof of Proposition 3 are contained in Section G.1.
6.2. Least Absolute Deviation Regression. As an example of $\sqrt{n}$-consistent estimators, we consider the least absolute deviation regression (see [26]). Let $z = (y, x) \in \mathbb{R} \times \mathbb{R}$, and consider the criterion function

$$f_\theta(y, x) = |y - x\theta|.$$  \hspace{1cm} (5)

It is easy to verify Assumption [1]. Suppose $\Theta$ and the support of $x$ are compact. Then the conditions (A1), (A3), and (A4) are satisfied. Also a simple calculation shows the condition (A2) holds with $q = c_\ell = 1$. Our condition (A2) holds even when $P_Z$ contains a discrete measure, e.g. $P_Z$ is a mixture of Gaussian and Dirac measure, while the existing general frameworks [19, 33, 21] is not applicable because $\mathbb{E}_Z[f_\theta(Z)]$ is not differentiable in $\theta$.

For Assumption [2], we analyze its covariance matrix by applying Proposition [3]. To specify $\text{Cov}_Z(f_\theta(Z), f_\theta'(Z))$, a detailed calculation shows that there exist positive functions $c_0(\theta), c_1(\theta)$, and $c_2(\theta)$ satisfying

$$\text{Cov}_Z(f_\theta(Z), f_\theta'(Z)) = c_0(\theta) - c_1(\theta)|\theta' - \theta| + c_2(\theta)|\theta' - \theta|^2 + c'|\theta' - \theta|^3,$$

with some constant $c' > 0$. Based on this formulation, we can decompose the covariance matrix $\Sigma$ to satisfy coherent positive definiteness, and then obtain the following corollary. Details are provided in Section [G.2] in Appendix.

**Corollary 6.** The criterion function (5) satisfies Assumptions [7] and [2] with $\kappa = 1/2$.

6.3. Minimum Volume Prediction. This subsection considers a minimum volume prediction region estimation developed by [27]. Let $\mathcal{Z} = [0, 1]^2$ be a sample space as a product of input and output spaces. Suppose we observe a random sample $Z_i = (X_i, Y_i), i = 1, \ldots, n$. A purpose of the method is to construct a prediction interval of the output $Y$ given $X = x$ with minimizing the volume of the interval. With a level $\alpha \in [0, 1]$, the estimated interval is defined as

$$\hat{I} = \arg\min_{S \subset [0, 1]} \hat{\lambda}(S), \text{ s.t. } \hat{P}(S) \geq \alpha,$$

where $\hat{\lambda}$ is the Lebesgue measure and $\hat{P}(S) = \sum_{i=1}^n 1(Y_i \in S)K((X_i - x)/h_n)/K((X_i - x)/h_n)$. Here, $K : \mathbb{R} \to \mathbb{R}$ is a kernel function and $h_n > 0$ is a bandwidth depending on $n$. We can rewrite the interval optimization problem by an optimization in terms of its center $\theta$ and width $t$ so that $I = [\theta - t, \theta + t]$. We set $\Theta = [0, 1]$ and consider the following problem

$$\min_{\theta \in \Theta} \hat{P}([\theta - t, \theta + t]), \text{ s.t. } t = \inf \left\{ t' > 0 : \sup_{\theta' \in \Theta} \hat{P}([\theta' - t', \theta' + t']) \geq \alpha \right\}.$$  

To discuss the property of the estimator, we assume that there exists the unique shortest interval $[\theta^* - t^*, \theta^* + t^*]$ the conditional density function $p(y|x)$ is bounded, positive, and symmetric, and satisfies $\partial p(\theta^* - t^*|x) > \partial p(\theta^* + t^*|x)$. Furthermore, $K(x') = o(1/x')$ holds as $x' \to \infty$.

We discuss approximation for the distribution the estimator of $\theta$ with given $t$. By [19], the estimator $t$ is known to converge to $t^*$ by the rate $O_P(1/(nh_n)^{1/2} + h_n^2)$. To discuss the approximation,
we can rewrite the estimation problem for $\theta$ as
$$\hat{\theta} = \arg\max_{\theta \in \Theta} n^{-1} \sum_{i=1}^{n} f_{\theta}(Z_i)$$
with the following criterion function with the fixed input $x$:
$$f_{\theta}(Z) = K \left( \frac{X - x}{h_n} \right) \mathbf{1}\{Y \in [\theta - \iota, \theta + \iota]\}.$$  \hspace{1cm} (6)

We verify this criterion satisfies the assumptions of our theory. For Assumption [1] the conditions (A1) and (A3) are satisfied trivially. The condition (A2) and (A4) are also verified, owing to finite of moments of $K((X_i - x)/h_n)$ by the condition of $K$. For Assumption [2] we derive the explicit covariance matrix and find that it has a similar form as (4). Hence, applying Proposition [3] provides the following result. Details are presented in Section G.3 in Appendix.

**Corollary 7.** The criterion function (6) satisfies Assumptions [1] and [2] with $\kappa = 1/2$.

### 7. Conclusion

This study develops a general non-asymptotic approximation theory which is applicable to a wide range of M-estimators. In some cases, derivations of limiting distributions for complicated M-estimators are highly non-trivial; to solve this issue, this study proposed a method that can be used universally. This result is based on the non-asymptotic approximation scheme proposed in a previous studies [3, 4]. We also develop several techniques to extend the scheme for distributional approximation of argmax values of empirical criteria.

### Appendix A. Proofs for Main Result (Section 2)

**A.1. Proof of Theorem [1].** For a measurable subset $A \subset \Theta$, let us introduce additional notations as $Q_{\hat{\Theta}} := \sup_{\theta \in A} Q_n(\theta)$, and $G_{\hat{\Theta}} := \sup_{\theta \in A} G(\theta)$.

To prove Theorem [1] we consider a discretized version of $Q_n(\theta)$. Firstly, we define a finite subset of $\Theta$. For each $\varepsilon > 0$, we consider a $2\varepsilon$-packing set as satisfying Assumption [2]. We note that $\hat{\Theta}$ is also a $\varepsilon$-covering set for $\Theta$. Here, cardinality of $\hat{\Theta}$ is described as
$$M(\varepsilon) := |\hat{\Theta}| \leq N(\varepsilon, \Theta, \| \cdot \|),$$
where the inequality follows that a cardinality of a $2\varepsilon$-packing set is bounded by that of a $\varepsilon$-covering set.

We also consider discretization of a subset of $A$. Consider a measurable non-empty set $A \subset \Theta$ and define $A^c := \Theta \setminus A$. If $A \cap \hat{\Theta} \neq \emptyset$, we set $A_M := A \cap \hat{\Theta}$ and $A_n := \hat{\Theta} \cap A^c$. Here, without loss of generality, we assume that $A \cap \hat{\Theta}$ is non-empty. If $A \cap \hat{\Theta} = \emptyset$, we set $A_M = \{\theta\}$ where $\theta$ is the closest element of $\hat{\Theta}$ to $A$, and define $A_n := \hat{\Theta} \setminus \{\theta\}$. For any $A$, we can find
$$|A_M| \leq M(\varepsilon).$$
Further, we consider discretization of a subset of $\Theta$. Let us define $\Theta_\varepsilon = \{ (\theta, \theta') \in \Theta \times \Theta \mid \| \theta - \theta' \| \leq \varepsilon \}$, and define a difference of stochastic processes as

$$\Delta Q_{\Theta_\varepsilon}^\vee := \sup_{(\theta, \theta') \in \Theta_\varepsilon} Q_n(\theta) - Q_n(\theta'),$$

and

$$\Delta G_{\Theta_\varepsilon}^\vee := \sup_{(\theta, \theta') \in \Theta_\varepsilon} G(\theta) - G(\theta').$$

We note that we can extend $G$ to the linear hull of $\Theta$ by Theorem 3.1.1 in [13]. Then, we define

$$\phi(\varepsilon) := \mathbb{E}_G \left[ |\Delta G_{\Theta_\varepsilon}^\vee| \right] \vee \mathbb{E}_Z \left[ |\Delta Q_{\Theta_\varepsilon}^\vee| \right].$$

We provide the following lemma to evaluate the effect.

**Lemma 1.** Suppose Assumption 1 holds. Then, for any measurable non-empty $A \subseteq \Theta$ and $\varepsilon, \tau > 0$ and any $n \geq 1$, we obtain

$$\mathbb{P}_G \left( G_{\Theta_\varepsilon}^\vee - G_{\Theta_{\varepsilon}^A}^\vee \leq \phi(\varepsilon) + \varepsilon b \sqrt{2 \tau \frac{1}{n^{1/2}}} \right) \geq 1 - \exp(-\tau),$$

and

$$\mathbb{P}_Z \left( Q_{\Theta_\varepsilon}^\vee - Q_{\Theta_{\varepsilon}^A}^\vee \leq \phi(\varepsilon) + \varepsilon b \sqrt{2 \tau \frac{1}{n^{1/2}}} + \frac{\sqrt{4b^2 \tau^2 + 2\tau b}}{n} \right) \geq 1 - \exp(-\tau).$$

**Proof of Lemma 1** Fix measurable $A \subseteq \Theta$ arbitrary. We bound the supremum of the stochastic process $Q_n$ on $A$ as

$$\sup_{\theta \in A} Q_n(\theta) \leq \max_{\theta \in A} Q_n(\theta) + \sup_{(\theta, \theta') \in \Theta_\varepsilon} Q_n(\theta) - Q_n(\theta').$$

Then, we evaluate a difference between $Q_{\Theta_\varepsilon}^\vee$ and $Q_{\Theta_{\varepsilon}^A}^\vee$ as

$$0 \leq Q_{\Theta_\varepsilon}^\vee - Q_{\Theta_{\varepsilon}^A}^\vee \leq |\Delta Q_{\Theta_\varepsilon}^\vee|.$$

Note that $0 \leq Q_{\Theta_\varepsilon}^\vee - Q_{\Theta_{\varepsilon}^A}^\vee$ holds since $A_{\varepsilon} \subseteq A$. To bound the term $|\Delta Q_{\Theta_\varepsilon}^\vee|$, we apply the Talagrand’s (of Bousquet’s) inequality (page 335, Theorem 12.5 in [6]) and achieve the following bound

$$|\Delta Q_{\Theta_\varepsilon}^\vee| \leq \mathbb{E}_Z \left[ |\Delta Q_{\Theta_\varepsilon}^\vee| \right] + \varepsilon \sqrt{2 \tau \frac{1}{n^{1/2}}} + \frac{2b\sqrt{\tau} + 2\tau b}{n} \leq \phi(\varepsilon) + \varepsilon \sqrt{2 \tau \frac{1}{n^{1/2}}} + \frac{2b\sqrt{\tau} + 2\tau b}{n},$$

with probability at least $1 - \exp(-\tau)$. Note that the condition (A3) in Assumption 1 holds. The second inequality follows the definition of $\phi(\varepsilon)$.

Similarly, we also bound $G_{\Theta_\varepsilon}^\vee - G_{\Theta_{\varepsilon}^A}^\vee$ by the Borel-TIS inequality (page 50, Theorem 2.1.1 in [2]) as

$$0 \leq G_{\Theta_\varepsilon}^\vee - G_{\Theta_{\varepsilon}^A}^\vee \leq |\Delta G_{\Theta_\varepsilon}^\vee| \leq \mathbb{E}_G \left[ |\Delta G_{\Theta_\varepsilon}^\vee| \right] + \varepsilon b \sqrt{2 \tau \frac{1}{n^{1/2}}} \leq \phi(\varepsilon) + \varepsilon b \sqrt{2 \tau \frac{1}{n^{1/2}}},$$
with probability at least $1 - \exp(-\tau)$.

Following the bounds in Lemma 1 we define

$$\nu(n, \varepsilon, \tau) := \phi(\varepsilon) + \varepsilon \sqrt{2\tau} \frac{1}{n^{1/2}} + \frac{2b\sqrt{\tau} + 2\tau b}{n},$$

and

$$\mu(n, \varepsilon, \tau) := \phi(\varepsilon) + \varepsilon b \sqrt{2\tau} \frac{1}{n^{1/2}},$$

for $\varepsilon > 0$ and $\tau > 0$.

We also develop the following lemma to evaluate $\arg\max$ of empirical processes.

**Lemma 2.** Suppose Assumption 1 and 2 are satisfied. Then, for any measurable non-empty $A \subset \Theta$, $\varepsilon, \tau > 0$ and any $n \geq n$ with an existing $n \in \mathbb{N}$, we obtain

$$|\mathbb{P}_Z (Q_A^\vee - Q_A^{\vee_m} \geq 0) - \mathbb{P}_G (G_A^\vee - G_A^{\vee_m} \geq 0)| \leq 3 \exp(-\tau) + \frac{C u (\tilde{M}_3 + 4, \tilde{M}_2, \tilde{M}_1) \log M(\varepsilon)}{4 n^{1/8}} \cdot \frac{1}{n^{1/2}} + \frac{2\nu(n, \varepsilon, \tau) + n^{-1/8} + 4\mu(\varepsilon, \tau)}{\sigma} (\sqrt{2 \log M(\varepsilon)} + 2).$$

**Proof of Lemma 2** For a preparation, we consider a discretization of $A \subset \Theta$. By Lemma 1 we have that

$$0 \leq Q_A^\vee - Q_A^{\vee_m} \leq \nu(n, \varepsilon, \tau), \text{ and } 0 \leq G_A^\vee - G_A^{\vee_m} \leq \mu(n, \varepsilon, \tau)$$

(7)

hold with probability at least $1 - \exp(-\tau)$ respectively. Note that $0 \leq Q_A^\vee - Q_A^{\vee_m}$ is obvious since $A \supset A_M$. Then, similarly, Lemma 1 yields

$$0 \leq Q_{A^c}^\vee - Q_{A^c}^{\vee_m} \leq \nu(n, \varepsilon, \tau), \text{ and } 0 \leq G_{A^c}^\vee - G_{A^c}^{\vee_m} \leq \mu(n, \varepsilon, \tau),$$

(8)

with probability at least $1 - \exp(-\tau)$. They enable us to bound the probability $\mathbb{P}_Z (Q_A^\vee - Q_A^{\vee_m} \geq 0)$ by $\mathbb{P}_G (G_A^\vee - G_A^{\vee_m} \geq 0)$ and an error term, through the discretization. We evaluate it as

$$\mathbb{P}_Z (Q_A^\vee - Q_A^{\vee_m} \geq 0) \leq \mathbb{P}_Z \left( Q_A^\vee - Q_A^{\vee_m} \geq 0 \right) \leq \mathbb{P}_Z \left( Q_A^\vee - Q_A^{\vee_m} + \nu(n, \varepsilon, \tau) \geq 0 \right) + \exp(-\tau),$$

where the first inequality follows $Q_{A^c}^\vee \geq Q_{A^c}^{\vee_m}$ by (8), and the second inequality follows (7).

Now, we will bound $\mathbb{P}_Z (Q_{A^c}^\vee - Q_{A^c}^{\vee_m} + \nu(n, \varepsilon, \tau) \geq 0)$ by the probability $\mathbb{P}_G (G_{A^c}^\vee - G_{A^c}^{\vee_m} + \nu(n, \varepsilon, \tau) \geq 0)$, by utilizing the technique developed in the proof of Proposition 4. Here, we can regard $Q_{A^c}^{\vee_m}$ and $Q_{A^c}^\vee$ as maximums of an empirical mean of independently and identically distributed random vectors, namely, we can define a random vector $\bar{Z}^{(G)} := (n^{-1} \sum_{i \in [n]} f_{\theta}(Z_i))_{\theta \in \Theta}$, and thus
\[ Q_{AM}^\vee = \max_{\theta \in AM} \tilde{Z}^{(G)} \] and \[ Q_{AMc}^\vee = \max_{\theta \in AMc} \tilde{Z}^{(G)} \]. Then, we utilize the function \( f_{\beta, \delta, AM} \) defined in (20) for a smooth approximation of \( Q_{AM}^\vee - Q_{AMc}^\vee \). As similar to the inequality in (21), we obtain

\[
P_Z \left( Q_{AM}^\vee - Q_{AMc}^\vee + \nu(n, \varepsilon, \tau) \geq 0 \right)
\]

\[
= P_Z \left( \max_{\theta \in AM} \tilde{Z}^{(G)} - \max_{\theta \in AMc} \tilde{Z}^{(G)} + \nu(n, \varepsilon, \tau) \geq 0 \right)
\]

\[
\leq E_Z \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{Z}^{(G)} \right) \right],
\]

with arbitrary parameters \( \beta, \delta > 0 \). Then, we bound the expectation by

\[
E_W \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{W}^{(G)} \right) \right]
\]

where \( \tilde{W}^{(G)} := (G(\theta))_{\theta \in \Theta} \) is a Gaussian random vector. By a centorization and Proposition 4, we obtain

\[
E_Z \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{Z}^{(G)} \right) \right]
\]

\[
\leq E_W \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{W}^{(G)} \right) \right]
\]

\[
+ \frac{C_S(\tilde{M}_3 + 4, \tilde{M}_2, \tilde{M}_1)}{4n^{1/2}} \sup_{x \in \mathbb{R}^M} m \in [M] \sum_x | \partial_m^3 f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)}(x) |
\]

\[
\leq E_W \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{W}^{(G)} \right) \right]
\]

\[
+ \frac{C_S(\tilde{M}_3 + 4, \tilde{M}_2, \tilde{M}_1)}{4n} (\delta^{-3} + \delta^{-2} \beta + \delta \beta^2)
\]

\[
\leq E_W \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{W}^{(G)} \right) \right]
\]

\[
+ \frac{C_S(\tilde{M}_3 + 4, \tilde{M}_2, \tilde{M}_1)}{4n^{1/8}} \log M(\varepsilon) \frac{1}{n^{1/2}}
\]

(10)

where the second inequality follows Lemma 6 and the last inequality follows by setting \( \beta = n^{1/8} \log^{1/2} M(\varepsilon) \) and \( \delta = n^{-1/8} \). Note that the error term follows \( \tilde{M}_3 \leq \tilde{M}_3, \tilde{M}_1 \leq \tilde{M}_1 \), and \( |AM \setminus AMc| \leq M(\varepsilon) \) by their definitions. Further, by the similar process in (22), the expectation

\[
E_W \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{W}^{(G)} \right) \right]
\]

is bounded as

\[
E_W \left[ f_{\beta, \delta, AM, \Xi(M, \beta) + \nu(n, \varepsilon, \tau)} \left( \tilde{W}^{(G)} \right) \right]
\]

\[
\leq P_W \left( \max_{\theta \in AM} \tilde{W}^{(G)} - \max_{\theta \in AMc} \tilde{W}^{(G)} + \delta + 2\nu(n, \varepsilon, \tau) \geq 0 \right).
\]

\[
= P_W \left( G_{AM}^\vee - G_{AMc}^\vee + n^{-1/8} + 2\nu(n, \varepsilon, \tau) \geq 0 \right).
\]

(11)

We combine (9), (10) and (11), then we obtain

\[
P_Z \left( Q_{AM}^\vee - Q_{AMc}^\vee + \nu(n, \varepsilon, \tau) \geq 0 \right)
\]

\[
\leq P_G \left( G_{AM}^\vee - G_{AMc}^\vee + n^{-1/8} + 2\nu(n, \varepsilon, \tau) \geq 0 \right) + \Upsilon(n, \varepsilon)
\]

(12)
Next, we will bound the probability $\mathbb{P}_G(G_{AM}^\nu - G_{AM}^\nu + n^{-1/8} + 2\nu(n, \varepsilon, \tau) \geq 0)$ in (12) by the conditional anti-concentration inequality. Since $G_{AM}^\nu$ and $G_{AM}^\nu$ are maximums of Gaussian random vectors, we can apply the conditional anti-concentration inequality in Lemma 5. For any $\zeta > 0$, applying the inequality yields that

$$\mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu + n^{-1/8} + 2\nu(n, \varepsilon, \tau) \geq 0\right)$$

$$= \mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu \geq \zeta\right) + \mathbb{P}_G\left(\zeta > G_{AM}^\nu - G_{AM}^\nu \geq - (n^{-1/8} + 2\nu(n, \varepsilon, \tau))\right)$$

$$\leq \mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu \geq \zeta\right) + \frac{\nu(n, \varepsilon, \tau) + n^{-1/8} + \zeta}{\sigma} \left(\sqrt{2\log M(\varepsilon)} + 2\right).$$

(13)

We note that $\nu(n, \varepsilon, \tau) > 0$ holds by its definition.

We will obtain a bound with the probability $\mathbb{P}_G(G_{A}^\nu - G_{A}^\nu \geq 0)$ by the derived inequality. Combining the results in (12) and (13) as

$$\mathbb{P}_Z\left(Q_{A}^\nu - Q_{A}^\nu \geq 0\right)$$

$$\leq \mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu \geq \zeta\right) + \nu(n, \varepsilon, \tau, \zeta) + Y(n, \varepsilon) + \exp(-\tau).$$

(14)

We bound the term $\mathbb{P}_G(G_{AM}^\nu - G_{AM}^\nu \geq \zeta)$ by evaluating an effect of the discretization of $G_{AM}^\nu - G_{AM}^\nu$ in the following:

$$\mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu \geq \zeta\right)$$

$$= \mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu \geq \zeta \text{ and } \left|\left(G_{A}^\nu - G_{A}^\nu\right) - \left(G_{AM}^\nu - G_{AM}^\nu\right)\right| \leq \zeta\right)$$

$$+ \mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu \geq \zeta \text{ and } \left|\left(G_{A}^\nu - G_{A}^\nu\right) - \left(G_{AM}^\nu - G_{AM}^\nu\right)\right| > \zeta\right)$$

$$\leq \mathbb{P}_G\left(G_{A}^\nu - G_{A}^\nu \geq 0\right) + \mathbb{P}_G\left(\left|\left(G_{A}^\nu - G_{A}^\nu\right) - \left(G_{AM}^\nu - G_{AM}^\nu\right)\right| > \zeta\right).$$

About the last term, by the relation (7) and (8), we obtain

$$\mathbb{P}_G\left(\left|\left(G_{A}^\nu - G_{A}^\nu\right) - \left(G_{AM}^\nu - G_{AM}^\nu\right)\right| > \zeta\right)$$

$$\leq \mathbb{P}_G\left(\left|G_{A}^\nu - G_{A}^\nu\right| > \zeta/2\right) + \mathbb{P}_G\left(G_{AM}^\nu - G_{AM}^\nu \geq \zeta/2\right)$$

$$\leq 2\exp(-\tau),$$

where the last inequality follows Lemma 1 and setting $\zeta = 2\mu(n, \varepsilon, \tau)$. Then, as substituting the result into (14), we obtain

$$\mathbb{P}_Z\left(Q_{A}^\nu - Q_{A}^\nu \geq 0\right)$$

$$\leq \mathbb{P}_G\left(G_{A}^\nu - G_{A}^\nu \geq 0\right) + 3\exp(-\tau) + \nu(n, \varepsilon, \tau) + Y(n, \varepsilon).$$
About an opposite inequality, we can bound it by the same way, hence we have
\[ |\mathbb{P}_Z(Q_A^\vee - Q_{A^c}^\vee \geq 0) - \mathbb{P}_G(G_A^\vee - G_{A^c}^\vee \geq 0)| \leq 3 \exp(-\tau) + 2\nu'(n, \epsilon, \tau, 2\mu(n, \epsilon, \tau)) + \Upsilon(n, \epsilon). \]

Then, we obtain the statement. \( \square \)

**Proof of Theorem 1.** If \( A = \emptyset \) or \( A = \Theta \), the result obviously holds. In the following, we consider \( A \subsetneq \Theta \) is non-empty.

At the beginning, we rewrite the probability of \( \mathbb{P}_Z(\hat{\theta} \in A) \) by the difference of the supreme values. We simply obtain
\[ \mathbb{P}_Z(\hat{\theta} \in A) = \mathbb{P}_Z(Q_A^\vee - Q_{A^c}^\vee > 0). \]

Similarly, we obtain
\[ \mathbb{P}_G(\hat{\theta}_G \in A) = \mathbb{P}_G(G_A^\vee - G_{A^c}^\vee > 0). \]

Then, from Lemma 2 we obtain
\[ |\mathbb{P}_Z(\hat{\theta} \in A) - \mathbb{P}_G(\hat{\theta}_G \in A)| = |\mathbb{P}_Z(Q_A^\vee - Q_{A^c}^\vee \geq 0) - \mathbb{P}_G(G_A^\vee - G_{A^c}^\vee \geq 0)| \leq 3 \exp(-\tau) + \Upsilon(n, \epsilon) + 2\nu'(n, \epsilon, \tau, 2\mu(n, \epsilon, \tau)), \]

where \( \Upsilon(n, \epsilon) \) is defined in the proof of Lemma 2. Note that \( H(\epsilon) \geq \log M(\epsilon) \) holds. Then, we obtain
\[ |\mathbb{P}_Z(\hat{\theta} \in A) - \mathbb{P}_G(\hat{\theta}_G \in A)| \leq C_U \left\{ \exp(-\tau) + \frac{C_\sigma H(\epsilon)}{n^{1/8}} \cdot \frac{1}{n^{1/2}} + \frac{C_\sigma}{\sigma} \left( \phi(\epsilon) + \frac{\epsilon b \sqrt{\tau n}}{n} + \frac{b \sqrt{\tau + \tau b}}{n} \right) \left( \sqrt{H(\epsilon)} + 1 \right) \right\}, \]

where \( C_U > 0 \) is a universal constant and \( C_\sigma > 0 \) is a constant depends on \( \tilde{M}_3, \tilde{M}_2 \) and \( \tilde{M}_1 \).

We will bound \( \phi(\epsilon) \) by the condition (A4) in Assumption 1. For \( q \in (0, 1] \) we define its integral as
\[ J_q(\epsilon) := \int_0^\epsilon \sqrt{1 + \log \mathcal{N}(\delta q, \Theta, ||\cdot||_\Theta)} d\delta. \]

By Lemma 7, we obtain
\[ \phi(\epsilon) \leq C_\Phi \left\{ \frac{J_1(b\epsilon) + J_q(2c_\ell b\epsilon)}{n^{1/2}} + c_\ell \epsilon \right\}. \]

for all \( \epsilon \in (0, 1] \), and a constant \( C_\Phi > 0 \).
We substitute the bound for $\phi(\varepsilon)$ and also set $\tau = \log(1/\varepsilon)$. Also, we apply $J_q(\varepsilon) \lesssim J_1(\varepsilon) = J(\varepsilon)$. Then, we apply Assumption 2 and obtain
\[
\left| \mathbb{P}_Z(\hat{\theta} \in A) - \mathbb{P}_G(\hat{\theta}_G \in A) \right| \\
\leq C_U C_\sigma C_\phi C_b \left\{ \varepsilon + \frac{H(\varepsilon)}{n^{1/8}} \cdot \frac{1}{n^{1/2}} \right. \\
+ \sqrt{\frac{H(\varepsilon) + 1}{\varepsilon^k}} \left( \varepsilon + \frac{J(\varepsilon) + \varepsilon \log(1/\varepsilon)}{n^{1/2}} + \frac{\log(1/\varepsilon) + \log(1/\varepsilon)}{n} \right) \},
\]
where $C_U$ is another universal coefficient and $C_b > 0$ is a constant depends on $b, c_\ell, q$ and $k$. Then, we obtain the statement.  

A.2. Finite Index Set Case. As a preparation, we consider a finitely approximated version of the empirical stochastic process. With $M \in \mathbb{N}_{\geq 2}$, let $\mathcal{F}_M \subset \mathcal{F}$ with $|\mathcal{F}_M| = M$ be a finite subset of $\mathcal{F}$, and consider a corresponding i.i.d. $M$-dimensional random vector $X_i$ for $i \in [n]$, which is analogous to $(f_1(Z_i), \ldots, f_M(Z_i)) \in \mathbb{R}^M$. Let $\Sigma_X$ is a covariance matrix of $X_i$, and for any sets $A, A' \subset [M]$, $\Sigma_A := \text{Var}_X((X_j)_{j \in A})$ and $\Sigma_{AA'} := \text{Cov}_X((X_j)_{j \in A}, (X_j')_{j' \in A'})$ sub-covariance matrices of $\Sigma_X$. Let us define $\overline{\Sigma}^2 := \max_{m \in [M]} \text{Var}(X_m)$. We consider approximating the distribution of the maximizer of the scaled empirical mean
\[
\hat{m}_X := \arg\max_{m \in [M]} \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{i,m}.
\] (16)
We introduce the following assumption on the covariance of $X$, which guarantees uniqueness of the above maximizer.

To approximate the distribution of $\hat{m}_X$, let $W_1, \ldots, W_n$ be i.i.d. Gaussian random vectors in $\mathbb{R}^M$ such that $\mathbb{E}_W[W_{i,m}] = \mathbb{E}_X[X_{i,m}]$ and $\text{Cov}_X(X_{i,m}, X_{i,m'}) = \text{Cov}_W(W_{i,m}, W_{i,m'})$ for all $m, m' \in [M]$. Define
\[
\hat{m}_W := \arg\max_{m \in [M]} \frac{1}{\sqrt{n}} \sum_{i=1}^n W_{i,m},
\]
and for $k > 0$,
\[
\mathcal{M}_k := \mathbb{E}_X \left[ \max_{m \in [M]} |X_{1,m} - \mathbb{E}_X[X_{1,m}]|^k \right].
\]
The distribution of $\hat{m}_X$ is approximated by that of $\hat{m}_W$ as follows.

**Proposition 4** (Argmax of Random Vectors). Suppose $\Sigma_X$ is a coherently positive definite. Then for any $A \subset [M]$ and $n \geq n$ with an existing $n \in \mathbb{N}$, it holds
\[
|\mathbb{P}_X(\hat{m}_X \in A) - \mathbb{P}_W(\hat{m}_W \in A)| \leq \frac{C_U \log M}{n^{1/8}} \left( \overline{\Sigma}^{-1} + \mathcal{M}_3 + \overline{\Sigma}^2 \mathcal{M}_1 \right),
\]
for a universal constant $C_U > 0$.  

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Proof of Proposition 4. If $A = \emptyset$ or $A = [M]$, the result obviously holds. In the following, we consider $A \subset [M]$ is non-empty.

This proof contains mainly three steps: 1) we prepare a smooth function to approximate the argmax operation, 2) we approximate the probability $\mathbb{P}(\hat{m}_X \in A)$, then 3) combine the results of all the steps. Several significant technical lemmas will be provided after this proof.

**Step 1: Smooth Approximation.**

In this step, we approximate a probability $\mathbb{P}_X(\hat{m}_X \in A)$ with any set $A \subset [M]$ into several terms. The approximation depends on a smooth representation of the argmax operation.

To the end, we rewrite an event with an argmax operation. Let $x = (x_m)_{m \in [M]} \in \mathbb{R}^M$ be an vector with no ties, i.e. $x_m \neq x_{m'}$ holds for any $m, m' \in [M], m \neq m'$. Then, for any non-empty $A \subset [M]$, we have

$$1 \left\{ \arg\max_{m \in [M]} x_m \in A \right\} = 1 \left\{ \max_{m \in A} x_m \geq \max_{m \in A^c} x_m \right\},$$

(17)

where $A^c := [M] \setminus A$. Using the relation, we utilize the following representation of the probability: for a random vector $X = (X_1, \ldots, X_M)$, we have

$$\mathbb{P}_X \left( \arg\max_{m \in [M]} X_m \in A \right) = \mathbb{P}_X \left( \max_{m \in A} X_m \geq \max_{m \in A^c} X_m \right).$$

To handle the probability, we provide a smooth approximation. Firstly, we define a *softmax function*. Let $\beta > 0$ be a parameter, $x = (x_m)_{m \in [M]} \in \mathbb{R}^M$ be a vector, and $A \subset [M]$ be a non-empty set. Then, a softmax function $h_{\beta, A}(x) : \mathbb{R}^{|A|} \rightarrow \mathbb{R}$ is defined as

$$h_{\beta, A}(x) := \beta^{-1} \log \left( \sum_{m \in A} \exp(\beta x_m) \right).$$

It asymptotically works as a max function, namely, the following holds $\lim_{\beta \rightarrow \infty} h_{\beta, A}(x) = \max_{m \in A} x_m$. It is obvious that $h_{\beta, A} \in C^\infty(\mathbb{R}^M)$. Now, we can obtain a bound

$$0 \leq h_{\beta, A}(x) - \max_{m \in A} x_m \leq \beta^{-1} \log M =: \Xi(M, \beta),$$

(18)

for any $x \in \mathbb{R}^M$.

Additionally, we define a *soft-step function*. For a parameter $\delta > 0$ and $z \in \mathbb{R}$, we define the function as

$$g_{\delta}(z) = \int_{\mathbb{R}} (1 - (z + \delta t)/\delta) \cdot C_{\varphi} \exp(1/(t^2 - 1)) dt,$$

where $C_{\varphi}$ is a normalizing constant. It works as a step function $1\{ \cdot \geq 0 \}$ as $\delta \rightarrow 0$, and it satisfies

$$1\{z \geq 0\} \leq g_{\delta}(z) \leq 1\{z \geq -\delta\},$$

(19)

for any $z \in \mathbb{R}$. We note that $g_{\delta} \in C^3(\mathbb{R})$ holds.
Finally, we define a smooth function to approximate an argmax operator. For a vector \( x \in \mathbb{R}^M \), parameters \( \beta, \delta \), a non-empty set \( A \subseteq [M] \), and \( \Delta > 0 \), we define a function \( f_{\beta,\delta,A,\Delta} : \mathbb{R}^M \rightarrow \mathbb{R} \) as
\[
f_{\beta,\delta,A,\Delta}(x) := g_{\delta} \left( h_{\beta,A}(x) - h_{\beta,A^c}(x) + \Delta \right). \tag{20}
\]
We will show that \( f_{\beta,\delta,A,\Delta}(x) \) approximates the event [17] with suitable selected parameters \( \beta, \delta \) and \( \Delta \) in the next step.

**Step 2: Approximate probability** \( \mathbb{P}(\tilde{m}_X \in A) \).

We rewrite the probability \( \mathbb{P}(\tilde{m}_X \in A) \) for an arbitrary fixed non-empty \( A \subseteq [M] \) by utilizing \( f_{\beta,\delta,A,\Delta} \). In this step, we provide an upper bound for \( \mathbb{P}(\tilde{m}_X \in A) \) associated with \( \mathbb{P}(\tilde{m}_W \in A) \). An opposite lower bound is yielded by the similar way, hence we omit it.

Let us introduce notations \( \tilde{X} := n^{-1/2} \sum_{i \in [n]} X_i \) and \( \tilde{W} := n^{-1/2} \sum_{i \in [n]} W_i \) as \( M \)-dimensional random vectors. Now, we provide an upper bound for \( \mathbb{P}(\tilde{m}_X \in A) \) as
\[
\mathbb{P}_X(\tilde{m}_X \in A) = \mathbb{P}_X \left( \text{argmax } \tilde{X}_m \in A \right) = \mathbb{P}_X \left( \max_{m \in A} \tilde{X}_m - \max_{m \in A^c} \tilde{X}_m \geq 0 \right)
\]
\[
\leq \mathbb{P}_X \left( h_{\beta,A}(\tilde{X}) - h_{\beta,A^c}(\tilde{X}) + \mathbb{E}(M,\beta) \geq 0 \right) = \mathbb{E}_X \left( \mathbb{1} \{ h_{\beta,A}(\tilde{X}) - h_{\beta,A^c}(\tilde{X}) + \mathbb{E}(M,\beta) \geq 0 \} \right)
\]
\[
\leq \mathbb{E}_X [ f_{\beta,\delta,A,\mathbb{E}(M,\beta)}(\tilde{X}) ], \tag{21}
\]
where the second equality follows [17], and the first inequality follows the bound [18] for the approximation error of \( h_{\beta,A} \). The last inequality follows [19] and the definition of \( f_{\beta,\delta,A,\Delta} \) with \( \Delta = \mathbb{E}(M,\beta) \).

Now, we approximate the bound \( \mathbb{E}_X [ f_{\beta,\delta,A,\mathbb{E}(M,\beta)}(\tilde{X}) ] \) with a transformation of the Gaussian \( \tilde{W} \), namely, \( \mathbb{E}_Y [ f_{\beta,\delta,A,\mathbb{E}(M,\beta)}(\tilde{W}) ] \). We apply an approximation technique associated with a Gaussian approximation and the Stein’s identity. For a preparation, we consider a modified version of \( f_{\beta,\delta,A,\Delta} \) and \( \tilde{X} \). Let \( \tilde{X} \) be a centered version of \( \tilde{X} \) as \( \tilde{X} := n^{-1/2} \sum_{i \in [n]} \tilde{X}_i \) where \( \tilde{X}_i = (\tilde{X}_i,1,\ldots,\tilde{X}_i,M) = (X_{i,m} - \mathbb{E}_X[X_{i,m}])_{m \in [M]} \). Similarly, we define \( \tilde{W} := n^{-1/2} \sum_{i \in [n]} \tilde{W}_i \) where \( \tilde{W}_i = (\tilde{W}_i,1,\ldots,\tilde{W}_i,M) = (W_{i,m} - \mathbb{E}_W[W_{i,m}])_{m \in [M]} \). Also, consider a mapping \( \tau : \mathbb{R}^M \rightarrow \mathbb{R}^M \) such as \( \tau(x) = (x + \mathbb{E}_X[X_{i,m}])_{m \in [M]} \). Let us define a modified function for the centered vectors as
\[
\tilde{f}_{\beta,\delta,A,\Delta}(x) := f_{\beta,\delta,A,\Delta} \circ \tau(x).
\]

Then, we utilize the centered random vectors and rewrite the following difference as
\[
\mathbb{E}_X [ f_{\beta,\delta,A,\mathbb{E}(M,\beta)}(\tilde{X}) ] - \mathbb{E}_W [ f_{\beta,\delta,A,\mathbb{E}(M,\beta)}(\tilde{W}) ]
= \mathbb{E}_X [ \tilde{f}_{\beta,\delta,A,\mathbb{E}(M,\beta)}(\tilde{X}) ] - \mathbb{E}_W [ \tilde{f}_{\beta,\delta,A,\mathbb{E}(M,\beta)}(\tilde{W}) ].
\]
Now, we are ready to apply Lemma 3. Since \( \tilde{X} \) satisfies the conditions of Lemma 3, we obtain that
\[
E_X[\tilde{f}_{\beta, \delta, A, \Xi(M, \beta)}(\tilde{X})]
\leq E_W[\tilde{f}_{\beta, \delta, A, \Xi(M, \beta)}(\tilde{W})] + \Delta_f
\leq E_W[\mathbb{1}\{h_{\beta, A}(\tilde{W}) - h_{\beta, A^c}(\tilde{W}) + \delta + \Xi(M, \beta) \geq 0\}] + \Delta_f
= P_W(h_{\beta, A}(\tilde{W}) - h_{\beta, A^c}(\tilde{W}) + \delta + \Xi(M, \beta) \geq 0) + \Delta_f
\leq P_W\left(\max_{m \in A} \tilde{W}_m - \max_{m \in A^c} \tilde{W}_m + \delta + 2\Xi(M, \beta)  \geq 0\right) + \Delta_f,
\]
where \( C_S, C_U > 0 \) are universal constants. The last inequality follows a bound for the derivatives for \( f_{\beta, \delta, A, \Xi(M, \beta)} \) which are derived in Lemma 6.

Then, by the property of \( g_\delta \) as (19), we continue the upper bound for \( P_X(\hat{m}_X \in A) \) as
\[
E_X[\tilde{f}_{\beta, \delta, A, \Xi(M, \beta)}(\tilde{X})]
\leq E_W[\tilde{f}_{\beta, \delta, A, \Xi(M, \beta)}(\tilde{W})] + \Delta_f
\]
where the last inequality follows the similar argument.

To bound the probability term in the last line (22), we apply the anti-concentration inequality for this setting. We apply Lemma 5 with setting \( r = -(\delta + 2\Xi(M, \beta)) / 2 \) and \( \varepsilon = (\delta + 2\Xi(M, \beta)) / 2 \), then it provides
\[
P_W\left(\max_{m \in A} \tilde{W}_m - \max_{m \in A^c} \tilde{W}_m + \delta + 2\Xi(M, \beta)  \geq 0\right)
= P_W\left(\max_{m \in A} \tilde{W}_m - \max_{m \in A^c} \tilde{W}_m \geq 0\right)
+ P_W\left(0 \geq \max_{m \in A} \tilde{W}_m - \max_{m \in A^c} \tilde{W}_m \geq -(\delta + 2\Xi(M, \beta))\right)
\leq P_W\left(\max_{m \in A} \tilde{W}_m - \max_{m \in A^c} \tilde{W}_m \geq 0\right) + \frac{1}{2}(\sqrt{2 \log M + 2})(\delta + 2\Xi(M, \beta)).
\]
Now, we can continue the inequality (22) as
\[
P_W\left(\max_{m \in A} \tilde{W}_m - \max_{m \in A^c} \tilde{W}_m + \delta + 2\Xi(M, \beta)  \geq 0\right) + \Delta_f
\leq P_W\left(\min_{m \in A} \tilde{W}_m - \max_{m \in A^c} \tilde{W}_m \geq 0\right) + \Delta_h(\delta + 2\Xi(M, \beta)) + \Delta_f
\]
\[
\begin{align*}
\mathbb{P}_W (\hat{m}_W \in A) + \Delta_h (\delta + 2\Xi (M, \beta)) + \Delta_f.
\end{align*}
\]

We can obtain an opposite version of the inequality to bound \( \mathbb{P} (m_W^0 \in A) \), then we obtain that
\[
|\mathbb{P}_W (\hat{m}_X \in A) - \mathbb{P}_W (\hat{m}_Y \in A)| \leq \Delta_h (\delta + 2\Xi (M, \beta)) + \Delta_f.
\]

**Step 3: Combine the results.**
Combining the result of the steps with the bounds for \( \Delta_f \) and \( \Delta_h \), we obtain that
\[
|\mathbb{P}_X (\hat{m}_X \in A) - \mathbb{P}_W (\hat{m}_W \in A)| \leq \frac{2(\sqrt{2\log M} + 2)}{\sigma} (\delta + 2\Xi (M, \beta)) + \frac{M_3 + 4\sigma^2 M_1}{n^{1/2}} (\delta^3 + \delta^{-2} \beta + \delta^{-1} \beta^2),
\]
where \( C_M > 0 \) be a constant depends on up to a third moment of \( \hat{X} \) and \( \hat{\sigma} \). We set
\[
\beta = n^{1/8} \log^{1/2} M, \quad \text{and} \quad \delta = n^{-1/8},
\]
then we obtain
\[
|\mathbb{P}_X (\hat{m}_X \in A) - \mathbb{P}_W (\hat{m}_W \in A)| \leq \frac{\log M}{n^{1/8}} \left( \frac{6\sqrt{2} + 12}{\sigma} + \frac{M_3 + 4\sigma^2 M_1}{2} \right).
\]

By adjusting coefficients, we obtain the statement. \( \square \)

**Appendix B. Technical Results: Stein Approximation and Anti-concentration**

We provide a Gaussian approximation technique for an expectation of smooth functions via the Stein’s identity. The technique has been developed by several studies \([3, 39]\). The following lemma is a straightforward application of the result by \([15]\).

**Lemma 3** (Approximation by Stein Identity). Let \( X_1, \ldots, X_n \) be independent \( M \)-dimensional random vectors such as \( \mathbb{E}_X [X_i, m] = 0 \) and \( \mathbb{E}_X [X_i^2, m] = \sigma_m^2 \) for all \( m \in [M] \). Define \( \mu_k := \max_{i \in [n]} \mathbb{E}_X [\max_{m \in [M]} |X_i, m|^k] \) for \( k = 1, 3 \), and \( \tilde{X} := n^{-1/2} \sum_{i=1}^n X_i \). Also, for each \( i = 1, \ldots, n \), let \( W_i \) be a \( M \)-dimensional Gaussian random vector such as \( \mathbb{E}_W [W_i] = \mathbb{E}_X [X_i] \) and \( \text{Var}_W (W_i) = \text{Var}_X (X_i) \), and we define \( \tilde{W} := n^{-1/2} \sum_{i=1}^n W_i \). Then, for any bounded \( f \in C^3 (\mathbb{R}^M) \), we have
\[
|\mathbb{E}_X [f(\tilde{X})] - \mathbb{E}_W [f(\tilde{W})]| \leq c_S (\mu_3 + 4\mu_1 \max_{m \in [M]} \sigma_m^2) \sup_{x \in \mathbb{R}^M} \sum_{m' \in [M]} \left| \frac{\partial^3}{\partial m' \partial m'' \partial m'''} f (x) \right|.
\]

**Proof of Lemma**\( \Box \) We apply the multivariate version of the Stein’s equation \([35]\) as
\[
\mathbb{E}_X [f(\tilde{X})] - \mathbb{E}_W [f(\tilde{W})] = \sum_{m' \in [M]} \mathbb{E}_X \left[ \sigma_m^2 \partial_{m'}^2 f (\hat{X}) - \hat{X}_{m'} \partial_{m'} f (\hat{X}) \right]. \tag{23}
\]
Here, \( \tilde{f} : \mathbb{R}^M \to \mathbb{R} \) be a solution of the identity which is induced from \( f \) as

\[
\tilde{f}(\tilde{w}) := -\int_0^{\infty} \mathbb{E}_\tilde{w} \left[ f(e^{-s} \tilde{w} + \sqrt{1 - e^{-2s}} \tilde{W}) \right] - \mathbb{E}_\tilde{W} [f(\tilde{W})] ds,
\]

for \( \tilde{w} \in \mathbb{R}^M \).

We evaluate the term \( \mathbb{E}_X [\tilde{\sigma}_{m'}^2 \partial_{m'}^2 \tilde{f}(\tilde{X}) - \tilde{X}_{m'} \partial_{m'} \tilde{f}(\tilde{X})] \) for each \( m' \in [M] \). Let us define \( \tilde{X}_{-i} := \tilde{X} - n^{-1/2} X_i \). Then, we consider the first-order Taylor expansion of \( \mathbb{E}_X [\partial_{m'}^2 \tilde{f}(\tilde{X})] \) around \( \tilde{X}_{-i} \) for all \( i \in [n] \) as

\[
\tilde{\sigma}_{m'}^2 \mathbb{E}_X [\partial_{m'}^2 \tilde{f}(\tilde{X})]
= \frac{\tilde{\sigma}_{m'}^2}{n} \sum_{i \in [n]} \mathbb{E}_X [\partial_{m'}^2 \tilde{f}_{\beta, \delta, A}(\tilde{X}_{-i})] + \sum_{i \in [n]} \frac{\tilde{\sigma}_{m'}^2 \mathbb{E}_X [X_{i,m'} \partial_{m'}^3 \tilde{f}(\tilde{X}_{i,m'})]}{n^{3/2}}.
\]

Note that \( \tilde{X}_{i,m'} \) is an existing random variable as an inter point between \( \tilde{X}_{-i} \) and \( \tilde{X} \). By an iterative use of the Hölder’s inequality, we obtain that

\[
\left| \sum_{m' \in [M]} R_{1,m',i} \right| \leq \frac{\tilde{\sigma}_{m'}^2 \mathbb{E}_X [\max_{m' \in [M]} |X_{i,m'}| \sum_{m' \in [M]} |\partial_{m'}^3 \tilde{f}(\tilde{X}_{i,m'})|]}{n^{3/2}}
\]

\[
\leq \frac{\tilde{\sigma}_{m'}^2 \mu_1 \sup_{x \in \mathbb{R}^M} \sum_{m' \in [M]} |\partial_{m'}^3 \tilde{f}(x)|}{n^{3/2}},
\]

for any \( i \in [n] \).

Similarly, the first-order Taylor expansion for \( \partial_{m'} \tilde{f}(\tilde{X}) \) around \( \tilde{X}_{-i} \) provides

\[
\mathbb{E}_X [\tilde{X}_{i,m'} \partial_{m'} \tilde{f}(\tilde{X})]
= \frac{1}{n^{1/2}} \sum_{i \in [n]} \mathbb{E}_X [X_{i,m'} \partial_{m'} \tilde{f}(\tilde{X})]
= \frac{1}{n^{1/2}} \sum_{i \in [n]} \mathbb{E}_X [X_{i,m'} \partial_{m'} \tilde{f}(\tilde{X}_{-i})] + \frac{1}{n} \sum_{i \in [n]} \mathbb{E}_X [X_{i,m'}^2 \partial_{m'}^2 \tilde{f}(\tilde{X}_{-i})]
+ \frac{1}{n^{3/2}} \sum_{i \in [n]} \frac{\mathbb{E}_X [X_{i,m'}^3 \partial_{m'}^3 \tilde{f}(\tilde{X}_{i,m'})]}{2!}
= \sum_{i \in [n]} \frac{1}{j! n^{j/2+1/2}} \mathbb{E}_X [X_{i,m'}^{j+1}] \mathbb{E}_X [\partial_{m'}^{j+1} \tilde{f}(\tilde{X}_{-i})]
+ \sum_{i \in [n]} \frac{\mathbb{E}_X [X_{i,m'}^3 \partial_{m'}^3 \tilde{f}(\tilde{X}_{i,m'})]}{4n^{3/2}}
=: R_{2,m',i},
\]

\[
= \frac{1}{n} \sum_{i \in [n]} \mathbb{E}_X [X_{i,m'}^2] \mathbb{E}_X [\partial_{m'}^2 \tilde{f}(\tilde{X}_{-i})] + \sum_{i \in [n]} R_{2,m',i}.
\]
Here, the second equality holds since \(X_i\) and \(\tilde{X}_{-i}\) are independent by its definition, and the last equality follows \(\mathbb{E}_X[X_{i,m}] = 0\). Similar to the bound for \(|\sum_{m' \in [M]} R_{2,m',i}|\), we obtain
\[
\left| \sum_{m' \in [M]} R_{2,m',i} \right| \leq \frac{\mu_3 \sup_{x \in \mathbb{R}^M} \sum_{m' \in [M]} |\partial^3_{m'} f'(x)|}{4n^{3/2}},
\]
for each \(i \in [n]\).

Substituting the Taylor expansions into (23). Let \(\mathcal{M}_k := \max_{m \in [M]} \mathbb{E}_X[|X_{1,m} - \mathbb{E}[X_{1,m}]|^k]\). We obtain
\[
\left| \mathbb{E}_X[f(\tilde{X})] - \mathbb{E}_W[f(\tilde{W})] \right| \\
\leq \sum_{m' \in [M]} \sum_{i \in [n]} \frac{1}{n^2} |\tilde{\sigma}^2_{m'} - \mathbb{E}_X[X^2_{1,m'}]| \mathbb{E}_X[|\partial^2_{m'} f(\tilde{X}_{-i})|] \\
+ \sum_{i \in [n]} \left( \left| \sum_{m' \in [M]} R_{1,m',i} \right| + \left| \sum_{m' \in [M]} R_{2,m',i} \right| \right) \\
\leq \frac{\mu_3 + 4\tilde{\sigma}^2_{m'} \mu_1}{4n^{1/2}} \sup_{x \in \mathbb{R}^M} \sum_{m' \in [M]} |\partial^3_{m'} f'(x)| \\
\leq \frac{c_S(\mu_3 + 4\tilde{\sigma}^2_{m'} \mu_1)}{4n^{1/2}} \sup_{x \in \mathbb{R}^M} \sum_{m' \in [M]} |\partial^3_{m'} f'(x)|,
\]
where \(c_S > 0\) is a universal constant. Here, the second inequality follows \(\mathbb{E}_X[X^2_{1,m'}] = \tilde{\sigma}^2_{m'}\), and the last inequality follows Proposition 2.1 in [15]. □

We further provide an anti-concentration inequality for a difference of two Gaussian maxima. Our result depends on the following inequality for a single Gaussian maxima by [10].

**Lemma 4** (Anti-Concentration of Gaussian Maxima: Lemma 4.3 in [11]). Let \((X_1, \ldots, X_M)^\top\) be a (possibly non-centered) \(M\)-dimensional Gaussian random vector such as \(\sigma^2_m := \text{Var}_X(X_m)\) for \(m \in [M]\). Also, we consider \(\underline{\sigma} := \min_{m \in [M]} \sigma_m\) and \(\overline{\sigma} := \max_{m \in [M]} \sigma_m\), then suppose \(0 < \underline{\sigma} < \overline{\sigma} < \infty\) holds. Then, for any \(\epsilon > 0\), we have
\[
\sup_{r \in \mathbb{R}} \mathbb{P}_X \left( r - \epsilon \leq \max_{m \in [M]} X_m \leq r + \epsilon \right) \leq \frac{2\epsilon}{\overline{\sigma}} (\sqrt{2 \log M} + 2).
\]

We apply the anti-concentration inequality to our setting.

**Lemma 5** (Conditional Anti-Concentration). Let \((W_1, \ldots, W_M)^\top\) be a (possibly non-centered) \(M\)-dimensional Gaussian random vector such as \(\sigma^2_m := \text{Var}_X(W_m)\) for \(m \in [M]\). Suppose that a covariance matrix \(\Sigma\) of \((W_1, \ldots, W_M)^\top\) is coherently positive definite (Definition 7). Then, for any \(\epsilon > 0\) and non-empty \(A \subsetneq [M]\), we obtain
\[
\sup_{r \in \mathbb{R}} \mathbb{P}_W \left( r - \epsilon \leq \max_{m \in A} W_m - \max_{m \in A^c} W_m \leq r + \epsilon \right) \leq \frac{2\epsilon}{\overline{\sigma}} (\sqrt{2 \log M} + 2),
\]

\(27\)
where $A^c := [M] \setminus A$.

**Proof of Lemma** Fix $\varepsilon > 0$ and an non-empty $A \subseteq [M]$ arbitrary. Let us define $\bar{W}_A := (W_m)_{m \in A}$ and $\bar{W}_{A^c} := (W_m)_{m \in A^c}$. For some $r \in \mathbb{R}$, we decompose the probability as

$$
\mathbb{P}_W \left( r - \varepsilon \leq \max_{m \in A} W_m - \max_{m \in A^c} W_m \leq r + \varepsilon \right) = \int_{\mathbb{R}^{|A^c|}} \mathbb{P}_{\bar{W}_A} \left( r - \varepsilon \leq \max_{m \in A} W_m - \max_{m \in A^c} \bar{w}_m \leq r + \varepsilon \mid \bar{W}_{A^c} = \bar{w}_{A^c} \right) \times \phi_{\bar{W}_{A^c}}(\bar{w}_{A^c}) d\bar{w}_{A^c},
$$

(24)

where $\phi_{\bar{W}_{A^c}}$ is a density function of $\bar{W}_{A^c}$, i.e., it is a density function of a multivariate Gaussian distribution with mean $\mu_{A^c} := \mathbb{E}_W[\bar{W}_{A^c}] \in \mathbb{R}^{|A^c|}$ and covariance matrix $\Sigma_{A^c} := \text{Var}_W(\bar{W}_{A^c}) \in \mathbb{R}^{|A^c| \times |A^c|}$.

Here, we set a non-random vector $\bar{w}_{A^c} := (w_m)_{m \in A^c} \in \mathbb{R}^{|A^c|}$.

We analyze a conditional distribution of $\bar{W}_A$ with fixed $\bar{W}_{A^c} = \bar{w}_{A^c}$. Let $\mu_A := \mathbb{E}_W[\bar{W}_A] \in \mathbb{R}^{|A|}$. Since $(\bar{W}_A, \bar{W}_{A^c})$ is a joint Gaussian random variable, the conditional distribution is regarded as a Gaussian distribution such as

$$
\bar{W}_A | \bar{W}_{A^c} \sim \mathcal{N}(\mu_A + \Sigma_{AA^c}^{-1} \Sigma_{A^c} (\bar{w}_{A^c} - \mu_{A^c}), \Sigma_A - \Sigma_{AA^c} \Sigma_{A^c}^{-1} \Sigma_{AA^c}).
$$

Note that $\Sigma_{A^c}^{-1}$ exists by the assumption of coherently positive definiteness. Then, we rewrite the probability (24) as

$$
\int_{\mathbb{R}^{|A^c|}} \mathbb{P}_{\bar{W}_A} \left( r - \varepsilon \leq \max_{m \in A} W_m - \max_{m \in A^c} \bar{w}_m \leq r + \varepsilon \mid \bar{W}_{A^c} = \bar{w}_{A^c} \right) \times \phi_{\bar{W}_{A^c}}(\bar{w}_{A^c}) d\bar{w}_{A^c} = \int_{\mathbb{R}^{|A^c|}} \mathbb{P}_{\bar{W}_A | \bar{W}_{A^c}} \left( r - \varepsilon + \max_{m \in A^c} \bar{w}_m \leq \max_{m \in A} W_m \leq r + \varepsilon + \max_{m \in A^c} \bar{w}_m \right) \times \phi_{\bar{W}_{A^c}}(\bar{w}_{A^c}) d\bar{w}_{A^c},
$$

(25)

where the last inequality follows since $\bar{W}_{A^c}$ and $\bar{W}_A | \bar{W}_{A^c}$ are independent, because they are uncorrelated Gaussian random variables. Then, we bound the probability in the integral. By Lemma[4], we obtain

$$
\mathbb{P}_{\bar{W}_A | \bar{W}_{A^c}} \left( r - \varepsilon + \max_{m \in A^c} \bar{w}_m \leq \max_{m \in A} W_m \leq r + \varepsilon + \max_{m \in A^c} \bar{w}_m \right) \leq \frac{2\sigma}{\sqrt{2 \log (|A|) + 2}} \leq \frac{2\varepsilon}{\sigma} (\sqrt{2 \log M} + 2).
$$

Substituting the inequality into (25), we obtain

$$
\mathbb{P}_W \left( r - \varepsilon \leq \max_{m \in A} W_m - \max_{m \in A^c} W_m \leq r + \varepsilon \right) \leq \frac{2\varepsilon}{\sigma} (\sqrt{2 \log M} + 2) \int_{\mathbb{R}^{|A^c|}} \phi_{\bar{W}_{A^c}}(\bar{w}_{A^c}) d\bar{w}_{A^c} = \frac{2\varepsilon}{\sigma} (\sqrt{2 \log M} + 2).
$$

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Then, we achieve the statement.

APPENDIX C. PROOF FOR BOOTSTRAP ALGORITHM (SECTION 3)

Proof of Theorem 2 We will consider the discretization of the process $B_n(f)$. To the end, for a set $A \subset \Theta$, we define notations as $B^\langle_A = \sup_{\theta \in A} B(\theta)$. Similarly, we define $P^\langle_A$ and $E^\langle_A$, respectively.

We bound the effect of discretization of $G(\theta)$ and $B_n(\theta)$.

From Lemma 1 we obtain

$$0 \leq G^\langle_A - G^\langle_{\bar{A}} \leq \mu(n, \varepsilon, \tau) \quad \text{and} \quad 0 \leq G^\langle_{A^c} - G^\langle_{\bar{A}} \leq \mu(n, \varepsilon, \tau),$$

with probability at least $1 - \exp(-\tau)$. To discretize the bootstrap process $B_n(f)$, we define

$$B^{\langle}_{\Theta} := \sup_{(\theta, \theta') \in \Theta} B_n(\theta) - B_n(\theta').$$

We also define $\Delta B^{\langle}_{\Theta}$ and $\Delta E^{\langle}_{\Theta}$, respectively. Then, we have

$$0 \leq B^\langle_A - B^\langle_{\bar{A}} \leq |\Delta B^\langle_{\Theta}| \leq |\Delta P^\langle_{\Theta}| + |\Delta E^\langle_{\Theta}|,$$  \hspace{1cm} (26)

by the definition of $B_n(\theta)$. We will bound the terms $|\Delta P^\langle_{\Theta}|$ and $|\Delta E^\langle_{\Theta}|$. About the first term $|\Delta P^\langle_{\Theta}|$, Lemma 1 yields

$$|\Delta P^\langle_{\Theta}| = |\Delta Q^\langle_{\Theta}| \leq n(\varepsilon, \tau),$$

with probability at least $1 - \exp(-\tau)$. About the second term $|\Delta E^\langle_{\Theta}|$, we define parts of the process as $E_n(\theta) = E_f(\theta) + E_p(\theta)$ such as

$$E_f(\theta) := \frac{1}{n} \sum_{i=1}^{n} e_i f_\theta(Z_i), \quad \text{and} \quad E_p(\theta) := \frac{1}{n} \sum_{i=1}^{n} e_i P_n(\theta).$$

Also, we define $\Delta E^\langle_{P,\Theta}$ and $\Delta E^\langle_{P,\Theta}$ respectively. Then, we evaluate $|\Delta E^\langle_{\Theta}|$ by the two parts as

$$|\Delta E^\langle_{\Theta}| \leq |\Delta E^\langle_{f,\Theta}| + |\Delta E^\langle_{P,\Theta}|.$$  

To bound $|\Delta E^\langle_{f,\Theta}|$, since it is a Gaussian process with fixed $Z_1, \ldots, Z_n$, the Borell-TIS inequality yields

$$|\Delta E^\langle_{f,\Theta}| \leq \mathbb{E}_{\mathcal{D}Z} \left[ |\Delta E^\langle_{f,\Theta}| \right] + \varepsilon b \sqrt{2\tau}$$

$$\leq C_K \frac{1}{n^{1/2}} \int_0^{b} \sqrt{\log \mathcal{M}(\delta, \Theta, ||\cdot||)} d\delta + \frac{1}{n^{1/2}} \varepsilon b \sqrt{2\tau}$$

$$\leq \frac{1}{n^{1/2}} J_1(\varepsilon) + \frac{1}{n^{1/2}} \varepsilon b \sqrt{2\tau},$$  \hspace{1cm} (27)

with probability at least $1 - \exp(-\tau)$. Here, the second inequality holds because $\Delta E^\langle_{f,\Theta}$ is a Gaussian process and thus we can apply the maximal inequality (Corollary 2.2.8 in [38]). $C_K > 0$.
is an existing constant. About the term $|\Delta E_{P,\Theta_e}^\vee|$, it is also a Gaussian process with fixed $Z_1, \ldots, Z_n$, then we have

$$|\Delta E_{P,\Theta_e}^\vee| \leq \mathbb{E}_{e|Z} \left[ |\Delta E_{P,\Theta_e}^\vee| \right] + \frac{1}{n^{1/2}} \varepsilon b \sqrt{2\tau}$$

$$\leq \frac{1}{n^{1/2}} \mathbb{E}_{e} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \right] |\Delta P_{\Theta_e}^\vee| + \frac{1}{n^{1/2}} \varepsilon b \sqrt{2\tau}$$

$$\leq \frac{\sqrt{2}}{\pi} |\Delta Q_{\Theta_e}^\vee| + \frac{1}{n^{1/2}} \varepsilon b \sqrt{2\tau}$$

$$\leq \frac{\sqrt{2} \nu(n, \varepsilon, \tau)}{\sqrt{\pi n}} + \frac{1}{n^{1/2}} \varepsilon b \sqrt{2\tau},$$

(28)

where the first inequality follows the Borel-TIS inequality, and the last inequality follows Lemma \[1\]. Combining (27) and (28) with (26), we obtain

$$0 \leq B_A^\vee - B_A^{\vee} \lesssim (1 + n^{-1}) \nu(n, \varepsilon, \tau) + \frac{1}{n^{1/2}} \varepsilon b \sqrt{2\tau} + \frac{1}{n^{1/2}} \vartheta_1(n, \varepsilon)$$

with probability at least $1 - 2 \exp(-\tau)$.

With the result with discretization, we apply the process in Lemma \[2\] and obtain

$$\left| \mathbb{P}_{e|Z} \left( \hat{\Theta}_B \in A \right) - \mathbb{P}_{G} \left( \hat{\Theta}_G \in A \right) \right|$$

$$\leq \mathbb{P}_{e|Z} \left( B_A^\vee - B_A^{\vee} \geq 0 \right) - \mathbb{P}_{G} \left( G_A^\vee - G_A^{\vee} \geq 0 \right)$$

$$\leq \mathbb{P}_{e|Z} \left( B_A^{\vee} - B_A^{\vee} + \nu(n, \varepsilon, \tau) \geq 0 \right) - \mathbb{P}_{G} \left( G_A^{\vee} - G_A^{\vee} + \mu(n, \varepsilon, \tau) \geq 0 \right)$$

$$+ 3 \exp(-\tau),$$

for any $\varepsilon, \tau > 0$. Then, we follow the same strategy of Theorem \[1\] with the smooth approximation. We apply the same procedure for (12), then obtain

$$\mathbb{P}_{Z} \left( Q_A^{\vee} - Q_A^{\vee} + \nu(n, \varepsilon, \tau) \geq 0 \right)$$

$$\leq \mathbb{P}_{G} \left( G_A^{\vee} - G_A^{\vee} + n^{-1/8} + \nu(n, \varepsilon, \tau) + \nu(n, \varepsilon, \tau) \geq 0 \right) + \varGamma(n, \varepsilon)$$

Further, we apply Lemma \[5\] as the process for (13) in the proof of Theorem \[1\]. We obtain that

$$\mathbb{P}_{e|Z} \left( G_A^{\vee} - G_A^{\vee} + n^{-1/8} + \nu(n, \varepsilon, \tau) + \nu(n, \varepsilon, \tau) \geq 0 \right)$$

$$\leq \mathbb{P}_{G} \left( G_A^{\vee} - G_A^{\vee} \geq 0 \right)$$

$$+ \frac{\nu(n, \varepsilon, \tau) + n^{-1/8} + \nu(n, \varepsilon, \tau) + \mu(n, \varepsilon, \tau)}{\sigma} \left( \sqrt{2 \log M(\varepsilon)} + 2 \right).$$

The opposite inequality is shown by a similar way.
We apply Lemma 7 to bound \( \phi(\varepsilon) \), then obtain
\[
\left| \mathbb{P}_{e|Z}(\widehat{\theta}_B \in A) - \mathbb{P}_G(\widehat{\theta}_G \in A) \right|
\leq 3 \exp(-\tau) + \Upsilon(n, \varepsilon)
+ \frac{\nu(n, \varepsilon, \tau) + n^{-1/8} + \nu(n, \varepsilon, \tau) + \mu(n, \varepsilon, \tau)}{n^\sigma}(\sqrt{2 \log M(\varepsilon)} + 2)
\lesssim \left\{ \varepsilon + \frac{H(\varepsilon)}{n^{1/8}} \cdot \frac{1}{n^{1/2}} + \frac{\varepsilon^{-2}J(\varepsilon)^2 + \log(1/\varepsilon)}{n^{\kappa}}(\sqrt{H(\varepsilon)} + 1)
+ \frac{J(\varepsilon)}{\varepsilon^\kappa} \cdot \frac{1}{n^{1/2}}(\sqrt{H(\varepsilon)} + 1) + \frac{1}{n^{1/2}} \varepsilon^{1-\kappa}(\sqrt{H(\varepsilon)} + 1) \sqrt{\log(1/\varepsilon)} \right\}. \tag{29}
\]

Here, we set \( \tau = \log(1/\varepsilon) \) and apply \( J_q(\varepsilon) \lesssim J_1(\varepsilon) \).

We substitute \( \varepsilon = n^{-1/8} \), then achieve the statement. Then, the same procedures of Corollary 1 and 2 provide the result.

**Proof of Proposition 2.** Since \{0\} \( \in \Theta \), we can rewrite the probability as
\[
\mathbb{P}_{Z', Z''}(\theta_0 \in \bar{A}_s^*) = \mathbb{P}_{Z', Z''}(0 \in \{ -\widehat{\theta} + \theta_s \mid \theta_s \in \bar{A}_s \}) = \mathbb{P}_{Z', Z''}(\widehat{\theta} \in \bar{A}_s).
\]

By combining Theorem 1 and Theorem 2 we can state \( \mathbb{P}_Z(\widehat{\theta} \in A) = \mathbb{P}_{e|Z}(\widehat{\theta}_B \in A) + o(1) \) for any measurable fixed \( A \subset \Theta \). Then, by the conditional we have
\[
\mathbb{P}_{Z', Z''}(\widehat{\theta} \in \bar{A}_s) = \mathbb{P}_{Z''|Z'}(\widehat{\theta} \in \bar{A}_s) \mathbb{P}_{Z'}(Z')
= \mathbb{P}_{e|Z', Z''}(\widehat{\theta}_B \in \bar{A}_s) \mathbb{P}_{Z'}(Z') + o(1)
= 1 - s + o(1),
\]
where the last equality follows the definition of \( \bar{A}_s \) with fixed \( Z' \).

**APPENDIX D. AUXILIARY RESULTS**

**Lemma 6.** For any \( \beta, \delta, \Delta > 0, A \subset [m] \) and \( x \in \mathbb{R}^M \), the following inequalities hold:
\[
\sum_{m \in [M]} |\partial_m f_{\beta, \delta, A, \Delta}(x)| \leq 2\delta^{-1},
\]
\[
\sum_{m \in [M]} |\partial_m^2 f_{\beta, \delta, A, \Delta}(x)| \leq 2C_u \delta^{-2} \beta + 2\delta^{-1} \beta,
\]
and
\[
\sum_{m \in [M]} |\partial_m^3 f_{\beta, \delta, A, \Delta}(x)| \leq C_u \delta^{-3} + 4C_u \delta^{-2} \beta + \delta^{-1} \beta^2.
\]
Proof. For a preparation, we bound $\|\partial^\ell g\|_{e^\ell}$ and $\sum_{m \in [M]} \|\partial^\ell h_{\beta,A}\|_{e^\ell}$ for $\ell = 1, 2, 3$. About $g$, some calculation yields

$$
\|\partial g\|_{e^1} \leq \delta^{-1}, \|\partial^2 g\|_{e^2} \leq C_u \delta^{-2}, \text{ and } \|\partial^3 g\|_{e^3} \leq C_u \delta^{-3},
$$

where $C_u > 0$ is an existing universal constant.

About $h_{\beta,A}$, we define a soft probability

$$
p_{\beta,m,A}(x) := \begin{cases} 
expt{\beta x_m} / \left(\sum_{m' \in A} \expt{\beta x_m'}\right), & \text{if } m \in A, \\ 0, & \text{otherwise.} \end{cases}
$$

Since $\partial_m h_{\beta,A} = p_{\beta,m,A}$, we have

$$
\sum_{m \in A} |\partial_m (h_{\beta,A}(x) - h_{\beta,A'}(x))| = \sum_{m \in A} p_{\beta,m,A}(x) + \sum_{m \in A'} p_{\beta,m,A'}(x) = 2.
$$

Also, we have

$$
\partial_m p_{\beta,m,A} = \begin{cases} 
\beta (p_{\beta,m,A} - p_{\beta,m,A}^2), & \text{if } m \in A, \\
0, & \text{otherwise},
\end{cases}
$$

hence it shows

$$
\sum_{m \in [M]} |\partial_m^2 (h_{\beta,A}(x) - h_{\beta,A'}(x))|
$$

$$
= \beta \left\{ \sum_{m \in A} |p_{\beta,m,A}(x) - p_{\beta,m,A}(x)^2| + \sum_{m \in A'} |p_{\beta,m,A'}(x) - p_{\beta,m,A'}(x)^2| \right\}
$$

$$
= \beta \left\{ \sum_{m \in A} p_{\beta,m,A}(x)(1 - p_{\beta,m,A}(x)) + \sum_{m \in A'} p_{\beta,m,A'}(x)(1 - p_{\beta,m,A'}(x)) \right\}
$$

$$
\leq 2\beta,
$$

here the last inequality follows $\sum_{m \in A} p_{\beta,m,A} = 1$ and $(1 - p_{\beta,m,A}) \leq 1$. About the second order, we obtain

$$
\partial_m^2 p_{\beta,m,A} = \begin{cases} 
\beta^2 (p_{\beta,m,A} - 3p_{\beta,m,A}^2 + 2p_{\beta,m,A}^3), & \text{if } m \in A, \\
0, & \text{otherwise.}
\end{cases}
$$

Then, we have

$$
\sum_{m \in [M]} |\partial_m^2 (h_{\beta,A}(x) - h_{\beta,A'}(x))|
$$

$$
\leq \beta^2 \sum_{m \in A} p_{\beta,m,A}(x)(1 - 3p_{\beta,m,A}(x) + 2p_{\beta,m,A}^2(x))
$$

$$
+ \beta^2 \sum_{m \in A'} p_{\beta,m,A'}(x)(1 - 3p_{\beta,m,A'}(x) + 2p_{\beta,m,A'}^2(x))
$$

$$
\leq 2\beta^2,
$$

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Assumption 1, we rewrite the values as \( \tilde{\theta} \) where both \( \tilde{G} \) and \( \tilde{Q} \) have zero mean. Using these processes and the condition (A2) in Assumption \( \Box \) we rewrite the values as

\[
\sum_{m \in [M]} |\partial_m f_{\beta, \delta, A}(x)| \leq \| \partial g_{\delta} \|_{L^\infty} \sum_{m \in [M]} |\partial_m (h_{\beta, A}(x) - h_{\beta, A^c}(x))| \leq 2\delta^{-1}.
\]

Also, about the second derivative,

\[
\sum_{m \in [M]} |\partial_m^2 f_{\beta, \delta, A}(x)| \leq \| \partial^2 g_{\delta} \|_{L^\infty} \sum_{m \in [M]} |\partial_m^2 (h_{\beta, A}(x) - h_{\beta, A^c}(x))| + 2\| \partial^2 g_{\delta} \|_{L^\infty} \sum_{m \in [M]} |\partial_m^2 (h_{\beta, A}(x) - h_{\beta, A^c}(x))| \leq 2C_u\delta^{-2} + 2\delta^{-1}\beta.
\]

About the third derivative, we obtain

\[
\sum_{m \in [M]} |\partial_m^3 f_{\beta, \delta, A}(x)| \leq \| \partial^3 g_{\delta} \|_{L^\infty} \sum_{m \in [M]} |\partial_m (h_{\beta, A}(x) - h_{\beta, A^c}(x))| + 2\| \partial^3 g_{\delta} \|_{L^\infty} \sum_{m \in [M]} |\partial_m^2 (h_{\beta, A}(x) - h_{\beta, A^c}(x))| + 2\| \partial^3 g_{\delta} \|_{L^\infty} \sum_{m \in [M]} |\partial_m^3 (h_{\beta, A}(x) - h_{\beta, A^c}(x))| \leq C_u\delta^{-3} + 4C_u\delta^{-2}\beta + \delta^{-1}\beta^2.
\]

Then, we obtain the statement.

\[\Box\]

**Lemma 7.** Suppose that the conditions (A1) and (A2) in Assumption \( \Box \) hold. Then, for \( \varepsilon \in (0, 1) \) and \( \delta \in (0, 1) \), there exists a constant \( C_\Phi > 0 \) which satisfies the following:

\[
\phi(\varepsilon) \leq C_\Phi \left\{ \frac{J_1(b\varepsilon) + J_1(2c_\ell\varepsilon)}{n^{1/2}} + (c_\ell\varepsilon)^q \right\}.
\]

**Proof of Lemma**\( \Box \) As preparation, we consider a centered version of \( Q_n(\theta) \) and \( G(\theta) \). We decompose the stochastic processes as

\[
G(\theta) = \tilde{G}(\theta) + \mathbb{E}_Z[f_\theta(Z)], \text{ and } Q_n(\theta) = \tilde{Q}_n(\theta) + \mathbb{E}_Z[f_\theta(Z)],
\]

where both \( \tilde{G}(\theta) \) and \( \tilde{Q}_n(\theta) \) have zero mean. Using these processes and the condition (A2) in Assumption \( \Box \) we rewrite the values as

\[
0 \leq \Delta G_{\Theta_e}^{\gamma} \leq \Delta \tilde{G}_{\Theta_e}^{\gamma} + \sup_{(\theta, \theta') \in \Theta_e} (\mathbb{E}_Z[f_\theta(Z) - f_{\theta'}(Z)]) \leq \Delta \tilde{G}_{\Theta_e}^{\gamma} + c_\ell\varepsilon,
\]

and

\[
0 \leq \Delta Q_{\Theta_e}^{\gamma} \leq \Delta \tilde{Q}_{\Theta_e}^{\gamma} + \sup_{(\theta, \theta') \in \Theta_e} (\mathbb{E}_Z[f_\theta(Z) - f_{\theta'}(Z)]) \leq \Delta \tilde{Q}_{\Theta_e}^{\gamma} + c_\ell\varepsilon.
\]
Here, we utilize the notations $\Delta G^\vee_{\Theta} := \sup_{(\theta, \theta') \in \Theta} \widetilde{G}(\theta) - \widetilde{G}(\theta')$ and $\Delta Q^\vee_{\Theta} := \sup_{(\theta, \theta') \in \Theta} \widetilde{Q}_n(\theta) - \widetilde{Q}_n(\theta')$.

With the results, we bound the term $\mathbb{E}_{G}[|\Delta G^\vee_{\Theta}|]$. By the maximal inequality (Corollary 2.2.8, page 101 in [38]), we obtain

$$\mathbb{E}_{G}[|\Delta G^\vee_{\Theta}|] \leq C_1 \frac{1}{n^{1/2}} \int_0^{b \varepsilon} \sqrt{\log \mathcal{N}(\delta, \Theta, \| \cdot \|)} d\delta \lesssim \frac{1}{n^{1/2}} J_1(b \varepsilon),$$

with an existing constant $C_1 > 0$.

Next, we bound the other term $\mathbb{E}_{Z}[|\Delta Q^\vee_{\Theta}|]$. As a preparation, we bound a covering number of $\Theta_{\varepsilon}$ in terms of a metric $\| \cdot \|_{\Theta_{\varepsilon}}$ such as $\|(\theta, \theta') - (\theta'', \theta''')\|_{\Theta_{\varepsilon}} := \|\theta - \theta''\| + \|\theta' - \theta''\|$ for $(\theta, \theta'), (\theta'', \theta''') \in \Theta_{\varepsilon}$. We obtain

$$\log \mathcal{N}(\delta, \Theta_{\varepsilon}, \| \cdot \|_{\Theta_{\varepsilon}}) \leq 2 \log \mathcal{N}(\delta/2, \Theta, \| \cdot \|). \quad (30)$$

Then, we rewrite $\Delta Q^\vee_{\Theta}$ by using $g_{\theta, \theta'} := f_{\theta}(\cdot) - f_{\theta'}(\cdot) - \mathbb{E}_{Z}[f_{\theta}(Z) - f_{\theta'}(Z)]$ as

$$\Delta Q^\vee_{\Theta} = \sup_{(\theta, \theta') \in \Theta_{\varepsilon}} \frac{1}{n} \sum_{i \in [n]} g_{\theta, \theta'}(Z_i).$$

Here, we define $\mathcal{G}_{\varepsilon} := \{g_{\theta, \theta'} \mid (\theta, \theta') \in \Theta_{\varepsilon}\}$. By the condition (A4) in Assumption 1, there exists an envelope function $F_{\Theta}$ for $\mathcal{G}_{\varepsilon}$ such as $\|F_{\Theta}\|_{L^\infty(p_2)} \vee \|F_{\Theta}\|_{L^2(p_2)} \leq 2b$.

By the standard Rademacher complexity technique (e.g. Section 7 in [36]) and the condition (A2) in Assumption 1, we bound the expectation of the empirical process as

$$\mathbb{E}_{Z} \left[ \sup_{g \in \mathcal{G}_{\varepsilon}} \frac{1}{n} \sum_{i \in [n]} g(Z_i) \right] \leq 12 \int_0^{2b} \sqrt{\frac{\log \mathcal{N}(\delta, \mathcal{G}_{\varepsilon}, \| \cdot \|_{L^1(p_2)})}{n}} d\delta$$

$$\leq 12 \int_0^{2be} \sqrt{\frac{\log \mathcal{N}(c_1 \delta, \Theta_{\varepsilon}, \| \cdot \|)}{n}} d\delta$$

$$\leq 12 \int_0^{2c_1be} \sqrt{\frac{2 \log \mathcal{N}(c_1 \delta/2, \Theta, \| \cdot \|)}{n}} d\delta$$

$$\leq C_2 n^{-1/2} J_1(2c_1 b \varepsilon).$$

where $C_2$ is an existing constant. The third inequality follows (30). Then, we obtain the statement.

Lemma 8 (Lemma 2.2 in [11]). If a parameter space $\Theta$ has an envelope function $F$ such that $\|F\|_{L^2(p_2)} \vee \|F\|_{L^\infty(p_2)} < \infty$ and

$$\int_0^1 \sqrt{\log \mathcal{N}(\varepsilon, \Theta, \| \cdot \|)} d\varepsilon < \infty$$

holds. Then, $\Theta$ is pre-Gaussian.
APPENDIX E. PROOF FOR SEMIPARAMETRIC M-ESTIMATOR (SECTION 4)

For convenience, we introduce additional notation $\mathcal{T} = \Theta \times \mathcal{H}$. As preparation for its proof, we consider a $2\varepsilon$-packing set $\hat{\mathcal{T}} := \{(\theta_j, \eta_j)\}_{j=1}^J$ of $\Theta \times \mathcal{H}$ as satisfying Assumption 4 such that $\| (\theta_j, \eta_j) - (\theta_j', \eta_j') \|_{\Theta \times \mathcal{H}} = \| \theta_j - \theta_j' \| + \| \eta_j - \eta_j' \|_{\mathcal{H}} \geq 2\varepsilon$ holds for $j, j' = 1, \ldots, J, j \neq j'$. We note that $J = P(\mathcal{E}, \mathcal{T}, \| \cdot \|)$ holds. Using the points for the packing set, we define $\hat{\Theta} := \{\theta_j\}_{j=1}^J$ and $\hat{\mathcal{H}} := \{\eta_j\}_{j=1}^J$. As a preparation, we define a covering number of the joint parameter space and its integral as

$$H_S(\varepsilon) := \log \mathcal{N}(\varepsilon, \mathcal{T}, \| \cdot \|_{\Theta \times \mathcal{H}}), \text{ and } J_S(\varepsilon) := \int_0^\varepsilon \sqrt{1 + H_S(\delta)} d\delta.$$

If $A \cap \hat{\Theta} \neq \emptyset$, we set $\mathcal{T}_A := \{ (\theta, \eta) \in \mathcal{T} | \theta \in A \}$, $\mathcal{T}_A^c := \{ (\theta, \eta) \in \mathcal{T} | \theta \in A^c \}$, $\hat{\mathcal{T}}_A := \{ (\theta, \eta) \in \hat{\mathcal{T}} | \theta \in A \}$ and $\hat{\mathcal{T}}_A^c := \{ (\theta, \eta) \in \hat{\mathcal{T}} | \theta \in A^c \}$. If $A \cap \hat{\Theta} = \emptyset$, we pick the closest element $\tilde{\Theta} \in \hat{\Theta}$ to $A$ and set $\mathcal{T}_A = \{ (\theta, \eta) \in \mathcal{T} | \theta = \tilde{\Theta} \}$ and $\hat{\mathcal{T}}_A = \{ (\theta, \eta) \in \hat{\mathcal{T}} | \theta \in \tilde{\Theta} \}$. We also provide several notions. For a set $\Omega \subset \Theta \times \mathcal{H}$, we define $\Gamma^{\vee}_{\Omega} := \max_{(\theta, \eta) \in \Omega} n^{-1/2} \Gamma(\theta, \eta)$ and $S^\vee_{\Omega} := \max_{(\theta, \eta) \in \Omega} S_n(\theta, \eta)$. Also, define a difference of stochastic processes as

$$\Delta_\varepsilon S_n := \sup_{(\theta, \eta) \times (\theta', \eta') \in \mathcal{T} \times \hat{\mathcal{T}} : \|(\theta, \eta) - (\theta', \eta')\|_{\Theta \times \mathcal{H}} \leq \varepsilon} S_n(\theta, \eta) - S_n(\theta', \eta'),$$

and

$$\Delta_\varepsilon \Gamma := \sup_{(\theta, \eta) \times (\theta', \eta') \in \mathcal{T} \times \hat{\mathcal{T}} : \|(\theta, \eta) - (\theta', \eta')\|_{\Theta \times \mathcal{H}} \leq \varepsilon} n^{-1/2} (\Gamma(\theta, \eta) - \Gamma(\theta', \eta')).$$

Also, we define

$$\phi(\varepsilon) := \mathbb{E}_Z [||\Delta_\varepsilon S_n||] \vee \mathbb{E}_\Gamma [||\Delta_\varepsilon \Gamma||].$$

We provide two lemmas, which are analogous to Lemma 1 and 2.

**Lemma 9.** Suppose Assumption 3 holds. Then, for any measurable non-empty $A \subset \Theta$ and $\varepsilon, \tau > 0$ and any $n \geq 1$, we obtain

$$\mathbb{P}_T \left( \Gamma^{\vee}_{\mathcal{T}_A} - \Gamma^{\vee}_{\hat{\mathcal{T}}_A} \leq \phi_S(\varepsilon) + \frac{b \sqrt{2\tau n}}{n} \right) \geq 1 - \exp(-\tau),$$

and

$$\mathbb{P}_Z \left( S^{\vee}_{\mathcal{T}_A} - S^{\vee}_{\hat{\mathcal{T}}_A} \leq \phi_S(\varepsilon) + \frac{\varepsilon \sqrt{2\tau n}}{n} + \frac{\sqrt{4\tau b^2 + 2\tau b}}{n} \right) \geq 1 - \exp(-\tau).$$

**Proof of Lemma 9** Fix measurable $A \subset \Theta$ arbitrary. Similar to the proof of Lemma 1, we can evaluate the discretization of $S_n$ as

$$0 \leq S^{\vee}_{\mathcal{T}_A} - S^{\vee}_{\hat{\mathcal{T}}_A} \leq |\Delta_\varepsilon S_n|.$$
To bound the term $\Delta e_n$, we apply the Talagrand (of Bousquet’s) inequality (page 335, Theorem 12.5 in [3]) and achieve the following bound

$$|\Delta e_n| \leq \mathbb{E}Z[|\Delta e_n|] + \frac{\epsilon \sqrt{2\tau n}}{n} + \frac{2b\sqrt{\tau} + 2\tau b}{n},$$

with probability at least $1 - \exp(-\tau)$.

About $\Gamma$, we apply the Borel-TIS inequality (page 50, Theorem 2.1.1 in [2]) and obtain

$$0 \leq \Gamma^\tau_{\mathcal{A}}(S_A^\tau - S_A^\tau) \leq |\Delta e\Gamma| \leq \mathbb{E}\Gamma[|\Delta e\Gamma|] + \frac{\epsilon \sqrt{2\tau n}}{n} \leq \phi_\epsilon + \frac{\epsilon b\sqrt{2\tau n}}{n},$$

with probability at least $1 - \exp(-\tau')$. Then, we obtain the statement. \hfill \square

Following the bounds in Lemma 9 we define

$$\nu_\epsilon(\epsilon, \tau) := \phi_\epsilon + \epsilon \frac{\sqrt{2\tau n}}{n} + \frac{2b\sqrt{\tau} + 2\tau b}{n},$$

and

$$\mu_\epsilon(\epsilon, \tau) := \phi_\epsilon + \epsilon \frac{b\sqrt{2\tau n}}{n}.$$

We also develop the following lemma to bound difference between the probabilities.

**Lemma 10.** Suppose Assumption 3 and 4 are satisfied. Then, for any measurable non-empty $A \subset \Theta, \epsilon, \tau > 0$ and any $n \geq n_\epsilon$ with an existing $n_\epsilon \in \mathbb{N}$, we obtain

$$\left| \mathbb{P}_Z\left( S_{\mathcal{A}}^\tau - S_{\mathcal{A}}^\tau \geq 0 \right) - \mathbb{P}_\Gamma \left( \Gamma^\tau_{\mathcal{A}} - \Gamma^\tau_{\mathcal{A}} \geq 0 \right) \right| \leq 3 \exp(-\tau) + \frac{C_{U,S}(\mathcal{A}, S_3 + \mathcal{A}, S_2, \mathcal{A}, S_1) H_\epsilon(\epsilon)}{4n^{1/8}} + \frac{2\nu_\epsilon(\epsilon, \tau) + \frac{n^{-1/8} + 4\mu_\epsilon(\epsilon, \tau)}{\sigma} (\sqrt{2H_\epsilon(\epsilon)} + 2)}{\sigma}.$$

**Proof of Lemma 10** We follow the line of Lemma 2. By Lemma 1 we can evaluate an effect of discretization as

$$\mathbb{P}_Z\left( S_{\mathcal{A}}^\tau - S_{\mathcal{A}}^\tau \geq 0 \right) \leq \mathbb{P}_Z\left( S_{\mathcal{A}}^\tau - S_{\mathcal{A}}^\tau + \nu_\epsilon(\epsilon, \tau) \geq 0 \right) + \exp(-\tau).$$

Now, we will bound $\mathbb{P}_Z(S_{\mathcal{A}}^\tau - S_{\mathcal{A}}^\tau + \nu_\epsilon(\epsilon, \tau) \geq 0)$ by the probability $\mathbb{P}_G(\Gamma^\tau_{\mathcal{A}} - \Gamma^\tau_{\mathcal{A}} + \nu_\epsilon(\epsilon, \tau) \geq 0)$ as Lemma 2. We define a random vector $\tilde{Z}(S) := (n^{-1/2} \sum_{i \in [n]} f_{\theta, \eta}(Z_i))_{(\theta, \eta) \in \mathcal{F}}$, and thus $S_{\mathcal{A}}^\tau = n^{-1/2} \max_{(\theta, \eta) \in \mathcal{F}} \tilde{Z}(S)$ and $S_{\mathcal{A}}^\tau = n^{-1/2} \max_{(\theta, \eta) \in \mathcal{F}} \tilde{Z}(S)$. Then, we utilize the function $f_{\beta, \delta, A, \Delta}$
defined in (20) for a smooth approximation of $S_{\overline{\mathcal{A}}} - S_{\mathcal{F}}$. As similar to the inequality in (21), we obtain

$$
\mathbb{P}_Z \left( S_{\overline{\mathcal{A}}} - S_{\mathcal{F}} + v_S(\varepsilon, \tau) \geq 0 \right)
$$

$$
= \mathbb{P}_Z \left( \max_{(\theta, \eta) \in \overline{\mathcal{A}}} \bar{Z}^{(S)} - \max_{(\theta, \eta) \in \mathcal{F}} \bar{Z}^{(S)} + v_S(\varepsilon, \tau) \geq 0 \right)
$$

$$
\leq \mathbb{E}_Z[f_\beta, \delta, A_M, \Xi(M, \beta) + v_S(\varepsilon, \tau)(\bar{Z}^{(S)})],
$$

(31)

with arbitrary parameters $\beta, \delta > 0$. Then, we bound the expectation by $\mathbb{E}_\Gamma[f_\beta, \delta, A_M, \Xi(M, \beta) + v_S(\varepsilon, \tau)(\bar{W}^{(\Gamma)})]$ where $\bar{W}^{(\Gamma)} := (\Gamma(\theta, \eta))_{(\theta, \eta) \in \overline{\mathcal{F}}}$ is a Gaussian random vector. By a centorization, Proposition [4] and Lemma [6] we obtain

$$
\mathbb{E}_Z[f_\beta, \delta, A_M, \Xi(M, \beta) + v_S(\varepsilon, \tau)(\bar{Z}^{(S)})]
$$

$$
\leq \mathbb{E}_\overline{W}[f_\beta, \delta, A_M, \Xi(M, \beta) + v_S(\varepsilon, \tau)(\bar{W}^{(\Gamma)})]
$$

$$
+ C_\Gamma(-\mathcal{H}, 3 + 4 \mathcal{H}, 1, 4) \left( \delta^{-3} + \delta^{-2} \beta + \delta^{-1} \beta^2 \right)
$$

$$
\leq \mathbb{E}_\overline{W}[f_\beta, \delta, A_M, \Xi(M, \beta) + v_S(\varepsilon, \tau)(\bar{W}^{(\Gamma)})]
$$

$$
+ C_\Gamma(-\mathcal{H}, 3 + 4 \mathcal{H}, 1, 4) \log M(\varepsilon) \frac{1}{4n^{1/8}} \left( \log \frac{1}{4n^{1/8}}, \gamma_S(n, \varepsilon) \right).
$$

(32)

where the last inequality follows by setting $\beta = n^{1/8} \log^{1/2} M(\varepsilon)$ and $\delta = n^{-1/8}$.

Further, we bound $\mathbb{E}_\overline{W}[f_\beta, \delta, A_M, \Xi(M, \beta) + v_S(\varepsilon, \tau)(\bar{W}^{(\Gamma)})]$ by the similar process in (22) as

$$
\mathbb{E}_W[f_\beta, \delta, A_M, \Xi(M, \beta) + v_S(\varepsilon, \tau)(\bar{W}^{(\Gamma)})]
$$

$$
\leq \mathbb{P}_{\overline{W}} \left( \max_{(\theta, \eta) \in \overline{\mathcal{A}}} \bar{W}^{(\Gamma)} - \max_{(\theta, \eta) \in \mathcal{F}} \bar{W}^{(\Gamma)} + \delta + 2v_S(\varepsilon, \tau) \geq 0 \right).
$$

$$
= \mathbb{P}_\Gamma \left( \frac{\Gamma^{(S)}_{\overline{\mathcal{A}}} - \Gamma^{(S)}_{\mathcal{F}} + n^{-1/8} + 2v_S(\varepsilon, \tau) \geq 0}{\gamma_S(n, \varepsilon)} \right).
$$

(33)

Combining (31), (32) and (33), we obtain

$$
\mathbb{P}_Z \left( S_{\overline{\mathcal{A}}} - S_{\mathcal{F}} + v_S(\varepsilon, \tau) \geq 0 \right)
$$

$$
\leq \mathbb{P}_\Gamma \left( \frac{\Gamma^{(S)}_{\overline{\mathcal{A}}} - \Gamma^{(S)}_{\mathcal{F}} + n^{-1/8} + 2v_S(\varepsilon, \tau) \geq 0}{\gamma_S(n, \varepsilon)} \right) + \gamma_S(n, \varepsilon)
$$

(34)

Next, we will bound the probability in (12) by the conditional anti-concentration inequality in Lemma[5]. For any $\zeta > 0$, Applying the inequality yields that

$$
\mathbb{P}_\Gamma \left( \Gamma^{(S)}_{\overline{\mathcal{A}}} - \Gamma^{(S)}_{\mathcal{F}} + n^{-1/8} + 2v_S(\varepsilon, \tau) \geq 0 \right)
$$
\[ = \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq \zeta \right) \]  
\[ + \mathbb{P}_\Gamma \left( \zeta > \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq -(n^{-1/8} + 2\nu_S(e, \tau)) \right) \]  
\[ \leq \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq 2\nu_S(e, \tau) + n^{-1/8} - \zeta \right) \]  
\[ + \left( \frac{2}{9} \sqrt{2H_5(e) + 2} \right). \]  

Then, we combine the results (34) and (36), then we obtain

\[ \mathbb{P}_S \left( S_{\mathcal{F}_A}^\vee - S_{\mathcal{F}_A}^\vee \geq \zeta \right) \]  
\[ \leq \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq \zeta \right) \]  
\[ + \nu_S(n, e, \tau, \zeta) + \Upsilon_S(n, e) + \exp(-\tau). \]  

We will bound the term \( \mathbb{P}_\Gamma(\Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq \zeta) \) by evaluating an effect of the discretization of \( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \). We obtain

\[ \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq \zeta \right) \]  
\[ = \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq \zeta \text{ and } \left| \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) - \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) \right| \leq \zeta \right) \]  
\[ + \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq \zeta \text{ and } \left| \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) - \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) \right| > \zeta \right) \]  
\[ \leq \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq 0 \right) + \mathbb{P}_\Gamma \left( \left| \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) - \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) \right| > \zeta \right). \]

About the last term, by Lemma\(^9\) we obtain

\[ \mathbb{P}_\Gamma \left( \left| \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) - \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right) \right| > \zeta \right) \]  
\[ \leq \mathbb{P}_\Gamma \left( \left| \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right| > \zeta / 2 \right) + \mathbb{P}_\Gamma \left( \left| \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \right| > \zeta / 2 \right) \]  
\[ \leq 2 \exp(-\tau), \]

where the last inequality follows Lemma\(^9\) and setting \( \zeta = 2\mu_S(e, \tau) \). Then, as substituting the result into (37), we obtain

\[ \mathbb{P}_Z \left( S_{\mathcal{F}_A}^\vee - S_{\mathcal{F}_A}^\vee \geq 0 \right) \]  
\[ \leq \mathbb{P}_\Gamma \left( \Gamma_{\mathcal{F}_A}^\vee - \Gamma_{\mathcal{F}_A}^\vee \geq 0 \right) + 3 \exp(-\tau) + \nu_S(n, e, \tau, 2\mu_S(e, \tau)) + \Upsilon_S(n, e). \]

About an opposite inequality, we can bound it by the same way, then we obtain the statement. \( \square \)
Proof of Theorem 3. If $A = \emptyset$ or $A = \Theta$, the result obviously holds. Hence, in the following, we consider $A \subseteq \Theta$ is non-empty. This proof follows the line in the proof of Theorem 1 with replacing $\Theta$ by $\mathcal{T} = \Theta \times \mathcal{H}$.

We consider an approximation for the distribution of $\hat{\Theta}$. For any measurable set $A \subset \Theta$, we have

$$\mathbb{P}_Z(\hat{\Theta} \in A) = \mathbb{P}_Z(S_{\hat{\mathcal{G}}_A}^\vee \geq S_{\hat{\mathcal{F}}_A}^\vee) = \mathbb{P}_Z(S_{\hat{\mathcal{G}}_A}^\vee - S_{\hat{\mathcal{F}}_A}^\vee \geq 0).$$

Similarly, we have

$$\mathbb{P}_T(\hat{\Theta} \in A) = \mathbb{P}_T(\Gamma_{\hat{\mathcal{G}}_A}^\vee \geq \Gamma_{\hat{\mathcal{F}}_A}^\vee) = \mathbb{P}_Z(\Gamma_{\hat{\mathcal{G}}_A}^\vee - \Gamma_{\hat{\mathcal{F}}_A}^\vee \geq 0).$$

We apply Lemma 10 and obtain

$$\left| \mathbb{P}_Z(\hat{\Theta} \in A) - \mathbb{P}_T(\hat{\Theta} \in A) \right| = \left| \mathbb{P}_Z(S_{\hat{\mathcal{G}}_A}^\vee - S_{\hat{\mathcal{F}}_A}^\vee \geq 0) - \mathbb{P}_T(\Gamma_{\hat{\mathcal{G}}_A}^\vee - \Gamma_{\hat{\mathcal{F}}_A}^\vee \geq 0) \right| \leq 3 \exp(-\tau) + \Upsilon(n, \varepsilon) + 2\nu_S(n, \varepsilon, \tau, 2\mu(n, \varepsilon, \tau)),$$

where $\Upsilon(n, \varepsilon)$ is defined in the proof of Lemma 10. Then, we obtain

$$\left| \mathbb{P}_Z(\hat{\Theta} \in A) - \mathbb{P}_G(\hat{\Theta}_G \in A) \right| \leq C_U \left\{ \exp(-\tau) + \frac{C_\sigma H_S(\varepsilon)}{n^{1/8}} + \frac{C_\sigma}{\sigma} \left( \phi_S(\varepsilon) + \frac{\varepsilon b \sqrt{\tau} n}{n} + \frac{b \sqrt{\tau} + \tau b}{n} \right) \left( \sqrt{H_S(\varepsilon)} + 1 \right) \right\},$$

where $C_U > 0$ is a universal constant and $C_\sigma > 0$ is a constant depends on $\bar{M}_{S,3}, \bar{M}_{S,2}$ and $\bar{M}_{S,1}$.

We will bound $\phi_S(\varepsilon)$ by the condition (A4) in Assumption 3. For $q \in (0, 1]$ we define its integral as

$$J_{S,q}(\varepsilon) := \int_0^\varepsilon \sqrt{1 + \log \mathcal{N}(\delta^q, \mathcal{T}, ||\Theta \times \mathcal{H}||)}d\delta.$$

By Lemma 7, we obtain

$$\phi(\varepsilon) \leq \frac{1}{n^{1/2}} \left( J_{S,1}(\varepsilon) + b J_{S,q}(2bc_\ell \varepsilon') + n^{-1/2} \varepsilon' - 2 J_{S,q}(2bc_\ell \varepsilon')^2 \right), \hspace{1cm} (38)$$

for all $\varepsilon, \varepsilon' \in (0, 1]$, and a constant $C_\Phi > 0$.

We substitute the bound for $\phi_S(\varepsilon)$ and also set $\tau = \log(1/\varepsilon)$. Then, we apply Assumption 4 and obtain

$$\left| \mathbb{P}_Z(\hat{\Theta} \in A) - \mathbb{P}_T(\hat{\Theta} \in A) \right| \leq C_U C_\sigma C_\Phi C_b \left\{ \varepsilon + \frac{H_S(\varepsilon)}{n^{1/8}} + \frac{1}{n^{1/2}} + \frac{\varepsilon' - 2 J_{S,q}(\varepsilon')^2 + \log(1/\varepsilon)}{n^{1/2} \varepsilon^\kappa} \left( \sqrt{H_S(\varepsilon)} + 1 \right) \right. \right. \right.$$  

$$\left. \left. + \frac{J_{S,1}(\varepsilon) + J_{S,q}(\varepsilon')}{\varepsilon^\kappa} \cdot \frac{1}{n^{1/2}} \left( \sqrt{H_S(\varepsilon)} + 1 \right) + \frac{1}{n^{1/2}} \varepsilon' \cdot \frac{1}{n^{1/2}} \varepsilon^{1-\kappa} \left( \sqrt{H_S(\varepsilon)} + 1 \right) \right\},$$

where $C_U'$ is another universal coefficient and $C_b > 0$ is a constant depends on $b, c_\ell, q$ and $\kappa$. 

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The ordinary calculation for the Dudley integrals yields
\[ J_{S,1}(\varepsilon) \lesssim \varepsilon^{1-\alpha/2}, \] and \[ J_{S,q}(\varepsilon') \lesssim \varepsilon'^{1-\alpha/2+2(1-q)/q}. \]

Here, we set \( \varepsilon' = \varepsilon \) and obtain \( J_S(\varepsilon) = J_{S,1}(\varepsilon) \lesssim J_{S,q}(\varepsilon) \) for any \( q \in (0,1] \). Then, we have the abstract statement
\[ |P_Z(\hat{\theta} \in A) - P_G(\hat{\theta} \in A)| \leq C \left\{ \varepsilon + \frac{H_S(\varepsilon)}{n^{5/8}} + \frac{\Delta_S(n,\varepsilon)}{n^{\varepsilon}} \sqrt{H_S(\varepsilon) + 1} \right\}, \]
where
\[ \Delta_S(n,\varepsilon) = \varepsilon^{-2}J_S(\varepsilon)^2 + \log(1/\varepsilon) + \sqrt{n}J_S(\varepsilon) + \varepsilon \sqrt{\log(1/\varepsilon)}. \]

The rest of this proof is same to the proof of Corollary 2. \( \square \)

APPENDIX F. PROOF FOR NON-DONSKER CLASS (SECTION 5)

Proof of Theorem 2 Fix \( A \subset \Theta \) and recall the notation \( A^c = \Theta \setminus A \). We define \( A_K = A \cap \Theta_K \) and \( A_K^c = A^c \cap \Theta_K \). We also define \( \tilde{\Theta}_{K,M} \) be a \( 2\varepsilon \)-packing set of \( \Theta_K \) with its cardinarity \( M \). Similarly, we also utilize \( A_{K,M} := \tilde{\Theta}_{K,M} \cap A \) and \( A_{K,M}^c := \tilde{\Theta}_{K,M} \cap A^c \).

We prepare some notations. We define a covering number of \( \Theta_K \) and its integral as
\[ H_K(\varepsilon) := \log \mathcal{N}(\varepsilon, \Theta_K, \| \|), \] and \( J_K(\varepsilon) := \int_0^\varepsilon \sqrt{1 + H_K(\delta)} d\delta. \)
We also define \( \nu_K(n,\varepsilon,\tau) \) and \( \mu_K(n,\varepsilon,\tau) \) as analogous to the proof of Theorem 1 by replacing \( \Theta \) by \( \Theta_K \). The existence of the Gaussian process is guaranteed by Lemma 8. We achieve the bound as follows:
\[
P_Z(\hat{\theta} \in A) \\
\leq P_Z \left( \max_{\theta \in A_{K,M}} Q_n(\theta) - \max_{\theta \in A_{K,M}^c} Q_n(\theta) + \Delta(\hat{K}) + \nu_K(n,\varepsilon,\tau) \geq 0 \right) + \exp(-\tau) \\
\leq P_G \left( \max_{\theta \in A_{K,M}} G(\theta) - \max_{\theta \in A_{K,M}^c} G(\theta) + \Delta(\hat{K}) + n^{-1/8} + \nu_K(n,\varepsilon,\tau) \geq 0 \right) \\
+ \Upsilon(n,\varepsilon) + \exp(-\tau) \\
\leq P_G \left( \max_{\theta \in A} G(\theta) - \max_{\theta \in A^c} G(\theta) + 2\Delta(\hat{K}) + n^{-1/8} + \nu_K(n,\varepsilon,\tau) + \mu_K(n,\varepsilon,\tau) \geq 0 \right) : \delta_K \\
+ \Upsilon(n,\varepsilon) + \exp(-\tau) + C\delta_K \\
\leq P_G \left( \hat{\theta} \in A_{\delta_K} \geq 0 \right) + \Upsilon(n,\varepsilon) + \exp(-\tau) + C\delta_K. \]
The second last inequality follows Assumption[6] Then, by the Strassen’s theorem (Lemma 4.1 in [11]), we obtain
\[ P \left( |\hat{\theta} - \theta_k| > \delta_K \right) \leq \gamma(n, \varepsilon) + \exp(-\tau) + C\delta_K. \]

We substitute \( \tau = \log(1/\varepsilon) \) and obtain
\[ P \left( |\hat{\theta} - \theta_k| > C'(K^{-\lambda} + n^{-1/8} + \sqrt{K\varepsilon} + K\varepsilon n^{-1/2}) \log(1/\varepsilon) \right) \]
\[ \leq \varepsilon + \frac{K\log(1/\varepsilon)}{n^{5/8}} C'(K^{-\lambda} + \sqrt{K\varepsilon} + K\varepsilon n^{-1/2}) \log(1/\varepsilon), \]
with sufficiently large \( C' > 0 \). We set \( \varepsilon = Kn^{-5/8} \) and \( K = n^{1/3} \), then we obtain the result. □

APPENDIX G. PROOF FOR EXAMPLES (SECTION 6)

We provide a technical details of Section[6] for the explicit M-estimators as example.

G.1. Cubic root estimator. We provide the proof for the cubic root estimator in Section[6]. Let \( \Theta \ni \theta \) be a compact interval \( \Theta = [0, 1] \) with \( \|\theta\| = |\theta| \) and assume \( P_Z \) as a positive finite density on \( \mathcal{X} = [0, 1] \).

Proof of Corollary[5] We verify that the assumptions are satisfied. About Assumption[1] \( \Theta \) is a subset of \( \mathbb{R} \), hence the conditions (A1) and (A2) hold. We have an envelope function \( F(x) = 1 \) such as \( \|F\|_{L^2(p)} \vee \|F\|_{L^n} \) by 1, hence the condition (A3) is also satisfied. About the condition (A4), a simple calculation yields \( \|f_\theta - f_{\theta'}\|_{L^1} = 2|\theta' - \theta| \) and it is satisfied with \( q = 1 \) and \( c = 2 \).

To verify the condition (A2) in Assumption[1] we calculate \( \|f_\theta - f_{\theta'}\|_{L^1} \) with \( \theta, \theta' \in \Theta \) with \( \theta \leq \theta' \) as
\[
\|f_\theta - f_{\theta'}\|_{L^1} = \int_{\mathcal{X}} |1\{\theta - 1 \leq z \leq \theta + 1\} - 1\{\theta' - 1 \leq z \leq \theta' + 1\}| dz
\]
\[ = 2|\theta' - \theta|. \]

Hence, the condition (A4) holds with \( q = 1 \) and \( c_\ell = 2 \).

We validate Assumption[2] As preparation, we consider there are \( p + 1 \) grids \( \{\theta_j\}_{j=1}^{p+1} \subset \Theta \) and they are equally spaced such as \( |\theta_j - \theta_j'| = \delta|j - j'| \) with \( \delta > 0, \delta p \leq 1 \). By the grids, we can specify the covariance matrix \( \Sigma \) whose (\( j, j' \))-th element is \( \text{Cov}_Z(f_{\theta_j}(Z), f_{\theta_j}(Z)) \) for \( j, j' \in [p + 1] \). The covariance it written as
\[
\text{Cov}_Z(f_{\theta_j}(Z), f_{\theta_j}(Z)) = \mathbb{E}_Z[(f_{\theta_j}(Z) - 1/2)(f_{\theta_j}(Z) - 1/2)]
\]
\[ = \mathbb{E}_Z[f_{\theta_j}(Z)f_{\theta_j}(Z)] - \mathbb{E}[f_{\theta_j}(Z) + f_{\theta_j}(Z)]/2 + 1/4
\]
\[ = \mathbb{E}_Z[1\{(\theta' \lor \theta) - 1 \leq Z \leq (\theta \land \theta') + 1\}] - 1/4
\]
\[ = 2 - |\theta_j - \theta_j| - 1/4
\]
\[ = 7/4 - \delta|j - j'|. \tag{39} \]

Then, we obtain the covariance matrix form[4]. By Proposition[3] we obtain the statement. □
Lemma 11 plays a critical role for the analysis on the cubic root estimator. To prove the result, our aim is to develop a lower bound for diagonal elements of a conditional covariance matrix by $\Sigma$.

**Proof of Proposition 3.** For brevity, we only investigate a conditional variance on the first grid $\theta_1$ as

$$\text{Var}(f_{\theta_1}(z)|f_{\theta_2}(z), \ldots, f_{\theta_{p+1}}(z)) = \text{Var}(f_{\theta_1}(z)) - \Sigma_{2:p+1,1}\Sigma_{2,p+1:1}^{-1}\Sigma_{1,p+1:1}.$$

Conditional variance on the other parameter grids is bounded by a similar way. For the derivation, we define a $p \times p$ matrix $M^{(p)}$ with its $(i, j)$-th element is

$$c - \delta|i - j|.$$

for $i, j = 1, \ldots, p$, and a vector $v^{(p)} \in \mathbb{R}^p$ as

$$v^{(p)} := \begin{pmatrix} c - \delta \\ c - 2\delta \\ \vdots \\ c - p\delta \end{pmatrix}.$$

Then, we can rewrite the conditional variance as

$$\text{Var}(f_{\theta_1}(z)|f_{\theta_2}(z), \ldots, f_{\theta_{p+1}}(z)) = c - (v^{(p)})^\top (M^{(p)})^{-1}v^{(p)} \quad (40)$$

To derive the inverse matrix $(M^{(p)})^{-1}$, we utilize a cofactor matrix based approach. Let $\hat{M}^{(p)} \in \mathbb{R}^{p \times p}$ be a sign-adjusted cofactor matrix of $M^{(p)}$, i.e., $\hat{M}^{(p)} = m^{(p)}_{i,j} := (-1)^{(i+j)}|M^{(p)}_{i,-j}|$ where $M^{(p)}_{i,-j}$ is a $(p - 1) \times (p - 1)$ sub-matrix of $M^{(p)}$ by eliminating its $i$-th row and $j$-th column. Then, we can reform the inverse matrix as

$$(v^{(p)})^\top (M^{(p)})^{-1}v^{(p)} = |M^{(p)}|^{-1}(v^{(p)})^\top \hat{M}^{(p)}v^{(p)},$$

for arbitrary $i \in [p]$. To achieve the value of $|M^{(p)}|$ and $\hat{M}^{(p)}$, we derive an explicit value of $m^{(p)}_{i,j}$ in the following result:

**Lemma 11.** For each integer $p \geq 3$, we obtain the following form:

- $m^{(p)}_{1,1} = m^{(p)}_{p,p} = 2^{p-2}c\delta^{p-2} - (p - 2)2^{p-3}\delta^{p-1},$
- $m^{(p)}_{i,i} = 2^{p-1}c\delta^{p-2} - (p - 1)2^{p-2}\delta^{p-1}, \quad (i = 2, \ldots, p - 1),$
- $m^{(p)}_{i,j} = -2^{p-2}c\delta^{p-2} + (p - 1)2^{p-3}\delta^{p-1}, \quad (|i - j| = 1),$
- $m^{(p)}_{1,p} = m^{(p)}_{p,1} = 2^{p-3}\delta^{p-1},$
- $m^{(p)}_{i,j} = 0, \quad \text{(otherwise)}.$
Furthermore, we obtain

$$|M^{(p)}| = 2^{p-1}c\delta^{p-1} - (p-1)2^{p-2}\delta^p.$$  

Proof of Lemma \[\text{II}\] The result is shown by the mathematical induction. When \(p = 3\), simple calculation yields

\[
\tilde{M}^{(3)} = \begin{pmatrix}
m^{(3)}_{1,1} & m^{(3)}_{1,2} & m^{(3)}_{1,3} \\
m^{(3)}_{2,1} & m^{(3)}_{2,2} & m^{(3)}_{2,3} \\
m^{(3)}_{3,1} & m^{(3)}_{3,2} & m^{(3)}_{3,3}
\end{pmatrix} = \begin{pmatrix}
2c\delta - \delta^2 & -2c\delta + 2\delta^2 & \delta^2 \\
-2c\delta + 2\delta^2 & 4c\delta - 4\delta^2 & -2c\delta + 2\delta^2 \\
\delta^2 & -2c\delta + 2\delta^2 & 2c\delta - \delta^2
\end{pmatrix}.
\]

Also, we have

$$|M^{(3)}| = cm^{(3)}_{1,1} + (c - \delta)m^{(3)}_{1,2} + (c - 2\delta)m^{(3)}_{1,3} = 4c\delta^2 - 4\delta^3.$$  

Hence, the statement holds with \(p = 3\).

Suppose that the statement is valid with \(p = k - 1\), then consider the case with \(p = k\). Then, troublesome but standard calculation yields

$$m^{(k)}_{1,1} = m^{(k)}_{k,k} = \sum_{i=1}^{k} (c - (i-1)\delta)m^{(k-1)}_{1,i} = 2^{k-2}c\delta^{k-2} - (k-2)2^{k-3}\delta^{k-1}.$$  

For the other terms \(m^{(k)}_{i,i}, m^{(k)}_{i,j}\) and \(m^{(k)}_{i,k}\), the same way yields the desired result. Also, we obtain

$$|M^{(k)}| = \sum_{i=1}^{k} (c - (i-1)\delta)m^{(k)}_{1,i} = 2^{k-1}c\delta^{k-1} - (k-1)2^{k-2}\delta^k.$$  

Then, we obtain the statement.

By this lemma, we have

\[
(b^{(p)})^T \tilde{M}^{(p)} b^{(p)} = \sum_{i=1}^{p} \sum_{j=1}^{p} (c - i\delta)(c - j\delta)m^{(p)}_{i,j}
\]

\[
= (c - \delta)^2(2^{p-1}c\delta^{p-2} - (p-2)2^{p-3}\delta^{p-1})
\]

\[
+ (c - p\delta)^2(2^{p-1}c\delta^{p-2} - (p-2)2^{p-3}\delta^{p-1})
\]

\[
+ \sum_{i=2}^{p-1} (c - i\delta)^2(2^{p-1}c\delta^{p-2} - (p-1)2^{p-2}\delta^{p-1})
\]

\[
+ 2\sum_{i=1}^{p-1} (c - i\delta)(c - (i+1)\delta) - 2^{p-2}c\delta^{p-2} + (p-1)2^{p-3}\delta^{p-1}
\]

\[
+ 2(c - \delta)(c - p\delta)2^{p-3}\delta^{p-1}
\]

\[
= 2^{p-2}\delta^{p-1}(2c^2 - (3 + p)c\delta + 2p\delta^2).
\]
By combining this result with (40), we achieve
\[
c - (b^{(p)})^\top (M^{(p)})^{-1} b^{(p)} = c - \frac{2^{p-2} \delta^{p-1} (2c^2 - (3 + p)c\delta + 2p\delta^2)}{|M^{(p)}|}
\]
\[
= c - \frac{2^{p-2} \delta^{p-1} (2c^2 - (3 + p)c\delta + 2p\delta^2)}{2^{p-1} c \delta^{p-1} - (p - 1) 2^{p-2} \delta^p}
\]
\[
= \frac{2\delta (2c - p\delta)}{2c - (p - 1)\delta}
\]
\[
\geq \delta.
\]

Then, we achieve the statement. □

G.2. Least Absolute Deviation Regression. We provide the details of the least absolute deviation regression problem in Section 6.2.

Proof of Corollary 6. We set \( \mathcal{Z} = \mathcal{Z} \times \mathcal{Y} = [-1, 1] \times [-1, 1] \) and set \( \Theta = [-1, 1] \). Without the loss of generality, we set \( Y \) is a uniform random variable on \([0, 1/2]\) and \( X = 1 \) almost surely for brevity. In this case, the conditions (A1), (A3), and (A4) in Assumption 1 are satisfied. In this setting, \( f_\theta \) is bounded by 2, so we obtain \( \|F\|_{L^2(P_2)} \leq \|F\|_{L^2(P_2)} \leq 2 \). For the condition (A2), we bound the difference \( \|f_\theta - f_{\theta'}\|_{L^2} \) for \( \theta, \theta' \in \Theta \). We have a sub-additivity of the norm \( |x + x'| \leq |x| + |x'| \), hence we can utilize the reverse triangle inequality \( ||x| - |x'|| \leq |x - x'| \). Then,

\[
\|f_\theta - f_{\theta'}\|_{L^1} = \int |y - x\theta| - |y - x\theta'| |dP_z(x, y)
\]
\[
\leq \int |x\theta - x\theta'| |d\lambda(x)
\]
\[
\leq |\theta - \theta'| \int |x| |d\lambda(x)
\]
\[
\leq ||\theta - \theta'||.
\]

Hence, the condition (A2) holds with \( q = 1 \) and \( c = 1 \).

We investigate Assumption 2 for this regression model. Similar to the previous section, we consider \( p + 1 \) grids \( \{\theta_j\}_{j=1}^{p+1} \subset \Theta \) and they are equally spaced such as \( |\theta_j - \theta_j'| = \delta |j - j'| \) with \( \delta > 0, \delta p \leq 1/2 \). By the grids, we can specify the covariance matrix \( \Sigma \) whose \((j, j')\)-th element is \( \text{Cov}_Z(f_{\theta_j}(Z), f_{\theta_{j'}}(Z)) \) for \( j, j' \in [p+1] \). Consider parameters \( \theta, \theta' \in \{\theta_j\}_{j=1}^{p+1} \) such that \( \theta' \geq \theta \). Its covariance is written as

\[
\text{Cov}_Z(f_{\theta_j}(Z), f_{\theta'_{j'}}(Z)) = \mathbb{E}_Y \left[ (Y - \theta_j) (Y - \theta'_{j'}) \right] - \mathbb{E}_Y \left[ (Y - \theta_j) \right] \mathbb{E}_Y \left[ (Y - \theta'_{j'}) \right].
\] (41)

A simple calculation yields

\[
\mathbb{E}_Y \left[ (Y - \theta_j) \right] = \mathbb{P}_Y (Y \leq \theta_j) \mathbb{E}_Y \left[ (Y - \theta) \mid Y \leq \theta_j \right] + \mathbb{P}_Y (Y > \theta_j) \mathbb{E}_Y \left[ (Y - \theta) \mid Y > \theta_j \right]
\]
\[
= \theta_j^2 - 3\theta_j + \frac{9}{2}.
\]
By a similar way, we obtain

\[
\mathbb{E}_Y [\|Y - \theta\|_Y - \theta'] = 9 - 9\theta + 3\theta^2 - \frac{9}{2}\theta - \theta' + \frac{1}{3}\theta' - \theta^3 + 3\theta|\theta - \theta'|.
\]

Substituting the results into (41), we obtain

\[
\text{Cov}_Z(f_{\theta}(Z), f_{\theta'}(Z)) = C_0(\theta) - C_1(\theta)|\theta' - \theta| + C_2(\theta)|\theta' - \theta|^2 + \frac{|\theta' - \theta|^3}{3},
\]

where \(C_0(\theta) = 576/64 - 71\theta/8 + 5\theta^2/2 + \theta^3 - 3\theta^4, C_1(\theta) = 71/16 - 5\theta/2 - 3\theta^2/2 + 2\theta^3, \) and \(C_2(\theta) = -1/8 + \theta/2 - \theta^2/2.\) We can simply rewrite the covariance as

\[
\text{Cov}_Z(f_{\theta}(Z), f_{\theta'}(Z)) = C_0(\theta) - C_1(\theta)/16 + (C_2(\theta) + 1/48)|\theta' - \theta|^2
\]

\[
+ (C_1(\theta) - |\theta' - \theta|^2/3)(1/16 - |\theta' - \theta|)
\]

\[
= C_3(\theta) + (5/48 - \theta^2/2 + \theta^3)(-\theta^2 + 2\theta')
\]

\[
= :I(\theta, \theta')
\]

\[
+ (C_4(\theta) - \theta^2/3 + 2\theta\theta'/3)(1/16 - |\theta' - \theta|),
\]

where \(C_3(\theta) = 2233/256 - 279\theta/32 - 241\theta^2/16 - 13\theta^3/32 - 3\theta^4/2\) and \(C_4(\theta) = 19/4 - 5\theta/2 - \theta^2 + 2\theta^3.\)

Now, we consider the covariance matrix \(\Sigma \in \mathbb{R}^{p \times p}\) by a matrix form. Based on the covariance form of \(I(\theta, \theta')\) and \(II(\theta, \theta')\), we utilize matrices \(M^A, M^B, M^C, M^D, M^E \in \mathbb{R}^{p \times p}\) and obtain

\[
\Sigma = M^A + (M^B + M^C + M^D)M^E,
\]

where there elements are \(M^E_{i,j} = 1/16 - |\theta_i - \theta_j|, M^A_{i,j} = I(\theta_i, \theta_j), M^B_{i,j} = 1(i = j)C_4(\theta_i), M^C_{i,j} = 1(i = j)(-\theta_j^2/3), \) and \(M^D_{i,j} = 2\theta_i\theta_j/3.\) We note that \(M^B\) and \(M^C\) are diagonal matrices, since their elements only depend on one of \(\theta\) or \(\theta'.\)

We will develop a lower bound of \(\lambda_{\min}(\Sigma)\) by evaluating an eigenvalue and positive definiteness of the matrices. For \(M^E,\) Proposition \(\square\) states that \(\lambda_{\min}(M^E) \geq \delta\) as \(\delta \to 0.\) With a vector \(b = (\theta_1, ..., \theta_p),\) we obtain a form \(M^D = bb^\top,\) hence \(M^D\) has one positive eigenvalue and other eigenvalues are zero. Then, it is positive semi-definite. Since \(M^B\) and \(M^D\) are diagonal, their minimum diagonal elements represent their minimum eigenvalue. Since \(C_4(\theta) > 2\) and \(-\theta^2/3 > -1/3,\) \(M^B + M^C\) is a strictly positive definite matrix. By the result, we obtain that \(\lambda_{\min}((M^B + M^C + M^D)M^E) \geq \delta.\) For the matrix \(M^A,\) by the similar discussion and the fact \(C_3(\theta) > 2,\) we can show that \(M^A\) is strictly positive definite. Finally, we have \(\lambda_{\min}(\Sigma) \geq \delta,\) and Proposition \(\square\) states the claim.

**G.3. Minimum Volume Prediction.** We verify the conditions with \(f_{\theta}(z)\) of the minimum volume prediction in Section \(\square\)
Proof of Corollary \[2\] Without loss of generality, we assume \( Y \vert X \) follows a uniform distribution on \([0, 1]\) for any \( X \) and \( t > 1/2 \). For Assumption \[1\] the conditions (A1) and (A3) hold immediately by the setting of \( \Theta \). The condition (A4) also holds by the boundedness and integrability of \( K \) which follows the setting of the kernel and \( K(x) = o(1/x) \). About the condition (A2), the positivity of the kernel yields

\[
\|f_\theta - f_{\theta'}\|_1
\]

\[
= \mathbb{E}_{X,Y} \left[ K \left( \frac{X_i - x}{h_n} \right) \left\{ |1\{Y_i \in [\theta - t, \theta + t]\} - 1\{Y_i \in [\theta' - t, \theta' + t]\}| \right\} \right]
\]

\[
= \mathbb{E}_X \left[ K \left( \frac{X - x}{h_n} \right) \mathbb{P}_{Y \mid X} \left| 1\{Y_i \in [\theta - t, \theta + t]\} - 1\{Y_i \in [\theta' - t, \theta' + t]\} \right| \right]
\]

\[
= C_K \mathbb{P}_{Y \mid X} \left( Y \in [\theta \land \theta' - t, \theta \lor \theta' - t] \cup [\theta \land \theta' + t, \theta \lor \theta' + t] \right)
\]

\[
= 2C_{K,n}|\theta - \theta'|,
\]

where \( C_K = \mathbb{E}_X[K((X - x)/h_n)] \) which is bounded by following the properties of \( K \). The first inequality follows the iterated expectation, and the last equality follows the distribution of \( Y \). Hence, the condition (A2) holds with \( q = 1 \) and \( c_\ell = 2C_K \). For Assumption \[2\] we obtain \( \mathbb{E}_Z[f_\theta(Z)] = 2C_{K,n}t, \forall \theta \in \Theta \) and

\[
\text{Cov}_Z(f_\theta(Z), f_{\theta'}(Z))
\]

\[
= \mathbb{E}_Z[f_\theta(Z)f_{\theta'}(Z)] - \mathbb{E}_Z[f_\theta(Z)]\mathbb{E}_Z[f_{\theta'}(Z)]
\]

\[
= \mathbb{E}_{X,Y} \left[ K \left( \frac{X - x}{h_n} \right)^2 1\{\theta \lor \theta' - t \leq Y \leq \theta \land \theta' + t\} \right] - 2C_{K,n}t^2
\]

\[
= \tilde{C}_{K,n}|\theta - \theta'| - 2C_{K,n}t^2,
\]

where \( \tilde{C}_{K,n} = \mathbb{E}_X[K((X - x)/h_n)^2] \) which is guaranteed to exist. By the form, we obtain that the covariance matrix \( \Sigma \) has the form \[4\] by substituting \( c = \tilde{C}_{K,n}/2C_{K,n} \). We note that \( C_{K,n} \) is strictly positive by the property of \( K \). Hence, by Proposition \[3\] Assumption \[2\] is satisfied. \( \Box \)

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