An essential singularity of the cotangent of the Coulomb-nuclear phase shift, and a finite limit of the nuclear part of the effective-range function derived at zero energy

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The Coulomb-nuclear phase shift $\delta_{l}^{(cs)}$, cot $\delta_{l}^{(cs)}$ and a finite limit of the nuclear part $\Delta_{l}(k)$ of the effective-range function (ERF) are derived for an arbitrary orbital momentum $l$ when energy $E \to 0$. It is proved that cot $\delta_{l}^{(cs)}$ has an essential singularity at zero energy, but $\Delta_{l}(k)$ does not. The explicit finite limit of $\Delta_{l}(0)$ is found. The property of $\Delta_{l}(k)$ as a meromorphic function makes possible the analytical continuation of a re-normalized scattering amplitude from the physical energy region to a bound state pole. Then the asymptotic normalization coefficients (ANC) can be deduced from experimental phase-shift data and applied to radiative capture processes which are important in nuclear astrophysics for new elements creation. Our results are in agreement with the results published for $S$ wave scattering.

I. INTRODUCTION

Many reactions in supernovae explosions proceed through bound states or low-energy resonance states. One needs to find the asymptotic normalization coefficient (ANC) of the radial wave function which can be used to calculate radiative capture cross sections. This process is one of the main sources of new element creation.

The conventional method of finding the ANC consists in fitting an experimental phase-shift $\delta_{l}^{(cs)}$ data presented by the effective-range function (ERF) to carry out an analytical continuation of the re-normalized (or effective) scattering amplitude from the physical region to a bound or resonance state pole. The transformation of the Coulomb-nuclear amplitude, which is not analytical, to a re-normalized one is shown in the well-known paper Ref. [1]. The re-normalized amplitude acquires analytical properties similar to those of an the amplitude for a short-range interaction.

Recently, a new $\Delta$ algorithm has been validated and applied to the ANC calculation in bound (Ref. [2]) and resonance (Ref. [3]) low-energy states instead of using the conventional effective-range function (ERF) method. In [4] the related form of the re-normalized scattering amplitude is proposed without proof. The $\Delta$ algorithm is much simpler than the ERF because it does not contain $h$, the Coulomb part of the ERF, including the psi-function. In fact, the function $h$ does not appear in the re-normalized scattering amplitude.

In contrast to the $\Delta$ method it was found that the ERF method is limited by the charge product of colliding particles. The $^{12}$C system is a good example of such a situation when the nuclear part of the ERF is on average three orders of magnitude smaller than the Coulomb part (see, for example, Ref. [3]). This problem of the ERF method appears in [5] where the ERF fittings for $^{12}$C behave similarly in the different $^{16}$O states.

However, in a number of publications (see [6], [7], [8]) the authors argue that the $\Delta$ method is not applicable to bound states since the Coulomb-nuclear re-normalized amplitude in this method does not allow an analytical continuation from the physical energy area to the region of negative energies. The authors of papers [6] and [7] claim that the $\Delta_{l}$ function (see Eq. (6) below) supposedly has an essential singularity at zero energy.

The main purpose of the present paper is to prove that the $\Delta_{l}(k)$ function has a finite limit at $E=0$. For the $S$ wave it was shown in the literature (see a special section below).

The re-normalized scattering amplitude has an explicit form because the only ingredient carrying experimental information is the experimental Coulomb(c)-nuclear(s) cot $\delta_{l}^{(cs)}$ which is included in the $\Delta_{l}$ function. Analytic properties of cot $\delta_{l}^{(cs)}$ are very important for the ANC deducing. In the next section the $\Delta_{l}^{(cs)}$, $\delta_{l}^{(cs)}$ and cot $\delta_{l}^{(cs)}$ are derived as the explicit functions of the relative momentum $k$ when $ka_{B} \to 0$. It is proved that the $\Delta_{l}$ has a finite limit when $E \to 0$. Thus the $\Delta_{l}(k)$ has no singularity at zero energy. This makes the $\Delta$ method no less mathematically strict than the ERF method.

II. DERIVATION OF THE COULOMB-NUCLEAR PHASE SHIFT WITH ITS COTANGENT AND THE NUCLEAR PART OF THE ERF NEAR ZERO ENERGY

For charge-less scattering, the ERF is written as

$$K_{l}^{(s)}(k^2) = k^{2l+1} \cot \delta_{l}^{(s)} = -\frac{1}{a_{l}^{(s)}} + \frac{1}{2} a_{l}^{(s)} k^{2} + ..., \quad (1)$$

where $a_{l}^{(s)}$ is the scattering length. The standard effective-range expansion is used in (1), although generally $K_{l}^{(s)}(k^2)$ is a meromorphic function. So the Padé approximant can be used instead of the effective-range...
expansion (ERE). From Eq. (1) for \( k \to 0 \) one gets
\[
\cot \delta_1^{(c)} = -1/a_1^{(c)} k^{2l+1}.
\]

Similarly to the procedure above, \( \cot \delta_2^{(c)} \), \( \delta_2^{(c)} \) near zero energy and at last \( \Delta_l(0) \) are derived from the ERF for charged particles which has the following well-known form (see, for example, [2]):
\[
K_l(k^2) = 2\xi D_l(k^2) \left[ C_0^2(\eta)(\cot \delta_l^{(c)} - i) + h(\eta) \right].
\] (3)

Here and below \( l \) is the orbital momentum, \( \xi = Z_1 Z_2 e^2 = 1/2\mu B \), \( \mu \) is the reduced mass, and \( \alpha \) is the fine-structure constant. All formulas are given in the unit system \( \hbar = c = 1 \).

The following notations are used:
\[
C_0^2(\eta) = \frac{\pi}{\exp(2\pi\eta) - 1},
\] (4)
\[
D_l(k^2) = \frac{l}{n+1} \left( k^2 + \frac{\eta^2}{n^2} \right), \quad D_0(k^2) = 1.
\] (5)
The \( \Delta_l(k) \) is defined as
\[
\Delta_l(k) = \frac{\pi \cot \delta_l^{(c)}}{\exp(2\pi\eta) - 1}.
\] (6)
The Coulomb part is
\[
h(\eta) = \psi(i\eta) + (2i\eta)^{-1} - \ln(i\eta),
\] (7)
where \( \psi(i\eta) \) is the digamma function.

Using the explicit expression for \( \psi(i\eta) \) in the form of the infinite sum, one can write \( h(\eta) \) as:
\[
h(\eta) = iC_0^2(\eta) + \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + \eta^2)} - \ln(\eta) - \zeta,
\] (8)
where \( \zeta \approx 0.5772 \) is the Euler constant (see, for example, [1]).

Thus in the physical energy region, the ERF has a form (see [3] and [4]) where its imaginary parts are mutually reduced:
\[
K_l(k^2) = 2\xi D_l(k^2) \left[ \Delta_l(k) + \text{Re} h(\eta) \right].
\] (9)

In my opinion, since \( \Delta_l(k) \) and \( \text{Re} h(\eta) \) are two terms in the same expression in square brackets, the following statement is very controversial: “However, the validity of employing \( \Delta_l(E) \) was not obvious since \( \Delta_l(E) \), in contrast to \( K_l(E) \), possesses an essential singularity at \( E = 0^+ \)” (page 2, section III, right column).

The limit of \( \Delta_l(k) \) when \( k \to 0 \) can be found directly from Eq. (9) for the ERF in the physical region. In [2] it is shown how the real function \( \text{Re} h(\eta) \) continues to the real function \( h(\eta) \) in the region \( E < 0 \). This function can be approximated by \( 1/12\eta^2 \) around zero energy (see [1] and Eqs. (6.3.18, 6.3.19) in [12]) which has equal limits for \( E \to 0 \). Then the Coulomb part in (9)
\[
\text{Re} h(\eta) \to 0, \text{when } \eta \to \infty \text{ or } E \to 0.
\] (10)

The function \( K_l^{(c)}(k^2) \) can be expand as usual in series of powers \( k^2 \). This leads to the relation
\[
K_l^{(c)}(k^2) = 2\xi D_l(k^2) \Delta_l(k) = -1/a_1^{(c)},
\] (11)
where only the first term is taken into account when \( k \to 0 \) in the ERE on the right side of the equation. After taking the limit of the \( D_l(k^2) \) when \( ka_B \to 0 \) (see Eq. (5)),
\[
D_l(k^2) \to \xi^{2l}/(l!)^2,
\] (12)
the following limiting relations are obtained:
\[
\Delta_l(k) \to \Delta_l(0) = \frac{-(l!)^2}{2a_1^{(c)} \xi^{2l+1}},
\] (13)
\[
\Delta_0(0) = -\frac{a_B}{2a_0^{(c)}}
\] (14)

As a result, the absence of the essential singularity in the \( \Delta_l(k) \) function at zero energy is proved. The \( \Delta_l(k) \) has a finite explicit limit which depends on \( l \), \( \xi \) and the scattering length \( a_1^{(c)} \). Eqs. (13) and (14) define the new relationship between \( \Delta_l(0) \) and the scattering length \( a_1^{(c)} \). For the Padé approximation, a combination of fitted constants appears instead of \( a_1^{(c)} \).

Next the explicit expressions for \( \cot \delta_1^{(c)} \) and \( \delta_1^{(c)} \) are derived. The following formula is obtained from the definition of \( \Delta_l(k) \) Eq. (10) and from the last relation Eq. (14) when \( ka_B \to 0 \) and \( l > 0 \):
\[
\cot \delta_1^{(c)}(k) = -\frac{(l!)^2}{2\pi a_0^{(c)}} \left[ \exp \left( \frac{2\pi}{ka_B} \right) - 1 \right].
\] (15)

One can ignore 1 in square brackets and get the final formulas:
\[
\cot \delta_1^{(c)}(k) = -\frac{(l!)^2}{2\pi a_0^{(c)}} \exp \left( \frac{2\pi}{ka_B} \right),
\] (16)
\[
\delta_1^{(c)}(k) = -\frac{2\pi (l!)^2}{(l!)^2} \exp \left( \frac{2\pi}{ka_B} \right),
\] (17)
when \( \eta \gg 1 \) or \( ka_B \ll 1 \) (\( \cot \delta \to 1/\delta \) when \( \delta \to 0 \)).

Using (14), the following expressions are derived:
\[
\cot \delta_0^{(c)}(k) = -\frac{1}{2\pi a_0^{(c)}} \exp \left( \frac{2\pi}{ka_B} \right),
\] (18)
\[
\delta_0^{(c)}(k) = -2\pi a_0^{(c)} \exp \left( -\frac{2\pi}{ka_B} \right).
\] (19)
So, the factor in the denominator of Eq. \(6\), which contains the essential singularity, is reduced with the same factor of \(\cot \delta_t\) Eq. \(10\) in the nominator of Eq. \(9\) when \(E \to 0\). This leads to the formulas \(13\) and \(14\) for the \(\Delta_t(k)\) function when \(ka_B \to 0\). Eqs. \(13\) to \(10\) complete our derivation of the formulas for \(\Delta_t(0)\), \(\cot \delta_t^{(cs)}(k)\) and \(\delta_t^{(cs)}(k)\) near zero energy.

### III. AGREEMENT OF THE OBTAINED RESULTS WITH INFORMATION FROM THE LITERATURE

The related expression for the limit of \(\delta_t^{(cs)}(k)\):

\[
\delta_t^{(cs)}(k) = -\frac{2\pi}{(l!)^2} e^{2l+1} \delta_t^{(cs)} \exp\left(-\frac{2\pi k a_B}{k a_B}\right),
\]

when \(\eta \gg 1\) or \(a_B k < 1\), is presented in Ref. \(13\) (Eq. \(1.5)\)) but its derivation is not given.

The Coulomb-nuclear phase shift \(\delta_t^{(cs)}\) is defined for a strong \((s)\) short-range interaction in the presence of the Coulomb \((c)\) force, while \(\delta_t^{(s)}\) is defined for a strong \((s)\) interaction. It is known that they are quite different functions of momentum \(k\). Eq. \(20\), which coincides with the derived above Eq. \(17\), shows that the \(\delta_t^{(cs)}(k) \to 0\) when \(k \to 0\), while the \(\delta_t^{(s)}(k) \to n\pi\) (\(n\) is an integer). The latter limit is related to the Levinson’s theorem. Due to this difference between the phase shifts \(\delta_t^{(cs)}\) and \(\delta_t^{(s)}\), which is stressed in Ref. \(14\), the Levinson’s theorem is applicable only to the phase shifts \(\delta_t^{(s)}\) for a short-range interaction and not to the phase shift \(\delta_t^{(cs)}\).

As it is noted in Refs. \(2\) and \(3\) and deduced in the previous section, the presence of the essential singularity in \(\cot \delta_t^{(cs)}\) compensates for the same singularity in the denominator of the \(\Delta_t\) function. Therefore an analytical continuation of the re-normalized scattering amplitude from the physical region to the negative energy is mathematically rigorous.

Our general results for the arbitrary \(l\) agree with those in the section “Resonance scattering of charged particles” of Ref. \(14\) where the \(S\)-wave resonance near a threshold is studied. There the related expression is presented for \(\cot \\
\delta_t^{(cs)}\) \(138.11\), taking into account only the first term in the ERF expansion. We note here only the presence of the factor \(\exp(2\pi \eta)\) which confirms the essential singularity of \(\cot \delta_t^{(cs)}\). It is peculiar that a typo in Eq. \(138.12)\) in the American third edition \(1977\) repeats that in Eq. \(136.12\) in the second Russian edition \(1963\): the exponent in the denominator is written as \(2\pi k a_c\) instead of \(2\pi /k a_c\) (in our notation \(a_c = a_B\)).

Last but not least, support to our results is given in subsection \(4.3\) “Effective-range theory for proton-proton scattering” in book \(15\) for the \(S\)-wave. The main conclusions, supported by the formulas that are written below in our notations, are as follows.

The function

\[
f(E) = \Delta_0(\eta) + \text{Re}h(\eta)
\]

should have a limit when \(E \to 0\) or \(\eta \to \infty\). Indeed, as is shown in Ref. \(16\), \(f(E)\) is a regular function of \(E\) for the Yukawa potential superposition in the half-plane \(ReE > 0\) and has a cut along the negative semi-axis when \(ReE \leq E_d\). \(E_d\) is the negative energy which connected to the radius of nuclear forces. Due to this, \(f(E)\) can be expanded into a Taylor series around the point \(E=0\). The limit of \(f(E)\) when \(E \to 0\) is defined as

\[
f(E = 0) = [\Delta_0(\eta) + \text{Re}h(\eta)] \big|_{E=0} = -\frac{a_B}{2a_p},
\]

where \(a_p = a_0^{(cs)}\) is the proton-proton scattering length. Thus, we obtain

\[
\Delta_0(\eta) + \text{Re}h(\eta) = \frac{1}{a_p} - \frac{k^2 r_0}{2} - PK^4 r_0^3 + ...
\]

Formula \(28\) completes the citation from book \(15\). The results outlined above have predecessors that have begun to study the problems under consideration (see, for example, Ref. \(17\)). Since \(Re h = 0\) at \(E = 0\), Eq. \(22\) coincides with Eq. \(14\) for \(l=0\). This is a direct proof for the essential singularity absence in the \(\Delta_0(\eta)\).

Note, that in \((4.22)\) \(h(\eta) = \text{Re} h(\eta)\) in our notations. Eq. \(10\) coincides with \((4.1 7b)\) in \(15\). There are no objections to this formula in the literature.

The typos in this subsection of Ref. \(15\) should be corrected. In \(C_0^2\) definition \((4.16)\) \([2\pi \eta/\exp(2\pi \eta) - 1)]^2\) should be replaced by \(2\pi \eta/\exp(2\pi \eta) - 1)\) and in \((4.23)\) \(C^2\) should be replaced by \(C_0^2\).

It follows from the above that the \(\Delta_0(\eta)\) is also a meromorphic function and therefore can be expanded into a Taylor series or represented by a Padé-approximant. There is no obstacle for the exclusion of the function \(\text{Re} h(\eta)\) from Eq. \(23\). After that, the coefficients in the right part of Eq. \(23\) will change. \(\text{Re} h(\eta)\) is not included in the expression for the re-normalized scattering amplitude, as explained in detail in Ref. \(3\).

Thus, the equations in the sections II and III, especially Eq. \(22\), validate our conclusion that there is no \(\Delta_t(k)\) essential singularity at zero energy. Otherwise not only \(\Delta_t(k)\) but the whole ERF function could not be considered as a meromorphic function at low energies and used to find the ANC.

### IV. CONCLUSION

In the present paper the explicit expressions of the Coulomb-nuclear phase shifts, their cotangents and the \(\Delta_t(k)\) function are derived using the ERF well-known formula. A finite explicit formula for \(\Delta_t(0)\) is obtained which clearly depends on \(l\), \(a_B\) and the scattering length. This means that \(\Delta_t(0)\) has no essential singularity and
there are no obstacles to applying the $\Delta$ method to deducing ANC values from an experimental phase-shift input.

The results of the present paper are in agreement with the literature including books considering the $S$-wave state. In book [15] for the proton-proton system an equation is given for $\Delta_0(0)$ which coincides with Eq. (15) derived in our work. The related results validates that the ERF is a meromorphic function and support the main point of the $\Delta$ method that the $\Delta_0(0)$ function is also meromorphic. Therefore $\Delta_0(0)$ can be expanded into a Taylor series or represented by a Padé-approximant.

One needs to highlight a reference to Levinson’s theorem in Ref. [2] for systems with two bound states with the same $J^\pi$ on the example of the nucleus $^{16}\text{O}$. The phrase “The pole at $E > 0$ is due to the Levinson theorem,” from Ref. [2] (see page 024602-5, left hand column in front of the subsection A) may indicate a possible error in the non-essential singularity of the $\Delta_l$ function. This theorem is valid only for a pure strong interaction without Coulomb repulsion. If $\cot \delta_l^{(cs)}$ in the nominator of (20) is changed to $\cot \delta_l^{(s)}$, then the non-essential singularity in $\Delta_l$ indeed will occur because of the denominator.

In [2] and [3], it is stated that the singularity in $C_6^l$ is compensated by the corresponding singularity in $\cot \delta_l^{(cs)}(k)$, due to the Coulomb-nuclear phase-shift $\delta_l^{(cs)}(k)$ behavior near zero (see (20)).

Thus, the claim in [2] and [7] about the supposed ‘essential $\Delta_l$ singularity’ is incorrect, as is the criticism of the Delta method which is formulated, validated and used in Ref. [2]. Note that book [18] includes a correct short description of the $\Delta$ method.

In sum, the present paper proves that there is no essential singularity in the $\Delta_l(k)$ function at zero energy. The $\Delta_0(0)$ finite explicit expression is found. In addition, Eq. (13) can be used to estimate the scattering length when the ERF method becomes invalid. If the scattering length is known then $\Delta_l(0)$ values can be included in the input experimental data for better fitting. The main conclusion of the present paper: there are no obstacles to applying the $\Delta$ method to deducing the ANC values, which are important for astrophysics and for the direct reaction theory using Feynman diagrams.

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