Scalar wave equation in Kasner spacetime

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Abstract

The scalar wave equation in Kasner spacetime is solved, for a particular choice of Kasner parameters, by first relating the integrand in the wave packet to the Bessel functions. An alternative integral representation is also displayed, which relies upon the method of integration in the complex domain for the solution of hyperbolic equations with variable coefficients. For generic values of the three Kasner parameters, the solution of the Cauchy problem is built through a pair of integral operators, where the amplitude and phase functions in the integrand solve a coupled system of quasi-linear partial differential equations. The resulting formulas can be used to build self-dual solutions to the field equations of noncommutative gravity, as has been shown in the recent literature.

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I. INTRODUCTION

The subject of partial differential equations has been always an interdisciplinary field where both theoretical physicists and mathematicians can play an active role as well as learn new fundamental properties of the phenomena they are studying. For example, one of the basic difficulties of hyperbolic equations consists of the interplay between the physical space and the properties intimately tied to oscillations, which are best studied in Fourier space [1]. One then builds the solutions (see Sec. V) with the help of Fourier-Maslov integral operators [2] for linear hyperbolic equations, which are applied to investigate the propagation of singularities. Interestingly, the solution can be built also when the hyperbolic equation has variable coefficients, i.e. when the Fourier transforms has to be extended to the case of curved Riemannian manifolds. Such a framework can be further extended to deal with nonlinear partial differential equations, by taking account of the way large and small frequencies interact [1].

Physicists are often more familiar with linear differential operators. For example, starting with the Euclidean space $\mathbb{R}^n$, the Laplacian $-\Delta$ is the simplest differential operator invariant under the group of distance-preserving maps of $\mathbb{R}^n$. The heat, Schrödinger, and wave operators, i.e. (we use units where the speed of light in vacuum is set to 1)

$$\partial_t - \Delta, \quad \frac{1}{i} \partial_t - \Delta, \quad \partial_t^2 - \Delta,$$

are the simplest evolution operators that one can form by using $\Delta$. Next, the wave operator

$$\Box = -\partial_t^2 + \Delta$$

is associated to the Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$ in the same way that $\Delta$ is associated to the Euclidean manifold $(\mathbb{R}^n, g_E)$, where $g_E$ is the Euclidean metric on $\mathbb{R}^n$. Moreover, the functions in the kernel of the Laplacian, i.e. the harmonic functions, which solve $\Delta f = 0$, can be seen as particular, time-independent solutions of the wave equation in Minkowski spacetime. At a deeper level, the metric $g$ can be viewed to determine the elliptic or hyperbolic nature of the operator $g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$, where $\nabla_{\mu}$ can denote covariant differentiation with respect to the Levi-Civita connection on spacetime, or on a vector bundle over spacetime,

\footnote{With our convention, the Laplacian is described by a positive-definite quadratic form in momentum space, through the symbol map which builds the characteristic polynomial by replacing $\frac{\partial}{\partial x^i}$ with $i\xi^i$.}
depending on our needs. When \( g \) is Riemannian, i.e. positive-definite, this operator is minus the Laplacian, whereas if \( g \) is Lorentzian, one gets the wave operator. Note also that, in four-dimensional manifolds, our Lorentzian world lies in between two other options, i.e. a Riemannian metric \( g \) with signature 4 and elliptic operator \( g^\mu\nu \nabla_\mu \nabla_\nu \), and a ultrahyperbolic metric \( g \) with signature 0 and ultrahyperbolic operator \( g^\mu\nu \nabla_\mu \nabla_\nu \).

In the so-called Euclidean (or Riemannian) framework used by quantum field theorists in functional integration, where the metric is positive-definite, the most fundamental differential operator is however the Dirac operator, obtained by composition of Clifford multiplication with covariant differentiation. Its leading symbol is therefore Clifford multiplication, and it generates all elliptic symbols on compact Riemannian manifolds \(^3\). This reflects the better known property according to which, out of the Dirac operator and its (formal) adjoint, one can define two operators of Laplace type, as well as powers of these operators.

Recent work on the self-dual road to noncommutative gravity with twist has found it useful to start from a classical, undeformed spacetime which is a self-dual solution of the vacuum Einstein equation, e.g. a Kasner spacetime \(^4\). Within that framework, it is of interest to solve first the scalar wave equation in such a Kasner background. Since such a task was only outlined in Ref. \(^4\), we find it appropriate to develop a systematic calculus in the present paper.

Relying in part upon Ref. \(^4\), we begin by considering the scalar wave equation \( \Box \phi = 0 \) for a classical scalar field \( \phi \) when the Kasner\(^2\) parameters \( p_1, p_2, p_3 \) take the values 1, 0, 0, respectively, i.e.

\[
\left( -\frac{\partial^2}{\partial t^2} - \frac{1}{t} \frac{\partial}{\partial t} + \frac{1}{t^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0,
\]

where \( \phi \) admits the integral representation

\[
\phi(t, x, y, z) = \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\xi_3 \, A(\xi_1, \xi_2, \xi_3, t) e^{i(\xi_1 x + \xi_2 y + \xi_3 z)}.
\]

One can then set \(^4\)

\[
A(\xi_1, \xi_2, \xi_3, t) = \frac{1}{\sqrt{t}} W(\xi_1, \xi_2, \xi_3, t),
\]

\(^2\) With this particular choice of parameters, we are working on an edge of Minkowski spacetime, i.e., Rindler space. The literature on quantum field theory and accelerated observers has considered in detail such a space, but in our paper the emphasis is on partial differential equations in classical physics, hence we do not strictly need ideas from quantum physics.
where \( W(\xi_1, \xi_2, \xi_3, t) \) has to solve, for consistency, the equation

\[
\left[ \frac{\partial^2}{\partial t^2} + \frac{(1 + (\xi_1)^2)}{t^2} + (\xi_2)^2 + (\xi_3)^2 \right] W(\xi_1, \xi_2, \xi_3, t) = 0, \tag{1.4}
\]

and the \( \frac{1}{\sqrt{t}} \) term in the factorization (1.3) ensures that, in Eq. (1.4), the first derivative of \( W \) is weighed by a vanishing coefficient. This is a sort of canonical form of linear second-order ordinary differential equations with variable coefficients (see Section 10.2 of Ref. [5]), and Eq. (1.4) can be viewed as a 3-parameter family of such equations, the parameters being the triplet \( \xi_1, \xi_2, \xi_3 \).

Section II relates Eq. (1.4) to the Bessel functions, while Sec. III studies a specific choice of Cauchy data. Section IV solves Eq. (1.1) through an integral representation that relies upon integration in the complex domain. Section V studies the most general scalar wave equation in Kasner spacetime, through a pair of integral operators where the integrand consists of amplitude and phase functions. Concluding remarks and open problems are presented in Sec. VI, while relevant background material is described in the Appendices.

II. RELATION WITH BESSEL FUNCTIONS

From now on, we therefore study the ordinary differential equation

\[
\left[ \frac{d^2}{dt^2} + \frac{(1 + (\xi_1)^2)}{t^2} + (\xi_2)^2 + (\xi_3)^2 \right] W(t) = 0. \tag{2.1}
\]

This is a particular case of the differential equation

\[
\left[ \frac{d^2}{dt^2} + \frac{(1 - 2\alpha)}{t} \frac{d}{dt} + \beta^2 + \frac{(\alpha^2 - \nu^2)}{t^2} \right] f(t) = 0, \tag{2.2}
\]

which is solved by the linear combination

\[
f(t) = C_1 t^\alpha J_\nu(\beta t) + C_2 t^\alpha Y_\nu(\beta t). \tag{2.3}
\]

By comparison of Eqs. (2.1) and (2.2) we find

\[
\alpha = \frac{1}{2}, \quad \beta = \sqrt{(\xi_2)^2 + (\xi_3)^2}, \quad \nu = i\xi_1, \tag{2.4}
\]

and hence, in light of what we pointed out at the end of Sec. I, our partial differential equation (1.4) is solved by replacing \( C_1 \) and \( C_2 \) in (2.8) by some functions \( Z_1(\xi_1, \xi_2, \xi_3) \) and
$Z_2(\xi_1, \xi_2, \xi_3)$, whose form depends on the choice of Cauchy data, i.e. (see Sec. III)

$$W(\xi_1, \xi_2, \xi_3, t) = Z_1(\xi_1, \xi_2, \xi_3) \sqrt{t} J_{\xi_1}(t \sqrt{(\xi_2)^2 + (\xi_3)^2}) + Z_2(\xi_1, \xi_2, \xi_3) \sqrt{t} Y_{\xi_1}(t \sqrt{(\xi_2)^2 + (\xi_3)^2}).$$

(2.5)

The Bessel function $Y_{\xi_1}$ is not regular at $t = 0$ and hence, by using this representation, we are considering an initial time $t_0 > 0$. We use the linearly independent Bessel functions $J_{\xi_1}$ and $Y_{\xi_1}$ which describe accurately the time dependence of the integrand in Eq. (1.2). Note that the three choices $(p_1 = 1, p_2 = p_3 = 0)$, $(p_2 = 1, p_1 = p_3 = 0)$, $(p_3 = 1, p_1 = p_2 = 0)$ are equivalent, since the three coordinates $x, y, z$ in the scalar wave equation [4] are on equal footing. Only the calculational details change. More precisely, on choosing $p_3 = 1, p_1 = p_2 = 0$, one finds

$$\beta = \sqrt{\xi_1^2 + (\xi_2)^2}, \quad \nu = i \xi_3,$$

whereas, upon choosing $p_2 = 1, p_1 = p_3 = 0$, one finds

$$\beta = \sqrt{\xi_1^2 + (\xi_3)^2}, \quad \nu = i \xi_2.$$

III. ROLE OF CAUCHY DATA

The task of solving our wave equation (1.1) can be accomplished provided that one knows the Cauchy data

$$\Phi_0 \equiv \phi(t_0, x, y, z), \quad \Phi_1 \equiv \frac{\partial \phi}{\partial t}(t = t_0, x, y, z).$$

(3.1)

Indeed, from our Eqs. (1.2), (1.3) and (2.5), one finds (denoting by an overdot the partial derivative with respect to $t$)

$$A(\xi_1, \xi_2, \xi_3, t_0) = (2\pi)^{-3} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \Phi_0 e^{-i(\xi_1 x + \xi_2 y + \xi_3 z)}$$

$$= Z_1(\xi_1, \xi_2, \xi_3) J_{\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}) + Z_2(\xi_1, \xi_2, \xi_3) Y_{\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}),$$

(3.2)

$$\dot{A}(\xi_1, \xi_2, \xi_3, t_0) = (2\pi)^{-3} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \dot{\Phi}_0 e^{-i(\xi_1 x + \xi_2 y + \xi_3 z)}$$

$$= \sqrt{(\xi_2)^2 + (\xi_3)^2} \left[ Z_1(\xi_1, \xi_2, \xi_3) J_{\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}) + Z_2(\xi_1, \xi_2, \xi_3) Y_{\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}) \right].$$

(3.3)
Equations (3.2) and (3.3) are a linear system of algebraic equations to be solved for \(Z_1\) and \(Z_2\), and they can be studied for various choices of Cauchy data. For example, inspired by the simpler case of scalar wave equation in two-dimensional Minkowski spacetime, we may consider the Cauchy data \(\Phi_0\) and \(\Phi_1\):

\[
\Phi_0 \equiv e^{-\frac{(x^2+y^2+z^2)}{2L^2} \left( \cos \gamma_1 x \right) \left( \cos \gamma_2 y \right) \left( \cos \gamma_3 z \right)},
\]

(3.4)

\[
\Phi_1 \equiv 0,
\]

(3.5)

where \(L\) has dimension of length. Thus, by virtue of the identity

\[
\int_{-\infty}^{\infty} dx \ e^{-ix} e^{-\frac{x^2}{2L^2} \left( \cos \xi_0 x \right)} = \sqrt{2\pi L} \left[ e^{-\frac{L^2}{2}(\xi-\xi_0)^2} + e^{-\frac{L^2}{2}(\xi+\xi_0)^2} \right],
\]

(3.6)

we obtain from (3.2) and (3.4)

\[
A(\xi_1, \xi_2, \xi_3, t_0) = (2\pi)^{-\frac{3}{2}} \left( \frac{L}{2} \right)^3 \prod_{i=1}^{3} \left[ e^{-\frac{L^2}{2}(\xi_i-\gamma)^2} + e^{-\frac{L^2}{2}(\xi_i+\gamma)^2} \right],
\]

(3.7)

while (3.3) and (3.5) yield

\[
\dot{A}(\xi_1, \xi_2, \xi_3, t_0) = 0.
\]

(3.8)

An interesting generalization of the Cauchy data (3.4) and (3.5) might be taken to be

\[
\Phi_0 \equiv e^{-\frac{(x^2+y^2+z^2)}{2L^2} \left( \cos \gamma_1 x \right) \left( \cos \gamma_2 y \right) \left( \cos \gamma_3 z \right) \left( \cos \gamma t_0 \right)},
\]

(3.9)

\[
\Phi_1 \equiv \pm \gamma e^{-\frac{(x^2+y^2+z^2)}{2L^2} \left( \cos \gamma_1 x \right) \left( \cos \gamma_2 y \right) \left( \cos \gamma_3 z \right) \left( \sin \gamma t_0 \right)},
\]

(3.10)

since it reduces to (3.4) and (3.5) at \(t_0 = 0\), which is indeed the value of initial time assumed in the Minkowski spacetime example considered in Ref. [6] (whereas in Kasner spacetime we take so far \(t_0 \neq 0\) to have enough equations to determine \(Z_1(\xi_1, \xi_2, \xi_3)\) and \(Z_2(\xi_1, \xi_2, \xi_3)\)).

Hereafter, to avoid cumbersome formulas, we keep choosing the Cauchy data (3.4) and (3.5).

At this stage, Eqs. (3.2), (3.3), (3.7) and (3.8) lead to

\[
Z_1(\xi_1, \xi_2, \xi_3) = \frac{\dot{Y}_{\xi_1}}{\left( J_{\xi_1} \dot{Y}_{\xi_1} - \dot{Y}_{\xi_1} J_{\xi_1} \right)_{(t_0 \sqrt{\left(\xi_1^2+\xi_2^2+\xi_3^2\right)}}} A(\xi_1, \xi_2, \xi_3, t_0),
\]

(3.11)

\[
Z_2(\xi_1, \xi_2, \xi_3) = -\frac{\dot{J}_{\xi_1}}{\left( J_{\xi_1} \dot{Y}_{\xi_1} - \dot{Y}_{\xi_1} J_{\xi_1} \right)_{(t_0 \sqrt{\left(\xi_1^2+\xi_2^2+\xi_3^2\right)}}} A(\xi_1, \xi_2, \xi_3, t_0),
\]

(3.12)

where (3.7) should be used to express \(A(\xi_1, \xi_2, \xi_3, t_0)\). The integral representation of the solution is therefore completely accomplished, at least in principle.
IV. ANOTHER INTEGRAL REPRESENTATION OF THE SOLUTION

Note now that the original hyperbolic equation (1.1) is a particular case of the general form

\[ L[u] \equiv \left[ \frac{\partial^2}{\partial t^2} - \left( \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \right) + b \frac{\partial}{\partial t} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c \right] u = 0, \quad (4.1) \]

where \( n = 3, x_1 = x, x_2 = y, x_3 = z \) and

\[ a_{jk} = \text{diag}(t^{-2}, 1, 1), \quad b = \frac{1}{t}, \quad b_j = 0, \quad c = 0. \quad (4.2) \]

Thus, for all \( t \neq 0 \) (as we said before, we avoid \( t = 0 \), which is a singularity of the Kasner coordinates), we can exploit the integral representation (see Appendix A) of the solution of hyperbolic equations with variable coefficients [7], while remarking that Eq. (1.1) is also of a type similar to other hyperbolic equations for which the mathematical literature (see Appendix A) has proved that the Cauchy problem is well posed [8, 9]. On referring the reader to chapters 5 and 6 of Ref. [7] for the interesting details, we simply state here the main result when Eqs. (4.1) and (4.2) hold.

**Theorem** The solution of the scalar wave equation (1.1) with Cauchy data (3.4) and (3.5) at \( t = t_0 \neq 0 \) admits the integral representation

\[ u(t, x_1, x_2, x_3) = \lim_{\partial D \to T} \int_{\partial D} B[u(\tau, y_1, y_2, y_3), S(\tau, y_1, y_2, y_3; t, x_1, x_2, x_3)], \quad (4.3) \]

where \( S \) is a fundamental solution (see Appendix B) of the adjoint equation

\[ M[S] = 0, \quad (4.4) \]

\( M \) being, in our case, the adjoint operator acting as

\[ M \equiv \frac{\partial^2}{\partial t^2} - \sum_{j,k=1}^{3} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \frac{db}{dt} + b \frac{\partial}{\partial t}, \quad (4.5) \]

while the integrand \( B[u, S] \) is the differential 3-form

\[ B[u, S] = \left[ buS - \left( S \frac{\partial u}{\partial \tau} - u \frac{\partial S}{\partial \tau} \right) \right] dy_1 \wedge dy_2 \wedge dy_3 + \sum_{j=1}^{3} \left( -1 \right)^j \sum_{k=1}^{3} a_{jk} \left( S \frac{\partial u}{\partial y_k} - u \frac{\partial S}{\partial y_k} \right) d\tau \wedge dy_1 \wedge \ldots \widehat{dy_j} \wedge \ldots dy_3. \quad (4.6) \]
With this notation, the hat upon \( dy_j \) denotes omission of integration with respect to that particular variable, and \( D \) is the region of integration viewed as a cell in the complex domain, with boundary \( \partial D \). Integration over \( \partial D \) should be therefore interpreted in the sense of the calculus of exterior differential forms. Our \( D \) is a manifold defined by the conditions

\[
\text{Im}(\tau^2) + \sum_{k=1}^{3} (y_k - x_k)^2 \leq \varepsilon^2, \tag{4.7}
\]

\[
\text{Re}(\tau - t) = \text{Im}(y_1) = \text{Im}(y_2) = \text{Im}(y_3) = 0, \tag{4.8}
\]

which describe a sphere of radius \( \varepsilon \) in the complex domain, centered at the real point \((t, x_1, x_2, x_3)\). Moreover, the symbolic notation \( \partial D \rightarrow T \) indicates the process of describing the boundary \( \partial D \) down around the domain of dependence on the space where the initial data (3.4) and (3.5) are assigned (such a space is a two-dimensional plane when the Kasner exponents \((1, 0, 0)\) are chosen, whereas, for more general exponents, it corresponds to a singular surface of infinite curvature). A quite complicated evaluation of residues is involved in Eq. (4.3), because the fundamental solution \( S \) of Eq. (4.4) is singular where the Hadamard-Ruse-Synge world function (see Appendix B) vanishes.

V. GENERAL FORM OF THE SCALAR WAVE EQUATION IN KASNER SPACETIME

Equation (1.1) is just a particular case of the following general form of scalar wave equation in Kasner spacetime:

\[
P \phi = \left( \frac{\partial^2}{\partial t^2} + \frac{1}{t} \frac{\partial}{\partial t} - \sum_{l=1}^{3} t^{-2\rho_l} \frac{\partial^2}{\partial x_l^2} \right) \phi = 0. \tag{5.1}
\]

In light of the technical results in Appendix A, it is rather important to study Eq. (5.1), which is what we do know.

Since we are studying a wave equation, we may expect that the solution formula involves amplitude and phase functions, as well as the Cauchy data (here, unlike Secs. II-IV, we exploit techniques that do not need to avoid \( t = 0 \))

\[
\phi(t = 0, x) \equiv u_0(x), \tag{5.2}
\]

\[
\frac{\partial \phi}{\partial t}(t = 0, x) \equiv u_1(x), \tag{5.3}
\]
which are again assumed to be Fourier transformable. However, the variable nature of the coefficients demands for a nontrivial generalization of the integral representation (1.2). This is indeed available, since a theorem guarantees that the solution of the Cauchy problem (5.1)-(5.3) can be expressed in the form

$$\phi(x,t) = \sum_{j=0}^{1} E_j(t) u_j(x),$$

where, on denoting by $\hat{u}_j$ the Fourier transform of the Cauchy data, the operators $E_j(t)$ act according to (hereafter, $(x) \equiv (x_1, x_2, x_3)$, with covariable $(\xi) \equiv (\xi_1, \xi_2, \xi_3)$)

$$E_j(t) u_j(x) = 2 \sum_{k=1}^{2} (2\pi)^{-3} \int e^{i\varphi_k(x,t,\xi)} a_{jk}(x,t,\xi) \hat{u}_j(\xi) d^3 \xi + R_j(t) u_j(x),$$

where the amplitude functions are expandable in the form

$$a_{jk}(x,t,\xi) = \sum_{q=0}^{\infty} a_{jkq}(x,t,\xi),$$

the $\varphi_k$ are real-valued phase functions which satisfy the initial condition

$$\varphi_k(t = 0, x, \xi) = x \cdot \xi = \sum_{s=1}^{3} x_s \xi_s,$$

and $R_j(t)$ is a regularizing operator which smoothes out the singularities acted upon by it. In other words, the Cauchy problem is here solved by a pair of Fourier-Maslov integral operators of the form (5.5), and such a construction (leaving aside, for the moment, its global version, which can be built as shown in Chapter VII of Ref.) generalizes the monochromatic plane waves for the d’Alembert operator from Minkowski spacetime to Kasner spacetime. Strictly, we are dealing with the approximate Green function for the wave equation, called the parametrix. In our case, since we know a priori that (5.4) and (5.5) yield an exact solution of (5.1)-(5.3), we can insert them into Eq. (5.1), finding that, for all $j = 0, 1$,

$$P[E_j(t) u_j(x)] \sim 2 \sum_{k=1}^{2} (2\pi)^{-3} \int P[e^{i\varphi_k} a_{jk}] \hat{u}_j(\xi) d^3 \xi,$$

where $PR_j(t) u_j(x)$ can be neglected with respect to the integral on the right-hand side of Eq. (5.8), because $R_j(t)$ is a regularizing operator. Next, we find from Eq. (5.1) that

$$P[e^{i\varphi_k} a_{jk}] = e^{i\varphi_k} (iA_{jk} + B_{jk}),$$
where

\[ A_{jk} \equiv \frac{\partial^2 \varphi_k}{\partial t^2} a_{jk} + 2 \frac{\partial \varphi_k}{\partial t} \frac{\partial a_{jk}}{\partial t} + \frac{1}{t} \frac{\partial \varphi_k}{\partial t} a_{jk} - \sum_{l=1}^{3} t^{-2p_l} \left( \frac{\partial^2 \varphi_k}{\partial x_l^2} a_{jk} + 2 \frac{\partial \varphi_k}{\partial x_l} \frac{\partial a_{jk}}{\partial x_l} \right), \tag{5.10} \]

\[ B_{jk} \equiv \frac{\partial^2 a_{jk}}{\partial t^2} - \left( \frac{\partial \varphi_k}{\partial t} \right)^2 a_{jk} + \frac{1}{t} \frac{\partial a_{jk}}{\partial t} - \sum_{l=1}^{3} t^{-2p_l} \left( \frac{\partial^2 a_{jk}}{\partial x_l^2} - \left( \frac{\partial \varphi_k}{\partial x_l} \right)^2 a_{jk} \right). \tag{5.11} \]

If the phase functions \( \varphi_k \) are real-valued, since the exponentials \( e^{i \varphi_k} \) can be taken to be linearly independent, we can fulfill Eq. (5.1), up to the negligible contributions resulting from \( PR_j(t)u_j(x) \), by setting to zero in the integrand (5.8) both \( A_{jk} \) and \( B_{jk} \). This leads to a coupled system of partial differential equations. They are quasilinear because they are always linear in the highest order derivative, but nonlinearities make themselves manifest in Eq. (5.11). Our Cauchy problem (5.1)-(5.3) is therefore equivalent to solving the equations

\[ A_{jk} = 0, \quad B_{jk} = 0, \tag{5.12} \]

where a recursive scheme \[2\] for the evaluation of amplitude functions \( a_{jk}(x,t,\xi) \) can be obtained by exploiting the expansion (5.6). Equation (5.12) is the dispersion relation for the scalar wave equation in Kasner spacetime.

More precisely, the initial condition (5.7) and the work in section 6.1 of Ref. \[2\] suggest looking for phase functions \( \varphi_k \) and amplitude mode functions \( a_{jkq} \) in (5.6) which, for \( |\xi| = \sqrt{(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2} \) large, are homogeneous with respect to \( \xi \), whereas what happens near the origin can only contribute regularizing terms in the operators \( E_j(t) \) of Eq. (5.5) and can therefore be neglected \[2\]. Hence we express the homogeneity assumption in the form (\( \xi_0 \) being a lower bound)

\[ \varphi_k(x,t,\chi\xi) = \varphi_k(x,t,\tilde{\xi}) = \chi\varphi_k(x,t,\xi) \text{ if } |\xi| > \xi_0, \tag{5.13} \]

\[ a_{jkq}(x,t,\chi\xi) = a_{jkq}(x,t,\tilde{\xi}) = \chi^{-q} a_{jkq}(x,t,\xi) \text{ if } |\xi| > \xi_0, \tag{5.14} \]

where \( \chi \) is a real parameter, and \( \tilde{\xi} \equiv \chi \xi \). By virtue of (5.6), (5.10), (5.11), (5.13) and (5.14), Eqs. (5.12) can be re-expressed in the form (with the understanding that \( a_{jkq} = 0 \forall q < 0)\)

\[ \sum_{q=0}^{\infty} \chi^q A_{jkq}(x,t,\tilde{\xi}) = 0 \forall \chi, \tag{5.15} \]

\[ \sum_{q=0}^{\infty} \chi^q B_{jkq}(x,t,\tilde{\xi}) = 0 \forall \chi, \tag{5.16} \]
where, on denoting again by $P$ the operator in round brackets in the scalar wave equation (5.1), we find

$$A_{jkq}(x, t, \tilde{\xi}) = a_{jkq}(x, t, \tilde{\xi}) P \varphi_k(x, t, \tilde{\xi}) + C_{jkq}(x, t, \tilde{\xi}),$$  \hspace{1cm} (5.17)

where

$$C_{jkq}(x, t, \tilde{\xi}) \equiv 2 \left[ \frac{\partial \varphi_k(x, t, \tilde{\xi})}{\partial t} \frac{\partial a_{jkq}(x, t, \tilde{\xi})}{\partial t} - \sum_{l=1}^{3} t^{-2p_l} \frac{\partial \varphi_k(x, t, \tilde{\xi})}{\partial x_l} \frac{\partial a_{jkq}(x, t, \tilde{\xi})}{\partial x_l} \right],$$  \hspace{1cm} (5.18)

and

$$B_{jkq}(x, t, \tilde{\xi}) = P a_{jk(q-2)}(x, t, \tilde{\xi}) + Q_k(x, t, \tilde{\xi}) a_{jkq}(x, t, \tilde{\xi}),$$  \hspace{1cm} (5.19)

having defined

$$Q_k(x, t, \tilde{\xi}) \equiv -\left( \frac{\partial \varphi_k(x, t, \tilde{\xi})}{\partial t} \right)^2 + \sum_{l=1}^{3} t^{-2p_l} \left( \frac{\partial \varphi_k(x, t, \tilde{\xi})}{\partial x_l} \right)^2.$$  \hspace{1cm} (5.20)

The recursive solution scheme that we assume requires therefore to impose, $\forall q = 0, 1, ..., \infty$, the vanishing of $A_{jkq}$ and $B_{jkq}$, i.e.

$$A_{jkq}(x, t, \tilde{\xi}) = 0, \quad B_{jkq}(x, t, \tilde{\xi}) = 0, \quad \forall q = 0, 1, ..., \infty,$$  \hspace{1cm} (5.21)

but the resulting set of equations goes beyond our present capabilities.

However, the original equations (5.12), despite their nonlinear nature, allow for nontrivial checks. For example, if one wants to test the viability of the ansatz

$$\varphi_k(x, t, \xi) = \sum_{s=1}^{3} x_s \xi_s - t \omega(\xi_1, \xi_2, \xi_3),$$  \hspace{1cm} (5.22)

which is suggested by the initial condition (5.7), one finds, by insertion into (5.10) and (5.11), the linear partial differential equations for $a_{jk}$

$$\left( \frac{\partial}{\partial t} + \frac{1}{2t} + \sum_{l=1}^{3} t^{-2p_l} \frac{\xi_l}{\omega} \frac{\partial}{\partial x_l} \right) a_{jk}(x, t, \xi, \omega(\xi)) = 0,$$  \hspace{1cm} (5.23)

$$\left[ \left( \frac{\partial^2}{\partial t^2} + \frac{1}{t} \frac{\partial}{\partial t} - \omega^2 \right) - \sum_{l=1}^{3} t^{-2p_l} \left( \frac{\partial^2}{\partial x_l^2} - \xi_l^2 \right) \right] a_{jk}(x, t, \xi, \omega(\xi)) = 0.$$  \hspace{1cm} (5.24)

The challenge is then to understand whether there exists a form of $a_{jk}$ which solves approximately both Eq. (5.23) and Eq. (5.24). The more (resp. less) the solutions of Eqs. (5.23) and (5.24) differ, the more (resp. less) one has to amend the ansatz (5.22).
VI. CONCLUDING REMARKS

The work in Ref. [4] succeeded in the difficult task of setting up a solution algorithm for defining and solving self-dual gravity field equations to first order in the noncommutativity matrix. However, precisely the first building block, i.e. the task of solving the scalar field equation in a classical self-dual background was only briefly described.

This incompleteness has been taken care of in the present paper for the case of Kasner spacetime, first with a particular choice of Kasner parameters: $p_1 = 1, p_2 = p_3 = 0$. The physics-oriented literature had devoted efforts to evaluating quantum propagators for a massive scalar field in the Kasner universe [16], but the relevance for the classical wave equation of the mathematical work in Refs. [5, 7–10, 17] had not been appreciated, to the best of our knowledge. There is however still a lot of work to do, i.e.

(i) How to evaluate in a generic Kasner model the world function $\sigma(x, \xi)$ and hence the fundamental solution $S(x, \xi)$ defined in Appendix B.

(ii) How to evaluate via residues the integral formula (4.3), overcoming the computational difficulties in Kasner geometry well discussed in Ref. [16].

(iii) How to study departures from the Huyghens’ principle and/or other qualitative features of our solution of the scalar wave equation.

(iv) How to solve the coupled equations (5.21), and hence complete the parametrix\(^3\) construction presented therein for the case of generic values of Kasner parameters.

This adds evidence in favour of noncommutative gravity needing the whole apparatus of classical mathematical physics for a proper solution of its field equations (see also the work in Ref. [19], where Noether-symmetry methods have been used to evaluate the potential term for a wave-type operator in Bianchi I spacetime).

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\(^3\) We note, incidentally, that rediscovering the versatility of parametrices might lead to important progress in canonical quantum gravity, since the work in Ref. [18] obtained diffeomorphism-invariant Poisson brackets on the space of observables, i.e. diff-invariant functionals of the metric, but this relied upon exact Green functions obeying advanced and retarded boundary conditions, whereas the parametrix is what is strictly needed in the applications, and it might prove more useful in defining and evaluating quantum commutators.
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Appendix A: Assessment of our wave equation and its solution

As we know from Sec. V, Eq. (1.1) is a particular case of the wave equation (5.1). The operator $P$ in Eq. (5.1) is an example of what is called, in the mathematical literature, a Fuchsian hyperbolic operator with weight 2 with respect to $t$. In general, the weight is $m-k$, and such Fuchsian hyperbolic operators read as (hereafter $(t, x) = (t, x_1, ..., x_n) \in [0, T] \times \mathbb{R}^n$)

\[
P(t, x, \partial_t, \partial_x) = t^k \partial_t^m + P_1(t, x, \partial_x) t^{k-1} \partial_t^{m-1} + ... + P_k(t, x, \partial_x) \partial_t^{m-k} + P_{k+1}(t, x, \partial_x) \partial_t^{m-k-1} + ... + P_m(t, x, \partial_x),
\]  

subject to 10 conditions stated in Ref. [10] which specify the relation between $k$ and $m$, the form of the coefficients, hyperbolicity, quadratic form associated to the operator (see below), estimates for principal part and lower order terms of the operator. When all these 10 conditions hold, one can prove the following theorem [10]:

**Theorem A1.** For any functions $u_0(x), ..., u_{m-k-1}(x) \in C^\infty(\mathbb{R}^n)$ and $f(t, x) \in C^\infty([0, T] \times \mathbb{R}^n)$, there exists a unique solution $u(t, x) \in C^\infty([0, T] \times \mathbb{R}^n)$ such that

\[
P(t, x, \partial_t, \partial_x)u(t, x) = f(t, x) \text{ on } [0, T] \times \mathbb{R}^n,
\]  

\[
\partial_i u(t, x) \big|_{t=0} = u_i(x) \text{ for } 0 \leq i \leq m - k - 1,
\]  

and the solution has a finite propagation speed.

For the operator in Eq. (5.1), the quadratic form of the general theory, obtained by replacing

\[
\frac{\partial}{\partial x_j} \rightarrow i\xi_j
\]

in all spatial derivatives of second order, reads as

\[
S(t, \xi) = \sum_{j=1}^{3} t^{-2p_j} \xi_j^2.
\]
According to Tahara, for Fuchsian hyperbolic operators, the quadratic form $S(t, \xi)$ leading to Theorem B1 should be positive-definite as a function of $\xi$ for any $t > 0$, with symmetric coefficients of class $C^1$ on $[0, T]$, and such that

$$\max_{|\xi|=1} \left| \frac{\partial}{\partial t} \log S(t, \xi) \right| = O \left( \frac{1}{t} \right) \text{ as } t \to 0^+. \quad (A5)$$

For the operator in Eq. (5.1) one finds indeed

$$\frac{\partial}{\partial t} \log S(t, \xi) = -\frac{2}{t} \sum_{j=1}^3 p_j t^{-2p_j} \xi_j^2, \quad (A6)$$

Thus, bearing in mind that, when the $p_j$ Kasner parameters are all nonvanishing, one of them is negative and the other two are positive, one obtains (on defining $p \equiv \max \{p_j\}$ for all $p_j > 0$)

$$\left| \frac{\partial}{\partial t} \log S(t, \xi) \right| \sim \frac{2p}{t} \text{ as } t \to 0^+, \quad (A7)$$

and hence condition (A5) of the general theory is fulfilled. This is also the case of the operator in Eq. (1.1), for which

$$S(t, \xi) = t^{-2} \xi_1^2 + \xi_2^2 + \xi_3^2, \quad (A8)$$

which implies that

$$\left| \frac{\partial}{\partial t} \log S(t, \xi) \right| = \left| \frac{2}{t} t^{-2} \xi_1^2 t^{-2} \xi_2^2 + \xi_3^2 \right| = O \left( \frac{1}{t} \right) \text{ as } t \to 0^+. \quad (A9)$$

In other words, the hyperbolic equation studied in our paper can always rely upon the Tahara theorem on the Cauchy problem.

If instead we resort to the Garabedian technique of integration in the complex domain, strictly speaking, we need to assume analytic coefficients \cite{7}, which is not fulfilled, for example, by $b = \frac{1}{t}$ in (4.2) if we replace $t$ by a complex $\tau = \tau_1 + i\tau_2$ and want to consider also the value $\tau_1 = \tau_2 = 0$. However, Ref. \cite{7} describes the way out of this nontrivial technical difficulty. For this purpose, one considers first a more complicated, inhomogeneous equation

$$L[u] = f \quad (A10)$$

with analytic coefficients and analytic right-hand side, from which one can write down a direct analogue of the solution (4.3) in the form

$$u(t, x) = \lim_{\partial D \to T} \left[ \int_{\partial D} B[u, \Pi] + \int_D (Pf - uM[\Pi])d\tau \wedge dy_1 \wedge dy_2 \wedge dy_3 \right], \quad (A11)$$
where $\Pi(x, y)$ is called a parametrix and is given by

$$
\Pi(x, y) = \sum_{l=0}^{\nu} U_l(x, y) \sigma^l m(x, y) + \sum_{l=0}^{\mu} V_l(x, y) \sigma^l (x, y) \log \sigma(x, y), \quad (A12)
$$

in terms of the world function $\sigma(x, y)$ of Appendix B. The notation $\partial D \rightarrow T$ means that the manifold of integration $D$ is supposed to approach the real domain in such a way that it folds around the characteristic conoid $\sigma = 0$ without intersecting it. Equation (A11) defines a Volterra integral equation for the solution of the Cauchy problem. It follows that $u$ varies continuously with the derivatives of the coefficients of Eq. (A10). Similarly, the second partial derivatives of $u$ depend continuously on the derivatives of the coefficients of a high enough order. Thus, when they are no longer analytic, we may replace these coefficients by polynomials approximating an appropriate set of their derivatives in order to establish the validity of (A11) in the general case by passage to the limit. Note also that the integral equation (A11) has a meaning in the real domain even where the partial differential equation (A10) is not analytic, since the construction of the parametrix $\Pi(x, y)$ and of the world function $\sigma(x, y)$ only requires differentiability of the coefficients of a sufficient order \[1\].

More precisely, for coefficients possessing partial derivatives of all orders, we introduce a polynomial approximation that includes enough of these derivatives to ensure that the solution of the corresponding approximate equation (A11) converges together with its second derivatives. The limit has therefore to be a solution of the Cauchy problem associated with the more general coefficients, and must itself satisfy the Volterra integral equation (A11).

**Appendix B: World function and fundamental solutions**

In his analysis of partial differential equations, Hadamard discovered the importance of the *world function* \[11\] \[13\], which can be defined as the square of the geodesic distance between two points with respect to the metric

$$
g = \sum_{i, j=1}^{n} g_{ij} \mathbf{dx}^i \otimes \mathbf{dx}^j. \quad (B1)
$$

In the analysis of second-order linear partial differential equations

$$
N[u] = \left[ \sum_{i, j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \right] u = 0, \quad (B2)
$$
the first-order nonlinear partial differential equation for the world function \( \sigma(x, y) \) reads as
\[
\sum_{i,j=1}^{n} a_{ij} \frac{\partial \sigma(x, y)}{\partial x_i} \frac{\partial \sigma(x, y)}{\partial x_j} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial \sigma(x, y)}{\partial y_i} \frac{\partial \sigma(x, y)}{\partial y_j} = 4\sigma(x, y), \tag{B3}
\]
where the coefficients \( a_{ij} \) are the same as those occurring in the definition of the operator \( N \) (this is naturally the case because the wave or Laplace operator can be always defined through the metric, whose signature determines the hyperbolic or elliptic nature of the operator, as we stressed in Sec. I). The world function can be used provided that the points \( x \) and \( y \) are so close to each other that no caustics occur.

A fundamental solution \( S = S(x, y) \) of Eq. (B2) is defined to be a solution of that equation in its dependence on \( x = (x_1, ..., x_n) \) possessing, at the parameter point \( y = (y_1, ..., y_n) \), a singularity characterized by the representation
\[
S(x, y) = \frac{U(x, y)}{(\sigma(x, y))^m} + V(x, y) \log(\sigma(x, y)) + W(x, y), \tag{B4}
\]
where \( U, V, W \) are supposed to be regular functions of \( x \) in a neighbourhood of \( y \), with \( U \neq 0 \) at \( x = y \), and where the exponent \( m \) depends on the spacetime dimension \( n \) according to \( m = \frac{(n-2)}{2} \). The sources of nonvanishing \( V \) are either a mass term in the operator \( N \) or a nonvanishing spacetime curvature. The term \( V \log(\sigma) \) plays an important role in the evaluation of the integral (4.3), as is stressed in Sec. 6.4 of Ref. [7].

In Kasner spacetime, the Hadamard Green function (B4) has been evaluated explicitly only with the special choice of parameters \( p_1 = p_2 = 0, p_3 = 1 \) in Ref. [16]. In that case, direct integration of the geodesic equation yields eventually an exact formula for the Hadamard-Ruse-Synge world function in the form
\[
\sigma = t_0^2 \left( r_{\perp}^2 - \tau^2 - \tau'^2 + 2\tau\tau' \cosh r_3 \right), \tag{B5}
\]
having defined
\[
\tau \equiv \frac{t}{t_0}, \quad \tau' \equiv \frac{t'}{t_0}, \tag{B6}
\]
\[
\tau \equiv \frac{t}{t_0}, \quad r_{\perp} \equiv \frac{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}}{t_0}, \quad r_3 \equiv \frac{(x_3-y_3)}{t_0}. \tag{B7}
\]
Following our remarks at the end of Sec. II, we expect that the choice of Kasner parameters made in Sec. II would still lead to a formula like (B5) for the world function, but with
\[
r_{\perp} \equiv \frac{\sqrt{(x_2-y_2)^2 + (x_3-y_3)^2}}{t_0}, \quad r_1 \equiv \frac{(x_1-y_1)}{t_0}. \tag{B8}
\]
However, as far as we know, the extension of these formulas to generic values of Kasner parameters is an open problem.

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