An analog of the Skewes number for twin primes

Marek Wolf

Institute of Theoretical Physics, University of Wrocław
Pl. Maxa Borna 9, PL-50-204 Wrocław, Poland
e-mail: mwolf@ift.uni.wroc.pl

Abstract

The results of the computer investigation of the sign changes of the difference between the number of twin primes \( \pi_2(x) \) and the Hardy–Littlewood conjecture \( c_2 \text{Li}_2(x) \) are reported. It turns out that \( \pi_2(x) - c_2 \text{Li}_2(x) \) changes the sign at unexpectedly low values of \( x \) and for \( x < 2^{42} \) there are over 90000 sign changes of this difference. It is conjectured that the number of sign changes of \( \pi_2(x) - c_2 \text{Li}_2(x) \) for \( x \in (1, T) \) is given by \( \sqrt{T} / \log(T) \).
Let \( \pi(x) \) be the number of primes less than \( x \) and let \( \text{Li}(x) \) denote the logarithmic integral:

\[
\text{Li}(x) = \int_2^x \frac{du}{\log(u)}.
\]  

(1)

The Prime Number Theorem tells us that \( \text{Li}(x)/\pi(x) \) tends to 1 for \( x \to \infty \) and the available data (see [2]) shows that always \( \text{Li}(x) > \pi(x) \). This last experimental observation was the reason of the common belief in the past, that the inequality \( \text{Li}(x) > \pi(x) \) is generally valid. However, in 1914 J.E. Littlewood has shown [1] (see also [10]) that the difference between the number of primes smaller than \( x \) and the logarithmic integral up to \( x \) infinitely often changes the sign. In 1933 S. Skewes [3] assuming the truth of the Riemann hypothesis has estimated that for sure \( d(x) = \pi(x) - \text{Li}(x) \) changes sign for some \( x_0 < 10^{10^{10^{34}}} \). The smallest value \( x_0 \) such that for the first time \( \pi(x_0) > \text{Li}(x_0) \) holds is called Skewes number. In 1955 Skewes [4] has found, without assuming the Riemann hypotheses, that \( d(x) \) changes sign at some

\[ x_0 < \exp \exp \exp \exp(7.705) < 10^{10^{10^{34}}} \]

This enormous bound for \( x_0 \) was reduced by Cohen and Mayhew [5] to \( x_0 < 10^{10^{529.7}} \) without using the Riemann hypothesis. In 1966 Lehman [6] has shown that between 1.53 \times 10^{1165} and 1.65 \times 10^{1165} there are more than \( 10^{500} \) successive integers \( x \) for which \( \pi(x) > \text{Li}(x) \). Following the method of Lehman in 1987 H.J.J. te Riele [7] has shown that between 6.62 \times 10^{370} and 6.69 \times 10^{370} there are more than \( 10^{180} \) successive integers \( x \) for which \( d(x) > 0 \). The lowest present day known value of the Skewes number is around \( 10^{316} \), see [8] and [9].

The number of sign changes of the difference \( d(x) = \pi(x) - \text{Li}(x) \) for \( x \) in a given interval \((1, T)\), which is commonly denoted by \( \nu(T) \), see [10], was treated for the first time by A.E. Ingham in 1935 [11] chapter V, [12] and next by S. Knapowski [13]. Regarding the number of sign changes of \( d(x) \) in the interval \((1, T)\), Knapowski [13] proved

\[
\nu(T) \geq e^{-35 \log \log \log \log T}
\]

provided \( T \geq \exp \exp \exp \exp(35) \). Further results about \( \nu(T) \) were obtained by J. Pintz [14] and J. Kaczorowski [15].

In this paper I am going to look for the analog of the Skewes number for the twin primes.

Let \( \pi_2(x) \) denote the number of twin primes smaller than \( x \). Then the unproved (see however [17]) conjecture B of Hardy and Littlewood [16] on the number of prime pairs \( p, p + d \) applied to the case \( d = 2 \) gives that

\[
\pi_2(x) \sim C_2 \text{Li}_2(x) \equiv C_2 \int_2^x \frac{u}{\log^2(u)} du,
\]

(3)

where \( C_2 \) is called “twin constant” and is defined by the following infinite product:

\[
C_2 \equiv 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 1.32032\ldots
\]

(4)
Usually the lower limit of integration $a$ in (3) is chosen 2, but the author believes that the proper choice for the lower limit of integration should be 5, not 2, because (3,5) is the first twin pair. (Analogously in (11) the lower limit of integration is 2, to ensure that $\text{Li}(2) = 0$.)

For the first time the conjecture (3) was checked computationally by R. P. Brent [18]. This author noticed the sign changes of the difference $\pi_2(x) - C_2\text{Li}_2(x)$ but did not elaborate about this further. I have looked on the difference $d_2(x) = \pi_2(x) - C_2\text{Li}_2(x)$ using the computer for $T$ up to $2^{42} \approx 4.4 \times 10^{12}$. Like for usual primes initially $C_2\text{Li}_2(x) > \pi_2(x)$, but surprisingly, it turns out that there is a lot sign changes of $d_2(x) = \pi_2(x) - C_2\text{Li}_2(x)$ for $x$ in the range $(1, 2^{42})$. For the case of the lower integration limit $a = 2$ in (3) the first sign change of $d_2(x)$ appears at the twin pair $(1369391, 1369393)$. However, for the choice $a = 5$ the first sign change of $d_2(x)$ appears already at the pair $(41, 43)!$ Next “Skewes” pairs for $a = 5$ are $(6959, 6961)$ and $(7127, 7129)$. After that there is a gap in the sign changes up to $(1353257, 1353259)$, what is comparable to the case of $a = 2$. Such a behavior is reasonably, because the difference between $a = 2$ and $a = 5$ is relevant only at low $x$ and the contribution stemming from the interval $(2,5)$ is becoming negligible for large $x$.

Let $\nu_2(T)$ denote, by analogy with usual primes, the number of sign changes of $d_2(x)$ in the interval $(1, T)$. The Table 1 contains the recorded number of sign changes of $\pi_2(x) - C_2\text{Li}_2(x)$ up to $T = 2^{22}, 2^{23}, \ldots, 2^{42}$ for both choices of the parameter $a$. The values of $T$ searched by the direct checking are of small magnitude from the point of view of mathematics, but large for modern computers. The observed numbers $\nu_2(T)$ behave very erratically, see Fig.1, in particular there are large gaps without any change of sign of the $d_2(x)$. If one assumes the power-like dependence of $\nu_2(T)$ then the fit by the least square method gives the function $\alpha T^\beta$, where $\alpha = 0.0766741$ and $\beta = 0.479031 \ldots$. Instead of such accidentally looking parameters for the pure power-like behavior (especially $\alpha$ has a very small value) after a few trials I have picked the function $\sqrt{T} / \log(T)$ as an approximation to $\nu_2(T)$ as a much nicer function without any random parameters involved and simultaneously taking values very close to the fit $\alpha T^\beta$, see Figure 1. Thus we state the conjecture:

$$\nu_2(T) = \sqrt{T} / \log(T) + \text{ error term}$$ \hspace{1cm} (5)

Let us stress in favor of (5) that there are 7 crosses of the curve $\sqrt{T} / \log(T)$ with the staircase-like plot of $\nu_2(T)$ obtained directly from the computer data. The last column in the Table 1 contains the values of the function $\sqrt{T} / \log(T)$. If the conjecture (5) is true, then there is infinity of twins. Also if (5) is valid, it means that the estimation (3) is in some sense more accurate than (11), because there are more points, where (3) exactly reproduces the actual number of twins — for (11) there is much less such values of $x$ that $\text{Li}(x)$ is equal to $\pi(x)$, see (2). However, presumably the (unknown) error term in (3) is larger than error term for $\pi(x)$.

The difference of many hundreds of orders between values of $x$ such that $\pi(x) - \text{Li}(x)$ and $\pi_2(x) - C_2\text{Li}_2(x)$ changes the sign for the first time seems to be very
astonishing. Let me give an example from physics: the energy of the ground states of the hydrogen and helium are respectively -13.6 eV and -79 eV and do not differ by hundreds of orders!

I have tested the numerical results using several different computers, programs and compilers. In particular, to calculate the integral $L_i(x)$ I have used the 8–point self–adaptive Newton–Cotes method and the 10–point Gauss method. This integral was calculated numerically in successive intervals between consecutive twins and added to the previous value. It seems to be natural that different methods gave exactly the same values of $L_i(x)$ since the integrand in (3) is a very well behaved function. At least up to $T = 2^{32}$ (this limitation stems from the fact that some compilers did not allow larger integers than $2^{32}$) all results obtained by different runs were exactly the same.
TABLE 1

The number of sign changes of $d_2(x)$. The case $a = 2$ is in second column, and third column contains data for $a = 5$, while the last column presents values obtained from (5).

| $T$ | $\nu_2(T)$ for $a = 2$ | $\nu_2(T)$ for $a = 5$ | $\sqrt{T\log(T)}$ |
|-----|------------------------|------------------------|------------------|
| $2^{22}$ | 29 | 32 | 134 |
| $2^{23}$ | 29 | 32 | 182 |
| $2^{24}$ | 29 | 32 | 246 |
| $2^{25}$ | 29 | 32 | 334 |
| $2^{26}$ | 238 | 269 | 455 |
| $2^{27}$ | 854 | 942 | 619 |
| $2^{28}$ | 1226 | 1401 | 844 |
| $2^{29}$ | 1226 | 1401 | 1153 |
| $2^{30}$ | 1226 | 1401 | 1576 |
| $2^{31}$ | 1226 | 1401 | 2157 |
| $2^{32}$ | 2854 | 3045 | 2955 |
| $2^{33}$ | 7383 | 7358 | 4052 |
| $2^{34}$ | 9115 | 8974 | 5562 |
| $2^{35}$ | 12682 | 12431 | 7641 |
| $2^{36}$ | 23634 | 23103 | 10505 |
| $2^{37}$ | 31641 | 30770 | 14455 |
| $2^{38}$ | 31641 | 30770 | 19905 |
| $2^{39}$ | 31641 | 30770 | 27428 |
| $2^{40}$ | 38899 | 37904 | 37819 |
| $2^{41}$ | 55106 | 54179 | 52180 |
| $2^{42}$ | 90355 | 89768 | 72037 |

References

[1] J.E. Littlewood, “Sur la distribution des nombres premiers”, Comptes Rendus 158 (1914), p. 1869–1872

[2] J.C. Lagarias, V.S. Miller and A.M. Odlyzko, “Computing $\pi(x)$: The Meissel–Lehmer Method” Mathematics of Computation 44 (1985), p. 537–560, Table I or check the table http://numbers.computation.free.fr/Constants/Primes/pixtable.html

[3] S. Skewes, “On the difference $\pi(x) - \text{Li}(x)$” J. London Math. Soc. 8 (1934), pp. 277–28

[4] S. Skewes “On the difference $\pi(x) - \text{Li}(x)$ (II)” Proc. London Math. Soc. 5 (1955), pp. 48–70
[5] A.M. Cohen and M.J.E. Mayhew, “On the difference $\pi(x) - \text{Li}(x)$”, \textit{Proc. London Math. Soc.} \textbf{18} (1968), pp. 691–713

[6] R.S. Lehman, “On the difference $\pi(x) - \text{Li}(x)$”, \textit{Acta Arithmetica} \textbf{XI} (1966), p.397–410

[7] H.J.J. te Riele, “On the difference $\pi(x) - \text{Li}(x)$”, \textit{Mathematics of Computation} \textbf{48} (1987), p.323–328

[8] Bays, C. and Hudson, R. H. “A New Bound for the Smallest $x$ with $\pi(x) > \text{Li}(x)$”, \textit{Mathematics of Computation} \textbf{69} (2000), p. 1285-1296

[9] Demichel, P. “The Prime Counting Function and Related Subjects.” available here \url{http://www.mybloop.com/dmlpat/maths/li_crossover_pi.pdf}

[10] W. and F. Ellison \textit{Prime Numbers} (John Wiley and Sons, New York, London, 1985)

[11] A.E. Ingham, “The distribution of prime numbers”, originally published in 1932, unchanged reprint: Hafner Publ. Comp. (New York, 1971)

[12] A.E. Ingham, “A note on the distribution of primes”, \textit{Acta Arithmetica} \textbf{I} (1936), p.201–211

[13] S. Knapowski, “On sign changes of the difference $\pi(x) - \text{Li}(x)$”, \textit{Acta Arithmetica} \textbf{VII} (1962), p.107–119

[14] J. Pintz, “On the remainder term of the prime number formula, III IV”, \textit{Studia Sci. Math. Hungar.} \textbf{12}(1977), pp.343-369, \textbf{13} (1978), pp.29-42

[15] J. Kaczorowski “On sign-changes in the remainder-term of the prime-number formula, I” \textit{Acta Arithmetica} \textbf{44} (1984), pp.365-377; Kaczorowski “On sign-changes in the remainder-term of the prime-number formula, II” \textit{Acta Arithmetica} \textbf{45} (1984), pp.65-74

[16] G.H.Hardy and J.E. Littlewood, ”Some problems of ‘Partitio Numerorum’ III: On the expression of a number as a sum of primes”, \textit{Acta Mathematica} \textbf{44} (1922), p.1-70

[17] M.Rubinstein, “A simple heuristic proof of Hardy and Littlewood conjecture B”, \textit{Amer. Math. Monthly} \textbf{100} (1993), p. 456–460

[18] R. P. Brent, “Irregularities in the distribution of primes and twin primes” \textit{Mathematics of Computation} \textbf{29} (1975), pp. 43-56
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Fig. 1 The plot showing the comparison of the actual values of $\nu_2(T)$ for $a = 2$ and $a = 5$ found by a computer search with the conjecture [5].