On Holographic Entanglement Entropy with Second Order Excitations

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We study the low-energy quantum excitation corrections to the holographic entanglement entropy of the boundary CFT from the bulk gravitational perturbations up to second order. By focusing on the case when the boundary subsystem is a strip, we show that the bulk minimal surface can be expanded in terms of the conserved charges such as the mass, angular momentum and electric charge of the bulk AdS black brane when the black brane is a slightly perturbed geometry deviates from the pure AdS spacetime. We also calculate the energy of the subsystem in the CFT and argue that the first law-like relation for the subsystem should be satisfied at second order when the bulk geometry is stable under fluctuations at the same order.

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I. INTRODUCTION

The quantum entanglement or entanglement entropy is of great importance in characterizing the correlation between non-local physical quantities or operators in quantum many-body systems. For example, it can be used as an order parameter to probe the quantum phase transitions of quantum fields at critical points [1–5]. For a pure state with density matrix \( \rho \), dividing the system into two spatial regions \( A \) and \( B \), the entanglement entropy \( S_A \) of the subsystem \( A \) can be calculated by the von Neumann entropy as

\[
S_A = -\text{tr}_A (\rho_A \ln \rho_A),
\]

where \( \rho_A = \text{tr}_B \rho \) is the reduced density matrix of \( A \) obtained by tracing out the degrees of freedom which belong to the Hilbert space of the subsystem \( B \). The entanglement entropy can also be described by a more general notion—the entanglement Rényi entropy, defined as

\[
S_n = \frac{1}{1-n} \ln \text{tr} (\rho^n_A),
\]

where the positive number \( n \) is called the order of the Rényi entropy, and when taking \( n \to 1 \), it returns to the von Neumann entropy. The important and unusual property of the entanglement entropy (contributed from many quantum fields) is that it is proportional to the area of the boundary surface dividing the subsystems \( A \) and \( B \), namely, the area law (there are indeed some exceptions to violate the area law such as entanglement entropy from fermions [6]). The area law relation, together with the fact that entanglement entropy also describes the lack of information, say, measured by observer in \( A \), inspired the studies of interpreting the black hole area entropy as the entanglement entropy between quantum states of certain quantum fields in and outside the black hole horizon [7–10].

In the framework of the AdS/CFT correspondence or the gauge/gravity duality [11–13], the entanglement entropy also gained a novel holographic interpretation, called the holographic entanglement entropy (HEE), proposed by Ryu and Takayanagi [14–15]. In this proposal, the entanglement entropy of the subsystem \( A \) in the boundary \( d \)-dimensional CFT can be calculated from the area \( A_{\gamma_A} \) of a co-dimensional-2 static minimal surface \( \gamma_A \) in the bulk gravity side. More explicitly, the HEE of \( A \) is

\[
S_A = \frac{A_{\gamma_A}}{4G_{d+1}},
\]
where the boundary of \( \gamma_A \) is \( \partial A \). Soon after, Fursaev gave a proof to the above HEE formulae eq. (1) by applying the off-shell Euclidean path integral approach to manifolds with conical singularities in the context of the AdS/CFT correspondence [10]. The proof has been improved and generalized from recent studies by Lewkowycz and Maldacena [17] by using the replica trick. Besides, another derivation of the HEE for spherical entangling surfaces has been provided by Casini, Huerta and Myers in [18]. So far there are also many evidences to check and confirm the HEE proposal within the AdS\(_3\)/CFT\(_2\) correspondence [19–21]. In addition to various applications of computing entanglement entropy holographically, such as the use of HEE to probe the confinement/deconfinement phase transition in the large \( N \) gauge theories [22] and the phase structures in condensed matter systems [23–26], the utility of HEE to study the renormalization group in quantum field theories [27–29], and many other interesting studies, see [30] for a recent review and references therein.

The original HEE suggestion in eq. (1) was made in the large \( N \) and large ’t Hooft coupling limit, which indicates that the dual gravitational theory is the purely classical Einstein theory. In the full version of the AdS/CFT correspondence or gauge/gravity duality, due to the strong/weak duality property, there are two typical higher order corrections to the theory, one is the higher curvature effects from bulk (super)gravity (which is the \( \alpha' \) correction to the boundary CFT), another is the quantum corrections \( (1/N) \) to the boundary field theory, associated with the perturbations from the bulk gravitational theory. For bulk effective gravitational theory with higher curvature or higher derivative terms, the corrections to the HEE area law formulae eq. (1) have been studied in the presence of the gravitational Chern-Simons term (with gravitational anomalies) [31, 32], and also in quantum field theory with the general Lovelock gravity dual [33–38].

While in quantum field theories, the universal behavior of entanglement entropy in low-energy excited states is also very important in understanding the quantum entanglement nature of the system. This topic has been studied by many authors, for example [39, 40]. Thus it is interesting and important to generalize the HEE proposal into the situation with quantum corrections. Usually it is difficult to investigate the properties of entanglement entropy of systems with generic configurations either from the pure field theory or from the holographic approaches. However, one can consider two different limits of the subsystem to analyze the properties of the entanglement entropy. The first limit is to study the entanglement entropy with small size for the subsystem, it has been shown in [39] that the entanglement entropy and Rényi entropy of the 2-dimensional CFT in vacuum and low-energy excited states are scaled with the primary operators in the theory. Later, the HEE proposal has been applied to study the related problem and the scaling relation for the first order quantum excited entanglement entropy has been reproduced. Furthermore, a
novel universal relation between the linearized variation of entanglement entropy and energy of the subsystem with small size in the boundary CFT has been found

$$\delta E = T_e \delta S,$$

(2)

where $T_e$ is an effective temperature called the entanglement temperature which is proportional to the reciprocal of the size of the CFT, and apparently eq.(2) is similar to the familiar first law of thermodynamics. On the other hand, one can also consider entanglement entropy of local excited states in the second limit in which the subsystem is of large size. There were various investigations on this direction from the quantum field theory side [42-44], in which the authors have proposed that entanglement entropy with large size subsystem corresponds to the logarithm of quantum dimension of local excitation of local operators. Inspired by the HEE approach, recently there were also some holographic analysis on the related problems [45, 46]. In addition, the first law-like relation which involve in other spatial components of the energy momentum tensor in the first order limit of subsystem have been discussed in [47-49]. Furthermore, it has been shown that the first law like relation can be written down in terms of ground state entanglement Hamiltonian [50], and a general relation between the ground state entanglement Hamiltonian and the stress tensor has been derived from the path integral formalism [51]. Interestingly, the authors in [52, 53] have found that the first law like relation was shown to be equivalent to the perturbative Einstein equations in pure AdS. The equivalence has been further supported by [54-56]. The HEE with first order quantum excitations has also been extended into theories containing higher derivative corrections [47] and the nonconformal cases [57, 58]. In addition, the 1-loop quantum correction to HEE of a 2-dimensional CFT has been calculated in [59] and the flavor corrections to HEE of $\mathcal{N} = 4$ super-Yang-Mills field has also been studied in [60].

Despite many interesting progresses have been made, previous studies on entanglement entropy with quantum excitations mostly focused on the linearized perturbation. In the present paper, we will further studying the low-energy excitations of the entanglement entropy caused by quantum fluctuations, from the viewpoint of the gauge/gravity duality. More specifically, we systematically study the first law-like relation between the variation of entanglement entropy and the variation of energy of the strip-like subsystem in the boundary CFT up to second order small perturbations of the source fields, based on the HEE proposal. We analyze the HEE and energy in three examples, $d$-dimensional CFT dual to the $d + 1$-dimensional Reissner-Nordström-Anti de Sitter black brane, and 2-dimensional CFTs dual to the BTZ black hole and its charged counterpart, and give evidences that, the first law-like relation should still be satisfied when taking into account of the second order
quantum corrections, while the associated entanglement temperature is modified, namely,

\[ T_e'(S^{(1)} + S^{(2)}) = E^{(1)} + E^{(2)}, \]

where \( S^{(1)} \), \( E^{(1)} \) and \( S^{(2)} \), \( E^{(2)} \) are the first order and second entanglement entropy and energy of the subsystem in the CFT. As has been pointed out in [61] that, the quantum corrections or loop corrections to the HEE contains two parts, one is from the entanglement from bulk regions separated by the bulk minimal surface, another comes from the variations from the area of the bulk minimal surface. While the quantum corrections considered in this paper is exactly of the second kind. Moreover, we also show that the second order HEE actually reflect the structure of the 2-point correlation function of the boundary renormalized stress tensor. When \( d = 2 \), it shows the correct scaling relation, consistent with the estimation from the dual CFT2 side. For other recent studies on second order quantum correction to entanglement entropy, see [62].

The rest parts of the paper are organized as follows: in section II we will describe the general formula for the area functional of bulk co-dimensional-2 surface up to second order perturbations and explain the logic that we will apply in the following sections. In sections III and IV we start from the bulk RN-AdS_{d+1} black brane, then calculate the second order HEE for a strip region and the second order stress tensor of the boundary CFT, and subsequently analyze the first law-like relation at second order. While in section V we continue studying the HEE with second order excitations for two interesting examples in the asymptotically AdS3 spacetime, one is the spinning BTZ black hole, another is the charged black hole in AdS3. Conclusions and discussions are drawn in Section VI.

II. PERTURBATIONS OF THE BULK CO-DIMENSIONAL-2 SURFACE

The area of the bulk spacelike co-dimensional 2 surface in \( d + 1 \)-dimensional spacetime is

\[ A = \int \sqrt{\gamma} d^{d-1} \xi, \]

where

\[ \gamma_{ij} = \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} g_{AB}. \]

is the induced metric and \( \xi^i \) is the coordinate on the surface, while \( X^A \) and \( g_{AB} \) are the coordinate and the metric of the bulk background spacetime, respectively. Then the variations of the area
functional are

$$\delta A = -\frac{1}{2} \int \sqrt{\gamma} \left( \gamma_{ij} \delta \gamma^{ij} + \frac{1}{2} \gamma_{ik} \gamma_{jl} \delta \gamma^{kl} + O((\delta \gamma)^3) \right) d^{d-1} \xi, \quad (6)$$

$$\delta^2 A = \frac{1}{2} \int \sqrt{\gamma} \left( \frac{1}{2} \gamma_{ij} \gamma_{kl} + \gamma_{ik} \gamma_{jl} \right) \delta \gamma^{ij} \delta \gamma^{kl} + O((\delta \gamma)^3) \right) d^{d-1} \xi. \quad (7)$$

where we have used $\delta \sqrt{\gamma} = \frac{\xi}{\sqrt{\gamma}} \left( \gamma_{ij} \delta \gamma_{ij} - \frac{1}{2} \gamma^{jk} \gamma^{jl} \delta \gamma_{kj} \delta \gamma_{li} \right)$ (up to the second order). Note that $\sqrt{\gamma}$ and $\gamma_{ij}$ in eqs. (6-7) are taken to be their zeroth order on-shell values $\sqrt{\gamma}^{(0)}$ and $\gamma_{ij}^{(0)}$. And in the small dimensionless parameter $\epsilon$ expansion

$$\delta \gamma_{ij} = \gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} + O(\epsilon^3), \quad (8)$$

$$\delta \gamma^{ij} = -\gamma^{(1)ij} + \gamma^{(1)ik} \gamma^{(1)kj} - \gamma^{(2)ij} + O(\epsilon^3), \quad (9)$$

where the indices are lowered and raised by the zeroth order metric $\gamma_{ij}^{(0)}$ and $\gamma^{(0)ij}$. Consequently,

$$A^{(1)} = \frac{1}{2} \int \sqrt{\gamma}^{(0)} \gamma_{ij}^{(1)} d^{d-1} \xi = \frac{1}{2} \int \sqrt{\gamma}^{(0)} \gamma^{(1)} d^{d-1} \xi, \quad (10)$$

$$A^{(2)} = \int \sqrt{\gamma}^{(0)} \left( -\frac{1}{2} \gamma^{(1)ij} \gamma_{ij}^{(1)} + \frac{1}{8} \left( \gamma^{(1)} \right)^2 + \frac{1}{2} \gamma^{(2)} \right) d^{d-1} \xi. \quad (11)$$

While the induced metric is expanded as

$$\gamma_{ij}^{(0)} = \left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(0)} g_{AB}, \quad (12)$$

$$\gamma_{ij}^{(1)} = \left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(0)} g_{AB}^{(1)} + \left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(1)} g_{AB}^{(0)}, \quad (13)$$

$$\gamma_{ij}^{(2)} = \left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(0)} g_{AB}^{(2)} + 2 \left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(1)} g_{AB}^{(1)} + \left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(2)} g_{AB}^{(0)}. \quad (14)$$

When only considering the first order perturbation, the following relation holds

$$\left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(1)} g_{AB}^{(0)} = \delta \left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right) g_{AB}^{(0)} = 0 \quad (15)$$

provided that the zeroth order EoM of the bulk co-dimensional 2 minimal surface is satisfied (on-shell). However, when doing the perturbation up to second order,

$$\left( \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right)^{(1)} g_{AB}^{(0)} \neq 0. \quad (16)$$

For the asymptotically AdS$_{d+1}$ spacetime in the Poincaré coordinates

$$ds^2 = \frac{L^2}{\xi^2} \left( d\xi^2 + \tilde{g}_{\mu \nu}(\xi, x) dx^\mu dx^\nu \right), \quad (17)$$
the general form of the bulk co-dimensional 2 static surface is \( \tilde{z} = \tilde{z}(x^i) \), then the area functional eq.\((\ref{eq:area_functional})\) becomes

\[
A = L^{d-1} \int \frac{d^{d-1}x}{\tilde{z}^{d-1}} \sqrt{\det \left( g_{ij}(\tilde{z}, x) + \frac{\partial \tilde{z}}{\partial x^i} \frac{\partial \tilde{z}}{\partial x^j} \right)}. \tag{18}
\]

Let us focus on the situation when eq.\((\ref{eq:perturbed_geometry})\) is a slightly perturbed geometry obtained from the pure AdS spacetime and study the small variation of eq.\((\ref{eq:area_functional})\) which deviates from its pure AdS counterpart. Generally speaking, in order to determine the shape of the bulk minimal surface, one needs to expand \( \bar{g}_{ij} = \eta_{ij} + \bar{g}^{(1)}_{ij} + \bar{g}^{(2)}_{ij} \) as well as \( \tilde{z} = \tilde{z}^{(0)} + \tilde{z}^{(1)} + \tilde{z}^{(2)} \) and solve the Euler-Lagrange equation for the bulk static minimal surface up to second order to determine \( \tilde{z}^{(1)} \) and \( \tilde{z}^{(2)} \). However, when the explicit integration form of the area functional (for the bulk minimal surface) in eq.\((\ref{eq:area_functional})\) is known, we can expand it around the minimal surface in pure AdS spacetime in terms of \( \varepsilon \) to arbitrary orders and the perturbed geometries already satisfy the Euler-Lagrange equation order by order. Applying this method, we can study the low-energy excitation corrections to the HEE in vacuum states from the expansion of the HEE with excited states.

**III. HOLOGRAPHIC ENTANGLEMENT ENTROPY IN ADS\(_{d+1}\) BLACK BRANE**

In the Einstein-Maxwell theory

\[
I = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{-g} \left( R + \frac{d(d - 1)}{L^2} - \frac{L^2}{g_s^2} F_{\mu\nu} F^{\mu\nu} \right), \tag{19}
\]

the charged black brane in AdS\(_{d+1}\), i.e. the Reissner-Nordström-Anti de Sitter (RN-AdS\(_{d+1}\)) black brane is \((d \geq 3)\)

\[
ds^2 = \frac{L^2}{r^2 f(r)} dr^2 + \frac{r^2}{L^2} \left( -f(r) dt^2 + dx_i^2 \right), \\
A = \mu \left( 1 - \frac{r_o^{d-2}}{r^{d-2}} \right) dt, \tag{20}
\]

where \( g_s \) is the dimensionless coupling constant of the U(1) gauge field, \( r_o \) is the radius of the outer horizon, \( L \) is the curvature radius of the AdS spacetime, \( M \) and \( Q \) are the mass and charge of the black brane, respectively, and

\[
f(r) = 1 - \frac{M}{r^d} + \frac{Q^2}{r^{2d-2}}, \quad \text{and} \quad \mu = \sqrt{\frac{d - 1}{2(d - 2)}} \frac{g_s Q}{L^2 r_o^{d-2}}. \tag{21}
\]

In the Poincaré coordinate \( z = L^2/r \), the metric is

\[
ds^2 = \frac{L^2}{z^2} \left( \frac{dz^2}{f(z)} - f(z) dt^2 + dx_i^2 \right), \tag{22}
\]
with \( f(z) = 1 - \frac{M_L^d}{L^d} + \frac{Q^2}{L^{d-2} z^{d-2}} = 1 - \tilde{M}^d_L + \tilde{Q}^2, \) where \( \tilde{M} \equiv M/L^d \) and \( \tilde{Q} \equiv Q/L^{d-1} \) are dimensionless mass and charge of the RN-AdS_{d+1} black brane.

To study the HEE in the background eq.(22) perturbatively, \( \tilde{M} \) and \( \tilde{Q} \) can be taken as the small parameters which act as the sources of the geometric perturbation away from the pure AdS_{d+1} spacetime (the perturbations can also be caused by adding external matter fields or the fluctuations from the metric itself). Taking \( \tilde{M} \) and \( \tilde{Q} \) as the first order perturbation, then both the metric components \( g_{tt} \) and \( g_{zz} \) undergo a second order variation as

\[
\delta g_{tt} = \frac{L^{d-2}}{r^{d-2}} \tilde{M} - \frac{L^{2d-4}}{r^{d-4}} \tilde{Q}^2 = \frac{z^{d-2}}{L^{d-2}} \tilde{M} - \frac{z^{2d-4}}{L^{2d-4}} \tilde{Q}^2 \equiv g_{(1)}^{(1)} + g_{(2)}^{(2)},
\]

\[
\delta g_{rr} = \frac{L^{d+2}}{r^{d+2}} \tilde{M} + \frac{L^{2d}}{r^{2d}} \tilde{M} - \frac{L^{2d+2}}{r^{2d+2}} \tilde{Q}^2 = \frac{z^{d+2}}{L^{d+2}} \tilde{M} - \frac{z^{2d+2}}{L^{2d+2}} \tilde{Q}^2 \equiv g_{(1)}^{(1)} + g_{(2)}^{(2)}. \quad (23)
\]

Dividing the boundary CFT into two intervals \( A + B \) and taking the subsystem \( A \) to be a strip in the region \( x_1 \in [-1/2, 1/2] \) and \( x_2 \in [-L_0/2, L_0/2] \) with \( b = 2, 3, \cdots x_{d-1} \). The bulk static minimal surface \( \gamma_A \) is a co-dimensional-2 surface, which is described by \( x_1 = x_1(z) \), then the bulk static hypersurface becomes

\[
ds^2 = \frac{L^2}{z^d} \left[ \left( \frac{1}{f(z)} + \left( \frac{1}{f(z)} \right)^2 \right) dz^2 + dx^2_a \right],
\]

and its area is

\[
A_{\gamma_A} = 2L^{d-1} L_0^{d-2} \int_1^2 \frac{dz}{z^{d-4}} \sqrt{ \frac{1}{f(z)} + \left( \frac{\partial x_1}{\partial z} \right)^2 },
\]

The bulk static minimal surface \( \gamma_A \) is determined from the Euler-Lagrange equation, i.e. by minimizing the area functional \( A_{\gamma_A} \)

\[
\frac{\delta A_{\gamma_A}}{\delta x_1} - \partial z \left( \frac{\delta A_{\gamma_A}}{\delta (\partial x_1)} \right) = 0,
\]

which results

\[
p(z) \equiv \frac{\partial z}{\partial x_1} = \sqrt{ f(z) \left( \frac{z^{2d-2}}{z^{d-2}} - 1 \right) },
\]

where \( z_\ast \) is the turning point of the bulk minimal surface at which \( p(z_\ast) = 0 \). Besides, it also reflects the embedding of the minimal surface into the bulk which will be affected by the bulk geometric perturbations. Consequently, the area of the bulk static minimal surface becomes

\[
A_{\gamma_A} = 2L^{d-1} L_0^{d-2} \int_1^{z_\ast} dx a^{d-2} \frac{dz}{z^{d-4}} \sqrt{ \frac{1}{f(z)} + \frac{1}{p(z)^2} },
\]

\[
= 2L^{d-1} L_0^{d-2} \int_1^{z_\ast} dx a^{d-2} \frac{dz}{z^{d-4}} \frac{1}{\sqrt{ f(z) \left( 1 - \frac{z^{2d-2}}{z^{d-2}} \right) }},
\quad (28)
\]
where $\epsilon$ is the geometric short distance cutoff which is related to the UV cutoff $a$ of the dual CFT via the UV/IR relation $\frac{L}{\epsilon} \simeq \frac{L_{A+B}}{a}$, with $L_{A+B}$ the total spatial length of the boundary CFT system $A + B$.

From eqs. (27) (28) we have

$$\frac{l}{2} = \int_{0}^{z_*} \frac{dz}{\sqrt{f(z) \left( \frac{z^{2d-2}}{z_*^{2d-2}} - 1 \right)}}. \quad (29)$$

Recall from eq. (1) that the variation of the area functional contains the contributions from the bulk metric perturbations as well as the changes in the embedding, both of which are controlled by the source fields, i.e. the conserved charges of the AdS black brane, consequently, the final expression of the area functional is the expansion in terms of these conserved charges. Defining $\frac{z^{2d-2}}{z_*^{2d-2}} = \xi$ (It is clear that here $\xi$ is different from $\xi^i$ in section II, which is the coordinates of the induced metric.), then up to order $O(\tilde{M}^3, \tilde{Q}^3)$

$$l = \frac{z_*}{d-1} \int_{0}^{1} d\xi \left( \frac{\tilde{M}}{2L^d} \frac{z^{\frac{d}{d-2}}}{z_*^{\frac{d}{d-2}}} + \frac{3\tilde{M}^2 z^{2d}}{8L^d} \frac{z^{\frac{d+2}{d-2}}}{z_*^{\frac{d+2}{d-2}}} - \frac{\tilde{Q}^2 z^{2d-2}}{2L^d} \frac{z^{\frac{d-2}{d-2}}}{z_*^{\frac{d-2}{d-2}}} \right) \left( 1 - \xi \right)^{\frac{1}{d-2}}$$

$$= \frac{z_* \sqrt{\pi}}{d-1} \left( \frac{\Gamma \left( \frac{d}{d-2} \right)}{\Gamma \left( \frac{d}{2d-2} \right)} + \frac{\tilde{M} z^{d}}{2L^d} \frac{\Gamma \left( \frac{d}{d-1} \right)}{\Gamma \left( \frac{d-1}{2d-2} \right)} + \frac{3\tilde{M}^2 z^{2d}}{8L^d} \frac{\Gamma \left( \frac{d+2}{d-1} \right)}{\Gamma \left( \frac{d}{2d+2} \right)} - \frac{d(d-1)\tilde{Q}^2 z^{2d-2}}{2(d-1)L^d} \frac{\Gamma \left( \frac{d}{2d-2} \right)}{\Gamma \left( \frac{1}{2d-2} \right)} \right) \left( 1 - \xi \right)^{\frac{1}{d-2}} \quad (30)$$

which in turn gives

$$z_* = z_*^{(0)} + z_*^{(1)} + z_*^{(2)} + O(\tilde{M}^3, \tilde{Q}^3), \quad (31)$$

where

$$z_*^{(0)} = \frac{l \Gamma \left( \frac{1}{2d-2} \right)}{2\sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)},$$

$$z_*^{(1)} = \frac{\tilde{M} \sqrt{\pi}}{(d+1)L^d \sqrt{\pi}} \frac{\Gamma \left( \frac{1}{2d-2} + 1 \right)}{2\sqrt{\pi} \Gamma \left( \frac{d+1}{2d-2} \right)} \frac{l \Gamma \left( \frac{1}{2d-2} \right)}{2\sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)} \frac{1}{d+1},$$

$$z_*^{(2)} = \frac{\tilde{M}^2}{L^d} \left( \frac{l \Gamma \left( \frac{1}{2d-2} \right)}{2\sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)} \right)^{2d+1} \frac{1}{4(d+1)} \frac{\Gamma \left( \frac{1}{2d-2} \right) \Gamma \left( \frac{1}{d-1} \right)}{\Gamma \left( \frac{d+1}{2d-2} \right) \Gamma \left( \frac{d+1}{2d-2} \right)} - \frac{3}{8(2d+1)} \frac{\Gamma \left( \frac{1}{2d-2} \right) \Gamma \left( \frac{3d}{2d-2} \right)}{\Gamma \left( \frac{d}{2d-2} \right) \Gamma \left( \frac{2d+1}{2d-2} \right)}$$

$$+ \frac{d\tilde{Q}^2}{2(2d-1)L^{2d-2}} \left( \frac{l \Gamma \left( \frac{1}{2d-2} \right)}{2\sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)} \right)^{2d-1}, \quad (32)$$
The corresponding area is

\[ A_{\gamma_A} = 2L^{d-1}L_0^{d-2} \int_{\epsilon}^{z_*} \frac{dz}{z^{d-1}} \frac{1}{\sqrt{f(z)(1 - \frac{2d-2}{z^2})}} \]

\[ = \frac{2L^{d-1}}{(d-2)} \left( \frac{L_0}{\epsilon} \right)^{d-2} \frac{2 \sqrt{\pi} L^{d-1} L_0^{d-2} \Gamma \left( \frac{d}{2d-2} \right)}{\Gamma \left( \frac{1}{2d-2} \right)} z_*^{2-d} + \frac{\tilde{M} \sqrt{\pi} L^{d-1} L_0^{d-2} \Gamma \left( \frac{1}{d-1} \right)}{2(d-1) \Gamma \left( \frac{d+1}{2d-2} \right)} z_*^2 \]

\[ + \frac{3 \tilde{M}^2}{8L^{2d}} \frac{\sqrt{\pi} L^{d-1} L_0^{d-2} \Gamma \left( \frac{d+2}{2d-2} \right)}{\Gamma \left( \frac{2d+1}{2d-2} \right)} z_*^{d+2} - \frac{\tilde{Q}^2 L_0^{d-1} \sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)}{L^{d-1} \Gamma \left( \frac{1}{2d-2} \right)} z_*^d + O(\tilde{M}^3, \tilde{Q}^3). \]

Substituting eq. (32) into eq. (33), we obtain

\[ A_{\gamma_A}^{(0)} = \frac{2L^{d-1}}{(d-2)} \left( \frac{L_0}{\epsilon} \right)^{d-2} - \frac{L^{d-1} L_0^{d-2}}{(d-2)d-2} \left( \frac{2 \sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)}{\Gamma \left( \frac{1}{2d-2} \right)} \right)^{d-1} \]

\[ A_{\gamma_A}^{(1)} = \frac{\tilde{M} L_0^{d-2} l^2}{8 \sqrt{\pi} (d+1) L \Gamma \left( \frac{d+1}{2d-2} \right)} \left( \frac{\Gamma \left( \frac{1}{d-1} \right)}{\Gamma \left( \frac{2d-2}{2d-2} \right)} \right)^2 \]

\[ A_{\gamma_A}^{(2)} = \frac{\tilde{M}^2 L^{d-2} \sqrt{\pi}}{4L^{d+1}} \left( \frac{l \Gamma \left( \frac{1}{d-2} \right)}{2 \sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)} \right)^{d-2} - \frac{1}{(d+1)^2(d-1) \Gamma \left( \frac{d}{2d-2} \right)} \left( \frac{\Gamma \left( \frac{1}{d-1} \right)}{\Gamma \left( \frac{d+1}{2d-2} \right)} \right)^2 \]

\[ + \frac{3}{2(2d+1) \Gamma \left( \frac{2d+1}{2d-2} \right)} \frac{\tilde{Q}^2 L_0^{d-2} l^2}{2L^{d-1}} \left( \frac{l \Gamma \left( \frac{1}{d-2} \right)}{2 \sqrt{\pi} \Gamma \left( \frac{d}{2d-2} \right)} \right)^{d-1} \] \hspace{1cm} (34)

Therefore, the corresponding HEE is

\[ S_{\gamma_A} = \frac{1}{4G_{d+1}} \left( A_{\gamma_A}^{(0)} + A_{\gamma_A}^{(1)} + A_{\gamma_A}^{(2)} \right) + O(\tilde{M}^3, \tilde{Q}^3). \] \hspace{1cm} (35)

Note that the first order value is always positive, while the second order value is always negative

\[ S_{\gamma_A}^{(1)} > 0, \quad \text{and} \quad S_{\gamma_A}^{(2)} < 0. \] \hspace{1cm} (36)

### IV. THE BOUNDARY STRESS TENSOR OF THE DUAL CFT\(_d\)

The \(d+1\) dimensional bulk spacetime can be written into the ADM form as

\[ ds^2 = N^2 dr^2 + g_{\mu\nu} (N^\mu dr + dx^\mu) (N^\nu dr + dx^\nu) \equiv g_{AB} dx^A dx^B \] \hspace{1cm} (37)

\(^1\) Note that eq. (34) is held for \(d > 2\), the special case is \(d = 2\), in which the zeroth order dominant term in the area is given by \(\lim_{\epsilon \to 0} 2L \int_{z_*}^z \frac{dz}{z^d} = 2L \ln \frac{1}{\epsilon} \simeq 2L \ln \frac{1}{\epsilon}\), i.e. the logarithmic divergence.
and the boundary stress tensor is calculated from the Brown-York formalism

\[ T_{\mu\nu} = \frac{1}{8\pi G_{d+1}} (K g^{\mu\nu} - K^{\mu\nu}) , \]  

where

\[ K_{\mu\nu} = N \Gamma_{\mu\nu} = \frac{N}{2} g^{r A} (g_{\mu A,\nu} + g_{A\nu,\mu} - g_{\mu\nu,A}) , \]

in which

\[ T_{\mu\nu} = \frac{1}{8\pi G_{d+1}} \left( \frac{1}{d+1} g^{r A} (g_{\mu A,\nu} + g_{A\nu,\mu} - g_{\mu\nu,A}) \right) , \]  

The variation of the stress tensor with respect to some dimensionless parameter \( \varepsilon \) is

\[ \delta T_{\mu\nu} = \frac{-1}{8\pi G_{d+1}} (\delta K g_{\mu\nu} + K \delta g_{\mu\nu} - \delta K_{\mu\nu}) \]

\[ = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + \mathcal{O}(\varepsilon^3), \]  

with

\[ T_{\mu\nu}^{(1)} = \frac{-1}{8\pi G_{d+1}} \left( K^{(0)} g_{\mu\nu}^{(1)} + K^{(1)} g^{(0)}_{\mu\nu} - K^{(1)}_{\mu\nu} \right) , \]

\[ T_{\mu\nu}^{(2)} = \frac{-1}{8\pi G_{d+1}} \left( K^{(0)} g_{\mu\nu}^{(2)} + 2K^{(1)} g_{\mu\nu}^{(1)} + \left( K^{(2)} - K^{(1)}_{\alpha\beta} g^{(1)\alpha\beta} \right) g_{\mu\nu}^{(0)} - K_{\mu\nu}^{(2)} \right) , \]

\[ K^{(1)} = g^{(0)\mu\nu} K^{(1)}_{\mu\nu} - g^{(1)\mu\nu} K^{(0)}_{\mu\nu} , \]

\[ K^{(2)} = g^{(0)\mu\nu} K^{(2)}_{\mu\nu} - g^{(1)\mu\nu} K^{(1)}_{\mu\nu} + g^{(1)\mu\lambda} g^{(1)\nu}_{\lambda} K^{(0)}_{\mu\nu} - g^{(2)\mu\nu} K^{(0)}_{\mu\nu} , \]

\[ K_{\mu\nu}^{(1)} = N^{(1)} \Gamma_{\mu\nu}^{(1)\rho} + N^{(0)} \Gamma_{\mu\nu}^{(1)\rho} , \]

\[ K_{\mu\nu}^{(2)} = N^{(2)} \Gamma_{\mu\nu}^{(0)\rho} + 2N^{(1)} \Gamma_{\mu\nu}^{(1)\rho} + N^{(0)} \Gamma_{\mu\nu}^{(2)\rho} . \]

When the boundary is spatially flat, such as the RN-AdS\(_{d+1}\) black brane, the boundary counterterm added to cancel the UV divergence is

\[ I_{ct} = \frac{-(d-1)}{8\pi G_{d+1}} \int_{\partial M} \frac{d^d x \sqrt{-g}}{L} , \]

which contribute to the boundary stress tensor as

\[ T_{\mu\nu}^{ct} = \frac{2}{\sqrt{-g} g^{\mu\nu}} I_{ct} = \frac{-1}{8\pi G_{d+1}} \frac{N^{(2)} \Gamma_{\mu\nu}^{(0)\rho}}{L} g_{\mu\nu} . \]

Then the first order renormalized stress tensor on the fixed \( r \) hypersurface is

\[ T_{\mu\nu}^{(1)} + T_{\mu\nu}^{ct(1)} = \frac{-1}{8\pi G_{d+1}} \left( \frac{g_{\mu\nu}^{(1)}}{L} + \frac{(d-1) r^2}{2L^3} g_{\mu\nu}^{(1)} + \frac{g_{\lambda\mu}^{(1)} g_{\lambda\nu}^{(1)}}{L^2} - \frac{r}{2L^2} g_{\alpha\beta}^{(0)} g_{\mu\nu}^{(1)} + \frac{r}{2L} g_{\mu\nu}^{(1)} \right) \]
While the expectation value of the renormalized stress tensor of the boundary CFT$_d$ is obtained from variation of the full action with respect to the metric $\bar{g}_{\mu\nu}$ on the conformal boundary (the asymptotic boundary is located at $z = \epsilon \to 0$) \[63, 64\].

$$\delta (I + I_{\text{bdy}} + I_{\text{ct}}) = \text{bulk terms} - \frac{1}{2} \int \sqrt{-g} d^d x \langle T_{\mu\nu} \rangle \delta \bar{g}^{\mu\nu},$$

where $\bar{g}_{\mu\nu} = \frac{k^2_{\text{eff}} g_{\mu\nu}}{L^2}$, $I_{\text{bdy}}$ is the boundary action required by a well-defined variation principle and $\langle T_{\mu\nu} \rangle = (\frac{L}{\pi G_N})^{d-2} (T_{\mu\nu} + T_{\mu\nu}^{\text{ct}}) = (\frac{L}{\pi G_N})^{d-2} (T_{\mu\nu} + T_{\mu\nu}^{\text{ct}})$. For the RN-AdS$_{d+1}$ black brane eq.\[20\], the nonvanishing components of $\langle T_{\mu\nu} \rangle$ are

$$\langle T^{(1)}_{tt} \rangle = \frac{(d-1)\bar{M}}{16\pi G_{d+1} L},$$

$$\langle T^{(1)}_{xx} \rangle = \frac{\bar{M}}{16\pi G_{d+1} L}.$$

Subsequently, the trace of the first order stress tensor

$$\langle T^{(1)}_{\lambda\lambda} \rangle = g^{(0)tt} \langle T^{(1)}_{tt} \rangle + g^{(0)xx} \langle T^{(1)}_{xx} \rangle = 0,$$

which indicates that the boundary dual CFT$_d$ is conformal anomaly free up to first order quantum corrections.

The second order renormalized stress tensor is

$$T^{(2)}_{\mu\nu} + T^{\text{ct}(2)}_{\mu\nu} = -\frac{1}{8\pi G_{d+1}} \left( - \frac{g^{(2)}_{\mu\nu}}{L} + \left( \frac{r^2}{L^3} g^{(1)}_{rr} + \frac{2g^{(1)}_{\lambda\lambda}}{L} - \frac{r}{L} g^{(0)\alpha\beta} g^{(1)}_{\alpha\beta, r} \right) g^{(1)}_{\mu\nu} ight) + \left( \frac{r^2}{2L^3} g^{(2)}_{rr} + \frac{(d-1)r^4}{8L^5} \left( g^{(1)}_{rr} \right)^2 - \frac{(d-1)g^{(1)\lambda\lambda}}{L} g^{(1)\alpha\beta} g^{(1)}_{\alpha\beta, r} - \frac{r^2}{L^3} g^{(1)}_{\lambda\lambda} g^{(1)}_{rr} + \frac{r}{L} g^{(1)\alpha\beta} g^{(1)}_{\alpha\beta, r} - \frac{g^{(1)\alpha\beta}}{L} g^{(2)\alpha\beta} g^{(1)}_{\mu\nu} + \frac{r}{2L} g^{(2)}_{\mu\nu, r} \right).$$

Thus the nonvanishing components of $\langle T^{(2)}_{\mu\nu} \rangle$ are

$$\langle T^{(2)}_{tt} \rangle = -\frac{1}{8\pi G_{d+1}} \left( \frac{(3d-11)z^d}{8L^{d+1}} \bar{M}^2 + \frac{(d-1)z^{d-2}}{2L^{d-1}} \bar{Q}^2 \right),$$

$$\langle T^{(2)}_{xx} \rangle = -\frac{1}{8\pi G_{d+1}} \left( \frac{(11-3d)z^d}{8L^{d+1}} \bar{M}^2 + \frac{(d-1)z^{d-2}}{2L^{d-1}} \bar{Q}^2 \right).$$

It is straightforward to check that

$$\langle T^{(2)\lambda} \rangle = g^{(0)\mu\nu} \langle T^{(2)}_{\mu\nu} \rangle - g^{(1)\mu\nu} \langle T^{(1)}_{\mu\nu} \rangle = \frac{1}{8\pi G_{d+1}} \left( \frac{(3d^2 - 15d + 4)z^{d+2}}{8L^{d+3}} \bar{M}^2 - \frac{(d-2)(d-1)z^d}{2L^{d+1}} \bar{Q}^2 \right).$$
When taking \( z = \epsilon \to 0 \) to the asymptotical boundary, \( \langle T_{\mu\nu}^{(2)} \rangle \to 0 \). With the explicit expressions of \( \langle T_{tt} \rangle \), we then can calculate the energy associated to the subsystem \( A \) in the boundary CFT, which is

\[
E = \int dx_b d^{d-2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \langle T_{tt} \rangle = E^{(1)} + E^{(2)} + \mathcal{O}(\tilde{M}^3, \tilde{Q}^3),
\]

where

\[
E^{(1)} = \frac{(d-1)L_0^{d-2}l\tilde{M}}{16\pi G_{d+1}L},
E^{(2)} = 0.
\]

Consequently, we can check that the first law-like relation for the boundary CFT holds at the first order perturbation, namely,

\[
T_e S_{\gamma A}^{(1)} = E^{(1)},
\]

with the entanglement temperature given by

\[
T_e = \frac{2(d^2 - 1)\Gamma \left( \frac{d+1}{2(d-2)} \right) \left( \Gamma \left( \frac{d}{2d-2} \right) \right)^2}{\sqrt{\pi} \Gamma \left( \frac{1}{d-1} \right) \Gamma \left( \frac{1}{d-1} \right)} l^{-1}.
\]

When including the contribution from the second order excitations, we have

\[
T_e \left( S_{\gamma A}^{(1)} + S_{\gamma A}^{(2)} \right) < E^{(1)} + E^{(2)}.
\]

Naively, eq.\( 57 \) may give a consistent entropy bound for the subsystem \( A \) resembling the Bekenstein bound 55. Similar related arguments can be found in the study of the excitation of relative entropy for spherical entangling surface 50. However, recall that the bulk gravity dual we studied is a stationary RN-AdS\(_{d+1}\) black brane, so it is expectable that the perturbed geometries are also stable, order by order away from the pure AdS\(_{d+1}\) spacetime under small fluctuations. Further supporting evidence is that we checked that the bulk perturbed Einstein equations are satisfied both at the first and second orders, as they should be. Further taking into account the equivalent relation between eq.\( 55 \) and the first order Einstein equation 52, 53, it is natural to expect that at higher orders perturbation, the first law-like relation should dual to bulk perturbed dynamical equations at the same order. That is to say, the first law-like relation eq.\( 55 \) should still be satisfied at the second order perturbation, namely,

\[
T_e' \left( S_{\gamma A}^{(1)} + S_{\gamma A}^{(2)} \right) = E^{(1)} + E^{(2)}.
\]
which indicates that the entanglement temperature gets modified as (up to second order)

\[ T'_e = T_e \left( 1 - a_1 \tilde{M} - a_2 \frac{\tilde{Q}^2}{M} + a_1^2 \tilde{M}^2 + 2a_1a_2\tilde{Q}^2 + a_2^2 \frac{\tilde{Q}^4}{M^2} \right), \quad (59) \]

where

\[ a_1 = \frac{(d+1)}{2} \left( \frac{l}{L} \right)^d \frac{(d+1)}{2(d-2)} \frac{\Gamma \left( \frac{1}{d-1} \right)}{\Gamma \left( \frac{d}{d-2} \right)} \left( \frac{\Gamma \left( \frac{1}{d-2} \right)}{2\sqrt{\pi} \Gamma \left( \frac{d}{d-2} \right)} \right)^d \frac{1}{\Gamma \left( \frac{1}{d-1} \right)} \left( \frac{1}{d-1} \right)^2 \right)^2 -\frac{3}{2(2d+1)} \frac{\Gamma \left( \frac{d+2}{2d-2} \right)}{\Gamma \left( \frac{2d+2}{2d-2} \right)}, \]

\[ a_2 = -\frac{(d+1)}{\sqrt{\pi}} \left( \frac{l}{L} \right)^{d-2} \frac{\Gamma \left( \frac{d+1}{2d-2} \right)}{\Gamma \left( \frac{1}{d-1} \right)} \left( \frac{\Gamma \left( \frac{1}{d-2} \right)}{2\sqrt{\pi} \Gamma \left( \frac{d}{d-2} \right)} \right)^{d-3}. \quad (60) \]

V. ASYMPTOTICALLY ADS\(_3\) SPACETIME

Let us study the holographic entanglement entropy of CFT\(_2\) with second order excitations in asymptotically AdS\(_3\) spacetime, following the spirit that they are formed by small spatially homogeneous metric perturbations from the pure AdS\(_3\) spacetime. The first example is the spinning BTZ black hole and the second one is the charged black hole in AdS\(_3\) spacetime.

A. Spinning BTZ black hole

For the spinning BTZ black hole \[66, 67\]

\[ ds^2 = -\frac{(r^2 - r^2_+)(r^2 - r^2_-)}{L^2 r^2} dt^2 + \frac{L^2 r^2}{(r^2 - r^2_+)(r^2 - r^2_-)} dr^2 + r^2(d\phi - \frac{r_+ - r_-}{L^2} dt)^2, \quad (61) \]

where the black hole is of mass \( M = (r^2_+ + r^2_-)/(8G_3L^2) \), angular momentum \( J = r_+ r_-/(4G_3L) \) and temperature \( T = (r^2_+ - r^2_-)/(2\pi r_+ L^2) \). The cosmic censorship requires \( ML \geq J \), and if we take \( ML \equiv \alpha \) to be the small parameter, the magnitude of \( J \) cannot be determined generally. In the following we will require that \( ML \) and \( J \) are of the same order, the special case is the near extreme BTZ black hole, in which \( ML \to J \), hence

\[ \delta g_{tt} = g^{(1)}_{tt} = 8G_3M, \quad \delta g_{tx} = g^{(1)}_{tx} = g^{(1)}_{xt} = -\frac{4G_3J}{L}, \]

\[ \delta g_{rr} = g^{(1)}_{rr} + g^{(2)}_{rr} = \frac{8G_3L^4 M}{r^4} + \frac{16G_3L^4}{r^6} (4M^2 L^2 - J^2), \quad \delta g_{xx} = 0. \quad (62) \]
So the components of the first and second order renormalized boundary stress tensor are

\[
\langle T^{(1)}_{tt} \rangle = \frac{\alpha}{2\pi L^2}, \quad \langle T^{(2)}_{tt} \rangle = \frac{G_3}{\pi L r^2} \left(5M^2L^2 + 3J^2\right),
\]

\[
\langle T^{(1)}_{tx} \rangle = \langle T^{(1)}_{xt} \rangle = \frac{J}{2\pi L^2}, \quad \langle T^{(2)}_{tx} \rangle = \langle T^{(2)}_{xt} \rangle = 0,
\]

\[
\langle T^{(1)}_{xx} \rangle = \frac{\alpha}{2\pi L^2}, \quad \langle T^{(2)}_{xx} \rangle = -\frac{G_3}{\pi L r^2} \left(5M^2L^2 + 3J^2\right). \quad (63)
\]

To calculate the holographic entanglement entropy in the background of the BTZ black hole, it is more convenient to convert eq. (61) into the Poincaré coordinate via \( z = \frac{L^2}{r^2} \), namely, the BTZ black hole becomes

\[
ds^2 = \frac{L^2}{z^2} \left( -\left(1 - \frac{z^2}{z_+^2}\right) (1 - \frac{z^2}{z_-^2}) dt^2 + \frac{dz^2}{(1 - \frac{z^2}{z_+^2})(1 - \frac{z^2}{z_-^2})} + \left(Ld\phi - \frac{z^2}{z_+z_-}dt\right)^2 \right), \quad (64)
\]

where \( z_{\pm} = \frac{L^2}{r_{\pm}^2} \). Denoting the bulk static codimensional-2 surface (curve) as \( x = L\phi = x(z) \), dividing the boundary CFT into two subsystems A and B, and requiring A is located in \( x_1 \in [-l/2, l/2] \), then following the steps in asymptotically AdS\(_{d+1}\) case, i.e. using eqs. (27) (29) we obtain

\[
l = \frac{1}{2} = \int_0^{z_*} dz \sqrt{\left(1 - \frac{8G_3\alpha z^2}{L^3} + \frac{16G_3^2J^2z^4}{L^6}\right)} \left(\frac{z^2}{z^4} - 1\right)
\]

\[
= z_* \left(1 + \frac{8G_3^2\alpha}{3L^3} z_*^2 + \frac{64G_3^2}{15L^6} \left(3\alpha^2 - J^2\right) z_*^4 + O(\alpha^3, J^3)\right). \quad (65)
\]

Subsequently, \( z_* \) is determined as

\[
z_* = \frac{l}{2} - \frac{G_3l^3}{3L^3\alpha} + \frac{4G_3^2l^5}{15L^6} \left(\alpha^2 + \frac{J^2}{2}\right) + O(\alpha^3, J^3)
\]

\[
= \frac{l}{2} - \frac{l^3}{2cL^2\alpha} + \frac{3l^5}{5c^2L^4} \left(\alpha^2 + \frac{J^2}{2}\right) + O(\alpha^3, J^3), \quad (66)
\]

in which \( c = \frac{3l}{2G_3} \) is the central charge of the boundary CFT\(_2\) dual to the bulk BTZ black hole.

Consequently, the area of the bulk minimal curve is

\[
A_{\gamma A} = 2L \int_a^{z_*} \frac{dz}{z} \sqrt{\left(1 - \frac{8G_3\alpha z^2}{L^3} + \frac{16G_3^2J^2z^4}{L^6}\right)} \left(1 - \frac{z^2}{z_*^2}\right)
\]

\[
= L \left(2\ln \frac{2z_*}{a} + \frac{8G_3\alpha}{L^3} z_*^2 + \frac{32G_3^2}{3L^6} \left(3\alpha^2 - J^2\right) z_*^4\right) + O(\alpha^3, J^3, a^2)
\]

\[
= L \left(2\ln \frac{l}{a} + \frac{2G_3\alpha}{3L^3} l^2 - \frac{2G_3^2}{45L^6} \left(\alpha^2 + 3J^2\right) l^4\right) + O(\alpha^3, J^3, a^2). \quad (67)
\]
Namely, we obtain the holographic entanglement entropy of the subsystem $A$ as
\[
S_{\gamma A} = \frac{A_{\gamma A}}{4G_3} = S_{\gamma A}^{(0)} + S_{\gamma A}^{(1)} + S_{\gamma A}^{(2)} + \mathcal{O}(\alpha^3, J^3, a^2)
\]
\[
= \frac{c}{3} \ln \frac{l}{a} + \frac{l^2 \alpha}{6L^2} - \frac{l^4}{60cL^4} (\alpha^2 + 3J^2) + \mathcal{O}(\alpha^3, J^3, a^2).
\] (68)

The energy of the subsystem $A$ in the boundary CFT$_2$ is
\[
E = \int_{-\pi}^{\pi} dx \langle T_{tt} \rangle
= E^{(0)} + E^{(1)} + E^{(2)} + \mathcal{O}(\alpha^3, J^3, a^2),
\] (69)

where
\[
E^{(0)} = 0,
E^{(1)} = \frac{\alpha l}{2\pi L^2},
E^{(2)} = 0.
\] (70)

From eqs. (68) (70) we see that the first order result gives the first law-like relation [41] as
\[
T_e S_{\gamma A}^{(1)} = E^{(1)},
\] (71)

with the entanglement temperature $T_e = \frac{3}{\pi l}$. Again, including the second order excitations, we have the inequality
\[
T_e \left( S_{\gamma A}^{(1)} + S_{\gamma A}^{(2)} \right) < E^{(1)} + E^{(2)}.
\] (72)

If $T_e$ is not modified, eq. (72) shows that the subsystem has reached a maximal entropy under small fluctuations. But similar to the RN-AdS$_{d+1}$ black brane case, the bulk BTZ black hole is an exact stationary solution. Regarding it as the slightly perturbed geometry from the pure AdS$_3$ spacetime, the perturbations should be stable under all orders of the coupling constants, thus at the second order, the first law-like relation should be
\[
T_e' \left( S_{\gamma A}^{(1)} + S_{\gamma A}^{(2)} \right) = E^{(1)} + E^{(2)},
\] (73)

with the second order modified entanglement temperature
\[
T_e' = \frac{3}{\pi l} \left( 1 + \frac{l^2}{10cL^2} \left( \alpha + \frac{3J^2}{\alpha} \right) + \frac{l^4}{100c^2L^4} \left( \alpha + \frac{3J^2}{\alpha} \right)^2 \right).
\] (74)

Furthermore, recall that the conformal dimension of the boundary stress tensor of the CFT$_2$ dual to the bulk massless graviton is $\Delta = d = 2$, and
\[
S_{\gamma A}^{(2)} = -\frac{l^4}{60cL^4} (M^2 L^2 + 3J^2) = -\frac{l^4}{60cL^4} (M^2 L^2 + 3J^2),
\] (75)

which is in accord with the estimation from the dual CFT$_2$ side [39].
B. Charged black holes in AdS$_3$

The charged black hole in AdS$_3$ obtained from the Einstein-Maxwell theory

\[
I = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left( R + \frac{2}{L^2} - \frac{L^2}{g_s^2} F_{\mu\nu} F^{\mu\nu} \right),
\]

has the following metric

\[
ds^2 = -\frac{r^2}{L^2} f(r) dt^2 + \frac{L^2}{r^2 f(r)} dr^2 + r^2 d\phi^2,
\]

with

\[
f(r) = 1 - \frac{M}{r^2} - \frac{Q^2}{4r^2} \ln \left( \frac{r^2}{L^2} \right) \quad \text{and} \quad A = \frac{g_s Q}{2L^2} \ln \frac{r}{r_h} dt,
\]

where $g_s$ is the coupling constant, $Q$ is the charge, and $r_h$ is the black hole outer horizon. When $\frac{M}{L^2} - \frac{Q^2}{4L^2} \geq 0$, the above metric describes a regular nonrotating charged black hole within the circle with radius $L$, previous studies on the CFT description of the black hole has been discussed in [68]. In the Poincaré coordinate eq. (77) is

\[
ds^2 = \frac{L^2}{z^2} \left( -\left( 1 - \frac{Mz^2}{L^4} + \frac{Q^2 z^2}{4L^4} \ln \left( \frac{z^2}{L^2} \right) \right) dt^2 + \frac{dz^2}{1 - \frac{Mz^2}{L^4} + \frac{Q^2 z^2}{4L^4} \ln \left( \frac{z^2}{L^2} \right)} + dx^2 \right),
\]

As before, let us treat the mass and charge of the black hole as small perturbations in the pure AdS$_3$ spacetime. When both $M$ and $Q$ are small, the cosmic censorship requires $\frac{M}{L^2}$ and $\frac{Q}{L^2}$ are of the same magnitudes. and define $\frac{M}{L^2} \equiv \beta$ and $\frac{Q}{L^2} \equiv \gamma$. Hence,

\[
\delta g_{tt} = g_{tt}^{(1)} + g_{tt}^{(2)} = \beta + \frac{\gamma^2}{4} \ln \frac{r^2}{L^2}, \quad \delta g_{xx} = 0,
\]

\[
\delta g_{rr} = g_{rr}^{(1)} + g_{rr}^{(2)} = \frac{L^4 \beta}{r^4} + \frac{L^4 \gamma^2}{4r^4} \ln \frac{r^2}{L^2} + \frac{L^6 \beta^2}{r^6}.
\]

Beside, the variation of the boundary holographic stress tensor is

\[
\langle T_{tt}^{(1)} \rangle = \frac{\beta}{16\pi G_3 L}, \quad \langle T_{tt}^{(2)} \rangle = \frac{5L^2 \beta^2}{64\pi G_3 L} + \frac{\gamma^2}{64\pi G_3 L} \ln \frac{r^2}{L^2},
\]

\[
\langle T_{tx}^{(1)} \rangle = \langle T_{xt}^{(1)} \rangle = 0, \quad \langle T_{tx}^{(2)} \rangle = \langle T_{xt}^{(2)} \rangle = 0,
\]

\[
\langle T_{xx}^{(1)} \rangle = -\frac{\beta}{16\pi G_3 L}, \quad \langle T_{xx}^{(2)} \rangle = -\frac{5L^2 \beta^2}{32\pi G_3 L} - \frac{\gamma^2}{64\pi G_3 L} + \frac{\gamma^2}{64\pi G_3 L} \ln \frac{r^2}{L^2}.
\]

Note that both $\langle T_{tt}^{(2)} \rangle$ and $\langle T_{xx}^{(2)} \rangle$ are divergent as $r$ approaches to the boundary. To cancel the divergence, the boundary counterterm term from the gauge field is required \(^2\), which is

\[
I_{ct}^{\text{gauge}} = c_1 \int dx^2 \sqrt{-g} F_{\tau\rho} F^\tau{}^\rho \ln \frac{r}{L},
\]

\(^2\) To make the variation of the action in eq. (76) to be well-defined, the Gibbons-Hawking boundary term and another boundary term for the bulk gauge field which satisfies the Neumann boundary condition should be added, which is $I_N = c_2 \int \sqrt{-g} dx^2 n^i F_{\tau\mu} A^\mu$, with $n^i = \frac{\sqrt{-g}}{L}$ to be the unit normal vector of the timelike boundary.
where $c_1$ is the constant which will be fixed by canceling the UV divergence. Eq. (82) will only contribute to the boundary stress tensor of the CFT$_2$ at the second order,

\[
T_{\mu\nu}^{\text{gauge}} = \frac{2}{\sqrt{g}} \frac{\delta T_{\mu\nu}^{\text{gauge}}}{\delta g^{\mu\nu}} = c_1 \left( \left( g_s Q \right)^2 \left( \frac{g_{\mu\nu}}{2L^2} \right)^2 + 2 \frac{r^2 f(r)}{L^2} F_{\mu\rho} F_{\nu} \right) \ln \frac{r}{L},
\]

then the total second order boundary stress tensor is

\[
\langle T_{tt}^{(2)} \rangle_{\text{total}} = \langle T_{tt}^{(2)} \rangle + T_{tt}^{\text{gauge}(2)} = \frac{5L\beta^2}{64\pi G_3 r^2} + \frac{\gamma^2}{12 \pi G_3 L} \ln \frac{r^2}{L^2} + c_1 \frac{g_s^2 \gamma^2}{4L^4} \ln \frac{r}{L},
\]

\[
\langle T_{xx}^{(2)} \rangle_{\text{total}} = \langle T_{xx}^{(2)} \rangle + T_{xx}^{\text{gauge}(2)} = -\frac{\gamma^2}{32 \pi G_3 L} - \frac{5L\beta^2}{64\pi G_3 r^2} + \frac{\gamma^2}{12 \pi G_3 L} \ln \frac{r^2}{L^2} + c_1 \frac{g_s^2 \gamma^2}{4L^4} \ln \frac{r}{L},
\]

which give the finite results for the total stress tensor when choosing $c_1 = \frac{L^3}{8\pi G_3 g_s^2}$. Finally,

\[
\langle T_{tt}^{(2)} \rangle_{\text{total}} = \frac{5L\beta^2}{64\pi G_3 r^2},
\]

\[
\langle T_{xx}^{(2)} \rangle_{\text{total}} = -\frac{\gamma^2}{32 \pi G_3 L} - \frac{5L\beta^2}{64\pi G_3 r^2},
\]

In the Poincaré coordinate, the bulk codimensional-2 surface is $x = x(z)$ and the boundary CFT is divided into two subsystems A and B, in which A is located in $x_1 \in [-l/2, l/2]$, then

\[
\frac{l}{2} = \int_0^{z_*} \frac{dz}{\sqrt{\left(1 - \frac{M^2 x^2}{L^2} + \frac{Q^2 x^2}{4L^2} \ln \left(\frac{z^2}{L^2}\right)\left(\frac{z^2}{L^2} - 1\right)\right)}} = z_* \left(1 + \frac{\beta^2 z_*^2}{3 L^2} + \frac{\gamma^2 z_*^2}{36 L^2} \left(5 - 6 \ln 2 + \frac{L}{z_*}\right) + \frac{\beta^2 z_*^4}{5 L^4}\right),
\]

and $z_*$ is obtained as

\[
z_* = \frac{l}{2} - \frac{l^3}{24 L^2} \beta + \frac{l^5}{240 L^4} \beta^2 - \frac{l^3}{288 L^2} (5 + 6 \ln \frac{L}{z_*}) \gamma^2 + O(\beta^3, \beta \gamma^2).
\]

While the area of the bulk minimal curve is

\[
A_{\gamma_A} = 2L \int_a^{z_*} \frac{dz}{z} \frac{1}{\sqrt{\left(1 - \frac{M^2 x^2}{L^2} + \frac{Q^2 x^2}{4L^2} \ln \left(\frac{z^2}{L^2}\right)\left(\frac{z^2}{L^2} - 1\right)\right)}} = L \left(2 \ln \frac{2z_*}{a} + \frac{\beta^2 z_*^2}{2 L^2} + \frac{\gamma^2}{2 L^2} \left(1 - \ln 2 + \frac{L}{z_*}\right) z_* + \frac{\beta^2 z_*^4}{2 L^4}\right) + O(\alpha^3, \beta \gamma^2, a^2)
\]

\[
= L \left(2 \ln \frac{1}{a} + \frac{l^2 \beta}{12 L^2} - \frac{l^4 \beta^3}{1440 L^4} + \left(\frac{l^2}{18 L^2} + \frac{l^4}{24 L^2} \ln \frac{L}{l}\right) \gamma^2\right) + O(\alpha^3, \beta \gamma^2, a^2).
\]

Finally, the holographic entanglement entropy of the subsystem A is

\[
S_{\gamma_A} = \frac{c}{3} \ln \frac{1}{a} + \frac{c l^2}{72 L^2} \beta - \frac{c l^4}{8640 L^4} \beta^2 + \frac{c l^2}{36 L^2} \left(\frac{1}{3} + \frac{\ln \frac{L}{l}}{4}\right) \gamma^2 + O(\alpha^3, \beta \gamma^2, a^2),
\]
in which
\[
S^{(1)}_{\gamma A} = \frac{c l^2}{72 L^2} \beta, \\
S^{(2)}_{\gamma A} = -\frac{c l^4}{8640 L^4} \beta^2 + \frac{c l^2}{36 L^2} \left( \frac{1}{3} + \frac{\ln \frac{L}{\beta}}{4} \right) \gamma^2,
\]
which shows that the second order excitation to HEE from the mass is negative. While the contribution from the charge is positive, in contrast to the \(d \geq 3\) charged black brane cases considered in Section III.

In addition, the energy of dual boundary CFT is
\[
E = \int dx \langle T_{tt} \rangle_{\text{total}} \\
= E^{(0)} + E^{(1)} + E^{(2)} + \mathcal{O}(\alpha^3, \alpha \gamma^2, a^2),
\]
where
\[
E^{(0)} = 0, \\
E^{(1)} = \frac{l \beta}{16 \pi G_3 L}, \\
E^{(2)} = 0.
\]
Consequently, it is straightforward to check the first law-like relation is held at first order when \(T_e\) is not corrected, i.e.
\[
T_e S^{(1)}_{\gamma A} = E^{(1)},
\]
with \(T_e = 3/(\pi l)\) as in the spinning BTZ black hole cases. However, when the contribution from the second order excitations are taken into account, especially for the small \(l\) limit we have
\[
T_e S^{(2)}_{\gamma A} \approx \frac{l \gamma^2}{12 \pi L^2} \left( \frac{1}{3} + \frac{\ln \frac{L}{\beta}}{4} \right) > E^{(2)},
\]
which is different from previous examples. Eq. (93) means that the entropy of the subsystem \(A\) is not bounded. This specific behaviour is caused by the logarithmic divergence both in the metric and the \(U(1)\) gauge field in the charged black hole in AdS\(_3\). Nevertheless, based on the same reason discussed in the previous sections, the first law-like relation for the boundary CFT should be satisfied at the second order, thus
\[
T'_{e} \left( S^{(1)}_{\gamma A} + S^{(2)}_{\gamma A} \right) = E^{(1)} + E^{(2)},
\]
in which the corrected entanglement temperature is
\[
T'_{e} = \frac{3}{\pi l} \left( 1 + \frac{l^2 \beta^2}{120 L^2} \left( \frac{2}{3} + \frac{\ln \frac{L}{\beta}}{2} \right) \gamma^2 + \frac{l^4 \beta^2}{14400 L^4} - \frac{4 + 3 \ln \frac{L}{\beta}}{360 L^2} + \frac{4 + 3 \ln \frac{L}{\beta}}{36} \gamma^4 \right).
\]
VI. CONCLUSIONS AND DISCUSSIONS

We studied the HEE of the boundary CFT with low-energy excited states up to second order of the gravitational perturbations (or geometric perturbations) when the spatial region of the boundary subsystem is a strip and analyzed the first law-like relation at the second order. Our strategy is to start from an exact bulk black brane solution in asymptotically AdS spacetime, and then regard the black brane as the perturbed geometry deviates from its ground state—the pure AdS spacetime through small fluctuations, caused by interactions from external fields or operators such as the adding of the mass and gauge fields. From the viewpoint of the dual boundary CFT side, this is equivalent to treat a thermodynamically stable finite temperature CFT (or grand canonical ensemble) with thermal and quantum excitations as the perturbed system deviates from its vacuum state CFT. Following this idea, we solved the bulk co-dimensional-2 static minimal surface up to second order in terms of the conserved charges of the AdS black brane, or more specifically to say, we obtained second order quantum excitation corrections to the HEE of vacuum state from the expansion of HEE of the excited states. Based on the facts that the bulk perturbed configurations (which are expansions from an exact stationary AdS black brane solution) should be stable and stationary order by order, namely, the bulk Einstein equation plus the EoMs for gauge fields are satisfied at all orders of the small perturbation, and the correspondence between the linearized Einstein equation and the first law-like relation of the boundary CFT at first order perturbation, we pointed out that the first law-like relation should also be held at the second order, which corresponds to the bulk perturbed Einstein equation at the same order. We want to say that since all of the perturbations (metric and world-sheet) are known from the starting point, so there is actually “no” dynamics for those perturbations in our calculations (for “no” dynamics we mean that we don’t need to solve the EoMs to obtain the solutions in the present cases). To study the second order dynamics of HEE, we need to consider the fluctuations from the bulk black brane, and then solve the EoMs for the bulk minimal surface and the perturbed Einstein equation at second order. In addition, the second order energy of the subsystem $A$ calculated from the boundary stress tensor gives a trivial contribution when the boundary goes to the spatial infinity. While from the field/operator duality and the structure of the area functional eq. (11), one can see that it includes the contributions from the 2-point correlation function of some quasi-local stress tensor, e.g., $\langle T^{(1)}T^{(1)} \rangle$ on the minimal surface. So it would be interesting to take the bulk minimal surface as an additional boundary for the bulk spacetime and calculate the quasi-local stress tensor on it, we will study this issue in another paper. Moreover, the expansion in terms of the conserved
charges is actually related to the Fefferman-Graham expansion, using this property, our method can also be used to study the dynamics of HEE with excited states in the asymptotically AdS spacetime.

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