Some Computational and Theoretical Aspects of the Exponentiated Generalized Nadarajah-Haghighi Distribution

VedoVatto, T.\(^1,2,\ast\), Nascimento, A. D. C.\(^2\), Miranda Filho, W. R.\(^2\), Lima, M. C. S.\(^2\), Pinho, L. G. B.\(^3\) and Cordeiro, G. M.\(^2\)

\(^1\)Instituto Federal de Educação, Ciência e Tecnologia de Goiás, Goiânia, Brazil
\(^2\)Departamento de Estatística, Universidade Federal de Pernambuco, Recife, Brazil
\(^3\)Departamento de Estatística e Matemática Aplicada, Universidade Federal do Ceará, Fortaleza, Brazil

(Received: 00 Month 200x · Accepted in final form: 00 Month 200x)

Abstract

Real data from applications in the survival context have required the use of more flexible models. A new four-parameter model called the Exponentiated Generalized Nadarajah-Haghighi (EGNH) distribution has been introduced in order to verify this requirement. We prove that its hazard rate function can be constant, decreasing, increasing, upside-down bathtub and bathtub-shape. Theoretical essays are provided about the EGNH shapes. It includes as special models the exponential, exponentiated exponential, Nadarajah-Haghighi’s exponential and exponentiated Nadarajah-Haghighi distributions. We present a physical interpretation for the EGNH distribution and obtain some of its mathematical properties including shapes, moments, quantile, generating functions, mean deviations and Rényi entropy. We estimate its parameters by maximum likelihood, on which one of the estimates may be written in closed-form expression. This last result is assessed by means of a Monte Carlo simulation study. The usefulness of the introduced model is illustrated by means of two real data sets. We hope that the new distribution could be an alternative to other distributions available for modeling positive real data in many areas.

Keywords: Exponentiated Exponential · Exponentiated generalized family · Nadarajah-Haghighi’s exponential · T-X family

Mathematics Subject Classification: Primary 62E10 · Secondary 62P30.

1. Introduction

In recent years, several ways of generating new continuous distributions have been proposed in survival analysis to provide flexibility and new forms for the hazard rate function (hrf). A detailed study about ‘the evolution of the methods for generalizing classic distributions’ was made by \cite{Leeetal2013}. Most of them are special cases of the \(T-X\) class defined by \cite{Alzaatrehetal2013}. This class of distributions extends some recent families such as the beta-G pioneered by \cite{Eugeneetal2002}, the gamma-G defined
The process to construct a generator based on the $T-X$ class is given as follows. Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a, b]$ for $-\infty < a < b < \infty$ and let $W[G(x)]$ be a mapping under the following conditions:

- $W[G(x)] \in [a, b]$;
- $W[G(x)]$ is monotonically non-decreasing and differentiable;
- $W[G(x)] \to a$ as $x \to -\infty$ and $W[G(x)] \to b$ as $x \to \infty$.

The new $T-X$ generator has cumulative distribution function (cdf) and pdf given, respectively, by

$$F(x) = \int_a^{W[G(x)]} r(t) dt \quad \text{and} \quad f(x) = \left\{ \frac{d}{dx} W[G(x)] \right\} r \{ W[G(x)] \} .$$

The choice of $W(\cdot)$ is intrinsically associated with the cdf of $T$. For example, if $T$ follows a beta distribution, $W(\cdot)$ can be the identity function, leading to the beta-G family. On the other hand, if $T$ follows a gamma distribution, $W[G(x)]$ must be a function such that $W:[0, 1] \to [0, \infty)$. Examples of functions that satisfy this restriction are: $W[G(x)] = -\log[1 - G(x)]$, $W[G(x)] = -\log[G(x)]$ and $W[G(x)] = G(x)/[1 - G(x)]$.

Cordeiro et al. (2013) proposed an interesting family in the $T-X$ class inspired by the second type Kumaraswamy distribution. We refer to this family as the exponentiated generalized (EG) family, which has cdf and pdf (for $x > 0$) given, respectively, by

$$F(x) = \{1 - [1 - G(x)]^\alpha\}^\beta \tag{1}$$

and

$$f(x) = \frac{\alpha \beta g(x)[1 - G(x)]^{\alpha - 1}}{\{1 - [1 - G(x)]^\alpha\}^{1-\beta}}, \tag{2}$$

where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters that can control both tail weights. One can note that equations (1) and (2) do not involve any complicated function, which is an advantage when compared to the beta and gamma families. Further, the EG family is nothing more than a sequential application of the Lehmann type 2 alternative to the exponentiated class. Setting $\alpha = 1$ gives the exponentiated-G (Lehmann type 1) class defined by Gupta et al. (1998).

The best benefit of the EG family is its ability to fitting skewed data. This class is also dual of the Kw-G family and has similar properties of the beta-G family and some advantages in terms of tractability. Some special models in this class have been studied recently. The EG-inverse Gaussian by Lemonte and Cordeiro (2011), the EG-generalized gamma by Silva et al. (2015), the EG-inverse Weibull by Elbatgal and Muhammed (2014) and the EG-Gumbel by Andrade et al. (2015) distributions are special cases obtained by taking $G(x)$ to be the cdf of the inverse Gaussian, generalized gamma, inverse Weibull and Gumbel distributions, respectively. Hence, each new EG distribution can be generated from a specified $G$ distribution.

A new generalization of the exponential law as an alternative to the gamma and Weibull models has been proposed by Nadarajah and Haghighi (2011), which model has cdf given by (for $x > 0$)

$$G(x) = 1 - e^{-(1+ax)^k}, \tag{3}$$
Table 1. Nadarajah-Haghighi extensions

| Distribution | Author(s) |
|--------------|-----------|
| GPW          | Nikulin and Haghighi (2006, 2009) |
| ENH          | Lemonte (2013) and Abdul Moniem (2015) |
| GNH          | Ortega et al. (2015) and Bourguignon et al. (2015) |
| PGNH         | Ortega et al. (2015) |
| TNH          | Ahmed et al. (2015) |
| KNH          | Lima (2015) |
| MNH          | El Dancese and Ramadan (2015) |
| MONH         | Lemonte et al. (2016) |
| KGPW         | Selim and Badr (2016) |
| BNH          | Dias (2016) |

where $a > 0$ and $b > 0$. Its pdf reduces to

$$g(x) = ab(1 + ax)^{b-1}e^{1-(1+ax)^b}. \quad (4)$$

This model is called the Nadarajah-Haghighi’s exponential (denoted by $Z \sim \text{NH}(a, b)$) because the exponential is a special case when $b = 1$. This bi-parametric model has been used for modeling lifetime data. The pdf (4) is always monotonically decreasing with $g(0) = ab$. Nadarajah and Haghighi (2011) pointed out that this distribution has an attractive feature of always having the zero mode. They also showed that larger values of $b$ lead to faster decay of the upper tail. Additionally, the hrf can be monotonically increasing for $b > 1$, and monotonically decreasing for $b < 1$. For $b = 1$, the hrf becomes constant. So, the major weakness of the NH distribution is its inability to accommodate non-monotone hrfs (i.e. bathtub-shaped and upside-down bathtub).

Several extensions of the NH model have been proposed in recent years such as the following distributions: the generalized power Weibull (GPW), Kumaraswamy generalized power Weibull (KGPW), exponentiated (Lehmann type 1) Nadarajah-Haghighi (ENH), gamma (Zografos-Balakrishnan type) Nadarajah-Haghighi (GNH), Poisson gamma (also of Zografos-Balakrishnan type) Nadarajah-Haghighi (PGNH), transmuted (quadratic random transmuted map) Nadarajah-Haghighi (THN), Kumaraswamy Nadarajah-Haghighi (KNH), exponentiated (Lehmann type 2) Nadarajah-Haghighi (E2NH) (submodel of the KNH), modified Nadarajah-Haghighi (MNH), Marshall-Olkin Nadarajah-Haghighi (MONH) and beta Nadarajah-Haghighi (BNH). These distributions and their corresponding authors are listed in Table 1.

In this paper, we obtain a natural extension of the NH distribution, which we refer to as the exponentiated generalized Nadarajah-Haghighi (EGNH) distribution. The first motivation for introducing the new distribution is based on the fact that it has simple expressions for the pdf and hrf and, as a consequence, a simpler statistical inference process. Second, the flexibility of the new distribution to describe complex positive real data is concluded since this hrf can present constant, decreasing, increasing, unimodal, bathtub-shaped and upside-down bathtub forms. Due to the great flexibility of its hrf, it can provide a good alternative to many existing lifetime distributions to model real data. Therefore, the beauty and importance of the new distribution lies in its ability to model monotone as well as nonmonotone hrfs, which are quite common in reliability and biological studies.

So, our main aim is to propose a new four-parameter distribution, which extends the exponential, NH and ENH distributions, with the hope that it may provide a ‘better fit’ compared to other lifetime distributions in certain practical situations. Additionally, we provide a comprehensive account of the mathematical properties of the introduced distribution. The formulae related with the new distribution are simple and manageable, and,
with the use of modern computer resources and their numerical capabilities, the proposed
distribution may prove to be an useful addition to the arsenal of applied statisticians in
areas like biology, medicine, economics, reliability, engineering, among others.

The paper is outlined as follows. In Section 2 we introduce the new distribution, some of
its basic properties and provide plots of the pdf and hrf. Section 3 deals with nonstandard
measures of skewness and kurtosis. Linear representations for the pdf and cdf are presented
in Section 4 and explicit expressions for the moments are provided in Section 5. Section
6 is devoted to the moment generating function (mgf). The mean deviations are obtained
in Section 7. The Rényi entropy is derived in Section 8. In Section 9 we present the
order statistics. A procedure to calculate the maximum likelihood estimators (MLEs) of
the model parameters is investigated in Section 10. A simulation study illustrating the
convergence and consistency of the MLEs is addressed in Section 11. Two applications to
real data are provided in Section 12. Finally, Section 13 offers some concluding remarks.

2. The proposed model

The EGNH cdf is obtained by applying (3) in Equation (1)

\[ F(x) = \left\{ 1 - \left[ e^{1-(1+ax)^b} \right] ^\alpha \right\} ^\beta , \quad x > 0, \]

where \( \alpha, \beta, a, b > 0 \) and \( b \neq 1 \). The EGNH pdf can be obtained by applying (3) and (4) in
(2) or by differentiating (5):

\[ f(x) = \frac{a\alpha b\beta(1 + ax)^{b-1} \left[ e^{1-(1+ax)^b} \right] ^\alpha}{\left\{ 1 - \left[ e^{1-(1+ax)^b} \right] ^\alpha \right\} ^{1-\beta}}. \]

Henceforth, we denote by \( X \sim \text{EGNH}(\alpha, \beta, a, b) \) a random variable having pdf (6). So,
the EGNH distribution is obtained by adding two shape parameters \( \alpha \) and \( \beta \) to the NH
distribution. It is necessary to consider \( b \neq 1 \), an additional restriction to the parametric
space, to give identifiability to the generated model.

The EGNH distribution has also an attractive physical interpretation whenever \( \alpha \) and \( \beta \)
are positive integers. Consider a device based on \( \beta \) independent components in a parallel
system. Further, each component is defined in terms of \( \alpha \) independent subcomponents
identically distributed following the NH model in a series system. Assume that the device
fails if all \( \beta \) components fail and each component fails if any subcomponent fails. Let
\( X_{j1}, \ldots, X_{j\alpha} \) denote the lifetimes of the subcomponents within the \( j \)th component, \( j = 1, \ldots, \beta \), with common cdf given by (3). Let \( X_j \) denote the lifetime of the \( j \)th component
and let \( X \) denote the lifetime of the device. Thus, the cdf of \( X \) is given by

\[ P(X \leq x) = P(X_1 \leq x, \ldots, X_\beta \leq x) = P(X_1 \leq x)^\beta = [1 - P(X_1 > x)]^\beta \\
= [1 - P(X_{11} > x, \ldots, X_{1\alpha} > x)]^\beta = [1 - P(X_{11} > x)^\alpha]^\beta \\
= \{1 - [1 - P(X_{11} \leq x)]^\alpha\}^\beta. \]

So, the lifetime of the device obeys the EGNH distribution.
The quantile function (qf) of $X$ is determined by inverting (5) (for $0 < p < 1$) as

$$Q(p) = \frac{1}{a} \left\{ \left[ 1 - \log \left( \left( 1 - p^{\frac{1}{a}} \right)^{\frac{1}{b}} \right) \right]^{\frac{1}{a}} - 1 \right\}. \quad (7)$$

Clearly, the median of the new distribution follows by setting $p = 1/2$ in (7). We can also use (7) for simulating EGNH random variables by the inverse transform sampling method, which works as follows:

(1) Let $u$ be an outcome of uniform distribution in the interval $[0, 1]$.
(2) $x = Q(u)$ is then an outcome of the EGNH distribution.

Finally, the survival function (sf) and hrf of $X$ are given, respectively, by

$$S(x) = 1 - \left\{ 1 - \left[ e^{1-(1+ax)^{\alpha}} \right]^{\beta} \right\} \quad \text{and}$$

$$h(x) = \frac{aabo(1+ax)^{b-1}\left[e^{1-(1+ax)^{\alpha}}\right]^\alpha}{1 - \left\{ 1 - \left[ e^{1-(1+ax)^{\alpha}} \right]^{\alpha} \right\}^\beta \left\{ 1 - \left[ e^{1-(1+ax)^{\alpha}} \right]^{\alpha} \right\}^{1-\beta}}. \quad (8)$$

The reverse hrf of $X$ is given by

$$r(x) = \frac{aabo(1+ax)^{b-1}\left[e^{1-(1+ax)^{\alpha}}\right]^\alpha}{\left\{ 1 - \left[ e^{1-(1+ax)^{\alpha}} \right]^{\alpha} \right\}^\beta} = \frac{\beta}{[F_{E2NH}(x)]^{\beta} f_{E2NH}(x)}$$

where $F_{E2NH}(x)$ and $f_{E2NH}(x)$ are, respectively, the cdf and pdf of E2NH (Lehmann type 2) distribution, which is a special case of the model proposed by Lima (2015) and Selim and Badr (2016). In the following, we address limit distributions, shape and special cases of the EGNH model.

### 2.1 Limiting Behavior of the Density

The pdf (6) can take various forms depending on the values of the $\alpha$ and $\beta$ shape parameters. It is easy to verify that

$$\lim_{x \to 0} f(x) = \begin{cases} \infty, & \beta < 1 \\ a\alpha, & \beta = 1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 0. \\ 0, & \beta > 1 \end{cases}$$

### 2.2 Shapes

In Figures 1 and 2, we plot some possible shapes of the pdf (6) and hrf (8), respectively, for selected parameter values. The pdf and hrf of $X$ can take various forms depending on the parameter values. It is evident that the EGNH distribution is much more flexible than the NH distribution. So, the new model can be very useful in many practical situations for modeling positive real data.
It is easy to prove that

\[
\frac{d \log f(x)}{dx} = \frac{a \alpha b (\beta - 1)(1 + ax)^{b-1} \left(e^{1-(1+ax)\beta}\right)^\alpha}{1 - \left(e^{1-(1+ax)\beta}\right)^\alpha} - a \alpha b (1 + ax)^{b-1} + \frac{a(b - 1)}{1 + ax}.
\]

The mode of the pdf (6) is the solution of the equation

\[
\frac{d \log f(x)}{dx} = 0.
\]

Proposition 2.1 For any \(\alpha > 0\) and \(a > 0\), the EGNH pdf is log-convex if \(\beta < 1\) and \(b < 1\), and it is log-concave if \(\beta > 1\) and \(b > 1\).

**Proof** The proof is analogous to that one given by [Lemonte (2013)](lemonte2013). Let \(z = (1 + ax)^b\). Then, \(x > 0\) implies \(z > 1\). Inverting for \(x\), we obtain \(x = (z^{1/b} - 1)/a\). The EGNH pdf rewritten as a function of \(z\) is given by

\[
\phi(z) = \frac{a \alpha b \beta z^{\frac{b-1}{b}} e^{\alpha (1 - e^{\alpha - a \alpha})^\beta}}{e^{a z} - e^\alpha}, \quad z > 1.
\]  

(9)

The result follows by nothing that the second derivative of \(\log[\phi(z)]\) is

\[
\frac{d^2 \log[\phi(z)]}{dz^2} = - \left( \frac{b - 1}{b z^2} + \frac{1}{4} \alpha^2 (\beta - 1) \text{csch}^2 \left( \frac{1}{2}(\alpha - a z) \right) \right).
\]

Proposition 2.2 For any \(\alpha > 0\) and \(a > 0\), the EGNH model has an increasing hrf if \(\beta > 1\) and \(b > 1\) and it has a decreasing hrf if \(\beta < 1\) and \(b < 1\). It is constant if \(\beta = b = 1\).

**Proof** First, note that \(\beta = b = 1\) implies \(h(x) = a \alpha\) (constant hrf).

Next, let \(z = (1 + ax)^b\). Notice that \(z > 1\) (for \(x > 0\)) and \(x = (z^{1/b} - 1)/a\). Rewriting the
EGNH hrf as a function of $z$, we have

$$\eta(z) = \frac{a\alpha b \beta z^{b-1}}{[1-e^{\alpha(1-z)}]^{\beta}} \left\{ [1-e^{\alpha(1-z)}]^{\beta} - 1 \right\}, \quad z > 1. \quad (10)$$

Next, taking the first derivative of the logarithm of both sides of equation (10), we obtain

$$\frac{d\log[\eta(z)]}{dz} = \frac{\eta'(z)}{\eta(z)} = \frac{b-1}{bz} + \alpha \left\{ \frac{\beta + [1-e^{\alpha(1-z)}]^{\beta} - 1}{[1-e^{\alpha(1-z)}] \left\{ [1-e^{\alpha(1-z)}]^{\beta} - 1 \right\}} - 1 \right\},$$
Table 2. Some special distributions

| Model   | α  | β  | a  | b  |
|---------|----|----|----|----|
| ENH     | 1  | -  | -  | -  |
| E2NH    | -  | 1  | -  | -  |
| NH      | 1  | 1  | -  | -  |
| EE      | 1  | -  | -  | 1  |
| EE      | -  | -  | 1  | 1  |
| exponential | 1  | 1  | -  | 1  |
| exponential | -  | 1  | 1  | 1  |

which implies that
\[
\eta'(z) = \eta(z) \left\{ \frac{b - 1}{bz} + \alpha \left\{ \frac{\beta + [1 - e^{\alpha(1-z)}]^\beta - 1}{[1 - e^{\alpha(z-1)}]\left\{[1 - e^{\alpha(1-z)}]^\beta - 1\right\}} - 1 \right\} \right\},
\]

Thus, \( \eta'(z) \) has the same sign of
\[
\xi(z) = \frac{b - 1}{bz} + \alpha \left\{ \frac{\beta + [1 - e^{\alpha(1-z)}]^\beta - 1}{[1 - e^{\alpha(z-1)}]\left\{[1 - e^{\alpha(1-z)}]^\beta - 1\right\}} - 1 \right\}, \quad z > 1,
\]

since \( \eta(z) > 0 \) for \( z > 1 \). Note that \( \xi(z) < 0 \) if \( b < 1 \) and \( \beta < 1 \) and \( \xi(z) > 0 \) if \( b > 1 \) and \( \beta > 1 \).

The last proposition is important because it anticipates behaviors of device lifetime on practical situations.

2.3 Special distributions

In Table 2, one can observe some EGNH special models. As expected, the exponential distribution is obtained as a special case, but surprisingly by two distinct ways. The exponentiated exponential (EE) is also obtained by two different settings. Any baseline distribution will also be a sub-model when all the additional parameters of the EG family are equal to one, and then the NH distribution is listed. Finally, the ENH (Lehmann type 1) and E2NH (Lehmann type 2) models are examples of distributions discussed early, which are really special cases of the EGNH model.

3. Quantile measures

The effects of the parameters \( \alpha \) and \( \beta \) on the skewness and kurtosis of \( X \) can be considered based on the qf given by (7). The Bowley skewness (B), proposed by Kenney and Keeping (1962), and the Moors kurtosis (M), proposed by Moors (1988), are defined, respectively, by

\[
B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)} \quad \text{and} \quad M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(3/4) - Q(1/4)}.
\]

The \( B \) and \( M \) measures are less sensitive to outliers and they exist even for distributions without moments.
In Figures 3 and 4, we plot the measures $B$ and $M$ for some parameter values of the EGNH distribution as functions of $\alpha$ and $\beta$ (for $a$ and $b$ fixed). These plots indicate that both measures can be very sensitive on these shape parameters and reveal the importance of the proposed distribution.

Figure 3. The Bowley skewness of the EGNH distribution as function of $\alpha$ and $\beta$.

Figure 4. The Moors kurtosis of the EGNH distribution as function of $\alpha$ and $\beta$.

4. Linear Representations

Some useful linear representations for (5) and (6) can be derived using the concept of exponentiated distributions. The properties of these distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) and Mudholkar et al. (1995) for exponentiated Weibull, Gupta et al. (1998) and Gupta and Kundu (2001) for exponentiated exponential, Nadarajah and Gupta (2007) for exponentiated gamma and
Cordeiro et al. (2011) for exponentiated generalized gamma distributions. For an arbitrary baseline cdf \( G(x) \), a random variable is said to have the exponentiated-G ("exp-G" for short) distribution with power parameter \( a \), say \( Y \sim \text{exp-G}(a) \), if its cdf and pdf are 
\[ H_a(x) = G(x)^a \quad \text{and} \quad h_a(x) = ag(x)G(x)^{a-1}, \]
respectively. For any real non-integer \( \beta \), we also consider the power series
\[ (1 - z)^\beta = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} z^i, \quad (11) \]
which holds for \( |z| < 1 \).

Using expansion (11) in equation (5), we can express the EGNH cdf as
\[ F(x) = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} \left\{ 1 - \left[ 1 - e^{1-(1+ax)^b} \right]^i \right\}^{i\alpha} \]
and applying again (11) in the equation above gives
\[ F(x) = \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{\beta}{i} \binom{i\alpha}{j} \left[ 1 - e^{1-(1+ax)^b} \right]^j. \]

Then, the following proposition holds.

Proposition 4.1 The EGNH cdf can be rewritten as
\[ F(x) = \sum_{j=0}^{\infty} w_j H_j(x), \quad (12) \]
where the coefficients (for \( j \geq 0 \)) are given by
\[ w_j = w_j(\alpha, \beta) = \sum_{i=0}^{\infty} (-1)^{i+j} \binom{\beta}{i} \binom{i\alpha}{j} \]
and \( H_j(x) = \{ 1 - \exp[1 - (1 + ax)^b] \}^j \) is the ENH\((j, a, b)\) cdf (for \( j \geq 0 \)).

We can prove using Mathematica that \( \sum_{j=0}^{\infty} w_j = 1 \) as expected.

By differentiating (12), we obtain
\[ f(x) = \alpha \beta g(x) \sum_{j=0}^{\infty} t_j G(x)^j \quad (13) \]
where the coefficients (for \( j \geq 0 \)) are given by
\[ t_j = t_j(\alpha, \beta) = \sum_{i=0}^{\infty} (-1)^{i+j} \binom{\beta - 1}{i} \binom{\alpha(i + 1) - 1}{j}. \quad (14) \]
Further, the corollary bellow follows:

Proposition 4.2 Equation (13) can be rewritten as

\[ f(x) = \sum_{j=0}^{\infty} u_j h_{j+1}(x), \]  

where the coefficients are

\[ u_j = u_j(\alpha, \beta) = \frac{\alpha \beta}{j+1} t_j \]

and

\[ h_{j+1}(x) = \frac{ab(j+1)(1+ax)^b \exp[1 - (1+ax)^b]}{\{1 - \exp[1 - (1+ax)^b]\}^{-j}} \]

is the ENH\((j+1, a, b)\) pdf.

Equation (15) reveals that the EGNH pdf is an infinite linear combination of ENH pdfs. Thus, some structural properties of the EGNH distribution, such as the ordinary and incomplete moments and generating function, can be determined from those of the ENH model, which have been studied by Lemonte (2013) and Abdul Moniem (2015).

5. Moments

In this section, we derive the moments of the EGNH distribution as well as other related measures. Let \( Y \sim \text{NH}(a, b) \) and \( X \sim \text{EGNH}(\alpha, \beta, a, b) \). Cordeiro et al. (2013) proved that the moments of the EG distribution can be expressed as an infinite weighted sum of the \((r, j)\)th probability weighted moments \( \tau_{r,j} \) of the baseline random variable. Following this result, we can write from (13)

\[ \mu'_r = E(X^r) = \alpha \beta \sum_{j=0}^{\infty} t_j \tau_{r,j}, \]

where \( t_j \) is given by (14) and

\[ \tau_{r,j} = E[Y^r G(Y)^j] = \int_{-\infty}^{\infty} x^r G(x)^j g(x) dx. \]

By simple algebraic manipulations,

\[ \tau_{r,j} = ab \int_0^{\infty} x^r (1+ax)^{b-1} \exp\left[1 - (1+ax)^b\right] \frac{1}{\{1 - \exp[1 - (1+ax)^b]\}^{-j}} dx. \]  

(17)

It is clear that \( \left| \exp\left[1 - (1+ax)^b\right]\right| < 1 \), and then applying (11) in (17) gives

\[ \tau_{r,j} = ab \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} \int_0^{\infty} x^r (1+ax)^{b-1} \left\{ \exp\left[1 - (1+ax)^b\right]\right\}^{k+1} dx. \]
Then, we can write
\[
\tau_{r,j} = a b \sum_{k=0}^{\infty} (-1)^k \left(\frac{j}{k}\right) e^{k+1} \int_{0}^{\infty} x^r (1 + ax)^{b-1} \exp \left[-(k + 1)(1 + ax)^b\right] \, dx.
\]

By setting \( y = (k + 1)(1 + ax)^b \), \( \tau_{r,j} \) reduces to
\[
\tau_{r,j} = \frac{1}{a^r} \sum_{k=0}^{\infty} (-1)^{k+r} \left(\frac{j}{k}\right) \frac{e^{k+1}}{k+1} \int_{k+1}^{\infty} \left[ 1 - \left(\frac{y}{k+1}\right)^{\frac{1}{b}} \right]^{r} e^{-y} \, dy.
\]

Taking \( r \) integer and applying the binomial expansion, the last expression becomes
\[
\tau_{r,j} = \frac{1}{a^r} \sum_{k=0}^{\infty} \sum_{l=0}^{r} (-1)^{k+r+l} \left(\frac{j}{k}\right) \left(\frac{r}{l}\right) \frac{e^{k+1}}{(k+1)^{l+b+1}} \Gamma \left(\frac{l}{b} + 1, k + 1\right),
\]
where \( \Gamma(a, x) = \int_{x}^{\infty} y^{a-1} e^{-y} \, dy \) denotes the complementary incomplete gamma function.

Then, the \( r \)th ordinary moment of \( X \) is given by
\[
\mu'_r = \alpha \beta \frac{\alpha^r}{a^r} \sum_{j=0}^{\infty} \sum_{k=0}^{r} (-1)^{k+r+l} t_j \left(\frac{j}{k}\right) \left(\frac{r}{l}\right) \frac{e^{k+1}}{(k+1)^{l+b+1}} \Gamma \left(\frac{l}{b} + 1, k + 1\right).
\]

For \( \alpha > 0 \) and \( \beta > 0 \) integers, the moments in (18) reduce to
\[
\mu'_r = \alpha \beta \frac{\alpha^r}{a^r} \sum_{j=0}^{\infty} \sum_{k=0}^{r} (-1)^{k+r+l} t_j \left(\frac{j}{k}\right) \left(\frac{r}{l}\right) \frac{e^{k+1}}{(k+1)^{l+b+1}} \Gamma \left(\frac{l}{b} + 1, k + 1\right),
\]
where \( t_j \) becomes
\[
t_j = \sum_{i=0}^{\beta-1} (-1)^{i+j} \binom{\beta-1}{i} \binom{\alpha(i+1)-1}{j}.
\]

Additionally, the expression for \( \mu'_r \) when \( \alpha = \beta = 1 \) takes the form
\[
\mu'_r = \frac{e}{a^r} \sum_{l=0}^{r} (-1)^{l+r} \binom{r}{l} \Gamma \left(\frac{l}{b} + 1, 1\right),
\]
which agrees with the moments given by Nadarajah and Haghighi (2011).

The central moments \( (\mu_r) \) of \( X \) are given by
\[
\mu_r = \sum_{m=0}^{r} (-1)^m \binom{r}{m} \mu'_m \mu^{r-m}_r.
\]

The cumulants \( (\kappa_r) \), skewness \( (\gamma_1) \) and kurtosis \( (\gamma_2) \) of \( X \) are determined from (18) using well-known relationships \( \kappa_r = \mu'_r - \sum_{m=0}^{r-1} \binom{r-1}{m-1} \kappa_m \mu'_m \), for \( r \geq 1 \), \( \gamma_1 = \kappa_3 / \kappa_2^{3/2} \) and \( \gamma_2 = \kappa_4 / \kappa_2^2 \), where \( \kappa_1 = \mu'_1 \).
6. Generating function

The mgf of $X$ can be obtained from (15) as

$$M(s) = \sum_{j=0}^{\infty} u_j M_{j+1}(s),$$

where $M_{j+1}(s)$ is the mgf of the ENH$(j+1, a, b)$ distribution (Bourguignon et al., 2015) given by

$$M_{j+1}(t) = \sum_{s,r=0}^{\infty} \eta_s g_{s,r} t^{j+1},$$

where $\eta_i = \sum_{k=0}^{\infty} (\frac{-1}{k!})^i, g_{i,r} = (r\zeta_0)^{-1} \sum_{n=1}^{r} [n(i+1) - r] \zeta_n g_{i,r-n}$ for $r \geq 1$, $g_{i,0} = \zeta_0$, $\zeta_r = \sum_{m=0}^{\infty} f_m d_{m,r}, f_m = \sum_{j=m}^{\infty} (-1)^{j-m} (\frac{j}{m}) (\alpha^{-1})_j/j!$ and $d_{m,r} = (ra_0)^{-1} \sum_{v=0}^{r} \nu(m+1) - v] a_v d_{m,r-v}$. Here, $(\alpha^{-1})_j = (\alpha^{-1}) \times (\alpha^{-1} - 1) \ldots (\alpha^{-1} - j + 1)$ is the descending factorial.

7. Mean deviations

The mean deviations about the mean and the median of $X$ can be expressed, respectively, as

$$\delta_1(X) = \frac{E(|X - \mu_1|)}{\mu_1}, \mu_1 = \frac{1}{f(\mu_1)}$$

and

$$\delta_2(X) = \frac{E(|X - \tilde{X}|)}{\mu_1}, \mu_1 = \frac{1}{f(\mu_1)}$$

where $\mu_1$ comes from (18), $F(\mu_1)$ is easily calculated from (5) and $m_1(z) = \int_{-\infty}^{z} x f(x) \, dx$ is the first incomplete moment.

A general equation for $m_1(z)$ can be derived from (13) as

$$m_1(z) = \sum_{j=0}^{\infty} u_j J_{j+1}(z),$$

where $J_{j+1}(z) = \int_{-\infty}^{z} h_{j+1}(x) \, dx$ and $h_{j+1}(x)$ is given by (10). After some algebra, we can prove that

$$J_{j+1}(z) = \frac{(j+1)}{a} \sum_{k=0}^{\infty} \sum_{l=0}^{1} \frac{(-1)^{k+l+1} e^{k+1}}{(k+1)^{1+\frac{1}{b}}} \left(\frac{j}{k}\right) \phi(a, b, k, l; z),$$

where

$$\phi(a, b, k, l; z) = \gamma \left(\frac{b+l+1}{b}, (k+1)(1+az)^b\right) - \gamma \left(\frac{b+l+1}{b}, k+1\right).$$

The Bonferroni and Lorenz curves are defined (for a given probability $\pi$) by $B(\pi) = m_1(q)/\pi\mu_1$ and $L(\pi) = m_1(q)/\mu_1$, respectively, where $q = Q(\pi)$ comes from (7). Using (21), we can easily obtain $B(\pi)$ and $L(\pi)$. 
8. Rényi Entropy

The entropy of a random variable $X$ with pdf $f(x)$ is a measure of variation of the uncertainty. For any real parameter $\lambda > 0$ and $\lambda \neq 1$, the Rényi entropy is given by

$$I_R(\lambda) = \frac{1}{1 - \lambda} \log \left( \int_{-\infty}^{\infty} f(x)^\lambda \, dx \right).$$

(22)

Inserting (3) in equation (22) and, after some algebra, we obtain

$$I_R(\lambda) = \frac{1}{1 - \lambda} \left\{ \lambda \log(ab\alpha\beta) + \alpha\lambda + \log \int_{0}^{\infty} \frac{(1 + ax)^{\lambda(b-1)} \exp \left[ -\alpha\lambda (1 + ax)^b \right]}{1 - \exp \left[ 1 - (1 + ax)^b \right]^\alpha} \left\{ \int_{-\infty}^{\infty} f(x) \right\}^{\lambda(1-\beta)} \, dx \right\}.$$ (23)

Applying the binomial expansion gives

$$I_R(\lambda) = -\log(ab) + \frac{1}{1 - \lambda} \left\{ \lambda \log(\alpha\beta) + \alpha\lambda + \log \Sigma(\alpha, \beta, b, \lambda) \right\},$$

where

$$\Sigma(\alpha, \beta, b, \lambda) = \sum_{i=0}^{\infty} \binom{\lambda(\beta-1)}{i} (-1)^i e^{i\alpha} \frac{\lambda(\beta-1) + 1}{\lambda(i + \alpha) b} \Gamma \left( \frac{\lambda(b-1) + 1}{b}, (\lambda + i)\alpha \right).$$

(23)

For $\beta > 0$ integer, Equation (23) takes the form

$$\Sigma(\alpha, \beta, b, \lambda) = \sum_{i=0}^{\beta-1} \binom{\lambda(\beta-1)}{i} (-1)^i e^{i\alpha} \frac{\lambda(\beta-1) + 1}{\lambda(i + \alpha) b} \Gamma \left( \frac{\lambda(b-1) + 1}{b}, (\lambda + i)\alpha \right).$$

For $\alpha = \beta = 1$, the expression above reduces to

$$I_R(\lambda) = -\log(ab) + \frac{1}{1 - \lambda} \left\{ \lambda - \frac{\lambda(b-1) + 1}{b} \log(\lambda) + \log \left[ \Gamma \left( \frac{\lambda(b-1) + 1}{b}, \lambda \right) \right] \right\},$$

which agrees with a result given by Nadarajah and Haghighi (2011).

Figure 5 displays plots of some Rényi entropy curves versus $\lambda$ for the EGNH, ENH and NH distributions with some parameter values. In this case, one can note that the NH entropy is fixed thus, we can measure the impact of the change of $\alpha$ and $\beta$ additional parameters on the EGNH entropy. This quantity increases when $\alpha$ and $\beta$ increases with respect the comparison between ENH and EGNH entropies, one can observe that the former is a lower bond of the second and the distance between them is higher when $\alpha$ increases.

9. Order statistics

The pdf $f_{i:n}(x)$ of the $i$th order statistic, say $X_{i:n}$, for $i = 1, \ldots, n$, from independent identically distributed random variables $X_1, \ldots, X_n$ is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} (1 - F(x))^{n-i-1}.$$
Following a result given by Cordeiro et al. (2013), the pdf $f_{i:n}(x)$ for the EGNH distribution can be expressed as

$$f_{i:n}(x) = \frac{\alpha \beta}{B(i, n - i)} \sum_{l=0}^{\infty} s_l h_{l+1}(x),$$  \hspace{1cm} (24)

where

$$s_l = \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} (-1)^{j+k+l} \left( \begin{array}{c} n - i \alpha \beta \\ j k \end{array} \right) \left( \begin{array}{c} \beta(j+1) - 1 \\ k \end{array} \right) \left( \begin{array}{c} \alpha(k+1) - 1 \\ l \end{array} \right)$$

and $h_{l+1}(x)$ is the ENH($l + 1, a, b$) pdf given by (16).

Equation (24) reveals that the pdf of the EGNH order statistic is a linear combination of ENH densities. A direct application of this expansion allows us to determine the ordinary moments and the mgf of the EGNH order statistics.

The $r$th ordinary moment of $X_{i:n}$ is given by

$$E(X_{i:n}^r) = \frac{\alpha \beta}{B(i, n - i + 1)} \sum_{l=0}^{\infty} s_l \int_0^{\infty} x^r h_{l+1}(x) \, dx.$$  \hspace{1cm} (25)

Equation (25) reveals that the moments of the order statistics depend on the moments of the ENH($l + 1, a, b$) distribution. Lemonte (2013) determined the moments of the ENH distribution as

$$\mu'_r = \frac{\beta}{a^r} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \binom{r}{m} \binom{\beta - 1}{p} (-1)^{r-m+p} e_p^{p+1} \frac{m}{b} + 1, p + 1.$$  \hspace{1cm} (26)

From this result, Equation (25) can be reduced to

$$E(X_{i:n}^r) = \frac{\alpha \beta^2}{B(i, n - i + 1)} \sum_{l=0}^{\infty} s_l^* \Gamma \left( \frac{m}{b} + 1, t + 1 \right),$$

where

$$s_l^* = \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} s_l \binom{r}{m} \binom{\beta - 1}{p} (-1)^{r-m+p} e_p^{p+1} \frac{m}{a^r(p+1)^{r+1}}.$$
Further, the mgf of $X_{i:n}$ can follow from (24) as

$$M(t) = \frac{\alpha \beta}{B(i, n-i+1)} \sum_{i=0}^{\infty} s_i \int_{0}^{\infty} e^{tx} h_{i+1}(x) \, dx = \frac{\alpha \beta}{B(i, n-i+1)} \sum_{l=0}^{\infty} s_l M_{l+1}(t),$$

where $M_{l+1}(t)$ is given by (20).

10. Maximum likelihood estimation

In this section, we provide an analytic procedure to obtain the MLEs for the EGNH parameters using the profile log-likelihood function (pllf). The MLEs are employed due to their interesting asymptotic properties. By means of such properties, one can construct confidence intervals for the model parameters.

Let $X$ be the EGNH distribution with vector of parameters $\theta = (\alpha, \beta, a, b)^\top$. The log-likelihood function for $\theta$, given the observed sample $x_1, \ldots, x_n$, due to (6) is given by

$$\ell(\theta) = n[\alpha + \log(aob\beta)] + (b - 1) \sum_{i=1}^{n} \log(ax_i + 1) - \alpha \sum_{i=1}^{n} (ax_i + 1)^b$$

$$+ (\beta - 1) \sum_{i=1}^{n} \log \left\{ 1 - \left[ e^{1-(ax_i+1)^b} \right]^\alpha \right\}. \quad (26)$$

The components of the associated score vector $U = \left( \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b} \right)^\top$ are

$$\frac{\partial \ell(\theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \left\{ 1 - (1 + ax_i)^b \right\} \left\{ 1 - \frac{(\beta - 1)\left\{ \exp \left[ 1 - (1 + ax_i)^b \right] \right\}^\alpha}{1 - \left\{ \exp \left[ 1 - (1 + ax_i)^b \right] \right\}^\alpha} \right\},$$

$$\frac{\partial \ell(\theta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log \left\{ 1 - \left\{ \exp \left[ 1 - (1 + ax_i)^b \right] \right\}^\alpha \right\},$$

$$\frac{\partial \ell(\theta)}{\partial a} = \frac{n}{a} + (b - 1) \sum_{i=1}^{n} \frac{x_i}{1 + ax_i} - \alpha \sum_{i=1}^{n} \frac{bx_i(1 + ax_i)^{b-1}}{1 + ax_i^{b-1}}$$

$$+ \alpha(\beta - 1) \sum_{i=1}^{n} \frac{bx_i(1 + ax_i)^b \exp \left\{ \alpha \left[ 1 - (1 + ax_i)^b \right] \right\}}{1 - \left\{ \exp \left[ 1 - (1 + ax_i)^b \right] \right\}^\alpha},$$

and

$$\frac{\partial \ell(\theta)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log(1 + ax_i) - \alpha \sum_{i=1}^{n} \left\{ \log(1 + ax_i) \right\} \left\{ 1 + ax_i^b \right\}$$

$$+ \alpha(\beta - 1) \sum_{i=1}^{n} \frac{[\log(1 + ax_i)](1 + ax_i)^b \exp \left\{ \alpha \left[ 1 - (1 + ax_i)^b \right] \right\}}{1 - \left\{ \exp \left[ 1 - (1 + ax_i)^b \right] \right\}^\alpha}.$$
The MLEs, \( \hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})^\top \), can be obtained numerically by solving the system of nonlinear equations

\[
\frac{\partial \ell(\theta)}{\partial \alpha} = \frac{\partial \ell(\theta)}{\partial \beta} = \frac{\partial \ell(\theta)}{\partial a} = \frac{\partial \ell(\theta)}{\partial b} = 0.
\]

Note that from (27) it is possible to obtain a semi-closed estimator of \( \beta \). From \( \frac{\partial \ell(\theta)}{\partial \beta} = 0 \), the estimator for \( \beta \) is given by

\[
\hat{\beta}(\hat{\alpha}, \hat{a}, \hat{b}) = -\frac{n}{\sum_{i=1}^{n} \log \left\{ 1 - \left\{ \exp \left[ 1 - (1 + ax_i) b \right] \right\} \right\}^\alpha}.
\] (28)

Based on (28), fixed on \( x_1, \ldots, x_n \), \( \hat{\beta} \rightarrow \hat{0}^+ \) when \( \hat{b} \rightarrow \hat{0}^+ \) and/or \( \hat{\alpha} \rightarrow \hat{0}^+ \). This behavior anticipates that estimates for smaller \( \alpha \) and/or \( b \) may require improved estimation procedures.

Another way to obtain the MLEs is using the pllf. We suppose \( \beta \) fixed and rewrite (26) as \( \ell(\theta) = \ell_\beta(\alpha, a, b) \) (to show that \( \beta \) is fixed but \( \theta_\beta = (\alpha, a, b)^\top \) varies). To estimate \( \theta_\beta \), we maximize \( \ell_\beta(\alpha, a, b) \) with respect to \( \theta_\beta \), i.e.,

\[
\hat{\theta}_\beta = \arg \max_{\theta_\beta} \ell_\beta(\alpha, a, b).
\]

By replacing \( \beta \) by \( \hat{\beta}(\alpha, a, b) \) in (26), we can obtain the pllf for \( \alpha, a \) and \( b \) as

\[
\ell_\beta(\alpha, a, b) = \left\{ \alpha + \log \left\{ \alpha - \frac{nab}{\sum_{i=1}^{n} \log \left[ 1 - \left( e^{1-(ax_i+1)b} \right)^{\alpha} \right] } \right\} \right\}
+ (b - 1) \sum_{i=1}^{n} \log (ax_i + 1) - \alpha \sum_{i=1}^{n} (ax_i + 1)^b
- \left\{ \frac{n}{\sum_{i=1}^{n} \log \left[ 1 - \left( e^{1-(ax_i+1)b} \right)^{\alpha} \right] } + 1 \right\} \sum_{i=1}^{n} \log \left( 1 - \left[ e^{1-(ax_i+1)b} \right]^\alpha \right) + 1 \right\} \sum_{i=1}^{n} \log \left( 1 - \left[ e^{1-(ax_i+1)b} \right]^\alpha \right). \] (29)

As a consequence MLEs for \( \alpha, a \) and \( b \) can also be determined by maximizing the pllf (29) with respect to these parameters. Another way to obtain the MLEs is to solve the profile log-likelihood equations

\[
\frac{\partial \ell_\beta(\alpha, a, b)}{\partial \alpha} = \frac{\partial \ell_\beta(\alpha, a, b)}{\partial a} = \frac{\partial \ell_\beta(\alpha, a, b)}{\partial b} = 0.
\]

It should be noted that the maximization of (29) is simpler than the maximization of (26) with respect to their respective parameters, since the pllf has three parameters, whereas the llf has four parameters.

For interval estimation on the model parameters, we require the \( 4 \times 4 \) observed information matrix, \( J(\theta) \), whose elements.

Under standard regularity conditions, the asymptotic distribution of \( \sqrt{n}(\hat{\theta} - \theta) \) is multivariate normal \( N_4(0, I(\theta)^{-1}) \), where \( I(\theta) \) is the expected information matrix. This approximated distribution holds when \( I(\theta) \) is replaced by \( J(\theta) \). The multivariate normal
\( N_d(0, J(\theta)^{-1}) \) distribution for \( \sqrt{n}(\hat{\theta} - \theta) \) can be used to construct approximate confidence intervals for the individual parameters.

11. Simulation studies

In this section, we assess the performance of the MLEs for \( \alpha, \beta, a \) and \( b \) by means of a Monte Carlo study. To that end, we adopt sample sizes \( n \in \{10, 15, \ldots, 250\} \) and the parameter values \( \theta = (\alpha, \beta, a, b) = (1.8 \times 10^{-3}, 2.83 \times 10^{-1}, 1.75 \times 10^{-3}, 47.066) \) (defined from the first application to real data discussed in Section 12).

Our simulation study consists of the following steps. For each pair \( (\theta, n) \), with \( \theta = (\alpha, \beta, a, b) \):

1. generate \( n \) outcomes of the EGNH distribution;
2. obtain the MLEs;
3. compute the average biases and standard errors.

The simulation study is performed using the \texttt{R} programming language and the BFGS (constrained) method. For maximizing the pllf (29), we use the MaxLik package (Heningsen and Toomet, 2011).

Figures 6 and 7 display results of this simulation study. As expected, both biases and standard errors decrease when the sample size increases. This proposed estimation procedure can be used effectively for estimating the parameters of the EGNH family. Further, it is important to note that the estimate for \( \beta \) are good, even before the small value considered of \( \alpha \).

![Figure 6. Std. errors of the estimates vs. sample size for \( (\alpha, \beta, a, b) = (1.8 \times 10^{-3}, 2.83 \times 10^{-1}, 1.75 \times 10^{-3}, 47.066) \)](image)

![Figure 7. Biases of the estimates vs. sample size for \( (\alpha, \beta, a, b) = (1.8 \times 10^{-3}, 2.83 \times 10^{-1}, 1.75 \times 10^{-3}, 47.066) \)](image)

12. Applications

In this section, we present two applications to real data in order to illustrate the EGNH potentiality. The first uncensored real dataset refers to the lifetimes of 50 components
The second uncensored dataset represents the stress-rupture life of Kevlar 49/epoxy strands, which were subjected to constant sustained pressure at 70% stress level until all units had failed (Andrews and Herzberg, 1985): 1051, 1337, 1389, 1921, 1942, 2322, 3629, 4006, 4012, 4063, 4921, 5445, 5620, 5817, 5905, 5956, 6068, 6121, 6473, 7501, 7886, 8108, 8546, 8666, 8831, 9106, 9711, 9806, 10205, 10396, 10861, 11026, 11214, 11362, 11604, 11608, 11745, 11762, 11895, 12044, 13520, 13670, 14110, 14496, 15395, 16179, 17092, 17568, 17568. For a previous study with this dataset, see Cooray and Ananda (2008).

Table 3 gives a descriptive summary of the two samples. For both datasets, the median is greater than the mean (indicating asymmetry). The Aarset dataset has negative skewness (left-skewed data), while the Andrews and Herzberg dataset has positive one (right-skewed data).

Table 3. Descriptive Statistics

| Datasets             | Mean   | Median | Variance  | Skew. | Kurt. | Min. | Max. |
|----------------------|--------|--------|-----------|-------|-------|------|------|
| Aarset               | 45.686 | 48.5   | 1078.153  | -0.1378 | 1.414 | 0.1  | 86   |
| Andrews and Herzberg | 8805.694 | 8831  | 20738145  | 0.097  | 2.172 | 1051 | 17568 |

In many applications empirical hrf shape can indicate a particular model. The plot of total time on test (TTT) (Aarset, 1987) can be useful in this sense. The TTT plot is obtained by plotting \( G(r/n) = (\sum_{i=1}^{r} y_{i:n} + (n-r)y_{n:n})/\sum_{i=1}^{n} y_{i:n}, \) where \( r = 1, \ldots, n \) and \( y_{i:n}(i = 1, \ldots, n) \) are the order statistics of the sample, against \( r/n. \) The TTT plots for the current datasets are presented in Figure 8. The TTT plot for the Aarset’s data indicates a bathtub-shaped hrf and the TTT plot for the Andrews and Herzberg’s data indicates an increasing hrf. Therefore, these plots reveal the appropriateness of the EGNH distribution to fit these datasets, since the new model can present bathtub-shaped and increasing hrfs.

We compare the fits of the EGNH distribution defined in (6) with the submodels NH, EE and ENH and additionally with some other lifetime models with four, three and two parameters, namely:

- The Kumaraswamy Nadarajah-Haghighi (KNH) distribution (Lima, 2015) having pdf
(for $x > 0$) given by

$$f_{KNH}(x) = \frac{ab\alpha\beta(1 + ax)^b \exp\left[1 - (1 + ax)^b\right]}{\left\{1 - \left\{1 - \exp\left[1 - (1 + ax)^b\right]\right\}^\alpha\right\}^{1-\beta}}.$$ 

- The beta Nadarajah-Haghighi (BNH) distribution \cite{Dias2016} having pdf (for $x > 0$) given by

$$f_{BNH}(x) = \frac{ab}{B(\alpha, \beta)}(1 + ax)^{b-1} \left\{1 - \exp\left[1 - (1 + ax)^b\right]\right\}^{\alpha-1} \left\{\exp\left[1 - (1 + ax)^b\right]\right\}^\beta.$$ 

- The Zografos-Balakrishnan Nadarajah-Haghighi (ZBNH) distribution \cite{Bourguignon2015} having pdf (for $x > 0$) given by

$$f_{ZBNH}(x) = ab\Gamma(\alpha)(1 + ax)^{b-1} \left\{1 - \frac{1 - \exp\left[1 - (1 + ax)^b\right]}{1 - (1 - \alpha) \exp\left[1 - (1 + ax)^b\right]}\right\}^\alpha.$$ 

- The Marshall-Olkin Nadarajah-Haghighi (MONH) distribution \cite{Lemonte2016} having pdf (for $x > 0$) given by

$$f_{MONH}(x) = ab\alpha(1 + ax)^{b-1} \exp\left[1 - (1 + ax)^b\right] \left\{1 - (1 - \alpha) \exp\left[1 - (1 + ax)^b\right]\right\}^2.$$ 

- The transmuted Nadarajah-Haghighi (TNH) distribution \cite{Ahmed2015} having pdf (for $x > 0$) given by

$$f_{TNH}(x) = ab(1 + ax)^{b-1} \exp\left[1 - (1 + ax)^b\right] \left\{(1 - \alpha) + 2\exp\left[1 - (1 + ax)^b\right]\right\}.$$ 

- The modified Nadarajah-Haghighi (MNH) distribution \cite{El2015} having pdf (for $x > 0$) given by

$$f_{MNH}(x) = \alpha(\lambda + 2\beta x)(1 + \lambda x + \beta x^2)^{a-1} \exp\left[1 - (1 + \lambda x + \beta x^2)^\alpha\right].$$ 

- The generalized power Weibull (GPW) distribution \cite{Nikulin2006} having pdf (for $x > 0$) given by

$$f_{GPW}(x) = abcx^{c-1}(1 + ax^c)^{b-1} \exp[1 - (1 + ax^c)^b].$$ 

- The classic Weibull distribution having pdf (for $x > 0$) given by

$$f_{Weibull}(x) = \left(\frac{a}{b}\right) \left(\frac{x}{b}\right)^{a-1} \exp\left[-\left(\frac{x}{b}\right)^a\right].$$ 

Tables 4 and 5 present the MLEs and their standard errors for some models fitted to the Aarset’s and Andrews and Herzberg’s datasets, respectively. In order to compare the competitive models, Tables 6 and 7 list the Goodness-of-Fit (GoF) statistics of the fitted models for the Aarset’s and Andrews and Herzberg’s datasets, respectively. We use the following GoF statistics: Cramér-von Mises ($W^*$), Anderson Darling ($A^*$), Kolmogorov-Smirnov (KS), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC),
Consistent Akaike Information Criterion (CAIC) and Hannan-Quinn Information Criterion (HQIC). They are used to verify the quality of the fits. In general, smaller GoF values are associated with better fits.

| model            | estimates                      |
|------------------|--------------------------------|
| EGNH(α, β, a, b) | 1.8 × 10^{-4}                 |
|                  | 2.83 × 10^{-4}                |
|                  | 1.75 × 10^{-3}                |
|                  | 47.066                         |
| (8.8 × 10^{-4})  | (1.19 × 10^{-2})              |
|                 | (2.47 × 10^{-4})              |
|                 | (3.69)                         |
| BNH(α, β, a, b)  | 2.55 × 10^{-1}                |
|                  | (5.2 × 10^{-2})               |
|                 | 5.225 × 10^{-3}               |
|                 | 9.753                          |
| (2.41 × 10^{-2}) | (1.93 × 10^{-2})              |
|                 | (5.018 × 10^{-4})             |
|                 | (3.20)                         |
| KNH(α, β, a, b)  | 1.22 × 10^{-1}                |
|                  | 6.27 × 10^{-2}                |
|                 | 5.85 × 10^{-4}                |
|                 | 71.05                          |
| (3.10 × 10^{-2}) | (2.91 × 10^{-2})              |
|                 | (3.98 × 10^{-5})              |
|                 | (1.26 × 10^{-1})              |
| MNH(α, a, b)     | 2.78 × 10^{-6}                |
|                  | 2.01 × 10^{-4}                |
|                 | 30.30                          |
| (3.98 × 10^{-7}) | (5.21 × 10^{-5})              |
|                 | (8.61 × 10^{-1})              |
| MONH(α, a, b)    | 4.36                           |
|                  | 4.61 × 10^{-5}                |
|                 | 370.406                        |
| (4.78 × 10^{-1}) | (4.92 × 10^{-6})              |
|                 | (60.91)                        |
| TNH(α, a, b)     | (−4.03 × 10^{-1})             |
|                  | 4.43 × 10^{-6}                |
|                 | 3080.45                        |
|                 | (1.01 × 10^{-1})              |
|                 | (3.50 × 10^{-6})              |
|                 | (112.90)                       |
| ENH(α, a, b)     | 8.29 × 10^{-1}                |
|                  | 2.32 × 10^{-5}                |
|                 | 509.597                        |
| (7.03 × 10^{-2}) | (2.06 × 10^{-6})              |
|                 | (60.71)                        |
| ZBNH(α, a, b)    | 8.91 × 10^{-1}                |
|                  | 3.41 × 10^{-6}                |
|                 | 3437.14                        |
| (7.55 × 10^{-2}) | (1.54 × 10^{-7})              |
|                 | (106.40)                       |
| GPW(a, b, c)     | 395.387                       |
|                  | 1.013                          |
|                 | (2.99 × 10^{-5})              |
| (49.66)          | (8.23 × 10^{-2})              |
|                 | (1.31 × 10^{-5})              |
| NH(a, b)         | 1.6 × 10^{-5}                 |
|                  | 775.132                       |
| (2.10 × 10^{-6}) | (97.86)                       |
| EE(α, β)         | 8.85 × 10^{-1}                |
|                  | 1.70 × 10^{-2}                |
| (8.74 × 10^{-2}) | (9.53 × 10^{-4})              |
| Weibull(a, b)    | 1.13                          |
|                  | 56.08                         |
| (9.29 × 10^{-2}) | (3.25)                        |

Based on the figures of merit in Table 6, the proposed EGNH distribution presents smallest GoF values. Figure 9(a) displays fitted and empirical densities for this dataset. We consider only EGNH, KNH and BNH models because these models present the smallest values of the KS statistic. The EGNH and KNH distributions seem to provide a closer fit to the histogram. The empirical cdf and the fitted cdfs of these models are displayed in Figure 10(a). From this plot, we note that the EGNH model provides a good fit.

On the other hand, with respect to Table 7, notice that the proposed EGNH model presents smallest values for the W*, A* and KS statistics. The fitted EGNH, MONH and Weibull pdfs in Figure 9(b) reveal that the proposed model fits current data better than others. The use of AIC, BIC, CAIC and HQIC is only recommended to compare nested models, then it is not adequate to compare the Weibull and MONH models with the EGNH using these criteria. In Figure 10(b) we can see that the empirical and fitted cdfs, which confirm that the EGNH model is one of the best candidates to describe the second dataset. In summary, the EGNH model may be an interesting alternative to other models available in the literature for modelling positive real data.

13. Concluding remarks

In this paper, we propose a new distribution, which is a simple generalization of the Nadarajah-Haghighi’s exponential proposed by Nadarajah and Haghighi (2011). Further,
the new model includes others known distributions as special cases. We refer to it as the 
exponentiated generalized Nararajah-Haghighi (EGNH) distribution. Some of its proper-
ties have been derived and discussed: ordinary and incomplete moments, mean deviations, 
Rényi entropy and order statistics. We present theoretical evidence that the new distri-
bution can take decreasing, increasing, bathtub-shaped and upside-down bathtub hazard 
rate functions. We provide a maximum likelihood procedure for estimating the EGNH 
parameters. A Monte Carlo study to assess the consistency of the maximum likelihood 
estimates is performed, which satisfies the expected asymptotic behavior. The usefulness 
of the new model is illustrated through two applications to real data. The results indicate 
that our proposal can furnish better performance than other extended Nadarajah-Haghighi 
models recorded in the survival literature. The formulae related with the new model are

---

| model               | estimates          | Andrews and Herzberg’s dataset | Aarset’s dataset |
|---------------------|--------------------|-------------------------------|-------------------|
| model               | W*                | A*              | KS    | AIC  | CAIC | BIC  | HQIC |
| EGNH(α, β, a, b)    | 2.41 × 10⁻¹       | 1.194            | 1.27 × 10⁻⁵ | 14.268 |
| BNH(α, β, a, b)     | 1.189             | 1.61 × 10⁻¹      | 2.48 × 10⁻⁵ | 6.886  |
| KNH(α, β, a, b)     | 2.202             | 9.559            | 6.629 × 10⁻⁶| 5.903  |
| MNH(α, a, b)        | 1.71 × 10⁻¹       | 3.44 × 10⁻¹      | 4.76 × 10⁻⁷ | 3.89 × 10⁻¹ |
| MONH(α, a, b)       | 6.08 × 10⁻⁷       | 2.55 × 10⁻⁴      | 0.157         |
| ZBNH(α, a, b)       | 4.65              | 6.24 × 10⁻⁶      | 19.46          |
| GPW(α, b, c)        | 1.916 × 10⁻¹⁴     | 6.13 × 10⁻¹³     | 4.07 × 10⁻⁸   |
| ENH(α, a, b)        | 1.992             | 5.18 × 10⁻⁶      | 20.139         |
| NH(α, b)            | 2.95 × 10⁻⁶       | 28.51            | 479.83         |
| EE(α, β)            | 3.02              | 2.16 × 10⁻⁴      | 480.08         |
| Weibull(a, b)       | 2.16              | 9411.52          | 487.46         |

---

Table 5. Estimates for Andrews and Herzberg’s dataset

Table 6. Goodness-of-fit tests for Aarset’s dataset
Table 7. Goodness-of-fit tests for Andrews and Herzberg’s dataset

| Model      | $W^*$ | $A^*$ | KS   | AIC  | CAIC | BIC  | HQIC |
|------------|-------|-------|------|------|------|------|------|
| $\text{EGNH}(\alpha, \beta, a, b)$ | 0.032 | 0.236 | 0.069 | 966.81 | 967.72 | 974.38 | 969.68 |
| $\text{MONH}(\alpha, a, b)$ | 0.033 | 0.246 | 0.072 | 965.42 | 965.95 | 971.09 | 967.57 |
| $\text{BNH}(\alpha, \beta, a, b)$ | 0.033 | 0.237 | 0.070 | 966.95 | 967.86 | 974.51 | 969.82 |
| $\text{ZBNH}(\alpha, a, b)$ | 0.054 | 0.323 | 0.079 | 965.49 | 966.02 | 971.17 | 967.64 |
| $\text{GPW}(a, b, c)$ | 0.057 | 0.361 | 0.191 | 968.59 | 969.12 | 974.26 | 970.74 |
| $\text{ENH}(\alpha, a, b)$ | 0.058 | 0.383 | 0.084 | 966.26 | 966.80 | 971.94 | 968.42 |
| $\text{NH}(a, b)$ | 0.059 | 0.388 | 0.186 | 973.48 | 973.75 | 977.27 | 974.92 |
| $\text{Weibull}(a, b)$ | 0.073 | 0.482 | 0.088 | 965.70 | 965.95 | 971.07 | 967.48 |
| $\text{KNH}(\alpha, \beta, a, b)$ | 0.075 | 0.496 | 0.089 | 969.79 | 970.70 | 977.36 | 972.66 |
| $\text{EE}(\alpha, \beta)$ | 0.164 | 1.061 | 0.118 | 972.14 | 972.40 | 975.92 | 973.58 |
| $\text{MNH}(a, b)$ | 0.342 | 2.137 | 0.277 | 1021.80 | 1022.34 | 1027.48 | 1023.96 |

Figure 9. Empirical and fitted pdfs

Figure 10. Empirical and fitted pdfs

manageable and may turn into adequate tools comprising the arsenal of applied statistics.
We hope that the proposed model may attract wider applications for modeling positive real data in many areas such as engineering, survival analysis, hydrology and economics.

References

Aarset, M. V. 1987. How to Identify a Bathtub Hazard Rate. *Reliability, IEEE Transactions on*, R-36, 106–108.

Abdul Monaem, I. B. 2015. Exponentiated Nadarajah and Haghighi’s exponential Distribution. *International Journal of Mathematical Analysis and Applications*, 2.

Ahmed, A., Muhammed, H. Z., and Elbatal, I. 2015. A New Class of Extension Exponential Distribution. *International Journal of Applied Mathematical Sciences*, 8, 13–30.

Alzaatreh, A., Lee, C., and Famoye, F. 2013. A new method for generating families of continuous distributions. *METRON - International Journal of statistics*, 71, 63–79.

Andrade, T., Rodrigues, H. M., Bourguignon, M., and Cordeiro, G. M. 2015. The exponentiated generalized Gumbel distribution. *Revista Colombiana de Estadística*, 38, 123–143.

Andrews, D. F. and Herzberg, A. M. 1985. *Stress-Rupture Life of Kevlar 49/Epoxy Spherical Pressure Vessels*, (pp. 181–186). New York, NY: Springer New York.

Bourguignon, M., Lima, M. C. S., Leão, J., Nascimento, A. D. C., Pinho, L. G. B., and Cordeiro, G. M. 2015. A new generalized gamma distribution with applications. *American Journal of Mathematical and Management Sciences*, 34, 309–342.

Bourguignon, M., Silva, R. B., and Cordeiro, G. M. 2014. The Weibull-G Family of Probability Distributions. *Journal of Data Science*, 12, 53–68.

Cooray, K. and Ananda, M. M. A. 2008. A Generalization of the Half-Normal Distribution with Applications to Lifetime Data. *Communications in Statistics - Theory and Methods*, 37, 1323–1337.

Cordeiro, G. M. and Castro, M. 2011. A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81, 883–898.

Cordeiro, G. M., Ortega, E. M. M., and Cunha, D. C. C. 2013. The Exponentiated Generalized Class of Distributions. *Journal of Data Science*, 1, 1–27.

Cordeiro, G. M., Ortega, E. M. M., and Silva, G. O. 2011. The exponentiated generalized gamma distribution with application to lifetime data. *Journal of Statistical Computation and Simulation*, 81, 827–842.

Dias, C. R. B. 2016. *New Continuous Distributions Applied To Lifetime Data And Survival Analysis*. Doctoral thesis, Universidade Federal de Pernambuco, Recife.

El Damcese, M. A. and Ramadan, D. A. 2015. Studies on Properties and Estimation Problems for Modified Extension of Exponential Distribution. *International Journal of Exponential Distribution*, 125.

Elbatal, I. and Muhammed, H. Z. 2014. Exponentiated Generalized Inverse Weibull Distribution. *Applied Mathematical Sciences*, 8, 3997–4012.

Eugene, N., Lee, C., and Famoye, F. 2002. Beta-normal Distribution And Its Applications. *Communications in Statistics - Theory and Methods*, 31, 497–512.

Gupta, R. C., Gupta, P. L., and Gupta, R. D. 1998. Modeling failure time data by lehman alternatives. *Communications in Statistics - Theory and Methods*, 27, 887–904.

Gupta, R. D. and Kundu, D. 2001. Exponentiated Exponential Family. *Biometrical Journal*, 43, 117–130.

Henningsen, A. and Toomet, O. 2011. maxLik: A package for maximum likelihood estimation in R. *Computational Statistics*, 26, 443–458.

Kenney, J. F. and Keeping, E. S. 1962. *Mathematics of statistics* (3 ed.), volume 1. Van Nostrand, New Jersey: Chapman & Hall Ltd.
Lee, C., Famoye, F., and Alzaatreh, A. Y. 2013. Methods for generating families of univariate continuous distributions in the recent decades. Wiley Interdisciplinary Reviews, 5, 219–238.

Lemonte, A. J. 2013. A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Computational Statistics & Data Analysis, 62, 149–170.

Lemonte, A. J. and Cordeiro, G. M. 2011. The exponentiated generalized inverse Gaussian distribution. Statistics & Probability Letters, 81, 506–517.

Lemonte, A. J., Cordeiro, G. M., and Moreno Arenas, G. 2016. A new useful three-parameter extension of the exponential distribution. Statistics, 50, 312–337.

Lima, S. R. L. 2015. The Half-Normal Generalized Family And Kumaraswamy Nadarajah-Haghighi Distribution. Master thesis, Universidade Federal de Pernambuco.

Moors, J. J. A. 1988. A Quantile Alternative for Kurtosis. Journal of the Royal Statistical Society. Series D (The Statistician), 37, 25–32.

Mudholkar, G. S. and Srivastava, D. K. 1993. Exponentiated Weibull family for analyzing bathtub failure-rate data. Reliability, IEEE Transactions on, 42, 299–302.

Mudholkar, G. S., Srivastava, D. K., and Freimer, M. 1995. The Exponentiated Weibull Family: A Reanalysis of the Bus-Motor-Failure Data. Technometrics, 37, 436–445.

Nadarajah, S. and Gupta, A. K. 2007. A product Pareto distribution. Metrika, 68, 199–208.

Nadarajah, S. and Haghighi, F. 2011. An extension of the exponential distribution. Statistics, 45, 543–558.

Nikulin, M. and Haghighi, F. 2006. A Chi-Squared Test for the Generalized Power Weibull Family for the Head-and-Neck Cancer Censored Data. Journal of Mathematical Sciences, 133, 1333–1341.

Nikulin, M. and Haghighi, F. 2009. On the power generalized Weibull family. METRON - International Journal of statistics, LXVII, 75–86.

Ortega, E. M. M., Lemonte, A. J., Silva, G. O., and Cordeiro, G. M. 2015. New flexible models generated by gamma random variables for lifetime modeling. Journal of Applied Statistics, 42, 2159–2179.

Selim, M. A. and Badr, A. M. 2016. The Kumaraswamy Generalized Power Weibull Distribution. Mathematical Theory and Modeling, 6, 110–124.

Silva, R. V., Gomes Silva, F., Ramos, M. W. A., and Cordeiro, G. M. 2015. A New Extended Gamma Generalized Model. International Journal of Pure and Applied Mathematics, 100.

Zografos, K. and Balakrishnan, N. 2009. On families of beta- and generalized gamma-generated distributions and associated inference. Statistical Methodology, 6, 344–362.