Energy methods for Cauchy problems of evolutions equations

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Abstract. In the present paper a numerical method based on minimizing energy functionals, is developed for solving Cauchy problem of evolution equations. Two cases are treated: the parabolic equations as the heat equations and the hyperbolic ones as the elastodynamics equations. The method is first presented in some details, then, illustrated on various applications.

1. Introduction

The recent and rapid development of full surface field imaging gives a new topicality to the classical problem, known as the Cauchy problem, of expanding a field, known together with its dual quantity field on a part of a boundary of a solid, towards the interior of it. In the applications, the problem is seen as a data completion problem, that is, a search for lacking information on a part of the boundary of a solid provided overspecified data are available on the remaining part of the boundary. Various applications can then be identified as inverse problem applications (identification of unknown boundary conditions, geometrical inverse problems, identification of material parameters in a geometrically know inclusion embedded in an elastic solid, estimation of contact zones...) or damage detection or distributed material parameters identification (the data completion being the first step of the identification procedure).

A two-fields based energy error method has been designed for general symmetric elliptic operators, leading to efficient numerical algorithms that have been implemented in various situations, including 3D heterogeneous ones with non linear boundary conditions, see Andrieux and Baranger [1-3]. The method is based on the definition of an appropriate energy error functional, as a function of the fields on the boundary that are out of reach. This functional turns out to be positive quadratic and its (null) minimum delivers the desired data when the available Cauchy data are compatible. Furthermore, the computation of the functional and its gradient is greatly simplified by an alternative expression involving only the boundary of the solid. An interpretation of the Kozlov-Maz'ya-Fomin (KMF) fixed point algorithm (Kozlov et al. [7] and Baumeister et al. [5]) usually used in numerical resolution of Cauchy problems can be given to explain the superior performances of the proposed approach. Namely, the KMF algorithm appears as an alternating direction minimization method of the error functional designed here, so that conjugate gradient methods or trust region methods dramatically improve the convergence speed.
Turning now towards evolutions equations on a time domain $[0, D]$ (parabolic equations such as the heat equation or hyperbolic ones such as elastodynamics equations) the question of extending the energy error functional is investigated. For the heat equation, the error functional is completed by a term involving the norm of the difference of the two-fields at the final instant $D$. In elastodynamics, two difficulties arise: first the energy has to be replaced by the power of the external forces, integrated over the time interval, secondly the exact controllability leads to supplement the error functional with an extra penalization term on the boundary to recover the property of identifiability of the lacking boundary conditions. These results are first presented in some details, then, illustrated on various applications.

2. A variational method for the Cauchy problem of the Heat equation

Consider a domain shown on the figure 1. The boundary of the domain, $\Gamma$, is splitted into two parts; $\Gamma = \Gamma_m \cup \Gamma_u$.

**Figure 1.** The domain and the boundary conditions.

Denoting:

$$Q = \Omega \times [0, D], \quad \Sigma_m = \Gamma_m \times [0, D], \quad \text{and} \quad \Sigma_u = \Gamma_u \times [0, D] \quad (1)$$

let us define the following transient heat conduction problem with overspecified boundary conditions on $\Sigma_m$,

$$
\begin{cases}
\rho \partial_t u - \text{div}(k \nabla u) = 0 & \text{in } Q \\
u = T_m & \text{on } \Sigma_m \\
k \frac{\partial u}{\partial n} = \Phi_m & \text{on } \Sigma_m \\
u(x, 0) = u_0(x) & \text{on } \Omega
\end{cases}
$$

(2)

Here, $u$ is the temperature field, $T_m$ and $\Phi_m$ are the known temperature and heat flux on $\Sigma_m$, $k$ is the thermal conductivity, $\rho$ the density and $c$ is the specific heat. The aim is to determine the temperature field in the domain $Q$ and to identify the Neumann and Dirichlet boundary conditions on $\Gamma_u$.

In the approach presented here, we consider, for a given pair $(\eta, \tau)$, the following two mixed and well posed problems, whose solutions are denoted by $u_1$ and $u_2$.3
The temperature fields \( u_1 \) and \( u_2 \) are equal when the pair \((\eta, \tau)\) meets the real data \((T_m, \Phi_m)\) on the boundary \(\Gamma_u\). Assuming that \(T_m\) and \(\Phi_m\) are compatible, we propose to solve the Cauchy problem via the minimization of the following gap or error functional, using the notation \(v = u_1 - u_2\):

\[
(\Phi_u, T_u) = \arg \min_{(v, \tau)} J(v, \eta, \tau)
\]

\[
J(v) = \int_0^\Omega k(\nabla v)^2 d\Omega dt + \frac{1}{2} \int_\Omega \rho c v^2 d\Omega
\]  

This functional is convex, quadratic and positive with a minimum equal to zero when the measured data \((\Phi_m, T_m)\) on \(\Sigma_m\) are compatible. Furthermore, using the equation (3) for \(u_1\) and \(u_2\), it is straightforward to derive an alternative expression of the functional:

\[
J(v) = \int_0^\Omega (u_1 - \tau)(-\eta - k\nabla u_2, n)d\Gamma dt + \int_0^\Omega (T_m - u_2)(k\nabla u_1, n + \Phi_u)d\Gamma dt
\]  

This expression shows that the error functional (4) can be expressed equivalently as a boundary integral on \(\Sigma_m \cup \Sigma_u\), as given by equation (5).

3. A variational method for the Cauchy problem in elastodynamics with damping

Provided the Hooke’s tensor \(A\), the viscosity tensor \(B\) and the damping \(c\) of the material that forms the solid are known, even if possibly dependent on space, with the usual properties on \(A\) ensuring existence and uniqueness of a classical linear elasticity problem, the Cauchy problem is written as follows:

\[
\begin{cases}
\rho \ddot{u} + c\dot{u} - \nabla(\varepsilon(u) + B : \varepsilon(\dot{u})) = 0 & \text{in } Q \\
[A : \varepsilon(u) + B : \varepsilon(\dot{u})]n = \Phi_m, u = T_m & \text{on } \Sigma_m
\end{cases}
\]

with the following initial values:

\[
\begin{cases}
u(x,0) = u_0^0(x) & \text{in } \Omega \\
\dot{u}(x,0) = u_0^1(x)
\end{cases}
\]  

Solving the above Cauchy problem can be stated as follows: given the surface traction field \(\Phi_m\) and displacement field \(T_m\) on \(\Sigma_m\), find the surface tractions \(\Phi_u\) and displacement \(T_u\) such that an elastic displacement field \(u\) exists and satisfies the following dynamic equilibrium equations:

\[
\begin{cases}
\rho \ddot{u} + c\dot{u} - \nabla(\varepsilon(u) + B : \varepsilon(\dot{u})) = 0 & \text{in } Q \\
[A : \varepsilon(u) + B : \varepsilon(\dot{u})]n = \Phi_u, u = T_u & \text{on } \Sigma_u
\end{cases}
\]  

The general method proposed here to solve the Cauchy problems (2) and (3) is to derive a functional of the unknown fields on \(\Sigma_u\), say, \((\tau, \nu)\), where (possibly local) minima deliver the desired pair of fields \((T_m, U_m)\). Building this functional follows the two steps below. First, the following elastic two displacement fields \(u_1\) and \(u_2\) are defined, as functions respectively of \((T_m, \tau)\) and \((\Phi_m, \nu)\).
They correspond to the solutions of two well-posed classical mixed elastodynamics problems. If the system has no damping (\(B\) and \(c\) are canceled) for instance, we obtain:

\[
\begin{align*}
\rho \ddot{u}_1 - \text{div}(A : \varepsilon(u_1)) &= 0 \quad \text{in } Q \\
\rho \ddot{u}_2 - \text{div}(A : \varepsilon(u_2)) &= 0 \quad \text{in } Q \\
u_1 &= T_m \quad \text{on } \Sigma_m \quad \text{and} \quad A : \varepsilon(u_2).n = \Phi_m \quad \text{on } \Sigma_m \\
A : \varepsilon(u_1).n &= \tau \quad \text{on } \Sigma_u \\
u_2 &= \nu \quad \text{on } \Sigma_u
\end{align*}
\]

with \(u_1(x,0) = u^0(x)\) and \(\dot{u}_i(x,0) = u^I(x)\) in \(\Omega\) for \(i = 1, 2\).

Then, the second step consists of the introduction of a functional measuring the gap \(v\) between the two fields. The choice of this functional here is the error in the semi-norm of energy, using the notation \(v = u_1 - u_2\):

\[
J(v) = \int_0^D \int_\Omega (A : \varepsilon(v) : \varepsilon(\dot{v}) + \rho \text{div}(v)) d\Omega dt
\]

that appears also as a control of the power of external forces on the boundary, during the whole time interval \([0, D]\):

\[
J(v) = \int_0^D \int_\Omega \sigma(n) \cdot \dot{v} d\Omega dt
\]

This functional can also be written as a control of the elastic and kinetic energies at the final time instant \(D\):

\[
0 \leq J(v) = \frac{1}{2} \int_\Omega (\rho \dot{v}^2 + A : \varepsilon(v) : \varepsilon(v)) d\Omega |_{t = D}
\]

Because of the exact controllability of the undamped elastodynamics equations for boundary control, the above functional doesn’t ensure the uniqueness of the solution: that is: non zero pairs \((\tau, \nu)\) can lead to vanishing value of \(J\). More precisely, Lions [6] showed that there exists an infinity of boundary controls (prescribed displacement of surface traction fields) that enable to lead an elastic solid at rest, starting from an equilibrium position at rest in a finite time, provided it is large enough. Vanishing value of the \(J\) functional (15) only prescribes that the final value (at time \(t = D\)) of the \(v\) displacement field is a rigid body motion and the rate field \(\dot{v}\) is zero. Then using any of the previous boundary controls it is possible to build a non-vanishing \(v\) field on \(\Omega \times [0, D]\) with \(J(v) = 0\) by solving an initial and boundary value problem. The minimum of \(J(J = 0)\) is not only reached by the vanishing field (leading to the desired equality between \(u_1\) and \(u_2\)) but also by non-vanishing fields. Hence, we introduce a supplementary cost term in the functional involving a boundary term on the whole time domain, an extra scalar \(\lambda\) and two pseudo material-parameters \(k\) and \(K\):

\[
0 \leq J_{\lambda}(v) = \frac{1}{2} \int_\Omega (\rho \dot{v}^2 + A : \varepsilon(v) : \varepsilon(v)) d\Omega |_{t = D} + \frac{\lambda}{2} \int_0^D \int_\Omega (k\dot{v}^2 + K\dot{v}^2) d\Omega dt
\]

When the system has nonzero significant damping properties \((c > 0\) or \(B\) positive tensor), the functional (13) can directly be used and turns out to be the sum of two terms. The first one expresses a control of the elastic and kinetic energies at the last final time instant \(D\) while the second one expresses a control of the dissipated power on the whole time domain.

\[
0 \leq J(v) = \frac{1}{2} \int_\Omega (\rho \dot{v}^2 + A : \varepsilon(v) : \varepsilon(v)) |_{t = D} + \int_0^D \int_\Omega [c\dot{v}^2 + B : \varepsilon(\dot{v}) : \varepsilon(\dot{v})]
\]

Then, the data \((T_\infty, \Phi_\infty)\) can be identified by solving the following optimization problem:

\[
(T_\infty, \Phi_\infty) = \arg \min_{(\tau, \nu)} J[v(\tau, \nu)]
\]
4. Applications

4.1. Heat conduction problem
To explore the efficiency of the proposed data matching procedure, we consider therefore the reconstruction of temperature and flux in a pipeline of infinite length as shown on figure 3. We assume that the temperature does not depend on the longitudinal coordinate. We deal, therefore with a two-dimensional problem.

\[
\text{Given temperature } T_m(t) \text{ and heat flux } \Phi_m(t)
\]

\[
\text{Heat flux } \Phi_d(t) \text{ and temperature } T_u(t) \text{ to be identified}
\]

\[\text{Known Initial Condition}\]

Figure 2. Two-dimensional model for pipeline with the inner radius \(R_i = 12\) cm and the outer radius \(R_o = 12.923\) cm. The material presents the following properties: \(k = 15.86 \times 10^{-2}\), \(\rho = 7.8 \times 10^{-3}\) Kg/cm\(^3\), \(c = 493.89\).

Figures 3 and 4 show the identified temperature on the inner boundary for two cases of transient heat problems, where the temperature presents discontinuity on short and medium interval of time. These results are obtained by using spectral method to solve direct problems and minimization algorithm based on BFGS and line search methods. The gradient of the functional is evaluated by finite differences method, which implies a very high computational cost. In order to improve the computational cost, an adjoint-based method is currently under development and validation. In the presented method regularization is not used. However, to be able to deal with noisy measurements it is necessary to introduce regularization techniques. This part of the project is also currently under development.

Figure 3. Identified internal temperature (stars line) and exact one (dashed line) for \(t \in [0, D]\).

Figure 4. Identified internal temperature (stars line) and exact one (dashed line) for \(t \in [0, D]\).
4.2. Elastodynamic problem

We consider the two-dimensional flexural beam problem as shown on the figure 5. The finite element model of this structure is considered to be an \( n \) degrees-of-freedom system. Therefore, the differential equations of motion of the system in terms of mass, stiffness and damping matrices are:

\[
[M]\ddot{U}(t) + [C]\dot{U}(t) + [K]U(t) = F(t)
\]

where \([M]\) denotes the \( n \times n \) mass matrix, \([C]\) is the \( n \times n \) damping matrix, \([K]\) is the \( n \times n \) stiffness matrix, \( F(t) \) is the \( n \times 1 \) input force vector, \( \dot{U}(t) \) and \( U(t) \) denotes the \( n \times 1 \) vectors of acceleration, velocity and displacement respectively.

The known data are measured at the extremity B and we want to identify the boundary condition at the point A. The beam is characterised by the Young modulus \( E \), the flexural inertia \( I \) and the Rayleigh damping expressed by the coefficients \( \alpha \) and \( \beta \). Then the damping matrix is \([C] = \alpha [K] + \beta [M] \). The boundary conditions at the point B can be parameterized by the stiffness \((K, C)\) of rotational and longitudinal springs.

![Figure 5. Cantilever beam with impulse Force \( F_m \) at the point \( B \) and elastic boundary conditions at the point \( A \). The material presents the following properties: \( E = 200 \times 10^9 \text{Pa}, \rho = 850 \text{kg/m}^3, \alpha = 3.529 \times 10^{-5}, \beta = 21.474 \). The geometry of the beam has the following values: \( L = 1 \text{ m}, I = 8.33 \times 10^{-9} \text{ m}^4, S = 10^{-5} \text{ m}^2 \).](image)

The space discretized form of the energy functional (16) is:

\[
J(X_\eta, X_\tau) = \frac{1}{2}\left( (U_1 - U_2)^T[K](U_1 - U_2) + (\dot{U}_1 - \dot{U}_2)^T[M](\dot{U}_1 - \dot{U}_2) \right)_{x-D} + \int_0^L (\dot{U}_1 - \dot{U}_2)^T[C](\dot{U}_1 - \dot{U}_2)
\]

Where \( X_\eta(t) \) and \( X_\tau(t) \) denotes the discretized Neumann and Dirichlet boundary conditions respectively. The identification is then performed by minimizing \( J(X_\eta(t), X_\tau(t)) \). The computation is performed by means of Matlab Software and by using Newmark integration time schema. The minimization procedure is hold on by means of BFGS and line search methods. The gradient of the functional is also evaluated by finite differences method, which implies a high computational cost.

Again, in order to improve the computational cost, an adjoint-based method is currently under development and validation. In the presented method regularization is not used. However, to be able to deal with noisy measurements it is necessary to introduce regularization techniques. This part of the project is also currently under development.

Figure 6 and 7 show the identified displacements and forces respectively obtained for spring value \( K = C = 1000 \text{ Pa} \). As can be observed from the figures the identified values and the exact ones are in good agreement when comparing the displacement and forces.
Figures 8 and 9 show the identified displacements and forces respectively obtained for spring stiffness \( K = C = 10^9 \text{ Pa} \). In this case the identified displacements present values around \( 10^{-8} \text{ m} \). Then we can deduce that this extremity is clamped.

5. Conclusion

A general method has been briefly presented in this paper to solve Cauchy problems for evolutions equations where initial conditions are known, and consequently to identify boundary conditions. This method, already presented for stationary problems in [1-4], is based on the idea of exploiting simultaneously and symmetrically the overspecified data measured on the accessible boundary. Next, two well-posed problems are defined. The first one involves the unknown Neumann boundary...
condition and the measured Dirichlet boundary condition. The second one involves the unknown Dirichlet boundary condition and the measured Neumann boundary condition. The two problems can also have the additional usual boundary conditions. Energy functionals are then established, for parabolic and hyperbolic problems, to measure the gap between the solutions of these two problems. Therefore, two numerical applications are presented to illustrate the method. An optimization approach is then used to minimize these functionals, which depend explicitly on the above fields and implicitly on the unknown boundary conditions. The presented method can be easily implemented in industrial finite element software in order to deal with industrial Cauchy problems.

However, in order to improve the optimization process an efficient numerical time integration schema, for forward and backward problems, must be established in order to derive the gradient of the functionals by the adjoint method. Similarly, the influence of noisy measurements should be considered and improved by means of regularization techniques. These aspects are currently under development and will be published soon.

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