Turán numbers of complete 3-uniform Berge-hypergraphs

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Abstract

Given a family $F$ of $r$-graphs, the Turán number of $F$ for a given positive integer $N$, denoted by $ex(N,F)$, is the maximum number of edges of an $r$-graph on $N$ vertices that does not contain any member of $F$ as a subgraph. For given $r \geq 3$, a complete $r$-uniform Berge-hypergraph, denoted by $K^r_N$, is an $r$-uniform hypergraph of order $n$ with the core sequence $v_1, v_2, \ldots, v_n$ as the vertices and distinct edges $e_{ij}$, $1 \leq i < j \leq n$, where every $e_{ij}$ contains both $v_i$ and $v_j$. Let $F^{(r)}_n$ be the family of complete $r$-uniform Berge-hypergraphs of order $n$. We determine precisely $ex(N,F^{(3)}_n)$ for $n \geq 13$. We also find the extremal hypergraphs avoiding $F^{(3)}_n$.

Keywords: Turán number, Extremal hypergraph, Berge-hypergraph.

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1 Introduction

A hypergraph $\mathcal{H}$ is a pair $\mathcal{H} = (V,E)$, where $V$ is a finite non-empty set (the set of vertices) and $E$ is a collection of distinct non-empty subsets of $V$ (the set of edges). We denote by $e(\mathcal{H})$ the number of edges of $\mathcal{H}$. An $r$-uniform hypergraph or $r$-graph is a hypergraph such that all its edges have size $r$. A complete $r$-uniform hypergraph of order $N$, denoted by $K^r_N$, is a hypergraph consisting of all the $r$-subsets of a set $V$ of cardinality $N$. For a family $\mathcal{F}$ of $r$-graphs, we say that the hypergraph $\mathcal{H}$ is $\mathcal{F}$-free if $\mathcal{H}$ does not contain any member of $\mathcal{F}$ as a subgraph.

Given a family $\mathcal{F}$ of $r$-graphs, the Turán number of $\mathcal{F}$ for a given positive integer $N$, denoted by $ex(N,\mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free $r$-graph on $N$ vertices. An $\mathcal{F}$-free $r$-graph $\mathcal{H}$ on $N$ vertices is extremal hypergraph for $\mathcal{F}$ if $e(\mathcal{H}) = ex(N,\mathcal{F})$. These are natural generalizations of the classical Turán number for 2-graphs [14]. For given $n, r \geq 2$, let $\mathcal{H}^{(r)}_n$ be the family of $r$-graphs $\mathcal{F}$ that have at most $\binom{n}{2}$ edges, and have some set $T$ of size $n$ such
that every pair of vertices in \( T \) is contained in some edge of \( F \). Let the \( r \)-graph \( H_n^{(r)} \in \mathcal{H}_n^{(r)} \) be obtained from the complete 2-graph \( K_2^n \) by enlarging each edge with a new set of \( r - 2 \) vertices. Thus \( H_n^{(r)} \) has \( (r - 2)\binom{n}{2} + n \) vertices and \( \binom{r}{2} \) edges. For given \( n \geq 5 \) and \( r \geq 3 \), a complete \( r \)-uniform Berge-hypergraph of order \( n \), denoted by \( K_n^{(r)} \), is an \( r \)-uniform hypergraph with the core sequence \( v_1, v_2, \ldots, v_n \) as the vertices and \( \binom{n}{2} \) distinct edges \( e_{ij}, 1 \leq i < j \leq n \), where every \( e_{ij} \) contains both \( v_i \) and \( v_j \). Note that a complete \( r \)-uniform Berge-hypergraph is not determined uniquely as there are no constraints on how the \( e_{ij} \)'s intersect outside \( \{v_1, v_2, \ldots, v_n\} \).

Extremal graph theory is that area of combinatorics which is concerned with finding the largest, smallest, or otherwise optimal structures with a given property. There is a long history in the study of extremal problems concerning hypergraphs. The first such result is due to Erdős, Ko and Rado [2].

In contrast to the graph case, there are comparatively few known results on the hypergraph Turán problems. In the paper in which Turán proved his classical theorem on the extremal numbers for complete graphs \([14]\), he posed the natural question of determining the Turán number of the complete \( r \)-uniform hypergraphs. Surprisingly, this problem remains open in all cases for \( r > 2 \), even up to asymptotics. Despite the lack of progress on the Turán problem for dense hypergraphs, there are considerable results on certain sparse hypergraphs. Recently, some interesting results were obtained on the exact value of extremal number of paths and cycles in hypergraphs. Füredi et al. [3] determined the extremal number of \( r \)-uniform loose paths of length \( n \) for \( r \geq 4 \) and large \( N \). They also conjectured a similar result for \( r = 3 \). Füredi and Jiang [4] determined the extremal function of loose cycles of length \( n \) for \( r \geq 5 \) and large \( N \). Recently, Kostochka et al. [11] extended these results to \( r = 3 \) for loose paths and \( r = 3, 4 \) for loose cycles. Győri et al. [6] found the extremal numbers of \( r \)-uniform hypergraphs avoiding Berge paths of length \( n \). Their results substantially extend earlier results of Erdős and Gallai [1] on extremal number of paths in graphs. Let \( C_n^{(r)} \) denote the family of \( r \)-graphs that are Berge cycles of length \( n \). Győri and Lemons [8, 7] showed that for all \( r \geq 3 \) and \( n \geq 3 \), there exists a positive constant \( c_{r,n} \), depending on \( r \) and \( n \), such that

\[
ex(N, C_n^{(r)}) \leq c_{r,n}N^{1 + \frac{1}{2r}}.
\]

Let \( N, n, r \) be integers, where \( N \geq n > r \) and \( r \geq 2 \). Also let \( T_r(N, n-1) \) be the complete \( r \)-uniform \((n-1)\)-partite hypergraph with \( N \) vertices and \( n-1 \) parts \( V_1, V_2, \ldots, V_{n-1} \) whose partition sets differ in size by at most 1. Suppose that \( t_r(N, n-1) \) denotes the number of edges of \( T_r(N, n-1) \). If \( N = \ell(n-1) + j \), where \( \ell \geq 1 \) and \( 1 \leq j \leq n-1 \), then it is straightforward to see that

\[
t_r(N, n-1) = \sum_{i=0}^{r} \ell^{r-i} \binom{j}{i} \binom{n-1-i}{r-i}.
\]

In 2006, Mubayi [12] showed that the unique largest \( \mathcal{H}_n^{(r)} \)-free \( r \)-graph on \( N \) vertices is \( T_r(N, n-1) \). Settling a conjecture of Mubayi in [12], Pikhurko [13] proved that there exists \( N_0 \) so that the Turán numbers of \( H_n^{(3)} \) and \( \mathcal{H}_n^{(r)} \) coincide for all \( N > N_0 \). Let \( \mathcal{F}_n^{(r)} \) be the family of complete \( r \)-uniform Berge-hypergraphs of order \( n \). Because \( H_n^{(3)} \in \mathcal{F}_n^{(3)} \), the Pikhurko’s result
[13] implies that \( \text{ex}(N, \mathcal{F}_n^{(3)}) \leq t_3(N, n-1) \) for sufficiently large \( N \). In this paper, for \( N \geq 13 \), we show that \( \text{ex}(N, \mathcal{F}_n^{(3)}) = t_3(N, n-1) \) and \( T_3(N, n-1) \) is the unique extremal hypergraph for \( \mathcal{F}_n^{(3)} \). More precisely, we prove the following theorem.

**Theorem 1.1** Let \( N, n \) be integers so that \( N \geq n \geq 13 \). Then

\[
\text{ex}(N, \mathcal{F}_n^{(3)}) = t_3(N, n-1).
\]

Furthermore, the unique extremal hypergraph for \( \mathcal{F}_n^{(3)} \) is \( T_3(N, n-1) \).

First we show that \( \text{ex}(N, \mathcal{F}_n^{(r)}) \geq t_r(N, n-1) \). To see that, consider an arbitrary sequence \( v_1, v_2, ..., v_n \) of the vertices of \( T_r(N, n-1) \). By the pigeonhole principle, there exists some part \( V_h, 1 \leq h \leq n-1 \), in \( T_r(N, n-1) \) containing at least two vertices of this sequence. Since every edge of \( T_r(N, n-1) \) includes at most one vertex of each part \( V_i, 1 \leq i \leq n-1 \), This sequence can not be the core sequence of a \( K_n^{(r)} \). Hence \( T_r(N, n-1) \) is \( \mathcal{F}_n^{(r)} \)-free and

\[
\text{ex}(N, \mathcal{F}_n^{(r)}) \geq t_r(N, n-1), \quad r \geq 3.
\]

Therefore, in order to clarify Theorem 1.1 it suffices to show that \( \text{ex}(N, \mathcal{F}_n^{(3)}) \leq t_3(N, n-1) \) and \( T_3(N, n-1) \) is the only \( \mathcal{F}_n^{(3)} \)-free hypergraph with \( N \) vertices and \( t_3(N, n-1) \) edges. Here, we give a proof by induction on the number of vertices. More precisely, we prove Theorem 1.1 in three steps. First, we show that Theorem 1.1 holds for \( N = n \) (see Theorem 2.2). Then, in Theorem 2.4, we demonstrate that it is true for \( n \leq N \leq 2n - 2 \). Finally, using Theorem 2.2 and Theorem 2.4 we show that the desired holds for all \( N \geq n \) (Section 3).

**Conventions and Notations:** For an \( r \)-uniform hypergraph \( H = (V, E) \), the complement hypergraph of \( H \), denoted by \( H^c \), is the hypergraph on \( V \) so that \( E(H^c) = \binom{V}{r} \setminus E \). Also we say that \( X \subseteq V \) is an independent set of \( H \) if for any pair \( v, v' \in X \), there is no edges in \( E \) containing both of \( v \) and \( v' \). For \( U \subseteq V \) we denote by \( H[U] \) the subgraph of \( H \) induced by the edges of \( U \). For \( U, W \subseteq V \), The hypergraph \( H[U, W] \) is the subgraph of \( H \) induced by the edges of \( H \) intersecting both \( U \) and \( W \). For a vertex \( v \in V \), the degree of \( v \) in \( H \), denoted by \( d_H(v) \), is the number of edges in \( H \) containing \( v \). Also \( H - v \) is the subhypergraph of \( H \) obtained by deleting of \( v \) and all the edges containing it.

## 2 Preliminaries

In this section, we present some results that will be used in the follow up section. Let \( A = \{A_1, A_2, ..., A_n\} \) be a family of subsets of a set \( X \). A system of distinct representatives, or SDR, for the family \( A \), is a set \( \{a_1, a_2, ..., a_n\} \) of elements of \( X \) satisfying two following conditions:

- \( a_i \in A_i \), \( i = 1, ..., n \),
- \( a_i \neq a_j \), \( i \neq j \).
Lemma 2.1 Let \( U = \{u_1, u_2, ..., u_m\} \), \( m \geq 5 \) and \( x \notin U \). Also, let \( A = \{A_1, A_2, ..., A_m\} \) be a family of sets so that \( |A_1| \leq |A_2| \leq ...|A_m| \) and \( A_i \subseteq \{B : B = \{x, u_i, u_k\}, k \neq i\} \) for \( 1 \leq i \leq m \). If \( A \) has no SDR, then

\[
| \bigcup_{i=1}^{m} A_i | \leq \binom{m-1}{2}
\]

and equality holds if and only if \( A_1 = \emptyset \) and

\[
A_i = \{B : B = \{x, u_i, u_k\}, k \neq 1, i\} \quad 2 \leq i \leq m.
\]

Proof. Since \( A \) contains no SDR, using the Hall’s theorem [9], for some \( q \), \( 1 \leq q \leq m \), we have \( | \bigcup_{i=1}^{q} A_i | \leq q - 1 \). So \( | \bigcup_{i=1}^{m} A_i | \leq f(q) \), where \( f(k) = k - 1 + \binom{m-k}{2} \), for \( 1 \leq k \leq m \). On the other hand, one can easily see that \( f(1) > f(k) \), for \( 2 \leq k \leq m \). Therefore

\[
| \bigcup_{i=1}^{m} A_i | \leq f(1) = \binom{m-1}{2}
\]

and the equality holds if and only if \( A_1 = \emptyset \) and

\[
A_i = \{B : B = \{x, u_i, u_k\}, k \neq 1, i\} \quad 2 \leq i \leq m.
\]

\[\blacksquare\]

In order to state our main results we need some definitions. Let \( \mathcal{H} = (V, E) \) be an \( r \)-uniform hypergraph, where \( V = \{v_1, v_2, ..., v_n\} \) and \( E = \{e_1, e_2, ..., e_m\} \). We denote by \( B(\mathcal{H}) \), the bipartite graph with parts \( X \) and \( Y \) so that \( X = \{v_i v_k : i < k \text{ and } v_i, v_k \in V(\mathcal{H})\}, Y = E(\mathcal{H}) \) and \( v_i v_k \) is adjacent to \( e_h \) if and only if \( \{v_i, v_k\} \subseteq e_h \), for every \( v_i v_k \in X \) and \( e_h \in Y \). For every \( v_i v_k \in X \), \( d_{B(\mathcal{H})}(v_i v_k) \) is the number of edges in \( B(\mathcal{H}) \) containing \( v_i v_k \). A matching of \( X \) in \( B(\mathcal{H}) \) is matching that saturates all vertices of \( X \). Note that, every matching of \( X \) in \( B(\mathcal{H}) \) is equivalent to a complete \( r \)-uniform Berge-hypergraph with core sequence \( v_1, v_2, ..., v_n \).

Now, we demonstrate that Theorem 1.1 holds for \( N = n \).

Theorem 2.2 Let \( n \geq 13 \) be an integer. The hypergraph \( T_3(n, n-1) \) is the only \( \mathcal{F}_n^{(3)} \)-free hypergraph with \( n \) vertices and \( ex(n, \mathcal{F}_n^{(3)}) \) edges.

Proof. Assume that \( \mathcal{H} \) is an \( \mathcal{F}_n^{(3)} \)-free hypergraph with \( n \) vertices and \( ex(n, \mathcal{F}_n^{(3)}) \) edges. Let \( V(\mathcal{H}) = \{v_1, v_2, ..., v_n\} \). First, suppose that there is a vertex \( v \in V(\mathcal{H}) \), say \( v_n \), so that \( d_{\mathcal{H}}(v_n) \leq \binom{n-2}{2} \). Therefore

\[
e(\mathcal{H}) = d_{\mathcal{H}}(v_n) + e(\mathcal{H} - v_n) \leq \binom{n-2}{2} + \binom{n-1}{3} = t_3(n, n-1).
\]

So by (1) and (2), we have

\[
ex(n, \mathcal{F}_n^{(3)}) = t_3(n, n-1).
\]
Therefore \( d_H(v_n) = \binom{n-2}{2} \) and \( e(H - v_n) = \binom{n-1}{3} \). So \( H - v_n \cong K_n^{(3)} \) and clearly there is a copy of \( K_n^{(3)} \) with the core sequence \( v_1, v_2, \ldots, v_{n-1} \) in \( H - v_n \). Set \( x = v_n, U = \{v_1, v_2, \ldots, v_{n-1}\} \) and \( A = \{A_1, A_2, \ldots, A_{n-1}\} \), where

\[
A_i = \{B : B \in E(H), \{x, v_i\} \subseteq B\} \quad 1 \leq i \leq n - 1.
\]

Note that \( d_H(v_n) = |\bigcup_{i=1}^{n-1} A_i| = \binom{n-2}{2} \). Since \( H \) is \( F_n^{(3)} \)-free and there is a copy of \( K_n^{(3)} \) in \( H - v_n \), \( A \) has no SDR. Now, using Lemma 2.1 we have \( H \cong T_k(n, n-1) \).

Now suppose that for every vertex \( v \in V(H) \), \( d_H(v) \geq \binom{n-2}{2} + 1 \). Set \( G = B(H) \). So we may assume that \( G = [X, Y] \), where

\[
X = \{u_{ik} = v_i v_k : i < k \quad \text{and} \quad v_i, v_k \in V(H)\}
\]

and \( Y = E(H) \). Since, by (1), \( |Y| \geq \binom{n-1}{3} + \binom{n-2}{2} \), we have \( |X| \leq |Y| \). Let \( X = X_1 \cup X_2 \), where \( X_1 = \{u \in X : d_G(u) \leq 4\} \) and \( X_2 = X \setminus X_1 \). Recall that every matching of \( X \) in \( G \) is equivalent to a \( K_n^{(3)} \) in \( H \). We have two following cases.

**Case 1.** \( X_1 = \emptyset \).

Since for every \( y \in Y \) and \( u \in X \), we have \( d_G(y) = 3 \) and \( d_G(u) \geq 5 \), the Hall’s theorem [9] guarantees the existence of a matching of \( X \), a contradiction.

**Case 2.** \( X_1 \neq \emptyset \).

Let \( X_1 = \{v_i, v'_i, v_i v''_i, \ldots, v_i v'_i\} \). We show that the following claim holds.

**Claim 2.3** The elements of \( X_1 \) are pairwise disjoint.

**Proof of Claim 2.3** Suppose to contrary that for \( 2 \leq s \leq t \), \( \{w v'_i, w v''_i, \ldots, w v'_i\} \subseteq X_1 \). So \( d_H(w) \leq f(s) \), where \( f(k) = 4k + \binom{n-k-1}{2} \) is a function on \( k \), \( 2 \leq k \leq t \leq n - 1 \). Using \( n \geq 13 \), it is straightforward to see that the absolute maximum of \( f(k) \) occurs in point \( k = 2 \). Hence

\[
d_H(w) \leq f(2) = 8 + \binom{n-3}{2}.
\]

Since \( 8 + \binom{n-3}{2} < \binom{n-2}{2} + 1 \) for \( n \geq 13 \), we have \( d_H(w) < \binom{n-2}{2} + 1 \). That is a contradiction to our assumption. \( \Box \)

Since for every vertex \( v \in V(H) \), we have \( d_H(v) \geq \binom{n-2}{2} + 1 \), so for any two vertices \( x, y \in V(H) \), there is at least one edge in \( E(H) \) containing both of \( x \) and \( y \). So \( d_G(v_i v'_i) \geq 1 \) for every \( 1 \leq l \leq t \). On the other hand, by Claim 2.3 the elements of \( X_1 \) are pairwise disjoint. Therefore \( G \) contains a matching \( M_1 \) of \( X_1 \). Suppose that \( G' = [X_2, Y'] \) is the subgraph of \( G \) so that \( Y' \subseteq Y \) is obtained by deleting the vertices of \( M_1 \). Note that for every \( u \in X_2 \) and \( y \in Y' \), we have \( d_{G'}(u) \geq 3 \) and \( d_{G'}(y) \leq 3 \). Therefore the Hall’s theorem [9] implies the existence of a matching \( M_2 \) of \( X_2 \) in \( G' \). This is a contradiction, since \( M_1 \cup M_2 \) is a matching of \( X \) in \( G \). This contradiction completes the proof. \( \Box \)
**Theorem 2.4** Let \( n \geq 13 \) and \( N, n \) be integers so that \( n \leq N \leq 2n - 2 \). Also, let \( \mathcal{H} \) be an \( \mathcal{F}_n^{(3)} \)-free hypergraph with \( N \) vertices and \( \text{ex}(N, \mathcal{F}_n^{(3)}) \) edges. Then \( e(\mathcal{H}) = t_3(N, n - 1) \) and \( \mathcal{H} \cong T_3(N, n - 1) \).

**Proof.** Let \( N = n - 1 + j \), where \( 1 \leq j \leq n - 1 \). We apply induction on \( j \). Using Theorem 2.2 the basic step \( j = 1 \) is true. For the induction step, let \( j > 1 \). Set
\[
d = \binom{n - 2}{2} + (j - 1)(n - 3) + \binom{j - 1}{2}.
\]

First suppose that there is a vertex \( x \in V(\mathcal{H}) \) so that \( d_H(x) \leq d \). So using the induction hypothesis, we have
\[
e(\mathcal{H}) = d_H(x) + e(\mathcal{H} - x) \leq d + t_3(N - 1, n - 1) = t_3(N, n - 1).
\]

Therefore by (1), we conclude that \( e(N, \mathcal{F}_n^{(3)}) = t_3(N, n - 1) \). Hence \( d_H(x) = d \) and \( e(\mathcal{H} - x) = t_3(N - 1, n - 1) \). So, using the induction hypothesis, \( \mathcal{H} - x \cong T_3(N - 1, n - 1) \). Hence we may assume that \( \mathcal{H} - x \) is a complete 3-uniform \((n - 1)\)-partite hypergraph with parts \( V_1, V_2, ..., V_{n-1} \), where \( V_i = \{v_i, x_i\} \) \( 1 \leq i \leq j - 1 \), \( \{v_i\} \) \( j \leq i \leq n - 1 \).

Let \( \mathcal{H}' \) be the induced subgraph of \( \mathcal{H} - x \) on \( \{v_1, v_2, ..., v_{n-1}\} \). According to the construction of \( \mathcal{H} - x \), we have \( \mathcal{H}' \cong K_{n-1}^{(3)} \) and so there is a copy of \( K_{n-1}^{(3)} \) with core sequence \( v_1, v_2, ..., v_{n-1} \) in \( \mathcal{H}' \). Set \( U = \{v_1, v_2, ..., v_{n-1}\} \) and \( A = \{A_1, A_2, ..., A_{n-1}\} \), where for \( 1 \leq i \leq n - 1 \),
\[
A_i = \{e \in E(\mathcal{H}) : e = \{x, v_i, v_k\}, \ k \neq i\}.
\]

For a vertex \( v \in V(\mathcal{H}) \), we denote by \( E_v \) the set of edges of \( \mathcal{H} \) containing \( v \). Clearly we have
\[
d_H(x) = |E_x| = |E_1| + |E_2| + |E_3|, \tag{3}
\]
where
\[
E_i = \{e \in E_x : \ |e \cap \{x_1, x_2, ..., x_{j-1}\}| = i - 1 \}, \quad 1 \leq i \leq 3.
\]

We have the following claim.

**Claim 2.5**

(i) \( |E_1| \leq \binom{n-2}{2} \).

(ii) \( |E_2| \leq (j - 1)(n - 3) \).

(iii) \( |E_3| \leq \binom{j-1}{2} \).

**Proof of Claim 2.5** (i) Clearly \( |E_1| = |\bigcup_{i=1}^{n-1} A_i| \). If \( A \) contains an SDR, then \( x, v_1, v_2, ..., v_{n-1} \) is the core sequence of a copy of \( K_{n-1}^{(3)} \) in \( \mathcal{H} \), a contradiction. So, using Lemma 2.1
\[
|E_1| = |\bigcup_{i=1}^{n-1} A_i| \leq \binom{n-2}{2}.
\]
(ii) For $1 \leq k \leq j - 1$, set

$$B_k = \{ e \in E_2 : \{ x, x_k \} \subseteq e \}.$$  

We demonstrate that for $1 \leq k \leq j - 1$, $|B_k| \leq n - 3$ and so

$$|E_2| = \left| \bigcup_{k=1}^{j-1} B_k \right| \leq (j - 1)(n - 3).$$

Because of the similarity, it suffices to show that $|B_1| \leq n - 3$. Suppose not. So $|B_1| \geq n - 2$. On the other hand, the construction of $H - x$ and the fact that $F^{(3)}_n \not\subseteq H$ imply that every edge in $E_x$ contains at most one vertex of each $V_i$, for $1 \leq i \leq n - 1$. Hence $|B_1| = n - 2$ and

$$B_1 = \{ \{ x, x_1, v_2 \}, \{ x, x_1, v_3 \}, \ldots, \{ x, x_1, v_{n-1} \} \}.$$  

In this case, there is no edge in $E(H) \setminus B_1$ containing both of $x$ and $v_i$, for $2 \leq i \leq n - 1$. To see it, suppose that $f = \{ x, v_2, u \} \in E(H) \setminus B_1$. Let $H''$ be the induced subgraph of $H - x$ on $\{ x_1, v_2, \ldots, v_{n-1} \}$. By the construction of $H - x$, we have $H'' \cong K^{3}_{n-1}$ and so $H''$ contains a $K^{(3)}_{n-1}$, say $K'$. Hence $x, x_1, v_2, \ldots, v_{n-1}$ represents the core sequence of a $K^{(3)}_n$ in $H$ with the following edge assignments. Set $e_{xx_1} = \{ x, x_1, v_2 \}, e_{xv_2} = f, e_{xv_1} = \{ x, x_1, v_1 \}$ for $3 \leq i \leq n - 1$ and other edges are selected from $E(K')$. That is a contradiction to our assumption. Therefore the set of edges in $H$ containing $x$ and $v_1$ is a subset of the following set:

$$S = \{ \{ x, v_1, x_2 \}, \{ x, v_1, x_3 \}, \ldots, \{ x, v_1, x_{j-1} \} \}.$$  

Hence

$$d_H(x) \leq |B_1| + |S| + \frac{(j - 1)}{2} = (n - 2) + (j - 2) + \frac{(j - 1)}{2} < d.$$  

This contradiction demonstrates that $|B_1| \leq n - 3$ and so $|E_2| \leq (j - 1)(n - 3)$.

(iii) This case is trivial. \( \square \)

Since $d_H(x) = d$, using (3) and Claim 2.5, we have

$$|E_1| = \binom{n - 2}{2}, \quad |E_2| = (j - 1)(n - 3), \quad |E_3| = \frac{(j - 1)}{2}.$$  

(4)

Since $|E_1| = \binom{n - 2}{2}$, using the proof of part (i) of Claim 2.5 and Lemma 2.1 for some $1 \leq i' \leq n - 1$, $A_{i'} = \emptyset$ and

$$A_i = \{ e \in E(H) : e = \{ x, v_i, v_l \}, \ l \neq i, i' \}, \quad 1 \leq i \leq n - 1 \quad \text{and} \quad i \neq i'.$$

If $j \leq i' \leq n - 1$, using (4), we have $H \cong T_3(N, n - 1)$. Hence we may assume that for some $1 \leq i' \leq j - 1$, say $i' = 1$, $A_1 = \emptyset$. By considering the sets $E_1$ and $E_2$ and using (4), it can be shown that $H[x, x_1, v_2, \ldots, v_{n-1}] \cong K^{3}_{n}$ and so it contains a copy of $K^{(3)}_{n}$. This contradiction completes the proof of the theorem.

Now we may assume that for every vertex $x \in V(H)$, $d_H(x) \geq d + 1$. Set $G = B(H)$. So we may assume that $G = [X, Y]$, where

$$X = \{ u_{ik} = v_i v_k : i < k \quad \text{and} \quad v_i, v_k \in V(H) \}$$

7
and $Y = E(\mathcal{H})$. Since, by \(1\), $|Y| \geq \sum_{i=0}^{3} \ell^{3-i} \binom{\ell}{i} \left(\frac{n-1-i}{3-i}\right)$, we have $|X| \leq |Y|$. Recall that every matching of $X$ in $G$ is equivalent to a $K_{N}^{(3)}$ in $\mathcal{H}$. Let $X = X_1 \cup X_2 \cup X_3$, where
\[
X_1 = \{u \in X : d_G(u) = 0\}, \\
X_2 = \{u \in X : 1 \leq d_G(u) \leq 4\}, \\
X_3 = \{u \in X : d_G(u) \geq 5\}.
\]
We have one of the following cases:

**Case 1.** $X_1 \cup X_2 = \emptyset$.
In this case, the Hall’s theorem $[9]$ guarantees the existence of a matching of $X$ in $G$. That is a contradiction.

**Case 2.** $X_1 \cup X_2 \neq \emptyset$.
Let $X_1 \cup X_2 = \{v_i, v'_i, v_{i2}, ..., v_{i}v'_{i}\}$. First we show that the following claim holds.

**Claim 2.6** The elements of $X_1 \cup X_2$ are pairwise disjoint.

**Proof of Claim 2.6** Suppose to contrary that for some $2 \leq s \leq t$, $\{w_{v'_i}, w_{v_{i2}}, ..., w_{v_{i}}\} \subseteq X_1 \cup X_2$. So we have $d_{\mathcal{H}}(w) \leq f(s)$, where $f(k) = 4k + \left(\frac{n+j-k-2}{2}\right)$ is a function on $k$, $2 \leq k \leq t \leq N-1$. It is straightforward to see that the absolute maximum of $f(k)$ occurs in point $k = 2$. Hence
\[
d_{\mathcal{H}}(w) \leq f(2) = 8 + \left(\frac{n+j-4}{2}\right).
\]
On the other hand, $8 + \left(\frac{n+j-4}{2}\right) < d + 1$ for $n \geq 13$. That contradiction completes the proof of our claim.

Also we have the following claim.

**Claim 2.7** $|X_1| \leq j - 1$.

**Proof of Claim 2.7** Suppose not. Therefore we may assume that $\{v_i, v'_i, v_{i2}, ..., v_{i}v'_{i}\} \subseteq X_1$. Set $L = \{v_{i2}, v_{i3}, ..., v_{ij}\}$. We have $E_{v_{i1}} = F_1 \cup F_2 \cup F_3$, where
\[
F_k = \{e \in E_{v_{i1}} : |e \cap L| = k - 1\}, \quad 1 \leq k \leq 3.
\]
Since, using Claim 2.6 the elements of $X_1$ are pairwise disjoint, the elements of $L$ are distinct. So, an easy computation shows that $|F_1| \leq \left(\frac{n-2}{2}\right)$, $|F_2| \leq (j-1)(n-3)$ and $|F_3| \leq \left(\frac{j-1}{2}\right)$. Therefore
\[
d_{\mathcal{H}}(v_{i1}) = |E_{v_{i1}}| \leq \left(\frac{n-2}{2}\right) + (j-1)(n-3) + \left(\frac{j-1}{2}\right) = d.
\]
This contradiction completes the proof of this claim.

Using the definition of $X_2$, for every $u_{ik} = v_{i}v_{k} \in X_2$, we have $d_G(u_{ik}) \geq 1$. On the other hand, by Claim 2.6 the elements of $X_2$ are pairwise disjoint. Therefore $G$ contains a matching
M_1 of X_2 in G. Suppose that G' = [X_3, Y'] is the induced subgraph of G so that Y' ⊆ Y is obtained by deleting the vertices of M_1. Note that for every u ∈ X_3 and y ∈ Y', we have d_{G'}(u) ≥ 3 and d_{G'}(y) ≤ 3. So the Hall’s theorem [9] guarantees the existence of a matching M_2 of X_3 in G'. Now, using Claim 2.7, we may suppose that X_1 = \{v_{i_1}, v_{i_1'}, v_{i_2}, ..., v_{i_t} \}, where t ≤ j - 1. Set V' = V(H) \ {v_{i_1'}, v_{i_2'}, ..., v_{i_t}}. Clearly |V'| ≥ n and M_1 ∪ M_2 induces a matching of X_2 ∪ X_3 in G. As every matching of X_2 ∪ X_3 in G is equivalent to a K^{(3)}_{|V'|} in H[V'], we have a copy of K^{(3)}_n in H. This is a contradiction to our assumption.

\[\Box\]

3 \ proof of Theorem 1.1

Let H be an \mathcal{F}^{(3)}_n-free hypergraph with N vertices and ex(N, \mathcal{F}^{(3)}_n) edges. Also let N = \ell(n-1) + j, where \ell ≥ 1 and 1 ≤ j ≤ n - 1. We use induction on \ell to show that ex(N, \mathcal{F}^{(3)}_n) = t_3(N, n-1).

Using Theorem 2.4 the basic step \ell = 1 is true. Now suppose that \ell > 1. Since at least one K^{(3)}_n is made by adding one edge to H, we deduce that H contains a K^{(3)}_{n-1}. Let K be such a K^{(3)}_{n-1} in H with the core sequence v_1, v_2, ..., v_{n-1} so that e(H[v_1, v_2, ..., v_{n-1}]) ∩ e(K) is maximum.

Let H_1 = H[V_1], H_2 = H[V_2] and H_3 = H[V_1 \cup V_2], where V_1 = V(K) = \{v_1, v_2, ..., v_{n-1} \} and V_2 = V(H) \ V_1. Also let N' = |V_2| = (\ell - 1)(n - 1) + j, where \ell > 1 and 1 ≤ j ≤ n - 1. Set

\[\begin{align*}
H_3 = &\{ e \in E(H_3) : |e \cap V_1| = 1 \text{ and } |e \cap V_2| = 2\}, \\
H_3^y = &\{ e \in E(H_3) : |e \cap V_1| = 2 \text{ and } |e \cap V_2| = 1\}.
\end{align*}\]

Note that E(H_3) = H_3^y \cup H_3^y. So

\[ e(H) = e(H_1) + e(H_2) + |H_3^y| + |H_3^y|. \tag{5} \]

By the induction hypothesis, we have

\[ e(H_2) ≤ t_3(N', n - 1) = \sum_{i=0}^{3}(\ell - 1)^{3-i}\binom{j}{i}\binom{n-1-i}{3-i}. \tag{6} \]

Moreover,

\[ |H_3^y| ≤ t_2(N', n - 1). \tag{7} \]

To see that, let G be a graph on V_2 so that the vertices u and v of V_2 are adjacent in G if and only if there exists the edge \{x, u, v\} ∈ H_3^y, for some x ∈ V_1. If there is a K_n in G, then we can find a K^{(3)}_n in H, a contradiction. Therefore, by Turán’s theorem [14], we have |H_3^y| ≤ t_2(N', n - 1).

Now we show that e(H_1) + |H_3^y| ≤ \binom{n-1}{3} + N'(n-2)^2. For this purpose, set

\[ B_1^y = \{ e \in H_3^y : e \in E(K) \} \]

and \[ B_2^y = H_3^y \setminus B_1^y. \] Clearly, we have

\[ |B_2^y| ≤ N'(n-2)^2. \tag{8} \]
To see that, choose an arbitrary vertex \( u \in V_2 \). Set \( x = u \) and \( U = V_1 = \{v_1, v_2, \ldots, v_{n-1}\} \) and \( A_u = \{A^1_u, A^2_u, \ldots, A^n_u\} \), where

\[
A^i_u = \{ e \in B^i_u : \{u, v_i\} \subset e \}.
\]

If \( A_u \) contains an SDR, then \( u, v_1, v_2, \ldots, v_{n-1} \) is the core sequence of a copy of \( K^{(3)}_n \) in \( H \), a contradiction. So using Lemma 2.1 we have \( |U| = \binom{n-1}{2} \). Since \( u \) is chosen as an arbitrary vertex of \( V_2 \), thus \( |B^3_u| \leq \binom{n-2}{2} \). Now we demonstrate that

\[
e(\mathcal{H}_1) + |B^3| \leq \binom{n-1}{3} + \binom{n-2}{2}.
\]

(9)

To see this, suppose that \( |B^3_u| = t \). If \( t \leq e(\mathcal{H}_1) \), then we are done. So we may assume that \( e(\mathcal{H}_1) \leq t-1 \). On the other hand, clearly \( K^{(3)}_{n-1} \) contains a copy of \( K^{(3)}_{n-1} \). Therefore, by the maximality of \( K \), at most \( t-1 \) edges of \( K \) are not in \( E(\mathcal{H}_1) \). This is a contradiction to the assumption that \( |B^3_u| = t \).

Therefore by (8) and (9), we have

\[
e(\mathcal{H}_1) + |B^3| \leq \binom{n-1}{3} + \binom{n-2}{2}.
\]

(10)

Now set

\[
B = \binom{n-1}{3} + \binom{n-2}{2} + t_3(N', n-1) + t_2(N', n-1).
\]

Hence by (5), (6), (7) and (10), we have

\[
|E(\mathcal{H})| \leq B = \left( \binom{n-1}{3} + ((\ell - 1)(n-1) + j) \binom{n-2}{2} + (\ell - 1)^3 \binom{n-1}{3} + j(\ell - 1)^2 \binom{n-2}{2} \right)
\]

\[
+ (\ell - 1)(n-3) \binom{j}{2} + \binom{j}{3} + (\ell - 1)^2 \binom{n-1}{2} + j(\ell - 1)(n-2) + \binom{j}{2}.
\]

To demonstrate that \( ex(N, k^{(3)}_n) \leq t_3(N, n-1) \), it suffices to show that

\[
B \leq t_3(N, n-1) = \ell^3 \binom{n-1}{3} + j\ell^2 \binom{n-2}{2} + \ell(n-3) \binom{j}{2} + \binom{j}{3}.
\]

By simplifying the above inequality, it suffices to show that

\[
3\ell \binom{n-1}{3} + (\ell - 1)^2 \binom{n-1}{2} + j\ell(\ell - 1)(n-2)
\]

\[
+ \binom{j}{2} + (\ell - 1)(n-1) \binom{n-2}{2} + 2j \binom{n-2}{2}
\]

\[
\leq 3\ell^2 \binom{n-1}{3} + 2j\ell \binom{n-2}{2} + \binom{j}{2}(n-3).
\]

But the above inequality is certainly true since \( n \geq 13, \ell > 1 \) and \( j \geq 1 \) imply
• (\(\binom{\ell}{2}\)) \leq (\(\binom{\ell}{2}\)(n - 3)).

• \(j(\ell - 1)(n - 2) + 2j(\binom{n-2}{2}) \leq 2j\ell(\binom{n-2}{2}).

• \(3\ell(\binom{n-3}{3}) + (\ell - 1)^2(\binom{n-2}{2}) + (\ell - 1)(n-1)(\binom{n-2}{2}) \leq 3\ell^2(\binom{n-3}{3}).

So

\[ \text{ex}(N, F_n^{(3)}) \leq t_3(N, n - 1) \]

and the equality follows by inspection of (11). Therefore,

(i) \(e(H_2) = t_3(N', n - 1).\)

(ii) \(|H_3^\delta| = t_2(N', n - 1).\)

(iii) \(|B_2^\eta| = N'(n - 2).\)

(iv) \(e(H_1) + |B_2^\eta| = (n - 1).\)

In the sequel, we demonstrate that \(H \cong T_3(N, n - 1).\) Since \(e(H_2) = t_3(N', n - 1),\) the induction hypothesis implies that \(H_2 \cong T_3(N', n - 1).\) Therefore \(H_2\) is a complete 3-uniform \((n - 1)\)-partite hypergraph on \(N'\) vertices whose partition sets differ in size by at most 1. Assume that \(U_1, U_2, ..., U_{n-1}\) are the partition sets of \(H_2.\) Recall that \(U = V_1 = \{v_1, v_2, ..., v_{n-1}\}\) and for every \(u \in V_2, A_u = \{A_u^1, A_u^2, ..., A_u^v\},\) where \(A_u^i = \{e \in B_2^\eta : \{u, v_i\} \subset e\}\) and \(|\bigcup_{i=1}^{n-1} A_u^i| \leq (n - 2).\) Since \(|B_2^\eta| = N'(n - 2),\)

\[ |\bigcup_{i=1}^{n-1} A_u^i| = \binom{n - 2}{2}, \quad \forall u \in V_2.\]

So using Lemma 2.1, there exists \(1 \leq q_u \leq n - 1,\) so that \(A_{q_u}^u = \emptyset\) and for every \(1 \leq i \leq n - 1\) and \(i \neq q_u,\) we have

\[ A_i^u = \{e \in B_2^\eta : \{u, v_i\} \subset e\} = \{\{u, v_i, v_k\} : k \neq i, q_u\}.\]

On the other words,

\[ \bigcup_{i \neq q_u} A_i^u = \{\{u, v_l, v_k\} : v_l, v_k \in V_1, l, k \neq q_u\}.\]

So we can partition the vertices of \(V_2\) into \(n - 1\) parts \(U_1', U_2', ..., U_{n-1}',\) so that for every \(x \in U_1', A_1^x = \emptyset\) and

\[ \bigcup_{i \neq m} A_i^x = \{\{x, v_l, v_k\} : v_l, v_k \in V_1, l, k \neq m\}.\]

Now we show that for \(1 \leq i \leq n - 1, U_i'\) is an independent set in \(H.\) Suppose not. By symmetry we may assume that for two vertices \(x, y \in U_1',\) the edge \(\{x, y, z\} \in E(H).\) It can be shown that \(x, y, v_2, v_3, ..., v_{n-1}\) represents the core sequence of a copy of \(K_n^{(3)}\) in \(H\) with the following edge assignments. Set \(e_{xy} = \{x, y, z\}, e_{xv_i} \in A_i^x\) for \(2 \leq i \leq n - 1, e_{yv_j} \in A_j^y\) for \(2 \leq i \leq n - 1\) and \(e_{v_i v_j} \in E(K)\) for \(2 \leq i, i' \leq n - 1.\) Hence \(U_i'\)'s, \(1 \leq i \leq n - 1,\) are independent sets in \(H.\)
Therefore \( \{U_1, U_2, \ldots, U_{n-1}\} = \{U'_1, U'_2, \ldots, U'_{n-1}\} \). With no loss of generality, we may suppose that
\[
U_i = U'_i \quad \text{for} \quad 1 \leq i \leq n-1.
\]
Now we demonstrate that for \( 1 \leq i \leq n-1 \), \( U_i \cup \{v_i\} \) is an independent set in \( \mathcal{H} \). Suppose to the contrary that for some \( 1 \leq h \leq n-1 \), \( U_h \cup \{v_h\} \) is not independent set. So for some \( u_h \in U_h \), \( f = \{u_h, v_h, w\} \in E(\mathcal{H}) \). Since \( U_h \) is an independent set in \( \mathcal{H} \), \( w \notin U_h \). Choose the vertices \( x_1, x_2, \ldots, x_{n-1} \) so that \( x_h = u_h \) and
\[
x_i \in U_i \quad \text{for} \quad i \neq h.
\]
Since \( \mathcal{H}[\{x_1, x_2, \ldots, x_{n-1}\}] \cong K_{n-1}^3 \), \( x_1, x_2, \ldots, x_{n-1} \) is the core sequence of a \( K_{n-1}^{(3)} \), say \( K' \), in \( \mathcal{H} \). Thus \( x_1, x_2, \ldots, x_{n-1}, v_h \) represents the core sequence of a \( K_{n-1}^{(3)} \) in \( \mathcal{H} \) with the following edge assignments. Set \( e_{x_h v_h} = f, e_{v_h x_i} \in A_h^{x_i} \) for \( i \neq h \) and \( e_{x_i x_i'} \in E(K') \) for \( i, i' \neq h \).
Therefore for \( 1 \leq i \leq n-1 \), \( W_i = U_i \cup \{v_i\} \) is an independent set in \( \mathcal{H} \). So \( \mathcal{H} \) is an \( n-1 \)-partite hypergraph with parts \( W_1, W_2, \ldots, W_{n-1} \) whose partition sets differ in size by at most 1. Since \( e(\mathcal{H}) = t_3(N, n-1) \), we deduce that \( \mathcal{H} \cong T_3(N, n-1) \).

\[\square\]

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