AXISYMMETRIC ROTATING FLUID EQUATIONS

OLFA BEJAOUI

Abstract. We investigate the equations of anisotropic axisymmetric incompressible viscous fluids in the exterior of a cylinder of $\mathbb{R}^3$, rotating around an inhomogeneous vector $B(t,r)$. We prove uniform local existence with respect to the Rossby number in suitable anisotropic Sobolev spaces. We also obtain the propagation of the isotropic Sobolev regularity. This extends the results of [25].

1. Introduction

The motion of incompressible rotating fluids in a domain $\Omega$ of $\mathbb{R}^3$ is described by the following system of equations

\[
(S^\varepsilon)
\begin{cases}
\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \nu_h \Delta_h u^\varepsilon - \nu_v \partial_{x_3}^2 u^\varepsilon + \frac{1}{\varepsilon} (u^\varepsilon \times B) + \nabla p^\varepsilon = 0 \text{ in } \Omega, \\
\text{div } u^\varepsilon = 0 \text{ in } \Omega, \\
 u^\varepsilon = 0 \text{ on } \partial \Omega, \\
 u^\varepsilon(0,x) = u_0(x),
\end{cases}
\]

where $u^\varepsilon$ is the velocity field and $p^\varepsilon$ is the pressure. The constants $\nu_h > 0$ and $\nu_v \geq 0$ represent respectively the horizontal and vertical viscosities. The symbol $\Delta_h$ stands for the horizontal Laplacian and the term $\frac{1}{\varepsilon} (u^\varepsilon \times B)$ represents the Coriolis force, where $B$ is the rotation vector and $\varepsilon$ is a small parameter. We assume that $B$ is a smooth function with bounded derivatives, depending on time $t$ and horizontal variables $x_h$ (that is $x = (x_h, x_3)$). Generally, it is a vector field directed along the $x_3$ coordinate. Additional assumptions on $B$ will be made later.

Notice that if $B = 0$ in the system $(S^\varepsilon)$, then we get the classical incompressible Navier-Stokes equations. It is well known (see [24]) that if $u_0$ is only in $L^2(\mathbb{R}^3)$ and $\nu_v > 0$, then a global weak solution exists. The uniqueness of such solution is an outstanding open problem. Concerning strong solutions, the pioneer work goes back to Fujita-Kato [12] where local existence and uniqueness were obtained in $H^{\frac{1}{2}}(\mathbb{R}^3)$. Moreover, if the initial data is small enough then the solution is global in time. We also refer to [19] for well-posedness in thin domains and in the anisotropic Sobolev spaces $H^{0,s}$. We recall that $H^{0,s}$ is the space of functions which are $L^2$ in the horizontal variables and $H^s$ in the vertical one.

Let us then consider the case $\nu_v = 0$. The anisotropic Navier-Stokes system with vanishing vertical viscosity was studied for the first time in [9], where local existence for large data and global existence for small data were obtained in the anisotropic Sobolev spaces...
$H^{0,s}(\mathbb{R}^3), s > \frac{1}{2}$. Note that in [9] the uniqueness was proved only for $s > \frac{3}{2}$. Later on, D. Iftimie ([21]) filled the gap between existence and uniqueness by proving uniqueness for $s > \frac{1}{2}$. Recently, M. Paicu obtains the uniqueness in the critical Besov space $B^{0,\frac{1}{2}}(\mathbb{R}^3)$ (see [27]).

Let us now recall some well-known facts about the constant case $B = e_3 := (0,0,1)$. For a physical motivation we refer the reader to the book of J. Pedlosky [30] as well as to [8] [15]. As the singular perturbation is a linear skew-symmetric operator, weak solutions can be constructed by the approximate scheme of Friedrichs when $\nu_\ell > 0$: approximate solutions are obtained by a standard truncation in high frequencies. In [9], J.-Y. Chemin et al. obtained local existence in the anisotropic Sobolev spaces $H^{0,1/2+\epsilon}(\mathbb{R}^3)$ and the global existence for data which are small compared to the horizontal viscosity. They also proved the global existence of the solution for anisotropic rotating fluids. In [28], global existence of the solution for rotating fluids with vanishing vertical viscosity was shown in the periodic case. Other related results can be found in [9] [10] [28].

Let us focus now on the variable case $B = B(t,x_h)$. As for the constant case, the classical proofs of existence of weak solutions for the Navier-Stokes equations can be extended to $(S^\varepsilon)$ when $\nu_\ell > 0$. The asymptotic of those solutions was investigated by I. Gallagher and L. Saint-Raymond in [16]. Using weak compactness arguments, they showed that weak solutions converge to the solution of a heat equation in the region when $B$ is non stationary.

The existence of strong solutions in Sobolev spaces was the main goal of a recent work of M. Majdoub and M. Paicu [27]. They obtained global existence for small initial data, and uniform local existence for large data under the assumption that the field $B$ depends on $t$ and $x_1$ (or $t$ and $x_2$).

In this paper, our main concern is to improve this assumption on the field $B$ in order to get uniform local existence in the general case. To do so, we restrict ourselves to the axisymmetric case. This means that we assume that the velocity $u^\varepsilon$ and the pressure $p^\varepsilon$ are axisymmetric (see Definition 2.3 below). We also assume that the domain $\Omega$ is the exterior of some cylinder. We obtain uniform local existence with respect to the Rossby number $\varepsilon$ as well as the propagation of the isotropic Sobolev regularity.

It is expected that similar results can be shown in the case of the whole space $\mathbb{R}^3$. This will be dealt with in a forthcoming work.

The paper unfolds as follows: section 2 contains some notations needed in the statement and the proofs of our results. In the third section, we present the functional spaces used along this paper and we state the main results. Section 4 is devoted to the proof of the uniform local existence and to the propagation of the isotropic Sobolev regularity. A few technical lemmas have been postponed in the final section.

Finally, $C$ will denote a constant that does not depend on $\varepsilon$ but that may change from line to line.

### 2. Definitions and notations

Here we give the definitions of axisymmetric domains and axisymmetric vector fields.

---

1This space is close to $H^{0,\frac{1}{2}}$. However, as far as we know, there is no result in $H^{0,\frac{1}{2}}$. \[\]
Definition 2.1. We denote by \((r, \theta, x_3) \in \mathbb{R}_+^* \times \pi, \pi \times \mathbb{R}\) the cylindrical coordinates in \(\mathbb{R}^3\), where \(r\) and \(\theta\) are defined by
\[
r = \sqrt{x_1^2 + x_2^2}, \quad x_1 = r \cos \theta \quad \text{and} \quad x_2 = r \sin \theta.
\]

Definition 2.2. An open domain \(\Omega\) of \(\mathbb{R}^3\) is said to be axisymmetric if for every rotation \(\Gamma\) around the vertical axis \(e_3\), we have \(\Gamma(\Omega) \subset \Omega\).

Definition 2.3. An axisymmetric vector field \(u\), defined on an open axisymmetric domain of \(\mathbb{R}^3\), is a field having the representation
\[
u(r, x_3) = u^r(r, x_3)e_r + u^\theta(r, x_3)e_\theta + u^3(r, x_3)e_3
\]
in the cylindrical coordinate system, where
\[
e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_3 = (0, 0, 1).
\]

Let us then introduce some useful notations. We denote by
\[
\mathcal{V}(\Omega) = \left\{ u \in (C_c^\infty(\Omega))^3, \quad \text{div}u = 0 \right\},
\]
where \(C_c^\infty(\Omega)\) is the space of smooth functions compactly supported in the domain \(\Omega := \Omega_h \times \mathbb{R}\). Here \(\Omega_h\) is given by
\[
\Omega_h := \{ (x_1, x_2) \in \mathbb{R}^2, \quad x_1^2 + x_2^2 > \rho^2 > 0 \},
\]
where \(\rho\) is a fixed positive number. The notation \((.,./)_{\mathcal{E}}\) corresponds to the inner product in the functional space \(\mathcal{E}\) and the symbol \(\partial_i\) stands for the partial derivative in the direction \(x_i\).

For an axisymmetric vector field \(u\) defined on \(\Omega\), we write \(u(x_h, x_3) = \bar{u}(r, x_3)\), and we define its vertical Fourier transform by
\[
\mathcal{F}^\vee(u)(x_h, \xi_3) = \int_{\mathbb{R}} u(x_h, x_3) e^{-i\xi_3 x_3} dx_3.
\]

The operator of localization in vertical frequencies \(S_N^{x_3}\) \((N \in \mathbb{N})\), is defined by
\[
\mathcal{F}^\vee(S_N^{x_3}u)(x_h, .) = \Psi(2^{-N}|.|) \mathcal{F}^\vee(u)(x_h, .),
\]
where \(\Psi\) is a smooth compactly supported function with values in \([0,1]\) such that
\[
\begin{align*}
\Psi(s) &= 1, \quad \text{if} \quad s \in [0,1] \\
\Psi(s) &= 0, \quad \text{if} \quad |s| \geq 2.
\end{align*}
\]

3. Functional spaces and statement of the results

In order to proceed in a more easy way, we give definitions and properties of some functional spaces used along this paper. In the frame of anisotropic Lebesgue spaces, the Hölder inequality reads.

Lemma 3.1. Hölder inequality

Let \(1 \leq p, p', p'', q, q', q'' \leq \infty\) be numbers such that \(\frac{1}{p'} = \frac{1}{p} + \frac{1}{p'} + \frac{1}{q'} = \frac{1}{q} + \frac{1}{q''}\). Then, we have
\[
\|uv\|_{L^p(\mathbb{L}_h^q)} \leq \|u\|_{L^{p'}(\mathbb{L}_h^{q'})} \|v\|_{L^{q''}(\mathbb{L}_h^{q''})}.
\]

\(^2 u^\theta\) is called the swirl component. If \(u^\theta = 0\), we say that \(u\) is an axisymmetric vector field without swirl.
Let us recall the definition of isotropic Sobolev spaces.

**Definition 3.1.** Let \( m \geq 1 \) be an integer. The space \( H_0^m(\Omega) \) is defined as the closure of \( \mathcal{V}(\Omega) \) in the space \( H^m(\Omega) \) for the norm \( \| \cdot \|_{H^m(\Omega)} \).

Considering the anisotropy of the problem, we use spaces of functions that take into account this anisotropy. More precisely, we use anisotropic Sobolev spaces. Such spaces have been introduced by D. Iftimie in [19].

**Definition 3.2.** Let \( s \) be a real number. The norm \( \| \cdot \|_{H^{0,s}(\Omega)} \) is defined by

\[
\|u\|_{H^{0,s}(\Omega)}^2 = \int_{\Omega_h} \|u(x_h,.)\|_{H^s}^2 dx_h,
\]

where

\[
\|u(x_h,.)\|_{H^s}^2 = \int_\mathbb{R} (1 + \xi_3^2)^s |\mathcal{F}^\gamma(u)(x_h,\xi_3)|^2 d\xi_3.
\]

We define the norm \( \| \cdot \|_{H^{1,s}(\Omega)} \) by

\[
\|u\|_{H^{1,s}(\Omega)} = \|\nabla_h u\|_{H^{0,s}(\Omega)}.
\]

Throughout this paper, we consider spaces constructed on the Sobolev space \( H^{0,s}(\Omega) \).

**Definition 3.3.** The space \( H_0^{0,s}(\Omega) \) is the closure of \( \mathcal{V}(\Omega) \) for the \( \| \cdot \|_{H^{0,s}(\Omega)} \) norm. The space \( H_0^{1,s}(\Omega) \) is the closure of \( \mathcal{V}(\Omega) \) for the \( \| \cdot \|_{H^{1,s}(\Omega)} \) norm.

Let us stress that in all what follows, we consider \( \Omega = \Omega_h \times \mathbb{R} \) and we suppose that \( B = B(t,r) \) is a \( C^\infty \) vector field defined on \( \Omega_h \).

Now, we are ready to state the main results of this paper.

**Theorem 3.1.** Assume that \( u_0 \in H_0^{0,s}(\Omega) \) is an axisymmetric vector field with \( s > \frac{1}{2} \). Then, there exists a time \( T > 0 \) independent of \( \varepsilon \) and a unique solution \( u_\varepsilon \) to \((S_\varepsilon)\) such that

\[
u_\varepsilon \in C([0,T],H_0^{0,s}(\Omega)) \cap L^2([0,T],H_0^{1,s}(\Omega)).
\]

**Theorem 3.2.** Let \( m \geq 1 \) be an integer and assume that \( u_0 \in H_0^m(\Omega) \) is an axisymmetric vector field. Then, there exists a time \( T > 0 \) independent of \( \varepsilon \) and a unique solution \( u_\varepsilon \) to \((S_\varepsilon)\) such that

\[
u_\varepsilon \in C([0,T],H_0^m(\Omega)) \cap \nabla_h u_\varepsilon \in L^2([0,T],H_0^m(\Omega)).
\]

We mention that Theorem 3.2 is a consequence of Theorem 3.1 and of the following result about the Navier-Stokes equations with vanishing vertical viscosity proved in [25] in the case of the whole space.

**Theorem 3.3.** Let \( m \geq 1 \) be an integer and assume that \( u_0 \) belongs to \( H_0^m(\Omega) \). Then, there exists a positive time \( T \) such that the Navier-Stokes equations with vanishing vertical viscosity admits a unique solution \( u \) satisfying

\[
u \in C([0,T],H_0^m(\Omega)) \cap \nabla_h u \in L^2([0,T],H_0^m(\Omega)).
\]

Let us denote by \( T^* \) the maximal time of existence; if \( T^* \) is finite, then

\[
\lim_{t \to T^*} \int_0^t \|\nabla_h u(\tau)\|_{L^\infty(L_h^2)^3}^2 (1 + \|u(\tau)\|_{L^\infty(L_h^2)^3}) d\tau = +\infty.
\]
The main goal of this section is to prove uniform local existence of strong solutions with respect to the Rossby number \( \varepsilon \). The first step consists in splitting the initial data into a small part and a regular one. Hence, we get two systems: a globally well-posed linear system associated to the regular part of \( u_0 \) with solution \( v^\varepsilon \) and a nonlinear one associated to the small part with solution \( w^\varepsilon := u^\varepsilon - v^\varepsilon \). We have only to show the local uniform existence of \( w^\varepsilon \). As in [25], we use energy estimate and a Gronwall’s lemma. The most delicate term to estimate is \((v^\varepsilon \cdot \nabla v^\varepsilon / w^\varepsilon)_{H^0,s}\), and here comes the importance of considering axisymmetric vector fields and the fact that the domain \( \Omega \) does not contain a neighborhood of zero. The key idea is, then, to use extension operators and Sobolev embeddings.

We now come to the details of the proof of Theorem 3.1.

**Proof of Theorem 3.1.**

First of all, using the operator of localization in vertical frequencies \( S^{x_3}_N \), we decompose \( u_0 \) into two parts. As \( u_0 \) belongs to the space \( \mathcal{H}^{0,s}_0(\Omega) \), we obtain

\[
\lim_{N' \to +\infty} \|(I - S^{x_3}_N)u_0\|_{H^0,s} = 0.
\]

Hence, there exists a positive integer \( N \) such that \( \|(I - S^{x_3}_N)u_0\|_{H^0,s} \leq c \nu h \), where \( c > 0 \) is a small constant. Then, we split the system \((S^\varepsilon)\) into

\[
(S_1^\varepsilon) \quad \begin{cases}
\partial_t v_N^\varepsilon - \nu_h \Delta_h v_N^\varepsilon + \frac{1}{\varepsilon} (v_N^\varepsilon \times B) + \nabla p_N^\varepsilon = 0 \quad \text{in } \Omega, \\
\operatorname{div} v_N^\varepsilon = 0 \quad \text{in } \Omega, \\
v_N^\varepsilon = 0 \quad \text{on } \partial\Omega, \\
v_N^\varepsilon(0, x) = S^{x_3}_N u_0,
\end{cases}
\]

and

\[
(S_2^\varepsilon) \quad \begin{cases}
\partial_t w_N^\varepsilon + (w_N^\varepsilon + v_N^\varepsilon) \cdot \nabla (w_N^\varepsilon + v_N^\varepsilon) - \nu_h \Delta_h w_N^\varepsilon + \frac{1}{\varepsilon} (w_N^\varepsilon \times B) + \nabla p_N^\varepsilon = 0 \quad \text{in } \Omega, \\
\operatorname{div} w_N^\varepsilon = 0 \quad \text{in } \Omega, \\
w_N^\varepsilon = 0 \quad \text{on } \partial\Omega, \\
w_N^\varepsilon(0, x) = (I - S^{x_3}_N) u_0.
\end{cases}
\]

Let us notice that \( S^{x_3}_N u_0 \) belongs to \( \mathcal{H}^{0,s}_0(\Omega) \). As \((S_1^\varepsilon)\) is a linear system with a regular initial data, there exists a unique global in time solution

\[
v_N^\varepsilon \in L^\infty(\mathbb{R}_+, \mathcal{H}^{0,s}_0(\Omega)) \cap L^2(\mathbb{R}_+, \mathcal{H}^{1,s}_0(\Omega)).
\]

Therefore, we need only to prove the uniform local existence of \( w_N^\varepsilon \). In other words, we will prove that \( w_N^\varepsilon \) which is small with respect to \( \nu h \) at \( t = 0 \) remains so for a certain time. Thus, let us define

\[
T_{\varepsilon,N} = \sup \left\{ t \geq 0 \mid \forall 0 \leq t' \leq t, \quad \|w_N^\varepsilon(t')\|_{H^0,s} \leq 2c \nu h \right\}.
\]

As \( w_N^\varepsilon \) is a divergence free vector field, we obtain

\[
(\nabla p_N^\varepsilon / w_N^\varepsilon)_{H^0,s} = -(p_N^\varepsilon / \operatorname{div} w_N^\varepsilon)_{H^0,s} = 0,
\]

and as the vector field \( B \) is independent of \( x_3 \), we get

\[
(w_N^\varepsilon \times B / w_N^\varepsilon)_{H^0,s} = 0.
\]
Hence, computing the $H^{0,s}$ scalar product of $(S_2^\epsilon)$ by $w_N^\epsilon$ leads to
\[
\frac{1}{2} \frac{d}{dt} \|w_N^\epsilon\|_{H^{0,s}}^2 + \nu_h \|\nabla_h w_N^\epsilon\|_{H^{0,s}}^2 \leq \tilde{T}_{N}^{\epsilon,1} + \tilde{T}_{N}^{\epsilon,2} + \tilde{T}_{N}^{\epsilon,3} + \tilde{T}_{N}^{\epsilon,4},
\]
where
\[
\begin{align*}
\tilde{T}_{N}^{\epsilon,1} &= |(w_N^\epsilon, \nabla w_N^\epsilon / w_N^\epsilon)|_{H^{0,s}}, \\
\tilde{T}_{N}^{\epsilon,2} &= |(v_N^\epsilon, \nabla w_N^\epsilon / w_N^\epsilon)|_{H^{0,s}}, \\
\tilde{T}_{N}^{\epsilon,3} &= |(w_N^\epsilon, \nabla v_N^\epsilon / w_N^\epsilon)|_{H^{0,s}}, \\
\tilde{T}_{N}^{\epsilon,4} &= |(v_N^\epsilon, \nabla v_N^\epsilon / w_N^\epsilon)|_{H^{0,s}}.
\end{align*}
\]

Let us now bound these four terms. As in [25], the following lemma is the main ingredient to estimate the first term. We postpone the proof to the fifth section.

**Lemma 4.1.** Let $s > 1/2$ be a real number. For any vector fields $u$ and $v$ belonging to $H_0^{\gamma,s}(\Omega) \cap H_0^{1,s}(\Omega)$, we have
\[
\|(u, \nabla v/v)\|_{H^{0,s}} \leq C \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}} \|v\|_{H^{0,s}} \|\nabla_h v\|_{H^{0,s}}.
\]

In particular, if $u = v$, then
\[
\|(u, \nabla u/u)\|_{H^{0,s}} \leq C \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}}^2.
\]

Thanks to this lemma, we have
\[
\tilde{T}_{N}^{\epsilon,1} \leq C \|w_N^\epsilon\|_{H^{0,s}} \|\nabla_h w_N^\epsilon\|_{H^{0,s}}^2.
\]

Hence, we get on the interval $[0, T_{\epsilon,N}[$
\[
\tilde{T}_{N}^{\epsilon,1} \leq \frac{\nu_h}{10} \|\nabla_h w_N^\epsilon\|_{H^{0,s}}^2.
\]

Applying again Lemma 4.1 for the second term, we obtain
\[
\tilde{T}_{N}^{\epsilon,2} \leq C \|v_N^\epsilon\|_{H^{0,s}}^2 \|\nabla_h v_N^\epsilon\|_{H^{0,s}} \|w_N^\epsilon\|_{H^{0,s}} \|\nabla_h w_N^\epsilon\|_{H^{0,s}}^2
\]
\[
+ C \|\nabla_h v_N^\epsilon\|_{H^{0,s}} \|w_N^\epsilon\|_{H^{0,s}} \|\nabla_h w_N^\epsilon\|_{H^{0,s}}.
\]

Using the convexity inequality $ab \leq \theta a^2 + (1 - \theta)b^2$, we get on the interval $[0, T_{\epsilon,N}[$
\[
\tilde{T}_{N}^{\epsilon,2} \leq C \|w_N^\epsilon\|_{H^{0,s}}^2 \|\nabla_h v_N^\epsilon\|_{H^{0,s}}^2 + \|v_N^\epsilon\|_{H^{0,s}}^2 + \frac{\nu_h}{50} \|\nabla_h w_N^\epsilon\|_{H^{0,s}}^2.
\]

Now we have to estimate the third term $\tilde{T}_{N}^{\epsilon,3}$. First of all, we split it into two parts
\[
\tilde{T}_{N}^{\epsilon,3} = (w_{N,h}^\epsilon, \nabla_h v_N^\epsilon / w_N^\epsilon)_{H^{0,s}} + (w_{N,h}^\epsilon, \nabla_h v_N^\epsilon / w_N^\epsilon)_{H^{0,s}}.
\]

The Cauchy-Schwarz inequality leads to
\[
|(w_{N,h}^\epsilon, \nabla_h v_N^\epsilon / w_N^\epsilon)_{H^{0,s}}| \leq \|w_{N,h}^\epsilon\|_{H^{0,s}} \|\nabla_h v_N^\epsilon\|_{H^{0,s}} \|w_N^\epsilon\|_{H^{0,s}}.
\]

The following lemma will be useful for the estimate of $\|w_{N,h}^\epsilon, \nabla_h v_N^\epsilon\|_{H^{0,s}}$. We refer to the fifth section for the proof.

**Lemma 4.2.** Let $s > 1/2$ be a real number. For any axisymmetric vector fields $u$ and $v$ such that $u \in H_0^{\gamma,s}(\Omega) \cap H_0^{1,s}(\Omega)$ and $v \in H_0^{\gamma,s}(\Omega)$, we have
\[
\|uv\|_{H^{0,s}} \leq C \|v\|_{H^{0,s}} (\|u\|_{H^{0,s}} + \|\nabla_h u\|_{H^{0,s}}).
\]
Hence, we get
\[ \| u^{\varepsilon}_{N,h} \nabla h v^{\varepsilon}_{N} \|_{H^{0,s}} \leq C \| \nabla h v^{\varepsilon}_{N} \|_{H^{0,s}} (\| u^{\varepsilon}_{N} \|_{H^{0,s}} + \| \nabla h u^{\varepsilon}_{N} \|_{H^{0,s}}), \]
and then, we obtain
\[ |(w^{\varepsilon}_{N,h} \cdot \nabla h v^{\varepsilon}_{N}/w^{\varepsilon}_{N})_{H^{0,s}}| \leq C \| \nabla h v^{\varepsilon}_{N} \|_{H^{0,s}} (\| u^{\varepsilon}_{N} \|_{H^{0,s}} + \| \nabla h u^{\varepsilon}_{N} \|_{H^{0,s}}) \]
\[ \leq \frac{\nu h}{100} \| \nabla h u^{\varepsilon}_{N} \|_{H^{0,s}}^{2} + C \| u^{\varepsilon}_{N} \|_{H^{0,s}}^{2} (\| \nabla h v^{\varepsilon}_{N} \|_{H^{0,s}} + \| \nabla h u^{\varepsilon}_{N} \|_{H^{0,s}}^{2}). \]
Using again the Cauchy-Schwarz inequality, we get
\[ |(w^{\varepsilon}_{N,3} \cdot \partial_{3} v^{\varepsilon}_{N}/w^{\varepsilon}_{N})_{H^{0,s}}| \leq \| w^{\varepsilon}_{N,3} \cdot \partial_{3} v^{\varepsilon}_{N} \|_{H^{0,s}} \| u^{\varepsilon}_{N} \|_{H^{0,s}}. \]
Lemma 4.2 implies that
\[ \| w^{\varepsilon}_{N,3} \cdot \partial_{3} v^{\varepsilon}_{N} \|_{H^{0,s}} \leq C \| \partial_{3} v^{\varepsilon}_{N} \|_{H^{0,s}} (\| u^{\varepsilon}_{N} \|_{H^{0,s}} + \| \nabla h u^{\varepsilon}_{N} \|_{H^{0,s}}). \]
To estimate the term \( \| \partial_{3} v^{\varepsilon}_{N} \|_{H^{0,s}} \), it is worth noting that \( S^{x_{3}} S^{x_{3}} v^{\varepsilon}_{N} = v^{\varepsilon}_{N} \). As \( B \) is independent of \( x_{3} \), we have
\[ S^{x_{3}} \left( v^{\varepsilon}_{N} \times B \right) = (S^{x_{3}} v^{\varepsilon}_{N}) \times B. \]
Thus, \( v^{\varepsilon}_{N} \) and \( S^{x_{3}} v^{\varepsilon}_{N} \) satisfy the same equation. Moreover, we notice that \( v^{\varepsilon}_{N/t=0} = S^{x_{3}} v^{\varepsilon}_{N/t=0} = S^{x_{3}} u_{0} \).
By uniqueness we get \( S^{x_{3}} v^{\varepsilon}_{N} = v^{\varepsilon}_{N} \). Consequently, we have
\[ \| \partial_{3} v^{\varepsilon}_{N} \|_{H^{0,s}} \leq C \| v^{\varepsilon}_{N} \|_{H^{0,s}}, \]
and then
\[ |(w^{\varepsilon}_{N,3} \cdot \partial_{3} v^{\varepsilon}_{N}/w^{\varepsilon}_{N})_{H^{0,s}}| \leq \frac{\nu h}{100} \| \nabla h u^{\varepsilon}_{N} \|_{H^{0,s}}^{2} + C \| v^{\varepsilon}_{N} \|_{H^{0,s}}^{2} (\| v^{\varepsilon}_{N} \|_{H^{0,s}}^{2} + \| v^{\varepsilon}_{N} \|_{H^{0,s}}^{2}). \]
So, it turns out that
\[ \tilde{T}_{N}^{x_{3}} \leq C \| w^{\varepsilon}_{N} \|_{H^{0,s}}^{2} (\| v^{\varepsilon}_{N} \|_{H^{0,s}}^{2} + \| v^{\varepsilon}_{N} \|_{H^{0,s}}^{2} + \| \nabla h v^{\varepsilon}_{N} \|_{H^{0,s}} + \| \nabla h v^{\varepsilon}_{N} \|_{H^{0,s}}^{2}). \]
Now, we deal with the fourth term \( \tilde{T}_{N}^{x_{4}} \). As \( v^{\varepsilon}_{N} \) is a divergence free vector field, an integration by parts leads to
\[ \tilde{T}_{N}^{x_{4}} = \left( \text{div}(v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N})/w^{\varepsilon}_{N} \right)_{H^{0,s}} \]
\[ = (v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N}/\nabla h w^{\varepsilon}_{N})_{H^{0,s}} + (\partial_{3} (v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N})/w^{\varepsilon}_{N})_{H^{0,s}}. \]
First, we estimate the term \( (v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N}/\nabla h w^{\varepsilon}_{N})_{H^{0,s}} \). Thanks to the Cauchy-Schwarz inequality, we obtain
\[ (v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N}/\nabla h w^{\varepsilon}_{N})_{H^{0,s}} \leq \| v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N} \|_{H^{0,s}} \| \nabla h w^{\varepsilon}_{N} \|_{H^{0,s}}. \]
The fact that \( S^{x_{3}} S^{x_{3}} v^{\varepsilon}_{N} = v^{\varepsilon}_{N} \) implies that
\[ \| v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N} \|_{H^{0,s}} \leq C \| v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N} \|_{L^{2}(\Omega)} \]
\[ \leq C \| v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N} \|_{L^{2}(L^{2}_{x}(L^{2}_{r}))}. \]
As \( \| v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N} \|_{L^{2}(L^{2}_{x}(L^{2}_{r}))} \) is equivalent to \( \| \tilde{v}^{\varepsilon}_{N} \otimes \tilde{v}^{\varepsilon}_{N} \|_{L^{2}(L^{2}_{x}(L^{2}(rdr)))} \), we get by the Hölder inequality
\[ \| v^{\varepsilon}_{N} \otimes v^{\varepsilon}_{N} \|_{L^{2}(L^{2}_{x}(L^{2}_{r}))} \leq \| v^{\varepsilon}_{N} \|_{L^{2}^{x}(L^{2}(rdr))} \| v^{\varepsilon}_{N} \|_{L^{2}(L^{2}(rdr))}. \]
Since we have \( S^{x_{3}} S^{x_{3}} v^{\varepsilon}_{N} = v^{\varepsilon}_{N} \), Bernstein lemma yields
\[ \| \tilde{v}^{\varepsilon}_{N} \|_{L^{2}^{x}(L^{4}(rdr))} \leq C \| \tilde{v}^{\varepsilon}_{N} \|_{L^{2}(L^{4}(rdr))}. \]
Remark 4.1. We can consider (for example) More general results about extension operators can be found in [5], for instance. Such that embeddings, we get

\[ \psi_{\varepsilon,N} \]

exists an extension operator \( R \) in order to use Sobolev embedding on the whole space \( \mathbb{R} \), we infer

\[ \| \psi_{\varepsilon,N} \|_{L^2(\Omega)} \leq \frac{1}{\rho^2} \| \tilde{\psi}_{\varepsilon,N} \|_{L^2(\Omega)} \]

and

\[ \| \partial_r \psi_{\varepsilon,N} \|_{L^2(\Omega)} \leq \frac{C}{\rho^2} \| \tilde{\psi}_{\varepsilon,N} \|_{L^2(\Omega)} + \frac{C}{\rho^2} \| \partial_r \psi_{\varepsilon,N} \|_{L^2(\Omega)}. \]

Since \( v_{\varepsilon} \) is axisymmetric, then \( \| \nabla_h v_{\varepsilon} \|_{L^2} \) and \( \| \partial_r v_{\varepsilon} \|_{L^2} \) are equivalent. But, as \( v_{\varepsilon} \) is in \( L^2(\Omega_h) \) with \( \nabla_h v_{\varepsilon} \) in \( L^2(\Omega_h) \), then we get \( \psi_{\varepsilon,N} \in H^1(\rho, +\infty) \).

In order to use Sobolev embedding on the whole space \( \mathbb{R} \), we need to extend the function \( \psi_{\varepsilon,N} \). The following extension lemma is needed.

**Lemma 4.3.** There exists an extension operator

\[ P : H^1([\rho, +\infty]) \to H^1(\mathbb{R}) \]

such that

\[ Pu_{[\rho, +\infty]} = u, \]

\[ \| Pu \|_{L^2(\mathbb{R})} \leq C \| u \|_{L^2([\rho, +\infty])}, \]

\[ \| Pu \|_{H^1(\mathbb{R})} \leq C \| u \|_{H^1([\rho, +\infty])}. \]

**Remark 4.1.** We can consider (for example)

\[ Pu(r, x_3) = \begin{cases} u(r, x_3), & \text{if } r > \rho \\ u(2\rho - r, x_3), & \text{if } r \leq \rho. \end{cases} \]

More general results about extension operators can be found in [5] for instance.

Now, we continue the study of the term \( (v_{\varepsilon} \otimes v_{\varepsilon}/\nabla v_{\varepsilon})_{H^0} \). Thanks to Sobolev embeddings, we get

\[ \| P\psi_{x_3} \|_{L^4(\mathbb{R})} \leq C \| P\psi_{x_3} \|_{\dot{H}^{1/2}(\mathbb{R})}. \]

By interpolation, we have

\[ \| P\psi_{x_3} \|_{\dot{H}^{1/2}(\mathbb{R})} \leq \| \partial_r (P\psi_{x_3}) \|_{L^2(\mathbb{R})} \| P\psi_{x_3} \|_{L^2(\mathbb{R})} \]

\[ \leq \| \psi_{x_3} \|_{H^1([\rho, +\infty])} \psi_{x_3} \|_{L^2([\rho, +\infty])} \]

\[ \leq \| \psi_{x_3} \|_{L^2(\mathbb{R})} + \| \psi_{x_3} \|_{L^2(\mathbb{R})} \| \partial_r \psi_{x_3} \|_{L^2(\mathbb{R})}. \]

Hence, we get

\[ \| \tilde{\psi}_{\varepsilon,N} \|_{L^4(\mathbb{R})} \leq \| \tilde{\psi}_{\varepsilon,N} \|_{L^2(\mathbb{R})} + \| \partial_r \tilde{\psi}_{\varepsilon,N} \|_{L^2(\mathbb{R})}. \]

Taking the \( L^2 \) norm, we obtain thanks to the Hölder inequality

\[ \| \tilde{\psi}_{\varepsilon,N} \|_{L^2(\mathbb{R})} \leq \| \partial_r \tilde{\psi}_{\varepsilon,N} \|_{L^2(\mathbb{R})} \| \tilde{\psi}_{\varepsilon,N} \|_{L^2(\mathbb{R})} + \| \tilde{\psi}_{\varepsilon,N} \|_{L^2(\mathbb{R})}. \]
Thanks to the Hölder inequality, we obtain
\[
(v^\varepsilon_N \otimes v^\varepsilon_N / \nabla_h w^\varepsilon_N)_{H^0, s} \leq \frac{\nu_h}{100} \| \nabla_h w^\varepsilon_N \|_{H^0, s}^2 + C \| \nabla_h v^\varepsilon_N \|_{L^2(\Omega)} \| v^\varepsilon_N \|_{L^2(\Omega)}^2 + C \| v^\varepsilon_N \|_{L^2(\Omega)}^4.
\]
As \( v^\varepsilon_N \) is localized in vertical frequencies, the term \( (\partial_3 (v^\varepsilon_N \otimes v^\varepsilon_N) / w^\varepsilon_N)_{H^0, s} \) is estimated as
\[
(\partial_3 (v^\varepsilon_N \otimes v^\varepsilon_N) / w^\varepsilon_N)_{H^0, s} \leq C \| w^\varepsilon_N \|_{H^0, s}^2 + C \| \nabla_h v^\varepsilon_N \|_{L^2(\Omega)} \| v^\varepsilon_N \|_{L^2(\Omega)}^3 + C \| v^\varepsilon_N \|_{L^2(\Omega)}^4.
\]
Considering all the above estimates, we get on the interval \([0, T_{\varepsilon, N}][\]
\[
\frac{1}{2} \frac{d}{dt} \| w^\varepsilon_N(t) \|_{H^0, s}^2 + \nu_h \| \nabla_h w^\varepsilon_N \|_{H^0, s}^2 \leq C \| w^\varepsilon_N(0) \|_{H^0, s}^2 + \int_0^t C (\| v^\varepsilon_N(\tau) \|_{H^0, s}^3 + \| \nabla_h v^\varepsilon_N(\tau) \|_{L^2(\Omega)} + \| v^\varepsilon_N(\tau) \|_{L^2(\Omega)}^4) d\tau
\]
\[
\times \exp \left[ \int_0^t C (1 + \| v^\varepsilon_N(\tau) \|_{H^0, s} + \| \nabla_h v^\varepsilon_N(\tau) \|_{H^0, s} + \| \nabla_h v^\varepsilon_N(\tau) \|_{H^0, s}^2)
\]
\[
+ \| v^\varepsilon_N(\tau) \|_{H^0, s}^2 \| \nabla_h v^\varepsilon_N(\tau) \|_{H^0, s}^2) d\tau \right].
\]

It is of interest to note that \( L^2 \) and \( H^0, s \) energy estimates on \( v^\varepsilon_N \) imply that
\[
\| v^\varepsilon_N(t) \|_{L^2(\Omega)}^2 + 2 \nu_h \int_0^t \| \nabla_h v^\varepsilon_N(\tau) \|_{L^2(\Omega)}^2 d\tau \leq \| u_0 \|_{L^2(\Omega)}^2,
\]
and
\[
\| v^\varepsilon_N(t) \|_{H^0, s}^2 + 2 \nu_h \int_0^t \| \nabla_h v^\varepsilon_N(\tau) \|_{H^0, s}^2 d\tau \leq \| u_0 \|_{H^0, s}^2.
\]
Thanks to the Hölder inequality, we obtain
\[
\int_0^t C (\| v^\varepsilon_N(\tau) \|_{L^2(\Omega)}^3 + \| \nabla_h v^\varepsilon_N(\tau) \|_{L^2(\Omega)} + \| v^\varepsilon_N(\tau) \|_{L^2(\Omega)}^4) d\tau \leq C \| u_0 \|_{L^2(\Omega)}^4 (t + \sqrt{t}).
\]
Finally, we get
\[
\| w^\varepsilon_N(t) \|_{H^0, s}^2 \leq \left[ \| w^\varepsilon_N(0) \|_{H^0, s}^2 + C \| u_0 \|_{L^2(\Omega)}^4 (t + \sqrt{t}) \right] \times \exp (C + C t + C \sqrt{t}),
\]
where \( C > 0 \) depends on \( \| u_0 \|_{H^0, s} \).

Let us consider a positive real number \( T \) such that
\[
\left[ \| w^\varepsilon_N(0) \|_{H^0, s}^2 + C \| u_0 \|_{L^2(\Omega)}^4 (T + \sqrt{T}) \right] \times \exp (C + C T + C \sqrt{T}) \leq \left( \frac{3}{2} C \nu_h \right)^2.
\]
Notice that \( T \) is independent of \( \varepsilon \). Therefore, \( w^\varepsilon \) exists on the time interval \([0, T] \). But as \( v^\varepsilon_N \) is global in time, then \( u^\varepsilon = v^\varepsilon_N + w^\varepsilon_N \) exists on the time interval \([0, T] \) and Theorem 3.1 is proved.

As said before, uniform local existence in isotropic Sobolev space is a consequence of Theorem 3.1. We note that the proof of this fact is contained in [25], but we give it for the convenience of the reader.
Proof of Theorem 3.2
As $u_0$ belongs to the space $H^m_0(\Omega)$, $m \geq 1$, then $u_0$ is in $H^0_{0,m}(\Omega)$. Hence, Theorem 3.1 yields the existence of a unique solution $u^\varepsilon$ for the system $(S^\varepsilon)$ on a uniform time interval $[0,T]$ such that $\|u^\varepsilon(t)\|_{H^{0,s}}$ and $\int_0^t \|\nabla_h u^\varepsilon(\tau)\|_{H^{0,s}}^2 \, d\tau$ are uniformly bounded on the time interval $[0,T]$. Let $T^\varepsilon$ be the maximal time of existence of $u^\varepsilon$ in $H^m_0(\Omega)$. Theorem 3.3 and the inclusion of $H^0_{0,m}(\Omega)$ in $L^\infty(L^2_h)$ imply that $T^\varepsilon > T$. Thus, we get the uniform local in time existence of $u^\varepsilon$ in the space $H^m_0(\Omega)$. □

5. Product laws

Before proving the technical lemmas, let us first recall some results about the anisotropic Littlewood Paley theory.

5.1. Anisotropic Littlewood Paley theory. Anisotropic Sobolev spaces can be characterized using a dyadic decomposition in the vertical frequency space. So, let us first recall some elements of the Littlewood-Paley theory, the details of which can be found in [19] for instance. Let $u$ be a function defined on $\Omega$, we have

$$\mathcal{F}^\vee(\Delta^\vee_q u)(x_h, \cdot) = \varphi\left(\frac{1}{2^q}\right)\mathcal{F}^\vee(\Delta^\vee u)(x_h, \cdot), \quad q \geq 0,$$

$$\mathcal{F}^\vee(\Delta^\vee_{-1} u)(x_h, \cdot) = \chi(\cdot)\mathcal{F}^\vee(u)(x_h, \cdot),$$

$\Delta^\vee_q u = 0, \quad q \leq -2.$

The positive functions $\varphi$ and $\chi$ represent a dyadic partition of unity in $\mathbb{R}$, that is to say they are smooth functions such that

$$\text{supp} \chi \subset B(0, \frac{4}{3}), \quad \text{supp} \varphi \subset C(0, \frac{3}{4}, \frac{8}{3}),$$

and $\forall t \in \mathbb{R}$

$$\chi(t) + \sum_{q \geq 0} \varphi(2^{-q} t) = 1.$$

Let us also define the operator

$$S^\vee_q u = \sum_{q' \leq q-1} \Delta^\vee_{q'} u.$$

Those definitions enable us to characterize anisotropic Sobolev spaces $H^{0,s}(\Omega)$. More precisely, a tempered distribution $u$ belongs to $H^{0,s}(\Omega)$ if and only if

$$\sum_{q} 2^{qs} \|\Delta^\vee_q u\|_{L^2_h(\Omega)}^2 < \infty.$$

Moreover, we have

$$\|u\|_{H^{0,s}(\Omega)}^2 \approx \sum_{q} 2^{qs} \|\Delta^\vee_q u\|_{L^2_h(\Omega)}^2.$$

The dyadic decomposition is also important for studying the product of two distributions thanks to Bony’s decomposition. Let $u$ and $v$ be two distributions. We have

$$u = \sum_{q \in \mathbb{Z}} \Delta^\vee_q u; \quad v = \sum_{q \in \mathbb{Z}} \Delta^\vee_q v.$$
We denote
\[ T_u v = \sum_q S_{q-1}^u \Delta_q^v v \]
\[ R(u, v) = \sum_{i \in \{0, \pm 1\}} \Delta_i^v u \Delta_q^{v, i} v. \]

Thus, we obtain
\[ u.v = T_u v + T_v u + R(u, v). \]

We have
\[ \Delta_q^{v}(uv) = \sum_{|q'| < 4} \Delta_q^{v}(S_{q-1}^uv, \Delta_q^v v) + \sum_{|q'| < 4} \Delta_q^{v}(S_{q-1}^vu, \Delta_q^v u) \]
\[ + \sum_{|q'| > |q| - 4} \Delta_q^{v}(\Delta_i^v u, \Delta_q^{v, i} v). \]

The dyadic decomposition is useful in the sense that the derivatives in vertical variable act in a very special way on functions localized in vertical frequencies in a ball or a ring. More precisely, we have the following lemma the proof of which can be found in [25], [27] for instance.

**Lemma 5.1. Bernstein lemma**

Let \( p, r \) and \( r' \) be numbers such that \( \infty \geq p \geq 1 \) and \( \infty \geq r \geq r' \geq 1 \). Then, there exists a constant \( C > 0 \) such that for any vector field \( u \) defined on \( \Omega \times \mathbb{R} \) with \( \text{supp} \mathcal{F} u \subset \mathbb{R}^2 \times 2^q \mathcal{C} \), where \( \mathcal{C} \) is a dyadic ring, we have
\[ 2^q C^{-k} ||u||_{L_p^r(L_r^q)} \leq ||\partial_x^k u||_{L_p^r(L_r^q)} \leq 2^q C^k ||u||_{L_p^r(L_r^q)}, \]
\[ 2^q C^{-k} ||u||_{L_r^q(L_r^q)} \leq ||\partial_x^k u||_{L_r^q(L_r^q)} \leq 2^q C^k ||u||_{L_r^q(L_r^q)}, \]
\[ ||u||_{L_r^q(L_r^q)} \leq C 2^{q \left( \frac{1}{r'} - \frac{1}{r} \right)} ||u||_{L_r^q(L_r^q)}, \]
\[ ||u||_{L_r^q(L_r^q)} \leq C 2^{q \left( \frac{1}{r'} - \frac{1}{r} \right)} ||u||_{L_r^q(L_r^q)}. \]

**5.2. Proofs of the technical lemmas.** In this part, we denote by \( (b_q)_{q \in \mathbb{Z}} \) and \( (c_q)_{q \in \mathbb{Z}} \) positive sequences such that
\[ \sum_{q \in \mathbb{Z}} b_q \leq 1 \quad \text{and} \quad \sum_{q \in \mathbb{Z}} c_q^2 \leq 1. \]

For the proof of Lemma 4.1, we proceed as in [27] where the critical Besov space \( B^{0, \frac{1}{2}} \) is used.

**Proof of Lemma 4.1.** The proof of Lemma 4.1 relies on basic inequalities.

**Proposition 5.1.**

For any vector field \( u \) in \( H^{0,s}(\Omega) \), we have
\[ \|\Delta_q^v u\|_{L^2} \leq C c_q 2^{-qs} \|u\|_{H^{0,s}}, \]
\[ \|u\|_{L^\infty(\Omega)} \leq C \|u\|_{H^{0,s}}, \quad s > \frac{1}{2}. \]
For any vector field \( u \) in \( \mathcal{H}_0^{0,s}(\Omega) \cap \mathcal{H}_0^{1,s}(\Omega) \), we have

\[
\| \Delta_q u \|_{L^2_h(L_h^4)} \leq C c_q 2^{-qs} \| u \|_{H^{0,s}}^{\frac{1}{2}} \| \nabla_h u \|_{H^{0,s}}^{\frac{1}{2}}.
\]

(3)

\[
\| u \|_{L^\infty(L_h^4)} \leq C \| u \|_{H^{0,s}}^{\frac{1}{2}} \| \nabla_h u \|_{H^{0,s}}^{\frac{1}{2}}, \quad s > \frac{1}{2}.
\]

(4)

Proof of Proposition 5.1.

Thanks to the Bernstein lemma, we get

\[
\| u \|_{L^\infty(L_h^4)} \leq C \sum q 2^{\frac{q}{2}} \| \Delta_q u \|_{L^2}.
\]

As \( s > \frac{1}{2} \), then the Cauchy-Schwarz inequality leads to (2).

To get (3), we just have to prove it for \( u \in C^\infty(\Omega) \). Sobolev embeddings imply that

\[
\| \Delta_q u(u, x) \|_{L_h^4} \leq C \| \Delta_q u(u, x) \|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \leq C \| \Delta_q u(u, x) \|_{L_h^4}^{\frac{1}{2}} \| \nabla_h \Delta_q u(u, x) \|_{L_h^2}^{\frac{1}{2}}.
\]

Taking the \( L^2_h \) norm and using (1), we get (3).

To prove (4), we use the Bernstein lemma to get

\[
\| u \|_{L^\infty(L_h^4)} \leq C \sum q 2^{\frac{q}{2}} \| \Delta_q u \|_{L^2(L_h^4)}.
\]

As \( s > \frac{1}{2} \), we get the result thanks to (3) and to the Cauchy-Schwarz inequality.

Let us go back to the proof of Lemma 4.1.

We have

\[
(u, \nabla v)_{H^{0,s}} = \sum q 2^{2qs} (\Delta_q^u(u, \nabla v))_{L^2}\]

\[
= \sum q 2^{2qs} (F^h_q / \Delta_q v)_{L^2} + \sum q 2^{2qs} (F^\nabla_q / \Delta_q v)_{L^2},
\]

where \( F^h_q = \Delta_q^u(u_h, \nabla_h v) \) and \( F^\nabla_q = \Delta_q^u(u_3, \partial_3 v) \).

For the term \((F^h_q / \Delta_q v)_{L^2}\), we have thanks to the Hölder inequality

\[
| (F^h_q / \Delta_q v)_{L^2} | \leq \| F^h_q \|_{L^2(L_h^4)} \| \Delta_q v \|_{L^2(L_h^4)}^{\frac{1}{2}}.
\]

Proposition 5.1 leads to

\[
\| \Delta_q v \|_{L^2(L_h^4)} \leq C c_q 2^{-qs} \| v \|_{H^{0,s}}^{\frac{1}{2}} \| \nabla_h v \|_{H^{0,s}}^{\frac{1}{2}}.
\]

Bony’s decomposition implies that

\[
F^h_q = T^h_{1,q} + T^h_{2,q} + T^h_{3,q},
\]
where
\[
T_{1,q}^h = \sum_{|q' - q| \leq 4} \Delta_q^v (S_{q'-1}^v u_h, \Delta_{q'}^v \nabla_h v),
\]
\[
T_{2,q}^h = \sum_{|q' - q| \leq 4} \Delta_q^v (S_{q'-1}^v \nabla_h v, \Delta_{q'}^v u_h),
\]
\[
T_{3,q}^h = \sum_{i \in \{0, \pm 1\}} \sum_{q' > q - 4} \Delta_q^v (\Delta_{q'}^v u_h, \Delta_{q'-1}^v \nabla_h v).
\]

The Hölder inequality leads to
\[
\|T_{1,q}^h\|_{L^2(L^4_h)} \leq \sum_{|q' - q| \leq 4} \|S_{q'-1}^v u\|_{L^\infty(L^4_h)} \|\Delta_{q'}^v \nabla_h v\|_{L^2}.
\]

But, we have
\[
\|S_{q'-1}^v u\|_{L^\infty(L^4_h)} \leq \|u\|_{L^\infty(L^4_h)},
\]
then, by Proposition 5.1, we obtain
\[
\|T_{1,q}^h\|_{L^2(L^4_h)} \leq C 2^{q s} \sum_{|q' - q| \leq 4} c_{q'} 2^{(q - q')s} \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}} \|\nabla_h v\|_{H^{0,s}}
\]
\[
\leq C c_q 2^{-q s} \|\nabla_h v\|_{H^{0,s}} \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}}.
\]

As for the term \(T_{2,q}^h\), the Hölder inequality implies that
\[
\|T_{2,q}^h\|_{L^2(L^4_h)} \leq \sum_{|q' - q| \leq 4} \|S_{q'-1}^v \nabla_h v\|_{L^\infty(L^2_h)} \|\Delta_{q'}^v u\|_{L^2(L^4_h)}
\]
\[
\leq \|\nabla_h v\|_{L^\infty(L^2_h)} \sum_{|q' - q| \leq 4} \|\Delta_{q'}^v u\|_{L^2(L^4_h)}.
\]

Thanks to Proposition 5.1, we get
\[
\|T_{2,q}^h\|_{L^2(L^4_h)} \leq C c_q 2^{-q s} \|\nabla_h v\|_{H^{0,s}} \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}}.
\]

For the term \(T_{3,q}^h\), the Bernstein lemma yields
\[
\|T_{3,q}^h\|_{L^2(L^4_h)} \leq C 2^q \sum_{i \in \{0, \pm 1\}} \sum_{q' > q - 4} \|\Delta_q^v u_h, \Delta_{q'-1}^v \nabla_h v\|_{L^1(L^4_h)}.
\]

By the Hölder inequality, we obtain
\[
\|T_{3,q}^h\|_{L^2(L^4_h)} \leq C 2^q \sum_{i \in \{0, \pm 1\}} \sum_{q' > q - 4} \|\Delta_q^v u\|_{L^2(L^4_h)} \|\Delta_{q'-1}^v \nabla_h v\|_{L^2}.
\]

Thanks to Proposition 5.1, we have
\[
\|T_{3,q}^h\|_{L^2(L^4_h)} \leq C c_q 2^{-q s} \|\nabla_h v\|_{H^{0,s}} \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}}.
\]

For the term \((F_q^v / \Delta_q^v v)_{L^2}\), we use the following decomposition
\[
(F_q^v (u_3, \partial_3 v) / \Delta_q^v v)_{L^2} = T_{1,q}^v + T_{2,q}^v + T_{3,q}^v + T_{4,q}^v,
\]
where

\[
T_{1,q}^\nu = \int_{\Omega_h \times \mathbb{R}} S_{q-1,u}^{\nu} \partial_3 \Delta_q^{\nu} v \Delta_q^{\nu} v \, dx,
\]

\[
T_{2,q}^\nu = \sum_{|q-q'| \leq 4} \int_{\Omega_h \times \mathbb{R}} \left[ \Delta_q^{\nu}; S_{q-1,u}^{\nu} \right] \partial_3 \Delta_q^{\nu} v \Delta_q^{\nu} v \, dx,
\]

\[
T_{3,q}^\nu = \sum_{|q-q'| \leq 4} \int_{\Omega_h \times \mathbb{R}} (S_{q-1,u}^{\nu} - S_{q-1,u}^{\nu}) \partial_3 \Delta_q^{\nu} \Delta_q^{\nu} v \Delta_q^{\nu} v \, dx,
\]

\[
T_{4,q}^\nu = \sum_{q' > q - 4} \int_{\Omega_h \times \mathbb{R}} \Delta_q^{\nu} (S_{q-1}^{\nu} (\partial_3 v), \Delta_q^{\nu} u_3) \Delta_q^{\nu} v \, dx.
\]

As \( u \) is divergence free, we get after integration by parts

\[
T_{1,q}^\nu = \frac{1}{2} \int_{\Omega_h \times \mathbb{R}} S_{q-1}^{\nu} (\text{div}_h u_\nu) \Delta_q^{\nu} v \Delta_q^{\nu} v \, dx.
\]

The Hölder inequality leads to

\[
|T_{1,q}^\nu| \leq C \|S_{q-1}^{\nu} (\text{div}_h u_\nu)\|_{L^2(L^2_h)} \|\Delta_q^{\nu} v\|_{L^2(L^2_h)}^2 \\
\leq C \|\nabla_h u\|_{L^\infty(L^2_h)} \|\Delta_q^{\nu} v\|_{L^2(L^2_h)}^2.
\]

Thanks to Proposition 5.1, we obtain

\[
|T_{1,q}^\nu| \leq C 2^{-2qs} b_q \|\nabla_h u\|_{H^{0,s}} \|v\|_{H^{0,s}} \|\nabla_h v\|_{H^{0,s}}.
\]

For the term \( T_{2,q}^\nu \), we use the following lemma (see [27] for the proof).

**Lemma 5.2.** Let \( p, r, s \) and \( t \) be real numbers such that

\[
1 \leq p, r, s \leq \infty \quad \text{and} \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s}.
\]

For any vector fields \( u \) and \( v \) the following inequality holds

\[
\|([\Delta_q^{\nu}; u] v)\|_{L^p(L^r_h)} \leq C 2^{-q} \|\partial_3 u\|_{L^\infty(L^2_h)} \|v\|_{L^p(L^r_h)}
\]

where \( [\Delta_q^{\nu}; u] v = \Delta_q^{\nu} (uv) - u \Delta_q^{\nu} v \).

Hence, we get thanks to the Hölder inequality

\[
|T_{2,q}^\nu| \leq \sum_{|q-q'| \leq 4} \|([\Delta_q^{\nu}; S_{q-1}^{\nu} u_3] \partial_3 \Delta_q^{\nu} v\|_{L^2(L^2_h)} \|\Delta_q^{\nu} v\|_{L^2(L^2_h)} \\
\leq C \sum_{|q-q'| \leq 4} 2^{-q} \|S_{q-1}^{\nu} \partial_3 u_3\|_{L^\infty(L^2_h)} \|\partial_3 \Delta_q^{\nu} v\|_{L^2(L^2_h)} \|\Delta_q^{\nu} v\|_{L^2(L^2_h)}.
\]

Since \( \Delta_q^{\nu} v \) is localized in vertical frequencies in the ring of size \( 2^{q'} \), we get

\[
\|\partial_3 \Delta_q^{\nu} v\|_{L^2(L^2_h)} \leq C 2^{q'} \|\Delta_q^{\nu} v\|_{L^2(L^2_h)}.
\]

As \( \partial_3 u_3 = -\text{div}_h u_\nu \), we obtain thanks to Proposition 5.1

\[
|T_{2,q}^\nu| \leq C 2^{-2qs} b_q \|\nabla_h u_\nu\|_{H^{0,s}} \|v\|_{H^{0,s}} \|\nabla_h v\|_{H^{0,s}}.
\]

As for the term \( T_{3,q}^\nu \), the Hölder inequality leads to

\[
|T_{3,q}^\nu| \leq \sum_{|q-q'| \leq 4} \|S_{q-1}^{\nu} - S_{q-1}^{\nu} u_3\|_{L^\infty(L^2_h)} \|\partial_3 \Delta_q^{\nu} \Delta_q^{\nu} v\|_{L^2(L^2_h)}.
\]
But, since \((S_{q-1}^{v}u_{3} - S_{q-1}^{v}u_{3})\) is localized in vertical frequencies in the ring of size \(2^{q}\), we get thanks to the Bernstein lemma
\[
\|S_{q-1}^{v}u_{3} - S_{q-1}^{v}u_{3}\|_{L_{h}^{\infty}(L_{h}^{2})} \leq C2^{-q}\|(S_{q-1}^{v} - S_{q-1}^{v})\partial_{3}u_{3}\|_{L_{h}^{\infty}(L_{h}^{2})}.
\]
As \(u\) is divergence free, we obtain by Proposition 5.1
\[
\|(S_{q-1}^{v} - S_{q-1}^{v})\partial_{3}u_{3}\|_{L_{h}^{\infty}(L_{h}^{2})} \leq C2^{-q}\|\nabla_{h}u\|_{H^{0,.}}.
\]
Applying again Bernstein lemma and Proposition 5.1, we get as in the above estimates
\[
|T_{3,q}^{v}| \leq C2^{-2qs}b_{q}\|\nabla_{h}u\|_{H^{0,.}}\|v\|_{H^{0,.}}\|\nabla_{h}v\|_{H^{0,.}}.
\]
Finally, for the term \(T_{4,q}^{v}\), the Hölder inequality implies that
\[
|T_{4,q}^{v}| \leq \sum_{q' > q-4} \|S_{q+1}^{v}v\|_{L_{h}^{\infty}(L_{h}^{2})}\|\Delta_{q}^{v}u_{3}\|_{L_{h}^{2}}\|\Delta_{q}^{v}v\|_{L_{h}^{2}(L_{h}^{2})}.
\]
Thanks to the Bernstein lemma, we get
\[
\|S_{q+1}^{v}(\partial_{3}v)\|_{L_{h}^{\infty}(L_{h}^{2})} \leq C2^{q'}\|v\|_{L_{h}^{\infty}(L_{h}^{2})}.
\]
Proposition 5.1 leads to
\[
\|S_{q+1}^{v}(\partial_{3}v)\|_{L_{h}^{\infty}(L_{h}^{2})} \leq C2^{2q'}\|v\|_{H^{0,.}}^{\frac{1}{2}}\|\nabla_{h}v\|_{H^{0,.}}^{\frac{1}{2}}.
\]
Using again the Bernstein lemma and the fact that \(u\) is divergence free, we infer
\[
\|\Delta_{q}^{v}u_{3}\|_{L_{2}} \leq C2^{-q'}\|\Delta_{q}^{v}\nabla_{h}u\|_{L_{2}}.
\]
We get thanks to Proposition 5.1
\[
|T_{4,q}^{v}| \leq C2^{-2qs}b_{q}\|\nabla_{h}u\|_{H^{0,.}}\|v\|_{H^{0,.}}\|\nabla_{h}v\|_{H^{0,.}}.
\]
Finally, we sum all the estimates to conclude the proof of Lemma 4.1.

Proof of Lemma 4.2.
The proof of Lemma 4.2 relies on some basic inequalities.

Proposition 5.2. For any axisymmetric vector fields \(u\) and \(v\) such that \(u\) belongs to \(H_{0}^{0,s}(\Omega) \cap H_{0}^{1,s}(\Omega)\) and \(v\) belongs to \(H_{0}^{0,s}(\Omega)\), we have
\[
\|u(., x_{3})v(., x_{3})\|_{L_{h}^{2}} \leq C\|v(., x_{3})\|_{L_{h}^{2}}(\|u(., x_{3})\|_{L_{h}^{2}} + \|\nabla_{h}u(., x_{3})\|_{L_{h}^{2}}).
\]

Proof of Proposition 5.2.
By density arguments, we may suppose that \(u\) are \(v\) are in \(C_{c}^{\infty}(\Omega)\).

We have
\[
\|u(., x_{3})v(., x_{3})\|_{L_{h}^{2}}^{2} \leq C\int_{\rho}^{+\infty} \tilde{u}^{2}(r, x_{3})v^{2}(r, x_{3})r dr.
\]
But, we have
\[
\tilde{u}^{2}(r, x_{3}) = \int_{\rho}^{r} 2\tilde{u}(\sigma, x_{3})\partial_{r}\tilde{u}(\sigma, x_{3}) d\sigma
\]
\[
\leq 2\left(\int_{\rho}^{r} |\tilde{u}(\sigma, x_{3})|^{2} d\sigma\right)^{\frac{1}{2}} \left(\int_{\rho}^{r} |\partial_{r}\tilde{u}(\sigma, x_{3})|^{2} d\sigma\right)^{\frac{1}{2}}
\]
\[
\leq 2\left(\int_{\rho}^{+\infty} |\tilde{u}(\sigma, x_{3})|^{2} \frac{1}{\sigma} d\sigma\right)^{\frac{1}{2}} \left(\int_{\rho}^{+\infty} |\partial_{r}\tilde{u}(\sigma, x_{3})|^{2} \frac{1}{\sigma^{2}} d\sigma\right)^{\frac{1}{2}}.
\]
Thus, we get
\[ \tilde{u}^2(r, x_3) \leq C \| \tilde{u}(., x_3) \|_{L^2(\partial\Gamma, \partial_r \tilde{u}(., x_3))} \] and then
\[ \| u(., x_3) v(., x_3) \|_{L_h^2} \leq C \| v(., x_3) \|_{L_h^2} (\| u(., x_3) \|_{L_h^2} + \| \nabla_h u(., x_3) \|_{L_h^2}). \]

Let us now prove Lemma 4.2. Bony’s decomposition implies that
\[ \Delta_q^\prec(uv) = T_{1,q} + T_{2,q} + T_{3,q}, \]
where
\[ T_{1,q} = \sum_{|q' - q| \leq 4} \Delta_q^\prec(S_{q'-1}^\prec u, \Delta_q^\prec v), \]
\[ T_{2,q} = \sum_{|q' - q| \leq 4} \Delta_q^\prec(S_{q'-1}^\prec v, \Delta_q^\prec u), \]
\[ T_{3,q} = \sum_{i \in \{0, \pm 1\}, q' > q - 4} \Delta_q^\prec(\Delta_q^\prec u, \Delta_q^\prec_{q'-i} v). \]

Proposition 5.2 leads to
\[ \| S_{q'-1}^\prec(u(., x_3), \Delta_q^\prec v(., x_3)) \|_{L_h^2} \leq C (\| S_{q'-1}^\prec u(., x_3) \|_{L_h^2} + \| \nabla_h S_{q'-1}^\prec u(., x_3) \|_{L_h^2}) \| \Delta_q^\prec v(., x_3) \|_{L_h^2}. \]

Taking the $L_h^2$ norm implies that
\[ \| S_{q'-1}^\prec u \Delta_q^\prec v \|_{L_h^2} \leq C (\| S_{q'-1}^\prec u \|_{L_h^\infty(L_h^2)} + \| \nabla_h S_{q'-1}^\prec u \|_{L_h^\infty(L_h^2)}) \| \Delta_q^\prec v \|_{L_h^2}. \]

By Proposition 5.1, we get
\[ \| T_{1,q} \|_{L_h^2} \leq C c_q 2^{-qs} \| v \|_{H^0, s} (\| u \|_{H^0, s} + \| \nabla_h u \|_{H^0, s}). \]

For the term $T_{2,q}$, Proposition 5.2 leads to
\[ \| S_{q'-1}^\prec v(., x_3), \Delta_q^\prec u(., x_3) \|_{L_h^2} \leq C (\| S_{q'-1}^\prec v(., x_3) \|_{L_h^2} + \| \nabla_h \Delta_q^\prec u(., x_3) \|_{L_h^2}). \]

Taking the $L_h^2$ norm implies that
\[ \| S_{q'-1}^\prec v \Delta_q^\prec u \|_{L_h^2} \leq C (\| S_{q'-1}^\prec v \|_{L_h^\infty(L_h^2)} (\| \Delta_q^\prec u \|_{L_h^2} + \| \nabla_h \Delta_q^\prec u \|_{L_h^2}). \]

Thanks to Proposition 5.1, we get
\[ \| T_{2,q} \|_{L_h^2} \leq C c_q 2^{-qs} \| v \|_{H^0, s} (\| u \|_{H^0, s} + \| \nabla_h u \|_{H^0, s}). \]

The term $T_{3,q}$ is estimated as follows. Thanks to the Bernstein lemma, we get
\[ \| T_{3,q} \|_{L_h^2} \leq C \sum_{i \in \{0, \pm 1\}, q' > q - 4} \| \Delta_q^\prec u \Delta_q^\prec_{q'-i} v \|_{L_h^2(L_h^2)}. \]

Proposition 5.2 implies that
\[ \| \Delta_q^\prec u(., x_3), \Delta_q^\prec_{q'-i} v(., x_3) \|_{L_h^2} \leq C (\| \Delta_q^\prec_{q'-i} v(., x_3) \|_{L_h^2} (\| \Delta_q^\prec u(., x_3) \|_{L_h^2} + \| \Delta_q^\prec \nabla_h u(., x_3) \|_{L_h^2}). \]

Taking the $L_h^1$ norm yields
\[ \| \Delta_q^\prec u \Delta_q^\prec_{q'-i} v \|_{L_h^1(L_h^1)} \leq C \| \Delta_q^\prec_{q'-i} v \|_{L^2} (\| \Delta_q^\prec u \|_{L^2} + \| \Delta_q^\prec \nabla_h u \|_{L^2}). \]

As $s > \frac{1}{2}$, Proposition 5.1 implies that
\[ \| T_{2,q} \|_{L^2} \leq C c_q 2^{-qs} \| v \|_{H^0, s} (\| u \|_{H^0, s} + \| \nabla_h u \|_{H^0, s}). \]
Finally, we sum all the above estimates to conclude the proof of Lemma 4.2.

Acknowledgements: The author would like to thank Mohamed Majdoub for suggesting her this problem and for his kind advice.

References

[1] A. Babin, A. Mahalov and B. Nicolaenko, Global splitting, integrability and regularity of 3D Euler and Navier-Stokes equations for uniformly rotating fluids, Eur. J. Mech, 15, p. 291-300, 1996.

[2] A. Babin, A. Mahalov and B. Nicolaenko, Global regularity of 3D rotating Navier-Stokes equations for resonant domains, Indiana Univ. Math. J, 48(3), p. 1133-1176, 1999.

[3] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Annales Scientifiques de l'École Normale Supérieure, 14, p. 209-246, 1981.

[4] D. Bresch, D. Gérard-Varet and E. Grenier, Derivation of the planetary geostrophic equations, Arch. Rat. Mech. Anal, 182(2), p. 387-413, 2006.

[5] H. Brézis, Analyse fonctionnelle Théorie et applications, Masson, Paris, 1983.

[6] J.-Y Chemin, A propos d'un problème de pénalisation de type antisymétrique, J. Math. Pures Appl. 76 (9) p. 739-755, 1997.

[7] J.-Y. Chemin, Calcul paradifférentiel précis et application à des équations aux dérivées partielles non semi-linéaires, Duke Mathematical Journal, 56, p. 431-469, 1988.

[8] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Mathematical Geophysics: An introduction to rotating fluids and the Navier-Stokes equations, Clarendon press, Oxford, 2006.

[9] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, M2AN. Math. Numer. Anal, 34(2), p. 315-335, 2000.

[10] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Ekman boundary layers in rotating fluids, ESAIM Contrôle optimal et Calcul des Variations, Special Tribute Issue to Jacques-Louis Lions, p. 441-466, 2002.

[11] C. Cheverry, Propagation of oscillations in real vanishing viscosity Limit, Comm. Math. Phys, 247, p. 655-695, 2004.

[12] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, Arch. Ration. Mech. Anal, 16, p. 269-315, 1964.

[13] I. Gallagher, Applications of Schochet’s Methods to parabolic equations, Journal de Mathématiques Pures Et Appliquées, 77, p. 989-1054, 1998.

[14] I. Gallagher, Asymptotics of the solutions of hyperbolic equations with a skew-symmetric perturbation, J. Differ. Équ, 150, p. 363-384, 1998.

[15] I. Gallagher and L. Saint-Raymond, On the influence of the Earth’s rotation on geophysical flows, Handbook of Mathematical Fluid Dynamics Vol 4, p. 201-329.

[16] I. Gallagher and Laure Saint-Raymond, Weak convergence results for inhomogeneous rotating fluid equations, Journal d’Analyse Mathématique, 99, p. 1-34, 2006.

[17] E. Grenier, Oscillatory perturbations of the Navier-Stokes equations, Journal de Mathématiques Pures Et Appliquées, 76, p. 477-498, 1997.

[18] E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, Comm. Partial Differ. Équ, 22(5-6), p. 953-975, 1997.

[19] D. Iftimie, Resolution of the Navier-Stokes equations in anisotropic spaces, Revista Matematica Iberoamericana, 15(1), p. 1-36, 1999.

[20] D. Iftimie, The 3D Navier-Stokes equations seen as a perturbation of the 2D Navier-Stokes equations, Bulletin de la Société Mathématique de France, 127, p. 473-517, 1999.

[21] D. Iftimie, A Uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity, SIAM J. Math. Anal, 33(6), p. 1483-1493, 2002.

[22] J.-L. Joly, G. Métivier, J. Rauch, Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves, Duke Math. J, 70, p. 373-404, 1993.

[23] S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters, and the incompressible limit of compressible fluids, Comm. Pure Appl. Math, 34, p. 481-524, 1981.

[24] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Matematica, 63, p. 193-248, 1933.
[25] **M. Majdoub and M. Paicu**, Uniform local existence for inhomogeneous rotating fluid equations, to appear in Journal of Dynamics and Differential Equations.

[26] **N. Masmoudi**, Ekman layers of rotating fluids: the case of general initial data, Comm. Pure Appl. Math, 53(4), p. 432-483, 2000.

[27] **M. Paicu**, Équation anisotrope de Navier-Stokes dans des espaces critiques, Rev. Mat. Iberoamericana, 21(1), p. 179–235, 2005.

[28] **M. Paicu**, Étude asymptotique pour les fluides anisotropes en rotation rapide dans le cas périodique, Journal de Mathématiques Pures Et Appliquées, 83, p. 163-242, 2004.

[29] **M. Paicu**, Équation périodique de Navier-Stokes sans viscosité dans une direction, Comm. Partial Differ. Equ, 30(7-9), p. 1107–1140, 2005.

[30] **J. Pedlosky**, Geophysical Fluid Dynamics, Springer, 1979.

[31] **J. Rauch and M. Reed**, Nonlinear microcal analysis of semilinear hyperbolic systems in one space dimension, Duke Math. J, 49, p. 397-475, 1982.

[32] **M. Sablé-Tourgeron**, Régularité microlocale pour des problèmes aux limites non linéaires, Ann. Inst. Fourier , 36, p. 39-82, 1986.

[33] **S. Schochet**, Fast singular limits of hyperbolic PDEs, J. Differ . Equ, 114, p. 476-512, 1994.

[34] **S. Schochet**, Asymptotics for symmetric hyperbolic systems with a large parameter, J. Differ . Equ, 75, p. 1-27, 1988.

[35] **R. Temam**, On the Euler Equations of incompressible perfect fluids, J. Functional Analysis, 20, p. 32-43, 1975.

Faculty of Sciences of Tunis, Department of Mathematics, ElManar 2092, Tunis, Tunisia.

E-mail address: bjaouiolfa@yahoo.fr