Spectrum generating algebra of the symmetric top

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We consider an algebraic treatment of a three-body system. We develop the formalism for a system of three identical objects and show that it provides a simultaneous description of the vibrational and rotational excitations of an oblate symmetric top.

1. INTRODUCTION

The study of few-body problems has played an important role in many areas of physics \cite{1}. Over the years accurate methods have been developed to solve the few-body equations. The degree of sophistication required depends on the physical system, \textit{i.e.} to solve the few-body problem in atomic physics requires a far higher accuracy than in hadronic physics.

In recent years the development and application of algebraic methods to the many-body problem (\textit{e.g.} collective excitations in nuclei \cite{2} and molecules \cite{3}) has received considerable attention. In spectroscopic studies these algebraic methods provide a powerful tool to study symmetries and selection rules, to classify the basis states, and to calculate matrix elements. In this paper we discuss an application of algebraic methods to the few-body problem. Especially in the area of hadronic physics, which is that of strong interactions at low energies, for which exact solutions of QCD are unavailable, these methods may become very useful \cite{4}.

2. THREE-BODY SYSTEM

The internal motion of a three-body system can be described in terms of Jacobi coordinates, \(\bar{\rho}\) and \(\bar{\lambda}\), which in the case of three identical objects are

\[
\bar{\rho} = (\vec{r}_1 - \vec{r}_2)/\sqrt{2} \quad \text{and} \quad \bar{\lambda} = (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)/\sqrt{6}.
\]

Instead of a formulation in terms of coordinates and momenta we use second quantization by introducing a dipole boson with \(L^P = 1^-\) for each independent relative coordinate, and an auxiliary scalar boson with \(L^P = 0^+\)

\[
p_{\rho,m}^\dagger, p_{\lambda,m}^\dagger, s^\dagger \quad (m = -1, 0, 1). \quad (1)
\]

The scalar boson does not represent an independent degree of freedom, but is added under the restriction that the total number of bosons \(N = n_\rho + n_\lambda + n_s\) is conserved. This procedure leads to a spectrum generating algebra of \(U(7)\). The model space is spanned by the symmetric irreducible representation \([N]\) of \(U(7)\), which contains the oscillator shells with \(n = n_\rho + n_\lambda = 0, 1, \ldots, N\). The value of \(N\) determines the size of the model space.
3. PERMUTATION SYMMETRY

For three identical objects (e.g., for $X_3$ molecules or nonstrange $qqq$ baryons) the Hamiltonian (or mass operator) has to be invariant under the permutation group $S_3$. The eigenstates of a $S_3$ invariant Hamiltonian are characterized by the irreducible representations of $S_3$. However, in anticipation of the discussion of the oblate top we prefer to use a labeling under the point group $D_3$ (which is isomorphic to $S_3$): $A_1$ and $A_2$ for the one-dimensional symmetric and antisymmetric representations, and $E$ for the two-dimensional representation. The scalar boson $s^\dagger$ transforms as the symmetric representation $A_1$, whereas the two dipole bosons $p_{\rho,m}^\dagger$ and $p_{\lambda,m}^\dagger$ transform as the two components of the mixed-symmetry representation, $E_\rho$ and $E_\lambda$, respectively.

The Hamiltonian is expressed in terms of the creation and annihilation operators of Eq. (1), such that the total number of bosons $N$ is conserved. All one- and two-body $S_3$ invariant interactions have, in addition to angular momentum $L$, parity $P$ and permutation symmetry $t$, still another symmetry. They commute with

$$
\hat{F}_2 = -i \sum_m \left( p_{\rho,m}^\dagger p_{\lambda,m} - p_{\lambda,m}^\dagger p_{\rho,m} \right).
$$

This gives rise to an extra quantum number $M_F$, which has a direct connection with the permutation symmetry $t$ and plays a role similar to that of the label $g$ in the context of a six-dimensional harmonic oscillator model and the label $G$ discussed in [7] for $X_3$ molecules. The corresponding $U(7)$ eigenstates are then labeled by

$$
\left\vert [N], \alpha, M_F, L_t^P \right\rangle,
$$

where $\alpha$ denotes the extra labels which are needed for a unique classification of the states.

4. OBLATE SYMMETRIC TOP

The general one- and two-body $S_3$ invariant Hamiltonian has a rich symmetry structure. Here we consider an intrinsic Hamiltonian

$$
H_{\text{int}} = \xi_1 \left( R^2 s^\dagger s^\dagger - p_{\rho}^\dagger p_{\rho}^\dagger - p_{\lambda}^\dagger p_{\lambda}^\dagger \right) (h.c.)
+ \xi_2 \left[ (p_{\rho}^\dagger p_{\rho}^\dagger - p_{\lambda}^\dagger p_{\lambda}^\dagger) (h.c.) + (p_{\rho}^\dagger p_{\lambda}^\dagger + p_{\lambda}^\dagger p_{\rho}^\dagger) (h.c.) \right],
$$

Figure 1. Geometry of a three-body system.
which describes the vibrational excitations of the configuration of Fig. 1, in which the 
geometry of the three objects is that of an equilateral triangle [4,5]. In the limit of a large 
model space ( \( N \) large), the vibrational energies are

\[
E_{\text{int}} = \epsilon_1 v_1 + \epsilon_2 v_2 ,
\]

with \( \epsilon_1 = 4N\xi_1 R^2 \) and \( \epsilon_2 = 4N\xi_2 R^2/(1 + R^2) \). Here \( v_1 \) denotes the number of quanta
in the symmetric stretching mode (\( \nu_1 \)) with \( A_1 \) symmetry, and \( v_2 \) the total number of
quanta in the antisymmetric stretching (\( \nu_{2a} \)) and the bending modes (\( \nu_{2b} \)), which form
a degenerate doublet with \( E \) symmetry (see Fig. 2), in agreement with a point group
classification of the fundamental vibrations for a nonlinear \( X_3 \) configuration [8]. Hence
\( H_{\text{int}} \) describes the vibrations of an oblate symmetric top.

In a geometric description, the excitations of an oblate top are labeled by

\[
| (v_1, v_2^l), K, L_t^P \rangle .
\]

The label \( l \) is associated with the degenerate vibration. It is proportional to the vibrational
angular momentum about the axis of symmetry and can have the values \( l = v_2, v_2 - 2, \ldots, 1 \)
or 0 for \( v_2 \) odd or even, respectively. The rotational states, which are characterized by
the angular momentum \( L \) and its projection \( K \) on the three-fold symmetry axis, are
arranged in bands built on top of each vibration. The projection \( K \) can take the values
\( K = 0, 1, 2, \ldots \), while the values of the angular momentum are \( L = K, K + 1, K + 2, \ldots \).
The parity is \( P = (-)^K \), and \( t \) denotes the transformation character of the total wave
function under \( D_3 \). On the other hand, the eigenstates of the Hamiltonian of Eq. (4) are
labeled by Eq. (2). A geometric analysis of the eigenstates of \( H_{\text{int}} \) shows, that the label
\( M_F \) is related to the geometric labels \( K \) and \( l \) of the oblate top [8]

\[
M_F = | K \mp 2l |.
\]

For \( l = 0 \) or \( K = 0 \) there is only one value of \( M_F \), but for \( l > 0 \) and \( K > 0 \) there are two
possible values of \( M_F \) for each \( K \).
$M_F = 2 \quad M_F = 1 \quad M_F = 3 \quad M_F = 0 \quad M_F = 4 \quad M_F = 1 \quad M_F = 5$

\begin{align*}
\begin{array}{cccc}
3^+_E & 3^-_E & 3^+_{A_1A_2} & 3^-_E \\
2^+_E & 2^-_E & 2^+_{A_1A_2} & 2^-_E \\
1^+_E & 1^-_E & 1^+_{A_1A_2} & \\
0^+_E & & & \\
\end{array}
\end{align*}

$K=0 \quad K=1 \quad K=2 \quad K=3$

Figure 3. Rotational spectrum of a $(v_1, v_2^{l=1})$ vibrational band with $t = E$ symmetry.

In order to analyze the rotational excitations of an oblate top we consider a rotational Hamiltonian

\[ \hat{H}_{\text{rot}} = \kappa_3 \hat{L} \cdot \hat{L} + \kappa_4 \hat{L}_z^2, \]  

(8)

with eigenvalues

\[ E_{\text{rot}} = \kappa_3 L(L+1) + \kappa_4 M_F^2 \]

\[ = \kappa_3 L(L+1) + \kappa_4 (K^2 + 4 Kl + 4l^2). \]

(9)

The last term contains the effects of the Coriolis force which gives rise to a $8\kappa_4 K l$ splitting between $+l$ and $-l$ levels, which increases linearly with $K$. In Fig. 3 we show the classification scheme for the rotational levels with $L \leq 3$ belonging to a $(v_1, v_2^{l=1})$ vibrational band with $E$ symmetry. The two $L_E$ levels with $K = 3$ are characterized by different values of $M_F$. This shows that $M_F$ provides an additional quantum number which is needed for a unique classification of the states of an oblate top. According to Eq. (8) the splitting of the levels with $K = 0$ is zero, whereas for $K > 0$ there remains a twofold degeneracy because of the two projections $\pm K$ on the symmetry axis. In particular, the rotational spectrum of Fig. 3 does not exhibit $l$-type doubling, which is reflected by the occurrence of degenerate doublets of $A_1$ and $A_2$ levels. This degeneracy can be lifted by introducing higher order interactions that break the $M_F$ symmetry. For example, there exist three-body $D_3$ invariant interactions that mix states with $\Delta M_F = \pm 6$.
5. CONCLUSIONS

We have presented an algebraic treatment of the three-body problem. The relative motion of the three-body system is treated by the method of bosonic quantization, which for the two relative vectors gives rise to a $U(7)$ spectrum generating algebra. The model space is spanned by the symmetric irreducible representation $[N]$ of $U(7)$.

In particular, we have studied the case of three identical objects and showed how the corresponding permutation symmetry can be taken into account exactly. For the special case of one- and two-body interactions, the eigenstates can be labeled by an additional quantum number $M_F$. This label plays a very interesting role. On the one hand, it has a direct connection to the permutation symmetry. On the other hand, in the limit of a large model space ($N$ large) it is directly related to the geometric labels $K$ and $l$ of the oblate top, and provides an extra label which is needed to classify the rotations and vibrations of the oblate top uniquely.

It was shown that $U(7)$ provides a unified treatment of both rotational and vibrational excitations of an oblate top. The ensuing algebraic treatment of the oblate top has found useful applications both in hadronic physics (nonstrange $qqq$ baryons [4]) and in molecular physics ($X_3$ molecules [5]).

Finally, we note that, although we have only discussed the case of three identical objects, the algebraic procedure is valid in general, both for three nonidentical objects (linear or bent) and for more complicated systems such as for example the four-body system with tetrahedral symmetry.

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