Hidden Algebras of the (super) Calogero and Sutherland models

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Abstract

We propose to parametrize the configuration space of one-dimensional quantum systems of $N$ identical particles by the elementary symmetric polynomials of bosonic and fermionic coordinates. It is shown that in this parametrization the Hamiltonians of the $A_N$, $BC_N$, $B_N$, $C_N$ and $D_N$ Calogero and Sutherland models, as well as their supersymmetric generalizations, can be expressed — for arbitrary values of the coupling constants — as quadratic polynomials in the generators of a Borel subalgebra of the Lie algebra $gl(N+1)$ or the Lie superalgebra $gl(N+1|N)$ for the supersymmetric case. These algebras are realized by first order differential operators. This fact establishes the exact solvability of the models according to the general definition given by one of the authors in 1994, and implies that the Calogero and Jack-Sutherland polynomials, as well as their supersymmetric generalizations, are related to finite-dimensional irreducible representations of the Lie algebra $gl(N+1)$ and the Lie superalgebra $gl(N+1|N)$.

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1 Introduction

Exact solutions of non-trivial problems are always of great importance. They give a hint about the structure of real problems and also provide a laboratory for testing approximate methods. The non-relativistic many-body Calogero-Sutherland models \cite{2,3} together with their supersymmetric extensions \cite{4,5,6} provide one of the most valuable sources of exact solutions to one-dimensional many-body quantum mechanical systems. The goal of this article is to try to uncover a hidden reason for the solvability of these models in order to answer the question what is special about them and whether there are other exactly-solvable or quasi-exactly-solvable many-body problems.

The remarkable discovery of the solvability of the bosonic $N$-body Calogero \cite{2} and Sutherland \cite{3} models was at the time a state-of-the-art achievement. Of course, this raised the question of whether there existed a regular procedure to generate these models. After a few years, it was found that the models are connected with the root systems of the $A_{N-1}$ Lie algebras \cite{7}, and can be obtained from the Laplace-Beltrami operators defined on the symmetric spaces; this procedure was called ‘the method of Hamiltonian reduction’ \cite{8,9,10}. The Hamiltonian reduction method provides a regular basis for an explanation of the solvability of the Calogero and Sutherland models, at least for a selected set of values of the coupling constant(s). Considerations of the root systems of the other simple Lie algebras made it possible to find several additional families of exactly-solvable multi-dimensional Schrödinger equations and to prove the complete integrability of all these models.

Recently, another explanation of the exact-solvability of bosonic many-body problems was presented, which was based on the finding that the eigenfunctions of the $N$-body $A_{N-1}$ Calogero-Sutherland models form a flag coinciding with the flag of the finite-dimensional representation spaces of the Lie algebra $gl(N)$ \cite{1,12}. It was shown that the Hamiltonians of the Calogero and Sutherland models are nothing but different non-linear elements of the universal enveloping algebra of a Borel subalgebra of the $gl(N)$ algebra. Unlike the “method of reduction”, in the second method the coupling constants appear as certain parameters fixing the element of the universal enveloping algebra. The second method can be used to explain the solvability of the above models for all allowed values of the coupling constant(s). In the present paper we show that the $BC_N$, $B_N$, $C_N$ and $D_N$ Calogero (rational) and Sutherland (trigonometric) Hamiltonians, for any values of the coupling constants, are second order polynomials in the generators of the $gl(N+1)$ algebra. These results were made possible by exploiting the fact that the configuration space of the above quantum-mechanical systems can be parametrized by variables in which the permutation symmetry is already encoded. The most suitable variables are given by the elementary symmetric polynomials of the coordinates of the particles. In these variables the eigenfunctions have an especially simple form.

Other approaches have also been used to study the Calogero and Sutherland models. Some of these have been directed towards obtaining closed expressions for the wave functions. The bosonic $A_N$ Calogero model was solved using an operator method in \cite{13}, a similar method was applied to the bosonic $A_N$ Sutherland model in \cite{14}. The eigenfunctions of the $BC_N$ Sutherland model has been studied in \cite{10,15} (ground-state) and \cite{16}.
It is worth mentioning the remarkable observation that the wave functions of the bosonic $A_N$ Sutherland models are in correspondence with singular vectors of $W$-algebras (see, for example, [17]).

Quite recently, supersymmetric extensions of the many-body $A_N$ Calogero [4, 5] and Sutherland [6] models were constructed using the standard prescription of supersymmetrization. In the present article we derive the supersymmetric extensions of the $BC_N$, $B_N$, $C_N$ and $D_N$ Calogero and Sutherland models and show that their solvability can be explained by the existence of the hidden Lie superalgebra $gl(N + 1|N)$. It is worth mentioning that a supersymmetric analogue of the Hamiltonian reduction method has not so far been constructed for these models.

The paper is organized as follows: In the next Section we briefly review the bosonic many-body $A_N$ Calogero and Sutherland models with emphasis on the property of exact solvability, and also set up our notation. In Section 3 we study the supersymmetric extension of the Calogero model and show that it is exactly solvable. In Section 4 the same analysis is carried out for the supersymmetric Sutherland model. Section 5 is devoted to the Calogero models connected with the Lie algebras $BC_N$, $B_N$, $C_N$ and $D_N$ as well as their supersymmetric extensions; it is shown that those models are exactly solvable. In Section 6 the $BC_N$, $B_N$, $C_N$ and $D_N$ Sutherland models and their supersymmetric extensions are explored and their exact solvability is established. The Conclusion contains a summary of the results obtained and a discussion of some possible directions for future investigations. In Appendix A we present a realization of the Lie algebra $gl(N)$ and the Lie superalgebra $gl(N|N)$ in terms of first order differential operators. Appendix B contains the Lie algebraic forms of the Hamiltonians discussed in this paper. Finally, in Appendix C we give some details on the derivation of the Lie algebraic forms of the Hamiltonians for the Calogero and Sutherland models.

2 Bosonic (many-body) $A_{N-1}$ Calogero and Sutherland models (review)

In this section we briefly review, following [12], the exact-solvability of the bosonic many-body Calogero and Sutherland models or, more precisely, the $A_{N-1}$ Calogero and Sutherland models.

The Calogero and Sutherland models describe systems of $N$ identical particles situated on the line and the circle, respectively. The degrees of freedom of these models are parametrized by $N$ real coordinates $x_i$. The Hamiltonian of the Calogero model is defined by

$$H_{\text{Cal}} = \frac{1}{2} \sum_{i=1}^{N} \left[ -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right] + g \sum_{i<j} \frac{1}{(x_i - x_j)^2},$$ (2.1)

where $g = \nu(\nu - 1) > -\frac{1}{4}$ is the coupling constant and $\omega$ is the harmonic oscillator frequency. For convenience only, we place the center-of-mass under the influence of the

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6For the sake of simplicity, in Sections 2-4 we refer to these models as the Calogero and Sutherland models.
harmonic oscillator force. The ground state eigenfunction is given by

$$\Psi^{(c)}_0(x) = \Delta^\nu(x)e^{-\omega X^2/2},$$

(2.2)

where $\Delta(x) = \prod_{i<j} |x_i - x_j|$ is the Vandermonde determinant and $X^2 = \sum x_i^2$. As has been shown by F. Calogero, any eigenstate of the system can be written in the form

$$\Psi(x) = \Psi^{(c)}_0(x)P_c(x),$$

(2.3)

where $P_c(x)$ is a certain polynomial in the $x_i$'s, symmetric under the permutation of any two particles. We refer to these polynomials as Calogero polynomials. The operator having the Calogero polynomials as eigenfunctions is obtained from the Hamiltonian (2.1) by a “gauge” rotation, by which we mean the similarity transformation

$$h_{\text{Cal}} = (\Psi^{(c)}_0)^{-1}H_{\text{Cal}}\Psi^{(c)}_0.$$  

(2.4)

The Hamiltonian of the Sutherland model is defined by

$$H_{\text{Suth}} = -\frac{1}{2N} \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} + \frac{g}{4N} \sum_{k<l} \frac{1}{\sin^2(\frac{1}{2}(x_k - x_l))}$$

(2.5)

where $g = \nu(\nu - 1) > -\frac{1}{4}$ is the coupling constant. The ground state eigenfunction is given by

$$\Psi^{(s)}_0(x) = (\Delta^{(\text{trig})}(x))^\nu,$$

(2.6)

where $\Delta^{(\text{trig})}(x) = \prod_{i<j} |\sin(\frac{1}{2}(x_i - x_j))|$ is a trigonometric analogue of the Vandermonde determinant. It was shown by B. Sutherland that similarly to the Calogero model any eigenfunction of the Hamiltonian (2.3) can be written in the form

$$\Psi(x) = \Psi^{(s)}_0(x)P_s(e^{ix}),$$

(2.7)

where $P_s(e^{ix})$ is a certain polynomial in $e^{ix}$, symmetric under the permutation of any two particles. These polynomials are the so called Jack-Sutherland polynomials [18, 3] (for a general description, see, for example, [19, 20]). Performing a gauge rotation of $H_{\text{Suth}}$ with the gauge factor $\Psi^{(s)}_0$ we arrive at the operator

$$h_{\text{Suth}} = (\Psi^{(s)}_0)^{-1}H_{\text{Suth}}\Psi^{(s)}_0,$$

(2.8)

which has the Jack-Sutherland polynomials as eigenfunctions.

In order to study the internal dynamics we introduce the center-of-mass coordinate $Y = \sum_{j=1}^N x_j$ and the translation-invariant relative Perelomov coordinates [21]:

$$y_i = x_i - \frac{1}{N}Y, \quad i = 1, 2, \ldots, N,$$

(2.9)

7One should stress the point that to a fixed value of the coupling constant $g$ there correspond two different values of the parameter $\nu$, namely, $\nu = \alpha$ and $\nu = (1 - \alpha)$, giving rise to two families of eigenfunctions. Of course, $\alpha$ should be chosen in such a way as to minimize the eigenvalue; then (2.2) corresponds to the ground state. The value $\nu = (1 - \alpha)$ inserted in (2.2) describes the ground state (if normalizable) but of another family. If $g = 0$, this family comprises the states of negative parity with respect to permutations.

8See footnote 7.
which obey the constraint \(\sum_{i=1}^{N} y_i = 0\). In order to incorporate the permutation symmetry and the translation invariance we consider two sets of coordinates \[12\]:

\[
(x_1, x_2, \ldots, x_N) \rightarrow (Y, \tau_n(x) = \sigma_n(y(x))) \mid n = 2, 3, \ldots, N,
\]

and

\[
(x_1, x_2, \ldots, x_N) \rightarrow (e^{iY}, \eta_n(x) = \sigma_n(e^{iy}(x))) \mid n = 1, 2, \ldots, (N - 1),
\]

where

\[
\sigma_k(x) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}
\]

are the elementary symmetric polynomials (see, for example, \[20\]). \(\sigma_k(y(x))\) are thus the elementary symmetric polynomials with translation-invariant arguments \[9\]. Then the following statement holds \[12\]:

After separation of the center-of-mass, the operators \(h_{Cal}\) and \(h_{Suth}\), when written in the coordinates \(\tau\) \((2.10)\) and \(\eta\) \((2.11)\), respectively, are quadratic polynomials in the generators of the Borel subalgebra of \(gl(N)\) spanned by the operators \(J_{ij}^0\) and \(J_i^-\) realized as first order differential operators (see Appendix A, eq. \((A.1)\)):

\[
h = \sum_{i,j,k,l=2}^{N} A_{ijkl} J_{ij}^0 J_{kl}^0 + \sum_{i,j,k=2}^{N} B_{ijk} J_{ij}^0 J_k^- + \sum_{i,j=2}^{N} C_{ij} J_{ij}^0 + \sum_{i=2}^{N} D_i J_i^-,
\]

(see Appendix B). The operators \(J_{ij}^0\) and \(J_i^-\) can be represented by triangular matrices and preserve the flag of spaces of inhomogeneous polynomials

\[
\mathcal{P}_n = \text{span}\{v_2^{n_2} v_3^{n_3} v_4^{n_4} \ldots v_N^{n_N} \mid 0 \leq \sum n_i \leq n\},
\]

where \(v_k = \tau_k\) and \(v_k = \eta_{k+1}\), respectively, and \(k = 2, \ldots, N\). The coupling constants \(g\) (see \((2.1)\) and \((2.3)\)) appear only in the coefficients \(C_{ij}\) and \(D_i\) and are not related to the dimension of the representation of \(gl(N)\). Consequently, each Hamiltonian \((2.1), (2.5)\) has one or several infinite families of polynomial eigenfunctions.

This statement leads to the important conclusion that the Calogero and Jack-Sutherland polynomials are intimately connected with the finite-dimensional irreducible representations of the Lie algebra \(gl(N)\). It also provides a simple computational tool for deriving the explicit form of the Calogero and Jack-Sutherland polynomials.

### 3 The supersymmetric many-body Calogero model

In order to proceed to the problem of the supersymmetric generalizations of \((2.1)\) and \((2.3)\), let us first recall \[9\] that in a supersymmetric system of particles each bosonic

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\(9\)Due to the constraint \(\sum_{i=1}^{N} y_i = 0\), the symmetric polynomials can be considered as defined in \(\mathbb{R}^N\) and then restricted to the hyperplane \(\sum_{i=1}^{N} y_i = 0\).
degree of freedom \( x_i \) is accompanied by the fermionic variables \( \vartheta_i \) and \( \vartheta_i^\dagger \), which obey the standard anti-commutation rules \( \{ \vartheta_i, \vartheta_j \} = \{ \vartheta_i^\dagger, \vartheta_j^\dagger \} = 0 \) and \( \{ \vartheta_i, \vartheta_j^\dagger \} = \delta_{ij} \). Throughout the paper we use a concrete realization of this algebra:

\[
\vartheta_i = \theta_i, \quad \vartheta_i^\dagger = \partial \frac{\partial}{\partial \theta_i}.
\]

It is convenient to introduce the ‘cmino’ \( \Psi \) — the fermionic analogue of the center-of-mass coordinate as

\[
\Psi = \sum_{i=1}^{N} \theta_i,
\]

which is the super-partner of \( Y \). We can also define the fermionic analogue of the Perelomov coordinates (2.9)

\[
\lambda_i = \theta_i - \frac{1}{N} \Psi.
\]

In order to construct the supersymmetric many-body Calogero model let us introduce the supercharges \( Q \) and \( Q^\dagger \) as defined in [4]:

\[
Q = \sum_k \frac{\partial}{\partial \theta_k} \left( p_k - i \frac{\partial W}{\partial x^k} \right), \quad Q^\dagger = \sum_k \theta_k \left( p_k + i \frac{\partial W}{\partial x^k} \right),
\]

where \( Q^2 = Q^\dagger 2 = 0 \), \( p_k = -i \frac{\partial}{\partial x^k} \), and the superpotential is

\[
W(x_1, x_2, \ldots, x_N) = -\frac{\omega}{2} \sum_{i=1}^{N} x_i^2 + \nu \sum_{i<j} \log |x_i - x_j|.
\]

Then the supersymmetric Hamiltonian \( \mathcal{H}_{\text{Cal}} = \frac{1}{2} \{ Q, Q^\dagger \} \) has the form [4]

\[
\mathcal{H}_{\text{Cal}} = \frac{1}{2} \sum_{i=1}^{N} \left[ -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right] + \omega \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i} + \sum_{i<j} \nu \left( \frac{\nu - 1}{(x_i - x_j)^2} \right) \left( \theta_i - \theta_j \right) \left( \frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j} \right) + C,
\]

where \( C = -\frac{1}{2} \nu \omega N(N - 1) - \frac{1}{2} N \omega \). The ground state eigenfunction remains the same (cf. (2.2)) as for the bosonic many-body Calogero model (2.1). It is easy to see that a gauge rotation of \( \mathcal{H}_{\text{Cal}} \) with the ground state wave function \( \Psi_0^{(c)} \) as a gauge factor affects only the bosonic part of the Hamiltonian. Defining

\[
\mathcal{H}_{\text{Cal}} = -2(\Psi_0^{(c)})^{-1} \mathcal{H}_{\text{Cal}} \Psi_0^{(c)}
\]

(cf. (2.4)), we obtain

\[
\mathcal{H}_{\text{Cal}} = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2 \omega \sum_{i=1}^{N} \left[ \theta_i \frac{\partial}{\partial x_i} + \theta_i \frac{\partial}{\partial \theta_i} \right] + 2 \nu \sum_{i<j} \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \left( \theta_i - \theta_j \right) \left( \frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j} \right).
\]

This expression can be called the rational form of the supersymmetric many-body Calogero model. Similarly to the bosonic case the eigenfunctions of the operator (3.7) after separation of the center-of-mass remain polynomials but now in \((y^i, \lambda^i)\). These polynomials
are symmetric under the permutation of any pair of particles \((x^i, \theta^i) \leftrightarrow (x^j, \theta^j)\), and can be considered as the supersymmetric analogue of the Calogero polynomials.

Now let us introduce permutation-symmetric variables. One can construct two sets of such variables: (i) a set analogous to the Newton polynomials

\[ s_n = \sum_{i=1}^{N} x_i^n, \quad \rho_n = \sum_{i=1}^{N} \theta_i x_i^{n-1} \] (3.8)

and (ii) a set analogous to the elementary symmetric polynomials

\[ \sigma_k = \sum_{1 < i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad \zeta_k = \frac{1}{(k-1)!} \sum_{i_1 \neq i_2 \neq \cdots \neq i_k} \theta_{i_1} x_{i_2} \cdots x_{i_k}. \] (3.9)

The variables \(s_n\) and \(\rho_n\) as well as \(\sigma_k\) and \(\zeta_k\) are symmetric under the permutation, \((x^i, \theta^i) \leftrightarrow (x^j, \theta^j)\). However, the property \(\sigma_{N+k} = 0, \zeta_{N+k} = 0, k = 1, 2, \ldots\) implies that the variables \(\sigma_k\) and \(\zeta_k\), has the advantage that they avoid the difficulties associated with the overcompleteness of the basis, which plague the variables \(s_n\) and \(\rho_n\). Therefore, in the sequel we make use of the variables \(\sigma_k\) and \(\zeta_k\).

It is worth mentioning the following relations between the two sets of variables

\[ \sum_{k=0}^{N} \sigma_k(x) t^k = \exp\left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_n(x) t^n \right], \]

\[ \sum_{k=0}^{N} \zeta_k(x) t^k = \left( \sum_{m=1}^{\infty} (-1)^{m+1} \rho_m(x, \theta) t^m \right) \exp\left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_n(x) t^n \right]. \] (3.10)

A convenient way of succinctly writing these relations is obtained if one introduces the “superspace” coordinates

\[ \chi_i = \sigma_i + \alpha \zeta_i, \quad \phi_i = x_i + \alpha \theta_i, \] (3.11)

where \(\alpha\) is a Grassmann number, \(\alpha^2 = 0\); then

\[ \sum_{k=0}^{N} \chi_k(\phi_i) t^k = \exp\left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_n(\phi_i) t^n \right], \] (3.12)

encodes both the relations (3.10).

The “superspace” formulation makes it possible to write formulas in a more compact way. The “supercoordinate” \(\phi_i\) is clearly an \(N = 1\) superfield. However, one should stress that the models we study possess \(N = 2\) supersymmetry, while the “superspace” formulation is not manifestly \(N = 2\) supersymmetric. Nevertheless, later on we will see explicitly that the superspace (we drop the quotation marks hereafter) formulation turns out to be a useful computational aid. For example, one can rewrite the Hamiltonian \(h_{\text{Cal}}\) in the superspace coordinates as follows

\[ h_{\text{Cal}} = \int d\alpha d\bar{\alpha} \delta_{\phi_i} \delta_{\phi_i} - \int d\alpha W - \int d\bar{\alpha} \bar{W}. \] (3.13)
Here $\bar{\alpha}$ is a Grassmann variable independent of $\alpha$ and
\[
\frac{\delta}{\delta \phi_i} = \frac{\partial}{\partial \theta_i} + \alpha \frac{\partial}{\partial x_i}, \quad \frac{\delta}{\delta \bar{\phi}_i} = \frac{\partial}{\partial \theta_i} + \bar{\alpha} \frac{\partial}{\partial x_i},
\] (3.14)

while
\[
\mathcal{W} = \omega \sum_{i=1}^{N} \phi_i \frac{\delta}{\delta \phi_i} - \nu \sum_{i<j} \frac{1}{\phi_i - \phi_j} \left( \frac{\delta}{\delta \phi_i} - \frac{\delta}{\delta \phi_j} \right), \quad \bar{\mathcal{W}} = \mathcal{W}(\bar{\phi}),
\] (3.15)

where $\bar{\phi}_i = x_i + \bar{\alpha} \theta_i$. Notice that we have not introduced any superderivatives; $\frac{\delta}{\delta \phi_i}$ and $\frac{\delta}{\delta \bar{\phi}_i}$ should be considered as “superfields” (since they do not contain derivatives with respect to $t$, $\alpha$ and/or $\bar{\alpha}$).

Like we did in Section 2 we would like to separate the dynamics of the center-of-mass and the relative motion, and encode in relative coordinates the translation invariance and permutation symmetry of the system. In order to implement these requirements we introduce the translation-invariant and permutation-symmetric coordinates of the relative motion
\[
\tau_k(y) = \sigma_k(y(x)), \quad \kappa_k(y, \lambda) = \zeta_k(y(x), \lambda(\theta)),
\] (3.16)

for example, in the symmetric polynomials (3.9) the arguments ($x$’s) are replaced by the translation-invariant $y$’s (2.9) and the $\theta$’s are replaced by the translation-invariant $\lambda$’s (3.2). Here $\tau_0 = 1$, $\tau_1 = 0$ and $\kappa_0 = \kappa_1 = 0$. The next step is to make the change of variables
\[
(x_i, \theta_i | i = 1, 2, \ldots, N) \rightarrow (Y, \tau_k; \Psi, \kappa_k | k = 2, 3, \ldots, N).
\] (3.17)

in the Hamiltonian $h_{scal}$. In general, this is a tedious and cumbersome calculation. Fortunately, the superspace formulation allows us to avoid most of the tiresome algebraic calculations. Let us define
\[
\tilde{\phi}_i = \phi_i - \frac{1}{N} \sum_{j=1}^{N} \phi_j.
\] (3.18)

Then the expression for $\psi_i = \tau_i + \alpha \kappa_i$ can be obtained from the relation
\[
\sum_{k=0}^{N} \psi_k(\tilde{\phi}_i)t^k = \exp[\sum_{n=1}^{\infty} \frac{(-1)^n}{n} s_n(\tilde{\phi}_i)t^n].
\] (3.19)

The derivative $\frac{\delta}{\delta \phi_1(\alpha)} = \frac{\partial}{\partial \theta_1} + \alpha \frac{\partial}{\partial x_1}$, satisfies
\[
\frac{\delta \phi_j(\beta)}{\delta \phi_1(\alpha)} = \delta_{ij}(\alpha - \beta) = \delta_{ij} \delta(\alpha - \beta),
\] (3.20)

where we have used that $\int d\alpha \delta(\alpha - \beta)f(\alpha) = f(\beta)$. We now make the change of variables
\[
(\phi_i | i = 1, \ldots, N) \rightarrow (\chi_1 = \sum_{k=1}^{N} \phi_k; \psi_j | j = 2, \ldots, N).
\]

Under such a change of variables the derivatives transform as\(^{10}\)
\[
\frac{\delta}{\delta \phi_1(\alpha)} = -\int d\beta \frac{\delta \psi_j(\beta)}{\delta \phi_1(\alpha)} \frac{\delta}{\delta \psi_j(\beta)},
\] (3.21)

\(^{10}\)From now on we suppress the center-of-mass coordinate $\chi_1$. 

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where \( \frac{\delta}{\delta \phi_i(\alpha)} = \frac{\partial}{\partial \phi_i} + \alpha \frac{\partial}{\partial \tau_i} \), and summation over \( j \) is assumed. To prove the relation (3.21), one simply writes both sides of the equality in components. It is convenient to perform the change of variables in two steps: to write, 
\[
\frac{\delta}{\delta \phi_i(\alpha)} = \int d\beta d\gamma \frac{\delta \tilde{\phi}_j(\gamma)}{\delta \tilde{\phi}_i(\alpha)} \frac{\delta \tilde{\psi}_k(\beta)}{\delta \tilde{\psi}_j(\gamma)} \frac{\delta}{\delta \tilde{\psi}_k(\beta)},
\]
(3.22)
and then use the definition of \( \tilde{\phi}_i \) together with (3.21) which leads to \( \frac{\delta \tilde{\phi}_j(\gamma)}{\delta \tilde{\phi}_i(\alpha)} = (\delta_{ij} - \frac{1}{N}) \delta(\alpha - \gamma) \). Let us give as an example the expression for the Laplace operator in the new coordinates

\[
\sum_{i=1}^{N} \int d\alpha d\beta \frac{\delta}{\delta \phi_k(\beta)} \frac{\delta}{\delta \phi_i(\alpha)} \rightarrow \sum_{i=1}^{N} \sum_{j=2}^{N} \sum_{l,m=1}^{N} \int d\alpha d\beta \int d\gamma (\delta_{il} - \frac{1}{N}) \frac{\delta \tilde{\psi}_k(\beta)}{\delta \tilde{\psi}_j(\gamma)} \int d\epsilon (\delta_{im} - \frac{1}{N}) \frac{\delta \tilde{\psi}_k(\beta)}{\delta \tilde{\psi}_m(\epsilon)}.
\]
(3.23)

The above considerations imply that within the framework of the superspace formalism the calculations for the supersymmetric models follow closely the calculations carried out in the bosonic case \(^{12}\). In fact, for most of the terms the supersymmetric results can be obtained from the bosonic ones by judiciously replacing the bosonic coordinates by superspace coordinates. More details can be found in Appendix C. The derivation makes extensive use of the generating function (3.12).

In the superspace coordinates the Laplace operator becomes (after reinserting the center-of-mass coordinate)

\[
\sum_{k=1}^{N} \int d\alpha d\tilde{\alpha} \frac{\delta}{\delta \tilde{\phi}_k} \frac{\delta}{\delta \tilde{\phi}_i} = \int d\alpha d\tilde{\alpha} \left[ \sum_{i=2}^{N} A_{ij} \frac{\delta}{\delta \tilde{\psi}_j} \frac{\delta}{\delta \tilde{\psi}_i} + \sum_{j=2}^{N} B_i \frac{\delta}{\delta \tilde{\psi}_i} \right],
\]
(3.24)
where

\[
\psi_i = \tau_i + \alpha \kappa_i, \quad \frac{\delta}{\delta \tilde{\psi}_i} = \frac{\partial}{\partial \tau_i} + \alpha \frac{\partial}{\partial \kappa_i},
\]
(3.25)
and

\[
A_{ij} = \left( \frac{N - i + 1}{N} \right) (j - 1) \psi_{i-1} \tilde{\psi}_{j-1} + \sum_{l \geq \max(1, j-i)} (j - i - 2l) \psi_{i+l-1} \tilde{\psi}_{j-l-1},
\]
(3.26)

\[
B_i(\psi) = -\frac{1}{2N} \sum_{i=2}^{N} (N - i + 2)(N - i + 1) \psi_{i-2} \quad \tilde{B}_i = B_i(\tilde{\psi}).
\]
(3.27)

The final expression for the Laplace operator in components is given by

\[
\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} = N \frac{\partial^2}{\partial Y^2} + \sum_{i,j=2}^{N} \left[ A_{ij}^{\tau \tau} \frac{\partial^2}{\partial \tau_i \partial \tau_j} + A_{ij}^{\tau \kappa} \frac{\partial^2}{\partial \tau_i \partial \kappa_j} + A_{ij}^{\kappa \kappa} \frac{\partial^2}{\partial \kappa_i \partial \kappa_j} + A_{ij}^{\kappa \tau} \frac{\partial^2}{\partial \kappa_i \partial \tau_j} + A_{ij}^{\tau \kappa} \frac{\partial^2}{\partial \tau_i \partial \kappa_j} + A_{ij}^{\kappa \kappa} \frac{\partial^2}{\partial \kappa_i \partial \kappa_j} \right] + \sum_{i=2}^{N} \left[ B_i^{\tau} \frac{\partial}{\partial \tau_i} + B_i^{\kappa} \frac{\partial}{\partial \kappa_i} \right],
\]
(3.28)
where

\[
A_{ij}^{\tau\tau} = \frac{(N - i + 1)(j - 1)}{N} \tau_{i-1} \tau_{j-1} + \sum_{l \geq \max(1, j-i)} (j - i - 2l) \tau_{i+l-1} \tau_{j+l-1},
\]

\[
A_{ij}^{\tau\kappa} = \frac{(N - i + 1)(j - 1)}{N} \kappa_{i-1} \tau_{j-1} + \sum_{l \geq \max(1, j-i)} (j - i - 2l) \tau_{i+l-1} \kappa_{j+l-1}
\]

\[
- \sum_{l \geq 1} l [\kappa_{i+l} \tau_{j-2-l} - \tau_{i+l} \kappa_{j-2-l}],
\]

\[
A_{ij}^{\kappa\tau} = \frac{(N - i + 1)(j - 1)}{N} \kappa_{i-1} \tau_{j-1} + \sum_{l \geq \max(1, j-i)} (j - i - 2l) \kappa_{i+l-1} \tau_{j+l-1}
\]

\[
+ \sum_{l \geq 1} l [\kappa_{i+l} \tau_{j-2-l} - \tau_{i+l} \kappa_{j-2-l}],
\]

\[
A_{ij}^{\kappa\kappa} = \frac{(N - i + 1)(j - 1)}{N} \kappa_{i-1} \kappa_{j-1} + \sum_{l \geq \max(1, j-i)} (j - i - 2l) \kappa_{i+l-1} \kappa_{j+l-1}
\]

\[
+ 2 \sum_{l \geq 1} l \kappa_{i+l} \kappa_{j-2-l},
\]

\[
B_i^\tau = -\frac{(N - i + 2)(N - i + 1)}{N} \tau_{i-2},
\]

\[
B_i^\kappa = -\frac{(N - i + 2)(N - i + 1)}{N} \kappa_{i-2}.
\]  \hspace{1cm} (3.29)

As expected, \( A_{ij}^{\tau\tau} \) and \( B_i^\tau \) coincide with the expressions found in \[12\] for the bosonic Calogero model. Let us note that the coefficient functions \( A_{ij} \) and \( B_i \) are second and first order polynomials in \( \tau_i \) and \( \kappa_i \), respectively.

Dropping the decoupled center-of-mass terms one can show that in superspace coordinates the remaining part of the Hamiltonian \( h^{(rel)}_{sCal} \), which describes the relative motion, becomes

\[
h^{(rel)}_{sCal} = \int d\alpha d\bar{\alpha} \sum_{i,j=2}^N A_{ij} \frac{\delta}{\delta \psi_j} \frac{\delta}{\delta \bar{\psi}_i} - \int d\alpha \mathcal{W} - \int d\bar{\alpha} \bar{\mathcal{W}},
\]  \hspace{1cm} (3.30)

where \( A_{ij} \) is given in \[3.26\] and

\[
\mathcal{W} = \omega \sum_{i=2}^N i \psi_i \frac{\delta}{\delta \psi_i} + \frac{1}{2} \left( \frac{1}{N} + \nu \right) \sum_{i=2}^N (N - i + 2)(N - i + 1) \psi_{i-2} \frac{\delta}{\delta \bar{\psi}_i}, \hspace{0.5cm} \bar{\mathcal{W}} = \mathcal{W}(\bar{\psi}).
\]  \hspace{1cm} (3.31)

Finally, in component form \( h^{(rel)}_{sCal} \) looks like

\[
h^{(rel)}_{sCal} = \sum_{i,j=2}^N \left[ A_{ij}^{\tau\tau} \frac{\partial^2}{\partial \tau_i \partial \tau_j} + A_{ij}^{\tau\kappa} \frac{\partial^2}{\partial \tau_i \partial \kappa_j} + A_{ij}^{\kappa\tau} \frac{\partial^2}{\partial \kappa_i \partial \tau_j} + A_{ij}^{\kappa\kappa} \frac{\partial^2}{\partial \kappa_i \partial \kappa_j} \right]
\]

\[-\left( \frac{1}{N} + \nu \right) \sum_{i=2}^N (N - i + 2)(N - i + 1) \left[ \tau_{i-2} \frac{\partial}{\partial \tau_i} + \kappa_{i-2} \frac{\partial}{\partial \kappa_i} \right]
\]

\[-2\omega \sum_{i=2}^N i \left[ \tau_i \frac{\partial}{\partial \tau_i} + \kappa_i \frac{\partial}{\partial \kappa_i} \right].
\]  \hspace{1cm} (3.32)
where $A_{ij}^{\tau\tau}, A_{ij}^{\tau\kappa}, A_{ij}^{\kappa\tau}, A_{ij}^{\kappa\kappa}, B_{i}^{\tau}, B_{i}^{\kappa}$ are given in (3.29). This form can be called the algebraic form of the supersymmetric many-body Calogero Hamiltonian.

It turns out that we are able to rewrite the supersymmetric Calogero Hamiltonian $H_{s\text{Cal}}^{(\text{rel})}$ in terms of generators of the Borel subalgebra of the algebra $gl(N|N - 1)$ spanned by first order differential operators in the variables $(\tau_k, \kappa_k), \ k = 2, \ldots N$ (see Appendix A.2). The result is given in Appendix B, eq. (B.2). Since the expression (B.2) contains no positive-root generators $J_i^+, Q_i^+$, then in accordance with the general definition given in \[1\] we conclude that the supersymmetric many-body Calogero model (3.5) is exactly solvable. The existence of the representation (B.2) proves that there are infinitely-many eigenfunctions of the operator (3.7) having the form of polynomials in the variables $(\tau_k, \kappa_k)$. It also implies that eigenfunctions of the supersymmetric many-body Calogero model (3.5) have a factorizable form being the product of the ground-state eigenfunction multiplied by a polynomial in the variables $(\tau_k, \kappa_k)$. These polynomials are related to finite-dimensional irreducible representations of the algebra $gl(N|N - 1)$ in the realization (A.3).

### 4 The supersymmetric many-body Sutherland model

In this section we analyze the exact-solvability of the supersymmetric Sutherland model.

In order to define the supersymmetric extension of the bosonic many-body Sutherland model (2.7) we use the standard prescription of supersymmetrization already used in the previous Section. Let us take the supercharges $Q, Q^\dagger$ (see (1.3)) with a superpotential

$$W(x_1, x_2, \ldots, x_N) = \nu \sum_{i<j} \log |\sin(\frac{1}{2}(x_i - x_j))| = \nu \log \Delta^{(\text{trig})}, \quad (4.1)$$

(cf. (1.4)) and then construct the supersymmetric Hamiltonian $H_{s\text{Suth}} = \frac{1}{2}\{Q, Q^\dagger\}$. After carrying out the calculation the Hamiltonian of the supersymmetric many-body Sutherland model emerges [4]:

$$H_{s\text{Suth}} = -\frac{1}{2} \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + \frac{1}{4} \sum_{k<l}^{\nu} \frac{\nu}{\sin^2(\frac{1}{2}(x_k - x_l))} \left[ \nu - 1 + (\theta_k - \theta_l)(\frac{\partial}{\partial \theta_k} - \frac{\partial}{\partial \theta_l}) \right] + C, \quad (4.2)$$

where $C = -N(N^2 - 1)\nu^2/3$. The ground state wave function remains the same as in the bosonic case, $\Psi_0^{(s)} = (\Delta^{(\text{trig})}(x))^\nu$, where $\Delta^{(\text{trig})}(x) = \prod_{k<l} |\sin(\frac{1}{2}(x_k - x_l))|$. Introducing the gauge-rotated Hamiltonian $h_{s\text{Suth}} = -2(\Psi_0^{(s)})^{-1}H\Psi_0^{(s)}$, we get

$$h_{s\text{Suth}} = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + \nu \sum_{k<l} \left[ \cot(\frac{1}{2}(x_k - x_l))(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_l}) - \frac{1}{2} \frac{(\theta_k - \theta_l)}{\sin^2(\frac{1}{2}(x_k - x_l))} \left( \frac{\partial}{\partial \theta_k} - \frac{\partial}{\partial \theta_l} \right) \right]. \quad (4.3)$$

It is worth mentioning that if the operator (1.3) is written in the coordinates $e^{ix_k}$, it appears in its rational form, which is a supersymmetric generalization of the rational form of the Sutherland Hamiltonian (see [4]).

One can show that like in the bosonic case (cf.(2.8)) after separation of the center-of-mass motion the eigenfunctions of the operator (1.3) remain polynomials but now in the coordinates $(e^{\nu_k}, \lambda_k)$. These polynomials are symmetric under the permutation of any
pair of particles \((x_i, \theta_i) \leftrightarrow (x_j, \theta_j)\) and can be considered as a supersymmetric analogue of the Jack-Sutherland polynomials.

In the superspace coordinates (3.11), (3.14) the Hamiltonian \(h_{\text{sSuth}}\) can be written

\[
h_{\text{sSuth}} = \int d\alpha d\bar{\alpha} \sum_{k=1}^{N} \frac{\delta}{\delta \phi_i} \frac{\delta}{\delta \bar{\phi}_i} + \nu \int d\alpha W + \nu \int d\bar{\alpha} \bar{W},
\]

where

\[
W(\phi) = \frac{1}{2} \sum_{k<l} \cot\left(\frac{1}{2} (\phi_k - \phi_l)\right) \left(\frac{\delta}{\delta \phi_k} - \frac{\delta}{\delta \phi_l}\right), \quad \bar{W} = W(\bar{\phi}).
\]

Next, we introduce the new variables (cf. (3.16))

\[
\eta_n + \alpha \mu_n = \sigma_n(\exp[i(y_k + \alpha \lambda_k)]), \quad n = 1, \ldots, N - 1,
\]

and

\[
\sigma_N + \alpha \zeta_N = \sigma_N(\exp[i(x_k + \alpha \theta_k)]), \quad n = N,
\]

where \(\alpha^2 = 0\). \(\sigma_n\) is defined in (3.9) and \(y_i, \lambda_i\) are given in (2.9), (3.2), respectively. Furthermore,

\[
\sum_{k=0}^{N} \eta_k t^k = \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_n (e^{iy_k}) t^n\right],
\]

\[
\sum_{k=0}^{N} \mu_k t^k = \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} s_m (e^{iy_k}) t^m\right) \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_n (e^{iy_k}) t^n\right],
\]

(cf. (3.10)), where we have set \(\eta_N = 1\) and \(\mu_N = 0\).

Following the same procedure as in Section 3 one can rewrite the Laplace operator in the new coordinates (4.6), (4.7):

\[
- \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} = N(\sigma_N \frac{\partial}{\partial \sigma_N})^2 + \sum_{i,j=1}^{N-1} \left[ A_{ij}^{mn} \frac{\partial^2}{\partial \eta_i \partial \eta_j} + A_{ij}^{mu} \frac{\partial^2}{\partial \eta_i \partial \mu_j} + A_{ij}^{mu} \frac{\partial^2}{\partial \eta_j \partial \mu_i} + A_{ij}^{mu} \frac{\partial^2}{\partial \mu_j \partial \mu_i} \right]
\]

\[
+ \sum_{i=1}^{N-1} \left[ B_{i}^{n} \frac{\partial}{\partial \eta_i} + B_{i}^{\mu} \frac{\partial}{\partial \mu_i} \right],
\]

where

\[
A_{ij}^{mn} = \frac{j(N-i)}{N} \eta_i \eta_j + \sum_{l \geq \max(1,j-i)} (j-i-2l) \eta_i \eta_{j-l},
\]

\[
A_{ij}^{mu} = \frac{j(N-i)}{N} \eta_i \mu_j + \sum_{l \geq \max(1,j-i)} (j-i-2l) \eta_i \mu_{j-l} - \sum_{l \geq 1} l \mu_{i+l-1} \eta_{j-l-1} - \eta_{i+l-1} \mu_{j-l-1},
\]

\[
A_{ij}^{nu} = \frac{j(N-i)}{N} \mu_i \eta_j + \sum_{l \geq \max(1,j-i)} (j-i-2l) \mu_i \eta_{j-l} + \sum_{l \geq 1} l \mu_{i+l-1} \eta_{j-l-1} - \eta_{i+l-1} \mu_{j-l-1},
\]

\[
A_{ij}^{nu} = \frac{j(N-i)}{N} \mu_i \mu_j + \sum_{l \geq \max(1,j-i)} (j-i-2l) \mu_i \mu_{j-l} + 2 \sum_{l \geq 1} l \mu_{i+l-1} \mu_{j-l+1},
\]

\[
B_{i}^{n} = \frac{i(N-i)}{N} \eta_i,
\]

\[
B_{i}^{\mu} = \frac{i(N-i)}{N} \mu_i.
\]
As expected, the coefficients $A_{ij}^{\eta\eta}$ and $B_i^{\eta}$ coincide with the expressions found in [12] for the bosonic Sutherland model. Furthermore, the coefficient functions $A_{ij}$ and $B_i$ are second and first order polynomials in $\eta_i$ and $\mu_i$, respectively. Similarly to what happens in the bosonic case the Laplace operator (4.3) possesses infinitely-many polynomial eigenfunctions [12]. These have the form of supersymmetric analogies of the Bethe-ansatz wave functions.

After omitting the center-of-mass motion associated with $\sigma_N$ and $\zeta_N$, the final expression for the supersymmetric many-body Sutherland Hamiltonian $h_{Suth}$ takes the form (cf. Section 3, eq.(3.32))

$$h_{Suth}^{(rel)} = \sum_{i,j=1}^{N-1} \left[ A_{ij}^{\eta\eta} \frac{\partial^2}{\partial \eta_i \partial \eta_j} + A_{ij}^{\eta\mu} \frac{\partial^2}{\partial \eta_i \partial \mu_j} + A_{ij}^{\mu\eta} \frac{\partial^2}{\partial \eta_j \partial \mu_i} + A_{ij}^{\mu\mu} \frac{\partial^2}{\partial \mu_j \partial \mu_i} \right] + \frac{1}{N + \nu} \sum_{i=1}^{N-1} i(N - i) \left[ \eta_i \frac{\partial}{\partial \eta_i} + \mu_i \frac{\partial}{\partial \mu_i} \right].$$  (4.11)

where $A_{ij}^{\eta\eta}, A_{ij}^{\eta\mu}, A_{ij}^{\mu\eta}, A_{ij}^{\mu\mu}, B_i^{\eta}, B_i^{\mu}$ are given in (4.10). This expression can be called the algebraic form of the supersymmetric many-body Sutherland Hamiltonian.

In superspace coordinates the Hamiltonian $h_{Suth}^{(rel)}$ describing the relative motion can be written

$$-h_{Suth}^{(rel)} = \int d\alpha d\bar{\alpha} \sum_{i,j=1}^{N-1} A_{ij} \delta \frac{\delta}{\delta \psi_j^i} + \delta \psi_i^j + \int d\alpha \mathcal{W} + \int d\bar{\alpha} \bar{\mathcal{W}},$$  (4.12)

where

$$\mathcal{W} = \frac{1}{2} \left( \frac{1}{N + \nu} \right) \sum_{i=1}^{N-1} i(N - i) \psi_i \frac{\delta}{\delta \psi_i}, \quad \bar{\mathcal{W}} = \mathcal{W}(\bar{\psi}),$$  (4.13)

$$\psi_i = \eta_i + \alpha \mu_i, \quad \frac{\delta}{\delta \psi_i} = \frac{\partial}{\partial \mu_i} + \alpha \frac{\partial}{\partial \eta_i},$$  (4.14)

(cf. (3.25)), while

$$A_{ij} = \frac{j(N - i)}{N} \psi_i \bar{\psi}_j - \sum_{l \geq \max(1,j-i)} (j - i - 2l) \psi_{i+l} \bar{\psi}_{j-l} + \sum_{l \geq 1} l \left( \psi_{i+l-1} \bar{\psi}_{j-l+1} - \psi_{i+l-1} \bar{\psi}_{j-l+1} \right),$$  (4.15)

(cf. (3.26)). The method of the calculation of the coefficients $A_{ij}$ is similar to that presented in App.C.

It turns out that the Hamiltonian governing the relative motion can be rewritten in terms of the generators of a Borel subalgebra of the Lie superalgebra $gl(N|N-1)$ (see Appendix A.2) after substituting $\kappa_i \rightarrow \mu_{i-1}$ and $\tau_i \rightarrow \eta_{i-1}$. The final result is given in Appendix B, eq. (B.4). Since the expression (B.4) contains no positive-root generators $T_i^+, Q_i^+$, then in accordance with the general definition [1] we conclude that the supersymmetric many-body Sutherland model (4.2) is exactly solvable. The existence of the representation (B.4) proves that there are infinitely-many eigenfunctions of the operator (4.3) having the form of polynomials in the variables $(\eta_k, \mu_k)$. It also leads to the conclusion that eigenfunctions of the supersymmetric many-body Sutherland model (4.2) have a factorizable form being the product of the ground-state eigenfunction multiplied by
a polynomial in the variables \((\eta_k, \mu_k)\). These polynomials are related to finite-dimensional irreducible representations of the algebra \(gl(N|N-1)\) in the realization (A.3).

To conclude, the supersymmetric many-body Sutherland models possess the algebraic form (4.11) and also the Lie-algebraic form (B.4) represented by second-order polynomials in the generators of the of the algebra \(gl(N|N-1)\) with certain coefficients.

5 The \(BC_N, B_N, C_N\) and \(D_N\) Calogero models and their supersymmetric extensions

It is well-known that there is a deep connection between the integrable Calogero-Sutherland systems and Lie algebras (see discussion in the Introduction). This connection is explicitly realized in the Hamiltonian reduction method [7, 8] (see also [9, 10, 11]). In particular, the celebrated many-body Calogero and Sutherland models discussed in Section 2 are related to the Lie algebras \(A_{N-1}\) and are therefore known as the Calogero and Sutherland systems of \(A_{N-1}\)-type. A natural question emerges: what are the integrable systems corresponding to the other simple Lie algebras like \(B_N, C_N, D_N\) etc. appearing in the Hamiltonian reduction method. The answer and a complete classification of these systems is given in [7] (for a review, see [9, 10]). In the present paper we focus on the quantum Calogero (rational) and Sutherland (trigonometric) systems of the \(BC_N, B_N, C_N\) and \(D_N\) types leaving for a future publication the case of the exceptional algebras.

5.1 The bosonic case

Unlike the \(A_{N-1}\) many-body Calogero model, the \(BC_N, B_N, C_N\) and \(D_N\) Calogero models are not translation invariant, and describe systems with boundaries. However, they are still permutation invariant with respect to the interchange of any pair of coordinates. The configuration space of these models is \(\{x_i | x_i > 0; j < k : x_j < x_k\}\). In this section we will show that like the \(A_{N-1}\) many-body Calogero system, all the \(BC_N, B_N, C_N, D_N\) quantum Calogero systems are not only completely integrable, but also exactly-solvable possessing a hidden algebra \(gl(N + 1)\).

It is well known that the Hamiltonians of the \(BC_N, B_N, C_N\) Calogero models are all given by (see [4])

\[
\mathcal{H}_{BCD}^{(c)} = \frac{1}{2} \sum_{i=1}^{N} \left[-\partial_i^2 + \omega^2 x_i^2\right] + g \sum_{i<j} \left[\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2}\right] + g_2 \frac{N}{2} \sum_{i=1}^{N} \frac{1}{x_i^2} \quad (5.1)
\]

where \(g = \nu(\nu - 1)\) and \(g_2 = \nu_2(\nu_2 - 1)\). When the coupling constant \(g_2\) tends to zero, the Hamiltonian \(\mathcal{H}_{BCD}^{(c)}\) degenerates to the Hamiltonian of the \(D_N\) Calogero model. The ground state eigenfunction of the Hamiltonian (5.1) is given by

\[
\Psi_0 = \left[\prod_{i<j} |x_i - x_j|^{\nu} |x_i + x_j|^{\nu_2} \prod_{i=1}^{N} |x_i|^2\right] e^{-\frac{\pi}{2} \sum_{i=1}^{N} x_i^2}, \quad (5.2)
\]

(cf. (2.3)).
One should stress the point\textsuperscript{11} that to fixed values of the coupling constants \(g, g_2\) there corresponds two different values of the parameter \(\nu(\nu_2)\): \(\nu(\nu_2) = \alpha(\alpha_2)\) and \((1 - \alpha(\alpha_2))\) giving rise to four families of eigenfunctions. Of course, \(\alpha(\alpha_2)\) should be chosen in such a way as to minimize the eigenvalue and then \((5.2)\) corresponds to the ground state. The ground state can be denoted as \((\alpha, \alpha_2)\). The other values \(\nu(\nu_2)\) inserted in \((5.2)\) describes the ground states (if normalizable) of three other families: \((1 - \alpha, \alpha_2), (\alpha, 1 - \alpha_2), (1 - \alpha, 1 - \alpha_2)\). If \(g_2 = 0\), the sectors \((\alpha, 1 - \alpha_2), (1 - \alpha, 1 - \alpha_2)\) correspond to totally antisymmetric states with respect to the reflections \(x_i \leftarrow -x_i\). In what follows, for the sake of simplicity, we call \((5.2)\) the ground state, of course keeping in mind the above discussion.

Following the same approach as in Sections 2-4, we make a gauge rotation of \((5.1)\) with the ground-state eigenfunction as the gauge factor; \(h^{(c)}_{BCD} = -2\Psi_0^{-1} \mathcal{H} \Psi_0\). A straightforward calculation gives (we omit an additive constant)

\[
h^{(c)}_{BCD} = \sum_{i=1}^{N} \partial_i^2 + 2\nu \sum_{i<j} \left[ \frac{1}{x_i - x_j} (\partial_i - \partial_j) + \frac{1}{x_i + x_j} (\partial_i + \partial_j) \right] + \nu_2 \sum_{i=1}^{N} \frac{1}{x_i} \partial_i - 2\omega \sum_{i=1}^{N} x_i \partial_i. \tag{5.3}
\]

We call this operator the rational form of the \(BC_N, B_N, C_N\) and \(D_N\) Calogero systems.

The Hamiltonian \((5.4)\) as well as the operator \((5.3)\) is invariant under the permutation of any pair of the coordinates \(x_i \leftrightarrow x_j\) and also under the action of the reflection group \(Z_2^\infty\): \(x_i \rightarrow -x_i, \ i = 1, 2, \ldots, N\). An infinite set of eigenfunctions of \((5.3)\) are given as polynomials in the \(x\)’s and are classified by the representations of the reflection group \(Z_2^\infty\). For the sake of simplicity, we consider in what follows only the eigenfunctions of the Hamiltonian \((5.3)\), which are totally symmetric under the reflection and permutation group actions. The totally (anti)symmetric eigenfunctions under the reflection and/or permutation group actions are reproduced by a change of parameters \(\nu, \nu_2\) in \((5.1)\). This fact implies that it is sufficient to look for eigenfunctions to the operator \((5.3)\) of the form

\[
\Psi^{\nu, \nu_2} = F(x_1^2, x_2^2, \ldots, x_N^2), \tag{5.4}
\]

Now let us construct variables which encode the permutation invariance of the system. Analogously to what was done for the \(A_{N-1}\) Calogero model \([12]\) we use as new variables the elementary symmetric polynomials \(\bar{\sigma}_k\) (see \((2.12)\)) but with \(x_i^2\) as arguments:

\[
\sum_{k=0}^{N} \bar{\sigma}_k(x_i^2)t^k = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_{2n}(x_i)t^n \right], \tag{5.5}
\]

(cf. \((2.10), (3.10)\)). Finally, in the new variables the Hamiltonian \(h^{(c)}_{BCD}\) becomes

\[
h^{(c)}_{BCD} = \sum_{i,j=1}^{N} A_{ij} \frac{\partial}{\partial \bar{\sigma}_i} \frac{\partial}{\partial \bar{\sigma}_j} + \sum_{j=1}^{N} B_j \frac{\partial}{\partial \bar{\sigma}_j}, \tag{5.6}
\]

where (cf.\((3.28)\))

\[
A_{ij} = 4 \sum_{l \geq 0} (2l + 1 + j - i) \bar{\sigma}_{i-1} \bar{\sigma}_{j+l}, \quad B_j = 2[1 + \nu_2 + 2\nu(N - j)](N - j + 1) \bar{\sigma}_{j-1} - 4\omega j \bar{\sigma}_j, \tag{5.7}
\]

\textsuperscript{11}See also the discussion in footnote 7.
and $\tilde{\sigma}_k = 0$, when $k > N$ or $k < 0$. The method of the calculation of the coefficients $A_{ij}, B_j$ is similar to that presented in App.C. This expression can be called the algebraic form of the $BC_N, B_N, C_N$ and $D_N$ Calogero Hamiltonians.

Similarly to what happened in all previously discussed cases the coefficients $A_{ij}, B_j$ are polynomials of second and first degree in $\tilde{\sigma}_k$, respectively. Hence, there exists a Lie algebraic form of the Hamiltonian and $h_{BCD}^{(c)}$ can be written in terms of generators of a Borel subalgebra of $gl(N + 1)$ (see Appendix A.1) as in Section 2. The result is presented in Appendix B, eq.(B.5). Then in accordance with the general definition given in [1] we can conclude that all $BC_N, B_N, C_N, D_N$ Calogero models (5.1) are exactly solvable. The existence of the representation (5.4) proves that there are infinitely-many eigenfunctions of the operator (5.3) having the form of polynomials in the variables $\tilde{\sigma}_k$. It also implies that totally (anti)symmetric-(anti)symmetric eigenfunctions with respect to permutations and reflections of the $BC$ and $A$ factorizable form being the product of the ground-state eigenfunction (5.2) multiplied by a polynomial in the variables $\tilde{\sigma}_k$. These polynomials are related to finite-dimensional irreducible representations of the Lie algebra $gl(N + 1)$ in the realization (A.1) and can be called the $BC_N$ Calogero polynomials.

### 5.2 The supersymmetric extensions

The $BC_N, B_N, C_N$ and $D_N$ Calogero models discussed above have also natural supersymmetric extensions. These can be constructed in a straightforward way. Let us consider the supercharges (5.3) with the superpotential

$$W = \nu \sum_{i<j} (\log |x_i - x_j| + \log |x_i + x_j|) + \nu_2 \sum_{i=1}^{N} \log |x_i| - \frac{\omega_2}{2} \sum_{i=1}^{N} x_i^2. \quad (5.8)$$

(cf.(3.4)). After some calculations the supersymmetric Hamiltonian $\mathcal{H}_{sBCD}^{(c)} = \frac{1}{2} \{ Q, Q^\dagger \}$ of the supersymmetric $BC_N, B_N, C_N, D_N$ Calogero models emerges:

$$\mathcal{H}_{sBCD}^{(c)} = \frac{1}{2} \sum_{i=1}^{N} \left[ - \frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right] + \nu \sum_{i<j} \frac{1}{(x_i - x_j)^2} \left[ \nu - 1 + (\theta_i - \theta_j)(\frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j}) \right] + \nu \sum_{i<j} \frac{1}{(x_i + x_j)^2} \left[ (\nu - 1) + (\theta_i + \theta_j)(\frac{\partial}{\partial \theta_i} + \frac{\partial}{\partial \theta_j}) \right] + C \quad (5.9)$$

where the constant $C = -2N[\nu(N - 1) - \nu_2 - 1]$. Again, like for the other supersymmetric extensions the ground-state eigenfunction remains the same as in the bosonic case and is given by (5.4). Making the gauge rotation of the Hamiltonian (5.9): $h_{sBCD}^{(c)} = -2\Psi_0^{-1} \mathcal{H}_{sBCD}^{(c)} \Psi_0$, we get

$$h_{sBCD}^{(c)} = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\nu \sum_{i<j} \left[ \frac{1}{x_i - x_j}(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}) - \frac{(\theta_i - \theta_j)}{(x_i - x_j)^2}(\frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j}) \right]$$
\[ + 2\nu \sum_{i<j} \left[ \frac{1}{x_i + x_j} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) - \frac{\left( \theta_i + \theta_j \right)}{(x_i + x_j)^2} \left( \frac{\partial}{\partial \theta_i} + \frac{\partial}{\partial \theta_j} \right) \right] \]
\[ - 2\omega \sum_{i=1}^{N} \left[ x_i \frac{\partial}{\partial x_i} + \theta_i \frac{\partial}{\partial \theta_i} \right] + \nu_2 \sum_{i=1}^{N} \left[ \frac{1}{x_i^2} - \frac{\theta_i}{x_i^2} \frac{\partial}{\partial \theta_i} \right]. \tag{5.10} \]

We call this operator the \textit{rational} form of the supersymmetric $BC_N, B_N, C_N$ Calogero systems.

The superspace coordinates $\chi_i$ which are invariant under the permutation and parity transformations are defined through the relation

\[ \sum_{k=0}^{N} \chi_k (\phi_i^2) t^k = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n} s_2 n (\phi_i^2) t^n \right], \tag{5.11} \]

(cf.\,(5.5) and \,(3.19)), where $\phi_i = x_i + \alpha \theta_i$ as in Sections 3 and 4, and $\chi_k (\phi_i^2) = \tilde{\sigma}_k (x_i^2) + \alpha \tilde{\zeta}_k (x_i^2, 2x_i \theta_i)$. Here $\tilde{\sigma}$ and $\tilde{\zeta}$ are the elementary symmetric polynomials defined in \,(3.9) but of the new arguments: $x \to x^2, \theta \to 2x \theta$. Using the same technique as in Sections 3-4, one can find the representation of the supersymmetric models directly from the analogous representation for the bosonic cases (for details see App.C). Finally, in the superspace coordinates \,(5.11) the Hamiltonian $h^{(c)}_{sBCD}$ has the form

\[ h^{(c)}_{sBCD} = \int d\alpha d\bar{\alpha} \sum_{i,j=1}^{N} A_{ij} \frac{\partial}{\partial \chi_i} \frac{\partial}{\partial \chi_j} + \int d\alpha \sum_{j=1}^{N} B_j \frac{\partial}{\partial \chi_j} \tag{5.12} \]

where

\[ A_{ij} = 2 \sum_{l \geq 0} \left\{ 2(2l + 1 + j - i) \chi_{i-l-1} \chi_{j+l} - l [ \chi_{j+l} \chi_{i-l-1} - \chi_{j+l} \chi_{i-l-1} \right. \]
\[ \left. + \chi_{i+l-1} \chi_{j-l} - \chi_{i+l-1} \chi_{j-l} ] \right\}, \]
\[ B_j = 2[1 + \nu_2 + 2\nu(N - j)](N - j + 1) \chi_{j-1} - 4\omega j \chi_{j}. \tag{5.13} \]

For completeness we give the form of $h^{(c)}_{sBCD}$ in components

\[ h^{(c)}_{sBCD} = \sum_{i,j=1}^{N} \left[ A_{ij} \tilde{\sigma} \frac{\partial^2}{\partial \tilde{\sigma}_i \partial \tilde{\sigma}_j} + A_{ij} \tilde{\zeta} \frac{\partial^2}{\partial \tilde{\sigma}_i \partial \tilde{\zeta}_j} + A_{ij} \tilde{\zeta} \frac{\partial^2}{\partial \tilde{\zeta}_i \partial \tilde{\sigma}_j} + A_{ij} \tilde{\zeta} \frac{\partial^2}{\partial \tilde{\zeta}_i \partial \tilde{\zeta}_j} \right] \]
\[ + \sum_{j=1}^{N} \left\{ 2[1 + \nu_2 + 2\nu(N - j)](j - N + 1) \left[ \tilde{\sigma}_{j-1} \frac{\partial}{\partial \tilde{\sigma}_j} + \tilde{\zeta}_{j-1} \frac{\partial}{\partial \tilde{\zeta}_j} \right] \right. \]
\[ \left. - 4\omega j \left[ \tilde{\sigma}_{j} \frac{\partial}{\partial \tilde{\sigma}_j} + \tilde{\zeta}_{j} \frac{\partial}{\partial \tilde{\zeta}_j} \right] \right\}. \tag{5.14} \]

where

\[ A_{ij} \tilde{\sigma} = 4 \sum_{l \geq 0} (2l + 1 + j - i) \tilde{\sigma}_{i+l-1} \tilde{\zeta}_{j+l} , \]
\[ A_{ij} \tilde{\zeta} = 2 \sum_{l \geq 0} \left[ 2(2l + 1 + j - i) \tilde{\sigma}_{i+l-1} \tilde{\zeta}_{j+l} - l [ \tilde{\sigma}_{j-l-1} \tilde{\zeta}_{i+l-1} - \tilde{\sigma}_{i+l-1} \tilde{\zeta}_{j-l-1} + \tilde{\sigma}_{j-l} \tilde{\zeta}_{i+l-1} - \tilde{\sigma}_{i+l-1} \tilde{\zeta}_{j-l} ] \right] \]
\[ A^{(c)}_{ij} = 2 \sum_{l \geq 0} \left[ 2(2l + 1 + j - i)\tilde{\sigma}_{i-l-1}\tilde{\sigma}_{j+l} + l[\tilde{\sigma}_{j-l-1}\tilde{\sigma}_{i+l-1} - \tilde{\sigma}_{j-l-1}\tilde{\sigma}_{i+l-1} - \tilde{\sigma}_{i+l-1}\tilde{\sigma}_{j-l}\right] \]

\[ A^{(c)}_{ij} = 4 \sum_{l \geq 0} \left[ (2l + 1 + j - i)\tilde{\sigma}_{i-l-1}\tilde{\sigma}_{j+l} - l\tilde{\sigma}_{i+l}\tilde{\sigma}_{j-l-1} - l\tilde{\sigma}_{i+l-1}\tilde{\sigma}_{j-l}\right] . \] (5.15)

From the expression (5.14) with the coefficients (5.15) it follows that \( h_{s_{BCD}}^{(c)} \) can be written as a quadratic polynomial in the generators of a Borel subalgebra of \( gl(N+1|N) \) and, hence, the \( BC_N \) Calogero model is exactly solvable.

The existence of the representation of (5.14) in terms of the generators of a Borel subalgebra of \( gl(N+1|N) \) (see (B.6)), proves that the operator \( h_{s_{BCD}}^{(c)} \) has infinitely-many eigenfunctions having the form of polynomials. These polynomials are related to finite-dimensional irreducible representations of the algebra \( gl(N+1|N) \) in the realization (A.3) and can be called the supersymmetric \( BC_N \) Calogero polynomials.

So the supersymmetric \( BC_N, B_N, C_N \) and \( D_N \) Calogero models possess the algebraic form and also the Lie-algebraic form (B.6) being represented by second-order polynomials in the generators of the of the algebra \( gl(N+1|N) \) with certain coefficients.

6 The \( BC_N, B_N, C_N \) and \( D_N \) Sutherland models and their supersymmetric extensions

Similarly to what was done in Section 5 for the \( BC_N, B_N, C_N \) and \( D_N \) Calogero models the present section is devoted to a consideration of the \( BC_N, B_N, C_N \) and \( D_N \) Sutherland models.

6.1 The bosonic case

The Hamiltonians for \( BC_N, B_N, C_N, D_N \) Sutherland models are special cases of the general \( BC_N \) Hamiltonian [7]

\[ H_{BCD}^{(s)} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g_2 \sum_{i<j} \left[ \frac{1}{\sin^2(\frac{1}{2}(x_i - x_j))} + \frac{1}{\sin^2(\frac{1}{2}(x_i + x_j))} \right] + g_3 \sum_{i=1}^{N} \frac{1}{\sin^2(\frac{x_i}{2})} \] (6.1)

where \( g = \nu(\nu - 1), g_2 = \nu_2(\nu_2 - 1) \) and \( g_3 = \nu_3(\nu_3 + 2\nu_2 - 1) \). From the general Hamiltonian the \( B_N, C_N \) and \( D_N \) cases are obtained as follows:

- \( B_N \) case: \( \nu_2 = 0 \),
- \( C_N \) case: \( \nu_3 = 0 \), and
- \( D_N \) case: \( \nu_2 = \nu_3 = 0 \).
The ground state wave function is \([10, 13]\)

\[
\Psi_0 = \prod_{i<j} \left| \sin \left( \frac{1}{2} (x_i - x_j) \right) \right|^{\nu_i} \prod_{i=1}^{N} \left| \sin \left( \frac{1}{2} x_i \right) \right|^{\nu_2} \sin \left( \frac{x_i}{2} \right)^{\nu_3} .
\] (6.2)

Again, one should emphasize that to a fixed value of the coupling constant \(g(g_2)g_3\) there corresponds two different values of the parameter \(\nu(g_2)\nu_3 : \nu(g_2)\nu_3 = \alpha(2)[\alpha_3]\) and \(1 - \alpha(1 - \alpha_2)(1 - \alpha_3)\) giving rise to eight families of eigenfunctions. In order to get the ground state the parameters \(\alpha, \alpha_2, \alpha_3\) should be chosen in such a way as to minimize the eigenvalue and then (6.2) corresponds to the ground state. It can be denoted \((\alpha, \alpha_2, \alpha_3)\).

The other values of \(\nu, \nu_2, \nu_3\), if inserted in (6.2), describe (provided that the corresponding wavefunctions are normalizable) the ground states of the remaining seven other families of eigenstates:

\[
(\alpha, \alpha_2, 1 - \alpha_3) , (\alpha, 1 - \alpha_2, \alpha_3) , (\alpha, 1 - \alpha_2, 1 - \alpha_3) ,
\]

\[
(1 - \alpha, \alpha_2, \alpha_3) , (1 - \alpha, 1 - \alpha_2, \alpha_3) , (1 - \alpha, \alpha_2, 1 - \alpha_3) ,
\]

\[
(1 - \alpha, 1 - \alpha_2, 1 - \alpha_3) .
\]

So taking different values of \(\nu, \nu_2, \nu_3\) in (6.2) one can exhaust all the types of ground states corresponding to the different families mentioned above.\(^{13}\)

Using the same approach as in Sections 2-4, we make a gauge rotation of (5.1) with the ground-state eigenfunction as gauge factor, \(h^{(s)}_{BCD} = -2\Psi_0^{-1}\mathcal{H}^{(s)}_{BCD}\Psi_0\). A straightforward calculation leads to the operator (we omit an additive constant)

\[
h^{(s)}_{BCD} = \sum_{i=1}^{N} \partial_i^2 + \nu \sum_{i<j} \left[ \cot \left( \frac{1}{2} (x_i - x_j) \right) (\partial_i - \partial_j) \right] + \cot \left( \frac{1}{2} x_i \right) \partial_i .
\] (6.3)

It is worth mentioning that if the operator (6.3) is written in the coordinates \(e^{ix}\), it appears in its rational form \([13]\) (cf. the rational form of the \(A_{N-1}\) Sutherland Hamiltonian in \([8]\)).

According to the above discussion about eight different families of eigenfunctions, it is sufficient to study the operator (6.3) for generic \(\nu, \nu_2, \nu_3\) and consider only eigenfunctions which are symmetric under reflections. In particular, this implies that eigenfunctions of (6.3) have the form

\[
P^{\nu, \nu_2, \nu_3} = F(\cos x_1, \cos x_2, \ldots, \cos x_N) ,
\] (6.4)

where \(F\) is permutationally symmetric.

Next we encode the symmetry properties of the problem studied by introducing the permutation and reflection-invariant periodic variables \(\tilde{\sigma}_k(\cos x)\), which satisfy

\[
\sum_{k=0}^{N} \tilde{\sigma}_k(\cos x_i)t^k = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n}s_n(\cos x_i)t^n \right] ,
\] (6.5)

\(^{12}\)See also the discussion in footnote 7 and in Section 5.1

\(^{13}\)In what follows, for the sake of simplicity, we continue to call (6.2) the ground state, of course, keeping in mind above discussion.
Let us emphasize that these variables (6.5) are characterized by the same period as the original Hamiltonian (6.1). In these variables the Hamiltonian $h_{BCD}^{(s)}$ becomes

$$h_{BCD}^{(s)} = \sum_{i,j=1}^{N} A_{ij} \frac{\partial}{\partial \hat{\sigma}_i} \frac{\partial}{\partial \hat{\sigma}_j} + \sum_{j=1}^{N} B_j \frac{\partial}{\partial \hat{\sigma}_j}$$

(6.6)

where

$$A_{ij} = \sum_{l \geq 0} \left[ (i - l)\hat{\sigma}_{i-l}\hat{\sigma}_{j+l} + (l + j - 1)\hat{\sigma}_{i-l-1}\hat{\sigma}_{j+l-1} - (i - 2 - l)\hat{\sigma}_{i-2-l}\hat{\sigma}_{j+l} - (l + j + 1)\hat{\sigma}_{i-l-1}\hat{\sigma}_{j+l+1} \right]$$

$$B_j = \nu_3 \left( j - N - 1 \right) \hat{\sigma}_{j-1} - \frac{\nu_3}{2} + 1 + \nu \left( 2N - j - 1 \right) \right] j \hat{\sigma}_j$$

$$- \nu \left( N - j + 1 \right) \left( N - j + 2 \right) \hat{\sigma}_{j-2},$$

(6.7)

and $\hat{\sigma}_k = 0$, when $k < 0$ or $k > N$. The method of the calculation of the coefficients $A_{ij}, B_j$ is similar to that presented in App.C. This expression can be called the algebraic form of the $BC_N, B_N, C_N$ and $D_N$ Sutherland Hamiltonians. Similarly to what happened in all previously discussed bosonic cases the coefficients $A_{ij}, B_j$ are polynomials of second and first degree in $\hat{\sigma}_k$, respectively. Hence, the Hamiltonian $h_{BCD}^{(s)}$ can be written in terms of generators of the Borel subalgebra of $gl(N+1)$ realized as first order differential operators (see Appendix A.1) as in Sections 2 and 5.1. The result is given in eq. (B.7) (see Appendix B).

Then in accordance with the general definition given in [1] we conclude that all $BC_N, B_N, C_N, D_N$ Sutherland models (6.1) are exactly solvable. The existence of the representation (6.6) proves that there are infinitely-many eigenfunctions of the operator (5.3) having the form of polynomials in the variables $\hat{\sigma}_k$. It also implies that totally (anti)symmetric-(anti)symmetric eigenfunctions with respect to permutations and reflections of the $BC_N, B_N, C_N$ and $D_N$ Sutherland models (6.1) have a factorizable form being the product of the ground-state eigenfunction (6.2) multiplied by a polynomial in the variables $\hat{\sigma}_k$. These polynomials are related to finite-dimensional irreducible representations of the Lie algebra $gl(N+1)$ in the realization (A.1). They can be called the $BC_N$ Jack-Sutherland polynomials.

### 6.2 The supersymmetric extensions

The bosonic $BC_N, B_N, C_N$ and $D_N$ Sutherland models (6.1) have natural supersymmetric extensions. In this subsection we will construct these models and show that, as was the case for the other supersymmetric extensions discussed in Sections 3, 4 and 5.2, these models are also exactly-solvable. Let us introduce the supercharges (3.3) with the superpotential $W$ given by

$$W = \nu \sum_{i<j} \left[ \log \left| \sin \left( \frac{1}{2} (x_i - x_j) \right) \right| + \log \left| \sin \left( \frac{1}{2} (x_i + x_j) \right) \right| \right] + \nu_2 \sum_{i=1}^{N} \log \left| \sin (x_i) \right|$$

$$+ \nu_3 \sum_{i=1}^{N} \log \left| \sin \left( \frac{x_i}{2} \right) \right|.$$  

(6.8)
In terms of these variables the Hamiltonian (6.10) becomes

\[ H^{(s)}_{sBCD} = -\frac{1}{2} \sum_{i=1}^{N} \partial^2_{x_i} + \nu \sum_{i<j} \left[ \cot \left( \frac{1}{2} (x_i - x_j) \right) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) - \frac{1}{2} \sin^2 \left( \frac{1}{2} (x_i - x_j) \right) \left( \frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j} \right) \right] + \nu \sum_{i<j} \left[ \cot \left( \frac{1}{2} (x_i + x_j) \right) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) - \frac{1}{2} \sin^2 \left( \frac{1}{2} (x_i + x_j) \right) \left( \frac{\partial}{\partial \theta_i} + \frac{\partial}{\partial \theta_j} \right) \right] + \nu_2 \sum_{i=1}^{N} \left[ \cot(x_i) \frac{\partial}{\partial x_i} - \frac{\theta_i}{\sin^2(x_i)} \frac{\partial}{\partial \theta_i} \right] + \nu_3 \sum_{i=1}^{N} \left[ \cot \left( \frac{x_i}{2} \right) \frac{\partial}{\partial x_i} - \frac{1}{2} \frac{\theta_i}{\sin^2 \left( \frac{x_i}{2} \right)} \frac{\partial}{\partial \theta_i} \right], \]  

(6.10)

where \( C = -\frac{1}{2} \sum_{i=1}^{N} \nu(N - i) + \nu_2 + \nu_3 \). As should be familiar by now, the next step is to introduce the gauge-rotated Hamiltonian \( h^{(s)}_{sBCD} = -2\Psi_0^{-1} H^{(s)}_{sBCD} \Psi_0 \). After some calculations we get

\[ h^{(s)}_{BCD} = \sum_{i=1}^{N} \partial^2_{\chi_i} + \nu \sum_{i<j} \left[ \cot \left( \frac{1}{2} (x_i - x_j) \right) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) - \frac{1}{2} \sin^2 \left( \frac{1}{2} (x_i - x_j) \right) \left( \frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j} \right) \right] + \nu \sum_{i<j} \left[ \cot \left( \frac{1}{2} (x_i + x_j) \right) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) - \frac{1}{2} \sin^2 \left( \frac{1}{2} (x_i + x_j) \right) \left( \frac{\partial}{\partial \theta_i} + \frac{\partial}{\partial \theta_j} \right) \right] + \nu_2 \sum_{i=1}^{N} \left[ \cot(x_i) \frac{\partial}{\partial x_i} - \frac{\theta_i}{\sin^2(x_i)} \frac{\partial}{\partial \theta_i} \right] + \nu_3 \sum_{i=1}^{N} \left[ \cot \left( \frac{x_i}{2} \right) \frac{\partial}{\partial x_i} - \frac{1}{2} \frac{\theta_i}{\sin^2 \left( \frac{x_i}{2} \right)} \frac{\partial}{\partial \theta_i} \right]. \]

(6.11)

It is worth mentioning that if the operator (6.10) is written in the coordinates \( e^{ix} \) it appears in its rational form \([3]\).

As in Sections 3, 4 and 5.2 it is convenient to use the superspace formalism and to introduce the superspace coordinates \( \chi_i \), which satisfies:

\[ \sum_{k=0}^{N} \chi_k(\cos(\phi_i)) t^k = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_n(\cos(\phi_i)) t^n \right]. \]

(6.11)

Here \( \chi_k(\cos(\phi_i)) = \delta_k(\cos x_i) + \alpha_k(\cos x_i, -\theta_i \sin x_i) \), where \( \delta_k \) and \( \alpha_k \) are the elementary symmetric polynomials defined in (3.3) but of new arguments: \( x \rightarrow \cos x, \theta \rightarrow -\theta \sin x \). In terms of these variables the Hamiltonian (6.10) becomes

\[ h^{(s)}_{sBCD} = \int d\alpha d\bar{\alpha} \sum_{i,j=1}^{N} A_{ij} \frac{\partial}{\partial \chi_i} \frac{\partial}{\partial \chi_j} + \int d\alpha \sum_{j=1}^{N} B_j \frac{\partial}{\partial \chi_j}, \]

(6.12)

where

\[ A_{ij} = N \chi_{i-1} \bar{\chi}_{j-1} - \sum_{l \geq 0} \left[ (i - l) \chi_{i-l} \bar{\chi}_{j+l} + (l + j - 1) \chi_{i-1} \bar{\chi}_{j+l-1} - (i - 2 - l) \chi_{i-2} \bar{\chi}_{j+l} - (l + j + 1) \chi_{i-l} \bar{\chi}_{j+l+1} + l[\chi_{i+l} \bar{\chi}_{j-l-1} - \chi_{i+l-1} \bar{\chi}_{j-l}] \right], \]

\[ B_j = \frac{\nu_3}{2} (j - N - 1) \chi_{j-1} - (\nu_2 + \frac{\nu_3}{2} + 1 + \nu(2N - j - 1)) j \chi_j - \nu(N - j + 1)(N - j + 2) \chi_{j-2}. \]

(6.13)

This expression should be compared to the rational form of the supersymmetric \( A_N \) Sutherland model (4.3) and the \( BC_N \) Sutherland model (6.3).
The method of the calculation of the coefficients \( A_{ij}, B_j \) is similar to that presented in App.C. In components \( h_{sBCD}^{(s)} \) becomes

\[
\begin{align*}
  h_{sBCD}^{(s)} = & \sum_{i,j=1}^{N} \left[ A_{ij}^\sigma \frac{\partial^2}{\partial \sigma_i \partial \sigma_j} + A_{ij}^\zeta \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} + A_{ij}^\nu \frac{\partial^2}{\partial \nu_i \partial \nu_j} + A_{ij}^{\sigma_\zeta} \frac{\partial^2}{\partial \sigma_i \partial \zeta_j} \right] \\
  & + \sum_{j=1}^{N} \left\{ \frac{\nu_3}{2} (j - N - 1) \left[ \sigma_{j-1} \frac{\partial}{\partial \sigma_{j-1}} + \zeta_{j-1} \frac{\partial}{\partial \zeta_{j-1}} \right] - \left( \nu_2 + \frac{\nu_3}{2} + 1 + \nu (2N - j - 1) \right) j \cdot \left[ \sigma_{j-1} \frac{\partial}{\partial \sigma_{j-1}} + \zeta_{j-1} \frac{\partial}{\partial \zeta_{j-1}} \right] \right\}, \quad (6.14)
\end{align*}
\]

where

\[
\begin{align*}
  A_{ij}^\sigma &= N \hat{\sigma}_{i-1} \hat{\sigma}_{j-1} - \sum_{l \geq 0} \left[ (i - l) \hat{\sigma}_{i-l} \hat{\sigma}_{j+l} + (l + j - 1) \hat{\sigma}_{i-l} \hat{\sigma}_{j+l} \right] \\
  A_{ij}^\zeta &= N \hat{\zeta}_{i-1} \hat{\zeta}_{j-1} - \sum_{l \geq 0} \left[ (i - l) \hat{\zeta}_{i-l} \hat{\zeta}_{j+l} + (l + j - 1) \hat{\zeta}_{i-l} \hat{\zeta}_{j+l} \right] \\
  A_{ij}^{\sigma_\zeta} &= N \hat{\sigma}_{i-1} \hat{\zeta}_{j-1} - \sum_{l \geq 0} \left[ (i - l) \hat{\sigma}_{i-l} \hat{\zeta}_{j+l} + (l + j - 1) \hat{\sigma}_{i-l} \hat{\zeta}_{j+l} \right] \\
  A_{ij}^{\zeta_\sigma} &= N \hat{\zeta}_{i-1} \hat{\sigma}_{j-1} - \sum_{l \geq 0} \left[ (i - l) \hat{\zeta}_{i-l} \hat{\sigma}_{j+l} + (l + j - 1) \hat{\zeta}_{i-l} \hat{\sigma}_{j+l} \right] \\
  A_{ij}^{\nu} &= N \hat{\nu}_{i-1} \hat{\nu}_{j-1} - \sum_{l \geq 0} \left[ (i - l) \hat{\nu}_{i-l} \hat{\nu}_{j+l} + (l + j - 1) \hat{\nu}_{i-l} \hat{\nu}_{j+l} \right]. \quad (6.15)
\end{align*}
\]

From the expression (6.14) with the coefficients (6.15) it follows, that \( h_{sBCD}^{(s)} \) can be written as a quadratic polynomial in the generators of a Borel subalgebra of \( gl(N + 1|N) \) and, hence, the \( BC_N \) Sutherland model is exactly solvable.

The existence of the representation of (6.14) in terms of the generators of a Borel subalgebra of \( gl(N + 1|N) \) (see (B.8)) proves that there are infinitely-many eigenfunctions of (6.14) having the form of polynomials. These polynomials are related to finite-dimensional irreducible representations of the algebra \( gl(N + 1|N) \) in the realization (A.3). These polynomials can be called the supersymmetric \( BC_N \) Sutherland polynomials.

So the supersymmetric \( BC_N, B_N, C_N \) and \( D_N \) Sutherland models possess an algebraic form and also the Lie-algebraic form (B.8) being represented by second-order polynomials in the generators of the of the algebra \( gl(N + 1|N) \) with certain coefficients.
7 Conclusion

In this paper we have described the rational, algebraic and Lie-algebraic forms of the integrable $A_N - B_N - C_N - D_N$ rational (Calogero) and trigonometric (Sutherland) Hamiltonians in addition to the superspace expressions for the supersymmetric generalizations of these models. We have shown that

All Hamiltonians of the integrable $A_N - B_N - C_N - D_N$ rational (Calogero) and trigonometric (Sutherland) models possess the same hidden algebra $gl(N + 1)$ and can be represented by second-degree polynomials in the generators of a Borel subalgebra of the $gl(N + 1)$-algebra. If the configuration space is parametrized by permutationally symmetric coordinates $v$, the Hamiltonians have a triangular form and preserve the flag of spaces of inhomogeneous polynomials

$$\mathcal{P}_n = \text{span}\{v_1^{n_1}v_2^{n_2}v_3^{n_3} \ldots v_N^{n_N}|0 \leq \sum n_i \leq n\},$$

Consequently, each Hamiltonian possesses one or several infinite families of polynomial eigenfunctions.

All Hamiltonians of the supersymmetric generalizations of the $A_N - B_N - C_N - D_N$ rational (Calogero) and trigonometric (Sutherland) models possess the same hidden algebra $gl(N + 1|N)$ and can be represented by second-degree polynomials in the generators of a Borel subalgebra of the $gl(N + 1|N)$-algebra. If the configuration space is parametrized by permutationally symmetric coordinates $v, \kappa$, the Hamiltonians have a triangular form and preserve the flag of spaces of inhomogeneous polynomials

$$\mathcal{P}_n = \text{span}\{v_1^{n_1}v_2^{m_2} \ldots v_N^{n_N} \kappa_1^{n_1} \kappa_2^{m_2} \ldots \kappa_N^{m_N}|0 \leq \sum n_i + \sum m_i \leq n, m_i = 0, 1\},$$

Consequently, each Hamiltonian has one or several infinite families of polynomial eigenfunctions. The integrability of the $A_N$ supersymmetric system was proven in \cite{6}; as for the $B_N - C_N - D_N$ systems, it is an open question.

As an interesting issue for future study we would like to mention the question about whether there exists a Lie algebraic description of the higher $A_N - B_N - C_N - D_N$ rational (Calogero) and trigonometric (Sutherland) Hamiltonians. It is very probable that this will be the case and, perhaps, even the Lax operator can be represented in terms of $gl(N)$-generators. There are almost no doubts that it will be possible to extend the analysis of Sections 2-6 to the case of the exceptional simple Lie algebras. In particular, a special case of the one-parametric $G(2)$ rational and trigonometric Hamiltonians corresponding to three-body interactions only \cite{22, 23} possesses a hidden $gl(3)$-algebra and, hence, is expressible in terms of the $gl(3)$ generators \cite{23, 24}.

The question about the integrability of the above supersymmetric models leads naturally to the question about the possibility to develop the Hamiltonian reduction method for the supersymmetric cases and attempt to connect these models to $2d$ supersymmetric gauge theories along the lines of \cite{11}.

\begin{itemize}
  \item[15] where $\kappa^2 = 0$.
  \item[16] Recently, a Lie algebraic form for the general $G(2)$ rational and trigonometric models was found \cite{24}.
\end{itemize}
An alternative description of the supersymmetric models discussed in this paper can be obtained by using the matrix representation of the $\theta^\alpha$’s in terms of the Pauli matrices, $\sigma^{\pm,3}$:

\[ \begin{align*}
\theta^1 &= \sigma^- \otimes 1 \otimes 1 \otimes \cdots \\
\frac{\partial}{\partial \theta^1} &= \sigma^+ \otimes 1 \otimes 1 \otimes \cdots \\
\theta^2 &= \sigma^3 \otimes \sigma^- \otimes 1 \otimes \cdots \\
\frac{\partial}{\partial \theta^2} &= \sigma^3 \otimes \sigma^+ \otimes 1 \otimes \cdots \\
&\vdots
\end{align*} \]

(7.1)

In this formalism our supersymmetric models become matrix generalizations of the ‘scalar’ $A_N - B_N - C_N - D_N$ Calogero and Sutherland models. For the $A_N$ case our matrix models are particular cases of the general construction with internal degrees of freedom proposed in [25].

The existence of the Lie algebraic form for the exactly-solvable $A_N - B_N - C_N - D_N$ Calogero and Sutherland models can be considered as a good starting point to investigate whether there exist other exactly-solvable problems and/or quasi-exactly-solvable generalizations (for definitions see, for instance, [26, 27]). The first step in applying the Lie algebraic formalism to the search for quasi-exactly-solvable problems was taken in [28].

As a final comment we would like to mention that a great challenge in the subject is the search for solutions of the $N$-body elliptic model (for a description see, for example, [10]). For the two-body case, the rational and algebraic forms have been known since Hermite’s days. Recently, it was found that the 2-body case (the Lame operator) admits a Lie algebraic form with the hidden algebra $gl(2)$ and is a quasi-exactly-solvable problem [29] (see also [1]). It gives a hint that the three-body (and more generally the $N$-body) elliptic problem has to be quasi-exactly-solvable if any analytic solution exists whatsoever. However, all attempts to find a rational, algebraic or Lie algebraic form even for the three-body elliptic model have failed so far.

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Appendices

A Representations of the Lie algebra $gl(N)$ and the Lie superalgebra $gl(N|M)$.

A.1 $gl(N)$

The generators of the Lie algebra $gl(N)$ can be realized by first order differential operators. For our purposes we need one of the simplest realizations of $gl(N)$ acting on the real (complex) space of dimension $(N-1)$. This is the space of minimal dimension where this algebra can act. The generators can be represented in the following form:

$$
E_{0i} = J_i^- = \frac{\partial}{\partial \tau_i}, \quad i = 2, 3, \ldots, N,
$$

$$
E_{ij} = J_{i,j}^0 = \tau_i J_j^- = \tau_i \frac{\partial}{\partial \tau_j}, \quad i, j = 2, 3, \ldots, N,
$$

$$
E_{00} = J^0 = n - \sum_{k=2}^{N} \tau_k \frac{\partial}{\partial \tau_k},
$$

$$
E_{i0} = J_i^+ = \tau_i J^0, \quad i = 2, 3, \ldots, N,
$$

where the parameter $n \in \mathbb{R} (\mathbb{C})$. One of the generators, namely $J^0 + \sum_{p=2}^{N} J_{p,p}^0$, is proportional to a constant and, if it is removed, we end up with the algebra $sl(N)$. The generators $J_{i,j}^0$ form the algebra of the vector fields of $sl(N - 1)$, which is a subalgebra of $gl(N)$. If $n$ is a non-negative integer, the representation (A.1) becomes the finite-dimensional representation acting on the space of polynomials in $(N - 1)$ variables of the following type

$$
V_n(t) = \text{span}\{\tau_2^{n_2} \tau_3^{n_3} \tau_4^{n_4} \cdots \tau_N^{n_N} | 0 \leq \sum n_i \leq n\}. \quad (A.2)
$$

This representation corresponds to a Young tableau with one row and $n$ blocks and is irreducible. The positive-root generators $J_i^+$ define the highest-weight vector. If the $J_i^+$’s are removed, the remaining generators form a Borel subalgebra.

A.2 $gl(N|M)$

Similarly to what was done for the algebra $gl(N)$ one can construct a representation of the Lie superalgebra $gl(N|M)$ in terms of first order differential operators over the direct sum of an even space and an odd space. Again the simplest realization of $gl(N|M)$ act on the space $C^{(N-1|M)}$ spanned by the even variables $(\tau^i | i = 1, 2, \ldots, N - 1)$ and the odd variables $(\kappa^\gamma | \gamma = 1, 2, \ldots, M)$ and is given by

$$
E_{ij} = J_{i,j}^0 = \tau^i \frac{\partial}{\partial \tau^j}, \quad E_{00} = J^0 = -n + \sum_i \tau^i \frac{\partial}{\partial \tau^j} + \sum_\gamma \kappa^\gamma \frac{\partial}{\partial \kappa^\gamma},
$$

$$
E_{i0} = J_i^+ = \tau^i J^0, \quad E_{0i} = J_i^- = \frac{\partial}{\partial \tau^i},
$$

$$
E_{\gamma i} = Q_{\gamma i} = \kappa^\gamma \frac{\partial}{\partial \tau^i}, \quad E_{i\gamma} = \bar{Q}_{i\gamma} = \tau^i \frac{\partial}{\partial \kappa^\gamma},
$$

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\[ E_{0\gamma} = Q^- = \frac{\partial}{\partial \kappa^\gamma}, \quad E_{\gamma 0} = Q^+ = \kappa^\gamma T^0 \]

\[ E_{\gamma \beta} = T^{\gamma \beta} = \kappa^\gamma \frac{\partial}{\partial \kappa^\beta} \]  \hspace{1cm} (A.3)

where the parameter \( n \in \mathbb{R} \) (C). Here the indices run over the following values, \( i = 1 \ldots N - 1 \) and \( \gamma = 1 \ldots M \). The subset \( \{E_{ij}, E_{\gamma i}, E_{i\gamma}, E_{\gamma \beta}\} \) form the “vector field” representation of \( gl(N - 1|1) \). The algebra has a \( \mathbb{Z}_2 \) gradation and it is clear that the elements with an odd number of Greek indices are odd and the elements with an even number are even. Using the composite index notation \( I = (0, i, \gamma) \), the commutation relations of the superalgebra can be written (cf. the commutation relations in the defining \( (N + M) \times (N + M) \) supermatrix representation, with \((E_{IJ})_{MN} = \delta_{IN}\delta_{JM}\))

\[
\begin{align*}
[E_{IJ}, E_{KL}] &= \delta_{JK} E_{IL} - \delta_{IL} E_{KJ}, \quad \text{(both even)} \\
[E_{IJ}, E_{KLL}] &= \delta_{JK} E_{IL} - \delta_{IL} E_{KJ}, \quad \text{(\( E_{IJ} \) even, \( E_{KL} \) odd)} \hspace{1cm} (A.4) \\
\{E_{IJ}, E_{KLL}\} &= \delta_{JK} E_{IL} + \delta_{IL} E_{KJ}, \quad \text{(both odd)}
\end{align*}
\]

The Lie superalgebra \( gl(N|M) \) is not simple. There are two different cases: (i) \( N \neq M \) and (ii) \( N = M \). Let us consider the first case, when \( N \neq M \). Removing the unit element \( \sum_{I=0}^{N+M} E_{II} \) we are left with the superalgebra \( sl(N|M) \), which is simple. The “diagonal” generators for this case are given by

\[
H_{II} = \begin{cases} 
E_{00} - E_{11}, & I = 0 \\
E_{II} - E_{I+1,I+1}, & 1 \leq I \leq N - 1 \\
E_{II} + E_{I+1,I+1}, & I = N \\
E_{II} - E_{I+1,I+1}, & N + 1 \leq I \leq N + M - 1
\end{cases} \]  \hspace{1cm} (A.5)

When \( N = M \) the situation is slightly more complicated. There are two one dimensional abelian ideals, which must be removed in order to make the algebra simple.

The generators of \( gl(N|M) \) defined in (A.3) act in the space spanned by the monomials \( \{\tau_1^{i_1} \ldots \tau_{N-1}^{i_{N-1}} \kappa_1^{\delta_1} \ldots \kappa_M^{\delta_M}\} \). Here \( \delta_k \) equals either 0 or 1. When \( n \) (in the expression for \( T^0 \) above) is an integer we have finite-dimensional highest weight representations whose highest weight vectors satisfy \( \sum_{k=1}^{N-1} i_k + \sum_{k=1}^{M} \delta_k = n \). These representations form a flag of spaces.

**B The Lie algebraic forms of the** \( A_{N-1}, BC_N, B_N, C_N \) **and** \( D_N \) **Hamiltonians**

In this Appendix we collect all Lie algebraic forms of the Hamiltonians for the (supersymmetric) \( A_{N-1}, BC_N, B_N, C_N \) and \( D_N \) Calogero and Sutherland models found in [12] and in the present paper.

- **\( A_{N-1} \) Calogero model**

The Lie algebraic form of the many-body Calogero model is the following [12]

\[
h_{\text{Cal}} = \sum_{j=2}^{N} \left\{ \frac{(N - j + 1)(j - 1)}{N} (J_{j-1,j})^2 - 2 \sum_{\ell=1}^{j-1} \ell J_{j+\ell-1,j} J_{j-\ell-1,j} \right\}
\]
where the following notations are introduced for the $gl(N)$ generators (A.1): $J_{ij} \equiv J^0_{ij}$, $J^0_{i,k} \equiv J^0_k$, $J^0_{1,k} \equiv 0$.

The supersymmetric Calogero model has the Lie algebraic form:

\[
\mathcal{h}_{\text{Cal}}^{(\text{rel})} = \sum_{i,j=2}^{N} \left\{ \frac{(N-i+1)(j-1)}{N} [J_{i-1,j}J_{j-1,i} + J_{i-1,i}T_{j-1,j} + J_{j-1,j}T_{i-1,i}] \\
- T_{i-1,j}T_{j-1,i} \right\} + \sum_{l>\max(1,j-i)} (j-i-2l)[J_{i+l-1,j}J_{j-l-1,i} + J_{i+l-1,i}T_{j-l-1,j} + J_{j-l-1,j}T_{i+l-1,i}] \\
+ \sum_{l=1}^{N} \{T_{j-2-l,i}J_{i+l,i} - T_{i+l,i}J_{j-2-l,i} + T_{i+l,i}J_{j-2-l,i} - T_{j-2-l,i}J_{i+l,i} \} \\
- \frac{1}{N} + \nu \sum_{j=2}^{N} (N-j+2)(N-j+1)[J_{j-2,j} + T_{j-2,j}],
\]

(B.2)

where the following conventions are used for the $gl(N|N-1)$ generators (A.3): $J_{ij} \equiv J^0_{ij} = 0$ and $T_{ij} \equiv T^0_{ij} = 0$, if $i$ and/or $j$ is less than 2 or greater than $N$.

- **$A_{N-1}$ Sutherland model**

The Lie algebraic form of the many-body Sutherland model in terms of the $gl(N)$ generators is the following (12)

\[
- \mathcal{h}_{\text{Suth}} = \sum_{j=1}^{N-1} \left\{ \frac{j(N-j)}{N} (J_{j,j})^2 - 2 \sum_{l=1}^{j} l J_{j+l,j}J_{j-l,j} \right\} + \nu \sum_{l=1}^{N-1} (N-l)J_{l,l} \\
+ 2 \sum_{1 \leq k < j \leq N-1} \left\{ \frac{k(N-j)}{N} J_{j,k}J_{k,j} - \sum_{l=1}^{k} (j-k+2l)J_{j+l,j}J_{k-l,k} \right\}.
\]

(B.3)

Here we have used the identifications $J_{ij} \equiv J^0_{ij}$, $J_{0,i} \equiv J^-_{i}$, $J_{N,i}$, $J_{1,i} = 0$. The supersymmetric many-body Sutherland model written in terms of the $gl(N|N-1)$ generators has the form:

\[
- \mathcal{h}_{\text{Suth}}^{(\text{rel})} = \sum_{i,j=1}^{N-1} \left\{ \frac{j(N-i)}{N} [J_{i,j}J_{j,i} + J_{i,i}T_{j,j} + J_{j,j}T_{i,i} - T_{i,j}T_{j,i}] \right\} \\
+ \sum_{l>\max(1,j-i)} (j-i-2l)[J_{i+l,j}J_{j-l,i} + J_{i+l,i}T_{j-l,j} + J_{j-l,j}T_{i+l,i} - T_{i+l,i}T_{j-l,i}] \\
+ \sum_{l=1}^{\infty} \{T_{j-l-1,i}J_{i+l-1,i} - T_{i+l-1,i}J_{j-l-1,i} + T_{i+l-1,i}J_{j-l-1,i} - T_{j-l-1,i}J_{i+l-1,i} \}
\]

\[+ 2 \sum_{1 \leq k < j \leq N-1} \left\{ \frac{k(N-j)}{N} J_{j,k}J_{k,j} - \sum_{l=1}^{k} (j-k+2l)J_{j+l,j}J_{k-l,k} \right\}.
\]
In (B.4) $J_{ij} \equiv j_{ij}^0 = 0$ and $T_{ij} \equiv T_{ij}^0 = 0$ are by definition zero when $i$ and/or $j$ is greater than $N - 1$ or less than 1.

• $BC_N$, $B_N$, $C_N$ and $D_N$ Calogero models

The bosonic $BC_N$, $B_N$, $C_N$ and $D_N$ Calogero models have the following Lie algebraic form in terms of the $gl(N + 1)$ generators

$$h_{BCD}^{(c)} = \begin{array}{c}
4 \sum_{i,j=1}^{N} \sum_{l \geq 0} (2l + 1 + j - i)J_{i-1,j}J_{j+1,i} - 4\nu \sum_{i=1}^{N} iJ_{i,i} \\
+ 2 \sum_{i=1}^{N} \left\{ [1 + \nu_2 + 2\nu(N - j)](N - j + 1) - \frac{N}{2}(N + 3 - 2i) \right\} J_{i-1,i}
\end{array}$$

(B.5)

and the supersymmetric extensions have the form

$$h_{sBCD}^{(c)} = \begin{array}{c}
2 \sum_{i,j=1}^{N} \sum_{l \geq 0} (2l + 1 + j - i)[J_{i-1,j}J_{j+1,i} + J_{i-1,j}T_{j+1,i} + T_{i-1,j}J_{j+1,i} \\
- T_{i-1,j}J_{j+1,i} + l[T_{j-1,j}J_{j+1,i} - T_{j+1,j}J_{j-1,i} - T_{i-1,j}J_{j-1,i} \\
+ T_{j-1,j}J_{j+1,i} - T_{j+1,j}J_{j-1,i} + T_{i+1,j}J_{j-1,i} - T_{j-1,j}J_{j-1,i} \\
+ T_{i+1,j}T_{j-1,i} + T_{i+1,j}T_{j-1,i}] + \sum_{i=1}^{N} 2[1 + \nu_2 + 2\nu(N - j)](N - j + 1) \\
\cdot \left\{ \frac{N}{2}(N + 3 - 2i)T_{i-1,i} - \frac{N}{2}(N + 3 - 2i)J_{i-1,i} \right\} - 4\omega \sum_{i=1}^{N} i[J_{i,i} + T_{i,i}].
\end{array}$$

(B.6)

in terms of the $gl(N + 1|N)$ generators. In the above two formulas, as well as in the two formulas below, we have used that $J_{ij} \equiv J_{ij}^0 = 0$ and $T_{ij} \equiv T_{ij}^0 = 0$ when $i$ and/or $j$ is greater than $N$ or less than 1.

• $BC_N$, $B_N$, $C_N$ and $D_N$ Sutherland models

Finally, the Lie algebraic form of the bosonic $BC_N$, $B_N$, $C_N$ and $D_N$ Sutherland models is

$$h_{BCD}^{(s)} = \begin{array}{c}
\sum_{i,j=1}^{N} \left[ NJ_{i-1,j}J_{j-1,i} - \sum_{l \geq 0} [(i - l)J_{i-1,j}J_{j+1,i} + (l + j - 1)J_{i-1,j}J_{j+1,i} \\
- (i - 2l - 1)J_{i-2,j}J_{j+1,i} - (l + j - 1)J_{i-1,j}J_{j+1,i}] \\
+ \sum_{j=1}^{N} \left\{ \frac{\nu_3}{2} (j - N - 1)J_{j-1,j} - (\nu_2 + \nu_3) + 1 + N + \nu(2N - j - 1))J_{j,j} \\
- \nu(N - j + 1)(N - j + 2) + 2N]J_{j-2,j} \right\}
\end{array}$$

(B.7)
in terms of the $gl(N + 1)$ generators, and the Lie algebraic form of their supersymmetric extensions is

$$
 h_{BCD}^{(s)} = \sum_{i,j=1}^{N} \left\{ N(J_{i-1,j}J_{j-1,i} - T_{i-1,j}T_{j-1,i}) - \sum_{l \geq 0} [(i - l)J_{i-l,j}J_{j+l,i} + T_{i-l,j}J_{j+l,i}]
 + J_{i-1,l}T_{j+l,i} - T_{i-1,l}T_{j+l,i} + (l + j - 1)[J_{i-l,l}J_{j+l,l} + J_{i-1,l}T_{j+l,l}]
 + T_{i-1,l}J_{j+l,l} - T_{i-1,l}T_{j+l,l} - (i - 2 - l)[J_{i-2,l}J_{j+l,l} + T_{i-2,l}J_{j+l,l}]
 + J_{i-2,l}T_{j+l,l} - T_{i-2,l}T_{j+l,l} - (l + j + 1)[J_{i-1,l}J_{j+l,l} + T_{i-1,l}J_{j+l,l}]
 + J_{i-1,l}T_{j+l,l} - T_{i-1,l}T_{j+l,l} + l[J_{i-1,l}T_{j+l,l} + T_{i+l,1}]
 + J_{i-1,l}T_{j+l,l} - T_{i-1,l}T_{j+l,l} - J_{i-1,l}(T_{i+l,1} + T_{i+l,1})
 - J_{i-1,l}(T_{i-1,3,i} - T_{i-1,1}) - J_{j-1,l}(T_{i+l,1} + T_{i+l,1})
 - J_{i-1,l}(T_{i-1,3,i} - T_{i-1,1}) - T_{i-1,1}(T_{j-1,1} - T_{j-3,i})
 - (T_{j-1,1} + T_{i+l,1})T_{j-1,1})j + \frac{\nu_3}{2}(j - N - 1)[J_{j-1,1} + T_{j-1,1}]
 - (\nu_2 + \frac{\nu_3}{2} + 1 + \nu(2N - j - 1))j[J_{j,j} + T_{j,j}] - NJ_{j,j} - T_{j,j}]
 - [\nu(N - j + 1)(N - j + 2)](J_{j-2,j} + T_{j-2,j}) - 2N(J_{j-2,j} - T_{j-2,j}) \right\}
$$

(B.8)

in terms of the $gl(N + 1|N)$ generators.

In conclusion, let us emphasize the fact that all Lie algebraic forms of the supersymmetric models (B.2), (B.4), (B.6), (B.8) contain no fermionic (odd) generators.

C Deriving the Lie algebraic forms: technical details

In this appendix we will give some details of the derivation of the Lie algebraic forms of the Hamiltonians discussed in this paper. We will exemplify the method for the $BC_N$ Calogero models. The other cases are tackled in a similar way.

(A) Bosonic $BC_N$ Calogero model.

Our final goal is to derive (5.6)–(5.7). Let us recall that the gauge-rotated $BC_N$ Hamiltonian is given by

$$
 h_{BCD}^{(c)} = \sum_{i=1}^{N} \partial_i^2 + 2\nu \sum_{i<j} \left[ \frac{1}{x_i - x_j}(\partial_i - \partial_j) + \frac{1}{x_i + x_j}(\partial_i + \partial_j) \right] + \nu_2 \sum_{i=1}^{N} \frac{1}{x_i} \partial_i - 2\omega \sum_{i=1}^{N} x_i \partial_i. \quad (C.1)
$$

After the change of variables $x_i \to \bar{\sigma}_i$ the Hamiltonian becomes

$$
 h_{BCD}^{(c)} = \sum_{i,k=1}^{N} \sum_{i=1}^{N} \frac{\partial \bar{\sigma}_i}{\partial x_i} \frac{\partial \bar{\sigma}_k}{\partial \bar{\sigma}_i} \frac{\partial}{\partial \bar{\sigma}_k} + \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 \bar{\sigma}_k}{\partial x_i^2} \frac{\partial}{\partial \bar{\sigma}_k} - 2\omega \sum_{k=1}^{N} \sum_{i=1}^{N} x_i \frac{\partial \bar{\sigma}_k}{\partial x_i} \frac{\partial}{\partial \bar{\sigma}_k}
$$

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\[ + \nu \sum_{k=1}^{N} \sum_{i \neq j} \left[ \frac{1}{x_i + x_j} \left( \frac{\partial \tilde{\sigma}_k}{\partial x_i} + \frac{\partial \tilde{\sigma}_k}{\partial x_j} \right) + \frac{1}{x_i - x_j} \left( \frac{\partial \tilde{\sigma}_k}{\partial x_i} - \frac{\partial \tilde{\sigma}_k}{\partial x_j} \right) \right] \frac{\partial}{\partial \tilde{\sigma}_k} \]

\[ + \nu_2 \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{1}{x_i} \frac{\partial \tilde{\sigma}_k}{\partial x_i} \frac{\partial}{\partial \tilde{\sigma}_k}. \]  

(C.2)

As an example, let us express

\[ \tilde{B}_k = \sum_{i \neq j}^{N} \left[ \frac{1}{x_i + x_j} \left( \frac{\partial \tilde{\sigma}_k}{\partial x_i} + \frac{\partial \tilde{\sigma}_k}{\partial x_j} \right) + \frac{1}{x_i - x_j} \left( \frac{\partial \tilde{\sigma}_k}{\partial x_i} - \frac{\partial \tilde{\sigma}_k}{\partial x_j} \right) \right] \]

(C.3)

and

\[ \tilde{A}_{\ell,k} = \sum_{i=1}^{N} \frac{\partial \tilde{\sigma}_\ell}{\partial x_i} \frac{\partial \tilde{\sigma}_k}{\partial x_i} \]  

(C.4)

in terms of the \( \tilde{\sigma} \) variables. The approach we will follow is to represent the generating functions of (C.3) and (C.4)

\[ \tilde{A}(t, p) = \sum_{k=0}^{N} \tilde{A}_{\ell,k} t^\ell p^k, \quad \tilde{B}(t) = \sum_{k=0}^{N} \tilde{B}_k t^k, \]

in terms of the generating function of the \( \tilde{\sigma} \) variables

\[ G(t) = \sum_{k=0}^{N} \tilde{\sigma}_k(t) t^k = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_{2n}(x) t^n \right]. \]  

(C.5)

Let us begin by calculating the generating function \( \tilde{B}(t) \) for (C.3)

\[ \tilde{B}(t) = \sum_{i \neq j} \left[ \frac{1}{x_i + x_j} \left( \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) + \frac{1}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \right] G(t) \]

\[ = 2 \sum_{i \neq j} \left\{ \sum_{m=1}^{\infty} (-1)^{m+1} \left[ \frac{x_i^{2m-1} + x_j^{2m-1}}{x_i + x_j} + \frac{x_i^{2m-1} - x_j^{2m-1}}{x_i - x_j} \right] t^m \right\} G(t). \]  

(C.6)

Using the relation

\[ \frac{x_i^{2n-1} + x_j^{2n-1}}{x_i + x_j} + \frac{x_i^{2n-1} - x_j^{2n-1}}{x_i - x_j} = 2 \sum_{l=0}^{n-1} x_i^{2l} x_j^{2(n-1)-2l}, \]  

(C.7)

and making rather obvious mathematical transformations, one can show that

\[ \tilde{B}(t) = \left[ 4t^3 \frac{\partial^2}{\partial t^2} - 4(2N - 2)t^2 \frac{\partial}{\partial t} + 4tN(N - 1) \right] G(t). \]  

(C.8)

Substituting (C.5) in (C.8) we find that \( \tilde{B}_k \) in (C.3) is

\[ \tilde{B}_k = 4(N - k + 1)(N - k)\tilde{\sigma}_{k-1}. \]  

(C.9)
Next we proceed to the calculation of \((C.4)\). The corresponding generating function can be written as

\[
\tilde{A}(t, p) = \sum_{i=1}^{N} \frac{\partial G(p)}{\partial x_i} \frac{\partial G(t)}{\partial x_i} = 4 \sum_{i=1}^{N} \sum_{k, \ell = 1}^{\infty} (-1)^{k+\ell} x_i^{2(k+\ell)} - 2 t p \ell G(t) G(p) .
\]

(C.10)

Changing the summation variables to \(r = k + \ell\) and \(u = k - \ell\) one obtains that

\[
\tilde{A}(t, p) = 4 \sum_{i=1}^{N} \sum_{r=2}^{\infty} (-1)^r s_{2(r-1)}(tp) \sum_{u} \left( \frac{t}{p} \right)^{r} G(t) G(p) .
\]

(C.11)

The sum over \(u\) is easily determined

\[
\sum_{u} \left( \frac{t}{p} \right)^{r} = \left( \frac{t}{p} \right)^{-r/2} \left( \frac{1 - \left( \frac{t}{p} \right)^{r-1}}{1 - \frac{t}{p}} \right) .
\]

(C.12)

Using this formula it is easy to transform (C.11) to the following expression for the generating function:

\[
\tilde{A}(t, p) = \frac{4 t}{1 - \frac{t}{p}} \left[ t \frac{\partial}{\partial t} - p \frac{\partial}{\partial p} \right] G(t) G(p) .
\]

(C.13)

Expanding the rhs of (C.13) we arrive at the final expression for (C.4)

\[
\tilde{A}_{\ell,k} = 4 \sum_{q \geq 0} (2q + 1 + l - k) \tilde{\sigma}_{k-q-1} \tilde{\sigma}_{\ell+q} .
\]

(C.14)

The other terms in (C.2) can be calculated using the same method which finally leads to the results presented in (5.6), (5.7).

(B) Supersymmetric BCN Calogero model.

For the supersymmetric models the calculations are similar to those carried out for the bosonic models. In terms of the superspace \(\phi\)-coordinates (3.11) the gauge rotated Hamiltonian (5.10) is written as

\[
\hbar_{sBCD} = \int d\alpha d\beta d\bar{\alpha} d\bar{\beta} \sum_{i=1}^{N} \frac{\delta}{\delta \phi_i(\bar{\alpha})} \frac{\delta}{\delta \phi_i(\alpha)} + 2 \nu \int d\alpha \sum_{i<j} \frac{1}{\phi_i(\alpha) - \phi_j(\alpha)} \left( \frac{\delta}{\delta \phi_i(\alpha)} - \frac{\delta}{\delta \phi_j(\alpha)} \right) + \nu_2 \int d\alpha \sum_{i=1}^{N} \frac{1}{\phi_i(\alpha)} \left( \frac{\delta}{\delta \phi_i(\alpha)} \right) + 2 \omega \int d\alpha \sum_{i=1}^{N} \phi_i(\alpha) \frac{\delta}{\delta \phi_i(\alpha)} .
\]

(C.15)

Now we make the change of variables \(\phi_i \rightarrow \chi_i\) (see (5.11)) using the relation (3.21). Finally, an expression similar to (C.2) emerges except for the fact that the bosonic coordinates are replaced by supercoordinates with integration over the Grassmann variables:

\[
\hbar_{sBCD} = \sum_{i=1}^{N} \int d\alpha d\beta d\bar{\alpha} d\bar{\beta} \frac{\delta \chi_i(\bar{\beta})}{\delta \phi_i(\alpha)} \frac{\delta \chi_i(\beta)}{\delta \phi_i(\alpha)} \frac{\delta}{\delta \chi_i(\beta)} \frac{\delta}{\delta \chi_i(\beta)}
\]

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\[-2\nu \int d\omega d\beta \sum_{i<j}^{N} \sum_{k=1}^{N} \frac{1}{\phi_i(\alpha) - \phi_j(\alpha)} \left( \frac{\delta \chi_k(\beta)}{\delta \phi_i(\alpha)} \right) \left( \frac{\delta \chi_j(\beta)}{\delta \phi_j(\alpha)} \right) + \frac{1}{\phi_i(\alpha) + \phi_j(\alpha)} \left( \frac{\delta \chi_k(\beta)}{\delta \phi_i(\alpha)} \right) \left( \frac{\delta \chi_j(\beta)}{\delta \phi_j(\alpha)} \right) \delta \chi_k(\beta) \delta \chi_j(\beta) \right) \delta \chi_k(\beta) \delta \chi_j(\beta). \]

For terms linear in derivatives the process of calculation is exactly the same as for the bosonic case. The final result can be obtained from the bosonic calculation simply by replacing the \(\sigma\)-coordinates by \(\chi\)-supercoordinates. However, for the term quadratic in derivatives:

\[
\sum_{i=1}^{N} \int d\omega d\alpha d\beta d\beta \frac{\delta \chi_k(\beta)}{\delta \phi_i(\alpha)} \frac{\delta \chi_j(\beta)}{\delta \phi_i(\alpha)} = \int d\beta d\bar{\beta} \bar{\mathcal{A}}^{(s)}_{k,\ell} \frac{\delta}{\delta \chi_k(\beta)} \frac{\delta}{\delta \chi_j(\beta)} ,
\]

a slight complication arises. Again our strategy is to make use of a generating function of the \(\chi\) coordinates

\[
\mathcal{G}(\beta, p) = \sum_{k=0}^{N} \chi_k(\beta) p^k = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_{2n}(\phi(\beta)) p^n \right] .
\]

in order to represent the generating function for \(\bar{\mathcal{A}}^{(s)}_{k,\ell} = \sum_{i=1}^{N} \int d\omega d\alpha \frac{\delta \chi_k(\beta)}{\delta \phi_i(\alpha)} \frac{\delta \chi_j(\beta)}{\delta \phi_i(\alpha)} \):

\[
\bar{\mathcal{A}}^{(s)}(t, p) = \sum_{k,\ell=1}^{N} \bar{\mathcal{A}}^{(s)}_{k,\ell} k^\ell .
\]

In what follows we use the notation \(\phi_i := \phi_i(\beta)\) and \(\bar{\phi}_i := \phi_i(\bar{\beta})\). Let us first note that

\[
\bar{\mathcal{A}}^{(s)}(t, p) = 4 \sum_{i=1}^{N} \sum_{m,n=1}^{\infty} (-1)^{n+m} \phi_i^{2n-1} \phi_i^{2m-1} t^m p^m \mathcal{G}(\beta, p) \mathcal{G}(\bar{\beta}, t) ,
\]

(cf.\(\text{(C.10)}\)). After some simple mathematical transformations similar to those leading to \(\text{(C.11)}\) \(\bar{\mathcal{A}}^{(s)}(t, p)\) can be rewritten as

\[
2 \sum_{r=2}^{\infty} (-1)^r (tp) \frac{\hat{w}}{p} \sum_{u} \left[ s_{2r-2}(\phi) + s_{2r-2}(\bar{\phi}) \right] \left( \frac{t}{p} \right)^{\hat{w}} \mathcal{G}(\beta, p) \mathcal{G}(\bar{\beta}, t) + 4 \sum_{r=2}^{\infty} (-1)^r (tp) \frac{\hat{w}}{p} \sum_{u} u \left[ \frac{s_{2r-2}(\phi) - s_{2r-2}(\bar{\phi})}{2(r-1)} \right] \left( \frac{t}{p} \right)^{\hat{w}} \mathcal{G}(\beta, p) \mathcal{G}(\bar{\beta}, t) .
\]

Using \(\text{(C.12)}\) together with the relation

\[
\sum_{u} u \left( \frac{t}{p} \right)^{\hat{w}} = \left( \frac{t}{p} \right)^{\hat{w}} \frac{1}{1 - \left( \frac{t}{p} \right)^2} \left[ -(r-2) + \frac{r^2}{p} - r \left( \frac{t}{p} \right)^{r-1} + (r-2) \left( \frac{t}{p} \right)^r \right] ,
\]

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\( (C.21) \) takes the form
\[
4 \frac{t}{1 - \frac{t}{p}} \sum_{r=1}^{\infty} (-1)^{r+1} \left[ s_{2r}(\phi)p^r - s_{2r}(\bar{\phi})t^r \right] \mathcal{G}(\beta, p) \mathcal{G}(\tilde{\beta}, t) \\
- 2 \frac{t}{p} \frac{t + p}{(1 - \frac{t}{p})^2} \sum_{r=1}^{\infty} (-1)^{r+1} \frac{1}{r} \left[ s_{2r}(\bar{\phi}) - s_{2r}(\phi) \right] (t^r - p^r) \mathcal{G}(\beta, p) \mathcal{G}(\tilde{\beta}, t). \tag{C.23}
\]

Evidently, the first term in \( (C.23) \) is equal to
\[
\frac{4t}{1 - \frac{t}{p}} \left[ t \frac{\partial}{\partial t} - p \frac{\partial}{\partial p} \right] \mathcal{G}(\beta, p) \mathcal{G}(\tilde{\beta}, t) \tag{C.24}
\]
(cf.\((C.13))\). In order to calculate the second term in \( (C.23) \) we need to use two relations:
\[
- \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left[ s_{2r}(\bar{\phi}) - s_{2r}(\phi) \right] (t^r - p^r) = \left[ e^{-\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left[ s_{2r}(\bar{\phi}) - s_{2r}(\phi) \right] (t^r - p^r)} - 1 \right] \tag{C.25}
\]
and
\[
\mathcal{G}(\beta, t) \mathcal{G}(\tilde{\beta}, p) = e^{\frac{1}{2} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left[ s_{2r}(\bar{\phi}) + s_{2r}(\phi) \right] (t^r + p^r)} e^{\frac{1}{2} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left[ s_{2r}(\bar{\phi}) - s_{2r}(\phi) \right] (t^r - p^r)}. \tag{C.26}
\]

Finally, the second term in \( (C.23) \) becomes
\[
2 \frac{t}{p} \frac{t + p}{(1 - \frac{t}{p})^2} \left[ \mathcal{G}(\beta, t) \mathcal{G}(\tilde{\beta}, p) - \mathcal{G}(\beta, p) \mathcal{G}(\tilde{\beta}, p) \right]. \tag{C.27}
\]

We should emphasize that the bosonic part of this expression vanishes. Combining \((C.23)\) and \((C.26)\) we obtain the final expression for the generating function \( \tilde{A}(s) (t, p) \) in terms of the generating function \( \mathcal{G}(\beta, p) \):
\[
\tilde{A}(s) (t, p) = \frac{4t}{1 - \frac{t}{p}} \left[ t \frac{\partial}{\partial t} - p \frac{\partial}{\partial p} \right] \mathcal{G}(\beta, p) \mathcal{G}(\tilde{\beta}, t) \tag{C.28}
- 2 \frac{t}{p} \frac{t + p}{(1 - \frac{t}{p})^2} \left[ \mathcal{G}(\beta, t) \mathcal{G}(\tilde{\beta}, p) - \mathcal{G}(\beta, p) \mathcal{G}(\tilde{\beta}, p) \right].
\]

(cf.\((C.13))\). Expanding \( \tilde{A}(s) (t, p) \) in powers of \( t, p \) we arrive at the explicit expression for the coefficients in front of the second derivative terms
\[
\tilde{A}_{\ell, k}^{(s)} = 2 \sum_{q \geq 0} \left\{ 2(2q + 1 + k - \ell) \chi_{\ell-q-1} \chi_{k+q} - l[\chi_{k+q} \chi_{\ell-q-1} - \chi_{k+q} \chi_{\ell-q-1}]
+ \chi_{\ell+q-1} \chi_{k-q} - \chi_{\ell+q-1} \chi_{k-q} \right\}. \tag{C.29}
\]

(cf.\((C.14))\) given in \((5.12)\). Making similar calculations for the coefficients in front of the terms linear in derivatives we come to the conclusion that these coefficients are equal to those given by \((5.7)\) with replacement of \( \sigma \)'s by \( \chi \)'s (see \((5.7)\) and \((5.13)\)).
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