THE AVERAGE NUMBER OF INTEGRAL POINTS ON THE CONGRUENT NUMBER CURVES

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Abstract. We show that the total number of non-torsion integral points on the elliptic curves \( E_D : y^2 = x^3 - D^2x \), where \( D \) ranges over positive squarefree integers less than \( N \), is \( O(N(\log N)^{-1/4+\varepsilon}) \). The proof involves a discriminant-lowering procedure on integral binary quartic forms and an application of Heath-Brown’s method on estimating the average size of the 2-Selmer group of the curves in this family.

1. Introduction

Given an elliptic curve over \( \mathbb{Q} \) with short Weierstrass model
\[
E : y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{Z},
\]
we study the quadratic twists of \( E \), with the model
\[
E_D : y^2 = x^3 + AD^2x + BD^3,
\]
where \( D \) denotes a positive squarefree integer. Consider the set of integral points
\[
E_D(\mathbb{Z}) := \{ (x, y) \in \mathbb{Z}^2 : y^2 = x^3 + AD^2x + BD^3 \}.
\]
It follows from a result of Mordell [17] that \( #E_D(\mathbb{Z}) \) is always finite.

We are interested in the distribution of the number of integral points \( #E_D(\mathbb{Z}) \) in quadratic twist families, when \( E_D \) are ordered according to the size of \( D \). If \( E(\mathbb{Q}) \) contains a 2-torsion point, this point must have the form \( (a, 0) \) for some integer \( a \) under the model (1), then \( (aD, 0) \in E_D(\mathbb{Z}) \) for all squarefree integers \( D \). Therefore we call an integral point non-trivial if it is not a 2-torsion point of \( E_D(\mathbb{Q}) \). Define the set of non-trivial integral points on \( E_D \) to be
\[
E_D^*(\mathbb{Z}) := \{ (x, y) \in E_D(\mathbb{Z}) : y \neq 0 \}.
\]

Define
\[
\mathcal{D} := \{ D \in \mathbb{Z}_{>0} : D \text{ squarefree} \},
\]
\[
\mathcal{D}_N := \{ D \in \mathcal{D} : D \leq N \}.
\]
Granville [9] conjectured that almost all curves within a quadratic twist family have no non-trivial integral point. We state the conjecture adapted to our model (2).

Key words and phrases. elliptic curve, quadratic twist, integral point.
Conjecture 1.1 (Granville [3]). Fix $A, B \in \mathbb{Z}$ such that $4A^3 + 27B^2 \neq 0$. Let $E_D : y^2 = x^3 + AD^2x + BD^3$, $D \in \mathcal{D}$. Then
\[
\#\{D \in \mathcal{D}_N : E_D^*(\mathbb{Z}) \neq \emptyset\} \sim C_{A,B}N^{\frac{4}{3}},
\]
where $C_{A,B}$ is a constant that depends only on $A, B$.

We note that Granville’s original conjecture considers a different model $Dy^2 = f(x)$, where $f \in \mathbb{Z}[x]$ and deg $f = 3$. When $f(x) = x^3 + Ax + B$, any point $(x, y) \in \mathbb{Z}^2$ satisfying $Dy^2 = f(x)$ corresponds to a point $(Dx, Dy) \in E_D(\mathbb{Z})$, so there are fewer integral points using the model $Dy^2 = f(x)$ when compared to our model (2). The exponent $\frac{1}{2}$ stated in Conjecture 1.1 replaces $\frac{1}{3}$ in the original conjecture because of this discrepancy. The exponent $\frac{1}{2}$ is suggested by some heuristics we gave in [3] p. 6677–6678 for the family $y^3 = x^3 - D^2x$.

In this direction, Matschke and Mudigonda [10] handled the case when $f(x)$ is reducible, assuming the abc conjecture.

Theorem 1.2 (Matschke–Mudigonda [16]). Assume that the abc conjecture is true. Suppose $f(x) = x^3 + Ax + B$, $A, B \in \mathbb{Z}$, such that $4A^3 + 27B^2 \neq 0$ and $f(x)$ is reducible over $\mathbb{Q}$. Then
\[
\#\{D \in \mathcal{D}_N : Dy^2 = f(x) \text{ for some } x, y \in \mathbb{Z}, \ y \neq 0\} \leq N^{\frac{4}{3} + o(1)}.
\]

Our goal here is to gain progress towards Conjecture 1.1 on a specific quadratic twist family. We restrict our attention to the congruent number curve $E : y^2 = x^3 - x$, and study its twists
\[
E_D : y^2 = x^3 - D^2x.
\]

It is well known that the torsion subgroup of $E_D(\mathbb{Q})$ is $\{O, (0, 0), (\pm D, 0)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see for example [13] Chapter I, Proposition 17), where $O$ denotes the point at infinity.

For this family $E_D$, we can deduce from existing results that all moments of $\#E_D(\mathbb{Z})$ are finite. The 2-Selmer groups of $E_D$, which we denote by $\text{Sel}_2(\mathcal{E}_D)$, is a finite group with order 2 that is defined via local conditions and admits an injection $E_D(\mathbb{Q})/2E_D(\mathbb{Q}) \hookrightarrow \text{Sel}_2(\mathcal{E}_D)$ (see for example [21] Chapter X). In particular, the 2-Selmer rank provides an upper bound to the rank rank($E_D(\mathbb{Q})$) of the Mordell–Weil group of $E_D(\mathbb{Q})$. It is usually easier to compute the 2-Selmer groups of elliptic curves with a torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Q}$, since then most of the work can be done over $\mathbb{Q}$. Heath-Brown [11] Theorem 1] computed all the moments of the size of the 2-Selmer groups of $E_D$. For any fixed positive integer $k$, he showed that
\[
\lim_{N \to \infty} \frac{1}{\#D_N} \sum_{D \in \mathcal{D}_N} (\#\text{Sel}_2(\mathcal{E}_D))^k = c_k + o_k(1),
\]
where $c_k$ are explicit constants that can be bounded by $3^{k(k+1)}$. Since the 2-Selmer rank provides an upper bound to the the rank of $E_D$, (3)
implies that
\[
\limsup_{N \to \infty} \frac{1}{\#D_N} \sum_{D \in D_N} 2^{k \text{rank} E_D(\mathbb{Q})} \leq 3^{k(k+1)}.
\]

Lang [14, page 140] conjectured that the number of integral points on a quasi-minimal Weierstrass equation of an elliptic curve $E$ should be bounded only in terms of $\text{rank } E(\mathbb{Q})$. For the family $E_D$, if follows from known results in this direction [20, Theorem A], [12, Theorem 0.7], that there exists some absolute constant $C_1$, such that
\[
\#E_D(\mathbb{Z}) \ll C_1^{\text{rank} E_D(\mathbb{Q})}.
\]

In [5], we showed that $C_1$ in (5) can be taken as $3.8$. Combining the upper bound (5) and (4), we can bound the $k$-th moment
\[
\limsup_{N \to \infty} \frac{1}{\#D_N} \sum_{D \in D_N} (\#E_D(\mathbb{Z}))^k \ll C_2^{k(k+1)},
\]
where $C_2$ is an absolute constant.

We will show that in fact the moments of $\#E_D(\mathbb{Z})$ should each tend to 0. The following is our main result.

**Theorem 1.3.** For any $\epsilon > 0$, we have
\[
\sum_{D \in D_N} \#E_D^*(\mathbb{Z}) \ll N(\log N)^{-\frac{1}{4}+\epsilon}.
\]

This shows that the average size of $\#E_D^*(\mathbb{Z})$ tends to 0 as $N$ tends to infinity, since $\#D_N \sim \frac{6}{\pi^2} N$.

Theorem 1.3 implies that
\[
\#\{D \in D_N : E_D(\mathbb{Z}) \neq \emptyset\} \ll N(\log N)^{-\frac{1}{4}+\epsilon}.
\]

An application of Hölder’s inequality using (6) and (7), gives
\[
\sum_{D \in D_N} (\#E_D(\mathbb{Z}))^k \leq \left( \sum_{D \in D_N} (\#E_D(\mathbb{Z}))^{\frac{k}{2}} \right)^{\frac{2}{k}} \left( \#\{D \in D_N : E_D(\mathbb{Z}) \neq \emptyset\} \right)^{1-\epsilon} \ll NC_2^{(\frac{k}{2}+1)k} (\log N)^{(-\frac{1}{4}+\epsilon)(1-\epsilon)}.
\]
Rescaling $\epsilon$ gives Corollary 1.4.

**Corollary 1.4.** For any $\epsilon > 0$ and $k > 0$, we have
\[
\sum_{D \in D_N} (\#E_D(\mathbb{Z}))^k \ll_{\epsilon,k} N(\log N)^{-\frac{1}{4}+\epsilon}.
\]

We now give an outline of the proof of Theorem 1.3. In Section 2, for each integral point $(x, y) \in E_D(\mathbb{Z})$, we use Mordell’s correspondence [18, Chapter 25] to construct a corresponding integral binary quartic form $f$ that represents 1 and has discriminant related to the discriminant of $E$. Then in Section 3, we show that by picking an auxiliary prime $p \mid D/\gcd(x, D)$, we can transform $f$ into an integral binary quartic
form $F$ that represents $p$ and has discriminant lowered by a factor of $p^6$. In Section 4, we show that $\gcd(x, D)$ can be controlled by the image of $(x, y)$ in the $2$-Selmer group of $E_D$ under the map

$$E_D(\mathbb{Z}) \hookrightarrow E_D(\mathbb{Q}) \twoheadrightarrow E_D(\mathbb{Q})/2E_D(\mathbb{Q}) \hookrightarrow \text{Sel}_2(E_D).$$

Then in Section 5, we extract some information about the distribution of $2$-Selmer elements from Heath-Brown’s work [10, 11] to show that for almost all $D$, we are able to pick a prime $p$ of a suitable size. In particular, this $p$ is not too small, so that there are $o(N)$ many discriminants for the discriminant-lowered quartic $F$ to take. At the same time, this $p$ is not too large, so that each $\text{GL}_2(\mathbb{Z})$-equivalence class of $F$ can only be the image of finitely many integral points by applying bounds on the number of solutions to Thue inequalities. In Section 6, we use Hölder’s theorem and (6) to bound the contribution from the exceptional curves to the number of integral points. In Section 7, we proceed to count the set of those quartics $F$ that were discriminant-lowered by some suitable $p$. We make use of the fact that every integral binary quartic form is $\text{SL}_2(\mathbb{Z})$-equivalent to at least one reduced form with bounded seminvariants [6]. Applying the syzygy satisfied by the seminvariants returns a set of integral points on twists of $E$ with bounded height. Then Theorem 1.3 follows from an application of an upper bound by Le Boudec [15].

2. INTEGER-MATRIX BINARY QUARTIC FORMS

We say that a binary quartic form is integer-matrix if it has the form

$$f(X, Y) = a_0X^4 + 4a_1X^3Y + 6a_2X^2Y^2 + 4a_3XY^3 + a_4Y^4, \quad a_i \in \mathbb{Z}.$$  

Given any integral binary quartic form $f$ and $(x_0, y_0) \in \mathbb{Z}^2$, define the action of

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$$

on the pair $(f, (x_0, y_0))$ by

$$\gamma \cdot (f(X, Y), (x_0, y_0)) = (f((X, Y) \cdot \gamma), (x_0, y_0) \cdot \gamma^{-1}),$$

where

$$(X, Y) \cdot \gamma = (aX + cY, bX + dY).$$

This action preserves the value of $f(x_0, y_0)$.

We recall some facts about the seminvariants of quartic forms [6, Section 4.1.1]. For our convenience, we choose to scale the seminvariants differently than in [6], since we will only be dealing with integer-matrix binary forms. The invariants of $f$ are

$$I = I(f) = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad \text{and}$$

$$J = J(f) = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3.$$
The discriminant of \( f \) is
\[
\Delta(f) := I^3 - 27J^2
\]
where
\[
I := a_0^3a_1^3 - 64a_0^4a_1^2a_2^2 - 128a_0a_1a_2^2a_3^2 - 6a_0a_1^2a_2^2a_4
- 180a_0a_1a_2^2a_3a_4 + 81a_0a_2^2a_4
+ 36a_1^2a_2^2a_3^2 - 27(a_0^2a_3^2 + a_1^2a_2^2)
+ 54a_2(-a_2^2 + 2a_1a_3 + a_0a_4)(a_4a_1^2 + a_0a_2^2)
\]
The seminvariants attached to the form are
\[
I, J, a = a(f) = a_0.
\]
Comparing to the formulas in [6, Section 4.1.1], here we have taken out a factor of \(-48\) from their \( H \), a factor of \( 327 \) from their \( R \), a factor of \( 12 \) from their \( I \), a factor \( 432 \) from their \( J \), and a factor of \( 256 \cdot 27 \) from their \( \Delta \). The seminvariants are related by the syzygy
\[
H^3 - \frac{I}{4}a^2H - \frac{J}{4}a^3 = \left( \frac{R}{2} \right)^2.
\]
Notice that \( (H, \frac{1}{2}R) \) defines an integral point on a twist of the elliptic curve \( y^2 = x^3 - \frac{7}{4} - \frac{J}{4} \).

2.1. Mordell’s correspondence. For integers \( A, B \in \mathbb{Z} \) such that \( 4A^3 + 27B^2 \neq 0 \), define an elliptic curve over \( \mathbb{Q} \) with affine integral Weierstrass model
\[
E_{A,B} : y^2 = x^3 + Ax + B.
\]
The discriminant of \( E_{A,B} \) is given by
\[
\Delta_{E_{A,B}} = -16(4A^3 + 27B^2).
\]
For integers \( c, d, e \in \mathbb{Z} \), define an integer-matrix binary quartic form
\[
f_{c,d,e}(X,Y) = X^4 + 6cX^2Y^2 + 8d + eY^4.
\]
Define
\[
A := \{(E_{A,B},(x_0,y_0)) : A, B \in \mathbb{Z}, \ 4A^3 + 27B^2 \neq 0, \ (x_0,y_0) \in E_{A,B}(\mathbb{Z})\},
\]
\[
B := \{f_{c,d,e} : c, d, e \in \mathbb{Z}, \ e \equiv c^2 \mod 4, \ \Delta(f) \neq 0\}.
\]
The following correspondence is given by Mordell [13, Chapter 25] (or see [3, Section 2.3] for a modern interpretation).

**Theorem 2.1** (Mordell). Fix an integer \( L \neq 0 \). There is a bijection
\[
A \to B
\]
given by
\[
(E_{A,B},(x_0,y_0)) \mapsto f,
\]
where
\[
f(X,Y) = X^4 - 6x_0X^2Y^2 + 8y_0XY^3 + (-4A - 3x_0^2)Y^4.
\]
Moreover, under this map, \( \Delta(f) = \Delta_{E_{A,B}}, \ I(f) = -4A \) and \( J(f) = -4B \).
The inverse map comes from the syzygy satisfied by the semi-invariants, but we will only make use of the the direction from \( A \) to \( B \) in Theorem 2.1.

### 3. Lowering the discriminant

We now fix an elliptic curve \( E : y^2 = x^3 + Ax + B, \ A, B \in \mathbb{Z} \) and consider its quadratic twists \( E_D : y^2 = x^3 + AD^2x + BD^3, \) where \( D \in D \). For each \( P = (c, d) \in E_D(\mathbb{Z}) \), Theorem 2.1 gives the binary quartic form

\[
(9) \quad f_P(X, Y) := X^4 - 6cX^2Y^2 + 8dXY^3 + (-4AD^2 - 3c^2)Y^4,
\]
which satisfies \( \Delta(f_P) = \Delta_E D^6 \), \( I(f_P) = -4AD^2 \) and \( J(f_P) = -4BD^3 \).

Denote the space of integer-matrix binary quartic forms by \( V \). Define

\[
(10) \quad \Psi : \bigcup_{D \in D} \left\{ (P, M, k) \in E_D(\mathbb{Z}) \times \mathbb{Z}_{>0} \times (\mathbb{Z}/M\mathbb{Z})^\times : M \mid D, \gcd(6 \cdot x(P), M) = 1, \right. \\
\left. k^2 \equiv x(P) \mod M \right\} \\
\rightarrow (V \times \mathbb{Z}^2) / GL_2(\mathbb{Z})
\]
given by

\[
(P, M, k) \mapsto (F, (1, 0)),
\]
where

\[
F(X, Y) = \frac{1}{M^3} f_P(MX + kY, Y).
\]

We will show that \( \Psi \) is well-defined in Lemma 3.1 and injective in Lemma 3.2.

In work of Bombieri and Schmidt, to bound the number of solutions to a Thue equation \( F_1(X, Y) = h \), they transformed the integral binary form \( F_1 \) to a different form \( F_2 \), whose discriminant is raised by a factor of \( h^6 \), and so that there is a solution to \( F_2(X, Y) = 1 \). Some applications of this idea can be found in \([2, 1]\). Here we attempt to carry out the reverse process on the integral quartic forms \( f_P \) to lower their discriminants.

**Lemma 3.1.** Let \( P = (c, d) \in E_D(\mathbb{Z}) \) and take \( f_P \) as defined in (9). Fix a positive squarefree integer \( M \) dividing \( D \) that is coprime to \( 6c \). Then \( c \) is a square modulo \( M \), and for any integer \( k \) such that \( k^2 \equiv c \mod M \), we have that

\[
F(X, Y) := \frac{1}{M^3} f_P \left( (X, Y) \cdot \begin{pmatrix} M & 0 \\ k & 1 \end{pmatrix} \right) = \frac{1}{M^3} f_P(MX + kY, Y)
\]
is an integer-matrix binary quartic form. Moreover, we have

1. \( F(1, 0) = M \),
2. \( I(F) = -4A(D/M)^2 \), \( J(F) = -4B(D/M)^3 \), and
3. \( \Delta(F) = \Delta(f_P)/M^6 \).
Proof. Since $M$ is squarefree, $M$ is a product of distinct prime factors. Since $(c, d) \in E_D(\mathbb{Z})$, we have $d^2 = c^3 - D^2c$. For each prime $p | M$, we see that $c^3 \equiv d^2 \mod p$, so $(\frac{c}{d})^2 = 1$, therefore $c$ is a square modulo $M$.

Taking any integer $k$ such that $k^2 \equiv c \mod M$, by Hensel’s lemma we can find a lift of $k$ such that $k \equiv K \mod M$ and

\begin{equation}
\gamma = K^2 \mod M^3.
\end{equation}

It suffices to show that $F$ is an integer-matrix binary quartic form with this choice of $K$, since otherwise $k = K + uM$ for some integer $u$, and we can consider $F(X - uY, Y)$ instead.

Putting (11) into $d^2 = c^3 + AD^2c + BD^3$, we see that

\begin{equation}
d \equiv K^3 + \frac{AD^2}{2K} \mod M^3.
\end{equation}

By (11) and (12), we see that the coefficients of

\[f_P(MX + kY, Y) = M^4X^4 + 4M^3KX^3y + 6M^2(K^2 - c)X^2y^2 + 4M(K^3 - 3cK + 2d)XY^3 + (K^4 - 6cK^2 + 8dK - 4AD^2 - 3c^2)Y^4\]

are all divisible by $M^3$. Therefore $F$ is integer-matrix from the coefficients. The remaining properties are then a straightforward check from the definition of $F$. \qed

**Lemma 3.2.** The map $\Psi$ is injective.

Proof. The value of $F(1, 0)$ determines $M$, and together with the discriminant of $F$, determines $D$. In the following, fix some $D \in \mathcal{D}$ and some $M | D$ such that $\gcd(6, M) = 1$. Suppose $P, Q \in E_D(\mathbb{Z})$ satisfy $\gcd(x(P), M) = \gcd(x(Q), M) = 1$ and write $\Psi(P, M, k_P) = (F_P, (1, 0))$ and $\Psi(P, M, k_Q) = (F_Q, (1, 0))$. Suppose $(F_P, (1, 0))$ and $(F_Q, (1, 0))$ are $\text{GL}_2(\mathbb{Z})$-equivalent, so $\gamma \cdot (F_P, (1, 0)) = (F_Q, (1, 0))$ for some $\gamma \in \text{GL}_2(\mathbb{Z})$. Then $(1, 0) \cdot \gamma^{-1} = (1, 0)$ implies that we can write $\gamma = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ for some $u \in \mathbb{Z}$. Recall that

\[F_P(X, Y) = \frac{1}{M^3} \cdot f_P \left( (X, Y) \cdot \begin{pmatrix} M \\ 0 \\ 1 \end{pmatrix} \right) \]

and

\[F_Q(X, Y) = \frac{1}{M^3} \cdot f_Q \left( (X, Y) \cdot \begin{pmatrix} M \\ 0 \\ 1 \end{pmatrix} \right) \]

From $F_P((X, Y) \cdot \gamma) = F_Q(X, Y)$, we get

\[f_P \left( (X, Y) \cdot \gamma \cdot \begin{pmatrix} M \\ 0 \\ k_P \end{pmatrix} \right) = f_Q \left( (X, Y) \cdot \begin{pmatrix} M \\ 0 \\ k_Q \end{pmatrix} \right) \cdot \gamma.\]

Then since

\[\begin{pmatrix} M & 0 \\ k_Q & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} M & 0 \\ k_P & 1 \end{pmatrix} = \begin{pmatrix} uM + k_P - k_Q & 0 \\ 1 & 0 \end{pmatrix},\]
we have
\[ f_P \left( (X,Y) \cdot \begin{pmatrix} 1 & 0 \\ uM + k_P - k_Q & 1 \end{pmatrix} \right) = f_Q(X,Y). \]
The \( X^3Y \)-coefficients of \( f_P \) and \( f_Q \) are both 0, it must be that \( uM + k_P - k_Q = 0 \) and \( f_P = f_Q \). Hence \( P = Q \) and \( k_P \equiv k_Q \mod M \).

Heath-Brown \([10, 11]\) computed the moments of the size of the 2-Selmer group modulo 2-torsion in this family is 3. We will extract some information about the 2-Selmer elements in this family from the argument in \([10, 11]\), in order to show that we can usually pick a suitable \( M \) to apply Lemma 3.1.

4. The 2-Selmer group of \( y^2 = x^3 - D^2x \)

In the following sections we will specialise in the case when \( A = -1 \) and \( B = 0 \), that is, the quadratic twists family

\[ \mathcal{E}_D : y^2 = x^3 - D^2x. \]

Heath-Brown \([10, 11]\) computed the moments of the size of the 2-Selmer group modulo 2-torsion in this family is 3. We will extract some information about the 2-Selmer elements in this family from the argument in \([10, 11]\), in order to show that we can usually pick a suitable \( M \) to apply Lemma 3.1.

The 2-Selmer group of \( \mathcal{E}_D \) is defined to be

\[ \text{Sel}_2(\mathcal{E}_D) := \ker \left( H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathcal{E}_D[2]) \to \prod_{p \text{ place of } \mathbb{Q}} H^1(\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p), \mathcal{E}_D) \right). \]

Since \( \mathcal{E}_D \) has full 2-torsion, there is an isomorphism \( H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathcal{E}_D[2]) \cong (\mathbb{Q}^\times/(\mathbb{Q}^\times)^2)^2 \), and it is possible to obtain explicit equations for the homogeneous spaces (See for example \([21]\) Chapter X, Proposition 1.4)). Explicit equations for homogeneous spaces for the curves \( \mathcal{E}_D \) were found as part of Heath-Brown’s argument \([10]\) Section 2]. As we will see, each 2-Selmer element of \( \mathcal{E}_D(\mathbb{Q}) \) corresponds to a system of two binary quadratic forms that is everywhere locally solvable. We will follow \([10]\) Section 2] to recover the equations.

Definition 4.1. For \( D \in \mathcal{D} \), define \( \mathcal{W}_D \) to be the set of all \( (D_1, D_2, D_3, D_4) \in \mathcal{D}^4 \) such that

1. the system
\[ D_1X^2 + D_4W^2 = D_2Y^2, \quad D_1X^2 - D_4W^2 = D_3Z^2; \]
is everywhere locally solvable, and
2. \( D_1D_2D_3D_4 = D \).
Consider the injective homomorphism
\[
\theta : \mathcal{E}_D(\mathbb{Q})/2\mathcal{E}_D(\mathbb{Q}) \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \times \mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \times \mathbb{Q}^\times/(\mathbb{Q}^\times)^2
\]
given by \((x, y, \theta)\) at non-torsion points. At torsion points \(\theta(O) = (1, 1, 1), \theta((0, 0)) = (D, -1, D), \theta((D, 0)) = (2, D, 2D), \theta((-D, 0)) = (-2D, -D, 1).\)

**Lemma 4.2.** The set \(\mathcal{W}_D\) is in bijection to
\[
\begin{cases}
\text{Sel}_2(\mathcal{E}_D)/\psi(\{(O, (0, 0), (\pm D, 0))\}) & \text{if } D \text{ is odd}, \\
\text{Sel}_2(\mathcal{E}_D)/\psi(\{(O, (0, 0))\}) & \text{if } D \text{ is even},
\end{cases}
\]
where \(\psi\) here denotes the natural map
\[\psi : \mathcal{E}_D(\mathbb{Q}) \to \mathcal{E}_D(\mathbb{Q})/2\mathcal{E}_D(\mathbb{Q}) \to \text{Sel}_2(\mathcal{E}_D)\].

In particular, if the image of \((c, d) \in \mathcal{E}_D(\mathbb{Z})\) under the map
\[\mathcal{E}_D(\mathbb{Z}) \to \mathcal{E}_D(\mathbb{Q}) \to \mathcal{E}_D(\mathbb{Q})/2\mathcal{E}_D(\mathbb{Q}) \to \text{Sel}_2(\mathcal{E}_D) \to \mathcal{W}_D\]
is \((D_1, D_2, D_3, D_4)\), then
\[
\gcd(c, d) \in \{D_1D_2D_3, D_2D_3D_4, D_1D_2D_4, D_1D_3D_4\}.
\]

**Proof.** We study \(\text{im } \theta\) following [10, Section 2]. Suppose \((x, y) \in \mathcal{E}_D(\mathbb{Q})\), and write \(x = r/s\) and \(y = t/u\), where \(r, s, t, u\) are integers and \(\gcd(r, s) = 1\) and \(\gcd(t, u) = 1\). Putting this into \(y^2 = x^3 - D^2x\), we have
\[r(r+sD)(r-sD)u^2 = t^2s^3.\]
Then since \(\gcd(t, u) = 1\) and \(\gcd(r, s) = 1\), we must have \(s^3 = u^2\), so \(s = W^2\) for some integer \(W\). Now write \(\gcd(r, D) = D_0\), and \(r = D_0r'\).

From
\[r(r+sD)(r-sD) = t^2\]
we see that \(D_0^2 | t^2\), hence \(D_0^2 | t\) since \(D_0\) is squarefree. Then writing \(D_4 = D/D_0\), we have \(\gcd(r', sD_4) = 1\) by construction, and we have
\[r'(r'+sD_4)(r'-sD_4) = D_0(t/D_0^2)^2.\]
The factors on the left are pairwise coprime except possibly a common factor of 2, which only occurs when \(r'\) and \(sD_4\) are both odd, in this case \(r', (r'+sD_4)/2, (r'-sD_4)/2\) are pairwise coprime. Now we can write \(r' = D_1X^2, r' + sD_4 = D_2Y^2\) and \(r' - sD_4 = D_3Z^2\), where \(D_1, D_2, D_3\) are squarefree integers such that \(D_1D_2D_3 = D_4 = D_0^2\).

In the first case \(D_1D_2D_3D_4 = D\) and in the second case \(D_1D_2D_3D_4 = 4D\) with \(D_1, D_4\) odd and \(D_2, D_3\) even. When \(D\) is even, the case \(D_1D_2D_3D_4 = 4D\) is not possible since \(D_1, D_4\) would need to be odd, and \(D_2, D_3\) would need to be even and squarefree. This produces a solution to the system
\[D_1X^2 + D_4W^2 = D_2Y^2, D_1X^2 - D_4W^2 = D_3Z^2.\]
An element in $\text{Sel}_2(\mathcal{E}_D)$ corresponds to a system of this form that is everywhere locally solvable. To fix the signs of $D_1, D_2, D_3, D_4$ and their valuations at 2, we will use the torsion points of $\mathcal{E}_D$.

If $D$ is odd, exactly one of $(x', y') \in \{(x, y), (x, y) + (0, 0), (x, y) + (D, 0), (x, y) + (-D, 0)\}$ satisfies $x' > 0$ and $v_2(x') \neq 0$. By looking at $\theta((x', y'))$, we see that the image of $(x, y)$ in $\mathcal{W}_D$ can be taken as

$$
\begin{align*}
(D_1, D_2, D_3, D_4) & \quad \text{if } v_2(x) \neq 0 \text{ and } x > 0, \\
(D_4, D_3, -D_2, -D_1) & \quad \text{if } v_2(x) \neq 0 \text{ and } x < 0, \\
(D_2/2, D_1, D_4, D_3/2) & \quad \text{if } v_2(x) = 0 \text{ and } x > 0, \\
(D_3/2, D_4, -D_1, -D_2/2) & \quad \text{if } v_2(x) = 0 \text{ and } x < 0.
\end{align*}
$$

If $D$ is even, exactly one of $(x', y') \in \{(x, y), (x, y) + (0, 0)\}$ satisfies $x' > 0$ and $v_2(x') \equiv 1 \mod 2$. We take the image of $(x, y)$ in $\mathcal{W}_D$ to be

$$
\begin{align*}
(D_1, D_2, D_3, D_4) & \quad \text{if and } x > 0, \\
(D_4, D_3, -D_2, -D_1) & \quad \text{if and } x < 0.
\end{align*}
$$

If $(x, y) \in \mathcal{E}_D(\mathbb{Z})$, by construction $\gcd(x, D) = D_0$, which is $D_1D_2D_3$ or $\frac{1}{2}D_1D_2D_3$. This gives \[\ref{thm:generic-2-selmer-elements}\] by checking each of the above cases and using the different labelling of $D_1, D_2, D_3, D_4$ in the statement of the lemma. $\square$

5. Generic 2-Selmer Elements

We want to show that those $(D_1, D_2, D_3, D_4)$ that appears usually satisfies some nice properties that will eventually allow us to pick $M$ for Lemma \[\ref{lem:selmer-criterium}\]. Take

$$
N^\dagger = \exp(4\kappa(\log \log N)^2),
$$

where $\kappa > 0$ is an absolute constant as defined in \cite[Lemma 7]{10} or \cite[Section 3]{11}. This $N^\dagger$ is $C^4$ in the notation of \cite{10}.

Henceforth $0 < \epsilon < \frac{1}{2}$ will be a fixed constant. Let $S$ be the interval

$$
S := \left[\exp((\log N)^{2\epsilon}), \exp((\log N)^{1-2\epsilon})\right].
$$

so that any $p \in S$ satisfies

$$
2\epsilon \log \log N < \log \log p < (1 - 2\epsilon) \log \log N.
$$

Define

$$
\omega(n) := \#\{p \text{ prime : } p \mid n\},
$$

$$
\omega_S(n) := \#\{p \text{ prime : } p \mid n, \ p \in S\}.
$$

We will prove the following.

**Theorem 5.1.** Define two properties on $(D_1, D_2, D_3, D_4) \in \mathcal{W}_D$:

$(S1)$ for each $i = 1, 2, 3, 4$, we have $D_i \geq N^\dagger$ and there exist some $p \mid D_i$ such that $p \in S$;

$(S2)$ $(D_1, D_2, D_3, D_4)$ corresponds to a torsion point on $\mathcal{E}_D(\mathbb{Q})$.  

Then
\[ \# \{ D \in D_N \colon (S1) \text{ or } (S2) \text{ fails for some } (D_1, D_2, D_3, D_4) \in \mathcal{W}_D \} \ll N(\log N)^{-\frac{1}{4} + \varepsilon}. \]

Define
\[ \delta_i(\eta) := \begin{cases} 1 & \text{if } \eta = i, \\ 0 & \text{otherwise}. \end{cases} \]

To prove Theorem 5.1, it suffices to bound the number of 4-tuples of positive odd integers \((D_1, D_2, D_3, D_4)\) satisfying the following conditions:

1. \((2^\delta_1(\eta)D_1, 2^\delta_2(\eta)D_2, 2^\delta_3(\eta)D_3, 2^\delta_4(\eta)D_4) \in \mathcal{W}_D\) for some \(D \in D_N\) and some \(\eta \in \{0, 1, 2, 3, 4\}\); and
2. one of the conditions \((W1)\) and \((W2)\) listed below.

\((W1)\) For some \(i = 1, 2, 3, 4\), we have \(D_i < N^\dagger\), and
\[ (D_1, D_2, D_3, D_4) \neq \begin{cases} (1, 1, 1, D) & \text{if } \eta = 0 \text{ or } 4, \\ (1, 1, D/2, 1) & \text{if } \eta = 2. \end{cases} \]

\((W2)\) We have \(D_i \geq N^\dagger\) for all \(i = 1, 2, 3, 4\), and there exists an \(i\) such that \(D_i\) has no prime factor in \(S\).

In the above notation, \(\eta = 0\) implies that \(D = D_1D_2D_3D_4\) is odd, and \(\eta = 1, 2, 3, 4\) implies that \(D = 2D_1D_2D_3D_4\) is even.

We first consider the case when \(D\) is odd. By [10, Lemma 3], considering the local solvability of \((13)\) at each prime, we can package the condition \((D_1, D_2, D_3, D_4) \in \mathcal{W}_D\) as a sum of product of Jacobi symbols. Write \(D = (D_{ij})\) as a 16-tuple of positive odd integers, where the indices are in the range
\[ 1 \leq i \leq 4, \ 0 \leq j \leq 4, \ i \neq j. \]

For odd \(D\), set
\[ g_0(D) = \left( -\frac{1}{\alpha} \right) \left( \frac{2}{\beta_0} \right) \prod_i 4^{-\omega(D_{i0})} \prod_{j \neq 0} 4^{-\omega(D_{ij})} \prod_{k \neq i, j} \prod_l \left( \frac{D_{kl}}{D_{lj}} \right), \]
where \(\alpha = D_{12}D_{14}D_{23}D_{21}\) and \(\beta_0 = D_{24}D_{21}D_{34}D_{31}\). Then
\[ G_0(D_1, D_2, D_3, D_4) := \sum_{D = \prod_{i \neq j, i, j} = D_i} g_\eta(D) = \begin{cases} 1 & \text{if } (D_1, D_2, D_3, D_4) \in \mathcal{W}_D, \\ 0 & \text{else}. \end{cases} \]

For even \(D\), since we are only aiming for an upper bound, we can ignore any local conditions at 2 when considering the solvability of \((13)\). Following the proof of [10, Lemma 3], the only difference being essentially pulling the factor of 2 in \(D\) into the \((2)\) term, set
\[ g_\eta(D) = \left( -\frac{1}{\alpha} \right) \left( \frac{2}{\beta_\eta} \right) \prod_i 4^{-\omega(D_{i0})} \prod_{j \neq 0} 4^{-\omega(D_{ij})} \prod_{k \neq i, j} \prod_l \left( \frac{D_{kl}}{D_{lj}} \right), \]
where
\[
\begin{align*}
\beta_1 &= D_{33}D_{32}D_{42}D_{43}D_{21}D_{31}, \\
\beta_2 &= D_{13}D_{14}D_{43}D_{24}D_{21}, \\
\beta_3 &= D_{12}D_{14}D_{41}D_{42}D_{34}D_{31}, \\
\beta_4 &= D_{12}D_{13}D_{23}D_{32}D_{24}D_{34}.
\end{align*}
\]

Then
\[
G_\eta(D_1, D_2, D_3, D_4) := \sum_{D: \prod_{i \neq j} D_{ij} = D_i} g_\eta(D)
\geq \begin{cases} 
1 & \text{if } (2^{\delta_1(n)} D_1, 2^{\delta_2(n)} D_2, 2^{\delta_3(n)} D_3, 2^{\delta_4(n)} D_4) \in \mathcal{W}_D \\
0 & \text{else.}
\end{cases}
\]

5.1. The case \(D_i < N^\frac{1}{4}\). For each \(\eta = 0, 1, 2, 3, 4\), we will want to estimate the sum
\[
\sum_{(D_1, D_2, D_3, D_4) \in \mathcal{W}_1} G_\eta(D_1, D_2, D_3, D_4)
\]
where the sum is taken over all positive odd integers \(D_1, D_2, D_3, D_4\) such that \(D_1D_2D_3D_4 \in \mathcal{D}_N\). Following \cite[Section 3]{10} and \cite[Section 3]{11}, dissect the sum according to the size of each \(D_{ij}\) in the factorisation. For each \((i, j)\), take \(A_{ij}\) to run over powers of 2. For \(A = (A_{ij})\), define the restricted sum
\[
|S_\eta(A)| = \sum_{A_{ij} < D_{ij} \leq 2A_{ij}} g_\eta(D),
\]
where the sum is taken over all 16-tuples of odd positive integers \(D = (D_{ij})\) such that \(\prod_{i,j} D_{ij} \in \mathcal{D}_N\) and \(A_{ij} < D_{ij} \leq 2A_{ij}\) for every \(i, j\).

Note that if \(A_{ij} = \frac{1}{2}\), the interval \(A_{ij} < D_{ij} \leq 2A_{ij}\) forces \(D_{ij} = 1\).

We can summarise the error term estimates in \cite{10, 11} as follows.

**Lemma 5.2** (\cite[Lemma 7, Lemma 11]{10}, \cite[Lemma 6]{11}).
\[
\sum_A |S_\eta(A)| \ll N (\log N)^{\frac{7}{4} + \epsilon},
\]
where the sum is over all \(A\) other than those that satisfy
\[
\begin{cases} 
A_u > N^\frac{1}{4} & \text{if } u \in \mathcal{U}, \text{ and} \\
A_u = \frac{1}{2} & \text{if } u \notin \mathcal{U},
\end{cases}
\]
for all indices \(u\), where \(\mathcal{U}\) is one of
\[
\begin{align*}
\{10, 20, 30, 40\}, \{40, 41, 42, 43\}, \{20, 12, 32, 42\}, \{30, 13, 23, 43\} & \quad \text{if } \eta = 0, \\
\{10, 20, 30, 40\}, \{40, 14, 24, 34\} & \quad \text{if } \eta = 1, \\
\{10, 20, 30, 40\}, \{20, 12, 22, 32\}, \{30, 31, 32, 34\} & \quad \text{if } \eta = 2, \\
\{10, 20, 30, 40\}, \{30, 13, 23, 33\} & \quad \text{if } \eta = 3, \\
\{10, 20, 30, 40\}, \{10, 11, 21, 31\}, \{40, 41, 42, 43\} & \quad \text{if } \eta = 4.
\end{align*}
\]
Moreover, the estimate still holds if we further impose that $\prod_j D_{ij} \leq N_i$ for each $i = 1, 2, 3, 4$.

Strictly speaking [10] only applies to the case for odd $D$, namely $\eta = 0$. The modification made in Lemma 5.2 for even $D$ was in the sets in (17), since the possible $\mathcal{U}$ depends on the variables in $\beta_\eta$. Following the case analysis in the proof of [10, Lemma 11] with this change, the sets in (17) can be found for each of $\eta = 1, 2, 3, 4$.

We are ready to bound the contribution from (W1).

**Lemma 5.3.** For each $\eta \in \{0, 1, 2, 3, 4\}$, we have

$$\# \{ (2^\varepsilon_1(n) D_1, 2^\varepsilon_2(n) D_2, 2^\varepsilon_3(n) D_3, 2^\varepsilon_4(n) D_4) \in W_D : (W1) \text{ holds} \} \ll N \log N^{1/4 + \epsilon}.$$

**Proof.** We want to bound

\[ \sum_{(D_1, D_2, D_3, D_4)} G_\eta(D_1, D_2, D_3, D_4) = \sum_{(D_1, D_2, D_3, D_4)} \sum_{\prod_j D_{ij} = D_i} g_\eta(D). \]

Dissect the sum using $A$, then apply Lemma 5.2 to (18). The only possibility that this $S_\eta(A)$ is not covered by Lemma 5.2 is if $A$ satisfies (16) with

$$\mathcal{U} = \begin{cases} \{40, 41, 42, 43\} & \text{when } \eta = 0, \\ \{30, 31, 32, 34\} & \text{when } \eta = 2, \\ \{40, 41, 42, 43\} & \text{when } \eta = 4. \end{cases}$$

This implies that

$$(D_1, D_2, D_3, D_4) = \begin{cases} (1, 1, 1, D) & \text{if } \eta = 0 \text{ or } 4, \\ (1, 1, D/2, 1) & \text{if } \eta = 2, \end{cases}$$

which are from the excluded torsion points. Therefore the required estimate follows from that in Lemma 5.2.

\[ \square \]

### 5.2. Prime divisor of a large $D_i$

We now bound the contribution from (W2). Assume that it is $D_4 \geq N^\delta$ that is not divisible by any prime in $S$. The cases with $D_4$ replaced by $D_1, D_2, D_3$ the same after relabelling. We want to bound

\[ \sum_{D_1, D_2, D_3, D_4 \in D_N} G_\eta(D_1, D_2, D_3, D_4) = \sum_{D_1, D_2, D_3, D_4 \in D_N} \sum_{\prod_j D_{ij} = D_i} g_\eta(D). \]

For each $D$ that appears in the sum of $g_\eta(D)$, we can find a set of indices $\mathcal{U} = \{1i, 2j, 3k, 4l\}$, where $i, j, k, l$ are not necessarily distinct, such that $D_{ij}, D_{2j}, D_{3k}, D_{4l} > (N^\delta)^{\delta}$. Lemmas 5.2 do not immediately apply because of the restrictions on the primes dividing $D_4$. However it will be sufficient to follow [10, Section 3].
Definition 5.4. We call two indices \((i, j)\) and \((k, l)\) linked if

\[ i \neq k \text{ and precisely one of the conditions } \begin{cases} l \neq 0, i, \\ j \neq 0, k \end{cases} \text{ holds.} \]

We will show that the indices in \(\mathcal{U}\) are pairwise unlinked with the number of exceptions contributing \(O(N(\log N)^{-\frac{1}{4} + \epsilon})\) to (19).

When there are linked indices \(u\) and \(v\), the Jacobi symbol between \(D_u\) and \(D_v\) appears in \(g_{\eta}\) non-trivially. When both \(D_u\) and \(D_v\) are large, then the large sieve ([10, Lemma 4]) can be applied to obtain cancellation. Imposing further restrictions to each of \(D_u, D_v, (D_w)_{w \neq u, v}\) do not affect the estimates in [10, Section 3]. We state the modified statement below.

Lemma 5.5 ([10, Lemma 5]). Suppose \(u\) and \(v\) are linked indices. Take \(C_1\) to be any collection of \(D_u \geq (\log N)^{544}\). Take \(C_2\) to be any collection of \(D_v \geq (\log N)^{544}\). Take \(C_3\) to be any collection of \(B = (D_w)_{w \neq u, v}\). Then

\[
\sum_{D_u \in C_1} \sum_{D_v \in C_2} \sum_{B \in C_3} g_{\eta}(D) \ll N(\log N)^{-17}.
\]

Suppose \(u\) and \(v\) are linked. When one of \(D_u\) is large, and \(D_v \neq 1\) is small, Siegel–Walfisz for character sums ([10, Lemma 6]) is used instead. For this we need that the sum over \(D_u\) is over an interval, but we can still impose restrictions on \(D_v\) and \(D_w)_{w \neq u, v}\) without affecting the estimates.

Lemma 5.6 ([10, Lemma 7]). Suppose \(u\) and \(v\) are linked indices. Take \(C_2\) to be any collection of \(1 < D_v < (\log N)^{544}\). Take \(C_3\) to be any collection of \(B = (D_w)_{w \neq u, v}\). Define

\[
S_{1, \eta}(A_u) := \sum_{A_u \leq D_u \leq 2A_u} \sum_{D_v \in C_2} \sum_{B \in C_3} g_{\eta}(D).
\]

Then

\[
\sum_{A_u} |S_{1, \eta}(A_u)| \ll N(\log N)^{-17},
\]

where the sum is over all \(A_u \geq (N^4)^{\frac{1}{2}}\) that are powers 2.

Combining Lemma 5.5 and Lemma 5.6 we have the following.

Lemma 5.7. Suppose \(u\) and \(v\) are linked indices. Take \(C_2\) to be any collection of \(D_v > 1\). Take \(C_3\) to be any collection of \(B = (D_w)_{w \neq u, v}\). Define

\[
S_{1, \eta}(A_u) := \sum_{A_u \leq D_u \leq 2A_u} \sum_{D_v \in C_2} \sum_{B \in C_3} g_{\eta}(D).
\]
Then
\[ \sum_{A_u} |S_{1,\eta}(A_u)| \ll N(\log N)^{-17}, \]
where the sum is over all \( A_u \geq (N^1)^\frac{1}{4} \) that are powers 2.

We can now return to the set \( U = \{1i, 2j, 3k, 4l\} \). Since by construction \( D_u > (N^1)^\frac{1}{4} \) for all \( u \in U \), we can assume that the indices \( 1i, 2j, 3k, 4l \) are pairwise unlinked by Lemma 5.5.

Now suppose \( v \notin U \). If \( v \) is not linked any one of \( 1i, 2j, 3k \), then \( \{1i, 2j, 3k, v\} \) is a set of unlinked indices. Comparing against the 24 possible sets of unlinked indices in [10, Lemma 9], if \( \{1i, 2j, 3k, v\} \) and \( \{1i, 2j, 3k\} \) are both sets of unlinked indices, they must be the same set. Therefore \( v \) must be linked to one of \( \{1i, 2j, 3k\} \), and this allows us to apply Lemma 5.7, so we are left with the terms in the sum (19) with \( D_v = 1 \) for all \( v \notin U \). Noting that there are only a maximum of 24 possible sets of unlinked indices \( U \), and putting in
\[ D_1 = D_{1i}, D_2 = D_{2j}, D_3 = D_{3k}, D_4 = D_{4l}, \]
the sum (19) is bounded by
\[ \ll \sum_{D_1, D_2, D_3, D_4 \in D_N} 4^{-\omega(D_1 D_2 D_3 D_4)} + O \left( N(\log N)^{-\frac{1}{4} + \epsilon} \right) \]
\[ \ll \sum_{D \in D_N} 4^{-\omega(D)} \sum_{(D_1, D_2, D_3, D_4) \atop D_1 D_2 D_3 D_4 = D} \sum_{\text{p} | D_i \Rightarrow p \notin S} 1 + O \left( N(\log N)^{-\frac{1}{4} + \epsilon} \right) \]
\[ = \sum_{D \in D_N} \left( \frac{3}{4} \right)^{\omega_s(D)} + O \left( N(\log N)^{-\frac{1}{4} + \epsilon} \right). \]

To bound the main term here we make use of the following result.

**Lemma 5.8 ([19, Theorem 1]).** Fix \( 0 < \epsilon < 1 \) and some positive constant \( C \). Let \( f \) be a multiplicative function such that \( f(p^\ell) \leq C \) for all prime \( p \) and \( \ell \geq 1 \). Then
\[ \sum_{x - y < n \leq x} f(n) \ll \frac{y}{\log x} \exp \left( \sum_{p \leq x} \frac{f(p)}{p} \right) \]
uniformly for \( 2 \leq X^{1-\epsilon} \leq Y < X \).

By Lemma 5.8 and Mertens’ theorem,
\[ \sum_{D_1, D_2, D_3, D_4 \in D_N} G_{\eta}(D_1, D_2, D_3, D_4) \ll \sum_{D \in D_N} \left( \frac{3}{4} \right)^{\omega_s(D)} \ll N(\log N)^{-\frac{1}{4} + \epsilon} \]
This gives the following estimate.
Lemma 5.9. For each $\eta \in \{0, 1, 2, 3, 4\}$, we have
\[
\# \{ (2^{\delta_1(\eta)} D_1, 2^{\delta_2(\eta)} D_2, 2^{\delta_3(\eta)} D_3, 2^{\delta_4(\eta)} D_4) \in W_D : \left[ \begin{bmatrix} W_2 \end{bmatrix} \right] \} \ll N(\log N)^{-\frac{1}{4} + \epsilon}.
\]

Combining Lemma 5.3 and Lemma 5.9 proves Theorem 5.1.

6. CONTRIBUTION FROM NON-GENERIC 2-SELMER ELEMENTS

Take $G_N$ to be the collection of $D \in D_N$ that satisfies one of the following

- (P1) $\omega(D) \geq 2 \log \log N,$
- (P2) $D < \exp(3(\log N)^{1-\epsilon}),$
- (P3) at least one of the conditions [S1] and [S2] fails.

Lemma 6.1. We have
\[
\sum_{D \in G_N} \# \mathcal{E}_D(\mathbb{Z}) \ll N(\log N)^{-\frac{1}{4} + 2\epsilon}.
\]

Proof. By the Erdős-Kac theorem [7], the numbers of $D \in D_N$ from (P1) is bounded by
\[
\# \{ D \in D_N : \omega(D) \geq 2 \log \log N \} \ll N(\log N)^{-\frac{1}{2}}.
\]
The number of $D \in D_N$ that satisfies (P2) is trivially bounded by $\exp(3(\log N)^{1-\epsilon})$. Theorem 5.1 bounds the number of $D \in D_N$ that satisfies (P3). Therefore
\[
\# G_N \ll N(\log N)^{-\frac{1}{4} + \epsilon}.
\]

We now bound the contribution from the curves in $G_N$ to the total number of integral points. Applying (6) with $k = 1/\epsilon$, we have
\[
\sum_{D \in D_N} (\# \mathcal{E}_D(\mathbb{Z}))^{\frac{1}{\epsilon}} \ll_{\epsilon} N.
\]
Using Hölder’s inequality,
\[
\sum_{D \in D_N} \# \mathcal{E}_D(\mathbb{Z}) \leq \left( \sum_{D \in D_N} (\# \mathcal{E}_D(\mathbb{Z}))^{\frac{1}{\epsilon}} \right)^{\epsilon} (\# G_N)^{1-\epsilon} \ll_{\epsilon} N^{\epsilon}(\# G_N)^{1-\epsilon}.
\]
The claims follow from putting $\# G_N \ll N(\log N)^{-\frac{1}{4} + \epsilon}$. \qed

7. COUNTING GENERIC POINTS

By Lemma 6.1 we may exclude any $D \in G_N$. Any integral point in $\mathcal{E}_D(\mathbb{Z})$ maps to 2-Selmer element, and hence to $W_D$ under the map
\[
\mathcal{E}_D(\mathbb{Z}) \hookrightarrow \mathcal{E}_D(\mathbb{Q}) \rightarrow \mathcal{E}_D(\mathbb{Q})/2\mathcal{E}_D(\mathbb{Q}) \rightarrow \text{Sel}_2(\mathcal{E}_D) \rightarrow W_D.
\]
Recall that
\[
\mathcal{E}_D^*(\mathbb{Z}) = \mathcal{E}_D(\mathbb{Z}) \setminus \{(0, 0), (\pm D, 0)\}.
\]
For the non-trivial integral points that has image of the type [S2] we have the following bound from [5, Theorem 1.4] and the discussion after [5, Theorem 10.1].
Lemma 7.1. We have
\[ \sum_{D \in \mathcal{D}_N} \sum_{T \in \{O, (0,0), (\pm D, 0)\}} \#(\mathcal{E}_D^* (\mathbb{Z}) \cap (T + 2\mathcal{E}_D(\mathbb{Q}))) \ll \sqrt{N} \log N. \]

Therefore it remains to handle the integral points that arise from (S1).

Define \( Z_N := \bigcup_{D \in \mathcal{D}_N \setminus \mathcal{G}_N} \{ P \in \mathcal{E}_D(\mathbb{Z}) : P \notin 2\mathcal{E}_D(\mathbb{Q}) + \{O, (0,0), (\pm D, 0)\} \}. \)

Then the image of \((x,y) \in Z_N\) corresponds to \((D_1, D_2, D_3, D_4) \in W_D\) of the type (S1). By Lemma 4.2,
\[ D/\gcd(x,D) \in \{D_1, D_2, D_3, D_4\}. \]

This implies that we can find a prime factor \( M \) of \( D/\gcd(x,D) \) of size
\[ (20) \exp((\log N)^2 \epsilon) < M < \exp((\log N)^1 - 2\epsilon). \]

Now \( M \) divides \( D \) but does not divide \( x \). Therefore we can carry out the transformation in Lemma 3.1 with this \( M \).

For each \( P \in \mathcal{E}_D(\mathbb{Z}) \cap S_N \), write \( \tilde{D} = D/M \), so \( \Delta(F) = (2\tilde{D})^6 \) if \( F = \Phi(P) \). Since \( D \geq \exp(3(\log N)^1 - 2\epsilon) \) by (P2) and \( M \) is in the range (20), we have
\[ \exp(2(\log N)^1 - 2\epsilon) < D \exp(-((\log N)^1 - 2\epsilon)) \leq \tilde{D} < D \exp(-((\log N)^2 \epsilon)). \]

7.1. Points lowered to the same quartic. We now show that each binary quartic form in the image of \( \Phi \) cannot arise from too many integral points.

Lemma 7.2. For any \( F \in \text{im} \Phi \), we have
\[ \#\Phi^{-1}(F) \ll 1, \]
where the implied constant is independent of \( F \).

Proof. Lemma 3.2 implies that \( \Psi \) is injective. Fix some \( F_0 \in \text{im} \Phi \). Suppose \((F, (1,0)) \in \text{im} \Psi\) is such that \( F \) and \( F_0 \) are \( \text{GL}_2(\mathbb{Z}) \)-equivalent, so we can write
\[ F_0(X,Y) = F((X,Y) \cdot \gamma) \]
for some \( \gamma \in \text{GL}_2(\mathbb{Z}) \). Then
\[ \gamma \cdot (F(X,Y), (1,0)) = (F((X,Y) \cdot \gamma), (1,0) \cdot \gamma^{-1}) = (F_0(X,Y), (1,0) \cdot \gamma^{-1}). \]
This gives a solution \((x, y) = (1, 0) \cdot \gamma^{-1}\) to the Thue inequality
\[(22) \quad 1 \leq |F_0(x, y)| \leq h,\]
where \(h := \exp((\log N)^{1-\epsilon})\) is taken so that \(h\) is greater than any \(M\) in \((20)\). In particular this solution is primitive \((x \text{ and } y \text{ coprime})\), since \(\gamma^{-1} \in \text{GL}_2(\mathbb{Z})\) has determinant \(\pm 1\) and entries in \(\mathbb{Z}\). The solutions to the Thue inequality constructed from different elements \((F, (1, 0)) \in \text{im } \Psi\) are distinct as long as \((F, (1, 0))\) are from different \(\text{GL}_2(\mathbb{Z})\)-equivalence classes. Indeed, suppose \(F_0(x, y) = F_1((X, Y) \cdot \gamma_1)\) and \(F_0(x, y) = F_2((X, Y) \cdot \gamma_2)\), then if the solutions produced on (22) are same, we also have \((1, 0) \cdot \gamma_1^{-1} = (1, 0) \cdot \gamma_2^{-1}\), so \(\gamma_1 \cdot (F_1(X, Y), (1, 0)) = \gamma_2 \cdot (F_2(X, Y), (1, 0))\).

A result by Evertse \cite{Evertse96} implies that when \(2^8 \Delta(F_0) \geq (13b)^{10}\), the number of solutions to (22) is bounded by some absolute constant. Since \(\Delta(F_0) = (2D)^6 \gg \exp(12(\log N)^{1-\epsilon})\) from (21), and \(h^{10} = \exp(10(\log N)^{1-\epsilon})\), we conclude that the number of possible classes \((F, (1, 0))\) associated to each class of \(F_0\) is absolutely bounded.

\[\square\]

7.2. Integral points with bounded height. Every integral binary quartic form is \(\text{SL}_2(\mathbb{Z})\)-equivalent to at least one reduced form \((\mathfrak{b})\). The seminvariant \(a, H\) of the reduced form are bounded in terms of \(I\) and \(J\). We restate a theorem in \cite{Evertse96} in terms of our rescaled seminvariants. The scale factors of the seminvariants can be found in Section 2.

**Theorem 7.3** (\cite{Evertse96} Proposition 11). Suppose \(F_0(X, Y) \in \mathbb{Z}[X, Y]\) is a \(\text{GL}_2(\mathbb{Z})\)-reduced quartic form, and \(\Delta(F_0) > 0\), with leading coefficient \(a\) and seminvariant \(H\). Order the three real roots \(\phi_1, \phi_2, \phi_3\) of \(X^3 - \frac{1}{3}X - \frac{1}{3}\) so that \(\phi_1 < \phi_2 < a\phi_3\). Then \((a, H)\) satisfies one of the following:

1. \(|a| \leq \frac{4}{3} \phi_1 - \phi_3|\) and \(\max\{a\phi_1, a\phi_3 - 4\phi_2^3 + \frac{1}{3}I\} \leq H \leq a\phi_2; or
2. \(|a| \leq \frac{4}{3} \phi_1 - \phi_2|\) and \(a\phi_3 \leq H \leq a\phi_2 - 4\phi_2^3 + \frac{1}{3}I\).

For \(\text{im } \Phi, \Delta(F) = (2D)^6 > 0\). Also \(I(F) = 4D^2\) and \(J(F) = 0\), so \(\{\phi_1, \phi_3\} = \{-D, D\}\) and \(\phi_2 = 0\). Suppose \(F_0\) is a reduced form of \(F\) with leading coefficient \(a\) and seminvariant \(H\). Then the two possible cases in Lemma 7.3 both lead to
\[(23) \quad |a| \leq \frac{8}{3}D \quad \text{and} \quad |H| \leq \frac{4}{3}D^2.\]

The syzygy in \cite{Evertse96} for \(F_0\) now takes the form
\[H^3 - (aD)^2H = \left(\frac{1}{2}R\right)^2.\]

Notice that this gives an integral point \((H, \frac{1}{2}R) \in E_{|aD|}(\mathbb{Z})\) when \(a \neq 0\). Below we show that the possibility that \(a = 0\) does not happen because we restricted our counting to \(\mathbb{Z}_N\).
Lemma 7.4. Suppose $F \in \text{im } \Phi$. Then any form in the $SL_2(\mathbb{Z})$-equivalence class of $F$ has non-zero leading coefficient.

Proof. Assume for contradiction that $\Phi(P) = F$ for some $P = (c, d) \in \mathbb{Z}_N$ and $F$ is equivalent to some quartic form with leading coefficient 0. Then there is a non-trivial integral solution to $F(X, Y) = 0$. From $\Phi(P) = F$, we know that $F(X, Y) = \frac{1}{M} f_P(MX + kY, Y)$ for some $M, k \in \mathbb{Z}$, so $f_P(X, Y) = 0$ also has a non-trivial solution, say $(x_0, y_0)$. Then from the expression of $f_P$ in (3),

$$f_P(x_0, y_0) = x_0^4 - 6cx_0^2y_0^2 + 8dx_0y_0^3 + (4D^2 - 3c^2)y_0^4 = 0.$$  

We see that $y_0 \neq 0$ since the solution is non-trivial. The roots of $f_P(X, 1)$ are

$$\frac{x_0}{y_0} = -\sqrt{c + \sqrt{c + D}} + \sqrt{c - \sqrt{c - D}}, \quad \frac{x_0}{y_0} = -\sqrt{c - \sqrt{c + D}} - \sqrt{c - D}, \quad \frac{x_0}{y_0} = \sqrt{c + \sqrt{c + D}} - \sqrt{c - D}, \quad \frac{x_0}{y_0} = \sqrt{c - \sqrt{c + D}} + \sqrt{c - D}.$$  

For $x_0/y_0$ to be rational, it must be that $\theta(P) = (1, 1, 1)$, where $\theta$ is the 2-descent homomorphism defined in (14). This implies that $P \in 2E_D(\mathbb{Q})$, but such points are not in $\mathbb{Z}_N$. \hfill \Box

The $SL_2(\mathbb{Z})$-equivalence class of $F_0$ is determined by $(a, \tilde{D}, H, R)$ and $R$ is fixed by $(a, \tilde{D}, H)$ up to $\pm$ sign, so it suffices to count the number of $(a, \tilde{D}, H)$ that arise from this reduction, with the bounds (23).

7.3. Torsion points. We have reduced the problem to counting integral points with bounded height on $E_{[a\tilde{D}]}$, but since there are 2-torsion points on every curve, we need to deal with this possibility separately.

Here we bound those $F \in \text{im } \Phi$ that reduces to a form whose syzygy produces a torsion point on some $E_{[a\tilde{D}]}$. From (21),

$$\tilde{D} \leq N \exp(-(\log N)^{2\epsilon}).$$

Lemma 7.5. Let $\tilde{N} = N \exp(-(\log N)^{2\epsilon})$. The total number of $SL_2(\mathbb{Z})$-equivalence classes of integer-matrix binary forms $F$ that satisfy $H(F) \in \{ -a(F)\tilde{D}, 0, a(F)\tilde{D} \}$, $I(F) = (2\tilde{D})^2$ and $J(F) = 0$ for some $\tilde{D} \in \mathcal{D}_\mathbb{Z}$, is bounded by

$$\ll N \exp(-(\log N)^{\epsilon}).$$

Proof. Suppose that $F(X, Y) = a_0X^4 + 4a_1X^3Y + 6a_2X^2Y^2 + 4a_3XY^3 + a_4Y^4$. Since $H(F) = a_1^2 - a_0a_2$, if $H(F) \in \{ -a_0\tilde{D}, 0, a_0\tilde{D} \}$, it must be that $a_0 | a_1^2$. Then $a_0 | \Delta(F) = (2\tilde{D})^6$ by the discriminant formula. Therefore for each $\tilde{D}$, there can only be a maximum of $7^{\epsilon(2\tilde{D})}$ possible $a_0$. Sum over $\tilde{D} \in \mathcal{D}_\mathbb{Z}$, and apply Lemma 5.5, we can bound the number of classes

$$\ll \sum_{\tilde{D} \leq \tilde{N}} 7^{\epsilon(\tilde{D})} \ll \tilde{N}(\log \tilde{N})^6.$$
Put in $\tilde{N} = N \exp(-\log N)^{2\varepsilon}$ proves Lemma 7.5. □

7.4. Non-torsion points. We now bound those $F \in \text{im } \Phi$ that reduces to a form whose syzygy $\mathcal{S}$ produces a non-torsion point on some $E_{\tilde{a}\tilde{D}}$ with (23). Since the $E_D$ that satisfies $[P1]$ have been removed, we can assume that $\omega(\tilde{D}) < \omega(D) < 2 \log \log N$. Also by Lemma 7.4, $a \neq 0$.

Lemma 7.6. Let $\tilde{N} = N \exp(-\log N)^{2\varepsilon})$. Then

$$\sum_{1 \leq |a| \leq \frac{2}{3} \tilde{N}} \sum_{\tilde{D} \in \mathcal{P}_\tilde{N}} \sum_{\omega(D) < 2 \log \log N} \# \left\{ \left( H, \frac{1}{2} R \right) \in \mathcal{E}_{|a\tilde{D}|}^*(\mathbb{Z}) : |H| \leq \frac{4}{3} \tilde{D}^2 \right\} \ll N \exp(-\log N)^{4\varepsilon}.$$ 

Proof. Write $n = |a\tilde{D}| \leq \frac{2}{3} \tilde{N}^2$. For each positive integer $n$, the number of ways to factorise $n$ into a product of $a$ and $\tilde{D}$ such that $\tilde{D}$ is squarefree and $\omega(\tilde{D}) < 2 \log \log N$, is bounded by

$$\sum_{k \leq \min\{\omega(n), 2 \log \log N\}} \left( \frac{\omega(n)}{k} \right) < \sum_{k \leq 2 \log \log N} (\omega(n))^k \ll e^{3(\log \log N)^2},$$

where we have used the fact that $\omega(n) \ll \log N$.

The number of integral points we are counting are of bounded height $|x(P)| \leq \frac{4}{3} \tilde{N}^2$, so applying a result by Le Boudec [15, Theorem 2] we get

$$\sum_{n \geq 1} \# \left\{ P \in \mathcal{E}_{a\tilde{D}}^*(\mathbb{Z}) : |x(P)| \leq \frac{4}{3} \tilde{N}^2 \right\} \ll \tilde{N}(\log \tilde{N})^6.$$ 

Now multiplying (24) and (25), we get that the total number of triples $(H, a, \tilde{D})$ is

$$\ll N \exp(-\log N)^{2\varepsilon} + 4(\log \log N)^{2\varepsilon}).$$

This proves the claim. □

Theorem 1.3 follows from Lemma 6.1, Lemma 7.1, Lemma 7.5 and Lemma 7.6.

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