SOME CARLESON MEASURES FOR THE
HILBERT-HARDY SPACE OF TUBE DOMAINS OVER
SYMmetric CONES

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Abstract. In this note, we obtain a full characterization of radial Carleson measures for the Hilbert-Hardy space on tube domains over symmetric cones. For large derivatives, we also obtain a full characterization of the measures for which the corresponding embedding operator is continuous. Restricting to the case of light cones of dimension three, we prove that by freezing one or two variables, the problem of embedding derivatives of the Hilbert-Hardy space into Lebesgue spaces reduces to the characterization of Carleson measures for Hilbert-Bergman spaces of the upper-half plane or the product of two upper-half planes.

1. Introduction

All over the text, $T_\Omega = V + i\Omega$ is the tube domain over the irreducible symmetric cone $\Omega$ (a symmetric Siegel domain of type I). We put $n = \dim V$ and we denote by $r$ the rank of the cone $\Omega$. For more on symmetric cones and on tube domains over these cones, we refer to \cite{8}. We shall adopt the notations of \cite{8} and call $\Delta$ the determinant function of the symmetric cone $\Omega$.

A typical example of an irreducible symmetric cone is the Lorentz cone $\Lambda_n$, $n \geq 3$, of $\mathbb{R}^n$, i.e. the set defined by

$$\Lambda_n = \{(y_1, \cdots, y_n) \in \mathbb{R}^n : y_1 + y_2 > 0 \text{ and } y_1^2 - \cdots - y_n^2 > 0\},$$

which is a symmetric cone of rank $r = 2$ and its determinant function is given by the Lorentz form

$$\Delta(y) = y_1^2 - \cdots - y_n^2.$$
The Hardy space $H^p(T_\Omega)$ is the space consisting of holomorphic functions $F$ on $T_\Omega$ which satisfy the estimate
\[ ||F||_{H^p(T_\Omega)} := \left( \sup_{y \in \Omega} \int_V |F(x + iy)|^p dx \right)^{\frac{1}{p}} < \infty. \]

For $\nu$ a real number, and $1 \leq p < \infty$, we recall that the Bergman space $A^p_\nu(T_\Omega)$ is the subspace of the Lebesgue space $L^p(T_\Omega, dV_\nu)$ consisting of holomorphic functions; here $dV_\nu(x + iy) = \Delta^{\nu - \frac{n}{r}}(y) dx dy$. We observe that $A^p_\nu(T_\Omega)$ is nontrivial only if $\nu > \frac{n}{r} - 1$ (see [4] and [1]).

We call $\langle \cdot, \cdot \rangle$ the scalar product in $V$ with respect to which $\Omega$ is self-dual. Recall that the Box operator $\Box = \Delta(\frac{1}{i} \frac{\partial}{\partial x})$ is the differential operator of degree $r$ in $\mathbb{R}^n$ defined by the equality:
\[ \Box [e^{i(x|\xi)}] = \Delta(\xi) e^{i(x|\xi)}, \quad x \in \mathbb{R}^n, \xi \in \Omega. \]

**Definition 1.1.** Let $k$ be a positive integer. A positive Borel measure $\mu$ on $T_\Omega$ is called a $k$-box Carleson measure for $H^p(T_\Omega)$, $0 < p < \infty$ if there exists a positive constant $C = C(p, k, \mu)$ such that
\[ \int_{T_\Omega} |\Box^k F(z)|^p d\mu(z) \leq C ||F||_{H^p(T_\Omega)} \]
for every $F \in H^p(T_\Omega)$. When $k = 0$, $\mu$ is just called a Carleson measure for $H^p(T_\Omega)$.

**Problem:** Characterize the $k$-box Carleson measures for the Hardy space $H^p(T_\Omega)$, $0 < p < \infty$.

In the one-dimensional case ($n = 1$, $\Omega = (0, \infty)$), the solution to this problem was provided by L. Carleson (cf. [10]) for $k = 0$, and for $k \neq 0$, the result is due to D. Luecking (cf. [13]). For tube domains over symmetric cones, the problem is still essentially open. Pretty recently, for values of $p, q$ such that $p < q$, a characterization of $q$-Carleson measures for $H^p(T_\Omega)$, that is the positive measures $\mu$ such that $H^p(T_\Omega)$ embeds continuously into $L^q(T_\Omega, d\mu)$, was obtained in [2] in terms of boundedness of a kind of balayage of the measure $\mu$. In this note, we are interested in the above question in the case $p = 2$. We do not provide a general characterization but restricting to the two following classes of measures:
- radial measures;
- products of a Dirac measure and a measure in the lower dimension, we provide a full characterization. In particular, we prove that when our measure is the product of the Dirac measure and a measure in dimension two, then the problem reduces to a characterization of Carleson measures for Hilbert-Bergman spaces of the product of two upper-half planes. For completeness of our paper, we characterize at the end of our work, Carleson embeddings for (vector) weighted Bergman spaces of the product of upper-half planes.

2. Preliminary results

We refer to [1]. The determinant function $\Delta$ of the symmetric cone $\Omega$ has a natural holomorphic extension $\Delta(z)$ to the tube domain $T_{\Omega}$ which does not vanish on $T_{\Omega}$. For $\alpha \in \mathbb{R}$, we shall denote $\Delta^{\alpha}(z)$ the holomorphic determination of the power which coincides with $\Delta^{\alpha}(y)$ when $z = iy \in i\Omega$.

Proposition 2.1. The integral
\[ I(y, w) := \int_{\Omega} \left| \Delta^{-\alpha} \left( \frac{x + iy - \bar{w}}{i} \right) \right|^2 \, dx \quad (y \in \Omega, \; w \in T_{\Omega}) \]
converges if and only if $\alpha > \frac{n}{r} - \frac{1}{2}$. In this case, there exists a positive constant $C_{\alpha}$ such that
\[ I(y, w) = C_{\alpha} \Delta^{-2\alpha + \frac{n}{r}} (y + \Im w). \]

Furthermore, the weighted Bergman kernel functions
\[ F(z) = \Delta^{-\alpha}(z - \bar{w}) \quad (w \in T_{\Omega}) \]
belong to $H^2(T_{\Omega})$ if and only if $\alpha > \frac{n}{r} - \frac{1}{2}$. In this case,
\[ \|F\|^2_{H^2(T_{\Omega})} = C_{\alpha} \Delta^{-2\alpha + \frac{n}{r}} (\Im w). \]

Let us recall the following characterization of $H^2(T_{\Omega})$ (cf. e.g. [8]).

Proposition 2.2. (Paley-Wiener) A holomorphic function $F$ on $T_{\Omega}$ belongs to the Hardy space $H^2(T_{\Omega})$ if and only if there exists a function $f \in L^2(\Omega)$ such that
\[ F(z) = \int_{\Omega} f(t) e^{it \cdot z} \, dt \quad (z \in T_{\Omega}). \]
In this case, \(|F|_{\mathcal{B}^2(T_\Omega)}^2 = (2\pi)^n \int_\Omega |f(t)|^2 dt\).

**Definition 2.3.** When a holomorphic function \(F\) on \(T_\Omega\) can be expressed as in (2.3), we say that \(F\) is the Laplace transform of \(f\).

Let us recall the following Paley-Wiener result for the Bergman spaces \(A^2_\nu(T_\Omega)\) (see [1] and [8]).

**Proposition 2.4. (Paley-Wiener)** Let \(\nu > \frac{n}{r} - 1\). A holomorphic function \(F\) on \(T_\Omega\) belongs to the Bergman space \(A^2_\nu(T_\Omega)\) if and only if there exists a function \(f \in L^2(\Omega, \frac{dy}{\Delta_\nu(y)})\) such that\n
\[(2.4) \quad F(z) = \int_\Omega f(t) e^{i\langle t, z \rangle} dt \quad (z \in T_\Omega).\]

In this case, \(|F|_{A^2_\nu(T_\Omega)}^2 = (2\pi)^n \Gamma_\Omega(\nu) \int_\Omega |f(t)|^2 \frac{dt}{\Delta_\nu(t)}\), where \(\Gamma_\Omega\) denotes the gamma function of the cone \(\Omega\).

We next focus on the upper half-plane \(\Pi^+\) of the complex plane \(\mathbb{C}\).

**Definition 2.5.** Let \(\alpha > -1\). We denote by \(A^2_\alpha(\Pi^+)\) the weighted Bergman space consisting of holomorphic functions \(G\) on \(\Pi^+\) satisfying the estimate\n
\[||G||_{A^2_\alpha(\Pi^+)} := \left( \int_{\Pi^+} |G(x + iy)|^2 y^\alpha dx dy \right)^{\frac{1}{2}} < \infty.\]

We next recall the analogue of the Paley-Wiener theorem for the Bergman space \(A^2_\alpha(\Pi^+)\) (cf. [1]).

**Proposition 2.6.** Let \(G\) be a holomorphic function on \(\Pi^+\). The following assertions are equivalent.

1. \(G\) belongs to the weighted Bergman space \(A^2_\alpha(\Pi^+)\).
2. There exists a function \(g : (0, \infty) \to \mathbb{C}\) satisfying the estimate \(\int_0^\infty |g(t)|^2 \frac{dt}{t^{\alpha+1}} < \infty\), such that\n
\[G(z) = \int_0^\infty g(t) e^{itz} dt \quad (z \in \Pi^+).\]

In this case, \(|G|_{A^2_\alpha(\Pi^+)}^2 = 2\pi \Gamma(\alpha + 1) \int_0^\infty |g(t)|^2 \frac{dt}{t^{\alpha+1}}\), where \(\Gamma\) is the usual gamma function.
Definition 2.7. Let $\vec{\alpha} = (\alpha_1, \alpha_2)$ where $\alpha_1, \alpha_2 > -1$. We denote by $A^2_{\vec{\alpha}}(\Pi^+ \times \Pi^+)$ the weighted Bergman space consisting of holomorphic functions $G$ on $\Pi^+ \times \Pi^+$, which satisfy the following estimate

$$||G||_{A^2_{\vec{\alpha}}(\Pi^+ \times \Pi^+)} = \left( \int_{\Pi^+ \times \Pi^+} |G(x_1 + iy_1, x_2 + iy_2)|^2 y_1^{\alpha_1} y_2^{\alpha_2} dx_1 dx_2 dy_1 dy_2 \right)^{\frac{1}{2}} < \infty.$$ 

When $\alpha_1 = \alpha_2 = \alpha$, for simplicity, we use the notation $A^2_{(\alpha, \alpha)}(\Pi^+ \times \Pi^+)$ for $A^2_{\vec{\alpha}}(\Pi^+ \times \Pi^+)$. 

We recall also the following Paley-Wiener result for Bergman spaces of the tube domain $\Pi^+ \times \Pi^+$ in $\mathbb{C}^2$ over the first octant $(0, \infty) \times (0, \infty)$ (cf. [18]).

Proposition 2.8. Let $\vec{\alpha} = (\alpha_1, \alpha_2)$, with $\alpha_1, \alpha_2 > -1$. Let $G$ be a holomorphic function on $\Pi^+ \times \Pi^+$. Then the following assertions are equivalent.

1. $G$ belongs to the weighted Bergman space $A^2_{\vec{\alpha}}(\Pi^+ \times \Pi^+)$. 
2. There exists a function $g : (0, \infty) \times (0, \infty) \to \mathbb{C}$ satisfying the estimate $\int_{(0, \infty) \times (0, \infty)} |g(t_1, t_2)|^2 \frac{dt_1}{t_1^{\alpha_1+1}} \frac{dt_2}{t_2^{\alpha_2+1}} < \infty$, such that

$$G(z) = \int_{(0, \infty) \times (0, \infty)} g(t) e^{i(z_1 t_1 + z_2 t_2)} dt_1 dt_2 \quad (z \in \Pi^+ \times \Pi^+).$$

In this case, $||G||_{A^2_{\vec{\alpha}}(\Pi^+ \times \Pi^+)} = c_{\vec{\alpha}} \left( \int_{(0, \infty) \times (0, \infty)} |g(t_1, t_2)|^2 \frac{dt_1}{t_1^{\alpha_1+1}} \frac{dt_2}{t_2^{\alpha_2+1}} \right)^{\frac{1}{2}}$ with $c_{\vec{\alpha}} = 2\pi \sqrt{\Gamma(\alpha_1+1) \Gamma(\alpha_2+1)}$.

3. Radial Carleson measures for $H^2(T_\Omega)$

In view of the second assertion of Proposition 2.1, testing on the weighted Bergman kernel functions

$$\Delta^{-\alpha}(z - \bar{w}) \quad (w \in T_\Omega),$$

we obtain at once the following necessary condition for $\mu$ be a Carleson measure for $H^2(T_\Omega)$ : there exists a positive constant $C = C(\alpha, \mu)$ such that

$$(3.5) \quad \int_{T_\Omega} |\Delta^{-\alpha}(z - \bar{w})|^2 d\mu(z) \leq C \Delta^{-2\alpha + \frac{n}{r}}(\Im w)$$

for every $w \in T_\Omega$, provided $\alpha > \frac{n}{r} - \frac{1}{2}$. 


The aim of the present section is to investigate whether this necessary condition is also sufficient as in the one-dimensional case. We consider particular radial measures \( \mu \) on \( T_\Omega \) of the form
\[
d\mu(x + iy) = \varphi(y)dx\,dy,
\]
where \( \varphi \) is a positive measurable function on \( \Omega \).

Using the Plancherel Theorem, one obtains that the Carleson measure property (1.2) may be expressed in this particular case as
\[
\int_\Omega \left( \int_\Omega |f(t)|^2e^{-2\langle t,y \rangle}dt \right) \varphi(y)dy = \int_\Omega \left( \int_\Omega \varphi(y)e^{-2\langle t,y \rangle}dy \right) |f(t)|^2dt
\leq C \int_\Omega |f(t)|^2dt.
\]
(3.6)

On the other hand, in view of the first assertion of Proposition 2.1, the necessary condition inequality (3.5) may be expressed as
\[
\int_\Omega \Delta^{-2\alpha + \frac{n}{2}}(y + t)\varphi(y)dy \leq C_{\alpha,\varphi} \Delta^{-2\alpha + \frac{n}{2}}(t)
\]
for every \( t \in \Omega \).

We shall prove the following theorem:

**Theorem 3.1.** Let \( \varphi \) be a positive measurable function on the cone \( \Omega \). The following three assertions are equivalent.

1. The measure \( d\mu(x + iy) = dx\varphi(y)dy \) is a Carleson measure for the Hardy space \( H^2(T_\Omega) \).
2. The function \( \varphi \) is integrable on \( \Omega \).
3. For some (all) \( \alpha > \frac{n}{2} - \frac{1}{2} \), there exists a positive constant \( C_{\alpha,\varphi} \) such that (3.7) holds for every \( t \in \Omega \).

**Proof.** We shall prove the following implications: \((2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (2)\).

We first show the implication \((2) \Rightarrow (1)\). In view of (3.6), the measure \( d\mu(x + iy) = dx\varphi(y)dy \), where \( \varphi \) is a positive measurable function on \( \Omega \), is a Carleson measure for \( H^2(T_\Omega) \) if and only if the function
\[
t \in \Omega \mapsto \left\{ \int_\Omega \varphi(y)e^{-2\langle t,y \rangle}dy \right\}^{\frac{1}{2}}
\]
is a multiplier of \( L^2(\Omega) \). The latter property is valid if and only if the relevant function is bounded on \( \Omega \). It is then clear that assertion (2) implies assertion (1).
Prior to the statement of the theorem, we proved the implication 
\((1) \Rightarrow (3)\) which amounts to the fact that \((3.7)\) is a necessary condition 
for \((1)\).

We finally show the implication \((3) \Rightarrow (2)\). We recall the following 
order relation \(\prec\) on \(\Omega\). We write 
\(x \prec y\) if 
\(y - x \in \Omega\). It is well known that 
\(\Delta(x) \leq \Delta(y)\) whenever \(x \prec y\). Consequently, if \(y \prec t\), then 
\(y + t \prec 2t\) and hence \(\Delta(t) \leq \Delta(y + t) \leq \Delta(2t) = C\Delta(t)\).

The assertion \((3)\) may be written as

\[
\sup_{t \in \Omega} \int_{\Omega} \frac{\Delta^{-2\alpha+\frac{n}{p}}(y + t)}{\Delta^{-2\alpha+\frac{n}{p}}(t)} \varphi(y) dy \leq C_{\alpha,\phi}.
\]

We obtain:

\[
\sup_{t \in \Omega} \int_{y \in \Omega: y \prec t} \varphi(y) dy \leq C_{\alpha} \sup_{t \in \Omega} \int_{\Omega} \frac{\Delta^{-2\alpha+\frac{n}{p}}(y + t)}{\Delta^{-2\alpha+\frac{n}{p}}(t)} \varphi(y) dy \leq C_{\alpha,\phi}.
\]

We call \(y_0\) a base point of \(\Omega\). Then

\[
\int_{\Omega} \varphi(y) dy = \lim_{N \to \infty} \int_{y \in \Omega: y \prec Ny_0} \varphi(y) dy \leq C_{\alpha} \sup_{N} \int_{y \in \Omega: y \prec Ny_0} \varphi(y) dy \leq C_{\alpha,\phi}.
\]

Here, the equality follows from the Lebesgue monotone convergence theorem. This concludes the proof of the implication \((3) \Rightarrow (2)\). \(\square\)

4. Embedding derivatives of Hardy spaces into Lebesgue spaces

In [13], D. Luecking characterized those measures \(\mu\) on the upper 
half-plane \(\Pi^+\) of the complex plane such that differentiation \(\frac{d^m}{dz^m}\) of order \(m = 0, 1, \cdots\) maps \(H^p(\Pi^+)\) boundedly into \(L^q(\Pi^+, d\mu)\), where 
\(0 < p, q < \infty\).

In our setting, this question is also still open. Note that in this case, 
we replace the differential operator \(\frac{d}{dz}\) by the box operator \(\Box\) defined 
in the introduction. We shall consider here only the case \(p = q = 2\).

4.1. Embedding large derivatives of \(H^2(T_\Omega)\) into Hilbert-Lebesgue spaces. We deduce the following corollary from Proposition 2.4.

**Corollary 4.1.** We suppose that \(m\) is an integer such that 
\(2m > \frac{n}{p} - 1\). Then the differential operator \(\Box^m\) is a bounded isomorphism from 
\(H^2(T_\Omega)\) to \(A_{2m}^2(T_\Omega)\).
Proof. First by Proposition 2.2 each $F \in H^2(T_{\Omega})$ is the Laplace transform of a measurable function $f : \Omega \to \mathbb{C}$ with the equality $||F||^2_{H^2(T_{\Omega})} = (2\pi)^n \int_{\Omega} |f(\xi)|^2 d\xi$. Applying the operator $\Box$ to $F$ $m$ times, we obtain

$$\Box^m F(z) = \int_{\Omega} \Delta^m(\xi) e^{i(z,\xi)} f(\xi) d\xi, \quad z \in T_{\Omega}.$$ 

Put $g(\xi) := \Delta^m(\xi) f(\xi)$. In view of Proposition 2.4, we have

$$||\Box^m F||^2_{\mathcal{A}^2_{2m}(T_{\Omega})} = (2\pi)^n \Gamma_{\Omega}(2m) \int_{\Omega} |g(\xi)|^2 \frac{d\xi}{\Delta^{2m}(\xi)} = (2\pi)^n \Gamma_{\Omega}(2m) \int_{\Omega} |f(\xi)|^2 d\xi = \Gamma_{\Omega}(2m) ||F||^2_{H^2(T_{\Omega})}.$$ 

Conversely, if $G \in \mathcal{A}^2_{2m}(T_{\Omega})$, then by Proposition 2.4

$$G(z) = \int_{\Omega} g(\xi) e^{i(z,\xi)} d\xi, \quad z \in T_{\Omega}$$

for some $g \in L^2\left(\Omega, \frac{d\xi}{\Delta^{2m}(\xi)}\right)$, with $||G||^2_{\mathcal{A}^2_{2m}(T_{\Omega})} = (2\pi)^n \Gamma_{\Omega}(2m) \int_{\Omega} |g(\xi)|^2 \frac{d\xi}{\Delta^{2m}(\xi)}$. Put $f(\xi) := \Delta^{-m}(\xi) g(\xi)$. Then $f \in L^2(\Omega, d\xi)$ and if we define $F$ by

$$F(z) = \int_{\Omega} f(\xi) e^{i(z,\xi)} d\xi, \quad z \in T_{\Omega},$$

then $F$ is well-defined and one easily checks that $\Box^m F = C_m G$, and $||F||^2_{H^2(T_{\Omega})} = (2\pi)^n \int_{\Omega} |f(\xi)|^2 d\xi = (2\pi)^n ||g||^2_{L^2(\Omega, \frac{d\xi}{\Delta^{2m}(\xi)})}$. The proof is complete. \hfill \Box

We then obtain the following characterization of $m$-box Carleson measures for $H^2(T_{\Omega})$ for large integer $m$.

**Theorem 4.2.** We suppose that $m$ is an integer such that $2m > \frac{n}{\gamma} - 1$. Let $\mu$ be a positive Borel measure on $T_{\Omega}$. Then the following two assertions are equivalent.

1. There exists a positive constant $A = A(m)$ such that

$$\int_{T_{\Omega}} ||\Box^m F||^2 d\mu \leq A ||F||^2_{H^2(T_{\Omega})}$$

for each $F \in H^2(T_{\Omega})$;

2. $\mu$ is a Carleson measure for the weighted Bergman spaces $\mathcal{A}^2_{2m}(T_{\Omega})$. 


Proof. According to the previous corollary, assertion (1) is equivalent to the following assertion: there exists a positive constant $C$ such that

$$\int_{T_\Omega} |G|^2 d\mu \leq C||G||^2_{A_{2m}^2(T_\Omega)}$$

for each $G \in A_{2m}^2(T_\Omega)$. The latter assertion is clearly assertion (2). □

We recall that the Carleson measures for Bergman spaces on tube domains over symmetric cones were characterized in [14] in terms of a geometrical condition on Bergman balls.

When the integer $m$ is such that $0 \leq m \leq \frac{1}{2}(\frac{n}{r} - 1)$, the above techniques do not provide any answer. Nevertheless, in the following, we provide a full characterization when considering only some restricted measures in the setting of the tube domain over the Lorentz cone of dimension three.

4.2. Two examples of $m$-box Carleson measures for the Hardy space $H^2$ on the tube domain over the Lorentz cone $\Lambda_3$ of $\mathbb{R}^3$.

1. First class of examples: measures of the form $\mu(z_1)\delta_O(z_2, z_3)$. We denote by $\delta_O(z_2, z_3)$ the Dirac measure at the origin in $\mathbb{C} \times \mathbb{C}$. We shall characterize the positive measures $\mu(z_1)$ on the upper half-plane $\Pi^+$ for which there exists a positive constant $C$ such that for each $F \in H^2(T_{\Lambda_3})$, the following estimate holds.

$$\int_{T_{\Lambda_3}} |\Box^m F(z_1, z_2, z_3)|^2 d\mu(z_1) d\delta_O(z_2, z_3) = \int_{\Pi^+} |(\Box^m F)(z_1, 0, 0)|^2 d\mu(z_1) \leq C||F||^2_{H^2(T_{\Lambda_3})}.$$

**Definition 4.3.** (a) Let $F$ be a complex-valued function defined on $T_{\Lambda_3}$. We call restriction of $F$ to $\Pi^+$, the function $RF : \Pi^+ \to \mathbb{C}$ defined by

$$(RF)(z_1) = F(z_1, 0, 0).$$

(b) Let $G$ be a complex-valued function defined on $\Pi^+$. We say that the function $F : T_{\Lambda_3} \to \mathbb{C}$ is an extension of $G$ if $RF = G$.

We shall use the following result.
Proposition 4.4.  

(1) The following estimate holds:

\[ \|R\Box^m F\|_{A^2_m(I^1)} \leq \frac{\Gamma(4m + 2)}{2^{4m+4}\pi (2m + 1)} \|F\|^2_{H^2(T_{\Lambda_3})} \]

for every \( F \in H^2(T_{\Lambda_3}) \).

(2) Conversely, for every function \( G \in A^2_{4m+1}(\Pi^+) \), there exists a function \( F \in H^2(T_{\Lambda_3}) \) such that \( \Box^m F \) is an extension of \( G \). Moreover,

\[ \|F\|^2_{H^2(T_{\Lambda_3})} = \frac{m + 1}{\pi \sqrt{2\Gamma(4m + 2)}} \|G\|_{A^2_{4m+1}(\Pi^+)} \].

Proof. (1): We have \( \langle z, t \rangle := z_1t_1 + z_2t_2 + z_3t_3 \). Let \( F \in H^2(T_{\Lambda_3}) \). In view of Proposition 2.2, there exists a measurable function \( f : \Lambda_3 \to \mathbb{C} \) such that

\[ F(z) = \int_{\Lambda_3} f(t) e^{i\langle z, t \rangle} dt \quad (z \in T_{\Lambda_3}), \]

with \( \|F\|^2_{H^2(T_{\Lambda_3})} = 8\pi^3 \int_{\Lambda_3} |f(t)|^2 dt \). Then

\[ (R\Box^m F)(x_1 + iy_1) = \int_{\Lambda_3} f(t_1, t_2, t_3) e^{it_1(x_1 + iy_1)} \Delta^m(t) dt_1 dt_2 dt_3 \]

\[ = \int_0^\infty \left( \int_{t_2 + t_3 < t_1^2} f(t_1, t_2, t_3) \Delta^m(t) dt_2 dt_3 \right) e^{it_1(x_1 + iy_1)} dt_1. \]

An application of the Plancherel formula implies

\[ \int_{-\infty}^{\infty} |(R\Box^m F)(x_1 + iy_1)|^2 dx_1 = 2\pi \int_0^\infty \left| \int_{t_2 + t_3 < t_1^2} f(t_1, t_2, t_3) \Delta^m(t) dt_2 dt_3 \right|^2 e^{-2t_1y_1} dt_1. \]

Next, when applying the Fubini Theorem, we obtain:

\[ \|R\Box^m F\|^2_{A^2_m(I^1)} = \int_0^\infty \left( \int_{-\infty}^{\infty} |(R\Box^m F)(x_1 + iy_1)|^2 dx_1 \right) y_1^{4m+1} dy_1 \]

\[ = 2\pi \int_0^\infty \left| \int_{t_2 + t_3 < t_1^2} f(t_1, t_2, t_3) \Delta^m(t) dt_2 dt_3 \right|^2 \]

\[ \left( \int_0^\infty e^{-2t_1y_1} y_1^{4m+1} dy_1 \right) dt_1 \]

\[ = \frac{\pi \Gamma(4m + 2)}{2^{4m+1}} \int_0^\infty \left| \int_{t_2 + t_3 < t_1^2} f(t_1, t_2, t_3) \Delta^m(t) dt_2 dt_3 \right|^2 \frac{dt_1}{t_1^{4m+2}}. \]

We finally applying the Schwarz inequality to the integral with respect to \( dt_2dt_3 \), we get
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We conclude that

\[ ||R \Box^m F||^2_{A^2_{4m+1}(\Pi^+)} \leq \frac{\pi^2 \Gamma(4m+2)}{2^{4m+1}(2m+1)} \int_0^\infty \left( \int_{t_2^2+t_3^2<t_1^2} |f(t_1,t_2,t_3)|^2 dt_2 dt_3 \right) dt_1 \]
\[ = \frac{\pi^2 \Gamma(4m+2)}{2^{4m+1}(2m+1)} \int_\Omega |f(t)|^2 dt \]
\[ = \frac{\Gamma(4m+2)}{2^{4m+1}(2m+1)} ||F||^2_{H^2(T_{\Lambda_3})}. \]

(2): Conversely, let $G \in A^2_{4m+1}(\Pi^+)$. In view of Proposition 2.6, there exists a measurable function $g : (0, \infty) \to \mathbb{C}$ such that

\[
G(z_1) = \int_0^\infty g(t_1) e^{it_1z_1} dt_1 \quad (z_1 \in \Pi^+),
\]

with $||G||^2_{A^2_{4m+1}(\Pi^+)} = 2\pi \Gamma(4m+2) \int_0^\infty |g(t_1)|^2 \frac{dt_1}{t_1^{2m+2}}$. Let

\[
F(z) := \frac{m+1}{\pi} \int_\Omega g(t_1) \frac{e^{i(z_1t_1+z_2t_2+z_3t_3)}}{t_1^{2m+2}} dt_1 dt_2 dt_3.
\]

We have

\[
R \Box^m F(z) = \frac{m+1}{\pi} \int_\Omega g(t_1) e^{i(z_1t_1+zt_2+zt_3)} \Delta^m(t) dt_1 dt_2 dt_3
\]
\[ = \frac{m+1}{\pi} \int_0^\infty g(t_1) \left( \int_{t_2^2+t_3^2<t_1^2} \Delta^m(t) dt_2 dt_3 \right) dt_1 \]
\[ = \int_0^\infty g(t_1) e^{it_1z_1} dt_1.
\]

That is, $\Box^m F$ is an extension of $G$ to $T_{\Lambda_3}$. Let us prove that $F \in H^2(T_{\Lambda_3})$. In view of Proposition 2.2, it suffices to prove that

\[
||F||^2_{H^2(T_{\Lambda_3})} = \frac{(m+1)^2}{\pi^2} \int_\Omega \frac{|g(t_1)|^2}{t_1^{4m+4}} dt_1 dt_2 dt_3 < +\infty.
\]
Proceeding as above, we obtain that
\[
\int_{\Omega} \left| g(t_1) \right|^2 \frac{dt_1 dt_2 dt_3}{t_1^{4m+4}} = \int_0^\infty \left| g(t_1) \right|^2 \left( \int_{t_2^4 + t_3^4 < t_1^4} dt_2 dt_3 \right) dt_1
\]
\[
= \pi \int_0^\infty \frac{\left| g(t_1) \right|^2}{t_1^{4m+2}} dt_1
\]
\[
= \frac{1}{2\Gamma(4m+2)} \|G\|^2_{A^2_{4m+2}(\Pi^+)},
\]
Hence
\[
\|F\|^2_{H^2(T_{\Lambda_3})} = \frac{(m+1)^2}{2\pi^2 \Gamma(4m+2)} \|G\|^2_{A^2_{4m+2}(\Pi^+)} < \infty.
\]

Our main result in this subsection is the following.

**Theorem 4.5.** Let \( \mu \) be a positive measure on \( \Pi^+ \), and \( m \geq 0 \) an integer. Then the following two properties are equivalent.

1. \( \mu(z_1)\delta_O(z_2, z_3) \) is a \( m \)-box Carleson measure for the Hardy space \( H^2(T_{\Lambda_3}) \);
2. \( \mu \) is the Carleson measure for the weighted Bergman space \( A^2_{4m+1}(\Pi^+) \).

**Proof.** (2) \( \Rightarrow \) (1) Let \( \mu \) be a Carleson measure for the weighted Bergman space \( A^2_{4m+1}(\Pi^+) \) with Carleson constant \( C \). Then for every \( F \in H^3(T_{\Lambda_3}) \), we have
\[
\int_{T_{\Lambda_3}} |\Box^m F(z)|^2 d\mu(z_1) d\delta_O(z_2, z_3) = \int_{\Pi^+} |(\Box^m F)(z_1, 0, 0)|^2 d\mu(z_1)
\]
\[
= \int_{\Pi^+} |(R \Box^m F)(z_1)|^2 d\mu(z_1)
\]
\[
\leq C \|R \Box^m F\|^2_{A^2_{4m+2}(\Pi^+)}
\]
\[
\leq \frac{C\Gamma(4m+2)}{2^{4m+4}\pi(2m+1)} \|F\|^2_{H^2(T_{\Lambda_3})},
\]
where the latter inequality follows from assertion (1) of Proposition 4.4.

(1) \( \Rightarrow \) (2) We suppose that \( \mu(z_1)\delta_O(z_2, z_3) \) is a \( m \)-box Carleson measure for the Hardy space \( H^2(T_{\Lambda_3}) \) with Carleson constant \( C \). It follows from assertion (2) of Proposition 4.4 that for every function

\[\Box^m F(z) = \frac{1}{2\pi^2 \Gamma(4m+2)} \|G\|^2_{A^2_{4m+2}(\Pi^+)} < \infty.\]
G \in A^2_{4m+1}(\Pi^+), there is an \( F \in H^2(T_{\lambda_3}) \) such that \( \Box^m F \) is an extension of \( G \) to \( T_{\lambda_3} \) and \( \|F\|_{H^2(T_{\lambda_3})} = \frac{2}{\pi \sqrt{2m+2}} \|G\|_{A^2_{4m+1}(\Pi^+)} \). Thus
\[
\int_{\Pi^+} |G(z_1)|^2 d\mu(z_1) = \int_{T_{\lambda_3}} |\Box^m F(z_1, z_2, z_3)|^2 d\mu(z_1) d\delta_0(z_2, z_3) 
\leq C \|F\|^2_{H^2(T_{\lambda_3})} = \frac{C(m+1)^2}{2m+2} \|G\|^2_{A^2_{4m+1}(\Pi^+)}.
\]
The proof is complete. \( \Box \)

We recall that a characterization of Carleson measures for standard Bergman spaces \( A^2_\alpha \) of the unit disc was provided by D. Stegenga \([17]\) in terms of a geometrical condition on Carleson sectors. For the unweighted case, refer to \([11]\). A characterization of Carleson measures for standard Bergman spaces \( A^2_\alpha(\Pi^+) \) in terms of a geometrical condition on Carleson rectangles can be found in \([5]\) (see also Section 5 below).

2. Second class of examples: measures of the form \( \mu(z_1, z_2)\delta_0(z_3) \).

We denote by \( \delta_0(z_3) \) the Dirac measure at the origin in \( \mathbb{C} \). We characterize the positive measures \( \mu(z_1, z_2) \) on the domain \( \mathcal{D} := \{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : y_1^2 > y_2^2, y_1 > 0\} \) of \( \mathbb{C}^2 \) for which there exists a positive constant \( C \) such that the following estimate holds:
\[
\int_{T_{\lambda_3}} |\Box^m F(z_1, z_2, z_3)|^2 d\mu(z_1, z_2) \delta_0(z_3) = \int_{\mathcal{D}} |(\Box^m F)(z_1, z_2, 0)|^2 d\mu(z_1, z_2) \leq C \|F\|^2_{H^2(T_{\lambda_3})}
\]
for every \( F \in H^2(T_{\lambda_3}) \). It is easily checked that the Lorentz cone \( \Lambda_3 \) is linearly equivalent to the spherical cone \( \Sigma \) in \( \mathbb{R}^3 \) defined by
\[
\Sigma := \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1y_2 - y_3^2 > 0, y_1 > 0\}
\]
Our problem takes the following form. Characterize the positive measures \( \mu(z_1, z_2) \) on the product \( \Pi^+ \times \Pi^+ \) of two upper half-planes (the tube domain over the first octant) for which there exists a positive constant \( C \) such that the following estimate holds:
\[
\int_{T_{\Sigma}} |\Box^m F(z_1, z_2, z_3)|^2 d\mu(z_1, z_2) \delta_0(z_3) = \int_{\Pi^+ \times \Pi^+} |(\Box^m F)(z_1, z_2, 0)|^2 d\mu(z_1, z_2) \leq C \|F\|^2_{H^2(T_{\Sigma})}
\]
for every \( F \in H^2(T_{\Sigma}) \). We note that in this case, the determinant function is now \( \Delta(t) = t_1t_2 - t_3^2 \).
Definition 4.6. (1) Let $F$ be a complex-valued function defined on $T_{\Sigma}$. We call restriction of $F$ to $\Pi^+ \times \Pi^+$ the function $R F : \Pi^+ \times \Pi^+ \rightarrow \mathbb{C}$ defined by

$$(RF)(z_1, z_2) = F(z_1, z_2, 0).$$

(2) Let $G$ be a complex-valued function defined on $\Pi^+ \times \Pi^+$. We say that a function $F : T_{\Sigma} \rightarrow \mathbb{C}$ is an extension of $G$ if $RF = G$.

Our result is the following.

Theorem 4.7. Let $\mu$ be a positive measure on $\Pi^+ \times \Pi^+$, and let $m \geq 0$ be an integer. Then the following two assertions are equivalent.

(1) $\mu(z_1, z_2)\delta_0(z_3)$ is a $m$-box Carleson measure for the Hardy space $H^2(T_{\Sigma})$

(2) $\mu$ is a Carleson measure for the weighted Bergman space $A_{2m-\frac{1}{2}}^2(\Pi^+ \times \Pi^+)$.

The proof is an easy adaptation of the proof of Theorem 4.5 with the help of the following result.

Proposition 4.8. (1) The following estimate holds:

$$||R \Box^m F||_{A_{2m-\frac{1}{2}}^2(\Pi^+ \times \Pi^+)}^2 \leq \frac{4\pi^2 (2m)!\Gamma \left(2m + \frac{1}{2}\right)}{2m + \frac{1}{2}} ||F||_{H^2(T_{\Sigma})}^2$$

for every $F \in H^2(T_{\Sigma})$.

(2) Conversely, for every function $G \in A_{2m-\frac{1}{2}}^2(\Pi^+ \times \Pi^+)$, there exists a function $F \in H^2(T_{\Sigma})$ such that $\Box^m F$ is an extension of $G$. Moreover,

$$||F||_{H^2(T_{\Sigma})}^2 = \frac{\left(\Gamma \left(m + \frac{3}{2}\right)\right)^2}{4\pi^2 \left(\Gamma \left(2m + \frac{1}{2}\right) m!\right)^2} ||G||_{A_{2m-\frac{1}{2}}^2(\Pi^+ \times \Pi^+)}^2.$$

Proof of Proposition 4.8. (1): We recall that for $z = (z_1, z_2, z_3) \in T_{\Sigma}$ and $t = (t_1, t_2, t_3) \in \Sigma$, we have $\langle z, t \rangle := \frac{1}{2}(z_1t_1 + z_2t_2) + z_3t_3$. Let $F \in H^2(T_{\Sigma})$. We recall with Proposition 2.2 that there exists a measurable function $f : \Sigma \rightarrow \mathbb{C}$ such that

$$F(z) = \int_{\Sigma} f(t)e^{i\langle z, t \rangle} dt \quad (z \in T_{\Sigma}),$$
with \( \|F\|_{H^2(T_2)}^2 = 8\pi^3 \int_0^\infty |f(t)|^2 dt \). Then

\[
(R \Box^m F)(z_1, z_2) = \int_\Sigma f(t_1, t_2, t_3) e^{\frac{i}{2}(t_1 z_1 + t_2 z_2)} \Delta^m(t) dt_1 dt_2 dt_3
\]

\[
= \int_{(0,\infty)^3} \left( \int_{t_3^2 < t_1 t_2} f(t_1, t_2, t_3) \Delta^m(t) dt_3 \right) e^{\frac{i}{2}(t_1 z_1 + t_2 z_2)} dt_1 dt_2.
\]

Using Plancherel formula, we obtain

\[
\int_{\mathbb{R}^2} \left| (R \Box^m F)(x_1 + iy_1, x_2 + iy_2) \right|^2 dx_1 dx_2
\]

\[
= 4\pi^2 \int_{(0,\infty)^2} \left| \int_{t_3^2 < t_1 t_2} f(t_1, t_2, t_3) \Delta^m(t) dt_3 \right|^2 e^{-(t_1 y_1 + t_2 y_2)} dt_1 dt_2.
\]

Next, applying the Fubini Theorem, we obtain:

\[
\|R \Box^m F\|_{L^2}^2 = \int_{(0,\infty)^2} \left( \int_{\mathbb{R}^2} \left| (R \Box^m F)(x_1 + iy_1, x_2 + iy_2) \right|^2 dx_1 dx_2 \right) \times
\]

\[
y_1^{2m-\frac{1}{2}} y_2^{2m-\frac{1}{2}} dy_1 dy_2
\]

\[
= 4\pi^2 \int_{(0,\infty)^2} \left| \int_{t_3^2 < t_1 t_2} f(t_1, t_2, t_3) \Delta^m(t) dt_3 \right|^2 \times
\]

\[
\left( \int_{(0,\infty)^2} e^{-(t_1 y_1 + t_2 y_2)} y_1^{2m-\frac{1}{2}} y_2^{2m-\frac{1}{2}} dy_1 dy_2 \right) dt_1 dt_2
\]

\[
= (2\pi \Gamma(2m + \frac{1}{2}))^2 \int_{(0,\infty)^2} \left| \int_{t_3^2 < t_1 t_2} f(t_1, t_2, t_3) \Delta^m(t) dt_3 \right|^2 \frac{dt_1}{t_1^{2m+\frac{1}{2}}} \frac{dt_2}{t_2^{2m+\frac{1}{2}}}.
\]

Apply the Schwarz inequality to the integral with respect to \( dt_3 \); we obtain

\[
\left| \int_{t_3^2 < t_1 t_2} f(t_1, t_2, t_3) \Delta^m(t) dt_3 \right|^2 \leq \left( \int_{t_3^2 < t_1 t_2} |f(t_1, t_2, t_3)|^2 dt_3 \right) \left( \int_{t_3^2 < t_1 t_2} \Delta^{2m}(t) dt_3 \right)
\]

\[
= \beta \left( 2m + 1, \frac{1}{2} \right) t_1^{2m+\frac{1}{2}} t_2^{2m+\frac{1}{2}} \int_{t_3^2 < t_1 t_2} |f(t_1, t_2, t_3)|^2 dt_3.
\]

In the last inequality, we have used that for \( k > 0 \) an integer, and \( a > 0 \),

\[
(4.8) \quad \int_0^{\sqrt{a}} (a - x^2)^k dx = c_k a^{k+\frac{1}{2}}.
\]
More precisely, we have \( c_k = \frac{1}{2} \beta \left( k + 1, \frac{1}{2} \right) \), where \( \beta \) denotes the usual beta function. We conclude that
\[
\|R^{\square m}F\|_{A^{2m-\frac{1}{2}}_{\Pi^+ \times \Pi^+}} ^2 \leq \frac{4\pi^2 (2m)!\Gamma(2m+\frac{1}{2})}{(2m+\frac{1}{2})} \int_{(0,\infty)^2} \left( \int_{|t_2|<t_1t_2} |f(t_1, t_2, t_3)|^2 dt_3 \right) dt_2 dt_1 \]
\[
= \frac{4\pi^2 (2m)!\Gamma(2m+\frac{1}{2})}{(2m+\frac{1}{2})} \int_{\Sigma} |f(t)|^2 dt = \frac{(2m)!\Gamma(2m+\frac{1}{2})}{2\sqrt{\pi}(2m+\frac{1}{2})} \|F\|_{H^2(T_\Sigma)} ^2.
\]

(2): Now \( G \in A^{2m-\frac{1}{2}}_{\Pi^+ \times \Pi^+} \). We know from Proposition 2.8 that there exists a measurable function \( g : (0, \infty) \times (0, \infty) \rightarrow \mathbb{C} \) such that
\[
G(z) = \int_{(0,\infty)^2} g(t_1, t_2) e^{i(t_1z_1+t_2z_2)} dt_1 dt_2 \quad (z = (z_1, z_2) \in \Pi^+ \times \Pi^+),
\]
with
\[
\|G\|_{A^{2m-\frac{1}{2}}_{\Pi^+ \times \Pi^+}} ^2 = 4\pi^2 \left( \Gamma \left( 2m + \frac{1}{2} \right) \right) ^2 \int_{(0,\infty)^2} |g(t_1, t_2)|^2 \frac{dt_1}{t_1^{2m+\frac{1}{2}}} \frac{dt_2}{t_2^{2m+\frac{1}{2}}}.
\]
Let
\[
F(z) := \frac{1}{2c_m} \int_{\Sigma} \frac{g(t_1, t_2)}{t_1^{m+\frac{1}{2}} t_2^{m+\frac{1}{2}}} e^{i(z_1t_1+z_2t_2)} dt_1 dt_2 dt_3,
\]
where \( c_m \) is the constant of (1.3). Using again (1.8), one easily obtains that
\[
R^{\square m}F(z_1, z_2) = G(z_1, z_2).
\]
That is, the function \( \square^m F \) is an extension of \( G \) to \( T_\Sigma \).

Now to conclude, in view of Proposition 2.2 it is enough to prove that
\[
\int_{\Sigma} \frac{|g(t_1, t_2)|^2}{t_1^{2m+1} t_2^{2m+1}} dt_1 dt_2 dt_3 < \infty.
\]
This is obvious since the left hand side of the latter is equal to
\[
2\int_{(0,\infty)^2} |g(t_1, t_2)|^2 \frac{dt_1}{t_1^{2m+\frac{1}{2}}} \frac{dt_2}{t_2^{2m+\frac{1}{2}}} = \frac{1}{2\pi^2 (\Gamma(2m+\frac{1}{2})) ^2} \|G\|_{A^{2m-\frac{1}{2}}_{\Pi^+ \times \Pi^+}} ^2 < \infty
\]
by Proposition 2.8. The proof is complete. \( \square \)

For completeness, we give a characterization of Carleson measures for weighted Bergman space of \( \Pi^+ \times \Pi^+ \) in the next section. For a characterization of Carleson measures on the bi-disc, the reader can consult
for unweighted Bergman spaces and \([12]\) for standard weighted Bergman spaces.

5. **Carleson measures for Bergman spaces of the tube over the first octant**

As in the one-parameter case \([5]\), we apply techniques of real harmonic analysis as developed for example in \([3, 9, 16]\). Let denote by \(I\) the set of all intervals of \(\mathbb{R}\), and by \(\mathcal{R}\) the set of all rectangles in \(\mathbb{R}^2\).

For \(R \in \mathcal{R}\), \(R = I_1 \times I_2\) where the \(I_j \in I, j = 1, 2\).

For \(I\) an interval, we recall that the Carleson square associated to \(I\) and denoted \(Q_I\) is the set defined by

\[ Q_I := \{ z = x + iy \in \Pi^+ : x \in I, \ \text{and} \ 0 < y < |I| \}. \]

The upper half of the Carleson square \(Q_I\) is the set

\[ T_I := \{ z = x + iy \in \Pi^+ : x \in I, \ \text{and} \ \frac{|I|}{2} < y < |I| \}. \]

Given \(\alpha > -1\), we define the maximal function \(M_\alpha\) on \(\Pi^+\) by

\[ M_\alpha f(z) := \sup_{I \in \mathcal{I}} \frac{\chi_{Q_I}(z)}{V_\alpha(Q_I)} \int_{Q_I} |f(z)| dV_\alpha(z) \]

where for simplicity, we have used the notation \(dV_\alpha(x + iy) = y^\alpha dx dy\).

It is not hard to prove that for any \(p \in (1, \infty]\) (see for example \([5]\)),

\[ \|M_\alpha f\|_{p,\alpha} \leq c_{p,\alpha} \|f\|_{p,\alpha}. \]

For \(R \in \mathcal{R}\), we define the Carleson box (rectangle) \(Q_R\) by

\[ Q_R = Q_{I_1} \times Q_{I_2}, \ \text{whenever} \ R = I_1 \times I_2. \]

On the product \(\Pi^+ \times \Pi^+\) of two upper-planes, we define the product measure \(V_{\tilde{\alpha}}\), \(\tilde{\alpha} = (\alpha_1, \alpha_2)\) by \(dV_{\tilde{\alpha}}(z_1, z_2) = dV_{\alpha_1}(z_1)dV_{\alpha_2}(z_2)\). The weighted Lebesgue space \(L^p_{\tilde{\alpha}}(\Pi^+ \times \Pi^+)\) is the space of measurable functions \(f\) on \(\Pi^+ \times \Pi^+\) such that

\[ \|f\|_{p,\tilde{\alpha}} := \left( \int_{\Pi^+ \times \Pi^+} |f|^p dV_{\tilde{\alpha}} \right)^{\frac{1}{p}} < \infty. \]

The weighted Bergman space \(A^p_{\tilde{\alpha}}(\Pi^+ \times \Pi^+)\) is the subspace of \(L^p_{\tilde{\alpha}}(\Pi^+ \times \Pi^+)\) consisting of holomorphic functions.
Definition 5.1. Given \( \vec{\alpha} = (\alpha_1, \alpha_2) \), \( \alpha_1, \alpha_2 > -1 \), and \( 1 < p \leq q < \infty \), we say a positive measure \( \mu \) defined on \( \Pi^+ \times \Pi^+ \) is a \( (q, \vec{\alpha}) \)-Carleson measure, if there is a constant \( C > 0 \) such that for any \( R \in \mathbb{R} \),

\[
\mu(Q_R) \leq C (V_{\vec{\alpha}}(Q_R))^\frac{q}{p}.
\]

Here \( V_{\vec{\alpha}}(Q_R) = V_{\alpha_1}(Q_{I_1})V_{\alpha_2}(Q_{I_2}) \) whenever \( R = I_1 \times I_2 \).

We have the following characterization of Carleson measures for the weighted Bergman spaces in the product of two upper-half planes.

Theorem 5.2. Let \( \vec{\alpha} = (\alpha_1, \alpha_2) \), with \( \alpha_1, \alpha_2 > -1 \), and let \( 1 < p \leq q < \infty \). Assume \( \mu \) is a positive measure on \( \Pi^+ \times \Pi^+ \). Then the following assertions are equivalent.

(a) \( \mu \) is \( (q, \vec{\alpha}) \)-Carleson measure.

(b) There exists a constant \( C > 0 \) such that for any \( f \in A^p_{\vec{\alpha}}(\Pi^+ \times \Pi^+) \),

\[
\int_{\Pi^+ \times \Pi^+} |f(z)|^p d\mu(z) \leq C \left( \int_{\Pi^+ \times \Pi^+} |f(z)|^p dV_{\vec{\alpha}}(z) \right)^\frac{q}{p}
\]

where \( dV_{\vec{\alpha}}(z_1, z_2) = dV_{\alpha_1}(z_1)dV_{\alpha_2}(z_2) \).

We define the strong maximal function on \( \Pi^+ \times \Pi^+ \) to be the operator

\[
\mathcal{M}_{\vec{\alpha}} f(z) := \sup_{R \in \mathbb{R}} \frac{\chi_{Q_R}(z)}{V_{\vec{\alpha}}(Q_R)} \int_{Q_R} |f(w)| dV_{\vec{\alpha}}(w).
\]

We observe that

\[
\mathcal{M}_{\vec{\alpha}} f \leq M_\alpha_1 \circ M_\alpha_2 f
\]

where \( M_\alpha_j \) is the one-parameter maximal function on \( \Pi^+ \). It follows from the boundedness of \( M_\alpha_j \) that for any \( 1 < p \leq \infty \),

\[
\| \mathcal{M}_{\vec{\alpha}} f \|_{p, \vec{\alpha}} \leq C_{p, \vec{\alpha}} \| f \|_{p, \vec{\alpha}}.
\]

Using the mean value formula in each variable, we obtain that there is a constant \( C > 0 \) such that for any \( z \in \Pi^+ \times \Pi^+ \),

\[
|f(z)| \leq \frac{C}{V_{\vec{\alpha}}(Q_R)} \int_{Q_R} |f(w)| dV_{\vec{\alpha}}(w)
\]

provided \( f \) is analytic on \( \Pi^+ \times \Pi^+ \); where for \( z = (z_1, z_2) \), \( Q_R \) is such that \( z_j \) is the centre of \( Q_{I_j} \), \( R = I_1 \times I_j \).
It follows in particular that there is a constant $C > 0$ such that for any $f$ analytic on $\Pi^+ \times \Pi^+$,
\[
|f(z)| \leq C \mathcal{M}_{\vec{\alpha}}f(z), \quad \text{for any } z \in \Pi^+ \times \Pi^+.
\]
Let us prove the following.

**Proposition 5.3.** Let $\vec{\alpha} = (\alpha_1, \alpha_2)$, with $\alpha_1, \alpha_2 > -1$, and let $1 < p < q < \infty$. Assume $\mu$ is $(\frac{q}{p}, \vec{\alpha})$-Carleson measure. Then there is a constant $C > 0$ such that for any $f \in L^p(\Pi^+ \times \Pi^+, dV_{\vec{\alpha}}(w))$,\[
\int_{\Pi^+ \times \Pi^+} (\mathcal{M}_{\vec{\alpha}}f(z))^q d\mu(z) \leq C \left( \int_{\Pi^+ \times \Pi^+} |f(z)|^p dV_{\vec{\alpha}}(z) \right)^{\frac{q}{p}}.
\]

**Proof.** Consider the following dyadic grids
\[
\mathcal{D}^\beta := \{2^j([0, 1) + m + (-1)^j \beta) : m, j \in \mathbb{Z} \} \quad \text{for } \beta \in \{0, \frac{1}{3}\}.
\]
For $\beta = 0$, $\mathcal{D}^\beta = \mathcal{D}^0$ is the standard dyadic grid of $\mathbb{R}$ denoted $\mathcal{D}$.

For $\vec{\beta} = (\beta_1, \beta_2)$, $\beta_j \in \{0, \frac{1}{3}\}$, we define the dyadic strong maximal function $\mathcal{M}^{d, \vec{\beta}}_{\vec{\alpha}}$ by
\[
\mathcal{M}^{d, \vec{\beta}}_{\vec{\alpha}}f(z) := \sup_{R \in \mathcal{D}^\vec{\beta}} \frac{\chi_{Q_R}(z)}{V_{\vec{\alpha}}(Q_R)} \int_{Q_R} |f(w)| dV_{\vec{\alpha}}(w),
\]
\[
\mathcal{D}^\vec{\beta} = \mathcal{D}^{\beta_1} \times \mathcal{D}^{\beta_2}.
\]
We recall for any interval $I$ of $\mathbb{R}$, there exists an interval $J \in \mathcal{D}^\vec{\beta}$ for some $\beta \in \{0, \frac{1}{3}\}$, such that $I \subseteq J$ and $|J| \leq 6|I|$ (see [15]). It follows that the proposition will follow if we can prove that for $\mu$ an $(\frac{q}{p}, \vec{\alpha})$-Carleson measure, we can find a constant $C > 0$ such that
\[
\int_{\Pi^+ \times \Pi^+} \left( \mathcal{M}^{d, \vec{\beta}}_{\vec{\alpha}}f(z) \right)^q d\mu(z) \leq C \left( \int_{\Pi^+ \times \Pi^+} |f(z)|^p dV_{\vec{\alpha}}(z) \right)^{\frac{q}{p}}
\]
for any $\vec{\beta} \in \{0, \frac{1}{3}\}^2$.

For simplicity of presentation and because our proof works the same for each product dyadic grid, we restrict to the case $\vec{\beta} = (0, 0)$ and denote by $\mathcal{M}^d_{\vec{\alpha}}$ the corresponding dyadic strong maximal function. For $R = I \times J \in \mathcal{D} \times \mathcal{D} = \mathcal{D}^{(0,0)}$, by top half of the Carleson rectangle $Q_R$, we will mean the set
\[
T_R := T_I \times T_J.
\]
We observe that unlike in the one parameter case, for $R_1 \in \mathcal{D} \times \mathcal{D}$ and $R_2 \in \mathcal{D} \times \mathcal{D}$, we have

$$R_1 \cap R_2 \in \{\emptyset, R_1, R_2, R'\}$$

with $R' \in \mathcal{D} \times \mathcal{D}$ such that $R' \nsubseteq R_1$ and $R' \nsubseteq R_2$. Nevertheless, we still have that the family $\{T_R\}_{R \in \mathcal{D} \times \mathcal{D}}$ forms a tiling of $\Pi^+ \times \Pi^+$. Indeed, one observes that even if two rectangles intersect with their intersection strictly contained in each of them, the corresponding top halves of the Carleson rectangles are still disjoint.

Let $f \in L^p(\Pi^+ \times \Pi^+, d\nu_\alpha(w))$. For each integer $k$, define

$$E_k := \{z \in \Pi^+ \times \Pi^+ : 2^k < \mathcal{M}_\alpha^d f(z) \leq 2^{k+1}\}.$$ 

Denote by $F_k$ the family of all rectangles $R \in \mathcal{D} \times \mathcal{D}$ such that

$$\frac{1}{\nu_\alpha(Q_R)} \int_{Q_R} |f(w)| d\nu_\alpha(w) > 2^k.$$ 

Then clearly, we have that

$$E_k \subseteq \bigcup_{R \in F_k \setminus F_{k+1}} Q_R. \quad (5.12)$$

Indeed, let $z \in E_k$ and suppose that there is no dyadic rectangle $R$ with $z \in Q_R$ such that

$$2^k < \frac{1}{\nu_\alpha(Q_R)} \int_{Q_R} |f(w)| d\nu_\alpha(w) \leq 2^{k+1}. \quad \text{Then for any } R \in \mathcal{D} \times \mathcal{D} \text{ such that } z \in Q_R, \text{ either}$$

$$\frac{1}{\nu_\alpha(Q_R)} \int_{Q_R} |f(w)| d\nu_\alpha(w) \leq 2^k$$

or

$$\frac{1}{\nu_\alpha(Q_R)} \int_{Q_R} |f(w)| d\nu_\alpha(w) > 2^{k+1}. \quad \text{Hence either } \mathcal{M}_\alpha^d f(z) \leq 2^k \text{ or } \mathcal{M}_\alpha^d f(z) > 2^{k+1}. \text{ This contradicts the fact that } z \in E_k.$$
It first follows that

\begin{align*}
L & := \int_{\Pi^+ \times \Pi^+} (\mathcal{M}_\alpha^q f(z))^q \, d\mu(z) \\
& = \sum_k \int_{E_k} (\mathcal{M}_\alpha^q f(z))^q \, d\mu(z) \\
& \leq 2^q \sum_k 2^{kq} \mu(E_k) \\
& \leq 2^q \sum_k \sum_{R \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}} 2^{kq} \mu(Q_R) \\
& \leq C 2^q \sum_k \sum_{R \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}} 2^{kq} (\mathcal{V}_\alpha(Q_R))^{q/p}.
\end{align*}

We next use the property (5.12) of rectangles in \( \mathcal{F}_k \) and the equivalence \( \mathcal{V}_\alpha(Q_R) \approx \mathcal{V}_\alpha(T_R) \) to obtain

\begin{align*}
L & \leq C 2^q \sum_k \sum_{R \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}} \left( \frac{1}{\mathcal{V}_\alpha(Q_R)} \int_{Q_R} \left| f(w) \right| d\mathcal{V}_\alpha(w) \right)^q (\mathcal{V}_\alpha(Q_R))^{q/p} \\
& \leq C 2^q \left( \sum_k \sum_{R \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}} \left( \frac{1}{\mathcal{V}_\alpha(Q_R)} \int_{Q_R} \left| f(w) \right| d\mathcal{V}_\alpha(w) \right)^p \mathcal{V}_\alpha(Q_R) \right)^{q/p} \\
& \approx C 2^q \left( \sum_k \sum_{R \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}} \left( \frac{1}{\mathcal{V}_\alpha(T_R)} \int_{Q_R} \left| f(w) \right| d\mathcal{V}_\alpha(w) \right)^p \mathcal{V}_\alpha(T_R) \right)^{q/p}.
\end{align*}

Recalling that \( \{T_R\}_{R \in \mathcal{D} \times \mathcal{D}} \) forms a tiling of \( \Pi^+ \times \Pi^+ \) and using (5.11), we finally obtain

\begin{align*}
L & := \int_{\Pi^+ \times \Pi^+} (\mathcal{M}_\alpha^q f(z))^q \, d\mu(z) \\
& \leq C 2^q \left( \sum_k \sum_{R \in \mathcal{F}_k \setminus \mathcal{F}_{k+1}} \int_{T_R} \left( \frac{1}{\mathcal{V}_\alpha(Q_R)} \int_{Q_R} \left| f(w) \right| d\mathcal{V}_\alpha(w) \right)^p d\mathcal{V}_\alpha(z) \right)^{q/p} \\
& \leq C 2^q \left( \sum_{R \in \mathcal{D} \times \mathcal{D}} \int_{T_R} (\mathcal{M}_\alpha^p f(z))^p d\mathcal{V}_\alpha(z) \right)^{q/p} \\
& = C 2^q \left( \int_{\Pi^+ \times \Pi^+} (\mathcal{M}_\alpha^p f(z))^p d\mathcal{V}_\alpha(z) \right)^{q/p} \\
& \leq C \left( \int_{\Pi^+ \times \Pi^+} \left| f(z) \right|^p d\mathcal{V}_\alpha(z) \right)^{q/p}.
\end{align*}

The proof of the proposition is complete.

Finally, we prove Theorem 5.2.

Proof of Theorem 5.2. That (a) \( \Rightarrow \) (b) follows from the observations made above and Proposition 5.3.
(b)⇒(a): Assume that there is a constant $C > 0$ such that for every $f \in A^p_{\vec{\alpha}}(\Pi^+ \times \Pi^+)$,

$$\int_{\Pi^+ \times \Pi^+} |f(z)|^q d\mu(z) \leq C \left( \int_{\Pi^+ \times \Pi^+} |f(z)|^p d\mathcal{V}_{\vec{\alpha}}(z) \right)^{\frac{q}{p}} \quad (5.13)$$

Let $Q_{I_1 \times I_2}$ be a fixed Carleson box, and let $w = (w_1, w_2)$, where $w_j$ is the centre of the Carleson square $Q_{I_j}$, $j = 1, 2$. We consider the function $f_w$ defined on $\Pi^+ \times \Pi^+$ by

$$f_w(z) := \left( \frac{(3m w_1)^{1+\frac{\alpha_1}{2}} (3m w_2)^{1+\frac{\alpha_2}{2}}}{(z_1 - \overline{w_1})^{2+\alpha_1} (z_2 - \overline{w_2})^{2+\alpha_2}} \right)^{\frac{2}{p}}, \quad z = (z_1, z_2) \in \Pi^+ \times \Pi^+. \quad \text{Then } f_w \text{ is uniformly in } A^p_{\vec{\alpha}}(\Pi^+ \times \Pi^+).$$

We observe for any $z_j \in Q_{I_j}$, $|z_j - \overline{w_j}| \approx 3m w_j$. Testing the inequality (5.13) with our choice $f_w$, we obtain

$$\frac{\mu(Q_{I_1 \times I_2})}{(3m w_1)^{(2+\alpha_1)} (3m w_2)^{(2+\alpha_2)}} \lesssim \int_{\Pi^+ \times \Pi^+} |f_w(z)|^q d\mu(z)$$

$$\leq C \left( \int_{\Pi^+ \times \Pi^+} |f_w(z)|^p d\mathcal{V}_{\vec{\alpha}}(z) \right)^{\frac{q}{p}} \leq C.$$

The proof is complete. \hfill \Box

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