On the fractional metric dimension of corona product graphs and lexicographic product graphs

Min Feng  Kaishun Wang

Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing, 100875, China

Abstract

A vertex $x$ in a graph $G$ resolves two vertices $u$, $v$ of $G$ if the distance between $u$ and $x$ is not equal to the distance between $v$ and $x$. A function $g$ from the vertex set of $G$ to $[0, 1]$ is a resolving function of $G$ if $g(R_G\{u, v\}) \geq 1$ for any two distinct vertices $u$ and $v$, where $R_G\{u, v\}$ is the set of vertices resolving $u$ and $v$. The real number $\sum_{v \in V(G)} g(v)$ is the weight of $g$. The minimum weight of all resolving functions for $G$ is called the fractional metric dimension of $G$, denoted by $\text{dim}_f(G)$. In this paper we reduce the problem of computing the fractional metric dimension of corona product graphs and lexicographic product graphs, to the problem of computing some parameters of the factor graphs.

Key words: fractional metric dimension; corona product; lexicographic product.

2010 MSC: 05C12.

1 Introduction

All graphs considered in this paper are finite, simple and undirected graph. Let $G$ be a graph. We often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For any two vertices $u$ and $v$ of $G$, denote by $d_G(u, v)$ the distance between $u$ and $v$ in $G$, and write $R_G\{u, v\} = \{w \mid w \in V(G), d_G(u, w) \neq d_G(v, w)\}$. A subset $W$ of $V(G)$ is called a resolving set of $G$ if $W \cap R_G\{u, v\} \neq \emptyset$ for any two distinct vertices $u$ and $v$. The metric dimension of $G$ is the minimum cardinality of all resolving sets of $G$. Metric dimension was first defined by Harary and Melter [9], and independently by Slater [14]. This parameter arises in various applications (see [3, 4] for more information).

The problem of finding the metric dimension of a graph was formulated as an integer programming problem by Chartrand et al. [5], and independently by Currie and Oellermann [6]. In graph theory, fractionalization of integer-valued graph theoretic concepts is an interesting area of research (see [13]). Currie and Oellermann [6] and Fehr et al. [7] defined fractional metric dimension as the optimal solution of the linear relaxation of the integer programming problem. Arumugam and Mathew

*Corresponding author. E-mail address: wangks@bnu.edu.cn
Let $g$ be a function assigning each vertex $u$ of a graph $G$ a real number $g(u) \in [0, 1]$. For $W \subseteq V(G)$, denote $g(W) = \sum_{v \in W} g(v)$. The weight of $g$ is defined by $|g| = g(V(G))$. We call $g$ a resolving function of $G$ if $g(R_G\{u, v\}) \geq 1$ for any two distinct vertices $u$ and $v$. The minimum weight of all resolving functions for $G$ is called the fractional metric dimension of $G$, denoted by $\dim_f(G)$.

Let $G$ and $H$ be two graphs. The corona product $G \circ H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $|V(H)|$ copies of $H$ and joining by an edge each vertex from the $i$th-copy of $H$ with the $i$th-vertex of $G$. The lexicographic product $G[H]$ is the graph with the vertex set $V(G) \times V(H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$, and the edge set $\{(u_1, v_1), (u_2, v_2) \mid d_G(u_1, u_2) = 1, \text{ or } u_1 = u_2 \text{ and } d_H(v_1, v_2) = 1\}$. In the rest of this paper, we always assume that $G$ and $H$ denote graphs with at least two vertices.

Yero et al. [15], and Jannesari and Omoomi [11] investigated the metric dimension of product graphs mentioned above. In this paper, we study the fractional metric dimension of these two product graphs. In Section 2, we introduce a new parameter $l_f(H)$ of a graph $H$ and calculate it when $H$ is a vertex-transitive graph. In Section 3, we discover the relationship between $l_f(H)$ and the fractional metric dimension of the corona product of two graphs $G$ and $H$. In Section 4, we express the fractional metric dimension of the lexicographic product graph in terms of some parameters of the factor graphs.

2 Locating function

Let $H$ be a graph. Assume that $N_H(v)$ is the set of all neighbors of the vertex $v$ in $H$. For $v_1, v_2 \in V(H)$, write

$$S_H\{v_1, v_2\} = \{v_1, v_2\} \cup (N_H(v_1) \triangle N_H(v_2)),$$

where the symbol $\triangle$ is the set symmetric difference operation.

A real value function $g : V(H) \rightarrow [0, 1]$ is called a locating function of $H$ if $g(S_H\{v_1, v_2\}) \geq 1$ for any two distinct vertices $v_1$ and $v_2$. Denote by $l_f(H)$ the minimum weight of all locating functions of $H$. Since $S_H\{v_1, v_2\} \subseteq R_H\{v_1, v_2\}$, we have $\dim_f(H) \leq l_f(H)$. If the diameter of $H$ is at most two, then $\dim_f(H) = l_f(H)$.

For a regular graph $H$, denote by $k(H)$ the degree of $H$. Let $\lambda(H)$ (resp. $\mu(H)$) denote the maximum number of common neighbors of any two distinct adjacent (resp. nonadjacent) vertices. For convenience, assume that $\mu(K_n) = 0$ and $\lambda(K_n) = -1$, where $K_n$ is the complete graph of order $n$ and $\overline{K_n}$ is the null graph of order $n$.

**Proposition 2.1** Let $H$ be a regular graph. If $H$ is not a complete graph, then $|V(H)| \geq 2k(H) - \min\{\lambda(H), \mu(H) - 2\}$.

**Proof.** If each connected component of $H$ is a complete graph, the desired result is directed. Suppose there exists a connected component $H_1$ of $H$ with diameter at least two. By computing the minimum size of $N_H(v_1) \cup N_H(v_2) \cup \{v_1, v_2\}$ for any two distinct vertices $v_1$ and $v_2$ of $H_1$, we obtain the desired inequality. $\square$
A graph is vertex-transitive if its full automorphism group acts transitively on the vertex set.

**Theorem 2.2** For a vertex-transitive graph $H$, we have

$$l_f(H) = \frac{|V(H)|}{2k(H) - \max\{2\lambda(H), 2\mu(H) - 2\}}.$$ 

**Proof.** Since $\lambda(H) \leq \mu(H)$, the desired result follows. \hfill $\square$

### 3 Corona product

In this section we express the fractional metric dimension of the corona product of two graphs in terms of some parameters of the factor graphs.

Recall that the corona product $G \circ H$ of graphs $G$ and $H$ has the vertex set $V(G) \cup (V(G) \times V(H))$, two vertices $x$ and $y$ are adjacent if and only if $x$ and $y$ are adjacent vertices of $G$, or $x \in V(G)$ and $y = (x, v)$, or $x = (u, v_1)$ and $y = (u, v_2)$ for two adjacent vertices $v_1, v_2$ of $H$. For $u_1, u_2 \in V(G)$ and $v_1, v_2 \in V(H)$, we have

1. $d_{G \circ H}(u_1, u_2) = d_G(u_1, u_2)$,
2. $d_{G \circ H}(u_1, (u_2, v_2)) = d_G(u_1, u_2) + 1$,
3. $d_{G \circ H}((u_1, v_1), (u_2, v_2)) = \begin{cases} d_H(v_1, v_2), & \text{if } u_1 = u_2 \text{ and } d_H(v_1, v_2) \leq 1, \\ d_G(u_1, u_2) + 2, & \text{otherwise}. \end{cases}$

**Lemma 3.1** Let $G$ be a connected graph and $H$ be a graph. Let $x$ and $y$ be two distinct vertices of the corona product graph $G \circ H$. Write $uH = \{(u, v) \mid v \in V(H)\}$ for $u \in V(G)$.

(i) If $\{x, y\} \subseteq uH$ for some $u \in V(G)$, write $x = (u, v_1)$ and $y = (u, v_2)$, then

$$R_{G \circ H}\{x, y\} = \bigcup_{v \in \mathcal{S}_H\{v_1, v_2\}} \{(u, v)\}.$$ 

(ii) If $\{x, y\} \not\subseteq uH$ for any $u \in V(G)$, then there exists a vertex $u_0$ of $G$ such that $u_0 H \subseteq R_{G \circ H}\{x, y\}$.

**Proof.** (i) Since $d_{G \circ H}(x, z) = d_{G \circ H}(y, z)$ for any $z \in V(G \circ H) \setminus uH$, we have $R_{G \circ H}\{x, y\} \subseteq uH$. Note that $(u, v) \in R_{G \circ H}\{x, y\}$ is equivalent to $v \in S_H\{v_1, v_2\}$. Hence, the desired result follows.

(ii) We divide our proof into four cases:

1. **Case 1.** $x, y \in V(G)$. Since $d_{G \circ H}(x, (x, v)) = 1 < d_{G \circ H}(y, (x, v))$ for any $v \in V(H)$, we have $xH \subseteq R_{G \circ H}\{x, y\}$.

2. **Case 2.** $x \in V(G)$ and $y \in uH$ for some $u_1 \in V(G)$. If $u_1 \neq x$, then $xH \subseteq R_{G \circ H}\{x, y\}$. If $u_1 = x$, choose $u_2 \in V(G) \setminus \{x\}$, then $d_{G \circ H}(y, (u_2, v)) = d_{G \circ H}(x, (u_2, v)) + 1$ for any $v \in V(H)$, which implies that $u_2 H \subseteq R_{G \circ H}\{x, y\}$.

3. **Case 3.** $y \in V(G)$ and $x \in uH$ for some $u_1 \in V(G)$. Similar to Case 2, the desired result follows.
Let $G$ be a connected graph and $H$ be a graph. Then $\dim_f(G \odot H) = |V(G)|l_f(H)$.

Proof. First, we prove that
$$\dim_f(G \odot H) \geq |V(G)|l_f(H). \tag{1}$$

Let $\mathcal{F}$ be a resolving function of $G \odot H$ with $|\mathcal{F}| = \dim_f(G \odot H)$. For each $u \in V(G)$, define
$$\mathcal{F}_u : V(H) \rightarrow [0, 1], \quad v \mapsto \mathcal{F}(u, v).$$

For any two distinct vertices $v_1$ and $v_2$ of $H$, by Lemma 3.1,
$$\mathcal{F}_u(S_H(v_1, v_2)) = \sum_{v \in S_H(v_1, v_2)} \mathcal{F}((u, v)) = \mathcal{F}(R_{G \odot H}\{(u, v_1), (u, v_2)\}) \geq 1,$$

which implies that $|\mathcal{F}_u| \geq l_f(H)$. Since $V(G) \subseteq V(G \odot H)$, we have $|\mathcal{F}| \geq \sum_{u \in V(G)} |\mathcal{F}_u|$. Hence, (1) holds.

Second, we prove that
$$\dim_f(G \odot H) \leq |V(G)|l_f(H).$$

Let $g$ be a locating function of $H$ with $|g| = l_f(H)$. Define a function
$$\overline{g} : V(G \odot H) \rightarrow [0, 1], \quad u \mapsto 0, \quad (u, v) \mapsto g(v),$$

where $u \in V(G), v \in V(H)$. Since $|\overline{g}| = |V(G)|l_f(H)$, it suffices to show that $\overline{g}$ is a resolving function of $G \odot H$. Pick any two distinct vertices $x$ and $y$ of $G \odot H$. If $x = (u, v_1)$ and $y = (u, v_2)$, by Lemma 3.1 (i) we have $\overline{g}(R_{G \odot H}\{x, y\}) = g(S_H(v_1, v_2)) \geq 1$. If $\{x, y\} \not\subseteq ^uH$ for any $u \in V(G)$, by Lemma 3.1 (ii) we have $\overline{g}(R_{G \odot H}\{x, y\}) \geq |g| \geq 1$. Hence, $\overline{g}$ is a resolving function of $G \odot H$, as desired. □

Combining Theorem 3.2 and Theorem 3.2, the following result is directed.

Corollary 3.3 Let $G$ be a connected graph. If $H$ is a vertex-transitive graph, then
$$\dim_f(G \odot H) = \frac{|V(G)||V(H)|}{2k(H) - \max\{2\lambda(H), 2\mu(H) - 2\}}.$$

Next, we consider graphs $K_1 \odot H$ and $G \odot K_1$.

Theorem 3.4 Let $G$ be a connected graph and $H$ be a graph. Then

$$l_f(H) \leq \dim_f(K_1 \odot H) \leq l_f(H) + 1, \quad \text{(2)}$$

$$\dim_f(G) \leq \dim_f(G \odot K_1) \leq \frac{|V(G)|}{2}. \quad \text{(3)}$$

Case 4. $x \in ^u_1H$ and $y \in ^u_2H$ for two distinct vertices $u_1, u_2 \in V(G)$. For any $v \in V(H)$, we have $d_{G \odot H}(x, (u_1, v)) \leq 2 < d_G(u_1, u_2) + 2 = d_{G \odot H}(y, (u_1, v))$. Hence $^u_1H \subseteq R_{G \odot H}\{x, y\}$.

We accomplish our proof. □
Proof. Since the inequality (1) holds for \( G = K_1 \), one has \( l_f(H) \leq \dim_f(K_1 \circ H) \). For any locating function \( g \) of \( H \), define a function

\[
\overline{g} : V(K_1 \circ H) \rightarrow [0,1], \quad u \mapsto 1, \quad (u,v) \mapsto g(v),
\]

where \( u \in V(K_1), \ v \in V(H) \). Then \( \overline{g} \) is a resolving function of \( K_1 \circ H \), which implies that \( \dim_f(K_1 \circ H) \leq l_f(H) + 1 \). Hence (2) holds.

For any two vertices \( u_1 \) and \( u_2 \) of \( G \), we have

\[
d_G(u_1, u_2) = d_{G \circ K_1}(u_1, u_2),
\]

which implies that \( \dim_f(G) \leq \dim_f(G \circ K_1) \). Note that

\[
\overline{T} : V(G \circ K_1) \rightarrow [0,1], \quad u \mapsto 0, \quad (u,v) \mapsto \frac{1}{2}
\]

is a resolving function of \( G \circ K_1 \), where \( u \in V(G), \ v \in V(K_1) \). Then \( \dim_f(G \circ K_1) \leq \frac{|V(G)|}{2} \). Hence (3) holds.

By [2, Theorem 2.2] the inequalities in (3) are tight. Observe that \( \dim_f(K_1 \circ K_{1,n}) = l_f(K_{1,n}) + 1 \) for \( n \geq 2 \). Next we show that the lower bound for \( \dim_f(K_1 \circ H) \) in (2) is tight.

**Proposition 3.5** If \( H \) is a disconnected graph without isolated vertices or a connected graph with diameter at least six, then \( \dim_f(K_1 \circ H) = l_f(H) \).

Proof. Let \( f \) be a locating function of \( H \) with \( |f| = l_f(H) \). Define

\[
\overline{f} : V(K_1 \circ H) \rightarrow [0,1], \quad u \mapsto 0, \quad (u,v) \mapsto f(v),
\]

where \( u \in V(K_1), \ v \in V(H) \). Since \( |\overline{f}| = l_f(H) \), by Theorem 3.4 it suffices to show that \( \overline{f} \) is a resolving function of \( K_1 \circ H \). Note that \( \overline{f}(R_{K_1 \circ H}\{u, v_1\}) = f(S_H\{v_1, v_2\}) \geq 1 \) for any two distinct vertices \( v_1, v_2 \in V(H) \). We only need to prove that, for any vertex \( v \in V(H) \),

\[
\overline{f}(R_{K_1 \circ H}\{u, (u,v)\}) \geq 1. \quad (4)
\]

Suppose \( H \) is a disconnected graph without isolated vertices. Denote by \( H_1 \) the connected component containing \( v \). Choose two distinct vertices \( v_1, v_2 \in V(H) \setminus V(H_1) \). Since \( S_H\{v_1, v_2\} \subseteq V(H) \setminus V(H_1) \) and \( uH_1 \subseteq R_{K_1 \circ H}\{u, (u,v)\} \), we obtain (4).

Suppose \( H \) is a connected graph with diameter at least six. We may pick two distinct vertices \( v_1 \) and \( v_2 \) with distance at least three from \( v \) in \( H \). Then \( S_H\{v_1, v_2\} \subseteq \{w \mid w \in V(H), d_H(v, w) \geq 2\} \). Since \( \{(u, w) \mid w \in V(H), d_H(v, w) \geq 2\} \subseteq R_{K_1 \circ H}\{u, (u,v)\} \), we obtain (4).

\[
\Box
\]

### 4 Lexicographic product

In this section we shall reduce the problem of computing the fractional metric dimension of the lexicographic product graph \( G[H] \) to the problem of computing the fractional metric dimension of the graph \( K_2[H] \).
Let $G$ be a graph. For $u \in V(G)$, write $N_G[u] = N_G(u) \cup \{u\}$. Two distinct vertices $u_1$ and $u_2$ of $G$ are called twins if $N_G[u_1] = N_G[u_2]$ or $N_G(u_1) = N_G(u_2)$. Define $u_1 \equiv u_2$ if $u_1$ and $u_2$ are twins or $u_1 = u_2$. Hernando et al. [10] proved that “≡” is an equivalent relation and the equivalence class of a vertex is of three types: a class with one vertex (type 1), a clique with at least two vertices (type 2), an independent set with at least two vertices (type 3). For $i = 1, 2, 3$, denote by $O_i$ the set of equivalence classes of type $i$; and write $m_i(G) = \sum_{O \in O_i} |O|$. Clearly, $|V(G)| = m_1(G) + m_2(G) + m_3(G)$.

For any two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ of $G[H]$, we observe that

$$d_{G[H]}((u_1, v_1), (u_2, v_2)) = \begin{cases} 1, & \text{if } u_1 = u_2, v_2 \in N_H(v_1), \\ 2, & \text{if } u_1 = u_2, v_2 \not\in N_H(v_1), \\ d_G(u_1, u_2), & \text{if } u_1 \not= u_2. \end{cases}$$

The following result is directed from the above observation.

**Lemma 4.1** Let $G$ be a connected graph and $H$ be a graph. Let $(u_1, v_1)$ and $(u_2, v_2)$ be two distinct vertices of the lexicographic graph $G[H]$.

(i) If $u_1 \not= u_2$, then there exists $u \in V(G)$ such that $uH \subseteq R_{G[H]}((u_1, v_1), (u_2, v_2))$.

(ii) If $u_1 \equiv u_2$, then

$$R_{G[H]}((u_1, v_1), (u_2, v_2)) = \begin{cases} \bigcup_{v \in S_H\{v_1, v_2\}} \{(u_1, v), (u_2, v)\}, & \text{if } u_1 = u_2, \\ \{(u_1, v)\} \cup \{(u_2, v)\}, & \text{if } N_G[u_1] = N_G[u_2], \\ \{(u_1, v)\} \cup \{(u_2, v)\}, & \text{if } N_G(u_1) = N_G(u_2). \end{cases}$$

where $\overline{H}$ is the complement graph of $H$.

For a function $\overline{f} : V(G[H]) \rightarrow [0, 1]$, let

$$\overline{f}_u : V(H) \rightarrow [0, 1], \quad v \mapsto \overline{f}(u, v).$$

**Lemma 4.2** Let $G$ be a connected graph and $H$ be a graph. If $\overline{f}$ is a resolving function of $G[H]$, then $\overline{f}_u$ is a locating function of $H$ for any $u \in V(G)$. In particular, we have $\dim_f(G[H]) \geq |V(G)|l_f(H)$.

Proof. For any two distinct vertices $v_1, v_2 \in V(H)$, by Lemma 4.1 we have

$$\overline{f}_u(S_H\{v_1, v_2\}) = \sum_{v \in S_H\{v_1, v_2\}} \overline{f}(u, v) = \overline{f}(R_{G[H]}((u, v_1), (u, v_2))) \geq 1,$$

so $\overline{f}_u$ is a locating function of $H$. \hfill \Box

In the remaining of this section, we shall calculate $\dim_f(G[H])$ in terms of some parameters of $G$, $H$ and $K_2[H]$.

**Lemma 4.3** Let $G$ be a connected graph and $H$ be a graph. Then

$$\dim_f(G[H]) \geq m_1(G)l_f(H) + \frac{m_2(G)}{2} \dim_f(K_2[H]) + \frac{m_3(G)}{2} \dim_f(K_2[\overline{H}]).$$
Proof. Let $\overline{f}$ be a resolving function of $G[H]$ with $|\overline{f}| = \dim_f(G[H])$. Note that

$$|\overline{f}| = \sum_{O \in \mathcal{O}_1} \sum_{u \in O} |\overline{f}_u| + \sum_{O \in \mathcal{O}_2} \sum_{u \in O} |\overline{f}_u| + \sum_{O \in \mathcal{O}_3} \sum_{u \in O} |\overline{f}_u|.$$  

By Lemma 4.2 we have $\sum_{O \in \mathcal{O}_1} \sum_{u \in O} |\overline{f}_u| \geq m_1(G)l_f(H)$, so it suffices to show that

$$\sum_{O \in \mathcal{O}_1} \sum_{u \in O} |\overline{f}_u| \geq \frac{m_2(G)}{2} \dim_f(K_2[H_1])$$

(5)

holds for $i \in \{2, 3\}$, where $H_2 = H$ and $H_3 = \overline{H}$.

Let $O \in \mathcal{O}_i$. Pick any two distinct vertices $u_1$ and $u_2$ in $O$. Write $V(K_2) = \{w_1, w_2\}$. Define

$$g_i : V(K_2[H_i]) \rightarrow [0, 1], \quad (w_j, v) \mapsto \overline{f}_{u_j}(v).$$

Next, we shall prove that $g$ is a resolving function of $K_2[H_i]$. Pick any two distinct vertices $(w_j, v_1)$ and $(w_k, v_2)$ of $K_2[H_i]$.

Case 1. $w_j = w_k$. Since $S_H\{v_1, v_2\} = S_H\{v_1, v_2\}$, by Lemmas 4.1 and 4.2 we have $g_i(R_{K_2[H_i]}((w_j, v_1), (w_j, v_2))) = \overline{f}_{u_j}(S_H\{v_1, v_2\}) \geq 1$.

Case 2. $w_j \neq w_k$. By Lemma 4.1 we get

$g_i(R_{K_2[H_i]}((w_1, v_1), (w_2, v_2))) = \overline{f}(R_G[H]\{u_1, v_1\}, (u_2, v_2)) \geq 1$.

By the above discussion, each $g_i$ is a resolving function of $K_2[H_i]$, which implies that $|\overline{f}_{u_1}| + |\overline{f}_{u_2}| \geq \dim_f(K_2[H_i])$. Note that

$$\sum_{u_1, u_2 \in O, u_1 \neq u_2} (|\overline{f}_{u_1}| + |\overline{f}_{u_2}|) = (|O| - 1) \sum_{u \in O} |\overline{f}_u|.$$  

Then $\sum_{u \in O} |\overline{f}_u| \geq \frac{|O|}{2} \dim_f(K_2[H_i])$ and (5) holds.

$\square$

**Theorem 4.4** Let $G$ be a connected graph and $H$ be a graph. Then

$$\dim_f(G[H]) = m_1(G)l_f(H) + \frac{m_2(G)}{2} \dim_f(K_2[H]) + \frac{m_3(G)}{2} \dim_f(K_2[H]).$$

In particular $\dim_f(G[H]) = |V(G)|l_f(H)$ when $G$ has no twins.

Proof. Write $H_2 = H, H_3 = \overline{H}$ and $V(K_2) = \{w_1, w_2\}$. For each $i = 2, 3$, assume that $\overline{f}_i$ is a resolving function of $K_2[H_i]$ with $|\overline{f}_i| = \dim_f(K_2[H_i])$. Define

$$f_i : V(H) \rightarrow [0, 1], \quad v \mapsto \frac{\overline{f}_i((w_1, v)) + \overline{f}_i((w_2, v))}{2}.$$  

Then $f_i$ is a locating function of $H$ with $|f_i| = \frac{1}{2} \dim_f(K_2[H_i])$. Let $f_1$ be a locating function of $H$ with $|f_1| = l_f(H)$. Define a function $\overline{f} : V(G[H]) \rightarrow [0, 1]$ by $\overline{f}((u, v)) = f_1(v)$ whenever $u$ belongs to the set $\cup_{O \in \mathcal{O}_1} O$, where $i = 1, 2, 3$. Note
that $\mathcal{T}_u$ is a resolving function of $H$ for any $u \in V(G)$. We shall prove that $\mathcal{T}$ is a resolving function of $G[H]$. Pick two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ of $G[H]$.

**Case 1.** $u_1 \neq u_2$. By Lemma 4.1 we get $\mathcal{T}(R_{G[H]}((u_1, v_1), (u_2, v_2))) \geq l_f(H) \geq 1$.

**Case 2.** $u_1 = u_2$. By Lemma 4.1 we have

$$\mathcal{T}(R_{G[H]}((u_1, v_1), (u_1, v_2))) = \mathcal{T}(S_H(v_1, v_2)) \geq 1.$$

**Case 3.** $u_1$ and $u_2$ are twins. By Lemma 4.1 we have

$$\mathcal{T}(R_{G[H]}((u_1, v_1), (u_2, v_2))) = \frac{1}{2} \left[ \mathcal{T}(R_{K_2[H]}((w_1, v_1), (w_2, v_2))) + \mathcal{T}(R_{K_2[H]}((w_1, v_2), (w_2, v_1))) \right] \geq 1.$$

Hence $\mathcal{T}$ is a resolving function of $G[H]$ such that $|\mathcal{T}|$ meets the bound in Lemma 4.3 and so the desired result follows.

**Theorem 4.5** Let $G$ be a connected graph. If $H$ is a vertex-transitive graph, then

$$\dim_f(G[H]) = |V(G)|l_f(H) = \frac{|V(G)||V(H)|}{2k(H) - \max\{2\lambda(H), 2\mu(H) - 2\}}.$$

**Proof.** Write $s = 2k(H) - \max\{2\lambda(H), 2\mu(H) - 2\}$. Combining Theorem 2.2 and Lemma 4.2 we only need to prove that $\dim_f(G[H]) \leq \frac{|V(G)||V(H)|}{s}$. Define

$$\mathcal{T} : V(G[H]) \rightarrow [0, 1], \quad (u, v) \mapsto \frac{1}{s}.$$

Since $|\mathcal{T}| = \frac{|V(G)||V(H)|}{s}$, it suffices to show that $\mathcal{T}$ is a resolving function of $G[H]$. Pick two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ of $G[H]$.

**Case 1.** $u_1 = u_2$. Note that $|S_H(v_1, v_2)| \geq s$. By Lemma 4.1 we get

$$\mathcal{T}(R_{G[H]}((u_1, v_1), (u_2, v_2))) = \frac{|S_H(v_1, v_2)|}{s} \geq 1.$$

**Case 2.** $d_G(u_1, u_2) = 1$. Then

$$R_{G[H]}((u_1, v_1), (u_2, v_2)) \supseteq \left( \bigcup_{v \in N_H[u_1]} \{(u_1, v)\} \right) \cup \left( \bigcup_{v \in N_H[u_2]} \{(u_2, v)\} \right). \quad (6)$$

Proposition 2.1 implies that $2(|V(H)| - k(H)) \geq s$. By (6), we have

$$\mathcal{T}(R_{G[H]}((u_1, v_1), (u_2, v_2))) \geq \frac{2(|V(H)| - k(H))}{s} \geq 1.$$

**Case 3.** $d_G(u_1, u_2) \geq 2$. Then

$$\mathcal{T}(R_{G[H]}((u_1, v_1), (u_2, v_2))) \geq \sum_{v \in N_H[u_1]} \mathcal{T}((u_1, v)) + \sum_{v \in N_H[u_2]} \mathcal{T}((u_2, v)) \geq 1.$$

Hence $\mathcal{T}$ is a resolving function of $G[H]$, as desired. \qed
Acknowledgement

This research was supported by NSF of China and the Fundamental Research Funds for the Central Universities of China.

References

[1] S. Arumugam and V. Mathew, The fractional metric dimension of graphs, Discrete Math. 312 (2012) 1584-1590.

[2] S. Arumugam, V. Mathew and J. Shen, On fractional metric dimension of graphs, preprint.

[3] R.F. Bailey and P.J. Cameron, Base size, metric dimension and other invariants of groups and graphs, Bull. London Math. Soc. 43 (2011) 209-242.

[4] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara and D.R. Wood, On the metric dimension of Cartesian products of graphs, SIAM J. Discrete Math. 21 (2007) 423-441.

[5] G. Chartrand, L. Eroh, M. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000) 99-113.

[6] J. Currie and O. R. Oellermann, The metric dimension and metric independence of a graph, J. Combin. Math. Combin. Comput. 39 (2001) 157-167.

[7] M. Fehr, S. Gosselin and O. R. Oellermann, The metric dimension of Cayley digraphs, Discrete Math. 306 (2006) 31-41.

[8] M. Feng, B. Lv and K. Wang, On the fractional metric dimension of graphs, arXiv:1112.2106v2.

[9] F. Harary and R.A. Melter, On the metric dimension of a graph, Ars Combin. 2 (1976) 191-195; 4 (1977) 318.

[10] C. Hernando, M. Mora, I.M. Pelayo, C. Seara and D. R. Wood, Extremal graph theory for metric dimension and diameter, Electron. Notes in Discrete Math. 29 (2007) 339-343.

[11] M. Jannesari and B. Omoomi, The metric dimension of lexicographic product of graphs, arXiv:1103.3336.

[12] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, Discrete Appl. Math. 70 (1996) 217-229.

[13] E.R. Scheinerman, D.H. Ullman, Fractional Graph Theory, in: Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1997.

[14] P.J. Slater, Leaves of trees, Congr. Numer. 14 (1975) 549-559.
[15] I.G. Yero, D. Kuziak, and J. A. Rodríguez-Velázquez, On the metric dimension of corona product graphs, Comput. Math. Appl. 61 (2011) 2793-2798