INTERPOLATING, MODULI-INTERPOLATING AND ULTRA-INTERPOLATING CURVES OF ANY GENUS ON FANO HYPERSURFACES, AND POSITIVITY OF SOME KONTSEVICH INTERSECTION NUMBERS

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ABSTRACT. On a general hypersurface of degree $d \leq n$ in $\mathbb{P}^n$ or $\mathbb{P}^n$ itself, we prove the existence of curves of any genus and high enough degree depending on the genus passing through the expected number $t$ of general points or meeting some general collection of linear subspaces; in some cases we also show that the family of curves through $t$ fixed points has general moduli as family of $t$-pointed curves. These results imply positivity of certain intersection numbers on Kontsevich spaces of stable maps.

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A curve $C$ on a variety $X$ is said to be interpolating or to have the interpolation property if $C$ can be deformed so as to go through the expected number of general points on $X$. Here ‘expected number’ means, in terms of the normal bundle $N_{C/X}$, the largest integer $t$ such that $(n-1)t \leq \chi(N_{C/X}), n = \dim(X)$ or explicitly, where $g$ denotes the genus of $C$,

$$t = \left\lceil s(N_{C/X}) \right\rceil + 1 - g = \left\lceil \frac{C(-K_X) + 2g - 2}{n-1} \right\rceil + 1 - g.$$

This makes most sense if $H^1(N_C) = 0$, so that $C$ moves in an unobstructed family of the expected dimension, i.e. $h^0(N_{C/X})$. The adjective ‘separable’ may be added if the appropriate correspondence is separable over the symmetric product $X^{(t)}$. A stronger property than interpolation, though equivalent in genus 0, is that of ultra-interpolation, where passage through points is generalized to incidence to a collection of generally positioned subvarieties of given codimensions. The existence of an interpolating or ultra-interpolating curve implies positivity of certain intersection numbers on Kontsevich spaces of stable maps, which measure the ‘virtual’ number of such curves.

A property related to interpolation is that of modular interpolation. Given $m$ fixed general points on $X$, the family of deformations of $C$ going through them yields a family of $m$-pointed curves of genus $g$ and one may inquire whether a general member has general moduli as such. When this holds for all $m$ up to the expected number, namely

$$t = \left\lceil \chi(T_X|_C)/n \right\rceil = \left\lceil (-C.K_X)/n \right\rceil + 1 - g,$$

we will say that $C$ is moduli-interpolating. Again the adjective ‘separable’ may be added if the appropriate map to the moduli of $t$-pointed curves is separable. Again there is an ultra version.

The various interpolation properties of a curve $C$ are implied by, and in char. 0 equivalent to, certain properties called balancedness or ultra-balancedness of either the normal bundle $N_{C/X}$ or the restricted ambient tangent bundle $T_X|_C$. When these bundle properties hold the curve is said to be (ultra) balanced or (ultra) ambient-balanced.

There is a fair amount of work on curve interpolation in the case where $C$ is rational and $X$ is a Fano manifold, e.g. $\mathbb{P}^n$, a Fano hypersurface in $\mathbb{P}^n$ or a Grassmannian, starting with the case of rational curves in $\mathbb{P}^n$, due to Sacchiero [11]; see [?], [2], [8] [10] [7] [9]. For curves of higher genus and $X = \mathbb{P}^n$, there are older results for elliptic curves due to Ellingsrud and Laksov [4], Hulek [5] and Ein and Lazarsfeld [3], and for $n = 3$ due to
Perrin [6]. More recently, comprehensive results for \( X = \mathbb{P}^n \), any \( n \), and C nonspecial of any genus were obtained by A. Atanasov, E. Larson and D. Yang [1]. To my knowledge there are no results in the literature on interpolation, much less ultra-interpolation, for higher-genus curves and ambient spaces other than \( \mathbb{P}^n \).

As for modular interpolation, in case \( X = \mathbb{P}^n \), \( g = 0 \) and any \( e \geq n \), it is easy to see that any sufficiently general rational curve of degree \( e \) is ambient-balanced. But already for \( X \) a Grassmannian, \( g = 0 \) and ‘most’ degrees \( e \), there are no moduli-interpolating curves of degree \( e \) (see Example 21). Thus for ‘most’ varieties \( X \) one would expect some topological obstructions in terms of degree and genus in order for a curve to be ambient-balanced.

In this paper we consider (separable) interpolation and modular interpolation in arbitrary genus on \( \mathbb{P}^n \) and on general Fano hypersurfaces, i.e. hypersurfaces \( X \) of degree \( \leq n \) in \( \mathbb{P}^n \), \( n \geq 4 \). Notably, we will show:

- (See §3) In \( \mathbb{P}^n \), the general curve of genus \( g \) and degree \( e \geq n + g(n - 2) \) is balanced, hence (separably) interpolating (see Corollary 28). For \( e \geq 2(g + 1)n \), the curve is ultra-interpolating and ultra ambient-interpolating as well (see Corollary 34). These results refine and extend the results of [1] albeit for a smaller set of curve degrees.
- (See §4) On a general hypersurface of degree \( n \) in \( \mathbb{P}^n \), \( n \geq 4 \), there exist ultra-balanced, ultra ambient-balanced curves of any genus \( g \geq 1 \) and degree \( e \geq 4g(n - 1) \) and of genus 0 and any degree \( e \geq n - 1 \).
- (See §5) On a general hypersurface of degree \( d < n \) in \( \mathbb{P}^n \), there exist balanced (resp. ambient-balanced) curves of any genus \( g \geq 0 \) and degree \( e \) provided \( e \) satisfies a certain arithmetical condition; as shown in the Appendix by M. C. Chang, for given \( n, d, g \), the condition is satisfied for all \( e \) in at least one arithmetic progression with difference \( d(n - 2) \) (resp. for infinitely many \( e \)) (see Theorem 41 and the ensuing examples).

The method of proof builds on the one used before in [8] for rational curves, and is likewise based on fans and fang degenerations, degenerating the curve together with its ambient space, be it \( \mathbb{P}^n \) or a hypersurface (which in turn degenerates together with its own ambient \( \mathbb{P}^n \)) to a reducible pair. Along the way we introduce notions of balanced and ultra-balanced bundle for curves of any genus, generalizing the usual balancedness notion for (semi-positive) bundles on rational curves. Elementary properties of balanced and ultra-balanced bundles are developed in §1. In §2 we study a relative version of the tangent bundle for a family of varieties degenerating to normal-crossing double points. This is useful in studying moduli-interpolating families.

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1. Balanced bundles in any genus

We work over an algebraically closed field or arbitrary characteristic.

1.1. Basics. Let $E$ be a vector bundle of slope $s = s(E)$ on a curve $C$ of genus $g$. We set

$$t(E) = s + 1 - g = \frac{\chi(E)}{\text{rk}(E)}$$

and call it the Euler slope or e-slope of $E$. Also let

$$r(E) = \deg(E) \text{ mod } \text{rk}(E) = \chi(E) \text{ mod } \text{rk}(E)$$

where $\text{mod}$ denotes remainder; this is called the remainder of $E$.

For an effective divisor $D$ on $C$ we denote by $\rho_{D}$ the restriction map

$$\rho_{D} : H^{0}(E) \to H^{0}(E \otimes \mathcal{O}_{D}).$$

If $D$ is general of degree $t$ we will denote $\rho_{D}$ by $\rho_{t}$. Here ‘general’ means, in case $C$ is reducible, general in some component of $C^{(t)}$.

**Definition 1.** A bundle $E$ is said to be regular if $H^{1}(E) = 0$. $E$ is semi-balanced if

(i) $E$ is generically generated;

(ii) $E$ is regular;

(iii) the restriction map $\rho_{t}$ is surjective for all $t \leq t(E)$.

A semi-balanced bundle is balanced if $\rho_{t}$ is moreover injective for all $t \geq t(E)$.

A balanced bundle is perfectly balanced if in addition $s$ is an integer.

The notion of balanced bundle can be generalized as follows.

**Definition 2.** Let $E$ be a regular, generically generated bundle. Given a weight vector $\underline{u} = (u_{1}, ..., u_{t}), 0 \leq u_{i} \leq \text{rk}(E)$, $E$ is said to be $\underline{u}$-balanced if there exist points $x_{1}, ..., x_{t}$, each general in some component of $C$, and for each $i$, a general skyscraper quotient $U_{i}$ of $E|_{x_{i}}$ of dimension $u_{i}$, such that the restriction map

$$\rho_{\underline{u}} : H^{0}(E) \to H^{0}(\bigoplus U_{i})$$

has maximal rank. $E$ is perfectly $\underline{u}$-balanced if $\rho_{\underline{u}}$ is an isomorphism.

$E$ is said to be ultra-balanced if it is $\underline{u}$-balanced for every $\underline{u}$. □

Obviously $\rho_{t}$ is just $\rho_{\text{rk}(E), ..., \text{rk}(E)},$ so $E$ is balanced iff it is $\underline{u}$-balanced for all scalar weight-vectors of the form $(\text{rk}(E), ..., \text{rk}(E)) \in \mathbb{Z}^{t}, \forall t$. Note that for $E$ regular, $\rho_{t}$ can be surjective only for $t \leq t(E)$.
Remark 3. Regarding balancedness vs. (semi) stability. For a bundle of slope $s$ on a curve of genus $g$, balancedness excludes subbundles of degree $s + 1 - g$ or less while stability excludes subbundles of degree $s$ or less. Thus balancedness seems not implied by stability if $g > 1$ though we don’t have an explicit example of an unbalanced stable bundle. Conversely there exist direct sums of lines bundles that are ultra-balanced but not stable (see Lemma 2).

Lemma 4. Suppose $E$ is generically generated. Then the following are equivalent:

(i) $E$ is semi-balanced;

(ii) for general points $x_1, ..., x_t \in C$ and $\forall t \leq t(E)$, we have $H^1(E(-x_1 - ... - x_t)) = 0$ or equivalently

$$h^0(E(-x_1 - ... - x_t)) = \chi(E(-x_1 - ... - x_t));$$

(iii) $h^0(E) = \chi(E)$ and $h^0(E(-x_1 - ... - x_t)) = h^0(E - t.rk(E)), \forall t \leq t(E)$.

Moreover, if $E$ is semi-balanced, then $E$ is balanced iff $H^0(E(-x_1 - ... - x_t)) = 0, \forall t \geq t(E)$. In particular, the condition that $\rho_t$ be injective or surjective depends only on the linear equivalence class of $\sum x_i$ hence only on $t$ if $g = 0$.

The proof may be left to the reader.

Lemma 5. A balanced bundle $E$ is ultra-balanced provided $\rho_u$ is an isomorphism for all weight-vectors $u$ of weight $\sum u_i = \chi(E)$.

Lemma 6. A generically generated bundle $E$ is $u$-balanced iff, in the above notations, the modified bundle

$$E_u = \ker(E \to \bigoplus U_i)$$

has natural cohomology, i.e. $h^0(E_u)h^1(E_u) = 0$.

For rational curves, the above notion of balanced coincides with the usual:

Lemma 7. If $g = 0$, $E$ is balanced iff $E$ is ultra-balanced iff $E \simeq b_1O(a + 1) \oplus b_0O(a)$ for some $a \geq 0, b_0 > 0, b_1$.

Proof. If $E$ has the form $b_1O(a + 1) \oplus b_0O(a)$ then so does a general modification of $E$, so $E$ is ultra-balanced. Conversely assume $E$ is balanced and let $a$ be the smallest degree of a line bundle quotient (= summand) of $E$. By semi-balancedness clearly $[s(E)] = a \geq 0, [t(E)] = [a] + 1$. If $E$ has a line bundle summand of degree $\geq a + 2$ then $H^0(E(-x_1 - ... - x_{t+1})) \neq 0$, contradicting balancedness.

Note that for $g = 0$ the ‘test’ divisor $\sum x_i$ may actually be an arbitrary effective divisor of degree $t$. For general $g$ the injectivity or surjectivity conditions for balancedness depend only on the linear equivalence class of $\sum x_i$. Also for general $g$, half the above characterization still holds:
Lemma 8. Suppose $E$ admits a filtration whose quotients $L_1, \ldots, L_r$ are line bundles such that $\deg(L_1), \ldots, \deg(L_r) \in [a, a+1]$ for some $a \geq 2g - 1$. Then $E$ is balanced.

Proof. If $D_t$ denotes a general effective divisor of degree $t$ then it is easy to check that
\[
H^1(E(-D_t)) = 0, \ t \leq g,
\]
\[
H^0(E(-D_t)) = 0, \ t \geq g + 1.
\]
\[\square\]

There is a version of this for ultra-balanced:

Lemma 9. Let $E$ be a direct sum of line bundles with degrees in $[a, a+1], a \geq 2g - 1$. Then $E$ is ultra-balanced.

Proof. As has been noted, if $L$ is a line bundle of degree $a \geq 2g - 1$ then
\[
H^1(L(-D_t)) = 0, \ t \leq a + 1 - g,
\]
\[
H^0(L(-D_t)) = 0, \ t \geq a + 1 - g.
\]
We can write
\[
E = L_1 \oplus \ldots \oplus L_s \oplus L_{s+1} \oplus \ldots \oplus L_r
\]
where
\[
\deg(L_i) = \begin{cases} a+1, & i \leq s; \\ a, & i > s \end{cases}
\]
and the subbundle $L_1 \oplus \ldots \oplus L_s \subset E$ is uniquely determined. Then we have $\chi(E) = ra + s$. If $u = (u_1, \ldots, u_t)$ is a weight vector, we have, by generality of the quotient involved,
\[
E^{u_1} = L_1(-p) \oplus \ldots \oplus L_{u_1}(-p) \oplus L_{u_1+1} \oplus \ldots \oplus L_r,
\]
where $p \in \mathbb{C}$ is a general point, and this is a direct sum of line bundles of degrees in $[a,a+1]$ if $u_1 \leq s$ or $[a-1,a]$ if $u_1 \geq s$. Then it is easy to check, e.g. by induction of the length of the weight-vector $u$, that
\[
H^1(E^u) = 0, \ |u| \leq \chi(E),
\]
\[
H^0(E^u) = 0, \ |u| \geq \chi(E).
\]
\[\square\]

We can similarly characterize semi-balanced bundles on $\mathbb{P}^1$:

Lemma 10. A globally generated bundle of slope $s$ on $\mathbb{P}^1$ is semi-balanced iff the smallest degree of its line bundle summands is $[s]$.

Example 11. The bundle $\mathcal{O}(2) \oplus 2\mathcal{O}$ on $\mathbb{P}^1$ is semi-balanced but not balanced.
There is a partial extension for elliptic curves:

**Lemma 12.** Assume \( g = 1 \), \( E \) is generically generated and regular, and and that \( E \) is either (1) poly-stable or (2) semi-stable of non-integer slope. Then \( E \) is balanced.

**Proof.** Here \( t(E) = s(E) \) and for \( t \leq t(E) \) (resp. \( t \geq t(E) \)), \( E(-x_1 - \ldots - x_t) \) has nonnegative (resp. nonpositive) slope so the conclusion is immediate. \( \square \)

For general \( g \) one might conjecture that if \( E \) is regular and generically generated then \( E \) is balanced iff the slopes of its Harder-Narasimhan graded pieces are all in some length-1 interval.

1.2. **Splitting, modifying and matching.** The following result is useful in constructing some semi-balanced and sometimes balanced bundles by smoothing from a bundle on a reducible curve.

**Lemma 13.** Let \( C = C_1 \cup C_2 \) be a nodal curve such that \( C_1 \cap C_2 \) consists of \( k \) general points on \( C_1 \). Let \( E \) be a bundle on \( C \). Assume

(i) \( E \) is regular and generically generated;
(ii) \( E_i = E_{C_i} \) are balanced, \( i = 1, 2 \);
(iii) the remainders satisfy \( r(E_1) + r(E_2) < r(E) \) (e.g. \( E_{C_1} \) or \( E_{C_2} \) is perfectly balanced);
(iv) \( t(E_1) \geq k \).

Then

(a) \( E \) is semi-balanced.
(b) Moreover if \( r(E_2) = 0 \), \( E \) is balanced.

**Proof.** The respective genera satisfy \( g = g_1 + g_2 + k - 1, k = C_1.C_2 \) hence for the Euler slopes

\[
t(E) = t(E_1) + t(E_2) - k.
\]

For \( t = \lfloor t(E) \rfloor \) write \( t = t_1 + t_2 \) where

\[
t_1 = \lfloor t(E_1) \rfloor - k, t_2 = \lceil t(E_2) \rceil.
\]

To prove \( E \) is semi-balanced, choose general points

\[
x_{11}, \ldots, x_{1t_1} \in C_1, x_{21}, \ldots, x_{2t_2} \in C_2.
\]

By balancedness of \( E_2 \), there is a section \( s_2 \) of \( E_2 \) with arbitrary assigned values at \( x_{21}, \ldots, x_{2t_2} \). By balancedness of \( E_1 \) there is a section \( s_1 \) of \( E_1 \) with arbitrary assigned values at \( x_{11}, \ldots, x_{1t_1} \) and matching \( s_2 \) on \( C_1 \cap C_2 \). Then \( s_1 \) and \( s_2 \) glue to a section of \( E \) with assigned values at all the \( x_{ij} \). This proves (a). Then the proof of (b) is similar. \( \square \)

**Remark.** Note the absence of a ’general gluing’ assumption over \( C_1 \cap C_2 \). The result will be used mainly in case \( E_2 \) is perfectly balanced.
The same argument also proves:

**Lemma 14.** Let \( C = C_1 \cup C_2 \) be a nodal curve such that \( C_1 \cap C_2 = \{p_1, \ldots, p_k\} \) consists of \( k \) general points on each component. Let \( E \) be a regular, rank-\( r \) bundle on \( C \) and \( u, v \) weight-vectors. Assume:

(i) \( E|_{C_1} \) is \( u \)-balanced;

(ii) \( E|_{C_2} \) is \( v \)-balanced;

(iii) The restriction map \( H^0(E|_{C_1}) \oplus H^0(E|_{C_2}) \to H^0(E|_{p_1, \ldots, p_k}) \) is surjective

Then \( E \) is \((u, v)\)-balanced.

**Proof.** \( H^0(E|_{C_1}(-\sum p_i)) \oplus H^0(E|_{C_2}(-\sum p_i)) \) is a subspace of \( H^0(E) \) which already surjects onto \( H^0(U_1 \oplus \ldots \oplus V_t) \).

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The following property of ultra-balanced bundles is immediate from the definition but worth noting:

**Lemma 15.** Let \( E \) be an ultra-balanced bundle and \( E' = E^u \subset E \) a general down modification, i.e. kernel of a general surjection \( E \to \bigoplus u_i k_{p_i} \), such that \( E' \) is regular and generically generated.

Then \( E' \) is ultra-balanced. In particular, if \( D_t = \sum_{i=1}^t p_i \) is a general effective divisor and \( E(-D_t) \) is regular and generically generated, then \( E(-D_t) \) is ultra-balanced.

The following two lemmas, which are analogues of simple facts in the case of rational curves, show that a general (up or down) elementary modification of a balanced bundle is balanced:

**Lemma 16.** Let \( E \) be a balanced bundle and \( E' \subset E \) a general locally corank-1 modification at some general points. Assume \( E' \) is regular and generically generated. Then \( E' \) is balanced.

**Proof.** It suffices to prove this for modification at a single point \( p \), so \( E' \subset E \) is the kernel of a general surjection \( E \to \bigoplus u_i k_{p_i} \). Now if \( t(E) < 1 \), the conclusion is obvious, so assume \( t(E) \geq 1 \). We first prove \( E' \) is semi-balanced. Let \( t = \lceil t(E) \rceil > 0 \). Assume first \( E \) is not perfect. This easily implies that \( \lceil t(E') \rceil = t \). Then for general \( x_1, \ldots, x_t \), we get a subsheaf

\[
H^0(E(-x_1 - \ldots - x_t)) \otimes O \subset E(-x_1 - \ldots - x_t)
\]

that is not contained in the kernel of the (general) modification at \( p \). Hence \( H^0(E'(-x_1 - \ldots - x_t)) \) has the expected dimension so that \( H^0(E') \to E'_{x_1, \ldots, x_t} \) is surjective so \( E' \) is semi-balanced.

If \( E \) is perfect then \( t(E') = t(E) - 1 \), therefore for a general divisor \( x_1 + \ldots + x_{t-1} \), \( H^0(E(-x_1 - \ldots - x_{t-1})) \) has the expected dimension and the restriction map

\[
H^0(E(-x_1 - \ldots - x_{t-1})) \to E(-x_1 - \ldots - x_{t-1})|_p
\]
is surjective. Therefore the kernel $H^0(E'(-x_1 - \ldots - x_t))$ of the restriction map has the expected dimension and semi-balancedness follows.

Now the injectivity statement required to show $E'$ balanced is obvious if $[t(E')] = [t(E)]$. Otherwise, $t := [t(E')] = [t(E)] - 1$ and the required injectivity for $E'$ follows from injectivity of $H^0(E) \rightarrow E_{x_1,\ldots,x_t,p}$.

There is a similar statement for up modifications:

**Lemma 17.** Let $E$ be a balanced bundle and $E \subset E^+$ a general locally rank-1 modification at some general points. Then $E^+$ is balanced.

**Proof.** First it is obvious that $E^+$ is regular and generically generated. For balancedness, it again suffices to prove it for the case of modification at a single point $p$, so $(E^+)^* \subset E^*$ is the kernel of a general surjection $E^* \rightarrow k_p$ and $E_p \rightarrow E_p^+$ has kernel a general 1-dimensional subspace. Now semi-balancedness is obvious if $[t(E)] = [t(E^+)]$. If not, then $t(E^+) = [t(E)] = [t(E)] + 1 := t + 1$ and in particular $t(E^+)$ is an integer. Now $H^0(E(-x_1 - \ldots - x_t)) \subset H^0(E^+(-x_1 - \ldots - x_t))$ injects to $E'(-x_1 - \ldots - x_t)|_p$ and its image is just the inverse image of the natural map $E' \rightarrow k_p$. Therefore the kernel of $H^0(E^+(-x_1 - \ldots - x_t)) \rightarrow k_p$ is contained in the latter image, hence must vanish because $H^0(E(-x_2 - \ldots - x_t - p)) = 0$. This proves $H^0(E^+(-x_1 - \ldots - x_t)) \rightarrow E_p^+$ is injective, i.e. surjective, so $E^+$ is semi-balanced.

Now to prove $E^+$ is balanced let $t + 1 := [t(E^+)] \geq [t(E)]$. Then $t(E) < t + 1$. Now the kernel of $H^0(E^+(-x_1 - \ldots - x_t)) \rightarrow E^+|_p$ corresponds to the intersection of the image of $H^0(E(-x_1 - \ldots - x_t)) \rightarrow E|_p$ with the kernel if $E|_p \rightarrow E^+|_p$ which is a general 1-dimensional subspace and the intersection is trivial because the latter image is a proper (maybe trivial) subspace thanks to $t(E) < t + 1$. Thus $H^0(E^+(-x_1 - \ldots - x_t - p)) = 0$ so $E^+$ is balanced.

The following Lemma strengthens Lemma 25 of [8] and generalizes it to arbitrary genus (note that Cases 2,3 are new even for genus 0):

**Lemma 18.** Let $E$ be an exact sequence of vector bundles on a curve such that $E_1, E_2$ are balanced of respective slopes $s_1, s_2$. Assume either:

**Case 1:**

$s_1 = [s_2];$

**or Case 2:**

$s_2 = [s_1] + 1;\tag{9}$
or Case 3:

\[ s_1 = \lfloor s_2 \rfloor + 1. \]

Then \( E \) is balanced. Moreover the slope \( s = s(E) \) satisfies:

- Case 1: \( [s] = [s_1] \);
- Case 2: \( [s] = s_2 \);
- Case 3: \( [s] = s_1 \).

**Proof.** Apply the Snake Lemma to the following (exact, since \( H^1(E_1) = 0 \)) diagram, in which \( D_m = p_1 + \ldots + p_m \) denotes a general effective divisor of degree \( m \):

\[
\begin{array}{ccccccc}
0 & \rightarrow & H^0(E_1) & \rightarrow & H^0(E) & \rightarrow & H^0(E_2) & \rightarrow & 0 \\
\rho_1 \downarrow & & \rho \downarrow & & \rho_2 \downarrow & & \\
0 & \rightarrow & E_1|_{D_m} & \rightarrow & E|_{D_m} & \rightarrow & E_2|_{D_m} & \rightarrow & 0
\end{array}
\]

(1)

Case 1: The assertion about \( s \) is obvious and implies

\[ t := [t(E)] = [t(E_1)] = [t(E_2)]. \]

Taking \( m = t \), we have \( \rho_1, \rho_2 \) surjective hence so is \( \rho \). Taking \( m = [t(E)] \), \( \rho_1, \rho_2 \) are injective hence so it \( \rho \).

Case 2: Note this case can occur only if \( s_2 \), hence \( t_2 = t(E_2) \) is an integer. Taking \( m = t_2 \), \( \rho_2 \) is an isomorphism and \( \rho_1 \) is injective, hence \( \rho \) is injective. Taking \( m = t_2 - 1 \), \( \rho_1 \) and \( \rho_2 \) are surjective hence so is \( \rho \).

Case 3 is similar to Case 2. \( \square \)

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1.3. **Balanced and ultra-balanced curves, Kontsevich intersections.** A lci curve \( C \rightarrow X \) is said to be separably regular or (semi-, perfectly) balanced if its normal bundle \( N_{C/X} \) has the corresponding property. Regularity means that \( C \) belongs to a smooth family of the expected dimension. Semi-balance implies (and in char. 0 is equivalent to) the semi-interpolating property, i.e. that \( C \) can be deformed to go through the expected number of general points of \( X \), and balance implies moreover that the subvariety of \( X \) filled up by the deformations through a fixed maximal collection of general points has the expected dimension. When \( X \) contains a (semi-) balanced curve we will say that \( X \) has the (semi-) interpolation property (for curves of genus \( g(C) \) and degree \( \deg(C) \) if understood).

If \( C \) is reducible and \( C_1 \subset C \) is a component, we will say \( E \) is (semi-) balanced around \( C_1 \) if \( H^1(E) = 0 \), \( E \) is generated by its sections at a general point of \( C_1 \), and the required surjectivity or injectivity statements as appropriate hold for general points of \( C_1 \).

If \( C \) has degree \( e \) and genus \( g \) in \( X = \mathbb{P}^n \) then

\[
t(C) = e + 1 - g + \left[ \frac{2e - 1 + g}{n - 1} \right].
\]
In particular if \( C \) is nondegenerate (so that \( e \geq n \)) and nonspecial (so that \( e + 1 - g = \chi(O_C(H)) \geq n + 1 \)), we have \( t(C) \geq n + 3 \).

See \[8\], especially §1 and §5 for various information on normal bundles and fangs.

A curve \( C \rightarrow X \) is said to be ultra-balanced if its normal bundle is. This condition has an interesting interpretation in terms of intersection numbers on Kontsevich spaces of stable maps. Thus let \( M_{g,t}(X) \) be the Kontsevich space of stable \( t \)-pointed maps \( C \rightarrow X \) where \( (C, x_1, \ldots, x_t) \) is a \( t \)-pointed stable curve of genus \( g \). Let

\[
\sigma_i : M_{g,t}(X) \rightarrow X, i = 1, \ldots, t
\]

be the natural maps. Let \( h \) be a birationally ample divisor on \( X \) and set

\[
\eta_i = \sigma_i^*(h).
\]

Define

\[
I_M(0, u_1, \ldots, u_t) = \int_M \eta_1^{u_1} \cdots \eta_t^{u_t}.
\]

This definition will shortly be extended to the case of a nonzero first argument.

**Proposition 19.** Let \( M \) be a component of \( M_{g,t}(X) \) whose general point has the form \( (C, x_1, \ldots, x_t) \) where \( C \) is ultra-balanced (resp. balanced). Then for all \( u_1, \ldots, u_t \) such that

\[
u_1 + \cdots + u_t = \chi(N_{C/X} = (C - K_X) + (n - 3)(1 - g),
\]

(resp. and such that \( u_2 = \ldots = u_t = n \)) we have

\[
I_M(0, u_1, \ldots, u_t) > 0.
\]

**Proof.** Considering \( X \subset \mathbb{P}^N \), there is a natural map

\[
F : M \rightarrow (\mathbb{P}^N)^t.
\]

Our ultra-balanced hypothesis implies that for \( Z = P^{N-u_1} \times \ldots \times \mathbb{P}^{N-u_t} \), \( F^{-1}(Z) \) contains an isolated reduced point. Therefore the intersection number \( F_*(M) \cdot Z > 0 \), which implies our result in the ultra-balanced case. The balanced case is similar. \( \square \)

1.4. **Ambient-balanced curves.** A curve \( C \rightarrow X \) of genus \( g \) is said to be ambient-balanced if the restricted tangent bundle \( T_X|_C \) is semi-balanced, i.e. for all

\[
t \leq t(T_X|_C) = (-K_X.C/n) + 1 - g, n = \dim(X),
\]

and general points \( x_1, \ldots, x_t \in C \), we have

\[
H^1(T_X|_C(-x_1 - \ldots - x_t)) = 0.
\]

Note that the vanishing \( (2) \) implies \( H^1(N_{C/X}(-x_1 - \ldots - x_t) = 0 \) so that a general deformation of \( C \) contains \( t \) general points of \( X \). However ambient balance does not imply balance because \( (2) \) is only assumed for \( t \leq t(T_X|_C) \) but usually \( t(N_{C/X}) > t(T_X|_C) \).
Now (2) also implies surjectivity the natural map induced by the normal sequence
\[ H^0(N_{C/X}(-x_1 - ... - x_t)) \to H^1(T_C(-x_1 - ... - x_t)). \]
Consequently we have

**Corollary 20.** If \( C \to X \) is ambient-balanced then \( C \) is separably moduli-interpolating, i.e. for \( t \leq (-C.K_X/n) + 1 - g \) and general points \( x_1, ..., x_t \in X \), the family of deformations of \( C \) in \( X \) passing through \( x_1, ..., x_t \) has separably general moduli as a family of \( t \)-pointed curves.

Thus, for an ambient-balanced curve \( C \) we are able to impose on deformations of \( C \) simultaneously a fixed set of \( t \) general points of \( X \) and fixed set of \( t \)-pointed moduli where \( t = \lceil -C.K_X/n \rceil + 1 - g \). Note that such moduli are nontrivial even if \( g = 0 \) provided \( t \geq 4 \).

For genus 0 and \( X = \mathbb{P}^n \), it follows easily from [8], Lemma 26 that a general deformation of any given curve \( C \) is ambient-balanced. For higher genus, see Corollary 34 below.

For example, the rational normal curve in \( \mathbb{P}^n \) is both perfectly balanced and perfectly ambient-balanced.

**Example 21.** To put matters in perspective consider the case of a Grassmannian \( X = G(k,n) \) with its tautological subbundle \( S \) and quotient bundle \( Q \) and tangent bundle \( T_X = S^* \otimes Q \). For a rational curve \( C \subset X \) of degree \( e \), it is easy to see that on a general deformation of \( C \), both \( S \) and \( Q \) will be balanced but, unless \( k|e \) or \( (n - k)|e \), both will be imperfect, hence \( T_X|_C \) will be unbalanced. Consequently, \( X \) contains an ambient-balanced rational curve of degree \( e \) iff either \( k|e \) or \( (n - k)|e \). In particular the set of degrees of ambient-balanced curves in \( X \) constitutes 2 arithmetic progressions.

As for balance, the normal sequence
\[ 0 \to O(2) \to S^* \otimes Q \to N_{C/X} \to 0 \]
plus Lemma 18 show that if the slope \( s = s(N_{C/X}) \) satisfies \( [s] = 2 \) and \( S^* \otimes Q \) is unbalanced, then so is \( N_{C/X} \). Explicitly, the slope condition is
\[ \left\lfloor \frac{en - 2}{k(n - k) - 1} \right\rfloor = 2. \]
So whenever this holds and \( e \) is not divisible by either \( k \) or \( n - k \), any rational curve of degree \( e \) in \( X \) is unbalanced. For example, when \( n = 2k \) the condition on \( e \) is
\[ k < e < 3k/2 - 1/2k. \]
A general rational curve with degree in this range will be nondegenerate (i.e. correspond to a nondegenerate scroll in \( \mathbb{P}^{n-1} \)), unbalanced and ambient-unbalanced.
Thus, for general Fano manifolds one may expect topological obstructions on a curve to be ambient-balanced or balanced, though there remains the possibility that all curves of sufficiently high degree are balanced. For Fano hypersurfaces of degree $d < n$ in $\mathbb{P}^n$ we will show below that the set of degrees of ambient-balanced or balanced curves contains some arithmetic progressions, resembling the situation for Grassmannians, while for $d = n$ this set contains all sufficiently large integers.

A curve $C \to X$ is said to be ultra ambient-balanced if $T_X|_C$ is ultra-balanced. Similarly as in Proposition 19, ultra ambient balance has an application to intersection numbers. Let

$$\phi : M_{g,t}(X) \to M_{g,t}$$

be the natural map and $\kappa = \phi^*(L)$ for some birationally ample $L$. Now define

$$I_M(u_0, u_1, ..., u_t) = \int_M \kappa^{u_0} \eta_1^{u_1} \cdots \eta_t^{u_t}.$$ 

**Proposition 22.** Notations as above, assume $C$ is ultra ambient-balanced (resp. ambient-balanced) rather than ultra-balanced and $t > 0$. Let

$$u_0 = \dim(M_{g,t}) = 3g - 3 + t$$

Then for all $u_1, ..., u_t$ such that

$$\sum u_i = \chi(N) - u_0 = (C - K_X) - n(g - 1) - t,$$

(resp. and $u_1 = ... = u_t = n$), we have

$$I_M(u_0, ..., u_t) > 0.$$ 

The proof is similar to that of Proposition 19. Note that the case of a general exponent vector $(u_0, ..., u_t)$ of weight $\chi(N)$ remains open.

2. **Relative and Log Tangent Bundles**

2.1. **Degeneration of tangent bundles.** We construct a relative version of the tangent bundle for a family of varieties degenerating to normal crossings of multiplicity 2. We begin with some local considerations. Consider the surface $X$ with equation $x_1x_2 = t$ in $\mathbb{A}^3$ with its $t$-projection $\pi : X \to \mathbb{A}^1$. There is an associated derivative map

$$d\pi : T_X \to \pi^* T_{\mathbb{A}^1}$$

which is clearly surjective except at the node, i.e. the origin, and has image $m \pi^* T_{\mathbb{A}^1}$, where $m$ is the ideal of the origin. Its kernel is invertible and locally generated by the vector field

$$v = (x_1 \partial_{x_1} + x_2 \partial_{x_2})/2 + t \partial_t.$$
Now working globally, let
\[ \pi : \mathcal{X} \to B \]
be a flat morphism of a smooth variety to a smooth curve whose general fibre is smooth and whose special fibres have at most normal crossing double points along a smooth subvariety \( \Delta \) of codimension 2 (codimension 1 in \( \pi^{-1}(\pi(\Delta)) \)). Again there is a derivative map
\[ d\pi : T\mathcal{X} \to \pi^* T_B. \]
Because \( \pi \) can be locally modelled by the above curve fibration, it follows that the image of \( d\pi \) is \( \mathcal{I}_\Delta \pi^* T_B \) and its kernel, denoted \( T\mathcal{X}/B \) and called the relative tangent bundle of the fibration \( \pi \), is locally free. Thus we have an exact sequence
\[ 0 \to T\mathcal{X}/B \to T\mathcal{X} \to \mathcal{I}_\Delta \pi^* T_B \to 0. \]
(3)
In fact \( T\mathcal{X}/B \) is locally near \( \Delta \) generated by \( \nu \) as above together with the complementary vector fields \( \partial_{x_3}, \ldots \) tangent to \( \Delta \). Note that for a smooth fibre \( \mathcal{X}_t \), we have
\[ T\mathcal{X}/B|_{\mathcal{X}_t} = T\mathcal{X}_t. \]
On the other hand for a singular fibre \( \mathcal{X}_0 \) with normalization \( \tilde{\mathcal{X}}_0 \) and double locus \( \Delta \subset \tilde{\mathcal{X}}_0 \), the pullback \( T\mathcal{X}_B|_{\tilde{\mathcal{X}}_0} \) is generated by \( x_1 \partial_{x_1} \) or \( x_2 \partial_{x_2} \) plus the complementary fields. Therefore we have
\[ T\mathcal{X}/B|_{\tilde{\mathcal{X}}_0} = T_{\tilde{\mathcal{X}}_0}(\langle -\log \Delta \rangle). \]
In particular if \( \mathcal{X}_0 = \mathcal{X}_1 \cup \mathcal{X}_2 \) is a union of smooth components then
\[ T\mathcal{X}/B|_{\mathcal{X}_i} = T_{\mathcal{X}_i}(\langle -\log \Delta \rangle), i = 1, 2. \]
Note the exact sequences
\[ 0 \to T_{\mathcal{X}_i}(-\Delta) \to T_{\mathcal{X}_i}(\langle -\log \Delta \rangle) \to T_\Delta \to 0, i = 1, 2 \]
which induce
\[ 0 \to \mathcal{O}_\Delta \to T_{\mathcal{X}_i}(\langle -\log \Delta \rangle)|_\Delta \to T_\Delta \to 0 \]
(4)
where the \( \mathcal{O}_\Delta \) subsheaf is locally generated by \( x_1 \partial_{x_1} \) or \( x_2 \partial_{x_2} \). The latter sequence is compatible with the identifications
\[ T_{\mathcal{X}_1}(\langle -\log \Delta \rangle)|_\Delta \simeq T_{\mathcal{X}_2}(\langle -\log \Delta \rangle)|_\Delta \simeq T\mathcal{X}/B|_\Delta. \]
2.2. Restriction on curves. Note that given a smooth pair $(X_i, \Delta)$ and a curve $C_i \subset X_i$ meeting $\Delta$ transversely in $\delta = \Delta \cap C_i$, the restriction $T_{X_i}(-\log \Delta)|_{C_i}$ is just the elementary corank-1 down modification of $T_{X_i}|_{C_i}$ at $\delta$ corresponding to the tangent hyperplanes $T_p\Delta \subset T_pX_i$, $p \in \delta$. This has the following immediate consequence

**Corollary 23.** In the above notations let $C/B \to X/B$ be a family of curves with special fibre $C_0 = C_1 \cup \delta C_2 \subset X_1 \cup \Delta X_2$. Then there is a bundle $T = T_{X/B}$ on $X$ such that for a general fibre $C_t \subset X_t$ we have

$$T|_{C_t} = T_{X_t}|_{C_t}$$

while on the special fibre, $T|_{C_i}$ for $i = 1, 2$ is the elementary corank-1 down modification of $T_{X_i}|_{C_i}$ at the points $p \in \delta$ corresponding to the hyperplanes $T_p\Delta \subset T_pX_i$.

**Example 24.** With notations as above, suppose $C_2$ is a $\mathbb{P}^1$ with trivial normal bundle $N_{C_2/X_2} = (n-1)O$ and $\delta = \{p\}$. Then $T_{X_2}|_{C_2} = T_C \oplus (n-1)O = O(2) \oplus (n-1)O$, so that

$$T|_{C_2} = T_{X_2}(-\log \Delta)|_{C_2} = O(1) \oplus (n-1)O$$

where the $(n-1)O$ quotient coincides at $p$ with the $T_\Delta$ quotient. There is an analogous and compatible quotient on the $X_1$ side. Then for a point $q \neq p \in C_2$, we can identify $H^0(T_{C_1 \cup C_2}(-q))$ with the kernel of the natural map

$$H^0(T_{X_1}(-\log \Delta)|_{C_1}) \to T_{p,\Delta}.$$

Therefore

$$H^0(T|_{C_1 \cup C_2}(-q)) = H^0(T_{X_1}|_{C_1}(-p)).$$

More is true. In fact as in [8], §1, there is a modification $T \to T'$ with cokernel on $C_2$ such that

$$T'|_{C_2} = nO$$

while $T'|_{C_i}$ is the elementary up modification of $T_{X_i}(-\log \Delta)|_{C_i}$ at $p$ corresponding to the $O_\Delta$ subsheaf as in [4], which clearly coincides with $T_{X_i}|_{C_i}$ itself, i.e.

$$T'|_{C_1} = T_{X_1}|_{C_1}.$$

In particular, given a point modification of $T'|_{C_2}$ leading to an exact sequence

$$0 \to K \to T'|_{C_1 \cup C_2} \to kO_q \to 0, q \neq p \in C_2$$

then there is a corresponding exact sequence

$$0 \to K_1 \to T_{X_1}|_{C_1} \to kO_p \to 0$$

such that

$$H^0(K) = H^0(K_1).$$
This argument evidently extends to the case where $C_2$ is a disjoint union of lines with trivial normal bundle. The upshot is that such components may effectively be ignored and the log tangent bundle $T_{X_1} \langle - \log \Delta \rangle_{|C_1}$ replaced by by $T_{X_1}|_{C_1}$ near $C_1 \cap C_2$. This situation occurs in the proof of Theorem 40 and Theorem 41.

2.3. Log tangents for projective bundle pairs. Let $\pi : X = \mathbb{P}(G) \to B$ be a projective bundle and let $Y = \mathbb{P}(G/A) \subset X$ be a codimension-1 projective subbundle, corresponding to a line subbundle $A \subset G$. Let $S_G$ be the kernel of the canonical surjection $\pi^* G \to O_X(1)$. Then we have the relative tangent bundle
\[ T_{X/B} = S_G^* \otimes O_X(1). \]
Note that $Y$ is the zero-divisor of the natural map $A \to O_X(1)$, hence
\[ N_{Y/X} = A^* \otimes O_Y(1) \]
where $O_Y(1)$ is the restriction of $O_X(1)$. Then we have an exact sequence
\[ 0 \to T_{X/B} \langle - \log Y \rangle \to S_G^* \otimes O_X(1) \to A^* \otimes O_Y(1) \to 0. \]
Now given a curve $C \to B$, a lifting $C \to X$ corresponds to a rank-1 surjection $G_C \to O_X(1)$. Assume that $A_C \to M$ is injective (i.e. $C \cap Y$ is finite). Then we get an exact sequence
\[ 0 \to T_{X/B} \langle - \log Y \rangle |_C \to S_G^* \otimes M \to A^* \otimes M |_{C \cap Y} \to 0. \]

2.4. Log tangents for blowups. Let $\pi : \hat{X} \to X$ be the blowup of a smooth subvariety $Y$ with normal bundle $N_Y$. Let $E = \mathbb{P}(\hat{N}_Y) \subset \hat{X}$ be the exceptional divisor. Then we have an exact diagram
\[ 0 \to T_{\hat{X}} \langle - \log E \rangle \to \pi^* T_X \to \pi^* N_Y \to 0 \]
\[ 0 \to T_{\hat{X}} \to \pi^* T_X \to O_E(1) \to 0 \]
For example, let $Y$ be a line in $X = \mathbb{P}^2$ so $E = Y, \hat{X} = X$. If $L \subset X$ is a general line then clearly
\[ T_X \langle - \log E \rangle |_L = O(2, 0) \]
with upper subbundle $O(2)$ corresponding to $T_L$. If $L_1, L_2$ are distinct lines then the $O(2)$ subspaces differ at the intersection point $L_1 \cap L_2$, hence
\[ T_X \langle - \log E \rangle |_{L_1 \cup L_2} = O(2, 2), \]
i.e. a direct sum of line bundles of total degree 2; therefore likewise for a general conic $C_2 \subset \mathbb{P}^2$. 

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Now let $Y$ be a line in $X = \mathbb{P}^3$ and $C_2$ a conic in a hyperplane $H \subset X$ containing $Y$, with birational transform $\hat{H} \subset \hat{X}$. Then letting $C_2' \subset \hat{H}$ denote the birational transform of $C_2$, we have $\mathcal{O}_{\hat{H}}(\hat{H})|_{C_2'} = \mathcal{O}_{C_2'}$, consequently

$$T_{\hat{X}}(-\log E)|_{C_2'} = \mathcal{O}(2,2,0)$$

with upper subsheaf $\mathcal{O}(2,2)$ coming from $T_{\hat{H}}(-\log Y)$. Now if $L \subset \hat{X}$ is the birational transform of a general line meeting $C_2'$ is a point then $T_{\hat{X}}|_L = T_{\hat{X}}(-\log E)|_L = \mathcal{O}(2,1,1)$. Therefore as above we get

$$T_{\hat{X}}(-\log E)|_{C_2' \cup L} = \mathcal{O}(3,3,2),$$

therefore likewise for $C_2' \cup L$ replaced by $C_3' \subset \hat{X}$, the birational transform of a twisted cubic meeting $Y$ in 2 points.

Continuing in the way, we can show that that if $\hat{X}$ is the blowup of $\mathbb{P}^n$ in a line $Y$ and $C_n'$ is the birational transform of a general rational normal curve 2-secant to $Y$, then

$$T_{\hat{X}}(-\log E)|_{C_n'} = 2\mathcal{O}(n) \oplus (n-2)\mathcal{O}(n-1).$$

In particular this bundle is balanced.

Now an argument similar to but simpler than that in the proof of Lemma 31 below shows that the balancedness result holds for $Y$ replaced by a linear subspace of any codimension $c \in [2, n-1]$ as well as $C_n$ replaced by higher-degree rational curves, so we may conclude:

**Lemma 25.** Let $A \subset \mathbb{P}^n$ be a linear subspace of codimension $c \in [2, n-1]$ and let $\mathcal{P} \to \mathbb{P}^n$ be the blowup of $A$ with exceptional divisor $E$. Let $C' \subset \mathcal{P}$ be the birational transform of a general rational curve $C \subset \mathbb{P}^n$ of given degree $e = n$ or $e \geq 2n - 1$ meeting $A$ in $m \leq 2$ points. Then $T_{\mathcal{P}}(-\log E)|_{C'}$ is balanced.

### 3. Curves in Projective Space

#### 3.1. Balanced

In [1], Atanasov, Larson and Yang construct many semi-balanced curves of any genus in projective space. Here we will refine their result, albeit for a mire restricted degree range, to construct balanced, rather than semi-balanced curves. The refined result will be used below, e.g. in the proof of Theorem 37.

**Theorem 26.** Let $C_1, C_2 \subset \mathbb{P}^n$, $n \geq 3$, be smooth balanced nondegenerate curves of respective degrees $e_1, e_2$, genera $g_1, g_2$, Euler slopes $t_1, t_2 > 0$ and remainders $r_1, r_2$. Assume

$$r_1 + r_2 < n - 1.$$

Then

(i) there exists a smooth balanced curve $C \subset \mathbb{P}^n$ of degree $e_1 + e_2 - 1$, genus $g_1 + g_2$ and remainder $r = r_1 + r_2$;
(ii) there exists a smooth balanced curve \( C' \subset \mathbb{P}^n \) of degree \( e_1 + e_2 - 2 \), genus \( g_1 + g_2 + 1 \) and remainder \( r = r_1 + r_2 \).

**Proof.** We begin with some numerology. Set \( g = g_1 + g_2, e = e_1 + e_2 - 1 \) and

\[
\begin{align*}
  s &= \frac{e(n + 1) + 2g - 2}{n - 1}, \\
  s_i &= \frac{e_i(n + 1) + 2g_i - 2}{n - 1}, i = 1, 2.
\end{align*}
\]

Thus \( s = [s] + r/(n - 1) \) and likewise for \( t_i, s_i, t_i \). Note that \( s = s_1 + s_2 - 1 \) hence \( [s] = [s_1] + [s_2] - g \) and

\[
t = t_1 + t_2 - 2.
\]

We use the same basic fang construction as in [8]. Let

\[
b_1 : \mathcal{P}(\ell) = B_{\mathbb{P}^{e} \times 0}(\mathbb{P}^{n}_1 \times \mathbb{A}^1) \to \mathbb{P}^n_1 \times \mathbb{A}^1
\]

be the blow up, which fibres \( \pi : \mathcal{P}(\ell) \to \mathbb{A}^1 \) with special fibre \( P_0 = \pi^{-1}(0) = P_1 \cup_E P_2 \) where

\[
P_1 = B_{\mathbb{P}^{e}_1} \mathbb{P}^n, P_2 = B_{\mathbb{P}^{e-1}_2} \mathbb{P}^n, E = \mathbb{P}^{\ell}_1 \times \mathbb{P}^{n-\ell-1}_2
\]

and general fibre \( \mathbb{P}^n \). \( P_0 \) is called a fang of type \( \ell \).

Now for (i), we let \( C_i \subset P_i, i = 1, 2 \) be the proper transform of a smooth curve of degree \( \ell \) and genus \( g_i \), such that \( C_1.E = C_2.E = p \) (transverse intersection) and \( C_0 = C_1 \cup_p C_2 \). Then the normal bundle \( N_{C_i/P_i}, i = 1, 2 \) is an elementary pointwise modification of \( N_{C_i/P_i} \) of colength \( n - 1 - \ell \) (resp \( \ell \)), and under the identification \( N_{C_i/P_i}|_E = T_{P_0}E \), the kernel of the natural map \( N_{C_i/P_i} \to N_{C_i/P_i} \) may be identified with \( T_{P_0} \mathbb{P}^{n-\ell-1}_2 \) (resp \( T_{P_0} \mathbb{P}^{\ell}_1 \)).

There is an exact sequence

\[
0 \to N_{C_0/P_0} \to N_{C_0/\mathcal{P}(\ell)} \to T^1 \to 0
\]

where \( N_{C_0/P_0}, N_{C_0/\mathcal{P}(\ell)} \) are the lci normal bundles, \( N_{C_0/P_0} = N_{C_1/P_1} \cup_{T_{P_0}E} N_{C_2/P_2} \) parametrizes compatible deformations of \( (C_1, C_2) \) and

\[
T^1 = T_{P_0}|_{C_0} = N_{P_0/\mathcal{P}(\ell)}|_{C_0} = T_{C_0}^1
\]

is a 1-dimensional skyscraper sheaf at \( p \).

As the equations defining \( C_0 \) on \( P_0 \) restrict to defining equations for each \( C_i \) on \( P_i \)

\[
N_{C_0/P_0}|_{C_i} = N_{C_i/P_i}, i = 1, 2.
\]

We have exact sequences

\[
0 \to N_{C_i/P_i} \to N_{C_i/\mathbb{P}^n} \to \tau_i \to 0, i = 1, 2,
\]

\[
0 \to N_{C_i/\mathbb{P}^n}(-p) \to N_{C_i/P_i} \to \sigma_i \to 0
\]
where \( \tau_i \) is a skyscraper sheaf at \( p \) of length \( \ell(\tau_i) = n - 1 - k, i = 1 \) or \( k, i = 2 \), and \( \ell(s_i) = n - 1 - \ell(\tau_i) \). We have canonical identifications

\[
N_{C_1/P_1}|_p \simeq N_{C_2/P_2}|_p \simeq T_p E.
\]

Note that we have subspaces

\[
V_i = N_{C_i/P^n}(-p)|_p \subset N_{C_i/P_i}|_p, i = 1, 2
\]

of codimensions \( k \) resp \( n - 1 - k \). The image of the restriction map

\[
N_{C_0/P_0} \to N_{C_1/P_1} \oplus N_{C_2/P_2}
\]

and the induced map

\[
H^0(N_{C_0/P_0}) \to H^0(N_{C_1/P_1}) \oplus H^0(N_{C_2/P_2})
\]

is the inverse image of the ‘diagonal’ \( \Delta \) under the above identification (10). There is a standard deformation \( \Delta_t \) of \( \Delta \) to a \( \Delta_0 \) which is union of subspaces, one of them being \( V_1 \times V_2 \). This implies firstly that \( N_{C_0/P_0} \) admits a specialization to a sheaf that contains \( N_{C_1/P^n}(-p) \oplus N_{C_2/P^n}(-p) \) as cotorsion subsheaf and since that latter sheaf has \( H^1 = 0 \) (because \( t_1, t_2 > 0 \)), so does \( N_{C_0/P_0} \), i.e.

\[
H^1(N_{C_0/X_0}) = 0.
\]

It also follows easily that \( N_{C_0/X_0} \) is generically generated.

Now the above \( H^1 \) vanishing implies that, possibly after an étale base change \( A \to \Delta^1, C_0 \subset P_0 \) extends to a surface \( S \) fibred over \( A \). Let \( C \) be its general fibre. Let \( x_{i1}, \ldots, x_{it_i-1}, i = 1, 2 \) be general sections of \( S \) specializing to general points of \( C_i \). Now as \( x_{i1}, \ldots, x_{it_i-1}, P \) for \( i = 1, 2 \) are general points on \( C \) and hence by our hypothesis on \( C_1 \) and \( C_2 \), the restriction map

\[
\rho_0 : V_1 \times V_2 \to N_{C_0/P_0}|_{\{x_{i1}, \ldots, x_{it_i-1}, x_{i21}, \ldots, x_{i2t_i-1}\}}
\]

is surjective. Therefore the same is true of \( \Delta_t \) for general \( t \) hence for \( \Delta \) itself if choose the above identifications generally. Therefore the same is true \( N_{C/P^n} \), which shows that \( C \) is semi-balanced.

For balancedness we argue similarly but, in case \( s \) is not an integer, add one more section \( y \) specializing to a general point on \( C_1 \). Because \( C_1 \) is balanced, the kernel of the map \( \rho_0 \) above injects into \( N_{C_1/P^n}(-p)|_y \). Therefore the same is true for the kernel of the analogous restriction map on \( H^0(N_{C_0/P_0}) \) therefore ditto for \( H^0(N_{C/P^n}) \), which proves the injectivity property yielding balancedness. This completes the proof of (i).

For (ii), we use the same construction except now \( C_i \subset P_i \) meet \( E \) and each other in 2 general points \( p, q \), so that

\[
C_0 = C_1 \cup_{\{p,q\}} C_2
\]
has genus \( g = g_1 + g_2 + 1 \) and 'degree' \( e = e_1 + e_2 - 2 \). Note in this case we have
\[
s = s_1 + s_2 - 2, \quad [s] = [s_1] + [s_2] - 2, \quad t = t_1 + t_2 - 4.
\]

We have subspaces
\[V_{ip} = N_{C_i/P_i}(-p - q) \subset N_{C_i/P_i}, \quad i = 1, 2\]
and likewise for \( q \), and the image of the restriction map
\[H^0(N_{C_0/P_0}) \to H^0(N_{C_1/P_1}) \oplus H^0(N_{C_2/P_2})\]
is the inverse image of the 'bidiagonal' \( \Delta_p \times \Delta_q \) under restriction to \( \bigoplus_{i=1,2} N_{C_i/P_i} \cup \{p, q\} \). As above, \( \Delta_p \times \Delta_q \) deforms to \( \Delta_{0,p} \times \Delta_{0,q} \) which contains \( W := V_{1,p} \times V_{2,p} \times V_{1,q} \times V_{2,q} \).

We consider general sections \( x_{ij}, i = 1, 2, j = 1, \ldots, t_i - 2 \). As above, \( W \) surjects to \( N_{C_0/P_0} \mid_{x_{11}, \ldots, x_{t_2-2}} \) which implies the required surjectivity for \( H^0(N_{C_0/P_0}) \) and hence for \( H^0(N_{C/P_{\nu}}) \) for the smoothing \( C \), which proves semi-balancedness.

Now the injectivity statement for balancedness is proven as in part (i). This completes the proof.

\[\Box\]

Example 27. (i) Taking \( e_1 = e + 2 - n, e_2 = n, g_1 = g_2 = 0 \) in Theorem 37 (ii) yields ultra-balanced elliptic curves in \( \mathbb{P}^n \) of any degree \( e \geq 2n - 2 \). In this case \( r_2 = 0, r_1 = r \).

In particular, the resulting curve is perfect when \( e = 2n - 2 \).

(ii) Using two ultra-balanced elliptic curves as above and combining them as in Theorem 37 (i) yields a balanced curves of genus 2 and any degree \( e \geq 2(2n - 2) - 2 = 4n - 6 \) in \( \mathbb{P}^n \). Continuing inductively, we get ultra-balanced curves of genus \( g \) and any degree \( e \geq g(2n - 4) + 2 \) in \( \mathbb{P}^n \).

(iii) Taking \( C_1 \) (ultra)- balanced and \( C_2 \) a rational normal curve (remainder 0) in Part (i) yields (ultra) balanced curves of degree \( e_1 + n - 1 \) and genus \( g_1 \). Taking such \( C_1, C_2 \) in Part (ii) yields balanced curves of degree \( e_1 + n - 2 \) and genus \( g_1 + 1 \).

Continuing inductively, this yields:

Corollary 28. For all \( g \geq 1, n \geq 3 \) and \( e \geq n + g(n - 2) \), a general curve of genus \( g \) and degree \( e \) in \( \mathbb{P}^n \) is balanced.

3.2. Ultra-balanced. Next we refine the result to yield ultra-balanced curves.

Theorem 29. For all \( g \geq 0 \) and \( e \geq 2(g + 1)n, n \geq 3 \) there exists an ultra-balanced curve of degree \( e \) and genus \( g \) in \( \mathbb{P}^n \).

Corollary 30. For \( e, g, n \) as in Theorem 29 the conclusion of Proposition 19 holds for \( X = \mathbb{P}^n \) and any \( t > 0 \).

Proof of Theorem. We begin with a lemma.
Lemma 31. Let $A \subset \mathbb{P}^n$ be a linear subspace of codimension $c \in [2, n-1]$ and let $P \to \mathbb{P}^n$ be the blowup of $A$. Let $C' \subset P$ be the birational transform of a general rational curve $C \subset \mathbb{P}^n$ of given degree $e = n$ or $e \geq 2n - 1$ meeting $A$ in $m \leq 2$ points. Then $C'$ is balanced in $P$.

Proof. The case $m = 0$, i.e. the assertion that $C$ is balanced in $\mathbb{P}^n$, originally due to Sacchiero, is reproved as Proposition 19 in [8] and the case $m = 1$ follows easily from that as $N_{C'/P}$ is a general modification of $N_{C/P^n}$. We will focus on the case $m = 2$ which is harder, as the modifications involved are not general. The proof will proceed analogously to the one in loc. cit.

Case 1: $e = n$, i.e. $C$ is a rational normal curve.

Consider first the case dim$(A) = 1$, i.e. $c = n - 1$, where the claim is that

$$N_{C'/P} = 2\mathcal{O}(n + 1) \oplus (n - 3)\mathcal{O}(n).$$

First, for $n = 3$, $A$ is a 2-secant line of the twisted cubic $C$ and $C \cup A$ is a (2,2) complete intersection, so $C'$ is a complete intersection of type $(\mathcal{O}(2) - E, \mathcal{O}(2) - E)$ in $P$, $E$ being the exceptional divisor, hence clearly $N_{C'/P} = 2\mathcal{O}(4)$ as desired.

For $n \geq 4$ we use induction on $n$ using a degenerated curve of the form $C = L \cup_p C_{n-1}$ where $C_{n-1}$ is a general rational normal curve in a hyperplane $H \subset \mathbb{P}^n$ and $A$ is a general 2-secant line to $C_{n-1}$ while $p$ is a general point on $C_{n-1}$ and $L$ is a general line through $p$. Let $C'_{n-1} \subset H' \subset P$ denote the proper transforms. By induction, we have

$$N_1 := N_{C'_{n-1}/H'} = 2\mathcal{O}(n) \oplus (n - 4)\mathcal{O}(n - 1),$$

hence

$$N_2 := N_{C'_{n-1}/P} = N_1 \oplus \mathcal{O}(n - 3)$$

where $N_1 \subset N_2$ is canonical but not the $\mathcal{O}(n - 3)$. Moreover, as in loc. cit. we have

$$N_{C'/P}|_{C'_{n-1}} = N_1 \oplus \mathcal{O}(n - 2)$$

and the image of $N_{C'_{n-1}/P}|_p \to N_{C'/P}|_p$ coincides with the image of $N_1$. On the other hand we have $N_{C'/P}|_L = \mathcal{O}(2) \oplus (n - 2)\mathcal{O}(1)$ and the upper subspace coming from the $\mathcal{O}(2)$ is clearly not in the image of $N_{L'/P} \to N_{C'/P}|_L$ at $p$, which coincides with the image of $N_{C'_{n-1}} \to N_{C'/P}|_{C_{n-1}}$ at $p$. The upshot is that, as in loc. cit. the $\mathcal{O}_{L'}(2)$ must be glued at $p$ to an $\mathcal{O}_{C'_{n-1}}(n - 2)$ and consequently we have

$$N_{C'/P} = 2\mathcal{O}(n + 1) \oplus (n - 3)\mathcal{O}(n),$$

as claimed.

Next consider the case $c + 1 \leq n \leq 2c - 1$ where we must show

$$N_{C'/P} = (2n - 2c)\mathcal{O}(n + 1) \oplus (2c - n - 1)\mathcal{O}(n).$$
Again the proof is by induction on $n$ fixing $c$, where the initial case $n = c + 1$ is where $A$ is a line which was just concluded. Thus assume $n > c + 1$ and consider a degenerated curve $C = C_{n-1} \cup_p L$ as above. Arguing as above we get
\[
N_{C'/P}|_{C_{n-1}'} = (2n - 2c - 2)\mathcal{O}(n) + (2c + 1 - n)\mathcal{O}(n - 1),
\]
\[
N_{C'/P}|_{L'} = \mathcal{O}(2) \oplus (n - 2)\mathcal{O}(1)
\]
where the $\mathcal{O}_L(2)$ must glue at $p$ to a general $\mathcal{O}_{C_{n-1}'}(n - 1)$ which implies $N_{C'/P}$ has the desired form.

Finally consider the case where $A$ has codimension $c$ with $n \geq 2c - 1$. Then the claim is
\[
N_{C'/P} = (n + 1 - 2c)\mathcal{O}(n + 1) \oplus (2c - 2)\mathcal{O}(n).
\]
Again we work by induction on $n$ where the initial case $n = 2c - 1$ is already known, so assume $n > 2c - 1$. Here a similar argument shows
\[
N_{C'/P}|_{C_{n-1}'} = (n - 2c)\mathcal{O}(n) \oplus (2c - 2)\mathcal{O}(n - 1),
\]
\[
N_{C'/P}|_{L'} = \mathcal{O}(2) \oplus (n - 2)\mathcal{O}(1)
\]
and again the $\mathcal{O}_{L'}(2)$ must glue at $p$ to a $\mathcal{O}_{C_{n-1}'}(n - 1)$, so we can conclude as above. This finally completes the proof of Case 1.

Note that what we have proven is equivalent to: if $C \subseteq \mathbb{P}^n$ is a rational normal curve with normal bundle $N \simeq (n - 1)\mathcal{O}(n + 2)$, $p, q \in C$ are general points, $A$ is a general linear space containing the line $\overline{pq}$, and $N'$ is the corresponding `$A$-modification', i.e.
\[
N' = \ker(N \to (N|_p/T_pA) \oplus (N|_q/T_q))) \subset N,
\]
then $N'$ is balanced.

**Case 2:** $e \geq 2n - 1$.

Notations as above, set
\[
n = n - 1 - ((n + 2)(n - 1) - 2(c - 1))\% (n - 1).
\]
Using a fang degeneration as in the first part of the proof, take a general $\mathbb{P}^\ell$ meeting the rational normal curve $C$ in 1 point and let $C_1 \subset P_1 = B_\ell\mathbb{P}^n$ be the birational transform of $C$; let $C_2 \subset P_2 = B_{\mathbb{P}^n - \ell - 1}\mathbb{P}^n$ be the birational transform of a general rational curve of degree $e - n + 1$, so that $C_1 \cap E = C_2 \cap E = \{y\}$ is 1 point, where $E$ is the exceptional divisor in $P_1$ and $P_2$. Then the appropriate $A$-modification of $N_{C_1/P_1}$ at $p, q$ (which is also a suitable modification of $N'$ above at $y$) is perfect, while $N_{C_2/P_2}$ is balanced. Then
\[
C_1 \cup_y C_2 \subset P_1 \cup E P_2
\]
smooths out to a rational curve of degree $e$ in $\mathbb{P}^n$ whose $A$-modification is balanced. This completes the proof of the Lemma.
Let \( P \) be balanced in \( B \). By choosing \( e \) for then the other equality in (11) is automatic. Let \( g_1 = g_2 = 0, e_1 + e_2 = e + 2 \), \( e_1, e_2 \geq 2n \) but with \( C_0 = C_1 \cup C_2, C_1 \cap E = C_2 \cap E = \{ p, q \} \).

\( e_1, e_2 \) and \( \ell \) are to be determined. By Lemma 31, we may assume each \( C_i \) is ultra-balanced in \( P_i \).

Let \( N = N_{C_0/P_0} \). Let \((u_1, ..., u_t)\) be any weight vector with each \( u_i \in [1, n - 1] \), such that

\[
\chi_1 + \cdots + u_t = \chi(N) = e(n + 1), e = e_1 + e_1 - 2, e_i = \deg(C_i).
\]

Let \( N^u = N^{(u_1, ..., u_t)} \). We will show \( H^0(N^u) = 0 \), so that \( N \) is \((u_1, ..., u_t)\)-balanced. Set

\[
N_i = N_{C_0/P_0} | C_i = N_{C_i/P_i},
\]

\[
\chi_i = \chi(N_{C_i} | P_i) = e_1(n + 1) + (n - 3), \quad \chi'_i = \chi(N_i) = e_1(n + 1) + (n - 3) - 2\ell_i, i = 1, 2,
\]

where \( \ell_1 = \ell - 1, \ell_1 = n - \ell \). Then

\[
u_1 + \cdots + u_t = \chi'_1 + \chi'_2 - 2(n - 1).
\]

**Lemma 32.** By choosing \( e_1, e_2 \) properly and relabeling \( u_1, ..., u_t \), we can arrange things so that

(11)

\[
u_1 + \cdots + u_s = \chi'_1 - (n - 1), u_{s+1} + \cdots + u_t = \chi'_2 - (n - 1).
\]

**Proof of Lemma.** It suffices to arrange that

\[
2\ell_1 = \chi_1 - (n - 1) - (u_1 + \cdots + u_s) = e_1(n + 1) - 2 - (u_1 + \cdots + u_s)
\]

for then the other equality in (11) is automatic. Let \( u_1 + \cdots + u_s \) be a maximal sub-sum that is \( \leq \chi_1 - (n - 1) = e_1(n + 1) - 2 \). Then

\[
\chi_1 - 2(n - 1) \leq u_1 + \cdots + u_s \leq \chi_1 - (n - 1),
\]

\[
\chi_2 - 2(n - 1) \leq u_{s+1} + \cdots + u_t \leq \chi_2 - (n - 1).
\]

If either \( d_1 := \chi_1 - (n - 1) - (u_1 + \cdots + u_s) \) or the analogous \( d_2 \) is even we can just set

\[
\ell_i = (\chi_1 - (n - 1) - (u_1 + \cdots + u_s))/2
\]
and (11) holds. Hence we may assume \( d_1 \) and \( d_2 \) are odd. Assume first that \( n \) is odd, hence we may also assume \( u_s \) is odd. If

\[
u_1 + \ldots + u_{s-1} \geq \chi_1 - 2(n-1)
\]

we may just replace \( s \) by \( s' = s - 1 \) and be done. If (12) fails we may replace \( e_1 \) by \( e'_1 = e_1 - 1 \) and \( e_2 \) by \( e'_2 = e_2 + 1 \) and then be done.

Now Assume \( n \) is even. If

\[
u_1 + \ldots + u_s \equiv e_1 \mod 2
\]

we can just let

\[
\ell_1 = (e_1(n + 1) - 2 - (u_1 + \ldots + u_s))/2.
\]

Otherwise, we just let \( e'_1 = e_1 + 1 \) and \( e'_2 = e_2 - 1 \) and work with \( e'_1, e'_2 \) instead. QED claim.

Now let \( \ell \in P, \ell = 1, 2 \) be the exceptional divisor (a copy of \( E \)). Then \( P_0 \) is constructed using an arbitrary isomorphism \( \phi : E_1 \to E_2 \) and I claim that by choosing \( \phi \) sufficiently general, we can ensure that

\[
H^0(N) = H^0(N_1 \cup_{p, q} N_2) = 0,
\]

i.e. no nonzero sections of \( N_1 \) and \( N_2 \) agree on \( p \) and \( q \). Now we have natural isomorphisms

\[
T_p E_i \cong N_i \mid_p \cong H^0(N_i) \cong N_i \mid_q \cong T_q E_i, i = 1, 2.
\]

It will suffice to choose the isomorphism \( \phi \), which may be identified as an arbitrary automorphism of \( \mathbb{P}^\ell \times \mathbb{P}^{n-1-\ell} \), so that the derivative \( d_p \phi - d_q \phi \) is nonsingular where \( d_p \phi : T_p E_1 \to T_p E_2 \) is the derivative and likewise for \( q \). By suitable identifications, we may assume \( d_p \phi \) is the identity \( I \) while \( d_q \phi \) is an arbitrary trace-0 matrix \( M \). Then clearly for suitable \( M \) (e.g. non-scalar diagonal), \( M - I \) is nonsingular. This completes the proof for genus 1.

Now for \( g > 1 \) we argue by induction, using a fang degeneration as above but with

\[
C_0 = C_1 \cup_p C_2 \subset P_0 = P_1 \cup P_2, g_1 = 1, g_2 = g - 1.
\]

Using notations as above, we let \( u = (u_1, \ldots, u_t) \) be any weight vector with \( \chi(N_u) = 0 \). We may assume \( C_1, C_2 \) are ultra-balanced and that \( p \) is general on \( C_1, C_2 \). An argument
similar to the proof of Lemma 32 above but simpler shows that we may assume by choosing the fang type (i.e. \(\ell\)) suitably that

\[
\chi(N_1^{(\mu_1, \ldots, \mu_s)}) = 0, \chi(N_2^{(\mu_{s+1}, \ldots, \mu_t)}) = n - 1.
\]

By ultra-balancedness we have first \(H^0(N_1^u) = 0\), then because \(\chi(N_2^u(-p)) = 0\), also \(H^0(N_2^u) = 0\).

\[+----------------------------------------+****************
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\(\square\)

3.3. Ambient-balanced. The analogue of Theorem 26 for ambient-balanced curves also holds:

**Theorem 33.** Let \(C_1, C_2\) be as in Theorem 26 and assume moreover

(i) \(C_1, C_2\) are ambient-balanced;

(ii) the ambient remainders \(r_1 = e_1 \% n, r_2 = e_2 \% n\) satisfy \(r_1 + r_2 < n\) (e.g. \(n \mid e_1\)).

Then

(i) there exists a smooth ambient-balanced curve \(C \subset \mathbb{P}^n\) of degree \(e_1 + e_2 - 1\), genus \(g_1 + g_2\) and ambient remainder \(r = r_1 + r_2\);

(ii) there exists a smooth ambient-balanced curve \(C' \subset \mathbb{P}^n\) of degree \(e_1 + e_2 - 2\), genus \(g_1 + g_2 + 1\) and ambient remainder \(r = r_1 + r_2\).

**Proof.** We follow the general outline of the proof of Theorem 26 but now taking \(C_1\) and \(C_2\) in the same \(\mathbb{P}^n\). By assumption \(t(N_{C_i}/\mathbb{P}^n) \geq 2, i = 1, 2\) so we may assume \(C_1 \cap C_2\) is exactly 1 general point (Case (i)) or 2 general points (Case (ii)). Then as in the above proof it follows that \(C_1 \cup C_2\) is smoothable in \(\mathbb{P}^n\). From Lemma 13 it follows that \(T_{\mathbb{P}^n|_{C_1 \cup C_2}}\) is semi-balanced, hence this is true for the smoothing as well. \(\square\)

**Corollary 34.** For all \(g \geq 0, n \geq 4\) and \(e \geq n + g(n - 2)\), there exists a balanced and ambient-balanced, hence moduli-interpolating curve of genus \(g\) and degree \(e\) in \(\mathbb{P}^n\).

**Proof.** The case \(g = 0\) is well known (balancedness by Sacchiero [11], ambient-balancedness e.g. by Lemma 26 of [8]), so assume \(g \geq 1\). By Corollary 28 there exists such a curve \(C'\) that is balanced. Using Theorem 33 with \(C_1\) a rational normal curve, it follows similarly using induction on \(g\) that there is such a curve \(C''\) that is ambient-balanced. Because \(C', C''\) are non-special, the family of curves of degree \(e\) and genus \(g\) in \(\mathbb{P}^n\) is irreducible, hence the general curve \(C\) in the family is balanced and ambient-balanced. \(\square\)

Finally, we will prove an analogue of Theorem 29 for ambient balanced curves.

**Theorem 35.** For \(e \geq 3g + n + 1, n \geq 3\), there exists an ultra ambient-balanced curve of degree \(e\) and genus \(g\) in \(\mathbb{P}^n\).
Using Theorem 29 we conclude

**Corollary 36.** For \( e \geq 2(g + 1)n, n \geq 3 \), a general curve of degree \( e \) and genus \( g \) in \( \mathbb{P}^n \) is ultra-balanced and ultra-ambient balanced.

**Proof of Theorem.** The proof is analogous to that of Theorem 29 and proceeds by induction on the genus. The case \( g = 0 \) follows from the fact that balanced = ultra balanced in genus 0.

We next take up the case \( g = 1, e \geq 3, n \geq 2 \), beginning with \( n = 2, e = 3 \). In this case what must be shown is that for a weight-vector \( u = (u_1, \ldots, u_t), u_i \in \{1, 2\}, |u| = 9 \), and for a general cubic \( C \), we have

\[
H^0(T_u^u|_{\mathbb{P}^2}) = 0.
\]

As this is an open property of \( C \) we may consider a reducible cubic \( C = C_2 \cup P_{p,q} \) with \( C_2 \) a conic and \( P \) a line. Then we have

\[
T_{\mathbb{P}^2}|_L = \mathcal{O}(2, 1), T_{\mathbb{P}^2}|_{C_2} = \mathcal{O}(3, 3).
\]

Now the weight vector \( u \) must have an odd number of components equal to 1, with the rest equal to 2, hence we may assume \( u = (u', u'') \) with \( |u'| = 5 \) and then \( H^0(T_{\mathbb{P}^2}^u|_L) = 0 \) and \( H^0(T_{\mathbb{P}^2}^u|_{C_2}(-p - q)) = 0 \). Consequently \( H^0(T_{\mathbb{P}^2}^u|_C) = 0 \), which proves the result for cubics in \( \mathbb{P}^2 \).

Next we will prove by induction on \( n \geq 2 \) that for a general cubic \( C \) in a plane in \( \mathbb{P}^n \), \( C \) is ultra ambient balanced in \( \mathbb{P}^n \). The proof is by induction on \( n \) with \( n = 2 \) already known so assume \( n \geq 3 \) and note that

\[
T_{\mathbb{P}^n}|_C = T_{\mathbb{P}^{n-1}}|_C \oplus L, L := \mathcal{O}(1)|_C.
\]

If \( u = (u_1, \ldots, u_t), u_i > 0 \) is a weight vector of weight \( |u| = 3(n + 1) \) then \( t \geq 3 \) so we can write \( u = u' + u'', u'' = (1, 1, 1, 0, \ldots, 0) \) and then

\[
H^0(T_{\mathbb{P}^n}^u|_C) = H^0(T_{\mathbb{P}^{n-1}}^u|_{C}) \oplus H^0(Lu'').
\]

Now the first summand vanishes by induction and the second by inspection. Thus \( T_{\mathbb{P}^n}|_C \) is ultra-balanced.

Next we consider the case of a general degree \( e \geq 3 \) and \( g = 1 \), working by induction on \( e \). Consider a curve of the form \( C_{e+1}^{t+1} = C_t^e \cup L \) where \( C_t^e \) is elliptic and \( L \) is a 1-secant line, and pick a weight vector \( u = (u_1, \ldots, u_t) \) with \( |u| = \chi(T_{\mathbb{P}^n}|_{C_{e+1}^{t+1}}) = (n + 1)(e + 1) \). Note that

\[
T_{\mathbb{P}^n}|_L = \mathcal{O}(2) \oplus (n - 1)\mathcal{O}(1).
\]
Write \( u = (u', u'') \) with \(|u'| \) maximal subject to \(|u'| \leq \chi(T_{\mathbb{P}^n}|_L) = 2n + 1 \), so that \(|u'| \geq n + 1 \) and also
\[(n + 1)e - n \leq u'' \leq (n + 1)e = \chi(T_{\mathbb{P}^n}|_{C^1}).\]
Write \( u' = (u_1, ..., u_s) \) and let the quotients \( U_1, ..., U_s \) be supported on \( L \). Then the restriction maps
\[
\rho_1 : H^0(T_{\mathbb{P}^n}|_{u'_1}) \to T_{\mathbb{P}^n}|_p, \rho_2 : H^0(T_{\mathbb{P}^n}|_{u''}) \to T_{\mathbb{P}^n}|_p
\]
are injective by inspection (resp. induction). Considering \( N_L(-1) \) trivialized, the quotients \( U_1, ..., U_s \) are general mod \( T_pL \) while \( T_pL \) itself may be chosen generally fixing \( C^1 \). Therefore the images of \( \rho_1, \rho_2 \) are in general position, i.e. complementary. Therefore \( H^0(T_{\mathbb{P}^n}|_{C^1+C^2}) = 0 \). This finally proves the Theorem for \( g = 1 \).

Now for \( g > 1 \) we argue by induction on \( g \) and can just copy over the last part of the proof of Theorem 29 using a fang curve
\[
C_1 \cup_p C_2 \subset P_1 \cup_E P_2
\]
with \( C_1 \) elliptic and \( C_2 \) of genus \( g - 1 \), and using the relative tangent bundle \( T_{\mathcal{P}(E)}/\mathbb{A}^1 \) discussed in \([22]\) instead of the relative normal bundle. \( C_1 \) and \( C_2 \) may be assumed ultra ambient-balanced in \( \mathbb{P}^n \) and consequently \( T_{\mathcal{P}(E)}(-\log E)|_{C^i}, i = 1, 2 \) is ultra-balanced as well. Appropriately distributing weights and degrees among \( C_1, C_2 \) as in the latter proof, it goes through essentially verbatim.

\[\square\]

4. CURVES IN ANTICANONICAL HYPERSURFACES

The purpose of this section is to prove our results constructing (ultra) balanced and ambient-balanced curves on anticanonical hypersurfaces. The construction is based on the following result:

**Theorem 37.** Suppose the exists a balanced (resp. ultra-balanced, resp. semi-balanced) curve of degree \( e_1 \) and genus \( g \) in \( \mathbb{P}^{n-1}, n \geq 4 \). Then for all \( e \) with \((n - 1)(e_1 - 1) \leq e \leq (n - 1)e_1 \) (resp. for \( e = (n - 1)e_1 \)), there exists a balanced (resp. ultra-balanced, resp. semi-balanced) curve of genus \( g \) and degree \( e \) on a general hypersurface of degree \( n \) in \( \mathbb{P}^n \).

**Proof.** We begin with the balanced and ultra-balanced cases. For \( g = 0 \) this is contained in Theorem 20 in \([13]\), and the proof for general \( g \) proceeds along similar lines, modulo the constructions of the last section for higher-genus curves in \( \mathbb{P}^n \).

Assume to begin with that \( C \subset \mathbb{P}^{n-1} \) is balanced (resp. ultra-balanced) of degree \( e_1 \) and genus \( g_1 \) as in Corollary 34. Write
\[
e = e_1(n - 1) - a, 0 \leq a \leq n - 1.
\]
We start with the same setup as in the proof of Theorem 26. Thus consider a fan
\[ \mathcal{P} = B_b(\mathbb{P}^n \times \mathbb{A}^1) \to \mathbb{A}^1 \]
with special fibre
\[ P_0 = P_1 \cup E, P_1 = B_b \mathbb{P}^n, P_2 = \mathbb{P}^n, E = \mathbb{P}^{n-1}. \]
Now in \( \mathcal{P} \) we consider a general relative hypersurface \( X \) of type \((n, n-1)\) with special fibre
\[ X_0 = X_1 \cup_F X_2 \]
where: \( X_1 \) is the blow up at \( b \in \mathbb{P}^n \) of a general hypersurface in \( \mathbb{P}^n \) of degree \( n \) and multiplicity \( n - 1 \) at \( b \), with exceptional divisor \( F \); and \( X_2 \) is a general hypersurface of degree \( n - 1 \) in \( \mathbb{P}^n \) with hyperplane section \( F \). Then, via projection from \( b \), \( X_1 \) is realized as \( \mathbb{P}^{n-1} \) blown up at a general \((n, n-1)\) complete intersection
\[ Y = F_{n-1} \cap F_n \]
where the exceptional divisor \( F \) becomes the birational transform of \( F_{n-1} \).

Now by the discussion in Case 1 of the proof of Theorem 20 of \cite{8}, which uses nothing about the genus of \( C \), we may assume \( Y \) meets \( C \) transversely in \( a \) general points \( p_1, ..., p_a \) and its tangents \( T_{p_i} Y \) yields general hyperplanes in the normal space \( N_{C_1}(p_i), i = 1, ..., a \). If \( C_1, F \) denotes the birational transform of \( C_1 \) resp. \( F_{n-1} \) in \( X_1 \), then \( N_{C_1/X_1} \) is a general down modification of \( N_{C_1/\mathbb{P}^{n-1}} \) at \( p_1, ..., p_a \), hence it is balanced by Lemma \cite{16} (resp. ultra-balanced by definition). Then set
\[ \{q_1, ..., q_e\} = C \cap F_{n-1} \setminus \{p_1, ..., p_a\} = C_1 \cap F \]
and
\[ C_0 = C_1 \cup (\bigcup_{i=1}^e L_i) \]
where \( L_i \) is a general line in \( X_2 \) through \( q_i \). Because \( N_{L_i/X_2} \) is a trivial bundle, it is easy to check that \( N_{C_0/X_0} \) is balanced (resp. ultra-balanced) around \( C_1' \). Therefore when \( (C_0, X_0) \) smooth out to a general \((C, X)\), \( X \) a general hypersurface of degree \( n \), the normal bundle \( N_{C/X} \) is likewise balanced (resp. ultra-balanced). This proves the assertion of the Theorem in the balanced and ultra-balanced cases.

Note that in the above argument, if \( C_1 \) is semi-balanced and \( a = 0 \), then \( C_0 \) is semi-balanced around \( C_1' \) hence its smoothing \( C \) is semi-balanced. This proves the assertion in the semi-balanced case. \( \Box \)

Now Theorem \cite{29} yields:

**Corollary 38.** For \( n \geq 4 \) a general hypersurface of degree \( n \) in \( \mathbb{P}^n \) contains ultra-balanced curves of genus \( g \) and degree \( e \) for all \( e \geq 2(g + 1)n(n - 1) \).
Remark 39. Trying to prove even semi-balancedness for $C_0$ when $e$ is not a multiple of $n - 1$ requires modifications of the normal bundle to $C_1$ and hence an assumption that $C_1$ be balanced, rather than weakly balanced.

A modification of this approach yields curves that are both balanced and ambient-balanced:

**Theorem 40.** A general hypersurface of degree $n$ in $\mathbb{P}^n$, $n \geq 4$, contains ultra-balanced and ultra ambient-balanced curves of degree $e$ and genus $g$ provided $g = 0, e \geq n - 1$ or $g \geq 1, e \geq 4g(n - 1)$.

**Proof.** We use the construction and notations in the proof of Theorem 37. Given Corollary 36, proving Theorem 40 is a matter of showing that the curves constructed in the latter proof may be assumed ultra ambient-balanced provided $C \subset \mathbb{P}^{n-1}$ is. We use the relative tangent bundle as discussed in §2, so the restricted tangent bundle $T_X|_C$ for a curve on $X$ specializes to

$$T_{X_1}(-\log E)|_{C_1} \cup T_{X_2}(-\log E)|_{C_2}, C_1 \cup C_1 \subset X_1 \cup X_2,$$

where $C_2 \subset X_2$ is a disjoint union of lines with trivial normal bundle. Now working as in Example 24 we modify the relative tangent bundle along $C_2$ so the specialized bundle becomes $T_{X_1}|_{C_1} \cup (n - 1)\mathcal{O}_{C_2}$. Then it is clearly sufficient to show that $C_1 \subset X_1$ is ultra ambient-balanced. But, with the above notations, $T_{X_1}|_{C_1}$ is a general corank-1 down modification of of the ultra-balanced bundle $T_{\mathbb{P}^{n-1}}|_C$ at $p_1, ..., p_a$, hence is ultra-balanced.

/******************** ****************************/
(ii) $X$ contains ambient-balanced curves $C$ of degree $e$ and genus $g$ provided there exists $e_0 \in [(g+1)n,e]$ such that

$$\left[\frac{-de_0 + e}{n-d}\right] + e = e_0 + \left[\frac{e_0}{d-1}\right]$$

In particular, given $g \geq 0$, there exist such ambient-balanced curves for infinitely many $e$.

For the ‘in particular’ portion of (i) see the Appendix by M. C. Chang below.

Remark 42. (i) Note that for $d > n/2$, eq. (13) already implies $e > e_0$.

(ii) In light of Example 21, it is not unreasonable to expect some obstructions in terms of $e$ to the existence of an ambient-balanced curve of degree $e$.

Example 43. Solving (14) is elementary. Write

$$e_0 = \alpha(d-1) + \beta, 0 \leq \beta < d-1, \alpha = \left[\frac{e_0}{d-1}\right],$$

$$e - de_0 = q(n-d) + r, 0 \leq r < n-d.$$  

Then an elementary calculation yields

$$d(d-2)\alpha + (d-1)\beta = (-q)(n-d+1) - r.$$  

This is solvable for $e$ iff

$$(d-1)e_0 - \left[\frac{e_0}{d-1}\right] \neq 1 \mod n-d+1.$$  

Explicitly, writing

$$(d-1)e_0 - \left[\frac{e_0}{d-1}\right] = u(n-d+1) + v, -(n-d) < v \leq 0,$$

the solution is

$$e = de_0 - u - v.$$  

Because $u \leq ((d-1)e_0 + n-d)/2$, clearly $e \to \infty$ as $e_0 \to \infty$ so there are infinitely many $e$ for given $n,d,g$.

Example 44. (M. C. Chang) For $d = n-1$, equation (13) reads

$$2e = ne_0 + \left[\frac{2e_0 + 2g - 2}{n-3}\right].$$

Write

$$g = x(d-2) + y, e_0 = (2k+r)(d-2) + c, 0 \leq y, c \leq d-3, r \in \{0,1\}.$$  

Then, setting $t = [(2c+2y-2)/(d-2)]$, we get

$$e = kd(d-1) + x + (t + r(d^2 - d + c(d+1)))/2.$$  

$e$ is an integer iff $t + c(d + 1)$ is even. Assuming $c > 0$, we have $t \in [0, 3]$. We try to count the ‘bad’ pairs $(c, r) \in [1, d - 3] \times [0, 1]$, i.e. those where $t + c(d + 1)$ is odd, with $y$ given. If $d$ is odd badness means $t$ is odd, i.e. $t \in \{1, 3\}$. The number of such $c$ is at most $d/2 - 1$. If $d$ is even badness means either $t \in \{1, 3\}, c$ even (at most $(d/2 - 1)/2$ solutions) or $t \in \{0, 2\}, c$ odd (again at most $(d/2 - 1)/2$ solutions). Thus if $d$ is either even or odd, there are at most $d/2 - 1$ bad $c$ values, hence the number of good pairs $(c, r)$ is at least $2(d - 3 - (d/2 - 1)) = d - 4$; i.e. there are at least $d - 4$ good congruence classes of $e_0 \mod 2(d - 2)$ hence at least $d - 4$ distinct arithmetical progressions for $e$ with difference $d(d - 1)$.

Similarly treating eq. (14) for $d = n - 1$ yields

$$e = (ne_0 + \left\lfloor \frac{e_0}{n-2} \right\rfloor)/2.$$  

When $n$ is even (resp. odd), this is an integer provided $\left\lfloor \frac{e_0}{n-2} \right\rfloor$ is even (resp. the remainder $e_0 \% (n - 2)$ is even). This leads to about $n - 2$ (resp. $(n - 3)/2$) arithmetic progressions of $e$ values with difference $n(n - 2)$ (resp. $(n - 1)^2/2$) for $n$ even (resp. odd). Note that the condition for (14) to hold is, in the above notations $2k + r \equiv c \mod d - 1$. This yields about $d - 4$ arithmetic progressions for $e$ with difference $d(d - 1)^2$.

*************

**Example 45.** When are the curves produced by Theorem 41 actually perfect? For perfect balance, it is a matter of replacing (13) by the ‘exact’ equation

$$(15) \quad \frac{-de_0 + e}{n-d} + e = e_0 + \frac{2e_0 + 2g - 2}{d-2}$$

together with the condition that both sides of (15) be integers. This is a sufficient condition that the curve $C$ is perfectly balanced. Assume first that $d$ is odd and write

$$(16) \quad e_0 = \lambda(d - 2) + 1 - g, \lambda \in \mathbb{Z}.$$  

Then the condition that (15) can be solved for an integer $e$ is

$$\lambda d(n - 2) + n(1 - g) \equiv 0 \mod n - d + 1$$

or equivalently

$$(17) \quad \lambda(n + 1)(n - 2) + n(1 - g) \equiv 0 \mod n - d + 1.$$  

At the upper end $d = n - 1$, $n$ even, (17) is automatic, so the curves produced by Theorem 41 are always perfectly balanced. A the lower end, if $d = 3$, eq. (17) becomes the condition $2 - 2g \equiv 0 \mod n - 2$. For $d > 3$ odd, (17) admits an arithmetic progression of solutions $\lambda$ (hence of $e$ values yielding perfectly balanced curves) provided

$$(d, n + 1) = 1 = (d - 3, n - 2).$$
For example when \( d = 5 \) this holds whenever \( n \) is odd and \( n \not\equiv 4 \pmod{5} \). Similarly analyzing the case \( d = 2d_0 \) even leads to
\[
(d^2 / 2 - 2d + 1)\lambda + (d - 1)(1 - g) \equiv 0 \pmod{n - d + 1}
\]
which admits an arithmetic progression of solutions \( \lambda \) provided
\[(d^2 / 2 - 2d + 1, n - d + 1) = 1.
\]
Similarly treating eq. (14), i.e. seeking \( C \) that is perfectly ambient-balanced, leads to
\[
e = \frac{(n - 1)d}{(d - 1)(n - d + 1)}e_0.
\]
This is solvable at least when \((d - 1)(n - d + 1)\) divides \( n_0 \), leading to at least one arithmetic progression of degrees for which there exists a perfectly ambient-balanced curve.

**Proof of Theorem.** The proof proceeds along similar lines as that of Theorem 31 of [8], using a relative fang. Thus let \( Z \to \mathbb{A}^1 \) be a relative fang of type \((n, m)\), \( m = d - 1 \geq 2 \), with special fibre
\[Z_0 = Z_1 \cup Z_2, Z_1 = \mathbb{P}_m(1, 0^{n-m}), Z_2 = \mathbb{P}_{m-1}(1, 0^{m-1}).\]
Let \( \mathcal{X} \subset Z \) be a general member of the linear system \(|dH - (d - 1)Z_2|\) where \( H \subset \mathbb{P}^n \) is a hyperplane. The \( \mathcal{X} \to \mathbb{A}^1 \) has special fibre
\[X_0 = X_1 \cup_E X_2.
\]
Here \( X_1 = \mathbb{P}_m(G) \) where \( G \) is a bundle on \( \mathbb{P}^m \) that fits in an exact sequence
\[
0 \to \mathcal{O}(-m) \to \mathcal{O}(1) \oplus (n - m)\mathcal{O} \to G \to 0
\]
in which the left map is general. Also \( X_2 \) fibres over \( \mathbb{P}^{n-m-1} \) with general fibre a general hypersurface of degree \( d - 1 = m \) in \( \mathbb{P}^{m+1} \). As in the above-referenced proof, we will construct a balanced curve in \( X_0 \) of the form \( C_1 \cup C_2 \) where \( C_1 \subset X_1 \) is balanced and \( C_2 \subset X_2 \) is a disjoint union of lines in fibres of \( X_2 \to \mathbb{P}^{n-m-1} \) and as such has trivial hence balanced normal bundle. Then \( X_0 \) will smooth along with \( Z_0 \) to a balanced curve in the general fibre of \( \mathcal{X} \to \mathbb{A}^1 \). It will suffice to construct \( C_1 \).

To this end, proceeding as in [8], proof of Theorem 31, we will start with a balanced curve \( C_0 \subset \mathbb{P}^m \) of genus \( g \) and degree \( e_0 \) and lift it to \( C_0 \simeq C_1 \subset \mathbb{P}(G) = X_1 \) using a general surjection
\[
\psi : G_{C_0} \to M
\]
where \( M = \mathcal{O}_{C_0}(H + A) \) with \( L = \mathcal{O}(H) \) being the hyperplane bundle from \( \mathbb{P}^m \) and \( A \) is a general effective divisor \( A \) of degree \( e - e_0 \), \( e_0 = \deg(L) \), which also coincides with \( C_1.E \). Such a map \( C_1 \to X_1 \) comes from a map \( \phi : C \to \mathbb{P}^n \) corresponding to \( n + 1 \)
sections of $L$ among which $m + 1$ vanish on $A$, and can be constructed by starting from $C_0 \to \mathbb{P}^m$ corresponding to $m + 1$ sections of $L$ and adding $n - m$ additional sections of $M = L(A)$.

Now setting $K = \ker(\psi)$, the vertical part of the normal bundle $N_{C_1/\mathbb{P}(G)}$ is $K^*(M)$, i.e. we have an exact normal sequence
\begin{equation}
0 \to K^*(M) \to N_{C_1/\mathbb{P}(G)} \to N_{C_0/\mathbb{P}^m} \to 0
\end{equation}
and the relation (13) means exactly that the slope matching condition of Lemma 18 and [8], eq. (10) holds. Thus will suffice to prove as in [8] that $K^*(M)$ is balanced. For $g = 0$ this is proved in [8], Lemma 33. In the general case we will use induction on $g$, starting with a reducible form of $C_0$ of the form
\begin{equation}
C_{00} = C_{01} \cup_{p,q} C_{02} \subset \mathbb{P}^m
\end{equation}
where $C_{01}$ is a rational normal curve (of degree $m$, $C_{02}$ is a balanced curve of genus $g - 1$ and degree $e_{02} \geq m + (g - 1)(m - 2)$ (see Corollary 34) and $p, q$ are general points. We then lift $C_{00}$ to
\begin{equation}
C_{10} = C_{11} \cup_{p,q} C_{12} \subset X_1
\end{equation}
using the surjection $\psi : G_{C_{00}} \to M_0$ to a line bundle of degree $e$ of the form $O_{C_0}(H + A_0)$ as above. We choose the line bundle $M_0$ on $C_{00}$ so that
\[ e_1 := \deg(M_0|_{C_{01}}) \equiv d(d - 1) \mod n - d, e_1 \geq m, e_2 := \deg(M_0|_{C_{02}}) \geq (g - 1)n \]
and
\[ e_1 + e_2 = e. \]
We may assume $e_1 \leq 2n$. Now we have analogues of the sequence (20) for $C_{11}, C_{12}$ and inductively both left and right members in those sequences have Euler slope $\geq 2$, and it follows that
\[ H^1(N_{C_{i1}/X_1}(-p - q)) = 0, i = 1, 2. \]
Because $N_{C_{10}/X_1}$ contains $N_{C_{i1}/X_1}(-p - q) \oplus N_{C_{i2}/X_1}(-p - q)$ as a subsheaf parametrizing deformations where $C_{i1}$ and $C_{i2}$ deform separately going through $p, q$, it follows easily that $C_{10}$ is smoothable in $X_1$ to a curve of genus $g$ and degree $e = e_1 + e_2$. Now the bundle $K^*(M)$ restricts to the analogous bundles on $C_{i1}, i = 1, 2$ which are balanced by induction and perfect for $i = 1$ by the congruence condition on $e_1$. Moreover as noted the Euler slope of $K^*(M)|_{C_{11}}$ is clearly at least $2$. Hence by Lemma 13 it follows that $K^*(M)$ is balanced on $C_{10}$, hence on its smoothing in $X_1$.

Finally for ambient-balancedness, we argue as in the proof of Theorem 40, noting that here again $C_2$ is a union of lines $L$ with trivial normal bundle, hence
\[ T_{X_2}(-\log E)|_L = O(1) \oplus (n - 2)O_L \]
where the \((n - 2)\mathcal{O}_L\) quotient coincides at \(p = L.C_1\) with \(T_{p,E}\). Moreover \(C_2 \cap C_1 = A\) is a general divisor on \(C_1\). As in the above proof, it will suffice to prove that \(T_{X_1}|_{C_1}\) is balanced. Note the exact sequence

\[
0 \rightarrow K^*(M) \rightarrow T_{\mathbb{P}(G)}|_{C_1} \rightarrow T_{\mathbb{P}^m}|_{C_1} \rightarrow 0,
\]

which identifies \(K^*(M)\) as the relative tangent bundle \(T_{X_1}/\mathbb{P}^m\).

Now (14) ensures that the slopes of \(K^*(M)\) and \(T_{\mathbb{P}^m}|_{C_1}\) have the same roundoff, so by Lemma [18] it will suffice to show \(K^*(M)\) and \(T_{\mathbb{P}^m}|_{C_1}\) are balanced. As for \(T_{\mathbb{P}^m}|_{C_1}\), it may be assumed balanced thanks to Corollary [34]. As for \(K^*(M)|_{C_1}\), we will use induction on \(g\). First for \(g = 0\), it is proven in [8], Lemma 33, p. +35, that \(K|_{C_1}\) is balanced, hence so is \(K^*(M)|_{C_1}\). Then the general case is proven by degeneration to \(C_{10} = C_{11} \cup C_{12}\) similarly to the above where \(K^*(M)|_{C_{11}}\) is perfect.

\[\square\]

Remark 46. The ultra version of the Matching Lemma [18] is not known. Therefore neither is the ultra version of Theorem [41].

Remark 47. There is a misprint in the proof of Lemma 33 in the journal version of [8] (p.+35, l.-11). The arxiv version is correct.

6. APPENDIX ON DEGREE ARITHMETIC

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In this appendix, we prove the following

**Theorem A.1.** Let \(3 \leq d \leq n - 1\), and \(g > 0\) be integers. Then the set of integers \(e\) such that for some integer \(e_0\), \(e \geq e_0 \geq (g + 1)n\), one has

\[
\left\lfloor \frac{-de_0 + e}{n - d} \right\rfloor + e = e_0 + \left\lfloor \frac{2e_0 + 2g - 2}{d - 2} \right\rfloor.
\]

contains at least one arithmetic progression with difference \(d(n - 2)\).

**Remark** In some cases the proof will actually yield more than 1 arithmetic progression. See Remark A.2 below.

**Proof.**

We write

\[\text{Research partially financed by the NSF Grants DMS 1764081.}\]
\[ g = x(d-2) + y, \text{ where } y \in [0, d-3], \quad (A.2) \]

and denote
\[ b = n - d + 1. \quad (A.3) \]

For \((c, r) \in [0, d-3] \times [0, b-1]\), let
\[ t = \left\lfloor \frac{2c + 2y - 2}{d-2} \right\rfloor, \quad (A.4) \]

and
\[ I = \frac{rd(d-3) + c(d-1) - 2x - t + A}{b}, \text{ with } A \in [0, b-2]. \quad (A.5) \]

Assume \(I \in \mathbb{Z}\).

For any \(k \in \mathbb{Z}^+_0\), let
\[ e_0 = (kb + r)(d-2) + c, \quad (A.6) \]

and
\[ e = kd(n - 2) + 2x + t + rd + c + I. \quad (A.7) \]

Substituting (A.2)-(A.7) in (A.1), with some work, we have both sides equal to
\[ kbd + rd + 2x + t + c. \]

Hence, \(e_0\) and \(e\) satisfy equation (A.1).

Next, we want to find \((c, r, A) \in [0, d-3] \times [0, b-1] \times [0, b-2]\) such that \(I \in \mathbb{Z}\), which is equivalent to
\[ rd(d-3) + c(d-1) - 2x - t + A \equiv 0 \mod b. \]

Namely,
\[ (d-1)c \equiv 2x + t - A - d(d-3)r \mod b. \quad (A.8) \]

Case (a). \(2x + t - d(d-3) \neq b - 1 \mod b\).

Let \((c, r) = (0, 1)\) in (A.8). We have
\[ A \equiv 2x + t - d(d-3) \mod b. \]

We can take \(A \in [0, b-2]\) and \(A \equiv 2x + t - d(d-3) \mod b\).
Our solution to (A.1) is
\[ e_0 = (kb + 1)(d - 2), \quad \text{and} \quad e = kd(n - 2) + 2x + t + d + I, \]
where
\[ I = \frac{d(d - 3) - 2x - t + A}{b}. \]

We will use the following fact about the solvability of congruence equations.

Fact. The congruence equation
\[ ax \equiv d \pmod{b} \]
is solvable if and only if \( g := \gcd(a, b) \) divides \( d \).

Case (b). \( 2x + t - d(d - 3) \equiv b - 1 \pmod{b} \).

We write (A.8) as
\[ d(d - 3)(r - 1) \equiv b - 1 - A - (d - 1)c \pmod{b}. \quad (A.9) \]
Let
\[ \alpha = \gcd(d(d - 3), b). \]

Case (b.1). \( \alpha < b \).
Let \( c = 0 \). We have
\[ d(d - 3)(r - 1) \equiv b - 1 - A \pmod{b}. \quad (A.10) \]
We can take \( A = b - 1 - \alpha \). By Fact, (A.10) has a solution \( r \in [0, b - 1] \).
Our solution to (A.1) is
\[ e_0 = (kb + r)(d - 2), \quad \text{and} \quad e = kd(n - 2) + 2x + t + rd + I, \]
where
\[ I = \frac{rd(d - 3) - 2x - t + b - 1 - \alpha}{b}. \]

Case (b.2). \( \alpha = b \). Namely,
\[ b | d(d - 3). \quad (A.11) \]
To solve (A.9) under condition (A.11) is to solve \( c \in [0, d - 3] \) in
\[ (d - 1)c \equiv b - 1 - A \pmod{b}. \quad (A.12) \]
Let
\[ \beta = \gcd(d - 1, b). \]

If we can choose \( A \in [0, b - 2] \) such that \( \beta | (b - 1 - A) \), then there is a solution \( c \in [0, b - 1] \). Furthermore, if \( b - 1 \leq d - 3 \), then this \( c \) is the solution. So we may assume
\[ d - 3 < b - 1. \]  

(A.13)

It is easy to see that
\[ \beta \neq b. \]  

(A.14)

Otherwise, we take a prime factor \( p \) of \( b \). From (A.11), we have either \( p | d \) and \( p | (d - 1) \), or \( p | (d - 3) \) and \( p | (d - 1) \). The former is clearly impossible. The latter gives \( p = 2, b = 2^m \), and \( d = 2^m X + 1 \). Since by (A.13), \( b \geq d - 1 \), we have \( b = 2, d = 3 \).

We finish the proof by the following two cases.

Case (b.2.i). \( b - 1 \leq d - 3 \).

Since \( \beta < b \), in (A.12) we can take \( A = b - 1 - \beta \in [0, b - 2] \). So (A.12) has a solution \( c \in [0, b / \beta - 1] \subset [0, b - 1] \subset [0, d - 3] \).

Case (b.2.ii). \( d - 1 \leq b \).

We note that by (A.11), \( d - 1 \neq b \). so we can choose \( c \) such that \( c(d - 1) \leq b - 1 \). Then if
\[ b \neq d(d - 3), \]

(A.11) implies
\[ c \leq \frac{1}{d - 1}\left(\frac{d(d - 3)}{2} - 1\right) \leq d - 3. \]

If \( b = d(d - 3) \), to solve the congruence equation
\[ (d - 1)c \equiv -1 - A \mod d(d - 3), \]
we can take \( c = d - 3 \) and \( A = d - 4 \in [0, d(d - 3) - 2] \).

Remark A.2. From our proof and Lemma A.3 below, we have the following minimal number of arithmetic progressions (AP) with difference \( d(n - 2) \) for each case.

Case (a). \( 2x + t - d(d - 3) \neq b - 1 \). There exists one AP represented by \( (0, 1, 2x + t - d(d - 3)) \).

Case (b). \( 2x + t - d(d - 3) \equiv b - 1 \).

Let \( \alpha = \gcd(d(d - 3), b) \).
Case (b.1). $\alpha < b$. There exist $\alpha$ many AP represented by $(0, r, b - 1 - \alpha)$, for some $r \in \left[0, \frac{b}{\alpha}\right]$.

Case (b.2). $\alpha = b$. i.e. $b|d(d - 3)$.

Let $\beta = \gcd(d - 1, b)$.

Case (b.2.i). $b - 1 \leq d - 3$. There exist $\beta b$ many AP represented by $(c, r, b - 1 - \beta)$, where $c \in \left[0, \frac{d - 3}{b}\right], r \in [0, b - 1]$.

Case (b.2.ii). $b \geq d - 1$.

If $b \neq d(d - 3)$. There are $b \left(\frac{b - 1}{d - 3}\right)$ many AP represented by $(c, r, A)$, where $c$ satisfies $c(d - 1) \leq b - 1$, and any $r \in [0, b - 1]$.

If $b = d(d - 3)$. There are $b$ many AP represented by $(d - 3, r, d - 4)$ with any $r \in [0, b - 1]$.

Presumably, using Lemma A.3 and Lemma A.4, one may give a much better estimate of the numbers of AP in Remark A.2.

**Lemma A.3.** Let

$$e = e(c, r, A) = kd(n - 2) + 2x + t + rd + c + I$$

be defined as in (A.7).

If $(c, r, A) \neq (c_1, r_1, A_1)$, then $e(c, r, A) \neq e(c_1, r_1, A_1) \mod d(n - 2)$.

**Proof.** Let

$$E(c, r, A) = 2x + t + rd + c + I.$$

Claim 1. $E(c, r, A) \neq E(c_1, r_1, A_1)$ as real numbers.

**Proof of Claim 1.**

First, we assume $r_1 - r \geq 1$, and $E(c, r, A) = E(c_1, r_1, A_1)$. Then

$$(r_1 - r) \left( d + \frac{d^2 - 3d}{b} \right) = (c - c_1) \left( 1 + \frac{d - 1}{b} \right) + (t - t_1) \left( 1 - \frac{1}{b} \right) + \frac{1}{b}(A - A_1). \quad (A.16)$$

By Lemma A.4 below, $t - t_1 \leq 2$. Also, in (A.4) and (A.5), we take $c \in [0, d - 3]$, and $A \in [0, b - 2]$. Hence, the right hand side of (A.16) is at most

$$(d - 3) \frac{b + d - 1}{b} + 2 \frac{b - 1}{b} + \frac{b - 2}{b},$$

which is less than

$$d + \frac{d^2 - 3d}{b} - \frac{d}{b} < d + \frac{d^2 - 3d}{b} \leq \text{the left hand side of (A.16)}.$$
This is a contradiction. Hence, \( r_1 = r \) and (A.16) is

\[
0 = (c - c_1) \left( 1 + \frac{d - 1}{b} \right) + (t - t_1) \left( 1 - \frac{1}{b} \right) + \frac{1}{b} (A - A_1). \tag{A.17}
\]

Next, we assume \( c_1 - c \geq 1 \). From the definition of \( t \) in (A.4), \( t_1 \geq t \), and we have

\[
b + d - 1 \leq (c - c_1)(b + d - 1) + (t - t_1)(b - 1) = A_1 - A \leq b - 2,
\]

which is a contradiction.

**Claim 2.** If \( E(c, r, A) \not\equiv E(c_1, r_1, A_1) \), then

\[
E(c, r, A) \not\equiv E(c_1, r_1, A_1) \mod d(n - 2).
\]

**Proof of Claim 2.** Assume \( E(c, r, A) \not\equiv E(c_1, r_1, A_1) > 0 \). Then

\[
E(c_1, r_1, A_1) - E(c, r, A) \leq (b - 1) \frac{bd + d^2 - 3d}{b} + (d - 3) \frac{b + d - 1}{b} + 2 \frac{b - 1}{b} + \frac{b - 2}{b} - d(n - 2).
\]

Hence \( E(c_1, r_1, A_1) - E(c, r, A) \) cannot be a multiple of \( d(n - 2) \). \( \square \)

For Lemma A.4, we need the following notations. For an integer \( y \in [0, d - 3] \), define the following (pairwise disjoint) integer intervals

\[
C_0 = \left[ 0, \frac{d}{2} - y \right] \cap \mathbb{Z} \cap [0, d - 3],
\]

\[
C_1 = \left[ \frac{d}{2} - y, d - y - 1 \right] \cap \mathbb{Z} \cap [0, d - 3],
\]

\[
C_2 = \left[ d - y - 1, \frac{3d}{2} - y - 2 \right] \cap \mathbb{Z} \cap [0, d - 3],
\]

\[
C_3 = \left[ \frac{3d}{2} - y - 2, 2d - y - 3 \right] \cap \mathbb{Z} \cap [0, d - 3].
\]

**Lemma A.4.** Let

\[
t = t(y, c) = \left\lfloor \frac{2c + 2y - 2}{d - 2} \right\rfloor,
\]

be as in (A.4). Then
∀i, if c ∈ C_i, then t(y, c) = i.
(ii) If 2 ≤ y < d/2, then C_3 = ∅.
(iii) If y ∈ [0, 1], then C_2 = C_3 = ∅.
(iv) If y ∈ [d/2 + 1, d − 3] then C_0 = ∅.
(v) If y = d/2 then C_0 = C_3 = ∅.

Lemma A.5.
(i) C_0 ≠ ∅ if and only if y < d/2.
(ii) C_2 ≠ ∅ if and only if y ≥ 2.
(iii) |C_1|, |C_2| ∼ d/2 − 1, ∑ |C_i| = d − 3, and the C_i are pairwise disjoint.
(iv) C_1 ≠ ∅.
(v) C_3 ≠ ∅ if and only if y ≥ d/2 + 1.

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