Minimum tree-stretch of Hamming graphs and higher-dimensional grids

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Abstract: The minimum stretch spanning tree problem for a graph $G$ is to find a spanning tree $T$ of $G$ such that the maximum distance in $T$ between two adjacent vertices is minimized. The minimum value of this optimization problem gives rise to a graph invariant $\sigma_T(G)$, called the tree-stretch of $G$. The problem has been studied in the algorithmic aspects, such as NP-hardness and fixed-parameter solvability. This paper presents the exact values $\sigma_T(G)$ of the Hamming graphs $K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}$ and the higher-dimensional grids $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$.

Keywords: spanning tree optimization, tree-stretch, tree-congestion, Hamming graphs, higher-dimensional grids.

1 Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Then $G$ contains a spanning tree $T$. For $uv \in E(G)$, let $d_T(u, v)$ denote the distance between $u$ and $v$ in $T$. The max-stretch of a spanning tree $T$ is defined by

$$\sigma_T(G, T) := \max_{uv \in E(G)} d_T(u, v).$$

The minimum stretch spanning tree problem (the MSST problem for short) is to find a spanning tree $T$ such that $\sigma_T(G, T)$ is minimized, where the minimum value

$$\sigma_T(G) := \min \{ \sigma_T(G, T) : T \text{ is a spanning tree of } G \}$$

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is called the tree-stretch of \( G \) (following the terminology of \[15\] and the notation \( \sigma_T(G) \) of \[10\]). A spanning tree \( T \) attaining this minimum value is called an optimal spanning tree.

With applications in distribution systems and communication networks, a series of tree spanner problems were intensively studied in the literature (see \[22, 7, 15\], etc). A basic decision version of these problems can be stated as follows: For a given integer \( k \), is there a spanning tree \( T \) of \( G \) (called a tree \( k \)-spanner) such that the distance in \( T \) between every pair of vertices is at most \( k \) times their distance in \( G \)? The MSST problem mentioned above is the optimization version of this decision problem. Moreover, it is worth pointing out that this graph embedding problem can be regarded as a variant of the bandwidth (dilation) problem when graph \( G \) is embedded into its spanning tree \( T \) (see survey \[9\]).

In the algorithmic aspect, the MSST problem has been proved NP-hard (see \[5, 6, 7, 10, 12\]), and fixed-parameter polynomial algorithms were discussed in details (see \[5, 6, 10, 11\]). Several exact results of \( \sigma_T(G) \) for special graphs were also investigated. For example, the characterization of \( \sigma_T(G) = 2 \) was given in \[2, 7\]. Besides, \( \sigma_T(G) \leq 3 \) for interval, split, and permutation graphs were showed in \[5, 14, 18, 23\]. Some formulas for basic families of special graphs, such as complete \( k \)-partite graphs \( K_{n_1,n_2,\ldots,n_k} \), rectangular grids \( P_m \times P_n \), torus grids \( C_m \times C_n \), triangular grids \( T_n \) and hypercubes \( Q_n \), can be seen in \[16, 17\].

The MSST problem has close relations to the minimum congestion spanning tree problem \[20\], which is to find a spanning tree \( T \) of \( G \) such that the size of the maximum fundamental edge-cut is minimized. The problem has been proved NP-hard in \[19\], and fixed-parameter polynomial algorithms were presented in \[4\]. Much interest was paid to the exact results for special graphs (see, e.g., \[3, 8, 13, 21\]). These results motivate our study on the MSST problem for typical graphs.

The embedding problems of Hamming graphs \( K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d} \) have significant applications in error-correcting code and multichannel communication. Even so, the bandwidth problem of Hamming graphs was a long-standing open problem in this field (see \[9\]). It is also unsolved for the minimum congestion spanning tree problem. On the other hand, various grid graphs are also appealing in graph embedding. For example, the minimum tree-congestion and the minimum tree-stretch of two-dimensional grids \( P_m \times P_n \) have been determined in \[13, 8, 16\], but the results for higher-dimensional grids \( P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d} \) are unknown yet. The goal of this paper is to determine the exact value \( \sigma_T(G) \) for the Hamming graphs and the higher-dimensional grids.

The paper is organized as follows. In Section 2, some definitions and elementary properties are introduced. In Section 3, we are concerned with the Hamming graphs. Section 4 is devoted to the higher-dimensional grids. We give a short summary in Section 5.
2 Preliminaries

We shall follow the graph-theoretic terminology and notation of [1]. Let $G$ be a simple connected graph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. For an edge $e \in E(G)$, denote by $G - e$ the graph obtained from $G$ by deletion of $e$. For an edge $e$ not in $E(G)$, denote by $G + e$ the graph obtained from $G$ by addition of $e$.

Let $P_n, C_n, K_n$ denote the path, the cycle, the complete graph, respectively, on $n$ vertices. The cartesian product of two graphs $G$ and $H$, denoted $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(u, v)$ and $(u', v')$ are adjacent if and only if either $[u = u'$ and $vv' \in E(H)]$ or $[v = v'$ and $uu' \in E(G)]$.

Let $T$ be a spanning tree of $G$. Usually, the spanning tree $T$ is regarded as a set of edges. The cotree $\overline{T}$ of $T$ is the complement of $T$ in $E(G)$, namely $\overline{T} = E(G) \setminus T$. For an edge $e \in \overline{T}$, the unique cycle in $T + e$ is called the fundamental cycle with respect to $e$. Moreover, the detour for an edge $uv \in E(G)$ is the unique $u-v$ path in $T$, denoted by $P_T(u, v)$. For $uv \in \overline{T}$, the fundamental cycle with respect to $uv$ is indeed the detour $P_T(u, v)$ plus edge $uv$. So, the MSST problem is equivalent to a problem of finding a spanning tree such that the length of maximum fundamental cycle is minimized, where the length of the longest cycle) in the cycle space.

Let $\sigma_T(G)$ be the vertex set of one of these components. Then $\partial(\chi_e) := \{uv \in E(G) : u \in \chi_e, v \notin \chi_e\}$ is called the fundamental edge-cut (or bond) with respect to the tree-edge $e$. Here, $|\partial(\chi_e)|$ is called the congestion of edge $e$. The minimum congestion spanning tree problem is to find a spanning tree $T$ of $G$ such that the maximum congestion in $T$ is minimized. This is an optimal basis problem in the cocycle space.

The duality relation of the above problems lies on the following fact: For $e' \in \overline{T}$, $e$ is contained in the fundamental cycle with respect to $e'$ if and only if $e'$ is contained in the fundamental edge-cut $\partial(\chi_e)$ with respect to $e$ (see [1]).

The following observation is immediate.

**Proposition 2.1** For a spanning tree $T$ of $G$, let $D(T)$ be the diameter of $T$ (i.e., the maximum distance between any two vertices of $T$). Then

$$\sigma_T(G, T) \leq D(T).$$

**Proof:** This is because for any $uv \in E(G)$, $d_T(u, v) \leq D(T)$. \(\square\)

In a Hamming graph $G = K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}$, each vertex can be represented by a $d$-dimensional vector $v = (x_1, x_2, \ldots, x_d)$ with $x_i \in \{0, 1, \ldots, n_i - 1\}$, $n_i \geq 2$ for $1 \leq i \leq d$.  

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Two vertices \( v = (x_1, x_2, \ldots, x_d) \) and \( u = (y_1, y_2, \ldots, y_d) \) are adjacent if they differ in exactly one coordinate. An illustration is shown in Figure 1.

![Illustration of Hamming graphs](image)

When \( n_1 = n_2 = \cdots = n_d = 2 \), \( Q_d = K_2 \times K_2 \times \cdots \times K_2 \) is called a \( d \)-dimensional hypercube or \( d \)-cube, whose vertex set is the set of all \( d \)-dimensional 0-1 vectors \( (x_1, x_2, \ldots, x_d) \) and two vertices are adjacent if they differ in exactly one coordinate.

In a \( d \)-dimensional grid \( P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d} \), each vertex can be represented by a \( d \)-dimensional vector \( v = (x_1, x_2, \ldots, x_d) \) with \( x_i \in \{1, 2, \ldots, n_i\} \), \( n_i \geq 2 \) for \( 1 \leq i \leq d \). Two vertices \( v = (x_1, x_2, \ldots, x_d) \) and \( u = (y_1, y_2, \ldots, y_d) \) are adjacent if they differ by 1 in exactly one coordinate. The illustration of higher-dimensional grids is similar to Figure 1 with \( K_n \) replaced by \( P_n \) for \( 1 \leq i \leq d \).

### 3 Hamming graphs

On the one hand, we shall show the upper bound of \( \sigma_T(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}) \).

**Lemma 3.1** For the Hamming graphs \( G = K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d} \) \((2 \leq n_1 \leq n_2 \leq \cdots \leq n_d)\), it holds that

\[
\sigma_T(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}) \leq \begin{cases} 2d - 1, & \text{if } n_1 = 2 \\ 2d, & \text{if } n_1 \geq 3. \end{cases}
\]

**Proof:** We first consider \( d = 2 \) and \( G = K_{n_1} \times K_{n_2} \). Suppose that \( V(G) := \{(i, j) : 0 \leq i \leq n_1 - 1, 0 \leq j \leq n_2 - 1\} \) and \((i, j)\) is adjacent to \((i', j')\) if \( i = i' \) or \( j = j' \). We may call \( R_i := \{(i, j) : 0 \leq j \leq n_2 - 1\} \) the \( i \)-th row for \( 0 \leq i \leq n_1 - 1 \), and \( C_j := \{(i, j) : 0 \leq i \leq n_1 - 1\} \) the \( j \)-th column for \( 0 \leq j \leq n_2 - 1 \), each of which is a clique. We construct a spanning tree \( T_2 \) as follows. First, take a star (regarded as \( T_1 \)) in each row \( R_i \) with the center at column \( C_1 \) \((0 \leq i \leq n_1 - 1)\). Then, take a star in column \( C_1 \) with center \( x_0 = (0, 0) \) to join the centers of stars in rows. An example is shown in Figure 2. For \( n_1 = 2 \), \( T_2 \) is a tree of diameter three (double star), and so by
Proposition 2.1, we have $\sigma_T(G) \leq \sigma_T(G, T_2) \leq 3$. For $n_1 \geq 3$, $T_2$ is a tree of diameter four (each leaf has distance at most two from $x_0$). By Proposition 2.1, it follows that $\sigma_T(G) \leq \sigma_T(G, T_2) \leq 4$. Therefore, by means of this spanning tree $T_2$, we have

$$\sigma_T(G) \leq \begin{cases} 
3, & \text{if } n_1 = 2 \\
4, & \text{if } n_1 \geq 3,
\end{cases}$$

and so the assertion is true for $d = 2$.

We proceed to construct a spanning tree $T_d$ by induction on $d$. For the spanning tree $T_2$ before, we call the vertex $x_0$ the center of $T_2$ and denoted $x_0^2$ henceforth. Assume that $d \geq 3$ and $T_{d-1}$ has been constructed. The graph $G$ now consists of $n_1$ copies of $K_{n_2} \times \cdots \times K_{n_d}$, each of which has a required spanning tree $T_{d-1}$. We construct a spanning tree $T_d$ of $G$ by joining a star between $n_1$ centers of the copies $T_{d-1}$; and let $x_0^d$ be the center of this star. Then $x_0^d$ is called the center of $T_d$.

This spanning tree $T_d$ has the property that each leaf of $T_d$ has distance at most $d$ from the center $x_0^d$. In fact, this is trivially true for $d = 1, 2$ (see Figure 2). If it is true for $T_{d-1}$, that is, each leaf has distance at most $d - 1$ from the center $x_0^{d-1}$, then, since $x_0^{d-1}$ and $x_0^d$ have distance one, the property follows for $T_d$.

To show the upper bound, we first consider the case $n_1 = 2$. There are only two copies of $T_{d-1}$ in $G$. Then $T_d$ is obtained by joining an edge between the centers of these two copies of $T_{d-1}$. Since each leaf of $T_{d-1}$ has distance at most $d - 1$ from the center, it follows that the diameter of $T_d$ is at most $2(d - 1) + 1 = 2d - 1$. By Proposition 2.1, we see that $\sigma_T(G) \leq \sigma_T(G, T_d) \leq D(T_d) \leq 2d - 1$.

We next consider the case $n_1 \geq 3$. Now, $T_d$ is constructed as follows: Among the $n_1$ centers of copies $T_{d-1}$, we choose one as the center $x_0^d$ of $T_d$, and join a star connecting to the other $n_1 - 1$ centers of copies $T_{d-1}$ (as the star in column $C_1$ of Figure 2). Since each leaf of $T_{d-1}$ has distance at most $d - 1$ from the center, it follows that the diameter of $T_d$ is at most $2(d - 1) + 2 = 2d$. By Proposition 2.1, we have $\sigma_T(G) \leq \sigma_T(G, T_d) \leq D(T_d) \leq 2d$. Thus the assertion is proved. $\Box$
On the other hand, we shall show the lower bound. In the Hamming graph $G = K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}$ ($n_i \geq 2$), for each vertex $v = (x_1, x_2, \ldots, x_d)$, the vertex $f(v) = (x'_1, x'_2, \ldots, x'_d)$ is called the antipodal vertex of $v$ if

$$x'_i = x_i + 1 \pmod{n_i}, \quad i = 1, 2, \ldots, d. \quad (3)$$

Obviously, $v$ and $f(v)$ have distance $d$ in $G$. Note that this definition is not symmetrical, as the antipodal vertex of $f(v)$ is not necessarily $v$.

**Theorem 3.2** For the Hamming graphs $G = K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}$ ($2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$), it holds that

$$\sigma_T(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}) = \begin{cases} 2d - 1, & \text{if } n_1 = 2 \\ 2d, & \text{if } n_1 \geq 3. \end{cases}$$

**Proof:** By Lemma 3.1, it suffices to show the lower bound. Suppose that $T$ is an arbitrary spanning tree of $G$. We distinguish two cases as follows.

**Case 1:** $n_1 = 2$. We shall show that $\sigma_T(G, T) \geq 2d - 1$.

For each vertex $v \in V(G)$, let $f(v)$ be the antipodal vertex of $v$. We denote by $P_T(v, f(v))$ the path in $T$ from $v$ to $f(v)$. Furthermore, we define the successor of $v$, denoted $s(v)$, by the next vertex of $v$ on the path $P_T(v, f(v))$. We claim that there exists an edge $uv \in T$ such that $s(u) = v$ and $s(v) = u$. Assume, to the contrary, that there is no such edge. Then we can start at a vertex $v_1$ and let $v_2 = s(v_1)$. Since $s(v_2) \neq v_1$, $v_3 = s(v_2)$ is a new vertex. In this way, we can define a sequence $(v_1, v_2, v_3, \ldots)$ by setting $v_{i+1} = s(v_i)$ in the spanning tree $T$. Since $T$ contains no cycles, each vertex in this sequence cannot repeat the ones previously visited. Hence this is indeed an infinite sequence, contradicting that $T$ is a finite tree.

Now we take an edge $uv \in T$ that $s(u) = v$ and $s(v) = u$. Then $f(u)$ and $f(v)$ belong to different components of $T - uv$. Thus the paths $P_T(u, f(u))$ and $P_T(v, f(v))$ have only the edge $uv$ in common. We take $P_T(f(u), f(v)) = P_T(f(u), u) \cup P_T(v, f(v))$ by joining these two paths. Suppose that

$$u = (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_d),$$

$$v = (x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_d),$$

where $a, b \in \{0, 1, \ldots, n_i - 1\}$, and $a \neq b$. Then

$$f(u) = (x_1 + 1, \ldots, x_{i-1} + 1, a + 1, x_{i+1} + 1, \ldots, x_d + 1),$$

$$f(v) = (x_1 + 1, \ldots, x_{i-1} + 1, b + 1, x_{i+1} + 1, \ldots, x_d + 1),$$

where the addition is modular as in (3). So $f(u)$ and $f(v)$ are adjacent in $G$. Furthermore, $uv \in T$ implies $f(u)f(v) \in T$ (for otherwise there would be a cycle in $T$). Note that the
lengths of $P_T(u, f(u))$ and $P_T(v, f(v))$ are at least $d$, and they have $uv$ in common. Hence the length of $P_T(f(u), f(v))$ is at least $2d - 1$, and so $d_T(f(u), f(v)) \geq 2d - 1$. Therefore, we deduce the lower bound $\sigma_T(G, T) \geq 2d - 1$.

**Case 2:** $n_1 \geq 3$. We shall show that $\sigma_T(G, T) \geq 2d$.

By the proof of Case 1, we found a path $P_T(f(u), f(v))$ with $d_T(f(u), f(v)) \geq 2d - 1$. If $d_T(f(u), f(v)) > 2d - 1$, then we are done. So we may assume that $d_T(f(u), f(v)) = 2d - 1$. Thus $d_T(u, f(u)) = d_T(v, f(v)) = d$. We observe the path $P_T(u, f(u))$ from $u = (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_d)$ to $f(u) = (x_1+1, \ldots, x_{i-1}+1, a+1, x_{i+1}+1, \ldots, x_d+1)$. Since its length is exactly $d$, it follows that each coordinate of the $d$-dimensional vector increases exactly by $1$ (in modular sense) successively along this path. Noting that $v = s(u)$ is the next vertex of $u$ on this path, we see that $b = a + 1 \pmod{n_i}$. However, as $n_i \geq n_1 \geq 3$, we know that $b + 1 \neq a \pmod{n_i}$.

Let us now observe another path $P_T(v, f(v))$ from $v = (x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_d)$ to $f(v) = (x_1+1, \ldots, x_{i-1}+1, b+1, x_{i+1}+1, \ldots, x_d+1)$. Here, the next vertex of $v$ on this path is $u$, instead of $(x_1, \ldots, x_{i-1}, b+1, x_{i+1}, \ldots, x_d)$ (note that $a \neq b + 1$). Along this path after vertex $u$, the coordinate $x_i = a$ of $u$ must have one more changing to $x_i = b+1$ somewhere. Consequently, the length of $P_T(v, f(v))$ is greater than $d$. That is to say, the assumption of $d_T(v, f(v)) = d$ is impossible. Therefore, $d_T(f(u), f(v)) > 2d - 1$, which gives the lower bound $\sigma_T(G, T) \geq 2d$.

Combining the lower bound here and the upper bound in Lemma 3.1 completes the proof of the theorem. □

As a special case, we derive the following result for hypercubes in $[17]$. In the above proof, we also generalize a property of R.L. Graham on hypercubes (every spanning tree of $Q_d$ has a fundamental cycle of length at least $2d$, see Exercise 4.2.15(d) of $[1]$) to the Hamming graphs.

**Corollary 3.3** For the hypercubes $Q_d$, it holds that $\sigma_T(Q_d) = 2d - 1$.

### 4 Higher-dimensional grids

We consider the $d$-dimensional grids $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$, in which each vertex is represented by $v = (x_1, x_2, \ldots, x_d)$ with $x_i \in \{1, 2, \ldots, n_i\}, n_i \geq 2$ for $1 \leq i \leq d$. Two vertices $v = (x_1, x_2, \ldots, x_d)$ and $u = (y_1, y_2, \ldots, y_d)$ are adjacent if they differ by $1$ in exactly one coordinate.

Suppose that $G_i = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_i}$ ($1 \leq i \leq d$). Then $G_1 = P_{n_1}$, $G_i = G_{i-1} \times P_{n_i}$ ($i \geq 2$), and $G_d = G$. When $d = 1$, it is trivial that $G_1 = P_{n_1}$ is a path and $T_1 = G_1$ is an optimal tree. In the sequel, for a path $P_n = (v_1, v_2, \ldots, v_n)$, the vertex $v_{[n/2]}$ is called the center of this path.

We begin with the case $d = 2$. For a 2-dimensional grid $G = P_{n_1} \times P_{n_2}$ ($2 \leq n_1 \leq n_2$),
let $V(G) := \{(i, j) : 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$ denote the vertex set, and $(i, j)$ is adjacent to $(i', j')$ if $|i - i'| + |j - j'| = 1$. We call $R_i := \{(i, j) : 1 \leq j \leq n_2\}$ the $i$-th row, and $C_j := \{(i, j) : 1 \leq i \leq n_1\}$ the $j$-th column.

We construct a spanning tree $T_2$ as follows. First, we take $n_2$ copies of $P_{n_1}$, namely, the $n_2$ columns. Let $v_1, v_2, \ldots, v_{n_2}$ be the centers of these columns in turn. Then we join a path $P_{n_2}$ to pass through these $n_2$ centers, namely, the row $R_{\lceil n_1/2 \rceil}$. An example is shown in Figure 3 (where the centers are marked by heavy dots). In this tree $T_2$, the path $P_{n_2}$ passing through $n_2$ centers of $P_{n_1}$’s is called the central path (see $R_2$ in Figure 3). The center of this central path is called the center of $T_2$.

Figure 3. Spanning tree for $P_4 \times P_5$.

As we shall see later in the general case, this spanning tree $T_2$ is optimal, and a detour in $T_2$ with the maximum stretch is as follows: Start at a leaf of column $C_i$ with distance $\left\lfloor \frac{n_1}{2} \right\rfloor$ to the central path, go to the central path, pass through an edge of it, and then go back to the leaf of column $C_{i+1}$ at the same side. So we have the following (see [16]):

$$\sigma_T(P_{n_1} \times P_{n_2}) = 2 \left\lfloor \frac{n_1}{2} \right\rfloor + 1.$$

Figure 4. Spanning tree for $P_3 \times P_4 \times P_4$.

We further consider the case $d \geq 3$. For this, a spanning tree $T_i$ of $G_i$ in constructed by induction on $i$. Assume that $i \geq 3$ and $T_{i-1}$ has been constructed. The graph $G_i$ now
consists of \(n_i\) copies of \(G_{i-1}\), denoted \(G^1_{i-1}, G^2_{i-1}, \ldots, G^{n_i}_{i-1}\), where \(G^j_{i-1}\) has a spanning tree \(T^j_{i-1}\) \((1 \leq j \leq n_i)\). We construct a spanning tree \(T_i\) of \(G_i\) by joining a central path \(P_{n_i}\) through \(n_i\) centers of \(T^1_{i-1}, T^2_{i-1}, \ldots, T^{n_i}_{i-1}\). The center of the central path \(P_{n_i}\) is the center of \(T_i\), denoted \(v_i^0\). An example of \(d = 3\) is shown in Figure 4 (where the centers are marked by heavy dots).

We proceed to show that \(T_d\) is an optimal spanning tree of \(G\). First, the following lemma gives the upper bound.

**Lemma 4.1** For the \(d\)-dimensional grids \(G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}\) \((2 \leq n_1 \leq n_2 \leq \cdots \leq n_d)\), it holds that

\[
\sigma(T(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d})) \leq 2 \left( \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_d-1}{2} \right\rfloor \right) + 1.
\]

**Proof:** We have defined the spanning tree \(T_i\) of \(G_i\) with center \(v_i^0\) inductively. The following property plays an important role.

**Claim** In the spanning tree \(T_i\) of \(G_i\), the distance between each vertex \(v\) and the center \(v_i^0\) of \(T_i\) satisfies

\[
d_{T_i}(v, v_i^0) \leq \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_i}{2} \right\rfloor.
\]

To see this, we use induction on \(i\). When \(i = 1\), \(T_1\) is a path and the distance between each vertex and the center is at most \(\lfloor n_1/2 \rfloor\). Suppose that \(i \geq 2\) and the assertion is true for smaller \(i\). Note that \(G_i = G_{i-1} \times P_{n_i}\) and \(G_i\) consists of \(n_i\) copies of \(G_{i-1}\), that is, \(G^1_{i-1}, G^2_{i-1}, \ldots, G^{n_i}_{i-1}\), where \(G^j_{i-1}\) has a spanning tree \(T^j_{i-1}\) \((1 \leq j \leq n_i)\).

Let \(v\) be a vertex of \(G_i\). Then \(v\) belongs to some copy \(G^j_{i-1}\). It follows that \(P_{T_i}(v, v_i^0) = P_{T_i}(v, v^0_{i-1}) \cup P_{T_i}(v^0_{i-1}, v_i^0)\), where \(P_{T_i}(v, v^0_{i-1}) = P_{T_{i-1}}(v, v^0_{i-1})\) and \(P_{T_i}(v^0_{i-1}, v_i^0)\) is contained in the central path \(P_{n_i}\). By the inductive hypothesis, we have

\[
d_{T_i}(v, v^0_{i-1}) \leq \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_{i-1}}{2} \right\rfloor.
\]

and

\[
d_{T_i}(v^0_{i-1}, v_i^0) \leq \left\lfloor \frac{n_i}{2} \right\rfloor.
\]

Combining the above two inequalities results in the claim.

Now consider the spanning tree \(T_d\) of \(G\). For any cotree-edge \(uv \in \overline{T}_d\), we may assume that \(uv\) is between two copies of \(G_{d-1}\). For otherwise \(uv\) is contained in a copy of \(G_{d-1}\), and we can get a smaller upper bound by the same method. Suppose that \(u^0_{d-1}\) and \(v^0_{d-1}\) are the centers of these two copies of \(G_{d-1}\). Then \(P_{T_d}(u, v) = P_{T_{d-1}}(u, u^0_{d-1}) \cup \{u^0_{d-1}, v^0_{d-1}\} \cup P_{T_{d-1}}(v, v^0_{d-1})\), where the edge \(u^0_{d-1}v^0_{d-1}\) is contained in the central path. By the above Claim, we have

\[
d_{T_d}(u, v) \leq 2 \left( \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_{d-1}}{2} \right\rfloor \right) + 1.
\]
Since this inequality holds for every cotree-edge $uv \in \overline{T}_d$, we deduce the upper bound in the theorem. 

It remains to show the lower bound. To this end, we pay attention to some special paths as follows. For $a_j \in \{1, n_j\}, 1 \leq j \leq d, j \neq i$, the path
\[
P^A_{n_i} := ((a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_d), (a_1, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_d),
\]
\[
\ldots, (a_1, \ldots, a_{i-1}, n_i, a_{i+1}, \ldots, a_d))
\]
is called a boundary path in $x_i$-coordinate. For all possible combinations of $a_j \in \{1, n_j\}, 1 \leq j \leq d, j \neq i$, there are $2^{d-1}$ boundary paths in $x_i$-coordinate. So there are totally $d2^{d-1}$ boundary paths in all coordinates. For example, in the case $d = 2$ of Figure 3, the 4 boundary paths are those on the 4 sides of the rectangular region. In the case $d = 3$ of Figure 4, the 4 boundary paths in $x_3$-coordinate are those on the horizontal dotted lines and all 12 boundary paths are those on the 12 edges of the parallelepiped.

Moreover, the antipodal boundary path of $P^A_{n_i}$ is defined by
\[
P^B_{n_i} := ((b_1, \ldots, b_{i-1}, 1, b_{i+1}, \ldots, b_d), (b_1, \ldots, b_{i-1}, 2, b_{i+1}, \ldots, b_d),
\]
\[
\ldots, (b_1, \ldots, b_{i-1}, n_i, b_{i+1}, \ldots, b_d))
\]
where $\{a_j, b_j\} = \{1, n_j\}$ for $1 \leq j \leq d, j \neq i$. For example, in the case $d = 2$ of Figure 3, two antipodal boundary paths are on the opposite sides of the rectangular region. In the case $d = 3$ of Figure 4, two antipodal boundary paths are on the opposite edges of the parallelepiped (two opposite edges of a polyhedron are two edges which are not contained in the same face).

By virtue of symmetry, the following antipodal boundary paths in $x_d$-coordinate are said to be in standard form:
\[
P^A_{n_d} := ((1, 1, \ldots, 1, 1), (1, 1, \ldots, 1, 2), \ldots, (1, 1, \ldots, 1, n_d)), \quad (4)
\]
\[
P^B_{n_d} := ((n_1, n_2, \ldots, n_{d-1}, 1), (n_1, n_2, \ldots, n_{d-1}, 2), \ldots, (n_1, n_2, \ldots, n_{d-1}, n_d)). \quad (5)
\]
In fact, any pair of antipodal boundary paths $P^A_{n_d}$ and $P^B_{n_d}$ can be transformed into this form by reversing the order of some $x_i$-coordinates from $(1, 2, \ldots, n_i-1, n_i)$ to $(n_i, n_i-1, \ldots, 2, 1)$ if necessary.

In the proof below, we mainly analyze the relationship of two antipodal boundary paths.

**Theorem 4.2** For the $d$-dimensional grids $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$ ($2 \leq n_1 \leq n_2 \leq \cdots \leq n_d$), it holds that
\[
\sigma_T(K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}) = 2 \left( \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_{d-1}}{2} \right\rfloor \right) + 1.
\]

**Proof:** By Lemma 4.1, it suffices to show the upper bound is also the lower bound. Suppose that $T$ is an arbitrary spanning tree of $G$. Note that $2 \leq n_1 \leq n_2 \leq \cdots \leq n_d$. 

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We consider \( n_d \) copies \( G_{d-1}^1, G_{d-1}^2, \ldots, G_{d-1}^{n_d} \) of \( G = G_d \). For clarity, we may denote these \((d - 1)\)-dimensional grids by \( H_1 = G_{d-1}^1, H_2 = G_{d-1}^2, \ldots, H_{n_d} = G_{d-1}^{n_d} \). Moreover, let \( P_T(v_1, v_k) = (v_1, v_2, \ldots, v_k) \) be a shortest path in \( T \) from \( H_1 \) to \( H_{n_d} \), where \( v_1 \in V(H_1), v_k \in V(H_{n_d}) \), and \( k \geq n_d \).

We take a pair of antipodal boundary paths \( P_{n_d}^A \) and \( P_{n_d}^B \) in \( x_d \)-coordinate as shown in (4) and (5). They are also from \( H_1 \) to \( H_{n_d} \), but not necessarily contained in \( T \). We distinguish three cases as follows.

**Case 1:** Neither \( P_{n_d}^A \) nor \( P_{n_d}^B \) is contained in \( T \).

For an edge \( v_jv_{j+1} \in E(T) \) in the path \( P_T(v_1, v_k) \), let \( S_j \) and \( \overline{S}_j \) be the vertex sets of the two components of \( T - v_jv_{j+1} \), where \( v_j \in S_j \) and \( v_{j+1} \in \overline{S}_j \). Then \( \partial(S_j) := \{uv \in E(G) : u \in S_j, v \in \overline{S}_j \} \) is a fundamental edge-cut with respect to the tree-edge \( v_jv_{j+1} \in T \). Let \( u_1 = (1, 1, \ldots, 1) \) be the first vertex of \( P_{n_d}^A \) and \( w_1 = (n_1, n_2, \ldots, n_{d-1}, 1) \) the first vertex of \( P_{n_d}^B \). Suppose that \( P_T(u_1, v_1) \) is the path in \( T \) from \( u_1 \) to the path \( P_T(v_1, v_k) \) and \( P_T(w_1, v_j) \) is the path in \( T \) from \( w_1 \) to the path \( P_T(v_1, v_k) \). Then when \( j \geq \max \{i_1, i_2 \} \), \( u_1 \) and \( w_1 \) are contained in \( S_j \). We take the edge \( v_jv_{j+1} \) in \( P_T(v_1, v_k) \) with such index \( j \). Since \( u_1, w_1 \in S_j \), it follows that there exists an edge \( u_hu_{h+1} \) of \( P_{n_d}^A \) which is contained in \( \partial(S_j) \) and there exists an edge \( wlw_{l+1} \) of \( P_{n_d}^B \) which is contained in \( \partial(S_j) \). Therefore, these edges \( u_hu_{h+1} \) and \( wlw_{l+1} \) are cotree edges in \( T \).

On the other hand, for the tree-edge \( v_jv_{j+1} \in E(T) \), we may assume that \( v_j \in V(H_a) \) and \( v_{j+1} \in V(H_{a+1}) \). For otherwise \( (v_j \text{ and } v_{j+1} \text{ belong to the same } H_a) \) we can take a greater \( j \). Then these two vertices can be represented by \( v_j = (x_1, x_2, \ldots, x_{d-1}, a) \) and \( v_{j+1} = (x_1, x_2, \ldots, x_{d-1}, a + 1) \). In the component \( T[S_j] \) of \( T - v_jv_{j+1} \), we have two paths \( P_T(u_h, v_j) \) and \( P_T(w_l, v_j) \) while in the component \( T[\overline{S}_j] \) of \( T - v_jv_{j+1} \), we have two paths \( P_T(u_{h+1}, v_{j+1}) \) and \( P_T(w_{l+1}, v_{j+1}) \). In this way, we obtain two fundamental cycles

\[
P_T(u_h, v_j) \cup \{v_jv_{j+1}\} \cup P_T(u_{h+1}, v_{j+1}) \cup \{u_hu_{h+1}\}
\]

and

\[
P_T(w_l, v_j) \cup \{v_jv_{j+1}\} \cup P_T(w_{l+1}, v_{j+1}) \cup \{w lw_{l+1}\}
\]

where \( u_hu_{h+1} \) and \( w lw_{l+1} \) are cotree edges. The maximum stretch incurred by these two fundamental cycles is at least

\[
\max \left\{ 2 \sum_{i=1}^{d-1} (x_i - 1) + 1, 2 \sum_{i=1}^{d-1} (n_i - x_i) + 1 \right\} \geq 2 \sum_{i=1}^{d-1} \left\lceil \frac{n_i - 1}{2} \right\rceil + 1
\]

\[= 2 \left( \left\lceil \frac{n_1}{2} \right\rceil + \left\lceil \frac{n_2}{2} \right\rceil + \cdots + \left\lceil \frac{n_{d-1}}{2} \right\rceil \right) + 1.\]

This yields the required lower bound.

**Case 2:** \( P_{n_d}^A \) is contained in \( T \) but \( P_{n_d}^B \) is not.

We use \( P_{n_d}^A \) in place of \( P_T(v_1, v_k) \). For an edge \( u_ju_{j+1} \in E(T) \) of \( P_{n_d}^A \), let \( S_j \) and \( \overline{S}_j \) be the vertex sets of the two components of \( T - u_ju_{j+1} \), where \( u_j \in S_j \) and \( u_{j+1} \in \overline{S}_j \). Then
∂(S_j) is a fundamental edge-cut with respect to u_ju_{j+1} ∈ T. We can choose such j that w_1 = (n_1, n_2, …, n_{d-1}, 1) is contained in S_j. Hence there exists an edge w_1w_{l+1} of P^B_n which is contained in ∂(S_j), and so it is a cotree edge in T. On the other hand, suppose that u_j = (1, 1, …, 1, a) and u_{j+1} = (1, 1, …, 1, a + 1). Then we can find a path P_T(w_1, u_j) in the component T[S_j] of T − u_ju_{j+1} and a path P_T(w_{l+1}, u_{j+1}) in the component T[S_j]. This results in a fundamental cycle

$$P_T(w_1, u_j) \cup \{u_ju_{j+1}\} \cup P_T(w_{l+1}, u_{j+1}) \cup \{w_1w_{l+1}\}.$$ 

Consequently, the stretch incurred by this fundamental cycle is at least

$$2 \sum_{i=1}^{d-1} (n_i - 1) + 1 > 2\left( \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_{d-1}}{2} \right\rfloor \right) + 1,$$

as required.

**Case 3:** Both P^A_{n_i} and P^B_{n_i} are contained in T.

If all pairs of antipodal boundary paths are contained in T, then there would be cycles in T, which contradicts that T is a spanning tree. Otherwise we can take a pair antipodal boundary paths P^A_{n_i} and P^B_{n_i} in some x_i-coordinate, not both of which are contained in T. Now suppose that

$$P^A_{n_i} := ((1, \ldots, 1, 1, \ldots, 1), (1, \ldots, 1, 2, 1 \ldots, 1), \ldots, (1, \ldots, 1, n_i, 1 \ldots, 1)),$$

$$P^B_{n_i} := ((n_1, \ldots, n_{i-1}, 1, n_{i+1} \ldots, n_d), (n_1, \ldots, n_{i-1}, 2, n_{i+1} \ldots, n_d), \ldots, (n_1, \ldots, n_{i-1}, n_i, n_{i+1} \ldots, n_d)).$$

By the same method of Case 1 and Case 2 for P^A_{n_i} and P^B_{n_i}, we can show that

$$\sigma_T(G) \geq 2 \sum_{1 \leq j \leq d, j \neq i} \left\lfloor \frac{n_j}{2} \right\rfloor + 1$$

$$\geq 2\left( \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_{d-1}}{2} \right\rfloor \right) + 1,$$

since n_i ≤ n_d. This completes the proof. □

5 Concluding remarks

The minimum stretch spanning tree problem and the minimum congestion spanning tree problem are two dual problems in spanning tree optimization, in which the fundamental cycles and the fundamental edge-cuts are considered respectively. They have applications in information science and have close relations to labeling and embedding for graphs (such as the bandwidth and cutwidth problem). It is meaningful to establish connections between these two problems. What we have seen from the above are the exact formulas
of $\sigma_T(G)$ for two families of graphs, the Hamming graphs and higher-dimensional grids. The corresponding results for the tree-congestion $c_T(G)$ have not been seen yet.

The study of these optimization problems gives rise to two graph-theoretic invariants, the tree-stretch $\sigma_T(G)$ and tree-congestion $c_T(G)$. From the perspective of graph theory, several aspects are worthwhile to explored. For example, exact representations for more graph families, relations with other parameters, extremal graph characterizations, duality, symmetry, decomposability, etc., are expected.

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