Nambu-Goto Like Action for the $AdS_5 \times S^5$ Superstrings in the Generalized Light-Cone Gauge

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Abstract

We reinvestigate the $\kappa$-symmetry-fixed Green-Schwarz action in the $AdS_5 \times S^5$ background in a version of the light-cone gauge. In the generalized light-cone gauge, the action has been written in the phase space variables. We convert it into the standard action written in terms of the fields and their derivatives. We obtain a Nambu-Goto type action which has the correct flat-space limit.

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1 Introduction and Summary

It is a challenging problem to quantize the Green-Schwarz (GS) action \cite{1,2} in the $AdS_5 \times S^5$ background \cite{3}. The knowledge of the spectrum will tell us the strong coupling dynamics of the large $N$ gauge theory through the AdS/CFT correspondence. One of the difficulties in covariant quantization of the GS action stems from the existence of the local $\kappa$-symmetry, which halves the fermionic degrees of freedom \cite{1,2,4}. One approach to this problem is to abandon the covariance and fix the $\kappa$-symmetry non-covariantly. But after the $\kappa$-symmetry fixing, the model is still a constrained system due to the world-sheet diffeomorphism. Various gauges \cite{5,6,7,8,9,10,11,12,13,14,15,16} have been proposed to fix these symmetries. Especially, the uniform light-cone gauge, a generalization of the flat-space light-cone gauge \cite{17} to the curved space background, has been extensively investigated in \cite{11,12,13,14,15,16}.

In our previous paper \cite{18}, we adopted the light-cone gauge and considered the Hamiltonian dynamics of the GS action by using the physical degrees of freedom. The action is formulated in the first order formalism, i.e., is written in terms of the phase space variables. This first-order formulation is suited for considering the problem of canonical quantization.

Unfortunately, the reduced action in the generalized light-cone gauge still has an involved form. Before considering the quantization problem of the action, we would like to investigate quantum fluctuation in various limits. Extensive study around the plane-wave region was done in \cite{11}. But, in general, the first-order Lagrangian is not so convenient to study the quantum spectrum in various limits, such as the BMN limit \cite{19}, the Hofman-Maldacena limit \cite{20}, or Maldacena-Swanson limit \cite{21}. They can be better investigated by using the Lagrangian written in terms of the fields and their derivatives. Therefore, in this paper, we reformulate the GS action to the standard form in the generalized light-cone gauge.

After $\kappa$-symmetry fixing and taking the generalized light-cone gauge

$$X^+ = \kappa \tau, \quad P_- = \sqrt{\lambda \omega},$$

(1.1)

with $\kappa$ and $\omega$ are constants, we find the Nambu-Goto like action for the GS action in the $AdS_5 \times S^5$ background. Its bosonic part has the following form:

$$S = \frac{1}{2\pi} \int d^2 \xi \left( \sqrt{-\lambda \det(J_{ij} + G_{ij})} + \sqrt{\lambda \kappa \omega} \frac{G_{++}}{G_{--}} \right).$$

(1.2)

Here $G_{ij} = G_{mn} \partial_i X^m \partial_j X^n$ is the induced metric, induced from the target space metric for the transverse spatial directions. (The indexes $m, n$ run over the transverse directions:...
The additional term $J_{ij}$ comes from the longitudinal directions. The 't Hooft coupling $\lambda$ is related to the radius $R$ of $AdS_5$ and $S^5$ as $\sqrt{\lambda} = R^2/\alpha'$. The full Lagrangian which includes the fermions will be given by (3.45). This is a Nambu-Goto type Lagrangian.

It is natural to appear the Nambu-Goto type action when the world-sheet diffeomorphism is fixed by certain gauge conditions other than the conformal gauge. Solving the equations of motion for the world-sheet metric yields the Nambu-Goto type action. For example, the Nambu-Goto type action for the GS model in $AdS_5 \times S^5$ in the static gauge can be found in [8].

In contract to ordinary Nambu-Goto actions, we have chosen the sign before the square root term to be positive. This comes from the requirement that the action must have the correct flat-space limit. Indeed, the Lagrangian (3.45) goes to the correct $\kappa$-symmetry fixed light-cone gauge Lagrangian in the limit.

The Lagrangian (3.45) will serves as a starting point for developing the various limits and investigating the quantum fluctuations.

This paper is organized as follow. In section 2, using the bosonic sigma model as an example, we explain the procedure to obtain the standard Lagrangian in the generalized light-cone gauge. In section 2.1, we start from the Lagrangian in the first-order formalism and arrive at the standard one. In section 2.2, we derive it without going to the first-order form. In section 3, we first briefly review our notation for the GS action. In section 3.1, the $\kappa$-symmetry fixing is done and the action in the $AdS_5 \times S^5$ background is given. In section 3.2, the $\kappa$-symmetry fixed GS action in the generalized light-cone gauge is obtained. This is our main result. In section 3.3, it is discussed that the action has the correct flat-space limit. Some of our notations are summarized in Appendix.

2 The bosonic sigma model

The action for the bosonic sigma model in the $D$-dimensional curved target space is given by

$$ S = \frac{1}{2\pi} \int d^2\xi \, \mathcal{L}, $$

where

$$ \mathcal{L} = -\frac{1}{2} \sqrt{\lambda} h_{ij} G_{\mu\nu} \partial_i X^\mu \partial_j X^\nu. $$

Here $m, n = 0, 1, \ldots, D - 1$, $\xi^0 = \tau$, $\xi^1 = \sigma$, $\partial_i = \partial/\partial \xi^i$, $h_{ij} = \sqrt{-g} g^{ij}$ ($i, j = 0, 1$).
The conjugate momenta are given by

\[ P_m = -\sqrt{\lambda} h^{0j} G_{mn} \partial_j X^n. \]  

(2.3)

The target space metric is assumed to have the following form

\[ G_{mn} dX^m dX^n = G_{ab} dX^a dX^b + G_{mn} dX^m dX^n. \]  

(2.4)

Here \( a, b = \pm \) denote the longitudinal directions, \( m, n = 1, 2, \ldots, D - 2 \) denote the transverse directions. We assume that \( \partial/\partial X^{\pm} \) is a Killing vector.

Let us decompose the Lagrangian into two pieces:

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \]  

(2.5)

\[ \mathcal{L}_1 = -\frac{1}{2} \sqrt{\lambda} h^{ij} G_{ab} \partial_i X^a \partial_j X^b, \]  

(2.6)

\[ \mathcal{L}_2 = -\frac{1}{2} \sqrt{\lambda} h^{ij} G_{mn} \partial_i X^m \partial_j X^n. \]  

(2.7)

The first part \( \mathcal{L}_1 \) and the second part \( \mathcal{L}_2 \) are related to the metric for the longitudinal directions and the metric for the transverse directions respectively.

Under the target space metric ansatz \([2.4]\), the momenta \( P_- \) which is conjugate to \( X^- \) is given by

\[ P_- = -\sqrt{\lambda} h^{0i} G_{-,a} \partial_j X^a. \]  

(2.8)

The generalized light-cone gauge is given by the following two conditions

\[ X^+ = \kappa \tau, \quad P_- = \sqrt{\lambda} \omega = \text{const}, \]  

(2.9)

which fixes the world-sheet diffeomorphism.

### 2.1 From the first order form to the standard Lagrangian

The reduced action in the generalized light-cone gauge is given by (we use the notation of \([18]\))

\[ S_{\text{red}} = \frac{1}{2\pi} \int d^2 \xi (P_m \dot{X}^m - \mathcal{H}_{\text{LC}}), \]  

(2.10)

\[ \text{1 Originally, the second condition was given by } \partial_1 (P_-) = 0. \text{ Most general solution is } P_- = P_- (\sigma). \text{ But without loss of generality, we can set } P_- \text{ to a constant by redefining the world-sheet space variable } \sigma \text{ and conjugate momenta such that } P_- (\sigma) d\sigma = P'_- d\sigma' \text{ with } P'_- \text{ constant. Therefore, we adopt the condition } P_- = \text{const as one of the gauge conditions.} \]
where
\[
\mathcal{H}_{\text{LC}} = -\kappa P_+.
\] (2.11)

This is the first-order Lagrangian \( \mathcal{L} = \mathcal{L}(X^m, P_m) \) written in terms of the transverse coordinates \( X^m \) and their conjugate momenta \( P_m \).

Here \( P_+ \) is a solution of
\[
G^{++}P_+^2 + 2\sqrt{\lambda} \omega G^{+-}P_+ + C = 0,
\] (2.12)

where
\[
C = \lambda \omega^2 G^{--} + \lambda G_{mn} \partial_1 X^m \partial_1 X^n + K^{mn} P_m P_n,
\] (2.13)
\[
K^{mn} := G^{mn} + \frac{1}{\omega^2} G_{-} \partial_1 X^m \partial_1 X^n.
\] (2.14)

Explicitly, \( P_+ \) is given by
\[
P_+ = \frac{\varepsilon}{G^{++}} \sqrt{\lambda}(G^{+-})^2 \omega^2 - G^{++} C - \sqrt{\lambda} \omega \frac{G^{+-}}{G^{++}},
\] (2.15)

where \( \varepsilon = \pm 1 \).

Now let us convert this first-order Lagrangian into the standard form. The equations of motion for \( P_m \)
\[
\dot{X}^m + \kappa \frac{\partial P_+}{\partial P_m} = 0
\] (2.16)
yield the following relations
\[
\varepsilon \sqrt{\lambda} \omega^2 (G^{+-})^2 - G^{++} C \dot{X}^m = \kappa K^{mn} P_n.
\] (2.17)

It is convenient to introduce \( \mathcal{J}_{ij} \) and \( \mathcal{G}_{ij} \) by
\[
\mathcal{J}_{00} := \frac{\kappa^2}{G^{++}}, \quad \mathcal{J}_{11} := \frac{\omega^2}{G_{-}}, \quad \mathcal{J}_{01} = \mathcal{J}_{10} := 0,
\] (2.18)
\[
\mathcal{G}_{ij} := G_{mn} \partial_i X^m \partial X^n, \quad i, j = 0, 1.
\] (2.19)

Let \( K_{mn} \) be the inverse of \( K^{mn} \):
\[
K_{mn} = G_{mn} - \frac{(G_{mm'} \partial_1 X^{m'}) (G_{nn'} \partial_1 X^{n'})}{\mathcal{J}_{11} + \mathcal{G}_{11}}.
\] (2.20)

Then
\[
P_m = \frac{\varepsilon}{\kappa} \sqrt{\lambda} \omega^2 (G^{+-})^2 - G^{++} C K_{mn} \dot{X}^n.
\] (2.21)

By substituting this relation into (2.13), we have
\[
C = \lambda \omega^2 G^{--} + \lambda \mathcal{G}_{11} + \frac{1}{\kappa^2} \left( \lambda \omega^2 (G^{+-})^2 - G^{++} C \right) K_{mn} \dot{X}^m \dot{X}^n.
\] (2.22)
Then we have
\[
C = \frac{\lambda \omega^2 G^{-} + \lambda G_{11} + (\lambda \omega^2 / \kappa^2) (G^{++})^2 K_{mn} \dot{X}^m \dot{X}^n}{1 + (1/\kappa^2) G^{++} K_{mn} \dot{X}^m \dot{X}^n}.
\]  
(2.23)

Note that
\[
K_{mn} \dot{X}^m \dot{X}^n = \frac{J_{11} G_{00} + \det(G_{ij})}{J_{11} + G_{11}}.
\]  
(2.24)

With some work, we find
\[
\lambda \omega^2 (G^{++})^2 - G^{++} C = -\frac{\lambda \kappa^2}{\det(J_{ij} + G_{ij})} (J_{11} + G_{11})^2.
\]  
(2.25)

Assuming \(\det(J_{ij} + G_{ij}) < 0\), we have
\[
\sqrt{\lambda \omega^2 (G^{++})^2 - G^{++} C} = \varepsilon' \sqrt{-\lambda \det(J_{ij} + G_{ij})} \kappa (J_{11} + G_{11}).
\]  
(2.26)

Here \(\varepsilon' = \text{sign}(\kappa (J_{11} + G_{11}))\).

Let \(J_{ij}\) be a matrix defined by
\[
J_{ij} := J_{ij} + G_{ij},
\]  
(2.27)

and \(J^{ij}\) be its inverse. We can see that
\[
K_{mn} \dot{X}^n = \frac{1}{J_{11} + G_{11}} G_{mn} [(J_{11} + G_{11}) \dot{X}^n - G_{01} \partial_i X^m] = \frac{\det(J_{ij} + G_{ij})}{J_{11} + G_{11}} G_{mn} J^{0j} \partial_j X^n.
\]  
(2.28)

Now we finally have
\[
P_m = -\varepsilon \varepsilon' \sqrt{-\lambda \det(J_{ij})} G_{mn} J^{0j} \partial_j X^n.
\]  
(2.29)

By substituting this expression into the first order form of the action, we get the reduced Lagrangian in the generalized light-cone gauge.

Summary: The Lagrangian of the bosonic sigma model in the generalized light-cone gauge is given by
\[
S_{\text{red}} = \frac{1}{2\pi} \int d^2 \xi \mathcal{L}_{\text{LC}},
\]  
(2.30)

where
\[
\mathcal{L}_{\text{LC}} = -\varepsilon \varepsilon' \sqrt{-\lambda \det(J_{ij})} + \sqrt{\lambda \kappa \omega} \frac{G^{++}}{G^{--}}.
\]  
(2.31)

Here
\[
J_{00} = \frac{\kappa^2}{G^{++}}, \quad J_{11} = \frac{\omega^2}{G^{--}}, \quad J_{01} = J_{10} = 0, \quad G_{ij} = G_{mn} \partial_i X^m \partial_j X^n.
\]  
(2.32)
2.2 Rederivation of the reduced Lagrangian

In this subsection, we rederive the reduced Lagrangian \((2.31)\) without bypassing the first-order formalism.

Let us restart from the Lagrangian \((2.2)\). Let us decompose it as follows
\[
L = L_1 + L_2, \tag{2.33}
\]
\[
L_1 = -\frac{1}{2} \sqrt{\lambda h^{ij}} G_{ab} \partial_i X^a \partial_j X^b, \quad L_2 = -\frac{1}{2} \sqrt{\lambda h^{ij}} G_{mn} \partial_i X^m \partial_j X^n. \tag{2.34}
\]

The generalized light-cone gauge conditions are given by
\[
X^+ = \kappa \tau, \quad P_- = -\sqrt{\lambda h^{0j}} G_{-a} \partial_j X^a = \sqrt{\lambda} \omega = \text{const}. \tag{2.35}
\]

We interpret the second condition as the following relation for \(\dot{X}^-\):
\[
\dot{X}^- = -\left(\frac{h^{01}}{h^{00}}\right) \partial_1 X^- - \frac{1}{\sqrt{\lambda} h^{00}} \left(\frac{G_{++}}{G_{--}} - \frac{\omega}{G_{--}}\right). \tag{2.36}
\]

With some work, we have
\[
L = L_1' + P_\dot{X}^- , \tag{2.37}
\]
where
\[
L_1' = \sqrt{\lambda} \kappa \omega \left(\frac{G_{++}}{G_{--}}\right) + \frac{1}{2} \sqrt{\lambda} \frac{\kappa^2}{h^{00} G_{++}} - \frac{1}{2} \sqrt{\lambda} h^{00} \frac{\omega^2}{G_{++}} \tag{2.38}
\]
\[
- \frac{1}{2} \sqrt{\lambda} h^{00} \frac{\kappa^2}{G_{++}} + \frac{1}{2} \frac{G_{--}}{h^{00}} (\partial_1 X^-)^2 + \sqrt{\lambda} \omega \left(\frac{h^{01}}{h^{00}}\right) \partial_1 X^- .
\]

Note that \(P_\dot{X}^-\) is a total \(\tau\)-derivative term. So, we use \(L_1'\) as the Lagrangian in the generalized light-cone gauge.

In \(L_1'\), the field \(X^-\) appears only through the form of \(\partial_1 X^-\). The field \(\partial_1 X^-\) plays the role of an auxiliary field. The equations of motion for \(\partial_1 X^-\) gives
\[
\partial_1 X^- = -\frac{\omega h^{01}}{G_{--}}. \tag{2.39}
\]

By substituting this solution into \(L_1'\), we have
\[
L_1' = \sqrt{\lambda} \kappa \omega \left(\frac{G_{++}}{G_{--}}\right) - \frac{1}{2} \sqrt{\lambda} h^{00} \frac{\kappa^2}{G_{++}} - \frac{1}{2} \sqrt{\lambda} h^{11} \frac{\omega^2}{G_{--}} . \tag{2.40}
\]

Let us introduce a world-sheet symmetric tensor \(J_{ij}\) by
\[
J_{00} := \frac{\kappa^2}{G_{++}}, \quad J_{11} := \frac{\omega^2}{G_{--}}, \quad J_{01} = J_{10} := 0. \tag{2.41}
\]
The reduced action now has the form
\[
\mathcal{L}' = \mathcal{L}'_1 + \mathcal{L}_2 = \sqrt{\lambda\kappa\omega} \left( \frac{G_{++}}{G_{--}} \right) - \frac{1}{2} \sqrt{h} h^{ij} (J_{ij} + G_{ij}), \tag{2.42}
\]
where
\[
G_{ij} = G_{mn} \partial_i X^m \partial_j X^n. \tag{2.43}
\]
Since the world-sheet diffeomorphism is fixed by the light-cone gauge conditions (2.35), \(h^{ij}\) are determined by solving the equations of motion for \(h^{ij}\):
\[
h^{ij} = \pm \sqrt{\frac{-\det(J_{ij})}{J_{ij}}} J^{ij}, \tag{2.44}
\]
where \(J_{ij} = J_{ij} + G_{ij}\), and \(J^{ij}\) is the inverse of \(J_{ij}\).

Then, we finally have the reduced Lagrangian in the generalized light-cone gauge
\[
\mathcal{L}' = \pm \sqrt{-\lambda \det(J_{ij} + G_{ij})} + \sqrt{\lambda\kappa\omega} \left( \frac{G_{++}}{G_{--}} \right). \tag{2.45}
\]

3 The GS action

Now let us consider the GS action in the \(AdS_5 \times S^5\) background. The GS action in the flat target space \([1, 2]\) is generalized in the curved supergravity background in \([22]\). More explicit GS action in the \(AdS_5 \times S^5\) background was constructed in \([3]\). (See also \([23, 24]\)). Originally, the Wess-Zumino term is written in the three-dimensional form. The manifestly two-dimensional form of the Wess-Zumino term was presented in \([25, 26, 27]\).

We write the GS action in the \(AdS_5 \times S^5\) background as follows:
\[
S_{\text{GS}} = \frac{1}{2\pi} \int d^2 \xi \mathcal{L}_{\text{GS}}, \tag{3.1}
\]
\[
\mathcal{L}_{\text{GS}} = -\frac{1}{2} \sqrt{\lambda h^{ij}} \eta_{\underline{a} \underline{b}} E_i^\underline{a} E_j^\underline{b} + \sqrt{\lambda} \epsilon^{ij} (E_i^a \eta_{a \underline{a}} E_j^\underline{a} - E_i^{\underline{a}} \eta_{\underline{a} \underline{a}} E_j^\underline{a}). \tag{3.2}
\]
Here \(a, b = 0, 1, \ldots, 9, \alpha, \beta, \bar{\alpha}, \bar{\beta} = 1, 2, \ldots, 16, h^{ij} = \sqrt{-gg^{ij}} \ (i, j = 0, 1), \epsilon^{01} = 1.
\]
\[
\eta_{\underline{a} \underline{b}} = \text{diag}(-1, 1, \ldots, 1). \tag{3.3}
\]

The induced vielbein for the type IIB superspace \(E^A_i\) is denoted by
\[
E^A_i = E^A_M \partial_i Z^M = E^A_m \partial_i X^m + E^A_\alpha \partial_i \theta^\alpha + \overline{E}^A_{\bar{\alpha}} \partial_i \bar{\theta}^{\bar{\alpha}}, \quad A = (a, \alpha, \bar{\alpha}). \tag{3.4}
\]
We use a Majorana-Weyl representation for the Gamma matrices:

\[
\Gamma^a = \begin{pmatrix} 0 & (\gamma^a)_{\dot{\alpha}\beta} \\ (\gamma^a)_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (\Gamma^a)^* = \Gamma^a, \quad \{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{1}_{32}, \quad (3.5)
\]

\(a = 0, 1, \ldots, 9, \alpha, \beta = 1, 2, \ldots, 16\). We denote the \(n \times n\) identity matrix by \(\mathbb{1}_n\). For our specific choice of the Gamma matrices, see appendix.

A 32-component Weyl spinor \(\Theta\) with positive chirality has the following form in the Majorana-Weyl representation:

\[
\Theta = \begin{pmatrix} \theta^\alpha \\ 0 \end{pmatrix}. \quad (3.6)
\]

Above, we have used the 16-component notation for the Weyl spinors. A spinor with upper index \(\alpha\) represent a Weyl spinor with positive chirality.

The constant matrix \(\varrho\) in the Wess-Zumino term is given by

\[
C\Gamma^{01234} = \begin{pmatrix} \varrho^\alpha_{\dot{\beta}} & 0 \\ 0 & \varrho^\alpha_{\dot{\beta}} \end{pmatrix}. \quad (3.7)
\]

Here \(C\) is the charge conjugation matrix.

### 3.1 \(\kappa\)-symmetry fixing

Let us decompose each of the two 16-component Weyl spinors into two 8-component \(SO(4) \times SO(4)\) spinors:

\[
\theta^\alpha = \begin{pmatrix} \theta^{+\alpha} \\ \theta^{-\dot{\alpha}} \end{pmatrix}, \quad \bar{\theta}^{\dot{\alpha}} = \begin{pmatrix} \bar{\theta}^{+\alpha} \\ \bar{\theta}^{-\dot{\alpha}} \end{pmatrix}, \quad (3.8)
\]

where \(\alpha = 1, 2, \ldots, 8, \dot{\alpha} = \hat{1}, \hat{2}, \ldots, \hat{8}\). \(\bar{\alpha} = \bar{1}, \bar{2}, \ldots, \bar{8}\) and \(\dot{\alpha} = \dot{\hat{1}}, \dot{\hat{2}}, \ldots, \dot{\hat{8}}\).

We first fix the \(\kappa\)-symmetry by setting \(\theta^{-\dot{\alpha}} = \bar{\theta}^{-\dot{\alpha}} = 0\). In the 32-component notation, these conditions are equivalent to the condition \(\Gamma^+\Theta = 0\).

To simplify expressions, we combine the remaining fermionic coordinates into \(\Psi^{\dot{\alpha}}\):

\[
(\Psi^{\dot{\alpha}}) = \begin{pmatrix} \theta^{+\alpha} \\ \bar{\theta}^{+\dot{\alpha}} \end{pmatrix}, \quad \dot{\alpha} = \hat{1}, \hat{2}, \ldots, \hat{16}. \quad (3.9)
\]

Let \(\mathcal{M}^2\) be a \(16 \times 16\) matrix

\[
\mathcal{M}^2 = \begin{pmatrix} (\mathcal{M}^2)^\alpha_{\beta} & (\mathcal{M}^2)^\alpha_{\dot{\beta}} \\ (\mathcal{M}^2)^{\dot{\alpha}}_{\beta} & (\mathcal{M}^2)^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix}. \quad (3.10)
\]
with elements constructed purely from the fermionic variables:

\[
(M^2)_{\alpha \beta} = \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}) - \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}),
\]

\[
(M^2)_{\alpha \beta} = \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}) + \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}),
\]

\[
(K_{11})^\alpha_\beta = \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}) - \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}),
\]

\[
(K_{21})^\alpha_\beta = \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}) + \frac{1}{2}(\theta^+ a_{\alpha \beta})(\bar{\theta}^+ a_{\alpha \beta}).
\]

The \(\kappa\)-symmetry fixed action in the \(AdS_5 \times S^5\) can be written as [18]:

\[
L_{GS} = -\frac{1}{2} \sqrt{\lambda} h^{ij} G_{mn} D_i X_m D_j X_n + \frac{1}{2} \sqrt{\lambda} \epsilon^{ijk} B_{\alpha \beta} D_i \Psi^\alpha D_j \Psi^\beta.
\]

Here \(m, n = 0, 1, \ldots, 9, \hat{\alpha}, \hat{\beta} = 1, 2, \ldots, 16\). The target space metric \(G_{\hat{m} \hat{n}}\) is the bosonic \(AdS_5 \times S^5\) metric, chosen as follows:

\[
ds^2 = ds^2_{AdS_5} + ds^2_{S^5} = G_{\hat{m} \hat{n}} dX_{\hat{m}} dX_{\hat{n}} = G_{\hat{a} \hat{b}} dX^\hat{a} dX^\hat{b} + G_{\hat{m} \hat{n}} dX^\hat{m} dX^\hat{n}.
\]

The \(AdS_5\) part metric is chosen as

\[
ds^2_{AdS_5} = -\left(\frac{1 + (z^2/4)}{1 - (z^2/4)}\right)^2 dt^2 + G_z \sum_{a=1}^4 (d z^a)^2,
\]

and the \(S^5\) part metric is chosen as

\[
ds^2_{S^5} = \left(\frac{1 - (y^2/4)}{1 + (y^2/4)}\right)^2 d\varphi^2 + G_y \sum_{s=1}^4 (d y^s)^2,
\]

where

\[
G_z = \frac{1}{(1 - (z^2/4))^2}, \quad G_y = \frac{1}{(1 + (y^2/4))^2}.
\]

Here

\[
z^2 = \sum_{a=1}^4 (z^a)^2, \quad y^2 = \sum_{s=1}^4 (y^s)^2.
\]
We choose the coordinates $X^m$ as follows:

$$X^\pm = \frac{1}{\sqrt{2}}(t \pm \varphi), \quad X^a = z^a, \quad X^{4+s} = y^s.$$  \hspace{1cm} (3.19)

The metric for the longitudinal directions $G_{ab} \ (a, b = \pm)$ is given by

$$G_{++} = G_{--} = -\frac{1}{2} \left( \left( \frac{1}{1 + (z^2/4)} \right)^2 + \frac{1}{2} \left( \frac{1 - (y^2/4)}{1 + (y^2/4)} \right)^2 \right),$$  \hspace{1cm} (3.20)

$$G_{+-} = G_{-+} = -\frac{1}{2} \left( \left( \frac{1}{1 + (z^2/4)} \right)^2 - \frac{1}{2} \left( \frac{1 - (y^2/4)}{1 + (y^2/4)} \right)^2 \right).$$  \hspace{1cm} (3.21)

For $G_{mn} \ (m, n = 1, 2, \ldots, 8)$, we have

$$G_{ab} = G_{z} \delta_{ab}, \quad G_{4+s,4+s'} = G_{y} \delta_{s,s'}, \quad G_{a,4+s} = 0.$$  \hspace{1cm} (3.22)

Here $a, b = 1, 2, 3, 4, s, s' = 1, 2, 3, 4$. Let us denote the inverse of $G_{mn}$ by $G^{mn}$. Note that $\partial/\partial X^\pm$ are Killing vectors.

The derivatives $\mathcal{D}_j$ in (3.13) are given by

$$\mathcal{D}_j X^+ = \partial_j X^+, \quad \mathcal{D}_j X^- = \partial_j X^- + \Lambda^- \mathcal{D}_j \Psi \hat{\alpha},$$

$$\mathcal{D}_j X^m = \partial_j X^m + (\Lambda^m_{n\hat{\alpha}} \mathcal{D}_j \Psi \hat{\alpha}) X^n,$$

$$\mathcal{D}_j \Psi \hat{\alpha} = \partial_j \Psi \hat{\alpha} + (\Lambda_{\hat{\beta} \hat{\alpha}} \partial_j X^+) \Psi \hat{\beta}.$$  \hspace{1cm} (3.23)

\(\hat{\alpha} = (\alpha, \bar{\alpha}), \hspace{1cm} \hat{\alpha} = 1, 2, \ldots, 16, \alpha, \bar{\alpha} = 1, 2, \ldots, 8\). The terms $\Lambda^- \mathcal{D}_j X^- \mathcal{D}_j$ are given by

$$\Lambda^\alpha = 2\sqrt{2} \left[ (\bar{\theta}^+ \gamma_+ K_{11})_\alpha + (\theta^+ \gamma_+ K_{21})_\alpha \right],$$

$$\Lambda^{\bar{\alpha}} = 2\sqrt{2} \left[ (\bar{\theta}^+ \gamma_+ K_{12})_{\bar{\alpha}} + (\theta^+ \gamma_+ K_{22})_{\bar{\alpha}} \right],$$  \hspace{1cm} (3.24)

where

$$\left( \gamma_+ \right)_{\alpha \beta} = \frac{1}{2} \left( \left( \gamma_0 \right)_{\alpha \beta} + (\gamma_9)_{\alpha \beta} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (3.25)

The terms $\Lambda^m_{n\hat{\alpha}} \mathcal{D}_j X^m$ are given by

$$\Lambda^a_{\alpha \bar{\alpha}} = -2 \left[ (\bar{\theta}^+ \gamma_{ab} \bar{\varrho} K_{11})_\alpha - (\theta^+ \gamma_{ab} \bar{\varrho} K_{21})_\alpha \right],$$

$$\Lambda^a_{\bar{\alpha} \bar{\alpha}} = -2 \left[ (\bar{\theta}^+ \gamma_{ab} \bar{\varrho} K_{12})_{\bar{\alpha}} - (\theta^+ \gamma_{ab} \bar{\varrho} K_{22})_{\bar{\alpha}} \right],$$

$$\Lambda^a_{(4+s)\hat{\alpha}} = 0.$$  \hspace{1cm} (3.26)
From the second equation, we have
\[ \Lambda_{\alpha \tilde{\alpha}}^{4+s} = 0, \]
\[ \Lambda_{(4+s')\alpha}^{4+s} = 2 \left[ (\theta^+ \gamma^{4+s+4+s'} g K_{11})_\alpha - (\theta^+ \gamma^{4+s+4+s'} g K_{21})_\alpha \right], \]
\[ \Lambda_{(4+s')\tilde{\alpha}}^{4+s} = 2 \left[ (\theta^+ \gamma^{4+s+4+s'} g K_{12})_{\tilde{\alpha}} - (\theta^+ \gamma^{4+s+4+s'} g K_{22})_{\tilde{\alpha}} \right]. \]  
(3.27)

The terms \( \Lambda_{\dot{\alpha} \dot{\beta}} \) in \( D_4 \Psi = (D_4 \theta^{+\alpha}, D_4 \tilde{\theta}^{+\dot{\alpha}}) \) are given by
\[ \Lambda_{\alpha \beta} = -\frac{i}{\sqrt{2}} (\gamma_+ \theta)^{\alpha \beta}, \quad \Lambda_{\dot{\alpha} \dot{\beta}} = \frac{i}{\sqrt{2}} (\gamma_+ \theta)^{\dot{\alpha} \dot{\beta}}, \quad \Lambda_{\dot{\alpha} \beta} = \Lambda_{\alpha \dot{\beta}} = 0. \]  
(3.28)

The fields \( B_{\alpha \dot{\beta}} \) in the Wess-Zumino term in (3.13) are defined by
\[ B_{\alpha \beta} = 2W_{\gamma \delta} \left( (L_{11})^{\gamma \alpha} (L_{11})^{\delta \beta} - (L_{21})^{\gamma \alpha} (L_{21})^{\delta \beta} \right), \]
\[ B_{\dot{\alpha} \dot{\beta}} = 2W_{\gamma \delta} \left( (L_{11})^{\gamma \dot{\alpha}} (L_{12})^{\delta \beta} - (L_{21})^{\gamma \dot{\alpha}} (L_{22})^{\delta \beta} \right), \]
\[ B_{\alpha \dot{\beta}} = 2W_{\gamma \delta} \left( (L_{12})^{\gamma \alpha} (L_{11})^{\delta \beta} - (L_{22})^{\gamma \alpha} (L_{21})^{\delta \beta} \right), \]
\[ B_{\dot{\alpha} \beta} = 2W_{\gamma \delta} \left( (L_{12})^{\gamma \dot{\alpha}} (L_{12})^{\delta \beta} - (L_{22})^{\gamma \dot{\alpha}} (L_{22})^{\delta \beta} \right), \]  
(3.29)

where
\[ W_{\alpha \beta} = \frac{[(1 + (z^2/4))(1 - (y^2/4))\bar{\epsilon}_{\alpha \beta} - z^a y^a (\gamma_a \gamma_{4+s} \theta)_{\alpha \beta}]}{(1 - (z^2/4))(1 + (y^2/4))}. \]  
(3.30)

### 3.2 Generalized light-cone gauge

Now let us consider the relevant part of (3.13)
\[ \mathcal{L}_1 := -\frac{1}{2} \sqrt{\lambda} h^{ij} G_{ab} D_i X^a D_j X^b. \]  
(3.31)

The generalized light-cone gauge is chosen as
\[ X^+ = \kappa \tau, \quad P_- = -\sqrt{\lambda} h^{0i} G_{-a} D_i X^a =: \sqrt{\lambda} \omega = \text{const.} \]  
(3.32)

From the second equation, we have
\[ D_0 X^- = -\frac{\omega}{h^{00} G_{--}} - \kappa \left( \frac{G_{+-}}{G_{--}} \right) - \left( \frac{h^{01}}{h^{00}} \right) D_1 X^-, \]  
(3.33)
or
\[ \dot{X}^- = -\Lambda_{-a} D_0 \dot{\Psi}^{\dot{a}} - \frac{\omega}{h^{00} G_{--}} - \kappa \left( \frac{G_{+-}}{G_{--}} \right) - \left( \frac{h^{01}}{h^{00}} \right) D_1 X^-. \]  
(3.34)

\[ \mathcal{L}_1 = \mathcal{L}_1' + P_- \dot{X}^-, \]  
(3.35)
Now $D_1 X^-$ is an auxiliary field. By solving the equations of motion, we have

$$D_1 X^- = -\frac{\omega h^{01}}{G_{--}}. \quad (3.37)$$

Substitution of this relation into $L'_1$ yields

$$L'_1 = \sqrt{\kappa} \kappa \omega \left( \frac{G_{--}}{G_{--}} \right) + \sqrt{\lambda} \omega \Lambda^- \hat{\alpha} D_0 \Psi^\hat{\alpha} \frac{1}{2} \sqrt{\lambda} h^{00} \left( \frac{\kappa^2}{G_{++}} \right) - \frac{1}{2} \sqrt{\lambda} \kappa \omega \Lambda^- \hat{\alpha} D_0 \Psi^\hat{\alpha}. \quad (3.38)$$

Let

$$J_{00} := \frac{\kappa^2}{G_{++}}, \quad J_{11} := \frac{\omega^2}{G_{--}}, \quad J_{01} = J_{10} := 0. \quad (3.39)$$

The GS action in the generalized light-cone gauge is given by

$$L'_{GS} = -\frac{1}{2} \sqrt{\lambda} h^{ij} (J_{ij} + G_{ij}) + \frac{1}{2} \sqrt{\lambda} \epsilon^{ij} B_{\hat{\alpha} \hat{\beta}} D_i \Psi^\hat{\alpha} D_j \Psi^\hat{\beta} \frac{1}{2} \sqrt{\lambda} \kappa \omega \left( \frac{G_{--}}{G_{--}} \right) + \sqrt{\lambda} \omega \Lambda^- \hat{\alpha} D_0 \Psi^\hat{\alpha}. \quad (3.40)$$

where

$$G_{ij} = G_{mn} D_i X^m D_j X^n. \quad (3.41)$$

By removing $h^{ij}$, we finally have the GS Lagrangian in the $AdS_5 \times S^5$ in the generalized light-cone gauge:

$$L'_{GS} = \sqrt{-\lambda} \det(J_{ij} + G_{ij}) + \frac{1}{2} \sqrt{\lambda} \epsilon^{ij} B_{\hat{\alpha} \hat{\beta}} D_i \Psi^\hat{\alpha} D_j \Psi^\hat{\beta} \frac{1}{2} \sqrt{\lambda} \kappa \omega \left( \frac{G_{--}}{G_{--}} \right) + \sqrt{\lambda} \omega \Lambda^- \hat{\alpha} D_0 \Psi^\hat{\alpha}. \quad (3.42)$$

Here

$$D_i \theta^+ = \partial_i \theta^+ - \frac{i \kappa}{\sqrt{2}} \delta_{i,0} (\theta \gamma_+ \theta^+), \quad D_i \theta^- = \partial_i \theta^- + \frac{i \kappa}{\sqrt{2}} \delta_{i,0} (\theta \gamma_- \theta^-), \quad (3.43)$$

$$D_i X^m = \partial_i X^m + (\Lambda^m_{\alpha} D_i \Psi^\hat{\alpha}) X^n. \quad (3.44)$$
This is the main result of this paper. It can be rewritten as follows:

\[
\mathcal{L}_{GS}' = \sqrt{-\lambda} \det(J_{ij} + G_{ij}) + \sqrt{\kappa} \omega \left( \frac{G_{+}}{G_{-}} \right) + \sqrt{\lambda} \kappa \omega \left( \sinh \frac{M}{M} \right) \left( \cosh \frac{M}{M} - \frac{1}{16} \right) D_{0} \hat{\Psi}.
\]

(3.45)

Here

\[
\left[ \sinh \frac{M}{M} \right]_{\hat{\alpha} \hat{\beta}} = \left[ \sinh \frac{M}{M} \right]_{\hat{\beta} \hat{\delta}} W_{\hat{\gamma} \hat{\delta}}^{\hat{\gamma}} \left( \frac{\cosh \frac{M}{M} - 1}{M^2} \right),
\]

(3.46)

\[
\left[ \sinh \frac{M}{M} \right]_{\hat{\alpha} \hat{\beta}} = \left[ \sinh \frac{M}{M} \right]_{\hat{\gamma} \hat{\delta}} W_{\hat{\gamma} \hat{\delta}} = \left( \frac{W_{\hat{\gamma} \hat{\delta}}}{0 \ 1} \right),
\]

(3.47)

with \( W_{\hat{\gamma} \hat{\delta}} = W_{\gamma \alpha} \), and

\[
(\gamma)_{\hat{\alpha} \hat{\beta}} = \left( \begin{array}{ll} 0 & (\gamma)_{\hat{\alpha} \hat{\beta}} \\ (\gamma)_{\hat{\alpha} \hat{\beta}} & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1_8 \\ 1_8 & 0 \end{array} \right).
\]

(3.48)

### 3.3 Flat space limit

In the last subsection, we show that the Green-Schwarz action in the generalized light-cone gauge (3.42) has the correct flat space limit.

In order to have consistent flat space limit, the following assumptions are necessary:

\[
\kappa < 0, \quad \omega > 0.
\]

(3.49)

Let us rescale the fields as follows:

\[
X^{m} = \lambda^{-1/4} \bar{X}^{m}, \quad m = 1, 2, \ldots, 8,
\]

(3.50)

\[
\theta^{+\alpha} = \frac{\lambda^{-1/8}}{\sqrt{2}} (S_{1}^{\alpha} + iS_{2}^{\alpha}), \quad \bar{\theta}^{+\alpha} = \frac{\lambda^{-1/8}}{\sqrt{2}} (S_{1}^{\alpha} - iS_{2}^{\alpha}), \quad \alpha = 1, 2, \ldots, 8.
\]

(3.51)

We also need to set the parameters as follows:

\[
\kappa = -\lambda^{-1/4} \bar{\kappa}, \quad \omega = \lambda^{-1/4} \bar{\kappa}, \quad \bar{\kappa} > 0.
\]

(3.52)

The flat space limit corresponds to \( \lambda \to \infty \) with \( \bar{\kappa} \) fixed. In the flat space, one of the light-cone gauge condition corresponds to \( \bar{X}^{+} = -\bar{\kappa} \tau \).
Taking $\lambda \to \infty$, and ignoring (divergent) surface terms, the GS Lagrangian in the generalized light-cone gauge arrives at

$$\mathcal{L}_{\text{GS}}^{\text{flat}} = \frac{1}{2} \sum_{m=1}^{8} \left( (\partial_0 \tilde{X}^m)^2 - (\partial_1 \tilde{X}^m)^2 \right) + \sqrt{2}i \kappa \sum_{\alpha=1}^{8} \left[ S^\alpha_1 (\partial_0 + \partial_1) S^\alpha_1 + S^\alpha_2 (\partial_0 - \partial_1) S^\alpha_2 \right].$$

(3.53)

This is the correct form of the Green-Schwarz action in the light-cone gauge.

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**A Notation**

**A.1 Gamma matrices**

Our choice of the $32 \times 32$ Gamma matrices is given by

$$\Gamma^a = \begin{pmatrix} 0 & (\gamma^a_{\alpha \beta}) \\ (\gamma^a_{\alpha \beta}) & 0 \end{pmatrix}, \quad \{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \mathbf{1}_{32},$$

(A.1)

with

$$((\gamma^a_{\alpha \beta}) = (1_{16}, \sigma^i), \quad ((\gamma^a_{\alpha \beta}) = (-1_{16}, \sigma^i),$$

(A.2)

where $\sigma^i$ are real symmetric $16 \times 16$ matrices which satisfy the $SO(9)$ Clifford algebra:

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}1_{16}, \quad i, j = 1, 2, \ldots, 9.$$  

(A.3)

The Gamma matrices in this “Majorana-Weyl” representation are real:

$$\left(\Gamma^a\right)^* = \Gamma^a, \quad a = 0, 1, \ldots, 9.$$  

(A.4)
A choice of $\sigma^i$ is [28]:

$$
\begin{align*}
\sigma^1 &= \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2, \\
\sigma^2 &= \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes 1_2, \\
\sigma^3 &= \tau_2 \otimes \tau_2 \otimes \tau_3 \otimes 1_2, \\
\sigma^4 &= \tau_2 \otimes 1_2 \otimes \tau_2 \otimes \tau_1, \\
\sigma^5 &= \tau_2 \otimes 1_2 \otimes \tau_2 \otimes \tau_3, \\
\sigma^6 &= \tau_2 \otimes \tau_1 \otimes 1_2 \otimes \tau_2, \\
\sigma^7 &= \tau_2 \otimes \tau_3 \otimes 1_2 \otimes \tau_2, \\
\sigma^8 &= \tau_1 \otimes 1_2 \otimes 1_2 \otimes \tau_2, \\
\sigma^9 &= \tau_3 \otimes 1_2 \otimes 1_2 \otimes \tau_2.
\end{align*}
$$

Here $\tau_k \ (k = 1, 2, 3)$ are the Pauli matrices.

The charge conjugation matrix $C$ is chosen as

$$
C = \begin{pmatrix}
0 & 1_{16} \\ -1_{16} & 0
\end{pmatrix}.
$$

The chirality matrix in $D = 9 + 1$ is defined by

$$
\Gamma := \Gamma^0 \Gamma^1 \ldots \Gamma^9 = \begin{pmatrix}
1_{16} & 0 \\ 0 & -1_{16}
\end{pmatrix}.
$$

Any 32-components spinor can be written as

$$
\Theta = \begin{pmatrix}
\theta_\alpha \\ \chi_\alpha
\end{pmatrix}.
$$

A Weyl spinor with positive chirality $\Gamma \Theta = +\Theta$ is given by

$$
\Theta = \begin{pmatrix}
\theta_\alpha \\ 0
\end{pmatrix},
$$

and a Weyl spinor with negative chirality $\Gamma \Theta = -\Theta$ is given by

$$
\Theta = \begin{pmatrix}
0 \\ \chi_\alpha
\end{pmatrix}.
$$

In the 16-components notation, an spinor $\theta_\alpha \ (\chi_\alpha)$ with the upper (lower) index $\alpha$ represents a Weyl spinor with positive (negative) chirality.
In this Majorana-Weyl representation, the matrix $\rho$ is given by

$$C\Gamma^{012345} = \begin{pmatrix} \rho_{\alpha\beta} & 0 \\ 0 & \rho^{\alpha\beta} \end{pmatrix},$$  \hspace{1cm} (A.11)$$

$$\rho_{\alpha\beta} = \tau_2 \otimes \tau_2 \otimes \tau_3, \hspace{1cm} \rho^{\alpha\beta} = \tau_2 \otimes \tau_2 \otimes \tau_3.$$  \hspace{1cm} (A.12)

An antisymmetric product of $\Gamma$’s is denoted by

$$\Gamma^{ab} = \Gamma^{[a} \Gamma^{b]} = \begin{pmatrix} (\gamma^{a \beta})_{\alpha \beta} & 0 \\ 0 & (\gamma^{a \beta})^{\alpha \beta} \end{pmatrix}.$$

(A.13)

### A.2 Indexes

We use the following conventions for the various indexes in $AdS_5 \times S^5$.

(i) The world sheet index: $i, j = 0, 1$.

$$\xi^0 = \tau, \hspace{1cm} \xi^1 = \sigma, \hspace{1cm} \epsilon^{01} = 1.$$  

(ii) The curved bosonic index: $m, n = 0, 1, \ldots, 9$.

$$m = (a, m) = (a, a, s): \hspace{1cm} a = \pm, \hspace{1cm} m = 1, 2, \ldots, 8, \hspace{1cm} a = 1, 2, 3, 4, \hspace{1cm} s = 1, 2, 3, 4.$$  

$$X^m: \hspace{1cm} X^0 = t, \hspace{1cm} X^9 = \varphi, \hspace{1cm} X^\pm = \frac{1}{\sqrt{2}}(t \pm \varphi), \hspace{1cm} X^a = z^a, \hspace{1cm} X^{4+s} = y^s.$$  

Also $a', b' = 5, 6, 7, 8$: $X^{a'} = y^{a'-4}$.

(iii) The local Lorentz index: $a, b = 0, 1, \ldots, 9$.

$$\eta_{ab} = \text{diag}(-1, +1, \ldots, +1).$$

(iv) The index for Weyl spinors: $\alpha, \beta, \bar{\alpha}, \bar{\beta} = 1, 2, \ldots, 16$.

The bar over $\alpha, \beta$ has no physical meaning, which are occasionally used to distinguish complex Weyl spinors from their conjugate:

$$\theta^\alpha = \frac{1}{\sqrt{2}}(\theta^{1\alpha} + i\theta^{2\alpha}), \hspace{1cm} \bar{\theta}^{\bar{\alpha}} = \frac{1}{\sqrt{2}}(\theta^{1\bar{\alpha}} - i\theta^{2\bar{\alpha}}).$$

Here $\theta^{I\alpha}$ ($I = 1, 2$) are Majorana-Weyl spinors: $(\theta^{I\alpha})^* = \theta^{I\alpha}$.  

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The decomposition of Weyl spinors:

\[ \theta^\alpha = \begin{pmatrix} \theta^+\alpha \\ \theta^-\tilde{\alpha} \end{pmatrix}, \quad \bar{\theta}^{\tilde{\alpha}} = \begin{pmatrix} \bar{\theta}^+\tilde{\alpha} \\ \bar{\theta}^-\alpha \end{pmatrix}, \]

\( \alpha = (\alpha, \tilde{\alpha}), \alpha = 1, 2, \ldots, 8, \tilde{\alpha} = \hat{1}, \hat{2}, \ldots, \hat{8}. \)

(v) The index for combined spinors: \( \hat{\alpha}, \hat{\beta} = 1, 2, \ldots, 16. \)

\[ (\Psi^{\hat{\alpha}}) = \begin{pmatrix} \theta^+\alpha \\ \bar{\theta}^+\tilde{\alpha} \end{pmatrix}. \quad (A.14) \]
References

[1] M.B. Green and J.H. Schwarz, “Covariant description of superstrings,” Phys. Lett. B136 (1984) 367-370.

[2] M.B. Green and J.H. Schwarz, “Properties of the covariant formulation of superstring theories,” Nucl. Phys. B243 (1984) 285-306.

[3] R.R. Metsaev and A.A. Tseytlin, “Type IIB superstring action in $AdS_5 \times S^5$ background,” Nucl. Phys. B533 (1998) 109-126, arXiv:hep-th/9805028.

[4] T. Hori and K. Kamimura, “Canonical Formulation of Superstring,” Prog. Theor. Phys. 73 (1985) 476-495.

[5] I. Pesando, “A $\kappa$ Gauge Fixed Type IIB Superstring Action on $AdS_5 \times S^5$,” JHEP 9811 (1998) 002, arXiv:hep-th/9808020;
I. Pesando, “All roads lead to Rome: Supersolvables and Supercosets,” Mod. Phys. Lett. A14 (1999) 343-348, arXiv:hep-th/9808146;
I. Pesando, “On the Fixing of the $\kappa$ Gauge Symmetry on $AdS$ and Flat Background: the Lightcone Action for the Type IIb String on $AdS_5 \otimes S_5$,” Phys. Lett. B485 (2000) 246-254, arXiv:hep-th/9912284.

[6] R. Kallosh, “Superconformal Actions in Killing Gauge,” arXiv:hep-th/9807206.

[7] R. Kallosh and J. Rahmfeld, “The GS String Action on $AdS_5 \times S^5$,” Phys. Lett. B443 (1998) 143-146, arXiv:hep-th/9808038.

[8] R. Kallosh and A.A. Tseytlin, “Simplifying superstring action on $AdS_5 \times S^5$,” JHEP 9810 (1998) 016, arXiv:hep-th/9808088.

[9] R.R. Metsaev and A.A. Tseytlin, “Superstring action in $AdS_5 \times S^5$: $\kappa$-symmetry light cone gauge,” Phys. Rev. D63 (2001) 046002, arXiv:hep-th/0007036.

[10] R.R. Metsaev, C.B. Thorn and A.A. Tseytlin, “Light-cone Superstring in AdS Space-time,” Nucl. Phys. B596 (2001) 151-184, arXiv:hep-th/0009171.

[11] C.G. Callan, T. McLoughlin and I. Swanson, “Holography beyond the Penrose limit,” Nucl. Phys. B694 (2004) 115-169, arXiv:hep-th/0404007.
[12] M. Kruczenski, A.V. Ryzhov and A.A. Tseytlin, “Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains,” Nucl. Phys. B692 (2004) 3-49, arXiv:hep-th/0403120.

[13] M. Kruczenski and A.A. Tseytlin, “Semiclassical relativistic strings in $S^5$ and long coherent operators in $\mathcal{N} = 4$ SYM theory,” JHEP 0409 (2004) 038, arXiv:hep-th/0406189.

[14] G. Arutyunov and S. Frolov, “Integrable Hamiltonian for Classical Strings on $AdS_5 \times S^5$,” JHEP 0502 (2005) 059, arXiv:hep-th/0411089.

[15] G. Arutyunov and S. Frolov, “Uniform Light-Cone Gauge for Strings in $AdS_5 \times S^5$: Solving $\mathfrak{su}(1|1)$ Sector,” JHEP 0601 (2006) 055, arXiv:hep-th/0510208.

[16] S. Frolov, J. Plefka and M. Zamaklar, “The $AdS_5 \times S^5$ Superstring in Light-Cone Gauge and its Bethe Equations,” J. Phys. A39 (2006) 13037-13082, arXiv:hep-th/0603008.

[17] P. Goddard, J. Goldstone, C. Rebbi and C.B. Thorn, “Quantum dynamics of a massless relativistic string,” Nucl. Phys. B56 (1973) 109-135.

[18] H. Itoyama and T. Oota, “The $AdS_5 \times S^5$ Superstrings in the Generalized Light-Cone Gauge,” Prog. Theor. Phys. 117 (2007) 957-972, arXiv:hep-th/0610325.

[19] D. Berenstein, J. Maldacena and H. Nastase, “Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills,” JHEP 0204 (2002) 013, arXiv:hep-th/0202021.

[20] D.M. Hofman and J.M. Maldacena, “Giant magnons,” J. Phys. A39 (2006) 13095-13118, arXiv:hep-th/0604135.

[21] J. Maldacena and I. Swanson, “Connecting giant magnons to the pp-wave: An interpolating limit of $AdS_5 \times S^5$,” Phys. Rev. D76 (2007) 026002, arXiv:hep-th/0612079.

[22] M.T. Grisaru, P. Howe, L. Mezincescu, B.E.W. Nilsson and P.K. Townsend, “$N = 2$ superstrings in a supergravity background,” Phys. Lett. B162 (1985) 116-120.

[23] R. Kallosh, J. Rahmfeld and A. Rajaraman, “Near horizon superspace,” JHEP 9809 (1998) 002, arXiv:hep-th/9805217.
[24] R.R. Metsaev and A.A. Tseytlin, “Supersymmetric D3 brane action in $AdS_5 \times S^5$,” Phys. Lett. B436 (1998) 281, arXiv:hep-th/9806095.

[25] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, “Superstring Theory on $AdS_2 \times S^2$ as a Coset Supermanifold,” Nucl. Phys. B567 (2000) 61-86, arXiv:hep-th/9907200.

[26] N. Berkovits, “Super-Poincaré Covariant Quantization of the Superstring,” JHEP 0004 (2000) 018, arXiv:hep-th/0001035.

[27] R. Roiban and W. Siegel, “Superstrings on $AdS_5 \times S^5$ supertwistor space,” JHEP 0011 (2000) 024, arXiv:hep-th/0010104.

[28] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, “Yang-Mills Theory as an Illustration of the Covariant Quantization of Superstrings,” in the proceedings of the Third Sacharov Conference, Moscow 2002, arXiv:hep-th/0211095.