On tilings defined by discrete reflection groups

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The recent articles of Waldspurger and Meinrenken contained the results of tilings formed by the sets of type \((1 - w)C^\circ\), \(w \in W\), where \(W\) is a linear or affine Weyl group, and \(C^\circ\) is an open kernel of a fundamental chamber \(C\) of the group \(W\). In this article we generalize these results to cocompact hyperbolic reflection groups. We also give more clear and simple proofs of the Waldspurger and Meinrenken theorems.

1 Introduction

Let \(X\) be a simply-connected space of a constant curvature and \(W\) be a cocompact discrete reflection group acting on this space. In a spherical case \(X \simeq S^n \subset V\) is a \(n\)-dimensional sphere embedded in \((n + 1)\)-dimensional Euclidean space \(V\) with the inner product \((\cdot, \cdot)\). In Euclidean case \(X\) is a \(n\)-dimensional affine Euclidean space, associated Euclidean vector space we denote by \(V\). In a hyperbolic case \(X \simeq \mathbb{H}^n \subset V\) is a connected component of \(n\)-dimensional hyperboloid embedded in \((n + 1)\)-dimensional Minkowskii space with the inner product \([x, y] = -x_0 y_0 + x_1 y_1 + \ldots + x_n y_n\) and defined by the equation \([x, x] = -1\). In the corresponding cases we call a group \(W\) spherical, euclidian or hyperbolic discrete reflection group.

The distance between points \(x, y \in X\) is denoted by \(\rho(x, y)\). By \(D\) denote a compact fundamental domain for the action of \(W\) on \(X\) and by \(D^\circ\) denote its interior. In spherical an hyperbolic cases let \(C\) be a fundamental chamber for the group \(W\) in the space \(V\) and \(C^\circ\) is its interior. (We can assume that \(D = C \cap X\) and \(D^\circ = C^\circ \cap X\).) Denote by \(X^W\) the space of \(W\)-fixed points (we note that \(X^W\) is not empty only in the spherical case) and by \(W_x\) we denote a stabilizer of a point \(x\) in the group \(W\). By \(W_{\text{reg}}\) we denote a set of elements in \(W\), those fixed point sets are equal to \(X^W\). We denote by \(s_H\) a reflection in a hyperplane \(H\).

This paper is organized as follows. In section 2 we prove a fundamental ”Fixed point lemma” for cocompact discrete reflection groups. In section 3 by means of this lemma we prove the theorems of Waldspurger [1] and Meinrenken [2] of tilings formed by the sets of type \((1 - w)C^\circ\), where \(w \in W\), and we also prove a corresponding theorem for hyperbolic reflection groups. In the last section 4 we consider the properties of these decompositions, we also relate them with the theorem of Kostant [3].

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2 Fixed point lemma

Lemma 1. For the points $x_0 \in D^o$ and $x \in D$ we have inequality $\rho(x_0, wx) \geq \rho(x_0, x)$, besides for every $w \not\in W_x$ the inequality is strict.

Proof. It is sufficient to prove the inequality $\rho(wx_0, x) \geq \rho(x_0, x)$. The proof is by the induction on the length $\ell(w)$ of the element $w$. We can find a wall $H$ of codimension one in the chamber $wD$ such that $\ell(s_Hw) < \ell(w)$. This means (cf. [4, 1, 5, 3]) that $D$ and $wD$ lie in the different half spaces cut by the hyperplane $H$. Let us denote by $x_H$ the intersection point of the hyperplane $H$ and a closed interval connecting $x$ and $wx_0$ (Fig. 1). Then we have $\rho(x_H, wx_0) = \rho(x_H, s_Hwx_0)$. By the induction assumption and a triangle inequality we have

$$\rho(x, x_0) \leq \rho(x, s_Hwx_0) \leq \rho(x, x_H) + \rho(x_H, s_Hwx_0) = \rho(x, wx_0).$$

A triangle inequality becomes an equality iff $x_H$ belongs to the closed interval $[x, s_Hwx_0]$. This is possible only when $x = x_H$, that implies $s_Hx = x$. Thus we have the equality $\rho(x, x_0) = \rho(x, wx_0)$ only when we have the equalities $\rho(x, s_Hwx_0) = \rho(x, x_0)$ and $s_Hx = x$. From the first equality and induction step we get $s_Hwx = x$, taking into account the second equality we obtain $w \in W_x$.

Lemma 2 (Fixed point lemma). Let $g \in \text{Isom}(X)$. Then there exists a unique element $w \in W$ such that the element $w^{-1}g$ has a fixed point in $D^o$.

Proof. 1. The proof is by the induction on $\dim X$. First we prove the existence of $w \in W$ such that $w^{-1}g$ has a fixed point in the closed chamber $D$. Since the group $W$ is cocompact from a topological viewpoint the chamber $D$ is a closed ball. Let $x \in D$ be an arbitrary point. Then the point $gx$ belongs to one of the chambers $wD$, where $w \in W$. Let us set $f(x) = w^{-1}gx \in D$. This defines a map $f$ from the ball $D$ into itself. It is clear that it is well defined (since the chamber $D$ is compact, the chamber $gD$ intersects only finite set of chambers of type $wD$) and continuous (indeed, if $x_1, x_2 \in D$ and $w_1x_1 = w_2x_2$ then $x_1 = x_2$). Applying the Brauer fixed point theorem we get that this map has a fixed point. Thus we obtain an element $w \in W$ and a point $x \in D$ such that $w^{-1}gx = x$.

For $x \in D^o$, there is nothing to prove. Let $x \in D \setminus D^o$. Consider a sphere $S(x)$ with the center in the point $x$, which doesn’t intersect the hyperplanes spanned by the faces of $D$ that do
not contain \(x\). We also consider the subgroup \(W_x \subset W\) that acts on the sphere \(S(x)\). The group \(W_x\) is generated by the reflections that fix the point \(x\). The chamber \(S(x) \cap D\) is fundamental domain for the group \(W_x\), since it is cut by those hyperfaces of the chamber \(D\) that contain \(x\). By the induction hypothesis applied to the sphere \(S(x)\), the element \(w^{-1}g\), the group \(W_x\) and the chamber \(S(x) \cap D\) there exists an element \(w' \in W_x\) and a point \(x' \in (S(x) \cap D)^0\) such that \(w'^{-1}w^{-1}gx' = x'\). Since the sphere \(S(x)\) does not intersect the hyperplanes spanned by those faces of the chamber \(D\) that do not contain \(x\) we have \(x' \in D^0\).

2. Let us prove the uniqueness of \(w\). Suppose there exist two elements \(w_1, w_2 \in W\) such that \(w_1^{-1}gx_1 = x_1\) and \(w_2^{-1}gx_2 = x_2\), where \(x_1, x_2 \in D^0\). Let us set \(w = w_1^{-1}w_2\), then we have \(\rho(gx_1, gx_2) = \rho(w_1x_1, w_1w_2x_2)\). If \(w \neq 1\) taking into account that \(W_x = 1\) we get the following inequalities from Lemma 11:

\[
\rho(gx_1, gx_2) = \rho(x_1, x_2) < \rho(x_1, wx_2) = \rho(w_1x_1, w_1wx_2).
\]

Thus we have \(w = 1\) and \(w_1 = w_2\).

**Remark.** In a more general case when we consider the fundamental domains of finite volume (i.e. the chamber \(D\) can have infinite points) Lemma 2 is not true. Indeed, consider a hyperbolic space \(X\) and a parallel transport \(g\) along the line containing the infinite point \(o\) of the chamber \(D\). Then there are no elements \(w \in W\) such that \(gx = wx\) for \(x \in D^0\). Assume the contrary, then the point \(gx = wx\) is contained in the horosphere \(O\) with the center in \(o\) that also contains \(x\), that is not true since \(gO \cap O = \emptyset\). Moreover one can prove that the equality \(gx = wx\) doesn’t hold for all points \(x \in D\).

**Corollary 1.** Let \(g : X \to X\) be a continuous map. Then there exists an element \(w \in W\) such that the transformation \(w^{-1}g\) has a fixed point in \(D\). Moreover, if we have \(\rho(gx_1, gx_2) < \rho(x_1, x_2)\) for all \(x_1, x_2 \in X\) and the fixed point belongs to \(D^0\) then such element \(w\) is unique.

**Proof.** The first claim follows from the proof of part 1 of “Fixed point lemma” 2. The second follows from Lemma 11 and the proof of second part of “Fixed point lemma”.

### 3 Tilings related to discrete reflection groups

In the spherical and hyperbolic cases let us define \(\Pi\) as a set of unite normals to the hyperfaces of the chamber \(C\) lying in the same halfspaces as \(C\). For every \(e \in \Pi\) denote by \(H_e\) a hyperface orthogonal to \(e\) and by \(s_e\) a reflection in \(H_e\). By \(C^*\) we denote the cone dual to \(C\) with respect to the corresponding inner product.

In a hyperbolic case we consider the cone \(\mathcal{K} = \{x \in V : [x, x] \leq 0\} = \mathcal{K}_+ \cup \mathcal{K}_-\), where \(\mathcal{K}_+ = \{x = (x_0, x_1, \ldots, x_n) \in \mathcal{K} : x_0 \geq 0\}\). Let us recall that in this case \([e, e] > 0\) for all \(e \in \Pi\), we also have the inclusions \(C \subseteq \mathcal{K}_+\) and \(\mathcal{K}_- \subseteq C^*\) (cf. [4, ex. 12–13 to §4 ch. 5]).

**Lemma 3.** For spherical or hyperbolic reflection group \(W\) we have the inclusion

\[
C^* \supseteq \bigcup_{w \in W} (1 - w)C^*.
\]

**Proof.** By Lemma 11 we have \(\rho(wx_0, x) \geq \rho(x_0, x)\) where \(x_0 \in C^*\) and \(x \in C\). In the case of spherical reflection group \(W\) we have:

\[
\frac{(wx_0, x)}{\sqrt{(wx_0, wx_0)(x, x)}} = \cos \rho(wx_0, x) \leq \cos \rho(x_0, x) = \frac{(x_0, x)}{\sqrt{(x_0, x_0)(x, x)}}.
\]
that implies that \((x, (1 - w)x_0) \geq 0\). In the case of hyperbolic group \(W\) the argument is similar (we have to remind the following formula for the distance in \(\mathbb{H}^n\): \(\text{ch} \rho(x_0, x) = -\frac{|x_0, x|}{\sqrt{|x_0, x_0||x, x|}}\).

**Theorem 1** (Waldspurger [2]). Let \(W\) be a spherical reflection group. We have a decomposition \(C^* = \bigcup_{w \in W} (1 - w)C^0\).

**Proof.** By “Fixed point lemma” [2] applied to the element \(g = s_u\), where \(u \in C^*\), there exists a point \(x \in C^0\) and a unique element \(w \in W\) such that \(w^{-1}s_u x = x\). Thus we have \(1 - w)x = \frac{2(u, x)}{(u, u)}u\). Setting \(v = \frac{(u, u)}{2(u, x)}x\), we get the equality \((1 - w)v = u\). Since \(x \in C^0\) and \((u, x) > 0\) we get \(v \in C^0\). Taking into account Lemma 3 this means that \(\bigcup_{w \in W} (1 - w)C^0 = C^*\).

**Theorem 2.** Let \(W\) be a Euclidian reflection group. Then for every \(h \in \text{Isom}(X)\) we have an equality \(V = \bigcup_{w \in W} (h - w)D^0\).

**Proof.** By Corollary 1 applied to the element \(g = t_{-u}h\) (where \(t_{-u}\) is a parallel transport by the vector \(-u\)) there exists a point \(v \in D\) and a unique element \(w \in W\) such that \(w^{-1}t_{-u}hv = v\). This is equivalent to \((h - w)v = u\).

**Corollary 2** (Meinrenken, [2], Thm.2). There is an equality \(V = \bigcup_{w \in W} (1 - w)D^0\).

**Corollary 3.** For every \(h \in GA(X)\) there is an equality \(V = \bigcup_{w \in W} (h - w)D\), moreover if \(\rho(hx_1, hx_2) \leq \rho(x_1, x_2)\) for all \(x_1, x_2 \in X\) the chamber \((h - w)D\) have pairwise intersections of codimension at least 1.

**Proof.** By Corollary 1 applied to the element \(g = t_{-u}h\) there exists a point \(v \in D\) and an element \(w \in W\) such that \(w^{-1}t_{-u}hv = v\). Thus we have \((h - w)v = u\). Moreover the element \(w\) is unique if \(v \in D^0\), that implies the second assertion.

**Remark.** Consider the spherical reflection subgroup \(W_0 \subset W\). From Theorem 2 we get that the set \(M = \bigcup_{w \in W_0} (1 - w)D^0\) is a fundamental domain for the subgroup of \(W\), generated by the parallel transports correspond to the simple coroots. The set \(N = \bigcup_{w \in W_0} wD\) is the closure of another fundamental domain for the subgroup of parallel transports in consideration. Thus we have \(\sum_{w \in W_0} \det(1 - w) \cdot \text{vol}(D) = \text{vol}(M) = \text{vol}(N) = |W_0| \cdot \text{vol}(D)\), that implies the formula \(\sum_{w \in W_0} \det(1 - w) = |W_0|\) (cf. [4], ex.3,§2,ch.5]). By the same argument one obtains \(\sum_{w \in W_0} \det(1 - hw^{-1}) = |W_0|\), where \(\rho(hx_1, hx_2) \leq \rho(x_1, x_2)\) for all \(x_1, x_2 \in X\).

**Theorem 3.** Let \(W\) be a hyperbolic reflection group. We have the following equalities: (i) \(C^* \setminus K_- = \bigcup_{w \in W} (1 - w)C^0\) (ii) \(K_- = \bigcup_{w \in W} (-1 - w)C^0\).

**Proof.** (i) First let us prove the inclusion \(\subseteq\). By Lemma 3 we have \(C^* \supseteq \bigcup_{w \in W} (1 - w)C^0\). Let us prove that \(\bigcup_{w \in W} (1 - w)C^0 \not\subseteq K_-\). The proof is by induction on the length \(\ell\) of a reduced decomposition of \(w\). The case \(\ell(w) = 0\) is obvious. If \(\ell(w) > 0\) we have an equality \(w = s_ew'\),
where \( \ell(w') = \ell(w) - 1 \). The latter means that the cones \( C \) and \( w'C \) lie in the same halfspace of \( H_e \), and in particular \( [w',e] > 0 \). We obtain

\[
[(1 - w)x, (1 - w)x] = [(1 - w')x, (1 - w')x] + 4\frac{[w'x,e][x,e]}{[e,e]} > 0
\]

for all \( x \in C^o \). This proves that \( C^* \setminus \mathcal{K}_- \supseteq \bigsqcup_{w \in W} (1 - w)C^o \).

Let us prove the opposite inclusion. Consider \( u \in C^* \setminus \mathcal{K}_- \). Let us apply “Fixed point lemma” \( \mathfrak{2} \) to the element \( g = s_u \). We obtain: \( wx = s_u x = x - 2\frac{[u,x]}{[u,u]} u \), hence \( (1 - w)x = 2\frac{[u,x]}{[u,u]} u \).

Setting \( v = \frac{[u,u]}{2[u,x]}x \) we get \( (1 - w)v = u \). It is obvious that \([u,x] > 0\), since \( u \not\in \mathcal{K}_- \) then \([u,u] > 0\). Thus \( v \in C^o \) that proves the desired decomposition.

(ii) First let us prove the inclusion \( \sqsupseteq \). As previously we argue by the induction on the length \( \ell \) of the shortest decomposition of \( w \). When \( \ell(w) = 0 \) we are done. If \( \ell(w) > 0 \) we have \( w = s_vw' \), where \( \ell(w') = \ell(w) - 1 \) and

\[
[(-1 - w)x, (-1 - w)x] = [(-1 - w')x, (-1 - w')x] - 4\frac{[w'x,e][x,e]}{[e,e]} < 0
\]

for all \( x \in C^o \). Besides it is obvious that \((-1 - w)x \not\in \mathcal{K}_+ \) for \( x \in C^o \). This implies \( \mathcal{K}_- \supseteq \bigsqcup_{w \in W} (1 - w)C^o \).

Let us prove the opposite inclusion. Consider \( u \in \mathcal{K}_o \). This time applying “Fixed point lemma” \( \mathfrak{2} \) to the element \( g = -s_u \), we get: \( wx = -s_u x = -x + 2\frac{[u,x]}{[u,u]} u \), thus \( (1 - w)x = -2\frac{[u,x]}{[u,u]} u \).

Setting \( v = -\frac{[u,u]}{2[u,x]}x \), we obtain \( (1 - w)v = u \). We have \([u,x] > 0\), and since \( u \in \mathcal{K}_o \), we have \([u,u] < 0\). This implies \( v \in C^o \) and proves the second decomposition.

\( \blacksquare \)

Remark. Theorem \( \mathfrak{3} \) cannot be generalized directly to the discrete hyperbolic groups with a fundamental domain of finite volume. We shall construct a point \( u \in C^* \setminus \mathcal{K}_- \) such that the equality \( u = (1 - w)v \) is impossible for all \( v \in C^o \) (and even for \( v \in C \)). Consider \( u \in (C^* \setminus \mathcal{K}_-) \cap \langle s_vo, o \rangle \), where \( e \in w'' \Pi, w'' \in W \) and \( o \in C \) is an infinite point, moreover \( H_u \cap C = \emptyset \). Let us prove that the equality \( u = (1 - w)v \) is impossible for every \( v \in C^o \).

Indeed there is \( \lambda > 0 \) such that \( \lambda s_vo = o \). Since \( H_u \cap C = \emptyset \), then \( \lambda < 1 \). This implies that if \( O \) is a horosphere with the center in \( o \), then \( s_voO \) lies inside the horosphere \( O \). Thus we get that \( u = (\lambda - w')p \), where \( w' = s_v \) and \( p = \frac{[v,u]}{2[u,o]}o \in C \). Assume that there exist \( w \in W \) and \( v \in C^o \) such that \( u = (1 - w)v \). Then we get \( (\lambda - w')p = (1 - w)v \), which implies \( \lambda p - v = w'p - wv \). Taking the norm of this equality we obtain \( (v, (\lambda - \tilde{w})p) = 0 \), where \( \tilde{w} = w^{-1}w' \). But we have \( (\lambda - \tilde{w})p = (1 - \tilde{w})p - (1 - \lambda)p \in C^* \) and the equality \( 2(v, (\lambda - \tilde{w})p) = 0 \) is possible only if \( (\lambda - \tilde{w})p = 0 \). But this is impossible due to limit argument. Indeed consider the sequence \( \{p_n\} \subseteq C^o \) such that \( p_n \to p \). If \( x \in C^o \) then \( 0 > [p_n,x] \geq [\tilde{w}p_n,x] \) and \( \lambda[p_n,x] < [p_n,x] \leq [\tilde{w}p_n,x] \). Thus \([\lambda p_n,x] < [p_n,x] \leq [\tilde{w}p_n,x] \). Taking the limit as \( n \to \infty \), we obtain \([\lambda p,x] < [p,x] \leq [\tilde{w}p,x] \), that contradicts the assumptions.

Thus we found a subset in \( C^* \setminus \mathcal{K}_- \) that is not covered by the cones of type \((1 - w)C^o \). The similar arguments also hold for the decomposition of \( \mathcal{K}_- \).

## 4 Properties of decompositions

In this paragraph we restrict ourselves to the case of finite reflection groups acting on Euclidian space \( V \). In the preceding paragraphs we obtained various theorems about decompositions
formed by the sets of type \((1 - w)C\), where \(w \in W\). One of the main questions related with these decompositions is the question which cones of type \(C_w = (1 - w)C\) are adjacent.

For the case when \(w, w' \in W_{\text{reg}}\), the answer was given in \([\mathbb{K}]\) (Figures 2 and 3 describe the sections of corresponding cones in the cases \(W = A_3\) and \(W = B_3\)). In case of finite reflection groups this question is related to the original proof of Walspurger theorem \([1]\) in \([\mathbb{I}]\). The main idea of the proof is to construct from an element \(w \in W_{\text{reg}}\) and a vector \(e \in \Pi\) the element \(w' \in W_{\text{reg}}\) such that the cones \(C_w\) and \(C_{w'}\) are adjacent and have intersection of codimension one contained in the wall \((1 - w)H_e\).

![Figure 2:](image-url)

![Figure 3:](image-url)

Before giving the answer to this question let us state two simple facts.

**Proposition 1.** Let \(g \in O(V)\) be an arbitrary orthogonal transformation and let \(s \in O(V)\) be a reflection. Then the dimensions of fixed point sets of \(g\) and \(gs\) differ exactly by 1.

**Proof.** The eigenvalues' norm of orthogonal transformation \(g\) is equal to 1, and each complex eigenvalue has a corresponding conjugated eigenvalue (counted with multiplicities). Thus we have \(\det g = (-1)^{n-k}\), where \(k = \dim \ker(1 - g)\). Since \(\det s = -1\) we get \(\det g = -\det(gs)\). This implies that \(\dim \ker(1 - g) = \dim \ker(1 - gs)\).

Without the loss of generality assume that \(\dim \ker(1 - g) > \dim \ker(1 - gs)\). On the other hand \(\ker(1 - g) \cap H \subset \ker(1 - gs)\) (where \(H\) is the reflection hyperplane) that implies \(\ker(1 - g) - 1 \leq \ker(1 - gs) < \ker(1 - g)\) and \(\ker(1 - gs) = \ker(1 - g) - 1\). \(\square\)

**Proposition 2.** If \(wC \cap w'C \neq \{0\}\), then \(C_w \cap C_{w'} \neq \{0\}\).

**Proof.** Indeed the equality \(wx = w'x'\) for \(x, x' \in C \setminus \{0\}\) implies that \(x = x'\) and \((1 - w)x = (1 - w')x'\). \(\square\)

For each \(e \in \Pi\) denote by \(\pi_e\) the vector orthogonal to \(V^W\) such that its inner product with \(f\) is equal to Kroneker symbol \(\delta_{ef}\) for all \(f \in \Pi\). Let us fix an element \(w \in W_{\text{reg}}\), for each vector \(v \in V\) we denote by \(v_u\) the vector orthogonal to \(V^W\) such that \((w^{-1} - 1)v_u = u\).

**Lemma 4** (cf. \([\mathbb{K}]\) Lemma 3). Let \(u, v \in V\) and \(e \in \Pi\). For vectors \(v_u, v_r\) and \(v_e\) we have the following properties: 1) \((v_u, r) + (v_r, u) = -(u, r);\) 2) \((v_u, u) = -\frac{1}{2}(u, u);\) 3) \(v_u = ws_u v_u;\) 4) \(v_u \perp (1 - w)H_u;\) 5) \(\ker(1 - ws_u) = (v_u);\) 6) \((v, (1 - w)\pi_e) < 0;\)

**Proof.** 1) Taking inner product of the equalities \(w^{-1}v_u = v_u + u\) and \(w^{-1}v_r = v_r + r\) we obtain \((v_u, v_r) = (w^{-1}v_u, w^{-1}v_r) = (v_u, v_r) + (v_u, r) + (v_r, u) + (u, r)\), that proves the equality. 3) From 2) we get that \(w^{-1}v_u = v_u + u = v_u - \frac{2\langle v_u, u \rangle}{\langle u, u \rangle}u = s_u v_u\). The proofs of other claims are evident. \(\square\)
Theorem 4 (cf. [3] Theorem 2). Let \( w \in W_{\text{reg}} \). The element \( w' \in W \) is regular and the cones \( C_w \) and \( C_{w'} \) are adjacent iff there exist vectors \( e, f \in \Pi \) such that \( w' = ws_ee_{\sigma}f \) and \( (v_e, f) > 0 \); in this case \( \text{codim}(1-w)H_e \cap (1-w')H_f = 1 \).

Figures 4 and 5 show the transversal sections of the cones considered in the theorem.

Let us consider the case when the cone of full dimension is adjacent to the cone of strictly smaller dimension. Let \( \dim C_{w'} < \dim C_w = n \). We say that the cone \( C_{w'} \) is adjacent to the cone \( C_w \) if \( \dim C_{w'} = \dim(C_{w'} \cap C_w) \).

Lemma 5. If \( w = w'\bar{w} \in W_{\text{reg}} \) and the cone \( C_{w'} \) is adjacent to \( C_w \) then we have \( \dim(1-w') + \dim(1-\bar{w}) = n \).

Proof. Consider the minimal set \( \{\pi_{i_1}, \ldots, \pi_{i_k}\} \) such that \( C_w \cap C_{w'} \subseteq (1-w)\langle \pi_{i_1}, \ldots, \pi_{i_k}\rangle = (1-w)\tilde{C} \). This implies \( \dim \ker(1-w') \geq n - \dim(1-w)\tilde{C} = n - k \).

There exist \( v \in \tilde{C} \) and \( v' \in C^\circ \) such that \((1-w)v = (1-w')v' \in C_w\). This implies \( v - v' = w'\bar{w}(v - v') \). Taking the scalar square of this equality after simplifications we obtain: \((1-\bar{w})v, v' = 0 \). Since \( v' \in C^\circ \) and \( (1-\bar{w})v \in C^* \) then we have \( v \in \ker(1-\bar{w}) \). By Steinberg fixed point theorem \( \langle \pi_{i_1}, \ldots, \pi_{i_k}\rangle \subseteq \ker(1-\bar{w}) \) that implies \( \dim \ker(1-\bar{w}) \geq k \) and \( \dim(1-w) + \dim(1-\bar{w}) \geq (n-k) + k = n \).

For the proof of the opposite inequality it is sufficient to note that \( \ker(1-w') \cap \ker(1-\bar{w}) = \{0\} \). Indeed if \( x \in \ker(1-w') \cap \ker(1-\bar{w}) \) then we have \( wx = w'(\bar{w}x) = x \) and \( x \in \ker(1-w) = \{0\} \). That implies \( \dim \ker(1-w') \leq n \). \( \square \)

Let \( R \) be the set of the unit normals to the reflection hyperplanes of group \( W \). It will be convenient for us to consider the decompositions of the elements of \( W \) in the products of the reflections \( s_u \) for \( u \in R \) where the reflection is not supposed to be simple (i.e. equal to \( s_e \) for \( e \in \Pi \)). The next theorem of Kostant (cf. [3] prop.5.1) describes this type of decompositions.

Theorem 5 (Kostant). Let \( w = s_{u_1}s_{u_2} \cdots s_{u_k} \) where \( u_1, u_2 \ldots u_k \in R \). Then the following assertions are equivalent: (i) \( \dim(1-w) = k \); (ii) \( w = s_{u_1}s_{u_2} \cdots s_{u_k} \) is the decomposition of minimal length; (iii) \( u_1, u_2 \ldots u_k \) are linearly independent.

Proof. (i)\( \Rightarrow (ii) \) and (i)\( \Rightarrow (iii) \). Since \( \ker(1-w) \supseteq H_{u_1} \cap H_{u_2} \cap \ldots \cap H_{u_k} \) we have \( k \geq \dim(1-w) \). In case of linear dependence of \( u_1, u_2 \ldots u_k \) the inequality is strict.

(ii)\( \Rightarrow (i) \). Arguing by the induction on \( \dim V \) let us show that there exists a decomposition of length not bigger than \( \dim(1-w) \). We may assume that the theorem is proved for all \( w \notin W_{\text{reg}} \). Indeed let \( V^w \neq 0 \) be the subspace of \( w \)-fixed vectors, we may apply the induction step to the subspace \( (V^w)\perp \) and the subgroup \( W_{V^w} \subseteq W \) that fixes \( V^w \) pointwise.
Let \( w \in W_{\text{reg}} \). Let us choose an arbitrary vector \( u \in R \). Since \( w \in W_{\text{reg}} \) from Proposition 1 we obtain \( \dim \ker(1-ws_u) > 0 \) (this also follows from Lemma 3). Thus we obtain \( ws_u \notin W_{\text{reg}} \) this assumes that the claim is proved for such elements, we are finished.

(iii) \( \Rightarrow \) (i). Assume that \( u_1, u_2, \ldots, u_k \) are linearly independent. Let us choose \( x_1 \in (H_{u_2} \cap \ldots \cap H_{u_k}) \setminus H_{u_1} \). Since \( (1-w)x_1 = \frac{2(x_1, u_1)}{(u_1, u_1)} u_1 \) we have \( u_1 \in (1-w)V \). Taking \( x_2 \in (H_{u_3} \cap \ldots \cap H_{u_k}) \setminus H_{u_2} \) we obtain \( (1-w)x_2 = c_{21}u_1 + \frac{2(x_2, u_2)}{(u_2, u_2)} u_2 \) that implies \( u_1, u_2 \in (1-w)V \). Taking \( x_i \in (H_{u_{i+1}} \cap \ldots \cap H_{u_k}) \setminus H_{u_i} \) we get \( (1-w)x_i = (1-s_{u_1} \ldots s_{u_i})x_i = c_{i1}u_1 + c_{i2}u_2 + \ldots + c_{i-1}u_{i-1} + \frac{2(x_i, u_i)}{(u_i, u_i)} u_i \), \( u_i \in (1-w)V \). The latter implies \( \dim(1-w)V \geq k \) and finishes the proof.

The decomposition of \( w \) into the product of reflections that satisfy the conditions of the Kostant theorem is called \textit{minimal}.

**Theorem 6.** Suppose we are given \( w = w'\tilde{w} \in W_{\text{reg}} \) such that \( \text{rk}(1-w') = k \) and the decomposition \( w' = s_{u_1} \ldots s_{u_k} \) is minimal. The cone \( C_w \) is adjacent to \( C_w \) iff \( \tilde{w} = s_{u_{k+1}} \ldots s_{u_n} \) where \( u_1, \ldots, u_n \) are linearly independent and \( u_{k+1}, \ldots, u_n \in \tilde{R} = R \cap \text{Im}(1-\tilde{w}) \). In particular if the cone \( C_w \) is adjacent to \( C_w \) then \( \tilde{w} \in \tilde{W}_{\text{reg}} \) where \( \tilde{W} \) is a reflection group generated by the reflections \( s_u \) for \( u \in \tilde{R} \).

**Proof.** By Lemma 3 we have: \( \text{rk}(1-\tilde{w}) = n - \text{rk}(1-w') = n - k \). Applying the Kostant theorem we can find the elements \( u_{k+1}, \ldots, u_n \) such that \( \tilde{w} = s_{u_{k+1}} \ldots s_{u_n} \) and the elements \( u_1, \ldots, u_n \) are linearly independent.

Let \( v \in C \) and \( v' \in C' \) be the vectors that satisfy \( (1-w)v = (1-w')v' \). From Theorem 3(iii) it follows that \( v \in \ker(1-\tilde{w}) = H_{u_{k+1}} \cap \ldots \cap H_{u_n} \). From the condition \( \dim C_w' = \dim(C_w \cap C_w') \) and the proof of Lemma 3 it follows that \( \tilde{C} \subset H_{u_i} \) for all \( i = k + 1, \ldots, n \). That implies \( u_{k+1}, \ldots, u_n \in \tilde{R} \) and proves the claim.

In opposite direction, if \( \tilde{w} = s_{u_{k+1}} \ldots s_{u_n} \) where \( u_1, \ldots, u_n \) are linearly independent, then by theorem of Kostant \( \text{rk}(1-w) = n \). Since \( u_{k+1}, \ldots, u_n \in \tilde{R} \), we have \( \tilde{C} \subset H_{u_{k+1}} \cap \ldots \cap H_{u_n} \) and \( \dim(H_{u_{k+1}} \cap \ldots \cap H_{u_n} \cap C) = k \). For every \( x \in H_{u_{k+1}} \cap \ldots \cap H_{u_n} \cap C \) we get \( (1-w)x = x - w'(s_{u_{k+1}} \ldots s_{u_n}x) = (1-w')x \) that implies \( \dim C_{w'} = \dim(C_{w'} \cap C_w) \).

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