Maximizing operating speed of an interval control system with a robust controller on a base of a root approach to synthesis

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Abstract. The paper is dedicated to a problem of improving operating speed of control systems with interval parameters. Solution of the problem reduces to finding values of controller parameters providing minimal setting time of a system with the worst combination of values of interval parameters. Controller synthesis is proposed to be performed on a base of interval characteristic polynomial of a system on a base of maximal stability degree criterion. The research resulted in a synthesis algorithm based on an interval extension of mathematical programming method. Example of synthesizing a robust controller providing maximal operating speed for a system with interval parameters via the algorithm is also given.

1. Introduction
Vast variety of industrial control systems (CS) include a control object with uncertain parameters varying within certain ranges according to unknown laws. In conditions of such interval parametric uncertainty a problem of providing maximal operating speed of an interval control system is highly relevant. Interval CS can be described as a set of CS with time-invariant parameters each of which can be considered as a separate operating mode of interval CS. Considering this, a problem of operating speed maximization can be reduced to maximizing operating speed in the worst mode of a system or, in other words, for the worst combination of values of interval parameters. Due to the fact, that CS operating speed is determined by its stability degree it is reasonable to apply a root approach [1]-[8] to synthesis and base proposed method of synthesis on a maximal stability degree criterion [9]-[15]. In [15] a method of finding parameters of a controller, which provide maximal stability degree for the time-invariant control system considered, is described. The problem of extending this method to interval control system synthesis is highly relevant.

2. Formulation of the problem
To formulate the problem of the research let us firstly consider time-invariant control systems described via its characteristic polynomial. There are methods [15,16] of operating speed maximization for such systems. It should be noticed, that problem in [15] is solved with a method of non-linear mathematical programming. It is highly relevant to find a solution for the similar problem for CS with interval parameters, which are included in coefficients of interval characteristic polynomial with parameters of a controller. Parametric variations within certain intervals cause variations of CS stability degree. There is a set of values of interval parameters which provides minimal value of stability degree. It was determined, that in order to find this set, it is enough to examine only vertices of interval CS polytope of parameters and not all combinations of their values.
[17]. Considering this, the problem of the research can be formulated as follows: to develop a method of finding a set of optimal values of linear controller parameters maximizing a minimal stability degree of a CS in vertices of its polytope of parameters.

3. Conditions of maximizing stability degree of a time-invariant control system

Let us define a characteristic polynomial of a time-invariant CS as follows:

\[ D(s, k) = \sum_{i=0}^{n} d_i(k) \cdot s^i, \]  

where \( k \) – is a vector of controller parameters. Let us assume, that a root of (1) determining a stability degree of a system is complex-conjugate \( s = \alpha + j\beta \), where \( \alpha \) – is a value of stability degree. After substituting the expression for root to (1), a characteristic polynomial can be rewritten as follows:

\[ D(\alpha, \beta, k) = \sum_{i=0}^{n} d_i(k) \cdot (\alpha + j\beta)^i. \]  

To find maximal stability degree \( \alpha \), a system of equations based on real and imaginary parts of (2) and their partial derivatives to \( \alpha \) must be solved:

\[
\begin{align*}
\text{Re} D\left( k, \alpha, \beta \right) & = 0; \\
\text{Im} D\left( k, \alpha, \beta \right) & = 0; \\
\frac{\partial \text{Re} D\left( k, \alpha, \beta \right)}{\partial \alpha} & = 0; \\
\frac{\partial \text{Im} D\left( k, \alpha, \beta \right)}{\partial \alpha} & = 0; \\
L \quad \frac{\partial^n \text{Re} D\left( k, \alpha, \beta \right)}{\partial \alpha^n} & = 0; \\
\frac{\partial^n \text{Im} D\left( k, \alpha, \beta \right)}{\partial \alpha^n} & = 0.
\end{align*}
\]

Number of equations in (3) is depends on number of variables of interest, which are \( \alpha, \beta \) and controller parameters.

4. Conditions of maximizing a stability degree of a control system with interval parameters

Let us designate interval parameters of a control object as \( T_j \), \( j = 1,r \); an adjustable parameter of a controller as \( k \). In this case, the characteristic polynomial can be written as follows:

\[ D(s, k, T_j) = \sum_{i=0}^{n} d_i(k, T_j) \cdot s^i. \]  

Interval parameters \( [T_j] \) compose a polytope \( P \) with a number of vertices equal to \( V = 2^r \), where \( r \) – is a number of interval parameters. Let us substitute \( s = \alpha + j\beta \) to (4) and derive an equation system for finding maximal stability degree in a vertex \( v \) of parametric polytope \( P \) on a base of (3):
where vertex index $v = 1, V$ corresponds to each of operating modes of the system. By solving (5) $V$ times, $V$ values of controller parameter can be obtained and also values of stability degree provided by them. From these data an operating mode with minimal stability degree $\alpha_{v_{\min}}$ and value of controller parameter $k^*$ corresponding to it must be chosen.

Then, value of stability degree $\alpha_v(k^*)$ must be found for all other vertices of parametric polytope $v = 1, V$, $v \neq q$ considering $k = k^*$. To do this, $V$ equation systems must be solved:

$$
\begin{align*}
\text{Re } D_v(k, \alpha, \beta) &= 0; \\
\text{Im } D_v(k, \alpha, \beta) &= 0; \\
\partial \text{Re } D_v(k, \alpha, \beta) / \partial \alpha &= 0,
\end{align*}
$$

(5)

and then compare found $\alpha_v(k^*)$ values with $\alpha_{v_{\min}}$. If $\alpha_v(k^*) > \alpha_{q_{\min}}$, then the problem of synthesis is solved and stability degree of the control system considered will be more than $\alpha_{q_{\min}}$ for all values of interval parameters. If $\alpha_v(k^*) < \alpha_{q_{\min}}$ in some vertex, then synthesis procedure must be continued by plotting graphs of stability degree function of controller parameter for all vertices of parametric polytope. Eventually, the worst operating mode of the system considered will be defined by intersection of two of these graphs.

In the paper it is proposed to find such mode in another way – by composing a set of conditions (5) for all pairs of vertices of parametric polytope $v = 1, V$, $v \neq q$ considering $k = k^*$. To do this, $V$ equation systems must be solved:

$$
\begin{align*}
\text{Re } D_{i}(\alpha, \beta)\big|_{k=k^*} &= 0; \\
\text{Im } D_{i}(\alpha, \beta)\big|_{k=k^*} &= 0,
\end{align*}
$$

(6)

where $\text{Re } D_{i}(\alpha, \beta)$ and $\text{Im } D_{i}(\alpha, \beta)$ – are real and imaginary parts of characteristic polynomial in $i$-th vertex; $\text{Re } D_{j}(\alpha, \beta)$ and $\text{Im } D_{j}(\alpha, \beta)$ – are real and imaginary parts of characteristic polynomial in $j$-th vertex. It is now necessary to solve system (7) $C_v^2$ times and choose one of all solutions which provides minimal value of stability degree $\alpha$. This value will be minimal value of stability degree in all vertices of parametric polytope and will correspond to the worst operating mode of the system considered. The solution chosen will also provide the value of controller parameter $k$ providing maximal operating speed of the system in the worst operating mode.

On a base of the research considered, an algorithm of synthesizing a controller with one adjustable parameter providing maximal operating speed of a control system with interval parameters was developed. Flowchart of the algorithm is shown in the figure 1.
Begin
Defining \([T_j]\), \(s, D(s, k, [T_j])\)
Finding \(V(T_j)\)
Finding \(s = \alpha + j\beta\)
Finding \(ReD_v(k, \alpha, \beta), ImD_v(k, \alpha, \beta)\)
\(\forall v = 1...2^r, v \neq l\)
Solving (5)
\(\alpha_{\text{min}}, \alpha_{\text{max}}\)
\(\alpha_{\text{min}} = \min \alpha_{\text{max}}\)
\(k^* = k\{z\}\)
Solving (6)
\(\forall v = 1...2^r\)
\(\alpha = \min \alpha_{\text{min}}\)
\(\alpha = \alpha_{\text{min}}\)
Yes
No
end

Figure 1. Flowchart of the algorithm of synthesizing a controller parameter \(k\) providing maximal operating speed of the control system synthesized.

5. Example
Let us apply the algorithm proposed to a problem of synthesizing a controller for a control system shown in the figure 2.

Figure 2. Structural diagram of the interval control system considered.

Let us assume, that the system considered is equipped with a proportional controller \(W_1(s) = k\). Control object of the system is described as follows: \(W_2(s) = 1/(\{T_1 s + 1\})\{T_2 s + 1\})\{T_3 s + 1\})s\), where \(T_1 = 0.01s\), \([T_2] = [0.02; 0.08]s\), \([T_3] = [0.05; 0.2]s\). Interval parameters \(T_2\) and \(T_3\) compose a parametric polytope which is a rectangle \(P\) with four vertices. The problem is to find a value of \(k\) providing maximal operating speed for all values of interval parameters. Let us derive a transfer function of the closed-loop system considered: \(W(s) = K/(\{T_1 s + 1\})\{T_2 s + 1\})\{T_3 s + 1\})s + k\); and its characteristic polynomial \(W_1 D(s, k, [T_j]) = T_1[T_2][T_3]s^4 + (T_1[T_2] + T_1[T_3] + [T_2][T_3])s^3 + (T_1 + [T_2] + [T_3])s^2 + s + k\). Then,
according to the synthesis algorithm proposed, let us substitute \( s = \alpha + j \beta \) to the characteristic polynomial \( D(s,k,\left[ T_j \right]) \) and find its real and imaginary part. On their base, equations systems (5) can be composed and solved for all four vertices of \( P \). Solutions obtained are given in the table 1.

**Table 1.** Solutions of equations system (5) for all four vertices of \( P \)

| Vertex coordinates | \( V_1 \) | \( V_2 \) | \( V_3 \) | \( V_4 \) |
|---------------------|----------|----------|----------|----------|
| \( T_1 \) = 0.02   | \( T_2 \) = 0.05 | \( T_1 \) = 0.05 | \( T_2 \) = 0.08 | \( T_1 \) = 0.02 |
| Maximal stability degree | 8.306 | 4.834 | 2.402 | 2.171 |
| Value of adjustable parameter \( k \) | 3.713 | 2.139 | 1.16 | 0.993 |

Table 1 shows, that the worst operating mode with minimal stability degree \( \alpha_{\text{min}} = 2.171 \) corresponds to vertex \( V_4 \). However, if \( k^* = 0.993 \), then stability degree in other vertices is less, than in vertex 4: \( V_1 - \alpha_{\text{min}}(k^*) = 1.085 \), vertex \( V_2 - \alpha_{\text{min}}(k^*) = 1.791 \) and vertex \( V_3 - \alpha_{\text{min}}(k^*) = 1.471 \). Considering this, according to the synthesis algorithm proposed it is necessary to derive and solve equations systems (7) for all pairs of vertices of parametric polytope. Solutions are provided in the table 2.

**Table 2.** Solutions of equations system (7)

| \( V_1 \) | \( V_2 \) | \( V_3 \) | \( V_4 \) |
|----------|----------|----------|----------|
| \( \alpha = 4.448 \) | \( \alpha = 4.448 \) | \( \alpha = 2.339 \) | \( \alpha = 1.982 \) |
| \( k = 3.011 \) | \( k = 1.923 \) | \( k = 1.649 \) | \( k = 1.681 \) |
| \( \alpha = 2.339 \) | \( \alpha = 2.339 \) | \( \alpha = 2.362 \) | \( \alpha = 2.030 \) |
| \( k = 3.011 \) | \( k = 1.649 \) | \( k = 1.649 \) | \( k = 1.681 \) |
| \( \alpha = 1.982 \) | \( \alpha = 2.030 \) | \( \alpha = 2.127 \) | \( \alpha = 2.127 \) |
| \( k = 1.681 \) | \( k = 1.497 \) | \( k = 1.145 \) | \( k = 1.145 \) |

Table 2 shows, that CS considered reaches maximal value of minimal stability degree equal to \( \alpha = 1.982 \) in vertices 1 and 4; \( k = 1.681 \). This value of controller parameter provides following values of stability degree in vertices 2 and 3: \( \alpha = 2.437 \), \( \alpha = 2.359 \). Considering this, \( k = 1.681 \) provides maximum of minimal stability degree of the CS considered; its stability degree will be more than 1.982 for all values of interval parameters.

6. Conclusion

Considered research resulted in a method of parametric synthesis of a robust controller allowing to improve operating speed of control systems with interval parameters. The approach proposed provides relatively simple procedure of finding values of controller parameters, which maximize stability degree of an interval control system synthesized. The procedure is based on interval extension of mathematical programming method. It was also tested on a numerical example of controller synthesis.

7. References

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