ON THE VECTOR BUNDLES FROM CHANG AND RAN’S PROOF OF THE UNIRATIONALITY OF $M_g$, $g \leq 13$

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Abstract. We combine the idea of Chang and Ran [Invent. Math. 76 (1984), 41–54] of using monads of vector bundles on the projective 3-space to prove the unirationality of the moduli spaces of curves of low genus with our classification of globally generated vector bundles with the first Chern class $c_1 = 5$ on the projective 3-space arXiv:1805.11336 to get an alternative argument for the unirationality of the moduli spaces of curves of degree at most 13 (following the general guidelines of the method of Chang and Ran but with quite different effective details).

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Introduction

The problem of the unirationality of the moduli space $M_g$ of curves of genus $g$ is a classical one: see Verra [18] for a recent survey. We have to mention, however, the papers of Arbarello and Sernesi [3] (who treated, in modern terms, the classical case $g \leq 10$), Sernesi [14] (who solved the case $g = 12$), Mori and Mukai [12] (who proved the uniruledness of $M_{11}$), Chang and Ran [7] (who showed that $M_{11}$, $M_{13}$ (and $M_{12}$) are unirational), and Verra [17] (who solved the case $g = 14$ and lower, using results of Mukai [13]). The subject came to our attention incidentally, as a consequence of our study [1], [2] of globally generated vector bundles with small $c_1$ on projective spaces. The connection with the above mentioned problem stemmed from the circumstance that Chang and Ran [7] use a certain class of vector bundles with $c_1 = 5$ on $\mathbb{P}^3$ to prove the unirationality of $M_g$, for $g \leq 13$. More precisely,

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Chang and Ran [7] show that, for \( g \leq 13 \), there exists a family of nonsingular space curves of genus \( g \) and degree \( d = \lceil (3g + 12)/4 \rceil \) having general moduli and other “good properties”. These curves, in case they exist, can be represented as the dependency locus of \( r - 1 \) global sections of certain vector bundles of rank \( r = g - d + 4 \) and, in turn, these vector bundles are the cohomology sheaves of some linear monads with terms depending only on \( g \) and \( d \). Finally, the good properties of the curves imply, quickly, that the space of these monads is rational. At this point, the vector bundles leave the stage and Chang and Ran concentrate on the existence of the above mentioned families of curves, which is the key and difficult point of their approach. In order to achieve this goal, the two authors use a method of Sernesi [15] which consists in successively attaching 4-secant conics to a curve of lower degree and genus, then showing that the resulting reducible curve has the necessary properties and, finally, smoothing that curve.

The alternative approach we propose in this paper is to concentrate on the vector bundles instead of the curves. More precisely, we show that the cohomology sheaf of a general monad of the above type is a vector bundle \( E \) with a number of good properties and that the dependency locus of \( r - 1 \) general global sections of \( E \) is a nonsingular curve which has the properties required by the approach of Chang and Ran. The difficult point becomes, this time, the proof of the fact that \( r - 1 \) general global sections of \( E \) are linearly dependent along a nonsingular curve. We verify this by showing that \( E \) is globally generated if \( g \leq 12 \), while for \( g = 13 \), where this fact is no longer true, we use a criterion of Martin-Deschamps and Perrin [10], recalled in Lemma A.6 from Appendix A. Actually, if one wants to stick to vector bundles, the method of Chang and Ran works only in the range \( 8 \leq g \leq 13 \). The approach with linear monads can be, however, extended to the range \( 5 \leq g \leq 7 \), replacing the vector bundles by rank 2 reflexive sheaves. This approach is due to Chang [6, Cor. 4.8.1].

The paper is organized as follows: we explain the method of Chang and Ran and our alternative approach in Section 1. We treat, then, in the next sections, the cases \( 8 \leq g \leq 11 \), \( g = 13 \), and \( g = 12 \). We say, in the final Section 5, a few words about the cases \( 5 \leq g \leq 7 \). We gather, in Appendix A some facts about monads and dependency loci while Appendix B contains a number of other auxiliary results.

**Note.** Our emphasis in this paper is on the usefulness of vector bundles on projective spaces in handling various geometric problems. The study of these bundles was an active area of research in the 1970s and 1980s but fall later into oblivion. One ignores, nowadays, many of the results and techniques developed during that period.

**Notation.** (i) We denote by \( \mathbb{P}^n \) the projective \( n \)-space over an algebraically closed field \( k \) of characteristic 0 and by \( S \simeq k[x_0, \ldots, x_n] \) its projective coordinate ring.
(ii) If \( \mathcal{F} \) is a coherent sheaf on \( \mathbb{P}^n \) and \( i \geq 0 \) is an integer, we denote by \( H^i(\mathcal{F}) \) the graded \( S \)-module \( \bigoplus_{l \in \mathbb{Z}} H^i(\mathcal{F}(l)) \) and by \( h^i(\mathcal{F}) \) the dimension of \( H^i(\mathcal{F}) \).
(iii) If \( E \) is a vector bundle (= locally free sheaf) on a variety \( X \), we denote its dual \( \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X) \) by \( E^\vee \).
(iv) If \( Y \subset X \subset \mathbb{P}^n \) are closed subschemes of \( \mathbb{P}^n \), defined by ideal sheaves \( \mathcal{I}_X \subset \mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n} \), we denote by \( \mathcal{I}_{Y,X} \) the ideal sheaf of \( Y \) as a subscheme of \( X \), that is \( \mathcal{I}_{Y,X} = \mathcal{I}_Y/\mathcal{I}_X \).
(v) If \( D \) is a Cartier divisor on an equidimensional projective scheme \( X \), we denote by \( \mathcal{O}_X[D] \) the associated invertible \( \mathcal{O}_X \)-module.

1. **The Method of Chang and Ran**

Let \( g \) be an integer with \( 8 \leq g \leq 13 \) and let \( Y \subset \mathbb{P}^3 \) be a nonsingular, connected, nondegenerate (that is, not contained in a plane) space curve, of genus \( g \) and (some) degree \( d \). Let \( H_{d,g} \) denote the open subset of the Hilbert scheme of subschemes of \( \mathbb{P}^3 \) parametrizing nonsingular, connected space curves of degree \( d \) and genus \( g \) and let \([Y]\) be the point of \( H_{d,g} \) corresponding to \( Y \).

**1.1. The map from the Hilbert scheme to the moduli space.** Consider the exact sequences:

\[
0 \longrightarrow T_Y \longrightarrow T_{\mathbb{P}^3} \mid Y \longrightarrow N_Y \longrightarrow 0,
\]
\[
0 \longrightarrow \mathcal{O}_Y \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee \otimes_k \mathcal{O}_Y(1) \longrightarrow T_{\mathbb{P}^3} \mid Y \longrightarrow 0,
\]

where \( N_Y := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}_Y, \mathcal{O}_Y) \) is the normal bundle of \( Y \) in \( \mathbb{P}^3 \) (and \( T_Y \) is the tangent bundle of \( Y \)). Notice that the map \( H^1(\mathcal{O}_Y) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee \otimes H^1(\mathcal{O}_Y(1)) \) is the dual of the multiplication map \( \mu : H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\omega_Y(-1)) \rightarrow H^0(\omega_Y) \). It follows that if \( \mu \) is injective then \( H^1(T_{\mathbb{P}^3} \mid Y) = 0 \) which implies that \( H^1(N_Y) = 0 \)
(hence \( H_{d,g} \) is nonsingular, of (local) dimension \( h^0(N_Y) = \chi(N_Y) = 4d \), at \([Y]\)) and that the connecting morphism \( \partial : H^0(N_Y) \rightarrow H^1(T_Y) \) is surjective. But \( H^0(N_Y) \) is the tangent space of \( H_{d,g} \) at \([Y]\) and \( \partial \) is the Kodaira-Spencer map of the universal family of curves over \( H_{d,g} \) at \([Y]\). One deduces that the restriction of the natural map \( H_{d,g} \rightarrow \mathcal{M}_g \) to a neighbourhood of \([Y]\) is dominant.

In order to use the intrinsic geometry of \( Y \), one assumes that \( Y \) is linearly normal (that is, \( H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(\mathcal{O}_Y(1)) \) or, equivalently, \( H^1(\mathcal{O}_Y(1)) = 0 \)). In this case, \( \mu \) is injective if and only if the multiplication map \( \mu_0(Y) : H^0(\mathcal{O}_Y(1)) \otimes H^0(\omega_Y(-1)) \rightarrow H^0(\omega_Y) \) is injective. Notice that \( h^1(\omega_Y(-1)) = h^0(\mathcal{O}_Y(1)) = 4 \) hence, by Riemann-Roch on \( Y \), \( h^0(\omega_Y(-1)) = g - d + 3 \) and the injectivity of \( \mu_0(Y) \) implies that the Brill-Noether number \( \rho(g, 3, d) = g - 4(g - d + 3) = 4d - 3g - 12 \) is non-negative. One assumes that \( d \) is the least integer satisfying this condition.

**1.2. Space curves and vector bundles with \( c_1 = 5 \).** If \( \omega_Y(-1) \) is globally generated then, putting \( r := 1 + h^0(\omega_Y(-1)) \), any epimorphism \( \delta : (r - 1)\mathcal{O}_{\mathbb{P}^3} \rightarrow \omega_Y(-1) \) defined by a \( k \)-basis of \( H^0(\omega_Y(-1)) \) determines, according to Serre’s method of extensions, an extension:

\[
0 \longrightarrow (r - 1)\mathcal{O}_{\mathbb{P}^3} \longrightarrow E \longrightarrow \mathcal{I}_Y(5) \longrightarrow 0,
\]

with \( E \) locally free of rank \( r \), such that, dualizing the extension, one gets the exact sequence:

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^\vee \longrightarrow (r - 1)\mathcal{O}_{\mathbb{P}^3} \overset{\delta}{\longrightarrow} \omega_Y(-1) \longrightarrow 0.
\]

One deduces, from the last exact sequence, that \( H^i(E^\vee) = 0, \ i = 0, 1 \). The Chern classes of \( E \) are \( c_1 = 5, c_2 = d, c_3 = \deg \omega_Y(-1) = 2g - 2 - d \). \( Y \) is linearly normal if and only if \( H^1(E(-4)) = 0 \) (which, by Serre duality, is equivalent to \( H^2(E^\vee) = 0 \)) and, in this case, \( E \) has rank \( r = g - d + 4 \). Under these assumptions, \( \mu_0(Y) \) is injective if and only if \( H^0(E^\vee(1)) = 0 \).
Conversely, let $E$ be a vector bundle of rank $r$, with $c_1 = 5$, and such that $H^i(E^\vee) = 0, i = 0, 1$. Let $W$ be an $(r - 1)$-dimensional vector subspace of $H^0(E)$ such that the degeneracy locus of the evaluation morphism $W \otimes_k \mathcal{O}_{P^3} \to E$ is a nonsingular curve $Y$. Then the Eagon-Northcott complex of this morphism provides an exact sequence:

$$0 \to W \otimes_k \mathcal{O}_{P^3} \to E \to \mathcal{I}_Y(5) \to 0,$$

which, by dualization, defines an epimorphism $\delta : W^\vee \otimes_k \mathcal{O}_{P^3} \to \omega_Y(-1)$ with $H^0(\delta)$ bijective. One gets, in this case, a family of nonsingular space curves parametrized by an open subset of the Grassmannian $\mathbb{G}_{r-1}(H^0(E))$ of $(r - 1)$-dimensional vector subspaces of $H^0(E)$.

Now, if $E$ varies in a family with irreducible, rational base, then, in order to get a family of nonsingular space curves with irreducible, rational base, one has to assume that $h^0(E)$ is constant in the family of vector bundles and that the evaluation morphisms $H^0(E) \otimes_k \mathcal{O}_{P^3} \to E$ can be patched together to a morphism of vector bundles on the total space of the family. This can be accomplished by assuming that $H^1(E) = 0$, for any bundle $E$ in the family, by Grothendieck’s cohomology and base change theorem (see, for example, Tengan [16, Cor. 3.1]). Notice, also, that $H^2(E) \simeq H^2(\mathcal{I}_Y(5)) \simeq H^1(\mathcal{O}_Y(5))$, that $H^1(\mathcal{O}_Y(5)) = 0$ if $5d \geq 2g - 1$, and that $H^2(E) \simeq H^0(E^\vee(-4))^\vee = 0$ hence, by Riemann-Roch, $h^0(E) = 2g - 6d + 58$ if $H^1(E) = 0$. Finally, the condition $H^1(E) = 0$ is equivalent to $H^1(\mathcal{I}_Y(5)) = 0$.

### 1.3. Linear monads.

Let $E$ be a vector bundle of rank $r = g - d + 4$ and Chern classes $c_1 = 5, c_2 = d, c_3 = 2g - 2 - d$. Applying Cor. [A.2] (and Remark [A.3] from Appendix A) to the vector bundle $F := E(-2)$ (as a matter of notation, this $F$ is what Chang and Ran denote by $E$) one gets that $E(-2)$ is the cohomology sheaf of a monad of the form:

$$0 \to \rho \mathcal{O}_{P^3}(-1) \to \sigma \mathcal{O}_{P^3} \to \tau \mathcal{O}_{P^3}(1) \to 0$$

if and only if $H^0(E(-3)) = 0$, $H^0(E^\vee(1)) = 0$, $H^1(E(-4)) = 0$ and $H^1(E^\vee) = 0$. Actually, the last condition is, in our case, a consequence of the first three because $\chi(E^\vee) = 0$ (one can use the convenient form of Riemann-Roch stated in Thm. 2.3 from [9]) and the first three conditions imply that $H^i(E^\vee) = 0$ for $i \neq 1$. According to the last part of Cor. [A.2] one must have $\tau = -\chi(E(-3)) = \chi(E^\vee(-1)) = 2d - g - 9$, $\rho = -\chi(E^\vee(1)) = 4d - 3g - 12$ and $\sigma = r - \tau - \rho = 5d - 3g - 17$. If $Y$ is a nonsingular curve that can be described as the dependency locus of $r - 1$ global sections of $E$ then the above conditions on $E$ are equivalent to: $H^0(\mathcal{I}_Y(2)) = 0$, $\mu_0(Y)$ injective and $H^1(\mathcal{I}_Y(1)) = 0$ (that is, $Y$ linearly normal).

It is easy to see that the monads of the above form with the property that $H^0(\beta^\vee(1)) : H^0(\sigma \mathcal{O}_{P^3}(1)) \to H^0(\rho \mathcal{O}_{P^3}(2))$ is surjective can be put together into a family with irreducible rational base. If $E(-2)$ is the cohomology sheaf of a monad of the above form, then $H^0(\beta^\vee(1))$ is surjective if and only if $H^1(E^\vee(3)) = 0$. In terms of the curve $Y$, the last condition is equivalent to the fact that the multiplication map $H^0(\omega_Y(-1)) \otimes H^0(\mathcal{O}_{P^3}(3)) \to H^0(\omega_Y(2))$ is surjective. Notice that this condition implies that $\omega_Y(-1)$ is globally generated. Notice also that, by Lemma [A.5] from Appendix A the condition $H^0(\beta^\vee(1))$ surjective is automatically satisfied if $\rho \leq 2$. 


1.4. The approach of Chang and Ran. Taking into account what has been said, in order to show that the moduli space $\mathcal{M}_g$ is unirational it suffices to prove the existence of nonsingular, connected space curves $Y \subset \mathbb{P}^3$, of genus $g$ and degree $d$, with the following properties:

(L) $Y$ is linearly normal;
(M) $\mu_0(Y)$ is injective;
(R) $H^0(\mathcal{I}_Y(2)) = 0$ and $H^1(\mathcal{I}_Y(5)) = 0$;
(S) The multiplication map $H^0(\omega_Y(-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\omega_Y(2))$ is surjective.

Actually, Chang and Ran use, instead of (R), the stronger condition asserting that “$Y$ has maximal rank” (which, in the cases $10 \leq g \leq 13$, means that $H^0(\mathcal{I}_Y(4)) = 0$ and $H^1(\mathcal{I}_Y(5)) = 0$) although they are aware of the fact that the above weaker condition is sufficient (see [7, Remark 3.1]). They prove the above existence result by starting with a curve of smaller degree and genus, having some “good properties”, and successively attaching 4-secant conics to it. They show that, if one is careful enough, the resulting reducible curve has the above properties and can be smoothed. This approach is based on results of Sernesi [15] and, in particular, on his results showing the injectivity of the $\mu_0$-map of a reducible curve obtained by attaching 4-secant conics.

1.5. An alternative approach. The first three subsections above show that the unirationality of $\mathcal{M}_g$, $8 \leq g \leq 13$, is a consequence of the following:

**Theorem 1.1.** Let $g$ be an integer with $8 \leq g \leq 13$, and let $d$ be the least integer for which $\rho := 4d - 3g - 12 \geq 0$. Consider, also, the integers $\sigma := 5d - 3g - 17$ and $\tau := 2d - g - 9$. Then there exist vector bundles $E$ on $\mathbb{P}^3$, of rank $r := g - d + 4$, subject to the following conditions:

(a) $F := E(-2)$ is the cohomology sheaf of a monad of the form:
\[
0 \rightarrow \rho \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\beta} \sigma \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha} \tau \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,
\]  
with $H^0(\beta^\vee(1))$ surjective;
(b) $H^1(E) = 0$;
(c) The dependency scheme of $r - 1$ general global sections of $E$ is a nonsingular curve.

In order to prove the theorem, we consider bundles $E$ with the property that their duals $E^\vee$ can be realized as extensions:
\[
0 \rightarrow A \rightarrow E^\vee \rightarrow \mathcal{I}_C \rightarrow 0,
\]  
where, for $8 \leq g \leq 11$ (when $d = g + 1$ and $r = 3$), $A = \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$ and $C$ is a rational curve of degree $g - 5$, while, for $g = 12, 13$ (when $d = g$ and $r = 4$), $A = \mathcal{O}_{\mathbb{P}^3}(-3) \oplus 2\mathcal{O}_{\mathbb{P}^3}(-1)$ and $C$ is a rational curve of degree $g - 7$. These bundles occurred naturally in our systematic study of globally generated vector bundles with $c_1 = 5$ on $\mathbb{P}^3$ from [2].

The extensions (1.2) can be constructed by elementary transformations as follows: assume that $A = \bigoplus_{i=0}^m \mathcal{O}_{\mathbb{P}^3}(-a_i)$, $a_0 \geq \ldots \geq a_m$, and that $C$ is a nonsingular connected curve on a nonsingular surface $X \subset \mathbb{P}^3$ of degree $d$. Let $\gamma$ be a global section of $\mathcal{O}_X[C]$ whose zero divisor is $C$ and let $\delta_i$ be a global section of $\mathcal{O}_X(d - a_i) \otimes \mathcal{O}_X[C]$, with zero divisor $D_i$, $i = 0, \ldots, m$, such that $C \cap D_0 \cap \ldots \cap D_m = \emptyset$. 


Let $\varepsilon: \mathcal{O}_{\mathbb{P}^3}(d) \oplus A^\vee \to \mathcal{O}_X(d) \otimes \mathcal{O}_X[C]$ be the epimorphism defined by $\gamma, \delta_0, \ldots, \delta_m$ and let $E$ be the kernel of $\varepsilon$.

Recall the following simple observation: if $\phi: \mathcal{F}_0 \oplus \mathcal{F}_1 \to \mathcal{G}$ is a morphism of sheaves, with components $\phi_i: \mathcal{F}_i \to \mathcal{G}$, $i = 0, 1$, then one has an exact sequence:

$$0 \to \text{Ker} \phi_0 \to \text{Ker} \phi \to \mathcal{F}_1 \xrightarrow{\overline{\phi}_1} \text{Coker} \phi_0 \to \text{Coker} \phi \to 0,$$

where $\overline{\phi}_1$ is the composite morphism $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{G} \to \text{Coker} \phi_0$. Using this observation and the adjunction formula one gets that $(\mathcal{O}_X(d) \otimes \mathcal{O}_X[C]) | C \simeq \omega_C(4)$ and an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E \to A^\vee \xrightarrow{\overline{\phi}} \omega_C(4) \to 0,$$

with $\overline{\phi}$ defined by $\delta_i | C$, $i = 0, \ldots, m$. Let $\mathcal{K}$ be the kernel of $\overline{\phi}$. Dualizing the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E \to \mathcal{K} \to 0,$$

one gets the extension (1.2) above.

We shall use two filtrations of $\mathcal{K}$. Firstly, applying the above observation, one gets an exact sequence:

$$0 \to \mathcal{I}_C(a_0) \to \mathcal{H} \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^3}(a_i) \xrightarrow{\overline{\phi}} (\mathcal{O}_X(d) \otimes \mathcal{O}_X[C]) | D_0 \cap C \to 0,$$

where $\overline{\phi}$ is defined by $\delta_i | D_0 \cap C$, $i = 1, \ldots, m$.

Secondly, let $\mathcal{K}$ be the kernel of the morphism $\delta: A^\vee \to \mathcal{O}_X(d) \otimes \mathcal{O}_X[C]$ defined by $\delta_0, \ldots, \delta_m$. Applying the Snake Lemma to the diagram:

$$\begin{array}{ccccccccc}
0 & \to & 0 & \to & A^\vee & \to & A^\vee & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{O}_X(d) & \to & \mathcal{O}_X(d) \otimes \mathcal{O}_X[C] & \to & \omega_C(4) & \to & 0
\end{array}$$

one gets an exact sequence:

$$0 \to \mathcal{K} \to \mathcal{H} \to \mathcal{I}_{D_0 \cap \ldots \cap D_m \cap X}(d) \to 0.$$

Notice that if $D_i = \Delta_i + B$, $i = 0, \ldots, m$, such that the scheme $W := \Delta_0 \cap \ldots \cap \Delta_m$ is 0-dimensional (or empty), then $\mathcal{I}_{D_0 \cap \ldots \cap D_m \cap X} \simeq \mathcal{I}_{W \cap X} \otimes \mathcal{O}_X[-B]$. On the other hand, applying the above observation to $\delta$ one obtains an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a_0 - d) \to \mathcal{K} \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^3}(a_i) \xrightarrow{\delta'} (\mathcal{O}_X(d) \otimes \mathcal{O}_X[C]) | D_0 ,

where $\delta'$ is defined by $\delta_i | D_0$, $i = 1, \ldots, m$. Notice that if $D_i = \Delta_i + B$, $i = 0, \ldots, m$, such that $\Delta_0 \cap B$ consists of finitely many points then the last exact sequence induces an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a_0 - d) \to \mathcal{K} \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^3}(a_i) \xrightarrow{\delta''} (\mathcal{O}_X(d) \otimes \mathcal{O}_X[C]) | \Delta_0 ,

where $\delta''$ is defined by $\delta_i | \Delta_0$, $i = 1, \ldots, m$. The reason is that $\mathcal{O}_{\Delta_0}$ embeds into $\mathcal{O}_{\Delta_0} \oplus \mathcal{O}_B$ and $\delta_i$ vanishes on $B$, $i = 1, \ldots, m$.

For our purposes, if $C$ is a rational curve of degree 6 (resp., 5) we take $X$ to be a cubic surface which is the blow-up $\pi: X \to \mathbb{P}^2$ of $\mathbb{P}^2$ in six general points $P_1, \ldots, P_6$, embedded in $\mathbb{P}^3$ such that $\mathcal{O}_X(1) \simeq \pi^* \mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_X[-E_1 - \ldots - E_6]$, where $E_i := \pi^{-1}(P_i)$, and we take $C$ to be the strict transform of a nonsingular
conic $C \subset \mathbb{P}^2$ containing none of the points $P_1, \ldots, P_6$ (resp., containing $P_1$ but none of the points $P_2, \ldots, P_6$), while if $C$ is a rational curve of degree 4 or 3, we take $X$ to be a quadric surface containing $C$. The divisors $D_0, \ldots, D_m$ will be specified during the proof.

Now, using the above construction, the properties (a) and (b) from the conclusion of the theorem can be easily checked. Actually, for $g = 12, 13$, there is a slight technical complication due to the fact that the bundles $E$ constructed as above have the property that $E(-2)$ is the cohomology sheaf of a minimal monad of the form:

$$0 \longrightarrow (\rho + 2)\mathcal{O}_{\mathbb{P}^3}(-1) \overset{\beta'}{\longrightarrow} \sigma\mathcal{O}_{\mathbb{P}^3} \oplus 2\mathcal{O}_{\mathbb{P}^3}(-1) \overset{\alpha'}{\longrightarrow} \tau\mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0, \quad (1.8)$$

with $H^0(\beta'\nu(1))$ surjective. This is, however, harmless because the monads of the form (1.8) with $H^0(\beta'\nu(1))$ surjective can be put together into a family with irreducible base. For a general monad of this type, the component $(\rho + 2)\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(-1)$ of $\beta'$ is surjective hence the cohomology sheaf of the monad is isomorphic to the cohomology sheaf of a monad of the form (1.1) with $H^0(\beta'(1))$ surjective. Moreover, the conditions (b) and "(b) + (c)" are open conditions in the family of vector bundles $E$ with the property that $E(-2)$ is the cohomology sheaf of a monad of the form (1.8) (because, in this case, $H^i(E) = 0$ for $i \geq 2$).

The non-trivial part of the proof is the verification of condition (c). We verify, for $g \leq 12$, the stronger condition "$E$ is globally generated". Actually, $E$ is 0-regular for $8 \leq g \leq 10$, while, for $g = 11, 12$, $E$ is 1-regular (this follows from (a) and (b)) and we show that "the multiplication map $H^0(E) \otimes S_1 \rightarrow H^0(E(1))$ is surjective". Here $S_1 := H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ is the space of linear forms on $\mathbb{P}^3$.

On the other hand, if $g = 13$ then $d = 13$, $E$ has rank 4 and, if it satisfies (a) and (b), one has $H^0(E) = 6$ hence the degeneracy locus of the evaluation morphism $6\mathcal{O}_{\mathbb{P}^3} \rightarrow E$ is non-empty. The best one can hope for in this case is that "the evaluation morphism $H^0(E) \otimes S_1 \mathcal{O}_{\mathbb{P}^3} \rightarrow E$ is an epimorphism except at finitely many points where it has corank 1" (this would, obviously, imply (c)). We were not able to verify this condition for the bundles $E$ constructed as above. We can, fortunately, show that some of the bundles $E$ constructed as above satisfy the weaker condition asserting that "the evaluation morphism of $E$ has corank at most 1 at every point and its degeneracy scheme is a curve contained in a nonsingular surface in $\mathbb{P}^3$". According to some results of Martin-Deschamps and Perrin [10], recalled in Lemma 4.6 from Appendix A, this condition still implies (c).

2. The Cases $8 \leq g \leq 11$

In these cases, $d = g + 1$, $r = 3$ and the monads from Theorem 1.1(a) have the form:

$$0 \longrightarrow (g - 8)\mathcal{O}_{\mathbb{P}^3}(-1) \overset{\beta}{\longrightarrow} 2(g - 6)\mathcal{O}_{\mathbb{P}^3} \overset{\alpha}{\longrightarrow} (g - 7)\mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0. \quad (2.1)$$

Using the notation from Subsection 1.5, one has $A = \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$ and $C \subset \mathbb{P}^3$ is a nonsingular rational curve of degree $g - 5$, contained in a nonsingular surface $X \subset \mathbb{P}^3$ of degree 3 or 2. We shall specify, now, the divisors $D_0$ and $D_1$.

- If $g = 11$ then $C$ has degree 6. $X$ is the unique effective divisor of degree 3 in $\mathbb{P}^3$ containing $C$ because $C$ admits six 4-secants (namely, the strict transforms...
of the conics in \( \mathbb{P}^2 \) containing five of the six points \( P_1, \ldots, P_6 \) and a complete intersection of type \((3, 3)\) in \( \mathbb{P}^3 \) containing \( C \) would contain all of these 4-secants. Since \( h^0(\mathcal{I}_C(3)) = 1 \) and \( h^0(\mathcal{I}_C(3)) = 19 \) it follows that \( H^1(\mathcal{I}_C(3)) = 0 \). Consider a general (nonsingular) conic \( \overline{C}_0 \subset \mathbb{P}^2 \), containing \( P_1, P_2, P_3 \) but none of the points \( P_4, P_5, P_6 \), and intersecting \( \overline{C} \) (the conic in \( \mathbb{P}^2 \) whose inverse image on \( X \) is \( C \)) in four distinct points. The strict transform \( C_0 \subset X \) of \( C_0 \) is a twisted cubic curve in \( \mathbb{P}^3 \). One has \( C_0 \sim C - E_1 - E_2 - E_3 \) as divisors on \( X \). Put \( Y := E_1 + E_2 + E_3 \).

By the adjunction formula, \( \mathcal{O}_X[C_0] | C_0 \sim \omega_{C_0}(1) \). Moreover, the restriction map \( H^0(\mathcal{O}_X(1) \otimes \mathcal{O}_X[C_0]) \to H^0((\mathcal{O}_X(1) \otimes \mathcal{O}_X[C_0]) | C_0) \) is surjective because its cokernel embeds into \( H^1(\mathcal{O}_X(1)) = 0 \). It follows that if \( D \) is a general member of the complete linear system \( | \mathcal{O}_X(1) \otimes \mathcal{O}_X[C_0] \| \) then the scheme \( W := D \cap C_0 \) consists of four \textit{general} simple points of \( C_0 \), none of them belonging to \( C \) or to \( Y \). We take \( D_0 := C_0 + Y \) and \( D_1 := D + Y \).

- If \( g = 10 \) then \( C \) has degree 5. Since \( H^1(\mathcal{I}_C(2)) = 0 \) (because, otherwise, \( C \) would be linked by a complete intersection of type \((2, 3)\) to a line and this would contradict the fact that \( C \) is not arithmetically Cohen-Macaulay) it follows that \( h^1(\mathcal{I}_C(2)) = 1 \). If \( H \subset \mathbb{P}^3 \) is a general plane then \( H \cap C \) consists of five points, no three collinear. In this case, \( \mathcal{I}_{H \cap C \cap H} \) is 3-regular. The Lemma of Le Potier (see, for example, [1, Lemma 1.22]) implies that \( H^1(\mathcal{I}_C(3)) = 0 \). Consider a general (nonsingular) conic \( \overline{C}_0 \subset \mathbb{P}^2 \), containing \( P_1, P_2, P_3 \) but none of the points \( P_4, P_5, P_6 \), and intersecting \( \overline{C} \) (the conic in \( \mathbb{P}^2 \) whose strict transform on \( X \) is \( C \)) in four distinct points (\( P_1 \) being one of them). The strict transform \( C_0 \subset X \) of \( C_0 \) is a twisted cubic curve in \( \mathbb{P}^3 \). One has \( C_0 \sim C - E_2 - E_3 \) as divisors on \( X \) and \( C_0 \cap C \) consists of three simple points. Let \( D \) be a general member of the complete linear system \( | \mathcal{O}_X(1) \otimes \mathcal{O}_X[C_0] | \) such that the scheme \( W := D \cap C_0 \) consists of four \textit{general} simple points of \( C_0 \), none of them belonging to \( C \) or to \( E_2 \cup E_3 \). We take \( D_0 := C_0 + E_2 + E_3 \) and \( D_1 := D + E_2 + E_3 \).

- If \( g = 9 \) (resp., \( g = 8 \)) then \( C \) is a divisor of type \((3, 1)\) (resp., \((2, 1)\)) on a nonsingular quadric surface \( X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) in \( \mathbb{P}^3 \). In both cases, \( H^1(\mathcal{I}_C(2)) = 0 \). Consider two lines \( L_0, L_1 \) of type \((1, 0)\) on \( X \). We take \( D_0 := L_0 + L_1 \) (resp., \( D_0 = L_0 \)) and \( D_1 \) any member of the complete linear system \( | \mathcal{O}_X[C] \| \) containing none of the points of \( C \cap D_0 \).

Now, in all of the above cases, \( D_0 \cap C \) consists of \( g - 7 \) simple points, no three collinear and no four coplanar (recall that, for \( g = 11 \) and \( g = 10 \), they belong to the twisted cubic curve \( C_0 \subset \mathbb{P}^3 \)). Using the exact sequence \((1.3)\) and the exact sequence \((1.4)\) that becomes:

\[
0 \longrightarrow \mathcal{I}_C(3) \longrightarrow \widetilde{\mathcal{H}} \longrightarrow \mathcal{I}_{C_0 \cap C}(2) \longrightarrow 0
\]

one deduces that \( H^0(E(−2)) = 0 \) (and \( H^0(E(−1)) = 0 \) if \( g = 11 \)) and that \( h^1(E(−3)) = h^1(\mathcal{I}_{D_0 \cap C}(−1)) = g - 7 \). On the other hand, using the extension \((1.2)\), one gets that \( H^0(E^\vee(1)) = 0 \), \( H^1(E(−4)) \simeq H^2(E^\vee)^\vee = 0 \), and \( H^1(E^\vee) = 0 \). Moreover, \( h^1(E^\vee(1)) = h^1(\mathcal{I}_C(1)) = g - 8 \). Cor. \([A.2]\) from Appendix [A] implies, now, that \( F := E(−2) \) is the cohomology sheaf of a monad of the form \((2.1)\). Notice, also, that \( H^1(E^\vee(3)) \simeq H^1(\mathcal{I}_C(3)) = 0 \), hence \( H^0(\beta^\vee(1)) \) is surjective (\( \beta \) being
the differential of the monad). Consequently, \( E \) satisfies condition (a) from Theorem 1.1. Using, again, the fact that \( H^1(\mathcal{J}_C(3)) = 0 \) and the exact sequences (1.3) and (1.4), one gets that \( H^1(E) = 0 \) hence \( E \) satisfies condition (b) from Theorem 1.1. Moreover, for \( g = 8, 9 \), one has \( H^1(E(-1)) = 0 \) hence \( E \) is 0-regular in those cases (actually, for \( g = 8 \) there is only one bundle \( E \) satisfying the conclusion of Theorem 1.1 namely \( E = \Omega_{p^3}(3) \); look at the monad (2.1)). It, consequently, remains to show that the multiplication map \( H^0(E) \otimes S_1 \rightarrow H^0(E(1)) \) is surjective if \( g = 11 \) and that \( H^1(E(-1)) = 0 \) if \( g = 10 \).

**Conclusion of the proof of Theorem 1.1 for \( g = 11 \).** For the above choice of the divisors \( D_0 \) and \( D_1 \), the exact sequences (1.5) and (1.7) become:

\[
0 \rightarrow \mathcal{H} \rightarrow \tilde{\mathcal{H}} \rightarrow \mathcal{I}_{Y \cup W, X}(3) \rightarrow 0 ,
0 \rightarrow \mathcal{O}_{p^3} \rightarrow \mathcal{H} \rightarrow \mathcal{I}_{C_0}(2) \rightarrow 0 .
\]

It follows that, in order to show that the multiplication map \( H^0(E) \otimes S_1 \rightarrow H^0(E(1)) \) is surjective, it suffices to check that \( E_1, E_2, E_3 \) and \( W \) satisfy the hypotheses of Lemma B.1 from Appendix B and that is exactly what we are going to do next.

Since \( W \) consists of four simple points on the twisted cubic curve \( C_0 \) it is not contained in a plane. We assert that, for \( 1 \leq l \leq 3 \), \((Y \setminus E_l) \cup C_0 \) is not contained in a quadric surface. Indeed, if it would be contained then the surface would be nonsingular, isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), \( C_0 \) would be a divisor of type \((2,1)\) on this surface and the two components of \( Y \setminus E_l \) would be divisors of type \((1,0)\) (because the intersection of each of them with \( C_0 \) is a simple point). But this would contradict the fact that \((Y \setminus E_l) \cup C_0 \) is contained in an irreducible cubic surface.

It follows that the restriction map \( H^0(\mathcal{I}_{Y \setminus E_l}(2)) \rightarrow H^0(\mathcal{O}_{C_0}(2)) \) is injective, \( l = 1, 2, 3 \). Since \( W \) consists of four general points of \( C_0 \), one can, consequently, assume that \( H^0(\mathcal{I}_{Y \setminus E_l \cup W}(2)) = 0 \), \( l = 1, 2, 3 \). Moreover, one can assume that none of the points of \( W \) belongs to the quadric surface containing \( Y \). This completes the verification of the hypotheses of Lemma B.1 and, with it, the proof of the assertion that the multiplication map \( H^0(E) \otimes S_1 \rightarrow H^0(E(1)) \) is surjective. \( \square \)

**Conclusion of the proof of Theorem 1.1 for \( g = 10 \).** For the above choice of the divisors \( D_0 \) and \( D_1 \), the exact sequences (1.5) and (1.7) become:

\[
0 \rightarrow \mathcal{H} \rightarrow \tilde{\mathcal{H}} \rightarrow \mathcal{I}_{E_2 \cup E_3 \cup W, X}(3) \rightarrow 0 ,
0 \rightarrow \mathcal{O}_{p^3} \rightarrow \mathcal{H} \rightarrow \mathcal{I}_{C_0}(2) \rightarrow 0 .
\]

As we saw in the above proof of the case \( g = 11 \) of Theorem 1.1 one can assume that \( E_2 \cup E_3 \cup W \) is contained in no quadric surface in \( \mathbb{P}^3 \). This implies that \( H^1(\mathcal{I}_{E_2 \cup E_3 \cup W}(2)) = 0 \) (because \( h^0(\mathcal{O}_{E_2 \cup E_3 \cup W}(2)) = 10 \) hence \( H^1(E(-1)) = 0 \). \( \square \)

3. **The Case \( g = 13 \)**

**Proof of Theorem 1.1 for \( g = 13 \).** In this case, \( d = 13, r = 4 \), and the monad from the statement of Theorem 1.1(a) has the form:

\[
0 \rightarrow \mathcal{O}_{p^3}(-1) \rightarrow 9 \mathcal{O}_{p^3} \rightarrow 4 \mathcal{O}_{p^3}(1) \rightarrow 0 .
\] (3.1)

Using the notation from Subsection 1.5, one has \( A = \mathcal{O}_{p^3}(-3) \oplus 2 \mathcal{O}_{p^3}(-1) \) and \( C \) is a nonsingular rational curve of degree 6 contained in a nonsingular cubic surface.
X in $\mathbb{P}^3$. We shall specify, now, the divisors $D_0, D_1, D_2$. Let $C_0 \subset X$ be a twisted cubic curve which is the strict transform of a general (nonsingular) conic $\overline{C}_0 \subset \mathbb{P}^2$, containing $P_1, P_2, P_3$ but none of the points $P_4, P_5, P_6$, and intersecting $\overline{C}$ (the conic in $\mathbb{P}^2$ whose inverse image on $X$ is $C$) in four distinct points. One has $C_0 \sim C - Y$, where $Y := E_1 + E_2 + E_3$. Let $\gamma_0$ be a global section of $\mathcal{O}_X[C_0]$ whose zero divisor is $C_0$. Complete $\gamma_0$ to a $k$-basis $\gamma_0, \gamma_1, \gamma_2$ of $H^0(\mathcal{O}_X[C_0])$ and let $C_i \subset X$ be the zero divisor of $\gamma_i, i = 1, 2$. Let, finally, $D$ be a general member of the linear system $|\mathcal{O}_X(2)|$ such that $W := D \cap C_0$ consists of six general simple points of $C_0$, not belonging to $C \cup Y$. We take $D_0 := C_0 + Y, D_1 := D + C_1 + Y, D_2 := D + C_2 + Y$.

**Claim 1.** $F := E(-2)$ is the cohomology sheaf of a monad of the form:

$$0 \rightarrow 3\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 9\mathcal{O}_{\mathbb{P}^3} \oplus 2\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 4\mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

with $H^0(\beta^{\vee}(1))$ surjective.

**Indeed,** the exact sequence (**1.4**) becomes, now:

$$0 \rightarrow \mathcal{I}_C(3) \rightarrow \overline{\mathcal{H}} \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\overline{\mathcal{S}'}} (\mathcal{O}_X(3) \otimes \mathcal{O}_X[C]) | C_0 \cap C \rightarrow 0,$$  \hspace{1cm} (**3.2**)

where $\overline{\mathcal{S}'}$ is defined by $\delta_i | C_0 \cap C, i = 1, 2$. Taking into account the exact sequence (**1.3**), one deduces immediately that $H^0(E(-2)) = 0$ (one can, actually, show that no non-trivial linear combination of $\delta_1 | C_0$ and $\delta_2 | C_0$ vanishes on $C_0 \cap C$ hence $H^0(E(-1)) = 0$ and that $H^1(E(-3)) = 4$. On the other hand, using the extension (**1.2**), one gets that $h^0(E^\vee(1)) = 2, H^1(E(-4)) \simeq H^2(E^\vee)^\vee = 0, H^1(E^\vee) = 0, H^1(E^\vee(-1)) = 0$ and $h^1(E^\vee(1)) \simeq h^1(\mathcal{I}_C(1)) = 3$. Lemma A1 from Appendix A shows, now, that $F := E(-2)$ is the cohomology of a monad of the above form. Since $H^1(E^\vee(3)) \simeq H^1(\mathcal{I}_C(3)) = 0$ it follows that $H^0(\beta^{\vee}(1))$ is surjective.

**Claim 2.** $H^1(E) = 0$.

**Indeed,** since the morphism $\overline{\mathcal{S}'}$ from the exact sequence (**3.2**) is an epimorphism, it follows that a general linear combination $a_1\delta_1 + a_2\delta_2$ vanishes at no point of $C_0 \cap C$. One deduces that the composite morphism:

$$\mathcal{O}_{\mathbb{P}^3}(\alpha_2) \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\overline{\mathcal{S}'(-1)}} (\mathcal{O}_X(2) \otimes \mathcal{O}_X[C]) | C_0 \cap C$$

is an epimorphism hence its kernel is isomorphic to $\mathcal{I}_{C_0 \cap C}$. Since $C_0 \cap C$ consists of four points that are not coplanar, one has $H^1(\mathcal{I}_{C_0 \cap C}(1)) = 0$. One deduces that $H^0(\overline{\mathcal{S}'})$ is surjective and this implies that $H^1(E) = 0$.

It remains to verify condition (c) from Thm. [A1.1](#). Recall, for this purpose, the exact sequences (**1.5**) and (**1.7**) from Subsection 1.5 that become, in our case:

$$0 \rightarrow \mathcal{H} \rightarrow \overline{\mathcal{H}} \rightarrow \mathcal{I}_{Y \cup W \cup X}(3) \rightarrow 0, $$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{H} \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\gamma'} (\mathcal{O}_X(1) \otimes \mathcal{O}_X[C_0]) | C_0 \rightarrow 0,$$

where $\gamma'$ is defined by $\gamma_i | C_0, i = 1, 2$. One deduces, from the second exact sequence, that $\mathcal{H}$ is 0-regular. Taking into account the exact sequence (**1.3**), one gets that the cokernel of the evaluation morphism of $E$ is isomorphic to the cokernel of the evaluation morphism of $\mathcal{I}_{Y \cup W \cup X}(3)$. 


Claim 3. There exists a unique divisor $\Delta$ in the complete linear system $|\mathcal{O}_X(3) \otimes \mathcal{O}_X[-Y]|$ such that $\Delta \cap C_0 = W$ as schemes.

Indeed, the restriction map $H^0(\mathcal{O}_X(3) \otimes \mathcal{O}_X[-Y]) \rightarrow H^0((\mathcal{O}_X(3) \otimes \mathcal{O}_X[-Y])|C_0)$ is bijective because its kernel is $H^0(\mathcal{O}_X(3) \otimes \mathcal{O}_X[-C])$ and its cokernel embeds into $H^1(\mathcal{O}_X(3) \otimes \mathcal{O}_X[-C])$ and these cohomology groups are both zero because $h^0(\mathcal{I}_C(3)) = 1$ and $H^1(\mathcal{I}_C(3)) = 0$. One uses, now, the fact that $(\mathcal{O}_X(3) \otimes \mathcal{O}_X[-Y])|C_0$ is a line bundle of degree 6 on $C_0 \simeq \mathbb{P}^1$ and $W$ is an effective divisor of degree 6 on $C_0$.

One deduces, from Claim 3, that the cokernel of the evaluation morphism of $\mathcal{I}_{Y \cup W,X}(3) \simeq \mathcal{I}_{W,X}(3) \otimes \mathcal{O}_X[-Y]$ is isomorphic to $\mathcal{I}_{W,\Delta} \otimes \mathcal{L}$, where $\mathcal{L}$ is the restriction of $\mathcal{O}_X(3) \otimes \mathcal{O}_X[-Y]$ to $\Delta$. Moreover, since $W$ consists of six simple points of $C_0$ and since $\Delta \cap C_0 = W$ as schemes, it follows that the points of $W$ are nonsingular points of $\Delta$, hence $\mathcal{I}_{W,\Delta} \otimes \mathcal{L}$ is an invertible $\mathcal{O}_\Delta$-module. The results of Martin-Deschamps and Perrin recalled in Lemma A.6 and Remark A.7 from Appendix A imply, now, that the dependency locus of three general global sections of $E$ is a nonsingular curve in $\mathbb{P}^3$.

4. The Case $g = 12$

Proof of Theorem 1.1 for $g = 12$. In this case, $d = 12$, $r = 4$, and the monads from the statement of Theorem 1.1 are of the form:

$$0 \rightarrow 0 \rightarrow 7\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha} 3\mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0. \quad (4.1)$$

Using the notation from Subsection 1.5, one has $A = \mathcal{O}_{\mathbb{P}^3}(-3) \oplus 2\mathcal{O}_{\mathbb{P}^3}(-1)$ and $C$ is a nonsingular rational curve of degree 5 contained in a nonsingular cubic surface $X$ in $\mathbb{P}^3$. We shall specify, now, the divisors $D_0$, $D_1$, $D_2$. Let $C_0 \subset X$ be a twisted cubic curve which is the strict transform of a general (nonsingular) conic $C_0 \subset \mathbb{P}^2$, containing $P_1$, $P_2$, $P_3$ but none of the points $P_4$, $P_5$, $P_6$, and intersecting $C$ (the conic in $\mathbb{P}^2$ whose strict transform on $X$ is $C$) in four distinct points (including $P_1$; it follows that $C_0 \cap C$ consists of three simple points). One has $C_0 \sim C - E_2 - E_3$. Let $\gamma_i$, $i = 0, 1, 2, C_1$, $C_2$, $D$ and $W := D \cap C_0$ be as at the beginning of the above proof of the case $g = 13$. We take $D_0 = C_0 + E_2 + E_3$ and $D_i = D + C_i + E_2 + E_3$, $i = 1, 2$.

One shows easily, as in Claim 1 and Claim 2 of the above proof of the case $g = 13$, that $F := E(-2)$ is the cohomology sheaf of a monad of the form:

$$0 \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\beta'} 7\mathcal{O}_{\mathbb{P}^3} \oplus 2\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha'} 3\mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

with $H^0(\beta'(1))$ surjective, and that $H^1(E) = 0$. (One can, moreover, show that $H^0(E(-1)) = 0$.) It thus remains to show that the multiplication map $H^0(E) \otimes S_1 \rightarrow H^0(E(1))$ is surjective. This is equivalent to the fact that the multiplication map $H^0(\mathcal{I}_{E_2 \cup E_3 \cup W}(3)) \otimes S_1 \rightarrow H^0(\mathcal{I}_{E_2 \cup E_3 \cup W}(4))$ is surjective (one uses arguments similar to those used in the above proof of the case $g = 13$).

Recall that $W$ consists of six general simple points of $C_0$. As we saw in the proof of the case $g = 11$ of Theorem 1.1 from Section 2 there is no quadric surface in $\mathbb{P}^3$ containing $E_2 \cup E_3 \cup C_0$. On the other hand, there is a unique quadric surface $Q_i$ in $\mathbb{P}^3$ containing $E_i \cup C_0$ (because $E_i$ intersects $C_0$ in one simple point), $i = 2, 3$.
Choose two general points $R_4$ and $R_5$ of $C_0$ such that the line $L_1 \subset \mathbb{P}^3$ joining them does not intersect $E_2$ and $E_3$ and is not contained in any of the surfaces $Q_2$ and $Q_3$. In this case $L_1 \cup E_i \cup C_0$ is contained in no quadric surface, $i = 2, 3$. One can choose, now, four general points $R_0, \ldots, R_3$ of $C_0 \setminus (L_1 \cup E_2 \cup E_3)$ such that $L_1, E_2, E_3$ and $\{R_0, \ldots, R_3\}$ satisfy the hypothesis of Lemma 3.1 from Appendix B (see the proof of the case $g = 11$ of Theorem 1.1 in Section 2). In this case, taking $W = \{R_0, \ldots, R_5\}$ and applying Cor. 3.2 one gets that the multiplication map $H^0(\mathcal{I}_{E_2 \cup E_3 \cup W}(3)) \otimes S_1 \to H^0(\mathcal{I}_{E_2 \cup E_3 \cup W}(4))$ is surjective. 

5. The Cases $5 \leq g \leq 7$

In these cases, the least integer $d$ for which $\rho(g, 3, d) \geq 0$ is $d = g + 2$. The method of Chang and Ran, as formulated in Section 1 does not work anymore because, assuming that the curve $Y$ is linearly normal, one has $h^0(\omega_Y(-1)) = h^1(\mathcal{O}_Y(1)) = 1$ and $\deg \omega_Y(-1) = g - 4 > 0$ hence $\omega_Y(-1)$ cannot be globally generated (a condition that was necessary for the construction of a vector bundle). Things can be, however, fixed by working with rank 2 reflexive sheaves instead of vector bundles (this is, actually, the approach from Chang [3 Cor. 4.8.1]).

More precisely, if $Y \subset \mathbb{P}^3$ is a linearly normal, nonsingular, connected curve of genus $g$, $5 \leq g \leq 7$, and degree $d = g + 2$ then, as we saw above, $h^0(\omega_Y(-1)) = 1$. This implies immediately that $\mu_0(Y)$ is injective. Moreover, for reasons of degree and genus, $H^0(\mathcal{I}_Y(2)) = 0$. A non-zero global section of $\omega_Y(-1)$ defines an extension:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{E} \to \mathcal{I}_Y(5) \to 0,$$

with $\mathcal{E}$ a rank 2 reflexive sheaf with $c_1(\mathcal{E}) = 5$. Consider the “normalized” rank 2 reflexive sheaf $\mathcal{F} := \mathcal{E}(-3)$, with Chern classes $c_1(\mathcal{F}) = -1$, $c_2(\mathcal{F}) = \deg Y - 6 = g - 4$, $c_3(\mathcal{F}) = \deg \omega_Y(-1) = g - 4$. One has $H^0(\mathcal{F}) = 0$ (because $H^0(\mathcal{I}_Y(2)) = 0$) and $H^1(\mathcal{F}(1)) = 0$ (because $H^1(\mathcal{I}_Y(1)) = 0$). Now, one has:

Lemma 5.1. Let $g$ be an integer with $5 \leq g \leq 7$ and let $\mathcal{F}$ be a rank 2 reflexive sheaf on $\mathbb{P}^3$ with Chern classes $c_1(\mathcal{F}) = -1$, $c_2(\mathcal{F}) = g - 4$, $c_3(\mathcal{F}) = g - 4$. If $H^0(\mathcal{F}) = 0$ and $H^1(\mathcal{F}(-1)) = 0$ then $\mathcal{F}(1)$ is the cohomology sheaf of a monad of the form:

$$0 \to (g - 4)\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\beta} (2g - 7)\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha} (g - 5)\mathcal{O}_{\mathbb{P}^3}(1) \to 0. \quad (5.1)$$

Here “monad” means that $\alpha$ is an epimorphism, $\beta^\vee$ is an epimorphism except at finitely many points, and $\alpha \circ \beta = 0$.

Proof. We use the properties of the spectrum of a stable rank 2 reflexive sheaf from Hartshorne [1 Sect. 7]. Since $H^0(\mathcal{F}) = 0$, $\mathcal{F}$ is stable. Since $H^1(\mathcal{F}(-1)) = 0$, the spectrum of $\mathcal{F}$ must be $k_\mathcal{F} = (-1, \ldots, -1)$ ($g - 4$ times). It follows that $H^1(\mathcal{F}(l)) = 0$ for $l \leq -1$ and $H^2(\mathcal{F}(l)) = 0$ for $l \geq -1$. Since $H^2(\mathcal{F}(-1)) = 0$ and $H^3(\mathcal{F}(-2)) \simeq H^0(\mathcal{F}^\vee(-2))^\vee \simeq H^0(\mathcal{F}(-1))^\vee = 0$, the Castelnuovo-Mumford lemma (in the form stated in [1 Lemma 1.21]) implies that the graded $\mathbb{S}$-module $H^1(\mathcal{F})$ is generated in degrees $\leq 0$, hence it is generated by $H^1(\mathcal{F})$. By Riemann-Roch, $h^1(\mathcal{F}) = -\chi(\mathcal{F}) = g - 5$. Consider, now, the universal extension:

$$0 \to \mathcal{F} \to \mathcal{G} \to (g - 5)\mathcal{O}_{\mathbb{P}^3} \to 0.$$
\[G\] is a rank 4 reflexive sheaf with \(H^1(G) = 0\) and with \(H^0(G) = 0\). Since \(H^2(G(-1)) \cong H^2(F(-1)) = 0\) and \(H^3(G(-2)) \cong H^3(F(-2)) = 0\), \(G\) is 1-regular. By Riemann-Roch:

\[ h^0(G(1)) = h^0(2O_{\mathbb{P}^3}(1)) + h^0(F(1)) - h^1(F(1)) = h^0(2O_{\mathbb{P}^3}(1)) + \chi(F(1)) = 2g - 7. \]

The kernel \(K\) of the evaluation epimorphism \((2g - 7)O_{\mathbb{P}^3} \rightarrow G(1)\) of \(G(1)\) is locally free of rank \(g - 4\). One has \(H^1(K) = 0\) and \(H^2(K) \cong H^1(G(1)) = 0\), hence \(K\) is a direct sum of line bundles. Since \(c_1(K) = -(g - 4)\) and \(H^0(K) = 0\) it follows that \(K \cong (g - 4)O_{\mathbb{P}^3}(-1)\).

Conversely, if \(F(1)\) is the cohomology sheaf of a monad of the form (5.1) then \(F\) is a rank 2 reflexive sheaf with \(H^0(F) = 0\) and \(H^1(F(-1)) = 0\). Moreover, since \(g - 5 \leq 2\), Lemma [A.5] from Appendix [A] implies that \(H^1(F(2)) = 0\) hence \(F\) is 3-regular. In particular, \(F(3)\) is globally generated. If, moreover, \(\text{Ext}^1_{\mathbb{P}^3}(F, O_{\mathbb{P}^3}) \cong O_{\Gamma}\), where \(\Gamma\) is a 0-dimensional subscheme of \(\mathbb{P}^3\) consisting of simple points, then the zero scheme of a general global section of \(F(3)\) is a nonsingular curve \(Y\) (see, for example, [H, Prop. 4]). \(Y\) has genus \(g\) and degree \(d = g + 2\), and is linearly normal.

Finally, the monads of the form (5.1) can be put together into a family with irreducible, rational base because, by Lemma [A.5] from Appendix [A] \(H^0(\alpha(1))\) is surjective, for any such monad (again, since \(g - 5 \leq 2\)). Consequently, in order to prove the unirationality of \(\mathcal{M}_g\), for \(5 \leq g \leq 7\), it suffices to show that there exist rank 2 reflexive sheaves \(F\), with \(c_1(F) = -1\), \(c_2(F) = g - 4\), \(c_3(F) = g - 4\), such that \(H^0(F) = 0\), \(H^1(F(-1)) = 0\) and such that \(\text{Ext}^1_{\mathbb{P}^3}(F, O_{\mathbb{P}^3})\) is as above. One can construct such sheaves as general extensions:

\[
0 \longrightarrow O_{\mathbb{P}^3}(-2) \longrightarrow F \longrightarrow F_C(1) \longrightarrow 0,
\]

with \(C\) a rational curve of degree \(g - 2\). Here general means that the extension is defined by a global section of \(\omega_C(1) \cong O_{\mathbb{P}^3}(g - 4)\) vanishing in \(g - 4\) distinct points.

**Remark 5.2.** In the case \(g = 4\) one gets \(d = 6\). Then \(\omega_Y(-1)\) has degree 0 and, since \(h^0(O_Y(1)) - h^0(\omega_Y(-1)) = 3\) and \(h^0(O_Y(1)) \geq 4\), \(h^0(\omega_Y(-1)) > 0\). It follows that \(\omega_Y(-1) \cong O_Y\). A non-zero global section of \(\omega_Y(-1)\) defines an extension \(0 \rightarrow O_{\mathbb{P}^3}(-3) \rightarrow F \rightarrow F_Y(2) \rightarrow 0\), with \(F\) locally free of rank 2, with \(c_1(F) = -1\), \(c_2(F) = 0\). Since \(H^0(F) \cong H^0(F_Y(2)) \neq 0\) (because \(H^1(O_Y(2)) = 0\) hence \(h^0(O_Y(2)) = 9\)) and \(H^0(F(-1)) = 0\), one gets that \(F \cong O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(-1)\) hence \(Y\) is a complete intersection of type \((2, 3)\). One can view, perhaps, the whole story above as a generalization of this simple fact.

**Appendix A. Monads and Dependency Loci**

**Lemma A.1.** Let \(F\) be a vector bundle on \(\mathbb{P}^3\). If \(H^0(F(-1)) = 0\), \(H^1(F(-2)) = 0\), \(H^2(F(-1)) = 0\), \(H^3(F(-2)) = 0\) and \(h^3(F(-3)) \leq 3\) then \(F\) is the cohomology sheaf of a monad of the form:

\[
0 \longrightarrow H^1(F^\vee(-1))^\vee \otimes O(-1) \longrightarrow aO \longrightarrow \rightarrow H^1(F(-1)) \otimes O(1) \longrightarrow 0.
\]
Proof. $h^0(F^\vee(-1)) = h^3(F(-3)) \leq 3$ implies that $H^0(F^\vee(-2)) = 0$ hence, by Serre duality, $H^3(F(-2)) = 0$. Now, the following assertions are consequences of the Castelnuovo-Mumford lemma (in the form stated in [11 Lemma 1.21]):

- $H^2(F(-1)) = 0$ and $H^3(F(-2)) = 0$ imply that the graded $S$-module $H^1_S(F)$ is generated in degrees $\leq 0$.
- $h^2(F^\vee(-2)) = h^1(F(-2)) = 0$ and $h^3(F^\vee(-3)) = h^0(F(-1)) = 0$ imply that $H^2(F^\vee(l)) = 0$ for $l \geq -2$ hence $H^1(F(l)) = 0$ for $l \leq -2$.
- $h^2(F^\vee(-2)) = h^1(F(-2)) = 0$ and $h^3(F^\vee(-3)) = h^0(F(-1)) = 0$ imply that the graded $S$-module $H^1_S(F^\vee)$ is generated in degrees $\leq -1$.
- $H^2(F(-1)) = 0$ and $H^3(F(-2)) = 0$ imply that $H^2(F(l)) = 0$ for $l \geq -1$ hence $H^1(F^\vee(l)) = 0$ for $l \leq -3$. Moreover, $H^1(F^\vee(-2)) \simeq H^2(F(-2))^\vee = 0$.

Consequently, $H^1_S(F)$ has $h^1(F(-1))$ minimal generators in degree $-1$ and some number $b$ of minimal generators in degree $0$, and $H^1_S(F^\vee)$ is generated by $H^1(F^\vee(-1))$. Now, applying Horrocks’ method of “killing cohomology” (explained in Barth and Hulek [5]), one gets that $F$ is the cohomology sheaf of a monad of the form:

$$0 \to H^1(F^\vee(-1))^\vee \otimes_k \mathcal{O}_{P^3}(1) \to A \xrightarrow{\alpha} H^1(F(-1))^\otimes_{k} \mathcal{O}_{P^3}(1) \to 0,$$

with $A$ is a direct sum of line bundles. One has $H^0(A(-1)) = 0$ (since $H^0(F(-1)) = 0$), $h^0(A^\vee(-1)) = h^0(F^\vee(-1))$ and $H^0(A^\vee(-2)) = 0$ (because $H^0(F^\vee(-2)) = 0$). One deduces that $A \simeq a\mathcal{O}_{P^3} \oplus H^1(F^\vee(-1))^\vee \otimes \mathcal{O}_{P^3}(1)$, for some integer $a$. Since the component $a\mathcal{O}_{P^3} \to b\mathcal{O}_{P^3}$ of $\alpha$ is 0 (because $H^1(F)$ has $b$ minimal generators of degree 0) and since there is no epimorphism $3\mathcal{O}_{P^3}(1) \to \mathcal{O}_{P^3}$ one deduces that $b = 0$. 

Corollary A.2. Let $F$ be a vector bundle on $P^3$. If $H^0(F(-1)) = 0$, $H^1(F(-2)) = 0$, $H^2(F(-2)) = 0$ and $H^3(F(-3)) = 0$ then $F$ is the cohomology sheaf of a linear monad of the form:

$$0 \to H^1(F^\vee(-1))^\vee \otimes_k \mathcal{O}_{P^3}(1) \to a\mathcal{O}_{P^3} \to H^1(F(-1))^\otimes_{k} \mathcal{O}_{P^3}(1) \to 0.$$ 

Moreover, $h^1(F(-1)) = -\chi(F(-1))$ and $h^1(F^\vee(-1)) = -\chi(F^\vee(-1))$.

Proof. Since $H^2(F(-2)) = 0$ and $H^3(F(-3)) = 0$, the Lemma of Castelnuovo-Mumford implies that $H^2(F(-1)) = 0$ (and $H^3(F(l)) = 0$ for $l \geq -3$). One can apply, now, Lemma A.1. For the last assertions from the statement, one notices that $H^i(F(-1)) = 0$ for $i \neq 1$ and that $H^i(F^\vee(-1)) = 0$ for $i \neq 1$ (because $F^\vee$ satisfies the hypothesis of the corollary, too).

Remark A.3. It is easy to see that, conversely, if a vector bundle $F$ is the cohomology sheaf of a linear monad $0 \to a\mathcal{O}_{P^3}(-1) \to b\mathcal{O}_{P^3} \to c\mathcal{O}_{P^3}(1) \to 0$ then it satisfies the hypothesis of Cor. A.2.

Lemma A.4. Let $E$ be a vector bundle on $P^n$, $n \geq 2$. If $H^1(E_H) = 0$, for every hyperplane $H \subset P^n$, then $h^1(E) \leq \max(0, h^0(E(-1)) - n)$.

Proof. If $H \subset P^n$ is a hyperplane of equation $h = 0$ then, using the exact sequence:

$$H^1(E(-1)) \xrightarrow{h} H^1(E) \longrightarrow H^1(E_H) = 0,$$

one deduces that the multiplication by the linear form $h: H^1(E(-1)) \to H^1(E)$ is surjective. Applying, now, the Bilinear Map Lemma [3 Lemma 5.1], to the bilinear
map $H^1(E) \times H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^1(E(-1))$ deduced from the multiplication map $H^1(E(-1)) \times H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^1(E)$, one gets the inequality from the statement. \qed

**Lemma A.5.** If $\phi: m\mathcal{O}_{\mathbb{P}^n} \to 2\mathcal{O}_{\mathbb{P}^n}(1)$ is an epimorphism of vector bundles on $\mathbb{P}^n$, $n \geq 1$, then $H^0(\phi(1)): H^0(m\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(2\mathcal{O}_{\mathbb{P}^n}(2))$ is surjective.

**Proof.** Since $\phi$ is an epimorphism, $H^0(\phi): H^0(m\mathcal{O}_{\mathbb{P}^n}) \to H^0(2\mathcal{O}_{\mathbb{P}^n}(1))$ must have rank $\geq n+2$. The kernel $K$ of $\phi$ is a vector bundle on $\mathbb{P}^n$ with $h^1(K) \leq 2(n+1)-(n+2) = n$. We shall prove, by induction on $n$, that $H^1(K(1)) = 0$.

The case $n = 1$ is clear because, in that case, $K \simeq (m-4)\mathcal{O}_{\mathbb{P}^1} \oplus 2\mathcal{O}_{\mathbb{P}^1}(-1)$ or $K \simeq (m-3)\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Assuming that our assertion is true on $\mathbb{P}^{n-1}$ let us prove it on $\mathbb{P}^n$. By the induction hypothesis, $H^1(K_H(1)) = 0$, for every hyperplane $H \subset \mathbb{P}^n$. As we saw at the beginning of the proof, $h^1(K) \leq n$. Applying Lemma A.4 to $E := K(1)$ one gets that $H^1(K(1)) = 0$. \qed

The next lemma is a weak variant of some results of Martin-Deschamps and Perrin [10]. This variant suffices for our purposes. We include, for the reader's convenience, an argument that we have extracted from (several places of) the paper of Martin-Deschamps and Perrin.

**Lemma A.6.** Let $E$ be a vector bundle on $\mathbb{P}^3$, of rank $r \geq 2$, such that the evaluation morphism $ev_E: H^0(E) \otimes_k \mathcal{O}_{\mathbb{P}^3} \to E$ has rank $\geq r-1$ at every point of $\mathbb{P}^3$. Let $\Delta$ be the degeneracy scheme of $ev_E$. If $\dim \Delta \leq 1$ and if there are only finitely many points $x \in \Delta$ for which $\mathcal{H}om(\bigwedge^{i+1} F, \bigwedge^i E) \simeq (\bigwedge^i F)^\vee \otimes \bigwedge^i E$. If $r k F \geq r k E =: r$ then the degeneracy scheme of $\phi$ is, by definition, $D_{r-1}(\phi)$. Let $\phi': F' \to E'$ be another morphism of vector bundles. If $\text{Coker } \phi'$ is isomorphic, locally on $\mathbb{P}^3$, to $\text{Coker } \phi$ then $D_{r-i}(\phi') = D_{r-i}(\phi), \forall i \geq 1, r'$ being the rank of $E'$; this follows from the basic property of Fitting ideals (see, for example, Eisenbud [8, § 20.2]).

Next, if $E$ is a rank $r$ vector bundle on $\mathbb{P}^3$ let $\mathbb{P}(H^0(E))$ denote the (classical) projective space of 1-dimensional $k$-vector subspaces of $H^0(E)$. Assuming that $h^0(\mathcal{O}(E)) = N + 1$, one has $\mathbb{P}(H^0(E)) \simeq \mathbb{P}^N$. Consider the following closed subscheme of $\mathbb{P}(H^0(E)) \times \mathbb{P}^3$:

$$Z := \{([s], x) \mid s(x) = 0\},$$

and the canonical projections $p: Z \to \mathbb{P}(H^0(E))$ and $q: Z \to \mathbb{P}^3$. The fiber $p^{-1}([s])$ can be identified with the zero scheme of the global section $s$ of $E$, while $q$ turns $Z$ into a $\mathbb{P}^{N-1}$-bundle over (the scheme) $D_i(ev_E) \setminus D_{i-1}(ev_E)$. In particular, the singular locus of $Z$ is contained in $q^{-1}(D_{r-1}(ev_E))$. We also notice that if one has an exact sequence:

$$0 \longrightarrow m\mathcal{O}_{\mathbb{P}^3} \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

with $E'$ a vector bundle of rank $r - m$ then $D_i(ev_{E'}) = D_{i+m}(ev_E), \forall i \geq 0$, because $\text{Coker } ev_{E'} \simeq \text{Coker } ev_E$. 


It follows, now, easily, by decreasing induction on \( r \geq 3 \), that if \( \text{ev}_E \) has rank \( \geq r - 2 \) at every point of \( \mathbb{P}^3 \), if \( \dim D_{r-1}(\text{ev}_E) \leq 1 \) and if \( \dim D_{r-2}(\text{ev}_E) \leq 0 \) then the dependency scheme \( \Gamma \) of \( r - 2 \) general global sections of \( E \) consists of finitely many simple points. In this case, one has an exact sequence:

\[
0 \rightarrow (r - 2)\mathcal{O}_{\mathbb{P}^3} \rightarrow E \rightarrow \mathcal{F} \rightarrow 0,
\]

where \( \mathcal{F} \) is a rank 2 reflexive sheaf with \( \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^3}}^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_{\Gamma} \). Let \( \sigma \) denote the restriction of the evaluation morphism \( \text{H}^0(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{F} \) of \( \mathcal{F} \) to \( \mathbb{P}^3 \setminus \Gamma \), let \( W \) be the closed subscheme of \( \mathbb{P}(\text{H}^0(\mathcal{F})) \times (\mathbb{P}^3 \setminus \Gamma) \) analogous to \( Z \) above, and let \( \pi: W \rightarrow \mathbb{P}(\text{H}^0(\mathcal{F})) \) and \( \rho: W \rightarrow \mathbb{P}^3 \setminus \Gamma \) be the canonical projections. Notice that \( \mathbb{P}(\mathcal{F}) \) has dimension \( N' + 1 \), where \( N' = N - r + 2 \).

Under the hypothesis of the lemma, \( D_1(\sigma) = \Delta \setminus \Gamma \) and \( D_0(\sigma) = \emptyset \). \( W \) is given, locally, by two equations. Since it is a \( \mathbb{P}^{N' - 2} \)-bundle over \( \mathbb{P}^3 \setminus (\Delta \cup \Gamma) \) and a \( \mathbb{P}^{N' - 1} \)-bundle over \( \Delta \setminus \Gamma \), one deduces that \( W \) is irreducible of dimension \( N' + 1 \) and \( \text{Sing} W \subseteq \rho^{-1}(\Delta \setminus \Gamma) \).

\( \Delta \setminus \Gamma \) can be covered with open subsets \( U \) of \( \mathbb{P}^3 \setminus \Gamma \) with the property that \( \mathcal{F} \mid U \) is trivial and there exists a global section \( s_0 \) of \( \mathcal{F} \) vanishing at no point of \( U \). Extend \( s_0 \mid U \) to a local frame \( (s_0 \mid U, t) \) of \( \mathcal{F} \mid U \) and \( s_0 \) to a \( k \)-basis \( s_0, \ldots, s_N \) of \( \mathbb{P}(\mathcal{F}) \). Then:

\[
s_i \mid U = f_i(s_0 \mid U) + g_i t, \quad \text{with } f_i, g_i \in \mathcal{O}_{\mathbb{P}^3}(U), \quad i = 1, \ldots, N'.
\]

One can assume that \( U \) is isomorphic to an open subset of the affine space \( \mathbb{A}^3 \) hence \( f_i \) and \( g_i \) are functions in three variables \( x_1, x_2, x_3 \). \( \text{W} \cap (\mathbb{P}(\text{H}^0(\mathcal{F})) \times U) \) is given by the equations:

\[
\lambda_0 + \sum_{i=1}^{N'} \lambda_i f_i(x) = 0, \quad \sum_{i=1}^{N'} \lambda_i g_i(x) = 0,
\]

\( \lambda_0, \ldots, \lambda_N \) being homogeneous coordinates on \( \mathbb{P}(\text{H}^0(\mathcal{F})) \simeq \mathbb{P}^N \). The Jacobian matrix of this system of equations is:

\[
\begin{pmatrix}
1 & f_1(x) & \cdots & f_{N'}(x) \\
0 & g_1(x) & \cdots & g_{N'}(x)
\end{pmatrix}
\begin{pmatrix}
\lambda(x, 0, \ldots, 0) \partial f_1/\partial x_1(x, 0, \ldots, 0) & \ldots & \lambda(x, 0, \ldots, 0) \partial f_{N'}/\partial x_1(x, 0, \ldots, 0) \\
\lambda(x, 0, \ldots, 0) \partial g_1/\partial x_2(x, 0, \ldots, 0) & \ldots & \lambda(x, 0, \ldots, 0) \partial g_{N'}/\partial x_2(x, 0, \ldots, 0)
\end{pmatrix}
\begin{pmatrix}
\lambda(x, 0, \ldots, 0) \partial f_1/\partial x_3(x, 0, \ldots, 0) \\
\lambda(x, 0, \ldots, 0) \partial g_1/\partial x_3(x, 0, \ldots, 0)
\end{pmatrix}.
\]

One deduces that if \( x \in U \) then \( \rho^{-1}(x) \subseteq \text{Sing} W \) if and only if:

\[
g_i(x) = 0, \quad (\partial g_i/\partial x_1(x) = 0, \quad (\partial g_i/\partial x_2(x) = 0, \quad (\partial g_i/\partial x_3(x) = 0, \quad i = 1, \ldots, N',
\]

and this is equivalent to \( (g_i)_x \in \mathfrak{m}_x^2, \quad i = 1, \ldots, N' \). Since the ideal \( \mathcal{I}_x \) of \( \mathcal{O}_{\mathbb{P}^3, x} \) is generated by \( (g_i)_x, \quad i = 1, \ldots, N' \), it follows that there are only finitely many points \( x \) of \( \Delta \setminus \Gamma \) for which the fiber \( \rho^{-1}(x) \) is entirely contained in \( \text{Sing} W \). This implies that \( \dim \text{Sing} W \leq N' - 1 \) hence \( \pi(\text{Sing} W) \) is not dense in \( \mathbb{P}(\text{H}^0(\mathcal{F})) \). Applying the Theorem of generic smoothness, one gets that the zero scheme of a general global section \( s \) of \( \mathcal{F} \) is a curve whose singular locus is contained in \( \Gamma \). (The above argument appears in the proof of [10, IV, Prop. 3.1].)

On the other hand, if \( s \in \mathbb{P}(\mathcal{F}) \) and \( x \in \Gamma \) then the zero scheme of \( s \) contains \( x \) and it is, locally at \( x \), a nonsingular curve if and only if \( s(x) \neq 0 \) in \( \mathcal{F}(x) := \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x \) (see, for example, the proof of [3, Prop. 3]). Since the evaluation map \( \mathbb{P}(\mathcal{F}) \rightarrow \mathcal{F}(x) \) is non-zero (it has, in fact, rank \( \geq 2 \)) the lemma is proven. \( \square \)

**Remark A.7.** (i) The hypothesis of Lemma [A.6] is verified if there exists a closed subscheme \( \Delta \) of \( \mathbb{P}^3 \), of dimension \( \leq 1 \), containing only finitely many points \( x \) for
which $\mathcal{I}_{\Delta,x} \subseteq m_x^2$, such that the cokernel of the evaluation morphism of $E$ is an invertible $\mathcal{O}_{\Delta}$-module.

(ii) $\mathcal{I}_{\Delta,x}$ is not contained in $m_x^2$ if and only if there exists an open neighbourhood $U$ of $x$ in $\mathbb{P}^3$ such that $U \cap \Delta$ is contained in a nonsingular surface.

**APPENDIX B. LINES AND POINTS IN $\mathbb{P}^3$**

**Lemma B.1.** Let $Y$ be the union of three mutually disjoint lines $L_1, L_2, L_3$ in $\mathbb{P}^3$ and let $P_0, \ldots, P_3$ be four points in $\mathbb{P}^3$ such that $W := \{P_0, \ldots, P_3\}$ is not contained in a plane. We assume that none of the four points belongs to the quadric surface $Q \subset \mathbb{P}^3$ containing $Y$ and that $H^0(\mathcal{I}_{Y \setminus L_i} \otimes W(2)) = 0$, $l = 1, 2, 3$. Then the homogeneous ideal of $Y \cup W$ is surjective. Using the commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & H^0(\mathcal{I}_{Y \cup W}(3)) \otimes S_1 & \rightarrow & H^0(\mathcal{I}_Y(3)) \otimes S_1 & \rightarrow & H^0(\mathcal{O}_W(3)) \otimes S_1 & \rightarrow & 0 \\
& & \downarrow{\mu} & & \downarrow{\mu_Y} & & \downarrow{\mu_W} & & \\
0 & \rightarrow & H^0(\mathcal{I}_{Y \cup W}(4)) & \rightarrow & H^0(\mathcal{I}_Y(4)) & \rightarrow & H^0(\mathcal{O}_W(4)) & \rightarrow & 0
\end{array}
$$

one sees that it suffices to show that the map $\text{Ker} \mu_Y \rightarrow \text{Ker} \mu_W$ induced by this diagram is surjective.

For $0 \leq i \leq 3$, let $h_i = 0$ be an equation of the plane containing $W \setminus \{P_i\}$ and let $e_i$ be the element of $H^0(\mathcal{O}_W)$ defined by $e_i(P_j) = \delta_{ij}$, $j = 0, \ldots, 3$. Since $\mu_W(e_i \otimes h_j) = h_j(P_i)e_i$ it follows that $\text{Ker} \mu_W$ has a $k$-basis consisting of the elements $e_i \otimes h_j$, with $0 \leq i \leq 3$, $0 \leq j \leq 3$ and $i \neq j$.

Take an $i \in \{0, \ldots, 3\}$. For $1 \leq l \leq 3$, let $h_{il} = 0$ be an equation of the plane containing $L_i \cup \{P_l\}$. We assert that $h_{i1}, h_{i2}, h_{i3}$ are linearly independent. Indeed, if they are linearly dependent then they vanish on a line $L \subset \mathbb{P}^3$ containing $P_i$. $L$ is, then, a 3-secant of $Y = L_1 \cup L_2 \cup L_3$ hence $L$ is contained in the quadric surface $Q \subset \mathbb{P}^3$ containing $Y$ and this contradicts the fact that $P_i \notin Q$. It, thus, remains that $h_{i1}, h_{i2}, h_{i3}$ are linearly independent.

For $1 \leq l \leq 3$, let $q_{il} = 0$ be an equation of the unique quadric surface containing $(Y \setminus L_i) \cup (W \setminus \{P_l\})$ (recall that $H^0(\mathcal{I}_{Y \setminus L_i}(2)) \cong H^0(\mathcal{O}_W(2))$). Choose, also, $h_{il} \in S_1$ vanishing on $L_i$ but not at $P_l$. Then:

$$h_{il}q_{il} \otimes h_{il} - h_{il}q_{il} \otimes h_{il}$$

belongs to $\text{Ker} \mu_Y$ and its image into $\text{Ker} \mu_W$ is $(h_{il}q_{il}(P_l)e_i \otimes h_{il})$. Since $h_{il}$ and $q_{il}$ do not vanish at $P_l$, one deduces that $e_i \otimes h_{il}$ belongs to the image of $\text{Ker} \mu_Y \rightarrow \text{Ker} \mu_W$.

Finally, if $j \in \{0, \ldots, 3\} \setminus \{i\}$ then $h_j(P_l) = 0$. Since $h_{i1}, h_{i2}, h_{i3}$ vanish at $P_l$ and are linearly independent, it follows that $h_j$ is a linear combination of $h_{i1}, h_{i2}, h_{i3}$ hence $e_i \otimes h_j$ belongs to the image of $\text{Ker} \mu_Y \rightarrow \text{Ker} \mu_W$. Since $i \in \{0, \ldots, 3\}$ and $j \in \{0, \ldots, 3\} \setminus \{i\}$ were arbitrary, the map $\text{Ker} \mu_Y \rightarrow \text{Ker} \mu_W$ is surjective. \[\square\]
Corollary B.2. Under the hypothesis of Lemma B.1, choose two more points \(P_4\) and \(P_5\) on the line \(L_1\) and put \(W' := \{P_0, \ldots, P_5\}\). Then the homogeneous ideal of \(L_2 \cup L_3 \cup W'\) is generated by cubic forms.

Proof. One has an exact sequence:

\[
0 \longrightarrow \mathcal{I}_{Y \cup W} \longrightarrow \mathcal{I}_{L_2 \cup L_3 \cup W'} \longrightarrow \mathcal{I}_{\{P_4, P_5\}, L_1} \longrightarrow 0
\]

and \(\mathcal{I}_{\{P_4, P_5\}, L_1} \cong \mathcal{O}_{L_1}(-2)\). Since, by the proof of Lemma B.1, \(H^1(\mathcal{I}_{Y \cup W}(3)) = 0\) and the multiplication map \(H^0(\mathcal{I}_{Y \cup W}(3)) \otimes S_1 \rightarrow H^0(\mathcal{I}_{Y \cup W}(4))\) is surjective it follows that \(H^1(\mathcal{I}_{L_2 \cup L_3 \cup W'}(3)) = 0\) and the multiplication map \(H^0(\mathcal{I}_{L_2 \cup L_3 \cup W'}(3)) \otimes S_1 \rightarrow H^0(\mathcal{I}_{L_2 \cup L_3 \cup W'}(4))\) is surjective. \(\square\)

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