THE BETTI SIDE OF THE DOUBLE SHUFFLE THEORY: A SURVEY

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ABSTRACT. This is a survey of [EF1, EF2, EF3]. The purpose of this series of papers is: (1) to give a proof that associator relations imply double shuffle relations, alternative to [F3]; (2) to make explicit the bitorsor structure on Racinet’s torsor of double shuffle relations. The main tool is the interpretation of the harmonic coproduct in terms of the topology of the moduli space \( \mathcal{M}_{0,4} \) and \( \mathcal{M}_{0,5} \), introduced in [DeT], and its extension to the Betti setup.

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INTRODUCTION

The multizeta values (MZVs) are the real numbers defined by the series

\[
\zeta(k_1, \ldots, k_m) := \sum_{n_1 > \cdots > n_m > 0} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}
\]

for \( k_1, \ldots, k_m \in \mathbb{Z}_{>0} \) and \( k_1 > 1 \). These numbers have recently garnered much interest due to their appearance in various fields of physics and mathematics ([BrKr]). They appear to be examples of periods ([KoZ]) and are, as such, related with motive theory ([De]). Using this

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theory, upper bounds for dimensions of spaces of MZVs have been obtained ([DeG, T]). A related problem is the identification of the algebraic and linear relations among MZVs.

A recent review of the available systems of relations can be found in [B]. Among them, we will focus on the interrelations between: (a) the associator system of relations ([Dr, LM]); (b) the regularized double shuffle relations ([IKaZ, R]).

Each system of relations gives rise to a $\mathbb{Q}$-scheme, defined as the spectrum of the free commutative $\mathbb{Q}$-algebra over formal variables $\zeta^f(k_1, \ldots, k_m)$ for $(k_1, \ldots, k_m) \in \mathbb{Z}_{>1} \times (\mathbb{Z}_{>0})^{m-1}$ by the corresponding ideal. These schemes are called the scheme of associators in case (a) and the double shuffle scheme in case (b); for a $\mathbb{Q}$-algebra, the sets of $\mathbb{k}$-points of these schemes are denoted $M(\mathbb{k})$ in case (a) and $\text{DMR}^B(\mathbb{k})$ in case (b). The definition of $M(\mathbb{k})$ can be found in [Dr], p. 848, and $\text{DMR}^B(\mathbb{k}) = \bigsqcup_{\mu \in \mathbb{k}^*} \text{DMR}_\mu(\mathbb{k})$, where $\text{DMR}_\mu(\mathbb{k})$ is as in [R], Déf. 3.2.1.

The structures of these schemes is elucidated by the following results.

**Theorem 0.1.** ([Dr]) (1) There are explicit $\mathbb{Q}$-group schemes $\{\mathbb{Q}\text{-algebras}\} \ni \mathbb{k} \mapsto GT(\mathbb{k})$, $\text{GRT}(\mathbb{k}) \in \{\text{groups}\}$, and for any $\mathbb{Q}$-algebra $\mathbb{k}$, commuting left and right free and transitive actions of $GT(\mathbb{k})$ and $GRT(\mathbb{k})$ on $M(\mathbb{k})$.

(2) These $\mathbb{Q}$-group schemes are extensions of the multiplicative group $\mathbb{G}_m$ by prounipotent group schemes. Their Lie algebras $\mathfrak{g}_t$ and $\mathfrak{g}_r$ are filtered, moreover $\mathfrak{g}_r$ is complete graded.

Note that the group $\text{Aut}_{GRT(\mathbb{k})}(M(\mathbb{k}))$ of permutations of $M(\mathbb{k})$ which commute with the action of $GRT(\mathbb{k})$ naturally acts on $M(\mathbb{k})$. (1) says that there is an isomorphism between this group and the explicit group $GT(\mathbb{k})$, which is compatible with their actions on $M(\mathbb{k})$.

**Theorem 0.2.** ([R]) (1) There is an explicit $\mathbb{Q}$-group scheme $\{\mathbb{Q}\text{-algebras}\} \ni \mathbb{k} \mapsto \text{DMR}^\text{DR}(\mathbb{k}) \in \{\text{groups}\}$, and for any $\mathbb{Q}$-algebra $\mathbb{k}$, a free and transitive left action of $\text{DMR}^\text{DR}(\mathbb{k})$ on $\text{DMR}^B(\mathbb{k})$.

(2) This $\mathbb{Q}$-group scheme is an extension of the multiplicative group $\mathbb{G}_m$ by a prounipotent group scheme. Its Lie algebra $\mathfrak{d}_m$ is complete graded.

This formulation is obtained in [EF2] using the main result of [R]. The best available result on the comparison of the associator and double shuffle schemes is as follows.

**Theorem 0.3.** ([F3]) For any $\mathbb{Q}$-algebra $\mathbb{k}$, there are compatible inclusions of sets $M(\mathbb{k}) \subset \text{DMR}^B(\mathbb{k})$ and of groups $\text{GRT}(\mathbb{k})^{\text{op}} \subset \text{DMR}^\text{DR}(\mathbb{k})$ (where $\text{op}$ denotes the opposite group).

The proof in [F3] relies on the construction, out of the family of multiple polylogarithm functions, of elements in the bar-complex of the moduli space $\mathcal{M}_{0,5}$, which are then viewed as linear forms on the enveloping algebra $U(p_5)$ (see §2.1), and on the study of the combinatorics of these linear forms. This result was also announced in the unfinished preprint [DeT], which contains in particular a description of one of the main actors of double shuffle theory, the ‘harmonic coproduct’, in terms of topology of the moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$.
The main objectives of the series of papers [EF1, EF2, EF3] are: (a) giving a new proof of
Theorem 0.3, based on the ideas of [DeT] ([EF2]); (b) making the group Aut_{DMR^{\kappa}}(DMR^{DR,R}(\kappa))
explicit, together with its action on DMR^{DR,R}(\kappa) ([EF3]). In order to reach them, we perform
an intermediate task: (c) constructing a ‘Betti’ version of the algebraic apparatus of double
shuffle theory and showing how it is related to the original one by an associator ([EF1]). The
material relative to objective (c) (resp., (a), (b)) is reviewed in §1 (resp. §2, §3).

1. The algebraic framework of the double shuffle theory

1.1. The de Rham side of double shuffle theory. Let \( \mathcal{V}^{DR} \) be the free associative \( \mathbb{k} \)-algebra
over generators \( e_0, e_1 \); it is \( \mathbb{Z}_{\geq 0} \)-graded, with \( e_0, e_1 \) being of degree 1. Let \( \mathcal{W}^{DR} := \mathbb{k}1 + \mathcal{V}^{DR}e_1 \);
this is a \( \mathbb{Z}_{\geq 0} \)-graded subalgebra of \( \mathcal{V}^{DR} \). Set \( \mathcal{M}^{DR} := \mathcal{V}^{DR}/\mathcal{V}^{DR}e_0 \); this is a \( \mathbb{Z}_{\geq 0} \)-graded left
\( \mathcal{V}^{DR} \)-module, therefore by restriction a left \( \mathcal{V}^{DR} \)-module, which is free of rank one, generated
by the class \( 1_{DR} \in \mathcal{M}^{DR} \) of the element \( 1 \in \mathcal{V}^{DR} \).

Let \( \Delta^{V,DR} : \mathcal{V}^{DR} \to (\mathcal{V}^{DR})^{\otimes 2} \) be the \( \mathbb{k} \)-algebra morphism such that \( e_i \mapsto e_i \otimes 1 + 1 \otimes e_i \) for
\( i = 0, 1 \). One shows that \( \mathcal{V}^{DR} \) is freely generated, as an associative algebra, by its elements
\( y_n := -e_0^{n-1}e_1 \), where \( n \geq 1 \). We denote by \( \Delta^{W,DR} : \mathcal{W}^{DR} \to (\mathcal{W}^{DR})^{\otimes 2} \) the \( \mathbb{k} \)-algebra morphism such that \( y_n \mapsto \sum_{i=0}^n y_i \otimes y_{n-i} \) for \( n \geq 1 \), where \( y_0 := 1 \), and by \( \Delta^{M,DR} : \mathcal{M}^{DR} \to (\mathcal{M}^{DR})^{\otimes 2} \) the \( \mathbb{k} \)-module morphism such that \( \Delta^{M,DR}(a \cdot 1_{DR}) = \Delta^{W,DR}(a) \cdot 1_{DR}^{\otimes 2} \), where \( \cdot \) denotes
the action of \( \mathcal{V}^{DR} \) on \( \mathcal{M}^{DR} \). The maps \( \Delta^{X,DR}, X \in \{V, W, M\} \) are all compatible with the
\( \mathbb{Z}_{\geq 0} \)-gradings.

Then \( (\mathcal{V}^{DR}, \Delta^{V,DR}) \) and \( (\mathcal{W}^{DR}, \Delta^{W,DR}) \) are cocommutative Hopf algebras, but the inclusion
\( \mathcal{W}^{DR} \subset \mathcal{V}^{DR} \) is not compatible with the coproducts; \( (\mathcal{M}^{DR}, \Delta^{M,DR}) \) is a cocommutative
coalgebra, and is a coalgebra-module over \( (\mathcal{V}^{DR}, \Delta^{V,DR}) \).

We will denote by \( \hat{X} \) (or \( X^\wedge \)) the completion of a \( \mathbb{Z}_{\geq 0} \)-graded \( \mathbb{k} \)-module \( X \), and use the same
notation for the completion of a morphism between such objects.

For \( g \in \hat{\mathcal{V}}^{DR} \), let \( \Gamma_g(t) := \exp(\sum_{n \geq 1}(1 - 1)^{n+1}(g|e_0^{n-1}e_1)t^{n}/n) / \kappa[[t]]^\wedge \), where \( w \mapsto (g|w) \) is the map \( \mathcal{V} \to \mathbb{k} \) such that \( g = \sum_{w \in \{e_0, e_1\}}^\wedge (g|w).w \).

Let \( w \mapsto w_{\text{reg}} \) be the composed map \( e_0 \mathcal{V}^{DR}e_1 \to \mathcal{V}^{DR} \to \mathcal{V}^{DR} \otimes_{\mathbb{Q}} \mathbb{Q}[a_0, a_1] \to \mathcal{V}^{DR} \), where
the second map is the \( \mathbb{k} \)-algebra morphism induced by \( e_i \mapsto e_i \otimes 1 - 1 \otimes e_i \) for \( i = 0, 1 \) and
the third map is the \( \mathbb{k} \)-module map induced by \( v \otimes a_0^a a_1^b \mapsto e_i^b v e_0^a \) for \( a, b \geq 0, v \in \mathcal{V}^{DR} \). For
\( w \in e_0\{e_0, e_1\}^\wedge e_1 \), let \( m(w) \) be the number of occurrences of \( e_1 \) in \( w \).

Let \( \zeta : e_0\{e_0, e_1\}^\wedge e_1 \to \mathbb{R} \) be the map defined by \( \zeta(e_0^{k_0}e_1 \cdots e_0^{k_m}e_1) = \zeta(k_1, \ldots, k_m) \) for
\( m \geq 1, (k_1, \ldots, k_m) \in \mathbb{Z}_{\geq 1} \times (\mathbb{Z}_{\geq 0})^m-1 \). Set\(^1\)

\[
\varphi_{KZ}^{\hat{\mathcal{V}}^{DR}} := \sum_{w \in e_0\{e_0, e_1\}^\wedge e_1} (-1)^{m(w)} \zeta(w) w_{\text{reg}} \in \hat{\mathcal{V}}^{DR};
\]

\(^1\)The notation \( \hat{\mathcal{V}}^{DR}, \mathcal{M}^{DR} \) stands for the specializations of \( \mathcal{V}^{DR}, \mathcal{M}^{DR} \) for \( \mathbb{k} = \mathbb{C} \).
this is a generating series for the MZVs, usually called the Knizhnik-Zamolodchikov associator
(see [Dr, LM, F1]).

The system of regularization and double shuffle relations ([IKaZ, R]) between MZVs can be
formulated as follows:

\begin{align}
\varphi_{KZ} & \in \mathcal{G}(\hat{\mathcal{V}}^{\text{DR}}), \quad (\Gamma_{\varphi_{KZ}}(-e_1)^{-1}\varphi_{KZ}) \cdot 1_{\text{DR}} \in \mathcal{G}((\hat{\mathcal{M}})^{\text{DR}}), \\
(\varphi_{KZ}|e_0) = (\varphi_{KZ}|e_1) = 0, (\varphi_{KZ}|e_0e_1) = (2\pi i)^2/24,
\end{align}

where \( \mathcal{G} \) denotes the set of group-like elements of \( \hat{\mathcal{V}}^{\text{DR}} \) (resp. \( \hat{\mathcal{M}}^{\text{DR}} \) for \( \hat{\Delta}^{\text{V,DR}} \) (resp. \( \hat{\Delta}^{\text{M,DR}} \)).

This formulation leads to the following definition:

**Definition 1.1.** For \( \mu \in k \), one defines \( \text{DMR}_\mu(k) \) to be the set of elements \( \Phi \in \hat{\mathcal{V}}^{\text{DR}} \) satisfying relations (1.1.1), (1.1.2), with \( \varphi_{KZ}, 2\pi i, \mathcal{C} \) replaced by \( \Phi, \mu, k \).

One has therefore \( \varphi_{KZ} \in \text{DMR}_{2\pi i}(\mathbb{C}) \).

**Remark 1.2.** The ‘double shuffle’ system of relations is the conjunction of the systems of
harmonic and shuffle relations, which follow from the expressions of the MZVs respectively as
iterated sums and as iterated integrals. An example of a harmonic relation is

\[ \forall a, b > 1, \quad \zeta(a)\zeta(b) = \zeta(a + b) + \zeta(a, b) + \zeta(b, a), \]

which follows from \( \zeta(a)\zeta(b) = \sum_{n,m>0} n^{-a}m^{-b} = (\sum_{n=m>0} + \sum_{n,m>0} + \sum_{m,n>0})n^{-a}m^{-b} = \zeta(a, b) + \zeta(a, b) + \zeta(b, a) \).

An example of a shuffle relation is

\[ \forall a, b > 1, \quad \zeta(a)\zeta(b) = \sum_{i+j=a+b} \binom{a-1}{i-1} \binom{b-1}{j-1} \zeta(i,j), \]

which follows from \( \zeta(c) = \int_{0<s_1<\ldots<s_n<1} \prod \frac{ds_1}{s_1} \cdots \frac{ds_n}{s_n} \zeta(i,j) = \int_{0<s_1<\ldots<s_{n+j}<1} \prod \frac{ds_1}{s_1} \cdots \frac{ds_{n+j}}{s_{n+j}} \zeta(i,j) \)

and the shuffle identity for products of iterated integrals.

**Remark 1.3.** The relation between the formalism of [EF1] and [R] is as follows: the
elements \( e_0, e_1 \) correspond to \( x_0, -x_1 \); the pair \( (\hat{\mathcal{V}}^{\text{DR}}, \hat{\Delta}^{\text{V,DR}}) \) corresponds to \( (k\langle X \rangle, \hat{\Delta}) \); the pairs
\( (\hat{\mathcal{V}}^{\text{DR}}, \hat{\Delta}^{\text{W,DR}}) \) and \( (\hat{\mathcal{M}}^{\text{DR}}, \hat{\Delta}^{\text{M,DR}}) \) both correspond to \( (k\langle Y \rangle, \hat{\Delta}_*) \); the map \( \hat{\mathcal{V}}^{\text{DR}} \to \hat{\mathcal{M}}^{\text{DR}} \),
\( a \mapsto a \cdot 1_{\text{DR}} \) corresponds to \( \pi_Y \).

1.2. The Betti side of double shuffle theory. Let \( \mathcal{V}^B \) be the \( k \)-algebra with generators
\( X_0^{\pm 1}, X_i^{\pm 1} \) and relations \( X_iX_j^{-1} = X_j^{-1}X_i = 1 \) for \( i = 0, 1 \). It is equipped with the filtration
\( \mathcal{V}^B = F^0\mathcal{V}^B \supset F^1\mathcal{V}^B \supset \cdots \), where \( F^k\mathcal{V}^B := (\mathcal{V}^B)^k \) for \( k \geq 0 \), where \( \mathcal{V}^B_+ \) is the kernel of the \( k \)-algebra morphism \( \mathcal{V} \to k \) given by \( X_i^{\pm 1} \mapsto 1 \) for \( i = 0, 1 \).

Define a \( k \)-subalgebra \( \mathcal{W}^B \) of \( \mathcal{V}^B \) by \( \mathcal{W}^B := k1 \oplus \mathcal{V}^B(X_1 - 1) \). It is equipped with the induced filtration \( F^k\mathcal{W}^B := \mathcal{W}^B \cap F^k\mathcal{V}^B \) for \( k \geq 0 \). One can show that \( \mathcal{W}^B \) is presented by generators
1.4.1. Let $\text{aut}^{1}(\hat{M}) := \text{aut}^{1}(\hat{M})$ be the topological $k$-algebra automorphism of $\hat{M}$ defined by $e_{0} \mapsto g \cdot \mu e_{0} \cdot g^{-1}$, $e_{1} \mapsto \mu e_{1}$. Let $\text{aut}^{1}(\hat{M})$ be the topological $k$-module automorphism of $\hat{M}$ defined by $\text{aut}^{1}(\hat{M}) := \text{aut}^{1}(\hat{M}) \cdot g$ for any $a \in \hat{M}$. One checks that $\text{aut}^{1}(\hat{M})$ restricts to a topological $k$-algebra automorphism of $\hat{M}$, denoted $\text{aut}^{1}(\hat{M})$, and that there is a unique topological $k$-module automorphism $\text{aut}^{1}(\hat{M})$ of $\hat{M}$, such that $\text{aut}^{1}(\hat{M}) := \text{aut}^{1}(\hat{M}) \cdot 1_{\hat{M}}$ for any $a \in \hat{M}$.

1.3. Filtrations and gradings. As $(\mathcal{V}^{B}, \Delta^{V,B})$, $(\mathcal{W}^{B}, \Delta^{W,B})$ and $(\mathcal{M}^{B}, \Delta^{M,B})$ are Hopf algebras and a coalgebra in the category of filtered $k$-modules, the associated graded objects have the same status in the category of $\mathbb{Z}_{\geq 0}$-modules; these objects are respectively isomorphic to $(\mathcal{V}^{DR}, \Delta^{V,DR})$, $(\mathcal{W}^{DR}, \Delta^{W,DR})$ and $(\mathcal{M}^{DR}, \Delta^{M,DR})$. The isomorphism $\text{gr}(\mathcal{V}^{B}) \simeq \mathcal{V}^{DR}$ is induced by $\text{gr}_{i}(\mathcal{V}^{B}) \ni (\text{class of } X_{i} - 1) \mapsto e_{i} \in \mathcal{V}^{DR}$ for $i = 0, 1$; it induces the isomorphism $\text{gr}(\mathcal{W}^{B}) \simeq \mathcal{W}^{DR}$. The isomorphism $\text{gr}(\mathcal{M}^{B}) \simeq \mathcal{M}^{DR}$ is based on the fact that the isomorphism $\mathcal{W}^{B} \rightarrow \mathcal{M}^{B}$, $a \mapsto a \cdot 1_{B}$, induces an isomorphism $\mathcal{W}^{B} \rightarrow \mathcal{M}^{B}$ for any $i \geq 0$, which induces the first map in the sequence of isomorphisms $\text{gr}_{i}(\mathcal{M}^{B}) \simeq \mathcal{W}^{DR} \rightarrow \mathcal{M}^{DR}$, the last map being $a \mapsto a \cdot 1_{\mathcal{M}^{DR}}$.

1.4. Comparison isomorphisms and geometric interpretations.

1.4.1. Automorphisms of the de Rham side. Set $G^{DR}(k) := k^{\times} \times G(\hat{V}^{DR})$. For $(\mu, g) \in G^{DR}(k)$, let $\text{aut}^{1}(\mu, g)$ be the topological $k$-algebra automorphism of $\hat{V}^{DR}$ defined by $e_{0} \mapsto g \cdot \mu e_{0} \cdot g^{-1}$, $e_{1} \mapsto \mu e_{1}$. Let $\text{aut}^{1}(\mu, g)$ be the topological $k$-module automorphism of $\hat{V}^{DR}$ defined by $\text{aut}^{1}(\mu, g)(a) := \text{aut}^{1}(\mu, g)(a) \cdot g$ for any $a \in \hat{V}^{DR}$. One checks that $\text{aut}^{1}(\mu, g)$ restricts to a topological $k$-algebra automorphism of $\hat{V}^{DR}$, denoted $\text{aut}^{1}(\mu, g)$, and that there is a unique topological $k$-module automorphism $\text{aut}^{1}(\mu, g)$ of $\hat{V}^{DR}$, such that $\text{aut}^{1}(\mu, g)(a) \cdot 1_{\hat{V}^{DR}} = \text{aut}^{1}(\mu, g)(a) \cdot 1_{\hat{V}^{DR}}$ for any $a \in \hat{V}^{DR}$.
One checks that \((\mu, g) \otimes (\mu', g') := (\mu \mu', \text{aut}^{\text{YDR},(10)}_{(\mu, g)}(g'))\) equips \(G^{\text{DR}}(k)\) with a group structure, of which \(G^{\text{DR}}(V)\) is a subgroup. For \(A\) an algebra and \(M\) an \(A\)-module, denote by \(\text{Aut}(A, M)\) the set of pairs \((\alpha, \theta)\), where \(\alpha\) is an algebra automorphism of \(A\), and \(\theta\) is an automorphism of \(M\), such that \(\theta(\alpha(m)) = \alpha(\theta(m)) + a \in A, m \in M;\) this is naturally a group. Then the map taking \((\mu, g)\) to \(\text{aut}^{\text{WDR},(1)}_{(\mu, g)}\), \text{aut}^{\text{MDR},(10)}_{(\mu, g)}\) is a group morphism from \((G^{\text{DR}}(k), \otimes)\) to \(\text{Aut}(V^{\text{DR}}, M^{\text{DR}})\).

The map \(\Gamma : G^{\text{DR}}(k) \to (V^{\text{DR}})^{\times}, (\mu, g) \mapsto \Gamma^{-1}(-e_1)\) satisfies the cocycle identity \(\Gamma((\mu, g) \otimes (\mu', g')) = \Gamma(\mu, g) \text{aut}^{\text{YDR},(10)}_{(\mu, g)}(\Gamma(\mu', g'))\). It follows that the map taking \((\mu, g)\) to \(\text{aut}^{\text{WDR},(1)}_{(\mu, g)}\), \(\text{aut}^{\text{MDR},(10)}_{(\mu, g)}\) is a group morphism from \((G^{\text{DR}}(k), \otimes)\) to \(\text{Aut}(V^{\text{DR}}, M^{\text{DR}})\), where \(\text{aut}^{\text{YDR},(10)}_{(\mu, g)} := \text{Ad}_{\Gamma, (\mu, g)} \circ \text{aut}^{\text{MDR},(10)}_{(\mu, g)}\).

\[\text{Remark 1.4.}\] For \(g \in G(V^{\text{DR}})\), the automorphisms \(S_g, S_{\theta(g)}, S_{g^Y}, S_{\theta(g)}^Y\) from \([R]\) correspond to \(\text{aut}^{\text{YDR},(10)}_{(1, g)}, \text{aut}^{\text{YDR},(10)}_{(1, g)}, \text{aut}^{\text{MDR},(10)}_{(1, g)}, \text{aut}^{\text{MDR},(10)}_{(1, g)}\).

\[1.4.2.\] Isomorphisms between the Betti and de Rham sides. There is a topological \(k\)-algebra isomorphism \(i^V : \hat{V}^B \to \hat{V}^{\text{DR}}\) defined by \(X_i \mapsto \exp(e_i)\) for \(i = 0, 1\). It restricts to a topological \(k\)-algebra isomorphism \(i^W : \hat{V}^B \to \hat{V}^{\text{DR}}\) and it induces a \(k\)-module isomorphism \(i^M : M^B \to M^{\text{DR}}\), defined by \(i^M(a \cdot 1) = i^V(a) \cdot 1\) for \(a \in \hat{V}^{\text{DR}}\).

One then defines the topological \(k\)-algebra isomorphisms \(\text{comp}^{V,(1)}_{(\mu, g)} : \hat{V}^B \to \hat{V}^{\text{DR}}\) and \(\text{comp}^{W,(1)}_{(\mu, g)} : \hat{V}^B \to \hat{V}^{\text{DR}}\) and the \(k\)-module isomorphisms \(\text{comp}^{V,(10)}_{(\mu, g)} : \hat{V}^B \to \hat{V}^{\text{DR}}\) and \(\text{comp}^{M,(10)}_{(\mu, g)} : M^B \to M^{\text{DR}}\) by \(\text{comp}^{V,(\alpha)}_{(\mu, g)} := \text{aut}^{M,(\alpha)}_{(\mu, g)} \circ i^V\) for \(X \in \{V, W, M\}\) and \(\alpha \in \{1, 10\}\).

One then defines \(\Gamma \text{comp}^{V,(1)}_{(\mu, g)} := \text{Ad}_{\Gamma, (\mu, g)} \circ \text{comp}^{W,(1)}_{(\mu, g)}\) and \(\Gamma \text{comp}^{M,(10)}_{(\mu, g)} := \ell_{\Gamma, (\mu, g)} \circ \text{comp}^{W,(10)}_{(\mu, g)}\).

\[1.4.3.\] Geometric interpretations. In \([D]\), prounipotent \(\mathbb{Q}\)-group schemes \(\pi^{\text{DR}}(X, x)\) (§12.4) and \(\pi^B(X, x)\) (§10.5, denoted there \(\pi^B_{\text{Lie, um}}(X, x)\)) are attached to particular schemes \(X\) with (tangential) base point \(x\); torsors \(\pi^{\text{DR}}_1(X; y, x)\) and \(\pi^B_1(X; y, x)\) are also attached to the datum of an additional (tangential) base point \(y\). One has compatible algebra and module identifications \(\text{ULie}\pi^B_1(\mathcal{M}_{0,4}, \bar{1}) \simeq \hat{V}^B\) and \(\mathcal{O}(\pi^B_1(\mathcal{M}_{0,4}, \bar{1}, 0))^{\vee} \simeq \hat{V}^B\) for \(\omega \in \{B, \text{DR}\}\); here \(\mathcal{M}_{0,4}\) is the moduli space of genus zero curves with four marked points and \(\bar{0}\) and \(\bar{1}\) are its tangential base points corresponding to \((0, \frac{\partial}{\partial z})\) and \((1, -\frac{\partial}{\partial z})\) under the identification \(\mathcal{M}_{0,4} \simeq \mathbb{P}^1\) with coordinate \(z\); \(\text{ULie}(-)\) is the comutative topological Hopf algebra attached to a \(\mathbb{Q}\)-group scheme and \(\mathcal{O}(-)\) be the ring of regular functions over a scheme. When \((\mu, g) = (2\pi i, \varphi_{KZ})\), then \(\text{comp}^{V,(1)}_{(\mu, g)}\) and \(\text{comp}^{W,(10)}_{(\mu, g)}\) can respectively be identified with the Betti-de Rham comparison algebra and module isomorphisms \(\text{ULie}\pi^B_1(\mathcal{M}_{0,4}, \bar{1}) \to \text{ULie}\pi^{\text{DR}}_1(\mathcal{M}_{0,4}, \bar{1})\) and \(\mathcal{O}(\pi^B_1(\mathcal{M}_{0,4}, \bar{1}, 0))^{\vee} \to \mathcal{O}(\pi^{\text{DR}}_1(\mathcal{M}_{0,4}, \bar{1}, 0))^{\vee}\) given by [De], §12.16.

\[\text{Remark 1.4.}\] For \(g \in G(V^{\text{DR}})\), the automorphisms \(S_g, S_{\theta(g)}, S_{g^Y}, S_{\theta(g)}^Y\) from \([R]\) correspond to \(\text{aut}^{\text{YDR},(10)}_{(1, g)}, \text{aut}^{\text{YDR},(10)}_{(1, g)}, \text{aut}^{\text{MDR},(10)}_{(1, g)}, \text{aut}^{\text{MDR},(10)}_{(1, g)}\).
2. Geometric interpretation of the harmonic coproducts

2.1. The topology of the moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. For $n \geq 4$, let $\mathcal{M}_{0,n}$ be the moduli space of complex curves of genus 0 with $n$ marked points. A contractible subspace of $\mathcal{M}_{0,n}$ is the space $B_n$ of marked curves given by $\mathbb{P}^1(\mathbb{C})$ with marked points $(x_1, \ldots, x_n)$ distributed on the subset $\mathbb{P}^1(\mathbb{R})$ in counterclockwise order. The corresponding fundamental group $P_n^* := \pi_1(\mathcal{M}_{0,n}, b_n)$ is the pure modular group of the sphere with $n$ marked points; one has $P_n^* \simeq K_{n-1}/Z(K_{n-1})$, where $K_{n-1}$ is the Artin pure braid group of $n-1$ strands on the plane, and $Z(K_{n-1})$ is its center, isomorphic to $\mathbb{Z}$. Recall $K_n$ is the kernel of the natural morphism $B_n \to S_n$, where $B_n$ is the Artin braid group with $n$ strands; let $\sigma_1, \ldots, \sigma_{n-1}$ be the Artin generators of $B_n$, satisfying the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i < n-1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$. Define the family of elements $x_{ij} := \sigma_j^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1}\cdots\sigma_{j-1}^{-1} \in B_n$ for $1 \leq i < j \leq n$; one shows that $x_{ij} \in K_n$, and that this family generates $K_n$. The center $Z(K_n)$ is generated by $x_{12} \cdot x_{13}x_{23} \cdots x_{1n} \cdots x_{n-1,n}$.

For $i \in [1, 5]$, let $pr_i : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$ be the map corresponding to the erasing of the point labeled $i$, and denote in the same way the induced morphism $P_n^* \to P_n^*$. The operation of replacing the point labeled 4 with two nearby points labeled 4 and 5 induces a morphism $\ell : P_n^* \to P_n^*$. We also denote by $pr_i \circ \ell$ the morphisms between group algebras induced by these group morphisms.

When $n = 4$, then $P_n^* \simeq K_3/Z(K_3)$ is freely generated by $x_{12}$ and $x_{23}$; we identify it with the group $F_2 = \mathcal{G}(\mathcal{V}^B)$ of group-like elements of $(\mathcal{V}^B, \Delta^{V,B})$ via $X_0 \simeq x_{23}$, $X_1 \simeq x_{12}$.

To any group $\Gamma$, one functorially associates the $\mathbb{Z}_{\geq 0}$-graded $\mathbb{k}$-Lie algebra $gr(\Gamma)$ attached to its lower central series, and therefore the $\mathbb{Z}_{\geq 0}$-graded $\mathbb{k}$-Lie algebra $gr(\Gamma) \otimes \mathbb{k}$. We set $p_n := gr(P_n^*) \otimes \mathbb{k}$. The Lie algebra $p_n$ is presented by generators $e_{ij}$, $1 \leq i \neq j \leq n$ subject to relations $e_{ji} = e_{ij}$, $[e_{ij}, e_{kl}] = 0$ for $i, j, k, l$ distinct, $\sum_{j \in [1,n]-\{i\}} e_{ij} = 0$ if $i < j$, then $e_{ij}$ is the image of $x_{ij}$ in $gr_1(P_n^*) \otimes \mathbb{k} \subseteq p_n$. The graded Lie algebra morphisms induced by $\ell$ and $pr_i$ will be denoted $\ell : p_4 \to p_5$ and $pr_i : p_5 \to p_4$. The corresponding bialgebra morphisms between universal enveloping algebras will be denoted in the same way.

When $n = 4$, then $p_4$ is freely generated by $e_{12}$ and $e_{23}$; it can be identified with the Lie algebra $f_2 = \mathcal{P}(\mathcal{V}^{DR})$ of primitive elements of $(\mathcal{V}^{DR}, \Delta^{DR})$ via $e_0 \simeq e_{23}$, $e_1 \simeq e_{12}$. The isomorphism $gr(F_2) \otimes \mathbb{k} \simeq f_2$ takes the image of $X_i - 1$ in $gr_1(F_2) \otimes \mathbb{k}$ to $f_2$ to $e_i$ for $i = 0, 1$.

2.2. Geometric interpretation of the Betti harmonic coproducts $\Delta^{X,B}$, $X \in \{W, M\}$.

2.2.1. Interpretation of $\Delta^{W,B}$. Let $J(pr_5)$ be the kernel of the algebra morphism $pr_5 : kP_5^* \to kP_4^* \simeq \mathcal{V}^B$. Then $J(pr_5)$ is a two-sided ideal of $kP_5^*$, freely generated by the family $(x_{5i}-1)_{1 \leq i \leq 3}$ both as a left and as a right $kP_5^*$-module. These properties imply that for any $a \in kP_5^*$, there is a unique element $\varpi(a) = (\varpi(a)_{ij})_{1 \leq i,j \leq 3} \in M_3(kP_5^*)$, such that $(x_{5i}-1)a = \sum_{j=1}^{3} \varpi(a)_{ij}(x_{5j}-1)$ for any $i \in [1, 3]$, and that the map $\varpi : kP_5^* \to M_3(kP_5^*)$ is an algebra morphism.
Let $\mathcal{V}^B[(X_1 - 1)^{-1}]$ be the localization of $\mathcal{V}^B$ with respect to $X_1 - 1$. It admits an algebra $\mathbb{Z}$-filtration given $F^n\mathcal{V}^B[(X_1 - 1)^{-1}] = \sum_{k \geq 0, n_0, \ldots, n_k | n_0 + \ldots + n_k - k = n} F^{n_0} \mathcal{V}^B(X_1 - 1)^{-1} \cdots (X_1 - 1)^{-1} F^{n_k} \mathcal{V}^B$ for $n \in \mathbb{Z}$. Define the elements $\text{row}_1^B := (1 \otimes (1 - X_1)^{-1} - (1 - X_1)^{-1} \otimes 1 \ 0) \in M_{1 \times 3}((\mathcal{V}^B(X_1 - 1)^{-1})^{\otimes 2})$, $\text{col}_1^B := \left( \begin{array}{c} (X_1 - 1) \otimes (X_1 - 1) \\ (X_1 - 1) \otimes (X_1^{-1} - 1) \\ 0 \end{array} \right) \in M_{3 \times 1}((\mathcal{V}^B)^{\otimes 2})$.

**Proposition 2.1.** (see [EF1], Proposition 8.6) The diagram

$$
\begin{array}{c}
\mathcal{V}^B \xrightarrow{\ell} kP^*_5 \xrightarrow{\mathcal{E}} M_3(kP^*_5) \xrightarrow{M_3(\text{diag})} M_3((\mathcal{V}^B)^{\otimes 2}) \xrightarrow{\text{row}_1^B \cdot (-) \cdot \text{col}_1^B} \mathcal{V}^B[(X_1 - 1)^{-1}]^{\otimes 2}
\end{array}
$$

is commutative, where $\text{pr}_{12} : kP^*_5 \to (\mathcal{V}^B)^{\otimes 2}$ is the morphism induced by the group morphism $P^*_5 \xrightarrow{\text{diag}} (P^*_5)^2 \xrightarrow{\text{pr}_{12} \times P^*_5} (P^*_5)^2$, where $\text{diag}$ is the diagonal morphism, where $\text{row}_1^B \cdot (-) \cdot \text{col}_1^B$ is the map $a \mapsto \text{row}_1^B \cdot a \cdot \text{col}_1^B$, where $\mathcal{V}_1^B$ is the subalgebra without unit $\mathcal{V}^B(X_1 - 1) \hookrightarrow \mathcal{V}^B$, and the left vertical map is $a \mapsto a(X_1 - 1)$. In this diagram, all the maps are compatible with the filtrations, except for the left vertical and the rightmost top horizontal maps, which increase the filtration degree by one.

**Idea of proof.** The proof is based on: (1) the fact that $\rho := M_3(\text{pr}_{12}) \circ \mathcal{E} \circ \ell$ is an algebra morphism such that $X_1 - 1 \mapsto \text{col}_1^B \cdot \text{row}_1^B$; (2) the fact that (1) implies that $(\text{row}_{1B}^B \cdot (-) \cdot \text{col}_{1B}^B) \circ \rho \circ ((-) \cdot (X_1 - 1))$ is an algebra morphism $\mathcal{W}^B_{1B} \to \mathcal{V}^B[(X_1 - 1)^{-1}]^{\otimes 2}$; (3) the identification of a generating family of $\mathcal{W}^B_{1B}$; (4) the identification of the images by $(\text{row}_{1B}^B \cdot (-) \cdot \text{col}_{1B}^B) \circ \rho \circ ((-) \cdot (X_1 - 1))$ and $\Delta^{\mathcal{W},B}$ of each element of the generating family of (3). \hfill \qed

2.2.2. Interpretation of $\Delta^{\mathcal{M},B}$. Set $\mathcal{M}^B[(X_1 - 1)^{-1}] := \mathcal{V}^B[(X_1 - 1)^{-1}] / \mathcal{V}^B[(X_1 - 1)^{-1}] \langle X_0 - 1 \rangle$. This is a left $\mathcal{V}^B[(X_1 - 1)^{-1}]$-module. There is a natural morphism $\mathcal{M}^B \to \mathcal{M}^B[(X_1 - 1)^{-1}]$ compatible with the algebra morphism $\mathcal{V}^B \to \mathcal{V}^B[(X_1 - 1)^{-1}]$, and which can be shown to be injective. We denote by $1_B \in \mathcal{M}^B[(X_1 - 1)^{-1}]$ the image of $1_B \in \mathcal{M}^B$. The module $\mathcal{M}^B[(X_1 - 1)^{-1}]$ is equipped with a $\mathbb{Z}$-filtration given by $F^n\mathcal{M}^B[(X_1 - 1)^{-1}] := F^n\mathcal{V}^B[(X_1 - 1)^{-1}] \langle 1_B \rangle$ for $n \in \mathbb{Z}$ and is then a filtered module over $\mathcal{V}^B[(X_1 - 1)^{-1}]$. Let $\text{col}_1^B := \left( \begin{array}{c} 0 \\ (((1 - X_1) \otimes X_1^{-1}) \cdot 1_{1B}^{\otimes 2}) \\ (((1 - X_1^{-1}) \otimes X_1^{-1}) \cdot 1_{1B}^{\otimes 2}) \end{array} \right) \in M_{3 \times 1}((M^B)^{\otimes 2})$.

**Proposition 2.2.** (see [EF1], Proposition 8.14) The diagram

$$
\begin{array}{c}
\mathcal{V}^B \xrightarrow{\ell} kP^*_5 \xrightarrow{\mathcal{E}} M_3(kP^*_5) \xrightarrow{M_3(\text{diag})} M_3((\mathcal{V}^B)^{\otimes 2}) \xrightarrow{\text{row}_1^B \cdot (-) \cdot \text{col}_1^B} \mathcal{M}^B[(X_1 - 1)^{-1}]^{\otimes 2}
\end{array}
$$

is commutative, where $\text{pr}_{12} : kP^*_5 \to (\mathcal{V}^B)^{\otimes 2}$ is the morphism induced by the group morphism $P^*_5 \xrightarrow{\text{diag}} (P^*_5)^2 \xrightarrow{\text{pr}_{12} \times P^*_5} (P^*_5)^2$, where $\text{diag}$ is the diagonal morphism, where $\text{row}_1^B \cdot (-) \cdot \text{col}_1^B$ is the map $a \mapsto \text{row}_1^B \cdot a \cdot \text{col}_1^B$, where $\mathcal{W}^B_{1B}$ is the subalgebra without unit $\mathcal{V}^B(X_1 - 1) \hookrightarrow \mathcal{V}^B$, and the left vertical map is $a \mapsto a(X_1 - 1)$. In this diagram, all the maps are compatible with the filtrations, except for the left vertical and the rightmost top horizontal maps, which increase the filtration degree by one.
is commutative, where \( \text{row}_1^B \cdot (-) \cdot \text{col}_0^B \) is the map \( a \mapsto \text{row}_1^B \cdot a \cdot \text{col}_0^B \). In this diagram, all the maps are compatible with the filtrations.

**Idea of proof.** The proof is based on: (1) the fact that \( \rho(X_0-1) \cdot \text{col}_0^B = 0 \), which implies that there exists a map \( \delta : \mathcal{M}^B \to \mathcal{M}^B [ (X_1 - 1)^{-1}] \otimes 2 \), such that \( \delta \circ ((-1_\mathcal{B}) \cdot \text{col}_0^B) = (\text{row}_1^B \cdot (-) \cdot \text{col}_0^B) \circ \rho \) (\( \rho \) being as in the proof of Proposition (2.1)); (2) Proposition (2.1), which implies that \( \delta \) is compatible with the module structures on both sides and with the morphism \( \Delta^{W, B} : W^B \to (W^B)^{\otimes 2} \); (3) the fact that \( \text{row}_1^B \cdot \text{col}_0^B = 1^B_\otimes 2 \), which implies that \( \delta(1_\mathcal{B}) = 1^B_\otimes 2 \) by (1), and therefore \( \delta = \Delta^{W, B} \) by (2).  

### 2.3. Geometric interpretation of the de Rham harmonic coproducts \( \Delta^{X, \text{DR}} \), \( X \in \{W, M\} \).

#### 2.3.1. Interpretation of \( \Delta^{W, \text{DR}} \). The morphism \( \varpi : kP_5^* \to M_3(kP_5^*) \) is compatible with the filtrations. Let us denote by \( \varpi : U(p_5) \to M_3(U(p_5)) \) the associated graded morphism. One can show that \( \varpi \) may be constructed as follows. Let \( J(p_5) \) be the kernel of the algebra morphism \( \text{pr}_5 : U(p_5) \to U(p_4) \); this is two-sided ideal of \( U(p_5) \), free and generated by \( (e_{i5})_{1 \leq i \leq 3} \) both as a left and right \( U(p_5) \)-module. For \( a \in U(p_5) \), the matrix \( \varpi(a) = (\varpi(a)_{ij})_{1 \leq i,j \leq 3} \) is uniquely determined by the identity \( e_{i5}a = \sum_{k=1}^{3} \varpi(a)_{ik} e_k \) for \( 1 \leq i \leq 3 \).

Let \( \nu_{\text{DR}[e_1^{-1}]} \) be the localization of \( \nu_{\text{DR}} \) with respect to \( e_1 \). This is a \( \mathbb{Z} \)-graded algebra, which can be identified with \( \text{gr}(\nu_{\mathcal{B}}[ (X_1 - 1)^{-1}]) \). Define the elements \( \text{row}_1^\text{DR} := (1 \otimes e_1^{-1} - e_1^{-1} \otimes 1 \otimes 0) \in M_{1 \times 3}(\nu_{\text{DR}[e_1^{-1}]}^{\otimes 2}) \) and \( \text{col}_1^\text{DR} := \left( \begin{array}{c} e_1 \otimes e_1 \\ -e_1 \otimes e_1 \\ 0 \end{array} \right) \in M_{1 \times 3}(\nu_{\text{DR}[e_1^{-1}]}^{\otimes 2}) \).

**Proposition 2.3.** (see [DeT] and [EF1], Proposition 6.3) The diagram

\[
\begin{array}{ccc}
\nu_{\text{DR}} & \xrightarrow{\ell} & U(p_5) \\
\downarrow \scriptstyle{\varpi} & & \downarrow \scriptstyle{M_3(\text{pr}_5)} \\
\nu^{\text{DR}}_{\text{DR}} & \xrightarrow{\Delta^{W, \text{DR}}} & \nu_{\text{DR}}^{\otimes 2} \\
\downarrow \scriptstyle{(-) \cdot e_1} & & \downarrow \scriptstyle{\nu_{\text{DR}}^{\otimes 2}} \\
W^{\text{DR}}_{\text{DR}} & & \end{array}
\]

is commutative, where \( \text{pr}_{12} : U(p_5) \to (\nu_{\text{DR}})^{\otimes 2} \) is the morphism induced by the group morphism \( p_5 \xrightarrow{\text{diag}} (p_5)^2 \xrightarrow{\text{pr}_1 \times \text{pr}_2} (p_4)^2 \), where \( \text{diag} \) is the diagonal morphism, where \( \text{row}_1^\text{DR} \cdot (-) \cdot \text{col}_1^\text{DR} \) is the map \( a \mapsto \text{row}_1^\text{DR} \cdot a \cdot \text{col}_1^\text{DR} \), where \( \nu_{\text{DR}} \) is the subalgebra without unit \( \nu_{\text{DR}}^{e_1^{-1}} \mapsto \nu_{\text{DR}}^{\otimes 2} \), and the left vertical map is \( a \mapsto ae_1 \). In this diagram, all the maps are of degree zero, except for the left vertical and the rightmost top horizontal maps, which are of degree one.

**Idea of proof.** This can be proved either by repeating the steps of the proof of Proposition 2.1, which is what is being done in [EF1], or by applying the associated graded functors to the diagram of this proposition. \qed
2.3.2. Interpretation of $\Delta^{M,\text{DR}}$. Set $M_{\text{DR}}[e_1^{-1}] := V_{\text{DR}}[e_1^{-1}] / V_{\text{DR}}[e_1^{-1}]e_0$. This is a left $V_{\text{DR}}[e_1^{-1}]$-module. There is a natural morphism $M_{\text{DR}} \to M_{\text{DR}}[e_1^{-1}]$ compatible with the algebra morphism $V_{\text{DR}} \to V_{\text{DR}}[e_1^{-1}]$, and which can be shown to be injective. We denote by $1_{\text{DR}} \in M_{\text{DR}}[e_1^{-1}]$ the image of $1_{\text{DR}} \in M_{\text{DR}}$. The module $M_{\text{DR}}[e_1^{-1}]$ is equipped with a $\mathbb{Z}$-grading defined by the condition that $1_{\text{DR}}$ has degree 0 and that $M_{\text{DR}}[e_1^{-1}]$ is graded as a $V_{\text{DR}}[e_1^{-1}]$-module. Let $col_0^{\text{DR}} := \begin{pmatrix} 0 \\ (e_1 \otimes 1) \cdot 1^{\otimes 2}_{\text{DR}} \end{pmatrix} \in M_{3 \times 1}((M^{\text{DR}}) \otimes 2).

Proposition 2.4. (see [EF1], Proposition 6.9) The diagram

\[
\begin{array}{c}
\begin{array}{ccc}
\gamma_{\text{DR}} & \xrightarrow{e} & U(p_5) \\
\downarrow (-) \cdot 1_{\text{DR}} & & \downarrow \pi & \downarrow M_3(U(p_5)) \xrightarrow{M_{3}(pr_{12})} M_3((\gamma_{\text{DR}}) \otimes 2) \\
M_{\text{DR}} & \xrightarrow{\Delta_{M,\text{DR}}} & (M^{\text{DR}}) \otimes 2 \\
\end{array}
\end{array}
\]

is commutative, where \(\text{row}_{1}^{\text{DR}} \cdot (-) \cdot \text{col}_{0}^{\text{DR}}\) is \(a \mapsto \text{row}_{1}^{\text{DR}} \cdot a \cdot \text{col}_{0}^{\text{DR}}\). In this diagram, all the maps have degree zero.

Idea of proof. This can be proved either by repeating the steps of the proof of Proposition 2.2, which is what is being done in [EF1], or by applying the associated graded functor to the diagram of that proposition.

3. Associators and double shuffle relations

3.1. Associators. The notion of associator was defined in [Dr]; it was then shown in [F2] that this definition can be formulated as follows. For \((I,J,K)\) one of the triples \((2,3,4), (12,3,4), (1,23,4), (1,2,3)\), define a Lie algebra morphism \(f_2 : p_5 \to x \mapsto x^{I,J,K}\) by \(e_0^{I,J,K} := e_{I,J}\) and \(e_{i,j,k}^{I,J,K} := e_{I,K}\), where \(e_{i,j,k} := e_{i,k} + e_{j,k}\) and \(e_{i,j,k} := e_{i,j,k}\). We denote the induced algebra morphisms \(\gamma_{\text{DR}} : U(p_5) \to (p_5)^{\wedge}\) in the same way.

Definition 3.1. For \(\mu \in k^{\times}\), one sets \(M_{\mu}(k) := \{\Phi \in \mathcal{G}(\gamma_{\text{DR}})|\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} = \Phi^{12,3,4}\Phi^{12,3,4}\}; (\Phi|e_0) = (\Phi|e_1) = 0\) and \((\Phi|e_0 e_1) = \frac{u^2}{2\pi}\), where \((\Phi|e_0 e_1)\) is the coefficient of \(e_0 e_1\) as an expansion in words in \(e_0, e_1\). The set of associators is \(M(k) := \sqcup_{\mu \in k^{\times}} M_{\mu}(k)\).

One has \(\varphi_{\text{KZ}} \in M_{2\pi i}(\mathbb{C})\).

3.2. Compatibility of the associators with the coproducts.

Theorem 3.2. ([EF1], Theorems 10.9 and 11.13) Let \(\mu \in k^{\times}\), and \(\Phi \in M_{\mu}(k)\).
1) The diagram of $k$-algebra morphisms

\[
\begin{array}{ccc}
\mathcal{W}^B & \xrightarrow{\Delta^A_{\mu, B}} & (\mathcal{W}^B)^{\otimes 2^\Lambda} \\
\gamma_{\text{comp}_{P,\Phi}}^W & \downarrow & \downarrow \gamma_{\text{comp}_{P,\Phi}}^W \\
\mathcal{M}^B & \xrightarrow{\Delta^A_{\mu, B}} & (\mathcal{M}^B)^{\otimes 2^\Lambda}
\end{array}
\]

is commutative.

2) The diagram of $k$-module morphisms

\[
\begin{array}{ccc}
\mathcal{M}^B & \xrightarrow{\Delta^A_{\mu, B}} & (\mathcal{M}^B)^{\otimes 2^\Lambda} \\
\gamma_{\text{comp}_{P,\Phi}}^M & \downarrow & \downarrow \gamma_{\text{comp}_{P,\Phi}}^M \\
\mathcal{M}^B & \xrightarrow{\Delta^A_{\mu, B}} & (\mathcal{M}^B)^{\otimes 2^\Lambda}
\end{array}
\]

is commutative.

**Sketch of proof of 1.** In [BN], one introduces the categories $\text{PaB}$ and $\text{PaCD}$ of parenthesized braids and parenthesized braid diagrams, whose sets of objects both coincide with the set of parenthesized words in one letter $\bullet$, and one attaches to each associator $(\mu, \Phi)$ a functor $\text{PaB} \to \text{PaCD}$. If $P, Q$ are two parenthesized words in $\bullet$ of length $n$, and $\mathcal{C}(X, Y)$ is the set of morphisms $X \to Y$, where $X, Y$ are objects of a category $\mathcal{C}$, one denotes by $\text{comp}_{P, \Phi}^Q : B_n \simeq \text{PaB}(P, Q) \to \text{PaCD}(P, Q) \simeq U(p_{n+1})^\wedge \rtimes S_n$ the resulting map; we also denote $\text{comp}_{P, \Phi}^Q$ (resp. $\text{PaB}(P)$, $\text{PaCD}(P)$) for $\text{comp}_{P, \Phi}^P$ (resp. $\text{PaB}(P)$, $\text{PaCD}(P)$).

Define the element $P_{\mu, \Phi} \in M_3((U_p)^{\otimes 2^\Lambda})$ by the equality

\[
M_{3 \times 1}(\text{comp}_{P, \Phi}^Q) \begin{pmatrix} x_{15} - 1 \\ x_{25} - 1 \\ x_{35} - 1 \end{pmatrix} = P_{\mu, \Phi} \begin{pmatrix} e_{15} \\ e_{25} \\ e_{35} \end{pmatrix}.
\]

and set

\[
\mathcal{P}_{\mu, \Phi} := M_3(\text{pr}_{12})(P_{\mu, \Phi}) \in M_3((U_p)^{\otimes 2^\Lambda}).
\]

In [EF1], Proposition 9.20 and Corollary 9.21, one gives explicit expressions for $P_{\mu, \Phi}$ and $\mathcal{P}_{\mu, \Phi}$.

Define also the following elements in $(U_p)^{\otimes 2^\Lambda} \times \subset (U_p)^{\otimes 2^\Lambda} \times$:

\[
B_\Phi := \frac{\Gamma(\Phi(-e_1) \otimes \Gamma(\Phi(-e_1))}{\Gamma(\Phi(-e_1) \otimes 1 - 1 \otimes e_1)}, \quad u_{\mu, \Phi} := \mu(1 \otimes e^{-\mu_{e_1}}) e^{\mu(e_1 \otimes 1 + 1 \otimes e_1)} - 1 \Gamma(\Phi(e_1 \otimes 1 - 1 \otimes e_1))
\]

\[
u_{\mu, \Phi} := \frac{1}{\mu}(1 \otimes e^{\mu_{e_1}}) \Gamma(\Phi(e_1) \otimes \Gamma(\Phi(e_1))) - \frac{1}{\mu}(1 \otimes e_{\mu_{e_1}}) \Gamma(\Phi(e_1) \otimes 1 - 1 \otimes e_1), \quad \kappa_{\mu, \Phi} := \mu(e_0, e_1) \otimes (e_{\mu/2}, e_1)
\]

Set also

\[
\text{row}_1 := ((X_1 - 1) \otimes (1 - X_1^{-1})) \cdot \text{row}_1^B \in M_{1 \times 3}((U_p)^{\otimes 2^\Lambda}), \quad \text{col}_1 := \text{col}_1^B \cdot ((X_1 - 1) \otimes (1 - X_1^{-1}))^{-1} \in M_{3 \times 1}((U_p)^{\otimes 2^\Lambda});
\]

\[
\text{row}_1 := (e_1 \otimes e_1) \cdot \text{row}_1^\text{DR} \in M_{1 \times 3}((U_p)^{\otimes 2^\Lambda}), \quad \text{col}_1 := \text{col}_1^\text{DR} \cdot (e_1 \otimes e_1)^{-1} \in M_{3 \times 1}((U_p)^{\otimes 2^\Lambda}).
\]
Define the following morphisms of topological $k$-algebras:

- $\hat{\rho} := (M_3(\text{pr}_{12}) \circ \Box \circ \ell)^\wedge : \hat{\nu}^{\text{DR}} \to M_3(\nu^{\text{DR}})^{\otimes 2}$, $\hat{\mu} := (M_3(\text{pr}_{12}) \circ \Box \circ \ell)^\wedge : \hat{\nu}^{\text{B}} \to M_3(\nu^{\text{B}})^{\otimes 2}$, where $(-)^\wedge$ indicates the completion with respect to the underlying filtrations.
- $\text{comp}_{(\mu,\Phi)}^{V,1} \overset{\cdot}{\rightarrow}$ is the extension of $\text{comp}_{(\mu,\Phi)}^{V,1}$ to a $k$-algebra isomorphism $\hat{\nu}^{\text{B}}[(X_1 - 1)^{-1}] \to \hat{\nu}^{\text{DR}}[e_1^{-1}]$.
- for $A$ a $k$-algebra and $a \in A$, $\cdot a$ is the product on $A$ defined by $x \cdot_a y = x ay$, and $\text{mor}_{A,a} : (A, \cdot_a) \to A$ is the algebra morphism given by $x \mapsto xa$.

Consider the diagram

where we write $\text{Ad}(a)$ for $\text{Ad}_a$. In this diagram, the commutativities of (A1) and (A7) follows from Proposition 2.1 and 2.3. The commutativities of (A2), (A5), (A6) and (A8) are immediate. The commutativity of (A3) follows from the definition of $\hat{P}_{(\mu,\Phi)}$, and more precisely from the relations between $P_{(\mu,\Phi)}$, $\text{comp}_{(\mu,\Phi)}^{(1)}$, $\hat{\omega}$, and $\hat{\omega}$. The commutativity of (A4) is a consequence of two equalities, one in $M_{1 \times 3}(\nu^{\text{DR}})^{\otimes 2\wedge}$ and the other in $M_{3 \times 1}(\nu^{\text{DR}})^{\otimes 2\wedge}$. Both follow from explicit computation based on the already mentioned computation of $\hat{P}_{(\mu,\Phi)}$. One easily derives the commutativity of the announced diagram.

**Sketch of proof of 2.** Set

$$R_{(\mu,\Phi)} := \hat{\rho}(\Phi)^{-1} P_{(\mu,\Phi)}^{-1} \Phi(\epsilon_0, e_1) \otimes \Phi(\epsilon_0, e_1) \in \text{GL}_3(\nu^{\text{DR}})^{\otimes 2\wedge}).$$

Define the element $Q_{(\mu,\Phi)} \in M_3((U_{p_5})^{\otimes 2\wedge})$ by the equality

$$M_{3 \times 1}(\text{comp}_{(\mu,\Phi)}^{(1)}) \begin{pmatrix} x_{15} - 1 \\ x_{25} - 1 \\ x_{35} - 1 \end{pmatrix} = Q_{(\mu,\Phi)} \begin{pmatrix} \epsilon_{15} \\ \epsilon_{25} \\ \epsilon_{35} \end{pmatrix}. $$
and set
\[ \overline{Q}_{(\mu,\phi)} := M_3(\text{pr}_{12})(Q_{(\mu,\phi)}) \in M_3((V_{\text{DR}})^{\otimes 2\wedge}). \]

In \([EF1]\), Proposition 9.23 and Corollary 9.24, one gives explicit expressions for \(Q_{(\mu,\phi)}\) and \(\overline{Q}_{(\mu,\phi)}\).

By the categorical origin of the morphisms \(\text{comp}^P_{(\mu,P)}\), the two algebra morphisms corresponding to \(P = ((\bullet\bullet)\bullet)\) and \((\bullet(\bullet)\bullet)\) are related by an inner conjugation. One derives from there an expression of \(Q_{(\mu,\phi)}\) in terms of \(P_{(\mu,\phi)}\) (see \([EF1]\), Lemma 11.6), which implies the relation
\[ \overline{Q}_{(\mu,\phi)} = \overline{P}_{(\mu,\phi)}\rho(\Phi) \]
([EF1], Lemma 11.7), from which one derives
\[ R_{(\mu,\phi)}^{-1} = (\Phi(e_0, e_1) \otimes \Phi(e_0, e_1))^{-1}\kappa_{(\mu,\phi)}\overline{Q}_{(\mu,\phi)} \]
([EF1], Corollary 11.8).

Consider the diagram

\[ \begin{array}{ccccccc}
\hat{\mathcal{M}}^B & \xrightarrow{\hat{\Delta}_{\mathcal{M},B}} & (\mathcal{M}^B)^{\otimes 2\wedge} & \xrightarrow{F^{-1}} & (\mathcal{M}^B_{[\frac{1}{1}]}{[1]}_{\geq -1})^{\otimes 2\wedge} \\
\hat{\mathcal{M}}^{\text{DR}} & \xrightarrow{\Delta_{\mathcal{M},\text{DR}}} & (\mathcal{M}^{\text{DR}})^{\otimes 2\wedge} & \xrightarrow{\text{comp}^{\text{M}_{\text{loc}}(10)}} & (\mathcal{M}^{\text{DR}}_{[\frac{1}{1}]}{[1]}_{\geq -1})^{\otimes 2\wedge} \\
\end{array} \]

where \(\text{comp}^{\text{M}_{\text{loc}}(10)} : \mathcal{M}^B([X_1 - 1]^{\wedge}) \rightarrow \mathcal{M}^{\text{DR}}[e_1^{-1}]^{\wedge}\) is the \(k\)-module isomorphism which both extends the \(k\)-module isomorphism \(\text{comp}_P^{\text{M}_{\text{loc}}(10)} : \hat{\mathcal{M}}^B \rightarrow \hat{\mathcal{M}}^{\text{DR}}\) and is compatible with the \(k\)-algebra isomorphism \(\text{comp}_{(\mu,\phi)}^{\text{V}_{\text{loc}}(1)} : V^B([X_1 - 1]^{\wedge}) \rightarrow V^{\text{DR}}[e_1^{-1}]^{\wedge}\).

In this diagram, the commutativity of (M2) is immediate. The commutativities of (M1) and (M5) are consequences of Proposition 2.2 and 2.4.

(M3) states the equality of two \(k\)-module morphisms \(\hat{\mathcal{M}}^B \rightarrow M_3((\mathcal{M}^{\text{DR}})^{\otimes 2\wedge}),\) which turn out to be free rank one module morphisms over the two algebra morphisms \(\hat{\mathcal{V}}^B \rightarrow M_3((V_{\text{DR}})^{\otimes 2\wedge})\) whose equality is stated by the commutativity of (A3); its equality is then a consequence of the fact that the images of the generator \(1_B \in \hat{\mathcal{M}}^B\) coincide, which is itself a consequence of the definition of \(R_{(\mu,\phi)}\).
The commutativity of (M4) is a consequence of two equalities, one in $M_{1\times 3}(V_{DR})^\otimes$ and the other in $M_{3\times 1}(M_{DR}[\epsilon_1^{-1}])^\otimes$. The first equality is a part of the proof of the commutativity of (A4). The second equality follows from explicit computation based on (3.2.1) and on the hexagon identities satisfied by $\Phi$. One easily derives the commutativity of the announced diagram. □

3.3. Inclusion of the scheme of associators in the double shuffle scheme.

**Theorem 3.3.** Let $\mu \in k^\times$. Then $M_\mu(k) \subset DMR_\mu(k)$.

**Proof.** Let $\Phi \in M_\mu(k)$. One has $(\Phi|e_0) = (\Phi|e_1) = 0$ and $(\Phi|e_0 e_1) = \mu^2/24$. Applying the equality $\hat{\Delta}^{M,DR} \circ r_{\text{comp}}^{M,(10)} = (r_{\text{comp}}^{M,(10)})^\otimes \circ \hat{\Delta}^{M,B}$ from Theorem 3.2, 2) to $1_B \in M_B$, and using $\hat{\Delta}^{M,B}(1_B) = 1_B^{\otimes 2}$, one obtains the group-likeness of $r_{\text{comp}}^{M,(10)}(1_B)$ for $\hat{\Delta}^{M,DR}$. One computes $r_{\text{comp}}^{M,(10)}(1_B) = (\Gamma_s(-e_1)^{-1} \hat{\Phi}) \cdot 1_{DR}$, which implies the result. □

4. Bitorsor structure on the double shuffle torsor

4.1. The torsors $DMR_0(k)$ and $DMR_{DR}(k)$.

**Definition 4.1.** A torsor $GX$ is the data of a group $G$, of a nonempty set $X$, and of a free and transitive action of $G$ on $X$.

The left regular action of a group $G$ on itself gives rise to the trivial torsor $GG$.

**Definition 4.2.** A torsor $GX'$ is called a subtorsor of the torsor $GX$ iff $G'$ (resp. $X'$) is a subgroup (resp. subset) of $G$ (resp. of $X$) and if the action of $G'$ on $X'$ is compatible with the action of $G$ on $X$.

**Theorem 4.3.** ([R], §3.2.3) $DMR_0(k)$ is a subgroup of $(G(\hat{Y}_{DR}), \oplus)$, and for any $\mu \in k^\times$, $DMR_0(k)DHR_{\mu}(k)$ is a subtorsor of $G(\hat{Y}_{DR})G(\hat{Y}_{DR})$.

**Definition 4.4.** One sets $DMR^{DR}(k) := k^\times \times DMR_0(k) \subset G^{DR}(k)$, $DMR^{DR,B}(k) := \{(\mu, g)\mid \mu \in k^\times, g \in DMR_0(k)\} \subset G^{DR}(k)$.

**Lemma 4.5.** (see [EF2], Lemma 2.13) $DMR^{DR}(k)$ is a subgroup of $(G^{DR}(k), \oplus)$, and $DMR_{DR}(k)DMR_{DR,B}(k)$ is a subtorsor of $G^{DR}(k)$.

4.2. Relation of $DMR_{DR}(k)DMR_{DR,B}(k)$ with a stabilizer subtorsor of $G^{DR}(k)$.

**Lemma 4.6.** (see [EF2], Lemma 2.3) If $G_1X_1$ and $G_2X_2$ are subtorsors of the torsor $GX$ such that $X_1 \cap X_2 \neq \emptyset$, then $G_1 \cap G_2X_1 \cap X_2$ is a subtorsor of $GX$, called the intersection of both subtorsors.
Lemma 4.7. (see [EF2], Lemma 2.10) Set $\Gamma^{\text{DR}}(\mathbf{k}) \coloneqq \{(\mu, g) \in \Gamma^{\text{DR}}(\mathbf{k}) | (g|e_0) = (g|e_1) = (g|e_0e_1) = 0\}$ and $\Gamma^{\text{DR},B}(\mathbf{k}) \coloneqq \{(\mu, \Phi) \in \Gamma^{\text{DR}}(\mathbf{k}) | (\Phi|e_0) = (\Phi|e_1) = 0, (\Phi|e_0e_1) = \mu^2/24\}$, then $\Gamma^{\text{DR}}(\mathbf{k})^{\text{DR},B}(\mathbf{k})$ is a subtorus of $\Gamma^{\text{DR}}(\mathbf{k})^{\text{DR},B}(\mathbf{k})$.

Definition 4.8. An action of a torsor $G \times X$ on a pair of isomorphic $\mathbf{k}$-modules $(V, V')$ is the data of a group morphism $\rho : G \to \text{Aut}_{\mathbf{k}\text{-mod}}(V)$ and of a map $\rho' : X \to \text{Iso}_{\mathbf{k}\text{-mod}}(V', V)$, such that $\rho'(g \cdot x) = \rho(g) \circ \rho'(x)$ for $g \in G, x \in X$.

One proves:

Lemma 4.9. (see [EF2], Lemma 2.6) Let $G \times X$ be a torsor and $(\rho, \rho')$ be an action on the pair $(V, V')$ of isomorphic $\mathbf{k}$-modules. Let $(v, v') \in V \times V'$. Let $\text{Stab}(v) \coloneqq \{g \in G | \rho(g)(v) = v\}$ and $\text{Iso}(v', v) \coloneqq \{x \in X | \rho'(x)(v') = v\}$. If $\text{Iso}(v', v)$ is nonempty, then $\text{Stab}(v) \text{Iso}(v', v)$ is a subtorus of $G \times X$.

Set $G \times X := \Gamma^{\text{DR}}(\mathbf{k})^{\text{DR}}(\mathbf{k})$ and $V^\omega := \text{Hom}_{\mathbf{k}\text{-mod}_{\text{op}}}(\hat{\mathcal{M}}^\omega, (\mathcal{M}^\omega)^{\otimes 2\omega})$ for $\omega \in \{\text{DR}, \text{B}\}$; here $\mathbf{k}\text{-mod}_{\text{op}}$ is the category of topological $\mathbf{k}$-modules, i.e., $\mathbf{k}$-modules equipped with a decreasing $\mathbb{Z}_{\geq 0}$-filtration, separated and complete for the corresponding topology. An action of $G \times X$ on $(V^\text{DR}, V^\text{B})$ is given by $\rho : (\mu, g) \mapsto (V^\text{DR} \ni f^\text{DR} \mapsto (t_{\text{Aut}^M_{\mu,g}})^{\otimes 2\omega} \circ f^\text{DR} \circ (t_{\text{Aut}^M_{\mu,g}})^{\otimes 1} \in V^\text{DR})$ and $\rho' : (\mu, g) \mapsto (V^\text{B} \ni f^\text{B} \mapsto (t_{\text{comp}_{\mu,g}})^{\otimes 2\omega} \circ f^\text{B} \circ (t_{\text{comp}_{\mu,g}})^{\otimes 1} \in V^\text{DR})$.

The stabilizer subtorus relative to the pair of vectors $(\hat{\Delta}^{\text{M,DR}}, \hat{\Delta}^{\text{M,B}})$ is denoted $\text{Aut}(\hat{\Delta}^{\text{DR}}) \text{Iso}(\hat{\Delta}^{\text{B/DR}})$.

Theorem 4.10. (see [EF2], Theorem 3.1) The subtorus $\text{DMR}^{\text{DR}}(\mathbf{k})^{\text{DR}}(\mathbf{k})$ coincides with the intersection of the subtoruses $\text{Aut}(\hat{\Delta}^{\text{M,DR}}) \text{Iso}(\hat{\Delta}^{\text{M,B/DR}})$ and $G^{\text{DR}}(\mathbf{k})^{\text{DR,B}}(\mathbf{k})$.

Sketch of proof. It follows from the proof of Theorem 3.3 that $\rho(\Phi(-e_1)^{-1}) \cdot 1_{\text{DR}} \in \mathcal{G}(\hat{\mathcal{M}}^{\text{DR}})$. Therefore $\text{Iso}(\hat{\Delta}^{\text{M,B/DR}}) \cap G^{\text{DR}}(\mathbf{k}) \subset \text{DMR}^{\text{DR,B}}(\mathbf{k})$. Both sides of this inclusion are subtorus of $G^{\text{DR}}(\mathbf{k})^{\text{DR}}(\mathbf{k})$, with underlying groups $\text{Aut}(\hat{\Delta}^{\text{M,DR}}) \cap G^{\text{DR}}(\mathbf{k})$ and $\text{DMR}^{\text{DR}}(\mathbf{k})$. It follows from [EF0] that these subgroups of $G^{\text{DR}}(\mathbf{k})$ are equal, which implies the equality of both torus.

4.3. Computation of $\text{Aut}_{\text{DMR}^{\text{DR}}(\mathbf{k})}^{\text{DMR}^{\text{DR,B}}(\mathbf{k})}$ and $\text{Aut}_{\text{DMR}^{\text{B}}(\mathbf{k})}^{\text{DMR}^{\text{B}}(\mathbf{k})}$.

4.3.1. Group corresponding to a torsor. For $G \times X$ a torsor, let $\text{Aut}_G(X)$ be the group of right-acting permutations of $X$ which commute with the action of $G$. This group acts simply and transitively on $X$. We will call it the group corresponding to $G \times X$.

Note that the choice of an element of $X$ induces an isomorphism between $G$ and $\text{Aut}_G(X)$, which however gets composed with an inner automorphism upon change of the element.

Lemma 4.11. (a) If $G \times X'$ is a subtorus of $G \times X$, then $\text{Aut}_{G'}(X')$ is canonically a subgroup of $\text{Aut}_G(X)$.

(b) If $G_1 X_1$ and $G_2 X_2$ are subtorus of $G \times X$ with $X_1 \cap X_2 \neq \emptyset$, then $\text{Aut}_{G_1 \cap G_2}(X_1 \cap X_2) = \text{Aut}_{G_1}(X_1) \cap \text{Aut}_{G_2}(X_2)$ (equality of subgroups of $\text{Aut}_G(X)$).
Proof. For $E$ a set, denote by $S_E$ the group of permutations of $E$, and if $e \in E$, let $S_{E,e}$ be the subgroup of permutations which take $e$ to itself. Then $X'$ may be viewed as an element of the quotient $G' \setminus X$, and there is a natural group morphism $\text{Aut}_G(X) \to S_{G' \setminus X}$. Let $\text{Aut}_G(X, X')$ be the preimage of $S_{G' \setminus X, X'}$. The natural map $\text{Aut}_G(X, X') \to \text{Aut}_G(X')$ is a group isomorphism.

The result follows from the diagram $\text{Aut}_G(X') \simeq \text{Aut}_G(X, X') \subset \text{Aut}_G(X)$. This implies (a). (b) then follows from $\text{Aut}_G(X, X_1 \cap X_2) = \text{Aut}_G(X, X_1) \cap \text{Aut}_G(X, X_2)$. □

4.3.2. The group $G^B(k)$ and its actions. Let $G^B(k) := k^\times \times G(V^B)$. For $(\mu, g) \in G^B(k)$, let $\text{aut}_{(\mu,g)}^{V,B}$ be the automorphism of the topological $k$-algebra $\hat{V}^B$ given by $X_0 \mapsto gX_0g^{-1}$, $X_1 \mapsto X_1^\mu$, where $a \mapsto a^\mu$ is the self-map of $G(\hat{V}^B)$ given by $a^\mu := \exp(\mu \log(a))$. Let $\text{aut}_{(\mu,g)}^{V,B}(10)$ be the topological $k$-module automorphism of $\hat{V}^B$ defined by $\text{aut}_{(\mu,g)}^{V,B}(10) := \text{aut}_{(\mu,g)}^{V,B}(10) \cdot g$ for any $g \in \hat{V}^B$. As in the de Rham situation, there is a unique topological $k$-module automorphism $\text{aut}_{(\mu,g)}^{M,B(10)}$ of $\hat{M}^B$, such that $\text{aut}_{(\mu,g)}^{M,B(10)}(a \cdot 1_B) = \text{aut}_{(\mu,g)}^{V,B}(10)(a) \cdot 1_B$ for any $a \in \hat{V}^B$.

One checks that $(\mu, g) \circ (\mu', g') := (\mu' \cdot \text{aut}_{(\mu,g)}^{V,B}(g'))$ equips $G^B(k)$ with a group structure, of which $G(\hat{V}^B)$ is a subgroup.

Then the map taking $(\mu, g)$ to $\text{aut}_{(\mu,g)}^{V,B}(10)$ (resp. $\text{aut}_{(\mu,g)}^{V,W,B}(10)$, $\text{aut}_{(\mu,g)}^{M,B}(10)$) is a group morphism from $(G^B(k), \oplus)$ to $\text{Aut}_{k\text{-alg}_{\text{top}}}^{(\hat{V}^B)}$ (resp. $\text{Aut}_{k\text{-mod}_{\text{cont}}}^{(\hat{V}^B)}$, $\text{Aut}_{k\text{-mod}_{\text{top}}}^{(\hat{M}^B)}$).

For $g \in \hat{V}^B$, let $\Gamma_g(t) := \exp(\sum_{n \geq 1}(1)^{n+1}(g(|\log X_0|)^n - 1 \log X_1)t^n/n) \in k[[t]]^\times$, where $w \mapsto (g/w)$ is the map $\{ \log X_0, \log X_1 \}^* \to k$ such that $g = \sum_{w \in \{ \log X_0, \log X_1 \}^*} (g/w)w$.

As in the de Rham case, the map $\Gamma : G^B(k) \to (\hat{V}^B)^\times$, $(\mu, g) \mapsto \Gamma_g(-\log X_1)$ satisfies a cocycle identity, which implies that the map taking $(\mu, g)$ to $\Gamma_{\text{aut}_{(\mu,g)}^{M,B(10)}} := \text{Ad}_{\Gamma_{(\mu,g)}} \circ \text{aut}_{(\mu,g)}^{M,B(10)}$ is a group morphism.

There is a unique isomorphism $i^V : \hat{V}^B \to \hat{V}^B_{\text{DR}}$, induced by $X_i \mapsto \exp(e_i)$ for $i = 0, 1$. It induces a group isomorphism $i^G : (G^B(k), \oplus) \to (G^{\text{DR}}(k), \oplus)$.

4.3.3. Subgroups of $G^B(k)$. One checks that $G^B_{\text{quad}}(k) := \{(\mu, g) \in G^B(k) | \mu^2 = 1 - 2\delta(\log X_0 \log X_1)\}$ is a subgroup of $(G^B(k), \oplus)$ (see [EF3], Lemma 3.2). On the other hand, its follows from the group morphism property of $(\mu, g) \mapsto \Gamma_{\text{aut}_{(\mu,g)}^{M,B(10)}}$ that $\text{Aut}(\hat{M}^B) := \{(\mu, g) \in G^B(k) | \hat{M}^B \circ \text{aut}_{(\mu,g)}^{M,B(10)} = (\text{aut}_{(\mu,g)}^{M,B(10)})^2 \circ \hat{M}^B\}$ is also a subgroup of $G^B(k)$. \oplus\)

We then define $\text{DMR}^B(k) := \text{Aut}(\hat{M}^B) \cap G^B_{\text{quad}}(k)$ and $\text{DMR}^B_0(k)$ as the intersection of $\text{DMR}^B(k)$ with $G(\hat{V}^B) \subset G^B(k)$. These are subgroups of $(G^B(k), \oplus)$.

One proves:

Proposition 4.12. ([EF3]) One has $\text{DMR}^B(k) = \{(\mu, g) \in G^B(k) | (g) \log X_0 = (g) \log X_1 = 0, \mu^2 = 1 - 2\delta(\log X_0 \log X_1), (\Gamma_g(-\log X_1)^{-1} \cdot g) \cdot 1_B \in G(\hat{M}^B)\}$ and $\text{DMR}^B_0(k) = \{g \in G(\hat{V}^B) | (1, g) \in \text{DMR}^B(k)\}$. 

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4.3.4. Computation of groups corresponding to torsors.

**Theorem 4.13.** There are compatible group isomorphisms of $\text{Aut}_{\text{DMR}}(k)(\text{DR}_B(k))$ with $\text{DMR}^B(k)$ and for any $\mu \in k^\times$, of $\text{Aut}_{\text{DMR}}(k)(\text{DR}_\mu(k))$ with $\text{DMR}^B_\mu(k)$.

**Proof.** The group $\text{Aut}_{\text{DR}}(k)(\text{DR}_B(k))$ corresponding to the trivial torsor $\text{DR}_B(k)$ is equal to $\text{DR}_B(k)$. According to Lemma 4.11 (a), $\text{Aut}_{\text{DR}}(k)(\text{DR}_B(k))$ is then a subgroup of $\text{DR}_B(k)$. One checks that its image under the isomorphism $(i^G)^{-1}$ is the subgroup $\text{DR}_B(k)$. In the same way, $\text{Aut}_{\text{DR}}(k)(\text{DR}_B(k))$ is a subgroup of $\text{DR}_B(k)$, whose image under $(i^G)^{-1}$ is the subgroup $\text{Aut}(\text{DR}_B(k))$. It then follows from Lemma 4.11, (b) that the image under $(i^G)^{-1}$ of $\text{DMR}_B(k)(\text{DR}_B(k))$ is equal to $\text{DMR}_B(k)$. This implies the first statement. The second statement follows from the fact that the natural map $\text{DMR}_B(k) \to k^\times$ is compatible with the group morphisms $\text{DMR}^\omega(k) \to k^\times$, $\omega \in \{B, \text{DR}\}$, and with the left and right actions. □

4.4. Bitorsor structures. A bitorsor $G \times H$ is a triple $(G, X, H)$ such that $G \times H$ is a torus, and $H$ is a group equipped with a simple and transitive right action on $X$, commuting with that of $G$. Like torsors, bitorsors from a category, and a category equivalence from torsors to bitorsors is given by $G \times H \mapsto G \times X_{\text{Aut}(G,H)}$. Theorem 4.13 may therefore be interpreted as an explicitation of the bitorsors corresponding to the torsors $\text{DMR}_B(k)\text{DMR}_B(k)$ and $\text{DMR}_0(k)\text{DMR}_0(k)$.

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