EUCLIDEAN MAXWELL-EINSTEIN THEORY*

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Abstract
After reviewing the context in which Euclidean propagation is useful we compare and contrast Euclidean and Lorentzian Maxwell-Einstein theory and give some examples of Euclidean solutions.

1. Introduction
The sourcefree Maxwell equations minimally coupled to Einstein gravity represent a classical field theory to which L. Witten has made important contributions\(^1\). Today this theory is still of considerable interest, not least because it contains features not present in pure Einstein gravity; for example, certain aspects of black holes can be simpler in Maxwell-Einstein than in pure Einstein theory. As we are today groping toward quantization of gravity, the classical equations and their solutions can give us new insights when considered in their imaginary-time or Euclidean form, because such solutions can be used as the basis for a WKB approximation to the as yet unknown quantum theory.

In the realm of gravitational physics, Euclidean solutions have been considered primarily in pure Einstein gravity and its higher-dimensional versions on the one hand, and in string theory on the other. But because we have a much better intuitive and practical understanding of electromagnetism than of other fields, it is appropriate to consider Maxwell-Einstein theory as a model, and it is interesting in particular to compare and contrast Euclidean electrodynamics with the standard (Lorentzian) version. That the differences are small and subtle is suggested by the fact that Maxwell’s equations are metric-independent if written in terms of two 2-forms (one containing $B$, $E$, the other $D$, $H$). The metric enters only in the “vacuum constitutive relations,” which demand that these tensors are duals of each other.

We first recall the relation between Euclidean time development and the WKB approximation in classically forbidden regions. Then we consider Euclidean vs. Lorentzian Maxwell theory and show that the only essential difference is the sign in Lenz’ law. We give examples of solutions that illustrate this difference, and comment on the peculiar role of duality transformations. We conclude by giving Einstein-Maxwell solutions that are amusing even if their physical significance may be a matter of speculation.

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2. Tunneling, Bounces and Instantons

The motivation for considering Euclidean theories can be explained by the simple example of one-dimensional particle motion \( x(t) \) in a potential. In regions where the potential energy \( V \) exceeds the kinetic energy \( T \), classical motion is forbidden, but quantum tunneling may take place. In a semiclassical description of tunneling one imagines that, for a fixed total energy, real time “freezes” and barrier penetration occurs by the particle “taking an excursion into complex time.”\(^2\) The complex-time motion can then be used as a basis for computing the WKB wavefunction.

The usual quadrature for the total time taken by a particle of total energy \( E = 0 \) to move between \( x_1 \) and \( x_2 \),

\[
t_{12} = \int_{x_1}^{x_2} \sqrt{-m/2V} \, dx
\]

(1)

shows that \( t \) will be complex if the interval between \( x_1 \) and \( x_2 \) includes a forbidden region (where \( V > 0 \)). If we allow \( t \) to vary from 0 to \( t_{12} \) along some path (contour) in the complex \( t \)-plane, there will typically be a complex solution \( x(t) \) to the equations of motion satisfying the boundary conditions \( x(0) = x_1 \), \( x(t_{12}) = x_2 \) and \( E = 0 \). This contour can then be deformed to consist of segments with either purely real or purely imaginary tangent, in such a way that \( x \) is purely real.\(^2,3\) At a (simple) zero of \( V \) the contour turns abruptly by 90°, so we still call these turning points (though it is the direction of \( t \) that turns, rather than the velocity of the classical motion).

A family of such solutions corresponds in the usual way to a solution \( S \) of the (time-independent) Hamilton-Jacobi equation, and the latter gives the “phase of the wavefunction” in the lowest-order WKB approximation. Because the momentum \( p = dS/dx \) will be imaginary in the classically forbidden region, \( S \) will generally be complex. Classically we can give no meaning to complex \( S \), but in the factor \( e^{iS/h} \) of the wavefunction, the imaginary part of \( S \) simply describes the exponential decrease associated with tunneling. As usual one does not have to solve the Hamilton-Jacobi equation to find the value of \( S \), but can instead evaluate the action integral

\[
I = \int \left( \frac{1}{2} m \dot{x}^2 - V(x) \right) dt
\]

(2)

“on shell”, i.e., for the solution \( x(t) \) and the same contour as described above.

Often one is interested only in the exponential factor that the wavefunction acquires between two adjacent turning points. One then needs to find the imaginary-time motion between these turning points, and evaluate the action for this motion. Although this is a needlessly arduous way to arrive at the simple formula, \( S = \int \sqrt{-2mV} \, dx \) in the one-dimensional case, it is the procedure that can be generalized to higher dimensions. The corresponding solutions of the equations of motion in imaginary time and with finite action are called instantons. One can show\(^3\) that at a turning point (which is a whole spacelike surface in field theory), \( \text{all} \) momenta should vanish (for example, because the momenta of the imaginary-time motion are imaginary, but should agree at the turning point with those of the real motion). Of particular importance are motions that start at a classically
static solution at $t = -\infty$, and that have precisely one turning point. Such a motion is called a bounce* and is useful, for example, in describing vacuum decay.\(^4\)

In an instanton we can avoid the one remaining source of imaginary quantities, the imaginary time $t$, by making a coordinate transformation to a real $\tau = t/i$. This changes the signature of the metric to positive definite, making the spacetime Euclidean (or Riemannian, if there is curvature). Covariant field equations have their old form in the new metric. Only those quantities change their form that are by convention not “analytically related”, i.e., not related by the above coordinate transformation. A typical example is the volume element, $\sqrt{-g} \, d^4x$ in Lorentzian spacetimes. If analytically continued to Euclidean it would become imaginary, but convention defines it to be real, $\sqrt{g} \, d^4x$ in Riemannian spaces. Another example is the length of timelike vectors. Quantities involving such objects, such as the duality (*) operator or the extrinsic curvature (and other quantities arising from a 3+1 split) similarly change their form, and are not analytically related. Where this is important one invents more or less ad hoc rules. For example, one defines the Euclidean action $S_E$ of a real Euclidean field history to be real (one uses an expression identical to the Lorentzian action $I$, except with an overall minus sign and the Euclidean metric and volume element) and remembers that it contributes a factor $e^{-S_E/\hbar}$ to the wavefunction.

3. The Euclidean Maxwell Equations

The difference between the Lorentzian and Euclidean Maxwell equations are not apparent in their (identical) covariant form, but do emerge in the usual 3+1 dimensional description. Contemplation of the Euclidean “dynamics” in this form is not only amusing, but may give at least some measure of satisfaction to those who would like to imagine, to the extent possible, what goes on during tunneling; at best the good physical intuition we have about Lorentzian electromagnetism may translate into a better understanding of Maxwell and Einstein-Maxwell instantons.

For simplicity we consider the Euclidean Maxwell equations in flat space (but much of what we will say will apply, mutatis mutandis, to curved space as well),

$$\text{div}E = 0, \quad \text{div}B = 0, \quad \text{curl}E = \partial B/\partial t, \quad \text{curl}B = \partial E/\partial t. \quad (3)$$

The notable difference from the usual vacuum Maxwell equations is that the minus sign, which goes by the name of Lenz’ law, is absent. Thus Euclidean electrodynamics is “just like” ordinary electrodynamics except for an “anti-Lenz” law. This one sign change has, however, far-reaching effects. It changes the equations from hyperbolic to elliptic, so there is no propagation with a finite speed in Euclidean spaces. On the other hand, there can be run-away solutions, because now induced (displacement) currents reinforce those that led to the induction. Thus one can have creation of Euclidean electromagnetic fields “from nothing”. Energy conservation does not prevent such solutions, because Euclidean energy,

$$E = \int T_{00} d^3V = (1/8\pi) \int (B^2 - E^2) \, d^3V \quad (4)$$

* In more precise language than used here, bounces and instantons are mutually exclusive categories.
is not positive definite.

Nonetheless such solutions cannot describe a decay of the vacuum: for this we would need a proper bounce solution with a turning surface on which all momenta, that is all $E$-fields, vanish. For such pure magnetic configurations Eq. (4) is positive definite. In other words, because tunneling conserves energy, and Lorentzian electromagnetic energy is positive definite, there is no Lorentzian state to which the vacuum could decay.*

The Euclidean action of a Euclidean solution is

$$S_E = \left(\frac{1}{16\pi}\right) \int F_{\mu\nu} F^{\mu\nu} \sqrt{g} \, d^4x = \left(\frac{1}{8\pi}\right) \int (E^2 + B^2) \, d^4V \quad (5)$$

and is manifestly positive, as is appropriate for a barrier penetration factor (and similarly for contributions to the path integral). It is interesting that this action is invariant both under the Lorentzian duality ($E \rightarrow B$, $B \rightarrow -E$), and under Euclidean duality ($E \rightarrow B$, $B \rightarrow E$), which preserves Euclidean solutions.† (In both cases the invariance is valid only on shell, i.e., for solutions of the field equations. This is so because $E$ and $B$ cannot be freely varied, but must obey flux conservation, $F_{[\alpha\beta\gamma]} = 0$, or be derived from a potential). This assures that physical quantities, such as barrier penetration factors, are also invariant under Lorentzian duality of the initial state. This last statement may seem surprising in view of our earlier statement that $E$ must vanish on a turning surface. If an initial (or final) Lorentzian $E$ were present, it might also seem that it should somehow contribute with the opposite sign to $S_E$ than $E$ fields induced in the Euclidean development. Nonetheless Eq. (6) is the correct expression in either case, and for any case that can be transformed to a pure $B$ field by duality transformation (and then forms a proper instanton). The reason is that the electromagnetic action is to be varied subject to flux conservation; the situation is analogous to barrier penetration in a spherically symmetric potential when there is a conserved angular momentum.5

4. Examples of Maxwell-Einstein Instantons

In gravitational physics Euclidean solutions of interest appear to be either tunneling “from nothing” (universe creation), bounces (decay of a classically static state), or instantons proper (fluctuations). Bounces have the clearest physical interpretation, but there are not many suitable initial states: the initial state must not only be static, but also not uniquely determined by its energy (so there is another, non-static state to which to tunnel). Therefore there are no examples in pure, asymptotically flat Einstein theory.

In some theories, such as higher-dimensional compactified pure Einstein or low-energy string theories, the positive-energy theorem does not hold. The principal example of a

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* This is of course well known, since we do know how to quantize electromagnetism not only in the WKB approximation, but in general. The point here is to learn something about instantons; also we will see that the story changes in a background field and in interaction with gravity.

† When written covariantly, either duality becomes† $F \rightarrow * F$ but, as mentioned above, the Euclidean * as conventionally defined is not the analytic continuation of the Lorentzian *. For example, $** = +1$ for Euclidean, and $-1$ for Lorentzian spaces.
bounce that signals vacuum decay in such a theory was given by the son of L. Witten. The analogous decay with an electromagnetic field exists in various dimensions. To illustrate Euclidean Maxwell theory we consider the four-dimensional case, which is somewhat unrealistic because one dimension must be compactified. So the initial state is the gravito-electro-magnetic vacuum with spacelike topology $R^2 \times S^1$. The amount of electromagnetic field that will be created “from nothing” is encoded in the initial state by a vector potential $A$, with $F = dA = 0$, but $\oint A$ non-vanishing around the $S^1$-direction. (This line integral has no local effect classically, but could in principle be measured by an Aharonov-Bohm interference between beams that traverse the $S^1$ in opposite directions.) If we slice up the bounce solution that describes the decay by a suitable set of 3-surfaces of constant Euclidean time, we find at large negative times a geometry that is still nearly flat, and an electromagnetic field close to one of the run-away solutions of flat space: There is a small but increasing $E$-field in the $S^1$-direction. Its displacement current generates closed circles of $B$-field lines in the orthogonal plane ($R^2$). This likewise increasing $B$-field induces further $E$-field, and so on. In flat space this would eventually lead to infinite fields, say at $\tau = 0$. But gravity simultaneously exhibits the Witten instability: a “hole” in space grows outward from a point at the center of the $B$-field circles. It pushes these field lines outward, which induces an $E$-field opposing the one originally present and reducing it to zero at the turn-around surface $\tau = 0$. On this surface the hole has reached its maximum size and is enclosed by the maximum $B$-field. If developed further in Euclidean time, the remaining history of the bounce reverses this time development, with the $B$-field decreasing, the $E$-field temporarily reappearing, and the hole vanishing, back to the vacuum state. But if we switch to Lorentzian time at the turn-around surface, the state on that surface is the initial state of the decay product, in which the hole later expands to infinity.

Another way to avoid the restrictions of the positive energy theorem is to start with a state other than the vacuum, so that background fields are present initially. Again pure Einstein gravity provides no truly static examples, but because Maxwell fields can be repulsive, static Maxwell-Einstein configurations do exist. But the repulsion acts only in two directions perpendicular to the field direction, so the examples we consider are homogeneous and extend to infinity in the field direction. (They would not be globally static, but collapse in the field direction, if made finite in that direction by identification via a discrete subgroup of the homogeneity.)

One such example is the Melvin universe, the closest analogy to a constant magnetic field allowed in General Relativity. Garfinkle and Strominger have found a bounce solution that connects the Melvin universe with a state that is asymptotically Melvin, but contains mouths of a maximally magnetically charged wormhole, which are then pulled apart by the global magnetic field — a pair creation of magnetic wormhole charges. When we slice this bounce by 3-surfaces to reveal the history of the tunneling, we can imagine the progress of the field topology as follows: Assume the original $B$-field is vertical. Cut a thin horizontal lens-shaped region out of space, but keep the top and bottom surfaces of the lens identified (so the topology has not changed). The lens intercepts some magnetic flux, which is subsequently kept constant. Blow up the lens to a sphere with top and bottom hemispheres identified, and linking the same flux. Then pinch the sphere inward at the equator until it pinches off into two spheres, changing the topology at the moment
of pinch-off, but keeping the linked flux unchanged. Now you have two spheres that are identified and link the flux — they are the wormhole mouths created at the turning surface. The rest of the bounce retraces this history back to the original Melvin state. This gives only a very crude picture of what is happening to the geometry and the field. (The exact solution of course contains all the details, but it is given in coordinates that require some ingenuity in finding a suitable 3+1 split.)

Another example of a static Maxwell-Einstein space is the Bertotti-Robinson universe,\textsuperscript{10} which is not only homogeneous in the $B$-field ($z$-) direction, but constant curvature spherical in the two orthogonal (transverse) spacelike directions. The Einstein equations then require that the $z,t$ subspace be of constant curvature. An instanton solution has been given\textsuperscript{11} that connects this with several Bertotti-Robinson universes. In a 3+1 split that appears natural the $E$-field vanishes on all slices. Thus this solution illustrates the interaction of Euclidean gravity and $B$-field, rather than Euclidean Maxwell theory. Again the history begins as a run-away solution: Near $\tau = -\infty$ suppose the $B$-field is a little weaker near the equator of the transverse $S^2$ than at the poles. This allows gravity to pinch in the sphere at the equator, forcing more of the flux to the poles. The equator pinches off at $\tau = 0$, forming two distorted spheres, each with more flux in one half than the other. The flux and curvature even out, and the two spheres become round at $\tau = \infty$. (The details of this are to be worked out to obtain a mini-superspace in which to solve the Wheeler-DeWitt equation and compare its results with those from the instanton.)

The duality transformation discussed in Section 3 allows us to change the $B$-field, in terms of which all these examples were discussed, into an $E$-field. For example, an electric Melvin universe will pair create electrically charged wormholes. The first and third examples have also been generalized to low-energy string theory.\textsuperscript{7,12}

5. Multitudes of Universes

In this final section we want to consider Euclidean Maxwell-Einstein solutions with more than one turning surface. Whether these can be interpreted as tunneling solutions is much less clear than those of the previous sections. If such interpretation is valid, then apparently the initial state, corresponding to one of the turning surfaces, has been (carefully) prepared to be time-symmetric, and the tunneling occurs when it has reached this moment of time symmetry. The examples concern universe models, which typically reach time-symmetry at the moment of maximum expansion — an unlikely era for quantum effects to play an important role. Nonetheless we describe these examples briefly, because they are extensions to 4-dimensional Maxwell-Einstein theory of an interesting geometrical technique developed for 2+1 gravity with a negative cosmological constant.\textsuperscript{13}

The Euclidean version of the Bertotti-Robinson solution is geometrically $S^2 \times H^2$, where $S^2$ is the round 2-sphere with electromagnetic field 2-form proportional to its area 2-form, and $H^2$ is a space of constant negative curvature. Rather than finding a Euclidean solution that is asymptotic to this one (as in ref. 11), we take exactly this solution and find turning surfaces (totally geodesic surfaces) in it. These have the form $S^2 \times$ geodesic of $H^2$. Now take a geodesic equilateral triangle in $H^2$ and move the corners to infinity. The sides are then infinitely long geodesics traversing the finite part of $H^2$, and they enclose a finite area. The 4-space enclosed by the corresponding turning surfaces, regarded as an
instanton, therefore has finite action. In the spirit of ref. 13 this instanton may describe the break-up of one Bertotti-Robinson universe into two, not by a transverse splitting as in Section 4, but by “snapping” longitudinally, like a rubber band that has been stretched too much. Or, for that matter it might be viewed as a creation of a triplet of universes from “nothing,” the “nothing” being the $S^2$ that carries the flux. (This construction can be generalized to involve more universes; it is amusing to observe that creation from “nothing” in this way must produce at least three universes!)

As another example, consider a geodesic regular hexagon in $H^2$. Because of the negative curvature it is possible to make it of such a size that all the interior angles are $90^\circ$. Lay an identical copy on top and sew together between top and bottom along sides 1, 3 and 5. Because the corresponding surfaces $S^2 \times$ geodesic are totally geodesic, this is a smooth identification. The remaining sides 2, 4 and 6 then form circles $S^1$ that are smooth due to the $90^\circ$ angles, and hence correspond to turning surfaces of topology $S^2 \times S^1$. If we follow the reasoning of ref. 13, one of these, say from side 2, would be the initial state of a tunneling that leads to two copies of $S^2 \times S^1$ (sides 4 and 6).

It is clear that these examples can be extensively generalized to involve a multitude of universes. This shows at least that there exist Euclidean solutions connecting widely different topologies in a 4-dimensional theory that is usually regarded as reasonable. Lou Witten indeed chose to work on a fascinating theory — it can still give us interesting lessons and even surprises!

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7. References

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