The Steiner \((n - 3)\)-diameter of a graph *

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Abstract

The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural generalization of the concept of classical graph distance. For a connected graph \(G\) of order at least 2 and \(S \subseteq V(G)\), the Steiner distance \(d(S)\) among the vertices of \(S\) is the minimum size among all connected subgraphs whose vertex sets contain \(S\). Let \(n\) and \(k\) be two integers with \(2 \leq k \leq n\). Then the Steiner \(k\)-eccentricity \(e_k(v)\) of a vertex \(v\) of \(G\) is defined by \(e_k(v) = \max\{d(S) \mid S \subseteq V(G), \ |S| = k, \ and \ v \in S\}\). Furthermore, the Steiner \(k\)-diameter of \(G\) is \(s\text{diam}_k(G) = \max\{e_k(v) \mid v \in V(G)\}\).

In 2011, Chartrand, Okamoto, Zhang showed that \(k - 1 \leq s\text{diam}_k(G) \leq n - 1\). In this paper, graphs with \(s\text{diam}_k(G) = \ell\) for \(k = n, n - 1, n - 2, n - 3\) and \(k - 1 \leq \ell \leq n - 1\) are characterized, respectively.

Keywords: diameter, Steiner tree, Steiner \(k\)-diameter.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to [4] for graph-theoretic notation and terminology not described here. We divide our introduction into the following four subsections to state the motivations and our results of this paper.

1.1 Distance and its generalizations

Distance is one of the most basic concepts of graph theory. If \(G\) is a connected graph and \(u, v \in V(G)\), then the distance \(d(u, v)\) between \(u\) and \(v\) is the length of a shortest path

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connecting $u$ and $v$. If $v$ is a vertex of a connected graph $G$, then the eccentricity $e(v)$ of $v$ is defined by $e(v) = \max\{d(u, v) \mid u \in V(G)\}$. Furthermore, the radius $\text{rad}(G)$ and diameter $\text{diam}(G)$ of $G$ are defined by $\text{rad}(G) = \min\{e(v) \mid v \in V(G)\}$ and $\text{diam}(G) = \max\{e(v) \mid v \in V(G)\}$. These last two concepts are related by the inequalities $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$. The center $C(G)$ of a connected graph $G$ is the subgraph induced by the vertices $u$ of $G$ with $e(u) = \text{rad}(G)$. Recently, Goddard and Oellermann gave a survey paper on this subject, see [19].

The distance between two vertices $u$ and $v$ in a connected graph $G$ also equals the minimum size of a connected subgraph of $G$ containing both $u$ and $v$. This observation suggests a generalization of distance. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an $S$-Steiner tree or a Steiner tree connecting $S$ (or simply, an $S$-tree) is a subgraph $T(V', E')$ of $G$ that is a tree with $S \subseteq V'$. Let $G$ be a connected graph of order at least 2 and let $S$ be a nonempty set of vertices of $G$. Then the Steiner distance $d_G(S)$ among the vertices of $S$ (or simply the distance of $S$) is the minimum size among all connected subgraphs whose vertex sets contain $S$. Note that if $H$ is a connected subgraph of $G$ such that $S \subseteq V(H)$ and $|E(H)| = d_G(S)$, then $H$ is a tree. Observe that $d_G(S) = \min\{e(T) \mid S \subseteq V(T)\}$, where $T$ is subtree of $G$. Furthermore, if $S = \{u, v\}$, then $d_G(S) = d(u, v)$ is nothing new but the classical distance between $u$ and $v$. Set $d_G(S) = \infty$ when there is no $S$-Steiner tree in $G$.

**Observation 1** Let $G$ be a graph of order $n$ and $k$ be an integer with $2 \leq k \leq n$. If $S \subseteq V(G)$ and $|S| = k$, then $d_G(S) \geq k - 1$.

Let $n$ and $k$ be two integers with $2 \leq k \leq n$. The Steiner $k$-eccentricity $e_k(v)$ of a vertex $v$ of $G$ is defined by $e_k(v) = \max\{d(S) \mid S \subseteq V(G), |S| = k, \text{ and } v \in S\}$. The Steiner $k$-radius of $G$ is $\text{srad}_k(G) = \min\{e_k(v) \mid v \in V(G)\}$, while the Steiner $k$-diameter of $G$ is $\text{sdiam}_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. Note for every connected graph $G$ that $e_2(v) = e(v)$ for all vertices $v$ of $G$ and that $\text{srad}_2(G) = \text{rad}(G)$ and $\text{sdiam}_2(G) = \text{diam}(G)$.

**Observation 2** Let $k, n$ be two integers with $2 \leq k \leq n$.

1. If $H$ is a spanning subgraph of $G$, then $\text{sdiam}_k(G) \leq \text{sdiam}_k(H)$.
2. For a connected graph $G$, $\text{sdiam}_k(G) \leq \text{sdiam}_{k+1}(G)$.

As a generalization of the center of a graph, the Steiner $k$-center $C_k(G)$ ($k \geq 2$) of a connected graph $G$ is the subgraph induced by the vertices $v$ of $G$ with $e_k(v) = \text{srad}_k(G)$. Oellermann and Tian [30] showed that every graph is the $k$-center of some graph. In particular, they showed that the $k$-center of a tree is a tree and those trees that are $k$-centers of trees are characterized. The Steiner $k$-median of $G$ is the subgraph of $G$
induced by the vertices of $G$ of minimum Steiner $k$-distance. For Steiner centers and Steiner medians, we refer to [28, 29, 30].

The average Steiner distance $\mu_k(G)$ of a graph $G$, introduced by Dankelmann, Oellermann and Swart in [12], is defined as the average of the Steiner distances of all $k$-subsets of $V(G)$, i.e.

$$\mu_k(G) = \frac{1}{\binom{n}{k}} \sum_{S \subseteq V(G), |S|=k} d_G(S).$$

For more details on average Steiner distance, we refer to [12, 13].

Let $G$ be a $k$-connected graph and $u, v$ be any pair of vertices of $G$. Let $P_k(u,v)$ be a family of $k$ vertex-disjoint paths between $u$ and $v$, i.e., $P_k(u,v) = \{P_1, P_2, \ldots, P_k\}$, where $p_1 \leq p_2 \leq \cdots \leq p_k$ and $p_i$ denotes the number of edges of path $P_i$. The $k$-distance $d_k(u,v)$ between vertices $u$ and $v$ is the minimum $|p_k|$ among all $P_k(u,v)$ and the $k$-diameter $d_k(G)$ of $G$ is defined as the maximum $k$-distance $d_k(u,v)$ over all pairs $u,v$ of vertices of $G$. The concept of $k$-diameter emerges rather naturally when one looks at the performance of routing algorithms. Its applications to network routing in distributed and parallel processing are studied and discussed by various authors including Chung [10], Du, Lyuu and Hsu [17], Hsu [21, 22], Meyer and Pradhan [27].

### 1.2 Application background and progress of Steiner distance

The Steiner tree problem in networks, and particularly in graphs, was formulated quite recently—i.e., 1971—by Hakimi (see [20]) and Levi (see [25]). In the case of an unweighted, undirected graph, this problem consists of finding, for a subset of vertices $S$, a minimal-size connected subgraph that contains the vertices in $S$. The computational side of this problem has been widely studied, and it is known that it is an NP-hard problem for general graphs (see [23]). The determination of a Steiner tree in a graph is a discrete analogue of the well-known geometric Steiner problem: In a Euclidean space (usually a Euclidean plane) find the shortest possible network of line segments interconnecting a set of given points. Steiner trees have application to multiprocessor computer networks. For example, it may be desired to connect a certain set of processors with a subnetwork that uses the least number of communication links. A Steiner tree for the vertices, corresponding to the processors that need to be connected, corresponds to such a desired subnetwork.

In [9], Chartrand, Okamoto, Zhang obtained the following result.

**Theorem 1** [9] Let $k, n$ be two integers with $2 \leq k \leq n$, and let $G$ be a connected graph of order $n$. Then $k - 1 \leq sdiam_k(G) \leq n - 1$. Moreover, the upper and lower bounds are sharp.

In [14], Dankelmann, Swart and Oellermann obtained a bound on $sdiam_k(G)$ for a graph $G$ in terms of the order of $G$ and the minimum degree of $G$, that is, $sdiam_k(G) \leq$
\[ \frac{3p}{q+1} + 3n \]. Later, Ali, Dankelmann, Mukwembi \cite{2} improved the bound of \( sdiam_k(G) \) and showed that \( sdiam_k(G) \leq \frac{3p}{q+1} + 2n - 5 \) for all connected graphs \( G \). Moreover, they constructed graphs to show that the bounds are asymptotically best possible.

In \cite{3}, Bloom characterized graphs with diameter 2. Mao \cite{26} characterized the graphs with \( sdiam_3(G) = 2, 3, n - 1 \), and obtained the Nordhaus-Gaddum-type results for the parameter \( sdiam_k(G) \). In this paper, graphs with \( sdiam_k(G) = \ell \) for \( k = n, n - 1, n - 2, n - 3 \) and \( k - 1 \leq \ell \leq n - 1 \) are characterized, respectively.

## 2 The case \( k = n, n - 1, n - 2 \)

To begin with, we show the following two lemmas, which will be used later.

**Lemma 1** Let \( k, n \) be two integers with \( 2 \leq k \leq n \), and let \( T \) be a tree of order \( n \). Then \( sdiam_k(T) = n - 1 \) if and only if \( r \leq k \), where \( r \) is the number of the leaves in \( T \).

**Proof.** Suppose \( r \leq k \). Let \( v_1, v_2, \ldots, v_r \) be all the leaves of \( T \). Choose \( S \subseteq V(T) \) and \( |S| = k \) such that \( v_1, v_2, \ldots, v_r \in S \). Then any tree connecting \( S \) must use all edges of \( T \). Since \( |E(T)| = n - 1 \), it follows that \( d_T(S) \geq |E(T)| = n - 1 \). From the arbitrariness of \( S \), we have \( sdiam_k(T) \geq n - 1 \). Combining this with Theorem \( \text{[1]} \) we have \( sdiam_k(T) = n - 1 \).

Conversely, suppose \( sdiam_k(T) = n - 1 \). If \( s \geq k + 1 \), then for any \( S \subseteq V(G) \) with \( |S| = k \), there exists a leaf \( v \in V(T) \) such that \( v \notin S \). Set \( T' = T \setminus v \). Then the tree \( T' \) is an \( S \)-Steiner tree and hence \( d_{T'}(S) \leq n - 2 \). From the arbitrariness of \( S \), we have \( sdiam_k(T) \leq n - 2 < n - 1 \), a contradiction. Therefore, \( s \leq k \). \( \blacksquare \)

**Lemma 2** Let \( k, n \) be two integers with \( 1 \leq k \leq n - 2 \). Let \( G \) be a connected graph of order \( n \). Then \( sdiam_{n-k}(G) = n - 1 \) if and only if \( G \) contains at least \( k \) cut vertices.

**Proof.** Suppose \( G \) contains at least \( k \) cut vertices; pick \( k \) of them, say \( v_1, v_2, \ldots, v_k \). Choose \( S = V(G) \setminus \{v_1, v_2, \ldots, v_k\} \). Then \( |S| = n - k \) and any \( S \)-Steiner tree \( T \) must occupy the vertices \( v_1, v_2, \ldots, v_k \), which implies that \( |V(T)| = n \) and \( e(T) = n - 1 \). Furthermore, \( d_G(S) \geq e(T) = n - 1 \) and hence \( sdiam_{n-k}(G) \geq d_G(S) \geq n - 1 \). Theorem \( \text{[1]} \) yields \( sdiam_{n-k}(G) = n - 1 \), as desired.

Conversely, suppose that \( sdiam_{n-k}(G) = n - 1 \). Assume to the contrary that \( G \) contains at most \( k - 1 \) cut vertices; let \( C \) be the set of all cut vertices in \( G \). Then for any \( S \subseteq V(G) \) with \( |S| = n - k \), we have \( |S \cup C| \leq n - 1 \), so we can find a vertex \( x \in V(G) \) such that \( x \) is not a member of \( S \) and not a cut vertex of \( G \). Therefore \( G \setminus x \) is connected and has a spanning tree \( T \). Observe that \( |V(T)| = n - 1 \), so \( d_G(S) \leq |e(T)| = n - 2 \). Since \( S \) was arbitrary, we have \( sdiam_{n-k}(G) \leq n - 2 \), a contradiction. So \( G \) contains at least \( k \) cut vertices, as desired. \( \blacksquare \)
Proposition 1 Let $k, n$ be two integers with $1 \leq k \leq n - 2$, and let $G$ be a graph of order $n$. Then $\kappa(G) \geq k$ if and only if $\text{sdiam}_{n-k+1}(G) = n-k$. 

Proof. For any $S \subseteq V(G)$ with $|S| = n-k+1$, we have $|V(G)\setminus S| = k-1$. Since $\kappa(G) \geq k$, it follows that $G[S]$ is connected. Therefore, $G[S]$ contains a spanning tree $T$ of order $n-k+1$ and hence $e(T) = n-k$. From the arbitrariness of $S$, we have $\text{sdiam}_{n-k+1}(G) \leq d_T(S) = e(T) = n-k$. From this together with Theorem 1 $\text{sdiam}_{n-k+1}(G) = n-k$.

Conversely, we suppose $\text{sdiam}_{n-k+1}(G) = n-k$. If $\kappa(G) \leq k-1$, then there exist a cut set $U \subseteq V(G)$ with $|U| = \kappa(G)$ such that $G \setminus U$ is disconnected. Let $C_1, C_2, \cdots, C_r$ be the connected components of $G \setminus U$. Note that $\left(\bigcup_{i=1}^{r} V(C_i)\right) \cup U = V(G)$. Since $|U| \leq k-1$, it follows that $|\bigcup_{i=1}^{r} V(C_i)| \geq n - (k-1) = n-k+1$. Pick up $n-k+1$ vertices from $\bigcup_{i=1}^{r} V(C_i)$. Let $S$ be the vertex set of these $n-k+1$ vertices. Then any $S$-Steiner tree $T$ must use at least one vertex of $U$, which implies that $|V(T)| \geq (n-k+1)+1 = n-k+2$. Thus, $d_G(S) \geq e(T) = |V(T)|-1 \geq n-k+1$ and hence $\text{sdiam}_{n-k+1}(G) \geq d_G(S) \geq n-k+1$, a contradiction. So $G$ is $k$-connected.

For $k = n, n-1$, we have $\text{sdiam}_n(G) = n-1$ and $n-2 \leq \text{sdiam}_{n-1}(G) \leq n-1$ by Theorem 1. The following two corollaries are immediate by Proposition 1.

Corollary 1 Let $G$ be a graph of order $n$. Then $\text{sdiam}_n(G) = n-1$ if and only if $G$ is connected.

Corollary 2 Let $G$ be a connected graph of order $n$. Then

1. $\text{sdiam}_{n-1}(G) = n-2$ if and only if $G$ is 2-connected;
2. $\text{sdiam}_{n-1}(G) = n-1$ if and only if $G$ contains at least one cut vertex.

From Theorem 1 we know $n-3 \leq \text{sdiam}_{n-2}(G) \leq n-1$. Let us now characterize the graphs with $\text{sdiam}_{n-2}(G) = n-3, n-2, n-1$.

Theorem 2 Let $G$ be a connected graph of order $n$ ($n \geq 4$). Then

1. $\text{sdiam}_{n-2}(G) = n-3$ if and only if $\kappa(G) \geq 3$.
2. $\text{sdiam}_{n-2}(G) = n-2$ if and only if $\kappa(G) = 2$ or $G$ contains only one cut vertex.
3. $\text{sdiam}_{n-2}(G) = n-1$ if and only if there are at least two cut vertices in $G$.

Proof. (1) The result follows by Proposition 1.

(2) By (1), we must have $\kappa(G) \leq 2$. If $\kappa(G) = 2$, we are done. If $\kappa(G) = 1$, then $G$ must contain at least one cut vertex, but fewer than two cut vertices by (3), so $G$ contains exactly one cut vertex. 


3 The case \( k = n - 3 \)

From Theorem 1, we know that \( n - 4 \leq \text{sdiam}_{n-3}(G) \leq n - 1 \). Graphs with \( \text{sdiam}_{n-3}(G) = n - 4, n - 3, n - 2, n - 1 \) are characterized in this section.

The following is immediate from Proposition 1 and Lemma 2.

**Proposition 2** Let \( G \) be a connected graph of order \( n \). Then \( \text{sdiam}_{n-3}(G) = n - 4 \) if and only if \( \kappa(G) \geq 4 \), and \( \text{sdiam}_{n-3}(G) = n - 1 \) if and only if \( G \) contains at least 3 cut vertices.

**Lemma 3** Let \( G \) be a connected graph of order \( n \). If \( \kappa(G) = 1 \), then \( \text{sdiam}_{n-3}(G) = n - 3 \) if and only if \( G \) satisfies the following two conditions.

1. \( G \) contains only one cut vertex \( u \);
2. for each connected component \( C_i \) of order at least 3 in \( G \setminus u \), \( G[V(C_i) \cup \{u\}] \) is 3-connected, or \( \kappa(G[V(C_i) \cup \{u\}]) = 2 \) and there exists a vertex \( v \in V(C_i) \) such that \( \{u, v\} \) is a vertex cut set of \( G[V(C_i) \cup \{u\}] \), and for each component \( C_j^i \) (1 \( \leq j \leq p \)) of \( G[V(C_i) \cup \{u\}] \setminus \{u, v\} \), one of the following conditions holds:
   - \( uv \in E(G) \);
   - \( p \geq 3 \);
   - \( p = 2 \), and \( |E_G[v, V(C_i^j)]| \geq 2 \) or \( |E_G[v, V(C_i^j)]| \geq 2 \).
   and one of the following conditions holds:
   - \( G[V(C_i^j) \cup \{u\}] \) is 3-connected;
   - \( G[V(C_i^j) \cup \{v\}] \) is 3-connected;
   - \( \kappa(G[V(C_i^j) \cup \{u\}]) = \kappa(G[V(C_i^j) \cup \{v\}]) = 2 \) and \( \{y, z\} \) is not a common vertex cut set of \( G[V(C_i^j) \cup \{u\}] \) and \( G[V(C_i^j) \cup \{v\}] \) where \( z', z'' \in V(C_i^j) \);
   - \( \kappa(G[V(C_i^j) \cup \{u\}]) = 2 \) and \( \kappa(G[V(C_i^j) \cup \{v\}]) = 1 \) and if \( \{z', z''\} \) is a vertex cut set of \( G[V(C_i^j) \cup \{u\}] \), then neither \( z' \) nor \( z'' \) is a cut vertex of \( G[V(C_i^j) \cup \{v\}] \).

**Proof.** In one direction, we suppose \( \text{sdiam}_{n-3}(G) = n - 3 \). Assume, to the contrary, that \( G \) contains only one cut vertex \( u \) such that there exists a connected component \( C_j \) of order at least 3 in \( G \setminus u \) satisfying one of the following.

1. \( \kappa(G[V(C_j) \cup \{u\}]) = 1 \);
2. \( \kappa(G[V(C_j) \cup \{u\}]) = 2 \) and \( \{v, u\} \) is not a vertex cut set of \( G[V(C_j) \cup \{u\}] \) for any \( v \in V(C_j) \);
3. \( \kappa(G[V(C_j) \cup \{u\}]) = 2 \), \( \{v, u\} \) is a vertex cut set of \( G[V(C_j) \cup \{u\}] \), and there exists a component \( C_j^i \) of \( G[V(C_j) \cup \{u\}] \setminus \{u, v\} \) satisfying one of the following conditions.
\begin{itemize}
  \item $\kappa(G[V(C_j' \cup \{u\})]) = 1$.
  \item $\kappa(G[V(C_j' \cup \{u\})]) = \kappa(G[V(C_j' \cup \{v\})]) = 2$ and \{y, z\} is a common vertex cut set of $G[V(C_j' \cup \{u\})]$ and $V(C_j' \cup \{v\})$ for any $y, z \in V(C_j')$.
  \item $\kappa(G[V(C_j' \cup \{u\})]) = 2$ and $\kappa(G[V(C_j' \cup \{v\})]) = 1$ and if \{y, z\} is a common vertex cut set of $G[V(C_j' \cup \{u\})]$ where $y, z \in V(C_j')$, then at least one of \{y, z\} is a cut vertex of $V(C_j' \cup \{v\})$.
  \item $uv \notin E(G)$, $p = 2$ and there is only one edge between $v$ and each connected component of $G[V(C_j) \cup \{u\}] \setminus \{u, v\}$.
\end{itemize}

Our aim is to show $sdiam_{n-3}(G) \geq n - 2$ and get a contradiction. Let $H_j = G[V(C_j) \cup \{u\}]$. If $\kappa(H_j) = 1$, then $u$ is not a cut vertex of $H_j$ since $C_j$ is a connected component of $G \setminus u$. Therefore, there exists a cut vertex of $H_j$, say $x$, such that $x \neq u$. Let $C_{j1}, C_{j2}, \ldots, C_{js}$ ($s \geq 2$) be the connected components of $H_j \setminus x$. Clearly, $u \in \bigcup_{i=1}^{s} V(C_{ji})$. Without loss of generality, let $u \in V(C_{j1})$. We claim that there exists a connected component $C_{j1i}$ ($i_1 \in \{2, 3, \ldots, s\}$) such that $|E_G[u, C_{j1i}]| = 0$. Assume, to the contrary, that $|E_G[u, C_{j1i}]| \geq 1$ for each $i \in \{2, 3, \ldots, s\}$. Then $H_j \setminus x$ is connected, a contradiction. Thus, $x$ is also a cut vertex in $G$, which contradicts to the fact that $G$ only contains one cut vertex $u$.

If $\kappa(H_j) = 2$ and for any $v \in V(C_j) \setminus \{u, v\}$ is not a vertex cut set of $H_j$. Then there exist a vertex cut \{x, y\} of $H_j$. Therefore, $H_j \setminus \{x, y\}$ is disconnected. Let $C_{j1}, C_{j2}, \ldots, C_{js}$ ($s \geq 2$) be the connected components of $H_j \setminus \{x, y\}$. Clearly, $u \in \bigcup_{i=1}^{s} V(C_{ji})$. Without loss of generality, let $u \in V(C_{j1})$. Then there exists a connected component $C_{j1i}$ ($i_1 \in \{2, 3, \ldots, s\}$) such that $|E_G[u, C_{j1i}]| = 0$. Choose $S = V(G) \setminus \{x, y, u\}$. Then $|S| = n - 3$. Obviously, any Steiner tree, say $T$, must contain the vertex $u$ and one of \{x, y\}, which implies $|V(T)| \geq n - 1$ and hence $d_G(S) \geq e(T) \geq n - 2$ and hence $sdiam_{n-3}(G) \geq d_G(S) \geq n - 2$, a contradiction.

Suppose $\kappa(H_j) = 2$ and there exists a vertex $v \in V(C_j)$ such that $\{u, v\}$ is a vertex cut set of $H_j$ and there exists a component $C_{jv}$ of $H_j \setminus \{u, v\}$ such that $\kappa(G[V(C_{jv} \cup \{u\})]) = 1$. Then there exists a cut vertex of $G[V(C_{jv} \cup \{u\})]$, say $z$. Since $C_{jv}$ is connected, it follows that $z \neq u$. Choose $\bar{S} = \{u, z, v\}$. Then any Steiner tree, say $T$, must contain the vertex $u$ and one of $\{z, v\}$, which implies $|V(T)| \geq n - 1$ and hence $d_G(S) \geq e(T) \geq n - 2$ and hence $sdiam_{n-3}(G) \geq d_G(S) \geq n - 2$, a contradiction.

Suppose $\kappa(H_j) = 2$ and there exists a vertex $v \in V(C_j)$ such that $\{u, v\}$ is a vertex cut set of $H_j$ and there exists a component $C_{jv}$ of $H_j \setminus \{u, v\}$ such that $\kappa(G[V(C_{jv} \cup \{u\})]) = \kappa(G[V(C_{jv} \cup \{v\})]) = 2$ and $\{y, z\}$ is a common vertex cut set of $G[V(C_{jv} \cup \{u\})]$ and $V(C_{jv} \cup \{v\})$ for any $y, z \in V(C_{jv})$. Choose $\bar{S} = \{y, z, u\}$. Then any Steiner tree, say $T$, must contain the vertex $u$ and one of $\{y, z\}$, which implies $|V(T)| \geq n - 1$ and hence $d_G(S) \geq e(T) \geq n - 2$ and hence $sdiam_{n-3}(G) \geq d_G(S) \geq n - 2$, a contradiction.

Suppose $\kappa(H_j) = 2$ and there exists a vertex $v \in V(C_j)$ such that $\{u, v\}$ is a vertex cut cut
set of \( H_j \) and there exists a component \( C'_j \) of \( H_j \setminus \{ u, v \} \) such that \( \kappa(G[V(C'_j) \cup \{ u \}]) = 2 \) and \( \kappa(G[V(C'_j) \cup \{ v \}]) = 1 \) and if \( \{ y, z \} \) is a common vertex cut set of \( G[V(C'_j) \cup \{ u \}] \) where \( y, z \in V(C'_j) \), then at least one of \( \{ y, z \} \) is a cut vertex of \( V(C'_j) \cup \{ v \} \). Without loss of generality, let \( y \) is a cut vertex of \( V(C'_j) \cup \{ v \} \). Choose \( \bar{S} = \{ y, z, u \} \). Then any \( S \)-Steiner tree, say \( T \), must contain the vertex \( u \) and one of \( \{ y, z \} \), which implies \( |V(T)| \geq n-1 \) and hence \( d_G(S) \geq e(T) \geq n-2 \) and hence \( \text{sdiam}_{n-3}(G) \geq d_G(S) \geq n-2 \), a contradiction.

Suppose that \( uv \notin E(G) \), \( p = 2 \) and there is only one edge between \( v \) and each connected component of \( G[V(C_j) \cup \{ u \}] \setminus \{ u, v \} \). Let \( C'_j, C''_j \) be the connected components of \( G[V(C_j) \cup \{ u \}] \setminus \{ u, v \} \), and let \( vv_p, vv_q \) be the edges between \( v \) and \( C'_j, C''_j \), respectively. Choose \( S = \{ u, v, v_p, v_q \} \). Recall that \( uv \notin E(G) \). For any \( S \)-Steiner tree \( T \), \( T \) must occupy the vertex \( u \). Also, in order to reach the vertex \( v \), \( T \) must contain the vertex one of \( \{ v_p, v_q \} \), which implies \( |V(T)| \geq n-1 \) and hence \( d_G(S) \geq e(T) \geq n-2 \) and hence \( \text{sdiam}_{n-3}(G) \geq d_G(S) \geq n-2 \), a contradiction.

Conversely, we suppose that \( G \) satisfies the conditions of this theorem. From Proposition \( 2 \) we know that \( \text{sdiam}_{n-3}(G) \geq n-3 \). So it suffices to show \( \text{sdiam}_{n-3}(G) \leq n-3 \).

In this case, each connected component of \( G \setminus u \) is a connected subgraph of order at least 3, or an edge of \( G \), or an isolated vertex. Let \( w_1, w_2, \ldots, w_r \) be the isolated vertices, \( e_1, e_2, \ldots, e_t \) be the edges, and \( C_1, C_2, \ldots, C_r \) be the connected components of order at least 3 in \( G \setminus u \). Set \( e_i = u_i v_i \ (1 \leq i \leq t) \), \( W = \{ w_1, w_2, \ldots, w_s \} \), \( U = \{ u_1, u_2, \ldots, u_t \} \), \( V = \{ v_1, v_2, \ldots, v_t \} \) and \( n_i = |V(C_i)| \). Obviously, \( uw_i, uu_i, uv_i \in E(G) \) and \( s + 2t + \sum_{i=1}^{r} n_i = n - 1 \). Since \( |S| = n - 3 \), there exists three vertices \( x, y, z \) such that \( x, y, z \notin S \) and \( x, y, z \in V(G) = W \cup U \cup V \cup (\cup_{i=1}^{r} V(C_i)) \cup \{ u \} \).

Suppose that at least two of \( \{ x, y, z \} \) belong to \( W \cup U \cup V \). Without loss of generality, let \( x, y \in W \cup U \cup V \). Then \( G \setminus \{ x, y \} \) is connected and hence \( G \setminus \{ x, y \} \) contains a spanning tree \( T \). Therefore, \( d_G(S) \leq e(T) = n - 3 \), as desired.

Suppose that only one of \( \{ x, y, z \} \) belongs to \( W \cup U \cup V \). Without loss of generality, let \( x \in W \cup U \cup V \) and \( x = w_1 \). Then \( y, z \in (\bigcup_{i=1}^{r} V(C_i)) \cup \{ u \} \). Without loss of generality, let \( y \in \bigcup_{i=1}^{r} V(C_i) \). Thus there exists some \( C_j \) such that \( y \in C_j \). Since \( G[V(C_j) \cup \{ u \}] \) is 2-connected, it follows that \( G[V(C_j) \cup \{ u \}] \setminus y \) is connected and hence \( G[V(C_j) \cup \{ u \}] \setminus y \) contains a spanning tree \( T_j \). Furthermore, \( G[V(C_i) \cup \{ u \}] \) contains a spanning tree \( T_i \) for each \( i \) \((1 \leq i \leq r, i \neq j) \). Then the tree \( T \) induced by the edges in \( \{ uw_i \mid 2 \leq i \leq s \} \cup \{ uu_i \mid 1 \leq i \leq t \} \cup \{ w_i \mid 2 \leq i \leq t \} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired \( S \)-Steiner tree. Therefore, \( d_G(S) \leq e(T) = (s - 1) + 2t + \sum_{i=1, i \neq j}^{r} n_i + (n_j - 1) = n - 3 \), as desired.

Suppose that none of \( \{ x, y, z \} \) belongs to \( W \cup U \cup V \). Then \( x, y, z \in (\bigcup_{i=1}^{r} V(C_i)) \cup \{ u \} \). Therefore, at least two of \( x, y, z \) belongs to \( \bigcup_{i=1}^{r} V(C_i) \). Without loss of generality, let \( x, y \in \bigcup_{i=1}^{r} V(C_i) \).

First, we consider the case that \( x, y \) belong to different connected components. Without
loss of generality, let \( x \in V(C_1) \) and \( y \in V(C_2) \). Since \( G[V(C_i) \cup \{ u \}] (i = 1, 2) \) is 2-connected, it follows that both \( G[V(C_1) \cup \{ u \}] \setminus x \) and \( G[V(C_2) \cup \{ u \}] \setminus y \) is connected. Therefore, \( G[V(C_1) \cup \{ u \}] \setminus x \) contains a spanning tree \( T_1 \) and \( G[V(C_2) \cup \{ u \}] \setminus y \) contains a spanning tree \( T_2 \). Furthermore, \( G[C_i] \cup \{ u \} \) contains a spanning tree \( T_i \) for each \( i (3 \leq i \leq r) \). Then the tree \( T \) induced by the edges in \( \{ uw_i | 1 \leq i \leq s \} \cup \{ uw_i | 1 \leq i \leq t \} \cup \{ uw_i | 1 \leq i \leq t \} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired \( S \)-Steiner tree. Therefore, \( d_G(S) \leq e(T) = s + 2t + \sum_{i=3}^{r} n_i + (n_1 - 1) + (n_2 - 1) = n - 3 \), as desired.

Next, we consider the case that \( x, y \) belong to one connected component \( C_{j_1} \), where \( j_1 \in \{1, 2, \ldots, r\} \). If \( G[V(C_{j_1}) \cup \{ u \}] \) is 3-connected, then \( G[V(C_{j_1}) \cup \{ u \}] \setminus \{ x, y \} \) is connected. Therefore, \( G[V(C_{j_1}) \cup \{ u \}] \setminus \{ x, y \} \) contains a spanning tree \( T_j \). For each \( i(i \neq j_1, 1 \leq i \leq r) \), \( G[V(C_i) \cup \{ u \}] \setminus \{ x, y \} \) contains a spanning tree, say \( T_i \). Then the tree \( T \) induced by the edges in \( \{ uw_i | 1 \leq i \leq s \} \cup \{ uw_i | 1 \leq i \leq t \} \cup \{ uw_i | 1 \leq i \leq t \} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired \( S \)-Steiner tree. Therefore, \( d_G(S) \leq e(T) = n - 3 \), as desired.

Suppose \( \kappa(G[V(C_{j_1}) \cup \{ u \}]) = 2 \) and there exists a vertex \( v \in V(C_{j_1}) \) such that \( \{ u, v \} \) is a vertex cut set of \( G[V(C_{j_1}) \cup \{ u \}] \) and for each component \( C_{j_1}^i \) \((1 \leq i \leq p)\) of \( G[V(C_{j_1}) \cup \{ u \}] \setminus \{ u, v \} \), one of the following conditions holds.

- \( G[V(C_{j_1}^i) \cup \{ u \}] \) is 3-connected;
- \( G[V(C_{j_1}^i) \cup \{ v \}] \) is 3-connected;
- \( \kappa(G[V(C_{j_1}^i) \cup \{ u \}]) = \kappa(G[V(C_{j_1}^i) \cup \{ v \}]) = 2 \) and \( \{ z', z'' \} \) is not a common vertex cut set of \( G[V(C_{j_1}^i) \cup \{ u \}] \) and \( G[V(C_{j_1}^i) \cup \{ v \}] \) where \( z', z'' \in V(C_{j_1}^i) \);
- \( \kappa(G[V(C_{j_1}^i) \cup \{ u \}]) = 2 \) and \( \kappa(G[V(C_{j_1}^i) \cup \{ v \}]) = 1 \) and if \( \{ z', z'' \} \) is a vertex cut set of \( G[V(C_{j_1}^i) \cup \{ u \}] \) then neither \( z' \) nor \( z'' \) is a cut vertex of \( G[V(C_{j_1}^i) \cup \{ v \}] \).

It is clear that \( |E_G[C_{j_1}^i, u]| \geq 1 \) and \( |E_G[C_{j_1}^i, v]| \geq 1 \) for each \( i (1 \leq i \leq p) \).

Suppose \( v \in \{ x, y \} \). Without loss of generality, let \( v = x \). Then \( y \in \bigcup_{i=1}^{p} V(C_{j_1}^i) \). Then there exists some component, say \( C_{j_1}^i \), such that \( y \in C_{j_1}^i \). Since \( G[V(C_{j_1}^i) \cup \{ u \}] \) is 2-connected, it follows that \( G[V(C_{j_1}^i) \cup \{ u \}] \setminus y \) is connected and hence \( G[V(C_{j_1}^i) \cup \{ u \}] \setminus y \) contains a spanning tree, say \( T_{j_1,i} \). For each \( i (1 \leq i \leq p, i \neq i_1) \), because \( G[V(C_{j_1}^i) \cup \{ u \}] \) is connected and hence \( G[V(C_{j_1}^i) \cup \{ u \}] \) contains a spanning tree, say \( T_{j_1,i} \). Since \( |E_G[C_{j_1}^i, v]| \geq 1 \) for each \( i (1 \leq i \leq p) \), it follows that there exists a component \( C_{j_1}^i \) such that \( y \in V(C_{j_1}^i) \) and there exists an edge \( vv_p \in E_G[C_{j_1}^i, v] \), where \( v_p \in V(C_{j_1}^i) \). Set \( T_{j_1} = (\bigcup_{i=1}^{p} T_{j_1,i}) \cup \{ vv_p \} \). One can see that \( T_{j_1} \) is a spanning tree of \( G[V(C_{j_1}) \cup \{ u \}] \). For each \( j (1 \leq i \leq r, j \neq j_1) \), because \( G[V(C_j) \cup \{ u \}] \) is connected and hence \( G[V(C_j) \cup \{ u \}] \) contains a spanning tree, say \( T_j \). Then the tree \( T \) induced by the edges in \( \{ uw_i | 1 \leq i \leq s \} \cup \{ uw_i | 1 \leq i \leq t \} \cup \{ uw_i | 1 \leq
\[ i \leq t \} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \] is our desired S-Steiner tree. Therefore, \( d_G(S) \leq e(T) = n - 3 \), as desired.

Suppose \( v \not\in \{x, y\} \). Then \( x, y \in \bigcup_{i=1}^{p} V(C_{j_i}^i) \). Consider the case that \( x, y \) belong to different components in \( \{C_{j_1}^1, C_{j_2}^1, \ldots, C_{j_p}^p\} \). Without loss of generality, let \( x \in V(C_{j_1}^1) \) and \( y \in V(C_{j_1}^1) \). Since \( G[V(C_{j_1}^1) \cup \{u\}] \) is 2-connected, it follows that \( G[V(C_{j_1}^1) \cup \{u\}] \) contains a spanning tree \( T_{j_1,1} \) and \( G[V(C_{j_1}^1) \cup \{u\}] \) contains a spanning tree \( T_{j_1,2} \). For each \( i \) (\( 3 \leq i \leq p \)), \( G[V(C_{j_i}^i) \cup \{u\}] \) is connected and hence \( G[V(C_{j_i}^i) \cup \{u\}] \) contains a spanning tree, say \( T_{j_i,i} \). We want to obtain a spanning tree of \( G[V(C_{j_1}^1) \cup \{u\}] \) by \( T_{j_1,1}, T_{j_1,2}, \ldots, T_{j_1,p} \) by adding one edge. If \( uv \in E(G) \), then we set \( T_{j_1} = (\bigcup_{i=1}^{p} T_{j_i,i}) \cup \{vu\} \). Suppose \( p \geq 3 \). Since \( |E_G(C_{j_1}^1,v)| \geq 1 \) for each \( i \) (\( 1 \leq i \leq p \)), it follows that there exists a component, say \( C_{j_1}^1 \), such that \( vu \in E_G(C_{j_1}^1,v) \), where \( v_p \in V(C_{j_1}^1) \). Then there exists two vertices \( v_p, v_q \in V(C_{j_1}^1) \) such that \( v_p \notin x \) or \( v_q \neq x \). Without loss of generality, let \( v_q \neq x \).

Then, the tree \( T \) induced by the edges in \( \{uv_1 | 1 \leq i \leq s\} \cup \{uv_2 | 1 \leq i \leq t\} \cup \{uv_3 | 1 \leq i \leq t\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired S-Steiner tree. Therefore, \( d_G(S) \leq e(T) = n - 3 \), as desired. Consider the case that \( x, y \) belong to same connected component. Without loss of generality, let \( x, y \in V(C_{j_1}^1) \). For \( i \) (\( 2 \leq i \leq p \)), since \( G[V(C_{j_i}^i) \cup \{u\}] \) is 2-connected, it follows that \( G[V(C_{j_i}^i) \cup \{u\}] \) contains a spanning tree, say \( T_{j_i} \). Then the tree \( T \) induced by the edges in \( \{uv_1 | 1 \leq i \leq s\} \cup \{uv_2 | 1 \leq i \leq t\} \cup \{uv_3 | 1 \leq i \leq t\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired S-Steiner tree. Therefore, \( d_G(S) \leq e(T) = n - 3 \), as desired.
Lemma 4 Let $G$ be a connected graph of order $n$. If $\kappa(G) = 2$, then $s\text{diam}_{n-3}(G) = n - 3$ if and only if $G$ contains a vertex cut set $\{u, v\}$ and for each connected component $C_i$ of order at least $3$ in $G \setminus \{u, v\}$, $C_i$ satisfies one of the following two conditions.

1. $G[V(C_i) \cup \{u, v\}]$ is 3-connected;
2. $\kappa(G[V(C_i) \cup \{u, v\}]) = 2$, both $G[V(C_i) \cup \{u\}]$ and $G[V(C_i) \cup \{v\}]$ are 2-connected, for any vertex cut $\{x, y\} \neq \{u, v\}$ of $G[V(C_i) \cup \{u, v\}]$ and any connected component $C^j_i (1 \leq j \leq s)$ of $G[V(C^j_i) \cup \{u, v\}] \setminus \{x, y\}$, one of the following conditions is true.
   1. $G[V(C^j_i) \cup \{x\}]$ is 2-connected;
   2. $G[V(C^j_i) \cup \{y\}]$ is 2-connected;
   3. $\kappa(G[V(C^j_i) \cup \{x\}]) = 1$, and for any cut vertex $z$ and any component $C^{j,k}_i (1 \leq k \leq t)$, $|E_G[C^{j,k}_i, x]| \geq 1$.
   4. $\kappa(G[V(C^j_i) \cup \{y\}]) = 1$, and for any cut vertex $z$ and any component $C^{j,k}_i (1 \leq k \leq t)$, $|E_G[C^{j,k}_i, x]| \geq 1$.

Proof. In one direction, we suppose $s\text{diam}_{n-3}(G) = n - 3$. For a vertex cut set $\{u, v\}$, there exists a connected component $C_j$ of order at least $3$ of the graph $G \setminus \{u, v\}$ such that

- $\kappa(G[V(C_j) \cup \{u, v\}]) = 1$;
- $\kappa(G[V(C_j) \cup \{u, v\}]) = 2$ and $\kappa(G[V(C_i) \cup \{u\}]) = 1$;
- $\kappa(G[V(C_j) \cup \{u, v\}]) = 2$ and $\kappa(G[V(C_i) \cup \{v\}]) = 1$;
- $\kappa(G[V(C_j) \cup \{u, v\}]) = \kappa(G[V(C_j) \cup \{u\}]) = \kappa(G[V(C_j) \cup \{v\}]) = 2$, and there exists a vertex cut set $\{x, y\}$ and a connected component $C^{i_1}_{i_j}$ of $G[V(C_j) \cup \{u, v\} \setminus \{x, y\}]$ such that $\kappa(G[V(C^{i_1}_{i_j})] \cup \{x\}) = 1$, and there exists a cut vertex $z$ and a connected component $C^{i_2}_{i_j}$ such that $|E_G[C^{i_2}_{i_j}, x]| = 0$.

Our aim is to get a contradiction. Let $H_j = G[V(C_j) \cup \{u, v\}]$. Suppose that $\kappa(H_j) = 2$ and $\kappa(H_j \setminus u) = 1$, or $\kappa(H_j) = 2$ and $\kappa(H_j \setminus v) = 1$. Without loss of generality, let $\kappa(H_j) = 2$ and $\kappa(H_j \setminus v) = 1$. Since $\kappa(H_j \setminus v) = 1$, it follows that there exists a cut vertex of $H_j \setminus v$, say $x$. One can see that $x \neq u$ since $H_j \setminus \{u, v\}$ is connected. Thus, $H_j \setminus \{x, v\}$ is disconnected. Let $C^1_j, C^2_j, \ldots, C^s_j (s \geq 2)$ be the connected components of $H_j \setminus \{x, v\}$. Clearly, $u \in \bigcup_{i=1}^{s} V(C^i_j)$. Without loss of generality, let $u \in V(C^1_j)$. Then there exists a connected component $C^{i_1}_{i_j}$ ($i_1 \in \{2, 3, \ldots, s\}$) such that $|E_G[u, C^{i_1}_{i_j}]| = 0$. 

\[\Rightarrow\]
Choose $S = V(G) \setminus \{u, v, x\}$. Then $|S| = n - 3$. Obviously, any Steiner tree connecting $S$, say $T$, must contain the vertices $u$ and $x$, which implies $|V(T)| \geq n - 1$ and hence $d_G(S) \geq e(T) \geq n - 2$ and hence $sdiam_{n-3}(G) \geq d_G(S) \geq n - 2$, a contradiction.

Suppose $\kappa(H_j) = 1$. If $v$ is a cut vertex of $H_j$, then $H_j \setminus v$ is disconnected. Then $u$ belongs to a connected component $C'_j$ of $H_j \setminus v$. If $|V(C'_j)| \geq 2$, then $H_j \setminus \{u, v\}$ is also disconnected, a contradiction. So we may assume that $V(C'_j) = \{u\}$. Then $v$ is a cut vertex of $G$, which contradicts to $\kappa(G) = 2$. We now suppose that neither $u$ nor $v$ is a cut vertex of $H_j$. Then there exists a cut vertex of $H_j$, say $x$, such that $x \neq u$ and $x \neq v$. If $u, v$ belongs to the same connected component of $H_j \setminus x$, then we choose $S = V(G) \setminus \{u, v, x\}$. Obviously, $|S| = n - 3$ and any Steiner tree connecting $S$, say $T$, must contain the vertex $x$ and one of $\{u, v\}$, which implies $|V(T)| \geq n - 1$ and hence $d_G(S) \geq e(T) \geq n - 2$ and hence $sdiam_{n-3}(G) \geq d_G(S) \geq n - 2$, a contradiction. Suppose $u, v$ belongs to the different connected component of $H_j \setminus x$. One can also check that $sdiam_{n-3}(G) \geq n - 2$ and get a contradiction.

Suppose that $\kappa(H_j) = \kappa(H_j \setminus \{u\}) = \kappa(H_j \setminus \{v\}) = 2$, and there exists a vertex cut set $\{x, y\}$ and a connected component $C^i_j$ of $H_j \setminus \{x, y\}$ such that $\kappa(G[V(C^i_j) \cup \{x\}] = 1$, and there exists a cut vertex $z$ and a connected component $C^j_{i_1^1, k_1^1}$ such that $|E_G[C^j_{i_1^1, k_1^1}, x]| = 0$. Set $S = V(G) \setminus \{x, y, z\}$. Obviously, $|S| = n - 3$ and any Steiner tree connecting $S$, say $T$, must contain the vertex $x$ and one element $\{x, y\}$, which implies $|V(T)| \geq n - 1$ and hence $d_G(S) \geq e(T) \geq n - 2$ and hence $sdiam_{n-3}(G) \geq d_G(S) \geq n - 2$, a contradiction. Suppose $u, v$ belongs to the different connected component of $H_j \setminus x$. One can also check that $sdiam_{n-3}(G) \geq n - 2$ and get a contradiction.

In another direction, it suffices to show that $d_G(S) \leq n - 3$ for any $S \subseteq V(G)$ and $|S| = n - 3$. Let $\bar{S} = V(G) \setminus S$. Clearly, $0 \leq |\{u, v\} \cap S| \leq 2$, and each connected component of $G \setminus \{u, v\}$ is a connected subgraph of order at least 3, or an edge of $G$, or an isolated vertex. Let $w_1, w_2, \ldots, w_s$ be the isolated vertices, $e_1, e_2, \ldots, e_t$ be the independent edges, and $C_1, C_2, \ldots, C_r$ be the connected components of order at least 3 in $G \setminus \{u, v\}$. Set $e_i = u_i v_i$ ($1 \leq i \leq t$), $W = \{w_1, w_2, \ldots, w_s\}$, $U = \{u_1, u_2, \ldots, u_t\}$, $V = \{v_1, v_2, \ldots, v_t\}$. Obviously, there is at least one edge between $u$ and $u_i$, $v_i$, and there is at least one edge between $v$ and $v_i$, $u_i$, and $uw_i, vw_i \in E(G)$, and $s + 2t + \sum_{i=1}^n n_i$ where $n_i = |V(C_i)|$. Clearly, $0 \leq |\bar{S} \cap \{u, v\}| \leq 2$.

Consider the case $|\bar{S} \cap \{u, v\}| = 2$. Since $|S| = n - 3$, there exists a vertex $x$ such that $x \notin \{u, v\}$ and $x \notin S$. Then $x \in W$ or $x \in U \cup V$ or $x \in \bigcup_{i=1}^r V(C_i)$.

Suppose $x \in W = \{w_1, w_2, \ldots, w_s\}$. Without loss of generality, let $x = w_1$. Clearly, $G \setminus \{x, v\}$ is connected. Then $G \setminus \{x, v\}$ contains a spanning tree $T$, which is a Steiner tree connecting $S$. Observe that $|V(T)| = n - 2$. Therefore, $d_G(S) \leq e(T) \leq n - 3$, as desired.

Suppose $x \in U \cup V$. Without loss of generality, let $x = v_1$ and $uv_1 \in E(G)$. Thus,
G \ \{x,v\} contains a spanning tree T, which is a Steiner tree connecting S. Observe that \(|V(T)| = n - 2\). Therefore, \(d_G(S) \leq e(T) \leq n - 3\), as desired.

Suppose \(x \in \bigcup_{i=1}^{r} V(C_i)\). Without loss of generality, let \(x \in V(C_1)\). Note that \(G[V(C_1) \cup \{u,v\}]\) is 3-connected, or \(\kappa(G[V(C_1) \cup \{u,v\}]) = 2\) and \(G[V(C_1) \cup \{u\}]\) is 2-connected. Then \(G[V(C_1) \cup \{u\}] \setminus x\) is connected and hence \(G[V(C_1) \cup \{u\}] \setminus x\) contains a spanning tree \(T_1\). For each \(C_i (2 \leq i \leq r)\), \(G[V(C_i) \cup \{u\}]\) contains a spanning tree \(T_i\). Therefore, the tree \(T\) induced by the edges in \(\{uw_1, uw_2, \ldots, uw_r\} \cup \{uw_1, uw_2, \ldots, uw_r\} \cup \{u_1v_1, u_2v_2, \ldots, u_rv_r\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r)\) form a Steiner tree connecting \(S\) and so \(d_G(S) \leq e(T) = \sum_{i=2}^{r} n_i + (n_1 - 1) + 2t + s = n - 3\), as desired.

Consider the case \(|S \cap \{u,v\}| = 1\). Without loss of generality, let \(v \in S\) and \(u \notin S\). Then \(v \notin S\) and \(u \in S\). Since \(|S| = n - 3\), there exists two vertices \(x,y\) such that \(x,y \notin \{u,v\}\) and \(x,y \notin S\). Then \(x,y \in V(G) \setminus \{u,v\} = W \cup U \cup V(\bigcup_{i=1}^{r} V(C_i))\).

Suppose that at least one of \(\{x,y\}\) belong to \(\bigcup_{i=1}^{r} V(C_i)\). Without loss of generality, let \(x \in W\) and \(x = w_1\). Note that for each \(C_i (1 \leq i \leq r)\), \(G[V(C_i) \cup \{u,v\}]\) is 3-connected, or \(\kappa(G[V(C_i) \cup \{u,v\}]) = 2\) and \(G[V(C_i) \cup \{u\}]\) is 2-connected. Furthermore, \(G[V(C_i) \cup \{u\}]\) contains a spanning tree, say \(T_i\). Then the tree \(T\) induced by the edges in \(\{uw_1 | 1 \leq i \leq s\} \cup \{uw_i | 1 \leq i \leq t\} \cup \{u_1v_1, u_2v_2, \ldots, u_rv_r\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r)\) is a Steiner tree connecting \(S\) and so \(d_G(S) \leq e(T) = (s-1)+2t+\sum_{i=1}^{r} n_i = n - 3\), as desired.

Suppose that at least one of \(\{x,y\}\) belong to \(\bigcup_{i=1}^{r} V(C_i)\). Without loss of generality, let \(x \in \bigcup_{i=1}^{r} V(C_i)\). Then there exists some \(C_j\) such that \(x \in V(C_j)\). Observe that \(G[V(C_j) \cup \{u\}] \setminus x\) contains a spanning tree, say \(T_j\). For each \(C_i (i \neq j, 1 \leq i \leq r)\), \(G[V(C_i) \cup \{u\}]\) contains a spanning tree, say \(T_i\). Then the tree \(T\) induced by the edges in \(\{uw_i | 1 \leq i \leq s\} \cup \{uw_i | 1 \leq i \leq t\} \cup \{u_1v_1, u_2v_2, \ldots, u_rv_r\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r)\) is a Steiner tree connecting \(S\) and so \(d_G(S) \leq e(T) = s+2t+\sum_{i=1}^{r} n_i + (n_j - 1) = n - 3\), as desired.

Suppose that neither \(x\) nor \(y\) belong to \(W \cup (\bigcup_{i=1}^{r} V(C_i))\). Then \(x \in U \cup V\), and \(G \setminus \{x,y\}\) is connected and hence \(G \setminus \{x,y\}\) contains a spanning tree \(T\). So \(d_G(S) \leq e(T) = n - 3\), as desired.

Consider the case \(|S \cap \{u,v\}| = 0\). Then \(u,v \in S\). Since \(|S| = n - 3\), there exists three vertices \(x,y,z\) such that \(x,y,z \notin \{u,v\}\) and \(x,y,z \notin S\). Then we have \(x,y,z \in V(G) \setminus \{u,v\} = W \cup U \cup V(\bigcup_{i=1}^{r} V(C_i))\).

Suppose that at least two of \(\{x,y,z\}\) belong to \(W \cup U \cup V\). Without loss of generality, let \(x,y \in W\). Then \(G \setminus \{x,y\}\) is connected and hence \(G \setminus \{x,y\}\) contains a spanning tree \(T\). So \(d_G(S) \leq e(T) = n - 3\), as desired.
Suppose that only one of \( \{x, y, z\} \) belong to \( W \cup U \cup V \). Without loss of generality, let \( x \in W \cup U \cup V \). Then \( y, z \in \cup_{i=1}^{r} V(C_i) \). Then there exists some \( C_j \) such that \( y \in V(C_j) \). Since \( G[V(C_j) \cup \{u\}] \) is 2-connected, it follows that \( G[V(C_j) \cup \{u\}] \) contains a spanning tree, say \( T_j \). Furthermore, for each \( C_i \) \((i \neq j, 1 \leq i \leq r)\), \( G[V(C_i) \cup \{u\}] \) contains a spanning tree, say \( T_i \). Suppose \( x \in W \). Without loss of generality, let \( x = w_1 \). Then the tree \( T \) induced by the edges in \( \{uw_{i} \mid 2 \leq i \leq s\} \cup \{uw_i \mid 1 \leq i \leq t\} \cup \{uv_{i} \mid 1 \leq i \leq t\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired Steiner tree connecting \( S \). Suppose \( x \in U \cup V \). Without loss of generality, let \( x = u_1 \). Then the tree \( T \) induced by the edges in \( \{uw_{i} \mid 2 \leq i \leq s\} \cup \{uw_i \mid 1 \leq i \leq t\} \cup \{uv_{i} \mid 2 \leq i \leq t\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired Steiner tree connecting \( S \). Therefore, \( d_G(S) \leq e(T) = n - 3 \), as desired.

Suppose that none of \( \{x, y, z\} \) belong to \( U \cup V \cup W \). Then \( x, y, z \in \cup_{i=1}^{r} V(C_i) \). Consider the case that \( x, y, z \) belong to three connected components, say \( C_1, C_2, C_3 \). Without loss of generality, let \( x \in V(C_1), y \in V(C_2) \) and \( z \in V(C_3) \). Since \( G[V(C_1) \cup \{u\}] \) \((i = 1, 2, 3)\) is 2-connected, it follows that \( G[V(C_1) \cup \{u\}] \) contains a spanning tree, say \( T_1 \). Without loss of generality, let \( x \in V(C_1) \) and \( z \in V(C_3) \). Since \( G[V(C_1) \cup \{u\}] \) \((i = 1, 2)\) is 2-connected, it follows that \( G[V(C_1) \cup \{u\}] \) and \( G[V(C_2) \cup \{u\}] \) \((i = 1, 2)\) are both connected. Therefore, \( G[V(C_1) \cup \{u\}] \) contains a spanning tree, say \( T_1 \), and \( G[V(C_2) \cup \{u\}] \) contains a spanning tree, say \( T_2 \). Furthermore, for each \( C_i \) \((3 \leq i \leq r)\), \( G[V(C_i) \cup \{u\}] \) contains a spanning tree, say \( T_i \). Then the tree \( T \) induced by the edges in \( E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired \( S \)-Steiner tree. Therefore, \( d_G(S) \leq e(T) = \sum_{i=1}^{r} n_i - 2 = n - 4 \), as desired. Consider the case that \( x, y, z \) belong to the same connected component, say \( C_1 \). If \( \{x, y\} \) or \( \{y, z\} \) or \( \{x, z\} \) is not a vertex cut set of \( G[V(C_1) \cup \{u, v\}] \), then \( G[V(C_1) \cup \{u, v\}] \) \((x, y)\) is connected, it follows that \( G[V(C_1) \cup \{u, v\}] \) \((x, y)\) contains a spanning tree, say \( T_1 \). Furthermore, for each \( C_i \) \((2 \leq i \leq r)\), \( G[V(C_i) \cup \{u\}] \) contains a spanning tree, say \( T_i \). Then the tree \( T \) induced by the edges in \( E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r) \) is our desired \( S \)-Steiner tree connecting \( S \). Therefore, \( d_G(S) \leq e(T) = \sum_{i=1}^{r} n_i - 2 = n - 3 \), as desired.

Suppose that \( \{x, y\}, \{y, z\} \) and \( \{x, z\} \) are all vertex cut sets of \( G[V(C_1) \cup \{u, v\}] \). We consider the connected component \( C_j \) \((1 \leq j \leq s)\) of \( G[V(C_1) \cup \{u, v\}] \) \((x, y)\). If \( G[V(C_j) \cup \{x\}] \) is 2-connected, then \( G[V(C_j) \cup \{x\}] \) \((x, y)\) contains a spanning tree, say \( T_{1,j} \). Furthermore, for each \( C_i \) \((1 \leq i \leq s, i \neq j)\), \( G[V(C_i) \cup \{y\}] \) contains a spanning tree, say \( T_{1,i} \).
Let $T_1$ denote the tree induced by the edges in $E(T_{1,1}) \cup E(T_{1,2}) \cup \cdots \cup E(T_{1,r})$. Furthermore, for each $C_i$ $(2 \leq i \leq r)$, $G[V(C_i) \cup \{u\}]$ contains a spanning tree, say $T_i$. Then the tree $T$ induced by the edges in $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r)$ is our desired $S$-Steiner tree. Therefore, $d_G(S) \leq e(T) = \sum_{i=1}^r n_i - 1 = n - 3$, as desired.

Suppose that $\kappa(G[V(C_i^j) \cup \{x\}]) = 1$ and for any cut vertex $z$ and any component $C_{i}^{j,k}$ $(1 \leq k \leq t)$, $|E_G[C_{i}^{j,k}, x]| \geq 1$. Then $G[V(C_i^j) \cup \{x\}] \setminus \{z\}$ is connected, and hence $G[V(C_i^j) \cup \{x\}] \setminus \{z\}$ contains a spanning tree, say $T_{1,j}$. Furthermore, for each $C_i^j$ $(1 \leq i \leq s, i \neq j)$, $G[V(C_i^j) \cup \{x\}]$ contains a spanning tree, say $T_{1,i}$. Let $T_1$ denote the tree induced by the edges in $E(T_{1,1}) \cup E(T_{1,2}) \cup \cdots \cup E(T_{1,r})$. Furthermore, for each $C_i$ $(2 \leq i \leq r)$, $G[V(C_i) \cup \{u\}]$ contains a spanning tree, say $T_i$. Then the tree $T$ induced by the edges in $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r)$ is our desired $S$-Steiner tree. Therefore, $d_G(S) \leq e(T) = \sum_{i=1}^r n_i - 1 = n - 3$, as desired.

From the above argument, $d_G(S) \leq n - 3$ for any $S \subseteq V(G)$ and $|S| = n - 3$. So $sdiam_{n-3}(G) \leq n - 3$. $\blacksquare$

**Proposition 3** Let $G$ be a connected graph of order $n$. Then $sdiam_{n-3}(G) = n - 3$ if and only if $G$ satisfies one of the following conditions.

1. $\kappa(G) = 3$;
2. $\kappa(G) = 2$ and $G$ contains a vertex cut set $\{u, v\}$ and for each connected component $C_i$ of order at least $3$ in $G \setminus \{u, v\}$, $C_i$ satisfies one of the following two conditions.
   1. $G[V(C_i) \cup \{u, v\}]$ is $3$-connected;
   2. $\kappa(G[V(C_i) \cup \{u, v\}]) = 2$, both $G[V(C_i) \cup \{u\}]$ and $G[V(C_i) \cup \{v\}]$ are $2$-connected, for any vertex cut $\{x, y\} \neq \{u, v\}$ of $G[V(C_i) \cup \{u, v\}]$ and any connected component $C_i^j$ $(1 \leq j \leq s)$ of $G[V(C_i^j) \cup \{u, v\}] \setminus \{x, y\}$, one of the following conditions is true.
      1. $G[V(C_i^j) \cup \{x\}]$ is $2$-connected;
      2. $G[V(C_i^j) \cup \{y\}]$ is $2$-connected;
   3. $\kappa(G[V(C_i^j) \cup \{x\}]) = 1$, and for any cut vertex $z$ and any component $C_i^{j,k}$ $(1 \leq k \leq t)$, $|E_G[C_i^{j,k}, x]| \geq 1$.
      4. $\kappa(G[V(C_i^j) \cup \{y\}]) = 1$, and for any cut vertex $z$ and any component $C_i^{j,k}$ $(1 \leq k \leq t)$, $|E_G[C_i^{j,k}, x]| \geq 1$.
3. $G$ contains only one cut vertex $u$; for each connected component $C_i$ of order at least $3$ in $G \setminus u$, $G[V(C_i) \cup \{u\}]$ is $3$-connected, or $\kappa(G[V(C_i) \cup \{u\}]) = 2$ and there exists a vertex $v \in V(C_i)$ such that $\{u, v\}$ is a vertex cut set of $G[V(C_i) \cup \{u\}]$, and for each component $C_i^j$ $(1 \leq j \leq p)$ of $G[V(C_i) \cup \{u\}] \setminus \{u, v\}$, one of the following conditions holds:
   - $uv \in E(G)$;
   - $p \geq 3$;

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• \( p = 2 \), and \(|E_G[v, V(C^1_i)]| \geq 2\) or \(|E_G[v, V(C^2_i)]| \geq 2\).

and one of the following conditions holds:

• \( G[V(C^1_i) \cup \{u\}] \) is 3-connected;
• \( G[V(C^2_i) \cup \{v\}] \) is 3-connected;
• \( \kappa(G[V(C^1_i) \cup \{u\}]) = \kappa(G[V(C^2_i) \cup \{v\}]) = 2 \) and \( \{y, z\} \) is not a common vertex cut set of \( G[V(C^1_i) \cup \{u\}] \) and \( G[V(C^2_i) \cup \{v\}] \) where \( z', z'' \in V(C^j_i) \);
• \( \kappa(G[V(C^2_i) \cup \{u\}]) = 2 \) and \( \kappa(G[V(C^1_i) \cup \{v\}]) = 1 \) and if \( \{z', z''\} \) is a vertex cut set of \( G[V(C^1_i) \cup \{u\}] \), then neither \( z' \) nor \( z'' \) is a cut vertex of \( G[V(C^1_i) \cup \{v\}] \).

Proof. In one direction, we suppose \( sdiam_{n-3}(G) = n - 3 \). From Lemma 2, we have \( \kappa(G) \leq 3 \). If \( \kappa(G) = 2 \), then the result holds by Lemma 4. If \( \kappa(G) = 1 \), then the result holds by Lemma 2.

Conversely, we suppose that \( G \) satisfies the conditions of this theorem. From Lemma 2 we know that \( sdiam_{n-3}(G) \geq n - 3 \). So it suffices to show that \( sdiam_{n-3}(G) \leq n - 3 \). If \( \kappa(G) = 3 \), then \( sdiam_{n-3}(G) \leq sdiam_{n-2}(G) = n - 3 \) by (2) of Observation 1 and (1) of Lemma 2. From (1) of this theorem, we have \( sdiam_{n-3}(G) \geq n - 3 \). Therefore, \( sdiam_{n-3}(G) = n - 3 \). If \( \kappa(G) = 2 \) and \( G \) satisfies Condition (2), then \( sdiam_{n-3}(G) \leq n - 3 \) by Lemma 3. If \( \kappa(G) = 1 \) and \( G \) satisfies Condition (3), then \( sdiam_{n-3}(G) \leq n - 3 \) by Lemma 3.

Corollary 3 Let \( G \) be a connected graph of order \( n \). Then \( sdiam_{n-3}(G) = n - 2 \) if and only if \( G \) satisfies one of the following conditions.

(1) \( \kappa(G) = 2 \); for a vertex cut set \( \{u, v\} \), there exists a connected component \( C_j \) of order at least 3 of the graph \( G \setminus \{u, v\} \) such that

• \( \kappa(G[V(C_j) \cup \{u, v\}]) = 1 \);
• \( \kappa(G[V(C_j) \cup \{u, v\}]) = 2 \) and \( \kappa(G[V(C_i) \cup \{u\}]) = 1 \);
• \( \kappa(G[V(C_j) \cup \{u, v\}]) = 2 \) and \( \kappa(G[V(C_i) \cup \{v\}]) = 1 \);
• \( \kappa(G[V(C_j) \cup \{u, v\}]) = \kappa(G[V(C_j) \cup \{u\}]) = \kappa(G[V(C_j) \cup \{v\}]) = 2 \), and there exists a vertex cut set \( \{x, y\} \) and a connected component \( C^{i_1}_{j_1} \) of \( G[V(C_j) \cup \{u, v\} \setminus \{x, y\}] \) such that \( \kappa(G[V(C_j^{i_1}_{j_1}) \cup \{x\}]) = 1 \), and there exists a cut vertex \( z \) and a connected component \( C^{i_1, k_1}_{j_1} \) such that \( |E_G[C^{i_1, k_1}_{j_1}, x]| = 0 \).

(2) there exist exactly two cut vertices in \( G \).

(3) \( G \) contains only one cut vertex \( u \) such that there exists a connected component \( C_j \) of order at least 3 in \( G \setminus u \) satisfying one of the following.
(3.1) \( \kappa(G[V(C_j) \cup \{u\}]) = 1; \)
(3.2) \( \kappa(G[V(C_j) \cup \{u\}]) = 2 \) and \( \{v, u\} \) is not a vertex cut set of \( G[V(C_j) \cup \{u\}] \) for any \( v \in V(C_j); \)
(3.3) \( \kappa(G[V(C_j) \cup \{u\}]) = 2, \) \( \{v, u\} \) is a vertex cut set of \( G[V(C_j) \cup \{u\}], \) and there exists a component \( C'_j \) of \( G[V(C_j) \cup \{u\} \setminus \{u, v\}] \) satisfying one of the following conditions.

- \( \kappa(G[V(C'_j) \cup \{u\}]) = 1. \)
- \( \kappa(G[V(C'_j) \cup \{u\}]) = \kappa(G[V(C'_j) \cup \{v\}]) = 2 \) and \( \{y, z\} \) is a common vertex cut set of \( G[V(C'_j) \cup \{u\}] \) and \( V(C'_j) \cup \{v\} \) for any \( y, z \in V(C'_j). \)
- \( \kappa(G[V(C'_j) \cup \{u\}]) = 2 \) and \( \kappa(G[V(C'_j) \cup \{v\}]) = 1 \) and if \( \{y, z\} \) is a common vertex cut set of \( G[V(C'_j) \cup \{u\}] \) where \( y, z \in V(C'_j), \) then at least one of \( \{y, z\} \) is a cut vertex of \( V(C'_j) \cup \{v\}. \)
- \( uv \notin E(G), \) \( p = 2 \) and there is only one edge between \( v \) and each connected component of \( G[V(C_j) \cup \{u\}] \setminus \{u, v\}. \)

From the above lemmas and corollary, we conclude the following theorem.

**Theorem 3** Let \( G \) be a connected graph of order \( n. \) Then

1. \( sdiam_{n-3}(G) = n - 4 \) if and only if \( \kappa(G) \geq 4. \)
2. \( sdiam_{n-3}(G) = n - 3 \) if and only if \( G \) satisfies the conditions of Proposition [5]
3. \( sdiam_{n-3}(G) = n - 2 \) if and only if \( G \) satisfies the conditions of Corollary [6]
4. \( sdiam_{n-3}(G) = n - 1 \) if and only if there are at least three cut vertices in \( G. \)

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