Differential equations properties and the Riemann hypothesis

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Abstract
Using the differential equations properties, we prove that any non-trivial zero of Riemann’s zeta function is of real part equal to $\frac{1}{2}$. In other words, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Zeta function, Abel summation formula.

1 Introduction
We consider the representation of Riemann’s zeta function [1, page 14 Equation 2.1.5] defined for all $s \in \mathbb{C}$ such that $\Re(s) > 0$ by the Abel summation formula as

$$\zeta(s) := -s \int_{0}^{+\infty} \frac{\{t\}}{t^{1+s}} dt,$$

where $\{t\}$ is the fractional part of the real $t$. In this paper, we focus on the non-trivial zeros of the function $\zeta$, defined by Equation (1), in the following sense

Definition 1. Let be $s \in \mathbb{C}$. We say that $s$ is a non-trivial zero of the function $\zeta$ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

The following theorem affirms the Riemann hypothesis.

Main Result. Consider the function $\zeta$ defined by Equation (1). Suppose that $s \in \mathbb{C}$ is a non-trivial zero of the function $\zeta$ then $\Re(s) = \frac{1}{2}$.
2 Main proposition

The Equation (1) can be written as

$$\zeta(s) = s \int_{1}^{+\infty} \frac{1}{t^{2-s}} - \frac{\{t\}}{t^{1+s}} dt, \quad \forall s \in \mathbb{C}, \quad \Re(s) \in (0, 1).$$

By a change of variable, $t \to \exp(u)$ we get

$$\forall s := x + i\tau \in \mathbb{C}, \quad x \in (0, 1), \quad \tau \in \mathbb{R}^*: \quad \zeta(s) := -s \int_{0}^{+\infty} \exp(-(1 - s)u) + \exp(-su)\{\exp(u)\} du.$$

For every $\tau \in \mathbb{R}^*$, define the function $\phi_\tau : (0, 1) \times \mathbb{R}_+ \to \mathbb{R}$ as

$$\phi_\tau(x, t) := -\int_{0}^{t} \left[ \exp(-(1 - x)u) - \exp(-xu)\{\exp(u)\} \right] \sin(\tau u) du. \quad (2)$$

Denote

$$\forall x \in (0, 1): \quad \phi_\tau(x) = \lim_{t \to +\infty} \phi_\tau(x, t).$$

By identification, for every $\tau \in \mathbb{R}^*$ and every $x \in (0, 1)$, we have $\phi_\tau(x) = \Re(\zeta(s)/s)$ where $s := x + i\tau$ and we have

$$\zeta(x + i\tau) = 0 \implies \phi_\tau(x) = 0. \quad (3)$$

**Proposition 2.** Let be $\tau \in \mathbb{R}^*$. Then,

$$\forall r \in (0, \frac{1}{2}): \quad \phi_\tau\left(\frac{1}{2} - r\right) \neq \phi_\tau\left(\frac{1}{2} + r\right).$$

**Lemma 3.** Let be $\tau \in \mathbb{R}^*$ and $z > 0$. Then there exists $\beta_{z, \tau} > 0$ such that for all $s > v \geq 0$ we have

$$\left| \int_{v}^{s} \exp(-zt)\{\exp(t)\} \sin(\tau t) dt - [w_\tau(z, s) - w_\tau(z, v)] \right| \leq h_\tau(z, \beta, v, s),$$

where

$$w_\tau(z, t) := -\frac{1}{2z^2 + \tau^2} \exp(-zt)[z \sin(\tau t) + \tau \cos(\tau t)], \quad \forall t \geq 0,$$

$$h_\tau(z, \beta, v, s) := \beta_{z, \tau} \left[ \exp(-(1 + z)s) + \exp(-(1 + z)v) \right], \quad \forall s > v.$$
Proof. In order to simplify the notation, denote
\[ I_\tau(z, s, v) := \int_v^s \exp(-zt) \{\exp(t)\} \sin(\tau t) dt. \]

Use the change of variable \(\exp(t) = u\) we get
\[ I_\tau(z, s, v) := \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \{t\} \sin(\tau \ln(t)) dt. \]

Since the function \(u \mapsto \{u\}\) is 1-periodic function and since
\[ \int_0^1 t dt = \frac{1}{2}, \]

Then there exists a 1-periodic function \(t \mapsto p(t)\) such that
\[ \int_s^t \{u\} du = \frac{1}{2} (t-s) + p(t) - p(s), \quad \forall t \geq s \geq 0. \]

Since the function \(u \mapsto \{u\}\) is continuous on \(\mathbb{R}/\mathbb{Z}\) then the function \(t \mapsto p(t)\) is continuous on \(\mathbb{R}_+\) and \(C^1\) on \(\mathbb{R}/\mathbb{Z}\). Use the integration part formula,
\[ I_\tau(z, s, v) = \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \]
\[ + \exp(-(1 + z)s) \sin(\tau s)(p(\exp(s)) - p(\exp(v))) \]
\[ + \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} f_\tau(z,t)(p(\exp(t)) - p(\exp(v))) dt, \]
\[ f_\tau(z,t) := (1 + z) \sin(\tau \ln(t)) + \tau \cos(\tau \ln(t)). \]

Since the function \(t \mapsto p(t)\) is 1-periodic, there exists \(\alpha > 0\) such that
\[ \sup_{t \geq s \geq 0} |p(t) - p(s)| < \alpha. \]

By Equation (4), we get
\[ \left| I_\tau(z, s, v) - \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \right| \]
\[ \leq \alpha \left[ \exp(-(1 + z)s) + [1 + z + |\tau|] \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} dt \right] \]
\[ \leq \beta_{|\tau|} \left[ \exp(-(1 + z)s) + \exp(-(1 + z)v) \right], \]
where
\[ \beta_{r,\tau} := \alpha [1 + \frac{|\tau|}{1 + z}]. \]

Use the fact
\[
\int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt = \Im \left[ \int_{\exp(v)}^{\exp(s)} t^{1-z+i\tau} dt \right] \\
- \frac{1}{z^2 + \tau^2} \exp(-zs)[z \sin(\tau s) + \tau \cos(\tau s)] \\
+ \frac{1}{z^2 + \tau^2} \exp(-zv)[z \sin(\tau v) + \tau \cos(\tau v)],
\]
to deduce the Lemma.

In order to simplify the notation, denote
\[ A := \{ t \geq 0, \ t \neq \ln(n), \ \forall n \geq 2 \}. \]

In order to simplify the notation we denote on all this Section,
\[ x_r := \frac{1}{2} - r. \]

**Lemma 4.** Let be \( \tau \in \mathbb{R}^* \) and \( r \in (0, \frac{1}{2}) \). Let \((\epsilon_n)_n, (v_n)_n, (s_n)_n \subset \mathbb{R}_+\) be a sequence such that
- \( \lim_{n \to +\infty} [\epsilon_n - \Im \left[ \frac{1}{x_r+i\tau} \exp(-i\tau v_n) \right]] \neq 0 \),
- \( \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} v_n = \lim_{n \to +\infty} s_n - v_n = +\infty \),
- \( s_n \leq 2v_n \).

For every \( n \in \mathbb{N} \), let \( t \mapsto \Delta_r(r, n, t) \) be the continuous function satisfying the following differential equation,

\[
\frac{d}{dt} \Delta_r(r, n, t) = x_r \Delta_r(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\
\Delta_r(r, n, v_n) = \epsilon_n, \ \ t \in A, \ t \geq v_n. \tag{5}
\]

Then,
\[ \lim_{n \to +\infty} |\Delta_r(r, n, s_n)| = +\infty. \]
**Proof.** Integrate Equation (5) and use the fact $\Delta_{\tau}(r, n, v_n) = \epsilon_n$, we get

\[
\Delta_{\tau}(r, n, s_n) = \exp\left(x_{r}(s_n - v_n)\right)\epsilon_n
\]

\[
+ \int_{s_n}^{v_n} \exp\left(x_{r}(s_n - t)\right)\left[1 + \{\exp(t)\}\right]\sin(\tau t)dt
\]

\[
= \exp\left(x_{r}(s_n - v_n)\right)\epsilon_n
\]

\[
+ \int_{s_n}^{v_n} \exp\left(x_{r}(s_n - t)\right)\sin(\tau t)dt
\]

\[
+ \int_{v_n}^{s_n} \exp\left(x_{r}(s_n - t)\right)\{\exp(t)\}\sin(\tau t)dt.
\]

By Lemma 3,

\[
\left|\int_{v_n}^{s_n} \exp(-x_{r}t)\{\exp(t)\}\sin(\tau t)dt - [w_{\tau}(x_{r}, s_n) - w_{\tau}(x_{r}, v_n)]\right|
\]

\[
\leq h_{\tau}(z, \beta, v_n, s_n),
\]

where

\[
w_{\tau}(z, t) := -\frac{1}{2} \frac{1}{z^2 + \tau^2} \exp(-zs)[z \sin(\tau s) + \tau \cos(\tau s)], \forall t \geq 0,
\]

\[
h_{\tau}(z, \beta, v, s) := \beta_{\tau, r}\left[\exp(-(1 + z)s) + 1 + \frac{|\tau|}{1 + z} \exp(-(1 + z)v)\right].
\]

By hypothesis of the present Lemma we have

\[
x_{r}s_n \leq 2x_{r}v_n \leq (2x_{r} + 1)v_n, \quad \text{and} \quad \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} v_n = +\infty,
\]

then

\[
\lim_{n \to +\infty} \exp(x_{r}(s_n - v_n))h_{\tau}(z, \beta, v_n, s_n) = 0.
\]

Use the fact

\[
\int_{v_n}^{s_n} \exp(-x_{r}t)\sin(\tau t)dt = 2[w_{\tau}(x_{r}, s_n) - w_{\tau}(x_{r}, v_n)]
\]

We have

\[
\lim_{n \to +\infty} \exp(-x_{r}(s_n - v_n))\left[\Delta_{\tau}(r, n, s_n) + \exp(x_{r}s_n)[w_{\tau}(x_{r}, s_n) - w_{\tau}(x_{r}, v_n)]\right] = 0.
\]
We have
\[ \exp(x_r v_n) [w_r(x_r, s_n) - w_r(x_r, v_n)] \]
\[ = -\frac{1}{2} x_r^2 \frac{1}{\tau^2} \exp(-x_r(s_n - v_n)) [x_r \sin(\tau s_n) + \tau \cos(\tau s_n)] \]
\[ + \frac{1}{2} x_r^2 \frac{1}{\tau^2} [x_r \sin(\tau v_n) + \tau \cos(\tau v_n)]. \]

By hypothesis of the present Lemma we have \( \lim_{n \to +\infty} s_n - v_n = +\infty \) and
\[ \alpha(\tau, r) := \lim_{n \to +\infty} [\epsilon_n - \frac{1}{2} \Im \left( \frac{1}{x_r + i\tau} \exp(-i\tau v_n) \right)] \neq 0, \]
thanks to Equation (6), we get
\[ \lim_{n \to +\infty} \exp(-x_r(s_n - v_n)) \Delta_r(r, n, s_n) = \alpha(\tau, r) \neq 0, \]
since \( \lim_{n \to +\infty} s_n - v_n = +\infty \), then
\[ \lim_{n \to +\infty} |\Delta_r(r, n, s_n)| = +\infty. \]

\[ \square \]

**Lemma 5.** Let be \( r \in (0, \frac{1}{2}) \) and \( t \mapsto \theta_r(t) \) be the continuous function
satisfying the following differential equation,
\[ \frac{d}{dt} \theta_r(t) = x_r \theta_r(t) - [1 - \exp(-2rt)][1 + \{\exp(t)\}] \sin(\tau t), \]
\[ \theta_r(t) = 0, \quad t \in A. \] (7)

Then,
\[ \sup_{t \geq 0} |\theta_r(t)| = +\infty. \]

**Proof.** By contradiction, suppose that there exists \( r \in (0, \frac{1}{2}) \) and \( \omega_r > 0 \) such that
\[ \sup_{t \geq 0} |\theta_r(t)| \leq \omega_r. \] (8)

Prove that there exists a sequence \((v_n)_n \subset \mathbb{R}_+\) such that
\[ \lim_{n \to +\infty} v_n = +\infty, \] (9)
and such that
\[ \lim_{n \to +\infty} \left[ \theta_r(t, v_n) - \frac{1}{2} \Im \left( \frac{1}{x_r + i\tau} \exp(-i\tau v_n) \right) \right] \neq 0. \] (10)
By contradiction, suppose that

$$\lim_{t \to +\infty} \left[ \theta(r, t) - \frac{1}{2} \Im \left[ \frac{1}{x_r + i \tau} \exp(-i \tau t) \right] \right] = 0. \quad (11)$$

Since $t \mapsto \theta(r, t)$ is a continuous function, integrate Equation (7), we obtain for all $s \geq t \geq 0$ the following equality

$$\theta(r, t) - \theta(r, s) = x_r \int_s^t \theta(r, u) du - \int_s^t \left[ 1 - \exp(-2ru) \right] \left[ 1 + \{\exp(u)\} \right] \sin(\tau u) du. \quad (12)$$

By the hypothesis (11), for every sequences $(u_n)_n, (\tilde{u}_n)_n \subset \mathbb{R}_+$ such that

$$\sup_{n \geq 0} |u_n - \tilde{u}_n| < +\infty \quad \text{and} \quad \lim_{n \to +\infty} u_n = +\infty,$$

we find

$$\lim_{n \to +\infty} \left[ g(r, u_n) - g(r, \tilde{u}_n) \right] = \lim_{n \to +\infty} \int_{\tilde{u}_n}^{u_n} \left[ 1 - \exp(-2ru) \right] \left[ 1 + \{\exp(u)\} \right] \sin(\tau u) du.$$

where the $C^\infty$ and $\frac{2\pi}{\tau}$-periodic function $s \mapsto g(r, s)$ is defined as

$$g(r, s) := \frac{1}{2} \Im \left[ \frac{\tau - ix_r}{\tau(x_r + i \tau)} \exp(-i \tau s) \right], \forall s \geq 0.$$

Contradiction. We deduce then Equation (11) and (10).

Now, for every $n \in \mathbb{N}$, let $t \mapsto \Delta(r, n, t)$ be the continuous function satisfying the following differential equation,

$$\frac{d}{dt} \Delta(r, n, t) = x_r \Delta(r, n, t) - \left[ 1 + \{\exp(t)\} \right] \sin(\tau t),$$

$$\Delta(r, n, v_n) = \theta(r, v_n), \quad t \in A, \ t \geq v_n. \quad (13)$$

Denote $\eta(r, n, t) := \theta(r, t) - \Delta(r, n, t)$. By Equations (7) and (13),

$$\frac{d}{dt} \eta(r, n, t) = x_r \eta(r, n, t) + \exp(-2rt) \left[ 1 + \{\exp(t)\} \right] \sin(\tau t), \ \eta(r, n, v_n) = 0.$$
where \( t \in A \) and \( t \geq v_n \). Since \( t \mapsto \eta_r(x, n, t) \) is a continuous function, integrate,

\[
|\eta_r(x, n, t)| \leq 2 \exp(-2v_n) \exp(x_r t) \int_{v_n}^t \exp(-x_r s) ds = \frac{2}{x_r} \exp(-2v_n) \left[ \exp(x_r(t - v_n)) - 1 \right].
\]

For \( s_n \in [v_n, v_n + \frac{1}{x_r} 2v_n] \), we get

\[
\sup_{n \geq 0} |\eta_r(x, n, s_n)| \leq \frac{2}{x_r}.
\]

By Equation (9), we have \( \lim_{n \to +\infty} v_n = +\infty \). Then we can choose \((s_n)_n\) such that

- \( \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} s_n - v_n = +\infty \),
- \( s_n \leq 2v_n \).

By Equations (10), (13) and by Lemma 4 we obtain

\[
\lim_{n \to +\infty} |\Delta_r(r, n, s_n)| = +\infty.
\]

Since \( \eta_r(x, n, t) := \theta_r(r, t) - \Delta_r(r, n, t) \), thanks to Equation (14) we find,

\[
\lim_{n \to +\infty} |\theta_r(r, t_n)| = +\infty.
\]

Contradiction with the hypothesis (8). \( \square \)

**Proof of Proposition 2.** Equation (2), implies that \( \forall t \geq 0 \) we have

\[
\phi_r(\frac{1}{2} + r, t) = - \int_0^t \exp(-\frac{1}{2} u) \left[ \exp(ru) - \exp(-ru) \{\exp(u)\} \right] \sin(\tau u) du,
\]

\[
\phi_r(\frac{1}{2} - r, t) = - \int_0^t \exp(-\frac{1}{2} u) \left[ \exp(-ru) - \exp(ru) \{\exp(u)\} \right] \sin(\tau u) du.
\]

Denote

\[
y_r(r, t) := \exp\left(\frac{1}{2} t\right) \left[ \phi_r\left(\frac{1}{2} + r, t\right) - \phi_r\left(\frac{1}{2} - r, t\right) \right], \quad \forall r \in \left(0, \frac{1}{2}\right), \forall t \geq 0.
\]
Then

\[ y_\tau(r, t) = -2 \int_0^t \exp\left(\frac{1}{2}(t - u)\right) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du. \quad (15) \]

The function \( t \mapsto y_\tau(r, t) \) is a continuous function and satisfies the following differential equation

\[
\frac{d}{dt} y_\tau(r, t) = \frac{1}{2} y_\tau(r, t) - 2 \sinh(rt)[1 + \{\exp(t)\}] \sin(\tau t), \\
y_\tau(r, 0) = 0, \ \forall t \in A. \quad (16)
\]

By contradiction, suppose that there exists \( r \in (0, \frac{1}{2}) \) such that

\[ \phi_\tau\left(\frac{1}{2} - r\right) = \phi_\tau\left(\frac{1}{2} + r\right). \]

Then

\[ \int_0^{+\infty} \exp(-\frac{1}{2} u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du = 0, \]

implies

\[
\int_0^t \exp(-\frac{1}{2} u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du \\
= - \int_t^{+\infty} \exp(-\frac{1}{2} u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du.
\]

Then

\[
2\left| \int_0^t \exp(-\frac{1}{2} u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du \right| \\
\leq \frac{2}{\frac{1}{2} - r} \exp(-\frac{1}{2} - r)t + \frac{2}{\frac{1}{2} + r} \exp(-\frac{1}{2} + r)t, \ \forall t \geq 0.
\]

Denote \( \theta_\tau(r, t) := \exp(-rt)y_\tau(r, t) \). By the last Inequality and Equation (15), we get

\[
\sup_{t \geq 0} |\theta_\tau(r, t)| = \sup_{t \geq 0} |\exp(-rt)y_\tau(r, t)| \leq \frac{2}{\frac{1}{2} - r^2}. \quad (17)
\]

By Equation (16), The function \( t \mapsto \theta_\tau(r, t) \) is a continuous function satisfies \( \theta_\tau(r, 0) = 0 \) and satisfies the following differential equation

\[
\frac{d}{dt} \theta_\tau(r, t) = \left(\frac{1}{2} - r\right) \theta_\tau(r, t) - [1 - \exp(-2rt)][1 + \{\exp(t)\}] \sin(\tau t), \ t \in A.
\]
By Lemma [5] we have $\sup_{t \geq 0} |\theta_{\tau}(r, t)| = +\infty$. Contradiction with Equation (17). Then
$$\forall r \in (0, \frac{1}{2}): \phi_{\tau}(\frac{1}{2} - r) \neq \phi_{\tau}(\frac{1}{2} + r).$$

3 Proof of the Main result

Proof of the Main result. Suppose that there exist $\tau \in \mathbb{R}^*$ and $r \in [0, \frac{1}{2})$ such that
$$\zeta(\frac{1}{2} - r + i\tau) = 0,$$
by Equation (3), we get
$$\phi_{\tau}(\frac{1}{2} - r) = 0.$$
Using the Riemann’s functional equation [1], page 13 Theorem 2.1], the non-trivial zeros of the function $\zeta$ are symmetric about the line $\Re(s) = \frac{1}{2}$. We get
$$\zeta(\frac{1}{2} + r + i\tau) = 0.$$
By Equation (3) we obtain
$$\phi_{\tau}(\frac{1}{2} + r) = 0.$$
By Proposition [2] we obtain $r = \frac{1}{2}$. \hfill \Box

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).