A non-perturbative study of the evolution of cosmic magnetised sources

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Abstract We undertake a hydrodynamical study of a magnetised cosmic fluid between the end of the leptonic era and the beginning of the radiation-dominated epoch. We assume this fluid to be the source of a Bianchi I model and to be a mixture of tightly coupled primordial radiation, neutrinos, baryons, electrons and positrons, together with a gas of already decoupled dark matter WIMPS and an already existing magnetic field. The interaction of this field with the tightly coupled gas mixture is described by suitable equations of state that are appropriate for the particle species of the mixture. Comparison of our results with those of previous studies based on an FLRW framework reveals that the effects of the anisotropy of the magnetic field on the evolution of the main thermodynamical variables are negligible, thus validating these studies, though subtle differences are found in the evolution of the magnetic field itself. For larger field intensities we find quantitative and qualitative differences from the FLRW based analysis. Our approach and our results may provide interesting guidelines in potential situations in which non–perturbative methods are required to study the interaction between magnetic fields and the cosmic fluid.

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1 Introduction

A full account of the origin and evolution of cosmic and astrophysical magnetic fields is a long standing problem in contemporary Theoretical Physics. Magnetic fields have been observed in all scales: our own galaxy [1], low [2] and high redshift [3] galaxies, and up to scales of galaxy clusters [4] and superclusters [5] (for comprehensive reviews see [6]). There is also indirect evidence, from gamma-ray observations of blazars [7], of coherent intergalactic magnetic fields in low density regions. While magnetic fields can result as a consequence of a variety of local astrophysical processes involving electric charges in motion (accretion into AGN’s or compact objects, ionization of intergalactic gas and interactions between baryons and CMB photons), all this observational evidence of their presence on all scales supports the hypothesis that (at least) some of these magnetic fields may have a primordial origin (see topic review in [8]). Evidently, in order to probe this possibility, we need to undertake a study of cosmic evolution that includes a fundamental (and physically motivated) magnetic interaction that can be incorporated in the usual hydrodynamical approach to cosmic sources at early times before and up to cosmic nucleosynthesis.

There is a large amount of diverse literature on the magnetic interaction in a cosmological context:

– Highly specialised articles looking at magnetic fields on early cosmic times from the perspective of high energy particle physics. This type of articles (see eg [9,10,11] and references therein) incorporate the detail and complexity of specific particle interactions, but lack a dynamical study of the evolution of these magnetic fields.

– Articles that fully incorporate classical Maxwellian magnetic fields (under the assumption of infinite conductivity [12]) in the dynamics of cosmic evolution in the context of General Relativity:
– An older literature [13,14,15] (see review in [16]) deals with solutions of Einstein’s equations for extremely idealised fluids (even dust) that are (somehow) coupled to the Maxwellian field, all of them taken as sources for the homogeneous but anisotropic Bianchi I model, which provides the simplest spacetime geometry compatible with the pressure anisotropy that necessarily arises when taking magnetic field as a source.
– A more recent body of literature [17,18,19,20,21] looks at more realistic magnetic Maxwellian fields in terms of a covariant formalism (the “1+3” formalism) that allows for non-perturbative and perturbative approaches, the latter in terms of a gauge invariant and covariant perturbations on an FLRW or Bianchi I backgrounds.
– Other authors \[8,22,23\] (see update in \[24\]) have examined an early
time magnetised fluid in which a “frozen” magnetic field appears in
a purely FLRW context. This can be justified from the high electric
conductivity of the cosmic fluid and the fact that anisotropic effects in
the spacetime geometry of early cosmic times are negligible.

Besides the methodological approach one may follow, it is very important in
any study of cosmic magnetised sources to place appropriate observational
bounds on magnetic fields. In earlier work these bounds were based on the
limits of the abundances of small anisotropic stresses from the COBE four-
year anisotropy data \[25,26\]. Lately, Bianchi I models have received renewed
attention \[27,28,29\], since present observational bounds from Planck \[30,31,
32\] and WMAP \[33\] cannot rule out (in principle) the possibility that the large
scale structure of the Universe is not perfectly isotropic from a statistical point
of view.

In the present article we fill a niche not covered by the previous literature
summarised above:
– instead of idealised unphysical fluid sources for the Bianchi I model (as
in \[13,14,15\]), we consider as field source a magnetic field together with
a more realistic mixture gas of baryons (neutrons and protons), leptons
(neutrinos, antineutrinos, electrons and positrons), dark matter WIMPS
and photons.
– instead of assuming a purely FLRW geometry (as in \[8,22,23,24\]), we con-
sider the full anisotropy of the magnetic field and the Bianchi I geometry
through a non–perturbative approach based on Einstein–Maxwell equa-
tions.
– instead of considering only strictly Maxwellian fields in the infinite con-
ductivity approximation (as in \[12,17,18,19,20,21\]), we consider physically
motivated equations of state for magnetised gases.

By assuming a Bianchi I geometry that becomes spatially flat FLRW when
the magnetic field vanishes, the magnetic field direction can always be locally
described as a uniform field directed along a space axis (say, the z-axis). Ev-
dently, this field breaks the local FLRW isotropy and introduces anisotropic
momentum fluxes that modify the energy–momentum tensor and thus affect
the evolution of the state variables. Hence, the anisotropic pressure terms asso-
ciated with the magnetic field necessarily contain a (“pure”) classical Maxwell
term \[34\], but must also modify the equations of state of the fluid sources. In
this context, is important to emphasise that we are considering realistic fluid
sources interacting with a magnetic field, namely: an ideal gas of magnetised
fermions \[35,36,37\] whose equation of state reflects the full anisotropic ef-
facts of the magnetic field. We remark that in previous work we have studied
the gravitational collapse of such magnetised sources in a Bianchi I geometry,
considering the case of zero \[38,39,40\] and finite temperatures \[41\].

In order to examine the hydrodynamical evolution of the magnetised fluid
mixture described above, we consider two separate cosmic epochs where the
effect of the magnetic field can be particularly relevant for a Universe full of
free electric charges: (i) the end of the leptonic era, and: (ii) the beginning of the radiation dominated era immediately before cosmic nucleosynthesis. Although all fermions, neutral and charged, interact with the magnetic field, we will assume an equation of state in which the contribution of electrons and positrons is dominant. This is a good approximation, since in the cosmic times we are interested the constraint $eB \ll m^2_{p,n}$ holds and the interaction of the magnetic field with protons and neutrons can be neglected from the statistical properties of these particles.

Besides introducing anisotropic stresses, primordial magnetic fields lead to various important effects on the evolution of early Universe sources. Hence, stringent bounds need to be imposed on their field strength. In particular, we are concerned on the effects of the magnetic field on cosmic nucleosynthesis, since it is the cosmological event in the cosmic stages under examination that is “closest” to us. Magnetic fields lead to various different effects on cosmic nucleosynthesis: (i) their contribution to the energy density content of the cosmic fluid affects the expansion rate; (ii) the electron-positron quantum statistics is modified, and so is the rate of neutron beta decay (see [8] for a review). Considering these effects together, an upper bound of $\langle B_0 \rangle \leq 3 \times 10^{-7} \text{G}$ is necessary at length scales of the order of the Hubble horizon size at BBN time [22,23]. An updated value of this bound is: $\langle B_0 \rangle \leq 1.5 \times 10^{-6} \text{G}$ (this is related to the local field amplitude $B$ contributed from all wavelengths) [24]. These bounds were obtained without considering the anisotropy introduced by the magnetic field. Without neglecting the importance of observational bounds, we will also consider strong magnetic fields that violate these bounds in order to highlight their effects. The obtained results may be useful in potential situations in which strong cosmic magnetic field need to be examined in a non-perturbative manner.

The paper is organised as follows: in Sec. 2 we examine the Einstein–Maxwell field equations that govern the dynamics of a magnetised cosmological fluid mixture in a Bianchi I geometry. The fluid sources and equations of state for the constituents of this magnetised fluid are discussed in detail in Sec. 3. The Einstein–Maxwell system is transformed into a first order system given in terms of dimensionless variables in Sec. 4. The perfect fluid and FLRW limits are examined in Sec. 5. In Sec. 6 we report our numerical results for the evolution of relevant thermodynamical and dynamical parameters. Finally, we provide our conclusions in Sec. 7.

2. Einstein–Maxwell equations for a Bianchi I model

Bianchi I models are among the simplest spacetime geometries compatible with the inherent anisotropy of a magnetic field. The latter can be introduced as a source without considering moving electric charges and currents under the infinite conductivity approximation [12], which is well justified given the high electric conductivity of the cosmic fluid before nucleosynthesis [8].
While Bianchi I models involve a great deal of idealisation, their exact spatial flatness and homogeneity allow for well motivated, fully general relativistic and non-perturbative, study of post-inflationary radiation dominated conditions characterised by small deviations from spatial flatness and homogeneity. In particular, the initial conditions that determine the parameters of the models can always be selected in order to “control” their deviation from isotropy to comply with the desired bounds.

Considering rectangular comoving coordinates \((t, x, y, z)\) with \(x, y, z\) aligned along the principal axes of the shear tensor, Bianchi I models are described by the Kasner metric

\[
ds^2 = -dt^2 + a_1^2(t)\,dx^2 + a_2^2(t)\,dy^2 + a_3^2(t)\,dz^2.
\]

In order to describe a matter-energy source that includes magnetic interaction for this metric we consider the following energy–momentum tensor

\[
T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + \Pi_{\mu\nu},
\]

where \(\rho = \rho(t), p = p(t)\) and \(\Pi_{\mu\nu}\) are, respectively, the energy density, isotropic pressure and the traceless spacelike symmetric part of the stress tensor (a nonzero \(\Pi_{\mu\nu}\) is necessary to describe a magnetic field, see equations (16) and (17) further ahead). In a comoving frame with 4-velocity \(u^\alpha = \delta^\alpha_t\) and considering a magnetic field locally aligned in the \(z\) direction, this stress-energy tensor takes the diagonal form:

\[
T^\nu_{\nu} = \text{diag} \left[ -\rho, P_\perp, P_\perp, P_\parallel \right],
\]

where \(P_\perp = P_\perp(t)\) and \(P_\parallel = P_\parallel(t)\) are the pressures (i.e. eigenvalues of the tensor \(\Pi_{\mu\nu}\)) in the directions perpendicular and parallel to the magnetic field in the frame defined by the coordinates \((t, x, y, z)\).

The dynamics of a cosmological fluid allowing for a magnetic interaction follows from the coupled Einstein-Maxwell field equations \((G_{\mu\nu} = \kappa T_{\mu\nu})\), which for the Bianchi I geometry \([1]\) take the form:

\[
-G^x_x = \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} = -\kappa P_\perp,
\]

\[
-G^y_y = \frac{\dot{a}_1 \dot{a}_3}{a_1 a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_3}{a_3} = -\kappa P_\perp,
\]

\[
-G^z_z = \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} = -\kappa P_\parallel,
\]

\[
-G^t_t = \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\dot{a}_3}{a_1 a_3} + \frac{\ddot{a}_3}{a_2 a_3} = \kappa \rho.
\]

where \(\dot{a} = u^\alpha a_\alpha = a_t\). The conservation equation \(T^{\mu\nu;\nu} = 0\) yields:

\[
\dot{\rho} = - \left( \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} \right) (P_\perp + \rho) - \frac{\dot{a}_3}{a_3} (P_\parallel + \rho),
\]

\(^1\) Unless specified otherwise, we use natural units.
together with the only non-trivial of Maxwell’s equations ($F^\mu{}_{\nu} = 0$ and $F[\mu\nu;\alpha] = 0$):

$$\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{B}}{B} = 0.$$  \hspace{1cm} (9)

Notice that $B$ is not a scalar. It can be characterised as the single nonzero component of the covariant magnetic field vector in the comoving frame: $B_\mu = B\delta_\mu^z = (1/2)\eta_{\mu\alpha\beta} F^{\alpha\beta}$, where $\delta_\mu^z$ and $\eta_{\mu\alpha\beta}$ are the Kronecker delta and Levi-Civita tensors.

3 The field sources

We will assume as the field sources of the Bianchi I model (i) a tightly coupled gas mixture of leptons, baryons and photons, (ii) an already decoupled non-interacting gas of cold dark matter WIMPS whose contribution is basically the rest-mass energy density (whose abundance value can be assumed to be very subdominant: $\Omega_{\text{CDM}} \sim 10^{-6}$), (iii) an already existing magnetic field $B$. We will follow a purely phenomenological approach to this magnetic field, assuming that it emerged from previous fundamental processes whose details are beyond the scope of this paper (see comprehensive discussion on this issue in [8]). We will also assume that the magnetic interaction is only significant with some of the particle species in (i). The total energy density and pressures for the source (2)–(3) have the form:

$$\rho = \rho^B + \rho^\gamma + \rho^\nu + \rho^{\text{CDM}} + \sum_k \rho^k,$$  \hspace{1cm} (10)

$$P_\perp = P_\perp^B + P^\gamma + P^\nu + \sum_k P^k_\perp,$$  \hspace{1cm} (11)

$$P_\parallel = P_\parallel^B + P^\gamma + P^\nu + \sum_k P^k_\parallel,$$  \hspace{1cm} (12)

where $k$ runs over all particles that could interact with the magnetic field, and the upper indices $B$, $\gamma$ and $\nu$ respectively denote the density and pressures associated with the magnetic field, the photons and the neutrinos.

Because of the symmetries of the Bianchi I model the magnetic field in a comoving frame is necessarily homogeneous (i.e. purely time dependent) and (as we have chosen it) to be locally oriented along the $z$-direction. The magnetic density and pressures are then

$$P_\perp^B = -P_\parallel^B = B^2/(8\pi) = \rho^B,$$  \hspace{1cm} (13)

leading to a positive pressure term along the $x$ and $y$-axes and a negative pressure along the field direction. This negative pressure can be interpreted as a tension or elasticity of the field lines, which tend to remain as “straight” as possible by reacting to any effect that distorts them [42,43]. Consequently, an anisotropic pressure must give rise to an anisotropic expansion law.
Photons contribute to the source with an isotropic pressure and energy density given by the equation of state

\[ p^\gamma(T) = \rho^\gamma(T)/3, \quad \rho^\gamma(T) = a_6 T^4, \tag{14} \]

where \( a_6 = \frac{\pi^2}{15} \) and \( T \) is the temperature. On the other hand the isotropic pressure and energy density of neutral leptons (neutrinos and antineutrinos) are given by

\[ p^\nu(T) = \rho^\nu(T)/3, \quad \rho^\nu(T) = 7g' a_6 T^4/16, \tag{15} \]

with \( g' \) equals to \( 2 \times 3 \), hence we have taken into account the three different species of neutrinos and antineutrinos.

It is safe to neglect the interaction of photons, neutrinos and WIMPs with the magnetic field. However, leptons and baryons could interact with the field through their charges (if they are charged) and via their anomalous magnetic moment (if they are neutral). In any case the interactions with the field (via charges or anomalous magnetic moment) lead to a momentum–energy tensor with anisotropic stresses \[35\]-\[37\]. We assume hereafter that the only particle species interacting with the magnetic field are electrons and positrons (charged leptons), protons (charged baryons) and neutrons (neutral baryons).

In general the equation of state for these species in presence of a time–dependent magnetic field can be written as follows \[35\]-\[37\]-\[45\]:

\[
\begin{align*}
    p^k_\perp &= -\Omega^k - BM^k, \\
    p^k_\parallel &= -\Omega^k, \\
    \rho^k &= -\Omega^k + TS^k + \mu_k N^k. 
\end{align*} \tag{16-18} 
\]

where the upper index \( k \) denotes generically the electron, positron, proton and neutron, \( M^k = -\left(\partial \Omega^k / \partial B\right) \) is the magnetisation, \( S^k = -\left(\partial \Omega^k / \partial T\right) \) is the entropy, \( N^k = -\left(\partial \Omega^k / \partial \mu_k\right) \) is the particle number density (with \( \mu_k \) the chemical potential), and \( \Omega^k \) the thermodynamical potential which has two contributions:

\[ \Omega^k = \Omega_{\text{SQFT}}^k(B, \mu_k, T) + \Omega_{\text{QFT}}^k(B), \tag{19} \]

where \( \Omega_{\text{SQFT}}^k \) is the statistical Quantum Field Theory contribution and \( \Omega_{\text{QFT}}^k \) (which does not depend on the temperature and the chemical potential) is the well-known Quantum Field Theory vacuum term given by \[2\]

\[ \Omega_{\text{QFT}}^k(B) = -\frac{1}{4\pi^2} \sum_{\eta=1,-1,-\infty} \int_{-\infty}^{\infty} dp \, d^2 p_\perp \varepsilon_k, \tag{20} \]

\[2\] See definitions in \[37\]. The term \( \Omega_{\text{QFT}}^k \) has non-field-dependent ultraviolet divergencies, after renormalisation the Schwinger expression is obtained \[35\].
and the statistical term:

\[ \Omega_k^{SQFT}(B, T, \mu) = -\frac{1}{4\pi^2\beta} \sum_{\eta = 1, -1} \int_{-\infty}^{\infty} dp_\parallel dp_\perp \ln \left( \left( 1 + e^{-\beta(\varepsilon_k - \mu)} \right) \left( 1 + e^{-\beta(\varepsilon_k + \mu)} \right) \right). \]  

(21)

In the previous equations \( \beta = 1/T, \eta = 1, -1 \) correspond to the two orientations of the magnetic moment (parallel and antiparallel) with respect to the field, while \( \varepsilon_k \) is the spectrum of the fermions given by:

\[ \varepsilon_k = \begin{cases} \sqrt{p_\parallel^2 + 2|eB|l + m_k^2} & \text{Charged fermions}, \\ \sqrt{p_\parallel^2 + \left( \sqrt{p_\perp^2 + m_k^2 + \eta q B} \right)^2} & \text{Neutral fermions}. \end{cases} \]  

(22)

In the equations above \( m_k \) denotes the fermion mass and \( q \) is the anomalous magnetic moment. For charged fermions we need to carry on the following substitution:

\[ \int \frac{d^2 p_\perp}{(2\pi)^2} \rightarrow |eB| \frac{2}{2\pi} \sum_{l = 0}^{\infty} (2 - \delta_{l0}), \]  

(23)

where \( d(l) = 2 - \delta_{l0} \) is the spin degeneracy of Landau levels with \( l \neq 0 \). For electron–positron pairs we assume a negligible chemical potential (\( \mu = 0 \)). Hence, the last term on the right-hand side of equation (18) vanishes.

For the temperature of the full duration of the stage of cosmic evolution we are interested the protons and neutrons satisfy \( eB \sim T^2 \ll m_{p,n}^2 \), so they contribute to the evolution mostly through their rest energy. Therefore, we assume that the only particles that contribute to the anisotropy in the pressures are the electrons and positrons.

### 4 Einstein–Maxwell equations as a first order system

For a hydrodynamical numeric framework it is necessary to transform the second order Einstein–Maxwell equations (4)–(9) into a first order system of evolution equations for local kinematic covariant objects (see A) and the relevant state variables. We also need to introduce the following dimensionless evolution parameters: \( \tau, \mathcal{H}, S_2 \) and \( S_3 \) given by

\[ \frac{d}{d\tau} = \frac{1}{H_*} \frac{d}{dt}, \quad \tau = H_* (t - t_0), \]  

(24)

\[ \mathcal{H} = \theta/3, \quad S_2 = \frac{\Sigma_2}{H_*}, \quad S_3 = \frac{\Sigma_3}{H_*}, \]  

(25)

where \( H_* \) is an inverse length defined by the condition \( 3H_*^2 = 8\pi G\lambda/c^4 \) and \( \theta = u^{\mu}_{\mu} \) is the expansion scalar, \( \Sigma_2, \Sigma_3 \) are the eigenvalues of the shear.
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Notice that $H_\star$ corresponds to the Hubble length in the outset of the radiation epoch (just before nucleosynthesis).

We also define for generic magnetised gas mixtures the following dimensionless variables

$$\mathcal{E} = \frac{\rho}{\lambda}, \quad \mathcal{P}_\perp = \frac{P_\perp}{\lambda}, \quad \mathcal{P}_\parallel = \frac{P_\parallel}{\lambda},$$

$$B = \frac{B}{B_c}, \quad T = \frac{T}{m_e},$$

where $\rho, P_\perp, P_\parallel$ are the total energy density and anisotropic pressures in (10)–(12),

$$\lambda = \frac{m_e}{\pi^2 \lambda_c^3} \left( \frac{\pi^2 \lambda_c^3}{\lambda^3} \right),$$

with $\lambda_c$ the Compton wavelength and $m_e$ the electron mass, $B_c = m_e^2/e = 4.41 \times 10^{13} G$ is the critical magnetic field for electrons.

Considering the variables (24)–(27), the Einstein-Maxwell field equations become the following first order system:

$$H_{\tau} = -\frac{1}{2} \left( 3\mathcal{E} + 2\mathcal{P}_\perp + \mathcal{P}_\parallel \right) - \frac{1}{2} \left( (S_2 + S_3)^2 - S_2 S_3 \right),$$

$$S_{2,\tau} = \mathcal{P}_\perp - \mathcal{P}_\parallel - 3\mathcal{H} S_2,$$

$$S_{3,\tau} = 2 \left( \mathcal{P}_\parallel - \mathcal{P}_\perp \right) - 3\mathcal{H} S_3,$$

$$B_{\tau} = B \left( S_3 - 2\mathcal{H} \right),$$

$$T_{\tau} = -\frac{1}{E_T} \left[ (2\mathcal{H} - S_3) \left( \mathcal{E} + \mathcal{P}_\perp \right) + (\mathcal{H} + S_3) \left( \mathcal{E} + \mathcal{P}_\parallel \right) - \mathcal{E} B B \left( S_3 - 2\mathcal{H} \right) \right],$$

together with the Hamiltonian constraint

$$3\mathcal{E} = -S_2^2 - S_3^2 - S_2 S_3 + 3\mathcal{H}^2,$$

where $E_T = \partial E/\partial T$ and $E_B = \partial E/\partial B$ in (32).

Since we have provided in (13)–(23) the equations of state for all the constituents of the total energy density and pressures in (10)–(12), we can now provide constraints that link $\mathcal{E}, B, P_\parallel, P_\perp$ (or $\rho, B, P_\parallel, P_\perp$), the differential equations (28)–(32) and the constraint (33) become a complete and self-consistent system whose numerical integration allows us to examine the dynamical evolution of a magnetised Universe in the cosmic stages we are concerned. Since all thermodynamical functions depend only on $B$ and $T$ (dimensionless magnetic field and temperature, respectively), from the numerical solutions for these two variables we can obtain the thermodynamical functions. On the other hand, the solutions for $S_2, S_3$ and $\mathcal{H}$ provide the necessary information to study the kinematical evolution of the cosmic fluid. The local proper volume can be expressed in terms of $\mathcal{H}$ as follows:

$$V(\tau) = V(\tau_i) \exp \left( 3 \int_{\tau_i}^\tau \mathcal{H} d\tau \right),$$

\(^3\) The critical magnetic field for an electron as defined above is the strength at which electron cyclotron energy equals its rest energy.
and the expression for metric coefficients reads:

\[
a_a(\tau) = a_a(\tau_i) \exp \left( \int_{\tau_i}^{\tau} (S_a + \mathcal{H}) \, d\tau \right), \quad a = 1, 2, 3
\]  

with \( S_1 = -(S_2 + S_3) \) (since the shear tensor is trace-free).

5 The FLRW and perfect fluid limits

If the magnetic field vanishes (\textit{i.e.} \( B \to 0 \)) we have \( P_\parallel = P_\perp \) and thus \( \Pi_{\mu\nu} \to 0 \) (from [3], [13] and [16]–[17]). As a consequence, (2) reduces to the momentum-energy tensor of a perfect fluid with isotropic pressure:

\[
T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}, \quad P = P_\parallel = P_\perp.
\]  

However, an isotropic pressure does not imply an isotropic geometry (\textit{i.e.} an FLRW geometry), as the energy–momentum tensor (36) can still be compatible with the inherent anisotropy of the Bianchi I geometry that is present in the different fluid expansion rates in [49] and the nonzero shear tensor in [50] and [51] (see Appendix A). Hence, a zero magnetic field in a Bianchi I model only leads to an FLRW geometry if besides the magnetic field the shear tensor vanishes as well.

The conditions for an evolution with zero or nonzero shear follow readily from the evolution equations (29)–(30), which if the source is a perfect fluid (\( P_\perp = P_\parallel \) from [26]) can be integrated formally as

\[
S_1(\tau) = S_1(\tau_i) \exp \left( -3 \int \mathcal{H} \, d\tau \right), \quad S_2(\tau) = S_2(\tau_i) \exp \left( -3 \int \mathcal{H} \, d\tau \right),
\]  

with \( S_3 = -(S_1 + S_2) \). Hence, when \( B = 0 \) and the energy–momentum tensor becomes (36) we can identify the following two possibilities:

- Evolution with nonzero shear. If the initial values \( S_a(\tau_i) \) for \( a = 1, 2, 3 \) of (at least) one of the shear eigenvalues is nonzero the shear tensor is nonzero for all \( \tau > \tau_i \). Notice that (for example) \( S_2(\tau_i) = 0 \) implies \( S_3(\tau_i) = -S_1(\tau_i) \), which is nonzero in general.
- Evolution with zero shear: if any two of the initial values \( S_a(\tau_i) \) (for \( a = 1, 2, 3 \)) vanishes, then \( S_1 = S_2 = S_3 = 0 \) holds for all \( \tau > \tau_i \). The shear tensor vanishes.

We examine below these two limit cases for a zero magnetic field and the possibility of studying the latter in a purely FLRW context.

5.1 Perfect fluid Bianchi I limit

If \( B = 0 \) but the shear tensor is nonzero \( (S_a(\tau_i) \neq 0 \) for at least one of the \( S_a) \) we have a Bianchi I model whose source is the same particle mixture described
in section 3 but without the magnetic interaction. This source is characterised by the equations of state (13), (16)–(18) with $B = 0$, leading for each particle species to

$$P^k = P^k_\perp = P^k_\parallel = -\Omega^k, \quad \rho^k = -\Omega^k + TS + \mu N,$$

where now the statistical thermodynamical potential $\Omega^k = \Omega^k_{\text{SQFT}}(\mu, T)$ in (21) does not depend on $B$. The remaining equation (20)–(21) remain valid with $B = 0$. The dynamics follows from Einstein’s equations, which is the system (28)–(32) and (33) with $P^B = P^B_\parallel = P^B_\perp = 0$ and without the Maxwell part: i.e. without (31) since $B = 0$.

### 5.2 FLRW limit model

If $B = 0$ and the shear tensor vanishes ($S_a(\tau_i) = 0$ for at least two of $S_a$), then $P^\perp = P^\parallel = P$ and $S_a = 0$ hold for all $a = 1, 2, 3$, hence $a_1 = a_2 = a_3 = a$. It is straightforward to show that (4)–(8) reduce to the well known spatially flat FLRW equations:

$$\ddot{a}^2 + \frac{2\dot{a}}{a} = -\kappa P, \quad \ddot{\eta} = \frac{\kappa}{3} \rho, \quad \dot{\rho} = -3(\rho + P)\dot{a},$$

or, in dimensionless form for the particle mixture:

$$\mathcal{H}_\tau = -\frac{3}{2} (\mathcal{E} + P), \quad T_{\tau} = -\frac{3\mathcal{H}(\mathcal{E} + P)}{\mathcal{E}_\tau}, \quad \mathcal{H}^2 = \mathcal{E},$$

where $P \equiv P/\lambda$.

### 5.3 Magnetic fields in an FLRW context

While magnetic fields are incompatible with a non-perturbed FLRW Universe, the high electric conductivity of the cosmic fluid after recombination [8] suggests considering the infinite conductivity approximation [12] to introduce the magnetic interaction through suitable vector field perturbations on an FLRW background. Such a perturbative approach is fully justified if the magnetic field is tangled on scales smaller than the Hubble horizon [12] and has lead to a comprehensive literature [18,19,20,21] in which the perturbations are covariant and gauge invariant.

However, given the fact that anisotropic effects in early times cosmic expansion are negligible, other authors [11,12,13,24] have avoided the usage of a full gauge invariant perturbative approach by considering a “frozen” magnetic field in a purely FLRW context. This approximation is further justified by arguing that the effects of the magnetic field on the spacetime geometry are negligible if this field is “not too tangled on scales smaller than the magnetic
dissipation scale” [49] (see detail in [8]). Under these assumptions, the magnetic field is introduced in [8,22,23,24] as a relativistic scalar correction to the energy density and the isotropic pressure of a perfect fluid particle mixture considered as source of an FLRW metric (see equations (3.26)–(3.29) of [8]). While the energy density of this perfect fluid fully coincides with our total energy density in equation (10), we consider anisotropic pressures in order to be consistent with the full non–perturbative treatment based on Einstein–Maxwell field equations. Under the approach of [8,22,23,24] the conservation of the frozen magnetic flux yields:

\[ \frac{\dot{B}}{B} + \frac{2\dot{a}}{a} = 0 \quad \Rightarrow \quad B(t) = \frac{B(t_i)}{a^2(t)}, \tag{42} \]

which can be compared with the exact equation (9) of the Einstein–Maxwell system obtained in a Bianchi I Universe (or equation (31) in dimensionless variables):

\[ \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{B}}{B} = 0 \quad \Rightarrow \quad B(t) = \frac{B(t_i)}{a_1(t)a_2(t)}. \tag{43} \]

While (42) can be regarded as an approximation of (43) if anisotropy is negligible (so that \(a_1 \approx a_2\)), our approach is different from that of [8,22,23,24]: we regard the Maxwell equation (43) as part of the coupled Einstein–Maxwell system in the Bianchi I geometry, whereas the flux conservation (42) in [8,22,23,24] merely provides a subsidiary condition for Einstein’s equations in the FLRW metric and \(B\) is taken as a scalar (which is not correct: it is the component of a frame–dependent magnetic field vector). Another important difference with respect to [8,22,23,24] is that these authors simply assume the FLRW radiation temperature for the magnetised mixture, whereas in our Einstein–Maxwell system this temperature needs to be obtained from the evolution equation

\[ \dot{T} = \frac{1}{\rho T} \left[ -\left( \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} \right) (P_\bot + \rho) - \frac{\dot{a}_3}{a_3} (P_\| + \rho) + \left( \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} \right) B\rho B \right], \tag{44} \]

which is coupled with the magnetic field and expressed in terms of dimensionless variables (26)–(27) becomes equation (32). However, if we assume an FLRW geometry \(a = a_1 = a_2 = a_3\), a perfect fluid source \(P_\bot = P_\|\) and the radiation equation of state, then (44) becomes the FLRW evolution equation for the radiation temperature: \(\dot{T} = -T\dot{a}/a\). Nevertheless, for very weak magnetic fields (32) should approximate this radiation temperature evolution law.

Therefore, the results of [8,22,23,24] can always be obtained from our Bianchi I based results if we assume the same approximations considered by these authors. While these approximations may be well justified for weak fields in an early time cosmic mixture, our non–perturbative approach allows us to examine magnetised cosmic fluids also when such assumptions cannot be justified, as we consider the coupled Einstein–Maxwell system that takes into full account the spacetime anisotropy, its effects on the energy density and pressures and the vectorial (frame–dependent) nature of the magnetic field.
6 Numerical results

6.1 Stages of cosmic evolution

We consider for the numerical study of the dynamics of a magnetised Universe the following two periods of the cosmic evolution before the nucleosynthesis:

- Epoch I: End of the leptonic era: \(100 \text{ MeV} > T > m_e \) (\(200 \gtrsim T > 1\))
  The constituents of the cosmic fluid were leptons (neutrinos-antineutrinos and electrons-positrons), baryons (neutrons and protons), photons and cold dark matter. We will assume besides these particles a homogeneous (time dependent) magnetic field. At \(T \approx 1 \text{ MeV}\) the neutrinos decouple, however for temperature values such that \(T > m_e\), \((m_e \approx 0.5 \text{ MeV})\), photons, neutrinos and electron-positrons continue with a unique temperature \(T_{e^\pm} = T_\gamma = T_\nu \equiv T\).

- Epoch II: Beginning of the radiation epoch: \(m_e > T > 0.06 \text{ MeV} \) (\(1 > T \gtrsim 0.12\)) \([44] - [47]\)
  When the temperature drops below \(m_e\), the electron-positron pairs are transformed into photons but not into decoupled neutrinos. After these annihilations the number of photons is therefore greater than the number of neutrinos. Since thermal equilibrium is maintained until very few electrons remain, the entropy of the photon-electron-positron system for \(T > m_e\) is nearly the entropy of the photon system for \(T < m_e\). Thus, after the electron-positron annihilation the photon and neutrino temperatures are related by:
  \[T_\nu = (4/11)^{1/3} T_\gamma.\] (45)

Since \(\tau \approx 0.029 \text{ s}^{-1} \times (t - t_i)\), the cosmic times for these epochs range from an initial time at the onset of the leptonic era \(t_i \approx 1 \text{ s} \Rightarrow \tau_i = 0\) at temperature \(T = 100 \text{ MeV}\), towards the final stage at the beginning of nucleosynthesis at \(t \approx 200 \text{ s} \Rightarrow \tau \approx 6\).

Having defined the cosmic eras we are interested in studying, we undertake in this section the numerical integration of the system (28)–(32) and (33) under initial conditions specified at an initial time \(\tau = \tau_i\) and following the assumptions summarised below:

- An initial temperature of \(T(\tau_i) \approx 200\) and ending our analysis when temperature drops below \(T(\tau_i) \approx 0.12\).
- Initial magnetic field is \(B(\tau_i) \sim T^2(\tau_i)\) (in Gauss \(B(t_i) \sim 5 \times 10^{17}\text{ G}\)).
- Initial shear is zero: \(S_1(\tau_i) = S_2(\tau_i) = S_3(\tau_i) = 0\). From (29)–(30) and (37) this choice implies that all the anisotropy introduced by a Bianchi I geometry can be ascribed exclusively to the magnetic interaction through \(P_\perp \neq P_\parallel\). In other words, we have a magnetised Bianchi Universe that approaches an FLRW as the magnetic field becomes weak or negligible, while the strict limit \(B = 0\) \((B = 0)\) yields a pure FLRW evolution (not a perfect fluid Bianchi I model). While the shear tensor in early cosmic times should be absolutely negligible but not strictly (mathematically) zero, the assumption \(S_a(\tau_i) = 0\) for \(a = 1, 2, 3\) is a reasonable approximation.
The proper volume as function of time for increasing values of the initial magnetic field. The panel in the left hand side shows that larger values of $B(\tau_i)$ produce faster expansion rates, while the curves of the panel in the right hand side show that the expansion rate of $V(\tau)$ is practically indistinguishable from an FLRW expansion if we consider values of $B(\tau_i)$ that fit observational constraints [23].

The magnetic field modifies cosmic dynamics by both, the pure magnetic field contribution (Maxwell term) and its inclusion in the statistical treatment of the electron-positron gas. While the Maxwell term is present during the whole evolution, the electron-positron gas is considered as a magnetised gas only during the leptonic era.

6.2 Kinematics

We examine in figure 1 the local proper volume for different initial values $B(\tau_i)$ of the magnetic field. The figure reveals a faster rate of expansion for larger $B(\tau_i)$. However, if we consider values of $B(\tau_i)$ compatible with observational bounds [23], then the volume expansion is practically coincident with the expansion rate of the FLRW model obtained when we set $B(\tau_i) = 0$ (see the grey line and the black triangles in the enclosed graph). In other words, the anisotropy effects on the volume expansion are completely negligible when the magnetic field complies with observational bounds. Since we are assuming zero initial shear, the magnitude of the initial magnetic field intensity $B(\tau_i)$ should determine the difference of the expansion rates for the three metric coefficients $a_1, a_2, a_3$. This can be appreciated in figure 2 which displays these functions for various values of $B(\tau_i)$, showing distinct curves for these metric functions (i.e. larger anisotropy) for larger values of $B(\tau_i)$ and an almost isotropic FLRW expansion (almost the same evolution for the three scale factors) for a weak field.
Fig. 2 Metric coefficients $a_1(\tau), a_2(\tau), a_3(\tau)$ for varied values of $B(\tau)$. Larger values of $B(\tau)$ lead to very different curves (large anisotropy) for the three scale factors, while smaller values lead to almost the same curves for them (almost isotropy).

6.3 Thermodynamics

The anisotropy inherent in the magnetic field can also be appreciated through the anisotropy of the pressures. The classical Maxwellian contribution $B^2$ produces a decelerated fluid expansion in the direction of the magnetic field ($z$), as the latter produces a negative pressure or tension in its direction, and an accelerated expansion in the directions perpendicular to it due to a positive pressure. However, we obtain the opposite effect from the statistical contribution of $B$ in the equation of state for the magnetised electron–positron gas mixture: the pressure in the direction parallel to the field increases and that in the perpendicular direction decreases [35,41,36,45]. Although the dominant effect in the dynamics comes from the contribution $B^2$, we can observe that including the magnetised electron–positron gas leads to a slight acceleration of the cosmic rate of expansion. In fact, the contribution of the electron–positron gas is of the order of $O(\alpha) \times B^2$ (where $\alpha$ is the constant of fine structure) [18], therefore this contribution will always be sub-dominant in comparison with that of the “pure” magnetic field $\sim B^2$.

The dominance of the Maxwellian contribution $B^2$ in the fluid expansion is consistent with the curves displayed in figure 2: the scale factor $a_3(\tau)$ (direction parallel to $B$) expands at a slower rate than $a_2(\tau)$ and $a_1(\tau)$ (directions perpendicular to $B$). This dominance is also displayed in figures 3 and 4 where we considered magnetic field values much larger than observational bounds to highlight this effect. Figure 3 shows that the pressure $P_\parallel$ is much smaller than $P_\perp$, while figure 4 depicts the growth of the volume expansion for different in-
Perpendicular and parallel pressures for each mixture component. We assumed $B(\tau) = 2 \times T^2(\tau)$. See further explanation in the text.

Contribution of each magnetic field effect in the volume expansion. We assumed $B(\tau) = 2 \times T^2(\tau)$.

So far, we have not mentioned the contribution of the magnetic field to the energy density and to the fluid dynamics. This contribution depends on its inter-
Fig. 5 Energy densities (total and for each mixture component). We assumed $B(\tau_i) = 2 \times T^2(\tau_i)$.

tial value $B(\tau_i)$. It is negligible when $B(\tau_i) \ll T^2(\tau_i)$ but becomes significant when $B(\tau_i) \sim T^2(\tau_i)$. In the former case $B$ and $T$ exhibit almost identical behaviour to that of the well known FLRW solutions (along the lines of [8]) that follow by assuming $T \sim 1/a$ and $B \sim 1/a^2$, where $a$ is the FLRW scale factor for a radiation dominated fluid. However, if $B(\tau_i) \sim T^2(\tau_i)$ there are significant differences with the FLRW evolution. In order to explore these differences we depict in Figure 5 the energy density for the different components of the fluid mixture during the leptonic era, assuming an (unrealistic) initial magnetic field $B(\tau_i) = 2 \times T^2(\tau_i)$ that is much larger than values allowed by observational bounds. As follows from these graphs, the curve that corresponds to the pure magnetic Maxwell term contribution ($\propto B^2$) decays very fast, but is overtaken by the curves for the neutrinos, the electrons–positrons and the photons. This behaviour is very different from the expected radiation–like FLRW behaviour $T \sim 1/a$ and $B \sim 1/a^2$ of much weaker magnetic fields (subjected to observational constraints). Hence, if $B(\tau_i) \sim T^2(\tau_i)$ the Bianchi I and FLRW dynamics lead to very different decay rates for both the temperature and the magnetic field (see Sec. 5, in particular equations (43)-(44)).

6.4 The magnetic field

The different behaviour of the pure magnetic energy density ($\propto B^2$) is illustrated in figure 6. The solid curves represent the solutions obtained from Bianchi I dynamics, whereas the dashed curves correspond to a magnetic field in an FLRW context ($B \propto 1/a^2$). In both cases we plotted in the left hand
Fig. 6 Pure magnetic energy in the magnetised Bianchi I model and in a FLRW context (assuming $B \propto a^{-2}$) for various values of $B(\tau_i)$ (left hand side panel). For magnetic fields complying with observational bounds $B$ and $B(\tau_i/a^2(\tau))$ are clearly distinguishable: the relative difference between them $\Delta_B(\tau) = [B(\tau) - B(\tau_i/a^2(\tau))] / B(\tau)$, plotted in the right hand side panel, can reach values of order $O(10^{-1})$ (see discussion in the text).

side panel curves for two values of the initial magnetic field: $B(\tau_i) = 2T^2(\tau_i)$ (black line) and $B(\tau_i) = T^2(\tau_i)/2$ (grey line). In the right hand side panel we plotted the relative error $\Delta_B(\tau)$ between $B$ obtained from Bianchi I dynamics and its FLRW equivalent $B(\tau)/a^2(\tau)$ for weak magnetic fields complying with observational bounds. As shown by this graph, $\Delta_B(\tau)$ increases as the fluid evolves and can reach values $\sim O(10^{-1})$. This is an interesting result: while the evolution of kinematic and state variables for initial values $B(\tau_i)$ of weak fields (compatible with observational constraints) is practically indistinguishable in Bianchi I and FLRW dynamics, the evolution of the magnetic field itself reveals non-negligible differences. The explanation for this effect is straightforward: while $a^2 \approx a_1 a_2$ holds for weak fields, each Bianchi I scale factor is related to the FLRW scale factor by a small (but time dependent) correction: $a_1(\tau) \simeq a(\tau) + \epsilon_1(\tau)$ and $a_2(\tau) \simeq a(\tau) + \epsilon_2(\tau)$, hence we have:

$$[B]_{\text{BI}} \approx [B]_{\text{FLRW}} - \left(\frac{\epsilon_1(\tau) + \epsilon_2(\tau)}{a^2(\tau)}\right) B_i,$$

(46)

where $[B]_{\text{FLRW}}$ and $[B]_{\text{BI}}$ are the FLRW and Bianchi I scaling laws given by (42) and (43). Since we are assuming that the shear tensor vanishes at $\tau = \tau_i$, then $\epsilon_1(\tau_i) = \epsilon_2(\tau_i) = 0$, and thus the Bianchi I and FLRW forms initially coincide. However, as the expansion proceeds the “error” introduced by $\epsilon_1(\tau) + \epsilon_2(\tau)$ in (46) (which enters in $\Delta_B(\tau)$) will be negligibly small only for $\tau \approx \tau_i$ but necessarily grows to small (but not negligible values) as the fluid expands. On the other hand, since even with these corrections the field itself is very weak, the evolution of the kinematic and state variables is practically insensitive to them.

Finally, we remark that larger initial values of the magnetic field affect the dynamics of the other thermodynamical functions. Since the latter depend
on the temperature, this can be illustrated by its different evolution with and without magnetic field. Hence, we depict in figure 7 the evolution of the quantity

$$\Delta T = \frac{T_{\text{FLRW}} - T}{T_{\text{FLRW}}} = \frac{T_{\text{FLRW}} - T}{T_{\text{FLRW}}},$$

which provides an estimation of the relative difference between the temperature in the magnetised Bianchi I mixture and the temperature in the non-magnetised FLRW model ($T_{\text{FLRW}}$) that results by setting $B = 0$. Since the initial temperature value is fixed, all curves depicted in the figure start at $\Delta T = 0$. As $\tau$ advances the evolution of the temperature of the magnetised mixture differs slightly (depending on the value of $B(\tau_i)$) from the FLRW evolution without magnetic field, with $\Delta T$ reaching a maximum value close to $\tau_i$ (when the magnetic field is stronger) and dropping as the expansion proceeds and the magnetic field decays. However, for weak fields values compatible with observational bounds $\Delta T \sim O(10^{-5})$, therefore $T$ is practically indistinguishable from $T_{\text{FLRW}}$.

7 Conclusions

We have examined the evolution of a magnetised cosmic fluid mixture between the leptonic and nucleosynthesis eras by means of a non-perturbative approach to the Einstein–Maxwell field equations. For this study we considered a Bianchi I model, as it provides one of the simplest geometries that is compatible with the inherent anisotropy of the magnetic field. We considered as field sources a mixture of baryons (neutrons and protons), leptons (neutrinos, antineutrinos,
electrons and positrons), cold dark matter WIMP’s and photons, together with an already existing time dependent magnetic field directed along the $z$-axis. For the dynamical study of this cosmic fluid we transformed the Einstein–Maxwell equations into a first order system amenable for numerical integration.

The general features of the behaviour of the physical quantities during the two epochs we are concerned can be summarised as follows. Temperature, magnetic field and thermodynamical functions depending on them, like pressures and energy densities of each species, start decaying at a fast rate and later at slower rates as the expansion proceeds. In epoch I at the end of the leptonic era the neutrinos provide the dominant contribution to the energy density and pressures, followed by the contribution of the electron–positron gas and photons. On the other hand, after the electron–positron annihilation the radiation dominates cosmic dynamics. These results agree with well known previous results [44,47].

We have found that cosmic dynamics in the epochs described above is modified by both, the Maxwell term ($\sim B^2$) and the presence of the magnetic field in the equations of state of a magnetised electron–positron gas. For weak magnetic fields compatible with observational constraints these modifications are negligible (though this is not true for the evolution of the field itself). For magnetic fields complying with $eB(t_i) \sim T^2(t_i)$ the main contribution to the dynamics comes from the Maxwell term, leading to an anisotropic expansion that can be appreciated by the different time growth of the metric coefficients and anisotropic pressures. The anisotropy caused by the Maxwell term produces an accelerated expansion in the direction perpendicular to the field and a decelerated one in the field direction. On the other hand, the dependence on $B$ in the electron–positron equation of state acts as a small correction $O(\alpha) \times B^2$ to the energy density where $\alpha$ is the fine structure constant (see further detail in [48]).

We found that the effect of weak magnetic fields compatible with observational bounds on the evolution of the main thermodynamical variables are negligible (for such magnetic fields the time decay of these variables is practically undistinguishable from their evolution under FLRW dynamics). These results seem to validate the approach of [8,22,23,24], who examined magnetic fields during nucleosynthesis under a purely FLRW framework based on the assumption that the spacetime geometry is not affected by magnetic fields in these cosmic times (hence they rule out the anisotropy in the momentum-energy tensor that includes the magnetic interaction). While the results of [8,22,23,24] can always be obtained from our approach under these same assumptions (which may be well justified for weak fields in a realistic cosmic fluid), our non–perturbative methodology allows us to examine magnetised cosmic fluids also under conditions in which such assumptions may not be justified.

It is important to mention that (even under the assumption of weak fields compatible with observational bounds) we also found subtle differences with the FLRW approach of [8,22,23,24], namely: the magnetic field itself (associated with an Einstein–Maxwell system) dilutes faster under Bianchi I dynamics
than the FLRW decay rate $B \propto 1/a^2$ considered in these references as a scalar constituent of a perfect fluid cosmic mixture.

A natural continuation of this study is revisiting the magnetic field constraints as well as the study of abundance of light elements, all under the framework of Bianchi I dynamics. Also, as an alternative to the usage of a Bianchi I model, it would be desirable to study the magnetic interaction in the same cosmic fluid mixture, but under the full formalism of relativistic gauge invariant scalar and vector perturbations in an FLRW background. These tasks will be attempted in future work.

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A Local kinematic variables

The kinematics of local fluid elements can be described through covariant objects defined by the 4-velocity field $u^\alpha$. For a Kasner metric in the comoving frame endowed with a normal geodesic 4-velocity, the only non-vanishing kinematic parameters are the expansion scalar, $\theta$, and the shear tensor $\sigma_{\alpha\beta}$:

$$\theta = u^{\alpha}_{\delta,\alpha}, \quad \sigma_{\alpha\beta} = u_{(\alpha,\beta)} - \frac{\theta}{3}h_{\alpha\beta},$$  \hspace{1cm} (48)

where $h_{\alpha\beta} = u_\alpha u_\beta + g_{\alpha\beta}$ is the projection tensor and rounded brackets denote symmetrization. For the Kasner metric these parameters take the form:

$$\theta = \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3},$$  \hspace{1cm} (49)

$$\sigma_{\beta\gamma} = \text{diag} [\sigma_{x,x}, \sigma_{y,y}, \sigma_{z,z}, 0] = \text{diag} [\Sigma_1, \Sigma_2, \Sigma_3, 0],$$  \hspace{1cm} (50)

where:

$$\Sigma_a = \frac{2}{3} \frac{\dot{a}_a}{a_a} - \frac{1}{3} \frac{\dot{a}_b}{a_b} - \frac{1}{3} \frac{\dot{a}_c}{a_c}, \quad a \neq b \neq c \ (a,b,c = 1,2,3).$$  \hspace{1cm} (51)

The geometric interpretation of these parameters is straightforward: $\theta$ represents the isotropic rate of change of the 3-volume of a fluid element, while $\sigma_{\beta\gamma}$ describes its rate of local deformation along different spatial directions given by its eigenvectors. Since the shear tensor is traceless: $\sigma_{\alpha\alpha} = 0$, it is always possible to eliminate any one of the three quantities $(\Sigma_1, \Sigma_2, \Sigma_3)$ in terms of the other two. We choose to eliminate $\Sigma_1$ as a function of $(\Sigma_2, \Sigma_3)$. Finally, the substitution of (49) and (50) into the Einstein equations, allow us to obtain a first order system of differential equations.
B Particles

At temperature values in the range $100 \text{ MeV} > T > m_e$ neutrons and protons are free in chemical equilibrium. The equilibrium is possible because of reactions transforming neutrons into protons and vice versa [47]:

$$\nu_e n \leftrightarrow e^- p, \quad \nu_e p \leftrightarrow e^+ n,$$

which implies the following variation of their numbers:

$$\frac{n_n}{n_p} = e^{-\Delta M/T}, \quad \Delta M = m_n - m_p. \quad (53)$$

On the other hand, for temperature values less than $m_e$ the neutrons decay freely. Hence the densities of neutrons and protons satisfy

$$\dot{n}_n = -\theta n_n - \Gamma n_n, \quad (54)$$

$$\dot{n}_p = -\theta n_p + \Gamma n_n, \quad (55)$$

where $\Gamma = 1/\tau_n$ is the decay rate ($\tau_n \approx 881.5 \text{ s}$). Also, from the charge neutrality we have $n_e = n_p$.

Finally, since we assume during the whole evolution a baryons/photon ratio $\eta = 5 \times 10^{-5}$, we have

$$\frac{n_b}{n_\gamma} = \frac{n_n + n_p}{n_\gamma} = \eta, \quad (56)$$

where $n_n$, $n_p$, and $n_\gamma$ are respectively the numbers density of neutrons, protons and photons. In this way we can roughly estimate the neutron and proton concentrations, a necessary task to obtain their rest energy. Since we are not interested in calculating element abundances, this rough approximated result is sufficient for our purposes. To improve these calculations the magnetic field should be included in the analysis [8], but this is beyond the scope of the present paper.

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