QSD III:
Quantum Constraint Algebra and Physical Scalar
Product in Quantum General Relativity

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Abstract

This paper deals with several technical issues of non-perturbative four-dimensional
Lorentzian canonical quantum gravity in the continuum that arose in connection
with the recently constructed Wheeler-DeWitt quantum constraint operator.
1) The Wheeler-DeWitt constraint mixes the previously discussed diffeomorphism
superselection sectors which thus become spurious,
2) Thus, the inner product for diffeomorphism invariant states can be fixed by re-
quiring that diffeomorphism group averaging is a partial isometry,
3) The established non-anomalous constraint algebra is clarified by computing com-
mutators of duals of constraint operators,
4) The full classical constraint algebra is faithfully implemented on the diffeomor-
phism invariant Hilbert space in an appropriate sense,
5) The Hilbert space of diffeomorphism invariant states can be made separable if a
natural new superselection principle is satisfied,
6) We propose a natural physical scalar product for quantum general relativity by
extending the group average approach to the case of non-self-adjoint constraint op-
erators like the Wheeler-DeWitt constraint and
7) Equipped with this inner product, the construction of physical observables is
straightforward.

1 Introduction

Recently [1, 2, 3], a new technique was successfully employed to quantize the Wheeler-
DeWitt quantum constraint operator of four-dimensional Lorentzian canonical gravity in
the continuum (see also [4] for a nice and compact non-technical exposition of the essential
results of these papers). More precisely, the results of these papers show that while using

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all the rigorous kinematical framework that has been developed in [5, 6, 7, 8, 9, 10], it is actually possible to define a densely defined operator on the underlying kinematical Hilbert space whose classical limit corresponds to the classical Wheeler-DeWitt (or Hamiltonian) constraint of canonical general relativity.

The Hilbert space itself provides a representation of the operator algebra in which an $SU(2)$ connection is diagonal, that is, it uses very crucially the fact that we are dealing with a connection dynamics rather than geometrodynamics formulation of general relativity. The advantages of the use of a connection as opposed to a three-metric has been discovered a decade ago by Ashtekar [11]. However, in contrast to the original point of view advocated in [11] and many later publications which made essential use of a complex valued connection, we are dealing with an entirely real formulation of Lorentzian general relativity, a point of view first stressed by Barbero [12]. Although the Hamiltonian constraint of general relativity reads much more complicated in terms of real than in terms of complex connections, it allows us to solve the difficult reality conditions which arise for the complex variables and which were one of the two major roadblocks to making progress with these (see, however, [13] for partial progress in this direction). Moreover, and most importantly, one is able to rigorously quantize the loop variables introduced by Gambini and Trias [14] and rediscovered for general relativity by Rovelli and Smolin [15], by means of techniques developed in [5, 6, 7, 8, 9, 10].

The virtue of [1, 2, 3] is that one can overcome the second major roadblock to quantizing the Wheeler-DeWitt constraint: its highly non-polynomial structure in terms of the basic variables which make seem it impossible to give corresponding quantum analogs any mathematical meaning (one can make the Wheeler-DeWitt constraint polynomial upon multiplying it by a density weighted overall factor but this is not allowed in a diffeomorphism invariant theory for reasons explained in [1, 2] and in fact leads to a highly singular operator). The results of [1, 2, 3] also show that the operator so constructed is anomaly-free in a suitable sense and in [3] the entire space of rigorous solutions to all constraints was constructed. However, the quantization programme was still not completed because the following questions were still unresolved:

1) In [10] the existence of superselection sectors for diffeomorphism invariant theories of connections was discovered (theories which do not have any further constraint). The superselection rule holds provided that one can deal with strongly diffeomorphism invariant observables only. Is this assumption justified in general relativity?

2) In [10] an infinite number of different inner products, appropriate for the general solution to the diffeomorphism constraint, all of which implement the reality conditions, were constructed. An infinite number of overall factors were kept undetermined for the inner product on each sector. Does the presence of the Hamiltonian constraint somehow fix those factors? In other words, is there a unique group averaging procedure [23, 24] (employed there to construct an inner product on diffeomorphism invariant states)?

3) The operator constructed in [1, 2] suffers from a huge amount of ambiguity: There is not only one operator but an uncountably infinite number of them, the freedom being captured by a regularization choice. The dual of that operator, determined by evaluating it on a diffeomorphism invariant distribution, however, was shown to be choice independent because the choice freedom is equivalent to a choice of (partial)
diffeomorphism gauge fixing. The problem then, however, was that the dual of the Hamiltonian constraint is not diffeomorphism invariant and so multiplying dual operators was not obviously defined. Can this be clarified?

4) In [2] the commutator of two Hamiltonian constraint operators was shown to vanish when evaluated on diffeomorphism invariant states. However, we did not compute the commutator directly between duals of these operators. More seriously, it is unexpected that the commutator just equals zero instead of being proportional to a diffeomorphism constraint operator. Finally, we also did not compute the commutator between the Hamiltonian and the Diffeomorphism constraint operators. Can these problems be at least partially solved?

5) The Hilbert space of diffeomorphism invariant states is non-separable in the present framework. Can one make it separable by imposing a supersection principle?

6) In [3] an algorithm was derived of how to find the general solution to all quantum constraints in principle (to find such solutions is a matter of computations and can be done in finite time). However, we did not provide an inner product on the space of solutions which would separate a general distribution from a normalizable physical state. The main problem is that the Hamiltonian constraint is, expectedly, not a self-adjoint operator and so the group averaging technique which was designed for self-adjoint constraint operators cannot be applied. Is it possible to supplement the solution space with an inner product?

7) Since in [3] no physical inner product was derived, the issue of Dirac observables was unresolved. Can we construct such observables?

The aim of this paper is to answer all these questions. Specifically we show that:

1) The superselection sectors are spurious: the Hamiltonian constraint mixes the sectors and therefore solutions are linear combinations of vectors from different sectors. Interesting physical observables therefore cannot leave these sectors invariant.

2) Since the superselection sectors are spurious the group averaging maps on each sector must come from one big averaging map. We show that there is only one such map if one wants to implement the group averaging such that strongly diffeomorphism invariant observables are self-adjoint and such that a natural class of orthonormal bases on the kinematical Hilbert space stays orthonormal after averaging.

3) As was shown in [2], arbitrary products of Hamiltonian operators have a dual which is again choice independent when evaluated on dioffeomorphism invariant states. Thus, one can define the product of duals to be the dual of the product in reverse order. Therefore commutators can be rigorously computed and trivially reproduce the results from [2].

4) As follows immediately from [2] the commutator algebra between dual Hamiltonian constraint operators is Abelian which is not expected. However, we prove that there exists an operator \( \hat{O}(M, N) \) corresponding to \( \int d^3x (MN_a - M_aN)q^{ab}V_b \) (which is the right hand side of the Poisson bracket \( \{H(M), H(N)\} \)) and whose dual is again choice independent and annihilates diffeomorphism invariant states. We therefore
have the trivial operator identity \([\hat{H}'(M), \hat{H}'(N)] = -\hat{O}'(M, N)\) on diffeomorphism invariant states (here the prime stands for dual) and it is in this sense that the Dirac algebra is faithfully implemented. Moreover, the commutator between Hamiltonian and diffeomorphism constraint closes in the expected way.

5) We argue that the present results and the structure of the Hamiltonian constraint (dynamics) suggest that the Hilbert space breaks into a continuous orthogonal sum of isomorphic sectors each of which is labelled by a countable number of moduli parameters and that no physical observable maps between those sectors. Thus, physically one may restrict to one of these sectors which turns out to be separable.

6) We extract the essential features of the group average map for self-adjoint operators and show that these can be naturally extended to any non-self-adjoint operator. We show that in quantum mechanical cases our proposal does lead to the known or expected result. Astonishingly we find that even for self-adjoint operators the group average proposal can give rise to solutions (obtained via the rigorously defined group averaging map) to the constraint which are in fact elements of the original Hilbert space, in spite of the fact that the constraint operator is unbounded! We provide quantum mechanics models which mimic the Euclidean or Lorentzian Hamiltonian constraint and find that all solutions are normalizable with respect to the original kinematical inner product. This suggests that there is nothing wrong with the fact that the basic solutions found in \([3]\) are in fact all normalizable with respect to the inner product obtained by group averaging the diffeomorphism constraint. Finally, we use our method to propose a physical inner product for quantum general relativity which is very natural. It is very different from the kinematical diffeomorphism invariant inner product and encodes the full dynamics of quantum general relativity.

7) We take the definition that a Dirac observable maps the physical Hilbert space to itself and is self-adjoint thereon. Given the inner product constructed, the construction of a complete set of at least symmetric operators is straightforward.

We will leave unresolved the very physical questions of quantum general relativity and leave it for future publications. An (incomplete) list of such questions is given by:

1) The problem of time, deparametrization of the theory (see \([16]\) for first attempts in this direction).
2) Physical interpretation of Dirac observables or how to represent known classical observables in the present framework.
3) Existence of a semi-classical limit \([23]\).
4) Black holes and the entropy puzzle from first principles (see \([18, 17, 19]\) for first success in this direction).
5) Connection or relation with string theory.
6) Matter coupling and Energy operator \([24, 21]\).

The organization of the article is that we answer the questions raised above in the chronological order displayed after introducing some basic notation at the beginning of the next section.
2 Diffeomorphism Superselection, Graph Symmetries and Diffeomorphism Invariant Scalar Product

The goal of this section is a) to show that there are no superselection sectors in quantum gravity once we take the Hamiltonian constraint into account and b) to select an inner product for diffeomorphism invariant states of quantum gravity. On the other hand, the superselection sectors arose by restricting the algebra of observables to those which strongly commute with all diffeomorphisms. We show that if we require that these observables are to be promoted to self-adjoint operators and that group averaging with respect to diffeomorphisms is a partial isometry, then there is a unique diffeomorphism invariant group averaging inner product.

We begin with a compact review of the relevant notions from [10]. The interested reader is urged to consult this paper and references therein, in particular [5, 6, 7, 8, 9].

By γ we will denote in the sequel a closed, piecewise analytic graph embedded into a d-dimensional smooth manifold Σ (the case of interest in general relativity is \(d = 3\)). The set of its edges will be denoted \(E(\gamma)\) and the set of its vertices \(V(\gamma)\). By suitably subdividing edges into two halves we can assume that all of them are outgoing from a vertex (the remaining endpoint of the so divided edges are not vertices of the graph because they are points of analyticity). Let \(A\) be a \(G\)-connection for a compact gauge group \(G\) (the case of interest in general relativity is \(G = SU(2)\)). We will denote by \(h_e(A)\) the holonomy of \(A\) along the edge \(e\). Let \(\pi_j\) be the (once and for all fixed) representant of the equivalence class of the set of independent \(j\)-th irreducible representations of \(G\) (in general relativity \(j\) is just a spin quantum number) and label each edge \(e\) of \(\gamma\) with a label \(j_e\). Let \(v\) be an \(n\)-valent vertex of \(\gamma\) and let \(e_1, \ldots, e_n\) be the edges incident at \(v\). Consider the decomposition of the tensor product \(\otimes_{e=1}^n \pi_{j_e}\) into irreducibles and denote by \(c_v\) the linearly independent projectors onto the singlets that appear.

Definition 2.1 A spin-network state is defined by

\[
T_{\gamma, \vec{j}, \vec{c}}(A) := \text{tr}(\otimes_{v \in V(\gamma)}[c_v \cdot \otimes_{e \in E(\gamma), v \in e} \pi_{j_e}(h_e(A))])
\]

(2.1)

where \(\vec{j} = \{j_e\}_{e \in E(\gamma)}\), \(\vec{c} = \{c_v\}_{v \in V(\gamma)}\). In what follows we will use a compound label \(I \equiv (\gamma(I), \vec{j}(I), \vec{c}(I))\).

Thus, a spin-network state is a particular gauge invariant function of smooth connections restricted to a graph. Their importance is that they are an orthonormal basis for a Hilbert space \(H \equiv H_{aux}\), called the auxiliary Hilbert space. Orthonormality means that

\[
< T_{\gamma, \vec{j}, \vec{c}}, T_{\gamma', \vec{j}', \vec{c}'} >_{aux} = \delta_{\gamma, \gamma'} \delta_{\vec{j}, \vec{j}'} \delta_{\vec{c}, \vec{c}'}.
\]

Another way to describe \(H\) is by displaying it as a space of square integrable functions \(L_2(\mathcal{A}/G, d\mu_0)\). Here \(\mathcal{A}/G\) is a space of distributional connections modulo gauge transformations, typically non-smooth and \(\mu_0\) is a rigorously defined, \(\sigma\)-additive, diffeomorphism invariant probability measure on \(\mathcal{A}/G\), called the Ashtekar-Lewandowski measure. The space \(\mathcal{A}/G\) is the maximal extension of the space \(A/G\) of smooth connections modulo gauge transformations such that (Gel’fand transforms of) spin-network functions are still continuous. The inner product can be extended, with the same orthonormality relations,
to any smooth graph with a finite number of edges. It can actually be extended to the full
smooth category \cite{26} but only at the expense of introducing a huge amount of additional
technicalities.

We will denote by \(\Phi\) the finite linear combinations of spin-network functions and call it
the space of cylindrical functions. A function \(f_\gamma\) is said to be cylindrical with respect to a
d graph \(\gamma\) whenever it is a finite linear combination of spin-network functions on that graph
such that \(\pi_{j_e}\) is not the trivial representation for no \(e \in E(\gamma)\). The space \(\Phi\) is equipped
with some topology, for instance the following “Fourier topology” \(|f|_1 = \sum_t |<T_t,f>|\)
(called this way for reasons explained in \cite{27}) which turns it into a topological vector
space. By \(\Phi'\) we mean the topological dual of \(\Phi\), that is, the bounded linear functionals
on \(\Phi\). By the Schwarz inequality we have the inclusion \(\Phi \subset \mathcal{H} \subset \Phi'\) (notice, however,
that this is not strictly speaking a Rigged Hilbert Space because the Fourier Topology is
not nuclear, see \cite{10} for another definition).

This will be enough background for the purpose of the present paper.

We now begin entering the diffeomorphism invariant regime.

**Definition 2.2**

i) The set of smooth diffeomorphisms which leave a piecewise analytic
d graph \(\gamma\) piecewise analytic is denoted by \(\text{Diff}_\gamma(\Sigma)\). The set of all smooth diffeomorphisms
will be denoted \(\text{Diff}(\Sigma)\).

ii) Let \(f_\gamma\) be a function cylindrical with respect to a graph \(\gamma\) and \(\varphi \in \text{Diff}(\Sigma)\). A unitary
representation of \(\text{Diff}(\Sigma)\) is defined by \(\hat{U}(\varphi)f_\gamma = f_{\varphi(\gamma)}\). We define also \(\hat{U}(\varphi)T_I =: T_{\varphi \cdot I}\).

iii) A strongly diffeomorphism invariant operator \(\hat{O}\) on \(\mathcal{H}\) is a self-adjoint operator on \(\mathcal{H}\)
such that \(\hat{U}(\varphi)\hat{O} = \hat{O}\hat{U}(\varphi)\) for any \(\varphi \in \text{Diff}(\Sigma)\).

Notice that in contrast to \cite{10} we do not work with analytical but with smooth diffeo-
morphisms.

The naive idea of how to obtain a diffeomorphism invariant state is to take an element of
\(\Phi\) and to consider its orbit under diffeomorphisms. More precisely we have the following :

Given a cylindrical function \(f \in \Phi\) we can uniquely decompose it as \(f = \sum_{i=1}^N f_{\gamma_i}\) where
\(f_{\gamma_i}\) is a function cylindrical with respect to the graph \(\gamma_i\) and all the graphs in the sum are
considered to be different. The first guess is to consider the orbit \(\{f\}\) of the state \(f\) under diffeomorphisms,
that is, all cylindrical functions \(f'\) such that there is a diffeomorphism \(\varphi\) with \(\hat{U}(\varphi)f = f'\) and then to just define \([f] := \sum_{f' \in \{f\}} f'\). We immediately see that
this does not work :

To see this consider the simple state \(f = T_I + T_J\) where \(\gamma(I), \gamma(J)\) are disjoint, say. Then
there are infinitely many analyticity preserving diffeomorphisms \(\varphi_I\) to be considered in the
orbit \(\{f\}\) which leave \(\gamma(I)\) invariant but not \(\gamma(J)\) and vice versa. As a result, \([f]\)
would contain the meaningless term \(\sum_{\varphi_1} 1|T_I\). The reason for this difficulty is that each
graph has an infinite-dimensional invariance group of diffeomorphisms. Therefore, functions
cylindrical with respect to non-coinciding graphs have to be averaged separately. Thus we are lead to define

\[
[f] := \sum_{i=1}^N \sum_{f'_i \in \{f_{\gamma_i}\}} f'_i = \sum_{i=1}^N [f_{\gamma_i}]
\]

where clearly \([f_{\gamma_i}]\) only depends on \([\gamma_i]\), the orbit (or generalized knot class) of the graph
\(\gamma\).
However, this still does not solve the problem because the orbit size of different states cylindrical with respect to the same graph can be different. To see what the problem is, consider the following instructive example:\footnote{This example is due to Don Marolf.} Let $\gamma$ be the figure-eight loop (with intersection) and consider two spin-network states on $\gamma$, $T_1 := T_{\gamma,j_1,j_2,c}$ with $j_1 \neq j_2$ and $T_2 := T_{\gamma,j_1,j_2,c'}$ where $c, c'$ are some contractors and $j_1, j_2, j$ are certain irreducible representations. Now notice that there is a diffeomorphism $\varphi_0$ (the one that rotates the graph by 180 degrees) which leaves $T_2$ invariant but not $T_1$. As a result we see that the orbit size of $T_1$ is double as large as that of $T_2$. Therefore we find that $[T_1 + T_2] = [T_1] + 2[T_2]$ so that the $[\cdot]$ operation is not at all linear!

Since we want a linear operation as otherwise we will not preserve the linear structure of quantum theory, we must modify the definition of $[f_\gamma]$ even further, namely, we must choose a basis, write $f_\gamma$ in terms of this basis and average each basis element separately. Thus, the group averaging becomes basis dependent, for each basis we get another average map.

Not every basis is allowed : we want the resulting average map to commute with the diffeomorphism group. Notice that this is not an entirely trivial requirement because given a basis element $B_I$ and a diffeomorphism $\varphi$ we are supposed to decompose $U(\varphi)B_I$ into basis elements $B_I'$ before averaging and then to average the $B_I'$. Thus, it is not clear that $[U(\varphi)B_I]$ is the same as $U(\varphi)[B_I] = [B_I]$. We will call a basis with this property an \textit{allowed basis}.\footnote{This example is due to Don Marolf.}

This basis dependence is an ugly feature because it implies that unitary operators are not promoted to unitary operators. However, we will see that the Hilbert space splits into two orthogonal subspaces $\mathcal{H}^S, \mathcal{H}^A$ which are basis independently defined : Namely, $\mathcal{H}^A$ is nothing else than the kernel of every group averaging map while $\mathcal{H}^S$ is its orthogonal complement which shows that both spaces are invariantly defined. In each of the subspaces the group averaging can be defined \textit{basis-independently} while being a linear a map. In other words, there are preferred allowed bases of $\mathcal{H}$ which respect the orthogonal split $\mathcal{H} = \mathcal{H}^S \oplus \mathcal{H}^A$ and the group average with respect to either of them is equivalent. As one might expect from the above analysis, the spin-network basis is \textit{not} a preferred basis, although it is closely related to one.

Let us make precise what we have just said.

\textbf{Definition 2.3} \textit{i)} Let $\{B_I\}_I$ be an orthonormal basis for $\mathcal{H}$ (for instance a spin-network state basis with $B_I = T_I$, without loss of generality we can assume that still $I = (\gamma(I), \lambda(I))$). Consider the orbit of a basis element $B_I$ under $\text{Diff}_\gamma(I)(\Sigma)$, that is, $$\{B_I\} := \{U(\varphi)B_I, \ \varphi \in \text{Diff}_\gamma(I)(\Sigma)\}.$$ The group average of $B_I$ is defined to be $$B_{[I]} := [B_I] := \sum_{B \in \{B_I\}} B \in \Phi'$$ \hspace{1cm} (2.2)

and the group average of any $f = \sum_{k=1}^N a_k B_{I_k} \in \Phi$ with respect to the basis $\{B_I\}_I$ is defined by $[f] := \sum_{k=1}^N a_k B_{[I_k]}$.

\textit{ii)} A basis $\{B_I\}_I$ is said to be \textit{allowed} if a) the corresponding group averaging commutes
with the diffeomorphism group, i.e. \([\hat{U}(\varphi)B_I] = \hat{U}(\varphi)[B_I] = [B_I]\) for all \(I, \varphi \in \text{Diff}_\gamma(\Sigma)\) and b) all the vectors are well-defined distributions, that is, \([B_I] \in \Phi'\). In the sequel we will only allow for average maps determined by allowed bases and without loss of generality we restrict to orthonormal bases.

iii) The diffeomorphism invariant inner product with respect to group averaging as determined by an allowed basis \(\{B_I\}\) is defined to be

\[
< [B_I], [B_J] >_{\text{Diff}'} := [B_I](B_J) := \sum_{B \in \{B_I\}} < B, B_J >_{\text{aux}}
\]

and extended by sesquilinearity. We will call the set of group averaged cylindrical functions \(\Phi'_{\text{Diff}'}\).

Notice that in contrast to [10] we do not average with respect to the whole state labelled by \(I\). This difference will have important consequences with respect to the so-called graph-symmetries to which we turn in a moment. Also notice that we average with respect to analyticity preserving diffeomorphisms rather than with respect to analytic ones: while this reduces the size of diffeomorphism invariant states (it increases the orbit and two analytic knot classes may now lie in the same orbit) it poses no problem and all the properties derived in [10] are preserved with the bonus that we do not have to deal with the subtle type I and II graphs any longer that were discussed there.

We now construct the subspaces \(\mathcal{H}^S, \mathcal{H}^A\) mentioned above.

**Definition 2.4**

i) Given a graph \(\gamma\) the set of graph symmetries of \(\gamma\) is the subset \(G(\gamma) \subset \text{Diff}_\gamma(\Sigma)\) of diffeomorphisms which leave \(\gamma\) as a whole invariant but permute its edges and vertices.

ii) Given the orbit \(\{T_I\}\) of a spin-network state \(T_I\) choose a set \(D(\{T_I\})\) of diffeomorphisms which are in one-to-one correspondence with the orbit. Let \(\varphi_1, \varphi_2 \in D(\{T_I\})\) and say that they are equivalent iff there is a \(\varphi_0 \in G(\varphi_1(\gamma(I)))\) such that \(\varphi_2 = \varphi_0 \circ \varphi_1\). We define \(D_0(\{T_I\})\) to be the set of equivalence classes of these diffeomorphisms. Clearly \(D_0(\{T_I\})\) is in bijection with \(D(\{\gamma(I)\})\) where \([\gamma]\) is the orbit of the graph \(\gamma\) under \(\text{Diff}_\gamma(\Sigma)\) and \(D([\gamma])\) is a set of diffeomorphisms in bijection with \([\gamma]\).

iii) Let \(T_I\) be a spin-network state. Let us write \(I = (\gamma(I), \lambda(I))\). Then define

\[
\Lambda([I]) := \{\lambda, \exists \varphi \in G(\gamma(I)) : \exists \hat{U}(\varphi)T_I = T_{\gamma(I), \lambda}\} \text{ and } n([I]) := |\Lambda([I])|.
\]

To see that \(\Lambda[I]\) interacts well with the diffeomorphism group we need the following lemma.

**Lemma 2.1** The set \(\Lambda([I])\) is diffeomorphism invariant.

Proof:

Notice that if \(\varphi_0(\gamma) = \gamma\) then \(\varphi \circ \varphi_0 \circ \varphi^{-1}(\varphi(\gamma)) = \varphi(\gamma)\) for any \(\varphi \in \text{Diff}_\gamma(\Sigma)\). We conclude that \(G(\varphi(\gamma)) = \varphi \circ G(\gamma) \circ \varphi^{-1}\). Let \(\varphi' \in D(\{T_I\})\). Then we may pick \(\varphi \in D_0(\{T_I\})\) such that \(\hat{U}(\varphi)T_I = T_{\varphi'(\gamma(I)), \lambda(I)}\). Since there is \(\varphi_0 \in G(\varphi'(\gamma(I)))\) such that \(\varphi' = \varphi_0 \circ \varphi\) we see that \(\Lambda([\varphi' \cdot I]) = \Lambda([I])\).

\square

The existence of a non-trivial graph-symmetry group implies that certain different labellings of the graph are related by diffeomorphisms.
Example:

Consider, again, the figure-8 loop $\gamma = \alpha_1 \circ \alpha_2$. Label the loop $\alpha_i$ with $j_i$, $i = 1, 2$ and the vertex with $c$. The figure-8 loop has a graph symmetry-group consisting of $\varphi_0$, id where $\varphi_0(\alpha_i) = \alpha_j$, $i \neq j$ and thus $\Lambda([I]) = \{(j_1, j_2, c_2), (j_2, j_1, c_1)\}$ provided $j_1 \neq j_2$ and $\Lambda([I]) = \{(j, j, c')\}$ if $j_1 = j_2 = j$.

We see that the orbit size of the state depends not only on the graph but also on the labels. It is for this reason that the spin-network states are not a well-adapted basis for the group-averaging with respect to diffeomorphisms. For example, consider a strongly diffeomorphism invariant operator $\hat{O}$ and the states $\psi_1 = T_{\gamma_1 j_1 j_2 c}, \psi_2 = T_{\gamma_2 j_2 j_1 c}, \psi_3 = T_{\gamma_3 j_3 j_4 c'}$ with $j_1 \neq j_2$ as above with $\hat{U}(\varphi_0)\psi_1 = \psi_2$, $\hat{U}(\varphi_0)\psi_2 = \psi_1$, $\hat{U}(\varphi_0)\psi_3 = \psi_3$. First of all we see that $[\psi_1] = [\psi_2]$ (here we are tentatively assuming that the spin-network basis is an allowed basis and average with respect to it). Moreover, suppose that $\hat{O}\psi_1 = a\psi_1 + b\psi_2 + c\psi_3$, $\hat{O}\psi_2 = b\psi_1 + a\psi_2 + c\psi_3$, $\hat{O}\psi_3 = d\psi_1 + d\psi_2 + c\psi_3$ with arbitrary complex coefficients $a, b, c, d, e$.

This is the most general strongly diffeomorphism invariant operator on these three states. If it is self-adjoint then $a, b, e$ are real and $c = d$. Now let us see whether with the present definition $\hat{O}$ is promoted to a self-adjoint operator on $\mathcal{H}_{\text{Diff}}$. The necessary and sufficient condition is that $[\psi_1](\hat{O}\psi_2) = [\hat{O}\psi_1](\psi_2)$. But $[\psi_1](\hat{O}\psi_3) = 2d$ because $[\psi_1] = [\psi_2]$ so that the orbit of $\psi_1$ also contains $\psi_2$ while $[\hat{O}\psi_1](\psi_3) = c = d$ contradicting symmetry.

On the other hand, this example shows how to cure the situation by a suitable change of basis: consider the orthonormal states $\psi_{\pm} := (\psi_1 \pm \psi_2)/\sqrt{2}$. Then by definition $[\psi_\pm] = \sqrt{2}[\psi_1], [\psi_-] = 0$, that is, the group average as defined by (2.2) has a huge kernel even when restricted to vectors defined on the same graph. Notice that group averaged states remain orthonormal if not zero, provided we divide the average of $\psi_+$ by the number of spin-network states involved (here two) since they come out of the averaging anyway: $\frac{1}{2}[\psi_+](\psi_+) = \frac{\sqrt{2}}{2}[\psi_1](\psi_1 + \psi_2) = 1$ (still averaging with respect to the spin-network basis) and the so defined inner product is hermitean. In terms of the new vectors the operator $\hat{O}$ becomes $\hat{O}\psi_+ = \frac{a+b}{\sqrt{2}}\psi_+ + \sqrt{2}c\psi_3$, $\hat{O}\psi_- = \frac{a-b}{\sqrt{2}}\psi_-$, $\hat{O}\psi_3 = d\sqrt{2}\psi_+ + c\psi_3$ and we see that it leaves the kernel of the group averaging map invariant and moreover does not have any matrix elements between the kernel and its complement. We now verify that $\frac{1}{2}[\psi_+](\hat{O}\psi_3) = \frac{d}{\sqrt{2}} = \frac{c}{\sqrt{2}} = \frac{1}{2}[\hat{O}\psi_+](\psi_3)$ is indeed self-adjoint. How did this happen? It happened because the group average was defined with respect to the spin-network basis and so one must expect a basis dependence of the resulting diffeomorphism invariant inner product. Indeed, the average with respect to the new basis $\psi_\pm$ modified the averaging as compared to the spin-network basis and in fact, the spin-network basis is not well-adapted to diffeomorphism invariance because the group average of two orthogonal states with the same graph on $\mathcal{H}$ can result in diffeomorphism invariant states which are identical and so have unit inner product. By changing to the basis involving $\psi_\pm$ we have done two things: first, the symmetric state $\psi_+$ is essentially what is important after group averaging while the anti-symmetric state $\psi_-$ is unimportant and becomes zero after averaging. Moreover, the representation of diffeomorphism invariant self-adjoint operators is completely reducible with symmetric and anti-symmetric states as invariant subspaces.
We will now show that these features of the example are true in general and that the generalization of the basis of the $\psi_\pm$ is preferred.

**Definition 2.5**  

i) Given a spin-network state $T_I$, label the $n=n([I])$ states $T_{\gamma(I),\lambda_k} := T_{I,k}$ where $\{\lambda_1, \ldots, \lambda_n\} = \lambda([I])$. Now construct the orthonormal states

$$T_I^S := \frac{1}{\sqrt{n}}\sum_{k=1}^n T_{I,k}$$

$$T_{I,k}^A := \frac{1}{\sqrt{k(k+1)}}\left[\sum_{l=1}^k T_{I,l} - kT_{I,k+1}\right] \text{ for } k = 1, \ldots, n-1. \quad (2.5)$$

ii) Let $H^S$ be (the completion of the) span of the $T_I^S$ and likewise for $H^A$. Then $H = H^S \oplus H^A$. $H^A$ is called the kernel of the average map [for reasons explained in the next theorem].

iii) A basis such that each basis vector lies either in $H^S$ or in $H^A$ is said to be adapted.

That these states are orthonormal is readily verified by induction over $k=1, \ldots, n$. We see that we arrive at a new orthonormal decomposition of the Hilbert space of states and of the space of functions cylindrical with respect to given graph $\gamma$.

**Theorem 2.1**  

i) The orbit size of any state of the form $T_I^S$ is given by $D([\gamma(I)])$, that is, it only depends on the orbit size of the graph and not on the state.

ii) The adapted basis given by the $T_I^S$ and $T_{I,k}^A$ is allowed.

iii) The space $[H^A]_{\gamma}$ is the kernel of every average map restricted to functions cylindrical with respect to a graph $\gamma$ for any $\gamma$ and therefore is invariantly defined.

iv) Two average maps defined by adapted bases of $H$ are identical. We will call the resulting average map $\eta_{Diff}$.

Proof:

i) Since the $T_I^S$ are invariant under the graph symmetry group $G(\gamma(I))$ they actually do not depend on $\lambda([I])$ but only on $\Lambda([I])$. Therefore, the orbit size of $T_I^S$ is that of the set $D_0\{T_I\}$ which in turn is in bijection with $D([\gamma(I)])$. Notice that therefore $\eta_{Diff} T_I^S = [T_{I,1}]/\sqrt{n([I])}$ where $[T_{I,1}]$ is the average with respect to the spin-network basis. This shows that

$$\eta_{Diff} T_I^S(T_J^S) = \frac{1}{\sqrt{n([I])n([J])}}[T_{I,1}][T_{J,1} + \ldots + T_{J,n([J])}]$$

$$= \frac{n([J])}{\sqrt{n([I])n([J])}}[T_{I,1}][T_{J,1}] = \frac{n([J])}{\sqrt{n([I])n([J])}} \delta_{\gamma(I)[\gamma(J)]} \delta_{\Lambda(I)\Lambda(J)}$$

$$=: \delta_{[I][J]} \quad (2.6)$$

so the states $\eta_{Diff} T_I^S$ are orthonormalized under the associated diffeomorphism invariant inner product.

ii) We need to show that $[T_I^S] = [\hat{U}(\varphi)T_I^S]$ and $[T_{I,k}^A] = 0 = [\hat{U}(\varphi)T_{I,k}^A]$ for each $\varphi, I, k$. However, by definition, each of the $T_I^S, T_{I,k}^A$ is nothing else than a particular linear combination of spin-network states $T_I$ which have the nice feature that a diffeomorphism just can be
translated into an action on the label \( I \) : \( \hat{U}(\varphi)T_I = T_{\varphi(I),\lambda(I)} = T_{\varphi,I} \). It follows that \( \hat{U}(\varphi)T^S_I = T^S_{\varphi,I} \) and \( \hat{U}(\varphi)T^A_{I,k} = T^A_{\varphi,I,\varphi,k} \). Thus the result follows from the fact that \( T^S_I \) only depends on \( \Lambda(I) \) which is diffeomorphism invariant and from the fact that \( [T^A_{I,k}] = 0 \) for any \( I,k \).

\[ \Rightarrow \]

First we show that every state of the form \( T_I - \hat{U}(\varphi)T_I \), where \( \varphi \in G(\gamma(I)) \), is averaged to zero by every averaging map, that is, by the averaging map determined by any allowed basis \( \{ B_I \} \). Because the averaging map is linear and these states span \( \mathcal{H}^A \), we will have shown that \( \mathcal{H}_A^A \) is in the kernel of every average map on \( \mathcal{H}_A^A \).

Since \( T_I \) and \( B_I \) belong orthonormal bases there is a unitary operator \( \hat{V} \) with matrix elements \( V_{iJ} \) such that \( T_I = \sum_j V_{iJ}B_J \) and the sum extends only over labels such that \( \gamma(J) = \gamma(I) \). Now the state \( \hat{U}(\varphi)B_J \) is a linear combination of states \( B_J' \) : in other words \( \hat{U}(\varphi)B_J = \sum_{J'} V_{J'J}B_{J'} \). Therefore by definition \([\hat{U}(\varphi)B_J] := \sum_{J'} L_{JJ'}[B_{J'}] = [B_J]\) because \( \{ B_I \} \) is an allowed basis. We conclude that \([T_I] = [\hat{U}(\varphi)T_I] \).

\[ \Leftarrow \]

Conversely, we need to show that every vector that is averaged to zero and that is cylindrical with respect to the same graph \( \gamma \) lies in \( \mathcal{H}_A^A \). Since we have shown already that each vector in \( \mathcal{H}_A^A \) has this property we need to show that no vector in \( \mathcal{H}_A^S \) has this property because the average map is linear.

Let us introduce a new notation : Let \( T^S_I := T^S_I \) and \( T^A_{I,k} := T^A_{\mu,k} \) and consider any allowed orthonormal basis \( B_I \). There exists a unitary operator \( \hat{V} \) such that

\[
T^S_I = \sum_i V_{iI}B_I, \quad T^A_{\mu} = \sum_i V_{\mu I}B_I \quad \text{and} \quad B_I = \sum_i \bar{V}_{iI}T^S_i + \sum_{\mu} \bar{V}_{\mu I}T^A_{\mu}.
\]

According to i) all vectors \( T^S_I \) are invariant under the graph symmetry group \( G(\gamma) \) while the vectors \( T^A_{\mu} \) are not. In fact, by definition of the graph symmetry group, their orbit size is an integral multiple of the orbit size of \( \gamma \) which is the orbit size of any \( T^S_i \). Therefore the orbit size of the state \( B_J \) is an integral multiple \( n(I) \geq 1 \) of the orbit size of \( \gamma \) the value of which depends on which and how many of the states \( T^A_{\mu} \) occur in its decomposition. Combining this with the first half of the proof of iii) we find \([B_J] = n(I) \sum_i \bar{V}_{iI}T^S_i \).

Certainly, \( n(I) \) is a bounded function of \( I \) as otherwise not all the \([B_J]\) would be in \( \Phi' \) (since \([\eta_{D_{i,j}}T^S_i](T^S_j) = \delta_{\Lambda(I),\Lambda(J)} \)).

We now ask whether there exist complex numbers \( a_i \) such that the vector \( \psi^S := \sum_i a_iT^S_i \in \mathcal{H}^S \) is averaged to zero. By definition we have

\[
[\psi^S] = \sum_i a_i \sum_I V_{iI}[B_I] = \sum_i a_i \sum_I V_{iI} \sum_j n(I)\bar{V}_{ji}\eta_{D_{i,j}}T^S_j = \sum_j \eta_{D_{i,j}}T^S_j \sum_i a_i \sum_I V_{iI}n(I)\bar{V}_{ji} = 0
\]

which has a non-trivial solution \( \{ a_i \} \), if and only if the operator \( \hat{M} \) on \( \mathcal{H} \cdot \gamma \) with matrix elements between the \( T^S_i \)

\[
M_{ij} = \sum_I V_{iI}n(I)\bar{V}_{ji} = [\hat{V}\hat{D}\hat{V}^\dagger]_{ij}
\]

is singular where we have defined the operator \( \hat{D} \) on \( \mathcal{H} \) defined by \( \hat{D}B_I = n(I)B_I \), that is, it has matrix elements \( D_{I,J} = n(I)\delta_{I,J} \) between \( B_I, B_J \). Notice that the operators \( \hat{D}, \hat{V} \)
are bounded so that the resummation that we performed above is justified. Since \( n(I) \geq 1 \), the operator \( \hat{D} \) has a square root \( \hat{R} \) and thus \( \hat{M} = [\hat{V}\hat{R}][\hat{V}\hat{R}]^{\dagger} \). Thus we are asking whether the projection to \( \mathcal{H}^S \) of a positive definite operator on \( \mathcal{H} \) is singular which is impossible.

This shows that \( [\psi^S] = 0 \) if and only if \( \psi^S = 0 \).

iv)

Let \( \{B_I\}_I \) be an allowed basis adapted to the orthogonal decomposition \( \mathcal{H}^S \oplus \mathcal{H}^A \). That is, each \( B_I \) lies entirely only in one of these spaces. Since the \( B_I \) define an orthonormal basis there is a unitary map \( V^S \oplus V^A \) such that \( (V^S)^\dagger V^A = (V^A)^\dagger V^S = 0 \), \((V^S)^\dagger V^S = \text{id}_S\), \((V^A)^\dagger V^A = \text{id}_A\) and \( T^S_I = \sum_J V^S_{IJ} B_J, T^A_I = \sum_J V^A_{IJ} B_J \) where the sum runs only over labels \( J \) such that \( \gamma(I) = \gamma(J) \) (we have written \( T^A_{I,k} = T^A_I \)) and also only \( B_J \) are involved which are either in \( \mathcal{H}^S \) or \( \mathcal{H}^A \) respectively. By definition

\[
[T^S_I] = \sum_J V^S_{IJ} \sum_{B'_J \in \{B_J\}} B'_J = \sum_J V^S_{IJ} \sum_{\varphi \in \mathcal{D}(\{B_J\})} \hat{U}(\varphi) B_J
\]

where, as before, \( \mathcal{D}(\{B_J\}) \) is a set of diffeomorphisms which are in one to one correspondence with the orbit of \( B_J \). Notice that \( B_J = \sum_K (V^S)^{-1}_{JK} T^S_K \) since the basis is adapted (there are no \( T^A_K \) on the right hand side). According to i) the orbit size of all the \( T^S_K \) is identical and in fact coincides with the orbit size of the graph \( \gamma(I) \). That means that each \( \varphi \in \mathcal{D}(\{B_J\}) \) which moves \( B_J \) at all must move all the \( T^S_K \) which appear on the right hand side of \( B_J = \sum_K (V^S)^{-1}_{JK} T^S_K \). It follows that the orbit size of \( B_J \) is the same as that of \( [\gamma(I)] \) as well. It follows that

\[
[T^S_I] = \sum_K (V^S)^{-1}_{JK} \eta_{\text{Diff}} T^S_K
\]

so that, putting these results together, we find

\[
[T^S_I] = \eta_{\text{Diff}} T^S_I
\]

where \( \eta_{\text{Diff}} \) is the average map defined by the basis \( T^S_I \).

\( \square \)

Let us verify that \( \eta_{\text{Diff}} T^S_I \in \Phi' \). If \( f \in \Phi \), then

\[
||[T^S_I](f)|| := \frac{1}{\sqrt{n(I)}} \sum_{T \in \{T_I\}} | \langle T, f >_{\text{aux}} | \leq \frac{1}{\sqrt{n(I)}} \sum_{T \in \{T_I\}} | \langle T, f >_{\text{aux}} | \leq \frac{1}{\sqrt{n(I)}} \sum_J | \langle T_J, f >_{\text{aux}} | = \frac{||f||_1}{\sqrt{n(I)}}
\]

which shows that \( \eta_{\text{Diff}} T^S_I \in \Phi'_{\text{Diff}} \).

Let us summarize the contents of Theorem (2.7) with regard to the uniqueness of the average map:

Every average map has the same kernel \( \mathcal{H}^A \) so that the kernel is defined independently of the allowed basis that was used to define the average map. Therefore the spaces \( \mathcal{H}^S, \mathcal{H}^A \) are invariantly defined.

Being invariantly defined one can consider all bases that are adapted to the associated orthogonal decomposition which is natural because the space \( \mathcal{H}^A \) does not play any role after moding out with respect to the diffeomorphism group. Moreover, as we will see below, the subalgebra of the observable algebra given by strongly diffeomorphism invariant observables leaves the subspaces \( \mathcal{H}^S, \mathcal{H}^A \) invariant so that we have an additional motivation to consider only averages with respect to adapted bases in the sequel.
Then, when restricting to adapted bases it turns out that there is actually only one averaging map.
Thus, among all the possible averagings, one of them, $\eta_{Diff}$, is singled out.

We could be content with this result, however, we now ask if we can do even better: Recall that we defined an average map by first decomposing a cylindrical function into components each of which was cylindrical with respect to a certain graph. Now diffeomorphic graphs certainly have identical orbit sizes but non-diffeomorphic ones do not have, in general, the same orbit size and as we have seen above, in general their orbit sizes are "infinitely different" in the sense that one gets in general not an element of $\Phi$ if summing over all vectors in the orbit of a vector which is a linear combination of vectors cylindrical with respect to non-diffeomorphic graphs. It seems therefore that we do not have any justification for a universal average map $\eta_{Diff}$, that is, for each diffeomorphism equivalence class of graphs $[\gamma]$ we could have a different average map $\eta_{\gamma_{Diff}}$. According to the above analysis, the difference between $\eta_{Diff}$ and $\eta_{\gamma_{Diff}}$ can really be only a pre-factor if we choose the canonical way of averaging and the prefactor is a positive real number which, heuristically speaking, accounts for the fact that each graph has different orbit size as compared to the orbit size of the "universal graph" which contains all graphs (and which therefore has the largest possible orbit size) and arises by somehow renormalizing the arising infinite pre-factor.

The justification for the prefactor in $\eta_{\gamma_{Diff}} = a([\gamma])\eta_{Diff}$ is even more serious given the superselection principle derived in [10]. Let us shortly review how strongly diffeomorphism invariant observables define a superselection rule: Assume that $\gamma(I)$ and $\gamma(J)$ are different graphs and that $\hat{O}$ is a strongly diffeomorphism invariant observable which is densely defined on $\Phi$. There are an at least countable number of diffeomorphisms $\phi_n$ which leave $\gamma(I)$ invariant but such that $\phi_n^{-1}(\gamma(J))$ are mutually different graphs for each $n$. It follows that

$$O_{IJ} := \langle T_I, \hat{O} T_J \rangle_{aux} = \langle \hat{U}(\varphi_n) T_I, \hat{O} T_J \rangle_{aux} = \langle T_I, \hat{U}(\varphi_n^{-1}) \hat{O} T_J \rangle_{aux}$$

and so $\hat{O}$ has the same matrix element $O_{IJ}$ between $T_I$ and an infinite number of mutually orthonormal states. Thus, since $\hat{O}$ was densely defined we conclude that $O_{IJ} = 0$.

This is the superselection rule: strongly diffeomorphism invariant observables cannot map between cylindrical functions defined on different graphs. In particular, they cannot map between graphs which are non-diffeomorphic. Therefore, the inner product between diffeomorphism invariant states which arise as group averages from different diffeomorphism invariant sectors are completely unrelated and we may therefore choose each of them separately.

Of course, the existence of the superselection rule rests on the assumption that strongly diffeomorphism invariant observables are the only ones that give rise to physical observables in quantum gravity. Now, from [1, 2] we know that the Hamiltonian constraint operator does not leave the superselected sectors invariant. In particular, the space of solutions consists of distributions which are linear combinations of diffeomorphism invariant states from different superselected sectors. It follows that any non-trivial physical observable, that is, one that maps the solution space to itself, and which differs from
the identity operator mixes different super-selected sectors. Clearly, such an observable cannot be strongly diffeomorphism invariant. Strongly diffeomorphism invariant operators are built purely from electric field operators (as they cannot change the graph) while weakly diffeomorphism invariant operators will also depend on the connection through loops. We conclude that the restriction to strongly diffeomorphism invariant observables was too restrictive and that the superselection sectors are therefore spurious.

Still there remains the issue which value of the constants \( a(\gamma) \) is the correct one. Is there is a physical motivation for \( a[\gamma] = 1 \)?

**Theorem 2.2** i) The representation of the algebra of strongly diffeomorphism invariant observables on \( \mathcal{H} \) is completely reducible with \( \mathcal{H}^S, \mathcal{H}^A \) as invariant subspaces.

ii) The inner product defined by \( \langle \eta_{\text{Diff}} f, \eta_{\text{Diff}} g \rangle_{\text{Diff}} := (\eta_{\text{Diff}} f)(g) \) for all \( f, g \in \Phi \), that is, \( a(\gamma) = 1 \), is uniquely selected by requiring that a) strongly diffeomorphism invariant self-adjoint operators on \( \mathcal{H} \) are promoted to self-adjoint operators on \( \mathcal{H}_{\text{Diff}} \) and b) \( \langle f, g \rangle_{\text{aux}} = \langle \eta_{\text{Diff}} f, \eta_{\text{Diff}} g \rangle_{\text{Diff}} \) for all \( f, g \in (\mathcal{H}^S \cap \Phi)_\gamma \).

**Proof:**

i) Notice that each element \( \varphi_0 \in G(\gamma(I)) \) permutes the edges and vertices of \( \gamma(I) \) in some particular way. This permutation induces a permutation \( \sigma_0 \) of the elements of the set \( \Lambda([I]) \). Thus, \( \sigma_0 \in S(n([I])) \) where \( S(n) \) denotes the symmetric group of \( n \) elements (the set of these permutations does not necessarily contain all elements of \( S(n) \) but is a subgroup thereof). The vectors \( T^S_I \) are therefore invariant under \( G(\gamma([I])) \) because they are completely symmetric under any element of \( S(n([I])) \).

We conclude that for a strongly diffeomorphism invariant operator \( \hat{O} \) the vector \( \hat{O} T^S_I \) is again invariant under \( G(\gamma(I)) \) (recall that a strongly diffeomorphism invariant operator must map functions cylindrical with respect to a graph to functions cylindrical with respect to the same graph). Next, notice that the graph symmetry group acts transitively and fixpoint freely on the set of vectors \( \{T_{I,k}\}_{k=1}^{n([I])} \) by its very definition (every vector can be mapped into any other). It follows that the induced subgroup \( S'(n([I])) \) of the permutation group on \( \Lambda([I]) \) contains a cyclic element \( \sigma \) of order \( n([I]) \) (that is, one with \( \sigma^{n([I])} = \text{id} \) and \( n = n([I]) \) is the smallest integer with this property).

To see this just decompose every \( \sigma \in S'(n([I])) \) into cycles. Take one element which contains a cycle of length \( k \) and compose it with another element which contains a cycle of length \( l \) that involves at least one element which is not involved in the first cycle and at least one that is involved in the first cycle. Such an element must exist because the graph symmetry group acts transitively on \( \Lambda([I]) \). The result is an element of \( S'(n([I])) \) which contains a cycle of length \( k + 1 \) at least. Proceeding inductively, the assertion follows.

Given such a cyclic element \( \sigma_0 \in S'(n([I])) \) corresponding to \( \varphi_0 \in G(\gamma(I)) \), let us label the states \( \hat{T}_{k,l} \) in such a way that \( \hat{U}(\varphi_0)^k \hat{T}_{l,1} = T_{l,k+1} \). Let us consider a new (non-orthonormal) basis of \( \text{span}\{T^A_{I,k}\}_{k=1}^{n([I])} \) given by \( A_{I,k} := T_{I,k} - T_{I,k+1}, k = 1, ..., n([I]) - 1 \). That indeed every \( T^A_{I,k} \) is a linear combination of these new vectors follows from the identities \( \sum_{l=1}^k T_{I,l} - kT_{I,k+1} = \sum_{l=1}^k[T_{I,l} - T_{I,k+1}] \) and \( T_{I,l} - T_{I,k+1} = A_{I,l} + ... + A_{I,k} \). The virtue of this new basis is that it allows us to readily verify that

\[
\hat{n}_l A_{I,l} := \left( \sum_{k=0}^{n([I])-1} \hat{U}(\varphi_0)^k \right) A_{I,l} = 0 \quad \forall \ l = 1, ..., n([I]) - 1
\]

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since $\hat{U}(\varphi_0)A_{I,l} = A_{I,l+1}$ with $T_{I,n([I]) + 1} := T_{I,1}$ and because $\sum_{k=1}^{n([I])} A_{I,k} = 0$. We conclude that $\eta_i T_{I,k}^A = 0$ for all $k = 1, \ldots, n([I])$.

We will show that a strongly diffeomorphism invariant operator $\hat{O}$ must leave the cylindrical subspaces of $\mathcal{H}^S$ and $\mathcal{H}^A$ invariant. To that end let be given arbitrary elements $f$ of $\mathcal{H}^S \cap \Phi$ and $g$ of $\mathcal{H}^A \cap \Phi$. This means that they are finite linear combinations of the form $f = \sum_{i=1}^M s_i T_i^S$ and $g = \sum_{j=1}^N a_j T_{j,k}$, where $k_j \in \{1, \ldots, n([J]) - 1\}$. We need to show that $<f, \hat{O}g> = <g, \hat{O}f> = 0$ for all $f \in \mathcal{H}^S \cap \Phi$, $g \in \mathcal{H}^A \cap \Phi$. This is equivalent to showing that $<T_i^S, \hat{O}T_{j,k}^A> = <T_{j,k}^A, \hat{O}T_i^S> = 0$ for all $I, J$ and $k = 1, \ldots, n([J]) - 1$.

Since a strongly diffeomorphism invariant operator leaves the graph of a spin-network state invariant these equalities follow trivially unless $\gamma(I) = \gamma(J)$ so let us assume that this is the case. We now use the fact proved above that $T_i^S$ is invariant under $G(\gamma(I))$. This implies that, since $\gamma(I) = \gamma(J)$, $\eta_i T_i^A = T_i^A$. Since by assumption $[\hat{\eta}_i^T, \hat{O}] = 0$ we have

$$<T_i^S, \hat{O}T_{j,k}^A> = <\hat{\eta}_i^T T_i^S, \hat{O}T_{j,k}^A> = <\hat{\eta}_j^T T_i^S, \hat{O}T_{j,k}^A> = 0$$

by construction of $\hat{\eta}_i$. Similarly

$$<T_{j,k}^A, \hat{O}T_i^S> = <\hat{\eta}_j T_{j,k}^A, \hat{O}T_i^S> = <\hat{\eta}_j T_{j,k}^A, \hat{O}T_i^S> = 0.$$

As an aside we notice that $G(\gamma(I))$ leaves also $\mathcal{H}^A$ invariant because the identity operator is obviously strongly diffeomorphism invariant.

ii) The most general group averaging with respect to adapted bases is given by $\eta_{Diff}^{(a)} := a(\gamma(I)) \eta_{Diff}$ and the associated most general inner product is given by $<\eta_{Diff}^{(a)} f, \eta_{Diff}^{(a)} g>_{Diff} = (\eta_{Diff}^{(a)} f, \eta_{Diff}^{(a)} g)(g)$ where $a(\gamma(I)) > 0$ depends only on the orbit of the graph. Let us see which restrictions are imposed by requiring properties a), b) stated in part ii) of the theorem. We begin with b). We have already seen in Theorem (2.1), i) that the $\eta_{Diff} T_i^S$ are orthonormal with respect to the inner product given by $a(\gamma(I)) = 1$ for all $\gamma$. It follows that for any $f, g \in \mathcal{H}_S^g$ we have with $f = \sum_l f_l T_l^S$, $g = \sum_l g_l T_l^S$

$$<\eta_{Diff}^{(a)} f, \eta_{Diff}^{(a)} g>_{Diff} = \sum_{l,j} \hat{f}_l g_j (\eta_{Diff}^{(a)} T_l^S)(T_j^S) = a(\gamma(I)) \sum_l \hat{f}_l g_l = a(\gamma(I)) <f, g>_{aux}$$

which fixes $a(\gamma(I)) = 1$.

We turn to a). Since a strongly diffeomorphism invariant operator $\hat{O}$ leaves $\mathcal{H}^A$ invariant the only non-trivial matrix elements involving $T_{j,k}^A$ are given by $\eta_{Diff}^{(a)} T_{j,k}^A(\hat{O} T_{j,k}^A) = 0$ since $\eta_{Diff}^{(a)} T_{j,k}^A = 0$ and $\eta_{Diff}^{(a)} (\hat{O} T_{j,k}^A) = 0$ since $\eta_{Diff}^{(a)} \hat{O} T_{j,k}^A = 0$ because the averaged vector lies in $\mathcal{H}^A$ again. So on $\mathcal{H}^A$ the operator $\hat{O}$ is trivially self-adjoint because after averaging it is the zero operator. Thus we need to check only whether or not it is true that

$$<\eta_{Diff}^{(a)} f, \eta_{Diff}^{(a)} g>_{Diff} = <\eta_{Diff}^{(a)} f, \eta_{Diff}^{(a)} \hat{O} g>_{Diff}$$

for any $f, g \in \mathcal{H}^S$. However, this is trivial: Since $\hat{O}$ cannot map between functions which are cylindrical with respect to diffeomorphism inequivalent graphs we may assume that $f, g$ are in fact cylindrical with respect to the same graph $\gamma$. Since both lie in $\mathcal{H}^S$ and $\hat{O}$ leaves $\mathcal{H}^S$ invariant all vectors in $\mathcal{H}^S$ have the orbit size of $\gamma$. Thus, since $\hat{O}$ commutes with all diffeomorphisms and is self-adjoint on $\mathcal{H}_{aux}$

$$<\eta_{Diff}^{(a)} \hat{O} f, \eta_{Diff}^{(a)} g>_{Diff} = \sum_{\varphi \in \mathcal{D}(\gamma)} \langle \hat{U}(\varphi) \hat{O} f, g \rangle$$
= \sum_{\varphi \in D(\gamma)} \langle \hat{U}(\varphi)f, g \rangle
\]
\[= \sum_{\varphi \in D(\gamma)} \langle \hat{U}(\varphi)f, \hat{O}g \rangle
\]
\[= \langle \eta_{\text{diff}}f, \eta_{\text{diff}}\hat{O}g \rangle_{\text{diff}}
\] (2.10)

which furnishes the proof.

\[\square\]

Notice that it was crucial in the proof for the satisfaction of the reality conditions that strongly diffeomorphism invariant observables leave the spaces $\mathcal{H}^S, \mathcal{H}^A$ invariant. Interestingly, the Hamiltonian constraint operator defined in [1, 2, 3] maps the space $\mathcal{H}^S$ also into itself because it treats all parts of the graph democratically.

The condition that fixes the value $a([\gamma]) = 1$ is that inner products between vectors on $\mathcal{H}^S_\gamma$ are preserved under group averaging. This implies that the $\eta_{\text{diff}}T^S_I = [T_I]/[\sqrt{n([I])}]$ are a natural orthonormal basis on $\mathcal{H}_{\text{diff}}$. What is the meaning of this condition? It means that the group averaging defined is a partial isometry from $\mathcal{H}_{\text{aux}}$ to $\mathcal{H}_{\text{diff}}$ (partial because $\eta_{\text{diff}}$ has a huge kernel even on $\mathcal{H}^S$). We can interpret this as saying that a reality condition selected the inner product because for a transformation to be a partial isometry means that it satisfies a (partial) adjointness condition with respect to the inner product to be determined.

### 3 The dual Constraint Algebra

Recall that the action of the Hamiltonian $\hat{H}(N)$ constraint on $\Phi$ was inherently equipped with a huge amount of ambiguity which had to do with the precise position of a particular edge to be attached to a graph. In other words, there is a certain infinite-dimensional family of prescriptions $p$ and associated Hamiltonian constraints $\hat{H}^p(N)$ such that if $f_\gamma$ is a function cylindrical with respect to a graph $\gamma$ then there are projections $\hat{H}^p(N)f_\gamma = \hat{H}_{\gamma}^p(N)f_\gamma = \sum_{v \in V(\gamma)} N(v)\hat{H}_{v,\gamma}^p$ where $\hat{H}_{\eta,\gamma}^p$ is an operator which attaches certain edges in between pairs of edges of $\gamma$ incident at the vertex $v$ in a prescription dependent and graph dependent way. However, that prescription is covariantly defined, that is, if we map the graph with a diffeomorphism $\varphi$ then the prescription $p(\varphi(\gamma))$ is such that the operators $\hat{U}(\varphi)\hat{H}_{\eta,\gamma}^p$ and $\hat{H}_{\eta,\varphi(\gamma)}^p$ are related by a diffeomorphism which in general differs from the identity. This has the important consequence that the dual of the operators $\hat{H}^p(N)$, when evaluated on diffeomorphism invariant distributions, are independent of $p$, at least within the same diffeomorphism equivalence class of prescriptions $p$. Thus, with this restriction, the dual operator $\hat{H}'(N)$ defined by $\hat{H}'(N)\Psi(f) := \Psi(\hat{H}^p(N)f)$ is completely independent of the choice of $p$, provided that $\Psi \in \Phi'$ is a diffeomorphism invariant distribution. Now, the problem is that while $\Psi$ is diffeomorphism invariant, $\hat{H}'(N)\Psi$ is not. In fact we have the following theorem which demonstrates that the commutator algebra between dual diffeomorphism and dual Hamiltonian constraint closes in the way expected from the classical Poisson constraint algebra $\{H(M), V(N)\} = H(L_{\tilde{N}M})$ where $V$ is the classical diffeomorphism constraint.

**Theorem 3.1** The quantum constraint algebra between the dual Hamiltonian constraint operator $\hat{H}'(N)$ and the exponentiated diffeomorphism constraint operator $\hat{U}(\varphi)$ is anomaly-
free on diffeomorphism invariant distributions $\Psi$, that is,

$$([\hat{H}(N), \hat{U}(\varphi)])'\Psi = \hat{H}'(\varphi^* N - N)\Psi \quad .$$

(3.1)

Proof:
Notice that it makes sense to check the constraint algebra only on diffeomorphism invariant elements of $\Phi'$ because this is where $\hat{H}'(N)$ is defined. Moreover, since $(\hat{U}(\varphi))'\Psi = \Psi$ it makes sense to compute the commutator.

So let $f_\gamma$ be a function in $\Phi$ cylindrical with respect to a graph $\gamma$. Then we have by definition

$$-(([\hat{H}(N), \hat{U}(\varphi)])'\Psi(f_\gamma) = ([\hat{H}'(N), \hat{U}'(\varphi)\hat{H}'(N)]\Psi(f_\gamma) = \Psi([\hat{H}'(N) - \hat{H}'(N)\hat{U}(\varphi)])f_\gamma)$$

$$= \Psi(\hat{H}'(N)f_\gamma - \hat{H}'(N)f_{\varphi(\gamma)})$$

$$= \sum_{v \in V(\gamma)} \{N(v)\Psi(\hat{H}_{v,\gamma}^{p(\gamma)} f_\gamma) - N(\varphi(v))\Psi(\hat{H}_{v,\varphi(\gamma)}^{p(\varphi(\gamma))} f_{\varphi(\gamma)})\}$$

$$= \sum_{v \in V(\gamma)} [N(v) - N(\varphi(v))\Psi(\hat{H}_{v,\gamma}^{p(\gamma)} f_\gamma)$$

$$= (\hat{H}'(N - \varphi^* N)\Psi)(f_\gamma) \quad .$$

(3.2)

The only non-trivial step in this computation has been the last one in which we have used the fact mentioned before that the prescription is covariantly defined and that the images of the Hamiltonian constraint on diffeomorphic vectors are diffeomorphic so that the difference is not seen by $\Psi$.

\square

We also recall the following theorem from [10].

**Theorem 3.2** The commutator algebra of the exponentiated diffeomorphism constraints closes, that is,

$$\hat{U}(\varphi)\hat{U}(\varphi')\hat{U}^{-1}(\varphi)\hat{U}^{-1}(\varphi') = \hat{U}(\varphi \circ \varphi' \circ \varphi^{-1} \circ (\varphi')^{-1}) \quad .$$

(3.3)

Proof:
The proof follows trivially from the fact that $\hat{U}(\varphi)f_\gamma = f_{\varphi(\gamma)}$ which in particular shows that $\hat{U}$ is a unitary representation of the diffeomorphism group on $\mathcal{H}$.

\square

Theorems (3.1),(3.2) mean that the quantum Dirac algebra closes in the appropriate sense between Diffeomorphism constraints among themselves on $\mathcal{H}$ and between Diffeomorphism constraints and the dual of the Hamiltonian constraint on $\mathcal{H}_{Diff}$.

What remains to be shown is that the algebra closes between two (duals of) Hamiltonian constraints. In fact, the following theorem was proved in [2].

**Theorem 3.3** The quantum constraint algebra of the Hamiltonian constraint closes, that is,

$$((\hat{H}(M), \hat{H}(N)))'\Psi(f_\gamma) = 0 \quad .$$

(3.4)

for any diffeomorphism invariant $\Psi \in \Phi'$, any $f_\gamma \in \Phi$ and any $M, N$. 

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Formula (3.4) is unsatisfactory for two reasons:
1) On the left hand side one would like to see something like \([\hat{H}'(M), \hat{H}'(N)]\) rather than \((\hat{H}(M), \hat{H}(N))]\).
2) On the right hand side one would like to see something like \(\int d^3 x(M_a N - MN_a)q^{ab}\hat{V}_b\) rather than just zero.

The problem with issue 1) is that the product operator \(\hat{H}'(M)\hat{H}'(N)\) does not exist on \(\mathcal{H}_{Diff}\). The problem with issue 2) is that none of the operators \(\hat{V}_b\) exist even on \(\mathcal{H}\). Moreover, even if the right hand side would be of that kind, how would one tell the difference between this operator and the zero operator? The point is that whenever the algebra is anomaly free then the right hand side must annihilate a diffeomorphism invariant state.

In the sequel we will somewhat improve (3.2) as follows:
First we emphasize that the prescription \(p\) of [2] was designed such that not only \(\hat{H}'(N)\Psi\) but also the dual of arbitrary products \(\hat{H}^p(N_1)\ldots\hat{H}^p(N_n)\), evaluated on diffeomorphism invariant states, for arbitrary \(N_i\) is independent of \(p\), that is, there exists a well-defined operator \([\hat{H}(N_1), \ldots, \hat{H}(N_n)]\)\). This follows immediately from the fact that in [2] the prescription was covariantly defined, that is, the prescription applied to graphs that differ by a diffeomorphism \(\varphi\) yields graphs that differ by some diffeomorphism \(\varphi'\) which does not need to equal \(\varphi\). We stress this point here one more time for the sake of clarifying things, however, it is already contained in [2]. This fact can be used to define products of dual operators indirectly to be the dual of the product and this was actually done in Theorem 3.1 and section 3.4 of [2]. (For a different approach that involves a certain enlargement of the space of diffeomorphism invariant distributions and therefore enables to define products of dual operators directly see [23]). Thus, we are able to remedy issue 1) above. As expected, the dual operator algebra turns out to be completely Abelian as one can immediately see from Theorem 3.3 and as pointed out in [2].

Secondly, it turns out that one can quantize the classical expression \(\int d^3 x(M_a N - MN_a)q^{ab}\hat{V}_b\) in the sense that one can turn it into a densely defined operator \(\hat{O}(M, N)\) on \(\mathcal{H}\). We then show that \(\hat{O}(M, N)\Psi = 0\) for any diffeomorphism invariant \(\Psi \in \Phi'\). This remedies issue 2) above.

In conclusion we can write the operator equality (on \(\mathcal{H}_{Diff}\)) \([\hat{H}'(M), \hat{H}'(N)] = \hat{O}(M, N)\) which is as close as we can get to representing faithfully the quantum Dirac algebra in the present framework which suffers from the unavoidable fact that \(\hat{H}(N)\) is state dependently defined (in other words, \(A_i^a\) is not defined, only holonomies along paths are defined and the choice of path is the source of the ambiguity).

We begin with issue 1).

**Theorem 3.4** Let \(\Psi \in \Phi'\) be a diffeomorphism invariant state and \(f \in \Phi\). Then \(\Psi(\hat{H}^p(N_1)\ldots\hat{H}^p(N_n)f)\) depends only on the diffeomorphism invariant properties of the prescription \(p\) for any \(N_1, \ldots, N_n\) and any \(n = 1, 2, \ldots\)

**Proof**: We can assume that \(f\) is cylindrical with respect to a graph \(\gamma\). Also, the ambiguity is already completely encoded in the prescription dependence of the Euclidean Hamiltonian constraint \(\hat{H}^p_E\) because the Lorentzian constraint is just a polynomial of these with operator valued coefficients that leave the graph invariant [2].

Now, whenever \(\hat{H}^p_E\) acts on a function cylindrical with respect to a graph \(\gamma\) then it reduces to a sum of terms \(\hat{H}^p_{E,v,v',\gamma}\), one for each vertex \(v \in V(\gamma)\) and one for each pair
\[ e, e' \in E(\gamma) \text{ of edges of } \gamma \text{ incident at } v. \] The prescription \( p \) assigns to each \( \hat{H}^p_{E,v,v'; e,e'} \) proper segments \( s \subset e, s' \subset e' \) incident at \( v \) and an arc \( a \) incident at the endpoints of \( s, s' \) and not intersecting the graph anywhere else together with a diffeomorphism invariant condition about the topology of the routing of \( a \) through the edges of \( \gamma \) incident at \( v \). The “length” of \( s, s' \) and the “shape” of \( a \) is the whole source of the prescription dependence. The point is that when \( \hat{H}^p_E \) acts again, the new arcs to be attached will never intersect the old arcs already attached because at each stage \( s, s' \) are always proper segments of \( e, e' \). Consider now a particular term

\[ \Psi(\hat{H}^p_{E,v_1,v_1',e_1,e_1'}, \hat{H}^p_{E,v_2,v_2',e_2,e_2'}, \ldots, \hat{H}^p_{E,v_n,v_n',e_n,e_n'}) \]  

in the expansion of \( \Psi(\hat{H}^p_E(N_1) \ldots \hat{H}^p_E(N_n)f) \). If it does not vanish then \( \Psi \) is a linear combination of group averaged cylindrical functions and one of them, \( \Psi' \), depends on the graph \( \gamma' = \gamma \cup a_1 \cup \ldots \cup a_n \) and it is the only term of \( \Psi \) that contributes to \( (3.3) \). Here \( \gamma' \subset \gamma \) is the graph \( \gamma \) with possibly some of the \( s_i, s'_i \) removed \([3, 3]\). Now, since we allow for smooth diffeomorphisms that leave a graph analytic, there are smooth diffeomorphisms involved in the definition of \( \Psi' \) that leave \( \gamma \) invariant and \( \gamma' \) analytic and that alter the length of \( s_i, s'_i \) and the shape of \( a_i \) arbitrarily while keeping the topology of \( \gamma' \) invariant (arcs cannot intersect each other or \( \gamma \) other than in their endpoints but they can slide up and down along \( e, e' \) and change their shapes). It follows that \((3.5)\) does not depend on \( p \) but only on its diffeomorphism class \([p]\).

\[ \square \]

**Corollary 3.1** The quantity \([\hat{H}(N_1) \ldots \hat{H}(N_n)]'\Psi \) defined by \( \Psi(\hat{H}^p(N_1) \ldots \hat{H}^p(N_n)f) \) is well-defined, that is, only depends on the (fixed) diffeomorphism class \([p]\).

**Definition 3.1** Let \( \mathcal{B} \) be the vector space of sums of products of operators \( \hat{H}^p(N) \) for some \( p \) and let \( \mathcal{B}' \) be the set of dual operators defined by \( \mathcal{B}' = \{B'; \exists B \in \mathcal{B} \exists [B'\Psi](f) = \Psi(Bf)\forall \Psi \in \Phi', f \in \Phi\} \) where again \( \Phi' \) is the diffeomorphism invariant subset of \( \Phi \). We turn the vector space \( \mathcal{B}' \) into an associative algebra by defining a product by \( B'_1B'_2 := (B_2B_1)' \). Notice that the product is a well-defined map from \( \mathcal{B}' \times \mathcal{B}' \) to \( \mathcal{B}' \) by Corollary \((3.4)\).

**Corollary 3.2** The algebra \( \mathcal{B}' \) is Abelian.

**Proof**:

Notice that \([B'_1,B'_2]' = ([B_2,B_1])' \) and write \( \Psi([B_2,B_1]f) \) as sums of expressions of the form \( \Psi(B[H^p(M),\hat{H}^p(N)]'f') \) where \( B \in \mathcal{B} \) and \( f' \in \Phi \). Now use Theorem \((3.4)\) in the argument of the proof of Theorem \((3.3)\) (or 3.1 in \([2]\)) in order to complete the argument. (Essentially, one needs to show that one can slide the arcs along \( e, e' \) despite the presence of the operator \( B \). This can be accomplished by straightforward application of Theorem \((3.4)\).

\[ \square \]

Next we turn to issue 2) : In the last part of this section we show that one can indeed define an operator corresponding to the right hand side of \( \{H(M), H(N)\} \) whose dual annihilates diffeomorphism invariant distributions.
Let $\omega_a := MN_a - M_aN$ and consider the triple of vector fields $v_a$ with components $(v_a)^b := \delta_b^a$ as well as the triple of one forms $\sigma^b$ with components $(\sigma^b)_a := \delta_b^a$. Let $\chi_\epsilon(x, y)$ be the characteristic function of a coordinate box with center $x$ and coordinate volume $\epsilon^3$. The volume of the box as measured by $q_{ab}$ is given by $V(x, \epsilon) := \int d^3y \chi_\epsilon(x, y) \sqrt{\det(q)}(y)$. Finally, let $V_a = \text{tr}(F_{ab}E^b)$ be the classical infinitesimal generator of diffeomorphisms. With this preparation we have the following classical identity

\[
O(M, N) := \int d^3x (\omega_a q^{ab}V_b)(x)
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{4\epsilon^6} \int d^3x (\omega_a \epsilon^{abe} \epsilon_{ijk} \frac{e^i_j e^k_e}{\sqrt{\det(q)}}(x) \times
\]

\[
\times \int d^3y ((\sigma^a)_d \epsilon^{def} \epsilon_{ilm} \frac{e^i_l e^m_j}{\sqrt{\det(q)}}(y) \int d^3z \chi_\epsilon(x, z) \chi_\epsilon(y, z)((v_a)^b V_b)(z)
\]

\[
= \lim_{\epsilon \to 0} \frac{\epsilon_{ijklm}}{4\epsilon^6} \int (\omega \wedge \frac{e^i_j e^k_e}{\sqrt{\det(q)}}(x) \times
\]

\[
\times ((\sigma^a) \wedge \frac{e^i_j e^m_k}{\sqrt{\det(q)}})(y) \int d^3z \chi_\epsilon(x, z) \chi_\epsilon(y, z)((v_a)^b V_b)(z)
\]

\[
= \lim_{\epsilon \to 0} \frac{\epsilon_{ijklm}}{16\epsilon^6} \int (\omega \wedge \frac{A^i_j V(x, \epsilon)}{V(x, \epsilon)} \wedge \frac{A^k_m V(x, \epsilon)}{V(x, \epsilon)}(x) \times
\]

\[
\times ((\sigma^a) \wedge \frac{A^i_j V(y, \epsilon)}{V(y, \epsilon)} \wedge \frac{A^m_k V(y, \epsilon)}{V(y, \epsilon)})(y) \times
\]

\[
\times \int d^3z \chi_\epsilon(x, z) \chi_\epsilon(y, z)((v_a)^b V_b)(z)
\]

\[
= \lim_{\epsilon \to 0} \frac{\epsilon_{ijklm}}{16\epsilon^6} \int (\omega \wedge \frac{A^i_j \sqrt{V(x, \epsilon)}}{\sqrt{V(x, \epsilon)}} \wedge \frac{A^k_m \sqrt{V(x, \epsilon)}}{\sqrt{V(x, \epsilon)}})(x) \times
\]

\[
\times ((\sigma^a) \wedge \frac{A^i_j \sqrt{V(y, \epsilon)}}{\sqrt{V(y, \epsilon)}} \wedge \frac{A^m_k \sqrt{V(y, \epsilon)}}{\sqrt{V(y, \epsilon)}})(y) \times
\]

\[
\times \int d^3z \chi_\epsilon(x, z) \chi_\epsilon(y, z)((v_a)^b V_b)(z)
\]

(3.6)

where in the first step we just have written $q^{ab}$ in terms of $\epsilon^{i}_a$, in the second step we have rewritten everything in terms of forms, in the third step we have used the key identity [4, 2]

\[2\sqrt{\text{sgn}(\det(\epsilon_e^i)))} \epsilon^i_a = \{A^i_a(x), V(x, \epsilon)\},\]

in the fourth step we used that $V(x, \epsilon)/\epsilon^3 \to \sqrt{\det(q)}(x)$ as $\epsilon \to 0$ and finally in the last step we just used $\{A_a(x), \sqrt{V(x, \epsilon)}\} = \{A_a(x), V(x, \epsilon)\}/(2\sqrt{V(x, \epsilon)})$.

Expression (3.3) is just one example of a general theorem proved in [20] which says that singular prefactors of $\sqrt{\det(q)}$ can always be absorbed into Poisson brackets which
then are subject to quantization as follows:

Given a function \( f_\gamma \) cylindrical with respect to a graph \( \gamma \) we triangulate \( \Sigma \) in adaption to \( \gamma \) as in [2]. We leave the the quantization \( \hat{V}_a \) of \( V_a \) open at this point and will specify it only later. For the moment being we just order it to the left hand side and replace \( \epsilon \) for each edge \( \gamma \) to \( \tau \) with the algebra \( [\tau_i, \tau_j] = \epsilon_{ijk}\tau_k \) so that \( \text{tr}(\tau, A) = -A^i/2 \). Each expression of the form \( \text{tr}(\tau_i h_s[h_s^{-1}, \hat{V}^{1/2}(v, \epsilon)])f_\gamma \), where \( v \) is a vertex of \( \gamma \) and \( s \) a segment of an edge starting at \( v \), does not depend on \( s \) because it must be gauge invariant at the endpoint of \( s \). Since the volume operator only acts at vertices of \( \gamma \) we find in complete analogy with computations in [2] the regulated operator

\[
\hat{O}_\epsilon(M, N)f_\gamma = \frac{16\epsilon_{ijkilm}}{h^2K^2} \sum_{v, v' \in V(\gamma)} \int d^3z \chi_\epsilon(v, z)\chi_\epsilon(v', z)((v_a)^b\hat{V}_b)(z) \times \\
\times \sum_{v(\Delta)=v} \epsilon_{npq}[M(v)N(s_n(\Delta)) - M(s_n(\Delta))N(v)] \times \\
\times \text{tr}(\tau_j h_{sp(\Delta)}[h_{sp(\Delta)}^{-1}, \hat{V}^{1/2}(v, \epsilon)])\text{tr}(\tau_k h_{sq(\Delta)}[h_{sq(\Delta)}^{-1}, \hat{V}^{1/2}(v, \epsilon)]) \times \\
\times \sum_{v(\Delta')=v'} \epsilon_{rst}(\sigma^d)_c[s_c^*(\Delta') - v'^c] \times \\
\times \text{tr}(\tau_i h_{s(\Delta')}[h_{s(\Delta')}^{-1}, \hat{V}^{1/2}(v', \epsilon)])\text{tr}(\tau_m h_{s(\Delta')}[h_{s(\Delta')}^{-1}, \hat{V}^{1/2}(v', \epsilon)])f_\gamma
\]

(3.7)

where in abuse of notation \( s_n(\Delta) \) denotes also the endpoint of the segment \( s_n(\Delta) \) whose starting point is \( v(\Delta) \).

Fix the value of \( a, v, v' \) and let now \( \xi^b(z) := (v_a)^b\chi_\epsilon(v, z)\chi_\epsilon(v', z) \). We replace \( E^a_i \) by \( \hat{E}^a_i(z) = -i\hbar\delta/\delta A^i_a \) in \( V_b \) and order \( \hat{E} \) to the right hand side. Let \( f'_\gamma \) be any function cylindrical with respect to \( \gamma \). Also we abbreviate

\[
X^i_e(t) := [h_x(0, t)\tau_i h_x(t, 1)]_{AB}\partial/\partial(h_x(0, 1))_{AB}
\]

for each edge \( e \) of \( \gamma \). Then we have

\[
\int d^3x \xi^a(x)\hat{V}_a(x)f'_\gamma = -i\hbar\kappa \int d^3x \xi^a(x)F^i_{ab}(x) \sum_{e \in E(\gamma)} \frac{\delta(h_x(0, 1))_{AB}}{\delta A^i_b(x)} \frac{\partial f'_\gamma}{\partial(h_x(0, 1))_{AB}} \\
= -i\hbar\kappa \sum_{e \in E(\gamma)} \int d^3x \xi^a(x)F^i_{ab}(x) \int_0^1 dt \delta(x, e(t))\epsilon^b(t)X^i_e(t)f'_\gamma \\
= -i\hbar\kappa \sum_{e \in E(\gamma)} \int_0^1 dt \xi^a(e(t))F^i_{ab}(e(t))\epsilon^b(t)X^i_e(t)f'_\gamma \\
= -i\hbar\kappa \sum_{e \in E(\gamma)} \lim_{n \to \infty} \sum_{k=1}^n \xi^a(e(t_k))F^i_{ab}(e(t_k))[\epsilon^b(t_k) - \epsilon^b(t_{k-1})]X^i_e(t_k)f'_\gamma \tag{3.8}
\]

for any partition \( 0 = t_0 < t_1 < .. < t_n = 1 \) of \( [0, 1] \) which we may choose individually for each edge \( e \).
We now combine (3.7) and (3.8). We choose to take first the $\epsilon \to 0$ limit. (This exchange of limits is justified by the fact that one can alternatively also keep $\epsilon \approx 1/n$ finite and and still arrive at the quantum result below, the virtue being that this way of guiding the limit also gives back the original function $O(M,N)$ on the classical side. See 22 for details). The product of characteristic functions $\chi_{\varepsilon}(v,e(t_k))\chi_{\varepsilon}(v',e(t_k))$ will vanish for any $k,n$ if $v \neq v'$ when choosing $\epsilon$ sufficiently small. We conclude that only the terms with $v = v'$ survive in (3.7). Next, since $v$ is a vertex for sufficiently small $\epsilon$, $\chi_{\varepsilon}(v,e(t_k))$ vanishes (keeping $n$ fixed) unless $k = 0$ because we have labelled edges $e$ such that only their starting point can be a vertex of $\gamma$. Therefore in the limit $\epsilon \to 0$ the sum over $k$ in (3.8) disappears and the dependence on the triangulation $\gamma$ vanishes (keeping $n$ fixed) unless $k = 0$ because we have labelled edges $e$ such that only their starting point can be a vertex of $\gamma$. Therefore in the limit $\epsilon \to 0$ the sum over $k$ in (3.8) disappears and the dependence on $\epsilon$ disappears when we replace $\hat{V}(v,e)$ by the operator $\hat{V}$ as explained in 22. Finally we contract the indices $a$ in $\xi^a(e(0)) = \delta^b_a$ from (3.8) and in $(\sigma^a)_c[s^c_r(\Delta') - v^c] = [s^a_r(\Delta') - v^a]$ and find that (3.7) becomes

$$\hat{O}_n(M,N)f\gamma = -i\frac{16\epsilon_{ijk}\epsilon_{ilm}}{h^3k_h} \sum_{v \in V(\gamma)} \sum_{e \in E(\gamma), e(0) = v} \sum_{v(\Delta) = v(\Delta')} \times$$

$$\times [s^a_r(\Delta') - v^a]F_{ab}(v)[s(e)^b - v^b]X^i_e[M(v)N(s_n(\Delta)) - M(s_n(\Delta))N(v)] \times$$

$$\times \text{tr}(\tau_\gamma h_{s_r(\Delta)}[h^{-1}_{s_r(\Delta)},\hat{V}_{v'}^{1/2}])\text{tr}(\tau_\gamma h_{s_r(\Delta)}[h^{-1}_{s_r(\Delta)},\hat{V}_{v'}^{1/2}])\times$$

$$\times \text{tr}(\tau_\gamma h_{s_r(\Delta)}[h^{-1}_{s_r(\Delta)},\hat{V}_{v'}^{1/2}])\text{tr}(\tau_\gamma h_{s_r(\Delta)}[h^{-1}_{s_r(\Delta)},\hat{V}_{v'}^{1/2}])f\gamma$$

(3.9)

where we have used that the operators in the second and third line do not change the graph $\gamma$ and $X^i_e = X^i_e(0)$ is the right invariant vector field on $SU(2)$ evaluated at $h_e(0,1)$. Finally, let us choose $t_1$ for each $\epsilon$ individually such that $e(t_1)$ coincides with the segment of $e$ that appears as an edge $s_r(\Delta)$ of one or several tetrahedrons $\Delta$ of the triangulation with $v(\Delta) = v$ and let us call this segment $e(t_1) = s_r(\Delta) =: s(e)$. Then (3.9) only depends on the triangulation $T$. The final step is to replace the curvature expression $[s^a_r(\Delta') - v^a]F_{ab}(v)[s(e)^b - v^b]$ by $h_{\alpha(s_r(\Delta'),s(e))} - 1_{SU(2)}$ where $\alpha(s_r(\Delta'),s(e))$ is the loop $s_1 \circ s_2 \circ s_3^{-1} \circ s_4^{-1}$ where $s_1 = s_r(\Delta')$, $s_4 = s(e)$ and $s_2$, $s_3$ respectively are some arbitrarily chosen segments which do not intersect $\gamma$ except in the end points of $s_1$, $s_4$ such that the loop $\alpha$ looks like a little parallelogramm in the coordinate frame under consideration. Moreover, the “arc with a corner” $s_2 \circ s_3^{-1}$ is subject to the same routing conditions as the arcs of tetrahedra (see [2] for details).

Now consider the combination

$$[s^a_r(\Delta') - v^a]F_{ab}(v)[s(e)^b - v^b]X^i_e \approx \text{tr}(h_{\alpha(s_r(\Delta'),s(e))}h_e(0,1)\partial_{h_e(0,1)}) - \text{tr}(h_e(0,1)\partial_{h_e(0,1)})$$

and compare it with the infiniteesimal variation of $f_\gamma'$ under the deformation of $e = s(e) \circ e'$ into $\tilde{e} = \alpha(s_r(\Delta'),s(e)) \circ e = s_1 \circ s_2 \circ s_3^{-1} \circ e'$. We find (we suppress all arguments different from $h_e$)

$$\delta f_\gamma' := f_\gamma'(h_e) - f_\gamma'(h_e) = f_\gamma'(h_{\alpha(s_r(\Delta'),s(e))}h_e) - f_\gamma'(h_e) =$$

$$= f_\gamma'(h_e + [h_{\alpha(s_r(\Delta'),s(e))} - 1_{SU(2)}]h_e) - f_\gamma'(h_e) =$$

$$= \text{tr}(h_{\alpha(s_r(\Delta'),s(e))} - 1_{SU(2)}h_e\partial_{h_e})f_\gamma' + o([h_{\alpha(s_r(\Delta'),s(e))} - 1_{SU(2)}]^2)$$

$$= [\hat{U}(\varphi(s_r(\Delta'),s(e))) - \text{id}_H]f_\gamma' + o([h_{\alpha(s_r(\Delta'),s(e))} - 1_{SU(2)}]^2)$$

(3.10)

where in the last step we have written the variation of $f_\gamma$ as the change under a corresponding diffeomorphism $\varphi(s_r(\Delta'),s(e))$. Clearly, such a diffeomorphism will not only
deform \( e \) but also the edge of \( \gamma \) which has \( s_r(\Delta') \) as a segment. However, one may choose \( \varphi(s_r(\Delta'), s(e)) \) such that the corresponding change of \( f'_\gamma \) is of higher order.

Expression (3.10) shows that (up to higher order terms which vanish in the continuum limit) the operator \( V_b \) can be multiplicatively regulated by multiplying it with a small but finite line element (here given by \( \xi^a \)) and writing it as a small but finite diffeomorphism minus the identity operator. This is because the infinitesimal operator does not make any sense on \( \mathcal{H} \) as mentioned earlier and since \( \xi^a \) comes from \( q^{ab} \), it happened what was to be expected: While \( q^{ab} V_b \) individually do not make sense as operators, the combination \( q^{ab} V_b \) in fact does make sense.

Thus, combining (3.9) and (3.10) we obtain the desired end result (\( T \) reminds us of the triangulation dependence)

\[
\hat{O}_T(M, N) f_\gamma = -i \frac{16 \epsilon_{ijk} \epsilon_{ilm} \epsilon_{rst} \epsilon_{npq}}{h^3} \sum_{v \in V(\gamma)} \sum_{e \in E(\gamma)} \sum_{\tau(v(\gamma)) = v} \sum_{e(0) = v} \sum_{u(\Delta) = v} \tau
\times \left[ \hat{U}(\varphi(s_r(\Delta'), s(e))) - \text{id}_\mathcal{U} \right] \left[ M(v) N(s_n(\Delta)) - M(s_n(\Delta)) N(v) \right] \times
\times \text{tr}(\tau \hat{h}_{s_p(\Delta)} \hat{V}_{s_p(\Delta)}^{1/2}) \text{tr}(\tau \hat{h}_{s_n(\Delta)} \hat{V}_{s_n(\Delta)}^{1/2}) \times
\times \text{tr}(\tau h_{s_p(\Delta)} \hat{V}_{s_p(\Delta)}^{1/2}) \text{tr}(\tau h_{s_n(\Delta)} \hat{V}_{s_n(\Delta)}^{1/2}) f_\gamma .
\]

The only point which is important is that that the diffeomorphism operator \( \hat{U}(\varphi(s_r(\Delta'), s(e))) \) stands to the left in (3.11) which implies that when evaluating (3.11) on a diffeomorphism invariant distribution \( \Psi \), the result vanishes. That such an ordering is possible while still obtaining a (prescription dependent) densely defined operator is nevertheless a non-trivial result.

Moreover, \( O_T(M, N) \) is clearly just like \( \hat{H}(M) \) prescription dependent but its dual \( \hat{O}'(M, N) \) is not. We may therefore simply set \( [\hat{H}'(N), \hat{H}'(M)] = \hat{O}'(M, N) \) on \( \Psi_{\text{Diff}} \).

Formula (3.11) deserves several comments:

1) The immediate question is: Can we do even better than this? That is, can one recover (3.11) by simply evaluating the commutator \([\hat{H}(M), \hat{H}(N)] f_\gamma \)? Comparing (3.11) with the expression found in [2] for the commutator shows some similarities but also some key differences:

The commutator, at least for the Euclidean Hamiltonian constraint, involves two factors of operators \( \hat{V} \) while (3.11) involves four factors of \( \hat{V}^{1/2} \). Thus, while the overall power is the same, the factor ordering is different. Next, the commutators involve two loops but (3.11) only one. Checking on generic states \( f_\gamma \) reveals that the two loops do not in general combine into only one loop. Trial and error reveals that one has a chance to combine the two loops into one if and only if the Hamiltonian constraint acts also at the vertices it creates. However, then one finds that when evaluating on diffeomorphism invariant states, the result does not vanish any longer.

This is therefore a major difference which seems insurmountable and shows that the commutator and (3.11) are simply very different quantizations of the same classical object (meaning that, since the classical expression \( O(M, N) \) can be written as a Poisson bracket, we may choose to quantize it through the corresponding commutator).

The deeper reason for this difference is the following: when one does the classical computation to prove that indeed \( \{H(M), H(N)\} = O(M, N) \), one has to do several rather intricate algebraic manipulations and integrations by part. In
particular, the reason why one curvature term disappears is because the Poisson bracket picks the partial derivative of $A$ in $F$. None of these manipulations can have a quantum analogue because $F, A$ are not well-defined operators on $\mathcal{H}$. It is therefore not surprising that the commutator does not give (3.11). Finally, it is reassuring that the combination $[M(v)N(s_n(\Delta)) - M(s_n(\Delta))N(v)]$ in (3.11) does appear in the commutator.

2) That $\hat{O}(M, N)$ can be given a meaning at all is astonishing at first because we know that the infinitesimal generator of $\hat{U}(\varphi)$ as well as $\hat{q}^{ab}$ do not exist on $\mathcal{H}$. However, since both operators combine to a density of weight one, the structure theorem proved in [20] applies and it is therefore clear that the integrated expression $O(M, N)$ can be promoted to a densely defined operator on $\mathcal{H}$.

3) Concluding, we see that the quantum Dirac algebra closes in the appropriate sense and gives a faithful representation of the classical Dirac algebra in the maximal possible way given the choice of representation $\mathcal{H}$.

4 Θ moduli, Superselection Sectors and Separability

In this short section we mainly wish to express a speculation which needs further investigation to make it definite. The issue is the following:

Notice that we have group averaged the states with respect to (subsets of) smooth diffeomorphisms and that graphs were piecewise analytic. Consider a graph with an $n$-valent vertex $v$. The diffeomorphism group at $v$ reduces to a $GL(3, \mathbb{R})$ matrix and so one has at most $d^2 = 9$ free parameters to change the tangent directions of all $n$ edges incident at $v$. The number of angles $\theta$ (with respect to some background metric) between those tangents is given by $n(n - 1)/2$ which for $n \geq 5$ exceeds 9: This is the first example of the appearance of the $\theta$ moduli, that is, there is diffeomorphism invariant information encoded in the intersections of analytic edges. These moduli take values in a compact subset of $\mathbb{R}$ with the cardinality of $\mathbb{R}$. It is these $\theta$ moduli which make $\mathcal{H}_{Diff}$ a non-separable Hilbert space! Other kinds of moduli which are invariant under smooth, that is, $C^\infty$ diffeomorphisms are the the degrees of differentiability of intersections of analytic edges, however, these moduli are integer valued and therefore do no spoil separability.

The question arises whether these moduli have any physical significance or are just an artefact of the fact that we are dealing with piecewise analytic graphs and smooth analyticity preserving diffeomorphisms. In fact, in [29] it is shown that if one works, for instance, with piecewise linear graphs or simply enlarges the diffeomorphism group, then the moduli disappear. Another striking feature of the formalism is that geometric operators like length, area and volume which one would expect to depend on those moduli actually do not depend on them as pointed out in [30] which is an immediate consequence of the fact that their spectrum is discrete. In fact, the only diffeomorphism invariant moduli that play a role are orientations of certain tangent directions and so are of the discrete type [31]. The speculation is that none of the physical operators can detect the continuous moduli. By this we mean the following:

Notice that the number of moduli parameters is countable: a concrete counting scheme is provided by numbering graphs by some integers corresponding to the number of vertices together with their valences (this fixes the number of edges) some integers that fix the
routing of edges between vertices and some integers \( k \) that label \( C^k \) properties of differentiable intersections of analytic edges (we do not need continuous parameters labelling the germs of analytic edges because we average with respect to analyticity preserving smooth diffeomorphisms rather than analytic diffeomorphisms). Let \( \theta := (\theta_1, \theta_2, ..) \) denote a sequence of such a complete set of moduli. Since functions cylindrical with respect to graphs labelled with different moduli parameters are orthogonal with respect to \( \mathcal{H} \) they continue to be orthogonal with respect to \( \mathcal{H}_{\text{Diff}} \) by the very definition of the group averaging proposal. Thus, the Hilbert space \( \mathcal{H}_{\text{Diff}} \) breaks up into a continuous direct sum

\[
\mathcal{H}_{\text{Diff}} = \bigoplus_{\theta} \mathcal{H}_{\text{Diff}}^\theta
\]

and each of the Hilbert spaces \( \mathcal{H}_{\text{Diff}}^\theta \) corresponds to a fixed sequence \( \theta \), all of them being naturally isomorphic to each other. Notice that the Hamiltonian constraint does not mix these sectors because it attaches arcs to the graph in such a way that the new vertices are tri-valent for which there are no continuous moduli. Moreover, \( \mathcal{H}_{\text{Diff}}^\theta \) is separable: we have indicated above how to count graphs and otherwise the set of labellings of edges and vertices with spins and contractors also is countable. Thus, the corresponding spin-network states provide an explicit orthonormal and countable basis.

**\( \Theta \)-Superselection Assumption**

*The continuous moduli are superselected, that is, no physical observable maps between states of different sectors \( \mathcal{H}_{\text{Diff}}^\theta \).*

Summarizing, if the \( \Theta \)-Superselection Assumption turns out to be correct (and the present results indicate that this might be the case) then we may restrict ourselves to a separable Hilbert space by fixing a value of \( \theta := \theta_0 \). This means, for instance, that if an operator creates new vertices on a graph with valence five or higher then it must create them with the correct value of the moduli corresponding to our choice \( \theta_0 \).

### 5 An inner product on the space of physical solutions

In [3] a general algorithm was displayed to construct all solutions of the Lorentzian Hamiltonian constraint (and diffeomorphism constraint) of pure quantum general relativity. A huge number of solutions have the surprising feature that they are normalizable with respect to the inner product for solutions of the diffeomorphism constraint. The treatment was incomplete in the sense that no inner product for the solutions to all constraints was constructed which is capable to render normalizable also solutions which are not normalizable with respect to the inner product to the diffeomorphism constraint. The reason for why such an inner product is not entirely straightforward to construct is because the favoured version of the Hamiltonian constraint operator constructed in [3] is not self-adjoint (also a symmetric version of the Hamiltonian constraint operator was constructed in [3] but it involves a rather unnatural technique which makes it less appealing). This fact makes the group averaging technique to construct an inner product as given in [10, 24, 23] inapplicable.

On the other hand, arguments first raised in [32] suggest that \( \hat{H}(M) \) must not be self-adjoint in order that the algebra closes. In fact, the symmetric operator \( \hat{H}'(M) + \hat{H}'(M)^\dagger \) is easily seen to be anomalous.
We therefore conclude that we need to deal with the not self-adjoint operator $\hat{H}(M)$
and are asked to invent a group averaging technique for not self-adjoint operators. The
following quantum mechanical example will hint how that might be done in the case of
interest.

## 5.1 A two-oscillator example

We consider the Hilbert space $\mathcal{H} := L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$ corresponding to a system with
two degrees of freedom. $\mathcal{H}$ plays the same role as $\Phi'_{\text{Diff}}$ does in quantum general relativity.
In order to model the Hamiltonian constraint of quantum gravity it will be sufficient for
our purposes to consider only its Euclidean part $\hat{H}^E(M)$ which is simpler, at least to start
with (we will generalize it later to the Lorentzian case). The most important property
of $\hat{H}^E(M)$ is that it creates new edges when applying it to a function cylindrical with
respect to a graph $\gamma$. Thus, we may think of $\hat{H}^E(M)$ as some kind of creation operator in
harmonic oscillator terminology where the excitation of the harmonic oscillator is replaced
by an excitation of the gravitational field along the new edge. We therefore study the
following oversimplified model operator corresponding to $\hat{H}^E(M)$

$$\hat{C} := \hat{a}^\dagger - \bar{c}\hat{b}^\dagger$$  (5.1)

where $c$ is an arbitrary complex number and $\hat{a}, \hat{b}$ respectively are the usual annihilation
operators associated with the $x, y$ coordinate respectively. Clearly the operator $\hat{C}$ is far
from being self-adjoint on $\mathcal{H}$ and so we cannot use the usual group averaging procedure
to construct an inner product on the space of its solutions.

However, this model is simple enough so that all the solutions to $\hat{C}$ can be constructed in
closed form, just as in the gravitational case. The complete solution space will suggest
a natural inner product theron.

So, let $|m, n> \text{ be the usual harmonic oscillator eigenfunctions, that is, } \hat{a}|m, n> = \sqrt{m}|m-1, n> \text{, } \hat{a}^\dagger|m, n> = \sqrt{m+1}|m+1, n> \text{ so that } [\hat{a}, \hat{a}^\dagger] = 1 \text{ and similarly for } \hat{b}$. In
fact, $\hat{a}^\dagger$ almost precisely models the operator $\text{tr}(h_\alpha - h_\alpha^{-1})h_s[h_s^{-1}, \hat{V}]$ which is the basic
building block of the Euclidean Hamiltonian constraint. Here $\alpha$ is a a loop bounding a face
of a tetrahedron $\Delta$ considered in previous sections and $s$ is an edge of $\Delta$. Namely, $\hat{a}^\dagger$ does
not only rise the excitation by one unit (which corresponds to $h_\alpha^{\pm1}$) but also multiplies by
a coefficient $\sqrt{m+1}$ which depends on the “spin” $m$ (corresponding to $h_s[h_s^{-1}, \hat{V}]$). Even
the power $1/2$ of the spin is correct : The volume operator $\hat{V}$ is bounded by $\sqrt{|\hat{J}|}$ where $\hat{J}$ is the total spin of the graph. Thus it should have eigenvalues of order $m^{3/2}$. Now the
commutator acts roughly like a spin derivative, leaving us with a leading order of $m^{1/2}$.

The span of these oscillator eigenfunctions constitutes a topological vector space $\Phi$ which
is dense in $\mathcal{H}$ and is the precise analogue of (diffeomorphism invariant) spin-network
states with “spins” $m, n$. We are interested in distributions $\Psi \in \Phi'$ (the dual of $\Phi$) where
$\Phi \subset \mathcal{H} \subset \Phi'$ such that $\Psi(\hat{C}|m, n>) = (\hat{C}'|\Psi)(|m, n>) = 0$ for all $m, n$. We are looking for
$\Psi$’s of the form $\Psi(f) := \int dx dy \bar{\Psi}f$ and then the dual operator can be explicitly computed
and is given by $\hat{C}' = \hat{a} - \bar{c}\hat{b}$. That is, it is a complex linear combination of annihilation
operators, just as in the gravitational case. We can immediately see why $\hat{C}'$ has a huge
number of solutions $\Psi$ which actually lie in $\Phi$ : Define the level of an oscillator state
$|m, n>$ to be the number $m+n$. Then, if $f \in \Phi$ is a finite linear combination of oscillator
states none of whose level exceeds a given number $N$, it follows that $(\hat{C}')^{N+1}f = 0$ because
\((\hat{C}'t)^{N+1}\) is a linear combination of monomials of the form \(\hat{a}^k\hat{b}^{N+1-k}\) all of which reduce the level of an oscillator state by \(N+1\) or annihilate it. In other words, \(\hat{C}'\) is nilpotent on states of a given level \(N\).

Explicitly, the complete solution space can be constructed as follows:

The most general state of level \(N\) is given by

\[
|N, \{c\} \rangle = \sum_{m=0}^{N} c_m |m, N-m \rangle \quad (5.2)
\]

and these states (over)span \(\Phi\) as the \(c_m\) take all complex values. We apply \(\hat{C}'\) and find

\[
\hat{C}'|N, \{c\} \rangle = \sum_{m=0}^{N} c_m \sqrt{m}|m-1, N-m \rangle - c\sum_{m=0}^{N-1} c_m \sqrt{N-m}|m, N-m-1 \rangle
\]

\[
= \sum_{m=0}^{N-1} c_{m+1} \sqrt{m+1}|m, N-1-m \rangle - c\sum_{m=0}^{N-1} c_m \sqrt{N-m}|m, N-1-m \rangle
\]

\[
= \sum_{m=0}^{N-1} [c_{m+1} \sqrt{m+1} - cc_m \sqrt{N-m}]|m, N-1-m \rangle . \quad (5.3)
\]

Since \(\hat{C}'\) reduces the level precisely by one unit it follows that the condition \((\hat{C}'\Psi)(|n, m \rangle) = 0\) is only a condition on those vectors involved in the definition of \(\Psi\) with the same level. Thus any solution \(\Psi\) is a (not necessarily finite) linear combination of vectors of the form (5.2). We find the condition

\[
c_{m+1} \sqrt{m+1} - cc_m \sqrt{N-m} = 0 \quad \text{or} \quad c_m = c^m \sqrt{\frac{N}{m}} c_0 \quad (5.4)
\]

and if we wish to normalize the solution then \(|c_0| = (1 + |c|^2)^{-N/2}\). Let us simply denote the solution of level \(N\) by \(|N \rangle\). Of course not every solution need to be normalizable since we check \((\hat{C}'\Psi)(f)\) only with elements \(f \in \Phi\) so that there is no growth restriction on the coefficients \(d_N\) in \(\Psi = \sum_{N=1}^{\infty} d_N |N \rangle\).

Another way of looking at what has happened is that we performed a Bogolubov transformation on the annihilation operators and require that the physical vacuum is annihilated by one of the two new annihilation operators. The states \(|N \rangle\) are therefore nothing else than the excited states corresponding to the other (independent) annihilation operator.

We now address the question which inner product to impose on the solutions \(\Psi\) so obtained. Usually one expects that a typical solution to the Hamiltonian constraint is not normalizable with respect to the inner product on \(\mathcal{H}\). The intuition behind this expectation is experience with typical solutions of self-adjoint quantum constraints that can be obtained by group averaging. Let us give an example:

Consider the same Hilbert space as above but instead of the constraint \(\hat{C}\) consider the constraint \(\hat{H} = \hat{p} - \hat{q}\) where \(p, q\) are the momenta conjugate to \(x, y\) respectively. Thus, \(\hat{H}\) is a self-adjoint operator on \(\mathcal{H}\) and we may follow the group average proposal to construct the space of solutions and an inner product thereon. Denote by \(|p, q \rangle\) the usual momentum
generalized eigenstates. Notice that these kinematical states, in contrast to the \(|m, n\rangle\), are already not normalizable. Let now \(|f\rangle \in \Phi\). Then the group average map is given by

\[
|\tilde{f}\rangle := \hat{\eta}|f\rangle := \int_{\mathbb{R}} \frac{dt}{2\pi} e^{it\hat{H}}|f\rangle = \int_{\mathbb{R}^2} dpdq \int_{\mathbb{R}} \frac{dt}{2\pi} e^{it\hat{H}} |p, q \rangle \langle p, q | f \rangle = \int_{\mathbb{R}^2} dpdq \delta(p, q) |p, q \rangle \langle p, q | f \rangle
\]

that is, \(\hat{\eta} = \int dp|p, p \rangle \langle p, p | f \rangle\). Notice that \(\hat{H}|p, p\rangle = 0\).

The group average inner product is defined by

\[
<\tilde{f}, \tilde{g}> := <\tilde{f}|g> = \int dp <f|p, p \rangle \langle p, p | g > .
\]

In particular, we have \(|| |\tilde{f}\rangle ||^2 = \int dp |p, p | f \rangle |f \rangle^2 = \int dpdq |p, q | f \rangle \langle p, q | f \rangle^2 <\infty\) (under suitable conditions on \(<p, q | f \rangle\) which defines \(\Phi\)) but \(|| |\tilde{f}\rangle ||^2 = \int dpdq |p, p | f \rangle \langle p, p | g >\) blows up.

This feature is shared by all models whose self-adjoint constraint operator has continuous and unbounded spectrum.

We will now study a second example with a self-adjoint, unbounded constraint operator whose spectrum is, however, discrete. Here we will encounter a surprise: all the group average solutions are normalizable elements of the Hilbert space.

Consider again the Hilbert space \(\mathcal{H}\) as above but now the self-adjoint constraint operator is given by \(\hat{H} := \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}\), that is, the difference between two harmonic oscillator Hamiltonians. Clearly, the spectrum is entirely discrete and is given by the set of integers. The oscillator eigenstates \(|m, n\rangle\) are eigenstates of \(\hat{H}\) with eigenvalue \(m - n\) and clearly any solution is a (not necessarily finite) linear combination of the states \(|m, m\rangle\) for arbitrary \(m \geq 0\). This is precisely what the group average map does as well: we have

\[
|\tilde{f}\rangle := \hat{\eta}|f\rangle := \sum_{m, n} \int_{-\pi}^{\pi} \frac{dt}{2\pi} e^{it\hat{H}} |m, n \rangle \langle m, n | f \rangle = \sum_{m, n} \delta_{m,n} |m, n \rangle \langle m, n | f \rangle
\]

that is, \(\hat{\eta} = \sum_m |m, m \rangle \langle m, m |\). The reason why \(t\) now only ranges over \(S^1\) is because the spectrum of \(\hat{H}\) is discrete in contrast to the above situation and so the group average must produce a Kronecker delta rather than a Dirac delta. Notice that the group averaging inner product is given by

\[
<\tilde{f}, \tilde{g}> := \sum_m <f|m \rangle \langle m |g >\]

so that in particular

\[
|| |\tilde{f}\rangle ||^2 = \sum_m |<m, m | f \rangle |^2 \leq \sum_{m,n} |<m, n | f \rangle |^2 = || |f\rangle ||^2
\]

and

\[
|| |\tilde{f}\rangle ||^2 = \sum_{m,n} <|m, m | f \rangle \langle m, m | g > = \delta_{0,0} || |\tilde{f}\rangle ||^2 !
\]

Not only is the norm of an averaged vector with respect to the average inner product
smaller than or equal to the norm of the original vector with respect to the original inner product but also the average inner product and the original inner product in fact coincide on averaged vectors, nothing blows up. The reason for this is that the spectrum is discrete so that generalized eigenfunctions are actually elements of the Hilbert space.

Let us return now to the constraint operator \( \hat{C}' \) above and let us set \( c = 1 \) for simplicity. We are actually able to describe explicitly the complete spectrum of \( \hat{C}' \), namely, it is the entire complex plane. Generalized eigenvectors are given by \(|z, z' > := |z > \otimes |z' > \) where \(|z > \) are the usual coherent states and the eigenvalue is given by \( z - z' \). Now, while the spectrum is not discrete just as in the case of \( \hat{H} = \hat{p} - \hat{q} \), still the generalized eigenvectors are proper, normalized (although not orthonormal) elements of the Hilbert space just as in the case of \( \hat{H} = \hat{a}^{\dagger} \hat{a} - \hat{b}^{\dagger} \hat{b} \) which is in contrast to the case \( \hat{H} = \hat{p} - \hat{q} \). As we will see, it is this difference that makes the group averaged solutions again normalizable with respect to the original inner product.

To see this we must first define a group averaging. To begin with notice that all vectors to the original inner product.

Thus, in analogy with \( \hat{H} = \hat{p} - \hat{q} \) we try to define an average map as follows:

\[
1_H = \int \frac{d\bar{z}dz}{2i\pi} \int \frac{d\bar{z}'dz'}{2i\pi} |z, z' >= z, z' =: \int d\mu(\bar{z}, z) \int d\mu(z', z') |z, z' >= z, z' .
\]

Thus, in analogy with \( \hat{H} = \hat{p} - \hat{q} \) we try to define an average map as follows:

\[
|\tilde{f} > := \tilde{\eta}|f > := \int \frac{dt}{2\pi} e^{i\tilde{C}'t} |f > = \int d\mu(\bar{z}, z) \int d\mu(z', z') \int \frac{dt}{2\pi} e^{i\tilde{C}'t} |z, z' > |z, z' > |f > = \int d\mu(\bar{z}, z) \int d\mu(z', z') \delta(z, z') |z, z' > |z, z' > |f > \tag{5.9}
\]

where \( \delta(z) := \delta(x)_{x=z} \) for complex \( z \) denotes the analytic continuation of the \( \delta \) distribution. Interpreting this formal object as \( \delta(x)\delta(y) \) with \( z = x + iy \) results in

\[
|\tilde{f} > := \int d\mu(\bar{z}, z) |z, z > |z, z > |f > \tag{5.10}
\]

that is, \( \tilde{\eta} = \int d\mu(\bar{z}, z) |z, z > |z, z > \). Another way to justify \( \tilde{\eta} \) is to interpret \( \delta(z) \) as a holomorphic \( \delta \) distribution in the sense \( \int dz \delta(z, z_0)f(z) = f(z_0) \) for holomorphic \( f \) and to drop the remaining \( d\bar{z} \) integral. A reader who does not like the formal derivation \( \tilde{\eta} \) may take \( \delta(z) \) as a definition which is very natural and motivated by all the examples above. In fact, apart from the explicit measure \( d\mu(\bar{z}, z) \), this \( \tilde{\eta} \) is the most general one that maps \( |f > \) to a solution. Notice that \( \tilde{\eta} \) is the precise analogue of the average maps \( \int dp |p, p > |p, p > \) and \( \sum_m |m, m > |m, m > \) encountered above, all of them are projectors on the solution space and the measures \( d\mu(\bar{z}, z), dp, 1 \) are the diagonal measures that survive from the insertion of the \( 1_H \) after evaluating the \( \delta \) distribution.

It turns out that our \( \eta \) can be written in a yet much simpler form in terms of the vectors \( |N > \). First, by expanding the coherent states in terms of oscillator eigenfunctions, we observe that

\[
|z, z > = e^{-|z|^2} \sum_{m,n=0}^{\infty} \frac{z^{m+n}}{\sqrt{m!n!}} |m, n >
\]
\[ e^{-|z|^2} \sum_{N=0}^{\infty} z^N \frac{1}{\sqrt{m!(N-m)!}} |m, N-m> \]

\[ = e^{-|z|^2} \sum_{N=0}^{\infty} \frac{z^N}{\sqrt{N!}} 2^{N/2} |N> \]  

(5.11)

where in the last step we have used definition (5.4) of \(|N>\). Formula (5.11) looks like the coherent state \(|\sqrt{2}z>\) except that the Fock states \(|n>\) are replaced by the solution states \(|N>\) of \(C'\) of level \(N\). This confirms our earlier statement that every solution of the constraint is a linear combination of solutions of the form \(|N>\).

Next we explicitly perform the \(\bar{z}, z\) integrals involved in \(\hat{\eta}\) and find (the calculation is exactly the same as the one for proving overcompleteness of coherent states)

\[ \hat{\eta} = \sum_{M,N=0}^{\infty} \frac{2^{(M+N)/2}}{\sqrt{M!N!}} |M><N| \int d\mu(z) e^{-|z|^2} z^M \bar{z}^N \]

\[ = \frac{1}{2} \sum_{M,N=0}^{\infty} \frac{1}{\sqrt{M!N!}} |M><N| \int d\mu(z) e^{-|z|^2} z^M \bar{z}^N \]

\[ = \frac{1}{2} \sum_{M,N=0}^{\infty} \frac{1}{\sqrt{M!N!}} |M><N| \delta_{M,N}N! \]

\[ = \frac{1}{2} \sum_{N=0}^{\infty} |N><N| \]  

(5.12)

Apart from the factor 1/2 expression (5.12) is the expected result : the \(\hat{\eta}\) operator is a projector on the solution space which can be more conveniently written in terms of the \(|N>\)'s. In the sequel we will disregard the 1/2. The group average inner product becomes

\[ <\tilde{f}, \tilde{g}> = \sum_{N=0}^{\infty} <f,N><N,g> \]  

(5.13)

Notice that the \(|N>\) are orthonormal among each other with respect to the original inner product. Let \(\hat{\eta}'\) be the projector on the subspace of \(H\) orthogonal to the completion of the span of the \(|N>\) so that \(1_H = \hat{\eta} + \hat{\eta}'\). Then, since both \(\hat{\eta}, \hat{\eta}'\) are positive operators we have

\[ \| |\tilde{f}>\| =<f, \hat{\eta} f> \leq< f, f> = \| f >\|^2 \]

and

\[ \| |\tilde{f}>\| =<f, \hat{\eta}' \hat{\eta} f> =< f, \hat{\eta} \hat{\eta} f> =< f, \hat{\eta} f> = \| |\tilde{f}>\| \]

and we encounter the same phenomenon as for the constraint \(\tilde{H} := a^\dagger a - b^\dagger b\) !

What we learn from these model investigations is the following :

1) The assumption that averaged vectors are in general not normalizable with respect to the original inner product is false. Roughly, it is false for models in which generalized eigenvectors are in fact elements of the Hilbert space. Examples include unbounded self-adjoint (with discrete spectrum) and non-self-adjoint constraint operators (with continuous spectrum) as we saw above.
2) In the examples above we saw that the following chain of inequalities holds

$$|| |\tilde{f} > ||^2 = || |\tilde{f} > ||^2 \leq || |f > ||^2 \tag{5.14}$$

that is, whenever $|f >$ has a norm, so does $|\tilde{f} >$ and moreover we can compute it by means of the original or the group average inner product. Thus, the group average map can be extended from $\Phi$ to all of $\mathcal{H}$.

3) However, it is not true that $< \tilde{f}, \tilde{g} > = < f, g >$ for $f \neq g$. Thus, the group average inner product is still needed and different from the original inner product. It is given by computing the inner products of the projections of general vectors into the solution space. The observation that $\hat{\eta}$ is a certain kind of projector is a general observation and can serve as an abstract definition of the average map for a general theory whose constraint operator is not self-adjoint.

4) One certainly can write down solutions to $\hat{C}'$ which are well-defined distributions on $\Phi$, however, unless they are actually elements of $\mathcal{H}$ they are not normalizable with respect to the average inner product. All distributions which do not have a norm with respect to the original inner product have to be discarded.

5) The importance of coherent states for the solution of the quantum constraint lets us expect that coherent states will also play a quite important role in quantum gravity. Notice that coherent states are infinite linear combinations of the basic states $|N >$ and so they are not cylindrical, however, they are elements of the Hilbert space while the states $|N >$ are in fact cylindrical.

6) We may view the definition

$$\hat{\eta} := \int_S d\nu(s) |s > < s|,$$

where $S$ is a complete set of labels of (generalized) solutions $|s >$ to a (not necessarily self-adjoint) dual constraint operator $\hat{C}'$ and $\nu$ is a “natural” measure thereon, as a more general definition of a group average map which in the case of self-adjoint $\hat{C}$ reduces to the usual definition. The measure $\nu$, as we have seen, arises typically as follows: one considers first the spectral measure $\mu$ for the constraint operator $\hat{C}'$ and then $d\nu = \int_S d\mu \delta(\hat{C}')$ is the measure induced by $\mu$ by deleting the integral over the spectrum $S$ of $\hat{C}'$. In case of a self-adjoint constraint operator this is precisely what happens via the traditional group average approach.

Finally, let us address the question of how to model the Lorentzian Hamiltonian constraint. Recall from [2] that the Lorentzian Hamiltonian constraint consists of two pieces $\hat{H}(N) = \hat{T}(N) - \hat{H}^E(N)$ where $\hat{H}^E(N)$ is the Euclidean Hamiltonian constraint which can be modeled by a linear combination $\hat{a}^\dagger - \bar{c}\hat{b}$ as above. The piece $\hat{T}(N)$ on the other hand is a complicated multiple commutator between two factors of $\hat{H}^E(N)$ and three factors of the volume operator $\hat{V}$ and several holonomy operators along open paths. More precisely, it contains two factors of the form $h_s^{-1}[h_s, [\hat{V}, \hat{H}^E(1)]]$ and one factor of the form $h_s^{-1}[h_s, \hat{V}]$ where $s$ is an open path. The operator $\hat{V}$ does not change the graph, it acts essentially by multiplication with functions that depend on the spin. The above combinations of holonomies along open paths and back also do not change the graph. Thus it is only the
two factors of $\hat{H}^E(N)$ that change the graph. Moreover, since every commutator decreases
the spin power by one unit we conclude that the spin power of $h^{-1}_s [h_s, [V, \hat{H}^E(1)]]$ is of
order zero (recall that the spin power of $\hat{H}^E(N)$ is $1/2$ to see this). Thus the spin power
of $\hat{T}(N)$ is $1/2$ as well. Since the “spin” power of $\hat{a}^\dagger$ is $1/2$ as well we conclude that
the bilinear operators $(\hat{a}^\dagger)^2, (\hat{b}^\dagger)^2, \hat{a}^\dagger \hat{b}^\dagger$ appropriately model $\hat{T}(N)$:
they create two new excitations and their spin power is even higher than that of $\hat{T}(N)$. In order to account
for the difference in the spin power one could choose to divide by a square root of the
number operator but we will not do that because that would push the analogy too far and
even if we would do it the argument displayed below would still apply. We therefore
propose to study the constraint operator

$$
\hat{C}' = \alpha (\hat{a}^2 + \hat{b}^2) + \beta \hat{a} \hat{b} - \gamma (\hat{a} + \hat{b})
$$

where the choice of coefficients respects the fact that the coordinates $x, y$ have to be
treated on equal footing in order to mirror the expression of $\hat{T}(N)$.

We can immediately solve $\hat{C}'$: consider a state $|z, z'\rangle$ as above, then
$\hat{C}' |z, z'\rangle = [\alpha (z^2 + (z')^2) + \beta zz' - \gamma (z + z')] |z, z'\rangle$ is an eigenfunction of $\hat{C}'$, again the operator $\hat{C}'$
is diagonalized by proper elements of $\mathcal{H}$. The condition that the eigenvalue vanishes
leads to a condition $z' = h(z)$ where $h(z)$ is a holomorphic function of $z$ (possibly
on a two-sheeted non-compact Riemann surface depending on the choice of the parameters
$\alpha, \beta, \gamma$). The same arguments as above now suggest to define the average operator by

$$
\hat{\eta} := \int d\mu \langle \bar{z}, z | h(z) \rangle < f, h(z) |
$$

Expression (5.15) is sufficient to prove that for any cylindrical state $|f\rangle$ (that is, they
are finite linear combinations of the states $|m, n\rangle$) the averaged state $|\bar{f}\rangle := \hat{\eta} |f\rangle$ is
still an element of the Hilbert space. Namely, since $|| |z, h(z) \rangle || = 1$ we have by the
Schwarz inequality

$$
|| |\bar{f}\rangle ||^2 \\
\leq \int d\mu \langle \bar{z}, z | f, h(z) \rangle \int d\mu \langle \bar{z}', z' | f, h'(z') \rangle \langle f, h(z) | z', h(z') \rangle \langle f, h(z) | h(z) \rangle \\
\leq (\int d\mu \langle \bar{z}, z | f, h(z) \rangle || f, h(z) ||^2.
$$

But $< f | h(z) > = p(z, h(z)) e^{-||z||^2 + |h(z)|^2}$ where $p(z, h(z))$ is a polynomial depending on
$|f|$ so that (5.14) obviously converges. We also have

$$
< \bar{f}, \bar{g} >_\sim = < f, \hat{\eta} g > \quad \text{and} \quad || |\bar{f}\rangle ||^2 = \int d\mu \langle \bar{z}, z | f, h(z) \rangle < f | h(z) > ||^2 < \infty \quad (5.17)
$$

for the same reason. This time we cannot find an obvious relation between $|| |\bar{f}\rangle ||^2$
though.

We conclude that the normalizability of averaged vectors with respect to the original inner
product carries over to the more complicated “Lorentzian” constraint.

### 5.2 A physical inner product for quantum general relativity

Let us first check whether the inner product that results from averaging the diffeomorphism
constraint actually is of the projector type mentioned above. This is easily seen to
be the case: if we write \( \hat{n}_{\text{Diff}} := \sum_{\{T\}_I} (T_I^S)[T_I^S]^I \) then we recover indeed \( \hat{n}_{\text{Diff}}(T_I^S) = [T_I^S] \), i.e. we have demonstrated that \( \hat{n}_{\text{Diff}} \) is of the general type encountered in the examples, namely it is a projector formed by elementary solutions which themselves are averages of a suitable basis of the original Hilbert space.

Next, let us recall the structure of the solution space of Lorentzian quantum gravity from \([3]\). Consider all possible labels \( \mathcal{I} \) of spin-network states and call the resulting set \( W \). The set \( W \) splits naturally into two parts: a piece \( W_0 \), called the set of sources and its complement \( \overline{W}_0 \) in \( W \). The set \( W_0 \) is characterized by the property that no vector \( T_I \) for \( I \in W_0 \) can occur in the (decomposition into spin-networks of the) image of the Euclidean Hamiltonian constraint \( \hat{H}^E(N) \) on \( \Phi \) for no choice of prescription \( p \). The set \( \overline{W}_0 \) therefore contains the labels which are sufficient to span the image of \( \hat{H}^E(N) \). It can be further decomposed as follows: for each \( I \in W_0 \) let \( W^{(n)}(I) \) be the set of labels that one finds by decomposing \( \{\hat{H}^E(N)\}^n T_I \) into spin-network states for all possible \( N \) and for all possible prescriptions \( p \). These are the states of level \( n \) with source \( I \) and they are the precise analogue of the states \( |m, N-m> \) of level \( N \) considered above. We now take the group average with respect to the diffeomorphism constraint and it then turns out that all the states with labels \( [J] \) coming from different \( W^{(n)}(I) \) (up to a diffeomorphism) are orthonormal. Thus the diffeomorphism invariant Hilbert space splits as

\[
\mathcal{H}_{\text{Diff}} = \oplus_{[I], I \in W_0} \mathcal{H}_{[I]} \quad \text{where} \quad \mathcal{H}_{[I]} = \oplus_{n=0}^{\infty} \mathcal{H}_{[W^{(n)}(I)]}
\]

and where \( [W^{(n)}(I)] \) denotes the group averaged labels of \( W^{(n)}(I) \). The space \( \mathcal{H}_{[W^{(n)}(I)]} \) is obviously a finite dimensional vector space and so \( \mathcal{H}_{[I]} \) is separable. Moreover, \( \hat{H}'(N) \) maps \( \mathcal{H}_{[I]} \) into itself, more precisely, it maps \( \mathcal{H}_{[W^{(n)}(I)]} \) into \( \mathcal{H}_{[W^{(n-1)}(I)]} \cup \mathcal{H}_{[W^{(n-2)}(I)]} \). Therefore the Hamiltonian constraint can be solved on each \( \mathcal{H}_{[I]} \) separately. Let us label the vectors of \( \mathcal{H}_{[W^{(n)}(I)]} \) by \( T_i^a \) (notice that all these vectors are diffeomorphism invariant). Then there are matrices \( a_{ij}^n, b_{ij}^n \) which depend on \([I]\) and on one of the (set of diffeomorphic) vertices \( v \) of the graph underlying \( I \) such that \( \hat{H}' T_i^n = a_{ij}^n T_j^{n-1} - b_{ij}^n T_j^{n-2} \). Writing a general element of \( \Phi \) as a linear combination of distributions of the form (one for each \([I]\)) \( \sum_n c_i^n T_i^n \), we find that \( c_i^{n+1} a_{ij}^{n+1} = c_i^{n+2} b_{ij}^{n+2} \) for each \([I], n, j, v\). It turns out that the matrices \( a_b, b^n \) are sufficiently degenerate as to allow for solutions \( c_i^n \) such that \( c_i^{n+1} a_{ij}^{n+1} = c_i^{n+2} b_{ij}^{n+2} = 0 \) for each \([I], n, j, v\) and so does not mix levels but a general solution will mix levels.

The precise growth of the coefficients of the matrices \( a_{ij}^n, b_{ij}^n \) is unknown (although the reasoning from above suggests that they do not grow worse than \( j^{1/2} \) where \( j \) is the total spin of a state). On the other hand, if \( b \) was identically zero as it is the case for the Euclidean Hamiltonian constraint then levels do not get mixed and then the example \( \hat{C}' = \hat{a} - \hat{b} \) considered above suggests (although we have no proof at present, of course) that a general solution will be normalizable with respect to the inner product on \( \mathcal{H}_{\text{Diff}} \).

The example \( \hat{C}' = \alpha (\hat{a}^2 + \hat{b}^2 + \beta \hat{a} \hat{b} - \gamma (\hat{a} + \hat{b}) \) studied above suggests that the same is true for the full Lorentzian constraint, i.e. \( b \neq 0 \) (again we have no proof for this).

Suppose now that we have explicitly solved the infinite number of linear equations \( c_i^{n+1} a_{ij}^{n+1} = c_i^{n+2} b_{ij}^{n+2} \) for each \([I], n, j, v\) exactly (this is actually possible, although quite complicated; we expect that one can invent some efficient code that lets us write down the solutions at least up to a controllable error). The examples studied above suggest that every solution is a kind of coherent state. The first possibility is that every solution lies in \( \mathcal{H}_{\text{Diff}} \). Then, since \( \mathcal{H}_{[I]} \) is separable, one can in principle find an orthonormal basis \( \left\{ T_I^{(n)} \right\}_{n=0}^{\infty} \) (with respect to \( <.,.>_{\text{Diff}} \)) on the solution space for each \([I]\) which defines actually a subspace
of \( \mathcal{H}_{[I]} \). If not every solution lies in \( \mathcal{H}_{\text{Diff}} \) then we can still find a countable generalized basis \( \{ T_{[I],\mu} \}_{\mu=0}^{\infty} \) built from infinite linear combinations of elements of \( \mathcal{H}_{[I]} \) which are orthonormal (with respect to \( \langle .., .. \rangle_{\text{Diff}} \)) in the sense of generalized eigenvectors, that is, \( \langle T_{[I],\mu}, T_{[J],\nu} \rangle_{\text{Diff}} = 0 \) unless \( [I] = [J] \) and \( \mu = \nu \) in which case this quantity diverges. (This is quite similar to the momentum eigenfunction “normalization” in the sense of \( \delta \) distributions : \( \langle p, p' \rangle = \delta(p, p') \). It is also precisely what happens with the group averaged spin-network states : with respect to the inner product on \( \mathcal{H} \) they are orthogonal in the sense that

\[
\langle [T_{[I]}], [T_{[J]}] \rangle := \sum_{T \in \{ T_{[I]} \}} [T_{[I]}](T) = 0
\]

unless \( [I] = [J] \) in which case this quantity blows up).

In either case the group average proposal then leads us to define the projectors

\[
\hat{\eta}_{\text{Ham}} := \sum_{[I]} \hat{\eta}_{[I]} \text{ where } \hat{\eta}_{[I]} = \sum_{\mu=0}^{\infty} T_{[I],\mu} T_{[I],\mu}^\dagger
\]

(5.18)

physical states as arising from group averaging are defined by \( T_{\text{phys}} := \hat{\eta}T \) for each \( T \in \Phi'_{\text{Diff}} \) and the physical inner product between such physical states is given by

\[
\langle T_{\text{phys}}, T'_{\text{phys}} \rangle_{\text{phys}} := \langle T, \hat{\eta}T' \rangle_{\text{Diff}} = T_{\text{phys}}(T')
\]

(5.19)

where the latter notation means evaluation of the distribution \( T_{\text{phys}} \in (\Phi'_{\text{Diff}})' \) on the “test function” \( T' \in \Phi'_{\text{Diff}} \) by means of the inner product of \( \mathcal{H}_{\text{Diff}} \).

Equation (5.19) is a physical inner product for Lorentzian quantum general relativity which is naturally suggested to us by group average kind of reasonings. It is precisely of the structure of the group average map for the diffeomorphism constraint \( \eta_{\text{Diff}} \) displayed at the beginning of this section.

Remarks :

1) We want to stress that the inner product (5.19) exists even if the solutions \( T_{[I],\mu} \) are not normalizable with respect to \( \langle .., .. \rangle_{\text{Diff}} \) although we expect this to be true. We have struggled to list arguments in favour of normalizability of \( T_{[I],\mu} \) only in order to remove the feeling of discomfort : The quantum mechanics example \( \hat{H} = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \) shows that physical states as obtained by rigorous group averaging can be normalizable with respect to the original inner product without that anything is wrong with that.

2) The fact that solutions to the Hamiltonian constraint are presumably normalizable with respect to the diffeomorphism invariant inner product can be interpreted as saying that the whole distributionality of a solution to all constraints is already captured by averaging with respect to the diffeomorphism constraint alone. In fact, if we had not split the solution of diffeomorphism and Hamiltonian constraint into two steps but had solved all constraints in one stroke without even constructing \( \eta_{\text{Diff}} \) then we would not have even noticed this fact.

3) As one can show, the “flat state” \( \delta(F) \) can be given rigorous meaning as a distribution on \( \Phi'_{\text{Diff}} \) and it solves \( \hat{H}^E(N) \) at least. In 2+1 Euclidean gravity the flat state is a physical state for the curvature constraint \( F = 0 \). On the other hand, to make 2+1 Euclidean gravity resemble 3+1 Lorentzian gravity one can replace \( F = 0 \)
by Diffeomorphism and Hamiltonian constraints [22]. One finds that in 2+1 it is indeed true that physical solutions as selected by the so defined Hamiltonian constraint (analogously constructed as in this paper for 3+1 Lorentzian gravity) are normalizable with respect to \(< \ldots >_{\text{Diff}}\). It follows then that the flat state \(\delta(F)\) is not normalizable with respect to \(< \ldots >_{\text{phys}}\) because it is an uncountably infinite linear combination of diffeomorphism invariant spin-network states with all coefficients of order unity [22] and so is not even normalizable with respect to \(< \ldots >_{\text{Diff}}\). Thus, the inner products of a topological quantum field theory (like 2+1 Euclidean gravity in the Witten formulation) and quantum general relativity in 3+1 dimensions have inner products which are presumably very singular with respect to each other as one would expect. To prevent wrong conclusions, notice that \(F = 0\), which can be solved, for instance, by \(A = 0\), does not mean that we are in Minkowski space: \(A = \Gamma + K = 0\) just does not have any obvious Minkowski space interpretation.

4) Notice that since the projector inner product (5.19) is induced from that of \(\mathcal{H}\) for which the classical reality conditions are implemented as adjointness relations, we can claim that (5.19) incorporates the physical reality conditions on Dirac observables provided a Dirac observable is self-adjoint on \(\mathcal{H}\) already. This is because a Dirac observable leaves the solution space invariant and is therefore already projected (“it commutes with the group averaging”), see below.

5) A complete set of Dirac observables in the sense that it be a self-adjoint and densely defined operator on \(\mathcal{H}_{\text{phys}}\) are trivial construct: Define

\[
\hat{O'}_{[I]\mu,[J]\nu} := T_{[I]\mu}[T_{[J]\nu}]^\dagger \quad \text{and} \quad \hat{O}_{[I]\mu,[J]\nu} := \frac{\hat{O'}_{[I]\mu,[J]\nu} + \hat{O'}_{[J]\nu,[I]\mu}}{2}
\]

(5.20)

where the dagger operation is with respect to \(< \ldots >_{\text{phys}}\) then linear combinations \(\hat{O}\) of symmetric operators of the form (5.20) form a complete set of Dirac observables (modulo domain questions).

Notice that in case that the \(T_{[I]\mu}\) are not \(< \ldots >_{\text{Diff}}\) normalizable then (5.20) has to be understood in the following sense: there are pre-images \(f_{[I]\mu} \in \Phi\) such that \(\hat{\eta} f_{[I]\mu} = T_{[I]\mu}\). Now the \(T_{[I]\mu}\) become normalizable with respect to \(< \ldots >_{\text{phys}}\) because by definition \(< T_{[I]\mu}, T_{[J]\nu} >_{\text{phys}} = T_{[I]\mu}[f_{[J]\nu}]\). In case of general relativity the \(f_{[I]\mu}\) are related to the “source states” constructed in [3] which are really elements of \(\Phi\) and orthonormal with respect to \(< \ldots >\) (after diffeomorphism group averaging they form an orthonormal system (but not a basis) contained in \(\mathcal{H}_{[I]}\)). This implies that the \(T_{[I]\mu}\) are orthonormal with respect to \(< \ldots >_{\text{phys}}\).

From this follows then immediately that \(\hat{\eta} \hat{O} = \hat{O} \hat{\eta}\). Also, \(\hat{O}\) is self-adjoint on \(\mathcal{H}\) if we define \(< f, \hat{O'}_{[I]\mu,[J]\nu} g > := T_{[I]\mu}[f]T_{[J]\nu}[g]\).

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