Embedding of LCK manifolds with potential into Hopf manifolds using Riesz-Schauder theorem

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Abstract
A locally conformally Kähler (LCK) manifold with potential is a complex manifold with a cover which admits a positive automorphic Kähler potential. A compact LCK manifold with potential can be embedded into a Hopf manifold, if its dimension is at least 3. We give a functional-analytic proof of this result based on Riesz-Schauder theorem and Montel theorem. We provide an alternative argument for compact complex surfaces, deducing the embedding theorem from the Spherical Shell Conjecture.

Contents
1 Introduction
2 Preliminaries of functional analysis
  2.1 Normal families of functions
  2.2 Topologies on spaces of functions
  2.3 Montel theorem for normal families
  2.4 The Banach space of holomorphic functions
    2.4.1 Compact operators
    2.4.2 Holomorphic contractions
    2.4.3 The Riesz-Schauder theorem
3 Proof of Theorem 1.1
4 Proof of the embedding theorem
  4.1 Theorem 4.2 implies Theorem 1.3
  4.2 Proof of Theorem 4.2

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5  Kato conjecture and non-Kähler surfaces

1  Introduction

A locally conformally Kähler (LCK) manifold is a Hermitian manifold \((M, J, g)\) such that the fundamental two-form \(\omega(X, Y) = g(JX, Y)\) satisfies the equation

\[d\omega = \theta \wedge \omega\]

for a closed one-form \(\theta\), see [DO].

The one-form \(\theta\) is called the Lee form, and it produces a twisted cohomology associated to the operator \(d_\theta := d - \theta \wedge\).

An equivalent definition requires the existence of a covering \(\Gamma \to \tilde{M} \to M\) endowed with a Kähler metric \(\tilde{g}\) with respect to which the deck group \(\Gamma\) acts by holomorphic homotheties. This gives rise to a character \(\chi : \Gamma \to \mathbb{R}^> 0\) which associates to a homothety \(\gamma \in \Gamma\) its scale factor \(c_\gamma\). A differential form \(\eta\) on \(\tilde{M}\) is then called automorphic if \(\gamma^* \eta = \chi(\gamma) \eta\). Clearly, \(\tilde{\omega}\) is automorphic.

An LCK manifold is called with potential if there exists a Kähler covering with Kähler form \(\tilde{\omega}\) having positive, automorphic global potential \(\psi\): \(\tilde{\omega} = dd^c \psi\). This is equivalent to the existence of a function \(\varphi\) on \(M\) such that \(\omega = d\theta + d_\theta \varphi\), see [OV3]. Note that \(\varphi\) is not a potential on \(M\).

Important examples are the Hopf surfaces and, more generally, the linear Hopf manifolds \(C^n \setminus \{0\}/\langle A\rangle\), where \(\langle A\rangle\) is the cyclic group generated by a linear operator \(A \in \text{GL}(n, \mathbb{C})\) with all eigenvalues \(|\alpha_i| < 1\), [OV4].

On the other hand, there exist compact complex manifolds which admit LCK metrics, but no LCK metric with potential: such are blow-ups of LCK manifolds ([Vu]) and the LCK Inoue surfaces, [O1] (see also [AD]).

We proved in [OV1] (see also [OV3]) that if \((M, \omega, \theta)\) is a compact LCK manifold with potential, there exists another LCK structure \((\omega', \theta')\), close to \((\omega, \theta)\) in the \(\mathcal{C}^\infty\)-topology, such that the corresponding \(\tilde{\omega}'\) has a proper potential, this being equivalent with the monodromy of the covering, \(\text{Im}(\chi)\), being isomorphic with \(\mathbb{Z}\).

Vaisman manifolds are LCK manifolds whose Kähler coverings are Riemannian cones over Sasakian manifolds (see [DO], and [Be] for the classification of Vaisman compact surfaces). All Vaisman manifolds are LCK with potential (represented by the squared Kähler norm of the pull-back of the Lee form). The converse is not true, as the example of non-diagonal Hopf manifolds shows, see [OV2]. Still, the covering of an LCK manifold with potential is very close to being a cone:

**Theorem 1.1**: ([OV2]) Let \(M\) be an LCK manifold with proper potential, \(\dim\mathbb{C} M \geq\)
3, and \(\tilde{M}\) its Kähler \(\mathbb{Z}\)-covering. Then the metric completion \(\tilde{M}_c\) admits a structure of a complex variety, actually Stein, compatible with the complex structure on \(\tilde{M} \subset \tilde{M}_c\). Moreover, \(\tilde{M}_c \setminus \tilde{M}\) is just one point.

**Remark 1.2:** The same result seems to be true for \(\dim \mathbb{C} M = 2\). In Section 5 we deduce it from classification of surfaces and the spherical shell conjecture on surfaces of Kodaira class VII (Conjecture 5.5).

The main property of an LCK manifold with potential is the following Kodaira type embedding result:

**Theorem 1.3:** (\cite{OV2}) A compact LCK manifold with proper potential, of complex dimension at least 3, can be holomorphically embedded in a linear Hopf manifold.

**Remark 1.4:** The hypothesis \(\dim \mathbb{C} M > 2\) in Theorem 1.1 is essential in order to apply a result in \cite{AS, Ro} (see Theorem 3.2 below) from which we deduce that the completion \(\tilde{M}_c\) is Stein. Once we know that the completion is Stein, Theorem 1.3 follows without further assumptions on the dimension.

The aim of this note is to give new proofs of the above two theorems, based on applications of Montel and Riesz-Schauder theorems. This will require several notions of functional analysis (see, e.g. \cite{Ko1}) that we recall for the reader’s convenience. In the last section we comment on the possible validity of the result for complex surfaces.

## 2 Preliminaries of functional analysis

### 2.1 Normal families of functions

**Definition 2.1:** Let \(M\) be a complex manifold, and \(\mathcal{F}\) a family of holomorphic functions \(f_i \in H^0(\mathcal{O}_M)\). \(\mathcal{F}\) is called **a normal family** if for each compact \(K \subset M\) there exists \(C_K > 0\) such that for each \(f \in \mathcal{F}\), \(\sup_K |f| \leq C_K\).

**Lemma 2.2:** Let \(M\) be a complex Hermitian manifold, \(\mathcal{F} \subset H^0(\mathcal{O}_M)\) a normal family, and \(K \subset M\) a compact subset. Then there exists a number \(A_K > 0\) such that \(\sup_K |f'| \leq A_K\).

**Proof:** By contradiction, suppose there exists \(x \in K\), \(\nu \in T_x M\), and a sequence \(f_i \in \mathcal{F}\) such that \(\lim_i |D_\nu f_i| = \infty\). We choose a disk \(\Delta \xrightarrow{i} M\) with compact clo-
sure in $M$, tangent to $\nu$ in $x$, such that $j(0) = x$. Let $w = t\nu$ have norm 1. Then $\sup_\Delta |f_i| < C_\Delta$ by the normal family condition. By Schwarz lemma (see [10]), this implies $|D_{w_i} f_i| < C_\Delta$. However, $t^{-1}\lim_i |D_{w_i} f_i| = \lim_i |D_{w_i} f_i| = \infty$, yielding a contradiction.

2.2 Topologies on spaces of functions

**Definition 2.3:** Let $C(M)$ be the space of functions on a topological space. The **topology of uniform convergence on compacts** (also known as **compact-open topology**), usually denoted as $C^0$, is the topology on $C(M)$ whose base of open sets is given by

$$U(X, C) := \{f \in C(M) \mid \sup_K |f| < C\},$$

for all compacts $K \subset M$ and $C > 0$.

A sequence $\{f_i\}$ of functions converges to $f$ if it converges to $f$ uniformly on all compacts.

**Remark 2.4:** In a similar way one defines the $C^0$-topology on the space of sections of a bundle.

**Definition 2.5:** Let $B$ be a vector bundle on a smooth manifold $M$, and $\nabla : B \rightarrow B \otimes \Lambda^1 M$ a connection. Define the $C^1$-topology on the space of sections of $B$ (denoted, as usual, by the same letter $B$) as one where a sub-base of open sets is given by $C^0$-open sets on $B$ and $\nabla^{-1}(W)$, where $W$ is an open set in $C^0$-topology in $B \otimes \Lambda^1 M$.

**Remark 2.6:** A sequence $\{f_i\}$ converges in the $C^1$-topology if it converges uniformly on all compacts, and the first derivatives $\{f_i'\}$ also converge uniformly on all compacts. This can be seen as an equivalent definition of the $C^1$-topology.

2.3 Montel theorem for normal families

**Theorem 2.7:** (Montel). Let $M$ be a complex manifold and $\mathcal{F} \subset H^0(\Theta_M)$ a normal family of functions. Denote by $\overline{\mathcal{F}}$ its closure in the $C^0$-topology. Then $\overline{\mathcal{F}}$ is compact and contained in $H^0(\Theta_M)$.

**Proof:** Let $\{f_i\}$ be a sequence of functions in $\mathcal{F}$. By Tychonoff’s theorem, for each compact $K$, there exists a subsequence of $\{f_i\}$ which converges pointwise on a dense countable subset $Z \subset K$. Taking a diagonal subsequence, we find a subsequence $\{f_{i_p}\} \subset \{f_i\}$ which converges pointwise on a dense countable subset...
Z \subset M$. Since $|f'_i|$ is uniformly bounded on compacts, the limit $f := \lim_i f_i$ is Lipschitz on all compact subsets of $M$. It is thus continuous, because a pointwise limit of Lipschitz functions is again Lipschitz.

Then, since $|f'_i|$ is uniformly bounded on compacts, we can assume that $f'_i$ also converges pointwise in $Z$, and $f := \lim_i f_i$ is differentiable. Since a limit of complex-linear operators is complex linear, $Df$ is complex linear, and $f$ is holomorphic. This implies that $\mathcal{F} \cap H^0(\mathcal{O}_M)$ is compact. ■

### 2.4 The Banach space of holomorphic functions

We begin by proving:

**Theorem 2.8:** Let $M$ be a complex manifold, and $H^0_b(\mathcal{O}_M)$ the space of all bounded holomorphic functions, equipped with the sup-norm $|f|_{\sup} := \sup_M |f|$. Then $H^0_b(\mathcal{O}_M)$ is a Banach space.  

**Proof:** Let \( \{f_i\} \in H^0_b(\mathcal{O}_M) \) be a Cauchy sequence in the sup-norm. Then \( \{f_i\} \) converges to a continuous function $f$ in the sup-topology.

Since \( \{f_i\} \) is a normal family, it has a subsequence which converges in \( C^0 \)-topology to $\tilde{f} \in H^0(\mathcal{O}_M)$, by Montel’s Theorem (Theorem 2.7). However, the \( C^0 \)-topology is weaker than the sup-topology, hence $\tilde{f} = f$. Therefore, $f$ is holomorphic. ■

#### 2.4.1 Compact operators

Recall that a subset of a topological space is called **precompact** if its closure is compact.

**Definition 2.9:** Let $V, W$ be topological vector spaces, and let $\varphi : V \rightarrow W$ be a continuous linear operator. It is called **compact** if the image of any bounded set is precompact.

**Remark 2.10:** Note that the notion of **bounded set** makes sense in all topological vector spaces $V$. Indeed, a set $K \subset V$ is called **bounded** if for any open set $U \ni 0$, there exists a number $\lambda_U \in \mathbb{R}^+ \setminus \{0\}$ such that $\lambda_U K \subset U$.

**Claim 2.11:** Let $V = H^0(\mathcal{O}_M)$ be a space of holomorphic functions on a complex manifold $M$ with $C^0$-topology. Then any bounded subset of $V$ is precompact. In this case, the identity map is a compact operator.

**Proof:** This is a restatement of Montel’s theorem (Theorem 2.7). ■
Remark 2.12: By Riesz theorem, a closed ball in a normed vector space $V$ is never compact, unless $V$ is finite-dimensional. This means that $(H^0(\mathcal{O}_M), \mathcal{C}^0)$ does not admit a norm. A topological vector space where any bounded subset is precompact is called **Montel space**.

2.4.2 Holomorphic contractions

Definition 2.13: A **contraction** of a manifold $M$ to a point $x \in M$ is a continuous map $\varphi: M \to M$ such that for any compact subset $K \subset M$ and any open set $U \ni x$, there exists $N > 0$ such that for all $n > N$, the map $\varphi^n$ maps $K$ to $U$.

Theorem 2.14: Let $X$ be a complex variety, and let $\gamma: X \to X$ be a holomorphic contraction such that $\gamma(X)$ is precompact. Consider the Banach space $V = H^0_b(\mathcal{O}_X)$ with the sup-metric. Then $\gamma^*: V \to V$ is compact, and its operator norm $\|\gamma^*\| := \sup_{|v| \leq C} |\gamma^*(v)|$ is strictly less than 1.

Proof: Let $B_C := \{v \in V | |v|_{\sup} \leq C\}$. Then

$$|\gamma^* f|_{\sup} = \sup_{x \in \gamma(X)} |f(x)|.$$  

Therefore, for any sequence $\{f_i\}$ converging in the $\mathcal{C}^0$-topology, the sequence $\{\gamma^* f_i\}$ converges in the sup-topology. However, $B_C$ is precompact in the $\mathcal{C}^0$-topology, because it is a normal family. Then $\gamma^* B_C$ is precompact in the sup-topology.

Since $\sup_X |\gamma^* f| = \sup_{x \in \gamma(X)} |f(x)| \leq \sup_X |f|$, one has $\|\gamma^*\| \leq 1$. If this inequality is not strict, for some sequence $f_i \in B_1$ one has $\lim \sup_{x \in \gamma(X)} |f_i(x)| = 1$. Since $B_1$ is a normal family, $f_i$ has a subsequence converging in $\mathcal{C}^0$-topology to $f$. Then $\gamma(f_i)$ converges to $\gamma(f)$ in sup-topology, giving

$$\lim \sup_i |f_i(x)| = \sup_{x \in \gamma(X)} |f(x)| = 1.$$  

Since, by the maximum principle, a holomorphic functions has no strict maxima, this means that $|f(x)| > 1$ somewhere on $X$. Then $f$ cannot be the $\mathcal{C}^0$-limit of $f_i \in B_1$. ■

2.4.3 The Riesz-Schauder theorem

The following result is a Banach analogue of the usual spectral theorem for compact operators on Hilbert spaces. It will be the central piece in our argument.
Theorem 2.15: (Riesz-Schauder, [Co]) Let \( F : V \to V \) be a compact operator on a Banach space. Then for each non-zero \( \mu \in \mathbb{C} \), there exists a sufficiently large number \( N \in \mathbb{Z} \) such that for each \( n > N \) one has

\[
V = \ker(F - \mu \Id)^n \oplus \overline{\text{im}(F - \mu \Id)^n},
\]

where \( \overline{\text{im}(F - \mu \Id)^n} \) is the closure of the image of \( (F - \mu \Id)^n \). Moreover, \( \ker(F - \mu \Id)^n \) is finite-dimensional and independent on \( n \).

3 Proof of Theorem 1.1

Recall that \( \tilde{M}_c \) denotes the metric completion of the \( \mathbb{Z} \)-cover \( \tilde{M} \) of \( M \).

Claim 3.1: The complement \( \tilde{M}_c \setminus \tilde{M} \) is just one point, called the origin.

Proof: Indeed, let \( z_i = \gamma^i(x_i) \) be a sequence of points in \( \tilde{M} \), with each \( x_i \) in the fundamental domain \( \varphi^{-1}([1, \lambda]) \) of the \( \Gamma = \mathbb{Z} \)-action. Clearly, the distance between two fundamental domains \( M_n := \gamma^n \varphi^{-1}([1, \lambda]) = \varphi^{-1}([\lambda^n, \lambda^{n+1}]) \) and \( M_{n+k+2} = \gamma^{n+k+2} \varphi^{-1}([1, \lambda]) \) is written as

\[
d(M_n, M_{n+k+2}) = \sum_{i=0}^{k} \lambda^{n+i} \nu,
\]

where \( \nu \) is the distance between \( M_0 \) and \( M_2 \). Then, \( z_i \) may converge only if \( \lim_{i} n_i = -\infty \) or if all \( n_i \), except finitely many, belong to the set \( (p, p+1) \) for some \( p \). The second case is irrelevant, because each \( M_i \) is compact, and in the first case, \( \{z_i\} \) is always a Cauchy sequence, as follows from (3.1). All such \( \{z_i\} \) are therefore equivalent, hence converge to the same point in the metric completion. 

Recall now the following result in complex analysis:

Theorem 3.2: ([AS], [Ro]) Let \( S \) be a compact strictly pseudoconvex CR manifold, \( \dim_{\mathbb{R}} S \geq 5 \), and let \( H^0(\mathcal{O}_S)_b \) the ring of bounded CR holomorphic functions. Then \( S \) is the boundary of a Stein manifold \( M \) with isolated singularities, such that \( H^0(\mathcal{O}_S)_b = H^0(\mathcal{O}_M)_b \), where \( H^0(\mathcal{O}_M)_b \) denotes the ring of bounded holomorphic functions. Moreover, \( M \) is defined uniquely, \( M = \text{Spec}(H^0(\mathcal{O}_S)_b) \).

The proof of Theorem 1.1 now goes as follows:

Step 1: Applying Rossi-Andreotti-Siu Theorem 3.2 to \( \varphi^{-1}([a, \infty[) \), we obtain a Stein variety \( \tilde{M}_a \) containing \( \varphi^{-1}([a, \infty[) \). Since \( \tilde{M}_a \) contains \( \varphi^{-1}([a_1, \infty[) \) for any
\[a_1 > a,\] and the Rossi-Andreotti-Siu variety is unique, one has \(\overline{M}_a = \overline{M}_a'.\) This implies that \(\overline{M}_a =: \overline{M}_c\) is independent from the choice of \(a \in \mathbb{R}^{>0}.\)

It remains to identify \(\overline{M}_c\) with the metric completion of \(\overline{M}.\) By \textbf{Claim 3.1} this is equivalent to the complement \(\overline{M}_c \backslash \overline{M}\) being a singleton.

**Step 2:** The monodromy group \(\Gamma = Z\) acts on \(\overline{M}_c\) by holomorphic automorphisms. Indeed, any holomorphic function (hence, any holomorphic map) can be extended from \(\overline{M}\) to \(\overline{M}_c\) uniquely.

**Step 3:** Denote by \(\gamma\) the generator of \(\Gamma\) which decreases the metric by \(\lambda < 1,\) and let \(\overline{M}_c^{\lambda}\) be the Stein variety associated with \(\varphi^{-1}([0, a]) \subset \overline{M}\) as above. Since \(\gamma(\overline{M}_c) = \overline{M}_c^{\lambda}\), for any holomorphic function \(f\) on \(\overline{M}_c\), one has

\[
\sup_{z \in \overline{M}_c} |f(\gamma^n(z))| = \sup_{z \in \overline{M}_c^{\lambda^n}} |f(z)| \leq \sup_{z \in \overline{M}_c} |f(z)| .
\]

Therefore, \(\{f(\gamma^n(z))\}\) is a normal family.

**Step 4:** Let \(f_{\lim}\) be any limit point of the sequence \(\{f(\gamma^n(z))\}\). Since the sequence \(t_i := \sup_{z \in \overline{M}_c^{\lambda^n}} |f(z)|\) is non-increasing, it converges, and \(\sup_{z \in \overline{M}_c} f_{\lim} = \lim t_i.\) Similarly, \(\sup_{z \in \overline{M}_c^{\lambda^n}} f_{\lim} = \lim t_i.\) By the strong maximum principle, [GT], a non-constant holomorphic function on a complex manifold with boundary cannot have local maxima (even non-strict) outside of the boundary. Since \(\overline{M}_c^{\lambda}\) does not intersect the boundary of \(\overline{M}_c^{\lambda^n}\), the function \(f_{\lim}\) must be constant.

**Step 5:** Consider now the complement \(V := \overline{M}_c \backslash \overline{M},\) and suppose it has two distinct points \(x\) and \(y.\) Let \(f\) be a holomorphic function which satisfy \(f(x) \neq f(y).\) Replacing \(f\) by an exponent of \(\mu f\) if necessarily, we may assume that \(|f(x)| < |f(y)|.\) Since \(\gamma\) fixes \(Z,\) which is compact, for any limit \(f_{\lim}\) of the sequence \(\{f(\gamma^n(z))\}\), supremum \(f_{\lim}\) on \(Z\) is not equal to infimum of \(f_{\lim}\) on \(Z.\) This is impossible, hence \(f = \text{const}\) on \(V,\) and \(V\) is one point.

This finishes the proof of [Theorem 1.1].

### 4 Proof of the embedding theorem

Theorem 1.3 is implied by Theorem 4.2. To see this, we need the following:

**Definition 4.1:** Let \(\gamma\) be an endomorphism of a vector space \(V.\) A vector \(v \in V\) is called \(\gamma\)-finite if the subspace \(\langle v, \gamma(v), \gamma^2(v), \ldots \rangle\) is finite-dimensional.

**Theorem 4.2:** Let \(M\) be an LCK manifold with potential, \(\dim_{\mathbb{C}} M > 2,\) and \(\overline{M}\) its Kähler \(Z\)-covering. Consider the metric completion \(\overline{M}\) with its complex structure and a contraction \(\gamma : \overline{M} \longrightarrow \overline{M}\) generating the \(Z\)-action. Let \(H^0(\mathcal{O}_{\overline{M}})_{\text{fin}}\)
be the space of functions which are $\gamma^*$-finite. Then $H^0(\mathcal{O}_{\widetilde{M}_c})_{\text{fin}}$ is dense in the sup-topology on each compact subset of $\widetilde{M}_c$.

### 4.1 Theorem 4.2 implies Theorem 1.3

**Step 1:** Let $W \subset H^0(\mathcal{O}_{\widetilde{M}_c})_{\text{fin}}$ be an $m$-dimensional $\gamma^*$-invariant subspace $W$ with basis $\{w_1, \ldots, w_m\}$. Then the following diagram is commutative:

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\Psi} & \mathbb{C}^m \\
\gamma \downarrow & & \downarrow \gamma^* \\
M & \xrightarrow{\Psi} & \mathbb{C}^m \\
\end{array}
$$

where $\Psi(x) = (w_1(x), w_2(x), \ldots, w_m(x))$.

Suppose that the map $\Psi$ associated with a given $W \subset H^0(\mathcal{O}_{\widetilde{M}_c})_{\text{fin}}$ is injective. Then the quotient map gives an embedding $\Psi : \widetilde{M}/Z \rightarrow (\mathbb{C}^m\setminus\{0\})/\gamma^*$; all eigenvalues of $\gamma^*$ are < 1 because its operator norm is < 1, by Theorem 2.14.

**Step 2:** To find an appropriate $W \subset H^0(\mathcal{O}_{\widetilde{M}_c})_{\text{fin}}$, choose a holomorphic embedding $\Psi_1 : \tilde{M}_c \hookrightarrow \mathbb{C}^n$, which exists because $\tilde{M}_c$ is Stein. Let $\bar{w}_1, \ldots, \bar{w}_n$ be the coordinate functions of $\Psi_1$. Theorem 4.2 allows one to approximate $\bar{w}_i$ by $w_i \in H^0(\mathcal{O}_{\widetilde{M}_c})_{\text{fin}}$ in $\mathcal{C}^0$-topology. Choosing $w_i$ sufficiently close to $\bar{w}_i$ in a compact fundamental domain of the $Z$-action, we obtain that $x \mapsto (w_1(x), w_2(x), \ldots, w_n(x))$ is injective in a compact fundamental domain of $Z$.

Finally, take $W \subset H^0(\mathcal{O}_{\widetilde{M}_c})_{\text{fin}}$ generated by the $\gamma^*$ from $w_1, \ldots, w_n$, and apply Step 1. $lacksquare$

### 4.2 Proof of Theorem 4.2

The core of our argument is an application of Riesz-Schauder theorem. First we prove:

**Proposition 4.3:** Fix a precompact subset $\tilde{M}_c^a := \varphi^{-1}([0, a[)$, where $\varphi : \tilde{M}_c : \longrightarrow \mathbb{R}^{>0}$ is the Kähler potential. Let $A$ be the ring of bounded holomorphic functions on $\tilde{M}_c^a$, and $m$ the maximal ideal of the origin point. Clearly, $\gamma^*$ preserves $m$ and all its powers. Let $P_k(t)$ be the minimal polynomial of $\gamma^*|_{A/mk}$. Then $\text{im}(P_k(\gamma^*)) \subset m^k(A)$, and $\ker P_k(\gamma^*)$ generates $A/m^k$.

**Proof:** Since $P_k(t)$ is a minimal polynomial of $\gamma^*$ on $A/m^k$, the endomorphism $P_k(\gamma^*)$ acts trivially on $A/m^k$, by Cayley-Hamilton theorem, hence it maps $A$ to $m^k$. 

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From Riesz-Schauder theorem applied to the Banach space $A$ and $F = P_k(\gamma^*) - P_k(0)$, with $\mu = -P_k(0)$, it follows that $A = \ker(P_k(\gamma^*) \oplus \text{im}(P_k(\gamma^*)))^\mathbb{N}$. Since $P_k(\gamma^*)$ acts trivially on $A/m^k$, its image lies in $m^k$. This gives a surjection of $\ker P_k(\gamma^*)$ onto $A/m^k$.

This implies:

**Proposition 4.4:** Let $H^0(\mathcal{O}_{\tilde{M}_c})_{\text{fin}} \subset H^0(\mathcal{O}_{\tilde{M}_c})$ be the set of $\gamma^*$-finite functions and $m$ the maximal ideal of the origin in $\tilde{M}_c$. Then $H^0(\mathcal{O}_{\tilde{M}_c})_{\text{fin}}$ is dense in $m$-adic topology.[1]

**Proof:** A subspace $V \subset A$ is dense in $m$-adic topology in $A$ if and only if the quotient $V/v \cap m^k$ surjects to $A/m^k$. This is proven in Proposition 4.3 for the ring of bounded holomorphic functions on $\tilde{M}_c^a$. However, any such function can be extended to $\gamma^*$-finite function on $\tilde{M}_c$ using the $\gamma^*$-action.

To finish the proof, observe that Theorem 4.2 is implied by the following:

**Claim 4.5:** Let $X$ be a connected complex variety, $A$ the ring of bounded holomorphic functions on $X$, $x \in X$ a point, $m \subset A$ its maximal ideal, and $R : A \rightarrow \hat{A}$ the natural map from $A$ to its $m$-adic completion. Then $R$ is continuous in $C^0$-topology and induces homeomorphism of any bounded set to its image.

**Proof:** Continuity is clear because the $C^0$-topology on holomorphic functions is equivalent to $C^1$-topology, $C^2$-topology and so on, by Montel Theorem (Theorem 2.7). Therefore, taking successive derivatives in a point is continuous in $C^0$-topology. However, $R$ takes a function and replaces it by its Taylor series.

To see that $R$ is a homeomorphism, notice that any bounded, closed subset of $A$ is compact, hence its image under a continuous map is also closed. Then $R$ induces a homeomorphism on all bounded sets. To see that the preimage of a converging sequence is converging, notice that any such sequence is bounded in $A$ by another application of Schwarz lemma.

## 5 Kato conjecture and non-Kähler surfaces

All surfaces in this section are assumed to be compact.

**Definition 5.1:** A complex surface $M$ with $b_1(M) = 1$ and Kodaira dimension $-\infty$.

[1] Recall that for a ring $A$ with a proper ideal $m$, the $m$-adic topology on $A$ is given by the base of open sets formed by $m^k$ and their translates.
is called **Kodaira class VII surface**. If it is also minimal, it is called **class VII$_0$ surface**.

**Remark 5.2:** Class VII surfaces are obviously non-Kähler. Indeed, their $b_1$ is odd.

The main open question in the classification of non-Kähler surfaces is the following conjecture, called **spherical shell conjecture**, or **Kato conjecture**. To state it, recall first:

**Definition 5.3:** Let $S \subset M$ be a real submanifold in a complex surface, diffeomorphic to $S^3$. We call $S$ **spherical shell** if $M \setminus S$ is connected, and $S$ has a neighbourhood which is biholomorphic to an annulus in $\mathbb{C}^2$. A class VII$_0$ surface which contains a spherical shell is called a **Kato surface**.

**Remark 5.4:** From [Kat], we know that any Kato surface contains exactly $b_2(M)$ distinct rational curves (the converse was proven in [DOT]).

**Conjecture 5.5:** (spherical shell conjecture) Any class VII$_0$ surface with $b_2 > 0$ is a Kato surface.

**Theorem 5.6:** Assume that the spherical shell conjecture is true. Then [Theorem 1.1] and [Theorem 1.3] are true in dimension 2.

**Proof:** The only part of the proof missing for dimension 2 is Rossi-Andreotti-Siu Theorem (Theorem 3.2), see also [Remark 1.4]. We used it to prove the following result (which is stated here as a conjecture, because we don’t know how to prove it for class VII$_0$ non-Kato surfaces).

**Conjecture 5.7:** Let $M$ be an LCK complex surface with proper potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-cover. Then the metric completion of $\tilde{M}$, realized by adding just one point, is a Stein variety.

**Remark 5.8:** For dim $M \geq 3$, this is [Theorem 1.1]

Conjecture 5.7 follows from the spherical shell conjecture and the classification of surfaces.

First of all, notice that an LCK surface $M$ with an LCK potential cannot contain rational curves. Indeed, if $M$ contains rational curves, by homotopy lifting $\tilde{M}$
would also contain rational curves, but $\tilde{M}$ is embedded to a Stein variety. This implies that $M$ cannot be a Kato surface, and that $M$ is minimal.

If the spherical shell conjecture is true, class VII surfaces which are not Kato have $b_2 = 0$. However, class VII surfaces with $b_2 = 0$ were classified by Bogomolov, Li, Yau, Zheng and Teleman ([Bo1, Bo1, LYZ, LY, Te1, Te2]). From these works it follows that any class VII surface with $b_2 = 0$ is biholomorphic to a Hopf surface or to an Inoue surface.

Inoue surfaces don’t have LCK potential for topological reasons ([Ot]). The Hopf surfaces are quotients of $\mathbb{C}^2 \setminus 0$ by an action of $\mathbb{Z}$, hence their 1-point completions are affine, and hence Stein.

The only non-Kähler minimal surfaces which are not of class VII are non-Kähler elliptic surfaces ([BHPV]). These surfaces are obtained as follows. Let $X$ be a 1-dimensional compact complex orbifold, and $L$ an ample line bundle on $X$. Consider the space $\tilde{M}$ of all non-zero vectors in the total space of $L^*$, and let $Z$ act on $\tilde{M}$ as $v \mapsto a v$, where $a \in \mathbb{C}$ is a fixed complex number, $|a| > 1$. Any non-Kähler elliptic surface is isomorphic to $\tilde{M}/Z$ for appropriate $a$, $X$ and $L$. However, the sections of $L^{an}$ define holomorphic functions on $\tilde{M} \subset \text{Tot}(L^*)$, identifying $\tilde{M}$ and the corresponding cone over $X$. As such, $M$ is Vaisman ([Be]), in particular LCK with potential. The completion of this cone is $\tilde{M}_c$, and it is affine ([EGA2 §8]), and hence Stein. This finishes the proof of Theorem 5.6.

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L. Ornea, M. Verbitsky
Embedding LCK manifolds with potential

– 14 –