A NOTE ON CONICAL KÄHLER-RICCI FLOW ON MINIMAL ELLIPTIC KÄHLER SURFACES

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Abstract. We prove that, under a semi-ampleness type assumption on the twisted canonical line bundle, the conical Kähler-Ricci flow on a minimal elliptic Kähler surface converges in the sense of currents to a generalized conical Kähler-Einstein on its canonical model. Moreover, the convergence takes place smoothly outside the singular fibers and the chosen divisor.

1. Introduction

The Kähler-Ricci flow has been studied extensively and become a powerful tool in Kähler geometry. Cao [2] proved that Kähler-Ricci flow will converge smoothly to Kähler-Einstein metrics on a compact Kähler manifold with numerically trivial or ample canonical line bundle. Tsuji [23] and Tian-Zhang [20] proved the convergence of Kähler-Ricci flow to singular Kähler-Einstein metrics on a smooth minimal model of general type. In [18, 19], Song-Tian obtained the weak convergence of Kähler-Ricci flow to generalized Kähler-Einstein metrics if the canonical line bundle is semi-ample.

The conical Kähler-Ricci flow was introduced in [3] and is expected to deform a conical Kähler metric to conical Kähler-Einstein metric. In fact the convergence of the conical Kähler-Ricci flow has been studied when the twisted canonical line bundle is positive or trivial [4], nef and big [17] or negative [13]. Also see, e.g., [14, 16, 24] for discussions of the conical Ricci flow on Riemann surfaces.

This paper aims to study the convergence of the conical Kähler-Ricci flow on a minimal elliptic Kähler surface of Kodaira dimension one, under a semi-ampleness type assumption (1.2) on the twisted canonical line bundle.

Throughout this paper, let \((X, \omega_0)\) be a minimal elliptic Kähler surface of Kodaira dimension \(\text{kod}(X) = 1\). By definition (see, e.g., Section I.3 of [15] or Section 2.2 of [18]), there exists a holomorphic map \(f : X \to \Sigma\), determined by the pluricanonical system \(|mK_X|\) for sufficiently large integer \(m\), from \(X\) onto a smooth projective curve \(\Sigma\) (i.e., the canonical model of \(X\)), such that the general fiber is a smooth elliptic curve and all fibers are free of \((-1)\)-curves. Set \(\Sigma_{\text{reg}} := \{s \in \Sigma | X_s := f^{-1}(s)\text{ is a nonsingular fiber}\}\) and \(X_{\text{reg}} = f^{-1}(\Sigma_{\text{reg}})\). Assume \(\Sigma \setminus \Sigma_{\text{reg}} = \{s_1, \ldots, s_k\}\) and let \(m_i F_i = X_{s_i}\) be the corresponding singular fiber of multiplicity \(m_i, i = 1, \ldots, k\). We refer readers to Section I.5 of [15] for several interesting examples of minimal elliptic surfaces.

Consider a fixed point \(r \in \Sigma_{\text{reg}}\), which can be seen as a divisor on \(\Sigma\), and the smooth divisor \(D := X_r\) on \(X\). Let \(S'\) be the defining section of \(r\) and \(h'\) be a fixed smooth Hermitian metric on the holomorphic line bundle associated to \(r\) with \(|S'|^2_{h'} \leq 1\). Then \(S := f^* S'\) is a defining section of \(D\) and \(h := f^* h'\) is a smooth Hermitian metric on the holomorphic line bundle associated to \(D\) with \(|S|^2_h = f^* |S'|^2_{h'} \leq 1\). Set \(\Gamma' = \{r, s_1, \ldots, s_k\}\) and \(\Gamma = f^{-1}(\Gamma')\).

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Fix an arbitrary \( \beta \in (0, 1) \). Then for any sufficiently small positive constant \( \delta \),

\[
\omega_0^* := \omega_0 + \delta \sqrt{-1} \partial \bar{\partial} |S|_h^{2\beta}
\]

is a conical Kähler metric with cone angle \( 2\pi \beta \) along \( D \) (see [5]), which means that \( \omega_0^* \) is a smooth Kähler metric on \( X \setminus D \) and is asymptotically equivalent along \( D \) to the local model metric

\[
\sqrt{-1} \left( \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + dz_2 \wedge d\bar{z}_2 \right),
\]
on \( \mathbb{C}^2 \), where \( (z_1, z_2) \) are local holomorphic coordinates such that \( D = \{ z_1 = 0 \} \) locally.

Consider the conical Kähler-Ricci flow

\[
\begin{aligned}
\partial_t \omega &= -\text{Ric}(\omega) - \omega + 2\pi(1-\beta)[D] \\
\omega(0) &= \omega_0^*,
\end{aligned}
\]

(1.1)

where \([D]\) is the current of integration associated to the divisor \( D \) on \( X \). By results in [17] we know that a solution \( \omega \) to (1.1) exists uniquely for all \( t \geq 0 \) if and only if the twisted canonical line bundle \( K_X + (1-\beta)L_D \) is nef, where \( L_D \) is the holomorphic line bundle associated to the divisor \( D \). In particular, under the assumption (1.2) below, (1.1) admits a unique long time solution. The following is our main result.

**Theorem 1.1.** Assume as above. Assume in addition that there exists a Kähler metric \( \theta' \) on \( \Sigma \) such that

\[
f^* \theta' \in -2\pi c_1(X) + 2\pi(1-\beta)[D].
\]

Then,

(1) As \( t \to \infty \), \( \omega(t) \to f^* \omega'_\infty \) as currents on \( X \) and the convergence takes place smoothly on \( X \setminus \Gamma \).

(2) \( \omega'_\infty \) is a positive closed \((1,1)\)-current on \( \Sigma \) such that it is a smooth Kähler metric on \( \Sigma \setminus \Gamma' \) and \( \text{Ric}(\omega'_\infty) = -\sqrt{-1} \partial \bar{\partial} \log \omega'_\infty \) is a well-defined current on \( \Sigma \) satisfying

\[
\text{Ric}(\omega'_\infty) = -\omega'_\infty + 2\pi(1-\beta)[\gamma] + \omega_{WP} + 2\pi \sum_{i=1}^{k} \frac{m_i-1}{m_i} [s_i],
\]

(1.3)

where \( \omega_{WP} \) is the induced Weil-Petersson metric.

**Remark 1.2.** Theorem 1.1 is a generalization of the results of Song-Tian [18] and Fong-Zhang [7] on the Kähler-Ricci flow to the conical setting. Following [18], we may call \( \omega'_\infty \) a generalized conical Kähler-Einstein metric on \( \Sigma \).

**Remark 1.3.** We should mention that in the setting of Theorem 1.1 it is proved in [6] that the scalar curvature of \( \omega(t) \) is uniformly bounded on \( (X \setminus D) \times [0, \infty) \).

The paper is organized as follows: In Section 2 we construct the generalized conical Kähler-Einstein metric \( \omega'_\infty \) on \( \Sigma \). In Section 3 we derive some necessary estimates and then give a proof of Theorem 1.1.

2. **Generalized conical Kähler-Einstein metrics**

In this section we construct the generalized conical Kähler-Einstein metric on \( \Sigma \). To begin with, let \( \omega_{SF} := \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho_{SF} \), where \( \rho_{SF} \) is a smooth function on \( X_{\text{reg}} \), be the semi-flat \((1,1)\)-form defined by Lemma 3.2 of [18]. Here the semi-flatness means that, for any \( s \in \Sigma_{\text{reg}} \), the restriction \( \omega_{SF}|_{X_s} \) is a Ricci-flat Kähler metric on the smooth fiber \( X_s \).
On the other hand, by the assumption (1.2) we can fix a smooth volume form \( \Omega \) on \( X \) satisfying
\[
\int_X \Omega = \int_X \omega_0^2.
\]
and
\[
\sqrt{-1}\partial\bar{\partial} \log \Omega = f^*\theta' - (1 - \beta)R_h
\]
where \( R_h = -\sqrt{-1}\partial\bar{\partial} \log h \) is the curvature form of \( h \). Note that if \( R_{h'} \) is the curvature form of \( h' \), then \( R_h = f^*R_{h'} \). Define
\[
F = \frac{\Omega}{2\omega_{SF} \wedge f^*\theta'}.
\]
(2.1)

Then \( F \) can be seen as a function on \( \Sigma \) and \( F \in L^p(\Sigma, \theta) \) for some \( p > 1 \) (see [18, 19, 9]). Without loss of any generality we assume \( p < \frac{1}{1 - \beta} \).

The following proposition is essentially contained in [18], which is also a special case of the general theory of complex Monge-Ampère equations (see [11, 12]).

**Proposition 2.1.** There exists a unique solution \( \varphi_\infty \) solving the following equation on \( \Sigma \)
\[
\theta' + \sqrt{-1}\partial\bar{\partial}(\varphi_\infty + \delta|S'|_{h'}^{2\beta}) = \frac{Fe^{\varphi_\infty + \delta|S'|_{h'}^{2\beta}}}{|S'|_{h'}^{2\beta(1 - \beta)}} \theta'.
\]
(2.2)

with \( \varphi_\infty + \delta|S'|_{h'}^{2\beta} \in PSH(\Sigma, \theta) \cap C^\alpha(\Sigma, \theta) \cap C^\infty(\Sigma \setminus \Gamma') \) for some \( \alpha \in (0, 1) \).

Moreover, if we define \( \omega_\infty = \theta' + \sqrt{-1}\partial\bar{\partial}(\varphi_\infty + \delta|S'|_{h'}^{2\beta}) \), then \( \omega_\infty \) satisfies all the properties stated in part (2) of Theorem 1.1.

**Remark 2.2.** \( \omega_\infty \) is a conical Kähler metric on \( \Sigma_{reg} \) with cone angle \( 2\pi\beta \) at a.

3. **Estimates and Convergence**

To prove part (1) of Theorem 1.1 we first reduce the conical Kähler-Ricci flow (1.1) to a parabolic complex Monge-Ampère equation, as in, e.g., [13, 17, 6].

Set \( \omega_t = e^{-t} \omega_0 + (1 - e^{-t}) f^*\varphi' \). Then \( \omega = \omega_t + \delta \sqrt{-1} \partial\bar{\partial}|S'|_{h'}^{2\beta} + \sqrt{-1} \partial\bar{\partial} \varphi \) solves the conical Kähler-Ricci flow (1.1) if \( \varphi = \varphi(t) \) solves the following parabolic complex Monge-Ampère equation
\[
\begin{cases}
\partial_t \varphi = \log \frac{e^t|S'|_{h'}^{2(1 - \beta)}(\omega_t + \delta \sqrt{-1} \partial\bar{\partial}|S'|_{h'}^{2\beta} + \sqrt{-1} \partial\bar{\partial} \varphi)^2}{\Omega} - \varphi - \delta|S'|_{h'}^{2\beta} \\
\varphi(0) = 0.
\end{cases}
\]
(3.1)

Note that in the above reduction we have used the Poincaré-Lelong formula
\[
\sqrt{-1} \partial\bar{\partial} \log |S'|_{h}^2 = -R_h + 2\pi[D].
\]

To obtain the estimates we need, we will make use of the approximation method developed in [1], which is also used in, e.g., [13, 17, 6]. Following [1] we define the smoothing metric by
\[
\omega_{t, \varepsilon} = \omega_t + \delta \sqrt{-1} \partial\bar{\partial} \chi(\varepsilon^2 + |S'|_{h}^2),
\]
where
\[
\chi(\varepsilon^2 + x) = \beta \int_0^x \frac{(\varepsilon^2 + r)^\beta - \varepsilon^{2\beta}}{r} dr.
\]
Note that for each $\epsilon > 0$, $\omega_{t,\epsilon}$ is a smooth Kähler metric and, as $\epsilon \to 0$, $\omega_{t,\epsilon}$ converges to $\omega_t + \delta \sqrt{-1} \partial \bar{\partial} |S|_h^2$ globally on $X$ in the sense of currents and in $C^\infty_{loc}(X \setminus D)$-topology. Moreover there exists a uniform constant $C > 1$ such that

$$0 \leq \chi(\epsilon^2 + |S|_h^2) \leq C$$

(3.2)

and

$$\omega_{0,\epsilon} \geq C^{-1} \omega_0.$$  

(3.3)

Consider the following smooth approximation equation of (3.1)

$$\left\{ \begin{array}{l}
\partial_t \varphi_\epsilon = \log \left( |S|_h^2 + \epsilon^2 \right)^{1-\beta} \left( \omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon \right)^2 - \varphi_\epsilon - \delta \chi(\epsilon^2 + |S|_h^2)
\varphi_\epsilon(0) = 0,
\end{array} \right.$$  

(3.4)

which is equivalent to the following generalized Kähler-Ricci flow

$$\left\{ \begin{array}{l}
\partial_t \omega_{\varphi_\epsilon} = - \text{Ric}(\omega_{\varphi_\epsilon}) - \omega_{\varphi_\epsilon} + (1 - \beta) \sqrt{-1} \partial \bar{\partial} (|S|_h^2 + \epsilon^2) + (1 - \beta) R_h \\
\omega_{\varphi_\epsilon}(0) = \omega_{0,\epsilon},
\end{array} \right.$$  

(3.5)

where $\omega_{\varphi_\epsilon} = \omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon$.

Since the twisted canonical line bundle $K_X + (1 - \beta) L_D$ is nef (see assumption (1.2)), the generalized Kähler-Ricci flow (3.4), or (3.5), has a unique smooth long time solution [2] [20] [23].

Set $\psi_\epsilon = \varphi_\epsilon + \delta \chi(\epsilon^2 + |S|_h^2)$ and $\omega_{\psi_\epsilon} = \omega_t + \sqrt{-1} \partial \bar{\partial} \psi_\epsilon = \omega_{\varphi_\epsilon}$. The following lemma is proved in [6].

**Lemma 3.1.** [6] There exists a uniform positive constant $C$ such that on $X \times [0, \infty)$,

$$|\psi_\epsilon| + |\partial_t \psi_\epsilon| + \text{tr}_{\omega_{\psi_\epsilon}} f^* \theta' \leq C.$$  

Following [21] we fix a smooth nonnegative function $\sigma$ on $X$, which vanishes exactly on $\Gamma$, such that $\sigma \leq 1$ and

$$\sqrt{-1} \partial \sigma \wedge \bar{\partial} \sigma \leq C f^* \theta', \quad -C f^* \theta' \leq \sqrt{-1} \partial \bar{\partial} \sigma \leq C f^* \theta'$$

(3.6)

on $X$ for some constant $C$. Define a function on $\Sigma$ by

$$\psi_\epsilon(s) = \frac{\int_{X_s} \psi_\epsilon \omega_0 |_{X_s}}{\int_{X_s} \omega_0 |_{X_s}}.$$  

In the following, we will also use $\bar{\psi_\epsilon}$ to denote the pull-back $f^* \bar{\psi_\epsilon}$ to $X$.

**Lemma 3.2.** There exist positive constants $C$ and $\lambda$ such that

$$|\psi_\epsilon - \bar{\psi_\epsilon}| \leq \frac{Ce^{-t}}{\sigma^\lambda}.$$  

(3.7)

**Proof.** Note that $\frac{1}{|S|_h^2 + \epsilon^2} \leq \frac{1}{\sigma}$ if we choose $\lambda$ large enough. Then combining the estimates in Lemma 3.2 and the uniform boundedness of $\chi(\epsilon^2 + |S|_h^2)$ mentioned in (3.2), this lemma can be checked by the same arguments as in the proofs of Corollary 5.1 and Corollary 5.2 of [18].

**Lemma 3.3.** There exist positive constants $C$ and $\lambda$ such that

$$(\partial_t - \Delta_{\omega_{\psi_\epsilon}})(e^t(\psi_\epsilon - \bar{\psi_\epsilon})) \geq \text{tr}_{\omega_{\psi_\epsilon}} \omega_0 - \frac{C}{\sigma^\lambda} - Ce^t.$$  

(3.8)
Proof. Firstly recall the inequality (5.22) in [18]:
\[
\Delta_{\omega_{\psi}} (e^t(\psi_{\epsilon} - \overline{\psi_{\epsilon}})) \leq - tr_{\omega_{\psi}} \omega_0 + \frac{1}{\int_{X_s} \omega_0 |_{X_s}} tr_{\omega_{\psi}} \left( \int_{X_s} \omega_0^2 \right) + 2e^t.
\]
Moreover,
\[
\frac{1}{\int_{X_s} \omega_0 |_{X_s}} tr_{\omega_{\psi}} \left( \int_{X_s} \omega_0^2 \right) \leq \frac{C}{\int_{X_s} \omega_0 |_{X_s}} tr_{\omega_{\psi}} \left( \int_{X_s} \Omega_{SF} \wedge f^*\theta' \right)
\]
\[
= C \left( \frac{\Omega}{\omega_{SF} \wedge f^*\theta'} \right) tr_{\omega_{\psi}} f^*\theta'
\]
\[
\leq \frac{C}{\sigma^\lambda}
\]
for some uniform positive constants \(C\) and \(\lambda\). Thus
\[
\Delta_{\omega_{\psi}} (e^t(\psi_{\epsilon} - \overline{\psi_{\epsilon}})) \leq - tr_{\omega_{\psi}} \omega_0 + \frac{C}{\sigma^\lambda} + 2e^t. \tag{3.9}
\]
On the other hand,
\[
\partial_t (e^t(\psi_{\epsilon} - \overline{\psi_{\epsilon}})) = e^t(\psi_{\epsilon} - \overline{\psi_{\epsilon}}) + e^t(\partial_t \psi_{\epsilon} - \partial_t \overline{\psi_{\epsilon}})
\]
\[
\geq - \frac{C}{\sigma^\lambda} - Ce^t, \tag{3.10}
\]
where we have used Lemma 3.1 and Lemma 3.2. Combining (3.9) and (3.10) we conclude (3.8).
\[\square\]

Now we are ready to prove the following key lemma in this section.

**Lemma 3.4.** There exist positive constants \(C\) and \(\lambda\) such that
\[
tr_{\omega_{\psi}} \omega_t \leq Ce^{\frac{C}{\sigma^\lambda}}. \tag{3.11}
\]

**Proof.** Recall that \(\omega_t = e^{-t} \omega_0 + (1 - e^{-t}) f^*\theta\). By Lemma 3.1 it suffices to show
\[
tr_{\omega_{\psi}} (e^{-t} \omega_0) \leq Ce^{\frac{C}{\sigma^\lambda}}. \tag{3.12}
\]
A direct computation gives
\[
(\partial_t - \Delta_{\omega_{\psi}}) tr_{\omega_{\psi}} (e^{-t} \omega_0)
\]
\[
\leq C_1 e^{-t} (tr_{\omega_{\psi}} \omega_0)^2 - e^{-t} g_{\psi}^i g_{\psi}^j (g_0)_{ij} \Theta_{ab} - g_{\psi}^i g_{\psi}^j (g_0)_{ij} \partial_k \nabla_i (g_0)_{pq} \nabla_j (g_0)_{cd}, \tag{3.13}
\]
where \(C_1\) is a positive constant only depending on the bi-sectional curvature of \(\omega_0\), and \(\Theta = (1 - \beta) \sqrt{\partial \overline{\partial} \log(|S|_h^2 + e^2)} + (1 - \beta) R_h\). Note that it was shown in [6] that there exists a uniform positive constant \(C\) such that
\[
\Theta \geq - Cf^*\theta',
\]
which implies that, increasing \(C\) if necessary,
\[
\Theta \geq - C \omega_0.
\]
Substituting the above inequality into \((3.13)\), we obtain
\[
(\partial_t - \Delta_{\omega_{\psi}}) tr_{\omega_{\psi}}(e^{-t}\omega_0) \\
\leq C_1 e^{-t}(tr_{\omega_{\psi}}\omega_0)^2 + C_2 e^{-t} g_{\psi_0}^i g_{\psi_0}^j (g_0)_{ij} (g_0)_{ab} - g_{\psi_0}^i g_{\psi_0}^j (g_0)_{ij} d\nabla_i (g_0)_{pd} \nabla_j (g_0)_{cq} \\
\leq C_1 e^{-t}(tr_{\omega_{\psi}}\omega_0)^2 - g_{\psi_0}^i g_{\psi_0}^j (g_0)_{ij} d\nabla_i (g_0)_{pd} \nabla_j (g_0)_{cq}.
\]
Hence we have
\[
(\partial_t - \Delta_{\omega_{\psi}}) \log tr_{\omega_{\psi}}(e^{-t}\omega_0) \leq C tr_{\omega_{\psi}}\omega_0
\]
for some uniform positive constant \(C\).

Set \(H = \sigma^{\lambda_1} \log tr_{\omega_{\psi}}(e^{-t}\omega_0) - A e^t(\psi_\ell - \bar{\psi}_\ell)\) for some large constants \(\lambda_1\) and \(A\). Using \((3.6)\) and the evolution inequalities \((3.8)\) and \((3.14)\) we have
\[
(\partial_t - \Delta_{\omega_{\psi}}) H \leq - \frac{A}{2} \sigma^{\lambda_1} tr_{\omega_{\psi}}\omega_0 + C_0 \sigma^{\lambda_1 - 2} \log tr_{\omega_{\psi}}(e^{-t}\omega_0) - 2 Re \left( \frac{\nabla H \nabla \sigma^{\lambda_1}}{\sigma^{\lambda_1}} \right) + 2 A e^t.
\]
Note that when \(tr_{\omega_{\psi}}\omega_0 > 1\) and \(\sigma > 0\),
\[
\log tr_{\omega_{\psi}}\omega_0 \leq 2 \sqrt{tr_{\omega_{\psi}}\omega_0} \leq \sigma^2 tr_{\omega_{\psi}}\omega_0 + \frac{1}{\sigma^2}.
\]
Substituting the above inequality into \((3.15)\), we have
\[
(\partial_t - \Delta_{\omega_{\psi}}) H \leq - \frac{A}{3} \sigma^{\lambda_1} tr_{\omega_{\psi}}\omega_0 - 2 Re \left( \frac{\nabla H \nabla \sigma^{\lambda_1}}{\sigma^{\lambda_1}} \right) + 3 A e^t.
\]
Now by the maximum principle argument we can conclude \((3.12)\). Lemma \(3.4\) is proved. \(\Box\)

By Lemma \(3.4\) we have
\[
tr_{\omega_{\psi}} \omega_0 \leq \left( tr_{\omega_{\psi}}\omega_t \right) \frac{\omega_0^2}{\omega_t^2} \\
\leq C \left( tr_{\omega_{\psi}}\omega_t \right) e^{-t} \left| S_{h_t}^{2(1-\beta)} \Omega \right| e^{-t} \omega_0 \wedge f^* \theta^t \\
\leq \frac{C}{\sigma^{\lambda_1} \left| S_{h_t}^{2(1-\beta)} \right|} e^{gt} \\
\leq C e^{\sigma^{\lambda_1} t}.
\]
In conclusion, we have obtained that
\[
C^{-1} e^{-\sigma^{\lambda_1} t} \omega_t \leq \omega_{\psi} \leq C e^{\sigma^{\lambda_1} t} \omega_t. \tag{3.17}
\]
Now, since all the smooth fibers of \(f\) are elliptic curves, we can apply an idea due to \(8, 10\) (see also \(7, 22\)) to obtain the local higher order estimates for \(\psi_t\).

**Proposition 3.5.** For any fixed \(K \subset X \setminus \Gamma\) and all \(k \in \mathbb{N}\), there exists a positive constant \(C_{K,k}\), which is independent of \(\epsilon\) and \(t\), such that
\[
\|\psi_t\|_{C^{k}(K,\omega_0)} \leq C_{K,k}. \tag{3.18}
\]

**Proof.** Note that on the approximation equation \((3.5)\), the extra term \((1-\beta)\sqrt{-1} \overline{\partial} \partial (|S_{h_t}^{2(1-\beta)} \omega_t^2 + \epsilon^2) + (1-\beta) R_{h_t} = f^*((1-\beta)\sqrt{-1} \overline{\partial} \partial (|S_{h_t}^{2(1-\beta)} \omega_t^2 + \epsilon^2) + (1-\beta) R_{h_t})\), which in particular implies that it does not depend on the variable in fiber direction. Thus we can apply the arguments in Theorem 5.24 of \(22\) to conclude this proposition. \(\Box\)
Now we choose a sequence $\epsilon_j \to 0^+$ such that $\varphi_{\epsilon_j}$ converges to the unique solution $\varphi$ of the conical Kähler-Ricci flow (3.1) in $L^1(X, \omega_0)$ and in $C^\infty_{\text{loc}}(X \setminus D)$-topology. From Proposition 3.5 we have

**Corollary 3.6.** For any fixed $K \subset X \setminus \Gamma$ and all $k \in \mathbb{N}$, there exists a positive constant $C_{K,k}$, such that for all $t \geq 0$,

$$\|\varphi + \delta S^2_h\|_{C^k(K, \omega_0)} \leq C_{K,k}. \quad (3.19)$$

Now we are going to show the uniform convergence at the level of potentials.

**Proposition 3.7.** For any fixed $K \subset X \setminus \Gamma$, $\varphi \to f^*\varphi_\infty$ in $L^\infty(K)$ as $t \to \infty$, where $\varphi_\infty$ is the function given by Proposition 2.7.

**Proof.** The proof will make use of a argument due to [18]. Firstly, by pulling back a defining section of $s_1 + \ldots + s_k + r$ on $\Sigma$, we may fix a defining section $\tilde{S}$ of the divisor $f^*([s_1] + \ldots + [s_k] + [r])$ vanishing exactly on $\Gamma$. Moreover, if we denote the holomorphic line bundle on $\Sigma$ associated to $s_1 + \ldots + s_k + r$ by $L'$, then by using $H^1(\Sigma) = \mathbb{R}$ we know $c_1(L') = \mu[\theta']$ for some constant $\mu$, where $[\theta']$ is the Kähler class of $\theta'$. Thus by pulling back a smooth Hermitian metric $h$ on $\Sigma$, we may fix a smooth Hermitian metric $\tilde{h}$ on the holomorphic line bundle associated to $f^*([s_1] + \ldots + [s_k] + [r])$ with $\text{Ric}(\tilde{h}) = \mu f^*\theta'$ for some $\mu \in \mathbb{R}$ and $|\tilde{S}|_{\tilde{h}}^2 \leq 1$.

Define $B_d(\Gamma') = \{s \in \Sigma | \text{dist}_\theta(s, \Gamma') \leq d\}$ and

$$B_d(\Gamma) = f^{-1}(B_d(\Gamma')).$$

Let $\epsilon$ be an arbitrary positive constant. Choose $d_K > 0$ such that $\overline{K} \subset X \setminus B_{d_K}(\Gamma)$ and for all $z \in B_{d_K}(\Gamma)$ and $t \geq 0$,

$$(\varphi - f^*\varphi_\infty + \epsilon \log |\tilde{S}|^2_{\tilde{h}})(z,t) < -1$$

and

$$(\varphi - f^*\varphi_\infty - \epsilon \log |\tilde{S}|^2_{\tilde{h}})(z,t) > 1.$$

Let $\eta_K$ be a smooth cut-off function on $\Sigma$ such that $\eta_K = 1$ on $\Sigma \setminus B_{d_K}(\Gamma')$ and $\eta_K = 0$ on $B_{d_K}(\Gamma')$.

Define $\rho_K = (f^*\eta_K)\rho_S$ and $\omega_{S,F,K} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho_K$. Then set

$$\psi_K = \varphi - f^*\varphi_\infty - e^{-t}\rho_K + \epsilon \log |\tilde{S}|^2_{\tilde{h}}$$

and

$$\psi_K^+ = \varphi - f^*\varphi_\infty - e^{-t}\rho_K - \epsilon \log |\tilde{S}|^2_{\tilde{h}}.$$

A direct computation gives

$$\partial_t \psi_K = \log \frac{e^t(e^{-t}\omega_{S,F,K} + f^*\omega_\infty + (\epsilon\mu - e^{-t})f^*\theta' + \sqrt{-1}\partial\bar{\partial}\psi_K^2)}{2\omega_{S,F} \wedge f^*\omega_\infty} - \psi_K + \epsilon \log |\tilde{S}|^2_{\tilde{h}}. \quad (3.20)$$

Now we can use the same argument as in the proof of Proposition 6.1 of [18] to conclude that there exist uniform constants $C$ and $T_1$ such that for all $t \geq T_1$,

$$\psi_K \leq C\epsilon$$

if $\epsilon$ is sufficiently small. In particular, on $K \times [T_1, \infty)$,

$$\varphi - f^*\varphi_\infty \leq C\epsilon \quad (3.21)$$
for some constant $C_K$ only depending on $K$. Similarly,

$$\varphi - f^*\varphi_\infty \geq -C_K \varepsilon$$  \hspace{1cm} (3.22)

on $K \times [T_1, \infty)$. Proposition 3.7 follows from (3.21) and (3.22).

Finally, we are able to prove Theorem 1.1.

**Proof of Theorem 1.1.** Combining Proposition 2.1, Corollary 3.6 and Proposition 3.7, Theorem 1.1 is proved.

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