Relations between multi-resolution analysis and quantum mechanics

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Abstract
We discuss a procedure to construct multi-resolution analyses (MRA) of $L^2(\mathbb{R})$ starting from a given seed function $h(s)$ which should satisfy some conditions. Our method, originally related to the quantum mechanical hamiltonian of the fractional quantum Hall effect (FQHE), is shown to be model independent. The role of a canonical map between certain canonically conjugate operators is discussed. This clarifies our previous procedure and makes much easier most of the original formulas, producing a convenient framework to produce examples of MRA.
I Introduction

In a series of papers, [1]-[7], we have discussed the relations between a generic MRA and the ground state of the free single-electron hamiltonian of the FQHE. In particular we proved that any MRA produces an orthonormal set of functions in the subspace of $L^2(\mathbb{R}^2)$ known as the lowest Landau level, see Section II, and that, vice-versa, any such a set produces a sequence of complex numbers related to a certain MRA.

In this paper we extend these results and propose a model-independent construction which still give rise to a MRA starting from a certain square integrable function, which we call seed function. Our extension clarifies the role of some canonical maps for a certain quantum hamiltonian, and for its related physical system, which is behind the construction.

We devote the rest of this Introduction to recall, just to fix the notation, few known facts about MRA which will be useful in the following.

A MRA of $L^2(\mathbb{R})$ is an increasing sequence of closed subspaces
\[ \ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots \subset L^2(\mathbb{R}), \tag{1.1} \]
with $\bigcup_{j \in \mathbb{Z}} V_j$ dense in $L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, and such that

1. $f(x) \in V_j \iff f(2x) \in V_{j+1}$

2. There exists a function $\phi \in V_0$, called scaling function, such that $\{\phi(x-k), k \in \mathbb{Z}\}$ is an o.n. basis of $V_0$.

From these two requirements clearly follows that, for any fixed $j \in \mathbb{Z}$, $\{\phi_{j,k}(x) \equiv 2^{j/2}\phi(2^jx-k), k \in \mathbb{Z}\}$ is an o.n. basis of $V_j$, which can be interpreted as an approximation space: the approximation of $f \in L^2(\mathbb{R})$ at the resolution $2^j$ is defined by its projection onto $V_j$. The additional details needed for increasing the resolution from $2^j$ to $2^{j+1}$ are given by the projection of $f$ onto the orthogonal complement $W_j$ of $V_j$ in $V_{j+1}$:

\[ V_j \oplus W_j = V_{j+1}, \tag{1.2} \]

and we have:

\[ \bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{R}). \tag{1.3} \]
Now, the main result of a MRA is that there exists a function \( \psi \), the mother wavelet, explicitly computable from \( \phi \), such that \( \{ \psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z} \} \) constitutes an orthonormal basis of \( L^2(\mathbb{R}) \).

The construction of \( \psi \) proceeds as follows. First, the inclusion \( V_0 \subset V_1 \) yields the relation

\[
\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \phi(2x - n), \quad h_n = \langle \phi_{1,n} | \phi \rangle.
\] (1.4)

Then one uses these coefficients to define the function \( \psi \) as

\[
\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^{n-1} h_{-n-1} \phi(2x - n).
\] (1.5)

As we see, the role of the coefficients \( h_n \) is quite important. For this reason we introduce the following definition:

**Definition 1:** We call **relevant** any sequence \( h = \{ h_n, n \in \mathbb{Z} \} \) which satisfies the following properties:

1. \( \sum_{n \in \mathbb{Z}} h_n h_{n+2l} = \delta_{l,0} \);  
2. \( h_n = O\left(\frac{1}{1+|n|^2}\right), \quad n \gg 1 \);
3. \( \sum_{n \in \mathbb{Z}} h_n = \sqrt{2} \);
4. \( H(\omega) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{-i\omega n} \neq 0 \quad \forall \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] \).

Using Mallat’s algorithm it is known that any relevant sequence produces a MRA. In particular, it produces a scaling function \( \Phi(x) \) and the related mother wavelets, [9].

The main goal of this paper is the construction of a quite non-standard procedure which helps in the production of relevant sequences and, as a consequence, of multi-resolutions of \( L^2(\mathbb{R}) \). More in details:

in the next section we briefly resume our original results in this direction related to the Hall effect.

In Section III we propose our more abstract approach, mainly regarding condition (r1) of Definition 1. This analysis will produce a so-called *orthonormality condition*,
ONC, for a certain seed function \( h(s) \) in \( L^2(\mathbb{R}) \). We will show how to use \( h(s) \) to construct a sequence \( \{h_n\} \) satisfying condition (r1).

In Section IV we show how to find easily solutions of the ONC and, as a consequence, how to produce sequences satisfying condition (r1).

In Section V we propose an orthonormalization trick, ONT, which generates more solutions of the ONC.

In Section VI we consider the other requirements contained in Definition 1, in connection with our approach.

Section VII contains our conclusions and plans for the future.

II The old results: FQHE

The many-body model of the FQHE consists simply in a two-dimensional electron gas, 2DEG, (that is a gas of electrons constrained in a two-dimensional layer) in a positive uniform background and subjected to an uniform magnetic field along \( z \), whose hamiltonian (for \( N \) electrons) is, [8],

\[
H^{(N)} = H_0^{(N)} + \lambda (H_C^{(N)} + H_B^{(N)}), \tag{2.1}
\]

where \( H_0^{(N)} \) is the sum of \( N \) contributions:

\[
H_0^{(N)} = \sum_{i=1}^{N} H_0(i). \tag{2.2}
\]

Here \( H_0(i) \) describes the minimal coupling of the \( i \)–th electron with the magnetic field:

\[
H_0 = \frac{1}{2} \left( p + A(r) \right)^2 = \frac{1}{2} \left( p_x - \frac{y}{2} \right)^2 + \frac{1}{2} \left( p_y + \frac{x}{2} \right)^2. \tag{2.3}
\]

\( H_C^{(N)} \) is the canonical Coulomb interaction between charged particles, \( H_C^{(N)} = \frac{1}{2} \sum_{i \neq j} \frac{1}{|r_i - r_j|} \), and \( H_B^{(N)} \) is the interaction of the charges with the background, [8].

We now consider \( \lambda (H_C^{(N)} + H_B^{(N)}) \) as a perturbation of the free hamiltonian \( H_0^{(N)} \), and we look for eigenstates of \( H_0^{(N)} \) in the form of Slater determinants built up with
single electron wave functions. The easiest way to approach this problem consists in introducing the new variables

\[ P' = p_x - y/2, \quad Q' = p_y + x/2. \] (2.4)

In terms of \( P' \) and \( Q' \) the single electron hamiltonian, \( H_0 \), can be written as

\[ H_0 = \frac{1}{2} (Q'^2 + P'^2). \] (2.5)

The transformation (2.4) can be seen as a part of a canonical map from \((x, y, p_x, p_y)\) into \((Q, P, Q', P')\) where

\[ P = p_y - x/2, \quad Q = p_x + y/2. \] (2.6)

These operators satisfy the following commutation relations:

\[ [Q, P] = [Q', P'] = i, \quad [Q, P'] = [Q', P] = [Q, Q'] = [P, P'] = 0. \] (2.7)

Using the results contained in [10], it can be deduced that a wave function in the \((x, y)\)-space is related to its \(PP'\)-expression by the formula

\[ \Psi(x, y) = \frac{e^{ixy/2}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(xP' + yP + PP')} \Psi(P, P') \, dP \, dP', \] (2.8)

which can be easily inverted:

\[ \Psi(P, P') = \frac{e^{-iPP'}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(xP' + yP + xy/2)} \Psi(x, y) \, dx \, dy. \] (2.9)

The usefulness of the \(PP'\)-representation stems from the expression (2.5) of \( H_0 \). Indeed, in this representation, the single electron Schrödinger equation admits eigenvectors \( \Psi(P, P') \) of \( H_0 \) of the form \( \Psi(P, P') = f(P')h(P) \). Thus the ground state of (2.5) must have the form \( f_0(P')h(P) \), where

\[ f_0(P') = \pi^{-1/4}e^{-P'^2/2}, \] (2.10)

while the function \( h(P) \) is arbitrary, which manifests the degeneracy of the lowest Landau level, LLL, i.e. the lowest eigenspace of \( H_0 \). The explicit expression of \( h(P) \) should be
fixed by the interaction, whose mean value should be minimized. With \( f_0 \) as above, formula (2.8) becomes
\[
\Psi(x,y) = \frac{e^{i xy/2}}{\sqrt{2 \pi^{3/4}}} \int_{-\infty}^{\infty} e^{iyP} e^{-(x+P)^2/2} h(P) \, dP,
\]
whose inverse is
\[
h(P) = \frac{e^{-i PP'+P'^2/2}}{2 \pi^{3/4}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(x P'+y P'+xy/2)} \Psi(x,y) \, dx \, dy.
\]
Let us now define the so-called magnetic translation operators \( T(\vec{a}_i) \) for a square lattice with basis \( \vec{a}_1 = a(1,0), \vec{a}_2 = a(0,1), a^2 = 2\pi, [4], \) by
\[
T_1 := T(\vec{a}_1) = e^{iaQ}, \quad T_2 := T(\vec{a}_2) = e^{iaP}.
\]
We see that, due to (2.7) and to the condition on the cell of the lattice, \( a^2 = 2\pi, \)
\[
[T(\vec{a}_1), T(\vec{a}_2)] = [T(\vec{a}_1), H_0] = [T(\vec{a}_2), H_0] = 0.
\]
The action of the \( T \)'s on a generic function \( f(x,y) \in \mathcal{L}^2(\mathbb{R}^2) \) is the following:
\[
f_{m,n}(x,y) := T_1^m T_2^n f(x,y) = (-1)^{mn} e^{i\alpha (my-nx)} f(x+ma, y+na).
\]
This formula shows that, if for instance \( f(x,y) \) is localized around the origin, then \( f_{m,n}(x,y) \) is localized around the site \( a(-m,-n) \) of the square lattice.

Now we have all the ingredients to construct the ground state of \( H_0^{(N)} \) mimicking the classical procedure. We simply start from the single electron ground state of \( H_0 \) given in (2.11), \( \Psi(x,y) \). Then we construct a set of copies \( \Psi_{m,n}(x,y) \) of \( \Psi(x,y) \) as in (2.15), with \( m, n \in \mathbb{Z} \). All these functions still belong to the LLL for any choice of the function \( h(P) \) due to (2.14). \( N \) of these wave functions \( \Psi_{m,n}(x,y) \) are finally used to construct a Slater determinant \( \Psi^{(N)} \) for the finite system in the usual way, which is normalized for all \( N \) if
\[
< \Psi_{m_1,n_1}, \Psi_{m_j,n_j} > = \delta_{m_1,m_j} \delta_{n_1,n_j}.
\]
Let \( \Psi(x,y) \) be as in (2.11) and \( \Psi_{m,n}(x,y) = (-1)^{mn} e^{i\alpha (my-nx)} \Psi(x+ma, y+na) \).
With the above definitions we find
\[
\tilde{S}_{1_1,1_2} = < \Psi_{0,0}, \Psi_{1_1,1_2} > = \int_{-\infty}^{\infty} dpe^{-i2\alpha P(h(p-P_{1_1})h(p))}.
\]
which restates the problem of the orthonormality of the wave functions in the LLL in terms of the unknown function $h(P)$.

In the construction above we are considering a square lattice in which all the lattice sites are occupied by an electron. We say that the filling factor $\nu$ is equal to 1. We have seen in [4] that, in order to construct an o.n. set of functions in the LLL corresponding to a filling $\nu = \frac{1}{2}$ (only half of the lattice sites are occupied), we have to replace (2.17) with the following slightly weaker condition,

$$S_{l_1,l_2} = \tilde{S}_{l_1,2l_2} = \int_{-\infty}^{\infty} dpe^{-2i2la\hat{h}(p-l_1a)h(p)} =$$

$$= \int_{-\infty}^{\infty} dpe^{il_1ap}\hat{h}(p-2l_2a)\hat{h}(p) = \delta_{l_1,0}\delta_{l_2,0},$$

(2.18)

for all $l_1, l_2 \in \mathbb{Z}$, where $\hat{h}(p) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ipx}h(x)dx$ is the Fourier transform of $h(x)$. If $h(x)$ satisfies (2.18), then, defining

$$h_n = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} dpe^{-inx}h(x),$$

(2.19)

it is easily checked that

$$\sum_{n \in \mathbb{Z}} h_n h_{n+2l} = \delta_{l,0}.$$  

(2.20)

The proof of this claim, contained in [4], is based on condition (2.18) and on the use of the Poisson summation formula (PSF) which we write here as

$$\sum_{n \in \mathbb{Z}} e^{inx} = \frac{2\pi}{|c|} \sum_{n \in \mathbb{Z}} \delta(x - n\frac{2\pi}{c}),$$

(2.21)

for any $c \in \mathbb{R}$. It is well known that the PSF does not always hold, see [11] p.298 and references therein, for instance. In this paper, however, we will always assume its validity.

In [7] we have also discussed a possible way to find solutions of the equation (2.18) starting from a generic seed function in $L^2(\mathbb{R})$: in particular we have shown how the assumption that this function produces a relevant sequence of complex numbers, i.e. a sequence satisfying Definition 1 above, produces many constraints on the seed function.
itself. We will now go back to this construction from a much more abstract point of view, showing that there exists a general framework behind the construction just sketched, construction which allows us to extract the really crucial ingredients of our method.

**Remark:** we also want to remind that the above construction has been extended to other shapes of the lattice and to different values of the filling. These results, contained in [5], are more relevant for concrete numerical applications to the FQHE, but exactly for this same reason, are harder to be concretely applied.

### III A more abstract point of view

In this section we will *embed* the above results in an abstract and more general framework. This will make our procedure more direct and much simpler, both from a theoretical and from a practical point of view.

Consider the operators \(((\hat{x}, \hat{p}_x), (\hat{y}, \hat{p}_y))\) and \(((\hat{x}_1, \hat{p}_1), (\hat{x}_2, \hat{p}_2))\), satisfying

\[
[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i,
\]

\[
[\hat{x}_1, \hat{p}_1] = [\hat{x}_2, \hat{p}_2] = i
\]

Let \(\xi_x\) and \(\eta_y\) be the generalized eigenstates of \(\hat{x}\) and \(\hat{y}\): \(\hat{x}\xi_x = x\xi_x\), \(\hat{y}\eta_y = y\eta_y\), and \(\xi'_{x_1}\) and \(\eta'_{x_2}\) the eigenstates of \(\hat{x}_1\) and \(\hat{x}_2\): \(\hat{x}_1\xi'_{x_1} = x_1\xi'_{x_1}\), \(\hat{x}_2\eta'_{x_2} = x_2\eta'_{x_2}\). We recall that all these vectors are \(\delta\)-like normalized, e.g., \(\langle \xi_x, \xi_{x'} \rangle = \delta(x - x')\), \(\langle \eta_y, \eta_{y'} \rangle = \delta(y - y')\), and produce resolutions of the identity:

\[
\int dx \int dy |\xi_{x,y}\rangle < \xi_{x,y}| = \int dx_1 \int dx_2 |\xi'_{x_1,x_2}\rangle < \xi'_{x_1,x_2}| = I, \tag{3.1}
\]

where \(\xi_{x,y} = \xi_x \otimes \eta_y\) and \(\xi'_{s,t} = \xi'_s \otimes \eta'_t\). In this section sometime we adopt the Dirac bra-ket symbols to simplify the notation. Any \(\Psi \in \mathcal{H}\), our Hilbert space, can be written in the \((x, y)\)-coordinates or in the \((x_1, x_2)\)-coordinates as

\[
\Psi(x, y) = <\xi_{x,y}|\Psi> \quad \text{and} \quad \Psi'(x_1, x_2) = <\xi'_{x_1,x_2}|\Psi>.
\]

which, because of (3.1), are related to each other as follows

\[
\Psi(x, y) = \int dx_1 \int dx_2 <\xi_{x,y}|\xi'_{x_1,x_2}> \Psi'(x_1, x_2) \tag{3.2}
\]
\[ \Psi'(x_1, x_2) = \int dx \int dy \langle \xi_{x_1, x_2}' | \xi_{x, y} \rangle \Psi(x, y) \] (3.3)

It is clear that these formulas are just the abstract versions of formulas (2.8) and (2.9), with a kernel

\[ K(x, y; x_1, x_2) := \langle \xi_{x, y} | \xi_{x_1, x_2}' \rangle \] (3.4)

which is easily identified.

We may interpret \((x, y)\) as the physical spatial coordinates (in analogy with the FQHE), while \((x_1, x_2)\) can be seen as a pair of fictitious coordinates and they are not required to have any physical meaning, in general. For this reason there is no objection in taking \(\Psi'(x_1, x_2)\) as a product function \(\Psi'(x_1, x_2) = \varphi(x_1)h(x_2)\) in (3.2), and we call \(\Psi(h)(x, y)\) the related function in the \((x, y)\)-space:

\[ \Psi(h)(x, y) = \int dx_1 \int dx_2 K(x, y; x_1, x_2) \varphi(x_1)h(x_2) \] (3.5)

As a matter of fact, \(\Psi(h)(x, y)\) clearly also depends on \(\varphi\). However, it will appear clear in the following that this dependence disappears in all the scalar products we will consider. For this reason we prefer to adopt this simpler but somehow misleading notation.

Also, we introduce three commuting operators: \(H = H(\hat{x}_1, \hat{p}_1) = H^\dagger\), \(T_1 = e^{ia\hat{x}_2}\) and \(T_2 = e^{ia\hat{p}_2}\). Here, for reasons that will appear clear in the following, we take \(a^2 = 4\pi\).

It is clear that, as for the FQHE, independently of the explicit definition of \(H\), we have

\[ [T_1, T_2] = [T_1, H] = [T_2, H] = 0. \] (3.6)

We still call the unitary operators magnetic translations and \(H\) the hamiltonian. Notice that, while the explicit expressions for \(T_1\) and \(T_2\) are fixed above, there is no need to fix the expression of \(H\), which will be kept general here. We will comment on possible explicit expressions of \(H\) several times along the paper and in the examples below. What this really means is that in our treatment there is no need of having any concrete physical system behind.

\(^1\)this is slightly different from what we have done in the previous section, where we had \(a^2 = 2\pi\) but where only one site of the lattice every two was occupied by an electron.
As for the FQHE, the main idea is to require orthonormality of the functions

$$\Psi^{(h)}_{\vec{l}}(x, y) = T_{1}^{l_1} T_{2}^{l_2} \Psi^{(h)}(x, y) = \int dx_1 \int dx_2 K_{\vec{l}}(x, y; x_1, x_2) \varphi(x_1) h(x_2), \quad (3.7)$$

where $\vec{l} = (l_1, l_2)$ and $K_{\vec{l}}(x, y; x_1, x_2) = T_{1}^{l_1} T_{2}^{l_2} K(x, y; x_1, x_2)$, which, all together, generate a (fictitious) lattice with cell area equal to $4\pi$. In particular, if the function $\varphi(x_1)$ is an eigenstate of $H(\hat{x}_1, \hat{p}_1)$ corresponding to an eigenvalue $\epsilon$, then each $\Psi^{(h)}_{\vec{l}}(x, y)$ is still an eigenstate of $H(\hat{x}_1, \hat{p}_1)$ with the same eigenvalue $\epsilon$. We can still speak of infinite degeneracy of the energetic levels, which we still call Landau levels. In this case we could think of $H$ as an operator like $\epsilon|\varphi><\varphi| + \tilde{H}$, where $\tilde{H}$ is again self adjoint, and contains the rest of the spectrum of $H$ rather than $\epsilon$.

**Remarks:**

1. In the FQHE the function $\varphi(x_1)$ was taken to be the ground state of $H = H_0$. In the rest of this section we will show that this is quite unessential.

2. It may be worthwhile to notice that the appearance of two commuting unitary operators like $T_1$ and $T_2$ strongly suggests the relevance of the $(k, q)$-representation behind our strategy. This is not surprising since the $(k, q)$-representation was exactly our starting point in our first approach to the problem of finding an orthonormal set in the LLL. This has been originally discussed in [8] and, more in connection with MRA, in [4] and [6]. However, how it will be clear from our treatment, our main results can be found without any use of this representation.

As for the FQHE we now compute the overlap between different wave functions, which can be written as follows:

$$S_{l_1, l_2}^{(h)} = <\Psi_{l_1, l_2}^{(h)}, \Psi_{0, 0}^{(h)}> = \int_{\mathbb{R}^2} dt \, dt' \, h(t) \, \Gamma_{\vec{l}}(t, t') \, h(t'), \quad (3.8)$$

where we have introduced the following quantities

$$\Gamma_{\vec{l}}(t, t') = \int_{\mathbb{R}^2} ds \, ds' \, \overline{\varphi(s)} \, Q_{\vec{l}}(s, t; s', t') \, \varphi(s') \quad (3.9)$$

and

$$Q_{\vec{l}}(s, t; s', t') = \int_{\mathbb{R}^2} dx \, dy \, \overline{K_{\vec{l}}(x, y; s, t)} \, K(x, y; s', t'). \quad (3.10)$$
It is now evident that, putting this in (3.9), we get, for any normalized \( \phi \),
\[
\Gamma (t, t') = e^{-iatl_1} \delta (t - al_2 - t'),
\]
(3.12)
which, in turns, produces
\[
S_{l_1, l_2}^{(h)} = \int_\mathbb{R} ds \, h(s) \, \overline{h(s + al_2)} \, e^{-isatl_1} = \int_\mathbb{R} dp \, \hat{h}(p) \, \overline{\hat{h}(p - al_1)} \, e^{-ipal_2},
\]
(3.13)
writing the result also in terms of the Fourier transform \( \hat{h}(p) \) of \( h(s) \).

**Remarks:**

1. This formula clearly shows what we have stated before: the overlap between differently localized wave functions constructed as shown above is independent of the particular physical model we may consider, as well as from the details of the canonical transformation mapping \( ((\hat{x}, \hat{p}_x), (\hat{y}, \hat{p}_y)) \) into \( ((\hat{x}_1, \hat{p}_1), (\hat{x}_2, \hat{p}_2)) \). Also, it does not depend on the explicit expression for \( \varphi (x_1) \), as far it is normalized in \( L^2(\mathbb{R}) \). This last result was already noticed in [4], where we proved that the ONC obtained from wave functions in the higher Landau levels coincides with (2.18).

2. It is also worthwhile to stress that the reason why we have chosen here \( a^2 = 4\pi \), instead of \( a^2 = 2\pi \) as in the previous section, is to avoid a unnatural asymmetry between the indices \( l_1 \) and \( l_2 \) (i.e., between the two orthogonal directions of the lattice), which is present, e.g., in formula (2.18) but not in our new approach, see (3.13). We recall that, see [7] and references therein, the factor 2 appearing in (2.18) has a double meaning: from one side, it refers to the value \( \frac{1}{2} \) of the filling factor \( \nu \) for the electron gas. From the other side, it corresponds to a 2-MRA, that is to a MRA with dilation parameter equal to 2. In [3] we have extended the procedure to a filling \( \nu = \frac{1}{d} \) or, equivalently, to a d-MRA, \( d \in \mathbb{N} \). The same extension can be performed here: indeed it would be sufficient to choose \( a = \sqrt{2\pi d} \).
(3) We finally want to stress that the differences arising between (2.18) and (3.13) are not only a consequence of the different choices of the value of $a^2$ in the two sections, but also follow from a different choice of the variables used to describe the wave function after the unitary transformation. Indeed, while in this section we have used as new coordinates the eigenvalues of the new position operators $\hat{x}_1$ and $\hat{x}_2$, in Section II, adopting the same choice made in [1]-[8] as well as in the paper where this canonical map was used for the first time in connection with the FQHE, [12], we have used as new coordinates the eigenvalues of the new momenta operators, but for a minus sign. We will come back on this point in Example 1 below, where we make uniform the notation.

Before going on with the relations of our procedure with relevant sequences, we briefly discuss three examples of the above construction.

**Example 1.**

As a first example we consider the FQHE already discussed in many details in the previous section. In order to uniform the notation, we rewrite the results using the approach discussed in this section. In particular we take $a^2 = 4\pi$ and we use $\Psi'(x_1, x_2)$, see (3.3), instead of $\Psi(P, P')$, see (2.9).

With this in mind we notice that the kernel of the transformation, which is slightly different from the one deduced by (2.8), is

$$K(x, y; s, t) = \frac{1}{2\pi} \exp \left\{ i(\frac{xt + ys - st - xy}{2}) \right\}$$

Using now the analogous of (2.15) and condition $a^2 = 4\pi$ we deduce that

$$K_\ell(x, y; s, t) = \frac{1}{2\pi} \exp \left\{ i((x + l_1 a)t + (y + l_2 a)s - st - \frac{xy}{2} - ixl_2 a) \right\}$$

It is now easy to check that

$$Q_\ell(s, t; s', t') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dx dy e^{-i((x+l_1 a)t+(y+l_2 a)s-st-\frac{xy}{2}-ixl_2 a)} e^{i(xt'+ys'-s't'-\frac{xy}{2})} =$$

$$= e^{-iat l_1} \delta(s - s') \delta(t - al_2 - t'),$$

exactly as in (3.11). The results for $\Gamma_\ell$ and $S_\ell^{(h)}$ are direct consequences of this one, and coincide with (3.12) and (3.13) respectively.

**Example 2.**
The second example was originally introduced in [3], as a prototype of the FQHE in which the single electron hamiltonian and the canonical transformations are different, and simpler, than those considered in Section II. In particular we have \( H = \frac{1}{2}(\hat{p}_x^2 + \hat{x}^2) + \frac{1}{2}\hat{p}_y^2 + \hat{p}_x \hat{p}_y \). The new variables are defined as \( \hat{x}_1 = \hat{p}_x + \hat{p}_y, \hat{p}_1 = -\hat{x}, \hat{x}_2 = \hat{p}_y \) and \( \hat{p}_2 = \hat{x} - \hat{y} \). Then we have \([\hat{x}_j, \hat{p}_j] = i\), for \( j = 1, 2 \), while all the other commutators among these new operators are zero. In terms of these operators the hamiltonian looks like \( H = \frac{1}{2}(\hat{x}_1^2 + \hat{p}_1^2) \), whose ground state is \( \varphi(x_1) = \frac{1}{\pi}\sqrt{\pi}e^{-x_1^2/2} \). We also introduce the unitary operators \( T_1 = e^{ia_1 \hat{x}_2} \) and \( T_2 = e^{ia_2 \hat{p}_2} \), with \( a_2 = 4\pi \). These operators commute between themselves and with \( H \), and act on a generic function \( f(x,y) \in L^2(\mathbb{R}^2) \) as follows: \( T_1^{l_1} T_2^{l_2} f(x,y) = e^{ial_2(x-y)}f(x,y + al_1) \). Following the procedure discussed in [10], we find that the kernel of the transformation is

\[
K(x,y; s,t) = \frac{1}{2\pi} \exp \left\{ i(x(s-t) + y(t+s)) \right\},
\]

so that \( K_l(x,y; s,t) = \frac{1}{2\pi} \exp \left\{ i(2al_2(x-y)) + i(s-t) + i(y+al_1)t \right\} \). Therefore

\[
Q_l(s,t; s', t') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dx \, dy \, e^{-ial_2(x-y)-ix(s-t)-i(y+al_1)t} e^{ix(s'-t') + iyt'} =
\]

\[
e^{-iatl_1} \delta(s-s') \delta(t - al_2 - t'),
\]

again as in (3.11). The results for \( \Gamma_l \) and \( S_l^{(h)} \) directly follow, and clearly coincide with the ones in (3.12) and (3.13).

**Example 3.**

We now consider a third example which differs from the previous ones since there is no hamiltonian structure behind. We consider the following canonical transformation: \( ((\hat{x}, \hat{p}_x), (\hat{y}, \hat{p}_y)) \rightarrow ((\hat{x}_1, \hat{p}_1), (\hat{x}_2, \hat{p}_2)) \), where \( \hat{x}_1 = \hat{x} - \hat{p}_x, \hat{p}_1 = \hat{p}_x, \hat{x}_2 = \hat{p}_y \) and \( \hat{p}_2 = -\hat{y} + \hat{p}_y \). Then \([\hat{x}_j, \hat{p}_j] = i\), for \( j = 1, 2 \), while all the other commutators among the new operators are zero. Following [10] we deduce the expression for the kernel of the transformation:

\[
K(x,y; s,t) = \frac{1}{2\pi} \exp \left\{ \frac{i}{2} \left( x^2 + 2(yt - xs) + s^2 - t^2 \right) \right\},
\]

so that, since \( T_1^{l_1} T_2^{l_2} f(x,y) = e^{-ial_2y}f(x,y + al_1 + al_2) \) and \( a_2 = 4\pi \), we get

\[
K_l(x,y; s,t) = \frac{e^{-ial_2y}}{2\pi} \exp \left\{ \frac{i}{2} \left( x^2 + 2((y - al_1 + al_2)t - xs) + s^2 - t^2 \right) \right\}.
\]
It is now a trivial exercise to check that, again, formulas (3.11), (3.12) and (3.13) are recovered.

It is now quite easy to check that any MRA produces solutions of the following orthonormality condition (ONC):

$$S_{l_1,l_2}^{(h)} = \delta_{l_1,0}\delta_{l_2,0}$$  \hspace{1cm} (3.14)

Indeed, any MRA is related to a relevant sequence $h = \{h_n, n \in \mathbb{Z}\}$ satisfying, among the others, property (r1) of Definition 1: $\sum_{n \in \mathbb{Z}} h_n \overline{h_{n+2l}} = \delta_{l,0}$. Modifying a little bit a result already contained in [7], we define now

$$h_2(s) = \begin{cases} \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}} h_n e^{-isna/2}, & s \in [0,a[, \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (3.15)

Then we deduce that $S_i^{(h)} = \int_{\mathbb{R}} h_2(s)h_2(s+al)e^{-isal_1}ds = \delta_{l_2,0}\int_{\mathbb{R}} |h_2(s)|^2 e^{-isal_1}ds$, because of the support of $h_2$, and we find $S_i^{(h)} = \delta_{l_2,0}\sum_{m \in \mathbb{Z}} h_m \overline{h_{m+2l_1}} = \delta_{l_2,0}$. Therefore $h_2(s)$ is a solution of the ONC above.

However, our main interest here is to proceed in the opposite direction: given a solution of the ONC (3.14) we would like to obtain a relevant sequence. This is also the only relevant point here, since we are in a more general settings than in [7], and there is no physical system behind our construction, in general. For this reason, it may be of no interest at all to obtain an o.n. basis in a fictitious LLL.

The way in which the elements of our tentative relevant sequence should be defined is suggested by formula (3.15): if we want to recover $h_n$ from $h_2(s)$ we have to compute the following integral: $\frac{1}{\sqrt{a}} \int_{\mathbb{R}} h_2(s) e^{isna/2} ds$. This suggests to take exactly this formula as our definition of $h_n$, given a generic $h(s)$, solution of the ONC. Therefore we put

$$h_n = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} h(s) e^{isna/2} ds.$$  \hspace{1cm} (3.16)

It is now with a simple application of the PSF that we can prove that the sequence $\{h_n\}$ satisfies condition (r1). Indeed we have

$$\sum_{n \in \mathbb{Z}} h_n \overline{h_{n+2l}} = \frac{1}{a} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} h(s) e^{isna/2} ds \int_{\mathbb{R}} \overline{h(t)} e^{-ita(n/2+l)} dt =$$
\[
\frac{1}{a} \int_{\mathbb{R}^2} ds \, dt \, h(s)\overline{h(t)} \, e^{-ia(t-s)} = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^2} ds \, dt \, h(s)\overline{h(t)} \, e^{-ia(t-s)} \delta(s - t - na) = \\
= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} ds \, h(s)\overline{h(s - na)} \, e^{-ias} = \sum_{n \in \mathbb{Z}} \delta_{n,0} \delta_{l,0} = \delta_{l,0},
\]
which is what we had to prove.

Before considering examples of our construction it is interesting to consider some points:

the first one is the following: the plus sign in the exponential in the definition of \( h_n \), formula (3.16), could be replaced by a minus sign; indeed the sequence that we obtain still satisfies condition (r1). This can be proved explicitly or simply noticing that, if \( \tilde{h}_n = h_{-n} \), then, introducing \( m = -n \),

\[
\sum_{n \in \mathbb{Z}} \tilde{h}_n \overline{h_{n+2l}} = \sum_{m \in \mathbb{Z}} \overline{h}_{-m} \overline{h_{m+2l}} = \sum_{m \in \mathbb{Z}} \overline{h}_m \overline{h_{m-2l}} = \delta_{-l,0} = \delta_{l,0}.
\]

The second remark is obvious: using \( \hat{h}(p) \) we can simply write equation (3.16) as

\[
h_n = \sqrt{\frac{2\pi}{a}} \hat{h} \left( -\frac{an}{2} \right) \quad \text{(3.17)}
\]

Finally, formula (3.13) for \( S_l^{(h)} \) has an interesting and useful consequence, as far as solutions of the ONC are concerned. It is clear indeed that if \( h(s) \) solves the ONC, then another solution of the ONC is a function \( m(s) \) which is the inverse Fourier transform of a function \( \hat{m}(p) := h(p) \). This is a trivial consequence of the expression of \( S_l^{(h)} \) given in terms of \( h(s) \) and \( \hat{h}(p) \). It is also clear that \( h(s) \) and \( m(s) \) produce different relevant sequences!

**IV Solutions of the ONC and consequences**

We already noticed that any MRA produces a solution of the ONC (3.14) as in (3.15). These are not the only solutions of the ONC. Different solutions are given in the table.
below, where we list the solution of the ONC in terms of \( h(s) \) or of its Fourier transform \( \hat{h}(p) \) and the coefficients that we find using (3.16) or (3.17).

| \( h(s) \) | \( \hat{h}(p) \) | \( h_n \) |
|---|---|---|
| \( \frac{1}{\sqrt{a}}, \ s \in [0,a[ \) | \( 0 \) | \( h_n = \delta_{n,0} \) |
| \( \frac{a}{\sqrt{a}}, \ s \in [0,a[ \) | \( 0 \) | \( h_n = i(1 - e^{-ia}) \frac{1}{2\pi n - a} \) |
| \( \frac{\sqrt{2}}{\pi}, \ s \in [0,a/2[, \cup [a/2,3a[ \) | \( \frac{1}{\sqrt{a}}, \ p \in [0,a[, \cup [a/2,3a[ \) | \( h_n = \frac{1}{\sqrt{2}}(\delta_{n,0} + \delta_{n,-1}) \) |
| \( \frac{\sqrt{2}}{\pi}, \ s \in [0,a/2[, \cup [a/2,a[ \) | \( -\frac{1}{\sqrt{2a}}, \ p \in [a/2,a[, \cup [a/2,3a[ \) | \( h_n = \frac{1}{2}(\delta_{n,0} + \delta_{n,-4} + \delta_{n,-5} - \delta_{n,-1}) \) |
| \( \frac{\sqrt{2a}}{\pi}, \ s \in [0,a/2[, \cup [a/2,a[ \) | \( 0 \) | \( h_n = \frac{1}{2}(\delta_{n,0} + \delta_{n,-2} + \delta_{n,-3} - \delta_{n,-1}) \) |

Moreover, it is also easy to check that the functions

\[
h_{e1}(s) = \begin{cases} \frac{1}{\sqrt{2a}}, & p \in [0,a/2[, \cup [2a,3a[ \\ -\frac{1}{\sqrt{2a}}, & p \in [a/2,a[, \cup [2a,3a[ \\ 0, & \text{otherwise} \end{cases} \quad h(s)_{e2} = \begin{cases} \frac{1}{\sqrt{2a}}, & p \in [0,a/2[, \cup [a,2a[ \\ -\frac{1}{\sqrt{2a}}, & p \in [a/2,a[, \cup [a,2a[ \\ 0, & \text{otherwise} \end{cases}
\]

both return the same coefficients as in the third row of the table above: \( h_0 = \frac{1}{\sqrt{2}}, h_{2n} = 0 \) if \( n \neq 0 \), and \( h_{2n+1} = \frac{i\sqrt{2}}{\pi(2n+1)} \).

This is a first example of an interesting feature of our procedure: the same sequence \( \{h_n\} \) can be obtained starting from very different functions \( h(s) \). Just to mention another, maybe more interesting, example of this fact let us consider the Haar multires-
olution. This can be obtained from the function

\[
\hat{h}(p) = \begin{cases} \frac{1}{\sqrt{a}}, & p \in [0, a[, \\
0, & \text{otherwise}, \end{cases}
\]

or by

\[
h_2(s) = \begin{cases} \frac{1}{\sqrt{2a}}(1 + e^{-isa/2}), & s \in [0, a[, \\
0, & \text{otherwise}, \end{cases}
\]

see the fifth row in the table above and formula \((3.15)\).

All the coefficients \(\{h_n\}\) found with our procedure satisfy condition (r1): this is trivially checked in some of the examples above while it is absolutely non trivial for the second, third and fourth examples listed in the table. Also, it is worth noticing that the fifth example produces the well known Haar MRA.

Moreover, we see from the first and the fifth row an example of the symmetry that was mentioned at the end of the previous section: we see that \(h(s)\) and \(\hat{h}(p)\) have the same dependence on their variables but they produce different coefficients. The same holds true for the seventh row and the example arising from the function \(h_{11}(s)\) above: they have the same dependence on, respectively, \(p\) and \(s\), but produce completely different coefficients.

It is not hard to check that these coefficients do not always generate relevant sequences: for instance, the sixth example of the table give rise to a sequence which surely satisfies (r1) and (r2), while condition (r3) does not hold. The fourth example, on the contrary, satisfies all the conditions of Definition 1. We will return on this point in Section VI.

V More solutions: the orthonormalization trick

The solutions of the ONC considered in the previous section all share a common feature: \(h(s)\) or \(\hat{h}(p)\) in the table are all compactly supported. Moreover, most of the time, the support is just contained in \([0, a[\). There is a reason for that: because of the expression \((3.13)\) of \(S_{l_i}^{(h)}\), these choices produce \(S_{l_i}^{(h)} = \delta_{l_2,0}s_{l_1}\) or \(S_{l_i}^{(h)} = \delta_{l_1,0}\tilde{s}_{l_2}\), with \(s_{l_1}\) or \(\tilde{s}_{l_2}\) to be computed, depending on which function, \(h(s)\) or \(\hat{h}(p)\), has compact support in \([0, a[\). One may wonder if other solutions of the ONC do exist and, if they exist, how they can be found.

In this section we will discuss an explicit construction which allows, given a generic function \(h(s)\) in \(L^2(\mathbb{R})\), to construct another function, \(H(s)\), which is still in \(L^2(\mathbb{R})\) and
satisfies the ONC. We call this technique \textit{orthonormalization trick} (ONT) in analogy to what is done in [13], where it is shown how to use spline functions to construct an orthonormal set in \(V_0\).

Let \(h(s) \in \mathcal{L}^2(\mathbb{R})\) be a generic function, \(\Psi_{l_1, l_2}^{(h)}\) the related set in \(\mathcal{L}^2(\mathbb{R}^2)\) constructed as discussed in Section III, and \(S_{l_1, l_2}^{(h)} = \langle \Psi_{l_1, l_2}^{(h)}, \Psi_{0, 0}^{(h)} \rangle\) the related overlap. If \(S_{l_1, l_2}^{(h)} \neq \delta_{l_1, 0} \delta_{l_2, 0}\) then the \(\Psi_{l_1, l_2}^{(h)}(\vec{r})\) generate a non o.n. lattice (in \(\mathcal{L}^2(\mathbb{R}^2)\)) and if we use \(h(s)\), as in (3.16), to construct a sequence \(\{h_n\}\), this will not satisfy condition (r1), in general. However, [8], we can find an o.n. set in \(\mathcal{L}^2(\mathbb{R}^2)\), still invariant under magnetic translations, simply by considering

\[
\Psi_{l_1, l_2}^{(H)}(\vec{r}) = T_{l_1} T_{l_2} \Psi_{0, 0}^{(H)}(\vec{r}), \quad \text{where} \quad \Psi_{0, 0}^{(H)}(\vec{r}) = \sum_{\vec{n} \in \mathbb{Z}^2} f_{\vec{n}} \Psi_{\vec{n}}^{(h)}(\vec{r}),
\]

and the coefficients \(f_{\vec{n}}\) are fixed by requiring that

\[
S_{l_1, l_2}^{(H)} = \delta_{l_1, 0} \delta_{l_2, 0} \tag{5.2}
\]

In these formulas a new function \(H(s)\) has been implicitly introduced: \(\Psi^{(H)}(\vec{r}) = \Psi_{0, 0}^{(H)}(\vec{r})\) is the function in \(\mathcal{L}^2(\mathbb{R}^2)\), (or in the LLL), which is generated via formula (3.5) by the function \(H(s)\). Again the role of the function \(\varphi\) is unessential, as far as the overlap between the functions \(\Psi_{l_1}^{(H)}(\vec{r})\) is concerned. Without giving the details of our construction, which are not really different from those in [7], defining the functions

\[
F(\vec{p}) = \sum_{\vec{n} \in \mathbb{Z}^2} f_{\vec{n}} e^{i \vec{p} \cdot \vec{n}} \quad \text{and} \quad S^{(h)}(\vec{p}) = \sum_{\vec{n} \in \mathbb{Z}^2} S_{\vec{n}}^{(h)} e^{i \vec{p} \cdot \vec{n}},
\]

the orthonormality requirement (5.2) produces

\[
\delta_{l_1, 0} = \sum_{\vec{n}, \vec{m} \in \mathbb{Z}^2} \frac{F_{\vec{n}}}{\sqrt{S^{(h)}(\vec{p})}} S_{\vec{n} + \vec{l} - \vec{m}}^{(h)} \rightarrow 1 = |F(\vec{p})|^2 S^{(h)}(\vec{p}) \rightarrow F(\vec{p}) = \frac{1}{\sqrt{S^{(h)}(\vec{p})}}.
\]

where we have chosen properly the phase in the solution (see [7] for a discussion concerning the effects of the phase). The coefficients \(f_{\vec{n}}\) are therefore given by

\[
f_{\vec{n}} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d^2 \vec{p} \frac{e^{-i \vec{p} \cdot \vec{n}}}{\sqrt{S^{(h)}(\vec{p})}}.
\]

and we find that \(H(s) = \sum_{\vec{n} \in \mathbb{Z}^2} f_{\vec{n}} e^{is \vec{n} \cdot \vec{a}} h(s + an)\). The related coefficients, \(H_n = \)
\[
\frac{1}{\sqrt{a}} \int_{\mathbb{R}} ds \, H(s) e^{ia sn/2} = \sqrt{\frac{2\pi}{a}} \mathcal{H} \left( -\frac{na}{2} \right), \text{ can be finally written as}
\]
\[
H_n = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \frac{ds}{\sqrt{S^{(h)}(as, 0)}} h(s) e^{ia sn/2}.
\]

**Remarks:**

1. This formula should be compared with equation (3.15) of [7], which looks more difficult than (5.4) to be concretely applied, since it involves an (infinite) sum of integrals! It is clear, therefore, that (5.4) represents a substantial improvement of our previous results concerning the ONT.

2. It is interesting to notice that, if from the very beginning \( S(h) \overset{\bullet}{\sim} \delta_{l_1,0}\delta_{l_2,0} \), then \( S^{(h)}(\vec{p}) = 1 \) and we get \( H_n = h_n \). In this case, therefore, the ONT does not modify the set of coefficients we get from our procedure.

3. It is a simple exercise to check that, using formula (5.4), condition (r1) directly follows:
\[
\sum_{n \in \mathbb{Z}} H_n H_{n+2l} = \delta_{l,0}, \text{ for all integer } l.
\]
This result can be deduced by making use of the PSF.

4. As it can be deduced from a previous remark, it turns out that it is sufficient to look to some mild version of the ONC (3.14), like
\[
S_{l_1,l_2}^{(h)} = \delta_{l_1} s_{l_2}, \quad (5.5)
\]
for no matter which \( l^2(\mathbb{Z}) \)-sequence \( \{s_{l_2}\} \). We call this MONC. We only need to require that this sequence belongs to \( l^1(\mathbb{Z}) \), as we will now see. Indeed if \( h(s) \) solves (5.5) above, then \( S^{(h)}(p,0) = \sum_{l_1,l_2 \in \mathbb{Z}} S_{l_1,l_2}^{(h)} e^{ipl_1} = \sum_{l_2 \in \mathbb{Z}} \sigma \), which is finite since \( \{s_{l_2}\} \in l^1(\mathbb{Z}) \) and does not depend on \( p \). Therefore equation (5.4) produces \( H_n = \frac{h_n}{\sqrt{\sigma}} \), for all \( n \in \mathbb{Z} \). This result has the following meaning: if we have a solution \( h(s) \) of the MONC then its related coefficients \( h_n \) also satisfy (r1) but for an overall normalization constant, \( \sigma \). In this case, therefore, the ONT reduces to a simple overall normalization of the coefficients.

**VI On the other conditions**

In this section we briefly analyze the other conditions which make of a sequence a relevant one. For all these conditions we will discuss separately the cases in which \( h(s) \) is a solution of the ONC or not, so that the ONT is needed.
VI.1 On the asymptotic behavior

Condition (r2) states that the asymptotic behavior of a relevant sequence should be as follows: \( h_n = O\left(\frac{1}{1+|n|}\right), \ n \gg 1 \). Suppose that \( h_n \) are generated as in (3.16) or, equivalently, as in (3.17), where \( h(s) \) solves the ONC. Then it is obvious that the requirement on the asymptotic behavior of \( h_n \) is surely satisfied if, e.g., the function \( \hat{h}(p) \) has compact support. Suppose that this is not the case but \( h(s) \) is differentiable and \( h'(s) \) still belongs to \( L^2(\mathbb{R}) \). Then it is a standard exercise to check that there exists a positive constant \( M \) such that \( |h_n| \leq \frac{M}{n} \ \forall n \in \mathbb{Z} \). It is clear then that the more regular \( h(s) \) is, the faster the coefficients \( h_n \)'s go to zero with \( n \). In particular then, in order (r2) to be satisfied, it is sufficient to look for solutions of the ONC for which the second derivative exists and still belongs to \( L^2(\mathbb{R}) \).

Suppose now that \( h(s) \) does not satisfy the ONC. Then we need to apply the ONT, which produces a new set of coefficients \( H_n \) as in (5.4). We can simply repeat the above considerations simply replacing \( h(s) \) with \( h(s)/\sqrt{S(h)(as,0)} \), or with its Fourier transform.

**Remark:** It may be worth noticing that these results essentially simplify the ones obtained in [7], where, in order to analyze the asymptotic behavior of the \( h_n \)'s, we needed to use some non trivial results on the convolution of sequences.

VI.2 Another sum rule

We want now to comment briefly on condition (r3) of a relevant sequence. As before, we first consider the case in which \( h(s) \) solves the ONC. In this case, due to (3.17), it is clear that (r3) is satisfied if and only if \( \sum_{n \in \mathbb{Z}} \hat{h} \left( \frac{ma}{2} \right) = \sqrt{\frac{2}{a}} \). Using the PSF we can also deduce that a necessary condition for (r3) to hold is that \( \sum_{n \in \mathbb{Z}} h(na) = \sqrt{\frac{2}{a}} \).

This condition is often deep in contrasts with condition (r1), how appears clear from the examples discussed in Section IV, where only one among the seven examples listed in the table satisfies also this requirement. For this reason it is worth finding more and more solutions of the ONC, in order to have a larger set of possible solutions of condition (r3). For that the ONT is clearly to be adopted. In this case, if we start with a generic square-integrable function \( h(s) \), we produce the set \( H_n \) as in (5.4), and we know that
the set \( \{ H_n \} \) surely satisfies condition (r1) for any given \( h(s) \). However, in order for \( \{ H_n \} \) to satisfy also condition (r3), not all the functions \( h(s) \) work equally well. In fact, it is possible to check that the following must hold

\[
\sum_{n \in \mathbb{Z}} h(na) = \sqrt{\sum_{\ell \in \mathbb{Z}^2} h\left(\frac{a l_1}{2}\right) h\left(\frac{a l_1}{2} + a l_2\right)}
\]  

(6.1)

or, in terms of the Fourier transform \( \hat{h}(p) \) of \( h(s) \),

\[
\sum_{n \in \mathbb{Z}} \hat{h}\left(\frac{a n}{2}\right) = \sqrt{2} \sum_{\ell \in \mathbb{Z}^2} \hat{h}\left(\frac{a l_2}{2}\right) \hat{h}\left(\frac{a l_2}{2} + a l_1\right)
\]

(6.2)

It is easy to check, for instance, that the Haar example as given in Section IV satisfies both these conditions.

To deduce condition (6.1) we start observing that, by means of the PSF,

\[
\sqrt{2} = \sum_{n \in \mathbb{Z}} H_n = \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{ds}{\sqrt{S^{(h)}(as, 0)}} h(s) e^{iasn/2} = \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{ds}{\sqrt{S^{(h)}(as, 0)}} h(s) a \delta(s - an) = \sqrt{a} \sum_{n \in \mathbb{Z}} h(na),
\]

where we have also used that \( a^2 = 4\pi \). Condition (6.1) follows now from an explicit computation of \( S^{(h)}(\vec{0}) \), again performed by means of the PSF:

\[
S^{(h)}(\vec{0}) = \sum_{\ell \in \mathbb{Z}^2} S^{(h)}_\ell = \sum_{\ell \in \mathbb{Z}^2} \int_{\mathbb{R}} ds h(s) \overline{h(s + a l_2)} e^{-is a l_1} = \frac{a}{2} \sum_{\ell \in \mathbb{Z}^2} \int_{\mathbb{R}} ds h(s) \overline{h(s + a l_2)} \delta(s - a \frac{1}{2} l_1) = \frac{a}{2} \sum_{\ell \in \mathbb{Z}^2} \left( \frac{a l_1}{2} \right) h\left(\frac{a l_1}{2} + a l_2\right).
\]

Equation (6.2) simply follows from the definition of the Fourier transform and, again, from the PSF.
VI.3 The last condition

The last step consists in rephrasing condition (r4), \( h(\omega) := \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{-i\omega n} \neq 0 \) for all \( \omega \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), in terms of the seed function. This can be easily done and the condition we get turns out to be independent of the fact that we need to use the ONT or not. Indeed, if \( h(s) \) is already a solution of the ONC, then \( h(\omega) \) can be rewritten as

\[
h(\omega) = \frac{1}{\sqrt{2a}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} h(s) e^{ina/2(s-2\omega/a)} ds = \sqrt{\frac{a}{2}} \sum_{n \in \mathbb{Z}} h \left( a \left( n + \frac{\omega}{2\pi} \right) \right),
\]

using again the PSF. It is clear, then, that in order for (r4) to hold, \( h \) must satisfy the following condition:

\[
\sum_{n \in \mathbb{Z}} h \left( a \left( n + \frac{\omega}{2\pi} \right) \right) \neq 0, \quad \omega \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]. \tag{6.3}
\]

It is interesting to observe that this same condition must be satisfied also when the seed function does not solve the ONC, at least under very general assumptions. Indeed, in this case, condition (6.3) should be replaced by \( \sum_{n \in \mathbb{Z}} \tilde{h} \left( a \left( n + \frac{\omega}{2\pi} \right) \right) \neq 0, \) for all \( \omega \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), where \( \tilde{h}(s) = \frac{h(s)}{\sqrt{S(h)(as,0)}} \). But, since \( S(h)(s,0) \) is \( 2\pi \)-periodic, our requirement is satisfied when (6.3) holds, at least if \( S(h)(2\omega,0) \) is bounded. This is always so in all the examples considered so far, as well as any time \( h(s) \) or \( \hat{h}(p) \) are compactly supported.

It is worth remarking that this result simplify in a significant way the original ones discussed in [7].

VII Conclusions

We have discussed in some details a procedure to construct relevant sequences, and MRA as a consequence, starting with a given seed function \( h(s) \in L^2(\mathbb{R}) \) which should satisfy some conditions. In particular, we have shown that no condition at all is required to \( h(s) \) as far as condition (r1) is concerned. However, if \( h(s) \) has to generate a sequence satisfying also (r2)-(r4), not all the choices are equivalently good. Our method, originally related to the quantum mechanical hamiltonian of the FQHE, has been shown to be model-independent. What is really relevant is the presence of a canonical map between certain canonically conjugate operators. This makes all the procedure much easier
and suggests that many other applications might exist, for example to the toy model discussed in [3]. Also, this map allows us to construct a sort of quantum mechanical interpretation of the ONT, relating this procedure to the construction of a two-dimensional (fictitious) lattice of orthonormal functions of $L^2(\mathbb{R}^2)$, which can be chosen to belong all to a same subspace of $L^2(\mathbb{R}^2)$, the eigenspace of a certain *Hamiltonian* corresponding to a fixed eigenvalue.

Many other things still has to be done:
first of all we should construct more explicit examples of our procedure. This will be done in a paper which is now in preparation, [14].

Secondly, it can be of some interest to deduce which conditions should be imposed on our seed function in order to have stronger conditions on the related mother wavelet.

Also, how it has been discussed in a rather recent paper by Ali and myself, [15], there exists an underlying modular structure connected to the hamiltonian structure of the Hall effect. This structure is still present in our present formulation of the problem, and we believe that it deserves a deeper investigation.

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