Regular Black Hole may act as Particle Accelerators

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Abstract

We investigate the collision of particles in the vicinity of the horizon of a regular Bardeen, Ayón-Beato and García, and Hayward black hole. We show that the center-of-mass energy of collisions of particles near the horizon of the extreme regular black holes are arbitrarily large while the non-extreme regular black holes have the finite energy. We also compute the equation of innermost stable circular orbit (ISCO), marginally bound circular orbit (MBCO) and circular photon orbit (CPO) of the above regular black holes.

1 Introduction

Until now our experimental setup in LHC do not reach the Planck scale energy, where gravity and quantum mechanics are coincides. Thus the theoretical search for possibility of such high energy is really an important issue today. The acceleration of particles in the vicinity of a horizon may provide such a possible approach.

In a recent work, Banados, Silk, West (henceforth BSW) [2] demonstrated that particles falling freely from rest exterior of a rotating extremal black hole can produce an infinite amount of high center of mass energy. In an semi-realistic setup, this energy can be higher than the Planckian energy, so that one might think about extremal black holes as super high energy particle accelerators. After appearance of this work in the literature several objections have been raised. Particularly in [3] the authors showed that there is an astrophysical bound i.e. maximal spin, back reaction effect and gravitational radiation etc. on that center of mass energy due to the Thorn’s bound [4] i.e. \( \frac{a}{M} = 0.998 \) (\( M \) is the mass and \( a \) is the spin of the black hole). There are lots of work describes BSW mechanism for different types of black holes and demonstrated that the center-of-mass (C.M.) energy is infinite for extremal black holes [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

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The aim of this work is to show that an analogous mechanism of particle collision with a high center-of-mass energy is also possible when a black hole is extremal regular (singularity-free) black holes.

This is why, we first examine the C.M. energy of collision for two neutral particles falling freely from rest at infinity in the horizon of a Bardeen black hole— the first regular (singularity-free) black hole model in General Relativity (GR) [17].

The spacetime has an interesting feature: it is interpreted as the gravitational field of a nonlinear monopole, i.e., as a magnetic solution to Einstein field equations coupled to a non-linear electrodynamics. This model also satisfies the weak energy condition (WEC) and their energy-momentum has the symmetry $T_{00} = T_{11}$. For extremal Bardeen spacetime, we find that the center of mass energy is arbitrarily large at the near horizon.

Next, we investigate the C.M. energy of collision for two neutral particles falling freely from rest at infinity in the horizon of an Ayón-Beato and García (ABG) black hole [18]. This spacetime is also a regular black hole spacetime and singularity free solutions of the coupled system of a non-linear electrodynamics and general relativity. The source is a nonlinear electrodynamics field satisfying the WEC, which in the limit of weak field becomes the Maxwell field. We find the center-of-mass energy for this spacetime can be infinitely high when the black hole is only extremal.

Finally, we take another simple regular (singularity-free) black hole which was suggested by Hayward in 2006 for the process of a regular black hole formation and evaporation [22]. What we now called it the Hayward black hole. It is also shown that the center-of-mass energy would be infinitely high when this black hole is extremal.

In each cases, we have also studied the equatorial circular geodesic motion by extremization of the effective potential for time-like circular orbits and null circular geodesics. We particularly focus on the innermost stable circular orbit (ISCO), marginally bound circular orbit (MBCO) and circular photon orbit (CPO).

2 C.M. energy of the collision near the horizon of the Bardeen Spacetime:

To compute the center-of-mass energy near the horizon of the regular Bardeen black hole, we first need to know the geodesic structure of the black hole and four velocity of the particles.

2.1 Equatorial circular orbit in the Bardeen spacetime:

The line element for the Bardeen spacetime [17, 18, 23] is given by

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$ (1)
where the function $F(r)$ is defined by

$$F(r) = 1 - \frac{2mr^2}{(r^2 + g^2)^{\frac{3}{2}}}, \quad (2)$$

where $m$ is the mass of the black hole and $g$ is the monopole charge of the non-linear self gravitating magnetic field.

The fact that the metric as well as the curvature invariants $R, R_{ab}R^{ab}, R_{abcd}R^{abcd}$ are regular everywhere in the spacetime. Hence in this sense it is called regular. It may be noted that this metric function asymptotically behaves as

$$F(r) \sim 1 - \frac{2m}{r} + \frac{3mg^2}{r^3}. \quad (3)$$

Furthermore, if $r \to 0$, the metric behaves as the de-Sitter spacetime.

The Bardeen black hole has an event horizon $(r_+)$ which occur at $F(r_+) = 0$. i.e.

$$r_+^6 + (3g^2 - 4m^2)r_+^4 + 3g^4r_+^2 + g^6 = 0. \quad (4)$$

The real positive root of the equation gives the event horizon $(r_+)$ of the Bardeen black hole is given by

$$r_+ = m\left[\frac{16 - 24 \left(\frac{g}{m}\right)^2 + \{4 - 3 \left(\frac{g}{m}\right)^2\} x^\frac{1}{3} + x^\frac{2}{3}}{\sqrt{3}x^\frac{2}{3}}\right]. \quad (5)$$

where

$$x = 64 - 144 \left(\frac{g}{m}\right)^2 + 54 \left(\frac{g}{m}\right)^4 + 6\sqrt{3} \left(\frac{g}{m}\right)^3 \sqrt{27 \left(\frac{g}{m}\right)^2 - 16}. \quad (6)$$

Similarly, using MAPLE software we could find the radius of the Cauchy horizon $(r_-)$ for Bardeen black hole is given by

$$r_- = -\frac{m}{\sqrt{3}}\left[\frac{16 - 24 \left(\frac{g}{m}\right)^2 + \{4 - 3 \left(\frac{g}{m}\right)^2\} x^\frac{1}{3} + x^\frac{2}{3}}{x^\frac{1}{3}}\right]. \quad (7)$$

The Bardeen spacetime represents a regular black hole when $27g^2 \leq 16m^2$.

When $27g^2 < 16m^2$, there are two horizons $r_\pm$ in the Bardeen spacetime, we may call it non-extremal Bardeen spacetime as in the non-extremal Reissner Nordstrøm spacetime.

When $27g^2 = 16m^2$, the two horizons are coincident, which correspond to an extreme Bardeen black hole as in the Reissner Nordstrøm black hole.
To derive the complete geodesic structure of the Bardeen black hole we shall follow the pioneer book of S. Chandrasekhar\[29\] and J. B. Hartle\[30\]. To compute the geodesic motion of the test particle in the equatorial plane we set $\theta = 0$ and $\theta = constant = \frac{\pi}{2}$. Since the spacetime admits two Killing vectors namely, $\xi \equiv \partial_t$ and $\chi \equiv \partial_\phi$. Therefore the quantities $E = -\xi \cdot u$ and $L \equiv \chi \cdot u$ are conserved quantities along the geodesics and $u$ is the four velocity of the particle. Where $E$ and $L$ can be interpreted as conserved energy and conserved angular momentum per unit mass respectively.

Thus in this coordinate chart, $E$ can be written as

\[ E = -\xi \cdot u = F(r) \, u^t. \] (8)

and $L$ can be expressed as in terms of the metric

\[ r^2 \, u^\phi = L. \] (9)

From the normalization condition of the four velocity for massive particles we find

\[ u^2 = \sigma . \] (10)

where $\sigma = -1$ for time-like geodesics, $\sigma = 0$ for light-like geodesics and $\sigma = +1$ for space-like geodesics. Solving (8) and (9) for $u^t$ and $u^\phi$ we get

\[ u^t = \frac{E}{F(r)} . \] (11)

\[ u^\phi = \frac{L}{r^2} . \] (12)

where $E$ and $L$ are the energy and angular momentum per unit mass of the test particle. Substituting these equations in (11) and (12) in (10), we obtain the radial equation for the Bardeen spacetime:

\[ (u^r)^2 = E^2 - V_{eff} = E^2 - \left( \frac{L^2}{r^2} - \sigma \right) F(r) . \] (13)

where the standard effective potential for the geodesic motion of the Bardeen spacetime is given by

\[ V_{eff} = \left( \frac{L^2}{r^2} - \sigma \right) \left( 1 - \frac{2mr^2}{(r^2 + g^2)^{\frac{3}{2}}} \right) . \] (14)
2.1.1 Particle orbits:

The effective potential for timelike geodesics can be obtained from the above equation (13) by putting $\sigma = -1$ as

$$V_{\text{eff}} = \left(1 + \frac{L^2}{r^2}\right) \left(1 - \frac{2mr^2}{(r^2 + g^2)^{3/2}}\right).$$  \hspace{1cm} (15)

To derive the circular geodesic motion of the test particle in Bardeen spacetime, we must impose the condition $\dot{r} = 0$ at $r = r_0$. Thus one gets from equation (13)

$$V_{\text{eff}} = E^2.$$  \hspace{1cm} (16)

and

$$\frac{dV_{\text{eff}}}{dr} = 0.$$  \hspace{1cm} (17)

Thus one can obtain the energy and angular momentum per unit mass of the test particle along the circular orbits are:

$$E_0^2 = \frac{\left[(r_0^2 + g^2)^{3/2} - 2mr_0^2\right]^2}{\sqrt{r_0^2 + g^2 \left[(r_0^2 + g^2)^{3/2} - 3mr_0^4\right]}}.$$  \hspace{1cm} (18)

and

$$L_0^2 = \frac{mr_0^4 (r_0^2 - 2g^2)}{\left[(r_0^2 + g^2)^{3/2} - 3mr_0^4\right]}.$$  \hspace{1cm} (19)

Circular motion of the test particle to be exists when both energy and angular momentum are real and finite.

Thus we must have

$$(r_0^2 + g^2)^{3/2} - 3mr_0^4 > 0 \text{ and } r_0 > \sqrt{2g}.$$  \hspace{1cm} (20)

Circular orbits do not exist for all values of $r$, so from Eq. (18) and Eq. (19), we can see that the denominator would be real only when

$$(r_0^2 + g^2)^{3/2} - 3mr_0^4 \geq 0.$$  \hspace{1cm} (21)

or

$$r_0^{10} + (5g^2 - 9m^2)r_0^8 + 10g^4 r_0^6 + 10g^6 r_0^4 + 5g^8 r_0^2 + g^{10} \geq 0.$$  \hspace{1cm} (22)
The limiting case of equality gives a circular orbit with infinite energy per unit rest mass i.e. a photon orbit. This photon orbit is the inner most boundary of the circular orbits for massive particles.

One can obtain marginally bound circular orbit (MBCO) for Bardeen spacetime can be written as:

\[ r_0^6 + (9g^2 - 16m^2)r_0^4 + 24g^4r_0^4 + 16g^6 = 0. \] (23)

Let \( r_0 = r_{mb} \) be the solution of the equation which gives the radius of MBCO.

From astrophysical point of view the most important class of orbits are ISCOs which can be derived from the second derivative of the effective potential of time-like case. i.e.

\[ \frac{d^2V_{eff}}{dr^2} = 0 \] (24)

Thus one may get the ISCO equation for the Bardeen spacetime reads as

\[ r_0^{14} + (19g^2 - 36m^2)r_0^{12} + 99g^4r_0^{10} + 65g^6r_0^8 - 160g^8r_0^6 - 144g^{10}r_0^4 + 64g^{12}r_0^2 + 64g^{14} = 0. \] (25)

Let \( r_0 = r_{ISCO} \) be the real solution of the equation (25) which gives the radius of the ISCO of Bardeen spacetime.

In the limit \( g \to 0 \), we obtain the radius of ISCO for Schwarzschild black hole which is \( r = 6m \).

2.1.2 Photon orbits:

For null circular geodesics, the effective potential becomes

\[ U_{eff} = \frac{L^2}{r^2} F(r) = \frac{L^2}{r^2} \left( 1 - \frac{2mr^2}{(r^2 + g^2)^{\frac{3}{2}}} \right) \] (26)

For circular null geodesics at \( r = r_c \), we find

\[ U_{eff} = E^2 \] (27)

and

\[ \frac{dU_{eff}}{dr} = 0 \] (28)

Thus one may obtain the ratio of energy and angular momentum of the test particle evaluated at \( r = r_c \) for circular photon orbits are:
\[
\frac{E_c}{L_c} = \pm \sqrt{\frac{(r_c^2 + g^2)^{\frac{3}{2}} - 2mr_c^2}{r_c^2(r_c^2 + g^2)^{\frac{3}{2}}}}
\]  

(29)

and

\[
r_c^{10} + (5g^2 - 9m^2)r_c^8 + 10g^4r_c^6 + 10g^6r_c^4 + 5g^8r_c^2 + g^{10} = 0.
\]

(30)

Let \( r_c = r_{ph} \) be the solution of the equation (30) which gives the radius of the circular photon orbit (CPO) of the Bardeen spacetime. In the limit \( g \to 0 \), we obtain the radius of photon orbit for Schwarzschild black hole which is \( r_c = 3m \).

Let \( D_c = \frac{L_c}{E_c} \) be the impact parameter for null circular geodesics then

\[
\frac{1}{D_c} = \frac{E_c}{L_c} = \frac{\sqrt{(r_c^2 + g^2)^{\frac{3}{2}} - 2mr_c^2}}{r_c^2(r_c^2 + g^2)^{\frac{3}{2}}}
\]

(31)

In the limit \( g \to 0 \), we obtain the impact parameter of the circular photon orbit (CPO) for the Schwarzschild black hole which is \( D_c = 3\sqrt{3}m \).

### 2.2 C.M. energy and Particle collision:

Now let us compute the C.M. energy for the collision of two neutral particles coming from infinity with \( \frac{E_1}{m_0} = \frac{E_2}{m_0} = 1 \) and approaching the Bardeen black hole with different angular momenta \( L_1 \) and \( L_2 \).

The center of mass energy is evaluated by using the formula which is first given by BSW [2] reads

\[
\left( \frac{E_{cm}}{\sqrt{2m_0}} \right)^2 = 1 - g_{\mu\nu}u_1^\mu u_2^\nu.
\]

(32)

We also assume throughout this work the geodesic motion of the colliding particles confined in the equatorial plane.

As we have previously said that the Bardeen spacetime admits a time-like isometry followed by the time-like Killing vector field \( \xi \) whose projection along the four velocity \( u \) of geodesics \( \xi \cdot u = -E \), is conserved along such geodesics. Similarly there is also the ‘angular momentum’ \( L = \chi \cdot u \) is conserved due to the rotational symmetry(where \( \chi \equiv \partial_{\phi} \)).

For time-like particles, the components of the four velocity are

\[
u^t = \frac{E}{F(r)}
\]

(33)
\[ u^r = \pm \sqrt{E^2 - F(r) \left(1 + \frac{L^2}{r^2}\right)} \]  
\[ u^\theta = 0 \]  
\[ u^\phi = \frac{L}{r^2}. \]  

and

\[ u_1^\mu = \left(\frac{E_1}{F(r)} - \sqrt{E_1^2 - F(r) \left(1 + \frac{L_1^2}{r^2}\right)} \right) \left(0, \frac{L_1}{r^2}\right). \]  
\[ u_2^\mu = \left(\frac{E_2}{F(r)} - \sqrt{E_2^2 - F(r) \left(1 + \frac{L_2^2}{r^2}\right)} \right) \left(0, \frac{L_2}{r^2}\right). \]  

Substituting this in (32), we get the center of mass energy:

\[ \left(\frac{E_{cm}}{\sqrt{2m_0}}\right)^2 = 1 + \frac{E_1E_2}{F(r)} - \frac{1}{F(r)} \sqrt{E_1^2 - F(r) \left(1 + \frac{L_1^2}{r^2}\right)} \sqrt{E_2^2 - F(r) \left(1 + \frac{L_2^2}{r^2}\right)} - \frac{L_1L_2}{r^2}. \]  

(39)

For simplicity, \( E_1 = E_2 = 1 \) and substituting the value of \( F(r) \), we obtain the CM energy near the event horizon \((r_+)^\) of the Bardeen spacetime:

\[ E_{cm} \bigg|_{r \to r_+} = \sqrt{2m_0} \sqrt{\frac{4r_+^2 + (L_1 - L_2)^2}{2r_+^2}}. \]  

(40)

where \( r_+ \) is described in equation (5)

Near the Cauchy horizon \((r_-)^\) the C.M. energy for Bardeen spacetime is given by

\[ E_{cm} \bigg|_{r \to r_-} = \sqrt{2m_0} \sqrt{\frac{4r_-^2 + (L_1 - L_2)^2}{2r_-^2}}. \]  

(41)

where \( r_- \) is described in equation (7)

The angular velocity of the Bardeen spacetime at the \( r_+ \) is given by

\[ \Omega_H = \frac{\dot{\phi}}{t} = \sqrt{\frac{m(r_0^2 - 2g^2)}{(r_0^2 + g^2)^2}}. \]  

(42)
The critical angular momenta $L_i$ can be written as

$$L_i = \frac{E_i}{\Omega_H}. \quad (43)$$

At the extremal cases, when $27g^2 = 16m^2$, the horizon is at $r_0 = 1.08m$ and the values of critical angular momenta are diverges i.e.

$$L_i \rightarrow \infty. \quad (44)$$

Therefore for extremal Bardeen spacetime, we get the infinite amounts of C.M. energy, i.e.

$$E_{cm} \rightarrow \infty. \quad (45)$$

3 CM energy of the collision near the horizon of the Ayón-Beato and García Spacetime:

Before computing the center-of-mass energy we shall demonstrate shortly the geodesic structure of the Ayón-Beato and García spacetime.

3.1 Equatorial circular orbit in the Ayón-Beato and García spacetime:

The metric of the Ayón-Beato and García spacetime \cite{20, 25} is given by

$$ds^2 = -G(r)dt^2 + \frac{dr^2}{G(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (46)$$

where the function $G(r)$ is defined by

$$G(r) = 1 - \frac{2mr^2}{(r^2 + q^2)^\frac{3}{2}} + \frac{q^2r^2}{(r^2 + q^2)^2}. \quad (47)$$

where $m$ is the mass of the black hole and $q$ is the monopole charge.

This is also a first regular black hole solution in general relativity. The source is a nonlinear electrodynamic field satisfying the WEC, which in the limit of weak field becomes the Maxwell field.

The metric as well as the curvature invariants $R, R_{ab}R^{ab}, R_{abcd}R^{abcd}$ and the electric field are regular everywhere in the spacetime. Hence in this sense it is also called regular in the Einstein-Maxwell gravity.
It may also be noted that this metric function asymptotically behaves as the Reissner Nordstrøm spacetime i.e.,

\[ G(r) \sim 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \]  

(48)

The ABG black hole has an event horizon \((r_+)\) which occur at \(G(r_+) = 0\). i.e.

\[ r_+^8 + (6q^2 - 4m^2)r_+^6 + (11q^4 - 4m^2q^2)r_+^4 + 6q^6r_+^2 + q^8 = 0. \]  

(49)

The ABG spacetime represents a regular black hole when \(|q| \leq q_c\). The value of \(q_c\) is \(\approx 0.634m\). When \(|q| \leq q_c\), there are two horizons \(r_\pm\) in the ABG spacetime, we call it non-extremal ABG spacetime as in the non-extremal Reissner Nordstrom spacetime.

When \(|q| = q_c\), the two horizons are coincident at \(r_+ \sim 1.005m\), which correspond to an extreme ABG black hole as in the Reissner Nordstrom black hole. The Carter-Penrose diagram of ABG spacetime is quite similar structure to the Reissner Nordstrom black hole.

Proceeding analogously as in section 2, the radial equation that govern the geodesic structure in the Ayón-Beato and García spacetime reads

\[ (u^r)^2 = \dot{r}^2 = E^2 - V_{\text{eff}} = E^2 - \left( \frac{L^2}{r^2} - \sigma \right) G(r). \]  

(50)

where the effective potential for the geodesic motion of the ABG spacetime is given by

\[ V_{\text{eff}} = \left( \frac{L^2}{r^2} - \sigma \right) \left( 1 - \frac{2mr^2}{(r^2 + q^2)^\frac{3}{2}} + \frac{q^2r^2}{(r^2 + q^2)^2} \right). \]  

(51)

### 3.1.1 Particle orbits:

The effective potential for timelike geodesics can be written as using the equation (50) by setting \(\sigma = -1\):

\[ V_{\text{eff}} = \left( 1 + \frac{L^2}{r^2} \right) \left( 1 - \frac{2mr^2}{(r^2 + q^2)^\frac{3}{2}} + \frac{q^2r^2}{(r^2 + q^2)^2} \right). \]  

(52)

To derive the circular geodesic motion of the test particle in ABG spacetime, we must use the condition \(\dot{r} = 0\) at \(r = r_0\). Thus one gets from equation (50)

\[ V_{\text{eff}} = E^2. \]  

(53)
and
\[ \frac{dV_{eff}}{dr} = 0 . \] (54)

Thus one can obtain the energy and angular momentum per unit mass of the test particle along the circular orbits:

\[ E_0^2 = \frac{(r_0^2 + q^2)^2 - 2mr_0^2 \sqrt{r_0^2 + q^2} + q^2 r_0^2}{(r_0^2 + q^2) \left[ (r_0^2 + q^2)^3 - 3mr_0^4 \sqrt{r_0^2 + q^2} + 2q^2 r_0^4 \right]} . \] (55)

and

\[ L_0^2 = \frac{r_0^4 \left[ m(r_0^2 - 2q^2) \sqrt{r_0^2 + q^2} - q^2 (r_0^2 - q^2) \right]}{\left[ (r_0^2 + q^2)^3 - 3mr_0^4 \sqrt{r_0^2 + q^2} + 2q^2 r_0^4 \right]} . \] (56)

Circular motion of the test particle to be exists for ABG spacetime when both energy and angular momentum are real and finite. Thus we get the inequality:

\[ (r_0^2 + q^2)^3 - 3mr_0^4 \sqrt{r_0^2 + q^2} + 2q^2 r_0^4 > 0 \text{ and } r_0 > q \sqrt{\frac{2mr_0^2 + q^2 - 2q^2}{m \sqrt{r_0^2 + q^2} - q^2}} . \] (57)

Circular orbits do not exist for all values of \( r \), so from Eq. (55) and Eq. (56), we can see that the denominator would be real only when

\[ (r_0^2 + q^2)^3 - 3mr_0^4 \sqrt{r_0^2 + q^2} + 2q^2 r_0^4 \geq 0 . \] (58)

or

\[ r_0^{12} + (10q^2 - 9m^2) r_0^{10} - (9m^2 q^2 - 31q^4) r_0^8 + 32q^6 r_0^6 + 19q^8 r_0^4 + 6q^{10} r_0^2 + q^{12} \geq 0 \] (59)

The limiting case of equality indicates a circular orbit with diverging energy per unit rest mass i.e. a photon orbit. This photon orbit is the inner most boundary of the circular orbits for timelike particles.

The equation of marginally bound circular orbit (MBCO) for ABG spacetime looks like:

\[ m^2 r_0^{10} - (16m^4 - 3m^2 q^2) r_0^8 + (99m^2 q^4 - 32m^4 q^2) r_0^6 \]

\[- (16m^4 q^4 - 23m^2 q^6 - 9q^8) r_0^4 + (72m^2 q^8 - 12q^{10}) r_0^2 + (16m^2 q^{10} - 4q^{12}) = 0 . \] (60)
Let $r_0 = r_{mb}$ be the solution of the equation which gives the radius of MBCO close to the black hole.

The ISCO equation can be obtained from the second derivative of the effective potential of time-like case. i.e.

$$\frac{d^2 V_{eff}}{dr^2} = 0 \quad (61)$$

Thus one may get the ISCO equation for the ABG spacetime reads

$$m^2 r_0^{18} - (36m^4 - 39m^2 q^2 + 4q^4)r_0^{16} + (97m^2 q^4 - 72m^4 q^2 + 40q^6)r_0^{14}$$

$$-(36m^4 q^4 - 97m^2 q^6 + 52q^8)r_0^{12} - (89m^2 q^8 + 216q^{10})r_0^{10} - (357m^2 q^{10} + 272q^{12})r_0^8$$

$$-(292m^2 q^{12} + 104q^{14})r_0^6 + (16m^2 q^{14} + 12q^{16})r_0^4 + (144m^2 q^{16} + 24q^{18})r_0^2 + 4q^{18}(16m^2 - q^2) = 0 \quad (62)$$

Let $r_0 = r_{ISCO}$ be the real solution of the equation (62) which gives the radius of the ISCO of ABG spacetime.

In the limit $q \to 0$, we obtain the radius of ISCO for Schwarzschild black hole which is $r_{ISCO} = 6m$.

### 3.1.2 Photon orbits:

For null circular geodesics, the effective potential becomes

$$U_{eff} = \frac{L^2}{r^2}G(r) = \frac{L^2}{r^2} \left( 1 - \frac{2mr^2}{(r^2 + q^2)^{\frac{3}{2}}} + \frac{q^2 r^2}{(r^2 + q^2)^{\frac{5}{2}}} \right) \quad (63)$$

For circular null geodesics at $r = r_c$, we find

$$U_{eff} = E^2 \quad (64)$$

and

$$\frac{dU_{eff}}{dr} = 0 \quad (65)$$

Thus one may obtain the ratio of energy and angular momentum of the test particle evaluated at $r = r_c$ for circular photon orbits(CPO) are:

$$\frac{E_c}{L_c} = \pm \sqrt{\frac{1}{r_c^2} \left( 1 - \frac{2mr_c^2}{(r_c^2 + q^2)^{\frac{3}{2}}} + \frac{q^2 r_c^2}{(r_c^2 + q^2)^{\frac{5}{2}}} \right)} \quad (66)$$
and
\[ r_c^{12} + (10q^2 - 9m^2)r_c^{10} - (9m^2q^2 - 31q^4)r_c^8 + 32q^6r_c^6 + 19q^8r_c^4 + 6q^{10}r_c^2 + q^{12} = 0. \] (67)

Let \( D_c = \frac{L_c}{E_c} \) be the impact parameter for null circular geodesics then
\[ \frac{1}{D_c} = \frac{E_c}{L_c} = \sqrt{\frac{1}{r_c^2} \left( 1 - \frac{2mr_c^2}{(r_c^2 + q^2)^2} + \frac{q^2r_c^2}{(r_c^2 + q^2)^2} \right)} \] (68)

Let \( r_c = r_{ph} \) be the solution of the equation (67) which gives the radius of the photon orbit of the ABG spacetime. In the limit \( q \to 0 \), we recover the circular photon orbit (CPO) of Schwarzschild black hole which is \( r_{ph} = 3m \).

3.2 Center-of-mass Energy for ABG spacetime:

Now let us compute the C.M. energy for the collision of two neutral particles coming from infinity with \( \frac{E_1}{m_0} = \frac{E_2}{m_0} = 1 \) and approaching the ABG spacetime with different angular momenta \( L_1 \) and \( L_2 \).

Since the ABG spacetime has also Killing symmetries followed by the Killing vector field thus energy (\( E \)) and angular momentum (\( L \)) are conserved quantities as we have defined in case of Bardeemn spacetime.

Therefore for massive particles of ABG spacetime, the components of the four velocity are
\[
\begin{align*}
    u^t &= \frac{E}{G(r)} \quad (69) \\
    u^r &= \pm \sqrt{E^2 - G(r) \left( 1 + \frac{L^2}{r^2} \right)} \quad (70) \\
    u^\theta &= 0 \quad (71) \\
    u^\phi &= \frac{L}{r^2}. \quad (72)
\end{align*}
\]

and
\[
\begin{align*}
    u_1^\mu &= \left( \frac{E_1}{G(r)}, -\sqrt{E_1^2 - G(r) \left( 1 + \frac{L_1^2}{r^2} \right)}, 0, \frac{L_1}{r^2} \right) \quad (73) \\
    u_2^\mu &= \left( \frac{E_2}{G(r)}, -\sqrt{E_2^2 - G(r) \left( 1 + \frac{L_2^2}{r^2} \right)}, 0, \frac{L_2}{r^2} \right) \quad (74)
\end{align*}
\]
Substituting this in (32), we get the center of mass energy for ABG spacetime:

$$\left( \frac{E_{cm}}{\sqrt{2m_0}} \right)^2 = 1 + \frac{E_1 E_2}{G(r)} - \frac{1}{G(r)} \sqrt{E_1^2 - G(r) \left( 1 + \frac{L_1^2}{r^2} \right)} \sqrt{E_2^2 - G(r) \left( 1 + \frac{L_2^2}{r^2} \right)} - \frac{L_1 L_2}{r^2}. \quad (75)$$

For simplicity, $E_1 = E_2 = 1$ and putting the value of $G(r)$, we obtain the CM energy near the event horizon ($r_+$) of the ABG spacetime:

$$E_{cm} \big|_{r \to r_+} = \sqrt{2m_0} \sqrt{\frac{4r_+^2 + (L_1 - L_2)^2}{2r_+^2}}. \quad (76)$$

where $r_+$ is the root of the following equation:

$$r_+^8 + (6q^2 - 4m^2)r_+^6 + (11q^4 - 4m^2q^2)r_+^4 + 6q^6r_+^2 + q^8 = 0. \quad (77)$$

Near the Cauchy horizon ($r_-$) the C.M. energy for ABG spacetime is given by

$$E_{cm} \big|_{r \to r_-} = \sqrt{2m_0} \sqrt{\frac{4r_-^2 + (L_1 - L_2)^2}{2r_-^2}}. \quad (78)$$

where $r_-$ is the root of the following equation:

$$r_-^8 + (6q^2 - 4m^2)r_-^6 + (11q^4 - 4m^2q^2)r_-^4 + 6q^6r_-^2 + q^8 = 0. \quad (79)$$

The angular velocity of the ABG spacetime at the event horizon $r_+$ is given by

$$\Omega_H = \frac{\dot{\phi}}{\dot{t}} = \sqrt{\frac{m(r_0^2 - 2q^2)}{(r_0^2 + q^2)^{3/2}}} - \frac{q^2(r_0^2 - q^2)}{(r_0^2 + q^2)^{3/2}}. \quad (80)$$

The critical angular momenta $L_i$ may be written as

$$L_i = \frac{E_i}{\Omega_H}. \quad (81)$$

At the extremal cases, when $q = q_c$, the horizon is at $r_0 = 1.005m$ and the values of critical angular momenta are diverges i.e.

$$L_i \to \infty. \quad (82)$$

Therefore for extremal ABG spacetime, we obtain the infinite amounts of C.M. energy, i.e.

$$E_{cm} \mapsto \infty. \quad (83)$$
4 CM energy of the collision near the horizon of the Hayward Black hole:

The line element for the Hayward black hole \[22\] is given by

\[
ds^2 = -H(r)dt^2 + \frac{dr^2}{H(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).
\]

where the function \( H(r) \) is defined by

\[
H(r) = 1 - \frac{2mr^2}{(r^3 + 2\alpha^2)}.
\]

and

\[
\alpha^2 = ml^2.
\]

where \( m \) is the mass of the black hole and \( l \) is a free parameter. Hayward first use such types of metric for the formation and evaporation of a non-singular black hole \[22\]. The metric describes a static, spherically symmetric and asymptotic flat, have regular centres and for which the resulting energy-momentum tensor satisfying the WEC. The Carter-Penrose diagram is similar to that of Reissner Nordstrøm space-time, with the internal singularities replaced by regular centers.

It is shown that the metric as well as the curvature invariants \( R, R_{ab}R^{ab}, R_{abcd}R^{abcd} \) for such spacetimes are regular everywhere in the manifold. Hence in this sense it is also regular space-time without singularity. It may be noted that this metric function asymptotically behaves as

\[
F(r) \sim 1 - \frac{2m}{r}.
\]

Furthermore, if \( r \to 0 \), the metric behaves as the de-Sitter spacetime:

\[
F(r) \sim 1 - \frac{r^2}{l^2}.
\]

The Hayward black hole has an event horizon \( (r_+) \) which occur at \( H(r_+) = 0 \). i.e.

\[
r_+^3 - 2mr_+^2 + 2ml^2 = 0.
\]

The largest real positive root of the equation is given by

\[
r_+ = \frac{m}{3} \left[ 2 + z^{\frac{1}{3}} + \frac{4}{z^{\frac{1}{3}}} \right].
\]
where

\[ z = 8 - 27 \left( \frac{l}{m} \right)^2 + 9\sqrt{3} \left( \frac{l}{m} \right) \sqrt{27 \left( \frac{l}{m} \right)^2 - 16} . \]  

(91)

when \( l = 0 \), we recover the Schwarzschild black hole horizon.

The Hayward space-time represents a regular black hole when \( 27l^2 \leq 16m^2 \).

When \( 27l^2 < 16m^2 \), there are two horizons \( r_\pm \) in the Hayward space-time, we call it non-extremal Hayward space-time.

When \( 27l^2 = 16m^2 \), the two horizons are merges, which correspond to an extreme Hayward black hole.

To study the geodesic motion for this space-time we shall perform similar analysis as we have done in section 2 using Killing symmetries.

Thus the radial equation that govern the geodesic motion in the equatorial plane for the Hayward space-time can be written as:

\[ (u^r)^2 = E^2 - V_{\text{eff}} = E^2 - \left( \frac{L^2}{r^2} - \sigma \right) H(r) . \]  

(92)

where the standard effective potential that describe the geodesic motion of the Hayward space-time is

\[ V_{\text{eff}} = \left( \frac{L^2}{r^2} - \sigma \right) \left( \frac{2mr^2}{r^3 + 2ml^2} \right) . \]  

(93)

4.0.1 Particle orbits:

The effective potential for time-like geodesics for the Hayward space-time becomes

\[ V_{\text{eff}} = \left( 1 + \frac{L^2}{r^2} \right) \left( 1 - \frac{2mr^2}{r^3 + 2l^2m} \right) . \]  

(94)

To derive the circular geodesic motion of the test particle in the Hayward space-time, we must have the condition \( \dot{r} = 0 \) at \( r = r_0 \). Thus one gets, from equation (92)

\[ V_{\text{eff}} = E^2 . \]  

(95)

and

\[ \frac{dV_{\text{eff}}}{dr} = 0 . \]  

(96)

A straightforward calculation implies that the energy and angular momentum per unit mass of the test particle along the circular orbits are:
\[ E_0^2 = \frac{[r_0^3 + 2ml^2 - 2mr_0^2]^2}{[r_0^6 - 3mr_0^5 + 4ml^2r_0^3 + 4m^2l^4]} . \]  

(97)

and

\[ L_0^2 = \frac{mr_0^4 (r_0^3 - 4ml^2)}{[r_0^6 - 3mr_0^5 + 4ml^2r_0^3 + 4m^2l^4]} . \]  

(98)

The condition for circular motion to be exists in the Hayward space-time when both energy \((E_0)\) and angular momentum \((L_0)\) are real and finite.

Thus we have the condition:

\[ r_0^6 - 3mr_0^5 + 4ml^2r_0^3 + 4m^2l^4 > 0 \text{ and } r_0 > (4ml^2)^\frac{1}{4} . \]  

(99)

Circular orbits do not exist for all radii, so from Eq. (97) and Eq. (98), we can find that the denominator would be real only when

\[ r_0^6 - 3mr_0^5 + 4ml^2r_0^3 + 4m^2l^4 \geq 0 . \]  

(100)

The limiting case of equality gives a circular orbit with infinite energy per unit rest mass i.e. a photon orbit. This photon orbit is the inner most boundary of the circular orbits for massive particles.

One can obtain marginally bound circular orbit (MBCO) for Hayward spacetime would be

\[ r_0^3 - 4mr_0^2 + 8ml^2 = 0 . \]  

(101)

Using MAPLE we can find the real positive root of the Eq.(101) which gives the radius of MBCO closest to the black hole is given by

\[ r_{mb} = \frac{m}{3} \left[ 4 + \frac{4}{y^{\frac{3}{4}}} + \frac{16}{y^{\frac{3}{2}}} \right] . \]  

(102)

where

\[ y = 64 - 108 \left( \frac{l}{m} \right)^2 + 12\sqrt{3} \left( \frac{l}{m} \right) \sqrt{27 \left( \frac{l}{m} \right)^2 - 32} . \]  

(103)

From an astrophysical significance the most important class of orbits are ISCOs which can be calculated from the second derivative of the effective potential of time-like case. i.e.

\[ \frac{d^2 V_{eff}}{dr^2} = 0 \]  

(104)
Thus one would obtain the ISCO equation for the Hayward space-time:

\[ r_0^9 - 6m r_0^8 + 24 m l^2 r_0^6 - 12 m l^2 r_0^5 + 12 m^2 l^4 r_0^3 - 64 m^3 l^6 = 0. \tag{105} \]

Let \( r_0 = r_{ISCO} \) be the real solution of the equation (105) which gives the radius of the ISCO of Hayward space-time.

In the limit \( l \to 0 \), we obtain the radius of ISCO for Schwarzschild black hole which is \( r_{ISCO} = 6m \).

### 4.0.2 Photon orbits:

For null circular geodesics, the effective potential becomes

\[ U_{eff} = \frac{L^2}{r^2} H(r) = \frac{L^2}{r^2} \left( 1 - \frac{2mr^2}{r^3 + 2ml^2} \right) \tag{106} \]

For circular null geodesics at \( r = r_c \), we find

\[ U_{eff} = E^2 \tag{107} \]

and

\[ \frac{dU_{eff}}{dr} = 0 \tag{108} \]

Thus one may obtain for Hayward space-time, the ratio of energy and angular momentum of the test particle evaluated at \( r = r_c \) for circular photon orbits are:

\[ \frac{E_c}{L_c} = \pm \sqrt{\frac{1}{r_c^2} \left( 1 - \frac{2mr_c^2}{r_c^3 + 2ml^2} \right)} \tag{109} \]

and

\[ r_c^6 - 3mr_c^5 + 4ml^2r_c^3 + 4m^3l^4 = 0. \tag{110} \]

Let \( D_c = \frac{L_c}{E_c} \) be the impact parameter for null circular geodesics then

\[ \frac{1}{D_c} = \frac{E_c}{L_c} = \sqrt{\frac{(r_c^3 + 2ml^2) - 2mr_c^2}{r_c^2(r_c^3 + 2ml^2)}} \tag{111} \]

Let \( r_c = r_{ph} \) be the solution of the equation (110) which gives the radius of the photon orbit of the Hayward space-time.
4.1 Center-of-mass Energy for Hayward spacetime:

The components of the four velocity in terms of the energy and angular momentum due to the Killing symmetries of the space-time for time-like particles are

\[ u^t = \frac{E}{H(r)} \]  \hspace{1cm} (112)

\[ u^r = \pm \sqrt{E^2 - H(r) \left( 1 + \frac{L^2}{r^2} \right)} \]  \hspace{1cm} (113)

\[ u^\theta = 0 \]  \hspace{1cm} (114)

\[ u^\phi = \frac{L}{r^2} \]  \hspace{1cm} (115)

and

\[ u_1^\mu = \begin{pmatrix} \frac{E_1}{H(r)} \ \ \ - \sqrt{E_1^2 - H(r) \left( 1 + \frac{L_1^2}{r^2} \right)} \ \ \ 0 \ \ \ \frac{L_1}{r^2} \end{pmatrix} \]  \hspace{1cm} (116)

\[ u_2^\mu = \begin{pmatrix} \frac{E_2}{H(r)} \ \ \ - \sqrt{E_2^2 - H(r) \left( 1 + \frac{L_2^2}{r^2} \right)} \ \ \ 0 \ \ \ \frac{L_2}{r^2} \end{pmatrix} \]  \hspace{1cm} (117)

Substituting this in (32), we find the center of mass energy for Hayward space-time:

\[
\left( \frac{E_{cm}}{\sqrt{2}m_0} \right)^2 = 1 + \frac{E_1 E_2}{H(r)} - \frac{1}{H(r)} \sqrt{E_1^2 - H(r) \left( 1 + \frac{L_1^2}{r^2} \right)} \sqrt{E_2^2 - H(r) \left( 1 + \frac{L_2^2}{r^2} \right)} - \frac{L_1 L_2}{r^2}.
\]

(118)

Taking, \( E_1 = E_2 = 1 \) and putting the value of \( H(r) \), we obtain the CM energy near the event horizon \( (r_+) \) of the Hayward space-time:

\[
E_{cm} \bigg|_{r \to r_+} = \sqrt{2m_0} \sqrt{\frac{4r_+^2 + (L_1 - L_2)^2}{2r_+^2}}.
\]

where \( r_+ \) is given in (90).

The angular velocity of the Hayward space-time at the \( r_+ \) is given by

\[
\Omega_H = \frac{\dot{\phi}}{\dot{t}} = \sqrt{\frac{m(r_0^3 - 4ml^2)}{(r_0^3 + 2ml^2)^2}}.
\]

(120)
The critical angular momenta $L_i$ can be written as

$$L_i = \frac{E_i}{\Omega_H}.$$  \hfill (121)

At the extremal cases, when $27l^2 = 16m^2$, the horizon is at $r_0 = \frac{4}{3}m$ and the values of critical angular momenta are diverges i.e.

$$L_i \rightarrow \infty.$$  \hfill (122)

Therefore for extremal Hayward space-time, we get the infinite amounts of C.M. energy, i.e.

$$E_{cm} \longrightarrow \infty.$$  \hfill (123)

### 5 Discussion:

In this note, we have demonstrated that the collision of two neutral particles falling freely from rest at infinity in the background of a regular Bardeen black hole, Ayón-Beato and García black hole, and Hayward black hole. It is shown that these regular black holes may act as particle accelerators with arbitrarily high center-of-mass energy.

Our analysis suggests that for non-extremal regular spacetime the CM energy is finite and depends upon the angular momentum parameter. For extremal regular black holes the CM energy is unlimited due to the diverging angular momentum of the colliding particles.

Thus the theoretical prediction of arbitrarily high energy issues in the near horizon of a extremal regular black hole is very interesting and crucial at least for astrophysical applications. In this connection, it might be interesting to investigate the acceleration of a particles in the vicinity of a horizon of the regular black hole.

It would be interesting to find the conditions of the ISCO, MBCO and CPO of these extremal regular black holes by analyzing the behaviour of the effective potential.

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