The Regularization of the Fermion Determinant 
in Chiral Quark Models

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Abstract

The momentum dependence of the quark self energy gives a physically motivated and consistent regularization of both the real and imaginary parts of the quark loop contribution to the meson action. We show that the amplitudes for anomalous processes are always reproduced correctly.

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The regularization of the fermion determinant in effective low energy chiral quark models has been the subject of much debate, frustration and confusion. The contribution of the fermion determinant to the meson effective action takes the form of a quark loop (see fig. 1) which induces the propagation and interactions of the local chiral field. If the couplings are local the loop integration is formally infinite, and the model is then only well defined when supplemented with some regularization and renormalization prescription. The standard procedure works only for a very limited set of models, such as the linear sigma–model for example, so in general it is necessary to introduce a finite cut–off scale and treat the meson effective action as an effective field theory. In such cases we are faced with the problem of determining not only the magnitude of the cut–off, but also its nature and origin. It is difficult to take seriously statements claiming that the cut–off must be of the order of the QCD scale, the chiral symmetry or the confinement scale; such easy statements fail to tell us how precisely to regulate the quark loop, or what the physical mechanism is which provides the cut–off. For some observables (meson mass ratios, soliton energies, the effective potential, etc.) it does not seem to matter very much which regularization is used. But many features, such as the rates of anomalous decays, can and indeed do depend crucially on the chosen regularization method.

Opinions differ as to which regularization is to be used. Nambu–Jona-Lasinio models, for example, have been regularized by a variety of methods: 3– or 4–momentum cut–off functions, proper–time cut–offs, Pauli–Villars regulators, etc. In fact proper–time regularization is useless if the fermions are chiral since in general only the real part of the determinant is then well defined. Pauli–Villars regularization, although leading to a well defined determinant, necessarily breaks both global and local axial symmetries; this then leads to all sorts of contradictions with current algebra. For example the rates for the anomalous processes $2K \to 3\pi$ and $\gamma \to 3\pi$ will be given incorrectly; nonanomalous processes will be similarly corrupted. However in strictly local chiral models no consistent regularization which preserves global chiral symmetry is known, apart from the rather artificial “LR–regularization” in which the Dirac spinors are split into their Weyl components, and all local symmetries are violated. Furthermore it has been known for some time that with such regularizations even the amplitude for $\pi^0 \to 2\gamma$, calculated from the famous ‘triangle’ diagram (see fig. 2) contradicts the rather general (and of course phenomenologically remarkably successful) result of Adler, Bell and Jackiw.

Momentum cut–offs can preserve global symmetries at the expense of non–locality. However even so there may be problems with anomalous decays. For example five years ago a paper appeared in which the authors calculated, among other things, the $\pi^0 \to 2\gamma$ decay rate, using the Nambu–Jona-Lasinio model. The paper rightly took the point of view that if a cut–off is introduced to regularize the normal parity part of the action, the abnormal parity part should also be calculated with the same cut–off. However using a sharp momentum cut–off of a reasonable magnitude they found a very substantial reduction in the contribution of fig. 2 to the decay amplitude, the result of only being recovered in the infinite cut–off limit. More recently however, a non–local model with a smooth

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1 A review of the definition and of various regularizations of chiral fermion determinants, and the structure of the resulting chiral anomalies, may be found in ref. [2].
momentum cut–off has been shown \cite{7} to give the correct result for $\pi^0 \rightarrow 2\gamma$ independently of the precise form of the momentum cut–off.

Because of, or perhaps in spite of, all these well–known results, many groups are still performing calculations with Nambu–Jona-Lasinio or chiral quark models in which the nonanomalous (and in many cases diverging) normal processes (resulting from terms in the real part of the fermion determinant) are regularized using typically proper–time regularization, with a finite cut–off, while the finite anomalous processes (resulting from terms in the imaginary part of the fermion determinant) are either added by hand, or calculated in the infinite cut–off limit (the problems described in \cite{3,4} being ignored). It seems to us that such calculations make little sense. Physics is an art. But to calculate according to the rule “wherever a divergent integral occurs, chop off the divergent part” is not. Clearly it must be possible to regulate the quark loop (both real and imaginary parts) with a finite cut–off scale in such a way that global chiral symmetry is preserved, and the correct rates for all the anomalous processes (and in particular $\pi^0 \rightarrow 2\gamma$) are reproduced, without the need for ad hoc subtractions.

In low energy effective theories, calculated observables are in general approximately independent of the form of the cut–off if they are non–anomalous and involve energy scales much lower than the cut–off scale. When the energy scale of the calculated observable (the inverse soliton radius, the mass, etc.) is of the same order of magnitude as the cut–off, or the observable is anomalous (as are the decay rates mentioned above), it will in general depend strongly on the value and nature of the cut–off; the cut–off thus becomes an intrinsic feature of the model which attempts to account for complicated interactions which would not otherwise be included. In spite of this, most discussions of the regularization of quark loops in low energy chiral theories have avoided the question of its physical origin. This question is addressed in this article. Its natural result, namely the regularization of the quark loop by the non–locality of its interaction with the chiral field, is found in fact to have precisely the right properties to completely resolve all the paradoxes related to anomalous decays.

1. Dynamically Regularized Quark Loops.

In this section we explain how the quark loop is regulated dynamically by the intrinsic non–locality of the quark–meson interaction. The chiral field, which in the vacuum gives the quarks their constituent mass through a dynamical chiral symmetry breaking mechanism, gives rise to a quark self energy which is not a constant but a non–local function of the quark momentum. This mass generation may be thought of schematically as a Schwinger–Dyson self–energy in the form of a ‘summation’ of rainbow diagrams, or equivalently of an insertion of a $q-q\bar{q}$ pair interacting through a Bethe–Salpeter ladder of gluon exchanges (see fig. 3).

Writing the quark propagator as $S^{-1}(p) = (\not{p} + \Sigma(p^2))^{-1}$, where $\Sigma(p^2)$ is the quark self energy, the most general effective quark–meson coupling is, by definition, the amputated Bethe–Salpeter amplitude (see fig. 4 for the notation)

$$\chi(k, k') \equiv S^{-1}(k)\langle 0|\psi(k)\bar{\psi}(k')|\pi(k - k')\rangle S^{-1}(k'),$$

(1.1)
where \(|\pi(k-k')\) is a pion state of momentum \(k-k'\), containing a \(q\bar{q}\) pair with momenta \(k\) and \(k'\) respectively. The quark–pion dynamics may then be summarized by writing an effective action

\[
S_{\text{eff}} = \int d^4x \bar{\psi}(x)iD\psi(x) + \int d^4x \int d^4x' \bar{\psi}(x)M(x,x')\psi(x'),
\]

where the covariant derivative \(D_\mu \psi \equiv (\partial_\mu + A_\mu)\psi\) couples the quarks locally to external gauge fields (such as the photon), while \(M(x,x')\) is the double Fourier transform of

\[
M(k,k') = f_\pi \chi(k,k')U(k-k').
\]

Here \(U(k)\) is the Fourier transform of the chiral field \(U(x) \equiv \exp(i\gamma_5\pi(x)/f_\pi)\) (\(f_\pi\) being the pion decay constant). Expanding \(U\) in powers of \(\pi\) gives the quark self energy and then couplings of the \(q\bar{q}\) pair to increasing numbers of pions \(\pi\). In the chiral limit (i.e. ignoring quark masses; we will do this throughout for simplicity) the quark self–energy and the on–shell coupling to pions are thus related by the chiral Ward–Takahashi identity (see for example [8])

\[
\Sigma(p^2) = f_\pi \chi(p,p).
\]

Effective actions such as eqn.(1.2) have long been used to discuss dynamical chiral symmetry breaking and to compute meson form factors and decay amplitudes [9–11]; more recently attempts have been made to find a formal justification for them within QCD ([12] and references therein) and to develop their low energy phenomenological implications more systematically [13]. In fact the action (1.2) is formally exact in the low energy limit, in the sense that interactions of pions mediated purely by gluons must be symmetric under a local chiral symmetry, and are therefore trivial since \(U^\dagger U = 1\) [14].

The quark self–energy \(\Sigma(p^2)\) may in principle be determined by solution of the Schwinger–Dyson equation. In practice uncertainties in the infrared behaviour of the gluon propagator and gluon–quark vertex function make such a determination impractical, except in the deep Euclidean region. From asymptotic freedom, we expect \(\Sigma(p^2)\) to be a decreasing function for large space–like \(p^2\). Indeed it is not difficult to show using either the operator product expansion [15] or (less trivially) the Schwinger–Dyson equation [16] that at large Euclidean momenta

\[
\Sigma(p^2) \sim \frac{(\log p^2)^{d-1}}{p^2},
\]

where \(d\) is a related to the anomalous dimension of \(\bar{\psi}\psi\). In the infrared \(\Sigma(p^2)\) remains finite, but is otherwise unknown. The scale of \(\Sigma(p^2)\) is again in principle given by solution of the (nonlinear) Schwinger–Dyson equation in terms of \(\Lambda_{QCD}\), but in practice it is better to use the Pagels–Stokar condition[14] which fits \(\Sigma(p^2)\) to the pion decay constant \(f_\pi = 93\) MeV:

\[
f_\pi^2 = \frac{N_c}{4\pi^2} \int_0^\infty dp^2 p^2 \Sigma(p^2) \frac{\Sigma(p^2) - \frac{1}{2}p^2 \frac{d\Sigma(p^2)}{dp^2}}{(p^2 + \Sigma(p^2))^2}.
\]
The amputated Bethe–Salpeter amplitude is rather more complicated. In what follows we will for simplicity approximate it by simply enforcing the identification even when \( k' \neq k \). In this way all the various derivative couplings of pions to quarks are suppressed, and the non–derivative coupling is given (in the chiral limit) simply by the quark self energy. We will explain elsewhere why, for the processes considered here, this approximation is in practice a very good one.

The soft asymptotic behaviour of the dynamically induced quark self–energy, and thus of the effective quark–pion coupling, means that if we use the effective action to calculate quark loops in which at least some of the external legs are pions, such loops will always be finite. In this way the form of the regularization of the quark loop (by which we mean both its real and imaginary parts) in the effective theory of quarks and pions is determined by the underlying theory from which it is in principle derivable, namely QCD. If this natural regularization is ignored (by, for example, replacing the non–local vertex functions with local ones) unphysical divergences are created, which must then be regulated by some ad hoc procedure; if this procedure is not chosen properly, the quark loop may not be well–defined, and inconsistencies of the type described in [3,6] can arise. Unpopular as non–local field theories may be (although as shown in [17], if the non–locality is particularly mild (i.e. analytic), many of the traditional objections to such theories may be shown rigorously to be without foundation), the non–locality of effective low energy Lagrangians is in general an unescapable fact that should be faced squarely. In the following sections we sketch one useful consequence of the non–locality; it allows us to recover the correct (and indeed experimentally confirmed) results for anomalous processes.

2. The Regularized Fermion Determinant

The inverse quark propagator is now described by a Dirac operator of the form:

\[
\mathcal{D} = -i\gamma_\mu \partial_\mu + M
\]

where \( M \) is the non–local self energy, which (using (1.3) and (1.4)) is assumed to be of the form (expressed for convenience in a (bra)(ket) notation, with \( \langle x|k \rangle = \Omega^{-d/2} e^{ix \cdot k} \))

\[
\langle x|M|y \rangle = \Sigma(x - y)U\left(\frac{1}{2}(x + y)\right),
\]

where \( \Sigma(x) \) is real and \( U(x)U^\dagger(x) = 1 \). The non–locality is thus represented by a self–energy function which is diagonal in momentum space:

\[
\langle q|\Sigma|k \rangle = \delta_{qk} \frac{1}{\Omega^d} \int d^dxe^{iq\cdot x} \Sigma(x) \equiv \delta_{qk} \frac{1}{\Omega^d} \Sigma(q),
\]

where \( \Omega^d \) is the volume element in \( d \)-dimensional space–time, while the dynamical chiral field \( U(x) \) is local (diagonal in \( x \)-space). A matrix element of \( M \) between plane wave states can then be expanded as follows:

\[
\langle q|M|k \rangle = \frac{1}{\Omega^d} \Sigma\left(\frac{1}{2}(q + k)\right)U((k - q) = \langle q\left|U\Sigma + \frac{1}{4!} U_{\mu} \Sigma_\mu + \frac{1}{2! (2i)^2} U_{\mu\nu} \Sigma_{\mu\nu} + \cdots\right|k\rangle,
\]
where $\Sigma(k)$ and $U(k)$ are the Fourier transforms $\Sigma(k) = \int d^4k e^{ikx} \Sigma(x)$, etc., and $\Omega^d$ is the volume element in $d$-dimensions. The second line of (2.4) is written as an operator product expansion with the notation

$$\Sigma_{\mu\nu \cdots}(k) = \frac{\partial \cdots}{\partial k_\mu \partial k_\nu \cdots} \Sigma(k), \quad U_{\mu\nu \cdots}(k) = \int d^4k e^{ikx} \frac{\partial \cdots}{\partial x_\mu \partial x_\nu \cdots} U(x). \quad (2.5)$$

From (2.4) we can derive an expansion of the Dirac operator (2.1) in powers of the gradients of the chiral field $U$.

The contribution of the fermion loop to the action is formally

$$S_D \equiv -\text{Tr} \ln D \equiv S^+_D + S^-_D. \quad (2.6)$$

In practice for local couplings to external fields $S_D$ may be defined by integrating its functional derivatives. This is only a good definition if the regularization of the fermion loop is chosen in such a way that this integration is path independent [18]. When this is the case the real (normal parity) part is given by

$$S^+_D = -\frac{1}{2} \text{Tr} \ln D \equiv \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \text{Tr} \langle x|e^{-\tau D^\dagger D}|x\rangle \quad (2.7)$$

in just the same way as for a bosonic field, since

$$D^\dagger D = -\partial^2 - \gamma_\mu [\partial_\mu, M] + M^\dagger M \quad (2.8)$$

is both hermitian and positive definite. The second two terms may be expanded in derivatives of $U$ in an analogous way to (2.4), and thus a derivative expansion for the real part of the action obtained from (2.7) in much the usual way [2]. The leading term in this expansion (the ‘Weinberg Lagrangian’) then yields precisely the expression (1.6) for the pion decay constant; the terms with four derivatives give the Gasser–Leutwyler coefficients [13].

An exact representation for the imaginary (or abnormal parity) part may also be constructed [19, 2]:

$$S^-_D = 2\pi i Q = 2\pi i \int_{M^{d+1}} d^{d+1}x j^V_\mu, \quad (2.9)$$

where $M^{d+1}$ is an open manifold in $d + 1$ dimensions whose boundary is the space–time $d$–dimensional manifold $M^d$, and $j^V_\mu$ is the abnormal parity part of the conserved vector current in $d + 2$ dimensions; $Q$ is then the conserved charge on the open manifold $M^{d+2}$. It is crucial that the current $j^V_\mu$ is conserved, since this guarantees that the action $S_D$ is independent, mod $2\pi i$, of the extension of $M^d$ to $M^{d+1}$ [20]. This in turn is closely related to the integrability condition [18]; when there is an integrability obstruction, the conservation of the current will be violated anomalously. The derivative expansion of the representation (2.9) may be constructed by expanding the current; the leading term is then the famous Wess–Zumino term, while all subsequent terms (those with at least $d + 3$ derivatives) may be shown to be total derivatives, and thus local.

A similar construction may be carried out when some of the couplings of the bosonic fields to the quark loop are non–local, as in (2.2) and (2.1). It turns out that the regularization due to the form–factor satisfies the integrability condition, while naturally preserving the global $U(n_f) \times U(n_f)$ chiral symmetries (though of course local chiral symmetries may still be broken anomalously). The representations (2.7) and (2.9) remain the same, although now more care must be taken to construct a conserved vector current; this will be considered in more detail in the following section.
3. The Vector Singlet Current

As explained in the previous section the calculation of the abnormal parity part of
the quark effective action can be achieved by first calculating the abnormal parity part
of the vector singlet current as an expansion in derivatives. The full calculation will be
explained in a forthcoming publication. We give here a simplified account which focusses
on the relevant features of the non–locality. We neglect here contributions from the current
quark masses, and set the external gauge fields to zero; we will explain how to extend our
results to the gauged current in the next section.

We construct the vector current using the Noether construction. When the Dirac
operator undergoes a gauge transformation,
\[ \mathcal{D} \rightarrow e^{-i\alpha(x)} \mathcal{D} e^{i\alpha(x)}, \quad (3.1) \]
the first order variation of the action is
\[ \delta S_D = i \text{Tr}(\mathcal{D}^{-1}[\alpha, \mathcal{D}]) \]
\[ = -\text{Tr}(\mathcal{D}^{-1}(\alpha \gamma_\mu - i[\alpha, M])) \equiv \int d^d x \alpha_\mu(x) j^V_\mu(x), \quad (3.2) \]
where \( j^V_\mu(x) \) is the vector current.

In local theories the commutator \([\alpha, M]\) appearing in \((3.2)\) vanishes and we recover
the familiar contribution from the operator \( \bar{q} \gamma_\mu q \). The commutator \([\alpha, M]\) thus gives
contributions to the current which have been induced by the nonlocality; indeed this extra
term makes an essential contribution, since without it the current would not be conserved.
It can be expanded in gradients of the chiral field \( U \); in the notation of eqn.\((2.4)\) we obtain
\[ [\alpha, M] = \alpha_\mu(iU \Sigma_\mu + \frac{1}{2} U_\alpha \Sigma_{\alpha \mu} + \cdots) + \frac{1}{2} \alpha_{\mu \nu}(U \Sigma_{\mu \nu} - \cdots) + \cdots. \quad (3.3) \]
We thus deduce the following expression for the vector singlet current;
\[ j^V_\mu(x) = -\text{Tr}\langle x | (\gamma_\mu + U \Sigma_\mu - \frac{i}{2} U_\nu \Sigma_{\nu \mu} + \cdots) \mathcal{D}^{-1} | x \rangle. \quad (3.4) \]

The current may be readily separated into real and imaginary parts:
\[ j^V_\mu(x) = \frac{1}{2} (j_\mu^V + (j_\mu^V)^*) + \frac{1}{2} (j_\mu^V - (j_\mu^V)^*) \equiv j_\mu^{V+} + j_\mu^{V-}. \quad (3.5) \]
Here we will consider only the imaginary part \( j_\mu^{V-} \) which describes abnormal parity processes. Noting that \( \mathcal{D}^{-1} = \mathcal{D}^\dagger(\mathcal{D}\mathcal{D}^\dagger)^{-1} \), and using \((3.4)\), we may write
\[ j_\mu^{V-} = \text{Tr} \int_0^\infty d\tau \langle x | (\gamma_\mu + U \Sigma_\mu - \frac{i}{2} U_\nu \Sigma_{\nu \mu} + \cdots)(\mathcal{D}^\dagger e^{-\tau \mathcal{D}\mathcal{D}^\dagger} + \mathcal{D} e^{-\tau \mathcal{D}^\dagger \mathcal{D}}) | x \rangle. \quad (3.6) \]

The expressions \((2.4)\) and \((3.6)\) can then be used to derive an expansion of the ab-
normal parity vector current in gradients of the chiral field \( U \). Collecting all the terms we
find that, to lowest order in the gradient of the chiral field \( U \), the current is given by
\[ j_\mu^{V-} = i^{d/2} N_c \epsilon_{\mu \nu_1 \nu_2 \cdots \nu_{d-1}} \text{tr}(\bar{\gamma} U^\dagger U_{\nu_1} U^\dagger_{\nu_2} \cdots U^\dagger_{\nu_{d-1}}) \frac{1}{(2\pi)^d} \int d^d k \frac{(\Sigma^d - 2\Sigma^{d-1} k^2 \frac{d\Sigma}{dk^2})}{(k^2 + \Sigma(k)^2)^d} + \cdots. \quad (3.7) \]
Now if we set \( y = k^2 \) and \( \Sigma(k)^2 = y\phi(y) \), the integral in (3.7) reduces to

\[
\frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty (k^2)^{d/2-1} dk^2 \frac{(\Sigma^d - 2\Sigma^{d-1}k^2 \frac{d\Sigma}{dk^2})}{(k^2 + \Sigma(k)^2)^d} = -\frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dy \frac{\phi^{-1}_2 \phi'}{(1 + \phi)^d} \tag{3.8}
\]

where in the second line we use the asymptotic behaviour (1.5) to show that \( \phi(y) \sim y^{-3} \to 0 \) as \( y \to \infty \), while we assumed (with little loss of generality) that as \( y \to 0 \), \( \Sigma(y) \) is strictly bounded below by \( \sqrt{y} \), so \( \phi(y) \to \infty \). From (3.8) it may be seen that the integrand was an exact differential; the coefficient of the topological current (3.7) is thus independent of the precise form of \( \Sigma(k) \), and is in fact the same as in the local model in which \( \Sigma(k) = m \). Note that, had we omitted the extra contribution \([\alpha, M]\) to the current (3.2), which arises from the non-locality of the mass operator \( \Sigma \), the integrand in (3.7) would not have been an exact differential and we would not have obtained a contribution which is independent of \( \Sigma(k) \).

In fact the integral is just one of the representations of the standard function \( \beta(z, w) \equiv \Gamma(z)/\Gamma(w)/\Gamma(z + w) \). The topological current (3.7) thus takes the usual form

\[
j_{\mu}^{-} = \frac{i^{d/2}(d-1)!}{(4\pi)^{d/2}(d-1)!}\epsilon_{\mu\nu_1\nu_2...\nu_{d-1}} \text{tr}\left(\bar{\gamma} U_{\nu_1} U_{\nu_2} \cdots U_{\nu_{d-1}}\right). \tag{3.9}
\]

Of course this is as it must be; if it were not the winding number of the chiral field \( U \) would not be equal to its topological charge. Substitution of (3.9) into (2.9) successfully reproduces Witten’s form [20] of the Wess–Zumino term, and thus the standard result for the anomalous process \( 2K \to 3\pi \).

4. Gauged Currents and the Wess–Zumino term

Since the gauged action is required when considering the processes involving external photons, and thus for the amplitudes of such processes such as \( \pi^0 \to 2\gamma \) and \( \gamma \to 3\pi \), it is necessary to extend the calculation described in the previous section to include local couplings to external gauge fields. Fortunately this may be done without too much difficulty.

We thus consider a more general Dirac operator of the form

\[
\mathcal{D} = -i\gamma_\mu D_\mu + M; \tag{4.1}
\]

(2.8) thus becomes

\[
\mathcal{D}^\dagger \mathcal{D} = -D^2 + M^\dagger M - \gamma_\mu [D_\mu, M] - \sigma_{\mu\nu} F_{\mu\nu}. \tag{4.2}
\]

These simple expressions belie the full complexity of the gauge field dependence, however, since as \( M \) is now non-local, it must also depend on the gauge field in order to maintain
gauge invariance. Expansions such as (2.4) will thus become rather more complicated. Although this must be taken into account when expanding the normal parity part of the action (2.7) [13], it is fortunately of no consequence for the leading term in the expansion of the imaginary part (2.9) since there each derivative of \( U \) must be contracted with a gamma–matrix if it is to lead to a non–vanishing contribution. Thus for present purposes we may ignore the gauge field dependence of \( M \).

A further complication is that for general vector and axial gauge fields \( A_\mu = v_\mu + \gamma a_\mu \), the integrability obstruction [13] will no longer always vanish. The representations (2.4) and (2.5) must then be supplemented by some consistent regularization scheme (for example Pauli–Villars) just as in ref. [19], resulting in the usual external field anomalies. It is not difficult to check that these anomalies depend only on the external fields; at the regulator scale the effects of the soft couplings to the pions (and indeed to any other strongly bound states) are effectively screened out by their form factors. In the standard model the anomalies are of course cancelled off by similar contributions from the leptons.

It follows that in the presence of the external gauge fields the gauged current corresponding to (1.3) is obtained by letting \( U_\mu = D_\mu U = \partial_\mu + A_\mu U - UA_\mu \) (where \( A_\mu \equiv v_\mu + \gamma a_\mu \)), adding the extra contributions due to the last term in (4.2), and finally (to ensure that the current is conserved and the determinant well defined) subtracting off the anomalous terms which depend only on the external fields. The only nontrivial step is thus the second, and the new contributions may be found by replacing either \( U_{\nu \rho} U_{\nu \rho+1}^\dagger \) by \( F_{\nu \rho \nu \rho+1} \) or \( U_{\nu \rho}^\dagger U_{\nu \rho+1} \) by \( U_{\nu \rho}^\dagger F_{\nu \rho \nu \rho+1} U \) in all possible ways. Since the couplings to the external fields are local, the number of factors of \( \Sigma \) under the integral is reduced by two for each replacement, while the number of factors of \( k^2 + \Sigma^2 \) is reduced by one; for a term with \( f \) factors of \( F_{\mu \nu} \) the integral is thus \( \beta \left( \frac{d}{2} - f \right) \), again completely independent of the form of \( \Sigma(k^2) \). Indeed using the symmetrical representation for \( \beta(m,n) \) the gauged current may be seen to take the form

\[
\frac{i d^{2} / 2 N_{c}}{(4\pi)^{d / 2} \Gamma(d / 2)} \epsilon_{\mu \nu \nu_{1} \nu_{2} \cdots \nu_{d-1} \nu_{d}} \frac{1}{2} \int_{-1}^{1} dt \, \text{tr} \left[ U_{\nu \rho} U_{\nu \rho+1} \prod_{p=1}^{d/2-1} \left( \frac{1}{2} (1 + t) U_{\nu \rho}^\dagger F_{\nu \rho \nu \rho+1} U + \frac{1}{2} (1 - t) U_{\nu \rho}^\dagger U_{\nu \rho+1} \right) \right].
\]

(4.3)

Not surprisingly this is just the standard expression for the Chern–Simons form \( \omega^{VA} \) (see for example eqn. (3.100) of ref. [2]). We thus recover the usual form for the gauged topological current in arbitrary even dimension \( d \).

It now follows immediately from the representation (2.9) that the leading term in the derivative expansion of the abnormal parity part of the effective action is indeed given by the standard Wess–Zumino term in (using Pauli–Villars to regulate the external currents) VA form. In four–dimensional space–time (which means we should take \( d = 6 \) in (4.3)) this means in particular that when the pion couplings to the quark loop are regulated naturally by their soft form factors, the amplitudes for the processes \( \pi^0 \to 2\gamma \) and \( \gamma \to 3\pi \) (and indeed that for \( KK \to 3\pi \) in \( SU(3) \)) are given to leading order in the derivative expansion and chiral perturbation theory by the results obtained in the local chiral quark model in the infinite cut–off limit.

The paradox noted in ref. [13] is resolved by noting that the sharp momentum cutoff used there is equivalent to taking \( \Sigma(k^2) = m \Theta (\Lambda^2 - k^2) \). Then \( d\Sigma / dk^2 = -m \delta (k^2 - \Lambda^2) \), and the second term in (3.7) results in a positive boundary contribution to the integral which vanishes as \( (m/\Lambda)^d \) in the limit \( \Lambda/m \to \infty \). When this contribution (which results from the extra commutator term in the current (1.2), and is necessary for its conservation) is ignored, the remaining piece is then too small to account for the full amplitude.
5. Conclusion

The non–locality of the quark mass term $\Sigma$, and thus of the quark–meson interactions, provides us with a consistent and physically motivated regularization of the quark loop in which the real and imaginary parts of the action are treated on the same footing and in which the leading term in the derivative expansion of the abnormal parity part of the action becomes independent of $\Sigma$ so that we recover the usual results \cite{21} for the anomalous decays $\pi^0 \rightarrow 2\gamma$ and $\gamma \rightarrow 3\pi$. Indeed, apart from mass corrections (due both to contributions from the current mass terms in the quark loop, and from higher order terms in the derivative expansion) and pion loop corrections (which only begin at next-to-leading order in the derivative expansion, at least at one loop \cite{23}) the naive calculation with a pointlike pion and bare quark loop gives the same result as the highly non–perturbative effective theory. The Adler–Bardeen theorem \cite{24} may thus be confirmed not only to all orders in perturbation theory, but non–perturbatively as well.

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Figure Captions

Fig. 1. The quark loop.

Fig. 2. The triangle diagram for $\pi^0 \rightarrow 2\gamma$.

Fig. 3. a) Rainbow graphs b) Ladder graphs.

Fig. 4. The non–local coupling of the chiral field to $q-\bar{q}$. 