Fermions in non-relativistic AdS/CFT correspondence

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Abstract

We extend the non-relativistic AdS/CFT correspondence to the fermionic fields. In particular we study the two point function of a fermionic operator in non-relativistic CFTs by making use of a massive fermion propagating in geometries with Schrödinger group isometry. Although the boundary of the geometries with Schrödinger group isometry differ from that in AdS geometries where the dictionary of AdS/CFT is established, using the general procedure of AdS/CFT correspondence, we see that the resultant two point function has the expected form for fermionic operators in non-relativistic CFTs, though a non-trivial regularization may be needed.
1 Introduction

In this paper, motivated by general idea of AdS/CFT correspondence [1], we would like to study non-relativistic fermions by making use of a gravity description. More precisely we will explore how to compute the two point function of a fermionic operator in a non-relativistic CFT generalizing the relativistic one first studied in [2].

Non-relativistic CFT in $d$ dimensions has $d-1$ dimensional Schrödinger symmetry, $\text{Sch}_{d-1}$. The generators of the Schrödinger algebra are spatial translations $P_i$, rotations $M_{ij}$, time translation $H$, Galilean boosts $K_i$, dilation $D$, number operator $N$ and special conformal transformation $C$. The algebra of $\text{Sch}_{d-1}$ is given by

$$
[M_{ij}, P_k] = -i(\delta_{ik} P_j - \delta_{jk} P_i), \quad [M_{ij}, K_k] = -i(\delta_{ik} K_j - \delta_{jk} K_i), \\
[M_{ij}, M_{kl}] = -i\delta_{ik} M_{jl} + \text{perms}, \quad [P_i, K_j] = -iN\delta_{ij}, \\
[D, P_i] = -iP_i, \quad [D, K_i] = iK_i, \quad [D, H] = -2iH, \\
[C, P_i] = iK_i, \quad [C, D] = -2iC, \quad [C, H] = -iD.
$$

(1.1)

Since $[D, N] = 0$ one may diagonalize them simultaneously leading to the fact that representations of the Schrödinger algebra may be labeled by two numbers; dimension $\Delta$ and a number $M$ which are the eigenvalues of $D$ and $N$, respectively.

Following the relativistic AdS/CFT correspondence one expects that if there are gravity duals to non-relativistic CFTs, the isometry of the relevant geometry must be Schrödinger group. In fact such gravity duals exist and the corresponding geometry is given by

$$
ds^2 = \rho^2(-\rho^2 dt^2 - 2dtd\xi + dx_i^2) + \frac{d\rho^2}{\rho^2}, \quad i = 1, \cdots, d-1,
$$

(1.2)

which could be thought of as a solution of a $d+2$ dimensional gravity coupled to massive gauge field [3, 4].

This geometry has been used to study different features of non-relativistic CFT. In particular utilizing a propagating massive scaler field in the bulk geometry (1.2) the corresponding point function in the dual non-relativistic CFT has been calculated [4]. Although the geometry (1.2) has a one dimensional time-like boundary\(^2\), it has been used to compute two point functions of $d$ dimensional non-relativistic CFT. In other words, in order to compute the correlation function in the non-relativistic AdS/CFT it was implicitly assumed in [4] that the procedure is the same as that in the relativistic case [5, 6] where the gravity description is given in terms of AdS geometry which has a well defined boundary\(^3\). Indeed following this assumption we will get the expected two point function of scalars in the non-relativistic CFT, though a non-trivial regularization seems to be needed [4]. One then may wonder if the need for such a regularization is due to the peculiar boundary of the geometry (1.2).

The aim of this paper is to further explore the AdS/CFT correspondence in the context of non-relativistic CFTs. We study correlation functions of fermionic operators in the non-relativistic CFT by making use of fermions propagating in the bulk geometry given by (1.2).

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\(^1\)This algebra can be obtained from the relativistic conformal algebra in $d+1$ dimensions by a contraction. In other words the Schrödinger group may be thought of as a subgroup of $SO(2, d+1)$ with fixed momentum along the null direction.

\(^2\)$g_{tt}$ grows faster than the other metric components for large $\rho$.

\(^3\)Such an assumption has generically been accepted in the literature. See for example [7–15]
More precisely we will consider a propagating massive fermion in the geometry (1.2) and compute the value of the action with a proper boundary term for a classical solution of the equation of motion. Then we shall identify this value with the generating function of a fermionic operator in the non-relativistic CFT.

Since the non-relativistic AdS/CFT correspondence relates two theories with dimensions $d$ and $d+2$, an immediate difficulty we face when we are considering fermions in this context is that the number of degrees of freedom of the spinors do not match for these two theories. Therefore there must be a condition which projects out half of the degrees of freedom. Actually this is the case. In fact the correspondence is smart enough to automatically project out half of the degrees of freedom due to its particular boundary interaction. We note, however, that although following the general procedure of AdS/CFT correspondence we will get the expected fermionic correlation function, a non-trivial regularization is needed to make the result finite.

The paper is organized as follows. In the next section we will study fermions in $d$ dimensional non-relativistic CFT by making use of free relativistic fermions in $d+1$ dimensions. In particular using the field theory method we compute two point function of the non-relativistic fermions. In section three we consider gravity description of the non-relativistic fermions where the fermion's two point function is found via gravity calculations. The last section is devoted to discussions.

## 2 Non-relativistic fermions

In this section we review non-relativistic fermions in $d$ dimensions and compute their two point function. First of all we note that the $d$ dimensional Galilean conformal symmetry may be obtained from conformal symmetry in $d+1$ dimensions [17–24]. Therefore to proceed we will start from the action of a massless relativistic fermion in $d+1$ dimensions whose symmetry is $SO(2,d+1)$, the conformal group in $d+1$ dimensions, then compactify it to $d$ dimensions along a null direction. In this way we will end up with the action of a non-relativistic fermion.

To be specific we will consider $d = 4$, though it can be easily generalized to other dimensions. The action of $d+1$ dimensional massless fermion is

$$S = \int d^5 x \bar{\psi} i \gamma^\mu \partial_\mu \psi.$$  \hspace{1cm} (2.1)

Let us decompose the coordinates as $(t, \xi, x_i)$, $i = 1, 2, 3$, where the light like coordinates $(t, \xi)$ are defined by

$$t = \frac{1}{\sqrt{2}}(x^0 + x^4), \quad \xi = \frac{1}{\sqrt{2}}(x^0 - x^4).$$ \hspace{1cm} (2.2)

Accordingly the gamma matrices can also be decomposed along the light like and spatial

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4 The non-relativistic fermion has been studied in the context of supersymmetric Schrödinger symmetry in [16].

5 Here we work in ($-,+,+,+,+$) signature.
the scaling that the equation is invariant under the non-relativistic conformal symmetry. Indeed under equation of motion for non-relativistic fermion which was studied in [25] where it was shown

Using this notation the action \((2.1)\) reads

\[
S = \frac{i}{\sqrt{2}} \int d^3x d\xi dt \psi^\dagger \left( \gamma_\xi \gamma_\eta \partial_\xi + \gamma_\xi \gamma_\eta \partial_\eta - (\gamma_\xi + \gamma_\eta) \gamma_i \partial_i \right) \psi. 
\]

(2.4)

Consider a single mode with definite momentum in the null direction \(\xi\). So that \(\psi(t, \xi, x) = e^{iM\xi} \psi_M(t, x)\). Therefore the action becomes

\[
S = \frac{i}{\sqrt{2}} \int d^3x dt \psi^\dagger_M \left( iM \gamma_\xi \gamma_t + \gamma_\xi \gamma_\eta \partial_t - (\gamma_\xi + \gamma_\eta) \gamma_i \partial_i \right) \psi_M. 
\]

(2.5)

Using the explicit representation for gamma matrices and setting

\[
\psi_M(x, t) = e^{i(Et + k_i x_i)} \begin{pmatrix} \phi \\ \chi \end{pmatrix},
\]

the equation of motion obtained from the above action is given by

\[
\begin{pmatrix} 2E \\ i\sqrt{2} \sigma_i k_i \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0,
\]

(2.7)

where \(\phi\) and \(\chi\) are two component Weyl spinors and \(\sigma_i\)’s are Pauli’s matrices. This is the equation of motion for non-relativistic fermion which was studied in [25] where it was shown that the equation is invariant under the non-relativistic conformal symmetry. Indeed under the scaling \(t \to \lambda^2 t, x_i \to \lambda x_i\), the \(\phi\) and \(\chi\) have scaling dimension \(-\frac{3}{2}\) and \(-\frac{5}{2}\), respectively.

The most general solution of the above equation is

\[
\psi_M(t, x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i(k^2t + k_i x_i)} \begin{pmatrix} \sum_{s=1}^2 a_s(k) \eta^s \\ i\sqrt{2} \sum_{s=1}^2 a_s(k) \eta^s \end{pmatrix},
\]

(2.8)

where

\[
\eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \{a_M^s(k), a^{r^r}_{-M}(k')\} = (2\pi)^3/2 \delta_{sr} \delta^3(k - k').
\]

(2.9)

Now we are ready to find two point function of the non-relativistic fermions. Using the explicit solution \((2.8)\) one finds

\[
\langle \psi_M(t, x) \bar{\psi}_{-M}(0, 0) \rangle = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{d^3k'}{(2\pi)^{3/2}} e^{i(k^2t + k_i x_i)} \mathcal{M},
\]

(2.10)

where

\[
\mathcal{M} = \begin{pmatrix}
\frac{\alpha_i k_i'}{\sqrt{2}M} \langle 0 | a_s^s(k) a^{r^r}_{-M}(k') | 0 \rangle \eta^s \eta^r \\
-i \langle 0 | a_s^s(k) a^{r^r}_{-M}(k') | 0 \rangle \eta^s \eta^r & -\frac{\alpha_i k_i'}{\sqrt{2}M} \langle 0 | a_s^s(k) a^{r^r}_{-M}(k') | 0 \rangle \eta^s \eta^r \\
\end{pmatrix}
\]

(2.11)
It is easy to evaluate the expectation values of \( a(k)'s \) using the anticommutator in (2.9). Then performing the integration over \( k' \) one arrives at

\[
\langle \psi_M(t,x) \bar{\psi}_M(0,0) \rangle = -\int \frac{d^3k}{(2\pi)^{3/2}} \left( \begin{array}{cc} -\frac{\sigma_i k_i}{\sqrt{2M}} & i \\ i \frac{k^2}{2M^2} & \frac{\sigma_i k_i}{\sqrt{2M}} \end{array} \right) e^{i \left( \frac{k^2}{2M} t + k_i x_i \right)}.
\]

which can be recast to the following form

\[
\langle \psi_M(t,x) \bar{\psi}_M(0,0) \rangle = \frac{i}{\sqrt{2M}} \left( i M \gamma_t + \gamma_\xi \partial_t - \gamma_i \partial_i \right) G(t,x;0,0),
\]

where

\[
G(t,x;0,0) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i \left( \frac{k^2}{2M} t + k_i x_i \right)} = e^{\frac{3\pi i}{4}} \left( \frac{M}{t} \right)^{3/2} e^{-\frac{im \xi^2}{2t}}\]

is the Green function of the non-relativistic scalar field propagating in four dimensions [4, 16, 21, 30].

### 3 Gravity description

In this section we will study non-relativistic fermions in \( d \) dimensions by making use of a \( d+2 \) dimensional gravity with Schrödinger group isometry. The corresponding gravity solution is given in (1.2). Setting \( r = \frac{1}{\rho} \) the metric (1.2) reads

\[
ds^2 = -\mu^2 \frac{dt^2}{r^4} + \frac{2dtd\xi + dx_i^2 + dr^2}{r^2}.
\]

Here we added the parameter \( \mu \) which parametrizes the deviation from \( AdS_{d+2} \) geometry.

The aim is to solve the Dirac equation in this background with an appropriate boundary condition. Following the AdS/CFT correspondence [5, 6] this may be used to evaluate two point function of an operator which would be dual to the bulk fermion. Due to the symmetry of the proposed background (3.1) one expects that the obtained two point function corresponds to the two point function of a fermionic operator in the dual non-relativistic CFT.

The Dirac equation in the background (3.1) is given by

\[
\left( r \Gamma_\xi \partial_\xi + r \Gamma_\xi \partial_t + r \Gamma_i \partial_i + r \Gamma_r \partial_r + \mu^2 \frac{\Gamma_\xi \partial_\xi}{2r} - \frac{d+1}{2} \Gamma_\xi - m \right) \Psi(x_i, t, \xi, r) = 0,
\]

We note, however, that due to the scaling symmetry \( t \rightarrow t/\beta, \xi \rightarrow \beta \xi \) of the geometry for \( \mu \neq 0 \) it can be rescaled to any value. On the other hand since \( \partial_\xi \) may be identify with number operator of the dual non-relativistic CFT, \( \xi \) direction should be periodic; \( \xi \equiv \xi + 2\pi L \). It is then natural to define the dimensionless parameter \( \mu L \) which may parametrize the non-relativistic nature of the theory [26]. Nevertheless we prefer to set it as a free parameter to follow the effects of deformation. For a nice discussion concerning this point see [27].
where $\Gamma^{'\hat A}$'s are $(d + 2)$-dimensional gamma matrices obeying\footnote{The hatted indices denote the coordinates of the tangent space.}

\[
\{\Gamma^{'\hat \xi}, \Gamma^{'\hat \xi}\} = 2, \quad \{\Gamma^{'\hat i}, \Gamma^{'\hat i}\} = 0, \quad \{\Gamma^{'\hat i}, \Gamma^{'\hat j}\} = 2\eta^{'\hat i}_{\hat j}, \quad \{\Gamma^{'\hat i}, \Gamma^{'\hat i}_{\hat \xi}\} = 0. \tag{3.3}
\]

We note, however, that it is not straightforward to solve the the Dirac equation (3.2) exactly. Nevertheless since we are interested in a solution near $r = 0$, one may try to find the solution perturbatively near $r = 0$. To proceed we multiply the Dirac equation (3.2) by

\[
r \Gamma^{'\hat i}_{\hat \xi} \partial_{\hat \xi} + r \Gamma^{'\hat i}_{\hat \xi} \partial_{t} + r \Gamma^{'\hat i}_{\hat \xi} \partial_{r} + \frac{\mu^{2}}{2r} \Gamma^{'\hat \xi}_{\hat \xi} \partial_{r}
\]

leading to the following Laplace equation

\[
\left( \mathbf{D}^{2} + m \Gamma^{'\hat \xi}_{\hat \xi} - \frac{\mu^{2}}{r} \Gamma^{'\hat \xi}_{\hat \xi} \partial_{\hat \xi} \right) \Psi(x_{i}, t, \xi, r) = 0, \tag{3.4}
\]

where

\[
\mathbf{D}^{2} = r^{2} \partial_{r}^{2} - r(d + 1) \partial_{r} + r^{2}(2 \partial_{\xi} \partial_{t} + \partial_{\xi}^{2}) + \left(\frac{(d + 1)^{2}}{4} + \frac{d + 1}{2} - m^{2} + \mu^{2} \partial_{\xi}^{2}\right). \tag{3.5}
\]

To find the solution it is useful to Fourier transform over $t$ and $x_{i}$ directions. Denoting by $M$ the momentum along the null direction $\xi$ one has

\[
\Psi(t, x_{i}, \xi, r) = e^{iM \xi} \int \frac{d\omega d^{d-1}q}{(2\pi)^{d/2}} e^{i\omega t + iq.x} \Psi_{M}(q, \omega, r), \tag{3.6}
\]

by which the equation (3.4) reads

\[
\left( \mathbf{D}^{2} + m \Gamma^{'\hat \xi}_{\hat \xi} - \frac{i\mu^{2}M}{r} \Gamma^{'\hat \xi}_{\hat \xi} \right)\Psi_{M}(q, \omega, r) = 0, \tag{3.7}
\]

where

\[
\mathbf{D}^{2} = r^{2} \partial_{r}^{2} - r(d + 1) \partial_{r} - r^{2}k^{2} + \left(\frac{(d + 1)^{2}}{4} + \frac{d + 1}{2} - m^{2} - \mu^{2}M^{2}\right), \tag{3.8}
\]

with $k^{2} = 2M\omega + q^{2}$. On the other hand since $\Gamma^{'\hat \xi}_{\hat \xi} = 0$ one finds

\[
(\mathbf{D}^{2} - m \Gamma^{'\hat \xi})\Psi_{M}(k, r) = 0. \tag{3.9}
\]

This equation can be exactly solved to find $\Gamma^{'\hat \xi}_{\hat \xi} \Psi_{M}(k, r)$. To do so, taking into account that $\Gamma^{'\hat \xi}_{\hat \xi} = 1$, one can decompose $\Psi_{M}(k, r)$ in terms of eigenvectors of $\Gamma^{'\hat \xi}$ as follows

\[
\Psi_{M}(k, r) = \Psi^{+}_{M}(k, r) + \Psi^{-}_{M}(k, r), \quad \Gamma^{'\hat \xi}_{\hat \xi} \Psi^{\pm}_{M} = \pm \Psi^{\pm}_{M}. \tag{3.10}
\]

So that $\Gamma^{'\hat \xi}_{\hat \xi} \Psi^{\pm}_{M}(k, r) = \mp \Psi^{\pm}_{M}(k, r)$. By making use of the above decomposition the equation (3.9) reduces to two equations for $\Gamma^{'\hat \xi}_{\hat \xi} \Psi_{M}(k, r)$

\[
(\mathbf{D}^{2} + m) \Gamma^{'\hat \xi}_{\hat \xi} \Psi^{+}_{M}(k, r) = 0, \quad (\mathbf{D}^{2} - m) \Gamma^{'\hat \xi}_{\hat \xi} \Psi^{-}_{M}(k, r) = 0, \tag{3.11}
\]
whose solutions are
\[ \Gamma_{\xi} \Psi_{M}^{+}(k, r) = r^{\frac{d}{2} + 1} K_{\nu^{-}}(kr) \Gamma_{\xi} u_{M}(k), \quad \Gamma_{\xi} \Psi_{M}^{-}(k, r) = r^{\frac{d}{2} + 1} K_{\nu^{+}}(kr) \Gamma_{\xi} v_{M}(k). \] (3.12)

Here \( v_{M}(k) \) and \( u_{M}(k) \) are constant spinors, \( K_{\nu}(z) \) is the Bessel function and
\[ \nu^{\pm} = \sqrt{(m \pm \frac{1}{2})^2 + \mu^2 M^2}. \] (3.13)

Plugging the solutions (3.12) into the equations (3.7) we get
\[ (D^2 + m) \Psi_{M}^{+}(k, r) = \frac{i\mu^2 M}{r} r^{\frac{d}{2} + 1} K_{\nu^{+}}(kr) \Gamma_{\xi} v_{M}(k), \]
\[ (D^2 - m) \Psi_{M}^{-}(k, r) = -\frac{i\mu^2 M}{r} r^{\frac{d}{2} + 1} K_{\nu^{-}}(kr) \Gamma_{\xi} u_{M}(k). \] (3.14)

The most general solution to the above equations are
\[ \Psi_{M}^{+}(k, r) = r^{\frac{d}{2} + 1} K_{\nu^{-}}(kr) u_{M}(k) + f(k, r) \Gamma_{\xi} v_{M}(k) \]
\[ \Psi_{M}^{-}(k, r) = r^{\frac{d}{2} + 1} K_{\nu^{+}}(kr) v_{M}(k) + g(k, r) \Gamma_{\xi} u_{M}(k) \] (3.15)

where \( f(k, r) \) and \( g(k, r) \) are solutions of the following differential equations
\[ (D^2 + m) f(k, r) = i\mu^2 M r^{\frac{d}{2}} K_{\nu^{+}}(kr), \quad (D^2 - m) g(k, r) = -i\mu^2 M r^{\frac{d}{2}} K_{\nu^{-}}(kr). \] (3.16)

In general it is not possible to solve the above equations exactly getting a closed form for the solution. Actually we do not even need to solve the equations exactly. The only thing we need is the asymptotic behavior of the solutions near \( r = 0 \). Therefore we will solve the above equations order by order near \( r = 0 \). Doing so, in the lowest order we arrive at
\[ f(k, r) = a k^{-\nu^{+}} r^{\frac{d}{2} - \nu^{+}} + a' r^{2 - \nu^{+} + 2} + \cdots \]
\[ g(k, r) = b k^{-\nu^{-}} r^{\frac{d}{2} - \nu^{-}} + b' r^{2 - \nu^{-} + 2} + \cdots \] (3.17)

where \( a, b, a', b', \cdots \) are numerical factors. In fact for our purpose we only need the first term in the expressions of \( g \) and \( f \) where the corresponding coefficients are given by
\[ a = i\mu^2 \frac{2\nu^{+} - 2M \Gamma(\nu^{+})}{\nu^{+} + m + \frac{1}{2}}, \quad b = -i\mu^2 \frac{2\nu^{-} - 2M \Gamma(\nu^{-})}{\nu^{-} - m + \frac{1}{2}}. \] (3.18)

Note that so far we have been trying to solve the equation (3.1) which is actually the square of the Dirac equation (3.2). Therefore a solution of a equation (3.7) is not necessarily a solution of the Dirac equation (3.2) as well. To find a solution of the Dirac equation one needs to plug the solution of (3.7) into the Dirac equation which in general leads to an extra condition on the constant spinors. In particular in our case we find
\[ v_{M} = -\frac{i}{2M} (i\kappa_{\bar{a}} \Gamma_{\bar{a}}) \Gamma_{\xi} v_{M}, \quad u_{M} = -\frac{i}{2M} (i\kappa_{\bar{a}} \Gamma_{\bar{a}}) \Gamma_{\xi} u_{M}. \] (3.19)
where \((\kappa_i,\kappa_{\xi},\kappa_\xi) = (M,\omega,q_i)\). It is worth mentioning that in these expressions there is a term, \(\omega\Gamma_{\xi}\), which has naively zero contribution due to the fact that \(\Gamma_{\xi}^2 = 0\). We note, however, that as we will see the physical spinors which inter in the dictionary of AdS/CFT correspondence in the non-relativistic case are \((\Gamma_{\xi}\mathbf{u}_M,\Gamma_{\xi}\mathbf{v}_M)\) and not \((\mathbf{u}_M,\mathbf{v}_M)\) themselves. Therefore it is important to keep this term too.

Similarly going through the above procedure for \(\Psi_M(k,r)\) one finds
\[
\Psi_M^+(k,r) = r^{d+d_f-1}\mathbf{K}_\nu^+(kr)\Psi_M(k) + g(k,r)\Psi_M(k)\Gamma_{\xi},
\]
\[
\Psi_M^-(k,r) = r^{d+d_f-1}\mathbf{K}_\nu^-(kr)\Psi_M(k) + f(k,r)\Psi_M(k)\Gamma_{\xi}. \tag{3.20}
\]

Now we have all ingredients to establish the AdS/CFT correspondence for the non-relativistic case. First of all we note that since in the non-relativistic case it is believed that \(d\) dimensional non-relativistic CFT is described by a \(d+2\) dimensional gravity, a Dirac spinor in the bulk is \(2[d_f+1]\) dimensional, though in the dual non-relativistic field theory it has \(2[d/2]\) dimensions. Therefore a priori it is not clear how to match the degrees of freedom in the bulk with that in the dual theory. So if the AdS/CFT works for the non-relativistic case, one would expect that there should be a constraint on the spinor in the bulk reducing its degrees of freedom with a factor of two. In fact, as we have already anticipated, the correspondence is clever enough to project out half of the degrees of freedom. To see this we note that at \(r \to 0\) we have
\[
\lim_{r \to 0} \Psi_M(k,r) \sim r^{d-d_f-1}\Gamma_{\xi}\mathbf{v}_M, \quad \lim_{r \to 0} \overline{\Psi}_M(k,r) \sim r^{d-d_f-1}\overline{\mathbf{u}}_M\Gamma_{\xi}, \tag{3.21}
\]
showing that if we want to follow the general procedure of AdS/CFT correspondence interpreting the asymptotic behavior of the spinor at \(r \to 0\) as the source for the dual operator in the non-relativistic field theory, the source should be given by \(\Gamma_{\xi}\mathbf{v}_M\) and \(\overline{\mathbf{u}}_M\Gamma_{\xi}\) rather than \(\mathbf{v}_M\) and \(\overline{\mathbf{u}}_M\). This is exactly what we need.

Indeed the spinors \(\Gamma_{\xi}\mathbf{v}_M\) and \(\overline{\mathbf{u}}_M\Gamma_{\xi}\) have the appropriate independent degrees of freedom to be identified as the sources for the dual operator \(\psi\) in non-relativistic CFT. More precisely to proceed we assume the following coupling in the non-relativistic CFT
\[
Z_{\text{CFT}} = \left\langle \exp \left[ \int d^d x \left( \overline{\Psi}_M \Gamma_{\xi}\mathbf{v}_M + \overline{\mathbf{u}}_M\Gamma_{\xi}\psi_M \right) \right] \right\rangle. \tag{3.22}
\]

Here \(\psi_M\) is a \(2[d/2]\) dimensional spinor in the non-relativistic CFT appropriately embedded into \(2[d/2+1]\) dimensional spinor representation by doubling the \(d\) dimensional spinor.

Therefore to evaluate the two point function \(\langle \overline{\psi}_M\overline{\Psi}_M \rangle\) one needs to perform the variations with respect to \(\overline{\mathbf{u}}_M\Gamma_{\xi}\) and \(\Gamma_{\xi}\mathbf{v}_M\). On the other hand we would like to identify the partition function \(Z_{\text{CFT}}\) of the non-relativistic CFT with \(\exp(I_{\text{AdS}})\), where \(I_{\text{AdS}}\) is the boundary term in the bulk action given by
\[
I_{\text{AdS}} = \int d\xi dt d^{d-1}x \sqrt{g} \overline{\Psi}(t,x,r,\xi)\Psi(t,x,r,\xi). \tag{3.23}
\]
Going to the momentum space and for \(r = \epsilon\) with \(\epsilon\) being an infinitesimal number, one gets
\[
I_{\text{AdS}} = \epsilon^{-d-1} \int \frac{d\omega dq^{d-1}}{(2\pi)^{d/2}} \left[ \overline{\Psi}_M^+(k,\epsilon)\Psi_M^+(k,\epsilon) + \overline{\Psi}_M^-(k,\epsilon)\Psi_M^-(k,\epsilon) \right]. \tag{3.24}
\]
On the other hand using the solutions (3.15) and (3.20) the integrand in leading order reads
\[
\left[\Psi_+^+(k, \epsilon) \Psi_+(k, \epsilon) + \Psi_-^-(k, \epsilon) \Psi_-(k, \epsilon) \right] \approx i \mu^2 C e^{d+1-2\nu^+} k^{-2\nu^+} u_{-M}(k) \Gamma_{\xi} v_M(k),
\]
where \( C = \frac{2^{2\nu^+ - 2} \Gamma^2(\nu^+)}{\nu^+ + m + 1/2} \). Therefore utilizing the equation (3.19) we arrive at
\[
I_{AdS} = \frac{\mu^2 C}{2M} e^{2\nu^+} \int \frac{dtd^{d-1}x}{(2\pi)^{d/2}} \frac{dt'd^{d-1}x'}{(2\pi)^{d/2}} \left[ e^{iq(x-x') + i\omega(t-t')} k^{-2\nu^+} u_{-M}(x', t') \Gamma_{\xi}(i\kappa_\alpha \Gamma_\alpha) \Gamma_{\xi} v_M(x, t) \right]
\]
which can be used to find the two point function as follows
\[
\langle \psi_M(x, t) \psi_{-M}(0, 0) \rangle = \frac{\mu^2 C}{2M} e^{2\nu^+} \int \frac{d\omega d^{d-1}q}{(2\pi)^{d/2}} e^{iqx + i\omega t} k^{-2\nu^+} (i\kappa_\alpha \Gamma_\alpha).
\]
Performing the integration one finds
\[
\langle \psi_M(x, t) \psi_{-M}(0, 0) \rangle = i \mu^2 B e^{\frac{\Delta}{2}} \sqrt{2M} e^{-2\nu^+} \left( i M \Gamma_i + \Gamma_{\xi} \partial_i + \Gamma_i \partial_{\xi} \right) \left( t^{-\Delta} e^{\frac{\Delta}{4}} \right)
\]
where \( \Delta = \frac{d+1}{2} - \nu^+ \). Here \( B \) is a numerical factor which can be absorbed in the definition of the coupling constant of the boundary term (3.23).

As we see this expression has the right form to be identified with the two point function of the fermion in the non-relativistic CFT. Indeed it is compatible with (2.13) for \( \nu^+ = 1 \) (\( m = -1/2 \pm \sqrt{1 - \mu^2 M^2} \)). We note, however, that it is naively divergent as \( \epsilon \to 0 \). This is exactly the same singular behavior appeared in the bosonic case [4]. Although we do not have a clear understanding for this behavior, one may wonder that this singularity may be removed by regularizing the fermionic operators in the non-relativistic CFT.

4 Discussions

In this paper we have studied fermions in non-relativistic CFT by making use of a gravity description. The gravity description is given by a geometry with Schrödinger isometry. We note, however, that although \( d + 2 \) dimensional geometries with Schrödinger isometry have one dimensional time-like boundary, it may be used to study non-relativistic CFT in \( d \) dimensions. Therefore unlike the relativistic case where the CFT lives on the boundary of AdS geometry, a priori, it is not obvious where the non-relativistic CFT is living. As a result it is not clear how to extend the dictionary of AdS/CFT correspondence to non-relativistic CFT.

Nevertheless using the general rules of the relativistic AdS/CFT correspondence we have been able to obtain two point function for non-relativistic fermions which agrees with that in [8].

Note that due to the definition of \( \psi_M \) the resultant two point function (3.28) is two copies of (2.13). Note also that these two equations are written in different signatures.

For a study of fermions in the context of dS/CFT correspondence see [29].
non-relativistic CFT. Of course the resultant two point function is not finite and a non-trivial regularization seems to be needed. This might be related to the fact that the interested geometry has one dimensional time like boundary though it has been used to study $d$ dimensional non-relativistic CFT. It would be interesting to understand this point better.

An alternative way to find a gravity dual for non-relativistic CFT has been considered in [30, 31]. It was proposed that in order to study non-relativistic CFT in the context of gauge-gravity duality one may start from an AdS bulk geometry and break the conformal symmetry of the boundary theory to non-relativistic conformal symmetry by imposing a specific boundary condition for the bulk fields. Using this approach the two point function of scalar fields has been computed which is is the same form as that in [4] up to a shift in the mass of the bulk field. For further discussions concerning the comparison between these two proposals see [27].

It is worth to apply this method to the fermionic fields and compare the result with that in previous section. Since in this case we work with AdS geometry one may borrow the results which already exist in the literature. More precisely following [2, 28] we have to solve the Dirac equation in the $AdS_{d+1}$ geometry in the light-cone coordinates

$$
\left( r(\gamma_i \partial_x + \gamma_\xi \partial_t + \gamma_\nu \partial_r + \gamma_\tilde{\nu} \partial_{\tilde{t}}) - \frac{d+1}{2} \gamma_\nu - m \right) \Psi = 0. \quad (4.1)
$$

The boundary condition we need to impose is

$$
\Psi(t, x_i, \xi, r) = e^{-iM\xi} \Psi_M(t, x_i, r), \quad (4.2)
$$

which breaks the conformal symmetry of the boundary theory to non-relativistic conformal symmetry. Following the notation of [28] we note that the two point function has the same expression as that in relativistic case except the momentum along $\xi$ direction is fixed. Therefore we get

$$
\langle \psi_M(x,t) \bar{\psi}_{-M}(0,0) \rangle = \lim_{\epsilon \to 0} i \epsilon^{-2m} \int \frac{d\omega d^{d-1}q}{(2\pi)^d} \frac{K_{m+\frac{1}{2}}(k\epsilon)}{K_{m+\frac{1}{2}}(k\epsilon)} \frac{\kappa_0 \gamma_0}{k} e^{i(\omega t + q \cdot x)} \left( \frac{2M}{t} \right)^{\Delta} e^{-\frac{\omega^2}{4M^2}}. \quad (4.3)
$$

leading to

$$
\langle \psi_M(x,t) \bar{\psi}_{-M}(0,0) \rangle = C(i \gamma_i M + \gamma_\xi \partial_t + \gamma_\nu \partial_r) \left( \frac{2M}{t} \right)^{\Delta} e^{-\frac{\omega^2}{4M^2}}. \quad (4.4)
$$

where $\Delta = m + d/2$. This has the same form as that in previous section except the dimension of the operators is different. Of course having the same expression is not surprising as the two point function can be fixed by the symmetry. The non-trivial point is that these two approaches lead to different scaling dimensions for operators. It would be extremely interesting to explore any possible connection between these two approaches.

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