Abstract. This paper aims to make a new contribution to the study of lifetime ruin problem by considering investment in two hedge funds with high-watermark fees and drift uncertainty. Due to multi-dimensional performance fees that are charged whenever each fund profit exceeds its historical maximum, the value function is expected to be multi-dimensional. New mathematical challenges arise as the standard dimension reduction cannot be applied, and the convexity of the value function and Isaacs condition may not hold in our ruin probability minimization problem with drift uncertainty. We propose to employ the stochastic Perron’s method to characterize the value function as the unique viscosity solution to the associated Hamilton–Jacobi–Bellman (HJB) equation without resorting to the proof of dynamic programming principle. The required comparison principle is also established in our setting to close the loop of stochastic Perron’s method.

Key words. Lifetime ruin problem, multiple hedge funds, high-watermark fees, drift uncertainty, stochastic Perron’s method, comparison principle

AMS subject classifications. Primary, 49L20, 49L25, 60G46; Secondary, 91G10, 93E20

1. Introduction. Hedge funds have existed for many decades in the financial market and have become increasingly popular in recent times. As opposed to the individual investment, a hedge fund pools capital and invests in a variety of assets and it is administered by professionals. Hedge fund managers charge performance fees for their service to individual investors as some regular fees proportional to fund’s component assets plus a fraction of the fund’s profits. The most common scheme entails annual fees of 2% of assets and 20% of fund profit whenever the profit exceeds its historical maximum—the so-called high-watermark. In the present paper, we are interested in investment opportunities among several hedge funds and we intend to study a stochastic control problem given the path-dependent trading frictions as multi-dimensional high-watermark fees.

The existing research on high-watermark fees mainly focused on the asset management problem from the point of view of the fund manager, see some examples by [21], [29], [1], [23] and [24]. Meanwhile, the high-watermark process is also mathematically related to wealth drawdown constraints studied in [22], [18], [20] and also discussed in [16] after the transformation into expectation constraint. Recently, the high-watermark fees have been incorporated also into Merton problem for individual investor together with consumption choice in [26] and [27]. In the presence with consumption control, analytical solutions can no longer be promised as in some of the previous work for fund managers. After identifying the state processes, the path-dependent feature from high-watermark fees can be hidden so that the dynamic programming argument can be recalled to derive the HJB equation heuristically. The homogeneity of power utility function in [26] and [27] enables the key dimension reduction of the value function and the associated HJB equations can be reduced into ODE problems. Although the regularity can hardly be expected, classical Perron’s method can be applied and the nice upgrade of regularity of the viscosity solution can be exercised afterwards using the convexity property of the transformed one-dimensional value function. As the last step, the verification theorem can be concluded with the aid of the smoothness of value function and standard Itô calculus.
In the present paper, we focus on the standpoint of the individual investor who confronts multiple hedge fund accounts in the market. However, we aim to minimize the probability that the investor outlives her wealth, also known as the probability of lifetime ruin, instead of the Merton problem on portfolio or consumption. We determine the optimal investment strategy of an individual among some hedge funds who targets a given rate of consumption by minimizing the probability that the ruin occurs before the death time. For the studies of lifetime ruin probability problem, readers can refer to [34, 10, 12, 11, 35]. In contrast to Merton problem, the dimension reduction of the value function will fail for our probability minimization problem. The auxiliary controlled state process, the so-called process of distance to pay performance fees defined in (2.7), can no longer be absorbed to simplify the PDE problem. Furthermore, comparing with [26] and [27] or the lifetime ruin problem with ambiguity aversion in [11], we need to handle a genuine multi-dimensional control problem with reflections as there exist multiple hedge funds in the market. In other words, the distance process itself is already multi-dimensional, which spurs many new mathematical challenges. To wit, one can still exploit the classical Perron’s method as in [26], [27] and [11], and obtain the existence of viscosity solution to the associated HJB equation. Nevertheless, the upgrade of regularity of the viscosity solution can hardly be attained for our multi-dimensional problem. Consequently, the proof of verification theorem, which requires certain regularity of the solution, cannot be completed. To relate the value function to the viscosity solution in our setting using classical Perron’s method, we have to provide the technical proof of dynamic programming principle at the beginning.

In addition, by observing that the individual investor usually cannot keep a real-time track of the performance of hedge funds from fund managers. Moreover, a reliable estimation of the return from hedge fund that consists of a bunch of various assets is almost impossible in practice. Even in the hedge fund performance report, the predicted future return in short term from fund manager is provided as a certain range instead of a fixed number. It is more realistic to assume that the investor allows drift misspecification and starts with a family of plausible probability measures of the underlying model. This leads to a robust investment strategy with Knightian model uncertainty. In particular, we assume that the investor would like to use the available data as a reference model and work on a robust control problem with the penalty on other plausible models based on the deviation from the reference one. One new mathematical challenge from this formulation is that the value function may lose convexity for some parameters and the Issacs condition may fail. Adding our previous difficulties coming from multi-dimensional performance fees, the feedback optimal investment strategy and the saddle point choice of probability measure cannot be obtained. The combination of market imperfections such as trading frictions together with model ambiguity renders many problems mathematically intractable. Some workable examples in this direction can only be found in robust Merton problem with proportional transaction costs, see [28], [15] and [19]. The methodology introduced in these paper may not work for our purpose with path-dependent high-watermark fees.

To tackle our stochastic control problem, we choose to employ the stochastic Perron’s method (SPM) and characterize the value function as the unique viscosity solution to the associated HJB equation. This stochastic version of Perron’s method, introduced by [7], can avoid the technical and lengthy proof of dynamic programming principle (DPP) and can obtain it as a by-product. We choose SPM over the weak DPP introduced in [14] because SPM can better handle the path-dependent structure of our control problem with additional model uncertainty. Let us note that the comparison principle is needed anyway in both methods. SPM requires the comparison principle to complete the squeeze argument and establish the equivalence between value function and the viscosity solution, while weak DPP needs the comparison principle to guarantee the uniqueness of the viscosity solution to the associated HJB equation. We actually find that the proof of comparison principle for SPM is relatively easier as the applicable class of state processes can be larger than
that of weak DPP. We refer a short list of previous work on stochastic control using SPM such as [7], [9], [5], [6], [8], [30], [31], [32], [4] and [33].

To establish the viscosity semisolution property of $v^\pm$, it is usually crucial to check the boundary viscosity semisolution property. In our framework, we can take advantage of the problem structure from lifetime ruin probability minimization and explicitly construct a stochastic super-solution and a stochastic sub-solution which satisfy the desired boundary conditions. We note that our arguments using stochastic Perron’s method differ from [12] that solves the lifetime ruin problem with transaction costs and [4] that examines the robust optimal switching problem. Some nontrivial issues need to be carefully addressed, which are caused by the uncertainty of drift term and the structure of the auxiliary state process defined as the distance to pay fees. The path-dependent running maximum part coming from high-watermark fees do not appear in [12] nor [4], which deserves some novel and tailor-made treatment in the present paper.

The rest of the paper is organized as follows. Section 2 introduces the market model with multiple hedge funds and related high-watermark fees, the default time as well as the set up with drift uncertainty. The robust lifetime ruin problem is defined afterwards. In Section 3, we derive the associated HJB equation for the control problem heuristically and define the viscosity solution accordingly. The main theorem to characterize the value function as the unique viscosity solution is presented. Section 4 provides the proof of all main results using stochastic Perron’s method. The proof of the comparison principle of the HJB equation is also reported therein.

2. Market Model and Problem Formulation.

2.1. Multiple Hedge Funds with High-watermark Fees. Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a filtered probability space such that $\mathbb{G}$ satisfies the usual conditions and $\mathbb{E}$ denote the expectation operator under $\mathbb{P}$. Let $(W_t)_{t \geq 0}$ denote an independent 2-dimensional Brownian motion and $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by $(W_t)_{t \geq 0}$ and it is assumed that $\mathcal{F}_t \subset \mathcal{G}_t$. Later, we will characterize $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ more precisely.

We consider the financial market consisting of one risk-less bond with interest rate $r \geq 0$ and two hedge fund accounts $(F^i_t)_{t \geq 0}$, $i \in \{1, 2\}$, described by

$$dF^i_t = \mu^i F^i_t dt + \sigma^i F^i_t dW_t,$$

for some constant $\mu^i \geq 0$ and constant vector $\sigma^i \in \mathbb{R}^2$. To simplify the presentation, we only focus on two hedge funds henceforth. The mathematical arguments and main results can be easily extended to the multi-dimensional case of $N \geq 2$ hedge funds without any technical difficulty. We shall denote

$$F := \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}, \quad \mu := \begin{bmatrix} \mu^1 \\ \mu^2 \end{bmatrix}, \quad \sigma := \begin{bmatrix} \sigma^1 \\ \sigma^2 \end{bmatrix},$$

and assume that $\sigma$ is invertible.

Contrary to some standard investment problems in liquid risky assets such as stocks, we are considering the model when the investor is facing the wealth allocation among some hedge fund accounts that charge proportional fees on the profit as trading frictions. In particular, the investor needs to pay some high-watermark fees to the fund manager whenever the accumulative profit reaches the highest value. The 2/20-rule is common for hedge funds in the sense that 2% per year of the total investment and 20% of the additional profits are paid to the fund manager whenever the high-watermark exceeds the previously attained profit maximum. To explain this in a more explicit manner, let $\pi = (\pi^1, \pi^2) \in \mathbb{R}^2$ denote the investment strategy in two hedge funds $F$. The accumulative profit $\mathcal{P}^\pi := [\mathcal{P}^{\pi^1}, \mathcal{P}^{\pi^2}]^T$ from the hedge fund before the deduction of the high-
watermark, is characterized by the stochastic integral

\[
\mathcal{P}^\pi_i := \int_0^t \pi_i^1 dP_t^i + \int_0^t \pi_i^2 \sigma dW_t^i.
\]

In practice, there is also a benchmark profit from which the manager’s performance is measured, see [27], and the high-watermark fee is deducted when the high-watermark of the fund is higher than the benchmark level. The initial high-watermark fee is denoted by some non-negative constant vector \( y = [y^1, y^2]^\top \). Let \( F^B \in \mathbb{R}^2 \) be the benchmark process given by

\[
dF^B_t = \text{diag}(F^B_t)\left[\mu^B dt + \sigma^B dW_t^i\right],
\]

for some \( \mu^B \in \mathbb{R}^2 \), \( \sigma^B \in \mathbb{R}^{2 \times 2} \). We denote by \( \mathcal{B}^i = [\mathcal{B}^{i,1,\pi}, \mathcal{B}^{i,2,\pi}]^\top \) the accumulated benchmark profit process if the same strategy \( \pi \) is adopted, i.e.,

\[
\mathcal{B}^i := \int_0^t \pi_s^1 dF^B_s + \int_0^t \pi_s^2 \sigma dW_s^i.
\]

Let \( q = [q^1, q^2]^\top \) represent the proportional rates of high-watermark fee of each hedge fund and \( P^{i,y,\pi} = [P^{i,1,y,\pi}, P^{i,2,y,\pi}] \) be the realized profit after charging the high-watermark fee. Moreover, we define \( M^{i,y,\pi} \) as the historical high-watermark of the \( i \)-th hedge fund. The realized profit process \( P^{i,y,\pi} \), \( i \in \{1, 2\} \), is given by

\[
\begin{align*}
\begin{cases}
dP_t^{i,y,\pi} := d\mathcal{P}^\pi_i - q^i dM_t^{i,y,\pi}, \\
M_t^{i,y,\pi} := \sup_{0 \leq s \leq t} \{(P_s^{i,y,\pi} - \mathcal{B}^i_s) \vee y^i\},
\end{cases}
\quad P_0^{i,y,\pi} = 0,
\end{align*}
\]

To represent (2.2) in a more convenient form, let us define

\[
\mathcal{M}^{i,y,\pi} := \sup_{0 \leq s \leq t} \{(\mathcal{P}^\pi_s - \mathcal{B}^i_s) \vee y^i\}, \quad i \in \{1, 2\}.
\]

Then by (2.2),

\[
\begin{align*}
\mathcal{M}^{i,y,\pi} - y^i &= \sup_{0 \leq s \leq t} \{(\mathcal{P}^\pi_s - \mathcal{B}^i_s) - y^i\}^+ \\
&= \sup_{0 \leq s \leq t} \left\{\left[P_s^{i,y,\pi} - \mathcal{B}^i_s\right] - y^i + q^i [M_s^{i,y,\pi} - y^i]\right\}^+ \\
&=(1 + q^i)(M_t^{i,y,\pi} - y^i).
\end{align*}
\]

Therefore, in view of (2.2) and (2.4), for \( i \in \{1, 2\} \), we have

\[
P_t^{i,y,\pi} := \mathcal{P}^\pi_t - \frac{q^i}{1 + q^i} [M_t^{i,y,\pi} - y^i],
\]

Equivalently, \( P^{i,y,\pi} \) can be rewritten as

\[
dP_t^{i,y,\pi} = \mu^i \pi_t^1 dt + \sigma^i \pi_t^2 dW_t - q^i (1 + q^i)^{-1} d\mathcal{M}^{i,y,\pi}.
\]

As the high-watermark fee is only deducted whenever \( \mathcal{M}^{i} - [\mathcal{P}^\pi - \mathcal{B}^i] = 0 \), the distance between \( \mathcal{M}^{i} \) and \( \mathcal{P}^\pi - \mathcal{B}^i \) will be considered in the investment decision. Therefore, let us introduce the distance process \( Y^{i,y,\pi} = [Y^{1,y,\pi}, Y^{2,y,\pi}]^\top \) as the difference

\[
Y_t^{i,y,\pi} := \mathcal{M}_t^{i,y,\pi} - [P_t^{i,y,\pi} - \mathcal{B}^i_t].
\]
In view of (2.4), (2.5), and (2.7), it clearly follows that \( Y^{y,\pi} = \mathcal{M}^{y,\pi} - [\mathcal{F}^\pi - \mathcal{B}^\pi] \). To facilitate the future analysis using dynamic programming argument, we expect to deal with a multiple dimensional value function of the control problem depending on the two dimensional initial distance \( Y_0 = (y^1, y^2) \) and the investor’s initial wealth \( x \). The precise formulation will be introduced later.

We continue to characterize the investor’s wealth more explicitly. The amount of the risky position (hedge funds) is \( 1 \uparrow \pi \) and the rest of the investor’s wealth is put into the risk-less bond. Furthermore, it is assumed that the investor consumes at a constant rate \( c \geq 0 \) all the time. Let \( X^{x,y,\pi} \) denote the process of investor’s wealth with initial value \( x \). Then the controlled state processes are given by

\[
\begin{align*}
\begin{cases}
\frac{dX_t^{x,y,\pi}}{dt} &= \left[ rX_t^{x,y,\pi} - c + \pi_t^r \mu^r_{\Delta} \right] dt + \pi_t^r \sigma dW_t - q^1 \frac{dM^{y,\pi}}{dt}, & X_0 = x, \\
\frac{dY_t^{y,\pi}}{dt} &= -\text{diag}(\pi_t)\left[ \mu^B_{\Delta} dt + \sigma^B_{\Delta} dW_t \right] + \text{diag}(1+q) \frac{dM^{y,\pi}}{dt}, & Y_0 = y.
\end{cases}
\end{align*}
\]

where we denote \( \mu^r_{\Delta} := [\mu^1 - r, \mu^2 - r] \top \), \( \mu^B_{\Delta} := \mu - \mu^B \), \( \sigma^B_{\Delta} := \sigma - \sigma^B \), and

\[ \text{diag}(1+q) := \begin{bmatrix} 1+q^1 & 0 \\ 0 & 1+q^2 \end{bmatrix}. \]

Sometimes, we omit the superscripts \( x, y, \pi \) for simplicity and we also denote

\[ Z := (X,Y^1,Y^2), \quad z := (x,y^1,y^2). \]

2.2. Default Time and Preliminaries. Another important ingredient of our model is the default time of the individual investor, such as the death time, which is defined as a random variable

\[ \tau_D : (\Omega, \mathcal{G}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \]

satisfying \( \mathbb{P}(\tau_D = 0) = 0 \) and \( \mathbb{P}(\tau_D > t) > 0 \), for any \( t \geq 0 \). From this point onward, the full market filtration \( \mathcal{G} \) is precisely defined by \( \mathcal{G} = (\mathcal{G}_t)_{t \geq 0} := (\mathcal{F}_t \vee \sigma(\{\tau_D \leq u\} : u \leq t))_{t \geq 0} \). It is worth noting that \( \tau_D \) is a \( \mathcal{G} \)-stopping time but may fail to be an \( \mathbb{F} \)-stopping time.

The following assumptions are assumed for the rest of the paper.

**Assumption 2.1.**

(i) There exists a constant intensity \( \lambda^D \geq 0 \) such that

\[
\begin{align*}
G^D_t := \mathbb{P}(\tau_D > t | \mathcal{F}_t) &= e^{-\lambda^D t}.
\end{align*}
\]

(ii) (H)-hypothesis (or immersion hypothesis) is assumed, i.e., every \( \mathbb{F} \)-square integrable martingale is a \( \mathcal{G} \)-square integrable martingale.

We call \( \lambda^D \) the intensity of default time \( \tau_D \) with respect to \( \mathbb{F} \). The constant intensity is assumed mainly for technical simplicity when we deal with an infinite time horizon stochastic control problem.

Given the previous setup and assumptions, the following properties hold. For the proof, readers can refer to [13].

**Lemma 2.2.**

(i) \( (\mathcal{M}^D_t)_{t \geq 0} \), where \( \mathcal{M}^D_t := 1_{\tau_D \leq t} - \lambda^D(t \wedge \tau_D) \), is a \( (\mathbb{F}, \mathcal{G}) \)-martingale.

(ii) Any \( \mathbb{F} \)-martingale stopped at \( \tau_D \) is a \( \mathcal{G} \)-martingale, i.e., for any \( \mathbb{F} \)-martingale \( X \), \( X^{\tau_D} \) is a \( \mathcal{G} \)-martingale.

**Remark 2.3.**

1. In view of the existence of the intensity, \( \tau_D \) is totally inaccessible. In other words, the default of the investor comes with total surprise. On the other hand, a *rain time*, which will be introduced later, is defined as a hitting time that the controlled wealth process crosses a given level and it is therefore predictable. In the present paper, we envision an individual investor who chooses her portfolio to minimize the probability involving the *rain time* before the default time occurs.
2. Although investment strategies are defined as $\mathcal{G}$-adapted processes, the full filtration $\mathcal{G}$ is not fully observable for the investor. However, in this filtration setup, for any $\mathcal{G}$-adapted process, we can find an $\mathcal{F}$-reduction, where $\mathcal{F}$ is the observable information. Therefore, the strictly $\mathcal{G}$-adapted strategies only describe an immediate action taken by the investor at the default time. Note that an $\mathcal{F}$-adapted process is not necessarily determined independently of the default time $\tau_D$, because the (constant) default intensity $\lambda_D$ is trivially $\mathcal{F}$-adapted.

2.3. Life Time Ruin Problem with Drift Uncertainty. Based on previous building blocks, we are ready to introduce the primary stochastic control problem that the investor confronts. In particular, the investor concerns the viability of her investment before the default time and she wishes to maintain the amount of her wealth above a certain level, say $R \geq 0$, before the default time happens. To this end, it is natural to introduce the so-called ruin time

$$\tau_{x,y,\pi}^R := \inf \{ t \geq 0 : X_{t,x,y,\pi} \leq R \}.$$ 

Mathematically speaking, the investor chooses $\pi$ from an admissible set $\mathcal{A}$ so that $\tau_R$ occurs as late as possible. As the investor cannot control the totally inaccessible time $\tau_D$, she aims to minimize the probability that the ruin occurs before the default time.

However, we consider a more practical scenario in the present paper that the return of hedge funds may not be revealed by fund manager to the investor very frequently. The investor usually can only get access to the performance of the fund from some reports on regular dates. Moreover, as the hedge fund consists of components from various assets, the estimation of return can hardly be provided on a timely basis. Based on these observations, it is reasonable to assume that the investor may not have a precise knowledge of the dynamics of hedge funds. This naturally leads to the so-called Knightian model uncertainty.

In this paper, we will only focus on the case with drift uncertainty, i.e. the investor conceives a family of plausible return terms from the hedge fund dynamics and proceeds to solve the control problem in a robust sense. Indeed, the precise estimation of the drift term is much more challenging than the estimation of volatility term, which motivates our research. In particular, we aim to minimize the probability of lifetime ruin by choosing wealth allocation among multiple hedge funds with high-watermark fees and drift uncertainty, which is new to the existing literature. To this end, let us first introduce a class of probability measures equivalent to the reference probability $\mathbb{P}$ and denote this class by $\mathcal{L}$.

**Definition 2.4.** $Q \in \mathcal{L}$ if for any $0 \leq t$,

$$\frac{dQ}{d\mathbb{P}}_{\mathcal{G}_t} = \exp \left( -\frac{1}{2} \int_0^t \|\theta_s\|^2 \, ds + \int_0^t \theta_s^\top dW_s \right),$$

for some $\mathcal{G}$-predictable process $\theta$ valued in a closed set $\mathcal{L} \subseteq \mathbb{R}^2$ containing $0$ such that

$$\mathbb{E}^Q \left[ \int_0^\infty e^{-\lambda_D s} \|\theta_s\|^2 \, ds \right] < \infty,$$

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^t \|\theta_s\|^2 \, ds \right) \right] < \infty, \text{ for any } t \geq 0.$$

In what follows, an equivalent measure $Q$ is generated by $\theta$ by the representation in (2.10), and we call $Q$ the $\theta$-measure. The investor intends to minimize the ruin probability under some $Q \in \mathcal{L}$, but the deviation of the measure from $\mathbb{P}$ is penalized by a relative entropy process up to the default time $\tau_D$:

$$H_t(Q|\mathbb{P}) := \mathbb{E}^Q \left[ \log \left( \frac{dQ}{d\mathbb{P}}_{\mathcal{G}_t} \right) \right], \text{ for } t \geq 0.$$
The investor's robust stochastic control problem is then defined by

\[ V(x, y; \varepsilon) := \inf_{\pi \in \mathcal{A}} \sup_{Q \in \mathcal{L}} \left\{ Q(\tau^x_{R}, y, \pi < \tau_D) - \frac{1}{\varepsilon} H_{\tau_D}(Q | \mathbb{P}) \right\}. \]  

Here \( \mathcal{A} \) denotes the set of all admissible controls defined in the following sense.

**Definition 2.5.** \( \pi \in \mathcal{A} \) if \( \pi \) is \( \mathcal{G} \)-predictable and valued in a compact set \( \mathcal{K} \subseteq \mathbb{R}^2 \) such that \((0,0) \in \mathcal{K}\).

**Remark 2.6.** The coefficient \( \varepsilon \) in the penalty term of (2.12) corresponds to the investor's level of model ambiguity about the reference probability \( \mathbb{P} \). For instance, the case \( \varepsilon \to 0 \) implies that

\[ \sup_{Q \in \mathcal{L}} \left\{ Q(\tau^x_{R}, y, \pi < \tau_D) - \frac{1}{\varepsilon} H_{\tau_D}(Q | \mathbb{P}) \right\} \to \mathbb{P}(\tau^x_{R}, y, \pi < \tau_D), \]

which indicates that the investor is completely confident about the probability measure \( \mathbb{P} \). On the other hand, if the agent is extremely uncertain as \( \varepsilon \to \infty \), we get that

\[ \sup_{Q \in \mathcal{L}} \left\{ Q(\tau^x_{R}, y, \pi < \tau_D) - \frac{1}{\varepsilon} H_{\tau_D}(Q | \mathbb{P}) \right\} \to \sup_{Q \in \mathcal{L}} Q(\tau^x_{R}, y, \pi < \tau_D), \]

which reduces to the best case scenario. It is worth noting that the formulation involving the penalty term only works for drift uncertainty. If some plausible probabilities are mutually singular due to volatility uncertainty, i.e. there is no dominating reference probability \( \mathbb{P} \), the entropy cannot be defined as in (2.11). Another interesting issue we can consider in the robust framework is to incorporate the investor's ambiguity attitude towards a given set of plausible priors. Similar to [25], one can employ the alpha-maxmin preference and formulate the ruin probability problem under model uncertainty as

\[ \inf_{\pi \in \mathcal{A}} \left[ \alpha \sup_{Q \in \mathcal{L}} Q(\tau^x_{R}, y, \pi < \tau_D) + (1 - \alpha) \inf_{Q \in \mathcal{L}} Q(\tau^x_{R}, y, \pi < \tau_D) \right]. \]

This formulation allows for both drift and volatility uncertainty and the constant coefficient \( \alpha \in [0,1] \) can represent how much ambiguity averse the investor is. Nevertheless, this problem becomes time inconsistent and we need to look for some equilibrium portfolio strategies instead of the optimal one, which is beyond the scope of this paper and will be left as future research.

**Remark 2.7.** The compactness of \( \mathcal{K} \) in the definition of admissible set \( \mathcal{A} \) can be understood that the investor does not take an extreme strategy and the immediate liquidation is also admissible. Moreover, as \( \pi \) is \( \mathcal{G} \)-predictable, it is also \( \mathcal{F} \)-predictable before \( \tau_D \). Therefore, there is a unique continuous \( \mathbb{P}^\pi \) satisfying (2.1). Thanks to (2.3) and (2.5), \( P^\pi \) is well-defined. More importantly, the compactness of \( \mathcal{K} \) is necessary for the associated HJB equation to be continuous. Otherwise, it becomes difficult to prove the comparison principle for its viscosity solutions because the typical doubling argument relies on Crandall-Ishii’s lemma and the closure of super/sub-jets, which require the compactness of \( \mathcal{K} \). In other words, if the comparison principle is already guaranteed, we can relax the conditions on \( \mathcal{A} \) only with care for \( P^\pi \) to be well-defined.

3. **Dynamic Programming Equation and Main Results.** In this section, we first heuristically derive the HJB equation associated with the value function using dynamic programming argument or martingale optimality principle. For technical reason, when default occurs, we assign a coffin state \( \Delta \) to the underlying process \( Z \). Moreover, for any domain in what follows, we consider its one point compactification and any function \( u \) is extended by assigning \( u(\Delta) = 0 \). Denote the
(Q, G)-Brownian motion by \( W^Q \), where \( Q \) is generated by \( \theta \). For \( t < \tau_D \), (2.8) can be written as

\[
\begin{aligned}
\begin{cases}
    dX^{x,y}_t = [rX^{x,y}_t - c + \pi^T_t (\mu^x + \sigma \theta)] dt + \pi^T_t \sigma dW^Q_t - q^T dM^x_t, & X_0 = x, \\
    dY^{y}_t = -\text{diag}(\pi_t)((\mu^y + \sigma^y \theta) dt + \sigma^y dW^Q_t] + \text{diag}(1 + q) dM^y_t, & Y_0 = y.
\end{cases}
\end{aligned}
\]

To obtain the associated HJB equation, we apply Itô’s formula to a smooth function \( \varphi \) that

\[
d\varphi(Z_t) - \frac{1}{2\varepsilon} \|\theta_t\|^2 dt = \left[ -\lambda \varphi(Z_t) - \varphi(\Delta) \right] dt + (rX_t - c)\varphi_x + A^{\pi_t, \theta_t} \varphi(Z_t) dt \]

\[
+ \sum_{i=1,2} \left( q^1 \varphi_x(Z_t) - (1 + q^1)\varphi_y(Z_t) \right) I_{Y_0 = 0} dM^i_t
\]

\[
+ \left[ \varphi_x(Z_t)\pi^T_t \sigma - \nabla_y \varphi(Z_t)^T \text{diag}(\pi_t)\sigma^y \right] dW^Q_t + [\varphi(\Delta) - \varphi(Z_{t-})] dM^D_t,
\]

where \( \nabla_y \varphi := [\partial_{y^1} \varphi, \partial_{y^2} \varphi]^T \) and

\[
A^{\pi, \theta}(\varphi)(x, y^1, y^2) := -\frac{1}{2\varepsilon} \|\theta\|^2 + b[\pi, \theta]^T \nabla\varphi + \frac{1}{2} \text{Tr}(\Sigma[\pi] \nabla^2 \varphi),
\]

\[
b[\pi, \theta] := \begin{bmatrix} \pi^T(\mu^x + \sigma \theta) \\ -\text{diag}(\pi)((\mu^y + \sigma^y \theta) \end{bmatrix},
\]

\[
\Sigma[\pi] := \begin{bmatrix} A^{x, \theta} & A^{y, \theta} \\ A^{y, \theta} & A^{y, \theta} \end{bmatrix}.
\]

Recall that \( \varphi(\Delta) = 0 \) in (3.2). Moreover, to deduce related boundary conditions, let us recall (2.12) and conclude that \( V(R, y^1, y^2) = 1 \) for any \( y^i \geq 0 \). In addition, if \( X_t = c/r \) at \( t \geq 0 \), the optimal strategy is liquidating the risky position so that \( X_s = c/r \) for any \( s \geq t \). Therefore, \( V(c/r, y^1, y^2) = 0 \) for any \( y^i \geq 0 \). Thus, motivated by these boundary conditions, we need to consider the following regions and boundaries

\[
\mathcal{O} := \{(x, y^1, y^2) : R < x < c/r, y^1 \geq 0, y^2 \geq 0\},
\]

\[
\mathcal{O}^+ := \{(x, y^1, y^2) : R < x < c/r, y^1 > 0, y^2 > 0\},
\]

\[
\partial\mathcal{O}^0 := \{(x, y^1, y^2) \in \mathcal{O} : R < x < c/r, y^1 = 0\},
\]

\[
\partial\mathcal{O}^0 := \{(x, y^1, y^2) \in \mathcal{O} : R < x < c/r, y^1 = 0 or y^2 = 0\},
\]

\[
\partial\mathcal{O}^0 := \{(R, y^1, y^2) : y^1 > 0, y^2 > 0\},
\]

\[
\partial\mathcal{O}^0 := \{(c/r, y^1, y^2) : y^1 > 0, y^2 > 0\}.
\]

Note that \( \partial\mathcal{O} = \partial\mathcal{O}^+ \cup \partial\mathcal{O}^0, \partial\mathcal{O} = \partial\mathcal{O}^+ \cup \partial\mathcal{O}^0 \cup \partial\mathcal{O}^0 \cup \partial\mathcal{O}^0 \), and \( \partial\mathcal{O}^0 = \partial\mathcal{O}^0 \cup \partial\mathcal{O}^0 \). Moreover, for any set \( A \), we let \( \text{cl}(A) \) denote the closure of \( A \) in what follows. We then consider the following operators

\[
\mathcal{F}[\varphi](\mathbf{z}) := A^{\pi} \varphi(\mathbf{z}) - (rX - c)\varphi_x(\mathbf{z}) - \inf_{\pi \in \mathcal{K}, \theta \in \mathcal{L}} A^{\pi, \theta}[\varphi](\mathbf{z}),
\]

\[
\mathcal{B}^i[\varphi](\mathbf{z}) := q^i \varphi_x(\mathbf{z}) - (1 + q^i)\varphi_y(\mathbf{z}), \quad i \in \{1, 2\},
\]

and the associated HJB equation can be (formally) written as

\[
\begin{cases}
    \mathcal{F}[\varphi](\mathbf{z}) = 0, & \text{on } \mathbf{z} \in \mathcal{O}^+,
    \\
    \mathcal{B}^1[\varphi](\mathbf{z}) = 0, & \text{on } \mathbf{z} \in \partial\mathcal{O}^0,
    \\
    \mathcal{B}^2[\varphi](\mathbf{z}) = 0, & \text{on } \mathbf{z} \in \partial\mathcal{O}^0,
    \\
    \varphi(\mathbf{z}) = 1, & \text{on } \mathbf{z} \in \partial\mathcal{O}^R,
    \\
    \varphi(\mathbf{z}) = 0, & \text{on } \mathbf{z} \in \partial\mathcal{O}_{c/r}.\n\end{cases}
\]
Our ultimate goal is to show that the value function $V$ defined in (2.12) is the unique viscosity solution of the HJB equation (3.3).

To this end, we first need to be careful for the boundary conditions on $\partial \Omega^0$, which should be defined using semi-continuous envelope of viscosity solutions. To be precise, we denote the lower (resp. upper) semi-continuous envelope of $B^i$, $i \in \{1, 2\}$, by $B_*$ (resp. $B^*$). On $\partial \Omega^0$, we will consider

$$B_*[\varphi] := \begin{cases} B^1[\varphi], & \text{on } \partial \Omega^0_1 \setminus \partial \Omega^0_2, \\ B^2[\varphi], & \text{on } \partial \Omega^0_2 \setminus \partial \Omega^0_1, \\ \min\{B^1[\varphi], B^2[\varphi]\}, & \text{on } \partial \Omega^0_1 \cap \partial \Omega^0_2, \end{cases}$$

and $B^*$ is defined in the same way by replacing $B_* = \min\{B^1, B^2\}$ using $B^* = \max\{B^1, B^2\}$ on the boundary $\partial \Omega^0_1 \cap \partial \Omega^0_2$. Furthermore, we denote

$$\text{USC}_b(A) := \{\text{bounded u.s.c functions on } A\}, \quad \text{LSC}_b(A) := \{\text{bounded l.s.c functions on } A\}.$$

The precise definition of viscosity sub/super solutions is given as below.

**Definition 3.1 (Viscosity solution).**

(i) $v \in \text{LSC}_b(\text{cl}(O))$ is a viscosity sub-solution of (3.3) if for any test function $\varphi$ such that $z \in O$ is a maximum point of $v - \varphi$ at zero, we have

$$\begin{aligned} 
& \left\{ \begin{array}{ll}
F[\varphi](z) \leq 0, & \text{on } z \in O^+,
\end{array} \right.
\end{aligned}$$

$$\min \{F[\varphi](z), B_*[\varphi](z)\} \leq 0, \quad \text{on } z \in \partial O^0,$$

$$v(z) \leq 1, \quad \text{on } z \in \partial O_R,$$

$$v(z) \leq 0, \quad \text{on } z \in \partial O_{c/r}.$$

(ii) $v \in \text{USC}_b(\text{cl}(O))$ is a viscosity super-solution of (3.3) if for any test function $\varphi$ such that $z \in O$ is a minimum point of $v - \varphi$ at zero, we have

$$\begin{aligned} 
& \left\{ \begin{array}{ll}
F[\varphi](z) \geq 0, & \text{on } z \in O^+,
\end{array} \right.
\end{aligned}$$

$$\max \{F[\varphi](z), B^*[\varphi](z)\} \geq 0, \quad \text{on } z \in \partial O^0,$$

$$v(z) \geq 1, \quad \text{on } z \in \partial O_R,$$

$$v(z) \geq 0, \quad \text{on } z \in \partial O_{c/r}.$$

(iii) $v$ is a viscosity solution of (3.3) if $v$ is both viscosity sub-solution and super-solution.

**Remark 3.2.** The definition of viscosity solutions is inextricably involved with min/max when the boundary conditions are given on derivatives. Consider $(p, X) \in \overline{\mathcal{F}^2_{\Omega}} \varphi(z)$ for some $z \in \partial O^0$, where $\overline{\mathcal{F}^2_{\Omega}}$ denote the closure of the second order superjet/subjet. Then there exists $(z_n, p_n, X_n) \in \mathcal{F}^2_{\Omega}$ such that $(z_n, p_n, X_n) \to (z, p, X)$. However, in this case, we cannot guarantee that $z_n \in \partial O^0$ for any $n \in \mathbb{N}$. For more detailed discussion, readers can refer to Section 7 in [17].

Now, we are ready to state the main result of this paper.

**Theorem 3.3 (The Main Theorem).** The value function $V$, defined at (2.12), is a unique viscosity solution of the HJB equation (3.3).

The proof of the theorem is split into several steps, which will be provided in the next sections. In summary, the first step is to define stochastic sub/super-solutions. We continue to show that supremum (resp. infimum) of stochastic sub-solutions (resp. stochastic super-solutions) is a viscosity super-solution (resp. sub-solution). Then the main theorem can be concluded with the help of the following comparison principle of the HJB equation, whose proof is reported in the next section.
3.3, which actually will using stochastic Perron’s method, which helps us to avoid the lengthy and technical proof of Definition 4.3 (4.2) (3.3)

\[ \text{Definition 4.3 (Stochastic sub-solutions).} \]

We call \((\tau, \xi)\) a random initial condition if \(\tau\) is a \(\mathbb{G}\)-stopping time valued in \([0, \tau_D]\), \(\xi = (\xi^X, \xi^Y, \xi^Z)\) is a \(\mathbb{G}_t\)-measurable random variable valued in \(\mathcal{O} \cup \{\Delta\}\), and \(\xi = \Delta\) if and only if \(\tau = \tau_D\). We denote \(\mathcal{R}\) as the set of all random initial conditions.

4. Stochastic Perron’s Method and Proofs. This section contributes to the proof of Theorem 3.3 using stochastic Perron’s method, which helps us to avoid the lengthy and technical proof of dynamic programming principle. To begin, we first need the concept of random initial conditions and exit times.

\[ \text{Definition 4.1.} \]

We call \((\tau, \xi)\) a random initial condition if \(\tau\) is a \(\mathbb{G}\)-stopping time valued in \([0, \tau_D]\), \(\xi = (\xi^X, \xi^Y, \xi^Z)\) is a \(\mathbb{G}_t\)-measurable random variable valued in \(\mathcal{O} \cup \{\Delta\}\), and \(\xi = \Delta\) if and only if \(\tau = \tau_D\). We denote \(\mathcal{R}\) as the set of all random initial conditions.

\[ \text{Definition 4.2.} \]

The exit time of \(X^{\tau, \xi, \pi}\) from \(\mathcal{O}\), denoted by \(\tau^{\tau, \xi, \pi}_E\), is defined by

\[ \tau^{\tau, \xi, \pi}_E := \inf \{t \geq \tau:\ X_t^{\tau, \xi, \pi} \notin \mathcal{O}\}. \]

4.1. Stochastic Sub-solutions. This subsection first introduces the definition of stochastic sub-solutions of (3.3) and establishes the result that the stochastic envelope of stochastic sub-solutions is a viscosity super-solution of (3.3). In a nutshell, stochastic sub-solutions are functions that become \(\mathbb{G}\)-submartingales by operating on \(Z = (X, Y)\). The purpose of defining the stochastic sub-solutions is to provide one direction of dynamic programming principle to some extent that

\[ \text{Proposition 3.4 (Comparison Principle).} \]

Assume \(u\) and \(v\) be a sub-solution and supersolution of (3.3), respectively and \(u \leq v\) on \(\partial O_R \cup \partial O_{c/R}\). Then \(u \leq v\) in \(\text{cl}(O)\).

\[ \text{Remark 4.4.} \]

Note that we do not impose the oblique-type boundary condition \(\mathcal{B}\) arising from the high-watermark fees in the definition of stochastic sub-solutions. The Dirichlet boundary conditions are from the associated financial problems, namely the ruin probability minimization problem. Such boundary conditions are invariant given the underlying processes, i.e., the same Dirichlet boundary conditions are imposed regardless of the SDE for \(Z = (X, Y)\). However, the oblique-type boundary condition \(\mathcal{B}\) comes from the structure of the process, the running maximum of the process, as \(\mathcal{F}\) does. Therefore, we can deal with \(\mathcal{B}\) and \(\mathcal{F}\) together in the same manner in applying SPM. This in turn shows another advantage of stochastic Perron’s method that is effective to handle control problem with high-watermark fee, especially with multiple hedge funds. Therefore, it is redundant to include the oblique-type boundary condition in Definition 4.3, which actually will
make the argument more complicated because it is difficult to verify that $\mathcal{V}^-$ is closed under the maximum operation with condition $\mathcal{B}$.

Our first task is to find one stochastic sub-solution so that $\mathcal{V}^-$ is not empty. One can think of (4.2) as an upper-bound, in other words, stochastic sub-solution can be found by considering a “better situation”. If there is no fee in reaching the high-watermark, the case is clearly better for the investor. The minimal ruin probability in this frictionless market was already studied by [34, 10], which will turn out to be a stochastic sub-solution in our case. Put

$$
\mathcal{U}(x) := \begin{cases} 
\left(\frac{c-rx}{c-r}\right)^\kappa, & R \leq x \leq c/r, \\
0, & c/r < x,
\end{cases}
$$

$$
\kappa := \frac{1}{2\lambda} \left[ (r + \lambda^D + R) + \sqrt{(r + \lambda^D + R)^2 - 4r\lambda^D} \right],
$$

$$
\Sigma := \frac{1}{2\mu_\Delta} (\sigma\sigma^\top)^{-1} \mu_\Delta.
$$

Before proceeding, note that $\mathcal{U}$ is a solution of the following differential equation:

$$
\begin{cases}
\lambda^D \mathcal{U}(x) + \Sigma [\mathcal{U}'(x)]^2 / \mathcal{U}''(x) + (c - rx) \mathcal{U}'(x) = 0, & R < x < c/r, \\
\mathcal{U}(R) = 1, & \mathcal{U}(c/r) = 0.
\end{cases}
$$

**Lemma 4.5.** Let $\psi^-(x, y) := \mathcal{U}(x)$. Then $\psi^- \in \mathcal{V}^-$.

**Proof.** To prove that $\psi^-$ is a stochastic sub-solution, let us consider an arbitrary random initial condition $(\tau, \xi)$, $\pi \in \mathcal{A}$, and a $\mathcal{G}$-stopping time $\rho \in [\tau, \tau^\xi_E]$. Then we will show that (SB2) in Definition 4.3 is satisfied with the reference measure $\mathbb{P}$. In other words, we choose

$$
\theta = 0
$$

in the representation of (2.10). For the rest of this proof, we omit the super-scripts $\tau, \xi, \pi$ for simplicity. Define a process $(\mathcal{X}_{t})_{t \geq 0}$ given by $\mathcal{X}_{t} = X_{\tau}$, $\mathcal{X}_{\tau_D} := \Delta$, and

$$
d\mathcal{X}_{t} = [r \mathcal{X}_{t} - c + \pi^\top_t \mu_\Delta] dt + \pi^\top_t \sigma dW_{t}, \quad \text{for} \quad t < \tau_D.
$$

In other words, $\mathcal{X}_{t}$ is a process without high-watermark fees, thus $X \leq \mathcal{X}_{t}$ on $[\tau, \tau_E]$. As $\mathcal{U}$ is non-increasing in $[R, \infty)$,

$$
\mathbb{E}[\psi^- (X_{\rho}, Y_{\rho}) | \mathcal{G}_\tau] = \mathbb{E}[1_{\rho < \tau_D} \mathcal{U}(X_{\rho}) | \mathcal{G}_\tau] \geq \mathbb{E}[1_{\rho < \tau_D} \mathcal{U}(\mathcal{X}_{\rho}) | \mathcal{G}_\tau] = \mathbb{E}[\mathcal{U}(X_{\rho}) | \mathcal{G}_\tau]
$$

It suffices to show $\mathbb{E}[\mathcal{U}(X_{\rho}) | \mathcal{G}_\tau] \geq \mathbb{E}[\mathcal{U}(\mathcal{X}_{\rho}) (\psi^- (\xi))]$. We first consider the event $U := \{\mathcal{X}_{\tau} \in [R, c/r] \} \in \mathcal{G}_\tau$ and let $\nu := \inf\{t \geq \tau : \mathcal{X}_{t} \geq c/r\}$. On the event $U$,

$$
\mathbb{E}[\mathcal{U}(X_{\rho}) | \mathcal{G}_\tau] \geq \mathbb{E}[\mathcal{U}(\mathcal{X}_{\rho}) | \mathcal{G}_\tau].
$$

Applying Itô’s formula on the event $U$ yields

$$
\mathcal{U}(\mathcal{X}_{\rho^\wedge\nu}) = \mathcal{U}(\mathcal{X}_{\tau}) + \int_{\tau}^{\rho^\wedge\nu} \left\{ \mathcal{U}'(\mathcal{X}_{t}) ([r \mathcal{X}_{t} - c] + \pi^\top_t \mu_\Delta) + \mathcal{U}''(\mathcal{X}_{t}) \frac{1}{2} \| \sigma\pi_t \| - \lambda^D \mathcal{U}(\mathcal{X}_{t}) \right\} dt
$$

$$
+ \int_{\tau}^{\rho^\wedge\nu} \mathcal{U}'(\mathcal{X}_{t}) \pi^\top_t \sigma dW_t - \int_{\tau}^{\rho^\wedge\nu} \mathcal{U}(\mathcal{X}_{t}) dM^D_t
$$
The $dt$-integral term is non-negative. Moreover, $u$, $u'$, and $\pi$ are bounded, so the local martingale terms are martingales. Therefore, we have

$$\mathbb{I}_U [E\{ u(X_{\rho \wedge \nu}) | \mathcal{G}_r \}] \geq \mathbb{I}_U u(X_r).$$  \hspace{1cm} (4.6)$$

On the other hand, on the event $U^c = \{X_r \in [c/r, \infty) \cup \{\Delta\}\}$, it clearly follows that $u(X_r) = 0 \leq E[u(X_{\rho})|\mathcal{G}_r]$. Therefore, thanks to (4.5)-(4.6), we obtain

$$E[\psi^-(X_\rho, Y_\rho)|\mathcal{G}_r] \geq E[u(X_\rho)|\mathcal{G}_r] \geq E[\mathbb{I}_A u(X_\rho) + \mathbb{I}_A u(X_\rho)|\mathcal{G}_r]$$

$$\geq \mathbb{I}_A u(X_r) = u(X_r)$$

$$\Rightarrow \psi^-(\xi). \hspace{1cm} (4.7)$$

Thus by (4.4), $\psi^-$ satisfies (SB2).

Let $(\tau, \xi) \in \mathcal{R}$, $\pi \in \mathcal{A}$, $\rho$ be a $\mathcal{G}$-stopping time valued in interval $[\tau, \tau^\varepsilon]$. Because $v^1$ and $v^2$ are stochastic sub-solutions, there exist $Q^1$ and $Q^2$ satisfying (SB2). We denote by $\theta^i$, $i \in \{1, 2\}$, the processes that generate $Q^i$. To find the measure satisfying (SB2) for $v^1 \vee v^2$, we define $\mathcal{A} := \{v^1(\xi) > v^2(\xi)\} \in \mathcal{G}_r$ and $\theta := I_{[\tau, \infty]}[I_{\mathcal{A}^1} + I_{\mathcal{A}^2}]$, i.e., on the stochastic interval $[\tau, \infty]$,

$$dQ \bigg|_{\mathcal{G}_r} = \mathbb{I}_{v^1} \frac{dQ^1}{d\mathbb{P}} g_r + \mathbb{I}_{v^2} \frac{dQ^2}{d\mathbb{P}} g_r.$$  \hspace{1cm} (4.8)$$

Moreover, let $Q$ denote the measure generated by $\theta$. Then as $v^1$ is a stochastic sub-solution and $\mathcal{A} \in \mathcal{G}_r$, we have

$$\mathbb{I}_\mathcal{A} v^1(\xi) \leq \mathbb{I}_\mathcal{A} E^{Q^1} \left[ v^1(Z_{\rho}^\tau, \xi) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda_D s} \|\theta_1\|^2 ds \right] \mid \mathcal{G}_r$$

$$= \left( \frac{dQ^1}{d\mathbb{P}} \right)_{\mathcal{G}_r}^{-1} E \mathbb{I}_\mathcal{A} \left[ \frac{dQ^1}{d\mathbb{P}} \right]_{\mathcal{G}_r} \left\{ v^1(Z_{\rho}^\tau, \xi) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda_D s} \|\theta_1\|^2 ds \right\} \mid \mathcal{G}_r$$

$$= E \left[ \frac{dQ^1}{d\mathbb{P}} \right]_{\mathcal{G}_r} \left\{ v^1(Z_{\rho}^\tau, \xi) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda_D s} \|\theta_1\|^2 ds \right\} \mid \mathcal{G}_r$$

$$\leq \mathbb{I}_\mathcal{A} E^{Q^1} \left[ v^1(\xi) \vee v^2(\xi)(Z_{\rho}^\tau, \xi) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda_D s} \|\theta_1\|^2 ds \right] \mid \mathcal{G}_r. $$

The second equality above is obtained by Definition 2.4 and boundness of $v^1$. Similarly, we obtain

$$\mathbb{I}_\mathcal{A} v^2(\xi) \leq \mathbb{I}_\mathcal{A} E^{Q^2} \left[ v^1(\xi) \vee v^2(\xi)(Z_{\rho}^\tau, \xi) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda_D s} \|\theta_1\|^2 ds \right] \mid \mathcal{G}_r. $$

Combining (4.8) and (4.9), we have

$$\frac{v^1 \vee v^2}(\xi) \leq E^{Q} \left[ (v^1 \vee v^2)(Z_{\rho}^\tau, \xi) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda_D s} \|\theta_1\|^2 ds \right] \mid \mathcal{G}_r.$$  \hspace{1cm} (4.9)$$

Thus, $(v^1 \vee v^2)$ satisfies (SB2) with $Q$. \hspace{1cm} \Box
In the next theorem, we will use Lemma 4.6 to construct a “bump” function to argue by contradiction.

**Theorem 4.7.** The lower stochastic envelope of $V^-$,

$$v^- := \sup_{v \in V^-} v,$$  \hspace{1cm} (4.10)

is a viscosity super-solution of (3.3).

**Proof.** Lemma 4.5 already asserts that $v^- \geq \psi^-$. Therefore, we have $v^- \geq 1$ on $\Omega_R$ and $v^- \geq 0$ on $\Omega_{r/R}$. It remains to show that for this $v^-$ and any test function $\varphi$ such that $z \in \Omega$ is a minimum point of $v^- - \varphi$ at zero, we have

\[
\begin{cases}
\mathcal{F}[\varphi](z) \geq 0, & \text{on } z \in \Omega^+,
\max \{ \mathcal{F}[\varphi](z), \mathcal{B}^t[\varphi](z) \} \geq 0, & \text{on } z \in \partial \Omega_0.
\end{cases}
\]

We first show the claim above holds on the boundary part $\partial \Omega_1 \cap \partial \Omega_2$.

Let us consider the region $B_a(z_0)$ of a ball with center $z_0 \in \text{cl}(\Omega)$ and the radius $a$ intersecting with $\Omega$ that

$$B_a(z_0) := \{ z \in \text{cl}(\Omega) : \| z - z_0 \| < a \}.$$

To argue by contradiction, we suppose that there exist $z_0 = (x_0, 0, 0) \in \partial \Omega_1 \cap \partial \Omega_2$ and some $\varphi \in C^2(\Omega)$ such that $v^- - \varphi$ attains its strict minimum of zero at $z_0$ and

$$\max \{ \mathcal{F}[\varphi](z_0), \mathcal{B}^t[\varphi](z_0), \mathcal{B}^2[\varphi](z_0) \} < 0.$$  \hspace{1cm} (4.11)

Using $\varphi$, we will construct a bump function that still is in $V^-$, in which it contradicts to (4.10). By continuity of $\mathcal{F}$ and $\mathcal{B}^t$, $i \in \{1, 2\}$, we can choose a small ball $B_{2a}(z_0)$, $a > 0$, such that for any $z \in \text{cl}(B_{2a}(z_0))$

$$\max \{ \mathcal{F}[\varphi](z), \mathcal{B}^t[\varphi](z), \mathcal{B}^2[\varphi](z) \} < 0.$$  \hspace{1cm} (4.12)

As $v^- - \varphi$ is l.s.c and $\text{cl}(B_{2a}(z_0)) \setminus B_a(z_0)$ is compact, there exists $\delta > 0$ satisfying

$$v^- - \varphi \geq \delta, \quad \text{on } \text{cl}(B_{2a}(z_0)) \setminus B_a(z_0).$$

As a result of Proposition 4.1 in [7] and Lemma 4.6, we can choose a non-decreasing sequence \{\nu_n\} $\subseteq V^-$ such that $\nu_n \searrow v^-$. By Lemma 2.4 in [9], we can pick $v := v_N$ such that

$$v - \varphi \geq \delta/2, \quad \text{on } \text{cl}(B_{2a}(z_0)) \setminus B_a(z_0).$$

Then we further choose $0 < \eta < \delta/2$ small enough such that $\varphi^\eta := \varphi + \eta$ satisfies

$$\max \{ \mathcal{F}[\varphi], \mathcal{B}^t[\varphi], \mathcal{B}^2[\varphi] \} < 0, \quad \text{on } \text{cl}(B_{2a}(z_0)).$$  \hspace{1cm} (4.13)

By this construction, we have

$$\varphi^\eta \leq v - \delta/2 \leq v, \quad \text{on } \text{cl}(B_{2a}(z_0)) \setminus B_a(z_0),$$  \hspace{1cm} (4.14)

$$\varphi^\eta(z_0) = \varphi^\eta(z_0) + \eta = v^-(z_0) + \eta > v^-(z_0).$$  \hspace{1cm} (4.15)

Let us define

$$v^\eta := \begin{cases} v \lor \varphi^\eta, & \text{cl}(B_{2a}(z_0)), \\ v, & \text{otherwise}. \end{cases}$$
Then we will show that \( v^\eta \in \mathcal{V}^\cdot \) and this is a contradiction by (4.10) and (4.15).

To this end, we consider an arbitrary \((\tau, \xi) \in \mathcal{R}, \pi \in \mathcal{A}\), and a \(\mathbb{G}\)-stopping time \(\rho \in \left[\tau, \tau_E^{\xi, \pi}\right]\). Our goal is to find a probability measure satisfying (SB2) for \( v^\eta \). As \( v \) is a stochastic sub-solution, for any strategy \( \pi \) we can find \((\theta^v_{t, \pi})_{t \geq 0}\) producing a probability measure \( Q^v_{\pi, \theta} \in \mathcal{L} \) that satisfies (SB2) for \( v \). Define

\[
\Gamma := \{ \xi \in B_a(z_0) \text{ and } v(\xi) < \varphi^\eta(\xi) \} \subseteq \mathcal{G}_\tau,
\]

and let \( \tau_a \) (resp. \( \xi_a \)) denote the exit time (resp. exit position) of the ball \( B_a(z_0) \), i.e.,

\[
\tau_a := \inf \{ t \in [\tau, \tau_E^{\xi, \pi}] : Z_t^{\tau, \xi, \pi} \notin B_a(z_0) \}, \\
\xi_a := Z_{\tau_a}^{\tau, \xi, \pi}.
\]

As \( \mathcal{A}^{\pi, \theta} \) is convex w.r.t \( \theta \), there exists one \( \theta^{\varphi, \pi} \) such that

\[
(4.16) \quad \theta^{\varphi, \pi} \in \arg \max_{\theta \in \mathcal{L}} \mathcal{A}^{\pi, \theta}[\varphi^\eta].
\]

Let us consider \((\tilde{\theta}_t)_{t \geq 0}\) defined by

\[
\tilde{\theta}^\pi := 1_{[\tau, \infty]}[\theta^{\varphi, \pi} 1_\Gamma + \theta^v_{\pi} 1_{\Gamma_c}]
\]

Note that \( \xi_a \in \partial B_a(z_0) \cup \{ \Delta \} \) and \( (\tau_a, \xi_a) \in \mathcal{R} \). Therefore, for \((\tau_a, \xi_a) \) and \( \pi \in \mathcal{A} \), there exists \( \theta^{v, \pi} \) producing \( Q^{v, \pi} \) given by (2.10) that satisfies (SB2) for \( v \). Then let

\[
(4.17) \quad \theta^\pi := 1_{[0, \tau_a]} \tilde{\theta}^\pi + 1_{[\tau_a, \infty]} \theta^{v, \pi},
\]

and \( Q^\pi \) be the measure by \( \theta^\pi \). Then for any \( \pi \in \mathcal{R} \), we show that \( Q^\pi \) is the measure for \( v^\eta \) to satisfy (SB2) from which we obtain the contradiction.

In particular, we can obtain a contradiction from the place where the measure by \( \theta^{\varphi, \pi} \) is taken.

Ito’s formula on the event \( \Gamma \) yields

\[
\varphi^\eta(Z_{\rho \wedge \tau_a}^{\tau, \xi, \pi}) - \varphi^\eta(Z_t^{\tau, \xi, \pi}) = \int^\rho_{\wedge \tau_a} \mathcal{A}^{\pi, \theta^\pi}[\varphi^\eta] + \frac{1}{2\varepsilon} ||\theta^\pi||^2 - \lambda^D \varphi^\eta + (r X_t^{\tau, \xi, \pi} - c) \varphi^\eta(Z_t^{\tau, \xi, \pi}) \, dt \\
- \sum_{i=1, \ldots, D} \int^\rho_{\wedge \tau_a} \mathcal{B}^i[\varphi^\eta](Z_t^{\tau, \xi, \pi}) \, dM^i_t \\
- \int^\rho_{\wedge \tau_a} \varphi^\eta(Z_t^{\tau, \xi, \pi}) \, dM^D_t \\
+ \int^\rho_{\wedge \tau_a} \left[ \varphi^\eta_Z(Z_t^{\tau, \xi, \pi}) \pi_t^\top \sigma - \nabla_y \varphi^\eta(Z_t^{\tau, \xi, \pi})^\top \text{diag}(\pi_t) \sigma^B \right] \, dW^Q_t.
\]

Since \( \varphi^\eta \) and \( \nabla \varphi^\eta \) are bounded on \( [\tau, \rho \wedge \tau_a] \) and \( \pi \) is valued in \( \mathcal{K} \), the last two terms in (4.18) are \( \mathbb{G} \)-martingales. Moreover, by (4.11),

\[
\mathbb{E}^{Q^\pi} \left[ 1_\Gamma v^\eta \left( Z_{\rho \wedge \tau_a}^{\tau, \xi, \pi} \right) | \mathcal{G}_\tau \right] \geq \mathbb{E}^{Q^\pi} \left[ 1_\Gamma \left\{ \varphi^\eta(Z_{\rho \wedge \tau_a}^{\tau, \xi, \pi}) - \frac{1}{2\varepsilon} \int^\tau_{\rho \wedge \tau_a} ||\theta^\pi||^2 \, dt \right\} | \mathcal{G}_\tau \right] \\
\geq 1_\Gamma \varphi^\eta(Z_\tau^{\tau, \xi, \pi}) = 1_\Gamma \varphi^\eta(\xi) = 1_\Gamma v^\eta(\xi).
\]
Note that at the last equality, we do not exclude the case that \( \tau = \tau_D \), i.e., \( \xi = \Delta \). Recall that on \( \Gamma^c \), we have \( v(\xi) = v^\eta(\xi) \) and \( \theta^\pi = \theta^{v,\pi} \) which is the \((\tau, \xi)\)-optimal control of \( v \). Let \( Q^{v,\pi} \) denote the \( \theta^{v,\pi} \)-measure. By (SB2), it follows that

\[
1_{\Gamma^c} v^\eta(\xi) = 1_{\Gamma^c} v(\xi) \leq \mathbb{E}^{Q^{v,\pi}} \left[ 1_{\Gamma^c} \left\{ v \left( Z_{\rho \wedge \tau_a}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_a \wedge \rho} \| \theta^\pi_t \|^2 \, dt \right\} \, \mathcal{G}_\tau \right] 
\leq \mathbb{E}^{Q^\eta} \left[ 1_{\Gamma^c} \left\{ v^\eta \left( Z_{\rho \wedge \tau_a}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_a \wedge \rho} \| \theta^\pi_t \|^2 \, dt \right\} \, \mathcal{G}_\tau \right].
\]

Hence, we obtain that

\[
(4.19) \quad v^\eta(\xi) \leq \mathbb{E}^{Q^\eta} \left[ v^\eta \left( Z_{\rho \wedge \tau_a}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_a \wedge \rho} \| \theta^\pi_t \|^2 \, dt \right] \mathcal{G}_\tau.
\]

Now, to replace \( \rho \wedge \tau_a \) with \( \rho \) in (4.19), we first consider the event \( \Lambda := \{ \rho > \tau_a \} \in \mathcal{G}_{\tau_a \wedge \rho} \). Since \( v = v^\eta \) at \( \partial B_a(z_0) \) and on \( [\tau_a, \rho] \cap (\Lambda \times \mathbb{R}_+) \), we have \( \theta^\pi = \theta^{v,\alpha,\pi} \). Then denoting by \( Q^{v,\alpha,\pi} \) the \( \theta^{v,\alpha,\pi} \)-measure,

\[
1_\Lambda v^\eta(\xi_a) = 1_\Lambda v(\xi_a) \leq \mathbb{E}^{Q^{v,\alpha,\pi}} \left[ 1_{\Lambda} \left\{ v \left( Z_{\rho}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} \| \theta^{v,\alpha,\pi}_t \|^2 \, dt \right\} \, \mathcal{G}_{\tau_a} \right] 
\leq \mathbb{E}^{Q^\eta} \left[ 1_{\Lambda} \left\{ v^\eta \left( Z_{\rho}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} \| \theta^\pi_t \|^2 \, dt \right\} \, \mathcal{G}_{\tau_a} \right].
\]

Moreover, by (4.19) together with (4.20), we can get

\[
(4.21) \quad v^\eta(\xi) \leq \mathbb{E}^{Q^\eta} \left[ v^\eta \left( Z_{\rho \wedge \tau_a}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho \wedge \tau_a} \| \theta^\pi_t \|^2 \, dt \right] \mathcal{G}_\tau 
= \mathbb{E}^{Q^\eta} \left[ 1_{\Lambda^c} \left\{ v^\eta \left( Z_{\rho}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} \| \theta^\pi_t \|^2 \, dt \right\} + 1_{\Lambda} \left\{ v^\eta(\xi_a) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_a} \| \theta^\pi_t \|^2 \, dt \right\} \right] \mathcal{G}_\tau.
\]

By (4.20), we have

\[
\mathbb{E}^{Q^\eta} \left[ 1_{\Lambda} \left\{ v^\eta(\xi_a) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_a} \| \theta^\pi_t \|^2 \, dt \right\} \right] \mathcal{G}_\tau 
= \mathbb{E}^{Q^\eta} \left[ \mathbb{E}^{Q^\eta} \left[ 1_{\Lambda} \left\{ v^\eta(\xi_a) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_a} \| \theta^\pi_t \|^2 \, dt \right\} \right] \mathcal{G}_\tau \right] 
\leq \mathbb{E}^{Q^\eta} \left[ 1_{\Lambda} \left\{ v^\eta \left( Z_{\rho}^{\tau,\xi,\rho} \right) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} \| \theta^\pi_t \|^2 \, dt \right\} \right] \mathcal{G}_\tau.
\]

Therefore, in view of (4.21) and (4.22), we deduce that \( v^\eta \in \mathcal{V}^- \), which clearly contradicts (4.10). Hence, it follows that \( v^- \) is a viscosity super-solution of (3.3) at \( z_0 \in \partial \mathcal{O}_1 \cap \partial \mathcal{O}_2 \).

We can deal with points in other regions \( z_0 \notin \partial \mathcal{O}_1 \cap \partial \mathcal{O}_2 \) in similar ways. To be more precise, for \( z_0 \in \mathcal{O}^+ \) (resp. \( z_0 \in \partial \mathcal{O}_i \), \( i \in \{ 1, 2 \} \)), we suppose that there exist a function \( \varphi \in C^2(\mathcal{O}) \) such that \( v^- - \varphi \) attains its strict minimum of zero at \( z_0 \) and

\[
\mathcal{F}[\varphi](z_0) < 0
\]

(resp. \( \max(\mathcal{F}[\varphi](z_0), B'[\varphi](z_0)) < 0 \)).

Then, by employing similar contradiction arguments, we can conclude that \( v^- \) is indeed a viscosity super-solution of (3.3). \( \square \)
4.2. Stochastic Super-solutions. Roughly speaking, stochastic super-solutions can be defined to facilitate the derivation of the other direction of DPP as

\[
\inf_{\pi \in \mathcal{A}} \sup_{Q \in \mathcal{L}} E^Q \left[ V(Z_{\rho}^{T,\xi,\pi}) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda s} \|\theta_s\|^2 \, ds \bigg| \mathcal{G}_\tau \right] \leq V(\xi).
\]

Note that the item (SP2) in the next definition is precisely motivated by the inequality above.

**Definition 4.8 (Stochastic super-solutions).** If \( v \in \text{USC}_b(\text{cl}(O)) \) satisfies

(SP1) \( v \geq 1 \) on \( \partial \mathcal{O}_R \) and \( v \geq 0 \) on \( \partial \mathcal{O}_{c,R} \),

(SP2) for any random initial condition \((\tau, \xi)\), there exists \( \pi \in \mathcal{A} \) such that for any \( \mathcal{G}\)-stopping time \( \rho \in [\tau, \tau^{\xi,\pi}_R] \) and \( Q \in \mathcal{L} \),

\[
E^Q \left[ v(Z_{\rho}^{\tau,\xi,\pi}) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} e^{-\lambda s} \|\theta_s\|^2 \, ds \bigg| \mathcal{G}_\tau \right] \leq v(\xi),
\]

where \( v \) in (4.2) is understood as its extension to \( \text{cl}(O) \cup \{\Delta\} \) by allocating \( v(\Delta) = 0 \).

Then \( v \) is called a stochastic super-solution of (3.3). In addition, we let \( \mathcal{V}^+ \) denote the class of all stochastic super-solutions of (3.3).

We can find a stochastic super-solution by considering a “worse scenario”. Consider a situation that the investor does not invest in the hedge funds, i.e., \( \pi = 0 \). Then, the investor’s wealth follows \( dX_t = [rX_t - c] \, dt \), \( X_0 = x \). We thus, can obtain that

\[
\psi(x) := \mathbb{P}(\tau^x_{\rho} < \tau_D) = \left( \frac{e^{-\lambda x} c - r x}{c - r\rho} \right)^{\frac{\rho}{\tau}}.
\]

**Lemma 4.9.** Let \( \psi^+(x, y) := \psi(x) \). Then \( \psi^+ \in \mathcal{V}^+ \).

**Proof.** It is obvious that \( \psi^+ \in \text{USC}_b(\text{cl}(O)) \) and satisfies (SP1). Let \((\tau, \xi)\) be a random initial condition and we choose \( \pi = 0 \) for the strategy. Thus, for \( \tau < \tau_D \),

\[
dX_t^{\tau,\xi,\pi} = [rX_t^{\tau,\xi,\pi} - c] \, dt.
\]

Consider \( \rho \in [\tau, \tau^{\xi,\pi}_R] \) as a \( \mathcal{G}\)-stopping time. In the rest of the proof, we suppress the superscripts \( \tau, \xi, \pi \). By Itô’s formula, we have

\[
\psi(X_\rho) - \psi(X_\tau) = \int_{\tau}^{\rho} \left\{ \psi'(X_t)[rX_t - c] - \lambda^D \psi(X_t) \right\} \, dt - \int_{\tau}^{\rho} \psi(X_{s-}) \, dM^D_s \\
= - \int_{\tau}^{\rho} \psi(X_{s-}) \, dM^D_s
\]

As for any equivalent probability measure \( Q \) given by (2.10), \( M^D \) is \((Q, \mathcal{G})\)-martingale, it follows that \( E^Q(\psi(X_\rho) | \mathcal{G}_\tau) = \psi(X_\tau) \) for any \( Q \in \mathcal{L} \). Therefore, for any \( \theta \)-measure \( Q \in \mathcal{L} \),

\[
E^Q \left[ \psi^+(Z_\rho) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} \|\theta_t\|^2 \, dt \bigg| \mathcal{G}_\tau \right] \leq E^Q \left[ \psi(Z_\rho) - \frac{1}{2\varepsilon} \int_{\tau}^{\rho} \|\theta_t\|^2 \, dt \bigg| \mathcal{G}_\tau \right] \leq E^Q(\psi(X_\rho) | \mathcal{G}_\tau) = \psi(X_\tau) = \psi^+(\xi).
\]

Therefore, \( \psi^+ \) satisfies (SP2), and we can deduce that \( \psi^+ \in \mathcal{V}^+ \). 

As in the previous section, we need to show \( \mathcal{V}^+ \) is stable under minimum operation. The proof follows closely the argument to prove Lemma 4.6, so we omit it.
Lemma 4.10. If \( v^1, v^2 \in \mathcal{V}^+ \), then \( v^1 \wedge v^2 \in \mathcal{V}^+ \).

Then Lemma 4.10 will be used to construct a bump function in the following theorem.

Theorem 4.11. The lower stochastic envelope of \( \mathcal{V}^+ \),

\[
v^+ := \inf_{v \in \mathcal{V}^+} v,
\]

is a viscosity sub-solution of (3.3).

Proof. By Lemma 4.9, \( v^+ \leq \psi^+ \). Therefore, we have \( v^+ \leq 1 \) on \( \mathcal{O}_R \) and \( v^+ \leq \chi_{\mathcal{O}_R^C} \). As in the proof of Theorem 4.7, it is sufficient to verify the sub-solution property of \( v^+ \) on the boundary part \( \partial \mathcal{O}_R^0 \cap \partial \mathcal{O}_R^2 \). Using the same notation of balls that intersect \( \mathcal{O} \), we again will prove by contradiction. Suppose that there exist \( z_0 = (x_0, 0, 0) \in \partial \mathcal{O}_R^0 \cap \partial \mathcal{O}_R^2 \) and \( \varphi \in C^2(\mathcal{O}) \) such that \( \varphi^+ - \varphi \) attains its strict maximum of zero at \( z_0 \) and

\[
\min \{ \mathcal{F}[\varphi](z_0), \mathcal{B}^1[\varphi](z_0), \mathcal{B}^2[\varphi](z_0) \} > 0.
\]

Again, as in the construction of a bump function in Theorem 4.7, we can choose \( \eta > 0, a > 0 \) and \( v \in \mathcal{V}^+ \) such that

\[
\begin{align*}
\varphi^n = \varphi + \eta & \geq v, & \text{on } \overline{\text{cl}(B_{2a}(z_0))} \setminus B_a(z_0), \\
\min \{ \mathcal{F}[\varphi^n], \mathcal{B}^1[\varphi^n], \mathcal{B}^2[\varphi^n] \} > 0, & \text{on } \overline{\text{cl}(B_{2a}(z_0))}, \\
\varphi^n(z_0) < v^-(z_0),
\end{align*}
\]

and we define

\[
v^n := \begin{cases} v \wedge \varphi^n, & \text{on } \overline{\text{cl}(B_{2a}(z_0))} \\
v, & \text{otherwise} \end{cases}
\]

Then we will show that \( v^n \in \mathcal{V}^+ \). To show that \( v^n \) satisfies (SP2), let \( (\tau, \xi) \in \mathcal{R} \). Since \( v \in \mathcal{V}^+ \), choose \( \pi^+ \) for \( v \) to satisfy (SP2) and by (4.26) choose \( \pi^- \) such that

\[
\lambda^D \varphi^0 - (rx - c) \varphi^0 - \sup_{\theta \in \mathcal{L}} \mathcal{A}^\pi^+, \theta [\varphi^n] > 0, \quad \text{where } (x, y^1, y^2) \in \overline{\text{cl}(B_{2a}(z_0))}.
\]

Let us denote

\[
\Gamma := \{ \xi \in B_a(z_0) \text{ and } v(\xi) < \varphi^n \},
\]

and define \( \bar{\pi} := 1_{[\tau, \infty]} [\pi^+ \mathbf{1} + \pi^- \mathbf{1} \tau] \). Let \( \tau_a \) (resp. \( \xi_a \)) denote the exit time (resp. exit position) of the ball \( B_a(z_0) \). Since \( (\tau_a, \xi_a) \in \mathcal{R} \) and \( v \in \mathcal{V}^+ \), we can choose \( \pi^+, \pi^- \in \mathcal{A} \) such that for any \( Q \in \mathcal{L} \) and \( \mathcal{G} \)-stopping time valued in \( \tau_a, \tau_{E, \xi, \pi} \), \( v \) satisfies (SP2). Finally, we let

\[
\pi := 1_{[0, \tau_a]} \bar{\pi} + 1_{[\tau_a, \infty]} [\pi^+, \pi^-].
\]

We will show that \( v^n \), with \( \pi \), satisfies (SP2). Consider an arbitrary \( \mathcal{G} \)-stopping time \( \rho \in [\tau, \tau_{E, \xi, \pi}] \) and \( \theta \)-measure \( Q \in \mathcal{L} \). Applying Itô’s formula on the event \( \Gamma \) yields, for any \( \theta \)-measure \( Q \),

\[
\begin{multline}
\varphi^n(Z_{\rho \wedge \tau}^\xi - Z_\tau^\xi) - \varphi^n(Z_{\tau}^\xi - Z_{\tau}^\xi) = \\
\int_\tau^{\rho \wedge \tau} \left[ \mathcal{A}^\pi, \theta [\varphi^n] + \frac{1}{2} ||v_t||^2 - \lambda^D \varphi^n + (r X_t^\xi, \xi - c) \varphi^n \right] (Z_t^\xi - Z_t^\xi) \, dt \\
- \sum_{i=1,2} \int_\tau^{\rho \wedge \tau} \mathcal{B}^i[\varphi^n] (Z_t^\xi) \, dM^i_t \\
- \int_\tau^{\rho \wedge \tau} \varphi^n(Z_t^\xi - Z_t^\xi) \, dM^D_s \\
+ \int_\tau^{\rho \wedge \tau} \left[ \varphi^n(Z_t^\xi, \xi_t) \sigma_t - \nabla_y \varphi^n(Z_t^\xi, \xi_t) \, \text{diag} \left( \text{diag} \left( \mathcal{A}^\pi \right) \right) \, dW^Q_t \right].
\end{multline}

Therefore, by (4.27), we have
\[
E^Q \left[ 1_{\Gamma^c} \veta^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) | \mathcal{G}_\tau \right] \geq E^Q \left[ 1_{\Gamma^c} \left\{ \varphi^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_c + \rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_\tau \\
\geq E^Q \left[ 1_{\Gamma^c} \varphi^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_c + \rho} \| \theta_t \|^2 \, dt \right] | \mathcal{G}_\tau \\
= 1_{\Gamma^c} \varphi^n(\xi).
\]
Recall that on \( \Gamma^c \), we have \( v(\xi) = \varphi^n(\xi) \) and \( \pi = \pi^v \). Since \( v \) is a stochastic super-solution by its construction, we have
\[
1_{\Gamma^c} \veta^n(\xi) = 1_{\Gamma^c} v(\xi) \geq E^Q \left[ 1_{\Gamma^c} \left\{ v(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_c + \rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_\tau \\
\geq E^Q \left[ 1_{\Gamma^c} \left\{ v^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_c + \rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_\tau.
\]
Thus, we deduce that
\[
v^n(\xi) \geq E^Q \left[ v^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau}^{\tau_c + \rho} \| \theta_t \|^2 \, dt \right] | \mathcal{G}_\tau.
\]
To replace \( \rho \wedge \tau_0 \) with \( \rho \), consider \( \Lambda := \{ \rho > \tau_0 \} \in \mathcal{G}_{\tau_0, \rho} \). Recall that \( v = v^n \) at \( \partial B(z_0) \) and on \( \{ \tau_0, \rho \} \cap (\Lambda \times \mathbb{R}_+) \), we have \( \pi = \pi^{v^n, a} \). It then follows that
\[
1_{\Lambda} \veta^n(\xi_\Lambda) = 1_{\Lambda} v(\xi_\Lambda) \geq E^Q \left[ 1_{\Lambda} \left\{ v(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau_0}^{\rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_{\tau_0} \\
\geq E^Q \left[ 1_{\Lambda} \left\{ v^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau_0}^{\rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_{\tau_0}.
\]
By (4.31), one can derive that
\[
E^Q \left[ 1_{\Lambda} \left\{ v^n(\xi_\Lambda) - \frac{1}{2\varepsilon} \int_{\tau_0}^{\rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_{\tau_0} = E^Q \left[ E^Q \left[ 1_{\Lambda} \left\{ v^n(\xi_\Lambda) - \frac{1}{2\varepsilon} \int_{\tau_0}^{\rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_{\tau_0} \right] \\
\geq E^Q \left[ 1_{\Lambda} \left\{ v^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau_0}^{\rho} \| \theta_t \|^2 \, dt \right\} \right] | \mathcal{G}_{\tau_0}.
\]
Therefore, thanks to (4.31) and (4.32), the inequality holds that
\[
1_{\Lambda} \veta^n(\xi) \geq 1_{\Lambda} E^Q \left[ v^n(\zeta^{\tau_c,\xi,\pi}_{\rho,\tau_0}) - \frac{1}{2\varepsilon} \int_{\tau_0}^{\rho} \| \theta_t \|^2 \, dt \right] | \mathcal{G}_{\tau_0}.
\]
We can obtain the inequality on \( \Lambda^c \) in the similar fashion as in the proof of Theorem 4.7. Hence, it can be shown that \( v^n \in V^+ \), which contradicts (4.26) and our claim holds.

**4.3. Proof of Comparison Principle.** Comparison principle with either Neumann or oblique-type boundary conditions was already studied; see, for example, [2, 3]. However, because we have both Dirichlet and oblique-type boundary conditions in our problem, some tailor made arguments need to be developed here.

We plan to apply a typical doubling argument, nevertheless, the additional difficulty by considering oblique-type conditions is that we need to construct a test function with care. We will choose a test function in a way that \( \mathcal{B} \neq 0 \) in a viscosity sense. Then by the definition of viscosity solution,
the test function should satisfy $\mathcal{F} = 0$ and this in turn will provide a contradiction. In what follows, we denote $q^1 := [q^1_1, -1 - q^1_2, 0]^\top$ and $q^2 := [q^2_1, 0, -1 - q^2_2]^\top$.

To explain the idea to choose a test function, let $z, z' \in \mathbb{R}^3$. As always, to push the variables into a diagonal entry, we need $\|z - z'|^2/\alpha$ for some $\alpha > 0$, in the test function. Moreover, since the domain $\mathcal{O}$ is not bounded, for the test function to have a maximum in a compact set, one may want to put $\frac{\beta}{2}(\|z\|^2 + \|z'\|^2)/2$ for some $\beta > 0$. If we stop here, the test function may or may not satisfy $\mathcal{B}^i, i \in \{1, 2\}$. To be more precise, for $z \in \partial \mathcal{O}^0$ or $z' \in \partial \mathcal{O}^0$, we can not guarantee that

$$
\nabla \left[ \frac{1}{\alpha} \|z - z'|^2 + \frac{\beta}{2}(\|z\|^2 + \|z'\|^2) \right] \cdot q^i > 0, \quad i \in \{1, 2\}.
$$

To eliminate the possibility to satisfy $\mathcal{B}^i$, i.e., to focus on $\mathcal{F}$, we seek to remedy the test function to meet (4.34). To this end, pick any $\nu^i > 0, i \in \{1, 2\}$, and choose $z_\nu := (R, \nu^1, \nu^2)$. Then for any $z = (x, 0, y^2) \in \partial \mathcal{O}^0_1$, we have $(z - z_\nu) \cdot q^1 = (x - R)q^1 + \nu^1(1 + q^1) > 0$. Likewise, we also have $(z - z_\nu) \cdot q^2 > 0$ for any $z \in \partial \mathcal{O}^0_2$. Therefore, instead of $\frac{\beta}{2}(\|z\|^2 + \|z'\|^2)/2$, we put

$$
\chi_{\beta}(z, z') := \frac{\beta}{2} \|z - z_\nu\|^2 + \frac{\beta}{2} \|z' - z_\nu\|^2.
$$

However, the effect of (4.34) is offset by the derivative of $\|z - z'|^2/\alpha$. Thus, to remove the derivative, we add additional terms and define

$$
\zeta_\alpha(z, z') := \frac{\|z - z'|^2}{2\alpha} + \sum_{i \in \{1, 2\}} \left\{ C^i_\alpha(z, z')[d^i(z) - d^i(z')] + \frac{\|q^i\|^2}{2\alpha(n^i \cdot q^i)^2}[d^i(z) - d^i(z')]^2 \right\},
$$

$$
C^i_\alpha(z, z') := (z - z') \cdot q^i/(\alpha n^i \cdot q^i),
$$

$$
d^i(z) := \text{dist}(z, \partial \mathcal{O}^0_i),
$$

$$
n^i := [0, -1, 0]^\top, \quad n^2 := [0, 0, -1]^\top.
$$

Note that $\nabla d^i = -n^i, i \in \{1, 2\}$. Then we define $\Psi_{\alpha, \beta} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ as

$$
(4.35) \quad \Psi_{\alpha, \beta}(z, z') := u(z) - v(z') - \psi_{\alpha, \beta}(z, z'),
$$

$$
(4.36) \quad \psi_{\alpha, \beta}(z, z') := \zeta_\alpha(z, z') + \chi_{\beta}(z, z').
$$

Now, we check some properties of $\psi$ by straightforward calculations. First, we can derive that

$$
\nabla_z \psi_{\alpha, \beta}(z, z') = \alpha^{-1}(z - z') + \sum_{i \in \{1, 2\}} \left\{ - C^i_\alpha(z, z') n^i + q^i(\alpha n^i \cdot q^i)^{-1}[d^i(z) - d^i(z')] \right\}
$$

$$
- \frac{\|q^i\|^2}{\alpha(n^i \cdot q^i)^2}[d^i(z) - d^i(z')]n^i + \beta(z - z_\nu)
$$

$$
- \frac{q^1 q^2}{\alpha(1 + q^1)(1 + q^2)} \left[ \sum_{i \in \{1, 2\}} \{d^i(z) - d^i(z')\} \right] [n^1 + n^2],
$$

$$
\nabla_z \psi_{\alpha, \beta}(z, z') = \alpha^{-1}(z' - z) + \sum_{i \in \{1, 2\}} \left\{ C^i_\alpha(z, z') n^i - q^i(\alpha n^i \cdot q^i)^{-1}[d^i(z) - d^i(z')] \right\}
$$

$$
+ \frac{\|q^i\|^2}{\alpha(n^i \cdot q^i)^2}[d^i(z) - d^i(z')]n^i + \beta(z' - z_\nu)
$$

$$
+ \frac{q^1 q^2}{\alpha(1 + q^1)(1 + q^2)} \left[ \sum_{i \in \{1, 2\}} \{d^i(z) - d^i(z')\} \right] [n^1 + n^2].
$$
Then we can observe that
\[
\nabla_z \alpha(z, z') = -\nabla_{z'} \alpha(z, z'),
\]
\[
\nabla_z \alpha, \beta(z, z') = -\nabla_{z'} \alpha, \beta(z, z') + \beta(z - z_v) + \beta(z' - z_v).
\]
Moreover, for any \( z \in \mathcal{O}, \ i \neq j, \ z_j \in \partial \mathcal{O}_j, \)
\[
(4.39) \quad \nabla_z \alpha, \beta(z_j, z) \cdot q^j = \beta(z_j - z_v) \cdot q^j + \frac{q^j q^2}{\alpha(1 + q^2)} q^j(z) > 0,
\]
\[
(4.40) \quad \nabla_{z'} (-\alpha, \beta)(z_j, z) \cdot q^j = -\beta(z_j - z_v) \cdot q - \frac{q^j q^2}{\alpha(1 + q^2)} q^j(z) < 0.
\]
(4.39)-(4.40) will be used later in the proof of Proposition 3.4. In addition, from (4.37) - (4.38), the second order derivative of \( \psi \) is obtained. Let
\[
A := I_3 + \sum_{i \in \{1, 2\}} \left\{ \frac{||q^i||^2 n^i (n^i)^\top}{(n^i \cdot q^i)^2} - \frac{n^i (q^i)^\top + q^i (n^i)^\top}{(n^i \cdot q^i)} \right\} + \frac{q^1 q^2}{(1 + q^1)(1 + q^2)} [n^1 + n^2][n^1 + n^2]^{\top},
\]
where \( I_3 \) is the \( 3 \times 3 \)-identity matrix. If \( q^i, i \in \{1, 2\} \), are not too big, we clearly have \( A \succeq 0 \). Then we can write
\[
\nabla^2 \alpha, \beta(z, z') = \frac{1}{\alpha} \begin{bmatrix} A & -A \\ -A & A \end{bmatrix} + \beta \begin{bmatrix} I_3 & 0 \\ 0 & I_3 \end{bmatrix}.
\]
We are ready to prove the comparison principle.

**Proof of Proposition 3.4.** We argue by contradiction. To this end, we suppose that for some \( z_e \in \text{cl}(\mathcal{O}), u(z_e) - v(z_e) = \delta > 0 \). Let us choose \( \beta \) small enough such that \( \delta > \chi_\beta(z_e, z_e) \), and choose \( \{\alpha_n\}_{n \in \mathbb{N}} \) such that \( \alpha_n \downarrow 0 \). Denote \( \Psi_n := \psi_{\alpha_n, \beta} \). As \( u \) and \( v \) are bounded, \( \chi_\beta \) dominates \( u - v \) outside a compact set. Therefore, for each \( n \in \mathbb{N}, \Psi_n \) has its maximum on \( \text{cl}(\mathcal{O}) \times \text{cl}(\mathcal{O}) \) in a compact set and we denote the maximal point by \( (z_n, z'_n) \), i.e.,
\[
\Psi_n(z_n, z'_n) = \sup_{(z, z') \in \text{cl}(\mathcal{O}) \times \text{cl}(\mathcal{O})} \Psi_n(z, z').
\]
The maximal point \( (z_n, z'_n) \) actually depends on \( \beta \) but we drop it for simplicity. As \( \{z_n, z'_n\}_{n \geq 1} \) lie in a compact set, we choose a convergent subsequence, still denoted by \( (z_n, z'_n) \), such that
\[
(z_n, z'_n) \to (z, z') = (\overline{x}, \overline{y}, \overline{x'}, \overline{y'}).
\]
As \( u \leq v \) on \( \partial \mathcal{O}_R \cup \partial \mathcal{O}_{c/R}, (z, z') \) must be in \( \mathcal{O} \times \mathcal{O} \). The previous assumption yields that
\[
\Psi_n(z_n, z'_n) \geq \sup_{z \in \text{cl}(\mathcal{O})} [u(z) - v(z) - \chi_\beta(z, z)] \geq \delta - \chi_\beta(z_e, z_e) > 0.
\]
Therefore, it follows that
\[
\zeta_{\alpha_n}(z_n, z'_n) \leq u(z_n) - v(z'_n) - \chi_\beta(z_n, z'_n) - \sup_{z \in \text{cl}(\mathcal{O})} [u(z) - v(z) - \chi_\beta(z, z)].
\]
In view that the right hand side is bounded above but \( \alpha_n \to 0 \) as \( n \to \infty \), \( (\overline{x}, \overline{y}) = (\overline{x'}, \overline{y'}) \). Moreover, the fact that \( u - v \) is u.s.c implies that
\[
0 \leq \limsup_{n \to \infty} [u(z_n) - v(z'_n) - \chi_\beta(z_n, z'_n) - \sup_{z \in \text{cl}(\mathcal{O})} [u(z) - v(z) - \chi_\beta(z, z)]] \leq 0.
\]
Hence, \( \lim_{n \to \infty} \zeta_n(z_n, z'_n) = 0 \).

By Crandall-Ishii’s lemma, for large \( n \in \mathbb{N} \), there exist \( A_n, B_n \in \mathcal{S}^3 \) such that
\[
(\nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n), A_n) \in \mathcal{F}^{\mathcal{D}^+}_0 u(z_n), \quad (-\nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n), B_n) \in \mathcal{F}^{\mathcal{D}^-}_0 v(z'_n)
\]
and that
\[
(4.41) \quad -\frac{10}{\alpha_n} \begin{bmatrix} 1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{bmatrix} < \begin{bmatrix} A_n & 0 & 0 \\
0 & -B_n & 0 \\
0 & -1 & 0 \end{bmatrix} + \frac{10}{\alpha_n} \begin{bmatrix} 1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 0 \end{bmatrix} + 2\beta \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}.
\]

We can calculate that
\[
\nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n) = \nabla_{z} \zeta_{\alpha_n}(z_n, z'_n) + \beta(z_n - z'_n)
\]
\[
-\nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n) = \nabla_{z} \zeta_{\alpha_n}(z_n, z'_n) - \beta(z'_n - z_n),
\]
Let \( F \) be the function such that \( F|c|(z) = F(z, \varphi(z), \nabla \varphi(z), \nabla^2 \varphi(z)) \). Then we have
\[
\lambda^D(u(z_n) - v(z'_n)) = F(z_n, u(z_n), \nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n), A_n) - F(z_n, v(z'_n), \nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n), A_n)
\leq F(z'_n, v(z_n), \nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n), B_n) - F(z_n, v(z'_n), \nabla_{z} \psi_{\alpha_n, \beta}(z_n, z'_n), A_n)
\leq F(z'_n, v(z_n), \nabla_{z} \zeta_{\alpha_n, \beta}(z_n, z'_n), B_n + 2\beta I_3)
-F(z_n, v(z'_n), \nabla_{z} \zeta_{\alpha_n, \beta}(z_n, z'_n), A_n - 2\beta I_3) + c(\beta),
\]
where \( c(\beta) \) is the modulus of continuity. The last inequality of (4.42) is obtained by the compactness of \( K \). By (4.41), we moreover, have \( A_n - 2\beta I_3 \prec B_n + 2\beta I_3 \). Therefore, we obtain
\[
F(z'_n, v(z_n), \nabla_{z} \zeta_{\alpha_n, \beta}(z_n, z'_n), B_n + 2\beta I_3)
\leq F(z_n, v(z'_n), \nabla_{z} \zeta_{\alpha_n, \beta}(z_n, z'_n), A_n + 2\beta I_3).
\]
By (4.42) and (4.43), taking \( n \uparrow \infty \) leads to \( \lambda^D \delta \leq c(\beta) \). Again taking \( \beta \downarrow 0 \), we have the desired contradiction, which completes the proof.

4.4. Proof of Theorem 3.3. Finally, we are ready to prove our main result of Theorem 3.3.

Proof of Theorem 3.3. Theorem 4.7, Theorem 4.11, together with Proposition 3.4 imply that \( v^+ \leq v^- \). Therefore, it suffices to show \( v^- \leq V \leq v^+ \). To show the first inequality, let us consider an arbitrary \( \phi \in \mathcal{V}^- \). It is obvious that \( \phi \leq V \) on \( \partial \mathcal{O}_R \cup \partial \mathcal{O}_{c/f} \). Let \((x, y) \in \mathcal{O}\) and take the random initial condition as \( \tau = 0 \) and \( \xi = (x, y) \). We fix some \( \pi \in \mathscr{R} \) and the hitting time defined by
\[
\tau^x_{c/f} := \inf \{ t \geq 0 : X^x_{t} \geq c/t \}.
\]
As there exists \( \theta \)-generated measure \( Q \) for \( \phi \) to satisfy (SB2), it follows that
\[
\phi(x, y) \leq \mathbb{E}^Q \left[ \phi(Z^x_{\tau^x_{c/f}, \pi}) - \frac{1}{2a} \int_{\tau^x_{c/f}}^{\tau^x_{c/f}} e^{-\lambda D s} \|\theta_s\|^2 ds \mid \mathcal{G}_{\tau} \right]
\leq \mathbb{E}^Q \left[ 1_{\tau^x_{c/f}, \pi = \tau^x_{c/f}, \pi} - \frac{1}{2a} \int_{\tau^x_{c/f}}^{\tau^x_{c/f}} e^{-\lambda D s} \|\theta_s\|^2 ds \mid \mathcal{G}_{\tau} \right] + 1_{\tau^x_{c/f}, \pi = \tau^x_{c/f}, \pi}.
\]
Moreover, we have
\[
\mathbb{E}^Q[1_{\tau^x_{c/f}, \pi = \tau^x_{c/f}, \pi}] = \mathbb{Q}[\tau^x_{c/f, \pi} < \tau_D \wedge \tau^x_{c/f, \pi}] \leq \mathbb{Q}[\tau^x_{c/f, \pi} < \tau_D].
\]
By combining (4.44) and (4.45), we have $\phi(x, y) \leq V(x, y)$, together with (4.10) yield $v^- \leq V$. In a similar fashion, we can show $V \leq v^+$ as well. Because $v^-$ is a viscosity super-solution, by Proposition 3.4, we have $v^+ \leq v^-$. It follows that $v^- \leq V \leq v^+ \leq v^-$, which readily implies our desired equality $v^- = V = v^+$ and hence the value function is the unique viscosity solution of the HJB equation (3.3).

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