Flux-noise spectra around the Kosterlitz-Thouless transition for two-dimensional superconductors

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The flux-noise spectra around the Kosterlitz-Thouless transition are obtained from simulations of the two-dimensional resistively shunted junction model. In particular the dependence on the distance \( d \) between the pick-up coil and the sample is investigated. The typical experimental situation corresponds to the large-\( d \) limit and a simple relation valid in this limit between the complex impedance and the noise spectra is clarified. Features, which distinguish between the large- and small-\( d \) limit, are identified and the possibility of observing these features in experiments is discussed.

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I. INTRODUCTION

Spontaneously created vortices drive the transition between the superconducting and normal state for thin-film superconductors and two-dimensional (2D) Josephson-Junction arrays (JJAs). This means that the physics of the vortices is responsible for the characteristic features in a region around the transition. One manifestation of this is the static characteristics of the phase transition, which is of the Kosterlitz-Thouless (KT) type. Another manifestation is the dynamical features around the transition. These are reflected in the flux-noise spectra and the complex impedance. In the present paper we investigate the flux-noise spectra through computer simulations of the resistively shunted Josephson junction (RSJ) model on a 2D square lattice. In particular we clarify the relation between the flux-noise spectra and the complex impedance.

There have been a number of recent experimental and theoretical studies dealing with the flux-noise spectra. The typical experimental setup measures the fluctuation of the magnetic flux through the pick-up coil situated at a distance above the sample. Many simulation studies on the other hand have measured the vorticity fluctuation associated with a fixed area of the sample itself. It has been assumed that this would roughly correspond to the magnetic flux spectra of the experiments. However, in the present investigation we show that there are significant differences. The typical experimental situation corresponds to the limit of large distance between the pick-up coil and the sample. In this limit there exists a simple relation between the flux-noise spectra and the complex impedance of the sample, which we here verify both directly from the simulations and through analytic calculations.

In Sec. II we describe how the flux-noise spectrum is obtained from simulations of the RSJ model. Section III clarifies the relation between the flux-noise spectrum and the complex impedance. The results from the simulations are described and discussed in Sec. IV. Particular attention is given to the implication for experimental measurements of flux noise and the complex impedance. Finally Sec. V contains some concluding remarks.

II. RSJ MODEL AND FLUX NOISE

A. RSJ model and numerical method

In our simulations we use the 2D RSJ model on a square lattice with periodic boundary conditions (PBC). This is usually assumed to be a good model of a Josephson junction array (JJA). In particular, from the point of view of vortex fluctuations and vortex dynamics, it is expected to have the same physics as a thin superconducting film, as well as a JJA. There might however be some differences in the level of vortex fluctuations around the transition.

The RSJ model incorporates the condition of the local current conservation and the equations of motion can be written as

\[
\dot{\theta}_i = - \sum_j G_{ij} \sum_k (\sin \phi_{jk} + \eta_{jk}),
\]

where \( G_{ij} \) is the square lattice Green function, the primed summation is over the four nearest neighbors of the site \( j \), and \( \phi_{jk} \equiv \theta_j - \theta_k \) with the phase \( \theta_j \) of the complex order parameter at site \( j \). Here we measure time \( t \) in units of \( \hbar/2eRI_c \), where \( R \) is the shunt resistance and \( I_c \) is the critical current of a single junction. The thermal noise current \( \eta_{jk} \) in units of \( I_c \) satisfies the conditions

\[
\langle \eta_{ij}(t) \rangle = 0 \quad \text{and} \quad \langle \eta_{ij}(t)\eta_{kl}(0) \rangle = 2T(\delta_{ik}\delta_{ji} - \delta_{il}\delta_{jk})\delta(t),
\]

where \( \langle \cdots \rangle \) is the ensemble average, and the temperature \( T \) is in units of the Josephson coupling strength \( J = \hbar I_c/2e \).

The RSJ model may be used to calculate the current distribution in the limit of a large perpendicular penetration length \( \Lambda = \Phi_0/4\pi I_c \), where \( \Phi_0 \) is the flux quantum. This is typically the case for a 2D superconductor. In this limit one may replace the gauge...
invariant phase difference $\phi_{jk} \equiv \theta_j - \theta_k - A_{jk}$, where
$$A_{jk} = \frac{2\pi}{\theta_0} \int_{J_0}^J A(r) \cdot dl$$
is the line integral of the vector potential, with $\phi_{jk} \equiv \theta_j - \theta_k$ because the coupling to the
electromagnetic self-field is in this limit so weak that it has little influence on the fluctuations of the supercurrent (we are here considering the situation without an
external magnetic field).

We measure the flux-noise spectrum $S(\omega)$ defined by the Fourier transformation:
$$S(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} S(t),$$
where $S(t)$ is the time-correlation function of the magnetic flux $\Phi(t)$ through a pick-up coil:
$$S(t) = \langle \Phi(t)\Phi(0) \rangle$$
(see Sec. II B for the calculation of $\Phi$). In addition, we calculate the dynamic dielectric function $\varepsilon(\omega)$ defined by
$$\varepsilon(\omega) = \frac{1}{\omega} \int_0^\infty dt \sin \omega t G(t),$$
where $G(t)$ is the correlation function of the magnetic flux $\Phi$. From Eqs. (7) and (9) to the average $\langle O(t+t')O(t') \rangle_\nu$ over time $t'$, and averages over more than $10^7$ time steps were typically performed.

**B. Flux noise and the dipole approximation**

As mentioned above, previous numerical simulations usually calculate the noise spectra from the fluctuation of the vorticity defined by the directional sum of the gauge-invariant phase difference around each plaquette. The fluctuations of the total vorticity over an area of the sample is then used to estimate the flux-noise spectrum. On the other hand in experiments the fluctuations of the magnetic flux penetrating a pick-up coil situated at a distance above the sample is measured. The relation between the vorticity noise and the flux noise has so far not been studied in detail and is the subject of the present paper.

We consider three distinct cases: The first one is the fluctuation of the magnetic flux associated with the vortices on a fixed area $A$ of the sample. The vortices describe the rotation of the supercurrent on the sample. This means that the magnetic flux associated with an elementary plaquette is proportional to the rotation of the supercurrent around the plaquette. For the RSJ model the magnetic flux for a plaquette is then given by
$$\Phi(t) = \int_A d^2r \sum_p \sin \phi_{ij},$$
in units of the flux quantum $\phi_0$. Here $r$ is the central position of the plaquette and the summation is around the plaquette $p$. Consequently the total magnetic flux associated with the rotation of the supercurrent at a given time $t$ is
$$\Phi(t) = \int_A d^2r \sum_p \sin \phi_{ij},$$
where the integral over position denotes the sum over all elementary squares inside the area $A$.

The second case is the fluctuation of the vorticity associated with a fixed area $A$ of the sample. The vorticity of an elementary plaquette is given by
$$v(r) = \frac{1}{2\pi} \sum_p \phi_{ij},$$
where the phase difference $\phi_{ij}$ is restricted to the interval $-\pi < \phi_{ij} \leq \pi$. The total vorticity $V(t)$ associated with the area $A$ is consequently
\[ V(t) = \int d^2r \, v(\mathbf{r}, t). \]

The third case corresponds to the experimental situation where one measures the magnetic flux through a pick-up coil situated at a distance \( d \) from the sample. Note that the first case corresponds to the case when \( d = 0 \). However, in typical experiments \( d \) is a macroscopic length (typically of the order of 0.1 mm).

In this section, we focus on the third case and use a dipole approximation to obtain an expression for the magnetic flux distribution from a vortex is of dipole form. In the continuum limit only the circulation of the supercurrent around a closed loop which encloses a vortex core region gives a finite value. This value can be positive or negative but has the same magnitude for all closed loops which enclose vortex cores. This means that one can estimate the magnetic field at \( d \gg A \) from the circulation of the supercurrent around small closed loops which cover the area of the superconductor by associating each of these loops with a dipole field where the strength of the dipole moment is proportional to the circulation of the supercurrent. This approximation is readily carried over to the array. The circulation of the supercurrent around a plaquette in units of \( I_c \) is given by [see Fig. 1(a)].

\[
\sin \left[ \phi \left( \mathbf{r} - \frac{x}{2} - \frac{\hat{y}}{2}, \mathbf{r} + \frac{x}{2} - \frac{\hat{y}}{2} \right) \right] + \sin \left[ \phi \left( \mathbf{r} + \frac{x}{2} - \frac{\hat{y}}{2}, \mathbf{r} + \frac{x}{2} + \frac{\hat{y}}{2} \right) \right] + \sin \left[ \phi \left( \mathbf{r} + \frac{x}{2} + \frac{\hat{y}}{2}, \mathbf{r} - \frac{x}{2} + \frac{\hat{y}}{2} \right) \right] + \sin \left[ \phi \left( \mathbf{r} - \frac{x}{2} + \frac{\hat{y}}{2}, \mathbf{r} - \frac{x}{2} - \frac{\hat{y}}{2} \right) \right], \tag{12}
\]

where \( \phi(\mathbf{r}', \mathbf{r}'') = \theta_{\mathbf{r}'} - \theta_{\mathbf{r}''} \) and \( \theta_{\mathbf{r}'} \) is the phase of the complex order parameter at site \( \mathbf{r}' \). The contribution to the magnetic field at large distances from this circulation can then approximately be described in terms of a dipole moment \( \mathbf{m}_r = m_r \hat{z} \) proportional to the circulation. The magnetic field at the observation point \( \mathbf{x} \) due to this vortex dipole moment \( \mathbf{m}_r \) is given by

\[
\mathbf{B}(\mathbf{x}) = \sum_r \frac{3e_{rx}(\mathbf{e}_{rx} \cdot \mathbf{m}_r) - \mathbf{m}_r}{|\mathbf{x} - \mathbf{r}|^3}. \tag{13}
\]

We consider the case \( d \gg a \) so that the magnetic field inside the pick-up coil does not vary much on a microscopic length scale \( a \). We may then replace the integral Eq. (14) by the discrete summation:

\[
\Phi \approx \sum_{\mathbf{r}' \in S} \mathbf{B}(\mathbf{r}' + d\hat{z}) \cdot \hat{z}
\]

\[
= \sum_{\mathbf{r}' \in S} \sum_{\mathbf{r} \in L \times L} \frac{3n_{rr'}(\mathbf{n}_{rr'} \cdot \mathbf{m}_r) - \mathbf{m}_r}{|\mathbf{r} + d\hat{z} - \mathbf{r}'|^3} \cdot \hat{z}
\]

\[
= \sum_{\mathbf{r}' \in S} \sum_{\mathbf{r} \in L \times L} \frac{3d^2 - |\mathbf{r}' + d\hat{z} - \mathbf{r}|^2}{|\mathbf{r}' + d\hat{z} - \mathbf{r}'|^5} m_{rr'}, \tag{15}
\]

where \( \mathbf{x} \) is decomposed into \( \mathbf{x} = \mathbf{r}' + d\hat{z} \), and the two 2D vectors \( \mathbf{r} \) and \( \mathbf{r}' \) denote positions on the array. The summations \( \sum_{\mathbf{r}'} \) and \( \sum_{\mathbf{r}} \) are performed on the \( l \times l \) pick-up coil and the \( L \times L \) whole array, respectively [see Fig. 4(b)]. Since Eq. (15) contains \( O(L^4) \) terms (in this work we choose \( l = L/2 \)), the calculation of the magnetic flux in this way requires most of the computer time in the numerical simulations. This time-consuming part is avoided in the approximate scheme we use in our simulation to obtain the results described in Sec. 1.

We use the following approximate scheme: The whole array is divided into stripes formed by elementary plaquettes which enclose the midpoint of the pick-up coil as the sides of a square. This is illustrated in Fig. 4(c). Each such collection of elementary plaquettes are denoted by \( S_n \) where \( 2n - 1 \) is the number of plaquettes of each of the four stripes forming the sides of the square. The magnetic flux in Eq. (15) is expressed as the summation of the contributions from each \( S_n \):
\[ \Phi = \sum_n q_n M_n, \]  

(16)

where \( M_n \) is the summation of the vortex dipole moments for the plaquettes forming \( S_n \) and \( q_n \) is the appropriate weight factor:

\[ q_n = \left( \sum_{r \in S_n} m_r \sum_{r' \in I \times l} \frac{3d^2 - \left| r' + d\mathbf{z} - r\right|^2}{\left| r' + d\mathbf{z} - r\right|^5} \right) \left( \frac{1}{\sum_{r \in S_n} m_r} \right), \]

(17)

\[ M_n = \sum_{r \in S_n} m_r. \]

(18)

We now assume that \( m_r \) in Eq. (17) can to good approximation be replaced by the average value for each \( S_n \), i.e., \( \bar{m}_r = \sum_{r \in S_n} m_r / A_n \) where \( A_n \) is the number of plaquettes contained in \( S_n \). This simplifies Eq. (17) to:

\[ q_n = \frac{1}{A_n} \sum_{r \in S_n} \sum_{r' \in I \times l} \frac{3d^2 - \left| r' + d\mathbf{z} - r\right|^2}{\left| r' + d\mathbf{z} - r\right|^5}. \]

(19)

The magnetic flux in Eq. (16) can be computed in \( O(L^2) \) operations, since \( q_n \) within this approximation is a purely geometric quantity which is independent of time. In Fig. 2 we compare the flux-noise spectrum and the complex impedance, or the real part of the conductivity \( \sigma(\omega) \) of the superconductor: The charge density correlation function is given by \( c(r, t) = (n(r, t)n(0, 0)) \). We will first relate \( c(r, t) \) to the dielectric function \( 1/\epsilon(\mathbf{k}, \omega) \) and the conductivity \( \sigma(\omega) \) of the superconductor:

\[ \text{Im} [\hat{g}(\mathbf{k}, \omega)] = \frac{\omega}{2TCG} \hat{\sigma}(\mathbf{k}, \omega), \]

(20)

where \( \hat{g} \) and \( \hat{\sigma} \) denote the Fourier transforms. The diellectric function \( 1/\epsilon(\mathbf{k}, \omega) \) is given by the usual linear response relation:

\[ \frac{1}{\epsilon(\mathbf{k}, \omega)} = 1 - \frac{2\pi}{k^2} \hat{g}(\mathbf{k}, \omega). \]

(21)

We define \( 1/\epsilon(\omega) \equiv 1/\epsilon(\mathbf{k} = 0, \omega) \) which means that Eqs. (20) and (21) for \( \mathbf{k} = 0 \) corresponds to Eqs. (8) and (9) for the RSJ model. From Eqs. (8), (20), and (21) we obtain a relation between the charge density correlations \( c(r, t) \) and the real part of the conductivity \( \sigma(\omega) \):

\[ \text{Re} [\sigma(\omega)] = -\frac{T}{2\pi TCG} \text{Im} \left[ \frac{1}{\epsilon(\omega)} \right], \]

(22)

and

\[ \text{Im} \left[ \frac{1}{\epsilon(\omega)} \right] = -\frac{\pi \omega}{T CG} \lim_{k \to 0} \frac{\hat{\sigma}(k, \omega)}{k^2}. \]

(23)

Next we relate the charge density correlation function \( c(r, t) \) to the flux-noise spectrum. From Eq. (15) we have that the magnetic flux measured by the pick-up coil is

\[ \Phi = \int_{\text{coil}} B_z(r) d^2r, \]

where the integral is over the area covered by the coil. The magnetic field \( B_z(r) \) can be expressed as

\[ B_z(r) = \int f(|r' - r|, d)n(r)d^2r', \]
the \( r' \)-integration is over the whole 2D plane, and from Eqs. (10) and (14), we have

\[
f(r, d) = \frac{T}{T_{CG}} \frac{3d^2 - (r^2 + d^2)}{(r^2 + d^2)^2}.
\] (24)

This means that the flux-noise spectrum \( S(t) = \langle \Phi(t)\Phi(0) \rangle \) is given by

\[
S(t) = \int_{\text{coil}} d^2r \int_{\text{coil}} d^2r' \int d^2r'' f(|r'' - r|, d)\langle n(r, t)n(r', 0) \rangle f(|r' - r''|, d)
\] (25)

We can now use the convolution theorem and express \( S(t) \) in terms of the Fourier transforms of \( f(r, d) \) and \( c(r, t) = \langle n(r, t)n(0, 0) \rangle \), i.e.,

\[
S(t) = \int_{\text{coil}} d^2r \int_{\text{coil}} d^2r' \int d^2k (\frac{1}{(2\pi)^2}e^{ik(\tau - \tau')}|\hat{f}(k, d)|^2\hat{c}(k, t).
\]

Taking the Fourier transform of \( S(t) \) and using the connection between \( \hat{c}(k, \omega) \) and \( 1/\hat{\epsilon}(k, \omega) \) given by Eqs. (21) and (21) yields

\[
S(\omega) = -\int_0^\infty dk \hat{F}(k, d)\text{Im}\left[\frac{1}{\hat{\epsilon}(k, \omega)}\right]
\] (26)

where

\[
\hat{F}(k, d) = \frac{2T_{CG}}{(2\pi)^2\omega} \int_{\text{coil}} d^2r \int_{\text{coil}} d^2r' e^{ik(r-r')}k^3|\hat{f}(k, d)|^2
\] (27)

and \( \hat{f}(k, d) \) is the Fourier transform of \( f(r, d) \) which describes the spreading of the magnetic field. Within the dipole approximation of Eq. (24) \( \hat{f}(k, d) \) is given by

\[
\hat{f}(k, d) = \frac{T}{T_{CG}} 2\pi ke^{-kd}.
\] (28)

The extreme case \( d = 0 \) is outside the dipole approximation and is given by

\[
\hat{f}(k) = \frac{T}{T_{CG}}.
\] (29)

The important point to note is that the flux-noise spectrum \( S(\omega) \) is directly related to the response function \( \text{Im}[1/\hat{\epsilon}(k, \omega)] \) through a function \( \hat{F}(k, d) \) which contains all the information of the spreading of the magnetic field above the superconductor as well as the geometry and position of the pick-up coil.

In Ref. 3 it was suggested that the relation between the flux noise \( S(\omega) \) and \( \text{Im}[1/\hat{\epsilon}(k, \omega)] \) could be further simplified to

\[
S_R(\omega) = \frac{8\pi^2T^2R^2}{T_{CG}^2\omega} \int_0^\infty dk k^3 e^{-2kd}[J_1(kR)]^2\text{Im}\left[\frac{1}{\hat{\epsilon}(k, \omega)}\right].
\] (32)
where the proportionality constant $C$ can be replaced by $\text{Im}[1/\epsilon(k, \omega)]$ which contributes to the integral in Eq. (23) because of the factor $e^{-2kd}$. This means that if $d$ is sufficiently large compared to the relevant microscopic length, then $\text{Im}[1/\epsilon(k, \omega)]$ can be replaced by $\text{Im}[1/\epsilon(0, \omega)]$, demonstrating that under these conditions $S(\omega)$ is indeed proportional to $\text{Im}[1/\epsilon(0, \omega)]$. In the present case of a circular pick-up coil we explicitly find

$$S_R(\omega) = -\frac{C}{\omega} \text{Im} \left[ \frac{1}{\epsilon(0, \omega)} \right],$$

where the proportionality constant $C$ is given by (after changing the integration variable to $x = kd$)

$$C = \frac{8\pi^2 T^2}{T_{CG}} \frac{R^2}{d^4} \int_0^\infty dx x^3 e^{-2x} [J_1(xR/d)]^2.$$  

In the limit $R/d \ll 1$ this reduces to

$$C \approx \frac{T^2}{T_{CG}} \frac{R^4}{d^6} \frac{15\pi^2}{4},$$

and in the limit $R/d \gg 1$ to

$$C \approx \frac{T^2}{T_{CG}} \frac{2\pi R}{d^3}.$$  

We conclude from Eqs. (33) and (34) that the proportionality constant $C$ will always contain a temperature-dependent factor $T^2/T_{CG}$ and a dependence on the size of the coil. This coil-size dependence reflects the relative magnitudes between the coil size and the distance from the sample: When $R$ is much larger than $d$, the noise amplitude is proportional to the perimeter of the coil $2\pi R$, and, when it is much smaller, it is proportional to square of the coil area $R^4$. In typical experiments $R$ is usually much larger than $d$.

In some previous simulations of the flux-noise spectrum, based on the $XY$ models, one has approximated the noise spectrum from the fluctuation of the total vorticity for a finite area of the model. This implies two differences in relation to the above dipole approximation: First of all it corresponds to $d = 0$ and consequently to a constant $f$ [see Eq. (20)]. Secondly it corresponds to changing the magnetic flux defined by Eq. (10) to vorticity defined by Eq. (11). Let us first consider the flux-noise spectrum [compare Eqs. (26), (29), and (31)]

$$S_R(\omega) = -\frac{2T^2}{\omega T_{CG}} R^2 \int_0^\infty dk k [J_1(kR)]^2 \text{Im} \left[ \frac{1}{\epsilon(k, \omega)} \right].$$

This means that, in this case, the noise spectrum depends on all the $k$-values of $\text{Im}[1/\epsilon(k, \omega)]$ and is not proportional to $\text{Im}[1/\epsilon(0, \omega)]$. For example the leading large-$R$ dependence of Eq. (35) is

$$S_R(\omega) \propto -\frac{R}{\omega} \int_0^\infty dk \text{Im} \left[ \frac{1}{\epsilon(k, \omega)} \right].$$

In this limit the flux noise is proportional to the perimeter of the pick-up area just as for the large-$R$ case, but instead of singling out the $k = 0$-contribution all $k$-values contribute. The proportionality between the flux noise $S(\omega)$ and $\text{Im}[1/\epsilon(0, \omega)] \propto \text{Re}[\sigma(\omega)]$ can be tested by experiments since both $S(\omega)$ and the conductivity $\sigma(\omega)$ can be independently measured.

The change from magnetic flux [defined by Eq. (10)] to vorticity [defined by Eq. (11)] influences the flux-noise spectrum in an additional significant way: Now the crossing of a vortex over the perimeter of the pick-up area is described as a discrete $\pm 2\pi$-change of the total vorticity of the pick-up area. The corresponding spectrum hence corresponds to a random walk of discrete events over a sharp boundary. This results in a $\omega^{-2}$-tail of the spectrum. However, this condition does not correspond to the experimental situation where the pick-up coil does not have a sharp boundary, is at a distance $d$ from the sample, and, most importantly, the magnetic field from a vortex is spread out.

In the next section we present numerical results of the flux-noise spectrum and its relation to the conductivity.

IV. SIMULATION RESULTS AND EXPERIMENTAL IMPLICATIONS

A. Comparison with previous works

We first relate our simulations results to earlier ones for the RSJ model. These earlier simulations calculated the noise spectrum of the vorticity [see Eq. (11)] over a fixed area of the system (the $d = 0$ case). As explained in the previous section, this corresponds to discrete events over a sharp boundary and implying a $\omega^{-3/2}$-tail. Such a tail has indeed been found in Ref. and is also verified in our simulations. This is apparent from Fig. which displays our data for a $64 \times 64$ array with a pick-up area of size $32 \times 32$ at $T = 1.1$; the lower dataset shows the vorticity-noise spectrum and the slope is $-3/2$. However, if we instead calculate the magnetic-flux-noise spectrum [see Eq. (10)], then the crossing of magnetic flux over the perimeter is not a discrete event. This means that there is no obvious reason for a $\omega^{-3/2}$-tail and nor do we find any such tail in the simulations, as is also apparent from Fig. The exponent in Fig. for this case is instead close to $-2$ as seen from the upper data-set in Fig.
B. Flux-noise spectra with a finite distance between pick-up coil and array

As mentioned in Sec. II, the typical experimental setup measures the magnetic-flux-noise spectrum with a finite distance $d$ between the array and the pick-up coil. Furthermore, the typical experimental setup corresponds to the large-$d$ limit where the noise spectrum $S(\omega)$ is proportional to the real part of the conductivity $\sigma(\omega)$ (see Sec. III).

In our simulations we investigate the flux-noise spectra as a function of the distance $d$ to the pick-up coil. Figure 3(a) shows the flux-noise spectra calculated from Eqs. (3), (4) with the magnetic flux given by Eqs. (16), (17) and (18) (see Sec. II B for details). The data sets are shown as $\omega S(\omega, d)$ against $\omega$ in a log-log plot. The vertical scale is adjusted in order to compare the shapes of the curves. One notices that the spectra for the different $d$'s all approach $\omega^{-1}$ for large $\omega$ and can be collapsed to a single curve in this large-$\omega$ region by a vertical adjustment. For small $\omega$ the curves becomes linear with $\omega$ which reflects a constant part (white noise) of $S(\omega, d)$.

As $d$ increases, the peak position of the $\omega S(\omega, d)$ curves moves to the left and the peak height increases. The uppermost curve in Fig. 3(a) is $\operatorname{Im}[1/\epsilon(\omega)]$ (full curve in the figure) and, as $d$ is increased, the flux-noise spectrum $\omega S(\omega, d)$ approaches this uppermost curve. This verifies that for large $d$ one has the simple connection $S(\omega) \propto \operatorname{Im}[1/\omega \epsilon(\omega)]$ as discussed in Sec. III.

Figure 3(b) shows that the characteristic frequency given by the peak position in Fig. 3(a) decreases with increasing $d$. In the limit of large $d$ the characteristic frequency of $S(\omega, d)$ agrees with the characteristic frequency of $1/\epsilon(\omega)$. Thus in this limit both the shape and the characteristic frequency of $S(\omega, d)$ and $\operatorname{Im}[1/\omega \epsilon(\omega)]$ are the same. This proportionality between the flux-noise spectrum and complex conductivity can be tested experimentally and indeed seems to be borne out.

The fact that the flux-noise spectrum in the large-$d$ limit is proportional to real part of the conductivity means that the characteristic features of the conductivity are reflected in the flux-noise spectrum. In case of a 2D superconductor the dynamical features of the conductivity $\sigma(\omega) \propto -1/\omega \epsilon(\omega)$, are well described by the response forms:

$$
\text{Re} \left[ \frac{1}{\epsilon(\omega)} \right] - \frac{1}{\epsilon(0)} = \frac{1}{\epsilon} \frac{\omega}{\omega + \omega_0},
$$

and

$$
\text{Im} \left[ \frac{1}{\epsilon(\omega)} \right] = -\frac{2}{\epsilon} \frac{\omega_0 \ln \omega}{\omega^2 - \omega_0^2},
$$

which catches the dynamics of vortex fluctuations in a region around the KT transition. From Eq. (35) one notices that the peak height is $1/\pi \epsilon$. Above the KT transition $1/\epsilon$ increases only weakly with increasing temperature and approaches unity for somewhat higher temperatures. For the flux-noise spectrum this means that $S(\omega) \propto T^2/T^{CG} \omega_0$. Now $T/T^{CG} \propto \rho_0(T)$ where $\rho_0(T)$ is the bare superfluid density which decreases slightly with temperature whereas $T$ increases so that also $T^2/T^{CG}$ depends only weakly on $T$. This means that to good approximation the flux-noise spectrum for different temperatures above the KT transition should have a common tangent $\propto 1/\omega$ which goes through all the points $S(\omega_0(T), T)$. This means that the common tangent in a log-log plot has the slope $-1$. This feature is illustrated in Fig. 3 which demonstrates the existence of a common tangent with the slope close to $-1$ both directly for $S(\omega, d)$ and for $\text{Re}[\sigma(\omega)] \propto -\text{Im}[1/\omega \epsilon(\omega)]$. The existence of a common tangent for the flux-noise spectra with the slope $-1$ can be readily tested in experiments and seems to be well borne out.

One should, however, notice that the argument for a common tangent with slope $-1$ does not single out the response form given by Eqs. (36) and (37). In fact the reasoning is also valid in a region above the KT transition where the response of the vortex fluctuations is given by the conventional Drude response form

$$
\text{Re} \left[ \frac{1}{\epsilon(\omega)} \right] = \frac{1}{\epsilon} \frac{\omega^2}{\omega^2 + \omega_0^2},
$$

and

$$
\text{Im} \left[ \frac{1}{\epsilon(\omega)} \right] = -\frac{1}{\epsilon} \frac{\omega_0 \omega}{\omega^2 + \omega_0^2},
$$

which gives the peak height $1/2\epsilon$. One expects that the response form Eqs. (36) and (37) describes the response from the vortex pairs in a region just above the KT transition whereas the conventional Drude response is obtained for higher temperatures where the response is dominated by free vortices. How wide these regions are depend on the details: for a real thin superconductor the vortex-pair response seems to dominate in a wide region. However, for the 2D RSJ model on a square lattice, which we are using here, the vortex-pair dominated region above the KT transition is narrow and the Drude response dominates in a broader region above the KT transition. Thus the data shown in Fig. 3 are predominantly Drude-like. A practical way of determining which response type is at hand is to measure the complex impedance and determine the peak ratio [i.e., the ratio $\text{Re}[\sigma]/\text{Im}[\sigma]$ at the peak position $\omega_0$ of $\text{Re}[\sigma/\omega]$ for the vortex pair dominated response, Eqs. (36) and (37)] this ratio is $2/\pi \approx 0.64$ and for free vortex Drude response, Eqs. (38) and (39), it is unity.

The essential point here is that, because the flux-noise spectrum for a large $d$ is proportional to $\text{Re}[\sigma/\omega]$, the noise spectra at a sequence of temperatures above the KT transition should in a log-log plot have a common tangent with slope $-1$. We can substantiate this claim further by simulating the noise spectra for a small $d$ where the flux-noise spectrum is not proportional to $\text{Re}[\sigma/\omega]$. In this
case there is no particular reason for a common tangent with any slope and, as apparent from the simulation results in Fig. 7, no such common tangent can be fitted to the data.

One may also notice from Fig. 8 that both $S(\omega, d)$ and Re[$\sigma(\omega)$] have intermediate regions with $\omega^{-1.5}$ followed by a $\omega^{-2}$-tail for even larger $\omega$. However, such an intermediate-$\omega^{-1.5}$ region appears to be less discernible for the small-$d$ case when $S(\omega, d)$ is not proportional to Re[$\sigma(\omega)$], as is apparent from Fig. 8.

C. Flux-noise spectrum below KT transition

Next we investigate what happens as the temperature is decreased towards the KT transition and below. Figure 8 shows data for Im[1/ε(ω)] and $\omega S(\omega, d = 20)$ over a wider range of temperatures (the data for $T \geq 1.1$ are the same as in Fig. 6). Again one observes that both quantities behave in precisely the same way over the whole temperature range, verifying that they are indeed proportional to very good approximation. Next one observes that the characteristic frequency of $\omega_0$ (the frequency of the peak position) decreases as the KT transition is approached from above. This suggests a critical slowing down at the KT transition to $\omega_0 = 0$. As the temperature passes through the transition the characteristic frequency starts to increase again (the full curves in Fig. 8 are below the KT transition).

The argument for a common 1/ω-tangent for the flux-noise spectra above the KT transition is related to the fact that the peak height for [Im[1/ε(ω)]] is 1/πε (1/2ε) for the vortex pair response, Eqs. (20) and (27) free vortex Drude response, Eqs. (25) and (33) together with the fact that 1/ε only increases very weakly with T above the KT transition and approaches unity for somewhat higher $T$. Figure 8 shows this weak increase in a region above the KT transition; the Drude value 1/2 is approached roughly like $\omega^{0.1}$ which explains the discrepancy between exponent 1 and the value 0.9 found in Fig. 8. Thus the existence of a common tangent 1/ω hinges on the weakness of the temperature dependent factor $T^2/T^{CG}(T)\epsilon(T)$ which in turn depends somewhat on the details of the system. However, since the temperature dependence of $\omega_0$ is dramatic just above the KT transition, the common tangent should to good approximation exist at least in a limited region above the KT transition.

Below the KT transition there are no free vortices and the response is given by the vortex pairs Eqs. (20) and (27). However, in this case the factor 1/ε ≈ 1 − 1/ε(ω = 0) decreases rapidly towards zero as the temperature is decreased below the KT transition. This means that while the characteristic frequency rapidly increases, as the temperature is decreased below the KT transition, the amplitude of the flux noise, which is proportional to 1/ε ≈ 1 − 1/ε(ω = 0), rapidly decreases. The KT transition is at $T \approx 0.9$ and already at $T = 0.85$ the amplitude of $\omega S(\omega, d)$ has dropped dramatically compared to the almost constant amplitude above the KT transition, as is apparent from Fig. 8(b).

Finally, there is in Fig. 8b an indication that the curves develop a plateau as the KT transition is approached from above (compare the curve for $T = 0.95$). Such a plateau would suggest that $S(\omega, d)$ is proportional to 1/ω in an intermediate region just above the KT transition. The same development of a plateau can be anticipated in Fig. 8a for [Im[1/ε(ω)]]] and has also been found for the XY model with the time-dependent Ginzburg-Landau (TDGL) dynamics [21, 22].

V. CONCLUDING REMARKS

In the present paper we have explored the fact that the typical experimental setup for measuring the magnetic flux noise for a superconductor, or JJA, corresponds to the case when the distance $d$ to the pick-up coil is much larger than the relevant microscopic lengths. In this limit the flux-noise spectrum and the real part of the conductivity are proportional. This proportionality seems first to have been anticipated in Ref. 7 and has also been experimentally verified. In this paper we have studied this connection in some more detail.

We have also demonstrated that both the shape and the characteristic frequency of the spectrum depend on the distance $d$ to the pick-up coil. This means that no detailed conclusions can be drawn from simulations which presumes $d = 0$. Furthermore, there is a qualitative difference between the vorticity-noise spectrum, which corresponds to discrete events over a sharp boundary, and the magnetic-flux-noise spectrum which corresponds to spread-out objects over a boundary. The experimental situation corresponds to a spread-out magnetic flux and a pick-up coil at a large distance $d$, which is very different from some earlier simulations which calculated the vorticity-noise spectrum for $d = 0$. Nevertheless the vorticity-noise spectrum for $d = 0$ has a $\omega^{-3/2}$ tail for higher frequencies which seems to match the experimental results whereas the magnetic-flux-noise spectra for $d = 0$ does not have such a tail. In accordance with the present simulations, we suggest that the resolution of this dichotomy is that in the large-$d$ limit the magnetic-flux-noise spectrum does have an intermediate region with a $\omega^{-3/2}$ behavior and that it is this intermediate region which is seen in the experiments.

The proportionality between the magnetic-flux-noise spectrum and the real part of the conductivity implies that the noise spectra for a sequence of temperatures just above the KT transition should in a log-log plot have a common tangent with the slope −1. The existence of such a common tangent has also been verified in experiments as well as in the present simulations. We also explicitly demonstrated through our simulations that for...
small $d$ there is no such common tangent.

The existence of a common tangent is by itself not necessarily conclusive. For example, the experimental data for the JJA’s in Ref. 5 correspond to the large-$d$ case and the data have indeed a common tangent with slope $-1$. However, the spectra at a fixed temperature seem to have a $1/\omega$ behavior over a very large region, which differ markedly from the spectra obtained in our simulations for the RSJ model.

The present simulations also suggest that immediately above the KT transition there should be a very small temperature region where the noise spectrum has an intermediate interval with a $1/\omega$ behavior. It has been suggested that the data in Ref. 5 might perhaps be related to this temperature region closest to the transition. However, at the moment there seem to be no accepted explanation for the $1/\omega$-behavior found in Ref. 5.

Finally, we showed that the amplitude of the flux-noise spectrum drops dramatically as the temperature is decreased below the KT transition, and that at the same time the characteristic frequency increases. It should also be possible to observe this effect in experiments.

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FIG. 1. (a) The vortex dipole moment associated with the dual lattice point $r$ is estimated by the circulating current around the plaquette. The magnetic field at the observation point $x$ is the summation of the contributions from all such vortex dipole moments. (b) The $l \times l$ pick-up coil is separated by the distance $d$ from the $L \times L$ square array of Josephson junctions. (c) The whole array is divided into quadratic enclosures. The plaquettes surrounding such an enclosure is denoted by $S_n$ (shaded area). The magnetic flux due the plaquettes belonging to $S_n$ is calculated.

FIG. 2. Comparison between the flux noise from the full calculation [Eq. (15)] and from the approximate scheme [Eqs. (16), (18), and (19)]. The figure shows the flux noise as a function of time $t$, $S(t, d) \equiv \langle \Phi(t, d)\Phi(0, d) \rangle$ for a $16 \times 16$ coil with the distance $d = 5$ from a $32 \times 32$ array at the temperature $T = 1.10$ (in units of $J/k_B$). As seen the approximation scheme reproduces the full calculation very accurately.

FIG. 3. The weight factor $q_n$ (in arbitrary unit) for $S_n$ plotted against the size $2n-1$ (in numbers of plaquettes per side). The data are for a $64 \times 64$ array with the coil size $32 \times 32$ for different distances $d$. When the distance $d$ between the coil and the array is very small, only plaquettes inside of the pick-up coil contribute to the magnetic flux, whereas all plaquettes contribute for larger $d$.

FIG. 4. Comparison between the vorticity-noise spectrum $S_1$ and the magnetic-flux-noise spectrum $S_2$ for $d = 0$. The data are for a $64 \times 64$ array at $T = 1.1$ with a $32 \times 32$ coil size. The vorticity spectrum has a $\omega^{-3/2}$ tail, whereas the magnetic-flux spectrum is closer to $\omega^{-2}$. 

10
FIG. 5. (a) The dependence of the flux-noise spectrum $S(\omega, d)$ on the distance $d$. The full drawn uppermost curve is the imaginary part of the dielectric function $|\text{Im}[1/\epsilon(\omega)]|$. The rest of the curves correspond to $d = 0.1, 5, 10, 20,$ and $40$ (from bottom to top) plotted as $\omega S(\omega, d)$ and the curves are shifted in the vertical direction for better comparison. The data are for a $64 \times 64$ array with a $32 \times 32$ pick-up coil at $T = 1.1$, except for $d = 40$ where a $128 \times 128$ array was necessary because plaquettes further away from the center contribute in this case. As $d$ is increased $\omega S(\omega, d)$ approaches $|\text{Im}[1/\epsilon(\omega)]|$. (b) The frequency at the maxima for the curves in (a) are plotted versus the distance $d$. As $d$ is increased, this frequency decreases and approaches the value for $|\text{Im}(1/\epsilon(\omega))|$. (The full line is a guide to the eye.)

FIG. 6. (a) The real part of the conductivity $\text{Re}[\sigma(\omega)]$ and (b) the flux-noise spectrum $S(\omega, d = 20)$ at temperatures above the KT transition. The curves for $\text{Re}[\sigma(\omega)]$ and $S(\omega, d = 20)$ have the same shape; for small frequencies they have a very weak $\omega$-dependence, for somewhat larger $\omega$ there is an approximate $\omega^{-3/2}$-behavior, whereas for even larger $\omega$ the behavior approaches $\omega^{-2}$. The curves in the log-log plot have a common tangent with the slope $-0.9$. 
FIG. 7. Flux-noise spectrum in the small-$d$ limit at temperatures above the KT transition [same as Fig. 6(b) but with $d = 0.1$ instead of $d = 20$]. In the case of small $d$ we find neither a common tangent nor an appreciable range of $\omega^{-1.5}$-behavior.

FIG. 8. (a) The imaginary part of the dielectric function, $\text{Im}[1/\epsilon(\omega)]$, and (b) the flux-noise spectrum multiplied by the frequency, $\omega S(\omega, d = 20)$, at temperatures above and below the KT transition. This again illustrates that both quantities behave in the same way. As the temperature is increased far above the KT transition, the maximum of $|\text{Im}[1/\epsilon(\omega)]|$ approaches the limit value $1/2$ (this corresponds to the Drude limit with $\tilde{\epsilon} = 1$) as denoted by the horizontal line in (a). The curves seem to develop a plateau as the KT transition is approached from above, as is suggested by the $T = 0.95$-curves. As the temperature drops below the KT transition, the amplitude of the flux noise rapidly decreases whereas the characteristic frequency increases, as is illustrated by the curves at $T = 0.85$. 