Unambiguous Forest Factorization*

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Abstract. In this paper, we look at an unambiguous version of Simon’s forest factorization theorem, a very deep result which has wide connections in algebra, logic and automata. Given a morphism $\varphi$ from $\Sigma^+$ to a finite semigroup $S$, we construct a universal, unambiguous automaton $A$ which is “good” for $\varphi$. The goodness of $A$ gives a very easy proof for the forest factorization theorem, providing a Ramsey split for any word in $\Sigma^\omega$ such that the height of the Ramsey split is bounded by the number of states of $A$. An important application of synthesizing good automata from the morphism $\varphi$ is in the construction of regular transducer expressions (RTE) corresponding to deterministic two-way transducers.

1 Introduction

In this paper, we revisit Simon’s forest factorization theorem, a central result in algebraic automata theory. In his seminal paper \cite{s}, Simon showed that, given a semigroup morphism $\varphi: \Sigma^+ \to S$, any word $w \in \Sigma^+$ admits a factorization tree $T(w)$ of height $\leq 9|S|$. Leaves of $T(w)$ are letters from $\Sigma$ and the yield of $T(w)$ is the word $w$. Internal nodes have arity at least two. Each node $x$ of $T(w)$ is labeled $F(x) = \varphi(u_x)$ where $u_x$ is the yield of the subtree rooted at $x$. The main constraint is that, if an internal node $x$ has arity $n > 2$ with children $x_1, \ldots, x_n$ then $F(x_1) = \cdots = F(x_n) = e$ is an idempotent of $S$. There are no constraints for binary nodes. Simon’s factorization theorem has many deep applications, see e.g., \cite{cs,csf}.

An easy consequence of Simon’s forest factorization theorem is that there is a regular expression $F = \bigcup F_i$ which is universal (the denoted language is $\mathcal{L}(F) = \Sigma^+$) and such that (1) for each subexpression $E$ of some $F_i$ the denoted language $\mathcal{L}(E)$ is mapped by $\varphi$ to a single semigroup element $s_E$, and (2) for each subexpression $E^+$ of some $F_i$ the associated element $s_E$ is an idempotent of $S$. In addition, the subexpressions $F_i$ do not use union and have $(+,+)$-depth at most $9|S|$ (the depth of $F_i$ is the longest chain of concatenations and Kleene plus, i.e., the height of the syntax tree of $F_i$). A similar statement is given in \cite{1,2,3}. Actually, the converse is also true. If $F = \bigcup F_i$ is a universal regular expressions satisfying (1) and (2), each word $w \in \Sigma^+$ can be parsed according to some $F_i$ and the parse tree is a factorization tree for $w$.

In this paper, we show how to construct a universal regular expression $F = \bigcup F_i$ satisfying (1) and (2) and which in addition is unambiguous. Therefore, each word $w \in \Sigma^+$ admits a unique parse tree according to $F$, which is indeed a factorization tree. The forest factorization theorem was extended to infinite words by Colcombet in \cite{2,3}. We also extend our unambiguous version to infinite words: we can construct an $\omega$-regular expression $\bigcup F_iG_i^\omega$ which is universal, unambiguous, and the subexpressions $F_i$, $G_i$ satisfy (1) and (2). We call these good expressions.

This work is motivated by \cite{5} in which regular transducer expressions (RTE) are defined and shown equivalent to deterministic two-way transducers (both for finite and infinite words in which case the transducer may use regular look-aheads). The universal good expression is used to parse the input word, and from the parse tree, the output is suitably computed. Since deterministic transducers define functions, it is essential that each input word has a unique parse.

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tree. This explains the need for an unambiguous extension of Simon’s forest factorization theorem. The other properties (1) and (2) are also essential to compute an RTE equivalent to the given deterministic transducer. We believe that the existence of good regular expressions may have several other applications.

After the initial bound of $9|S|$ by Simon, there have been follow ups. In [2], Colcombet extended Simon’s result to infinite words and reduced the bound to $3|S|$. He used a new proof technique, constructing Ramsey splits from which the factorization trees can be easily derived. Kufleitner [3] also improved the bound on the height to $3|S| - 1$. A variant of Kufleitner’s proof can be found in [1]. The bound on the height of factorization trees was further improved in [4] to $3|N(S)| - 1$, where $N(S)$ is the maximum over all chains $D_1 <_D \cdots <_D D_k$ of $D$-classes of the sum $\sum_{\ell=1}^k N(D_\ell)$ and $N(D)$ is 1 if $D$ is irregular, else $N(D)$ is the number of elements of $D$ which are $H$-equivalent to an idempotent. The proofs above are based on Green’s relations. Subsequently, a simplified proof not based on Green’s relations was given in [4], using the local divisor technique. Also in [3], a deterministic version of Simon’s forest factorization is given, but to achieve the determinism, conditions (1) and (2) had to be weakened.

The main contributions of this paper are as follows. Given a semigroup morphism $\varphi: \Sigma^+ \to S$, we construct a universal, unambiguous automaton $A$ that we call “$\varphi$-good”. The goodness of $A$ is determined by the following conditions (i) it is unambiguous and universal (it accepts all words in $\Sigma^\infty$), (ii) it has a unique initial state $\varepsilon$ with no incoming transitions to it, (iii) it has a unique final state $f$ with no outgoing transitions from it, (iv) there is a total ordering on the states of the automaton such that $Q \setminus \{\varepsilon, f\} < f < \varepsilon$, and (v) for each state $q$, the set of words that start at $q$, and come back to it, visiting only lower ranked states than $q$, must be mapped to a unique idempotent $e_q \in S$. These properties of $A$ are crafted in such a manner that given any word $w \in \Sigma^\infty$, the unique accepting run of $w$ on $A$ easily produces a Ramsey split of $w$ (in the sense of Colcombet), the height of the split being bounded above by the number of states of $A$.

We construct a $\varphi$-good automaton by induction on $(|S|, |\varphi(\Sigma)|)$ with a lexicographic ordering, a technique introduced by Wilke [10] and that is very close in spirit to the local divisor technique of [7]. The easy base cases of the induction are when $S$ is a group, and when $|\varphi(\Sigma)| = 1$. The inductive cases are when we consider a semigroup element $c \in S$ such that $Sc \subseteq S$ or $cS \subseteq S$. The inductive cases are technically involved. The case $Sc \subseteq S$ is a bit simpler than the other one. When one deals with commutative semigroups, we could therefore, simply use this case. We call the automaton weakly good if we drop condition (iii) which introduces non-determinism. Up to the first inductive case, we can obtain a weakly good automaton which is deterministic. But with the second inductive case $cS \subsetneq S$, things get more complex, and we show that it is not possible to obtain deterministic weakly good automata. In a way, the price we pay in obtaining Ramsey splits is the non-determinism. This must be contrasted with the construction of Colcombet [2], where a forward Ramsey split is obtained, while retaining determinism in the automaton. One way we can avoid non-determinism is to allow look-aheads in the constructed good automata. It turns out however that, it is not possible to obtain a bounded look-ahead, and in general, one needs a regular look-ahead in the constructed good automaton.

The good automata, though challenging in its construction and proof of correctness, has some nice take-aways: (1) It provides a very simple proof of the forest factorization theorem, and (2) it allows us to synthesize good expressions [5] by a standard elimination of states in $A$. The properties imposed on $A$ which make it good, helps significantly in both these cases: (1) in the case of the forest factorization, the states of $A$ are used in labelling the positions of the word; whenever a state repeats, we declare then equivalent, as long as no higher state has been seen in between. This trivially gives a Ramsey split, with the height being the number of states of $A$. (2) The synthesis of good expressions follows very easily thanks to the unambiguity of $A$, the ordering on the states, and the condition of obtaining a unique idempotent while returning to a state without seeing a higher state.
Our construction of good automata is in general exponential in the size of the semigroup. It would be interesting to study how this construction can be optimized.

2 Unambiguous Forest Factorization

Let $\Sigma$ denote a finite alphabet. $\Sigma^\infty$ represents $\Sigma^* \cup \Sigma^\omega$, the set of finite or infinite words. Given a word $w = a_1a_2 \ldots$ with $a_i \in \Sigma$, $w[x, \ldots, y]$ denotes the word $a_x \ldots a_y$. For rational expressions over $\Sigma$ we will use the following syntax:

$$F ::= \emptyset | \varepsilon | a | F \cup F | F \cdot F | F^+$$

where $a \in \Sigma$. For reasons that will be clear below, we prefer to use the Kleene-plus instead of the Kleene-star, hence we also add $\varepsilon$ explicitly in the syntax. An expression is said to be $\varepsilon$-free if it does not use $\varepsilon$. We denote by $\mathcal{L}(E)$ the regular language denoted by $E$.

Let $(S, \cdot, 1_S)$ be a finite monoid and $\varphi: \Sigma^* \to S$ be a morphism. We say that a rational expression $F$ is $\varphi$-good (or simply good when $\varphi$ is clear from the context) when

1. the rational expression $F$ is unambiguous,
2. for each subexpression $E$ of $F$ we have $\mathcal{L}(E) \subseteq \varphi^{-1}(s_E)$ for some $s_E \in S$.
3. for each subexpression $E^+$ of $F$ we have $\mathcal{L}(E) \subseteq \varphi^{-1}(s_E)$ for some idempotent $s_E \in S$.

Notice that the classical rewrite rules used to simplify expressions using $\emptyset$ preserve good expressions. These rewrite rules are $\emptyset^+ \Rightarrow \emptyset$, $\emptyset \cdot F \Rightarrow \emptyset$, $F \cdot \emptyset \Rightarrow \emptyset$, $\emptyset \cup F \Rightarrow F$, $F \cup \emptyset \Rightarrow F$. Hence, each good expression is equivalent to a good expression which is either simply $\emptyset$, or does not use $\emptyset$ as a subexpression. Also, $\varepsilon$-freeness is preserved by this simplification.

**Theorem 1** (Unambiguous Forest Factorization). Let $\varphi: \Sigma^* \to S$ be a morphism to a finite monoid $(S, \cdot, 1_S)$.

(T1) For each $s \in S$, there is an $\varepsilon$-free good rational expression $F_s$ such that $\mathcal{L}(F_s) = \varphi^{-1}(s) \setminus \{\varepsilon\}$.

Therefore, $G = \bigcup_{s \in S} F_s$ is an unambiguous rational expression over $\Sigma$ such that $\mathcal{L}(G) = \Sigma^\omega$.

(T2) There is an unambiguous rational expression $G = \bigcup_{k=1}^m F_k \cdot G^k$ over $\Sigma$ such that $\mathcal{L}(G) = \Sigma^\omega$ and for all $1 \leq k \leq m$ the expressions $F_k$ and $G_k$ are $\varepsilon$-free $\varphi$-good rational expressions and $s_k$ is an idempotent, where $\mathcal{L}(G_k) \subseteq \varphi^{-1}(s_k)$.

The good regular expressions will be obtained using the classical translation of automata to regular expressions by successive state eliminations. To this aim, the automaton should have several properties. Mainly it should be unambiguous and there should be a total order on states which is used in the state elimination. We study these properties in the next section.

3 Good Automata

Let $A = (Q, \Sigma, \Delta, \iota, F, R, <)$ be an automaton where $Q$ is the finite set of states, $\Sigma$ the alphabet, $\Delta \subseteq Q \times \Sigma \times Q$ the transition relation, $\iota \in Q$ is the initial state, $F, R \subseteq Q$ are the subsets of final and repeated (Büchi) states, and $<$ is a total order on $Q$. For $p, q \in Q$ and $w \in \Sigma^+$ we write $p \xrightarrow{w} q$ when there is a run in $A$ from $p$ to $q$ reading $w$. We let $L_{p,q}$ be the set of nonempty words $w \in \Sigma^+$ such that $p \xrightarrow{w} q$. If $X \subseteq Q$ then we write $p \xrightarrow{w} X q$ if there is such a run where all intermediary states are in $X$. We let $L_{p,X,q}$ be the set of nonempty words $w \in \Sigma^+$ such that $p \xrightarrow{w} X q$. Hence, we have $L_{p,Q,q} = L_{p,q}$ and $L_{p,\emptyset,q} \subseteq \Sigma$. We simply write $L_q = L_{q,q,q} \subseteq L_{q,q}$ where $L_q = \{p \in Q \mid p < q\}$.

Let $\varphi: \Sigma^* \to S$ be a semigroup morphism. The automaton $A$ is $\varphi$-good (or simply good) if it satisfies the following properties:

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1. We may start from a monoid morphism but during the induction we will have to consider semigroups.
Unambiguous Forest Factorization

(G1) \( \mathcal{A} \) is unambiguous and universal (accepts all words). For each word \( w \in \Sigma^+ \cup \Sigma^* \) there is one and only one accepting run for \( w \) in \( \mathcal{A} \).

(G2) For all \( q \in Q \), there is an idempotent \( e_q \in S \) such that \( L_q \subseteq \varphi^{-1}(e_q) \), i.e., all words in \( L_q \) (if any) are mapped by \( \varphi \) to the same semigroup element \( e_q \), which is an idempotent.

(G3) The initial state \( \iota \) has no incoming transitions and is maximal: \( q < \iota \) for all \( q \in Q \setminus \{ \iota \} \).

(G4) There is only one final state \( F = \{ f \} \) and \( f \) has no outgoing transitions. Moreover, the total order on states satisfies \( Q \setminus \{ \iota, f \} < f < \iota \).

We say that \( \mathcal{A} \) is weakly-good if it satisfies \((G_1 G_2 G_3)\).

Lemma 2. From a weakly-good automaton, we can construct an equivalent good automaton.

Proof. Let \( \mathcal{A} = (Q, \Sigma, \Delta, \iota, F, R, <) \) be a weakly-good automaton for the morphism \( \varphi: \Sigma^+ \rightarrow S \). Let \( f \notin Q \) be a new state and let \( Q' = Q \cup \{ f \} \). We define \( \mathcal{A}' = (Q', \Sigma, \Delta \cup \Delta', \iota, \{ f \}, R, <') \) as follows: \( \Delta' \) is the set of transitions \((q,a,f)\) such that there is a transition \((q,a,q') \in \Delta \) with \( q' \in F \). The ordering \(<'\) coincides with \(<\) on \( Q \) and satisfies \( Q \setminus \{ \iota \} < f < \iota \).

Clearly, \((G_1)\) holds for \( \mathcal{A}' \). Notice that \( f \) has no outgoing transitions, hence \((G_4)\) is satisfied. Also, \( L_q(\mathcal{A}') = \emptyset \) and \( L_q(\mathcal{A}) = L_q(\mathcal{A}) \) for all \( q \in Q \), hence \((G_2)\) holds for \( \mathcal{A}' \). Finally, \( \mathcal{A} \) and \( \mathcal{A}' \) have the same infinite runs and there is a bijection between the finite accepting runs of \( \mathcal{A} \) and the finite accepting runs of \( \mathcal{A}' \). We deduce easily that \((G_3)\) is satisfied. \( \square \)

Example 3. Consider the morphism \( \varphi: \Sigma^+ \rightarrow S = \{ a, b \} \) defined by \( \varphi(a) = a \) and \( \varphi(b) = b \). The product in \( S \) is so that \( a \) and \( b \) are both right absorbing (\( as = a \) and \( bs = b \) for all \( s \in S \)) and hence idempotents. The automaton \( \mathcal{A} \) (left in Figure 1) is \( \varphi \)-good. The ordering on states is \( n_b < n_a < f < \iota \), \( F = \{ f \} \), \( R = \{ n_a, n_b \} \). The states \( n_a, n_b \) determine the next symbol to be read as \( a \) and \( b \) respectively. It is easy to see that \((G_4)\) is true: consider a word \( w \in \Sigma^\omega \). For all \( i > 1 \), the \( i \)th symbol of \( w \) is \( x \in \{ a, b \} \) iff the \( i \)th state in the unique accepting run from \( \iota \) is \( n_x \).

By the ordering of states, \( L_{n_a} = \emptyset = L_{n_b} \). \( L_{n_a} = \{ b \} \subseteq \varphi^{-1}(\beta) \) and \( L_{n_b} = ab^* \subseteq \varphi^{-1}(\alpha) \). Since \( a \) and \( \beta \) are idempotents, \((G_1)\) holds good. \((G_2)\) and \((G_3)\) also hold good easily. Figure 1 also depicts on the right \( \psi \)-good automaton \( \mathcal{B} \) for the morphism \( \psi: \{ a, b \}^+ \rightarrow S = \{ a, b, ab, \alpha, \beta, \alpha \beta, a, b, \alpha \beta \} \) with \( \psi(a) = a \) and \( \psi(b) = b \) and the product in \( S \) is so that the semigroup elements \( \alpha a, \alpha \beta, \beta a, \alpha \beta \) are right absorbant. The repeated states of \( \mathcal{B} \) are \( R = \{ n_{aa}, n_{ab}, n_{bb}, n_{ba} \} \). Notice that merging \( n_{aa}, n_{ba} \) (or \( n_{ab}, n_{bb} \)) violates \((G_2)\) (the idempotents in \( S \) are \( \alpha a, \alpha \beta, \beta a, \alpha \beta \); the merge will result in \( a \in L_{n_{aa}} \), but \( \psi(a) = a \) is not idempotent.)

We now move towards the main result. Let \( \varphi: \Sigma^+ \rightarrow S \) be a semigroup morphism.

Theorem 4. Given \( \varphi \) as above, we can construct a \( \varphi \)-good automaton \( \mathcal{A}_\varphi \).

The proof is by induction on \( |S| \) with lexicographic ordering. Wilke [10] used this kind of induction while obtaining a temporal logic formula from a counter-free \( \omega \)-automata. The survey of Kufleitner and Diekert on local divisor technique [7] uses a similar induction to prove Simon’s Forest factorisation theorem. See also the survey [3] where the local divisor technique was used to obtain an LTL formula from an aperiodic monoid.

Base Cases

A first basic case is when \( S \) is a group, which is in particular the case when \( |S| = 1 \).

Lemma 5. If \( S \) is a group, we can construct a deterministic and complete weakly-good automaton for the morphism \( \varphi: \Sigma^+ \rightarrow S \).

Proof. We let \( Q = S \cup \{ \iota \} \). The initial state is \( \iota \). All other states are accepting: \( F = R = S \). The deterministic transition function is defined by \( \iota \xrightarrow{a} \varphi(a) \) and \( s \xrightarrow{a} s \cdot \varphi(a) \) for all \( s \in S \) and \( a \in \Sigma \). \((G_1)\) holds trivially since the automaton is deterministic and complete. Now, for \( s, t \in S \),
we check that \( L_{q, q} = \varphi^{-1}(0, 0) \), and \((0, 0)\) is the unit.

![Figure 1](image1.png)

**Figure 1** On the left, a \( \varphi \)-good automaton \( A \) for \( \{a, b\}^+ \to \{\alpha, \beta\} \), \( \varphi(a) = \alpha \), \( \varphi(b) = \beta \) and \( xy = x \) for all \( x, y \in S \). In the middle, a weakly-good automaton for \( \varphi \) which is deterministic and complete with one letter look-ahead. Here, the label \( x?y \) means reading \( x \) with look-ahead \( y \). On the right, is automaton \( B \) which is \( \psi \)-good for the morphism \( \psi: \{a, b\}^+ \to \{\alpha, \beta, \alpha\alpha, \alpha\beta, \beta\alpha, \beta\beta\} \) with \( \psi(a) = \alpha \), \( \psi(b) = \beta \) and \( xyz = xy \) for all \( x, y, z \in \{\alpha, \beta\} \). In all the figures, we use double circle to denote final states.

![Figure 2](image2.png)

**Figure 2** The \( \varphi \)-weakly good automaton \( A \), where \( \varphi: \{a, b\}^+ \to S \), \( S \) is the group \( ((\mathbb{Z}/2\mathbb{Z})^2, +) \). It is easy to see that for all states \( q \), \( L_{q, q} = \varphi^{-1}(0, 0) \), and \((0, 0)\) is the unit. We check that \( L_{s, s} = \varphi^{-1}(s^{-1}t) \). In particular, if \( s = t \) then \( L_{s, s} = \varphi^{-1}(1_S) \) where \( 1_S \) is the unit of \( S \) which is indeed idempotent. We deduce that \( (G_2) \) is also satisfied whichever total order \( \prec \) is chosen on \( Q \). We assume \( S < \iota \) so that \( (G_3) \) is also satisfied.

**Example 6.** As an example illustrating Lemma 5, consider the morphism \( \varphi: \{a, b\}^+ \to S \), where \( S = ((\mathbb{Z}/2\mathbb{Z})^2, +) \), the group of pairs \( (x, y) \in \{0, 1\}^2 \) with component wise addition, defined by \( \varphi(a) = (1, 0) \) and \( \varphi(b) = (0, 1) \). \((0, 0)\) is the unit element. The \( \varphi \)-weakly good automaton \( A \) is given in Figure 2. Since the automaton is deterministic and complete, \( (G_1) \) is easy. To see \( (G_2) \), observe that for any state \( q \), \( L_{q, q} \) is the set of all words with even number of \( a \) and \( b \), and indeed, \( \varphi(L_{q, q}) \) is \((0, 0)\), the unit element. This shows that \( L_q \subseteq L_{q, q} \subseteq \varphi^{(-1)}((0, 0)) \), satisfying \( (G_3) \). Finally, \( (G_3) \) holds trivially by construction on choosing an ordering of states respecting \( q < \iota \) for all \( q \neq \iota \).

The second basic case is when all letters from \( \Sigma \) are mapped to the same semigroup element, i.e., when \( |\varphi(\Sigma)| = 1 \).

**Lemma 7.** If all letters are mapped to the same semigroup element \( s \), i.e., \( \varphi(\Sigma) = \{s\} \), we can construct a deterministic and complete weakly-good automaton for the morphism \( \varphi: \Sigma^+ \to S \).

**Proof.** Since \( S \) is finite, there are integers \( k, \ell \geq 1 \) such that \( s^k = s^{k+\ell} \). We fix the least such pair for the lexicographic order. Also, since \( S \) is finite, we find \( n \geq 1 \) such that \( s^n \) is an idempotent.
Again, we fix the least such $n$. It is easy to see that $k \leq n$. Also, $n < k + \ell$ by minimality of $n$, since otherwise we have $s^n = s^{n-\ell}$. Further, from $s^n = s^{n+n}$ we deduce that $\ell$ divides $n$.

Now, we define the automaton. The set of states is $Q = \{0, 1, \ldots, k + n - 1\}$ and the initial state is $i = 0$. All states are accepting: $F = R = Q$. The deterministic and complete transition function is defined as expected: for all $a \in \Sigma$ and $i \in Q$ we let $i \xrightarrow{a} j$ where $j = i + 1$ if $i < k + n - 1$ and $j = k$ otherwise (see Figure 3).

Figure 3 The $\varphi$-weakly good automaton $A$ where $\varphi: \{a, b\}^+ \to S$, $S = \{s, s^2, s^3, s^4\}$, and $\varphi(a) = \varphi(b) = s$. All states are accepting.

Example 8. As an example illustrating Lemma 7, consider the morphism $\varphi: \{a, b\}^+ \to S$ defined by $\varphi(a) = \varphi(b) = s$ where $S$ is the finite semigroup $S = \{s, s^2, s^3, s^4\}$ with $s^4 = s^3$. We have $k = 3$ and $\ell = 2$ and the idempotent is $s^n = s^4$. The automaton $A$ for $\varphi$ is in Figure 3.

Inductive Steps

The other two cases are inductive. First, assume that there is some semigroup element $c \in \varphi(\Sigma)$ such that $Sc \subseteq S$. Then $(Sc, \cdot)$ is a strict subsemigroup of $(S, \cdot)$, i.e., $|Sc| < |S|$. Let $\Sigma_2 = \Sigma \cap \varphi^{-1}(c)$ be the set of all letters mapped to $c$ and $\Sigma_1 = \Sigma \setminus \Sigma_2$. If $\Sigma_1 = \emptyset$ then we are in the second basic case above. Hence we assume $\Sigma_1 \neq \emptyset$ and since $c \in \varphi(\Sigma) \setminus \varphi(\Sigma_1)$ we have $|\varphi(\Sigma_1)| < |\varphi(\Sigma)|$ so by induction hypothesis we can construct a good automaton $A_1 = (Q_1, \Sigma_1, \Delta_1, i_1, F_1, R_1, <_1)$ for the morphism $\varphi$ restricted to $\Sigma_1$. Each nonempty word $w \in \Sigma^+ \cup \Sigma^w$ has a unique factorization $w = (a_1u_1c_1)(a_2u_2c_2)(a_3u_3c_3)\cdots$ with $a_i \in \Sigma$, $u_i \in \Sigma_1^* \cup \Sigma_2^*$ and $c_i \in \Sigma_2$. If the word $w \in \Sigma^+$ is finite then the factorization has $n \geq 1$ blocks, the last block being either $a_nu_n$ or $a_nu_n\epsilon$. If $w \in \Sigma^w$ is infinite, the factorization has infinitely many blocks when $w$ has infinitely many letters from $\Sigma_2$, otherwise the factorization has $n \geq 1$ blocks and the last block is $a_nu_n$ with $u_n \in \Sigma_2^*$.

We view $B = \varphi(\Sigma_1^* \Sigma_2^*) \subseteq Sc$ as an alphabet and we consider the evaluation semigroup morphism $\psi: B^+ \to Sc$ defined by $\psi(b) = b$ for all $b \in B \subseteq Sc$. Let $b_i = \varphi(a_iu_ic_i) \in B$. The factorization of $w$ yields the word $b_1b_2b_3\cdots$ over $B$. Moreover, for $i \leq j$ we have $\psi(b_i \cdots b_j) = \varphi(a_iu_ic_i \cdots a_ju_jc_j)$. Since $|Sc| < |S|$, we can construct a good automaton $A_2 = (Q_2, B, \Delta_2, i_2, F_2, R_2, <_2)$ for the morphism $\psi: B^+ \to Sc$ by induction hypothesis.

Example 9. We give an example illustrating the first inductive case $Sc \not\subseteq S$. Consider the finite semigroup $S = \{s, s^2, s^3, s^4\}$ with $s^5 = s^2$ and $s^4$ idempotent. Consider the morphism $\varphi: \{a, b\}^+ \to S$ with $\varphi(a) = s$, $\varphi(b) = s^2$. Choosing $c = s^2$, we see that $Ss^2 = \{s^3, s^4\} \subseteq S$. It can be seen that $Ss^2$ is a group with unit element $s^4$. Considering $\Sigma_2 = \{b\}$ and $\Sigma_1 = \{a\}$ we have $\varphi(\Sigma_2) = s^2$, and $s^2 \not\in \varphi(\Sigma_1)$. The inductive hypothesis applies to $\varphi_1: \Sigma_1^* \to S$; since

Notice that, if $S$ is a monoid with unit $1_S$ then $1_S \not\in Sc$ (otherwise $Sc = S$). Hence $Sc$ is not a submonoid of $S$. Moreover, $Sc$ may not have a unit element. This is why we consider semigroup morphisms. Another possibility would be the local divisor technique described in [2] which allows to get a smaller monoid.
We now show how to construct a weakly-good automaton $A$ for $\varphi$ from $A_1$ and $A_2$. Consider the word $w = (aab)(baaabb)(ab) \in (\Sigma_1^* \cup \Sigma_2^*)^+$. Let $b_1 = s^3$, $b_2 = s^4$. Then $\varphi(w) = b_2 b_1 b_1 \in B^+$. Figure 4 depicts some example runs of $A$.

We now show how to construct a weakly-good automaton $A$ for $\varphi: \Sigma^* \to S$. Intuitively, we use $A_1$ to scan the words $u_i$ over $\Sigma_1$ and we use $A_2$ to scan the sequence of blocks $a_i u_i c_i$ represented by the letters $b_i$ in $B$. The set of states of $A$ is $Q = Q_2 \cup (Q_2 \times S \times Q_1)$. The initial state is $i = i_2$. The transitions are defined below in such a way that:

1. If $i_2 \xrightarrow{b_1} q_1 \xrightarrow{b_2} q_2 \xrightarrow{b_3} q_1 \cdots$ is a run of $A_2$ then we will have in $A$ the run

$$i \xrightarrow{a_1 u_1 c_1} q_1 \xrightarrow{a_2 u_2 c_2} q_2 \xrightarrow{a_3 u_3 c_3} q_3 \cdots$$

2. Now, zooming in some factor $a_i u_i c_i$ with $u_i = d_1 d_2 \cdots d_m$, if $i_1 \xrightarrow{d_1} p_1 \xrightarrow{d_2} p_2 \cdots \xrightarrow{d_m} p_m$ is a run of $A_1$ then, with $q = q_{i_{i-1}}$, we will have in $A$ the run

$$q \xrightarrow{a_i} (q, \varphi(a_i), i_1) \xrightarrow{d_1} (q, \varphi(a_i d_1), p_1) \xrightarrow{d_2} (q, \varphi(a_i d_1 d_2), p_2) \cdots \xrightarrow{d_m} (q, \varphi(a_i u_i), p_m) \xrightarrow{c_i} q_i$$

Formally, the transitions of $A$ are defined as follows:

- $q \xrightarrow{a \in \Sigma} (q, \varphi(a), i_1)$ for $q \in Q_2$,
- $(q, s, p) \xrightarrow{a \in \Sigma} (q, \varphi(a), p)$ if $p \xrightarrow{a_1} p'$ in $A_1$,
- $(q, s, p) \xrightarrow{a \in \Sigma} q'$ if $p \in F_1 \cup \{i_1\}$ and $q \xrightarrow{c} q'$ in $A_2$.

Notice that if $A_1$ and $A_2$ are deterministic and complete then so is the automaton $A$.

The total order $<$ on $Q$ is defined so that $Q_2 \times S \times Q_1 < Q_2$, and $< \text{ coincides with } <_2$ on $Q_2$, and $p < q'$ implies $(q, s, p) < (q, s', p')$. Notice that the initial state $i = i_2$ is the maximal state in $Q$ and has no incoming transitions, so $(G_3)$ holds. Figure 5 describes the

![Figure 4](image-url)
ordering. While summarizing the runs of \(A_1\) on \(u_i \in \Sigma_1^+\), all states are ranked strictly lower than the states of \(A_2\); hence, between two consecutive visits to \(Q_2\), all states seen are strictly lower. Intuitively this suggests that \(\varphi(L_q(A))\) for \(q \in Q_2\) is same as \(\psi(L_q(A_2))\). Likewise, while staying in \(Q_2 \times S \times Q_1\), the ordering of states is that in \(A_1\). Hence, while considering \(L_{(q,s,p)}(A_2)\), we cannot see any \(r \in Q_2\) in the loop; hence, \(\varphi(L_{(q,s,p)}(A_2))\) must be same as \(\varphi(L_q(A_1))\). This ensures \([\mathcal{L}_2]\). The final and repeated states of \(A\) are given by \(F = F_2 \cup \{(F_2 \cup \{t_2\}) \times S \times (F_1 \cup \{t_1\})\} \cup R = R_2 \cup \{(F_2 \cup \{t_2\}) \times S \times R_1\}\

**Lemma 10.** The automaton \(A\) defined above is weakly-good for \(\varphi: \Sigma^+ \to S\).

**Proof.** We have already seen that \(A\) satisfies \([G_3]\). We show that \(A\) satisfies \([G_1]\).

Consider a word \(w \in \Sigma^+ \cup \Sigma^\omega\) and its unique factorization \(w = (a_1u_1c_1)(a_2u_2c_2)(a_3u_3c_3)\cdots\) with \(a_i \in \Sigma\), \(u_i \in \Sigma_1^+ \cup \Sigma_1^\omega\) and \(c_i \in \Sigma_2\). Let \(b_i = \varphi(a_iu_ic_i) \in B\). There is a unique empty or accepting run \(\tau = t_2 \overset{b_i}{\rightarrow} q_1 \overset{b_1}{\rightarrow} q_2 \overset{b_2}{\rightarrow} q_3 \cdots\) of \(A_2\). For each \(i \geq 1\), assuming that \(u_i = d_1d_2\cdots\), there is a unique empty or accepting run \(\sigma_i = t_1 \overset{d_1}{\rightarrow} p_1 \overset{d_2}{\rightarrow} p_2 \cdots\) of \(A_1\). We construct the corresponding subrun \(\rho_i = q_{i-1} \overset{a_i}{\rightarrow} (q_{i-1}, \varphi(a_i), t_1) \overset{d_1}{\rightarrow} (q_{i-1}, \varphi(a_i), d_1) \overset{d_2}{\rightarrow} (q_{i-1}, \varphi(a_i, d_1), d_2) \cdots\) of \(A\). If \(u_i\) is finite with length \(m \geq 0\) then the last state of \(\rho_i\) is \((q_{i-1}, \varphi(a_iu_i), p_m)\) with \(p_m \in F_1 \cup \{t_1\}\) (we let \(p_0 = t_1\)). In this case, if \(c_i\) exists there is a transition \((q_{i-1}, \varphi(a_iu_i), p_m) \overset{c_i}{\rightarrow} q_i\) in \(A\) since \(b_i = \varphi(a_iu_i)c\) and \(q_{i-1} \overset{b_i}{\rightarrow} q_i\) is a transition of \(A_2\). Therefore, \(\rho_{i-1} \overset{c_i}{\rightarrow} q_i\) is a subrun of \(A\) reading \(a_iu_ic_i\).

When \(w\) contains infinitely many letters from \(\Sigma_2\), the factorization is infinite and we obtain the run \(\rho = \rho_1 \overset{c_1}{\rightarrow} \rho_2 \overset{c_2}{\rightarrow} \rho_3 \cdots\) of \(A\) reading \(w\). Since \(\tau\) is accepting in \(A_2\) we have \(q_i \in R_2\) for infinitely many \(i\)’s. Therefore, \(\rho\) is accepting in \(A\).

Assume now that \(w\) contains finitely many letters from \(\Sigma_2\). Then the factorization is finite, say of length \(n > 0\). If the last factor \(a_nu_nc_n\) is complete then \(\rho = \rho_1 \overset{c_1}{\rightarrow} \rho_2 \overset{c_2}{\rightarrow} \cdots \overset{c_{n-1}}{\rightarrow} \rho_n\) is a run of \(A\) reading \(w\) which is accepting since \(\tau\) is accepting.

When the last factor is of the form \(a_nu_n\) then \(\rho = \rho_1 \overset{c_1}{\rightarrow} \rho_2 \overset{c_2}{\rightarrow} \cdots \overset{c_{n-1}}{\rightarrow} \rho_n\) is a run of \(A\) reading \(w\). Since \(\tau\) is empty or accepting, we have \(q_{n-1} \in F_2 \cup \{t_2\}\). Since \(\sigma_n\) is an empty or an accepting finite or infinite run of \(A_1\), we deduce that \(A\) is accepting. We have proved that the automaton \(A\) accepts all words in \(\Sigma^+ \cup \Sigma^\omega\).

We show now that \(A\) is unambiguous. Let \(\rho'\) be an accepting run of \(A\) on \(w\). We have to show that \(\rho' = \rho\) where \(\rho\) is the accepting run for \(w\) defined above. By definition of \(A\), the run \(\rho'\) induces the very same factorization of \(w = (a_1u_1c_1)(a_2u_2c_2)\cdots\). Moreover, we can write \(\rho' = \tau \overset{a_1u_1c_1}{\rightarrow} q_1' \overset{a_2u_2c_2}{\rightarrow} q_2' \cdots\) and \(\tau' = \tau \overset{a_1u_1c_1}{\rightarrow} q_1' \overset{a_2u_2c_2}{\rightarrow} q_2' \cdots\) is a run of \(A_2\). We show that \(\tau' = \tau\).

If \(w\) has infinitely many letters from \(\Sigma_2\) then the run \(\tau'\) is infinite and none of the states \(q_i\) belongs to \(F_2 \cup \{t_2\}\) since \(A_2\) is good. Now \(\rho'\) is accepting in \(A\) and by definition of \(R\) we deduce
that \( q'_i \in R_2 \) for infinitely many \( i \)'s. Therefore \( \tau' \) is accepting in \( A_2 \). Since \( A_2 \) satisfies \( G_1 \), we deduce that \( \tau' = \tau \), i.e., \( q'_i = q_i \) for all \( i \).

If \( w \) has finitely many letters from \( \Sigma_2 \) and the last factor is of the form \( a_n u_n c_n \) then \( \rho' \) ends in state \( q'_n \in F \). We deduce that \( q'_n \in F_2 \) and \( \tau' \) is accepting in \( A_2 \). As above, we deduce that \( \tau' = \tau \). If the last factor of the factorization is \( a_n u_n \) then \( \rho' \) ends in some state \( (q'_{n-1}, s, p) \in F \) and \( q'_{n-1} \) is the last state of \( \tau' \). By definition of \( F \), we deduce that \( \tau' \) is empty (if \( n = 1 \)) or accepting (if \( n > 1 \)). Again, we obtain \( \tau' = \tau \).

It remains to show that, for each \( i \), the subrun \( \rho'_i \) of \( \rho' \) reading \( a_i u_i \) equals \( \rho_i \). Assuming that \( u_i = d_1 d_2 \cdots \) by definition of \( A \) we deduce that \( \rho'_i = q_{i-1} \xrightarrow{a_i} (q_{i-1}, \varphi(a_i), t_i) \xrightarrow{d_i} (q_{i-1}, \varphi(a_i d_i), p'_i) \xrightarrow{d_i} (q_{i-1}, \varphi(a_i d_i), p'_2) \cdots \) and \( \sigma'_i = \rho_i \xrightarrow{d_i} p'_i \xrightarrow{d_i} p'_2 \cdots \) is a run of \( A_1 \). If \( u_i \) is infinite, since \( \rho' \) is accepting in \( A \) we deduce that \( \sigma'_i = \sigma_i \), hence also \( \rho'_i = \rho_i \). Assume now that \( u_i \) is finite with length \( m \geq 0 \). Clearly, if \( m = 0 \) then we have \( \rho'_i = \rho_i \). We assume now that \( m > 0 \) and we show that the last state \( p'_m \) of \( \sigma'_i \) is final. If \( c_i \) exists in the factorization then \( (q_{i-1}, \varphi(a_i u_i), p'_m) \xrightarrow{d_i} q_i \) is a transition in \( A \) which implies \( p_m \in F_1 \cup \{1\} \). If the last factor is \( a_i u_i \) then, since \( \rho' \) is accepting, we deduce that \( p'_m \in F_1 \cup \{1\} \). Now, \( m > 0 \) and \( G_1 \) implies that \( p_m \neq \{1\} \). Therefore, \( \sigma'_i \) is accepting in \( A_1 \) and we deduce as above that \( \sigma'_i = \sigma_i \), hence also \( \rho'_i = \rho_i \). Since this holds for all \( i \)'s, we have shown that \( \rho' = \rho \).

**A satisfies \( G_2 \)**

Let \( r \in Q \) be a state of \( A \) and \( w \in L_r(A) = L_{r, \psi, \varphi}(A) \). So we have in \( A \) a run \( \rho = r \xrightarrow{w} r \) using intermediary states strictly less than \( r \).

Assume first that \( r = q \in Q_2 \). Then, the run \( \rho \) of \( A \) induces the following factorization \( w = (a_1 u_1 c_1)(a_2 u_2 c_2) \cdots (a_n u_n c_n) \) with \( n > 0 \). We have \( \rho = q \xrightarrow{a_1 u_1 c_1} q_1 \xrightarrow{a_2 u_2 c_2} q_2 \cdots q_{n-1} \xrightarrow{a_n u_n c_n} q \) and the states \( q_1, \ldots, q_{n-1} \) are all less than \( q \). Therefore, with \( b_i = \varphi(a_i u_i c_i) \), we deduce that \( \tau = q \xrightarrow{b_1} q_1 \xrightarrow{b_2} q_2 \cdots q_{n-1} \xrightarrow{b_n} q \) is a run of \( A_2 \). Since the order \( < \) restricted to \( Q_2 \) equals \( <_2 \) we deduce that \( b_1 b_2 \cdots b_n \in L_q(A_2) = L_{q, \psi, \varphi}(A_2) \). Since \( A_2 \) satisfies \( G_2 \), we obtain \( \psi(b_1 \cdots b_n) = e_q \) where \( e_q \) is the idempotent associated with state \( q \) for \( A_2 \). Now, \( \varphi(w) = b_1 \cdots b_n = \psi(b_1 \cdots b_n) = e_q \) and we get \( L_r(A) \subseteq \varphi^{-1}(e_q) \).

The second case is when \( r = (q, s, p) \in Q_2 \times S \times Q_1 \). Since \( Q_2 \times S \times Q_1 < Q_2 \times Q_1 < Q_2 \), we deduce that \( \rho = (q, s, p) \xrightarrow{d_i} (q, s_i, p_i) \cdots (q, s_m, p_m) \xrightarrow{d_m} (q, s, p) \) for some \( m > 0 \) and the intermediary states \( (q, s_i, p_i) \) are all less than \( (q, s, p) \) in \( A \). By definition of the order \( < \) in \( A \) we deduce that \( p_i \leq p \) in \( A_1 \) for all \( 1 \leq i < m \). Therefore, \( p \xrightarrow{d_i} p_1 \cdots p_{m-1} \xrightarrow{d_m} p \) is a run of \( A_1 \) and \( w \in (L_p(A_1))^+ \). Let \( e_p \) be the idempotent associated with state \( p \) of \( A_1 \) by \( G_2 \). We have shown that \( L_r(A) \subseteq L_p(A_1)^+ \subseteq \varphi^{-1}(e_q) \) since \( e_p \) is an idempotent.

The second inductive case is when there is some semigroup element \( c \in \psi(S) \) such that \( c S \subseteq S \).

The proof is along the same lines as the previous one but the construction turns out to be more complicated. Again \( (c S, \cdot) \) is a strict subsemigroup of \( (S, \cdot) \), i.e., \(|cS| < |S| \). Let \( \Sigma = \Sigma \cap \psi^{-1}(c) \) be the set of all letters mapped to \( c \) and \( \Sigma' = \Sigma \setminus \Sigma_2 \). If \( \Sigma_1 = \emptyset \) then we are in the second basic case above. Hence we assume \( \Sigma_1 \neq \emptyset \) and since \( c \in \psi(S) \setminus \psi(\Sigma_1) \) we have \( |\psi(\Sigma_1)| < |\psi(S)| \) so by induction hypothesis we can construct a good automaton \( A_1 = (Q_1, \Sigma_1, \Delta_1, t_1, F_1, R_1, \psi, e_1) \) for the morphism \( \varphi \) restricted to \( \Sigma_1 \).

Each nonempty word \( w \) has a unique factorization \( w = a_0 u_0(c_1 a_1 u_1)(c_2 a_2 u_2)(c_3 a_3 u_3) \cdots \) with \( a_i \in \Sigma \), \( u_i \in \Sigma_1^* \cup \Sigma_2^* \) and \( c_i \in \Sigma_1 \). If \( w \in \Sigma^* \) is infinite, the factorization has infinitely many blocks when \( w \) has infinitely many letters from \( \Sigma_2 \), otherwise the factorization ends with some \( a_n u_n \in \Sigma_2 \Sigma_1^* \) with \( n \geq 0 \). If the word \( w \in \Sigma^* \) is finite then the factorization ends with \( a_n u_n \in \Sigma_2 \Sigma_1^* \) with \( n \geq 0 \) or it ends with \( c_n \) with \( n \geq 1 \).

We view \( B = \psi(\Sigma_2 \Sigma_1^*) \subseteq c S \) as an alphabet and we consider the evaluation semigroup morphism \( \psi: B^+ \to c S \). Let \( \psi = \psi(c a_i u_i) \in B \). The factorization of \( w \) yields the word \( b_1 b_2 b_3 \cdots \) over \( B \). Moreover, for \( i < j \) we have \( \psi(b_i \cdots b_j) = \psi(c a_i u_i \cdots c_j a_j u_j) \).
Intuitively, the first choice (1(a),3(a),4) has to be taken when the next letter is in $\perp$ or $\perp/5$. Hence, even if the automata

$$\text{Formally, the transitions of } \varphi \text{ (automaton} \ \mathcal{A} \ \text{morphism} \ \psi \text{) are defined below so that:}
$$

1. If $t_2 \xrightarrow{a} q_1 \xrightarrow{b} q_2 \xrightarrow{b} q_3 \cdots$ is a run of $\mathcal{A}_2$ then we will have in $\mathcal{A}$ the run

$$t \xrightarrow{a_0 u_0} (t_2, \perp, \perp) \xrightarrow{c_0 a_1 u_1} (q_1, \perp, \perp) \xrightarrow{c_2 a_2 u_2} (q_2, \perp, \perp) \cdots$$

2. Now, zooming in the initial factor $a_0 u_0$ with $u_0 = d_1 d_2 d_3 \cdots$, if $t_1 \xrightarrow{d_1} p_1 \xrightarrow{d_2} p_2 \xrightarrow{d_3} p_3 \cdots$ is a run of $\mathcal{A}_1$ then, we will have in $\mathcal{A}$ the run

$$t \xrightarrow{a_0} (\perp, \varphi(a_0), t_1) \xrightarrow{d_1} (\perp, \varphi(a_0 d_1), p_1) \xrightarrow{d_2} (\perp, \varphi(a_0 d_1 d_2), p_2) \cdots$$

3. Finally, zooming in some factor $c_1 a_1 u_1$ with $u_1 = d_1 d_2 d_3 \cdots$, if $t_1 \xrightarrow{d_1} p_1 \xrightarrow{d_2} p_2 \xrightarrow{d_3} p_3 \cdots$ is a run of $\mathcal{A}_1$ then, with $q = q_{i-1}$, we will have in $\mathcal{A}$ the run

$$(q, \perp, \perp) \xrightarrow{c_0} (q, c \varphi(a_0), t_1) \xrightarrow{d_1} (q, c \varphi(a_0 d_1), p_1) \xrightarrow{d_2} (q, c \varphi(a_0 d_1 d_2), p_2) \cdots$$

Formally, the transitions of $\mathcal{A}$ are defined as follows:

1. (a) $t \xrightarrow{a \in \Sigma} (\perp, \varphi(a), t_1)$, (b) $t \xrightarrow{a \in \Sigma} (t_2, \perp, \perp)$,
2. $\langle q, \perp, \perp \rangle \xrightarrow{a \in \Sigma} \langle q, c, \perp \rangle$ for $q \in Q_2$,
3. (a) $\langle q, c, \perp \rangle \xrightarrow{a \in \Sigma} \langle q, c \varphi(a), t_1 \rangle$ for $q \in Q_2$, (b) $\langle q, c, \perp \rangle \xrightarrow{a \in \Sigma} \langle q', \perp, \perp \rangle$ if $q \xrightarrow{c \varphi(a)} q'$ in $\mathcal{A}_2$,
4. $\langle q, s, p \rangle \xrightarrow{a \in \Sigma} \langle q, s \varphi(a), p' \rangle$ if $q \in Q_2^i$ and $p \xrightarrow{a} p' \notin F_1$ in $\mathcal{A}_1$,
5. $\langle q, s, p \rangle \xrightarrow{a \in \Sigma} \langle q', \perp, \perp \rangle$ if $p \xrightarrow{a} p' \in F_1$ in $\mathcal{A}_1$ and $q \xrightarrow{c \varphi(a)} q'$ in $\mathcal{A}_2$ or $(q = \perp \land q' = t_2)$.

Notice that there are non-deterministic choices between transitions of type 1(a)/1(b), or 3(a)/3(b) or 4/5. Hence, even if the automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are deterministic, the automaton $\mathcal{A}$ constructed in this second inductive case is non-deterministic. We will see below that it is unambiguous. Intuitively, the first choice (1(a),3(a),4) has to be taken when the next letter is in $\Sigma_1$ while the second choice (1(b),3(b),5) has to be taken when the next letter is in $\Sigma_2$.

The total order $<$ on $Q$ is defined so that $Q_2 \times \{ \perp \} \times \{ \perp \} < Q_2^i \times S \times Q_1 < Q_2 \times \{ \perp \} \times \{ \perp \} < t$ and $(q, \perp, \perp) < (q', \perp, \perp)$ if $q < q'$, and $p \xrightarrow{a} p'$ implies $(q, s, p) < (q, s', p')$ for all $s, s' \in S$ and $q \in Q_2^i$. Notice that the initial state $t = (\perp, \perp, \perp)$ is the maximal state in $Q$ and has no incoming transitions, so $[t]$ holds.

The final and repeated states of $\mathcal{A}$ are given by $F = (F_2 \cup \{ t_2 \}) \times \{ \perp, c \} \times \{ \perp \}$, and $R = (R_2 \times \{ \perp \} \times \{ \perp \}) \cup (F_2^i \cup \{ t_4 \}) \times S \times R_1$. 

![Figure 6 Run for the second inductive case: $cS \subseteq S$.](image)
Consider a word \( w \in \Sigma^+ \cup \Sigma^* \) and its unique factorization \( w = a_0u_0(c_1a_1u_1)(c_2a_2u_2)(c_3a_3u_3) \cdots \) with \( a_i \in \Sigma, u_i \in \Sigma^* \cup \Sigma^*_1 \) and \( c_i \in \Sigma_2 \). Let \( b_i = \varphi(c_ia_iu_i) \in B \). There is a unique empty or accepting run \( \tau = q_2 \xrightarrow{b_1} q_1 \xrightarrow{b_2} q_2 \xrightarrow{b_3} q_3 \cdots \) of \( A_2 \). For each \( i \geq 0 \), assuming that \( u_i = d_1d_2 \cdots \) is a unique empty or accepting run \( \sigma_i = q_1 \xrightarrow{d_1} p_1 \xrightarrow{d_2} p_2 \cdots \) of \( A_1 \). We construct the corresponding subruns of \( A \). When \( i = 0 \) we define

\[
\rho_0 = \iota \xrightarrow{a_0} (q_2, \bot, \bot) \quad \text{(if } u_0 = \varepsilon) \\
\rho_0 = \iota \xrightarrow{a_0} (\bot, \varphi(a_0), \iota_1) \xrightarrow{d_1} (\bot, \varphi(a_0d_1), p_1) \xrightarrow{d_2} (\bot, \varphi(a_0d_1d_2), p_2) \cdots \quad \text{(if } |u_0| = m > 0) \\
\rho_0 = \iota \xrightarrow{a_0} (\bot, \varphi(a_0), \iota_1) \xrightarrow{d_1} (\bot, \varphi(a_0d_1), p_1) \xrightarrow{d_2} (\bot, \varphi(a_0d_1d_2), p_2) \cdots \quad \text{(if } u_0 \in \Sigma_2^*)
\]

Notice that when \( |u_0| = m > 0 \) then the last state of \( \sigma_0 \) is accepting, hence the last transition of \( \rho_0 \) in this case is well-defined. When \( i > 0 \), we define (with \( q_0 = \tau_2 \)):

\[
\rho_i = (q_{i-1}, \bot, \bot) \xrightarrow{c_i} (q_{i-1}, c_1, \bot) \xrightarrow{a_0} (q_i, \bot, \bot) \quad \text{(if } u_i = \varepsilon) \\
\rho_i = (q_{i-1}, \bot, \bot) \xrightarrow{c_i} (q_{i-1}, c_1, \bot) \xrightarrow{a_0} (q_{i-1}, c\varphi(a_i), \iota_1) \xrightarrow{d_1} (q_{i-1}, c\varphi(a_id_1), p_1) \xrightarrow{d_2} (q_{i-1}, c\varphi(a_id_1d_2), p_2) \cdots \quad \text{(if } u_i = \varepsilon) \\
\rho_i = (q_{i-1}, \bot, \bot) \xrightarrow{c_i} (q_{i-1}, c_1, \bot) \xrightarrow{a_0} (q_{i-1}, c\varphi(a_i), \iota_1) \xrightarrow{d_1} (q_{i-1}, c\varphi(a_id_1), p_1) \xrightarrow{d_2} (q_{i-1}, c\varphi(a_id_1d_2), p_2) \cdots \quad \text{(if } u_i \in \Sigma_2^*)
\]

Notice that when \( |u_i| = m > 0 \) then the last state of \( \sigma_i \) is accepting and we have \( b_i = c\varphi(a_iu_i) \), hence the last transition of \( \rho_i \) in this case is well-defined.

When \( w \) contains infinitely many letters from \( \Sigma_2 \), the factorization is infinite and each \( u_i \) is finite. We obtain a run \( \rho = \rho_0\rho_1\rho_2 \cdots \) of \( A \) for \( w \). Since \( \tau \) is accepting in \( A_2 \), we deduce that \( \rho \) uses infinitely many states from \( R_2 \times \{\bot\} \times \{\bot\} \). Therefore, \( \rho \) is accepting in \( A \).
Assume now that $w$ contains finitely many letters from $\Sigma_2$. Then the factorization is finite. If the last factor is $c_nu_au_a$, with $n > 0$ then $\rho = \rho_0\rho_1\rho_2\cdots\rho_n$ is a run of $A$ for $w$.

- If $u_a$ is finite then the last state of $\rho$ is $r = (q_a, \perp, \perp)$. Since $\tau$ is accepting, we deduce that $q_a \in F_2$ and therefore $r \in F$ and $\rho$ is accepting.
- If $u_a \in \Sigma_1^*$ is infinite then the run $\tau$ of $A_2$ ends in state $q_{n-1}$ (recall that $q_0 = \tau_2$). Since $\tau$ is empty or accepting, we have $q_{n-1} \in F_2 \cup \{\tau_2\}$. Now, the run $\sigma_n$ of $A_1$ reading $u_a$ is accepting, hence it uses infinitely many states from $R_1$. We deduce that $\rho_n$ uses infinitely many states in $\{q_{n-1}\} \times S \times R_1$ and $\rho$ is accepting.

If the last factor is $c_n$ with $n > 0$ then the run $\tau$ of $A_2$ ends in state $q_{n-1} \in F_2 \cup \{\tau_2\}$. Therefore, $\rho = \rho_0\rho_1\rho_2\cdots\rho_{n-1} \xrightarrow{c_n} (q_{n-1}, c, \perp)$ is a run of $A$ for $w$ which is accepting by definition of $F$.

The last case is when $w = a_0u_0$ with $u_0 \in \Sigma_1 \cup \Sigma_2^*$. Then $\rho = \rho_0$ is a run of $A$ for $w$. If $u_0$ is finite, then $\rho = \rho_0$ ends in state $(i_2, \perp, \perp) \in F$ and $\rho$ is accepting. If $u_0$ is infinite, then $\rho = \rho_0$ uses infinitely many states in $\{\perp\} \times S \times R_1$ since $\sigma_0$ is accepting in $A_1$. Again, $\rho$ is accepting.

We have proved that the automaton $A$ accepts all words in $\Sigma^+ \cup \Sigma^\omega$.

We show now that $A$ is unambiguous. Let $\rho'$ be an accepting run of $A$ on $w$. We have to show that $\rho' = \rho$ where $\rho$ is the accepting run for $w$ defined above. By definition of $A$, the run $\rho'$ induces the very same factorization of $w = a_0u_0(c_1a_1u_1)(c_2a_2u_2)\cdots$ with $a_i \in \Sigma$, $u_i \in \Sigma_1 \cup \Sigma_2^*$ and $c_i \in \Sigma_2$. Moreover, we can write

$$\rho' = \tau \xrightarrow{a_0u_0} (t_2, \perp, \perp) \xrightarrow{c_1a_1u_1} (q'_1, \perp, \perp) \xrightarrow{c_2a_2u_2} (q'_2, \perp, \perp) \cdots$$

We denote by $\rho'_i$ the subrun of $\rho'$ reading $a_0u_0$ and by $\rho'_i$ the subrun of $\rho'$ reading $c_1a_1u_1$ for $i > 0$.

From the definition of the transistions in $A$, it is easy to check that $\tau' = t_2 \xrightarrow{a_0u_0} q'_1 \xrightarrow{c_1a_1u_1} q'_2 \cdots$ is a run of $A_2$. We first show that $\tau' = \tau$.

If $w$ has infinitely many letters from $\Sigma_2$ then the run $\tau'$ is infinite and none of the states $q'_i$ belongs to $F_2 \cup \{\tau_2\}$ since $A_2$ is good. Now $\rho'$ is accepting in $A$ and by definition of $R$ we deduce that $q'_i \in R_2$ for infinitely many $i$’s. Therefore $\tau'$ is accepting in $A_2$. Since $A_2$ satisfies $[\mathcal{G}_1]$, we deduce that $\tau' = \tau$, i.e., $q'_i = q_i$ for all $i$.

If $w$ has finitely many letters from $\Sigma_2$ and the last factor is $c_n$ with $n > 0$ then $\rho'$ ends in state $(q'_{n-1}, c, \perp) \in F$. We deduce that $q'_{n-1} \in F_2 \cup \{\tau_2\}$ and $\tau'$ is empty (if $n = 1$) or accepting (if $n > 1$). As above, we deduce that $\tau' = \tau$.

Assume now that $w$ has finitely many letters from $\Sigma_2$ and the factorization ends with $a_0u_0a_1u_1\cdots$ $(n \geq 0)$. If $n = 0$ then $\tau'$ is empty and we get $\tau' = \tau$. So we assume $n > 0$. If $u_n$ is infinite, then by definition of $R$ we have $q_{n-1} \in F_2 \cup \{\tau_2\}$ with $q_0 = \tau_2$. We deduce that $\tau'$ is empty when $n = 1$ or accepting ending with $q'_{n-1} \in F_2$ when $n > 1$. Again we deduce that $\tau' = \tau$. The last case is when $u_n$ is finite. Since $\rho'$ is accepting, it ends in some state $r \in F$. Due to the letter $a_n$, $r = (q, c, \perp)$ is not possible. Therefore, $r = (q'_n, \perp, \perp) \in F$ and $q'_n \in F_2 \cup \{\tau_2\}$. We deduce that $\tau'$ is accepting and again $\tau' = \tau$.

It remains to show that $\rho'_i \equiv \rho_i$ for all $i$. Assume that $u_i = d_1d_2\cdots$. We start with the case $i = 0$. There are three cases depending on whether $u_0$ is empty, finite of length $m > 0$, or infinite.

- If $u_0 = \varepsilon$, by definition of $A$ we deduce that $\rho'_0 = \tau \xrightarrow{a_0} (t_2, \perp, \perp)$. Indeed, either $w = a_0$ and $\rho'_0 = \rho'$ which is accepting, which implies that the last state of $\rho'$ is $(i_2, \perp, \perp)$ by definition of $F$.
- Or the letter $c_1$ exists and the second transition of $\rho'$ must be of type 3, which implies again that the first transition of $\rho'$ is of type 2. In both cases, $\rho'_0 = \tau \xrightarrow{a_0} (t_2, \perp, \perp) = \rho_0$.
- If $u_0$ is of length $m > 0$ then by definition of $A$ we deduce that

$$\rho'_0 = \tau \xrightarrow{a_0} (\perp, \varphi(a_0), t_1) \xrightarrow{d_1} (\perp, \varphi(a_0d_1), p'_1) \cdots (\perp, \varphi(a_0d_1 \cdots d_{m-1}), \rho'_{m-1}) \xrightarrow{d_m} (t_2, \perp, \perp)$$

As above, we can check that the last state of $\rho'_0$ must be $(i_2, \perp, \perp)$ either because $w = a_0u_0$ and $\rho'$ is accepting, or because $c_1$ exists and the transition reading $c_1$ must start from $(t_2, \perp, \perp)$. Therefore, the last transition of $\rho'_0$ is of type 7 and we deduce that $\rho'_{m-1} \xrightarrow{d_m} \rho'_m \in F_1$ in $A_1$
As above, we can check that the last state of \( \sigma_0' = \sigma_0 \) and it follows \( \rho_0' = \rho_0 \).

If \( u_0 = \varepsilon \), by definition of \( A \) we deduce that

\[
\rho_0' = \varepsilon \xrightarrow{a_0} (\varepsilon, \varepsilon, a_0, t_0) \xrightarrow{d_1} (\varepsilon, \varepsilon, a_0 d_1, p'_1) \xrightarrow{d_2} (\varepsilon, \varepsilon, a_0 d_1 d_2, p'_2) \cdots
\]

and \( \sigma_0' = t_1 \xrightarrow{d_1} p'_1 \xrightarrow{d_2} p'_2 \cdots \) is a run of \( A_1 \) for \( u_0 \). Since \( \rho' \) is accepting in \( A \) we deduce that \( \sigma_0' \) is accepting in \( A_1 \). Since \( A_1 \) is unambiguous, we deduce that \( \sigma_0' = \sigma_0 \), hence also \( \rho_0' = \rho_0 \).

The case \( i > 0 \) is handled similarly. Again, there are three cases depending on whether \( u_i \) is empty, finite of length \( m > 0 \), or infinite.

If \( u_i = \varepsilon \), by definition of \( A \) we deduce that \( \rho_i' = (q_{i-1}, \perp, \perp) \xrightarrow{c_i} (q_{i-1}, c, \perp) \xrightarrow{a_i} (q_{i-1}, c\varphi(a_i), t_i) \xrightarrow{d_1} (q_{i-1}, c\varphi(a_i d_1), p'_1) \cdots \)

\[
(q_{i-1}, \varphi(a_i d_1 \cdots d_{m-1}), p'_{m-1}) \xrightarrow{d_m} (q_i, \perp, \perp)
\]

As above, we can check that the last state of \( \rho_i' \) must be \( (q_i, \perp, \perp) \) either because \( c_i a_i u_i \) is the last factor of \( w \) and \( \rho' \) is accepting, or because \( c_{i+1} \) exists and the transition reading \( c_{i+1} \) must start from \( (q_i, \perp, \perp) \). Therefore, the last transition of \( \rho_i' \) is of type 7 and we deduce that \( p'_{m-1} \xrightarrow{d_m} p'_m \in F_1 \) in \( A_1 \) (with \( \rho'_0 = t_1 \)). Therefore, \( \sigma_i' = t_1 \xrightarrow{d_1} p'_1 \xrightarrow{d_2} p'_2 \cdots \xrightarrow{d_{m-1}} p'_m \) is an accepting run of \( A_1 \) for \( u_i \). Since \( A_1 \) is unambiguous, we obtain \( \sigma_i' = \sigma_i \), and it follows \( \rho_i' = \rho_i \).

If \( u_i \) is of length \( m > 0 \) then by definition of \( A \) we deduce that

\[
\rho_i' = (q_{i-1}, \perp, \perp) \xrightarrow{c_i} (q_{i-1}, c, \perp) \xrightarrow{a_i} (q_{i-1}, c\varphi(a_i), t_i) \xrightarrow{d_1} (q_{i-1}, c\varphi(a_i d_1), p'_1) \xrightarrow{d_2} \cdots
\]

and \( \sigma_i' = t_1 \xrightarrow{d_1} p'_1 \xrightarrow{d_2} p'_2 \cdots \) is a run of \( A_1 \) for \( u_i \). Since \( \rho' \) is accepting in \( A \) we deduce that \( \sigma_i' \) is accepting in \( A_1 \). Since \( A_1 \) is unambiguous, we deduce that \( \sigma_i' = \sigma_i \), hence also \( \rho_i' = \rho_i \).

We have proved that \( \rho_i' = \rho_i \) for all \( i \). Therefore, \( \rho' = \rho_0 \rho_1 \rho_2 \cdots = \rho_0 \rho_1 \rho_2 \cdots = \rho \).

\( \mathcal{A} \) satisfies \( \Box(3) \)

Let \( r \in Q \) be a state of \( \mathcal{A} \) and \( w \in L_r(\mathcal{A}) = L_{r,\perp,r}(\mathcal{A}) \). So we have in \( \mathcal{A} \) a run \( \rho = r \xrightarrow{w} r \) using intermediary states strictly less than \( r \). Note that \( r = \varepsilon = (\perp, \perp, \perp) \) is not possible. Also, \( r \in Q_2 \times \{c\} \times \{\perp\} \) is not possible since between two occurrences of such states, we must use a transition of type 3, hence we must have a state from \( Q_2 \times \{\perp\} \times \{\perp\} \) which is strictly above for \( < \) in \( \mathcal{A} \).

Assume first that \( r = (q, \perp, \perp) \) with \( q \in Q_2 \). Then, the run \( \rho \) of \( \mathcal{A} \) induces the following factorization \( w = (c_1 a_1 u_1) (c_2 a_2 u_2) \cdots (c_n a_n u_n) \) with \( n > 0 \). We have

\[
\rho = (q, \perp, \perp) \xrightarrow{c_1 a_1 u_1} (q_1, \perp, \perp) \xrightarrow{c_2 a_2 u_2} (q_2, \perp, \perp) \cdots (q_{n-1}, \perp, \perp) \xrightarrow{c_n a_n u_n} (q, \perp, \perp)
\]

and the intermediary states \( (q_i, \perp, \perp) \) are all less than \( (q, \perp, \perp) \) in \( \mathcal{A} \). Therefore, with \( b_i = \varphi(c_i a_i u_i) \), we deduce that \( \tau = q \xrightarrow{b_1} q_1 \xrightarrow{b_2} q_2 \cdots \xrightarrow{b_{n-1}} q_n \xrightarrow{b_n} q \) is a run of \( \mathcal{A}_2 \). By definition of the order \( < \), we deduce that \( q_i < q \) for \( 1 \leq i < n \). Therefore, \( b_1 b_2 \cdots b_n \in L_q(\mathcal{A}_2) = L_{q,\perp,q}(\mathcal{A}_2) \).

Since \( \mathcal{A}_2 \) satisfies \( \Box(2) \), we obtain \( \psi(b_1 \cdots b_n) = e_q \) where \( e_q \) is the idempotent associated with state \( q \) for \( \mathcal{A}_2 \). Now, \( \varphi(w) = b_1 \cdots b_n = \psi(b_1 \cdots b_n) = e_q \) and we get \( L_\varphi(\mathcal{A}) \subseteq \varphi^{-1}(e_q) \).
The second case is when \( r = (q, s, p) \in Q_2^+ \times S \times Q_1 \). Since \( Q_2^+ \times S \times Q_1 \subseteq Q_2 \times \{ \bot \} \times \{ \bot \} \) in \( A \), the run \( p \) may only use transitions of type 6. We deduce that

\[
p = (q, s, p) \xrightarrow{d_1} (q, s_1, p_1) \cdots (q, s_{m-1}, p_{m-1}) \xrightarrow{d_m} (q, s, p)
\]

for some \( m > 0 \) and the intermediary states \((q, s_i, p_i)\) are all less than \((q, s, p)\) in \( A \). By definition of the order \( \prec \) in \( A \) we deduce that \( p_i \leq p \) in \( A_1 \) for all \( 1 \leq i < m \). Therefore, \( p \xrightarrow{d_1} p_1 \cdots p_{m-1} \xrightarrow{d_m} p \) is a run of \( A_1 \) and \( w \in (L_p(A_1))^+ \). Let \( e_p \) be the idempotent associated with state \( p \) of \( A_1 \) by \((G_2)\). We have shown that \( L_r(A) \subseteq (L_p(A_1))^+ \subseteq \varphi^{-1}(e_p) \) since \( e_p \) is an idempotent. This concludes the proof. \( \square \)

**Example 12.** To illustrate the second inductive case, consider the morphism \( \varphi: \Sigma^+ \to S = \{\alpha, \beta\} \) defined in Example **3**. The first inductive case does not apply to \( \varphi \) since \( S_\alpha = S_\beta = S \). The second inductive case can be used here: \( \alpha S = \{\alpha\} \subseteq S \). Then, \( \Sigma_1 = \{b\} \) and \( \Sigma_2 = \{a\} \). We have the morphism \( \varphi_1: \Sigma_1^+ \to \{\beta\} \) to which, applying the first or second basic case gives us the automaton \( A_1 \). Also, \( B = \varphi(\Sigma_2 \Sigma_1^+ \Sigma_1^*) = \{\alpha\} \) and we have the morphism \( \psi: B^+ \to \alpha S = \{\alpha\} \).

Both \( A_1, A_2 \) are automata with three states (see Figure **7**). We can apply the construction explained above on \( A_1, A_2 \) to obtain \( A \) as in Figure **10**. Notice that the automaton \( A \) in Figure **1** (manually crafted) is also \( \varphi \)-good.

**Why the weakly-good \( A \) cannot be deterministic.** On this example, we now explain why a weakly-good automaton \( A \) for \( \varphi \) must be non-deterministic. Towards a contradiction, assume that there exists a \( \varphi \)-weakly-good deterministic automaton. Let \( A \) be such an automaton with a minimal number of states. Let \( q \) be the highest ranked state of \( A \) reachable from \( i \) such that \( L_q \neq \emptyset \). Consider \( v \in L_q \). Without loss of generality, assume that \( v \in a \Sigma^* \). Then, we claim that \( L_q \cap b \Sigma^* = \emptyset \). If not, we will have words \( v \in a \Sigma^* \) and \( v' \in b \Sigma^* \) both in \( L_q \). Since \( A \) satisfies \((G_2)\), we know that \( L_q \subseteq \varphi^{-1}(e) \) for some idempotent \( e \). The claim follows since \( \varphi(a \Sigma^*) = \{\alpha\} \), \( \varphi(b \Sigma^*) = \{\beta\} \) and \( \alpha \neq \beta \). Thus \( L_q \subseteq \varphi \Sigma^* \). Since \( A \) is deterministic and universal \((G_1)\), we have an outgoing transition on \( b \) from \( q \). Let \( q \xrightarrow{b} q' \). Then \( q \) cannot be reached from \( q' \). Indeed, assume there is a run \( p = q' \xrightarrow{w} q \) from \( q' \) to \( q \) and let \( q'' \) be the highest state in \( p \). From the run \( q' \xrightarrow{w} q \xrightarrow{b} q' \xrightarrow{w} q \) we deduce that \( L_{q''} \neq \emptyset \), which implies \( q'' \preceq q \) by choice of \( q \). From the run \( q \xrightarrow{b} q' \xrightarrow{w} q \) we deduce that \( L_q \cap b \Sigma^* \neq \emptyset \), a contradiction. This means that in any run of \( A \), \( b \xrightarrow{u} q \xrightarrow{w} q \xrightarrow{b} q' \xrightarrow{w} q' \cdots \) there is no occurrence of \( q \) after \( q' \). This allows us to construct an automaton \( A' \) from \( A \) with a new initial state \( q' \) having transitions \( \delta_{A'}(q', x) = \delta_{A}(q', x) \) for \( x \in \{a, b\} \), and \( \delta_{A'}(r, x) = \delta_{A}(r, x) \) for all \( r \notin \{i, q\} \) and \( x \in \{a, b\} \). We can check that \( A' \) is deterministic and \( \varphi \)-weakly-good, and has at least one less state than \( A \). This contradicts the minimality of \( A \).

Notice that if we allow a look-ahead of size 1, we can obtain a \( \varphi \)-weakly-good deterministic automaton; the automaton \( A \) of Figure **1** (middle) is in fact one such. The good automaton \( B \) on the right Figure **1** can also be converted into a weakly-good, deterministic automaton with look-ahead two. Notice that we can generalize this example to show that in general, to obtain weakly-good and deterministic automata, a bounded look-ahead will not suffice. Below, we generalize this argument.

**The inherent non-determinism of weakly-good automata**

In this section, we show that the \( \varphi \)-weakly-good automata cannot be made deterministic even with bounded look-ahead. The second inductive case is the reason why this cannot be. If we look at the construction in the second inductive case, we give a split of \( w \in \Sigma^+ \) into chunks of \( \Sigma_2 \Sigma_1^* \Sigma_1^* \). Even if \( A_1, A_2 \) are both deterministic and complete weakly-good, \( A \) introduces non-determinism since we have to guess whether the next symbol is in \( \Sigma_2 \) or in \( \Sigma_1 \), each time we process \( u_i \in \Sigma_1^* \). This may give an impression that we can get rid of the non-determinism by
using a look-ahead of size 1, which simply checks if the next symbol is in $\Sigma_1$ or $\Sigma_2$. While this is true for Example 12 (see the left and middle automata in Figure 1), in general it is not possible to construct a $\varphi$-weakly-good automaton which is deterministic, and has a bounded look-ahead.

Consider the morphism $\varphi : \Sigma^+ \rightarrow S$ where $\Sigma = \{a, b\}$, $S = \Sigma^{\leq k} = \{u \in \Sigma^+ \mid |u| \leq k\}$, $\varphi(x) = x$ for all $x \in \Sigma$ and the product in $S$ is so that the elements in $\Sigma^k$ are right-absorbing: $\alpha \cdot \beta = \alpha$ for all $\alpha \in \Sigma^k$ and $\beta \in S$. The morphism $\varphi$ is a generalization of the morphism in Example 12. Notice that the idempotents of $S$ are all elements of $\Sigma^k$. It is easy to see that one can construct a $\varphi$-weakly-good automaton which is deterministic with a $k$-look-ahead (generalizing Figure 1). We show that it is not possible to construct a $\varphi$-weakly-good automaton $A$ which is deterministic with a $(k - 1)$-look-ahead. Let us assume that we can indeed do this, and let $A$ be such an automaton with a minimal number of states.

Since $A$ satisfies $\mathbf{(G)}$, we know that for each state $q$, there is an idempotent $v_q \in \Sigma^k$ such that $L_q \subseteq \varphi^{-1}(e_q) = v_q \Sigma^*$. Let $q$ be the highest ranked state of $A$ which occurs at least twice on some infinite accepting run $\rho$ of $A$. We may write $\rho = \iota \xrightarrow{u_1} q \xrightarrow{v_1au_2} q \xrightarrow{v_2w} \rho$ where $u_1, u_2 \in \Sigma^*, v_1, v_2 \in \Sigma^{k-1}, a \in \Sigma$ and $w \in \Sigma^\omega$. The unique accepting run on $u_1(v_1au_2)(v_1au_2)v_2w$ must start with $\iota \xrightarrow{u_1} q \xrightarrow{v_1au_2} q \xrightarrow{v_2w} \cdots$ since $A$ is deterministic with $(k - 1)$-look-ahead. By choice of $q$ we deduce that all states $q'$ occurring in the subrun $q \xrightarrow{v_1au_2} q$ satisfy $q' \leq q$. We deduce that $v_1au_2 \in L_q^+ \subseteq v_q \Sigma^*$ and therefore $v_1a = v_q$.

Since $A$ is universal $\mathbf{(U)}$ and deterministic with $(k - 1)$-look-ahead, there are accepting runs for all words in $u_1v_1au_2v_2\Sigma^k$ and all these runs start with $\iota \xrightarrow{u_1} q \xrightarrow{v_1au_2} q \xrightarrow{v_2w}$. Along these runs, the state $q$ cannot be reached again. Otherwise, we would have a run $\iota \xrightarrow{u_1} q \xrightarrow{v_1au_2} q \xrightarrow{v_2w}$.

As above, we would get $v_2bu_3 \in L_q^+ \subseteq v_q \Sigma^*$. This is a contradiction since $v_q = v_1a \neq v_2b$.

This allows us to construct from $A$, an automaton $A'$ as follows. Let $\iota'$ be the initial state of $A'$, and define $\delta_A'(\iota', x?v) = \delta_A(q, v_2bx?v)$, for $x \in \Sigma, v \in \Sigma^{k-1}$. Note that $\delta_A'(\iota', x?v)$ is a state of $A$ other than $q$. Further, $\delta_A'(p, x?v) = \delta_A(p, x?v)$ for all $p \neq q, x \in \Sigma, v \in \Sigma^{k-1}$. This makes $A'$ a strictly smaller deterministic, $\varphi$-weakly-good automaton with $(k - 1)$-look-ahead whenever $A$ is $\varphi$-weakly-good, contradicting the minimality of $A$.

Remark. Notice that if we are dealing with commutative semigroups, then $Sc = cS$ for any $c \in S$. In this case, the second inductive case $cS \subseteq S$ coincides with the first one. The difficulty occurs when dealing with non-commutative semigroups, and in this case, the proof is much more challenging for the case $cS \not\subseteq S$ as seen above.

Wrapping Up

Now we show that we have covered all cases. Let $\varphi : \Sigma^+ \rightarrow S$ be a semigroup morphism such that for all $c \in \varphi(\Sigma)$ we have $cS = S = Sc$, i.e., neither of the two inductive cases may be applied. Wlog, we assume that $\varphi(\Sigma^+)$ = $S$, otherwise we restrict $S$ to its sub-semigroup $\varphi(\Sigma^+)$. Hence, each element $s \in S$ can be written as a product $s = c_1 \cdots c_k$ where $c_1, \ldots, c_k \in \varphi(\Sigma)$. From the hypothesis it follows that $sS = S = Ss$ for all $s \in S$. Using Lemma 13 below we deduce that $S$ is a group so that we are in the first basic case (Lemma 5). Lemma 13 is a folklore result, and also works for infinite semi-groups.

Lemma 13. If $S$ is a finite semigroup such that $sS = S = Ss$ for all $s \in S$ then $S$ is a group.

Proof. We show first that $S$ contains a unit element and next that all elements have an inverse.

Since $S$ is a finite semigroup, it contains some idempotent $e$. Now, $Se = S$ implies that the right multiplication by $e$ defines a permutation $\sigma_e$ of $S$. We obtain $\sigma_e = \sigma_e \circ \sigma_e$ since $e$ is an idempotent. We deduce that $\sigma_e = \text{Id}$ is the identity since permutations of $S$ with composition form a group. Therefore, $s = \sigma_e(s) = se$ for all $s \in S$ and $e$ is a right unit. Using $eS = S$ we deduce similarly that $e$ is a left unit and therefore a unit of $S$.

Finally, let $s \in S$. From $Ss = S = sS$ we deduce that $rs = e = st$ for some $r, t \in S$. It follows that $r = re = r(st) = (rs)t = et = t$ which is the inverse of $s$. □
4 Applications

We now focus on two applications obtained from synthesizing good automata. Given a morphism \( \varphi : \Sigma^+ \to S \) for a semi-group \( S \), we first derive the forest factorization theorem from the \( \varphi \)-good automaton \( A \) constructed above.

**Theorem 14 (Forest Factorization Derived).** Let \( \varphi : \Sigma^+ \to S \) be a morphism. For each finite or infinite word \( w \in \Sigma^\infty \), we can construct a Ramsey split \( \sigma \) whose height is bounded by the number of states of a weakly-good automaton for \( \varphi \).

**Proof.** Let \( A \) be a weakly-good automaton for the morphism \( \varphi \). In particular \( A \) satisfies \([G]\) and \([S]\). Let \( h : (Q, \prec) \to (\{1, \ldots, |Q|\}, \prec) \) be a monotone bijection. To define the split \( \sigma \) of \( w = a_1 a_2 a_3 \cdots \in \Sigma^\infty \), consider the unique accepting run \( \rho = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} q_3 \cdots \) of \( w \) in \( A \) and define \( \sigma(i) = h(q_i) \) for all positions \( i \geq 0 \) of \( w \). Notice that two positions \( i < j \) are \( \sigma \)-equivalent \((i \sim j)\) iff \( q_i = q_j \) and \( q_i \leq q_j \) for all \( i \leq k \leq j \). We deduce that \( w(i, j) = a_{i+1} \cdots a_j \in L_{q_i}^\infty \). Hence, \( \varphi(w(i, j)) = c_{q_i} \) is the idempotent associated with state \( q_i \). Therefore, the split \( \sigma \) for \( w \) is Ramsey. \( \square \)

Some remarks on the height of the factorisation tree

**Theorem 14** gives an easy proof for the existence of a Ramsey split where the height is bounded by the number of states of the weakly-good automaton for \( \varphi \). Notice that this bound on the height is rather loose and can be optimized. To get an idea of this height \( H \), we look at the basic and inductive cases. In the first base case when \( S \) is a group, we know by construction that \( H = |S| + 1 \). In the second basic case where \( |\varphi(\Sigma)| = 1 \), we know that \( |S| = k + \ell - 1 \), and the automaton had \( k + n \) states where \( k \leq n \leq k + \ell \), obtaining \( H \leq 2|S| \).

Now let us turn to the inductive cases. Let \( H_1, H_2 \) respectively be the number of states of \( A_1 \) and \( A_2 \). The monotone bijection \( h \) for \( A \) is defined using the monotone bijections \( h_1 : (Q_1, \prec_1) \to (\{1, \ldots, |Q_1|\}, \prec_1) \) and \( h_2 : (Q_2, \prec_2) \to (\{1, \ldots, |Q_2|\}, \prec_2) \) obtained from \( A_1, A_2 \).

Assuming we are in the first inductive case, the number of states \( |Q| \) of the constructed \( A \) is \(|Q| = |Q_1| + |Q_2| \times |S| \times |Q_1| \). Actually, one can check that we can save on the height of the split by defining \( h(q) = h_1(q) + h_2(q) \) for \( q \in Q_2 \), and \( h((q, s, p)) = h_1(p) \) for \( (q, s, p) \in Q_2 \times S \times Q_1 \). The map \( h \) is not a bijection anymore, but a careful analysis shows that the split as defined in the proof of **Theorem 14** is Ramsey.

Now assume we are in the second inductive case. The number of states \( |Q| \) of the constructed \( A \) is \(|Q| = (|Q_2| + 1) \times (|S| + 1) \times (|Q_1| + 1) \). Since the states of \( Q_2 \) are the lowest in the ordering, and since we know that we cannot revisit any \((q, c, \perp)\) without seeing a higher state, we can safely assign the same height to all of them: \( h(Q_2 \times \{c\} \times \perp) = 1 \). As in the first inductive case, we can also define \( h(q, s, p) = 1 + h_1(p) \) for \( (q, s, p) \in Q_2 \times S \times Q_1 \), \( h(\perp, \perp, \perp) = 1 + H_1 + h_2(q) \) for \( q \in Q_2 \) and \( h(\perp, \perp, \perp) = H_1 + H_2 + 2 \). The split obtained in this way is Ramsey.

Note that in both cases, \( h \) is indeed monotone, respecting the ordering of states in \( A \) (see Figures 5 \& 6). Moreover, the bound on the height \( H \) that we require is \( H \leq H_1 + H_2 + 2 \).

4.1 Good Automata to Good Expressions

In this section, we show how we can use the \( \varphi \)-good automaton to obtain good expressions. We start from a **good** automaton \( A = (Q, \Sigma, \Delta, i, f, R, \prec) \) for a semigroup morphism \( \varphi : \Sigma^+ \to S \). Wlog, we assume that \( A \) is reduced, i.e., all states in \( A \) belong to some accepting run. We construct the good expressions by state elimination. For all \( p, q \in Q \) and \( X \subseteq Q \) such that \( X < (p, q) \), and for all \( s \in S \), we construct a \( \varphi \)-good expression \( F_{p,X,q}^s \) such that \( L(F_{p,X,q}^s) = L_{p,X,q} \cap \varphi^{-1}(s) \) (recall that \( \emptyset \) is a good expression). The construction is by induction.
The base case is when $X = \emptyset$. Then, $L_{p,\emptyset,q} \subseteq \Sigma$ so $F^{s}_{p,\emptyset,q}$ is either empty or a finite union of letters from $\Sigma$, which is indeed $\varphi$-good.

Let $r \in Q$, $X = \downarrow r = \{ r' \in Q \mid r' < r \}$ and $Y = X \cup \{ r \}$. Assume by induction that for all $p, q$ such that $X < \{ p, q \}$ and all $s \in S$ we have already constructed good expressions $F^{s}_{p,r,q}$.

In particular, we have already computed the good expressions $F^{w}_{e,r,\varphi^{-1}(e_{r})}$. Since $e_{r}$ is an idempotent, $F^{e_{r},r}_{r,r}$ is also a good expression. We can also check that each sub-expression maps to the same semigroup element, which could be $s_{1}$ or some $s_{1}e_{r}$ in the last union. In particular, we have $F^{s}_{p,Y,q} = \varphi^{-1}(s)$ and we have $F^{s}_{p,Y,q} \in \varphi^{-1}(s)$. Therefore, the unambiguous expression is $\bigcup_{r \in R} F_{e,r,\varphi^{-1}(r)}^{e_{r}}(F^{e_{r},r}_{e,r,\varphi^{-1}(r)})^{\omega}$. Therefore the unambiguous expression is $G = \bigcup_{r \in R} F_{e,r,\varphi^{-1}(r)}^{e_{r}}(F^{e_{r},r}_{e,r,\varphi^{-1}(r)})^{\omega}$.

This conclude the proof. 

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