POLARIZED RIGID DEL PEZZO SURFACES IN LOW CODIMENSION

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Abstract. We classify wellformed and quasismooth polarized del Pezzo surfaces having basket of rigid orbifold points of type \( \{ k_i \times \frac{1}{r_i}(1, a) : 3 \leq r_i \leq 10, k_i \in \mathbb{Z}_{\geq 0} \} \), such that their images under their anti-canonical embedding in a weighted projective space can be described by using one of the following sets of equations: a single equation, two linearly independent equations, maximal Pfaffians of \( 5 \times 5 \) skew symmetric matrix and \( 2 \times 2 \) minors of size 3 square matrix. The classification is complete under certain carefully chosen bounds on the weights of ambient weighted projective spaces and it is largely based on detailed computer assisted searches by using the computer algebra system Magma.

1. Introduction

A del Pezzo surface is a two dimensional algebraic variety with an ample anti-canonical divisor class. The classification of smooth del Pezzo surfaces is well known and there are ten deformation families of such surfaces: \( \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2 \) and blow up of \( \mathbb{P}^2 \) in \( d \) general points for \( 1 \leq d \leq 8 \). An orbifold del Pezzo surface \( X \) is a del Pezzo surface with at worst isolated orbifold points, classically known as log del Pezzo surface with cyclic quotient singularities. We call \( X \) to be locally qGorenstein(qG)-rigid if it contains only rigid isolated orbifold points, i.e. the orbifold points are rigid (not smoothable) under qG-deformations. If it admits a qG-degeneration to a normal toric del Pezzo surface then it is called a del Pezzo surface of class TG.

Recently the classification of orbifold del Pezzo surfaces has received much attention, primarily due to the mirror symmetry program for Fano varieties by Coates, Corti et al \([CCG+12]\). The mirror symmetry for orbifold del Pezzo surface has been formally formulated in \([ACC+16]\) in the form of a conjecture expecting a one to one correspondence between mutation equivalence classes of Fano polygons with the (qG)-deformation equivalence classes of locally qG-rigid del Pezzo surfaces of class TG. Therefore the construction of rigid orbifold del Pezzo surfaces has important links with the mirror symmetry due to this conjecture. The conjecture has been proved for smooth del Pezzo surfaces by Kasprzyk, Nills and Prince in \([KNP17]\). Corti and Heuberger \([CH17]\) gave the classification of locally qG-rigid del Pezzo surfaces with \( \frac{1}{3}(1, 1) \) singular points. The del Pezzo surfaces with single orbifold point of type \( \frac{1}{3}(1, 1) \) have been classified by Cavey and Prince.

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The mutation equivalence classes of Fano polygons with rigid singularities of type
\[
\left\{ k_1 \times \frac{1}{3}(1,1), k_2 \times \frac{1}{6}(1,1) : k_1 > 0, k_2 \geq 0 \right\} \text{ and } \left\{ k \times \frac{1}{5}(1,1) : k > 0 \right\}
\]
have been computed in [CK17]. This is equivalent to classification of del Pezzo surfaces of class TG with the above given baskets; though it may be missing surfaces which do not admit toric degeneration and having one of the above type of baskets of singularities. By using birational techniques, the classification of orbifold del Pezzo surface with basket consisting of combination of $\frac{1}{3}(1,1)$ and $\frac{1}{6}(1,1)$ orbifold points was given by Miura [Miu19].

In [CH17] the classification gave a total of 29 deformation families of del Pezzo surfaces with $\frac{1}{3}(1,1)$ orbifold points which were divided into 6 different cascades; one of the cascades was first studied by Reid and Suzuki in [RS04]. Moreover, good model constructions for all 29 surfaces were presented as complete intersections inside the so called rep-quotient varieties (mainly toric varieties). Among those, six of them can be described as hypersurface in $\mathbb{P}^3(a_i)$ or as complete intersection in $\mathbb{P}^4(a_i)$ or as complete intersection in weighted Grassmannian $\text{wGr}(2,5)$ [CR02]. This motivated us to classify rigid del Pezzo surfaces with certain basket of singularities which can be described by relatively small sets of equations.

1.1. Summary of results. We classify polarized rigid del Pezzo surfaces, under the bounds chosen in [3.2] which contain baskets of orbifold points
\[
\left\{ k_i \times \frac{1}{r_i}(1, a) : 3 \leq r \leq 10, k_i \geq 0 \right\}
\]
such that their images under their anti-canonical embedding can be described by one of the following ways.

(i) as hypersurface, i.e. by a single weighted homogenous equation: $X_d \hookrightarrow \mathbb{P}^3(a_i)$.
(ii) as codimension 2 weighted complete intersection, i.e. by 2 weighted homogeneous equations: $X_{d_1, d_2} \hookrightarrow \mathbb{P}^4(a_i)$.
(iii) as codimension 3 variety described by using five maximal Pfaffians of $5 \times 5$ skew symmetric matrix;
\[
X_{d_1, \ldots, d_5} \hookrightarrow \mathbb{P}^5(a_i).
\]
In other words they are weighted complete intersections in weighted Grassmannian $\text{wGr}(2,5)$ or (weighted) projective cone(s) over it [CR02, QS11, QS12].
(iv) as codimension 4 variety described by using nine $2 \times 2$ minors of size 3 square matrix
\[
X_{d_1, \ldots, d_9} \hookrightarrow \mathbb{P}^6(a_i).
\]
Equivalently they are weighted complete intersections in some weighted $\mathbb{P}^2 \times \mathbb{P}^2$ variety or (weighted) projective cone(s) over it [BKQ18].

We summarize it in form of the following theorem.

Theorem 1.1. Let $X$ be an orbifold del Pezzo surface containing a basket
\[
B = \left\{ k_i \times \frac{1}{r_i}(1, a) : 3 \leq r \leq 10, k_i \geq 0 \right\}
\]
of rigid orbifold points and the image of $X \hookrightarrow \mathbb{P}(a_i)$ under their anti-canonical embedding can be described as a hypersurface or as a codimension 2 complete intersection or as a weighted complete intersection in $\text{wGr}(2,5)$ or as a weighted complete intersection of weighted $\mathbb{P}^2 \times \mathbb{P}^2$ variety. Then, under the bounds chosen in 3.2, $X$ is one of the del Pezzo surfaces listed in Tables 2–5. In total there are 147 such families of such del Pezzo surfaces, divided as follows in each codimension.

| Hypersurface | Complete intersection | $4 \times 4$ Pfaffians | $2 \times 2$ Minors |
|--------------|-----------------------|------------------------|--------------------|
| 81           | 25                    | 21                     | 20                 |

We construct these examples by first computing all possible candidate varieties with required basket of orbifold points using algorithmic approach developed in [BKZ19, Qur17], under the bounds given in 3.2. In case of codimension 1 and 2, the equations of these varieties are generic weighted homogeneous polynomials of given degrees. In cases of codimension 3 and 4 they are induced from the equations of the corresponding ambient weighted projective variety. We perform a detailed singularity analysis of equations of these candidate varieties to prove the existence or non-existence of given candidate surface. We calculate the $qG$-deformation invariants like the anti-canonical degree $-K_X^2$ and first plurigenus $h^0(-K_X)$ in all cases. We calculate their Euler number and Picard rank in hypersurface case. In complete intersection case, we were able to calculate their Euler number and identify the non-prime examples, i.e. those with the Picard rank greater than 1 by computing their orbifold Euler number.

In each codimension and for each Fano index $I$, we provide complete classification of rigid del Pezzo surfaces $X \subset \mathbb{P}(a_i)$ satisfying

$$W - I \leq N$$

where $W = \sum a_i, N \geq 50$, see 3.2 for more details. We do not provide a completely theoretical proof of completeness of classification, which is free from bounds on $W$ and $I$. However, it is obvious that for larger values of $W$ most weights will be larger than 10; highest local index of allowed orbifold points in our classification, consequently the basket of orbifold points will very likely contain orbifold points of local index $r \geq 11$. In cases of hypersurfaces and complete intersections, the classifications of tuples $(d_i; a_i)$ which give rise to a quasismooth del Pezzo surfaces can be found in [Pae18] and [May16], where $d_i$s denote the degrees of the defining equations and $a_i$ are weights of the ambient weighted projective space. These classifications of tuples can perhaps be analyzed to give the bound free proof of completeness of our results in codimension 1 and 2. But their classification neither contains computation of any of the invariants like $h^0(-K_X), -K_X^2$ and $e(X)$ nor they compute basket of orbifold points lying on those surfaces.

1.2. Links with existing literature. A part of our search results recovers some existing examples in the literature, though a major part of them had not been described earlier in terms of equations. For example, the classification of Fano polygons (equivalently of rigid del Pezzo surfaces of class TG) with basket of orbifold points is given in [CK17]. We give descriptions in terms of equations for six of their examples; listed as
14, 16, 23, 85, 109 and 130 in our tables. We also recover the classical smooth del Pezzo surfaces of degrees 1, 2, 3, 4, 5 and 8; listed as 3, 2, 1, 82, 107, 128 and 12 respectively in Table 5. Moreover, 7 of the 29 examples from [CH17] also appear in our list with one of them seemingly having a new description as complete intersection in a $w(P^2 \times P^2)$ variety, listed as 129 in Table 5. A part of examples of Fano index 1 and 2 in codimension 3 and 4 given in Table 4 and 5 can be found in [Qur19], primarily appearing implicitly as a part of some infinite series of orbifold del Pezzo surfaces.

2. Background and notational conventions

2.1. Graded rings and Polarized varieties. We call a pair $(X, D)$ to be a polarized variety if $X$ is a normal projective algebraic variety and $D$ a $\mathbb{Q}$-ample Weil divisor on $X$, i.e., some integer multiple of $D$ is a Cartier divisor. One gets an associated finitely generated graded ring $R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$.

A surjective morphism from a free graded ring $k[x_0, ..., x_n]$ to $R(X, D)$ gives the embedding

$$i : X = \text{Proj}R(X, D) \hookrightarrow \mathbb{P}(a_0, \cdots, a_n)$$

where $a_i = \deg(x_i)$ and with the divisorial sheaf $\mathcal{O}_X(D)$ being isomorphic to $\mathcal{O}_X(1) = i^*\mathcal{O}_\mathbb{P}(1)$.

The Hilbert series of a polarized projective variety $(X, D)$ is given by

$$P_{(X,D)}(t) = \sum_{m \geq 0} h^0(X, mD) \ t^m,$$

where $h^0(X, mD) = \dim H^0(X, \mathcal{O}_X(mD))$. We usually write $P_X(t)$ for the Hilbert series for the sake of brevity. By the standard Hilbert–Serre theorem [AM69, Theorem 11.1], $P_X(t)$ has a compact form

$$P_X(t) = \frac{N(t)}{\prod_{i=0}^{n} (1 - t^{a_i})}.$$

2.2. Rigid del Pezzo surfaces.

Definition 2.1. An isolated orbifold point $Q$ of type $\frac{1}{r}(a_1, \ldots, a_n)$ is the quotient of $\mathbb{A}^n$ by the cyclic group $\mu_r$,

$$\epsilon : (x_1, \ldots, x_n) \mapsto (\epsilon^{a_1}x_1, \ldots, \epsilon^{a_n}x_n)$$

such that $\text{GCD}(r, a_i) = 1$ for $1 \leq i \leq n$ and $\epsilon$ is a primitive generator of $\mu_r$.

A del Pezzo surface $X$ is a two dimensional algebraic variety with an ample anticanonical divisor class $-K_X$. If, at worst, $X$ contains isolated orbifold points then we call it an orbifold or a log Del Pezzo surface. The Fano index $I$ of $X$ is the largest positive integer $I$ such that $-K_X = I \cdot D$ for some divisor $D$ in the divisor class group of $X$. An orbifold del Pezzo surfaces $X \subset \mathbb{P}(a_i)$ of codimension $c$ is well-formed if the singular
Then the quotient of affine cone over Grassmannian minus the origin $\widetilde{Q}$ is called a $T$-singularity. A singularity which is not smoothable under $\mathbb{Q}$-Gorenstein smoothing is called a rigid or $R$-singularity. Any normal surface $X$ with cyclic quotient singularities admits a $\mathbb{Q}$-Gorenstein partial smoothing to a surface with only rigid singularities by [KSB88]. We use the following characterization of a cyclic quotient singularity to be a $T$-singularity and $R$-singularity, appeared in [CP20].

**Definition 2.2.** Let $Q = \frac{1}{r}(a, b)$ be an orbifold point and take $m = \text{GCD}(a + b, r)$, $s = (a + b)/m$ and $k = r/m$ then $Q$ has a form $\frac{1}{mk}(1, ms - 1)$. Moreover $Q$ is called a $T$-singularity if $k | m$ [KSBSS] and an $R$-singularity if $m < k$ [AK14].

In the two dimensional case, any orbifold point $\frac{1}{r}(a, b)$ can be represented as $\frac{1}{r}(1, a')$ by choosing a different primitive generator of the cyclic group $\mu_r$ and the following Lemma follows from it.

**Lemma 2.3.** Let $Q_1 = \frac{1}{r}(1, a)$ and $Q_2 = \frac{1}{r}(1, b)$ be isolated orbifold points then $Q_1 = Q_2$ if and only if $ab \equiv 1 \mod r$.

By using the fact that each orbifold point on a surface can be written as $\frac{1}{r}(1, a)$ and by applying Lemma 2.3 on the all possible isolated rigid orbifold points of type $\frac{1}{r}(1, a); 3 \leq r \leq 10$, we get to the following Lemma.

**Lemma 2.4.** Let $3 \leq r \leq 10$ then any isolated rigid orbifold point $\frac{1}{r}(a, b)$ is equivalent to one of the following.

$$\left\{ \frac{1}{3}(1, 1), \frac{1}{3}(1, 1), \frac{1}{3}(1, 2), \frac{1}{5}(1, 1), \frac{1}{5}(1, 5), \frac{1}{10}(1, 1), \frac{1}{10}(1, 2) \right\}$$

2.3. **Ambient varieties.** In this section we briefly recall the definition of weighted Grassmannian $w\text{Gr}(2, 5)$ and $w(\mathbb{P}^2 \times \mathbb{P}^2)$ which we use, apart from weighted projective spaces, as rep-quotient varieties for the construction of our rigid orbifold del Pezzo surfaces; following the notion introduced in [CH17].

2.3.1. **Weighed Grassmannian** $w\text{Gr}(2, 5)$. This part is wholly based on material from [CR02], Sec.2]. Let $w := (w_1, \cdots, w_5)$ be a tuple of all integers or all half integers such that $w_i + w_j > 0, 1 \leq i < j \leq 5$,

Then the quotient of affine cone over Grassmannian minus the origin $\text{Gr}(2, 5) \setminus \{0\}$ by $\mathbb{C}^\times$ given by:

$$\epsilon : x_{ij} \mapsto \epsilon^{w_i + w_j} x_{ij}$$

is called weighted Grassmannians $w\text{Gr}(2, 5)$ where $x_{ij}$ are Plücker coordinates of the embedding $\text{Gr}(2, 5) \hookrightarrow \mathbb{P} \left( \mathbb{A}^2 \otimes \mathbb{C}^5 \right)$. Therefore we get the embedding $w\text{Gr}(2, 5) \hookrightarrow \mathbb{P}(\{a_{ij} : 1 \leq i < j \leq 5, a_{ij} = w_i + w_j\})$. 

The image of Gr(2, 5) and wGr(2, 5) under the Plücker embedding is defined by five 4 × 4 Pfaffians of 5 × 5 skew symmetric matrix

\[ \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \\ x_{34} & x_{35} \\ x_{45} \end{pmatrix}, \]

where we only write down the upper triangular part. Explicitly the defining equations are:

\[ \text{Pf}_i = x_{jk}x_{lm} - x_{jl}x_{km} + x_{jm}x_{lk}, \]

where 1 ≤ j < k < l < m ≤ 5 are four integers and i makes up the fifth one in \{1, 2, 3, 4, 5\}. In examples we usually write down the corresponding matrix of weights, replacing \(x_{ij}\) with \(a_{ij}\); to represent the given wGr(2, 5).

If wGr(2, 5) is wellformed then the orbifold canonical divisor class is

\[ K_{wGr(2,5)} = -\frac{1}{2} \sum_{1 \leq i < j \leq 5} a_{ij} D. \]

2.3.2. Weighted \(\mathbb{P}^2 \times \mathbb{P}^2\). This section recalls the definition of weighted \(\mathbb{P}^2 \times \mathbb{P}^2\) from [Sze05, BKQ18]. Let \(b = (b_1, b_2, b_3)\) and \(c = (c_1, c_2, c_3)\) be two integer or half integer vectors satisfying

\[ b_1 + c_1 > 0, b_i \leq b_j \text{ and } c_i \leq c_j \text{ for } 1 \leq i \leq j \leq 3, \]

Let \(\Sigma_P\) be the Segre embedding \(\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8(x_{ij})\). Then the weighted \(\mathbb{P}^2 \times \mathbb{P}^2\) variety, denoted by \(w\Sigma_P\), is the quotient of the punctured affine cone \(\Sigma_P^1\{0\}\) by \(\mathbb{C}^\times\):

\[ \epsilon : x_{ij} \mapsto \epsilon^{b_i + c_j}x_{ij}, \ 1 \leq i, j \leq 3. \]

So we get the embedding of

\[ w\Sigma_P \hookrightarrow \mathbb{P}^8(a_{ij} : a_{ij} = b_i + c_j; 1 \leq i, j \leq 3) \]

for a choice of \(b, c\), written together as a single input parameter \(p = (b_1, b_2, b_3; c_1, c_2, c_3)\). The equations are defined by 2 × 2 minors of the 3 × 3 matrix which we usually refer to as weight matrix and write it as

\[ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \]

where \(a_{ij} = b_i + c_j; 1 \leq i, j \leq 3\).

If \(w\Sigma_P\) is wellformed then the canonical divisor class is given by

\[ K_{w\Sigma_P} = -\sum_{i=j} a_{ij} D. \]

Notation and Conventions

- We work over the field of complex numbers \(\mathbb{C}\).
For two orbifold points where $r(1,a)=r(1,b)$ we choose a presentation with the minimum of $a$ and $b$.

In all the tables, integers appearing as subscripts of $X$ denote the degree of the defining equations of the given variety, where $d^m$ means that there are $m$ equations of degree $d$. Similarly, $\mathbb{P}(\cdots, a^m_i, \cdots)$ means that there are $m$ weights of degree $a_i$.

We use the same notation for canonical divisor class $K_X$ and canonical sheaf $\omega_X$, if no confusion can arise. We usually write $K_X = \mathcal{O}(n)$ to represent $K_X = nD$.

3. Computational steps of the proof

In this section we provide details of various steps of our calculations which together provide the proof of Theorem 1.1. In summary, for each codimension and Fano index, we first search for the list of candidate varieties using the algorithmic approach of [Qur17, BKZ19]. The candidate lists comes with a suggestive basket(s) of orbifold points and invariants. Then we perform theoretical analysis of each candidate to establish the existence or non-existence of candidate surfaces with given basket and invariants.

3.1. Algorithm. We briefly recall the algorithm from [Qur17] which we used to compute the candidate lists of examples. The key part of it is based on the orbifold Riemann–Roch formula of Bukcley, Reid and Zhou [BRZ13] which provides a decomposition of Hilbert series of $X$ into a smooth part and a singular part. It roughly states that if $X$ is an algebraic variety with basket $B = \{m_i \times Q_i : m_i \in \mathbb{Z}_{>0}\}$ of isolated orbifold points then its Hilbert series has a decomposition into a smooth part $P_I(t)$ and orbifold part $\sum m_i P_{Q_i}(t)$;

$$P_X(t) = P_I(t) + \sum m_i P_{Q_i}(t).$$

The algorithm search for all orbifolds of fixed dimension $n$ having fixed orbifold canonical sheaf $\mathcal{O}_X(k)$ in a given ambient rep-quotient variety. The algorithm has the following steps.

(i) Compute the Hilbert series and orbifold canonical class of ambient rep-quotient variety.

(ii) Find all possible embeddings of $n$-folds $X$ with $\omega_X = \mathcal{O}(k)$ by applying the adjunction formula.

(iii) For each possible $n$-fold embedding of $X$, compute the Hilbert series $P_X(t)$ and the smooth term $P_I(t)$.

(iv) Compute the list of all possible $n$-fold isolated orbifold points from the ambient weighted projective space containing $X$.

(v) For each subset of the list of possible orbifold points determine the multiplicities $m_i$ given in equation (8) of the orbifold terms $P_{Q_i}(t)$.

(vi) If $m_i \geq 0$ then $X$ is a candidate $n$-fold with suggested basket of isolated orbifold points.

(vii) Repeat step (iii) to (vi) for each possible $n$-fold embedding computed at step (ii).
3.2. **Bounds on search parameters.** We perform our search in the order of increasing sum of the weights on the ambient weight projective space $\mathbb{P}(a_0, \ldots, a_n)$ containing $X$. The search is theoretically unbounded in each codimension in two directions: there is no bound on the sum of weights $W = \sum a_i$ of the ambient weighted projective space containing $X$ and the Fano index $I$ is also unbounded.

In each codimension, we at least search for polarized rigid del Pezzo surfaces $X \hookrightarrow \mathbb{P}(a_i)$ such that $N = W - I \leq 50$, for $1 \leq I \leq 10$. If the last candidate example is found where $\sum a_i - I = q > 25$ then we further search for all possible cases such that $\sum a_i - I \leq 2q$, to further minimize the possibility of any missing examples. Similarly in each codimension if we find the last example in search domain $\sum a_i - I \leq 50$ for index $I = k > 5$ then we search for all cases with $I \leq 2k$. For examples, in the hypersurface case the last example is found for $I = 8$ case so we searched till index 16 in this case. Similarly, for index 2 hypersurfaces we got the last candidate when $\sum a_i - 2 = 36$ so we searched for all cases such that $\sum a_i - 2 \leq 72$.

**Table 1.** The following table summarises the number of surfaces we obtained for each Fano index $I$ in each codimension and exact search domain in each case. First column contains the codimension of each surface and the rest of the columns contain a pair of numbers. First number is the number of examples of given index and the second one gives the maximum value of $N = W_{\text{max}} - I$ for which the last candidate surface was found; the classification is complete till max($50, 2N$). The entries with no second number means that no examples were found for $N \leq 50$.

| Codimension | Fano Index, $(N)$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9–16 |
|-------------|------------------|----|----|----|----|----|----|----|----|------|
| 1           |                  | 11 | 44 | 6  | 6  | 6  | 2  | 2  | 4   | 0    |
| 2           |                  | 15 | 8  | 1  | 1  | 1  | 0  | 0  | 0   | 0    |
| 3           |                  | 12 | 7  | 1  | 1  | 0  | 0  | 0  | 0   | 0    |
| 4           |                  | 12 | 6  | 0  | 0  | 1  | 0  | 0  | 1   | 0    |

3.3. **Computing invariants.** We describe how we calculate each of the following invariants appearing in tables 2–5.

(i) $h^0(-K_X)$, is an invariant under $qG$-deformations. If it is equal to zero then we can easily conclude that $X$ does not admit a $qG$-deformation to a toric variety and such surfaces are not of class $TG$. We compute it as the coefficient of $t^I$ in the Hilbert series where $I$ is the Fano index of $X$.

(ii) $-K_X^2$, It is the anti-canonical degree of $X$ which we calculate from the Hilbert series; following [Qur19 3.1, 4.1].
(iii) \(e(X)\): We were able to compute the Euler characteristics of \(X\) in hypersurface and complete intersection cases by using Blache’s formula [Bla96, 2.11-14]:

\[
e(X) = e_{\text{orb}}(X) + \sum_{r(Q) \in B} \frac{r - 1}{r}
\]

where \(r\) is the local index of each orbifold point. It was applied by Brown and Fatighenti in the Appendix of [BF17] to illustrate the computation for a hypersurface. The formula has natural generalization to the cases of complete intersections \(X_{d_1, \ldots, d_k} \subset \mathbb{P}(a_0, \ldots, a_n)\) in higher codimension. We can computer \(e_{\text{orb}}(X)\) as:

\[
e_{\text{orb}}(X) = \text{coefficient of } t^{n-k} \text{ in the series expansion of } \left(\prod (1 + a_it) \prod (1 + d_it) \deg(X)\right).
\]

(iv) \(\rho(X)\), is the Picard rank of \(X\). We were able to calculate it explicitly when \(X\) is a hypersurface in \(\mathbb{P}^3(a_i)\) by using [Dol81, 4.4.1]. Given a hypersurface

\[X_d \hookrightarrow \mathbb{P}(a_0, a_1, a_2, a_3),\]

let

\[l = \text{coefficient of } t^{2d-\sum a_i} \text{ in the series expansion of } \left(\prod t^{d-a_i} - 1\right),\]

then \(\rho(X) = l + 1\). In cases of complete intersection examples we were able to identify those examples which are not prime, i.e. the Picard rank greater than 1. From [Hwa14], we know that if the Picard rank of a log del Pezzo surface is 1 then \(0 < e_{\text{orb}}(X) \leq 3\). Therefore for each codimension 2 complete intersection in Table 3 we list \(e_{\text{orb}}(X)\) and those with \(e_{\text{orb}}(X) > 3\) have Picard rank greater than 1.

3.4. **Theoretical singularity analysis.** The last step of the calculation is the theoretical singularity analysis of each candidate orbifold. We prove that the general member \(X\) in each family is wellformed and quasismooth for candidate family of orbifold surface. We first compute the dimensions of intersection of all orbifold strata with \(X\) to establish that \(X\) is wellformed. This should be less than or equal to zero for a surface to be wellformed, i.e. it does not contain any singular lines.

The next step is to show that \(X\) is quasismooth. It is not so difficult when \(X\) is a hypersurface or complete intersection: one can use the criteria given in [BF00, 8]. In cases of codimension 3 and 4 examples, we consider \(X\) as complete intersections \(\text{wGr}(2, 5)\) or in the Segre embedding of weighted \(\mathbb{P}^2 \times \mathbb{P}^2\) or in some projective cone(s) over either of those ambient varieties. So \(X\) may not only have singularities from the ambient weighted projective but it may also contain singularities on the base loci of linear systems of the intersecting weighted homogeneous forms. In such cases we mostly prove the quasismoothness on the base locus by using computer algebra system MAGMA [BCP97]. We write down explicit equations for \(X\) over the rational numbers and show that it is smooth, see [Qur19, 2.3] for more details. To prove quasismoothness on an orbifold point \(Q\) of type \(r(a, b),\)
which is mostly a coordinate point corresponding to some variables \(x_i\) with \(\deg(x_i) = r\), we proceed as follows. If \(c\) is the codimension of \(X\) then we find \(c\) tangent variables \(x_m\) \[BZ10\], i.e. we find \(c\) polynomials having monomial of type \(x_i^r x_m\). We can locally remove these variables by using the implicit function theorem. Moreover, if two other variables have weights \(a\) and \(b\) modulo \(r\) then \(Q\) is a quasismooth point of type \(1/r(a,b)\).

4. Sample Calculations

In this section we provide sample calculations of examples given in tables \[2\, 5\]

Example 4.1. #81 Consider the weighted projective space \(\mathbb{P}(1, 5, 7, 10)\) with variables \(x, y, z\) and \(w\) respectively, then the canonical class \(K_{\mathbb{P}} = \mathcal{O}(-23)\). The generic weighted homogenous polynomial of degree 15,

\[f_{15} = k_1 x^{15} + k_2 y^3 + k_3 yw + k_3 x z^2 + \cdots, \quad k_i \in \mathbb{C};\]
defines a del Pezzo surface \(X_{15} \hookrightarrow \mathbb{P}(x, y, z, w)\) of Fano index 8, i.e. \(K_X = \mathcal{O}(-8)\). \(f_{15}\) does not contain monomials of pure power in \(w\) and \(z\) so \(X\) contains the orbifold points \(p_1 = (0, 0, 0, 1)\) and \(p_2 = (0, 0, 1, 0)\). By applying the implicit function theorem we can remove the variable \(y\) near the point \(p_1\) by using the monomial \(yw\) and \(x, z\) are local variables near this point. Therefore \(X\) contains an orbifold point of type \(\frac{1}{10}(1, 7)\) = \(\frac{1}{10}(1, 3)\)(Lemma \[2, 3\]). Similarly near \(p_2\) the local variables are \(y\) and \(w\), so we get an orbifold point of type \(\frac{1}{7}(5, 10)\) = \(\frac{1}{7}(3, 5)\) = \(\frac{1}{7}(1, 4)\) = \(\frac{1}{7}(1, 2)\).

The coordinate point of weight 5 does not lie on \(X\) but one dimensional singular stratum \(\mathbb{P}^1(y, w)\) intersects with \(X\) non-trivially and by \[IF00\, Lemma 9.4\] they are two points. One of them is \(p_1\) and other can be taken as \(p_3 = (0, 1, 0, 0)\) which corresponds to weight 5 variable. By using the above arguments we can show that it is a singular point of type \(\frac{1}{10}(1, 2)\). Thus \(X\) contains exactly the same basket of singularities as given by the computer search and it is a wellformed and quasismooth rigid del Pezzo surface of Fano index 8. Moreover the vector space \(H^0(X, -K_X) = H^0(X, 8D) = < x^8, x^3 y, xz >,\)
so \(h^0(-K_X) = 3.\)

Example 4.2. #126 Consider the weighted Grassmannian \(\text{wGr}(2, 5)\)

\[\text{wGr}(2, 5) \hookrightarrow \mathbb{P}(1^2, 3^3, 5^4, 7)\]
with weight matrix

\[
\begin{pmatrix}
1 & 1 & 3 & 3 \\
3 & 5 & 5 & \\
5 & 5 & \\
7 & \\
\end{pmatrix}
\]

Then by equation \[15\] the canonical divisor class \(K_{\text{wGr}(2, 5)} = \mathcal{O}(-19)\). The weighted complete intersection of \(\text{wGr}(2, 5)\) with forms of degree 3 and two form of degree 5;

\[X = \text{wGr}(2, 5) \cap (f_3) \cap (g_3) \cap (f_5) \cap (g_5) \hookrightarrow \mathbb{P}(1^2, 3^3, 5^2, 7) = \mathbb{P}(x_1, x_2, y_1, z_1, z_2, w_1)\]
is a del Pezzo surface with $K_X = \mathcal{O}(-19 + (3 + 3 + 5 + 5)) = \mathcal{O}(-3)$. We can take $X$ to be defined by the maximal Pfaffians of

$$
\begin{pmatrix}
x_1 & x_2 & f_3 & g_3 \\
y_1 & f_5 & g_5 \\
z_1 & z_2 \\
w_1
\end{pmatrix},
$$

where $f_3, g_3, f_5$ and $g_5$ are general weighted homogeneous forms in given variables. The coordinate point corresponding to $w_1$ lies on $X$. From the equations we have $x_1, x_2$ and $y_1$ as tangent variables and $z_1, z_2$ as local variables. Therefore it is an orbifold point of type $\frac{1}{7}(1,1)$. The locus $X \cap \mathbb{P}(5,5)$ is locally a quadric in $\mathbb{P}^1$ which defines two points. By similar application of implicit function theorem we can show that each of it is an orbifold point of type $\frac{1}{5}(1,2)$. The restriction of $X$ to weight 3 locus is an empty set, so $X$ contains no further orbifold points. To show the quasismoothness on the base locus we use the computer algebra and write down equations for $X$. For example, if we choose

$$f_3 = 3x_1^3 + 3x_2^3, \quad f_5 = x_1^5 + x_2^2y_1 + x_2^2y_1 + z_1 + z_2,$$

$$g_3 = x_2^3 + y_1, \quad g_5 = x_1^5 + 2x_1^2y_1 + 3x_1^2y_1 + 3z_2$$

then the Pfaffians of (11) gives a quasismooth surface. Thus $X$ is an orbifold del Pezzo surface of Fano index 3 with singular points; $2 \times \frac{1}{5}(1,2)$ and $\frac{1}{7}(1,1)$.

As we mentioned in section 3.4 that we prove the existence of given orbifold del Pezzo surface by theoretical singularity analysis. Then only those which are quasismooth and having correct basket of singularities appear in tables of examples. The following is an example of candidate which we obtain via computer search but fails to be quasismooth, hence not included in tables of examples.

Example 4.3. (Non working candidate) A computer search also gives a candidate complete intersection orbifold del Pezzo surface of Fano index 2 given by

$$X_{6,30} \hookrightarrow \mathbb{P}(1,3,9,10,15) = \mathbb{P}(x, y, z, t, u).$$

Then $F_6 = f(x, y)$ (since other variables have weight higher than 6) and

$$F_{30} = x^{30} + x^{27}y + yz^3 + \cdots$$

are the defining equations of $X$. The coordinate point $p = (0, 0, 1, 0, 0)$ lies on $X$ as no pure power of $z$ appear in $F_{30}$. Now we can not find two tangent variables to $z$ in the equations of $X$ which implies that the rank of the Jacobian matrix of $X$ at $p$ is equal to 1 which is less than its codimension, so $X$ is not quasismooth at $p$. Thus $X$ is a del Pezzo surface which is not quasismooth and does not appear in the following tables.

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Notations in Tables.

- The column \( dP_2 \) represents a del Pezzo surface \( X \) and the corresponding weighted projective space containing \( X \); the subscripts give the equation degrees of \( X \). The column \( I \) lists the Fano index of \( X \).
- The next two columns contain the anti-canonical degree \( -K_X^2 \) and the first plurigenus \( h^0(-K_X) \). If \( h^0(-K_X) = 0 \) the \( X \) is not of class \( TG \).
- \( e(X) \) denotes the topological Euler characteristics of \( X \), \( \rho(X) \) is the rank of Picard group of \( X \), and \( \epsilon_{\text{orb}}(X) \) denotes the orbifold Euler number of \( X \). \( \rho(X) \) is only listed in table 2 and \( \epsilon_{\text{orb}}(X) \) only in table 3.
- The column \( B \) represents the basket of singular points of \( X \).
- In Table 4 and 5, the last column represents the matrix of weights, which provides weights of ambient weighted projective space containing \( w\text{Gr}(2,5) \) or weighted \( \mathbb{P}^2 \times \mathbb{P}^2 \) variety.
- We provide references to those examples which appeared in [CH17] and [CK17], primarily in a toric setting.

### Table 2: Hypersurfaces in \( w\mathbb{P}^3 \)

| S.No | \( dP_2 \) | \( I \) | \(-K_X^2\) | \( h^0(-K_X)\) | \( e(X) \) | \( \rho(X) \) | Basket \( B \) |
|------|-----------|-------|-----------|-------------|-------|-------|-----------|
| 1    | \( X_3 \subset \mathbb{P}(1^4) \) | 1     | 3         | 4           | 9     | 7     |           |
| 2    | \( X_4 \subset \mathbb{P}(1^3,2) \) | 1     | 2         | 3           | 10    | 8     |           |
| 3    | \( X_6 \subset \mathbb{P}(1^2,2,3) \) | 1     | 1         | 2           | 11    | 9     |           |

Continued on next page
|   |     |     |     |  |     |     |
|---|-----|-----|-----|---|-----|-----|
| 4 | $X_{10} \subset \mathbb{P}(1,2,3,5)$ | 1   | 1/3 | 1 | 11  | 9   |
| 5 | $X_{12} \subset \mathbb{P}(2,3^2,5)$ | 1   | 2/15| 0 | 10  | 8   |
| 6 | $X_{15} \subset \mathbb{P}(1,3,5,7)$ | 1   | 1/7 | 1 | 11  | 9   |
| 7 | $X_{15} \subset \mathbb{P}(3^2,5^2)$ | 1   | 1/15| 0 | 11  | 9   |
| 8 | $X_{16} \subset \mathbb{P}(1,3,5,8)$ | 1   | 2/15| 1 | 12  | 10  |
| 9 | $X_{18} \subset \mathbb{P}(2,3,5,9)$ | 1   | 1/15| 0 | 9   | 7   |
|10 | $X_{20} \subset \mathbb{P}(2,5^2,9)$ | 1   | 2/45| 0 | 10  | 8   |
|11 | $X_{28} \subset \mathbb{P}(3,5,7,14)$| 1   | 2/105|0 | 8   | 6   |
|12 | $X_{2} \subset \mathbb{P}(1^4)$    | 2   | 8   | 9 | 4   | 2   |
|13 | $X_{4} \subset \mathbb{P}(1^3,3)$ | 2   | 16/3| 6 | 6   | 4   |
|14 | $X_{6} \subset \mathbb{P}(1^3,5)$ | 2   | 24/5| 6 | 8   | 6   |
|15 | $X_{6} \subset \mathbb{P}(1^2,3^2)$ | 2   | 8/3 | 3 | 8   | 6   |
|16 | $X_{7} \subset \mathbb{P}(1^3,6)$ | 2   | 14/3| 6 | 9   | 7   |
|17 | $X_{8} \subset \mathbb{P}(1^2,3,5)$ | 2   | 32/15|3 | 10  | 8   |
|18 | $X_{8} \subset \mathbb{P}(1^3,7)$ | 2   | 32/7 | 6 | 10  | 8   |
|19 | $X_{9} \subset \mathbb{P}(1^2,3,6)$ | 2   | 2   | 3 | 11  | 9   |
|20 | $X_{9} \subset \mathbb{P}(1^3,8)$  | 2   | 9/2 | 6 | 11  | 9   |
|21 | $X_{10} \subset \mathbb{P}(1^2,3,7)$| 2   | 40/21|3 | 12  | 10  |
|22 | $X_{10} \subset \mathbb{P}(1^3,9)$ | 2   | 40/9 | 6 | 12  | 10  |
|23 | $X_{10} \subset \mathbb{P}(1^2,5^2)$| 2   | 8/5  | 3 | 12  | 10  |
|24 | $X_{11} \subset \mathbb{P}(1^3,10)$| 2   | 22/5 | 6 | 13  | 11  |
|25 | $X_{11} \subset \mathbb{P}(1^2,5,6)$| 2   | 22/15|3 | 13  | 11  |
|26 | $X_{11} \subset \mathbb{P}(1^2,3,8)$| 2   | 11/6 | 3 | 13  | 11  |
|27 | $X_{12} \subset \mathbb{P}(1^2,5,7)$| 2   | 48/35|3 | 14  | 12  |
|28 | $X_{12} \subset \mathbb{P}(1^2,6^2)$| 2   | 4/3  | 3 | 14  | 12  |
|29 | $X_{12} \subset \mathbb{P}(1^2,3,9)$| 2   | 16/9 | 3 | 14  | 12  |

Continued on next page
Table 2 continued from previous page

|   | \(X_{13} \subset \mathbb{P}(1^2, 5, 8)\) |   | 2 | \(13/10\) | 3 | 15 | 13 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|---|---|---|---|---|---|---|---|---|
|   | \(X_{13} \subset \mathbb{P}(1^2, 3, 10)\) |   | 2 | \(26/15\) | 3 | 15 | 13 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{13} \subset \mathbb{P}(1^2, 6, 7)\) |   | 2 | \(26/21\) | 3 | 15 | 13 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{14} \subset \mathbb{P}(1^2, 6, 8)\) |   | 2 | \(7/6\) | 3 | 16 | 14 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{14} \subset \mathbb{P}(1^2, 7^2)\) |   | 2 | \(8/7\) | 3 | 16 | 14 | \(2 \times \frac{1}{7}(1, 1)\) |
|   | \(X_{14} \subset \mathbb{P}(1^2, 5, 9)\) |   | 2 | \(56/45\) | 3 | 16 | 14 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{15} \subset \mathbb{P}(1^2, 6, 9)\) |   | 2 | \(10/9\) | 3 | 17 | 15 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{15} \subset \mathbb{P}(1^2, 5, 10)\) |   | 2 | \(6/5\) | 3 | 17 | 15 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{15} \subset \mathbb{P}(1^2, 7^2)\) |   | 2 | \(1\) | 3 | 18 | 16 | \(2 \times \frac{1}{7}(1, 1)\) |
|   | \(X_{15} \subset \mathbb{P}(1^2, 5, 10)\) |   | 2 | \(17/42\) | 1 | 11 | 9 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1), \frac{1}{3}(1, 5)\) |
|   | \(X_{17} \subset \mathbb{P}(1^2, 7, 10)\) |   | 2 | \(34/35\) | 3 | 19 | 17 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{17} \subset \mathbb{P}(1^2, 7, 8)\) |   | 2 | \(17/18\) | 3 | 19 | 17 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{18} \subset \mathbb{P}(1^2, 8, 10)\) |   | 2 | \(9/10\) | 3 | 20 | 18 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{18} \subset \mathbb{P}(1^2, 8^2)\) |   | 2 | \(8/9\) | 3 | 20 | 18 | \(2 \times \frac{1}{7}(1, 1)\) |
|   | \(X_{19} \subset \mathbb{P}(1^2, 9, 10)\) |   | 2 | \(38/45\) | 3 | 21 | 19 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 1)\) |
|   | \(X_{20} \subset \mathbb{P}(1^2, 10^2)\) |   | 2 | \(4/5\) | 3 | 22 | 20 | \(2 \times \frac{1}{7}(1, 1)\) |
|   | \(X_{21} \subset \mathbb{P}(3, 6, 7^2)\) |   | 2 | \(2/21\) | 0 | 9 | 7 | \(3 \times \frac{1}{7}(1, 1), \frac{1}{7}(1, 1), 3 \times \frac{1}{7}(1, 2)\) |
|   | \(X_{21} \subset \mathbb{P}(1, 3, 9, 10)\) |   | 2 | \(14/45\) | 1 | 13 | 11 | \(2 \times \frac{1}{7}(1, 1), \frac{1}{7}(1, 1), \frac{1}{7}(1, 3)\) |
|   | \(X_{22} \subset \mathbb{P}(1^2, 5, 7, 11)\) |   | 2 | \(8/35\) | 1 | 10 | 8 | \(\frac{1}{5}(1, 1), \frac{1}{5}(1, 3)\) |
|   | \(X_{24} \subset \mathbb{P}(3, 7, 8^2)\) |   | 2 | \(1/14\) | 0 | 7 | 5 | \(\frac{1}{7}(1, 1), 3 \times \frac{1}{7}(1, 5)\) |
|   | \(X_{30} \subset \mathbb{P}(3, 9, 10^2)\) |   | 2 | \(2/45\) | 0 | 9 | 7 | \(3 \times \frac{1}{5}(1, 1), \frac{1}{5}(1, 1), 3 \times \frac{1}{10}(1, 3)\) |
|   | \(X_{36} \subset \mathbb{P}(1, 7, 12, 18)\) |   | 2 | \(2/21\) | 1 | 11 | 9 | \(\frac{1}{7}(1, 1), \frac{1}{7}(1, 3)\) |

Continued on next page
| Index | Index Intersection & Field | Card. | 5 | 6 | 7 | 8 |
|-------|---------------------------|-------|---|---|---|---|
| 56    | $X_6 \subseteq \mathbb{P}(1^2, 2, 5)$ | 32/5 | 6 | 5 | 3 | $\frac{1}{5}(1, 2)$ |
| 57    | $X_8 \subseteq \mathbb{P}(1^2, 2, 7)$ | 36/7 | 6 | 6 | 4 | $\frac{1}{7}(1, 2)$ |
| 58    | $X_{10} \subseteq \mathbb{P}(1, 2, 5^2)$ | 9/5 | 2 | 7 | 5 | $2 \times \frac{1}{5}(1, 2)$ |
| 59    | $X_{12} \subseteq \mathbb{P}(1, 2, 5, 7)$ | 54/35 | 2 | 8 | 6 | $\frac{3}{5}(1, 2), \frac{1}{7}(1, 2)$ |
| 60    | $X_{14} \subseteq \mathbb{P}(1, 2, 7^2)$ | 9/7 | 2 | 9 | 7 | $2 \times \frac{1}{7}(1, 2)$ |
| 61    | $X_{15} \subseteq \mathbb{P}(1, 5^2, 7)$ | 27/35 | 1 | 9 | 3 | $3 \times \frac{1}{5}(1, 2), \frac{1}{5}(1, 1)$ |
| 62    | $X_6 \subseteq \mathbb{P}(1^2, 3, 5)$ | 32/5 | 7 | 4 | 2 | $\frac{1}{5}(1, 2)$ |
| 63    | $X_{10} \subseteq \mathbb{P}(1, 3, 5^2)$ | 32/15 | 2 | 6 | 4 | $\frac{1}{3}(1, 1), 2 \times \frac{1}{5}(1, 2)$ |
| 64    | $X_{12} \subseteq \mathbb{P}(1, 3, 5, 7)$ | 64/35 | 2 | 6 | 4 | $\frac{1}{5}(1, 2), \frac{1}{7}(1, 3)$ |
| 65    | $X_{15} \subseteq \mathbb{P}(1, 3, 5, 10)$ | 8/5 | 2 | 7 | 5 | $\frac{1}{5}(1, 2), \frac{1}{10}(1, 3)$ |
| 66    | $X_{15} \subseteq \mathbb{P}(3, 5^2, 6)$ | 8/15 | 0 | 7 | 5 | $2 \times \frac{1}{3}(1, 1), 3 \times \frac{1}{5}(1, 2), \frac{1}{5}(1, 1)$ |
| 67    | $X_{21} \subseteq \mathbb{P}(1, 7^2, 10)$ | 24/35 | 1 | 9 | 7 | $3 \times \frac{1}{5}(1, 3), \frac{1}{10}(1, 1)$ |
| 68    | $X_{8} \subseteq \mathbb{P}(1, 2, 3, 7)$ | 100/21 | 5 | 4 | 2 | $\frac{1}{3}(1, 1), \frac{1}{7}(1, 3)$ |
| 69    | $X_{8} \subseteq \mathbb{P}(1^2, 4, 7)$ | 50/7 | 8 | 4 | 2 | $\frac{1}{7}(1, 2)$ |
| 70    | $X_{12} \subseteq \mathbb{P}(1, 3, 4, 9)$ | 25/9 | 3 | 5 | 3 | $\frac{1}{3}(1, 1), \frac{1}{9}(1, 4)$ |
| 71    | $X_{14} \subseteq \mathbb{P}(2, 3, 7^2)$ | 25/21 | 1 | 5 | 3 | $\frac{1}{3}(1, 1), 2 \times \frac{1}{7}(1, 3)$ |
| 72    | $X_{16} \subseteq \mathbb{P}(1, 4, 7, 9)$ | 100/63 | 2 | 6 | 4 | $\frac{1}{7}(1, 2), \frac{1}{9}(1, 4)$ |
| 73    | $X_{21} \subseteq \mathbb{P}(3, 7^2, 9)$ | 25/63 | 0 | 7 | 5 | $2 \times \frac{1}{3}(1, 1), 3 \times \frac{1}{7}(1, 3), \frac{1}{9}(1, 1)$ |
| 74    | $X_{15} \subseteq \mathbb{P}(1, 5, 7, 8)$ | 27/14 | 2 | 5 | 3 | $\frac{1}{7}(1, 3), \frac{1}{9}(1, 5)$ |
| 75    | $X_{16} \subseteq \mathbb{P}(1, 5, 8^2)$ | 9/5 | 2 | 6 | 4 | $\frac{1}{5}(1, 1), 2 \times \frac{1}{9}(1, 5)$ |
| 76    | $X_{10} \subseteq \mathbb{P}(1, 2, 5, 9)$ | 49/9 | 6 | 3 | 1 | $\frac{1}{5}(1, 4)$ |
| 77    | $X_{12} \subseteq \mathbb{P}(2, 3, 5, 9)$ | 98/45 | 2 | 4 | 2 | $\frac{1}{3}(1, 1), \frac{1}{5}(1, 2), \frac{1}{9}(1, 4)$ |
| 78    | $X_{8} \subseteq \mathbb{P}(1, 3, 5, 7)$ | 512/105 | 5 | 4 | 2 | $\frac{1}{3}(1, 1), \frac{1}{7}(1, 2), \frac{1}{9}(1, 2)$ |
| 79    | $X_{14} \subseteq \mathbb{P}(1, 5, 7, 9)$ | 128/45 | 3 | 4 | 2 | $\frac{1}{5}(1, 2), \frac{1}{9}(1, 4)$ |
| 80    | $X_{15} \subseteq \mathbb{P}(1, 6, 7, 9)$ | 160/63 | 3 | 5 | 3 | $\frac{1}{5}(1, 1), \frac{1}{7}(1, 3), \frac{1}{9}(1, 4)$ |
| 81    | $X_{15} \subseteq \mathbb{P}(1, 5, 7, 10)$ | 96/35 | 3 | 5 | 3 | $\frac{1}{5}(1, 2), \frac{1}{7}(1, 2), \frac{1}{10}(1, 3)$ |
| S.No | $dP_2$ | $I$ | $-K^2$ | $h^0(-K)$ | $e(X)$ | $e_{\text{orb}}(X)$ | Basket $\mathcal{B}$ |
|------|--------|-----|---------|-----------|-------|----------------|----------------------------|
| 82   | $X_{2,2} \subset \mathbb{P}(1^5)$ | 1   | 4       | 5         | 8     | 8              | $\frac{1}{3}(1,1)$ [CH17] |
| 83   | $X_{4^2} \subset \mathbb{P}(1^2, 2^2, 3)$ | 1   | 4/3     | 2         | 10    | 28/3          | $\frac{1}{5}(1,1)$ [CH17] |
| 84   | $X_{4,0} \subset \mathbb{P}(1, 2^2, 3^2)$ | 1   | 2/3     | 1         | 10    | 26/3          | $2 \times \frac{1}{5}(1,1)$ [CH17] |
| 85   | $X_{6^2} \subset \mathbb{P}(1^2, 3^2, 5)$ | 1   | 4/5     | 2         | 12    | 56/5          | $\frac{1}{5}(1,1)$ [CK17] |
| 86   | $X_{6^2} \subset \mathbb{P}(2^2, 3^3)$ | 1   | 1/3     | 0         | 9     | 19/3          | $4 \times \frac{1}{5}(1,1)$ [CH17] |
| 87   | $X_{6,7} \subset \mathbb{P}(1, 2, 3^2, 5)$ | 1   | 7/15    | 1         | 11    | 133/15        | $2 \times \frac{1}{5}(1,1), \frac{1}{5}(1,1)$ |
| 88   | $X_{6,8} \subset \mathbb{P}(1, 2, 3, 4, 5)$ | 1   | 2/5     | 1         | 10    | 46/5          | $\frac{1}{5}(1,2)$ |
| 89   | $X_{8^2} \subset \mathbb{P}(1^2, 4^2, 7)$ | 1   | 4/7     | 2         | 14    | 92/7          | $\frac{1}{7}(1,1)$ |
| 90   | $X_{6,10} \subset \mathbb{P}(1, 3^2, 5^2)$ | 1   | 4/15    | 1         | 12    | 136/15        | $2 \times \frac{1}{5}(1,1), 2 \times \frac{1}{5}(1,1)$ |
| 91   | $X_{8,10} \subset \mathbb{P}(2, 3, 4, 5^2)$ | 1   | 2/15    | 0         | 8     | 86/15         | $\frac{1}{5}(1,1), 2 \times \frac{1}{5}(1,2)$ |
| 92   | $X_{9,10} \subset \mathbb{P}(2, 3^2, 5, 7)$ | 1   | 1/7     | 0         | 9     | 43/7          | $3 \times \frac{1}{5}(1,1), \frac{1}{5}(1,2)$ |
| 93   | $X_{10^2} \subset \mathbb{P}(1^2, 5^2, 9)$ | 1   | 4/9     | 2         | 16    | 136/9         | $\frac{1}{9}(1,1)$ |
| 94   | $X_{10,11} \subset \mathbb{P}(1, 2, 5^2, 9)$ | 1   | 11/45   | 1         | 13    | 473/45        | $2 \times \frac{1}{5}(1,2), \frac{1}{5}(1,1)$ |
| 95   | $X_{10,12} \subset \mathbb{P}(3^2, 5^2, 7)$ | 1   | 8/105   | 0         | 10    | 512/105       | $4 \times \frac{1}{5}(1,1), 2 \times \frac{1}{5}(1,1), \frac{1}{5}(1,2)$ |
| 96   | $X_{10,12} \subset \mathbb{P}(2, 3, 5, 6, 7)$ | 1   | 2/21    | 0         | 8     | 122/21        | $2 \times \frac{1}{5}(1,1), \frac{1}{5}(1,3)$ |
| 97   | $X_{6,8} \subset \mathbb{P}(1, 3^2, 4, 5)$ | 2   | 16/15   | 1         | 8     | 88/15         | $2 \times \frac{1}{5}(1,1), \frac{1}{5}(1,2)$ |
| 98   | $X_{8,10} \subset \mathbb{P}(1, 3, 4, 5, 7)$ | 2   | 16/21   | 1         | 8     | 136/21        | $\frac{1}{5}(1,1), \frac{1}{5}(1,3)$ |
| 99   | $X_{8,12} \subset \mathbb{P}(1, 3, 5, 6, 7)$ | 2   | 64/105  | 1         | 10    | 736/105       | $2 \times \frac{1}{5}(1,1), \frac{1}{5}(1,1), \frac{1}{5}(1,2)$ |
| 100  | $X_{10,12} \subset \mathbb{P}(3, 4, 5^2, 7)$ | 2   | 8/35    | 0         | 6     | 124/35        | $2 \times \frac{1}{5}(1,2), \frac{1}{5}(1,3)$ |
| 101  | $X_{9,14} \subset \mathbb{P}(1, 3, 6, 7, 8)$ | 2   | 1/2     | 1         | 10    | 61/8          | $\frac{1}{8}(1,1), \frac{1}{8}(1,1), \frac{1}{8}(1,1)$ |
| 102  | $X_{12,14} \subset \mathbb{P}(3, 4, 5, 7, 9)$ | 2   | 8/45    | 0         | 6     | 164/45        | $\frac{1}{9}(1,1), \frac{1}{9}(1,2), \frac{1}{9}(1,4)$ |
| 103  | $X_{14,15} \subset \mathbb{P}(3, 6, 7^2, 8)$ | 2   | 5/42    | 0         | 8     | 545/168       | $2 \times \frac{1}{5}(1,1), \frac{1}{5}(1,1), 2 \times \frac{1}{5}(1,2), \frac{1}{5}(1,2)$ |
| 104  | $X_{11,18} \subset \mathbb{P}(1, 3, 8, 9, 10)$ | 2   | 11/30   | 1         | 12    | 1067/120      | $2 \times \frac{1}{5}(1,1), \frac{1}{5}(1,1), \frac{1}{5}(1,3)$ |
| 105  | $X_{12,14} \subset \mathbb{P}(4, 5, 6, 7^2)$ | 3   | 9/35    | 0         | 5     | 87/35         | $\frac{1}{5}(1,2), 2 \times \frac{1}{5}(1,3)$ |

*Continued on next page*
Table 4: Codimension 3 Pfaffians

| S.No | $dP_2$ | $I$  | $-K^2$ | $h^0(-K)$ | Basket $B$ | Weight Matrix |
|------|--------|------|--------|-----------|------------|---------------|
| 107  | $X_{2,2,2,2,2}$ $\subset \mathbb{P}(1^9)$ | 1    | 5      | 6         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$ |
| 108  | $X_{3,3,4,4,4}$ $\subset \mathbb{P}(1^3,2^2,3)$ | 1    | 7/3    | 3         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{array}$ |
| 109  | $X_{4,4,6,6,6}$ $\subset \mathbb{P}(1^3,3^2,5)$ | 1    | 9/5    | 3         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 \end{array}$ |
| 110  | $X_{4,5,6,6,7}$ $\subset \mathbb{P}(1^2,2,3^2,5)$ | 1    | 17/15  | 2         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \end{array}$ |
| 111  | $X_{5,5,8,8,8}$ $\subset \mathbb{P}(1^3,4^2,7)$ | 1    | 11/7   | 3         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \end{array}$ |
| 112  | $X_{6,7,8,9,10}$ $\subset \mathbb{P}(1,2,3^2,5,7)$ | 1    | 10/21  | 1         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array}$ |
| 113  | $X_{6,6,10,10,10}$ $\subset \mathbb{P}(1^3,5^2,9)$ | 1    | 13/9   | 3         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \end{array}$ |
| 114  | $X_{7,8,8,9,10}$ $\subset \mathbb{P}(2,3^2,4,5^2)$ | 1    | 1/5    | 0         | $3/1(1,1)$ | $\begin{array}{cccc} 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \end{array}$ |
| 115  | $X_{6,7,10,10,11}$ $\subset \mathbb{P}(1^2,2,5^2,9)$ | 1    | 38/45  | 2         | $1/1(1,1)$ | $\begin{array}{cccc} 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \\ 1 & 1 & 4 & 5 \end{array}$ |

Continued on next page.
|   | $X_{6,8,10,10,12}$ | $X_{10,10,12,14}$ | $X_{11,12,12,15,16}$ | $X_{4,7,8,9}$ | $X_{4,8,9,9,10}$ | $X_{4,10,11,11,12}$ | $X_{8,9,12,13,14}$ | $X_{12,12,14,15,15}$ | $X_{14,14,15,15,16}$ | $X_{16,17,17,18,18}$ | $X_{6,8,8,10}$ |
|---|-------------------|-------------------|-------------------|--------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|--------------|
|   | $\subset \mathbb{P}(1,3^2,5^2,7)$ | $\subset \mathbb{P}(3^2,5^2,7^2)$ | $\subset \mathbb{P}(2,5^2,6,7,9)$ | $\subset \mathbb{P}(1^2,2,3,6,7)$ | $\subset \mathbb{P}(1^2,2,3,7,8)$ | $\subset \mathbb{P}(1^2,2,3,9,10)$ | $\subset \mathbb{P}(1,3,5,6,7,8)$ | $\subset \mathbb{P}(4,5^2,7^2,8)$ | $\subset \mathbb{P}(3,6,7^2,8^2)$ | $\subset \mathbb{P}(3,7,8^2,9,10)$ | $\subset \mathbb{P}(1^2,3,5^2,7)$ |
|   | $1$               | $1$               | $1$               | $2$           | $2$               | $2$               | $2$               | $2$               | $2$               | $2$               | $3$           |
|   | $29/105$          | $3/35$            | $23/315$          | $22/7$        | $43/14$           | $134/45$          | $19/30$           | $1170$            | $18/80$           | $11/105$          | $153/35$      |
|   | $1$               | $0$               | $0$               | $4$           | $4$               | $4$               | $1$               | $0$               | $0$               | $0$               | $4$           |
|   | $\frac{1}{3}(1,1), \frac{1}{5}(1,1), \frac{1}{7}(1,4)$ | $3 \times \frac{1}{3}(1,1), \frac{1}{5}(1,1), 2 \times \frac{1}{7}(1,4)$ | $3 \times \frac{1}{5}(1,2), \frac{1}{7}(1,3), \frac{1}{9}(1,1)$ | $\frac{1}{5}(1,1), \frac{1}{7}(1,1), \frac{1}{10}(1,2)$ | $\frac{1}{5}(1,1), \frac{1}{7}(1,1), \frac{1}{10}(1,3)$ | $\frac{1}{3}(1,1), \frac{1}{5}(1,1), \frac{1}{7}(1,5)$ | $2 \times \frac{1}{5}(1,2), 2 \times \frac{1}{7}(1,3), \frac{1}{8}(1,1)$ | $\frac{1}{5}(1,1), \frac{1}{7}(1,1), \frac{1}{8}(1,2)$ | $\frac{1}{5}(1,1), \frac{1}{7}(1,1), \frac{1}{8}(1,5), \frac{1}{10}(1,3)$ | $2 \times \frac{1}{5}(1,2), \frac{1}{7}(1,1)$ |
### Table 5: Codimension 4 $\mathbb{P}^2 \times \mathbb{P}^2$

| S. No | $dP_2$ | $I$ | $-K^2$ | $h^0(-K)$ | Basket $B$ | Weight Matrix |
|-------|--------|-----|--------|-----------|------------|---------------|
| 128   | $X_{29} \subset \mathbb{P}(17)$ | 1   | 6      | 7         | $\frac{1}{8}(1,1)$ | [CH17]        |
| 129   | $X_{234,44} \subset \mathbb{P}(14,22,3)$ | 1   | 10/3   | 4         | $\frac{1}{5}(1,1)$ | [CH17]        |
| 130   | $X_{244,64} \subset \mathbb{P}(14,3^2,5)$ | 1   | 14/5   | 4         | $\frac{1}{8}(1,1)$ | [CK17]        |
| 131   | $X_{254,84} \subset \mathbb{P}(14,4^2,7)$ | 1   | 18/7   | 4         | $\frac{1}{7}(1,1)$ |               |
| 132   | $X_{452,63,72,8} \subset \mathbb{P}(1,2^2,3^3,5)$ | 1   | 4/5    | 1         | $3 \times \frac{1}{5}(1,1), \frac{1}{8}(1,1)$ |               |
| 133   | $X_{264,10^4} \subset \mathbb{P}(14,5^2,9)$ | 1   | 22/9   | 4         | $\frac{1}{9}(1,1)$ |               |
| 134   | $X_{5,62,72,82,9,10} \subset \mathbb{P}(1,2,3^2,4,5^2)$ | 1   | 8/15   | 1         | $\frac{1}{5}(1,1), \frac{1}{6}(1,2), \frac{1}{7}(1,1)$ |               |
| 135   | $X_{4,72,82,10,11^2,12} \subset \mathbb{P}(1,2^2,3,5^2,9)$ | 1   | 26/45  | 1         | $\frac{1}{5}(1,1), 2 \times \frac{1}{5}(1,2), \frac{1}{9}(1,1)$ |               |
| 136   | $X_{6,82,10^4,12^2,14} \subset \mathbb{P}(1,3^2,5^2,7^2)$ | 1   | 2/7    | 1         | $2 \times \frac{1}{7}(1,4)$ |               |
| 137   | $X_{7,8,10,11^2,12^2,15,16} \subset \mathbb{P}(1,2,5^2,6,7,9)$ | 1   | 86/315 | 1         | $\frac{1}{5}(1,2), \frac{1}{7}(1,3), \frac{1}{9}(1,1)$ |               |
| 138   | $X_{10,11^2,12^2,13^2,14} \subset \mathbb{P}(3,4,5^2,6,7^2)$ | 1   | 3/35   | 0         | $2 \times \frac{1}{6}(1,2), 2 \times \frac{1}{4}(1,4)$ |               |
| 139   | $X_{8,9,11,12^3,13,15,16} \subset \mathbb{P}(2,3,5^2,6,7,9)$ | 1   | 38/315 | 0         | $\frac{1}{5}(1,1), 3 \times \frac{1}{6}(1,2), \frac{1}{4}(1,4), \frac{1}{9}(1,1)$ |               |

Continued on next page
|   | $X_{4,8,9,10,11,12,16,17^2,18}$ | 2 | 20/21 | 2 | 1/3(1,1), 2 × 1/6(1,1), 2 × 1/9(1,1) | 1 | 2 | 6 |
|---|---------------------------------|---|--------|---|--------------------------------------|---|---|---|
| 140 | $X_{4,8,9,10,11,12,13,14^2,15}$ | 2 | 37/42 | 2 | 1/6(1,1), 1/7(1,1), 1/7(1,2), 1/8(1,5) | 1 | 2 | 6 |
| 141 | $X_{4,8,9,11,12,15,16^2,17}$ | 2 | 248/315 | 2 | 1/3(1,1), 1/6(1,1), 1/7(1,2), 1/9(1,3) | 1 | 2 | 6 |
| 142 | $X_{4,9,10,11,12,16,17^2,18}$ | 2 | 451/630 | 2 | 1/7(1,1), 1/5(1,5), 1/7(1,1), 1/9(1,3) | 1 | 2 | 6 |
| 143 | $X_{4,11,12,18,19^2,20}$ | 2 | 28/45 | 2 | 1/3(1,1), 2 × 1/6(1,1), 2 × 1/9(1,3) | 1 | 2 | 6 |
| 144 | $X_{4,11,12,16,17^2,18}$ | 2 | 16/105 | 0 | 3 × 1/3(1,1), 1/5(1,1), 2 × 1/7(1,2), 1/9(1,3) | 1 | 2 | 6 |
| 145 | $X_{4,11,12,16,17^2,18}$ | 2 | 250/63 | 4 | 2 × 1/3(1,1), 2 × 1/7(1,3), 1/9(1,1) | 1 | 3 | 5 |
| 146 | $X_{6,8^2,10^2,11^2,14}$ | 2 | 256/315 | 1 | 1/6(1,1), 2 × 1/7(1,3), 2 × 1/9(1,4), 1/9(1,1) | 6 | 7 | 9 |
| 147 | $X_{6,8^2,10^2,11^2,14}$ | 2 | 256/315 | 1 | 1/6(1,1), 2 × 1/7(1,3), 2 × 1/9(1,4), 1/9(1,1) | 6 | 7 | 9 |

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