Radiative corrections to the Casimir effect for the massive scalar field

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We compute the first radiative correction to the Casimir energy of a massive scalar field with a $\lambda \phi^4$ self-interaction in the presence of two parallel plates. Three kinds of boundary conditions are considered: Dirichlet-Dirichlet, Neumann-Neumann and Dirichlet-Neumann. We use dimensional and analytical regularizations to obtain our physical results.

1. INTRODUCTION

In 1948 Casimir predicted an unexpected quantum field theory (QFT) effect: as a consequence of distorting the vacuum of the quantized electromagnetic field, two conducting and neutral parallel plates should attract each other with a force proportional to the inverse of the fourth power of their separation \[1\] (for general reviews on the Casimir effect, see Refs. \[2\]; for an introductory guide, see Ref. \[3\]). The first attempt to observe this phenomenon was made only in 1958 by Sparnaay \[4\]. However, the accuracy achieved by Sparnaay allowed him only to conclude that the experimental data was compatible with Casimir’s theoretical predictions. For different reasons, new experiments involving directly metal bodies were performed only very recently \[5\], but this time the high accuracy obtained and the excellent agreement between the experimental data and theory permit us to state safely that the Casimir effect is well established nowadays.

The Casimir effect is not a peculiarity of the quantized electromagnetic field. In fact, the vacuum state (and its energy) of any relativistic quantum field, bosonic or fermionic, depends on the boundary conditions (BC) imposed on the fields or, more generally, on the classical background with which the fields interact. This makes the Casimir effect an important topic of research, with applications in many branches of physics (see Bordag et al. in \[2\]).

Even though QFT is mainly concerned with interacting fields, most papers on the Casimir effect deals with non-interacting fields. The computation of radiative corrections to the Casimir energy of interacting fields has been performed in relatively few papers. (See, for instance, Refs. \[6\] in the context of QED, and Refs. \[7,8,9\] for scalar fields.)

For non-interacting fields, the Casimir energy is given by the sum, properly regularized and renormalized, of the zero-point energy of the normal modes of the fields, which behave as a collection of independent harmonic oscillators. Hence, at the one-loop level the Casimir energy is sensitive only to the fields’ eigenfrequencies, but not to their eigenmodes. This is the reason why, for instance, the Casimir energy of a scalar field confined between two parallel plates is the same for Dirichlet or Neumann BC on both plates.\footnote{To be precise, there are modes in the Neumann BC case that are not present in the Dirichlet BC case, but since their zero-point energy does not depend on the distance between the plates, they do not contribute to the Casimir force and can be discarded.}

In the case of interacting fields the situation is more complicated: the independent harmonic oscillators become anharmonic and coupled when the interaction is turned on. Therefore, one has to take into account not only the oscillators’ zero-point energy — which are modified by the interaction, as in the Lamb shift —, but also the interaction en-
energy among the oscillators. This should make the radiative corrections to the Casimir energy sensitive to the form of the eigenmodes; in particular, it should depend on the BC imposed on the fields. In spite of this, Krech and Dietrich showed that the $O(\lambda)$ correction to the Casimir energy of the massless $\lambda\phi^4$ theory is the same forDirichlet and Neumann BC on a pair of plates.

Our purpose here is to report the results of our investigation on whether that equality is also true for a massive field. We computed the $O(\lambda)$ radiative correction to the Casimir energy of the massive $\lambda\phi^4$ theory subject to Dirichlet and Neumann BC on a pair of parallel plates. Our results show that the mentioned equality is valid only in the massless cases. We also extend our calculations to the case in which the field is subject to one of the three BC already considered before. (However, we shall present the calculations only for the DN case, as the other two BC were consid-

ered in detail in [9].) Using perturbation theory, the $O(\lambda)$ correction to the previous results can be written as

$$E^{(1)} = \int_a^a dz \left[ \frac{\lambda}{8} G^2(x,x) + \frac{\delta m^2}{2} G(x,x) + \delta \Lambda \right],$$

(8)

where $G(x,x')$ is the Green function of the non-interacting theory, but obeying the BC, $\delta m^2$ is the radiatively induced shift in the mass parameter, and $\delta \Lambda$ is the shift in the cosmological constant (i.e., the change in the vacuum energy which

**2. ONE-LOOP CASIMIR EFFECT**

In order to introduce some notation and basic ideas, we briefly sketch in this section some results for the Casimir energy of a non-interacting massive scalar field submitted to BC at $z = 0$ and $z = a$. We shall consider three distinct BC, denoted by DD, NN and DN, and given, respectively, by: (i) $\phi(z = 0) = \phi(z = a) = 0$; (ii) $\partial \phi/\partial z|_{z=0} = \partial \phi/\partial z|_{z=a} = 0$, and (iii) $\phi(z = 0) = \partial \phi/\partial z|_{z=a} = 0$.

The Casimir energy per unit area when Dirichlet or Neumann BC are used on the two planes is given by

$$E^{(0)}_{DD} = E^{(0)}_{NN} = -\frac{m^2}{8\pi^2 a} \sum_{n=1}^{\infty} \frac{K_2(2nma)}{n^2}.$$  

(1)

For the mixed BC (Dirichlet-Neumann), we get

$$E^{(0)}_{DN} = -\frac{m^2}{16\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{n^2} [K_2(4nma) - 2K_2(2nma)].$$  

(2)

The small mass limit ($ma \ll 1$) of these expressions is given by

$$E^{(0)}_{DD} = E^{(0)}_{NN} = -\frac{1}{1440} \frac{\pi^2}{a^3} + \frac{1}{96} \frac{m^2}{a} + O(m^3),$$  

(3)

$$E^{(0)}_{DN} = \frac{7}{8} \frac{\pi^2}{1440} \frac{1}{a^3} - \frac{1}{192} \frac{m^2}{a} + O(m^3).$$  

(4)

On the other hand, in the large mass limit ($ma \gg 1$) Eqs. (1) and (2) yield

$$E^{(0)}_{DD} = E^{(0)}_{NN} \approx -\frac{1}{16} \left( \frac{m}{\pi a} \right)^{3/2} \exp(-2ma),$$  

(5)

$$E^{(0)}_{DN} \approx \frac{1}{16} \left( \frac{m}{\pi a} \right)^{3/2} \exp(-2ma).$$  

(6)

It is worth noting that the first term on the r.h.s. of Eq. (5) is precisely half the Casimir energy per unit area for the electromagnetic field between two perfectly conducting (or infinitely permeable) parallel plates, while the first term on the r.h.s. of Eq. (6) is half the Casimir energy per unit area for the electromagnetic field between a perfectly conducting plate and an infinitely permeable one.

**3. TWO-LOOP CASIMIR EFFECT**

Now we shall consider the $\lambda\phi^4$ model, defined by the Euclidean Lagrangian density

$$\mathcal{L}_E = \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + \mathcal{L}_{CT},$$

(7)

where $\mathcal{L}_{CT}$ contains the usual renormalization counterterms. The interacting field is submitted to one of the three BC already considered before. (However, we shall present the calculations only for the DN case, as the other two BC were considered in detail in [10].) Using perturbation theory, the $O(\lambda)$ correction to the previous results can be written as

$$E^{(1)} = \int_0^a dx \left[ \frac{\lambda}{8} G^2(x,x) + \frac{\delta m^2}{2} G(x,x) + \delta \Lambda \right],$$

(8)

where $G(x,x')$ is the Green function of the non-interacting theory, but obeying the BC, $\delta m^2$ is the radiatively induced shift in the mass parameter, and $\delta \Lambda$ is the shift in the cosmological constant (i.e., the change in the vacuum energy which
is due solely to the interaction, and not to the confinement.

The spectral representation of the Euclidean Green function in \((d + 1)\)-dimensions is given by

\[
G(x, x') = \int \frac{d\omega}{2\pi} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{-i\omega(r-r')} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \times \sum_n \frac{\varphi_n(z) \varphi_n^\ast(z')}{\omega^2 + k^2 + m^2 + k_n^2},
\]

where, in the DN case,

\[
\varphi_n(z) = \sqrt{\frac{2}{a}} \sin(k_n z),
\]

\[
k_n = (n + \frac{1}{2}) \frac{\pi}{a} \quad (n = 0, 1, 2, \ldots).
\]

Note that \(G(x, x')\) diverges when \(x' \to x\) for \(d \geq 1\). Therefore, a regularization prescription is needed. Using dimensional regularization we obtain

\[
G(x, x) = \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \omega_n^{d-2} \varphi_n(z) \varphi_n^\ast(z),
\]

where \(\omega_n = \sqrt{m^2 + k_n^2}\).

Now we need to compute the terms appearing in Eq. (10). Using the explicit form of \(\varphi_n(z)\) given in Eqs. (10) and (11) one obtains

\[
\int_0^a dz G(x, x) = \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \omega_n^{d-2},
\]

\[
\int_0^a dz G^2(x, x) = \frac{\Gamma^2(1 - d/2)}{(4\pi)^d a} \left[ \left( \sum_{n=0}^{\infty} \omega_n^{d-2} \right)^2 + \frac{1}{2} \sum_{n=0}^{\infty} \omega_n^{2d-4} \right].
\]

Collecting terms, we obtain

\[
E^{(1)}_{DN} = \frac{\lambda}{8a} \left[ \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} \right] [F(2 - d, 2a) - F(2 - d, a)] + \frac{2a \delta m^2 \gamma^2}{\lambda}
\]

\[
+ \frac{\lambda}{16a} \left[ \frac{\Gamma(1 - d/2)}{(4\pi)^d} \right] [F(4 - 2d, 2a) - F(4 - 2d, a)] + \left[ \delta \Lambda - \frac{\delta m^2 \gamma^2}{\lambda} \right] a,
\]

where \(F(s, a)\) is defined as

\[
F(s, a) := \sum_{n=1}^{\infty} \left[ m^2 + \left( \frac{n\pi}{a} \right)^2 \right]^{-s/2},
\]

for \(\Re(s) > 1\), and can be extended analytically to the whole complex \(s\)-plane via the identity

\[
F(s, a) = -\frac{1}{2} m^{-s} + \frac{am^{1-s}}{2\pi^{1/2} \Gamma(s/2)} \left[ \Gamma\left( s - \frac{1}{2} \right) \right]
\]

\[
+ 4 \sum_{n=1}^{\infty} \frac{K_{(1-s)/2}(2nma)}{(nma)^{(1-s)/2}}.
\]

In the above equation, \(K_s\) denotes the modified Bessel function of second kind. The structure of poles of \(F(s, a)\) is dictated by the \(\Gamma\) function: there are simple poles at \(s = 1, -1, -3, -5, \ldots\).

Let us now choose the renormalization conditions for \(\delta m^2\) and \(\delta \Lambda\). With this goal, recall that up to first order in \(\lambda\) the self-energy is given by \(\Sigma(x) = (\lambda/2)G(x, x) + \delta m^2\). We shall fix \(\delta m^2\) by imposing the following conditions on \(\Sigma(x)\): (i) \(\Sigma(x) < \infty\) (except possibly at some special points); (ii) \(\Sigma(x)\) vanishes away from the plates when \(a \to \infty\): \(\lim_{a \to \infty} \Sigma(z = \gamma a) = 0\) for \(0 < \gamma < 1\), and (iii) \(\delta m^2\) must be independent of \(a\). These conditions are fulfilled by taking \(\delta m^2 = -(\lambda/2) G_0(0)\), where \(G_0(z)\) denotes the non-interacting Green function without boundary conditions evaluated at the point \(x = (0, 0, z)\).

Computing \(G_0(0)\) within dimensional regularization, we get

\[
\delta m^2 = -\frac{\lambda \Gamma((1 - d)/2)}{2(4\pi)^{(d+1)/2}} m^{d-1}.
\]

For the shift in the cosmological constant we shall take \(\delta \Lambda = (\delta m^2)^2/2\Lambda\). With this choice one eliminates the term proportional to \(a\) in Eq. (15), which does not contribute to the force between the plates (the linear dependence on \(a\) is canceled by similar terms when one adds the energy of the regions \(z < 0\) and \(z > a\)). Collecting all these results, we obtain

\[
E^{(1)}_{DN} = \frac{2\lambda m^{2d-2}}{(4\pi)^{d+1}} \left[ \sum_{n=1}^{\infty} \frac{2K_{(d-1)/2}(4nma)}{(2nma)^{(d-1)/2}} \right]^2
\]

\[
- \frac{K_{(d-1)/2}(2nma)}{(nma)^{(d-1)/2}}.
\]
Taking Eqs. (20)–(22), obtaining results given by [9]. These results are graphically illustrated in Fig. 1.

The results for the DD and NN BC (in d = 3) are given by [7]

\[
E_{\text{DD}}^{(1)} = \frac{\lambda m^2}{512\pi^2a^2} \left( \sum_{n=1}^{\infty} \frac{K_1(2nma)}{n} \right)^2, \quad (21)
\]

\[
E_{\text{NN}}^{(1)} = \frac{\lambda m^2}{512\pi^2a^2} \left( 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{K_1(2nma)}{n} \right)^2. \quad (22)
\]

These results are graphically illustrated in Fig. 1.

It follows from Eqs. (21) and (22) that \( E_{\text{DD}}^{(1)} \) and \( E_{\text{NN}}^{(1)} \) are not equal, except in the zero mass case. Indeed, using the expansion [12]

\[
\sum_{n=1}^{\infty} \frac{K_1(nz)}{n} = \frac{\pi^2}{6z} - \frac{\pi}{2} + O(z \ln z), \quad (23)
\]

valid for small \( z \), we can take the limit \( m \to 0 \) in Eqs. (20)–(22), obtaining

\[
E_{\text{DN}}^{(1)} = \frac{\lambda m^2}{213\pi^2a^2}, \quad E_{\text{DD}}^{(1)} = E_{\text{NN}}^{(1)} = \frac{\lambda}{213\pi^2a^2}, \quad (24)
\]

which agree with the results obtained by Krech and Dietrich [7] for \( d = 3 \).

In the large mass limit \( (ma \gg 1) \) we have

\[
E_{\text{DN}}^{(1)} \approx \frac{\lambda m}{512\pi^2a^2} \exp(-4ma), \quad (25)
\]

\[
E_{\text{DD}}^{(1)} \approx -E_{\text{NN}}^{(1)} \approx \frac{\lambda}{256\pi} \left( \frac{m}{\pi a} \right)^{3/2} \exp(-2ma). \quad (26)
\]

Two aspects of the results above are worth of mention: (i) while for \( m = 0 \) \( E_{\text{NN}}^{(1)} \) is positive

\\[\text{Note that their } d \text{ equals our } d \text{ plus one.}\]

Figure 1. \((512\pi^2/\lambda)a^3E^{(1)}\) (in \( d = 3 \)) as a function of \( ma \) for three kinds of boundary conditions: Dirichlet-Dirichlet (upper curve), Neumann-Neumann (the curve that crosses the horizontal axis), and Dirichlet-Neumann.

for all \( a \), for \( m \neq 0 \) it eventually becomes negative for sufficiently large \( a \) (more precisely, for \( a \approx 0.2m^{-1} \)); (ii) while \( E^{(1)} \) decays with distance (or mass) as fast as \( E^{(0)} \) for DD or NN boundary conditions [cf. Eqs. (6) and (25)], the former decays faster than the latter in the DN case [cf. Eqs. (6) and (28)].

4. FINAL REMARKS

We have computed the first radiative correction to the Casimir energy of the massive \( \lambda\phi^4 \) model subject to three distinct BC on a pair of parallel plates. We showed that while for a massless field DD and NN boundary conditions lead to the same \( O(\lambda) \) radiative correction (an unexpected result), this is not true for a massive field. In addition, that correction presents very distinct behavior as a function of the distance \( a \) between the plates: while \( a^3E_{\text{DD}}^{(1)} \) first increases and then decreases with \( a \), \( a^3E_{\text{NN}}^{(1)} \) first decreases and then increases with \( a \) (see Fig. 1). As a consequence, a pair of DD plates is more attracted to each other than a pair of NN plates when the distance between the plates is sufficiently small; the opposite occurs
when the plates are far apart.
We also computed for the first time the $O(\lambda)$ radiative correction to the Casimir energy for that model subject to DN boundary conditions (i.e., Dirichlet BC on one plate and Neumann BC on the other). Our results show that in this case the correction to the one-loop result is much smaller than in the DD or NN cases for large separations between the plates (i.e., for $a \gg m^{-1}$).

Results for other kinds of BC (periodic and anti-periodic) will be presented elsewhere.

ACKNOWLEDGMENTS

This work was supported by CNPq and CAPES.

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