Double Counting Ambiguities in the Linear $\sigma$ Model

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Abstract

We study the dynamical consequences imposed on effective chiral field theories such as the quark-level SU(2) linear $\sigma$ model (L$\sigma$M) due to the fundamental constraints of massless Goldstone pions, the normalization of the pion decay constant and form factor, and the pion charge radius. We discuss quark-level double counting L$\sigma$M ambiguities in the context of the Salam-Weinberg $Z = 0$ compositeness condition. Then SU(3) extensions to the kaon are briefly considered.

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1. Introduction

Most physicists believe that the chiral theory of QCD is Nature’s way of binding quarks (and gluons) into observable hadrons. Although there is no double-counting problem in this QCD scheme, it is a nonlinear and nonperturbative theory which has not yet been solved in the low energy region. Consequently it is still reasonable to consider an effective field theory—the quark-level linear $\sigma$ model (L$\sigma$M)—involving both quark and meson loops. In fact, the particle data group has just reinstated the nonstrange $\sigma$ in the 1996 tables [1]. Such a L$\sigma$M might suggest double counting problems, because at first glance it is not clear when a (bound state) $\bar{q}q$ pion or sigma meson should be treated as an SU(2) L$\sigma$M elementary particle. In this paper we attempt to resolve the L$\sigma$M double-counting ambiguities while focusing on the calculation of the (i) pion mass in the chiral limit (CL), (ii) pion decay constant and pion form-factor normalizations, (iii) pion charge radius.

The first test of any chiral field theory is its ease in satisfying the Goldstone theorem [2] in the chiral limit,

$$m_\pi = 0.$$  \hspace{1cm} (1)

A second test involves the double-counting issue for the pion decay constant and pion form factor normalizations. A third test concerns the pion charge radius, which is now measured respectively to be [3,4]

$$r_\pi = (0.66 \pm 0.02) \text{ fm}, \quad r_\pi = (0.63 \pm 0.01) \text{ fm}.$$ \hspace{1cm} (2)

The latter value matches perfectly with the highly successful phenomenology of vector meson dominance (VMD), which predicts [5]

$$r_{\pi}^{VMD} = \sqrt{6}/m_\rho \approx 0.63 \text{ fm}.$$ \hspace{1cm} (3)
In fact the original chiral field theories of the 1960’s, the SU(2) nucleon-level \( L_{\sigma M} \) [6] and the Nambu-Jona-Lasinio four-fermion model (NJL [7]) recovered \( m_\pi = 0 \) in (1). Indeed, earlier Nambu [8] dynamically invoked axial current conservation when \( m_\pi = 0 \) to induce the extremely useful notion of partial conservation of axial currents (PCAC) for \( m_\pi \neq 0 \), which assumes a more quantitative form for the \( L_{\sigma M}[6] \). But it took until 1979 for physicists to obtain the \( quark \)-level \( L_{\sigma M} \) chiral-limiting prediction [9]

\[
 r_{\pi}^{L_{\sigma M}} = \sqrt{N_c/2\pi f_{\pi,CL}} \approx 0.60 \text{ fm}, \tag{4}
\]

for \( N_c = 3 \) and the chiral-limiting pion decay constant [10] \( f_{\pi,CL} \approx 90 \text{ MeV} \).

In Sec. 2 we first study the SU(2) quark-level \( L_{\sigma M} \) in one-loop order and remind the reader how the null pion mass Goldstone theorem (1) is satisfied. Next we introduce the external axial-vector and vector (photon) fields and demonstrate how the pion decay constant and pion form factor are self-consistently determined for quark loops (only) up to cutoff \( \Lambda \approx 750 \text{ MeV} \). Then we allow for the smaller meson (\( \vec{\pi} \) and \( \sigma \)) one-loop order graphs and show that the cutoff \( \Lambda' \approx 660 \text{ MeV} \) is reduced to the value where the \( \sigma(650) \) (see eq. (17)) is almost no longer an elementary field but instead is a \( q\bar{q} \) bound state. This speaks directly to the problem of double-counting the \( \vec{\pi} \) and \( \sigma \) particles in the \( L_{\sigma M} \) context of \( Z = 0 \) compositeness conditions.

In Sec. 3 we begin by showing that the quark loops (only) pion charge radius (for \( \Lambda \approx 750 \text{ MeV} > m_\sigma(650) \)) indeed recovers eq.(4). However adding in the much smaller meson loops (for \( \Lambda' \approx 660 \text{ MeV} \sim m_\sigma(650) \)) instead leads to an infrared singularity, so apparently the \( \sigma(650) \) has become a \( q\bar{q} \) bound state when computing \( r_{\pi}^{L_{\sigma M}} \). At this point we dynamically generate the entire \( L_{\sigma M} \) beginning from a simpler chiral quark model theory. This naturally links \( r_{\pi}^{L_{\sigma M}} \) in eq.(4) with \( r_{\pi}^{V_{MD}} \) in eq.(3).
We extend this quark-level L\sigma M theory to SU(3) states in Sec. 4, and show how the null kaon Goldstone theorem operates in one (quark)-loop order. Also the connection between \( r_K^{VMD} \) and \( r_K^{\sigma M} \) continues to hold as for the pion charge radius above. We draw our conclusions in Sec. 5. In particular we suggest that the quark-level L\sigma M with these double-counting issues resolved in the context of the \( Z = 0 \) compositeness condition has experimental relevance as a broad \( \sigma \) meson of mass below 1 GeV. Such a \( \sigma \) meson has in fact been detected in recent data analyses [1].

2. Quark-level SU(2) linear sigma model

Shifting to the true (chiral) vacuum with expectation values \( \langle \sigma \rangle = \langle \vec{\pi} \rangle = 0 \), the interacting part of the SU(2) quark-level L\sigma M lagrangian density is

\[
L_{\text{int}}^{\sigma M} = g'\sigma(\sigma^2 + \vec{\pi}^2) - (\lambda/4)(\sigma^2 + \vec{\pi}^2)^2 + g\bar{\psi}(\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi})\psi,
\]

with meson-quark and meson-meson chiral couplings for quark mass \( m_q \) (and again \( f_\pi \approx 93 \text{ MeV} \)),

\[
g = m_q/f_\pi, \quad g' = (m_\sigma^2 - m_\pi^2)/2f_\pi = \lambda f_\pi.
\]

While (5) and (6) are indeed the original tree level results of refs.[6] (but now with quarks replacing nucleons), the Goldstone pion \( m_\pi = 0 \) should be reviewed in a dynamical context as determined by the lagrangian (5).

At tree level the axial currents are conserved \( \partial \vec{A} = 0 \) because the axial current divergence term \( \partial_\mu \gamma^\mu \gamma_5/2 \) interferes destructively against the massless pion pole \( gf_\pi \gamma_5 \) due to the quark-level Goldberger-Treiman relation (GTR)

\[
m_q = f_\pi g.
\]
This dynamical approach of Nambu [8] has a tree-level $L\sigma M$ Gell-Mann-Levy version [6] due to eqs. (6).

In one-loop-order, the quark-level $L\sigma M$ obeys the Goldstone theorem in an interesting manner. Specifically, the quark bubble and tadpole graphs contributing to the pion mass are depicted in Fig.1a, while the corresponding $\pi$ and $\sigma$ meson bubble, quartic loops and tadpole graphs are depicted in Fig.1b. In the chiral limit (CL), the quark bubble plus tadpole loops (ql) for $u$ and $d$ quark flavors occurring in $N_c$ colors contribute to the pion mass (squared) according to

$$m_{\pi,ql}^2 = i8Ncg\left(-g + \frac{2g'm_q}{m_{\sigma}^2}\right)\int \frac{d^4p}{p^2 - m_q^2}, \quad (8a)$$

with $d^4p = (2\pi)^{-4}d^4p$. Regardless of the overall quadratically divergent integral in (8a), the quark loop component of $m_{\pi}^2$ vanishes if $g'$ is dynamically fixed to $g' = m_{\sigma}^2/2f_{\pi}$ in (6) when $m_{\pi}^2 = 0$ because of the GTR eq.(7).

To handle the meson loops (ml) depicted in Fig.1b, one must first invoke the partial fraction identity

$$\frac{g'^2}{(p^2 - m_{\sigma}^2)(p^2 - m_{\pi}^2)} = \frac{1}{2} \left[ \frac{1}{p^2 - m_{\sigma}^2} - \frac{1}{p^2 - m_{\pi}^2} \right]$$

with coupling coefficients again related using eqs.(6). Then the quadratically divergent $\pi$ and $\sigma$ parts separate into two vanishing integrals

$$m_{\pi,ml}^2 = (-2\lambda + 5\lambda - 3\lambda)i\int \frac{d^4p}{p^2 - m_{\pi}^2} + (2\lambda + \lambda - 3\lambda)i\int \frac{d^4p}{p^2 - m_{\sigma}^2}. \quad (8b)$$

Again the coupling coefficients multiplying these two divergent integrals in (8b) identically vanish. Therefore the complete one-loop level Goldstone theorem in the $L\sigma M$ becomes in the chiral limit [11]

$$m_{\pi}^2 = 0\mid_{\text{quark loops}} + 0\mid_{\pi \text{ loops}} + 0\mid_{\sigma \text{ loops}} = 0. \quad (9)$$
While the LσM Goldstone theorem (9) is not surprising, the vanishing of (9) places no constraints on $m_\sigma$ or on the cutoff involved. As we shall see, however, the analogue LσM one-loop versions of the pion form factor normalization and the pion charge radius will put severe constraints on the (ultraviolet) cutoff. This will in turn instruct us when the scalar $\sigma$ particle can indeed be taken as an elementary particle, or should instead be treated as a $\bar{q}q$ bound state.

But first we consider the pion decay constant $f_\pi = m_q/g$ in (7) to one-loop order in the LσM as given by the quark loop (ql) and meson loop (ml) diagrams depicted in Fig.2a,b respectively. The dominant $u$ and $d$ quark loops of Fig.2a generate a fermion trace $4m_qq_\mu$ in the chiral limit, so the (constituent) quark mass factor cancels out, leading to the “log-divergent gap” equation

$$1 = -i4N_cg^2 \int_0^\Lambda d^4p (p^2 - m_q^2)^{-2} = \ln(X + 1) - [1 + X^{-1}]^{-1}. \quad (10a)$$

Here $X = \Lambda^2/m_q^2$, so for $N_c = 3$ and $g \approx 3.6$ (as we shall later show but estimate here from the GTR $g \approx 320$ MeV/90 MeV $\sim 3.6$), the numerical solution of (10a) is $X \approx 5.3$ or $\Lambda \approx \sqrt{5.3}$ $m_q \approx 750$ MeV for $m_q \approx 325$ MeV (as we shall later find). This self-consistent cutoff separates the elementary $\sigma$ meson $<750$ MeV from the $\bar{q}q$ bound states: $\rho(770), \omega(783), A_1(1260) > \Lambda \approx 750$ MeV.

If we add to the quark loops of Fig.2a the (smaller) meson loop of Fig.2b, the cutoff $\Lambda$ in (10a) remains essentially unchanged. More specifically, the pion to vacuum matrix element of the axial-vector current $if_\pi q_\mu$ is incrementally shifted to

$$i\delta f_\pi q_\mu = 2g' \int \frac{d^4l(2l + q)_\mu}{(l^2 - m_\sigma^2)(l + q)^2 - m_\pi^2}. \quad (10b)$$
Changing Feynman variables to $l \rightarrow l - qx$ in (10b), and accounting for the (linearly divergent) surface term [12], the net dimensionless shift of $f_\pi$ due to the meson loop of Fig. 2b is

$$\delta f_\pi / f_\pi = \lambda / 16\pi^2 - i2\lambda \int_0^1 dx (1 - 2x) \int \frac{d^4 l}{[l^2 - m_\pi^2 (1 - x)]^2}. \quad (10c)$$

However explicit evaluation of the latter (log-divergent) integral in (10c) when folded into the vanishing integral $\int_0^1 dx (1 - 2x)$ in fact leads to the finite contribution $-\lambda / 16\pi^2$, which precisely cancels the surface term $+\lambda / 16\pi^2$. Thus one is led back to the log-divergent gap equation (10a) even after meson (as well as quark) loops are included in Figs. 2. This minimal shift in the PCAC relation is of course expected, and in fact has already been used when analyzing the $\pi^+ \rightarrow e^+ \nu \gamma$ form factors in a quark-level LσM context [13].

This log-divergent gap equation (10a) is in fact recovered from the quark loop version of the pion form factor normalization $F_\pi(q^2 = 0) = 1$. Then the quark loops (ql) probed by the off-shell (vector) photon in Fig. 3a lead to the pion form factor in the chiral limit [14]

$$F_{ql}^{\pi}(q^2) = -i4N_c g^2 \int_0^1 dx \int d^4 p [p^2 - m_\pi^2 + x(1 - x)q^2]^{-2}. \quad (11a)$$

The apparent linear divergence of $F_{ql}^{\pi}(q^2)$ is removed in (11a) by rerouting one-half of the loop momentum in the opposite direction [14], with the pion form factor covariant defined as $< \pi^+|V_\mu|\pi^+ > = F_\pi(q^2)(p' + p)\mu$. Then at $q^2 = 0$, the quark loop pion form factor in (11a) is normalized by (10a):

$$F_{\pi}^{ql}(q^2 = 0) = -i4N_c g^2 \int_0^\Lambda d^4 p [p^2 - m_\pi^2]^{-2} = 1, \quad (11b)$$

provided the cutoff is $\Lambda \approx 750$ MeV as found from the pion decay constant combined with the quark-level GTR in eqs. (10).

However this satisfying result (11b) is significantly altered when the LσM meson loops (ml) are included in the pion form factor as depicted in Fig. 3b.
While the second (quartic) pion loop vanishes, the first $\pi^+\sigma$ loop in Fig. 3b contributes to the pion form factor covariant as

$$F^{\text{ml}}_\pi(q^2)(p' + p)_\mu = (2g')^2i \int \frac{d^4l(2l + p' + p)_\mu}{(l^2 - m^2_\sigma)[(l + p')^2 - m^2_\sigma][(l + p)^2 - m^2_\pi]}. \quad (12a)$$

Since the meson loop integral in (12a) is convergent, one can shift Feynman variables to $l \to l + (qx - p'y)$ and pick out the $(p' + p)_\mu$ covariant from (12a) to obtain the meson loop component of the pion form factor,

$$F^\text{ml}_\pi(q^2) = 2m^2_\sigma \lambda i \int_0^1 dx \int_0^1 dy 2y(1 - y) \int d^4l[l^2 + q^2x(1 - x)y^2 - m^2_\sigma(1 - y)]^{-3}, \quad (12b)$$

in the chiral limit. Here we have used $(2g')^2 = 2m^2_\sigma\lambda$ from eq.(6). Finally invoking the dynamically generated meson coupling strength [15] $\lambda = 2g^2 = 8\pi^2/3$ (to which we shall later return in Sec. 3), one obtains the overall pion form factor normalization adding (11a) to (12b) at $q^2 = 0$ and using the cutoff $\Lambda'$ version of (10a):

$$1 = F^q_\pi(0) + F^{\text{ml}}_\pi(0) = \ln(X' + 1) - [1 + X'^{-1}]^{-1} + \frac{1}{6}. \quad (12c)$$

Here $X' = \Lambda'^2/m^2_q$ and the $m^2_\sigma$ mass term in (12b) cancels out [12], resulting in the additional ml factor $\lambda/16\pi^2 = 1/6$ in (12c).

The numerical solution of (12c) is $X' \approx 4.15$ or $\Lambda' \approx \sqrt{4.15}$ $m_q \approx 660$ MeV. This latter cutoff scale is dangerously close to the NJL $\sigma$ mass scale $m_\sigma = 2m_q \approx 650$ MeV (this latter $\sigma$ mass also holds in the dynamically generated version of the L$\sigma$M [15]). It suggests that for the pion form factor, the “elementary” $\sigma$ meson at $\sigma(650)$ may be double-counting its $\bar{q}q$ bound state version. This cutoff problem becomes a major issue when the derivative of the pion form factor is taken as needed for computing the pion charge radius in Sec. 3 to follow.
The above form factor normalization problem based on the log-divergent
gap equation (10a) or (11b) vs. (12c) is an example of the Salam-Weinberg
\cite{16} $Z = 0$ compositeness condition. This condition gives a self-consistent
field-theoretic interpretation of a $\bar{q}q$ pion and sigma meson treated either
as elementary particles (as in the L$\sigma$M) or as bound states (as in the NJL
model). In this case the resulting inequalities $m_\sigma \approx 650$ MeV $\approx \Lambda' \approx 660$
MeV $< \Lambda \approx 750$ MeV $< m_\omega \approx 780$ MeV speak to the Salam-Weinberg
compositeness condition.

3. Pion charge radius in L$\sigma$M and VMD theories

Having reconfirmed the exact pion form factor normalization due to the dom-
inant quark loops and also the approximate normalization for the meson loop
corrections of order 15\%, we now focus on the pion charge radius $r_\pi$, where
differentiation of $F_\pi(q^2)$ at $q^2 = 0$ will lead to no inconsistencies for pure
quark loops ($ql$) since then $\Lambda \approx 750$ MeV. However, when meson loops are
included, $F_\pi(0) = 1$ only when $\Lambda \rightarrow \Lambda' \approx 660$ MeV, which means $\sigma(650)$ is
on the verge of becoming a $\bar{q}q$ bound state.

Specifically for quark loops ($ql$) alone and $\Lambda \approx 750$ MeV, the pion charge
radius in the chiral limit is

$$r_{\pi,ql}^2 = \left. \frac{6dF_{\pi,ql}(q^2)}{dq^2} \right|_{q^2=0} = \frac{-i4N cg^2(-2)}{(2\pi)^4} \int_0^1 dx 6x(1-x) \int \frac{d^4p}{(p^2 - m_q^2)^3}$$

$$= \frac{N_c}{4\pi^2 f_{\pi,CL}^2}, \quad (13a)$$

upon using the GTR $g^2/m_q^2 = 1/f_{\pi,CL}^2$. This of course is the result of refs.\cite{9},
which can also be obtained via a once-subtracted dispersion relation evalu-
ated at $q^2 = 0$:

$$r_{\pi,ql}^2 = \frac{6}{\pi} \int_0^\infty dq^2 \frac{Im F_\pi(q^2)}{(q^2)^2} = \frac{N_c}{4\pi^2 f_{\pi,CL}^2}. \quad (13b)$$

We stress again the uniqueness and finiteness of $r_\pi$ in eqs. (13) for quark loops and for the VMD value of $r_\pi$ in (3). Since the (quark model) cutoff $\Lambda \approx 750$ MeV in (10a) has shifted to the lower value of $\Lambda' \approx 660$ MeV in (12c), it is legitimate to consider only quark loops even in the quark-level LσM when computing $r_\pi$ via differentiation of $F_\pi(q^2)$ as in (13). In effect, the shift of $\Lambda > m_\sigma$ to $\Lambda' \approx m_\sigma$ means that the elementary $\sigma(650)$ meson in the LσM is becoming a $\bar{q}q$ bound state as in the NJL scheme. As such, the LσM picture with $m_\sigma = 2m_q$ is merging into a pure quark model or NJL picture, again with $m_\sigma = 2m_q$.

Next we turn to $r_\pi$ as obtained from the meson loops (ml) of Fig.3b and eq.(12b). Differentiating (12b) with respect to $q^2$ and afterwards setting $q^2 = 0$, one obtains the meson loop (ml) contribution to the pion charge radius in the chiral limit,

$$r_{\pi,ml}^2 = \frac{6(-2m_\sigma^2\lambda)}{(2\pi)^4} \int_0^1 dx x(1-x) \int_0^1 dy 2y(1-y)y^2 \int \frac{d^4l}{[l^2 - m_\sigma^2(1-y)]^4} = \frac{1}{36} \frac{1}{m_\sigma^2} \int_0^1 dy 2y^3(1-y)^{-1}, \quad (14)$$

where the latter Feynman integral in (14) is $i\pi^2[6m_\sigma^4(1-y)^2]^{-1}$, and again $\lambda = 8\pi^2/3$. Although the squared length scale in (14) is 150 times smaller than the VMD scale $r_\pi^2 = 6/m_\rho^2$, the infrared singularity in (14) is signaling that the derivative of the meson loop form factor $F_{\pi,ml}^\sigma(q^2)$ (with normalization cutoff $\Lambda' \approx 660$ MeV $\approx m_\sigma$) has finally led to an inconsistency because this $\sigma(650)$ must then be treated as a $\bar{q}q$ bound state when computing $r_{\pi,\sigma}^{L\sigma M}$. Stated another way, the formal infrared singularity of $r_{\pi,ml}$ in (14) characterizes
chiral symmetry *breakdown* since then the log divergence in (14) corresponds to \( \ln \frac{m_\pi}{m_\sigma} \to \infty \) as \( m_\pi \to 0 \). This is a second signal (along with (12c)) that meson loops in a L\( \sigma \)M may lead to a double-counting inconsistency (resolved by a \( Z = 0 \) compositeness condition). This justifies the pure quark loop treatment of \( r_\pi \) in eqs.(13a) and (13b), and it should not be surprising that the \( r^{L\sigma M}_\pi \) is then in close agreement with experiment.

As for the relation between the one-loop order L\( \sigma \)M approach and tree-level VMD, with quark loops alone for \( \Lambda \approx 750 \text{ MeV} < m_\rho \), the rho meson can be taken as an external (bound state) particle and then the log-divergent gap equation in eq.(10a) leads to [15,17]

\[
g_{\rho\pi\pi} = g_\rho [-i4N_c g^2 \int_{-\Lambda}^\Lambda d^4p (p^2 - m_q^2)^{-2}] = g_\rho. \tag{15a}
\]

This is Sakurai’s [5] VMD universality condition. Equation (15a) can also be interpreted as a \( Z=0 \) compositeness condition [16] for the L\( \sigma \)M. If meson loops such as in Fig.3b (with \( \gamma \to \rho^0 \)) are included in eqs.(12), the extension of (15a) is

\[
g_{\rho\pi\pi} = g_\rho [-i4N_c g^2 \int_{-\Lambda}^\Lambda d^4p (p^2 - m_q^2)^{-2}] + \frac{1}{6} g_{\rho\pi\pi} \tag{15b}
\]

in a dynamically generated L\( \sigma \)M context. Here [15] \( \lambda/16\pi^2 = \frac{1}{6} \) as in eq.(12c). However since the (external field) rho meson is still a \( \bar{q}q \) bound state, we still maintain that the (quark model) cutoff is \( \Lambda \approx 750 \text{ MeV} \) as in (10a) or (10c). Then eq.(15b) becomes

\[
g_{\rho\pi\pi} = g_\rho + \frac{1}{6} g_{\rho\pi\pi} \quad \text{or} \quad g_{\rho\pi\pi}/g_\rho = \frac{6}{5} \approx 1.2, \tag{15c}
\]

and the latter ratio is in good agreement with data: \( g_{\rho\pi\pi} \approx 6.1 \) follows from the \( \rho \) width and \( g_\rho \approx 5.1 \) follows from the \( \rho^0 \to e^+e^- \) decay rate.

Since \( r^{VMD}_\pi \approx 0.63 \text{ fm} \) in eq.(3) (for \( \rho \) as an external field with \( \Lambda < m_\rho \)) and \( r^{L\sigma M}_\pi \approx 0.60 \text{ fm} \) in eq. (4) (then for quark loops alone with UV cutoff
Λ ≈ 750 MeV < m_ρ), there may be even a deeper link between r_π^{VMD} and r_π^{LσM} in the chiral limit. We now probe for such a connection.

To study the Goldstone theorem for m_π^2 and also the pion form factor F_π(q^2 = 0) in Sec. 2 and the pion charge radius r_π in Sec. 3, we have used only the original LσM lagrangian in eqs.(5,6) (but for quark rather than for nucleon fields). We have alluded to the dynamically generated LσM [15] only to streamline the results. Besides eqs.(5,6), the dynamically generated LσM appeals to dimensional regularization to obtain the two additional relations [15]

\[ m_\sigma = 2m_q, \quad g = 2\pi/\sqrt{N_c} \approx 3.6276 \]  

(16)

for N_c = 3. Of course the former equation in (16) is the famous NJL relation [7], while the latter together with the GTR (7) predicts a sensible chiral-limiting nonstrange quark mass

\[ m_q = f_{π,CL}2\pi/\sqrt{3} \approx (90\text{MeV})(3.6276) \approx 325\text{MeV}, \]  

(17a)

and in turn a scalar sigma mass

\[ m_\sigma = 2m_q \approx 650\text{MeV}. \]  

(17b)

The log-divergent gap equation (10a) helps to dynamically generate the rho couplings in (15) to one loop-order by invoking the VMD version of the ρ to vacuum matrix element of the em vector current < 0|V_{em}^\mu|ρ_o> = e k^2 ε_μ)/g_ρ for k^2 = m_ρ^2. Then the quark loop for the latter γ - ρ transition in the soft limit leads to [15]

\[ g_{ρππ} = g_ρ = \sqrt{3}g = 2\pi, \]  

(18)

a result also obtained by other methods [18]. Note that (18) is numerically compatible with the measured ρππ coupling constant extracted from the ρ width, giving g_{ρππ}^2/4π ≈ 3.0 or |g_{ρππ}| \approx 6.1.
Furthermore recall Sakurai’s derivation [19] of the KSRF relation from VMD of the $I = 1\ \pi N \rightarrow \pi N$ scattering amplitude: $M^{(-)} = g^{2}_{\rho}/m^{2}_{\rho}$. This is to be equated with the current algebra form $M^{(-)} = 1/2f^{2}_{\pi}$, leading to the KSRF relation [20] (ignoring the slight 15% correction from (15))

$$m^{2}_{\rho} = 2f^{2}_{\pi}g^{2}_{\rho}, \quad (19a)$$

which is empirically accurate to 10%. This is justified for $\rho \rightarrow \pi \pi$ because momentum conservation requires $p_{\rho} \rightarrow 0$ when $p_{\pi}$ and $p'_{\pi} \rightarrow 0$. Combining (19a) with the dynamical generated L$\sigma$M scale (18) and the quark level GTR (7) then converts the KSRF relation to

$$m_{\rho} = \sqrt{2}f_{\pi}g_{\rho} = \sqrt{6}f_{\pi}g = \sqrt{6}m_{q} \approx 795\text{MeV}. \quad (19b)$$

Moreover using $m_{\rho} = \sqrt{6}m_{q}$ in (19b) then transforms the VMD relation for the pion charge radius in (3) to

$$r^{VMD}_{\pi} = \sqrt{6}/m_{\rho} = 1/m_{q} = \sqrt{3}/2f_{\pi,CL} = r^{L\sigma M}_{\pi}. \quad (20)$$

Thus we have dynamically linked $r^{VMD}_{\pi}$ to $r^{L\sigma M}_{\pi}$, as anticipated.

Although $r^{VMD}_{\pi}$ in (3) and $r^{L\sigma M}_{\pi}$ in (4) appear numerically close, the dynamically generated versions of $r^{VMD}_{\pi}$ and $r^{L\sigma M}_{\pi}$ become identical in the chiral limit (CL) due to the blending together of the L$\sigma$M with VMD via KSRF. The expression $r_{\pi} = (1/m_{q})$ in (20) suggests a quark model interpretation of the pion charge radius for a $\bar{q}q$ Goldstone pion. Namely when $m_{\pi} \rightarrow 0$ the quarks fuse together with (Coulombic-type) potential $1/r$ and relativistic (Compton-type) wave length $r_{\pi} = (1/m_{q}) \approx 0.60$ fm, in close agreement with observation. As such, eq.(20) places a tight VMD-KSRF-L$\sigma$M constraint on other models purporting to contain all the richness of chiral symmetry.
4. Extension to SU(3) linear σ model and VMD

Here we show that the natural generalization of the SU(2) linear σ model (LσM) discussed in Secs. 2 and 3 but now for the SU(3) LσM also driven by the quark-level Goldberger-Treiman relation (GTR) gives [9,21]

\[ f_\pi g = \hat{m}, \quad f_K g = \frac{1}{2}(m_s + \hat{m}). \]  

(21)

Then the ratio of the two GTRs in (21) eliminates the meson-quark coupling \( g \) and predicts the empirical ratio

\[ \frac{f_K}{f_\pi} = \frac{1}{2}\left(\frac{m_s}{\hat{m}} + 1\right) \approx 1.22 \quad \text{or} \quad \frac{m_s}{\hat{m}} \approx 1.44 \]  

(22)

Indeed, this latter strange to nonstrange constituent quark mass ratio is approximately obtained from baryon magnetic moments [22], meson charge radii [23] and from \( K^* \to K\gamma \) decays [24].

Note that we have not passed to the SU(3) × SU(3) chiral limit in (21) or (22). But we do so now when studying the SU(3) generalization of the Goldstone theorem for \( m_K^2 = 0 \) in a LσM context. Then the quark loops (ql) in Fig.4 generate a straightforward extension of the SU(2) result in eq.(8):

\[ m_{K,ql}^2 = i4N_c g \int d^4p \left[ -\frac{2g(p^2 - m_s\hat{m})}{(p^2 - m_s^2)(p^2 - \hat{m}^2)} + \frac{2g'_{NS}\hat{m}}{m_{\sigma_{NS}}^2(p^2 - \hat{m}^2)} + \frac{\sqrt{2}g'_{S}m_s}{m_{\sigma_{S}}^2(p^2 - m_s^2)} \right]. \]  

(23)

Here \( \sigma_{NS} \) represents the SU(2) nonstrange σ meson and \( \sigma_{S} \) is the SU(3) \( s\bar{s} \) extension. That the integrand in (23) in fact vanishes can be seen from the partial fraction identity

\[ \frac{(m_s + \hat{m})(p^2 - m_s\hat{m})}{(p^2 - m_s^2)(p^2 - \hat{m}^2)} = \frac{\hat{m}}{p^2 - \hat{m}^2} + \frac{m_s}{p^2 - m_s^2}, \]  

(24)
combined with the natural [25] SU(3) extensions of the LσM meson-meson coupling in (6), \( g' = m^2_\pi/2f_\pi \):

\[
\begin{align*}
g'_\text{NS} &= \frac{m^2_{\sigma_{\text{NS}}}}{2f_K}, & g'_S &= \frac{m^2_{\sigma_S}}{\sqrt{2}f_K}.
\end{align*}
\]

The same is true for the meson loop graphs for \( m^2_K \). Thus the SU(3) Goldstone theorem \( m^2_K = 0 \) indeed holds in a straightforward manner to one-loop order in the SU(3) LσM. Note that as in the SU(2) LσM version of the Goldstone theorem (8) and (9), the vanishing of \( m^2_K \) in (23) is independent of any precise values of the scalar mesons.

The SU(3) analysis for the kaon charge radius \( r_{K^+} \) is even more transparent than the Goldstone vanishing of \( m^2_K \) in (23) for the LσM. Since \( r^{-1}_\pi = \hat{m} \) in (20) naturally corresponds to the kaon extension \( (m_s + \hat{m})/2 \), use of the SU(3) GTRs in (21) along with \( g = 2\pi/\sqrt{3} \) from (16) leads to the charge radius

\[
r^{L_{\sigma M}}_{K^+} = \sqrt{3}/2\pi f_K \approx 0.49 \text{ fm}
\]

for \( f_K \approx 110 \text{ MeV} \) in the chiral limit. On the other hand the SU(3) VMD extension of eq.(3) is

\[
r^{V_{\text{MD}}}_{K^+} = \sqrt{6}/m_{K^*} \approx 0.54 \text{ fm}.
\]

Not only are (26) and (27) in close proximity, but both are near the observed value \( \langle r^2_{K^+} \rangle \approx 0.28 fm^2 \) or \( r_{K^+} \approx 0.53 fm \).

Stated another way, the authors in ref.[23] develop a constituent quark mass expansion for the (LσM) quark loop version of the \( r^+ \) and \( r_{K^+} \) charge radii ratio:

\[
\frac{\langle r^2_{K^+} \rangle}{\langle r^2_\pi^+ \rangle} \approx 1 - (5/6)\delta + (3/5)\delta^2 \approx 0.75,
\]

for \( \delta = (m_s/\hat{m}) - 1 \approx 0.44 \) from eq.(22). This compares quite well with the measured ratio 0.70 ± 0.12. The extension to the (neutral) kaon charge

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radius in ref. [23] is also reasonable

\[
\frac{<r_{K^0}^2>}{<r_{\pi^+}^2>} \approx (-1/3)\delta + (1/2)\delta^2 \approx -0.05, \tag{29}
\]

whereas data finds the latter ratio to be $-0.12 \pm 0.06$.

5. Summary

We have attempted to resolve the apparent ambiguities arising in a quark-level linear sigma model (L$\sigma$M) field theory to one-loop order. In Sec. 2 we have shown that both quark and meson loops in the SU(2) L$\sigma$M manifest the Goldstone theorem and the pion decay constant combined with the quark-level GTR. While the same is true for quark loops generating the pion form factor (and the log-divergent gap equation for cutoff $m_\sigma < \Lambda \approx 750$ MeV $< m_\rho$), adding in meson loops to $F_\pi(q^2)$ reduces the cutoff to $\Lambda' \approx 660$ MeV $\approx m_\sigma$. This suggests that the $\bar{q}q \sigma$ is shifting from an elementary L$\sigma$M particle to a NJL bound state when one extracts $F_\pi(q^2)$ at $q^2 = 0$.

We began Sec. 3 by computing the quark loop version of the pion charge radius $r_\pi$, and the result is of course finite and in good agreement with experiment. However when L$\sigma$M meson loops are included, $r_\pi$ develops an infrared singularity. This just means that the $\sigma$ meson must be taken as a $\bar{q}q$ bound state (since the pion form factor cutoff is $\Lambda' \approx 660$ MeV $\approx m_\sigma$) when computing the pion charge radius. Then we invoked the dynamically generated L$\sigma$M theory finding $g_{\rho\pi\pi} = 2\pi$ (also compatible with data) together with the chiral KSRF relation $m_\rho = \sqrt{2}f_\pi g_\rho$ and showed that in the CL, $r_{\pi,QL}^{L\sigma M} = r_{\pi,VM}^{VD}$ exactly.

The above calculations employed the SU(2) dynamically generated $L\sigma M$ requiring [15] $m_\sigma = 2m_q, \quad g = 2\pi/\sqrt{3}$. Since the former relation also follows
from the four-fermion theory of NJL [7] where the $\vec{\pi}$ and $\sigma$ are $\bar{q}q$ bound states (so there is no meson ambiguity), it should not be surprising that the quark-level $L\sigma M$ also has no $\vec{\pi}, \sigma$ elementary particle–bound state ambiguity (due to the $Z = 0$ compositeness condition).

Finally in Sec. 4 we extended the $L\sigma M$ to SU(3) and demonstrated that the kaon Goldstone theorem for quark loops is again manifest. We also computed the $K^+$ charge radius $r_{K^+}^{VMD}$ in tree order and $r_{K^+}^{L\sigma M}$ in one-loop order. Both are compatible with data.

With hindsight, double-counting problems never arise in QCD or in the NJL four-quark pictures because only quarks (and gluons in the former case) are elementary while mesons are treated as $\bar{q}q$ bound states. The quark-level $L\sigma M$ in one-loop order (but with the double-counting issues discussed in this paper taken into consideration) has the additional scales of $m_q \approx 325$ MeV and $m_\sigma = 2m_q \approx 650$ MeV dynamically generated [15]. As pointed out in the latter reference, the log-divergent gap equation for the pion decay constant in our (10a) (or the pion form factor normalization in (11b)) can be taken as a $Z = 0$ compositeness condition [16] characterizing the $\pi$ and $\sigma$ particles as not elementary, but bound states of more basic fields (as in QCD or in NJL). This $Z = 0$ compositeness condition ((10a), (11b) or (15a)) in turn bootstraps quark loop graphs to tree diagrams. Such a “nonperturbative shrinkage” justifies not adding contact terms to one-loop terms as one would do in a (multiple-counting) perturbative field theory.

While focusing on this issue of double counting in the quark-level $L\sigma M$, we have also obtained new one-loop order results: (1) the pion decay constant involving both quark and meson loops; (2) the normalization of the pion form factor $F_\pi(q^2 = 0) = 1$ involving both quark and meson loops; (3) recovering the VMD universality relation $g_{\rho\pi\pi} = g_\rho$ due to quark loops only.
and extending it including also meson loops to the coupling ratio $g_{\rho\pi\pi}/g_{\rho} = 6/5$ which is in empirical agreement with the observed $\rho\pi\pi$ and $\rho\pi\nu$ decay rates; (4) using the KSRF relation to link the L$\sigma$M pion charge radius $r_\pi$ to the VMD version of $r_\pi$; (5) empirically extending $r_\pi$ to $r_K$ in the L$\sigma$M and VMD models. One might objectively question why we have worked so hard to repair this “toy theory”—the quark-level linear $\sigma$ model—using the Salam-Weinberg $Z = 0$ compositeness condition. Our answer is that it is now becoming experimentally clear in [1] and [26–30] that a broad $\sigma$ meson of mass below $1$ GeV is emerging from data (just as the dynamically generated L$\sigma$M requires).

Specifically, the DM2 collaboration [26] obtained a low mass $\pi\pi$ scalar $M \approx 420$ MeV from $J/\psi \rightarrow \omega\pi\pi$, while a reanalysis [27] of CERN-Munich data for $\pi^-p \rightarrow \pi^-\pi^+n$ found a $\sigma$ mass near 850 MeV. More recently a Törnqvist and Roos data analysis [28] finds a very broad $\sigma$ meson at $f_0(400–900)$, with an 860 MeV mass coming from a Breit-Wigner background and its pole at 400–900 GeV. Also Svec [29] studied polarized target $\pi N \rightarrow \pi\pi N$ data and detected a $\sigma(750)$. Finally, Ishida et. al. [30] analyzed the $\pi\pi$ scattering phase shifts and introduced a negative background phase and found a $\sigma(555)$ scalar meson. Also see ref. [31].

We conclude that a chiral $\sigma$ meson may indeed exist and that the quark-level L$\sigma$M with a $\sigma(650)$ may not be simply a “toy” model but in fact may reflect the real world.
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References

[1] Particle data group, R. M. Barnett et al, Phys. Rev. D 54, 1 (1996).

[2] J. Goldstone, Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127, 965 (1962).

[3] E.B. Dally et.at, Phys. Rev. Lett. 48, 375 (1982).

[4] A. F. Grashin and M. V. Lepeshkin, Phys. Lett. B146, 11 (1984).

[5] J. J. Sakurai, Ann. Phys. NY 11, 1 (1960).

[6] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960); also see V. DeAlfaro, S. Fubini, G. Furlan and C. Rossetti, Currents in Hadron Physics, North Holland Publ. (1973) Amsterdam, Chap. 5.

[7] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).

[8] Y. Nambu, Phys. Rev. Lett. 4, 380 (1960).

[9] R. Tarrach, Z. Phys. C2, 221 (1979); S. B. Gerasimov, Sov. J. Nucl. Phys. 29, 259 (1979); Also see V.Bernard, B.Hillar and W.Weise, Phys. Lett. B 205, 16 (1988).

[10] S.A. Coon and M.D. Scadron, Phys. Rev. C23, 1150 (1981) show that $1 - f_{\pi,CL}/f_\pi = m_\pi^2/8\pi^2 f_\pi^2 \approx 0.03$, so that $f_{\pi,CL} \approx 90$ MeV for $f_\pi \approx 93$ MeV.

[11] T. Hakioglu and M. D. Scadron, Phys. Rev. D42, 941 (1990); also see T.P. Cheng and L.F. Li, Gauge Theories of Elementary Particles, Oxford Press (1986) p. 187 for a simpler U(1) L\sigma M version of the Goldstone theorem.
[12] See e.g. appendix of J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, 2nd Ed. Berlin. Springer-Verlag, 1976.

[13] P. Pascual and R. Tarrach, Nucl. Phys. B146, 520 (1978); A. Bramon and M. D. Scadron, Europhys. Lett. 19, 663 (1992).

[14] N. Paver and M. D. Scadron, Nuovo Cimento, 78A, 159 (1983) noted that this rerouting regularization procedure leading to our eq.(11a) is unique and justified by the gauge invariance of the resulting vector amplitude as found from (11a).

[15] R. Delbourgo and M.D. Scadron, Mod. Phys. Lett. A 10, 251 (1995).

[16] A. Salam, Nuovo Cimento 25, 224 (1962); S. Weinberg, Phys. Rev. 130, 776 (1963).

[17] T. Hakiooglu and M. D. Scadron, Phys. Rev. D43, 2439 (1991).

[18] L. H. Chan, Phys. Rev. Lett. 39, 1125 (1977); 55, 21 (1985); V. Novozhilov, Phys. Lett. B228, 240 (1989).

[19] J.J. Sakurai, Phys. Rev. Lett. 17, 552 (1966).

[20] K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett 16, 255 (1966); Ri-azuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966).

[21] R. Delbourgo and M. D. Scadron, J.Phys. G5, 1621 (1979).

[22] A. De Rujula, H. Georgi and S. Glashow, Phys. Rev. D12, 147 (1975).

[23] See e.g. C. Ayala and A. Bramon, Europhys. Lett. 4, 777 (1987) and references therein.
[24] A. Bramon and M. D. Scadron, Phys. Rev. D40, 3779 (1989).

[25] J. Schechter and Y. Ueda, Phys. Rev. D36, 2874 (1971).

[26] DM2 Collab., J. Augustin et. al.: Nucl. Phys. B320, 1 (1989).

[27] B. S. Zou and D. V. Bugg: Phys. Rev. D48, 3948 (1993); D50, 591 (1994).

[28] N. A. Törnqvist and M. Roos: Phys. Rev. Lett. 76, 1575 (1996).

[29] M. Svec: Phys. Rev. D 53, 2343 (1996).

[30] S. Ishida et. al.: Prog. Theor. Phys. 95, 745 (1996).

[31] M. Harada, F. Sannino and J. Schechter, Phys. Rev.D54,1991 (1996).
Figure Captions

Fig. 1a. Quark bubble plus tadpole graphs for $m^2_\pi$.

Fig. 1b. Meson bubble plus quartic loop plus tadpole graphs for $m^2_\pi$.

Fig. 2. Quark (a) and meson (b) loops contributing to $f_\pi$.

Fig. 3a. Quark loops contributing to the pion form factor $F^{ql}_\pi(q^2)$.

Fig. 3b. Meson loops contributing to $F^{ml}_\pi(q^2)$.

Fig. 4. Quark bubble plus tadpole graphs for $m^2_K$. 