Towards an Algebraic Characterization of 3-dimensional Cobordisms

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**Abstract:** The goal of this paper is to find a close to isomorphic presentation of 3-manifolds in terms of Hopf algebraic expressions. To this end we define and compare three different braided tensor categories that arise naturally in the study of Hopf algebras and 3-dimensional topology. The first is the category $\mathbf{Cob}$ of connected surfaces with one boundary component and 3-dimensional relative cobordisms, the second is a category $\mathbf{Tgl}$ of tangles with relations, and the third is a natural algebraic category $\mathbf{Alg}$ freely generated by a Hopf algebra object. From previous work we know that $\mathbf{Tgl}$ and $\mathbf{Cob}$ are equivalent. We use this fact and the idea of Heegaard splittings to construct a surjective functor from $\mathbf{Alg}$ onto $\mathbf{Cob}$. We also find a map that associates to the generators of the mapping class group in $\mathbf{Cob}$ preimages in $\mathbf{Alg}$. The single block relations in the mapping class group are verified for these expressions. We propose to find a version of $\mathbf{Alg}$ with possibly additional relations to obtain isomorphic algebraic presentations of the mapping class groups and eventually of $\mathbf{Cob}$.

1. **Introduction**

For some time it has been a puzzling question whether the appearance of Hopf algebras in the world of quantum invariants of 3-manifolds as in [18] is a lucky coincidence that makes computations work or if these structures arise in more fundamental ways out of 3-dimensional topology.

It was soon understood that such algebraic structures are in fact inherent in the category of cobordisms $\mathbf{Cob}$ between connected surfaces with one boundary component. Specifically, the torus with one hole as an object in $\mathbf{Cob}$ was discovered to admit the structure of a braided Hopf algebra in the sense of [16].

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The three-dimensional pictures of the cobordisms representing the products and coproducts have been found by Crane and Yetter and proven to satisfy the relevant axioms, see [20] and [4]. The same picture emerged independently in investigations by the author via the route of tangle presentations, see [8]. An interpretation of braided Hopf algebra structures in linear abelian braided tensor categories was found by Lyubashenko in [15]. This combined with the equivalence of cobordisms categories with certain tangle categories given in [12] leads to the three dimensional interpretation. The cobordisms are easily worked out to be the same as the ones in [20] and [4], see [8]. The tangle pictures are also used as examples for integrals in braided categories in [3].

The purpose of this note is to prove a theorem that was already stated in [10], which not only asserts the existence of cobordisms representing structure morphisms such as products and coproducts of a Hopf algebra but also that these generate the entire cobordism category. In more formal terms we will define an algebraic category $\mathcal{A}_{lg}$ via generators and relations representing the axioms for a braided Hopf algebra in a braided tensor category. The existence of the special cobordisms is implied by the existence of a functor and their generating property by surjectivity of this functor.

**Theorem 1** Let $\mathcal{A}_{lg}$ be the braided tensor category freely generated by a Hopf algebra object as defined in Section 4, and $\mathcal{C}_{ob}$ the cobordism category defined in Section 2. There exists a surjective functor of braided tensor categories

$$\mathcal{G} : \mathcal{A}_{lg} \longrightarrow \mathcal{C}_{ob}.$$

The proof of this theorem as we present it here uses and demonstrates several techniques in graphical and diagrammatic categorical calculations. Particularly, we give a description of $\mathcal{A}_{lg}$ as a category of directed trivalent graphs with crossings and special types of endpoints modulo relations. Also, in Section 3 we recall how the cobordism category $\mathcal{C}_{ob}$ can be presented in terms of a category of tangles modulo relations $\mathcal{T}_{gl}$. The constructions of functors and assignments in Section 5 and Section 6 are done entirely in these graphical languages.

In Sections 6.2 and 6.3 we also discuss the problem of modifying $\mathcal{A}_{lg}$ so that $\mathcal{G}$ becomes an isomorphism of categories. This means we would have to find an assignment of generators of $\mathcal{C}_{ob}$ to generators of $\mathcal{A}_{lg}$ such that the relations in $\mathcal{C}_{ob}$ are also respected in $\mathcal{A}_{lg}$. For the genus one relations this is in fact true but for higher genera it is likely that we will have to impose more relations on $\mathcal{A}_{lg}$. We summarize next the observations we will make on this question.

**Theorem 2**

1. Let $\Gamma^*_{1,1}$ be the central extension of the mapping class group of the torus with one hole. Let $A \in \mathcal{A}_{lg}$ be the generating Hopf algebra object. Then there exists a homomorphism $\mathfrak{M}^{[1]} : \hat{\Gamma}^*_{1,1} \rightarrow Aut(A)$, such that the composite

$$\Gamma^*_{1,1} \xrightarrow{\mathfrak{M}^{[1]}} Aut(A) \xrightarrow{\mathcal{G}} \hat{\Gamma}^*_{1,1} \subset \mathcal{C}_{ob}$$

is the identity. Here $Aut(A)$ denotes the invertible morphisms $A \rightarrow A$ in $\mathcal{A}_{lg}$, and we use that $\Gamma^*_{1,1}$ is identical with the group of invertible cobordisms on the surface of genus one.

2. There is a natural set $Gen[\mathcal{C}_{ob}]$ of generators of $\mathcal{C}_{ob}$ and an assignment $\mathfrak{W} : Gen[\mathcal{C}_{ob}] \rightarrow \mathcal{A}_{lg}$ such that $\mathcal{G} \circ \mathfrak{W}$ is the identity on $Gen[\mathcal{C}_{ob}] \subset \mathcal{C}_{ob}$.

3. Suppose it is possible to find additional relations on $\mathcal{A}_{lg}$, yielding a subquotient category $\overline{\mathcal{A}_{lg}}$ such that $\mathcal{G}$ factors into $\mathcal{A}_{lg}$ and $\mathfrak{W}$ extends to a functor $\mathfrak{W}$ on $\overline{\mathcal{A}_{lg}}$.

Then $\mathfrak{W}$ is a two-sided inverse of $\mathcal{G}$, and hence $\overline{\mathcal{A}_{lg}} \cong \mathcal{C}_{ob}$.

The challenging question now is to find the additional relation that define $\overline{\mathcal{A}_{lg}}$. This would imply a characterization of three dimensional topology in purely algebraic terms!
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2. THE CATEGORY \textbf{Cob}:

\textbf{2.1 Category of 2-framed, relative cobordisms} For every integer \(g \geq 0\) construct a model surface \(\Sigma_g\) of genus \(g\) with \(\partial \Sigma_g = S^1\). This can be done as follows. Pick a disc \(D^2\) and cut out \(2g\) small discs along a diameter of \(D^2\). Along the new boundary components glue in \(g\) cylinders such that the ends of each cylinder are glued to two consecutive holes. The result is depicted below.

\[
\begin{array}{c}
\ldots \\
S^1 \\
\end{array}
\]

For two surfaces \(\Sigma_h\) and \(\Sigma_g\) let \(\Sigma_{[h,g]} = -\Sigma_h \cup S^1 \times [0,1] \cup S^1 \times 1 \Sigma_g\) be the closed oriented surfaces of genus \(g + h\) obtained by gluing the original surfaces together along their boundaries with a cylinder inserted. A relative cobordism is a compact oriented manifold \(M\), together with a homeomorphism \(\psi: \partial M \rightarrow \Sigma_{[h,g]}\).

We consider two pairs of data \((M, \psi)\) and \((M', \psi')\) as equivalent if there is a homeomorphism \(h: \partial M \rightarrow \partial M'\) such that \(\psi' \circ h = \psi\) on \(\partial M\). We write the equivalence class \([M, \psi]\), with minor abuse of notation, in the morphism form:

\[
M : \Sigma_h \rightarrow \Sigma_g
\]

It is not hard to see (e.g., [13]) that \([M, \psi]\) does not change under isotopies of \(\psi\).

For two cobordisms \(M: \Sigma_g \rightarrow \Sigma_h\) and \(N: \Sigma_h \rightarrow \Sigma_k\) we define a composite \(N \circ M: \Sigma_g \rightarrow \Sigma_k\) by gluing two cobordisms together along the common boundary piece \(\Sigma_h\) using the coordinate maps on \(\partial M \rightarrow \Sigma_{[h,g]}\) and \(\partial N \rightarrow \Sigma_{[g,k]}\). The cylindrical part of length 2 is then monotonously shrunk to a cylinder of height 1. It is easy to see that the resulting homeomorphism class does not depend on the choices of \(M\) and \(N\) in their classes.

Thus we obtain a category, denoted \(\textbf{Cob}_0\), which has the surfaces \(\Sigma_g\) as objects and the classes \([M, \psi]\) as morphisms.

For a relative cobordism \(M: \Sigma_h \rightarrow \Sigma_g\) denote by \(M_0\) the cobordism obtained by gluing in a full cylinder \(D^2 \times [0,1]\) along the \(S^1 \times [0,1]\) part of the boundary so that \(M_0\) is a cobordism between closed surfaces \(\Sigma_g = \Sigma_g \cup D^2\). Consider an unknotted embedding of \(\Sigma_g \times [0,\epsilon]\) into \(S^3\). A framing on \(S^3\) hence induces a standard framing on the collar \(\Sigma_g \times [0,\epsilon]\).

In addition to the topological structure from above we consider now also manifolds with 2-framings, i.e., isotopy classes of trivializations \(TM \oplus TM \rightarrow \mathbb{R}^6 \otimes M\), and assume standard trivializations on standard handle bodies bounding the surfaces. We restrict to those that are compatible with the standard 2-framings on the collars \(\Sigma_g \times [0,\epsilon]\). As a result the gluing composition operation extends to the 2-framed cobordisms.

It is standard knowledge, see for example [12], that this information is equivalent to the signature of a bounding 4-manifold. We thus obtain an exact sequence in the sense of group theory:

\[
1 \rightarrow \mathbb{Z} \rightarrow \textbf{Cob} \rightarrow \textbf{Cob}_0 \rightarrow 1
\]
2.2 Braided tensor category: The category $\text{Cob}_0$ has a natural tensor product. It is given on the objects by $\Sigma_g \otimes \Sigma_h = \Sigma_{g+h}$. In order to describe $\otimes$ on the morphisms we choose a disc $P = D^2 - (D^2 \cup S^2)$ with two holes. The two surfaces $\Sigma_g$ and $\Sigma_h$ are sewn them into the two holes of $P$ such that their handles are aligned as depicted below. Upto isotopy there is then a unique homeomorphism $\lambda_{g,h} : \Sigma_g \cup S^1 P \cup S^1 \Sigma_h \to \Sigma_{g+h}$ which maps the corresponding handles in order onto each other.

![Diagram](image)

For two cobordisms $M : \Sigma_h \to \Sigma_g$ and $L : \Sigma_p \to \Sigma_q$ the tensor product is obtained by gluing the cobordisms into $P \times [0,1]$ and using the $\lambda_{g,h}$ to adjust the boundary identifications so that $M \otimes N : \Sigma_{h+p} \to \Sigma_{g+q}$.

Since $\lambda_{a,b+c}(1 \otimes \lambda_{b,c})$ is isotopic to $\lambda_{a+b,c}(\lambda_{a,b} \otimes 1)$ this product is strictly associative. In [13] we describe a procedure by which this tensor product lifts to the 2-framing structure so that we still have $(M \otimes L) \circ (N \otimes K) = (M \circ N) \otimes (L \circ K)$ for compatible cobordisms.

Finally, we obtain a family of isomorphisms $c_{g,h} : \Sigma_g \otimes \Sigma_h \to \Sigma_{g+h}$ from the cylinder $P \otimes [0,1]$ by twisting the two ends by $\pi$ relative to each other so that opposite holes in $P$ are connected to each other by boundary cylinders. The morphism $c_{g,h}$ is then obtained by gluing the cylinders $\Sigma_g \times [0,1]$ and $\Sigma_h \times [0,1]$ into this twisted version of $P$. See [13] again for more details. By construction $c_{g,h}$ is a natural isomorphism from $\otimes T$ to $\otimes$ with all properties of a braiding. Here $T : \text{Cob} \times \text{Cob} \to \text{Cob} \times \text{Cob}$ is the transposition and $\otimes : \text{Cob} \times \text{Cob} \to \text{Cob}$ as defined.

The remarks so far are summarized as follows:

Theorem 3 $\text{Cob}$ is a braided tensor category with objects $\Sigma_g = \Sigma_1^{\otimes g}$.

2.3 Generators of $\text{Cob}$: We will describe here the set of generators of $\text{Cob}$ that comes from Heegaard splittings of cobordisms.

Let $H_0^+ : \Sigma_0 \to \Sigma_1$ be the cobordism, obtained by attaching a full handle $D^2 \times [0,1]$ to a thickening $D^2 \times [0,\varepsilon]$ of the disc from 2.1 along the holes in $D^2 \times 0$. The boundary identification is such that the restriction of the gluing construction to $S^1 \times [0,1]$ is precisely the construction of $\Sigma_1$ described in Section 2.1. Hence $H^+ \cong D^2 \times S^1$ is itself a full torus. We introduce the cobordisms

$$H_{g,k}^+ : \Sigma_g \to \Sigma_{g+k} \quad \text{with} \quad H_{g,k}^+ = id_{\Sigma_g} \otimes \underbrace{H_1^+ \otimes \ldots \otimes H_1^+}_{k}.$$ (3)

They describe the addition of $k$ 1-handles to a surface of genus $g$.

Conversely, we have a cobordisms $H_0^- : \Sigma_1 \to \Sigma_0$ where we glue a thickened disc to $\Sigma_1 \times [0,1]$ along the longitude of $\Sigma_1 \times 1$ so that $H_0^- \circ H_0^+ = id_{\Sigma_0} \cong D^3$. The cobordisms $H_{g,k}^- : \Sigma_{g+k} \to \Sigma_g$ are obtained analogously.

The second type of morphisms arise from the mapping class groups. Consider the group $\text{Homeo}^+(\Sigma_g)$ of orientation preserving homeomorphims of $\Sigma_g$ to itself, which leave the boundary pointwise fixed. The mapping class group of $\Sigma_g$ is thus the group of path connected components, that is $\Gamma_{g,1} = \pi_0(\text{Homeo}^+(\Sigma_g))$. 


To an element \( \psi \in \text{Homeo}^+(\Sigma_g) \) we assign a cobordism \( I_\psi \) as follows. The representative cobordism is given by the cylinder \( \Sigma_g \times [0, 1] \). The boundary identification with \( \Sigma_{[g,g]} \) is the canonical map (identity) on \( \Sigma_g \times 0 \) and it is given by \( \psi \) on \( \Sigma_g \times 1 \). The cobordism \( I_\psi \) only depends on the isotopy class of \( \psi \) in \( \Gamma_{g,1} \), which we abusively denote by the same letter. Now if \( \text{Aut}(\Sigma) \) denotes the group of invertible cobordisms on \( \Sigma \) in \textbf{Cob} we have the following result:

**Theorem 4** ([13]) *The following map is an isomorphims of groups.*

\[
\Gamma^*_{g,1} \rightarrow \text{Aut}(\Sigma_g) : \psi \mapsto I_\psi
\]

Here \( \Gamma^*_{g,1} \) is the central extension of \( \Gamma_{g,1} \), which carries the corresponding framing information.

A Heegaard splitting of a cobordism is given now as follows:

**Theorem 5** *Every cobordism \( M : \Sigma_h \rightarrow \Sigma_g \) in \textbf{Cob} is given as a composite

\[
M = H^-_{g,N-g} \circ I_\psi \circ H^+_{h,N-h}
\]

for some \( N \geq \max(g, h) \) and some \( \psi \in \Gamma^*_{N,1} \).

**Proof:** Consider the space of differentiable functions \( f : M \rightarrow [0, 1] \) such that \( f^{-1}(\Sigma_h) = \{0\} \), \( f^{-1}(\Sigma_g) = \{1\} \), and on the cylindrical piece \( \cong S^1 \times [0, 1] \subset \partial M \) \( f \) coincides with the canonical projection. Assume some metric on \( M \). By standard arguments from differential topology we can assume that \( f \) is a Morse function and has singularities only of index 1 or 2. Further more we can assume that the critical values of index 2 all lie in \( (\frac{1}{2}, 1] \) and the critical values of index 1 in \( [0, \frac{1}{2}) \) and \( \frac{1}{2} \) is a regular value. With \( \Sigma_{\text{int}} = f^{-1}(\{\frac{1}{2}\}) \) we have cobordisms \( A = f^{-1}([0, \frac{1}{2})) \) and \( B = f^{-1}([\frac{1}{2}, 1]) \) with \( M = B \circ A \). The gradient flow of \( f \) identifies \( \Sigma_h \) with 2l discs removed with a submanifold \( \phi : \Sigma_h - 2lD^2 \hookrightarrow \Sigma_{\text{int}} \). Here \( l = N - h \) is the number of index 1 critical points and the locations of the discs are given by their critical manifolds. We can always from a map \( \tau \in \text{Homeo}^+(\Sigma_h) \) isotopic to the identity, which maps these discs into the standard position of the holes for the next \( l \) handle attachments for the construction of \( \Sigma_{h+l} \). There is up to isotopy a unique map that extends \( \phi \circ \tau^{-1} : \Sigma_g - 2lD^2 \hookrightarrow \Sigma_{\text{int}} \) to a map \( \phi : \Sigma_{g+l} \hookrightarrow \Sigma_{\text{int}} \) over the additional glued in cylinders. Using the gradient flow and its behavior around the singularities the class of cobordism \( A \) is given by \( I_\phi \circ H^+_{h,l} \). By an analogous procedure we find \( B = H^-_{g,k} \circ I_\phi' \), which implies the claim if we set \( \psi = \phi' \circ \phi \). The framing of \( M \) is adjusted by choosing the appropriate extension class in \( \Gamma^*_{N,1} \).

The set of generators of \textbf{Cob} can be broken down even further using the special generators of the mapping class groups. It is a well known fact that \( \Gamma_{g,1} \) is generated by a finite set of Dehn twists. They are denoted by capital letters \( A_j, B_j \) and \( C_j \), for Dehn twists along the curves depicted in (4) labeled by the corresponding lower case letters.

\[
\begin{align*}
\text{A} & \quad \text{B} & \quad \text{C} \\
| & | & |
\shortcircled{a_1} & \shortcircled{b_1} & \shortcircled{c_1} \\
| & | & |
\shortcircled{a_2} & \shortcircled{b_2} & \shortcircled{c_2} \\
| & | & |
\shortcircled{a_g} & \shortcircled{b_g} & \shortcircled{c_{g-1}}
\end{align*}
\]

A slightly more convenient set of generators is given by the set \( \{A_j, S_j, D_j\} \) where

\[
D_j = A_j^{-1}A_{j+1}^{-1}C_j \quad \text{and} \quad S_j = A_jB_jA_j \quad \text{for } j = 1, \ldots, g.
\]
In terms of cobordisms we can write $I_{A_j} = id_{\Sigma_{j-1}} \otimes I_{A_1} \otimes id_{\Sigma_{g-j}}$, where $A_1 \in \Gamma_{1,1}$. Similar formulae exists for $I_{S_j}$ and $I_{D_j}$. We imply here some specific representative in $\Gamma_{1,1}^\ast$. We also introduce the cobordism $Z : 0 \to 0$, which is topologically the identity cylinder over $S^2$ but has framing changed by one. We find the following.

**Corollary 6** As a tensor category $\textbf{Cob}$ is generated by the cobordisms $H^+_0, H^-_0, I_{A_1}, I_{D_1}, I_{S_1},$ and $Z^{\pm 1}$.

3. **PRESENTATION OF $\textbf{Cob}$ BY $\mathcal{T}gl$**

We recall a variant of the tangle presentation of $\textbf{Cob}$ given in [12].

3.1 **Admissible tangles and moves:** First we define the category $\mathcal{T}gl$. Its objects are non-negative integers. A morphism $T : k \to l$ is obtained from a framed tangle in $R^2 \times [0,1]$ with $2k$ end points $1^+, 1^-, \ldots, k^+, k^-$ at the top line $R_x \times 1$ and $2l$ end points $1^+, 1^-, \ldots, l^+, l^-$ at the bottom line $R_x \times 0$, where $R_x \subset R^2$ is a given axis. An admissible tangle is one which has top, bottom, closed or through strands. A top strand is a component of the tangle that starts at $j^+ \in R_x \times 1$ for some $j$ and ends at the corresponding $j^- \in R_x \times 1$, and a bottom strand does the same thing at $R_x \times 0$. A closed strand is a component $\cong S^1$ in the interior of $R^2 \times [0,1]$. A through strand is a pair of components where one component starts at $j^+ \in R_x \times 1$ and ends in $k^+ \in R_x \times 1$ and the other starts at $j^- \in R_x \times 1$ and ends in $k^- \in R_x \times 1$ for some $k$ and $j$.

We depict an admissible framed tangle by a generic projection, subject to the second and third Reidemeister move as well as the usual moves for maxima and minima. We will assume the framing to be in the plane of projection. $2\pi$-twists in the framing along a strand are depicted by full or empty blobs as follows:

$$= \quad = \quad = \quad = \quad =$$

The admissible tangles are subject to the following relations generalizing Kirby’s calculus of links [14]:

1. A Hopf link that is isolated from the rest of the diagram with one component 0-framed and the other either 1- or 0-framed can be added or removed from a diagram:

$$$$

2. Any strand $R_1$ can be slid over a closed component $R_2$ by a 2-handle slide. This means that $R_1$ is replaced by a connected sum $R_1 \# R'_2$, where $R'_2$ is a push-off of $R_2$ along its framing.

$$$$
3. The boundary move, given by introducing two additional components in a vicinity of points \(\{j^+, j^-\}\) at the top line. One is an arc connecting \(j^+\) to \(j^-\), another an annulus going through that arc and, finally, the outgoing strands are connected through that annulus.

![Boundary Move Diagram]

\begin{equation}
(9)
\end{equation}

Let us record also a few moves that are implied by the above. The first two can be found in [6]. The third follows from the second, the 2-handle slide and the boundary move above.

1. The Fenn-Rourke Move in which a bunch of parallel strands are slid over a 1-framed annulus surrounding them. As a result the group of strands incurs a \(2\pi\)-twist and a shift in framing.

![Fenn-Rourke Move Diagram]

\begin{equation}
(10)
\end{equation}

2. The \(\beta\)-Move. If 0-framed annulus bounds a disc which intersects the tangle exactly once with a closed strand then the annulus together with the closed component can be removed.

![\(\beta\)-Move Diagram]

\begin{equation}
(11)
\end{equation}

3. The connecting annulus move. Two different components \(R_1\) and \(R_2\) of the tangle are linked together by an annulus \(A\) as shown. An equivalent configuration is the one where \(A\) is removed and the two other strands replaced by \(R_1\#R_2\).

![Connecting Annulus Move Diagram]

\begin{equation}
(12)
\end{equation}

3.2 Equivalence of braided tensor categories: There is an obvious way in which tangles can be made into a braided monoidal category. Two tangles \(t : k \to l\) and \(s : l \to m\) can be composed by stacking \(t\) on top of \(s\) connecting the \(2l\) intermediate points with each other. It is clear that the composite of admissible tangles is again admissible, and it is also not hard to prove that this composition factors into the equivalence classes defined by the moves.

In addition we have a tensor product \(t \otimes u : (k+p) \to (l+q)\) of tangles by putting them side by side into one diagram, that is by juxtaposing them. Again it follows easily that this operation closes in the admissible tangles and factors into the equivalence classes. Clearly, this tensor product is strictly associative.
The identity tangle $id : k \rightarrow k$ is given by $2k$ parallel vertical strands. A braiding $c_{k,l} : (k+l) \rightarrow (l+k)$ is given by taking a simple crossing, as, e.g., in [17], and replacing one strand by $2l$ parallel strands and the other by $2k$ parallel strands. All the axioms of a braided monoidal category are easily verified.

**Theorem 7 ([12])** There is an isomorphism of braided tensor categories

$\text{Surg} : \mathcal{T}_{gl} \cong \mathcal{Cob}$.

The assignment of surfaces to numbers is obvious. To produce a cobordism from a tangle $t : k \rightarrow l$ observe first that by compactness the tangle must be over some disc $D^2 \subset \mathbb{R}^2$ to which we restrict. Next we add 1-handles to one side of $D^2 \times [0,1]$ at respective end point pairs and continuing the strand through those handles. Moreover, we bore holes into $D^2 \times [0,1] \cup \{\text{handles}\}$ along the strands that start and end on the other side. As a result we obtain a 3-manifold $X$ for which $\partial X \cong \Sigma_{[k,l]}$. The end points of the tangle have now disappeared so that we have a link inside of $X$. The desired cobordism is obtained by performing surgery inside of $X$ along that particular link. More details of the construction and the fact that this functor is well defined and an isomorphism are given in [12].

### 3.3 Generating Tangles:

Here we list tangles in $\mathcal{T}_{gl}$ which are mapped to the generators in $\mathcal{Cob}$ from Corollary 6. The mapping class group generators are as follows:

$$A_j = \begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array}
\end{array} \quad (13)$$

$$S_j = \begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array}
\end{array} \quad (14)$$

$$D_j = \begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array}
\end{array} \quad (15)$$

They are the same as in [17]. The other three generators for handle additions and framing shift are as follows.

$$H^+_0 = \begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array}
\end{array} \quad H^-_0 = \begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array}
\end{array} \quad Z = \begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array}
\end{array} \quad (16)$$
4. The Category $\mathcal{Alg}$

The category $\mathcal{Alg}$ is described entirely in algebraic terms.

**Definition 1** $\mathcal{Alg}$ is the free braided tensor category freely generated by a braided Hopf algebra object $A$ with two-sided integral and a ribbon element, which induces a non-degenerate Hopf paring.

In particular the set of objects is $\{1, A^\otimes n \text{ with } n \in \mathbb{N}\}$. In this section we give a more explicit definition in terms of a category of planar diagrams. Hence a morphism from $A^\otimes n$ to $A^\otimes m$ in $\mathcal{Alg}$ is represented by a diagram in the strip $[0, 1] \times \mathbb{R}$ with $n$ endpoints at the top end $1 \times \mathbb{R}$ and $m$ endpoints at the bottom end $0 \times \mathbb{R}$. The composite of two morphisms is given by stacking the diagrams on top of each other, and the tensor product of two morphisms by their juxtaposition.

**4.1 The Generators for $\mathcal{Alg}$**

Every morphisms in $\mathcal{Alg}$ is the product and tensor product of a set of generators given by elementary diagrams. They are the units, $1 : 1 \to A$ and $\varepsilon : A \to 1$, multiplications, $m : A \otimes A \to A$ and $\Delta : A \to A \otimes A$, an invertible ribbon element $v : 1 \to A$, an invertible antipode, $\Gamma^\pm : A \to A$, an $S$-invariant integral $\mu : A \to 1$, and an invertible braid isomorphism $c : A \otimes A \to A \otimes A$. The elementary pictures are the following:

$$
\begin{align*}
I &= \bullet \\
\varepsilon &= \bullet \\
\mu &= \bullet \\
v &= \star \\
v^{-1} &= \star \\
\Gamma &= \pmb{+} \\
\Gamma^{-1} &= \pmb{-} \\
m &= \pmb{-} \\
\Delta &= \pmb{-} \\
c &= \pmb{-} \\
c^{-1} &= \pmb{-}
\end{align*}
$$

**4.2 The Relations for $\mathcal{Alg}$**

The relations for the generating morphisms that define $\mathcal{Alg}$ are mostly the usual Axioms for braided Hopf algebras. With the conventions as above we can express them as identities between diagrams.

The first set of identities are those resulting from general isotopies. This means the Artin braid relations, and the fact that a crossing can be moved over a fork representing one of the multiplications or over an endpoint representing one of elements in $\text{Hom}(1, A)$ or $\text{Hom}(A, 1)$.

To make $A$ an algebra and coalgebra we have to require axioms for associativity and units,
which translate into diagrams as follows.

\[
\begin{align*}
\text{\includegraphics{diagram1.png}}
\end{align*}
\]

Next the pictures that make \( A \) into a braided bialgebra:

\[
\begin{align*}
\text{\includegraphics{diagram2.png}}
\end{align*}
\]

The axioms for a braided Hopf algebra require the following identities for an invertible antipode.

\[
\begin{align*}
\text{\includegraphics{diagram3.png}}
\end{align*}
\]

The defining formula for the right integral and its \( S \)-invariance also have diagrammatic forms.

\[
\begin{align*}
\text{\includegraphics{diagram4.png}}
\end{align*}
\]

The ribbon element is firstly required to be central and invertible, which is given by the following pictures.
We denote the operator in the middle picture by \( V = m(v \otimes 1) = m(1 \otimes v) \). Another property of the ribbon element is that the associated element \( \omega = V^{-1} \otimes V^{-1} \Delta(v) : 1 \to A \otimes A \) is a Hopf pairing at least on one side.

\[ \omega = \]

In this language modularity means that \( \omega \) is also non-degenerate. There are several ways to express non-degeneracy. We will have to require only a relatively weak version, namely that \((f \otimes 1)\omega = (g \otimes 1)\omega\) implies \( f = g \). This will imply the existence of a side-inverse, which is the stronger version. In diagrams this looks as follows.

\[ f = g \]

Finally, we require a number of normalization conditions, which imply that some morphisms in \( \text{Hom}(1, 1) \) are 1, meaning their diagrams can be eliminated.

\[ = \]

A consequence of this normalization and the previous axioms are the following identities.

\[ = \]

**4.3 Identities for antipode and pairing:** A useful tool to derive Hopf algebra relation is an algebra structure in \( \text{Hom}(A^\otimes n, A^\otimes m) \) given by the convolution product. For two morphisms \( \alpha, \beta \in \text{Hom}(A^\otimes n, A^\otimes m) \) we define the product \( \alpha \ast \beta = m^\otimes m \circ b_n \circ (\alpha \otimes \beta) \circ b_n \circ \Delta^\otimes n \), where \( b_n \) are braid morphisms as indicated in the following diagram.
It is easily checked that the convolution product makes $\text{Hom}(A^\otimes n, A^\otimes m)$ into an associative algebra with unit $I = 1^\otimes m e^\otimes n$. Note that the antipode axiom in (20) means that $\Gamma$ is the convolution inverse of the identity, i.e., $\Gamma \ast (id) = (id) \ast \Gamma = I$. Three other natural convolution products can be found by using inverse braids at the top or bottom half of the diagram. The first application is a generalization Theorem 2.1.4 from [1]:

**Lemma 1**

\[
\begin{align*}
\alpha \ast \beta = & \quad = \quad \alpha \quad \beta \\
\end{align*}
\]

(27)

\[
\begin{align*}
\text{Proof: As in [1] we consider the operations } L = m \circ c \circ (\Gamma \otimes \Gamma) \text{ and } R = \Gamma \circ m \text{ on the right and left hand side of the equation. In the diagram below we compute the convolution products of these operations in } \text{Hom}(A^\otimes 2, A) \text{ with } m \text{ and find } L \ast m = I = m \ast R. \text{ On the left side we use (18) then twice (20) and on the right side (19) and (20).}
\end{align*}
\]

\[
\begin{align*}
L \ast m = & \quad = \quad 1 \quad 1 \\
\end{align*}
\]

(28)

This implies $R = L$. The relation for the coproduct follows similarly.

In addition there are relations for the inverse antipode, in which the crossings are exactly opposite. Here the first easy observation, which follows directly from Lemma [1] and (22).
Lemma 2  *The inverse of the ribbon element is central and \( \Gamma \)-invariant as well.*

Next let us prove several identities for the form \( \omega \).

Lemma 3  *The pairing \( \omega \) is braided skew and a two-sided Hopf pairing. In diagrams we have*

\[
\begin{array}{c}
\quad = \\
\quad = \\
\quad = \\
\quad =
\end{array}
\]

(30)

*Proof: The skew identity for the antipode is a direct consequence of Lemma [1], Lemma [3] and [22] applied to (23). For the second identity multiply \((1 \otimes c^{-1}) \circ (c^{-1} \otimes 1) \circ (1 \otimes c^{-1}) \circ (\Gamma \otimes \Gamma \otimes \Gamma)\) to the two sides of the identity in (23). Applying Lemma [1] and the skew relation that we just proved to both sides separately yields the two diagrams above.*

Lemma 4  *The pairing \( \omega^\dagger \) defined as \( \omega \) but with \( v \) and \( v^{-1} \) exchanged is given by \((\Gamma^{-1} \otimes 1) \circ \omega\). Moreover, the antipode is self conjugate with respect to \( \omega \). This is summarized in the following picture:*

\[
\begin{array}{c}
\quad = \\
\quad = \\
\quad = \\
\quad = \\
\end{array}
\]

(31)

*Proof: The first part of the proof is to show that \( \omega^\dagger \) is a two-sided convolution inverse of \( \omega \). That is

\[
\omega^\dagger * \omega = \omega * \omega^\dagger = I.
\]

This follows directly from the definition of the convolution product in \( Hom(1, A \otimes A) \), (19), Lemma [1] and (22).

We also find for the convolution product \( \omega * ((1 \otimes \Gamma^{-1}) \circ \omega) = I \) by the following diagrammatic calculation. In order we use an isotopy, (23), Lemma [1] (20), and (18).

\[
\begin{array}{c}
\quad = \\
\quad = \\
\quad = \\
\quad = \\
\quad = I
\end{array}
\]

(33)

In the same way we find \(((\Gamma^{-1} \otimes 1) \circ \omega) * \omega = I\). Hence \( \omega^\dagger = (1 \otimes \Gamma^{-1}) \circ \omega = (\Gamma^{-1} \otimes 1) \circ \omega \).
4.4. The dual integrals: We start with a direct consequence of Lemma 1.

**Lemma 5** \( \mu \) is a two-sided integral. That is the reflection of (21) along a vertical line also holds.

Next we infer the existence of a dual integral \( \lambda : 1 \rightarrow A \). We define it as \( \lambda = (1 \otimes \mu) \omega \) and use the following graphical notation.

\[
\lambda = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

(34)

**Lemma 6** Given that \( \omega \) is non-degenerate then \( \lambda \) is an \( \Gamma \)-invariant two-sided integral, and \( \lambda = (\mu \otimes 1) \omega \). In pictures

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

(35)

**Proof:** The fact that \( \lambda \) is \( \Gamma \)-invariant follows from self conjugacy of the antipode, see Lemma 4, and \( \Gamma \)-invariance of \( \mu \), see (21).

The opposite formula \( \lambda = (\mu \otimes 1) \omega \) is now a consequence of using the \( S \)-invariance of \( \mu \) and \( \lambda \) at the same time and the skew relation from Lemma 3.

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

(36)

In the following pictorial calculation we use (23) and (21). The integral identity for \( \lambda \) is then a consequence of the non-degeneracy condition (24).

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

(37)

Let us next record an identity that is useful for later calculations. It is obtained by considering the identity \((m \otimes 1) \circ (\Gamma \otimes L) \circ (\Delta \otimes 1) = (m \otimes 1) \circ (\Gamma \otimes R) \circ (\Delta \otimes 1)\), where \( L \) and \( R \) are the diagrams on the left and right side of the identity on the left side of (19).
4.5. **Inverse pairings:** A morphisms $F \in \text{Hom}(1, A \otimes^2)$ is the left (right) side-inverse of some $G \in \text{Hom}(A^{\otimes^2}, 1)$ if $(1 \otimes G) \circ (F \otimes 1) = 1 ((G \otimes 1) \circ (1 \otimes F) = 1)$. It follows easily that if a left and a right side-inverse exist they must be equal. We determine side-inverses related to the integrals and Hopf pairings.

**Lemma 7** Let $\Lambda = \Delta \circ \lambda \in \text{Hom}(1, A^{\otimes^2})$ and $U = \mu \circ m \in \text{Hom}(A^{\otimes^2}, 1)$.

Then $U(\Gamma \otimes 1) = U(1 \otimes \Gamma)$ is the two-sided side-inverse of $\Lambda$. Conversely, $U$ is the two-sided side-inverse of $(\Gamma \otimes 1)\Lambda = (1 \otimes \Gamma)\Lambda$. In pictures this becomes

\[
\begin{array}{c}
\text{Diagram 38}
\end{array}
\]

**Proof:** We multiply to the left identity of (38) $\lambda$ to the upper left end and $\mu$ to the lower right end. This is the middle step in the following diagrammatical calculation, which shows that $(\Gamma \otimes 1)\Lambda$ is a left side-inverse of $U$.

\[
\begin{array}{c}
\text{Diagram 39}
\end{array}
\]

The other steps are applications of the unit and integral axioms. It follows immediately that also $\Lambda$ is left side-inverse of $U(1 \otimes \Gamma)$. By an analogous calculation using the second relation in (38) we find that $(1 \otimes \Gamma)\Lambda$ is a right side-inverse of $U$, and, further, that $\Lambda$ is a right side-inverse of $U(\Gamma \otimes 1)$.

\[
\begin{array}{c}
\text{Diagram 40}
\end{array}
\]
We define a morphism $S \in \text{End}(A)$ by $S = (1 \otimes U) \circ (\omega \otimes 1)$. As a picture we have

$$S = \phantom{\text{picture}}$$

\[ (41) \]

**Lemma 8**

$$S^2 = \Gamma^{-1}$$

*Proof:* This follows from the diagrammatic computation depicted below. The first picture is the picture of $S^2$. The second is obtained by applying Lemma 3. Next the antipode on the right is slid through the $U$ and then the $\omega$ pairing using Lemma 4 and Lemma 7. An additional isotopy yields the third picture. The forth follows by an application of the second part of Lemma 3. In the fifth picture we substitute the definition of $\lambda$ from (34) and use of Lemma 1. The last step uses $\Gamma$-invariance of $\lambda$ and Lemma 2 and the pairing identity in Lemma 7.

\[ (42) \]

Note that Lemma 8 implies the existence of a side-inverse to $\omega$, which is a slightly stronger property than the non-degeneracy from (24). Conversely, observe that the only place we really used the non-degeneracy was in the proof of the fact in Lemma 3 that $\lambda$ as defined in (35) is an integral. If we had required instead from the start that $\lambda$ is an integral this would have implied non-degeneracy. Hence the two requirements are interchangeable as axioms!

Let us finally record the following actions of $S$ on elements in $\text{Hom}(1, A)$ and $\text{Hom}(A, 1)$. They are worked out easily using the fact that $\omega$ maps units and integrals to each other. The fact that the inverse $S^{-1}$ acts on these elements is a consequence of their $\Gamma$-invariance and Lemma 8.

\[ S^{\pm 1} \circ \lambda = 1 \]  \[ (43) \]

\[ \mu \circ S^{\pm 1} = \epsilon \]  \[ (44) \]

\[ S^{\pm 1} \circ v = (\mu \circ v)v^{-1} \]  \[ (45) \]
5. A functor from $\textbf{Alg}$ to $\textbf{Cob}$

We construct here the functor $\mathcal{G} : \textbf{Alg} \rightarrow \textbf{Cob}$. This we do by assigning to each generator $g : n \rightarrow m$ of $\textbf{Alg}$ an admissible tangle $T_g : n \rightarrow m$ in $\text{Tgl}$. The cobordism is then given by $\mathcal{G}(g) = \mathcal{S}_{\text{urg}}([T_g])$, where $[T]$ denotes the equivalence class of the tangle $T$ in $\text{Tgl}$ and $\mathcal{S}_{\text{urg}}$ is as in Theorem 7.

5.1 Assignment of Generators: We begin with the assignments of the product. It is given by the three component $2 \rightarrow 1$ tangle written below. We can apply the boundary move (9) to the left pair of strands of the left picture so that another arc and annulus are introduced. The resulting tangle is symmetric and hence equivalent to the right picture for $m$.

\[ m = \]

\[ \Gamma = \]

The coproduct is given by the next four component $1 \rightarrow 2$ tangle.

\[ \Delta = \]

The antipode $S$ is assigned to the following $1 \rightarrow 1$ tangle with only one through pair. Note that the loop can be moved to the other strand by applying again the boundary move (9) on one side, sliding the loop through and reversing the move (9). The picture on the right is already the inverse as a braid and so, in particular, as a morphism in $\text{Tgl}$.

\[ \Gamma^{-1} = \]

The generating braid isomorphism $c_{1,1} : 1 \otimes 1 \rightarrow 1 \otimes 1$ is assigned to the crossing of pairs of strands as follows. Note that, in our convention, overcrossing strands are mapped to undercrossing ones. For the ribbon elements the tangles are given by $0 \rightarrow 1$ arcs with a framing loop.
The unit elements and integrals are also mapped to arcs as in the following pictures. We already listed the picture for $\lambda$ here although it follows from previous assignments.

As cobordisms the units, integrals, and ribbon elements are all homeomorphic to full tori.

**Lemma 9** The pairings are mapped by $\Theta$ to the cobordism with the following $0 \to 2$ tangles.

Proof: We start with the tangle associated to $V = m \circ (v \otimes 1) : 1 \to 1$. By composition of the tangles in (46) and (49) we obtain the left tangle in the picture below. The next two pictures follow by a reverse application of (9) with or without sliding the framing blob through the annulus.

Next we compute $\Delta \circ v : 0 \to 2$. The composition of (47) and (49) yields diagram (a). The top annulus is slid off using (47) to give (b). Picture (c) is obtained by an isotopy. We apply (10) again by sliding the two arcs at the bottom over the annulus with the black blob to give (d). Finally, we obtain (e) by cancelling the isolated annuli as in (7) and isotopy.
If we put the expressions from (52) and (53) together as in the definition (23) and cancel opposite loops against each other we find the tangle in (51) for $\omega$. The proof for $\omega^\dagger$ is analogous.

In the diagrammatic composition below we see now immediately that the assignment of $\lambda$ is the one resulting from Lemma 9.

5.2 Relations: In order for the assignments of tangles to Hopf algebra generators to give rise to a functor $\mathcal{Alg} \to \mathcal{Tgl}$ we have to verify that all the relations in $\mathcal{Alg}$ are also satisfied in $\mathcal{Tgl}$.

We begin with the unit axioms from (18) for the coproduct. The diagram for $(\epsilon \otimes 1) \circ \Delta$ below is obtained by composition. The pair of annuli $A$ and $R$ is removed with the $\beta$-Move from (11). The resulting three component tangle is equivalent to two parallel strands by (9) which represents the identity. The other three unit axioms follows similarly.

For the associativity axiom in (18) we pick again only the case of the coproduct. The product case is analogous. The pictures for the two composition $(1 \otimes \Delta) \circ \Delta$ and $(\Delta \otimes 1) \circ \Delta$ on the left and right. They are equivalent to the diagram in the middle by the connecting annulus move from (12).
The bialgebra axiom in (19) is verified next. For $m \otimes m \circ (1 \otimes c \otimes 1) \circ \Delta \otimes \Delta$ we obtain the tangle in (a) of (57) by composition. (b) follows by applying (12) to each of the annuli in the picture for the $m$ part. An isotopy yields diagram (c) in (58). From this we obtain (d) by a 2-handle slide as in (8) of the component labeled $R$ over the component $S$. The picture in (e) follows again by an isotopy and is precisely the tangle assigned to $\Delta \circ m$.

For the bialgebra axiom with the units in (19) we consider only the product case. The first diagram in (59) is the composite of the tangles for $e \circ m$. The second is a result of the boundary move (9), and the third follows by applying the $\beta$-Move from (11) to the pair $A$ and $R$. 
The antipode axiom is proven in (60). To the annulus in the m-tangle part we apply (9). The remaining diagrams follow by isotopies.

\[ \text{Diagram (60)} \]

The axioms for the integrals are verified similarly to those of the units. One example is given next, which uses again the $\beta$-Move (11).

\[ \text{Diagram (61)} \]

The pairing axiom from (23) follows from (62). The tangle for the expression in \((1 \otimes \Delta) \circ \omega\) is depicted in (a). We obtain (b) by application of (9) or (12). (c) is a result of isotopy and is equivalent to (d) again by (12). The picture in (d) is also the tangle expression for \((m \otimes 1 \otimes 1) \circ (1 \otimes \omega \otimes 1) \circ \omega\) from (23).

\[ \text{Diagram (62)} \]

Non-degeneracy follows easily either by the observation that the tangle for $\lambda$ has the properties of an integral or by giving the side-inverse of tangle for $\omega$ explicitly, namely the reflection of the tangle for $\omega^\dagger$ along the horizontal axis.
The axioms (22) for the ribbon element follow already from the presentations in (52) and the first Reidemeister Move for framed tangles, which allows us to cancel a full with an empty blob on the same strand.

The normalization conditions in (25) are a result of the moves in (7).

**Corollary 8** The assignments of tangle classes by representing tangles (46) through (50) from Section 5.1 factor through the relations in $A_{lg}$.

Hence we have a well defined functor $\mathcal{X} : A_{lg} \to T_{gl}$.

Compatibility with the composition and tensor operations are obvious. The functor from Theorem 1 is thus defined as

$$\mathcal{G} = \mathcal{S}urg \circ \mathcal{X} : A_{lg} \to Cob$$

(63)

### 6. From Generators of $Cob$ to Generators of $A_{lg}$

Although we cannot construct an inverse functor we will define an assignment on the sets of generating morphisms

$$W : Gen[Cob] \to A_{lg}$$

(64)

For the cobordism category we choose them according to Corollary 3, that is, $Gen[Cob] = \{H^\pm_0, I^\pm_1, I^\pm_D, I^\pm_S, Z^\pm\}$. At least on the set of generators it will be a right inverse for $\mathcal{G}$.

**6.1 Assignment of Generators and Surjectivity:** We give the values of $W(g)$ for each $g \in Gen[Cob]$ and verify immediately that for this choice $\mathcal{G}(W(g)) = g$.

The morphism in $A_{lg}$ associated to the generator $I^\pm_{A}$ is given by $m \circ (v \otimes 1)$ as depicted in (65). The tangle associated to this is by (52) to be exactly the one that represents the $A_1$-Dehn-twist as in (13).

To the mapping class group generators $I_S$ we associate the morphism $\mathcal{S}$ from (41) with an additional normalization factor. The tangle assigned to this composition is depicted in the middle of (66). An application of (9) shows that this is equivalent to the tangle in (14). The morphism in $A_{lg}$ that is assigned below to $S^{-1}$ is the inverse of $W(I_S)$. This follows from Lemma 3 and (26).
The last mapping class group generator $I_D$ is mapped to the morphism depicted in (67). The next diagram shows the associated tangle for this morphism. Applying again (12) or (9) to the annuli $A$ and $B$ we obtain the tangle (15). The morphism we associate to $I_D^{-1}$ is again the inverse of $W(I_D)$ in $\text{Alg}$. This is a straightforward exercise using (23) and then (20).

Finally, we list the assignments for the handle attachments. They are given by the integral and normalization pictures. The associated tangles are immediately identified with the pictures in (16).

\begin{equation}
W(I_S) = \quad = \quad W(I_S^{-1}) = \quad = \quad \quad (66)
\end{equation}

\begin{equation}
W(I_D) = \quad = \quad W(I_D^{-1}) = \quad = \quad \quad (67)
\end{equation}

\begin{equation}
W(Z) = \quad = \quad W(Z^{-1}) = \quad = \quad \quad (68)
\end{equation}

\begin{equation}
W(H_n^+) = \quad = \quad W(H_n^-) = \quad = \quad \quad (68)
\end{equation}
In summary we found an assignment \( W \) such that
\[
\mathcal{G} \circ W = \text{Id} \quad \text{on} \quad \text{Gen[Cob]}.
\] (69)

Particularly, well definedness of the functor in (53), the generators of \( \text{Cob} \) in Corollary 8, and the map \( W \) with the inverse property (59) imply now the second part of Theorem 2 as well as Theorem 0.

6.2 A Braid Relation: A nearby question is whether \( W \) extends to a functor, and hence a right inverse to \( \mathcal{G} \), meaning \( \mathcal{G} \circ W \) is the identity on all of \( \text{Cob} \). The question is thus, whether all the relations that the generators from \( \text{Gen[Cob]} \) fulfill in \( \text{Cob} \) are also fulfilled by the images in \( \mathcal{A}[\text{Alg}] \) or if we have to introduce additional relations in \( \mathcal{A}[\text{Alg}] \).

Among the set of relations there have to be the relations for the mapping class group generators. They have been worked explicitly for example in [19]. Moreover, we need relations expressing the fact that some mapping class group generators can be extended to the full handles, as well as relations for Smale-cancellations of handles. Unfortunately, we do not know of any systematic presentation of \( \text{Cob} \) in this way.

Although it might be too optimistic to expect that there are no further relations in \( \mathcal{A}[\text{Alg}] \) we demonstrate next that some non-trivial relation in \( \Gamma_{g,1} \) can indeed be inferred from the relations in \( \mathcal{A}[\text{Alg}] \). Recall, that the generators \( A_j \) and \( B_j \) of the mapping class group satisfy the braid relation
\[
A_j B_j A_j = B_j A_j B_j.
\]
Using the definition in (5) for \( S_j \) this relation translates to
\[
S_j A_j^{-1} S_j = A_j S_j A_j.
\] (70)

**Lemma 10** The morphisms \( W(I_A) \) and \( W(I_S) \) assigned to the generators \( A_1 \) and \( S_1 \) fulfill the braid relation (72).

**Proof:** The proof is a diagrammatic calculation. The expression for the left hand side of (70) is given in diagram (a) of (71) below. In (b) we use centrality of the ribbon elements (22) and associativity to change the order of products. Moreover, we apply the first relation in (30). The next diagram (c) is the result of an isotopy and the second relation in (30). Next, in (e), we use the identity (31) between the regular and the opposite pairing and in (f) the explicit form of \( \omega^{-1} \) is inserted. In addition we make use of (28). Diagram (g) in (73) follows then by first cancelling the two right most ribbon elements, and then applying (21). We obtain an additional factor \( \mu \circ v^{-1} \), which we cancel with one of the \( \mu v \) factors in the next diagram (h) using (26). Note that it follows from (24) and (28) that \( V \) (multiplication with \( v \)) commutes with the application of the antipode. Hence we can introduce factors \( v^{-1} \) and \( v \) as indicated in (h). The resulting configuration in the identified in (i) with \( \omega \). The last diagram (j) follows now by another application of (31). It is readily identified with the composite on the right side of (70).

\[
\begin{align*}
(a) & \quad = \quad (b) & \quad = \quad (c)
\end{align*}
\] (71)
The commutation relations for Dehn twist along disjoint are also fulfilled for obvious reasons. The braid relations between the \( B_j \) and \( C_j \) are more difficult to verify and it seems like additional relations have to be imposed on \( \mathcal{A}lg \). The result is summarized in the first part of Theorem 2.

6.3 Heegaard decompositions of generators in \( \mathcal{A}lg \):

**Lemma 11** The images of the generators under \( \mathfrak{G} \) have the following Heegaard decompositions.

\[
\begin{align*}
\mathfrak{G}(\lambda) &= H_0^+ \otimes Z & \mathfrak{G}(\mu) &= H_0^- \otimes Z^{-1} & \mathfrak{G}(v) &= I_A \circ I^{-1}_S \\
\mathfrak{G}(\epsilon) &= H_0^- \circ I_S & \mathfrak{G}(1) &= I^{-1}_S \circ H_0^+ \\
\mathfrak{G}(m) &= (id \otimes H_0^-) \circ I_D \circ (id \otimes I_S) \\
\mathfrak{G}(\Delta) &= (I_S \otimes id) \circ (I_D^{-1}) \circ (I_S^{-1} \otimes I_S^{-1}) \circ (id \otimes H_0^+)
\end{align*}
\]

**Proof:** Verification by composition of the associated tangles.

**Proposition 9** For each of the products of generators of \( \mathcal{C}ob \) in Lemma \([7]\) we have that the corresponding product of the images under \( \mathcal{W} \) reproduces the generators in \( \mathcal{A}lg \).
Proof: As a first example we have for the product for $G(\lambda)$ that $W(H^+_0) \circ W(Z) = \lambda(\mu \circ v)(\mu \circ v^{-1})$, the one for $G(\mu)$ is similar. For $G(1)$ we have $W(I^{-1}_S) \circ W(H^+_0) = (\mu \circ v^{-1})S^{-1} \circ \lambda(\mu \circ v) = S^{-1} \circ \lambda = 1$ by (43). The relations for $G(1)$ and $G(v)$ follow similarly from (43) and (45).

For the multiplication the product $(id \otimes W(H^{-1} _0)) \circ W(I_D) \circ (id \otimes W(I_S))$ is depicted in (74). The two normalization elements cancel. We also identify two pictures for $S$ from (41). Lemma 8 tells us that we can cancel them together with the antipode. What is left is $m$.

\[\text{(74)}\]

The situation for the coproduct is more involved. First note that $W(I^{-1}_S) \circ W(H^+_0) = 1$. As a result it suffices to show that

\[(W(I_S)^{-1} \otimes id) \triangle = W(I_D^{-1}) \circ (W(I_S)^{-1} \otimes 1).\]  

(75)

The pictures for the left and right hand side of this equation are given by the second and third diagram in (76) below. Each of them is equivalent to the first and fourth by (51) and (23) respectively.

\[\text{(76)}\]

To check identity of the first and fourth picture in (76) we can remove the pairing $\omega$ on both sides. Instead let us apply the pairing $\Delta \circ \lambda$ to the right most strands on both sides. We obtain the second and third diagram in the next picture (76). Using Lemma 7 we find that the expressions on both sides are equivalent to $\Delta$.

\[\text{(77)}\]

This proves the equality in (75).

This lemma is useful once we have found a version of $\mathsf{Alg}$ on which $\mathcal{W}$ extends to a functor $\mathcal{M}$. We know by (49) that in this case $\mathcal{G} \circ \mathcal{M} = id$ but only Lemma 11 guarantees that that $\mathcal{M} \circ \mathcal{G} = id$. 

26
Corollary 10 Let $\overline{\text{Alg}}$ be a quotient of $\text{Alg}$, obtained from the same generators but additional relations. Suppose further that $\mathcal{G}$ factors into a functor $\overline{\text{Alg}} \to \text{Cob}$, and that $\mathcal{W}$ extends to a (unique) functor $\mathcal{W}$ on $\overline{\text{Alg}}$. Then $\mathcal{G}$ and $\mathcal{W}$ are two-sided inverses and hence $\overline{\text{Alg}}$ must be isomorphic to $\text{Cob}$.

This is the same statement as in the last part of Theorem 2.
Bibliography

[1] E. Abe: Hopf algebras. *Cambridge Tracts in Mathematics*, **74*. Cambridge University Press, 1980.

[2] M. Atiyah: Topological quantum field theories, Inst. Hautes Études Sci. Publ. Math. **68** (1988), 175–186.

[3] Yu. Bespalov, T. Kerler, V. Lyubashenko, V. Turaev: Integrals for braided Hopf algebras. *J. Pure Appl. Algebra* **148** (2000) no. 2, 113–164.

[4] L. Crane, D. Yetter: On algebraic structures implicit in topological quantum field theories. *J. Knot Theory Ramifications* **8** (1999) no. 2, 125–163.

[5] V.G. Drinfeld: *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990) no. 2, 321–342.

[6] R. Fenn, C. Rourke: On Kirby’s calculus of links. *Topology* **18** (1979), no. 1, 1–15.

[7] L.H. Kauffman, D.E. Radford: Invariants of 3-manifolds derived from finite-dimensional Hopf algebras, *J. Knot Theory Ramifications* **4** (1995) no. 1, 131–162.

[8] T. Kerler: A Topological Hopf Algebra. Talk presented at University of North Carolina, 1994, unpublished.

[9] T. Kerler: Mapping class group actions on quantum doubles, *Commun. Math. Phys.* **168** (1994) 353-388.

[10] T. Kerler: Genealogy of nonperturbative quantum-invariants of 3-manifolds: The surgical family. In ‘Geometry and Physics’, *Lecture Notes in Pure and Applied Physics* **184**, Marcel Dekker (1997) 503-547.

[11] T. Kerler: On the connectivity of cobordisms and half-projective TQFT’s, *Commun. Math. Phys.* **198** No. 3 (1998) 535-590.

[12] T. Kerler: Bridged links and tangle presentations of cobordism categories, *Adv. Math.* **141** (1999) 207-281.

[13] T. Kerler, V.V. Lyubashenko: Non-semisimple topological quantum field theories for 3-manifolds with corners. *Lecture Notes in Mathematics* Springer Verlag 2001. (To appear) 375 pages, over 200 illustrations.

[14] R. Kirby: A calculus for framed links in $S^3$, *Invent. Math.* **45** (1978), 35-56.

[15] V.V. Lyubashenko: Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity, *Commun. Math. Phys.* **172** (1995) 467–516.
[16] S. Majid: Braided groups, J. Pure Appl. Algebra 86 (1993) n. 2, 187–221.

[17] S. Matveev, M. Polyak: A geometrical presentation of the surface mapping class group and surgery, Commun. Math. Phys. 160 (1994) 537–550.

[18] N.Yu. Reshetikhin, V.G. Turaev: Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991), 547–597.

[19] B. Wajnryb: A simple presentation for the mapping class group of an orientable surface, Israel J. Math. 45 (1983) no. 2-3, 157–174. B. Wajnryb: An elementary approach to the mapping class group of a surface, Geom. Topol. 3 (1999) 405–466 (electronic).

[20] D. Yetter: Portrait of the handle as a Hopf algebra. Geometry and physics. (Aarhus, 1995), Lecture Notes in Pure and Appl. Math. 184, Dekker, New York, 1997, pp. 481–502.

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