ISOMETRY-INVARIANT GEODESICS AND NONPOSITIVE DERIVATIONS OF THE COHOMOLOGY

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Abstract. We introduce a new class of zero-dimensional weighted complete intersections, by abstracting the essential features of $\mathbb{Q}$-cohomology algebras of equal rank homogeneous spaces of compact connected Lie groups. We prove that, on a 1-connected closed manifold $M$ with $H^*(M, \mathbb{Q})$ belonging to this class, every isometry has a non-trivial invariant geodesic, for any metric on $M$. We use $\mathbb{Q}$-surgery to construct large classes of new examples for which the above result may be applied.

1. Introduction

1.1. Complexes which look like homogeneous spaces. Let $\mathcal{A}$ be a weighted, zero-dimensional (that is, artinian), complete intersection (WACI), i.e., a commutative graded $\mathbb{Q}$-algebra of the form

$$\mathcal{A} = \mathbb{Q}[x_1, \ldots, x_n]/I,$$

where the variables $x_i$ have positive even weights, $w_i := |x_i|$, and the ideal $I$ is generated by a regular sequence,

$$I = (f_1, \ldots, f_n),$$

of weighted-homogeneous polynomials, $f_i$.

We are going to introduce a special class of WACI’s, with an eye for applications in Riemannian geometry.

One may speak about the pseudo-homotopy groups of a 1-connected, finitely generated, graded-commutative $\mathbb{Q}$-algebra, $\mathcal{A}$:

$$\pi_*^*(\mathcal{A}) = \bigoplus_{i \geq 0, j > 1} \pi_i^j(\mathcal{A}).$$

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These algebraic invariants of $\mathcal{A}$ are finite-dimensional graded $\mathbb{Q}$-vector spaces, $\pi_i(\mathcal{A}) := \oplus_j \pi^j_i(\mathcal{A})$, for all $i \geq 0$. See [9].

When $\mathcal{A}$ is a WACI, as above, one knows that $\pi_{i+1}(\mathcal{A}) = 0$. Moreover, for any 1-connected CW-complex, $S$, such that $H^*(S, \mathbb{Q}) = \mathcal{A}^*$, as graded algebras, one has the following topological interpretation (see Sullivan [18]):

$$
\begin{aligned}
\pi^*_0(\mathcal{A}) &= \text{Hom}_\mathbb{Q}(\pi_{=\text{even}}(S) \otimes \mathbb{Q}, \mathbb{Q}), \\
\pi^*_1(\mathcal{A}) &= \text{Hom}_\mathbb{Q}(\pi_{=\text{odd}}(S) \otimes \mathbb{Q}, \mathbb{Q}).
\end{aligned}
$$

Assuming $\mathcal{A} \neq \mathbb{Q}$, that is, $\pi^*_1(\mathcal{A}) \neq 0$, set now:

$$k_{\mathcal{A}} := \max \{ k \mid \pi^{2k-1}_1(\mathcal{A}) \neq 0 \}.
$$

Denote by $\text{Der}^*(\mathcal{A})$ the graded Lie algebra of homogeneous derivations of $\mathcal{A}^*$, with Lie bracket given by graded commutator, and note that $\text{Der}^*(\mathbb{Q}) = 0$.

**Definition 1.2.** We shall say that the WACI $\mathcal{A}$ is simple if:

(i) $\text{Der}^{-0}(\mathcal{A}) = 0$.

(ii) $\dim \mathbb{Q} \text{Der}^0(\mathcal{A}) = 1$.

(iii) $\dim \mathbb{Q} \pi^{2k-1}_1(\mathcal{A}) = 1$.

**Definition 1.3.** A 1-connected CW-complex $S$ is said to be homologically homogeneous (HH) if

$$H^*(S, \mathbb{Q}) = \bigotimes_{j \in J} \mathcal{A}_j^*,$$

as graded algebras, where the index set $J$ is finite and the algebra $\mathcal{A}_j$ is a simple WACI, for every $j \in J$.

Both definitions are motivated by geometry. Firstly, it is well-known that the $\mathbb{Q}$-cohomology algebra of an equal rank homogeneous space, $G/K$, of compact connected Lie groups, is a WACI; see Borel [3]. Moreover, $G/K = \prod_{j \in J} G_j/K_j$, where in addition each Lie group $G_j$ is simple. Now, for $G$ simple and $\mathcal{A}^* = H^*(G/K, \mathbb{Q})$, properties (i) and (ii) from Definition 1.2 were proved by Shiga and Tezuka in [17]. Property (iii) follows from [15, Theorem 1.3], via (1.4). Therefore, all equal rank homogeneous spaces $G/K$ mentioned above are HH, in the sense of our Definition 1.3.

We exhibit in Section 4 (Theorems 4.4 and 4.8) families of new examples of HH closed manifolds. Their $\mathbb{Q}$–cohomology algebras are actually simple WACI’s, not isomorphic to the $\mathbb{Q}$–cohomology algebra of any equal rank homogeneous space $G/K$, with $G$ not necessarily simple. We first use tools from our previous work [16], to obtain simple WACI’s from 1-dimensional reduced weighted complete intersections. Rational surgery methods, due to Sullivan [18] and Barge [1], enable us to realize them by closed manifolds.
1.4. **Isometry–invariant geodesics.** Let $M^m$ be a 1-connected, closed, Riemannian $m$-manifold. A basic problem is to find topological conditions on $M$ guaranteeing that

\[(1.6) \quad \text{every isometry } a \text{ has a non-trivial } a\text{-invariant geodesic.}\]

Recall that a geodesic curve $\gamma$ is called $a$-invariant if there exists a period $t$ such that $a(\gamma(x)) = \gamma(x + t)$, for any $x$. When $a = \text{id}$, one recovers the classical notion of closed geodesic. Strong existence results for closed geodesics were obtained, under simple topological hypotheses on $M$, by Sullivan and Vigué–Poirrier in [19], using rational homotopy methods.

Further refinements led to various answers to the general fundamental question (1.6). Recall that a 1-connected finite complex $S$ is called **elliptic** if $\dim_{\mathbb{Q}} \pi_*(S) \otimes \mathbb{Q} < \infty$, and **hyperbolic** otherwise. Grove and Halperin proved in [7] that property (1.6) holds for all hyperbolic $M$, and for all elliptic $M$ with $m$ odd.

On the other hand, $HH$ manifolds are elliptic (see (1.4)) and even-dimensional (see Halperin [8]). In the particular case of equal rank homogeneous spaces, $M = G/K$, property (1.6) holds, when $K$ is a maximal torus ([14, Theorem 1.4(i)]) or $G$ is simple ([15, Corollary 1.4(i)]).

Our main result below significantly enlarges the class of manifolds $M$ having the ideal property (1.6).

**Theorem 1.5.** If the closed manifold $M$ is homologically homogeneous (in the sense of Definition 1.3), then, given an arbitrary metric on $M$, there is a non-trivial $a$-invariant geodesic, for every isometry, $a$.

Along the way, we shall also clarify the structure of the automorphism group of certain related graded algebras, $\mathcal{B}^*$. More precisely, assume that $\mathcal{B}^* = \otimes_{j \in J} \mathcal{B}_j^*$, where $J$ is finite, and each connected, commutative, evenly-graded $\mathbb{C}$-algebra $\mathcal{B}_j^*$ is artinian and satisfies properties (i) and (ii) from Definition 1.2 (over $\mathbb{C}$). Then the connected component of 1 in the (linear algebraic) group of graded algebra automorphisms of $\mathcal{B}^*$ is an algebraic torus, $(\mathbb{C}^*)^r$, where $r = |J|$. See Corollary 2.6.

2. **Proof of the main result**

If $a$ has no invariant geodesic, then both $\pi_*(a) \otimes \mathbb{C}$ and $\text{id} - \pi_*(a) \otimes \mathbb{C}$ are unimodular elements of $GL(\pi_*(M) \otimes \mathbb{C})$; see [6]. Set $k_j = k_{A_j}$, for $j \in J$, and $k = \max \{k_j \mid j \in J\}$. We will derive a contradiction, for $* = 2k - 1$.

2.1. **Dual homotopy representations.** Rational homotopy theory methods [18] may be used to get information on $\pi_{2k-1}(a) \otimes \mathbb{C}$. Tools from the representation theory of linear algebraic groups (see e.g. [10]) will provide additional information, so we will use $\mathbb{C}$-coefficients.
Set $A^* = H^*(M, \mathbb{Q})$. Since $M$ is $HH$, $A^*$ is a $WACI$. As shown in [18], this implies that the $\mathbb{C}$-minimal model of $M$, $(M, d)$, has the following properties. It is a bigraded algebra of the form $M = \mathbb{C}[Z_0] \otimes \bigwedge Z_1$, where $Z_0 = \mathbb{C} - \text{span}\{x_1, \ldots, x_n\}$, with $x_i$ of even upper degree, and $Z_1 = \mathbb{C} - \text{span}\{y_1, \ldots, y_n\}$, with $y_i$ of odd upper degree. The differential $d$ is homogeneous, with upper degree +1 and lower degree −1. More precisely, $dx_i = 0$ and $dy_i \in \mathbb{Q}[x_1, \ldots, x_n]$ is a polynomial with no linear part, for all $i$. Moreover, $H^+(M, d) = 0$, and $Z_j^* = \pi_j^*(A) \otimes \mathbb{C}$; see [9].

Denote by $E$ the group of self-homotopy equivalences of $M$. By (1.4), the natural action of $E$ on $\pi_2k−1(M)$ gives rise to a dual homotopy (anti)representation,

\[(2.1) \hspace{1cm} \tau : E \longrightarrow GL(Z_1^{2k−1}).\]

Our next goal is to relate (2.1) and (2.2), by constructing an algebraic representation,

\[(2.3) \hspace{1cm} \nu : \text{Aut}_\mathbb{C}(A) \longrightarrow GL(Z_1^{2k−1}),\]

called the pseudo–homotopy representation. To do this, we first consider the linear algebraic group $\text{Aut}_\mathbb{C}(\mathbb{C}[Z_0])$, consisting of those graded algebra automorphisms, $\alpha$, of $\mathbb{C}[Z_0]$, which leave the ideal $I$ invariant. From Lemma 2.2 we get a natural algebraic representation,

\[(2.4) \hspace{1cm} \nu : \text{Aut}_I(\mathbb{C}[Z_0]) \longrightarrow GL(Z_1^{2k−1}).\]
Note that $\mathcal{A}^* \otimes \mathbb{C} = H^*(\mathcal{M}, d) = H^\mathbb{C}_\bullet(\mathcal{M}, d) = \mathbb{C}[Z_0]/\mathcal{I}$. It follows that $\text{Aut}_\mathbb{C}(\mathcal{A})$ is the quotient of $\text{Aut}_\mathbb{I}(\mathbb{C}[Z_0])$ by the subgroup of those elements $\alpha$ having the property that

$$
\alpha(x_i) \equiv x_i \pmod{\mathcal{I}}, \quad \text{for all } i.
$$

It is easy to infer from (2.5) that $\alpha$ induces the identity on $\mathcal{I}/\mathbb{C}^+[Z_0] \cdot \mathcal{I}$, by resorting to the minimality property of $(\mathcal{M}, d)$. Thus, (2.4) factors to give the desired representation, (2.3).

**Lemma 2.3.** The representations (2.1), (2.2) and (2.3) are related by:

$$
\tau = \nu \circ h.
$$

**Proof.** For $a \in \mathcal{E}$, denote by $\hat{a}: (\mathcal{M}, d) \to (\mathcal{M}, d)$ the Sullivan minimal model of $a$. It is a differential graded algebra (DGA) automorphism, with the property that $H^*(\hat{a}) = h(a)$.

Recall from [18] that $\tau(a) = \pi^{2k-1}(\hat{a})$, where $\pi^{2k-1}(\hat{a}): Z_1^{2k-1} \to Z_1^{2k-1}$ denotes the map induced by the restriction of $\hat{a}$ to $Z_1^{2k-1}$, modulo decomposable elements of $\mathcal{M}$.

To compute $\nu(h(a))$, we need to lift $H^*(\hat{a})$ to a graded algebra map, $\mathbb{C}[Z_0] \to \mathbb{C}[Z_0]$, leaving $\mathcal{I}$ invariant. To do this, we may proceed as follows. Start by noting that $\hat{a}$ increases lower degrees. It is enough to check this on algebra generators of $\mathcal{M}$, which is obvious on $Z_0$; on $Z_1$, this follows from the fact that $\hat{a}(y_i)$ has odd upper degree, for all $i$. Write then $\hat{a} = \sum_{j \geq 0} \hat{a}_j$, with $\hat{a}_j \in \text{Hom}_\mathbb{C}(\mathcal{M}_*, \mathcal{M}_{*, j})$. An easy lower degree argument, together with the bihomogeneity property of $d$, show that $\hat{a}_0$ is a bigraded differential algebra map. Set $\alpha := \hat{a}_0: \mathbb{C}[Z_0] \to \mathbb{C}[Z_0]$. Since $H_\bullet(\mathcal{M}, d) = 0$, it is easily seen that $\alpha \in \text{Aut}_\mathbb{I}(\mathbb{C}[Z_0])$ lifts $H^*(\hat{a}) \in \text{Aut}_\mathbb{C}(\mathcal{A})$.

To finish our proof, let us check that

$$
d\pi^{2k-1}(\hat{a})(y_i) \equiv \hat{a}_0(dy_i) \pmod{\mathbb{C}^+[Z_0] \cdot \mathcal{I}},
$$

for $y_i$ of (upper) degree $2k - 1$; see Lemma 2.2. Note that $\pi^{2k-1}(\hat{a})(y_i)$ has odd upper degree; hence, $\pi^{2k-1}(\hat{a})(y_i) \equiv \hat{a}_0(y_i)$, modulo $\mathbb{C}^+[Z_0] \otimes Z_1$. Applying $d$, we get (2.6). $\square$

### 2.4. Derivations and the structure of $\text{Aut}_\mathbb{C}(\mathcal{A})$.

Denote by $\text{Aut}_\mathbb{C}^\mathbb{I}(\mathcal{A})$ the connected component of $1 \in \text{Aut}_\mathbb{C}(\mathcal{A})$. We are going to exploit conditions [11] and [11] from Definition [12] to give a complete description of the algebraic group $\text{Aut}_\mathbb{C}^\mathbb{I}(\mathcal{A})$.

In this subsection, all graded-commutative algebras $\mathcal{B}^\mathbb{I}$ will be supposed to be connected, finite-dimensional over a field $\mathbb{K}$, and evenly-graded. The examples we have in mind are $\mathcal{B} = \mathcal{A} \otimes \mathbb{C}$, where $\mathcal{A}$ is a $WACI$. For $p \in \mathbb{Z}$, recall that

$$
\text{Der}^p(\mathcal{B}) := \{ \theta \in \text{Hom}_\mathbb{K}(\mathcal{B}^\mathbb{I}, \mathcal{B}^{\mathbb{I} + p}) \mid \theta(ab) = \theta a \cdot b + a \cdot \theta b, \forall a, b \in \mathcal{B} \}.
$$
We will be particularly interested in the Lie algebra $\text{Der}^0(\mathcal{B})$. In our basic examples, one knows that $\text{Der}^0(\mathcal{A} \otimes \mathbb{C})$ is the Lie algebra of $\text{Aut}^0(\mathcal{A})$; see [10, 12.5 and 13.2]. We first explore the consequences of the vanishing of $\text{Der}^0$ on the structure of $\text{Der}^0$.

**Lemma 2.5.** Assume that $\text{Der}^0(\mathcal{B}_j) = 0$, for $j = 1, \ldots, r$. Then the Lie algebra $\text{Der}^0(\otimes_{j=1}^r \mathcal{B}_j)$ is isomorphic to the direct product $\prod_{j=1}^r \text{Der}^0(\mathcal{B}_j)$.

**Proof.** Consider $\theta \in \text{Der}^p(\mathcal{B}_1 \otimes \mathcal{B}_2)$, with $\mathcal{B}_1$, $\mathcal{B}_2$ and $p$ arbitrary. Pick $\{b_\beta\}_\beta$, a homogeneous $\mathbb{K}$-basis of $\mathcal{B}_2$, comprising 1. Write $\theta|_{\mathcal{B}_1 \otimes 1} = \sum \theta_\beta \otimes b_\beta$. A straightforward computation shows that $\theta_\beta \in \text{Der}^{p-|b_\beta|}(\mathcal{B}_1)$, for every $\beta$. This shows inductively that $\text{Der}^0(\otimes_{j=1}^r \mathcal{B}_j) = 0$, if $\text{Der}^0(\mathcal{B}_j) = 0$, for all $j$.

Let again $\mathcal{B}_1$ and $\mathcal{B}_2$ be arbitrary. It is easy to check that the map

$$\Phi: \text{Der}^0(\mathcal{B}_1) \times \text{Der}^0(\mathcal{B}_2) \longrightarrow \text{Der}^0(\mathcal{B}_1 \otimes \mathcal{B}_2),$$

defined by sending $(\theta_1, \theta_2)$ to $\theta_1 \otimes \text{id} + \text{id} \otimes \theta_2$, is a Lie algebra monomorphism. We claim that $\Phi$ is an isomorphism, whenever $\text{Der}^0(\mathcal{B}_1) = \text{Der}^0(\mathcal{B}_2) = 0$. To check surjectivity, pick $\theta \in \text{Der}^0(\mathcal{B}_1 \otimes \mathcal{B}_2)$ arbitrary. By the discussion from the preceding paragraph, $\theta|_{\mathcal{B}_1 \otimes 1} = \theta_1 \otimes 1$, and $\theta|_{1 \otimes \mathcal{B}_2} = 1 \otimes \theta_2$, with $\theta_j \in \text{Der}^0(\mathcal{B}_j)$, for $j = 1, 2$. We infer that $\theta = \theta_1 \otimes \text{id} + \text{id} \otimes \theta_2$, which proves our claim. By making repeated use of the isomorphism (2.7), one verifies inductively the assertion of the Lemma. $\square$

Let now $\mathbb{K}$ be algebraically closed, of characteristic zero. For arbitrary $\mathcal{B}_1, \ldots, \mathcal{B}_r$, one has an algebraic group morphism, with source the algebraic $r$–torus, $(\mathbb{K}^*)^r$,

$$\rho: (\mathbb{K}^*)^r \longrightarrow \text{Aut}^1(\bigotimes_{j=1}^r \mathcal{B}_j),$$

defined by $\rho(t) = \otimes_{j=1}^r \rho_j(t_j)$, for $t = (t_1, \ldots, t_r)$, where each grading automorphism $\rho_j(t_j)$ acts on $\mathcal{B}_j^q$ as $t_j^q \cdot \text{id}$.

Our next result, which extends Corollary (ii) from [10], will provide an important step in the proof of Theorem 1.3.

**Corollary 2.6.** Assume that each $\mathcal{B}_j$ satisfies properties (i) and (iii) from Definition 1.2 (over $\mathbb{K}$). Then the above morphism $\rho$ is onto, with finite kernel. In particular, $\text{Aut}^1(\otimes_{j=1}^r \mathcal{B}_j)$ is an $r$–torus.

**Proof.** Property (iii) forces $\mathcal{B}_j \neq \mathbb{K}$, for all $j$. This readily implies that the kernel of $\rho$ is finite. Together with Lemma 2.5, this in turn guarantees the surjectivity of $\rho$, given that $\text{Der}^0(\otimes_{j=1}^r \mathcal{B}_j)$ is the Lie algebra of $\text{Aut}^1(\otimes_{j=1}^r \mathcal{B}_j)$ [10], via a dimension argument. The last assertion of the Corollary follows from [10] 16.1–2. $\square$
2.7. A convenient $Q$-basis. To prove Theorem \[1.5\] we have seen, at the beginning of this section, that it is enough to show that one cannot have simultaneously $\det \tau(a) = \pm 1$ and $\det(\text{id} - \tau(a)) = \pm 1$, where $a \in E$ is an arbitrary self-homotopy equivalence of $M$. To this end, the major step consists in finding a $Q$-basis of $\pi_1^{2k-1}(A)$, the $Q$-form of $Z_1^{2k-1}$, with respect to which $\tau(a)$ has a particularly simple matrix, for every $a \in E$. This will be done by resorting to the last property from Definition \[1.2\] and then using Corollary \[2.6\].

Set $J' := \{ j \in J \mid k_j = k \}$. One knows that $\pi_1^{2k-1}(A) = \oplus_{j \in J_1^{2k-1}}(A_j)$; see \[9\], \[18\]. With the notation from § 2.1 pick $y_j \in \pi_1^{2k-1}(A_j)$, $y_j \neq 0$, for $j \in J'$. Then obviously $\{y_j\}_{j \in J'}$ will be a basis of the $Q$-form of $Z_1^{2k-1}$.

**Lemma 2.8.** For any $a \in E$, there exist $\{\lambda_j \in Q\}_{j \in J'}$, and a permutation $\sigma$ of $J'$, such that

$$\tau(a)y_j = \lambda_j y_{\sigma(j)}, \quad \text{for} \quad j \in J'.$$

**Proof.** By Lemma \[2.3\] it suffices to prove the above statement, replacing $\tau(a)$ with $\nu(\alpha)$, where $\alpha \in \text{Aut}_C(A)$ is arbitrary, and without demanding the rationality of the $\lambda$-vector. (For $\alpha = h(a)$, the rationality property easily follows, since $\tau(a)$ respects by construction the $Q$-structure.)

Since $\text{Aut}_C(A)$ obviously normalizes $\text{Aut}_C^1(A)$, the map on characters induced by conjugation by $\alpha^{-1}$ permutes the weights of the restriction of the linear representation $\nu$ to $\text{Aut}_C^1(A)$, and $\nu(\alpha)$ permutes the corresponding weight spaces; see \[10\], 11.4. Due to the surjectivity of $\rho$ from \[2.8\], these weights coincide with the weights of $\nu \rho$, and the corresponding weight spaces are equal; see again \[10\], 11.4.

At the same time, the structure of the representation $\nu \rho$ is easy to describe. Indeed, we claim that, for $j \in J'$, $y_j$ is a weight vector of $\nu \rho$, with weight $t_j^{2k}$. This may be checked without difficulty, starting from the fact that $(M, d)$, the minimal model of the DGA $(A \otimes C, d = 0)$, splits as $\otimes_{j \in J}(M_j, d_j)$, where $(M_j, d_j)$ is the minimal model of $(A_j \otimes C, d = 0)$, according to \[18\].

We infer that the (distinct) weights of $\nu \rho$ are $\{t_j^{2k}\}_{j \in J'}$, with one–dimensional corresponding weight spaces, spanned by $\{y_j\}_{j \in J'}$. The desired form of $\nu(\alpha)$ is thus obtained. \[\square\]

2.9. End of proof of Theorem \[1.5\]. We may now compute the characteristic polynomial of $\tau(a)$, $P(z)$, as follows. Set $s = |J'|$, let $\sigma_1, \ldots, \sigma_m$ be the associated cycles of $\sigma$, with lengths $s_1, \ldots, s_m$, and put $\gamma_i := \prod_{j \in \text{supp}(\sigma_i)} \lambda_j$, for $i = 1, \ldots, m$. Then:

$$P(z) = \prod_{i=1}^{m} (z^{s_i} - \gamma_i).$$

We claim that

$$\gamma_i \in \mathbb{Z}, \quad \text{for all} \quad i.$$
Granting the claim for the moment, we may quickly finish the proof of Theorem 1.5. From the assumption that the isometry $a$ has no non-trivial invariant geodesic, we have deduced that both $P(0)$ and $P(1)$ are equal to $\pm 1$. Due to (2.10), this is impossible, and we are done.

Coming back to the claim, let us remark that the polynomial $P(z)$ from (2.9) actually belongs to $\mathbb{Z}[z]$, since $\tau(a)$ also preserves the natural $\mathbb{Z}$-structure of $\pi_1^{2k-1}(A)$; see (1.4). The following elementary lemma will thus complete our proof.

**Lemma 2.10.** Let $P(z)$ be given by (2.9). If $P(z) \in \mathbb{Z}[z]$, then $\gamma_i \in \mathbb{Z}$, for all $i$.

**Proof.** Write $\gamma_i = \frac{u_i}{v_i}$, with $u_i$ and $v_i$ relatively prime. We have

$$v_1 \cdots v_m P(z) = \prod_{i=1}^{m} (v_i z^{s_i} - u_i).$$

Applying the Gauss Lemma (see e.g. [11, p.127]) in the above equality, we infer that $v_1 \cdots v_m = \pm 1$, since the content of the monic polynomial $P(z)$ is 1. \[\square\]

3. **Reduced complete intersection and smoothing**

To construct new examples of application of Theorem 1.5 we will proceed in two steps. We will first present a general way of obtaining simple WACI’s. Secondly, we will review the obstruction theory associated to the problem of realizing Artinian graded $\mathbb{Q}$-algebras by smooth manifolds.

3.1. **Reduced complete intersection.** The construction of simple WACI’s is based on the Corollary on page 597 of our previous work [16].

**Proposition 3.2.** Let $A$ be a WACI, as in (1.1)–(1.2), where the $f$-weights satisfy $0 < | f_1 | \leq \cdots \leq | f_{n-1} | < | f_n |$, and the polynomial $f_n$ has no linear part. If $\mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1})$ is reduced (over $\mathbb{C}$), then $A$ is simple.

**Proof.** Conditions (i) and (ii) follow directly from the abovementioned Corollary, and its proof; see [16].

To check the last condition (iii), one has just to recall from [18] the recipe for computing $\pi^*_1$ of an arbitrary WACI. Set $X := \mathbb{Q} - \text{span} \{x_1, \ldots, x_n\}$ and $Y := \mathbb{Q} - \text{span} \{y_1, \ldots, y_n\}$, with $| y_i | = | f_i | - 1$. Then:

$$\pi^*_1(A) = \ker (L),$$

where $L : Y \to X$ sends $y_i$ to the linear part of $f_i$. \[\square\]
3.3. The smoothing problem. Let $A^*$ be a 1-connected Poincaré duality algebra (PDA) over $\mathbb{Q}$, of formal dimension $m$. That is, a finite-dimensional graded-commutative $\mathbb{Q}$-algebra with $A^1 = 0$, $A^m \cong \mathbb{Q}$ and $A^{>m} = 0$, for which the multiplication gives a duality pairing, $A^i \otimes A^j \to \mathbb{Q}$, for $i + j = m$. We shall say that $A$ is smoothable if $A^* \cong H^\ast(M^m, \mathbb{Q})$, as graded $\mathbb{Q}$-algebras, where $M$ is a 1-connected closed smooth $m$-manifold. A natural source of 1-connected PDA’s of even formal dimension is provided by WACI’s; see [8, Theorem 3].

Let $A$ be an arbitrary 1-connected PDA, of formal dimension $m$. If $A$ is smoothable, we may pick an orientation class, $[M] \in H^m(M, \mathbb{Z})$, which defines an algebraic orientation, $\omega \in A^{4k} \setminus \{0\}$. Likewise, the Pontrjagin classes $p_i(M)$ define a total algebraic Pontrjagin class, $q = 1 + \sum_{i \geq 1} q_i$, with $q_i \in A^{4i}$. If $m = 4k$, $\omega$ and $q$ may be used to describe the following three well-known obstructions to smoothability (see [13]).

By construction, $\{\langle q_1 \cdots q_k, \omega \rangle \mid \sum_{i=1}^k ie_i = k\}$ are the Pontrjagin numbers of a closed oriented smooth $4k$-manifold.

Secondly, the nondegenerate quadratic form $\varphi(A, \omega)$, defined on $A^{2k}$ by Poincaré duality and the orientation, must be a sum of signed squares, over $\mathbb{Q}$ (the integrality property).

Finally, the signature formula must hold:

$$\sigma(\varphi(A, \omega)) = \langle L_k(q_1, \ldots, q_k), \omega \rangle,$$

where $\sigma$ denotes the signature and $L_k$ is the $k$-th Hirzebruch polynomial.

It turns out that these obstructions completely control the smoothing problem, as explained in the proposition below, which is an immediate consequence of a basic $\mathbb{Q}$-surgery result due to D. Sullivan [18, Theorem 13.2] (see also [1, Theorem 8.2.2]).

**Proposition 3.4.** Let $A$ be a 1-connected $\mathbb{Q}$–PDA, of formal dimension $m$. If $m \neq 4k$, then $A$ is smoothable. If $m = 4k$, then $A$ is smoothable if and only if there exist $\omega \in A^{4k} \setminus \{0\}$ and $\sum_{i \geq 1} q_i \in \bigoplus_{i \geq 1} A^{4i}$, which verify the abovementioned properties, concerning Pontrjagin numbers, integrality, and the signature formula.

**Proof.** Realize $A$ by a 1-connected, $\mathbb{Q}$-local, Poincaré complex $S$. Everything follows then by applying the results quoted from [18] and [1] to $S$, except the fact that, for $m = 4$, smoothability holds, as soon as the obstructions are satisfied. In this very easy case, the Poincaré quadratic form on $A^2$ is a sum of $r$ squares minus a sum of $s$ squares, by integrality. Then plainly $A^* = H^\ast((\#, \mathbb{C}P^2)\#(\#, \overline{\mathbb{C}P^2}), \mathbb{Q})$ (where $\overline{\mathbb{C}P^2}$ denotes $\mathbb{C}P^2$ with the opposite complex orientation), for $r + s > 0$, and $A^* = H^\ast(S^4, \mathbb{Q})$, for $r + s = 0$. □

4. **$HH$ manifolds which are not homogeneous**

According to the signature of their cohomology algebras, our new $HH$ manifolds fall into two classes. We begin with the (easier) zero signature case.
4.1. Split simple algebras. There is a large freedom of choice of (discrete) parameters for our construction. More precisely: pick any \( n \geq 2, \ k \geq 1 \), then arbitrary positive even weights, \( \{w_i\}_{1 \leq i \leq n} \), and integers \( a_i \geq 2 \) such that \( w_i a_i = d \), for \( i = 1, \ldots, n \). Set:

\[
\begin{align*}
  f_1 &= x_1^{a_1} - x_2^{a_2} \\
  \vdots \\
  f_{n-1} &= x_{n-1}^{a_{n-1}} - x_n^{a_n} \\
  f_n &= x_n^{2ka_n}
\end{align*}
\]  

(4.1)

Proposition 4.2. Define \( A := \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \), where \( f_1, \ldots, f_n \) are given by (4.1) above. Then:

1. \( A \) is a simple WACI.
2. \( A \) is a 1-connected Poincaré duality algebra, with even formal dimension.
3. The non-trivial odd pseudo-homotopy groups of \( A \) are:

\[
\begin{align*}
  \pi_{d-1}^A &= \mathbb{Q}^{n-1}, \quad \text{and} \\
  \pi_{2kd-1}^A &= \mathbb{Q}
\end{align*}
\]

Proof. Part (1) follows from Proposition 3.2 via Lemma 2.1(ii) from [16]. Part (3) is a direct consequence of (3.1). For Part (2), see [8, Theorem 3].

Smoothability will follow from the next well-known lemma, whose proof is included for the reader’s convenience.

Lemma 4.3. Let \( A \) be a 1-connected, commutative, Poincaré duality algebra over \( \mathbb{Q} \), with even formal dimension. Assume that there is a \( \mathbb{Q} \)-subspace, \( N \subset A \), with \( \dim \mathbb{Q} A = 2 \dim \mathbb{Q} N \), and such that \( N \cdot N = 0 \), where the dot denotes the Poincaré inner product. Then there is a 1-connected, closed, smooth manifold \( M \), with zero signature, such that \( A^* = H^*(M, \mathbb{Q}) \), as graded algebras.

Proof. Immediate, using Proposition 3.1. When the formal dimension of \( A \) is 4k, we may use any orientation, \( \omega \in A^{4k} \setminus \{0\} \), and the trivial Pontrjagin class, \( q = 1 \). Now, Poincaré duality implies that the inner product space \( (A, \cdot) \) equals \( (A^{2k}, \cdot) \), in the Witt group \( W(\mathbb{Q}) \). On the other hand, our assumptions on \( N \) mean precisely that \( (A, \cdot) \) equals zero in \( W(\mathbb{Q}) \). See [12, I.6–7]. In particular, \( \sigma(A^{2k}, \cdot) = 0 \). With these remarks, it is now easy to check the three obstructions to smoothing.

Theorem 4.4. Let \( A \) be a simple WACI as in Proposition 4.2. Then:

1. \( A^* \) is the \( \mathbb{Q} \)-cohomology algebra of a 1-connected, closed smooth manifold, \( M \). The signature of \( M \) is zero.
2. If \( n \geq 4 \), \( A^* \) is not isomorphic to the cohomology algebra of any equal rank homogeneous space, \( G/K \).
Proof. Part (1). According to Lemma 4.3, we are done, as soon as we find an isotropic subspace, $N \subset \mathcal{A}$, having the appropriate dimension. This in turn may be done in the following way.

Set $\mathcal{B} = \mathbb{Q}[y_1, \ldots, y_n]/(y_1 - y_2, \ldots, y_{n-1} - y_n, y_{2k})$, where $|y_i| = d$, for all $i$. Plainly, $\mathcal{B} = \mathbb{Q}[y_0]/(y_0^{2k})$, with $|y_0| = d$. Consider the graded algebra map, $\phi : \mathcal{B}^* \to \mathcal{A}^*$, which sends $y_i$ to $x_i^{2n}$, for $i = 1, \ldots, n$. A Poincaré series argument shows that $\mathcal{A}^*$ is isomorphic to $\mathcal{B}^* \otimes F^*$, as a graded $\mathcal{B}^*$-module, where $F^* = \bigotimes_{i=1}^n \mathbb{Q}[x_i]/(x_i^{a_i})$. Indeed, one may extend $\phi$ to a surjective map of graded $\mathcal{B}^*$-modules,

\[(4.2) \quad \Phi : \mathcal{B}^* \otimes F^* \to \mathcal{A}^*,\]

by sending the elements of the monomial $\mathbb{Q}$-basis of $F^*$, \(\{x_1^{c_1} \cdots x_n^{c_n} \mid 0 \leq c_i < a_i, \forall i\}\), to their classes modulo $I$. By Corollary 2 on p.198 from [8], the Poincaré series of $\mathcal{A}^*$ is

\[(1 - t^d)^{n-1} (1 - t^{2kd}) \prod_{i=1}^n (1 - t^{w_i})^{-1},\]

which is equal to the Poincaré series of $\mathcal{B}^* \otimes F^*$. Therefore, $\Phi$ from (4.2) above is actually an isomorphism, which will be used to identify $\mathcal{A}^*$ with $\mathcal{B}^* \otimes F^*$.

Set now $V := \mathbb{Q} - \text{span} \{y_0^s \mid 0 \leq s < k\} \subset \mathcal{B}$, and $N = V \otimes F$. Clearly, $\dim \mathcal{A} = 2 \dim \mathcal{Q} \cdot N$. The proof ends by checking that $N \cdot N = 0$, which is straightforward.

Part (2). Assume that $\mathcal{A}^* = H^*(G/K, \mathbb{Q})$, as graded algebras. Write $G/K = \prod_{i=1}^k G_i/K_i$, with $G_i$ simple, for all $i$. Compute $\text{Der}^0$-dimensions to infer that $G$ must be simple, since $\mathcal{A}$ is a simple $WACI$; see Proposition 4.2(1) and Lemma 2.5.

By our assumption, $\pi_i^*(\mathcal{A}) = \pi_i^*(H^*(G/K, \mathbb{Q}))$, as graded vector spaces. One also knows [3] that $H^*(G/K, \mathbb{Q})$ is a $WACI$ for which the weights of the variables are equal to 2 times the degrees of the fundamental polynomial invariants of the Weyl group $W_K$, and likewise for the degrees of the defining relations of $H^*(G/K, \mathbb{Q})$.

Case by case inspection of the tables in [4] reveals that the degree list of the fundamental $W_G$-invariants contains no element of multiplicity $> 2$, if $G$ is simple. For $n - 1 > 2$, this contradicts the computation from Proposition 4.2(1); see [3].

\[\square\]

4.5. Simple algebras with non-zero signature. The non-zero signature smooth- ing problem is more subtle. For instance, the answer may depend not only on the algebra structure, but also on the weights, as seen in the following example. Let $\mathcal{A}_k$ be $\mathbb{Q}[x]/(x^3)$, where the weight of $x$ is $2k$, with $k$ odd. It is easy to check that each $\mathcal{A}_k$ is a simple $WACI$, with formal dimension $4k$ and signature $\pm 1$. If $\mathcal{A}_k$ is smoothable, then we infer from [11, Proposition B.5] (see also [2]) that $\sigma(\mathcal{A}_k)$ must be a multiple of $2^{2k-1} - 1$. Thus, $\mathcal{A}_k^*$ is not smoothable, for $k > 1$, while $\mathcal{A}_1^* = H^*(\mathbb{C}P^2, \mathbb{Q})$ is obviously smoothable.
Let us consider the Eisenbud–Levine \( WACI \)'s from [3, p.24], with generators \( \{ x_i \}_{1 \leq i \leq n} \) of weight 2, and defining relations
\[
\begin{aligned}
f_1 &= x_1^2 - x_2^2 \\
\vdots \\
f_{n-1} &= x_{n-1}^2 - x_n^2 \\
f_n &= x_1 \cdots x_n
\end{aligned}
\] (4.3)

Given a \( \mathbb{Q} \)-PDA, \( A \), of formal dimension \( 4k \), we shall denote in the sequel by \( (A^{2k}, \omega) \in W(\mathbb{Q}) \) the symmetric inner product space over \( \mathbb{Q} \) obtained from the multiplication map, \( A^{2k} \otimes A^{2k} \to A^{4k} \), via a choice of orientation, \( \omega \in A^{2k} \setminus \{0\} \).

**Proposition 4.6.** If \( A \) is defined by (4.3), where \( n \geq 3 \), then:

1. \( A \) is a simple \( WACI \).
2. \( A \) is a 1-connected Poincaré duality algebra, with formal dimension \( 4(n-1) \).
3. There is an orientation, \( \omega \in A^{4(n-1)} \setminus \{0\} \), such that \( (A^{2(n-1)}, \omega) \in W(\mathbb{Z}) \) and \( \sigma(A^{2(n-1)}, \omega) = 2n-1 \).
4. The non-trivial odd pseudo-homotopy groups of \( A \) are:

\[
\begin{aligned}
\pi^3_1(A) &= \mathbb{Q}^{n-1}, \\
\pi^{2n-1}_1(A) &= \mathbb{Q}.
\end{aligned}
\]

**Proof.** Part (1) follows from Proposition [8, Theorem 3], and formula (3.1) respectively. Part (2) and (3) are direct consequences of [8, Theorem 3], and formula (3.1) respectively.

For the proof of Part (4), we will need a convenient \( \mathbb{Q} \)-basis of \( A^* \). To obtain it, we first set \( B := \mathbb{Q}[y_1, \ldots, y_n]/(y_1 - y_2, \ldots, y_{n-1} - y_n) = \mathbb{Q}[y_0] \), where \( |y_i| = 4 \), for all \( i \), and define a graded algebra map, \( \phi: B \to \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1}) \), by \( \phi(y_i) = x_i^2 \), for \( i = 1, \ldots, n \). Like in the proof of Theorem [10, 11], \( \phi \) extends in an obvious way to a surjective map of graded \( B^* \)-modules,
\[
(4.4) \quad \Phi: B^* \otimes F^* \to \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1}),
\]
where \( F^* = \bigotimes_{i=1}^n \mathbb{Q}[x_i]/(x_i^2) \). A standard argument, involving regular sequences and Poincaré series, shows that the Poincaré series of \( \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1}) \) is \( (1 - t^4)^{n-1}(1 - t^2)^{-n} \), which is clearly equal to the Poincaré series of \( B^* \otimes F^* \).

In this way, we see that the above map (4.4) is an isomorphism. We will use it to identify \( A^* \) with a quotient of \( B^* \otimes F^* \). To be more precise, we will need the following notation. For \( I = \{ i_1 < \cdots < i_k \} \subset \{1, \ldots, n\} \), we put \( x_I := x_{i_1} \cdots x_{i_k} \in F^{2k} \), and we denote by \( \overline{I} \) the complement of \( I \) in \( \{1, \ldots, n\} \). It is easy to check in \( B^* \otimes F^* \) the equalities
\[
f_n \cdot x_I = y_0^{|I|} \otimes x_T, \quad \forall I;
\]
they imply that the images of the monomials
\[
(4.5) \quad \{ y_0^r \otimes x_I \mid 0 \leq r < |\overline{I}| \}
\]
give a $\mathbb{Q}$-basis of $\mathcal{A}$. Using this basis, we will show that, if one takes $\omega = y_0^{n-1}$, then $(\mathcal{A}, \omega) \in W(\mathbb{Z})$ and $\sigma(\mathcal{A}, \omega) = 2^{n-1}$, which clearly proves Part (3), via Poincaré duality.

It is not difficult to check that $(\mathcal{A}, \omega)$ splits as an orthogonal direct sum, $\bigoplus_{I \neq 0} \mathcal{A}_I$, where $\mathcal{A}_I$ denotes the $\mathbb{Q}$-span of the monomials (4.5) with $I$ fixed. For fixed $I$, it is equally easy to check that the inner product of $y_0^r \otimes x_I$ and $y_0^s \otimes x_I$ equals

\[
\begin{cases}
1, & \text{if } r + s = n - 1 - |I| \\
0, & \text{otherwise}
\end{cases}
\]

One may use (4.6) to further decompose $\mathcal{A}_I$ into a sum of hyperbolic planes and one–dimensional summands belonging to $W(\mathbb{Z})$, thus checking that $(\mathcal{A}, \omega) \in W(\mathbb{Z})$.

Moreover,

\[
\sigma(\mathcal{A}_I) = \begin{cases}
0, & \text{if } n - 1 - |I| \text{ is odd} \\
1, & \text{if } n - 1 - |I| \text{ is even}
\end{cases}
\]

which proves that $\sigma(\mathcal{A}, \omega) = 2^{n-1}$.

**Remark 4.7.** Let $\mathcal{A}$ be a $\mathbb{Q}$–PDA, of formal dimension $4k$, with orientation $\omega \in \mathcal{A}^{4k} \setminus \{0\}$. Then: the quadratic form $\varphi(\mathcal{A}, \omega)$ has the integrality property described in §3.3 if and only if $(\mathcal{A}^{2k}, \omega) \in W(\mathbb{Z})$; see [12, Corollary IV.2.6]. We should also point out that this integrality result is the key fact from our Proposition 4.6(3) above (the value of the signature was computed in [5], using a different method).

**Theorem 4.8.** Let $\mathcal{A}$ be a simple WACI as in Proposition 4.6. Then:

1. $\mathcal{A}^*$ is the $\mathbb{Q}$-cohomology algebra of a 1-connected, smooth closed manifold, $M$. The signature of $M$ equals $\pm 2^{n-1}$.
2. $\mathcal{A}^*$ is not isomorphic to $H^*(G/K, \mathbb{Q})$, for any equal rank homogeneous space, $G/K$.

**Proof.** Part (1). We have to show that $\mathcal{A}$ is smoothable. The integrality condition is satisfied; see Remark 4.7.

To check the other two $\mathbb{Q}$-surgery obstructions from Proposition 3.3 we will construct the total Pontrjagin class, $q$, as follows. Set $k = n - 1$. Let $p(\mathbb{CP}^{2k}) = 1 + \sum_{i=1}^k c_i u^{2i}$ be the total Pontrjagin class of $\mathbb{CP}^{2k}$, where $u \in H^2(\mathbb{CP}^{2k}, \mathbb{Q})$ is the canonical generator. We will take

\[
q := 1 + \sum_{i=1}^k 2^i c_i y_0^i \in \bigoplus_{i \geq 0} \mathcal{A}^{4i}
\]

(see (4.5)). It readily follows from (4.7) that $\mathcal{A}$ has the same Pontrjagin numbers as $2^k \mathbb{CP}^{2k}$. Therefore, the Hirzebruch signature formula (3.2) is also verified for $\mathcal{A}$, since $\sigma(\mathcal{A}^{2k}, \omega) = 2^k$, by Proposition 4.6(3); see [13]. Hence, $\mathcal{A}$ is smoothable.
Part (2). The same argument as in the proof of Theorem 4.4(2) gives the result, for $n \geq 4$. For $n = 3$, one may notice that the degree list of $\pi_1^*(A)$ is (3, 3, 5), with $5 \not\equiv -1 \pmod{4}$, and resort again to the tables in [4].

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