Turán’s Theorem for random graphs

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Abstract

For a graph $G$, denote by $t_r(G)$ (resp. $b_r(G)$) the maximum size of a $K_r$-free (resp. $(r-1)$-partite) subgraph of $G$. Of course $t_r(G) \geq b_r(G)$ for any $G$, and Turán’s Theorem says that equality holds for complete graphs. With $G_{n,p}$ the usual (“binomial” or “Erdős-Rényi”) random graph, we show:

**Theorem** For each fixed $r$ there is a $C$ such that if

$$p = p(n) > Cn^{-\frac{2}{r+1} \log \left(\frac{r+1}{r-2}\right)} n,$$

then $\Pr(t_r(G_{n,p}) = b_r(G_{n,p})) \to 1$ as $n \to \infty$.

This is best possible (apart from the value of $C$) and settles a question first considered by Babai, Simonovits and Spencer about 25 years ago.

1 Introduction

Write $t_r(G)$ for the maximum size of a $K_r$-free subgraph of a graph $G$ (where a graph is $K_r$-free if it contains no copy of the complete graph $K_r$ and size means number of edges), and $b_r(G)$ for the maximum size of an $(r-1)$-partite subgraph of $G$. Of course $t_r(G) \geq b_r(G)$ for any $G$, while the classic theorem of Turán [35]—commonly held to have initiated extremal graph theory—says that equality holds when $G$ is the complete graph $K_n$.

Here we are interested in understanding, for a given $r$, when equality is likely to hold for the usual (“binomial” or “Erdős-Rényi”) random graph $G = G_{n,p}$—that is, for what $p = p(n)$ one has

$$t_r(G_{n,p}) = b_r(G_{n,p}) \quad w.h.p. \quad (1)$$

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(An event holds *with high probability* (w.h.p.) if its probability tends to 1 as \( n \to \infty \). Note (1) holds for sufficiently small \( p \) for the silly reason that \( G \) is itself likely to be \((r-1)\)-partite, but we are thinking of more interesting values of \( p \).

First results on this problem were given by Babai, Simonovits and Spencer [2] (apparently in response to a conjecture of Paul Erdős [25]). They showed that for \( r = 3 \)—in which case Turán’s Theorem is actually Mantel’s [23]—(1) holds when \( p > 1/2 \) (more precisely, when \( p > 1/2 - \varepsilon \) for some fixed \( \varepsilon > 0 \)), and asked whether their result could be extended to \( p > n^{-c} \) for some fixed positive \( c \). This was finally accomplished (with \( c = 1/250 \)) in an ingenious paper of Brightwell, Panagiotou and Steger [5], which actually proved a similar statement for every (fixed) \( r \):

Theorem 1.1 ([5]). For each \( r \) there is a \( c > 0 \) such that if \( p > n^{-c} \) then w.h.p. every largest \( K_r \)-free subgraph of \( G_{n,p} \) is \((r-1)\)-partite.

(Actually [2] considers the problem with a general forbidden graph \( H \) in place of a clique—though the discussion there is mostly confined to \( H \)'s of chromatic number three—and [5] also suggests that Theorem 1.1 may hold for more than cliques; see Section 13 for a little more on this.)

It was also suggested in [5] that when \( r = 3 \), \( p > n^{-1/2+\varepsilon} \) might suffice for (1), and the precise answer in this case—(1) holds for \( p > Cn^{-1/2} \log^{1/2} n \)—was proved in [9]. (The more conservative suggestion in [5] seems due to an excess of caution on the part of the authors, who surely realized that \( \Theta(n^{-1/2} \log^{1/2} n) \) is the natural guess [34].)

Here we settle the problem for every \( r \):

Theorem 1.2. For each \( r \) there is a \( C \) such that if

\[
p > Cn^{\frac{2}{r+1}} \log^{\frac{2}{(r+1)(r-2)}} n,
\]

then w.h.p. every largest \( K_r \)-free subgraph of \( G_{n,p} \) is \((r-1)\)-partite.

This is best possible (apart from the value of \( C \)), basically because (formal proof omitted) for smaller \( p \) there are usually edges of \( G := G_{n,p} \) not lying in \( K_r \)'s; and while these automatically belong to all largest \( K_r \)-free subgraphs of \( G \), there’s no reason to expect that they are all contained in every largest \((r-1)\)-partite subgraph (and if they are not, then \( t_r(G) > b_r(G) \)).

Context. One of the most interesting combinatorial directions of the last few decades has been the study of “sparse random” versions of classical
results (e.g. the theorems of Ramsey, Szemerédi and Turán)—that is, of the extent to which these results remain true in a random setting. These developments, initiated by Frankl and Rödl [12] and the aforementioned Babai et al. [2] and given additional impetus by the ideas of Rödl and Ruciński [27, 28] and Kohayakawa, Luczak and Rödl [21, 22], led in more recent years to a number of major results, beginning with the breakthroughs of Conlon and Gowers [7] and Schacht [32]. The following are special cases, the second of which will be needed for the proof of Theorem 1.2.

**Theorem 1.3** ([7, 32]). For each \( \vartheta > 0 \) there is a \( K \) such that if

\[
p > Kn^{-2/(r+1)}
\]

then w.h.p. \( t_r(G_{n,p}) < (1 - \frac{1}{r} + \vartheta)|G_{n,p}| \).

**Theorem 1.4** ([7]). For each \( \vartheta > 0 \) there is a \( K \) such that if

\[
p > Kn^{-2/(r+1)}
\]

then w.h.p. each \( K_r \)-free subgraph of \( G = G_{n,p} \) of size at least \( (1 - \frac{1}{r-1})|G| \) can be made \( (r-1) \)-partite by deletion of at most \( \vartheta n^2 p \) edges.

These may be considered sparse random versions of Turán’s Theorem and the “Erdős-Simonovits Stability Theorem” [10, 33] respectively. Both were conjectured by Kohayakawa et al. [22], who proved Theorem 1.4 for \( r = 3 \), the weaker Theorem 1.3 for \( r = 3 \) having been proved earlier by Frankl and Rödl [12]. (See also [14, 13] for further progress preceding Theorems 1.3 and 1.4, and [30] for a common generalization of [7] and [32].)

Even more recently, related (but independent) papers of Balogh, Morris and Samotij [3] and Saxton and Thomason [31] prove remarkable “container” theorems—more asymptotic counting than probabilistic methods—which, once established, yield surprisingly simple proofs of many of the very difficult results mentioned above. See also [29] for a survey of these and related developments.

Though it does finally establish the “true” random analogue of Turán’s Theorem, one cannot really say that Theorem 1.2 is the culmination of some of this earlier work. First, it does not quite imply Theorem 1.3 whose conclusion holds for \( p \) in a somewhat larger range, and its conclusion is not comparable to that of Theorem 1.4. (Of course it is much stronger than Theorem 1.3 in the range where it does apply.) Second, apart from a black-box application of Theorem 1.4, the problem addressed by Theorem 1.2 seems immune to the powerful ideas developed to prove the aforementioned
results. (Conversations with several interested parties support this opinion and suggest that the paucity of results in the direction of Theorem 1.2 is not due to lack of effort.)

Plan. We prove Theorem 1.2 only for \( r \geq 4 \); the proof could presumably be adapted to \( r = 3 \), but this seems pointless given that we already have the far simpler argument of [9].

We begin with terminology and such in Section 2, but defer further preliminaries in order to give an early idea of where we are headed. Thus Section 3 just states the main points—Lemmas 3.1 and 3.2—underlying Theorem 1.2 and shows how they imply the theorem.

Section 4 then collects machinery needed for the arguments to come. One new item here is Lemma 4.13, an extension of the recent Riordan-Warnke generalization [26] of the Janson Inequalities [16], that seems likely to be useful elsewhere.

We next, in Sections 5 and 6, outline the proofs of Lemmas 3.1 and 3.2 again meaning we state main points and derive the lemmas from them. The assertions underlying Lemma 3.1 are proved in Sections 7-9 and those underlying Lemma 3.2 in Sections 10-12. (The two parts both require Lemma 5.1 and the material of Section 4 but are otherwise independent.) Finally, Section 13 mentions a few related questions.

Discussion. The basic structure of the argument—deriving Theorem 1.2 from Lemmas 3.1 and 3.2—seems natural and is similar to that in [23]. (See the remark following Lemma 3.2. The reader familiar with [23] may notice that the rather ad hoc conditions around the analogue of \( Q(\Pi) \)—here defined in the second paragraph after (6)—have now disappeared.) It should, however, be stressed that the nature and difficulty of the problem undergo a drastic change when we move from \( r = 3 \) to \( r \geq 4 \), and that most of the ideas of [23] are pretty clearly useless for present purposes. (This feels akin to the familiar jump in difficulty when one moves from graphs to hypergraphs.) In the event, most of the key ideas in what follows are without much in the way of antecedents, the most notable exception being that the uses of Harris’ Inequality in Section 10 were inspired by a related use in [3].

We will try to say a little more about various aspects of the argument when we are in a position to do so intelligibly. The most interesting points are the proof of Lemma 5.3 (the last of the lemmas supporting Lemma 3.1) what’s most interesting here is how tricky this innocent-looking statement was to prove) and, especially, the several ideas developed in Sections 10-12 to deal with Lemma 3.2.
2 Usage and definitions

For integers $a \leq b$, we use $[a, b]$ for $\{a, \ldots, b\}$ and $[b]$ for $[1, b]$ (assuming $b \geq 1$). As usual, $2^X$ and $\binom{X}{k}$ are the collections of subsets and $k$-subsets of the set $X$. We write $\alpha = (1 \pm \delta)\beta$ for $(1 - \delta)\beta < \alpha < (1 + \delta)\beta$ and log for natural logarithm. Following a common abuse, we usually omit superfluous floor and ceiling symbols.

We use $B(n, p)$ for a random variable with the binomial distribution $\text{Bin}(n, p)$. In line with recent practice, we occasionally use $X_p$ for the “binomial” random subset of $X$ given by

$$\Pr(X_p = A) = p^{|A|}(1 - p)^{|X\setminus A|} \quad (A \subseteq X). \quad (4)$$

Throughout the paper $V = [n]$ is our default vertex set. The random graphs $G_{n,p}$ are defined as usual; see e.g. [17]. We will usually use $G$ as an abbreviation for $G_{n,p}$, so for the present discussion use $H$ for a general graph (on $V$).

We use $|H|$ for the size (i.e. number of edges) of $H$, $N_H(x)$ for the set of neighbors of $x$ in $H$, $d_H(x)$ for the degree of $x$ in $H$ (i.e. $|N_H(x)|$, $d_H(x, y)$ for $|N_H(x) \cap N_H(y)|$ and so on. When the identity of $H$ is clear—usually meaning $H = G = G_{n,p}$—we will sometimes drop the subscript (thus $N(x)$ or $N_x$, $d(x)$ etc.) and may then, a little abusively, use, for example, $N_B(x)$ for the set of neighbors of $x$ in $B \subseteq V$ or $N_L(x)$ for the set of vertices joined to $x$ by members of $L \subseteq \binom{V}{2}$. We use $\Delta_H$ for the maximum degree of $H$.

As usual, $H[A]$ is the subgraph of $H$ induced by $A \subseteq V$. For disjoint $A_1, \ldots, A_k \subseteq V$, we use $\nabla(A_1, \ldots, A_k)$ for the set of pairs $\{x, y\}$ meeting two distinct $A_i$’s, and often write $\nabla_H(A_1, \ldots, A_k)$ for $H \cap \nabla(A_1, \ldots, A_k)$. We will tend to use $xy$ ($= yx$), rather than $\{x, y\}$, for an element of $\binom{V}{2}$. Unless stated otherwise, $V(L)$ is the set of vertices belonging to members of $L \subseteq \binom{V}{2}$.

A cut is an ordered $(r-1)$-partition $\Pi = (A_1, \ldots, A_{r-1})$ of $V$. (The order of $A_2, \ldots, A_{r-1}$ isn’t important, but $A_1$ will play a special role.) Throughout the paper $\Pi$ will denote a cut. We say $\Pi$ is balanced if each of its blocks has size $(1 \pm \delta)n/(r-1)$, where $\delta$ is a small (positive) constant (see the discussion at the end of this section).

For $\Pi$ as above we sometimes use $\text{ext}(\Pi)$ for $\nabla(A_1, \ldots, A_{r-1})$ and $\text{int}(\Pi)$ for $\binom{V}{2} \setminus \text{ext}(\Pi)$ (and give $\text{ext}_H(\Pi)$, $\text{int}_H(\Pi)$ their obvious meanings). We will also use $|\Pi|$ for $|\nabla(A_1, \ldots, A_{r-1})|$ and $|\Pi_H|$ for $|\nabla_H(A_1, \ldots, A_{r-1})|$; thus $b_r(H) = \max|\Pi_H|$. The defect of $\Pi$ with respect to $H$ is

$$\text{def}_H(\Pi) = b_r(H) - |\Pi_H|.$$
and the defect of $\Pi$ is its defect with respect to $G = G_{n,p}$.

Though it may take some getting used to, the following notation will be quite helpful. Suppose that for $i = 1, \ldots, s$, $X_i$ is a collection of $a_i$-subsets of $V$ (we will usually have $a_i \leq 2$) and that $\sum a_i \leq r$. We then write $\kappa_H(X_1, \ldots, X_s)$ for the number of ways to choose disjoint $Y_1 \in X_1, \ldots, Y_s \in X_s$ and an $(r - \sum a_i)$-subset $Z$ of $V \setminus \bigcup Y_i$ so that

$$\left( Y_1 \cup \cdots \cup Y_s \cup Z \right) \setminus \bigcup_{i=1}^s \binom{Y_i}{2} \subseteq H.$$ 

When $X_i$ consists of a single set, say $\{x_1, \ldots, x_{a_i}\}$, we omit set brackets and commas in the specification; for example: (i) $\kappa_H(xy)$ counts choices of $Z \in \binom{V}{r-2}$ with all pairs from $Z \cup \{x, y\}$ belonging to $H$, and (ii) $\kappa_H(x_1x_2x_3, T)$, with $T \subseteq \binom{V}{2}$, counts choices of $\{x_4, x_5\} \in T$ with $\{x_4, x_5\} \cap \{x_1, x_2, x_3\} = \emptyset$ and $\{x_6, \ldots, x_r\} \subseteq V \setminus \{x_1, \ldots, x_5\}$ (with $x_4 \neq x_5$ and $x_6, \ldots, x_r$ distinct) such that all members of $\left( \binom{x_1, \ldots, x_r}{2} \right)$ other than those in $\left( \binom{x_1, \ldots, x_3}{2} \right) \cup \{x_4, x_5\}$ lie in $H$.

In one special case, when $s = r - 1$, $a_1 = 2$, $X_2, \ldots, X_s$ are disjoint and no pair from $X_1$ meets $X_2 \cup \cdots \cup X_s$, we will on a few occasions use $K_H(X_1, \ldots, X_s)$ for the collection counted by $\kappa_H(X_1, \ldots, X_s)$ (members of which may be thought of as copies of $K_r^-$, the graph obtained from $K_r$ by deleting an edge).

When $H = G = G_{n,p}$, we will tend to drop subscripts and write simply $\kappa(\cdots)$ and $K(\cdots)$.

The quantity

$$\Lambda_r(n, p) := n^{r-2}p^{\binom{r}{2}}^{-1},$$

which, up to scalar, is the expectation of $\kappa(xy)$ (for given $x, y$), will appear frequently (so we give it a name).

**Constants.** There will be quite a few of these, but not so many that are more than local. The most important are $\delta$ (see the above definition of a balanced cut); $\gamma$ (used in the definition of a “bad” pair for a given cut following (3)); $\alpha$ (see “rigidity” in Section 10); and $C$ (in (2)). The few constants that are given explicitly will, superfluously, be subscripted by $r$.

For the main constants, apart from an explicit constraint on $\gamma$ in Section 6 (see (52)), we will not bother with actual values, but the hierarchy is (of course) important: we assume $C^{-1} < \delta < \alpha, \gamma$ (where, just for the present discussion, “$a \prec b$” means $a$ is small enough relative to $b$ to support our arguments), with $\alpha$ (and $\gamma$, but this will follow from (52)) small relative to
We could take $\alpha = \gamma$, but prefer to distinguish them to emphasize their separate roles.) In particular the constant $C$—and $n$—are always assumed to be large enough to support our various assertions.

3 Skeleton

In this section we state the main points underlying Theorem 1.2 and derive the theorem from these (with a small assist from one of the standard large deviation assertions of Section 4).

We fix $r \geq 4$ and assume $p$ is as in (2) with $C$ a suitable constant (and, as always, $n$ large enough to support our arguments). Though not really necessary, it will also be convenient to assume, as we may by Theorem 1.1, that

$$p = o(1).$$

(6)

Fix $\gamma > 0$ to be specified below. (The specification will make more sense in Section 6 where we outline the proof of Lemma 3.2 so we postpone it until then.)

For disjoint $S_1, \ldots, S_{r-2} \subseteq V$, a pair $\{x, y\}$ is bad for $(S_1, \ldots, S_{r-2})$ in $G$ if $\kappa_G(xy, S_1, \ldots, S_{r-2}) < \gamma \Lambda_r(n, p)$. For $\Pi = (A_1, \ldots, A_{r-1})$, we write $Q_G(\Pi)$ for the set of pairs from $A_1$ that are bad for $(A_2, \ldots, A_{r-1})$ in $G$, and for $F \subseteq (V)_{2}$ let

$$\varphi(F, \Pi) = (r - 1)|F[A_1]| + |F \cap \text{ext}(\Pi)|.$$

We now write $G$ for $G_{n,p}$. The next two statements are our main points.

**Lemma 3.1.** There is an $\eta > 0$ such that w.h.p.

$$\varphi(F, \Pi) < |\Pi_G|$$

whenever $\Pi = (A_1, \ldots, A_{r-1})$ is balanced and $F \subseteq G$ is $K_r$-free and satisfies $F \not\supseteq \Pi_G$, $F \cap Q_G(\Pi) = \emptyset$,

$$|F[A_1]| < \eta n^2 p,$$

and

$$|N_F(x) \cap A_1| = \min\{|N_F(x) \cap A_i| : i \in [r-1]\} \quad \forall x \in A_1.$$  

(8)

**Lemma 3.2.** W.h.p. $\text{def}_G(\Pi) \geq r|Q|$ whenever $\Pi = (A_1, \ldots, A_{r-1})$ is balanced, $Q \subseteq Q_G(\Pi)$ and

$$d_Q(x) \leq \min\{|N_G(x) \cap A_i| : 2 \leq i \leq r - 1\} \quad \forall x \in A_1.$$  

(9)
(The lemma holds with any constant in place of $r$ in the defect bound, but $r$ (actually $r-1$) is what’s needed in the final inequality (13) below.)

**Remark.** Technicalities aside, the dichotomy embodied in Lemmas 3.1 and 3.2 is quite natural. If $\Pi = (A_1, \ldots, A_{r-1})$ is a cut and $xy \in G[A_1]$ (say), then any $K_r$-free $F \subseteq G$ containing $xy$ must miss at least one edge of each member of $K_G(xy,A_2,\ldots,A_{r-1})$. For a typical $xy$ there are (by our choice of $p$) many of these, and one may hope that this forces $\text{ext}_G(\Pi) \setminus F$ to be (much) larger than the number of such $xy$’s in $F$, which gives $|F| < |\Pi_G| (\leq b_r(G))$ provided a decent fraction of the edges of $F \cap \text{int}(\Pi)$ are “typical” edges of $G[A_1]$. Something of this sort is shown in Lemma 3.1.

Of course for a general $\Pi$ and $xy$ as above, $\kappa_G(xy,A_2,\ldots,A_{r-1})$ need not be large, or even positive. This more interesting situation—in which membership of $xy$ in $F$ says less about $\text{ext}_G(\Pi) \setminus F$—is handled by Lemma 3.2, which says, roughly, that the defect of $\Pi$ is large relative to the number of pairs $x,y$—or edges, but adjacency of $x,y$ is irrelevant here—from $A_1$ for which $\kappa_G(xy,A_2,\ldots,A_{r-1})$ is small.

Notice, for example, that $t_r(G) > b_r(G)$ whenever there are a maximum cut $(A_1,\ldots,A_{r-1})$ and $xy \in G[A_1]$ with $\kappa_G(xy,A_2,\ldots,A_{r-1}) = 0$; thus a baby requirement for Theorem 1.2 is that this situation be unlikely, and in fact we don’t know how to show even this much without some portion of the machinery of Sections 10-12.

At any rate, given Lemmas 3.1 and 3.2 we finish easily, as follows. Let $F_0$ be a largest $K_r$-free subgraph of $G$ and $\Pi = (A_1,\ldots,A_{r-1})$ a cut maximizing $|F_0 \cap \text{ext}(\Pi)|$, with $|F_0[A_1]| = \max_i |F_0[A_i]|$. Then (5) holds with $F_0$ in place of $F$ (if it did not, we could move a violating $x$ from $A_1$ to some other $A_i$ to increase $|F_0 \cap \text{ext}(\Pi)|$), and Theorem 1.4 implies that w.h.p. $F_0$ also satisfies (7) (actually with $o(1)$ in place of $\delta$ in the definition of balance) w.h.p., since (w.h.p.)

\[
|\nabla(A_1,\ldots,A_{r-1})|p + o(n^2p) > |\Pi_G| \tag{10}
\]

\[
\geq |F_0 \cap \text{ext}(\Pi)| \tag{11}
\]

\[
> |F_0| - o(n^2p) \tag{11}
\]

\[
> (r - 2)|G|/(r - 1) - o(n^2p) \tag{12}
\]

\[
> (r - 2)n^2p/[2(r - 1)] - o(n^2p), \tag{13}
\]

so that

\[
|\nabla(A_1,\ldots,A_{r-1})| > (r - 2)n^2/[2(r - 1)] - o(n^2),
\]


which easily gives $|A_i| = (1 \pm o(1))n/(r-1) \forall i$. Here (10) and (13) are easy applications of “Chernoff’s Inequality” (Theorem 4.1); (11) follows from Theorem 1.4 and (12) is the “standard observation” mentioned above.

Let $F_1 = F_0[A_1] \cup (F_0 \cap \text{ext}(\Pi))$ and $F = F_1 \setminus Q_G(\Pi)$. Noting that these modifications introduce no $K_r$’s and preserve (7) and (8), we have, w.h.p.,

$$t_r(G) = |F_0|$$
$$\leq \varphi(F_1, \Pi)$$
$$= \varphi(F, \Pi) + (r-1)|F_1 \cap Q_G(\Pi)|$$
$$\leq |\Pi_G| + (r-1)|F_1 \cap Q_G(\Pi)|$$  (14)
$$\leq b_r(G),$$  (15)

where (14) is given by Lemma 3.1 (note that if $F \supseteq \Pi_G$ then $F \cap Q_G(\Pi) = \emptyset$ implies $F = \Pi_G$) and (15) by Lemma 3.2 (applied with $Q = F_1 \cap Q_G(\Pi)$, noting that (9) follows from the fact that (8) holds for $F_1$).

This gives (1). For the slightly stronger assertion of Theorem 1.2, notice that we have strict inequality in (14) unless $F \supseteq \Pi_G$ and in (15) unless $F_1 \cap Q_G(\Pi) = \emptyset$. Thus $|F_0| = b_r(G)$ implies $F_0[A_1] = F[A_1] \cup (F_1 \cap Q_G(\Pi)) = \emptyset$, so also $F_0[A_i] = \emptyset$ for $i \geq 2$ (since we assume $|F_0[A_1]| = \max_i |F_0[A_i]|$).

4 Preliminaries

The following version of Chernoff’s Inequality may be found, for example, in [17, Theorem 2.1].

**Theorem 4.1.** For $\xi = B(m, p)$, $\mu = mp$ and any $\lambda \geq 0$,

$$\Pr(\xi > \mu + \lambda) < \exp\left(-\frac{\lambda^2}{2(\mu + \lambda/3)}\right),$$
$$\Pr(\xi < \mu - \lambda) < \exp\left(-\frac{\lambda^2}{2\mu}\right).$$

We will also need the following inequality for weighted sums of Bernoullis, which can be derived from, for instance, [4, Lemma 8.2].

**Lemma 4.2.** Suppose $w_1, \ldots, w_m \in [0, z]$. Let $\xi_1, \ldots, \xi_m$ be independent Bernoullis, $\xi = \sum \xi_i w_i$, and $E\xi \leq \psi$. Then for any $\eta \in [0, 1]$ and $\lambda \geq \eta \psi$,

$$\Pr(\xi \geq \psi + \lambda) \leq \exp[-\eta \lambda/(4z)].$$
We record a few easy consequences of Theorem 4.1 in which we again take \(G = G_{n,p}\) (with \(p\) as in \(2\), which is more than is needed here).

**Proposition 4.3.** W.h.p. for all \(x,y \in V\),
\[
    d(x) = (1 \pm o(1))np \quad \text{and} \quad d(x,y) = (1 \pm o(1))np^2.
\] (16)

**Proposition 4.4.**

(a) For each \(\varepsilon > 0\) there is a \(K\) such that w.h.p.
\[
    ||G[X] \|- |X|^p/2 \leq \max\{\varepsilon |X|^p, K|X|\log n\} \quad \forall X \subseteq V.
\]

(b) There is a fixed \(\varepsilon > 0\) such that w.h.p.
\[
    |G[X]| < |X|\log n \quad \forall X \subseteq V \text{ with } |X| < \varepsilon^{-1} \log n.
\]

**Proposition 4.5.** For all \(\varepsilon > 0\) and \(c\) there is a \(K\) such that w.h.p.
\[
    |\nabla G(S,T)| > (1 - \varepsilon)|S||T|^p
\]
for all disjoint \(S,T \subseteq V\) with \(|S| > cn \text{ and } |T| > K/p\).

We omit the straightforward proofs.

### 4.1 Polynomial concentration

We will need two instances of the “polynomial concentration” machinery of J.H. Kim and V. Vu \([20, 36, 37]\). Here we omit the polynomial language and just recall what we actually use, for which we assume the following setup. Let \(\mathcal{H}\) be a collection of \(d\)-subsets of \(X = [N]\), \(w : \mathcal{H} \to \mathbb{R}^+\), \(Y = X_p\) (see \([11]\) for \(X_p\)) and
\[
    \xi = \sum\{w_A : A \in \mathcal{H}, A \subseteq Y\}. \quad \text{(17)}
\]

For \(L \subseteq X\)
\[
    E_L = \sum\{w_A p^{(|A|-|L|)} : L \subseteq A \in \mathcal{H}\}
\]
and \(E_l = \max_{|L|=l} E_L\) for \(0 \leq l < d\) (e.g. \(E_0 = E\xi\)).

We will need the following particular consequences of \([20, 37, 39]\). (The first—as observed in \([19, \text{Cor. 5.5}]\), here slightly rephrased—follows easily from results of \([20]\) and \([37]\), and the second is contained in \([36, \text{Cor. 2.6}]\).)

**Lemma 4.6.** For each fixed \(d, \varepsilon > 0\), \(b\) and \(M\) there is a \(J\) such that if
\[
    \max w_A \leq b \text{ and } B \geq \max\{(1 + \varepsilon)E_0, J\log N, N^\varepsilon \max_{0<j<d} E_j\},
\]
then
\[
    \Pr(\xi > B) < N^{-M}.
\]
Lemma 4.7. For each fixed $d$, $\varepsilon > 0$, $b$ and $M$ there is a $J$ such that if $\max w_A \leq b$ and $\max_{0 \leq l < d} E_l < N^{-\varepsilon}$, then

$$\Pr(\xi > J) < N^{-M}.$$  

We will apply these results in the following setting. (There is nothing surprising here—e.g. similar applications of the above machinery appear in [19]—but, lacking a reference, we include a few details.)

Define a rooted graph to be a graph $H = (V(H), E(H))$ with members of some $R = R(H) \subset V(H)$ designated “roots.” In what follows it will be convenient to fix some ordering of $R$ and speak of the root sequence, $(u_1, \ldots, u_s)$, of $H$.

Though we allow edges between the roots, they play no role here and we set $E'(H) = \{ e \in E(H) : e \not\subseteq R \}$, $v_H = |V(H) \setminus R|$, $e_H = |E'(H)|$ and $\rho(H) = e_H/v_H$; this last quantity called the density of $H$. (For typographical reasons we will sometimes use $v(H)$ and $e(H)$ in place of $v_H$ and $e_H$.)

For the purposes of this limited discussion a subgraph of a rooted $H$ is a subgraph (in the ordinary sense) with the same roots. We say $H$ is balanced if $\rho(H') \leq \rho(H)$ for all subgraphs $H'$ of $H$ with $v_{H'} \neq 0$ and strictly balanced if the inequality is strict whenever $E'(H') \neq E'(H)$.

A copy of a rooted graph $H$ in a graph $G$ is an injection $\varphi : V(H) \to V(G)$ such that $\varphi(u)\varphi(v) \in E(G) \forall uv \in E'(H)$. (Note that here, for once, we are not assuming $G = G_{n,p}$; and that when we do assume this below (that is, until the end of this section) we are not placing any restriction on $p$.)

We use $\Phi(H, G)$ for the set of copies $\varphi$ of $H$ in $G$. If $(u_1, \ldots, u_s)$ is the root sequence of $H$ and $x_1, \ldots, x_s$ are distinct vertices of $G$, then we set

$$\Phi(H, G; x_1, \ldots, x_s) = \{ \varphi \in \Phi(H, G) : \varphi(u_i) = x_i \ \forall i \in [s] \}$$

and

$$N(H, G; x_1, \ldots, x_s) = |\Phi(H, G; x_1, \ldots, x_s)|.$$

(If $x_1, \ldots, x_n$ are not all distinct then we set $N(H, G; x_1, \ldots, x_s) = 0$.)

We now take $G = G_{n,p}$. Then $N(H, G; x_1, \ldots, x_s)$ is a random variable of the type treated in Lemmas 4.6 and 4.7, namely, with $X = \left( [n] \right) \setminus \{ x_1, \ldots, x_s \}$ (so $Y = G_{n,p} \cap X$), $d = e_H$ and

$$\mathcal{H} = \{ \varphi(E') : \varphi \in \Phi(H, K_n; x_1, \ldots, x_s) \},$$

we have

$$\xi := N(H, G; x_1, \ldots, x_s) = w|\{ A \in \mathcal{H}, A \subseteq Y \}|,$$  

(18)
where \( w \) is the number of automorphisms of the rooted graph \( H \) (that is, permutations of \( V(H) \) that are automorphisms in the usual sense and fix all roots). Of course if we set \( w_A = w \forall A \in \mathcal{H} \), then (18) agrees with (17).

Notice that with these definitions we have

\[
E_0 = E_\xi \left\{ \frac{v_H}{v_L} n^{v_H p^e H} \right\} \quad (19)
\]

and, for any \( L \subseteq X \),

\[
E_L < (v_H)_{v_L} n^{v_H - e_L p^e H - e_L}, \quad (20)
\]

with \( v_L = |V(L)| \) (where \( V(L) \) is the set of vertices of \( \{x_1, \ldots, x_s\} \) incident with edges of \( L)\), \( e_L = |L| \) and, as usual, \( (j)_i = j(j-1) \cdots (j-i+1) \).

(The unnecessarily precise \((v_H)_{v_L}\) bounds the number of possibilities for a bijection from some \( v_L \)-subset of \( V(H) \setminus R \) to \( V(L) \).)

Notice that \( v_L = v(H_L) \) and \( e_L = e(H_L) \), where \( H_L \) is the rooted graph with vertex set \( \{x_1, \ldots, x_s\} \cup V(L) \), edge set \( L \) and root sequence \( (x_1, \ldots, x_s) \), and that \( E_L = 0 \) unless

\[
H_L \text{ is isomorphic (as a rooted graph) to some subgraph of } H. \quad (21)
\]

On the other hand, for \( L \) with \( e_L < e_H \) satisfying \( 21 \),

\[
v_L \geq \begin{cases} v_H e_L / e_H & \text{if } H \text{ is balanced,} \\ v_H e_L / e_H + e_H & \text{if } H \text{ is strictly balanced,} \end{cases} \quad (22)
\]

where \( \psi_H \) is some positive constant (depending on \( H)\); thus, recalling (20) and writing \( z \) for the constant \((v_H)_{v_L}\) appearing there, we have (for such \( L \))

\[
E_L < \begin{cases} z(n^{v_H p^e H})^{1 - e_L / e_H} & \text{if } H \text{ is balanced,} \\ z(n^{v_H p^e H})^{1 - e_L / e_H n^{-\theta H}} & \text{if } H \text{ is strictly balanced.} \end{cases} \quad (23)
\]

In the next two propositions we assume the above setup and mainly aim for statements that suffice for our purposes. The quantity \( E_0 \) is, of course, the same for all choices of \( x_1, \ldots, x_s \) and we now use it for this common value. The propositions and the corollary that follows them are trivial when the \( x_i \)'s in question are not all distinct, and the proofs accordingly ignore this possibility.

**Proposition 4.8.** If \( H \) is balanced, then for each \( \theta > 0 \) there is a \( K \) such that if \( S > K \log n \) and \( E_0 < n^{-\theta} S \), then w.h.p.

\[
N(H, G; x_1, \ldots, x_s) < \theta S / \log n \quad \forall x_1, \ldots, x_s \in [n].
\]
Proof. It is enough to show that, for any $M$, we can choose $K$ so that (for any $x_1, \ldots, x_s$)

$$\Pr(N(H, G; x_1, \ldots, x_s) \geq \theta S/ \log n) < n^{-M}.$$  \hfill (24)

To see this, we fix $x_1, \ldots, x_s$, set $\xi = N(H, G; x_1, \ldots, x_s)$ and follow the notation introduced above. We first claim that (24) will follow if we show there is a fixed $\varepsilon > 0$ such that

$$E_l < n^{-\varepsilon}S \quad \forall 0 \leq l < e_H.$$  \hfill (25)

To see that this is enough, suppose first that $S \geq n^\varepsilon/2$. We may then apply Lemma 4.6 with, for example, $B = n^{-\varepsilon/4}S$ to say that with probability at least $1 - n^{-M}$ for any fixed $M$,

$$\xi \leq B = o(S/ \log n).$$

If, on the other hand, $S < n^\varepsilon/2$, then Lemma 4.7 gives

$$\Pr(\xi > N) < n^{-M}$$

for a suitable $N$, and taking $K = N/\theta$ gives (24).

Finally, for the proof of (25), we have, using (23), (19) and our hypotheses (with $z$ as in (18)) and $L$ of size less than $e_H$,

$$E_L \leq (1 + o(1))z(E_0)1^{-\varepsilon L/\varepsilon H} < (1 + o(1))z(n^{-\varepsilon}S)1^{-\varepsilon L/\varepsilon H} < n^{-\theta(1-\varepsilon L/\varepsilon H)}S,$$

which gives (24) with $\varepsilon = \theta/e_H$.

Proposition 4.9. If $H$ is strictly balanced, then for any $\beta > 0$ and $M$ there is a $K$ such that for any $x_1, \ldots, x_s \in [n]$, with probability at least $1 - n^{-M}$,

$$N(H, G; x_1, \ldots, x_s) < \begin{cases} K & \text{if } E_0 < n^{-\beta}, \\ \max\{(1 + \beta)E_0, K \log n\} & \text{otherwise.} \end{cases}$$  \hfill (26)

In particular, w.h.p. (26) holds for all $x_1, \ldots, x_s \in [n]$.

Proof. We will apply one of Lemmas 4.6, 4.7 with $\xi = N(H, G; x_1, \ldots, x_s)$, $N = \binom{n}{s} - \binom{s}{s}$, $d = e_H$, $b = w$ (w as in (18)) and $\varepsilon = \frac{1}{4}\min\{\beta, \theta_H\}$.

Suppose first that $E_0 \leq 1$ and let $K$ be the $J$ of Lemma 4.6. By (19) and (23) we have

$$E_l \leq (1 + o(1))zn^{-\theta H} \leq N^{-\varepsilon}$$  \hfill (27)
for all $0 < l < d$; so may take $B$ in Lemma 4.1 to be $K \log n$, and then the lemma gives $\Pr(\xi > K \log n) < N^{-M} < n^{-M}$ as desired. If, a fortiori, $E_0 < n^{-\beta}$, then we also have (27) for $l = 0$, so with $K$ equal to the $J$ of Lemma 4.7, that lemma gives $\Pr(\xi > K) < N^{-M}$.

Finally, if $E_0 > 1$, then (19) and (23) give $E_0 < n - \beta$, then we also have (27) for $l = 0$, so with $K$ equal to the $J$ of Lemma 4.7, that lemma gives $\Pr(\xi > K) < N^{-M}$.

In fact all our applications of Proposition 4.9 will be instances of the next assertion (so we really only use the proposition with $H = K_r$).

**Corollary 4.10.** For all $s < r$, $\beta > 0$ and $M$ there is a $K$ such that, with $Z = n^{r-s} p^{(r)}_{(2)}(2)$: for any $x_1, \ldots, x_s \in [n]$, with probability at least $1 - n^{-M}$,

$$\kappa(x_1 \cdots x_s) < \begin{cases} K & \text{if } Z < n^{-\beta}, \\ \max\{ (1 + \beta) Z, K \log n \} & \text{otherwise} \end{cases}$$

(28)

(where $\kappa = \kappa_G$). In particular, w.h.p. (28) holds for all $x_1, \ldots, x_s \in [n]$.

**Proof.** It is easy to see that all rooted versions of $K_r$ are strictly balanced. Note also that, again taking $\xi = N(K_r, G; x_1, \ldots, x_s)$ (for some specified choice of roots $u_1, \ldots, u_s$ for $K_r$), we have $\kappa(x_1 \cdots x_s) = \xi / (r-s)!$ and $E_0 := E_\xi < Z$ (see (19)). The assertion thus follows from Proposition 4.9.

**4.2 Harris**

Before continuing we quickly recall the seminal correlation inequality of T.E. Harris [15]. Fix a set $I$ and set $\Omega = \{0, 1\}^I \equiv 2^I$ (where we make the usual identification of a set with its indicator). For $f : \Omega \to \mathbb{R}$, recall that $f$ is **increasing in** $J \subseteq I$ if $f(x) \geq f(y)$ whenever $i \in J, x_i \geq y_i$ and $x_j = y_j$ for $j \neq i$ (decreasing in $J$ is defined similarly), and is determined by $J$ if $f(x) = f(y)$ whenever $x_i = y_i \forall i \in J$. An event (i.e. subset of $\Omega$) $F$ is increasing in $J$ if its indicator is, and similarly for “decreasing in” and “determined by.”

Harris’ Inequality (for Bernoullis) says that, with expectations taken with respect to some product measure on $\Omega$, if $f$ and $g$ are increasing (i.e. in $I$), then $f$ and $g$ are positively correlated (that is, $E f g \geq E f E g$), while if one of $f, g$ is increasing and the other is decreasing then they are negatively
correlated. Though this will be used in the proof of Theorem 4.13, it is familiar enough that a formal statement seems unnecessary; but we do record the following, perhaps less familiar variant, for use in the crucial applications of Harris’ Inequality in the proof of Lemma 3.2 (see Section 10).

**Theorem 4.11.** Suppose \( \xi_i, i \in I \), are independent Bernoullis and \( f, g : \Omega \to \mathbb{R} \) with \( f \) decreasing in and determined by \( J \subseteq I \) and \( g \) increasing in \( J \). Then \( f \) and \( g \) are negatively correlated.

To get this from Harris’ Inequality as given above, set \( \xi = (\xi_i : i \in J) \) and \( \lambda = (\xi_i : i \in I \setminus J) \), write \( f(\xi) \) for the common value of \( f(\xi, \lambda) \) and set \( g_\lambda(\xi) = g(\xi, \lambda) \). Then

\[
E fg = E_\lambda E_\xi f(\xi) g_\lambda(\xi) \leq E_\lambda [E_\xi f(\xi) E_\xi g_\lambda(\xi)] = E_\xi f(\xi) E_\lambda E_\xi g_\lambda(\xi) = EfEg,
\]

where the inequality follows from Harris since, given \( \lambda \), the functions \( f(\xi) \) and \( g_\lambda(\xi) \) are decreasing and increasing (respectively).

### 4.3 Lower tails

We will make substantial use of the following fundamental lower tail bound of Svante Janson ([16] or [17, Theorem 2.14]), for which we need a little notation. Suppose \( A_1, \ldots, A_m \) are subsets of the finite set \( \Gamma \). For \( j \in [m] \), let \( I_j \) be the indicator of the event \( \{ \Gamma_p \supseteq A_j \} \), and set \( X = \sum I_j \), \( \mu = E X = \sum \mathbb{E} I_j \) and

\[
\Delta = \sum \sum \{ \mathbb{E} I_i I_j : A_i \cap A_j \neq \emptyset \}. \tag{29}
\]

(Note this includes diagonal terms.)

**Theorem 4.12.** With notation as above, for any \( t \in [0, \mu] \),

\[
\Pr(X \leq \mu - t) \leq \exp[-t^2/(2\Delta)].
\]

A surprising recent result of O. Riordan and L. Warnke [26] shows that Theorem 4.12 continues to hold when the events \( \{ \Gamma_p \supseteq A_i \} \) are replaced by members of some union- and intersection-closed family \( \mathcal{I} \) of events (in some probability space) satisfying

\[
\Pr(B \cap C) \geq \Pr(B) \Pr(C) \quad \forall B, C \in \mathcal{I}, \tag{30}
\]

and “\( A_i \cap A_j \neq \emptyset \)” in (29) is replaced by dependence of the corresponding events. (For Theorem 4.12—which really applies to general product
measures on \(2^\Gamma - \mathcal{I}\) is the family of increasing events and \((30)\) is Harris' Inequality.) One crucial ingredient in the proof of Lemma \(3.1\) (see Section \(9\)) will be an application of a further generalization, which we state only for the Harris context (but see Remark 2 below).

Consider some product probability measure on \(2^\Gamma\), and suppose \(B_{ij} \subseteq 2^\Gamma\) are increasing and \(B_i = \cup_j B_{ij}\). Write \((i,j) \sim (k,l)\) if \(B_{ij}\) and \(B_{kl}\) are dependent. (Note that, unlike \([26]\), we take \((i,j) \sim (i,j)\).) Let \(I_{ij}\) and \(I_i\) be the indicators of \(B_{ij}\) and \(B_i\) and set \(X = \sum I_i\),

\[
\mu = \sum_{i,j} EI_{ij},
\]

\[
\Theta = \sum_{i,j} \sum_k \Pr(B_{ij} \cap (\cup_l \{B_{kl} : (k,l) \sim (i,j)\})),
\]

\[
\Delta = \sum \sum \{EI_{ij} I_{kl} : (i,j) \sim (k,l)\} \quad (\geq \Theta)
\]

and

\[
\gamma = \sum_i \sum_{(j,k)} EI_{ij} I_{ik},
\]

with the inner sum over (unordered) pairs with \(j \neq k\).

Specializing the next statement to the case when there is just one \(j\) for each \(i\) yields the result of \([26]\).

**Theorem 4.13.** With notation as above, for any \(t \in [\gamma, \mu]\),

\[
\Pr(X \leq \mu - t) \leq \exp\left[-(t - \gamma)^2/(2\Theta)\right] \leq \exp\left[-(t - \gamma)^2/(2\Delta)\right].
\]

(31)

**Remarks.** 1. We could, of course, replace \(\mu\) in \((31)\) by \(\mathbf{E}X\), yielding a more natural, if slightly weaker statement. We will find the theorem useful when (roughly speaking) \(\Pr(B_i) \approx \sum_j \Pr(B_{ij})\); that is, when the probability of seeing at least two \(B_{ij}\)'s for a given \(i\) is small relative to the probability of seeing just one. In this case there is not much difference between \(\Theta\) and \(\Delta\), and in fact the main reason for bothering with \(\Theta\) here is that it is needed in the proof.

2. As noted above, Theorem 4.13 is actually valid in the same generality as \([26]\)—that is, with \(B_{ij}\)'s from some \(\mathcal{I}\) as in the paragraph containing \((30)\)—this extension requiring only formal changes in the proof (\((30)\) in place of Harris and use of a nice observation from \([26]\) to give the independence of \(I_{ij}\) and \(Z_{ij}\) below). As in Theorem 4.12 and \([26]\), the bound in the first line of \((31)\) may be replaced by the slightly smaller \(\exp[-\varphi((\gamma - t)/\mu)\mu^2/\Theta]\), where \(\varphi(x) = (1 + x) \log(1 + x) - x\).
Proof of Theorem 4.13. This is mostly as in [16] (again, see [17]) and [26], so we aim to be brief. (We are basically copying the proof of Theorem 2.14 on pp. 32-33 of [17], adding one nice idea ((32) below) from [26] and taking account of the extra terms corresponding to $\gamma$.)

Let $I_{ijk}$ and $J_{ijk}$ be the indicators of $\cup\{B_{kl} : (k,l) \sim (i,j)\}$ and $\cup\{B_{kl} : (k,l) \not\sim (i,j)\}$ (so $I_k \leq I_{ijk} + J_{ijk}$ for any $i,j,k$), and set $Y_{ij} = \sum_k I_{ijk}$ and $Z_{ij} = \sum_k J_{ijk}$. Note that $I_{ij}$ and $Z_{ij}$ are independent (since increasing events are independent—that is, Harris’ Inequality holds with equality—iff they depend on disjoint subsets of $\Gamma$).

Set $\Psi(s) = Ee^{-sX}$ ($s \geq 0$). The main point is to give a lower bound on

$$-\Psi'(s) = EXe^{-sX} = \sum I_i e^{-sX}.$$ 

Using $I_i \geq \sum_j I_{ij} - \sum_{\{j,k\}} I_{ij}I_{ik}$ and $X \leq Y_{ij} + Z_{ij}$ (for any $i,j$), we have

$$\sum I_i e^{-sX} \geq \sum_j E[I_{ij}e^{-sY_{ij}}e^{-sZ_{ij}}] - \sum_{\{j,k\}} E[I_{ij}I_{ik}e^{-sX}].$$

The key observation from [26] (adapted to our setting) is

$$E[I_{ij}e^{-sY_{ij}}e^{-sZ_{ij}}] = E[I_{ij}e^{-sZ_{ij}}] = E[e^{-sZ_{ij}}(1 - e^{-sY_{ij}})I_{ij}]$$

$$\geq E[I_{ij}e^{-sZ_{ij}}] - E[e^{-sZ_{ij}}(1 - e^{-sY_{ij}})I_{ij}]$$

$$= E[I_{ij}e^{-sY_{ij}}]E[e^{-sZ_{ij}}]$$

$$\geq E[I_{ij}e^{-sY_{ij}}]E[e^{-sX}],$$

(32)

where the first inequality follows from the independence of $I_{ij}$ and $Z_{ij}$ (which gives $E[I_{ij}e^{-sZ_{ij}}] = E[I_{ij}]E[e^{-sZ_{ij}}]$) together with Harris’ Inequality (and the observation that $f := e^{-sZ_{ij}}$ and $g := (1 - e^{-sY_{ij}})I_{ij}$ are, respectively, decreasing and increasing).

On the other hand, again using Harris, we have

$$E[I_{ij}I_{ik}e^{-sX}] \leq E[I_{ij}]E[e^{-sX}].$$

Combining the preceding observations gives

$$-(log \Psi(s))' = -\frac{\Psi'(s)}{\Psi(s)} \geq \sum_{i,j} E[I_{ij}e^{-sY_{ij}}] - \sum_i \sum_{\{j,k\}} E[I_{ij}I_{ik}]$$

$$\geq \mu e^{-s\Theta/\mu} - \gamma.$$  

(33)

The lower bound $\mu \exp[-s\Theta/\mu]$ on the first sum in (33) is obtained via two applications of Jensen’s Inequality as in the last four lines of [17, p. 32].
We then have
\[- \log \Psi(s) \geq \int_0^s (\mu e^{-u\Theta} - \mu - s\gamma) du = \frac{\mu^2}{\Theta} (1 - e^{-s\Theta}/\mu) - s\gamma,
\]
yielding (by Markov’s Inequality)
\[
\log \Pr(X \leq \mu - t) \leq \log \mathbb{E} e^{-sX} + s(\mu - t) \leq -\frac{\mu^2}{\Theta} (1 - e^{-s\Theta}/\mu) + s(\mu - (t - \gamma)),
\]
and applying this with \(s = -\log(1 - (t - \gamma)/\mu)/\Theta\) gives (31) (actually the slightly better bound mentioned in Remark 2 above; again, cf. [17, p. 33]).

4.4 A calculation

The following observation will be needed twice below (see the proofs of Lemmas 9.2 and 12.2), so we include it in these preliminaries.

**Lemma 4.14.** For each \(\xi > 0\) there is a \(\vartheta > 0\) so that the following is true (as usual, provided \(p\) is as in (2) with a large enough \(C\)). Let \(R \subseteq \binom{V}{2}\) satisfy
\[
\Delta_R < \vartheta np/\log n
\]
and let \(H\) consist of all sets of the form \(K(xy, Z) := (\{x, y\} \cup Z) \setminus \{xy\}\) with \(xy \in R\) and \(Z \in \binom{V \setminus \{x, y\}}{r - 2}\). Let \(I_K\) be the indicator of \(\{K \subseteq G\}\) and
\[
\Delta = \sum \sum \{E I_K I_L : K, L \in H, K \cap L \neq \emptyset\}. \tag{35}
\]
Then
\[
\Delta < \xi |R| \Lambda_r(n, p)^2/\log n. \tag{36}
\]

**Remark.** The \(H\)'s in our applications will be subsets of the one here, which, of course, only shrinks \(\Delta\).

**Proof.** For \(K = K(xy, Z)\) as in the lemma, let \(V(K) = \{x, y\} \cup Z\) and \(e_K = \{x, y\}\). We organize \(G := \{(K, L) : K, L \in H, K \cap L \neq \emptyset\}\) as follows. For \((K, L) \in G\) set \(a(K, L) = |e_K \cap e_L| (\in \{0, 1, 2\})\) and \(b(K, L) = |V(K) \cap V(L)| (\in \{2, \ldots, r\})\). Notice that if \(a(K, L) = 2\) (that is, if \(e_K = e_L\)), then \(K \cap L \neq \emptyset\) implies \(b(K, L) \geq 3\).

Let
\[
N(a, b) = |\{(K, L) \in G : a(K, L) = a, b(K, L) = b\}|.
\]
Then, since $|V(K) \cup V(L)| = 2r - b(K, L)$, there is a fixe $B = B_r$ (e.g., crudely, $B = r!$) such that

$$N(0, b) < B|R|^2 n^{2r-4-b},$$

$$N(1, b) < B \sum_x d^2_R(x) n^{2r-3-b} \leq B|R| \Delta_R n^{2r-3-b}$$

and

$$N(2, b) < |R| n^{2r-2-b}.$$

On the other hand,

$$EI_K I_L = p^{r^2-r-2-|K \cap L|}$$

and

$$|K \cap L| \begin{cases} = \frac{b(K, L)}{2} - 1 & \text{if } a(K, L) = 2, \\ \leq \frac{b(K, L)}{2} & \text{otherwise}. \end{cases}$$

Combining these observations we have

$$\Delta \leq \sum_{b=3}^r N(2, b) p^{r^2-r-1-\binom{b}{2}} + B \sum_{a=0}^{1} \sum_{b=2}^r N(a, b) p^{r^2-r-2-\binom{b}{2}}$$

$$< |R|^n^{2r-4} p^{r^2-r-2} \left[ \sum_{b=3}^r n^{2-b} p^{1-\binom{b}{2}} + B(|R| + \Delta_R n) \sum_{b=2}^r n^{-b} p^{\binom{b}{2}} \right],$$

so for (36) would like to say that the expression in square brackets is less than $\xi \log^{-1} n$, which a little checking—using (34) with a small enough $\vartheta$ (something like $\xi/B$ will do) and our lower bound on $p$—shows to be true. (The largest contributions are (i) $n^{2-r} p^{1-\binom{b}{2}} = \Lambda_r(n, p)^{-1}$ corresponding to $b = r$ in the first sum, and (ii) $B\Delta_R n^{-1} p^{-1}$, which is the $(b = 2)$-term from the second sum multiplied by $B \Delta_R n$.)

4.5 Miscellaneous

In closing these preliminaries we mention two last, easy points. First, we recall just one detail (borrowed from [19, Lemma 4.1]), of the connection between $G_{n,p}$ and $G_{n,M}$:

**Lemma 4.15.** Let $n^\Omega(1) = M \leq \binom{n}{2}$ be an integer and $p = M/\binom{n}{2}$. If an event $\mathcal{E}$ holds with probability at least $1 - \varepsilon$ in $G_{n,p}$, then it holds with probability at least $1 - O(n \varepsilon)$ in $G_{n,M}$. 

(The extra $n$ in the conclusion won’t be a problem, since our exceptional probabilities will be much smaller than $1/n$. We will also want something in the other direction, but defer the trivial statement until needed; see [147].)

We will also find a couple of uses for the following observation [38, 24], in which we call a coloring equitable if the sizes of the color classes differ by at most one.

**Proposition 4.16.** For any $m \geq \Delta + 1$, the edges of any simple graph of maximum degree at most $\Delta$ can be equitably colored with $m$ colors.

5 Main points for the proof of Lemma 3.1

Here we derive Lemma 3.1 from the following three assertions, which are proved in Sections 7-9. We use $\tau(A_1, \ldots, A_{r-1})$ for the number of choices of distinct $x_1, \ldots, x_{r-1}$ with $x_i \in A_i$ and all pairs from $\{x_1, \ldots, x_{r-1}\}$ belonging to $G$, and also write $\tau(A, B, C)$ for $\tau(A, B, C, \ldots, C)$ (with $r-3$ copies of $C$).

**Lemma 5.1.** For fixed $\theta, \varrho > 0$, w.h.p.

$$\tau(S_1, \ldots, S_{r-1}) > (1 - \varrho)|S_1| \cdots |S_{r-1}| p^{(r-1)/2}$$

whenever $v \in V$ and $S_1, \ldots, S_{r-1}$ are disjoint subsets of $N_v$ with each of $|S_2|, \ldots, |S_{r-1}|$ at least $\theta np$ and

$$|S_1| > \varrho^{-2} 6r \log r \cdot 2^r \max \left\{ \frac{1}{\theta p}, \frac{np}{\theta r - 2 \Lambda_r(n, p)} \right\}.$$  

**Remarks.** For the proof of Lemma 5.1 we could replace the lower bound in (38) by $\theta np$, but the present stronger version (with the weaker lower bound on $|S_1|$) will be needed in the proof of Lemma 3.2 The constants in (38) (i.e. the $\theta$’s and the expression preceding the “max”) are unimportant.

**Lemma 5.2.** For fixed $\pi \geq \varepsilon > 0$, w.h.p.

$$\tau(S, T, R) \leq 8\pi |T| \Lambda_r(n, p)$$

whenever $v \in V$; $S$ and $T$ are disjoint subsets of $N_v$ with $|T| < \varepsilon np$ and $|S| < \pi |T| / \varepsilon$; and $R \subseteq N_v \setminus (S \cup T)$.

(Of course there is no change in content if we say “$R = N_v \setminus (S \cup T)$,” but the stated version will be convenient.)
Lemma 5.3. For each \( \pi > 0 \) there is an \( \varepsilon > 0 \) such that w.h.p.

\[
\kappa(S,T) < \pi |S| \Lambda_r(n,p)
\]  

whenever \( S \subseteq \left( \binom{V}{2} \right) \) and \( T \subseteq G \) satisfy \( \Delta_S \leq 2np \), \( V(S) \cap V(T) = \emptyset \) and \( |T| \leq |S| < \varepsilon n^2 p \).

Remarks. Each of Lemmas 5.2 and 5.3 bounds the quantity in question by something like its natural value; namely, the r.h.s. of (39) is, up to scalar, the natural value of the l.h.s. when \( |T| \approx \varepsilon np \) and \( |S| \approx \pi np \) (and \( R = N_v \setminus (S \cup T) \)), while (40) says that for a small \( T \subseteq G \), \( \kappa(S,T) \) can account for only a small fraction of \( \kappa(S) \approx |S| \Lambda_r(n,p) \). (It is easy to see that it is not enough to bound \( T \) without reference to \( |S| \).)

The proof of Lemma 5.3 turns out to be much less straightforward than one might expect, and a small puzzle may be worth mentioning. The bound on \( \Delta_S \), which will eventually come for free because we will have \( S \subseteq G \), happens to be just what’s needed for the current proof of the lemma, but we don’t know that it is really necessary. When \( r = 4 \) the lemma can fail with \( \Delta_S \) as small as \( n^2 p^3 \) (note that by (2), \( \Delta_S \approx n^2 p^3 \) requires \( r \leq 4 \) since \( \Delta_S < n \)): for some \( x \in V \), take \( T = G[N_G(x)] \) and let \( S \) consist of \( n^2 p^3 \) pairs containing \( x \) and avoiding \( N_G(x) \); then \( |S| \ll n^2 p \) and (typically) \( |T| \approx n^2 p^3/2 < |S| \), while \( \kappa(S,T) \approx |S||T|p^2 \approx |S| \Lambda_r(n,p)/2 \). But we don’t know whether \( \Delta_S \ll n^2 p^3 \) suffices when \( r = 4 \) or whether any bound (on \( \Delta_S \)) is needed for larger \( r \). It would be interesting to understand what’s going on here, and so see whether this seemingly innocuous lemma can be proved less circuitously.

We will also need the fact, contained in Proposition 4.4(a), that for some fixed \( M \), w.h.p.

\[
|G[X]| < \max\{|X|^2p, M|X| \log n\} \quad \forall X \subseteq V.
\]  

(41)

Proof of Lemma 3.1. Recalling \( \gamma \) from the definition of \( Q_G(\Pi) \) preceding Lemma 3.1, choose constants \( \rho, \zeta, \eta > 0 \) with \( \gamma^{r-2} \gg \rho \gg \zeta \) and \( \eta \ll \zeta^2 \) small enough so that the conclusion of Lemma 5.3 holds when \( \pi = \zeta \) and \( \varepsilon = (r-1)\eta \). We assert that Lemma 3.1 holds with this value of \( \eta \).

What we actually show is that the “w.h.p.” statement in Lemma 3.1 is true provided (16), (41) and the conclusions of Lemmas 5.1-5.3 hold for suitable values of the parameters. To say this properly, define properties:

(A) (37) holds for \( \theta = (2\rho/(1-\rho))^{1/(r-2)} \) and all \( v, S_1, \ldots, S_{r-1} \) as in Lemma 5.1.
(B) \((39)\) holds whenever \((\varepsilon, \pi) \in \{(\zeta, \varrho/(8(r - 2))), (\theta, (\gamma - 2\varrho)/(8(r - 2)))\}\) (with \(\theta\) as in \((A)\)) and \(v, S, T, R\) are as in Lemma \(5.2\).

(C) \((40)\) holds for \((\pi, \varepsilon) = (\zeta, (r - 1)\eta)\) and \(S, T\) as in Lemma \(5.3\).

(D) \((41)\) holds (for some fixed \(M\), whose value will be irrelevant here).

By Lemmas \(5.1-5.3\) and Propositions \(4.3\) and \(4.4(a)\), it is enough to show that the conclusion of Lemma \(3.1\) holds whenever \((16)\) (we just need degrees bounded by \(2np\)) and \((A)-(D)\) are satisfied, which we now assume.

Fix \(F\) and \(\Pi\) as in Lemma \(3.1\) and set \(I = F[A_1]\) and \(L = \Pi_{C \setminus F}\). We should show (provided \(I \neq \emptyset\))

\[|L| > (r - 1)|I|,\]  

which with \((43)\) gives the desired contradiction.

Set \(X = \{x : d_I(x) > \zeta np\}\) \((\subseteq A_1)\) and \(Y = A_1 \setminus X\). We begin with the observation that not many edges of \(I\) lie in \(X\): noting that \(|I| > |X|\zeta np/2\) and (consequently) \(|X| < 2(\eta/\zeta)n\), and using \((D)\), we have

\[|G[X]| < 2\frac{\zeta}{\zeta} \max\{\frac{|X|}{n}, \frac{M \log n}{np}\}|I| < 4\eta\zeta^{-2}|I|.\]  

We now use \(K\) for members of \(K(I, A_2, \ldots, A_{r-1})\) (recall this was defined near the end of Section \(2\)). Say such a \(K\) is covered at \(v \in A_1\) if it contains an edge of \(L(v) := \{e \in L : v \in e\}\) (so in particular contains \(v\)), is covered at \(W \subseteq A_1\) if it is covered at some \(v \in W\), and is covered off \(A_1\) if it contains an edge of \(L \cap \nabla(A_2, \ldots, A_{r-1})\). Let \(I_1 = \bigcup_{y \in Y} I_1(y)\), where (for \(y \in Y\))

\[I_1(y) = \{yw \in I : \text{at least } \varrho\Delta_r(n, p) \text{ of the } K\text{'s on } yw \text{ are covered at } y\};\]

\[I_2 = \{e \in I : \text{at least } \varrho\Delta_r(n, p) \text{ of the } K\text{'s on } e \text{ are covered off } A_1\};\]

and \(I_3 = I \setminus (G[X] \cup I_1 \cup I_2)\). Note that each \(e \in I_3\) is of the form \(xy\) with \(x \in X\), \(y \in Y\) and at least \((\gamma - 2\varrho)\) \(K\text{'s from } K(e, A_2, \ldots, A_{r-1})\) covered at \(x\).

In what follows we show that each of \(I_1\), \(I_2\), \(I_3\) is small compared to \(I\), which with \((43)\) gives the desired contradiction. We first assert that

\[|I_1| \leq 8(r - 1)(r - 2)\zeta |I|/\varrho,\]  

which follows from

\[|L(y)| \geq \varrho |I_1(y)|/(8(r - 2)\zeta) \quad \forall y \in Y\]  

(45)
and our assumption that (42) fails. For (45), notice that if \(|N_L(y) \cap A_i| < \varrho |I_1(y)|/(8(r-2)\zeta)|\) for \(i = 2, \ldots, r-1\), then (B), applied, for each \(i \in [2, r-1]\), with \((\varepsilon, \pi) = (\zeta, \varrho/(8(r-2)))\), \(v = y\) and

\[
(S, T, R) = (N_L(y) \cap A_i, N_I(y), N_y \setminus (A_1 \cup A_i))
\]

(note \(|I_1(y)| < \zeta np\) bounds the number of \(K\)'s from \(K(I_1(y), A_2, \ldots, A_{r-1})\) that are covered at \(y\) by \(\varrho |I_1(y)|\Lambda_r(n, p)\), contradicting the assumption that each \(e \in I_1(y)\) lies in more than \(\varrho \Lambda_r(n, p)\) such \(K\)'s.

We next show that \(I_2\) is small. Set \(J = L \cap \nabla(A_2, \ldots, A_{r-1})\). By the definition of \(I_2\) we have \(\kappa(I_2, J) \geq \varrho |I_2|\Lambda_r(n, p)\). But, as we will show in a moment, (C) together with \(|J| \leq |L| \leq (r-1)|I|\) gives

\[
\kappa(I_2, J) \leq \kappa(I, J) < (r-1)\zeta |I|\Lambda_r(n, p), \quad (46)
\]

so that

\[
|I_2| < (r-1)\zeta |I|/\varrho. \quad (47)
\]

For (46) we use (C) with \((S, T) = (I_2, J)\) if \(|J| \leq |I_2|\) and \((S, T) = (J, I_2)\) otherwise. In either case we have \(\Delta_S \leq 2np\) (by (16)) and \(|S| \leq (r-1)|I| < (r-1)\eta n^2p\), so that (C) gives \(\kappa(S, T) < \zeta |S|\Lambda_r(n, p) \leq (r-1)\zeta |I|\Lambda_r(n, p)\).

Finally, we show \(I_3\) is small. Set \(\theta = (2\varrho/(1-\varrho))^{1/(r-2)}\) (as in (A)). We first observe that

\[
|I_3(x)| \leq \theta np \quad \forall x \in X. \quad (48)
\]

To see this, suppose instead that \(|I_3(x)| = d > \theta np\) for some \(x \in X\). By (8) we have

\[
d \leq |N_F(x) \cap A_i| \leq \min\{|N_F(x) \cap A_i| : 2 \leq i \leq r-1\};
\]

so according to (A)—note \(\theta np\) is larger than the bound in (8)—there are at least \((1-\varrho)d^{r-1}p^{(r-1)/z}\) \(K\)'s containing \(x\) and one vertex from each of \(N_{I_3}(x), N_F(x) \cap A_2, \ldots, N_F(x) \cap A_{r-1}\)—that is,

\[
\kappa(x, N_{I_3}(x), N_F(x) \cap A_2, \ldots, N_F(x) \cap A_{r-1}) \geq (1-\varrho)d^{r-1}p^{(r-1)/z} \quad (49)
\]

—each of which must be covered either at \(Y\) or off \(A_1\). But since an edge of \(I_3(x)\) is contained in at most \(2\varrho \Lambda_r(n, p)\) \(K\)'s that are covered in one of these ways, the l.h.s. of (49) is at most \(2d\varrho \Lambda_r(n, p)\), which is less than the r.h.s. Thus (48) does hold.
We may now proceed as we did in bounding \(|I_1|\): for \(x \in X\), each \(e \in I_3(x)\) lies in at least \((\gamma - 2\theta)A_r(n, p)\) \(K\)'s from \(K(e, A_2, \ldots, A_{r-1})\) that are covered at \(x\), which by (B) (with \((\varepsilon, \pi) = (\theta, (\gamma - 2\theta)/(8(r - 2)))\) implies

\[
|L(x)| \geq \max\{|N_L(x) \cap A_i| : i \in [2, r - 1]\} \geq \frac{\gamma - 2\theta}{8(r - 2)p} |I_3(x)| \quad \forall x \in X
\]

and (again using failure of (12))

\[
|I_3| \leq 8(r - 1)(r - 2)\theta|I|/(|\gamma - 2\theta|) \quad (50)
\]

(a small multiple of \(|I|\) because of our choice of \(\theta\).)

Finally, combining (13), (14), (47) and (50), we have the contradiction

\[
|I| \leq |G[X]| + |I_1| + |I_2| + |I_3| < |I|.
\]

6 Main points for the proof of Lemma 3.2

Here we just state the two main assertions underlying Lemma 3.2 and show that they suffice. The assertions themselves are proved in Sections 11 and 12, with both arguments rooted in the observations of Section 10.

Let

\[
a_r = \frac{r - 4}{2(r - 3)}, \quad b_r = \frac{r(r - 3)}{2(r - 1)^2} \quad \text{and} \quad c_r = (a_r + b_r)/2. \quad (51)
\]

We can now, finally, say something about \(\gamma\) (the parameter in the definition of a bad pair in the passage preceding Lemma 3.1). Here it is not necessary (or desirable) to specify an actual value; we just stipulate that

\[
0 < \gamma < \frac{1}{2} \left( \frac{c_r - a_r}{4r^2 + 6} \right)^{r - 2} \quad (52).
\]

For \(x \in V\) and disjoint \(A_1, \ldots, A_{r-1} \subseteq V\), set

\[
D(x; A_1, \ldots, A_{r-1}) = \sum \{d_{A_i}(x)d_{A_j}(x) : 1 \leq i < j \leq r - 1\}. \quad (53)
\]

For a cut \(\Pi = (A_1, \ldots, A_{r-1})\), we also write \(D_{\Pi}(x)\) for \(D(x; A_1, \ldots, A_{r-1})\). We say \(x\) is bad for \(\Pi\) if \(x \in A_1\) and \(D_{\Pi}(x) < c_r n^2 p^2\).

Remark. When \(d(x) \approx np\) and \(\Pi = (A_1, \ldots, A_{r-1})\), \(b_r n^2 p^2\) is essentially the minimum possible value of \(D_{\Pi}(x)\) if at least \(r - 2\) of the \(d_{A_i}(x)\)'s are at least \(np/(r - 1)\), and \(a_r n^2 p^2\) is (essentially) the maximum possible value if at least two of the \(d_{A_i}(x)\)'s are zero. While the inequality \(a_r < b_r\) is easily verified,
we don’t see any intuitive reason why it should be true; yet our proof (of Theorem 1.2) collapses if it is not.

We will use \( c_r > a_r \) in the present section (see (61)) and \( c_r < b_r \) twice in Section 11 (see the proof of Proposition 11.1 and 131).

**Lemma 6.1.** There is a fixed \( \nu > 0 \) such that w.h.p.: for every \( t > 0 \), every balanced cut with at least \( t \) bad vertices has defect at least \( \nu tn^{3/2}p^2 \).

Set \( \zeta = (2\gamma)^{1/(r-2)} \)—thus (52) is

\[
0 < \zeta < \frac{c_r - a_r}{4r^2 + 6}
\]  

—and

\[
\Sigma = 24r \log r \cdot 2^r \max\{ (\zeta p)^{-1}, np(\zeta^{-2} \Lambda_r(n,p))^{-1} \}.
\]

\[
= \begin{cases} 
\Theta(np/\Lambda_r(n,p)) & \text{if } p < n^{-2/(r+2)}, \\
\Theta(p^{-1}) & \text{otherwise.}
\end{cases}
\]

**Lemma 6.2.** W.h.p. \( \text{def}_G(\Pi) \geq 2r^2|Q| \) whenever \( \Pi = (A_1, \ldots, A_{r-1}) \) is a balanced cut and \( Q \subseteq Q_G(\Pi) \) satisfies

\[ d_Q(x) < \Sigma \ \forall x. \]

Remarks. As in Lemma 3.2 the factor \( 2r^2 \) in the defect bound is what’s needed below, but could actually be any constant. The relatively severe constraint on \( \gamma \) in (52) is needed for the derivation of Lemma 3.2 not for Lemma 6.2 itself.

**Proof of Lemma 6.2.** We show that the conclusion of Lemma 3.2 holds whenever we have: the conclusion of Lemma 5.1 for \( \theta = \zeta \) and \( \varrho = 1/2 \); the conclusions of Lemmas 6.1 and 6.2; and

\[ d(x) < (1 + o(1))np \ \forall x. \]

This is enough since (by the lemmas just mentioned and Proposition 4.3) these assumptions hold w.h.p.

Suppose we have the stated conditions and \( \Pi, Q \) are as in Lemma 3.2. We may of course assume \( Q \neq \emptyset \). We first show that, for any \( x \in A_1 \),

\[ \min\{ d_{A_i}(x) : 2 \leq i \leq r - 1 \} > \zeta np \Rightarrow d_Q(x) < \Sigma \]  

and

\[ d_Q(x) \leq \zeta np. \]
Proof. Notice that $\Sigma$ is simply the r.h.s. of (38) for $\theta = \zeta$ and $\rho = 1/2$. So if (58) fails—that is, if $d_{A_i}(x) > \zeta np$ for $i \in \{2, \ldots, r-1\}$ and $d_Q(x) \geq \Sigma$—then the conclusion of Lemma 5.1 applied with $v = x$, $S_1 = N_Q(x)$ and $S_i = N_{A_i}(x)$ for $i \in \{2, r-1\}$, gives

$$\sum_{y \in N_Q(x)} \kappa(xy, A_2, \ldots, A_{r-1}) = \tau(N_Q(x), N_{A_2}(x), \ldots, N_{A_{r-1}}(x)) \geq (1/2)d_Q(x)(\zeta np)^{r-2}p^{(r-1)/2} = d_Q(x)\gamma \Lambda_r(n, p),$$

a contradiction since $xy \in Q$ implies $\kappa(xy, A_2, \ldots, A_{r-1}) < \gamma \Lambda_r(n, p)$. This gives (58), and (59) follows easily: if (59) fails then (9) implies $\min\{d_{A_i}(x) : 2 \leq i \leq r-1\} > \zeta np$, and then (58) gives $d_Q(x) < \Sigma$; but $\Sigma < \zeta np$, so we again have a contradiction.

Let $X$ be the set of vertices that are bad for $\Pi$ (so $X \subseteq A_1$) and set

$$Z_i = \{x \in A_1 : d_{A_i}(x) \leq \zeta np\} \quad 2 \leq i \leq r-1$$

and $Y = A_1 \setminus (X \cup Z_2 \cup \cdots \cup Z_{r-1})$.

Let $Q_v$ be the set of pairs from $Q$ meeting $X$. Then $|Q_v| \leq |X|\zeta np$ (by (59)), which with the conclusion of Lemma 6.1 gives

$$\text{def}_G(\Pi) \geq \nu|X|n^{3/2}p^2 \geq (\nu/\zeta)n^{1/2}p|Q_v|.$$ 

So we may assume $|Q_v| < r\zeta|Q|/(\nu n^{1/2}p) \ll |Q|$. We may further assume that $|Q[Y]| < (2r)^{-1}|Q|$; otherwise—in view of (58), which implies $d_{Q[Y]}(x) < \Sigma \forall x$ (note $d_{Q[Y]}(x) = 0$ if $x \notin Y$)—we may apply the conclusion of Lemma 6.2 to $Q[Y]$ to obtain

$$\text{def}_G(\Pi) \geq 2r^2|Q[Y]| \geq r|Q|.$$

We may thus assume (w.l.o.g.) that at least (say) $|Q|/r$ of the edges of $Q$ meet $Z := Z_2 \setminus X$, which with (59) gives

$$|Q| \leq r|Z|\zeta np. \quad (60)$$

But we will show that if this is true then we can obtain a cut significantly larger than $\Pi$ by moving an appropriate subset of $Z$ to $A_2$. The main point
here is that vertices of $Z$ must have many neighbors in $A_1$. Set $\lambda = \lambda_r = (c_r - a_r - 2\zeta)$. We assert that

$$d_{A_1}(x) > \lambda np \quad \forall x \in Z.$$  \hspace{1cm} (61)

Proof. For $x \in Z$ we have $D_{\Pi}(x) \geq c_r n^2 p^2$ (since $x \not\in X$) and (a little crudely, using (57) to bound $d(x)$ and $x \in Z$ to bound $d(A_2(x))$)

$$D_{\Pi}(x) \leq \left( (d_{A_1}(x) + d_{A_2}(x))d(x) + \left( \frac{r-3}{r-3} \right) \left( \frac{d(x)}{r-3} \right)^2 \right)
\leq (1 + o(1))[(d_{A_1}(x) + \zeta np)np + a_r n^2 p^2]
\leq \left[ \frac{d_{A_1}(x)}{np} + a_r + 2\zeta \right] n^2 p^2;$$

so we have (61). \hfill \blacksquare

Now choose $W \subseteq Z$ so that $\nabla_G(W, A_1 \setminus W)$ contains at least half the edges of $G[A_1]$ meeting $Z$. (This is true on average for $W$ chosen uniformly from the subsets of $Z$, so such a choice is possible.) Let

$$\Pi' = (A_1 \setminus W, A_2 \cup W, A_3, \ldots, A_{r-1}).$$

Then

$$|\Pi'| - |\Pi| = |\nabla_G(W, A_1 \setminus W)| - |\nabla_G(W, A_2)| \geq \sum_{x \in Z} (d_{A_1}(x)/4 - d_{A_2}(x))$$
$$\geq |Z|(\lambda/4 - \zeta) np.$$  \hspace{1cm} (62)

Thus $\text{def}_G(\Pi)$ is at least the r.h.s. of (62), and according to (60) (and (54)) this is larger than $r|Q|$. This completes the proof of Lemma 3.2

Remark. Constants aside, the value of $\Sigma$ in (55) cannot be increased without invalidating the proof of Lemma 6.2 (see the bound on $\Delta$ in the proof of Lemma 12.2), while Lemma 5.1 (not just its proof) is false for smaller values of the bound in (38). But the above proof of Lemma 3.2 uses Lemma 5.1 to bound degrees in $Y$ by an instance, $\Sigma$, of the latter bound, supporting application of Lemma 6.2 so the fact that the $\Sigma$ needed for this application is not less than what’s affordable in (38) is crucial, and it would be nice to somehow understand that this is not just a lucky accident.
7 Proof of Lemma 5.1

For given disjoint \( S_1, \ldots, S_{r-1} \subseteq V \) with \( |S_i| = s_i \), let
\[
\mathcal{B}(S_1, \ldots, S_{r-1}) = \{ \tau(S_1, \ldots, S_{r-1}) < (1 - \rho) s_1 \cdots s_{r-1} p^{(r-1)/2} \}.
\]
We will show that for any \( S_1, \ldots, S_{r-1} \) with sizes as in Lemma 5.1
\[
\Pr(\mathcal{B}(S_1, \ldots, S_{r-1})) < \exp[-(3 \log r) np], \quad (63)
\]
but first show that this does give the lemma. By Proposition 4.3 it is enough to bound the probability that the conclusion of Lemma 5.1 fails at some \( v \) with \( d(v) \leq 2np \); this probability is at most
\[
\sum_{W} \Pr(N_v = W) \sum_{S_1, \ldots, S_{r-1}} \Pr(\mathcal{B}(S_1, \ldots, S_{r-1})), \quad (64)
\]
with the first sum over \( W \subseteq V \setminus \{v\} \) of size at most \( 2np \) and the second over disjoint \( S_1, \ldots, S_{r-1} \subseteq W \) obeying the size requirements of the lemma. But according to (63) the expression in (64) is less than
\[
\binom{n}{2np}(r-1)^{2np} \exp[-(3 \log r) np] < \exp[-(\log r) np] = o(1/n)
\]
(which is needed since we multiply by \( n \) to account for the choice of \( v \)).

Proof of (63). This is a straightforward application of Theorem 4.12, the notation of which we now follow. With \( \Gamma = \nabla(S_1, \ldots, S_{r-1}) \) and \( A_1, \ldots, A_m \) the (edge sets of) copies of \( K_{r-1} \) in \( \Gamma \), we have
\[
\mu = s_1 \cdots s_{r-1} p^{(r-1)/2} \quad (65)
\]
and
\[
\Delta < \sum_{i=2}^{r-1} \sum_{I \in \binom{[r-1] \setminus i}{i}} \prod_{j \in I} s_j^{i-2} \prod_{j \in [r-1] \setminus I} s_j^{-1} \cdot p^{(r-1)/2} - \binom{i}{2}.
\]
For (63) (via Theorem 4.12) we need \((\rho \mu)^2 > 3 \log r \cdot np \cdot 2 \Delta\), or (equivalently)
\[
\rho^2 > (6 \log r) np \sum_{i=2}^{r-1} \sum_{I \in \binom{[r-1] \setminus i}{i}} \prod_{j \in I} s_j^{-1} \cdot p^{-\binom{i}{2}}. \quad (66)
\]
Setting $s_1^* = \min\{s_1, \theta np\}$ and using $s_i > \theta np$, we find that the r.h.s. of (66) is less than
\[
(6 \log r)2^{r} np \sum_{i=2}^{r-1} \left[ s_1^* (\theta np)^{i-1} p^{(i+1)/2} \right]^{-1}
= (6 \log r)2^{r} \sum_{i=2}^{r-1} (s_1^*)^{-1} \theta n^{2-i} p^{-(i+1)/2},
\]
so that the inequality holds provided
\[
s_1^* > \theta (6 \log r)2^{r} \sum_{i=2}^{r-1} \theta n^{2-i} p^{-(i+1)/2}.
\]
It is also easy to see (e.g. by considering ratios of consecutive terms) that the largest summand in (67) is either the first or the last; so, again without being too careful, we may (upper) bound the entire r.h.s. of (67) by the expression in (38) (which gives the lemma since, as already noted, this expression is less than $\theta np$).

8 Proof of Lemma 5.2

(A reminder: rooted graphs and some of the other notions and notation used here were introduced in Section 4.1.)

Set $\mathcal{I} = \{(i,j) : 1 \leq i < j \leq r-1\}$ and write “$\prec$” for “reverse lexicographic” order on $\mathcal{I}$; that is, $(i,j) \prec (k,l)$ if either $j < l$ or $j = l$ and $i < k$. For $(i,j) \in \mathcal{I}$, write $\varsigma(i,j)$ for the index of $(i,j)$ under “$\prec$”; for example $\varsigma(2,3) = 3$ and $\varsigma(r-2, r-1) = \binom{r-2}{2}$.

For $(i,j) \in \mathcal{I}$, let $H_{ij}$ be the rooted graph with vertex set $\{u_0, u_i, \ldots, u_j\}$, edge set
\[
\{u_0 u_k : k \in [j]\} \cup \{u_k u_l : (k,l) \in \mathcal{I}, (k,l) \prec (i,j)\}
\]
(so all edges except those joining $j$ to $[i,j-1]$) and root sequence $(u_0, u_i, u_j)$.

Set
\[
S_{ij} = n^{j-1} p^{c(i,j)+j-1}
\]
and notice that
\[
S_{ij} \geq n^{j-1} p^{(i+1)/2} = (np^{(j+2)/2})^{j-1} = \begin{cases} \Lambda_r(n,p) & \text{if } j = r - 1, \\ n^{\Omega(1)} & \text{otherwise}. \end{cases}
\]

We need one auxiliary result:
Proposition 8.1. For any \( \vartheta > 0 \), w.h.p.

\[
N(H_{ij}, G; v, x, y) < \vartheta S_{ij}/\log n
\]

for all \((i, j) \in \mathcal{I}\) and \(v, x, y \in V\).

Proof. This is an application of Proposition 4.8, in which, having fixed \((i, j) \in \mathcal{I}\), we set \(\theta = \min\{\vartheta, (r - 3)/(r + 1)\} \), \(H = H_{ij}\) (so \(s = 3\)), \(S = S_{ij}\) and \((x_1, x_2, x_3) = (v, x, y)\).

From (69) we have \(S > K \log n\) for any fixed \(K\) (and large enough \(C\)), while, since \((v_H, e_H) = (j - 2, \varsigma(i, j) + j - 3)\), the combination of (68) and (19) (which said \(E_0 \sim n^{v_H} p^{e_H}\)) gives

\[
E_0 < (1 + o(1))(np^2)^{-1}S < n^{-\theta}S.
\]

Thus Proposition 8.1 will follow from \(H\) is balanced. (70)

(As will appear below, \(H\) is strictly balanced unless \((i, j) = (2, 4)\).)

Proof. This is a routine verification and we aim to be brief. It is enough to show

\[
\rho(H) \geq \rho(H[k]) \quad \forall 1 \leq k \leq j - 2,
\]

where we write \(H[k]\) for the subgraph of \(H\) induced by \(\{0, \ldots, k\} \cup \{i, j\}\) (so \(H[j - 1] = H\)) and exclude the case \(k = i = 1\) since it gives \(v(H[k]) = 0\).

One easily checks that

\[
v(H[k]) = \begin{cases} 
k & \text{if } 1 \leq k < i, 
\end{cases}
\]

\[
e(H[k]) = \begin{cases} 
\binom{k+1}{2} + 2k & \text{if } 1 \leq k < i, 
\binom{k+1}{2} + i - 2 & \text{if } i \leq k < j,
\end{cases}
\]

and (consequently)

\[
\rho(H[k]) = \begin{cases} 
\frac{k+5}{2(k-1)} & \text{if } 1 \leq k < i, 
\end{cases}
\]

\[
\text{or } f_i(k) & \text{if } i \leq k < j.
\]

It follows (with a tiny calculation for the third assertion) that (71) holds: strictly if \(k \leq i - 2\); with equality if \(k = i - 1\) or \(k = i = 2\); and strictly otherwise (so if \(k \geq i\) and \((k, i) \neq (2, 2)\)). This completes the proofs of (71) and (70), and also shows that we have strict inequality in the former unless \(k = i = 2\) and \(j = 4\), so, as mentioned earlier, strict balance in the latter unless \((i, j) = (2, 4)\).
Proof of Lemma 5.2. To somewhat lighten the notational load, set
\[ \alpha_r = \frac{2^r - 3}{(r-3)!}, \ \beta_r = \frac{(r-3)!}{(r-3)!!}, \ \text{and} \ \gamma_r = \alpha_r \beta_r = \left( \frac{2}{r-3} \right)^{r-3}. \]

In what follows, we assume that \( v \in V \) and that \( S, T \) are disjoint subsets of \( V \setminus \{v\} \) satisfying the size requirements of Lemma 5.2. Of course we may also assume \( T \neq \emptyset \), since (39) is vacuous if \( T = \emptyset \). Let
\[ \mathcal{T}(v, S, T) = \{ \tau(S, T, N_v \setminus (S \cup T)) \geq 4\alpha_r \pi |T| \Lambda_r(n, p) \} \] (72)
and
\[ \mathcal{R} = \bigcup \left( \{S, T \subseteq N_v\} \wedge \mathcal{T}(v, S, T) \right), \]
with the union over \( v, S, T \) as above. Notice that \( \alpha_r \leq 2 \) for all \( r \), so that the expression \( 4\alpha_r \pi |T| \Lambda_r(n, p) \) in (72) is at most the bound in (39); thus to prove Lemma 5.2 it is enough to show
\[ \Pr(\mathcal{R}) = o(1). \] (73)

Set \( \vartheta = \frac{2^{r-2}}{80} \left( \begin{array}{c} r-1 \\ 2 \end{array} \right)^{-2} \) and let
\[ \mathcal{Q} = \{ N(H_{ij}, G; v, x, y) < \vartheta S_{ij} / \log n \ \forall i, j, v, x, y \} \wedge \{ d(v) < 2np \ \forall v \}. \]
According to Proposition 8.1 and Lemma 4.3 we have
\[ \Pr(\mathcal{Q}) = o(1). \] (74)

Now
\[ \mathcal{R} \subseteq \mathcal{Q} \bigcup \bigcup_{v, W, S, T} \{ N_v = W \} \wedge \mathcal{T}(v, S, T) \wedge \mathcal{Q} \] (75)
and
\[ \Pr(\mathcal{R}) \leq \Pr(\mathcal{Q}) + \sum_v \sum_W \Pr(N_v = W) \sum_{S, T} \Pr(\mathcal{T}(v, S, T) \wedge \mathcal{Q} | N_v = W), \] (76)
where (in both (75) and (76)) \( W \) runs over subsets of \( V \setminus \{v\} \) of size at most \( 2np \) and \( (S, T) \) over pairs of disjoint subsets of \( W \) with sizes as in Lemma 5.2.

Thus, in view of (74), we will have (73) if we can show that the inner sums in (76) are small; for example, it is enough to show that for fixed \( v, W, S, T \) (as above), with \( R := W \setminus (S \cup T) \) and
\[ \mathcal{T}(S, T, R) := \{ \tau(S, T, R) \geq 4\alpha_r \pi |T| \Lambda_r(n, p) \}, \]
...
we have
\[ \Pr(\tau(S, T, R) \land Q|N_v = W) < \exp[-4(\pi/\varepsilon)|T| \log n]. \]  
(77)

This suffices since for each \( t > 0 \) the number of choices for \( S, T \) with \(|T| = t\) is less than \( \binom{n}{t} \left( \frac{\pi}{\varepsilon} \right)^t \) \( \exp[2(\pi/\varepsilon)t \log n] \), which with (77)—recall we assume \( T \neq \emptyset \)—bounds the inner sums in (76) by \( \sum_{t \geq 1} n^{-t} \).

**Remark.** The bound on the number of \((S, T)\)'s could be made a little smaller since \( S, T \) are chosen from \( W \) rather than from all of \( V \), but there is little to be gained by this (unlike in the proof of Lemma 5.1 where the difference was crucial); rather, the point here is that, since \( W, S, T \) determine \( R \), we avoid paying an unaffordable \( \exp[\Omega(|R| \log n)] \) to account for choices of \( R \).

To slightly streamline some of our expressions, we now set, for an event \( A \), \( \Pr(A) = \Pr(A \land Q|N_v = W) \). For reasons that will appear below (see (84) and the lines immediately following it), we will derive (77) from the following multipartized version.

**Lemma 8.2.** For any \( v, W, S, T \) as above and partition \( R_1 \cup \cdots \cup R_{r-3} \) of \( R = W \setminus (S \cup T) \),
\[ \Pr(\tau(S, T, R_1, \ldots, R_{r-3}) > 2\gamma_r \pi|T|\Lambda_r(n, p)) < \exp[-5(\pi/\varepsilon)|T| \log n]. \]  
(78)

**Remark.** Note—cf. the preceding remark—this does not say that w.h.p. (under \( \Pr \)) we have \( \tau(S, T, R_1, \ldots, R_{r-3}) \leq 2\gamma_r \pi|T|\Lambda_r(n, p) \) for all relevant choices of \( S, T \) and \( R_i \)'s, since for small \( T \) the number of choices for the \( R_i \)'s overwhelms the bound in (78).

To see that Lemma 8.2 implies (77), choose, independently of \( G \), a random (uniform) ordered partition \( R_1 \cup \cdots \cup R_{r-3} \) of \( R = W \setminus (S \cup T) \). Given any specification of \( G \), say with \( \tau(S, T, R) = \tau \), we have
\[ \mathbb{E}\tau(S, T, R_1, \ldots, R_{r-3}) = \beta_r \tau, \]
whence, by Markov’s Inequality,
\[ \Pr(\tau(S, T, R_1, \ldots, R_{r-3}) < \varsigma \tau) = \Pr(\tau - \tau(S, T, R_1, \ldots, R_{r-3}) > (1 - \varsigma)\tau) \]
\[ < \frac{(1 - \beta_r)/(1 - \varsigma)}{1 - (\beta_r - \varsigma)/(1 - \varsigma)} \]
for any \( \varsigma \in (0, \beta_r) \), where “Pr” now refers only to the choice of partition.
Applying this with $\varsigma = \beta_r/2$ (and recalling $\gamma_r = \alpha_r\beta_r$) gives, with the natural extension of $P$ to probabilities involving both $G$ and the random partition,

$$P(\tau(S, T, R_1, \ldots, R_{r-3}) > 2\gamma_r\pi t \Lambda_r(n, p))$$

$$> (\beta_r/2)P(\tau(S, T, R) > 4\alpha_r\pi t \Lambda_r(n, p)),$$

and combining this with (78) we have

$$P(\tau(S, T, R) > 4\alpha_r\pi t \Lambda_r(n, p)) < (2/\beta_r) \exp\{-5(\pi/\varepsilon)t \log n\}$$

(which is less than the bound in (77)).

**Proof of Lemma 8.2.** Fix $v, S, T, W$ and $R_1, \ldots, R_{r-3}$ as in the lemma and set $|S| = s$, $|T| = t$, $|R_i| = r_i$, $\Psi = \tau(S, T, R_1, \ldots, R_{r-3})$ and $T = \{\Psi > 2\gamma_r\pi t \Lambda_r(n, p)\}$.

In what follows we will usually be considering variants of $Q$ rather than $Q$ itself, so abandon the notation $P$ used above; but we do continue to condition on $\{N_v = W\}$ and omit this conditioning from the notation. Thus our target inequality (78) (for our specified $v, S, T, W, R_1, \ldots, R_{r-3}$) becomes

$$\Pr(T \land Q) < \exp\{-5(\pi/\varepsilon)t \log n\}. \quad (79)$$

Set $R_{r-2} = S$, $R_{r-1} = T$ and, for $(i, j) \in I$,

$$J_{ij} = G \cap \nabla(R_i, R_j).$$

Note that $T$ is determined by $\cup J_{ij}$.

We choose the $J_{ij}$’s in the order given by $\prec$ and set

$$\Psi_{\varsigma} = \mathbb{E}[\Psi | (J_{ij} : \varsigma(i, j) \leq \varsigma)];$$

in particular $\Psi_0 = \mathbb{E}\Psi$ and $\Psi_{\binom{r-1}{2}} = \Psi$. Notice that

$$\Psi_0 = st\prod_{i=3}^{r-3} r_i \cdot p^{\binom{r-1}{2}} \leq st(|W|/(r-3))^{r-3} p^{\binom{r-1}{2}}$$

$$\leq \pi t np(2np/(r-3))^{r-3} p^{\binom{r-1}{2}} = \gamma_r\pi t \Lambda_r(n, p) =: \mu. \quad (80)$$

Given

$$G_{ij} := \nabla(v, W) \cup \bigcup\{J_{kl} : (k, l) \prec (i, j)\}$$

33
(Note \( \nabla(v, W) \subseteq G \)), we may write
\[
\Psi_{\varsigma(i,j)} = \sum \{ \xi_{xy} w_{xy} : (x, y) \in R_i \times R_j \},
\]
where the \( \xi_{xy} \)'s are independent Ber\( (p) \) r.v.'s and
\[
w_{xy} = M_{xy} \prod_{l=j+1}^{r-1} r_l \cdot p^{\left( \frac{r-1}{2} \right) - \varsigma(i,j)},
\]
with \( M_{xy} \) the number of copies \( \phi \) of \( H_{ij} \) in \( G_{ij} \) having \( \phi(u_0) = v \), \( \phi(u_l) = x \), \( \phi(u_l) \in R_l \) for \( l \in [j-1] \setminus \{i\} \). (The exponent of \( p \) in (82) is the number of \( J_{kl}'s \) that are chosen after \( J_{ij} \).) Of course \( M_{xy} \leq N(H_{ij}; G_{ij}, v, x, y) \) (83).

Define events \( Q_\nu \) (0 \( \leq \nu < \left( \frac{r-1}{2} \right) \)) by
\[
Q_{\varsigma(i,j)-1} = \{ N(H_{ij}, G_{ij}; v, x, y) < \nu S_{ij} / \log n \ \forall (x, y) \in R_i \times R_j \} \quad (84)
\]
(with \( Q_0 \) the full probability space). Then \( Q_{\varsigma} \supseteq Q \) for all \( \varsigma \) and—the point of the fussy definitions—\( J_{ij} \) is independent of \( \{ N_v = W \} \land Q_{\varsigma(i,j)-1} \).

If \( Q_{\varsigma(i,j)-1} \) holds, then (82) and (83) (and the definition of \( S_{ij} \) in (88)) give
\[
w_{xy} < 2 \vartheta \Lambda_r(n, p) / \log n \ \forall (x, y) \in R_i \times R_j
\]
(85)

For \( \varsigma \in \left[ \left( \frac{r-1}{2} \right) \right] \), let
\[
T_{\varsigma} = \{ \Psi_{\varsigma} - \Psi_{\varsigma-1} > \left( \frac{r-1}{2} \right) - 1 \mu \}.
\]
In view of (80), we have \( T \subseteq \bigcup T_{\varsigma} \) (with the union over \( \varsigma \in \left[ \left( \frac{r-1}{2} \right) \right] \)), whence, using \( Q \subseteq Q_{\varsigma} \),
\[
T \land Q \subseteq \bigcup \{ T_{\varsigma} \ land T_1 \ land \cdots \ land T_{\varsigma-1} \} \ land Q
\]
\[
\subseteq \bigcup \{ T_{\varsigma} \ land Q_{\varsigma-1} \ land T_1 \ land \cdots \ land T_{\varsigma-1} \};
\]
so we will have (78) if we show, for \( \varsigma \in \left[ \left( \frac{r-1}{2} \right) \right] \),
\[
Pr(T_{\varsigma} | Q_{\varsigma-1} \land T_1 \land \cdots \land T_{\varsigma-1}) < \left( \frac{r-1}{2} \right) - 1 \exp[-5(\pi/\varepsilon)t \log n].
\]
(86)
(Of course the first factor on the r.h.s. is unimportant.)

The main effect of the conditioning in (86) is the inequality (85) implied by $Q_{ς-1}$. A second effect is that nonoccurrence of earlier $T_{̺}'s bounds $Ψ_{ς-1}$ above by $2µ$.

Now let $ψ = 2µ$, $λ = (r-1)^{-1}µ$, $η = (2(r-1))^{-1}$ and $z = 2θΛ_r(n,p)/\log n$ (the bound in (85)). Then $T_ς$ is the event that $Ψ_ς$, which (again, given $G_{ij}$ where $ς = ς(i,j)$) is just the sum in (81), is greater than $Ψ_{ς-1} + λ$, where we have $EΨ_ς = Ψ_{ς-1} ≤ ψ$. We may thus apply Lemma 4.2 to bound the l.h.s. of (86) by

$$
\exp[-ηλ/(4z)] = \exp[-5(π/ε)t \log n].
$$

9 Proof of Lemma 5.3

As noted earlier, proving Lemma 5.3 turned out to be quite a bit trickier than seemed likely on first inspection. Most interesting here are the roundabout approach via Lemma 9.2 (discussed a bit below) and the use of Lemma 4.13 in the proof of Lemma 9.2. (While it had seemed to us since [9] that a proof of Theorem 1.2 for $r = 4$ would extend fairly automatically to general $r$, this turned out to be not quite true, the one point requiring significant additional ideas being the proof of Lemma 5.3)

One curious point here is that, while one expects things to get easier as $p$ grows, our main line of argument runs into difficulties when $p$ is too far above the lower bound in (2). On the other hand—now more in line with intuition—the statement for larger $p$ follows quite easily once we have the following quantified version for small $p$.

**Lemma 9.1.** For each $λ > 0$ there is a $q > 0$ such that for each $L$, if

$$
p = Cn^{-2(r+1)}(r+1)(r-2)n^2,
$$

with a sufficiently large (fixed) $C$, then with probability at least $1 - n^{-L}$,

$$
κ(S,T) < λ|S|Λ_r(n,p)
$$

whenever $S \subseteq \binom{V}{ς}$ and $T \subseteq G$ satisfy $ΔS \leq 2np$, $V(S) \cap V(T) = \emptyset$ and $|T| ≤ |S| ≤ qn^2p$.

**Proof of Lemma 5.3 given Lemma 9.1** This is similar to the derivation of (77) from Lemma 8.2. Fix $π$ as in Lemma 5.3 and let $q$ be the value corresponding to $λ = 2^{-(ς^2+2)π}$ in Lemma 9.1. We show that Lemma 5.3 holds with $ε = q/2$. 

35
For \( p \) as in [87] and \( q > p \), let \( \zeta = p/q \), \( G = G_{n,q} \) and \( G' = G'_{\zeta} \) \( \sim \) \( G_{n,p} \), and write \( \kappa(\cdot) \) and \( \kappa'(\cdot) \) for \( \kappa_G(\cdot) \) and \( \kappa_{G'}(\cdot) \) respectively. We first observe (this is just for convenience) that we may confine our attention to \( T' \)'s that are not too small. According to Corollary 4.10 there is a fixed \( K \) so that w.h.p. \( G \) satisfies

\[
\kappa(S, T) < |S||T| \max\{2n^{r-4}q|T|^{-6}, K \log n\} \quad \forall S, T \subseteq \binom{V}{2}.
\]

But if (89) holds then (40) is automatic whenever

\[
|T| \leq \frac{\pi \Lambda_r(n, q)}{\max\{2n^{r-4}q|T|^{-6}, K \log n\}}.
\]

Note also that the bound here is fairly large compared to \( \zeta^{-1} = p/q \), e.g. since each of \( \Lambda_r(n, q) \cdot p/q > \Lambda_r(n, p) \) and \( n^2 q^5 \cdot p/q > n^2 p^5 \) is at least a large multiple of \( \log n \).

Thus, in view of Lemma 9.1 it is enough to show that if \( G \) violates the conclusion of Lemma 5.3 (with \( q \) in place of \( p \)) at some \( S, T \) of size at least the r.h.s. of (90)—so in particular with \( \zeta|T| \) slightly large—then \( G' \) violates the conclusion of Lemma 9.1 with probability at least (say) \( n^{-r} \).

To see this, suppose a violation for \( G \) occurs at \((S, T)\) with sizes as above. We then observe that we may choose \( S' \subseteq S \) with (say)

\[
\Delta_{S'} \leq 2np, \quad \zeta|S|/2 < |S'| < 2\zeta|S| \quad \text{and} \quad \kappa(S', T) \geq \zeta \kappa(S, T)/2 \quad (91)
\]

(so \( |S'| < 2\zeta \pi n^2 q = 2\pi n^2 p \)). Existence of \( S' \) is given by Proposition 4.16 as follows. Set \( m = \Delta_{S} + 1 \) and let \( S_1, \ldots, S_m \) be (the color classes of) an equitable \( m \)-coloring of \( S \), with \( \kappa(S_1, T) \geq \cdots \geq \kappa(S_m, T) \). Then \( S' = S_1 \cup \cdots \cup S_{[\zeta m - 1]} \) satisfies (91).

Now set \( u = \min\{\lceil \zeta|T| \rceil, |S'|\} \) and \( v = \binom{u}{2} - 1 \), and let \( T' = G' \cap T \). We claim that with probability at least (say) \( n^{-r} \),

\[
|T'| = u \quad \text{and} \quad \kappa'(S', T') \geq (\zeta/2)^v \kappa(S', T)/2, \quad (92)
\]

in which case \( S', T' \) (which clearly satisfy the conditions following (88)) violate (88), since

\[
(\zeta/2)^v \kappa(S', T)/2 \geq (\zeta/2)^v \kappa(S, T)/4 \geq \lambda |S'| \Lambda_r(n, p)
\]

(where we used \( \kappa(S, T) \geq \pi |S| \Lambda_r(n, q) \geq \pi (2\zeta)^{-1} |S'| |S| \Lambda_r(n, q) \)).

For the claim, set \( \kappa' = \kappa'(S', T'), \mu = \zeta^v \kappa(S', T) = E \kappa', \mu' = 2^{-v}\mu \) and \( \mathcal{Q} = \{|T'| = u\} \). The probability in question is

\[
\Pr(\mathcal{Q}) \Pr(\kappa' \geq \mu'/2|\mathcal{Q}) > n^{-1} \Pr(\kappa' \geq \mu'/2|\mathcal{Q})
\]
(since $\Pr(Q) = \Omega((n^2p)^{-1/2}) > 1/n$). We also have
\[
\mathbb{E}[\kappa'|Q] \geq \frac{(u)^{n_m}}{(m)!} \frac{1}{n} \kappa(S', T) > (1 - \zeta)\mu'
\]
for some small constant $\zeta$. (Here we use the assumption that $u$ is fairly large. The expectation is typically more like $\zeta v \kappa(S', T)$, since most relevant edges are in $G \setminus T$, whereas the bound allows them to all be drawn from $T$.) Markov’s Inequality thus gives
\[
\Pr(\kappa' < \mu'/2|Q) = \Pr(\kappa(S', T) - \kappa' > \kappa(S', T) - \mu'/2|Q) < (\kappa(S', T) - \mu'/2)^{-1}(\kappa(S', T) - \mathbb{E}[\kappa'|Q]) < 1 - \mu'/(3\kappa(S', T)),
\]
so (92) holds with probability at least $n^{-1}(\zeta/2)^{\binom{r}{2}} > n^{-1}p^{\binom{r}{2}} > n^{-r}$.

We now turn to the proof of Lemma 9.1. Though the statement here, like that of Lemma 5.3, is natural, it seems resistant to frontal assault (the difficulties are reminiscent of those associated with upper tail bounds—see e.g. [6, 8] and the history reviewed in [18] or [17, Sec. 2.6] for a sort of case study of one such question—though nothing we know from that arena seems to help here); so the argument will be somewhat indirect.

Recall that $\kappa(S)$ counts choices of $xy \in S$ and $Z \in \binom{V \setminus \{x,y\}^2}{2} \setminus \{\{x,y\}\} \subseteq G$. The value of $\kappa(S)$ is (essentially) known (cf. (90)), so we also know the size of the multiset, say $M$, of edges appearing in the various $Z$’s (namely $|M| = \kappa(S)^{r-2}$). The sense of Lemmas 5.3 and 9.1 is that if $T$ is only a small part of $G$ then few of these edges should come from $T$. (Note, though, that if $|S| < n^2p/\Lambda_r(n,p)$ then all of the edges in question lie in some $T$ of size $\Theta(|S|\Lambda_r(n,p)) < n^2p$; so, as noted in Section 5 bounding $|T|$ without reference to $|S|$ will not do here.)

It is thus natural to try to prove Lemma 9.1 by controlling pairs that appear too often in $M$. When $r = 4$ there is in fact a (rather long, martingale-based) argument along these lines that does work; but we were unable to get that argument, or any such “natural” approach to work for larger $r$, and instead approach the problem from the other direction, showing (roughly) that most of $M$ is spread among edges that do not have very high multiplicities. This works best when $S$ (called $R$ below) is small enough that few of the edges of $G$ are counted even once in $M$. In this case we are able to show, using Theorem 4.13 that the number counted at least once is close
to \(|M|\), leaving little room for higher multiplicities. (This also requires a stricter bound on \(\Delta_S\).) The result for larger \(S\) then follows easily, though the partitioning step that accomplishes this seems rather wasteful.

For \(R \subseteq \binom{V}{2}\) and \(\{u,v\} \in \binom{V}{2}\), set \(\sigma_R(u,v) = 1_{\{\kappa(R, u, v) > 0\}}\) (thus \(\sigma_R(u,v) = 1\) iff there are \(xy \in R\) with \(\{x, y\} \cap \{u, v\} = \emptyset\) and \(Z \in \binom{V \setminus \{x, y, u, v\}}{r-4}\) such that all pairs from \(\{u, v, x, y\} \cup Z\) other than \(xy\) are edges of \(G\); in particular \(\sigma_R(u,v) = 0\) if \(uv \not\in G\)) and

\[
\sigma(R) = \sum \{\sigma_R(u,v) : \{u,v\} \in \binom{V}{2}\}.
\]

The next assertion easily implies Lemma 9.1.

**Lemma 9.2.** For each \(\beta > 0\) there is an \(\eta > 0\) such that for each \(L\), if \(p\) is as in (87) with a sufficiently large (fixed) \(C\), then with probability at least \(1 - n^{-L}\)

\[
\sigma(R) > \frac{1-\beta}{2(r-4)!}|R|\Lambda_r(n,p)
\]

whenever \(R \subseteq \binom{V}{2}\),

\[
|R| < \eta n^2 p / \Lambda_r(n,p) = \eta n^{-(r-4)} p^{-(r^2-r-4)/2},
\]

and

\[
\Delta_R \leq \beta^{-1} np / \Lambda_r(n,p) = \beta^{-1} n^{-(r-3)} p^{-(r^2-r-4)/2}.
\]

**Proof of Lemma 9.1 given Lemma 9.2.** Let \(\beta = \lambda/5\) and let \(\eta\) be the value corresponding to this \(\beta\) in Lemma 9.2. We show that \(\varrho = \beta \eta\) meets the requirements of Lemma 9.1.

By Corollary 4.10 we know that for any \(M\) we have

\[
\kappa(xy) < \frac{1+\beta}{(r-2)!} \Lambda_r(n,p) \quad \forall x, y
\]

with probability at least \(1 - n^{-M}\), provided \(C\) (in (87)) is sufficiently large (namely, large enough so that the bound in (90) is at least \(K \log n\), with \(K\) as in the corollary). So in view of Lemma 9.2, it is enough to show that (88) holds (for \(S, T\) as in Lemma 9.1) whenever (96) is true and (93) holds for \(R\) as in Lemma 9.2. We therefore assume these and proceed.

Set \(m = 4\beta \Lambda_r(n,p)\) and let \(R_1 \cup \cdots \cup R_m\) be a partition of \(S\) with, for each \(i\) (and with more precision than is really necessary),

\[
|R_i| < \lceil 2|S|/m \rceil < \eta n^2 p / \Lambda_r(n,p)
\]
and
\[ \Delta R_i \leq \left(\frac{(\Delta S + 1)/m}{\Lambda_r(n,p)} \right) < \beta^{-1} np/\Lambda_r(n,p). \] (98)

(Again, existence of such a partition is given by Proposition 4.16, briefly as follows. If \( m \geq \Delta S + 1 \), we can take the \( R_i \)'s themselves to be matchings; otherwise we equipartition \( S \) into \( \Delta S + 1 \) matchings and take each \( R_i \) to be a union of \( \lfloor (\Delta S + 1)/m \rfloor \) or \( \lceil (\Delta S + 1)/m \rceil \) of the matchings (whence (98)), with \( \max |R_i| - \min |R_i| \leq \lfloor |S|/(\Delta S + 1) \rfloor \), which is easily seen to imply (97).)

Then (93) gives
\[ \sigma(R_i) > 1 - \beta \frac{(r-2)!}{2(r-4)!} |R_i| \Lambda_r(n,p) \quad \forall i, \]
while from (96) we have
\[ \kappa(R_i) \left( \frac{r-2}{2} \right) = \left( \frac{r-2}{2} \right) \sum_{xy \in R_i} \kappa(xy) < \frac{1+\beta}{2(r-4)!} |R_i| \Lambda_r(n,p) \quad \forall i. \]

Thus (again, for each \( i \))
\[ \kappa(R_i, T) \leq |T| + \sum \{(\kappa(R_i, u, v) - 1)^+ : \{u, v\} \in \binom{V}{2}\} \]
\[ = |T| + \kappa(R_i) \left( \frac{r-2}{2} \right) - \sigma(R_i) < |T| + \beta |R_i| \Lambda_r(n,p) \]

(instead of \( \beta \) we could write \( \beta/(r-4)! \)) and, finally,
\[ \kappa(S, T) = \sum \kappa(R_i, T) \leq \beta \Lambda_r(n,p)(4|T| + \sum |R_i|) \leq 5\beta \Lambda_r(n,p)|S|. \]

Proof of Lemma 9.2. It is enough to show that for any \( R \subseteq \binom{V}{2} \) satisfying (94) and (95), and any fixed \( K \),
\[ \Pr(\sigma(R) < \frac{1-\beta}{2(r-4)!} |R| \Lambda_r(n,p)) < \exp[-K|R| \log n], \] (99)
provided \( C \) is large enough.

This is the promised application of Theorem 4.13. Following the notation used there, we let \( i \) run over \( \binom{V}{2} \) and, for a given \( i = uv \), let \( j \) run over pairs \((xy, W)\) with
\[ xy \in R, \{x, y\} \cap \{u, v\} = \emptyset \] and \( W \in \binom{V \setminus \{x, y, u, v\}}{r-4} \).

For such \( i = uv \) and \( j = (xy, W) \), we set
\[ B_{ij} = \{K(xy; W \cup \{uv\}) \subset G\} \]
where, as in Lemma 4.14, \( K(xy, Z) := \binom{\{x,y\} \cup Z \setminus \{xy\}}{2} \) and observe that \( I_i = \sigma_R(u, v) \). We then need estimates for \( \mu, \Delta \) and \( \gamma \), the first of which is easy:

\[
\mu (= \sum_{i,j} E I_{ij}) = |R| \binom{n-2}{r-4} \binom{r}{2} \frac{p}{(r-4)!} |R| \Lambda_r(n, p).
\]

The second is not quite so easy, but has essentially already been worked out in Lemma 4.14, the only difference being that each pair \((K, L)\) appearing in the \( \Delta \) of (35) is counted \((r-2)^2\) times in the present 

\[
\Delta := \sum \sum \{ E I_{ij} I_{kl} : (i, j) \sim (k, l) \}.
\]

(Each \( K = K(xy, Z) \) in (35) corresponds to \((r-2)^2\) pairs \((i, j)\) with \( i = uv \) for some \( \{u, v\} \in (Z/2) \) and \( j = (xy, Z \setminus \{u, v\}) \).) Lemma 4.14 thus implies that for (the present) \( \Delta \) we have

\[
\Delta < \zeta |R| n^{2r-4} p^{r^2-r-2} / \log n \tag{101}
\]

for any desired \( \zeta > 0 \) provided \( C \) is sufficiently large. (In more detail: given \( \zeta \), take \( \xi = \binom{r-2}{2} \zeta \), let \( \vartheta \) be the value associated with this \( \xi \) in Lemma 4.14, and choose \( C \) so that \( C(\xi)^{-1} > (\beta \vartheta)^{-1} \) (see (98)) and \( C \) is large enough to support Lemma 4.14 (with this \( \xi \) and \( \vartheta \)).)

For consideration of \( \gamma \) (= \( \sum_i \sum_{\{j,k\}} E I_{ij} I_{ik} \)), we temporarily fix \( i = uv \). A relevant \( \{j, k\} \) is then an (unordered) pair of distinct pairs of the form \((xy, W)\) as in (100). (The two pairs may, for example, use the same \( r-2 \) vertices, but must then have different \( xy \)'s.)

The argument here is similar to that for Lemma 4.14. We may classify a pair \( \{j, k\} \) with \( j = (xy, W), k = (x'y', W') \), according to

\[
a(j, k) = |\{x, y\} \cap \{x', y'\}| \in \{0, 1, 2\}
\]

and

\[
b(j, k) = |(W \cup \{x, y\}) \cap (W' \cup \{x', y'\})| \in \{0, 1, \ldots, r-2\},
\]

noting that if \( a(j, k) = 2 \) then \( b(j, k) \leq r-3 \) (and, of course, \( b(j, k) \geq a(j, k) \)). It will also be helpful to set \( K_j = K(xy; W \cup \{uv\}) = (W \cup \{u, v, x, y\}) \setminus \{xy\} \). Write \( \mathcal{G}_i \) for the set of relevant \( \{j, k\}'s \) and let

\[
N(a, b) = |\{(j, k) \in \mathcal{G}_i : a(j, k) = a, b(j, k) = b\}|.
\]
Then
\[ N(0, b) < B|R|^2n^{2r-8-b} \] (102)
and
\[ N(1, b) < B\sum_x d_R^2(x)n^{2r-7-b} \leq B|R|\Delta_R n^{2r-7-b} \] (103)
for a suitable \( B = B_r \), and
\[ N(2, b) < |R|n^{2r-6-b}. \]

(These are the same as the bounds we used for the \( N(a, b) \)'s in the proof of Lemma 4.14, except that what was \( r \) is now \( r - 2 \), since we have set aside the common pair \( i = uv \). Note also that we have slightly different constraints on the possibilities for \( (a, b) \): in the earlier discussion \( (a, b) = (2, 2) \) was excluded because we wanted only overlapping pairs \( (K, L) \) (that is, pairs sharing an edge not in \( R \)); in the present situation \( (2, 2) \) is allowed, but we exclude the possibility \( j = k \), a.k.a. \( (a, b) = (2, r - 2) \).)

On the other hand,
\[ E_I_{ij}I_{ik} = p^{r^2-r-2-|K_j \cap K_k|} \]
and, for \( b(j, k) = b \),
\[ |K_j \cap K_k| \begin{cases} = \binom{b+2}{2} - 1 & \text{if } a(j, k) = 2, \\ \leq \binom{b+2}{2} & \text{otherwise}. \end{cases} \]

(E.g. if \( b = r - 2 \) the truth is \( |K_j \cap K_k| = \binom{r}{2} - 2 \), but we don’t need this.)

Let \( \mu' = \binom{n}{2}^{-1}\mu \) (the average over \( i \) of \( \sum_j EI_{ij} \)) and \( D = 2B(r - 4)! \)
(with \( B \) as in (102), (103), chosen so that
\[ \mu'D > B|R|n^{r-4}\binom{r}{2}^{-1}. \]

Setting (again with \( i \) fixed)
\[ S(a, b) = \sum \{EI_{ij}I_{ik} : a(j, k) = a, b(j, k) = b \} \]
and combining the above observations (and little calculations) yields
\[ S(0, b) < B|R|^2n^{2r-8-b}p^{r^2-r-2-\binom{b+2}{2}} \]
\[ < \mu' \cdot D|R|n^{r-b-4}\binom{r}{2}^{-\binom{b+2}{2}} - 1 \]
\[ < \mu' \cdot Dnp^{(b+3)/2} - b \]
(using (94)),

\[ S(1,b) \leq B|R|\Delta R n^{2r-7-b}p^{r^2-r-2-(b+2)} \]
\[ \leq \mu' \cdot D\Delta R n^{r-b-3}p^{r_2-b+2} \leq 1 \]
\[ \leq \mu' \cdot D\beta^{-1}(np^{b+3/2})^{-b} \]

(using (95)), and

\[ S(2,b) \leq |R|n^{2r-6-b}p^{r^2-r-2-(b+2)} \leq 1 \]
\[ \leq \mu' \cdot Dn^{r-b-2}p^{r-b-2}(r+b+1)/2 \]
\[ = \mu' \cdot D(np^{r+b+1/2})^{-b-2}. \]

In particular, recalling (87) and the exclusion of \((a,b) = (2, r-2)\) (and the trivial \(b \geq a\)), we have

\[ S(a,b) < \begin{cases} 
D\eta \mu' & \text{if } (a,b) = (0,0), \\
o(\mu') & \text{otherwise},
\end{cases} \]

and, now letting \(i\) vary and summing over \(i\) and \((a,b)\),

\[ \gamma \leq O(\eta \mu) \]  
(104)

(where the implied constant doesn’t depend on \(\eta\)).

Finally, taking \(t = \beta \mu/2\), and applying Theorem 4.13 (using the bounds (101) and (104) and noting that \(X := \sum I_i = \sigma(R)\)), we find that the l.h.s. of (99) is less than

\[ \Pr(\sigma(R) < \mu - t) < \exp[-(t - \gamma)^2/(2\Delta)] \]
\[ < \exp[-(\beta/2 - O(\eta))^2/4\zeta(r-4)!^2|R|\log n], \]

which is less than the r.h.s. of (99) for suitable \(\eta\) and \(\zeta\). (Recall—see (101)—\(\zeta\) can be made as small as we like via a suitable choice of \(C\).

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10 Rigidity and correlation

One reason for the difficulty of the problem treated in this paper is surely the difficulty of understanding maximum cuts themselves, an issue whose
centrality is perhaps clearer in [3]; our ways of dealing with (or avoiding) it in Sections 11 and 12 are based on the notions and soft observations developed here.

We again use \( H \) for a general graph on \( V \) and \( G \) for \( G_{n,p} \). Let \( C \) be a collection of balanced cuts. The discussion in this section makes sense more generally, but all \( C \)'s used in what follows will be of the type

\[
C(X) := \{\Pi = (A_1, \ldots, A_{r-1}) : \Pi \text{ is balanced}, X \subseteq A_1\}
\tag{105}
\]

for some \( X \subseteq V \). We will often use this with \( X = V(Q) \) for some \( Q \subseteq (V)^2 \), in which case we also write \( C(Q) \) for \( C(X) \).

For a graph \( H \), we use

\[
b(C, H) = \max\{|\Pi_H| : \Pi \in C\}
\]

and

\[
\max(C, H) = \{\Pi \in C : |\Pi_H| = b(C, H)\}
\]

—we will speak of “max cuts”—and, for \( \Pi \in C \), define the defect of \( \Pi \) relative to \( (C, H) \) to be

\[
def_{C,H}(\Pi) = b(C, H) - |\Pi_H|.
\]

Given \( C \) and \( H \), we define an equivalence relation “\( \equiv \)” (or “\( \equiv_{(C,H)} \)” ) by:

\[
x \equiv y \text{ iff } \Pi(x) = \Pi(y) \forall \Pi \in \max(C, H),
\]

where \( \Pi(x) \) is the block of \( \Pi \) containing \( x \). Equivalence classes are \( (C, H) \)-components, or simply components if the identities of \( C \) and \( H \) are clear. (Of course if \( C = C(X) \) then \( X \) is automatically contained in some component, whatever the value of \( H \).)

Given \( C \), say \( H \) is rigid if the number of equivalent pairs under \( \equiv_{(C,H)} \) is at least \( (1 - \alpha)n^2/2(r-1) \). (Recall \( \alpha \) was one of the basic constants previewed at the end of Section 2.)

**Proposition 10.1.** If \( H \) is rigid then there are distinct \( (C, H) \)-components \( S_1, \ldots, S_{r-1} \) of size greater than \( n/r \).

For a rigid \( H \) we will call the (necessarily unique) collection \( \{S_1, \ldots, S_{r-1}\} \) in Proposition 10.1 the core of \( H \). (Note that, in contrast to our usage for cuts, we think of the core as unordered.) Of course a nonrigid \( H \) may also admit \( S_1, \ldots, S_{r-1} \) as in the proposition; but it will be convenient in what follows to regard only rigid graphs as having cores, so if we speak of the core of \( H \), then \( H \) is rigid by definition.
**Proof of Proposition 10.1.** This is given by the following assertion, applied when $H$ is rigid with ($C, H$)-components $S_1, \ldots, S_m$.

**Claim.** If $S_1 \cup \cdots \cup S_m$ is a partition of $V$ with $s_i := |S_i| \leq (1 + \delta)n/(r - 1)$ for all $i$ and $\sum s_i \geq (1 - \alpha)n^2/(2(r - 1))$, then some $r - 1$ of the $s_i$'s are greater than $(1 - r\alpha)n/(r - 1)$.

(So we actually get the proposition with $(1 - r\alpha)n/(r - 1)$ in place of $n/r$; but $n/r$ is convenient and sufficient for our purposes.)

For the proof of the claim, set $\lambda = r\alpha$. Among $(S_1, \ldots, S_m)$'s for which the conclusion fails, $\sum s_i$ is maximum when $m = r$, $s_1 = \cdots = s_{r-2} = (1 + \delta)n/(r - 1)$ and $s_{r-1} = (1 - \lambda)n/(r - 1)$ (so $s_m = n - (s_1 + \cdots + s_{r-1})$).

This gives

\[
(1 - \alpha)n^2 \frac{n^2}{2(r - 1)} < \sum s_i^2 < \left[ (r - 2)(1 + \delta)^2 + (1 - \lambda)^2 + (\lambda - (r - 2)\delta)^2 \right] \frac{n^2}{2(r - 1)} < (1 - \alpha)n^2 \frac{n^2}{2(r - 1)},
\]

a contradiction (where we used $\alpha \gg \delta$ for the final inequality).

For (rigid) $H$ with core $\{S_1, \ldots, S_{r-1}\}$, we say $Q \subseteq \binom{V}{2}$ is in the core if $V(Q)$ is contained in one of $S_1, S_2, \ldots, S_{r-1}$. (We will only use this with $C = C(Q)$, but note—a point that will cause some trouble below—this does not guarantee that $Q$ is in the core.)

For any $H$, set

\[
\text{crit}(H) \ (= \text{crit}_C(H)) = H \cap \bigcap \{ \text{ext}(\Pi) : \Pi \in \max(C, H) \}; \quad (106)
\]

thus $e \in H$ is in $\text{crit}(H)$ iff $b(C, H - e) < b(C, H)$. Notice in particular that if $\{S_1, S_2, \ldots, S_{r-1}\}$ is the core of $H$, then $\nabla(S_1, S_2, \ldots, S_{r-1}) \subseteq \text{crit}(H)$.

The next two lemmas are the promised applications of Harris’ Inequality (see Section 4.2). As mentioned earlier, these were suggested by the way Harris is used in [5]: the crucial new idea here appears in [110], where uniqueness of the core bounds the sum of probabilities by 1 (and again in the proof of Lemma 10.3, where uniqueness is arranged in a simpler way).

We again write $G$ for $G_{n,p}$.
Lemma 10.2. Fix $X \subseteq V$. Suppose that for each collection $\{T_1, \ldots, T_{r-2}\}$ of disjoint subsets of $V \setminus X$, the event $F(T_1, \ldots, T_{r-2}) = F(\{T_1, \ldots, T_{r-2}\})$ is decreasing in and determined by $\nabla(X, T_1, \ldots, T_{r-2})$, and that
\[
\Pr(F(T_1, \ldots, T_{r-2})) < \xi \quad \text{whenever } |T_1|, \ldots, |T_{r-2}| > n/r. \tag{107}
\]
Given $C$, let $R$ be the event that $G$ is rigid, say with core $\{S_1, \ldots, S_{r-1}\}$, $X \subseteq S_1$, and $F(S_2, \ldots, S_{r-1})$ holds. Then $\Pr(R) < \xi$.

Proof. For disjoint $S_1, \ldots, S_{r-1} \subseteq V$ with $X \subseteq S_1$, set
\[
E(S_1, \ldots, S_{r-1}) = \{G \text{ has core } \{S_1, \ldots, S_{r-1}\}\}.
\]
The main point (justified below) is that, for any such $S_1, \ldots, S_{r-1}$,
\[
E(S_1, \ldots, S_{r-1}) \text{ is increasing in } \nabla(X, S_2, \ldots, S_{r-1}), \tag{108}
\]
whence, by Theorem 4.11 (applied to the indicators of $E$ and $F$),
\[
\Pr(E(S_1, \ldots, S_{r-1}) \land F(S_2, \ldots, S_{r-1})) \leq \Pr(E(S_1, \ldots, S_{r-1})) \Pr(F(S_2, \ldots, S_{r-1})). \tag{109}
\]
This gives the lemma, since
\[
\Pr(R) = \sum \Pr(E(S_1, \ldots, S_{r-1}) \land F(S_2, \ldots, S_{r-1})) < \xi \sum \Pr(E(S_1, \ldots, S_{r-1})) \leq \xi, \tag{110}
\]
where the sums are over $(S_1, \ldots, S_{r-1})$ as above (that is, the $S_i$’s are disjoint with $X \subseteq S_1$) and the first inequality uses (107) and (109) (the latter applicable because $E(S_1, \ldots, S_{r-1})$ implies $|S_i| > n/r \forall i$).

The reason for (108) is simply that if $E(S_1, \ldots, S_{r-1})$ holds, then adding a pair from $\nabla(X, S_2, \ldots, S_{r-1})$ (or, for that matter, $\nabla(S_1, \ldots, S_{r-1})$) to $G$ doesn’t affect the set of max cuts: any such pair is in $\text{ext}(\Pi)$ for every $\Pi \in \max(C, G)$, so each such $\Pi$ remains a max cut, and, moreover (since $b$ increases), no new cuts are added to $\max(C, G)$.

Lemma 10.3. Fix $X \subseteq V$ and an order “$\prec$” on $C = C(X)$. Suppose that for each $(r-2)$-tuple $(B_1, \ldots, B_{r-2})$ of disjoint subsets of $V \setminus X$, $F(B_1, \ldots, B_{r-2})$ is an event decreasing in and determined by $\nabla(X, B_1, \ldots, B_{r-2})$, and that
\[
\Pr(F(B_1, \ldots, B_{r-2})) < \xi \quad \text{whenever } |B_1|, \ldots, |B_{r-2}| > (1-\delta)n/(r-1). \tag{111}
\]
Let $R$ be the event that for the first member (under $\prec$), say $(A_1, \ldots, A_{r-1})$, of $\max(C, G)$, $F(A_2, \ldots, A_{r-1})$ holds. Then $\Pr(R) < \xi$. 

45
Proof. This is similar to the proof of Lemma 10.2. For $A_1,\ldots,A_{r-1}$ partitioning $V$, the event

$$E(A_1,\ldots,A_{r-1}) = \{(A_1,\ldots,A_{r-1}) \text{ is the first member of } \max(\mathcal{C},G)\}$$

(which in particular implies $X \subseteq A_1$) is increasing in $\nabla_G(X,A_2,\ldots,A_{r-1})$. (If $E(A_1,\ldots,A_{r-1})$ holds, then adding a pair from $\nabla_G(X,A_2,\ldots,A_{r-1})$ doesn’t remove $(A_1,\ldots,A_{r-1})$ from, or add any new members to, the set of max cuts (though here the set of max cuts may shrink).

The rest of the earlier argument applies without modification. (Note that, since members of $\mathcal{C}$ are required to be balanced, $E(A_1,\ldots,A_{r-1}) = \emptyset$ unless $|A_i| > (1-\delta)n/(r-1)$ $\forall i$.)

\[\square\]

11 Proof of Lemma 6.1

Note. Here and in Section 12 it will sometimes be better to speak of a set of graphs rather than an event; in particular this will be helpful when the discussion involves more than one random graph. The default remains our usual $G = G_{n,p}$; that is, when we say without qualification that some event holds, we mean it holds for $G$.

We first observe that it is enough to prove Lemma 6.1 for

$$t < K p^{-1}$$

for a suitable fixed $K$:

**Proposition 11.1.** There is a $K$ such that w.h.p. no balanced cut admits more than $Kp^{-1}$ bad vertices.

**Proof.** By Proposition 11.3 it is enough to bound the probability that some balanced $\Pi$ admits $Kp^{-1}$ bad vertices $x$ with

$$d(x) = (1 \pm o(1))np.$$  \hspace{1cm} (113)

Here we use $c_r < b_r$. (Recall these were defined in (51).) If $x$ satisfying (113) is bad for $\Pi = (A_1,\ldots,A_{r-1})$ (so $x \in A_1$), then we assert that, writing $d_i$ for $d_{A_i}(x)$, at least one of $d_2,\ldots,d_{r-1}$ is less than $(1-\varsigma)np/(r-1)$, for some (fixed) $\varsigma = \varsigma_r > 0$. To see this without too much calculation, consider the “ideal” version in which $d(x) = np$ and each of $d_2,\ldots,d_{r-1}$ is at least
In this case $D_\Pi(x)$ (defined following (53)) is minimum when $d_2 = \cdots = d_{r-2} = np/(r-1)$ and $d_{r-1} = 2np/(r-1)$, in which case (cf. the remark preceding Lemma 6.1)

\[ D_\Pi(x) = \frac{(r-3)}{2} \left( \frac{np}{r-1} \right)^2 + (r-3) \cdot 2 \left( \frac{np}{r-1} \right)^2 = \frac{r(r-3)}{2(r-1)} n^2 p^2 = b_r n^2 p^2. \]

It is then clear that for a small enough $\varsigma (= \varsigma_r > 0)$, replacing these ideal assumptions by (113) and $d_i > (1-\varsigma)np/(r-1)$ ($i \in \{2, r-1\}$) still forces $D_\Pi(x) > c_r n^2 p^2$, contradicting the assumption that $x$ is bad for $\Pi$.

So if $\Pi$ admits at least $t$ bad vertices, then there are an $i \in \{2, r-1\}$ and some $T \subseteq A_i$ of size at least $t/(r-2)$, each of whose vertices $x$ has $d_{A_i}(x) < (1-\varsigma)np/(r-1)$. But if $\Pi$ is balanced (so $|A_i| > (1-\delta)n/(r-1)$), then this implies (say)

\[ |\nabla(T, A_i)| < (1-\varsigma + 2\delta)|T||A_i|p, \]

which, for the $K$ corresponding to $\epsilon = \varsigma - 2\delta$ and $c = (1-\delta)/(r-1)$ in Proposition 4.3, violates the conclusion of that proposition if $t > Kp^{-1}$.

We assume for the rest of this section that $t$ ranges over values satisfying (112) and $X$ over $t$-subsets of $V$. Set

\[ \vartheta = (b_r - c_r)/3. \]

We will prove Lemma 6.1 with $\nu = \vartheta \epsilon$, with $\epsilon$ as in the paragraph following the proof of Lemma 11.2.

Let $Q$ be the event that the conclusions of Propositions 4.3 and 4.4(b) hold and

\[ R_X = \{ \exists \Pi \in \mathcal{C}(X) : \text{def}_G(\Pi) < \vartheta tn^{3/2}p^2 \text{ and each } x \in X \text{ is bad for } \Pi \}; \]

so we should show $\Pr(\cup R_X) = o(1)$. We have

\[ \Pr(\cup R_X) \leq \Pr(\overline{Q}) + \sum \Pr(R_X \cap Q) < o(1) + \sum \Pr(R_X \cap Q), \quad (114) \]

so will be done if we show, for each $t$ and $X$,

\[ \Pr(R_X \cap Q) < \exp[-\Omega(tnp)]. \quad (115) \]

From this point we fix ($t$ and) $X$ and set $W = V \setminus X$, $R = R_X$. The strategy will involve choosing $G$ in two stages. We hope to arrange that the
output, $G'$, of the first stage admits some "reference" cut, say $\Pi^*$, that is both maximum in $G'$ and poised to gain many edges in the second stage, whereas it is likely that each "bad" cut (i.e. one for which $X$ is bad) sees significantly fewer additions. If this does happen then $\Pi^*$ will (typically) be significantly larger than any bad cut once we add the contributions of the second stage. (At a nontechnical level this echoes what was perhaps the main idea of [9]; see the proof of (32) there and item D in Section 13 below.)

The second stage will be confined to the (random) set of pairs $uv$ having at least—usually exactly—one common neighbor in $X$, so that the likely number of additions to a particular $\Pi$ grows with the number of such pairs in $\text{ext}(\Pi)$, roughly the sum over $x \in X$ of the $D_{\Pi}(x)$’s. Thus a $\Pi$ for which $X$ is bad will tend to suffer in the second stage; the more interesting question is, how do we know that there is some max cut for which the $D_{\Pi}(x)$’s are large? The answer is provided by Lemma 10.3, but circuitously.

The lemma easily gives the desired cut for our usual $G = G_{n,p}$, or, more generally, for a $G$ with edges chosen independently with large enough probabilities. But $G'$ will not be of this type and the lemma seems not to apply directly; instead we apply it to an (auxiliary) copy, $H$, of $G_{n,p}$ and then couple $H$ with $G'$. This looks unpromising since the distributions are not at all close, but succeeds roughly because even the tiny probability that the two graphs coincide is much larger than the probability that the event of interest fails for $H$.

Fix some order “$\prec$” on $C := C(X)$ and let $T$ be the set of graphs $H$ (on $V$) for which the first member, say $(A_1, \ldots, A_{r-1})$, of $\text{max}(C, H)$ satisfies

$$|\{x \in X : \min\{|N_H(x) \cap A_i| : i \in [2, r-1]\} < (1-\vartheta)np/(r-1)\}| > \theta t. \ (116)$$

**Remark.** As suggested earlier, the argument will now involve some interplay of different random graphs, and we need to be clear as to which graph is meant when we speak of membership in $T$. In particular the next lemma refers to a *generic* copy of $G_{n,p}$; it will be used to prove that a similar statement holds for a slightly mongrelized version of the graph we’re really interested in.

**Lemma 11.2.** $\Pr(G_{n,p} \in T) < \exp[-\Omega(tnp)]$

(where the implied constant depends on $\vartheta$).

**Proof.** For disjoint $B_1, \ldots, B_{r-2} \subseteq W$, let $F(B_1, \ldots, B_{r-2})$ be the event

$$\{|\{x \in X : \min\{d_{B_i}(x) : i \in [r-2]\} < (1-\vartheta)np/(r-1)\}| > \theta t\}.$$
According to Lemma 10.3 it is enough to show that (111) holds for some $\xi = \exp[-\Omega(tnp)]$.

To see this, fix $B_1, \ldots, B_{r-2}$ as above with $|B_i| > (1 - \delta)n/(r - 1)$ $\forall i$. If $F(B_1, \ldots, B_{r-2})$ holds then there are $i \in [r - 2]$ and $Y \subseteq X$ with $|Y| > \vartheta t/(r - 2)$ such that $d_{B_i}(x) < (1 - \vartheta)np/(r - 1)$ $\forall x \in Y$. By Theorem 4.1 the probability that this occurs for a given $i$ and $Y$ is less than $\exp[-\Omega(tnp)]$ (where, again, the implied constant—roughly $\vartheta^2/(2r)$—depends on $\vartheta$); so, accounting for the number of possibilities for $i$ and $Y$, we have

$$\Pr(F(B_1, \ldots, B_{r-2})) < r2^t \exp[-\Omega(tnp)] = \exp[-\Omega(tnp)].$$

We now generate $G$ (the version of $G_{n,p}$ in which we’re really interested) in stages. For $L \subseteq \nabla(X,W)$, set $P_L = \cup_{x \in X} \binom{N_L(x)}{2}$ and $Q_L = \binom{X}{2} \cup (W \setminus P_L)$. Fix $\varepsilon > 0$ with $\varepsilon^2$ small compared to the implied constant in Lemma 11.2 and set $q = \varepsilon n^{-1/2}$. We choose edges of $G$ in the following order.

(i) Choose $L = \nabla_G(X,W)$ and set $P_L = P$ and $Q_L = Q$.

(ii) Choose $G \cap Q$.

(iii) Choose edges in $P$ with probability $p'$, where $1 - p' = (1 - p)/(1 - q)$ (so $p' \approx p - q$), these choices made independently.

(iv) Choose additional edges in $P$ (again independently) with probability $q$.

(Note that the resulting $G$ is indeed a copy of $G_{n,p}$.) Let $G'$ be the output of (i)-(iii), and

$$S = \{|G' \cap P| \leq 2|P|p\}. \quad (117)$$

Let $Q^* \supseteq Q$ be the event that $G$ satisfies the conditions:

$$d(x) = (1 \pm o(1))np \; \forall x \in X; \quad (118)$$

$$d(x,y) = (1 \pm o(1))np^2 \; \forall x, y \in X \; (x \neq y); \quad (119)$$

$$|G[X]| < t \log n. \quad (120)$$

Note that membership of $G$ in $Q^*$ depends only on the edges chosen in (i) and (ii), so $G \in Q^*$ is the same as $G' \in Q^*$; this allows us to continue to use notation like $d(x)$, $d_A(x,y)$, $N(x)$ for $x, y \in X$ without ambiguity.

We need two easy consequences of $Q^*$ (actually of (118) and (119)): first,
\[ |P| = (1 \pm o(1))tn^2p^2/2, \tag{121} \]

and second, for any disjoint \( S, T \subseteq W \),
\[ |P \cap \nabla(S, T)| > \sum_{x \in X} d_S(x)d_T(x) - o(tn^2p^2). \tag{122} \]

**Proof of (121).** We have
\[ |P| \leq \sum_{x \in X} \left( d_W(x) \right)^2 \leq \sum_{x \in X} \left( d(x) \right)^2 < (1 + o(1))tn^2p^2/2 \tag{123} \]

(with the last inequality given by (118)). For a lower bound we may use
\[ |P| \geq \sum_{x \in X} \left( d(x) \right)^2 - \sum_{\{x, y\} \in \binom{X}{2}} \left( d(x, y) \right) - tn. \]

By (118) and (119) the first sum is \((1 \pm o(1))tn^2p^2/2\) and the second is at most \((1 + o(1))t^2n^2p^4/4\). Combining these observations (and recalling (112) and (6)) gives (121).

**Proof of (122).** This is similar. We have
\[ |P \cap \nabla(S, T)| = \left| \bigcup_{x \in X} \nabla(N_S(x), N_T(x)) \right| \]
\[ > \sum_{x \in X} d_S(x)d_T(x) - \sum_{\{x, y\} \in \binom{X}{2}} \left( d_S(x, y)d_T(x, y) : \{x, y\} \in \binom{X}{2} \right), \]

and (again using (119), (112) and (6)) the subtracted term is less than \(|X|^2n^2p^4 = o(tn^2p^2)\).

Returning to (115), we have
\[ \Pr(\mathcal{R} \land \mathcal{Q}) \leq \Pr(\mathcal{R} \land \mathcal{Q}^*) \leq \Pr(\mathcal{S} \land \mathcal{Q}^*) + \Pr(\mathcal{Q}^* \land (G' \in \mathcal{T})) + \Pr(\mathcal{R} | \mathcal{Q}^* \land \mathcal{S} \land (G' \not\in \mathcal{T})), \]

(recall \( \mathcal{S} \) was defined in (117)) and, from (121) and Theorem 4.1,
\[ \Pr(\mathcal{S} \land \mathcal{Q}^*) \leq \Pr(\mathcal{S} | \mathcal{Q}^*) < \exp[-\Omega(tn^2p^3)]. \]

50
Thus \[ \text{(115)} \] (and Lemma 6.1) will follow from the next two assertions, the more interesting of which is the first.

**Claim 1.** \( \Pr(Q^* \land (G' \in T)) < \exp[-\Omega(\varepsilon^2 np)]. \)

**Claim 2.** \( \Pr(R|Q^* \land S \land (G' \notin T)) < \exp[-\Omega(tn^{3/2}p^2)]. \)

(Note \( n^{3/2}p^2 \gg np \). The implied constant in Claim 1 doesn’t depend on \( \varepsilon \); it could of course absorb the \( \varepsilon^2 \), but we prefer the current form as it better reflects the source of the bound.)

**Proof of Claim 1.** This is achieved by a comparison (coupling) of \( G' \) and \( G_{n,p} \). Let \( H \) consist of the edges chosen in (i) and (ii) together with edge \( s \) in \( P \) chosen independently (of \( G' \) and each other), each with probability \( p \).

Then \( H \sim G_{n,p} \), so by Lemma 11.2 we have

\[
\Pr(H \in T) < \exp[-\Omega(tnp)]. \tag{124}
\]

Let \( \mathcal{G} = \{ K \in Q^* : |K \cap P| < |P|(p - 2q) \} \). (We again note that membership of \( K \) in \( Q^* \) depends only on the edges of \( K \) incident with \( X \).) By Theorem 4.1 (recalling that \( Q^* \) implies (121)) we have

\[
\Pr(G' \in \mathcal{G}) < \exp[-\Omega(\varepsilon^2 np)]. \tag{125}
\]

On the other hand, we assert,

\[
K \in Q^* \setminus \mathcal{G} \Rightarrow \Pr(G' = K) < \exp[O(\varepsilon^2 ntp)] \Pr(H = K). \tag{126}
\]

**Proof.** Fix \( K \in Q^* \setminus \mathcal{G} \), say with \( \nabla_K(X,W) = L \), and let \( P = P_L \), \( m = |P| \) \((\sim tn^2p^2/2\) since \( K \in Q^* \)) and \( k = |K \cap P| (> m(p - 2q)) \). Then

\[
\Pr(G' = K) = \Pr(G' \setminus P = K \setminus P) \Pr(G' = K | G' \setminus P = K \setminus P)
\]

and

\[
\Pr(G' = K | G' \setminus P = K \setminus P) = \Pr(B(m,p') = k)(\binom{m}{k})^{-1}.
\]

Repeating this with \( H \) in place of \( G' \) and using \( \Pr(G' \setminus P = K \setminus P) = \Pr(H \setminus P = K \setminus P) \) gives

\[
\frac{\Pr(G' = K)}{\Pr(H = K)} = \frac{\Pr(B(m,p') = k)}{\Pr(B(m,p) = k)}.
\]

The r.h.s. is less than 1 if \( k > mp \), and otherwise is less than

\[
[\Pr(B(m,p) = k)]^{-1} = \exp[O(\varepsilon^2 ntp)] \tag{127}
\]

(routine calculation omitted), so we have (126).
Finally, (125), (126) and (124) give
\[
\Pr(Q^* \land (G' \in T)) \leq \Pr(G' \in G) + \sum \left\{ \Pr(G' = K) : K \in T \cap (Q^* \setminus G) \right\} \\
< \Pr(G' \in G) + \exp[O(\varepsilon^2 ntp)] \Pr(H \in T \cap (Q^* \setminus G)) \\
< \exp[-\Omega(\varepsilon^2 ntp)]
\]
(where the last inequality uses our assumption on \( \varepsilon \)).

**Proof of Claim 2.** Fix \( G' \in Q^* \land T \) satisfying \( S \) and let \( \Pi = (A_1, \ldots, A_{r-1}) \) be the first member of \( \max(C, G') \); so we are assuming (116) fails (with \( G' \) in place of \( H \)). Let \( G'' = G \setminus G' \). We have
\[
|\Pi_G| = |\Pi_{G'}| + |G'' \cap \nabla(A_1, \ldots, A_{r-1})|
\]
and, for any \( \Pi' = (S_1, \ldots, S_{r-1}) \in C \),
\[
|\Pi'_{G'}| = |\Pi_{G'}| + |G'' \cap \nabla(S_1, \ldots, S_{r-1})| \\
\leq |\Pi_{G'}| + |G'' \cap \nabla(S_1, \ldots, S_{r-1})|,
\]
whence
\[
def_G(\Pi') \geq |\Pi_G| - |\Pi'_{G'}| \\
\geq |G'' \cap \nabla(A_1, \ldots, A_{r-1})| - |G'' \cap \nabla(S_1, \ldots, S_{r-1})|.
\]  
(128)

So—as we will explain in a moment—it is enough to show

**Lemma 11.3.** With probability \( 1 - \exp[-\Omega(tn^{3/2}p^2)] \) (where the implied constant depends on \( \vartheta \) and \( \varepsilon \)),
\[
|G'' \cap \nabla(A_1, \ldots, A_{r-1})| > (1 - 3\vartheta)b_rtn^2 p^2 q
\]  
(129)

and
\[
|G'' \cap \nabla(S_1, \ldots, S_{r-1})| < |P \cap \nabla(S_1, \ldots, S_{r-1})|q + \vartheta tn^2 p^2 q \\
\forall (S_1, \ldots, S_{r-1}) \in C.
\]  
(130)

To see that Lemma 11.3 implies Claim 2, notice that for any \( \Pi' = (S_1, \ldots, S_{r-1}) \in C \) for which all vertices of \( X \) are bad, we have
\[
|P \cap \nabla(S_1, \ldots, S_{r-1})| \leq \sum_{x \in X} |\binom{N(x)}{2} \cap \nabla(S_1, \ldots, S_{r-1})| \\
= \sum_{x \in X} D_{\Pi'}(x) < tc_r n^2 p^2.
\]

52
So if (129) and (130) hold then in view of (128) we have, for every such $\Pi'$,

$$\text{def}_C(\Pi') > [(1 - 3\vartheta)b_r - c_r - \vartheta]tn^2p^2q > \vartheta tn^2p^2q = \nu tn^{3/2}p^2. \quad (131)$$

Thus (more or less repeating), failure of $R$ implies that either (129) or (130) is violated, which by Lemma 11.3 occurs with probability at most $\exp[-\Omega(tn^{3/2}p^2)]$, as required for Claim 2.

\[\square\]

**Proof of Lemma 11.3.** Notice that for any $\Pi' = (S_1, \ldots, S_{r-1}) \in C$,

$$G'' \cap \nabla(S_1, \ldots, S_{r-1}) = (P \cap \nabla(S_1, \ldots, S_{r-1})) \setminus G'. \quad (132)$$

(Recall the r.h.s. was defined in (4). It may be helpful to observe that we could replace $S_1$ by $S_1 \setminus X$ on the r.h.s. of (132), since $P$ does not contain pairs meeting $X$.)

We first consider (129). Set $D(x) = D(x; A_1 \setminus X, A_2, \ldots, A_{r-1})$ (recalling that this notation was introduced in (53)). From (122) we have $|P \cap \nabla(A_1, \ldots, A_{r-1})| > \sum_{x \in X} D(x) - o(tn^2p^2)$, so also (since $|G' \cap P| = O(tn^2p^2) = O(tn^2p^2)$, as follows from (121), $\mathcal{S}$ and (5)),

$$|(P \cap \nabla(A_1, \ldots, A_{r-1})) \setminus G'| > \sum_{x \in X} D(x) - o(tn^2p^2). \quad (133)$$

Let $m = (1 - \vartheta)np/(r - 1)$ and

$$Y = \{x \in X : \min\{|N(x) \cap A_i| : i \in [2, r - 1]\} > m\}$$

(the complement in $X$ of the set in (116) when $H = G'$). We assert that for $x \in Y$ we have

$$D(x) > (1 - \vartheta)b_rn^2p^2. \quad (134)$$

To see this, notice that

$$D(x) \geq \binom{d_W(x)}{2} - (r - 3) \binom{m}{2} - \binom{d_W(x)-(r-3)m}{2},$$

since we minimize $D(x)$ (subject to $x \in Y$) by taking $r - 3$ of the sets $N(x) \cap A_i$ ($i \in [2, r-1]$) to be of size $m$ and one to be of size $d_W(x)-(r-3)m$ (and $N(x) \cap (A_1 \setminus X)$ to be empty). A little straightforward calculation, using

$$d_W(x) > (1 - o(1))np - |X| = (1 - o(1))np$$

(which follows from $G' \in Q^*$ and (112)), then gives (134) (with the “$\vartheta$” actually about $2\vartheta/r$.)

\[\square\]
Thus, since $G' \notin \mathcal{T}$ (that is, $|Y| > (1 - \vartheta)t$), we have $\sum_{x \in X} D(x) > (1 - \vartheta)^2 b_r t^2 n^2 p^2$, which with (133) gives (say)

$$|(P \cap \nabla(A_1, \ldots, A_{r-1})) \setminus G'| > (1 - 2\vartheta)b_r t^2 n^2 p^2.$$  

Finally, (132) and Theorem 4.1 give

$$\Pr(|G'' \cap \nabla(A_1, \ldots, A_{r-1})| < (1 - 3\vartheta)b_r t^2 n^2 p^2) < \exp[-\vartheta^2 b_r t^2 n^2 p^2 q/2] = \exp[-\vartheta^2 b_r t^2 n^3/2p^2].$$

(Here we are actually using an easy consequence/extension of Theorem 4.1 for $\xi = B(n, p)$, $s \leq np$ and any $\lambda \geq 0$,

$$\Pr(\xi < s - \lambda) < \exp[-\lambda^2/(2s)].$$

To get this from Theorem 4.1 we may, for example, take $\xi' = B(n, q)$ with $q = s/n \leq p$ and use $\Pr(\xi < s - \lambda) \leq \Pr(\xi' < s - \lambda) < \exp[-\lambda^2/(2s)].$)

We now turn to (130). This is just a union bound but we have to be a little careful since the most naive bound, $\exp[O(n)]$, on the number of choices for the $S'_i$’s is unaffordable for small $t$. But this is an overcount: since $P \subseteq (\binom{N(X)}{2})$ (where $N(X) = \cup_{x \in X} N(x)$), we have (130) provided

$$|G'' \cap \nabla(T_1, \ldots, T_{r-1})| < |P \cap \nabla(T_1, \ldots, T_{r-1})|q + \vartheta t n^2 p^2 q$$

(135)

whenever $(T_1, \ldots, T_{r-1})$ is a partition of $N(X) \cap W$ (rather than a member of $\mathcal{C}$); and, since $|N(X) \cap W| < (1 + o(1))tnp$ (using $G' \in Q^*$), the number of possibilities for such a $(T_1, \ldots, T_{r-1})$ is less than $\exp_{r-1}[(1 + o(1))tnp]$.

On the other hand, for a particular $(T_1, \ldots, T_{r-1})$ we have

$$|P \cap \nabla(T_1, \ldots, T_{r-1})| \leq \sum_{x \in X} \binom{d_{W(x)}}{2} < tn^2 p^2$$

(again using $G' \in Q^*$ to bound the $d_{W(x)}$’s), which with Theorem 4.1 implies that the probability of violating (135) for a given $(T_1, \ldots, T_{r-1})$ is less than (say) $\exp[-\vartheta^2 t n^2 p^2 q/3] = \exp[-(\vartheta^2/3)tn^3/2p^2]$. The union bound (and the fact that $np = o(n^{3/2}p^2)$) now completes the argument.

12 Proof of Lemma 6.2

Write $Q$ for the collection of nonempty $Q \subseteq \binom{V}{2}$ satisfying (56). Lemma 6.2 says that w.h.p. if $Q \in Q$ and all pairs in $Q$ are bad for the balanced cut $\Pi = (A_1, \ldots, A_{r-1})$ (so by definition $Q \subseteq \binom{A_r}{2}$), then $\def_G(\Pi) \geq 2\vartheta^2|Q|$. 

54
We will show something a little stronger. For \( Q \in \mathcal{Q} \), let \( \mathcal{B}_Q \) be the set of graphs \( H \) for which there is some \( \Pi = (A_1, \ldots, A_{r-1}) \in \mathcal{C}(Q) \) (defined following (105)) with
\[
Q \subseteq Q_H(\Pi) \text{ and } \text{def}_{\mathcal{C}(Q),H}(\Pi) < 2r^2|Q|.
\] (136)
We show
\[
\Pr(\cup_Q \mathcal{B}_Q) = o(1)
\] (137)
(with the union over \( Q \in \mathcal{Q} \)). This is stronger than Lemma 6.2 because \( \text{def}_{\mathcal{C}(Q),\mathcal{G}}(\Pi) \) may be smaller (and is not larger) than \( \text{def}_{\mathcal{C}}(\Pi) \).

Again we can only afford a union bound after restricting the range of discourse. Let \( \mathcal{A} \) be the set of graphs \( H \) satisfying
\[
d_H(x) = (1 \pm \delta)np \quad \forall x \in V
\] (138)
(note this implies
\[
|H| = (1 \pm \delta)\binom{n}{2}p,
\] (139)
\[
|H[S] - |S|^2p/2| < \delta n^2p \quad \forall S \subseteq V
\] (140)
and
\[
\forall s \in [3, r] \text{ and } \{x_1, \ldots, x_s\} \in \binom{V}{s}, \kappa(x_1 \ldots x_s) = o(\Lambda_r(n,p)).
\] (141)
Then \( \Pr(\mathcal{A}) = o(1) \) by Propositions 4.3 and 4.4(a) and Corollary 4.10 (When \( s = r, \) (141) just says that \( \Lambda_r(n,p) = \omega(1) \)). For smaller \( s \) we use Corollary 4.10 in which, with \( \beta = (r - s)(s - 2)/[2(r + 1)] \), we have
\[
Z = \left[np^{(s+1)/2} - (s-2)\Lambda_r(n,p) < n^{-2\beta}\Lambda_r(n,p)
\] (using (2)). Then either \( \Lambda_r(n,p) < n^\beta \), implying \( Z < n^{-\beta} \), and the bound \( K (= o(\Lambda_r(n,p))) \) in (28) applies, or \( \Lambda_r(n,p) \geq n^\beta \), in which case the second bound in (28) applies and is \( o(\Lambda_r(n,p)) \).

We thus have
\[
\Pr(\cup_Q \mathcal{B}_Q) < o(1) + \sum_Q \Pr(\mathcal{A} \cap \mathcal{B}_Q),
\] and for (137) it’s enough to show that for each \( Q \in \mathcal{Q} \),
\[
\Pr(\mathcal{A} \cap \mathcal{B}_Q) < \exp[-3|Q| \log n].
\] (142)
(As elsewhere we just give the bound we need, but the 3 could be replaced by any constant if \( C \) (in (2)) is large enough.)

For the rest of this discussion we fix \( Q \in \mathcal{Q} \) and set \( \mathcal{C}(Q) = \mathcal{C} \); in particular “rigid,” “core,” \( b(H) := b(\mathcal{C}, H) \) and \( \text{def}_H := \text{def}_{\mathcal{C},H} \) now refer to this \( \mathcal{C} \). The main line of argument in this section will work with \( G_{n,M} \) rather than \( G_{n,p} \), but for the moment we stick with the latter.

Set \( \gamma' = 2\gamma \). (The difference between \( \gamma \) and \( \gamma' \) should be ignored; the extra 2, which could really be \( 1 + o(1) \), is needed to cover a minor detail at (159).)

With \( Q' \) a particular subset of \( Q \) to be specified below, set, for disjoint \( T_1, \ldots, T_{r-2} \subseteq V \setminus V(Q) \),

\[
F(T_1, \ldots, T_{r-2}) = \{ \kappa(Q', T_1, \ldots, T_{r-2}) < \gamma' | Q' | \Lambda_r(n, p) \}.
\]

Set

\[
K = 50\alpha^{-1}r^3.
\] (143)

In a sense our argument attempts—not always successfully—to reduce (142) to a situation where the following statement applies.

**Lemma 12.1.** Let \( \mathcal{R} \) be the set of graphs \( H \) satisfying: \( H \) is rigid, say with core \( \{ S_1, \ldots, S_{r-1} \} \), \( V(Q) \subseteq S_1 \), and \( F(S_2, \ldots, S_{r-1}) \) holds in \( H \). Then for any \( q > (1 - 2\delta)p \), \( \Pr(G_{n,q} \in \mathcal{R}) < \exp[-(10K \log r + 1) | Q' | \log n] \).

**Remarks.** We will make sure that \( Q' \) is a reasonably large subset of \( Q \)—in some cases it will be \( Q \) itself—so that the probability here will be smaller than the \( \exp[-3|Q|\log n] \) of (148). The reason for the \( q \) is that we will see some graphs \( G_{n,M} \) with \( M \) slightly smaller than \( \binom{n}{2}p \). The reason for the silly “+1” will appear in Lemma 12.3.

**Proof.** This follows immediately, via Lemma 10.2 from the next assertion, which is an easy consequence of Theorem 4.12 and Lemma 4.14.

**Lemma 12.2.** If \( T_1, \ldots, T_{r-2} \subseteq V \setminus V(Q) \) are disjoint with \( |T_1|, \ldots, |T_{r-2}| > n/r \), then for any \( q > (1 - 2\delta)p \),

\[
\Pr(G_{n,q} \models F(T_1, \ldots, T_{r-2})) < \exp[-(10K \log r + 1) | Q' | \log n].
\]

**Proof.** It is of course enough to show this when \( q = (1 - 2\delta)p \). Notice first that for any fixed \( \theta \) we have \( \Sigma < \theta nq/\log n \) for large enough \( C \) (\( \Sigma \) as in (55), \( C \) as in (2)). In particular we may assume that \( \Delta Q' < \theta nq/\log n \), where \( \theta \) is chosen so that the conclusion of Lemma 4.14 holds with \( \xi = \frac{1}{12}(10K \log r + 1)^{-1}r^{-2(r-2)} \) and \( q \) in place of \( p \).
Let \( H \) consist of all sets of the form 
\[
K(xy, Z) = \left( \{x, y\} \cup Z \right) \setminus \{xy\}
\]
with \( xy \in Q' \) and \( Z \in \binom{V}{r-2} \) meeting each of \( T_1, \ldots, T_{r-2} \). For \( K \in H \), let \( I_K \) be the indicator of \( \{ K \subseteq G \} \). Then

\[
\mu := \sum E I_K = |Q'| \prod_{i=1}^{r-2} |T_i| q^{-\binom{r}{2}} > |Q'| r^{-2} \Lambda_r(n, q)
\]
and, by our choice of \( \vartheta \),

\[
\Delta := \sum \sum \{ E I_K I_L : K, L \in H, K \cap L \neq \emptyset \} < \xi |Q'| \Lambda_r(n, q)^2 / \log n. \tag{144}
\]

Thus, since \( F(T_1, \ldots, T_{r-2}) = \{ \sum I_K < \gamma |Q'| \Lambda_r(n, p) \} \) (and \( \gamma |Q'| \Lambda_r(n, p) \) is much smaller than \( \mu \); see (52)), Theorem 4.12 gives (e.g.)

\[
\Pr(G_{n, q} \mid F(T_1, \ldots, T_{r-2})) < \exp[-\mu^2/(3\Delta)] < \exp[-\xi^{-1} r^{-2}(r-2)|Q'| \log n].
\]

We now define \( Q' \) and associated sets \( W_Q, Z_Q \subseteq V(Q) \); these will be used to deal with a minor technical point involving steps of “type B” below. (See the second paragraph of the proof of Lemma 12.5.) The choice here depends on the size of \( Q \). If \( Q \) is very small, say \( |V(Q)| \leq 13 \), then we take \( Q' = Q \) and \( W_Q = Z_Q = V(Q) \). Otherwise, we choose \( W_Q \subseteq V(Q) \) with \( |W_Q| \leq |V(Q)|/2 \) and \( |Q[W_Q]| \geq |Q|/5 \) (which is possible because, as is easily verified, if \( W \) is chosen uniformly from the \( |V(Q)|/2 \)-subsets of \( V(Q) \) then \( E|Q[W]| \geq |Q|/5 \), and take \( Q' = Q[W_Q] \) and \( Z_Q = V(Q) \setminus W_Q \).

From this point we switch to \( G_{n, M} \), noting, to begin, that Lemma 12.15 allows transfer of Lemma 12.1 to this setting:

**Lemma 12.3.** For \( \mathcal{R} \) as in Lemma 12.7 and any \( M \geq (1 - 2\delta) \binom{n}{2} p \),

\[
\Pr(G_{n, M} \in \mathcal{R}) < \exp[-10K \log r|Q'| \log n].
\]

We assume for the rest of this section that

\[
M = (1 \pm \delta) \binom{n}{2} p. \tag{145}
\]

We will show

\[
\Pr(G_{n, M} \in \mathcal{A} \cap B_Q) < \exp[-3|Q| \log n]. \tag{146}
\]
Of course this gives (142), since

$$\Pr(G_{n,p} \in A \cap B_Q) \leq \max \{ \Pr(G_{n,M} \in A \cap B_Q) : M = (1 \pm \delta)(\binom{n}{2})p \}. \quad (147)$$

For the proof of (146) we will prefer counting. Having specified $M$, let $G = G_Q$ be the set of $M$-edge graphs in $A \cap B$. We may rewrite (146) as

$$|G| < \exp[-3|Q| \log n](\binom{n}{2})^M. \quad (148)$$

Set

$$L = K|Q| \log n, \quad d = 2r^2|Q| \quad (149)$$

(so $d$ is the defect bound in (136); recall $K$ was defined in (143)), and

$$\beta = [r\delta n^2p + L](r - 1)/M. \quad (150)$$

We will need some weak constraints on $\beta$, e.g.

$$\delta < \beta < 2r^2\delta \quad (151)$$

(the upper bound since $L < Kn\Sigma \log n$ is much smaller than $\delta n^2p$; see (55)).

Fix some order “≺” on $C (= C(Q))$. We will be interested in sequences $G_0, \ldots, G_T$ with $G_0 \in G$, $T \leq L$, and, for $\Pi = (A_1, \ldots, A_{r-1})$ the first cut as in (136) (with $H = G_0$) and $1 \leq i \leq L$,

(a) if $\text{crit}(G_{i-1}) \cap \text{int}(\Pi) \neq \emptyset$, then $G_i = G_{i-1} - e$ for some $e \in \text{crit}(G_{i-1}) \cap \text{int}(\Pi)$ (recall “crit” was defined in (106)); otherwise:

(b) if $G_{i-1}$ is rigid with core $\{S_1, \ldots, S_{r-1}\}$ and $Q$ is not in the core, then $G_i = G_{i-1} + e$ for some $e \in (\nabla(ZQ, U) \cap \text{ext}(\Pi)) \setminus G_{i-1}$ with $Q \sim U \in \{S_1, \ldots, S_{r-1}\}$, where $Q \sim U$ means $V(Q)$ and $U$ are in the same block of some max cut;

(c) if $G_{i-1}$ is not rigid then $G_i = G_{i-1} - e$ for some $e \in G_{i-1} \cap \text{int}(\Pi)$;

(d) If $G_{i-1}$ is rigid with $Q$ in the core (and $\text{crit}(G_{i-1}) \cap \text{int}(\Pi) = \emptyset$), then $T = i - 1$ (and the rest of the sequence is vacuous).

We call sequences as above legal. The transition from $G_{i-1}$ to $G_i$ is the $i$th step of the sequence. A deletion as in (a) is a step of type $A$, an addition as in (b) is a step of type $B$, and a deletion as in (c) is a step of type $C$.

In what follows we will show, roughly, that each $G_0 \in G$ is the starting point of “many” legal sequences of some length, whereas the total number of legal sequences of each length is “small” (so $G$ is small).
For the first of these objectives, we show that there are many choices for $G_i$ whenever $G_{i-1}$ is as in (e), and at least one such choice if $G_{i-1}$ is as in (b). (In reality there are also many choices in the second case—though not necessarily as many as are guaranteed in the first—but all we need here is that the process doesn’t get stuck in situations that demand steps of type B. Of course it cannot get stuck at a step of type A.)

**Lemma 12.4.** In any legal sequence, fewer than $rd$ steps are of types A and B, and all but at most $rd$ indices $i$ satisfy

$$G_i \text{ is not rigid and step } i \text{ is of type } C.$$  \(152\)

**Proof.** This will follow from the next two assertions.

**Claim 1.** Each step of type A reduces $\text{def}(\Pi)$ (that is, if step $i$ is of type A then $\text{def}_{G_i}(\Pi) < \text{def}_{G_{i-1}}(\Pi)$) and no step increases $\text{def}(\Pi)$.

**Claim 2.** If the $i$th step is of type B then either

(i) for some $j \in [r-2]$, steps $i+1, \ldots, i+j-1$ are of type B and step $i+j$ is of type A, or

(ii) $T \in \{i, \ldots, i+r-3\}$.

**Proof of Claim 1.** Deletion of an edge in $\text{int}(\Pi)$ (as happens in all steps not of type B) doesn’t affect $|\Pi|$ (that is, $|\Pi_{G_i}| = |\Pi_{G_{i-1}}|$) and doesn’t increase $b$ (that is, $b(G_i) = b(G_{i-1})$), so doesn’t increase $\text{def}(\Pi)$. A step of type A decreases $b$ (and doesn’t affect $|\Pi|$), so decreases $\text{def}(\Pi)$. A step of type B increases each of $|\Pi|$ and $b$ by 1, so doesn’t affect $\text{def}(\Pi)$.  

**Proof of Claim 2.** If $G_{i-1}$ is rigid with core $\{S_1, \ldots, S_{r-1}\}$ and $Q$ is not in the core, then (i) for each $U \in \{S_1, \ldots, S_{r-1}\}$ there is some max cut with $V(Q)$ and $U$ contained in different blocks, and (ii) $Q \sim U$ for at least two choices of $U \in \{S_1, \ldots, S_{r-1}\}$. Say step $i$ is of type $B_j$ if there are exactly $j \in [2, r-1]$ such $U$’s. It is enough to show that if this is the case, then step $i+1$, if taken (i.e. if $T \neq i$), is either of type $B_l$ for some $l \leq j-1$, or of type A.

Suppose (w.l.o.g.) that $G_i = G_{i-1} + e$ with $e \in \nabla(Z_Q, S_1) \cap \text{ext}(\Pi)$. Then $G_i$ is rigid with core $\{S_1', \ldots, S_{r-1}'\}$ satisfying (i) $S_k' \supseteq S_k$ for each $i$;

(ii) $Q \not\sim S_1'$ (in $G_i$); and (iii) $\{k : Q \sim S_k' \text{ in } G_i\} \subseteq \{k : Q \sim S_k \text{ in } G_{i-1}\}$.

(Because: addition of $e$ increases $b$ (as noted in the proof of Claim 1), so does not increase the set of max cuts; this gives rigidity, (i) and (iii), and also (ii) once we observe that addition of $e$ doesn’t increase the size of any
cut with \( V(Q) \) and \( S_1 \) in the same block. Actually, for \( k > 1 \), \( Q \sim S_k' \) in \( G_i \)
iff \( Q \sim S_k \) in \( G_{i-1} \), but we don’t need this.)

In particular, since \( G_i \) is rigid, the \((i + 1)\)st step, if taken, must be of
type A or B; and if it is of type B, then (ii) and (iii) imply that it is type
B\( _i \) for some \( l \leq j - 1 \).

Now to complete the proof of Lemma 12.4 just notice that Claim 1 (with
the assumption \( \text{def}_{G_0}(\Pi) < d \)) guarantees that there are at most \( d - 1 \) steps
of type A, and this together with Claim 2 implies that all steps of types A
and B are contained in at most \( d \) intervals of length at most \( r - 1 \) (the extra
interval corresponding to (ii) in Claim 2). This gives the first assertion of
Lemma 12.4 (actually with \( i \) in place of \( r \)), and the second follows since
if \( i \) violates (152) then either step \( i \) or step \( i + 1 \) is of type A or B. ■

Lemma 12.5. For each \( G_0 \in \mathcal{G} \), there is some \( T \in \{0, \ldots, L\} \) for which
the number of legal sequences \( G_0, \ldots, G_T \) is at least \( L^{-1}(1 - \beta)M/((r - 1))^T - rd \).
(Recall \( \beta \) was defined at (150). Of course for small enough \( T \) this just says
that there is a legal sequence.)

Proof. Let \( G_0, \ldots, G_{i-1} \) be a legal initial segment (defined in the obvious
way). If \( G_{i-1} \) is as in (c) then the number of possibilities for \( G_i \) is

\[
|G_{i-1} \cap \text{int}(\Pi)| > |G_0 \cap \text{int}(\Pi)| - L
\]

\[
> \frac{n^2}{2(r-1)} - (r - 1)\delta n^2 p - L
\]

\[
> \frac{M}{r-1} - \frac{\delta n^2}{2(r-1)} - (r - 1)\delta n^2 p - L
\]

\[
> \frac{M}{r-1} - r\delta n^2 p - L = (1 - \beta)M/((r - 1)),
\]

where the second inequality uses (130) and the third uses \( M < (1 + \delta)n^2p/2 \).

If \( G_{i-1} \) is as in (b) then, as noted earlier, we just want to say there is some
legal choice for \( G_i \). Since \( Q \) is not in the core, we have \( Q \sim U \) for at least two
choices of \( U \in \{S_1, \ldots, S_{r-1}\} \), say \( S_1 \) and \( S_2 \), and \( e \) (the edge to be added
to \( G_{i-1} \)) can be any member of \( \nabla(Z_Q, (S_1 \cup S_2) \cap (A_2 \cup \cdots \cup A_{r-1})) \) \( \setminus G_{i-1} \);
so we just need to say this set is nonempty, which is true because:

\[
|\nabla(Z_Q, (S_1 \cup S_2) \cap (A_2 \cup \cdots \cup A_{r-1}))| > |Z_Q| \left[ \frac{2}{r} + (r - 2) (1 - \delta) \frac{1}{r} - 1 \right] n = |Z_Q| \cdot \frac{r - 2}{r - 1} (1 - \delta) n,
\]

(153)
since \( |S_1|, |S_2| > n/r \) and \( |A_j| > (1 - \delta)n/((r - 1)) \) for each \( j \) (since the \( S_j \)’s
form a core and \( \Pi \) is balanced), while, using Lemma 12.3 and (138),

\[
|G_{i-1} \cap \nabla(Z_Q, V \setminus Z_Q)| \leq \sum_{x \in Z_Q} d_{G_0}(x) + rd < |Z_Q|(1 + \delta)np + rd,
\]

60
which is (much) smaller than the bound in (153) (using (6) and $d = 2r^2 |Q| < 2r^2 |V(Q)| \Sigma \leq 4r^2 |Z_Q| \Sigma$).

Thus, again using Lemma 12.4 (to say a legal sequence involves at most $rd$ steps that are not of type C), Lemma 12.5 follows from the next little (presumably known) observation.

Lemma 12.6. Suppose $T$ is a tree of depth at most $L > 0$ and $W$ is a subset of the internal vertices of $T$ such that each internal vertex not in $W$ has at least $\Delta$ children and each path from the root contains at most $s$ vertices of $W$. Then there is some $T \in \{0, \ldots, L\}$ for which the number of leaves at depth $T$ is at least $L^{-1} \Delta^{L-s}$.

Proof. For each $T < L$ and leaf $w$ at depth $T$, add (to $T$) a $\Delta$-branching subtree of depth $L - T$ rooted at $w$, forming a tree $T'$. Then $T'$ has at least $\Delta^{L-s}$ leaves (which are, of course, all at depth $L$), e.g. since the natural root-leaves random walk down $T'$ reaches no leaf with probability more than $\Delta^{-(L-s)}$. On the other hand, the number of leaves in $T'$ is precisely $\sum_T m_T \Delta^{L-T}$, where $m_T$ is the number of leaves at depth $T$ in $T$. The lemma follows.

Let $\bigcup_{T=0}^{T=L} G_T$ be a partition of $G$ such that for each $T$ and $G_0 \in G_T$ the number of legal sequences $G_0, \ldots, G_T$ is at least $L^{-1} (1 - \beta) M / (r - 1)^{T-rd}$.

We next give upper bounds on the numbers of legal sequences $G_0, \ldots, G_T$ for $T \in \{0, \ldots, L\}$. For this part of the argument we think of starting with $G_T$ and moving (now mostly by adding edges) to $G_0$. For typographical reasons we now set $\binom{N}{2} = N$.

Notice that if $G_0, \ldots, G_T$ is a legal sequence then, by Lemma 12.4 (and the fact that only steps of type B add edges), $M - T \leq |G_T| < M - T + 2rd$. Note also—just to keep things slightly cleaner—that (using (6))

$$\sum_{0 \leq i < 2rd} \binom{N}{M - T + i} < \binom{N}{M - T + 2rd}$$

(154)

(so the r.h.s. bounds the number of possibilities for $G_T$ for a given $T$).

We first consider $T = L$. Here we use the second assertion of Lemma 12.4 If $i$ satisfies (152) then $G_i = G_{i-1} - e$ with $e$ contained in some $(C, G_i)$-component (since adding $e$ to $G_i$ doesn’t increase $b$); so, since $G_i$ is non-rigid,
the number of possibilities for $G_{i-1}$ (given $G_i$) is at most $(1-\alpha)n^2/(2(r-1))$. This bounds the total number of legal sequences $G_0, \ldots, G_L$ by

$$\binom{N}{M-L+2rd} L^{rd} n^{2rd} [(1-\alpha)n^2/(2(r-1))]^{L-rd}. \quad (155)$$

Here the first term counts choices of $G_L$, and the term $L^{rd}$ is for specification of a set of at most $rd$ indices $i$ for which (152) fails (and for which we use the trivial bound $n^2$ on the number of possibilities for $G_{i-1}$ given $G_i$).

We next consider $T < L$. Here, in contrast to what we did for $T = L$, our goal is to say that the number of possibilities for $G_T$ is small. Suppose $G_T$ has core $\{S_1, \ldots, S_{r-1}\}$ with $V(Q) \subseteq S_1$. We show that in this case

$$G_T \text{ satisfies } F(S_2, \ldots, S_{r-1}). \quad (156)$$

**Proof.** Notice that

$$G_T \cap \nabla(V(Q), S_2, \ldots, S_{r-1}) \subseteq \text{ext}(\Pi) \quad (157)$$

(since

$$G_T \cap \nabla(V(Q), S_2, \ldots, S_{r-1}) \subseteq G_T \cap \nabla(S_1, \ldots, S_{r-1}) \subseteq \text{crit}(G_T)$$

and $\text{crit}(G_T) \cap \text{int}(\Pi) = \emptyset$). We consider the cases $|V(Q)| > 13$ and $|V(Q)| \leq 13$ separately.

If $|V(Q)| > 13$ then

$$G_T \cap \nabla(W_Q, S_2, \ldots, S_{r-1}) \subseteq G_0 \quad (158)$$

(since edges added in moving from $G_0$ to $G_T$ meet $V(Q)$ only in $Z_Q = V(Q) \setminus W_Q$). Combining this with (157), which in particular implies that

$$\nabla_{G_T}(V(Q), S_2 \cup \cdots \cup S_{r-1}) \cap \nabla(V(Q), A_1 \setminus V(Q)) = \emptyset,$$

we have

$$\kappa_{G_T}(Q', S_2, \ldots, S_{r-1}) \subseteq \kappa_{G_0}(Q', A_2, \ldots, A_{r-1})$$

and thus

$$\kappa_{G_T}(Q', S_2, \ldots, S_{r-1}) \leq \kappa_{G_0}(Q', A_2, \ldots, A_{r-1}).$$

Since $Q' \subseteq Q \subseteq Q_{G_0}(\Pi)$, this gives (156).

If $|V(Q)| \leq 13$ (in which case $Q' = Q$ and $W_Q = V(Q)$), then we don’t quite have (158), but can (we assert) say

$$\kappa_{G_T}(Q, S_2, \ldots, S_{r-1}) \leq \kappa_{G_0}(Q, A_2, \ldots, A_{r-1}) + o(A_r(n, p)), \quad (159)$$

62
which again gives $F(S_2, \ldots, S_{r-1})$ for $G_T$.

For (159), notice that $|G_T \setminus G_0| \leq rd = O(1)$ (by Lemma 12.4 since all edges of $G_T \setminus G_0$ are added in steps of type B). On the other hand, because of (157), each member of $K_{G_T}(Q, S_2, \ldots, S_{r-1}) \setminus K_{G_0}(Q, A_2, \ldots, A_{r-1})$ uses one of the $O(1)$ pairs from $Q$, together with at least one of the at most $rd = O(1)$ edges of $G_T \setminus G_0$, so uses two vertices of $V(Q)$ plus, for some $s \in [3, r]$, precisely $s$ other vertices incident with edges of $G_T \setminus G_0$. Thus, since the number of possibilities for these $s$ vertices is $O(1)$, (159) follows from (141).

Lemma 12.3 (applicable since $|G_T| \geq M - L > (1 - 2\delta)(\frac{n}{2})p$) and (154) now bound the number of choices for $G_T$ by

$$\xi = \exp[-10K \log r|Q'||\log n] \leq \exp[-2K \log r|Q||\log n],$$

so we may crudely bound the number of legal sequences of length $T$ by

$$\xi (N - T + 2rd) N^T.$$  \hspace{1cm} (161)

(The $N^T$ could of course be improved along the lines of the above discussion for $T = L$.)

Combining the bounds in (155) and (161) with the fact that each $G_0 \in G_T$ is the first term of at least $L^{-1}|(1 - \beta)M/(r - 1)|^{T-rd}$ legal sequences $(G_0, \ldots, G_T)$, we have

$$|G_L| \leq L \left[\frac{r-1}{(1-\beta)M}\right]^{L-rd} \left(\frac{N}{M-L+2rd}\right) L^{rd} n^{2rd} \left[\frac{N}{2(r-1)}\right]^{L-rd}.$$ \hspace{1cm} (162)

and, for $T < L$ (with $\xi$ as in (160)),

$$|G_T| \leq L \left[\frac{r-1}{(1-\beta)M}\right]^{T-rd} \xi \left(\frac{N}{M-T+2rd}\right) N^T.$$ \hspace{1cm} (163)

Thus, noting that (6) implies, for any $-2rd \leq i \leq M$,

$$\binom{N}{M-i} < \left[(1 + o(1))M/N\right]^i \binom{N}{M},$$

(164)
we have
\[ |G_L| < n^{4d}(N/M)^{2d}(1 - \alpha + \beta)^L \left( \frac{N}{M} \right) < n^{6d}(1 - \alpha + \beta)^L \left( \frac{N}{M} \right) \]  
and, for \( T < L \),
\[ |G_T| < n^{2d}(N/M)^{2d} \left[ \frac{r-1+o(1)}{1-\beta} \right]^T \xi \left( \frac{N}{M} \right) < n^{4d} \left[ \frac{r-1}{1-\beta} \right]^T \xi \left( \frac{N}{M} \right) \]
(166)
(where, to make things a little easier to look at, we used (151) in (165) (to say \((1 - \alpha + o(1))/(1 - \beta) < 1 - \alpha + \beta)\) and \((1 + o(1))^L < n^{o(d)}\) in (166)).

Finally, summing these bounds and using (143), (149), (151) and (160) gives (148):
\[ |G| < \left[ n^{6d}(1 - \alpha + \beta)^L + Ln^{4d}\left( \frac{r-1}{1-\beta} \right)^L \xi \left( \frac{N}{M} \right) \right] \left( \frac{N}{M} \right) < \exp\left[ -3|Q| \log n \right] \left( \frac{N}{M} \right). \]
(Here \((1 - \alpha + \beta)^L \approx \exp\left[ -50r^3|Q| \log n \right]\) dominates \( n^{6d} = \exp\left[ 12r^3|Q| \log n \right]\), and in

\[ Ln^{4d}\left( \frac{r-1}{1-\beta} \right)^L \xi \ < \ \exp[8r^3|Q| \log n + L \log r - 2K \log |Q| \log n] \]
\[ = \ \exp[8r^3|Q| \log n - K \log r|Q| \log n], \]
the term \( 8r^3|Q| \log n \) in the exponent is negligible.)

13 Remarks

A. An obvious (but probably formidable) challenge is to prove Theorem 1.2 with the correct \( C \). The natural guess is that
\[ C > \left[ 2r/(r + 1) \right]^2 \]
suffices, this being what’s needed to guarantee that (w.h.p.) all edges lie in \( K_r \)'s. Note, though, that the even more precise “stopping time” version—which says that if we choose \( e_1, \ldots \in E(K_n) \), with \( e_i \) uniform from edges as yet unchosen, and stop as soon as every \( e_i \) is in a \( K_r \), then w.h.p. the resulting \( G \) satisfies \( t_r(G) = b_r(G) \)—is not correct. To see this (informally), suppose \( xy \) is the last edge added in forming \( G \). There is then some \( uv \in G \) such that every \( K_r \) on \( uv \) also contains \( xy \). But in this case \( t_r(G) > b_r(G) \) whenever there is a maximum cut with (for example) \( u, v \) and \( x \) in a single block, and this is not a low probability event.
B. For a property (i.e. isomorphism-closed collection) $\mathcal{F}$ of graphs on $[n]$, set $\mu_p(\mathcal{F}) = \Pr(G_{n,p} \in \mathcal{F})$, and define the *threshold* for $\mathcal{F}$ to be

$$p_c(\mathcal{F}) := \min\{p : \mu_q(\mathcal{F}) \geq 1/2 \quad \forall q \geq p\}.$$ 

If $\mathcal{F}$ is increasing (preserved by addition of edges) then $\mu_p(\mathcal{F})$ is increasing in $p$ and $p_c(\mathcal{F})$ is that $p$ (unique except in trivial cases) for which $\mu_{p_c}(\mathcal{F}) = 1/2$. (This is not the original definition of threshold in [11] but is more or less equivalent. Of course these notions make sense more generally, but for this brief discussion we stick to graph properties.)

The property $\mathcal{F}_r := \{t_r(G) = b_r(G)\}$ of Theorem 1.2 is not increasing and $\mu_p(\mathcal{F}_r)$ is not increasing in $p$ (for a given $n$); rather it is close to 1 for $p$ either large enough or quite small, and is easily seen to be close to 0 for some intermediate values. Still, there is an interesting possibility (which for $r = 3$ was suggested in [9]), namely, *could it be that, for given $r$ and $n$, $\mu_p(\mathcal{F}_r)$ has just one local minimum?* In fact it would seem to be interesting to prove such a statement for *any* (natural) nonincreasing property, and similarly interesting to identify some natural situation(s) in which $\mu_p(\mathcal{F})$ is increasing although $\mathcal{F}$ is not; might this, for example, be true of the property $\{t_r(G) < (1 - 1/(r - 1) + \varepsilon)|G|\}$ (cf. Theorem 1.4)?

C. Another obvious question is, does Theorem 1.2 extend to graphs $H$ other than cliques; that is, if $t_H(G)$ and $b_H(G)$ are the maximum values of $|K|$ for $K$ ranging over, respectively, $H$-free and $(\chi(H) - 1)$-partite subgraphs of $G$ (where $\chi$ is chromatic number), when is $G_{n,p}$ likely to satisfy

$$\mathcal{F}_H := \{t_H(G) = b_H(G)\}?$$

It is easy to see that the question only makes sense when $H$ is critical, that is, contains a (color-critical) edge $e$ such that $\chi(H - e) < \chi(H)$. As noted in Section 1 the result of [2] mentioned there holds in this generality, and it is suggested by the authors of [5] that their main result (Theorem 1.1 above) should as well. Here again there is a natural guess. Say $\mathcal{G}_H$ holds for $G$ if each $e \in E(G)$ is color-critical in some copy of $H$ in $G$.

**Conjecture 13.1.** For any $H$ with a color-critical edge, $p_c(\mathcal{F}_H) = O(p_c(\mathcal{G}_H))$.

(An old theorem of M. Simonovits [33] says that if $H$ is critical then $K_n$ satisfies $\mathcal{F}_H$ for large enough $n$. For $H = K_r$, Conjecture 13.1 is Theorem 1.2. The threshold for $\mathcal{G}_H$ is not a mystery, but takes some space and is omitted here. One may also guess (cf. A above) that in fact $p_c(\mathcal{F}_H) \sim p_c(\mathcal{G}_H)$.)
D. Trivially necessary for \( t_r(G) = h_r(G) \) is that every max(imum) cut be maximal \( K_r \)-free; thus Theorem 1.2 implies that for \( p \) in our range this condition holds w.h.p. This is in fact an instance of Lemma 3.2 but is there an easier way to prove it? For \( r = 3 \) (where the requirement is that for each max cut \((A,B)\) and \(xy \in G[A]\), \(x\) and \(y\) share a neighbor in \(B\)), the proof implicit in [9] is simple once found; but finding it was the real key to that paper.

Here we are back to the difficulties associated with max cuts (cf. the beginning of Section 10). On this theme, a simple question suggested by the present work is: for what \( p \) is it true that \( G_{n,p} \) (w.h.p.) admits no max cut \((A_1,\ldots,A_{r-1})\) such that some \(x\) has all its neighbors in a single \(A_i\)?

When \( r \geq 4 \), the proof of Lemma 6.1 can be adapted to give this for \( p > C_r n^{-1/2} \), but it should really be both easier and true for considerably smaller \( p \), perhaps requiring only \( p \gg n^{-1} \log n \). For \( r = 3 \) we don’t even know that \( p > Cn^{-1/2} \) is enough, though the same guess seems reasonable:

**Conjecture 13.2.** If \( p \gg n^{-1} \log n \), then w.h.p. no (ordinary) max cut of \( G_{n,p} \) contains all (or even 51\% of) the edges at any vertex.

Thus \( p \) should be large enough that a typical cut contains only about half the edges at any vertex; a max cut will of course tend to contain more, but the guess is that this effect is relatively mild. (It follows from [5] Theorem 1.4 that the conclusion holds for \( p \) at least about \( n^{-1/3} \log^{2/3} n \).)

E. Long as the above argument is, the full proof of Theorem 1.2 is longer still, in that we begin with the already very difficult assertion (Theorem 1.4) that every large enough \( F \subseteq G = G_{n,p} \) is nearly \((r - 1)\)-partite. Note, though, that we really only need this when \( F \) is maximum \( K_r \)-free (and for \( p \) slightly larger than what’s specified in [3]). In fact both [2] and [5] (which of course preceded [7]) begin with such limited versions of Theorem 1.4, and it would be interesting to see whether a version adequate to present purposes could be proved relatively easily.

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