CONSISTENT AND COVARIANT COMMUTATOR ANOMALIES IN THE CHIRAL SCHWINGER MODEL

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Abstract

We derive all covariant and consistent divergence and commutator anomalies of chiral QED$_2$ within the framework of canonical quantization of the fermions. Further, we compute the time evolution of all occurring operators and find that all commutators evolve canonically. We comment on the relation of our results to the finding of a nontrivial $U(1)$-curvature in gauge-field space.

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I. INTRODUCTION

Chiral gauge theories are anomalous when the fermions are quantized. These anomalies have several, well-known consequences. The divergence of the gauge current deviates from its canonical value by a certain polynomial in the gauge field ("anomaly") [1]-[4]. Further, the commutators of charge densities [5] and of the generators of time-independent gauge transformations (the "Gauss law operators") acquire anomalous contributions, too [6]-[8]. In addition, there exist two versions, namely the consistent and covariant ones, of both the divergence and Gauss law commutator anomalies [9,4]. All these anomalies are determined by geometrical or cohomological considerations [10]-[12].

However, whereas the complete Gauss law commutator is fixed in this way, the situation is less clear for the individual components of the Gauss law operator \(G\),

\[
G(x^1) = \delta(x^1) - iJ_0^0(x^1), \quad \delta(x^1) = \partial_{x^1} \frac{\delta}{\epsilon \delta A_1(x^1)}
\]  

where \(\delta(x^1)\) generates gauge transformations on the gauge fields, and \(J_0^0(x^1)\), the zero component of the chiral gauge current, acts on fermions (we have already restricted to our two-dimensional, abelian model in (1)). The results for the commutators of the individual components of \(G\) depend both on the computational scheme and on whether one is dealing with the consistent or covariant case. Further, in most computations VEVs are computed instead of the operator relations themselves (e.g. by the use of the BJL limit, or by covariantly regularized gauge current VEVs, or by a generalized point splitting method [13]-[25]).

In this paper we shall follow a different approach, by using a method that was introduced in [26,27] for the computation of the covariant anomaly. We shall construct the second quantized fermionic operators in an external gauge field both in the interaction and Heisenberg pictures. This will enable us to compute the consistent and covariant divergence anomalies and all the commutators of the Gauss law operator components. We shall find that all commutators evolve canonically under time evolution. Finally, we shall comment on the relation of our results to the findings of a nontrivial \(U(1)\)-connection and curvature for the functional derivative \((\delta/\delta A_1(x^1))\) acting on fermionic Fock states [28]-[31].

At first sight, it might seem to be a strange idea to discuss consistent and covariant anomalies for the simple model of chiral QED\(_2\), where both the consistent and covariant anomalies are gauge invariant expressions, but, nevertheless, there is a difference [26,27]. This may be easily inferred, e.g., from the effective action \(W[A]\) of the chiral Schwinger model [8,32,33],

\[
W[A] = \frac{i e^2}{4 \pi} \int d^2x d^2y A_\mu(x) \frac{1}{\Box}(x - y)[\partial^\mu \partial^\nu - \Box g^{\mu\nu}]
\]

\[
-\frac{1}{2} (\epsilon^{\mu\alpha} \partial^\mu \partial_\alpha + \epsilon^{\nu\alpha} \partial^\nu \partial_\alpha) A_\mu(y) + \int d^2x A_\mu(x) C^{\mu\nu} A_\nu(x).
\]  

Here \(C^{\mu\nu}\) accounts for the possibility of adding local counterterms to the effective action, and we choose \(C^{\mu\nu}\) symmetric, because only the symmetric part of \(C^{\mu\nu}\) will contribute to VEVs upon functional differentiation w.r.t. \(A_\mu\). We shall not assume \(C^{\mu\nu}\) to be Lorentz covariant in general, as we want to relate to the canonical formalism where manifest Lorentz
covariance is absent. Setting $C^{\mu\nu} = 0$ for the moment we find for the VEV of the consistent current

$$\langle J^\mu_{\text{cons.}}(x) \rangle \equiv \frac{i}{e} \frac{\delta W}{\delta A_\mu(x)} = -\frac{e}{2\pi} \int d^2 y \frac{1}{\Box} (x - y) [\partial^\mu \partial^\nu - \Box g^{\mu\nu}]$$

$$-\frac{1}{2} (\epsilon^{\nu\alpha} \partial^\mu \partial_\alpha + \epsilon^{\mu\alpha} \partial^\nu \partial_\alpha) A_\nu(y) =: -\frac{e}{2\pi} \int d^2 y K^{\mu\nu}(x, y) A_\nu(y). \quad (3)$$

The nonlocal kernel $K^{\mu\nu}(x, y)$ is Bose symmetric, $K^{\mu\nu}(x, y) = K^{\nu\mu}(y, x)$, because it was derived as the functional derivative of an effective action; but $\langle J^\mu_{\text{cons.}}(x) \rangle$ is not invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$.

On the other hand, look at the expression

$$\langle J^\mu_{\text{cov.}}(x) \rangle = -\frac{e}{2\pi} \int d^2 y \frac{1}{\Box} (x - y) [\partial^\mu \partial^\nu - \Box g^{\mu\nu} - \epsilon^{\nu\alpha} \partial^\mu \partial_\alpha] A_\nu(y)$$

$$= -\frac{e}{2\pi} \int d^2 y K^{\mu\nu}(x, y) A_\nu(y) \quad (4)$$

which differs from (3) by a local polynomial (see (8)). Expression (4) is gauge invariant but not Bose symmetric (i.e. it may not be obtained as the functional derivative of some effective action). The VEVs of the two currents lead to the consistent and covariant anomalies

$$\partial_\mu \langle J^\mu_{\text{cons.}}(x) \rangle = -\frac{e}{4\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu(x) = \frac{e}{4\pi} (\partial_0 A_1(x) - \partial_1 A_0(x)) \quad (5)$$

$$\partial_\mu \langle J^\mu_{\text{cov.}}(x) \rangle = -\frac{e}{2\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu(x) \quad (6)$$

which differ by a factor of 2 but are both given by the same gauge invariant expression. However, the addition of a local counterterm $\int d^2 x A_\mu(x) C^{\mu\nu} A_\nu(x)$ to the effective action allows for a change of $\langle J^\mu_{\text{cons.}}(x) \rangle$ (where $C^{\mu\nu} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$),

$$\langle J^0_{\text{cons.}}(x) \rangle \to \langle J^0_{\text{cons.}}(x) \rangle + a A_0(x) + b A_1(x)$$

$$\langle J^1_{\text{cons.}}(x) \rangle \to \langle J^1_{\text{cons.}}(x) \rangle + b A_0(x) + c A_1(x) \quad (7)$$

where $a, b, c$ are arbitrary numbers. This allows for consistent anomaly expressions like e.g. $\frac{e}{2\pi} \partial_1 A_0 + c \partial_1 A_1$ (see (52)). On the other hand, the covariant current and anomaly are fixed by gauge invariance, as we shall see.

The consistent and covariant current VEVs (3) and (4) differ by the (local) Bardeen-Zumino polynomial $P^\mu(x)$

$$P^\mu(x) = \langle J^\mu_{\text{cov.}}(x) \rangle - \langle J^\mu_{\text{cons.}}(x) \rangle$$

$$= \frac{e}{4\pi} \int d^2 y \frac{1}{\Box} (x - y) (\epsilon^{\nu\alpha} \partial^\mu \partial_\alpha - \epsilon^{\mu\alpha} \partial^\nu \partial_\alpha) A_\nu(y)$$

$$= -\frac{e}{4\pi} \epsilon^{\mu\nu} A_\nu(x). \quad (8)$$
Other choices (7) for the consistent current VEV change $P^\mu$ by the local and symmetric term $C^{\mu\nu}A_\nu$.

Altogether, we find that the consistent current has to obey Bose symmetry, which is the abelian version of the Wess-Zumino consistency condition [34], whereas the covariant current is determined by gauge invariance, and there is no choice that obeys both Bose symmetry and gauge invariance.

Next we have to fix our conventions. We are in two-dimensional Minkowski space-time $(x^0, x^1) \equiv x$ with the conventions

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon_{01} = 1, \quad x^\pm = x^0 \pm x^1.$$  

(9)

Our Lagrangian density is

$$\mathcal{L} = \bar{\Psi}(i\partial - eA\gamma_5)\Psi,$$  

(10)

and we use the $\gamma$ matrix conventions

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(11)

$$\gamma_5 = \gamma^0 \gamma^1,$$  

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5).$$  

(12)

and the currents ($J^\mu = \bar{\Psi}\gamma^\mu\Psi$)

$$J^0 = (\Psi_+^\dagger, \Psi_-^\dagger)\gamma_0\gamma^0 \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \Psi_+^\dagger\Psi_+ + \Psi_-^\dagger\Psi_-$$

$$J^1 = (\Psi_+^\dagger, \Psi_-^\dagger)\gamma^1 \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \Psi_+^\dagger\Psi_+ - \Psi_-^\dagger\Psi_-.$$  

(13)

$$J_+^\mu = \bar{\Psi}\gamma^\mu P_+\Psi = \frac{1}{2}(g^{\mu\nu} + \epsilon^{\mu\nu})J^\nu.$$  

$$J_\pm^0 = J_+^1 = \Psi_+^\dagger\Psi_+ = J_+.$$  

(14)

This leads to the Hamiltonian density

$$\mathcal{H} = i\Psi_+^\dagger\partial_0\Psi_+ + i\Psi_-^\dagger\partial_0\Psi_- - \mathcal{L}$$

$$= -i\Psi_+^\dagger\partial_1\Psi_+ + i\Psi_-^\dagger\partial_1\Psi_- + eA_+J_+ + eA_-J_-$$

(15)

$$(\partial_\mu \equiv (\partial/\partial x^\mu)),$$ where

$$A_+ := A_0 + A_1.$$  

(16)

Observe that in the Lagrangian and Hamiltonian no kinetic terms for the gauge field occur, i.e., we shall treat $A_\mu$ as an external field throughout the article.

Further, we shall frequently use the Baker-Campbell-Hausdorff (BCH) formula for operators

$$e^{A}Be^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$$  

(17)
II. WHAT TO EXPECT

It is a well-known fact that both divergence anomaly and anomalous Gauss law commutator may be derived – up to an overall constant – by cohomological methods via the so-called descent equations. For the consistent anomaly this was done, e.g., in [10,11], and for the covariant case in [12]. Here we want to review these results briefly for \( d = 1 + 1 \) dimensions, because they will tell us what to expect in the forthcoming computations.

All the two-dimensional anomalies and anomalous commutators may be derived from the 3-dimensional Chern-Simons form

\[
Q_3^0(A, F) = \text{tr} \left( AF - \frac{1}{3} A^3 \right)
\]  

(18)

where we deal with the general nonabelian case for the moment and the trace is in color space. Here \( A \) and \( F = dA + A^2 \) are differential forms on coordinate space (\( A = A_\mu dx^\mu \) etc.)

By substituting \( A \rightarrow A + v \), where \( v \) is a one-form in group space, and by collecting powers in \( v \)

\[
Q_3^0(A + v, F) = Q_3^0(A, F) + \text{tr} \, v dA - \text{tr} \, v^2 A - \frac{1}{3} \text{tr} \, v^3 =: \sum_{k} Q_{3-k}^k
\]  

(19)

(where \( k \) counts the degree in \( v \)), one obtains the expressions for the consistent anomaly \( Q_2^1 \) and Gauss law commutator \( Q_1^2 \). However, this result is not unique. The cohomological information is encoded in the relation

\[
\delta_v Q_j^i = -dQ_{j-1}^{i+1}
\]  

(20)

where \( \delta_v \) is the exterior derivative on group space that generates gauge transformations on \( A, F \) and \( v \) [4]. As each \( Q_j^i \) is only fixed up to a total derivative, this, in turn, makes \( Q_{j-1}^{i+1} \) ambiguous. In our case we have, e.g.,

\[
Q_2^1 = \text{tr} \, v dA \quad \ldots \quad Q_2^1 = -\text{tr} \, v^2 A
\]  

\[
Q_2^1 = \text{tr} \, dv A \quad \ldots \quad Q_2^1 = \text{tr} \, v dv
\]  

(21)

where the two versions of \( Q_2^1 \) differ by a total derivative. In the abelian case all higher powers of the one-forms \( A, v \) vanish, and we find for the divergence anomaly \( A \) and Gauss law commutator \( S \)

\[
A(x) \sim dA(x) , \quad S(x^1, y^1) \sim \delta'(x^1 - y^1) \sim 0
\]  

(22)

where \( S \) is ambiguous and cohomologically equivalent to zero.

On the other hand, the covariant anomalies may be found by the expansion of

\[
Q_3^0(A + v, F + Dv) = Q_3^0(A, F) + 2\text{tr} \, v F + \text{tr} \, (v dv + v^2 A) - \frac{1}{3} \text{tr} \, v^3 =: \sum_{k} \tilde{Q}_{3-k}^k
\]  

(23)

\((D = d + [A, _])\), giving rise to the covariant anomalies

\[
\tilde{A}^a(x) \sim 2F^a(x) , \quad \tilde{S}^{ab}(x^1, y^1) \sim D^{ab}(x^1)\delta(x^1 - y^1)
\]  

(24)

\((a \ldots \text{color index}, D^{ab} \ldots \text{covariant derivative})\). In the abelian case this simplifies to

\[
\tilde{A}(x) \sim 2dA , \quad \tilde{S}(x^1, y^1) \sim \delta'(x^1 - y^1).
\]  

(25)

Observe that, again, the covariant anomaly is twice the consistent one. Further, the \( \tilde{Q}_{3-k}^k \) are uniquely fixed by the requirement of gauge covariance.
Next we should briefly review the second quantization of the free theory. In this section we closely follow the discussion of [26]. The free spinors obey the free Dirac equation
\[
(\partial_0 \pm \partial_1)\Psi_\pm = 0
\] (26)
and are therefore given by
\[
\Psi_\pm(x^0, x^1) = \int_{-\infty}^{\infty} \frac{dk^1}{\sqrt{2\pi}} b_\pm(k^1)e^{-ik^1x^+}
\] (27)
with dispersion \(k^0 = \pm k^1\). Here the \(b_\pm\) are the usual annihilation operators obeying the CAR
\[
\{b_+(k^1), b_\dagger_+(k'^1)\} = \{b_-(k^1), b_\dagger_-(k'^1)\} = \delta(k^1 - k'^1)
\] (28)
and all other anticommutators vanish. The free Hamiltonian reads
\[
H_0 = \int dx^1(-i\Psi_+^\dagger \partial_1 \Psi_+ + i\Psi_-^\dagger \partial_1 \Psi_-) = \int dk^1 k^1(b_+^\dagger(k^1)b_+(k^1) - b_-^\dagger(k^1)b_-(k^1))
\] (29)
and is unbounded from below. This necessitates the introduction of the Dirac vacuum
\[
b_\pm(k^1)|\rangle_D = 0 \quad \ldots \quad \pm k^1 > 0
\]
\[
b_\dagger_\pm(k^1)|\rangle_D = 0 \quad \ldots \quad \pm k^1 < 0
\] (30)
and the normal ordering w.r.t. the Dirac vacuum,
\[
Nb_\dagger_\pm(k^1)b_\pm(k'^1) = b_\dagger_\pm(k^1)b_\pm(k'^1)|_{\pm k'^1 > 0} - b_\pm(k'^1)b_\dagger_\pm(k^1)|_{\pm k'^1 < 0}
\] (31)
This normal ordering has the consequence of inducing the Schwinger term in the commutator of normal-ordered currents (a fact that was already known in the thirties [35]),
\[
NJ_+(x^-) = \int \frac{dk^1dk'^1}{2\pi} e^{-i(k'^1-k^1)x^-}Nb_\dagger_\pm(k^1)b_\pm(k'^1)
\] (32)
and its Fourier transforms
\[
N\tilde{J}_+(p^1) = \int dx^- e^{ip^1x^-}NJ_+(x^-) = \int dk^1 Nb_\dagger_\pm(k^1-p^1)b_+(k^1).
\] (33)
A straightforward computation reveals
\[
[NJ_+(p^1), N\tilde{J}_+(q^1)] = \int_{-\infty}^{\infty} dk^1 (b_\dagger_+(k^1 - q^1)b_+(k^1 + p_1) - b_\dagger_+(k^1 - p^1 - q^1)b_+(k^1)).
\] (34)
One possibility for the further evaluation of (34) is to rewrite it as its normal-ordered version plus some remainder. In the normal-ordered expression the shift is legitimate, and the remainder precisely gives the Schwinger term (see, e.g., [36]–[38]):

\[ [N\tilde{J}_+(p^1), N\tilde{J}_+(q^1)] = p^1\delta(p^1 + q^1). \] (35)

Further possibilities for the evaluation of (34) are given e.g. in [26,39] and lead to the same result. In coordinate space the Schwinger term reads

\[ [NJ+(x), NJ+(y)] = -\frac{i}{2\pi}\partial_{x^1}\delta(x^- - y^-). \] (36)

For \( NJ_- \) an analogous result may be obtained (differing in sign from (36)), but as \( \Psi_- \) is noninteracting, it is unimportant in the sequel.

For the interacting, positive chirality sector we shall identify the normal-ordered current \( NJ^+ \) with the consistent current (this identification will be justified in Section 5). One immediate consequence is that \( NJ^+ \) cannot be gauge invariant (see Section 4).

Therefore, next we should find a candidate for the covariant current. Precisely this was done in [26] by introducing the concept of kinetic normal ordering, what we want to review now.

The Dirac vacuum is introduced by splitting the fermionic Hilbert space into eigenstates of the free Hamiltonian \(-i\partial_1\) with positive and negative energy. Instead, one could split into eigenstates of the kinetic momentum operator \(-i\partial_1 + eA_1\) with positive and negative kinetic energy. These eigenvalues (the kinetic energy) are gauge invariant, measurable quantities, and the corresponding kinetic normal ordering will indeed lead to gauge invariant operators (see Section 4). So let us expand the free spinor \( \Psi^+_+ \) into annihilation operators of the free and kinetic momentum:

\[ \Psi^+(x) = \int \frac{dk^1}{\sqrt{2\pi}} e^{ik^1 x^1}b^+(k^1, x^0) = \int \frac{dk^1}{\sqrt{2\pi}} e^{ik^1 x^1 - i e\lambda(x)} \tilde{b}^+(k^1, x^0) \] (37)

\[ \lambda(x) := \int_{-\infty}^{x^1} d\bar{x}^1 A_1(x^0, \bar{x}^1) \] (38)

\( b^+(k^1, x^0) \) is the free time evolution of \( b^+(k^1) \), (27); \( \exp(ik^1 x^1 - i e\lambda(x)) \) is an eigenfunction of the kinetic momentum \((-i\partial_1 + eA_1)\) with eigenvalue \( k^1 \).

\( b^+ \) and \( \tilde{b}^+ \) (and their Fourier transforms \( \Psi^+_+ \) and \( \tilde{\Psi}^+_+ \)) are related by a gauge transformation,

\[ \tilde{\Psi}^+_+(x) = \int \frac{dk^1}{\sqrt{2\pi}} e^{ik^1 x^1} \tilde{b}^+_+(k^1, x^0) = e^{ie\lambda(x)}\Psi^+_+(x) \]

\[ \equiv \Lambda^+_+(x^0)\Psi^+_+(x)\Lambda^+_+(x^0) = \int \frac{dk^1}{\sqrt{2\pi}} e^{ik^1 x^1} \Lambda^+_+(x^0) b^+_+(k^1, x^0) \Lambda^+_+(x^0) \] (39)

where \( \Lambda^+_+(x^0) \) implements the gauge transformation on Fock space and is given by

\[ \Lambda^+_+(x^0) = e^{ie\int dx^1 \lambda(x)NJ^+(x)} \] (40)
as may be shown easily by using the BCH formula (17) and the ETC relation

\[
[NJ_+(x^1_n), \ldots [NJ_+(x^1_1), \Psi_+(y^1)] \ldots ] = (-1)^n \Psi_+(y^1) \prod_{i=1}^{n} \delta(x^1_i - y^1).
\] (41)

Actually, \(\bar{\Psi}[A_1 = 0] = \Psi\), and therefore \(\Lambda_+^\dagger(x^0)\) implements the gauge transformation that goes to Coulomb gauge \(A_1 = 0\).

Now, following [26], we define kinetic normal ordering in complete analogy with (31) as

\[
\bar{N} \bar{b}_+(k^1, x^0) \bar{b}_+(k'^1, x^0) = \bar{b}_+(k^1, x^0) \bar{b}_+(k'^1, x^0)|_{k'^1 > 0} - \bar{b}_+(k'^1, x^0) \bar{b}_+(k^1, x^0)|_{k'^1 < 0} = \Lambda_+^\dagger(x^0) N \bar{b}_+(k^1, x^0) \bar{b}_+(k'^1, x^0) \Lambda_+(x^0),
\] (42)

where the last equality follows at once. Using it we find for the kinetically normal ordered current and free Hamiltonian

\[
\bar{N} J_+(x) = \Lambda_+^\dagger(x^0) N J_+(x) \Lambda_+(x^0)
\] (43)

\[
\bar{N} H_0(x^0) = \Lambda_+^\dagger(x^0) N H_0(x^0) \Lambda_+(x^0) - e \int dx^1 A_1(x) \Lambda_+^\dagger(x^0) N J_+(x) \Lambda_+(x^0).
\] (44)

With the help of the BCH formula and the identity

\[
[N H_0(x^0), NJ_+(x)] = -i \partial_0 N J_+(x) = i \partial_1 N J_+(x)
\] (45)

(where we used the fact that \(N J_+(x)\) is a Heisenberg operator of the free theory in the first step, and the conservation of the free current in the second step) we finally get

\[
\bar{N} J_+(x) = NJ_+(x) + \frac{e}{2\pi} A_1(x)
\] (46)

\[
\bar{N} H_+(x^0) = NH_+(x^0) + \frac{e^2}{4\pi} \int dx^1 (A^2_1(x) + 2A_0(x)A_1(x))
\] (47)

where \(H_+\) is the Hamiltonian of the \(\Psi_+\) field,

\[
H_+(x^0) = \int dx^1 (\Psi_+^\dagger(x)(-i \partial_1) \Psi_+(x) + e A_+(x) J_+(x)).
\] (48)

The reordering just adds local polynomials in the external gauge field and, therefore, \(N J_+\) and \(\bar{N} J_+\) have the same Schwinger term. We shall identify \(\bar{N} J_+\) with the covariant current in the forthcoming sections.

### IV. GAUSS LAW OPERATOR

Before continuing, we want to emphasize again that the operators in the last section were in the interaction picture of the full theory, and we shall remain in the interaction picture in this section.
The Gauss law operator $G$ implements time independent gauge transformations and may be found e.g. by requiring a covariant transformation of the time independent Dirac equation,

$$\int dx^1 \lambda(x^0, x^1) G(x^0, x^1), (-i \partial_{y^1} + eA_1(x^0, y^1)) \Psi_+(x^0, y^1) =$$

$$i \lambda(x^0, y^1)(-i \partial_{y^1} + eA_1(x^0, y^1)) \Psi_+(x^0, y^1).$$

(49)

It reads

$$G(x) = \partial_1 \frac{\delta}{e \delta A_1(x)} - i N J_+ (x), \quad \bar{G}(x) = \partial_1 \frac{\delta}{e \delta A_1(x)} - i \tilde{N} J_+(x)$$

(50)

where we defined the consistent ($G$) and covariant ($\bar{G}$) Gauss law operators. Here $A_1(x)$ is treated as a function of space only and the time variable $x^0$ as a parameter, i.e. $(\delta/\delta A_1(x^0, x^1)) A_1(x^0, y^1) = \delta(x^1 - y^1)$.

For the consistent Gauss law commutator we find from the Schwinger term (36) (because $(\delta/\delta A_1(x^0, x^1)) N J_+ = 0$)

$$[G(x^0, x^1), G(x^0, y^1)] = \frac{i}{2\pi} \partial_{x^1} \delta(x^1 - y^1).$$

(51)

Further, we are able to reproduce the Fujikawa relation [40] that relates the commutator of Gauss law and Hamiltonian operators to the consistent anomaly

$$[G(x), N H_+(x^0)] = [\partial_{x^1} \frac{\delta}{e \delta A_1(x)} - i N J_+(x), N H_0(x^0) + e \int dy^1 A_+(x^0, y^1) N J_+(x^0, y^1)] =$$

$$\partial_{x^1} \int dy^1 \delta(x^1 - y^1) N J_+(x^0, y^1) - i[N J_+(x), N H_0(x^0)] -$$

$$- ie \int dy^1 A_+(x^0, y^1)[N J_+(x^0, x^1), N J_+(x^0, y^1)] = - \frac{e}{2\pi} \partial_1 A_+(x)$$

(52)

where we used (45) and (36). This result is equal to the consistent anomaly (5) of the introduction up to a local (but Lorentz-noncovariant) counterterm.

Next let us turn to the covariant Gauss law operator. First we observe that the covariant current is indeed gauge invariant (in contrast to the consistent one),

$$[\bar{G}(x^0, x^1), \tilde{N} J_+(x^0, y^1)] =$$

$$[\partial_{x^1} \frac{\delta}{e \delta A_1(x^0, x^1)} - i N J_+(x^0, x^1), N J_+(x^0, y^1) + \frac{e}{2\pi} A_1(x^0, y^1)] = 0.$$

(53)

For the covariant Gauss law commutator we obtain

$$[\bar{G}(x^0, x^1), \bar{G}(x^0, y^1)] = - \frac{i}{2\pi} \partial_{x^1} \delta(x^1 - y^1),$$

(54)
i.e., it is minus the consistent Gauss law commutator (51).

In addition, we find that the covariantly regularized Hamiltonian $\tilde{N}H_+$ is gauge invariant, too,

$$[\tilde{G}(x), \tilde{N}H_+(x^0)] =$$

$$[G(x^0, x^1), NH_+(x^0) + \frac{e^2}{4\pi} \int dy^1 (A^2_1(x^0, y^1) + 2A_0(x^0, y^1)A_1(x^0, y^1))] = 0.$$ (55)

Therefore, at least for the external field problem, the covariant anomaly cannot be inferred from a covariant version of the Fujikawa relation.

Remark: the covariant anomaly may be found from $[\tilde{G}(x), \tilde{N}H_+(x^0)]$, when we treat $A_\mu$ as a dynamical field, i.e. include the gauge field kinetic energy $H_g = (-1/4) \int dx^1 F_{\mu\nu}F^{\mu\nu} = (1/2) \int dx^1 E^2$, $E = \partial_0 A_1 - \partial_1 A_0$, into the Hamiltonian. Using $[E(x^1), A_1(y^1)] = -i\delta(x^1 - y^1)$ we find

$$[\tilde{G}(x), H_g(x^0)] = \frac{e}{2\pi} E(x) = -\frac{e}{2\pi} \epsilon^{\mu\nu}\partial_\mu A_\mu(x),$$ (56)

which is precisely the covariant anomaly (6). This shows that in ETCs the consistent and covariant anomalies have a somewhat different origin (observe that $[G(x), NH_+(x^0)]$ is not changed by the inclusion of $H_g(x^0)$, as $G(x)$ does not depend on $A_1$). However, we shall continue to treat $A_\mu$ as an external, nondynamical field.

V. TIME EVOLUTION AND HEISENBERG CURRENT OPERATORS

In the sequel we shall assume that the gauge field $A_\mu(x)$ vanishes in the remote past, $\lim_{x^0 \to -\infty} A_\mu(x) = 0$. The time evolution operator is given by (see [20])

$$U(x^0, -\infty) = T \exp(-i \int_{-\infty}^{x^0} dx^0 H_1(x^0)) = \exp(-i \int_{-\infty}^{x^0} dx^0 H_1(x^0) - iC(x^0))$$ (57)

where, in the consistent case,

$$H_1(x^0) = e \int dx^1 A_+(x)NJ_+(x)$$ (58)

$$C(x^0) = \frac{1}{2} i \int_{-\infty}^{x^0} dy^0 \int_{-\infty}^{y^0} dz^0[H_1(z^0), H_1(y^0)]$$

$$= -\frac{1}{4\pi} \int d^2 y d^2 z \theta(x^0 - y^0)\theta(x^0 - z^0)\theta(y^0 - z_0)\delta(y^- - z^-)A_+(y)\partial_z A_+(z).$$ (59)

The perturbative expansion of the time-ordered exponential into ordinary exponentials in (57) terminates at the quadratic order, because the commutator of two interaction Hamiltonians $H_1(x^0)$ is a c-number for all times. Therefore, (57) is an exact result [20].
So let us compute the consistent current in the Heisenberg picture with the help of the BCH formula (17)

\[ NJ^H_+(x) = U^\dagger(x^0, -\infty)NJ_+(x)U(x^0, -\infty) \]

\[ = NJ_+(x) + i \int_{-\infty}^{x^0} dy^0[H_1(y^0), NJ_+(x)] \]

\[ = NJ_+(x) - \frac{e}{2\pi} \int d^2y \theta(x^0 - y^0)\delta(x^- - y^-)\partial_y A_+(y). \] 

(60)

First, let us prove that the current \( NJ_+ \) is indeed the consistent current. Within perturbation theory, the VEV of the consistent current is defined as the normalized functional derivative of the vacuum functional,

\[ \langle J_\text{cons}^\mu(x) \rangle := \frac{i}{e} \frac{\delta}{\delta A_\mu(x)} \ln Z[A_\mu] \] 

(61)

where

\[ Z[A_\mu] = \langle 0, \text{out} | 0 \rangle = \langle 0, \text{in} | U(\infty, -\infty) | \text{in}, 0 \rangle. \] 

(62)

In our case, \( | \text{in}, 0 \rangle \) is just the Dirac vacuum of the free theory, and \( A_\mu(x) \) is now interpreted as a space-time function, \( (\delta/\delta A_\mu(x))A_\nu(y) = \delta^\mu_\nu \delta^2(x - y) \). We find e.g. for \( (\delta/\delta A_0(x)) \) (using again the BCH formula)

\[ \frac{1}{\langle 0, \text{out} | \text{in}, 0 \rangle} i \langle 0, \text{in} | \frac{\delta}{\delta A_0(x)} U(\infty, -\infty) | \text{in}, 0 \rangle = \]

\[ \frac{1}{\langle 0, \text{out} | \text{in}, 0 \rangle} i \langle 0, \text{in} | U(\infty, -\infty) \left( \frac{\delta}{\delta A_0(x)} + [(i \int dy^0 H_1(y^0) + iC(\infty)), \frac{\delta}{\delta A_0(x)}] \right) \]

\[ + \frac{1}{2} [i \int dy^0 H_1(y^0), [i \int dz^0 H_1(z^0), \frac{\delta}{\delta A_0(x)}] ] | \text{in}, 0 \rangle = \]

\[ \frac{1}{\langle 0, \text{out} | \text{in}, 0 \rangle} \langle 0, \text{out} | NJ_+(x) - \frac{e}{2\pi} \int d^2y \theta(x^0 - y^0)\delta(x^- - y^-)\partial_y A_+(y) | \text{in}, 0 \rangle \]

\[ \equiv \frac{1}{\langle 0, \text{out} | \text{in}, 0 \rangle} \langle 0, \text{out} | NJ^H_+(x) | \text{in}, 0 \rangle \] 

(63)

where

\[ [i \int dy^0 H_1(y^0), \frac{\delta}{\delta A_0(x)}] = -iNJ_+(x) \] 

(64)

\[ [iC(\infty), \frac{\delta}{\delta A_0(x)}] = \frac{ie}{4\pi} \int d^2y e(x^0 - y^0)\delta(x^- - y^-)\partial_y A_+(y) \] 

(65)
For the adjoint action on fermionic operators the gauge field dependent phases and we find a completely identical result for the other component \((\delta / \delta A_1(x))\) (remember that \(J^\mu_+ = J^\mu_+ = 0\)). Actually, the derivation (63) remains the same for general in and out states, and therefore the identification \(J^\mu_+ \text{cons.} = N J^\mu_+\) holds for all S-matrix elements.

Next we want to compute the consistent anomaly

\[
\partial_\mu N J^\mu_{+H}(x) = (\partial_0 + \partial_1)(N J_+(x) - \frac{e}{2\pi} \int d^2 y \theta(x^0 - y^0) \delta(x^- - y^-) \partial_y A_+(y))
\]

where we used the fact that \(\theta(x^0) \delta(x^-)\) is the (retarded) Green function of the operator \((\partial_0 + \partial_1),\)

\[
(\partial_{x^0} + \partial_{x^1}) \theta(x^0 - y^0) \delta(x^- - y^-) = \delta^2(x - y).
\]

This result precisely coincides with the consistent anomaly (52) of Section 4.

Next we want to discuss the covariant current operator in an analogous manner (here we just review the discussion of [26], where the covariant Heisenberg current and anomaly were already derived).

All the fermionic operators of Sections 3, 4 were Heisenberg operators of the free Hamiltonian \(NH_0\), therefore all the additional parts of \(\tilde{N}H\) must be treated as interaction terms,

\[
\tilde{H}_1(x^0) = \tilde{N}H(x^0) - NH_0(x^0) = H_1(x^0) + \frac{e^2}{4\pi} \int dx^1 (\partial^2_1(x) + 2A_0(x)A_1(x))
\]

leading to the time evolution operator

\[
\tilde{U}(x^0, -\infty) = \exp(-i \int_{-\infty}^{x^0} dy^0 H_1(y^0) - iC(x^0) - iD(x^0))
\]

\[
D(x^0) = \frac{e^2}{4\pi} \int d^2 y \theta(x^0 - y^0)(\partial^2_1(y) + 2A_0(y)A_1(y)).
\]

For the adjoint action on fermionic operators the gauge field dependent phases \(iC(x^0), iD(x^0)\) are irrelevant, and we find for the covariant current in the Heisenberg picture

\[
\tilde{N}J^H_{+}(x) = \tilde{U}(x^0, -\infty) \tilde{N}J_+(x) \tilde{U}(x^0, -\infty) =
\]

\[
U(x^0, -\infty) NJ_+(x) U(x^0, -\infty) + \frac{e}{2\pi} A_1(x) = N J^H_{+}(x) + \frac{e}{2\pi} A_1(x)
\]

and for the covariant anomaly

\[
\partial_\mu \tilde{N}J^\mu_{+H}(x) = -\frac{e}{2\pi} \partial_1 A_+(x) + \frac{e}{2\pi} (\partial_0 + \partial_1)A_1(x) = \frac{e}{2\pi} (\partial_0 A_1(x) - \partial_1 A_0(x)).
\]

This is precisely the gauge and Lorentz invariant result (6) of the introduction.
VI. TIME EVOLUTION OF THE GAUSS LAW OPERATORS

In Section 4 the Gauss law operator was defined in (50), and there the gauge field was treated as a function of space only. For the time evolution we need a generalization to space-time functions. Following [22, 23] we define

\[ G(x) = \delta(x^1) - iNJ_+(x), \quad \delta(x^1) = \int_{-\infty}^{\infty} dx^0 \frac{\delta}{e\delta A_1(x^0, x^1)} \] (75)

where \( A_1(x) \) is now a space-time function, i.e. \( \delta(\delta/\delta A_1(x^1))A_1(y) = \delta^2(x - y) \). Obviously, \( \delta(x^1) \) is just the generalization of \( \partial_1(\delta/\delta A_1(x^1)) \) to space-time functions.

We are now in a position to compute the time evolution of the Gauss law operator, which in Section 4 the Gauss law operator was defined in (50), and there the gauge field was treated as a function of space only. For the time evolution we need a generalization to space-time functions. Following [22, 23] we define

\[ G(x) = \delta(x^1) - iNJ_+(x), \quad \delta(x^1) = \int_{-\infty}^{\infty} dx^0 \frac{\delta}{e\delta A_1(x^0, x^1)} \] (75)

where \( A_1(x) \) is now a space-time function, i.e. \( \delta(\delta/\delta A_1(x^1))A_1(y) = \delta^2(x - y) \). Obviously, \( \delta(x^1) \) is just the generalization of \( \partial_1(\delta/\delta A_1(x^1)) \) to space-time functions.

We are now in a position to compute the time evolution of the Gauss law operator, which we want to do for the consistent Gauss law operator (75) first. For the time evolution of \( \delta(x^1) \) we find

\[ U^1(x^0, -\infty)\delta(x^1)U(x^0, -\infty) = \delta(x^1) + [(ie \int d^2y \theta(x^0 - y^0)A_+(y)NJ_+(y) + iC(x^0)), \delta(x^1)] \]

\[ + \frac{(ie)^2}{2!} i \int d^2y \theta(x^0 - y^0)A_+(y)NJ_+(y), [\int d^2z \theta(x^0 - z^0)A_+(z)NJ_+(z), \delta(x^1)] = \]

\[ \delta(x^1) - i \int d^2y \theta(x^0 - y^0)\partial_{x^1} \delta(x^1 - y^1)NJ_+(y) + \]

\[ \frac{ie}{2\pi} \int d^2yd^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\theta(z^0 - y^0)\delta(x^1 - z^1)\delta(y^- - z^-)\partial_{y^1}^2 A_+(y) \] (76)

where

\[ [ie \int d^2y \theta(x^0 - y^0)A_+(y)NJ_+(y), \delta(x^1)] = -i \int d^2y \theta(x^0 - y^0)\partial_{x^1} \delta(x^1 - y^1)NJ_+(y) \] (77)

\[ [iC(x^0), \delta(x^1)] = \]

\[ - \frac{ie}{4\pi} \int d^2yd^2z \theta(x^0 - y^0)\theta(x^0 - z^0)e(y^0 - z^0)\delta(x^1 - z^1)\delta(y^- - z^-)\partial_{y^1}^2 A_+(y) \] (78)

\[ \frac{(ie)^2}{2} [\int d^2y \theta(x^0 - y^0)A_+(y)NJ_+(y), [\int d^2z \theta(x^0 - z^0)A_+(z)NJ_+(z), \delta(x^1)] = \]

\[ \frac{ie}{4\pi} \int d^2yd^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\delta(x^1 - z^1)\delta(y^- - z^-)\partial_{y^1}^2 A_+(y). \] (79)

I.e., under time evolution the operator \( \delta(x^1) \) acquires a contribution proportional to the fermionic current operator and a further contribution that is a nonlocal functional of the gauge field. This latter contribution itself consists of a trivial part, stemming from the \([iC(x^0), \delta(x^1)]\)
commutator, and a nontrivial part from the $[\mathcal{H}_1, \mathcal{H}_1, \delta(x^1)]$ double commutator (what we mean by “trivial” and “nontrivial” will become clear in the sequel).

Now we want to investigate the time evolution of the commutator

$$\{\delta(x^1), \delta(x'^1)\} = 0. \tag{80}$$

We find ($U(x^0) \equiv U(x^0, -\infty)$)

$$[U^\dagger(x^0)\delta(x^1)U(x^0), U^\dagger(x^0)\delta(x'^1)U(x^0)] =$$

$$(-i)^2 \int d^2y d^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\partial_{x^1}\delta(x^1 - y^1)\partial_{x'^1}\delta(x'^1 - z^1)\{NJ_+(y), NJ_+(z)\}$$

$$+ \frac{ie}{2\pi} \{\delta(x^1), \int d^2y d^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\theta(z^0 - y^0)\delta(x'^1 - z^1)\delta(y^1 - z^1)\partial_\nu^2 A_+(y)\}$$

$$- \frac{ie}{2\pi} \{\delta(x'^1), \int d^2y d^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\theta(z^0 - y^0)\delta(x^1 - z^1)\delta(y^1 - z^1)\partial_\nu^2 A_+(y)\} =$$

$$= \ldots \equiv 0, \tag{81}$$

where the sum of the two commutators containing the $\delta(x^1)$, $\delta(x'^1)$ precisely cancels the $[NJ_+, NJ_+]$ term. Therefore, the commutator (80) remains unchanged under time evolution! This happens, because the time evolution $U^\dagger\delta U$ of $\delta$ contains two nontrivial pieces that give nonzero contributions to the commutator (81), namely a piece containing the fermionic current $NJ_+$, (77), and a gauge field piece that stems from the double commutator $[\mathcal{H}_1, \mathcal{H}_1, \delta(x^1)]$, (79). These two nontrivial contributions to the commutator precisely cancel each other and make the commutator (81) vanish. There is another gauge field piece in $U^\dagger\delta U$ stemming from the $[\mathcal{C}(x^0), \delta(x^1)]$ commutator, (78), but this is trivial and gives no contribution to the $[U^\dagger\delta U, U^\dagger\delta U]$ commutator, because

$$\{\delta(x^1)\delta(x'^1) - \delta(x'^1)\delta(x^1)\}C(x^0) \equiv 0. \tag{82}$$

The time evolution of the consistent current was already derived in the last section, therefore we may now compute the time evolution of the anomalous Gauss law commutator

$$[U^\dagger(x^0)(\delta(x^1) - iNJ_+(x^0, x^1))U(x^0), U^\dagger(x^0)(\delta(x'^1) - iNJ_+(x^0, x'^1))U(x^0)] =$$

$$(-i)^2 [NJ_+(x^0, x^1), NJ_+(x^0, x'^1)]$$

$$- i[U^\dagger(x^0)\delta(x^1)U(x^0), U^\dagger(x^0)NJ_+(x^0, x'^1)U(x^0)] - (x^1 \leftrightarrow x'^1)$$

$$= \ldots = \frac{i}{2\pi}\partial_{x^1}\delta(x^1 - x'^1). \tag{83}$$
Therefore, we find that the commutator of the Gauss law operator as well as the commutators of all its components remain unchanged under time evolution (see (51)).

Now let us briefly turn to the covariant Gauss law operator. The covariant time evolution operator \( \tilde{U} \) differs from the consistent one by the phase factor \( \exp(-iD(x^0)) \), (72), and, therefore, \( \tilde{U}^\dagger(x^0)\delta(x^1)\tilde{U}(x^0) \) acquires an additional term

\[
\tilde{U}^\dagger(x^0)\delta(x^1)\tilde{U}(x^0) = U^\dagger(x^0)\delta(x^1)U(x^0) - \frac{ie}{2\pi} \int d^2y \theta(x^0 - y^0)\delta(x^1 - y^1)\partial_y A_+(y). \tag{84}
\]

However, this additional term, stemming from a phase factor, cannot change the commutator. Therefore, we find that for the covariant Gauss law operator, too, the full commutator as well as the commutators of all components remain invariant under time evolution.

At first sight, this result may seem surprising. After all, it is a well-known fact that the anomaly in anomalous gauge theories is related to a nontrivial action of the functional derivative \( \delta/\delta A_1(x) \) on the fermionic Fock space [28] - [31].

But we shall find that we precisely recover these features within our approach. Let us look at the “field strength” operator \( E \) itself,

\[
E(x^1) := \int_{-\infty}^{\infty} dx^0 \frac{\delta}{\delta A_1(x^0, x^1)} \tag{85}
\]

(instead of its derivative \( \delta(x^1) = \partial_1 E(x^1) \) in the Gauss law). Analogous to (76) it has the following consistent time evolution

\[
U^\dagger(x^0)E(x^1)U(x^0) = E(x^1) - ie \int d^2y \theta(x^0 - y^0)\delta(x^1 - y^1)NJ_+(y)
\]

\[
+ \frac{ie^2}{4\pi} \int d^2yd^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\epsilon(z^0 - y^0)\delta(x^1 - z^1)\delta(y^- - z^-)\partial_y A_+(y)
\]

\[
+ \frac{ie^2}{4\pi} \int d^2yd^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\delta(x^1 - z^1)\delta(y^- - z^-)\partial_y A_+(y) \tag{86}
\]

where the second line is from the \([iC(x^0), E(x^1)]\) commutator and the third line is from the \([f H_1, [f H_1, E(x^1)]]\) double commutator.

The essential point now is that \( E(x) \) has a nonvanishing VEV

\[
\langle E(x) \rangle \equiv \langle 0, S|E(x^1)|S, 0 \rangle = \langle 0, IP|E(x^1)|IP, 0 \rangle = \langle 0, H|U^\dagger(x^0)E(x^1)U(x^0)|H, 0 \rangle \tag{87}
\]

where the transformation from Schrödinger to interaction picture acts, of course, trivially on \( E(x^1) \). In our case the Heisenberg vacuum is just the Dirac vacuum of the free theory, and therefore we get

\[
\langle E(x) \rangle = \frac{ie^2}{2\pi} \int d^2y d^2z \theta(x^0 - y^0)\theta(x^0 - z^0)\theta(z^0 - y^0)\delta(x^1 - z^1)\delta(y^- - z^-)\partial_y A_+(y). \tag{88}
\]

I.e., only the nonlocal gauge field part of the nontrivial time evolution \( U^\dagger EU \) occurs in the VEV, whereas the fermion current part has zero VEV. Now the procedure in [28] consists in defining a new “field strength” operator
\begin{equation}
\bar{E}(x) := E(x^1) + \mathcal{A}(x), \quad \mathcal{A}(x) := -\langle E(x) \rangle
\end{equation}

As \( E(x^1) \) is just an ordinary functional derivative, \( \bar{E}(x) \) may be interpreted as a covariant functional derivative that is invariant under \( U(1) \) phase transformations of the time evolution operator, \( U(x^0) \rightarrow \exp(i[f[A](x^0)] U(x^0)) \), where \( f[A](x^0) \) is an arbitrary functional of \( A_\mu \). \( \mathcal{A} \) is the corresponding \( U(1) \)-connection and gives rise to the curvature

\[ \mathcal{F}(x^1, y^1) := [\bar{E}(x^0, x^1), \bar{E}(x^0, y^1)] = \frac{ie^2}{4\pi\epsilon}(x^1 - y^1). \] 

\( \mathcal{F}(x^1, y^1) \) is solely determined by the double commutator contribution to (86) (the third line), because the other part stems from a pure phase \( iC(x^0) \). Generally \( \mathcal{A}(x^1) = \int d^2y \mathcal{A}(x, y)A_+(y) \), and only the antisymmetric part of the kernel \( \mathcal{A}(x, y) \) determines \( \mathcal{F}(x^1, y^1) \), whereas the symmetric part is a pure (functional \( U(1) \)) gauge.

When we use this new, covariant functional derivative \( \bar{E} \) in the Gauss law operator,

\[ \delta(x^1) \rightarrow \bar{\delta}(x) = \frac{1}{e} \partial_1 E(x), \] 

then the consistent commutator anomaly is just doubled,

\[ [\bar{G}(x^0, x^1), \bar{G}(x^0, y^1)] = \frac{i}{\pi} \partial_1 \delta(x^1 - y^1), \quad \bar{G}(x) := \bar{\delta}(x) - ieNJ_+(x). \]

On the other hand, the covariant commutator anomaly vanishes,

\[ [\bar{\tilde{G}}(x^0, x^1), \bar{\tilde{G}}(x^0, y^1)] = 0, \quad \bar{\tilde{G}}(x) := \bar{\tilde{\delta}}(x) - ie\tilde{N}J_+(x). \]

This means that the Gauss law operator \( \bar{\tilde{G}}(x) \) itself is gauge invariant. Completely analogous results were found in [28].

Remark: As \( [\bar{G}, \bar{G}] = 0, [\bar{\tilde{G}}, \bar{\tilde{N}}H] = 0 \), the full theory (including the gauge field) may be quantized. This was done in [28], and the resulting quantum field theory was found to break Lorentz invariance even at a physical level. However, this discussion is beyond the scope of our article, where the gauge field is treated as an external field throughout.

The main result of this investigation that we want to emphasize again is the fact that the nontrivial VEV of the “field strength” \( E \), (88), and the resulting functional curvature (90) are perfectly compatible with the canonical time evolution of the “field strength” commutator

\[ [U^\dagger(x^0)E(x^1)U(x^0), U^\dagger(x^0)E(y^1)U(x^0)] = U^\dagger(x^0)[E(x^1), E(y^1)]U(x^0) = 0. \]

\textbf{VII. SUMMARY}

We have discussed all the anomalous structure of chiral QED₂ by applying the formalism of canonical second quantization to the fermion field. The introduction of the Dirac vacuum and normal ordering, and the resulting Schwinger term in the current-current commutator
were the essential steps in this procedure, and the consistent and covariant divergence and commutator anomalies are just consequences of these fundamental concepts. By splitting the fermionic Hilbert space into positive and negative energy sectors w.r.t. the free and kinetic momentum, respectively, we could identify the consistent and covariant currents and anomalies (where we used the results of [26] for the kinetic normal ordering and covariant current).

Further, we computed the consistent and covariant Gauss law operators and the commutators of all its components both in the interaction and Heisenberg pictures. We found that the time evolution of the commutators is canonical. Especially the “field strength” commutator $[E(x^1), E(y^1)] = 0$ remains zero under time evolution. This is compatible with a nontrivial VEV $\langle E(x) \rangle \neq 0$, because the time evolution of $E(x^1)$ contains two nontrivial pieces (a gauge field piece and a fermion current piece) that cancel each other in the commutator.

In addition, we found that the consistent and covariant time evolution of the “field strength” $E(x^1)$ and of the gauge field part of the Gauss law, $\partial_1 E(x^1)$, only differ by a trivial term (a phase in the time evolution operator). Therefore, we obtained the same results for their consistent and covariant commutators.

Here, of course, the question arises what can be learned from our computations for more difficult chiral gauge theories. For chiral QCD$_2$ it remains true that normal ordering is sufficient to render all operators and VEVs finite. E.g., the time evolution operator may be computed analogously to (57). The perturbation series does not terminate for chiral QCD$_2$, but only the first few terms are relevant for anomalies. Therefore, an analogous discussion should be possible for chiral QCD$_2$, and it should lead to analogous results (see [27] for the covariant current and anomaly).

On the other hand, for $d = 4$ the situation is more involved. There, even after normal ordering some operator products remain singular and need regularization. This regularization has to be performed e.g. for the time evolution operator and prevents a direct applicability of our simple computations and conclusions.

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