PROPERTIES D AND aD ARE DIFFERENT

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Abstract. Under (♦∗) we construct a locally countable, locally compact, 0-dimensional T2 space X of size ω1 which is aD however not even linearly D. This consistently answers a question of Arhangel’skii, whether aD implies D. Furthermore we answer two problems concerning characterization of linearly D-spaces, raised by Guo and Junnila.

1. Introduction

The notion of a D-space was probably first introduced by E.K. van Douwen. We recommend G. Gruenhage’s paper [6] which gives a full review on what we know and do not know about D-spaces. A.V. Arhangel’skii and R. Buzyakova defined in [1] a weakening of property D, called aD. In [2] Arhangel’skii asked the following:

Problem 4.6. Is there a Tychonoff aD-space which is not a D-space?

In Section 5 we construct such a space under (♦∗). Before that we consider another weakening of property D. Recently H. Guo and H.J.K. Junnila in [7] introduced the notion of linearly D-spaces and proved several nice results concerning the topic. In Sections 3 and 4 we answer the following two questions from [7] in negative (in ZFC):

Problem 2.5. Let X be a T1 (linearly) D-space and let A ⊆ X have uncountable regular cardinality. Does A either have a complete accumulation point or a subset of size |A| which is closed and discrete in X?

Problem 2.6. Is a T1-space X linearly D provided that, for every set A ⊆ X of uncountable regular cardinality, either A has a complete accumulation point or A has a subset of size |A| which is closed and discrete in X?

As we will see these questions rise naturally. The constructions from Sections 3 and 4 can be considered as preparation to the space defined in Section 5. Our construction to Problem 4.6 also answers (consistently) the following

2000 Mathematics Subject Classification. 54A25,54A35.

Key words and phrases. linearly D-spaces, aD-spaces, Martin’s Axiom, guessing sequences.
Problem 2.12. Is every aD-space linearly D?

2. Definitions

An open neighborhood assignment (ONA, in short) on a space \((X, \tau)\) is a map \(U : X \to \tau\) such that \(x \in U(x)\) for every \(x \in X\). \(X\) is said to be a D-space if for every neighborhood assignment \(U\), one can find a closed discrete \(D \subseteq X\) such that \(X = \bigcup_{d \in D} U(d) = \bigcup U[D]\) (such a set \(D\) is called a kernel for \(U\)). In \([1]\) the authors introduced property aD:

Definition 2.1. A space \((X, \tau)\) is said to be aD iff for each closed \(F \subseteq X\) and for each open cover \(U\) of \(X\) there is a closed discrete \(A \subseteq F\) and \(\phi : A \to U\) with \(a \in \phi(a)\) such that \(F \subseteq \bigcup \phi[A]\).

It is clear that D-spaces are aD. A space \(X\) is irreducible iff every open cover \(U\) has a minimal open refinement \(U_0\); meaning that no proper subfamily of \(U_0\) covers \(X\). Later in \([2]\) Arhangel’skii showed the following equivalence.

Theorem 2.2 ([2, Theorem 1.8]). A \(T_1\)-space \(X\) is an aD-space if and only if every closed subspace of \(X\) is irreducible.

Another generalization of property D is due to Guo and Jumnila \([7]\). For a space \(X\) a cover \(U\) is monotone iff it is linearly ordered by inclusion.

Definition 2.3. A space \((X, \tau)\) is said to be linearly D iff for any ONA \(U : X \to \tau\) for which \(\{U(x) : x \in X\}\) is monotone, one can find a closed discrete \(D \subseteq X\) such that \(X = \bigcup U[D]\).

We cite two results from \([7]\). A set \(D \subseteq X\) is said to be \(U\)-big for a cover \(U\) iff there is no \(U \in U\) such that \(D \subseteq U\).

Theorem 2.4 ([7, Theorem 2.2]). The following are equivalent for a \(T_1\)-space \(X\):

1. \(X\) is linearly D.
2. For every non-trivial monotone open cover \(U\) of \(X\), there exists a closed discrete \(U\)-big set in \(X\).
3. For every subset \(A \subseteq X\) of uncountable regular cardinality \(\kappa\), there is a closed discrete subset \(B\) of \(X\), such that for every neighborhood \(U\) of \(B\), we have \(|U \cap A| = \kappa\).

In Problem 2.5 the authors ask whether condition (3) can be made stronger.

Theorem 2.5 ([7, Proposition 2.4]). A \(T_1\)-space \(X\) is linearly D if, and only if, for every set \(A \subseteq X\) of uncountable regular cardinality, either the set \(A\) has a complete accumulation point or there exists a closed discrete set \(D\) of size \(|A|\) and a disjoint family \(\{A_d : d \in D\}\) of subsets of \(A\) such that \(d \in A_d\) for every \(d \in D\).

In Problem 2.6 the authors ask whether the second condition of this dichotomy can be weakened.
3. On Problem 2.5 from [7]

In this section we give a negative answer to Problem 2.5. For this let us use the following notion for a space $X$. We say that $X$ satisfies (\*) iff

(*) for every regular, uncountable cardinal $\kappa$ and $A \in [X]^\kappa$ there is a complete accumulation point of $A$ or $A$ has a subset of size $\kappa$ which is closed discrete in $X$.

Problem 2.5 can be rephrased as whether property D implies (\*)? We will show the following:

1. There exists a locally countable $T_2$ D-space $X$ with cardinality $\omega_1$ which does not satisfy (\*).
2. The existence of a locally countable, locally compact, $T_2$ D-space $X$ with cardinality $< 2^{\omega_1}$ which does not satisfy (\*) is independent of ZFC.
3. There exists a locally countable, locally compact $T_3$ (even 0-dimensional) D-space $X$ with cardinality $2^{\omega_1}$ which does not satisfy (\*).

First let us observe the following.

**Proposition 3.1.** Suppose that the space $X$ is the union of a closed discrete set and a D-subspace. Then $X$ is a D-space.

**Proof.** Let $X = Y \cup Z$ such that $Y$ is closed discrete, $Z$ is a D-space. Let $U$ be an ONA on $X$. $Z_0 = Z \setminus \bigcup \{U(y) : y \in Y\}$ is a closed subspace of the D-space $Z$, thus $Z_0$ is a D-space either. There is a closed discrete kernel $D_0$ for the ONA $U|Z_0$ on $Z_0$. Then $D = D_0 \cup Y$ is a closed discrete kernel for $U$. \qed

**Proposition 3.2.** There exists a locally countable $T_2$ D-space $X$ with cardinality $\omega_1$ which does not satisfy (\*).

**Proof.** Let $X = \omega_1 \times 2$. We define the topology on $X$ as follows. Let $\omega_1 \times \{0\}$ be discrete. For $\alpha < \omega_1$ let $(\alpha, 1)$ have the following neighborhood base:

$$\{(\alpha, 1) \cup ((\beta, \alpha) \times \{0\}) : \beta < \alpha\}.$$

Clearly, $X$ is a locally countable, $T_2$ space. Observe that $\omega_1 \times \{1\} \subseteq X$ is closed discrete, $\omega_1 \times \{0\} \subseteq X$ is discrete, hence $D$. Thus $X$ is a D-space by Proposition 3.1. Let $A = \omega_1 \times \{0\}$. Then clearly any infinite subset of $A$ has an accumulation point in $X$. Thus $X$ does not satisfy (\*), since there is no full accumulation point of $A$ and any infinite subset of $A$ is not closed discrete in $X$. \qed

This answers the Problem 2.5 in the negative direction by a $T_2$ counterexample.

* * *

The question whether a regular space with this property exists is natural. First we will show, that the existence of a "nice" regular counterexample with cardinality below $2^{\omega_1}$ is independent. We will need a weakening of the axiom (t) which was introduced by I. Juhász in [9].
Definition 3.3. The weak (t) axiom: there exists a weak (t)-sequences \( \{A_\alpha : \alpha \in \text{lim}(\omega_1)\} \), meaning that \( A_\alpha \subseteq \alpha \) is an \( \omega \)-sequence converging to \( \alpha \) and for every \( X \in [\omega_1]^{\omega_1} \) there is a limit \( \alpha \) such that \( |X \cap A_\alpha| = \omega \).

The existence of such sequences is independent of ZFC. Under MA there is no weak (t)-sequence sequence and adding one Cohen real to any model adds a (weak) (t)-sequence either (see [9]).

Proposition 3.4. Suppose the weak (t)-axiom. Then there exists a locally compact, locally countable, 0-dimensional \( T_2 \) D-space \( X \) which does not satisfy \( (\ast) \).

Proof. Suppose that \( A = \{A_\alpha : \alpha \in \text{lim}(\omega_1)\} \) is a weak (t)-sequence. Let \( X = \omega_1 \times 2 \). Define the topology on \( X \) as follows. Let \( \omega_1 \times \{0\} \) be discrete. For \( \alpha \in \text{lim}(\omega_1) \) let \( (\alpha, 1) \) have the following neighborhood base:

\[ \{(\alpha, 1) \cup ((A_\alpha \setminus \beta) \times \{0\}) : \beta < \alpha\} \]

For successor \( \alpha < \omega_1 \) let \( (\alpha, 1) \) be discrete. Clearly, \( X \) is a locally countable, locally compact, 0-dimensional \( T_2 \) space. Notice that \( \omega_1 \times \{1\} \subseteq X \) is closed discrete, \( \omega_1 \times \{0\} \subseteq X \) is discrete, hence \( D \). Thus \( X \) is a D-space by Proposition 3.1. Let \( A = \omega_1 \times \{0\} \). We prove that any uncountable \( B \subseteq A \) is not closed discrete in \( X \), hence \( X \) does not satisfy \( (\ast) \). Let \( B_0 = \{\alpha < \omega_1 : (\alpha, 0) \in B\} \in [\omega_1]^{\omega_1} \). Since \( A \) is a weak (t)-sequence, there is \( \alpha \in \text{lim}(\omega_1) \) such that \( |A_\alpha \cap B_0| = \omega \). Clearly, \( B \) accumulates to \( (\alpha, 1) \), thus \( B \) is not closed discrete in \( X \). \( \square \)

Remark: In [8] T. Ishiu uses guessing sequences to refine the standard topology on an ordinal.

Now our aim is to prove Proposition 3.7 which implies that under MA there is no such space. The following was proved by Z. Balogh (actually more, but we only need this):

Theorem 3.5 ([3, Theorem 2.2]). Suppose MA. Then for any locally countable, locally compact space \( X \) of cardinality \( < 2^\omega \) exactly one of the following is true:

- \( X \) is the countable union of closed discrete subspaces,
- \( X \) contains a perfect preimage of \( \omega_1 \) with the order topology.

From this and the following observation we can deduce Proposition 3.7.

Proposition 3.6 ([4, Proposition 7.1]). If the space \( X \) is the countable union of closed D-subspace then \( X \) is a D-space.

Proposition 3.7. Suppose MA. Then for any locally countable, locally compact space \( X \) of cardinality \( < 2^\omega \) the following are equivalent:

1. \( X \) is \( \sigma \)-closed discrete,
2. \( X \) is a D-space,
3. \( X \) is a linearly D-space,
Theorem 3.9.

\( (3) \) \( X \) satisfies \((*)\).

Proof. The implications \((0) \Rightarrow (1) \Rightarrow (2)\) (by Proposition \ref{prop:3.6} and \((0) \Rightarrow (3)\) are straightforward. \((3)\) implies \((2)\) by Theorem \ref{thm:2.4} We only need to show \((2) \Rightarrow (0)\). Suppose that \( X \) is linearly \( D \)-space, by Theorem \ref{thm:3.5} we need to show that \( X \) does not contain any perfect preimage of \( \omega_1 \).

Claim 3.8. \( (i) \) If the space \( F \) is a perfect preimage of \( \omega_1 \) then \( F \) is countably compact, non compact.

\( (ii) \) If \( X \) is first-countable and \( F \subseteq X \) is a perfect preimage of \( \omega_1 \) then \( F \) is closed in \( X \).

Proof. \( (i) \) It is known that under perfect mappings, the preimage of a compact space is compact (see [5, Theorem 3.7.2]). Take any countably infinite \( \alpha < 2 \) points of \( X \) that are straightforward.

\( (ii) \) If the space \( F \) is a perfect preimage of \( \omega_1 \) then \( F \) cannot be a perfect preimage of \( \omega_1 \).

Finally we give a regular counterexample to the problem in ZFC without any further set-theoretic assumptions.

Theorem 3.9. There exists a locally countable, locally compact, 0-dimensional \( T_2 \) \( D \)-space \( X \) with cardinality \( 2^\omega \) such that \( X \) does not satisfy \((*)\).

Proof. Let \( \{ C_\alpha : \alpha < 2^\omega \} \) denote an enumeration of the closed dense in itself subsets of \( \mathbb{R} \). Let \( \{ Q_\alpha^\beta : \beta < 2^\omega \} \) denote an enumeration of all countable subsets of \( \mathbb{R} \) such that \( C_\alpha \subseteq Q_\alpha^\beta \) (Euclidean closure taken). Enumerate the pairs \((\alpha, \beta)\) from \( 2^\omega \times 2^\omega \) in order type \( 2^\omega \): \( \{ p_\delta : \delta < 2^\omega \} \). We define a topology on \( X = \mathbb{R} \times 2 \) as follows. Let \( \mathbb{R} \times \{ 0 \} \) be discrete and we define the topology on \( \mathbb{R} \times \{ 1 \} \) by induction. In step \( \delta \) for \( p_\delta = (\alpha, \beta) \), pick a point \( x_\delta \in C_\alpha \setminus \{ x_\delta' : \delta' < \delta \} \) and let \((x_\delta, 1)\) have the following neighborhood base:

\[ \{ (x_\delta, 1) \} \cup \{ (x_\delta, 0) : n \geq m \} : m < \omega \]

where \( \{ x_\delta^n : n < \omega \} \subseteq Q_\alpha^\beta \setminus \{ x_\delta \} \) is any sequence converging to \( x_\delta \) in the Euclidean sense. Let the remaining points \((\mathbb{R} \setminus \{ x_\delta : \delta < 2^\omega \}) \times \{ 1 \}\) be discrete. Clearly, this gives us a locally countable, locally compact, 0-dimensional \( T_2 \) space. \( \mathbb{R} \times \{ 1 \} \) is closed discrete and \( \mathbb{R} \times \{ 0 \} \) is discrete, hence a \( D \)-space. Thus \( X \) is a \( D \)-space by Proposition \ref{prop:3.1}.

We claim that there is no uncountable subset of \( A = \mathbb{R} \times \{ 0 \} \subseteq X \) such that it is closed discrete in \( X \) with this topology; this implies that \( X \) does not satisfy \((*)\). Let \( B \in [A]^{\omega_1} \) and \( B_0 = \{ x \in \mathbb{R} : (x, 0) \in B \} \). Then there is \( \alpha < 2^\omega \) such that \( C_\alpha \subseteq B_0 \) (where \( B_0 \) denotes the Euclidean accumulation points of \( B_0 \)) and \( \beta < 2^\omega \) such that \( Q_\alpha^\beta \subseteq B \). By definition in step \( \delta \) where
Now our aim is to answer Problem 2.6 in negative. For this we will say that a space $X$ satisfies (**) iff

$$(**) \text{ for every regular, uncountable cardinal } \kappa \text{ and } A \subseteq [X]^\kappa \text{ there is a full accumulation point of } A \text{ or there is } D \subseteq [A]^\kappa \text{ which is closed discrete in } X.$$ 

Problem 2.6 can be rephrased as whether (**) implies linearly D. We prove the following:

(1) The existence of a locally countable, locally compact, non linearly D space $X$ with cardinality $< 2^\omega$ which satisfies (**) is independent of ZFC.

(2) There is a locally countable, locally compact, 0-dimensional Hausdorff space $X$ with cardinality $2^\omega$ which satisfies (**) however not linearly D.

We will use the following notations in this section. $\{A_\alpha : \alpha \in \text{lim}(\omega_1)\}$ denotes a ♦-sequence: for every $\alpha < \omega_1$ $A_\alpha \subseteq \alpha$ is an $\omega$-type sequence converging to $\alpha$ and for every $X \in [\omega_1]^{\omega_1}$ there is some $\alpha < \omega_1$ such that $A_\alpha \subseteq X$. Supposing ♦ means that there is a ♦-sequence. For every $\alpha \in \omega_1$ enumerate increasingly $A_\alpha$ as $\{a_\alpha^n : n \in \omega\}$. Let $\{M_\beta : \beta \in \omega_1\} \subseteq [\omega]^{\omega}$ be an arbitrary almost disjoint family on $\omega$.

**Theorem 4.1.** Suppose ♦. Then there is a 0-dimensional $T_2$ space $X$ of cardinality $\omega_1$ such that $X$ is not linearly D, however satisfies (**).

**Proof.** First we introduce some further notations for the intervals between the points in the $A_\alpha$’s. For each $\alpha \in \omega_1$ let $\{I_\alpha^n : n \in \omega\}$ denote the following disjoint open sets in $\omega_1$: $I_\alpha^0 = (0, a_\alpha^0]$ and $I_\alpha^{n+1} = (a_\alpha^n, a_\alpha^{n+1}]$ for $n \in \omega$. We will define a topology on $X = \omega_1 \times \omega_1$. Let $\{\alpha\} \times \omega_1$ be discrete for successor $\alpha$. For $\alpha \in \text{lim}(\omega_1)$ and $\beta \in \omega_1$, a neighborhood base for the point $(\alpha, \beta)$ consists of sets:

$$U((\alpha, \beta), E) = \{(\alpha, \beta)\} \cup \bigcup \{I_\alpha^n \times \omega_1 : n \in M_\beta \setminus E\}$$

where $E \in [\omega]^{<\omega}$. Observe that if $\beta, \beta' \in \omega_1$ and $\beta \neq \beta'$ then $E = M_\beta \cap M_\beta'$ is finite, thus $U((\alpha, \beta), E) \cap U((\alpha, \beta'), E) = \emptyset$. This way we defined a 0-dimensional $T_2$ topology. Note that the set $X_\alpha = \{\alpha\} \times \omega_1$ for $\alpha \in \omega_1$ is closed discrete. Let $\pi(A) = \{\alpha \in \omega_1 : A \cap X_\alpha \neq \emptyset\}$ for $A \subseteq X$. 

Thus $B$ is not closed in $X$. □
Claim 4.2. If $|\pi(A)| = \omega_1$ for $A \subseteq X$ then there are stationary many $\alpha \in \omega_1$ such that $X_\alpha \subseteq A'$.

Proof. Since $\{A_\alpha : \alpha \in \text{lim}(\omega_1)\}$ is a ♦-sequence, the set $S = \{\alpha \in \omega_1 : A_\alpha \subseteq \pi(A)\}$ is stationary. For $\alpha \in S$ we clearly have $X_\alpha \subseteq A'$, since by definition for any $U$ neighborhood of any point $(\alpha, \beta) \in X_\alpha$, $U$ intersects $A$ in infinitely many points.

This claim has the following corollaries.

Claim 4.3. $X$ satisfies $(\ast \ast)$.

Proof. Let $A \in [X]^{\omega_1}$. If there is an $\alpha \in \omega_1$ such that $|A \cap X_\alpha| = \omega_1$ we are done. Otherwise for all $\alpha \in \omega_1$ we have $|A \cap X_\alpha| \leq \omega$ so $|\pi(A)| = \omega_1$. By Claim 4.2 there is an $\alpha \in \text{lim}(\omega_1)$ such that $X_\alpha \subseteq \overline{A}$ and $X_\alpha$ is closed discrete.

Claim 4.4. $X$ is not linearly $D$.

Proof. Suppose that $D \subseteq X$ is closed discrete. Then $\pi(D)$ is countable by Claim 1.2. Hence there is no closed discrete set which is big for the open cover $\{\alpha \times \omega_1 : \alpha < \omega_1\}$. Thus $X$ is not linearly $D$ by Theorem 2.4.

This completes the proof of this theorem.

Remark: If we modify the neighborhoods to be the following for $(\alpha, \beta)$ where $\beta \in \omega_1$ and $\alpha \in \text{lim}(\omega_1)$:

$$\{((\alpha, \beta)) \cup \bigcup\{I_\alpha^n \times [0, \beta] : n \in M^\beta \setminus E\} : E \in [\omega]^{<\omega}\}$$

then we obtain a topology which is locally countable, not linearly $D$, satisfies $(\ast \ast)$ however not even regular.

With some further set-theoretic assumptions we can improve the above construction.

Theorem 4.5. Suppose ♠ and CH (equivalently ♦). Then there is a locally countable, locally compact, 0-dimensional $T_2$ space $X$ of cardinality $\omega_1(= 2^\omega)$ such that $X$ is not linearly $D$, however satisfies $(\ast \ast)$. 


This way we will get a topology \( \tau \) on \( X \) as \( \{\alpha\} \times \omega_1 \), \( X_{<\alpha} = \alpha \times \omega_1 \). Define neighborhoods for points in \( X_\alpha \) by induction on \( \alpha \). Let \( (X_{<\alpha}, \tau_{<\alpha}) \) denote the topology defined by the induction till step \( \alpha \). We have the following conditions which we will preserve during each step:

(i) \( (X_{<\alpha}, \tau_{<\alpha}) \) is locally countable, locally compact, 0-dimensional,

(ii) \( X_{<\beta} \) is open in \( X_{<\alpha} \) for \( \beta < \alpha \),

(iii) for every \( \beta < \alpha \) and \( (\beta, \gamma) \in X_\beta \) there is some neighborhood \( G \) of \( (\beta, \gamma) \) such that \( G \setminus \{(\beta, \gamma)\} \subseteq X_{<\beta} \),

(iv) for every \( \beta' < \beta < \alpha \) the set \( \{\beta', \beta\} \times \omega_1 \) is clopen in \( (X_{<\alpha}, \tau_{<\alpha}) \).

This way we will get a topology \( \tau \) on \( X \) by taking \( \bigcup \{\tau_{<\alpha} : \alpha < \omega_1\} \) as a base.

For successor \( \alpha < \omega_1 \) just let \( (\alpha, \beta) \) be a discrete point for any \( \beta < \omega_1 \). Clearly this way the inductive hypothesis will hold. Suppose now that \( \alpha \) is limit, we define a neighborhood for \( (\alpha, \beta) \) (for any \( \beta < \omega_1 \)) as follows. Suppose that \( h(\beta) = (\delta, \gamma) \). For each \( n \in \omega \) take a countable, compact, clopen \( G_n \subseteq (a_\alpha^{n-1}, a_\alpha^n] \times \omega_1 \) such that \( (a_\alpha^n, F_\delta(n)) \in G_n \); this can be done by (i) and (iv). Then define the neighborhoods of \( (\alpha, \beta) \) by the following base:

\[
\{(\alpha, \beta)\} \cup \bigcup \{G_n : n \in M^\beta \setminus E \} : E \in [\omega]^{<\omega}.
\]

It is clear that the inductive assumptions will hold for the resulting topology. It follows from the construction that \((X, \tau)\) is locally countable, locally compact and 0-dimensional. Thus we constructed a space \((X, \tau)\) which refines the topology from the previous theorem, hence \( X \) is \( T_2 \) either. Let \( \pi(A) = \{\alpha \in \omega_1 : A \cap X_\alpha \neq \emptyset\} \) for \( A \subseteq X \).

**Claim 4.6.** If \( |\pi(A)| = \omega_1 \) for \( A \subseteq X \) then there are stationary many \( \alpha \in \omega_1 \) such that \( |X_\alpha \cap A'| = \omega_1 \).

**Proof.** Since \( \{A_\alpha : \alpha \in \text{lim}(\omega_1)\} \) is a \( \clubsuit \)-sequence, the set \( S = \{\alpha \in \omega_1 : A_\alpha \subseteq \pi(A)\} \) is stationary. Fix an \( \alpha \in S \). Define \( F : \omega \to \omega_1 \) such that \( (a_\alpha^n, F(n)) \in A \) where \( A_\alpha = \{a_\alpha^n : n \in \omega\} \). So there is some \( \delta \) for which we have \( F_\delta = F \). We claim that \( \{(\alpha, \beta) \in X_\alpha : \exists \gamma < \omega_1 : h(\beta) = (\delta, \gamma)\} \subseteq A' \).

Clearly for such an \( (\alpha, \beta) \) we used \( F_\delta = F \) in the induction to define the neighborhoods, from which we see that the set \( \{(a_\alpha^n, F(n)) : n \in \omega\} \subseteq A \) accumulates to \((\alpha, \beta)\). \( \square \)

**Claim 4.7.** \( X \) satisfies \((**)\).

**Proof.** Let \( A \in [X]^{\omega_1} \). If there is an \( \alpha \in \omega_1 \) such that \( |A \cap X_\alpha| = \omega_1 \) we are done. Otherwise since \( |A \cap X_\alpha| \leq \omega \) we have \( |\pi(A)| = \omega_1 \). By Claim 4.6 \( A \) intersects (stationary) many closed discrete sets \( X_\alpha \) in \( \omega_1 \) many points. \( \square \)

**Claim 4.8.** \( X \) is not linearly D.
Proof. Suppose that \( D \subseteq X \) is closed discrete. Then \( \pi(D) \) is countable by Claim 3.6. Hence there is no closed discrete set which is big for the open cover \( \{ \alpha \times \omega_1 : \alpha < \omega_1 \} \). Thus \( X \) is not linearly \( D \) by Theorem 2.4. \( \square \)

This completes the proof of this theorem. \( \square \)

We can further extend the equivalences of Proposition 3.7 using Theorem 3.5.

**Proposition 4.9.** Suppose \( MA \). Suppose that the space \( X \) is locally countable, locally compact of cardinality less than \( 2^\omega \). Then the following are equivalent:

1. \( X \) is (linearly) \( D \),
2. \( X \) satisfies (**).

Proof. (1) implies (4) trivially. Suppose (4), then a closed uncountable subspace of \( X \) cannot be countably compact. Then by Claim 3.8 there is no perfect preimage of \( \omega_1 \) in \( X \). By Theorem 3.5 this implies that \( X \) is \( \sigma \)-closed-discrete, hence (linearly) \( D \). \( \square \)

* Under ZFC, without any further set-theoretic assumptions we can give a counterexample.

**Theorem 4.10.** There is a locally countable, locally compact, 0-dimensional \( T_2 \) space \( X \) such that \( X \) is not linearly \( D \) however satisfies (**).

Proof. We will use the following notations: let \( \{ C_\alpha : \alpha < 2^\omega \} \) be an enumeration of uncountable closed dense in itself subsets of \( \mathbb{R} \) and enumerate \( \{ Q \in [R \setminus Q]^\omega : C_\alpha \subseteq \overline{Q} \} \) as \( \{ Q_{\alpha,\beta}^\delta : \beta < 2^\omega \} \). Enumerate the triples \( (C_\alpha, Q_{\alpha,\beta}^\delta, \gamma) \) for \( \alpha, \beta, \gamma < 2^\omega \) in order type \( 2^\omega \): \( \{ t_\delta : \delta < 2^\omega \} \). We need an enumeration of all functions \( F : \omega \to 2^\omega \), \( \{ F_{\varphi} : \varphi < 2^\omega \} \). Fix an \( h : 2^\omega \to 2^\omega \times 2^\omega \) bijection. Furthermore, let \( \{ M_\epsilon : \epsilon < 2^\omega \} \subseteq [\omega]^\omega \) be an almost disjoint family on \( \omega \).

We define a topology on \( X = 2^\omega \times 2^\omega \) by induction. Let \( X_\delta = \{ \delta \} \times 2^\omega \) for \( \delta < 2^\omega \). Let \( (X_{<\delta}, \tau_{<\delta}) \) denote the topology defined by the induction till step \( \delta < 2^\omega \) where \( X_{<\delta} = \bigcup\{ X_\delta : \delta < \delta \} \). In step \( \delta \) we pick a point \( x_\delta \) from the real line which will help us define the neighborhoods of points in \( X_\delta \). We have the following conditions which we preserve during the induction:

1. \( (X_{<\delta}, \tau_{<\delta}) \) is locally countable, locally compact, 0-dimensional,
2. \( X_{<\delta'} \) is open in \( X_{<\delta} \) for \( \delta' < \delta \),
3. for every \( \delta' < \delta \) and \( (\delta', \epsilon) \in X_{\delta'} \) there is some neighborhood \( G \) of \( (\delta', \epsilon) \) such that \( G \setminus \{ (\delta', \epsilon) \} \subseteq X_{<\delta'} \),
4. **property (E):** suppose \( \delta' < \delta \) and \( x_{\delta'} \in B \) where \( B \) is Euclidean open. If \( (\delta', \epsilon) \in X_{\delta'} \) then there is some compact, countable and clopen neighborhood \( G \) of \( (\delta', \epsilon) \) such that \( G \subseteq \bigcup\{ X_{\delta''} : x_{\delta''} \in B \} \).

This way we will get a topology \( \tau \) on \( X \) if we take \( \bigcup\{ \tau_{<\delta} : \delta < 2^\omega \} \) as a base.

Suppose we are in step \( \delta \in 2^\omega \), where \( t_\delta = (C_\alpha, Q_{\alpha,\gamma}^\delta) \). We do the following:
• pick a point $x_\delta \in C_\alpha \setminus \{x_\delta' : \delta' < \delta\} \cup \mathbb{Q}$,
• if the set $Q_\alpha^0 \cap \{x_\delta : \delta' < \delta\}$ does not accumulate to $x_\delta$ just let each point of $X_\delta$ be discrete,
• if the set $Q_\alpha^0 \cap \{x_\delta : \delta' < \delta\}$ accumulates to $x_\delta$, choose a sequence 
  \( \{x_\delta_n : n \in \omega\} \subseteq Q_\alpha^0 \cap \{x_\delta : \delta' < \delta\} \) converging to $x_\delta$,
• take disjoint open intervals $B_n$ with rational endpoints, containing $x_\delta_n$.

Now we are ready to define a neighborhood of a point $(\delta, \epsilon)$. Suppose $h(\epsilon) = (\varphi, \rho)$.

• Consider the points $(\delta'_n, F_\varphi(n))$ in $X_{\delta'_n}$,
• by property (E) we can take compact, countable and clopen neighborhoods $G_n$ of $(\delta'_n, F_\varphi(n))$ such that $G_n \subseteq \bigcup \{X_{\delta''} : x_{\delta''} \in B_n\}$.

Observe that $\bigcup \{G_n : n \in \omega\}$ is closed in $(X_{<\delta}, \tau_{<\delta})$.

Let

\[ U((\delta, \epsilon), E) = \{(\delta, \epsilon)\} \cup \bigcup \{G_n : n \in M^\epsilon \setminus E\} \]

for $E \in [\omega]^{<\omega}$ and let the following be a neighborhood base for $(\delta, \epsilon)$

\[ \{U((\delta, \epsilon), E) : E \in [\omega]^{<\omega}\}. \]

Note that if $\epsilon \neq \epsilon' < 2^\omega$ and $E = M^\epsilon \cap M^{\epsilon'}$ then $U((\delta, \epsilon), E) \cap U((\delta, \epsilon'), E) = \emptyset$; this yields that the resulting topology will be $T_2$. We need to check that the induction assumptions still hold. Clearly $U((\delta, \epsilon), E)$ is countable and compact, we need to check that it is clopen. Since $U((\delta, \epsilon), E) \cap X_{<\delta} = \bigcup \{G_n : n \in \omega\}$ is closed in $X_{<\delta}$, we only need to check that $(\delta, \epsilon') \notin U((\delta, \epsilon), E)$ for $\epsilon \neq \epsilon' < 2^\omega$. Let $F = M^\epsilon \cap M'^\omega \subseteq [\omega]^{<\omega}$ then $U((\delta, \epsilon), E) \cap U((\delta, \epsilon'), F) = \emptyset$. Properties (ii) and (iii) will clearly hold. We need to check (iv), property (E). For points in $X_{<\delta}$ this will still hold. Consider a Euclidean open $B$ such that $x_\delta \in B$ and a new point: $(\delta, \epsilon) \in X_\delta$. Using the notations of the definition of a basic neighborhood for $(\delta, \epsilon)$, there is some $m \in \omega$ such that $\bigcup \{B_n : n \geq m\} \subseteq B$. So for the following neighborhood:

\[ G = \{(\delta, \epsilon)\} \cup \bigcup \{G_n : n \in M^\epsilon, n \geq m\} \]

we have that $G \subseteq \bigcup \{X_{\delta''} : x_{\delta''} \in B\}$, since for $n \geq m$ $B_n \subseteq B$ thus $G_n \subseteq \bigcup \{X_{\delta''} : x_{\delta''} \in B_n\} \subseteq \bigcup \{X_{\delta''} : x_{\delta''} \in B\}$.

It is clear that $(X, \tau)$ is a locally countable, locally compact and 0-dimensional space. It is straightforward by property (iii) that each $X_\delta$ is closed discrete. Let $\pi(A) = \{\delta < 2^\omega : A \cap X_\delta \neq \emptyset\}$ and $\pi_0(A) = \{x_\delta : \delta \in \pi(A)\} \subseteq \mathbb{R}$ for $A \subseteq X$.

Claim 4.11. If $|\pi(A)| > \omega$ for $A \subseteq X$ then there are $2^\omega$ many $\delta < 2^\omega$ such that $|X_\delta \cap A| = 2^\omega$. 
Proof. There is $\alpha < 2^\omega$ such that $C_\alpha \subseteq \pi_0(A)$ (Euclidean closure taken) and $\beta < 2^\omega$ such that $Q_\alpha^\beta \subseteq \pi_0(A)$. Let

$$D = \{ \delta < 2^\omega : \exists \gamma < 2^\omega (t_\delta = (C_\alpha, Q_\alpha^\beta, \gamma)) \}$$

and $\forall \delta' < 2^\omega (x_{\delta'} \in Q_\alpha^\beta \Rightarrow \delta' < \delta)$. 

Take a $\delta \in D$. Clearly we did not defined $X_\delta$ to be discrete since all points in $Q_\alpha^\beta$ are of the form $x_{\delta'}$ where $\delta' < \delta$. So at step $\delta$ in the induction we chose some convergent sequence $\{ x_{\delta_n} : n \in \omega \}$ from $Q_\alpha^\beta$ where $\delta_n < \delta$. Let $F : \omega \to 2^\omega$ such that $(\delta_n, F(n)) \in A$. There is some $\varphi \in 2^\omega$ such that $F = F_\varphi$. We claim that $\{ (\delta, \epsilon) \in X_\delta : \exists \rho < 2^\omega : h(\epsilon) = (\varphi, \rho) \} \subseteq A'$. For such a point $(\delta, \epsilon)$, we used $F_\varphi = F$ for the definition of basic neighborhoods, thus the set $\{ (\delta_n, F(n)) : n \in \omega \} \subseteq A$ accumulates to $(\delta, \epsilon)$. 

Claim 4.12. $X$ satisfies (**) .

Proof. Let $A \in [X]^\kappa$ such that $\kappa$ is an uncountable, regular cardinal. If there is a $\delta < 2^\omega$ such that $|A \cap X_\delta| = \kappa$ we are done. Otherwise $|\pi(A)| > \omega$ since $|A \cap X_\delta| < \kappa$ for all $\delta < 2^\omega$. By Claim 4.11 $A$ intersects (continuum) many closed discrete sets $X_\delta$, in $2^\omega$ many points. 

Claim 4.13. $X$ is not linearly $D$.

Proof. Suppose that $D \subseteq X$ is closed discrete. Then $\pi(D)$ is countable by Claim 4.11. Hence there is no closed discrete set which is big for the open cover $\{ X_{< \delta} : \delta < 2^\omega \}$. Thus $X$ is not linearly $D$ by Theorem 2.4. 

This completes the proof of this theorem. 

Remark: P. Nyikos gave an example of a space $T$ which is not a $D$-space, however for every closed $F \subseteq T$: $e(F) = L(F)$. From [10, Theorem 1.11] from his article, one can see that $T$ is linearly $D$ (use the characterization of linear $D$ property by Theorem 2.4). Applying Claim 4.11 to our last construction we get the following:

Corollary 4.14. There exists a Hausdorff space $X$ of cardinality $2^\omega$ such that $X$ is locally countable, locally compact, 0-dimensional, not linearly $D$ however $e(F) = L(F)$ for every closed subset $F \subseteq X$. 

5. Consistently on property $D$ and $aD$

Our main goal in this section is to construct a space which is not linearly $D$, however every closed subset of it is irreducible; hence $aD$ by Theorem 2.2.

We will use the following set-theoretical assumption:

$(\Diamond^*)$ there is a $\Diamond^*$-sequence, meaning that there exists an $\{ A_\alpha : \alpha \in \text{lim}(\omega_1) \}$ such that $A_\alpha \subseteq [\alpha]^{\omega}$ is countable and for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ such that $X \cap \alpha \in A_\alpha$ for all $\alpha \in C$.

Before proving the theorem we need the following easy claim about maximal almost disjoint families (MAD, in short).
Claim 5.1. If \( \{ N_i : i \in \omega \} \subseteq [\omega]^\omega \) then there is a MAD family \( \mathcal{M} \subseteq [\omega]^\omega \) of size \( 2^\omega \) such that for all \( M \in \mathcal{M} \) and \( i \in \omega \): \( |M \cap N_i| = \omega \).

Proof. We will construct the MAD family \( \mathcal{M} \) on \( \mathbb{Q} \). We can suppose that each \( N_i \) is dense in \( \mathbb{Q} \). Let \( R = \{ x_\alpha : \alpha < 2^\omega \} \) and for all \( \alpha < 2^\omega \) let \( S_\alpha \subseteq \mathbb{Q} \) such that \( S_\alpha \) is a convergent sequence with limit point \( x_\alpha \) and \( |S_\alpha \cap N_i| = \omega \) for all \( i \in \omega \). Then \( S = \{ S_\alpha : \alpha < 2^\omega \} \) is almost disjoint, let \( T = \{ T_\alpha : \alpha < \lambda \} \subseteq [\mathbb{Q}]^\omega \) such that \( S \cup T \) is MAD. Then \( \mathcal{M} = \{ S_\alpha \cup T_\alpha : \alpha < \lambda \} \) is a MAD family with the desired property. \( \square \)

Theorem 5.2. Suppose \((\diamond^*)\). There is a locally countable, locally compact, 0-dimensional \( T_2 \) space \( X \) of size \( \omega_1 \) such that \( X \) is not linearly D, however every closed subset \( F \subseteq X \) is irreducible; equivalently \( X \) is an aD-space.

Proof. We will define a topology on \( X = \omega_1 \times \omega_1 \). Let \( X_\alpha = \{ \alpha \} \times \omega_1 \) and \( X_{<\alpha} = \alpha \times \omega_1 \) for \( \alpha < \omega_1 \).

Definition 5.3. The set \( A \subseteq [X]^\omega \) runs up to \( \alpha < \omega_1 \) iff \( A = \{ (\alpha_n, \beta_n) : n \in \omega \} \subseteq X_{<\alpha} \) such that \( \alpha_0 < \ldots < \alpha_n < \ldots \) and \( \sup \{ \alpha_n : n \in \omega \} = \alpha \).

Note that if \( A \subseteq X \) runs up to some \( \alpha < \omega_1 \) then \( A \cap X_\beta \) is finite for all \( \beta < \omega_1 \).

We need the following consequence of \((\diamond^*)\). Let \( \pi(A) = \{ \alpha \in \omega_1 : A \cap X_\alpha \neq \emptyset \} \) for \( A \subseteq X \).

Claim 5.4. \((\diamond^*)\) There exists a sequence \( \{ A_\alpha : \alpha \in \lim(\omega_1) \} \subseteq [X]^\omega \) with \( A_\alpha = \bigcup \{ A_\alpha^n : n \in \omega \} \) for all \( \alpha \in \lim(\omega_1) \) such that

1. \( |A_\alpha^n| = \omega \) for all \( n \in \omega \),
2. \( A_\alpha \) runs up to \( \alpha \),
3. for all \( Y \subseteq X \) if \( |\pi(Y)| = \omega_1 \) then

\[ \exists \text{ club } C \subseteq \omega_1 \text{ such that } \forall \alpha \in C \exists n \in \omega \{ A_\alpha^n \subseteq Y \}. \]

Proof. Let \( \{ A_\alpha : \alpha \in \lim(\omega_1) \} \) denote a \( \diamond^* \)-sequence. Let \( i : \omega_1 \times \omega_1 \to \omega_1 \) denote a bijection which maps \( (\omega_1 \times (\omega_1 + 1)) \setminus (\omega_1 \times \alpha) \) to \( \omega_1 \cdot (\omega_1 + 1) \setminus \omega \cdot \alpha \). Let \( \tilde{A}_\alpha = \{ i^{-1}(A) : A \in A_\omega, \sup (\pi(i^{-1}(A))) = \alpha \} \)

and let \( A_\alpha = \bigcup \{ A_\alpha^n : n \in \omega \} \) such that

1. \( |A_\alpha^n| = \omega \) for all \( n \in \omega \),
2. \( A_\alpha \) runs up to \( \alpha \),
3. for all \( B \in \tilde{A}_\alpha \) there is \( n \in \omega \) such that \( A_\alpha^n \subseteq B \),

for all \( \alpha \in \lim(\omega_1) \). We claim that the sequence \( \{ A_\alpha : \alpha \in \lim(\omega_1) \} \) has the desired properties. Let \( Y \subseteq X \) such that \( |\pi(Y)| = \omega_1 \). There is some club \( C_0 \subseteq \omega_1 \) such that \( Y \cap X_{<\alpha} \subseteq \alpha \times \alpha \) for all \( \alpha \in C_0 \). There is some club \( C_1 \subseteq \omega_1 \) such that \( \alpha \cap i[Y] \in \tilde{A}_\alpha \) for all \( \alpha \in C_1 \). Let \( C_2 = \{ \alpha < \omega_1 : \omega \cdot \alpha \in C_1 \} \); clearly, \( C_2 \) is a club. Let \( C = C_0 \cap C_2 \cap \pi(Y)' \). Fix some \( \alpha \in C \). Then \( \omega \cdot \alpha \cap i[Y] = A \) for some \( A \in \tilde{A}_\alpha \), thus \( i[Y \cap X_{<\alpha}] = A \) since \( \omega \cdot \alpha = i[\alpha \times \alpha] \) and \( Y \cap X_{<\alpha} \subseteq \alpha \times \alpha \). Hence \( i^{-1}(A) = Y \cap X_{<\alpha} \) and \( i^{-1}(A) \in \tilde{A}_\alpha \) because \( \alpha \in \pi(Y)' \). Thus there is \( n \in \omega \) such that \( A_\alpha^n \subseteq Y \) by (3)'.
Let \( \{A_\alpha : \alpha \in \text{lim}(\omega_1)\} \subseteq [X]^{\omega} \) denote a sequence with \( A_\alpha = \bigcup \{A_\alpha^n : n \in \omega\} \) for \( \alpha \in \text{lim}(\omega_1) \) from Claim 5.1. We want to define the topology on \( X \) such that

- \( X_\alpha \) is closed discrete for all \( \alpha < \omega_1 \),
- \( X_{<\alpha} \) is open for all \( \alpha \in \omega_1 \),
- if \( A \in [X]^{\omega} \) runs up to \( \alpha \) then \( A \) has an accumulation point in \( X_\alpha \),
- \( X_\alpha \subseteq A_\alpha^\omega \) for all \( \alpha \in \text{lim}(\omega_1) \) and \( n \in \omega \).

Let \( M_\alpha \subseteq [A_\alpha]^{\omega} \) denote a MAD family on \( A_\alpha \) for \( \alpha \in \text{lim}(\omega_1) \) such that \( |M \cap A_\alpha^n| = \omega \) for all \( M \in M_\alpha \) and \( n \in \omega \); such an \( M_\alpha \) exists by Claim 5.1.

Enumerate \( M_\alpha = \{M_\alpha^\beta : \beta < \omega_1\} \).

We define topologies \( \tau_{<\alpha} \) on \( X_{<\alpha} \) by induction on \( \alpha < \omega_1 \) such that \( \tau_{<\alpha} \cap \mathcal{P}(X_{<\beta}) = \tau_{<\beta} \) for all \( \beta < \alpha < \omega_1 \). This way we will get a topology \( \tau \) on \( X \) if we take \( \{\tau_{<\alpha} : \alpha < \omega_1\} \) as a base.

Suppose \( \alpha < \omega_1 \) and we have defined the topology \( (X_{<\alpha}, \tau_{<\alpha}) \) such that

(i) \( (X_{<\alpha}, \tau_{<\alpha}) \) is a locally countable, locally compact, 0-dimensional \( T_2 \) space,
(ii) for all \( \alpha' < \alpha \) and \( x \in X_{\alpha'} \) there is some neighborhood \( G \) of \( x \) such that \( G \cap X_{\alpha'} = \{x\} \),
(iii) \( (\alpha_0, \alpha_1) \times \omega_1 \subseteq X_{<\alpha} \) is clopen for all \( \alpha_0 < \alpha_1 < \alpha \).

If \( \alpha \in \omega_1 \setminus \text{lim}(\omega_1) \) then let \( X_\alpha \) be discrete. Suppose \( \alpha \in \text{lim}(\omega_1) \) and let us enumerate \( \{F \subseteq X_{<\alpha} \setminus A_\alpha : F \) runs up to \( \alpha \} \) as \( \{F_\alpha^\beta : \beta < \omega_1\} \).

**Definition 5.5.** A subspace \( A \subseteq T \) of a topological space \( T \) is completely discrete iff there is a discrete family of open sets \( \{G_a : a \in A\} \) such that \( a \in G_a \) for all \( a \in A \).

The following claim will be useful later.

**Claim 5.6.** Suppose that \( A = \{(\alpha_0, \beta_0) : n \in \omega\} \subseteq X \) runs up to \( \alpha \). Then \( A \) is completely discrete in \( X_{<\alpha} \); hence closed discrete either.

**Proof.** Let \( G_0 = (0, \alpha_0] \times \omega_1 \) and \( G_{n+1} = (\alpha_n, \alpha_{n+1}] \times \omega_1 \) for \( n \in \omega \). \( G_n \) is open for all \( n \in \omega \) by inductional hypothesis (iii). Note that \( \{G_n : n \in \omega\} \) is a discrete family of open sets such that \( A \cap G_n \) is finite for all \( n \in \omega \). Let \( \mathcal{G}_n \) denote a finite, disjoint family of clopen subsets of \( G_n \) such that for all \( a \in A \cap G_n \) there is exactly one \( G \in \mathcal{G}_n \) such that \( a \in G \). Then the discrete family \( \bigcup \mathcal{G}_n : n \in \omega \) shows that \( A \) is completely discrete. \( \Box \)

In step \( \alpha \in \text{lim}(\omega_1) \) we define the neighborhoods of points in \( X_\alpha = \{(\alpha, \beta) : \beta < \omega_1\} \) by induction on \( \beta < \omega_1 \) such that:

(a) \( X_{<\alpha} \cup \{(\alpha, \beta') : \beta' \geq \beta\} \) is locally countable, locally compact and 0-dimensional \( T_2 \),
(b) there is some neighborhood \( U \) of \( (\alpha, \beta) \) such that \( U \cap A_\alpha \subseteq M_\alpha^\beta \),
(c) \( M_\alpha^\beta \) converges to \( (\alpha, \beta) \),
(d) \( F_\alpha^\beta \) accumulates to \( (\alpha, \beta') \) for some \( \beta' \leq \beta \).

We need the following lemma to carry out the induction on \( \beta < \omega_1 \).
Lemma 5.7. Suppose that $(T \cup S, \tau)$ is a locally countable, locally compact and 0-dimensional $T_2$ space such that $T$ is open and $S$ is countable. Let $D = \{d_n : n \in \omega\} \subseteq T$ closed discrete in $T \cup S$ and completely discrete in $T$. Let $r \notin T \cup S$. Then there is a topology $\rho$ on $R = T \cup S \cup \{r\}$ such that

- $(R, \rho)$ is locally countable, locally compact and 0-dimensional $T_2$;
- $\rho|_{(T \cup S)} = \tau$;
- $D$ converges to $r$ and $r \notin \overline{S}$ in $(R, \rho)$.

Proof. Suppose that $d_n \in G_n$ such that $\{G_n : n \in \omega\}$ is a family of open sets which is discrete in $T$. For each $n \in \omega$ let $\{B^n_i : i \in \omega\}$ denote a neighborhood base of $d_n$ such that

- $G_n \supseteq B^n_0 \supseteq B^n_1 \supseteq \ldots$ and
- $B^n_i$ is countable, compact and clopen for all $n, i \in \omega$.

Since $S \cap D = \emptyset$ there is some clopen neighborhood $U_s$ of each $s \in S$ such that $U_s \cap D = \emptyset$. There is $g_s : \omega \rightarrow \omega$ such that

$$U_s \cap B^n_{g_s(n)} = \emptyset \text{ for all } n \in \omega.$$  

Since $S$ is countable, there is $g : \omega \rightarrow \omega$ such that for all $s \in S$ there is some $N \in \omega$ such that $g_s(n) \leq g(n)$ for all $n \geq N$. Define the topology $\rho$ on $R$ as follows. Let

$$B_N = \{r\} \cup \bigcup\{B^n_{g(n)} : n \geq N\} \text{ and } B = \{B_N : N \in \omega\}.$$ 

Let $\rho$ be the topology on $R$ generated by $\tau \cup B$.

Clearly $\rho|_{(T \cup S)} = \tau$. We claim that $(R, \rho)$ is locally countable, locally compact and 0-dimensional. Since $B$ is a neighborhood base for $r$, it suffices to prove that each $B \in B$ is countable, compact (trivial) and clopen. Let $N \in \omega$ then $B_N$ is clopen in $T$ since $\bigcup\{B^n_{g(n)} : n \in \omega\}$ is a family of clopen sets which is discrete in $T$ guaranteed by the discrete family $\{G_n : n \in \omega\}$. Let $s \in S$. There is $N \in \omega$ such that $U_s \cap B^n_{g(n)} = \emptyset$ for $n \geq N$. There is some neighborhood $V \in \tau$ of $s$ such that $V \cap \bigcup\{B^n_{g(n)} : n < N\} = \emptyset$ since $s$ is not in the closed set $\bigcup\{B^n_{g(n)} : n < N\}$. Thus $(U_s \cap V) \cap B_N = \emptyset$. This proves that $B_N$ is clopen.

We claim that $(R, \rho)$ is $T_2$. Let $s \in S$, then there is $N \in \omega$ such that $U_s \cap B^n_{g(n)} = \emptyset$ for $n \geq N$, thus $B_N \cap U_s = \emptyset$. As noted before $B_N \cap T$ is closed and clearly $\bigcap\{B_N \cap T : N \in \omega\} = \emptyset$. This yields that any point $t \in T$ and $r$ can be separated, thus $(R, \rho)$ is $T_2$.

Clearly $D$ converges to $r$ and $S \cap B = \emptyset$ for any $B \in B$ thus $r \notin \overline{S}$. \hfill $\square$

Suppose we are in step $\beta < \omega_1$ and we defined the neighborhoods of points in $X_{< \alpha} \cup \{(\alpha, \beta') : \beta' < \beta\}$. We use Lemma 5.7 to define the neighborhoods of $r = (\alpha, \beta)$. Let $T = X_{< \alpha}$ and $S = \{(\alpha, \beta') : \beta' < \beta\} \cup (A_\alpha \setminus M_\beta^3)$. Note that $F^3_\alpha \cup M_\beta^3$ runs up to $\alpha$ thus closed and completely discrete in $T$ by Claim 5.6. Also, $M_\beta^3$ is closed discrete in $T \cup S$ by inductional hypothesis (b) for $(\alpha, \beta')$ where $\beta' < \beta$. 
Claim 5.9. If \( F^\beta_\alpha \) accumulates to \( x_{\beta'} \) for some \( \beta' < \beta \) then let \( D = M^\beta_\alpha \).

Claim 5.10. If \( F^\beta_\alpha \) is closed discrete in \( T \cup S \) then let \( D = M^\beta_\alpha \cup F^\beta_\alpha \).

Note that \( D \) is closed discrete in \( T \cup S \). By Claim 5.7 we can define the neighborhoods of \( r = (\alpha, \beta) \) such that the resulting space satisfies conditions (a), (b), (c) and (d). After carrying out the induction on \( \beta \), the resulting topology on \( X_\alpha \) clearly satisfies conditions (i), (ii) and (iii). This completes the induction.

As a base, the family \( \bigcup \{ \tau_\alpha : \alpha \in \lim(\omega_1) \} \) generates a topology \( \tau \) on \( X \) which is locally countable, locally compact and 0-dimensional \( T_2 \). Observe that \( X_\alpha \) is closed discrete and \( X_{<\alpha} \) is open for all \( \alpha < \omega_1 \) (by inducational hypotheses (ii) and (iii)).

Claim 5.8. Suppose that \( F \subseteq X \) runs up to some \( \alpha \in \lim(\omega_1) \). Then there is some \( \beta < \omega_1 \) such that \( F \) accumulates to \( (\alpha, \beta) \). Equivalently, if \( G \subseteq X \) is open and \( X_\alpha \subseteq G \) then there is some \( \alpha' < \alpha \) such that \( (\alpha', \alpha] \times \omega_1 \subseteq G \).

Proof. There is some \( \beta < \omega_1 \) such that \( F = F^\beta_\alpha \). Thus by inducational hypothesis (d) there is some \( \beta' \leq \beta \) such that \( F \) accumulates to \( (\alpha, \beta') \). □

Claim 5.9. \( X \) is not linearly \( D \).

Proof. If \( D \subseteq X \) is closed discrete then \( \pi(D) \) is finite by Claim 5.8. Thus there is no big closed discrete set for the cover \( \{ X_{<\alpha} : \alpha < \omega_1 \} \). □

Our next aim is to prove that all closed subspaces of \( X \) are irreducible.

Claim 5.10. If \( |\pi(F)| = \omega \) for a closed \( F \subseteq X \) then \( F \) is a \( D \)-space, hence irreducible.

Proof. Since \( F = \bigcup \{ F \cap X_\alpha : \alpha \in \pi(F) \} \) is a countable union of closed discrete sets, \( F \) is a \( D \)-space by Proposition 3.6. We mention that if the ONA \( U \) on \( F \) has closed discrete kernel \( D \) then we get an irreducible cover by taking the following open refinement: \( \{ (U(d) \setminus D) \cup \{ d \} : d \in D \} \). □

Claim 5.11. If \( |\pi(A)| = \omega_1 \) for \( A \subseteq X \) then there is a club \( C \subseteq \omega_1 \) such that \( C \times \omega_1 \subseteq A' \). As a consequence, if \( \pi(U) \) is stationary for the open \( U \subseteq X \) then there is some \( \alpha < \omega_1 \) such that \( X \setminus U \subseteq \alpha \times \omega_1 \).

Proof. There is a club \( C \subseteq \omega_1 \) by Claim 5.4 such that for all \( \alpha \in C \) there is \( n \in \omega \) such that \( A^n_\alpha \subseteq A \). We will prove that \( X_\alpha \subseteq A' \) for all \( \alpha \in C \). Take any point \( (\alpha, \beta) \in X_\alpha \). \( |M^\beta_\alpha \cap A^n_\alpha| = \omega \) for all \( \beta < \omega_1 \) by the construction of the MAD family \( M_\alpha \) and \( M^\beta_\alpha \) converges to \( (\alpha, \beta) \) by inducational hypothesis (c). Thus \( A^n_\alpha \) accumulates to \( (\alpha, \beta) \), hence \( X_\alpha \subseteq A' \). □

Claim 5.12. If \( |\pi(F)| = \omega_1 \) for a closed \( F \subseteq X \) then \( F \) is irreducible.

Proof. Take an open cover of \( F \), say \( U \). We can suppose that we refined it to the form \( U = \{ U(x) : x \in F \} \), where \( U(x) \) is a neighborhood of \( x \in F \). From Claim 5.11 we know that there is some club \( C \subseteq \omega_1 \) such that \( C \times \omega_1 \subseteq F \). For \( \alpha \in C \) define the open set \( G_\alpha = \bigcup \{ U(x) : x \in X_\alpha \} \). For every \( \alpha \in C \)
there is some $\delta(\alpha) < \alpha$ such that $(\delta(\alpha), \alpha] \times \omega_1 \subseteq G_\alpha$; by Claim 5.8. So there is some $\delta < \omega_1$ and a stationary $S \subseteq C$ such that $(\delta, \alpha] \times \omega_1 \subseteq G_\alpha$ for all $\alpha \in S$. Fix some $\delta_0 > \delta$ such that $X_{\delta_0} \subseteq F$. Let $S_0 = S \setminus (\delta_0 + 1)$. For all $\alpha \in S_0$ there is $d_\alpha \in X_\alpha \subseteq F$ such that $(\delta_0, \alpha) \in U(d_\alpha)$. Let us refine these sets: $U_0(d_\alpha) = (U(d_\alpha) \setminus (\{\delta_0\} \times S_0)) \cup \{(\delta_0, \alpha)\}$ for all $\alpha \in S_0$; let $U_0 = \{U_0(d_\alpha) : \alpha \in S_0\}$. Clearly $U_0$ is an open refinement of $U$ which is minimal and $\{d_\alpha : \alpha \in \omega_1\} \subseteq \cup U_0$. Since $S_0$ is stationary and $S_0 \subseteq \pi[\cup U_0]$ we get that there is some $\gamma < \omega_1$ such that $F_1 = F \cup U_0 \subseteq \gamma \times \omega_1$ by Claim 5.11. So by Claim 5.10 the closed set $F_1$ is a D-space, hence irreducible. Take a minimal open refinement of the cover $\{U(x) \setminus (\{\delta_0\} \times S_0) : x \in F_1\}$, let this be $U_1$. The union $U_0 \cup U_1$ is an open refinement of $U$ which covers $F$ and minimal.

This proves that all closed subspaces of $X$ are irreducible. Hence $X$ is an aD-space by Theorem 2.2.

Using again the strong result of Balogh we can observe the following.

**Proposition 5.13.** Suppose MA. Let $X$ be a locally countable, locally compact space of cardinality $< 2^{\omega_1}$. Then the following are equivalent:

1. $X$ is a (linearly) D-space,
2. $X$ is an aD-space.

**Proof.** (1) implies (5) trivially. Suppose that $X$ is an aD-space. It is enough to show that $X$ does not contain any perfect preimage of $\omega_1$. Since property aD is hereditary to closed sets, any closed countably compact subspace is compact. By Claim 3.8 there is no perfect preimage of $\omega_1$ in $X$.

**Corollary 5.14.** The existence of a locally countable, locally compact space $X$ of size $\omega_1$ which is aD and non (linearly) D is independent of ZFC.

However, the following remains open.

**Problem 5.15.** (1) Is it consistent with ZFC that there exists a locally countable, locally compact space $X$ of cardinality $< 2^{\omega_1}$ such that $X$ is not (linearly) D however aD?

2. Is there a ZFC example of a Tychonoff space $X$ such that $X$ is not (linearly) D however aD?

2'. Is there a ZFC example of a locally countable, locally compact (0-dimensional) $T_2$ space $X$ such that $X$ is not (linearly) D however aD?

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