Product Representation of Dyon Partition Function in CHL Models

Justin R. David, Dileep P. Jatkar and Ashoke Sen

Harish-Chandra Research Institute,
Chhatnag Road., Jhunsi,
Allahabad 211019, India.
justin,dileep,sen@mri.ernet.in

Abstract: A formula for the exact partition function of 1/4 BPS dyons in a class of CHL models has been proposed earlier. The formula involves inverse of Siegel modular forms of subgroups of $Sp(2, \mathbb{Z})$. In this paper we propose product formulae for these modular forms. This generalizes the result of Borcherds and Gritsenko and Nikulin for the weight 10 cusp form of the full $Sp(2, \mathbb{Z})$ group.
1. Introduction and Summary

There exists a proposal for the exact degeneracy of dyons in toroidally compactified heterotic string theory\cite{1, 2, 3, 4, 5} and also in toroidally compactified type II string theory\cite{6}. These formulæ are invariant under the S-duality transformations of the theory, and also reproduce the entropy of a dyonic black hole in the limit of large charges\cite{2}. In \cite{7} this proposal was generalized to a class of CHL models\cite{8, 9, 10, 11, 12, 13}, obtained by modding out heterotic string theory on \(T^2 \times T^4\) by a \(\mathbb{Z}_N\) transformation that involves \(1/N\) unit of translation along one of the circles of \(T^2\) and a non-trivial action on the internal conformal field theory (CFT) describing heterotic string compactification on \(T^4\). The values of \(N\) considered in \cite{7} were \(N = 2, 3, 5, 7\). Using string-string duality\cite{14, 15, 16, 17, 18} one can relate these models to \(\mathbb{Z}_N\) orbifolds of type IIA string theory on \(T^2 \times K3\), with the \(\mathbb{Z}_N\) transformation acting
as $1/N$ unit of shift along a circle of $T^2$ together with an action on the internal CFT describing type IIA string compactification on $K3$.

The proposal of [7] may be summarized as follows. If we denote by $Q_e$ and $Q_m$ the electric and the magnetic charge vectors then the degeneracy $d(Q_e, Q_m)$ of dyons carrying charges $(Q_e, Q_m)$ is of the form

$$d(Q_e, Q_m) = g \left( \frac{1}{2}Q^2_m, \frac{1}{2}Q^2_e, Q_e : Q_m \right),$$

where $g(m, n, p)$ is defined through the Fourier expansion

$$\frac{1}{\Phi_k(U, T, V)} = C_0 \sum_{m,n,p} e^{2\pi i(mU+nT+pV)}g(m, n, p).$$

Here $C_0$ is a numerical constant and $\Phi_k(U, T, V)$ is a modular form of weight $k$ under a subgroup $\tilde{G}$ of $Sp(2, \mathbb{Z}) \equiv SO(2, 3; \mathbb{Z})$ where

$$k = \frac{24}{N+1} - 2.$$  

An explicit algorithm for constructing the Fourier expansion of $\Phi_k$ in the variables $T, U$ and $V$ was given in [7].

The degeneracy $d(Q_e, Q_m)$ defined through eqs. (1.1), (1.2) is invariant under the T- and S-duality symmetries of the theory. Furthermore it generates integer results for the degeneracies and its behaviour for large charges is consistent with the black hole entropy calculation [7, 19].

In this paper we use the method of [20, 21] to propose an alternative form of $\Phi_k$ as an infinite product:

$$\tilde{\Phi}_k(U, T, V) = -(i\sqrt{N})^{-k-2} \exp \left( \frac{2\pi i}{N} \left( T + U + V - 2 \sum_{s=0}^{N-1} e^{-2\pi i s/N} c^{(r,s)}(4lk'^2-b^2) \right) \right)$$

$$\prod_{r=0}^{N-1} \prod_{l,b \in \mathbb{Z},k',l,b>0} \left\{ 1 - \exp(2\pi i(k'T + lU + bV)) \right\}^{\frac{1}{2}} \prod_{r=0}^{N-1} \prod_{l,b \in \mathbb{Z},k',l,b>0} \left\{ 1 - \exp(2\pi i(k'T + lU + bV)) \right\}^{\frac{1}{2}}$$

where $(k', l, b) > 0$ means $k' > 0, l > 0, b \in \mathbb{Z}$ or $k' = 0, l > 0, b \in \mathbb{Z}$ or $k' = 0, l = 0, b < 0$ and $c^{(r,s)}(n)$ are some calculable coefficients related to the twisted elliptic
genus of $K3$. If $\tilde{g}$ denotes the generator of the $\mathbb{Z}_N$ action on $K3$ that is used in the construction of the CHL model, then we define the twisted elliptic genus of $K3$ as

$$F^{(r,s)}(\tau, z) = \frac{1}{N} Tr_{\text{RR}\tilde{g}^r} \left( (-1)^{F_{K3}} (-1)^{\bar{F}_{K3}} \tilde{g}^s e^{2\pi i z F_{K3}} q^{L_0} q^{\bar{L}_0} \right), \quad 0 \leq r, s \leq (N - 1),$$

(1.5)

where $Tr_{\text{RR}\tilde{g}^r}$ denotes trace in the superconformal field theory associated with target space $K3$ in the $\tilde{g}^r$ twisted RR sector, $q = e^{2\pi i \tau}$, and $F_{K3}, \bar{F}_{K3}$ denote the left- and right-handed world-sheet fermion numbers in this theory. Here and throughout the rest of the paper $L_0$ and $\bar{L}_0$ include an additive factor of $-c/24$ so that the RR sector ground state has $L_0 = \bar{L}_0 = 0$. The coefficients $c^{(r,s)}(n)$ are then defined through the Fourier expansion of $F^{(r,s)}(\tau, z)$:

$$F^{(r,s)}(\tau, z) = \sum_{b \in \mathbb{Z}, n} c^{(r,s)}(4n - b^2) q^n e^{2\pi i z b}.$$

(1.6)

Furthermore for the $N = 2, k = 6$ case we were able to explicitly compute the functions $F^{(r,s)}(\tau, z)$. They are given by

$$F^{(0,0)}(\tau, z) = 4 \left[ \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right],$$

$$F^{(0,1)}(\tau, z) = 4 \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2}, \quad F^{(1,0)}(\tau, z) = 4 \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2}, \quad F^{(1,1)}(\tau, z) = 4 \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2}.$$

(1.7)

For higher values of $N$ we did not evaluate the functions $F^{(r,s)}(\tau, z)$ directly, but were able to guess their forms from general considerations. The results are:

$$F^{(0,0)}(\tau, z) = \frac{8}{N} A(\tau, z),$$

$$F^{(0,s)}(\tau, z) = \frac{8}{N(N + 1)} A(\tau, z) - \frac{2}{N + 1} B(\tau, z) E_N(\tau) \quad \text{for } 1 \leq s \leq (N - 1),$$

$$F^{(r, rk)}(\tau, z) = \frac{8}{N(N + 1)} A(\tau, z) + \frac{2}{N(N + 1)} E_N \left( \frac{\tau + k}{N} \right) B(\tau, z),$$

for $1 \leq r \leq (N - 1), 0 \leq k \leq (N - 1),$

(1.8)

where

$$A(\tau, z) = \left[ \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right].$$

(1.9)
\[ B(\tau, z) = \eta(\tau)^{-6} \partial_1(\tau, z)^2, \]  
(1.10)

and

\[
E_N(\tau) = \frac{12i}{\pi(N-1)} \partial_\tau \left[ \ln \eta(\tau) - \ln(\eta(N\tau)) \right] = 1 + \frac{24}{N-1} \sum_{n_1, n_2 \geq 1, n_1 \neq 0 \text{ mod } N} n_1 e^{2\pi in_1 n_2 \tau}. \]
(1.11)

Eq. (1.4) gives a generalization of Borcherds and Gritsenko and Nikulin’s result \[22, 23\] of the product representation of \( \tilde{\Phi}_{10} \), – the unique cusp form of weight 10 of the group \( Sp(2, \mathbb{Z}) \). A systematic procedure for arriving at the product representation for \( \tilde{\Phi}_{10} \) was given in \[20\]. Our construction of \( \tilde{\Phi}_k \) is essentially based on a generalization of the techniques of \[20\].

Given the two different constructions of \( \tilde{\Phi}_k \), – one given in \[7\] and one in the present paper, it is natural to ask if they are the same. For the \( N = 2, k = 6 \) case we have compared 31 different Fourier expansion coefficients of the two proposals and found them to be the same.\(^1\) For other values of \( N \) we have compared the expansions up to order \( e^{4\pi iT} e^{4\pi iU} \) and all powers of \( e^{2\pi iV} \). For general \( N \) we also verify that the behaviour of \( \tilde{\Phi}_k \) (and of \( \Phi_k \) introduced in footnote \[1\]) in the \( V \to 0 \) limit as well as in the \( U \to i\infty \) limit agrees with the results found in \[7\].

The rest of the paper is organized as follows. In section 2 we outline the strategy that we shall be using for finding \( \tilde{\Phi}_k \). Sections 3 and 4 involve detailed calculations leading to the determination of \( \tilde{\Phi}_6 \) associated with the \( \mathbb{Z}_2 \) orbifold theory. In section 5 we give the final form of \( \tilde{\Phi}_6 \) and compare some of its properties with those found in \[7\]. Section 6 is devoted to the construction of the related quantity \( \Phi_6 \) described in footnote \[1\] and its comparison with the corresponding quantity calculated in \[7\]. In section 7 we describe the construction of \( \tilde{\Phi}_k \) and \( \Phi_k \) for a general \( k \) given in (1.3). The three appendices contain some technical details which were omitted from discussion in the main body of the paper.

2. The Strategy

Our goal is to find a product representation for \( \tilde{\Phi}_k \). In attaining this goal we shall proceed as in the case of ordinary toroidal compactification of heterotic string theory

\(^1\)Actually we compare not the Fourier expansion coefficients of \( \tilde{\Phi}_k \) but those of a closely related object \( \Phi_k(U, T, V) = T^{-k} \tilde{\Phi}_k(U - T^{-1}V^2, -T^{-1}, T^{-1}V) \).
or equivalently type II string theory on $T^2 \times K3$. This corresponds to the case $N = 1$, $k = 10$ and the associated modular form $\tilde{\Phi}_{10}$ is the unique weight 10 cusp form of the Siegel modular group $Sp(2; \mathbb{Z})$. The steps leading to a systematic construction of the product representation of $\tilde{\Phi}_{10}$ are as follows:

1. We consider a superconformal $\sigma$-model with target space $T^2 \times K3$ with $y^1, y^2$ denoting the $T^2$ coordinates. We denote by $F_{K3}$ and $F_{T2}$ the holomorphic parts of the world-sheet fermion number associated with the $K3$ and the $T^2$ parts and by $\bar{F}_{K3}$ and $\bar{F}_{T2}$ the anti-holomorphic parts of the world-sheet fermion number associated with the $K3$ and the $T^2$ parts. We shall be considering an arbitrary $T^2$ parametrized by the Kähler modulus $T$ and complex structure modulus $U$, and arbitrary Wilson lines $A_1, A_2$ corresponding to deforming the world-sheet theory by the marginal operator

$$\sum_{i=1}^{2} A_i \int d^2 z \bar{\partial} Y^i J_{K3}, \quad (2.1)$$

where $J_{K3}$ is the $U(1)$ current corresponding to the charge $F_{K3}$. We shall denote by $V$ the complex combination $A_2 - iA_1$. $V$ is normalized so that $V \equiv V + 1$.

This theory has an $SO(2, 3; \mathbb{Z})$ T-duality group. If we denote by $(m_1, m_2)$ the integers labeling momenta along $y^1, y^2$, by $(n_1, n_2)$ the integers labeling winding along $y^1, y^2$, and by $b$ the $F_{K3}$ charge, then the $SO(2, 3; \mathbb{Z})$ transformation $S$ acts on these charges and the parameters $T, U, V$ as

$$\begin{pmatrix} m'_1 \\ m'_2 \\ n'_1 \\ n'_2 \\ b' \end{pmatrix} = S \begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \\ b \end{pmatrix}, \quad \begin{pmatrix} T \\ T'U' - V'^2 \\ -U' \\ 1 \\ 2V' \end{pmatrix} = \lambda S \begin{pmatrix} T \\ T'U - V^2 \\ -U \\ 1 \\ 2V \end{pmatrix} \quad (2.2)$$

where $S$ is a $5 \times 5$ matrix with integer entries, satisfying

$$S^T LS = L, \quad L = \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad (2.3)$$

and $\lambda$ is a number to be adjusted so that the fourth element of the vector on the right hand side of (2.2) is 1. $I_n$ denotes $n \times n$ identity matrix.
Using the equivalence between $SO(2, 3)$ and $Sp(2)$ we can represent the T-duality group elements by $Sp(2, \mathbb{Z})$ matrices of the form \[
abla = \begin{pmatrix} A & B \\
C & D \end{pmatrix}
\] where $A$, $B$, $C$ and $D$ are each $2 \times 2$ matrix with integer entries satisfying
\[
AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I_2.
\]
(2.4)
If we define
\[
\Omega = \begin{pmatrix} U & V \\
V & T \end{pmatrix},
\]
(2.5)
then the duality group acts on $\Omega$ as
\[
\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}.
\]
(2.6)

2. In this theory we define:
\[
\mathcal{I}_0(U, T, V) = \int_F \frac{d^2 \tau}{2\pi} \frac{T_{RR}}{\tau} \left( (-1)^{(F_{K3}+F_{T^2})} (-1)^{(F_{K3}+\bar{F}_{T^2})} F_{T^2} F_{T^2} q^{L_0} \bar{q}^{L_0} \right)
\]
where $F$ is the fundamental domain of $SL(2, \mathbb{Z})$ and $q = e^{2\pi i \tau}$. $\mathcal{I}(U, T, V)$ is expected to be invariant under $SO(2, 3; \mathbb{Z})$ transformation.

3. Analysis of the integral given in (2.7) shows that it can be expressed in the form
\[
\mathcal{I}_0 = -20 \ln \det \text{Im} \Omega - 20 \ln \tilde{\Phi}_{10}(\Omega) - 20 \ln \tilde{\Phi}_{10}(\bar{\Omega}) + \text{constant}
\]
(2.8)
where $\tilde{\Phi}_{10}(\Omega)$ is a holomorphic function of $T$, $U$ and $V$ with a product representation. Since under the duality transformation (2.6)
\[
\det \text{Im} \Omega \rightarrow (\det(C\Omega + D))^{-1} (\det(C\bar{\Omega} + D))^{-1} \det \text{Im} \Omega,
\]
and $\mathcal{I}_0$ is invariant, we must have\(^2\)
\[
\tilde{\Phi}_{10} \left( (A\Omega + B)(C\Omega + D)^{-1} \right) = (\det(C\Omega + D))^{10} \tilde{\Phi}_{10}(\Omega).
\]
(2.10)
Thus $\tilde{\Phi}_{10}(\Omega)$ must be a Siegel modular form of weight 10. This leads to the construction of the product representation of $\tilde{\Phi}_{10}$.\(^2\)

\(^2\)In principle there could be $\Omega$ independent phases on the right hand side of (2.11), but it is known that they are absent in this case.
Our goal is to construct a modular form $\tilde{\Phi}_k$ of weight $k$ of an appropriate subgroup $\tilde{G}$ of $SO(2, 3; \mathbb{Z})$ for $k$ given in (1.3). The subgroup $\tilde{G}$ is the T-duality group of the superconformal field theory with target space $(T^2 \times K3)/\mathbb{Z}_N$ where the $\mathbb{Z}_N$ acts as a $1/N$ unit of shift along a circle on $T^2$ and as a geometric transformation of order $N$ on $K3$.\(^3\) Thus only those $SO(2, 3; \mathbb{Z})$ transformations which commute with the $1/N$ unit of shift along $T^2$ will be symmetries of the resulting theory.

We shall try to construct $\tilde{\Phi}_k$ by first defining an analog of the integral $I_0$ invariant under this subgroup and then splitting it into a sum of an holomorphic piece, an anti-holomorphic piece and a term proportional to $\ln \det \text{Im} \Omega$ as in (2.8). A natural candidate integral is

$$I(U, T, V) = \int F d^2 \tau \text{Tr}_{RR} \left( (-1)^{(F_{K3} + F_{T^2})} (-1)^{(\bar{F}_{K3} + \bar{F}_{T^2})} F_{T^2} \bar{F}_{T^2} q^{L_0} \bar{q}^{\bar{L}_0} \right) \quad (2.11)$$

where the trace is taken over the states in this orbifold superconformal field theory.

For $V = 0$ this integral has been calculated for the $\mathbb{Z}_2$ orbifold model in [24]. In the next few sections we shall describe computation of this integral for the $N = 2$ case for non-zero $V$. This will enable us to determine the product form of $\tilde{\Phi}_6$. Later we shall discuss generalization of this analysis to other values of $N$.

### 3. The Integrand for the $\mathbb{Z}_2$ Orbifold Theory

In this section we shall analyze the integrand in eq.(2.11) for the $\mathbb{Z}_2$ orbifold conformal field theory described earlier. We can decompose the contribution to the trace in (2.11) as a sum of the contribution from different sectors characterized by the five charges $(m_1, n_1, m_2, n_2, b)$ introduced earlier.\(^4\) In this case we can factor out the $T$, $U$ and $V$ dependence of the trace into an overall factor of $q^{p^2_R/2} \bar{q}^{\bar{p}^2_R/2}$ where

$$\frac{1}{2} p^2_R = \frac{1}{4 \det \text{Im} \Omega} \left| -m_1 U + m_2 + n_1 T + n_2 (TU - V^2) + b V \right|^2,$$

$$\frac{1}{2} \bar{p}^2_L = \frac{1}{2} p^2_R + m_1 n_1 + m_2 n_2 + \frac{1}{4} b^2. \quad (3.1)$$

\(^3\)In order to preserve the $\mathcal{N} = 4$ target space supersymmetry, the $\mathbb{Z}_N$ action on $K3$ must commute with the $(4,4)$ superconformal symmetry possessed by a supersymmetric $\sigma$-model with target space $K3$.

\(^4\)Note that now the twisted sector states carry half integer winding number $n_1$ along $y^1$. 

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Thus $I(U, T, V)$ has the form

$$I(U, T, V) = \int_\mathcal{F} \frac{d^2 \tau}{\tau_2} \sum_{m_1, m_2, n_1, n_2, b} q^{p_h^2/2 - b^2/4} \bar{q}^{p_h^2/2} F_{m_1, m_2, n_1, n_2, b}(\tau)$$

(3.2)

where $F_{m_1, m_2, n_1, n_2, b}(\tau)$ is independent of $T$, $U$ and $V$ and is given by

$$F_{m_1, m_2, n_1, n_2, b}(\tau) = Tr_{m_1, m_2, n_1, n_2; b, RR} \left( (-1)^{F_{K3} + F_{T^2}} (-1)^{F_{K3} + F_{T^2}} F_{T^2} \bar{F}_{T^2} q^{L_0} \bar{q}^{L_0'} \right).$$

(3.3)

Here

$$L_0' = L_0 - \frac{p_L^2}{2} + \frac{b^2}{4}, \quad \bar{L}_0' = \bar{L}_0 - \frac{p_B^2}{2},$$

(3.4)

are independent of $T$, $U$ and $V$ and $Tr_{m_1, m_2, n_1, n_2; b}$ denotes trace over a subspace of the Hilbert space carrying momentum $(m_1, m_2)$ and winding $(n_1, n_2)$ along $T^2$ and $F_{K3}$ charge $b$. Note that we have included the $b^2/4$ term in $L_0'$ so that for $V = 0$ when the conformal field theories associated with $K3$ and $T^2$ parts decouple, $L_0'$ and $\bar{L}_0'$ describe complete contribution from the CFT associated with $K3$ and oscillator contribution from the CFT associated with $T^2$. Since $F_{m_1, m_2, n_1, n_2; b}(\tau)$ is independent of $T$, $U$ and $V$, we can set $V = 0$ while evaluating (3.3).

Let us define

$$F_{m_1, m_2, n_1, n_2}(\tau, z) = \sum_b F_{m_1, m_2, n_1, n_2, b}(\tau) e^{2\pi i b z}. \quad (3.5)$$

It then follows from (3.3) that

$$F_{m_1, m_2, n_1, n_2}(\tau, z) = Tr_{m_1, m_2, n_1, n_2; RR} \left( (-1)^{F_{K3} + F_{T^2}} (-1)^{F_{K3} + F_{T^2}} F_{T^2} \bar{F}_{T^2} e^{2\pi i z F_{K3}} q^{L_0} \bar{q}^{L_0'} \right).$$

(3.6)

We shall first compute $F_{m_1, m_2, n_1, n_2}(\tau, z)$ and then extract $F_{m_1, m_2, n_1, n_2; b}(\tau)$ using eq.(3.5). Since the contribution to (3.4) from the $T^2$ part is somewhat trivial, it is useful to separate out this contribution. For this we denote by $g'$ the generator of the $\mathbb{Z}_2$ group with which we take the orbifold of $K3 \times T^2$. Then

$$F_{m_1, m_2, n_1, n_2}(\tau, z) = \frac{1}{2} \sum_{r,s=0}^1 Tr_{m_1, m_2, n_1, n_2; RR; (g')^r} \left( (-1)^{F_{K3} + F_{T^2}} (-1)^{F_{K3} + F_{T^2}} F_{T^2} \bar{F}_{T^2} e^{2\pi i z F_{K3}} q^{L_0} \bar{q}^{L_0'} (g')^s \right),$$

(3.7)
where the superscript $K^3 \times T^2$ in $Tr$ indicates that the trace is taken in the superconformal field theory with target space $K^3 \times T^2$, and the subscript $(g')^r$ in $Tr$ indicates that the trace is over the sector twisted by $(g')^r$. We now split $g'$ as

$$g' = \hat{g} \tilde{g}, \quad (3.8)$$

where $\hat{g}$ and $\tilde{g}$ represent the action of $g'$ on the $T^2$ and $K^3$ parts respectively. Twisting by $\hat{g}$ makes the winding number $n_1 \in \mathbb{Z} + \frac{r}{2}$, and hence the right hand side of (3.7) vanishes unless $n_1 - \frac{r}{2} \in \mathbb{Z}$. The $(\hat{g})^s$ factor inside the trace produces a factor of $(-1)^{m_1 s}$. The non-zero mode bosonic and fermionic oscillator contributions from the $T^2$ factor always cancel since they are neutral under $\hat{g}$. The fermion zero modes associated with $T^2$ give a factor of 4 due to 2-fold degeneracy each from the holomorphic and anti-holomorphic sectors, but this cancels with the factor of $1/4$ coming from the $F_{T^2} F_{T^2}$ factor inside the trace. Thus we can write

$$F_{m_1, m_2, n_1, n_2}(\tau, z) = \frac{1}{2} \sum_{s=0}^{1} (-1)^{m_1 s} F^{(r,s)}(\tau, z) \quad \text{for} \quad n_1 \in \mathbb{Z} + \frac{r}{2}, \quad r = 0, 1 \quad (3.9)$$

where

$$F^{(r,s)}(\tau, z) = \frac{1}{2} T_{r}^{K^3}_{R \bar{R} \bar{g}' - \bar{g}} \left( (-1)^{F_{K^3}} (-1)^{F_{K^3}} \tilde{g}' e^{2\pi i z F_{K^3} \bar{q} L_0 \bar{\bar{q}} \bar{L}_0} \right). \quad (3.10)$$

Here $T_{r}^{K^3}_{R \bar{R} \bar{g}'}$ denotes trace in the superconformal field theory associated with target space $K^3$ in the $\tilde{g}'$ twisted RR sector, and $L_0, \bar{L}_0$ inside the trace now includes contribution from $K^3$ only. This is twisted elliptic genus of $K^3$. These quantities were introduced in [25] in order to calculate the elliptic genus of $\tilde{g}$ orbifold of $K^3$. This would be given by $\sum_{r,s=0}^{1} F^{(r,s)}(\tau, z)$. Here however we need the individual $F^{(r,s)}(\tau, z)$ since we shall be using them for a different purpose.

From the definitions given in (3.10) it follows that [25]

$$F^{(r,s)} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = \exp \left( 2\pi i \frac{c z^2}{c\tau + d} \right) F^{(cs + ar, ds + br)}(\tau, z), \quad (3.11)$$

for

$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \quad (3.12)$$

In (3.11) the indices $cs + ar$ and $ds + br$ are to be taken mod 2.
\( F_{m_1,m_2,n_1,n_2}(\tau, z) \) has been calculated in appendix Appendix A using an orbifold description of K3 and the result is given in eq.(3.15). Comparing this with eq.(3.9) we get

\[
F^{(0,0)}(\tau, z) = 4 \left[ \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right], \\
F^{(0,1)}(\tau, z) = 4 \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2}, \quad F^{(1,0)}(\tau, z) = 4 \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \quad F^{(1,1)}(\tau, z) = 4 \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2}.
\]

(3.13)

Using the known modular transformation laws of \( \vartheta_i(\tau, z) \) we can verify that \( F^{(r,s)}(\tau, z) \) given in (3.13) satisfy (3.11).

We now use the relations:

\[
\begin{align*}
\vartheta_1^2(\tau, z) &= \vartheta_2(2\tau, 0)\vartheta_3(2\tau, 2z) - \vartheta_3(2\tau, 0)\vartheta_2(2\tau, 2z) \\
\vartheta_2^2(\tau, z) &= \vartheta_2(2\tau, 0)\vartheta_3(2\tau, 2z) + \vartheta_3(2\tau, 0)\vartheta_2(2\tau, 2z) \\
\vartheta_3^2(\tau, z) &= \vartheta_3(2\tau, 0)\vartheta_3(2\tau, 2z) + \vartheta_2(2\tau, 0)\vartheta_2(2\tau, 2z) \\
\vartheta_4^2(\tau, z) &= \vartheta_3(2\tau, 0)\vartheta_3(2\tau, 2z) - \vartheta_2(2\tau, 0)\vartheta_2(2\tau, 2z)
\end{align*}
\]

(3.14)

to rewrite (3.13) as

\[
F^{(r,s)}(\tau, z) = h_0^{(r,s)}(\tau) \vartheta_3(2\tau, 2z) + h_1^{(r,s)}(\tau) \vartheta_2(2\tau, 2z)
\]

(3.15)

where

\[
\begin{align*}
h_0^{(0,0)}(\tau) &= 8 \frac{\vartheta_3(2\tau, 0)^3}{\vartheta_3(\tau, 0)^2\vartheta_4(\tau, 0)^2} + 2 \frac{1}{\vartheta_3(2\tau, 0)} \\
h_1^{(0,0)}(\tau) &= -8 \frac{\vartheta_2(2\tau, 0)^3}{\vartheta_3(\tau, 0)^2\vartheta_4(\tau, 0)^2} + 2 \frac{1}{\vartheta_2(2\tau, 0)} \\
h_0^{(0,1)}(\tau) &= 2 \frac{1}{\vartheta_3(2\tau, 0)}, \quad h_1^{(0,1)}(\tau) = 2 \frac{1}{\vartheta_2(2\tau, 0)}, \\
h_0^{(1,0)}(\tau) &= 4 \frac{\vartheta_3(2\tau, 0)}{\vartheta_4(\tau, 0)^2}, \quad h_1^{(1,0)}(\tau) = -4 \frac{\vartheta_2(2\tau, 0)}{\vartheta_4(\tau, 0)^2}, \\
h_0^{(1,1)}(\tau) &= 4 \frac{\vartheta_3(2\tau, 0)}{\vartheta_3(\tau, 0)^2}, \quad h_1^{(1,1)}(\tau) = 4 \frac{\vartheta_2(2\tau, 0)}{\vartheta_3(\tau, 0)^2}.
\end{align*}
\]

(3.16)

Since

\[
\begin{align*}
\vartheta_3(2\tau, 2z) &= \sum_{b \in 2\mathbb{Z}} e^{2\pi ibz} q^{b^2/4}, & \vartheta_2(2\tau, 2z) &= \sum_{b \in 2\mathbb{Z} + 1} e^{2\pi ibz} q^{b^2/4}.
\end{align*}
\]

(3.17)
we see, by comparing (3.5) and (3.9), (3.15) that

\[ F_{m_1,m_2,n_1,n_2,b}(\tau) = q^{b^2/4} \sum_{s=0}^{1} (-1)^{m_1 s} h_i^{(r,s)}(\tau) \quad \text{for } n_1 \in \mathbb{Z} + \frac{r}{2}, \ b \in 2\mathbb{Z} + l \quad r, l = 0, 1. \tag{3.18} \]

Using (3.18) the original integral \( \mathcal{I}(U, T, V) \) given in eq.(3.2) may be written as

\[ \mathcal{I}(U, T, V) = \sum_{l,r,s=0}^{1} \mathcal{I}_{r,s,l} \tag{3.19} \]

where

\[ \mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \sum_{m_1, m_2, n_1, n_2 \in \mathbb{Z}} q^{\frac{r_2}{2} q^{\frac{l^2}{2}}} (-1)^{m_1 s} h_i^{(r,s)}(\tau). \tag{3.20} \]

From this we see that those \( SO(2, 3; \mathbb{Z}) \) transformations which, acting on a vector \((m_1, m_2, n_1, n_2, b)\) with \(m_1, m_2, n_2, b\) integers and \(n_1\) half-integer, preserves \(m_1\) modulo 2, \(n_1\) modulo 1 and \(b\) modulo 2, will be symmetries of \( \mathcal{I} \). This defines the subgroup \( \tilde{G} \).

For later use we define the coefficients \( c^{(r,s)}(4n) \) through the expansion

\[ h_0^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n, \quad h_1^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n. \tag{3.21} \]

By examining (3.16) we see that in the expansion of \( h_i^{(r,s)}(\tau) \), \( n \in \mathbb{Z} - \frac{l}{4} \) for \( r = 0 \) and \( n \in \frac{1}{2} \mathbb{Z} - \frac{l}{4} \) for \( r = 1 \). Note that we have used the same symbol \( c^{(r,s)}(4n) \) for describing the expansion of \( h_0^{(r,s)}(\tau) \) and \( h_1^{(r,s)}(\tau) \). This is possible since \( c^{(r,s)}(4n) \) has different support for \( l = 0 \) and \( l = 1 \).

Using eq.(3.13) and the Fourier expansion (3.17) of \( \vartheta_3 \) and \( \vartheta_2 \) we can write the double Fourier expansion of \( F^{(r,s)}(\tau, z) \)

\[ F^{(r,s)}(\tau, z) = \sum_{b \in \mathbb{Z}, n} c^{(r,s)}(4n - b^2) q^n e^{2\pi i z b}, \tag{3.22} \]

where \( n \in \mathbb{Z} \) for \( r = 0 \) and \( \frac{1}{2} \mathbb{Z} \) for \( r = 1 \).

4. The Integral

We shall now proceed to evaluate the integral (3.20). We define

\[ Y = \det \Im \Omega = T_2 U_2 - (V_2)^2, \quad T_2 > 0, \quad U_2 > 0, \quad Y > 0. \tag{4.1} \]
where for any complex number $u$, we denote by $u_1$ and $u_2$ its real and imaginary parts respectively. Substituting the values of $p^2_L$ and $p^2_R$ from (3.1) into (3.20) we obtain

$$I_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,m_2,n_1\in \mathbb{Z}, n_1 \in \mathbb{Z} + \frac{r}{2}, b\in \mathbb{Z} + l} \exp \left[ 2\pi i\tau (m_1 n_1 + m_2 n_2 + \frac{b^2}{4}) \right] \times \exp \left[ \frac{-\pi T_2}{Y} \left| n_2 (TU - V^2) + bV + n_1 T - Um_1 + m_2 \right|^2 \right] (-1)^{m_1 s} h^{(r,s)}_l(\tau).$$

(4.2)

To evaluate the integral we first perform the Poisson resummation over the momenta $m_1, m_2$. The basic formula for Poisson resummation we will use is

$$\sum_{m\in \mathbb{Z}} f(m) e^{2\pi ism/N} = \sum_{k\in \mathbb{Z} + \frac{s}{N}} \int_{-\infty}^{\infty} du f(u) \exp(2\pi iku)$$

(4.3)

for any integer $N$. Now performing the Poisson resummation over $m_1, m_2$ and performing the Gaussian integration over the corresponding variables $u_1, u_2$, we obtain the following

$$I_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{Y}{U_2} \sum_{n_2,k_2\in \mathbb{Z}, n_1\in \mathbb{Z} + \frac{r}{2}, k_1\in \mathbb{Z} + \frac{s}{2}, b\in \mathbb{Z} + l} h^{(r,s)}_l(\tau) \exp \left[ \mathcal{G}(\vec{n}, \vec{k}, b) \right]$$

(4.4)

where

$$\mathcal{G}(\vec{n}, \vec{k}, b) = -\frac{\pi Y}{U_2^2 \tau_2} |A|^2 - 2\pi iT \det A + \frac{\pi b}{U_2} V_2 (V \bar{A} - \bar{V} A) - \frac{\pi n_2}{U_2} (V^2 \bar{A} - \bar{V}^2 A) + \frac{2\pi i V_2^2}{U_2^2} (n_1 + n_2 \bar{U}) A + 2\pi i \tau \frac{b^2}{4},$$

(4.5)

$$A = \begin{pmatrix} n_1 & k_1 \\ n_2 & k_2 \end{pmatrix},$$

(4.6)

$$\mathcal{A} = (1, U) A \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \quad \bar{\mathcal{A}} = (1, \bar{U}) A \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$  

(4.7)

Using (4.3) we can represent the sum over $b$ in (4.4) as

$$\sum_{b\in \mathbb{Z} + l} e^{2\pi i \frac{b^2}{4} + \frac{ib}{2} (V \bar{A} - \bar{V} A)} = \begin{cases} \theta_3(2\tau, -i \frac{V \bar{A} - \bar{V} A}{U_2}) & \text{for } l = 0 \\ \theta_2(2\tau, -i \frac{V \bar{A} - \bar{V} A}{U_2}) & \text{for } l = 1 \end{cases}$$

(4.8)
Substituting this into (4.4) and using (3.15) we get

\[ I \equiv \sum_{l,r,s=0}^{1} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \sum_{r,s=0}^{1} \sum_{n_2, k_2 \in \mathbb{Z}, n_1 \in \mathbb{Z}, k_1 \in \mathbb{Z} + \frac{1}{2}} \mathcal{J}(A, \tau), \]  

(4.9)

where

\[ \mathcal{J}(A, \tau) = \frac{Y}{U_2} \exp \left( -\frac{\pi Y}{U_2^2 \tau_2} |A|^2 - 2\pi iT \det A \right. \]

\[ -\frac{\pi n_2}{U_2} (V^2 \bar{A} - \bar{V}^2 A) + \frac{2\pi i V^2}{U_2^2} (n_1 + n_2 \bar{U}) A \right) \]

\[ F^{(r,s)} \left( \tau, -i \frac{V \bar{A} - \bar{V} A}{2 U_2} \right) \]

(4.10)

In order to interpret the right hand side as a function of the matrix \( A \) we need to use eqs. (4.6), (4.7). We may now interpret the sum over \( r, s \) and \( \overrightarrow{n}, \overrightarrow{k} \) in the right hand side of eq.(4.9) as a sum over all matrices \( A \) of the form (4.6) with \( n_2, k_2 \) integer, and \( n_1, k_1 \) integer or half-integer. (4.9) may then be rewritten as

\[ I = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \sum_{A} \mathcal{J}(A, \tau). \]  

(4.11)

Now it follows from the modular transformation laws (3.11) and the definition of \( \mathcal{J}(A, \tau) \) given in (4.10) that

\[ \mathcal{J} \left( A, \frac{a \tau + b}{c \tau + d} \right) = \mathcal{J} \left( A \left( \begin{array}{cc} a & b \\ c & d \end{array} \right); \tau \right). \]  

(4.12)

Using this symmetry, we can extend the integration over the fundamental domain to its images under \( SL(2, \mathbb{Z}) \) and at the same time restrict the summation over \( A \) to summation over inequivalent \( SL(2, \mathbb{Z}) \) orbits. If we denote by \( \sum'_{A} \) the sum over inequivalent \( SL(2, \mathbb{Z}) \) orbits then we can express \( I \) as

\[ I = \sum'_{A} \int_{\mathcal{F}_A} \frac{d^2 \tau}{\tau_2^2} \frac{Y}{U_2} \exp \left( -\frac{\pi Y}{U_2^2 \tau_2} |A|^2 - 2\pi iT \det A \right. \]

\[ -\frac{\pi n_2}{U_2} (V^2 \bar{A} - \bar{V}^2 A) + \frac{2\pi i V^2}{U_2^2} (n_1 + n_2 \bar{U}) A \right) \]

\[ F^{(r,s)} \left( \tau, -i \frac{V \bar{A} - \bar{V} A}{2 U_2} \right) \]  

(4.13)

where now \( r, s \) in the label of \( F^{(r,s)} \) are to be interpreted as \( 2n_1 \) mod \( 2 \) and \( 2k_1 \) mod \( 2 \) respectively. The region of integration \( \mathcal{F}_A \) depends on the orbit represented by \( A \).
Following the same procedure as in [28] we now split the integration into the three orbits. These are the zero orbit

\[ A = 0, \quad (4.14) \]

the non-degenerate orbit

\[ A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad 2k - 1 \geq 2j \geq 0, \quad p \neq 0, \quad k, j \in \frac{1}{2} \mathbb{Z}, \quad p \in \mathbb{Z}, \quad (4.15) \]

and the degenerate orbit

\[ A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}, \quad (j, p) \neq (0, 0), \quad j \in \frac{1}{2} \mathbb{Z}, \quad p \in \mathbb{Z}. \quad (4.16) \]

The contribution from these orbits has been evaluated in appendix B. The final result, as given in (B.39), takes the form

\[
I = -2 \ln \left[ \kappa (\det \text{Im} \Omega)^6 \right] \exp \left( 2\pi i \left( \frac{1}{2} T + U + V \right) \right) \\
\prod_{r,s=0}^{1} \prod_{l, b \in \mathbb{Z}, k \in \mathbb{Z} + \frac{r}{2}, (k,l,b) > 0} \left\{ (1 - \exp (2\pi i (k T + l U + b V))) (-1)^{l_s c_{r,s}} (4 k l - b^2) \right\}^2 \\
\kappa = \left( \frac{8 \pi}{3 \sqrt{3}} e^{1-\gamma E} \right)^6 \quad (4.17)
\]

and \((k, l, b) > 0\) means \(k > 0, l \geq 0, b \in \mathbb{Z}\) or \(k = 0, l > 0, b \in \mathbb{Z}\) or \(k = 0, l = 0, b < 0\).

5. \(\tilde{\Phi}_6\) and its \(V \to 0\) Limit

Eq. (4.17) can be written as

\[ I = -2 \left[ 6 \ln \text{det} \text{Im} \Omega + \ln \tilde{\Phi}_6 + \ln \tilde{\Phi}_6 + \ln \kappa + 8 \ln 2 \right], \quad (5.1) \]

where

\[
\tilde{\Phi}_6(\Omega) = \frac{1}{16} \exp \left( 2\pi i \left( \frac{1}{2} T + U + V \right) \right) \\
\prod_{r,s=0}^{1} \prod_{l, b \in \mathbb{Z}, k \in \mathbb{Z} + \frac{r}{2}, (k,l,b) > 0} \left[ 1 - \exp \{ 2\pi i (k T + l U + b V) \} \right] (-1)^{l_s c_{r,s}} (4 k l - b^2). \quad (5.2)
\]

Note that we have normalized \(\tilde{\Phi}_6\) so that the coefficient of \(\exp(2\pi i (\frac{1}{2} T + U + V))\) is 1/16. This agrees with the normalization convention of [4].
Since under a duality transformation by an element \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) of \( \tilde{G} \subset Sp(2, \mathbb{Z}) \)
\[
\det \text{Im} \Omega \to |\det(C\Omega + D)|^{-2} \det \text{Im} \Omega ,
\]
we must have
\[
\tilde{\Phi}_6((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^6 \tilde{\Phi}_6(\Omega) ,
\]
in order that \( I \) given in (5.1) is invariant under this transformation. Thus \( \tilde{\Phi}_6 \) transforms as a modular form of weight 6 under \( \tilde{G} \).

We shall now analyze the \( V \to 0 \) limit of (5.2) and compare this with the corresponding result in [7]. This analysis is facilitated by examining the relation (3.22) at \( z = 0 \):
\[
\sum_n \sum_b c^{(r,s)}(4n - b^2)q^n = F^{(r,s)}(r,0) = \left\{ \begin{array}{ll} 12 & \text{for } (r,s) = (0,0) \\ 4 & \text{for } (r,s) \neq (0,0) \end{array} \right. .
\]
This gives
\[
\sum_b c^{(r,s)}(4n - b^2) = \left\{ \begin{array}{ll} 12 \delta_{n,0} & \text{for } (r,s) = (0,0) \\ 4 \delta_{n,0} & \text{for } (r,s) \neq (0,0) \end{array} \right. .
\]
Taking \( V \to 0 \) limit in (5.2) we now get
\[
\tilde{\Phi}_6(U, T, V) \simeq -\frac{4\pi^2 V^2}{16} e^{2\pi i(\frac{1}{2}T+U)} \prod_{k=1}^{\infty} \left\{ (1 - e^{2\pi ikT})^8 (1 - e^{\pi ikT})^8 \right\} \prod_{l=1}^{\infty} \left\{ (1 - e^{2\pi ilU})^8 (1 - e^{4\pi ilU})^8 \right\}
\]
where the \(-4\pi^2 V^2\) term comes from the \( k = l = 0, b = -1 \) term. This can be rewritten as
\[
\tilde{\Phi}_6(U, T, V) \simeq -\frac{1}{4} \pi^2 V^2 \eta(T/2)^8 \eta(T)^8 \eta(U)^8 \eta(2U)^8 .
\]
This factorization property, including the overall normalization of \(-\frac{1}{4} \pi^2\), agrees with that found in [7].

6. Construction of \( \Phi_6 \)

In the analysis of [7] we introduced another function \( \Phi_6 \) related to \( \tilde{\Phi}_6 \) by:
\[
\tilde{\Phi}_6(U, T, V) = T^{-6} \Phi_6 \left( U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T} \right) .
\]
or equivalently
\[ \Phi_6(U, T, V) = T^{-6} \tilde{\Phi}_6 \left( U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T} \right). \] (6.2)

From the expressions for \( I_{r,s,l} \) given in (4.2) we see that this transformation may be implemented by
\[ m_2 \rightarrow n_1, \quad n_1 \rightarrow -m_2, \quad m_1 \rightarrow -n_2, \quad n_2 \rightarrow m_1. \] (6.3)

Thus in order to find an expression for \( \Phi_6 \) we can replace \( I_{r,s,l} \) given in (4.2) by \( I'_{r,s,l} \) in which we sum over \( m_2 \in \mathbb{Z} + r \) instead of \( n_1 \in \mathbb{Z} + r \), and replace the \((-1)^{m_1 s}\) factor in the summand by \((-1)^{n_2 s}\):
\[ I'_{r,s,l} = \int_{F} \frac{d^2 \tau}{T_2} \sum_{m_1, n_2 \in \mathbb{Z}, m_2 \in \mathbb{Z} + r, b \in 2 \mathbb{Z} + l} \exp \left[ 2\pi i (m_1 n_1 + m_2 n_2 + \frac{b^2}{4}) \right] \times \exp \left[ -\frac{\pi T_2}{Y} \left| n_2 (TU - V^2) + bV + n_1 T - Um_1 + m_2 \right|^2 \right] (-1)^{n_2 s} h^{(r,s)}_l(\tau). \] (6.4)

After Poisson resummation this amounts to summing over only integer values of \( n_1, n_2, k_1, k_2 \) and including a factor of
\[ (-1)^{k_2 r} (-1)^{n_2 s}, \] (6.5)
in the summand. The integral can now be evaluated following exactly the same procedure as in appendix B, the only difference being that the sum over \( p \) in eqs. (B.11), (B.21), (B.26) will contain an additional factor of \((-1)^{pr} \). The net contribution to the full integral comes out to be
\[ \mathcal{I}' = -2 \ln \left[ 2^8 \kappa (\det \text{Im}\Omega)^{6} \right] \exp(2\pi i(T + U + V)) \prod_{r,s=0}^{1} \prod_{(k,l,b) \in \mathbb{Z}} \left\{ 1 - (-1)^r \exp(2\pi i(kT + lU + bV) c^{(r,s)}(4kl-b^2) \right\}^2 \]. (6.6)

\(^5\)An apparent additional complication arises due to the fact that the Fourier expansions of \( F^{(1,0)} \) and \( F^{(1,1)} \) as given in (3.22) have half integer powers of \( q \). Thus the sum over \( j \) in eq. (B.8) will not vanish for non-integer \( n/k \). However since \( F^{(1,0)} + F^{(1,1)} \) is invariant under \( \tau \rightarrow \tau + 1 \) due to the modular properties described in (3.11), it has Fourier expansion in integer powers of \( q \). Thus if in analyzing the sum over \( j \) in (3.8) we consider the contribution from \( F^{(1,0)} \) and \( F^{(1,1)} \) together, the sum over \( j \) will force \( n \) to be a multiple of \( k \).
We can rewrite this as

\[ T' = -2 \left[ 6 \ln \det \text{Im} \Omega + \ln \Phi_6 + \ln \bar{\Phi}_6 + \ln \kappa + 8 \ln 2 \right], \tag{6.7} \]

where

\[
\Phi_6(\Omega) = -\exp(2\pi i(T + U + V)) \prod_{r,s=0}^{1} \prod_{(k,l,b) \in \mathbb{Z}, (k,l,b) > 0} \left\{ 1 - (-1)^r \exp(2\pi i(kT + lU + bV)) \right\} e^{c(r,s)(4kT - b^2)}. \tag{6.8}
\]

The normalization of \( \Phi_6 \) is not arbitrary; it has been chosen so that we have the same additive constant \( 8 \ln 2 \) in (6.7) as in (5.1). The phase of \( \Phi_6 \) can be adjusted. With the choice of phase given in (6.8) the coefficient of the \( e^{2\pi i(T + U + V)} \) term matches with that of the corresponding expression in [7]. Following the same argument as in the case of \( \tilde{\Phi}_6 \) we can argue that \( \Phi_6 \) transforms as a modular form of weight 6 under a subgroup \( G \) of \( Sp(2, \mathbb{Z}) \) which is related to the earlier subgroup \( \tilde{G} \) by the conjugation described in (6.3).

Study of the \( V \to 0 \) limit of this expression is also straightforward. Using the relations (5.6) and the explicit expressions for the coefficients \( c^{(r,s)}(0) \) and \( c^{(r,s)}(-1) \) given in (B.35), we get

\[
\Phi_6(U, T, V) \simeq 4\pi^2 V^2 \eta(T)^8 \eta(2T)^8 \eta(U)^8 \eta(2U)^8. \tag{6.9}
\]

This is the same behaviour as found in [7].

We can also carry out a more detailed comparison between the \( \Phi_6 \) defined here and those in [7]. The algorithm given in [2] goes as follows:

- We first define a set of coefficients \( f_n \ (n \geq 1) \) through the relation:

\[
\sum_{n \geq 1} f_n e^{2\pi i \tau(n - \frac{1}{4})} = \eta(\tau)^2 \eta(2\tau)^8, \tag{6.10}
\]

where \( \eta(\tau) \) is the Dedekind function.

- Next we define the coefficients \( C(m) \) through

\[
C(m) = (-1)^m \sum_{s,n \in \mathbb{Z}} f_n \delta_{4n+s^2-1,m}. \tag{6.11}
\]
• $\Phi_6$ is now given by

$$
\Phi_6(U, T, V) = \sum_{n, m, r \in \mathbb{Z}} a(n, m, r) e^{2\pi i (nU + mT + rV)},
$$

(6.12)

where

$$
a(n, m, r) = \sum_{\alpha \in 2\mathbb{Z} + 1, \alpha \mid (n, m, r), \alpha > 0} \alpha^{-1} C \left( \frac{4mn - r^2}{\alpha^2} \right),
$$

(6.13)

We have compared 31 different coefficients $a(n, m, r)$ defined in (6.13) with the ones obtained from (6.8) and found them to be the same. These results for $a(n, m, r)$ are given in appendix C.

7. Construction of $\Phi_k$ and $\tilde{\Phi}_k$

Generalization of the modular form $\tilde{\Phi}_6$ to describe the degeneracy of dyons in a $\mathbb{Z}_N$ orbifold of $T^2 \times K3$ for $N = 2, 3, 5, 7$ was also introduced in [6]. The generator $g'$ of the $\mathbb{Z}_N$ is given by

$$
g' = \hat{g} \tilde{g},
$$

(7.1)

where $\hat{g}$ represents $1/N$ unit of shift along $T^2$ (which we shall take to be in the $y_1$ direction) and $\tilde{g}$ denotes an appropriate $\mathbb{Z}_N$ action on $K3$. $\tilde{g}$ preserves the harmonic (0,0)-form, (2,2)-form, (0,2)-form and (2,0)-form. Furthermore for each $r \neq 0$, there are $24/(N + 1)$ (1,1)-forms with $\tilde{g}$ eigenvalue $e^{2\pi i r/N}$. The rest of the $20 - 24(N - 1)/(N + 1)$ of the (1,1)-forms are invariant under $\tilde{g}$.

The generating function for the degeneracy is given by $(\tilde{\Phi}_k)^{-1}$ where

$$
k = \frac{24}{N + 1} - 2,
$$

(7.2)

and $\tilde{\Phi}_k$ is a weight $k$ modular form of a subgroup $\tilde{G}$ of $Sp(2, \mathbb{Z}) = SO(2, 3; \mathbb{Z})$ that commutes with $1/N$ unit of shift along a circle of $T^2$. Associated with $\tilde{\Phi}_k$ there is a modular form $\Phi_k$ of a different subgroup $G$ of $Sp(2, \mathbb{Z})$, related to $G$ by conjugation described in (6.2):

$$
\Phi_k(U, T, V) = T^{-k} \tilde{\Phi}_k \left( U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T} \right).
$$

(7.3)

Our goal is to find a product representation of $\Phi_k$ and $\tilde{\Phi}_k$. For this we shall start with an analog of eq.(2.11) for the superconformal field theory associated with
the \( \mathbb{Z}_N \) orbifold of \( K^3 \times T^2 \) and express it as a sum of a holomorphic and an anti-holomorphic term and a term proportional to \( \ln \det \text{Im}\Omega \). The holomorphic part can then be identified with \( \Phi_k \). Proceeding as in section 2 we arrive at the analog of eq.(3.9), (3.10)

\[
F_{m_1,m_2,n_1,n_2}(\tau, z) = \sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} F^{(r,s)}(\tau, z) \quad \text{for } n_1 \in \mathbb{Z} + \frac{r}{N}, \quad r = 0, 1, \ldots (N-1),
\]

(7.4)

where

\[
F^{(r,s)}(\tau, z) = \frac{1}{N} T r^{K^3}_{RR,\tilde{g}} \left( (-1)^{F_{K^3}} (-1)^{\tilde{F}_{K^3}} e^{2\pi i z F_{K^3}} q^{L_0} \bar{q}^{\bar{L}_0} \right).
\]

(7.5)

From these definitions it follows that

\[
F^{(r,s)}(a\tau + b, c\tau + d, z) = \exp \left( 2\pi i \frac{cz^2}{c\tau + d} \right) F^{(cs+ar,ds+br)}(\tau, z),
\]

(7.6)

for

\[
a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.
\]

(7.7)

In (7.6) the indices \( cs + ar \) and \( ds + br \) are to be taken mod \( N \). Thus for each \( (r, s) \), \( F^{(r,s)}(\tau, z) \) transforms as a weak Jacobi form\(^{[26]}\) of weight zero and index 1 under the group \( \Gamma(N) \).

We can now define the coefficients \( c^{(r,s)}(n) \) in a manner analogous to (3.22)

\[
F^{(r,s)}(\tau, z) = \sum_{b \in \mathbb{Z}, n \in \mathbb{Z}/N} c^{(r,s)}(4n - b^2) q^n e^{2\pi i z b}.
\]

(7.8)

In order that \( F^{(r,s)}(\tau, z) \) has an expansion of the form given in (7.8) we need to ensure that this can be expressed as a linear combination of \( \vartheta_3(2\tau, 2z) \) and \( \vartheta_2(2\tau, 2z) \) with \( z \)-independent coefficients as in (3.15). This follows from the fact that the \( z \)-dependence of \( F^{(r,s)}(\tau, z) \) comes from the SU(2) current algebra associated with the superconformal field theory, and this commutes with the \( \mathbb{Z}_N \) generator \( \tilde{g} \). \( \vartheta_3(2\tau, 2z) \) and \( \vartheta_2(2\tau, 2z) \) simply represent the contributions from the even and odd \( F_{K^3} \) charge sector of this SU(2) sector of the theory.
$K3$. These can be easily computed from the $\tilde{g}$ action of the cycles described earlier, and we get

$$
c^{(0,0)}(0) = \frac{20}{N}, \quad c^{(0,0)}(-1) = \frac{2}{N},
$$

$$
c^{(0,s)}(0) = \frac{1}{N} \left( 20 - \frac{24N}{N+1} \right), \quad c^{(0,s)}(-1) = \frac{2}{N}, \quad \text{for } s = 1, 2, \ldots (N-1).
$$

(7.9)

Several other useful properties of $c^{(r,s)}$ may be derived without explicitly computing $F^{(r,s)}(\tau, z)$. First note that $F^{(0,0)}(\tau, z)$ is $1/N$ times the elliptic genus of $K3$. Hence it is given by

$$
F^{(0,0)}(\tau, z) = \frac{8}{N} \left[ \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right]. \quad (7.10)
$$

Next it follows from the definition (7.5) that $F^{(0,s)}(\tau, 0)$ is $\tau$ independent since it receives contribution only from the $L_0 = \bar{L}_0 = 0$ states. The modular transformation laws (7.6) together with (7.9) then imply that

$$
F^{(r,s)}(\tau, 0) = F^{(0,t)}(\tau, 0)|_{t = g.c.d.(r,s)} = c^{(0,t)}(0) + 2 c^{(0,t)}(-1) = \frac{24}{N(N+1)}
$$

for $(r, s) \neq (0, 0).$ \quad (7.11)

Substituting (7.10), (7.11) into the expansion (7.8) we get the analog of eq.(5.6)

$$
\sum_{b} c^{(r,s)}(4n - b^2) = \begin{cases} 
\frac{24}{N} \delta_{n,0} & \text{for } (r, s) = (0, 0) \\
\frac{24}{N(N+1)} \delta_{n,0} & \text{for } (r, s) \neq (0, 0),
\end{cases} \quad (7.12)
$$

Further information about these coefficients comes from the fact that $\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z)$ represent the elliptic genus of the super-conformal $\sigma$-model with target space $K3/\mathbb{Z}_N$ with the $\mathbb{Z}_N$ generated by $\tilde{g}$. However for any $N$ this gives us back the superconformal field theory with target space $K3$, and hence $\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z)$ must give us the elliptic genus of $K3$. This in turn is just $NF^{(0,0)}(\tau, z)$. Thus we have

$$
\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = N F^{(0,0)}(\tau, z). \quad (7.13)
$$

Furthermore the contribution $\sum_{s=0}^{N-1} F^{(r,s)}(\tau, z)$ for a fixed $r$ may be interpreted as the contribution to the elliptic genus from the sector twisted by $\tilde{g}^r$. For prime values
of $N$, $\tilde{g}^r$ is an order $N$ transformation for all $r \neq 0 \mod N$. Hence we expect the sectors twisted by $\tilde{g}^r$ to give the same contribution to the elliptic genus for all $r \neq 0 \mod N$. This, together with (7.13), gives

$$N - 1 \sum_{s=0}^{N-1} F^{(r,s)}(\tau, z) = \frac{1}{N-1} \left[ NF^{(0,0)}(\tau, z) - \sum_{s=0}^{N-1} F^{(0,s)}(\tau, z) \right] \quad r \neq 0 \mod N .$$

Translated to a condition on the coefficients $c^{(r,s)}(m)$, this gives

$$N - 1 \sum_{s=0}^{N-1} c^{(r,s)}(m) = \frac{1}{N-1} \left[ Nc^{(0,0)}(m) - \sum_{s=0}^{N-1} c^{(0,s)}(m) \right] \quad \text{for any } m, \quad r \neq 0 \mod N .$$

(7.14)

For $m = 0, -1$ we can explicitly evaluate the right hand side of this equation using (7.9). In particular setting $m = -1$ we get

$$N - 1 \sum_{s=0}^{N-1} c^{(r,s)}(-1) = 0 , \quad \text{for } r \neq 0 \mod N .$$

(7.15)

Although for $N = 3, 5, 7$ we have not been able to compute $F^{(r,s)}(\tau, z)$ directly, a set of $F^{(r,s)}(\tau, z)$ satisfying the requirements given above are as follows. Let us define

$$A(\tau, z) = \left[ \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right] ,$$

(7.17)

$$B(\tau, z) = \eta(\tau)^{-6} \vartheta_1(\tau, z)^2 ,$$

(7.18)

and

$$E_N(\tau) = \frac{12i}{\pi(N-1)} \partial_\tau \left[ \ln \eta(\tau) - \ln \eta(N\tau) \right] = 1 + \frac{24}{N-1} \sum_{n_1, n_2 \geq 1 \atop n_1 \neq 0 \mod N} n_1 e^{2\pi in_1 n_2 \tau} .$$

(7.19)

Then under an $SL(2, \mathbb{Z})$ transformation $A(\tau, z)$ transforms as a weak Jacobi form of weight 0 and index 1 and $B(\tau, z)$ transforms as a weak Jacobi form of weight $-2$ and index 1. Furthermore

$$E_N(\tau + 1) = E_N(\tau), \quad E_N(-1/\tau) = -\tau^2 \frac{1}{N} E_N(\tau/N) .$$

(7.20)

From this it follows that $E_N(\tau)$ is a modular form of weight 2 of the group $\Gamma_0(N)$ and hence also of $\Gamma(N)$[27]. Using these properties one can show that the following
choice of $F^{r,s}(\tau, z)$ satisfy all the requirements described above:

$$
F^{(0,0)}(\tau, z) = \frac{8}{N} A(\tau, z),
$$

$$
F^{(0,s)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) - \frac{2}{N+1} B(\tau, z) E_N(\tau) \quad \text{for } 1 \leq s \leq (N-1),
$$

$$
F^{(r,r_k)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) + \frac{2}{N(N+1)} E_N \left( \frac{\tau + k}{N} \right) B(\tau, z),
$$

for $1 \leq r \leq (N-1)$, $0 \leq k \leq (N-1)$.

(7.21)

The rest of the analysis now proceeds exactly as in the $N=2$ case. We arrive at an analog of eq. (4.2) for $I_{r,s,l}$:

$$
I_{r,s,l} = \int d^2 \tau \sum_{m_1, m_2, n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z} + \frac{r}{N}, b \in 2\mathbb{Z} + l} \exp \left[ 2\pi i \tau (m_1 n_1 + m_2 n_2 + \frac{b^2}{4}) \right] \times \exp \left( \frac{-\pi \tau^2}{Y} \left| n_2 (TU - V^2) + bV + n_1 T - Um_1 + m_2 \right|^2 \right) e^{2\pi i m_1 s/N} h_{l}^{(r,s)}(\tau),
$$

$$
0 \leq r, s \leq (N-1).
$$

(7.22)

This can then be Poisson resummed and analyzed using the techniques described in appendix B and be split into holomorphic and anti-holomorphic parts to extract the expression for $\tilde{\Phi}_k$. On the other hand if we want information about $\Phi_k$ we need to use the operation eq. (6.3) to consider a new integral

$$
I_{r,s,l}' = \int d^2 \tau \sum_{m_1, m_2, n_2 \in \mathbb{Z}, m_1 \in \mathbb{Z} - \frac{r}{N}, b \in 2\mathbb{Z} + l} \exp \left[ 2\pi i \tau (m_1 n_1 + m_2 n_2 + \frac{b^2}{4}) \right] \times \exp \left( \frac{-\pi \tau^2}{Y} \left| n_2 (TU - V^2) + bV + n_1 T - Um_1 + m_2 \right|^2 \right) e^{-2\pi i m_1 s/N} h_{l}^{(r,s)}(\tau),
$$

$$
0 \leq r, s \leq (N-1).
$$

(7.23)

In this case Poisson resummation over $m_1, m_2$ will give rise to an additional factor of $\exp(2\pi ik_2 r/N)$ and the final sum will be over integer values of $n_1, n_2, k_1, k_2$. This can again be analyzed using the techniques described in appendix B.

We shall not give the details of the analysis but write down the final expression.

The expressions for $\Phi_k$ and $\tilde{\Phi}_k$ obtained this way are:

$$
\Phi_k(U, T, V) = - \exp \{ 2\pi i (T + U + V) \}.
$$
\[
\tilde{\Phi}_k(U, T, V) = -(i\sqrt{N})^{-k-2} \exp \left(2\pi i \left(\frac{1}{N} T + U + V\right)\right)
\]
\[
\prod_{r=0}^{N-1} \prod_{l,b \in \mathbb{Z}, k', l, b > 0} \left\{ 1 - \exp(2\pi i(k'T + lU + bV)) \right\} \frac{1}{2} \sum_{s=0}^{N-1} e^{-2\pi is/N} e^{c(r,s)(4k'l-b^2)}
\]
\[\prod_{r=0}^{N-1} \prod_{l,b \in \mathbb{Z}, k', l, b > 0} \left\{ 1 - \exp(2\pi i(k'T + lU + bV)) \right\} \frac{1}{2} \sum_{s=0}^{N-1} e^{2\pi is/N} e^{c(r,s)(4k'l-b^2)}
\]  

\[\Phi_k(U, T, V) \simeq 4\pi^2 V^2 (\eta(T)\eta(NT))^{k+2}(\eta(U)\eta(NU))^{k+2}, \quad (7.27)\]

\[\tilde{\Phi}_k(U, T, V) \simeq (i\sqrt{N})^{-k-2} 4\pi^2 V^2 (\eta(T)\eta(T/N))^{k+2}(\eta(U)\eta(NU))^{k+2}, \quad (7.28)\]
in agreement with \[\text{[7]}\].

Another important consistency check for eqs. (7.24), (7.25) comes from looking at the coefficient of the terms involving a single power of \(e^{2\pi iU}\) and all powers of \(T\)
and $V$. For $\Phi_k$ this is given by
\[ e^{2\pi i U} \eta(T)^{k-4} \eta(N T)^{k+2} \vartheta_1(T, V)^2, \] (7.29)
and for $\tilde{\Phi}_k$ this is given by
\[ (i \sqrt{N})^{-k-2} e^{2\pi i U} \eta(T)^{k-4} \eta(T/N)^{k+2} \vartheta_1(T, V)^2. \] (7.30)
These agree with the corresponding expressions found in \[7\].

We have also compared a few terms in the expansion of $\Phi_k$ given in (7.24) with the one given in \[7\]. The results are given in appendix C.

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A. Calculation of the Elliptic Genus

In this appendix we shall calculate
\[ F_{m_1, m_2, n_1, n_2} (\tau, z) = T_{RR, m_1, m_2, n_1, n_2} \left( (-1)^{F_{K3} + F_{T^2}} (-1)^{\bar{F}_{K3} + \bar{F}_{T^2}} F_{T^2} \bar{F}_{T^2} e^{2\pi iz \bar{F}_{K3}} q^{L_0} \bar{q}^{\bar{L}_0} \right), \] (A.1)
in the superconformal field theory with target space $(K3 \times T^2)/\mathbb{Z}_2$. For this we shall use an orbifold description of $K3$. We consider a superconformal $\sigma$-model with target space $T^2 \times T^4$ with $y^1, y^2$ denoting the $T^2$ coordinates and $y^3, y^4, y^5, y^6$ denoting the $T^4$ coordinates, and mod out the theory by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by elements $g$ and $g'$. The action of $g$ and $g'$ are given by:
\[ g : (y^1, y^2, y^3, y^4, y^5, y^6) \rightarrow (y^1, y^2, -y^3, -y^4, -y^5, -y^6) \]
\[ g' : (y^1, y^2, y^3, y^4, y^5, y^6) \rightarrow (y^1 + \pi, y^2, y^3 + \pi, y^4, y^5, y^6). \] (A.2)
Orbifolding by $g$ produces a $K3 \times T^2$ manifold. Further orbifolding by $g'$ produces $(K3 \times T^2)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ generator involves a shift along $T^2$ and a $\mathbb{Z}_2$ involution.
in $K3$ that preserves the $(4,4)$ superconformal symmetry of the corresponding worldsheet theory. We denote by $F_{T^4}$ and $F_{T^2}$ holomorphic parts of the worldsheet fermion number associated with the $T^4$ and the $T^2$ parts and by $\bar{F}_{T^4}$ and $\bar{F}_{T^2}$ the anti-holomorphic parts of the worldsheet fermion number associated with the $T^4$ and the $T^2$ parts. We shall be considering an arbitrary $T^2$ parametrized by the Kähler modulus $T$ and complex structure modulus $U$, and arbitrary Wilson lines $A_1, A_2$ corresponding to deforming the worldsheet theory by the marginal operator

$$
\sum_{i=1}^{2} A_i \int d^2 z \bar{\partial} Y^i J_{T^4},
$$

(A.3)

where $J_{T^4}$ is the U(1) current corresponding to the charge $F_{T^4}$. We shall denote by $V$ the complex combination $A_2 - iA_1$.

We now define

$$F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z) = T_{m_1,m_2,n_1,n_2;RR;g^a,g^b} \left( (-1)^{(F_{T^4}+F_{T^2})} (-1)^{(\bar{F}_{T^4}+\bar{F}_{T^2})} F_{T^2} \bar{F}_{T^4} e^{2\pi iz} F_{T^4} q^{L_0} \bar{q}^{\bar{L}_0} g^c g^d \right),
$$

(A.4)

where $L'_0, \bar{L}'_0$ have been defined in eqs. (3.1), (3.4). Here $a, b, c, d$ take values 0 or 1. $T_{m_1,m_2,n_1,n_2;RR;g^a,g^b}$ denotes trace in the original CFT associated with the $T^2 \times T^4$ target space over RR sector states twisted by $g^a g^b$ and carrying $(m_1, m_2)$ units of momentum and $(n_1, n_2)$ units of winding along $(y^1, y^2)$. The quantity $F_{m_1,m_2,n_1,n_2}(\tau,z)$ is then given by

$$F_{m_1,m_2,n_1,n_2}(\tau,z) = \frac{1}{4} \sum_{a,b,c,d=0} F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z).
$$

(A.5)

We shall now calculate $F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z)$. First we note that

$$F_{m_1,m_2,n_1,n_2}(0,0;0,d;\tau,z) = 0 \quad \text{for } d = 0, 1
$$

(A.6)

due to the fermion zero modes associated with the 3, 4, 5, 6 directions.

Next we have

$$F_{m_1,m_2,n_1,n_2}(0,0;1,d;\tau,z) = (-1)^{m_1 d} 4 \left( 1 + e^{2\pi iz} \right) \left( 1 + e^{-2\pi iz} \right) \prod_{n=1}^{\infty} \left( 1 + q^n e^{2\pi iz} \right)^2 \left( 1 + q^n e^{-2\pi iz} \right)^2 \prod_{n=1}^{\infty} (1 + q^n)^4
$$

$$= (-1)^{m_1 d} 16 \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}.
$$

(A.7)
In the first line the factor of 4 comes from the anti-holomorphic fermion zero modes associated with the 3,4,5,6 directions and the factor of \((1 + e^{2\pi i z})(1 + e^{-2\pi i z})\) comes from the holomorphic fermion zero-modes. In the second line the numerator comes from the holomorphic non-zero mode fermionic oscillators associated with the 3,4,5,6 directions and the denominator comes from the holomorphic non-zero mode bosonic oscillators associated with the same directions. The contribution from the bosonic and fermionic oscillators associated with the 1 and 2 directions cancel. Also the contributions from all the non-zero mode fermion and bosonic oscillators in the anti-holomorphic sector always cancel. In arriving at (A.7) we have used that the action of \(g'\) on the state carrying \(m_1\) units of momentum along \(y^1\) gives a factor of \((-1)^{m_1}\) and the action of \(g\) changes the signs of the fermionic and the bosonic oscillators associated with \(T^4\). Also since the action of \(g\) reverses the direction of momentum along the 3,4,5,6 directions, only states carrying zero momentum along \(T^4\) contributes to the trace and hence the result is independent of the moduli of \(T^4\). This will be a generic feature of all the terms; either they will vanish due to fermion zero modes or only the zero momentum mode will contribute due to either a \(g\) insertion or a twist under \(g\).

Let us now turn to the twisted sector states. First note that there are 16 twisted sector states under \(g\), located as \(y^m = 0, \pi\) for \(m = 3, 4, 5, 6\). \(g'\) (and also \(gg')\) exchanges these states pairwise. Thus the action of \(g'\) and \(gg'\) on these states is off-diagonal and hence the trace of \(g'\) and \(gg'\) over these states vanish. This gives

\[
F_{m_1,m_2,n_1,n_2}(1,0; c, 1; \tau, z) = 0 \quad \text{for } c = 0, 1. \quad (A.8)
\]

On the other hand we have

\[
F_{m_1,m_2,n_1,n_2}(1,0; c, 0; \tau, z) = \frac{16 \prod_{n=0}^{\infty} (1 - q^{n + \frac{1}{2}} e^{2\pi i z + i\pi c})^2 (1 - q^{n + \frac{1}{2}} e^{-2\pi i z + i\pi c})^2}{\prod_{n=0}^{\infty} (1 - e^{i\pi c} q^{n + \frac{1}{2}})^4} = \begin{cases} 16 \vartheta_4(\tau, z)^2/\vartheta_4(\tau, 0)^2 & \text{for } c = 0 \\ 16 \vartheta_3(\tau, z)^2/\vartheta_3(\tau, 0)^2 & \text{for } c = 1 \end{cases}. \quad (A.9)
\]

The factor of 16 is due to the existence of 16 twisted sector states.

Next we consider sectors twisted by \(g'\). In this case the winding number \(n_1\) along \(y^1\) must be half integer and similarly the winding number along \(y^3\) must also
be half integer. Since the $g'$ twist just involves a shift and does not affect the worldsheet fermions, the fermion zero modes associated with the 3-6 directions make the contribution vanish unless the $g$ projection is inserted into the trace. This gives:

$$F_{m_1,m_2,n_1,n_2}(0,1;0,d;\tau, z) = 0 \quad \text{for } d = 0,1.$$  \hspace{1cm} (A.10)

On the other hand the action of $g$ as well as of $gg'$ reverses the sign of the winding number along $y^3$ and hence these elements are off-diagonal in the sector twisted by $g'$. This gives

$$F_{m_1,m_2,n_1,n_2}(0,1;1,d;\tau, z) = 0 \quad \text{for } d = 0,1.$$  \hspace{1cm} (A.11)

Finally let us turn to the sector twisted under $gg'$. Action of $gg'$ on $y^3, y^4, y^5, y^6$ gives fixed points at $y^3 = \pi/2, 3\pi/2, y^m = 0, \pi$ for $m = 4, 5, 6$. Although this are not real fixed points due to the shift action $y^2 \rightarrow y^2 + \pi$, we can label the 16 twisted sectors by these would be fixed points. Both $g$ and $g'$ exchange these fixed points pairwise and hence are represented by off-diagonal matrices. This gives

$$F_{m_1,m_2,n_1,n_2}(1,1;0,0;\tau, z) = 0 ,$$

$$F_{m_1,m_2,n_1,n_2}(1,1;0,1;\tau, z) = 0.$$  \hspace{1cm} (A.12)

On the other hand both the identity element and $gg'$ leave the fixed points invariant and give non-zero answers. We have

$$F_{m_1,m_2,n_1,n_2}(1,1;0,0;\tau, z) = 16 \prod_{n=0}^{\infty} (1 - q^{n+\frac{1}{2}} e^{2\pi i z})^2 (1 - q^{n+\frac{1}{2}} e^{-2\pi i z})^2 \prod_{n=0}^{\infty} (1 - q^{n+\frac{1}{2}})^4$$

$$= 16 \frac{\varphi_4(\tau, z)^2}{\varphi_4(\tau, 0)^2},$$  \hspace{1cm} (A.13)

and

$$F_{m_1,m_2,n_1,n_2}(1,1;1,\tau; z) = 16 (-1)^{m_1} \prod_{n=0}^{\infty} (1 - q^{n+\frac{1}{2}} e^{2\pi i z + i\pi})^2 (1 - q^{n+\frac{1}{2}} e^{-2\pi i z + i\pi})^2 \prod_{n=0}^{\infty} (1 - e^{i\pi} q^{n+\frac{1}{2}})^4$$

$$= 16 (-1)^{m_1} \frac{\varphi_3(\tau, z)^2}{\varphi_3(\tau, 0)^2}.$$  \hspace{1cm} (A.14)

Using eqs.(A.4)-(A.14) we now get

$$F_{m_1,m_2,n_1,n_2}(\tau, z) = 4 \left[ \frac{\varphi_2(\tau, z)^2}{\varphi_2(\tau, 0)^2} + \frac{\varphi_3(\tau, z)^2}{\varphi_3(\tau, 0)^2} + \frac{\varphi_4(\tau, z)^2}{\varphi_4(\tau, 0)^2} \right] + 4 (-1)^{m_1} \frac{\varphi_2(\tau, z)^2}{\varphi_2(\tau, 0)^2}$$

$$= 4 \frac{\varphi_4(\tau, z)^2}{\varphi_4(\tau, 0)^2} + 4 (-1)^{m_1} \frac{\varphi_3(\tau, z)^2}{\varphi_3(\tau, 0)^2}$$  \hspace{1cm} (A.15)
B. Evaluation of the Integral

In this appendix we shall evaluate the integral (4.13)

\[ I = \sum_A \int_{\mathcal{F}_A} \frac{d^2 \tau}{\tau_2^2} \frac{Y}{U_2} \exp \left( -\frac{\pi Y}{U_2 \tau_2} |A|^2 - 2\pi iT \det A \right) \]

\[ - \frac{\pi n_2}{U_2} (V^2 \tilde{A} - \bar{V}^2 A) + \frac{2\pi i V^2}{U_2} (n_1 + n_2 \bar{U}) A \right) F^{(r,s)} \left( \tau, -i \frac{V \tilde{A} - \bar{V} A}{2 U_2} \right). \]

(B.1)

The sum over \( A \) runs over all integer valued 2 \( \times \) 2 matrices of the form (4.6) which are not related to each other by an \( SL(2, \mathbb{Z}) \) transformation acting from the right. \( \mathcal{F}_A \) is the union of images of the fundamental region \( \mathcal{F} \) under \( SL(2, \mathbb{Z}) \) transformations which act non-trivially on \( A \). \( A, \tilde{A} \) are defined in (4.7) and \( (r, s) = (2n_1, 2k_1) \) mod 2.

In carrying out the integral we need to introduce some regularization and subtraction scheme. Following [28] we regularize possible divergences in the integral by including a factor of \((1 - \exp(-\Lambda/\tau_2))\) in the integrand. For \( \tau_2 \ll \Lambda \) this factor is close to unity, but for \( \tau_2 >> \Lambda \) it is close to zero. We also add to the integral a term

\[ - \left( c^{(0,0)}(0) + c^{(0,1)}(0) \right) \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \left( 1 - \exp(-\Lambda/\tau_2) \right). \]

(B.2)

As we shall see, this is necessary for getting a finite \( \Lambda \to \infty \) limit.

Following the same procedure as in [28] we split the integration into the three orbits.

1. Contribution \( \mathcal{I}_1 \) from the zero orbit

For \( A = 0 \) we have \( (r, s) = (0, 0) \) and \( \mathcal{F}_A = \mathcal{F} \), – the fundamental region of \( SL(2, \mathbb{Z}) \).

The integral (4.13) reduces to

\[ \mathcal{I}_1 = \frac{Y}{U_2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} F^{(0,0)}(\tau, 0) = \frac{Y}{U_2} \frac{\pi}{3} 12; \]

(B.3)

using the expression for \( F^{(0,0)}(\tau, z) \) given in (3.13).

2. Contribution \( \mathcal{I}_2 \) from the non-degenerate orbit

Here we consider \( A \) to be

\[ A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad 2k - 1 \geq 2j \geq 0, \quad p \neq 0, \quad k, j \in \frac{1}{2} \mathbb{Z}, \quad p \in \mathbb{Z}. \]

(B.4)
In this case the region \( \mathcal{F}_A \) corresponds to two copies of the upper-half plane (coming from \( A \) and \(-A\)) and the indices \( (r, s) \) in (B.1) are given by
\[
(r, s) = (2k \text{ mod } 2, 2j \text{ mod } 2).
\] (B.5)

Note that for the above form of \( A \),
\[
\det A = kp, \quad \mathcal{A} = k\tau + j + pU, \quad \tilde{\mathcal{A}} = k\tau + j + p\bar{U}.
\] (B.6)

Let us first consider the case \( k \in \mathbb{Z}, j \in \mathbb{Z} \). In this case \( j \) runs from 0 to \( k - 1 \) in steps of 1. The relevant \( F^{(r,s)} \) is \( F^{(0,0)} \). In order to carry out the integral we replace \( F^{(0,0)}(\tau, z) \) in (B.1) by its Fourier expansion (3.22). If we now change the integration variable from \( \tau_1 \) to
\[
\tau'_1 = k\tau_1 + j + pU_1,
\] (B.7)
then \( \mathcal{A}, \tilde{\mathcal{A}} \) and hence also the exponential factor in (B.1), expressed as a function of \( \tau'_1 \) and \( \tau_2 \), will have no \( j \) dependence. The only \( j \) dependence comes from the term
\[
\exp(2\pi i n\tau_1) = \exp\left(2\pi i n\frac{1}{k}(\tau'_1 - j - pU_1)\right)
\] (B.8)
which arises from the factor \( c^{(0,0)}(4n - b^2) \exp(2\pi i \tau n) \) in the expansion (3.22) of \( F^{(0,0)}(\tau, z) \). Since in this case \( n \) is an integer, the summation over \( j \) from 0 to \( k - 1 \) in steps of 1 imposes the condition \( n = n'k \) where \( n' \) is an integer. Furthermore since \( n \geq 0 \) and \( k > 0 \), we have \( n' \geq 0 \). The summation over \( j \) also produces a factor of \( k \) which cancels the \( 1/k \) factor arising due to the change of variables from \( \tau_1 \) to \( \tau'_1 \) in the measure.

Using (B.6)-(B.8) we see that the integration over \( \tau'_1 \) in (B.1) is just a Gaussian integration. The result of carrying out this integral is
\[
\mathcal{I}_{2,k,j;\mathbb{Z}} = \sum_{n' \in \mathbb{Z}, b,p \in \mathbb{Z}, n' \geq 0, k > 0, p \neq 0} \sqrt{Y} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \exp(\mathcal{F}) c^{(0,0)}(4n'k - b^2)
\]
\[
\mathcal{F} \equiv -2\pi\tau_2 n'k - \frac{\pi Y}{U_2^2 \tau_2} (k\tau_2 + pU_2)^2 - 2\pi iT kp - 2\pi i pn'U_1 + \frac{\pi b}{U_2} (-2V_2 k\tau_2 - 2ipU_2 V_1) - \frac{2\pi V_2^2}{U_2^2} (k^2\tau_2 + kpU_2) - \frac{\pi B^2 U_2^2 \tau_2}{Y}
\]
\[
B \equiv n' + \frac{bV_2}{U_2} + \frac{V_2^2}{U_2^2} k
\] (B.9)
The \( \tau_2 \) integral is of the Bessel form and can be performed using

\[
\int_0^\infty \frac{du}{u^{3/2}} e^{-au - bu^{-1}} = e^{-2\sqrt{ab}} \sqrt{\frac{\pi}{b}}.
\]  

(B.10)

This gives

\[
\mathcal{I}_{2,k,j \in \mathbb{Z}} = \sum_{n', k, b \in \mathbb{Z}, p \in \mathbb{Z}, n' \geq 0, k > 0, p \neq 0} \frac{1}{|p|} e^{(0,0)(4n'k - b^2)} \exp \left\{ -2\pi i T k p - 2\pi k |p| T_2 - 2\pi k p T_2 - 2\pi i n' U_1 - 2\pi |p| U_2 n' - 2\pi i b p V_1 - 2\pi |p| b V_2 \right\}
\]

\[
= -\ln \prod_{n', k, b \in \mathbb{Z}, n' \geq 0, k > 0} \left\{ \left| 1 - \exp(2\pi i (kT + n'U + bV)) \right|^{2e^{(0,0)(4n'k-b^2)}} \right\}
\]  

(B.11)

Next we consider the contribution from the \( k \in \mathbb{Z}, j \in \mathbb{Z} + \frac{1}{2} \) terms. In this case \( j \) takes values from \( \frac{1}{2} \) to \( k - \frac{1}{2} \) in steps of 1 and \((r, s) = (0, 1)\). The analysis proceeds as in the previous case, the only difference being that the sum over \( j \) of (B.8) gives an additional factor of \((-1)n'\) besides forcing the condition \( n = n'k \) with \( n' \in \mathbb{Z} \).

The analog of eq.(B.11) is then

\[
\mathcal{I}_{2,k \in \mathbb{Z}, j \in \mathbb{Z} + \frac{1}{2}} = -\ln \prod_{n', k, b \in \mathbb{Z}, n' \geq 0, k > 0} \left\{ \left| 1 - \exp(2\pi i (kT + n'U + bV)) \right|^{2(-1)^{n'}e^{(1,0)(4n'k-b^2)}} \right\}
\]  

(B.12)

Finally let us consider the case \( k \in \mathbb{Z} + \frac{1}{2} \). In this case instead of letting \( j \) run from 0 to \( k - \frac{1}{2} \) in steps of \( \frac{1}{2} \) we can let it run from 0 to \((2k-1)\) in steps of 1 by means of a further SL(2, \( \mathbb{Z} \)) duality transformation. For each of these terms the relevant \((r, s)\) are \((1, 0)\). Proceeding as in the \( k, j \in \mathbb{Z} \) case we now see that the sum over \( j \) in (B.8) forces the condition \( n = 4n'k \) with \( n' \in \mathbb{Z} \) and when this condition is satisfied we get a factor of \( 2k \).\(^7\) The rest of the analysis proceeds as in the previous case and we obtain

\[
\mathcal{I}_{2,k \in \mathbb{Z} + \frac{1}{2}} = -2\ln \prod_{n', k, b \in \mathbb{Z}, k \in \mathbb{Z} + \frac{1}{2}, n' \geq 0, k > 0} \left\{ \left| 1 - \exp(2\pi i (kT + n'U + bV)) \right|^{2e^{(1,0)(4n'k-b^2)}} \right\}
\]  

(B.13)

\(^7\)Note that in this case \( n \) is either an integer or a half integer, but the sum over \( j \) still forces \( n \) to be an integer multiple of \( k \) since the sum runs over \( 2k \) values instead of \( k \) values.
Thus the net contribution to the integral from the non-degenerate orbits take the form

\[
I_2 = - \ln \left[ \prod_{n', k, b \in \mathbb{Z}, n' \geq 0, k > 0} \left( 1 - \exp\left(2\pi i (kT + n'U + bV)\right) \right)^{2^{c(0,0)(4n'k-b^2)+2(-1)^{n'}c(0,1)(4n'k-b^2)}} \right].
\]  

(B.14)

3. Contribution \( I_3 \) from the degenerate orbit

Here we consider \( A \) to be of the form

\[
A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}, \quad (j, p) \neq (0, 0), \quad j \in \frac{1}{2} \mathbb{Z}, \quad p \in \mathbb{Z}.
\]  

(B.15)

In this case the integration region \( \mathcal{F}_A \) corresponds to the strip

\[
-1/2 \leq \tau_1 \leq 1/2, \quad \tau_2 \geq 0.
\]  

(B.16)

Also we have

\[
(r, s) = (0, 0) \quad \text{for} \quad j \in \mathbb{Z}, \quad (r, s) = (0, 1) \quad \text{for} \quad j \in \mathbb{Z} + \frac{1}{2}.
\]  

(B.17)

For \( A \) given in (B.15)

\[
\mathcal{A} = j + pU, \quad \tilde{A} = j + p\bar{U}, \quad \det A = 0,
\]  

(B.18)

are independent of \( \tau \). Thus the exponential factor in (B.13) is independent of \( \tau_1 \) and the only dependence on \( \tau_1 \) of the integrand comes from the \( \exp(2\pi i \tau n) \) term in the expansion of \( F^{(r,s)}(\tau, z) \). The \( \tau_1 \) integration now forces \( n \) to vanish and the coefficients \( c^{(r,s)}(4n - b^2) \) multiplying the integrand reduces to \( c^{(r,s)}(-b^2) \). It follows from the definition of \( c^{(r,s)}(m) \) that these coefficients are non-zero only for \( b = 0 \) and \( b = \pm 1 \).

We first consider the case \( j \in \mathbb{Z} \). We begin with the contribution from the \( n = 0, b = 0 \) term and proceed as in [28]. We multiply the integrand with the regulating factor \( (1 - \exp(-\Lambda/\tau_2)) \), then integrate over \( \tau_2 \) and finally perform the sum over \( j \) and \( p \). Integrating over \( \tau_2 \) we obtain

\[
I_{3,b=0; j\in\mathbb{Z}} = c^{(0,0)}(0) \left[ \frac{U_2}{\pi} \sum_{(j,p) \neq (0,0)} \left( \frac{1}{|j+U p|^2} - \frac{1}{|j+U p|^2 + \Lambda U_2^2/\pi Y} \right) \right.
\]

\[
- \int_{\mathcal{F}} d^2 \tau \frac{1 - \exp(-\Lambda/\tau_2)}{\tau_2}, \quad \text{for} \quad j \in \mathbb{Z}.
\]  

(B.19)
Note that we have introduced a subtraction term proportional to \( \int F d^2 \tau \frac{1-\exp(-\Lambda/\tau_2)}{\tau_2} \) in eq.(B.19), – this is one of the two terms appearing in (B.2). This is necessary in order to get a finite value of the integral in the \( \Lambda \to \infty \) limit. The result of the integration in the second terms inside the square brackets is \( \ln \Lambda + \gamma_E + 1 + \ln(2/3\sqrt{3}) \).

To evaluate the summation we use \(^{29}\)

\[
\sum_{j \in \mathbb{Z}} \frac{\exp(i \theta j)}{(j + B)^2 + C^2} = \frac{\pi \exp(-i \theta(B - iC))}{C \exp(-2\pi i(B - iC))} \frac{1}{1 - \exp(-2\pi i(B - iC))} \\
+ \frac{\pi \exp(-i \theta(B + iC))}{C \exp(2\pi i(B + iC))} \frac{\exp(2\pi i(B + iC))}{1 - \exp(2\pi i(B + iC))}
\]

for \( C > 0, \ 0 \leq \theta \leq 2\pi \). \( (B.20) \)

We now regroup the summation in (B.19) as \( \sum_{p=0,j\neq0} + \sum_{j=-\infty,p\neq0} \) and use (B.20) at \( \theta = 0 \) to obtain

\[
\mathcal{I}_{3,b=0;j \in \mathbb{Z}} = c^{(0,0)}(0) \left[ \frac{\pi}{3} U_2 + \sum_{p \neq 0} \left\{ \frac{2}{p} \frac{e^{-2\pi ipU}}{1 - e^{-2\pi ipU}} + \frac{2}{p} \frac{e^{2\pi ipU}}{1 - e^{2\pi ipU}} \right\} + \left( \frac{2}{p} - \frac{2}{\sqrt{p^2 + \Lambda/\pi Y}} \right) \right] - \left( \ln \Lambda + \gamma_E + 1 + \ln(2/3\sqrt{3}) \right).
\]

\( (B.21) \)

Next we expand

\[
\frac{x}{1 - x} = \sum_{i=1}^{\infty} x^i , \quad (B.22)
\]

for \( x = e^{-2\pi ipU} \) and \( x = e^{2\pi ip\tilde{U}} \) in (B.21) and perform the sum over \( p \) in the first two terms. Finally we use

\[
\sum_{p \neq 0} \left( \frac{2}{p} - \frac{2}{\sqrt{p^2 + \Lambda/\pi Y}} \right) = -\ln \frac{\pi Y}{\Lambda} + 2\gamma_E - \ln 4 \quad \text{as} \ \Lambda \to \infty , \quad (B.23)
\]

to obtain

\[
\mathcal{I}_{3,b=0;j \in \mathbb{Z}} = c^{(0,0)}(0) \left( \frac{\pi}{3} U_2 - \ln Y + \kappa' \right) - \ln \prod_{l \in \mathbb{Z},l>0} \left\{ 1 - \exp(2\pi i l U) \right\}^{4c^{(0,0)}(0)} \quad (B.24)
\]
where

$$\kappa' = \gamma_E - 1 - \ln(8\pi/3\sqrt{3})\,.$$  \hspace{1cm} (B.25)

We now evaluate the contribution of $n = 0, b = \pm 1$. The corresponding coefficient is $c^{(0,0)}(-1)$. Integrating over $\tau_2$ we obtain

$$I_{3,b=\pm 1; j \in \mathbb{Z}} = c^{(0,0)}(-1) \frac{U_2}{\pi} \sum_{(j,p) \neq (0,0), j,p \in \mathbb{Z}} \frac{1}{|j+pU|^2} \exp\left(\frac{2\pi ib}{U_2}(jV_2 + p(V_2U_1 - V_1U_2))\right)$$  \hspace{1cm} (B.26)

We split this summation as before $\sum_{p=0,j \neq 0} + \sum_{p \neq 0,j}$. We shall assume, for definiteness, that $V_2 < 0$. \hspace{1cm} (B.27)

For the $p = 0$ one can apply the second formula in (B.20) to obtain

$$4\pi c^{(0,0)}(-1) \left(\frac{V_2^2}{U_2} + V_2 + \frac{U_2}{6}\right)$$  \hspace{1cm} (B.28)

Let us now turn to the contribution from the $p \neq 0$ terms. Since (B.26) contains the contribution for both $b = 1$ and $b = -1$, care should be taken so that the $\theta$ in (B.20) is between $0 \leq \theta \leq 2\pi$. Here $\theta = -2\pi V_2/U_2 \leq 1$. For the $p \neq 0$ case one splits the summation for $p > 0, b = \pm 1$ and $p < 0, b = \pm 1$, then one changes $j \rightarrow -j$ or $p \rightarrow -p$ so that one can always apply the formula in (B.20). Carefully taking all these contributions into account one obtains, after using (B.20), the total contribution from the $p \neq 0$ terms to be

$$-\ln \prod_{l \in \mathbb{Z}, l > 0, b = \pm 1} |1 - \exp(2\pi i(lU + bV))|^{4c^{(0,0)}(-1)} - \ln |1 - \exp(-2\pi iV)|^{4c^{(0,0)}(-1)}$$  \hspace{1cm} (B.29)

Thus the net contribution from the $b = \pm 1, j \in \mathbb{Z}$ terms are

$$I_{3,b=\pm 1; j \in \mathbb{Z}} = 4\pi c^{(0,0)}(-1) \left(\frac{V_2^2}{U_2} + V_2 + \frac{U_2}{6}\right)$$

$$-\ln \prod_{l \in \mathbb{Z}, l > 0, b = \pm 1} \left\{|1 - \exp(2\pi i(lU + bV))|^{4c^{(0,0)}(-1)}\right\}$$

$$- \ln |1 - \exp(-2\pi iV)|^{4c^{(0,0)}(-1)}$$  \hspace{1cm} (B.30)

Note that the last term in the above equation is singular as $V \rightarrow 0$. 

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Next we turn to the contribution from the \( j \in \mathbb{Z} + \frac{1}{2} \) terms. In this case \( (r, s) = (0, 1) \). The analog of (B.20) is obtained by replacing \( B \to B + \frac{1}{2} \) in this formula and multiplying the resulting equation by a factor of \( e^{i\theta/2} \) on both sides:

\[
\sum_{j \in \mathbb{Z} + \frac{1}{2}} \frac{\exp(i\theta j)}{(j + B)^2 + C^2} = \frac{\pi}{C} \frac{\exp(-i\theta (B - iC))}{1 + \exp(-2\pi i(B - iC))} \cdot \frac{1}{1 + \exp(2\pi i(B + iC))} - \frac{\pi}{C} \frac{\exp(-i\theta (B + iC))}{1 + \exp(2\pi i(B + iC))}
\]

for \( C > 0, \ 0 \leq \theta \leq 2\pi \) \quad (B.31)

Using this result we can get the analogs of (B.24) and (B.30):

\[
\mathcal{I}_{3, b=0; j \in \mathbb{Z} + \frac{1}{2}} = c^{(0,1)}(0) \left( \pi U_2 - \ln Y + \kappa' \right) - \ln \prod_{l \in \mathbb{Z}, l > 0} \left\{ \left| 1 - \exp(2\pi iU) \right|^4 \right\}
\]

\[
\mathcal{I}_{3, b=\pm 1; j \in \mathbb{Z} + \frac{1}{2}} = 4\pi c^{(0,1)}(-1) \left( V_2 + \frac{U_2}{2} \right) - \ln \prod_{l \in \mathbb{Z}, l > 0} \left\{ \left| 1 - \exp(2\pi i(U + bV)) \right|^4 \right\}
\]

Adding all the contributions we obtain.

\[
\mathcal{I}_3 = \mathcal{I}_{3, b=0; j \in \mathbb{Z}} + \mathcal{I}_{3, b=\pm 1; j \in \mathbb{Z}} + \mathcal{I}_{3, b=0; j \in \mathbb{Z} + \frac{1}{2}} + \mathcal{I}_{3, b=\pm 1; j \in \mathbb{Z} + \frac{1}{2}}
\]

\[
= c^{(0,0)}(0) \left( \frac{\pi}{3} U_2 - \ln Y + \kappa' \right) + 4\pi c^{(0,1)}(-1) \left( \frac{V_2}{U_2} + V_2 + \frac{U_2}{6} \right) + c^{(0,1)}(0) \left( \pi U_2 - \ln Y + \kappa' \right) + 4\pi c^{(0,1)}(-1) \left( \frac{V_2}{U_2} + \frac{U_2}{2} \right)
\]

\[
- \ln \prod_{l \in \mathbb{Z}, l > 0} \left\{ \left| 1 - \exp(2\pi iU) \right|^4 \right\} - \ln \left\{ \left| 1 - \exp(-2\pi iV) \right|^4 \right\}
\]

\[
- \ln \prod_{l \in \mathbb{Z}, l > 0} \left\{ \left| 1 - \exp(2\pi i(U + bV)) \right|^4 \right\} - \ln \left\{ \left| 1 - \exp(-2\pi iV) \right|^4 \right\}
\]

\[
- \ln \prod_{l \in \mathbb{Z}, l > 0} \left\{ \left| 1 - \exp(2\pi i(U + bV)) \right|^4 \right\} - \ln \left\{ \left| 1 - \exp(-2\pi iV) \right|^4 \right\}
\]

\[
(B.34)
\]

Combining the contribution from all the orbits and noting that

\[
c^{(0,0)}(0) = 10, \quad c^{(0,0)}(-1) = 1, \quad c^{(0,1)}(0) = 2, \quad c^{(0,1)}(-1) = 1,
\]
we can now express the full integral as
\[
\mathcal{I} = \mathcal{I}_1 + 2\mathcal{I}_2 + \mathcal{I}_3,
\]
\[
= -2 \ln \left[ \kappa (\det \text{Im}\Omega)^6 \exp(2\pi i (\frac{1}{2} T + U + V)) \prod_{(k,l,b) \in \mathbb{Z} \atop (k,l,b) > 0} (1 - \exp(2\pi i (kT + lU + bV))) c^{(0,0)}(4kl - b^2) + (-1)^l c^{(0,1)}(4kl - b^2) \right]
\]
\[
\prod_{i,b \in \mathbb{Z}, k \in \mathbb{Z} + \frac{1}{2}} \left\{ |1 - \exp(2\pi i (kT + lU + bV))|^{2c^{(1,0)}(4lk - b^2)} \right\}^2 \]
\]
\[
\text{(B.36)}
\]
where
\[
\kappa = \left( \frac{8\pi}{3\sqrt{3}} e^{1-\gamma_E} \right)^6 \text{ (B.37)}
\]
and \((k, l, b) > 0\) means \(k > 0, l \geq 0, b \in \mathbb{Z}\) or \(k = 0, l > 0, b \in \mathbb{Z}\) or \(k = 0, l = 0, b < 0\).

Note that we have \(2\mathcal{I}_2\) because of the two copies of the upper half plane.

From the modular transformation laws (3.11) and the series expansion (3.22) it follows that
\[
c^{(1,1)}(4lk - b^2) = (-1)^l c^{(1,0)}(4lk - b^2) \quad \text{for } k \in \mathbb{Z} + \frac{1}{2}, l \in \mathbb{Z}. \quad \text{(B.38)}
\]

Using this we can reexpress (B.36) in a more symmetric fashion:
\[
\mathcal{I} = -2 \ln \left[ \kappa (\det \text{Im}\Omega)^6 \exp \left( 2\pi i \left( \frac{1}{2} T + U + V \right) \right) \prod_{r,s=0}^{1} \prod_{(i,b) \in \mathbb{Z}, k \in \mathbb{Z} + \frac{1}{2}} \left\{ (1 - \exp(2\pi i (kT + lU + bV)))^{(-1)^l c^{(r,s)}(4lk - b^2)} \right\}^2 \right].
\]
\[
\text{(B.39)}
\]

C. Explicit Results for \(a(n, m, r)\)

In this appendix we present the results of explicit computation of the coefficients \(a(n, m, r)\) for \(\Phi_k\). These were calculated using the expression given in [7] as well as the expression found in the present paper and found to be the same. To write the expansion of \(\Phi_k\) in a convenient way we define \(t = \exp(2\pi iT), u = \exp(2\pi iU), v = \exp(2\pi iV)\). Then for \(N = 2\)
\[
\Phi_6 = \left[ (2 - \frac{1}{v} - v)u + (-4 + \frac{2}{v^2} + 2v^2)u^2 + (-16 - \frac{1}{v^3} - \frac{4}{v^2} + \frac{13}{v} + 13v - 4v^2 - v^3)u^3 \right] t
\]
\[
\left.\begin{array}{l}
+ \left[ (-4 + \frac{2}{v^2} + 2v^2)u + (32 - \frac{16}{v^2} - 16v^2)u^2 + (-72 - \frac{4}{v^4} + 40v^2 - 4v^4)u^3 \right] t^2 \\
+ \left[ (-16 - \frac{1}{v^3} - \frac{4}{v^2} + \frac{13}{v} + 13v - 4v^2 - v^3)u \\
+ (-72 - \frac{4}{v^4} + \frac{40}{v^2} + 40v^2 - 4v^4)u^2 \\
+ \left( 336 + \frac{13}{v^5} + \frac{40}{v^4} - \frac{87}{v^3} - \frac{64}{v^2} - \frac{70}{v} - 70v - 64v^2 - 87v^3 + 40v^4 + 13v^5 \right)u \right] t^3 \\
+ \cdots
\end{array}\right)
\] (C.1)

For \( N = 3 \)

\[
\Phi_4 = \left[ (2 - \frac{1}{v} - v)u + \left( \frac{2}{v^2} - \frac{2}{v} - 2v + 2v^2 \right)u^2 \right] t \\
+ \left( \frac{2}{v^2} - \frac{2}{v} - 2v + 2v^2 \right)u + \left( 4 - \frac{2}{v^3} - \frac{6}{v^2} + \frac{6}{v} + 6v - 6v^2 - 2v^3 \right)u^2 \right] t^2 + \cdots 
\] (C.2)

For \( N = 5 \)

\[
\Phi_2 = \left[ (2 - \frac{1}{v} - v)u + \left( 4 + \frac{2}{v^2} - \frac{4}{v} - 4v + 2v^2 \right)u^2 \right] t \\
+ \left( 4 + \frac{2}{v^2} - \frac{4}{v} - 4v + 2v^2 \right)u + \left( 28 - \frac{4}{v^3} + \frac{10}{v^2} - \frac{20}{v} - 20v + 10v^2 - 4v^3 \right)u^2 \right] t^2 + \cdots 
\] (C.3)

For \( N = 7 \)

\[
\Phi_1 = \left[ (2 - \frac{1}{v} - v)u + \left( 6 + \frac{2}{v^2} - \frac{5}{v} - 5v + 2v^2 \right)u^2 \right] t \\
+ \left( 6 + \frac{2}{v^2} - \frac{5}{v} - 5v + 2v^2 \right)u + \left( 52 - \frac{5}{v^3} + \frac{19}{v^2} - \frac{40}{v} - 40v + 19v^2 - 5v^3 \right)u^2 \right] t^2 + \cdots 
\] (C.4)

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