ROBIN HEAT KERNEL COMPARISON ON MANIFOLDS

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Abstract. We investigate the heat kernel with Robin boundary condition and prove comparison theorems for heat kernel on geodesic balls and on minimal submanifolds. We also prove an eigenvalue comparison theorem for the first Robin eigenvalues on minimal submanifolds. This generalizes corresponding results for the Dirichlet and Neumann heat kernels.

1. Introduction

Let $\Omega$ be an $m$-dimensional compact Riemannian manifold with smooth boundary $\partial \Omega$. Let $o \in \Omega$, and $H_\alpha(o, x, t)$ be the $o$-centered Robin heat kernel of $\Omega$, i.e. $H_\alpha(o, x, t)$ solves the heat equation

$$ u_t(x, t) - \Delta u(x, t) = 0, \quad (x, t) \in \Omega \times (0, \infty), $$

with the Robin boundary condition

$$ \frac{\partial u}{\partial \nu}(x, t) + \alpha u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, \infty), $$

and the initial condition

$$ u(x, 0) = \delta_o(x). $$

Here $\nu$ denotes the outward unit normal vector field on $\partial \Omega$, $\alpha \in \mathbb{R}$ is called the Robin parameter, and initial condition (1.3) is interpreted as that

$$ \lim_{t \to 0^+} \int_{\Omega} H_\alpha(o, x, t) \varphi(x) \, dx = \varphi(o), $$

for every continuous function $\varphi(x)$ on $\Omega$.

Let $M^m(\kappa)$ be the $m$-dimensional simply-connected space form of constant sectional curvature $\kappa$. Denote by $\bar{B}_\circ(R)$ the geodesic ball of radius $R$ centered at $\bar{o}$ in $\mathbb{R}^m$.

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\( M^m(\kappa) \) and by \( \bar{H}_\alpha(\bar{o}, x, t) \) the \( \bar{o} \)-centered Robin heat kernel of \( \bar{B}_\alpha(R) \). In other words, \( \bar{H}_\alpha(\bar{o}, x, t) \) satisfies

\[
\begin{aligned}
\begin{cases}
\partial_t \bar{H}_\alpha(\bar{o}, x, t) - \Delta \bar{H}_\alpha(\bar{o}, x, t) = 0, & x \in \bar{B}_\alpha(R), t > 0, \\
\bar{H}_\alpha(\bar{o}, x, 0) = \delta_\bar{\bar{o}}(x), & x \in B_\alpha(R), \\
\frac{\partial}{\partial \nu} \bar{H}_\alpha(\bar{o}, x, t) + \alpha \bar{H}_\alpha(\bar{o}, x, t) = 0, & x \in \partial \bar{B}_\alpha(R), t > 0.
\end{cases}
\end{aligned}
\]  

(1.4)

Since the metric on \( \bar{B}_\alpha(R) \) is rotational invariant, so is system (1.4). Then \( \bar{H}_\alpha \) is radially symmetric in the space variable by the uniqueness of the solution to system (1.4). Here and thereafter we rewrite \( \bar{H}_\alpha \) as \( \bar{H}_\alpha(\bar{r}_\bar{o}(x), t) \), where \( \bar{r}_\bar{o}(x) \) is the distance function from \( \bar{o} \) on \( M^m(\kappa) \).

The Robin boundary condition generates a global picture of the boundary value problems. Indeed, the Neumann (\( \alpha = 0 \)) and the Dirichlet (\( \alpha \rightarrow \infty \)) boundary conditions are all special cases of the Robin boundary conditions. Hence, existing results on the Dirichlet, or Neumann heat kernels naturally motivate the investigation on the Robin heat kernel. The first main result of this article is the following comparison theorem for the Robin heat kernel.

**Theorem 1.1.** Suppose \( M \) is an \( m \)-dimensional complete Riemannian manifold, \( o \in M \) and \( B_o(R) \subset M \) is the geodesic ball of radius \( R \) centered at \( o \). Let \( \alpha > 0 \). Denote by \( H_\alpha(o, x, t) \) the \( o \)-centered Robin heat kernel of \( B_o(R) \), and by \( \bar{H}_\alpha(\bar{r}, t) \) be the \( \bar{\bar{o}} \)-centered Robin heat kernel of \( \bar{B}_\alpha(R) \).

1. If the Ricci curvature of \( B_o(R) \) is bounded from below by \( (m - 1)\kappa \), then

\[
H_\alpha(o, x, t) \geq \bar{H}_\alpha(\bar{r}_\bar{o}(x), t)
\]

for all \( x \in \Omega \) and \( t > 0 \).

2. If the sectional curvature of \( B_o(R) \) is bounded from above by \( \kappa \) and \( R < \text{inj}(o) \), the injectivity radius of \( o \), then

\[
H_\alpha(o, x, t) \leq \bar{H}_\alpha(\bar{r}_\bar{o}(x), t)
\]

for all \( x \in \Omega \) and \( t > 0 \).

Where \( \bar{r}_\bar{o}(x) \) is the distance function from \( \bar{o} \) on \( M \).

**Remark 1.2.** When \( \alpha = +\infty \), i.e. \( H_\alpha \) is Dirichlet heat kernel, inequality (1.5) was proved under a lower bound on sectional curvature by Debiard, Gaveau and Mazet in [7]; when \( \alpha = 0 \) or \( \alpha = +\infty \), i.e. \( H_\alpha \) is Neumann or Dirichlet heat kernel, estimates (1.5) and (1.6) were proven by Cheeger and Yau in [4].

**Remark 1.3.** It is known from [11] Section 10] that for \( \alpha > 0 \) the Sturm-Liouville decomposition of the Robin heat kernel on \( \Omega \) is given by

\[
H_\alpha(x, y, t) = \sum_{i=1}^{\infty} \exp(-\lambda_{i,\alpha}t)\varphi_i(x)\varphi_i(y)
\]

(1.7)
with convergence absolute and uniform for each \( t > 0 \). Where \( \lambda_{i,\alpha} \) are Robin eigenvalues of Laplacian on \( M \) and \( \varphi_i(x) \) are corresponding eigenfunctions, defined as in (1.12). Then as \( t \to +\infty \), the expression (1.7) and Theorem 1.1 implies the following Cheng’s type eigenvalue comparison for Robin eigenvalue: if \( \text{Ric} \geq (m - 1)\kappa \) on \( B_o(R) \), then

\[
\lambda_{1,\alpha}(B_o(R)) \leq \lambda_{1,\alpha}(\bar{B}_o(R)); \tag{1.8}
\]

if \( \text{Sect} \leq \kappa \) on \( B_o(R) \) and \( R < \text{inj}(o) \), then

\[
\lambda_{1,\alpha}(B_o(R)) \geq \lambda_{1,\alpha}(\bar{B}_o(R)). \tag{1.9}
\]

Estimates (1.8) and (1.9) were proved by Savo in [19]. Recently, the authors [13] extended (1.8) and (1.9) to the first Robin eigenvalue of the \( p \)-Laplacian for \( p \in (1, +\infty) \).

We set up some notations before stating the next theorem. Let \( M \) be an immersed submanifold of a complete Riemannian manifold \((N, g_N)\). Denote by \( D_o(R) \) the extrinsic ball of radius \( R \) centered at \( o \in M \), i.e. the smooth connected component of \( \{ x \in M, \text{d}_N(o, x) \leq R \} \) which contains \( o \), where \( \text{d}_N \) is the distance function induced by the metric \( g_N \).

Our second main result is a Robin heat kernel comparison theorem for minimal submanifolds.

**Theorem 1.4.** Let \( \alpha > 0 \) and \( M^m \) be an \( m \)-dimensional minimally immersed submanifold of \( N^n \). Let \( D_o(R) \) be the extrinsic ball of radius \( R \) centered at \( o \in M \). Suppose the sectional curvature of \( N \) is bounded from above by \( \kappa \). If \( \kappa > 0 \), we assume further that

\[
R \leq \min\{i_N(o), \frac{1}{\sqrt{\kappa}} \arctan \frac{\alpha}{\sqrt{\kappa}} \}. \tag{1.10}
\]

Then the Robin heat kernel \( H_\alpha(o, x, t) \) of \( D_o(R) \) satisfies

\[
H_\alpha(o, x, t) \leq \bar{H}_\alpha(d_N(o, x), t) \tag{1.11}
\]

for all \( x \in D_o(R) \) and \( t > 0 \), where \( i_N(o) \) is the injectivity radius of \( N \) from \( o \).

**Remark 1.5.** For Neumann and Dirichlet heat kernels, Theorem 1.4 was first proved by Cheng, Li and Yau in [6] for space form ambient spaces \( N \), and by Markvorsen in [14] for ambient spaces with sectional curvature bounded from above. Besides, for the heat kernel of the Bergmann metric on algebraic varieties, a similar comparison as inequality (1.11) was proved by Li and Tian in [12].

Next, let’s turn to the eigenvalue problem with Robin boundary condition. Let \( M \) be an \( m \)-dimensional smooth compact Riemannian manifold with non-empty smooth
boundary. We consider the following Robin eigenvalue problem
\[
\begin{aligned}
-\Delta u &= \lambda u, \quad x \in \Omega, \\
\frac{\partial u}{\partial \nu} + \alpha u &= 0, \quad x \in \partial \Omega.
\end{aligned}
\tag{1.12}
\]
The first Robin eigenvalue for the Laplace operator, denoted by \(\lambda_{1,\alpha}(\Omega)\), is the smallest number such that (1.12) admits a solution. Moreover, it can be characterized as
\[
\lambda_{1,\alpha}(\Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^2 d\mu_g + \alpha \int_{\partial \Omega} u^2 dA : u \in W^{1,2}(\Omega), \int_{\Omega} u^2 d\mu_g = 1 \right\},
\tag{1.13}
\]
where \(d\mu_g\) is the Riemannian measure induced by the metric \(g\) and \(dA\) is the induced measure on \(\partial \Omega\).

Recall that the classical eigenvalue comparison theorem of Cheng \[5\] states that the first Dirichlet eigenvalue of a geodesic ball in an \(m\)-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below by \((m-1)\kappa\) is less than or equal to that of a geodesic ball in a space form of constant sectional curvature \(\kappa\), and that the reverse inequality holds if the Ricci lower bound is replaced by the sectional curvature upper bound \(\text{Sect}_g \leq \kappa\) and the radius of the geodesic ball is no larger than the injectivity radius at its center. For minimal submanifolds of spaces forms, Cheng’s type comparison theorems for Dirichlet eigenvalues were obtained by Cheng, Li and Yau \[6\]. By the expression (1.7) of \(H_\alpha\) and letting \(t \to +\infty\), estimate (1.11) yields

**Corollary 1.6.** With the same assumptions as in Theorem 1.4, we have
\[
\lambda_{1,\alpha}(D_o(R)) \geq \lambda_{1,\alpha}(\bar{B}_\alpha(R)),
\]
where \(\lambda_{1,\alpha}(D_o(R))\) is the first Robin eigenvalue of \(\Delta_M\) on \(D_o(R)\), and \(\lambda_{1,\alpha}(\bar{B}_\alpha(R))\) is the first Robin eigenvalue of Laplacian on geodesic ball \(\bar{B}_\alpha(R)\) in \(M^m(\kappa)\).

In the present paper, we prove a more general eigenvalue comparison theorem of Cheng’s type for the Robin eigenvalue on minimal submanifolds. Let \(M\) be a smooth compact submanifold of \((N, g_N)\), and denote by \(R\) the outer radius of \(M\) defined by
\[
R := \inf \sup_{p \in M, x \in M} d_N(p, x),
\tag{1.14}
\]
where \(d_N\) is the distance function induced by the metric \(g_N\). Our next result states that

**Theorem 1.7.** Let \(M^m\) be an \(m\)-dimensional compact, connected and minimally immersed submanifold of \(N^n\) \((m < n)\) with smooth boundary and the outer radius \(R\). Suppose the sectional curvature of the ambient space \(N\) is bounded from above by \(\kappa\) and \(\alpha > 0\). If \(\kappa > 0\), we assume further that \(R \leq \min\left\{ \frac{1}{\sqrt{\kappa}} \arctan \frac{\alpha}{\sqrt{\kappa}}, i_N(M) \right\}\). Then
\[
\lambda_{1,\alpha}(M) \geq \lambda_{1,\alpha}(\bar{B}_\alpha(R)).
\tag{1.15}
\]
Where \( i_N(M) = \inf_{x \in M} i_N(x) \) and \( i_N(x) \) is the injectivity radius of \( N \) from \( x \). Moreover, the equality holds if and only if \( M \) is isometric to \( B_0(R) \).

**Remark 1.8.** Corollary 1.6 is a special case of Theorem 1.7. In fact Corollary 1.6 is a direct consequence of heat kernel comparison \((1.11)\) as \( t \to \infty \); while in Theorem 1.7, we use another proof by translating eigenfunctions and using Barta’s inequality, see Section 7.

In addition, we also prove the following heat kernel comparison theorem for Kähler manifolds.

**Theorem 1.9.** Let \((M^m, g, J)\) be a Kähler manifold of complex dimension \( m \) whose holomorphic sectional curvature is bounded from below by \( 4\kappa \) and orthogonal Ricci curvature is bounded from below by \( 2(m-1)\kappa \). Let \( B_0(R) \subset M \) be the geodesic ball of radius \( R \) centered at \( o \). Let \( \alpha > 0 \) and \( H_\alpha(o, x, t) \) be the \( o \)-centered Robin heat kernel of \( B_0(R) \). Then

\[
H_\alpha(o, x, t) \geq \bar{H}_\alpha(r_o(x), t),
\]

where \( \bar{H}_\alpha(r, t) \) is the Robin heat kernel of a metric ball of radius \( R \) in the Kähler model of holomorphic sectional curvature \( 4\kappa \).

For quaternion Kähler manifolds, we prove

**Theorem 1.10.** Let \((M^m, g, I, J, K)\) be a quaternion Kähler manifold of complex quaternion dimension \( m \) whose scalar curvature is bounded from below by \( 4\kappa \) and orthogonal Ricci curvature is bounded from below by \( 16m(m+2)\kappa \). Let \( B_0(R) \subset M \) be the geodesic ball of radius \( R \) centered at \( o \). Let \( \alpha > 0 \) and denote by \( H_\alpha(o, x, t) \) the \( o \)-centered heat kernel on \( B_0(R) \) with Robin boundary condition. Then

\[
H_\alpha(o, x, t) \geq \bar{H}_\alpha(r_o(x), t),
\]

where \( \bar{H}_\alpha(r_o(x), t) \) is the Robin heat kernel of a metric ball of radius \( R \) in the quaternion Kähler model of scalar curvature \( 16m(m+2)\kappa \).

**Remark 1.11.** In a recent paper \([1]\), Baudoin and Yang proved that for Dirichlet heat kernel, the same results as in Theorem 1.9 and Theorem 1.10 hold (i.e. the case of \( \alpha = +\infty \)), via the study of the radial parts of the Brownian motions.

**Remark 1.12.** We note that Riemannian model spaces (spheres and hyperbolic spaces) are not (quaternionic) Kähler manifolds, so Theorem 1.9 and Theorem 1.10 are sharper than Theorem 1.7 on (quaternionic) Kähler manifolds.

This article is organized as follows. In Section 2, we study the Robin eigenvalue problem on geodesic balls in space of constant sectional curvature. In Section 3, we show the positivity of the Robin heat kernel using the maximum principle. Section 4 is devoted to the study of the Robin heat kernel on model spaces. The proofs of
Theorems 1.1, 1.4 and 1.7 are given in Sections 5, 6 and 7, respectively. In Section 8, we present the proofs of Theorems 1.9 and 1.10.

2. Robin eigenvalue on geodesic balls in model spaces

In this section, we set up the notation and recall some facts on the eigenfunctions for the first Robin eigenvalue on geodesic balls in space forms.

The first Robin eigenvalue \( \lambda_{1,\alpha}(\Omega) \) of the Laplacian is simple and its associated eigenfunction has a constant sign, thus can always be chosen to be positive. It follows from (1.13) that

\[
\lambda_{1,\alpha}(\Omega) = 0 \quad \text{if} \quad \alpha = 0, \\
\lambda_{1,\alpha}(\Omega) > 0 \quad \text{if} \quad \alpha > 0, \\
\lambda_{1,\alpha}(\Omega) < 0 \quad \text{if} \quad \alpha < 0.
\]

We denote by \( \overline{B}_{\bar{o}}(\mathbb{R}) \) the geodesic ball centered at \( \bar{o} \) in the \( m \)-dimensional space form \( M^m(\kappa) \) of constant sectional curvature \( \kappa \) and by \( \lambda_{1,\alpha}(\overline{B}_{\bar{o}}(\mathbb{R})) \) the first Robin eigenvalue for \( \overline{B}_{\bar{o}}(\mathbb{R}) \) with Robin parameter \( \alpha \in \mathbb{R} \). We write \( \lambda_{1,\alpha}(\overline{B}_{\bar{o}}(\mathbb{R})) \) as \( \bar{\lambda}_1 \) for short.

We collect some facts about the first Robin eigenfunctions associated with \( \bar{\lambda}_1 \). The eigenfunction associated to \( \bar{\lambda}_1 \) is radial due to the radial symmetry of \( \overline{B}_{\bar{o}}(\mathbb{R}) \). So we can choose a positive and radial function \( u(r(x)) \) on \( \overline{B}_{\bar{o}}(\mathbb{R}) \) as the first Robin eigenfunction associated to \( \bar{\lambda}_1 \), where \( r(x) \) is the distance function from \( \bar{o} \) in \( M^m(\kappa) \).

Then we deduce from the eigenvalue problem (1.12) that \( u(r) \) solves the ODE initial value problem

\[
\begin{cases}
  u''(r) + (m-1)\frac{sn'(r)}{sn(r)}u'(r) = -\bar{\lambda}_1 u(r), & r \in (0, R), \\
  u'(0) = 0, \\
  u'(R) + \alpha u(R) = 0.
\end{cases}
\]

Throughout the paper, we use the function \( sn_\kappa \) defined by

\[
sn_\kappa(r) = \begin{cases}
  \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa}r, & \kappa > 0, \\
  r, & \kappa = 0, \\
  \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa}r, & \kappa < 0.
\end{cases}
\]

It is easily seen from (2.1) that \( \bar{\lambda}_1 \) is characterized by

\[
\bar{\lambda}_1 = \inf \left\{ \int_0^R |u'|^2 sn_k^{m-1} \, dr + \alpha |u(R)|^2 : u \in C^\infty([0, R]), \int_0^R u^2 sn_k^{m-1} \, dr = 1 \right\}.
\]

We need the following properties of \( u(r) \), see for example [19, Lemma 8] and [13, Proposition 2.1].

**Proposition 2.1.** Let \( \alpha > 0 \) and \( u(r) \) be a positive first eigenfunction associated to \( \bar{\lambda}_1 \). Then

1. \( u'(r) < 0 \) on \((0, R]\).
(2) \((\log u)'\) is monotone decreasing on \((0, R]\). Particularly, \(u'(r) \geq -\alpha u(r)\) on \((0, R]\).

For \(\kappa > 0\), we have following lower bound for \(\bar{\lambda}_1\), which will be used later.

**Lemma 2.1.** If \(\alpha > 0, \kappa > 0\) and \(\sqrt{\kappa} \tan(\sqrt{\kappa} R) \leq \alpha\). Then

\[
\bar{\lambda}_1 \geq m\kappa. \tag{2.3}
\]

**Proof.** Let \(u(r)\) be the positive eigenfunction associated to \(\bar{\lambda}_1\). Using Bochner formula, we estimate that

\[
\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle \\
\geq \frac{(\Delta u)^2}{m} + \text{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle \\
= \frac{\bar{\lambda}_1^2 u^2}{m} + \left( (m - 1)\kappa - \bar{\lambda}_1 \right)|\nabla u|^2,
\]

where we used inequality \(|\nabla^2 u|^2 \geq \frac{(\Delta u)^2}{m}\) in the inequality, and equation \(\Delta u = -\bar{\lambda}_1 u\) and \(\text{Ric} = (m - 1)\kappa\) in the last equality. Integrating above inequality over \(\bar{B}_\delta(R)\) yields

\[
\int_{\partial B_\delta(R)} \frac{1}{2} \frac{\partial}{\partial \nu} |\nabla u|^2 \geq \frac{\bar{\lambda}_1^2}{m} \int_{B_\delta(R)} u^2 + \left( (m - 1)\kappa - \bar{\lambda}_1 \right) \int_{B_\delta(R)} |\nabla u|^2. \tag{2.4}
\]

Using ODE (2.1), we calculate

\[
\frac{1}{2} \frac{\partial}{\partial \nu} |\nabla u|^2 = u'' u' = -\left( \frac{(m - 1) \text{sn}'_\kappa u' + \bar{\lambda}_1 u'}{\text{sn}_\kappa} \right) u' \\
= \left( - (m - 1)\alpha \frac{\sqrt{\kappa}}{\tan(\sqrt{\kappa} R)} + \bar{\lambda}_1 \right) \alpha u^2 \\
\leq \alpha \left( - (m - 1)\kappa + \bar{\lambda}_1 \right) u^2 \tag{2.5}
\]

at \(r = R\), where we used the assumption \(\sqrt{\kappa} \tan(\sqrt{\kappa} R) \leq \alpha\) in the last inequality. Combining (2.4) with (2.5), we conclude

\[
\alpha \bar{\lambda}_1 \int_{\partial B_\delta(R)} u^2 + \bar{\lambda}_1 \int_{B_\delta(R)} |\nabla u|^2 \\
\geq \frac{\bar{\lambda}_1}{m} \int_{B_\delta(R)} u^2 + \left( m - 1 \right) \kappa \int_{B_\delta(R)} |\nabla u|^2 + \alpha \left( m - 1 \right) \kappa \int_{\partial B_\delta(R)} u^2. \tag{2.6}
\]

Recall from (1.13) that

\[
\alpha \int_{\partial B_\delta(R)} u^2 + \int_{B_\delta(R)} |\nabla u|^2 = \bar{\lambda}_1 \int_{B_\delta(R)} u^2,
\]
then inequality (2.6) gives
\[ \bar{\lambda}^2 \int_{B_0(R)} u^2 \geq \frac{\bar{\lambda}^2}{m} \int_{B_0(R)} u^2 + (m - 1) \kappa \bar{\lambda} \int_{B_0(R)} u^2, \]
proving the lemma. \( \square \)

3. Asymptotic and positivity of Robin heat kernels

In this section, we will recall some basic properties of Laplace heat kernels on
manifolds. It is well known that Laplace heat kernels on manifolds are smooth in
\( \Omega \times \Omega \times \mathbb{R}_+ \) and have an asymptotic expansion as \( t \to 0 \) and near the diagonal of the form
\[ \frac{\exp\left(-\frac{d^2(x,y)}{4t}\right)}{(4\pi t)^{n/2}} \left( \sum_{i=0}^{\infty} a_i(x,y)t^i \right) \] (3.1)
by Minakshisundaram-Pleijel’s construction, where \( a_i(x,y) \) are smooth with \( a_0(x,x) = 1 \) and \( d(x,y) \) is the distance function on \( \Omega \), see [4, Formula 1.14], [3, Sections 3 and 4
of Chapter VI], [10 Section 7.5], [2] [16 Formula (1.1)] and [20]. Similar as Dirichlet
and Neumann heat kernels, Robin heat kernel is also positive when Robin parameter
is positive, see [8, 9]. For the readers’ convenience, we give a direct proof of the posi-
tivity of Robin heat kernels on manifolds by adopting Cheeger and Yau’s arguments
in [4, Lemma 1.1].

Lemma 3.1. Let \( H_\alpha(x,y,t) \) be the Robin heat kernel defined by (1.1), (1.2) and (1.3). If \( \alpha > 0 \), then we have
\[ H_\alpha(x,y,t) > 0 \] (3.2)
for \( t > 0 \) and \( x, y \in \Omega \).

Proof. In view of the asymptotic expansion (3.1) for \( H_\alpha(x,y,t) \), there exists \( \varepsilon_0 > 0 \) such that
\[ H_\alpha(x,y,t) > 0 \] (3.3)
for \( d(x,y) < \varepsilon_0 \) and \( 0 < t < \varepsilon_0 \). Now we fix \( x \), and set
\[ h(y,t) = \frac{1}{2} \left( H_\alpha(x,y,t) - |H_\alpha(x,y,t)| \right) ; \] (3.4)
and for any \( T > \varepsilon_0 \) denote
\[ \Omega_T := \Omega \times (0, T] - B_x(\varepsilon_0) \times (0, \varepsilon_0) . \]
Clearly \( h(y,t) \leq 0 \) in \( \Omega \times (0,T) \), \( h(y,t) = 0 \) in \( B_2(\varepsilon_0) \times [0,\varepsilon_0] \) due to (3.3), and \( h(y,0) = 0 \) in \( \Omega_T \) due to the initial condition. Then using integration by parts, we compute that
\[
\int_{\Omega_T} \langle \nabla h, \nabla H_\alpha \rangle = \int_{\varepsilon_0}^{T} \int_{\Omega} \langle \nabla h, \nabla H_\alpha \rangle + \int_{0}^{\varepsilon_0} \int_{\Omega \setminus B_2(\varepsilon_0)} \langle \nabla h, \nabla H_\alpha \rangle \\
= \int_{\varepsilon_0}^{T} dt \int_{\partial \Omega} h \frac{\partial H_\alpha}{\partial \nu} d\sigma_y - \int_{\varepsilon_0}^{T} \int_{\Omega} h \Delta H_\alpha + \int_{0}^{\varepsilon_0} dt \int_{\partial \Omega} h \frac{\partial H_\alpha}{\partial \nu} d\sigma_y \\
- \int_{0}^{\varepsilon_0} dt \int_{\partial B_2(\varepsilon_0)} h \frac{\partial H_\alpha}{\partial \nu} d\sigma_y - \int_{0}^{\varepsilon_0} \int_{\Omega \setminus B_2(\varepsilon_0)} h \Delta H_\alpha \\
= -\alpha \int_{0}^{T} dt \int_{\partial \Omega} h H_\alpha d\sigma_y - \int_{\varepsilon_0}^{T} \int_{B_2(\varepsilon_0)} h \partial_t H_\alpha - \int_{0}^{T} \int_{\Omega \setminus B_2(\varepsilon_0)} h \partial_t H_\alpha,
\]
where we used the Robin boundary condition \( \partial H_\alpha / \partial \nu = -\alpha H_\alpha \) and \( h = 0 \) on \( \partial B_2(\varepsilon_0) \times (0,\varepsilon_0) \) in the last equality, and \( d\sigma_y \) denotes the induced measure on \( \partial \Omega \). Observing from the definition (3.4) of \( h \) that
\[
h H_\alpha = h^2, \quad h \partial_t H_\alpha = h \partial_t h \quad \text{and} \quad \langle \nabla h, \nabla H_\alpha \rangle = \vert \nabla h \vert^2,
\]
then equality (3.5) becomes
\[
\int_{\Omega_T} \vert \nabla h \vert^2 = -\alpha \int_{0}^{T} dt \int_{\partial \Omega} h^2 d\sigma_y - \frac{1}{2} \int_{\varepsilon_0}^{T} \int_{B_2(\varepsilon_0)} \partial_t h^2 - \frac{1}{2} \int_{0}^{T} \int_{\Omega \setminus B_2(\varepsilon_0)} \partial_t h^2.
\]
Therefore, using \( \alpha > 0 \), we have that
\[
\int_{\Omega_T} \vert \nabla h \vert^2 \leq -\frac{1}{2} \int_{\varepsilon_0}^{T} \int_{B_2(\varepsilon_0)} \partial_t h^2 - \frac{1}{2} \int_{0}^{T} \int_{\Omega \setminus B_2(\varepsilon_0)} \partial_t h^2 \\
= \frac{1}{2} \int_{B_2(\varepsilon_0)} h^2(y,\varepsilon_0) - \frac{1}{2} \int_{\Omega} h^2(y, T) \\
= -\frac{1}{2} \int_{\Omega} h^2(y, T) \\
\leq 0,
\]
where in the first inequality we used \( h(y,0) = 0 \) for \( y \in B_2(\varepsilon_0) \), and in the last equality we used fact that \( h(y,\varepsilon_0) = 0 \) for \( y \in B_2(\varepsilon_0) \) due to (3.3). Therefore
\[
\int_{\Omega_T} \vert \nabla h \vert^2 = 0
\]
implying \( h \equiv 0 \), so \( H_\alpha(x,y,t) \geq 0 \) on \( \Omega_T \). Since \( H_\alpha \) satisfies heat equation, thus
\[
H_\alpha(x,y,t) > 0
\]
in \( \Omega \times \Omega \times (0,\infty) \) by strong maximum principle.
\[\square\]
4. Robin heat kernel on model spaces

In this section we will examine the general properties of the Robin heat kernel on model spaces for later use.

Recall that $\bar{H}_\alpha(r, x) = \bar{\alpha}(r) V^\alpha(r)$ denotes the $\bar{\alpha}$-centered Robin Laplace heat kernel for geodesic ball $V^\alpha(\bar{\alpha}, \kappa, R)$ of radius $R$ centered at $\bar{\alpha}$ in space form $M^\kappa$. Then PDE (1.4) gives that $\bar{H}_\alpha$ satisfies

$$\frac{\partial \bar{H}_\alpha}{\partial t}(r, t) - \frac{\partial^2 \bar{H}_\alpha}{\partial r^2}(r, t) - (m - 1) \frac{\text{sn}_\kappa'(r)}{\text{sn}_\kappa(r)} \frac{\partial \bar{H}_\alpha}{\partial r}(r, t) = 0$$

with initial value

$$\bar{H}_\alpha(r(x), 0) = \delta(x),$$

and Robin boundary condition

$$\frac{\partial \bar{H}_\alpha}{\partial r}(R, t) + \alpha \bar{H}_\alpha(R, t) = 0$$

for $t > 0$. Now we rewrite Robin heat kernel $\bar{H}_\alpha$ on $V^\alpha(\bar{\alpha}, \kappa, R)$ as a function $\phi(s, t)$, where

$$s(r) = \begin{cases} \frac{1 - \cos(\sqrt{\kappa} r)}{\kappa}, & \kappa > 0, \\ \frac{r^2}{2}, & \kappa = 0, \\ \frac{\cosh(\sqrt{-\kappa} r) - 1}{-\kappa}, & \kappa < 0. \end{cases}$$

Clearly $s'(r) = \text{sn}_\kappa(r)$. Then equation (4.1) becomes

$$\frac{\partial}{\partial t} \phi(s, t) = \text{sn}_\kappa^2(r) \phi''(s, t) + m \text{sn}_\kappa'(r) \phi'(s, t)$$

for $(s, t) \in (0, s(R)) \times (0, \infty)$, and the Robin boundary (4.3) becomes

$$\text{sn}_\kappa(R) \phi'(s(R), t) + \alpha \phi(s(R), t) = 0$$

for $t > 0$. Here and thereafter we denote $\frac{\partial^k \phi}{\partial s^k}$ by $\phi^{(k)}$ for short. Differentiating equation (4.5) in $s$ twice yields

$$\frac{\partial^t \phi'}{\partial s^2}(s, t) = \text{sn}_\kappa^2(r) \phi''(s, t) + (m + 2) \text{sn}_\kappa'(r) \phi'(s, t) - \kappa m \phi'(s, t),$$

and

$$\frac{\partial^t \phi''}{\partial s^2}(s, t) = \text{sn}_\kappa^2(r) \phi''(s, t) + (m + 4) \text{sn}_\kappa'(r) \phi'(s, t) - \kappa (2m + 2) \phi''(s, t).$$

Differentiating Robin condition (4.6) in $t$ and applying equations (4.5) and (4.7), we have

$$\text{sn}_\kappa^3 \phi^{(3)} + ((m + 2) \text{sn}_\kappa \text{sn}_\kappa' + \alpha \text{sn}_\kappa^2) \phi'' + m (\alpha \text{sn}_\kappa' - \kappa \text{sn}_\kappa) \phi' = 0$$

at $s = s(R)$ and $r = R$. 
Lemma 4.1. Let $\alpha > 0$. Then
\[
\varphi'(s, t) < 0
\] (4.10)
for $s < s(R)$ and $t > 0$.

Remark 4.1. Because $\partial_r \bar{H}_\alpha(r, t) = \varphi'(s, t) \sin r$, so Lemma 4.1 gives
\[
\partial_r \bar{H}_\alpha(r, t) < 0
\] (4.11)
for $(r, t) \in (0, R] \times (0, \infty)$.

Proof of Lemma 4.1. By asymptotic expansion formula (3.1), we have following expansion formula
\[
\varphi(s, t) \sim \frac{\exp(-s^4 t)}{(4\pi t)^{m/2}} \left( \sum_{i=0}^{\infty} b_i(s) t^i \right),
\] (4.12)
near $s = 0$ as $t \to 0$, where $b_i$ are smooth with $b_0(0) = 1$. Thus we have
\[
\varphi'(s, t) \sim \frac{\exp(-s^4 t)}{(4\pi t)^{m/2}} \left( -\frac{1}{4t} \sum_{i=0}^{\infty} b_i(s) t^i + \sum_{i=0}^{\infty} b'_i(s) t^i \right),
\]
and the dominate term is
\[
-\frac{1}{4t} \exp(-s^4 t) b_0(s)
\]
as $t \to 0$, see also [4, (2.13)]. Thus there exists $\varepsilon_0 > 0$ such that
\[
\varphi'(s, t) < 0
\] (4.13)
for all $(s, t) \in [0, \varepsilon_0] \times (0, \varepsilon_0)$.

Let
\[
\varphi_1(s, t) = e^{e_m t} \varphi'(s, t),
\]
and
\[
\varphi_1(s, t) = \frac{1}{2} \left( \varphi_1(s, t) + |\varphi_1(s, t)| \right),
\]
then PDE (4.7) becomes
\[
\partial_t \varphi_1(s, t) = \sin^2 r \varphi_1''(s, t) + (m + 2) \sin' r \varphi_1'(s, t)
\] (4.14)
For $T > \varepsilon_0$, denote
\[
I_T := [0, s(R)] \times (0, T] - [0, \varepsilon_0] \times (0, \varepsilon_0].
\]
Using integration by parts, we have
\[
\int_{I_T} \varphi'_1(s, t) \bar{\varphi}'_1(s, t) \, ds \, dt
\]
\[
= \int_{I_T} \int_0^{s(R)} \varphi'_1(s, t) \bar{\varphi}'_1(s, t) \, ds \, dt
+ \int_{I_T} \int_{s(R)}^s \varphi'_1(s, t) \bar{\varphi}'_1(s, t) \, ds \, dt
\]
\[
= \int_0^T \bar{\varphi}_1(s(R), t) \varphi'_1(s(R), t) \, ds \, m^2(R) - \int_{I_T} \bar{\varphi}_1(s, t) \varphi'_1(s, t) \, ds \, m^2(R) + \bar{\varphi}_1(s, t) \varphi'_1(s, t) \, ds \, m^2(r) \, dt
\]
\[
- \int_{I_T} (m + 2) \varphi'_1(s, t) \bar{\varphi}'_1(s, t) \, ds \, m^2(r) + \bar{\varphi}_1(s, t) \varphi'_1(s, t) \, ds \, m^2(r) \, dt.
\]
Noticing from (4.6) and Lemma 3.1 that \( \varphi_1(s(R), t) < 0 \), so \( \bar{\varphi}_1(s(R), t) = 0 \). Then we have
\[
\int_{I_T} \varphi'_1(s, t) \bar{\varphi}'_1(s, t) \, ds \, m^2(r) \, dt
\]
\[
\leq - \int_{I_T} \varphi''_1(s, t) \bar{\varphi}_1(s, t) \, ds \, m^2(r) - (m + 2) \varphi'_1(s, t) \bar{\varphi}'_1(s, t) \, ds \, m^2(r) \, dt
\]
\[
= - \int_{I_T} \bar{\varphi}_1(s, t) \varphi'_1(s, t) \, ds \, m^2(r) \, dt
\]
\[
= - \frac{1}{2} \int_0^s \varphi^2(s, T) \, ds \, m^2(r) \, ds,
\]
where we used inequality (4.13) and equation (4.14). Since
\[
\varphi'_1(s, t) \bar{\varphi}'_1(s, t) = (\varphi'_1(s, t))^2,
\]
then
\[
\int_{I_T} (\bar{\varphi}'_1(s, t))^2 \, ds \, m^2(r) \, dt \leq 0
\]
implying \( \bar{\varphi}_1(s, t) \equiv 0 \), so
\[
\varphi_1(s, t) \leq 0
\]
for \( t > 0 \). Since \( \varphi_1(s, t) \) satisfies equation (1.14), then \( \varphi_1(s, t) < 0 \) for \( t > 0 \) follows from the strong maximum principle. \( \square \)

**Proposition 4.1.** Let \( \alpha > 0 \) and \( \lambda > 0 \). Suppose \( u(r) : [0, R] \to \mathbb{R} \) is a solution to
\[
u''(r) + (m - 1) \frac{\text{sn}'_\lambda(r)}{\text{sn}_\lambda(r)} u'(r) = -\lambda u(r)
\]
in \( (0, R) \) with \( u'(0) = 0 \). Let \( g(r) = m \frac{\text{sn}'_\lambda(r)}{\text{sn}_\lambda(r)} u'(r) + \lambda u(r) \). Then
\[
\lim_{r \to 0} g(r) = \lim_{r \to 0} g'(r) = 0,
\]
and
\[ \lim_{r \to 0} g''(r) = -\frac{2\lambda(\lambda - \kappa m)}{m(m + 2)} u(0). \] (4.17)

**Proof.** As \( r \to 0 \), equation (4.15) gives
\[ \lim_{r \to 0} u''(r) = -\frac{\lambda}{m} u(0), \] (4.18)
then we have
\[ \lim_{r \to 0} g(r) = \lim_{r \to 0} m \frac{\text{sn}'_\kappa(r)}{\text{sn}_\kappa(r)} u'(r) + \lambda u(0) = m \lim_{r \to 0} u''(r) + \lambda u(0) = 0. \] (4.19)
Differentiating \( g \) in \( r \) and using equality \((\text{sn}'_\kappa)^2 - \text{sn}_\kappa \text{sn}''_\kappa = 1\), we have
\[ g'(r) = -\frac{m}{\text{sn}_\kappa^2(r)} u'(r) + m \frac{\text{sn}'_\kappa(r)}{\text{sn}_\kappa(r)} u''(r) + \lambda u'(r), \] (4.20)
then using L'Hopital's rule and \( u'(0) = 0 \), we compute that
\[
\lim_{r \to 0} g'(r) = \lim_{r \to 0} \left( \frac{-m}{\text{sn}_\kappa(r)} u'(r) + m \frac{\text{sn}'_\kappa(r)}{\text{sn}_\kappa(r)} u''(r) \right)
\]
\[
= m \lim_{r \to 0} \left( \frac{-\kappa \text{sn}_\kappa^2(r) - 1}{\text{sn}_\kappa(r) \text{sn}'_\kappa(r)} u''(r) + \frac{1}{\text{sn}_\kappa^2(r)} u'(r) + u''(r) \right)
\]
\[
= m \lim_{r \to 0} \left( \frac{-2\kappa \text{sn}_\kappa^2(r) - (\text{sn}'_\kappa(r))^2}{\text{sn}_\kappa(r) \text{sn}'_\kappa(r)} u''(r) + \frac{1}{\text{sn}_\kappa^2(r)} u'(r) \right) + m \lim_{r \to 0} u''(r)
\]
\[
= m \lim_{r \to 0} \left( \frac{-\text{sn}'_\kappa(r)}{\text{sn}_\kappa(r)} u''(r) + \frac{1}{\text{sn}_\kappa^2(r)} u'(r) \right) + m \lim_{r \to 0} u''(r)
\]
\[
= -\lim_{r \to 0} g'(r) + m \lim_{r \to 0} u''(r),
\]
which implies
\[
\lim_{r \to 0} g'(r) = \frac{m}{2} \lim_{r \to 0} u''(r). \] (4.21)
Differentiating ODE (4.15) of \( u \) in \( r \) and and using equality \((\text{sn}'_\kappa)^2 - \text{sn}_\kappa \text{sn}''_\kappa = 1\), we obtain
\[ u''(r) + (m - 1) \frac{\text{sn}'_\kappa(r)}{\text{sn}_\kappa(r)} u''(r) - \frac{(m - 1)}{\text{sn}_\kappa^2(r)} u'(r) = -\lambda u'(r). \] (4.22)
As \( r \to 0 \), equation (4.22) gives
\[ \lim_{r \to 0} u''(r) = -\frac{m - 1}{m} \lim_{r \to 0} g'(r), \] (4.23)
where we used equality (4.20) and \( u'(0) = 0 \). Combining (4.23) with (4.21), we obtain
\[ \lim_{r \to 0} g'(r) = \lim_{r \to 0} u''(r) = 0. \] (4.24)
Differentiating (4.20) in $r$ yields
\[ g''(r) = \frac{2m \text{sn}'(r)}{\text{sn}^3(r)} u'(r) - \frac{2m}{\text{sn}^2(r)} u''(r) + m \frac{\text{sn}'(r)}{\text{sn}(r)} u'''(r) + \lambda u''(r). \] (4.25)

Using L’Hospital’s rule, (4.18), (4.24) and (4.25), we have
\[
\lim_{r \to 0} g''(r) = -\frac{\lambda^2}{m} u(0) + m \lim_{r \to 0} \left( u^{(4)}(r) - \kappa \frac{\text{sn}(r)}{\text{sn}(r)} u''(r) - 2 \frac{\text{sn}'(r)}{\text{sn}^2(r)} u'(r) \right)
+ 4 \frac{\kappa^2}{\text{sn}^2(r)} u''(r) - \frac{2 \kappa}{\text{sn}(r)} \frac{\text{sn}'(r)}{\text{sn}^2(r)} u'(r) - 4 \frac{\text{sn}'(r)}{\text{sn}^2(r)} u'(r)
= -\frac{\lambda^2}{m} u(0) + m \lim_{r \to 0} u^{(4)}(r) - 2m \lim_{r \to 0} u^{(4)}(r) - 2m \kappa \lim_{r \to 0} u''(r)
- 2 \lim_{r \to 0} (g''(r) - \lambda u''(r) - m u^{(4)}(r))
\]
\[
= -\frac{3 \lambda^2}{m} u(0) + m \lim_{r \to 0} u^{(4)}(r) + 2 \kappa \lambda u(0) - 2 \lim_{r \to 0} g''(r),
\]
which is equivalent to
\[ 3 \lim_{r \to 0} g''(r) = -\frac{3 \lambda^2}{m} u(0) + 2 \kappa \lambda u(0) + m \lim_{r \to 0} u^{(4)}(r). \] (4.26)

Differentiating equation (4.20), we have
\[ u^{(4)}(r) + (m - 1) \frac{\text{sn}'(r)}{\text{sn}(r)} u''(r) - \frac{2(m - 1)}{\text{sn}^2(r)} u''(r) - \frac{2(m - 1) \text{sn}'(r)}{\text{sn}^3(r)} u'(r) = -\lambda u''(r). \] (4.27)

Letting $r \to 0$, (4.27) implies
\[
\lim_{r \to 0} u^{(4)}(r) + \frac{m - 1}{m} \lim_{r \to 0} g''(r) + \frac{m - 1}{m} \frac{\lambda^2}{m} u(0) = \frac{\lambda^2}{m} u(0),
\] (4.28)
where we have used (4.18) and (4.25). Then equality (4.17) follows from equalities (4.26) and (4.28).

\[ \square \]

**Lemma 4.2.** Let $\alpha > 0$. Suppose $R \leq \frac{1}{\sqrt{\kappa}} \arctan \frac{\alpha}{\sqrt{\kappa}}$. For any $t > 0$, it holds
\[ \varphi''(0, t) > 0. \] (4.29)

**Proof.** Using $s'(r) = \text{sn}(r)$ and $\partial_r \bar{H}_\alpha = -|\nabla \bar{H}_\alpha|$, we have
\[
\varphi''(s, t) = \frac{1}{\text{sn}^2(\kappa)} \left( \partial_r^2 \bar{H}_\alpha(r, t) - \frac{\text{sn}'(r)}{\text{sn}(r)} \partial_r \bar{H}_\alpha(r, t) \right)
= \frac{1}{\text{sn}^2(\kappa)} \left( \Delta \bar{H}_\alpha + m \frac{\text{sn}'(r)}{\text{sn}(r)} |\nabla \bar{H}_\alpha| \right).
Denote by $\bar{\lambda}_{i,\alpha}$ Laplace Robin eigenvalues of geodesic ball $\bar{B}_0(R)$ and by $\phi_i$ the associated orthogonal eigenfunctions, satisfying either $\phi_i(\bar{\partial}) = 0$ or $\phi_i$ is radial (rewritten as $\phi_i(r_0(x)))$, see [6, Lemma 7]. Then the Sturm-Liouville decomposition (1.7) gives

$$\bar{H}_\alpha(r_0(x), t) = \sum_{i=1}^\infty e^{-\bar{\lambda}_{i,\alpha} t} \phi_i(\bar{\partial}) \phi_i(r_0(x)) = \sum_\lambda e^{-\lambda t} \phi_\lambda(0) \phi_\lambda(r),$$

where the summation is taken over all $\lambda$ such that $\phi_\lambda(0) \neq 0$. Then direct calculation gives

$$\Delta \bar{H}_\alpha + \frac{m \text{sn}'(r)}{\text{sn}(r)} |\nabla \bar{H}_\alpha| = \sum_\lambda e^{-\lambda t} \phi_\lambda(0) \left( -\lambda \phi_\lambda(r) + m \frac{\text{sn}'(r)}{\text{sn}(r)} |\phi'_\lambda(r)| \right).$$

We choose $\phi_\lambda > 0$. Noticing

$$-\lambda \phi_\lambda = \Delta \phi_\lambda = \phi''_\lambda + (m - 1) \frac{\text{sn}'(r)}{\text{sn}(r)} \phi'_\lambda$$

and $\phi'_\lambda(0) = 0$, we have $\phi''_\lambda(0) < 0$, so $\phi_\lambda(r) < 0$ for $r$ near zero. Then we conclude

$$\Delta \bar{H}_\alpha + \frac{m \text{sn}'(r)}{\text{sn}(r)} |\nabla \bar{H}_\alpha| = \sum_\lambda e^{-\lambda t} \phi_\lambda(0) \left( -\lambda \phi_\lambda(r) - m \frac{\text{sn}'(r)}{\text{sn}(r)} \phi'_\lambda(r) \right)$$

for all $r$ near 0. Therefore applying Proposition 4.1, we get

$$\varphi''(0, t) = \lim_{r \to 0} \sum_\lambda e^{-\lambda t} \phi_\lambda(0) \frac{-\lambda \phi_\lambda(r) - m \frac{\text{sn}'(r)}{\text{sn}(r)} \phi'_\lambda(r)}{\text{sn}^2_\lambda(r)}$$

$$= \sum_\lambda e^{-\lambda t} \phi^2_\lambda(0) \frac{\lambda(\lambda - \kappa m)}{m(m + 2)} > 0,$$

where we used Lemma 2.1 if $\kappa > 0$. We complete the proof of the lemma. \hfill \Box

**Lemma 4.3.** For $t > 0$, $0 < s < s(R)$, we have

$$\varphi''(s, t) > 0.$$  \hfill (4.30)

**Proof.** Recall from the asymptotic expansion (4.12) for $\varphi$ that

$$\varphi''(s, t) \sim (4\pi t)^{-m/2} \exp(-\frac{s}{4t}) \left( \frac{1}{16t^2} \sum_{i=0}^\infty b_i(s)t^i - \frac{1}{2t} \sum_{i=0}^\infty b'_i(s)t^i + \sum_{i=0}^\infty b''_i(s)t^i \right)$$

near $s = 0$ as $t \to 0$, then there exists $\varepsilon_0 > 0$ such that

$$\varphi''(s, t) > 0$$  \hfill (4.31)

for all $(s, t) \in [0, \varepsilon_0] \times (0, \varepsilon_0)$. Let

$$\varphi_2(s, t) = e^{(2m+2)t} \varphi''(s, t),$$

where we used Lemma 2.1 if $\kappa > 0$. We complete the proof of the lemma. \hfill \Box
Using equation (4.9) and the assumption $R \phi''$, we have

\[ \partial_t \phi_2(s, t) = \text{sn}^2_k(r) \phi_2''(s, t) + (m + 4) \text{sn}'_k(r) \phi_2'(s, t). \]  

(4.32)

For any $T > \varepsilon_0$, denote

\[ I_T := [0, s(R)] \times (0, T] - [0, \varepsilon_0] \times (0, \varepsilon_0]. \]

Using integration by parts and $\phi_2(\varepsilon_0, t) > 0$ for $t < \varepsilon_0$, we calculate that

\[
\int_{I_T} \phi_2'(s, t) \phi_2'(s, t) \text{sn}^{m+4}_k(r) \, ds \, dt \\
= \int_{\varepsilon_0}^T \int_0^{s(R)} \phi_2'(s, t) \phi_2'(s, t) \text{sn}^{m+4}_k(r) \, ds \, dt + \int_0^{\varepsilon_0} \int_{0}^{s(R)} \phi_2'(s, t) \phi_2'(s, t) \text{sn}^{m+4}_k(r) \, ds \, dt \\
= \int_0^T \phi_2'(s(R), t) \phi_2(s(R), t) \text{sn}^{m+4}_k(R) \, dt - \int_{I_T} \phi_2(s, t) \phi_2'(s, t) \text{sn}^{m+4}_k(r) \, ds \, dt \\
- \int_{I_T} (m + 4) \phi_2(s, t) \phi_2'(s, t) \text{sn}^m_k(r) \text{sn}^{m+2}_k(r) \, ds \, dt.
\]

(4.33)

Using equation (4.9) and the assumption $R \leq \frac{1}{\sqrt{\kappa}} \arctan \frac{\alpha}{\sqrt{\kappa}}$, we estimate that

\[
\phi_2'' \phi_2 = -\frac{\phi_2}{\text{sn}^2_k} \left( \left((m + 2) \text{sn} \text{sn}' + \alpha \text{sn}^2_k\right) \phi_2 + e^{(2m+2)\kappa t} m (\alpha \text{sn}' - \kappa \text{sn}) \phi' \right) \\
\leq - \frac{(m + 2) \text{sn} \text{sn}^2_k + \alpha \text{sn}^2_k}{\text{sn}^3_k} \phi_2^2
\]

for $r = R$, where in the inequality we used $\phi_2 \leq 0$ and $\phi'(s, t) < 0$. Then equality (4.33) yields

\[
\int_{I_T} \phi_2'(s, t) \phi_2'(s, t) \text{sn}^{m+4}_k(r) \, ds \, dt \\
\leq - \int_{I_T} \phi_2'' \phi_2 \text{sn}^{m+4}_k - (m + 4) \phi_2' \text{sn} \text{sn}^{m+2}_k \, ds \, dt \\
= - \int_{I_T} \phi_2 \partial_t \phi_2 - (m + 4) \text{sn}' \phi_2' \text{sn}^{m+4}_k + (m + 4) \phi_2' \text{sn} \text{sn}^{m+2}_k \, ds \, dt \\
= - \int_{I_T} \phi_2(s, t) \partial_t \phi_2(s, t) \text{sn}^{m+2}_k(r) \, ds \, dt \\
= - \frac{1}{2} \int_0^{s(R)} \phi_2^2(s, T) \text{sn}^{m+2}_k \, ds,
\]
where we used equation (4.32) in the first equality, \( \bar{\varphi}_2(s, \varepsilon_0) = 0 \) for \( s < \varepsilon_0 \), and \( \varphi_2(s, 0) = 0 \) for \( s > \varepsilon_0 \) from the initial condition in the last equality. Since

\[
\bar{\varphi}_2'(s, t) \varphi_2'(s, t) = (\varphi_2'(s, t))^2,
\]

we conclude

\[
\int_{I_T} (\bar{\varphi}_2'(s, t))^2 \, ds \, dt \leq 0
\]

implying \( \bar{\varphi}_2 \equiv 0 \), so

\[
\bar{\varphi}''(s, t) \geq 0
\]

for \( s < s(R) \) and \( t > 0 \). Then \( \varphi''(s, t) > 0 \) follows from the strong maximum principle for the heat equation (4.8).

5. Proof of Theorem

\textbf{Proposition 5.1.} Let \( M \) be a compact manifold with smooth boundary, \( o \in M \) and \( H_\alpha(o, y, t) \) be the \( o \)-centered Robin heat kernel of \( M \). Assume \( F(y, t) \) is a function defined on \( M \times (0, \infty) \) satisfying \( F(y, t) \geq 0 \) for all \( (y, t) \in M \times (0, \infty) \) and \( F(y, 0) = \delta_o(y) \).

1. If \( \partial_t F - \Delta F \geq 0 \) in \( M \times (0, \infty) \), and \( \partial F/\partial \nu + \alpha F \geq 0 \) on \( \partial M \times (0, \infty) \). Then

\[
H_\alpha(o, y, t) \leq F(y, t)
\]

in \( M \times (0, \infty) \).

2. If \( \partial_t F - \Delta F \leq 0 \) in \( M \times (0, \infty) \), and \( \partial F/\partial \nu + \alpha F \leq 0 \) on \( \partial M \times (0, \infty) \). Then

\[
H_\alpha(o, y, t) \geq F(y, t)
\]

in \( M \times (0, \infty) \).

\textit{Proof of (1).} Note from Duhamel’s principle that

\[
F(y, t) - H_\alpha(o, y, t)
= \int_M F(x, t) H_\alpha(x, y, 0) \, dx - \int_M F(x, 0) H_\alpha(x, y, t) \, dx
= \int_0^t \frac{\partial}{\partial \tau} \int_M F(x, \tau) H_\alpha(x, y, t - \tau) \, dx \, d\tau
= \int_0^t \int_M F_\tau(x, \tau) H_\alpha(x, y, t - \tau) \, dx \, d\tau - \int_0^t \int_M F(x, \tau) \partial_\tau H_\alpha(x, y, t - \tau) \, dx \, d\tau,
\]
see for example [12, Page 865]. Then using the assumptions \( \partial_t F - \Delta F \geq 0 \) in \( M \times (0, \infty) \) we estimate that
\[
F(y, t) - H_\alpha(o, y, t)
\geq \int_0^t \int_M \Delta_x F(x, \tau) H_\alpha(x, y, t - \tau) \, dx \, d\tau - \int_0^t \int_M F(x, \tau) \Delta_x H_\alpha(x, y, t - \tau) \, dx \, d\tau
= \int_0^t \int_{\partial M} \frac{\partial F}{\partial \nu_x}(x, \tau) H_\alpha(x, y, t - \tau) - F(x, \tau) \frac{\partial H_\alpha}{\partial \nu_x}(x, y, t - \tau) \, d\sigma_x \, d\tau
\geq -\alpha F(x, \tau) H_\alpha(x, y, t - \tau) + \alpha F(x, \tau) H_\alpha(x, y, t - \tau) \, d\sigma_x \, d\tau
= 0,
\]
where we used the boundary condition \( \partial F/\partial \nu + \alpha F \geq 0 \) on \( \partial M \times (0, \infty) \) in the last inequality. This completes the proof of (5.1).

The proof of (5.2) is similar, we omit the detail here. \( \square \)

An immediately consequence of Proposition 5.1 is the following monotonicity result on Robin parameter for the Robin heat kernel.

**Corollary 5.1.** Let \( M \) be a compact manifold with smooth boundary \( \partial M \).

(1) Denote by \( G(x, y, t) \), \( H_\alpha(x, y, t) \) and \( K(x, y, t) \) be the Dirichlet, Robin (with positive Robin parameter \( \alpha \)) and Neumann heat kernels respectively. Then
\[
G(x, y, t) < H_\alpha(x, y, t) < K(x, y, t)
\]
in \( M \times M \times (0, \infty) \).

(2) Robin heat kernel is monotone decreasing in Robin parameter. Namely for \( 0 < \alpha_1 < \alpha_2 \), it holds
\[
H_{\alpha_2}(x, y, t) < H_{\alpha_1}(x, y, t)
\]
in \( M \times M \times (0, \infty) \).

**Proof of Theorem 1.1.** (1) Suppose \( \text{Ric} \geq (m - 1)\kappa \) on \( B_o(R) \), then we have
\[
\Delta r_\alpha(x) \leq (m - 1) \frac{sn'_\alpha(r)}{sn_\alpha(r)} \tag{5.3}
\]
for all \( x \in M \setminus \{o, C(o)\} \) by the Laplace comparison for distance function. We transplant Robin heat kernel \( \bar{H}_\alpha(r_{\bar{o}}, t) \) on the model space \( V(\bar{o}, \kappa, R) \) to the geodesic ball \( B_o(R) \) by
\[
F(x, t) := \bar{H}_\alpha(r_\alpha(x), t),
\]
for \( x \in B_o(R) \) and \( t > 0 \). Clearly \( F(x, t) > 0 \) for \( t > 0 \) and \( F(x, 0) = \delta_o(x) \). Using inequalities (4.11) and (5.3), we estimate

\[
\partial_t F(x, t) - \Delta F(x, t) = \partial_t H_\alpha(r_o(x), t) - H'_\alpha(r_o(x), t) \Delta r_o(x) - H''_\alpha(r_o(x), t)
\]

\[
\leq \partial_t H_\alpha(r_o(x), t) - (m - 1) \frac{\text{sn}'(r)}{\text{sn}(r)} H'_\alpha(r_o(x), t) - H''_\alpha(r_o(x), t)
\]

\[
= 0
\]

for all \( x \in M \setminus \{o, C(o)\} \), where the last equality follows from (4.11). By the boundary condition (4.3), it follows that \( F(x, t) \) satisfies

\[
\partial_\nu F(x, t) + \alpha F(x, t) = \bar{H}'_\alpha(R, t) + \alpha \bar{H}_\alpha(R, t) = 0
\]

for \( x \in \partial B_o(R) \). Since the cut locus is a null set, standard argument via approximation shows that \( \bar{H}(r_o(x), t) \) satisfies

\[
\begin{cases}
\partial_t F(x, t) - \Delta F(x, t) \leq 0, & (x, t) \in B_o(R) \times (0, \infty), \\
\partial_\nu F(x, t) + \alpha F(x, t) = 0, & (x, t) \in \partial B_o(R) \times (0, \infty),
\end{cases}
\]

in the distributional sense. It then follows from part (2) of Proposition 5.1 that \( F(x, t) \leq H_\alpha(o, x, t) \), proving

\[
\bar{H}_\alpha(r_o(x), t) \leq H_\alpha(o, x, t)
\]

for \( (x, t) \in B_o(R) \times (0, \infty) \).

(2) Suppose \( \text{Sect} \leq \kappa \) on \( B_o(R) \). Then we have \( \Delta r(x) \geq (m - 1) \frac{\text{sn}'(r)}{\text{sn}(r)} \) for all \( x \in M \setminus \{o, C(o)\} \) by the Laplace comparison for distance function. Same argument as in the proof of (1) show that

\[
\begin{cases}
\partial_t F(x, t) - \Delta F(x, t) \geq 0, & (x, t) \in B_o(R) \times (0, \infty), \\
\partial_\nu F(x, t) + \alpha F(x, t) = 0, & (x, t) \in \partial B_o(R) \times (0, \infty),
\end{cases}
\]

in the distributional sense. The desired estimate \( \bar{H}_\alpha(r_o(x), t) = F(x, t) \geq H_\alpha(o, x, t) \) holds from part (1) of Proposition 5.1.

\[
\square
\]

6. Proof of Theorem 1.4

Let \( M \) be an \( m \)-dimensional compact minimal submanifolds of \( N^n \). We denote by \( \nabla^M \) and \( \nabla^N \) the covariant derivative on \( M \) and \( N \) respectively, and by \( \Delta_M \) and \( \Delta_N \) the Laplace-Beltrami operator on \( M \) and \( N \) respectively. We recall the following Laplace comparison for distance function \( d_N(o, x) \), see for example [15] Lemma 1 and [18] Lemma 2.1.
Proposition 6.1. Let $M^m$ be an $m$-dimensional minimally immersed submanifold of $N^n$. Suppose the sectional curvature of the ambient space $N$ is bounded from above by $\kappa$ from above. Let $p \in M$, and $\Omega_p(R)$ be any smooth connected component of $B_p(R) \cap M$. If $\kappa > 0$, assume further that $R \leq \min\{i_N(p), \frac{1}{\sqrt{\kappa}}\}$. Let $F : [0, R] \rightarrow \mathbb{R}$ be a smooth function with $F' \geq 0$. Then

$$\Delta_M F(r) \geq \left(F''(r) - \frac{\frac{r'}{\kappa} F'(r)}{\frac{r}{\kappa}} F'(r)\right) |\nabla^M r|^2 + m F'(r) \frac{\frac{r'}{\kappa} F'(r)}{\frac{r}{\kappa}}$$

(6.1)
on $\Omega_p(R) \setminus p$. Where $i_N(p)$ is the injectivity radius of $N$ from $p$ and $r(x) := d_N(o, x)$.

Proof of Theorem 1.4. We transplant the Robin heat kernel $\bar{H}(r_o, t)$ on the model space $V(o, \kappa, R)$ to the extrinsic geodesic ball $D_o(R)$ of radius $R$ centered at $o$ by

$$v(x, t) := \bar{H}_o(r, t) = \varphi(s(r), t),$$

where $s = s(r)$ defined by (6.4). Clearly $v(x, t) > 0$ for $t > 0$. Note that the extrinsic distance function $d_N(o, x)$ is asymptotic to the intrinsic distance function as $t \to 0$, hence $v(x, t) \to d_o(x)$.

Since the sectional curvature of the ambient space $N$ is bounded from above by $\kappa$ and $s'(r) = \frac{r}{\kappa}(r) > 0$, then applying Proposition 6.1 we have

$$\Delta_M s(r) \geq \left(s''(r) - \frac{\frac{r'}{\kappa} s'(r)}{\frac{r}{\kappa}} s'(r)\right) |\nabla^M r|^2 + m s'(r) \frac{\frac{r'}{\kappa} s'(r)}{\frac{r}{\kappa}} = m \frac{r'}{\kappa} s'(r).$$

(6.2)

Direct calculation gives

$$\frac{\partial}{\partial t} v - \Delta_M v = \frac{\partial}{\partial t} \varphi - \varphi'(s, t) \Delta_M s - \varphi''(s, t) |\nabla^M s|^2$$

$$\geq \frac{\partial}{\partial t} \varphi - m \frac{r'}{\kappa} s'(r) \varphi'(s, t) - \varphi''(s, t) \frac{r'}{\kappa} s'(r) |\nabla^M d_N(o, x)|^2$$

$$\geq \frac{\partial}{\partial t} \varphi - m \frac{r'}{\kappa} s'(r) \varphi'(s, t) - \varphi''(s, t) \frac{r'}{\kappa} s'(r)$$

$$= 0,$$

(6.3)

where we used inequality (6.2) and $\varphi'(s, t) < 0$ in the first inequality, $\varphi''(s, t) > 0$ and $|\nabla^M d_N(o, x)| \leq |\nabla^N d_N(o, x)| = 1$ in the second inequality, and the PDE (1.5) of $\varphi$ in the last equality.

For $x \in \partial M$, using $\varphi' < 0$ and $|\nabla^M d_N(o, x)| \leq 1$, we estimate

$$\frac{\partial v}{\partial r} = \varphi'(s(R), t) \frac{r'}{\kappa} s'(R) (\nabla^M d_N(o, x), \nu)$$

$$\geq \varphi'(s(R), t) \frac{r'}{\kappa} s'(R)$$

$$= - \alpha \varphi(s(R), t),$$

where we used boundary condition (1.6) in the last inequality. Thus

$$\frac{\partial v}{\partial r} + \alpha v \geq 0,$$

(6.4)
for \((x, t) \in \partial M \times (0, \infty)\). In view of (6.3) and (6.4), part (1) of Proposition 5.1 gives
\[
H_\alpha(o, x, t) \leq v(x, t),
\]
proving Theorem 1.4. 

7. Proof of Theorem 1.7

Proof of Theorem 1.7. Let \(\bar{\lambda}\) be the first Laplace Robin eigenvalue for the geodesic ball \(V(\bar{o}, \kappa, R)\) in space form \(M^m(\kappa)\), and \(u(r, o(x))\) be a positive corresponding eigenfunction satisfying the ODE (2.1). We rewrite \(u(r)\) as \(w(s)\), where \(s(r)\) is defined by (4.4) again. Therefore equation (2.1) gives
\[
\text{sn}_\kappa^2(r)w''(s) + m \text{sn}_\kappa'(r)w'(s) = -\bar{\lambda}_1 w(s) \tag{7.1}
\]
for \(s \in [0, s(R))\) with boundary condition
\[
\text{sn}_\kappa(R)w'(s(R)) + \alpha w(s(R)) = 0. \tag{7.2}
\]
It follows from (1) of Proposition 2.1 and \(w'(s) \text{sn}_\kappa(r) = u'(r)\) that
\[
w'(s) < 0, \quad s \in (0, s(R)]. \tag{7.3}
\]
Direct calculation gives
\[
w''(s) = -\frac{1}{\text{sn}_\kappa^2(r)} \left( m \frac{\text{sn}_\kappa'(r)}{\text{sn}_\kappa(r)} u'(r) + \bar{\lambda}_1 u(r) \right), \tag{7.4}
\]
thus Lemma 2.1 and Proposition 4.1 implies
\[
\lim_{s \to 0} w''(s) > 0. \tag{7.5}
\]
Differentiating the equation (7.1) in \(s\) yields
\[
\text{sn}_\kappa^2(r)w'''(s) = - (\bar{\lambda}_1 - \kappa m) \text{sn}_\kappa(r)w'(s). \tag{7.6}
\]
Then we conclude from (7.5) and (7.6) that
\[
w''(s) > 0 \tag{7.7}
\]
for \(s \in [0, s(R)]\). In fact, if \(w''(s_0) = 0\) for some \(s_0 = s(r_0) \in [0, s(R)]\), then using equation (7.6) and \(w'(s) < 0\) we have \(w''(s_0) > 0\). Therefore (7.7) holds by (7.5).

Now we transplant the function \(w\) to \(M\) by
\[
v(x) := w\left(s(r(x))\right)
\]
for $x \in M$, where $r(x) = d_N(o, x)$. Applying Proposition 6.1, we estimate that
\[
\Delta_M v(x) = w'(s)\Delta_M s + w''(s)|\nabla^M s|^2 \\
\leq m s'_n(r)w'(s) + w''(s) s_n^2(r)|\nabla^M d_N(o, x)|^2 \\
\leq m s'_n(r)w'(s) + w''(s) s_n^2(r) \\
= -\bar{\lambda}_1 w(s),
\]
where we used inequality (7.3) in the first inequality, inequality (7.7) and $|\nabla^M d_N(o, x)| \leq 1$ in the second inequality and equation (7.1) of $w$ in the last equality. Thus
\[
-\Delta_M v(x) \geq \bar{\lambda}_1 v(x) \quad \text{in} \quad M.
\] (7.9)

For $x \in \partial M$, we estimate
\[
\frac{\partial v(x)}{\partial \nu} = w'(s) s_n(r)\langle \nabla^M r, \nu \rangle \\
\geq w'(s) s_n(r) \\
\geq -\alpha w(s),
\]
where we used (7.3) and $|\langle \nabla^M r, \nu \rangle| \leq 1$ in the first inequality, and (2) of Proposition 2.1 in the last inequality. Thus
\[
\frac{\partial v(x)}{\partial \nu} + \alpha v(x) \geq 0 \quad (7.10)
\]
on $\partial M$. Then we conclude from inequalities (7.9), (7.10) and Barta’s inequality (see for example [13, Theorem 3.1]) that
\[
\lambda_{1, \alpha}(M) \geq \bar{\lambda}_1,
\]
proving inequality (1.15). Moreover if the equality occurs, $v(x)$ is a constant multiple of the first eigenfunction. In this case, all inequalities above hold as equalities. Hence $\nabla^M d_N(o, x) = \nu$ on $\partial M$ and $|\nabla^M d_N(o, x)| = 1$ in $M$. Therefore $M$ is a minimal cone, implying $M$ is isometric to $\bar{B}_o(R)$. We complete the proof of Theorem 1.7. □

8. Kähler and Quaternion Kähler manifolds

Recall from system (1.4) that the $o$-centered Robin heat kernel of the metric ball $\bar{B}_o(R)$ of Kähler model space satisfies one dimensional equation
\[
\partial_t \bar{H}_o(r, t) - \frac{\partial^2 \bar{H}_o(r, t)}{\partial r^2} - \left( \frac{(2m-2) s_n'(r)}{s_n(r)} + \frac{s'_n(r)}{s_{4n}(r)} \right) \frac{\partial \bar{H}_o}{\partial r}(r, t) = 0. \quad (8.1)
\]
Set
\[
\varphi(s, t) = \bar{H}_o(r, t),
\]
for $s \in [0, s(R)]$, where

$$s(r) = \int_0^r \left( \frac{\text{sn}_\kappa^{2m-2}(r) \text{sn}_{4\kappa}(r)}{2m-1} \right) \frac{1}{\text{sn}_\kappa^{2m-2}(r) \text{sn}_{4\kappa}(r)} \frac{1}{2m-1} dr.$$

Let

$$\eta(r) = s'(r) = \left( \frac{\text{sn}_\kappa^{2m-2}(r) \text{sn}_{4\kappa}(r)}{2m-1} \right),$$

then equation (8.1) becomes

$$\varphi_t(s, t) = \eta^2(r) \varphi''(s, t) + 2m \eta'(r) \varphi'(s, t) \quad \text{(8.2)}$$

for $(s, t) \in (0, s(R)) \times (0, \infty)$. Differentiating equation (8.2) in $s$ yields

$$\partial_s \varphi' = \eta^2 \varphi'' + (2m + 2) \eta' \varphi'' - \frac{2m}{2m-1} \left( (2m+2)\kappa + \frac{2m-2}{2m-1} \left( \frac{\text{sn}_\kappa'}{\text{sn}_\kappa} - \frac{\text{sn}_{4\kappa}'}{\text{sn}_{4\kappa}} \right)^2 \right) \varphi'. \quad \text{(8.3)}$$

**Lemma 8.1.** Let $\alpha > 0$. Then

$$\varphi'(s, t) < 0. \quad \text{(8.4)}$$

for $s < s(R)$ and $t > 0$.

**Proof.** Similarly as in the proof of Lemma 4.1, we choose $\varepsilon_0 > 0$ such that

$$\varphi'(s, t) < 0 \quad \text{(8.5)}$$

for all $(s, t) \in [0, \varepsilon_0] \times (0, \varepsilon_0)$.

Let

$$\varphi_1(s, t) = e^{\frac{2m(2m+2)}{2m-1} \varepsilon} \varphi'(s, t),$$

and

$$\tilde{\varphi}_1(s, t) = \frac{1}{2} \left( \varphi_1(s, t) + |\varphi_1(s, t)| \right),$$

then equation (8.3) becomes

$$\partial_t \tilde{\varphi}_1 = \eta^2(r) \tilde{\varphi}_1'' + (2m + 2) \eta'(r) \tilde{\varphi}_1' - \frac{2m(2m-2)}{(2m-1)^2} \left( \frac{\text{sn}_\kappa'}{\text{sn}_\kappa} - \frac{\text{sn}_{4\kappa}'}{\text{sn}_{4\kappa}} \right)^2 \varphi_1. \quad \text{(8.6)}$$

For any $T > \varepsilon_0$, denote

$$I_T := [0, s(R)] \times (0, T] - [0, \varepsilon_0] \times (0, \varepsilon_0).$$
Using integration by parts, we have

\[
\int_{I_T} \varphi_1'(s,t) \tilde{\varphi}_1'(s,t) \eta^{m+2}(r) \, ds \, dt
\]

\[
= \int_0^T \int_0^{s(R)} \varphi_1'(s,t) \varphi_1'(s,t) \eta^{m+2}(r) \, ds \, dt + \int_0^{\varepsilon} \int_{s(R)}^{s(R)} \varphi_1'(s,t) \varphi_1'(s,t) \eta^{m+2}(r) \, ds \, dt
\]

\[
= \int_0^T \tilde{\varphi}_1(s,R) \varphi_1'(s,R,t) \eta^{m+2}(R) \, dt - \int_{I_T} \varphi''_1(s,t) \tilde{\varphi}_1(s,t) \eta^{m+2}(r) \, ds \, dt
\]

\[
- \int_{I_T} (2m+2) \varphi_1'(s,t) \tilde{\varphi}_1'(s,t) \eta'(r) \eta^{m}(r) \, ds \, dt.
\]

Noticing from the Robin boundary condition (4.6) and Lemma 3.1 that \( \varphi_1(s,R,t) < 0 \), so \( \tilde{\varphi}_1(s,R,t) = 0 \). Then using equation (8.6), we estimate

\[
\int_{I_T} \varphi_1'(s,t) \tilde{\varphi}_1'(s,t) \eta^{m+2}(r) \, ds \, dt
\]

\[
\leq - \int_{I_T} \varphi''_1(s,t) \tilde{\varphi}_1(s,t) \eta^{m+2}(r) + (2m+2) \varphi_1'(s,t) \tilde{\varphi}_1(s,t) \eta'(r) \eta^{m}(r) \, ds \, dt
\]

\[
= - \int_{I_T} \tilde{\varphi}_1[(\varphi_1)_t + \frac{2m(2m-2)}{(2m-1)^2} \left( \frac{\text{sn'}_\kappa}{\text{sn}_\kappa} - \frac{\text{sn'}_{4\kappa}}{\text{sn}_{4\kappa}} \right)^2 \varphi_1] \eta^{m}(r) \, ds \, dt
\]

\[
\leq - \int_{I_T} \tilde{\varphi}_1(s,t) (\varphi_1)_t(s,t) \eta^{m}(r) \, ds \, dt
\]

\[
= - \frac{1}{2} \int_0^{s(R)} \varphi_1^2(s,T) \eta^{m}(r) \, ds,
\]

where we have used inequality (8.5). Since

\[
\varphi_1'(s,t) \tilde{\varphi}_1'(s,t) = (\tilde{\varphi}_1'(s,t))^2,
\]

then

\[
\int_{I_T} (\tilde{\varphi}_1'(s,t))^2 \eta^{m+2}(r) \, ds \, dt \leq 0
\]

implying \( \tilde{\varphi}_1(s,t) \equiv 0 \), so

\[
\varphi_1(s,t) \leq 0
\]

for \( s < s(R) \) and \( t > 0 \). Since \( \varphi_1(s,t) \) satisfies equation (8.6), then \( \varphi'(s,t) < 0 \) for \( t > 0 \) follows by strong maximum principle. \( \square \)
Proof of Theorem 1.9. Let \( r(x) = d_M(o, x) \) be the distance function on \( M \). By the curvature assumptions, it follows that
\[
\Delta r(x) \leq 2(m - 1) \frac{\text{sn}'(r)}{\text{sn}(r)} + \frac{\text{sn}_4'(r)}{\text{sn}_4(r)} \tag{8.7}
\]
for all \( x \in M \setminus \{o, C(o)\} \) from Ni-Zheng’s comparison [17, Theorem 1.1].

We transplant Robin heat kernel \( \bar{H}_\alpha(r_o, t) \) on model space \( V(\bar{o}, \kappa, R) \) to geodesic ball \( B_o(R) \) by
\[
F(x, t) := \bar{H}_\alpha(r_o(x), t),
\]
for \( x \in B_o(R) \) and \( t > 0 \). Clearly \( F(x, t) > 0 \) for \( t > 0 \) and \( F(x, 0) = \delta_o(x) \). Applying inequality (8.7) and Lemma 8.1, we get
\[
\partial_t F(x, t) - \Delta F(x, t) = \partial_t \bar{H}_\alpha(r_o(x), t) - \bar{H}_\alpha'(r_o(x), t) \Delta r_o(x) - \bar{H}_\alpha''(r_o(x), t)
\leq \partial_t \bar{H}_\alpha(r_o(x), t) - \left(2(m - 1) \frac{\text{sn}'(r)}{\text{sn}(r)} + \frac{\text{sn}_4'(r)}{\text{sn}_4(r)}\right) \bar{H}_\alpha'(r_o(x), t) - \bar{H}_\alpha''(r_o(x), t) = 0
\]
for all \( x \in M \setminus \{o, C(o)\} \), where the last equality follows from equation (8.1). Using the Robin boundary condition, we have
\[
\partial_\nu F(x, t) + \alpha F(x, t) = \bar{H}_\alpha'(R, t) + \alpha \bar{H}_\alpha(R, t) = 0
\]
for \( x \in \partial B_o(R) \). Since the cut locus is a null set, standard argument via approximation shows that \( \bar{H}_\alpha(r_o(x), t) \) satisfies
\[
\begin{cases}
\partial_t F(x, t) - \Delta F(x, t) \leq 0, & (x, t) \in B_o(R) \times (0, \infty), \\
\partial_\nu F(x, t) + \alpha F(x, t) = 0, & (x, t) \in \partial B_o(R) \times (0, \infty),
\end{cases}
\]
in the distributional sense. It then follows from part (2) of Proposition 5.1 that \( F(x, t) \leq H_\alpha(o, x, t) \), namely
\[
\bar{H}_\alpha(r_o(x), t) \leq H_\alpha(o, x, t)
\]
for \( (x, t) \in B_o(R) \times (0, \infty) \). This completes the proof of Theorem 1.9 \( \square \)

For quaternion Kähler manifolds, we use similar arguments as in Theorem 1.9 to prove Theorem 1.10

Proof of Theorem 1.10. On quaternion Kähler model space, (1.4) gives that
\[
\frac{\partial \bar{H}_\alpha}{\partial t} - \frac{\partial^2 \bar{H}_\alpha}{\partial r^2} - \left(\frac{(4m - 4) \text{sn}'(r)}{\text{sn}(r)} + 3 \frac{\text{sn}_4'(r)}{\text{sn}_4(r)}\right) \frac{\partial \bar{H}_\alpha}{\partial r} = 0. \tag{8.8}
\]
Set \( \varphi(s, t) = \bar{H}_\alpha(r, t) \) with 
\[
\xi(r) := s'(r) = \left( \text{sn}_\kappa^{4m-4} \text{sn}_{4\kappa}^3 \right)^{\frac{1}{4m-1}},
\]
where \( r(x) = d_M(o, x) \) is the distance function of \( M \). Then equation \( (8.8) \) is equivalent to
\[
\partial_t \varphi(s, t) = \xi^2(s, t) \varphi''(s, t) + 4m \xi'(r) \varphi'(s, t)
\]
for \((s, t) \in (0, s(R)) \times (0, \infty)\). Differentiating equation \( (8.9) \) in \( s \) yields
\[
\partial_t \varphi'(s, t) = \xi^2 \varphi'(s, t) + (4m + 2) \xi' \varphi'' - \frac{4m}{4m-1} \left( (4m + 8) \kappa + \frac{12m - 12}{4m-1} \left( \frac{\text{sn}'_\kappa}{\text{sn}_\kappa} - \frac{\text{sn}'_{4\kappa}}{\text{sn}_{4\kappa}} \right)^2 \right) \varphi'.
\]
Then by the similar argument as in Lemma \( 8.1 \) we obtain
\[
\varphi'(s, t) < 0,
\]
for \( s < s(R) \) and \( t > 0 \). Then from the Laplace comparison for distance function on quaternion Kähler manifold (see \[ \text{[1, Theorem 3.2]} \])
\[
\Delta r(x) \leq \frac{(4m - 4) \text{sn}'_\kappa(r)}{\text{sn}_\kappa(r)} + \frac{3 \text{sn}'_{4\kappa}(r)}{\text{sn}_{4\kappa}(r)},
\]
and inequality \( (8.10) \), we conclude similarly as in the proof of Theorem \( 1.9 \) that
\[
\bar{H}_\alpha(r_o(x), t) \leq H_\alpha(o, x, t)
\]
for \((x, t) \in B_o(R) \times (0, \infty)\). We have completed the proof of Theorem \( 1.10 \). \( \square \)

As consequences of Theorem \( 1.9 \) and Theorem \( 1.10 \) we observe the following eigenvalue comparisons of Cheng’s type for the first Robin eigenvalue.

**Corollary 8.1.** Let \((M^m, g, J)\) be a Kähler manifold of complex dimension \( m \) whose holomorphic sectional curvature is bounded from below by \( 4\kappa \) and orthogonal Ricci curvature is bounded from below by \( 2(m - 1)\kappa \) for some \( \kappa \in \mathbb{R} \). Let \( B_o(R) \subset M \) be the geodesic ball of radius \( R \) centered at \( o \). Let \( \alpha > 0 \). Then
\[
\lambda_{1, \alpha}(B_o(R)) \leq \bar{\lambda}_1(m, \kappa, \alpha, R),
\]
where \( \bar{\lambda}_1(m, \kappa, \alpha, R) \) denotes the first eigenvalue of one-dimensional eigenvalue problem
\[
\begin{cases}
\varphi'' - \left( 2(m - 1) \frac{\text{sn}'_\kappa(r)}{\text{sn}_\kappa(r)} + \frac{\text{sn}'_{4\kappa}(r)}{\text{sn}_{4\kappa}(r)} \right) \varphi' = -\lambda \varphi, \\
\varphi'(0) = 0, \quad \varphi'(R) + \alpha \varphi(R) = 0.
\end{cases}
\]

**Corollary 8.2.** Let \((M^m, g, I, J, K)\) be a quaternion Kähler manifold of complex quaternion dimension \( m \) whose scalar curvature is bounded from below by \( 16m(m+2)\kappa \).
for some $\kappa \in \mathbb{R}$. Let $B_o(R) \subset M$ be the geodesic ball of radius $R$ centered at $o$. Let $\alpha > 0$. Then

$$\lambda_{1,\alpha}(B_o(R)) \leq \bar{\lambda}_1(m, \kappa, \alpha, R),$$

(8.13)

where $\bar{\lambda}_1(m, \kappa, \alpha, R)$ denotes the first eigenvalue of one-dimensional eigenvalue problem

$$\begin{cases}
\varphi'' - \left((4m - 4)\frac{\text{sn}'(r)}{\text{sn}(r)} + 3\frac{\text{sn}_m'(r)}{\text{sn}_m(r)}\right)\varphi' = -\lambda \varphi, \\
\varphi'(0) = 0, \quad \varphi'(R) + \alpha \varphi(R) = 0.
\end{cases}$$

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