CONTINUOUS SOLUTIONS FOR DIVERGENCE-TYPE EQUATIONS ASSOCIATED TO ELLIPTIC SYSTEMS OF COMPLEX VECTOR FIELDS

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Abstract. In this paper, we characterize all the distributions $F \in \mathcal{D}'(U)$ such that there exists a continuous weak solution $v \in C(U, \mathbb{C}^n)$ (with $U \subset \Omega$) to the divergence-type equation

$$L_1^*v_1 + \ldots + L_n^*v_n = F,$$

where $(L_1, \ldots, L_n)$ is an elliptic system of linearly independent vector fields with smooth complex coefficients defined on $\Omega \subset \mathbb{R}^N$. In case where $(L_1, \ldots, L_n)$ is the usual gradient field on $\mathbb{R}^N$, we recover the classical result for the divergence equation proved by T. De Pauw and W. Pfeffer.

1. Introduction

Recently a series of new results on the classical divergence equation have been published. In the original paper due to J. Bourgain and H. Brezis [BB1] the authors presented new developments for the solvability of the equation

$$(1) \quad \text{div} \, v = F,$$

when $F \in L^p_p(\mathbb{T}^N) = \{ f \in L^p(\mathbb{T}^N) \mid \int_{\mathbb{T}^N} f = 0 \}$, in the special limiting case $p = N$. A surprising result [BB1 Theorem 1'] asserts that for every $f \in L^N_p(\mathbb{T}^N)$ there exists a continuous solution of $\text{(1)}$.

Concerning continuous solutions to $\text{(1)}$ in the whole Euclidean space, T. de Pauw and W. Pfeffer [DPP] characterized the (real) distributions $F$ for which the equation $\text{(1)}$ has a continuous solution, i.e. there exists $v \in C(\mathbb{R}^N, \mathbb{R}^N)$ such that the following holds:

$$F(\varphi) = -\int_{\mathbb{R}^N} v \cdot \nabla \varphi,$$

for every test function $\varphi \in \mathcal{D}(\mathbb{R}^N)$. They show such distributions are exactly the ones satisfying a particular continuity property: for each $\varepsilon > 0$ there should exist a constant $\theta > 0$ such that one has:

$$(2) \quad |F(\varphi)| \leq \theta \| \varphi \|_1 + \varepsilon \| \nabla \varphi \|_1,$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$ supported in the ball centered at the origin with radius $1/\varepsilon$. As a particular case, they show that (the distribution associated to) any function $F \in L^N(\mathbb{R}^N)$ enjoys that property, so that in particular $\text{(1)}$ is continuously solvable for all $F \in L^N(\mathbb{R}^N)$.

Integral estimates in $L^1$ norm like $\text{(2)}$ have been studied in several settings, among which div-curl and elliptic-canceling operators, measure and divergence-free vector fields, nilpotent groups, CR complexes and applications to fluid dynamics. We refer to [VS5] for an overview and development of these subjects.

The results obtained previously for $\text{(1)}$ are closely related to the gradient $\nabla$ generated by the canonical vector fields $L_j = \partial_{x_j}$ for $j = 1, \ldots, N$. Suppose now that $\mathcal{L} := \{L_1, \ldots, L_n\}$ is a system of linearly independent vector fields with smooth complex coefficients defined on an open set $\Omega \subset \mathbb{R}^N$. Analogously,
we may consider the gradient associated to the system $L$ defined by $\nabla L u := (L_1 u, \ldots, L_n u)$, for $u \in C^\infty(\Omega)$ and its formal complex adjoint operator

$$\text{div}_L^*: v := \sum_{j=1}^n L_j^* v_j, \quad v \in C^\infty(\Omega, \mathbb{C}^n),$$

which are precisely the operators $\nabla$ and $\text{div}$ when $n = N$ and $L_j = \partial_{x_j}$. We use the notation $L_j^* := \overline{L_j}$ where $\overline{L_j}$ denotes the vector field obtained from $L_j$ by conjugating its coefficients and $L_j^*$ is the formal transpose of $L_j$ for $j = 1, \ldots, n$ — namely this means that, for all (complex valued) $\varphi, \psi \in \mathcal{D}(\Omega)$, we have:

$$\int_\Omega (L_j \varphi) \overline{\psi} = \int_\Omega \varphi L_j^* \psi.$$

The following version of the $L^1$ Sobolev-Gagliardo-Nirenberg theorem associated to $\nabla L$ was proved in [HPT]

**Theorem 1.1.** Assume that the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is linearly independent and elliptic. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that

$$\|\varphi\|_{L^{N/N-1}_U} \leq C \sum_{j=1}^n \|L_j \varphi\|_{L^1}, \quad \forall \varphi \in \mathcal{D}(U),$$

holds for $C = C(U) > 0$. Conversely, if (4) holds then the system must be elliptic on $U$.

In this work we are interested to study the (local) continuous solvability of the equation:

$$\text{div}_L^* v = F.$$

Our main result is the following.

**Theorem 1.2.** Assume that the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is linearly independent and elliptic. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for any $F \in \mathcal{D}'(U)$, the equation (5) is continuously solvable in $U$ if and only if $F$ is an $\mathcal{L}$-charge in $U$, meaning that for every $\varepsilon > 0$ and every compact set $K \subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that one has:

$$|F(\varphi)| \leq C\|\varphi\|_1 + \varepsilon \|\nabla L \varphi\|_1,$$

for all $\varphi \in \mathcal{D}_K(U)$ — the latter being the set of all smooth functions in $U$ supported inside $K$.

One simple argument (see Section 3) shows that the above continuity property on $F$ is a necessary condition for the continuous solvability of equation (5) in $U$. Theorem 1.2 asserts that the continuity property (6) is also sufficient, under the ellipticity assumption on the system of vector fields.

The organization of the paper is as follows. In Section 2, we study some properties of elliptic systems of complex vector fields. Section 3 is devoted to the definition and some properties of the space $BV_{\mathcal{L},c}$ of functions with bounded $\mathcal{L}$-variation. In Section 4, we discuss linear functionals on $BV_{\mathcal{L},c}$ called $\mathcal{L}$-charges.

The proof of our main result is presented in Section 5. The Appendix is concerned with technical results on pseudodifferential operators, mainly on their boundedness and compactness.

**Notations.** We always denote by $\Omega$ an open set of $\mathbb{R}^N$, $N \geq 2$. Unless otherwise specified, all functions are complex valued and the notation $\int_A f$ stands for the Lebesgue integral $\int_A f(x) dx$. As usual, $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are the spaces of complex test functions and distributions, respectively. When $K \subset \subset \Omega$ is a compact subset of $\Omega$, we let $\mathcal{D}_K(\Omega) := \mathcal{D}(\Omega) \cap \mathcal{E}'(K)$, where $\mathcal{E}'(K)$ is the space of all distributions with compact support in $K$. Since the ambient field is $\mathbb{C}$, we identify (formally) each $f \in L^1_{\text{loc}}(\Omega)$ with the distribution $T_f \in \mathcal{D}'(\Omega)$ given by $T_f(\varphi) = \int_\Omega f \varphi$. We consider $C(\Omega, \mathbb{C}^n)$ the space of all continuous vector-valued functions $v : \Omega \to \mathbb{C}^n$. We also introduce the notation $\|\nabla L \varphi\|_p := \sum_{j=1}^n \|L_j \varphi\|_p$ (where $\| \cdot \|_p$ is the standard norm in $L^p(\Omega)$) for $1 \leq p \leq \infty$. Finally we use the notation $f \lesssim g$ to indicate the existence of an universal constant $C > 0$, independent of all variables and unmentioned parameters, such that one has $f \leq C g$. 

2. Ellipticity and its consequences

Consider \( n \) complex vector fields \( L_1, \ldots, L_n \), \( n \geq 1 \), with smooth coefficients defined on a neighborhood \( \Omega \) of the origin in \( \mathbb{R}^N \), \( N \geq 2 \). We will assume that the vector fields \( L_1, \ldots, L_n \) do not vanish in \( \Omega \), in particular, they may be viewed as nonvanishing sections of the vector bundles \( \mathbb{C}T\Omega \) as well as first order differential operators of principal type.

In the sequel, we will always assume (unless otherwise mentioned) that the following two properties hold:

(a) \( L_1, \ldots, L_n \) are everywhere linearly independent;
(b) the system \( \{L_1, \ldots, L_n\} \) is elliptic.

The latter means for any 1-form \( \omega \) (i.e. any section of \( T^*\Omega \)), the equality \( \langle \omega, L_j \rangle = 0 \) for \( 1 \leq j \leq n \) implies that one has \( \omega = 0 \). Consequently, the number \( n \) of vector fields must satisfy \( \frac{N}{2} \leq n \leq N \). Alternatively the assumption (b) is equivalent to require that the second order operator

\[
\Delta_L := L_1^x L_1 + \cdots + L_n^x L_n = \text{div}_L \nabla_L
\]

is elliptic. Using a representation of vector fields in local coordinates \((x, 1, \ldots, x_N)\) we can assume that one has:

\[
L_j = \sum_{k=1}^N c_{jk} \partial_x^k \quad j = 1, \ldots, n,
\]

with smooth coefficients globally defined on \( \mathbb{R}^N \) that possess bounded derivatives of all orders. A simple computation implies then that one has \( L_j^x = -\sum_{k=1}^N c_{jk} \partial_x c_{jk} \); the (uniform) ellipticity means that there exists \( c > 0 \) such that one has

\[
\sum_{j=1}^n \left| \sum_{k=1}^N c_{jk}(x) \xi_k \right|^2 \geq c|\xi|^2,
\]

for all \( x, \xi \in \mathbb{R}^N \).

The second-order (elliptic) operator \( \Delta_L \) may be regarded as an elliptic pseudodifferential operator with symbol in the Hörmander class \( S^{1,0}_2(\Omega) \). Hence there exist scalar-valued properly supported pseudodifferential operators \( q(x, D) \in OpS^{1,0}_2(\Omega) \) and \( r(x, D) \in OpS^{-\infty}(\Omega) \) such that one has:

\[
\Delta_L q(x, D)f + r(x, D)f = f \in C^\infty(\Omega).
\]

Writing \( \Delta_L q(x, D)f = \text{div}_L \nabla_L u \) for \( u_j = L_j q(x, D)f \) we then get:

\[
\text{div}_L u - f = r(x, D)f
\]

for every \( f \in C^\infty(\Omega) \).

As application from the previous identity we present the following a priori estimates

**Proposition 2.1.** Assume that the system of vector fields \( L_1, \ldots, L_n \), \( n \geq 2 \), is linearly independent and elliptic. Then for every point \( x_0 \in \Omega \) and \( 0 < \beta < 1 \), there exist an open neighborhood \( U \subset \Omega \) and a constant \( C = C(U) > 0 \) such that, for all \( \varphi \in \mathcal{D}(U) \), one has:

\[
\|\varphi\|_{1-\beta, 1} := \|J^{1-\beta} \varphi\|_1 \leq C\|\nabla_L \varphi\|_1.
\]

In the above statement, the operator \( J_\alpha := J_\alpha(x, D) \) for \( \alpha > 0 \) is the pseudodifferential operator, called Bessel potential, defined by

\[
J_\alpha f(x) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} b(x, \xi) \hat{f}(\xi) d\xi, \quad f \in S'(\mathbb{R}^N),
\]

where the symbol \( b(x, \xi) = (\xi)^\alpha := (1 + 4\pi^2 |\xi|^2)^{-\alpha/2} \) independent of \( x \), belongs to the Hörmander class \( S^{1,0}_\alpha(\mathbb{R}^N) \). The operator \( J_{-\alpha} \), usually denoted by \( (1 - \Delta)^{\alpha/2} \), allows us to introduce a nonhomogeneous

\[\text{In fact, if one writes } L_j = X_j + i Y_j \text{ where } \{X_j, Y_j\} \text{ are real vector fields, then } 2n \geq N. \text{ Suppose instead that } \# \{X_j, Y_j\} = 2n < N. \text{ Then there exist } f \notin \text{span } \{X_j, Y_j\} \text{ and } \omega := d_f \neq 0 \text{ such that } \omega(L_j) = 0 \text{ for } j = 1, \ldots, n \text{ but } \omega \neq 0; \text{ that is a contradiction, since the system } \{L_1, \ldots, L_n\} \text{ is supposed to be elliptic. Clearly, on the other hand, we have } n \leq N.\]
fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ for $1 \leq p < \infty$, defined as the set of tempered distributions $u \in S'(\mathbb{R}^N)$ such that $J_{-\alpha}u \in L^p(\mathbb{R}^N)$, endowed with the norm $\|u\|_{\alpha,p} := \|J_{-\alpha}u\|_{L^p}$. As a consequence of the continuity property of the action of the Bessel potential on Lebesgue spaces (see for instance [AH, Theorem 2.5]), the inclusion $W^{\alpha,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous for all $1 \leq p < \infty$.

**Proof.** Let $h = \nabla_{\mathcal{L}} \varphi$. Thanks to identity (9) we have

$$J_{\beta-1} \varphi = p(x,D)h + r'(x,D) \varphi,$$

where $r'(x,D) = J_{\beta-1}r(x,D)$ is a regularizing operator and $p(x,D) = J_{\beta-1}(x,D)\eta_1(x,D) \div_{\mathcal{L}}$ is a vector-valued pseudodifferential operator of negative order $-\beta$. As a consequence of Theorem 6.1 we have

$$\|J_{\beta-1} \varphi\|_{L^1} \leq \|\nabla_{\mathcal{L}} \varphi\|_{L^1} + \|r'(x,D) \varphi\|_{L^1}.$$  

As the second term on the right side may be absorbed (see [HP1, p. 798]), shrinking the neighborhood if necessary, we obtain the estimate (10). \hfill \blacksquare

The boundedness in $L^1$ norm of the pseudodifferential operators with negative order follow from the integrability property of the kernel due itself to a pointwise control obtained in [AH]. Another fundamental tool from pseudodifferential operators theory, inspired in the recent results obtained in [HKP], asserts that the embedding $W^{-1,1}(\mathbb{R}^N) \subset W^{1,1}(\mathbb{R}^N) \cap \mathcal{E}'(B)$, where $B$ is a generic ball, into $L^1(\mathbb{R}^N)$ is compact. These results are stated in the Appendix and will be proved there for sake of completeness.

### 3. Functions of bounded $\mathcal{L}$-variation

Throughout this section, we consider $L_1, \ldots, L_n$ a system of complex vector fields with smooth coefficients on $\Omega$.

**3.1. Basic definitions; approximation and compactness.** Let $L^1_\mathcal{L}(\Omega)$ be the linear space of all complex functions in $L^1(\Omega)$ whose support is a compact subset of $\Omega$.

The following definition of $\mathcal{L}$-variation of $g \in L^1_\mathcal{L}(\Omega)$ recalls the classical definition of variation in case $n = N$ and $L_j = \partial x_j$ for each $j = 1, 2, \ldots, N$. It has been formulated for (real) vector fields by N. Garofalo and D. Nhieu [GN].

**Definition 3.1.** Given $g \in L^1_\mathcal{L}(\mathbb{R}^n)$ and $U \subseteq \Omega$ an open set, one calls the extended real number:

$$\|D_{\mathcal{L}}g\|(U) := \sup \left\{ \int_\Omega g \div_{\mathcal{L}} \nu : \nu \in C_c^\infty(\Omega, \mathbb{C}^n), \supp \nu \subseteq U, \|\nu\|_\infty \leq 1 \right\},$$

the (total) $\mathcal{L}$-variation of $g$ in $U$ and we let $\|D_{\mathcal{L}}g\| := \|D_{\mathcal{L}}g\|(\Omega)$ in case there is no ambiguity on the open set $\Omega$. We denote by $BV_{\mathcal{L},c}(\Omega)$ the set of all $g \in L^1_\mathcal{L}(\Omega)$ with $\|D_{\mathcal{L}}g\| < +\infty$.

Given $g \in BV_{\mathcal{L},c}(\Omega)$, we denote by $D_{\mathcal{L}}g$ the unique $\mathbb{C}^n$-valued Radon measure satisfying:

$$\int_\Omega g \div_{\mathcal{L}} \nu = \int_\Omega \nu \cdot [D_{\mathcal{L}}g],$$

for all $\nu \in C_c^\infty(\Omega, \mathbb{C}^n)$. It is clear by definition that $\|D_{\mathcal{L}}g\|$ is also the total variation in $\Omega$ of $D_{\mathcal{L}}g$.

The next proposition allows us to define a vector-valued Radon measure $D_{\mathcal{L}}g$ for any $g \in BV_{\mathcal{L},c}(\Omega)$.

**Remark 3.2.** Given $g \in BV_{\mathcal{L},c}(\Omega)$, one has supp $D_{\mathcal{L}}g \subseteq$ supp $g$. Indeed, given $x \in \Omega \setminus$ supp $g$, find a radius $r > 0$ for which one has $B(x,r) \subseteq \Omega \setminus$ supp $g$. It is clear according to (11) that for any $v \in C_c^\infty(B(x,r), \mathbb{C}^n)$ we then have $D_{\mathcal{L}}g(v) = 0$. Hence we also get $D_{\mathcal{L}}g(v) = 0$ for all $v \in C_c(B(x,r), \mathbb{C}^n)$, which ensures that one has $x \notin$ supp $D_{\mathcal{L}}g$ and finishes to show the inclusion supp $D_{\mathcal{L}}g \subseteq$ supp $g$.

**Remark 3.3.** It follows readily from the previous definition that, as in the classical case, if $(g_i) \subseteq BV_{\mathcal{L},c}(\Omega)$ converges in $L^1$ to $g \in L^1_\mathcal{L}(\Omega)$, one then has $g \in BV_{\mathcal{L},c}(\Omega)$ and:

$$\|D_{\mathcal{L}}g\| \leq \liminf_i \|D_{\mathcal{L}}g_i\|.$$

We shall refer to this in the sequel as the lower semi-continuity of the $\mathcal{L}$-variation.
We say that a sequence \((f_i)_{i}\) of functions with complex values defined on open set \(\Omega \subset \mathbb{R}^N\) is compactly supported in \(\Omega\) if there is a compact set \(K \Subset \Omega\) such that one has \(\text{supp} f_i \subseteq K\) for every \(i\).

We shall make an extensive use of the following concept of convergence.

**Definition 3.4.** Given \(g \in L^1_\text{loc}(\Omega)\) and a sequence \((\varphi_i)_{i} \in \mathcal{D}(\Omega)\) we shall write \(\varphi_i \rightharpoonup g\) in case the following conditions hold:

(i) \((\varphi_i)\) converges to \(g\) in \(L^1\) norm;
(ii) \((\varphi_i)\) is compactly supported in \(\Omega\);
(iii) \(\sup_i \|\nabla \varphi_i\|_1<+\infty\).

Using a Friedrich’s type decomposition due to N. Garofalo and D. Nhieu \([\text{GN}\, \text{Lemma A.3}]\) in the real case, we obtain an analogous result, in BV\(_{\L,\epsilon}\), to the standard approximation theorem for BV\(_{\L}\) functions.

**Lemma 3.5.** Assume that \(L_1,\ldots,L_n\) have locally Lipschitz coefficients. For any \(g \in BV_{\L,\epsilon}(U)\), there exists a sequence \((\varphi_i)\subset \mathcal{D}(U)\) such that one has \(\varphi_i \rightharpoonup g\) and, moreover:

\[\|D_{\L} g\| = \lim_i \|\nabla \varphi_i\|_1.\]

**Proof.** Fix \(\eta \in \mathcal{D}(\mathbb{R}^n)\) a radial function with nonnegative values, satisfying \(\text{supp} \eta \subseteq B[0,1]\) and \(\int \eta = 1\), and, for each \(\epsilon > 0\), define \(\eta_\epsilon \in \mathcal{D}(\mathbb{R}^n)\) by \(\eta_\epsilon(x) := \epsilon^{-n} \eta(x/\epsilon)\).

Fix now \(g \in BV_{\L,\epsilon}(\Omega)\) and define for \(0<\epsilon<\text{dist}(\text{supp} g, \partial \Omega)\) a function \(g_\epsilon \in \mathcal{D}(\Omega)\) by the formula:

\[g_\epsilon := \eta_\epsilon \ast g.\]

For each \(i = 1,\ldots,n\), denote by \(D_{L_i} g\) the compactly supported distribution defined by:

\[D_{L_i} g(\varphi) := \int_{\Omega} g \overline{L_i \varphi},\]

let \(D_{\L} g\) denote the vector-valued distribution \((D_{L_1} g,\ldots,D_{L_n} g)\) and observe that according to N. Garofalo and D. Nhieu \([\text{GN}\, \text{Lemma A.3}]\), one can write:

\[\nabla_{\L} (\eta_\epsilon \ast g) = \eta_\epsilon \ast (D_{\L} g) + H_\epsilon(g),\]

where also \(\|H_\epsilon(g)\|_1 \to 0\), \(\epsilon \to 0\).

Fix now \(v \in C^\infty_c(\Omega, \mathbb{C}^n)\) a smooth vector field satisfying \(\|v\|_\infty \leq 1\) and compute:

\[
\left| \int_{\Omega} \nabla_{\L} (\eta_\epsilon \ast g) \cdot \overline{v} \right| \leq \left\| \sum_{i=1}^n D_{L_i} g(\eta_\epsilon \ast v_i) \right\| + \|H_\epsilon(g)\|_1
\]

\[= \left| \int_{\Omega} g \text{div}_{\L}^\epsilon(\eta_\epsilon \ast v) \right| + \|H_\epsilon(g)\|_1 \leq \|D_{\L} g\| + \|H_\epsilon(g)\|_1.\]

We hence get, by duality:

\[\|\nabla_{\L} (\eta_\epsilon \ast g)\|_1 \leq \|D_{\L} g\| + \|H_\epsilon(g)\|_1,\]

and the result follows from the aforementioned property of \(H_\epsilon(g)\) when \(\epsilon\) approaches 0. \(\blacksquare\)

The following proposition is a compactness result in BV\(_{\L}\).

**Proposition 3.6.** Assume that the open set \(U \subseteq \Omega\) supports a Sobolev-Gagliardo-Nirenberg inequality of type \([3]\) as well as an inequality of type \([10]\) for some \(\epsilon > 0\). If \((g_i) \in BV_{\L,\epsilon}(U)\) is compactly supported in \(U\) and if moreover one has:

\[\sup_i \|D_{\L} g_i\| < +\infty,\]

then there exists \(g \in BV_{\L,\epsilon}(U)\) and a subsequence \((g_{i_k}) \subseteq (g_i)\) converging to \(g\) in \(L^1\) norm.

**Proof.** Choose a compact set \(K \Subset U\) for which one has \(\text{supp} g_i \subseteq K\) for all \(i\), and let \(\chi \in \mathcal{D}(U)\) be such that \(\chi K \subseteq \chi \subseteq 1\) on \(U\). Choose also, according to Lemma 3.3 a sequence \((\varphi_i) \subseteq \mathcal{D}(U)\) satisfying the following conditions for all \(i\):

\[\|g_i - \varphi_i\| \leq 2^{-k} \quad \text{and} \quad \|\nabla_{\L} \varphi_i\|_1 \leq \|D_{\L} g_i\| + 1.\]
Define now, for each $i$, $\psi_i := \varphi_i \chi$ and compute using Hölder’s inequality together with \eqref{eq:holder}:

\[ \|\nabla \psi_i\|_1 \leq \|\varphi_i\|_{N/N-1} \|\nabla \chi\|_N + \|\nabla \varphi_i\|_1 \leq (C\|\nabla \chi\|_N + 1)\|\nabla \varphi_i\|_1. \]

We hence have $\sup_i \|\nabla \psi_i\|_1 < +\infty$ while it is clear that $(\psi_i)$ is compactly supported and satisfies $\|g_i - \psi_i\|_1 \to 0$, $i \to \infty$.

Now fix $0 < \beta < 1$ and observe that the sequence $(\psi_i)_i$ also satisfies, according to \eqref{eq:holder}:

\[ \sup_i \|\psi_i\|_{1-\beta,1} = \sup_i \|J_{\beta-1} \psi_i\|_1 \leq C \sup_i \|\nabla \psi_i\|_1 < +\infty. \]

It hence follows from the compactness of the inclusion of $W^{1-\beta,1}_c(U) \subset L^1(U)$ (see Theorem 6.2 in Appendix) that there exists $g \in L^1(U)$ and a subsequence $(\psi_{i_k}) \subset (\varphi_i)$ converging to $g$ in $L^1(U)$. On the other hand it is clear that one has $\sup g \leq K$ as well as $\psi_{i_k} \to g$. We hence have, by lower semicontinuity:

\[ \|D_L g\| \leq \liminf_k \|\nabla \psi_{i_k}\|_1, \]

which ensures that one has $g \in BV_{L,c}(U)$.

\[ \square \]

Remark 3.7. According to Theorem \ref{thm:main} and Proposition \ref{prop:subsequence} we see that if one assumes $L_1, \ldots, L_n$ to be everywhere linearly independent and elliptic, each point $x_0 \in \Omega$ is contained a neighborhood $U \subseteq \Omega$ satisfying the hypotheses of the previous proposition.

3.2. A Sobolev-Gagliardo-Nirenberg inequality in $BV_L$. As announced we get the following result:

**Proposition 3.8.** Assume that the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is linearly independent and elliptic. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subseteq \Omega$ such that the inequality:

\[ \|g\|_{N/N-1} \leq C\|D_L g\|, \]

holds for all $g \in BV_{L,c}(U)$, where $C = C(U) > 0$ is a constant depending only on $U$.

**Proof.** Fix $x_0 \in \Omega$. It follows from Theorem \ref{thm:main} that there exists a neighborhood $U \subset \Omega$ of $x_0$ and $C = C(U) > 0$ such that, for all $\varphi \in \mathcal{D}(U)$, one has:

\[ \|\varphi\|_{N/N-1} \leq C\|\nabla \varphi\|_1. \]

Then given $g \in BV_{L,c}(U)$ consider the sequence $\{\varphi_i\} \subset \mathcal{D}(U)$ satisfying (i)-(iii) by Lemma \ref{lem:sequence}. As a consequence of Fatou Lemma and the previous estimate we conclude that

\[ \|g\|_{N/N-1} \leq \lim_{i \to \infty} \|\varphi_i\|_{N/N-1} \leq C \lim_{i \to \infty} \|\nabla \varphi_i\|_1 \leq C\|D_L g\|. \]

The proof is complete. \[ \square \]

Remark 3.9. The converse of proposition is true, namely if the inequality \eqref{eq:sobolev} holds then the system must be elliptic on $U$ (see \cite{HP1} for details).

4. $L$-charges and their extensions to $BV_{L,c}$

We now get back to the original problem of finding, locally, a continuous solution to \eqref{eq:system}.  

4.1. $L$-fluxes and $L$-charges. Distributions which allow, in an open set $\Omega$, to solve continuously \eqref{eq:system}, will be called $L$-fluxes.

**Definition 4.1.** A distribution $F \in \mathcal{D}'(\Omega)$ is called an $L$-flux in $\Omega$ if the equation \eqref{eq:system} has a continuous solution in $\Omega$, i.e. if there exists $v \in C(\Omega, \mathbb{C}^n)$ such that one has, for all $\varphi \in \mathcal{D}(\Omega)$:

\[ F(\varphi) = \int_\Omega \overline{v} \cdot \nabla \varphi, \quad \forall \varphi \in \mathcal{D}(\Omega). \]

$L$-fluxes satisfy the following continuity condition.

**Lemma 4.2.** If $F$ is an $L$-flux then $\lim_i F(\varphi_i) = 0$ for every sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega)$ verifying $\varphi_i \to 0$.  

Proof. Let $F$ be an $\mathcal{L}$-flux and let $v \in C(\Omega, \mathbb{C}^n)$ be such that (14) holds. Fix a sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega)$ verifying $\varphi_i \to 0$, let $c := \sup_i \|D_L \varphi_i\|_1 < +\infty$ and choose a compact set $K \subset \subset \Omega$ for which one has $\text{supp} \varphi_i \subseteq K$ for all $i$.

Fix now $\varepsilon > 0$. According to Weierstrass’ approximation theorem, choose a vector field $w \in C^\infty_c(\Omega, \mathbb{C}^n)$ for which one has $\sup_K |v - w| \leq \varepsilon$ and compute, for all $i$:

$$|F(\varphi_i)| \leq \int_\Omega (\bar{\varphi} - \bar{w}) \cdot \nabla_L \varphi_i + \int_\Omega \bar{w} \cdot \nabla_L \varphi_i \leq \|\nabla_L \varphi_i\|_1 + \int_\Omega \bar{\varphi_i} |\nabla L \cdot w| \leq \varepsilon + \|\nabla L \cdot w\|_\infty \|\varphi_i\|_1.$$ 

We hence get $\lim_i |F(\varphi_i)| \leq \varepsilon$, and the result follows for $\varepsilon > 0$ is arbitrary.

The property above suggest the following definition of linear functionals associated to $\mathcal{L}$.

**Definition 4.3.** A linear functional $F : \mathcal{D}(\Omega) \to \mathbb{C}$ is called an $\mathcal{L}$-charge in $\Omega$ if $\lim F(\varphi_i) = 0$ for every sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega)$ satisfying $\varphi_i \to 0$. The linear space of all $\mathcal{L}$-charges in $\Omega$ is denoted by $CH_\mathcal{L}(\Omega)$.

The following characterization of $\mathcal{L}$-charges will be useful in the sequel.

**Proposition 4.4.** If $F : \mathcal{D}(\Omega) \to \mathbb{C}$ is a linear functional, then the following properties are equivalent

(i) $F$ is an $\mathcal{L}$-charge,

(ii) for every $\varepsilon > 0$ and each compact set $K \subset \subset \Omega$ there exists $\theta > 0$ such that, for any $\varphi \in \mathcal{D}_K(\Omega)$, one has:

$$|F(\varphi)| \leq \theta \|\varphi\|_{L^1} + \varepsilon \|\nabla L \varphi\|_1.$$ 

**Proof.** We proceed as in [DPP, Proposition 2.6].

Since (ii) implies trivially (i), it suffices to show that the converse implication holds. To that purpose, assume (i) holds, i.e. suppose that $F$ is an $\mathcal{L}$-charge. Fix $\varepsilon > 0$ and a compact set $K \subset \subset \Omega$. By hypothesis, there exists $\eta > 0$ such that for every $\varphi \in \mathcal{D}_K(\Omega)$ satisfying $\|\varphi\|_1 \leq \eta$ and $\|D_L \varphi\|_1 \leq 1$, we have $|F(\varphi)| \leq \varepsilon$.

We now define $\theta := \varepsilon/\eta$.

Fix now $\varphi \in \mathcal{D}_K(\Omega)$ and assume by homogeneity that one has $\|\nabla_L \varphi\|_1 = 1$. If moreover one has $\|\varphi\|_1 \leq \eta$, then one computes $|F(\varphi)| \leq \varepsilon = \|\nabla L \varphi\|_1$. If on the contrary we have $\|g\|_{L^1} > \eta$, we define $\tilde{\varphi} = \varphi \eta/\|\varphi\|_1$. We then have $\|\tilde{\varphi}\|_1 = \eta$ as well as $\|\nabla_L \varphi\|_1 < 1$, and hence also $|F(\tilde{\varphi})| \leq \varepsilon$; this yields finally $|F(\varphi)| = \|\varphi\|_1|f(\tilde{\varphi})|/\eta \leq \varepsilon \|\varphi\|_1/\eta = \theta \|\varphi\|_1$.

As we shall see now, $\mathcal{L}$-charges can be extended in a unique way to linear forms on $BV_L, e^c$.

**Proposition 4.5.** An $\mathcal{L}$-charge $F$ in $\Omega$ extends in a unique way to a linear functional $\tilde{F} : BV_L c^e(\Omega) \to \mathbb{C}$ satisfying the following property: for any $\varepsilon > 0$ and each compact set $K \subset \subset \Omega$, there exists $\theta > 0$ such that for any $g \in BV_L, c^e(\Omega)$ one has:

$$|\tilde{F}(g)| \leq \theta \|g\|_{L^1} + \varepsilon \|D_L g\|.$$ 

**Proof.** Given $g \in BV_L c^e(\Omega)$, fix $(\varphi_i)_i \subseteq \mathcal{D}(\Omega)$ satisfying $\varphi_i \to g$ and observe that it follows from (14) that $(F(\varphi_i))_i$ is a Cauchy sequence of complex numbers whose limit does not depend on the choice of sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega)$ satisfying $\varphi_i \to g$. We hence define $\tilde{F}(g) := \lim_i F(\varphi_i)$. It now follows readily from (14) and Lemma 3.3 that $\tilde{F}$ satisfies the desired property.

**Remark 4.6.** If $\tilde{F} : BV_L c^e(\Omega) \to \mathbb{C}$ extends the $\mathcal{L}$-charge $F$, it is easy to see from the previous proposition that for any compactly supported sequence $(g_i)_i \subseteq BV_L, c^e(\Omega)$ satisfying $g_i \to 0$, $i \to \infty$ in $L^1(\Omega)$ and $\sup_i \|D_L g_i\| < +\infty$, one has $F(g_i) \to 0$, $i \to \infty$.

From now on, we shall identify any $\mathcal{L}$-charge with its extension to $BV_L, c$ and use the same notation for the two linear forms.
4.2. Two important examples of $\mathcal{L}$-charges. Let us define two important classes of $\mathcal{L}$-charges.

**Example 4.7.** In case $F$ is the $\mathcal{L}$-flux associated to $v \in C(\Omega, \mathbb{C}^n)$ according to (14), its unique extension to $BV_{\mathcal{L},e}(\Omega)$ is the $\mathcal{L}$-charge:

$$\Gamma(v) : BV_{\mathcal{L},e}(\Omega) \rightarrow \mathbb{C}, g \mapsto \int_{\Omega} \tilde{v} \cdot d[D_L g].$$

To see this, fix $g \in BV_{\mathcal{L},e}(\Omega)$ together with a sequence $(\varphi_i)_i \subseteq \mathscr{P}(\Omega)$ satisfying (i)-(iii) in Lemma 3.5 and choose $\text{supp} g \subseteq K \subseteq \Omega$ a compact set for which one has $\text{supp} \varphi_i \subseteq K$ for all $i$. Given $\varepsilon > 0$, choose $w \in C^\infty(\Omega, \mathbb{C}^n)$ a smooth vector field satisfying $\text{supp}_K |v - w| \leq \varepsilon$ and compute:

$$\left| \Gamma(v)(g) - \int_{\Omega} \tilde{v} \cdot d[D_L g] \right| = \lim_{i} \left| \int_{\Omega} \tilde{v} \cdot \nabla_L \varphi_i - \int_{\Omega} \tilde{v} \cdot d[D_L g] \right|.$$

On the other hand we have for all $i$:

$$\left| \int_{\Omega} \tilde{v} \cdot \nabla_L \varphi_i - \int_{\Omega} \tilde{v} \cdot d[D_L g] \right| \leq \int_{\Omega} (\tilde{v} - \tilde{w}) \cdot \nabla_L \varphi_i + \int_{\Omega} (\tilde{v} - \tilde{w}) \cdot d[D_L g]$$

$$+ \int_{\Omega} \tilde{w} \cdot \nabla_L \varphi_i - \int_{\Omega} \tilde{w} \cdot d[D_L g] \leq \varepsilon\|\nabla_L \varphi_i\|_1 + \varepsilon\|D_L g\| + \int_{\Omega} \nabla_L \varphi_i \cdot \nabla_L w - \int_{\Omega} \tilde{w} \cdot d[D_L g].$$

Using the properties of $(\varphi_i)_i$ and Lebesgue’s dominated convergence theorem, we thus get:

$$\lim_{i} \left| \int_{\Omega} \tilde{v} \cdot \nabla_L \varphi_i - \int_{\Omega} \tilde{v} \cdot d[D_L g] \right| \leq 2\varepsilon\|D_L g\| + \int_{\Omega} g \cdot \nabla_L \varphi_i \cdot \nabla_L w - \int_{\Omega} \tilde{w} \cdot d[D_L g] = 2\varepsilon\|D_L g\|,$$

according to (11). The result follows, for $\varepsilon > 0$ is arbitrary.

**Example 4.8.** Assume that $U$ supports a Sobolev-Gagliardo-Nirenberg inequality of type (18) for $BV_{\mathcal{L}}$ functions in $U$. Define then, for any $f \in L^N(U)$, a map $\Lambda(f) : BV_{\mathcal{L},e}(U) \rightarrow \mathbb{C}$ by:

$$\Lambda(f)(g) := \int_U \tilde{f} g.$$ 

Fix $\varepsilon > 0$ and choose $\theta > 0$ large enough for $\int_{\{\tilde{f} > \theta\}} |f|^N \leq \varepsilon^N$ to hold. We then compute:

$$\int_{\Omega} |\tilde{f} g| \leq \theta \int_{\{\tilde{f} > \theta\}} |g| + \int_{\{\tilde{f} \leq \theta\}} |f g|,$$

$$\leq \theta \|g\|_1 + \left( \int_{\{\tilde{f} > \theta\}} |f|^N \right)^\frac{1}{N} \|g\|_{N/N-1},$$

$$\leq \theta \|g\|_{L^1} + \varepsilon\|D_L g\|.$$ 

for appropriated choice of $\theta$. Hence $\Lambda(f)$ defines an $\mathcal{L}$-charge.

**Remark 4.9.** It is easy to see that for any $x_0 = (x_0^1, \ldots, x_0^n) \in \Omega$, there exists an open set $x_0 \in U \subseteq \Omega$ such that one has $\Lambda[\mathscr{P}(U)] \subseteq \Gamma[C^\infty(U, \mathbb{C}^n)]$.

Given $\varphi \in \mathscr{P}(U)$, thanks to the local solvability of the elliptic equation (7) (see [GS Corollary 4.8]), there exists $u \in C^\infty(U)$ a smooth solution to $\Delta_L u = \varphi$ in $U$. Let $v := \nabla L u$. This yields, for any $g \in BV_{\mathcal{L},e}(U)$:

$$\Lambda(\varphi)(g) = \int_U \tilde{\varphi} g = \int_U g \cdot \nabla_L \varphi = \int_U \tilde{v} \cdot d[D_L g] = \Gamma(v)(g),$$

for we could, in the computation above, replace $v$ by $v \chi$ where $\chi \in \mathscr{P}(U)$ satisfies $\chi = 1$ in a neighborhood of $\text{supp} g$.

It turns out that a linear functional on $BV_{\mathcal{L},e}$ is an $\mathcal{L}$-charge if and only if it is continuous with respect to some locally convex topology on $BV_{\mathcal{L},e}$. 
4.3. Another characterization of $\mathcal{L}$-charges. In the sequel, a \textit{locally convex space} means a Hausdorff locally convex topological vector space. For any family $\mathcal{A}$ of sets and any set $E$ we denote $\mathcal{A} \downarrow E := \{A \cap E : A \in \mathcal{A}\}$. Following [DPMP] Theorem 3.3 we define the following topology on $BV_{\mathcal{L},c}(\Omega)$ (note that this result remains valid in the complex framework).

\textbf{Definition 4.10.} Let $\mathcal{T}_C$ be the unique locally convex topology on $BV_{\mathcal{L},c}(\Omega)$ such that

(a) $\mathcal{T}_C \subseteq BV_{\mathcal{L},K,\lambda} \subseteq \mathcal{T}_L \subseteq BV_{\mathcal{L},K,\lambda}$ for all $K \subset \subset \Omega$ and $\lambda > 0$ where we let:

$$BV_{\mathcal{L},K,\lambda} = \{g \in BV_{\mathcal{L},c}(\Omega) : \text{supp } g \subseteq K, \|D_{\mathcal{L}}g\| \leq \lambda\},$$

and where $\mathcal{T}_L$ is the $L^1$-topology;

(b) for every locally convex space $Y$, a linear map $f : (BV_{\mathcal{L},c}; \mathcal{T}_C) \rightarrow Y$ is continuous if only if $f \upharpoonright BV_{\mathcal{L},K,\lambda}$ is $L^1$ continuous for all $K \subset \subset \Omega$ and $\lambda > 0$.

$L$-charges are the $\mathcal{T}_C$-continuous linear functionals, as it readily follows from Remark 4.6.

\textbf{Proposition 4.11.} A linear functional $F : BV_{\mathcal{L},c}(\Omega) \rightarrow \mathbb{C}$ is an $\mathcal{L}$-charge if and only if it is $\mathcal{C}_\mathcal{L}$-continuous.

We now turn to proving the key result for obtaining Theorem 1.2.

5. Towards Theorem 1.2

Throughout this section, we assume that $L_1, \ldots, L_n$ is a system of linearly independent vector fields in $\Omega$, and that the open set $U \subseteq \Omega$ supports inequalities of type (1) and (10); we also assume that one has $\mathcal{A}[\mathcal{P}(U)] \subseteq \mathcal{C}(U, \mathbb{C}^n)$.

\textbf{Remark 5.1.} It follows from Theorem 1.1, Proposition 2.1 and Remark 4.9 that for any $x_0 \in \Omega$, one can find an open neighborhood $U$ of $x_0$ in $\Omega$ satisfying all the above assumptions.

Our intention is to prove the following result.

\textbf{Theorem 5.2.} If $F : BV_{\mathcal{L},c}(U) \rightarrow \mathbb{C}^n$ is an $\mathcal{L}$-charge in $U$, then there exists $v \in C(U, \mathbb{C}^n)$ for which one has $F = \Gamma(v)$, i.e. such that one has, for any $g \in BV_{\mathcal{L},c}(U)$:

$$F(g) = \int_U \bar{v} \cdot d[D_{\mathcal{L}}g].$$

To prove this theorem, we have to show that the map

$$\Gamma : C(U, \mathbb{C}^n) \rightarrow CH_{\mathcal{L}}(U), v \mapsto \Gamma(v),$$

is surjective. In order to do this, we endow $C(U, \mathbb{C}^n)$ with the usual Fréchet topology of uniform convergence on compact sets, and $CH_{\mathcal{L}}(U)$ with the Fréchet topology associated to the family of seminorms $(\|\cdot\|_K)_{K}$ defined by:

$$\|F\|_K := \sup \{\|F(g)\| : g \in BV_{\mathcal{L},K}(U), \|D_{\mathcal{L}}g\| \leq 1\},$$

where $K$ ranges over all compact sets $K \subset \subset U$. The surjectivity of $\Gamma$ will be proven in Theorem 1.2. In case we show that $\Gamma$ is continuous and verifies the following two facts:

(a) $\Gamma[C(U, \mathbb{C}^n)]$ is dense in $CH_{\mathcal{L}}(U)$.

(b) $\Gamma^*[CH_{\mathcal{L}}(U)]^*$ is sequentially closed in the strong topology of $C(U, \mathbb{C}^n)^*$.

Indeed, it will then follow from the Closed Range Theorem [EDW] Theorem 8.6.13 together with [DPMP] Proposition 6.8 and (b) that $\Gamma[C(U, \mathbb{C}^n)]$ is closed in $CH_{\mathcal{L}}(U)$. Using (a) we shall then conclude that one has:

$$\Gamma[C(U, \mathbb{C}^n)] = CH_{\mathcal{L}}(U),$$

i.e. that $\Gamma$ is surjective.

The strategy of the proof of (b) follows the lines of De Pauw and Pfeffer’s proof in [DPP]. For the proof of (a), however, the proof presented below is slightly different from their approach; we namely manage to avoid an explicit smoothing process and choose instead to use an abstract approach similar to the one used in [M] in order to solve the equation $d\omega = F$.

Let us start by showing that $\Gamma$ is continuous.
Lemma 5.3. The map $\Gamma : C(U, C^\alpha) \to CH_L(U)$ is linear and continuous.

Proof. Indeed given a compact set $K \subset U$ and $g \in BV_{K,L}(U)$ we have:

$$|\Gamma(v)(g)| = \left| \int_U \bar{v} \cdot d[D_L g] \right| \leq \|D_L g\| \|v\|_{\infty,K},$$

which implies $\|\Gamma(v)\|_K \leq \|v\|_{\infty,K}$.

First we have to identify the dual space $CH_L(U)^*$.

5.1. Identifying the dual space $CH_L(U)^*$. The following result is the identification we need.

Proposition 5.4. The map $\Phi : BV_{L,c}(U) \to CH_L(U)^*$ given by $\Phi(g)(F) := F(g)$ is a linear bijection.

The proof of the previous proposition is quite delicate. We shall proceed in several steps which will be interesting as such.

First let us check that $\Phi$ is well defined. In fact, given $K \subset U$ and $g \in BV_{L,c}(U)$ we have

$$|\Phi(g)(F)| = |F(g)| \leq \|D_L g\| \|F\|_K,$$

according to the definition of $\| \cdot \|_K$. Hence $\Phi(g)$ is continuous and $\Phi(g) \in CH_L(U)^*$.

To show that $\Phi$ is injective, let $g \in BV_{L,c}(U)$ be such that $\Phi(g) = 0$. Then for any $B \subset U$ measurable and bounded we have:

$$\int_B g = \int_U \chi_B g = \Lambda(\chi_B)(g) = \Phi(g)[\Lambda(\chi_B)] = 0.$$

Thus $g = 0$ a.e. in $U$, which implies that $\Phi$ injective.

The next step is to prove that $\Phi$ is surjective. To show this property we shall define a right inverse for $\Phi$, called $\Psi$.

Let $\Psi : CH_L(U)^* \rightarrow 2^U$ be defined by:

$$\Psi(\alpha)[\varphi] := \alpha[\Lambda(\varphi)].$$

We claim that $\Psi$ is well defined, i.e. that for $\alpha \in CH_L(U)^*$, we have $\Psi(\alpha) \in BV_{L,c}(U)$. Indeed, given $\alpha \in CH_L(U)^*$ there exist $C > 0$ and $K \subset U$ such that for all $F \in CH_L(U)$ we have $|\alpha(F)| \leq C \|F\|_K$. In particular, for every $\varphi \in 2^U$ we have:

$$|\Psi(\alpha)(\varphi)| \leq C \|\Lambda(\varphi)\|_K,$$

which implies that $\Psi(\alpha) \in L^{\infty}_c(U)$ by Riesz Representation theorem. Note that if $\varphi \in 2^U$ satisfies $(\text{supp } \varphi) \cap K = \emptyset$ then one has $|\Psi(\alpha)[\varphi]| = 0$, which implies $\sup \{\Psi(\alpha)\} \subset K$. Moreover, for any $v \in C_c^\infty(U, C^\alpha)$ we have:

$$|\Psi(\alpha)[\text{div}_L \cdot v]| = |\alpha[\Lambda(\text{div}_L \cdot v)]|,$$

which implies that $\Psi(\alpha) \in L^{\infty}_c(U)$ by Riesz Representation theorem. Note that if $\varphi \in 2^U$ satisfies $(\text{supp } \varphi) \cap K = \emptyset$ then one has $|\Psi(\alpha)[\varphi]| = 0$, which implies $\sup \{\Psi(\alpha)\} \subset K$. Moreover, for any $v \in C_c^\infty(U, C^\alpha)$ we have:

$$|\Psi(\alpha)[\text{div}_L \cdot v]| = |\alpha[\Lambda(\text{div}_L \cdot v)]|,$$
so that one has \( \Psi(\alpha) \in BV_{\mathbb{L},c}(U) \).

**Lemma 5.5.** The maps \( \Phi \) and \( \Psi \) defined above are inverses, i.e., we have:

(i) \( \Psi \circ \Phi = Id_{BV_{\mathbb{L},c}(U)} \);
(ii) \( \Phi \circ \Psi = Id_{CH_{\mathbb{L}}(U)} \), (in particular, \( \Phi \) is surjective).

In order to prove the previous lemma, we shall need some observations concerning the polar sets of some neighborhoods of the origin in \( CH_{\mathbb{L}}(U) \). First, observe that the family of all sets \( V(K, \varepsilon) \) (where \( K \) ranges over all compact subsets of \( U \), and \( \varepsilon \) over all positive real numbers) defined by:

\[
V(K, \varepsilon) := \{ F \in CH_{\mathbb{L}}(U) : \| F \| \leq \varepsilon \},
\]

is a basis of neighborhoods of the origin in \( CH_{\mathbb{L}}(U) \).

**Claim 5.6.** Fix \( K \subset U \) a compact set and a real number \( \varepsilon > 0 \). For any \( \alpha \in V(K, \varepsilon)^c \), one has:

(i) \( \text{supp } \Psi(\alpha) \subseteq K \);
(ii) \( \| D_{\mathbb{L}} \Psi(\alpha) \| \leq \frac{1}{\varepsilon} \).

**Proof.** To prove (i), assume that \( \varphi \in \mathcal{D}(\mathbb{C}^n) \) satisfies \( K \cap \text{supp } \varphi = \emptyset \). Then, we get for \( \lambda > 0 \):

\[
\| \lambda \Lambda(\varphi) \|_K = \sup \left\{ \lambda \left| \int_U \varphi g \right| : g \in BV_{\mathbb{L},c}(U), \| D_{\mathbb{L}} g \| \leq 1 \right\} = 0.
\]

In particular this yields \( \lambda \Lambda(\varphi) \in V(K, \varepsilon) \). We hence obtain:

\[
\lambda |\alpha| \Lambda(\varphi) = |\alpha| |\Lambda(\varphi)| \leq 1,
\]

for any \( \lambda > 0 \). Since \( \lambda > 0 \) is arbitrary, this implies that one has \( |\alpha| \Lambda(\varphi) = 0 \), i.e. that \( \Psi(\alpha)(\varphi) = 0 \).

We may now conclude that \( \text{supp } \Psi(\alpha) \subseteq K \). In order to obtain statement (ii), fix \( v \in \mathcal{D}(U, \mathbb{C}^n) \) satisfying \( \| v \|_{\infty} \leq 1 \) and compute:

\[
\| \varepsilon \Lambda(\text{div}_{\mathbb{L}} v) \|_K = \varepsilon \| \Lambda(\text{div}_{\mathbb{L}} v) \|_K = \varepsilon \sup \left\{ \int_U g \text{div}_{\mathbb{L}} v : g \in BV_{\mathbb{L},c}, \| D_{\mathbb{L}} g \| \leq 1 \right\} \leq \varepsilon \sup \left\{ \int_U \| D_{\mathbb{L}} g \| : g \in BV_{\mathbb{L},c}, \| D_{\mathbb{L}} g \| \leq 1 \right\} \leq \varepsilon,
\]

so that one has \( \varepsilon \Lambda(\text{div}_{\mathbb{L}} v) \in V(K, \varepsilon) \). It hence follows that:

\[
\varepsilon |\alpha| \Lambda(\text{div}_{\mathbb{L}} v) = |\alpha| \varepsilon \Lambda(\text{div}_{\mathbb{L}} v) \leq 1,
\]

and we thus get:

\[
|\Psi(\alpha)(\text{div}_{\mathbb{L}} v)| \leq \frac{1}{\varepsilon}.
\]

Since \( v \in \mathcal{D}(U, \mathbb{C}^n) \) is an arbitrary vector field satisfying \( \| v \|_{\infty} \leq 1 \), this yields \( \| D_{\mathbb{L}} \Psi(\alpha) \| \leq \frac{1}{\varepsilon} \), and concludes the proof of the claim.

We now turn to proving Lemma 5.4.

**Proof of Lemma 5.4.** To prove part (i), fix \( g \in BV_{\mathbb{L},c}(U) \) and compute, for \( \varphi \in \mathcal{D}(U) \):

\[
\Psi[\Phi(g)](\varphi) := \Phi(g)[\Lambda(\varphi)] = \Lambda(\varphi)(g) = \int_U g \varphi,
\]

that is, \( \Psi(\Phi(g)) = g \) in the sense of distributions.

In order to prove part (ii), fix \( \alpha \in CH_{\mathbb{L}}^e(U) \). We have to show that, for any \( F \in CH_{\mathbb{L}}(U) \), we have:

\[
\Phi[\Psi(\alpha)](F) = \alpha(F),
\]

i.e. that for any \( F \in CH_{\mathbb{L}}(U) \), one has:

\[
F[\Psi(\alpha)] = \alpha(F).
\]
To this purpose, define for any $F \in CH_L(U)$ a map:
$$\Delta_F : CH_L(U)^* \to \mathbb{C}, \alpha \mapsto \Delta_F(\alpha) := F[\Psi(\alpha)].$$

**Claim 5.7.** Given $F \in CH_L(U)$, the map $\Delta_F$ is weakly*–continuous on $V(K, \varepsilon)$ for all $K \subset U$ and $\varepsilon > 0$.

To prove this claim, fix $K \subset U$, $\varepsilon > 0$ and assume that $(\alpha_i)_{i \in I} \in C$ is a net weak*–converging to 0. In particular, one gets:

(a) for any $\varphi \in \mathcal{D}(U)$, we have $\Lambda(\varphi) \in CH_L(U)$ and hence the net $(\Psi(\alpha_i(\varphi)))_{i \in I} = (\alpha_i[\Lambda(\varphi)])_{i \in I}$ converges to 0.

According to Claim 5.6, we moreover have:

(b) $\text{supp} \Psi(\alpha_i) \subseteq K$ for each $i \in I$;

(c) $c := \sup_{i \in I} \|D_L \Psi(\alpha_i)\| \leq \frac{1}{\varepsilon}.$

It hence follow from Proposition 3.6 that the net $(\alpha_i)_{i \in I}$ converges to 0. From the fact that $F$ is an $L$–charge, we see that the net $(F[\Psi(\alpha_i)])_{i \in I}$ converges to 0 as well. This means, in turn, that $(\Delta_F(\alpha_i))_{i \in I}$ converges to 0, which shows that $\Delta_F$ is weak*–continuous on $V(K, \varepsilon)$.

**Claim 5.8.** For any $\alpha \in CH_L(U)^*$, we have $\Delta_F(\alpha) = \alpha(F)$.

To prove the latter claim, observe that according to Claim 5.7 and to the Banach-Grothendieck theorem [EDW, Theorem 8.5.1], there exists $\tilde{F} \in CH_L(U)$ such that for any $\alpha \in CH_L(U)^*$, we have:

$$\Delta_F(\alpha) = \alpha(\tilde{F}).$$

Yet given $g \in BV_{L,c}(U)$, we then have, according to [Lemma 5.3 (i)]:

$$F(g) = F[\Phi(g)] = \Delta_F[\Phi(g)] = \Phi(g)(\tilde{F}) = \tilde{F}(g),$$

i.e. $F = \tilde{F}$, which proves the claim.

It now suffices to observe that Lemma 5.8 is proven for we have established the equality $F[\Psi(\alpha)] = \alpha(F)$ for any $F \in CH_L(U)$ and $\alpha \in CH_L(U)^*$. ■

As a corollary, we get a proof of the density of $\Gamma[C(U, C^n)]$ in $CH_L(U)$.

**Corollary 5.9.** The space $\Lambda[\mathcal{D}(U)]$ is dense in $CH_L(U)$.

**Proof.** Assuming that $\alpha \in CH_L(U)^*$ satisfies $\alpha \upharpoonright \Lambda[\mathcal{D}(U)] = 0$, we compute for any $\varphi \in \mathcal{D}(U)$:

$$\Psi(\alpha)(\varphi) := \alpha[\Lambda(\varphi)] = 0.$$  

This means that $\Psi(\alpha) = 0$, and implies that $\alpha = \Phi \circ \Psi(\alpha) = \Phi(0) = 0$. The result then follows from the Hahn-Banach theorem. ■

**Corollary 5.10.** The space $\Gamma[C(U, C^n)]$ is dense in $CH_L(U)$.

**Proof.** It follows from the previous corollary that $\Lambda[\mathcal{D}(U)]$ is dense in $CH_L(U)$. Since by hypothesis we also have $\Lambda[\mathcal{D}(U)] \subseteq \Gamma[C(U, C^n)] \subseteq CH_L(U)$, it is clear that $\Gamma[C(U, C^n)]$ is dense in $CH_L(U)$. ■

In order to study the range of $\Gamma^*$, we introduce the following linear operator:

$$\Xi : BV_{L,c}(U) \to C(U, C^n)^*, g \mapsto \Xi(g),$$

defined by $\Xi(g)(v) := \Gamma(v)(g)$ for any $v \in C(U, C^n)$.

**Claim 5.11.** We have $\text{im} \Gamma^* = \text{im} \Xi$.

**Proof.** To prove this claim, fix $\mu \in C(U, C^n)$. If one has $\mu = \Gamma^*(\alpha)$ for some $\alpha \in CH_L(U)^*$, then we compute for $v \in C(U, C^n)$:

$$\Xi[\Psi(\alpha)](v) = \Gamma(v)[\Psi(\alpha)] = \Phi[\Psi(\alpha)][\Gamma(v)] = \alpha[\Gamma(v)] = \Gamma^*(\alpha)(v) = \mu(v),$$

so that one has $\mu = \Xi[\Psi(\alpha)] \in \text{im} \Xi$. Conversely, if one has $\mu = \Xi(g)$ for some $g \in BV_{L,c}(U)$, then we compute for $v \in C(U, C^n)$:

$$\Gamma^*[\Phi(g)](v) = \Phi(g)[\Gamma(v)] = \Gamma(v)(g) = \Xi(g)(v) = \mu(v),$$

so that one has $\mu = \Gamma^*[\Phi(g)] \in \text{im} \Gamma^*$. ■
Consider the set
\[ B := \{ v \in C(U, \mathbb{C}^n) : \| v \|_{\infty} \leq 1 \}. \]
It is clear that \( B \) is bounded in \( C(U, \mathbb{C}^n) \). Hence the seminorm:
\[ p : C(U, \mathbb{C}^n)^* \to \mathbb{R}_+, \mu \mapsto p(\mu) := \sup_{v \in B} |\mu(v)|, \]
is strongly continuous (i.e. continuous with respect to the strong topology) on \( C(U, \mathbb{C}^n)^* \). Observe now that one has, for \( g \in BV_{\mathbb{C}}(U) \):
\[
p[\Xi(g)] = \sup_{v \in B} |\Xi(g)(v)|, \\
= \sup\{ |\Gamma(v)(g) : v \in B \}, \\
= \|D\mathcal{L}g\|. \\
\]

**Lemma 5.12.** The set \( \im \Xi \) is strongly sequentially closed in \( C(U, \mathbb{C}^n)^* \).

**Proof.** Fix a sequence \( (\Xi(g_k))_{k \in \mathbb{N}} \subseteq \im \Xi \) and assume that, in the strong topology, one has:
\[
\Xi(g_k) \to \mu \in C(U, \mathbb{C}^n)^*, \quad k \to \infty.
\]
The strong continuity of \( p \) then yields:
\[
c := \sup_{k \in \mathbb{N}} \|D\mathcal{L}g_k\| = \sup_{k \in \mathbb{N}} p[\Xi(g_k)] < +\infty.
\]

**Claim 5.13.** There exists a compact set \( K \subset U \) such that one has \( \supp g_k \subseteq K \) for each \( k \in \mathbb{N} \).

To prove this claim, let us first prove that the sequence \( (\supp D\mathcal{L}g_k)_{k \in \mathbb{N}} \) is compactly supported in \( U \) (i.e. that there is a compact subset of \( U \) containing \( \supp Dg_k \) for all \( k \)). To this purpose, we proceed towards a contradiction and assume that it is not the case. Let then \( U = \bigcup_{j \in \mathbb{N}} U_j \) be an exhaustion of \( U \) by open sets satisfying, for each \( j \in \mathbb{N} \), \( \bar{U}_j \subset U_{j+1} \) and such that \( \bar{U}_j \) is a compact subset of \( U \) for each \( j \in \mathbb{N} \). Since \( (\supp D\mathcal{L}g_k)_{k \in \mathbb{N}} \) is not compactly supported, there exist increasing sequences of integers \( (j_k)_{k \in \mathbb{N}} \) and \( (k_j)_{j \in \mathbb{N}} \) satisfying, for any \( l \in \mathbb{N} \):
\[
\supp(D\mathcal{L}g_{k_l}) \cap (U_{j_{l+1}} \setminus \bar{U}_{j_l}) \neq \emptyset.
\]
In particular, there exists for each \( l \in \mathbb{N} \) a vector field \( v_l \in C_c(U_{j_{l+1}} \setminus \bar{U}_{j_l}, \mathbb{C}^n) \) with \( \| v_l \|_{\infty} \leq 1 \) and:
\[
a_l := \left| \int_{\bar{U}_l} \bar{v}_l \cdot d[Dg_{k_l}] \right| > 0.
\]
Let now, for \( l \in \mathbb{N} \), \( b_l := \max_{0 \leq k \leq l} \frac{1}{a_k} \) and define a bounded set \( B' \subseteq C(U, \mathbb{C}^n) \) by:
\[
B' := \{ v \in C(U, \mathbb{C}^n) : \| v \|_{\infty, \bar{U}_{j_{l+1}}} \leq b_l a_l \text{ for each } l \in \mathbb{N} \}.
\]
It follows from the construction of \( B \) that one has \( w_l := b_l v_l \in B \) for any \( l \in \mathbb{N} \). Moreover the seminorm
\[
p' := C(U, \mathbb{C}^n)^* \to \mathbb{R}_+, \mu \mapsto \sup_{v \in B'} |\mu(v)|,
\]
is strongly continuous. Yet we get for \( l \in \mathbb{N} \):
\[
p'[\Xi(g_{k_l})] \geq |\Xi(g_{k_l})(w_l)| = |\Gamma(w_l)(g_{k_l})| = lb_l \left| \int_{\bar{U}_l} \bar{v}_l \cdot d(Dg_{k_l}) \right| = lb_l a_l \geq l.
\]
Since this yields \( p'[\Xi(g_{k_l})] \to \infty, l \to \infty \), we get a contradiction with the fact that \( p' \) is strongly continuous (recall that \( (\Xi(g_k))_{k \in \mathbb{N}} \) converges in the strong topology).

Now fix \( k \in \mathbb{N} \) and \( x \in U \setminus \supp(D\mathcal{L}g_k) \); choose an open set \( V \subseteq \Omega \) such that one has \( V \cap \supp(D\mathcal{L}g_k) = \emptyset \) and observe that one has \( \|g_k\|_{L^{N/2-1}}(V) \leq \|D\mathcal{L}g_k\|V = 0 \). It hence follows that \( g_k \) is a.e. equal to 0 on \( V \), and hence that \( x \notin \supp g_k \). This proves the inclusion \( \supp g_k \subseteq K \) for all \( k \), which establishes the claim.

Getting back to the proof of Lemma 5.12 observe that, according to Proposition 5.10 there exists a subsequence \( (g_{k_l}) \subseteq (g_k) \), \( L^1 \)-converging to \( g \in BV_{\mathbb{C}}(U) \). Using the fact that \( \Gamma(v) \) is an \( \mathcal{L} \)-charge, we compute:
\[
\mu(v) = \lim_{l \to \infty} \Xi(g_{k_l})(v) = \lim_{l \to \infty} \Gamma(v)(g_{k_l}) = \Gamma(v)(g) = \Xi(g)(v),
\]
and hence we get $\mu = \Xi(g) \in \text{im} \Xi$. 

We hence proved the following theorem.

**Theorem 5.14.** We have $CH_\xi(U) = \Gamma[C(U, \mathbb{C}^n)]$. 

6. Appendix

**Theorem 6.1.** Let $p(x, D)$ be a pseudodifferential operator with symbol in the Hörmander class $S_{1,0}^m(\mathbb{R}^N)$ and consider $k(x, y)$ the distribution kernel of $p(x, D)$ defined by the oscillatory integral

$$k(x, y) = \int e^{2i\pi(x-y)\xi}p(x, \xi)d\xi.$$ 

If $m < 0$ then $p(x, D)$ maps continuously $L^1(\mathbb{R}^N)$ onto itself.

*Proof.* Writing $p(x, D)u = (k(x, \cdot) * u)(x)$ it is sufficient to prove that $k(x, y) \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$ using a pointwise control of the kernel due to Álvarez and Hounie in [AH]. In order to prove the boundedness in $L^1$ norm, we first localize the kernel in the diagonal region. Let $A = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| < 1\}$ be a neighborhood of the diagonal. If $m < -N$ then $k$ is bounded and clearly the property follows. If $0 < m + N$ then there exists $C > 0$ such that $|k(x, y)| \leq C|x - y|^{-(m+1)}$, and then $k$ is integrable on $A$, since $m < 0$. The limiting case occurs when $m = -N$, which implies $|k(x, y)| \leq C \log |x - y|$ from which the property follows. On the other hand, by the pseudo-local property (see [AH], Theorem 1.1), we see that there exists $L_0 \in \mathbb{Z}^*$ such that $|k(x, y)| \leq |x - y|^{-L}$ for $L \geq L_0$ and $|x - y| > 1$; hence $k$ is integrable on $\mathbb{C} A$, since $L \geq \max\{N, L_0\}$. Combining all those cases we conclude that $k(x, y) \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$.

Consider a class of pseudodifferential operators, called Bessel potential $J_\beta$ for $\beta > 0$, defined by

$$J_\beta f(x) = \int_{\mathbb{R}^N} e^{2i\pi x \cdot \xi}b(x, \xi)\hat{f}(\xi)d\xi, \quad f \in S'(\mathbb{R}^N),$$

where $b(x, \xi) = \langle \xi \rangle^\beta := (1 + 4\pi^2|\xi|^2)^{-\beta/2}$ belongs to the Hörmander class $S_{1,0}^{-\beta}(\mathbb{R}^N)$. We define the non-homogeneous Sobolev space $W^{\beta, p}(\mathbb{R}^N)$ for $\beta > 0$ and $1 \leq p < \infty$ as

$$W^{\beta, p}(\mathbb{R}^N) = \{f \in S'(\mathbb{R}^N) : J_{-\beta} f \in L^p(\mathbb{R}^N)\}$$

with associated norm $\|f\|_{\beta, p} := \|J_{-\beta} f\|_p$. As a consequence of Theorem 3.5 in [AH] and Theorem 6.1 when $p = 1$, it follows that $W^{\beta, p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ continuously, i.e. that one has:

$$\|u\|_p = \|J_{\beta}(J_{-\beta} u)\|_p \leq C\|J_{-\beta} u\|_p = C\|u\|_{\beta, p}.$$ 

For $B = B(x_0, \ell)$ a fixed ball let $B' = B(x_0, 2\ell)$ the ball with the same center as $B$ but twice its radius. Let $\psi \in C_c^\infty(\overline{B})$ satisfy $\psi(x) \equiv 1$ on $B$ and define $A_\beta := A_\beta(x, D)$ the pseudodifferential operator with symbol $A_\beta(x, \xi) = \psi(x)\langle \xi \rangle^\beta$. Denote by $W^{\beta, p}(B)$ the set of distributions $f \in \mathcal{E}'(B)$ such that $A_\beta f \in L^p(\mathbb{R}^N)$, endowed with the semi-norm $\|f\|_{\beta, p}(B) := \|A_\beta u\|_p$. Note that the space $W^{\beta, p}(B)$ is independent of the choice of $\psi$. In view of (19), we have the continuous inclusion:

$$W^{\beta, p}(B) \subset L^p(\mathbb{R}^N),$$

for $1 \leq p < \infty$. Next we present a version of the Rellich-Kondrachov compactness for $W^{\beta, 1}_{\mathcal{E}}(B)$.

**Theorem 6.2.** Let $0 < \beta < 1$. The embedding $W^{\beta, 1}_{\mathcal{E}}(B) \subset L^1(\mathbb{R}^N)$ is compact.

The proof follows the same strategy as the proof of Theorem A in [HKP] and will be presented for the sake of completeness. The compact embedding of $W^{\beta, p}_{\mathcal{E}}(B)$ in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$ could be established by analogous means.

*Proof.* According to the previous comments on continuity, it is enough to verify the compactness. We will show that if $(u_m)$ is a bounded sequence in $W^{\beta, 1}_{\mathcal{E}}(B)$ then there exist a subsequence $(u_{m_j})_j$ which converges in $L^1(\mathbb{R}^N)$. Consider the regularizations $u_m = \eta \ast u_m$ where $\eta \in C_c^\infty(B^0_1)$, $\int_{\mathbb{R}^N} \eta = 1$, $\eta(x) = \varepsilon^{-N}\eta(x/\varepsilon)$ and $0 < \varepsilon \leq 1$. It is enough to show that the family $\{u_m\}_{\varepsilon, m}$ has the following two properties:
(i) for any fixed $0 < \varepsilon < 1$, the sequence $(u^\varepsilon_m)_{m \in \mathbb{N}}$ is a relatively compact subset of $L^1_c(B') := L^1(\mathbb{R}^N) \cap \mathcal{E}(B')$;

(ii) $u^\varepsilon_m \to u^\varepsilon_m$ in $L^1_c(B')$ uniformly in $m$ as $\varepsilon \searrow 0$.

where $B'$ is a closed ball that contains the support of all $u^\varepsilon_m$.

Since the inclusion $C_c(B') \subset L^1_c(B')$ is continuous, property (i) will follow once we shall have proven that $(u^\varepsilon_m)_{m}$ is a precompact subset of $C_c(B')$. We claim that for each $\varepsilon > 0$, $(u^\varepsilon_m)_{m}$ is uniformly bounded and equicontinuous. In fact, one has for $x \in B'$:

$$|u^\varepsilon_m(x)| = |(u_m, \eta \varepsilon(x - \cdot))|,$$
$$\leq \|\Lambda_{\beta} \Lambda_{-\beta} u_m\|_1 \|\eta\|_{\infty},$$
$$\leq C(B) \|\Lambda_{-\beta} u_m\|_{L^1(B)} \|\eta\|_{\infty},$$
$$\leq C(B)\varepsilon^{-N} \|u_m\|_{\beta,1},$$

and analogously

$$|\nabla u^\varepsilon_m(x)| \leq (\Lambda_{-\beta} u_m)(\Lambda_{\beta} \circ \nabla)\eta\varepsilon \leq C(B)\varepsilon^{-N} \|u_m\|_{\beta,1}.$$

The conclusion follows from Arzelà-Ascoli theorem.

To prove (ii) we will first consider the identity :

$$u^\varepsilon_m(x) - u_m(x) = \int_0^\varepsilon \frac{\partial}{\partial s} (u_m \ast \eta_s)(x) ds,$$
$$= - \int_0^\varepsilon \{u_m \ast \nabla \cdot [x \eta_s]\} (x) ds,$$
$$= - \int_0^\varepsilon \{\Lambda_{-\beta} u_m \ast (\Lambda_{\beta} \circ \nabla) \cdot [x \eta_s]\} (x) ds.$$

But from the equalities $\Gamma_{\beta}(t, \xi) := 2i\pi \psi(x) \sum_{k=1}^N \xi_k (t^2 + 4\pi^2|\xi|^2)^{-\frac{\beta}{2}}$ we get:

$$(\Lambda_{\beta} \circ \nabla) \cdot g_s(x) = s^{-1}[\Gamma_{\beta}(s, D)g_s](x),$$

after which we compute, using Fubini’s theorem:

$$\int_{\mathbb{R}^N} |u^\varepsilon_m - u_m|(x) \leq \int_{\mathbb{R}^N} \int_0^\varepsilon s^{-1} |\Lambda_{-\beta} u_m \ast (\Gamma_{\beta}(s, D)[y \eta_s])| s(x) dsdx,$$
$$\leq \int_{\mathbb{R}^N} \int_0^\varepsilon \left( \int_{K \subset \mathbb{R}^N} \|\Gamma_{\beta}(s, D)[y \eta_s](\cdot)\|_{\infty} \right) |\Lambda_{-\beta} u_m(x - y)|dy dsdx,$$
$$\leq \int_0^\varepsilon \int_{K \subset \mathbb{R}^N} s^{-1} \|\Gamma_{\beta}(s, D)[y \eta_s](\cdot)\|_{\infty} \left( \int_{\mathbb{R}^N} |\Lambda_{-\beta} u_m(x - y)|dy \right) dy ds,$$
$$\leq C\varepsilon^\gamma \|u_m\|_{\beta,1}.$$

To obtain the latter inequalities, we observe (defining $\tilde{\Gamma}_{\alpha}(t, \xi) := 2i\pi \sum_{k=1}^N \xi_k (t^2 + 4\pi^2|\xi|^2)^{-\frac{\alpha}{2}}$ and letting $B$ be the ball defined above):

$$\left| \int_0^\varepsilon \int_{K \subset \mathbb{R}^N} s^{\alpha-1} \|\tilde{\Gamma}_{\alpha}(s, D) g_s(y)\|_{\infty} dy ds \right| = \left| \int_0^\varepsilon \int_B \psi\left( \frac{y}{s} \right) \tilde{\Gamma}_{\alpha}(1, D) g_s(y) dy ds \right|,$$
$$\leq C \int_0^\varepsilon \|g_s\|_{\infty} ds,$$
$$\leq C'\varepsilon^\gamma,$$

where $C = C(\eta, K) > 0$ and $\gamma = \frac{N}{2} - N + \alpha > 0$ are constants for $1 < r < N/(N - \alpha)$.

To finish the proof, we claim that, for a given $\delta > 0$, there exists a subsequence $(u_{m_j})_j \subset (u_m)_m$ such that one has:

$$\lim_{j,k \to \infty} \|u_{m_j} - u_{m_k}\|_1 \leq \delta.$$
Indeed, for $\varepsilon > 0$ sufficiently small, we have:

$$\|u_m^\varepsilon - u_m\|_1 \leq \frac{\delta}{2}$$

uniformly in $m$. Since $(u_m)$ and $(u_m^\varepsilon)$ are supported in a closed ball $B'$, by Arzelà-Ascoli’s theorem there exists a subsequence $(u_m^\varepsilon)_{j,k}$ which converges uniformly in $B'$. In particular, this yields:

$$\lim_{j,k \to \infty} \|u_m^\varepsilon - u_m\|_1 = 0.$$ 

Note that (20) is a consequence of (21) and (22). Using (20) for $\delta = 1/n$ for $n = 1, 2, 3, \ldots$ and the diagonal process we can extract a convergent subsequence $(u_m)_\ell$.

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