Pythagorean fuzzy soft rough sets and their applications in decision-making

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ABSTRACT
This paper aims to originate two new notions that are soft rough Pythagorean fuzzy set and Pythagorean fuzzy soft rough set, and investigate some important properties of soft rough Pythagorean fuzzy set and Pythagorean fuzzy soft rough set in detail. Furthermore, classical representations of Pythagorean fuzzy soft rough approximation operators are presented. Then the proposed operators are applied on decision-making problem in which the experts provide their preferences in Pythagorean fuzzy soft rough environment. Finally through an illustrative example, it is shown that how the proposed operators work in decision-making problems.

1. Introduction

In last few years, many complicated models in engineering, economics, medical sciences, social sciences and many other fields including uncertainties and many other well-known theories have been introduced for the exploration of uncertainties. Among these theories, the most popular theories are fuzzy set [1], soft set [2] and rough set [3]. Different complications are pointed out in this existing literature as mentioned in [2]. To tackle these complications the new approach of SS was originated by Moldstove [2], which is different from the other existing theories due to its parameterization tools. SS theory is free from the inherent complications and handles the vague and uncertain knowledge easily. Due to these characteristics, the theory of SS is famous among scholars in the recent era. This theory has strong concepts in various directions such as Riemann integration, game theory, perron integration, smoothness of functions, operational research, the theory of measure and probability theory [2,4].

In the recent scenario, SS is one of the most significant areas of research. Maji et al. [5] initiated various operations on SS. Through these operations, the theoretical study of SS becomes improved and emerges very rapidly in research area. On the basis of [5], Ali et al. [6] improved the existing literature and presented some new operations on SS. They developed the concept of complement and asserted that certain De Morgan’s laws are satisfied on the basis of these newly developed operations on SS. The generalization of SS is improving very rapidly in the research area to hybridize the different mathematical structures with SS. The combined structure of fuzzy set and SS were presented by Maji et al. [7] to initiate the new concept of fuzzy SS (FSS). Yang et al. [8] further generalized the concept of FSS and originated the new idea of multi FSS and under an imprecise situation, they provided their applications in decision-making (in short DM).

The fundamental approach of rough sets theory (RST) attracted the attention of scholars towards this topic, and had been applied in various directions like machine learning, data analysis, knowledge discovery, pattern recognition and information process, etc. with successful results. RST by Pawlak is based on indiscernibility relation. Actually, indiscernibility relation is also known as an equivalence relation and we do not have enough knowledge about the universe of discourse, which makes it difficult for us to define the target. So, different authors defined various structures of RST with less restrictions. Feng et al. [9] defined the hybrid theory of fuzzy set, rough set and SS to get several important theories like, soft rough set, rough SS and soft rough fuzzy set. Mahmood et al. [10] presented the concept of generalized roughness with the help of isotone and monotone mappings in ordered semigroups and for details see [11,12]. The detail of combining the study of rough set and SS are presented in [9,13,14]. Ma [15] defined two new types of fuzzy covering-based rough set models by the new concepts of fuzzy β-covering and fuzzy β-neighbourhood, gave the properties of the two models, and revealed the relationships between the two models and some others. For the fuzzy β-covering approximation space proposed in [15], Yang et al. [16] introduced the concept of fuzzy β-neighbourhood family, fuzzy β-cover and consistent fuzzy β-covering, and gave some propositions

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about the fuzzy \( \beta \)-cover and the consistent fuzzy \( \beta \)-
covering. Zhang [17] presented the novel fuzzy rough
set models and corresponding applications to multici-
teria decision-making (MCDM). Zhang and Zhan [18]
proposed the concepts of fuzzy soft \( \beta \)-coverings, fuzzy
soft \( \beta \)-neighbourhoods and fuzzy soft complement \( \beta \)-
neighbourhoods and some related properties are stud-
ied. Zhan and Sun [19] initiated the notion of covering-
based soft fuzzy rough theory and its application to
MCDM.

Intuitionistic fuzzy set (IF set) [20] consist of two
functions which are the generalization of fuzzy set and
like a fuzzy set, these two functions also map on unit
closed interval. Here, the first function represents the
membership grade and the second function represents
non-membership grade. IF set has a limitation that is
the sum of both these functions does not exceed 1. IF
set plays a vital role for tackling vagueness and uncer-
tainty. Many authors developed the new notions by
combining rough set and SS with IF set and proposed
the new results, such as Maji et al. [21,22] originated
the notion of intuitionistic fuzzy SS (IFSS). Karaaslan [23]
presented the concept of generalized IFSS and studied their
applications. Recently Feng et al. [24] clarified and reformu-
lated the existing concepts of generalized IFSS and studied their
applications in MCDM. Furthermore, they improved some existing
concepts and results in the literature through these
new notions. Jiang et al. [25] presented the applica-
tions of DM in IF soft environment. The concept of IF
rough set was originated by Xie et al. [26], which
depends on two universes and general IF relation. The
notions of rough IF set and IF rough set were initi-
bated by Zhou and Wu [27], and they studied the construc-
tive and axiomatic approaches in detail. Zhang et al.
in [28] investigated the notion of IF soft rough set
(IFSR) and proposed its application in DM. The con-
cept of generalized IFSRS was presented by Zhang et al.
[29] and studied their applications in DM. Zhan and
Sun [30] introduced three classes of coverings based IF
rough set model via IF- \( \beta \)-neighbourhoods and IF com-
plementary \( \beta \)-neighbourhood and studied their appli-
cation in MCDM. Zhang et al. [31] proposed the concept of
covering-based general multigranulation IF rough
set and presented their corresponding application to
MCDM. Yager [32] provided more comfortability and relaxa-
tion in limitation of values for the decision mak-
ers that is, the square sum of membership and non-
membership grades must not exceed 1, and he named
this notion as Pythagorean fuzzy set (PFS), which got
more attention of scholars in the recent era. Hussain
et al. [33] proposed the concept of rough Pythagorean
fuzzy ideals in semigroups. By viewing existing lit-
erature, it is clear that there does not exist any con-
cepts of soft rough Pythagorean fuzzy set (SRPFS) and
Pythagorean fuzzy soft rough set (PFSSRS). From the
investigation of existing study, it appears a lack of appli-
cation in DM problems with the evaluation of PF infor-
mation by means of PFSSRS. This motivates the present
research to present the novel approach of SRPFSS and
PFSSRS and presenting its application in DM problems
with the evaluation of PF information.

The arrangement of the rest of portion of the
manuscript is given as Section 2 consists of the brief
review of IF sets, Pythagorean fuzzy sets (PFSs), \((\alpha, \beta)\)-level
cut sets, rough sets and SSs. Furthermore, in
Section 3 the new structure of soft rough Pythagorean
fuzzy set (SRPFS) is presented and the related properties
of SRP fuzzy approximation operators are studied. Section 4
consists of the study of Pythagorean fuzzy soft rough
set (PFSSRS) and investigates their related properties in
detail. Section 5 consists of the technique for DM pro-
cess and its algorithm. In the final Section 6, we pre-
sented an illustrative example on the proposed meth-
ods SRPFSSRS and PFSSRS, and show how the proposed
operators work in DM problems.

2. Preliminaries

This section consists of a brief review of IF set,
Pythagorean fuzzy set (PFS), RST, SS and soft rough set.
These concepts will help us in next sections.

**Definition 2.1** ([20]): For a universal set \( \hat{U} \), an IF set \( \cal{A} \)
on \( \hat{U} \) is denoted and defined as
\[
\cal{A} = \{ (\kappa, \mu_{\cal{A}}(\kappa), \psi_{\cal{A}}(\kappa)) \mid \kappa \in \hat{U} \},
\]
where \( \mu_{\cal{A}} : \hat{U} \to [0, 1] \) represents the membership de-
gree and \( \psi_{\cal{A}} : \hat{U} \to [0, 1] \) represents the non-member-
ship degree of \( \kappa \in \hat{U} \) to the set \( \cal{A} \), satisfying that \( 0 \leq \mu_{\cal{A}}(\kappa) + \psi_{\cal{A}}(\kappa) \leq 1 \). Further \( \pi_{\cal{A}}(\kappa) = 1 - (\mu_{\cal{A}}(\kappa) + \psi_{\cal{A}}(\kappa)) \) is known as indeterminacy or hesitancy.

**Definition 2.2** ([33]): Let us consider a universal set \( \hat{U} \). A PFS \( \cal{A} \) on set \( \hat{U} \) is denoted and defined as
\[
\cal{A} = \{ (\kappa, \mu_{\cal{A}}(\kappa), \psi_{\cal{A}}(\kappa)) \mid \kappa \in \hat{U} \},
\]
where \( \mu_{\cal{A}} : \hat{U} \to [0, 1] \) represents the membership de-
gree and \( \psi_{\cal{A}} : \hat{U} \to [0, 1] \) represents the non-member-
ship degree of \( \kappa \in \hat{U} \) to the set \( \cal{A} \), satisfying that \( 0 \leq (\mu_{\cal{A}}(\kappa))^2 + (\psi_{\cal{A}}(\kappa))^2 \leq 1 \). Here \( \pi_{\cal{A}}(\kappa) = \sqrt{1 - ((\mu_{\cal{A}}(\kappa))^2 + (\psi_{\cal{A}}(\kappa))^2)} \) represents the indetermi-

The collection of all Pythagorean fuzzy subsets of \( \hat{U} \) is
represented by \( \text{PFS}(\hat{U}) \).

Consider \( \cal{A}_1 = \{ (\kappa, \mu_{\cal{A}_1}(\kappa), \psi_{\cal{A}_1}(\kappa)) \mid \kappa \in \hat{U} \} \) and \( \cal{A}_2 = \{ (\kappa, \mu_{\cal{A}_2}(\kappa), \psi_{\cal{A}_2}(\kappa)) \mid \kappa \in \hat{U} \} \) as the two PFSs. Then the basic operations on PFS defined by Yager [35,36] are given as:
Definition 2.3: Consider $S^* = \{(\kappa_1, \kappa_2) : (\kappa_1, \kappa_2) \in [0, 1] \times [0, 1] \text{ with } k_1^2 + k_2^2 \leq 1 \}$ with the ordered relation $\preceq$ denoted as

$$(\kappa_1, \kappa_2) \preceq (b_1, b_2) \iff k_1 \leq b_1 \quad \text{and} \quad k_2 \geq b_2 \quad \forall (k_1, k_2), (b_1, b_2) \in S^* \quad (1)$$

For any two $(k_1, k_2), (b_1, b_2)$ are incomparable if (1) does not hold.

Lemma 2.1: The ordered set $S^*$ with respect to the ordered relation $\preceq$ is a complete lattice.

Now for all $(k_1, k_2), (b_1, b_2) \in S^*$ the operation $\land$ and $\lor$ on $(S^*, \preceq)$ are defined as:

$$(k_1, k_2) \land (b_1, b_2) = [\min(k_1, b_1), \max(k_2, b_2)]$$

$$(k_1, k_2) \lor (b_1, b_2) = [\max(k_1, b_1), \min(k_2, b_2)]$$

Definition 2.4 ([2]): Suppose a universal set $\hat{U}$ and $\mathcal{E}$ be the initial set of parameters. Suppose that a function $F : \mathcal{E} \rightarrow P(\hat{U})$, then the pair $(F, \mathcal{E})$ is known to be a SS on $\hat{U}$, where the family of all subsets of a universal set is denoted by $P(\hat{U})$.

Definition 2.5 ([37]): Consider a SS $(F, \mathcal{E})$ on $\hat{U}$. Then a relation $\Omega$ from $\hat{U} \times \mathcal{E}$ is known to be a crisp soft relation from a set $\hat{U}$ to $\mathcal{E}$, which is given by

$$\Omega = \{(\kappa, x), \mu_\Omega(\kappa, x)\} \mid (\kappa, x) \in \hat{U} \times \mathcal{E} \}$$

where $\mu_\Omega : \hat{U} \times \mathcal{E} \rightarrow [0, 1]$ such that $\mu_\Omega(\kappa, x) = 1_{(\kappa, x) \in \Omega}$

Definition 2.6 ([7]): Suppose a universal set $\hat{U}$ and $\mathcal{E}$ be the set of parameters. Consider a mapping $F : \mathcal{E} \rightarrow E(\hat{U})$, then a fuzzy SS on $\hat{U}$ is denoted by a pair $(F, \mathcal{E})$ such that $F(\hat{U})$ is the family of all fuzzy subsets of $\hat{U}$.

Definition 2.7 ([38]): Suppose a fuzzy SS $(F, \mathcal{E})$ on a universal set $\hat{U}$. Then a relation $\Omega$ from $\hat{U} \times \mathcal{E}$ is known to be a fuzzy soft relation (FSR) from a set $\hat{U}$ to $\mathcal{E}$ and is given as:

$$\Omega = \{(\kappa, x), \mu_\Omega(\kappa, x)\} \mid (\kappa, x) \in \hat{U} \times \mathcal{E} \}$$

where $\mu_\Omega : \hat{U} \times \mathcal{E} \rightarrow [0, 1]$ and $\mu_\Omega(\kappa, x) = \mu_E(\kappa)(x)$.

Consider if set $\hat{U} = \{\kappa_1, \kappa_2, \ldots, \kappa_m\}$ and $\mathcal{E} = \{x_1, x_2, \ldots, x_n\}$, then the FSR $\Omega$ from set $\hat{U}$ to $\mathcal{E}$ is given in the following table:

| $\kappa$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ |
|----------|-------|-------|-----------|-------|
| $\kappa_1$ | $\mu_\Omega(\kappa_1, x_1)$ | $\mu_\Omega(\kappa_1, x_2)$ | $\cdots$ | $\mu_\Omega(\kappa_1, x_n)$ |
| $\kappa_2$ | $\mu_\Omega(\kappa_2, x_1)$ | $\mu_\Omega(\kappa_2, x_2)$ | $\cdots$ | $\mu_\Omega(\kappa_2, x_n)$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\kappa_m$ | $\mu_\Omega(\kappa_m, x_1)$ | $\mu_\Omega(\kappa_m, x_2)$ | $\cdots$ | $\mu_\Omega(\kappa_m, x_n)$ |

Definition 2.8 ([39,40]): Consider a universal set $\hat{U}$ and $\Omega \subseteq \hat{U} \times \hat{U}$ be a crisp relation. Now, consider a mapping $\Omega^* : \hat{U} \rightarrow P(\hat{U})$ defined by $\Omega^*(\kappa) = \{x \in \hat{U} \mid (\kappa, x) \in \Omega\}$, $\kappa \in \hat{U}$ is known to be the set-valued mapping (SVM). Then $(\hat{U}, \Omega^*)$ is known to be an approximation space. Consider for a nonempty subset $\exists \subseteq \hat{U}$, then the lower and upper approximation of $\exists$ with respect to $(\hat{U}, \Omega^*)$ are represented by $\Omega(\exists)$ and $\Omega^*(\exists)$ and are defined as follows:

$$\Omega(\exists) = \{x \in \hat{U} \mid \Omega^*(x) \subseteq \exists\}$$

and $$\Omega^*(\exists) = \{x \in \hat{U} \mid \Omega^*(x) \cap \exists \neq \emptyset\}$$

Then $(\Omega(\exists), \Omega^*(\exists))$ is known to be a crisp rough set where $\Omega(\exists) \neq \emptyset$. Thus the operators $\Omega(\exists), \Omega^*(\exists) : P(\hat{U}) \rightarrow P(\hat{U})$ are known to be lower and upper crisp approximation operators with respect to $(\hat{U}, \Omega^*)$.

Definition 2.9 ([29]): Let $\hat{U}$ be universal set and $\mathcal{E}$ be a set of parameters. Let $\Omega \subseteq \hat{U} \times \mathcal{E}$ be a crisp soft relation over $\hat{U} \times \mathcal{E}$. Consider a SVM $\Omega^* : \hat{U} \rightarrow P(\mathcal{E})$ defined by $\Omega^*(\kappa) = \{x \in \mathcal{E} \mid (\kappa, x) \in \Omega\}$, $\kappa \in \hat{U}$, where $P(\mathcal{E})$ consists of all subsets of $\mathcal{E}$.

The relation $\Omega$ is known to be serial if for all $\kappa \in \hat{U}$, $\Omega^*(\kappa) \neq \emptyset$. Then $(\hat{U}, \Omega, \mathcal{E})$ is known to be a crisp soft approximation space. Consider for a nonempty subset $\exists \subseteq \mathcal{E}$, then the lower and upper soft approximations of $\exists$ with respect to $(\hat{U}, \Omega, \mathcal{E})$ are represented by $\Omega(\exists)$ and $\Omega^*(\exists)$ as defined as:

$$\Omega(\exists) = \{x \in \hat{U} \mid \Omega^*(x) \subseteq \exists\}$$

and $$\Omega^*(\exists) = \{x \in \hat{U} \mid \Omega^*(x) \cap \exists \neq \emptyset\}$$

Then a crisp soft rough set is the pair $(\Omega(\exists), \Omega^*(\exists))$ where $\Omega(\exists) \neq \emptyset$. Thus the operators $\Omega(\exists), \Omega^*(\exists) : P(\mathcal{E}) \rightarrow P(\hat{U})$ are known to be a lower and upper crisp soft rough approximation operators with respect to $(\hat{U}, \mathcal{E}, \Omega)$.

3. Soft rough Pythagorean fuzzy set (SRPFS)

In this section, we will present the concept of SRPFS by combining the crisp soft relation from $\hat{U}$ to $\mathcal{E}$ with the rough Pythagorean fuzzy set. Furthermore, the concept of soft rough Pythagorean fuzzy (SRPF) approximation...
operators are investigated, and some basic properties of proposed operators are also discussed.

**Definition 3.1:** Consider a crisp soft approximation space \((\hat{U}, \tilde{\Omega}, \Omega)\). Let \(\mathfrak{A} = \{(k, \mu, \nu) \mid k \in \hat{U}\}\) where \(\mathfrak{A} \in \text{PSFS}(\mathcal{E})\). Then the lower and upper soft approximations of \(\mathfrak{A}\) with respect to \((\hat{U}, \tilde{\Omega}, \Omega)\) are represented by \(\Omega(\mathfrak{A})\) and \(\tilde{\Omega}(\mathfrak{A})\) as defined as:

\[
\Omega(\mathfrak{A}) = \{(k, \mu, \nu) \mid k \in \hat{U}\}
\]

\[
\tilde{\Omega}(\mathfrak{A}) = \{(k, \mu, \nu) \mid k \in \hat{U}\}
\]

where

\[
\mu(\mathfrak{A}) = \bigwedge_{k \in \hat{U}} \mu(k), \quad \text{and} \quad \nu(\mathfrak{A}) = \bigvee_{k \in \hat{U}} \nu(k)
\]

which satisfies that \(0 \leq (\mu(\mathfrak{A}))^2 + (\nu(\mathfrak{A}))^2 \leq 1\)

and

\[
\mu(\mathfrak{A}) = \bigwedge_{k \in \hat{U}} \mu(k), \quad \text{and} \quad \nu(\mathfrak{A}) = \bigvee_{k \in \hat{U}} \nu(k)
\]

with the condition that \(0 \leq (\mu(\mathfrak{A}))^2 + (\nu(\mathfrak{A}))^2 \leq 1\).

Then the pair \((\Omega(\mathfrak{A}), \tilde{\Omega}(\mathfrak{A}))\) is known to be an SRPFSS of \(\mathfrak{A}\) with respect to \((\hat{U}, \tilde{\Omega}, \Omega)\) where \(\Omega(\mathfrak{A}) \neq \tilde{\Omega}(\mathfrak{A})\). Thus, the operators \(\Omega(\mathfrak{A}), \tilde{\Omega}(\mathfrak{A})\) : \(\text{PSFS}(\mathcal{E}) \to \text{PSFS}(\hat{U})\) are known to be lower and upper SRPF approximation operators with respect to \((\hat{U}, \tilde{\Omega}, \Omega)\).

**Remark 3.1:** Suppose a crisp soft approximation space \((\hat{U}, \tilde{\Omega}, \Omega)\) and let \(\mathfrak{A} = \{(k, \mu, \nu) \mid k \in \hat{U}\}\) where \(\mathfrak{A} \in \text{PSFS}(\mathcal{E})\) is the family of fuzzy sets. Then the defined SRPF approximation operators \(\Omega(\mathfrak{A})\) and \(\tilde{\Omega}(\mathfrak{A})\) degenerate into the soft rough fuzzy set, that is:

\[
\Omega(\mathfrak{A}) = \{(k, \mu, \nu) \mid k \in \hat{U}\}
\]

\[
\tilde{\Omega}(\mathfrak{A}) = \{(k, \mu, \nu) \mid k \in \hat{U}\}
\]

where \(\mu(\mathfrak{A}) = \bigwedge_{k \in \hat{U}} \mu(k), \nu(\mathfrak{A}) = \bigvee_{k \in \hat{U}} \nu(k)\). Hence the pair \((\Omega(\mathfrak{A}), \tilde{\Omega}(\mathfrak{A}))\) is known to be soft rough fuzzy set.

**Remark 3.2:** Suppose a crisp soft approximation space \((\hat{U}, \tilde{\Omega}, \Omega)\) and for a crisp set \(\mathfrak{A} \in \text{PSFS}(\mathcal{E})\) of \(\mathcal{E}\). Then the defined SRPF approximation operators \(\Omega(\mathfrak{A})\) and \(\tilde{\Omega}(\mathfrak{A})\) degenerate into soft rough approximation operators as defined in Definition 2.9. Hence, it is clear that Definition 3.1 is the generalization of Definition 2.9.

**Example 3.1:** Suppose a universal set \(\hat{U} = \{x_1, x_2, x_3, x_4, x_5\}\) and let \(\mathcal{E} = \{x_1, x_2, x_3, x_4\}\) be initial set of parameters. Then a SS \((E, \mathcal{E})\) over \(\hat{U}\) defined as follows

\[
E(x_1) = \{k_1, k_2, k_3\}, \quad E(x_2) = \emptyset,
\]

\[
E(x_3) = \{k_2, k_4\}, \quad E(x_4) = \hat{U}
\]

Now to define crisp soft relation \(\Omega\) on \(\hat{U} \times \mathcal{E}\), that is

\[
\Omega = \{(k_1, x_1), (k_2, x_1), (k_3, x_1), (k_2, x_3), (k_4, x_3), (k_1, x_4), (k_2, x_4), (k_3, x_4), (k_4, x_4), (k_5, x_4)\}
\]

From Definition 2.9 to get the SVM \(\Omega^*\), that is

\[
\Omega^*(k_1) = \{x_1, x_4\}, \quad \Omega^*(k_2) = \{x_1, x_3, x_4\},
\]

\[
\Omega^*(k_3) = \{x_1, x_4\}, \quad \Omega^*(k_4) = \{x_3, x_4\},
\]

\[
\Omega^*(k_5) = \{x_4\}
\]

Now to define a Pythagorean fuzzy set \(\mathfrak{A} \in \text{PFS}(\mathcal{E})\) as follows

\[
\mathfrak{A} = \{(x_1, 0, 0.3), (x_2, 0.8, 0.5), (x_3, 0.4, 0.7), (x_4, 0.7, 0.5)\}
\]

From Definition 3.1 to get the lower and upper soft approximation \(\Omega(\mathfrak{A})\) and \(\tilde{\Omega}(\mathfrak{A})\), that are

\[
\Omega(\mathfrak{A}) = \{(k_1, 0.7, 0.5), (k_2, 0.4, 0.7), (k_3, 0.7, 0.5), (k_4, 0.4, 0.7), (k_5, 0.7, 0.5)\}
\]

\[
\tilde{\Omega}(\mathfrak{A}) = \{(k_1, 0.9, 0.3), (k_2, 0.9, 0.3), (k_3, 0.9, 0.3), (k_4, 0.7, 0.5), (k_5, 0.7, 0.5)\}
\]

**Theorem 3.1:** Consider a crisp soft approximation space \((\hat{U}, \tilde{\Omega}, \Omega)\). Then the following conditions are satisfied for the lower and upper SRPF approximation operators \(\Omega(\mathfrak{A})\) and \(\tilde{\Omega}(\mathfrak{A})\), for any \(\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2 \in \text{PSFS}(\mathcal{E})\).

(i) \(\Omega(\mathfrak{A}) = \sim \tilde{\Omega}(\sim \mathfrak{A})\)

(ii) \(\Omega(\mathfrak{A}_1 \cap \mathfrak{A}_2) = \Omega(\mathfrak{A}_1) \cap \Omega(\mathfrak{A}_2)\)

(iii) \(\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \Rightarrow \Omega(\mathfrak{A}_1) \subseteq \tilde{\Omega}(\mathfrak{A}_2)\)

(iv) \(\Omega(\mathfrak{A}_1) \cup \mathfrak{A}_2 \supseteq \Omega(\mathfrak{A}_1) \cup \tilde{\Omega}(\mathfrak{A}_2)\)

(v) \(\tilde{\Omega}(\mathfrak{A}) = \sim \Omega(\sim \mathfrak{A})\)

(vi) \(\tilde{\Omega}(\mathfrak{A}_1) \cup \mathfrak{A}_2 = \tilde{\Omega}(\mathfrak{A}_1) \cup \tilde{\Omega}(\mathfrak{A}_2)\)

(vii) \(\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \Rightarrow \tilde{\Omega}(\mathfrak{A}_1) \subseteq \tilde{\Omega}(\mathfrak{A}_2)\)

(viii) \(\tilde{\Omega}(\mathfrak{A}_1) \cup \mathfrak{A}_2 \supseteq \tilde{\Omega}(\mathfrak{A}_1) \cup \tilde{\Omega}(\mathfrak{A}_2)\).

**Proof:** (i) By Definition 3.1, we have

\[
\sim \Omega(\sim \mathfrak{A}) = \{(k, \mu(\mathfrak{A}), \nu(\mathfrak{A})) \mid k \in \hat{U}\}
\]

\[
= \left\{(k, \bigwedge_{k \in \hat{U}} \mu(k), \bigvee_{k \in \hat{U}} \nu(k)) \mid k \in \hat{U}\right\}
\]

\[
= \left\{(k, \bigwedge_{k \in \hat{U}} \mu(k), \bigvee_{k \in \hat{U}} \nu(k)) \mid k \in \hat{U}\right\}
\]

implies \(\sim \Omega(\sim \mathfrak{A}) = \Omega(\mathfrak{A})\)
(ii) Now to prove that $\Omega(\mathfrak{A}_1 \cap \mathfrak{A}_2) = \Omega(\mathfrak{A}_1) \cap \Omega(\mathfrak{A}_2)$, consider

$$
\Omega(\mathfrak{A}_1 \cap \mathfrak{A}_2) = \left\{ \langle \kappa, \mu_{\mathfrak{A}_1 \cap \mathfrak{A}_2}(\kappa), \psi_{\mathfrak{A}_1 \cap \mathfrak{A}_2}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

$$
= \left\{ \langle \kappa, \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_1}(\kappa) \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_2}(\kappa), \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_1}(\kappa) \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_2}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

$$
= \left\{ \langle \kappa, \mu_{\mathfrak{A}_1}(\kappa) \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_1}(\kappa) \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_2}(\kappa), \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_1}(\kappa) \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_2}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

implies $\Omega(\mathfrak{A}_1 \cap \mathfrak{A}_2) = \Omega(\mathfrak{A}_1) \cap \Omega(\mathfrak{A}_2)$

(iii) Now to prove that if $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ then $\Omega(\mathfrak{A}_1) \subseteq \Omega(\mathfrak{A}_2)$

$$
\Omega(\mathfrak{A}_1) = \left\{ \langle \kappa, \mu_{\mathfrak{A}_1}(\kappa), \psi_{\mathfrak{A}_1}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

$$
\subseteq \left\{ \langle \kappa, \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_1}(\kappa) \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_2}(\kappa), \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_1}(\kappa) \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_2}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

implies $\Omega(\mathfrak{A}_1) \subseteq \Omega(\mathfrak{A}_2)$

(iv) Next to prove that $\Omega(\mathfrak{A}_1 \cup \mathfrak{A}_2) \supseteq \Omega(\mathfrak{A}_1) \cup \Omega(\mathfrak{A}_2)$

$$
\Omega(\mathfrak{A}_1 \cup \mathfrak{A}_2) = \left\{ \langle \kappa, \mu_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(\kappa), \psi_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

$$
= \left\{ \langle \kappa, \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(\kappa) \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_2}(\kappa), \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_1}(\kappa) \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_2}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

$$
= \left\{ \langle \kappa, \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_1}(\kappa) \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_1}(\kappa) \bigwedge_{x \in 2^\Omega} \mu_{\mathfrak{A}_2}(\kappa), \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_1}(\kappa) \bigvee_{x \in 2^\Omega} \psi_{\mathfrak{A}_2}(\kappa) \rangle \mid \kappa \in \hat{U} \right\}
$$

implies $\Omega(\mathfrak{A}_1 \cup \mathfrak{A}_2) \supseteq \Omega(\mathfrak{A}_1) \cup \Omega(\mathfrak{A}_2)$

Proofs (v) to (viii) are straightforward and follows the above results and Definition 3.1.

Here, by counter example we will show that the equality does not hold in parts (iv) and (viii)

**Example 3.2:** Consider the crisp soft relation $\Omega$ on $\hat{U} \times \mathfrak{E}$ from Example 3.1, that is $\Omega = \{(\kappa_1, x_1), (\kappa_2, x_1), (\kappa_3, x_1), (\kappa_2, x_2), (\kappa_4, x_3), (\kappa_1, x_4), (\kappa_2, x_4), (\kappa_3, x_4), (\kappa_4, x_4), (\kappa_5, x_4)\}$

From Definition 2.9 to get the set-valued functions $\Omega^*$, that are

$$
\Omega^*(\kappa_1) = \{x_1, x_4\}, \quad \Omega^*(\kappa_2) = \{x_1, x_2, x_3, x_4\},
$$

$$
\Omega^*(\kappa_3) = \{x_1, x_4\}, \quad \Omega^*(\kappa_4) = \{x_3, x_4\}, \quad \Omega^*(\kappa_5) = \{x_4\}
$$

Now to define a Pythagorean fuzzy sets $\mathfrak{A}_1$ and $\mathfrak{A}_2 \in \text{PFS}(\mathfrak{E})$ as follows

$$
\mathfrak{A}_1 = \{(x_1, 0.9, 0.3), (x_2, 0.8, 0.5), (x_3, 0.4, 0.7), (x_4, 0.7, 0.5)\}
$$

$$
\mathfrak{A}_2 = \{(x_1, 0.85, 0.5), (x_2, 0.9, 0.4), (x_3, 0.5, 0.6), (x_4, 0.4, 0.3)\}
$$

Next consider

$$
\mathfrak{A}_1 \cup \mathfrak{A}_2 = \{(x_1, 0.9, 0.3), (x_2, 0.9, 0.4), (x_3, 0.5, 0.6), (x_4, 0.7, 0.3)\}
$$

To find

$$
\Omega(\mathfrak{A}_1) = \{(\kappa_1, 0.7, 0.5), (\kappa_2, 0.4, 0.7), (\kappa_3, 0.7, 0.5), (\kappa_4, 0.4, 0.7), (\kappa_5, 0.7, 0.5)\}
$$

$$
\Omega(\mathfrak{A}_2) = \{(\kappa_1, 0.4, 0.5), (\kappa_2, 0.4, 0.6), (\kappa_3, 0.4, 0.5), (\kappa_4, 0.4, 0.6), (\kappa_5, 0.4, 0.3)\}
$$

$$
\Omega(\mathfrak{A}_1 \cup \mathfrak{A}_2) = \{(\kappa_1, 0.7, 0.5), (\kappa_2, 0.4, 0.6), (\kappa_3, 0.7, 0.5), (\kappa_4, 0.4, 0.7), (\kappa_5, 0.7, 0.3)\}
$$

$$
\Omega(\mathfrak{A}_1 \cup \mathfrak{A}_2) = \{(\kappa_1, 0.7, 0.3), (\kappa_2, 0.5, 0.6), (\kappa_3, 0.7, 0.3), (\kappa_4, 0.5, 0.6), (\kappa_5, 0.7, 0.3)\}
$$

Therefore, it is clear that $\Omega(\mathfrak{A}_1 \cup \mathfrak{A}_2) \not\subseteq \Omega(\mathfrak{A}_1) \cup \Omega(\mathfrak{A}_2)$ because

$$
\{(\kappa_1, \mu_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(\kappa_1), \psi_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(\kappa_1)) \mid \kappa_1 \in \hat{U} \}
$$

$$
\not\subseteq \{(\kappa_1, \mu_{\mathfrak{A}_1}(\kappa_1), \psi_{\mathfrak{A}_1}(\kappa_1)) \mid \kappa_1 \in \hat{U} \}
$$

implies $(\kappa_1, 0.7, 0.3) \not\subseteq (\kappa_1, 0.7, 0.5) \Rightarrow 0.7 \leq 0.7$ but $0.3 \not\leq 0.5$. Similarly, we can show that

$$
\Omega(\mathfrak{A}_1 \cap \mathfrak{A}_2) \not\subseteq \Omega(\mathfrak{A}_1) \cap \Omega(\mathfrak{A}_2)
$$
4. Pythagorean fuzzy soft rough set (PFSRS)

In this section, on the bases of IF soft rough set [29] we originate the new notion of PFSRS. Moreover, some fundamental properties of Pythagorean fuzzy soft rough (PFSR) approximation operators are also discussed in detail.

**Definition 4.1:** Let $\mathcal{E}$ be a set of parameter and $\hat{U}$ be a universal set. Then $(F, \mathcal{E})$ is known to be a Pythagorean fuzzy SS over $\hat{U}$ if $F : \mathcal{E} \rightarrow \text{PFS}(\hat{U})$ such that $\forall x \in \mathcal{E}, F(x) = \{[\mu, \nu] | x \in \mathcal{E} \} \in \text{PFS}(\hat{U})$ where $\mu_{\mathcal{E}(x)}(x) \in [0, 1]$ and $\nu_{\mathcal{E}(x)}(x) \in [0, 1]$ represent the membership grade and non-membership grade of $x \in \mathcal{E}$ respectively, which satisfies that $0 \leq (\mu_{\mathcal{E}(x)}(x))^2 + (\nu_{\mathcal{E}(x)}(x))^2 \leq 1$.

**Definition 4.2:** Consider a set of parameters $\mathcal{E}$ on a universal set $\hat{U}$. Suppose $\Omega$ be an arbitrary FSR over $\hat{U}$ to $\mathcal{E}$. Then the triplet $(\hat{U}, \mathcal{E}, \Omega)$ is known to be a fuzzy soft approximation space. Now for any $\Xi = \{(\mu_{\mathcal{E}(x)}, \nu_{\mathcal{E}(x)}(x)) | x \in \mathcal{E} \} \in \text{PFS}(\mathcal{E})$, the lower and upper soft approximation of $\Xi$ w.r.t to $(\hat{U}, \mathcal{E}, \Omega)$ are represented by $\Omega_{\Xi}(\hat{U})$ and $\Xi_{\hat{U}}(\Omega)$ as follows:

- $\Omega_{\Xi}(\hat{U}) = \{(\mu_{\mathcal{E}(x)}, \nu_{\mathcal{E}(x)}(x)) | x \in \hat{U} \}$
- $\Xi_{\hat{U}}(\Omega) = \{(\mu_{\mathcal{E}(x)}, \nu_{\mathcal{E}(x)}(x)) | x \in \hat{U} \}$

where

- $\mu_{\Omega_{\Xi}(\hat{U})}(\kappa) = \bigwedge_{x \in \mathcal{E}} \{[1 - \mu_{\Xi}(x)] \lor \mu_{\Xi}(x)\}$, and
- $\nu_{\Xi_{\hat{U}}(\Omega)}(\kappa) = \bigvee_{x \in \mathcal{E}} \{[\mu_{\Xi}(x)] \land \nu_{\Xi}(x)\}$

and

- $\mu_{\Xi_{\hat{U}}(\Omega)}(\kappa) = \bigvee_{x \in \mathcal{E}} \{[\mu_{\Xi}(x)] \land \nu_{\Xi}(x)\}$
- $\nu_{\Omega_{\Xi}(\hat{U})}(\kappa) = \bigwedge_{x \in \mathcal{E}} \{[1 - \mu_{\Xi}(x)] \lor \nu_{\Xi}(x)\}$

Then the pair $(\Omega_{\Xi}(\hat{U}), \Xi_{\hat{U}}(\Omega))$ is known to be a PFSRS of $\Xi$ with respect to $(\hat{U}, \mathcal{E}, \Omega)$ where $\Omega_{\Xi}(\hat{U}) \neq \Xi_{\hat{U}}(\Omega)$.

Here, it is shown that $\Omega_{\Xi}(\hat{U})$ and $\Xi_{\hat{U}}(\Omega)$ are FFS(\hat{U}), that is

$$
(\mu_{\Omega_{\Xi}(\hat{U})}(\kappa))^2 + (\nu_{\Omega_{\Xi}(\hat{U})}(\kappa))^2
= \bigvee_{x \in \mathcal{E}} \{([1 - \mu_{\Xi}(x)] \lor \mu_{\Xi}(x))^2 \}
+ \bigvee_{x \in \mathcal{E}} \{([\mu_{\Xi}(x)] \land \nu_{\Xi}(x))^2 \}
= 1 - \bigwedge_{x \in \mathcal{E}} \{([\mu_{\Xi}(x)] \lor [1 - \mu_{\Xi}(x)])^2 \}
+ \bigwedge_{x \in \mathcal{E}} \{([\mu_{\Xi}(x)] \land [1 - \mu_{\Xi}(x)])^2 \}
\leq 1 - \bigwedge_{x \in \mathcal{E}} \{([\mu_{\Xi}(x)] \lor [1 - \mu_{\Xi}(x)])^2 \}
$$

**Table 1. Fuzzy soft relation $\Omega$ from set $\mathcal{E}$**

| $\kappa$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---------|-------|-------|-------|-------|
| $\kappa_1$ | 0.8   | 0.3   | 0.5   | 0.7   |
| $\kappa_2$ | 0.9   | 0.6   | 0.4   | 0.6   |
| $\kappa_3$ | 0.7   | 0.1   | 0.8   | 0.3   |
| $\kappa_4$ | 0.2   | 0.9   | 0.3   | 0.6   |
| $\kappa_5$ | 0.8   | 0.3   | 0.9   | 0.5   |

Therefore, $\Omega_{\Xi}(\hat{U}) \in \text{PFS}(\hat{U})$. Similarly, we can prove that $\Xi_{\hat{U}}(\Omega) \in \text{PFS}(\hat{U})$. Thus the operators $\Omega_{\Xi}(\hat{U}), \Xi_{\hat{U}}(\Omega) : \text{PFS}(\mathcal{E}) \rightarrow \text{PFS}(\hat{U})$ are known to be lower and upper PFSR approximation operators with respect to $(\hat{U}, \mathcal{E}, \Omega)$.

**Remark 4.1:** By taking crisp soft approximation space $(\hat{U}, \mathcal{E}, \Omega)$ and let $\Xi \in \text{PFS}(\mathcal{E})$. Then the PFSR approximation operators $\Omega_{\Xi}(\hat{U})$ and $\Xi_{\hat{U}}(\Omega)$ in Definition 4.2, degenerate into soft rough approximation operators $\Omega_{\Xi}(\hat{U})$ and $\Xi_{\hat{U}}(\Omega)$ in Definition 3.1. Hence it is clear that PFSR approximation operators in Definition 4.2, is the generalization of SRP approximation operators in Definition 3.1.

**Remark 4.2:** By taking fuzzy soft approximation space $(\hat{U}, \mathcal{E}, \Omega)$ and let $\Xi \in \text{E}(\mathcal{E})$ where $E(\mathcal{E})$ is the collection of fuzzy sets. Then the PFSR approximation operators $\Omega_{\Xi}(\hat{U})$ and $\Xi_{\hat{U}}(\Omega)$ in Definition 4.2, degenerate into soft fuzzy rough approximation operators defined by Sun and Ma [41]. Hence, it is clear that PFSR approximation operators in Definition 4.2, is the generalization of soft fuzzy rough approximation operators defined by Sun and Ma [41].

**Example 4.1:** For a universal set $\hat{U} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ and initial set of parameters $\mathcal{E} = \{x_1, x_2, x_3, x_4\}$. Consider fuzzy SS $(F, \mathcal{E})$ over $\hat{U}$, then a FSR $\Omega$ from $\hat{U} \times \mathcal{E}$, which is defined in the following Table 1.

Now to define a Pythagorean fuzzy set $\Xi \in \text{PFS}(\mathcal{E})$ as follows

$$
\Xi = \{(x_1, 0.8, 0.4), (x_2, 0.9, 0.2), (x_3, 0.7, 0.6), (x_4, 0.6, 0.3)\}
$$

Next to find $\Omega_{\Xi}(\hat{U})$ and $\Xi_{\hat{U}}(\Omega)$, that is

- $\Omega_{\Xi}(\hat{U}) = \{(\kappa_1, 0.6, 0.5), (\kappa_2, 0.6, 0.4), (\kappa_3, 0.7, 0.6), (\kappa_4, 0.6, 0.3), (\kappa_5, 0.6, 0.6)\}$
- $\Xi_{\hat{U}}(\Omega) = \{(\kappa_1, 0.8, 0.3), (\kappa_2, 0.8, 0.4), (\kappa_3, 0.7, 0.4), (\kappa_4, 0.9, 0.2), (\kappa_5, 0.8, 0.4)\}$

**Theorem 4.1:** Consider a fuzzy soft approximation space $(\hat{U}, \mathcal{E}, \Omega)$. Then the following properties are satisfied in lower and upper SRP approximation operators $\Omega_{\Xi}(\hat{U})$ and
Proof: Proofs are straightforward as the proof of Theorem 3.1.

Definition 4.3: Consider $\mathcal{A} = \{(\kappa, \mu_{L}(\kappa), \psi_{L}(\kappa)) | \kappa \in \hat{U} \} \in \text{PFS}(\hat{U})$ and $(\alpha, \beta) \in [0, 1]$, with $\alpha + \beta \leq 1$. Then the $(\alpha, \beta)$-level cut set on $\mathcal{A}$ is represented as follows:

$$\mathcal{A}^\alpha = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) > \alpha, \psi_{L}(\kappa) \leq \beta \}$$

The set $\mathcal{A}_{\alpha} = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \}$ is a membership set of $\alpha$-level cut which is generated by $\mathcal{A}$ and similarly, $\mathcal{A}_{\alpha}^\alpha = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) > \alpha \}$ is a membership set of strong $\alpha$-level cut which is generated by $\mathcal{A}$. The set $\mathcal{A}_{\beta} = \{ \kappa \in \hat{U} | \psi_{L}(\kappa) \leq \beta \}$ is a membership set of $\beta$-level cut generated by $\mathcal{A}$ and on the same way, $\mathcal{A}_{\beta}^\beta = \{ \kappa \in \hat{U} | \psi_{L}(\kappa) < \beta \}$ is a membership set of strong $\beta$-level cut which is generated by $\mathcal{A}$.

Analogously, the other cut sets of a PFS $\mathcal{A}$ are represented as:

$$\mathcal{A}^\beta = \mathcal{A}_{\alpha} \cap \mathcal{A}_{\beta},$$

which is called $(\alpha^+, \beta)$-level cut set of $\mathcal{A}$;

$$\mathcal{A}^\alpha = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha, \psi_{L}(\kappa) < \beta \},$$

which is called $(\alpha, \beta^+)$-level cut set of $\mathcal{A}$;

$$\mathcal{A}_{\alpha}^\beta = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) > \alpha, \psi_{L}(\kappa) < \beta \},$$

which is said to be $(\alpha^+, \beta^+)$-level cut set of $\mathcal{A}$.

Theorem 4.2: The cut sets of PFSs hold the following properties, that are for all $\mathcal{L}, \mathcal{A} \in \text{PFS}(\hat{U}), \alpha, \beta \in [0, 1]$ with $\alpha^2 + \beta^2 \leq 1$:

(i) $\mathcal{A}^\alpha \cap \mathcal{A}^\beta = \mathcal{A}_{\alpha} \cap \mathcal{A}_{\beta}$,

(ii) $(\sim \mathcal{A}) = \sim (\mathcal{A}^\alpha)$,

(iii) $\mathcal{A} \cap \mathcal{L} = \mathcal{A}^\alpha \cap \mathcal{L}^\beta$,

(iv) $\mathcal{A} \cup \mathcal{L} = \mathcal{A}^\alpha \cup \mathcal{L}^\beta$,

(v) $\mathcal{A} \cup \mathcal{L} = \mathcal{A}^\alpha \cup \mathcal{L}^\beta$,

(vi) $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2 \Rightarrow \mathcal{A}_{\alpha_1} \subseteq \mathcal{A}_{\alpha_2}, \mathcal{A}_{\beta_1} \subseteq \mathcal{A}_{\beta_2}$.

Proof: The proof of (i) and (iii) are straightforward and follow the Definition 4.3.

(ii) If $\mathcal{A} = \{(\kappa, \mu_{L}(\kappa), \psi_{L}(\kappa)) | \kappa \in \hat{U} \}$, then we have $(\sim \mathcal{A}) = \{(\kappa, \psi_{L}(\kappa), \mu_{L}(\kappa)) | \kappa \in \hat{U} \}$.

Now,

$$\mathcal{A}_{\alpha} = \{ \kappa \in \hat{U} | \psi_{L}(\kappa) \geq \alpha \} \quad (i)$$

and $$(\sim \mathcal{A}) = \{ \kappa \in \hat{U} | \psi_{L}(\kappa) > \alpha \} \quad (i)$$

Similarly,

$$\mathcal{A}_{\alpha}^\beta = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) > \alpha, \psi_{L}(\kappa) < \beta \} \quad \text{and} \quad \sim \mathcal{A}_{\alpha}^\beta = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) > \alpha, \psi_{L}(\kappa) < \beta \}$$

So

$$\sim \mathcal{A}_{\alpha}^\beta = \{ \kappa \in \hat{U} | \psi_{L}(\kappa) > \alpha \} \quad (ii)$$

Therefore, from (i) and (ii) we have

$$(\sim \mathcal{A}) = \sim \mathcal{A}_{\alpha}^\beta$$

Similarly, we can show that

$$(\sim \mathcal{A}) = \sim \mathcal{A}_{\alpha}^\beta$$

(iv) Consider,

$$\mathcal{L} \cap \mathcal{A} = \{ \kappa, \min(\mu_{L}(\kappa), \mu_{L}(\kappa)), \max(\psi_{L}(\kappa), \psi_{L}(\kappa)) | \kappa \in \hat{U} \}$$

Now,

$$\mathcal{L} \cap \mathcal{A} = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \} \quad \cap \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \}$$

and

$$\mathcal{L} \cap \mathcal{A} = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \} \quad \cap \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \}$$

Next, by using (i) we get

$$\mathcal{L} \cap \mathcal{A} = \mathcal{L} \cap \mathcal{A} \cap \mathcal{L} \cap \mathcal{A}$$

and

$$(\mathcal{L} \cap \mathcal{A}) \cap \mathcal{L} \cap \mathcal{A} = \{ \kappa \in \hat{U} | \max(\psi_{L}(\kappa), \psi_{L}(\kappa)) \leq \beta \} \quad \cap \{ \kappa \in \hat{U} | \psi_{L}(\kappa) \leq \beta \}$$

Now,

$$\mathcal{L} \cup \mathcal{A} = \{ \kappa \in \hat{U} | \max(\mu_{L}(\kappa), \mu_{L}(\kappa)) \geq \alpha \} \quad \cup \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \}$$

and

$$(\mathcal{L} \cup \mathcal{A} \cap \mathcal{L} \cap \mathcal{A} \cap \mathcal{L} \cap \mathcal{A}) = \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \} \quad \cup \{ \kappa \in \hat{U} | \mu_{L}(\kappa) \geq \alpha \}$$

Next, by using (i) we get

$$\mathcal{L} \cap \mathcal{A} = \mathcal{L} \cap \mathcal{A} \cap \mathcal{L} \cap \mathcal{A}$$

and

$$(\mathcal{L} \cap \mathcal{A}) \cap \mathcal{L} \cap \mathcal{A} = \{ \kappa \in \hat{U} | \max(\psi_{L}(\kappa), \psi_{L}(\kappa)) \leq \beta \} \quad \cap \{ \kappa \in \hat{U} | \psi_{L}(\kappa) \leq \beta \}$$

Now,
\[(\mathcal{L} \cup \mathcal{S})^\beta = \{\kappa \in \hat{U} \mid \min(\psi_\mathcal{L}(\kappa), \psi_\mathcal{S}(\kappa)) \leq \beta\} \]
\[= \{\kappa \in \hat{U} \mid \psi_\mathcal{L}(\kappa) \leq \beta\} \cup \{\kappa \in \hat{U} \mid \psi_\mathcal{S}(\kappa) \leq \beta\} \]
\[= \mathcal{L}^\beta \cup \mathcal{S}^\beta \]

As we know that
\[\mathcal{L} \subseteq \mathcal{L} \cup \mathcal{S} \quad \text{and} \quad \mathcal{S} \subseteq \mathcal{L} \cup \mathcal{S} \]

Then by using (iii) we have:
\[\mathcal{L}^\beta_a \subseteq (\mathcal{L} \cup \mathcal{S})^\beta_a \quad \text{and} \quad \mathcal{S}^\beta_a \subseteq (\mathcal{L} \cup \mathcal{S})^\beta_a \]

this implies that
\[\mathcal{L}^\beta_a \cup \mathcal{S}^\beta_a \subseteq (\mathcal{L} \cup \mathcal{S})^\beta_a \]

(iv) Consider for any \(\kappa \in \mathcal{S}_a\), then by Definition 4.3, we get \(\mu_3(\kappa) \geq \alpha_1 \geq \alpha_2 \Rightarrow \mu_3(\kappa) \geq \alpha_2\), hence \(\kappa \in \mathcal{S}_a\), therefore we have \(\mathcal{S}_a \subseteq \mathcal{S}_2\). Similarly, we can get the second one \(\mathcal{S}_a \subseteq \mathcal{S}_2\). Consequently \(\mathcal{S}_a \cap \mathcal{S}_a \subseteq \mathcal{S}_a \cap \mathcal{S}_a\), then by using property (i) we get \(\mathcal{S}_a^\beta \subseteq \mathcal{S}_a^\beta\).

Consider a FSR \(\Omega\) from \(\hat{U}\) to \(\mathcal{E}\), represented by
\[\Omega_\alpha(\kappa) = \{x \in \mathcal{E} \mid \mu_\Omega(x, x) \geq \alpha\}, \quad \text{for } \alpha \in [0, 1]\]

\[\Omega_\alpha^+(\kappa) = \{x \in \mathcal{E} \mid \mu_\Omega(x, x) > \alpha\}, \quad \text{for } \alpha \in [0, 1]\]

\[\Omega_\alpha^-(\kappa) = \{x \in \mathcal{E} \mid \psi_\Omega(x, x) < \alpha\}, \quad \text{for } \alpha \in (0, 1]\]

Then \(\Omega_\alpha, \Omega_\alpha^+, \Omega_\alpha^-\) and \(\Omega_\alpha^+\) are crisp soft relations on \(\hat{U} \times \mathcal{E}\).

In Theorems 4.3 and 4.4, it is presented that FFSR approximation operators can be denoted by crisp soft rough approximation operators.

**Theorem 4.3:** Consider a fuzzy soft approximation space \((\hat{U}, \mathcal{E}, \Omega, \mathcal{S})\) and \(\mathcal{S} \in \text{PFS}(\mathcal{E})\). Then the upper FFSR approximation operator can be shown as follows, for all \(\kappa \in \hat{U}\):

(i)
\[\mu_\Omega(\kappa) = \bigvee_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha(\kappa)]\]
\[= \bigvee_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha^+(\kappa)]\]
\[= \bigvee_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha^-\beta(\kappa)]\]
\[= \bigvee_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha^+(\kappa)]\]

(ii) The upper crisp soft rough approximation operator according to Definition 2.9, we have
\[\psi_\Omega(\kappa) = \bigwedge_{\alpha \in [0, 1]} [\alpha \vee (1 - \Omega\alpha^-\beta(\kappa)]]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \vee \Omega\alpha^-\beta(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \vee \Omega\alpha^-\beta(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \vee \Omega\alpha^-\beta(\kappa)]\]

and more over for any \(\alpha \in [0, 1]\),

(iii)
\[\Omega\alpha(\kappa) = \bigwedge_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha^+(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha^-\beta(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha^+(\kappa)]\]

**Proof:** (i) For any \(\kappa \in \hat{U}\), we have
\[\bigvee_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha(\kappa)]\]
\[= \sup_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha(\kappa)]\]

(ii) The upper crisp soft rough approximation operator according to Definition 2.9, we have
\[\bigvee_{\alpha \in [0, 1]} [\alpha \wedge \Omega\alpha(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \vee (1 - \Omega\alpha^-\beta(\kappa)]]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \vee \Omega\alpha^-\beta(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \vee \Omega\alpha^-\beta(\kappa)]\]
\[= \bigwedge_{\alpha \in [0, 1]} [\alpha \vee \Omega\alpha^-\beta(\kappa)]\]
\[ = \bigwedge_{x \in \mathcal{E}} \{1 - \mu_{\Omega}(\kappa, x) \} \lor \psi_{\Theta}(x) \]

\[ = \psi_{\Pi(\Theta)}(\kappa) \]

On the same way we can prove that

\[ \psi_{\Pi(\Theta)}(\kappa) = \bigwedge_{\alpha \in [0,1]} [\alpha \lor \Omega_{1-\alpha}^+(\Theta^+)(\kappa)] \]

\[ = \bigwedge_{\alpha \in [0,1]} [\alpha \lor \Omega_{1-\alpha}^-(\Theta^-)(\kappa)] \]

\[ = \bigwedge_{\alpha \in [0,1]} [\alpha \lor \Omega_{1-\alpha}^-(\Theta^-)(\kappa)] \]

(iii) It is easily verified that \( \Omega_{\alpha}^+(\Theta^+) \subseteq \Omega_{\alpha}^-(\Theta^+) \subseteq \Omega_{\alpha}(\Theta) \). We have just to prove that \( \Omega_{\alpha}(\Theta) \subseteq \Omega_{\alpha}^- \subseteq \Omega_{\alpha}^-(\Theta^-) \) and \( \Omega_{\alpha}(\Theta) \subseteq \Omega_{\alpha}^- \).

Let for all \( \kappa \in \Omega_{\alpha}(\Theta) \) implies \( \mu_{\Pi(\Theta)}(\kappa) > \alpha \). Now according to Definition 4.2, \( \mu_{\Pi(\Theta)}(\kappa) = \bigwedge_{x \in \mathcal{E}} [\mu_{\Omega}(\kappa, x) \land \mu_{\rho}(\kappa)] > \alpha \) holds. So there exist \( x_0 \in \mathcal{E} \) such that \( \mu_{\Omega}(\kappa, x_0) \land \mu_{\rho}(\kappa) > \alpha \), this implies that \( \mu_{\Omega}(\kappa, x_0) > \alpha \) and \( \mu_{\rho}(\kappa) > \alpha \). So \( x_0 \in \Omega_{\alpha}(\Theta) \) and \( x_0 \in \Omega_{\alpha}^- \).

Next for any \( \kappa \in \Omega_{\alpha}(\Theta) \), we have \( \Omega_{\alpha}(\Theta) \subseteq \Omega_{\alpha}^-(\Theta^-) \).

Since \( \mu_{\Pi(\Theta)}(\kappa) = \bigvee_{\beta \in [0,1]} [\beta \land \Omega_{\beta}(\Theta) \geq \alpha \land \Omega_{\beta}(\Theta) = \alpha \Rightarrow \mu_{\Pi(\Theta)}(\kappa) \geq \alpha \), thus \( \kappa \in \Omega_{\alpha}^-(\Theta^-) \). Therefore \( \Omega_{\alpha}^-(\Theta^-) \subseteq \Omega_{\alpha}^-(\Theta^-) \).

Theorem 4.4: Consider a fuzzy soft approximation space \( (\mathcal{U}, \mathcal{E}, \Xi) \) and \( \Theta \in \text{PFS}(\mathcal{E}) \). Then the lower PFS approximation operator can be shown as: for all \( \kappa \in \mathcal{U} ; \)

(iii) \[ [\Theta(\Theta)]^{-\alpha} \subseteq \Omega_{1-\alpha}^-(\Theta^-) \subseteq \Omega_{1-\alpha}^-(\Theta^-) \]

\[ \subseteq \Omega_{1-\alpha}^-(\Theta^-) \subseteq \Omega_{1-\alpha}^-(\Theta^-) \]

Proof: The proofs of (i) and (ii) according to Theorems 4.2 and 4.3. Now for any \( \kappa \in \mathcal{U} \), consider

\[ \mu_{\Pi(\Theta)}(\kappa) = \bigvee_{\alpha \in [0,1]} [\alpha \land \Omega_{\alpha}^-(\Theta^-)(\kappa)] \]

\[ = \bigvee_{\alpha \in [0,1]} [\alpha \land \Omega_{\alpha}^-(\Theta^-)(\kappa)] \]

\[ = \bigvee_{\alpha \in [0,1]} [\alpha \land \Omega_{\alpha}^-(\Theta^-)(\kappa)] \]

\[ = \bigvee_{\alpha \in [0,1]} [\alpha \land \Omega_{\alpha}^-(\Theta^-)(\kappa)] \]

\[ = \bigvee_{\alpha \in [0,1]} [\alpha \land \Omega_{\alpha}^-(\Theta^-)(\kappa)] \]

\[ = \bigvee_{\alpha \in [0,1]} [\alpha \land \Omega_{\alpha}^-(\Theta^-)(\kappa)] \]

The proofs of (iii) and (vi) are straightforward to the proofs of (iii) and (vi) of Theorem 4.3.
5. Application of Pythagorean fuzzy soft rough set (PFSRS) in decision-making

Here in this section, the technique for the DM process is constructed on the approach of PFSRS. For this, we will define the ring sum and ring product operations on PFSSs. By the operation, the basic concept of this method and approach to DM is given, which is based on the PFSRS approach.

Definition 5.1 ([42]): Let \( \mathcal{A}_1 = (\mu_{A_1}(\kappa), \psi_{A_1}(\kappa)) \), \( \mathcal{A}_2 = (\mu_{A_2}(\kappa), \psi_{A_2}(\kappa)) \) ∈ PFS(\( \hat{U} \)). Then the ring sum for PFSSs \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) can be defined as follows:

\[
\mathcal{A}_1 \oplus \mathcal{A}_2 = \{ (\kappa, \sqrt{\mu_{A_1}(\kappa)^2 + (\mu_{A_2}(\kappa))^2 - \mu_{A_1}(\kappa) \mu_{A_2}(\kappa) - \psi_{A_1}(\kappa) \psi_{A_2}(\kappa)} ) \mid \kappa \in \hat{U} \}
\]

Definition 5.2 ([42]): Let \( \mathcal{A}_1 = (\mu_{A_1}(\kappa), \psi_{A_1}(\kappa)) \), \( \mathcal{A}_2 = (\mu_{A_2}(\kappa), \psi_{A_2}(\kappa)) \) ∈ PFS(\( \hat{U} \)). Then the ring product for PFSSs \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) can be defined as follows:

\[
\mathcal{A}_1 \otimes \mathcal{A}_2 = \{ (\kappa, \mu_{A_1}(\kappa) \mu_{A_2}(\kappa), \sqrt{\psi_{A_1}(\kappa)^2 + (\psi_{A_2}(\kappa))^2 - \psi_{A_1}(\kappa) \psi_{A_2}(\kappa) \mu_{A_1}(\kappa) \mu_{A_2}(\kappa)} ) \mid \kappa \in \hat{U} \}
\]

The final decision is only one, one may go back to the second step and change the optimum decision object in the final step of the given algorithm, when there exist too many “optimal choices” to be chosen.

The concept of the proposed algorithm is illustrated with the help of the following example.

6. Illustrative example

For a certain senior position of a doctor in the Pakistan Institute of Medical Sciences (PIMS) Cardiac Centre, the appointment of new faculty has to face a very complex evaluation and DM process. The skill and ability of a candidate may be judged with respect to various attributes like “physical and surgical productivity” “managerial skill” “ability to work under pressure” “research productivity”, etc. In order to take the right decision about the candidate, the professional experts’ opinions are needed for these criteria.

Consider that \( \hat{U} = \{ \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5 \} \) be set of five candidates who fulfill the requirements for the senior faculty position in PIMS. In order to appoint the most qualified and suitable candidates for the required position, a team of experts is organized and chaired by Prof. Z as a director. The team of experts will judge the candidates upon the criteria in the parameter set \( \mathcal{E} = \{ x_1, x_2, x_3, x_4, x_5, x_6 \} \) where

\[
\begin{align*}
x_1 &= \text{physical and surgical productivity} \\
x_2 &= \text{managerial skill} \\
x_3 &= \text{experience and research productivity} \\
x_4 &= \text{ability to work under pressure} \\
x_5 &= \text{academic leadership qualities} \\
x_6 &= \text{contribution to PIMS}
\end{align*}
\]

According to the background and experience, the team of experts wants to appoint the candidate which qualifies with the parameters of \( \mathcal{E} \) who deserves extremely from candidate in \( \hat{U} \).
For PFSRS

**step i** Consider that the experts explain the gorgeous/attractiveness of the candidates by calculating an FSR \( \Omega \) from \( \hat{U} \times E \) which is given in the following Table 2.

**step ii** Now consider the team of experts present the optimum normal decision object \( Z \) which is a Pythagorean fuzzy subset over the a set \( E \) as follows:

\[
Z = \{ (x_1, 0.5, 0.3), (x_2, 0.6, 0.4), (x_3, 0.7, 0.2),
(x_4, 0.4, 0.5), (x_5, 0.5, 0.4), (x_6, 0.8, 0.2) \}.
\]

Therefore, the characteristics of the candidates upon the criteria of the given parameters can be described by the PFS. For example, the standard of the candidate under criteria/parameter \( x_1 \) is \((0.5, 0.3)\). The value 0.5 is the degree of membership and the value 0.3 is the degree of non-membership of candidate under criteria \( x_1 \) respectively. In other words, candidates is qualified for the grade of membership is 0.5 and disqualified for the grade of non-membership is 0.3.

**step iii** Next to calculate \( \Omega(\mathcal{H}) \) and \( \Omega(\mathcal{H}) \)

\[
\Omega(\mathcal{H}) = \{ (k_1, 0.4, 0.5), (k_2, 0.5, 0.4), (k_3, 0.5, 0.5), (k_4, 0.2, 0.5), (k_5, 0.5, 0.4) \}
\]

**step iv** By ring sum operation calculate the choice set.

\[
\mathcal{H} = \{ (k_1, 0.75604, 0.1), (k_2, 0.78581, 0.12), (k_3, 0.72111, 0.2), (k_4, 0.52915, 0.2), (k_5, 0.8544, 0.08) \}
\]

**step v** Compute the top level threshold value \( \lambda = (\mu, \psi) \in S^3 \) such that \( \mu = \max_{1 \leq i \leq n} \mu_X(k_i) \) and \( \psi = \min_{1 \leq i \leq n} \psi_X(k_i) \). It is clear that in choice set \( \mathcal{H} \) the Pythagorean fuzzy value \( \lambda \) is the maximum choice value. If \( \mu_X(k_i) \geq S^3 \mu \) and \( \psi_X(k_i) \geq S^3 \psi \), then the optimum decision value is \( k_i \). Hence, the optimal decision is \( \lambda = k_5 \) = \( (0.8544, 0.08) \).

**Ring Product Operator**

Now to calculate the optimal decision through ring product operator, we have

\[
\mathcal{H} = \{ (k_1, 0.28, 0.52915), (k_2, 0.35, 0.48539), (k_3, 0.3, 0.60828), (k_4, 0.4, 0.60828), (k_5, 0.4, 0.44) \}.
\]

Therefore, the optimal decision is \( \lambda = k_5 \) = \( (0.4, 0.44) \). Therefore, the most qualified and suitable candidate for the required position is \( k_5 \).

For SRPFS

**step i** Consider that the experts explain the gorgeous/attractiveness of the candidates through the proposed model of SRPFS. Consider a SS \((F, E)\) over \( \hat{U} \) defined as follows

\[
F(x_1) = \{ k_1, k_6 \}, F(x_2) = \varphi,
F(x_3) = \{ k_2, k_4, k_5 \}, F(x_6) = \{ k_1, k_3, k_4 \},
F(x_5) = \{ k_2, k_3, k_6 \}, F(x_8) = \{ k_1, k_3, k_5 \}
\]

Now to define the crisp soft relation \( \Omega \) on \( \hat{U} \times E \), that is

\[
\Omega = \{ (k_1, x_1), (k_6, x_1), (k_2, x_3), (k_3, x_3), (k_4, x_3), (k_5, x_3), (k_1, x_4), (k_3, x_4), (k_4, x_4), (k_2, x_5), (k_3, x_5), (k_5, x_5), (k_1, x_6), (k_3, x_6), (k_5, x_6) \}
\]

From Definition 2.9 to get the SVM \( \Omega^* \), that is

\[
\Omega^*(k_1) = \{ x_1, x_4, x_6 \},
\Omega^*(k_2) = \{ x_3, x_5 \}, \Omega^*(k_3) = \{ x_3, x_4, x_5, x_6 \},
\Omega^*(k_4) = \{ x_3, x_4 \}, \Omega^*(k_5) = \{ x_3, x_5, x_6 \}
\]

**step ii** Now consider the team of experts present the optimum normal decision object \( Z \) which is a Pythagorean fuzzy subset over the a set \( E \) as follows:

\[
Z = \{ (x_1, 0.5, 0.3), (x_2, 0.6, 0.4), (x_3, 0.7, 0.2), (x_4, 0.4, 0.5), (x_5, 0.5, 0.4), (x_6, 0.8, 0.2) \}
\]

Therefore, the characteristics of the candidates upon the criteria/parameter \( x_1 \) is \((0.5, 0.3)\). The value 0.5 is

---

**Table 2. Fuzzy soft relation \( \Omega \) from set \( \hat{U} \times E. \)**

| \( \Omega \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) |
|-------------|--------|--------|--------|--------|--------|--------|
| \( k_1 \)   | 0.6    | 0.5    | 0.8    | 0.7    | 0.9    | 0.4    |
| \( k_2 \)   | 0.8    | 0.4    | 0.7    | 0.6    | 0.5    | 0.3    |
| \( k_3 \)   | 0.6    | 0.3    | 0.4    | 0.5    | 0.1    | 0.6    |
| \( k_4 \)   | 0.4    | 0.5    | 0.1    | 0.3    | 0.8    | 0.2    |
| \( k_5 \)   | 0.5    | 0.4    | 0.8    | 0.7    | 0.3    | 0.9    |

**Table 3. Comparitive study of the proposed method with existing literature.**

| Methods        | Ranking         |
|----------------|-----------------|
| IFRS [43]     | Failed to handle|
| SRIFS [29]    | \( k_5 > k_2 > k_1 \quad \text{and} \quad k_3 > k_2 \) |
| SRIFS [29]    | \( k_5 > k_2 > k_1 \quad \text{and} \quad k_3 > k_2 \) |
| PFSRS         | \( k_5 > k_2 > k_1 \quad \text{and} \quad k_3 > k_2 \) |
the degree of membership and the value 0.3 is the degree of non-membership of candidate under criteria $x_1$ respectively. In other words, candidates is qualified for the grade of membership is 0.5 and disqualified for the grade of non-membership is 0.3.

**step iii** Next to calculate $\Omega(\mathcal{H})$ and $\Omega^c(\mathcal{H})$

$$\Omega(\mathcal{H}) = \{(k_1, 0.4, 0.5), (k_2, 0.5, 0.4), (k_3, 0.4, 0.5), (k_4, 0.4, 0.5), (k_5, 0.5, 0.4)\}$$

$$\Omega^c(\mathcal{H}) = \{(k_1, 0.8, 0.2), (k_2, 0.7, 0.2), (k_3, 0.8, 0.2), (k_4, 0.7, 0.2), (k_5, 0.8, 0.2)\}$$

**step iv** By ring sum operation calculate the choice set.

$\mathcal{H} = \{\Omega(\mathcal{H}) \oplus \Omega^c(\mathcal{H})\}$

$\mathcal{H} = \{(k_1, 0.83522, 0.01), (k_2, 0.78581, 0.08), (k_3, 0.83522, 0.1), (k_4, 0.75604, 0.1), (k_5, 0.8544, 0.08)\}$

**step v** Compute the top level threshold value $\lambda = (\mu, \psi) \in \mathbb{S}^*$ such that $\mu = \max_{1 \leq s \leq n} \mu_{\mathbb{H}}(k_s)$ and $\psi = \min_{1 \leq s \leq n} \psi_{\mathbb{H}}(k_s)$. It is clear that in choice set $\mathcal{H}$ the Pythagorean fuzzy value $\lambda$ is the maximum choice value. If $\mu_{\mathbb{H}}(k_j) \geq S^* \mu$ and $\psi_{\mathbb{H}}(k_j) \geq S^* \psi$, then the optimum decision value is $k_j$.

Hence, the optimal decision is $\lambda = k_5 = (0.8544, 0.08)$.

**Ring Product Operator**

Now to calculate the optimal decision through ring product operator, we have

$\mathcal{H} = \{\Omega(\mathcal{H}) \odot \Omega^c(\mathcal{H})\}$

$\mathcal{H} = \{(k_1, 0.32, 0.52915), (k_2, 0.35, 0.44), (k_3, 0.32, 0.52915), (k_4, 0.28, 0.52915), (k_5, 0.4, 0.44)\}$

Hence, the optimal decision is $\lambda = k_5 = (0.4, 0.44)$. Therefore, the most qualified and suitable candidate for the required position is $k_5$.

### 6.1. Comparative study

From the above analysis, it is clear, that the proposed approach is better than an intuitionistic fuzzy rough set (IFRS) [43], soft rough intuitionistic fuzzy set (SRIFS) and intuitionistic fuzzy soft rough set (IFSR) [29]. The advantages of the proposed method with existing literature are given below.

**Advantages**

(a) Suppose a crisp soft approximation space $\hat{U}, E, \Omega$ and let $\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) \mid x \in E\} \in E(\mathcal{E})$ where $E(\mathcal{E})$ is the family of fuzzy sets. Then the defined SRPF approximation operators $\Omega(\mathcal{H})$ and $\Omega^c(\mathcal{H})$ degenerate into the soft rough fuzzy set.

(b) Suppose a crisp soft approximation space $(\hat{U}, E, \Omega)$ and for a crisp set $\mathcal{A} \in P(\mathcal{E})$ of $\mathcal{E}$. Then the defined SRPF approximation operators $\Omega(\mathcal{H})$ and $\Omega^c(\mathcal{H})$ degenerate into soft rough fuzzy approximation operators as defined in Definition 2.9.

(c) By taking crisp soft approximation space $(\hat{U}, E, \Omega)$ and let $\mathcal{A} \in PFS(\hat{E})$. Then the PFSR approximation operators $\Omega(\mathcal{H})$ and $\Omega^c(\mathcal{H})$ in Definition 4.2, degenerate into SRPF approximation operators $\Omega(\mathcal{H})$ and $\Omega^c(\mathcal{H})$ in Definition 3.1.

(d) By taking fuzzy soft approximation space $(\hat{U}, E, \Omega)$ and let $\mathcal{A} \in E(\mathcal{E})$ where $E(\mathcal{E})$ is the collection of fuzzy sets. Then the PFSR approximation operators $\Omega(\mathcal{H})$ and $\Omega^c(\mathcal{H})$ in Definition 4.2, degenerate into soft fuzzy rough approximation operators defined by Sun and Ma [41].

Now to verify the effectiveness of the developed approach with some existing methods are presented in Table 3 by considering the above Illustrative Example of Section 6. IFRS [43] having no information about parameterization tools, so due to lack of this information the method developed in [43] failed to handle the proposed example. On the other hand, if the sum of PF value $(\mu(k), \psi(k))$ is greater than 1, that is $\mu(k) + \psi(k) > 1$ in optimum normal decision object $\mathcal{A}$ of Step ii. So, in this case, the method presented in [29] failed to tackle the situation. Thus from the comparative study, it is clear that the proposed method is more superior and provides more freedom to the decision-makers for the selection of membership and non-membership degrees as compared to existing literature.

### 7. Conclusion

The theories of rough set, Soft set, IF set and PFS all are important mathematical tools for dealing with uncertainties. In this manuscript, we have presented two new concepts: SRPS and PFSRS, which can be seen as two new generalizations of soft rough set models. Then we investigated some important properties of SRPS and PFSRS in detail. Moreover, classical representations of PFSR approximation operators are presented. In addition, the validity and effectiveness of the proposed operators are checked by applying them to the problems of DM in which the experts provide their preferences in the PFSR environment. Finally, through a numerical example, it is demonstrated that how the proposed operators work in DM problems. By comparative analysis, we find that it is more effective to deal with DM problem with the evaluation of PF information based on SRPS and PFSRS models than DM problems with the evaluation of SRIFS and IFSRS models.
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