On the problem of modeling the acoustic radiation pattern of source for the 2D first-order system of hyperbolic equations

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Abstract. In this paper the problem of modeling the acoustic radiation pattern of source is considered. The problem is formulated in the form of control problem for the 2D first-order system of hyperbolic equations. This problem is related to ultrasound tomography for early breast cancer detection. The modelling of the acoustic radiation patterns of sources allows us to improve the resolution of acoustic tomography.

1. Introduction

In this paper we consider the problem of modeling the acoustic radiation pattern of source which is formulated in the form of control problem for the 2D first-order system of hyperbolic equations. This problem is related to ultrasound tomography [1, 2, 3, 4, 5, 6, 20], aimed at the detection of inclusions in the soft human tissue, which is connected with the development of methods and instruments for early breast cancer detection. The modelling of the acoustic radiation patterns of sources allows us to improve the resolution of acoustic tomography.

We use the hyperbolic first-order system to describe the propagation of the acoustic waves. On the one hand, this allows to propose a more realistic model from the physical point of view. On the other hand, we can apply an optimization approach for recovering coefficients of such system, like the density of the medium, the speed of the wave propagation or the absorption coefficient.

We consider the system of hyperbolic equations of the first order as the mathematical model of the acoustic tomograph, because these equations are obtained directly from the conservation laws of continuum mechanics. It allows us to control the preservation of the basic invariants during the solution of direct and inverse problems. This is important for solving unstable problems, as the conservation laws of the main invariants are the only criterion of the well-posedness of the solution. When we use the hyperbolic system to formulate the problem, we can ensure that the numerical solution is close to the physical solution of the process. Moreover it is possible to formulate the problem of modeling the acoustic radiation pattern instead of the case when we deal with one acoustic equation of the second order.
We apply the numerical algorithm for solving direct problems, based on the S. K. Godunov scheme [8]. The numerical methods, based on the Godunov approach, allows to find the balance between mathematical modelling of physical process and the effective numerical realization.

Some considerations for choosing the model, based on the first-order system and the method for solving such system, can be found in [7, 21, 28].

The new numerical model of directional radiation pattern, in which part of energy is emitted in side and back lobs, has been presented in [24]. The model input values are determined on the base of primary parameters that can be read from the datasheet of used antennas. The elaborated model, program and output files can be easily implemented into an analysis of radio wave propagation phenomenon in any algorithms and numerical calculations. The comparison of the graphical plots that have been obtained on the base of measurements, producers’ data specification notes and modelling results confirms the model correctness.

The Characteristic Basis Function Pattern method was presented as a generic modeling technique for obtaining highly accurate approximations of antenna radiation patterns using very few measurements in [25]. The method is described in detail and illustrated for a reflector antenna system as an application example. Future challenges facing the further development and application of the method are also discussed.

It was demonstrated in [27] the possibility of designing a radiator using structural-acoustic interaction by predicting the pressure distribution and radiation pattern of a structural-acoustic coupling system that was composed by a wall and two spaces. Authors developed an equation that predicts the energy distribution and energy flow in the two spaces separated by a wall, and its computational examples are presented. Three typical radiation patterns that include steered, focused, and omnidirected are presented. A designed radiation pattern is also presented by using the optimal design algorithm.

A database of acoustic radiation patterns was recorded, modelled, and analysed for musical instruments in [26]. The generation of this database included recordings of each instrument over the entire chromatic tone range in an anechoic chamber using a surrounding spherical microphone array. Acoustic source centering was applied in order to align the acoustic center of the sound source to the physical center of the microphone array. The acoustic radiation pattern is generated in the spherical harmonics domain at each harmonic partial of each played tone. An analysis of the acoustic radiation pattern complexity has been performed in terms of the number of excitation points using the centering algorithm.

Inverse problems for hyperbolic systems were investigated theoretically in [22].

2. Problem formulation

Let us consider the following integral equalities:

\[ \oint p u dxdy + p dydt = 0; \]  
(1)

\[ \oint p v dxdy + p dxdt = 0; \]  
(2)

\[ \oint p dxdy + \rho c^2 (u dy dt + v dx dt) = 0. \]  
(3)

Here \( u = u(x, y, t) \), \( v = v(x, y, t) \) are components of the velocity vector, with respect to \( x \) and \( y \) respectively, \( p = p(x, y, t) \) is the acoustic pressure. The parameters of the system describes the properties of the medium: \( \rho(x, y) \) denotes the density of the medium, and \( c(x, y) \) is the speed of the wave propagation.

Equations (1) and (2) represent the conservation laws of impulse in direction \( x \) and \( y \), while Equation (3) is the conservation law of mass. The continuum mechanics equations
are usually derived in terms of integral conservation laws, and only after that it turns to differential equations.

2.1. Wavefield examples
In this subsection we present the numerical results of solving the direct problem. For solving the direct problem, based on the equations (1)—(3), we use the numerical method, proposed by S.K. Godunov [8]. The main element of this method is the solving of Riemann problem, which consists of an initial value problem composed of a conservation equation together with piece-wise constant data having a single discontinuity. The snapshots of the pressure wave yield are presented on the figures 1, 2 for different number of sources. The sources were considered to have the form of Ricker wavelet, with the frequency $\nu = 20kHz$. We add the non-reflecting boundary conditions to the formulation, that would be presented in the next section.

3. The optimization approach to the problem of radiation pattern
Let us formulate the problems of the modeling of the radiation patterns of the source of the acoustic waves. Usually such term is related to the dependence of the strength of the sounding impulse on the direction of propagation. The choice of the optimal sounding impulse (in terms of the temporal form and location) is important due to the applied nature of the problem considered. On the one hand, one could use modelling to obtain more accurate description of the real sounding pulse - that would decrease the error, introduced into the consideration by the inaccuracy of the model. On the other hand, improving the angular dependence of the system of sources would allow to increase the amount of data, registered in the receivers, which would lead to more accurate solution. In this paper we considered two approaches to the mentioned
problem, and both of them could be considered as the source inverse problem. The example of desired angular radiation pattern is presented on the figure 3.

Let us consider the problem with complex sources, introduced into the first and the second equation. More specifically, we consider the following problem:

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = q_1(x, y, t); \quad (4)
\]

\[
\frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial y} = q_2(x, y, t); \quad (5)
\]

\[
\frac{\partial p}{\partial t} + \frac{\rho c^2}{\rho} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \theta_{\Omega}(x, y) I(t); \quad (6)
\]

\[
u, v, p|_{(x,y) \in \partial \Omega} = 0; \quad (7)
\]

\[
u, v, p|_{t=0} = 0. \quad (8)
\]

Here \( \Omega = (x, y) \in [0, L] \times [0, L] \), function \( \theta_{\Omega}(x, y) \) describes the location of the source, \( I(t) \) describes the temporal form of the sounding wave.

The connection between differential and integral formulations based on the fact that, on the one hand, one can integrate equations (4)—(6) in an arbitrary domain to obtain conversation laws (1)—(3).

The inverse problem of recovering the coefficients \( \rho(x, y) \) and \( c(x, y) \) described the acoustic properties of the object were considered in [19, 9, 10, 11, 12].

Here functions \( q_1(x, y), q_2(x, y) \) are the unknown functions of right-hand side of the system. The goal is to find these control functions to obtain the desired form of pressure, that corresponds to the specific radiation pattern of the sound pulse.

Let us consider two inverse problems:

(i) the additional information is given in final moment (control problem) [13, 14, 15, 16]:

\[
p(x, y, T) = F_T(x, y), \quad (x, y) \in \Omega. \quad (9)
\]

(ii) the additional information is given for all time (specific radiation pattern of the sound pulse):

\[
p(x, y, t) = F(x, y, t), \quad (x, y) \in \Omega, \quad t \in (0, T). \quad (10)
\]

Thus, the problem is to obtain \( q_1(x, y, t), q_2(x, y, t) \), using the data \( F_T(x, y, t) \) or \( F(x, y, t) \).

We reduce the inverse problem (4)—(8) and (9) to minimizing the cost functional [17, 19, 18]:

\[
J(q) = \int \int_{\Omega} \left[ p(x, y, T) - F_T(x, y) \right]^2 \, d\Omega \rightarrow \min_q. \quad (11)
\]

Here vector-function \( q = (q_1, q_2) \). The gradient of the functional (11) could be calculated by the following formula

\[
J'(q) = \begin{pmatrix} J'_1(q_1) \\ J'_2(q_2) \end{pmatrix} = \begin{pmatrix} \psi_1(x, y, t) \\ \psi_2(x, y, t) \end{pmatrix}. \quad (12)
\]

Here \( \psi_1(x, y, t) \) and \( \psi_2(x, y, t) \) are the solution of the following adjoint problem

\[
\rho \psi_{1t} + \psi_{3x} = 0; \quad (13)
\]

\[
\rho \psi_{2t} + \psi_{3y} = 0; \quad (14)
\]

\[
\frac{1}{\rho c^2} \psi_{3t} + (\psi_{1x} + \Psi_{2y}) = 0; \quad (15)
\]

\[
\psi_1(x, y, T) = 0, \quad i = 1, 2, \quad \psi_3(x, y, T) = 2\rho(x, y)c^2(x, y) \left[ p(x, y, T) - F_T(x, y) \right]; \quad (16)
\]

\[
\psi_{1i}(x, y) \in \partial \Omega = 0, \quad i = 1, 2, 3. \quad (17)
\]
As for the problem of specific radiation pattern, the inverse problem (4)—(8) and (10) is reduced to minimizing the following cost functional:

$$J(q) = \int_{0}^{T} \int_{\Omega} \left[ p(x,y,t) - F(x,y,t) \right]^2 d\Omega dt \rightarrow \min_q.$$  \hspace{1cm} (18)

The gradient of the cost functional (18) could be written as follows:

$$J'(q) = \begin{pmatrix} J'(q_1) \\ J'(q_2) \end{pmatrix} = \begin{pmatrix} \psi_1(x,y,t) \\ \psi_2(x,y,t) \end{pmatrix}.$$  \hspace{1cm} (19)

Here $\psi_1(x,y,t)$ and $\psi_2(x,y,t)$ hold the equations (13), (14), the boundary conditions (17) and the following equation and initial conditions:

$$\frac{1}{\rho c^2} \psi_3 + \left( \psi_{1x} + \Psi_{2y} \right) + 2 \left[ p(x,y,t) - F(x,y,t) \right] = 0; \hspace{1cm} (20)$$

$$\psi_i(x,y,T) = 0, \hspace{1cm} i = 1, 2, 3. \hspace{1cm} (21)$$

One can mention, that the similar structure of both gradients relies on the fact, that the unknown functions in both problems (11) and (18) are the same source functions $q_1(x,y,t)$, $q_2(x,y,t)$. The difference in adjoint problem’s formulation reflects different type of inverse problem’s data, taken into consideration by (9), (10).

To minimize the functionals (18) and (11) we apply the Landweber iteration method and we use a priori information to decrease the iterations [23].

**Conclusion**

We consider the problem of modeling the acoustic radiation pattern which is formulated in the form of control problem for the 2D first-order system of hyperbolic equations which is connected with the development of methods and instruments for early breast cancer detection. The modelling of the acoustic radiation patterns of sources allows us to improve the resolution of acoustic tomography. In future work we will improve the quality of inverse problem data by adding the acoustic radiation patterns of receivers.
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