FOUNDATION OF COMPUTER ALGEBRA ANALYSIS SYSTEMS:
SEMANTICS, LOGIC, PROGRAMMING, VERIFICATION

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Abstract. We propose a semantics of operating on real numbers that is sound, Turing-complete, and practical. It modifies the intuitive but super-recursive Blum-Shub-Smale model (formalizing Computer Algebra Systems), to coincide in power with the realistic but inconvenient ‘stream’ Turing machine underlying Computable Analysis: reconciling both, as foundation to a Computer Analysis System. Several examples illustrate the elegance of rigorous numerical coding in this framework, formalized as a simple imperative programming language \textsc{ERC} with denotational semantics for realizing a real function \( f \): arguments \( x \) are given as exact real numbers, while values \( y = f(x) \) suffice to be returned approximately up to absolute error \( 2^p \) with respect to an additionally given integer parameter \( p \to -\infty \).

Real comparison (necessarily) becomes partial, possibly ‘returning’ the lazy Kleenean value \texttt{unknown} (subtly different from \( \perp \) for classically undefined expressions like \( 1/0 \)). This asserts closure under composition, and in fact yields ‘Turing-completeness over the reals’: All and only functions computable in the sense of Computable Analysis can be realized in \( \text{ERC} \). Programs thus operate on a many-sorted structure involving real numbers and integers, connected via the ‘precision’ embedding \( \mathbb{Z} \ni p \mapsto 2^p \in \mathbb{R} \), whose first-order theory we prove to be decidable and model-complete. This logic serves for formally specifying and formally verifying correctness of \( \text{ERC} \) programs.

1. Introduction

A rich and rigorous Theory of Computing underlies and enables nowadays state-of-the-art Software Engineering. For example the concept of an Abstract Data Type (ADT) allows and encourages to break down large-scale programming endeavours into smaller parts to be treated independently: one team developing and implementing the algorithms realizing said ADT, while another team may simultaneously already use the convenience and concise elegance of said ADT in order to solve the original computational problem—or to in turn develop yet another intermediate ADT. Writing a program thus means to combine certain designated lower-level operations (‘small-step’) in a way to realize some higher-level functionality (‘big-step’) whose correctness and reliability thus depends crucially on the previous ones’ to indeed provide their promised semantics.

Example 1.1. Christofides Algorithm provides one way for efficiently approximating the metric Travelling Salesperson Problem. It builds on the ADT of edge-weighted graphs with minimum spanning tree as ‘primitive’ operation. This in turn might (among several possible choices) be implemented using Prim’s Algorithm; which itself uses the ADT priority queue, that might (again among several alternatives) be implemented using a Fibonacci Heap; which in turn uses a doubly-linked list and totally-ordered keys: gradually climbing down layers of abstraction (and continued further to compiler, operating system, CPU, circuits, transistors.)

Observe that these ADTs are concerned with various cases of discrete data. Computer Algebra Systems implement similar hierarchies of algebraic structures [96]: building up from (sequences of) bytes via integers and integer polynomials to quotient fields and finite extensions. Such a hierarchical build-up also underlies mathematical Calculus in \textit{Analysis}: from real numbers via converging sequences and smooth/\textit{integrable} functions to bounded operators on compact domains [90]. So how about the algorithmic processing of \textit{continuous} data, for example in Numerics: from computing \( \pi \) and \( \exp(x) \) via dynamical systems, root finding and quadratures and ODE/PDE solving up to topology/shape optimization? Subsection 1.1 argues that common conceptions underlying imperative numerical programming actually
do not seem to admit a simultaneously consistent and sound semantics, already for real
numbers.

The present work formalizes precisely such a rigorous semantics of imperatively operating
on real numbers such as to compute possibly transcendental functions:
• Said semantics is carefully designed to support and justify programmers’ intuition,
• namely arguably closest to the model-theoretic algebraic approach (aka Blum-Shub-Smale
model aka real-RAM)
• yet of expressive power coinciding with (i.e. not exceeding nor lacking compared to)
Computable Analysis
• and in particular satisfying closure properties that enable modularly building advanced
real functionality from basic one, while avoiding the hassles of Turing machines,
• enabling a natural and elegant approach to formal program verification based on real
number axioms (of which for instance Distributive and Associative Laws are violated by
floating point numbers).

Rigorous implementation of this programming paradigm is possible by construction. It
paves the path to a library of computable ADTs corresponding to the structures comprising
Calculus, as backend to a Computer Analysis System and counterpart to contemporary
Computer Algebra Systems [96]. Implementation details and efficiency considerations are
deferred to successor work.

1.1. Motivation. Programming means combining lower-level operations such as to realize
some higher-level functionality, as in modular software engineering and state of the art for
discrete problems like Example 1.1. The present subsection explores the similar relation
between lower-level operations and higher-level functionality in numerics. We recall several
popular numerical methods, as well as various common (often implicit) conceptions of the
operational primitives. And we argue that no single consistent and computable semantics
of the primitives agrees with the purported behaviour (=specification) of said algorithms.
This discrepancy seriously hampers the formal specification and verification of numerical
programs.

Specifically, consider the Bisection Method, the Trapezoid Rule, Gaussian Elimination,
computing the exponential function, and simply evaluating an inductively defined real
sequence such as the following example due to Jean-Michel Muller [63, p. 48]:

Example 1.2. As a case of experimental analytic mathematics, evaluate the following
sequence to determine its asymptotic behaviour:

\[ a_0 := \frac{11}{2}, \quad a_1 := \frac{61}{11}, \quad a_{m+1} := 111 - \frac{(1130 - 3000/a_{m-1})}{a_m}. \quad (1.1) \]

Regarding the (usually implicitly understood) specification of this and the following examples,
inputs arguably are expected exact while outputs are approximate — a discrepancy which
destroys closure under composition [104, p.325]:
• The Trapezoid Rule, Bisection Method, and Taylor series (e.g. of the exponential function)
do not produce the real result (integral, root, exp) exactly, but rather approximately.
• Bisection and (pivot search in) Gaussian Elimination on the other hand suppose the
argument function / matrix entries be given exactly:

Indeed, regarding computable counterparts to the Intermediate-Value Theorem, even if
continuous \( f : [0; 1] \rightarrow \mathbb{R} \) satisfies \( f(0) < 0 < f(1) \) and has a unique and simple root
\( x_0 \in (0; 1) \), infinitesimal perturbation of \( f \) can flip the outcome of a sign test in the Bisection
Method—and drastically change the rest and result of the computation [97, p.175]; similarly for pivot search during Gaussian Elimination of the given matrix $A$.

Regarding the semantics of the basic operations comprising the above numerical algorithms, recall some popular possible conceptions:

- The IEEE 754 standard [62] does assign a rigorous semantics to floating point operations—however one different from mathematical reals. For example the former incur rounding errors [47]; violate the Distributive Law; and comparison “$x > y$” of ‘close’ but large values is delicate, to say the least. Moreover floating point computation makes the sequence from Example 1.2:

\[
(a_n)_{n \geq 2} = \left( \frac{341}{61}, \frac{1921}{341}, \frac{10901}{1921}, \frac{62281}{10901}, \ldots \right);
\]

converge to 100 instead of 6: Even with 1000 digits working precision, $a_{1000}$ is close to 100 and not at all to 6. Floating point-based semantics thus differs significantly from that of actual real numbers; and this discrepancy yields to the rigorous specification (for example of Bisection or Gaussian Elimination) to become rather involved [47].

- The mathematical semantics of rational numbers on the other hand does realize arithmetic operations exactly [36]. However, (1) rational numbers are not closed under trigonometric/exponential functions. (2) They do not support, say, matrix arguments with irrational/transcendental entries nor exact (recall the above discrepancy) evaluation of general function arguments—as employed in the Trapezoid rule and Bisection. (3) The lengths of numerators/denominators tend to blow up: For example Newton’s Method to approximate $\sqrt{2}$ incurs exponential bit-cost. And (4) the first-order theory of the field of rational numbers is undecidable, see Section 4. Algebraic numbers [59] salvage (4) but still suffer from (1) and a counterpart of (3), see [57].

- The Blum-Shub-Smale model of computation [12] underlies Algebraic Complexity Theory [23] and, under the name real-RAM, also Computational Geometry [28, §1.2]. It axiomatically supports arithmetic and comparison/tests whose semantics coincides with the model-theoretic structure of real numbers [72, 68]: one of the few having a decidable first-order theory [58]. It is praised for formally capturing the common conception of a numerical algorithm [24]. However testing in-/equality of real numbers is known (Turing-)undecidable [97, Exercise 4.2.9], and gives this machine model super-recursive power [13].

- Computable Analysis [73, 97] provides a sound and realistic algorithmic foundation to the digital processing of continuous data [21]. Moreover, it leads to a computational (bit-)complexity theory [48] whose predictions [16] agree with the performance of practical implementations in reliable high-precision numerics [42]. However, based on the inconvenient Type-2 (variant of the Turing) machine, it is rarely used in practice [78].

- Similar issues apply to Domain Theory as formalization of interval computation: Its semantics is rigorous [81, 1], but operating on ‘partial’ information differs so much from the intuition and prevalent conceptions in numerics that, after 40 years, it still awaits to be put into practice.

- The functional programming language realPCF for processing continuous data provides elegant abstraction from Turing machine-based Computable Analysis [31], but after 25 years still waits for acceptance by practitioners. ERC is deliberately imperative; it formalizes and unifies existing software (see Remark 1.9).
To summarize, imperative numerical programming seems to lack a simultaneously consistent, computable, and practical semantics. This hampers rigorous numerical software engineering [53, p.412], impedes formal verification [14, 6], forces resorting from algorithms to heuristics [74] with vague specification [66, e04bbc], and causes errors with serious consequences [84, 41, 39].

1.2. Intuition and Contribution. We collect a sound formalization, called Exact Real Computation (ERC), as justification of the implicit intuition underlying common imperative numerical programming. It reconciles the realistic but inconvenient Type-2 (variant of the Turing) machine model of real computation with the convenient but infeasible Blum-Shub-Smale model. Both agree that arithmetic on real numbers is computable, but disagree concerning comparison: inequality “$x < 0$” of real numbers is known to be equivalent to the Halting problem [97, Exercise 4.2.9], that is, semi-decidable but not decidable. Put differently, a computably modified semantics of this test returns true in case $x < 0$, false in case $x > 0$, but does not return at all in case $x = 0$: thus rendering this primitive partial—and, like using division $1/x$, putting responsibility on the programmer to avoid invoking it with $x = 0$.

In ERC this is reflected by a ternary logical data type KLEENEAN with values true, false, and unknown forming a Kleene Algebra. In order to enable writing total programs, this data type is lazy in the sense that it keeps the ‘result’ of a real comparison un-evaluated, and hence safe for assignment to a variable KLEENEAN $b = (x \leq 0)$ even when $x = 0$. This differs from an expression like 1/0 being classically undefined, i.e., of ‘value’ ⊥. Of course evaluating (including branching or looping in dependence on) $b$ will still stall in case it has value unknown. To avoid this, ERC includes the ‘parallel OR’ operation from [91, §4]: Intuitively, “choose($b_0, b_1, \ldots$)” returns some $j \in \mathbb{N}$ for which $b_j$ is true—even if some $b_j$ are unknown—provided that at least one $b_j$ is true; see Example 1.11 below. No further specification, nor reproducibility condition, is imposed on which $j$ gets returned in case several $j$ satisfy $b_j = \text{true}$. Such ‘hidden’ dependence beyond the values of the arguments is known as non-extensionality or multi-valuedness and is inherent to real computation [55]; see Subsubsection 1.4.1. Multi-valuedness of choose() carries over to compound terms with a subtle semantics, carefully designed to ensure real Turing-computability (i.e., in the sense of Computable Analysis) and formalized over certain powerdomains [3, 60]. Apart from providing a rigorous foundation, these semantics are essential for formal specification and verification to build on; whereas for actual programming purposes, the above naïve understanding should suffice and enable intuitive coding in agreement with common constructions from Calculus.

ERC thus offers three distinct basic data types: real numbers REAL, integers INTEGER, and logic KLEENEAN = {true, false, unknown}. Unlike hardware double or byte, these correspond exactly to the mathematical structures $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{K}$, respectively: REAL and INTEGER are devoid of rounding or wrapping. Instead (only) the semantics of real comparison changes from Boolean-valued $\leq$ to KLEENEAN-valued $\preceq$. KLEENEAN corresponds to the generalized three-point Sierpiński/Kleene space $\mathbb{K} = \{ff, tt, uk\}$ with the topology from Remark 1.5. Integers serve for counting, as array indices, and to parameterize real error bounds $i(p) = 2^p$ ($\mathbb{Z} \ni p \rightarrow -\infty$). However, conversion from integer to real numbers (other than via $i$) is not natively supported, as this would render the underlying first-order logic undecidable according to Theorem 4.4c). In addition to choose(), ERC provides another primitive to circumvent partial/diverging computation, namely an imperative adaptation of the ‘parallel if’ from functional programming [32]: For KLEENEAN $b$ and REAL $x, y$, “$b \ ? \ x : y$” evaluates to $x$ in case $b = \text{true}$, to $y$ in case $b = \text{false}$, as well as in case $x = y \land b = \text{unknown}$; see
Example 1.17 below. We call it continuous conditional, since the third case indeed arises from the continuous extension of the first two with respect to the topology (1.2).

Note that all operations thus carefully defined are real Turing-computable in the sense of Computable Analysis. Arithmetic operations and tests can express only piecewise rational functions; real transcendence is attained in the limit of a sequence (of real numbers, sequences, functions, or more generally elements from a complete metric space). To maintain Turing-computability, the rate of convergence must be known [85]. We choose to incorporate said limit implicitly with approximate return values up to given absolute error bound:

**Paradigm 1.3.** An ERC realizer of a mathematical real (single-valued) function $f : \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a (possibly multi-valued) program $F$ of type \texttt{REAL F(INTEGER p; REAL x)} with dedicated\footnote{This constitutes a preliminary design choice—which an actual production programming language may or may not approach differently; see Remark 1.4.} precision parameter $p \in \mathbb{Z}$. $F$ performs arithmetic operations (addition, subtraction, multiplication, division) on real and integer arguments as well as on intermediate results exactly. Real number comparison (less, greater) is exact, too, but permitted only when both sides differ: Don’t test for equality! Kleene’s ternary logic captures the ‘lazy’ value of such a partial comparison. The result can be safely tested using a multi-valued ‘Parallel-Or’ that captures dove-tailing as an algebraic operation. The value $y$ finally returned by $F(p;x)$ need not be exact but must merely approximate $f(x)$ with an error up to $2^p$, $p \rightarrow -\infty$.

Put differently, in ERC, arguments $x$ are provided exactly, return values are approximate\footnote{Like \texttt{argc} in \texttt{main} of C++ programs, the formal precision parameter (here called $p$) may be given any name or even be omitted — for example when the result can be computed exactly; cmp. Subsection 2.6.} up to any given error bound $2^p$, $p \in \mathbb{Z}$: Note that this is indeed the semantics underlying common numerical methods, such as the aforementioned Taylor expansion, Bisection Method, and Trapezoid Rule. In spite of the discrepancy between exact input and approximate output, and unlike in Geometric Computation [104, p.325], closure of ERC-realizable functions under composition does not leave the realm of real Turing-computability; see [97, Theorem 4.3.8]. This means that mathematical functions (like exp), once realized in ERC, become a part of ERC and can be called as new primitives with both argument and return value as exact real numbers; see Remark 2.3 below. It also implies by induction that every ERC-realizable function is real Turing-computable; and Theorem 2.4 below asserts that the converse holds as well: every Turing-computable real function can be realized in ERC.

**Remark 1.4.** The present work focuses on the basic concepts of ERC as a rigorous foundation to imperative computation with real numbers and functions: data types with Turing-computable operations, denotational semantics, logic for specification and verification, etc. Some design choices involve trade-offs. For example we consider integer multiplication and the embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$ as user functions. This yields Theorem 4.4(a) asserting decidability of the logical theory of the three data types $\mathbb{K}, \mathbb{Z}, \mathbb{R}$ supported by ERC, and thus guarantees the feasibility of formal program verification. A practitioner on the other hand might relinquish logical decidability and instead prefer integer multiplication and type casting to be primitive operations.

a) Limits are essential to leave the constraint of algebraic computing [17]. We here consider them implicitly as part of our notion of a realizer $F$ of a real function $f$: with the requirement to return an approximation $y$ to $f(\vec{x})$ up to error $2^p$, given $p \in \mathbb{Z}$ and $\vec{x} \in \mathbb{R}$. Turning limits into explicit operators requires sequences (of single reals, of
sequences of reals pointwise/uniformly, of continuous/Lipschitz/smooth real functions w.r.t. uniform/Sobolev norm etc.) as higher type and is deferred to future work.

b) Variables here have to be initialized during declaration; and arrays are one-dimensional of constant size: Again we leave it to practitioners to relax and extend those restrictions. We choose to avoid discussing arrays of multi-valued size. Also high-dimensional arrays can be expressed as one-dimensional ones without leaving decidable Presburger Arithmetic, see Subsection 3.4(f) below.

c) Similarly one may—although for conciseness we choose not to—distinguish between transcendental/analytic (single-valued) functions computed by approximation and algebraic (multi-valued) functions with exact return value. (A suitable semantics of multi-valued limits is still sought for, see Remark 1.10 below.)

d) ERC as introduced here formalizes real-valued functions and integer-valued multifunctions (the latter in order to cover integer rounding in Subsection 2.5) with mixed real and integer arguments: Data type KLEENEAN is restricted to local variables. (Multi)functions with KLEENEAN type arguments and return values are beyond the scope of the present work but definitely on the agenda, see Section 5.

e) Our proposed transition from double to REAL parallels the transition from byte to INTEGER as natively supported by modern programming languages. INTEGER may be implemented in software operating on a finite sequence of byte of length determined during execution [36]. Performing such simulation of INTEGER (multiplication, say) on top of byte with optimal efficiency has become a research topic of its own. Similarly, REAL may be implemented in various ways on top of byte or double: using finite precision, carefully chosen such that the user program’s behaviour—including branches and propagating rounding errors—remains indistinguishable from execution in infinite precision. Our Theorem 2.4 asserts that ERC is real Turing-complete and thus can indeed be implemented. Again, determining such sufficient precision automatically and transparently and efficiently is a separate question: see Remark 1.9 below.

To summarize our main contributions:

• A choice of operations over real numbers and their rigorous, real Turing-computable semantics
• A small imperative programming language with three basic data types REAL and INTEGER and KLEENEAN that fully coincide with R and Z and K with respect to the said semantics.
• Examples demonstrating programming with these operations and semantics, such as: multi-valued integer rounding, determinant via Gaussian elimination with full pivoting (subject to full-rank promise), and simple unique root finding.
• Proof that this programming language is Turing-complete over the reals: those and only those real functions that can be realized in ERC are computable in the sense of Computable Analysis.
• A many-sorted structure with decidable (see Remark 1.5) and ‘model-complete’ first-order theory for the formal specification of the properties of real functions in ERC.
• Concise and elegant rules extending classical Floyd-Hoare Logic to enable formal verification of ERC programs.

ERC thus compromises between the elegant but unrealistic Blum-Shub-Smale model (supporting tests for equality but no transcendental functions) and the realistic but inconvenient Type-2 (variant of the Turing) machine model of real computation as imperative program. Subsubsection 1.2.1 provides an overview of this work.
Remark 1.5. Inequality of analytic real numbers is known (semi-decidable but) undecidable [97, Exercise 4.2.9]. In/equality of real algebraic numbers on the other hand is decidable [96, §21]. The latter corresponds to the mathematical predicate “$x < y$” being total; and in order to reflect the former, we define the computational comparison “$x \preceq y$” as having value $\text{uk}$ in case $x = y$ holds mathematically, where $\mathbb{K} = \{\text{tt}, \text{ff}, \text{uk}\}$ denotes Kleene’s ternary logic equipped with the generalized Sierpiński topology

$$\{\emptyset, \{\text{tt}\}, \{\text{ff}\}, \{\text{uk}, \text{tt}, \text{ff}\}\}.$$ (1.2)

Recall that, in the (semi-)algebraic setting, also quantified formulas are decidable according to Tarski-Seidenberg. And Theorem 4.4(a) below asserts that quantified formulas do remain decidable even when involving reals and Kleene logic and integers; integers without multiplication but with embedding $\mathbb{Z} \ni p \mapsto 2^p \in \mathbb{R}$.

Note the difference between a formal logical statements arguing about, and performing some actual, computation involving such data; see also Remark 4.1 below. Also observe that type conversion $\mathbb{Z} \hookrightarrow \mathbb{R}$ would allow to recover integer multiplication via real multiplication and thus lead to Gödel undecidability of quantified formulas: see Theorem 4.4(c).

1.2.1. Overview. Related work and concepts are discussed in Subsection 1.3: Classical Numerics (Subsubsection 1.3.1), Models of Real Computation (Subsubsection 1.3.3), Logic in Analysis (Subsubsection 1.3.4). Subsection 1.4 recalls the basics of Computable Analysis: notions, properties and examples (Subsubsection 1.4.2) of computable partial real multi-/functions (Subsubsection 1.4.1) that justify the formal semantics of ERC (Subsubsection 1.4.3). Section 3 formally specifies the syntax (Subsection 3.1), type system (Subsection 3.2), and denotational semantics (Subsection 3.3) of ERC. Aiming at numerical practitioners and following pedagogical best practice [7], we deliberately present intuitive example programs before introducing abstract program syntax and semantics: Section 2 collects ERC programs computing inductively defined real sequences, square root using Heron’s Methods (Subsection 2.2), exponential function via Taylor expansion (Subsection 2.3) and iterative (Subsection 2.4); integer rounding (Subsection 2.5), matrix determinants via Gauss Elimination (Subsection 2.6), and root finding (Subsection 2.7). Subsection 2.8 exhibits ERC as ‘sound’ and ‘adequate’, namely Turing-complete over the reals.

We introduce in Section 4 a three-sorted logical structure (Subsection 4.1) for rigorously specifying and arguing about such non-extensional programs (Subsection 4.2): Theorem 4.4 shows its first-order theory to be decidable and model-complete. Subsection 4.3 extends classical Floyd-Hoare Logic from INTEGER to (KLEENEAN and) REAL: conveniently adding only few concise rules, reflecting the elegant mathematical properties exhibited by $\mathbb{R}$ as opposed to double [15, 6]. Subsection 4.4 demonstrates this with a (toy) formal verification of the aforementioned trisection algorithm for root finding. We conclude this work with Section 5 suggesting future research, such as extending ERC from operating on real numbers in order to realize functions to operating on functions in order to realize operators and other higher types.

Main results are Theorems 2.4 (real Turing-completeness of ERC) and 4.4 (decidability of the first-order logic of ERC) and 4.10 (soundness of extension of Floyd-Hoare logic). Arguably equally important are the definitions in Section 3 that identify and formalize a consistent semantics of numerical programming, motivated by examples in Section 2 and that lead to said theorems.
1.3. Related Work and Concepts. This subsection briefly reviews, and relates our contribution to, previous work. The existing literature is considered in three categories: numerics (Subsubsection 1.3.1), computable analysis (Subsubsection 1.3.2), and models of computation (Subsubsection 1.3.3).

1.3.1. Numerics. Numerical calculations, which are as old as Mathematics itself, have experienced a significant boost with the advent of digital computers since the 1960s; and another one in 1985 with the introduction of the IEEE-754 floating point standard and highly efficient hardware support of data type double. Such numbers share a subset of rationals; they come with a carefully ‘engineered’ yet involved semantics. In addition to including subnormals, not-a-numbers (NaN) and ±∞ floating point numbers violate central mathematical properties such as the Associative/Distributive Laws, Intermediate-Value Theorem, algebraic and logical completeness. Naïvely discretizing continuous data introduces numerical artifacts [47] and destroys underlying symmetries that instead may need to be re-introduced explicitly using tailored methods [2].

Remark 1.6. Folklore commonly claims two justifications for double as the state-of-the-art in mainstream numerics since 35 years:
   i) A digital computer can handle only finite information, hence discretization is unavoidable.
   ii) Without hardware support, computations become prohibitively slow.

Computable Analysis disproves (i); cmp. [100]. A common ‘trick’ is to execute the user program transparently out of order (similarly to speculative execution in modern CPUs), that is, to proceed from operational to declarative/denotational semantics; see Remark 1.9.

Concerning (ii) recall the similar conception from the 1980ies regarding hardware bytes versus software bignum: Nowadays the elegance of the latter and their full coincidence with the mathematical data type $\mathbb{Z}$ devoid of wrap-around errors$^3$ gained is generally agreed to outweigh the constant-factor loss in efficiency for many calculations. Of course implementations of bignum still use hardware data types; but such details are hidden, following the paradigm of Object-Oriented Programming with Abstract Data Types [54].

The same applies to IEEE-754 double ‘versus’ ERC REAL: The latter adds a carefully designed layer of abstraction [65, p.169 level 4], allowing to compute on actual real numbers exactly and elegantly. Its advantages — simplified rigorous algorithm design and verification — outweigh the loss in absolute performance for a growing number of applications. □

For example, in addition to the above Example 1.2, the following iterations may run fast when using double but silently produce ‘garbage’ (i.e., computational output far from to the exact mathematical value at every iterate).

Example 1.7. Iterating the Tent Map
$$[0; 1] \ni x \mapsto 1 - 2 \cdot |x - 1/2| \in [0; 1]$$

Similarly for iterating the Logistic Map [10]:
$$[0; 1] \ni x \mapsto r \cdot x \cdot (1 - x) \in [0; 1]$$
in the chaotic region $3.57 \leq r \leq 4$.

$^3$underlying the “Nuclear Gandhi” in the computer game Civilization
Example 1.8. Analytic continuation [44, 45]: \((c_j)_j \mapsto (c'_k)_k\), where

\[
    c_k(z) := \sum_j c_{j+k} \left( \binom{j+k}{j} \right) z^j
\]

High-precision numerics adds floating-point data types with mantissae of arbitrary but a-priori fixed length; cmp. [42, 33]. These defer, but cannot avoid, the aforementioned fundamental deficiencies of double and in particular their deviation from, and violation of essential properties of, mathematical real numbers. Interval arithmetic replaces the original point semantics of reals as numbers with that of intervals. Interval arithmetic and interval methods solve some of the fundamental problems of floating-point arithmetic by rigorously computing an error bound for each real value, but these methods usually do not focus on exact real values. Arbitrary-precision numerics and interval arithmetic underlie common approaches implementing ERC.

Remark 1.9. Several software libraries and program packages already support exact real computation to varying extent.

- AERN [51] is a suite of Haskell packages for computing with exact real numbers and various classes of real functions, based on Computable Analysis.
- Ariadne [25] is a C++ library for verification of dynamic systems with a kernel based on Computable Analysis. It implements INTEGER, KLEENEAN and REAL data types supporting the operations of ERC, including choose() and a version of the continuous conditional when, and also provides explicit limit operations. In addition to logical and numerical types, Ariadne also has data types for functions and sets in \(\mathbb{R}^n\).
- The package Clerical [5] specifies a language similar to ERC, but with enriched expressions such as an explicit limit construct.
- iRRAM [64] is arguably the precursor of all the above libraries and packages, spiritus rector of ERC.

Among further libraries and packages we mention, without claiming completeness: REAL-LIB [52], CORE2 [105], ROSA [27], Cdar [9].

Algorithmically determining a (nearly optimal) finite precision sufficient for simulating infinite precision usually proceeds by tracing the execution of the user code in order to symbolically record (e.g. in CORE2) or transparently re-execute (e.g. in iRRAM) its calculations. This can be regarded as proceeding from operational to a denotational semantics in imperative programming.

Popular recent systems, like Octave or Julia, on the other hand leave it to the user to choose the working precision. So, in spite of the benefits of operating on continuous data exactly and in agreement with Calculus, the various software mentioned in Remark 1.9 remain to reach numerical mainstream. The present work formalizes and ‘standardizes’ the diverse approaches from Remark 1.9: in order to join forces and increase impact, similar to IEEE754 having standardized the diverse proprietary floating point formats prior to 1985.

1.3.2. Computable Analysis. Computability investigations concerning real numbers date back to at least Turing’s famous 1937/38 paper [92, 93] which spawned the field of Computable Analysis. It formalizes computing a real number (an information-theoretically infinite object) on Turing machines by approximation up to guaranteed absolute error \(2^p, p \rightarrow -\infty\); and computing a real function [37] means to convert such a sequence of input approximations (not necessarily one by one) to a sequence of output approximations; see Subsubsection 1.4.2
This notion is closed under composition and renders both transcendental $\pi$ and the exponential function computable. It formally confirms numerical gospel to ‘avoid’ test for equality [97, Exercise 4.2.9]; and has refined computability [73] to efficiency investigations [48]. Type-2 machines are thus suitable for theoretical investigations, but inconvenient to program in practice [78].

1.3.3. Models of Real Computation. The Blum-Shub-Smale machine [12] comes arguably closest to the (often implicit) conception of practitioners in numerical programming: It considers real numbers as entities, operated on exactly and tested with a total comparison operation “$x < y$”. This has been generalized to computation over structures in the sense of logic/model theory [72, 68].

Also known as realRAM in Computational Geometry [28, §1.2], this model underlies Algebraic Complexity Theory [24] and Information-Based Complexity [102], but suffers from two drawbacks: (i) It cannot compute transcendental functions [17], and (ii) its semantics of comparisons makes them equivalent to the (complement of the undecidable) Halting Problem [13]. Variants therefore restrict to algebraic reals [59], where equality is indeed decidable. This provides an algorithmic foundation to Computer Algebra [96].

Alternatively, considering inputs as exact justifies testing them for equality. But then permitting output approximations either forfeits closure under composition [104, p. 325] or climbs up the arithmetical hierarchy of uncomputability [35].

Realizing the gap between the intuitive but unrealistic Blum-Shub-Smale model and the realistic but impractical Turing machine, several suggestions have been put forth to reconcile both and provide a sound semantical foundation to imperative numerical programming.

Brattka and Hertling [18, p.491] modify the classical semantics of real comparison to a multi-valued soft test:

"$x <_n 0$" returns: \( \text{true} \) in case \( x < -2^{-n} \),

\( \text{false} \) in case \( x > 2^{-n} \), and

either \( \text{true} \) or \( \text{false} \) in case \( -2^{-n} \leq x \leq 2^{-n} \).

In ERC this can be expressed by combining our partial but single-valued comparison with the lazy KLEENEAN and multi-valued choose(); and compare [103, §6] for a numerical variant.

[91, §2] extends the classical while programming language [79, §2.3] to real numbers with partiality and multi-valuedness. Like ERC it explicitly adds a data type for mathematical integers (but no KLEENEAN) as well as arrays [91, §3.3+§3.4]. [91, §4] considers a variant of ERC’s \texttt{choose()} and programs computing (necessarily piecewise algebraic) multifunctions; in order to cover also transcendental single-valued functions, [91, Definition 4.5.1] defines programs with approximate computation.

[101] considers flowcharts with indirect addressing instead of while programs, while we formalize a full-fledged imperative programming language. [31] considers functional programming over the reals.

The present work proceeds from [91] towards numerical practice: ERC (i) adds the logical type KLEENEAN and (ii) proposes a first-order logic over the many-sorted structure comprising KLEENEAN, INTEGER, REAL for formally specifying hybrid real/integer/logically-valued multifunctions with hybrid logical, integer, and real arguments. (iii) We prove this logic to be decidable, and ‘model-complete’ in the sense of admitting quantifier elimination up to possibly one existential block. (iv) We extend classical Hoare Logic for formal verification
of thus specified ERC realizers. Finally (v) we generalize the above items to higher type: from receiving and operating on real number arguments in order to compute real functions, to receiving and operating on continuous real function arguments in order to compute real functionals.

1.3.4. Logic in Analysis. Real numbers are infinite objects, while both mathematical proofs and Turing machines are finite. This discrepancy has been a subject of research in Mathematical Logic at least since Turing’s famous work [92]. For example the Tarski-Seidenberg Theorem asserts that the first-order theory of real closed fields is (Turing-)decidable and admits a complete recursively enumerable axiomatization; while the rational numbers do not [77]. The many similarities between Computability Theory and Constructive Mathematics are well-known [85, 87, 22, 40]; and so is the unavoidability of non-extensionality [55, 18, 32, 101]. Powerdomains are commonly employed to formalize the semantics of such ‘non-deterministic’ operations [3]. There, ‘fairness’ means that every permitted return value must actually occur—and is commonly imposed in the discrete/countable setting, but not in Computable Analysis; see Subsubsection 1.4.1.

Remark 1.10. The semantics of multi-valued limits is still under debate [34, 49]. Definitions 4.5.3+6.1.10 for example incorporates fairness as above, resulting in closure under composition in a sense different from Computable Analysis. It also renders Specker’s counter-example to root finding [86] multi-valued approximably computable.

1.4. Preliminaries. ERC provides exact operations on real numbers as entities, and allows to combine them (i.e., to program): such as to realize precisely those real functions that are computable in the sense of Computable Analysis, yet without the hassles of Turing machines. The present section recalls the underlying theoretical background: partial multi-functions (Subsubsection 1.4.1), Turing-computability of real and hybrid functions and functionals (Subsubsection 1.4.2), and computable primitives (Subsubsection 1.4.3) that Section 3 proceeds to axiomatize as constituents of ERC.

Notation. In the rest of the paper, the following conventions are used unless stated differently:

• $\vec{k}, \vec{a}, \ldots$ denote elements of $\mathbb{Z}^e$ for some $e \in \mathbb{N}$.
• $\vec{x}, \vec{y}, \ldots$ denote elements of $\mathbb{R}^d$ for some $d \in \mathbb{N}$.
• $\vec{b}$ denotes elements of $\mathbb{K}^\ell$ for some $\ell \in \mathbb{N}$.
• $f$ denotes a continuous (single-valued) real function.
• $f$ denotes as well an integer-valued multifunction.
• The maximum norm of $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is $\|\vec{x}\| := \max\{|x_1|, \ldots, |x_d|\}$, where $|x_i|$ denotes the absolute value of $x_i$.
• $\lfloor x \rfloor \in \mathbb{Z}$ denotes the integer closest to $x \in \mathbb{R}$, rounding down to break ties in case $x \in \mathbb{Z} + \frac{1}{2}$.
• $\mathbb{D}_n = \mathbb{Z}/2^n$ and $\mathbb{D} = \bigcup_n \mathbb{D}_n$ denote sets of dyadic rationals, the former those of order $n$.

The function $\lfloor \cdot \rfloor$ which is mathematical but discontinuous/uncomputable is distinguished from Round in Subsection 2.5 as computable multifunction.
1.4.1. Partial Multifunctions. A partial multi-valued function (aka multifunction) \( f \) between sets \( X, Y \) is simply a relation \( f \subseteq X \times Y \). It mathematically models a search problem: Given (any code of) \( x \in X \), return (some code of) some \( y \in Y \) with \( (x,y) \in f \). One may identify the relation \( f \) with the single-valued total function \( f : X \ni x \mapsto \{ y \in Y \mid (x,y) \in f \} \) from \( X \) to the powerset \( 2^Y \); but we prefer the notation \( f : X \Rightarrow Y \) to emphasize that not every \( y \in f(x) \) necessarily occurs as output. Alternatively, \( \text{graph}(f) = \{(x,y) : y \in f(x)\} \) stresses the perspective of \( f \) as a static object. Mathematically all three are of course equivalent.

Allowing a computation to return different values \( y \in f(x) \) for the same argument \( x \) (technically: allowing dependence on different possible codes of the same \( x \)) means dropping the requirement for \( f \) to be extensional; hence oxymoronically such \( f \) is sometimes called a non-extensional function. Note that no output is feasible in case \( f(x) = \emptyset \); we write \( \text{dom}(f) = \{ x \mid f(x) \neq \emptyset \} \) for the domain of \( f \); and \( f \) is total in case \( \text{dom}(f) = X \). We say that \( f(x) \) has or includes a value \( y \), to mean \( y \in f(x) \). A total function \( f : X \rightarrow Y \) is a selection of a total multifunction \( F : X \Rightarrow Y \) if it holds: \( \forall x \in X. f(x) \in F(x) \).

Example 1.11. The \( d \)-ary multifunction \( \text{choose} : \subseteq \mathbb{K}^d \Rightarrow \mathbb{N} \)

is the relation
\[
\{(b_0, b_1, \ldots, b_{d-1}; j) \mid b_j = \text{true} \}.
\]

If \( f(x) \) is a singleton for every \( x \in \text{dom}(f) \), \( f \) is single-valued, i.e., a partial function denoted \( f : \subseteq X \rightarrow Y \) and written \( y = f(x) \) instead of \( f(x) = \{ y \} \). Call \( f \) compact if \( f(U) \subseteq Y \) is compact for every compact \( U \subseteq X \).

Remark 1.12. Every continuous single-valued function (possibly partial with closed domain) is compact. The composition of compact multifunctions is again compact. A compact partial multifunction \( F \) can be considered as a total mapping to the powerset \( \mathbb{P}(\mathbb{Z}_\bot) \); see Subsection 3.3 below.

The composition of multi-valued functions is defined as follows.

Definition 1.13. The composition of \( f : \subseteq X \Rightarrow Y \) and \( g : \subseteq Y \Rightarrow Z \) is
\[
g \circ f := \{(x,z) \mid f(x) \subseteq \text{dom}(g) \land \exists y \in Y. (x,y) \in f \land (y,z) \in g \}
\]

Observe that it agrees with the usual composition of relations in case \( g \) is total. Note that identifying a sequence-valued mapping \( f : \subseteq X \rightarrow Y^Z \) with \( g : \subseteq X \times Z \rightarrow Y \) via currying fails in the multi-valued case [20].

1.4.2. Turing-Computing Real and Hybrid Function(al)s. Since real numbers are infinite objects (in terms of information content), Computable Analysis considers them encoded as infinite sequences of integers: of numerators of dyadic rational approximations with given error bounds. The following definition summarizes [97, §4 and Definitions 2.4.1+3.1.3] for the special case of real and integer multifunctions and predicates, extended to include the ‘lazy’ data type \( \mathbb{K} \).

Definition 1.14. 

a) A name of \( \vec{x} \in \mathbb{R}^d \) is an integer vector sequence \( \vec{a} = (\vec{a}_n) \) with \( \| \vec{x} - \vec{a}_n / 2^n \| \leq 2^{-n} \).

b) A name of \( \vec{b} \in \mathbb{K} \) is a sequence \( \vec{b} = (b_n) \) with \( b_n \in \mathbb{K} \) such that:
\[
\forall n. b_n \equiv u_k \text{ if } b = u_k, \text{ else } \exists N. b_N = b \land \forall n < N. b_n = u_k.
\]
c) A Type-2 Machine is a Turing machine \( M \) processing infinite streams of bits, read from the input tape and written to a dedicated one-way (i.e. append-only) output tape.

d) Computing \( \vec{x} \in \mathbb{R}^d \) means to output a name of \( \vec{x} \), encoded as infinite bit sequence.

e) A partial real multifunction \( f: \subseteq \mathbb{Z}^e \times \mathbb{K}^e' \times \mathbb{R}^d \Rightarrow \mathbb{R}^d \) is (Turing-) computable if some Type-2 Machine \( M \), when executed with the binary encoding of \( \vec{k} \in \mathbb{Z}^e \) and any name of \( \vec{x} \in \mathbb{R}^d \) and any name of \( \vec{b} \in \mathbb{K}^e' \) on its input tape for \((\vec{k}, \vec{b}, \vec{x}) \in \text{dom}(f)\), outputs a name of some \( \vec{y} \in f(\vec{x}, \vec{b}, \vec{k}) \); in all other cases, \( M \) may behave arbitrarily.

f) A partial integer multifunction \( g: \subseteq \mathbb{Z}^e \times \mathbb{K}^e' \times \mathbb{R}^d \Rightarrow \mathbb{Z}^d \) is (Turing-) computable if some Type-2 Machine \( M \), whenever executed with the binary encoding of \( \vec{k} \in \mathbb{Z}^e \) and any name of \( \vec{x} \in \mathbb{R}^d \) and any name of \( \vec{b} \in \mathbb{K}^e' \) on its input tape for \((\vec{k}, \vec{b}, \vec{x}) \in \text{dom}(g)\), outputs some \( \vec{c} \in g(\vec{x}, \vec{b}, \vec{k}) \) in binary; in all other cases, \( M \) may behave arbitrarily.

g) A partial Kleenean multifunction \( h: \subseteq \mathbb{Z}^e \times \mathbb{K}^e' \times \mathbb{R}^d \Rightarrow \mathbb{K} \) is (Turing-) computable if some Type-2 Machine \( M \), whenever executed with the binary encoding of \( \vec{k} \in \mathbb{Z}^e \) and any name of \( \vec{x} \in \mathbb{R}^d \) and any name of \( \vec{b} \in \mathbb{K}^e' \) on its input tape for \((\vec{k}, \vec{b}, \vec{x}) \in \text{dom}(h)\), outputs a name of some \( \vec{e} \in h(\vec{x}, \vec{b}, \vec{k}) \) in binary; in all other cases, \( M \) may behave arbitrarily.

h) A subset \( P \subseteq \mathbb{Z}^e \times \mathbb{K}^e' \times \mathbb{R}^d \) is semi-decidable if there exists a Type-2 Machine \( M \) which terminates whenever executed with the binary encoding of \( \vec{k} \in \mathbb{Z}^e \) and any name of \( \vec{x} \in \mathbb{R}^d \) and any name of \( \vec{b} \in \mathbb{K}^e' \) on its input tape in case \((\vec{k}, \vec{b}, \vec{x}) \in P \) and does not terminate in case \((\vec{k}, \vec{x}) \notin P \); in all other cases, \( M \) may behave arbitrarily.

Terminology here deliberately deviates from [97, §5.1]. Note the subtle difference—for example in Items (e)+(f)+(g)—between not terminating and behaving arbitrarily. Ternary logic \( \mathbb{K} \) captures the former with ‘value’ \( uk \), whereas the latter corresponds multi-valuedly to \{uk, tt, ff, ⊥\}.

**Fact 1.15.**

a) Computable partial multifunctions are closed under composition in the sense of Definition 1.13 [100].

b) Any single-valued computable function is continuous; and any semi-decidable subset must be open [97, Theorems 4.3.1+3.2.11].

1.4.3. **Computable Primitives.** Here we collect some arithmetic and further operations on real numbers and their Turing-computability [97, Theorem 4.3.2]: This will justify their axiomatization as ERC primitives in Section 3 below.

**Example 1.16 .** The following operations are Turing-computable:

1. **negation:** \( \mathbb{R} \ni t \mapsto -t \in \mathbb{R} \).
2. **Unary non-zero reciprocal:** \( \mathbb{R} \setminus \{0\} \ni t \mapsto 1/t \in \mathbb{R} \).
3. **Binary addition:** \( \mathbb{R} \times \mathbb{R} \ni (s, t) \mapsto s + t \in \mathbb{R} \).
4. **Binary multiplication:** \( \mathbb{R} \times \mathbb{R} \ni (s, t) \mapsto s \cdot t \in \mathbb{R} \).
5. **Comparison, identified with the set \{ \{x, y\} \mid x < y \} \subseteq \mathbb{R}^2 \}, is semi-decidable.
6. **Fix semi-decidable \( P_0, P_1, \ldots, P_{d-1} \subseteq \mathbb{Z} \). There is a computable compact partial multi-valued mapping \( \subseteq \mathbb{Z}^d \Rightarrow \{0, 1, \ldots, d - 1\} \), assigning to \((k_0, k_1, \ldots, k_{d-1})\) some \( j \in \{0, 1, \ldots, d - 1\} \) such that \( k_j \in P_j \), provided such \( j \) exists.
The function is undefined (has 'value' undetectable case).

We now show that • outputs an integer sequence integer sequences such that Proof of Example 1.17. Fix (not necessarily disjoint) semi-decidable sets $P, Q \subseteq \mathbb{Z}$. The following partial (possibly multi-valued) real function is compact and Turing-computable:

$$Z \times \mathbb{R} \times \mathbb{R} \ni (k, x, y) \implies (k \ ? \ x : y) := \begin{cases} 
  x & \text{if } k \in P \\
  y & \text{if } k \in Q \\
  x & \text{if } k \notin P \cup Q \land x = y.
\end{cases}$$

The function is undefined (has 'value' $\perp$, the computation does not return) in case $k \notin P \cup Q \land x \neq y$.

The function is single-valued in case $P \cap Q = \emptyset$. The third case in its definition is the computable/continuous extension of the first two cases; cmp. [97, Theorem 2.3.8]. The proof reveals that “$k \ ? \ x : y$” can also be regarded as axiomatizing dovetailing, but differently from choose():

Proof of Example 1.17. Let $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ be given. Moreover, let $(a_m)_m$ and $(b_m)_m$ be integer sequences such that $|x - a_m/2^m| \leq 1/2^n$ and $|y - b_m/2^m| \leq 1/2^n$ for all $m \in \mathbb{N}$. Now we describe the behavior of a Type-2 Turing machine $M$ that, on input of $k$ and $(a_m), (b_m)$, outputs an integer sequence $(c_n)$ with $|z - c_n/2^n| \leq 1/2^n$ for $z = (k \ ? \ x : y)$ when defined.

For each $p = 0, 1, 2, \ldots$:

- If $|a_{n+3}/2^{n+3} - b_{n+3}/2^{n+3}| \leq 1/2^{n+2}$ holds for all $n \leq p$, then output $[a_{p+3}/8]$. Note that rational numbers can be rounded exactly.

- As soon as $n \leq p$ is encountered with $|a_{n+3}/2^{n+3} - b_{n+3}/2^{n+3}| > 1/2^{n+2}$, switch to simultaneously (dovetailing) search for a witness that $k \in P$ and for a witness that $k \in Q$.

When $k \in P$ is asserted, then output $a_p$, and when $k \in Q$ is asserted, then output $b_p$.

We now show that $M$ computes the given multifunction.

- $x = y$: Then $|a_{n+3}/2^{n+3} - b_{n+3}/2^{n+3}| \leq 1/2^{n+2}$ holds for all $n$. So, given $p, M$ always outputs $[a_{p+3}/8]$. Moreover, the following holds:

$$|x - [a_{p+3}/8]/2^p| \leq |x - a_{p+3}/2^{p+3}| + |a_{p+3}/2^{p+3} - [a_{p+3}/8]/2^p| \leq 1/2^{p+3} + |a_{p+3}/8 - [a_{p+3}/8]|/2^p \leq 1/2^{p+3} + 1/2^{p+1} \leq 1/2^p.$$

- $x \neq y$: Then $|a_{n+3}/2^{n+3} - b_{n+3}/2^{n+3}| > 1/2^{n+2}$ for some $n$. Let $N$ be the least such.

- $k \in P$ and $k \notin Q$: $M$ outputs $[a_{p+3}/8]$ when $p < N$ and $a_p$ otherwise. As shown above, both cases provide an approximation to $x$ up to $1/2^p$.

- $k \notin P$ and $k \in Q$: $M$ outputs $[a_{p+3}/8]$ for $p < N$ and $b_p$ otherwise. When $p < N$, the following holds:

$$|y - [a_{p+3}/8]/2^p| \leq |y - a_{p+3}/2^{p+3}| + |a_{p+3}/2^{p+3} - [a_{p+3}/8]/2^p| \leq |y - b_{p+3}/2^{p+3}| + |b_{p+3}/2^{p+3} - a_{p+3}/2^{p+3}| + 1/2^{p+1} \leq 1/2^{p+3} + 1/2^{p+2} + 1/2^{p+1} \leq 1/2^p.$$

If $p \geq N$, then $|y - b_p/2^p| \leq 2^{-p}$ holds by definition.
– \( k \in P \) and \( k \in Q \): \( \mathcal{M} \) outputs \( \lfloor a_p + 3/8 \rfloor \) for \( p < N \), and \( a_p \) or \( b_p \) for \( p \geq N \) depending on what comes first during checking \( k \in P \) and \( k \in Q \). As the previous two cases have shown, \( \mathcal{M} \) computes then an approximation to \( x \) or \( y \) up to \( 1/2^p \).

– \( k \not\in P \) and \( k \not\in Q \): In this case, \( \mathcal{M} \) does not halt.

2. Programming in Exact Real Computation

Before formalizing the syntax, type system and semantics of ERC as programming language in Section 3, we collect (Subsection 2.1) several examples of (compact multi-)functions realized in ERC. They illustrate programming in this paradigm: both the convenience as well as the challenge and techniques connected to partial comparison and multi-valued operations. Subsection 2.8 asserts that precisely the Turing-computable partial real functions can be expressed in ERC.

2.1. Example Programs in ERC. Most ‘algorithms’ processing floating point numbers strictly speaking constitute heuristics and are difficult to even specify rigorously [47]. The purpose of ERC is to allow and justify naïvely implementing numerical algorithms, with Computable Analysis as rigorous but hidden theoretical foundation. The necessarily modified (namely partial) semantics of real comparisons is easy to get used to. Examples\(^4\) below illustrate both the elegance and convenience of, and the changes towards, realizing common classical algorithms in ERC: inductively defined real sequences, square root using Heron’s Methods (Subsection 2.2), exponential function via Taylor expansion (Subsection 2.3) and iteratively (Subsection 2.4); integer rounding (Subsection 2.5), matrix determinants via Gauss Elimination (Subsection 2.6), and root finding (Subsection 2.7).

The multi-valued \texttt{choose}() and the continuous conditional in ERC are new to classically-trained programmers. They are crucial to writing total programs, regarding that real comparison is only partial. The examples in this Subsection demonstrate how to incorporate such considerations.

\textbf{Remark 2.1.} Note that the original mathematical but numerically naïve problem specification may require modification, in order to become (Turing-)computable and thus admit rigorous realization in ERC. Two general approaches can help identify such a reasonable but computable, weaker specification of a problem \( f : X \to Y \):

(a) Dropping extensionality, that is, proceeding from the function problem \( f \) to a multifunction/search problem \( F : X \Rightarrow Y \) [70, 69]; consider for example the Soft Test (1.3) above or integer rounding (Subsection 2.5) below.

(b) Enriching the arguments \( x \in X \) to \( f \) with certain integers \( k \in \mathbb{Z} \), that is, replacing the domain \( \text{dom}(f) = X \) with a suitable subset of \( X \times \mathbb{Z} \) [19, 106]; consider for example the (already multi-valued) problem of finding a non-zero vector in the kernel of a given singular matrix \( A \in \mathbb{R}^{d \times d} =: X \), which is uncomputable but becomes computable when given \( k = \text{rank}(A) \in \mathbb{Z} \) in addition to \( A \) [106, §4.1].

\(^4\)These examples have been implemented using a shallow embedding of ERC into Haskell: https://github.com/michalkonecny/aern2/blob/master/aern2-erc/src/ERC/Examples.hs
All arithmetic operations provided to the user as exact, ERC significantly simplifies and rigorously justifies implementing many naïve numerical algorithms. For instance the real sequence from Example 1.2 can conveniently be realized in ERC literally in the way it is mathematically defined, as shown in Fig. 1.

Figure 1: $\text{JMMuller} : \subseteq \text{INTEGER} \rightarrow \text{REAL}$

```
input $n : \text{INTEGER}$  // $n \geq 0$
let $a : \text{REAL} = 11/2$;
let $b : \text{REAL} = 61/11$;
while $n > 0$ do
  let $c : \text{REAL} = 111 - (1130 - 3000/a)/b$;
  $a := b$; $b := c$; $n := n - 1$
end while
return $a$
```

2.2. Square Root Function via Heron’s Method. Heron’s Method approximates the square root of a given real number $x \geq 0$ up to any desired absolute error $2^p$, $p \in \mathbb{Z}$. To this end calculate a contracting sequence of upper and lower approximations $x_n \geq \sqrt{x} \geq y_n := z_n$ by iteratively taking the average $y_{n+1} := (y_n + z_n)/2$.

Recall that the arguments to choose are indexed from 0, so choose$(b_0, b_1) = 1$ means that the second argument $b_1$ must evaluate to $tt$.

Figure 2: $\text{HeronSqrt} : \subseteq \text{REAL} \rightarrow \text{REAL}$

```
input $p : \text{INTEGER}; x : \text{REAL}$  // $x \geq 0$
let $y : \text{REAL} = 1$; let $z : \text{REAL} = x/y$;  // $y \geq \sqrt{x} \geq z$
while choose $(\iota(p) \triangleright y - z, y - z \triangleright \iota(p-1)) = 1$ do
  $y := (y + z)/2$; $z := x/y$;  // $y \geq \sqrt{x} \geq z$
end while
return $y$  // $|y - \sqrt{x}| < 2^p$
```

Recall that $\iota$ denotes the ‘binary precision’ embedding $\mathbb{Z} \ni p \mapsto 2^p \in \mathbb{R}$. Since real comparison is only partial, the ERC program employs the multi-valued compact choose() operation applied to two tests: “$\iota(p) \triangleright y - z$” and “$y - z \triangleright \iota(p-1) = 2^{p-1}$”. The semantics of “$>$” (formalized in Subsubsection 3.3.1 and as opposed to $>$) is deliberately ‘lazy’ such that program execution continues even in case $\iota(p) = y - z$ or $y - z \triangleright \iota(p-1)$; imagine the actual evaluation to happen only inside of choose(). Note that at least one of both inequalities “$\iota(p) \triangleright y - z$” and “$y - z \triangleright \iota(p-1) = 2^{p-1}$” is always true (and in particular defined, i.e., not unknown), resulting in a total condition for the while loop; and when the loop terminates, the first test (corresponding to return value 0) must (while the second test, corresponding to return value 1, may or may not) be true: guaranteed to return an approximation $y$ to $\sqrt{x}$ up to absolute error $2^p$. 
2.3. Exponential Function via Taylor Expansion. The exponential function has a globally converging Taylor expansion

\[
\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}
\]  

(2.1)

A realizer in ERC according to Paradigm 1.3 must return, given a dedicated precision parameter \( p \in \mathbb{Z} \) and argument \( x \in \mathbb{R} \) exactly, an approximation to \( \exp(x) \) up to error \( 2^p = \pi(p) \). For \( |x| \leq 1 \) and positive \( n \in \mathbb{N} \), the tail bound

\[
\left| \sum_{j>n} \frac{x^j}{j!} \right| \leq \sum_{j>n} \left| \frac{x^j}{j!} \right| \leq \sum_{j>n} 2^{-j+1} = 2^{-n+1}
\]  

(2.2)

justifies the straightforward algorithm—without the need for rounding/cancellation error and propagation considerations required for floating point numbers, since in ERC all arithmetic operations are exact:

Figure 3: \( \text{Exp}' : \subseteq \text{REAL} \rightarrow \text{REAL} \)

| input \( p : \text{INTEGER}; \ x : \text{REAL} \) // \(-1 \leq x \leq 1\) |
|---|
| let \( j : \text{INTEGER} = 1 \); let \( j_r : \text{REAL} = 1 \) let \( f : \text{REAL} = 1 \); // \( j \equiv j_r, \ f \equiv j! \) |
| let \( y : \text{REAL} = 1 \); let \( z : \text{REAL} = x \); // \( z \equiv x^j \) |
| while \( j \leq -p + 1 \) do |
| \( y := y + z/f \); |
| \( j := j + 1 \); |
| \( j_r := j_r + 1 \); |
| \( z := z \times x \); |
| \( f := f \times j_r \); |
| end while |
| return \( y \) // \( |y - \exp x| \leq 2^p \) |

Observe that the two variables \( j : \text{INTEGER} \) and \( j_r : \text{REAL} \) in Fig. 4 retain identical values but have carefully distinguished types: ERC allows casting integers to reals only ‘exponentially’ via \( \pi : \mathbb{Z} \ni p \mapsto 2^p \in \mathbb{R} \), since this asserts the two-sorted structure \( \mathbb{Z} \uplus \mathbb{R} \) to exhibit a decidable first-order theory; see Section 4.

The following wrapper for \( \text{Exp}' \) removes the restriction \( |x| \leq 1 \) and instead works for all \( x \geq -1 \):

Figure 4: \( \text{Exp} : \subseteq \text{REAL} \rightarrow \text{REAL} \)

| input \( x : \text{REAL} \) // \(-1 \leq x\) |
|---|
| let \( z : \text{REAL} = \text{Exp}'(1/2) \); |
| let \( y : \text{REAL} = 1 \); |
| while \( \text{choose}(x < 1, x > 1/2) = 1 \) do |
| \( y := y \times z \); |
| \( x := x - 1/2 \); |
| end while |
| return \( y \times \text{Exp}'(x) \) |
Like Heron’s Method in ERC (Subsection 2.2), the loop employs “choose()” with overlapping conditions \( x < 1 \) and \( x > 1/2 \) in order to guarantee totality. Also, since for \( x < 0 \) it holds that \( \exp(x) = 1/\exp(-x) \), and \( \exp(y) = 1/\exp(y) \) at \( y = 0 \), we can construct a procedure that computes \( \exp(x) \) for all \( x \in \mathbb{R} \) simply by ‘\( x \geq 0 \) ? \( \text{Exp}'(x) : 1/\text{Exp}'(-x)’.

2.4. Exponential Function via Iteration. \( \text{Exp}' \) in Subsection 2.3 employs unbounded sums, which leave the realm of the first-order logic considered in Section 4 for formal specification and verification purposes. Here we consider an alternative, iterative approach to realize the exponential function in ERC. To this end recall for every \( x \in [0; 2] \) it holds [61, §3.6.3]:

\[
\exp(x) \xrightarrow{n \to \infty} (1 + \frac{x}{n})^n \leq \exp(x) \leq (1 + \frac{x}{n})^{n+1} \xrightarrow{n \to \infty} \exp(x)
\]

This suggests the following iterative ERC program:

```
input p : INTEGER; x : REAL // 0 \leq x \leq 2
let n : INTEGER = 1;
let n_r : REAL = 1;
let c : REAL = 1 + x; let a : REAL = c; let b : REAL = a \times c;
while choose(\( \nu(p) > b - a, b - a > \nu(p - 1) \)) = 1 do
    n := n + n;
    n_r := 2 \times n_r;
    c := 1 + x/n_r; a := c^n; b := a \times c
end while
return a
```

Note that, similarly to Figure 3, variables \( n \equiv n_r \) of different types are maintained to coincide in value. Moreover “\( a := c^n \)” is to be understood as abbreviating a loop of \( n \) real multiplications; see Subsection 3.4(d) below.

Program 5 supposes \( 0 \leq x \leq 2 \), but can be extended to the entire real line similar to Subsection 2.3. Note that the while loop is guaranteed to terminate since \( b - a \) converges to 0 as \( n \) grows. The while condition is total: At least one of ‘\( \nu(p) > b - a \)’ and ‘\( b - a > \nu(p - 1) \)’ is always true. And when the while loop terminates, then it must hold \( \nu(p) > b - a \), hence \( |a - \exp(x)| \leq |b - a| \leq 2^p \).

2.5. Integer Rounding Multifunction. All real-to-integer rounding functions (up, down, to nearest) are discontinuous and therefore not Turing-computable; recall Fact 1.15(b). We thus relax the specification according to Remark 2.1(a) and instead consider the following compact multifunction with ‘overlap’:

\[
\text{Round} : \mathbb{R} \ni x \mapsto \{ k \in \mathbb{Z} \mid x - 1 < k < x + 1 \} \subseteq \mathbb{Z} \quad (2.3)
\]

The below ERC program \( \text{Round1} \) realizes \( \text{Round} \): It returns, given a (not necessarily positive) real \( x \), some integer \( k \) such that \( |x - k| < 1 \).
Figure 6: Round 1 : REAL ⇒ INTEGER

1: input x : REAL
2: let k : INTEGER = 0;
3: while choose(x < 1, x > 1/2) = 1 do
4:   k := k + 1; x := x - 1;
5: end while;
6: while choose(x ≥ −1, x ≤ −1/2) = 1 do
7:   k := k - 1; x := x + 1;
8: end while;
9: return k

Intuitively, in view of Remark 1.4(e), the ‘number of steps made’ by Round1 are proportional to the value of the argument x, that is, exponential in its binary (≈output) length: because Round1 essentially counts up to x.

Recovering the rounded integer bitwise via binary search seems exponentially more efficient, but fails due to lack of continuity: Extracting any digit of ‘the’ binary expansion (or one of the at most two possible ones) of a given real number is uncomputable [92]. Instead, the following algorithm Round2 realizes the idea that some signed-digit expansion [97, Definition 7.2.4] of a given real number x ∈ ℝ can be determined computably:

Figure 7: Round 2 : REAL ⇒ INTEGER

1: input x : REAL
2: let k, j, b : INTEGER = 0; let y : REAL = x;
3: while choose(|y| < 1, |y| > 1/2) = 1 do
4:   j := j + 1; y := y/2;
5: end while;
6: while j > 0 do
7:   y := y × 2
8:   b := choose(y < 0, −1 < y < 1, y ≥ 1) − 1;
9:   y := y - b; k := 2 · k + b; j := j - 1
10: end while
11: return k

Loop invariants have been included into the code as informal comments in order to convey partial correctness with respect to the specified postcondition; separate work will expand these for formal verification. Due to multi-valuedness of the test, after the real number while loop (lines 3 to 5) has ended, the second argument “y > 1/2” may still be true, whereas the first “|y| < 1” must be true; but always at least one of both is valid, thus guaranteeing total correctness. In the integer loop (lines 6 to 10), multi-valuedness ‘strikes’ only at line 8 which
employs \texttt{choose()} with trinary argument. Recall that ERC avoids mixing real and integer arithmetic (Section 4); hence "$y - b$" in line 9 is to be understood as abbreviation of the following code:

\begin{verbatim}
if b = -1 then y := y + 1; if b = 1 then y := y - 1;
\end{verbatim}

2.6. Determinant Function via Gaussian Elimination. The determinant of a $d \times d$ matrix $A = (a_{ij})_{i,j}$ is given by Leibniz’ formula

$$
\text{det}(A) = \sum_{\pi} \text{sign}(\pi) \cdot \prod_{j=1}^{d} a_{j,\pi(j)}
$$

(2.4)

where the sum ranges over all $d!$ permutations $\pi : \{1, \ldots, d\} \to \{1, \ldots, d\}$. Since ERC conveniently relieves the programmer from numerical issues like cancellation, this formula gives rise to a straight-forward ERC program—however one executing a number of arithmetic operations (Remark 1.4e) exponential in $d$.

Common numerical approaches therefore transform $A$ to triangular form, whose determinant is simply the product of its diagonal elements [74, §2.3.3]. More formally, following Turing [94], apply Gaussian Elimination in order to determine a \textit{LU factorization with full pivoting} $P \cdot A \cdot Q = L \cdot U$ of $A$, where $P, Q$ denote permutation matrices and $L$ and $U$ are lower and upper triangular matrices, respectively. Error propagation through Gaussian Elimination with its variants are non-trivial topics in numerical analysis [89, §IV] that hamper rigorous specification when implemented in floating point arithmetic. Exact arithmetic in ERC on the other hand eliminates (pun!) the need for such considerations—except for pivot search, which involves real comparisons that are only partially defined. In Gauss’ original algorithm, such search either (i) returns the index of a non-zero entry in the given sub-matrix or (ii) asserts that said sub-matrix is identically zero. By iterating this process, Gaussian Elimination determines the rank $k \in \mathbb{N}$ of the original matrix $A \in \mathbb{R}^{d \times d}$—which depends discontinuously on $A$’s entries and hence cannot be Turing-computed by Fact 1.15(b).

\textbf{Remark 2.2.} Following Remark 2.1 we thus (must) change the specification of the determinant (and of the LUPQ factorization\footnote{LUPQ factorization $A \mapsto (L, U, P, Q)$ is not unique, hence a real matrix-tuple-valued \textit{multi}function, recall Remark 1.4(d).} it builds on): with the promise for the argument matrix $A$ to have full rank—otherwise its determinant will vanish, anyway.

$$
\text{Det}_d : \subseteq \text{GL}(\mathbb{R}^d) \ni A \mapsto \text{det}(A) \neq 0
$$

(2.5)

We also relax pivot search (within the lower-right submatrix) to become a compact multi-valued problem:

\begin{equation}
\text{Pivot}_d = \{(A, k, i, j) : A \in \mathbb{R}^{d \times d} \land k \leq i, j < d \land A_{i,j} \neq 0\} \subseteq \mathbb{R}^{d \times d} \times \mathbb{Z}^3
\end{equation}

(2.6)

So the argument is a real $d \times d$ matrix $A$ and an integer $k$, indicating that a pivot is to be sought for in the $(d-k) \times (d-k)$ non-zero sub-matrix $A[k \ldots d-1, k \ldots d-1] := (A[i,j])_{k \leq i,j < d}$.

Note that technically ERC (as formalized here) knows only single integer return values; recall Remark 1.4(d). The pair $(i_0, j_0)$ of return values here is thus understood as abbreviation of the single integer $i_0 + j_0 \cdot d$, similarly to the two-dimensional array access $A[i, j]$ as abbreviation of $A[i + j \cdot d]$; see Subsection 3.4(f) below.

Based on $\text{Pivot}_d$, the ERC program in Fig. 9 computes the non-zero determinant $\text{Det}_d$ from Equation (2.5) via LUP decompositon with full pivoting. The precision parameter
\begin{figure}[h]
\centering
\begin{algorithmic}
\State \textbf{input} $A : \text{REAL}[d \times d], k : \text{INTEGER}$
\State let $i_0 : \text{INTEGER} = k; \text{ let } j_0 : \text{INTEGER} = k; \text{ let } x : \text{REAL} = 0;$
\For{$i : \text{INTEGER} = k$ \text{ to } $d - 1$}
\For{$j : \text{INTEGER} = k$ \text{ to } $d - 1$}
\State $x := \max (x, \text{abs}(A[i,j])); \text{ end for; end for; end for; end for; end for; return } (i_0, j_0)$
\For{$i : \text{INTEGER} = k$ \text{ to } $d - 1$}
\For{$j : \text{INTEGER} = k$ \text{ to } $d - 1$}
\State if \text{choose}(\text{abs}(A[i,j]) < x, \text{abs}(A[i,j]) \geq x/2) = 1$
\State \quad then $i_0 := i; j_0 := j;$
\EndIf; \text{ end for; end for;}
\State return $(i_0, j_0)$
\end{algorithmic}
\caption{Pivot$_d : \subseteq \text{REAL}[d \times d] \times \mathbb{N} \Rightarrow \text{INTEGER} \times \text{INTEGER}$}
\end{figure}

\begin{figure}[h]
\centering
\begin{algorithmic}
\State \textbf{input} $p : \text{INTEGER}; A : \text{REAL}[d \times d]$ \quad \text{// A invertible, } p \text{ ignored}
\State let $i : \text{INTEGER} = 0; \text{ let } j : \text{INTEGER} = 0; \text{ let } k : \text{INTEGER} = 0; \text{ let } p_i : \text{INTEGER} = 0; \text{ let } p_j : \text{INTEGER} = 0; \text{ let } det : \text{REAL} = 1; \text{ ret.val}$
\State for $k := 0$ \text{ to } $d - 2$ \text{ do}
\State \quad \text{// Convert } A[k,.d - 1,k..d - 1] \text{ to reduced row echelon form:}
\State \quad $(p_i, p_j) := \text{Pivot}(A, k)$; \quad \text{// } p_i, p_j \geq k \text{ s.t. } A[p_i, p_j] \neq 0.$
\State \quad det := det $\times A[p_i, p_j]$;
\State \quad for $j := 0$ \text{ to } $d - 1$ \text{ do swap}(A[k, j], A[p_i, j]) \text{ end for;}
\State \quad \quad \text{// Exchange rows } \#k \text{ and } \#p_i.$
\State \quad if $k \neq p_i$ then det := $-det;$ \quad \text{// flip sign}
\State \quad for $i := 0$ \text{ to } $d - 1$ \text{ do swap}(A[i, k], A[i, p_j]) \text{ end for;}
\State \quad \quad \text{// Exchange columns } \#k \text{ and } \#p_j.$
\State \quad if $k \neq p_j$ then det := $-det;$ \quad \text{// flip sign}
\State \quad for $j := k + 1$ \text{ to } $d - 1$ \text{ do}
\State \quad \quad \quad A[k, j] := A[k, j] / A[k, k]; \quad \text{// and subtract the } A[i, k]-\text{fold from}
\State \quad \quad \quad for $i := k + 1$ \text{ to } $d - 1$ \text{ do}
\State \quad \quad \quad \quad \quad \text{// from rows } \#i = k + 1 \ldots d - 1.
\State \quad \quad \quad \quad \quad \text{do } A[i, j] := A[i, j] - A[i, k] \times A[k, j] \text{ end for}
\State \quad \quad \text{end for; } A[k, k] := 1; \text{ for } i := k + 1 \text{ to } d - 1 \text{ do } A[i, k] := 0 \text{ end for}
\State \quad \text{end for;}
\State \quad det := det $\times A[d - 1, d - 1];$
\State return det
\end{algorithmic}
\caption{Det : $\subseteq \text{REAL}[d \times d] \rightarrow \text{REAL}$}
\end{figure}

$p \in \mathbb{Z}$ is present but ignored since the result gets computed exactly; recall Paradigm 1.3(c) and Remark 1.4(c).

Note that pivot search in line 6 of program Det is guaranteed to succeed in that the $(d - k) \times (d - k)$ submatrix $A[k \ldots d - 1, k \ldots d - 1]$ under consideration will indeed contain
These hypotheses also avoid common counterexamples like [86] or [97, Theorem 6.3.2]. To the above hypotheses assert Turing-computability of \( f \) holds.

Indeed returns an approximation to the root up to error in an invariant \( \text{cont} \). See that \( \text{cont} \) is commonly treated using Bisection: Determine the sign of \( f(x) \) at the interval mid point \( x := (a + b)/2 \) and recurse to either \([a, x]\) or to \([x, b]\) accordingly. However, since equality is undecidable, the sign test fails in case \( f(x) = 0 \). Trisection [38, p. 336] instead considers the signs of \( f \) at both one third \( x' := (2a + b)/3 \) and at two third \( x'' := (a + 2b)/3 \) of the interval, in parallel; and recurses to either \([a, x'']\) or to \([x', b]\) accordingly: Now at most one of the two sign tests at \( f(x') \) and \( f(x'') \) can fail, provided that \( f \)'s root is unique and \( f(a) \cdot f(b) < 0 \). These hypotheses also avoid common counterexamples like [86] or [97, Theorem 6.3.2]. To summarize, we consider the (single-valued) root finding problem \( \text{Root} : f \mapsto x \) with \( f(x) = 0 \) for \( f \) satisfying the following first-order predicates:

\[
\text{cont}(f, a, b) := \\
\forall \epsilon > 0. \exists \delta > 0. \forall x, x'. a \leq x \leq x' \leq x + \delta \leq b \Rightarrow |f(x) - f(x')| \leq \epsilon
\]

\[
\text{uniq}(f, a, b) := f(a) \cdot f(b) < 0 \land \exists!x. a < x < b \land f(x) = 0
\]

See that \( \text{cont}(f, a, b) \) says \( f \) is continuous in the usual sense on the interval \([a; b]\) and \( \text{uniq}(f, a, b) \) says \( f \) admits a unique root in the interval \((a; b)\) and the signs of \( f(a) \) and \( f(b) \) are different.

\( \text{Root} : \{ f : [0; 1] \to \mathbb{R} : \text{cont}(f, a, b) \land \text{uniq}(f, a, b) \} \ni f \mapsto x : f(x) = 0 \)

The above hypotheses assert Turing-computability of \( \text{Root} \) [97, Corollary 6.3.5].

The ERC program in Subsection 4.4 computing \( \text{Root} \) is annotated with precondition \( \text{cont}(f, 0, 1) \land \text{uniq}(f, 0, 1) \) and postcondition \( \text{uniq}(f, a, b) \land |b - a| \leq 2^p \), as well as loop invariant \( 0 \leq a < b \leq 1 \land \text{uniq}(f, a, b) \). Given \( p \), the postcondition guarantees that \( \text{Root} \) indeed returns an approximation to the root up to error \( 2^p \) when \( \text{cont}(f, 0, 1) \land \text{uniq}(f, 0, 1) \) holds.

**Figure 10:** \( \text{Root} : \subseteq \text{INTEGER} \times (\text{REAL} \to \text{REAL}) \to \text{REAL} \)

```plaintext
1: input p : INTEGER, f : REAL \to REAL  // cont(f, 0, 1) \land \text{uniq}(f, 0, 1)
2: let a : REAL = 0; let b : REAL = 1;  // cont(f, 0, 1) \land \text{uniq}(f, 0, 1)
3: while choose (\( \downarrow p \) \( \triangleright b - a, b - a \triangleright \downarrow (p - 1) \)) = 1 do
   // 0 \leq a < b \leq 1 \land \text{cont}(f, a, b) \land \text{uniq}(f, a, b) \land b - a > 2^{p-1}
   if choose (\( \downarrow b/3 + 2 \times a/3 \) \( \triangleright f(b), 0 \triangleright f(a) \times f(2 \times b/3 + a/3) \)) = 1
      then b := 2 \times b/3 + a/3
   end if
4: else a := b/3 + 2 \times a/3  // \text{uniq}(f, a, b) \land |b - a| \leq 2^p
5: end while
6: return a
```
As ERC prohibits type conversion $\mathbb{Z} \rightarrow \mathbb{R}$, “$b/3$” is to be understood as abbreviation for “$b/(t(0) + t(0) + t(0))$”; similarly for $2 \times a/3$.

**Remark 2.3.** Note that $f : [0; 1] \rightarrow \mathbb{R}$ here represents an arbitrary continuous ‘external’ function that Root treats and calls as black box, with arguments passed and values returned exactly. It may for example (but does not have to) be realized in ERC itself, such as the square root (Subsection 2.2) or the exponential function (Subsections 2.3 and 2.4): recall Fact 1.15(a).

Logically speaking, this amounts to an expansion of ERC in the sense of Model Theory, namely with function symbol $f$ added to the operations provided by the structure. More generally, adapting the paradigm of computing over structures [72, 68], fix a collection $\mathcal{F}$ of partial (single-valued) real functions $f : \subseteq \mathbb{Z}^{e_f} \times \mathbb{R}^{d_f} \rightarrow \mathbb{R}$ and $\mathcal{G}$ a collection of partial multi-valued integer functions $g : \subseteq \mathbb{Z}^{e_g} \times \mathbb{R}^{d_g} \Rightarrow \mathbb{Z}$, the latter supposed to be compact according to Remark 1.12. Then the expansion $\text{ERC}(\mathcal{F}, \mathcal{G})$ allows the user program to make calls to any $f \in \mathcal{F}$ and/or $g \in \mathcal{G}$; see Section 3.

2.8. **Real Turing-Completeness of ERC.** Extending and building on the above examples of programming in ERC, we now establish that ERC is Turing-complete over the real numbers.

**Theorem 2.4** Turing-Completeness over the Reals.

a) Every partial single-valued real function $f$ Turing-computable in the sense of Computable Analysis (Subsubsection 1.4.2) can be realized in ERC.

b) Fix a finite and possibly empty collection $\mathcal{F}$ of Turing-computable partial (single-valued) real functions of various arities, and $\mathcal{G}$ a finite and possibly empty collection of Turing-computable partial compact multi-valued integer functions, also of various arities. Then

a) Every partial single-valued real function $f$ realizable over $\text{ERC}(\mathcal{F}, \mathcal{G})$ is Turing-computable in the sense of Computable Analysis (Subsubsection 1.4.2) and compact.

b) Similarly, every partial integer multi-function $g$ realizable over $\text{ERC}(\mathcal{F}, \mathcal{G})$ is Turing-computable and compact.

Before proceeding to the proof, let us record two more simple ERC programs. Firstly integer multiplication, although excluded from the primitives in ERC (Subsubsection 3.1.2), can be computationally realized, for example by repeated addition in a loop in Figure 11.

**Figure 11: MULT : INTEGER × INTEGER → INTEGER**

```
1: input N, M : INTEGER,  output N × M : INTEGER
2: let Z : INTEGER = 1;
3: if M < 0 then N := −N; M := −M; end if
4: while M > 0 do
5: Z := Z + N; M := M − 1;
6: end while
7: return Z
```
Secondly, although ERC deliberately prohibits direct casting from integers to reals (Section 4), the ‘precision’ embedding \( \iota: \mathbb{Z} \ni p \mapsto 2^p \in \mathbb{R} \) can be realized; see Figure 12. The same applies to type conversion \( j: \mathbb{Z} \hookrightarrow \mathbb{R} \).

Figure 12: \( \iota: \subseteq \text{INTEGER} \rightarrow \text{REAL} \)

```plaintext
1: input \( p: \text{INTEGER}, \) output \( \iota(p) = 2^p: \text{REAL} \)
2: let \( R: \text{REAL} = 1; \) let \( T: \text{REAL} = 2; \)
3: if \( P < 0 \) then \( P := -P; \ T := 1/T; \) end if
4: while \( P > 0 \) do
5: \( R := R \times T; \ P := P - 1; \)
6: end while
7: return \( R \)
```

**Proof of Theorem 2.4.** a) The data type \text{INTEGER} of ERC immediately yields the simulation of a Counter Machine [95, pp.32–35] with a finite number of registers \( r \) and instructions

\[
\text{CLR}(r), \ \text{INC}(r), \ \text{DEC}(r), \ \text{CPY}(r), \ \text{JZ}(r,z), \ \text{JE}(ri,rj,z) \quad (2.7)
\]

known equivalent to the Turing machine model—on integer inputs. Regarding real arguments and integer values, recall (Subsection 2.5) that the multi-valued rounding ‘function’ \( \text{Round}: \mathbb{R} \Rightarrow \mathbb{Z} \) can be expressed in ERC.

Together with the ‘precision’ embedding \( \iota: \mathbb{Z} \ni p \mapsto 2^p \in \mathbb{R} \) from Algorithm 12, ERC thus allows to extract numerators \( a_n := \text{Round}(x \cdot \iota(n)) \in \mathbb{Z} \) of dyadic approximations to any given \( x \in \mathbb{R} \), and to process them in any way a Type-2 Machine can according to Definition 1.14. Any Turing-computable single-valued function is well-known continuous [97, Theorem 3.2.11] and in particular compact.

b) Intuitively, the basic operations in ERC are designed as Turing-computable; cmp. Examples 1.16 and 1.17. And so are the expansions \( \mathcal{F} \) and \( \mathcal{G} \) by hypothesis. Finally, Turing-computable (compact multi) functions are closed under composition according to Fact 1.15(a). The precise statement is deferred to Proposition 3.15 below in terms of the formal semantics according to Subsection 3.3.

b) According to Paradigm 1.3, a real-valued ERC program produces, in dependence on the arguments and indexed by the integer error parameter \( p \rightarrow -\infty \), a real sequence \( (y_p)_p \) of approximations up to error \( 2^p \) to some \( y \in \mathbb{R} \). Computable Analysis asserts that the limit \( y \) is again computable, uniformly in said arguments [97, Theorems 4.2.3 and 4.3.8].

3. ERC as Formal Programming Language

Having provided intuition and motivation for ERC (Subsection 1.2, Subsubsection 1.4.3), and having seen several examples of ERC programs (Subsection 2.1), we finally proceed to provide a formalization of the programming language: syntax (Subsection 3.1), type system (Subsection 3.2), and multi-valued denotational semantics (Subsection 3.3); for terms (Subsections 3.1.2, 3.2.1, 3.3.1), commands (Subsections 3.1.3, 3.2.2, 3.3.2), and for programs (Subsections 3.1.4, 3.2.3, 3.3.3).
A particular benefit of ERC is closure under composition: Any real function \( f \) (or integer multifunction \( g \)), realized as some ERC program \( F(G) \), can be invoked by another ERC program \( H \) to receive an exact (rather than approximate) return value. Recall for instance Trisection (Subsection 2.7) invoking the function whose root is to be computed. This is formalized as expansion \( \text{ERC}(F, G) \), for example with \( F = \{ f \} \) and \( G = \{ g \} \).

**Convention 3.1.** For the sequel fix a (possibly empty) finite set \( F \) of Turing-computable (and thus automatically compact) real-valued partial functions as well as a (possibly empty) finite set \( F \) of Turing-computable compact integer-valued partial multi-functions.

Each realizer \( F \) of a real function \( f \) in the sense of Paradigm 1.3 is a separate program; same for realizers \( G \) of compact integer-valued multifunctions \( g \) in the sense of Remark 1.4 (d): see Subsubsection 3.3.3.

**Remark 3.2.** The purpose of this section is to provide a fully detailed formalization: particularly regarding (the propagation of) multi-valuedness and of \( \bot \) in careful distinction from unknown; see Remark 3.6. This involves abstract concepts like Smyth’s powerdomains in Subsection 3.3. It should serve as reference, for instance for programming language theorists. For practical programming, on the other hand, ERC is deliberately designed to appeal to, and suffice with, intuition.

### 3.1. Syntax

ERC is an imperative programming language comprising of the following axiomatized constituents: data types (Subsubsection 3.1.1), terms (Subsubsection 3.1.2), commands (Subsubsection 3.1.3), and programs (Subsubsection 3.1.4).

#### 3.1.1. Data Types of ERC

The data types that ERC provides are as follow:

\[
\tau ::= \text{KLEENEAN} | \text{INTEGER} | \text{REAL} | \text{REAL}[n]
\]

for each natural number \( n \). See that ERC provides countably many data types: for each natural number \( n \), there is a data type \( \text{REAL}[n] \), which represents the set of arrays of real numbers of length \( n \). We prohibit implicit type conversion, in order for the underlying logic to be decidable; see Section 4.

#### 3.1.2. Terms

A term represents values of a certain type. To define terms inductively, we assume that the following sets are fixed according to Convention 3.1: a set \( F \) of partial continuous (single-valued) real functions \( f : \mathbb{Z}^e_f \times \mathbb{R}^d_f \to \mathbb{R} \), and a set \( G \) of partial compact multi-valued integer functions \( g : \mathbb{Z}^e_g \times \mathbb{R}^d_g \Rightarrow \mathbb{Z} \).

Although the type of a term is not determined syntactically, we here follow the following conventions for convenience: Write \( z, z_i \) to denote terms which should be typed \( \text{INTEGER} \), \( x, y, x_i, y_i \) to denote terms which should be typed \( \text{REAL} \), and \( b, b_i \) to denote terms which should be typed \( \text{KLEENEAN} \). Arbitrary terms are denoted by \( t, t_i \). Moreover, \( f, g \) stand for function symbols with certain arity representing (multi-) functions from \( F \) and \( G \), respectively. Terms are defined inductively as follow:

\[
t, t_i, z, z_i, x, y ::= \\
\bar{z} \quad z \in \mathbb{Z}; \text{INTEGER literal} \\
| \text{true} | \text{false} | \text{unknown} \quad \text{KLEENEAN literal}
\]
Again, there are infinitely many term constructs: for each $v \in V$, $v$ is a term construct, for each $z \in Z$, $\bar{z}$ is a term construct, and for each natural number $n$, $\text{choose}_n$ is a term construct. Here, since the $n$ is determined syntactically by the number of the arguments, we may omit it and instead simply refer to $\text{choose}()$. Throughout this paper, we will write $t_1 - t_2$ as an abbreviation for $t_1 + -t_2$, , $t_1/t_2$ as an abbreviation for $t_1 \times /t_2$, $z_1 \geq z_2$ as an abbreviation of $z_2 \leq z_1$, $z_1 > z_2$ as an abbreviation of $-(z_2 \geq z_1)$, and $z_1 < z_2$ as an abbreviation of $z_2 > z_1$. We introduce $z_1 = z_2$ as a term construct instead of as an abbreviation for $z_1 \leq z_2 \land z_2 \leq z_1$; see Remark 3.6(4)

### 3.1.3. Commands
A command provides means of computation. Commands in ERC are inductively constructed as follow.

$$S, S_i ::= \begin{array}{l}
\text{skip} \\
\mid v := t \\
\mid v[z] := t \\
\mid \text{let } v : \tau = t \\
\mid S_1; \ S_2 \\
\mid \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ end if} \\
\mid \text{while } b \text{ do } S \text{ end while}
\end{array}$$

Note that although we use the identifier \textit{let}, it is not the usual ‘let binding’ for creating a local variable.
3.1.4. Programs. Having data types, terms and commands defined, we can finally define what a program in ERC is. Note that \( u_i \) varies over \( V \).

\[
\mathcal{P} := \text{input } u_1 : \tau_1, u_2 : \tau_2, \cdots, u_n : \tau_n \\
S \\
\text{return } t
\]

3.2. Typing Rules. Not all terms, commands, or programs have a mathematical meaning. Here we define well-typedness of terms (Subsubsection 3.2.1), commands (Subsubsection 3.2.2), and programs (Subsubsection 3.2.3) under a context. ERC is deliberately designed such that well-typed terms, statements and programs of ERC have semantics coinciding with their mathematical meaning (unlike, say, double).

ERC is strongly typed in the sense that typing in ERC propagates purely syntactically. Consider, for example, a term \( x + y \) where \( x \) and \( y \) are variables. Then, obviously, the type of the term \( x + y \) depends on the type of the variables; e.g., when \( x \) is of type REAL and \( y \) is of type KLEENEAN, the term \( x + y \) itself does not make sense; it is ill-typed under the mapping \( x \) to REAL and \( y \) to KLEENEAN. As it is seen from the example, well-typedness of terms (and hence commands and programs) depends on the data types of the declared variables. The context records each variable’s data type, and both well-typedness and the type of a term depend on its context.

Formally speaking, a context is a mapping from a finite set of variables to their corresponding types. \( \Gamma := u_1 : \tau_1, u_2 : \tau_2, \cdots, u_n : \tau_n \) denotes the context mapping \( u_i \) to \( \tau_i \) for \( i = 1, \ldots, n \). Declaring a new variable \( u \) of type \( \tau \) amounts to extending \( \Gamma \); we write this extension as \( \Gamma, u : \tau \). ERC only considers pure functions: a program does not have a global context.

3.2.1. Well-Typed Terms. Well-typedness of a term \( t \) in ERC(\( F, G \)) to \( \tau \) under a context \( \Gamma \) is written as \( \Gamma \vdash t : \tau \). Figure 13 shows ERC’s type inference rules.

Type checking is formally a function that tells whether a term \( t \) is well-typed and, if so, what type it has. Note that type checking under a context \( \Gamma \) is well-defined and computable.

3.2.2. Well-Typed Commands. Unlike terms, a command in ERC may modify contexts. Let us denote a command \( S \) under a context \( \Gamma \) being well-typed and yielding a new context \( \Gamma' \) as \( \Gamma \vdash S \triangleright \Gamma' \). Well-typedness of a command is defined with the inference rules in Figure 14.

See that the only construct that modifies a context is let \( v : \tau = t \) (variable declaration). When it is executed under a context \( \Gamma \), the command is well-typed if \( v \) is not already included in \( \Gamma \) and the type of the initializing term \( t \) is of the declared type \( \tau \). After the execution, we get the new context \( \Gamma, v : \tau \).

To keep things simple (Remark 1.4), we refrain from allowing the declaration of new variables inside of a branch or a loop (as common for example in C++).
Figure 13: Typing rules for terms

| Rule                                                                 | Type       |
|----------------------------------------------------------------------|------------|
| $\Gamma \vdash \tilde{z} : \text{INTEGER}$                         | $\Gamma \vdash \text{true} : \text{KLEENEAN}$    |
| $\Gamma \vdash \text{false} : \text{KLEENEAN}$                      | $\Gamma \vdash t_i : \text{REAL}$                 |
| $\Gamma \vdash \text{unknown} : \text{KLEENEAN}$                    | $\Gamma \vdash [t_1, \ldots, t_n] : \text{REAL}[n]$|
| $\Gamma \vdash [z] : \text{REAL}[n]$                                | $\Gamma \vdash x : \text{REAL}$                   |
| $\Gamma \vdash x, y : \text{REAL}$                                  | $\Gamma \vdash t_1 + t_2 : \text{INT}$            |
| $\Gamma \vdash t \cdot x : \text{REAL}$                             | $\Gamma \vdash -t : \text{REAL}$                  |
| $\Gamma \vdash t, z \cdot x, y : \text{REAL}$                       | $\Gamma \vdash b : \text{KLEENEAN}$               |
| $\Gamma \vdash (b \mid x : y) : \text{REAL}$                        | $\Gamma \vdash \text{choose}_{n}(b_0, b_1, \ldots, b_{n-1}) : \text{INTEGER}$|
| $\Gamma \vdash (b ? x : y) : \text{REAL}$                           | $\Gamma \vdash \text{choose}_{n}(b_0, b_1, \ldots, b_{n-1}) : \text{INTEGER}$|

Figure 14: Typing rules for commands

| Rule                                                                 | Type       |
|----------------------------------------------------------------------|------------|
| $\Gamma \vdash \text{skip} \triangleright \Gamma$                   | $\Gamma \vdash t : \tau$     |
| $\Gamma \vdash v : \tau \triangleleft \Gamma$                      | $\Gamma \vdash v : \tau$     |
| $\Gamma \vdash t : \tau$                                             | $\Gamma \vdash v : \tau$     |
| $\Gamma \vdash z : \text{INTEGER}$                                  | $\Gamma \vdash S \triangleright \Gamma$            |
| $\Gamma \vdash T[z] := t \triangleright \Gamma$                     | $\Gamma \vdash S \triangleright \Gamma$            |
| $\Gamma \vdash S_1 \triangleright \Gamma_1 \quad \Gamma_1 \vdash S_2 \triangleright \Gamma_2$ | $\Gamma \vdash b : \text{KLEENEAN}$               |
| $\Gamma \vdash \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ end } if \triangleright \Gamma$ | $\Gamma \vdash S \triangleright \Gamma$            |
| $\Gamma \vdash \text{while } b \text{ do } S \text{ end } \text{ while} \triangleright \Gamma$ | $\Gamma \vdash S \triangleright \Gamma$            |
3.2.3. **Well-Typed Programs.** An ERC program

\[ P := \text{input } u_1 : \tau_1, u_2 : \tau_2, \cdots, u_n : \tau_n \]

\[ S \]

\[ \text{return } t \]

is well-typed if there is a context \( \Gamma' \) and a data type \( \tau \) such that

\[ u_1 : \tau_1, u_2 : \tau_2, \cdots, u_n : \tau_n \vdash S \triangleright \Gamma' \quad \text{and} \quad \Gamma' \vdash t : \tau \]

where either (i) \( \tau_1 = \text{INTEGER}, \tau = \text{REAL} \) or (ii) \( \tau = \text{INTEGER} \). In the first case, we say \( P \) is a real program of arity \((\tau_2, \cdots, \tau_n)\) and in the second case, we say \( P \) is a (multi-valued) integer program of arity \((\tau_1, \cdots, \tau_n)\).

3.3. **Denotational Semantics.** We are going to define a multi-valued semantics for well-typed terms (Subsubsection 3.3.1), commands (Subsubsection 3.3.2), and programs (Subsubsection 3.3.3). In this semantics, the objects of ERC are assigned mathematical meanings that are arguably (i) closest possible to the intuition of real numbers as entities to be operated on exactly while simultaneously featuring (ii) Turing-completeness. (Recall [97, Exercise 4.2.9] that total real comparison violates computability.)

We start with denotations for data types and the definition of states.

**Definition 3.3.**

(1) Data types are interpreted as intended:

\[ [\text{KLEENEAN}] = \mathbb{K}, \quad [\text{INTEGER}] = \mathbb{Z}, \quad [\text{REAL}] = \mathbb{R}, \quad [\text{REAL}^n] = \mathbb{R}^n. \]

(2) Contexts are interpreted as sets of assignments: Given \( \Gamma = u_1 : \tau_1, \ldots, u_n : \tau_n \), then

\[ [\Gamma] := \prod_i \{(u_i, w) \mid w \in [\tau_i] \}. \]

(3) Given a context \( \Gamma \), an element \( \sigma \in [\Gamma] \) is called state. That is, states are specific assignments of variables contained in the domain of a context.

In order to make the denotations of well-typed terms, commands, and programs capture the multi-valuedness in ERC, we consider Smyth’s variant of the powerdomain introduced by Plotkin [71]. For any topological space \( A \), its induced powerdomain is the set of those nonempty subsets \( p \) of the domain \( A_\bot := A \cup \{\bot\} \) that are compact or contain \( \bot \):

\[ P(A_\bot) := \{p \subseteq A_\bot \mid p \neq \emptyset, \text{ and } p \text{ is compact or contains } \bot\}. \]

The powerdomain \( P(A_\bot) \) ordered by Egli-Milner ordering is known to be a domain:

\[ P \sqsubseteq Q \quad \text{if and only if} \quad (\bot \in P \land P \subseteq Q \cup \{\bot\}) \lor (\bot \notin P \land P = Q). \]

---

6A domain \( D \) is a partially ordered set with an ordering \( \sqsubseteq \) such that a least element denoted by \( \bot \) exists and every increasing chain of elements of \( D \) has a limit in \( D \) [75, Ch. 2.3]. It is also often called to be a \( \omega \)-CPO with a least element.
3.3.1. Denotations of Terms. Formalizing the multi-valued semantics of ERC, a term’s meaning under a state is a non-empty subset of a certain set. Intuitively, the denotation of a well-typed term is the set of all values that the term could evaluate to under a state. For example, a syntactically well-typed term $\Gamma \vdash t : \text{REAL}$ can ‘have’ multiple values under a state $\sigma$: Its denotation contains a subset of $\mathbb{R}$. Moreover if 0 is among these values, then the compound term $1/t$ could be undefined—in addition to its defined values derived from non-zero values of $t$. This is reflected by including the special symbol $\bot$ in the denotation, which is thus an element of $\mathcal{P}([\text{REAL}]_{\bot})$.

Remark 3.4. Observe the subtle but important difference between $\bot$ and unknown: the latter is an element of $\mathbb{K}$ and represents being constructively unknown but classically defined. The former is not part of the structure of ERC and expresses classical mathematical ill-definition. Note that $\mathcal{P}([\text{REAL}]_{\bot})$ does not contain the empty set: the denotation of a term is never empty; if it is undefined, it contains $\bot$.

A well-typed term $t$ such that $\Gamma \vdash t : \tau$ denotes a function of the following type:

$$[\Gamma \vdash t : \tau] : [\Gamma] \rightarrow \mathcal{P}([\tau]_{\bot})$$

That is, given a state $\sigma \in [\Gamma]$, $[\Gamma \vdash t : \tau](\sigma)$ is a subset of $[\tau]_{\bot}$. E.g., if $\tau = \text{INTEGER}$, $[\Gamma \vdash t : \tau](\sigma)$ is a subset of integers possibly including $\bot$. To ease the following description, we write $[\Gamma \vdash t : \tau] \sigma$ instead of $[\Gamma \vdash t : \tau](\sigma)$. Moreover, we simply write $[t]$ and $[t] \sigma$ instead of $[\Gamma \vdash t : \tau]$ and $[\Gamma \vdash t : \tau] \sigma$, respectively, omitting $\Gamma$ and $\tau$ when they are obvious or irrelevant.

Note that $[t] \sigma$ will never have empty set of values, but instead may or may not have $\bot$ among them. It means that a multi-valued term can be defined and undefined simultaneously and corresponds to the fact that a partial multi-valued function could have un-/defined value at an argument. We synonymously say $t$ evaluates to $v$ under $\sigma$, or $t$ has/contains the element $v \in [t] \sigma$.

Definition 3.5. Given a well-typed term $t$ such that $\Gamma \vdash t : \tau$, we define its interpretation as a function $[\Gamma \vdash t : \tau] : [\Gamma] \rightarrow \mathcal{P}([\tau]_{\bot})$ inductively as follows:

$$[\Gamma \vdash z : \text{INTEGER}] \sigma := \{z\}$$
$$[\Gamma \vdash \text{true} : \text{KLEENEAN}] \sigma := \{tt\}$$
$$[\Gamma \vdash \text{false} : \text{KLEENEAN}] \sigma := \{ff\}$$
$$[\Gamma \vdash \text{unknown} : \text{KLEENEAN}] \sigma := \{uk\}$$

$$[\Gamma \vdash [t_1, \cdots, t_n] : \text{REAL}[n]] \sigma := \bigcup_{x_i \in [t_i] \sigma} \begin{cases} \{(x_1, \cdots, x_n)\} & \text{if } x_i \neq \bot \text{ for all } i; \\ \{\bot\} & \text{otherwise} \end{cases}$$

$$[\Gamma \vdash v : \tau] \sigma := \{\sigma(v)\}$$

$$[\Gamma \vdash t[z] : \tau] \sigma := \bigcup_{m \in [z] \sigma} \bigcup_{r \in [t] \sigma} \begin{cases} \{\pi_m(r)\} & \text{if } \Gamma \vdash t[z] : \tau[n], 0 \leq m < n, \\ \{\bot\} & \text{otherwise} \end{cases}$$

where $\pi_m$ denotes the projection on the $m$-th component.
\[
\begin{align*}
\Gamma \vdash x < y : \text{KLEENEAN} \quad &\text{if } x' < y'; \\
&\{ \text{tt} \} \quad \text{if } x' > y'; \\
&\{ \text{ff} \} \quad \text{if } x' = y' \neq \bot; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash z_1 \leq z_2 : \text{KLEENEAN} \quad &\text{if } z_1^1 \leq z_2^1; \\
&\{ \text{tt} \} \quad \text{if } z_1^1 > z_2^1; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash z_1 = z_2 : \text{KLEENEAN} \quad &\text{if } z_1^1 = z_2^1 \text{ and } z_1^2, z_2^2 \neq \bot; \\
&\{ \text{ff} \} \quad \text{if } z_1^1 \neq z_2^1 \text{ and } z_1^2, z_2^2 \neq \bot; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash g(t_1, \ldots, t_n) : \text{INTEGER} \quad &\text{if } \forall i. \ w_i \neq \bot \text{ and } \quad (w_1, \ldots, w_n) \in \text{dom}(g); \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash f(t_1, \ldots, t_n) : \text{REAL} \quad &\text{if } \forall i. \ w_i \neq \bot \text{ and } \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash t_1 + t_2 : \tau \quad &\{ w_1 + w_2 \} \quad \text{if } w_1 \text{ and } w_2 \text{ are not } \bot; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash \neg t : \tau \quad &\{ \neg w \} \quad \text{if } w \text{ is not } \bot; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash x \times y : \text{REAL} \quad &\{ w_1 \cdot w_2 \} \quad \text{if } w_1 \text{ and } w_2 \text{ are not } \bot; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash / x : \text{REAL} \quad &\{ 1/w \} \quad \text{if } w \notin \{ 0, \bot \}; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash \neg b : \text{KLEENEAN} \quad &\{ \text{tt} \} \quad \text{if } l = \text{ff}; \\
&\{ \text{ff} \} \quad \text{if } l = \text{tt}; \\
&\{ \text{uk} \} \quad \text{if } l = \text{uk}; \\
&\{ \bot \} \quad \text{otherwise} \\
\Gamma \vdash b_1 \land b_2 : \text{KLEENEAN} \quad &\{ \text{tt} \} \quad \text{if } l_1 = l_2 = \text{tt}; \\
&\{ \text{ff} \} \quad \text{if } l_1 = \text{ff} \text{ or } l_2 = \text{ff}; \\
&\{ \text{uk} \} \quad \text{if } (l_1, l_2) \in \{ (\text{tt, uk}), (\text{uk, tt}), (\text{uk, uk}) \}; \\
&\{ \bot \} \quad \text{otherwise}
\end{align*}
\]
Due to multi-valuedness, the denotation of integer equality 
\[ z_1 = z_2 \] is subtly different from that of 
\[ z_1 \leq z_2 \land z_2 \leq z_1 \]. Consider a state \( \sigma \) where \( [z_1] \sigma = \{0, 2\} \) and \( [z_2] \sigma = \{1, 3\} \). Then, \( [z_1 \leq z_2 \land z_2 \leq z_1] \sigma = \{tt, ff\} \) whereas \( [z_1 = z_2] \sigma = \{ff\} \).

We add some explanation:

**Remark 3.6.**

(1) Although the denotations of terms are multi-valued, every variable only stores one specific value: in an assignment possibly selected non-deterministically, see Subsubsection 3.3.2.

(2) As mentioned in Remark 3.4, \( \perp \) represents mathematical ill-definition. For a well-typed term, if \( \perp \) is in its denotation, it means that its value could be meaningless. Observe that \( \perp \) propagates up; i.e., for a term, if the denotation of a sub-term contains \( \perp \), then the denotation of the term itself also contains \( \perp \). For example, the denotation of \( [t_1, \ldots, t_4] \) contains \( \perp \) on a state when the denotation of \( t_i \) contains \( \perp \) on the state.

(3) The denotation of comparison \( "x \leq y" \) between real terms \( x, y \) formalizes the return value \( tt \) if the value of \( x \) is smaller than the value \( y \), return value \( ff \) if the value of \( x \) is bigger than the value \( y \), and ‘return’ value \( uk \) when their values are equal: indicating constructive indefiniteness. Independently, comparison is ill-defined \( (\perp) \) if any of its sub-terms is.

(4) Due to multi-valuedness, the denotation of integer equality \( "z_1 = z_2" \) is subtly different from that of \( "z_1 \leq z_2 \land z_2 \leq z_1" \). Consider a state \( \sigma \) where \( [z_1] \sigma = \{0, 2\} \) and \( [z_2] \sigma = \{1, 3\} \). Then, \( [z_1 \leq z_2 \land z_2 \leq z_1] \sigma = \{tt, ff\} \) whereas \( [z_1 = z_2] \sigma = \{ff\} \).

(5) The denotations of compound terms \( \neg b_1 \land b_2, b_1 \lor b_2 \) follow Kleene/Priest logic.

(6) The denotation of \( \textbf{choose}_n(b_0, \ldots, b_{n-1}) \) corresponds to Example 1.16.(6).

(7) The denotation of \( "b ? x : y" \) corresponds to Example 1.17. It means that \( "b ? x : y" \) may still be defined in case \( b \) is neither \( tt \) nor \( ff \), provided both \( x \) and \( y \) are single-valued and agree. However, if \( x \) and \( y \) are not single-valued, then the result is undefined even if they agree as sets: the operational semantics may ‘select’ values from \( x \) and \( y \) independently.
A d-ary partial function \( f : S_1 \times \cdots \times S_d \rightarrow T \) can be lifted to \( f^\dagger : \mathcal{P}((S_1)_\perp) \times \cdots \times \mathcal{P}(T_\perp) \) by

\[
\bigcup_{(x_1, \ldots, x_d) \in S_1 \times \cdots \times S_d} \begin{cases} 
\{ f(x_1, \ldots, x_d) \} & \text{if } \forall i. \ x_i \neq \perp \land (x_1, \ldots, x_d) \in \text{dom } f, \\
\{ \perp \} & \text{otherwise.}
\end{cases}
\]

Note that \( f^\dagger(S_1, \ldots, S_d) \) is indeed compact or contains \( \perp \) whenever \( S_1, \ldots, S_d \) are compact or contain \( \perp \). See that we have defined the denotations of all constructs (except for \texttt{choose()}, \( b ? x : y \), and function calls by \( \mathcal{G} \)) by lifting the intended mathematical functions.

For example, \([t_1 + t_2] \sigma \) is \([t_1] \sigma +^\dagger [t_2] \sigma \). \( \square \)

Note that \texttt{choose()} and functions from \( \mathcal{G} \) constitute atomic constructs that yield multi-valuedness, all other atomic constructs do not generate any multi-valuedness. Multi-valuedness, however, does propagate through composite terms: only in special cases may the result again be a singleton. The following is easy (but tedious) to see:

Lemma 3.7. Fix \( \mathcal{F} \) and \( \mathcal{G} \) according to Convention 3.1. Each of the well-typed terms from Subsubsection 3.3.1, equipped with the semantics from Definition 3.5, constitutes a compact Turing-computable (partial multi-)function the sense of Definition 1.14.

3.3.2. Denotations of Commands. As the denotation of a well-typed term was the set of all possible values that the term may evaluate to, considering multi-valuedness in ERC, the denotation of a well-typed command is the set of all possible states that the command may result in. Hence, we let a well-typed command denote a function from the set of states to the restricted powerset of the resulting states:

\[
[\Gamma \vdash S \triangleright \Gamma'] : [\Gamma] \rightarrow \mathcal{P}(\{\Gamma'\}_\perp)
\]

Notation 3.8.

1. Given a state \( \sigma \) and a variable \( v \), \( \sigma[v \mapsto w] \) denotes the state which assigns \( w \) to \( v \), otherwise behaves the same as \( \sigma \).
2. Given a state \( \sigma \) and an array \( T \), \( \sigma[T \rightarrow_n w] \) denotes the state which substitutes \( w \) for the \( n \)-th element of \( T \) in the state \( \sigma \) if \( 0 \leq n < \text{len}(T) \), otherwise be \( \perp \). Here, \( \text{len}(T) \) denotes the fixed dimension of the one-dimensional array \( T \).
3. We simply write \([S] \) and \([S] \sigma \) instead of \([\Gamma \vdash S \triangleright \Gamma'] \) and \([\Gamma \vdash S \triangleright \Gamma'] \sigma \), respectively, omitting \( \Gamma \) and \( \Gamma' \) when they are obvious or irrelevant.

Definition 3.9. Given a well-typed command \( S \) such that \( \Gamma \vdash S \triangleright \Gamma' \), we define the mapping \([\Gamma \vdash S \triangleright \Gamma'] : [\Gamma] \rightarrow \mathcal{P}(\{\Gamma'\}_\perp) \) inductively as follows:

\[
[\Gamma \vdash \texttt{skip} \triangleright \Gamma] \sigma := \{ \sigma \}
\]
\[
[\Gamma \vdash v := t \triangleright \Gamma] \sigma := \bigcup_{w \in [t] \sigma} \begin{cases} 
\{ \sigma[v \mapsto w] \} & \text{if } w \neq \perp; \\
\{ \perp \} & \text{otherwise}
\end{cases}
\]
\[
[\Gamma \vdash v[z] := t \triangleright \Gamma] \sigma := \bigcup_{n \in [z] \sigma} \bigcup_{w \in [t] \sigma} \begin{cases} 
\{ \sigma[v \mapsto_n w] \} & \text{if } w \neq \perp; \\
\{ \perp \} & \text{otherwise}
\end{cases}
\]
The semantic of a while loop is defined in a way that it satisfies the recurrence relation:

\[
\begin{align*}
\Gamma \vdash \text{let } v : \tau = t \triangleright \Gamma' \quad &\Rightarrow \quad \sigma := \bigcup_{w \in [t]_{\sigma}} \begin{cases} 
\{ \sigma \cup (v \mapsto w) \} & \text{if } w \neq \bot; \\
\{ \bot \} & \text{otherwise}
\end{cases} \\
\Gamma \vdash S_1; S_2 \triangleright \Gamma' \quad &\Rightarrow \quad \sigma := \bigcup_{\delta \in [S_1]_{\sigma}} \begin{cases} 
[S_2]_{\delta} & \text{if } \delta \neq \bot; \\
\{ \bot \} & \text{otherwise}
\end{cases} \\
\Gamma \vdash \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ end } \triangleright \Gamma \quad &\Rightarrow \quad \sigma := \bigcup_{l \in [b]_{\sigma}} \begin{cases} 
[S_1]_{\sigma} & \text{if } l = tt; \\
[S_2]_{\sigma} & \text{if } l = ff; \\
\{ \bot \} & \text{if } l = \bot \lor l = uk.
\end{cases}
\end{align*}
\]

**Remark 3.10.** (1) Regarding assignment “\(v := t\)”: For an initial state \(\sigma\), \([v]_{\sigma}\) is the set of all possible values that \(t\) may evaluate to. Hence, for each \(w \in [t]_{\sigma}\), if \(w \neq \bot\), the state \(\sigma[v \mapsto w]\) is a possible resulting state.

(2) Regarding assignment “\(v[z] := t\)”: For an initial state \(\sigma\), \([z]_{\sigma}\) is the set of all possible indices that \(z\) may evaluate to and \([t]_{\sigma}\) is the set of all possible values that \(t\) can have. Hence, the possible resulting states are \(\sigma[v \rightarrow_{n} w]\) for each \(n \in [z]_{\sigma}\) and \(w \in [t]_{\sigma}\) if \(n\) is a proper index, \(n\) is not \(\bot\), and \(w\) is not \(\bot\). If \(n\) is \(\bot\) or \(w\) is \(\bot\), it means that an error occurs in the evaluation of the terms, the denotation of the assignment also contains \(\bot\).

(3) “if \(b\) then \(S_1\) else \(S_2\) end if”: for an initial state \(\sigma\), \([b]_{\sigma}\) is the set of all possible values that \(b\) can have. Hence, for each \(l \in [b]_{\sigma}\) that is not \(\bot\), the denotation of the command is the denotation of \(S_1\) if \(l = tt\) and \(S_2\) if \(l = ff\). It is the union of the two denotations if \([b]_{\sigma}\) is \(\{tt, ff\}\). If \(l = \bot\) or \(uk\), the denotation of the command contains \(\bot\).

In each case, \(\bot \in [t]_{\sigma}\) means that (some sub-term of) \(t\) can be mathematically undefined in the current multi-valued state \(\sigma\); hence, by construction, \(\bot\) is also among the possible resulting states.

**Remark 3.11** Denotation of the while loop.

(1) The least fixed point is well defined. This follows from the fact that the operator is monotone and continuous: The argument proceeds similarly to the case of the semantics of bounded nondeterminism in Dijkstra’s guarded command [29], see for example the proof in the textbook [75, §7].

(2) The semantic of a while loop is defined in a way that it satisfies the recurrence relation:

\[
[\text{while } b \text{ do } S \text{ end } \text{while}]_{\sigma} = [\text{if } b \text{ then } S; (\text{while } b \text{ do } S \text{ end } \text{while}) \text{ else skip } \text{end if}]_{\sigma}
\]
\[
\begin{align*}
&= \bigcup_{l \in \{\text{true}, \text{false}\}} \begin{cases} 
[S; (\text{while } b \text{ do } S \text{ end while})] \sigma & \text{if } l = \text{true} \\
\{\sigma\} & \text{if } l = \text{false} \\
\{\bot\} & \text{otherwise}
\end{cases} \\
&= \bigcup_{l \in \{\text{true}, \text{false}\}} \begin{cases} 
[\text{while } b \text{ do } S \text{ end while}] \delta & \text{if } \delta \neq \bot, l = \text{true} \\
\{\sigma\} & \text{if } l = \text{false} \\
\{\bot\} & \text{otherwise}
\end{cases}
\end{align*}
\]

3) Consider the sequence of commands:
- \(A^n_0 : \equiv \text{while } true \text{ do skip end while}\)
- \(A^n_{b,S} : \equiv \text{if } b \text{ then } S; A^n_{b,S} \text{ else skip end if}\)

The sequence, intuitively, represents the process of unrolling the loop. The sequence of the denotations \(\langle [A^n_{b,S}] \rangle_{n \in \mathbb{N}}\) forms a chain whose supremum is the denotation of \(\text{while } b \text{ do } S \text{ end while}\).

4) For a well-typed command \(\Gamma \vdash \text{while } b \text{ do } S \text{ end while} \triangleright \Gamma\) and for any two states \(\sigma, \delta \in \Gamma\), \(\delta\) is in \(\Gamma \vdash \text{while } b \text{ do } S \text{ end while} \triangleright \Gamma\) \(\sigma\) if and only if there is a natural number \(n\) where \(\delta \in \Gamma \vdash A^n_{b,S} \sigma\). Hence, instead of working with the subtle fixed point for the denotation of a while loop, one can do reasoning on \(A^n_{b,S}\) when an appropriate \(n\) is found.

Following up on Lemma 3.7, the following is equally tedious but easy to verify:

**Lemma 3.12.** States \(\sigma\) are functions on the finite domain of contexts. The mapping from Definition 3.9 transforming states according to well-typed commands is compact and Turing-computable in the sense of Definition 1.14; cmp. [99].

Note that ‘unrolled’ execution of loops is Turing-computable according to Remark 3.11; and termination is semi-decidable, whereas the ‘value’ of a non-terminating loop is \(\bot\) by definition.

3.3.3. Denotations of Programs. Having defined how to evaluate terms and commands, we are now ready to define denotations of well-typed ERC programs.

**Definition 3.13.** Let an ERC program
\[
P \colonequals \text{input } u_1 : \tau_1, u_2 : \tau_2, \ldots, u_n : \tau_n \\
S \\
\text{return } t
\]
be well-typed such that \(\Gamma \vdash S \triangleright \Gamma'\) and \(\Gamma' \vdash t : \tau\) for some context \(\Gamma'\) and some data type \(\tau\) where \(\Gamma := u_1 : \tau_1, u_2 : \tau_2, \ldots, u_n : \tau_n\). Then \(P\) denotes the function
\[
[P] : [\tau_1] \times \cdots \times [\tau_n] \rightarrow P([\tau]_\bot)
\]
defined by
\[
[P] (x_1, \cdots, x_n) \colonequals \bigcup_{\delta \in [S](\cup_i (u_i \mapsto x_i))} \begin{cases} 
[t] \delta & \text{if } \delta \neq \bot, \\
\{\bot\} & \text{otherwise}
\end{cases}
\]
The domain of the program is \( \text{dom}(P) := \{(x_1, \cdots, x_n) \mid \bot \notin [P](x_1, \cdots, x_n)\} \). A program \( P \) is total if its domain coincides with \( \prod_i [\tau_i] \), and single-valued if it holds \( \forall \vec{x} \in \text{dom}(P) \). Card([P](\vec{x})) = 1.

**Definition 3.14.**

1. A real program \( P \) over \( \text{ERC}(\mathcal{F}, \mathcal{G}) \) of arity \((\tau_1, \cdots, \tau_n) \) realizes a partial function \( f : [\tau_1] \times \cdots \times [\tau_n] \to \mathbb{R} \) if (i) \((x_1, \cdots, x_n) \in \text{dom}(f) \) implies \( \forall p \in \mathbb{Z}. (p, x_1, \cdots, x_n) \in \text{dom}(P) \), and (ii) for any \((x_1, \cdots, x_n) \in \text{dom}(f) \) and \( p \in \mathbb{Z} \), and for all \( z \in [P](p, x_1, \cdots, x_n) \), it holds that \( |f(x_1, \cdots, x_n) - z| \leq 2^p \).
2. An integer program \( P \) over \( \text{ERC}(\mathcal{F}, \mathcal{G}) \) of arity \((\tau_1, \cdots, \tau_n) \) realizes a multi-function \( g : [\tau_1] \times \cdots \times [\tau_n] \to \mathbb{Z} \) if (i) \( \text{dom}(g) \subseteq \text{dom}(\text{dom}(P)) \) and (ii) \( g(x_1, \cdots, x_n) \supseteq [P](x_1, \cdots, x_n) \) holds for all \((x_1, \cdots, x_n) \in \text{dom}(g)\).

Lemma 3.12 immediately implies (in view of Paradigm 1.3 in the real case):

**Proposition 3.15.** For \( \mathcal{F} \) and \( \mathcal{G} \) according to Convention 3.1, any partial real function realized by a program \( P \) over \( \text{ERC}(\mathcal{F}, \mathcal{G}) \) is Turing-computable (and in particular continuous and compact). And any partial integer function realized by a program \( P \) over \( \text{ERC}(\mathcal{F}, \mathcal{G}) \) is Turing-computable and compact.

### 3.4. Programming Abbreviations

As mentioned in Remark 1.4, the design of ERC is deliberately kept simple, but emphasises user extensions such as the exponential function in Subsection 2.3. Similarly, the following implicitly abbreviations have already been used in the above program examples:

- a) For \( \text{INTEGER} \) \( j \) and a constant \( d \in \mathbb{Z} \), \( “d \cdot j” \) and \( “j \cdot d” \) both mean \( “j + j + \cdots + j” \) in case \( d \geq 0 \), \( “-j \cdot (-d)” \) otherwise.
- b) \( \text{max}(x, y) := (y < x \iff x : y) \) and \( \text{abs}(z) := (0 < z \iff z : -z) \). Also, \( “y > x” \) means \( “x < y” \).
- c) \( \text{“for } N : \text{INTEGER} = K \text{ to } L \text{ do } C” \) is short for
  
  \begin{verbatim}
  let N : INTEGER = K; while (N < L) do C; N := N + 1 end while
  \end{verbatim}
  
  with \( N, L \) not modified in \( C \).
- d) For \( x : \text{REAL} \) and \( J : \text{INTEGER} \), \( x^J \) is short for the result \( y \) of the following code fragment:
  
  \begin{verbatim}
  let y : REAL = 1; for N : INTEGER = 1 to J do y := y \times x end for;
  for N : INTEGER = 1 to -J do y := y/x end for
  \end{verbatim}
  
  Note that \( \text{for} \) only counts \( u \)p \) according to (c); hence at most one of the two loops gets executed.
- e) \( \text{“if } B \text{ then } Q \text{ end if”} \) is abbreviation for
  
  \begin{verbatim}
  if B then Q else skip end if
  \end{verbatim}
  
  where \( \text{skip} \) is an instruction for doing nothing.
- f) Arrays in ERC are one-dimensional of constant size with indices starting at 0; recall Remark 1.4(b). We implicitly simulate and identify a two-dimensional array \( A[] = (A[i, j])_{i, j} \), having first dimension of size \( d > i \geq 0 \), with the one-dimensional array \( A[i + j \cdot d] \); similarly for higher dimensions.
4. Logic of Exact Real Computation

Operating on real numbers exactly significantly simplifies reliable numerical software development. Subsection 4.1 proposes a three-sorted structure for rigorously specifying (multi)functions with real arguments in Subsection 4.2. It is carefully designed rich enough to allow arguing about computations in ERC yet restricted such as to assert logical decidability as guarantee to formal program verification. For this purpose we (have to) degrade some operations from native primitives to user programs. Such trade-offs are unavoidable:

**Remark 4.1.** Consider the following three desirable features:

i) a programming language being Turing-complete over integers, and thus able to realize an algorithm whose termination is co-r.e. hard

ii) a logic sufficiently rich to express the termination of said algorithm

iii) a sound and complete r.e. deductive system of said logic.

Obviously not all three are simultaneously feasible. For instance integer WHILE programs / Peano arithmetic satisfy (i) / (ii), but not (iii) [26, §6]. We here choose (i) and (iii) over (ii); recall Remark 1.5. However it is equally reasonable (Remark 1.4b) to relinquish (iii) decidability in order to increase expressivity (ii).

The question of which sub-classes of ERC programs have decidable termination is a subject of ongoing research [11, 67].

4.1. Three-Sorted Structure with Decidable Theory. ERC natively supports three basic data types and operations: INTEGER coinciding with the Presburger structure \( \mathbb{Z} \), REAL coinciding with the ordered field \( \mathbb{R} \), and KLEENEAN coinciding with the three-valued Kleene Logic \( \{tt, ff, uk\} \) extending Boolean \( \mathbb{B} = \{tt, ff\} \). ERC being closed under composition, further operations and functions can be added and then called freely by the user, such as real rounding (Subsection 2.5) or exponential function (Subsection 2.3) or matrix determinant (Subsection 2.6) or integer multiplication (Figure 11). In Model Theory this is called expanding a structure.

**Definition 4.2** Theory of ERC. The Structure of ERC is the three-sorted structure \( \mathcal{S} \) combining the Kleene Algebra \( \langle \mathbb{K}, ff, tt, uk, \land, \lor, \neg = \rangle \) with Presburger Arithmetic \( \langle \mathbb{Z}, 0, 1, +, -, \leq, 2\mathbb{Z}, 3\mathbb{Z}, 4\mathbb{Z}, \ldots \rangle \) and ordered field \( \langle \mathbb{R}, 0, 1, +, -, \times, < \rangle \). They are connected via the two comparison operations \( <: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{B} \) and \( \leq: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{B} \) as well as via the binary precision embedding \( \iota: \mathbb{Z} \ni p \mapsto 2^p \in \mathbb{R} \) and its partial half-inverse \( \lceil \log_2 \circ \text{abs} \rceil : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{Z} \). Here \( k\mathbb{Z} \) denotes the predicate on \( \mathbb{Z} \) which is \( tt \) precisely for all integer multiples of \( k \in \mathbb{N} \).

The Logic of ERC is the first-order language of the structure \( \mathcal{S} \); the Theory of ERC is the complete first-order theory \( \mathcal{T} \) of the structure \( \mathcal{S} \).

More generally fix \( \mathcal{F} \) and \( \mathcal{G} \) according to Convention 3.1. The Structure of ERC(\( \mathcal{F}, \mathcal{G} \)) is the expansion \( \mathcal{S}(\mathcal{F}, \mathcal{G}) \) of \( \mathcal{S} \) with the partial real functions \( f \in \mathcal{F} \) of various arities and with the partial integer multi-functions \( g \in \mathcal{G} \), also of various arities. The Logic of ERC(\( \mathcal{F}, \mathcal{G} \)) is the first-order language of \( \mathcal{S}(\mathcal{F}, \mathcal{G}) \); the Theory of ERC(\( \mathcal{F}, \mathcal{G} \)) is the complete first-order theory \( \mathcal{T}(\mathcal{F}, \mathcal{G}) \) of \( \mathcal{S}(\mathcal{F}, \mathcal{G}) \).

We take the inequality on \( \mathbb{Z} \) as non-strict \( \leq \), but that on \( \mathbb{R} \) strict \( < \) as in [56]. This mathematical predicate \( <: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{B} \) is used for instance to define the denotational semantics in Subsection 3.3 and not to be confused with the computational, partial real
comparison $<$ : $\text{REAL} \times \text{REAL} \rightarrow \text{KLEENEAN}$; recall (Remark 1.5). The symbol $\bot$ which represents ill-definiteness is not an element of the structure of ERC.

This structure and its logic are ‘adequate’ for ERC as formal programming language:

**Lemma 4.3.** To every well-typed term $t$ in $\text{ERC}(\mathcal{F}, \mathcal{G})$ with $\Gamma \vdash t : \tau$ according to Subsubsection 3.1.2, there exists a formula $\langle \Gamma \vdash t : \tau \rangle_v$ in the logic of $\text{ERC}(\mathcal{F}, \mathcal{G})$ with free variables that are in the domain of $\Gamma$ and $v$ such that for any $\sigma \in [\Gamma]$ and $w \in [\tau]$,

$$\sigma \cup (v \mapsto w) \models \langle \Gamma \vdash t : \tau \rangle_v \text{ if and only if } w \in [\Gamma \vdash t : \tau]_\sigma \land \bot \notin [\Gamma \vdash t : \tau]_\sigma.$$

Here, $\sigma \models \psi$ denotes that the assignment $\sigma$ validates $\psi$ in the structure of $\text{ERC}(\mathcal{F}, \mathcal{G})$. Let us write simply $\langle \emptyset \rangle$ for $\langle \Gamma \vdash t : \tau \rangle$ when the particular context $\Gamma$ is obvious or irrelevant.

**Proof.** Consider a term $t = t(t_1, \ldots, t_n)$ such that $x_1 : \tau_1, \ldots, x_d : \tau_d \vdash t : \tau$ where $t$ represents a term construct and $t_1, \ldots, t_d$ are the subterms of $t$. If $t$ is not of the form $\text{choose}(t_0, \ldots, t_{n-1}), b ? t_1 : t_2$, or $g(t_1, \ldots, t_d)$ for any $g \in \mathcal{G}$, there is a partial function $f : S_1 \times \cdots \times S_d \rightarrow T$ where $T = [\tau], S_i = [\tau'_i]$, and $x_1 : \tau_1, \ldots, x_d : \tau_d \vdash t_i : \tau'_i$ such that

$$[t]_\sigma = f^\dagger([t_1]_\sigma, \ldots, [t_n]_\sigma).$$

Here, $f^\dagger$ is the lifting from Remark 3.6 8. Note that the graph of $f$ is definable in $\text{ERC}(\mathcal{F}, \mathcal{G})$; i.e., there is a formula $\psi$ with free variables $y_1, \ldots, y_n, y$ such that

$$(y_1 \mapsto w_1, \ldots, y_n \mapsto w_n, y \mapsto w) \models \psi \text{ if and only if } w = f(w_1, \ldots, w_n).$$

Assume that the denotations of $t_i$ are definable in the sense of the lemma; i.e., there are $\langle t_i \rangle_{y_i}$ such that

$$\sigma \cup (y_i \mapsto w_i) \models \langle t_i \rangle_{y_i} \text{ if and only if } w_i \in [t_i]_\sigma \land \bot \notin [t_i]_\sigma$$

for any $\sigma \in [x_1 : \tau_1, \ldots, x_d : \tau_d]$ and $w_i \in [\tau'_i]$. Then, the denotation of $t$ is definable by

$$\langle \emptyset \rangle_y = \exists y_1, \ldots, y_d. \langle t_1 \rangle_{y_1} \land \cdots \land \langle t_n \rangle_{y_n} \land \psi.$$

In order to complete the inductive proof, we need to show there are formulae of the form $\langle \text{choose}(t_0, \ldots, t_{n-1}), b ? t_1 : t_2 \rangle$ and $b ? t_1 : t_2$. Assuming the denotations of the subterms are definable in the sense of the lemma, they can be defined by

$$\langle \text{choose}(t_0, \ldots, t_{n-1}) \rangle_y = (\exists z. \langle t_0 \rangle_z) \land \cdots \land (\exists z. \langle t_{n-1} \rangle_z) \land ((y = 0 \land (\langle t_0 \rangle_z[tt/z] \lor \cdots \lor (y = n - 1 \land (\langle t_{n-1} \rangle_z[tt/z])))$$

and $\langle b ? t_1 : t_2 \rangle_y = \langle b \rangle_y[t/t] \Rightarrow (\exists z. \langle t_1 \rangle_z) \land \langle b \rangle_y[ff/z] \Rightarrow (\exists z. \langle t_2 \rangle_z) \land \langle b \rangle_y[uk/z] \Rightarrow ((\exists z. \langle t_1 \rangle_z) \land (\exists z. \langle t_1 \rangle_z) \land \forall z_1, z_2. \langle t_1 \rangle_z, z_1 \Rightarrow z_1 = z_2) \land ((\langle t_1 \rangle_y \land \langle b \rangle_y[t/t]) \lor (\langle t_2 \rangle_y \land \langle b \rangle_y[ff/z]) \lor ((\langle t_1 \rangle_y \land \langle b \rangle_y[uk/z]))$.

We regard $\langle \emptyset \rangle$ as a meta-predicate and write $\langle \emptyset \rangle_{[v/v]}$ for $\langle \emptyset \rangle_{[v/v]}$. The formula $\langle \emptyset \rangle$ expresses whether (the multi-valued denotation of) $t$ under a state $\sigma \in [\Gamma]$ is well-defined (i.e. does not contain $\bot$) and contains the value $v$; i.e. $[t]_\sigma$ is ill-defined iff there is no $v \in [\tau]$ that makes $\langle \emptyset \rangle_{[v/v]}$ valid under $\sigma$. And, in the other case, $v \in [t]_\sigma$ iff $\langle \emptyset \rangle_{[v/v]}$ is valid under $\sigma$. Hence, we can express the well-definiteness as a formula

$$\langle \emptyset \rangle_{[v/v]} \models \psi.$$
$\exists v. \{t\}(v)$ and write $\forall v. \{t\}(v) \Rightarrow \psi(v)$ to say all values in the denotation, if it is well-defined, satisfies $\psi$.

Theorem 4.4(a) shows that $T = T(\emptyset, \emptyset)$ is decidable: every first-order sentence about $S = S(\emptyset, \emptyset)$ can be formally either verified or refuted. This applies for example to pre/post conditions or loop invariants of ‘plain’ ERC programs; see Remarks 4.3 and 4.7. This is a significant advantage of ERC, compared to traditional programming languages for discrete data: Classical WHILE programs over integers with multiplication for instance do suffer from Gödel undecidability [26, §6].

**Theorem 4.4** Un-/Decidability of the Logic of ERC. a) The Theory $T = T(\emptyset, \emptyset)$ of ‘plain’ ERC is decidable.

b) $T$ is also ‘model complete’ in that it admits elimination of quantifiers up to one (by choice either existential or universal) block ranging over integers.

c) Each of the following expansions destroys decidability of the first-order theory of the structure $S$:

- expanding with integer multiplication
- expanding with the unary predicate $\mathbb{Z}$ on $\mathbb{R}$, or with ‘type casting’ $\mathbb{Z} \hookrightarrow \mathbb{R}$
- replacing the binary precision embedding $i$ with its unary counterpart $j : \mathbb{N} \ni n \mapsto 1/n \in \mathbb{R}$
- expanding simultaneously with the real exponential and sine function and with transcendental constants $\pi$ and $\ln 2$

**Proof.** c) Including integer multiplication recovers Peano arithmetic and Gödel undecidability via Robinson’s Theorem [77]. A unary predicate $\mathbb{Z}$ on $\mathbb{R}$ allows to express integer multiplication via the reals; similarly for (any total extension of) the unary precision embedding $j$. Finally the real transcendental functions and constants make the theory undecidable according to Richardson’s Theorem [76].

a)+b) A celebrated result of van den Dries [30] extends classical Tarski-Seidenberg quantifier elimination from the first-order theory of real-closed fields to the expanded structure

$$\left(\mathbb{R}, 0, 1, +, -, \times, <, 2^{k\mathbb{Z}} : k \in \mathbb{N}, 2^{\log_2 \circ \abs{\cdot}}\right) \quad (4.1)$$

with axiomatized additional predicates $2^{k\mathbb{Z}}$, $k \in \mathbb{N}$, and truncation function to binary powers $2^{\log_2 \circ \abs{\cdot}}$, see also [4].

Note that both the real-closed field $(\mathbb{R}, 0, 1, +, -, \times, <)$ and Presburger Arithmetic can be embedded into the expanded structure from Equation (4.1); the latter interpreted as its multiplicative variant $(2^{2\mathbb{Z}}, 1, 2, \times, <, 2^{k\mathbb{Z}} : k \in \mathbb{N})$ is called Skolem Arithmetic [8]:

- Replace quantifiers over Skolem integers with real quantifiers subject to the predicate $2^{k\mathbb{Z}}$ for $k := 1$;
- Consider $i : \mathbb{Z} \to \mathbb{R}$ as the restricted identity $id_{\mathbb{Z}}$ in $\mathbb{R}$.

Then every formula $\varphi$ with or without parameters in our two-sorted structure translates signature by signature to an equivalent one $\tilde{\varphi}$ over the expanded theory where quantifiers can be eliminated, yielding equivalent decidable $\tilde{\psi}$ which may involve binary truncation $2^{\log_2 \circ \abs{\cdot}}$.

To translate this back to some equivalent $\psi$ over the two-sorted structure, while reintroducing only one type of quantifiers, observe that for real $x$:

$$x \in 2^{k\mathbb{Z}} \iff \exists z \in \mathbb{Z}. z \in k\mathbb{Z} \land x = i(z);$$

$$x \notin 2^{k\mathbb{Z}} \iff \exists z \in \mathbb{Z}. z \in k\mathbb{Z} \land i(z) < x < i(z + k).$$
Similarly, replace real binary truncation $2^{\lfloor \log_2 \|x\|_\mathbb{R} \rfloor}$ with "$\iota(z)$" for some/every $z \in \mathbb{Z}$ s.t. $\iota(z) \leq |x| < \iota(z) + 1$ in case $x > 0$, with 0 otherwise.

The Kleene Algebra $\mathbb{K}$ as third sort is finite and does not affect decidability; recall Remark 1.5.

4.2. Logic of ERC as Specification Language. Logic is an invaluable tool for specifying a computational problem, and for arguing about a program composed from well-specified primitives. The denotational semantics introduced in Section 3 provides the latter for ERC; and the logic of ERC suggests use as specification language to formally describe the required/actual behavior of an ERC program. For example, one may want to state:

For any input $\vec{x} = (x_1, \cdots, x_d)$ satisfying $\phi(\vec{x})$, the ERC program under consideration returns $y$ such that $(\vec{x}, y)$ satisfies $\psi(\vec{x}, y)$

where $\phi, \psi$ are formulas according to Definition 4.2.

Thus rises the question about the expressive power of this logic. For example it can express the denotations of terms from Subsubsection 3.3.1—provided that no additional (multi-) function symbols are present, that is, in the case $\mathcal{F} = \emptyset = \mathcal{G}$. On the other hand even a computable/realizable real function may be undefinable in our logic:

Example 4.5. The restricted exponential function $\exp : I = [0; 1] \to \mathbb{R}$ is uniquely characterized by the following formula:

$$\forall x, y \in I. \ x + y \in I \Rightarrow \exp(x + y) = \exp(x) \cdot \exp(y)$$

$$\forall x, y \in I. \ |\exp(x) - \exp(y)| \leq 3 \cdot |x - y|$$

$$\exp(1) = \lim_{n} (1 + 1/n)^n = \sum_{n} 1/n!$$

The first line is the well-known functional equation, and the second one captures Lipschitz-continuity.

Although $\exp$ can be realized in $\text{ERC} = \text{ERC}(\emptyset, \emptyset)$ as in Subsection 2.3, the defining Formula (4.3) exceeds the Logic of ERC by involving the transcendental constant $e$, which cannot be characterized algebraically.

More generally, propositional formulas in the Logic of ERC can only define semi-algebraic subsets of Euclidean space; and, according to Tarski-Seidenberg, real quantification does not increase the expressive power. Integer quantification can define countable unions of semi-algebraic subsets, but no more according to Theorem 4.4(b). According to Lindemann-Weierstrass, the graph of $\exp : [0; 1] \to \mathbb{R}$ is no countable union of semi-algebraic Euclidean sets, hence impossible to define in $S(\emptyset, \emptyset)$.

The expansion $S(\{\exp\}, \emptyset)$ on the other hand makes exp trivial to define—but its first-order theory may violate decidability. Such trade-offs are unavoidable according to Remark 4.1.

On the other hand, specification and formal verification may suffice with less than definability of the function under consideration: applications tend to interested in solutions that satisfy given algebraic properties expressible in ERC—such as the exponential functional Equation 4.2—but do not necessarily make them unique, particularly transcendental or multi-valued ones.
4.3. Hoare Logic for ERC. With (multi)values of terms expressible in the Logic of ERC (Remark 4.3) and its generalization to ERC(\(F, G\)), we can now proceed to formally reason about commands and their denotations. Hoare Logic is a well-known tool for formally proving total correctness of a program and agreement with the problem specification. The following considerations are guided by [75, §3], adapted and extended to ERC with its three-sorted structure and multi-valued semantics. Both complicate matters since, for instance, a real guard variable in a while loop may strictly decrease during each iteration yet remain bounded forever; furthermore merely evaluating the loop condition can cause lack of termination when real equality occurs; see Remark 4.9 below. Our language being simple imperative, we adapt the following notion of “(total correctness) specification”:

**Definition 4.6.** For a well-typed command \(S\) in ERC(\(F, G\)) with \(\Gamma \vdash S \triangleright \Gamma'\), a (total correctness) specification is of the following form:

\[
\Gamma \vdash \begin{bmatrix} \phi \end{bmatrix} S \begin{bmatrix} \psi \end{bmatrix} \triangleright \Gamma'
\]

where \(\phi, \psi\) are formulae in the logic of ERC(\(F, G\)). In the precondition \(\phi\), only the variables in \(\Gamma\) appear free and in the postcondition \(\psi\), only the variables in \(\Gamma'\) appear free. The notation says, for any \(\sigma \in [\Gamma]\) which validates \(\phi\) (i) \(\perp \notin [S] \sigma\) and (ii) any \(\delta \in [S] \sigma\) validates \(\psi\).

The purpose of (Hoare) logic is to replace ‘semantic’ arguments with formal proofs: sequences of purely syntactic manipulations, starting with the axioms and following certain inference rules, that for example a computer can verify. Classical Hoare logic contains one exception:

**Remark 4.7.** The rule of consequence for precondition-strengthening and postcondition-weakening

\[
\Gamma \vdash \begin{bmatrix} \phi \end{bmatrix} C \begin{bmatrix} \psi \end{bmatrix} \triangleright \Gamma' \quad \phi \Rightarrow \phi' \text{ and } \psi' \Rightarrow \psi
\]

depends on the semantic side-conditions \(\phi \Rightarrow \phi'\) and \(\psi' \Rightarrow \psi\) which may or may not be feasible to verify algorithmically. Over integers, algorithmic verifiability can fail according to Gödel [26, §6], but not in the Logic of ‘pure’ ERC according to Theorem 4.4(a).

A side condition is also present in the rule for while loop termination.

**Definition 4.8.** A Hoare triple for ERC(\(F, G\)) is of the form \(\Gamma \vdash \begin{bmatrix} \phi \end{bmatrix} S \begin{bmatrix} \psi \end{bmatrix} \triangleright \Gamma'\) where \(S\) is a well-typed command in ERC(\(F, G\)) such that \(\Gamma \vdash S \triangleright \Gamma'\), and \(\phi, \psi\) are formulae in the Logic of ERC(\(F, G\)) such that only the variables in \(\Gamma\) are free in \(\phi\) and only the variables in \(\Gamma'\) are free in \(\psi\). Hoare Logic of ERC(\(F, G\)) is a formal system which consists of the inference rules and axioms for constructing Hoare triples defined in Figure 15.

**Remark 4.9.**

1. The first rule is called ‘pre/postcondition strengthening/weakening’ which says we can replace the precondition and the postcondition to some stronger and weaker ones, respectively.
2. In the axiom for assignments, the precondition \(\exists w. (\{t\}(w))\) ensures that the denotation of \(t\) is well-defined. And, \(\forall w. (\{t\}(w) \Rightarrow \psi[w/v])\) says that for each value in the denotation of \(t\), \(\psi\) holds when we replace the variable \(v\) with the value.
3. In the while loop case,
In the rule for loops, \( \Gamma' = \Gamma, \xi, \xi' : \text{REAL} \). And, the rule has the side-conditions:

\[
I \land (\llbracket b \rrbracket (tt) \land I \rightarrow \xi \land L = \xi') \\
S \left[ I \land V \leq \xi - \xi' \land L = \xi' \right] \\
\Gamma' \vdash [I] \text{ while } b \text{ do } S \text{ end while } [I \land (\llbracket b \rrbracket (ff))] \vdash \Gamma
\]

In the case of array assignment \texttt{ArrPre}(P, v, z, t) is defined as follows:

\[
\text{ArrPre}(P, v, z, t) := \exists m. (\llbracket t \rrbracket (w) \land (\llbracket z \rrbracket (m))) \\
\land \forall m. (\llbracket z \rrbracket (m) \Rightarrow 0 \leq m < \text{ dimension of } v) \\
\land \forall w, m. (\llbracket t \rrbracket (w) \land (\llbracket z \rrbracket (m) \Rightarrow P[v \leftarrow_m w])
\]

assuming \( \Gamma \vdash v[z] := t \vdash \Gamma \).

(a) the formula \( I \) is the loop invariant and the term \( V \) is the loop variant. The term \( L \) is some invariant quantity that bounds by how much \( V \) decreases in each iteration.

(b) \( \xi, \xi' \) are ghost variables that do not appear in \( \Gamma \). They can be understood as meta-level universally quantified variables.

(c) The side condition says (i) each loop decreases \( V \) by some positive invariant quantity \( L \); (ii) as long as \( I \) holds, the evaluation of \( b \) is either \( tt \) or \( ff \) (but not \( uk \)); and (iii) when \( V \) is negative, it is guaranteed that the evaluation of \( b \) is \( ff \).

(4) In the rule of \texttt{if} conditionals, the precondition \((\llbracket b \rrbracket (tt) \lor \llbracket b \rrbracket (ff)) \land \neg \llbracket b \rrbracket (uk)\) says that the evaluation of \( b \) is either \( tt \) or \( ff \) (but not \( uk \)).
**Theorem 4.10.** The Hoare logic of ERC(\(F, G\)) is sound; i.e., for any Hoare triple \(\Gamma \vdash [\phi] S [\psi] \Gamma'\), it holds that \(\Gamma \models [\phi] S [\psi] \Gamma'\).

*Proof.* See Appendix A.

4.4. **Example Formal Verification.** The present section picks up from Subsection 2.7 to illustrate formal verification in ERC. To emphasize, our purpose here is not to actually establish correctness of the long-known Trisection method, but to demonstrate the extended Hoare logic from Section 4.3 using a toy example. Since Trisection relies on the Intermediate Value Theorem, any correctness proof must make full use of real (as opposed to, say, floating point, rational, or algebraic) numbers.

Let us define some abbreviations such that the algorithm in Figure 10, which is a program in ERC(\(\{f\}, \emptyset\)) becomes of the form **input** \(p : \text{INTEGER}\) \(C_1; C_2\) **return** \(a\).

\[
\begin{align*}
\tilde{t}_1 &\equiv \imath(p) > b - a, \quad \tilde{t}_2 := b - a > \imath(p - 1) \\
t_1 &\equiv 0 > f(b/3 + 2a/3) \times f(b), \quad t_2 := 0 > f(a) \times f(2b/3 + a/3) \\
b_1 &\equiv \text{choose}(\tilde{t}_1, \tilde{t}_2) = 1, \quad b_2 := \text{choose}(t_1, t_2) = 1 \\
C_1 &\equiv \text{let } a : \text{REAL} = 0; \text{let } b : \text{REAL} = 1 \\
C_2 &\equiv \text{while } b_1 \text{ do } C_3 \text{ end while} \\
C_3 &\equiv \text{if } b_2 \text{ then } C_4 \text{ else } C_5 \text{ end if} \\
C_4 &\equiv b := 2b + 3a + a/3 \\
C_5 &\equiv a := b/3 + 2a/3
\end{align*}
\]

We want the program to realize a real function that computes a root of \(f\), provided that \(f\) is continuous, has a unique root in \((0; 1)\) and that the signs of \(f(0)\), \(f(1)\) are different. In order to verify that the program meets the desired property, we need show that under the condition, the following hold: for any initial state \(\sigma \in [p : \text{INTEGER}], (i) \not\in \mathcal{F}_{\Gamma}; C_1; C_2] \sigma\) and (ii) for all resulting states \(\delta \in [C_1; C_2] \sigma\), the return value \(\delta(a)\) is a \(\imath(\sigma(p))\) approximation of the unique root of \(f\).

The specification language we work on is the logic of ERC(\(\{f\}, \emptyset\)) where the theory of ERC(\(\{f\}, \emptyset\)) contains some sentences saying that \(\hat{f}\) is (the graph of) a continuous function. Although in our specification language \(\hat{f}\) is a binary relation, we can make a convention thus that we may still write \(f\) as a function: for a formula \(\phi\) that consists of function applications \(f(t_1), \ldots, f(t_n)\), we interpret the formula as \(\forall z_1, \ldots, z_n. \hat{f}(t_1, z_1) \land \cdots \land \hat{f}(t_n, z_n) \Rightarrow \phi[z_1/f(t_1), \ldots, z_n/f(t_n)]\). For example, we may write \(f(0) > 0\) to mean \(\forall z. \hat{f}(0, z) \Rightarrow z > 0\).

The specification we wish to have is as follows:

\[
\Gamma \vdash [p = p] C_1; C_2[\exists! z. f(z) = 0 \land 0 < z < 1 \land |a - z| \leq 2p'] \Gamma'
\]

where \(\Gamma = p, p' : \text{INTEGER}, \Gamma' = p, p' : \text{INTEGER}, a, b : \text{REAL}\). The ghost variable \(p'\) captures the initial value that the variable \(p\) stores, considering that the value \(p\) stores may vary (though it does not in this specific example) at the end state. The post condition says, when \(C_1; C_2\) terminates, the return value \(a\) is a \(\imath(p')\) approximation of the unique root of \(f\). Hence, according to Definition 3.14, the specification ensures that the program realizes a function that computes the root.

Implicitly replacing \(\ll\) with \(<\), the terms \(t_1, t_2, \tilde{t}_1, \tilde{t}_2\) can be interpreted as formulae in our specification language. See that \([\langle b_1 \rangle(tt) \leftrightarrow \tilde{t}_2, [\langle b_1 \rangle(ff) \leftrightarrow \tilde{t}_1, [\langle b_1 \rangle(uk), [b_2](tt) \leftrightarrow t_2, [b_2](ff) \leftrightarrow t_1, and [\neg b_2](uk)\) hold.
Recall that \( \text{uniq}(f, x, y) \equiv f(x) \cdot f(y) < 0 \land \exists z. \ x < z < y \land f(z) = 0 \). Let us define
\[
I := p = p' \land 0 \leq a < b \leq 1 \land \text{uniq}(f, a, b)
\]
as a candidate for the loop invariant, \( V := b - a - 2^{p-1} \) as a candidate for the loop variant, \( L := 2^{p-2} \) be a candidate for a lower bound decrements, \( \bar{P} := t_2 \land I \land V = \xi \land L = \xi' \), and \( \bar{Q} := I \land V \leq \xi - \xi' \land L = \xi' \) in our specification language with variables \( \xi, \xi' \) of the sort \( \mathbb{R} \). Let \( \Delta := p, p', \text{INTEGER}, a, b, \xi, \xi' : \text{REAL} \).

From the axiom for assignments, we have the triples:
\[
\Delta \vdash \exists \omega. \ (a/3 + 2 \times b/3) \omega \land \forall \omega. \ (a/3 + 2 \times b/3) \omega \Rightarrow \bar{Q}[\omega/b] \quad C_4 \quad [\bar{Q}] \triangleright \Delta
\]
\[
\Delta \vdash \exists \omega. \ (2 \times a/3 + b/3) \omega \land \forall \omega. \ (2 \times a/3 + b/3) \omega \Rightarrow \bar{Q}[\omega/a] \quad C_5 \quad [\bar{Q}] \triangleright \Delta
\]

See that we can apply the rule of precondition weakening to get the following triples derived:
\[
\Delta \vdash \bar{Q}[(a/3 + 2 \times b/3)/b] \quad C_4 \quad [\bar{Q}] \triangleright \Delta, \quad \Delta \vdash \bar{Q}[(2 \times a/3 + b/3)/a] \quad C_5 \quad [\bar{Q}] \triangleright \Delta.
\]
When we unwrap the abbreviations, we have
\[
\bar{Q}[(2 \times a/3 + b/3)/a] \iff
\begin{align*}
& p = p' \land 0 \leq (2 \times a/3 + b/3) < b \leq 1 \land \text{uniq}(f, (2 \times a/3 + b/3), b) \\
& \land \quad b - (2 \times a/3 + b/3) - 2^{p-1} \leq \xi - \xi'
\end{align*}
\]
\[
\bar{P} \land t_1 \iff
\begin{align*}
& p = p' \land 0 \leq a < b \leq 1 \land \text{uniq}(f, a, b) \land f(b/3 + 2 \times a/3) \times f(b) < 0 \\
& \land \quad 2^{p-1} < b - a \land b - a - 2^{p-1} = \xi
\end{align*}
\]
\[
\bar{P} \land t_1 \Rightarrow \bar{Q}[(2 \times a/3 + b/3)/a] \text{ holds using intermediate value theorem that if an interval } (a; b) \text{ contains a root of } f \text{ uniquely, and if } f(x) < 0 \text{ for } a \leq x < y \leq b, \text{ then } (x; y) \text{ also contains the root of } f \text{ uniquely. And, similarly, } \bar{P} \land t_2 \Rightarrow \bar{Q}[(a/3 + 2 \times b/3)/b] \text{ holds.}
\]

After having the implications proven, we can use the rule of precondition strengthening on the triples of \( C_4, C_5 \), and apply the rule for conditionals to get the triple:
\[
\Delta \vdash [\bar{I} \land (V = \xi) \land L = \xi'] \quad C_3 \quad [I \land (V \leq \xi - \xi') \land L = \xi'] \triangleright \Delta
\]
The side-conditions of the rule for while loops are quite trivial. Hence, assuming that they are proven, we apply the rule of while loops, apply the rules of assignments and sequential compositions, and we get the following triple:
\[
\Gamma \vdash [I[0/a, 1/b]; C_1; C_2 [I \land \bar{t}_2] \triangleright \Gamma'
\]
Using the rule of pre/postcondition strengthening/weakening, we can get the originally desired specification.

5. Extensions and Perspectives

We have formalized Exact Real Computation (ERC) following Paradigm 1.3: to combine, and reconcile between, the realistic but inconvenient Type-2 (variant of the Turing) machine model underlying Computable Analysis and the convenient but super-recursive algebraic model (aka Blum-Shub-Smale Machine aka real-RAM) underlying Computer Algebra Systems. Carefully chosen Turing-computable operations on real numbers and rigorous denotational
multivalued semantics in powerdomains formally justify common basic numerical methods and turn heuristics into algorithms, matching the intuition of operating on continuous data exactly without rounding errors and in agreement with (proofs in) Calculus. This enables a natural and elegant approach to formal program verification by adding real number axioms to Floyd/Hoare Logic.

The following considerations naturally suggest future further investigations:

- **Computational Cost:** After the design of an algorithm comes its analysis, in terms of computational cost as quantitative indicator of its practical performance. For realistic predictions, Real Complexity Theory [48, 98] employs bit-cost, as opposed to unit cost common in Algebraic Complexity Theory [23]. In [18, Definition 2.4] it is suggested that a logarithmic cost measure where each operation is supposed to take time according to the binary length of the integer (part of the real) to be processed. More accurate predictions take into account the precision parameter \(p\) from Paradigm 1.3; and for real number comparisons “\(x \gg y\)” the logarithm of the difference \(|x - y|\).

- **Full Mixed Data Types:** As pointed out in Remark 1.4(d), ERC as introduced here formalizes computing with data types \(\text{INTEGER}, \text{REAL}, \text{KLEENEAN}\) as counterparts to mathematical \(\mathbb{Z}, \mathbb{R}, \mathbb{K}\): for now permitting type \(\text{KLEENEAN}\) only for expressions and local variables: A future extension will include also (multi)functions with \(\text{KLEENEAN}\) type arguments and return values as well as arrays and a dedicated limit operator.

- **Multi-valued Real Functions:** As mentioned in Remark 1.4, the present version of ERC formalizes computing mappings from reals to integers in the multi-valued sense: because any single-valued, and necessarily continuous [97, Theorems 4.3.1+3.2.11], function with connected domain and discrete range must be constant. On the other hand Paradigm 1.3 of computing real values is deliberately restricting to the single-valued case. Defining approximate computation of real multi-valued functions is delicate and still under exploration [34, 49].

- **Functionals and Operators:** The algorithm in Figure 10 receives a continuous function as argument, accessible by pointwise blackbox evaluation, and is thus of higher type: a functional. To extend Theorem 2.4 (real Turing-completeness of ERC) from functions to functionals requires function arguments be enriched with quantitative continuity information, such as a modulus of continuity [43].

- **Automated Formal Verification:** The decidability of the Theory of ERC according to Theorem 4.4 includes its complete (and actually elegant) axiomatization: Guaranteed to yield convenient automated formal verification, for example in the Coq Proof Assistant [50, 88].

- **Computable ADTs beyond the Reals:** Real Computability Theory has been extended to topological \(T_0\) spaces, Real Complexity Theory to co-Polish spaces [46, 80]. Current and future works similarly extend ERC to continuous abstract data types beyond real numbers/functions, such as the Grassmannian, tensors [83] and groups [82].

The latter direction of research leads to a Computer Analysis System, complementing Computer Algebra and its restriction to symbolic manipulations.

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Appendix A. Proof of the Soundness of the Hoare Logic of ERC

We start the proof with the lemma:

**Lemma A.1.** For a well-typed command $\Gamma \vdash \text{while } b \text{ do } S \text{ end while} \triangleright \Gamma$, define the sequences of set-valued functions on $[\Gamma]$:

- $B^0_b,S(\sigma) \equiv \{\sigma\}$
- $C^0_b,S(\sigma) \equiv \emptyset$
- $B^{n+1}_b,S(\sigma) \equiv \bigcup_{\delta \in B^n_b,S(\sigma)} \bigcup_{\delta' \in [b]_\delta} \begin{cases} \{\delta'\} & \text{if } l = tt \land \delta' \neq \bot, \\ \emptyset & \text{otherwise.} \end{cases}$
Then, for all $n \in \mathbb{N}$, it holds that
\[
A^n_{b,S} \sigma = C^n_{b,S}(\sigma) \cup \{ \bot \mid \exists x. x \in B^n_{b,S}(\sigma) \}.
\]

Intuitively, $B^n_{b,S}(\sigma)$ is the set of states that requires further execution after running the while loop on $\sigma$ for $n$ times. $C^n_{b,S}(\sigma)$ is the set of states that have escaped from the loop (either because $\sigma f f$ has been evaluated or $\bot$ has occurred) during running the loop for $n$ times.

**Proof.** Let us drop the subscripts $b, S$ for the convenience in the presentation. We first prove the following alternative characterization of the sequence of sets:

\[
B^{n+1}(\sigma) = \bigcup_{\ell \in [0] \sigma \atop \delta \in [S] \sigma} \begin{cases} 
B^n(\delta) & \text{if } \ell = tt \wedge \delta \neq \bot, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

It is trivial when $n = 0$. Now, suppose the equation holds for all $\sigma$ and for all $n$ up to $m$. Then the following derivation shows that the characterization is valid for $n = m + 1$ as well.

\[
B^{m+2}(\sigma) = \bigcup_{\gamma \in B^{m+1}(\sigma) \atop \delta \in [S] \gamma} \begin{cases} 
\emptyset & \text{otherwise.}
\end{cases}
\]

We now show the following characterization:

\[
C^{n+1}(\sigma) = \bigcup_{\ell \in [0] \sigma \atop \delta \in [S] \sigma} \begin{cases} 
C^n(\delta) & \text{if } \ell = tt \wedge \delta \neq \bot, \\
\{ \sigma \} & \text{if } \ell = ff, \\
\{ \bot \} & \text{otherwise.}
\end{cases}
\]

It is easy to show that the equation holds for $n = 0$. Now, assume the equation holds for all $n$ up to $m$. Then,

\[
C^{m+2}(\sigma) = C^{m+1}(\sigma) \cup \bigcup_{\delta \in B^{m+1}(\sigma) \atop \ell \in [0] \delta \atop \delta' \in [S] \delta} \begin{cases} 
\emptyset & \text{if } \ell' = ff, \\
\{ \delta \} & \text{if } \ell' = tt \wedge \delta' \neq \bot, \\
\{ \bot \} & \text{otherwise.}
\end{cases}
\]
Then, Proof. We can prove the statement by checking the soundness of each rule. A.1. Proof of Theorem 4.10.

Now, using the suggested characterization, we prove $\left[ A_{b,S}^n \right] \sigma = C_{b,S}^n(\sigma) \cup \{ \perp \mid \exists x \in B_{b,S}^n(\sigma) \}$ for all $n \in \mathbb{N}$. When $n = 0$, both are $\{ \perp \}$. Suppose the equation holds for $n = m$. Then,

$$\left[ A_{b,S}^{m+1} \right] \sigma = \bigcup_{\ell \in \{t,b\}^{|b|}\sigma} \bigcup_{\delta \in \{\text{true, false}\}^{|b|}\sigma} \left\{ \begin{array}{ll} \left[ A^m \right] \delta & \text{if } \ell = tt \wedge \delta \neq \perp, \\ \{ \sigma \} & \text{if } \ell = ff, \\ \{ \perp \} & \text{otherwise.} \end{array} \right. \bigcup_{\ell \in \{t,b\}^{|b|}\sigma} \bigcup_{\delta \in \{\text{true, false}\}^{|b|}\sigma} \left\{ \begin{array}{ll} \{ \perp \} & \text{if } \ell = tt \wedge \delta \neq \perp, \\ \{ \exists \gamma. \gamma \in B^m(\delta) \} & \text{if } \ell = tt \wedge \delta \neq \perp, \\ \{ \perp \} & \text{if } \ell = ff, \\ \emptyset & \text{otherwise.} \end{array} \right. \bigcup_{\ell \in \{t,b\}^{|b|}\sigma} \bigcup_{\delta \in \{\text{true, false}\}^{|b|}\sigma} \left\{ \begin{array}{ll} \emptyset & \text{if } \ell = ff, \\ \emptyset & \text{otherwise.} \end{array} \right.$$

$$= C_{b,S}^{m+1}(\sigma) \cup \left\{ \perp \mid \exists \gamma. \gamma \in B_{b,S}^{m+1}(\sigma) \right\}$$

\[ \square \]

A.1. Proof of Theorem 4.10.

Proof. We can prove the statement by checking the soundness of each rule.
(1) (Assignment):
Consider any state \( \sigma \) which validates \( \exists w. (b)(w) \wedge \forall w. (b)(w) \Rightarrow \psi[w/v] \). Then, \( \perp \notin [t] \sigma \) and for any \( w \in [t] \sigma \), \( \psi[w/v] \) holds.

Now, see that \( [v \Leftarrow t] \sigma = \bigcup_{w \in [t] \sigma} \{ \sigma[v \mapsto w] \} \) since \( \perp \notin [t] \sigma \) and for all \( w \in [t] \sigma \), \( \sigma[v \mapsto w] \) validates \( \psi \).

(2) The rule variable declarations and the rule of array assignments can be verified in very similar manner as above and the rules of pre/postcondition strengthening/weakening, skip, and sequential compositions can be verified quite trivially.

(3) (Conditional):
Consider any state \( \sigma \) which validates \( \phi \wedge (b)(tt) \lor (b)(ff) \wedge \neg b(uk) \). Then, \( [b] \sigma = \{ tt, ff \}, \{ tt \}, \) or \( \{ ff \} \). Let us check the three cases:
(a) when \( [b] \sigma = \{ tt, ff \} \):
Then, \( \sigma \) validates \( \phi \wedge (b)(tt) \) and \( \phi \wedge (b)(ff) \). Therefore, (i) \( \perp \notin [S_1] \sigma \), (ii) for all \( \delta \in [S_1] \sigma \) it holds that \( \delta \vdash \psi \), (iii) \( \perp \notin [S_2] \sigma \), and (iv) for all \( \delta \in [S_2] \sigma \) it holds that \( \delta \vdash \psi \).

Since \( \perp \notin [b] \sigma \) and \( uk \notin [b] \sigma \), the denotation becomes \( [if \ b \ then \ S_1 \ else \ S_2 \ end \ if] \sigma = [S_1] \sigma \cup [S_2] \sigma \). Hence, the denotation does not contain \( \perp \) and any resulting state \( \delta \) validates \( \psi \).
(b) when \( [b] \sigma = \{ tt \} \):
Then, \( \sigma \) validates \( \phi \wedge (b)(tt) \). Hence, (i) \( \perp \notin [S_1] \sigma \), (ii) for all \( \delta \in [S_1] \sigma \) it holds that \( \delta \vdash \psi \). Since \( [b] \sigma = \{ tt \} \), the denotation becomes \( [if \ b \ then \ S_1 \ else \ S_2 \ end \ if] \sigma = [S_1] \sigma \). Therefore, \( \perp \) is not in the denotation and any resulting state \( \delta \) validates \( \psi \).
(c) when \( [b] \sigma = \{ ff \} \), it can be done very similarly to the above item.

(4) (Loop):
Consider any state \( \sigma \) that validates \( I \). Then, by the side-conditions, it also validates \( (b)(tt) \lor (b)(ff) \). Hence, \( [b] \sigma = \{ tt, ff \}, \{ tt \}, \) or \( \{ ff \} \) for any state \( \sigma \) that validates \( I \). Now, we fix a state \( \sigma \) which validates \( I \) hence satisfies the precondition.

The core part of the proof is the statement: for any natural number \( n \), it holds that (i) \( \perp \notin B_{bs}^n(\sigma) \), (ii) \( \perp \notin C_{bs}^n(\sigma) \), (iii) all \( \delta \) in either \( B_{bs}^n(\sigma) \) or \( C_{bs}^n(\sigma) \) validates \( I \), and (iv) all \( \delta \) in \( C_{bs}^n(\sigma) \) validates \( (b)(ff) \).

At the moment, suppose that the above statement is true. Then, all we have to show is that \( B_{bs}^n(\sigma) \) becomes empty as \( m \in \mathbb{N} \) increases. Let us define \( \ell_n := \max \{ V(\delta) \mid \delta \in B_{bs}^n(\sigma) \} \) and show that \( \ell_n \) is strictly decreasing by some quantity that is bounded below, as \( n \) increases. See that if it holds, there will be some \( m \) that for all \( \delta \in B_{bs}^m(\sigma) \), \( [b] \delta = \{ ff \} \) and hence \( B_{bs}^{m+1}(\sigma) = \emptyset \).

In order to prove it, we take the two steps:
(a) If \( B_{bs}^n(\sigma) \neq \emptyset \), then for all \( n \in \mathbb{N} \) and for all \( \delta \in B_{bs}^n(\sigma) \), it holds that \( L(\delta) = L(\sigma) > 0 \). In this case, let us write \( \ell_0 = L(\sigma) \).
(b) If \( B_{bs}^{n+1}(\sigma) \neq \emptyset \), it holds that \( \ell_{m+1} \leq \ell_m - \ell_0 \).

Now, we prove each statement:
(a) \( B_{bs}^n(\sigma) \neq \emptyset \) only if \( tt \in [b] \sigma \) and there is some non-bottom \( \delta \in [S] \sigma \). Therefore, by the side-condition, \( L(\sigma) > 0 \).

Suppose any \( \delta \in B_{bs}^{n+1}(\sigma) \) for any \( m \in \mathbb{N} \). See that it happens only if there is \( \delta' \in B_{bs}^n(\sigma) \) such that \( tt \in [b] \delta' \) and \( \delta \in [S] \delta' \). Together with Item (iii), \( \delta' \) validates \( I \) and \( (b)(tt) \). Let us define \( \tilde{\delta}' := \delta' \cup (\xi \mapsto V(\delta') \cup \xi' \mapsto L(\delta')) \). Since \( \tilde{\delta}' \) validates
the precondition in the premise, we have that for any \( \hat{\delta} \in [S] \delta' \), \( \hat{\delta} \) validates \( I \) and \( V \leq \xi - \xi' \) and \( L = \xi' \). Hence, \( L(\hat{\delta}) = L(\delta') \). Since \( \xi', \xi \) are ghost variables, \( L(\hat{\delta}) = L(\delta) = L(\delta') \). In conclusion, for any \( \delta \in B_{b,S}^{m+1}(\sigma) \), the quantity \( L(\delta) \) is identical to the quantity \( L(\delta') \) for some \( \delta' \in B_{b,S}^m(\sigma) \). Since, \( B_{b,S}^0(\sigma) = \{ \sigma \} \), we conclude that they are all identical to \( L(\sigma) \).

(b) Suppose any \( \delta \in B_{b,S}^{m+1}(\sigma) \) for any \( m \in \mathbb{N} \). See that it happens only if there is \( \delta' \in B_{b,S}^m(\sigma) \) such that \( tt \in [b] \delta' \) and \( \delta \in [S] \delta' \). Together with Item (iii), \( \delta' \) validates \( I \) and \( \{b\}(tt) \). Consider \( \delta : = \delta' \cup (\xi \mapsto V(\delta') \cup \xi' \mapsto L(\delta')) \) which validates the precondition of the premise. Hence, \( \delta \cup (\xi \mapsto V(\delta') \cup \xi' \mapsto L(\delta')) \) validates the postcondition. Hence, \( V(\delta) \leq V(\delta') - L(\delta) = V(\delta') - \ell_0 \). Hence, \( \ell_{m+1} \leq \ell_m - \ell_0 \).

Now, we need to prove the aforementioned statement on \( B_{b,S}^m \) and \( C_{b,S}^m \):

(a) (Base case): Recall that \( B_{b,S}^0(\sigma) = \{ \sigma \} \neq \{ \perp \} \) and \( C_{b,S}^0(\sigma) = \{ \} \). Hence, the four conditions are all satisfied.

(b) (Induction step): Recall \( B_{b,S}^{m+1}(\sigma) : = \bigcup_{\delta \in B_{b,S}^m(\sigma)} \bigcup_{l \in [b] \delta} \bigcup_{\delta' \in [S] \delta} \left\{ \{ \delta' \} \right\} \) if \( l = tt \wedge \delta' \neq \perp \) and \( \emptyset \) otherwise.

Since all \( \delta \in B_{b,S}^m(\sigma) \) validates \( I \), \([b] \delta = \{tt\}, \{ff\}, \{tt, ff\}\). In the case of \( tt \in [b] \delta \), \( \delta \) validates the precondition of the premise. Hence, for all \( \delta' \in [S] \delta \), \( \delta' \) is not \( \perp \) and also validates \( I \). The case of \([b] \delta = \{ff\}\) is not of interest.

Recall \( C_{b,S}^{m+1}(\sigma) : = C_{b,S}^m(\sigma) \bigcup_{\delta \in B_{b,S}^m(\sigma)} \bigcup_{l \in [b] \delta} \bigcup_{\delta' \in [S] \delta} \left\{ \{ \delta \} \right\} \) if \( l = ff \) and \( \emptyset \) otherwise.

Since all \( \gamma \in C_{b,S}^m(\sigma) \) validates \( I \) and \( \{b\}(ff) \), we only need to care the rightmost part of the construction. Since all \( \delta \in B_{b,S}^m(\sigma) \) validates \( I \), by the side-condition, \( uk \) and \( \perp \) are not in \([b] \delta \). The \( \delta \) is added to \( C_{b,S}^{m+1}(\sigma) \) only if \( ff \in [b] \delta \). Therefore, \( \delta \) validates both \( I \) and \( \{b\}(ff) \).

Also, in the case of \( tt \in [b] \delta \), since \( \delta \) validates the precondition in the premise, \( \perp \notin [S] \delta \). Therefore, \( \perp \notin C_{b,S}^{m+1} \).

\[ \square \]