Couplings between a single massless tensor field with the mixed symmetry (3,1) and one vector field

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February 1, 2008

Abstract
Under the hypotheses of smoothness in the coupling constant, locality, Lorentz covariance, and Poincaré invariance of the deformations, combined with the preservation of the number of derivatives on each field, the consistent interactions between a single free massless tensor gauge field with the mixed symmetry of a two-column Young diagram of the type (3,1) and one Abelian vector field have been investigated. The computations are done with the help of the deformation theory based on a cohomological approach, in the context of the antifield-BRST formalism. The main result is that there exist nontrivial cross-couplings between these types of fields in five spatiotemporal dimensions, which break the PT invariance and allow for the deformation of the gauge transformations of the vector field, but not of the gauge algebra.

PACS number: 11.10.Ef

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1 Introduction

Tensor fields in “exotic” representations of the Lorentz group, characterized by a mixed Young symmetry type $[1, 2, 3, 4, 5, 6, 7]$, held the attention lately on some important issues, like the dual formulation of field theories of spin two or higher [8, 9, 10, 11, 12, 13, 14], the impossibility of consistent cross-interactions in the dual formulation of linearized gravity [15], or a Lagrangian first-order approach [16, 17] to some classes of massless or partially massive mixed symmetry type tensor gauge fields, suggestively resembling to the tetrad formalism of General Relativity. An important matter related to mixed symmetry type tensor fields is the study of their consistent interactions, among themselves as well as with higher-spin gauge theories [18, 19, 20, 21, 22, 23, 24, 25, 26]. The most efficient approach to this problem is the cohomological one, based on the deformation of the solution to the master equation [27]. The purpose of this paper is to investigate the consistent interactions between a single free massless tensor gauge field $t_{\lambda \mu \nu |\alpha}$ with the mixed symmetry of a two-column Young diagram of the type $(3, 1)$ and one Abelian vector field $A_\mu$. It is worth mentioning the duality of the free massless tensor gauge field $t_{\lambda \mu \nu |\alpha}$ to the Pauli-Fierz theory in $D = 6$ dimensions and, in this respect, the recent developments concerning the dual formulations of linearized gravity from the perspective of $M$-theory [28, 29, 30]. Our analysis relies on the deformation of the solution to the master equation by means of cohomological techniques with the help of the local BRST cohomology, whose component in the $(3, 1)$ sector has been reported in detail in [31]. Under the hypotheses of smoothness in the coupling constant, locality, Lorentz covariance, and Poincaré invariance of the deformations, combined with the preservation of the number of derivatives on each field, we prove that there exists a case where the deformation of the solution to the master equation provides nontrivial cross-couplings. This case corresponds to a five-dimensional space-time and is described by a deformed solution that stops at order two in the coupling constant. The interacting Lagrangian action contains only mixing-component terms of order one and two in the coupling constant. At the level of the gauge transformations, only those of the vector fields are modified at order one in the coupling constant with a term linear in the antisymmetrized first-order derivatives of some gauge parameters from the $(3, 1)$ sector such that the gauge algebra and the reducibility structure of the coupled model are not modified during the deformation procedure, being the same like in the case of the starting free...
action. It is interesting to note that if we require the PT invariance of the deformed theory, then no interactions occur. Although it is not possible to construct interactions that deform the gauge algebra, our result is interesting since this seems to be the first case where mixed symmetry type tensor fields allow nontrivial cross-couplings.

2 Free model. BRST symmetry

We begin with the Lagrangian action

$$S_0 \left[ t_{\lambda\mu\nu|\alpha}, A_\mu \right] = \int d^Dx \left\{ \frac{1}{2} \left[ (\partial^\rho t^{\lambda\mu\nu|\alpha}) (\partial_\rho t_{\lambda\mu\nu|\alpha}) - (\partial_\alpha t^{\lambda\mu\nu|\alpha}) (\partial^\beta t_{\lambda\mu\nu|\beta}) \right] \\
- \frac{3}{2} \left[ (\partial_\lambda t^{\lambda\mu\nu|\alpha}) (\partial^\rho t_{\rho\mu\nu|\alpha}) + (\partial^\rho t^{\lambda\mu}) (\partial_\rho t_{\lambda\mu}) \right] + 3 (\partial_\alpha t^{\lambda\mu\nu|\alpha}) (\partial_\lambda t_{\mu\nu}) \right\} \equiv S_0^t \left[ t_{\lambda\mu\nu|\alpha} \right] + S_0^A \left[ A_\mu \right].$$

(1)

in $D \geq 5$ spatiotemporal dimensions. The massless tensor field $t_{\lambda\mu\nu|\alpha}$ has the mixed symmetry $(3,1)$ and hence transforms according to an irreducible representation of $GL(D, \mathbb{R})$ corresponding to a 4-cell Young diagram with two columns and three rows. It is thus completely antisymmetric in its first three indices and satisfies the identity $t_{\lambda\mu\nu|\alpha} \equiv 0$. The field strength of the vector field $A_\mu$ is defined in the standard manner by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv \partial_{[\mu} A_{\nu]}.$$  

(2)

Everywhere in this paper it is understood that the notation $[\lambda \cdots \alpha]$ signifies complete antisymmetry with respect to the (Lorentz) indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The trace of $t_{\lambda\mu\nu|\alpha}$ is defined by $t_{\lambda\mu} = \sigma^{\mu\nu} t_{\lambda\mu\nu|\alpha}$ and it is obviously an antisymmetric tensor. Everywhere in this paper we employ the flat Minkowski metric of ‘mostly plus’ signature $\sigma^{\mu\nu} = \sigma_{\mu\nu} = (-, + + + + \cdots)$.

A generating set of gauge transformations for the action (1) can be taken of the form

$$\delta_\epsilon t_{\lambda\mu\nu|\alpha} = -3 \partial_{[\lambda} \epsilon_{\mu\nu]\alpha] + 4 \partial_{[\lambda} \epsilon \mu_{\nu]} \alpha + \partial_{[\lambda} \chi_{\mu\nu]} |\alpha],$$

(3)

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon,$$  

(4)

3
where the gauge parameters $\epsilon_{\lambda\mu\nu}$ determine a completely antisymmetric tensor, the other set of gauge parameters displays the mixed symmetry (2, 1), such that they are antisymmetric in the first two indices and satisfy the identity $\chi_{[\mu\nu][\alpha]} \equiv 0$, and the gauge parameter $\epsilon$ is a scalar. The generating set of gauge transformations (3)–(4) is off-shell, second-stage reducible, the accompanying gauge algebra being obviously Abelian. More precisely, the gauge transformations (3) are off-shell, second-stage reducible. This is because: 1.

1. If in (3) we make the transformations

$$\begin{align*}
\epsilon_{\mu\nu\alpha} & \to \epsilon^{(\omega, \psi)}_{\mu\nu\alpha} = -\frac{1}{2} \partial_{[\mu} \omega_{\nu]\alpha], \\
\chi_{\mu\nu|\alpha} & \to \chi^{(\omega, \psi)}_{\mu\nu|\alpha} = \partial_{[\mu} \psi_{\nu]\alpha} + 2 \partial_{\alpha} \omega_{\mu\nu} - \partial_{[\mu} \omega_{\nu]\alpha],
\end{align*}$$

with $\omega_{\nu\alpha}$ antisymmetric and $\psi_{\nu\alpha}$ symmetric (but otherwise arbitrary), then the gauge variation of the tensor field identically vanishes $\delta_{\epsilon^{(\omega, \psi)}, \chi^{(\omega, \psi)}} t_{\lambda\mu\nu|\alpha} \equiv 0$. 2. If in (5)–(6) we perform the changes

$$\begin{align*}
\omega_{\nu\alpha} & \to \omega^{(\theta)}_{\nu\alpha} = \partial_{[\mu} \theta_{\nu]\alpha], \\
\psi_{\nu\alpha} & \to \psi^{(\theta)}_{\nu\alpha} = -3 \partial_{[\mu} (\theta_{\nu}\alpha),
\end{align*}$$

with $\theta_\nu$ an arbitrary vector field, where $(\nu \mu \cdots)$ signifies symmetrization with respect to the indices between parentheses without normalization factors, then the transformed gauge parameters (5)–(6) identically vanish $\epsilon^{(\omega^{(\theta)}, \psi^{(\theta)})}_{\mu\nu\alpha} \equiv 0, \chi^{(\omega^{(\theta)}, \psi^{(\theta)})}_{\mu\nu|\alpha} \equiv 0$. 3. There is no non-vanishing local transformation of $\theta_\nu$ that simultaneously annihilates $\omega^{(\theta)}_{\nu\alpha}$ and $\psi^{(\theta)}_{\nu\alpha}$ of the form (7)–(8) and hence no further local reducibility identity.

The field equations associated with action (11) are

$$\begin{align*}
\Sigma : \frac{\delta S_0}{\delta t_{\lambda\mu\nu|\alpha}} & \equiv -T_{\lambda\mu\nu|\alpha} \approx 0, \\
\frac{\delta S_0}{\delta A_\mu} & \equiv \partial_\nu F^{\nu\mu} \approx 0,
\end{align*}$$

where (minus) the Euler-Lagrange derivatives $T^{\lambda\mu\nu|\alpha}$ have the form

$$T^{\lambda\mu\nu|\alpha} = \square t^{\lambda\mu\nu|\alpha} - \partial_\rho \left( \partial^{[\lambda} t^{\mu\nu]\rho|\alpha} + \partial^\alpha t^{\lambda\mu\nu|\rho} \right) + \partial^\alpha \partial^{[\lambda} t^{\mu\nu]} + 3 \partial_\rho \left( \partial^{[\lambda} t^{\mu\nu]\rho|\beta} - \partial^\beta t^{\lambda\nu]\rho} \right) - \square t^{\nu\mu|\alpha}. $$

The notation $\approx$ means here the weak equality symbol. The tensor $T^{\lambda\mu\nu|\alpha}$ has the same properties like the field $t^{\lambda\mu\nu|\alpha}$, being antisymmetric in its first three indices and satisfying the identity $T^{[\lambda\mu\nu]|\alpha} \equiv 0$. Its trace

$$T^\lambda \equiv \sigma_{\nu\alpha} T^{\lambda\mu\nu|\alpha} = (4 - D) \left( \square t^\lambda + \partial_\rho \left( \partial^{[\lambda} t^{\mu\rho]\beta} - \partial^\beta t^{\lambda\rho\beta} \right) \right),$$
is an antisymmetric tensor.

The most general quantities invariant under the gauge transformations (3)–(4) are functions of the curvature tensor

\[ K^{\lambda \mu \nu \xi | \alpha \beta} = \partial^\alpha \partial^{[\lambda} t^{\mu \nu \xi]}|^{\beta} - \partial^\beta \partial^{[\lambda} t^{\mu \nu \xi]}|^{\alpha}, \]

of the field strength (2) as well as of their spatiotemporal derivatives of all orders. The curvature tensor exhibits the mixed symmetry (4, 2), such that it is separately antisymmetric in its first four indices and respectively in the last two ones and fulfills the (algebraic) Bianchi I identity

\[ K^{[\lambda \mu \nu \xi | \alpha} \beta]} \equiv 0. \]

Meanwhile, the curvature tensor obeys two types of (differential) Bianchi II identities

\[ \partial^\lambda [K^{\lambda \mu \nu \xi | \alpha \beta}] \equiv 0, \]

\[ K^{\lambda \mu \nu \xi | \alpha \beta, \gamma} \equiv \partial^\gamma K^{\lambda \mu \nu \xi | \alpha \beta}. \]

Its traces are defined through

\[ K^{\lambda \mu | \alpha} \equiv \sigma_{\nu \alpha} K^{\lambda \mu \nu \xi | \alpha} = 2 \left( \square t^{\lambda \mu | \alpha} + \partial^\rho \partial^{[\lambda} t^{\mu \nu]}|^{\rho \mid \alpha} \right), \]

where we made the notation

\[ K^{\lambda \mu \nu | \alpha} \equiv \sigma_{\nu \alpha} K^{\lambda \mu \nu \xi | \alpha} = \partial^\xi \partial^{\beta} \Phi^{\lambda \mu \nu \xi | \alpha \beta}. \]

It is interesting to note that if \( \bar{T}^{\lambda \mu \nu | \alpha} \) is a covariant tensor field with the mixed symmetry (3, 1), which simultaneously satisfies the equations

\[ \partial^\lambda \bar{T}^{\lambda \mu \nu | \alpha} = 0, \quad \partial^\alpha \bar{T}^{\lambda \mu \nu | \alpha} = 0, \]

then there exists a tensor \( \Phi^{\rho \lambda \mu \nu | \beta \alpha} \) with the mixed symmetry of the curvature tensor, such that

\[ \bar{T}^{\lambda \mu \nu | \alpha} = \partial^\xi \partial^\beta \Phi^{\lambda \mu \nu \xi | \alpha \beta}. \]

(A constant solution \( C^{\lambda \mu \nu | \alpha} \) is excluded from covariance arguments due to the mixed symmetry (3, 1).)

The construction of the antifield-BRST symmetry for this free theory de-buts with the identification of the algebra on which the BRST differential acts. The generators of the BRST algebra are of two kinds: fields/ghosts and antifields. The ghost spectrum for the model under study comprises the fermionic ghosts \( \{ \eta_{\lambda \mu \nu}, G_{\mu \nu | \alpha, \eta} \} \) associated with the gauge parameters
\{\epsilon_{\mu\nu}, \chi_{\mu\nu|\alpha}, \epsilon\} \text{ from (3)–(4), the bosonic ghosts for ghosts } \{C_{\mu\nu}, C_{\nu\alpha}\} \text{ due to the first-stage reducibility parameters } \{\omega_{\mu\nu}, \psi_{\nu\alpha}\} \text{ in (5)–(6), and also the fermionic ghost for ghost for ghost } C_{\nu} \text{ corresponding to the second-stage reducibility parameter } \theta_{\nu} \text{ in (7)–(8). In order to make compatible the behavior of the gauge and reducibility parameters with that of the accompanying ghosts, we ask that } \eta_{\lambda\mu\nu} \text{ and } C_{\mu\nu} \text{ are completely antisymmetric, } G_{\mu\nu|\alpha} \text{ obeys the analogue of the properties fulfilled by the gauge parameters } \chi_{\mu\nu|\alpha}, \text{ and } C_{\nu\alpha} \text{ is symmetric. The antifield spectrum is organized into the antifields } \{t^{*\lambda\mu\nu|\alpha}, A^{*\mu}\} \text{ of the original tensor fields, together with those of the ghosts, } \{\eta^{*\lambda\mu\nu}, G^{*\mu\nu|\alpha}, \eta^{*}\}, \{C^{*\mu\nu}, C^{*\nu\alpha}\}, \text{ and respectively } C^{*\nu}, \text{ of statistics opposite to that of the associated fields/ghosts. It is understood that } t^{*\lambda\mu\nu|\alpha} \text{ exhibits the same mixed-symmetry properties like } t_{\lambda\mu\nu|\alpha} \text{ and similarly with respect to } \eta^{*\lambda\mu\nu}, G^{*\mu\nu|\alpha}, C^{*\mu\nu}, \text{ and } C^{*\nu\alpha}. \text{ For subsequent purpose, we denote the trace of } t^{*\lambda\mu\nu|\alpha} \text{ by } t^{*\lambda\mu}, \text{ being understood that it is antisymmetric.}

Since both the gauge generators and reducibility functions for this model are field-independent, it follows that the BRST differential } s \text{ simply reduces to}

\[ s = \delta + \gamma, \tag{18} \]

where } \delta \text{ represents the Koszul-Tate differential, graded by the antighost number } \text{agh (agh (} \delta \text{) = −1) and } \gamma \text{ stands for the exterior derivative along the gauge orbits, whose degree is named pure ghost number } \text{pgh (pgh (} \gamma \text{) = 1). These two degrees do not interfere (agh (} \gamma \text{) = 0, pgh (} \delta \text{) = 0). The overall degree that grades the BRST complex is known as the ghost number } \text{gh and is defined like the difference between the pure ghost number and the antighost number, such that } \text{gh (} s \text{) = gh (} \delta \text{) = gh (} \gamma \text{) = 1. According to the standard rules of the BRST method, the corresponding degrees of the generators from the BRST complex are valued like}

\[ \begin{align*}
\text{pgh (} t_{\lambda\mu\nu|\alpha} \text{) = 0 = pgh (} A_{\mu} \text{), } & \text{pgh (} \eta_{\lambda\mu\nu} \text{) = pgh (} G_{\mu\nu|\alpha} \text{) = pgh (} \eta \text{) = 1,} \\
\text{pgh (} C_{\mu\nu} \text{) = 2 = pgh (} C_{\nu\alpha} \text{), } & \text{pgh (} C_{\nu} \text{) = 3, } \text{pgh (} t^{*\lambda\mu\nu|\alpha} \text{) = pgh (} A^{*\mu} \text{) = 0,} \\
\text{pgh (} \eta^{*\lambda\mu\nu} \text{) = pgh (} G^{*\mu\nu|\alpha} \text{) = pgh (} \eta^{*} \text{) = 0,} & \\
\text{pgh (} C^{*\mu\nu} \text{) = pgh (} C^{*\nu\alpha} \text{) = pgh (} C^{*\nu} \text{) = 0,} & \\
\text{agh (} t_{\lambda\mu\nu|\alpha} \text{) = agh (} A_{\mu} \text{) = 0, } & \text{agh (} \eta_{\lambda\mu\nu} \text{) = agh (} G_{\mu\nu|\alpha} \text{) = agh (} \eta \text{) = 0,} \\
\text{agh (} C_{\mu\nu} \text{) = agh (} C_{\nu\alpha} \text{) = agh (} C_{\nu} \text{) = 0, } & \text{agh (} t^{*\lambda\mu\nu|\alpha} \text{) = 1 = agh (} A^{*\mu} \text{),}
\end{align*} \]
agh \left( \eta^{\lambda \mu \nu} \right) = agh \left( G^{\mu \nu | \alpha} \right) = agh \left( \eta^* \right) = 2,
agh \left( C^{* \mu \nu} \right) = 3 = agh \left( C^{* \nu | \alpha} \right), \quad agh \left( C^{* \nu} \right) = 4.

Actually, (18) is a decomposition of the BRST differential according to the antighost number and it shows that \( s \) contains only components of antighost number equal to minus one and zero. The Koszul-Tate differential is imposed to realize a homological resolution of the algebra of smooth functions defined on the stationary surface of field equations (9) and the exterior longitudinal derivative is related to the gauge symmetries (3)–(4) of the action (11) through its cohomology at pure ghost number zero computed in the cohomology of \( \delta \), which is required to be the algebra of physical observables for the free model under consideration. The actions of \( \delta \) and \( \gamma \) on the generators from the BRST complex, which enforce all the above mentioned properties, are given by

\[
\begin{align*}
\gamma t_{\lambda \mu | \alpha} &= -3 \partial_\lambda \eta_{\mu | \alpha} + 4 \partial_\lambda \eta_{\mu | \alpha} + \partial_\lambda G_{\mu | \alpha}, \quad \gamma A_\mu = \partial_\mu \eta, \\
\gamma G_{\mu | \alpha} &= 2 \partial_\lambda C_{\mu | \alpha} - 3 \partial_\lambda C_{\mu | \alpha} + \partial_\lambda C_{\mu | \alpha}, \\
\gamma C_{\mu \nu} &= \partial_\lambda C_{\mu | \alpha}, \\
\gamma A^{* \mu} &= \gamma \eta^{* \lambda \mu \nu} = \gamma G^{* \mu | \alpha} = \gamma \eta^* = 0, \\
\gamma C^{* \mu \nu} &= \gamma C^{* \nu | \alpha} = \gamma C^{* \nu} = 0,
\end{align*}
\]

\[
\begin{align*}
\delta t_{\lambda \mu | \alpha} &= \delta A_\mu = \delta \eta_{\mu | \alpha} = \delta G_{\mu | \alpha} = \delta \eta = 0, \\
\delta C_{\mu \nu} &= \delta C_{\nu | \alpha} = \delta C_\nu = 0, \\
\delta t^{* \lambda \mu | \alpha} &= T^{* \lambda \mu | \alpha}, \quad \delta A^{* \mu} = - \partial_\mu F^{* \mu}, \quad \delta \eta^{* \lambda \mu \nu} = - 4 \partial_\alpha t^{* \lambda \mu | \alpha}, \\
\delta G^{* \mu | \alpha} &= - \partial_\lambda \left( 3 t^{* \lambda \mu | \alpha} - t^{* \mu | \alpha | \lambda} \right), \quad \delta \eta^* = - \partial_\mu A^{* \mu}, \\
\delta C^{* \mu \nu} &= 3 \partial_\lambda \left( G^{* \mu | \lambda} - \lambda \mu \nu \right), \quad \delta C^{* \nu | \alpha} = \partial_\mu G^{* \mu | \nu | \alpha}, \\
\delta C^{* \nu} &= 6 \partial_\mu \left( C^{* \mu | \lambda} - \frac{1}{3} C^{* \mu \nu} \right),
\end{align*}
\]

where \( T^{* \lambda \mu | \alpha} \) is given in (10). By convention, we take \( \delta \) and \( \gamma \) to act like right derivations. We note that the action of the Koszul-Tate differential on the
antifields with the antighost numbers equal to two and three gains a simpler expression if we perform the changes of variables

\[ G^{\prime*}_{\mu\nu||\alpha} = G^{*\mu\nu||\alpha} + \frac{1}{4} t^{*\mu\nu||\alpha}, \quad C^{*\nu\alpha} = C^{*\nu\alpha} - \frac{1}{3} C^{*\nu\alpha}, \quad (31) \]

where \( G^{*\mu\nu||\alpha} \) is still antisymmetric in its first two indices (but no longer fulfills the identity \( G^{*\mu\nu||\alpha} \equiv 0 \)) and \( C^{*\nu\alpha} \) has no definite symmetry or antisymmetry properties. With the help of (27)–(30), we find that \( \delta \) acts on the transformed antifields through the relations

\[ \delta G^{*\mu\nu||\alpha} = -3 \partial_\lambda t^{*\lambda\mu\nu||\alpha}, \quad \delta C^{*\nu\alpha} = 2 \partial_\mu G^{*\mu\nu||\alpha}, \quad \delta C^{*\nu\alpha} = 6 \partial_\mu C^{*\nu\mu}. \quad (32) \]

The same observation is valid with respect to \( \gamma \) if we make the changes

\[ G^{\prime}_{\mu\nu||\alpha} = G_{\mu\nu||\alpha} + 4 \eta_{\mu\nu||\alpha}, \quad C^{\prime}_{\nu\alpha} = C_{\nu\alpha} - 3 C_{\nu\alpha}, \quad (33) \]

in terms of which we can write

\[ \gamma t_{\lambda\mu\nu||\alpha} = \frac{1}{4} \partial_\lambda G^{\prime}_{\mu\nu||\alpha} + \partial_\lambda G^{\prime}_{\mu\nu||\alpha}, \quad \gamma C^{\prime}_{\mu\nu||\alpha} = \partial_\mu C^{\prime}_{\nu\alpha}, \quad \gamma C^{\prime}_{\nu\alpha} = -6 \partial_\nu C_{\alpha}. \quad (34) \]

Again, \( G^{\prime}_{\mu\nu||\alpha} \) is antisymmetric in its first two indices, but does not satisfy the identity \( G^{\prime}_{\mu\nu||\alpha} \equiv 0 \), while \( C^{\prime}_{\nu\alpha} \) has no definite symmetry or antisymmetry. We have deliberately chosen the same notations for the transformed variables (31) and (33) since they actually form pairs that are conjugated in the antibracket.

The Lagrangian BRST differential admits a canonical action in a structure named antibracket and defined by decreeing the fields/ghosts conjugated with the corresponding antifields, \( s^* = (\cdot, S) \), where \( (\cdot) \) signifies the antibracket and \( S \) denotes the canonical generator of the BRST symmetry. It is a bosonic functional of ghost number zero (involving both field/ghost and antifield spectra) that obeys the master equation

\[ (S, S) = 0. \quad (35) \]

The master equation is equivalent with the second-order nilpotency of \( s \), where its solution \( S \) encodes the entire gauge structure of the associated theory. Taking into account the formulas (19)–(30) as well as the standard
actions of $\delta$ and $\gamma$ in canonical form we find that the complete solution to
the master equation for the free model under study is given by

$$
S = S_0 \left[ t_{\lambda \mu \nu | \alpha}, A_\mu \right] + \int d^D x \left[ t^{** \lambda \mu \nu | \alpha} \left( 3 \partial_\alpha \eta_{\lambda \mu \nu} + \partial_{[\lambda} \eta_{\mu \nu | \alpha} + \partial_{[\lambda} G_{\mu \nu | \alpha} \right) \\
- \frac{1}{2} \eta^{* \lambda \mu \nu} \partial_{[\lambda} C_{\mu \nu]} + G^{* \mu \nu | \alpha} \left( 2 \partial_\alpha C_{\mu \nu} - \partial_{[\mu} C_{\nu | \alpha} + \partial_{\mu} C_{\nu | \alpha} \right) \\
+ C^{* \mu \nu \partial_{[\mu} C_{\nu ]} - 3 C^{* \nu | \alpha} \partial_{(\nu} C_{\alpha)} + A^{** \mu \partial_\mu} \right] ,
$$

such that it contains pieces with the antighost number ranging from zero to three.

3 Brief review of the deformation procedure

There are three main types of consistent interactions that can be added to
a given gauge theory: The first type deforms only the Lagrangian action,
but not its gauge transformations. The second kind modifies both the action
and its transformations, but not the gauge algebra. The third, and certainly
most interesting category, changes everything, namely, the action, its gauge
symmetries, and the accompanying algebra.

The reformulation of the problem of consistent deformations of a given ac-
tion and of its gauge symmetries in the antifield-BRST setting is based on the
observation that if a deformation of the classical theory can be consistently
constructed, then the solution to the master equation for the initial theory
can be deformed into the solution of the master equation for the interacting
theory

$$
\bar{S} = S + g S_1 + g^2 S_2 + O \left( g^3 \right) , \quad \varepsilon \left( \bar{S} \right) = 0, \quad gh \left( \bar{S} \right) = 0,
$$

such that

$$
\left( \bar{S}, \bar{S} \right) = 0.
$$

Here and in the sequel $\varepsilon \left( F \right)$ denotes the Grassmann parity of $F$. The pro-
jection of (37) on the various powers of the coupling constant induces the
following tower of equations:

\[ g^0 : (S, S) = 0, \]  
\[ g^1 : (S_1, S) = 0, \]  
\[ g^2 : \frac{1}{2} (S_1, S_1) + (S_2, S) = 0, \]  
\[ g^3 : (S_1, S_2) + (S_3, S) = 0, \]  
\[ g^4 : \frac{1}{2} (S_2, S_2) + (S_1, S_3) + (S_4, S) = 0, \]

(39) \hspace{2cm} (40) \hspace{2cm} (41) \hspace{2cm} (42) \hspace{2cm} (43)

The first equation is satisfied by hypothesis. The second equation governs the first-order deformation of the solution to the master equation \((S_1)\) and it shows that \(S_1\) is a BRST co-cycle, \(sS_1 = 0\). This means that \(S_1\) pertains to the ghost number zero cohomological space of \(s\), \(H^0(s)\), which is generically non-empty because it is isomorphic to the space of physical observables of the free theory. The remaining equations are responsible for the higher-order deformations of the solution to the master equation. No obstructions arise in finding solutions to them as long as no further restrictions, such as spatiotemporal locality, are imposed. Obviously, only nontrivial first-order deformations should be considered, since trivial ones \((S_1 = sB)\) lead to trivial deformations of the initial theory and can be eliminated by convenient redefinitions of the fields. Ignoring the trivial deformations, it follows that \(S_1\) is a nontrivial BRST-observable, \(S_1 \in H^0(s)\). Once that the deformation equations (40)–(43), etc., have been solved by means of specific cohomological techniques, from the consistent nontrivial deformed solution to the master equation one can extract all the information on the gauge structure of the resulting interacting theory.

## 4 Main results

The aim of this paper is to investigate the consistent interactions that can be added to the action \((1)\) without modifying either the field/ghost/antifield spectrum or the number of independent gauge symmetries. This matter is addressed in the context of the antifield-BRST deformation procedure described in the above and relies on computing the solutions to the Eqs. (40)–(43), etc., from the cohomology of the BRST differential. For obvious reasons,
we consider only smooth, local, and manifestly covariant deformations and, meanwhile, restrict to Poincaré-invariant quantities, i.e. we do not allow explicit dependence on the spatiotemporal coordinates. The smoothness of deformations refers to the fact that the deformed solution to the master equation, (37), is smooth in the coupling constant $g$ and reduces to the original solution (36) in the free limit ($g = 0$). Moreover, we ask that the deformed gauge theory preserves the Cauchy order of the uncoupled model, which enforces the requirement that the interacting Lagrangian is of maximum order equal to two in the spatiotemporal derivatives of the fields at each order in the coupling constant. Here, we present the main result without insisting on the cohomology tools required by the technique of consistent deformations. All cohomological proofs that lead to the main result are assembled in two appendices. The first one deals with the construction of the general form of the first-order deformation of the solution to the classical master equation and the second investigates the higher-order deformations. As it is shown in the end of Appendix B there appear two distinct solutions to (38) that exclude each other.

The first type of deformed solution to the master equation (38) that is consistent to all orders in the coupling constant stops at order one in the coupling constant and reads as

$$\bar{S} = S + \frac{g}{3 \cdot 4!} \int d^5 x \varepsilon^{\lambda \mu \nu \rho \kappa} F_{\lambda \mu} F_{\nu \rho} A_\kappa,$$

where $S$ is given in (36) in $D = 5$. This case is not interesting since it provides no cross-couplings between the vector field and the tensor field with the mixed-symmetry $(3, 1)$. It simply restricts the free Lagrangian action (11) to evolve on a five-dimensional space-time and adds to it the second term on the right-hand side of (44), without changing the original gauge transformations (3)–(4) and, in consequence, neither the original Abelian gauge algebra nor the reducibility structure.

The second type of full deformed solution to the master equation (38) ends at order two in the coupling constant and is given by

$$S = S + g \int d^5 x \varepsilon^{\lambda \mu \nu \rho \kappa} \left( A_\lambda^* F_{\mu \nu \rho \kappa} - \frac{2}{3} F_{\lambda \mu} \partial_\nu t_{\nu \rho \kappa}^\theta \sigma^\theta \xi \right)$$

$$+ \frac{16 g^2}{3} \int d^5 x \left( \partial_\xi t_{\nu \rho \kappa}^\theta \sigma^\theta \xi \right) \partial^{[\xi \nu \rho \kappa]} \partial^{\theta \xi \theta \xi'},$$

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We observe that this solution ‘lives’ also in a five-dimensional space-time, just like the previous one. From (45) we read all the information on the gauge structure of the coupled theory. The terms of antighost number zero in (45) provide the Lagrangian action. They can be equivalently organized as

\[ \tilde{S}_0^0 \left[ t_{\lambda\mu|\alpha}, A_\mu \right] = S_0^0 \left[ t_{\lambda\mu|\alpha} \right] - \frac{1}{4} \int d^5x \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}, \]  

(46)
in terms of the deformed field strength

\[ \tilde{F}^{\mu\nu} = F^{\mu\nu} + \frac{4g}{3} \epsilon^{\mu\nu\alpha\beta\gamma} \partial^{[\rho} t_{\alpha\beta\gamma]|\rho]} \]  

(47)

where \( S_0^0 \left[ t_{\lambda\mu|\alpha} \right] \) is the Lagrangian action of the massless tensor field \( t_{\lambda\mu|\alpha} \) appearing in (1) in \( D = 5 \). We observe that the action (46) contains only mixing-component terms of order one and two in the coupling constant. The piece of antighost number one appearing in (45) gives the deformed gauge transformations in the form

\[ \tilde{\delta}_{\epsilon,\chi} t_{\lambda\mu|\alpha} = -3\partial_{[\lambda} \epsilon_{\mu\nu|\alpha]} + 4\partial_{[\lambda} \epsilon_{\mu\nu|\alpha]} + \partial_{[\lambda} \chi_{\mu\nu]|\alpha], \]  

(48)

\[ \tilde{\delta}_{\epsilon,\chi} A_\mu = \partial_\mu \epsilon + 4g \epsilon_{\mu\alpha\beta\gamma} \delta^\rho \epsilon^{\beta\gamma\delta}. \]  

(49)

It is interesting to note that only the gauge transformations of the vector field are modified during the deformation process. This is enforced at order one in the coupling constant by a term linear in the antisymmetrized first-order derivatives of some gauge parameters from the (3, 1) sector. At antighost numbers strictly greater than one (45) coincides with the solution (36) corresponding to the free theory. Consequently, the gauge algebra and the reducibility structure of the coupled model are not modified during the deformation procedure, being the same like in the case of the starting free action (1) with the gauge transformations (3)–(4). It is easy to see from (46) and (48)–(49) that if we impose the PT-invariance at the level of the coupled model, then we obtain no interactions (we must set \( g = 0 \) in these formulas).

It is important to stress that the problem of obtaining consistent interactions strongly depends on the spatiotemporal dimension. For instance, if one starts with action (1) in \( D > 5 \), then one inexorably gets \( \tilde{S} = S \), so no term can be added to either the original Lagrangian or its gauge transformations.
5 Conclusion

In this paper we have discussed a cohomological approach to the problem of constructing consistent interactions between a single massless tensor field $t_{\lambda\mu\nu|\alpha}$ with the mixed symmetry $(3, 1)$ and one vector field. Under the general assumptions of smoothness of the deformations in the coupling constant, locality, (background) Lorentz invariance, Poincaré invariance, and preservation of the number of derivatives on each field, we have exhausted all the consistent, nontrivial couplings. Our final result is rather surprising since it enables nontrivial cross-couplings between these types of fields in five dimensions and also allows the deformation of the gauge transformations of the vector field. Although the cross-couplings break the PT invariance and are merely mixing-component terms, still this is the first situation encountered so far where there exist nontrivial deformations involving mixed-symmetry tensor fields complying with the hypothesis on the preservation of the number of derivatives on each field. If we relax this condition, it is possible that other interactions become consistent as well.

Acknowledgment

The authors are partially supported by the European Commission FP6 program MRTN-CT-2004-005104 and by the type A grant 304/2004 with the Romanian National Council for Academic Scientific Research and the Romanian Ministry of Education and Research.

A First-order deformation

Here, we determine the most general form of the first-order deformation of the solution to the master equation that complies with all the hypotheses exposed at the beginning of Sec. 4 (smoothness in the coupling constant, locality, Lorentz-covariance, Poincaré invariance, and preservation of the Cauchy order of the uncoupled model). In view of this, we initially compute the main cohomological ingredients necessary at the construction of the local cohomology of the BRST differential at ghost number zero. If we make the notation $S_1 = \int d^D x a$, with $a$ a local function, then the local form of the Eq. (40), which we have seen that controls the first-order deformation of the solution...
to the master equation, becomes

\[ sa = \partial_\mu m^\mu, \quad \text{gh}(a) = 0, \quad \varepsilon(a) = 0, \quad (50) \]

for some local \( m^\mu \), and it shows that the non-integrated density of the first-order deformation pertains to the local cohomology of \( s \) at ghost number zero, \( a \in H^0(s|d) \), where \( d \) denotes the exterior differential in space-time. In order to analyze the above equation, we develop \( a \) according to the antighost number

\[ a = \sum_{k=0}^{I} a_k, \quad \text{agh}(a_k) = k, \quad \text{gh}(a_k) = 0, \quad \varepsilon(a_k) = 0, \quad (51) \]

and assume, without loss of generality, that the above decomposition stops at some finite value of the antighost number, \( I \). By taking into account the splitting (18) of the BRST differential, the Eq. (50) is equivalent to a tower of local equations, corresponding to the different decreasing values of the antighost number

\[ \gamma a_I = \partial_\mu m^\mu, \quad (52) \]
\[ \delta a_I + \gamma a_{I-1} = \partial_\mu m^{(I-1)}_\mu, \quad (53) \]
\[ \delta a_k + \gamma a_{k-1} = \partial_\mu m^{(k-1)}_\mu, \quad I-1 \geq k \geq 1, \quad (54) \]

where \( \left( \frac{(k)_\mu}{m} \right)_{k=0,I} \) are some local currents with \( \text{agh}\left( \frac{(k)_\mu}{m} \right) = k \). It can be proved that we can replace the Eq. (52) at strictly positive antighost numbers with

\[ \gamma a_I = 0, \quad \text{agh}(a_I) = I > 0. \quad (55) \]

The proof can be done like in the Appendix A, Corollary 1 from [31], with the precaution to include in an appropriate manner the vector field sector. In conclusion, under the assumption that \( I > 0 \), the representative of highest antighost number from the non-integrated density of the first-order deformation can always be taken to be \( \gamma \)-closed, such that the Eq. (50) associated with the local form of the first-order deformation is completely equivalent to the tower of equations (53)–(54) and (55).

Before proceeding to the analysis of the solutions to the first-order deformation equation, we briefly comment on the uniqueness and triviality of
such solutions. Due to the second-order nilpotency of $\gamma (\gamma^2 = 0)$, the solution to the top equation, (55), is clearly unique up to $\gamma$-exact contributions, $a_I \rightarrow a_I + \gamma b_I$. Meanwhile, if it turns out that $a_I$ reduces to $\gamma$-exact terms only, $a_I = \gamma b_I$, then it can be made to vanish, $a_I = 0$. In other words, the nontriviality of the first-order deformation $a$ is translated at its highest antighost number component into the requirement that $a_I \in H^I (\gamma)$, where $H^I (\gamma)$ denotes the cohomology of the exterior longitudinal derivative $\gamma$ at pure ghost number equal to $I$. At the same time, the general condition on the non-integrated density of the first-order deformation to be in a nontrivial cohomological class of $H^0 (s|d)$ shows on the one hand that the solution to (50) is unique up to $s$-exact pieces plus divergences and on the other hand that if the general solution to (50) is found to be completely trivial, $a = sb + \partial_\mu n^\mu$, then it can be made to vanish, $a = 0$.

A.1 Basic cohomologies

In the light of the above discussion, we pass now to the investigation of the solutions to the Eqs. (55) and (53)–(54). We have seen that the solution to (55) belongs to the cohomology of the exterior longitudinal derivative, such that we need to compute $H (\gamma)$ in order to construct the component of highest antighost number from the first-order deformation. This matter is solved with the help of the definitions (19)–(24). As it has been shown in [31], the most general, nontrivial representative from $H (\gamma)$ in the $t$-sector is written like

$$a_I^t = \alpha_I \left( \left[ \Pi^{(t)\Delta} \right], \left[ K_{\lambda \mu \nu |\alpha \beta} \right] \right) \omega^{(t)I} (F_{\lambda \mu \nu \alpha}, C_\nu), \quad I > 0,$$

where we employed the notation

$$F_{\lambda \mu \nu \alpha} \equiv \partial \left( \lambda \eta_{\mu \nu \alpha} \right).$$

In the above $\Pi^{(t)\Delta}$ denote the antifields from the $t$-sector, the notation $f ([q])$ means that the function $f$ depends on the variable $q$ and its subsequent derivatives up to a finite number, and $\omega^{(t)I}$ are the elements of pure ghost number $I$ (and obviously of antighost number zero) of a basis in the space of polynomials in $F_{\lambda \mu \nu \alpha}$ and $C_\nu$, which is finite-dimensional since all these variables anticommute. (The spatiotemporal derivatives of $F_{\lambda \mu \nu \alpha}$ and $C_\nu$ have been shown in [31] to be $\gamma$-exact and the same is valid with respect to the derivatives of the ghosts for ghosts for ghosts $C_\nu$. Regarding the ghosts
for ghosts $C_{\mu\nu}$ and $C_{\nu\alpha}$, in the same paper we have proved that there is no linear combination of these undifferentiated ghosts in $H(\gamma)$ and all the elements from $H(\gamma)$ involving their derivatives are $\gamma$-exact.) At the level of the vector theory, all the corresponding antifields, the Abelian field strength $\mathcal{F}$, and their spatiotemporal derivatives are nontrivial objects from $H^0(\gamma)$, while the non-differentiated ghost $\eta$ is the sole nontrivial element from $H(\gamma)$ at strictly positive values of the pure ghost number. (Its spatiotemporal derivatives of any order are $\gamma$-exact, according to the second definition in (19).) Combining (56) with the above argument, we can state that the most general, nontrivial representative from $H(\gamma)$ for the overall theory (11) reads as

$$a_I = \alpha_I \left( [\Pi^{*\Delta} \right], [K_{\lambda\mu\nu\xi\alpha\beta}], [F_{\mu\nu}] \right) \omega^I \left( \mathcal{F}_{\lambda\mu\nu\alpha}, \eta, C_{\nu} \right), \quad I > 0, \quad (58)$$

where here $\Pi^{*\Delta}$ denote all the antifields and $\omega^I$ are now the elements of pure ghost number $I$ (and obviously of antighost number zero) of a basis in the space of polynomials in $\mathcal{F}_{\lambda\mu\nu\alpha}, \eta$, and $C_{\nu}$ (which is again finite-dimensional). The objects $\alpha_I$ (obviously nontrivial in $H^0(\gamma)$) were taken to have a bounded number of derivatives, and therefore they are polynomials in the antifields $\Pi^{*\Delta}$, in the curvature tensor $K_{\lambda\mu\nu\xi\alpha\beta}$, in the field strength $F_{\mu\nu}$, and in their derivatives. They are required to fulfill the property $\text{agh} (\alpha_I) = I$ in order to ensure that the ghost number of $a_I$ is equal to zero. Due to their $\gamma$-closeness, $\alpha_I$ will be called “invariant polynomials”. At zero antighost number, the invariant polynomials are polynomials in the curvature $K_{\lambda\mu\nu\xi\alpha\beta}$, in the field strength $F_{\mu\nu}$, and in their derivatives.

Replacing the solution (58) into the Eq. (53) and taking into account the definitions (25)–(30), we remark that a necessary (but not sufficient) condition for the existence of (nontrivial) solutions $a_{I-1}$ is that the invariant polynomials $\alpha_I$ are (nontrivial) objects from the local cohomology of the Koszul-Tate differential $H(\delta[d]$) at antighost number $I > 0$ and pure ghost number equal to zero, i.e.,

$$\delta \alpha_I = \partial_{j}^{(I-1)^{\mu}} j, \quad \text{agh} \left( j^{(I-1)^{\mu}} \right) = I - 1 \geq 0, \quad \text{pgh} \left( j^{(I-1)^{\mu}} \right) = 0. \quad (59)$$

The above notation is generic, in the sense that $\alpha_I$ and $j^{(I-1)^{\mu}}$ may actually

---

1We recall that the local cohomology $H(\delta[d]$ is completely trivial at both strictly positive antighost and pure ghost numbers (for instance, see [32], Theorem 5.4, [33], and [34]).
carry supplementary Lorentz indices. Consequently, we need to investigate some of the main properties of the local cohomology of the Koszul-Tate differential \( H(\delta|d) \) at pure ghost number zero and strictly positive antighost numbers in order to fully determine the component \( a_I \) of highest antighost number from the first-order deformation. As the free model under study is a linear gauge theory of Cauchy order equal to four, the general results from [32, 33] (also see [15, 35, 36]) ensure that \( H(\delta|d) \) (at pure ghost number zero) is trivial at antighost numbers strictly greater than its Cauchy order

\[
H_I(\delta|d) = 0, \; I > 4. \tag{60}
\]

Moreover, if the invariant polynomial \( \alpha_I \), with \( \text{agh} (\alpha_I) = I \geq 4 \), is trivial in \( H_I(\delta|d) \), then it can be taken to be trivial also in \( H^{\text{inv}}_I(\delta|d) \)

\[
\left( \alpha_I = \delta b_{I+1} + \partial_\mu \gamma_\mu^{(I)} c^\nu, \; \text{agh} (\alpha_I) = I \geq 4 \right) \Rightarrow \alpha_I = \delta \beta_{I+1} + \partial_\mu \gamma_\mu^{(I)} c^\nu, \tag{61}
\]

with \( \beta_{I+1} \) and \( \gamma_\mu^{(I)} \) invariant polynomials. [An element of \( H^{\text{inv}}_I(\delta|d) \) is defined via an equation similar to (59), but with the corresponding current also an invariant polynomial.] The result (61) can be proved like in the Appendix B, Theorem 3, from [31], up to the observation that the vector field sector must also be considered. This is important since together with (60) ensures that the entire local cohomology of the Koszul-Tate differential in the space of invariant polynomials (characteristic cohomology) is trivial in antighost number strictly greater than four

\[
H^{\text{inv}}_I(\delta|d) = 0, \; I > 4. \tag{62}
\]

Looking at the definitions (32) involving the transformed antifields (31) and taking into account the formulas (25)–(28) with respect to the vector field theory, we can organize the nontrivial representatives of \( H_I(\delta|d) \) (at pure ghost number equal to zero) and of \( \left( H^{\text{inv}}_I(\delta|d) \right)_{I \geq 2} \) in the following array

\[
\begin{array}{c|c}
\text{agh} & \text{nontrivial representatives spanning } H_I(\delta|d) \text{ and } H^{\text{inv}}_I(\delta|d) \\
\hline
I > 4 & \text{none} \\
I = 4 & C^{\nu} \\
I = 3 & C^{\nu\mu\alpha} \\
I = 2 & \mathcal{J}^{\mu\nu\rho\sigma}, \eta^* \\
\end{array} \tag{63}
\]
We remark that in $\left(H_1^I (\delta|d)\right)_{I \geq 2}$ or $\left(H^{inv}_1 (\delta|d)\right)_{I \geq 2}$ there is no nontrivial element that effectively involves the curvature tensor $K_{\lambda\mu\xi|\alpha\beta}$, the field strength $F_{\mu\nu}$, and/or their derivatives, and the same stands for the quantities that are more than linear in the antifields and/or depend on their derivatives. It is also important to note that the vector field sector brings no contribution to $\left(H_1^I (\delta|d)\right)_{I \geq 2}$. In contrast to the groups $\left(H_1^I (\delta|d)\right)_{I \geq 2}$ and $\left(H^{inv}_1 (\delta|d)\right)_{I \geq 2}$, which are finite-dimensional, the cohomology $H_1^1 (\delta|d)$ at pure ghost number zero, that is related to global symmetries and ordinary conservation laws, is infinite-dimensional since the theory is free.

The previous results on $H (\delta|d)$ and $H^{inv} (\delta|d)$ at strictly positive antighost numbers are important because they control the obstructions to removing the antifields from the first-order deformation. Indeed, due to (62), it follows that we can successively eliminate all the pieces with $I > 4$ from the non-integrated density of the first-order deformation by adding only trivial terms (the proof is similar to that from the Appendix C in [31] modulo the inclusion of the vector field sector), so we can take, without loss of nontrivial objects, the condition $I \leq 4$ in the decomposition (51). The last representative is of the form (58), where the invariant polynomial is necessarily a nontrivial object from $H^{inv}_I (\delta|d)$ for $I = 2, 3, 4$ and respectively from $H_1 (\delta|d)$ for $I = 1$.

A.2 Computation of first-order deformations

Now, we have at hand all the necessary ingredients for computing the general form of the first-order deformation of the solution to the master equation as solution to the local equation (50). In view of this, we decompose the first-order deformation like

$$ a = a^t + a^A + a^{t-A}, $$

where $a^t$ denotes the part responsible for the self-interactions of the field $t_{\lambda\mu|\alpha}$, $a^A$ is related to the deformations of the vector field, and $a^{t-A}$ signifies the component that describes only the cross-couplings between $t_{\lambda\mu|\alpha}$ and the vector field. Obviously, the Eq. (50) becomes equivalent with three equations, one for each component

$$ sa^t = \partial_\mu m^\mu_t, \quad sa^A = \partial_\mu m^\mu_A, \quad sa^{t-A} = \partial_\mu m^\mu_{t-A}. $$

The solutions to the first equation have been studied in [31] and were proved to be trivial

$$ a^t = 0. $$
The solutions to the second equation have been investigated in [37, 38]. If we select among them only the nontrivial solutions in $D \geq 5$ space-time dimensions containing at most two derivatives of the vector field, then we are left with one candidate living precisely in $D = 5$

$$a^A = \frac{c}{3 \cdot 4!} \delta_{D5} \varepsilon^{\lambda \mu \nu \rho \sigma} F_{\lambda \mu} F_{\nu \rho} A_{\kappa},$$  \hspace{1cm} (67)$$

with $c$ an arbitrary, real constant.

In order to solve the third equation in (65) we decompose $a^{t-A}$ along the antighost number like in (51) and stop at $I = 4$

$$a^{t-A} = a^{t-A}_0 + a^{t-A}_1 + a^{t-A}_2 + a^{t-A}_3 + a^{t-A}_4,$$  \hspace{1cm} (68)$$

where $a^{t-A}_4$ can be taken as solution to the equation $\gamma a^{t-A}_4 = 0$, and therefore it is of the form (58) for $I = 4$, with $a^{t-A}_4$ an invariant polynomial from $H_{\text{inv}}^4(\delta | d)$. Because $H_{\text{inv}}^4(\delta | d)$ is spanned by $C^{*\nu}$ (see (63)) and $a^{t-A}_4$ must yield cross-couplings between $t_{\lambda \mu \nu | \alpha}$ and $A_\mu$ with maximum two spatiotemporal derivatives, it follows that the eligible basis elements at pure ghost number equal to four remain $\omega^4_\mu (F_{\lambda \mu \nu | \alpha}, \eta, C_\nu) = C_\mu \eta$, so we have (up to trivial, $\gamma$-exact contributions) that

$$a^{t-A}_4 = c_1 C^{*\mu} C_\mu \eta.$$  \hspace{1cm} (69)$$

Replacing $a^{t-A}_4$ into an equation similar to (53) for $I = 4$ and computing $\delta a^{t-A}_4$, it follows that

$$a^{t-A}_3 = c_1 C^{*\mu} (C_\mu \eta + 6 A_\mu C_\nu).$$  \hspace{1cm} (70)$$

In the right-hand side of (70) one should add $a^{t-A}_3$ as solution to the `homogeneous' equation $\gamma a^{t-A}_3 = 0$. In this particular case one has $a^{t-A}_3 = 0$ because there are no basis elements $\omega^3(F_{\lambda \mu \nu | \alpha}, \eta, C_\nu)$ enabling cross-couplings with at most two derivatives in the corresponding $a^{t-A}_0$. Indeed, such elements must include precisely one fermionic ghost $\eta$ since $H_{\text{inv}}^3(\delta | d)$ is spanned only by antifields from the $t$-sector, and therefore they are forced to be quadratic in the combination $F_{\lambda \mu \nu \alpha}$ defined in (57). A simple estimation shows that even if consistent, the associated $a^{t-A}_0$ would contain three derivatives of the fields, which is unacceptable. Acting with $\delta$ on (70) we arrive at

$$\delta a^{t-A}_3 = \gamma \left[ -c_1 G^{t^{*\mu \nu} | \alpha} (G'_{\mu \nu | \alpha} \eta - A_{[\mu} C'_{\nu | \alpha]} \right] + 6 c_1 G^{t^{*\mu \nu} | \alpha} F_{\mu \nu} C_\alpha + \partial_\mu (2)_{t-A}.$$  \hspace{1cm} (71)$$
Comparing (71) with an equation of the form (54) for $k = 3$, we infer that $a^{t-A}t - A^2$ exists if and only if the second term in the right-hand side of (71) can be written in a $\gamma$-exact modulo $d$ form. However, this is impossible since this term is a nontrivial element from $H^3(\gamma)$ that cannot be represented in a divergence-like form. Consequently, we must set $c_1 = 0$ in (69)–(70), so $a^{t-A}$ can stop earliest at antighost number two.

In this case we have that

$$a^{t-A} = a^{t-A}_0 + a^{t-A}_1 + a^{t-A}_2.$$  

(72)

Here, $a^{t-A}_2$ is solution to the equation $\gamma a^{t-A}_2 = 0$, and thus of the type (58) for $I = 2$, with $\omega^2$ an invariant polynomial from $H_{inv}^2(\delta|d)$. The basis elements $\omega^2$ at pure ghost number two are in this case spanned by $F^\lambda\mu\nu\alpha F^\lambda\mu\nu\alpha$ and $F^\lambda\mu\nu\alpha\eta$. The former is forbidden by the hypothesis on the maximum number of derivatives in $a^{t-A}_0$ being equal to two (if consistent, then it would produce an $a^{t-A}_0$ with three derivatives), so only the latter is allowed. Inspecting next the formula (63) at $I = 2$, we remark that although this is the first place where an antifield from the vector sector may appear, it is impossible to construct a Lorentz scalar in $D \geq 5$ dimensions by ‘gluing’ $\eta$ to $F^\lambda\mu\nu\alpha\eta$. In this way we remain with two possible pieces in $a^{t-A}_2$

$$a^{t-A}_2 = (\delta_Dc_2\varepsilon_{\lambda\mu\alpha\beta\gamma}G^{t*\lambda\mu\nu} + \delta_Dc_3\varepsilon_{\lambda\alpha\beta\gamma}G^{t*\lambda\mu\nu}\sigma_{\mu\nu}) F^{\alpha\beta\gamma\delta} \eta,$$  

(73)

where $\delta_D$ is the Kronecker symbol. Substituting the above $a^{t-A}_2$ into an equation similar to (53) for $I = 2$ and computing $\delta a^{t-A}_2$, it results that

$$a^{t-A}_1 = \left(\delta_Dc_2\varepsilon_{\lambda\mu\alpha\beta\gamma}t^{\lambda\mu\nu\theta} + \delta_Dc_3\varepsilon_{\lambda\alpha\beta\gamma}t^{\lambda\mu\nu\sigma}\right) \left[(\delta^{\alpha}t^{\beta\gamma\delta}\theta)\sigma_{\theta\rho}\eta - 3A^\rho F^{\alpha\beta\gamma\delta}\right] + \bar{a}^{t-A}_1,$$  

(74)

where $\bar{a}^{t-A}_1$ is the general solution to the ‘homogeneous’ equation $\gamma \bar{a}^{t-A}_1 = 0$. The solution $\bar{a}^{t-A}_1$ requires special attention since although it is still of the form (58) with $I = 1$, the corresponding invariant polynomial $\alpha_1$ is no longer restricted to belong to $H_{inv}^1(\delta|d)$; it pertains to the larger, infinite-dimensional space $H_1(\delta|d)$. However, if we select from (58) with $I = 1$ the representatives that comply with all the requirements, like the restrictions on the dimension of the space-time, on the maximum number of derivatives in the corresponding $a^{t-A}_0$, and on the interaction vertices to generate cross-couplings between the two types of fields (and not self-interactions), then we
obtain just two independent contributions
\[ \tilde{a}_{1-A} = c_4 t^\alpha F_{\lambda\mu} \eta + \delta D_5 c_5 \varepsilon_{\lambda\alpha\beta\gamma\delta} A^{\alpha\beta} f_{\lambda\mu\eta}, \] (75)
where $F_{\lambda\mu}$ is the field strength of the vector field, $[2]$. Applying $\delta$ on (74) and taking into account (75) we arrive at
\[ \delta \tilde{a}_{1-A} = -\gamma \left( \frac{2c_5}{3} \varepsilon_{\mu\nu\alpha\beta\gamma} F^{\mu\nu} \partial_{[\rho t} A_\alpha]^{\beta] \| \theta} \sigma_{\theta\rho} + \tilde{a}_{1-A} \right) + \partial_{\mu} \tilde{z}_{1-A} \]
\[ + \left( \delta D_7 c_2 \varepsilon_{\lambda\mu\alpha\beta\gamma\delta} T^{\lambda\mu\| \theta} + \delta D_5 c_3 \varepsilon_{\lambda\alpha\beta\gamma\delta} T_{\theta\lambda} \right) \left[ \left( \partial_{[\alpha t} A_{\beta] \| \rho} \right) \sigma_{\theta\rho} \eta ight] 
- 3 A_{\theta} F^{\alpha\beta\gamma\delta} \right] + c_4 T^{\lambda\mu} F_{\lambda\mu}. \] (76)

Comparing (76) with an equation similar to (54) for $k=1$ it is now clear that $a_{1-A}^t$ exists if and only if the last terms in the right-hand side of (76) can be written in a $\gamma$-exact modulo $d$ form. Moreover, the terms proportional with $c_2$, $c_3$, and respectively $c_4$ must individually satisfy a $\gamma$-exactness modulo $d$ condition. This is because the first two kinds of terms involve Levi-Civita symbols in different dimensions, the third type contains no such symbol, and the definitions (19)–(24) of $\gamma$ acting on the fields/ghosts are also free of Levi-Civita symbols. This means that the equations
\[ \delta D_7 c_2 \varepsilon_{\lambda\mu\alpha\beta\gamma\delta} T^{\lambda\mu\| \theta} \left[ \left( \partial_{[\alpha t} A_{\beta] \| \rho} \right) \sigma_{\theta\rho} \eta \right] - 3 A_{\theta} F^{\alpha\beta\gamma\delta} \] =
\[ \gamma a_{0c_2} - \partial_{\mu} m_{t-Ac_2}, \] (77)
\[ \delta D_5 c_3 \varepsilon_{\lambda\alpha\beta\gamma\delta} T_{\theta\lambda} \left[ \left( \partial_{[\alpha t} A_{\beta] \| \rho} \right) \sigma_{\theta\rho} \eta \right] - 3 A_{\theta} F^{\alpha\beta\gamma\delta} \] =
\[ \gamma a_{0c_3} - \partial_{\mu} m_{t-Ac_3}, \] (78)
\[ c_4 T^{\lambda\mu} F_{\lambda\mu} \eta = \gamma a_{0c_4} - \partial_{\mu} m_{t-Ac_4}, \] (79)
are necessary and sufficient in order to ensure the existence of $a_{1-A}^t$. However, it can be shown that neither of the quantities from the left-hand sides of (77)–(79) can be set in a $\gamma$-exact modulo $d$ form. The proofs are given below.

We assume that $a_{1-A}^t$ as solution to the Eq. (77) exists. Then, from the left-hand of this equation combined with the formula (19) and the second relation from (20) it follows that it can be represented as a sum of terms, each of them being linear in the vector field, quadratic in $t_{\lambda\mu\|\alpha}$, and containing exactly two spatiotemporal derivatives. Up to irrelevant, total divergences, we can always move the derivatives such as to act only on the $t$-fields
\[ a_{0c_2}^{t-A} = c_2 \delta D_7 A_{\beta} f_{\lambda\mu\nu} (\partial t \partial t, t \partial t \partial t), \] (80)
where \( f_{\text{lin}}^{\mu} \) is linear in its arguments and contains one seven-dimensional Levi-Civita symbol. Acting with \( \gamma \) on (80), we deduce

\[
\gamma a_{0c_2}^{t-A} = -c_2 \delta_{D7} \left[ \partial_\mu f_{\text{lin}}^{\mu} (\partial t \partial t, t \partial \partial t) \right] \eta + c_2 \delta_{D7} A_\mu \gamma f_{\text{lin}}^{\mu} (\partial t \partial t, t \partial \partial t) \\
+ \partial_\mu \left[ c_2 \delta_{D7} f_{\text{lin}}^{\mu} (\partial t \partial t, t \partial \partial t) \eta \right].
\]

(81)

Comparing the first term from the right-hand side of (81) with the first term from the left-hand side of (77), we obtain a necessary condition for the existence of \( a_{0c_2}^{t-A} \)

\[
\varepsilon_{\lambda \mu \nu \alpha \beta \gamma \delta} T^{\lambda \mu \nu [\theta \partial_\alpha t \beta \gamma \delta]} |^\rho \sigma_{\theta \rho} = \partial_\mu N^\mu.
\]

(82)

By direct computation we have that

\[
\varepsilon_{\lambda \mu \nu \alpha \beta \gamma \delta} T^{\lambda \mu \nu [\theta \partial_\alpha t \beta \gamma \delta]} |^\rho \sigma_{\theta \rho} = -2 \varepsilon_{\lambda \mu \nu \alpha \beta \gamma \delta} T^{\lambda \mu \nu [\theta \partial_\alpha t \beta \gamma \delta]} |^\rho \sigma_{\theta \rho}
\]

\[
+ \partial_\mu \left( 4 \varepsilon_{\lambda \mu \nu \alpha \beta \gamma \delta} T^{\lambda \mu \nu [\theta \partial_\alpha t \beta \gamma \delta]} |^\rho \sigma_{\theta \rho} \right),
\]

(83)

where we used the generic notation \( k_\mu = \partial k/\partial x^\mu \). Obviously, the condition (82) does not hold since the first term in the right-hand side of (83) does not reduce to a total divergence. As a consequence, (77) is satisfied only for the trivial choice \( c_2 = 0 \). Passing to the next equation, (78), an absolutely similar argument leads to the conclusion that a necessary condition for the existence of \( a_{0c_3}^{t-A} \) is

\[
\varepsilon_{\lambda \alpha \beta \gamma \delta} T^{\theta \lambda} \partial^{[\alpha t \beta \gamma \delta]} |^\rho \sigma_{\theta \rho} = \partial_\mu N^\mu,
\]

(84)

which again is not satisfied since

\[
\varepsilon_{\lambda \alpha \beta \gamma \delta} T^{\theta \lambda} \partial^{[\alpha t \beta \gamma \delta]} |^\rho \sigma_{\theta \rho} = 2 \varepsilon_{\lambda \alpha \beta \gamma \delta} \left[ \left( \partial^\lambda t^{\alpha \xi} \right) \left( \partial_\xi \partial_\theta t^{\beta \gamma \delta | \theta} \right) \right.
\]

\[
+ \left( \partial^\lambda t^{\alpha \beta} \right) \left( \partial_\xi \partial_\theta t^{\xi | \beta \gamma \delta | \theta} \right) \right] + \partial_\mu \left\{ \varepsilon_{\lambda \alpha \beta \gamma \delta} \left[ \left( - \partial^\mu t^{\lambda \alpha} + 2 \partial^\lambda t^{\alpha \mu} \right) \right.
\]

\[
- \frac{1}{2} \partial_\xi \partial^\lambda t^{\alpha \mu | \xi} \right) \partial_\theta t^{\beta \gamma \delta | \theta} - \frac{3}{2} \left( \left( \partial^\mu t^{\lambda \alpha} \right) \partial^3 t^{\gamma \delta} - t^{\lambda \alpha} \partial_\rho \partial^3 t^{\gamma \delta} + \right.
\]

\[
\left. \sigma^{\beta \mu} t^{\lambda \alpha} \Box t^{\gamma \delta} \right) - 6 \sigma^{\beta \mu} t^{\lambda \alpha} \partial_\theta \partial^\gamma t^{\theta \delta} \right] \right\},
\]

(85)

and the first two terms in the right-hand side of (85) do not reduce to a full divergence. In conclusion, (78) takes place only for \( c_3 = 0 \). Suppose now that \( a_{0c_4}^{t-A} \) as solution to the Eq. (79) exists. The definition (19) yields that it can be represented as a sum of terms, each of them being quadratic in
the vector field, linear in $t_{\lambda\mu\nu|\alpha}$, and containing exactly two spatiotemporal derivatives. Moreover, each term may depend on $t_{\lambda\mu\nu|\alpha}$ only through gauge-invariant combinations since otherwise $\gamma a_{0c4}^{t-A}$ would imply ghosts of pure ghost number one from the $t$-sector, which is forbidden by the expression of the left-hand side of (79). But the most general, gauge-invariant quantities built out of $t_{\lambda\mu\nu|\alpha}$ with precisely two derivatives are proportional with the components of the curvature tensor (12). Consequently, we can write (up to insignificant, full divergences) that

$$a_{0c4}^{t-A} = c_4 K_{\lambda\mu\nu|\alpha\beta} A_{\theta} A_{\xi} f^{\theta\xi}_{\lambda\mu\nu|\alpha\beta},$$

(86)

where $f^{\theta\xi}_{\lambda\mu\nu|\alpha\beta}$ are some non-derivative constants, symmetric in their upper indices. Acting with $\gamma$ on (86), replacing the resulting expression in (79), recalling that $f^{\theta\xi}_{\lambda\mu\nu|\alpha\beta}$ cannot include Levi-Civita symbols (due to the Bianchi I identities for the curvature tensor), and using the relations

$$T^{\lambda\mu} = \frac{(4 - D)}{2} K^{\lambda\mu\nu|\alpha\beta} \sigma_{\rho\beta} \sigma_{\nu\alpha}, \quad \partial_{\lambda} T^{\lambda\mu} \equiv 0,$$

(87)

combined with the Bianchi II identities (13), we obtain that (79) is satisfied if and only if $c_4 = 0$.

Based on the last results (setting $c_2 = c_3 = c_4 = 0$ in (72)–(76)) we can state that $a^{t-A}$ actually stops at antighost number one

$$a^{t-A} = a_{0}^{t-A} + a_{1}^{t-A},$$

(88)

with

$$a_{1}^{t-A} = \delta_{D5} c_5 \varepsilon_{\lambda\alpha|\beta|\delta} A^{*|\lambda} \mathcal{F}^{\alpha|\beta|\delta},$$

(89)

$$a_{0}^{t-A} = 2 c_5 \frac{3}{\delta_{D5} \varepsilon_{\mu\nu\alpha|\beta|\gamma}} F^{\mu\nu|\rho\beta|\gamma|\theta} \sigma_{\theta\rho} + \bar{a}_{0}^{t-A},$$

(90)

where $\bar{a}_{0}^{t-A}$ is the general solution to the ‘homogeneous’ equation

$$\gamma \bar{a}_{0}^{t-A} = \partial_{\mu} \bar{m}_{t-A}^{(0)}.$$  

We stress that here, at antighost number zero, we cannot replace the equation $\gamma \bar{a}_{0}^{t-A} = \partial_{\mu} \bar{m}_{t-A}^{(0)}$ with the simpler one $\gamma \bar{a}_{0}^{t-A} = 0$ as we did before at strictly positive values of the antighost number. For details, see Corollary 1 from the Appendix A in [31].

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Next, we investigate the solutions to (91). There are two main types of solutions to this equation. The first type, to be denoted by \(a_0^{\mu} - A\), corresponds to \(\bar{m}_{t-A} = 0\) and is given by gauge-invariant, non-integrated densities constructed out of the original fields and their spatiotemporal derivatives, which, according to (58), are of the form \(\bar{a}_0^{\mu} = \bar{a}_0^{\mu} = \left[ K_{\lambda\mu\xi\alpha\beta}, \left[ F_{\mu\nu} \right] \right] \), up to the condition that they effectively describe cross-couplings between the two types of fields and cannot be written in a divergence-like form. Such a solution implies at least three derivatives of the fields and consequently must be forbidden by setting \(\bar{a}_0^{\mu} = 0\) since it breaks the hypothesis on the maximum derivative order.

The second kind of solutions is associated with \(\bar{m}_{t-A} \neq 0\) in (91), being understood that we discard the divergence-like quantities and maintain the condition on the maximum derivative order of the interacting Lagrangian being equal to two. In order to solve this equation we start from the requirement that \(\bar{a}_0^{\mu} - A\) may contain at most two derivatives, so it can be decomposed like

\[
\bar{a}_0^{\mu} - A = \omega_0 + \omega_1 + \omega_2,
\]

where \((\omega_i)_{i=0,1,2}\) contains \(i\) derivatives. Due to the different number of derivatives in the components \(\omega_0, \omega_1, \) and \(\omega_2\), the Eq. (92) is equivalent to three independent equations

\[
\gamma \omega_k = \partial \lambda \gamma_k^{\mu}, \quad k = 0, 1, 2.
\]

For \(k = 0\) the Eq. (93) implies the (necessary) conditions

\[
\partial \lambda \left( \frac{\partial \omega_0}{\partial t_{\lambda \mu \nu |\alpha}} \right) = 0, \quad \partial \alpha \left( \frac{\partial \omega_0}{\partial t_{\lambda \mu \nu |\alpha}} \right) = 0, \quad \partial \mu \left( \frac{\partial \omega_0}{\partial A_{\mu}} \right) = 0,
\]

whose solutions read as

\[
\frac{\partial \omega_0}{\partial t_{\lambda \mu \nu |\alpha}} = 0, \quad \frac{\partial \omega_0}{\partial A_{\mu}} = 0,
\]

so \(\omega_0\) provides no cross-couplings between \(t_{\lambda \mu \nu |\alpha}\) and \(A_{\mu}\), and therefore we can take

\[
\omega_0 = 0
\]

in (92). The solution to the more general equations (16) is of the form (17), but it does not apply to the first two equations in (94) as \(\partial \omega_0 / \partial t_{\lambda \mu \nu |\alpha}\) are
by hypothesis derivative-free. The same observation is valid with respect to the equation \( \partial_\mu M^\mu = 0 \), whose solution reads as \( M^\mu = \partial_\nu N^{\nu\mu} \), with \( N^{\nu\mu} \) antisymmetric, so it cannot enter the second solution from (95) since \( \partial_\omega / \partial A_\mu \) has no derivatives.

For \( k = 1 \) the Eq. (93) leads to the requirements

\[
\partial_\lambda \left( \frac{\delta \omega_1}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\alpha \left( \frac{\delta \omega_1}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\mu \left( \frac{\delta \omega_1}{\delta A_\mu} \right) = 0, \quad (97)
\]

where \( \delta \omega_1 / \delta t_{\lambda\mu\nu|\alpha} \) denote the Euler-Lagrange derivatives of \( \omega_1 \). Because \( \omega_1 \) is by hypothesis of order one in the spatiotemporal derivatives of the fields, the arguments presented in relation with the case \( k = 0 \) provide the solutions

\[
\frac{\delta \omega_1}{\delta t_{\lambda\mu\nu|\alpha}} = 0, \quad \frac{\delta \omega_1}{\delta A_\mu} = \partial_\nu B^{\nu\mu}, \quad (98)
\]

where the antisymmetric functions \( B^{\nu\mu} \) have no derivatives. The first solution forbids the cross-couplings between the two types of fields, allowing only the self-interactions of the vector field with precisely one derivative, so we can safely take

\[
\omega_1 = 0. \quad (99)
\]

We pass now to the Eq. (93) for \( k = 2 \), which produces the restrictions

\[
\partial_\lambda \left( \frac{\delta \omega_2}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\alpha \left( \frac{\delta \omega_2}{\delta t_{\lambda\mu\nu|\alpha}} \right) = 0, \quad \partial_\mu \left( \frac{\delta \omega_2}{\delta A_\mu} \right) = 0, \quad (100)
\]

whose solution, by virtue of the discussion made at the case \( k = 0 \), is

\[
\frac{\delta \omega_2}{\delta t_{\lambda\mu\nu|\alpha}} = \partial_\rho \partial_\beta U^{\lambda\mu\nu\rho|\alpha\beta}, \quad \frac{\delta \omega_2}{\delta A_\mu} = \partial_\nu \Phi^{\nu\mu}, \quad (101)
\]

where \( U^{\lambda\mu\nu\rho|\alpha\beta} \) has the mixed symmetry of the curvature tensor (12) and has no derivatives, while \( \Phi^{\nu\mu} \) is antisymmetric and comprises one spatiotemporal derivative of the fields. At this stage it is useful to introduce a derivation in the algebra of the fields and of their derivatives, which counts the powers of the fields and their derivatives

\[
N = \sum_{k \geq 0} \left( (\partial_{\mu_1} \cdots \partial_{\mu_k} t_{\lambda\mu\nu|\alpha}) \frac{\partial}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} t_{\lambda\mu\nu|\alpha})} \right. \\
+ (\partial_{\mu_1} \cdots \partial_{\mu_k} A_\mu) \frac{\partial}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_k} A_\mu)}, \quad (102)
\]

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so for every non-integrated density $\chi$ we have that

$$N\chi = t_{\lambda\mu\nu|\alpha} \frac{\delta\chi}{\delta t_{\lambda\mu\nu|\alpha}} + A_{\mu} \frac{\delta\chi}{\delta A_{\mu}} + \partial_{\mu} s^{\mu}. \quad (103)$$

If $\chi^{(l)}$ is a homogeneous polynomial of order $l > 0$ in the fields $\{t_{\lambda\mu\nu|\alpha}, A_{\mu}\}$ and their derivatives, then $N\chi^{(l)} = l\chi^{(l)}$. Using (101) and (103), we find that

$$N\omega_{2} = \frac{1}{8} K_{\lambda\mu\nu\rho|\alpha\beta} U^{\lambda\mu\nu\rho|\alpha\beta} - \frac{1}{2} F_{\mu\nu} \Phi^{\mu\nu} + \partial_{\mu} \tilde{v}^{\mu}. \quad (104)$$

We expand $\omega_{2}$ according to the various eigenvalues of $N$ like

$$\omega_{2} = \sum_{l>0} \omega_{2}^{(l)}, \quad (105)$$

where $N\omega_{2}^{(l)} = l\omega_{2}^{(l)}$, such that

$$N\omega_{2} = \sum_{l>0} l\omega_{2}^{(l)}. \quad (106)$$

Comparing (104) with (106), we reach the conclusion that the decomposition (105) induces a similar decomposition with respect to $U^{\lambda\mu\nu\rho|\alpha\beta}$ and $\Phi^{\mu\nu}$

$$U^{\lambda\mu\nu\rho|\alpha\beta} = \sum_{l>0} U^{\lambda\mu\nu\rho|\alpha\beta}_{(l-1)}, \quad \Phi^{\mu\nu} = \sum_{l>0} \Phi^{\mu\nu}_{(l-1)}. \quad (107)$$

Substituting (107) into (104) and comparing the resulting expression with (106), we obtain that

$$\omega_{2}^{(l)} = \frac{1}{8l} K_{\lambda\mu\nu\rho|\alpha\beta} U^{\lambda\mu\nu\rho|\alpha\beta}_{(l-1)} - \frac{1}{2l} F_{\mu\nu} \Phi^{\mu\nu}_{(l-1)} + \partial_{\mu} \tilde{v}^{\mu}_{(l)}. \quad (108)$$

Introducing (108) in (105), we arrive at

$$\omega_{2} = K_{\lambda\mu\nu\rho|\alpha\beta} \bar{U}^{\lambda\mu\nu\rho|\alpha\beta} + F_{\mu\nu} \bar{\Phi}^{\mu\nu} + \partial_{\mu} \tilde{v}^{\mu}, \quad (109)$$

where

$$\bar{U}^{\lambda\mu\nu\rho|\alpha\beta} = \sum_{l>0} \frac{1}{8l} U^{\lambda\mu\nu\rho|\alpha\beta}_{(l-1)} \quad \bar{\Phi}^{\mu\nu} = -\sum_{l>0} \frac{1}{2l} \Phi^{\mu\nu}_{(l-1)}. \quad (110)$$
Applying $\gamma$ on (109), after long and tedious computations we infer that a necessary condition for the existence of solutions to the equation $\gamma \omega_2 = \partial_\mu j_2^\mu$ is that the functions $\bar{U}^{\lambda\mu\rho|\alpha\beta}$ and $\bar{\Phi}^{\mu\nu}$ entering (109) have the expressions

$$\bar{U}^{\lambda\mu\rho|\alpha\beta} = C^{\lambda\mu\rho|\alpha\beta}, \quad \bar{\Phi}^{\mu\nu} = \bar{k}^{\mu\nu|\gamma\lambda} \partial_\rho j_2^\rho |_{\alpha\beta\gamma\lambda},$$

(111)

where $C^{\lambda\mu\rho|\alpha\beta}$ and $\bar{k}^{\mu\nu|\gamma\lambda}$ are non-derivative constants, antisymmetric in the indices followed by or between semicolons. Substituting (111) in (109) we deduce

$$\omega_2 = C^{\lambda\mu\rho|\alpha\beta} K^{\lambda\mu\rho|\alpha\beta} A_\sigma + \partial_\rho \left( F^{\mu\rho|\gamma\lambda} j_2^\rho |_{\alpha\beta\gamma\lambda} + \bar{v}^\rho \right).$$

(112)

Applying once more $\gamma$ on (112), we find that the equation $\gamma \omega_2 = \partial_\mu j_2^\mu$ holds if and only if

$$C^{\lambda\mu\rho|\alpha\beta} \partial_\sigma K^{\lambda\mu\rho|\alpha\beta} = 0.$$  

(113)

Taking into account the fact that the only vanishing combinations constructed from the first-order derivatives of the curvature tensor are the Bianchi II identities (13) and their traces, we find that the constants $C^{\lambda\mu\rho|\alpha\beta}$ must simultaneously ensure (113) and also a non-vanishing term in (112). In $D \geq 5$ dimensions there are no such constants, so we must take $C^{\lambda\mu\rho|\alpha\beta} = 0$. Eliminating the (trivial) divergence from (112), we can state that

$$\omega_2 = 0.$$  

(114)

Replacing (96), (99), and (114) in (92), we finally have that

$$\bar{a}_0^{t-A} = 0$$  

(115)

in (90).

Inserting now the results (89)–(90) and (115) in the relation (88) and the resulting expression together with (66) and (67) into the formula (64), we obtain that the most general form of the first-order deformation associated with the free theory (1) is given by

$$S_1 = \int d^5 x \varepsilon^{\lambda\mu\rho\kappa} \left[ c_5 \left( A^*_\lambda F_{\mu\rho\kappa} - \frac{2}{3} F_{\lambda\mu} \partial_{[\xi} t_{\nu\rho\kappa]|_{\theta\sigma\xi]} \right) + \frac{c}{3 \cdot 4!} F_{\lambda\mu} F_{\nu\rho} A_{\kappa} \right],$$

(116)

being parametrized by just two real (and so far arbitrary) constants.
Higher-order deformations

In the sequel we approach the higher-order deformation equations. The second-order deformation is controlled by the Eq. (41), whose solution, on behalf of the result (116), is expressed by

\[
S_2 = \int d^5 x \left[ \frac{16}{3} c^2 \left( \partial_{\xi} t_{\nu \rho \kappa} |_{\theta} \sigma^{\theta \xi} \partial_{[\xi} t_{\nu \rho \kappa]} |_{\theta} \sigma^{\theta \xi'} \right) - \frac{2}{3} c c_5 F^{\lambda \mu} \left( A^\rho \partial_{[\xi} t_{\lambda \mu \rho]} |_{\theta} \sigma^{\theta \xi} - t_{\lambda \mu}^* \eta \right) \right].
\] (117)

Using (116)–(117) in (42) we infer the third-order deformation as

\[
S_3 = \frac{4}{3} c c_5^2 \int d^5 x \varepsilon^{\lambda \mu \rho \kappa} \left[ \left( \frac{1}{3} G^{*}_{\lambda \alpha \beta \sigma} A^\alpha + t_{\alpha \lambda} A^\alpha \right) F_{\lambda \mu \rho \kappa} \right.
\]
\[
+ \left. \frac{2}{3} \left( 2 t_{\lambda \mu}^* \eta - A^\alpha \partial_{[\xi} t_{\alpha \lambda \mu]} |_{\theta} \sigma^{\theta \xi'} \partial_{[\xi} t_{\gamma \delta \eta]} |_{\theta} \sigma^{\theta \xi} \right) \right] \ .
\] (118)

Substituting the expressions (116)–(118) into the Eq. (43), we obtain the equivalent relation

\[
64 c c_5^3 \int d^5 x \left( 2 t_{\lambda \mu}^* F^{\lambda \alpha \beta \gamma} F^{\mu}_{\alpha \beta \gamma} - \partial_{[\xi} t_{\alpha \beta \mu]} |_{\theta} \sigma^{\theta \xi'} \partial_{[\xi} t_{\gamma \delta \eta]} |_{\theta} \sigma^{\theta \xi} \right) \sigma^{\mu \nu} F^{\alpha \beta \gamma \delta} + s \left( \frac{4}{9} c c_5^2 \int d^5 x \left( \frac{c}{3} A^{[\lambda} F^{\mu \nu]} A_\lambda F_{\mu \nu} - 32 c_5 t_{\lambda \mu}^* F^{\lambda \mu \nu} \right) + S_4 \right) = 0.
\] (119)

If we make the notations \( S_4 = \int d^5 x b \),

\[
64 c c_5^3 \left( 2 t_{\lambda \mu}^* F^{\lambda \alpha \beta \gamma} F^{\mu}_{\alpha \beta \gamma} - \partial_{[\xi} t_{\alpha \beta \mu]} |_{\theta} \sigma^{\theta \xi'} \partial_{[\xi} t_{\gamma \delta \eta]} |_{\theta} \sigma^{\theta \xi} \right) \sigma^{\mu \nu} F^{\alpha \beta \gamma \delta} \equiv \Lambda,
\] (120)

and

\[
\frac{4}{9} c c_5^2 \left( \frac{c}{3} A^{[\lambda} F^{\mu \nu]} A_\lambda F_{\mu \nu} - 32 c_5 t_{\lambda \mu}^* F^{\lambda \mu \nu} \right) + b \equiv \bar{b},
\] (121)

then (119) takes the local form

\[
\Lambda + \partial_{\mu} w^\mu + s \bar{b} = 0,
\] (122)

with

\[
gh (\Lambda) = 1, \quad gh (\bar{b}) = 0, \quad gh (w^\mu) = 1.
\] (123)
From (120) we see that $\Lambda$ decomposes like

$$\Lambda = \Lambda_0 + \Lambda_1,$$

with

$$\Lambda_0 = -64 cc^3 \partial_\xi t_{\alpha\beta\mu}^\sigma t_{\sigma\tau\nu}^\theta \sigma^{\theta\xi} \sigma^{\mu\nu} F^{\alpha\beta\gamma\delta},$$

$$\Lambda_1 = 128 cc^3 t_{\lambda\mu}^* F^{\lambda\alpha\beta\gamma} F^{\mu\alpha\beta\gamma}.$$  

Employing the decomposition (18) of the BRST differential and (124), it results that in (122) we can take (without loss of generality) $\tilde{b}$ and $w^\mu$ to stop at antighost number two

$$\tilde{b} = \tilde{b}_0 + \tilde{b}_1 + \tilde{b}_2, \quad \text{agh} (\tilde{b}_i) = i, \quad i = 0, 1, 2,$$

$$w^\mu = w_0^\mu + w_1^\mu + w_2^\mu, \quad \text{agh} (w_i^\mu) = i, \quad i = 0, 1, 2.$$  

By projecting the Eq. (122) on the various values of the antighost number, we infer an equivalent tower of equations

$$0 = \gamma \tilde{b}_2 + \partial_\mu w_2^\mu,$$

$$\Lambda_1 = - \left( \delta \tilde{b}_2 + \gamma \tilde{b}_1 \right) - \partial_\mu w_1^\mu,$$

$$\Lambda_0 = - \left( \delta \tilde{b}_1 + \gamma \tilde{b}_0 \right) - \partial_\mu w_0^\mu.$$  

According to the general result from Corollary 1, Appendix A in [31], we can always replace (129) with the simpler equation

$$\gamma \tilde{b}_2 = 0$$  

and take the current $w^\mu$ from (128) to stop at antighost number one, $w_2^\mu = 0$. Looking at (126), we can state that both $\tilde{b}_2$ and $\tilde{b}_1$ must contain only BRST generators from the $(3, 1)$ sector. The solution to (132) results from (56) at antighost number $I = 2$. However, there is no a priori reason to consider the corresponding invariant polynomial entering $\tilde{b}_2$ to pertain to $H_2 (\delta | d)$. On the other hand, (126) emphasizes that in order to render $\tilde{b}_2$ to contribute nontrivially to $\Lambda_1$ through the Eq. (130), it is necessary that the above mentioned invariant polynomial contains only the antifields $G^{\mu \lambda \nu | \sigma}$ (and neither products of two antifields $t^{\lambda \mu | \alpha}$ nor the curvature tensor $K_{\lambda \mu \nu | \alpha \beta}$). Due to the fact that $G^{\mu \lambda \nu | \sigma}$ belongs to $H_2^{\text{inv}} (\delta | d)$, it follows that we have reduced the problem of solving the Eq. (132) to the corresponding problem.
from the case of constructing self-interactions for the tensor $t_{\lambda \mu \nu |\alpha}$. As it has been shown in [31] under the same hypotheses like those employed here, the solution to (132) reads
\[ \bar{b}_2 = 0, \]  
so the Eq. (130) takes the form
\[ \Lambda_1 = -\gamma \bar{b}_1 - \partial_\mu w^\mu_1. \]  
The formula (126) emphasizes that $\Lambda_1$ is a nontrivial co-cycle from $H(\gamma)$, which does not reduce to a $\gamma$-exact modulo $d$ term, such that we must take its coefficient to vanish
\[ cc^3_5 = 0. \]  
So far, we have shown that the existence of $S_4$ as solution to the Eq. (119) requires the condition (135).

There appear two main cases related to the solutions of (135). If we take
\[ c_5 = 0 \]  
and leave $c$ to be an arbitrary, real constant (for definiteness, we fix this constant to unity, $c = 1$), then from (119) it follows that we can set
\[ S_4 = 0. \]  
In the meantime, from (117)–(118) we get that
\[ S_2 = 0 = S_3. \]  
The results (137) and (138) ensure that we can actually take all the other higher-order deformations to vanish,
\[ S_k = 0, \quad k > 4. \]  
In this situation (116) and (136) produce the first-order deformation like
\[ S_1 = \frac{1}{3 \cdot 4!} \int d^5 x \varepsilon^{\lambda \mu \nu \rho \kappa} F_{\lambda \mu} F_{\nu \rho} A_\kappa. \]  
The second possibility is to work with
\[ c = 0 \]
and take \( c_5 \) to be an arbitrary real constant (for definiteness, we fix this constant to unity, \( c_5 = 1 \)). Consequently, from (118)–(119) we find that

\[
S_k = 0, \quad k \geq 3.
\]

(142)

The first- and second-order deformations result from (116) and (117) where we set (141) and \( c_5 = 1 \), and take the form

\[
S_1 = \int d^5x \varepsilon^{\lambda\mu
\nu\rho\kappa} \left( A_{\lambda}^* F_{\mu\nu\rho\kappa} - \frac{2}{3} F_{\lambda\mu} \partial_{\xi} t_{\nu\rho\kappa||\sigma} \sigma^\theta \xi \right),
\]

(143)

\[
S_2 = \frac{16}{3} \int d^5x \left( \partial_{\xi} t_{\nu\rho\kappa||\sigma} \sigma^\theta \xi \right) \partial^{\xi'} t_{\nu'\rho'\kappa||\sigma'} \sigma^\xi'.
\]

(144)

Formulas (137)–(140) lead to the full deformed solution of the classical master equation as in (44). Similarly, relations (142)–(144) yield the overall deformed solution of the form (45).

References

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