TOPICS IN HIDDEN SYMMETRIES

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ABSTRACT. These three topics are an attempt to explicate some curiosities of the inverse problem of representation theory (i.e. having a set of operators to describe the "correct" algebraic object, which is represented by them) on simple examples related to the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \).

INTRODUCTION

The representation theory has, at least, two faces. The first one is related to the direct problem of the representation theory i.e. having an abstract object to describe its representations. Being in general completed for classical structures (Lie groups and Lie (super)algebras) this face has now its renaissance provoked by the discovering of a large scope of new algebraic structures in the modern quantum field theory (quantum groups, Zamolodchikov algebras, operator algebras of quantum field theory and their variations, \( W \)-algebras and their generalizations, bordism categories and trains of finite and infinite dimensional groups, homotopy Lie algebras, Batalin–Vilkovisky algebras, etc.). The second face is related to the inverse problem of the representation theory: having a set of operators to describe the "correct" algebraic object, which is represented by them. It should be marked that the most of new algebraic structures mentioned above was discovered as solutions of such inverse problem. The second face being very popular among physicists has no attracted a lot of attention of mathematicians. These topics on hidden symmetries are an attempt to explicate some curiosities of this face on simple examples related to the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \).

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TOPIC ONE: SETTING HIDDEN SYMMETRIES FREE
BY THE NONCOMMUTATIVE SEMICUBIC MAPPING

As it was marked above one of the main problems to deal with hidden symmetries is to set them free "correctly", i.e. to find a "correct" algebraic structure, which is represented by hidden symmetries. It is very convenient to consider this problem in the framework of noncommutative geometry [1]. Such approach to hidden symmetries in Verma modules over \( \mathfrak{sl}(2, \mathbb{C}) \) was adopted in [2], where hidden symmetries were set free by the noncommutative Veronese mapping. Another way to set hidden symmetries free, related to the noncommutative semicubic mapping, is described in the present topic.

**Definition 1.**

**A.** [2]. Let \( \mathfrak{g} \) be a Lie algebra and \( \mathcal{A} \) be an associative algebra such that \( \mathfrak{g} \subseteq \text{Der}(\mathcal{A}) \); a linear subspace \( V \) of \( \mathcal{A} \) is called a space of hidden symmetries iff (1) \( V \) is a \( \mathfrak{g} \)-submodule of \( \mathcal{A} \), (2) the Weyl symmetrization defines a surjection \( W : S(V) \twoheadrightarrow \mathcal{A} \) (the elements of \( V \) are called hidden symmetries with respect to \( \mathfrak{g} \)). An associative algebra \( \mathcal{F} \) such that \( \mathfrak{g} \subseteq \text{Der}(\mathcal{F}) \) is called an algebra of the set free hidden symmetries iff \( \mathcal{F} \) is generated by \( V \), (2) there exists a \( \mathfrak{g} \)-equivariant epimorphism of algebras \( \mathcal{F} \twoheadrightarrow \mathcal{A} \), (3) the Weyl symmetrization defines an isomorphism \( S(V) \twoheadrightarrow \mathcal{F} \) of \( \mathfrak{g} \)-modules.

Let \( \mathfrak{g} \) be a Lie algebra, \( V \) be a certain \( \mathfrak{g} \)-module, \( \mathcal{A}_s \) be a family of associative algebras, parametrized by \( s \in S \) such that \( \mathfrak{g} \subseteq \text{Der}(\mathcal{A}_s) \), \( \pi_s : V \hookrightarrow \mathcal{A}_s \) be a family of \( \mathfrak{g} \)-equivariant imbeddings such that \( \pi_s(V) \) is a space of hidden symmetries in \( \mathcal{A}_s \) with respect to \( \mathfrak{g} \) for a generic \( s \) from \( S \). An associative algebra \( \mathcal{F} \) is called an algebra of the \( \mathcal{A}_s, s \in S \)-universally set free hidden symmetries iff \( \mathcal{F} \) is an algebra of the set free hidden symmetries corresponding to \( V \cong \pi_s(V) \) for generic \( \mathcal{A}_s \) \((s \in S)\).

**B.** Let \( V \) be a space of hidden symmetries in algebra \( \mathcal{A} \) with respect to the Lie algebra \( \mathfrak{g} \); hidden symmetries from \( V \) will be called of semicubic type iff there exist (1) the nontrivial decomposition of \( V \) into the direct sum \( V_2 \oplus V_3 \) of its subspaces \( V_2 \) and \( V_3 \), (2) an irreducible \( \mathfrak{g} \)-module \( V_1 \) such that the epimorphisms \( S^i(V_1) \twoheadrightarrow V_i \) \((i = 2, 3)\) of \( \mathfrak{g} \)-modules are defined; in this case the mapping \( \bigoplus_{i=2,3} S^i(V_1) \twoheadrightarrow \mathcal{F} \), a composition of mappings \( S^i(V_1) \twoheadrightarrow V_i \) and the imbedding of \( V \) into \( \mathcal{F} \), is called the noncommutative semicubic mapping.

**Theorem 1.** The tensor operators of type \( \pi_2 \) and \( \pi_3 \) in the Verma module \( V_h \) over the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) \((\pi_i \text{ is a finite-dimensional representation of } \mathfrak{sl}(2, \mathbb{C}) \text{ of dimension } 2i + 1)\) form a space of hidden symmetries of semicubic type; there exist exactly one quadratic (homogeneous) algebra\(^3\) \( \mathfrak{A}_h \) of the set free hidden symmetries for each \( h \).

Let \( L_i \) be a basis in \( \mathfrak{sl}(2, \mathbb{C}) \) such that

\[
[L_i, L_j] = (i - j)L_{i+j}, \quad (i, j = -1, 2, 3)
\]

and \( d^i_j \) \((-k \leq j \leq k)\) be bases in \( \pi_k \), in which the \( \mathfrak{sl}(2, \mathbb{C}) \)-action has the form

\[
L_i(d^k_j) = (ki - j)d^k_{i+j}.
\]

\(^3\) Such algebras form a very natural class of associative algebras. Their exploration was initiated independently and simultaneously by several authors, the ref [3] should be marked as less known than others. One should also find ref [4] among more recent ones.
The corresponding tensor operators in the Verma modules $V_h$ will be denoted by the capitals. If the Verma module $V_h$ is realized in the space $\mathbb{C}[z]$ of polynomials of a complex variable $z$, where the generators of $\mathfrak{sl}(2, \mathbb{C})$ act as

$$L_{-1} = z, \quad L_0 = z\partial_z + h, \quad L_1 = z\partial_z^2 + 2h\partial_z,$$

then the tensor operators $D^k_i$ ($k = 2, 3$) are defined by the formulas

\[
\begin{align*}
D^2_{-2} &= z^2 \\
D^2_{-1} &= z(\xi + h + \frac{1}{2}) \\
D^2_0 &= \xi^2 + 2h\xi + \frac{3}{2}h(2h + 1) \\
D^2_1 &= (\xi + 2h)(\xi + h + \frac{1}{2})\partial_z \\
D^2_2 &= (\xi + 2h)(\xi + 2h + 1)\partial_z^2 \\
D^3_{-3} &= z^3 \\
D^3_{-2} &= z^2(\xi + h + 1) \\
D^3_{-1} &= z(\xi^2 + (2h + 1)\xi + \frac{2}{5}(h + 1)(2h + 1)) \\
D^3_0 &= \xi^3 + 3h\xi^2 + (2h^2 + \frac{1}{5}(h + 1)(2h + 1))\xi + \frac{1}{5}h(h + 1)(2h + 1) \\
D^3_1 &= (\xi + 2h)(\xi^2 + (2h + 1)\xi + \frac{2}{5}(h + 1)(2h + 1))\partial_z \\
D^3_2 &= (\xi + 2h)(\xi + 2h + 1)(\xi + h + 1)\partial_z^2 \\
D^3_3 &= (\xi + 2h)(\xi + 2h + 1)(\xi + 2h + 2)\partial_z^3
\end{align*}
\]

where $\xi = z\partial_z$.

**Proof of the Theorem.** Let’s denote the mapping $d^k_i \mapsto D^k_i$ by $\mathfrak{O}$ then the tensor product

$$\pi_i \otimes \pi_j = \bigoplus_{k = |i - j|}^{i+j} \pi_k$$

maybe divided in two parts:

$$\pi_i \otimes \pi_j = \pi_i \lor \pi_j \oplus \pi_i \land \pi_j,$$

where

$$\mathfrak{O}(\pi_i \lor \pi_j) = [\mathfrak{O}(\pi_i), \mathfrak{O}(\pi_j)]_+, \quad \mathfrak{O}(\pi_i \land \pi_j) = [\mathfrak{O}(\pi_i), \mathfrak{O}(\pi_j)]_-$$

or explicitly

$$\pi_i \lor \pi_j = \bigoplus_{k = |i - j|, i + j - k \in 2\mathbb{Z}} \pi_k,$$

$$\pi_i \land \pi_j = \bigoplus_{k = |i - j|, i + j - k \in 2\mathbb{Z} + 1} \pi_k.$$

The $\mathfrak{sl}(2, \mathbb{C})$–invariant structure of the homogeneous quadratic algebra is defined by the expression of commutators $[\pi_2, \pi_2]_-$, $[\pi_2, \pi_3]_-$ and $[\pi_3, \pi_3]_-$ via anticommutators $[\pi_1, \pi_1]_-$, $[\pi_1, \pi_3]_-$ and $[\pi_2, \pi_3]_-$ respectively (here $\pi_i$ is a collective.
Theorem 2. Universal enveloping algebras. Such expression is corresponded to the imbeddings

\[ \pi_2 \wedge \pi_2 = \pi_3 \oplus \pi_1 \subseteq \pi_5 \oplus \pi_3 \oplus \pi_1 = \pi_2 \vee \pi_3, \]
\[ \pi_2 \wedge \pi_3 = \pi_4 \oplus \pi_2 \subseteq (\pi_4 \oplus \pi_2) + (\pi_6 \oplus \pi_4 \oplus \pi_2) = \pi_2 \vee \pi_2 + \pi_3 \vee \pi_3, \]
\[ \pi_3 \wedge \pi_3 = \pi_5 \oplus \pi_3 \oplus \pi_1 = \pi_2 \vee \pi_3; \]

so the general commutation relations contain 4 indeterminate constants, two of them (the first and the fourth) maybe normalized by the scaling redefinition of tensor operators \( D_i^k: D_2^2 \rightarrow \lambda_2 D_2^2; D_i^3 \rightarrow \lambda_3 D_3^3 \), the second and the third constants are fixed by the Jacobi identities for triples from \( \pi_2 \otimes \pi_2 \otimes \pi_3 \) and \( \pi_3 \otimes \pi_3 \otimes \pi_2 \); Jacobi identities for triples from \( \pi_2 \otimes \pi_2 \otimes \pi_2 \) and \( \pi_3 \otimes \pi_3 \otimes \pi_3 \) do not put any new conditions \( \square \)

Remark. Algebras \( \mathfrak{A}_h \) are not isomorphic to each other for different \( h \) and there is no any quadratic (homogeneous as well as non–homogeneous) algebra, which set the described hidden symmetries \( \text{End}(V_h) \)–universally.

Hypothesis 1. The statement of the Theorem 1 remains true after the change of \( \pi_2 \) and \( \pi_3 \) on \( \pi_n \) and \( \pi_{n+1} \) \( (n \geq 3) \).

Question 1: Are the algebras \( \mathfrak{A}_h \) of the Theorem 1 Koszul algebras [5]?

Remark. It seems that the investigation of the set free hidden symmetries, defined by tensor operators, may enlight some additional relations in the theory of Clebsch–Gordan coefficients [6].

**TOPIC TWO: MHO–ALGEBRAS**

**Definition 2.** Let \( \mathfrak{g} \) be a Lie algebra and \( W \) be its (irreducible) representation. Mho–algebra \( \mathcal{U}(\mathfrak{g}, W) \) is an associative algebra such that (1) \( \mathcal{U}(\mathfrak{g}) \) is a subalgebra of \( \mathcal{U}(\mathfrak{g}, W) \), so \( \mathfrak{g} \) naturally acts in \( \mathcal{U}(\mathfrak{g}, W) \) by the adjoint action, (2) there exist a \( \mathfrak{g} \)–equivariant imbedding of \( W \) into \( \mathcal{U}(\mathfrak{g}, W) \), so one may consider \( W \) as a subspace of \( \mathcal{U}(\mathfrak{g}, W) \), (3) the \( \mathfrak{g} \)–equivariant imbedding of \( W \) into \( \mathcal{U}(\mathfrak{g}, W) \) is extented to a \( \mathfrak{g} \)–equivariant imbedding of \( S'(W) \) into \( \mathcal{U}(\mathfrak{g}, W) \) defined by the Weyl symmetrization mapping, so one may consider \( S'(W) \) as a subspace of \( \mathcal{U}(\mathfrak{g}, W) \); (4) the isomorphisms of \( \mathfrak{g} \)–modules \( \mathcal{U}(\mathfrak{g}) \) and \( S'(\mathfrak{g}) \otimes S'(W) \) holds, here the isomorphism of subalgebra \( \mathcal{U}(\mathfrak{g}) \) of \( \mathcal{U}(\mathfrak{g}, W) \) and \( S'(\mathfrak{g}) \) as \( \mathfrak{g} \)–modules is used; (5) let’s fix an arbitrary basis \( w_i \) in \( W \) then the commutator of two elements of the basis in the algebra \( \mathcal{U}(\mathfrak{g}, W) \) maybe represented as \( [w_i, w_j] = f_{ij}^k w_k \), where the ”noncommutative structural functions” \( f_{ij}^k \) are certain elements of \( \mathcal{U}(\mathfrak{g}) \).

The notation \( \mathcal{U}(\mathfrak{g}, W) \) should symbolize an analogy between mho–algebras and universal enveloping algebras.

**Theorem 2.** There exists exactly one mho–algebra \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C}), \pi_n) \) \( (n = 1, 2, 3) \) with ”noncommutative structural functions” \( f_{ij}^k \) of degree \( n - 1 \), which is an algebra of \( \text{End}(V_h) \)–universally set free hidden symmetries.

In the case \( n = 1 \) the mho–algebra \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C}), \pi_n) \) is the squashed tensor product of two copies of \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C})) \) (i.e., the universal envelopping algebra of the squashed
sum of two copies of \( \mathfrak{sl}(2, \mathbb{C}) \), the first copy acts on the second one by the adjoint action. In the case \( n = 2 \) the mho-algebra \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C}), \pi_n) \) is just the Racah--Wigner algebra \( RW(\mathfrak{sl}(2, \mathbb{C})) \) of \([2]\). Let’s describe the mho-algebra \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C}), \pi_3) \) explicitly.

Let’s fix a basis \( u_k \) \((-3 \leq k \leq 3\) in \( \pi_3 \) such that \( L_i(u_k) = (3i - k)u_{i+k} \). Let’s also introduce the elements \( \Omega_j \) \((-2 \leq j \leq 2; [L_i, \Omega_j] = (2i - j)\Omega_{i+j}) \) in \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C})) \):
\[
\begin{align*}
\Omega_{-2} &= L_{-1}^2, \\
\Omega_{-1} &= \frac{1}{5}(L_0 L_{-1} + L_{-1} L_0), \\
\Omega_0 &= \frac{1}{5}(L_1 L_{-1} + 4L_0^2 + L_{-1} L_1), \\
\Omega_1 &= \frac{1}{5}(L_1 L_0 + L_0 L_1), \\
\Omega_2 &= L_1^2.
\end{align*}
\]
Let’s also introduce the expressions \( A_j \) \((-5 \leq j \leq 5; [L_i, A_j] = (5i - j)A_{i+j}) \), \( B_j \) \((-3 \leq j \leq 3; [L_i, B_j] = (3i - j)B_{i+j}) \), \( C_j \) \((j = -1, 0, 1; [L_i, C_j] = (i - j)C_{i+j}) \):
\[
\begin{align*}
A_{-5} &= -\Omega_{-2} u_{-3} \\
A_{-4} &= -\frac{1}{5}(2\Omega_{-1} u_{-3} + 3\Omega_{-2} u_{-2}) \\
A_{-3} &= -\frac{1}{15}(2\Omega_0 u_{-3} + 8\Omega_{-1} u_{-2} + 5\Omega_{-2} u_{-1}) \\
A_{-2} &= -\frac{1}{10}(\Omega_1 u_{-3} + 9\Omega_0 u_{-2} + 15\Omega_{-1} u_{-1} + 5\Omega_{-2} u_{0}) \\
A_{-1} &= -\frac{1}{210}(\Omega_2 u_{-3} + 24\Omega_1 u_{-2} + 90\Omega_0 u_{-1} + 80\Omega_{-1} u_{0} + 15\Omega_{-2} u_{1}) \\
A_0 &= -\frac{1}{42}(2\Omega_2 u_{-2} + 10\Omega_1 u_{-1} + 20\Omega_0 u_{0} + 10\Omega_{-1} u_{1} + \Omega_{-2} u_{2}) \\
A_1 &= -\frac{1}{210}(15\Omega_2 u_{-1} + 80\Omega_1 u_{0} + 90\Omega_0 u_{1} + 24\Omega_{-1} u_{2} + 2\Omega_{-2} u_{3}) \\
A_2 &= -\frac{1}{30}(5\Omega_2 u_{0} + 15\Omega_1 u_{1} + 9\Omega_0 u_{2} + \Omega_{-1} u_{3}) \\
A_3 &= -\frac{1}{15}(5\Omega_2 u_{1} + 8\Omega_1 u_{2} + 2\Omega_0 u_{3}) \\
A_4 &= -\frac{1}{3}(3\Omega_2 u_{2} + 2\Omega_1 u_{3}), \\
A_5 &= -\Omega_{-2} u_{3}
\end{align*}
\]
\[
\begin{align*}
B_{-3} &= \Omega_0 u_{-3} - 2\Omega_{-1} u_{-2} + \Omega_{-2} u_{-1} \\
B_{-2} &= \frac{1}{5}(\Omega_1 u_{-3} - 3\Omega_{-1} u_{-1} + 2\Omega_{-2} u_{0}) \\
B_{-1} &= \frac{1}{10}(\Omega_2 u_{-3} + 6\Omega_1 u_{-2} - 9\Omega_0 u_{-1} - 4\Omega_{-1} u_{0} + 6\Omega_{-2} u_{1}) \\
B_0 &= \frac{1}{5}(\Omega_2 u_{-2} + \Omega_1 u_{-1} - 4\Omega_0 u_{0} + \Omega_{-1} u_{1} + \Omega_{-2} u_{2}) \\
B_1 &= \frac{1}{15}(6\Omega_2 u_{-1} - 4\Omega_1 u_{0} - 9\Omega_0 u_{1} + 6\Omega_{-1} u_{2} + \Omega_{-2} u_{3}) \\
B_2 &= \frac{1}{3}(2\Omega_2 u_{0} - 3\Omega_1 u_{1} + \Omega_{-1} u_{3}) \\
B_3 &= \Omega_2 u_{1} - 2\Omega_1 u_{2} + \Omega_0 u_{3},
\end{align*}
\]
\[
\begin{align*}
C_{-1} &= \Omega_2 u_{-3} - 4\Omega_1 u_{-2} + 6\Omega_0 u_{-1} - 4\Omega_{-1} u_{0} + \Omega_{-2} u_{1} \\
C_0 &= \Omega_2 u_{-2} - 4\Omega_1 u_{-1} + 6\Omega_0 u_{0} - 4\Omega_{-1} u_{1} + \Omega_{-2} u_{2} \\
C_1 &= \Omega_2 u_{1} - 4\Omega_1 u_{0} + 6\Omega_0 u_{1} - 4\Omega_{-1} u_{2} + \Omega_{-2} u_{3}.
\end{align*}
\]

The commutation relations in \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C}), \pi_3) \) have the form
\[
\begin{align*}
[u_{-3}, u_{-2}] &= 3A_{-5} \\
[u_{-3}, u_{-1}] &= 6A_{-4}.
\end{align*}
\]
\[ [u_{-3}, u_0] = 9A_{-3} - \frac{3}{2}D_{-3} \]
\[ [u_{-3}, u_1] = 12A_{-2} - 6D_{-2} \]
\[ [u_{-3}, u_2] = 15A_{-1} - 6D_{-1} - \frac{3}{7}C_{-1} \]
\[ [u_{-3}, u_3] = 18A_0 - 42D_0 - \frac{36}{7}C_0 \]
\[ [u_{-2}, u_{-1}] = 3A_{-3} + D_{-3} \]
\[ [u_{-2}, u_0] = 6A_{-2} + \frac{3}{2}D_{-2} \]
\[ [u_{-2}, u_1] = 9A_{-1} - 3D_{-1} + \frac{1}{7}C_{-1} \]
\[ [u_{-2}, u_2] = 12A_0 - 11D_0 + \frac{4}{7}C_0 \]
\[ [u_{-2}, u_3] = 15A_1 - 6D_1 - \frac{3}{7}C_1 \]
\[ [u_{-1}, u_0] = 3A_{-1} + \frac{3}{2}D_{-1} - \frac{3}{35}C_{-1} \]
\[ [u_{-1}, u_1] = 6A_0 + 2D_0 - \frac{4}{35}C_{-1} \]
\[ [u_{-1}, u_2] = 9A_1 - 3D_1 + \frac{1}{7}C_1 \]
\[ [u_{-1}, u_3] = 12A_2 - 6D_2 \]
\[ [u_0, u_1] = 3A_1 + \frac{3}{2}D_1 - \frac{3}{35}C_1 \]
\[ [u_0, u_2] = 6A_2 + \frac{3}{2}D_2 \]
\[ [u_0, u_3] = 9A_3 - \frac{3}{2}D_3 \]
\[ [u_1, u_2] = 3A_3 + D_3 \]
\[ [u_1, u_3] = 6A_4 \]
\[ [u_2, u_3] = 3A_5 \]

where \( D_k = B_k - 2u_k + \kappa(4KN_k + 66u_k - 15B_k) \) (\( K \) is \( sl(2, \mathbb{C}) \)-Casimir element: \( K = L_{-1}L_1 - 2L_0^2 + L_1L_{-1} \); the constant \( \kappa \) is fixed by the Jacobi identities and maybe found verifying them for the triple \( u_{-2}, u_{-1}, u_0 \)).

**Hypothesis 2.** The statement of the Theorem remains true for \( n > 3 \).

**TOPIC THREE: LIE \( g \)-BUNCHES AND RELATED HIDDEN SYMMETRIES**

**Definition 3.**

A. Let \( g \) be a Lie algebra. A **Lie \( g \)-bunch** is a \( g \)-module \( W \) such that there exists a \( g \)-equivariant mapping \( g \otimes \Lambda^\ell(W) \rightarrow W \), which defines a structure of Lie algebra in \( W \) when the first argument is fixed in an arbitrary way; we shall denote this mapping by \( \cdot, \cdot \)\(_L \), \( L \in g \).

B. The family (linear space) of operators is called the **isocommutator algebra** (or **Lie isoalgebra** [7]) of operators if there exist an operator \( A \), which is called an **isotopic element** or shortly **isotopy**, such that for all \( X \) and \( Y \) from the family the expression \( XAY - YAX \) also belongs to the family\(^4\). It should be mentioned that all isotopic elements for a fixed family of operators form a linear space\(^5\).

\(^4\) It should be mentioned that an isocommutator algebra of operators is a Lie algebra as abstract one. The correspondence of an isocommutator algebra of operators to the abstract Lie algebra is called an **isorepresentation** of the least.

\(^5\) If \( A \) is an isotopic element for a family (linear space) of operators and \( L \) is an operator such
Let $\mathfrak{g}$ be a Lie algebra and $H$ be its module, if $L$ belongs to $\mathfrak{g}$ then its image in $\text{End}(H)$ will be denoted by $\pi(L)$. The family (linear space) of operators in $H$ invariant with respect to the adjoint action of $\mathfrak{g}$ is called the isocommutator algebra (or Lie isoalgebra) of hidden symmetries if it is the isocommutator algebra of operators for an arbitrary $\pi(L)$, $L \in \mathfrak{g}$ as an isotopic element.

C. Let $\mathfrak{g}$ be a Lie algebra and $W$ be a Lie $\mathfrak{g}$–bunch. An isorepresentation of $W$ is a $\mathfrak{g}$–equivariant mapping $T$ from $W$ to $\text{End}(H)$, where $H$ is a certain $\mathfrak{g}$–module, such that $T([X,Y]_L) = T(X)\pi(L)T(Y) - T(Y)\pi(L)T(X)$, in other words $T$ corresponds an isocommutator algebra of hidden symmetries to the Lie $\mathfrak{g}$–bunch $W$.

It should be specially mentioned that the considered case is linear, so the Lie $\mathfrak{g}$–bunch maybe straightforwardly restored from an isocommutator algebra of operators in a similar way as the abstract Lie algebra is restored from a commutator algebra of operators. Otherwords, there is no problem of the setting hidden symmetries free in such situation.

Let’s consider several examples, $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

**Example 1.** There exists the unique structure of Lie $\mathfrak{sl}(2, \mathbb{C})$–bunch in $\pi_1$.

Namely, if one fixes a basis $m_i$ ($i = -1, 0, 1$) in $\pi_1$ such that $L_i(m_j) = (i-j)m_{i+j}$ then the isocommutators in the Lie $\mathfrak{sl}(2, \mathbb{C})$–bunch will have the form

\[
\begin{align*}
[m_{-1}, m_0]_{L_0} &= m_{-1} & [m_{-1}, m_0]_{L_{-1}} &= 0 & [m_{-1}, m_0]_{L_1} &= 2m_0 \\
[m_1, m_{-1}]_{L_0} &= 0 & [m_1, m_{-1}]_{L_{-1}} &= 2m_{-1} & [m_1, m_{-1}]_{L_1} &= -2m_1 \\
[m_1, m_0]_{L_0} &= m_1 & [m_1, m_0]_{L_{-1}} &= 2m_0 & [m_1, m_0]_{L_1} &= 0
\end{align*}
\]

**Example 2.** Let $\pi_{1/2}$ be the 2–dimensional fundamental representation of $\mathfrak{sl}(2, \mathbb{C})$ then operators from $\text{End}(\pi_{1/2})$ form naturally an isocommutator algebra of hidden symmetries. The corresponding Lie $\mathfrak{sl}(2, \mathbb{C})$–bunch is realized in the direct sum $\pi_0 \oplus \pi_1$ of the trivial and adjoint representations of $\mathfrak{sl}(2, \mathbb{C})$. If one fixes a basis $m_i$ in $\pi_1$ as above and an element $c$ of $\pi_0$ (which is mapped to the identity via an isorepresentation) then the isocommutators in the Lie $\mathfrak{sl}(2, \mathbb{C})$–bunch will have the form

\[
\begin{align*}
[m_{-1}, m_0]_{L_0} &= 0 & [m_{-1}, m_0]_{L_{-1}} &= 0 & [m_{-1}, m_0]_{L_1} &= -\frac{1}{2}c \\
[m_1, m_{-1}]_{L_0} &= -\frac{1}{2}c & [m_1, m_{-1}]_{L_{-1}} &= 0 & [m_1, m_{-1}]_{L_1} &= 0 \\
[m_1, m_0]_{L_0} &= 0 & [m_1, m_0]_{L_{-1}} &= \frac{1}{2}c & [m_1, m_0]_{L_1} &= 0 \\
[c, m_1]_{L_0} &= -m_1 & [c, m_1]_{L_{-1}} &= -2m_0 & [c, m_1]_{L_1} &= 0 \\
[c, m_0]_{L_0} &= 0 & [c, m_0]_{L_{-1}} &= -2m_0 & [c, m_0]_{L_1} &= 0 \\
[c, m_{-1}]_{L_0} &= m_{-1} & [c, m_{-1}]_{L_{-1}} &= -m_{-1} & [c, m_{-1}]_{L_1} &= m_{-1}
\end{align*}
\]

**Example 3.** There are exactly four types of structure of a Lie $\mathfrak{sl}(2, \mathbb{C})$–bunch in $\pi_1 \oplus \pi_2$.

*Case 1: Enlarged example 1.*

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that $\text{ad} L$ preserves this family then $[L, A]$ is also an isotopic element. Also if $A$ and $B$ are two isotopic elements and $X$ is an arbitrary operator from the family then $AXB - BXA$ is also an isotopic element. (Such “isotopic duality” claims a special and very serious attention).
\[ [m_{-1}, m_0]_{L_0} = am_{-1} \]
\[ [m_1, m_{-1}]_{L_0} = 0 \]
\[ [m_1, m_0]_{L_0} = \lambda c \]
\[ [c, m_1]_{L_0} = 0 \]
\[ [c, m_0]_{L_0} = \mu c \]
\[ [c, m_{-1}]_{L_0} = 0 \]

Case 2: Example 2.

\[ [m_{-1}, m_0]_{L_0} = 0 \]
\[ [m_1, m_{-1}]_{L_0} = 0 \]
\[ [c, m_1]_{L_0} = -\lambda c \]
\[ [c, m_0]_{L_0} = 0 \]
\[ [c, m_{-1}]_{L_0} = 0 \]

Case 3.

\[ [m_{-1}, m_0]_{L_0} = am_{-1} \]
\[ [m_1, m_{-1}]_{L_0} = 0 \]
\[ [m_1, m_0]_{L_0} = \lambda c \]
\[ [c, m_1]_{L_0} = 0 \]
\[ [c, m_0]_{L_0} = 2ac \]
\[ [c, m_{-1}]_{L_0} = 0 \]

Case 4.

\[ [m_{-1}, m_0]_{L_0} = am_{-1} \]
\[ [m_1, m_{-1}]_{L_0} = 0 \]
\[ [m_1, m_0]_{L_0} = am_1 \]
\[ [c, m_1]_{L_0} = -dm_1 \]
\[ [c, m_0]_{L_0} = 0 \]
\[ [c, m_{-1}]_{L_0} = dm_{-1} \]

Example 4. Let \( \mathcal{A} \) be an arbitrary associative algebra with an involution \( * \), \( \mathfrak{g} = \{ A \in \mathcal{A} : A^* = -A \} \) is a Lie algebra. Let \( W = \{ B \in \mathcal{A} : B^* = B \} \) then there exists a natural structure of the isocommutator algebra of hidden symmetries in \( W \).

In particular, let’s considered \( \mathcal{A} \) being \( \text{End}(\pi_1) \), \( \mathfrak{g} = \mathfrak{so}(3, \mathbb{C}) \subset \mathcal{A} \) – the set of skew-symmetric matrices \( 3 \times 3 \) (\( \mathfrak{so}(3, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}) \)), \( W \) – the set of symmetric matrices \( 3 \times 3 \) (it should be marked that \( \mathfrak{sl}(2, \mathbb{C}) \) acts in \( W \) and \( W \) is isomorphic to \( \pi_0 \oplus \pi_2 \) as \( \mathfrak{sl}(2, \mathbb{C}) \)-module). So \( W \) is an isocommutator algebra of hidden symmetries and therefore \( \pi_0 \oplus \pi_2 \) possesses a structure of Lie \( \mathfrak{sl}(2, \mathbb{C}) \)-bunch.

This construction of Lie \( \mathfrak{sl}(2, \mathbb{C}) \)-bunch maybe straightforwardly generalized on the direct sums \( \bigoplus_{i=1}^{k} \pi_{2i} \simeq S^2(\pi_k) \).

Remark. \( \pi_k \) does not admit any structure of a Lie \( \mathfrak{sl}(2, \mathbb{C}) \)-bunch.
Example 5 (A geometric example of Lie $\mathfrak{g}$–bunch). Let $\mathcal{M}$ be a smooth manifold. $C^\infty(\mathcal{M})$ is a Lie $\text{Vect}(\mathcal{M})$–bunch.

Definition 4.
A. The algebra $\mathfrak{A}$ is called Lie–admissible [8] if its commutator algebra $\mathfrak{A}(\langle \cdot,\cdot \rangle)$ is a Lie algebra. The main example of Lie–admissible algebras is an arbitrary associative algebra supplied by the operation $(X, Y) \mapsto XRY - YSX$ (for arbitrary fixed elements of $R$ and $S$).

B. Let $\mathfrak{g}$ be a Lie algebra. A Lie–admissible $\mathfrak{g}$–bunch is a $\mathfrak{g}$–module $W$ such that there is defined a $\mathfrak{g}$–equivariant mapping $\mathfrak{g} \otimes T^2(W) \mapsto W$ which defines a structure of Lie–admissible algebra in $W$, when the first argument is fixed in any arbitrary way, we shall denote this mapping by $\langle \cdot, \cdot \rangle_L$, $L \in \mathfrak{g}$. It will be called a Lie–admissible $\mathfrak{g}$–bunch of the first kind if the operation $X \circ_L Y = \frac{1}{2}(\langle X, Y \rangle_L - \langle Y, X \rangle_L)$ does not depend on $L$ and a Lie–admissible $\mathfrak{g}$–bunch of the second kind if the operation $[X, Y]_L = \langle X, Y \rangle_L - \langle Y, X \rangle_L$ does not depend on $L$.

A Lie–admissible $\mathfrak{g}$–bunch $W$ will be called quadratic (homogeneous or nonhomogeneous, respectively) if (1) there exists a $\mathfrak{g}$–submodule $W_0$ (coordinate base) in $W$ such that the mapping from $\mathfrak{g} \rightarrow W_0$ into $W$ defined by the Weyl symmetrizations is a $\mathfrak{g}$–equivariant isomorphism; (2) for a basis $w_m$ in $W_0$ the following relations hold $[w_i, w_j]_L = S^k_{ij} w_k \circ_L w_l$ or $[w_i, w_j]_L = S^k_{ij} w_k \circ_L w_l + R^k_{ij} w_k$, respectively.

C. Let $\mathfrak{g}$ be a Lie algebra. An isorepresentation of a Lie–admissible $\mathfrak{g}$–bunch $W$ of the first kind is a $\mathfrak{g}$–equivariant mapping from $W$ to $\text{End}(H)$, where $H$ is a certain $\mathfrak{g}$–module $(\pi : \mathfrak{g} \mapsto \text{End}(H))$, such that $T([X, Y]_L) = T(X)\pi(L)T(Y) - T(Y)\pi(L)T(X)$, $T(X \circ Y) = \frac{1}{2}(T(X)T(Y) + T(Y)T(X))$. An isorepresentation of a Lie–admissible $\mathfrak{g}$–bunch $\mathcal{W}$ of the second type is a $\mathfrak{g}$–equivariant mapping from $W$ to $\text{End}(H)$, where $H$ is a certain $\mathfrak{g}$–module $(\pi : \mathfrak{g} \mapsto \text{End}(H))$, such that $T([X, Y]) = [T(X), T(Y)]$, $T(X \circ_L Y) = \frac{1}{2}(T(X)\pi(L)T(Y) + T(Y)\pi(L)T(X))$.

Example 6 (cf. Example 4 above). Let $\mathcal{A}$ be an arbitrary associative algebra with an involution $\ast$, $\mathfrak{g} = \{ A \in \mathcal{A} : A^* = -A \}$. The space $W = \{ B \in \mathcal{A} : B^* = B \}$ possesses a natural structure of Lie–admissible $\mathfrak{g}$–bunch of the first kind, whereas $\mathfrak{g}$ itself possesses a natural structure of Lie admissible $\mathfrak{g}$–bunch of the second kind.

So a classical Lie algebra $\mathfrak{g}$ is supplied by a natural structure of Lie–admissible $\mathfrak{g}$–bunch.

In particular, $\pi_1$ admits a structure of Lie–admissible $\mathfrak{sl}(2, \mathbb{C})$–bunch of the second kind, whereas $\pi_0 \oplus \pi_2$ admits a structure of Lie–admissible $\mathfrak{g}$–bunch of the first kind. Moreover, structures of Lie–admissible $\mathfrak{g}$–bunches of the first kind are defined in $\bigoplus_{i=0}^k \pi_{2i}$, whereas structures of Lie–admissible $\mathfrak{sl}(2, \mathbb{C})$–bunches of the second kind are defined in $\bigoplus_{i=0}^k \pi_{2i+1}$.

Question 2: Are there any quadratic Lie–admissible $\mathfrak{sl}(2, \mathbb{C})$–bunches of the first and second types with $\pi_2$ as a coordinate base, which admit isorepresentations realizing elements of the coordinate base by the tensor operators $D_i^2 (-2 \leq i \leq 2)$ in the Verma modules $V_h$?

Example 7 (A geometric example of Lie–admissible $\mathfrak{g}$–bunch). Let $\mathcal{M}$ be a smooth manifold. The space of differential operators $\text{DOP} (\mathcal{M})$ on $\mathcal{M}$ is a Lie–admissible $\text{Vect}(\mathcal{M})$–bunch of the first kind.
CONCLUSIONS

Thus, some curiosities of the inverse problem of representation theory were considered on simple finite dimensional examples related to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Such consideration maybe regarded as toy one for analogous infinite dimensional problems in the modern quantum field theory (conformal field theory, integrable models, field theory in non–trivial backgrounds, etc.; see f.e. [9]). Some constructions inspired by these topics and related to infinite dimensional hidden symmetries, which are produced by vertex operator fields, will be discussed in the forthcoming article [10].

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