A Lagrangian for Electromagnetic Solitary Waves in Vacuum

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Abstract

A system of equations, describing the evolution of electromagnetic fields, is introduced and discussed. The model is strictly related to Maxwell’s equations. As a matter of fact, the Lagrangian is the same, but the variations are subjected to a suitable constraint. This allows to enlarge the space of solutions, including for example solitary waves with compact support. In this way, without altering the physics, one is able to deal with vector waves as they were massless particles. The main properties of the model, together with numerous exact explicit solutions are presented.

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1 The modelling equations

In [13], the theory of electromagnetism has been reviewed by introducing a suitable nonlinear set of equations, allowing for a very extended space of solutions, basically including any type of solitary wave. The new formulation provides a far more accurate description of wave phenomena, including their self-interactions. We reorganize here part of the material, providing additional insight.
We are mainly concerned with the evolution of pure electromagnetic waves in vacuum. Denoting by $c$ the speed of light, in Minkowski space the model equations read as follows:

$$\frac{\partial E}{\partial t} = c^2 \text{curl} B - (\text{div} E)V$$  \hspace{1cm} (1.1)$$

$$\frac{\partial B}{\partial t} = - \text{curl} E$$  \hspace{1cm} (1.2)$$

$$E + V \times B = 0$$  \hspace{1cm} (1.3)$$

Here, $E$ and $B$ are the usual electric and magnetic fields, while $V$ is a new velocity field. Note that there are 3 vector unknowns and 3 vector equations. We will require $V$ to be of constant norm. More exactly, we impose:

$$\|V\| = c$$  \hspace{1cm} (1.4)$$

The above condition can be eliminated in view of a more general formulation (see section 5). By taking the divergence of equation (1.2) we get for any $t$:

$$\text{div} B = 0$$  \hspace{1cm} (1.5)$$

provided the initial data are compatible with this constraint.

By defining $\rho = \text{div} E$ and by taking the divergence of equation (1.1) it is straightforward to get the continuity equation:

$$\frac{\partial \rho}{\partial t} = - \text{div}(\rho V)$$  \hspace{1cm} (1.6)$$

Equation (1.6) is the first one of many other conservation laws associated with the above system of equations. This is quite an important prerogative, since, in the search of solitonic solutions, all the possible quantities must be preserved in the evolution. The important fact is that, even in vacuum $\rho$ is allowed to be different from zero. Evidently, by enforcing $\rho = 0$, we come back with (1.1) and (1.2) to the standard Maxwell’s equations, but, certainly, this is not the case we would like to discuss. As a matter of fact, it is known that solitary waves do not belong to the space of solutions of Maxwell’s equations (we come back on this issue later in section 4). In the past, this led to the conclusion that quantum phenomena cannot be modelled by the classical equations of electromagnetism, so that, as we shall mention in the coming section, numerous variants of the Maxwell’s system have been taken into account. The model we are describing here differs from the others for its simplicity and for many additional reasons we are going to detail.
2 Lagrangian analysis

We start by introducing the potentials $A$ and $\Phi$ such that:

$$
B = \frac{1}{c} \text{curl} A \quad E = - \frac{1}{c} \frac{\partial A}{\partial t} - \nabla \Phi
$$

(2.1)

In this way we guarantee equations (1.1) and (1.5). Then, we consider the standard Lagrangian of electromagnetism:

$$
\mathcal{L} = \|E\|^2 - c^2 \|B\|^2
$$

(2.2)

It is known that Maxwell’s equations are linked to the stationary points of the action function of $\mathcal{L}$. To check this, one writes $\mathcal{L}$ in terms of $\Phi$ and $A$ and then differentiate with respect to the variations $\delta \Phi$ and $\delta A$, having compact support in space and time. A typical way to carry out this proof (that we do not report here) passes through the construction of the electromagnetic tensor $F_{ik}$ (see for instance [17]). Thus, one discovers that (1.2) must be true together with the condition: $\rho = \text{div} E = 0$.

Unfortunately, imposing $\rho = 0$ brings to a subspace of solutions that do not include, for example, solitary waves with compact support. Extensions are then necessary and the literature is rich of results. An usual approach is based on a modification of the Lagrangian, by adding a further term, on the basis of physical considerations. With this respect, we mention the pioneering paper of [6] (see also [18] and [19]). For a more recent general viewpoint we refer for instance to [2], [3] (see also [4] for a general review).

The procedure of adapting the Lagrangian is sometimes successful, but the new corresponding system of equations turns out to be heavily non-linear and some of the good invariance properties of Maxwell’s equations might be lost. Theoretical results are usually addressed to the existence of stable solitonic solutions. Another approach is to couple some equations, directly derived from Maxwell’s system, with other type of equations, such as Schrödinger or Klein-Gordon (see for example [1], [9], [11]).

The path followed here is to recover the set of model equations (1.1)-(1.2)-(1.3) from the stationary points of the action function associated with the standard Lagrangian in (2.2). In order to avoid the condition $\rho = 0$, we put a constraint to the variations. In this way, since there are less degrees of freedom for the test functions, we come out with a larger space of solutions.
To this end, let us impose the following relation:

\[ cA = \Phi \mathbf{V} \]  

(2.3)

Note that the above is not a gauge condition. As a matter of fact it will be somehow stronger (see later). From (2.3), due to (1.4), we easily get the scalar constraint:

\[ c \Phi = \mathbf{V} \cdot \mathbf{A} \]  

(2.4)

With the help of (2.3), we can actually obtain our set of equations. The formal proof of this fact is given in [13], using standard variational arguments, after writing \( \mathcal{L} \) in terms of the electromagnetic tensor \( F_{ik} \). Here we add some heuristic considerations.

By standard calculus one obtains:

\[
\mathcal{L} = \|E\|^2 - c^2 \|B\|^2 = E \cdot \left( -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \Phi \right) - c \mathbf{B} \cdot \text{curl} A
\]

\[
= -\frac{1}{c} \frac{\partial}{\partial t} (E \cdot A) + \frac{1}{c} \frac{\partial E}{\partial t} \cdot A - E \cdot \nabla \Phi - c A \cdot \text{curl} B - c \text{div}(A \times B)
\]

\[
= -\frac{1}{c} \frac{\partial}{\partial t} (E \cdot A) + \frac{1}{c} \left( \frac{\partial E}{\partial t} - c^2 \text{curl} B \right) \cdot A
\]

\[ - \text{div}(\Phi E) + \Phi \text{div} E - c \text{div}(A \times B) \]  

(2.5)

At this point, by imposing condition (2.3) and by noting that \( E \cdot A = 0 \) (obtained by multiplying (1.3) by \( A \)), the above equation becomes:

\[
\mathcal{L} = \|E\|^2 - c^2 \|B\|^2 = -\text{div}(\Phi(E + \mathbf{V} \times \mathbf{B}))
\]

\[ + \frac{1}{c} \left( \frac{\partial E}{\partial t} - c^2 \text{curl} B \right) \cdot A \]  

(2.6)

We now observe that \( \mathcal{L} \) vanishes when:

\[ \|E\| = c\|B\| \]  

(2.7)

Consequently, if (1.3) and (1.1) hold true, we are exactly in the situation \( \mathcal{L} = 0 \), that is: all our waves will have the norm of \( E \) equal to that of \( cB \).
More technical (but not too difficult) is to show that $L = 0$ is actually a stationary point of the action function associated with $L$.

Finally, we remark that the constraint (2.4) might be included in the Lagrangian, as a penalty term, in the following way:

$$L = \|E\|^2 - c^2\|B\|^2 + \frac{\rho}{c}(A \cdot V - c\Phi)$$

(2.8)

The added term is similar to the one we would have in presence of moving charged particles in an electromagnetic field (see, e.g.: [17]).

## 3 Eikonal equation

With a simple analysis, more general properties of the solutions can be recovered. For example, by expressing $E$ and $B$ in term of the potentials, one can prove that:

$$c^2(E + V \times B) = -\frac{\partial}{\partial t}(cA - \Phi V) + V \text{div}(cA - \Phi V)$$

$$+ V \times \text{curl}(cA - \Phi V) - cV \cdot \frac{DA}{Dt} + \frac{\Phi}{2} \nabla\|V\|^2$$

(3.1)

where $DA/Dt = \partial A/\partial t + (V \cdot \nabla)A$ is the substantial derivative of $A$ along the velocity vector field $V$. Thanks to (2.3) and (1.4), one concludes that:

$$E + V \times B = 0 \iff V \cdot \frac{DA}{Dt} = 0$$

(3.2)

Moreover, the following relations hold:

$$E \cdot V = 0 \quad E \cdot B = 0 \quad V \cdot B = 0 \quad V = c \frac{E \times B}{\|E \times B\|}$$

(3.3)

The first one is obtained by scalarly multiplying (1.3) by $V$. The second one is similarly obtained by multiplying (1.3) by $B$. The third and the fourth ones are consequence of the two previous orthogonality relations together with conditions (1.4) and (2.7). Thus $(E, B, V)$ is a right-handed orthogonal triplet.
Thanks to the above relations one can prove a classical results regarding the time derivative of the energy:

$$\frac{1}{2} \frac{\partial}{\partial t} \left( \|E\|^2 + c^2 \|B\|^2 \right) = c^2 (\text{curl}B \cdot E - \text{curl}E \cdot B) = -c^2 \text{div}(E \times B) \quad (3.4)$$

where $E \times B$ is the Poynting vector.

Additional conclusions are obtained by imposing conditions on the field $V$. For example, one may require that $V$ is not subjected to transversal acceleration:

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V \cdot \nabla)V = 0 \quad (3.5)$$

A way to satisfy (3.5) is to suppose that $V$ is an irrotational stationary field. This means that $V = \nabla \Psi$ for some scalar potential $\Psi$. In this way: $DV/Dt = \nabla(\partial \Psi/\partial t + \frac{1}{2} \nabla \|\Psi\|^2) = 0$, yielding (3.5). With this choice, the most important consequence is the following eikonal equation:

$$\|\nabla \Psi\| = c \quad (3.6)$$

directly obtainable from (1.4). This ensures that our solutions develop according to the rules of geometrical optics. Note also that, if we use (2.3) and (3.5), one gets: $cD\Phi/Dt = D(\Phi V)/Dt = V(D\Phi/Dt)$. Therefore, the condition at the right-hand side of (3.2) can be replaced by:

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + V \cdot \nabla \Phi = 0 \quad (3.7)$$

In conclusion, when $V$ is irrotational and $D\Phi/Dt = 0$, we automatically have (1.3) and (3.6), showing that the wave-fronts evolve as prescribed by geometrical optics.

If, in addition to these hypotheses, one also requires the following Lorenz gauge conditions on the potentials:

$$\text{div}A + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0 \quad (3.8)$$

from (2.3), we also get another continuity equation:

$$\frac{\partial \Phi}{\partial t} = - \text{div}(\Phi V) \quad (3.9)$$
which is stronger than demanding \( D\Phi/Dt = 0 \).

Our final purpose is to show that it is possible to explicitly compute infinite interesting solutions of the system (1.1)-(1.2)-(1.3), realizing an extended range of conservation laws.

4 Explicit solutions

In the cartesian reference frame \((x, y, z)\), let us take the following solution candidates:

\[
\begin{align*}
\mathbf{E} &= \left( E_1(x, y)g(z - ct), \; E_2(x, y)g(z - ct), \; 0 \right) \\
\mathbf{B} &= \left( B_1(x, y)g(z - ct), \; B_2(x, y)g(z - ct), \; 0 \right)
\end{align*}
\]

(4.1)

representing two field distributions, modulated by the function \( g \), laying on parallel planes and shifting in the direction of the \( z \)-axis at the speed of light. The functions \( E_1, E_2, B_1 \) and \( B_2 \) are smooth on the whole plane \((x, y)\) (they may allowed, for example, to be zero outside a domain of finite measure). The function \( g \) is also smooth.

By setting \( \mathbf{V} = (0, 0, c) \) and by direct substitution into the equations (1.1)-(1.2)-(1.3), the vector fields shown in (4.1) are solutions under the following assumptions:

\[
\begin{align*}
E_1 &= cB_2 & E_2 &= -cB_1 & \text{div}\mathbf{B} = \partial B_1/\partial x + \partial B_2/\partial y = 0
\end{align*}
\]

(4.2)

for any choice of \( g \). With this setting, one can actually find a function \( A_3 \) and construct the two potentials:

\[
\Phi = A_3(x, y)g(z - ct) \quad \mathbf{A} = \left( 0, \; 0, \; A_3(x, y)g(z - ct) \right)
\]

(4.3)

It is straightforward to check that all the conditions (2.3), (2.7), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9) are verified. Thus, depending on the arbitrary functions \( A_3 \) and \( g \), we can build infinite solutions of our system. If \( A_3 \) and \( g \) have compact support, we get solitary electromagnetic waves shifting unperturbed at the speed of light. Note that \( \rho = \text{div}\mathbf{E} \neq 0 \) and (1.6) becomes the trivial transport equation:

\[
\frac{\partial \rho}{\partial t} = -c \frac{\partial \rho}{\partial z}
\]

(4.4)
If we instead start from the general setting in (4.1), and try to solve the full set of Maxwell’s equations (i.e., including the additional condition div$E = 0$), there are no chances of getting interesting nontrivial solutions. In this case we can forget about the velocity vector $V$. By direct substitution, one easily deduces that both the functions $E_1 - iE_2$ and $B_1 - iB_2$, where $i$ is the imaginary unit, must be holomorphic (entire) on the whole complex plane $x + iy$ (see, for instance [10]). By the Liouville’s theorem: *if an entire holomorphic function is bounded, then it is a constant*, we entail that there are no bounded continuous electromagnetic fields of the form (4.1), having finite energy and solving the whole set of Maxwell’s equations, with the exception of $E = 0$ and $B = 0$ (note that classical plane waves have not finite energy). We recall that solutions may exist if we assume that $E$ and $B$ do not belong to the tangent plane of the advancing front. In this situation, however, the Poynting vector $E \times B$, indicating the direction of the energy flow (see (3.4)), is not lined up with the direction of movement, in contrast with the rules of geometrical optics. This brings to a diffusive behavior.

We can now transform the equations (1.1)-(1.2)-(1.3) in spherical coordinates $(r, \theta, \phi)$ and take the following fields, distributed on the tangent planes of spherical wave-fronts:

$$E = \frac{1}{r} \left( 0, E_2(\theta, \phi)g(r - ct), E_3(\theta, \phi)g(r - ct) \right)$$

$$B = \frac{1}{r} \left( 0, B_2(\theta, \phi)g(r - ct), B_3(\theta, \phi)g(r - ct) \right)$$

(4.5)

With this choice, the energy density $\|E\|^2 + c^2\|B\|^2$ remains constant when integrated over any spherical surface. Similarly to the previous case, by defining $V = (c, 0, 0)$, we can get infinite solutions, provided they satisfy:

$$E_3 = -cB_2 \quad E_2 = cB_3 \quad \text{div} B = \partial(B_2 \sin \theta)/\partial \theta + \partial B_3/\partial \phi$$

(4.6)

allowing us to build the potentials:

$$\Phi = A_1(\theta, \phi)g(r - ct) \quad A = \left( A_1(\theta, \phi)g(r - ct), 0, 0 \right)$$

(4.7)

where $A_1$ and $g$ are arbitrary. As before all the conditions (2.3), (2.7), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9) are verified. By taking $A_1(\theta, \phi) = -\cos \theta$ and $g(s) = \sin \omega s$, among the solutions we recognize the one, corresponding
to the monochromatic field generated by an infinitesimal dipole, usually employed in applications (see [5], [7], [15]):

\[
E = \left( 0, -\frac{\sin \theta}{r} \sin \omega (r - ct), 0 \right) \quad B = \left( 0, 0, -\frac{\sin \theta}{r} \sin \omega (r - ct) \right) \quad (4.8)
\]

Searching instead for solutions of the Maxwell’s system, one finds out that the complex functions \( E_2 - iE_3 \sin \theta \) and \( B_2 - iB_3 \sin \theta \) should be holomorphic on the Riemann sphere. Hence, \( E \) and \( B \) are bounded if and only if they are zero. Therefore, the only possible solutions of Maxwell’s equations in vacuum, having bounded electromagnetic fields laying on tangent planes of spherical fronts are identically zero. As a consequence the one in (4.8) cannot be solution (in fact \( \text{div} E \neq 0 \)). The same arguments can be applied to any closed, bounded, compact, oriented surface. Non trivial solutions, for example the celebrated Hertz solutions (see [16], [7], [15]), are instead possible by allowing the fields to have radial components \( E_1 \) and \( B_1 \). In this way, however, the Poynting vector \( E \times B \) is not aligned with the direction of movement and the evolution of the corresponding wave-fronts do not comply with the rules of geometrical optics (i.e.: the surfaces obtained as the envelope of the electromagnetic vectors, representing the wave-fronts, are not spherical), so that many of the conservation laws reported here are not satisfied.

5 Concluding remarks

With the new model equations we have an extended range of solutions, not available in the Maxwellian case. In particular, this includes wave-packets of almost any form, both from the viewpoint of the shape of the wave-fronts and the information written on them. These wave-packets travel unperturbed at speed \( c \), along the direction of the vector \( \mathbf{V} \), having constant norm. If \( \mathbf{V} \) is stationary and irrotational, then the eikonal equation is verified and the wave-fronts perfectly follow the laws of geometrical optics. The equations are compatible with the stationary points of the standard Lagrangian of the electromagnetism. In this new framework, the differentiation has to be taken with respect to potentials subject to a certain constraint, that, thanks (3.1), is equivalent in practice to impose (1.3).
At this point, something has to be said about the physical implications. First of all, we recall that in [13] equation (1.3) is generalized as follows:
\[
\frac{DV}{Dt} = -\mu (E + V \times B) - \nabla p \rho
\] (5.1)
which is the Euler equation for compressible fluids (recall the continuity equation (1.6)), with an electromagnetic type forcing term. The scalar \( p \) is a suitable pressure and \( \mu \) is a constant whose dimension is charge/mass. This approach combines the evolution of electromagnetic entities with that of a (non material) inviscid fluid. Some numerical experiments, regarding the interaction of waves with matter (i.e., the diffraction due to the passage of a photon through a small hole) are examined in [14]. Other numerical experiments, concerning electromagnetic waves trapped in bounded regions of space (basically, forming vortex rings) are reported in [8]. In this new context, that is better suited for a general relativity framework, we must drop condition (1.4) and replace it with an eikonal equation in a suitable metric space. In order to maintain the exposition at basic level, we do not add further considerations concerning this generalization.

So far, we studied the special case when \( p = 0 \) and \( DV/Dt = 0 \). In this circumstance, the fluid moves unperturbed. From the electromagnetic point of view, equation (1.3) says that the “Lorentz force” acting on the wave is null. This is in agreement with the fact that our solitons freely travel, without the influence of external factors.

Equation (1.1) is the Ampère law, without explicit external currents, but with a sort of electric density developing together with the wave and satisfying automatically the continuity equation (1.6). The presence of a nonvanishing \( \rho \) is not in contrast with the Gauss’s divergence theorem. If our travelling soliton has a bounded support included in a solid region \( \Omega \), then the integral \( \int_{\partial \Omega} E \cdot n = \int_{\Omega} \rho \) is equal to zero. Therefore, seen as a whole, our soliton is not a charge (it does not even emit electromagnetic fields during its movement), although, inside, there are points where \( \rho \neq 0 \).

In order to deal with equation (5.1) we may generalize the Lagrangian in (2.8) by setting:
\[
\mathcal{L} = \|E\|^2 - c^2 \|B\|^2 + \frac{\rho}{c}(A \cdot V - c\Phi) + \frac{c\rho}{\mu} \sqrt{c^2 - \|V\|^2} - \frac{p}{\mu} \] (5.2)
We come back to the special case of equations (2.8) by imposing $p = 0$ and recalling (1.4). This approach takes also into account relativistic effects. We did not carry out any analysis concerning the above Lagrangian. Perhaps, the techniques introduced in [4] to study the general Lagrangian $\mathcal{L}_{u,S,A,\Phi}$ may also be applied in our situation.

To conclude we say that the most important achievement, realized with the new model equations, is to be able to produce, maintaining a physical meaning, vector waves that display all the peculiarities of standard electromagnetic emissions. In addition, waves with the characteristics of a photon are allowed by the model, providing the link between the evolution of electromagnetic entities and that of classical mechanical bodies. Finally, let us point out that the new system of equations can be proved to be invariant under Lorentz transformations (see [13], section 2.6), can be written in covariant form (see [13], chapter 4) and is naturally linked to the derivative of the electromagnetic stress tensor (see [13], section 4.2, and [12]).

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