An achievable region for the double unicast problem based on a minimum cut analysis

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Abstract—We consider the multiple unicast problem under network coding over directed acyclic networks when there are two source-terminal pairs, \( s_1 - t_1 \) and \( s_2 - t_2 \). Current characterizations of the multiple unicast capacity region in this setting have a large number of inequalities, which makes them hard to explicitly evaluate. In this work we consider a slightly different problem. We assume that we only know certain minimum cut values for the network, e.g., mincut\((S, T_i)\), where \( S \subseteq \{s_1, s_2\} \) and \( T_i \subseteq \{t_1, t_2\} \) for different subsets \( S \) and \( T_i \). Based on these values, we propose an achievable rate region for this problem based on linear codes. Towards this end, we begin by defining a base region where both sources are multicast to both the terminals. Following this we enlarge the region by appropriately encoding the information at the source nodes, such that terminal \( t_i \) is only guaranteed to decode information from the intended source \( s_i \), while decoding a linear function of the other source. The rate region takes different forms depending upon the relationship of the different cut values in the network.

I. INTRODUCTION

The problem of characterizing the utility of network coding for multiple unicasts is an intriguing one. In the multiple unicast problem there is a set of source-terminal pairs in a network that wish to communicate messages. This is in contrast to the multicast problem where each terminal requests exactly the same set of messages from the source nodes. The multicast problem under network coding is very well understood. In particular, several papers [1][2][3] discuss the exact capacity region and network code construction algorithms for this problem.

However, the multiple unicast problem is not that well understood. A significant amount of previous work has attempted to find inner and outer bounds on the capacity region for a given instance of a network. In [4], an information theoretic characterization for directed acyclic networks is provided. However, explicit evaluation of the region is computationally intractable for even small networks due to the large number of constraints. The authors in [5] propose an outer bound on the capacity region. Price et al. [6] provide an outer bound on the capacity region in a two unicast session network, and provided a network structure in which their outer bound is the exact capacity region. The work of [7] forms a linear optimization to characterize an achievable rate region by packing butterfly structures in the original graph. This approach is limited since only the XOR operation is allowed in each butterfly structure.

In this work we propose an achievable region for the two-unicast problem using linear network codes. Our setup is somewhat different from the above-mentioned works in that we consider directed acyclic networks with unit capacity edges and assume that we only know certain minimum cut values for the network, e.g., mincut\((S, T_i)\), where \( S \subseteq \{s_1, s_2\} \) and \( T_i \subseteq \{t_1, t_2\} \) for different subsets \( S \) and \( T_i \). This is related to the work of Wang and Shroff [8] (see also [9]) for two-unicast that presented a necessary and sufficient condition on the network structure for the existence of a network coding solution that supports unit rate transmission for each \( s_i - t_i \) pair. In this work we consider general rates. Reference [10] is related in the sense that they give an achievable rate region for this problem based on the number of edge disjoint paths for \( s_i - t_i \) pair. In our work we propose a new achievable rate region given additional information about the network resources. The work of [11] considered the three unicast session problem in which each source is transmitting at unit rate. Finally, reference [12] applies the technique of interference alignment in the case of three unicast sessions and shows that communication at half the mincut of each source-terminal pair is possible.

This paper is organized as follows. Section II introduces the system model under consideration. Section III contains the precise problem formulation and the derivations of our proposed achievable rate region. Section IV compares our achievable region to existing literature. Due to space limitations, some of the lemma proofs are not given and can be found in [13].

II. SYSTEM MODEL

We consider a network represented by a directed acyclic graph \( G = (V, E) \). There is a source set \( S = \{s_1, s_2\} \in V \) in which each source observes a random process with a discrete integer entropy, and there is a terminal set \( T = \{t_1, t_2\} \in V \) in which \( t_i \) needs to uniquely recover the information transmitted from \( s_i \) at rate \( R_i \). Each edge \( e \in E \) has unit capacity and can transmit one symbol from a finite field of size \( q \). If a given edge has a higher capacity, it can be divided into multiple parallel edges with unit capacity. Without loss of generality (W.l.o.g.), we assume that there is no incoming edge into source \( s_i \), and no outgoing edge from terminal \( t_i \). By Menger’s theorem, the minimum cut between sets \( S_{N_1} \subseteq S \) and \( T_{N_2} \subseteq T \) is the number of edge disjoint paths from \( S_{N_1} \) to \( T_{N_2} \), and will be denoted by \( k_{N_1 - N_2} \) where \( N_1, N_2 \subseteq N = \{1, 2\} \). For two unicast sessions, we define the cut vector as the vector of the
cut values \(k_1 - 1, k_2 - 2, k_1 - 2, k_2 - 1, k_1 - 1, k_2 - 2, k_1 - 1, k_2 - 2\) and \(k_1 - 1, k_2 - 2\).

The network coding model in this work is based on \cite{2}. Assume source \(s_1\) needs to transmit at rate \(R_{t_1}\). Then the random variable observed at \(s_1\) is denoted as \(X_i = (X_{i1}, X_{i2}, \ldots, X_{iR_i})\), where each \(X_{ij}\) is an element of \(GF(q)\); the \(X_i\)'s are assumed to be independent. For linear network codes, the signal on an edge \((i, j)\) is a linear combination of the signals on the incoming edges at \(i\) or a linear combination of the source signals at \(i\). Let \(Y_{en}(\xi_n) = k\) and \(\text{head}(e_n) = l\) denote the signal on edge \(e_n \in E\). Then,

\[
Y_{en} = \sum_{\{e_m | \text{head}(e_m) = k\}} f_{m,n} Y_{em} \quad \text{if} \quad k \in V \setminus \{s_1, s_2\},
\]

\[
Y_{en} = \sum_{j=1}^{R_1} a_{ij,n} X_{ij} \quad \text{if} \quad X_i \text{ is observed at } k.
\]

The local coding vectors \(a_{ij,n}\) and \(f_{m,n}\) are also chosen from \(GF(q)\). We can also express \(Y_{en}\) as \(Y_{en} = \sum_{j=1}^{R_1} a_{ij,n} X_{ij} + \sum_{j=1}^{R_2} \beta_{jn} X_{2j}\). Then the global coding vector of \(Y_{en}\) is \([\alpha_n, \beta_n] = [\alpha_{1,n}, \alpha_{2,n}, \ldots, \beta_{1,n}, \beta_{2,n}, \ldots, \beta_{2,n}]\). We are free to choose an appropriate value of the field size \(q\).

In this work, we present an achievable rate region given a subset of the cut values in the cut vector; namely, \(k_1 - 1, k_2 - 2, k_1 - 2, k_2 - 1, k_1 - 1, k_1 - 2, k_2 - 2\). W.l.o.g, we assume there are \(k_{ij}\) outgoing edges from \(s_1\) and \(k_{ij}\) incoming edges to \(t_2\). If this is not the case one can always introduce an artificial source (terminal) node connected to the original source (terminal) node by \(k_{ij}\) edges. It can be seen that the new network has the same cut vector as the original network.

III. ACHIEVABLE RATE REGION FOR A GIVEN CUT VECTOR

First, suppose that only \(t_1\) is interested in recovering the random variables \(X_1\) and \(X_2\) which are observed at \(s_1\) and \(s_2\) respectively. Denote the rate from \(s_1\) to \(t_1\) and \(s_2\) to \(t_1\) as \(R_{11}\) and \(R_{12}\). Then the capacity region \(C_{t_1}\), that is achieved by routing will be

\[
\begin{align*}
R_{11} &\leq k_{1-1}, \\
R_{12} &\leq k_{2-1}, \\
R_{11} + R_{12} &\leq k_{1-2} - 1.
\end{align*}
\]

The capacity region \(C_{t_2}\) for \(t_2\) can be drawn in a similar manner. This is shown in Fig. 1(a). We also find the boundary points \(a, b, c, d\) such that their coordinates are \(a = (k_{2-1} - k_{2-1}, k_{2-1}), b = (k_{1-2}, k_{1-2} - k_{2-1}), c = (k_{1-1}, k_{2-1}, k_{2-1}), d = (k_{1-2} - k_{2-2}, k_{2-2}).\) A simple achievable rate region for our problem can be arrived at by multicasting both sources \(X_1\) and \(X_2\) to both the terminals \(t_1\) and \(t_2\).

**Theorem 3.1:** Rate pairs \((R_{11}, R_{12})\) belonging to the following set \(B\) can be achieved for two unicast sessions.

\[
B = \{R_1 \leq \min(k_{1-2}, k_{1-1}), \\
R_2 \leq \min(k_{2-2}, k_{2-2}), \\
R_1 + R_2 \leq \min(k_{1-2} - 1, k_{2-2})\}.
\]

**Proof:** We multicast both the sources to each terminal. This can be done using the multi-source multi-sink multicast result (Thm. 8) in \cite{2}.

![Fig. 1](image-url) (a) An example of a capacity region. (b) Base region for the example.

Subsequently we will refer to region \(B\) achieved by multicast as the base rate region (the grey region in Fig. 1(b)).

We now move on to precisely formulating the problem. Let \(Z_i\) denote the received vector at \(t_i\), \(X_i\) denote the transmitted vector at \(s_i\), and \(H_{ij}\) denote the transfer function from \(s_j\) to \(t_i\). Let \(M_i\) denote the encoding matrix at \(s_i\), i.e., \(M_i\) is the transformation from \(X_i\) to the transmitted symbols on the outgoing edges from \(s_i\). In our formulation, we will let the length of \(X_i\) to be \(k_i\) (i.e., the maximum possible). For transmission at rates \(R_{11}\) and \(R_{12}\), we introduce precoding matrices \(V_j, i = 1, 2\) of dimension \(R_i \times k_i\), so that the overall system of equations is as follows.

\[
\begin{align*}
Z_1 &= H_{11} M_1 V_1 X_1 + H_{12} M_2 V_2 X_2, \\
Z_2 &= H_{21} M_1 V_1 X_1 + H_{22} M_2 V_2 X_2.
\end{align*}
\]

(1)

We say that \(t_2\) can receive at rate \(R_{11}\) from \(s_1\) if it can decode \(V_1 X_1\) perfectly. The row dimension of the \(V_j\)'s can be adjusted to obtain different rate vectors. For \((R_{11}, R_{12}) \in B\), it can be shown that there exist local coding vectors over a large enough field such that the ranks of the different matrices in the first column of Table II are given by the corresponding entries in the third column, which correspond to the maximum possible. Furthermore, by the multi-source multi-sink multicast result, these matrices are such that \([H_{11} M_1, H_{12} M_2]\) is a full rank matrix of dimension \(k_{1-1} \times (R_1 + R_2)\), and \([H_{21} M_1, H_{22} M_2]\) is a full rank matrix of dimension \(k_{1-2} - 1 \times (R_1 + R_2)\). In Table II for instance since the minimum cut between \(s_1\) and \(t_1\) is \(k_{1-1}\), we know that the maximum rank of \(H_{11}\) is \(k_{1-1}\). Using the formalism of \cite{2}, we can conclude that there is a square submatrix of \(H_{11}\) of dimension \(k_{1-1} \times k_{1-1}\) whose determinant is not identically zero. Such appropriate submatrices can be found for each of the matrices in the first column of Table II. This in turn implies that their product is not identically zero and therefore using the Schwartz-Zippel lemma, we can conclude that there exists an assignment of local coding vectors so that the rank of all the matrices is simultaneously the maximum possible. For the rest of the paper, we assume that such a choice of local coding vectors has been made. Our arguments will revolve around appropriately modifying the source encoding matrices \(M_1\) and \(M_2\).

Note that there are two boundary points of the base region (the two boundary points may overlap). At point \(Q_1\), we denote the achievable rate pair by \((R_1^*, R_2^*)\) where

\[
R_1^* = \min(k_{1-2}, k_{1-1}), \quad \text{and} \quad R_2^* = \min(\min(k_{2-2}, k_{2-2}), \min(k_{1-2} - 1, k_{1-2} - 1)).
\]
Indeed a high value of $R_2$ though that this is nomenclature used for ease of presentation. Since $R_2 = \min(\min(k_2-1, k_2-2), \min(k_1-1, k_1-2) - R_2^{**})$.

In Fig. 1(a) these boundary points are $Q_1 = b$ and $Q_2 = e$. In what follows, we will present our arguments towards increasing the value of $R_1$ to be larger than $R_1^*$ (these arguments can be symmetrically applied for increasing $R_2$ as well). For this purpose, we will start with the point $Q_1$ and attempt to achieve points that are near it but do not belong to $B$. At $Q_1$, if $R_1^* = k_1-1$, then we cannot increase $R_1$ due to the cut constraints. Hence, we assume $R_1^* = k_1-2$. Furthermore, since $k_2-2 \geq k_2-2 - k_1-2 \geq \min(k_1-1, k_1-2) - k_2-1$, $R_2^{**} = \min(\min(k_2-2, k_2-2), \min(k_1-1, k_1-2) - R_1^*) = \min(k_2-2, \min(k_1-1, k_1-2) - k_2-1)$.

In this paper we refer to $k_1-2 + k_2-1$ as a measure of the interference in the network and in the subsequent discussion present achievable regions based on its value. We emphasize though that this is nomenclature used for ease of presentation. Indeed a high value of $k_1-2$ does not necessarily imply that there is a lot of interference at $t_2$, since the network code itself dictates the amount of interference seen by $t_2$. The following lemma will be used extensively.

**Lemma 3.2**: Consider a system of equations $Z = H_1 X_1 + H_2 X_2$, where $X_1$ is a vector of length $l_1$ and $X_2$ is a vector of length $l_2$ and $Z \in \text{span}(H_1, H_2)$.

The matrix $H_1$ has dimension $z_t \times l_1$ and rank $l_1 - \sigma$, where $0 \leq \sigma \leq l_1$. The matrix $H_2$ is full rank and has dimension $z_t \times l_2$ where $z_t \geq (l_1 + l_2 - \sigma)$. Furthermore, the column spans of $H_1$ and $H_2$ intersect only in the all-zeros vectors, i.e., $\text{span}(H_1) \cap \text{span}(H_2) = \{0\}$. Then there exists a unique solution for $X_2$.

**A. Low Interference Case**

This is the case when $k_1-2 + k_2-1 \leq \min(k_2-1, k_2-2)$. At $Q_1$, from the assumption, it follows that $R_1^* = k_1-2$, $R_2^{**} = \min(k_2-2, k_1-2) - k_2-1 = k_2-1$. An example is shown in Fig. 2(a). Furthermore, $Q_1 = Q_2 = e$.

Our solution strategy is to consider the encoding matrices $M_1$ and $M_2$ at the point $Q_1$, and to introduce a new encoding matrix at $s_1$, denoted $M_1'$ (with $R_1^* + \delta$ columns) such that $\text{span}(H_1 M_1') \cap \text{span}(H_1 M_2) = \{0\}$. As shown below, this will allow $t_1$ to decode from $s_1$ at rate $R_1^* + \delta$ and $t_2$ to decode from $s_2$ at rate $R_2^*$. After the modification, each $t_i$ is guaranteed to decode at the appropriate rate from $s_i$. A similar argument can then be applied for $R_2^*$ to arrive at the achievable rate region in this case.

At the point $Q_1$, the rates are $R_1^* = k_1-2$, $R_2^* = k_2-1$. Since both terminals can decode both sources, it holds that $\text{rank}(H_1 M_1) = k_1-2$, $\text{rank}(H_2 M_2) = k_2-1$, and $\text{span}(H_1 M_1) \cap \text{span}(H_2 M_2) = \{0\}$ for $i = 1, 2$.

By analyzing the properties of the above matrices, we have Theorem 3.4. Before we state the theorem, we first give the following lemma which will be used in proving Theorem 3.4.

**Lemma 3.3: Rate Increase Lemma.** In the base region, denote the achievable rates at $Q_1$ as $R_1^*$ and $R_2^*$, and the corresponding encoding matrices as $M_1$ and $M_2$. Let $\text{rank}([H_1, H_2 M_2]) = r \geq R_1^* + R_2^*$. There exist a series of full rank matrices $M_1^{(n)} = [M_1^{(n)} \ M_2^{(n)}]$ of dimension $k_1-12 \times ((n + R_1^*)$ such that $\text{rank}([H_1, H_2 M_2^{(n)}]) = R_1^* + R_2^* + n$, $0 \leq n \leq (r - R_1^* - R_2^*)$.

**Theorem 3.4**: Given a cut vector, if $k_1-2 + k_2-1 \leq \min(k_1-2, k_1-2)$, then the rate pair in the following region can be achieved.

**Region 1**:

$$R_1 \leq k_1-2 - k_2-1,$$

$$R_2 \leq k_1-2 - k_1-2,$$

which is shown in Fig. 2(b).

**Proof**: In this case, $R_1^* = k_1-2$ and $R_2^* = k_2-1$ is the boundary point $Q_1 = Q_2$. We will try to find full rank matrix $M_1'$ of dimension $k_1-12 \times (k_1-2 - k_2-1)$ and full rank matrix $M_2'$ of dimension $k_2-12 \times (k_2-2 - k_1-2)$ such that the system of equations can be written as

$$Z_1 = H_1 M_1' V_1 X_1 + H_2 M_2' V_2 X_2,$$

$$Z_2 = H_1 M_1' V_1 X_1 + H_2 M_2' V_2 X_2,$$

and $V_1 X_1$ can be decoded at $t_1$, $V_2 X_2$ can be decoded at $t_2$. The following are the achievable rate region.
First, note that $\text{rank}(H_{12}M_2) = \text{rank}(H_{12})$, which implies that $\text{span}(H_{12}) = \text{span}(H_{12}M_2)$. Therefore $\text{rank}([H_{11} H_{12}]) = \text{rank}([H_{11} H_{12} H_{12} M_2] = \text{rank}([H_{11} H_{12} M_2])$. Together, this implies that $\text{rank}([H_{11} H_{12} M_2]) = k_{1-2}$. Using the Rate Increase Lemma, we can find the matrix $M_1'$ such that the following conditions are satisfied: (i) $M_1'$ is a full rank matrix of dimension $k_{1-2} \times (k_{1-2} - k_{1-2})$, (ii) $\text{rank}(H_{11}M_1') = k_{1-2} - k_{2-1}$, and (iii) $\text{span}(H_{11}M_1') \cap \text{span}(H_{12}) = \{0\}$. (i) is from the Rate Increase Lemma. (ii) and (iii) hold because of the following argument. From Rate Increase Lemma and the fact that $\text{rank}(H_{12}M_2) = \text{rank}(H_{12}) = k_{2-1}$, we will have

$$k_{1-2} = \text{rank}(H_{11}M_1' H_{12}M_2)$$

$$= \text{rank}(H_{11}M_1') + \text{rank}(H_{12}M_2)$$

$$\leq \text{rank}(H_{11}M_1') + \text{rank}(H_{12}M_2)$$

$$\leq \text{rank}(M_1') + \text{rank}(H_{12})$$

$$= k_{1-2} - k_{2-1} + k_{2-1} = k_{1-2}.$$ Then all the inequalities become equalities. (ii) and (iii) are satisfied. Likewise, $M_2'$ can be found with similar conditions.

Next, since $\text{span}(H_{11}M_1') \cap \text{span}(H_{12}) = \{0\}$ and $\text{span}(H_{12}M_2) \subseteq \text{span}(H_{12})$, we will have $\text{span}(H_{11}M_1') \cap \text{span}(H_{12}M_2) = \{0\}$. By Lemma 3.2 and the above three conditions, $t_1$ can decode $V_1'X_1$ at rate $k_{1-2} - k_{2-1}$, but cannot decode $V_2'X_2$. By a similar argument, $t_2$ can decode $V_2'X_2$ at rate $k_{1-2} - k_{1-2}$, but cannot decode $V_1'X_1$.

**B. High Interference Case**

This is the case when $k_{1-2} + k_{2-1} \geq \min(k_{1-2}, k_{1-2})$. Recall that we also assume that $k_{2-1} \leq k_{1-1}$. At $Q_1$, $R_1' = k_{1-2} - k_{2-1} - k_{1-2} = \min(k_{1-2}, k_{1-2}) - k_{2-1}$. This means that $Q_1$ and $Q_2$ are two separated points. An example is shown in Fig. 1(a). In particular, when $C_{t_1}$ is contained in $C_{t_2}$ or vice versa, the achievable region is described by this case.

Our strategy is similar to the one for the previous case, but with important differences. We begin with the rate vector at point $Q_1$ and then attempt to increase $R_1$. However, in this particular case we will not be able to increase $R_2$ and in fact may need to reduce it. This is because at point $Q_1$, we have $R_2'' = \text{rank}(H_{12}M_2) < k_{2-1} = \text{rank}(H_{12})$, i.e., the encoding matrix $M_2$ is such that $\text{rank}(H_{12}M_2)$ is strictly less than the maximum possible. Therefore, if we augment $M_2$ with additional columns to arrive at $M_2'$, it is not possible to assert as before that the $\text{span}(H_{11}M_1') \cap \text{span}(H_{12}M_2') = \{0\}$. Hence, it may be possible that $s_1$ cannot be decoded at $t_1$, (after augmenting $M_2$ to $M_2'$). In this situation, we have the following result.

**Theorem 3.5:** Given a cut vector, if $k_{1-2} + k_{2-1} \geq \min(k_{1-2}, k_{1-2})$ and $k_{1-2} \leq k_{1-1}$, then the rate pair in the following region can be achieved.

Then in the above characterization, the sum rate constraint depends on $\text{rank}([H_{11} H_{12} M_2])$, we show a lower bound on $\text{rank}([H_{11} H_{12} M_2])$ in [H1-B1]. The following lemma that discusses situations in which rates can be traded off between the two unicast sessions is needed for the proof of Thm. 3.5

**Lemma 3.6: Rate Exchange Lemma.** Given that $\text{rank}([H_{11} M_1 H_{12} M_2]) = \text{rank}([H_{11} M_1 H_{12} M_2]) = r$, where $M_1$ is a full rank matrix of dimension $k_{1-2} \times (r - R_2)$, $M_2$ is a full rank matrix of dimension $k_{2-2} \times R_2$. If $M_1' = [\alpha M_1]$ where $\alpha$ is a vector of length $k_{1-2}$ and $\text{rank}(H_{11} M_1') = r - R_2 + 1$, then there exists an $M_2'$ such that $\text{span}(H_{11} M_1') \cap \text{span}(H_{12} M_2') = \{0\}$ where $M_2'$ is a full rank submatrix of $M_2$ of dimension $k_{2-2} \times (R_2 - 1)$.

**Proof of Theorem 3.5.** Given that $k_{2-2} + k_{2-1} \geq \min(k_{1-2}, k_{2-2})$ and $k_{1-2} \leq k_{1-1}$, we will extend the rate region from $Q_1$ where $R_1 = k_{1-2} - R_2' = \min(k_{1-2}, k_{1-2}) - k_{1-2}$. At $Q_1$, we need to increase $R_1$ while keeping $R_2$ as large as possible. By the Rate Increase Lemma, we can achieve the rate point $R_1' = \text{rank}([H_{11} H_{12} M_2]) - R_2'$, $R_2' = R_2$. The corresponding encoding matrices are $M_1'$ and $M_2'$. When we want to further increase $R_1'$, we could use the Rate Exchange Lemma repeatedly. Hence, when $R_1'$ is increased by $\delta$, $R_2'$ is decreased by $\delta$ where $0 \leq \delta \leq \min(R_2', k_{1-1} - R_1')$ ($\delta \leq k_{1-1} - R_1'$ comes from the fact that $R_1'$ can be increased to at most $k_{1-1}$). Terminal $t_1$ can decode messages both from $s_1$ at rate $R_1'' = R_1' + \delta$ and $s_2$ at rate $R_2'' = R_2' - \delta$. Denote the new set of encoding matrices as $M_1''$ and $M_2''$.

At $t_2$, because $M_2''$ is a submatrix of $M_2$, $\text{span}(H_{22} M_2'') \subseteq \text{span}(H_{22} M_2)$. Furthermore, we have $\text{span}(H_{21} M_1'') \subseteq \text{span}(H_{21} M_1) = \text{span}(H_{22} M_2)$, since $R_1' = k_{1-2}$. Hence, from the above argument, we will have $\text{span}(H_{21} M_1'') \cap \text{span}(H_{22} M_2'') = \{0\}$ since $\text{span}(H_{21} M_1) \cap \text{span}(H_{22} M_2) = \{0\}$. Then by Lemma 3.2, we can decode at $R_2'' = R_2' - \delta$ from $s_2$, but not decode any messages from $s_1$.

A similar analysis for $Q_2$ allows us to increase $R_2$, resulting in the following extended region.

**Corollary 3.7:** Given a cut vector, if $k_{1-2} + k_{2-1} \geq \min(k_{1-2}, k_{1-2})$ and $k_{2-1} \leq k_{2-2}$, then the rate pair in the following region can be achieved.

**Region 3:**

$$R_1 \leq R_2 \leq k_{2-2},$$

$$R_1 + R_2 \leq \text{rank}([H_{21} M_1 H_{22}]).$$

The overall rate region is the convex hull of base region, Region 2 and Region 3 which is shown in Fig. 3(a) where boundary segment $d - f$ is achieved via timesharing.

We note that the idea of increasing one rate while decreasing the other can also be applied to the region obtained in low interference case. Since $\text{rank}([H_{11} H_{12} M_2]) = k_{1-2} - 1$ and $\text{rank}([H_{21} M_1 H_{22}]) = k_{1-2} - 1$, we can obtain the following two new regions for low interference case.

**Region 2:**

$$R_1 \leq k_{1-1},$$

$$R_2 \leq \min(k_{1-2}, k_{1-2}) - k_{1-2},$$

$$R_1 + R_2 \leq \text{rank}([H_{11} H_{12} M_2]).$$

Note that in the above characterization, the sum rate constraint depends on $\text{rank}([H_{11} H_{12} M_2])$; we show a lower bound on $\text{rank}([H_{11} H_{12} M_2])$ in [H1-B1]
Finally, the achievable rate region for low interference case is the convex hull of the region 1, 2' and 3' shown in Fig. 3(b) where the boundary segment $d-f$ and $f-c$ is achieved via timesharing.

1) Lower bound of $\text{rank}([H_{11} H_{12} M_2])$: Next, we investigate the lower bound of $\text{rank}([H_{11} H_{12} M_2])$. In the following argument, $R_1$ and $R_2$ denote the rate at boundary point $Q_1$, and $M_1$ and $M_2$ denote the corresponding encoding matrices. First note that $\text{rank}([H_{11} H_{12} M_2]) \geq \text{rank}(H_{11}) = k_{1-1}$ and $\text{rank}([H_{11} H_{12} M_2]) \geq \text{rank}([H_{11} M_1 H_{12} M_2]) = R_1 + R_2$. Next we will also find another nontrivial lower bound of $\text{rank}([H_{11} H_{12} M_2])$ by the following lemma.

Lemma 3.8: Given $\text{rank}([H_{11} H_{12}]) = k_{1-2}$, $\text{rank}(H_{12}) = k_{2-1}$ and $\text{rank}([H_{12} M_2]) = l$, we have $\text{rank}([H_{11} H_{12} M_2]) \geq k_{1-2} - k_{2-1} + l$.

Proof: By the assumed conditions, there are $k_{2-1}$ columns in $H_{12}$ that are linearly independent, and in $H_{11}$, we can find a subset of $k_{1-2} - k_{2-1}$ columns denoted $H'_{11}$ such that $\text{span}(H'_{11}) \subseteq \text{span}(H_{12}) = \{0\}$ and $\text{rank}(H'_{11}) = k_{1-2} - k_{2-1}$, which further imply that $\text{rank}([H'_{11} H_{12}]) = k_{1-2}$.

Since $\text{span}(H_{12} M_2) \subseteq \text{span}(H_{12})$, this means that $\text{span}(H'_{11}) \cap \text{span}(H_{12} M_2) = \{0\}$. Therefore, $\text{rank}([H'_{11} H_{12} M_2]) = \text{rank}(H'_{11}) + \text{rank}(H_{12} M_2) - 0 = k_{1-2} - k_{2-1} + l$. Hence, $\text{rank}([H_{11} H_{12} M_2]) \geq \text{rank}([H'_{11} H_{12} M_2]) = k_{1-2} - k_{2-1} + l$.

Together with the two lower bounds above, we have $\text{rank}([H_{11} H_{12} M_2]) \geq \max(k_{1-1}, k_{1-2} - k_{2-1} + R_2^*, R_1^* + R_2^*)$, where $\max(k_{1-1}, k_{1-2} - k_{2-1} + R_2^*, R_1^* + R_2^*) = k_{1-2} - k_{2-1} + R_2^*$ is shown in Fig. 3(a).

Fig. 3. (a) The extended rate region for high interference case. (b) The final extended rate region for low interference case. For each point in the shaded grey area, both terminals can recover both sources. In the hatched grey area, for a given rate point, its $x$-coordinate is the rate for $s_1-t_1$ and its $y$-coordinate is the rate for $s_2-t_2$; the terminals are not guaranteed to decode both sources in this region.

IV. COMPARISON WITH EXISTING RESULTS

The authors in [8] and [12] explore the case when each source transmits one symbol at a time, or equivalently, $R_1 = R_2 = 1$ in detail, whereas we allow arbitrary rate pairs. Reference [10], also consider the scenario where the rates are arbitrary. Assuming that $k_{2-2} \leq k_{1-1}$, the basic region in [10] is

Region 2':
\[
\begin{align*}
R_1 &\leq k_{1-1} \\
R_2 &\leq k_{2-2} \\
R_1 + R_2 &\leq k_{1-2} - k_{2-2}
\end{align*}
\]

Region 3':
\[
\begin{align*}
R_1 &\leq k_{1-2} \\
R_2 &\leq k_{2-2} \\
R_1 + R_2 &\leq k_{1-2} - k_{2-2}
\end{align*}
\]

Region EF09:
\[
\begin{align*}
R_1 + 2R_2 &\leq k_{1-1} \\
R_2 &\leq k_{2-2}
\end{align*}
\]

They also extend the region using the knowledge of $k_{1-2}, k_{2-2}$ and other cut conditions arising from the network topology (see section IV of [10]). A comparison between our region and theirs indicates that there are example networks where there exist rate points that belong to our region but not to Region EF09. Conversely, there are instances of networks where points that belong to Region EF09, do not fall within our region. The work of [10] can be interpreted in part as an interference nulling scheme, and in future work it may be possible to incorporate this within our approach. The work of [6] considers several different cuts defined in the graph and propose an outer bound for the network capacity. Moreover, they provide certain network structures where the outer bound is tight. Since our work deals with an inner bound, it is qualitatively different. Finally Das et al. [12] have used interference alignment for the case of three unicast sessions, and are able to achieve a rate that is half the mincut for each unicast session. While this is an interesting result for a harder problem, the case of two unicast sessions considered here is different since each connection has only one interferer and the alignment problem does not exist. Moreover, achieving half the mincut for each session can be trivially achieved by timesharing in our problem. In that sense a comparison between our results and theirs is not possible.

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