AN ACTION OF A LIE ALGEBRA ON THE HOMOLOGY GROUPS OF MODULI
SPACES OF STABLE SHEAVES

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Abstract. We construct an action of a Lie algebra on the homology groups of moduli spaces of stable sheaves on K3 surfaces under some technical conditions. This is a generalization of Nakajima’s construction of $sl_2$-action on the homology groups. In particular, for an $A, D, E$-configuration of $(-2)$-curves, we shall give a collection of moduli spaces such that the associated Lie algebra acts on their homology groups.

0. Introduction

Let $X$ be a smooth projective surface defined over $\mathbb{C}$ and $H$ an ample divisor on $X$. Assume that $X$ is a K3 surface. Let $M_H(v)$ be the moduli space of $H$-stable sheaves $E$ with the Mukai vector $v(E) = v$ (cf. [11]). In [Y2], we studied a special kind of Fourier-Mukai transform called $(-2)$-reflection. For this purpose, we introduced the Brill-Noether locus on the moduli space and studied its properties. Similar results are obtained by Markman [M]. We fix a vector bundle $G$ on $X$. A stable sheaf $E_0$ is called exceptional, if $\text{Ext}^1(E_0, E_0) = 0$. Then $v(E_0)$ is a $(-2)$-vector, that is, $(v(E_0))^2 = -2$. We assume that the twisted degree $\text{deg}_G(E_0) := \text{deg}(G^\vee \otimes E_0) = 0$. Let $v \in H^*(X, \mathbb{Z})$ be a Mukai vector such that

$$\text{deg}_G(E) = \min\{\text{deg}_G(E') > 0 | E' \in K(X)\}$$

for $E \in M_H(v)$. Let

$$M_H(v)_{E_0, n} := \{E \in M_H(v) | \dim \text{Hom}(E_0, E) = n\}$$

be the Brill-Noether locus with respect to $E_0$. Under the condition (0.1), we showed that $M_H(v)_{E_0, n}$ is a Grassmannian bundle over a smooth manifold such that the relative cotangent bundle is isomorphic to the normal bundle $N_{M_H(v)_{E_0, n}/M_H(v)}$. Similar Grassmannian structure appears in Nakajima’s quiver varieties [N2]. By using this structure, he constructed a Lie algebra action on the (Borel-Moore) homology groups of quiver varieties. Based on our description of the Brill-Noether locus, recently Nakajima [N6] constructed an $sl_2$-action on the homology groups of moduli spaces $\bigoplus_v H_*(M_H(v), \mathbb{C})$, where $v$ runs a suitable set of Mukai vectors satisfying minimality condition (0.1).

In this note, under the same condition, we shall generalize Nakajima’s result. Thus we shall construct a Lie algebra action on the homology groups of moduli spaces of stable sheaves (Theorem 2.1): For a collection of exceptional sheaves $E_i, i = 1, 2, \ldots, s$ which satisfy some technical conditions, we shall construct operators $h_i, e_i, f_i, i = 1, 2, \ldots, s$ and show that they satisfy the commutation relations for Chevalley generators. In particular, we show that $[e_i, f_j] = h_i$ and $[e_i, f_j] = 0, i \neq j$. Since the first relation is proved by Nakajima, we only need to show the second one. For this purpose, we introduce the notion of universal extension (resp. division) with respect to $E_i, i = 1, 2, \ldots, s$ (see, sect. 1.3). This is our main idea and the other arguments are included in Nakajima’s papers. Since the action is defined by algebraic correspondences, we also have an action on the rational Chow groups. In section 3 we give some examples of actions.

Replacing $E_0$ by a purely 1-dimensional exceptional sheaf and the minimality condition by $\chi(E) = 1$, our construction also works for moduli spaces of purely 1-dimensional stable sheaves. In particular, we shall construct an action of the affine Lie algebra associated to a singular fiber of an elliptic surface. On an elliptic surface, purely 1-dimensional sheaves are related to torsion free sheaves of relative degree 0 via the relative Fourier-Mukai transform. Moreover purely 1-dimensional sheaves are related to the enumerative geometry of curves on $X$ (cf. [YZ]). Thus the moduli spaces of purely 1-dimensional stable sheaves are important objects to study. For a rational elliptic surface $X$, it is observed in [MNWV] that the Euler characteristics of the moduli spaces are $W(E^{(1)}_0)$-invariant, where $W(E^{(1)}_0)$ is the Weyl group associated to the $E^{(1)}_8$-lattice $K_2 \subset H^2(X, \mathbb{Z})$. An explanation is given in terms of the monodromy action, that is, we use the invariance of the homology groups of the moduli spaces under the deformation of $X$. Our construction of the Lie algebra gives another explanation of this invariance. These are treated in section 4. In section 5 we give a remark on the case of $G$-equivariant sheaves.

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1. Moduli of stable sheaves of minimal degree

Notation.
Let $X$ be a smooth projective surface. Let $\text{Coh}(X)$ be the category of coherent sheaves on $X$ and $K(X)$ the Grothendieck group of $X$. In this paper, we use the Borel-Moore homology groups. For an algebraic set $M$, $H_*(M, \mathbb{C})$ denotes the Borel-Moore homology group of $M$. If $M$ is compact, then $H_*(M, \mathbb{C})$ coincides with the usual singular homology group of $M$.

Let $D(X):=D^b(\text{Coh}(X))$ be the bounded derived categories of $\text{Coh}(X)$. For complexes $E,F \in D(X)$, we set $\text{Ext}^i(E,F):=\text{Hom}(E,F[i])$. We usually denote $\text{Ext}^0(E,F)$ by $\text{Hom}(E,F)$. For a morphism $\phi : E \to F$, $[E,F]$ denotes the mapping cone of a representative of $\phi$. If $H^i([E \to F])=0$ for all $i$, then we write $E \cong F$. We usually denote $\text{Ext}^i([E_1 \to E_2],F)$ (resp. $\text{Ext}^i(F,[E_1 \to E_2])$) by $\text{Ext}^i(E_1 \to E_2,F)$ (resp. $\text{Ext}^i(F,E_1 \to E_2)$).

Let $H$ be an ample divisor on $X$ and $G$ an element of $K(X)$ with $\text{rk}G>0$. For a coherent sheaf $E$ on $X$, we set $\deg_G(E):=\deg(G^i \otimes E)$ and $\chi_G(E):=\chi(G^i \otimes E)$.

1.1. Technical lemmas. In this subsection, we introduce some technical conditions (1.1), (1.7), and under these conditions we give some technical lemmas. These will play important roles for our construction of the action.

Definition 1.1. A purely 1-dimensional sheaf $E$ is $\mu$-stable, if the scheme-theoretic support $\text{Div}(E)$ of $E$ is reduced and irreducible.

We fix an ample divisor $H$ on $X$. Let $G$ be an element of $K(X)$ with $\text{rk} G>0$. In this note, we treat $\mu$-semi-stable sheaves $E$ with

$$\deg_G(E) = \min \{ \deg_G(E') > 0 \mid E' \in K(X) \}. $$

This is a fairly strong condition for $E$, but such $E$ behave very well.

Lemma 1.1. Let $G$ be an element of $K(X)$ with $\text{rk} G>0$ and $E_i$, $i=1,2,\ldots,s$ be $\mu$-stable vector bundles with $\deg_G(E_i)=0$. Let $E$ be a $\mu$-semi-stable sheaf satisfying (1.1).

(1) Then $E$ is $\mu$-stable.
(2) Every non-trivial extension

$0 \to E_1 \to F \to E \to 0$

defines a $\mu$-stable sheaf.

(3) Let $V_i$ be subspaces of $\text{Hom}(E_i,E)$, $i=1,2,\ldots,s$. Then $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective or surjective in codimension 1. Moreover,

(3-1) if $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective, then the cokernel is $\mu$-stable.

(3-2) if $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is surjective in codimension 1, then $\ker \phi$ is $\mu$-stable. In particular $D(E):=\text{Ext}^1(\bigoplus_{i=1}^s V_i \otimes E_i \to E, \mathcal{O}_X)$ is $\mu$-stable.

Since $\deg_G(E)/\text{rk}(E) = \deg(G)(\deg(E)/\text{rk}E - \deg(G)/\text{rk}G)$, the $\mu$-stability can be defined by using the $G$-twisted slope $\deg_G(E)/\text{rk}(E)$. By using the following lemmas, the proof of \cite[Lem. 2.1]{Y} implies our lemma. So we only give a proof of (1), (3). We first note the following easy lemmas.

Lemma 1.2. A purely 1-dimensional sheaf $E$ with (1.1) is $\mu$-stable.

Lemma 1.3. Let $r,d,x$ be positive integers. Let $y$ be an integer such that $y \in d\mathbb{Z}$. If $0 < y/x < d/r$, then $y \geq d \times r$.

Proof of Lemma 1.2 (1), (3). Let $E'$ be a subsheaf of $E$ with $\deg_G(E')/\text{rk}E = \deg_G(E')/\text{rk}E'$. Then $1 \geq \deg_G(E')/\deg_G(E') = \text{rk}E'/\text{rk}E' \geq 1$. Hence $\text{rk}E' = \text{rk}E$ and $\deg_G(E') = \deg_G(E)$, which implies that $E$ is $\mu$-stable. Thus (1) holds. We shall prove (3). We first assume that $\text{rk}E > 0$. By the $\mu$-stability of $E$, we have

$$0 \leq \frac{\deg_G(\text{im} \phi)}{\text{rk}(\text{im} \phi)} \leq \frac{\deg_G(E)}{\text{rk}E}.$$

By Lemma 1.2 (i) $\deg_G(\text{im} \phi) = 0$ or (ii) $\deg_G(\text{im} \phi)/\text{rk}(\text{im} \phi) = \deg_G(E)/\text{rk}E$. In the first case, $\deg_G(\ker \phi) = 0$. Assume that $\ker \phi \neq 0$. Let $F$ be a $\mu$-stable locally free subsheaf of $\ker \phi$ with $\deg_G(F) = 0$. Then there is a non-zero homomorphism $F \to E_1$, which is isomorphic. Hence $\text{Hom}(E_1, \ker \phi) \neq 0$, which is a contradiction. Therefore $\ker \phi = 0$. We shall show that $E' := \text{coker} \phi$ is $\mu$-stable. We note that $E'$ does not have a 0-dimensional subsheaf and $\deg_G(E') = \deg_G(E)$. We first assume that $\text{rk}E' > 0$. If $E'$ is not $\mu$-stable, then (1) implies that $E'$ is not $\mu$-semi-stable. Then there is a quotient $E' \to F$ with $\deg_G(F)/\text{rk}F < \deg_G(E')/\text{rk}E'$. By Lemma 1.2, $\deg_G(F) \leq 0$, which implies that $\deg_G(F)/\text{rk}F < \deg_G(E)/\text{rk}E$. This is a contradiction. Therefore $E'$ is $\mu$-stable. If $\text{rk}E' = 0$, then $E'$ is of pure dimension 1. Then Lemma 1.2 implies that $E'$ is $\mu$-stable.
We next treat the second case: \( \deg_G(\text{im} \phi)/\text{rk}(\text{im} \phi) = \deg_G(E)/\text{rk} E \). In this case, \( \phi \) is \( \mu \)-stable. Assume that there is a locally free subsheaf \( F \) of \( \text{ker} \phi \) with \( \deg_G(F)/\text{rk} F > \deg_G(\text{ker} \phi)/\text{rk}(\text{ker} \phi) = -\deg_G(E)/\text{rk}(\text{ker} \phi) \). Then we get that \( \deg_G(F) \leq 0 \). If \( \deg_G(F) = 0 \), then \( \text{Hom}(F,E_i) \neq 0 \) for an \( i \). Since \( E_i \) and \( F \) are \( \mu \)-stable sheaves with the same slope, non-trivial homomorphism \( F \to E_i \) is isomorphic in codimension 1. Since \( F \) is locally free, we conclude that \( F \cong E_i \). Then \( \text{Hom}(E_i,F) \neq 0 \), which is a contradiction. Hence \( \deg_G(F) < 0 \), which means that \( 0 < -\deg_G(F)/\text{rk} F < \deg_G(E)/\text{rk}(\text{ker} \phi) \). Then Lemma 1.3 implies that \( -\deg_G(F) \geq \deg_G(E) \) and \( \text{rk} F > \text{rk}(\text{ker} \phi) \), which is a contradiction. Therefore \( \text{ker} \phi \) is \( \mu \)-stable.

If \( \text{rk} E = 0 \), then since \( E \) is \( \mu \)-stable, we get \( \phi = 0 \) or \( \phi \) is surjective in codimension 1. Then by the same arguments as above, we see that \( \text{ker} \phi \) is \( \mu \)-stable. \( \square \)

Besides the condition for \( \mu \)-semi-stable sheaves (1.1), we also introduce similar conditions and lemmas for Gieseker (twisted) semi-stabilities.

**Definition 1.2.** Let \( G \) be an element of \( K(X) \) with \( \text{rk} G > 0 \). A torsion free sheaf \( E \) is \( G \)-twisted stable, if

\[
\frac{\chi_G(F(nH))}{\text{rk} F} < \frac{\chi_G(E(nH))}{\text{rk} E}, n \gg 0
\]

for all proper subsheaf \( F(\neq 0) \) of \( E \).

As in the proof of Lemma 1.1 we also have the following assertions.

**Lemma 1.4.** Let \( G \) be an element of \( K(X) \) with \( \text{rk} G > 0 \) and \( E_i, i = 1, 2, \ldots, s \), be \( G \)-twisted stable sheaves with \( \deg_G(E_i) = \chi_G(E_i) = 0 \). Let \( E \) be a \( G \)-twisted stable torsion free sheaf with \( \deg_G(E) = 0 \) and

\[
\chi_G(E) = \min \{ \chi_G(E') > 0 \mid E' \in \text{Coh}(X), \deg_G(E') = 0 \}
\]

or \( E = \mathbb{C}_P, P \in X \) with \( P \notin G \).

1. Then every non-trivial extension

\[
0 \to E_1 \to F \to E_1 \to 0
\]

defines a \( G \)-twisted stable sheaf.

2. Let \( V_i \) be a subspace of \( \text{Hom}(E_i, E) \). Then \( \phi : \bigoplus^s_{i=1} V_i \otimes E_i \to E \) is injective or surjective. Moreover,

- (2-1) if \( \phi : \bigoplus^s_{i=1} V_i \otimes E_i \to E \) is injective, then the cokernel is a \( G \)-twisted stable torsion free sheaf or \( \mathbb{C}_P, P \in X \),

- (2-2) if \( \phi : \bigoplus^s_{i=1} V_i \otimes E_i \to E \) is surjective, then \( \text{ker} \phi \) is \( G \)-twisted stable.

**Lemma 1.5.** Let \( G \) be an element of \( K(X) \) with \( \text{rk} G > 0 \) and \( E_i, i = 1, 2, \ldots, s \), be \( G \)-twisted stable sheaf with \( \deg_G(E_i) = \chi_G(E_i) = 0 \). Let \( E \) be a \( G \)-twisted stable torsion free sheaf with \( \deg_G(E) = 0 \) and

\[
\chi_G(E) = \max \{ \chi_G(E') < 0 \mid E' \in \text{Coh}(X), \deg_G(E') = 0 \}
\]

1. Then every non-trivial extension

\[
0 \to E \to F \to E_1 \to 0
\]

defines a \( G \)-twisted stable sheaf.

2. Let \( V_i \) be a subspace of \( \text{Hom}(E, E_i) \). Then \( \phi : E \to \bigoplus^s_{i=1} V_i \otimes E_i \) is injective or surjective. Moreover,

- (2-1) if \( \phi : E \to \bigoplus^s_{i=1} V_i \otimes E_i \) is injective, then the cokernel is a \( G \)-twisted stable torsion free sheaf or \( \mathbb{C}_P, P \in X \),

- (2-2) if \( \phi : E \to \bigoplus^s_{i=1} V_i \otimes E_i \) is surjective, then \( \text{ker} \phi \) is \( G \)-twisted stable.

1.2. Basic properties of stable sheaves of minimal degree. Assume that \( K_X \) is numerically trivial. We define a bilinear form \( \langle \ , \ \rangle \) on \( H^*(X, \mathbb{Q}) := \bigoplus^s_{i=0} H^{2i}(X, \mathbb{Q}) \) by

\[
\langle x, y \rangle := \int_X x_1 \wedge y_1 - x_0 \wedge y_2 - x_2 \wedge y_0
\]

where \( x_i \in H^{2i}(X, \mathbb{Q}) \) (resp. \( y_i \in H^{2i}(X, \mathbb{Q}) \)) is the \( 2i \)-th component of \( x \) (resp. \( y \)).

For an object \( E \in D(X) \), we define the Mukai vector of \( E \) by

\[
v(E) = \sum_i (-1)^i v(H^i(E))
\]

\[
= \sum_i (-1)^i \text{ch}(H^i(E)) \sqrt{t \text{d}_X} \in H^*(X, \mathbb{Q}),
\]

where \( t \text{d}_X \) is the todd class of \( X \). We have a map \( v : D(X) \to H^*(X, \mathbb{Q}) \). We call an element of \( v(D(X)) \) a Mukai vector. For \( E, F \in D(X) \), we define the Riemann-Roch number by

\[
\chi(E, F) := \sum_i (-1)^i \text{dim Ext}^i(E, F).
\]
Then the Riemann-Roch theorem says the following.

**Proposition 1.6.**

\[(1.12) \quad \chi(E,F) = -(v(E),v(F)).\]

By a similar way, we also define the rank \(\text{rk} E\) and other invariants. We fix an element \(G \in K(X)\) with \(\text{rk} G > 0\). For an object \(E \in \mathbf{D}(X)\) such that \(\text{deg}_G(E)\) satisfies \([1.1]\), we define a stability condition.

**Definition 1.3.** Let \(E \in \mathbf{D}(X)\) be an object such that \(\text{deg}_G(E)\) satisfies \([1.1]\). Then \(E\) is stable, if

\[(1.13) \quad H^i(E \otimes \mathbb{C}_P) = 0, i \neq -1,0\]

for all \(P \in X\) and one of the following conditions holds:

(i) \(H^i(E) = 0, i \neq 0\) and \(H^0(E)\) is a stable sheaf.

(ii) \(H^0(E) = 0, i \neq -1,0, H^{-1}(E)^v\) is a stable sheaf and \(H^0(E)\) is a 0-dimensional sheaf.

**Remark 1.1.**

(i) The condition \([1.1]\) implies that there is a complex \(C_{-1} \to C_0\) of locally free sheaves which is quasi-isomorphic to \(E\).

(ii) If \(\text{rk} E < 0\), then \(H^i(D(E)) = 0, i \neq 1\) and \(H^1(D(E))\) is a stable sheaf, where \(D(E) := R\operatorname{Hom}(E,F_X)\) is the dual of \(E\). Since we want to treat two cases simultaneously, we use \(E\) instead of using \(D(E)\).

**Definition 1.4.** For a Mukai vector \(v \in H^*(X,\mathbb{Q})\) with the property \([1.1]\), let \(M_H(v)\) be the moduli space of \((\text{quasi-isomorphism classes of})\) stable complexes \(E\) with \(v(E) = v\).

If \(\text{rk} v < 0\), then by Remark \([1.1]\) \(M_H(v)\) has a scheme structure. The Zariski tangent space of \(M_H(v)\) at \(E\) is \(\operatorname{Ext}^1(E,E)\) and the obstruction for the infinitesimal liftings belongs to the kernel of the trace map

\[(1.14) \quad \text{tr} : \operatorname{Ext}^2(E,E) \to H^2(X,\mathcal{O}_X).\]

In this paper, we assume that the trace map

\[(1.15) \quad \text{tr} : \operatorname{Ext}^2(E,E) \to H^2(X,\mathcal{O}_X)\]

is isomorphic.

By Lemma \([1.1]\) and the condition \([1.1]\), we get the following assertions.

**Lemma 1.7.** Assume that \(v \in H^*(X,\mathbb{Q})\) satisfies \([1.1]\).

(i) If \(M_H(v) \neq \emptyset\), then \(\dim M_H(v) = \langle v^2 \rangle + 1 + p_g\). In particular, if there is a stable complex \(E\) with \(v(E) = v\), then \(\langle v(E)^2 \rangle \geq -(p_g + 1)\).

(ii) Assume that \(X\) is a K3 surface. Then there is a stable complex \(E\) with \(v(E) = v\) if and only if \(\langle v^2 \rangle \geq -2\).

For the proof of (ii), we also use [1.2] Thm. 0.2.

Let \(S := \{E_1, E_2, \ldots, E_n\}\) be a finite set of \(\mu\)-stable vector bundles such that \(\text{deg}_G(E_i) = 0, 1 \leq i \leq n\). We assume that

\[(1.16) \quad E_i \otimes K_X \cong E_i, E_i \in S.\]

Let \(S\) be a subcategory of \(\operatorname{Coh}(X)\) consisting of semi-stable sheaves \(F\) whose Jordan-Hölder grading is \(\bigoplus_i E_i^{\oplus n_i}\).

**Lemma 1.8.** \(\operatorname{Hom}(E,F) = 0\) and \(\operatorname{Hom}(F[1],E) = 0\) for \(F \in S\).

**Proof.** We use the spectral sequence

\[(1.17) \quad E_2^{p,q} = \bigoplus_{q'+q'' = q} \operatorname{Ext}^p(H^{-q'}(*),H^{q''}(**)) \implies E_\infty^{p+q} = \operatorname{Ext}^{p+q}(*,**).\]

Since \(H^i(E) = 0, i \neq -1,0\), \(\operatorname{Hom}(E,F) = \operatorname{Hom}(H^0(E),F)\). If \(\text{rk} E \geq 0\), then \(H^0(E)\) is a stable sheaf of positive \(G\)-twisted degree. Hence \(\operatorname{Hom}(H^i(E),F) = 0\). If \(\text{rk} E < 0\), then \(H^0(E)\) is a 0-dimension sheaf. Hence \(\operatorname{Hom}(H^0(E),F) = 0\). Therefore the first claim holds. Since \(\operatorname{Hom}(F[1],E) = \operatorname{Hom}(F,H^{-1}(E))\), we also get the second claim. 

\(\square\)
1.3. A universal division and a universal extension.

**Definition 1.5.** An exact triangle

\[
F \rightarrow E \rightarrow \tilde{E} \rightarrow F[1]
\]

is a universal division of \(E\) with respect to \(\{E_1, E_2, \ldots, E_n\}\), if \(F \in \mathcal{S}\) and \(\tilde{E}\) is a stable complex such that \(\text{Hom}(E_i, \tilde{E}) = 0, 1 \leq i \leq n\).

For an exact triangle

\[
F' \rightarrow E \rightarrow E' \rightarrow F'[1],
\]

we have an exact sequence

\[
\text{Hom}(F'[1], \tilde{E}) \rightarrow \text{Hom}(E', \tilde{E}) \rightarrow \text{Hom}(E, \tilde{E}) \rightarrow \text{Hom}(F', \tilde{E}).
\]

By our assumption and Lemma 1.8, \(\text{Hom}(E', \tilde{E}) \rightarrow \text{Hom}(E, \tilde{E})\) is an isomorphism. Hence we have a unique morphism \(E' \rightarrow \tilde{E}\) in \(\mathcal{D}(X)\) which induces a commutative diagram of exact triangles (in \(\mathcal{D}(X)\)):

\[
\begin{array}{cccc}
F' & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & F'[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
F & \longrightarrow & E & \longrightarrow & \tilde{E} & \longrightarrow & F[1].
\end{array}
\]

In particular, a universal division of \(E\) is unique (up to isomorphism class in \(\mathcal{D}(X)\)). Since \(\text{Hom}(\tilde{E}, \tilde{E}) \cong \mathbb{C}\) and \(\text{Hom}(F, \tilde{E}) = 0\), we get

\[
\text{Hom}(E, \tilde{E}) \cong \mathbb{C}.
\]

Since \(E_i \otimes K_X \cong E_i\), we see that \(\text{Hom}(F \otimes K_X^r, \tilde{E}) = \text{Hom}(F \otimes K_X^r[1], \tilde{E}) = 0\). Hence we also get that

\[
\text{Hom}(E, \tilde{E} \otimes K_X) \cong \text{Hom}(\tilde{E}, \tilde{E} \otimes K_X) \cong H^0(X, K_X).
\]

**Definition 1.6.** An exact triangle

\[
F \rightarrow \tilde{E} \rightarrow E \rightarrow F[1]
\]

is a universal extension of \(E\) with respect to \(\{E_1, E_2, \ldots, E_n\}\), if \(F \in \mathcal{S}\) and \(\tilde{E}\) is a stable complex such that \(\text{Ext}^1(\tilde{E}, E_i) = 0, 1 \leq i \leq n\).

For an exact triangle

\[
F' \rightarrow E' \rightarrow E \rightarrow F'[1],
\]

we have an exact sequence

\[
\text{Hom}(\tilde{E}, F') \rightarrow \text{Hom}(\tilde{E}, E') \rightarrow \text{Hom}(\tilde{E}, E) \rightarrow \text{Ext}^1(\tilde{E}, F').
\]

By our assumption and Lemma 1.8, \(\text{Hom}(\tilde{E}, E') \rightarrow \text{Hom}(\tilde{E}, E)\) is an isomorphism. Hence we have a unique morphism \(\tilde{E} \rightarrow E'\) which induces a commutative diagram of exact triangles

\[
\begin{array}{cccc}
F' & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & F'[1] \\
\uparrow & & \| & & \uparrow & & \uparrow \\
F & \longrightarrow & \tilde{E} & \longrightarrow & E & \longrightarrow & F[1].
\end{array}
\]

In particular, a universal extension of \(E\) is unique. For a universal extension, we also see that

\[
\text{Hom}(\tilde{E}, E) \cong \mathbb{C},
\]

\[
\text{Hom}(\tilde{E}, E \otimes K_X) \cong \text{Hom}(\tilde{E}, \tilde{E} \otimes K_X) \cong H^0(X, K_X).
\]

1.3.1. **Condition for the existence.**

**Lemma 1.9.**

(i) If \(S\) defines a negative definite lattice, then a universal extension and a universal division exist for \(E\).

(ii) Assume that \(S\) defines a negative semi-definite lattice of affine type. Let \(\delta := \sum_i a_i v(E_i)\) satisfy \(\langle \delta, v(E_i) \rangle = 0\) for all \(E_i \in S\). If \(\langle v(E), \delta \rangle \neq 0\), then a universal extension or a universal division exist for \(E\).

**Proof.**
Claim 1.1. For a non-zero morphism $\psi : E_{n_1} \to E$, $E^{(1)} := [E_{n_1} \to E]$ is also stable.

Proof of Claim 1.1 For a non-zero morphism $\psi : E_{n_1} \to E$, we have an exact sequence

$$\text{Hom}(E_{n_1}, \mathbb{E}[−1]) \to \text{Hom}(E_{n_1}, E^{(1)}[−1]) \to \mathbb{C} \xrightarrow{\psi} \text{Hom}(E_{n_1}, \mathbb{E}).$$

By Lemma 1.8, $\text{Hom}(E_{n_1}, \mathbb{E}[−1]) = 0$, and hence we get $\text{Hom}(E_{n_1}, E^{(1)}[−1]) = 0$. We note that $E^{(1)}$ satisfies \ref{1.3} and we have the following exact sequence

$$0 \longrightarrow H^{-1}(E) \longrightarrow H^{-1}(E^{(1)}) \longrightarrow$$

$$E_{n_1} \longrightarrow H^{0}(E) \longrightarrow H^{0}(E^{(1)}) \longrightarrow 0.$$

Then we get $0 = \text{Hom}(E_{n_1}, E^{(1)}[−1]) \cong \text{Hom}(E_{n_1}, H^{-1}(E^{(1)}))$. If $H^{-1}(E) = 0$, then $E_{n_1} \to H^{0}(E)$ is a non-zero homomorphism. By Lemma 1.1 $E^{(1)}$ is stable. Assume that $H^{-1}(E^{(1)}) \neq 0$. Since $H^{-1}(E^{(1)})$ is locally free and $\text{Hom}(E_{n_1}, H^{-1}(E^{(1)})) = 0$, the extension

$$0 \to H^{-1}(E) \to H^{-1}(E^{(1)}) \to E_{n_1} \to 0$$

does not split, where $E'_{n_1}$ is a subsheaf of $E_{n_1}$ with $(E'_{n_1})^\vee = E_{n_1}$. By Lemma 1.1 $E^{(1)}$ is a stable complex. Thus the claim holds.

If there is a non-zero morphism $E_{n_2} \to E^{(1)}$, then we set $E^{(2)} := [E_{n_2} \to E^{(1)}]$. Then we have an exact triangle

$$F^2 \to E \to E^{(2)} \to F^2[1],$$

where $F^2$ fits in an exact sequence

$$0 \to F_{n_1} \to F^2 \to F_{n_2} \to 0.$$

Continuing this procedure, we get a sequence of stable complexes

$$E = E^{(0)}, E^{(1)}, \ldots, E^{(s)}, \ldots,$$

where $E^{(s)}$ fits in an exact triangle

$$F^s \to E \to E^{(s)} \to F^s[1],$$

$F^s \in S$. Since $v(E^{(s)}) = v(E^{(0)}) - \sum v(E_{n_i})$, if $S$ generate a negative definite lattice or $\langle \delta, v(E) \rangle > 0$, then $\langle v(E^{(s)})^2 \rangle = - (1 + p_g)$ for some $s$. By Lemma 1.7 this is impossible. Hence $\text{Hom}(E_{n_i}, E^{(s)}) = 0$, $1 \leq i \leq n$ for some $s$.

For a non-zero morphism $\psi : E \to E_{n_0}[1]$, we set $E^{(-1)}[1] := [E \to E_{n_0}[1]]$. Then $E^{(-1)}$ fits in an exact triangle:

$$E_{n_0} \to E^{(-1)} \to E \to E_{n_0}[1].$$

Claim 1.2. $E^{(-1)}$ is a stable complex.

Proof of Claim 1.2 For a non-zero morphism $\psi : E \to E_{n_0}[1]$, we have an exact sequence

$$\text{Hom}(E[1], E_{n_0}[1]) \to \text{Hom}(E^{(-1)}[1], E_{n_0}[1]) \to \mathbb{C} \xrightarrow{\psi} \text{Hom}(E, E_{n_0}[1]).$$

By Lemma 1.8, $\text{Hom}(E[1], E_{n_0}[1]) = 0$, and hence $\text{Hom}(E^{(-1)}[1], E_{n_0}[1]) = 0$. By our assumption, $E^{(-1)}$ satisfies \ref{1.3} and $H^{0}(E^{(-1)})$, $i = -1, 0$ fits in the exact sequence

$$0 \longrightarrow H^{-1}(E^{(-1)}) \longrightarrow H^{-1}(E) \longrightarrow$$

$$E_{n_0} \longrightarrow H^{0}(E^{(-1)}) \longrightarrow H^{0}(E) \longrightarrow 0.$$

If $H^{-1}(E) = 0$, then since $\text{Hom}(H^{0}(E^{(-1)}), E_{n_0}) = \text{Hom}(E^{(-1)}[1], E_{n_0}[1]) = 0$, Lemma 1.1 (2) implies that $H^{0}(E^{(-1)})$ is stable. Assume that $H^{-1}(E) \neq 0$. If $H^{-1}(E) \to E_{n_0}$ is a zero map, then since $H^{0}(E)$ is of 0-dimensional and $E_{n_0}$ is locally free, we get $\text{Ext}^{1}(H^{0}(E), E_{n_0}) = 0$. Hence the second line splits, which is a contradiction. Thus $\xi : H^{-1}(E) \to E_{n_0}$ is non trivial. Then by applying Lemma 1.1 (3) to $\xi^{\vee} : E_{n_0}^{\vee} \to H^{-1}(E)\vee$, we see that (1) $\xi^{\vee}$ is injective except finite subset of $X$ and $\text{coker}(\xi^{\vee})$ is $\mu$-stable torsion free sheaf, or (2) $\xi^{\vee}$ is injective except a divisor of $X$ and $\text{coker}(\xi^{\vee})$ is $\mu$-stable purely 1-dimensional sheaf, or (3) $\xi^{\vee}$ is surjective in codimension 1 and ker $\xi^{\vee}$ is a $\mu$-stable sheaf. In the case of (1), $H^{0}(E^{(-1)})$ is 0-dimensional and $H^{-1}(E^{(-1)})$ is a $\mu$-stable sheaf. If the case (2) occur, then $H^{0}(E^{(-1)})$ is a $\mu$-stable 1-dimensional sheaf and $H^{-1}(E^{(-1)}) = 0$. In the last case, $H^{0}(E^{(-1)})$ is a $\mu$-stable torsion free sheaf and $H^{-1}(E^{(-1)}) = 0$. Therefore $E^{(-1)}$ is a stable complex and we complete the proof of the claim.

If there is a non-zero homomorphism $E^{(-1)} \to E_{n-1}[1]$, we set $E^{(-2)}[1] := [E^{(-1)} \to E_{n-1}[1]]$. Continuing this procedure, we get a sequence of stable complexes

$$\ldots, E^{(-t)}, \ldots, E^{(-1)}, E^{(0)}.$$
Since $v(\mathbb{E}^{(-t)}) = v(\mathbb{E}(0)) + \sum v(E_n)$, if $S$ generate a negative definite lattice or $\langle \delta, v(\mathbb{E}) \rangle < 0$, then we see that $\text{Hom}(E_i, \mathbb{E}^{(-t)}) = 0, 1 \leq i \leq n$ for some $-t$. Therefore Lemma 1.9 holds. □

**Lemma 1.10.** Assume that $S$ satisfies the condition of (i) or (ii) in Lemma 1.9. If there is an exact triangle

\[(1.40) \quad F \to E \to E' \to F[1],\]

where $F \in \mathcal{S}$. Then $\text{Hom}(E, E') \cong \mathbb{C}$ and $\text{Hom}(E, E' \otimes K_X) \cong H^0(X, K_X)$.

**Proof.** We only show the first assertion. We assume that there is a universal division $\mathbb{E}$. Since $\text{Hom}(E', \mathbb{E}) \to \text{Hom}(\mathbb{E}, \mathbb{E})$ is surjective, we have an exact triangle

\[(1.41) \quad F' \to E' \to \mathbb{E} \to F'[1],\]

where $F' \in \mathcal{S}$. By the exact sequence

\[(1.42) \quad \text{Hom}(E, F') \to \text{Hom}(E, E') \to \text{Hom}(E, \mathbb{E}) = \mathbb{C}\]

and Lemma 1.8, we get our claim. If there is a universal extension $\mathbb{E}$, we also see that $\text{Hom}(E, E') \cong \mathbb{C}$. □

### 1.4 Coherent systems

We set

\[(1.43) \quad \mathcal{P}_E^{(n)}(v) := \{(E, U) | E \in M_H(v), U \subset \text{Hom}(E_i, E), \dim U = n\}.\]

$\mathcal{P}_E^{(n)}(v)$ is the moduli space of coherent systems. For the construction of $\mathcal{P}_E^{(n)}(v)$, see section 6.4. The Zariski tangent space of $\mathcal{P}_E^{(n)}(v)$ at $(E, U)$ is

\[(1.44) \quad \text{Ext}^1(U \otimes E_i \to E, E)/\text{End}(U \otimes E_i)\]

and the obstruction for the infinitesimal deformation belongs to the kernel of

\[(1.45) \quad \tau : \text{Ext}^2(U \otimes E_i \to E, E) \to \text{Ext}^2(E, E) \xrightarrow{i} H^2(X, \mathcal{O}_X).\]

By Lemma 1.10 and the Serre duality, $\ker \tau = 0$. Thus $\mathcal{P}_E^{(n)}(v)$ is a smooth scheme with

\[(1.46) \quad \dim \mathcal{P}_E^{(n)}(v) = \dim \text{Ext}^1(U \otimes E_i \to E, E)/\text{End}(U \otimes E_i) = \langle v - nv_i, v \rangle - n^2 + (1 + p_g) = \frac{1}{2}(\dim M_H(v) + \dim M_H(v - nv_i)).\]

For $(E, U) \in \mathcal{P}_E^{(n)}(v)$, $E$ and $[U \otimes E_i \to E]$ are stable. Hence we have morphisms $\pi : \mathcal{P}_E^{(n)}(v) \to M_H(v)$ and $\varpi : \mathcal{P}_E^{(n)}(v) \to M_H(v - nv_i)$.

**Remark 1.2.** We set $F := [U \otimes E_i \to E]$. Since $\text{Hom}(E[1], E_i[1]) = \text{Hom}(E_i \otimes U, E_i[1]) = 0$, by the exact triangle

\[(1.47) \quad U \otimes E_i \to E \to F \to U \otimes E_i[1],\]

we have an exact sequence

\[(1.48) \quad 0 \to U^\vee \to \text{Hom}(F, E_i[1]) \to \text{Hom}(E, E_i[1]) \to 0.\]

Thus we have

\[(1.49) \quad \mathcal{P}_E^{(n)}(v) := \{(F, U^\vee) | F \in M_H(v - nv_i), U^\vee \subset \text{Hom}(F, E_i[1]), \dim U = n\}.\]

We set $F_i := U \otimes E_i$. Then we have the following exact and commutative diagram:

\[(1.50) \quad \begin{array}{cccc}
\text{Hom}(F_i, F_i) & \downarrow & \text{Ext}^1(F_i \to E, F_i) & \downarrow \\
\text{Hom}(F_i \to E, F_i \to E) & \longrightarrow & \text{Ext}^1(F_i \to E, F_i) & \longrightarrow & \text{Ext}^1(F_i \to E, F_i \to E) & \longrightarrow & \text{Ext}^1(F_i \to E, F_i \to E) & \longrightarrow & \text{Ext}^1(F_i \to E, F_i \to E) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Hom}(E, F_i \to E) & \longrightarrow & \text{Ext}^1(E, F_i) & \longrightarrow & \text{Ext}^1(E, E) & \longrightarrow & \text{Ext}^1(E, F_i) & \longrightarrow & \text{Ext}^1(E, F_i) \\
\end{array}\]

By Lemma 1.10, we see that $\text{Ext}^1(E, F_i) \to \text{Ext}^1(E, E)$ is injective, which implies that

\[(1.51) \quad \text{Ext}^1(F_i \to E, E)/\text{End}(F_i) \to \text{Ext}^1(F_i \to E, F_i \to E) \oplus \text{Ext}^1(E, E)\]

is injective. Therefore $\pi \times \varpi : \mathcal{P}_E^{(n)}(v) \to M_H(v) \times M_H(v - nv_i)$ is a closed immersion.
Definition 1.7.

(1.52) \[ M_H(v)_{E_1,n} := \{ E \in M_H(v) \mid \dim \text{Hom}(E_1, E) = n \} \]

Then \( \pi_* \left( \Psi^{(n)}_{E_1}(v) \right) = \bigcup_{k \geq n} M_H(v)_{E_1,n} \).

2. AN ACTION OF A LIE ALGEBRA

We define a lattice

(2.1) \[ L(S) := \left( \sum_{i=1}^{n} \mathbb{Z}v(E_i), -\langle \ , \ \rangle \right) \]

Let \( g \) the Lie algebra associated to \( L(S) \), that is, the Cartan matrix of \( g \) is \((-\langle v(E_i), v(E_j) \rangle)_{i,j=1}^{n} \). In the same way as in [\text{N2}] and [\text{N6}], we shall construct an action of \( g \) on \( \bigoplus_j H_*(M_H(v), \mathbb{C}) \), where \( v \) runs a suitable set of Mukai vectors with \([13] \).

The fundamental class of \( \Psi^{(n)}_{E_1} \) defines an operator \( f^{(n)}_{vi} \):

(2.2) \[ H_*(M_H(v-nv_i), \mathbb{C}) \to H_*(M_H(v), \mathbb{C}) \]

\[ x \mapsto p_{2*}(p_1^*x \cap [\Psi^{(n)}_{E_1}(v)]) \]

where \( p_1, p_2 \) are the first and the second projections of \( M_H(v-nv_i) \times M_H(v) \). We also define the operator \( e^{(n)}_{vi} \):

(2.3) \[ H_*(M_H(v), \mathbb{C}) \to H_*(M_H(v-nv_i), \mathbb{C}) \]

\[ x \mapsto (-1)^{nr(v)p_1*}(p_2^*(x) \cap [\Psi^{(n)}_{E_1}(v)]) \]

where \( r(v) = \frac{1}{2}(\dim M_H(v-v_i) - \dim M_H(v)) = -\langle v_i, v \rangle - 1 \). We set \( e_{vi} := e^{(n)}_{vi} \) and \( f_{vi} := f^{(n)}_{vi} \). We also set

(2.4) \[ h_{vi}|H_*(M_H(v), \mathbb{C}) = \langle v_i, v \rangle \text{id}_{H_*(M_H(v), \mathbb{C})} \]

Theorem 2.1. Assume that \( S \) satisfies the assumptions in (i) or (ii) of Lemma [13]. Then \( e_{vi}, f_{vj}, h_{vk} \) satisfy the following relations:

(2.5) \[ [h_{vi}, e_{vj}] = -\langle v_i, v_j \rangle e_{vj} \]

(2.6) \[ [h_{vi}, f_{vj}] = \langle v_i, v_j \rangle f_{vj} \]

(2.7) \[ [e_{vi}, f_{vj}] = \delta_{ij}h_{vi} \]

(2.8) \[ \text{ad}(e_{vi})^{1+(n, v_i)}(e_{vj}) = \text{ad}(f_{vj})^{1+(n, v_i)}(f_{vj}) = 0, \quad i \neq j, \]

where \( \text{ad} \) means the adjoint action \( \text{ad}(x)(y) := [x, y] = xy - yx \).

Since \( \langle (v \pm nv_i)^2 \rangle = -(1 + p_g) \) for \( n \gg 0 \), \( e_{vi} \) and \( f_{vi} \) are locally nilpotent. Therefore we get an integral representation of \( g \).

2.1. Proof of Theorem 2.1 The proof is similar to [\text{N2}] and [\text{N6}]. We first note that the Serre relations [26] follows from the other relations and \( \langle (v \pm nv_i)^2 \rangle < -(1 + p_g) \) for \( n \gg 0 \): Let \( L \) be the subspace of \( \operatorname{Hom}(\bigoplus_{k \in \mathbb{Z}} H_*(M_H(v+kv_i)), \bigoplus_{k \in \mathbb{Z}} H_*(M_H(v + v_j + kv_i))) \) generated by \( \text{ad}(e_{vi})^{n}(e_{vj}), \ n \geq 0 \). Then \( s_{12} \) generated by \( e_{vi}, f_{vj}, h_{vi} \) acts on \( L \). Since \( \langle (v \pm nv_i)^2 \rangle < -(1 + p_g) \) for \( n \gg 0 \), \( L \) is of finite dimension. By the theory of the \( s_{12} \)-representation, we get \( \text{ad}(e_{vi})^{1+(n,v_i)}(e_{vj}) \). The proof of the other relation is the same.

Hence we only need to show relations (2.5), (2.6), and (2.7). The proof of (2.5), (2.6) are easy. We shall prove (2.7). For this purpose, we shall study the convolution products:

(2.9) \[ p_{13*} \left( p_{12}^* \left[ \omega(\Psi^{(n)}_{E_1}(v)) \right] \cap p_{23}^* \left[ \Psi^{(n)}_{E_1}(v) \right] \right), \]

\[ q_{13*} \left( q_{12}^* \left[ \Psi^{(n)}_{E_1}(v-nv_i) \right] \cap q_{23}^* \left[ \omega(\Psi^{(n)}_{E_1}(v-nv_i)) \right] \right), \]

where \( p_{ij} \) and \( q_{ij} \) are projections to the product of \( i \)-th and \( j \)-th factors in

\[ M_H(v-nv_i) \times M_H(v) \times M_H(v-nv_j), \quad M_H(v-nv_i) \times M_H(v-nv_i-nv_j) \times M_H(v-nv_j) \]

respectively, and \( \omega \) is the exchange of the factor. The both products have degree \( \frac{1}{2}(\dim M_H(v-nv_i) + \dim M_H(v-nv_j)) \).

(1) We first study the case where \( i \neq j \).

Lemma 2.2. We have an isomorphism over \( M_H(v-nv_i) \times M_H(v-nv_j) \):

(2.10) \[ p_{12}^{-1}(\omega(\Psi^{(n)}_{E_1}(v))) \cap p_{23}^{-1}(\Psi^{(n)}_{E_1}(v)) \to q_{12}^{-1}(\Psi^{(n)}_{E_1}(v-nv_i)) \cap q_{23}^{-1}(\omega(\Psi^{(n)}_{E_1}(v-nv_j))). \]
Proof. For $F_i \to E_1$ and $[F_i \to E_1] \to F_j[1]$, we set

$$\begin{align*}
E_2 & := [[F_i \to E_1] \to F_j[1]][-1], \\
E & := [E_1 \to F_j[1]][-1].
\end{align*}$$

(2.11)

Applying the Octahedral axiom to $E_1 \to [F_i \to E_1] \to F_j[1]$, we have a commutative diagram of exact triangles:

$$\begin{array}{cccc}
F_i & \longrightarrow & F_j & \\
\downarrow & & \downarrow & \\
E & \longrightarrow & E_1 [\longrightarrow & F_j[1] \\
\downarrow & & \downarrow & \\
F_i & \longrightarrow & E_2 [\longrightarrow & F_j[1] \\
\downarrow & & \downarrow & \\
F_i[1] & \longrightarrow & F_i[1].
\end{array}$$

(2.12)

Hence $E_1 \cong [F_j \to E], E_2 \cong [F_i \to E]$ and $[F_i \to E_1] \cong [F_i \to F_j \to E]$. Conversely for $E := [E_1 \to F_j[1]][-1]$ and $E_2 := [F_i \to E]$, we get the commutative diagram of exact triangles 2.12. Since the correspondence is functorial, we have a desired isomorphism

$$p_{12}^{-1}(\omega(\mathcal{P}^{(n)}_{E_i}(v))) \cap p_{23}^{-1}(\mathcal{P}^{(n)}_{E_j}(v)) \rightarrow q_{12}^{-1}(\mathcal{P}^{(n)}_{E_j}(v-n_i v_1)) \cap q_{23}^{-1}(\omega(\mathcal{P}^{(n)}_{E_i}(v-n_j v_j))).$$

(2.13)

Lemma 2.3.

(i) $p_{12}^{-1}(\omega(\mathcal{P}^{(n)}_{E_i}(v))) \cap p_{23}^{-1}(\mathcal{P}^{(n)}_{E_j}(v)) \rightarrow M_H(v-n_i v_1) \times M_H(v-n_j v_j)$ is injective.

(ii) $q_{12}^{-1}(\mathcal{P}^{(n)}_{E_j}(v-n_i v_1)) \cap q_{23}^{-1}(\omega(\mathcal{P}^{(n)}_{E_i}(v-n_j v_j))) \rightarrow M_H(v-n_i v_1) \times M_H(v-n_j v_j)$ is injective.

Proof. We shall prove (i). The proof of (ii) is similar. Assume that we have isomorphisms in the derived category:

$$\begin{align*}
[F_i \to E] & \cong [F_i \to E'], \\
[F_i \to E] & \cong [F_i \to E'].
\end{align*}$$

(2.14)

We shall show that there is an isomorphism $\phi : E \to E'$ which is compatible with the morphisms $F_i \to E$, $F_i \to E'$. Applying $\text{Hom}(E', \cdot)$ to the exact triangles

$$\begin{array}{cccc}
F_i & \longrightarrow & E & \longrightarrow & [F_i \to E] & \longrightarrow & F_i[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_i & \longrightarrow & [F_2 \to E] & \longrightarrow & [F_i \oplus F_2 \to E] & \longrightarrow & F_i[1],
\end{array}$$

(2.15)

we get a commutative diagram

$$\begin{array}{cccc}
\text{Hom}(E', F_i) & \longrightarrow & \text{Hom}(E', E) & \longrightarrow & \text{Hom}(E', F_i \to E) & \longrightarrow & \text{Ext}^1(E', F_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Hom}(E', F_i) & \longrightarrow & \text{Hom}(E', F_2 \to E) & \longrightarrow & \text{Hom}(E', F_i \oplus F_2 \to E) & \longrightarrow & \text{Ext}^1(E', F_i).
\end{array}$$

(2.16)

Since $\text{Hom}(E', F_i) = 0$, Lemma 1.10 implies that

$$\text{Hom}(E', F_2 \to E) \rightarrow \text{Hom}(E', F_i \oplus F_2 \to E)$$

(2.17)

is an isomorphism. Hence $\text{Hom}(E', E) \rightarrow \text{Hom}(E', F_i \to E) \cong C$ is also an isomorphism. We also have an isomorphism

$$\text{Hom}(E, E') \rightarrow \text{Hom}(E, F_i \to E').$$

(2.18)

Then the claim easily follow from these isomorphisms. Hence

$$p_{12}^{-1}(\omega(\mathcal{P}^{(n)}_{E_i}(v))) \cap p_{23}^{-1}(\mathcal{P}^{(n)}_{E_j}(v)) \rightarrow M_H(v-n_i v_1) \times M_H(v-n_j v_j)$$

(2.19)

is injective.

\[\square\]

Lemma 2.4. If $i \neq j$, then $p_{12}^{-1}(\omega(\mathcal{P}^{(n)}_{E_i}(v)))$ and $p_{23}^{-1}(\mathcal{P}^{(n)}_{E_j}(v))$ intersect transversely and $q_{12}^{-1}(\mathcal{P}^{(n)}_{E_j}(v-n_i v_1))$ and $q_{23}^{-1}(\omega(\mathcal{P}^{(n)}_{E_i}(v-n_j v_j)))$ intersect transversely.
Proof. We set $F_i := U_i \otimes E_i$ and $F_j := U_j \otimes E_j$. We shall show that the map of the tangent spaces

$$\text{Ext}^1(F_i \to E, E)/\text{End}(F_i) \oplus \text{Ext}^1(F_j \to E, E)/\text{End}(F_j)$$

is surjective and

$$\text{Ext}^1(F_i \oplus F_j \to E, F_j \to E)/\text{End}(F_i) \oplus \text{Ext}^1(F_i \oplus F_j \to E, F_i \to E)/\text{End}(F_j)$$

is surjective.

We shall only prove (2.20). By (1.50) and Lemma 1.10 it is sufficient to show that the natural homomorphism

$$\text{Ext}^1(F_i \to E, E) \to \text{Ext}^1(F_i, E) \to \text{Ext}^1(F_j, E)$$

is surjective. Since $\text{Ext}^2(F_i \oplus F_j \to E, E) \cong \text{Ext}^2(F_i \to E, E) \cong H^2(X, \mathcal{O}_X)$, the exact triangle

$$F_j \to [F_i \to E] \to [F_i \oplus F_j \to E] \to F_j[1]$$

implies that this homomorphism is surjective. \qed

By Lemmas 2.22, 2.23, 2.41 we obtain that

$$p_{13}^* \left( \omega(\mathcal{P}_i^{(1)}(v)) \right) \cap p_{23}^* \left[ \mathcal{P}_j^{(1)}(v) \right] = q_{13}^* \left( \pi_{12}^* \left[ \mathcal{P}_i^{(1)}(v_n) \right] \cap q_{23}^* \left[ \omega(\mathcal{P}_j^{(1)}(v_n)) \right] \right).$$

Hence we get

$$[e_{v_i}, f_{v_i}] = 0, \ i \neq j.$$

(II) We next treat the case where $i = j$. This case was treated by Nakajima [N6]. For convenience of the reader, we write a self-contained proof. We assume that $n = 1$. If $i = j$, then $p_{12}^*(\omega(\mathcal{P}_i^{(1)}))$ and $p_{23}^*(\omega(\mathcal{P}_j^{(1)}))$ intersect transversely outside $p_{13}^*(\Delta_{M_H(v_n-v)})$, and $q_{12}^*(\omega(\mathcal{P}_i^{(1)}(v_n-v)))$ and $q_{23}^*(\omega(\mathcal{P}_j^{(1)}(v_n-v)))$ intersect transversely outside $p_{13}^*(\Delta_{M_H(v_n-v)})$. Then we see that

$$p_{13}^* \left( \omega(\mathcal{P}_i^{(1)}(v)) \right) \cap p_{23}^* \left[ \mathcal{P}_j^{(1)}(v) \right] = q_{13}^* \left( \pi_{12}^* \left[ \mathcal{P}_i^{(1)}(v_n-v) \right] \cap q_{23}^* \left[ \omega(\mathcal{P}_j^{(1)}(v_n-v)) \right] + c \Delta_{M_H(v_n-v)} \right)$$

for some integer $c$. In order to compute $c$, we may restrict to a suitable open neighbourhood of the generic point of $\Delta_{M_H(v_n-v)}$. We set $w := v_n-v$.

(II-1) Assume that $-\langle v_i, w \rangle \geq 0$. We set

$$M_H(w) := M_H(w)_{E_i(-v_i,w)},$$

$$M_H(w-v_i) := M_H(w-v_i) \setminus \pi(\mathcal{P}_i^{(1)}(v_n-w)).$$

Then $\mathcal{P}_i^{(1)}(w) := \pi^{-1}(M_H(w))$ is a projective bundle over $M_H(w)$ and $\mathcal{P}_i^{(1)}(w) \to M_H(w-v_i)$ is a closed immersion. We have a fiber product diagram:

$$\begin{array}{ccc}
\mathcal{P}_i^{(1)}(w) & \longrightarrow & q_{23}^{-1}(\omega(\mathcal{P}_i^{(1)}(w))) \\
\downarrow & & \downarrow \\
q_{12}^{-1}(\mathcal{P}_i^{(1)}(w)) & \longrightarrow & M_H(w) \times M_H(w-v_i) \times M_H(w). 
\end{array}$$

By the excess intersection theory, we get that

$$q_{12}^* \left[ \mathcal{P}_i^{(1)}(w) \right] \cap q_{23}^{-1} \left[ \omega(\mathcal{P}_i^{(1)}(w)) \right] = c_{\text{top}}(N_{\mathcal{P}_i^{(1)}(w)/M_H(w-v_i)}).$$

We take $E \in M_H(w)$ with $\text{Ext}^1(E_i, E) = 0$. We set $V := \text{Hom}(E_i, E)$. Let $\mathbb{P} := \mathbb{P}(V^\vee)$ be the fiber of $\pi$. Then

$$q_{13}^* \left( \pi_{12}^* \left[ \mathcal{P}_i^{(1)}(w) \right] \cap q_{23}^{-1} \left[ \omega(\mathcal{P}_i^{(1)}(w)) \right] \right) = \left( \int_{\mathbb{P}} c_{\text{top}}(N_{\mathcal{P}_i^{(1)}(w)/M_H(w-v_i)}) \right) \Delta_{M_H(w)}. $$

We have a family of non-trivial homomorphisms:

$$\mathcal{O}_p(-1) \otimes E_i \to \mathcal{O}_p \otimes E.$$
We set
\[(2.33) \quad \mathcal{E} := [\mathcal{O}_p(-1) \boxtimes E_i \to \mathcal{O}_p \boxtimes \mathcal{E}].\]

We have an exact sequence
\[(2.34) \quad \begin{array}{c}
\text{Ext}^1_{p,p}(\mathcal{E}, \mathcal{O}_p \boxtimes \mathcal{E}) \\
\text{Ext}^2_{p,p}(\mathcal{E}, \mathcal{O}_p \boxtimes \mathcal{E})
\end{array} \quad \begin{array}{c}
\longrightarrow
\longrightarrow
\end{array} \quad \begin{array}{c}
\text{Ext}^1_{p,p}(\mathcal{E}, \mathcal{E}) \\
\text{Ext}^2_{p,p}(\mathcal{E}, \mathcal{O}_p \boxtimes \mathcal{E})
\end{array} \quad \begin{array}{c}
\longrightarrow
\longrightarrow
\end{array} \quad \begin{array}{c}
\text{Ext}^2_{p,p}(\mathcal{E}, \mathcal{O}_p(-1) \boxtimes E_i) \\
\text{Ext}^2_{p,p}(\mathcal{E}, \mathcal{E}).
\end{array}\]

The restriction of the normal bundle \((N_{E_i}^{(1)}(w')/M_H(w-v_i))\) is
\[(2.35) \quad \text{Ext}_{p,p}(\mathcal{E}, \mathcal{O}_p(-1) \boxtimes E_i) = \text{Hom}_{p,1}(\mathcal{O}_p(-1) \boxtimes E_i, \mathcal{E})\]

By the exact triangle
\[(2.36) \quad \mathcal{O}_p(-1) \boxtimes E_i \to \mathcal{O}_p \boxtimes \mathcal{E} \to \mathcal{E} \to \mathcal{O}_p(-1) \boxtimes E_i[1],\]
we get an exact sequence
\[(2.37) \quad 0 \to \mathcal{O}_p \to V \otimes \mathcal{O}_p(1) \to \text{Hom}_{p,1}(\mathcal{O}_p(-1) \boxtimes E_i, \mathcal{E}) \to 0.\]

Hence \(\text{Hom}_{p,1}(\mathcal{O}_p(-1) \boxtimes E_i, \mathcal{E}) = \Omega_{p,1}^1.\) Therefore
\[(2.38) \quad \int_p c_{top}(N_{E_i}^{(1)}(w')/M_H(w-v_i)) = (-1)^{\dim p}(\dim \mathbb{P} + 1) = (-1)^{-\langle v_i, w \rangle} \langle v_i, w \rangle.\]

Since \(E_{E_i}^{(1)}(v')\) does not meet \(M_H(v) \times M_H(w)\),
\[(2.39) \quad p_{13*} \left( \int_p p_{12}^* \left[ \omega(\mathcal{E}^{(1)}_E(v)) \right] \cap p_{23}^* \left[ \mathcal{P}^{(1)}_E(v') \right] \right) = 0\]
on \(M_H(w') \times M_H(w').\) Hence we see that
\[(2.40) \quad [e_{i,v}, f_{i,v}]|_{H_c(M_H(w), \mathcal{C})} = \langle v_i, w \rangle \text{id}|_{H_c(M_H(w), \mathcal{C})} = \langle v_i, w \rangle.\]

(II-2) Assume that \(\langle v_i, w \rangle \geq 0.\) We set \(M_H(v)^{v} := M_H(v) \setminus \pi(\mathcal{E}^{(1)}_E)\) and \(M_H(w)^{w} := M_H(w)_{E_i, 0}.\) For \(E \in M_H(v)^{v}\), we set \(V := \text{Ext}^1(E, E_i).\) We have a family of exact triangles:
\[(2.41) \quad \mathcal{O}_p \boxtimes E_i \to E' \to \mathcal{O}_p(-1) \boxtimes \mathcal{E} \to \mathcal{O}_p \boxtimes E_i[1].\]
The restriction of the normal bundle \((N_{E_i}^{(1)}(w')/M_H(v))\) is
\[(2.42) \quad \text{Ext}^1_{p,p}(\mathcal{O}_p \boxtimes E_i, \mathcal{E}') = \text{Ext}^1_{p,p}(\mathcal{E}', \mathcal{O}_p \boxtimes E_i)^{v}.\]

We have an exact sequence
\[(2.43) \quad 0 \to \mathcal{O}_p = \text{Hom}_{p,1}(\mathcal{O}_p \boxtimes E_i, \mathcal{O}_p \boxtimes E_i) \to \text{Ext}^1_{p,p}(\mathcal{O}_p(-1) \boxtimes \mathcal{E}, \mathcal{O}_p \boxtimes E_i) \to \text{Ext}^1_{p,p}(\mathcal{E}', \mathcal{O}_p \boxtimes E_i) \to 0.\]

Hence \(\text{Ext}^1_{p,p}(\mathcal{O}_p \boxtimes E_i, \mathcal{E}') = \Omega_{1}^1.\) Therefore
\[(2.44) \quad \int_p c_{top}(N_{E_i}^{(1)}(w)/M_H(v)) = (-1)^{\dim p}(\dim \mathbb{P} + 1) = (-1)^{-\langle v, w \rangle} \langle v_i, w \rangle.\]

By using this equality, we see that \((2.40)\) also holds.

\[\text{2.2. The case where the twisted degree is zero.}\]

Let \(G\) be an element of \(K(X).\)

**Definition 2.1.** Let \(\mathcal{E} \in \text{D}(X)\) be an object such that \(\deg_G(\mathcal{E}) = 0\) and
\[(2.45) \quad \chi_G(\mathcal{E}) = \min\{\chi_G(E') > 0 | E' \in \text{Coh}(X), \deg_G(E') = 0\}.\]
\(\mathcal{E}\) is \(G\)-twisted stable, if
\[(i) \quad H^i(\mathcal{E}) = 0, i \neq 0 \text{ and } H^0(\mathcal{E}) \text{ is } G\text{-twisted stable}, \text{ or}\]
\[(ii) \quad H^i(\mathcal{E}) = 0, i \neq -1 \text{ and } H^{-1}(\mathcal{E}) \text{ is } G\text{-twisted stable}.\]

Let \(M_H^{G}(v)\) be the moduli space of \(G\)-twisted stable complex \(\mathcal{E}\) with \(v(\mathcal{E}) = v.\)

**Remark 2.1.** If (ii) holds, then
\[(2.46) \quad \chi(H^{-1}(\mathcal{E})) = \max\{\chi_G(E') < 0 | E' \in \text{Coh}(X), \deg_G(E') = 0\}.\]

Let \(E_i, i = 1, \ldots, n\) be a collection of \(G\)-twisted stable vector bundles with \(\deg_G(E_i) = \chi_G(E_i) = 0\) and \((v(E_i))^2 = -2.\) Assume that \(E_i\) satisfies the condition \((2.10)\). By using Lemmas \((2.3)\) and \((2.5)\) we also obtain the same assertions in Lemma \((1.4)\). Hence we also get an action of the Lie algebra associated to \(E_i, i = 1, \ldots, n.\)
3. Examples

3.1. Stable sheaves on a K3 surface. Let $X$ be a K3 surface and $H$ an ample divisor on $X$. Let $G$ be a semi-stable vector bundle with respect to $H$ such that $\langle v(G)^2 \rangle = 0$. Assume that $G = \bigoplus_{i=0}^n E_i^{\oplus a_i}$, where $E_i$ is a $G$-twisted stable vector bundle such that

\[
\frac{\deg(E_i)}{\text{rk } E_i} = \frac{\deg(G)}{\text{rk } G},
\]

\[
\frac{\chi_G(E_i)}{\text{rk } E_i} = \frac{\chi_G(G)}{\text{rk } G} = 0.
\]

By [O-Y, Thm. 0.1], $v(E_0), v(E_1), \ldots, v(E_n)$ generate a lattice of affine type. We may assume that $a_0 = 1$. We set

\[
l := \min\{\deg_G(E) > 0 | E \in \text{Coh}(X)\}.
\]

We set $v_i := v(E_i), i = 0, 1, \ldots, n$. Let $\mathfrak{g}$ be the affine Lie algebra associated with $v_i, i = 0, 1, \ldots, n$ and $\overline{\mathfrak{g}}$ the finite Lie algebra associated with $v_i, i = 1, \ldots, n$. Let $\mathfrak{g}$ be the Cartan subalgebra of $\overline{\mathfrak{g}}$. For a root $\alpha$, $\overline{\mathfrak{g}}_\alpha$ denotes the root space of $\alpha$. $\theta := \sum_{i=1}^n a_i v_i$ denotes the highest root of $\overline{\mathfrak{g}}$. Then $\mathfrak{g}$ has the following standard expression:

\[
g = \mathbb{C}[t, t^{-1}] \otimes \overline{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d
\]

where

\[
e_i = 1 \otimes \varphi_{v_i}, \quad f_i = 1 \otimes \varphi_{-v_i}, \quad h_i = 1 \otimes \varphi_{v_i} \quad 1 \leq i \leq n,
\]

\[
e_0 = t \otimes \varphi_{\theta}, \quad f_0 = t^{-1} \otimes \varphi_{\theta}, \quad h_0 = -\sum_{i=1}^n a_i h_{v_i} + c,
\]

$\varphi_{\alpha} \in \overline{\mathfrak{g}}_\alpha, \varphi_{v_i} \in \mathfrak{h}$ and (3.4) are the Chevalley generator of $\overline{\mathfrak{g}}$. Hence we get

\[
c = \sum_{i=0}^n a_i h_{v_i}.
\]

The action of $d$ on $H_*(M_H(v), \mathbb{C})$ is defined as follows: We take $w \in H^*(X, \mathbb{Q})$ such that $\langle w, v(E_i) \rangle = \delta_{i,0}, i = 0, 1, \ldots, n$ and set

\[
d_{H_*(M_H(v), \mathbb{C})} := \langle w, v \rangle \text{id}_{H_*(M_H(v), \mathbb{C})}.
\]

Then we have a desired properties:

\[
[d, e_i] = \delta_{i,0} e_i, \quad [d, f_i] = -\delta_{i,0} f_i.
\]

**Proposition 3.1.** Let $\mathfrak{g}$ be the affine Lie algebra associated to $E_0, E_1, \ldots, E_n$. Assume that $E_i$ are $\mu$-stable for all $i$. Then we have an action of $\mathfrak{g}$ on $\bigoplus_{i=0}^n H_*(M_H(v), \mathbb{C})$ such that the center $c$ acts as a scalar multiplication $\langle v, v(G) \rangle$, where $v$ is a Mukai vector with $\deg_G(v) = l$.

**Example 3.1.** Let $C := (-a_{i,j})_{i,j=0}^n$ be a Cartan matrix of affine type and $\delta := (a_0, a_1, \ldots, a_n), a_i \in \mathbb{Z}_{>0}$ the primitive vector with $\delta C = 0$. Let $(X, H)$ be a polarized K3 surface such that

\[
\begin{align*}
(\text{i}) & \quad \text{Pic}(X) = \bigoplus_{i=0}^n \mathbb{Z} \xi_i, \quad (\xi_i, \xi_j) = -a_{i,j} + 2ra \quad \text{and} \\
(\text{ii}) & \quad H = \sum_{i=0}^n a_i \xi_i.
\end{align*}
\]

For an existence of $(X, H)$, see [O-Y] sect. 3. We set $v_0 := r + \xi_i + \alpha p$. Then

\[
(\text{i}) \quad \langle (v_i, v_j) \rangle_{i,j=0}^n = -C, \\
(\text{ii}) \quad \deg(v_i) = (\xi_i, H) = 2ra(\sum_{i=0}^n a_i) \quad \text{and} \\
(\text{iii}) \quad v := \sum_{i=0}^n a_i v_i \text{ is a primitive isotropic Mukai vector}.
\]

**Lemma 3.2.** Let $E_i$ be a $v$-twisted stable vector bundle with respect to $H$ with $v(E_i) = v_i$. Then $E_i$ is $\mu$-stable.

**Proof.** For a coherent sheaf $F$, we set $c_1(F) := \sum_i x_i \xi_i$, Then $\deg(F) = (\sum_i x_i) 2ra(\sum_i a_i)$. Since $\text{rk } E_i = r$ and $\deg E_i = 2ra(\sum_i a_i)$ for all $i$, if $\deg(F) / \text{rk } F = \deg(E_i) / \text{rk } E_i = 2a(\sum_i a_i)$, then $\text{rk } F = (\sum_i x_i)^r \geq \text{rk } E_i$. Therefore $E_i$ are $\mu$-stable.

We set $w := (r(\sum_i x_i) - 1) + \sum_i x_i \xi_i + \alpha p, x_i \in \mathbb{Z}$. Then

\[
\langle w, v \rangle = \min\{\deg_G(E) > 0 | E \in \text{Coh}(X)\},
\]

where $G \in M_H(v)$. Hence we have an action of $\mathfrak{g}$ on $\bigoplus_w H_*(M_H(w), \mathbb{C})$, where $w = \sum_i x_i v_i - 1 + \alpha p, x_i, \alpha \in \mathbb{Z}$. 

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Let $G$ be a vector bundle such that $\operatorname{rk} G = (H^2)$ and $c_1(G) = H$. For a Mukai vector $v := (1 + (D, H)) - D + a\rho$, we get
\begin{align*}
\deg_G(\nu) &= (H, H)(-D, H) - (1 + (D, H))(-H, H) \\
&= (H, H) \\
&= \min \{ \deg_G(E') > 0 | E' \in \operatorname{Coh}(X) \}.
\end{align*}
Let $C_1, C_2, \ldots, C_n$ be irreducible $(-2)$-curves on $X$. We set $v_i := (C_i, H) - C_i$.

**Lemma 3.3.** There is a stable vector bundle $E_i$ with $v(E_i) = v_i$. Moreover if $H = nH'$ and $(C_i, H') < 2(n - 1)(H'^2)$, then $E_i$ is $\mu$-stable.

**Proof.** There is a semi-stable sheaf $E_i$ with $v(E_i) = v_i$. We shall show that $E_i$ is stable. Let $\bigoplus_{j=1}^n E_{i,j}$ be the Jordan-Hölder grading of $E_i$ with respect to the Gieseker stability. We set $v(E_{i,j}) := r_j - D_j + a_j\rho$. Then $(D_j, H)/r_j = 1$ and $a_j/r_j = 0$, and hence $(D_j, H) > 0$ and $(D_j^2) = -2$, which implies that $D_j$ is effective. By our assumption on $C_i$, $s = 1$. Thus $E_i$ is stable. Assume that $H = nH'$ and $(C_i, H') < 2(n - 1)(H'^2)$. Let $\bigoplus_{j=1}^n E_{i,j}$ be the Jordan-Hölder grading of $E_i$ with respect to the $\mu$-stability. We set $v(E_{i,j}) := r_j - D_j + a_j\rho$. Then $r_j = (D_j, H) = n(D_j, H')$, and hence $(D_j, H') > 0$. By the stability of $E_{i,j}$, $(\deg(E_{i,j})^2) = (D_j^2) - 2r_ja_j \geq -2$. By the Hodge index theorem, $(D_j, H')^2 \geq (D_j^2)(H'^2)$. If $a_j > 0$, then we see that $(C_i, H') > (D_j, H') \geq 2(n - 1)(H'^2)$. Therefore $a_j \leq 0$. Since $\sum_j a_j = 0$, $a_j = 0$ for all $j$. Since $E_i$ is stable, $s = 1$. Thus $E_i$ is $\mu$-stable.

**Proposition 3.4.** Assume that $E_i$ are $\mu$-stable. Then we have an action of the Lie algebra $\mathfrak{g}$ associated to $C_i$, $i = 1, 2, \ldots, n$ on $\bigoplus_{v} H_s(M_{H}(v), \mathbb{C})$, where $v = (1 + (D, H)) - D + a\rho$, $D \in \operatorname{Pic}(X)$, $a \in \mathbb{Z}$.

**Example 3.2.** Let $\pi : X \to \mathbb{P}^1$ be an elliptic $K3$ surface with a section $C_0$. Let $C_1, \ldots, C_n$ be smooth $(-2)$-curves on fibers of $\pi$. We set $v_i := (C_i, H) - C_i$, $i = 0, 1, \ldots, n$. Then $(v, v_i)_{ij} = ((C_i, C_j)_{ij})$. We assume that $(C_i, C_j) \leq 1$. Hence we get an action of the Lie algebra generated by $C_i, 0 \leq i \leq n$ on $\bigoplus_{v} H_s(M_{H}(v), \mathbb{C})$, where $v = (1 + (D, H)) - D + a\rho$, $D \in \operatorname{Pic}(X)$, $a \in \mathbb{Z}$.

**Example 3.3.** In the notation of Example 3.1 we set $v_i := v(E_i)$. Then we see that
\begin{align*}
\{ \chi_G(w) | &w \in v(D(X)), \deg_G(w) = 0 \} \\
&= \{ \chi_G(w) | w = \sum_i x_i v_i + \rho \} \\
&= Z(v(G), \rho).
\end{align*}
Hence we have an action of $\mathfrak{g}$ on $\bigoplus_{w} H_s(M_{H}(w), \mathbb{C})$, where $w = \sum_i x_i v_i + \rho$, $x_i \in \mathbb{Z}$.

Let $\pi : X \to \mathbb{P}^1$ be the elliptic $K3$ surface as in Example 3.2. Let $G$ be an element of $K(X)$ with $v(G) = (H, f) - f$. We set $v_D := (H, C_0 + D) - (C_0 + D)$. Then $\deg_G(v) = 0$ and $\chi_G(v) = -(1 + (D, f))$. We assume that $E_i$ are $\mu$-stable. Let $\mathfrak{g}'$ be the Lie algebra generated by $C_1, \ldots, C_n$. By the remarks in section 2.2 we can construct an action of $\mathfrak{g}'$ on $\bigoplus_{D} H_s(M_{H}(v_D), \mathbb{C})$, where $D$ is an effective divisor with $(D, f) = 0$.

### 3.2 Stable sheaves on an Enriques surface.
Let $X$ be an Enriques surface and $\pi : Y \to X$ be the covering $K3$ surface of $X$. Assume that $X$ contains a smooth $(-2)$ curve. Let $C'$ be a connected component of $\pi^{-1}(C)$. Let $H'$ be an ample divisor on $Y$ and set $H := \pi_*(H')$. Then $H$ is an ample divisor on $X$ with $(H, C) = 2(H', C')$. We take a semi-stable sheaf $E'$ on $Y$ with $v(E') = (H', C') - C'$. $E$ is a rigid vector bundle. If $H'$ is sufficiently ample, then Lemma 3.3 implies that $E'$ is $\mu$-stable.

**Proposition 3.5.** We set $E := \pi_*(E')$. Then $E$ is a $\mu$-stable vector bundle with the Mukai vector $(H, C) - C$ which satisfies $E \otimes K_X \cong E$ and
\begin{align*}
\text{Hom}(E, E) &= \mathbb{C} \\
\operatorname{Ext}^1(E, E) &= 0 \\
\operatorname{Ext}^2(E, E) &= \mathbb{C}.
\end{align*}
If there is a configuration of $(-2)$-curves, then as in the $K3$ surface case, we have an action of the Lie algebra associated to $(−2)$-curves on $\bigoplus_{v} H_s(M_{H}(v), \mathbb{C})$, where $v = (1 + (D, H)) + D + a\rho$, $D \in \operatorname{Pic}(X)$, $a - 1/2 \in \mathbb{Z}$. 

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4. Actions associated to purely 1-dimensional exceptional sheaves

4.1. Purely 1-dimensional sheaves. In this section, we shall consider Lie algebra actions associated to purely 1-dimensional exceptional sheaves such as line bundles on $(-2)$-curves. Unfortunately we cannot construct the action for the moduli spaces of stable torsion free sheaves in general. Instead, we can construct it for the moduli spaces of purely 1-dimensional sheaves. In some cases, the moduli spaces of stable torsion free sheaves are deformation equivalent to moduli spaces of purely 1-dimensional sheaves. In this sense, we have an action for the moduli spaces of stable torsion free sheaves. This will be explained in 4.3. We also explain a partial result on the moduli spaces of stable torsion free sheaves in 4.4.

Let $(X, H)$ be a pair of a smooth projective surface $X$ and an ample divisor $H$ on $X$.

**Definition 4.1.** Let $G$ be an element of $K(X)$ with $\text{rk} G > 0$. A purely 1-dimensional sheaf $E$ is $G$-twisted stable, if

$$
\chi_G(F) < \chi_G(E)
$$

for all proper subsheaf $F(\neq 0)$ of $E$.

We have the following result whose proof is similar to Lemma 1.2.

**Lemma 4.1.** Let $G$ be an element of $K(X)$ with $\text{rk} G > 0$ and $E_i$, $i = 1, 2, \ldots, s$, be purely 1-dimensional $G$-twisted stable sheaves with $\chi_G(E_i) = 0$. Let $E$ be a purely 1-dimensional $G$-twisted stable sheaf with

$$
\chi_G(E) = \min\{\chi_G(E') > 0 | E' \in \text{Coh}(X), \text{rk} E' = 0\}
$$

or $E = \mathbb{C}_P$, $P \in X$ with the condition 1.2.

(1) Then every non-trivial extension

$$
0 \to E_1 \to F \to E \to 0
$$

defines a $G$-twisted stable sheaf.

(2) Let $V_i$ be a subspace of $\text{Hom}(E_i, E)$. Then $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective or surjective. Moreover,

$$(2-1)$$ If $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is injective, then the cokernel is a $G$-twisted stable purely 1-dimensional sheaf or $\mathbb{C}_P$, $P \in X$,

$$(2-2)$$ If $\phi : \bigoplus_{i=1}^s V_i \otimes E_i \to E$ is surjective, then $\ker \phi$ is $G$-twisted stable.

**Lemma 4.2.** Let $G$ be an element of $K(X)$ with $\text{rk} G > 0$ and $E_i$, $i = 1, 2, \ldots, s$, be purely 1-dimensional $G$-twisted stable sheaves with $\chi_G(E_i) = 0$. Let $E$ be a purely 1-dimensional $G$-twisted stable sheaf with

$$
\chi_G(E) = \max\{\chi_G(E') < 0 | E' \in \text{Coh}(X), \text{rk} E' = 0\}.
$$

(1) Then every non-trivial extension

$$
0 \to E \to F \to E_1 \to 0
$$

defines a $G$-twisted stable sheaf.

(2) Let $V_i$ be a subspace of $\text{Hom}(E, E_i)$. Then $\phi : E \to \bigoplus_{i=1}^s V_i^\vee \otimes E_i$ is injective or surjective. Moreover,

$$(2-1)$$ If $\phi : E \to \bigoplus_{i=1}^s V_i^\vee \otimes E_i$ is injective, then the cokernel is a $G$-twisted stable purely 1-dimensional sheaf or $\mathbb{C}_P$, $P \in X$,

$$(2-2)$$ If $\phi : E \to \bigoplus_{i=1}^s V_i^\vee \otimes E_i$ is surjective, then $\ker \phi$ is $G$-twisted stable.

**Remark 4.1.** We set

$$
d := \min\{\chi_G(E') > 0 | E' \in \text{Coh}(X), \text{rk} E' = 0\}.
$$

For a purely 1-dimensional sheaf $E$ with $\chi_G(E) = d$, $E$ is $G$-twisted stable if and only if $\chi_G(F) \leq 0$ for all proper subsheaf $F$ of $E$. Thus the $G$-twisted stability does not depend on the choice of $H$.

**Definition 4.2.** For a complex $E$ with $\text{rk}(E) = 0$, we set

$$
v(E) := (c_1(E), \chi(E)) \in H^2(X, \mathbb{Z}) \times \mathbb{Z}.
$$

We define a pairing of $v_i := (\xi_i, a_i) \in H^2(X, \mathbb{Z}) \times \mathbb{Z}$, $i = 1, 2$ by

$$
(v_1, v_2) := (\xi_1, \xi_2) \in \mathbb{Z}.
$$

Then the Riemann-Roch theorem says that

$$
\chi(E, F) = -\langle v(E), v(F) \rangle
$$

for $E, F \in D(X)$ with $\text{rk}(E) = \text{rk}(F) = 0$. We set $\rho := v(\mathbb{C}_P) = (0, 1)$. 

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Definition 4.3. Let $E \in D(X)$ be an object such that $\mathrm{rk}(E) = 0$ and
\begin{equation}
\chi_G(E) = \min \{ \chi_G(E') > 0 | E' \in D(X), \mathrm{rk}(E') = 0 \}.
\end{equation}

$E$ is $G$-twisted stable, if
\begin{enumerate}[(i)]
  \item $H^i(E) = 0$, $i \neq 0$ and $H^0(E)$ is $G$-twisted stable, or
  \item $H^i(E) = 0$, $i \neq -1$ and $H^{-1}(E)$ is $G$-twisted stable.
\end{enumerate}

Let $M^G_H(v)$ be the moduli space of $G$-twisted stable complexes $E$ with $v(E) = v$.

Let $E_i$, $i = 1, 2, \ldots, n$ be $G$-twisted stable purely 1-dimensional sheaves such that $\chi_G(E_i) = 0$, $E_i \otimes K_X \cong E_i$ and $(\psi(E_i)) = -2$. We set $v_i := v(E_i)$. Let $g$ be the Lie algebra associated to $E_i$, $i = 1, \ldots, n$. By using Lemma \[.12\] we get the following similar results to the results in section \[2\].

Proposition 4.3. For all $E \in M^G_H(v + \sum_i x_i v_i)$, $x_i \in \mathbb{Z}$, we assume that
\begin{equation}
\chi_G(E) = \min \{ \chi_G(E') > 0 | E' \in D(X), \mathrm{rk}(E') = 0 \}
\end{equation}
and $E$ satisfies \[11\]. Then we have an action of $g$ on $\bigoplus_{x_i \in \mathbb{Z}} H_*(M^G_H(v + \sum_i x_i v_i), \mathbb{C})$.

Let $C$ be an irreducible $(-2)$-curve on $X$. If $G = \mathcal{O}_X$, then $\mathcal{O}_C(-1)$ is a stable sheaf with $\chi(\mathcal{O}_C(-1)) = 0$. Then we can apply Proposition 4.3.

Corollary 4.4. Let $X$ be a K3 surface. Then $M_H((D, 1)) \neq \emptyset$ for all $H$.

Proof. By Proposition 4.3 we have isomorphisms
\begin{equation}
H_*(M_H((D, 1)), \mathbb{C}) \cong H_*(M_H((D + (D, C)C, 1)), \mathbb{C})
\end{equation}
for all irreducible $(-2)$-curves $C$. Hence we can reduce the proof to the case where $D$ is a nef divisor or $D$ is a smooth rational curve. If $D$ is nef, then \[Y2\] Rem. 3.4 implies that $M_H((D, 1)) \neq \emptyset$. Therefore our claim holds.

Remark 4.2. We can show that $M^G_H((D, n)) \neq \emptyset$ for a general $(H, G)$ by a different method.

Lemma 5.5. Let $X$ be a 9 point blow-up of $\mathbb{P}^2$ and assume that $\{-D\} X$ contains a reducible curve $Y = \sum_{i=0}^n a_i C_i$, where $C_i$ are smooth $(-2)$-curves. Then every $G$-twisted stable purely 1-dimensional sheaf $E$ with $(c_1(E), K_X) \leq 0$ satisfies \[14\].

Proof. Assume that there is a non-zero map $\psi : E \rightarrow E(K_X) = E(Y)$. By the homomorphism $\mathcal{O}_X(-Y) \rightarrow \mathcal{O}_X$, we have a homomorphism $E(-Y) \rightarrow E$, which is isomorphic on $\text{Div}(E) \setminus Y \neq \emptyset$. If $E \rightarrow E(-Y) \rightarrow E$ is a zero map, then $F := \psi(E)(-K_X)$ satisfies $\text{Supp}(F(K_X)) \subset Y$ and $\chi_G(E)/c_1(K_X, H) < \chi_G(E)/c_1(F, H)$. On the other hand, since $F$ is a proper subsheaf of $E$, we have $\chi_G(F)/c_1(F, H) < \chi_G(E)/c_1(F, H)$. Since $(C_i, K_X) = 0$, we get $(c_1(F), K_X) = 0$. This means that $\chi_G(F(K_X)) = \chi_G(F)$. Then we get $\chi_G(E)/c_1(F, H) < \chi_G(E)/c_1(F, E)$. This is a contradiction. Therefore $E \rightarrow E(-Y) \rightarrow E$ is a zero map. Then by using the stability of $E$ and $(\text{Div}(E), Y) > 0$, we get a contradiction. Hence we conclude that $\text{Hom}(E, E(K_X)) = 0$.

Corollary 4.6. Under the assumption in Lemma \[4.3\] we have an action of the affine Lie algebra associated to $C_i$, $0 \leq i \leq n$ on $\bigoplus_D H_*(M_H((D, 1)), \mathbb{C})$, where $(D, K_X) < 0$.

Proposition 4.7. Let $C_i$, $i = 0, 1, \ldots, n$ be a configuration of smooth $(-2)$-curves of $ADE$ or affine type such that $K_X$ is trivial in a neighbourhood of $\cup_i C_i$ and $(C_i, C_j) \leq 1$ for $i \neq j$. Let $D := \sum_i 0 = b_i C_i$, $b_i > 0$ be an effective divisor such that $(D^2) = -2$. There is a $G$-twisted stable sheaf $E$ with $(c_1(E), E) = (D, m)$ for a general $(H, G)$.

Proof. If $n = 0$, then $D = C_0$ and obviously the claim holds. Hence we may assume $n > 0$ and $\cup_i C_i$ is connected. We set $v_i := v(\mathcal{O}_C(-1))$. We first show that $M_H(\rho + \sum_i b_i v_i) \neq \emptyset$. Assume that there is a stable sheaf $E$ such that $\text{Supp}(E) \subset \cup_i C_i$. Since $K_X$ is trivial in a neighbourhood of $\cup_i C_i$, $\text{Ext}^2(E, E) \cong H^0(E, E \otimes K_X)^\vee = \mathbb{C}$. Hence we see that $c_1(E^2) \geq -2$ and the equality holds when $\text{Ext}^1(E, E) = 0$. In particular $M_H(\rho + \sum_i b_i v_i)$ is smooth. Let $v_i$ be the $(-2)$-reflection defined by $v_i$ and $W$ the Weyl group generated by $v_i$, $i = 0, 1, \ldots, n$. Then by the action of $W$, we have an isomorphism $M_H(\rho + \sum_i b_i v_i) \rightarrow M_H(\rho + v_j)$ for some $j$. Obviously $M_H(\rho + v_j) = \{ \mathcal{O}_C \}$. Therefore $M_H(\rho + \sum_i b_i v_i) \neq \emptyset$.

We shall treat the general cases. Since $K_X$ is trivial in a neighbourhood of $\cup_i C_i$, by using \[Y3\] Prop. 2.7, we see that the non-emptyness of $M_H(\rho)$ does not depend on the choice of a general $(H, G)$. There is a divisor $C$ such that $(C, D) = 1$. Indeed we take an element $w \in W$ such that $w(D) = C_i$ for some $i$. Then $1 = (C_j, C_i) = (w(C_j), D)$ for $j$. Let $E$ be a stable sheaf with $(c_1(E), E) = (D, 1)$. Then $E(nC)$ is a $\mathcal{O}_X(nC)$-twisted stable sheaf with $\chi(E(nC)) = 1 + n$. Therefore our claim holds for general cases.

□
Example 4.1. Let $Y$ be a germ of a rational double point and $\pi : X \to Y$ the minimal resolution. Let $H$ be a $\pi$-ample divisor on $X$. Let $C_i$, $i = 1, 2, \ldots, n$ be irreducible components of the exceptional divisor. We set $v_i := v(\mathcal{O}_{C_i}(-1))$. Let $g$ be the Lie algebra associated to $C_i$. We note that $K_X \cong \mathcal{O}_X$. For a coherent sheaf $E$ on $X$ with a compact support, we can define the stability with respect to $H$. For a stable sheaf $E$ with $v(E) = \rho + \sum_i n_i v_i$, $\dim \text{Ext}^1(E, E) = \langle \rho + \sum_i n_i v_i \rangle^2 + 2$. Hence we get

$$\dim \text{Ext}^1(E, E) = \begin{cases} 2, & \text{if } v = \rho, \\ 0, & \text{if } \langle \sum_i n_i v_i \rangle^2 = -2. \end{cases}$$

If $v = \rho$, then all stable sheaves are of the form $\mathbb{C} \cdot H_i, P \in X$. Hence $M_H(\rho)$ has a coarse moduli space which is isomorphic to $X$. Hence $M_H(\rho)$ is smooth. If $\langle \sum_i n_i v_i \rangle^2 = -2$, then the proof of Proposition 4.2 implies that $M_H(\rho + \sum_i n_i v_i)$ is not empty and consists of a stable sheaf on the exceptional divisors. Then we have an action of $g$ on $\bigoplus_{n_i \in \mathbb{Z}} H_*(M_H(\rho + \sum_i n_i v_i), \mathbb{C})$. Indeed the submodule consisting of the middle degree homology groups is isomorphic to $g$. For the structure of $M_H(\rho + \sum_i n_i v_i)$, we get the following: Let $D = \sum_i n_i C_i$ be an effective divisor with $(D^2) = -2$. Then $M_H(\rho + \sum_i n_i v_i) = \{ \mathcal{O}_D \}$ and $M_H(\rho - \sum_i n_i v_i) = \{ \mathcal{O}(D) \}$.

Proof of the claim: We note that $\chi(\mathcal{O}_D) = -(D^2)/2 = 1$. If there is a quotient $\mathcal{O}_D \to \mathcal{O}_{P}$, then since $(D^2) < 0$, we have $\chi(\mathcal{O}_{P}) = -(D^2)/2 > 1$. Therefore $\mathcal{O}_D$ is stable. We note that $\mathcal{O}_D(D)$ is the derived dual of $\mathcal{O}_D$. By using this fact, we can easily see the stability of $\mathcal{O}_D(D)$.

Example 4.2. Let $C$ be a germ of a curve at $P$ and $\pi : X \to C$ an elliptic surface with a section $\sigma$. Let $H$ be a $\pi$ ample divisor on $X$. Assume that $\pi^{-1}(P)$ is reducible and consists of smooth $(-2)$-curves $C_i$, $i = 0, 1, \ldots, n$: $\pi^{-1}(P) = \sum_{i=0}^n a_i C_i$. We may assume that $a_0 = 1$ and $(\sigma, C_0) = 1$. We set $v_i := v(\mathcal{O}_{C_i}(-1))$. Then we see that $M_H(\rho + \sum_i n_i v_i)$ is smooth with

$$\dim M_H(\rho + \sum_i n_i v_i) = \langle \sum_i n_i v_i \rangle^2 + 2$$

and $M_H(\rho + \sum_i n_i v_i) \cong X$, if $\langle \sum_i n_i v_i \rangle^2 = 0$. We also have an action of affine Lie algebra $g$ associated to $v_i$ on $\bigoplus_{n_i \in \mathbb{Z}} H_*(M_H(\rho + \sum_i n_i v_i), \mathbb{C})$. Indeed, if $(C_i, C_j) \leq 1$ for all $i \neq j$, then the result obviously holds. If $(C_i, C_j) = 2$, then we can directly check the commutation relation (2.4). We set $\delta := v(\mathcal{O}_{-1}(P)) = \sum_{i=0}^n a_i v_i$. If $\sum_i n_i v_i = m\delta$, $m \in \mathbb{Z}$, then under an identification $M_H(\rho + m\delta) \cong X$, we have an isomorphism

$$H_2(M_H(\rho + m\delta), \mathbb{C}) \cong \mathbb{C}[\sigma] \oplus \bigoplus_{i=1}^n \mathbb{C}[C_i].$$

Let

$$g = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

be the standard expression of the affine Lie algebra, where $c$ is the center of $g$. Then we have an exact sequence of $g$-modules:

$$0 \to \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \to \bigoplus_{n_i \in \mathbb{Z}} H_{mid}(M_H(\rho + \sum_i n_i v_i), \mathbb{C}) \to \mathbb{C}[t, t^{-1}] \to 0,$$

where $H_{mid}(\ast)$ is the middle degree homology group of $\ast$ and $\mathbb{C}[t, t^{-1}]$ is the $g/[g, g] = \mathbb{C}d$-module.

4.2. Moduli of stable sheaves on elliptic surfaces. We first collect some basic facts on the moduli spaces of stable sheaves on elliptic surfaces $X$. If $K_X$ is not numerically trivial, then we do not have a good invariant of torsion free sheaf $E$ which is a suitable generalization of the Mukai vector. In these cases, we shall use $\gamma(E) := (\text{rk}(E), c_1(E), \chi(E)) \in H^*(X, \mathbb{Z})$ as an invariant of $E$. If $\text{rk} E = 0$, then $\gamma(E) = (0, c_1(E), \chi(E))$ is the same as the Mukai vector $v(E)$ defined in Definition 4.1. We denote the moduli space of $G$-twisted stable sheaves $E$ on $X$ with $\gamma(E) = (r, \xi, \chi)$ by $M_H^G(\rho, \xi, \chi)$. We also denote the moduli of $G$-twisted semi-stable sheaves by $\mathcal{M}_H^G(\rho, \xi, \chi)$. Let $\pi : X \to C$ be an elliptic surface. Let $f$ be a smooth fiber and $L$ a nef and big divisor on $X$. Since $(L, f) > 0$, replacing $L$ by $L + nf$, $n > 0$, we may assume that $(L, C') > 0$ unless $C'$ is a $(-2)$-curve in a fiber of $\pi$. Let $G$ be a locally free sheaf on $X$ such that $\text{rk} G = r$ and $c_1(G) = dr$ with $\gcd(r, d) = 1$. We first study the stability condition, when the polarization is sufficiently close to $f$.

Lemma 4.8. For $(\xi, \chi) \in \text{NS}(X) \times \mathbb{Z}$ with $r \chi - (c_1(G), \xi) > 0$, we take $(n, e) \in \mathbb{Z} \times \text{NS}(X) \otimes \mathbb{Q}$ such that $n \gg 0$ and $e$ is an ample $\mathbb{Q}$-divisor with $|e| \ll 1$. Let $E$ be a purely $1$-dimensional sheaf with $v(\mathcal{E}) = (\xi, \chi)$. (i) $E$ is $G$-twisted stable with respect to $L + nf + e$ if and only if for all proper subsheaf $F$ of $E$, one of the following holds

(a) $\chi_G(E)/(c_1(E), f) > \chi_G(F)/(c_1(F), f)$ or
(b) $\chi_G(E)/(c_1(E), f) = \chi_G(F)/(c_1(F), f)$ and $\chi_G(E)/(c_1(E), L) > \chi_G(F)/(c_1(F), L)$ or
(c) \( \chi_G(E)/(c_1(E), f) = \chi_G(F)/(c_1(F), f) \), \( \chi_G(E)/(c_1(E), L) = \chi_G(F)/(c_1(F), L) \) and \( \chi_G(E)/(c_1(E), \epsilon) > \chi_G(F)/(c_1(F), \epsilon) \).

(ii) Moreover if \( L + nf \) is ample and \( \gcd((c_1(E), f), (c_1(E), L), \chi_G(E)) = 1 \), then there is no properly \( G \)-twisted semi-stable sheaf \( E \) with respect to \( L + nf \). In particular, \( E \) is \( G \)-twisted stable with respect to \( L + nf \) if and only if \( E \) is \( G(\eta) \)-twisted stable with respect to \( L + nf + \epsilon' \), where \( \eta, \epsilon' \) are sufficiently small \( Q \)-divisors.

**Proof.** We note that \( \chi_G(E) = r + \sigma((1 + (1 - \sigma))^2) \neq 0 \). If \( \chi_G(E)(c_1(E), f) - \chi_G(F)(c_1(E), f) < 0 \), then we see that

\[
(4.18) \quad \chi_G(E)(c_1(E), L + nf + \epsilon) - \chi_G(F)(c_1(E), L + nf + \epsilon) \\
\leq n(\chi_G((c_1(E), f) - \chi_G(F)(c_1(E), f)) + \chi_G(E)(c_1(E), L + \epsilon).
\]

If \( \chi_G(E)(c_1(E), f) - \chi_G(F)(c_1(E), f) \geq 0 \) and \( \chi_G(F) \geq 0 \), then

\[
(4.19) \quad \chi_G(E)(c_1(E), f) - \chi_G(F)(c_1(E), f) \geq n(\chi_G(E)(c_1(E), f) - \chi_G(F)(c_1(E), f)) \\
+ \chi_G(E)(c_1(E), f) ((c_1(E), f)(c_1(F), L + \epsilon) - (c_1(F), f)(c_1(E), L + \epsilon)) \\
\geq n(\chi_G(E)(c_1(E), f) - \chi_G(F)(c_1(E), f)) - \chi_G(E)(c_1(E), L + \epsilon).
\]

By using these inequalities, we can show claim (i). Moreover, if \( L + nf \) is ample, then the equalities

\[
(4.20) \quad \chi_G(E)(c_1(E), f) - \chi_G(F)(c_1(E), f) = 0, \\
\chi_G(E)(c_1(E), f) - \chi_G(F)(c_1(E), f) = 0
\]

imply that \( (c_1(F), f) = (c_1(E), f) = \chi_G(F) = 0 \) or \( (c_1(F), f) = (c_1(E), f) = \chi_G(F) = 0 \). By the ampleness of \( L + nf \), we get \( F = 0 \) or \( E/F = 0 \). Therefore the claim (ii) holds. \( \square \)

**Lemma 4.9.** Let \( \pi : X \to C \) be an elliptic surface and \( f \) a fiber of \( \pi \). If \( (D, f) = 1 \), then \( E \in M_H(0, D, 1) \) satisfies \( (1.15) \).

For the proof, see [Y3 Prop. 3.18]. Let \( \pi^{-1}(p) = \sum_{i=0}^{n} a_i C_i \) be a singular fiber of \( \pi \) such that \( (C_i, C_j) \leq 1 \). We may assume that \( (C_0, \sigma) = a_0 = 1 \).

**Lemma 4.10.** There is a \( G \)-twisted stable sheaf \( E_0 \) with \( c_1(E_0) = (r - 1)f + C_0 \) and \( \chi_G(E_0) = 0 \).

**Proof.** By Proposition 4.7 there is a \( G \)-twisted semi-stable sheaf \( E_0 \) with \( c_1(E_0) = (r - 1)f + C_0 \) and \( \chi_G(E_0) = 0 \). Assume that \( E_0 \) is \( S \)-equivalent to \( \bigoplus_i F_i \), where \( F_i \) are \( G \)-twisted stable sheaves with \( \chi_G(F_i) = 0 \). Since \( \text{Supp}(F_i) \) does not contain \( \tau \), \( (c_1(F_i), \sigma) \geq 0 \). Since \( \chi_G(F_i) = r(\chi_F(F_i) - d(\sigma, c_1(F_i))) \), there is an integer \( i_0 \) such that \( (\sigma, c_1(F_{i_0})) = r \) and \( (\sigma, c_1(F_i)) = 0 \) for \( i \neq i_0 \). Thus \( \text{Supp}(F_i), i \neq i_0 \) do not contain \( C_0 \), which implies that \( (c_1(F_i), C_0) \geq 0, i \neq i_0 \). Then we see that \( (c_1(F_{i_0})^2 = (c_1(E_0))^2 + (\sum_{i \neq i_0} c_1(F_i))^2 < -2 \).

This is a contradiction. Therefore \( E_0 \) is \( G \)-twisted stable. \( \square \)

**Lemma 4.11.** We set \( E_i := \mathcal{O}(C_i, -1), i > 0 \). Let \( E \) be a properly \( G \)-twisted semi-stable sheaf with \( \gamma(E) = (0, r, d) \) and \( \text{Supp}(E) = \sum_i a_i C_i \). Then \( E \) is \( S \)-equivalent to \( \bigoplus_i E_i^{a_i} \).

**Proof.** By the proof of Lemma 4.10 it is sufficient to prove the following. Let \( F \) be a \( G \)-twisted stable sheaf with \( c_1(F) = \sum_{i \geq 0} n_i C_i \) and \( \chi_G(F) = 0 \). Then \( F = E_i \) for an \( i > 0 \):

Since \( (c_1(F)^2 < 0 \), we can choose an integer \( i \) such that \( (c_1(F), C_i) < 0 \). Then \( \chi(E_i, F) > 0 \), which implies that \( \text{Hom}(E_i, F) = 0 \) or \( \text{Hom}(F, E_i) = 0 \). By the stability of \( E_i \) and \( F \), we see that \( E_i \approx F \). Thus the claim holds.

We take a sufficiently small \( Q \)-divisor \( \eta \) such that \( (\sigma, \eta) = (f, \eta) = 0 \) and \( \chi_G(\eta)(E_i) < 0 \) for \( i > 0 \). Then in the same way as in [OLY], we see that \( Y := M_H^{(\eta)}(0, r, d) \) is a resolution of \( M_H^{(\eta)}(0, r, d) \) at \( \bigoplus_i E_i^{a_i} \), and the exceptional divisors are

\[
(4.21) \quad C_i' := \{ E \in Y \mid \text{Ext}^2(E, E_i) \neq 0 \} \approx \mathbb{P}^1, i > 0
\]

and \( (C_i', C_j') = (C_i, C_j) \).

Let \( g \) (resp. \( \bar{g} \)) be the affine Lie algebra associated to \( E_i, i \geq 0 \) (resp. the finite Lie algebra associated to \( E_i, i \geq 1 \)).

**Proposition 4.12.** Let \( \pi : X \to C \) be an elliptic surface with a section \( \sigma \) as above. Assume that (1) \( l = 1 \), or (2) \( X \) is rational or of type \( K3 \). We set \( L := \sigma + (1 - (\sigma^2))f \). Let \( G \) be a locally free sheaf on \( X \) such that \( \text{rk} G = r \) and \( c_1(G) = \sigma f \) with \( \gcd(r, d) = 1 \). Then \( \mathfrak{g} \) acts on \( \bigoplus_D H_* (M^p_{l+nf+k}(0, l\sigma + D, k), C) \), where \( D \) is an effective divisor on fibers with \( (l + D)^2 + p_0 + 1 \geq 0 \) and \( k \) is an integer with \( \chi_G := \text{rk} - (c_1(G), l\sigma + D) > 0 \) and \( \gcd(l, \sigma + D, \chi_G) = 1 \). Moreover we also have an action of \( \mathfrak{g} \) if \( \gcd(l, \chi_G) = 1 \).
Proof. We first note that (1.13) holds, under (1) or (2). We first note that \( (c_1(E) + c_1(E_i), \sigma) = (c_1(E), \sigma) \) for \( i > 0 \). If \( \gcd(\chi_G(E), (c_1(E), f), (c_1(E), \sigma)) = 1 \), then the statements in Lemma 4.1 hold, where \( E_1 \) in Lemma 4.1 corresponds to \( E_i, i > 0 \). Hence we get our claim for \( \gamma \). Moreover if \( \gcd(\chi_G(E), (c_1(E), f)) = 1 \), then we can apply the results in Lemma 4.1 for \( E_0 \). Therefore our claim also holds for \( g \). \( \square \)

**Corollary 4.13.** Under the same notations as above, the Poincaré polynomial \( P(M^G_{L+n+f+e}(0, l\sigma + D, k)) \) is \( W(g) \)-invariant:

\[
P(M^G_{L+n+f+e}(0, w(l\sigma + D, k)) = P(M^G_{L+n+f+e}(0, l\sigma + D, k)), \quad w \in W(g),
\]

where \( W(g) \) is the Weyl group of \( g \).

Let \( X \) be a rational elliptic surface with a singular fiber of type \( E_8^{(1)} \). As we shall see in subsection 4.3, \( M^G_{L+n+f+e}(0, l\sigma + D, k) \) is related to a moduli space of torsion free sheaves. In [MNW], [Y1] and [Iq], it is observed that the Euler characteristic of \( M^G_{L+n+f+e}(0, l\sigma + D, k) \) is Weyl group invariant. Proposition 4.12 gives an explanation of this invariance.

### 4.3. Moduli of stable sheaves on rational elliptic surfaces.

Let \( \pi : X \to \mathbb{P}^1 \) be a rational elliptic surface with a surface \( \sigma \). Then there is a family of elliptic surfaces \( \pi : \mathcal{X} \to \mathbb{P}^1 \) over a scheme \( T \) such that

(i) \( \mathcal{X}_t \cong X, t \in T \);

(ii) there is a section \( \sigma \) of \( \pi \) with \( \sigma_t = \sigma \) and

(iii) for a general point \( t \in T \), \( \mathcal{X}_t \) is a nodal elliptic surface, that is, all singular fibers are irreducible nodal curves.

Let \( T_0 \) be the open subset of \( T \) consisting of nodal elliptic surfaces. Replacing \( T \) by a suitable covering of \( T_0 \), we may assume that \( \text{Pic}(\mathcal{X}/T) \cong R^2\phi_*\mathcal{O}_Z \) is a trivial local system, where \( \phi : \mathcal{X} \to T \) is the projection. Hence there is a relatively ample \( \mathbb{Q} \)-divisor \( H \) on \( \mathcal{X} \). Moreover, by adding \( m\sigma + nf \), we may assume that

\[
H = n\sigma + m\sigma + e, \quad e \in (\mathbb{Q} \sigma + \mathbb{Q}f)^+, \quad n \gg m \gg (|e|^2),
\]

where we use the identification \( R^2\phi_*\mathcal{O}_Z \cong H^2(X, \mathbb{Z}) \). For positive integers \( r, d \) with \( \gcd(r, d) = 1 \), we take a vector bundle \( G \) of rank \( r \) and \( c_1(G) = dr \) on \( \mathcal{X} \). We set \( \gamma := (r, \xi, \chi) \in R^*\phi_*\mathcal{O}_Z \). Then we have a family of moduli spaces of semi-stable sheaves \( \psi : \overline{\mathcal{M}}^G_{(\mathcal{X}, H)/T}(\gamma) \to T \), which is smooth on the locus of stable sheaves. Assume that \( \gamma \) is primitive and \( G \) general with respect to \( \gamma \). Then \( \mathcal{M}^G_{(\mathcal{X}, H)/T}(\gamma) \) consists of stable sheaves. For the existence of stable sheaves, see Appendix 6.2.

From now on, we assume that \( \gamma = (0, \xi, \chi) \), where \( \xi = l\sigma + kf + D, \quad l > 0, \quad \gcd(l, (\xi, \sigma), r\chi - (c_1(G), \xi)) = 1 \) and \( (D, f) = (D, \sigma) = 0 \). We take a sufficiently small \( \mathbb{Q} \)-divisor \( \eta \) such that \( \chi_{G(\eta)}(\mathcal{O}_{C_t}(-1)) < 0 \) for all \( i > 0 \).

We set \( \mathcal{Y} := \mathcal{M}^G_{(\mathcal{X}, H)/T}(0, r\sigma, d) \). Then \( \mathcal{Y} \), \( t \in T \) are smooth projective surface isomorphic to \( \mathcal{X}_t \). Hence \( \mathcal{Y} \to T \) is a smooth morphism. We have an isomorphism \( \mathcal{Y} \times_T T_0 \cong \mathcal{X} \times_T T_0 \) over \( T_0 \) (cf. [Y3]). Let \( H' \) be a relatively ample \( \mathbb{Q} \)-divisor on \( \mathcal{Y} \) whose restriction to \( \mathcal{Y} \times_T T_0 \) corresponds to a divisor on \( \mathcal{X} \times_T T_0 \) which is sufficiently close to \( m\sigma + nf \). By Lemma 4.3, we have an isomorphism

\[
M^G_{(\mathcal{X}, H)/T}(\gamma) \cong M^G_{(\mathcal{X}, H, H')/T}(\gamma').
\]

By our assumption, there is a universal family \( \mathcal{E} \) on \( \mathcal{X} \times_T \mathcal{Y} \). We consider a family of Fourier-Mukai transforms \( \Phi^E_{\mathcal{X} \to \mathcal{Y}} : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Y}) \). If \( r\chi - (c_1(G), \xi) > 0 \), then \( \Phi^E_{\mathcal{X} \to \mathcal{Y}} \) induces a birational map

\[
\zeta : M^G_{(\mathcal{X}, H)/T}(\gamma) \cdots \to M^G_{(\mathcal{Y}, H')/T}(\gamma')
\]

which is an isomorphism over \( T_0 \), where \( G' := \Phi^E_{\mathcal{X} \to \mathcal{Y}}(\mathcal{O}_\sigma) \) (see [Y3] Thm. 3.13, Rem. 3.1). Let \( Z \) be the graph of this birational correspondence. Then the cycle \( [Z]_{T_0} \) induces an isomorphism of the homology groups

\[
H_*(M^G_{(\mathcal{X}, H)/T}(\gamma), \mathbb{Z}) \to H_*(M^G_{(\mathcal{X}, H')/T}(\gamma'), \mathbb{Z})
\]

via the convolution product. Let \( E_i, \quad i = 0, 1, \ldots, n \) be \( G \)-twisted stable sheaves on \( X \) in section 4.2.

We set \( Y := \mathcal{Y}_{T_0} \) and

\[
\rho := \Phi^E_{\mathcal{X} \to \mathcal{Y}}(\mathcal{O}_{E_x}) \in K(Y), \quad x \in X
\]

\[
u_i := \Phi^E_{\mathcal{X} \to \mathcal{Y}}(E_i) \in K(Y), \quad i = 0, 1, \ldots, n.
\]

Then \( \sum_{i=0}^n \nu_i = \Phi^E_{\mathcal{X} \to \mathcal{Y}}(\mathcal{O}_x)(\bigoplus_{i=0}^n E_i) \cong \mathcal{C}_Y, y \in Y \). By Proposition 4.12 we get the following:

**Proposition 4.14.** We have an action of \( \mathfrak{g} \) on the homology groups

\[
\bigoplus_{n_i} H_*(M^G_{H}(\gamma(lG + \sum n_i u_i + k\rho)), n_i \in \mathbb{Q},
\]

where \( \sum_{i} n_i u_i \in K(Y) \), \( H \) is sufficiently close to \( f \), \( \chi_G(\mathcal{O}_\sigma) + kr > 0 \) and \( \gcd(l, r\sigma_0, kr) = 1 \). Moreover if \( \gcd(l, k) = 1 \), then we have an action of \( \mathfrak{g} \).
4.4. Moduli of stable vector bundles on an ADE-configuration. In this subsection, we explain a relation with a paper by Nakajima [N2]. Let $X$ be a smooth projective surface containing an ADE-type configuration of smooth rational curves $C_i, i = 1, 2, \ldots, n$. Assume that there is a nef and big divisor $H$ such that $(C_i, H) = 0$ for all $C_i$.

For $\xi \in NS(X)$ and $d \geq 0$, we set

\[(4.28) \quad B_{(\xi,d)} := \{ x \in \oplus_{i=1}^{n} \mathbb{Z}C_i \mid (x^2) - 2(\xi,x) + d \geq 0 \}. \]

Since $\oplus_{i=1}^{n} \mathbb{Z}C_i$ is negative definite, $B_{(\xi,d)}$ is a finite set. Let $r$ be a positive integer such that $2r > (x^2) - 2(\xi,x) + d$ for all $x \in B_{(\xi,d)}$. Assume that there is an integer $\chi_0$ such that $d = (\xi^2) - 2r\chi_0 - r(K_X, \xi) + (r^2 + 1)\chi_0$.\]

**Definition 4.4.** Let $M_H(r,\xi + x, \chi)^{\mu}, x \in \oplus_{i=1}^{n} \mathbb{Z}C_i, \chi \in \mathbb{Z}$ be the moduli space of $\mu$-stable sheaves $E$ with respect to $H$ such that $\gamma(E) = (r,\xi + y, \chi)$.

$M_H(r,\xi + y, \chi)^{\mu}$ is contained in a moduli space of $\mu$-stable sheaves with respect to an ample divisor $H'$ which is sufficiently close to $H$. If $\gcd(r,\xi, H)) = 1$, then $M_H(r,\xi + y, \chi)^{\mu}$ is projective. We assume that $[1,1,1]$ holds for all $E \in M_H(r,\xi + y, \chi)^{\mu}, y \in \oplus_{i=1}^{n} \mathbb{Z}C_i, \chi \in \mathbb{Z}$. Then $M_H(r,\xi + y, \chi)^{\mu}, y \in \oplus_{i=1}^{n} \mathbb{Z}C_i, \chi \in \mathbb{Z}$ is a smooth scheme of dimension $(y^2) - 2(\xi,y) + d + 2r(\chi_0 - \chi) + q, q = \dim H^1(X,\mathcal{O}_X)$, if it is not empty.

**Lemma 4.15.**

(i) $M_H(r,\xi + x, \chi)^{\mu}, x \in B_{(\xi,d)}$ consists of locally free sheaves.

(ii) $H^1(C_i, E_{(C_i)}) = 0$ for all $E \in M_H(r,\xi + x, \chi)^{\mu}, x \in B_{(\xi,d)}$.

**Proof.** We prove the second claim. The proof of the first one is similar. If $H^1(C_i, E_{(C_i)}) \neq 0$, then there is a surjective homomorphism $\phi : E \to \mathcal{O}_{C_i}(-1-k), k > 0$. By our assumption on $E$, $F := \ker \phi$ is a $\mu$-stable sheaf with $\gamma(F) = \gamma(E) - (0,C_i,-k)$. Then we have

\[(4.29) \quad M_H(\gamma(F)) = ((x-C_i)^2) - 2(\xi,x-C_i) + d - 2rk + q < 2r + q - 2rk \leq q.\]

This is impossible. Hence the claim holds.

**Corollary 4.16.** Let $E$ be an element of $M_H(r,\xi + x, \chi)^{\mu}, x \in B_{(\xi,d)}$. For a subspace $V \subset \Hom(E,\mathcal{O}_{C_i}(-1))$, $\phi : E \to V^\vee \otimes \mathcal{O}_{C_i}(-1)$ is surjective and $\ker \phi$ is a $\mu$-stable locally free sheaf with the Chern character $\text{ch}(F) = \text{ch}(E) - (\dim V)(0,C_i, 0)$.

We set

\[(4.30) \quad \mathcal{P}_{\mathcal{O}_{C_i}(-1)}^{(n)}(r,\xi + x, \chi) := \{(E,U^\vee) \mid E \in M_H(r,\xi + x, \chi)^{\mu}, U^\vee \subset \Hom(E,\mathcal{O}_{C_i}(-1)), \dim U = n \}
\]

and define operators $e_i, f_i, h_i$. Then we have the following which is due to Nakajima [N2] sect. 5.

**Proposition 4.17.** Let $g$ be a finite Lie algebra generated by $\mathcal{O}_{C_i}(-1)$. Then $g$ acts on $\oplus_{x \in B_{(\xi,d)}} H^1(M_H(r,\xi + x, \chi)^{\mu})$, provided the non-emptyness of the moduli spaces.

**Remark 4.3.** In order to compare the correspondence in Theorem [23], we need to set $F := E[1]$ (cf. (1.39)).

**Example 4.3.** Let $\pi : X \to \mathbb{P}^1$ be an elliptic K3 surface with a section $\sigma$. Let $f$ be a fiber of $\pi$. Then $H := \pi^* + tf, t \gg 0$ is a nef and big divisor on $X$. Let $C_i, i = 1, 2, \ldots, n$ be $(2)$-curves contracted by $\pi H$. Assume that $\gcd(r, \xi, f) = 1$. Then for $y \in NS(X)$ with $(y,f) = 0$ and $k \in \mathbb{Z}$,

\[(4.31) \quad M_H(r,\xi + y, \chi)^{\mu} := \left\{ E \mid E \text{ is a torsion free sheaf with } \gamma(E) = (r,\xi + y, \chi) \text{ such that } E_{(\pi^{-1}(p))} \text{ is stable for a general point } p \in \mathbb{P}^1 \right\}. \]

In particular, $M_H(r,\xi + y, \chi)^{\mu}$ is projective and coincides with $M_H(r,\xi + y, \chi)$ where $H'$ is an ample divisor which is sufficiently close to $H$. Therefore $M_H(r,\xi + y, \chi)^{\mu}$ is not empty, provided the expected dimension is non-negative (cf. [Y2]). Thus all the requirements are satisfied and we have an action of finite Lie algebra.

**Remark 4.4.** Let $X \to C$ be an elliptic surface in section [122]. We use the same notations. Let $E$ be the universal family on $Y \times X$. Then we have a Fourier-Mukai transform $\Phi_{X \to Y} : D(X) \to D(Y)$. We set $C' := \Phi_{X \to Y}(\mathcal{O}_\sigma)$. Let $\xi := \sigma + D, (f,D) = 0$ be an effective divisor such that $(\xi^2) = (\sigma^2)$. Assume that

\[(4.32) \quad (\xi,x) + (x^2) < 2r \text{ for all } x \in \oplus_{i=1}^{n} \mathbb{Z}C_i. \]
Then $\Phi^{\vee}_{X \to Y}$ induces an isomorphism $M^G_{H}((0, \xi + x, \chi)) \cong M^G_{H}(r, \xi', \chi')$, where $r\chi - d(\xi + x, \sigma) > 0$, $(c_1(G'), f) = (\xi', f)$ and $(\langle \xi \rangle + x) = (\langle \xi' \rangle) - 2r\chi - r(K_X, \xi') + r^2 \chi(O_X)$. Moreover $\Phi^{\vee}_{X \to Y}(O_{C_i}(-1)) \cong O_{C'_i}(k_i)$ for some $k_i$ with $\chi_G^\vee(O_{C'_i}(k_i)) = 0$. The action of $g$ generated by $\Phi^{\vee}_{X \to Y}(O_{C_i}(-1))$ is similar to the action in this section. Indeed if $(C_i, C_{i,j})$ is of type $E_8$, then there is a divisor $D = \sum_{i=1}^8 b_i C'_i$ such that $r(D, C'_i) = -(c_1(G'), C'_i)$. Then replacing $E$ be $E \otimes O_Y(-D)$, we may assume that $k_i = -1$ for all $i > 0$.

**Remark 4.5.** Let $Y$ be a projective surface with rational double points as singularities and $H'$ an ample Cartier divisor on $Y$. Assume that there is a morphism $\phi: X \to Y$ which gives the minimal resolution and $H = \phi^{-1}(H')$. For simplicity, we assume that there is a unique singular point $p \in Y$. Let $Z := \phi^{-1}(p)$ be the fundamental cycle. For $E \in M_H(r, \xi + x, \chi)^\mu$, we have an exact sequence

$$0 \to E' \to E \to F \to 0$$

such that $F$ is a successive extensions of $O_{C_i}(-1)$ and $\text{Hom}(E', O_{C_i}(-1)) = 0$, $i = 1, 2, \ldots, n$. Then we have $E'_{i,C_i} \cong O_{C_i}(1)^{a_i} \oplus O_{C_i}^{b_i}$. For all $C_i$, there is an exact sequence

$$0 \to G \to \text{Hom}(E', O_{C_i}) \to 0,$$

where $G$ is a successive extensions of $O_{C_i}(-1)$ (cf. Example 4.1). Hence we see that $H^1(Z, E'_i|Z) = 0$. Then we see that $R^1\phi_* F = 0$. Since $R^1\phi_* F = 0$, we get that $\phi_* (E) \cong \phi_* (E')$ and $R^1\phi_* (E) \cong R^1\phi_* (E') = 0$. Therefore we have a morphism

$$\phi_* : M_H(r, \xi + x, \chi)^\mu \to M_H(r, \xi + x, \chi)^\mu$$

where $M_H(r, \xi + x, \chi)^\mu$ is the moduli space of $\mu$-stable sheaves on $Y$. By this morphism, we have a contraction of the Brill-Noether locus. We can also show that $R^1\phi_*(E''') = 0$ and $E''|Z$ is generated by global sections. Thus $\phi_* (E) \cong \phi_* (E')$ is a reflexive sheaf and $E'$ is a full sheaf. Hence the local structure of this contraction map is an example of the studies of Ishii [11, 12]. More generally, for each moduli space $M_H(r, \xi + x, \chi)^\mu$, let $M_H(r, \xi + x, \chi)^\#$ be the open subset consisting of $E$ such that $E$ is locally free, \[4.15\] holds and $R^1\phi_* (E) = R^1\phi_* (E') = 0$. Then we see that $H^1(C_i, E|C_i) = H^1(C_i, E'|C_i) = 0$ for all $i$ and Corollary \[4.16\] holds. Since $R^0\phi_*(O_{C_i}(-1)) = 0$ for all $j$ and $\text{Ext}^1_{O_X}(O_{C_i}(-1), O_X) \cong O_{C'_i}(-1)$, ker $\phi$ belongs to $M_H(r, \xi - (\text{dim } V)C_i, \chi)^\#$. Therefore we also have similar claims for $M_H(r, \xi + x, \chi)^\#$.

5. Equivariant sheaves

In this section, we give a remark for the moduli of equivariant sheaves. Let $G$ be a finite group acting on $X$. Let $E_0$ be an irreducible $G$-sheaf of dimension 0, i.e. $E_0$ does not have a non-trivial $G$-subsheaf. Then $\text{Hom}(E_0, E_0)^G = C$.

**Lemma 5.1.** Let $E_0$ be an irreducible $G$-sheaf of dimension 0. Let $E$ be a torsion free (resp. purely 1-dimensional) $G$-sheaf.

1. Then every non-trivial extension

$$0 \to E \to F \to E_0 \to 0$$

defines a torsion free (resp. purely 1-dimensional) $G$-sheaf.

2. Let $V$ be a subspace of $\text{Hom}(E, E_0)$. Then $\phi : E \to V^\vee \otimes E_0$ is surjective. Moreover, ker $\phi$ is a torsion free (resp. purely 1-dimensional) $G$-sheaf.

Let $H$ be a $G$-equivariant line bundle on $X$ which is ample.

**Definition 5.1.** A $G$-sheaf $E$ is $\mu$-stable, if $E$ is torsion free and

$$\frac{(c_1(F), H)}{\text{rk } F} < \frac{(c_1(E), H)}{\text{rk } E}$$

for all $G$-subsheaf $F$ of $E$ with $0 < \text{rk } F < \text{rk } E$.

For a $G$-sheaf $E$ on $X$, $v(E)$ denotes the class of $E$ in $K^G(X)$. For a $v \in K^G(X)$, $M_H(v)^\mu$ is the moduli of $\mu$-stable $G$-sheaves $E$ with $v(E) = v$. Assume that

$$\text{Ext}^2(E, E)^G \to H^2(X, O_X)^G$$

is an isomorphism for all $E \in M_H(v)^\mu$. We set

$$\langle v(E), v(F) \rangle := -G_\chi(E, F) = -\sum_i (-1)^i \dim \text{Ext}^i(E, F)^G.$$
Let $E_1, E_2, \ldots, E_s$ be a configuration of irreducible $G$-sheaves of dimension 0 such that

$$E_i \otimes K_X \cong E_i,$$

$$(5.5) \quad \text{Ext}^1(E_i, E_i)^G = 0.$$ 

Then $v(E_i)$ are $(-2)$-vectors. We set

$$\mathcal{P}^{(n)}_E(v) := \{(E, U'')|E \in M_H(v)^\mu, U'' \subset \text{Hom}(E[1], E_i[1]), \dim U = n\}$$

and define operators $e_i, f_i, h_i$. Then we have an action of the Lie algebra $\mathfrak{g}$ generated by $v(E_1), v(E_2), \ldots, v(E_s)$ on $\bigoplus H^*(M_H(v)^\mu, \mathbb{C})$.

**Remark 5.1.** Let $X$ be an abelian surface or a K3 surface with a symplectic $G$-action. Assume that there is a fixed point. By the McKay correspondence [BKR], we have an equivalence $\Phi : D^b(H(X)) \cong D^b(X/G)$, where $X/G$ is the minimal resolution of $X/G$. Hence $M_H(v)^{\mu}$ is isomorphic to a moduli space of objects in $D^b(X/G)$. If $v = (r, \xi, a)$ with $\text{gcd}(r, \xi, H) = 1$, then $M_H(v)^{\mu}$ is projective. Hence $M_H(v)^{\mu}$ is a holomorphic symplectic manifold which is birationally equivalent to a moduli space $M_H(v)$ of stable sheaves on $X/G$, where $w$ is the Mukai vector corresponding to $v$ via $\Phi$. By a result of Huybrechts [H1], [H2], there is an isomorphism $H_v(M_H(v)^{\mu}, \mathbb{Z}) \cong H_v(M_H(v), \mathbb{Z})$ via a convolution product by an algebraic cycle. Hence by fixing this identification for each $H_v(M_H(v), \mathbb{Z})$, we also have an action of $\mathfrak{g}$ on $\bigoplus H_v(M_H(v), \mathbb{C})$. 

**Remark 5.2.** Assume that $X = \mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$ with an action of a Klein group $G \subset SL(\mathbb{C}^2)$. Let $W$ be a $G$-vector space. We consider the moduli of framed $G$-sheaves $(E, \Phi)$, where $E$ is a torsion free $G$-sheaf on $\mathbb{P}^2$ and $\Phi : E_i \otimes \mathcal{O}_{\ell_\infty} \otimes W$ is a $G$-isomorphism. This is an example of Nakajima’s quiver variety and we have an action of affine Lie algebra associated to $G$ on the homology groups $[N2]$. In this case, we set $\langle v(E), v(F) \rangle := -G(\chi(E, F(-\ell_\infty)))$ and we use the vanishing $\text{Ext}^2(E, E \rightarrow (\mathcal{O}_{\ell_\infty} \otimes W \oplus E_i)) = 0$ to show the smoothness of $\mathcal{P}^{(n)}_E(v)$.

6. Appendix

**6.1. Moduli of coherent systems.** In this subsection, we shall explain how to construct the moduli space of coherent systems $\mathcal{P}^{(n)}_{E_i}(v)$. We start with a definition of a flat family.

**Definition 6.1.** Let $S$ be a scheme and $\mathcal{E}_0 : \cdots \rightarrow \mathcal{E}_{-1} \rightarrow \mathcal{E}_0 \rightarrow \cdots$ a bounded complex on $S \times X$.

(i) $\mathcal{E}_s$ is a flat family of stable complexes, if $\mathcal{E}_s$ are coherent sheaves on $S \times X$ which are flat over $S$ and $(\mathcal{E}_s)_a$ are stable complexes for all $a \in S$.

(ii) $(\mathcal{E}_s, \mathcal{U})$ is a family of coherent systems, if $\mathcal{E}_s$ is a flat family of stable complexes and $\mathcal{U}$ is a locally free subsheaf of $\text{Hom}_{ps}(O_S \otimes E_i, \mathcal{E}_s)$ of rank $n$ such that $\mathcal{U}_s \rightarrow \text{Hom}(E_i, (\mathcal{E}_s)_a)$ is injective for all $a \in S$. In this case, we have a resolution of $E_i$:

$$W_s : W_{-2} \rightarrow W_{-1} \rightarrow W_0$$

with a morphism $\mathcal{U} \otimes W_s \rightarrow \mathcal{E}_s$ as complexes which induces the inclusion $\mathcal{U} \rightarrow \text{Hom}_{ps}(O_S \otimes E_i, \mathcal{E}_s)$.

For a quasi-isomorphism $\mathcal{E}_s \rightarrow \mathcal{E}_s'$ of families of stable complexes over $S$, we take a resolution of $E_i$

$$W_s : W_{-2} \rightarrow W_{-1} \rightarrow W_0$$

such that $\text{Ext}^p(W_j, (\mathcal{E}_s)_a) = 0$, $p > 0$ for $j = 0, -1$, $k \in \mathbb{Z}$ and all $a \in S$. Then we see that $\text{Ext}^p(W_{-2}, (\mathcal{E}_s)_a) = 0$, $p > 0$ for $k \in \mathbb{Z}$ and all $a \in S$. By this choice of $W_s$, we have an isomorphism

$$(6.3) \quad \text{Hom}_{K(S \times X)}(O_S \otimes W_s, \mathcal{E}_s[p]) \rightarrow \text{Hom}_{K(S \times X)}(O_S \otimes W_s, \mathcal{E}_s'[p])(\cong \text{Ext}^p(O_S \otimes E_i, \mathcal{E}_s'))$$

where $K(Z)$ is the homotopy category of complexes on $Z$. Hence for a family of coherent systems $(\mathcal{E}_s, \mathcal{U})$, there is a resolution of $E_i$ and a family of coherent systems $(\mathcal{E}_s, \mathcal{U})$ such that we have a homotopy commutative diagram:

$$\begin{array}{c}
\mathcal{U} \otimes W_s \longrightarrow \mathcal{E}_s \\
\downarrow \phi \downarrow \\
\mathcal{U} \otimes W_s \longrightarrow \mathcal{E}_s'.
\end{array}$$

(6.4)

The choice of $\phi$ is unique, up to homotopy equivalence. In this case, we say that $(\mathcal{E}_s, \mathcal{U})$ is equivalent to $(\mathcal{E}_s', \mathcal{U})$.

Let $q : Q_H(v) \rightarrow M_H(v)$ be a standard $PGL(N)$-covering of $M_H(v)$ which is an open subscheme of a suitable quotient scheme and satisfies the following properties:

(i) There is a flat family of stable complexes $\mathcal{V}_s : \mathcal{V}_{-1} \rightarrow \mathcal{V}_0$ on $Q_H(v) \times X$, which is $GL(N)$-equivariant.
(ii) For a flat family of stable complexes $E_*$ parametrized by $S$, if we take a suitable open covering $S = \cup \lambda S_\lambda$, then we have a morphism $f_\lambda : S_\lambda \rightarrow Q_H(v)$ such that $E_\lambda|S_\lambda$ is quasi-isomorphic to $f_\lambda^*(V_s)$. In particular $(q \circ f_\lambda)|S_\lambda \cap S_\mu = (q \circ f_\mu)|S_\lambda \cap S_\mu$ and we have a morphism $f : S \rightarrow M_H(v)$.

We take a locally free resolution of $E_i$

\begin{equation}
0 \rightarrow W_{-2} \rightarrow W_{-1} \rightarrow W_0 \rightarrow E_i \rightarrow 0
\end{equation}

such that $\text{Ext}^p(W_j, (V_k)_t) = 0, p > 0$ for $j = 0, -1, k = -1, 0$ and all $t \in Q_H(v)$. Then $\text{Ext}^p(W_{-2}, (V_k)_t) = 0, p > 0$ for $k = -1, 0$ and all $t \in Q_H(v)$. We set

\begin{equation}
\mathcal{H}_n := \bigoplus_{-j + k = n} \text{Hom}_{pD_H(v)}(O_{Q_H(v)} \otimes W_j, V_k).
\end{equation}

$\mathcal{H}_n, n \in \mathbb{Z}$ are locally free sheaves on $Q_H(v)$. We take a complex

\begin{equation}
0 \rightarrow \mathcal{H}_{-1} \xrightarrow{\psi_1} \mathcal{H}_0 \xrightarrow{\psi_0} \mathcal{H}_1 \xrightarrow{\psi_1} \cdots
\end{equation}

associated to $R \text{Hom}_{pD_H(v)}(O_{Q_H(v)} \otimes E_i, V_s)$. Since $\ker(\psi_1) = \text{Hom}(E_i, E_i[-1]) = 0$ for all $t \in Q_H(v)$, $\psi_1$ is injective as a vector bundle homomorphism. Hence $\mathcal{H}_0 := \text{coker}\psi_1$ is a locally free sheaf on $Q_H(v)$. For the morphism $f_\lambda : S_\lambda \rightarrow Q_H(v)$ and a locally free subsheaf $U \subset \text{Hom}_{pD}(O_S \otimes E_i, E_s)$ such that $U \rightarrow \text{Hom}(E_i, (E_s)_s)$ is injective for all $s \in S$, we have an inclusion as a vector bundle homomorphism:

\begin{equation}
U|_{S_\lambda} \rightarrow \text{Hom}_{pD}(O_S \otimes E_i, E_s)|_{S_\lambda} = \ker(f_\lambda^*(\mathcal{H}_0) \rightarrow f_\lambda^*(\mathcal{H}_1)) \rightarrow f_\lambda^*(\mathcal{H}_0).
\end{equation}

We take a Grassmann bundle $Gr(H_0', n) \rightarrow Q_H(v)$ over $Q_H(v)$ parametrizing $n$-dimensional subspaces $U$ of $(H_0')_t, t \in Q_H(v)$. Then we have a lifting $f_\lambda : S_\lambda \rightarrow Gr(H_0', n)$ of $f_\lambda$ and an equivalence between $(E_*|_{S_\lambda})$ and $(f_\lambda^*(V_s), U|_{S_\lambda})$. Hence $\mathcal{P}_{E_i}(v)$ is constructed as a closed subscheme of $Gr(H_0', n)/\text{PGL}(N)$.

6.2. The existence of stable sheaves on a rational elliptic surface. We shall find the conditions for the existence of stable sheaves on a rational elliptic surface $\pi : X \rightarrow \mathbb{P}^1$ with a section $\sigma$. We first note that a divisor $C$ with $(C^2) = (C, K_X) = -1$ is effective. Indeed since $(K_X - C, f) = -1, H^2(X, O_X(C)) = 0$. By the Riemann-Roch theorem, $\dim H^0(X, O_X(C)) \geq \chi(O_X(C)) = 1$. The following is the result for the case of rank 0.

**Proposition 6.1.** Let $X$ be a rational elliptic surface with a section $\sigma$. Let $D$ be a divisor with $(D^2) \geq 0$. Assume that $(0, D, \chi)$ is primitive. Then $M_H(0, D, \chi)$ is not empty for a general $H$ and $G$ if and only if $(D, C) \geq 0$ for all divisor $C$ with $(C^2) = (C, K_X) = -1$.

**Proof.** We use the notation in subsection 4.3. Since $\mathcal{M}_{(X,H)}(0, D, \chi) \rightarrow T$ is smooth, it is sufficient to prove the claim for a nodal rational elliptic surface $X$. Let $C$ be a divisor with $(C^2) = (C, K_X) = -1$. Since every fiber is irreducible, $C$ must be a section of $\pi$. If $(D, C) < 0$, then $\chi(O_C(k, E)) = -D, C) > 0$ for all sheaf $E$ with $c_1(E) = D$. We set $n := \max\{k \mid \text{Hom}(O_C(k), E) \neq 0\}$. Then $\text{Hom}(O_C(n), E) \neq 0$ and $\text{Hom}(E, O_C(n)) = \text{Ext}^2(O_C(n + 1), E) \neq 0$. This means that $E$ is not semi-stable, unless $E \cong O_C(n)$.

Conversely, we assume that $(D, C) \geq 0$ for all section $C$ with $(C^2) = (C, K_X) = -1$. Then $D$ is a nef divisor. If $(D, f) = 1$, then there is a section $\tau$ of $\pi$ such that $D = \tau + nf, n > 0$. In this case, $M_H(0, \tau + nf, \chi) \cong \text{Hilb}^{nf}_X \neq \emptyset$ via the relative Fourier-Mukai transform. Since the non-emptyness does not depend on the choice of $G$, we get our claim. Hence we may assume that $(D, f) \geq 2$. We shall show that there is a reduced and irreducible curve $C \subset |D|$. Then a line bundle $E$ on $C$ with $\chi(E) = \chi$ belongs to $M_H(0, D, \chi)$. If $(D^2) \geq 1$ or $(D, f) \geq 3$, then $D' := D - K_X$ is a nef divisor with $(D'^2) \geq 5$. Assume that there is an effective divisor $B$ with $(D', B) \leq 1$. Since $0 \geq (D, B) \leq (D', B) \leq 1, (i)$ $(f, B) = 0$ and $(D, B) \leq 1$ or (ii) $(f, B) = 1$ and $(D, B) = 0$. In the first case, $B = nf$. Since $(D, f) \geq 2$, this is impossible. In the second case, there is a section $\tau$ and $B = \tau + nf$. Then $(B^2) = 2n - 1 \neq 0$. By the Reider’s result $D = D' + K_X$ is base point free.

If $(D^2) = 0$ and $(D, f) = 2$, then $D = 2\tau_1 + f$ or $D = \tau_1 + \tau_2$ with $(\tau_1, \tau_2) = 1$, where $\tau_1, \tau_2$ are sections of $\pi$. In the first case, $(D, \tau_1) = -1$, which is a contradiction. In the second case, $D$ is connected and $D$ is base point free. By Bertini’s theorem, there is a reduced and irreducible curve $C \subset |D|$.

**Definition 6.2.** We set

\begin{equation}
C := \left\{ D \in \text{Pic}(X) \mid \begin{array}{c}
(D, C) \geq 0 \text{ for all divisors } C \\
\text{with } (C^2) = (C, K_X) = -1
\end{array}\right\}.
\end{equation}

Let $W := W(E_{8}^{(1)})$ be the Weyl group of the sublattice $f^+ \cong E_{8}^{(1)}$ of $\text{Pic}(X)$. $W$ acts on $\text{Pic}(X)$ and $C$ is a $W$-invariant subset of $\text{Pic}(X)$. Let $C^+ \subset C$ be the set of nef divisors. If $X$ is nodal, then $C^+ = C$. 

**Theorem 6.2.** Let $r$ and $d$ be relatively prime integers with $r \geq 0$. 

(i) For any $D \in \langle \sigma, f \rangle^\perp$, there is a stable vector bundle $E_D$ such that $\text{rk}(E_D) = r$, $c_1(E_D) \equiv d\sigma + D \mod \mathbb{Z}f$ and $\chi(E_D, E_D) = 1$. $E_D$ is unique up to $E_D(nf)$, $n \in \mathbb{Z}$. We set

$$E(r, d) := \{ E_D([D, \sigma] = ([D, f] = 0) \}. \tag{6.10}$$

(ii) Let $F \in K(X)$ be a primitive class with $\text{rk}(F) = lr$ and $(c_1(F), f) = ld$. Assume that $\chi(F, F) \leq 0$. We take an ample divisor $H$ which is sufficiently close to $f$. Then $F$ is represented by a stable sheaf if and only if $\chi(E_D, F) \leq 0$ for all $E_D \in E(r, d)$. Moreover $F$ is represented by a $\mu$-stable vector bundle, if $lr > 1$.

Proof. We may assume that $lr > 0$. By the deformation argument in the proof of Proposition 6.1, we may assume that $X$ is nodal. We first prove (i). We note that $M_H(0, rf, -d) \cong X$. Let $\mathcal{E}$ be a universal family on $X \times X$. Since every fiber is irreducible, we have $\sigma - D = \tau - ((\sigma, \tau) + 1)f$ where $\tau$ is a section of $\pi$. Then $\mathcal{E}_{X \times \tau}$ is a stable sheaf with the desired invariant. We next prove (ii). The proof of the necessary condition is similar to the proof of Proposition 6.1. We shall show that the condition is sufficient. Let $\Phi^\vee_{X \rightarrow X} : D(X) \rightarrow D(X)$ be the relative Fourier-Mukai transform defined by the sheaf $\mathcal{E}$. Then $\Phi^\vee_{X \rightarrow X}(E_D)[1] = \mathcal{O}_\tau$, where $\tau$ is a section of $\pi$ such that $-\sigma \equiv -D \mod \mathbb{Z}f$. Hence $\text{rk}(\Phi^\vee_{X \rightarrow X}(F)[1]) = 0$ and $c_1(\Phi^\vee_{X \rightarrow X}(F)[1]) \in C$. Therefore $\Phi^\vee_{X \rightarrow X}(F)[1]$ is represented by a line bundle $L$ on a reduced and irreducible curve. Then the inverse $\Phi^\vee_{X \rightarrow X}(L)[1]$ is a $\mu$-stable sheaf. $\square$

By the proof of the theorem, we also get the following.

Corollary 6.3. If $\text{gcd}(r, (\xi, f)) = 1$ and the expected dimension is non-negative, then $M_H(r, \xi, \chi)$ is not empty, where $H$ is sufficiently close to $f$.

Let $X$ be a rational elliptic surface with a section $\sigma$ such that there is a singular fiber $\pi^{-1}(o) = \sum_{i=0}^{8} a_i C_i$, $o \in \mathbb{P}^1$ of type $E_8^{(1)}$, where $C_i$ are smooth $(-2)$-curves. We assume that $a_0 = 1$. Let $C$ be a divisor with $(C^2) = (C, K_X) = -1$. Then $C = \sigma + \sum_{i=0}^{8} n_i C_i$, $n_i \geq 0$. Hence

$$C^+ = \{ D \in \text{Pic}(X) | (D, \sigma) \geq 0, (D, C_i) \geq 0, 0 \leq i \leq 8 \}. \tag{6.11}$$

Thus $D := rs + n + f + \xi, \xi \in \oplus_{i=1}^{8} \mathbb{Z} C_i$, is nef if and only if

$$\begin{cases} n \geq r, \\ \langle \xi, C_i \rangle \geq 0, 1 \leq i \leq 8 \\ \sum_{i=1}^{8} a_i \langle \xi, C_i \rangle \leq r. \end{cases} \tag{6.12}$$

Let $W$ be the affine Weyl group of $E_8^{(1)}$. Then $M_H(0, D', \chi) \neq \emptyset$ if and only if $D' = w(D)$ with $D \in C^+, w \in W$.

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