ALL COUPLINGS LOCALIZATION FOR QUASIPERIODIC OPERATORS
WITH LIPSCHITZ MONOTONE POTENTIALS

SVETLANA JITOMIRSKAYA, ILYA KACHKOVSKIY

Abstract. We establish Anderson localization for quasiperiodic operator families of the form

\((H(x)\psi)(m) = \psi(m + 1) + \psi(m - 1) + \lambda v(x + m\alpha)\psi(m)\)

for all \(\lambda > 0\) and all Diophantine \(\alpha\), provided that \(v\) is a 1-periodic function satisfying a Lipschitz monotonicity condition on \([0, 1)\). The localization is uniform on any energy interval on which Lyapunov exponent is bounded from below.

1. Introduction

Ever since the Nobel-prize winning discovery of Anderson that even weakly coupled 1D structures with random impurities exhibit insulator behavior (manifested mathematically as Anderson localization) a paradigm has been that the phenomenon of localization at small couplings is a signature of “randomness”, (see e.g. [17]). Localization at large couplings is intuitive and can be approached perturbatively in a variety of settings. However, small-coupling localization, i.e. the fact that there is pure point spectrum even for arbitrary small perturbations of the Laplacian, is not expected at all in dimensions higher than two, and even in random 1D is significantly more subtle than the corresponding high coupling fact. Indeed this phenomenon is not present for analytic quasiperiodic potentials which have purely absolutely continuous spectrum at small couplings for all phases and frequencies [21, 12, 3, 1], however is expected (yet apparently difficult to establish) even for the mildly random underlying dynamics such as skew shifts [23, 35]. So far however it was only proved in the random or quasirandom cases [15, 14].

In this paper we show that localization at all couplings, that is, “random-like” behavior, holds for all quasiperiodic potentials that are monotone (in a Lipschitz way) on the period. Such monotonicity of course implies discontinuity. It was shown in [19] (proving a conjecture made in [36]) that this already leads to a.e. positivity of the Lyapunov exponents as discontinuity makes the potentials non-deterministic in the Kotani sense. Yet the question of localization is still very subtle. First it is not a priori clear if the Lyapunov exponents are positive on the spectrum (or equivalently whether the spectrum has positive measure). Indeed, in the most well studied quasiperiodic model with discontinuity, the Fibonacci potential (see, for example, the recent preprint [20] for the most comprehensive results and references) the spectrum is a Cantor set of zero Lebesgue measure, and is purely singular continuous for all \(\lambda > 0\). The same holds, more generally, for Sturmian potentials, for all of which Lyapunov exponents are positive almost everywhere (by the same discontinuity reason) but not on the zero measure spectrum. Second, even under the condition of positivity of the Lyapunov exponents, establishing Anderson localization is a known difficult problem [9]. While monotonicity results in lack of resonances and thus should lead to obvious advantages in the perturbative arguments,
all the existing non-perturbative ones\footnote{We employ the word non-perturbative in the widely used by now sense of “obtained as a corollary of positive Lyapunov exponents without further largeness/smallness assumptions”\cite{9,28}.} have so far required (near) analyticity and used a powerful analytic apparatus, thus are not applicable here. Indeed the problem remains extremely difficult even for the smooth potentials (e.g. \cite{33,22,34}) and not much is known beyond (near) analyticity. In contrast, our results only require Lipschitz monotonicity and even that can be somewhat relaxed.

More precisely we consider quasiperiodic operator families in $l^2(\mathbb{Z})$ of the form

\begin{equation}
(H_{\alpha,\lambda}(x)\psi)(m) = \psi(m + 1) + \psi(m - 1) + \lambda v(x + m\alpha)\psi(m), \quad m \in \mathbb{Z},
\end{equation}

where $v$ is a 1-periodic function on $\mathbb{R}$, continuous on $[0, 1)$, satisfying $v(0) = 0$, $v(1 - 0) = 1$, and having the following Lipschitz monotonicity property:

\begin{equation}
\gamma_-(y - x) \leq v(y) - v(x) \leq \gamma_+(y - x), \quad \gamma_+, \gamma_- > 0.
\end{equation}

Hence, $v$ is strictly increasing on $[0, 1)$ and has jump discontinuities at integer points. A typical example of such function would be $v(x) = \{x\}$, for which we have $\gamma_+ = \gamma_- = 1$. We show (see Corollary 3.5) that for for all $\lambda$ and almost all $\alpha$, for any $v$ satisfying (1.2) and a.e. $x$, the spectrum of the operator (1.1) is purely point and the eigenfunctions decay exponentially at the Lyapunov rate.

Additionally, we show that for any operator (1.1) satisfying (1.2) the integrated density of states is absolutely continuous and the Lyapunov exponent is (almost Lipshitz) continuous. So far, results of this nature have been proved only for either random (where Wegner’s lemma is available) or analytic quasiperiodic potentials with Diophantine frequencies. In contrast, our result does not require any condition on frequency.

Finally, we show that Anderson localization for any operator (1.1) satisfying (1.2) is uniform on any interval on which Lyapunov exponent is uniformly positive (such positivity is established in Corollary 3.2 for large $\lambda$, and the bound depends only on $v$ for almost every $\alpha$). Uniform localization, while often considered a feature of localization in physics literature was shown in \cite{29,26} to not hold for random or analytic quasiperiodic models as it is incompatible with generic singular continuous spectrum that occurs in many ergodic families. In fact, so far the only known example is in the context of limit-periodic operators, see \cite{18}. Operators (1.1) with $v$ satisfying (1.2) provide therefore the first explicit example of a uniformly localized family\footnote{Maryland model exhibits uniform localization for energies restricted to a finite interval, but not overall.}. We note that our method covers a wide class of potentials compared to very concrete Fibonacci or Maryland models.

The proof is based on studying the restrictions of the operator (1.1) onto intervals of the form $[0, q_k - 1]$, where $q_k$ are denominators of the continued fraction approximation of $\alpha$, with certain properties. It is possible to show that the spectra of these restrictions are almost invariant under the transformation $x \mapsto x + \alpha$. Due to local monotonicity of the eigenvalues as functions of $x$ and using some finite rank perturbation arguments, this leads to linear repulsion of eigenvalues and to a Lipschitz bound on the integrated density of states\footnote{Even though the technical analysis only holds for a sparse sequence of scales, this is sufficient for a conclusion on the IDS. This is what allows to obtain the result without any Diophantine conditions.}. As a consequence, the Lyapunov exponent is continuous and, for large $\lambda$, is uniformly bounded from below. A more careful study of the spectra of $H_{q_k}(x)$ leads to a large deviation theorem. The rest of the proof follows the
non-perturbative scheme of proving Anderson localization in [27], with large deviation estimate replacing the analytic part of the argument. Uniformity of localization follows from uniformity in the large deviation theorem.

It is interesting how our results compare with the recent work [5] on so-called monotonic cocycles, i.e. cocycles \( A : \mathbb{T}^d \to \text{SL}(2, \mathbb{R}) \) for which one can find \( w \in \mathbb{R}^d \) such that the map \( t \mapsto \text{arg}\{A(x+tw) \cdot y\} \) has positive derivative in \( t \) for all \( x \in \mathbb{T}^d, y \in \mathbb{R}^2 \setminus \{0\} \). It was discovered that the Lyapunov exponents of such cocycles behave in a highly regular way; for example, they are analytic (resp. \( C^\infty \)) in any parameter as long as the cocycle analytically (resp. \( C^\infty \)) depends on the parameter. Nothing like that is true for general analytic cocycles: one can establish continuity in the analytic case [10, 13], but it can be seen that any prescribed continuity modulus on the parameter. Nothing like that is true for general analytic cocycles: one can establish continuity in the analytic case [10, 13], but it can be seen that any prescribed continuity modulus on the parameter. Nonetheless, in the \( C^\infty \) case even the continuity may fail [39]. The price to pay for regularity is that continuous monotonic cocycles are never homotopic to the identity, and hence there are no examples of monotonic Schrödinger cocycles. However, for the Schrödinger cocycle \( S_{v,E}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix} \) one can establish the following identity:

\[
\left\langle \frac{d}{dx} \left\{ S_{v,E}(x + \alpha)S_{v,E}(x) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S_{v,E}(x + \alpha)S_{v,E}(x) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle = v'(x)u_1^2 + v'(x + \alpha) ((v(x) - E)u_1 + u_2)^2 > 0
\]

whenever \( u_1^2 + u_2^2 > 0 \). This implies that the second iterate of the Schrödinger cocycle with \( v \) satisfying (1.2) is locally monotonic in \( x \) at the points where \( v'(x), v'(x + \alpha) \) exist\(^4\). In other words, we avoid topological obstruction to monotonicity by introducing a discontinuous potential. Thus, it may be natural to expect that some of the properties of monotonic cocycles (such as regularity of the Lyapunov exponent) survive to some extent in our case, and the advantage is that we remain in the framework of Schrödinger cocycles. We see that, while our method is completely different from that of [5], regarding continuity (and even some Hölder continuity), this is indeed the case. Moreover, it suggests that some general results of [5] may hold true even if discontinuity is allowed.

2. Preliminaries: density of states and Lyapunov exponent

Let \( \tilde{H}_n(x) \) denote the restriction of \( H_{\alpha,\lambda}(x) \) to \( l^2[0, n-1] \) with periodic boundary conditions. We will denote integer intervals by \( [0, n-1] \) instead of \( [0, n-1] \cap \mathbb{Z} \) where it is clear from the context. We have, therefore,

\[
(\tilde{H}_n(x)\psi)(m) = \psi((m+1) \text{ mod } n) + \psi((m-1) \text{ mod } n) + \lambda v(x + m\alpha), \quad m \in \{0, \ldots, n-1\}.
\]

The density states measure \( N(dE) \) can be defined as the following functional on continuous functions with compact support:

\[
\int_{\mathbb{R}} f(E)N(dE) = \lim_{n \to \infty} \frac{1}{n} \int_{[0,1]} \text{tr} f(\tilde{H}_n(x)) \, dx.
\]

\(^4\)In [5], monotonicity with respect to a parameter was also studied. In particular, it is mentioned that the second iterate of \( S_{v,E} \) is monotonic in \( E \), rather than the first iterate.
The measure $N(dE)$ is a continuous probability measure, and its distribution function is called the \textit{integrated density of states} (IDS) and is defined by

$$N(E) := N((-\infty, E)) = N((-\infty, E]).$$

Approximating the characteristic function of $(-\infty, E]$ from above and from below by continuous functions, one can easily see that

$$(2.1) \quad N(E) = \lim_{n \to \infty} \frac{1}{n} \int_{[0,1)} \tilde{N}_n(x, E) \, dx,$$

where

$$(2.2) \quad \tilde{N}_n(x, E) = \# \sigma(\tilde{H}_n(x)) \cap (-\infty, E]$$

is the counting function of the periodic restriction.

Let $H_n(x)$ be the \textit{Dirichlet} restriction of $H_{\alpha,\lambda}(x)$ onto $[0, n-1]$, and let $P_n(x, E) = \det(H_n(x) - E)$. The $n$-step transfer matrix is defined by

$$M_n(x, E) := \prod_{l=(n-1)}^{0} \begin{pmatrix} E - \lambda v(x + l\alpha) & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_n(x, E) & -P_{n-1}(x + \alpha, E) \\ P_{n-1}(x, E) & -P_{n-2}(x + \alpha, E) \end{pmatrix},$$

and the standard definition of the Lyapunov exponent is given by

$$(2.3) \quad \gamma(E) = \lim_{n \to \infty} \frac{1}{n} \int_{[0,1)} \ln \|M_n(x, E)\| \, dx = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_{[0,1)} \ln \|M_n(x, E)\| \, dx.$$ 

Thouless formula relates the Lyapunov exponent and the density of states measure:

$$(2.4) \quad \gamma(E) = \int_{\mathbb{R}} \ln |E - E'| \, N(dE').$$

The expression (2.1) also holds for $N_n$ instead of $\tilde{N}_n$ because $H_{\alpha,\lambda}(x)$ is a rank 2 perturbation of $\tilde{H}_{\alpha,\lambda}(x)$.

### 3. Main results

An irrational frequency $\alpha$ is called \textit{Diophantine} if there exist $C, \tau > 0$ such that for all $n \in \mathbb{N}$ we have $\|n\alpha\| \geq C|n|^{-\tau}$, where $\|x\| = \min(\{x\}, \{1 - x\})$. Let

$$\varepsilon(\alpha) = \lim_{k} \inf \frac{q_{k-1}}{q_{k+1}},$$

where $q_k$ are denominators of the continued fraction approximants of $\alpha$. Note that $\varepsilon(\alpha) \leq \frac{1}{2}$ for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

**Theorem 3.1.** The \textit{integrated density of states} $N(E)$ of the operator family $H_{\alpha,\lambda}$ is Lipschitz continuous and satisfies

$$(3.1) \quad |N(E') - N(E)| \leq \frac{|E' - E|}{\lambda(1 - \varepsilon(\alpha))\gamma_-}.$$

As a consequence, density of states measure is absolutely continuous, and the spectrum of the operator $H_{\alpha,\lambda}(x)$ has positive Lebesgue measure.
Corollary 3.2. The Lyapunov exponent of $H_{\alpha,\lambda}$ is continuous in $E$ and the set of zeros of $\gamma(E)$ is a closed subset of zero measure. It also admits a lower bound

$$\gamma(E) \geq \max \left\{ \ln \lambda - \ln \frac{2e}{(1 - \varepsilon(\alpha))\gamma_-}, 0 \right\}.$$  

Hence, $\gamma(E)$ is uniformly positive for large $\lambda$.

The continuity of $\gamma(E)$ immediately follows from Lipschitz continuity of $N(E)$. The fact that the zero set of $\gamma(E)$ has measure zero follows from the general result \[19\].

Remark 3.3. Due to \[33, Theorem 29\], we have $\varepsilon(\alpha) = 0$ for a full measure set of $\alpha$, with obvious implications for (3.1) and (3.2). If $\nu(x) = \{x\}$, then (3.1) and (3.2) hold for all irrational $\alpha$ with $\varepsilon(\alpha)$ replaced by 0, see Remark 5.5.

Operator $H$ exhibits uniform localization if it has pure point spectrum and there exist $C, c$ such that for any eigenfunction $\psi$ there exists $n_0(\psi)$ so that we have

$$|\psi(n)| \leq Ce^{-c|n-n_0|}.$$  

It is known that for ergodic families $H(x)$ uniform localization implies pure point spectrum for every $x$ with eigenfunctions satisfying (3.3) where $C, c$ are uniform in $x$ \[29, 26, 24\].

Here we introduce a new related notion. We will say that operator $H$ exhibits uniform Lyapunov localization if it has pure point spectrum with all eigenfunctions decaying exponentially and at the Lyapunov rate: for any $\delta > 0$, there exists $C(\delta)$ such that for any eigenfunction $\psi$ satisfying $H\psi = E\psi$, $||\psi||_{l^\infty} = 1$, there exists $n_0(\psi)$ so that we have

$$|\psi(n)| \leq C(\delta)e^{-(\gamma(E) - \delta)|n-n_0|}.$$  

The bound is uniform in the sense that there is no dependence on $E$ other than via $\gamma(E)$. Clearly, whenever the Lyapunov exponent $\gamma$ depends continuously on $E$, uniform localization is equivalent to uniform Lyapunov localization plus nonvanishing of the Lyapunov exponent $\gamma$.

Our main theorem is

**Theorem 3.4.** Let $\alpha$ be Diophantine. Then for any $\lambda > 0$, $H_{\alpha,\lambda}(x)$ has pure point spectrum, and moreover exhibits uniform Lyapunov localization for a.e. $x$.

An immediate corollary of Theorem 3.4, Theorem 7.1, and Corollary 3.2 is

**Corollary 3.5.** Suppose $\alpha$ is Diophantine and $\lambda > \frac{2e}{(1 - \varepsilon(\alpha))\gamma_-}$. Then $H_{\alpha,\lambda}(x)$ has uniform localization for all $x$.

**Remark 3.6.** It is a very interesting question whether or not $\gamma(E) > c(\lambda) > 0$ (and therefore whether uniform localization holds for all $x$), for all $\lambda > 0$.

**Remark 3.7.** Some Diophantine condition in Theorem 3.4 is necessary by a Gordon-type argument. We conjecture that the threshold between pure point and singular continuous spectrum lies at $\gamma(E) = \beta(\alpha)$ where $\beta(\alpha) = \limsup \frac{\ln q_{n+1}}{q_n}$, just like in the almost Mathieu case \[6, 31\].

Theorem 3.4 is proved in Sections 7 and 8.
4. Trajectories of irrational rotations

Let \( \alpha = [a_0; a_1, \ldots] \), and let \( \frac {p_n}{q_n} = [a_0; \ldots, a_k] \); note that \( q_0 = 1 \). We have (see, for example, \[33\])

\[
(4.1) \quad q_n \alpha - p_n = \frac{(-1)^n}{t_{n+1}q_n + q_{n-1}}, \quad \text{where} \quad t_n = [a_n; a_{n+1}, \ldots].
\]

The following is established in \[37\].

**Proposition 4.1.** Let \( k \geq 1 \). The points \( \{ja\} \), \( j = 0, \ldots, q_k - 1 \), split \([0, 1)\) to \( q_{k-1} \) “large” gaps with length \( \| (q_k - q_{k-1})\alpha \| \), and \( q_k - q_{k-1} \) “small” gaps with lengths \( \| q_{k-1}\alpha \| \).

We will also need the following elementary two-sided bounds on the lengths of these intervals.

**Proposition 4.2.** The lengths of the intervals from Proposition 4.1 satisfy

\[
(4.2) \quad \frac{1}{q_k} - \frac{q_{k-1}}{q_k q_{k+1}} \leq \| q_{k-1}\alpha \| \leq \frac{1}{q_k},
\]

\[
(4.3) \quad \frac{1}{q_k} \leq \| (q_k - q_{k-1})\alpha \| \leq \frac{1}{q_k} + \frac{1}{q_{k+1}}.
\]

**Proof.** The upper bound in (4.2) follows from (4.1):

\[
\| q_{k-1}\alpha \| = \frac{1}{t_{k}q_{k-1} + q_{k-2}} \leq \frac{1}{q_k},
\]

since \( t_k = a_k + \frac{1}{t_{k+1}} \). For the lower estimate, we have

\[
\frac{1}{q_k} - \frac{1}{t_kq_{k-1} + q_{k-2}} = \frac{t_kq_{k-1} + q_{k-2} - q_k}{q_k(t_kq_{k-1} + q_{k-2})} = \frac{q_k - q_{k-2}}{q_k(q_k + \frac{q_{k-1}}{t_{k+1}})} = \frac{q_k - q_{k-1}}{q_k(q_{k+1} + \frac{q_{k-1}}{t_{k+1}})} \leq \frac{q_k - q_{k-1}}{q_k q_{k+1}},
\]

so that

\[
\| q_{k-1}\alpha \| \geq \frac{1}{q_k} - \frac{q_{k-1}}{q_k q_{k+1}}.
\]

As for (4.3), since \( \| q_{k-1}\alpha \| \leq \frac{1}{q_k} \), we must have \( \| (q_k - q_{k-1})\alpha \| \geq \frac{1}{q_k} \) because the total length of the intervals is 1. The upper estimate in (4.3) follows from

\[
\| (q_k - q_{k-1})\alpha \| \leq \frac{1}{t_kq_{k-1} + q_{k-2}} + \frac{1}{t_{k+1}q_k + q_{k-1}} \leq \frac{1}{q_k} + \frac{1}{q_{k+1}}. \]

**Lemma 4.3.** Suppose that \( \{x\} \notin (1 - \frac{1}{q_{k+1}}, 1) \) for \( k \) even and \( \{x\} \notin [0, \frac{1}{q_{k+1}}) \) for \( k \) odd. Then

\[
|\{x + q_k\alpha\} - \{x\}| \leq \frac{1}{q_{k+1}}.
\]

**Proof.** Due to (4.1), we have \( \| q_k\alpha \| \leq \frac{1}{q_{k+1}} \). The choice of \( x \) guarantees that \( \{x + q_k\alpha\} \) and \( \{x\} \) are both close to 0 or close to 1. Hence,

\[
|\{x + q_k\alpha\} - \{x\}| = \| q_k\alpha \| \leq \frac{1}{q_{k+1}}.
\]
Recall that $\varepsilon(\alpha) = \liminf_k \frac{q_{k-1}}{q_{k+1}}$. For $\varepsilon(\alpha) < \varepsilon < 1$, define

$$Q(\alpha, \varepsilon) = \left\{ q_k : \frac{q_{k-1}}{q_{k+1}} \leq \varepsilon \right\},$$

For any $\varepsilon > \varepsilon(\alpha)$, the set $Q(\alpha, \varepsilon)$ is infinite.

## 5. Lipschitz Continuity of the IDS

Recall that $\tilde{H}_n(x)$ is the periodic restriction of $H_{\alpha, \lambda}(x)$ onto $l^2[0, n - 1]$. For a fixed $n$, let $\tilde{\mu}_l(x), 0 \leq l \leq n - 1$, be the eigenvalues of $\tilde{H}_n(x)$ in the increasing order, counted with multiplicities. The functions $\tilde{\mu}(x)$ are $1$-periodic and continuous on $[0; 1)$ except for the finite set of points $0 = \beta_0 < \beta_1 < \ldots < \beta_{n-1} < 1$, where

$$\{\beta_0, \beta_1, \ldots, \beta_{n-1}\} = \{0, \{-\alpha\}, \{-2\alpha\}, \ldots, \{-(n-1)\alpha\}\}$$

is the part of the trajectory of the irrational rotation. We have

$$\text{rank}(\tilde{H}_n(\beta_k) - \tilde{H}_n(\beta_k - 0)) = 1, \quad \text{tr}(\tilde{H}_n(\beta_k) - \tilde{H}_n(\beta_k - 0)) = -\lambda, \quad 0 \leq k \leq n - 1,$$

so that all “jumps” are negative rank one perturbations caused by discontinuity of $v$ at 1. Hence, we have

$$\tilde{\mu}_l(\beta_k - 0) \leq \tilde{\mu}_{l-1}(\beta_k) \leq \tilde{\mu}_{l+1}(\beta_k - 0), \quad 0 \leq l \leq n - 2.$$  

From (1.2), it follows that the eigenvalues are locally monotonic functions of $x$, and we have

$$\gamma_-(y - x) \leq \tilde{\mu}_l(x) \leq \gamma_+(y - x), \quad \forall x, y \in [\beta_k, \beta_{k+1}).$$

The goal of this section is to study the behavior of these eigenvalues for $n = q_k$ and obtain conclusions for the density of states and the Lyapunov exponent.

**Lemma 5.1.** Suppose that $0 \leq r \leq q_k - 1$ and that $x, x - \alpha, \ldots, x - (r - 1)\alpha$ satisfy the assumptions of Lemma 4.3. Then

$$|\tilde{\mu}_m(x) - \tilde{\mu}_m(x - r\alpha)| \leq \frac{\lambda \gamma_+}{q_{k+1}}, \quad \text{for} \quad 0 \leq m \leq q_k - 1.$$  

**Proof.** Let $\{e_0, \ldots, e_{q_k - 1}\}$ be the standard basis in $l^2[0, 1, \ldots, q_k - 1]$, and $T e_l = e_{(l+1) \mod q_k}$ be the unitary shift operator in this space. We compare the spectra of $\tilde{H}_{q_k}(x)$ and $\tilde{H}_{q_k}(x - r\alpha)$. Let us replace the first operator by unitary equivalent $T^r \tilde{H}_{q_k}(x) T^{-r}$. It is easy to see that

$$T^r \tilde{H}_{q_k}(x) T^{-r} - \tilde{H}_{q_k}(x - r\alpha) = \lambda \text{diag}\{w_0, w_1, \ldots, w_{q_k - 1}\},$$

where

$$w_l = \lambda [v(x + (l - r) \mod q_k \alpha) - v(x + (l - r)\alpha)]
= \begin{cases}
\lambda [v(x + (l - r + q_k \alpha) - v(x + (l - r)\alpha)], & 0 \leq l < r \\
0, & r \leq l \leq q_k - 1.
\end{cases}$$

From Lemma 4.3 and the Lipschitz bound on $v$, we get that

$$||T^r \tilde{H}_{q_k}(x) T^{-r} - \tilde{H}_{q_k}(x - r\alpha)|| \leq \frac{\lambda \gamma_+}{q_{k+1}},$$

which gives (5.5).
Theorem 5.2. Suppose that \( q_k \in Q(\alpha, \varepsilon) \) where \( \varepsilon(\alpha) < \varepsilon < 1 \). Then, for any \( K \le q_k - 1 \) and \( 0 \le m \le q_k - K - 1, 0 \le l \le q_k - 1 \), we have
\[
|\hat{\mu}_{m+K}(\beta_l) - \hat{\mu}_m(\beta_l)| \geq \lambda \left( \frac{K(1 - \varepsilon)\gamma_-}{q_k} - 3\frac{\gamma_+}{q_k+1} \right)
\]
for \( k \) odd and the same with \( \beta_l \) replaced by \( \beta_l - 0 \) for \( k \) even.

Proof. Start from considering the case \( l + K \le q_k - 1 \). Due to (5.3) and (5.4), it is easy to see that
\[
\hat{\mu}_{s+1}(\beta_{j+1}) \geq \hat{\mu}_s(\beta_{j+1} - 0) \geq \hat{\mu}_s(\beta_j) + \lambda(\beta_{j+1} - \beta_j)\gamma_-,
\]
and thus, by iteration,
\[
\hat{\mu}_{m+K}(\beta_{l+K}) \geq \hat{\mu}_m(\beta_l) + \lambda(\beta_{l+K} - \beta_l)\gamma_- \geq \hat{\mu}_m(\beta_l) + \lambda \frac{K(1 - \varepsilon)\gamma_-}{q_k},
\]
because \( \beta_{l+K} - \beta_l \geq \frac{K(1 - \varepsilon)}{q_k} \), see Proposition 4.1. Combining it with
\[
|\hat{\mu}_{m+K}(\beta_{l+K}) - \hat{\mu}_m(\beta_l)| \leq \frac{\lambda \gamma_-}{q_k+1}
\]
from Lemma 5.1 we get (5.7) (with the coefficient 1 instead of 3 in the last term). The case \( l > q_k - K - 1 \) follows from the case \( l \leq q_k - K - 1 \) also due to Lemma 5.1 with an additional error of \( \frac{2\lambda \gamma_-}{q_k+1} \). Depending on whether \( k \) is even or odd, we apply Lemma 5.1 to the points \( \beta_l \) of \( \beta_l - 0 \) in order to satisfy the assumptions of Lemma 4.3.

Corollary 5.3. Let \( \tilde{N}_{q_k}(x, E) \) be the counting function (2.2). Suppose that \( \varepsilon(\alpha) < \varepsilon < 1 \) and \( q_k \in Q(\alpha, \varepsilon) \). Then, for any \( \delta > 0 \) and \( E \leq E' \), we have
\[
(5.8) \quad \tilde{N}_{q_k}(x, E') - \tilde{N}_{q_k}(x, E) \leq \frac{(E' - E)q_k}{\lambda(1 - \varepsilon)\gamma_-}(1 + \delta) - C(\delta, \varepsilon, \gamma_-, \gamma_+).
\]
The same holds for the Dirichlet eigenvalue counting function \( N_{q_k}(x, E) \).

Proof. For \( x = \beta_l \) for \( k \) odd and \( x = \beta_l - 0 \) for \( k \) even, it is a direct consequence of (5.7), because one can choose \( K \) large enough (depending on \( \gamma_-, \gamma_+, \varepsilon \), but not \( q_k \)) and split the eigenvalues from \([E, E']\) into clusters of length \( K \). The factor \((1 + \delta)\) appears because of the second term in (5.7), and we have \( \delta \sim \frac{1}{K} \). If \( x \in (\beta_l, \beta_{l+1}) \), then, due to monotonicity,
\[
\tilde{N}_{q_k}(\beta_l - 0, E) + 1 \geq \tilde{N}_{q_k}(\beta_l, E) \geq \tilde{N}_{q_k}(x, E) \geq \tilde{N}_{q_k}(\beta_{l+1} - 0, E) \geq \tilde{N}_{q_k}(\beta_{l+1}, E) - 1.
\]
From Lemma 5.1 and Theorem 5.2, it also follows that
\[
|\tilde{N}_{q_k}(\beta_l, E) - \tilde{N}_{q_k}(\beta_{l+1}, E)| \leq C(\varepsilon, \gamma_-, \gamma_+),
\]
from which (5.8) follows. The Dirichlet restriction is a rank 2 perturbation of the periodic restriction, so the claim also holds in that case.

Proof of Theorem 3.1. The estimate (3.1) follows from the definition (2.1) and Corollary 5.3 since it holds for any \( \varepsilon(\alpha) < \varepsilon < 1 \) and any \( \delta > 0 \), it also holds for \( \varepsilon = \varepsilon(\alpha) \) and \( \delta = 0 \). The Lebesgue measure of the spectrum is positive because \( \sigma(H) \) is the essential support of the absolutely continuous measure \( N(dE) \).
Lemma 5.4. Let $0 \leq f(x) \leq a$ for all $x \in \mathbb{R}$, and $\int_{\mathbb{R}} f(x) \, dx = 1$. Then

$$\int_{\mathbb{R}} f(x) \ln |x| \, dx \geq \frac{1}{a} \int_{-a/2}^{a/2} \ln |x| \, dx = \ln(a/2e).$$

Proof. By rescaling, we can assume $a = 1$. Then

$$\int_{\mathbb{R}} f(x) \ln |x| \, dx - \ln(1/2e) = \int_{-1/2}^{1/2} (f(x) - 1) \ln |x| \, dx + \int_{\mathbb{R}\setminus[-1/2,1/2]} f(x) \ln |x| \, dx \geq$$

$$\geq \ln 2 \int_{-1/2}^{1/2} (1 - f(x)) \, dx - \ln 2 \int_{\mathbb{R}\setminus[-1/2,1/2]} f(x) \, dx = 0. \blacksquare$$

Proof of Corollary 3.2. The lower bound (5.2), which is the only thing remaining to prove, immediately follows from Corollary 3.1, Lemma 5.4 and the Thouless formula (2.4). $\blacksquare$

Remark 5.5. In the special case $v(x) = \{x\}$, the eigenvalues $\mu_m(x)$ of $H_n(x)$ are piecewise linear functions of $x$, and we can integrate them explicitly. Denote

$$P_n(x,E) := \det(H_n(x) - E) = \prod_{i=0}^{n-1} (\mu_i(x) - E).$$

Using the definition of $\gamma(E)$, we can get the following lower bound.

$$\gamma(E) \geq \limsup_{n \to \infty} \frac{1}{n} \int_{0}^{1} |P_n(x,E)| \, dx = \limsup_{n \to \infty} \frac{1}{n} \int_{0}^{1} \text{tr} \ln |H_n(x) - E| \, dx$$

$$= \limsup_{n \to \infty} \sum_{i=0}^{n-1} \frac{1}{n\lambda} \text{tr} \left[ g(H(\beta_{i+1} - 0) - E) - g(H(\beta_i - E)) \right],$$

where $g(\mu) = \mu \ln |\mu| - \mu$ is the antiderivative of $\ln |\mu|$. Regrouping the terms, we have $\gamma(E)$ is bounded from below by

$$\limsup_{n \to \infty} \frac{1}{n\lambda} \sum_{i=0}^{n-1} \int_{\Sigma_i} \ln |\mu| + E \, d\mu,$$

where $\Sigma_i = \cup_m [\mu_m(\beta_i), \mu_m(\beta_i - 0)]$ is the support of the difference of counting functions of $H(\beta_i - 0) - E$ and $H(\beta_i) - E$. Since $\text{tr}(H(\beta_i - 0) - H(\beta_i)) = \lambda$, we have $|\Sigma_i| = \lambda$ and, by Lemma 5.4,

$$\int_{\Sigma_i} \ln |\mu| + E \, d\mu \geq \int_{-\lambda/2}^{\lambda/2} \ln |\mu| \, d\mu = \lambda(\ln(\lambda/2) - 1),$$

so that

$$\gamma(E) \geq \max\{0, \ln(\lambda/2e)\}.$$
6. Large Deviation Theorem for \( P_{q_k}(x, E) \)

Recall that \( H_{q}(x) \) is the Dirichlet restriction of \( H_{\alpha,\lambda}(x) \) onto \( l^2[0, n - 1] \). The following two relations are well known and can be easily checked using properties of determinants.

\[
(6.1) \quad \tilde{P}_n(x, E) + 2(-1)^n = P_n(x, E) - P_{n-2}(x + \alpha, E), \quad n \geq 3,
\]

\[
(6.2) \quad P_n(x, E) + P_{n-2}(x, E) = (\lambda v(x + (n - 1)\alpha) - E)P_{n-1}(x, E), \quad n \geq 2.
\]

Here \( \tilde{P}_n(x, E) = \det(\tilde{H}_n(x) - E) \), and \( P_0(x, E) = 1 \). The following result is obtained in [32]. It holds for arbitrary \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and arbitrary piecewise continuous potentials.

**Theorem 6.1.** For any \( \varepsilon > 0 \) and \( E \in \mathbb{R} \) there exists an \( N \in \mathbb{N} \) such that \( |P_n(x, E)| \leq e^{n(\gamma(E)+\varepsilon)} \) for all \( n > N \). Moreover, \( N \) can be chosen uniformly in \( E \in [E_1, E_2] \) as long as \( \gamma(E) \) is continuous on this interval.

The following large deviation theorem is the main technical part in the proof of localization.

**Theorem 6.2.** Fix \( E, \lambda \in \mathbb{R} \) such that \( \gamma(E) > 0 \), and fix \( \varepsilon(\alpha) < \varepsilon < 1 \). For any \( \delta > 0 \), there exists \( q_0 > 0 \) such that for all \( q_k \in Q(\alpha, \varepsilon) \), \( q_k \geq q_0 \), we have

\[
(6.3) \quad |\{ x \in [0, 1) : |P_{q_k}(x, E)| < e^{q_k(\gamma(E)-\delta)} \}| < e^{-C\delta q_k},
\]

where \( c_1, c_2 \) may also depend on \( \gamma_-, \gamma_+, \varepsilon, \lambda \), but can be chosen uniformly in \( E \) on any compact interval. In addition, the set in the left hand side can be covered by at most \( q_k \) intervals of size \( e^{-C\delta q_k} \).

We need several preparatory lemmas.

**Lemma 6.3.** Under the assumptions of Theorem 6.2, the number of zeros of \( P_{q_k}(x, E) \) (counted with multiplicities) is equal to the number of points from \( \{ \beta_0, \ldots, \beta_{q_k-1} \} \) such that \( N_{q_k}(\beta_i - 0, E) < N_{q_k}(\beta_i, E) \). The same holds for \( \tilde{P}_{q_k}, \tilde{N}_{q_k} \).

**Proof.** Consider the function \( N_{q_k}(x, E) \) (or \( \tilde{N}_{q_k} \)) as \( x \) goes from 0 to 1. It decreases by 1 at each zero of \( P_{q_k}(x, E) \) (or \( \tilde{P} \), respectively) and can increase at most by 1 at each point \( \beta_i \). Since \( N_{q_k}(x, E) = N_{q_k}(x + 1, E) \), the statement follows. \( \blacksquare \)

**Lemma 6.4.** Suppose that \( A_1, A_2 \) are two finite subsets of \( [m, M] \) of the same cardinality, \( m > 0 \), and that \( f \) is a nondecreasing function on \( [m, M] \). Assume that the difference of counting functions of \( A_1 \) and \( A_2 \) is bounded by \( N \). Then

\[
\left| \sum_{a \in A_1} f(a) - \sum_{a \in A_2} f(a) \right| \leq 2N \max\{|f(m)|, |f(M)|\}.
\]

(Note that the values of \( f \) may be negative, hence we need to take both \( m \) and \( M \) into account.)

**Proof.** Obviously, the worst case is when \( (A_1 \setminus A_2) \cup (A_2 \setminus A_1) \) contains \( 2N \) points. \( \blacksquare \)

**Lemma 6.5.** Let \( a, b > 0 \). Then

\[
\sum_{j=1}^{n} (\ln(a j + b) - \ln(a j)) \leq \frac{b}{a} \ln(n + 1).
\]
Proof. The left hand side is
\[
\sum_{j=1}^{n} \ln \left( 1 + \frac{b}{a_j} \right) = \ln \prod_{j=1}^{n} \left( 1 + \frac{b}{a_j} \right) \leq \ln \prod_{j=1}^{n} \exp \left( \frac{b}{a_j} \right) \leq \ln \exp \left\{ \frac{b}{a} \ln (n+1) \right\}
\]
\[\blacksquare\]

Proof of Theorem 6.2. We study the behavior of the function
\[
\frac{1}{q_k} \ln |P_{q_k}(x, E)| = \frac{1}{q_k} \sum_{j=0}^{q_k-1} \ln |\mu_j(x) - E|.
\]
In the sequel, all constants are allowed to depend on \( \varepsilon, \gamma_-, \gamma_+ \). From Theorem 5.2, Corollary 5.3 and the fact that \(|N_{q_k}(x, E) - \tilde{N}_{q_k}(x, E)| \leq 2\), one can split the eigenvalues \( \mu_j(x) \) on each interval into three clusters: above \( E \), around \( E \), and below \( E \), such that

1. The eigenvalues \( \nu_j^+(x) \) in the cluster above \( E \), taken in the increasing order as \( j = 1, 2, \ldots \), admit a lower bound \( \nu_j^+(x) \geq E + j \frac{C_1}{q_k} \).

2. The eigenvalues \( \nu_j^-(x) \) in the cluster below \( E \), taken in the decreasing order as \( j = 1, 2, \ldots \), admit an upper bound \( \nu_j^-(x) \leq E - j \frac{C_1}{q_k} \).

3. There are \( C_2 \) eigenvalues in the remaining cluster around \( E \).

4. The same holds for the eigenvalues of the periodic restriction. We will denote them by \( \tilde{\nu}_j^+(x) \).

Let us also decompose \( P_{q_k}, \tilde{P}_{q_k} \) in the same way.

\[
P_{q_k}(x, E) = P_{q_k}^+(x, E) P_{q_k}^0(x, E) P_{q_k}^-(x, E), \quad \tilde{P}_{q_k}(x, E) = \tilde{P}_{q_k}^+(x, E) \tilde{P}_{q_k}^0(x, E) \tilde{P}_{q_k}^-(x, E),
\]
where the factors are formed by \( (\mu_j(x) - E) \) from the respective clusters. We now claim that, for any \( x, y \in [0, 1) \), we have
\[
|\ln |P_{q_k}^\pm(x, E)|| - \ln |P_{q_k}^\pm(y, E)|| | \leq C \ln q_k, \quad |\ln |\tilde{P}_{q_k}^\pm(x, E)|| - \ln |\tilde{P}_{q_k}^\pm(y, E)|| | \leq C \ln q_k.
\]
We start from proving (6.5) for \( x = \beta_t, y = \beta_m \). Due to Lemma 5.1, we have \( |\tilde{\nu}_j^+(\beta_t) - \tilde{\nu}_j^+(\beta_m)| \leq \frac{\lambda \gamma_+}{q_{k+1}} \), from which it follows that
\[
|\ln |\tilde{P}_{q_k}^\pm(\beta_t, E)|| - \ln |\tilde{P}_{q_k}^\pm(\beta_m, E)|| | \leq \sum_{j=1}^{q_k} \left\{ \ln \left( \frac{j C_1}{q_k} + \frac{\lambda \gamma_+}{q_{k+1}} \right) - \ln \frac{j C_1}{q_k} \right\} \leq C \ln q_k.
\]
by Lemma 6.5. Let us now consider the case \( [x, y) \in [\beta_t, \beta_{t+1}) \). Due to monotonicity, we can assume that \( x = \beta_t, y = \beta_{t+1} - 0 \), and then the statement also follows from
\[
|\ln |\tilde{P}_{q_k}^\pm(\beta_t, E)|| - \ln |\tilde{P}_{q_k}^\pm(\beta_{t+1} - 0, E)|| | \leq \sum_{j=1}^{q_k} \left\{ \ln \left( \frac{j C_1}{q_k} + \frac{\lambda \gamma_+}{(1 - \varepsilon)q_k} \right) - \ln \frac{j C_1}{q_k} \right\} \leq C \ln q_k
\]
by Lemma 6.5. The claim also holds for \( P^\pm \) by Lemma 6.4.

The above computations imply that there exists \( \gamma_{q_k}(E) \) such that
\[
\gamma_{q_k}(E) \leq \frac{1}{q_k} \ln |P_{q_k}^-(x, E) P_{q_k}^+(x, E)| \leq \gamma_{q_k}(E) + C \frac{\ln q_k}{q_k}.
\]
(6.7) \[ \gamma_{q_k}(E) \leq \frac{1}{q_k} \ln |\tilde{P}_{q_k}^{-}(x, E)\tilde{P}_{q_k}^{+}(x, E)| \leq \gamma_{q_k}(E) + C \frac{\ln q_k}{q_k}. \]

This means that, if $|P_{q_k}(x, E)| < e^{q_k(\gamma_{q_k}(E) - \delta)}$, we must have $|P_0(x, E)| \leq e^{-\gamma_{q_k} \delta}$, and hence for some $l$ we have $|\mu_l(x) - E| \leq e^{-Cq_k \delta}$, which implies that $x$ is either exponentially close to a zero of $P_{q_k}(\cdot, E)$, or to $\beta_l$ for some $l$. In the second case, if $x$ is not close to an “actual” root of $P_{q_k}$, then $N_{q_k}(\beta_l - 0, E) = N_{q_k}(\beta_l, E)$. Hence, by Lemma 6.3, we can add all these $\beta_l$ to the set of roots and still get a set with at most $q_k$ points to which $x$ must be exponentially close. We thus have verified the statement of the theorem, but for $\gamma(E)$ replaced by $\gamma_{q_k}(E)$.

We now claim that $\gamma_{q_k}(E) = \gamma(E) + o(1)$ as $q_k \to \infty$, uniformly in $E$. Fix $\varepsilon > 0$. From Theorem 6.1 we get that, for $n > N(\varepsilon)$, $|P_n(x, E)| \leq e^{n(\gamma(E) + \varepsilon)}$. From the definition (2.3) of $\gamma(E)$, it follows that, for all $k$, we must have $|P_k(x, E)| \geq Ce^{n(\gamma(E) - \varepsilon)}$ for $n = q_k, q_k - 1$ or $q_k - 2$, on a subset of $[0, 1]$ of measure at least 1/4. If $n = q_k - 1$, then (6.2) implies that it should hold for $P_{q_k}$ or $P_{q_k-2}$ on a set of sufficiently large measure (bounded from below by positive universal constant). Finally, if it holds for $P_{q_k-2}$, then (6.1) implies the similar statement for $P_{q_k}$, and the case $\tilde{P}_{q_k}$ implies the case of $P_{q_k}$ because of (6.6), (6.7); thus, Theorem 6.2 follows.

7. Localization

A generalized eigenfunction of $H$ is, by definition, a polynomially bounded solution of the equation $H\psi = E\psi$. The corresponding $E$ is called a generalized eigenvalue. We first prove

Theorem 7.1. Suppose that $\alpha$ is Diophantine, $E$ is a generalized eigenvalue of $H_{\alpha, \lambda}(x)$, and that $\gamma(E) > 0$. Then the corresponding generalized eigenfunction belongs to $l^2(\mathbb{Z})$.

From now on, let us drop the dependence on $\alpha$ and $\lambda$ from all the notation, assuming that they are fixed. By $G_{[a,b]}(x; m, n)$ we denote the $(m, n)$-matrix element of $\left( (H(x) - E)|_{[a,b]} \right)^{-1}$ with Dirichlet boundary conditions. Note that

\[ G_{[a,b]}(x + k\alpha; m, n) = G_{[a+k,b+k]}(x; m + k, n + k), \]

so it is sufficient to consider the intervals $[0, n]$.

Assume that $x$ is fixed. Following [27], let us call a point $m \in \mathbb{Z}$ $(\mu, q)$-regular if there exists an interval $[n_1, n_2]$ such that $n_2 = n_1 + q - 1, m \in [n_1, n_2], |m - n_i| \geq q/5$ for $i = 1, 2$, and

\[ |G_{[n_1,n_2]}(x; m, n_i)| < e^{-\mu|m-n_i|}. \]

Otherwise, $m$ is called $(\mu, q)$-singular. Any formal solution $H(x)\psi = E\psi$ can be reconstructed from its values at two points,

\[ \psi(m) = -G_{[n_1,n_2]}(x; m, n_1)\psi(n_1 - 1) - G_{[n_1,n_2]}(x; m, n_2)\psi(n_2 + 1), \quad m \in [n_1, n_2]. \]

If $\mu$ is fixed, then any point $m$ such that $\psi(m) \neq 0$ is $(\mu, q)$-singular for sufficiently large $q$.

Theorem 7.2. Under the assumptions of Theorem 7.1, let $\varepsilon(\alpha) < \varepsilon < 1$. For any $0 < \delta < \gamma(E)$ there exists $q_0$ such that if $q_k > q_0, q_k \in Q(\alpha, \varepsilon)$, and $n, m$ are both $(\gamma(E) - \delta, q_k)$-singular with $|m - n| > \frac{q_k+1}{2}$, then $|m - n| > e^{C(\alpha, \varepsilon)q_k}$. 

Proof. The proof follows the scheme from [27]. We have the following expressions for Green’s function matrix elements if \( b = a + q_k - 1 \), \( a \leq l \leq b \).

\[
(7.2) \quad |G_{[a,b]}(x; a, l)| = \frac{|P_{b-l}(x + (l + 1)\alpha)|}{|P_{q_k}(x + a\alpha)|}.
\]

\[
(7.3) \quad |G_{[a,b]}(x; l, b)| = \frac{|P_{l-a}(x + a\alpha)|}{|P_{q_k}(x + a\alpha)|}.
\]

Suppose that \( m - [3q_k/4] \leq l \leq m - [3q_k/4] + [(q_k + 1)/2] \). Since \( m \) is \((\gamma(E) - \delta, q_k)\)-singular, we either have

\[
(7.4) \quad |G_{[a,b]}(x; a, l)| > e^{-(l-a)(\gamma(E)-\delta)} \quad \text{or} \quad |G_{[a,b]}(x; l, b)| > e^{-(b-l)(\gamma(E)-\delta)}
\]

for all intervals \([a, b]\) such that \( |a - l|, |b - l| \geq q_k/5 \) and \( b = a + q_k - 1 \). From Theorem 6.1 and since \( q_k \geq q_0 \), we can choose a sufficiently large \( q_0 \) (depending only on \( \delta \)) such that

\[
|P_{b-l}(x + (l + 1)\alpha)| \leq e^{(b-l)(\gamma(E)+\delta/32)}, \quad |P_{l-a}(x + a\alpha)| \leq e^{(l-a)(\gamma(E)+\delta/32)}.
\]

In other words, the numerators of (7.2), (7.3) cannot get very large. Hence, the only possibility for (7.4) is for one of the denominators to become exponentially small. This means that if \( m - [3q_k/4] \leq a \leq m - [3q_k/4] + [(q_k + 1)/2] \), we have (without loss of generality, in the case of the first denominator)

\[
|P_{q_k}(x + a\alpha, E)| \leq \frac{e^{(b-l)(\gamma(E)+\delta/32)}}{e^{-(l-a)(\gamma(E)-\delta)}} = e^{(\gamma(E)(b-a) + (b-l)\delta - (l-a)\delta} \leq e^{\gamma(E)-\delta/16}.
\]

Suppose that the points \( m_1 \) and \( m_2 = m_1 + r \) are both \((\gamma(E) - \delta, q_k)\)-singular, \( r > 0 \). Let

\[
x_j = \{x + (m_1 - [3q_k/4] + (q_k - 1)/2 + j)\alpha\}, \quad j = 0, \ldots, [(q_k + 1)/2] - 1,
\]

\[
x_j = \{x + (m_2 - [3q_k/4] + (q_k - 1)/2 + j - [(q_k + 1)/2])\alpha\}, \quad j = [(q_k + 1)/2], \ldots, q_k.
\]

If \( r > \frac{q_k+1}{2} \), then all these points are distinct, and we have \( |P_{q_k}(x, E)| \leq e^{\gamma(E)-\delta/16} \). From Theorem 6.2, we get that, for sufficiently large \( q_k \), at least two of the points should be \( e^{-Cq_k} \)-close to each other, and so we get that \( |r'\alpha| \leq e^{-Cq_k} \) for some \( r' \leq r \). From Diophantine condition, we get that \( r \geq e^{-Cq_k''/r} \). This completes the proof.  

**Proof of Theorem 7.1.** Suppose that \( \psi \) is a generalized eigenfunction, so that \( |\psi(m)| \leq C(1 + |m|^p) \). Fix \( \varepsilon = 1/2 \), then \( Q(\alpha, \varepsilon) \) consists of all denominators of \( \alpha \). Without loss of generality, we may assume that \( \psi(0) \neq 0 \). Fix \( 0 < \delta < \gamma(E) \). The point \( 0 \) is \((\gamma(E) - \delta, q_k)\)-singular for sufficiently large \( q_k \in Q(\alpha, \varepsilon) \). Hence, the interval \( [q_k, e^{C(\alpha)q_k}] \) contains only \((\gamma(E) - \delta, q_k)\) regular points.

Take \( n \in \mathbb{N} \), and find \( k \) such that \( n \in [q_k, q_{k+1}] \). Since \( \alpha \) is Diophantine, we have \( q_{k+1} \leq q_k^{C''(\alpha)} \). Together with the previous observation, if \( n \) is sufficiently large, it is contained in an interval \([q_k, q_k^{C''(\alpha)}]\) consisting of \((\gamma(E) - \delta, q_k)\)-regular points. Hence there exist \( n_1, n_2 \) satisfying \( n_1 \leq n \leq n_2 \) and \( q_k/5 \leq |n_2 - n_1| \leq 4q_k/5 \), such that

\[
G_{[n_1, n_2]}(x; n_1, n_2) \leq e^{-(\gamma(E)-\delta)|n-n_1|}.
\]

From (7.1), we obtain

\[
|\psi(n)| \leq C(1 + |n|^p)e^{-\frac{\gamma(E)-\delta}{5}q_k} \leq C(1 + |n|^p)e^{-C_{1n^1/C''(\alpha)}},
\]
which holds for sufficiently large $n$. The case $n < 0$ is similar, and thus $\psi \in L^2(\mathbb{Z})$.

8. Proof of Theorem 3.4: exponential decay of eigenfunctions

Due to [8, Chapter VII], the spectral measure of $H_{a,\lambda}(x)$ is supported on the set of its generalized eigenvalues. The zero set of $\gamma(E)$ has Lebesgue and density of states measure zero due to Theorem 3.1 and Corollary 3.2. Hence, for almost every $x$, the set of energies $E$ for which the statement of Theorem 7.1 holds has full spectral measure, implying the pure point spectrum. Thus it remains to prove uniform Lyapunov localization for this full measure set of $x$. If $H_{a,\lambda}(x)\psi = E\psi, \psi \in L^2(\mathbb{Z})$, let $n_0(\psi)$ be the leftmost point where $|\psi(n)|$ attains its maximal value (which obviously exists). Then we assume that $\psi(n_0) = 1$. Suppose that $\alpha$ is Diophantine and that $\gamma(E) > 0$. Our goal is to show that, if $0 < \delta < \gamma(E)$, then

$$|\psi(n)| \leq C(\delta)e^{-\gamma(E)-\delta)|n-n_0(\psi)|},$$

where the constant $C$ does not depend on $E$ and $x$.

Fix any $\delta < \gamma(E)$. Similarly to the previous section, let us also fix $\varepsilon = 1/2$, so that $Q(\alpha, 1/2)$ contains all denominators of $\alpha$.

Without loss of generality, we can assume that $n_0(\psi) = 0$, otherwise we can shift $x$ to $x+n_0(\psi)\alpha$ and get a unitary equivalent operator whose eigenfunctions are translated by $n_0(\psi)$. There exists $q_0(\delta) > 0$ such that $0$ is $(\gamma(E) - \delta/2, q_k)$-singular for all denominators $q_k > q_0$, and that the statement of Theorem 7.2 holds for $q_0$ and $\delta/2$. Note that this choice is uniform in $x$ and $E$ as $E$ must belong to $\sigma(H_{a,\lambda}(x))$ which is contained in a uniformly bounded interval.

The rest of the proof follows the method of [27]. If $q_k > q_0$, then $[q_k, e^{c(\alpha)q_k}]$ must consist of $(\gamma(E) - \delta/2, q_k)$-regular points. Again, from the Diophantine condition, the intervals $[q_k, q_k^{C(\alpha)}]$ cover all sufficiently large integer points. For some $\theta > 0$, it also holds for the intervals $[q_k^{1+\theta}, q_k^{C(\alpha)}]$, and they also consist of $(\gamma(E) - \delta/2, q_k)$-regular points. Let $n \in [q_k^{1+\theta}, q_k^{C(\alpha)}]$. The fact that $n$ is $(\gamma(E) - \delta/2, q_k)$-regular implies existence of a certain interval $[n_1, n_2]$ with Green function’s decay from $n$ to the edges of the interval. Apply (7.1) on this interval, thus expanding $\psi(n)$ in terms of $\psi(n_1 - 1)$ and $\psi(n_2 + 1)$. The points $n_1 - 1$ and $n_2 + 1$ are also regular, and hence we can repeat the procedure and get an expansion involving the values of $\psi$ at four points. Let us repeat the procedure of finding a suitable interval and expressing each $\psi(n)$ using (7.1) until we get $n_1 < q_k$ at some stage, or the depth of the expansion reaches $[5n/q_k]$, whichever comes first. The last condition guarantees that $n_2$ will never be singular, as the maximal possible value of $n_2$ on the last step does not exceed $2n$, since $n_2 \leq n + 4q_k/5$ on each step.

The result can be written in the following form:

$$(8.1) \quad \psi(n) = \sum_{s \in S} G_{1,s}G_{2,s} \ldots G_{p(s),s}\psi(n_s),$$

where $S \subset \bigcup_{p=0}^{[5n/q_k]}\{0,1\}^p$ indicates all possible sequences of choices between $n_1$ and $n_2$ in applying (7.1) at each step. There are at most $2^{[5n/q_k]+1}$ terms, and at each term we either have $n_s < q_k$ or $p(s) \geq [5n/q_k] + 1$. $G_{i,s}$ are matrix elements of $G$ appearing in (7.1). Note that the only way to reach $(\gamma(E) - \delta/2, q_k)$-singular point in this construction is to get $n_s < q_k$ in which case the process stops and we no longer need to apply (7.1).
Let us first consider the case \( n_s < q_k \). Using the definition of regularity, we have

\[
|G_{1,s}G_{2,s} \ldots G_{p(s),s}| \leq e^{-\gamma(E)-\delta/2(n-n_s)},
\]

and so

\[
|G_{1,s}G_{2,s} \ldots G_{p(s),s}\psi(n_s)| \leq e^{-\gamma(E)-\delta/2(n-q_k)} \leq e^{-n(\gamma(E)-\delta/2)(1-q_k^{-\theta})} \leq e^{-\gamma(E)-\delta/2-\delta_1)n},
\]

where \( \delta_1 \) can be made smaller than, say, \( \delta/3 \) by choosing a sufficiently large \( q_k \). Note that this choice would not be uniform if \( \gamma(E) \) could become uncontrolably large, but, since \( \lambda \) and \( \alpha \) are fixed, \( \gamma(E) \) is continuous, and \( E \) belongs to the spectrum, this is not the case.

Let us now assume that the number of factors is at least \( \lceil 5n/q_k \rceil + 1 \). Then we can use the facts that \( |G_{1,s}| \leq e^{-\gamma(E)-\delta/2}q_k/5 \) and \( |\psi(n_s)| \leq 1 \), and obtain

\[
|G_{1,s}G_{2,s} \ldots G_{p(s),s}\psi(n_s)| \leq e^{-\gamma(E)-\delta/2}q_k^{\delta/2}/5 \leq e^{-\gamma(E)-\delta/2n}.
\]

Combining everything into (8.1), we get

\[
\psi(n) \leq 2^{\lceil 5n/q_k \rceil + 1}e^{-\gamma(E)-\delta/2-\delta_1)n} \leq e^{-\gamma(E)-\delta)n}
\]

for sufficiently large \( q_0 \) and \( n > q_0 \). \( \blacksquare \)

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