DEGENERATION OF KÄHLER POLARIZATIONS TO MIXED POLARIZATIONS ON TORIC VARIETIES

DAN, WANG

Abstract. Let \((X, \omega, J)\) be a toric variety of dimension \(2n\) determined by a Delzant polytope. In this paper, we first construct the polarizations \(P_k\) by the Hamiltonian \(T^k\)-action on \(X\) (see Theorem 3.11). We will show that \(P_k\) is a singular mixed polarization for \(1 \leq k < n\), and \(P_n\) is a singular real polarization which coincides with the real polarization studied in \([3]\) on the open dense subset of \(X\). Then for each \(1 \leq k \leq n\), we will find a one-parameter family of Kähler polarizations \(P_{k,t}\) on \(X\) that converges to \(P_k\) (see Theorem 3.12). Finally, we will show that \(H_{k,t}\) the space of \(T^k\)-invariant \(J_{k,t}\)-holomorphic sections converges to \(H_k^0\) (see Theorem 3.18).

1. Introduction

Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold equipped with a prequantum line bundle \((L, h, \nabla)\) which consists of a smooth complex line bundle \(L \to M\) with hermitian structure \(h\) and hermitian connection \(\nabla\) with curvature \(F_\nabla = -2\pi i \omega\). Geometric quantization is a procedure to assign a certain vector space, which is called a quantum Hilbert space, to \((M, \omega)\). In order to perform a geometric quantization procedure, we need to choose a polarization (see definition 2.3.) \(P \subset TM \otimes \mathbb{C}\), which is an integrable Lagrangian subbundle of the complexification of the tangent bundle \(TM\) of \(M\). The quantum Hilbert space \(H_P\) associated to a polarization \(P\) is the subspace of \(\Gamma(M, L)\) defined by:

\[
H_P = \{ s \in \Gamma(M, L) \mid \nabla_\xi s = 0, \forall \xi \in \Gamma(M, P) \}.
\]

A central problem in geometric quantization is the study of the dependence of \(H_P\) on the choice of \(P\). A common choice is Kähler polarization \(P_J\) which comes from an integrable complex structure \(J\) on \(M\) such that \((M, \omega, J)\) is a Kähler manifold. In fact, \(P_J = T^{0,1}M\) with respect complex structure \(J\) and note that \(P_J \cap \overline{P_J} = 0\). In this case, the quantum space \(H_{P_J}\) is the space of \(J\)-holomorphic sections:

\[
H_{P_J} = \{ s \in \Gamma(M, L) \mid \bar{\partial}_J s = 0 \}.
\]

Another common choice is real polarization \(P_R\) satisfying \(P_R = \overline{P_R}\). A completely integrable system \(f : M \to \mathbb{R}^n\) gives a singular real polarization \(P_R = \text{Ker} df\). In this case, the quantum space \(H_{P_R}\) consists of distributional sections which are covariant constant along fiber with supports on Bohr-Sommerfeld fibers. The remaining type of polarization is called mixed...
polarization $\mathcal{P}_{\text{mix}}$, which satisfies the condition that the rank of $\mathcal{P}_{\text{mix}} \cap \mathcal{P}_{\text{mix}} \cap TM$ is a constant $k$ with $0 < k < n$.

The relation between quantizations associated to different polarizations has been studied in several cases. Using the space of holomorphic symmetric tensor on the moduli space of stable bundles over a Riemann surface, Hitchin in [18] constructed a projectively flat connection on a vector bundle over Teichmüller space. In [22], Kirwin and Wu identified the holomorphic quantizations along a geodesic in the space of almost complex structures compatible with the symplectic structure on symplectic vector space by parallel transport. Hall in [16], and Florentino, Matias, Mourão, and Nunes in [8, 9] studied the equivalence of quantizations of the cotangent bundle of the compact Lie group. Explicit geometro-analytic relations between real polarization wave functions and holomorphic ones via degenerating families of complex structures have been found for theta functions on abelian varieties referring to [10, 4].

Baier, Florentino, Mourão, and Nunes in [3] investigated a one-parameter family of complex structures $J_t$ degenerating to the real polarization $\mathcal{P}_R$ defined by kernel of moment map on toric varieties. Hamilton and Konno in [17] constructed a family of complex structures on a complex flag manifold that converges to the real polarization coming from the Gelfand-Cetlin integrable system. Other related works on toric varieties refer to [5, 19, 20].

In this paper, we first construct the (singular) mixed polarizations $\mathcal{P}_k$ on toric variety (see subsection 3.1). Then we find a one-parameter family of Kähler polarizations, which degenerates to the mixed polarizations (see subsection 3.2). Finally we investigate the limit of Kähler polarizations. The following setting will be used frequently, so we write it down here.

\begin{itemize}
\item[(*)] Let $(X, \omega, J)$ be a $2n$-dimensional toric variety determined by a Delzant polytope $P$ with moment map $\mu_P : X \to P \subset (\mathbb{A}^n)^*$. Let $\hat{X} = \mu_P^{-1}(\check{P})$ be the open dense orbit with $\check{P}$ being the interior of the polytope $P$. We consider the Hamiltonian $k$-dimensional torus action $\rho_k : T^k \to \text{Diff}(X, \omega, J)$ with moment map $\mu_k : X \to \Delta_k \subset (\mathbb{A}^k)^*$.
\end{itemize}

A step in the process of geometric quantization is to choose a polarization. In general, polarizations may not exist. In this paper, we will focus on the toric variety $(X, \omega, J)$. It’s easy to see that there exist Kähler polarization $\mathcal{P}_J$ and singular real polarization $\mathcal{P}_R$ defined by $\text{Ker} \, d\mu_P$ on $X$. A natural question to ask is whether singular mixed polarization exists on toric varieties. The answer is affirmative. In subsection 3.1, we construct singular distribution $\mathcal{P}_k$ (See Definition 3.7) coming from Hamiltonian subtorus $T^k$-action on $X$ under the assumption (*). Then we proved that $\mathcal{P}_k$ is a singular polarization on $X$ and smooth on $\check{X}$ (see Theorem 3.11). Moreover $\mathcal{P}_k$ is a singular mixed polarization, for $1 \leq k < n$. And $\mathcal{P}_n$ is a singular real polarization on $X$, which coincide with real polarization $\mathcal{P}_R$ (investigated in [3]) defined by the moment map on $\check{X}$, i.e. $\mathcal{P}_n|_{\check{X}} = \mathcal{P}_R|_{\check{X}}$.

**Theorem 1.1.** (Theorem 3.11) Under the assumption (*), the singular distribution $\mathcal{P}_k = (\mathcal{D}_k \cap \mathcal{P}_J) \oplus \mathcal{I}_C^k$ is a singular polarization on $X$ and smooth on $\hat{X}$. Moreover,
(1) if $k = n$, $\mathcal{P}_n$ is a singular real polarization and $\mathcal{P}_n|_{\tilde{X}} = \mathcal{P}_R|_{\tilde{X}}$; 
(2) if $1 \leq k < n$, then $\mathcal{P}_k$ is a singular mixed polarization with $\text{rank}(\mathcal{P}_k \cap \mathcal{P}_k \cap TM) = k$ on $\tilde{X}$.

From the viewpoint of physics, the quantum theory should be independent of the choice of polarization. Under the assumption (*), we showed that there exist singular mixed polarizations $\mathcal{P}_k$ on $X$. Motived by [18][8][3][17], we wonder to know whether there exists a one-parameter family of Kähler polarizations $\mathcal{P}_{k,t}$ degenerating to the singular polarization $\mathcal{P}_k$, for $1 \leq k \leq n$. The reasons why we expect to find such degenerating Kähler structures are as follows. Guillemin in [13] stated that the canonical toric Kähler metric and symplectic form on $(X,\omega,J)$ are determined by a symplectic potential $g_0 : P \to \mathbb{R}$. In [1][2], Abreu provided the description of all toric complex structures by symplectic potentials $g = g_0 + \varphi$ satisfying certain convexity condition (see Definition 2.2). For any $k$-dimensional subtorus $T^k$ of $T^n$, let $i^*_k : (t^n)^* \to (t^k)^*$ be the dual map of the inclusion of the Lie algebra $i_k : t^k \to t^n$. Let $\varphi_k : \Delta_k \to \mathbb{R}$ be strictly convex function. We consider a family of symplectic potentials $g_{k,t} = g_0 + t(\varphi_k \circ i^*_k) : P \to \mathbb{R}$.

Here $g_0$ is the symplectic potential determining the canonical toric complex structure $J$.

In subsection 3.2 we find a one-parameter family of Kähler polarizations $\mathcal{P}_{k,t}$ given by a family of symplectic potentials $g_{k,t}$ based on the construction due to [3] and [17]. Then we show that the Kähler polarizations $\mathcal{P}_{k,t}$ degenerate to mixed polarizations $\mathcal{P}_k$ constructed in Theorem 3.11.

Theorem 1.2. (Theorem 3.12) Under the assumption (*), we have

$$\lim_{t \to \infty} \mathcal{P}_{k,t} = \mathcal{P}_k,$$

where the limit is taken in the positive Lagrangian Grassmannian of the complexified tangent space at each point in $X$.

When $k = n$, the family of Kähler polarizations $\mathcal{P}_{n,t}$ coincides with the family of Kähler polarizations $\mathcal{P}_C^n$ studied in [3 Theorem 1.2]. But our result states that not only $\Gamma(X, \lim_{t \to \infty} \mathcal{P}_{n,t}) = \Gamma(X, \mathcal{P}_n)$ but as distribution $\lim_{t \to \infty} \mathcal{P}_{n,t} = \mathcal{P}_n$. It follows $\Gamma(X, \mathcal{P}_n) = \Gamma(X, \mathcal{P}_R)$, where $\mathcal{P}_R$ is a singular polarization defined by $\text{Ker} \, d\mu_P$ and studied in [3].

Denote the quantum space of the Kähler polarization $\mathcal{P}_{k,t}$ by $\mathcal{H}_{k,t}$. The dimension of $\mathcal{H}_{k,t}$ is equal to the number of holomorphic sections of the prequantum line bundle $L$, which is the number of integer point in Delzant polytope $P$ (see [7][14]). Denote the $T^k$-invariant subspace of $\mathcal{H}_{k,t}$ by $\mathcal{H}^T_{k,t}$. Guillemin and Sternberg in [15] investigated that the geometric quantization commute with symplectic reduction. Motived by [3] and inspired by [15], we aim to study the relationship between $\mathcal{H}^T_{k,t}$ and $\mathcal{H}_t^0$.

In subsection 3.3, we first introduce how

\[ \mathcal{H}_t^0 \text{ is defined by equation 1.1} \]
to extend the operator of covariant differentiation from smooth to distributional sections in order to define $\mathcal{H}_k$ and $\mathcal{H}_k^0$. Then we see that the distributional section in $\mathcal{H}_k$ support on the pre-image of integer values of moment map $\mu_k$ (see Proposition 3.15). And each $T^k$-invariant holomorphic section $\sigma_{k,0}^m \in \mathcal{H}_{k,0} = H^0_{\partial_\psi}(X, L)$ associate with a distributional section $\delta_k^m$ that lies in $\mathcal{H}_k^0$ (see Theorem 3.17). Finally, we show that $\mathcal{H}_{k,t}^T$ converge to $\mathcal{H}_k$ in the sense that the family of $L^1$-normalized $T^k$-invariant $J_{k,t}$-holomorphic sections $\frac{\sigma_{k,t}^m}{||\sigma_{k,t}^m||_{L_1}}$ converges to $\delta_k^m$, as $t$ goes to $\infty$ (see Theorem 3.18).

Let $(L, h, \nabla)$ be the prequantum line bundle on $(X, \omega, J)$ determined by the moment polytope $P \in \mathbb{R}^n$ with curvature $F_\nabla = -i\omega$. Denote the space of distributional sections on $L$ by $\Gamma_c(U, L^{-1})'$. The quantum spaces $\mathcal{H}_k$ associated to polarizations $\mathcal{P}_k$ is defined by

$$\mathcal{H}_k = \{ \delta \in \Gamma_c(X, L^{-1})' \mid \nabla_\xi \delta = 0, \forall \xi \in \Gamma(X, \mathcal{P}_k) \}.$$ 

Assume that $0 \in \Delta_k$ is a regular value of moment map $\mu_k$, we define the subspace $\mathcal{H}_k^0$ of $\mathcal{H}_k$ by:

$$(1.1) \quad \mathcal{H}_k^0 = \{ \delta \in \mathcal{H}_k \mid \text{supp} \delta \subset \mu_k\{0\} \}.$$ 

**Proposition 1.3.** (Proposition 3.15) Take any $\delta \in \mathcal{H}_k$, we have:

$$(\text{supp} \delta) \subset \bigcup_{q \in \Delta_k \cap \mathbb{Z}^k} \mu_k^{-1}(q).$$

Under assumption $(\ast)$, for any $m \in (i_k^*)^{-1}(0) \cap (\mathbb{R}^n)^*$, (see Definition 3.16) we define a distributional section $\delta_k^m \in \Gamma_c(X, L^{-1})'$ associate to $\sigma_{k,0}^m$ by:

$$\delta_k^m(\phi) = \frac{1}{c_k^m} \int_{\mu_k^{-1}(0) \cap X} \langle \sigma_{k,0}^m, \phi \rangle_{\mu_k^{-1}(0)} \text{vol}_k,$$

where $c_k^m$ is a constant defined by:

$$c_k^m = \int_{\mu_k^{-1}(0) \cap X} ||\sigma_{k,0}^m||_{\mu_k^{-1}(0)} \text{vol}_k.$$

**Theorem 1.4.** (Theorem 3.17) Under the assumption $(\ast)$, we have $\delta_k^m \in \mathcal{H}_k^0$.

Finally, we will show that $\mathcal{H}_{k,t}^T$ degenerate to $\mathcal{H}_k^0$ in the following theorem, where $\mathcal{H}_{k,t}^T$ is defined by $\mathcal{H}_{k,t}^T = \{ \sigma \in \mathcal{H}_{k,t} \mid \sigma \text{ is } T^k\text{-invariant} \}$.

**Theorem 1.5.** (Theorem 3.18) Assume that $0 \in \Delta_k$ is regular and $(\ast)$, taking any strictly convex function $\varphi_k$ in a neighborhood of $\Delta_k$ and $m \in (i_k^*)^{-1}(0) \cap (\mathbb{R}^n)^*$, then

1. for any $t > 0$, $\sigma_{k,t}^m = \sigma_{k,0}^m$ on $\mu_k^{-1}(0)$; and

---

1. the operator $\nabla_\xi$ of covariant differentiation to distributional sections defined by equation 3.6.
2. $\Gamma(X, \mathcal{P}_k)$: space of the smooth sections of singular distribution $\mathcal{P}_k$ (see Definition 3.3).
3. $\mathcal{H}_{k,t}^T = \text{span}\{\sigma_{k,t}^m := (\tilde{\chi}_{g_{\varphi_k}})^*\sigma_m\}_{m \in \mu_k^{-1}(0) \cap (\mathbb{R}^n)^*}$, see equation (3.7).
(2) considering the family of $L^1$-normalized $T^k$-invariant $J_{k,t}$-holomorphic sections \( \frac{\sigma_{k,t}^m}{||\sigma_{k,t}^m||_{L^1}} \), under the injection of smooth in distributional section of $L$:

\[
i : \Gamma(X, L) \to \Gamma_c(X, L^{-1})'
\]

we have \( i(\frac{\sigma_{k,t}^m}{||\sigma_{k,t}^m||_{L^1}}) \) converges to \( \delta_{k,t}^m \), as \( t \) goes to \( \infty \), in the sense that for any \( \phi \in \Gamma_c(X, L^{-1})' \),

\[
\lim_{t \to \infty} \int_X \langle \sigma_{k,t}^m, \phi \rangle e^{\omega} = \frac{1}{c_k^m} \int_{\mu_k^{-1}(0)} \langle \sigma_{k,0}^m, \phi \rangle |\mu_k^{-1}(0)| \text{vol}_k.
\]

1.1. **Further works.** Leung and Wang will generalize the constructions and results in this paper to the general case of the Kähler manifolds with $T$-symmetry in the forthcoming papers [23] and [24].

1.2. **Acknowledgements.** The author is grateful to Naichung Conan Leung for explaining the construction of mixed polarizations on toric varieties to her several years ago, and for fruitful discussions. The author appreciates Siye Wu for insightful comments and discussions during her visit to NTHU in 2019. The author thanks the hospitality of YMSC of Tsinghua during her visit in the summer of 2019. The author would like to thank her fellow colleagues at IMS for many discussions, especially Qingyuan Jiang, Yutung Yau and Ki Fung Chan.

2. **Preliminaries**

2.1. **Prequatum data of toric varieties.** In this section, we recall the toric Kähler structures of a toric variety \((X, \omega, J)\) determined by Delzant polytope $P$ together with the Hamiltonian $T^n$-action and moment map $\mu_P : X \to P$. Denote the Lie algebra of $T^n$ by $t^n$. We recall the construction of Toric variety $X$ from symplectic point of view (see [12]) and complex point of view (see [11], [6]) following the convention in [17]. We identify them according to a choice of symplectic potentials due to [1, 2, 12, 13] in subsection 2.3. In addition, the polytope $P$ determines an equivariant $J$-holomorphic line bundle \((L, h, \nabla)\) with curvature $-i\omega$ (see [25], [21]). Let

\[
P = \{ p \in (t^n)^* \mid \langle p, r_j \rangle + \lambda_j \geq 0, j = 1, \ldots, d \} \subset (t^n)^*
\]

be a bounded Delzant polytope, where \( \langle \cdot, \cdot \rangle : (t^n)^* \times t^n \to \mathbb{R} \) is the natural pairing and \( r_j \) is a primitive vector in the lattice $t^n_\mathbb{Z} \subset t^n$ for \( j = 1, \ldots, d \). We assume $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$. We set

\[
l_j(p) = \langle p, r_j \rangle + \lambda_j, \quad F_j = \{ p \in (t^n)^* \mid l_j(p) = 0 \}, j = 1, \ldots, d.
\]

Let $T^d$ be a real torus with the Lie algebra $t^d = \mathbb{R}^d$ with $X_1, \ldots, X_d \in t^d_\mathbb{Z}$ the standard basis of $t^d$, and let $T^n$ be a real torus with Lie algebra $t^n$. Let $\pi : t^d \to t^n$ be the surjective Lie algebra homomorphism defined by $\pi(X_j) = r_j$, for $j = 1, \ldots, d$. This gives rise to an exact sequence:

\[
0 \to K \to T^d \to T^n \to 0.
\]
Here $K \subset T^d$ is a connected subtorus which is the kernel of the corresponding Lie group homomorphism $\tilde{\pi} : T^d \to T^n$ with the Lie algebra $\mathfrak{k}$. Let $u_1, \ldots, u_d \in (t^d)^*$ be the dual basis of $X_1, \ldots, X_d \in t^d_\mathbb{Z}$. We denote $\lambda_P = \lambda_1 u_1 + \cdots + \lambda_d u_d \in (t^d)^*_\mathbb{Z}$. We will construct toric varieties from Delzant polytope subsection 2.1.1 and 2.1.2 following the convention in the paper [17].

2.1.1. Construct toric variety $(X, \omega_P)$ from symplectic point of view. Let $\omega_{\text{std}} = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j = \sum_j dx_j \wedge dy_j$ be the standard symplectic form on $\mathbb{C}^d$, where $x_j, y_j$ are the real and imaginary part of $z_j$ respectively. The natural action of $T^d$ on $(\mathbb{C}^d, \omega_{\text{std}})$ gives rise to a moment map $\mu_{T^d} : \mathbb{C}^d \to (t^d)^*$ by $\mu_{T^d}(z) = \frac{1}{2} \sum_{j=1}^d |z_j|^2 u_j$, where $z = (z_1, \ldots, z_d)$. Let $\iota^* : (t^d)^* \to \mathfrak{t}^*$ be the dual map of the inclusion $\iota : \mathfrak{t} \to t^d$. Then the moment map $\mu_K : \mathbb{C}^d \to \mathfrak{t}^*$ for the action of the subtorus $K$ on $(\mathbb{C}^d, \omega_{\text{std}})$ is given by $\mu_K(z) = \frac{1}{2} \sum_{j=1}^d |z_j|^2 \iota^* u_j$. The compact symplectic toric manifold $X$ is defined as the symplectic quotient $X = \mu_K^{-1}(\iota^* \lambda_P)/K$ and the symplectic structure $\omega_P \in H^2(X_P, \mathbb{Z})$ is determined by $\pi^* \omega_P = \omega_{\text{std}}|_{\mu_K^{-1}(\iota^* \lambda_P)}$. The quotient torus $T^n = T^d/K$ acts on $(X, \omega_P)$ with the moment map $\mu_P : X \to P \subset (t^n)^*$. This gives rise to the following diagram:

\[
\begin{array}{ccc}
1 & \downarrow & 0 \\
\downarrow & & \downarrow \\
K & \xrightarrow{\mu_K} & (\mathbb{R}^k)^* \\
\downarrow & & \downarrow \\
T^d & \xrightarrow{\mu_{T^d}} & (\mathbb{R}^d)^* \\
\downarrow & & \downarrow \\
T^n & \xrightarrow{\mu_P} & (\mathbb{R}^n)^* \\
\downarrow & & \downarrow \\
1 & & 0
\end{array}
\]

Take any $z \in \mu_K^{-1}(\iota^* \lambda_P)$, we have $\iota^*(\mu_{T^d}(z) - \lambda_P) = 0$. It turns out that

$$\mu_{T^d}(z) - \lambda_P \in \text{Ker}(\iota^*) = \text{Image}(\pi^*),$$

and $\mu_P([z])$ is given by $\mu_P([z]) = (\pi^*)^{-1}(\mu_{T^d}(z) - \lambda_P) \in (\mathbb{R}^n)^* = (t^n)^*$. Note that,

$$\mu_P(X_P) = (\pi^*)^{-1}(\text{Im} \mu_{T^d} \cap (\iota^*)^{-1}(\iota^* \lambda_P)) = P.$$

Let $\tilde{L}_{\text{sym}} = \mathbb{C}^d \times \mathbb{C}$ be the trivial line bundle with the standard hermitian metric $\tilde{h}$ and a hermitian connection $\tilde{\nabla}$, where $\tilde{\nabla}$ is defined by $\tilde{\nabla} = d - \sqrt{-1} \pi \sum_{i,j}^d (x_j dy_j - y_j dx_j)$ with
the curvature $F_{\nabla} = -i2\pi\omega$. Let $\text{Exp}_{T^n} : t^d \to T^n$ be the exponential map. Define the action of $T^n$ on $\tilde{L}_{\text{symp}}$ by: for any $(z,\nu) \in \mathbb{C}^d \times \mathbb{C}$ and $\xi \in t^d$,

$$(z,\nu)\text{Exp}_{T^n}\xi = (z\text{Exp}_{T^n}{\xi}, v e^{\sqrt{-1}(\lambda \nabla)}(\xi))$$

where $\lambda_P = \lambda_1 u_1 + \cdots + \lambda_d u_d \in (t^d)^*_\mathbb{C}$. It’s easy to check that this action preserve the hermitian metric $\tilde{\nabla}$ and hermitian connection $\nabla$. Then we define a prequantum line bundle $(\tilde{L}_{\text{symp}}, \tilde{\nabla}, \tilde{\nabla})$ on $(X, \omega_P)$ as the quotient of the restriction of $\tilde{L}_{\text{symp}}$ to $\mu_K^{-1}(i^*\lambda_{\Delta})$ by the action of the subtorus $K$. It’s not hard to see that the quotient torus $T^n = T^d / K$ acts on $L_{\text{symp}}$, preserving $h$ and $\nabla$. Let $[z]_K \in X_P$ denote a point represented by $z \in \mu_K^{-1}(i^*\lambda_P)$ and let $[z,\nu]_K$ denote a point in $L_{\text{symp}}$ represented by $(z,\nu) \in \mu_K^{-1}(i^*\lambda_P) \times \mathbb{C}$. Let $\hat{X} = \mu_P^{-1}(\hat{P})$ be the open dense subset of $X$, where $\hat{P}$ is the interior of the Delzant polytope $P$. Then it is not hard to see that $(\sqrt{2l_1(p)}, \ldots, \sqrt{2l_d(p)}) \in \mu_K^{-1}(i^*\lambda_P)$ for any $p \in \hat{P}$. The map $\psi : \hat{P} \times t^n / T^n \to \hat{X}$ defined by

$$(2.3) \quad \psi(p, [q]) = [(\sqrt{2l_1(p)}, \ldots, \sqrt{2l_d(p)})]_K \text{Exp}_{T^n}(q)$$

is a diffeomorphism, where $\bar{q} \in t^d$ is a lifting of $q$ so that $\pi(\bar{q}) = q$. It’s easy to see that

$$\mu_P \circ \psi(p, [q]) = \mu_P([(\sqrt{2l_1(p)} e^{\sqrt{-1}(\lambda_1, \bar{q})}, \ldots, \sqrt{2l_d(p)} e^{\sqrt{-1}(\lambda_d, \bar{q})})]_K)$$

$$= (\pi^*)^{-1}(l_1(p) - \lambda_1, \ldots, l_d(p) - \lambda_d) = p.$$

for any $(p, [q]) \in \hat{P} \times t^n / T^n$.

Then we take a trivialization of the prequantum line bundle $L_{\text{symp}}$ on $\hat{X}$ given by a section $\mathbb{1}$ of $L_{\text{symp}}|_{\hat{X}}$, which is defined by

$$\mathbb{1}(p, [q]) = [(\sqrt{2l_1(p)}, \ldots, \sqrt{2l_d(p)})]_K \text{Exp}_{T^n}(q) \in L_{\text{symp}}.$$

Choose a $\mathbb{Z}$-basis $p_1, \ldots, p_n \in (t^n)^*_\mathbb{Z}$ and its dual basis $q_1, \ldots, q_n \in t^n$. Let $\hat{P} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i p_i \in \hat{P}\}$ be the interior of polytope $P$. Then we obtain a symplectic coordinate $(x, [\theta]) \in \hat{P} \times \mathbb{R}^n / \mathbb{Z}^n$ on $\hat{P} \times t^n / T^n$, which can be regard as a coordinate on $\hat{X}$ under the map $\psi$. It is easy to check the following proposition (refer to [3] or [17]).

**Proposition 2.1.** Let $(x, [\theta]) \in \hat{P} \times \mathbb{R}^n / \mathbb{Z}^n$ be the coordinate on $\hat{X}$ associated to the fixed basis $p_1, \ldots, p_n \in (t^n)^*_\mathbb{Z}$. Then the symplectic form $\omega_P$ on $\hat{X}$ and the connection $\nabla$ on $L_{\text{symp}}|_{\hat{X}_P}$ are described as follows.

$$(1) \quad \omega_P|_{\hat{X}} = \sum_{i=1}^n dx_i \wedge d\theta_i,$$

$$(2) \quad \nabla|_{\hat{X}} = -2\pi i \sum_{i=1}^n x_i d\theta_i$$

with respect to the unitary trivialization defined by the section $\mathbb{1}$ on $\hat{X}$. 

7
2.1.2. Construct toric variety \((W_p, J)\) from complex point of view. Let \(P\) be a Delzant polytope defined by \((2.1)\) and let \(P^d\) be its set of vertices. Let \(F_j \subset (t^n)^*\) be the hyperplane defined by \(l_j = 0\), for \(j = 1, \ldots, d\). We set \(C_p = \bigcup_{p \in P^o} C_p^d\), where \(C_p^d = \{z \in \mathbb{C}^d \mid z_j \neq 0\} \text{ if } j \in \{1, \ldots, d\} \setminus \Lambda_v\} \text{ with } \Lambda_v = \{j \mid v \in F_j\} \text{ for each vertex } v \in P^o\). Then we define the compact complex toric variety \(W_p\) as the quotient space \(W_p = \mathbb{C}^d_p/K_C\) with induced complex structure \(J\), where \(K_C\) is the complexification of the subtorus \(K\). Let \(T^d_C\) be the complexification of the torus \(T^d\). The quotient torus \(T^d_C = T^d_C/K_C\) acts on \((W_p, J)\), it’s not hard to see that \(T^d\)-action preserve the complex structure \(J\). Let \(\tilde{L}_{\text{comp}} = \mathbb{C}^d \times \mathbb{C}\) be a trivial holomorphic line bundle on \(\mathbb{C}^d\). Define the action of \(T^d_C\) on \(\tilde{L}_{\text{comp}}\) determined by Delzant polytope \(P\) as follows: for any \((z, v) \in \mathbb{C}^d \times \mathbb{C}\) and \(\xi \in \mathfrak{t}^d_{\mathbb{C}} = \mathfrak{t}^d \otimes \mathbb{C}\),

\[
(z, v) \exp_{T^d_C} \xi = (z \exp_{T^d_C} \xi, v e^{\sqrt{-1} \tau(\lambda_P, \xi)}),
\]

where \(\lambda_P = \lambda_1 u_1 + \cdots + \lambda_d u_d\). Then we define a holomorphic line bundle \(L_{\text{comp}}\) on \((W_p, J)\) as the quotient of the restriction of \(\tilde{L}_{\text{comp}}\) to \(\mathbb{C}^d_p\) by the action of \(K_C\). Then the quotient torus \(T^d_C = T^d_C/K_C\) acts on \(L_{\text{comp}}\), preserving its holomorphic structure \(\tilde{\partial}_J\). Let \([z]_{K_C} \in W_p\) denote a point represented by \(z \in \mathbb{C}^d_p\) and let \([z, v]_{K_C}\) denote a point in \(L_{\text{comp}}\) represented by \((z, v) \in \mathbb{C}^d_p \times \mathbb{C}\). There exists a meromorphic section \(s_\lambda\) of \(L_{\text{comp}}\) on \(W_p\) associated to \(\lambda_P\) defined by

\[
s_\lambda([z]_{K_C}) = [z, \prod_{j=1}^d z_j^{\lambda_j}]_{K_C} \in L_{\text{comp}} \text{ for } z \in \mathbb{C}^d_p.\]

Note that \((\mathbb{C}^*)^d = \{z \in \mathbb{C}^d \mid z_i \neq 0 \text{ for } i = 1, \ldots, d\} \subset \mathbb{C}^d_p\). So for \(z \in (\mathbb{C}^*)^d\), we have \(z_j \neq 0, j = 1 \cdots d\). It implies the section \(s_\lambda\) is holomorphic and non-zero on \(\tilde{W} = (\mathbb{C}^*)^d/K_C\). Therefore \(\tilde{s}_\lambda = s_\lambda|_{\tilde{W}}\) induces a holomorphic trivialization of \(L_{\text{comp}}\) on \(\tilde{W}\). For each \(m \in P \cap (t^n)^{\mathbb{Z}}\) we define a section \(\sigma^m\) of \(L_{\text{comp}}\) by

\[
\sigma^m([z]_{K_C}) = [z, \prod_{j=1}^d z_j^{l_j(m)}]_{K_C} \in L_{\text{comp}} \text{ for } z \in \mathbb{C}^d_p.\]

Since \(l_j(m) \in \mathbb{Z}\), for \(j = 1, \ldots, d\), we have \(\sigma^m\) is homomorphic section. In fact, \(\{\sigma^m\}_{m \in P \cap (t^n)^{\mathbb{Z}}}\) form a basis of the space of holomorphic sections \(H^0(L_{\text{comp}}, \tilde{\partial}_J)\).

Now we provide a complex coordinate on the open dense subset \(\tilde{W}\) of \(W_p\). Choose a \(\mathbb{Z}\)-basis \(p_1, \ldots, p_n \in (t^n)^{\mathbb{Z}}\) and its dual basis \(q_1, \ldots, q_n \in \mathfrak{t}^d_{\mathbb{Z}}\) as in Subsection 2.1.1. Then we define a complex coordinate \(\tilde{\psi} : \tilde{W} \to (\mathbb{C}^*)^n\) by

\[
\tilde{\psi}([z]_{K_C}) = (\prod_{j=1}^d z_j^{(p_1, r_j)}, \ldots, \prod_{j=1}^d z_j^{(p_n, r_j)}),
\]

where \(r_j \in \mathfrak{t}^d_{\mathbb{Z}}\) is the vector in \((2.1)\) for \(j = 1, \ldots, d\). Since \(\prod_{j=1}^d z_j^{(p_i, r_j)}\) is a \(K_C\)-invariant meromorphic function on \(\mathbb{C}^d\), it descends to a meromorphic function on \(W_p\). If we set
(w_1, \ldots, w_n) = \tilde{\psi}(\lfloor z \rfloor_{K_C}), \text{ then we have}

(2.7) \quad \sigma^m(\lfloor z \rfloor_{K_C}) = (\prod_{i=1}^{n} w_i^{(m,q_i)})^{\hat{s}_{\lambda}}(\lfloor z \rfloor_{K_C}) \text{ on } \tilde{W}.

### 2.2. Symplectic potentials on toric varieties.

In Subsections 2.1.1 and 2.1.2, we reviewed the construction of the symplectic toric manifolds \((X,\omega_P)\) and the complex toric manifolds \((W_P,J)\) from a Delzant polytope \(P\) defined in (2.1) respectively. In this section we identify them by canonical symplectic potentials \(g_0\) due to Abreu [1, 2] and Guillemin [12, 13] and then denote \((X,\omega,J)\) the toric variety determined by \(P\) with symplectic structure \(\omega\) and canonical complex structure \(J\).

The inclusion \(\mu_{K^{-1}}(\iota^*\lambda_P) \subset \mathbb{C}_P^d\) induces a diffeomorphism \(\chi_{g_0} : X_P \to W_P\) given by a canonical symplectic potential \(g_0 : \tilde{P} \to \mathbb{R}\) as described in [12, 13] with the following form:

\[
g_0(p) = \frac{1}{2} \sum_{j=1}^{d} l_j(p) \log l_j(p) + ( \text{ a linear function on } (t_n)^* ) \text{ for } p \in \tilde{P}.
\]

It’s easy to see that \(g_0\) can be extended continuously to the boundary of \(P\).

Fix a \(\mathbb{Z}\)-basis \(p_1, \ldots, p_n \in (t_n)^*\) and its dual basis \(q_1, \ldots, q_n \in t_n^\ast\) as in Subsections 2.1.1 and 2.1.2. Fix \(\tilde{q}_i \in t_n^\ast\) such that \(\pi(\tilde{q}_i) = q_i\) for \(i = 1, \ldots, n\). Let \((x, [\theta])\) be the symplectic coordinate on \(\tilde{X}\) and let \((w_1, \ldots, w_n)\) be the complex coordinate on \(\tilde{W}\) induced by \(p_1, \ldots, p_n \in (t_n)^*\) respectively. If we write \(p = \sum_{i=1}^{n} x_i p_i\), then, by (2.3) and (2.6) we have

\[
\chi_{g_0}(x, [\theta])) = \prod_{j=1}^{d} \left( \sqrt{2l_j(p)} e^{\sqrt{-1} \sum_{i=1}^{n} (u_j, \tilde{q}_i)}(p_n, r_j) \right) = e^{\left( \frac{\partial \ln \beta}{\partial x_i} + \sqrt{-1} \theta_i \right)},
\]

In [1, 2] Abreu showed that all toric complex structures can be given by a symplectic potential satisfying the conditions as in the following definition.

**Definition 2.2.** A function \(g \in C^0(P)\) is a **symplectic potential** if and only if it satisfies the following condition:

1. \(g - g_0 \in C^\infty(P)\),
2. The Hessian \(\text{Hess}_p g\) of \(g\) at \(p\) is positive definite for any \(p \in \tilde{P}\),
3. there exists a strictly positive function \(\beta \in C^\infty(P)\) such that

\[
\det(\text{Hess}_p g) = [\beta(p)] \prod_{j=1}^{d} l_j(p)^{-1}, \forall p \in \tilde{P}.
\]

The set of symplectic potentials is denoted by \(S(P)\).

The following results are due to [12, 13] [1, 2], supplemented by [3] and reviewed in [17].

**Theorem 2.3.** [17] Theorem 5.3 Let \(P \subset (t_n)^*\) be a Delzant polytope. Let \((X,\omega_P)\) be a symplectic toric manifold and \((W_P,J)\) a complex toric variety constructed from \(P\). Let
\((L_{\text{symp}}, h, \nabla)\) be a prequantum line bundle on \(X\) and \((L_{\text{comp}}, \bar{\partial})\) a holomorphic line bundle on \(W_P\) constructed from \(P\). Fix a \(\mathbb{Z}\)-basis \(p_1, \ldots, p_n \in (t^s)^*_\mathbb{Z}\). Let \((x, [\theta])\) be the symplectic coordinate on \(\hat{X}\) and \(w = (w_1, \ldots, w_n)\) the complex coordinate on \(\hat{W}\) induced by \(p_1, \ldots, p_n \in (t^s)^*_\mathbb{Z}\) respectively. Then each \(g \in S(P)\) defines a \(T^n\)-equivariant diffeomorphism \(\chi_g : X \to WP\) and a \(T^n\)-equivariant bundle isomorphism \(\tilde{\chi}_g : L_{\text{symp}} \to L_{\text{comp}}\) such that the following holds:

1. The following diagram commutes:

\[
\begin{array}{ccc}
(L_{\text{symp}}, h, \nabla) & \xrightarrow{\tilde{\chi}_g} & (L_{\text{comp}}, \bar{\partial}) \\
\downarrow & & \downarrow \\
(X, \omega_P) & \xrightarrow{\chi_g} & (W_P, J)
\end{array}
\]

2. \((X, \omega_P, \chi_g^*J)\) is a Kähler manifold.
3. \(\nabla\) is the Chern connection of the Hermitian holomorphic line bundle \((L_{\text{symp}}, h, \tilde{\chi}_g^*\bar{\partial})\).
4. \(\chi_g|_{\hat{X}} : \hat{X} \to \hat{W}\) is a diffeomorphism given by

\[
w_i(\chi_g(x, [\theta])) = e^{\left(\frac{\partial g}{\partial x_i} + \sqrt{-1} \theta_i\right)} i = 1, \ldots, n.
\]

The map \(\chi_g\) is independent of the choice of the basis \(p_1, \ldots, p_n \in (t^s)^*_\mathbb{Z}\). Moreover, if we write \(w_i = e^{y_i + \sqrt{-1} \theta_i}\) for \(i = 1, \ldots, n\), then the inverse mapping \((\chi_g|_{\hat{X}})^{-1} : \hat{W} \to \hat{X}\) is given by

\[
x_i((\chi_g)^{-1}(w)) = \frac{\partial h}{\partial y_i}, \quad \theta_i((\chi_g)^{-1}(w)) = \theta_i \text{ for } i = 1, \ldots, n,
\]

where \(h(y) = -g(x(y)) + \sum_{i=1}^n x_i(y) y_i\) is given by Legendre transformation.

5. \(\tilde{\chi}_g^*\delta_x = e^{g - \sum_{i=1}^n x_i \frac{\partial g}{\partial x_i} \delta_x}\) on \(\hat{X}\).

On the other hand, if \(\chi : X \to WP\) is a \(T^n\)-equivariant diffeomorphism such that \((X, \omega_P, \chi^*J)\) is a Kähler manifold and that \(\chi\) is homotopic to \(\chi_{g_0}\), then there exists \(g \in S(P)\) such that \(\chi = \chi_g\).

### 2.3. Polarizations

A step in the process of geometric quantization is to choose a polarization. We first recall the definitions of distribution and polarization on symplectic manifolds \((M, \omega)\) (See [26]). All polarizations discussed in this section are smooth.

**Definition 2.4.** A complex distribution \(\mathcal{P}\) on a manifold \(M\) is a sub-bundle of the complexified tangent bundle \(TM \otimes \mathbb{C}\), such that for every \(x \in M\), the fiber \(\mathcal{P}_x\) is a subspace of \(T_xM \otimes \mathbb{C}\).

We denote \(\Gamma(M, \mathcal{P})\) as the space of all vector fields of \(M\) tangent to \(\mathcal{P}\). Here a vector field \(u\) of \(M\) is tangent to \(\mathcal{P}\) if for every \(x \in M\), \(u_x \in \mathcal{P}_x\). Then we can define the complex polarizations on the symplectic manifolds.
**Definition 2.5.** A complex polarization $\mathcal{P}$ of a symplectic manifold $(M, \omega)$ is a complex distribution $\mathcal{P} \subset TM \otimes \mathbb{C}$ satisfying the following conditions:

(a) $\mathcal{P}$ is involutive, i.e. if $u, v \in \Gamma(M, \mathcal{P})$, then $[u, v] \in \Gamma(M, \mathcal{P})$;
(b) for every $x \in M$, $\mathcal{P}_x \subseteq T_xM \otimes \mathbb{C}$ is Lagrangian; and
(c) $\text{rank}(\mathcal{P} \cap \overline{\mathcal{P}} \cap TM)$ is constant.

Let $(M, \omega)$ be a symplectic manifold with $\dim \mathbb{R} M = 2n$ and let $\mathcal{P}$ be a real polarization on $X$. The complexification of $\mathcal{P}$ denoted by $\mathcal{P} \otimes \mathbb{C} = (\mathcal{P})_C$ is a complex polarization on $M$. Hence $(\mathcal{P})_C \cap (\overline{\mathcal{P}})_C = (\mathcal{P})_C$, the rank of $(\mathcal{P})_C \cap (\overline{\mathcal{P}})_C \cap TM = (\mathcal{P})_C \cap TM = \mathcal{P}$ is a constant $n$. And $(\mathcal{P})_C$ is integrable and Lagrangian. Conversely, if $\mathcal{P}$ is a complex polarization on the symplectic manifold $(M, \omega)$ such that $\mathcal{P} = \overline{\mathcal{P}}$, then $\mathcal{P}$ is the complexification of a real polarization. Hence, when $\mathcal{P} = \overline{\mathcal{P}}$, $\mathcal{P}$ is abusively called a real polarization.

Let $(M, \omega, J)$ be a Kähler manifold of $\dim \mathbb{R} M = 2n$. We can decompose the complexification $T_C M$ of tangent bundle $TM$ into its holomorphic part $T^{1,0}_C$ and anti-holomorphic part $T^{0,1}$ associated to $J$. The condition that $J$ is compatible with $\omega$ implies $(T^{1,0})_x \subseteq T_xM \otimes \mathbb{C}$ and $(T^{0,1})_x \subseteq T_xM \otimes \mathbb{C}$ are Lagrangians for any point $x \in M$. $T^{1,0} = \mathcal{P}_J$ and $T^{0,1} = \mathcal{P}_J$ are complex polarizations. Moreover $\mathcal{P}_J \cap \overline{\mathcal{P}_J} = 0$, $\mathcal{P}_J$ and $\overline{\mathcal{P}_J}$ are called Kähler polarizations.

**Remark 2.6.** Let $\mathcal{P}$ be a complex polarization on a symplectic manifold $(M, \omega)$, then $\mathcal{P}$ is called:

1. real polarization, if $\mathcal{P} = \overline{\mathcal{P}}$;
2. Kähler polarization, if $\mathcal{P} \cap \overline{\mathcal{P}} = 0$;
3. mixed polarization, if $0 < \dim \mathbb{R}(\mathcal{P} \cap \overline{\mathcal{P}} \cap TM) < n$.

**3. Main results**

**3.1. Construction of polarizations $\mathcal{P}_k$ from Hamiltonian $T^k$-action.** A step in the process of geometric quantization is to choose a polarization. In the previous section, we introduced smooth real polarization, Kähler polarization and mixed polarization on symplectic manifolds $(M, \omega)$. In general, polarizations may not exist. In this section, we will focus on the toric variety $(X, \omega, J)$. It’s easy to see that there exist Kähler polarization $\mathcal{P}_J$ and singular real polarization $\mathcal{P}_R$ defined by $d \mu_p$ on $X$. A natural question to ask is whether singular mixed polarization exists on toric varieties. The answer is affirmative. Under the assumption $(\ast)$, we construct a singular distribution $\mathcal{P}_k$ (See Definition 3.7) coming from the Hamiltonian subtorus $T^k$-action on $X$. Then we proved that $\mathcal{P}_k$ is a singular polarization on $X$ and smooth on $\check{X}$ (see Theorem 3.11). Moreover $\mathcal{P}_k$ is a singular mixed polarization, for $1 \leq k < n$. And $\mathcal{P}_n$ is a singular real polarization on $X$, which coincides with the real polarization $\mathcal{P}_R$ (investigated in [3]) defined by moment map on $\check{X}$, i.e. $\mathcal{P}_n|_{\check{X}} = \mathcal{P}_R|_{\check{X}}$. 

11
In the previous section, both distributions and polarizations defined on symplectic manifolds are smooth. Since the torus action on toric varieties determined by Delzant polytopes is not free, \( \text{Ker } d\mu_P \) is not a smooth distribution. In this section, we first give the definition of singular distributions and polarizations. Then we construct the singular complex polarization \( \mathcal{P}_k \) on \( X \). Finally, we will show that \( \mathcal{P}_k \) is a singular polarization on \( X \) and smooth on \( \tilde{X} \).

**Definition 3.1.** A singular complex distribution on \( X \) is a subset \( \mathcal{P} \subset TX \otimes \mathbb{C} \) such that for all point \( p \in X \), \( \mathcal{P}_p \) is a vector subspace of \( T_pX \otimes \mathbb{C} \).

**Definition 3.2.** A singular complex distribution \( \mathcal{P} \subset TX \otimes \mathbb{C} \) on \( X \) is said to be smooth on \( \tilde{X} \) if \( \mathcal{P}|_{\tilde{X}} \) is a sub-bundle of tangent bundle \( T\tilde{X} \otimes \mathbb{C} \).

**Definition 3.3.** Let \( \mathcal{P} \) be a singular complex distribution on \( X \). For any open subset \( U \) of \( X \), the space \( \Gamma(U, P) \) of smooth sections of \( \mathcal{P} \) on \( U \) consists of the smooth sections of \( TX \otimes \mathbb{C} \) with values in \( \mathcal{P} \), that is

\[
\Gamma(U, \mathcal{P}) = \{ v \in \Gamma(U, TX \otimes \mathbb{C}) \mid v_p \in P_p \}.
\]

In particular,

\[
\Gamma(X, \mathcal{P}) = \{ v \in \Gamma(X, TX \otimes \mathbb{C}) \mid v_p \in P_p \}.
\]

**Definition 3.4.** Let \( \mathcal{P} \) be a singular complex distribution on \( X \). \( \mathcal{P} \) is involutive if it satisfies

\[
[u, v] \in \Gamma(X, \mathcal{P}), \quad \forall u, v \in \Gamma(X, \mathcal{P}).
\]

**Definition 3.5.** A singular complex distribution \( \mathcal{P} \) on \( X \) is a singular complex polarization on \( X \) and smooth on \( \tilde{X} \) if \( \mathcal{P} \) satisfies the following conditions:

(a) \( \mathcal{P} \) is involutive, i.e. \([u, v] \in \Gamma(X, \mathcal{P}), \forall u, v \in \Gamma(X, \mathcal{P})\);

(b) for every \( x \in X \), \( \mathcal{P}_x \subseteq T_xX \otimes \mathbb{C} \) is Lagrangian; and

(c) \( \mathcal{P}|_{\tilde{X}} \) is a smooth subbundle of \( T\tilde{X} \otimes \mathbb{C} \) and \( \text{rank}(\mathcal{P} \cap \overline{\mathcal{P}} \cap TM) \) is constant on \( \tilde{X} \).

**Remark 3.6.** If \( \mathcal{P} \) is a singular polarization on \( X \) and smooth on \( \tilde{X} \), \( \mathcal{P}|_{\tilde{X}} \) is a smooth polarization on \( \tilde{X} \). then \( \mathcal{P} \) is called

1. a singular real polarization, if \( \mathcal{P} = \overline{\mathcal{P}} \) on \( \tilde{X} \);
2. a singular mixed polarization, if \( 0 < \text{rank}(\mathcal{P} \cap \overline{\mathcal{P}} \cap TX)|_{\tilde{X}} < n \).

\((*)\): Let \((X, \omega, J)\) be a 2n-dimensional toric variety determined by a Delzant polytope \( P \) with moment map \( \mu_P : X \rightarrow P \subset (t^n)^* \). Let \( \tilde{X} = \mu_P^{-1}(\bar{P}) \) be the open dense orbit with \( \bar{P} \) being the interior of the polytope \( P \). We consider the Hamiltonian \( k \)-dimensional torus action \( \rho_k : T^k \rightarrow \text{Diff}(X, \omega, J) \) with moment map \( \mu_k : X \rightarrow \Delta_k \subset (t^k)^* \).

Taking any \( k \)-dimensional subtorus \( T^k \) of \( T^n \), let \( i_k^* : (t^n)^* \rightarrow (t^k)^* \) be the dual map of the inclusion of the Lie algebra \( i_k : t^k \rightarrow t^n \). Under the assumption \((*)\), it’s easy to check that:

\[
\mu_k = i_k^* \circ \mu_P.
\]
For any point \( p \in X \), consider the map \( (\rho_k)_p : T^k \rightarrow X \) defined by \( (\rho_k)_p(g) = (\rho_k(g))(p) \). Let \( T^k_X \subset TX \) be defined by: \((T^k_X)_p = \text{Im } d(\rho_k)_p \). It’s easy to see that \( T^k_X \) is a singular real distribution on \( X \). Let \( D^k_C = (\text{Ker } d\mu_k) \otimes \mathbb{C} \subset TX \otimes \mathbb{C} \) be the complexification of \( \text{Ker } d\mu_k \subset TX \) and let \( T^k_C = T^k_X \otimes \mathbb{C} \subset TX \otimes \mathbb{C} \) be the complexification of \( T^k_X \subset TX \).

We construct the singular distribution \( \mathcal{P}_k \) from the Hamiltonian action as follows.

**Definition 3.7.** Under the assumption \((*)\), we define the distribution \( \mathcal{P}_k \subset TX \otimes \mathbb{C} \) as:

\[
\mathcal{P}_k = (D^k_C \cap \mathcal{P}_J) \oplus T^k_C.
\]

**Lemma 3.8.** Under the assumption \((*)\), \( T^k_C = T^k_X \otimes \mathbb{C} \) is involutive on \( X \) and is smooth on \( \hat{X} \).

**Proof.** Since \( T^k_X = T^k_X \otimes \mathbb{C} \subset TX \otimes \mathbb{C} \) is the complexification of \( T^k_X \subset TX \), it’s enough to show \( T^k_X \subset TX \) is involutive on \( X \) and is smooth on \( \hat{X} \). Note that the subtorus \( T^k \) freely acts on \( \hat{X} \), it turns out that the distribution \( T^k_X \) generated by fundamental vector fields is smooth on \( \hat{X} \). Now we are going to show \( T^k_X \subset TX \) is involutive on \( X \). Taking any \( f\xi^\#\), \( g\eta^\# \in \Gamma(X, T^k_X) = \{ v \in \Gamma(X, TX) \mid v(p) \in T^k_X, \forall p \in X \} \) with \( f, g \in C^\infty(X) \) and \( \xi^\#, \eta^\# \in \text{Im } d\rho_k \), observe that:

\[
[f\xi^\#, g\eta^\#] = f\xi^\#(g)\eta^\# - g\eta^\#(f)\xi^\# + fg[\xi^\#, \eta^\#].
\]

To obtain \([f\xi^\#, g\eta^\#] \in \Gamma(X, T^k_X)\), it’s enough to show \([\xi^\#, \eta^\#] \in \text{Im } d\rho_k \subset \Gamma(X, T^k_X)\). This is reduced to show that \( \text{Im } d\rho_k \) is involutive. Recall that: \( \rho_k : T^k \rightarrow \text{Diff}(X, \omega, J) \) is group homeomorphism given by the Hamiltonian subtorus \( T^k \)-action and \( d\rho_k : T^k \rightarrow \text{Vect}(X, \omega, J) \) is a Lie algebra homomorphism. It follows \( \text{Im } d\rho_k \) is involutive and \([\xi^\#, \eta^\#] = d\rho_k[\xi, \eta] \in \text{Im } d\rho_k \). \( \square \)

**Lemma 3.9.** Under the assumption \((*)\), \( D^k_C = \text{Ker } d\mu_k \otimes \mathbb{C} \) is involutive on \( X \).

**Proof.** Note that \( D^k_C = \text{Ker } d\mu_k \), where \( \mu_k : X \rightarrow \Delta_k \) is the moment map. Since \( D^k_C = D^k_C \otimes \mathbb{C} \) is the complexification of \( \text{Ker } d\mu_k \), it is sufficient to prove that \( D^k_C \) is involutive. For any \( u, v \in \Gamma(X, D^k_C) \), we want to show \([u, v] \in \Gamma(X, D^k_C)\), i.e \( d\mu_k([u, v]) = 0 \). Taking any smooth function \( f \in C^\infty(X) \), we have

\[
d\mu_k([u, v])f = [u, v](f \circ \mu_k) = uv(f \circ \mu_k) - vu(f \circ \mu_k) = 0.
\]

It turns out that \( D^k_C \) is involutive. \( \square \)

**Remark 3.10.** \( D^k_C \cap \mathcal{P}_J \) is involutive on \( X \) and smooth on \( \hat{X} \). The reason is that \( D^k_C \) and \( \mathcal{P}_J \) are involutive on \( X \) and smooth on \( \hat{X} \).

**Theorem 3.11.** Under the assumption \((*)\), the singular distribution \( \mathcal{P}_k = (D^k_C \cap \mathcal{P}_J) \oplus T^k_C \) is a singular polarization on \( X \) and smooth on \( \hat{X} \). Moreover,

1. if \( k = n \), then \( \mathcal{P}_n \) is a singular real polarization and \( \mathcal{P}_n|_{\hat{X}} = \mathcal{P}_R|_{\hat{X}} \);
(2) If $1 \leq k < n$, then $\mathcal{P}_k$ is a singular mixed polarization with $\text{rank}(\mathcal{P}_k \cap \bar{\mathcal{P}}_k \cap TM) = k$ on $\check{X}$.

**Proof.** We first show that $\mathcal{P}_k$ is involutive case by case as follows.

1. If $u, v \in \Gamma(X, \mathcal{D}_C^k \cap \mathcal{P}_J)$, one has $[u, v] \in \Gamma(X, \mathcal{D}_C^k \cap \mathcal{P}_J)$ because $\mathcal{D}_C^k \cap \mathcal{P}_J$ is involutive (see remark 3.10). It follows $[u, v] \in \Gamma(X, \mathcal{P}_k)$.

2. If $u, v \in \Gamma(X, \mathcal{I}_C^k)$, then $[u, v] \in \Gamma(X, \mathcal{I}_C^k)$ by Lemma 3.8. It turns out $[u, v] \in \Gamma(X, \mathcal{P}_k)$.

3. If $u \in \Gamma(X, \mathcal{D}_C^k \cap \mathcal{P}_J)$ and $v \in \Gamma(X, \mathcal{I}_C^k)$, then $[u, v] \in \Gamma(X, \mathcal{P}_k)$. This is because the $T^k$-action preserves the complex structure $J$ and $\mathcal{D}_C^k$ is involutive. More precisely,

   (i) $[u, v] \in \Gamma(X, \mathcal{P}_J)$ because the $T^k$-action preserves the complex structure $J$.

   (ii) $[u, v] \in \Gamma(X, \mathcal{D}_C^k)$ since $\mathcal{I}_C^k \subseteq \mathcal{D}_C^k$ and $\mathcal{D}_C^k$ is involutive (by Lemma 3.9).

Now we restrict our attention to show that $(\mathcal{P}_k)_x \subset T_x X \otimes \mathbb{C}$ is Lagrangian. Note that $\mathcal{P}_J$ is Kähler polarization, in particular $\mathcal{P}_J$ is a Lagrangian subbundle. So $\omega(u, v) = 0$, for any $u, v \in \Gamma(X, \mathcal{P}_J)$.

Since the Hamiltonian $T^k$-action preserves $\omega$, $\omega(u, v) = 0$, for all $u, v \in \mathcal{I}_C^k$. It is clear that $\omega(u, v) = 0$, for $u \in \mathcal{D}_C^k \cap \mathcal{P}_J$, $v \in \mathcal{I}_C$.

This implies $(\mathcal{P}_k)_x \subset T_x X \otimes \mathbb{C}$ is Lagrangian, for all $x \in X$. It remains to prove $\mathcal{P}_k$ is smooth on $\check{X}$ and $\text{rank}(\mathcal{P}_k \cap \bar{\mathcal{P}}_k \cap TX) = k$. Recall that $T^n$ acts freely on $\check{X}$. In particular, $T^k$ acts freely on $\check{X}$. This implies $\mathcal{I}_C^k$ and $\mathcal{D}_C^k$ is smooth on $\check{X}$ and

$$\text{rank}(\mathcal{P}_k \cap \bar{\mathcal{P}}_k \cap TX)|_{\check{X}} = \text{rank}(\mathcal{I}_C^k \cap \mathcal{I}_C^k \cap TX)|_{\check{X}} = \text{rank}(I^k)|_{\check{X}} = k.$$ 

Therefore we conclude that $\mathcal{P}_k$ is a singular polarization on $X$ and smooth on $\check{X}$. Moreover,

1. If $k = n$, then $\text{Ker}d\mu_{\mathcal{P}} = \text{Ker}d\mu_n$ and $\text{rank}(\mathcal{P}_k \cap \bar{\mathcal{P}}_k \cap TX) = n$, i.e. $\mathcal{P}_k = \bar{\mathcal{P}}_k$ on $\check{X}$.

   It follows $\mathcal{P}_n$ is a singular real polarization and $\mathcal{P}_n|_{\check{X}} = \mathcal{P}_\mathbb{R}|_{\check{X}}$.

2. If $1 \leq k < n$, then $\text{rank}(\mathcal{P}_k \cap \bar{\mathcal{P}}_k \cap TX) = k$ on $\check{X}$. So $\mathcal{P}_k$ is a singular mixed polarization with $\text{rank}(\mathcal{P}_k \cap \bar{\mathcal{P}}_k \cap TM) = k$ on $\check{X}$. 

\[\square\]

### 3.2. Degenerating Kähler polarizations to mixed polarizations $\mathcal{P}_k$. \(\star\):

Let $(X, \omega, J)$ be a 2n-dimensional toric variety determined by a Delzant polytope $P$ with moment map $\mu_P : X \rightarrow P \subset (\mathbb{R}^n)^*$. Let $\check{X} = \mu_P^{-1}(\check{P})$ be the open dense orbit with $\check{P}$ being the interior of the polytope $P$. We consider the Hamiltonian $k$-dimensional torus action $\rho_k : T^k \rightarrow \text{Diff}(X, \omega, J)$ with moment map $\mu_k : X \rightarrow \Delta_k \subset (\mathbb{R}^k)^*$.

From the point of view of physics, the quantum space should be independent of the choice of polarizations. In the previous section, under the assumption \(\star\), we showed that there exists singular mixed polarizations $\mathcal{P}_k$ on $X$. In [3], Baier, Florentino, Mourão, and Nunes studied a one-parameter family of Kähler polarizations degenerating to the singular real polarization $\mathcal{P}_\mathbb{R}$ defined by $\text{Ker}d\mu_P$. This guarantees that there exists a one-parameter
family of Kähler polarizations degenerates to the singular real polarization $\mathcal{P}_n$. In order to investigate the relationship between the Kähler polarization $\mathcal{P}_J$ and the mixed polarization $\mathcal{P}_k$, motived by [3], we wonder to know whether there exists a one-parameter family of Kähler polarizations $\mathcal{P}_{k,t}$ degenerating to singular mixed polarization $\mathcal{P}_k$, for $1 \leq k < n$. In this section, we construct a one-parameter family of Kähler polarizations $\mathcal{P}_{k,t}$ given by a family of symplectic potentials $g_{k,t}$ (see equation 3.1) based on the construction due to [3] and [17]. Then (See Theorem 3.12) we show that the Kähler polarizations $\mathcal{P}_{k,t}$ degenerates to mixed polarization $\mathcal{P}_k$ constructed in the last section. Let $(X, \omega, J)$ be the toric variety determined by moment polytope $\mathcal{P}$ with moment map $\mu: X \to \mathcal{P}$, where $\mathcal{P}$ is Delzant polytope defined by $l_j \geq 0$ as (2.1), $j = 1, \cdots, d$. Recall that the canonical complex structure $J$ can be written by a symplectic potential $g_0: \mathcal{P} \to \mathbb{R}$ as described in [1][2], and on $\check{X}$ under the symplectic coordinates $(x, \theta) \in \check{P} \times \mathbb{R}^n/\mathbb{Z}^n$, the canonical complex structure and Kähler metric $\gamma = \omega(\cdot, J \cdot)$ are given by:

$$J = \begin{pmatrix} 0 & -G_0^{-1} \\ G_0 & 0 \end{pmatrix}; \quad \gamma = \begin{pmatrix} G_0 & 0 \\ 0 & G_0^{-1} \end{pmatrix}$$

where $G_0 = \text{Hess}g_0$ is the Hessian of $g_0$ such that $G_0(x)$ is positive definite on $\check{P}$ and satisfies the regularity conditions:

$$\det G_0(x) = [\beta(x) \prod_{j=1}^{d} l_j(x)]^{-1},$$

for $\beta$ smooth and strictly positive on $P$.

Taking any $k$-dimensional subtorus $T^n_k$ of $T^n$, let $i^*_k: (t^k)^* \to (t^n)^*$ be the dual map of the inclusion of the Lie algebra $i_k: t^k \to t^n$. Under the assumption ($\ast$), it can be seen that: $\mu_k = i^*_k \circ \mu_P$. Let $\varphi_k: \Delta_k \to \mathbb{R}$ be a strictly convex function. We consider a one-parameter family of symplectic potentials

(3.1) $$g_{k,t} = g_0 + t(\varphi_k \circ i^*_k): P \to \mathbb{R}.$$  

It can be checked that $G_{k,t} = \text{Hess}(g_{k,t})$ is positive definite on $\check{P}$ and satisfies the regular conditions:

$$\det G_{k,t}(x) = [\beta_{k,t}(x) \prod_{j=1}^{d} l_j(x)]^{-1},$$

for $\beta_{k,t}$ smooth and strictly positive on $P$ as in [1][2]. Therefore the family of Kähler potentials $g_{k,t}$ determines a one-parameter family of toric complex structures denoted by $J_{k,t}$. Let $\mathcal{P}_{k,t}$ be the Kähler polarizations associated to the complex structures $J_{k,t}$. We show that the Kähler polarizations $\mathcal{P}_{k,t}$ converge to mixed polarization $\mathcal{P}_k$ in the following theorem.

**Theorem 3.12.** Under the assumption ($\ast$), we have

$$\lim_{t \to \infty} \mathcal{P}_{k,t} = \mathcal{P}_k,$$
where the limit is taken in the positive Lagrangian Grassmannian of the complexified tangent space at each point in \( X \).

Proof. Let \( i_k^* : (t^n)^* \to (t^k)^* \) be the dual map of the inclusion \( i_k : t^k \to t^n \). We observe that \( i_k^* (P) = \Delta_k \) due to \( \mu_k = (i_k)^* \circ \mu_p \). Now we take a basis of \((t^n)^*_\mathbb{Z}\) denoted by \( \vec{p}_1, \cdots, \vec{p}_n \) such that \( i^* (\vec{p}_1), \cdots, i^* (\vec{p}_k) \) is a \( \mathbb{Z}\)-basis of \((t^k)^*_\mathbb{Z}\) and \( i^* (\vec{p}_j) = 0 \) for \( j = k + 1, \cdots, n \). \( \vec{p}_1, \cdots, \vec{p}_n \) induces the complex coordinate \( \vec{w} = (\vec{w}_1, \cdots, \vec{w}_n) \) and symplectic coordinate \((\vec{x}, \vec{\theta})\) on the open dense subset \( \tilde{X} \) of \( X \). Moreover, the complex coordinate and symplectic coordinate are related by:

\[
\vec{w}^*_{k,t} = e^{2\pi i \frac{\partial g_{k,t}}{\partial x_j} + \sqrt{-1} \theta_j}, \quad i = 1, \cdots, n,
\]

where \( g_{k,t} \) is given by \( g_{k,t} = g_0 + t (\varphi_k \circ i_k^*) \). This implies \( \frac{\partial g_{k,t}}{\partial x_j} = \frac{\partial g_0}{\partial x_j} \) and \( \vec{w}^j_{k,t} = \vec{w}_j \), for \( j = k + 1, \cdots, n \). We therefore obtain that:

\[
P_{k,t} = \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^i_{k,t}}, i = 1, \cdots, n\} \\
= \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^i_{k,t}}, i = 1, \cdots, k\} \oplus \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^j_{k,t}}, j = k + 1, \cdots, n\} \\
= \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^j_{k,t}}, i = 1, \cdots, k\} \oplus \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^j_{k,0}}, j = k + 1, \cdots, n\} \\
= \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^j_{k,j}}, i = 1, \cdots, k\} \oplus \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{x}_j}, j = k + 1, \cdots, n\}.
\]

Recall that \( \mathcal{D}_k^\circ = \text{Ker} \, d\mu_k \otimes \mathbb{C} \) is the complexification of \( \text{Ker} \, d\mu_k \subset TX \), where \( \mu_k : X \to \Delta_k \) is the moment map associated to the Hamiltonian \( T^k \)-action on \( X \). Then by direct computation, the polarizations \( \mathcal{D}^k_{\mathcal{C}}|_{\tilde{X}} \) are given by:

\[
\mathcal{D}^k_{\mathcal{C}}|_{\tilde{X}} = \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^i_{k,0}}, i = 1, \cdots, n\} \oplus \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{x}_j}, j = k + 1, \cdots, n\}.
\]

Hence, on the open dense subset \( \tilde{X} \), we have:

\[
\mathcal{D}^k_{\mathcal{C}} \cap P_{k,t} = \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^j_{k,0}}, j = k + 1, \cdots, n\} = \mathcal{D}^k_{\mathcal{C}} \cap P_{\mathcal{J}}.
\]

It is easy to see that \( \mathcal{I}^k_{\mathcal{C}}|_{\tilde{X}} = \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^i_{k,j}}, i = 1, \cdots, n\} \). In combination with equations (3.3) and (3.2), it follows

\[
\mathcal{P}_{k,t} \cap \tilde{X} = (\mathcal{D}^k_{\mathcal{C}} \cap P_{\mathcal{J}}) \oplus \mathcal{I}^k_{\mathcal{C}} = \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{w}^j_{k,j}}, j = k + 1, \cdots, n\} \oplus \text{span}_\mathbb{C}\{\frac{\partial}{\partial \vec{x}_j}, j = 1, \cdots, k\}.
\]

On open dense subset \( \tilde{X} \), according to equation (3.2), it can be seen that \( \frac{\partial}{\partial \vec{x}_j} = \frac{\partial}{\partial y_{k,t}^j} - i \frac{\partial}{\partial \theta_j} = (G_{k,t}^{-1})_{j} \frac{\partial}{\partial \varphi_k} - i \frac{\partial}{\partial \theta_j} \), where \( y_{k,t}^j = \frac{\partial g_{k,t}}{\partial x_j} \) and \( G_{k,t}^{-1} = (\text{Hess} \, g_{k,t})^{-1} \). By the convexity of \( \varphi_k \), we
obtain: $\left(G^{-1}_{k,t}\right)_{jl} \to 0$ as $t \to \infty$, for $j = 1, \ldots, k$. This implies that

$$
\lim_{t \to \infty} \mathcal{P}_{k,t}|_{\tilde{X}} = \lim_{t \to \infty} \text{span}_C\left\{\frac{\partial}{\partial \tilde{w}_i^j} \mid i = 1, \ldots, k\right\} \oplus \text{span}_C\left\{\frac{\partial}{\partial \tilde{w}_j} \mid j = 1, \ldots, k\right\} \oplus \text{span}_C\left\{\frac{\partial}{\partial \theta^j} \mid j = 1, \ldots, k\right\}
$$

Finally, according to [3, Lemma 4.2], we have $\mathcal{D}_C^k \cap \mathcal{P}_{k,t} = \mathcal{D}_C^k \cap \mathcal{P}_J$ and $\lim_{t \to \infty} \mathcal{P}_{k,t} = \mathcal{P}_k$ on the point outside the open orbit $\tilde{X}$. To summarize the above discussion, we have:

$$
\lim_{t \to \infty} \mathcal{P}_{n,t} = \mathcal{P}_n.
$$

**Remark 3.13.** When $k = n$, the family of Kähler polarizations $\mathcal{P}_{n,t}$ coincide with the family of Kähler polarizations $\mathcal{P}_C^k$ studied in [3, Theorem 1.2]. But our result stated that not only $\Gamma(X, \lim_{t \to \infty} \mathcal{P}_{n,t}) = \Gamma(X, \mathcal{P}_n)$ but as distribution $\lim_{t \to \infty} \mathcal{P}_{k,t} = \mathcal{P}_k$.

**Remark 3.14.** The symplectic potential $g_{k,t}$ is defined by:

$$
g_{k,t} = g_0 + t(\varphi_k \circ \hat{i}_k^*): P \to \mathbb{R}.
$$

It follows $g_{k,0} = g_0$ for any $1 \leq k \leq n$. Therefore the one-parameter family of Kähler polarizations $\mathcal{P}_{k,t}$ relate Kähler polarization $\mathcal{P}_J = \mathcal{P}_{k,0}$ to the polarization $\mathcal{P}_k$.

### 3.3. Large limit of Kähler polarizations

$(\ast)$: Let $(X, \omega, J)$ be a $2n$-dimensional toric variety determined by a Delzant polytope $P$ with moment map $\mu_P: X \to P \subset (t^n)^*$. Let $\tilde{X} = \mu_P^{-1}(P)$ be the open dense orbit with $\tilde{P}$ being the interior of the polytope $P$. We consider the Hamiltonian $k$-dimensional torus action $\rho_k : T^k \to \text{Diff}(X, \omega, J)$ with moment map $\mu_k : X \to \Delta_k \subset (t^k)^*$.

Taking any $k$-dimensional subtorus $T^k$ of $T^n$, let $\hat{i}_k^*: (t^n)^* \to (t^k)^*$ be the dual map of the inclusion of the Lie algebra $i_k : t^k \to t^n$. We consider the one-parameter family of symplectic potentials $g_{k,t} = g_0 + t(\varphi_k \circ \hat{i}_k^*) : P \to \mathbb{R}$, where $\varphi_k : \to \mathbb{R}$ is a strictly convex function on $\Delta_k$. We denote the family of toric complex structures determined by the family of Kähler potential $g_{k,t}$ by $J_{k,t}$. Let $\mathcal{P}_{k,t}$ be the Kähler polarization associated to the complex structure $J_{k,t}$. We show that the Kähler polarizations $\mathcal{P}_{k,t}$ converge to mixed polarization $\mathcal{P}_k$ in the last section. Denote the quantum space of the Kähler polarization $\mathcal{P}_{k,t}$ by $\mathcal{H}_{k,t}$ and denote the $T^k$-invariant subspace of $\mathcal{H}_{k,t}$ by $\mathcal{H}_{k,t}^T$. We are going to study the relationship between $\mathcal{H}_{k,t}^T$ and $\mathcal{H}_{k,t}^T$ inspired by the results in [15]. We organised this section as follows. We first introduce how to extend the operator of covariant differentiation from smooth to distributional sections in order to define $\mathcal{H}_k$ and $\mathcal{H}_k^0$. Then we see that the
distributional section in $\mathcal{H}_k$ support on the pre-image of integer values of moment map $\mu_k$ (see Proposition 3.15). And each $T^n$-invariant holomorphic section $\sigma_{k,0}^m \in \mathcal{H}_k^0 = H^0_{\partial_J}(X, L)$ generate a distributional section $\delta_k^m \in \mathcal{H}_k^0$ (see Theorem 3.17). Finally, we show that $\mathcal{H}_{k,t}^T$ converge to $\mathcal{H}_k^0$ in the sense that the family of $L^1$-normalized $T^k$-invariant $J_{k,t}$-holomorphic sections $\frac{\sigma_{k,t}}{||\sigma_{k,t}||_{L^1}}$ converges to $\delta_k^m$, as $t$ goes to $\infty$ (see Theorem 3.18).

Let $(L, h, \nabla)$ be the prequantum line bundle on $(X, \omega, J)$ determined by the moment polytope $P \in (t^n)^*$ with curvature $F_\nabla = -i\omega$. To treat the polarization $P_k$ defined as above, we should extend the operator of covariant differentiation from smooth to distributional sections as in [3]. We consider the injection of smooth in distributional sections of $L$ determined by Liouville measure, on any open set $U \subset X$,

$$i : \Gamma(U, L) \rightarrow \Gamma_c(U, L^{-1})'$$

$$s \mapsto i(s)(\phi) = \int_U \langle s, \phi \rangle e^\omega$$

for any $\phi \in \Gamma_c(U, L^{-1})'$. We extend the operator $\nabla_\xi$ on smooth sections to an operator on distributional sections using the same notation $\nabla_\xi$ by abuse of notation such that the following diagram is commutative:

$$\begin{array}{ccc}
\Gamma(U, L) & \xrightarrow{i} & (\Gamma_c(U, L^{-1}))' \\
\downarrow \nabla_\xi & & \downarrow \nabla_\xi \\
\Gamma(U, L) & \xrightarrow{i} & (\Gamma_c^{\infty}(U, L^{-1}))'
\end{array}$$

To determine $\nabla_\xi \delta$ for a general distributional section $\delta \in \Gamma_c(U, L^{-1})'$ not of the form $i(s)$, we establish what its transpose is by integrating the operator $\nabla_\xi$ by parts. This gives, for any smooth section $s \in \Gamma(U, L)$ and smooth test section $\phi \in \Gamma_c(U, L^{-1})$,

$$(\nabla_\xi i(s))(\phi) = \int_U \langle (\nabla_\xi s), \phi \rangle e^\omega = \int_U \langle s, -(\text{div}_\xi \phi + \nabla^{-1}_\xi \phi) \rangle e^\omega.$$ 

Therefore, $\nabla_\xi \delta$ can be characterized by its transpose:

(3.6) $$(\nabla_\xi \delta)(\phi) = \delta(i' \nabla_\xi \phi),$$

for any $\phi \in \Gamma_c(U, L^{-1})$, where $i' \nabla_\xi \phi = -(\text{div}_\xi \phi + \nabla^{-1}_\xi \phi)$.

The quantum spaces $\mathcal{H}_k$ associated to polarizations $\mathcal{P}_k$ is defined by

$$\mathcal{H}_k = \{ \delta \in \Gamma_c(X, L^{-1})' \mid \nabla_\xi \delta = 0, \forall \xi \in \Gamma(X, \mathcal{P}_k) \}.$$ 

**Proposition 3.15.** Take any $\delta \in \mathcal{H}_k$, we have:

$$\text{supp } \delta \subseteq \bigcup_{q \in \Delta_k \cap \mathbb{Z}^k} \mu_k^{-1}(q).$$ 


Proof. For a loop $\gamma_b \subset T^k$ specified by a vector $b \in t^k_Z$, for any test function $\phi \in \Gamma_c(X, L^{-1})$, parallel transporting $\phi(p)$ with respect to the connection $\nabla$ around a loop $\gamma_b \cdot p \subset X$ results in multiplication of $\phi(p)$ by $e^{-2i\pi(m_{\mu}(p), b)}$, where $\langle , \rangle : (t^k)^* \times t^k \to \mathbb{R}$ is the natural pairing between $(t^k)^*$ and $t^k$. Since parallel transporting $\phi$ with respect to the connection $\nabla$ and a vector $b \in t^k_Z$ results in a section $Q_b(\phi) \in \Gamma_c(M, L^{-1})$, which is the multiplication of $\phi$ by smooth function $e^{-2i\pi(m_{\mu}, b)}$. That is,

$$Q_b(\phi) = e^{-2i\pi(m_{\mu}, b)} \phi.$$ 

This gives a linear map

$$Q_b : \Gamma_c(X, L^{-1}) \to \Gamma_c(X, L^{-1}), \phi \mapsto Q_b(\phi).$$

Its dual map

$$Q^T_b : \Gamma_c(X, L^{-1})' \to \Gamma_c(X, L^{-1})', \delta \mapsto Q^T_b(\delta)$$

is given by $(Q^T_b(\delta))(\phi) = \delta(Q_b(\phi))$, for any $\phi \in \Gamma_c(X, L^{-1})$. It implies:

$$Q^T_b(\delta) = e^{2i\pi(m_{\mu}, b)} \delta, \forall b \in t^k_Z.$$ 

But on the other hand, for $\delta \in \mathcal{H}_k$, it is unchanged by the parallel transport around any loop $\gamma_b \cdot p$ specified by a loop $\gamma_b$ in $T^\mathbb{R}$, since $\nabla_{\xi^b} \delta = 0, \forall \xi \in t^k$. Therefore we conclude that $\delta$ should be supported in the set where $m_{\mu}$ takes integral value. That is,

$$\text{supp} \delta \subset \bigcup_{q \in \Delta_k \cap \mathbb{Z}^k} \mu_k^{-1}(q).$$

Recall that for any $m \in P \cap (P')_Z^\mathbb{Z}$, there is a holomorphic section $\sigma^m$ of $L_{\text{comp}}$ defined by equation (2.23). Moreover $\{\sigma^m\}_{m \in P \cap (P')_Z^\mathbb{Z}}$ form a basis of $H^0(W_P, L_{\text{comp}})$. By theorem 2.3, $\{\sigma^m_{k,t} := (\tilde{\chi}_{g_k,t})^* \sigma_m\}_{m \in P \cap (P')_Z^\mathbb{Z}}$ form a basis of $\mathcal{H}_{k,t}$, where $\mathcal{H}_{k,t}$ is the quantum space with respect to the polarization $\mathcal{P}_{k,t}$. By the equations 2.4, 2.5, and Theorem 2.3, one has $\{\sigma^m_{k,t} := (\tilde{\chi}_{g_k,t})^* \sigma_m\}_{m \in \mu^{-1}(0) \cap (P')_Z}$ form a basis of the space $\mathcal{H}_{k,t}^T$ of the $T^k$-invariant $J_{k,t}$ holomorphic section, i.e.

$$(3.7) \quad \mathcal{H}_{k,t}^T = \{ \sigma \in \mathcal{H}_{k,t} | \sigma \text{ is } T^k\text{-invariant} \} = \text{span}\{\sigma^m_{k,t} := (\tilde{\chi}_{g_k,t})^* \sigma_m\}_{m \in \mu^{-1}(0) \cap (P')_Z}.$$ 

Taking any $k$-dimensional subtorus $T^k$ of $T^\mathbb{R}$, let $i_k^*(t^k)^* \to (t^k)^*$ be the dual map of the inclusion of the Lie algebra $i_k : t^k \to t^\mathbb{R}$. Now we fix a basis of $(t^k)^*_Z$ denoted by $\tilde{p}_1, \cdots, \tilde{p}_n$ such that $i_k^*(\tilde{p}_j) = i_k^*(\tilde{p}_j) = 0$ for $j = k+1, \cdots, n$. $\tilde{p}_1, \cdots, \tilde{p}_n$ induces the complex coordinate $\tilde{w} = (\tilde{w}_1, \cdots, \tilde{w}_n)$ and symplectic coordinate $(\tilde{x}, \tilde{\theta})$ on the open dense subset $\tilde{X}$ of $X$.

Define the volume form $\text{vol}_k$ on $\mu_k^{-1}(0) \cap \tilde{X}$ by:

$$(3.8) \quad \text{vol}_k = \frac{1}{n!} (-1)^{k^2} d\tilde{x}_{k+1} \wedge \cdots \wedge d\tilde{x}_n \wedge d\tilde{\theta}_1 \wedge \cdots \wedge d\tilde{\theta}_n.$$
Then the \( J_{k,0} \)-holomorphic section \( \sigma_{k,0}^m \) with \( m \in (i_k^*)^{-1}(0) \cap (t^n)^* \) generate a distribution section \( \delta_k^m \in \Gamma_c(X, L^{-1})' \) defined as follows.

**Definition 3.16.** Assume that \( 0 \in \Delta_k \) is regular and \((*)\), for any \( m \in (i_k^*)^{-1}(0) \cap (t^n)^* \), we define the distributional section \( \delta_k^m \in \Gamma_c(X, L^{-1})' \) associated to \( \sigma_{k,0}^m \) by:

\[
\delta_k^m(\phi) = \frac{1}{c_k^m} \int_{\mu_k^{-1}(0) \cap X} \langle \sigma_{k,0}^m(\phi) \rangle_{\mu_k^{-1}(0)} \text{vol}_k.
\]

where \( c_k^m \) is a constant defined by:

\[
c_k^m = \int_{\mu_k^{-1}(0) \cap X} \| \sigma_{k,0}^m \|_{\mu_k^{-1}(0)} \text{vol}_k.
\]

**Theorem 3.17.** Under the assumption \((*)\), we have \( \delta_k^m \in \mathcal{H}_k^0 \).

*Proof.* By the definition of \( \delta_k^m \), we have \( \delta_s^* \in \Gamma_c(X, L^{-1})' \) and \( \text{supp} \delta_k^m \subset \mu^{-1}(0) \). It remains to show that \( \nabla \xi \delta_k^m = 0 \), for any \( \xi \in \Gamma(X, \mathcal{P}_k) \). Take any test section \( \phi \in \Gamma_c(X, L^{-1}) \), according to equation \((3.6)\), we have:

\[
(\nabla \xi \delta_k^m) (\phi) = \delta_k^m (\nabla \xi \phi) = \int_{\mu_k^{-1}(0) \cap X} \langle \nabla \xi \sigma_{k,0}^m, \phi \rangle_{\mu_k^{-1}(0)} \text{vol}_k.
\]

Since \( \sigma_{k,0}^m \) is \( T^k \)-invariant section, we have \( \nabla \xi \sigma_{k,0}^m = 0 \), \( \forall \xi \in \Gamma(X, \mathcal{I}_k^c) \).

On the other hand \( \nabla \xi \sigma_{k,0}^m = 0 \), \( \forall \xi \in \Gamma(X, \mathcal{D}_k^c \cap \mathcal{P}_J) \), since \( \sigma_{k,0}^m \) is holomorphic section with respect to \( J \). This implies \( \nabla \xi \sigma_{k,0}^m = 0 \), for any \( \xi \in \Gamma(M, \mathcal{P}_k) \). We are then able to conclude that, for all \( \phi \in \Gamma_c(M, L^{-1}) \) and \( \xi \in \Gamma(X, \mathcal{P}_k) \),

\[
(\nabla \xi \delta^s) (\phi) = \int_{\mu_k^{-1}(0) \cap X} \langle \nabla \xi \sigma_{k,0}^m, \phi \rangle_{\mu_k^{-1}(0)} \text{vol}_k.
\]

Therefore we have: \( \delta_k^m \in \mathcal{H}_k^0 \). \( \square \)

Let \( \varphi_k : \Delta_k \to \mathbb{R} \) be a strictly convex function. We consider the one-parameter family of symplectic potentials \( g_{k,t} = g_0 + t(\varphi_k \circ i_k^*) : P \to \mathbb{R} \) as in the last section. We find a one-parameter family of Kähler polarizatios \( \mathcal{P}_{k,t} \) degenerating to \( \mathcal{P}_k \) in Theorem 3.12. Remember that

\[
\mathcal{H}_{k,t}^T = \{ \sigma \in \mathcal{H}_{k,t} | \sigma \text{ is } T^k \text{-invariant} \} = \text{span} \{ \sigma_{k,t}^m := (\check{x}_{g_{k,t}})^* \sigma_m \}_{m \in (\mu_k^{-1}(0) \cap (t^n)^*)}.
\]

Finally, we show that \( \mathcal{H}_{k,t}^T \) degenerate to \( \mathcal{H}_k^0 \) in the following theorem.

**Theorem 3.18.** Assume that \( 0 \in \Delta_k \) is regular and \((*)\), taking any strictly convex function \( \varphi_k \) in a neighborhood of \( \Delta_k \) and \( m \in (i_k^*)^{-1}(0) \cap (t^n)^* \), then

1. for any \( t > 0 \), \( \sigma_{k,t}^m = \sigma_{k,0}^m \) on \( \mu_k^{-1}(0) \); and
2. considering the family of \( L^1 \)-normalized \( T^k \)-invariant \( J_{k,t} \)-holomorphic sections \( \frac{\sigma_{k,t}^m}{\| \sigma_{k,t}^m \|_{L^1}} \),

under the injection of smooth in distributional section of \( L \):

\[
i : \Gamma(X, L) \to \Gamma_c(X, L^{-1})'
\]

20
we have $i(\sigma_{k,t}^m)_{|t\sigma_{k,t}^m|L_1}$ converges to $\delta_k^m$, as $t$ goes to $\infty$, in the sense that for any $\phi \in \Gamma_e(X, L^{-1})$,

$$
\lim_{t \to \infty} \int_X \langle \sigma_{k,t}^m, \phi \rangle e^\omega = \frac{1}{c_k^m} \int_{\mu_k^{-1}(0)} \langle \sigma_{k,0}^m, \phi \rangle |_{\mu_k^{-1}(0)} \, \text{vol}_k.
$$

**Proof.** The following proof is a slight modification of the argument in [3] and [17]. Recall that $i_k^* : (t^n)^* \to (t^k)^*$ is the dual Lie algebra homomorphism of $i_k : t^k \to t^n$. We observe that $i_k^*(P) = \Delta_k$ due to $\mu_k = i_k^* \circ \mu_p$. Now we take a basis of $(t^n)^*_Z$ denoted by $\tilde{p}_1, \ldots, \tilde{p}_n$ such that $i_k^*(\tilde{p}_1), \ldots, i_k^*(\tilde{p}_k)$ is the a $Z$ basis of $(t^k)_Z$ and $i_k^*(\tilde{p}_j) = 0$ for $j = k + 1, \ldots, n$. $\tilde{p}_1, \ldots, \tilde{p}_n$ induces the complex coordinate $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$ and symplectic coordinate $(\tilde{x}, \tilde{\theta})$ on the open dense subset $\tilde{X}$ of $X$.

If we write $m = \sum_{i=1}^n \tilde{m}_i \tilde{p}_i \in (t^n)^*_Z$, due to (2.7) and Theorem 2.3 we have:

$$
(\sigma_{k,t}^m)|_{\tilde{X}} = (\tilde{x}_k, t \sigma^m)|_{\tilde{X}} = (\tilde{x}_k, t \sum_{i=1}^n (\tilde{w}_i)^{\tilde{m}_i})|_{\tilde{X}}
$$

$$
= \left\{ \prod_{i=1}^n e^{\tilde{m}_i \frac{\partial}{\partial x_i}} \right\} e^{(g_0, -\sum_{i=1}^n \tilde{m}_i \frac{\partial}{\partial x_i})} \frac{1}{1} = e^{(g_0, -\sum_{i=1}^n (\tilde{x}_i - \tilde{m}_i) \frac{\partial}{\partial x_i})} \frac{1}{1} = e^{(g_0, -m \frac{\partial}{\partial x_i})} \frac{1}{1} = e^{-t \alpha_k(\tilde{x})}(\sigma_{k,0}^m)|_{\tilde{X}} = e^{-t \alpha_k(\tilde{x})} f_{\tilde{m},0}^m \frac{1}{1},
$$

where

$$
\sigma_{k,0}^m = \tilde{x}_k^* \sigma^m = e^{(g_0, -\sum_{i=1}^n (\tilde{x}_i - \tilde{m}_i) \frac{\partial}{\partial x_i})} \frac{1}{1},
$$

$$
\sigma_{k,0}^m = e^{(g_0, -\sum_{i=1}^n (\tilde{x}_i - \tilde{m}_i) \frac{\partial}{\partial x_i})} \frac{1}{1},
$$

and

$$
\alpha_k(\tilde{x}) = \sum_{i=1}^n (\tilde{x}_i - \tilde{m}_i) \frac{\partial}{\partial x_i} \left( \tilde{x}_k \frac{\partial}{\partial \tilde{x}_i} \right) - (\tilde{x}_k \frac{\partial}{\partial \tilde{x}_i}) (\tilde{x}_k).
$$

(1) fixing the basis $\tilde{p}_1, \ldots, \tilde{p}_n$ of $(t^n)^*_Z$ as above, we have: $i_k^*(\tilde{p}_j) = \tilde{x}_j$ for $j = 1, \ldots, k$ and $i_k^*(\tilde{p}_j) = 0$ for $j = k + 1, \ldots, n$. Let $\tilde{\alpha}_m : \Delta_k \to \mathbb{R}$ be defined by

$$
\tilde{\alpha}_m(\tilde{x}_1, \ldots, \tilde{x}_k) = \sum_{i=1}^k (\tilde{x}_i - \tilde{m}_i) \frac{\partial}{\partial \tilde{x}_i} (\tilde{x}_1, \ldots, \tilde{x}_k) - \varphi_k(\tilde{x}_1, \ldots, \tilde{x}_k).
$$

It follows that: $\alpha_m(\tilde{x}) = \tilde{\alpha}_m(\tilde{m}(\tilde{x})) = \alpha_m(\tilde{x}_1, \ldots, \tilde{x}_k)$. In particular, for $m \in (i_k^*)^{-1}(0) \cap \mathbb{Z}^n$ and $(\tilde{x}, \tilde{\theta}) \in \tilde{X} \cap \mu_k^{-1}(0)$, we have $\alpha_m(\tilde{x}) = 0$. This implies that:

$$
(\sigma_{k,t}^m)|_{\mu_k^{-1}(0) \cap \tilde{X}} = (\sigma_{k,0}^m)|_{\mu_k^{-1}(0) \cap \tilde{X}}, \forall t > 0.
$$
Note that $\mu_k^{-1}(0) \cap \hat{X} \subset \mu_k^{-1}(0)$ is a open dense subset of $\mu_k^{-1}(0)$. It follows

\[
(\sigma_{k,t}^m)|_{\mu_k^{-1}(0)} = (\sigma_{k,0}^m)|_{\mu_k^{-1}(0)}, \quad \forall t > 0.
\]

(2) for any test section $\phi \in \Gamma_c(X, L^{-1})$, we have:

\[
\int_X \langle \sigma_{k,t}^m, \phi \rangle e^\omega = \int_X \langle \sigma_{k,t}^m, \phi \rangle e^\omega
\]

\[
= \frac{1}{||\sigma_{k,t}||_{L_1}} \int_X \langle \sigma_{k,t}^m, \phi \rangle e^\omega
\]

\[
= \frac{||e^{-t\omega_m}||_{L_1}}{||\sigma_{k,t}^m||_{L_1}} \int_X \langle e^{-t\omega_m} \sigma_{k,t}^m, \phi \rangle e^\omega
\]

\[
= \frac{||e^{-t\omega_m}||_{L_1}}{||\sigma_{k,t}^m||_{L_1}} \int_{\Delta_k} \langle e^{-t\omega_m} \sigma_{k,t}^m \mu_k^{-1}(\hat{x}_1, \ldots, \hat{x}_k) \cap X, \phi \rangle e^\omega
\]

\[
= \frac{||e^{-t\omega_m}||_{L_1}}{||\sigma_{k,t}^m||_{L_1}} \int_{\Delta_k} \langle e^{-t\omega_m} \sigma_{k,t}^m \mu_k^{-1}(\hat{x}_1, \ldots, \hat{x}_k) \cap X, \phi \rangle e^\omega
\]

\[
Q_k^\diamond \hat{\phi}(\hat{x}_1, \ldots, \hat{x}_k) = \frac{\int_{\mu_k^{-1}(\hat{x}_1, \ldots, \hat{x}_k) \cap \hat{X}} \hat{\phi}(\hat{x}_1, \ldots, \hat{x}_k) e^\omega}{d\hat{x}_1 \wedge \cdots \wedge d\hat{x}_k}.
\]

Similarly, we have:

\[
\frac{||\sigma_{k,t}^m||_{L_1}}{||e^{-t\omega_m}||_{L_1}} = \frac{1}{||e^{-t\omega_m}||_{L_1}} \int_X \langle \sigma_{k,t}^m, e^\omega \rangle
\]

\[
= \frac{1}{||e^{-t\omega_m}||_{L_1}} \int_X \langle e^{-t\omega_m} \sigma_{k,t}^m, e^\omega \rangle
\]

\[
= \int_X \langle e^{-t\omega_m} \sigma_{k,t}^m, \mu_k^{-1}(\hat{x}_1, \ldots, \hat{x}_k) \cap X, e^\omega \rangle
\]

\[
= \int_{\Delta_k} \langle e^{-t\omega_m} \sigma_{k,t}^m \mu_k^{-1}(\hat{x}_1, \ldots, \hat{x}_k) \cap X, \phi \rangle e^\omega
\]

\[
Q_k^m(\hat{x}_1, \ldots, \hat{x}_k) = \frac{\int_{\mu_k^{-1}(\hat{x}_1, \ldots, \hat{x}_k) \cap \hat{X}} \phi^m e^\omega}{d\hat{x}_1 \wedge \cdots \wedge d\hat{x}_k}.
\]

Here $Q_k^m(\hat{x}_1, \ldots, \hat{x}_k)$ is determined by:

By Lemma 5.1 and direct computations, we have:
\[
\lim_{t \to \infty} \frac{||\sigma_{k,t}^m||}{L_1} = \lim_{t \to \infty} \frac{e^{-t\alpha_m}}{\Delta_k} \int_{Q_k} Q_k^m(\bar{x}_1, \cdots, \bar{x}_k) d\bar{x}_1 \land \cdots \land d\bar{x}_k = Q_k^m(0, \cdots, 0)
\]

\[
\int_{\mu^{-1}_k(0) \cap X} \frac{S_k^m c^m \omega}{d\bar{x}_1 \land \cdots \land d\bar{x}_k} = \int_{\mu^{-1}_k(0) \cap X} S_k^m \omega \mu_k = \frac{c_k^m}{\mu_k(0)} \in \mathbb{R},
\]

where

(3.9)
\[
c_k^m = \int_{\mu^{-1}_k(0) \cap X} ||\sigma_{k,0}^m|| \mu^{-1}(0) || \omega \mu_k.
\]

It follows that:

\[
\lim_{t \to \infty} \int_{\Delta_k} \frac{e^{-t\alpha_m}}{L_1} \int_{\mu^{-1}_k(1) \cap X} S_k^m \omega \mu_k = \int_{\Delta_k} e^{-t\alpha_m} \int_{\mu^{-1}_k(1) \cap X} S_k^m \omega \mu_k
\]

Finally, we obtain:

\[
\lim_{t \to \infty} \int_X \frac{\langle \sigma_{k,t}^m, \phi \rangle}{L_1} e^\omega = \lim_{t \to \infty} \int_X \frac{\langle \sigma_{k,t}^m, \phi \rangle}{L_1} e^\omega = \lim_{t \to \infty} \frac{1}{\|\sigma_{k,t}^m\| L_1} \int_X \langle \sigma_{k,t}^m, \phi \rangle e^\omega
\]

Denote by \((t^k)^*_{Z,reg}\) the set of regular integral value of moment map \(\mu_k\), i.e.

\[
(t^k)^*_{Z,reg} = \{ \lambda \in (t^k)^* | \lambda \text{ is the regular value of } \mu_k \}.
\]

Let \(\lambda \in (t^k)^*_{Z,reg}\) and \(m \in (t^k)^* \cap (t^m)^*\), we define the distributional section by

\[
\delta_{k,\lambda}^m(\phi) = \frac{1}{c_k^m} \int_{\mu^{-1}_k(\lambda) \cap X} \langle \sigma_{k,0}^m, \phi \rangle \mu^{-1}_k(\lambda) \omega \mu_k.
\]
Similar to the proof of (2) of Theorem 3.18, we obtain the following corollary.

**Corollary 3.19.** The family of $L^1$-normalized weight-$\lambda$ $J_{k,t}$-holomorphic sections $\frac{\sigma_{k,t}^m}{||\sigma_{k,t}^m||_1}$, under the injection of smooth in distributional section of $L$: $i : \Gamma(X, L) \to \Gamma_c(X, L^{-1})'$ converges to $\delta_{k,\lambda}^m$, as $t$ goes to $\infty$.

4. Appendix

In this appendix, we review the following two lemmata used in the proof of Theorem 3.12 and Theorem 3.18 respectively.

**Lemma 4.1** (Lemma 4.2, [3]). At a fixed point $v \in P$, we consider two open charts around $v$, $w_j, w_j' : V_v \to \mathbb{C}^n$, which are specified by two different symplectic potentials $g$ and $g'$ respectively. Whenever $w_j = 0$, also $w_j' = 0$, and at there points

$$\text{span}_\mathbb{C}\{\frac{\partial}{\partial w_j}\} = \text{span}_\mathbb{C}\{\frac{\partial}{\partial w_j'}\}, j = 1, \cdots, n.$$ 

*Proof.* According to the description of the charts, $w_j = w_j'f$, where $f$ is a real-valued function, smooth in $P$, and that factorizes through $\tilde{\mu} : X_P \to P$, that is through $|w_1'|, \cdots, |w_n'|$. By direct calculation,

$$dw_j = \sum_i [(\delta_{i,j} + w_j'\frac{\partial f}{\partial w_i'})dw_i' + w_j'\frac{\partial f}{\partial w_i'}d\bar{w}_i'].$$ 

We therefore obtain at point with $w_j' = 0$,

$$\text{span}_\mathbb{C}\{\frac{\partial}{\partial w_j}\} = \text{span}_\mathbb{C}\{\frac{\partial}{\partial w_j'}\}.$$ 

\[\square\]

**Lemma 4.2.** [3, Lemma 3.7.] For any $\psi$ strictly convex function in a neighbourhood of the moment polytope $P$ and any $m \in P \cap \mathbb{Z}^n$, the function:

$$f_m : P \to \mathbb{R}$$

$$x \mapsto t (x - m)\frac{\partial \psi}{\partial x} - \psi(x)$$

has a unique minimum at $x = m$ and

$$\lim_{t \to \infty} \frac{e^{-tf_m}}{||e^{-tf_m}||_1} \to \delta(x - m),$$

in the sense of distributions.
REFERENCES

1. M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math. 9 (1998) 641-651.
2. M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, in "Symplectic and Contact Topology: Interactions and Perspectives" (eds. Y. Eliashberg, B. Khesin and F. Lalonde), Fields Institute Communications 35, Amer. Math. Soc., 2003.
3. T. Baier, C. Florentino, J. M. Mourão and J. P. Nunes, *Toric Kähler metrics seen from infinity, quantization and compact tropical amoebas*, J. Diff. Geom., 89 (3), 411-454, 2011.
4. T. Baier, J. Mourão and J. P. Nunes, *Quantization of abelian varieties: distributional sections and the transition from Kähler to real polarizations*, J. Funct. Anal., 258 (2010) 3388-3412.
5. K. Chan and Y. H. Suen *Geometric quantization via SYZ transforms*, Adv. Theor. Math. Phys., Vol. 24, No.1 (2020), pp. 25-66.
6. D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Amer. Math. Soc., 2011.
7. V. I. Danilov, *The geometry of toric varieties*, Russ. Math. Surveys, 33 (1978) 97-154.
8. C. Florentino, P. Matias, J. M. Mourão and J. P. Nunes, *Geometric quantization, complex structures and the coherent state transform*, J. Funct. Anal., 221 (2005) 303-322.
9. C. Florentino, P. Matias, J. M. Mourão and J. P. Nunes, *On the BKS pairing for Kähler quantizations of the cotangent bundle of a Lie group*, J. Funct. Anal., 234 (2006) 180-198.
10. V. Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n- spaces*, Progress in Math., 122, Birkhäuser, 1994.
11. V. Guillemin, *Kähler structures on toric varieties*, J. Diff. Geom., 40(1994), 285-309.
12. V. Guillemin, V. Ginzburg and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, 98, Amer. Math. Soc., Providence, RI, 2002.
13. V. Guillemin and S. Sternberg, *Geometric Quantization and Multiplicities of Group Representations*, Inventiones mathematicae, 67.3 (1982): 515-538.
14. B. C. Hall, *Geometric quantization and the generalized Segal-Bargmann transform for Lie group of compact type*, Comm. Math. Phys. 226 (2002) 233-268.
15. M. D. Hamilton and H. Konno, *Convergence of Kähler to real polarization on flag manifolds via toric degenerations*, J. Sympl. Geo., 12 (3), 473-509, 2014.
16. N. J. Hitchin, *Flat connections and geometric quantization*, Comm. Math. Phys., 131 (1990) 347-380.
17. W. D. Kirwin, J. M. Mourão and J. P. Nunes, *Complex symplectomorphisms and pseudo-Kähler islands in the quantization of toric manifolds*, Math. Ann., (2016) 364 (1-2): 1-28.
18. W. D. Kirwin and S. Wu, *geometric Quantization, Parallel Transport and the Fourier Transform*, Comm. Math. Phys., 266 (2006) 577-594.
23. N.C. Leung and D. Wang *Geodesic rays in space of Kähler metrics with T-symmetry*, in preparation (2022).

24. N.C. Leung and D. Wang *Limit of geometric quantizations on Kähler manifolds with T-symmetry*, in preparation (2022).

25. T. Oda, *Convex Bodies and Algebraic geometry: An Introduction to the Theory of Toric Varieties*, Springer-Verlag, 1988.

26. N. M. J. Woodhouse, *Geometric quantization*, Second Edition, Clarendon Press, Oxford, 1991.

**The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong**

*Email address: dwang@math.cuhk.edu.hk*