Characteristic functions and Hamilton-Cayley theorem for left eigenvalues of quaternionic matrices

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Abstract

We introduce the notion of characteristic function of a quaternionic matrix, whose roots are the left eigenvalues. We prove that for all $2 \times 2$ matrices and for $3 \times 3$ matrices having some zero entry outside the diagonal there is a characteristic function which is a polynomial. For the other $3 \times 3$ matrices the characteristic function is a rational function with one point of discontinuity. We prove that Hamilton-Cayley theorem holds in all cases.

Keywords: quaternion, left eigenvalue, characteristic function, Hamilton-Cayley theorem

MSC: 15A33, 15A18

1 Introduction

Very little is known about left eigenvalues of $n \times n$ quaternionic matrices. F. Zhang’s papers [9, 10] review their main properties as well as some pathological examples, see also [3]. For $n = 2$ the explicit computation of the left spectrum is due to L. Huang and W. So [4], while the authors studied the symplectic group in [5, 6].

In 1985, R. M. W. Wood [8] proved, by using homotopic methods, that every quaternionic matrix has at least one left eigenvalue. At the end of his paper, Wood notes that “in the $2 \times 2$ case of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ there is a

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partially defined determinant $b - ac^{-1}d$ and partially defined characteristic equation

$$\lambda c^{-1} - \lambda c^{-1}d - ac^{-1} - b + [ac^{-1}d] = 0 \quad (1)$$

which reduces the eigenvalue problem to the fundamental theorem of algebra. The difficulties start with $3 \times 3$ matrices.

In this paper we introduce a definition of characteristic function for a quaternionic matrix, which generalizes the usual characteristic polynomial in the real and complex setting. In particular, its roots are the left eigenvalues. Explicitly, we say that $\mu: H \to H$ is a characteristic function of the matrix $A \in M(n, H)$ if, up to a constant, its norm verifies that $|\mu(\lambda)| = Sdet(A - \lambda I)$ for all $\lambda \in H$, where $Sdet: M(n, H) \to [0, +\infty)$ is Study’s determinant. As we shall see, this definition fits naturally with Equation (1), as well as with the method proposed by W. So in [7] to compute the left eigenvalues when $n = 3$.

Then we discuss Hamilton-Cayley theorem in this setting. Our main result is as follows.

**Theorem A.** For any quaternionic matrix $A \in M(n, H)$, $n \leq 3$, there exists a characteristic function $\mu$ whose extension to a map $\mu: M(n, H) \to M(n, H)$ verifies Hamilton-Cayley, that is $\mu(A) = 0$.

For $n = 2$, a characteristic function like that in (1) is a polynomial $\mu(\lambda)$ for which it is easy to check that $\mu(A) = 0$. It follows that

$$Ac^{-1}A = Ac^{-1}d + ac^{-1}A + (b - ac^{-1}d)I,$$

which generalizes the well known formula $A^2 = (\text{tr}A)A - (\text{det}A)I$ in the commutative setting. When $n = 3$ and the matrix has some zero entry outside the diagonal, we shall find a polynomial characteristic function that verifies Hamilton-Cayley. Otherwise, there is a characteristic function which is, outside a point of discontinuity, a rational function. We are able to extend it to a map $\mu: M(n, H) \to M(n, H)$ and we prove by brute force that Hamilton-Cayley is verified too.

At the end of the paper we discuss another possible definition of characteristic function.

## 2 Preliminaries

We consider the quaternionic space $H^n$ as a right vector space over $H$. Two square matrices $A, B \in M(n, H)$ are similar if $B = PAP^{-1}$ for some invertible square matrix $P$. 
If $A$ is a quaternionic $n \times n$ matrix, let us write $A = X + jY$, with $X,Y \in \mathcal{M}(n, \mathbb{C})$, and let
\[
c(A) = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \in \mathcal{M}(2n, \mathbb{C})
\]
be its complex form. We have $c(A \cdot B) = c(A) \cdot c(B)$, $c(A + B) = c(A) + c(B)$ and $c(tA) = tc(A)$ if $t \in \mathbb{R}$. In particular, $A$ is invertible if and only if $c(A)$ is invertible. Moreover, $\det c(A) \geq 0$ is a nonnegative real number, so we can define the Study’s determinant of $A$ as
\[
\text{Sdet}(A) = (\det c(A))^{1/2} \geq 0. \tag{2}
\]
For complex matrices, Sdet equals the norm of the complex determinant, see \[1, 2\] for a general discussion of quaternionic determinants. The following properties are immediate:

1. $\text{Sdet}(A \cdot B) = \text{Sdet}(A) \cdot \text{Sdet}(B)$;
2. $A$ is invertible if and only if $\text{Sdet}(A) > 0$;
3. if $A, B$ are similar matrices then $\text{Sdet}(A) = \text{Sdet}(B)$.

We also need the following result.

Lemma 2.1. For a matrix with boxes $M, N$ of size $m \times m$ and $n \times n$ respectively we have
\[
\text{Sdet} \begin{bmatrix} 0 & M \\ N & * \end{bmatrix} = \text{Sdet}(M) \cdot \text{Sdet}(N).
\]

It follows that $\text{Sdet}(A) = |q_1 \cdots q_n|$ when $A$ is a triangular matrix, with $q_1, \ldots, q_n$ being the elements of the diagonal.

Sometimes we shall permute two columns or rows of the matrix $A$. Or we shall add to a column a right linear combination of the columns. This will not affect the value of the determinant because the matrices of the type
\[
P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } P = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}
\]
verify $\text{Sdet}(P) = 1$.

Remark 2.2. Up to the exponent $1/2$ in (2), this is the same determinant that the one in Theorem 8.1 of \[9\] that we shall refer to later in Sect. 7. The exponent is normalized in order to have $\text{Sdet}(A) = |q_1 \cdots q_n|$ for a diagonal matrix $A = \text{diag}(q_1, \ldots, q_n)$. 

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3 Left eigenvalues and characteristic functions

A quaternion \( \lambda \in \mathbb{H} \) is said to be a left eigenvalue of the matrix \( A \in \mathcal{M}(n, \mathbb{H}) \) if \( Av = \lambda v \) for some vector \( v \in \mathbb{H}^n \), \( v \neq 0 \). Equivalently, the matrix \( A - \lambda I \) is not invertible, that is \( \text{Sdet}(A - \lambda I) = 0 \), where Sdet is Study’s determinant defined in Section 2.

**Definition 3.1.** A map \( \mu : \mathbb{H} \rightarrow \mathbb{H} \) is a characteristic function of the matrix \( A \in \mathcal{M}(n, \mathbb{H}) \) if, up to a constant, \( |\mu(\lambda)| = \text{Sdet}(A - \lambda I) \) for all \( \lambda \in \mathbb{H} \).

Notice that \( \lambda \) is a left eigenvalue of \( A \) if and only if \( \mu(\lambda) = 0 \).

**Remark 3.2.** It is well known that the left spectrum is not invariant under similarity. However, if \( P \) is a real invertible matrix then \( \text{Sdet}(PAP^{-1} - \lambda I) = \text{Sdet}(A - \lambda I) \), so \( A \) and \( PAP^{-1} \) have the same characteristic functions.

**Example 3.3.** Diagonal and triangular matrices.

If \( A = \text{diag}(q_1, \ldots, q_n) \) then \( \mu(A) = (q_n - \lambda) \cdots (q_1 - \lambda) \) is a characteristic function. Analogously for triangular matrices.

**Example 3.4.** \( 2 \times 2 \) matrices.

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}(2, \mathbb{H}) \). If \( b = 0 \) then \( \text{Sdet}(A) = |da| \) and the map \( \mu(\lambda) = (d - \lambda)(a - \lambda) \) is a characteristic function. If \( b \neq 0 \) we have

\[
A \sim \begin{bmatrix} 0 & b \\ c - \frac{1}{db}a & d \end{bmatrix}
\]

so

\[
\text{Sdet}(A) = |b| |c - \frac{1}{db}a|.
\]

Consequently, we consider the characteristic function

\[
\mu(\lambda) = c - (d - \lambda)b^{-1}(a - \lambda).
\]

(3)

Obviously, the characteristic function of a matrix is not unique. For instance, by permuting rows and columns we can obtain \( \mu(\lambda) = b - (a - \lambda)c^{-1}(d - \lambda) \) which is Wood’s function in Equation (1) (there is a misprint in the original article). However, as we shall see in Section 4 it is preferable to take minors starting from the top right corner as we do.
4 Characteristic function of $3 \times 3$ matrices

Now let $A = \begin{bmatrix} a & b & c \\ f & g & h \\ p & q & r \end{bmatrix}$ be a $3 \times 3$ quaternionic matrix. The computation of $\text{Sdet}(A)$ can be done as follows (a similar algorithm is valid for any $n > 3$).

4.1 Case $n = 3, c \neq 0$

First we consider the generic case when $c \neq 0$. In this case we can create zeroes in the first row,

$$A \sim \begin{bmatrix} 0 & 0 & c \\ f - hc^{-1}a & g - hc^{-1}b & h \\ p - rc^{-1}a & q - rc^{-1}b & r \end{bmatrix}.$$ 

By Lemma 2.1 and the $2 \times 2$ case, it follows:

**Proposition 4.1.** If $c \neq 0$, then $\text{Sdet}(A)$ is given:

1. when $g - hc^{-1}b \neq 0$, by

$$|c| \cdot |g - hc^{-1}b| \cdot |p - rc^{-1}a - (q - rc^{-1}b)(g - hc^{-1}b)^{-1}(f - hc^{-1}a)|;$$

2. when $g - hc^{-1}b = 0$, by

$$|c| \cdot |q - rc^{-1}b| \cdot |f - hc^{-1}a|.$$

**Corollary 4.2.** Let us call $\lambda_0 = g - hc^{-1}b$ the pole of $A$. Then

$$\text{Sdet}(A - \lambda_0 I) = |c| \cdot |q - (r - \lambda_0)c^{-1}b| \cdot |f - hc^{-1}(a - \lambda_0)|.$$ (4)

By applying Prop. 4.1 and Cor. 4.2 to $A - \lambda I$ we find the following characteristic function of $A$.

**Definition 4.3.** When $c \neq 0$, a characteristic function for the $3 \times 3$ matrix $A$ can be defined as follows:

1. if $\lambda_0 = g - hc^{-1}b$ is the pole of $A$,

$$\mu(\lambda_0) = (q - (r - \lambda_0)c^{-1}b) \left( f - hc^{-1}(a - \lambda_0) \right);$$
2. otherwise,

\[ \mu(\lambda) = (\lambda_0 - \lambda) \left[ \left( p - (r - \lambda)c^{-1}(a - \lambda) \right) - \left( q - (r - \lambda)c^{-1}b \right) (\lambda_0 - \lambda)^{-1} \left( f - hc^{-1}(a - \lambda) \right) \right]. \]

**Remark 4.4.** In [7], W. So proved that the left eigenvalues of a 3×3 matrix can be computed as roots of certain polynomials of degree ≤ 3. Even though our computation is different from his, we obtain that the function in Def. 4.3 is exactly So’s formula in [7, p. 563]. This is why we have chosen to compute determinants starting from the top right corner.

### 4.2 Case \( n = 3, c = 0 \)

We briefly review what happens when \( c = 0 \). First, if both \( b, h = 0 \) we have a triangular matrix, then we can take

\[ \mu(\lambda) = (r - \lambda)(g - \lambda)(a - \lambda). \]  \hspace{1cm} (5)

If \( b = 0 \) but \( h \neq 0 \) we can reduce to the 2×2 case by Lemma 2.1, so we take

\[ \mu(\lambda) = \left( q - (r - \lambda)h^{-1}(g - \lambda) \right) (a - \lambda). \]  \hspace{1cm} (6)

Finally, if \( b \neq 0 \) we can (see the proof of Theorem 6.3) create a zero in the left top corner of \( A - \lambda I \) and then permute the second and last column, in order to reduce the matrix \((A - \lambda I)P\) to the 2×2 case. Alternatively, we can simply permute the second and last column and the second and last row of \( A \), in order to obtain a matrix \( PAP^{-1} \) with the same characteristic function, to which Subsection 4.1 applies. Notice however that with the latter method we obtain a rational function, not a polynomial.

### 5 Continuity

The following example shows that the characteristic function \( \mu \) in Definition 4.3 may not be continuous, even if its norm \(|\mu|\) is a continuous map.

Let

\[ A = \begin{bmatrix} 0 & i & 1 \\ 3i - k & 0 & 1 \\ k & -1 + j + k & 0 \end{bmatrix}. \]

Its pole (see Cor. 4.2) is \( \lambda_0 = -i \) and

\[ \mu(\lambda_0) = (j + k)(2i - k) = 1 - i + 2j - 2k. \]
However, for \( \lambda \neq \lambda_0 \) we have
\[
\mu(\lambda) = (-i - \lambda) \left( k - \lambda^2 - (-1 + j + k + \lambda i)(-i - \lambda)^{-1}(3i - k + \lambda) \right),
\]
and by taking \( \lambda = -i + \varepsilon j, \varepsilon \in \mathbb{R}, \) with \( \varepsilon \to 0, \) we obtain
\[
\lim_{\varepsilon \to 0} \mu(-i + \varepsilon j) = 1 + i + 2j + 2k \neq \mu(\lambda_0).
\]
In fact, the limit
\[
\lim_{\varepsilon \to 0} \mu(-i + \varepsilon q) = -q(j + k)q^{-1}(2i - k)
\]
depends on \( q, \) so \( \lim_{\lambda \to \lambda_0} \mu(\lambda) \) does not exist.

It is an open question whether it is always possible to find a continuous characteristic function.

6 Hamilton-Cayley theorem

We now discuss Hamilton-Cayley theorem.

6.1 Case \( n = 2 \)

Theorem 6.1. Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be a \( 2 \times 2 \) quaternionic matrix. Let \( \mu(\lambda) = c - (d - \lambda)b^{-1}(a - \lambda) \) be the characteristic function defined in (3). Then \( \mu(A) = 0. \)

Proof. We have
\[
\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} - \begin{bmatrix} d - a & -b \\ -c & 0 \end{bmatrix} \begin{bmatrix} b^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & -b \\ -c & a - d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Corollary 6.2. \( Ab^{-1}A = Ab^{-1}a + db^{-1}A + (c - db^{-1}a)I. \)

6.2 Case \( n = 3, \ c = 0 \)

For \( n = 3, \) a direct computation will show that Hamilton-Cayley theorem is true when \( c = 0 \) (see Section 4).

Proposition 6.3. Let \( A = \begin{bmatrix} a & b & 0 \\ f & g & h \\ p & q & r \end{bmatrix} \). Let \( \mu(\lambda) \) be the characteristic function defined in Subsection 4.3. Then \( \mu(A) = 0. \)
Proof. If $b, h = 0$ we take formula \((5)\), so $\mu(A)$ equals
\[
\begin{bmatrix}
  r - a & 0 & 0 \\
  -f & r - g & 0 \\
  -p & -q & 0 \\
\end{bmatrix}
\begin{bmatrix}
  g - a & 0 & 0 \\
  -f & 0 & 0 \\
  -p & -q & g - r \\
\end{bmatrix}
\begin{bmatrix}
  0 & 0 & 0 \\
  -f & a - g & 0 \\
  -p & -q & a - r \\
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
\end{bmatrix}.
\]

If $b = 0, h \neq 0$ we take formula \((6)\), then we check
\[
\begin{bmatrix}
  r - a & 0 & 0 \\
  -f & r - g & -h \\
  -p & -q & 0 \\
\end{bmatrix}
\begin{bmatrix}
  g - a & 0 & 0 \\
  -f & 0 & 0 \\
  -p & -q & (g - r) \\
\end{bmatrix}
\begin{bmatrix}
  0 & 0 & 0 \\
  -f & a - g & -h \\
  -p & -q & a - r \\
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 0 \\
  q & -f & a - g & -h \\
  -p & -q & a - r \\
\end{bmatrix},
\]
that is, $(rI - A)h^{-1}(gI - A)(aI - A) = q(aI - A)$, hence $\mu(A) = 0$.

If $b \neq 0$, we have
\[
\text{Sdet}(A - \lambda I) = \text{Sdet}
\begin{bmatrix}
  0 & 0 & b \\
  f - (g - \lambda)b^{-1}(a - \lambda) & h & g - \lambda \\
  p - qb^{-1}(a - \lambda) & r - \lambda & q \\
\end{bmatrix},
\]
so we are in the $2 \times 2$ situation (see Lemma 2.1). First, assume $h = 0$ and let us take $\mu(\lambda) = (r - \lambda)(f - (g - \lambda)b^{-1}(a - \lambda))$. We check
\[
\begin{bmatrix}
  r - a & -b & 0 \\
  -f & r - g & 0 \\
  -p & -q & 0 \\
\end{bmatrix}
\begin{bmatrix}
  g - a & -b & 0 \\
  -f & 0 & 0 \\
  -p & -q & g - r \\
\end{bmatrix}
\begin{bmatrix}
  0 & -b & 0 \\
  -f & a - g & 0 \\
  -p & -q & a - r \\
\end{bmatrix} =
\begin{bmatrix}
  r - a & -b & 0 \\
  -f & r - g & 0 \\
  -p & -q & 0 \\
\end{bmatrix}
\begin{bmatrix}
  0 & 0 & b \\
  f & h & 0 \\
  q & bh^{-1} & 0 \\
\end{bmatrix},
\]
that is $(rI - A)(gI - A)b^{-1}(aI - A) = (rI - A)f$, hence $\mu(A) = 0$.

On the other hand, if $h \neq 0$ we take
\[
\mu(\lambda) = p - qb^{-1}(a - \lambda) - (r - \lambda)h^{-1}(f - (g - \lambda)b^{-1}(a - \lambda)).
\]

Then we compute
\[
\begin{bmatrix}
  p - (r - a)h^{-1}f & q + bh^{-1}f & 0 \\
  qb^{-1}f + fh^{-1}f & p - qb^{-1}(a - g) - (r - g)h^{-1}f & qb^{-1}h + f \\
  qb^{-1}p + ph^{-1}f & qb^{-1}q + qh^{-1}f & p - qb^{-1}(a - r) \\
\end{bmatrix}
\]

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and we check it equals
\[
- \begin{bmatrix}
  r - a & -b & 0 \\
  -f & r - g & -h \\
  -p & -q & 0
\end{bmatrix}
\begin{bmatrix}
  g - a & -b & 0 \\
  -f & a - g & -h \\
  -p & -q & a - r
\end{bmatrix}
\begin{bmatrix}
  0 & -b & 0 \\
  -f & a - g & -h \\
  -p & -q & a - r
\end{bmatrix}
\begin{bmatrix}
  0 & -b & 0 \\
  -f & a - g & -h \\
  -p & -q & a - r
\end{bmatrix}
= -(rI - A)R^{-1}(gI - A)b^{-1}(aI - A),
\]
hence \( \mu(A) = 0 \).

**Lemma 6.4.** Let \( A \) be a quaternionic matrix such that \( \mu(\lambda) = 0 \) for some quaternionic polynomial \( \mu(\lambda) \). Let \( B = PAP^{-1} \) be a similar matrix, with \( P \) a real matrix. Then \( \mu(B) = 0 \).

**Proof.** Let \( \mu(\lambda) = q_1\lambda q_2\lambda \cdots q_k\lambda q_{k+1} \) be a monomial. Then \( \mu(B) = P\mu(A)P^{-1} \).

Notice that the same result is true when \( \mu(\lambda) \) is a rational function.

By permuting rows and columns (see Remark 3.2) we deduce:

**Corollary 6.5.** Let \( A \) be a 3 \( \times \) 3 quaternionic matrix with some zero entry outside the diagonal. Then there exists a polynomial characteristic function \( \mu \) such that \( \mu(A) = 0 \).

**Example 6.6.** Let us consider the matrix
\[
A = \begin{bmatrix}
  1 & i & i \\
  i & j & k \\
  0 & -1 & j
\end{bmatrix}
\]
whose characteristic function is given by formula (7), that is
\[
\mu(\lambda) = i + i(j - \lambda) + (1 - \lambda)i(k + (j - \lambda)^2).
\]
Then the following equation holds:
\[
-AiA^2 + AiAj + AkA + iA^2 - iAj + A(i + j) - (i + k)A + (k - j)I = 0.
\]

**6.3 Case \( n = 3, c \neq 0 \)**

When \( c \neq 0 \), the characteristic function of the matrix \( A \) is a rational function with a pole. We shall extend it to a map in the space of matrices in the following natural way.

Let \( \lambda_0 = g - hc^{-1}b \) be the pole of \( A \). Let
\[
f_0 = f - hc^{-1}(a - \lambda_0),
\]
\[ q_0 = q - (r - \lambda_0)c^{-1}b. \]

**Lemma 6.7.** The matrix \( \lambda_0 I - A \) is invertible if and only if \( f_0, q_0 \neq 0 \).

**Proof.** By Corollary [1.2] \( \text{Sdet}(\lambda_0 I - A) = |c||q_0 f_0| \). \qed

**Definition 6.8.** We define \( \mu : \mathcal{M}(n, \mathbb{H}) \rightarrow \mathcal{M}(n, \mathbb{H}) \) as follows (see Definition [4.3]):

1. if \( \lambda_0 I - B \) is invertible, then \( \mu(B) = q_0 f_0 I \);
2. otherwise,

\[
\mu(B) = (\lambda_0 I - B) \left[ (p I - (r - B)c^{-1}(a I - B)) - (q I - (r I - B)c^{-1}b)(\lambda_0 - B)^{-1}(f I - hc^{-1}(a I - B)) \right].
\]

The following Proposition completes the proof of Theorem A.

**Proposition 6.9.** The map \( \mu \) in Def. [6.8] satisfies Hamilton-Cayley theorem, that is \( \mu(A) = 0 \)

**Proof.** If \( \lambda_0 I - A \) is not invertible, then \( \mu(A) = q_0 f_0 I = 0 \) by Lemma [6.7]. Otherwise it suffices to prove that

\[ p I - (r I - A)c^{-1}(a I - A) \] \hspace{1cm} (8)

equals

\[ (q I - (r I - A)c^{-1}b)(\lambda_0 I - A)^{-1}(f I - hc^{-1}(a I - A)). \] \hspace{1cm} (9)

**Lemma 6.10.** A direct computation shows that the first term (8) is

\[
\begin{bmatrix}
-\lambda_0 c^{-1}f & -q + (r - a)c^{-1}b + bc^{-1}(a - g) & -bc^{-1}h \\
+(r - g)c^{-1}f + hc^{-1}p & -f + (r - g)c^{-1}h + hc^{-1}(a - r) & -bc^{-1}h \\
-qc^{-1}f & -pc^{-1}b + qc^{-1}(a - g) & -qc^{-1}h \\
\end{bmatrix}.
\]

We now want to compute the term (9). We start by computing \( (\lambda_0 I - A)^{-1} \) by Gaussian elimination. Let

\[
P_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
c^{-1}(\lambda_0 - a) & 0 & 1
\end{bmatrix}
\]
and

\[
P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c^{-1}b & 1 \end{bmatrix}
\]

Then

\[
(\lambda_0 I - A)P_1 P_2 = \begin{bmatrix} 0 & 0 & -c \\ -f_0 & 0 & -h \\ -p^* & -q_0 & \lambda_0 - r \end{bmatrix},
\]

where

\[
p^* = p - (\lambda_0 - r)c^{-1}(\lambda_0 - a).
\]

The inverse of the matrix \((\lambda_0 I - A)P_1 P_2\) in (10) can be computed by hand; it is

\[
B = \begin{bmatrix} f_0^{-1}hc^{-1} & -f_0^{-1} & 0 \\ -q_0^{-1}n^* & q_0^{-1}p^*f_0^{-1} & -q_0^{-1} \\ -c^{-1} & 0 & 0 \end{bmatrix}
\]

where

\[
n^* = p^*f_0^{-1}hc^{-1} - (\lambda_0 - r)c^{-1}.
\]

It follows that

\[
(\lambda_0 I - A)^{-1} = P_1 P_2 B =
\]

\[
\begin{bmatrix} f_0^{-1}hc^{-1} & -f_0^{-1} & 0 \\ -q_0^{-1}n^* & q_0^{-1}p^*f_0^{-1} & -q_0^{-1} \\ c^{-1}(\lambda_0 - a)f_0^{-1}hc^{-1} + c^{-1}bq_0^{-1}n^* - c^{-1} & -c^{-1}(\lambda_0 - a)f_0^{-1} - c^{-1}bq_0^{-1}p^*f_0^{-1} + c^{-1}bq_0^{-1} \end{bmatrix}.
\]

Moreover

\[
F = fI - hc^{-1}(aI - A) = \begin{bmatrix} f & hc^{-1}b & h \\ hc^{-1}f & f - hc^{-1}(a - g) & hc^{-1}h \\ hc^{-1}p & hc^{-1}q & f - hc^{-1}(a - r) \end{bmatrix},
\]

while

\[
Q = qI - (rI - A)c^{-1}b = \begin{bmatrix} q - (r - a)c^{-1}b & bc^{-1}b & b \\ fc^{-1}b & q - (r - g)c^{-1}b & hc^{-1}b \\ pc^{-1}b & qc^{-1}b & q \end{bmatrix}.
\]

We have to compute (11), that is \(QP_1 P_2 BF\).

First we compute \((P_1 P_2 B)F\). For instance, its first column is given by

\[
[(P_1 P_2 B)F] = \begin{bmatrix} 0 \\ -q_0^{-1}n^* f + q_0^{-1}p^*f_0^{-1}hc^{-1}f - q_0^{-1}hc^{-1}p \\ +c^{-1}bq_0^{-1}n^* f - c^{-1}f - c^{-1}bq_0^{-1}p^*f_0^{-1}hc^{-1}f + c^{-1}bq_0^{-1}hc^{-1}p \end{bmatrix}.
\]
Now we check for instance the entry \((1, 1)\) of the matrix \(Q(P_1P_2BF)\). We have

\[
\begin{align*}
[Q(P_1P_2BF)]_{11} &= bc^{-1}b\left(-q_0^{-1}n^*f + q_0^{-1}p^*f_0^{-1}hc^{-1}f - q_0^{-1}hc^{-1}p\right) + \\
b\left(+c^{-1}bq_0^{-1}n^*f - c^{-1}f - c^{-1}bq_0^{-1}p^*f_0^{-1}hc^{-1}f + c^{-1}bq_0^{-1}hc^{-1}p\right) &= -bc^{-1}f
\end{align*}
\]

which indeed is the entry \((1, 1)\) in Corollary 6.10. The other entries are computed in a similar way.

**Example 6.11.** Let \(A = \begin{pmatrix} 1 & i & -j \\ i & -1 & k \\ 1 & -1 & j \end{pmatrix}\). The pole is \(\lambda_0 = -2\) and 

\[
\mu(\lambda_0) = -5 + 8j.\]

For \(\lambda \neq 2\), the characteristic function is

\[
\mu(\lambda) = -(2 + \lambda)(2 + \lambda(-1 + j) - \lambda j \lambda + (-1 + i - \lambda k)(2 + \lambda)^{-1}i(2 - \lambda)).
\]

With the notations of the proof of Proposition 6.9, it is

\[
(\lambda_0 I - A)^{-1} = (1/12) \begin{pmatrix}
-3 & 3i & 0 \\
2i - j - k & -8 + 2i + j + 3k & 2 + 2i + 4k \\
1 + i - j & -3 - i + 2j + k & 2 - 2 + j + k
\end{pmatrix}.
\]

\[
P = \begin{pmatrix}
-j & 1 - i + 3k & 1 \\
-k & 3 - i - 3j & -i \\
-k & -2j + k & i
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
-1 + i - k & j & i \\
j & -1 + i + k & 1 \\
k & k & -1
\end{pmatrix}
\]

and

\[
F = \begin{pmatrix}
i & 1 & k \\
1 & 3i & j \\
-i & i & 2i - k
\end{pmatrix}.
\]

We have \(P - Q(\lambda_0 I - A)^{-1}F = 0\).
7 Final remarks

In this Section we discuss a different approach to the definition of characteristic functions for left eigenvalues.

In order to clarify concepts, let us briefly comment the same problem but for right eigenvalues. Let $c(A) \in \mathcal{M}(2n, \mathbb{C})$ be the complex form of the matrix $A \in \mathcal{M}(n, \mathbb{H})$ (see Section [2]). Then, as it is well known, the right eigenvalues of $A$ are the quaternions $qzq^{-1}$, where $q \in \mathbb{H}$, $q \neq 0$, and $z$ is a complex eigenvalue of $c(A)$. It follows:

**Theorem 7.1 ([9]).** Let $p(z) = \det(c(A) - zI) = \sum_{k=0}^{2n} c_k z^k$, $c_k \in \mathbb{R}$, be the characteristic polynomial of $c(A)$. Then $p(A) = \sum_{k=0}^{2n} c_k A^k = 0$.

Now, let $\lambda = x + jy$, with $x, y \in \mathbb{C}$, be a left eigenvalue of $A$. Equivalently, the matrix $c(A - \lambda I)$ is not invertible. It follows that the left eigenvalues are the roots of the function $\sigma: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ given by

$$\sigma(x, y) = \det \begin{bmatrix} X - xI & -Y + \overline{y}I \\ Y - yI & \overline{X} - \overline{y}I \end{bmatrix}. \quad (11)$$

Let $A = X + jY$, with $X, Y \in \mathcal{M}(n, \mathbb{C})$. Then Hamilton-Cayley theorem could be stated as $\sigma(X, Y) = 0$, provided this has a meaning. However we have the following counterexample even for $n = 2$.

**Example 7.2.** Let $A = \begin{bmatrix} 0 & i \\ j & 0 \end{bmatrix}$. Let $x = x_1 + ix_2$, $y = y_1 + iy_2$. Then

$$\sigma(x, y) = 1 + (x_1^2 + x_2^2 + y_1^2 + y_2^2) - 4x_2y_1.$$ 

On the other hand, it is $X = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so $X_1 = 0$, $X_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $Y_2 = 0$, then $\sigma(X, Y) = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$.

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