An Affine-Based Algorithm for Registrating Non-rigid 3D Models

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Abstract. While exploiting isometric method to calculate the correspondence between two non-rigid 3D models, it is difficult to obtain the correct matching relations and tend to fall into local optimum by iterative optimization strategy when using the inappropriate initial spectral value. To address this issue, an affine-based algorithm for labeling non-rigid 3D model is focused on in this paper. First, the approximate rotation and translation parameters between different models are calculated and the model coordinate system is unified through SVD method with over 4 pairs of roughly matched points. Next, the far sampling point algorithm is chosen to obtain a fixed number of uniform discrete 3D points on the two meshes respectively, and then for each sampling point, their neighbor points and corresponding distances are solved by kNN algorithm, after that, the shortest distance between each sampling point pair is calculated via Dijkstra algorithm and is viewed as geodesic distance. Besides, the discrete MRF model is established to optimize the isometric error of different models. Experiment results on both standard datasets and real-life collected models show that the proposed method can avoid local optimum in the whole optimization process. But for the existing method based on isometric optimization, if the initialization is not good, the optimization procedure may negatively get in local optimum and cannot gain the correct model registration and mapping.

Keywords. 3D models; registration; SVD method; point sampling; MRF optimization.

1. Introduction
Non-rigid 3D shape registration is a key problem in computer vision and image processing communities [1], meeting with rapid increasing attention due to the growing amounts of 3D data in artificial application communities, such as virtual reality, robotics, and multimedia. Non-rigid 3D registration can be stated as follows [2]: for given two surfaces, $S, T \in \mathbb{R}^3$, we need to solve the problem for computing a perceptually or physically meaningful mapping $f : S \rightarrow T$. In order to find the accurate matching relationship between different 3D points on surface, the mainstream technologies usually use isometric correspondence [3] as the metric to describe the correspondence quality between different models. Isometry is a conversion of similar class models or the same model to the transformed model that the distance between the point elements in the new metric space is equal to the distance between the elements in the original metric space. A 3D object, that uses it, can preserve the same lengths of curves with non-rigid transformation. To better solve the mapping problem, geodesic distance [4, 5] is introduced to accurately represent the intrinsic attribute of a 3D model. While the 3D model can be represented by triangular meshes, the distance of point $s_i$ and point $s_j$ in a curve is equal to searching the shortest path in an undirected graph, and can be solved via Dijkstra [6] algorithm.
However, the sampling points on a 3D surfaces are discrete, and the geodesic distances between two pairs of discrete corresponding points cannot maintain absolutely invariant (i.e., there is no absolute equidistance between two or more different models). To obtain a better mapping result, the equidistant error is then applied for measuring the quality of geodesic distance correspondence. Mathematically, we need to solve the following optimization problem, equivalently:

$$E(l) = \sum_{i,j=1}^{n} \rho(d_{s}(s_{i},s_{j}),d_{t}(t_{i},t_{j})) + \lambda \sum_{i,j} |s_{i} - t_{j}|$$ \hspace{1cm} (1)

The above function aims to solve the best correspondence mapping between two 3D surfaces, $l : S \rightarrow T$, $l = l(s)$, by making the energy function minimized. Here, $d_{s},d_{t}$ are the geodesic distance functions on surface $S,T$; $s_{i},s_{j}$ and $t_{i},t_{j}$ are sampling points from $S,T$; $S = \{ s_{i} | i = 1,2,3,...,N \}$ and $T = \{ t_{i} | i = 1,2,3,...,N \}$ are both sampling points set with fixed number; $\rho(*,*)$ is the error function.

Where the first term imposes data errors that the original geodesic distance should be similar to the transformed value; the second term is used to regularize the solution that disables some of the distortion elements (i.e. the sample points in inaccurate surface of near-isometric shapes and intrinsic symmetries). It can be seen that the optimization of the above model is an NP-hard problem, and the global optimization method with good initialization can achieve ideal results.

However, the 3D models captured by scanning systems aren’t perfect, and if they are applied in 3D fitting room system or ReID system (the inputs that are captured by high and low cameras), registration algorithms should be robust to the drawbacks. Some factors may have a slightly negative influence on model matching, such as occlusion gaps and surface noise, or severe, under the condition of large-scale fusion of adjoining parts \[2\]. When using holistic model to match the part model, the precision is usually rather poor due to large appearance differences of two models, and this problem is especially severe when using different scale models as inputs for challenging labeling work. Some of the similar violations usually lead to registration failure.

Generally, optimization-based algorithm as a powerful tool provides an effective approach to the NP-hard problem (as shown in function 1). Nonetheless, the NP-hard problem has many local optimal solutions, and we have to solve each latent solution and compare different local optimum to obtain the global optimal solution. This approach is obviously not applicable to the non-rigid 3D model labeling problem, due to large computational complexity. Equivalently, in nonrigid registration of 3D surfaces, we need a series of roughly aligned point pairs for initialization. If most of the initialized corresponding mapping pairs are with the low matching degree (i.e., the initialization $l$ is chosen inappropriately), the algorithm would like to converge to a local minimum while optimizing above objective function, and thus we cannot get a physically or perceptually meaningful mapping $l$.

To deal with the mismatch problems, many methods have been reported, and a thorough comparison into the algorithms can be found in \[7\], where the universality of the algorithms were also shown. Zhang et al. \[8\] use tree search methods to seek correspondence mapping based on local feature descriptor. Refs \[9, 10\] further use similar meta element to seek the corresponding relationship. However, these methods are effective for rigid matching; but less competent for non-rigid correspondence, due to the descriptors, which are similar in the local part, but not represent for the holistic features. Thus, we cannot obtain the perpetually meaningful visual correspondence for non-rigid 3D matching results via these methods. Bronstein et al. propose to use generalized multidimensional scaling method (GMDS) to make the 3D model embedded into a low-dimensional space and exploit the rigid matching method to register them. Some methods \[11-13\] optimize the non-rigid matching correspondence based on intrinsic descriptors, and the effectiveness of these algorithms are also showed in research papers. However, Ref. \[2\] points out that the downside of their invariance to isometric deformations is their sensitivity to gross topological inconsistencies.

In order to get more stable nonrigid registration of 3D surfaces compared with the existing non-rigid matching method, an affine-based algorithm via convex optimization \[2\] for labeling 3D model is developed. First, the approximate rotation and translation parameters among different models are
calculated and the model coordinate system is unified through SVD method with several pairs of corresponding point pairs. Second, the MRF algorithm is applied for boosting the robustness of points labeling. If necessary, the obtained results will be optimized via up-sampling and refinement.

2. Our Method

2.1. Process Overview

Figure 1 shows the overall process of our 3D model registration method, which builds upon the iterative optimization framework commonly adopted in recent methods. To progressively match between two 3D models with different scale, our method contains three key steps: (1) Unifying different coordinate systems; (2) Computing the geodesic distances of different point pairs; (3) Optimization, upsampling and refinement (if necessary). In the first step, we apply SVD method and the given 4 pairs of corresponding points to unify the model coordinate system as a preprocess to decrease the negative influence caused by the scale factor to the most extent, by helping to give a relatively normal matching for the initialization of global optimization algorithm. In the second step, we extract the points and faces information from the models of the previous step, and then apply Poisson surface reconstruction [14] to make the meshes watertight and select some sampling points for computing the geodesic distances; use $k$NN algorithm to construct connection edges from the sampling points; compute the geodesic distance of two points with Dijkstra algorithm. In the final step, we apply the global optimization method (MRF method) to optimize isometric correspondence model to obtain a more perceptual mapping result.

2.2. Unifying the Coordinate System

The objective function (1) is NP-hard in general, but for some applications, good results can be obtained with optimization algorithms, such as graph cut and $TRW - S$ (sequence tree-reweighted message passing) [15, 16]. In this paper, we exploit $TRW - S$ in Ref. [2] to optimize the objective function (1). While using the $TRW - S$ method to approximately solve this LP relaxation of the problem, the convergence process and behavior are unpredictable. Because, the $TRW - S$ optimization methods are mainly based on linear programming relaxations; as for the initial mapping point pairs, we don’t know whether most of the mappings are within the search range. In general, if most of the corresponding relations exceed the search range, the algorithm convergence would occur when the energy value $E(l)$ minimizes, during the iterative optimization process. But, it may be more likely to converge to a local minimum, resulting in terrible visual mapping effect.

Hence, it’s necessary to add the preprocessing step for solving non-rigid registration problem. In the corresponding relations initialization, we aim to compute a rough version of matching relation for two or more input models, and thus we need to unify different coordinate systems into the common coordinate system. Initialization based on unifying coordinates contains two main steps, i.e., marking the same semantic dots, computing the transforming and scaling matrices with the marked dots, and unifying different coordinates with transforming and scaling matrices. The appropriate initial mapping relations can effectively avoid the local minimum during the iterative optimization process.

Given two or more 3D models, we need to obtain the roughly matching points, similar to [2], we don’t assume that the input models are watertight. To simplify the computation of transformation matrix, we begin by applying Poisson reconstruction to the meshes. The corresponding semantic sample sets $S_{\text{sample}} = \{s_1, s_2, ..., s_n\}, T_{\text{sample}} = \{t_1, t_2, ..., t_n\}, n \geq 4$ from two surfaces can be achieved by 3D registration algorithm [1, 17] or manual labeling. However, these methods, mainly based on Iterative Closest Point, cannot always be effective for different non-rigid 3D models due to the inconsistency of geometric structure. In this paper, we use manual labeling for obtaining initial corresponding pairs.
Figure 1. The work flow of non-rigid 3D labeling method.

Inspired by 3D-3D matching method [18], we show the method of unifying the coordinate system under different metrics. First, given the corresponding point pairs \( S_{\text{sample}} \), \( T_{\text{sample}} \), we aim to compute the transformed parameters for \( S \rightarrow T \), including the rotation matrix \( R \), translation vector \( t \), as well as scaling factor \( \mu \). Because there is equivalence for scale in the process of coordinate system transformation, the scaling factor can be implicitly integrated into \( R \), \( t \). The goal of optimization is to find a Euclidean transformation \( R, t \), and let the transformed points be consistent with the target points:

\[
\forall i, t_i = R s_i + t
\]  

(2)

The affine transformation parameters can be solved via SVD method. To achieve it, the error term is defined as, \( e_i = t_i - (R s_i + t) \) for any matching pairs; then, we construct the Least Squares model,

\[
\min_{s, t} J = \frac{1}{2} \sum_{i=1}^{n} \| t_i - (R s_i + t) \|^2
\]

(3)

Here, we also show how to solve \( R, t \) effectively: first, the two centroids of point sets are defined as, \( \bar{s} = \frac{1}{n} \sum_{i=1}^{n} (s_i) \) and \( \bar{t} = \frac{1}{n} \sum_{i=1}^{n} (t_i) \), respectively. Equation (3) can be equivalent to:

\[
\frac{1}{2} \sum_{i=1}^{n} \| t_i - (R s_i + t) \|^2 = \frac{1}{2} \sum_{i=1}^{n} \| t_i - R s_i - t + t R s + t - R s - t \|^2
\]

(4)
The summary of the last term \((t_i - \hat{t} - R(s_i - \bar{s}))\) in equation (4) is 0, and thus the primitive objective function can be transformed into:

\[
\min_{R,t} J = \sum_{i} \left( \|q_i - \hat{t} - R(s_i - \bar{s})\|^2 + \|\hat{t} - R\bar{t} - t\|^2 \right)
\]  

(5)

From equation (5), we could see that the first term in the right side of the equation is only related to the variable \(R\). In addition, if we want to minimize \(J\), the two terms in the right side of the equation should be both minimized. The minimized value of the second term is 0. In summary, we only need to compute \(R\), and then make the second term be 0 to obtain \(t\). By using the above results, the pseudocode for solving \(R, t\) is described in algorithm 1.

Algorithm 1: Compute the \(R, t\) for unifying different coordinate system

1. Compute the centroids of the marked point sets \(\bar{s}, \bar{t}\), compute the relative coordinates for two point sets: \(q_i = t_i - \bar{t}, p_i = s_i - \bar{s}\)
2. Optimize the following Least Squares model to obtain \(R\):
3. By using the result of step 2, solve \(t\), via \(t = \bar{t} - R\bar{t}\)

In algorithm 1, the detail analysis of solving \(R^*\) is also given:

\[
\sum_{i=1}^{n} \left\| q_i - R p_i \right\|^2 = \sum_{i=1}^{n} (q_i^T q_i + p_i^T R^T R p_i - 2 q_i^T R p_i)
\]  

(6)

From equation (6), we could see that the first two terms are both independent of \(R\), due to \(R^T R = I\), and the objective function is further simplified to:

\[
\sum_{i=1}^{n} -q_i^T R p_i = \sum_{i=1}^{n} -tr(R p_i q_i^T) = -tr(R \sum_{i=1}^{n} p_i q_i^T)
\]  

(7)

While using SVD method [19] for computing optimized \(R\), we define the auxiliary matrix \(W = \sum_{i=1}^{n} p_i q_i^T\) (\(W\) is a 3-by-3 matrix), and apply SVD method for decomposing \(W\) (i.e., \(W = UV^T\)), \(U\) is rectangular diagonal matrix with non-negative real numbers in descending order by value. Where \(U, V\) are both diagonal matrix, \(R = UV^T\) is established while \(W\) is a full rank matrix. Given \(R\), we can solve \(t\) via step 3 in algorithm 1.

Next, by using the above results (i.e. \(R, t\)), we can transform the sampling point \(src\) of surface \(S\) to the desired coordinate \(res\), (i.e. the coordinate system determined by \(T\)), and keep the triangular mesh of surface invariant:

\[
res = R * src + t
\]  

(8)

2.3. Discrete Point Sampling

Uniform sampling can provide a possible correspondence candidate point set for the sparse correspondence problem. Compared with other sampling algorithms, such as [3], the farthest point algorithm can avoid the prior knowledge requirements (e.g. curvature and search radius) for obtaining the uniform point set. Here, adaptively selecting the fixed number of sampling points can be achieved by this method. For a reconstructed mesh by Poisson reconstruction, the fixed number points can be obtained via [20]. First, randomly select a point as the first sampling point \(sp_1\). Then, calculate the distances between the point to the rest points of the model via Dijkstra algorithm, and choose the farthest point as the next sampling point \(sp_2\). Do the same operation for \(sp_2\), and then save the
distances from \( sp_2 \) to other selected points (i.e. \( sp_1 \)). Repeat above operation as \( sp_2 \) until getting enough points or attain the threshold \( pt\_num \). In this section, we choose three examples for experiment via the embedded point sampling method and show it in figure 2 (It’s noted that the points number \( pt\_num \) is set to 5000 in our experiment), where the input 3D models are reconstructed from the depth data (scanned by a Kinect) of real-world objects.

For each sampling point, their neighbor points and corresponding distances are obtained by using \( kNN \) algorithm, after that, the shortest distance between each point pair is calculated via Dijkstra algorithm, and is viewed as geodesic distance.

### 2.4. MRF-Based Optimization and Post-Process

In the non-rigid registration of 3D surfaces, we aim to compute a coarse version of the mapping result. Given the transformed \( T \) and \( S \), we need to solve equation (1) via LP relaxation-based MRF optimization [2]:

\[
\min_{\mathbf{z}} \sum_{a \in L} \sum_{i,j} \xi_{ij}(a,b)z_i^a \quad s.t. \sum_{i} z_i^a = z_i^* \quad \forall i, j \in S, \forall a \in L
\]

\[
\sum_{i} z_i^* = 1 \quad \forall i \in S
\]

\[
\xi_{ij}(l_i, l_j) = \rho(d_i(s_i, s_j), d_t(t_i, t_j)) = \lambda \|d_i(s_i, s_j) - d_t(t_i, t_j)\|^p
\]

Here, \( S, T \) are the point sets in surfaces, and \( TRW\_S \) is employed to obtain the solution of equation (9), (i.e. the \( l:S \rightarrow T \), \( l = l(s) \) in equation (1)). In general, the MRF optimization problem is NP-hard, and the approach to deriving approximate algorithms is to represent the problem as an integer linear program bounded by integer constraints. \( TRW\_S \) [15, 16] is a powerful tool for optimizing Markov random fields. If necessary, the generated mapping would be upsampling based on the coarse point sets and the corresponding relations would also be refined, and then the refined relations are viewed as the final registration results.

![Figure 2](image-url)

**Figure 2.** Three examples of point sampling for 3D meshes with the interim sampling strategy.

### 3. Experiments and Discussions

The developed method was implemented in MATLAB and was tested in PC with Intel Core i5-7300 CPU at 2.50 GHz and 8G RAM. The two modules, including unifying different model coordinate systems and calculating geodesic distance are implemented in C/C++, and the remaining modules are implemented using MATLAB; the \( TRW\_S \) optimizing part is implemented according to the method of [2]. During our experiment, the significant parameters setting is as follows: in the unifying coordinate module, the size \( n \) of \( S\_\text{sample} \) is set to 4, because it is the least number of 3D points pairs to determine the transformation parameters. In geodesic distance computing module, the searching radius threshold of \( kNN \) is set to 0.007, and the near neighbor points number is set to 25 based on our experiments. In the discrete points sampling module, the number of sampling points in a mesh is set to 5000. In this section, we first test whether the unifying coordinate module is effective, and then, the overall performance of the improved algorithm is validated by testing on the 3D face and head meshes.
We conduct our experiment in the FAUST dataset introduced by [21]. The FAUST dataset contains 100 scans of 10 human subjects, as well as 10 scans per subject, each in a different pose. The partial models are scanned by a Kinect. To enable comparison and further testify to the validity of our method, we would make our models available online.

3.1. The Influence of Unifying Coordinate System
To validate the effectiveness of the affine-based algorithm, we conduct the experiment of registrating partial model with a complete model via our algorithm and compare the results with the results obtained by the method without unifying the different coordinates. Figure 3 shows the comparison results by different methods. Nevertheless, only using qualitative comparisons cannot sufficiently manifest the effectiveness of the algorithm, so we use isometric errors to measure the quality of registration models. In table 1, $\text{Error}_{\text{iso}}$ denotes the isometric Error, which is defined as:

$$
\text{error}_{\text{iso}}(s,t) = \frac{1}{|C|} \sum_{(s_i,t_j) \in C} (d_i(s_i,s_j) - d_j(t_i,t_j))^2
$$

$$
\text{Error}_{\text{iso}} = \frac{1}{|C|} \sum_{(s_i,t_j) \in C} \text{error}_{\text{iso}}(s,t)
$$

From figure 3, we could see that, as for the two 3D models in different coordinate systems, wide range of differences can result in different matching effects: there exists translation error $\Delta T$ between two models in figure 3a; for figure 3b, there exists both translation error $\Delta T$ and the rotation error $\Delta R$; there exists zoom difference $\mu$ between two different models in figure 3c. The results obtained by using the method in Ref. [2] directly are not perceptually or physically meaningful mapping. The ICP fine-tuning cannot improve the negative situation, when a number of wrong perceptual correspondences are chosen for initialization, and thus, the subsequent TRW – S optimization is ineffective with roughly wrong correspondences ultimately. As for (c), that's mainly because, there exist large differences between the two input models, resulting in the topologies of the meshes that do not match with each other, to some extent. Similarly, accurate correspondences via ICP and TRW – S cannot be achieved, either. From table 1, we could see that the isometric Error measurements are consistent with the 3D model matching effects as a whole. However, in group (b), the isometric Error obtained by the method in Ref. [2] is slightly smaller than the result obtained in this paper, due to the local optimum problem. Although the convergence of $E(l)$ can be obtained by using TRW – S optimization, the registration relationship result is only a local optimum in this scenario and does not conform to the best visual matching mapping between the two models. This paper uses preprocessing to obtain roughly correct correspondence relations by avoiding falling into local optimum, making it suitable for the real applications, such as Person Re-identification (ReID system) using the different scale 3D data as input (reconstructed by the images captured via different height cameras).

Besides, the speed of the proposed method is compared with the state-of-the-art method, using sample examples in figure 3. Table 2 shows the comparison of running time under the same environment configuration. Compared with method in Ref. [2], our method needs much less computational time. That’s mainly because the affine-based algorithm is more likely to go to exact convergence with less iteration, while the other not.

3.2. The Labeling Results of Face and Head Examples
Finally, the 3D models reconstructed from the depth data scanned via Kinect by us are used to test the proposed algorithm, and the labeling results are shown in figure 4. Besides, the isometric Error of group (c) (i.e. $\text{Error}_{\text{iso}}$ ) is 0.3143. As for group (a), (b), (d), we cannot obtain $\text{Error}_{\text{iso}}$ for the present, because the faces and points are too many to cause the memory overflow while running the program. However, we believe that the memory requirement of our algorithm can be mild by using CNN
implementation of part modules. As can be seen from the experimental results, the non-rigid 3D labeling method in this paper has good performance.

![Image](image1.png)

**Figure 3.** Three examples for 3D points registering via different methods.

| Group   | Error   | Methods |
|---------|---------|---------|
| Ref. [2]| 0.3346  | 0.2733  | 1.094 |
| Ours    | 0.3117  | 0.3090  | 0.1443|

**Table 1.** Comparison of Isometric Error in figure 3.

| Group   | Time     | Methods |
|---------|----------|---------|
| Ref. [2]| 569.7    | 604.1   | 2208.5|
| Ours    | 497.8    | 551.5   | 1974.4|

**Table 2.** Comparison of Runtime in figure 3.

![Image](image2.png)

**Figure 4.** Sample input models and corresponding registration results in the experiment.

4. Conclusion

In order to get rid of the poor stability in the existing non-rigid matching method, an affine-based algorithm for labeling.

In (a) and (b), the 3D models from left to right are template, scanned face, registration result; In (c), the models are watertight model, scanned face, registration result, respectively; In (d), they are one human model in FAUST, head reconstructed using depth data from Kinect, registration result, respectively.

3D model is proposed. To obtain similar topologies for different 3D meshes, the approximate rotation and translation parameters between two models are calculated and the model coordinate system is unified through SVD method with several roughly matched points. And then, to sample the uniform points on the surface, the far sampling point algorithm is applied for obtaining the fixed number of points. Finally, the pretreated model registration problem is formulated by LP relaxation-
based MRF optimization (i.e., an integer linear program bounded by integer constraints), and is optimized by the \( TRW - S \) method to get the final mapping relations. The experimental results show that the improved registration method is competitive in both registration accuracy and computational efficiency, due to avoiding local optimum in the whole optimization process.

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References
[1] Van Kaick O, Zhang H, Hamarneh G, et al. 2011 A survey on shape correspondence computer Graphics Forum (Oxford, UK: Blackwell Publishing Ltd.) pp 1681-1707.
[2] Chen Q and Koltun V 2015 Robust nonrigid registration by convex optimization Proceedings of the IEEE International Conference on Computer Vision pp 2039-2047.
[3] Sahillioglu Y and Yemez Y 2010 3D shape correspondence by isometry-driven greedy optimization 2010 IEEE Computer Society Conference on Computer Vision and Pattern Recognition pp 453-458.
[4] Sahillioglu Y and Yemez Y 2011 Coarse-to-fine combinatorial matching for dense isometric shape correspondence Computer Graphics Forum (Oxford, UK: Blackwell Publishing Ltd.) pp 1461-1470.
[5] Hamza A B and Krim H 2006 Geodesic matching of triangulated surfaces IEEE Transactions on Image Processing 15 (8) 2249-2258.
[6] Jasika N, Alispahic N, Elma A, et al. 2012 Dijkstra’s shortest path algorithm serial and parallel execution performance analysis 2012 Proceedings of the 35th International Convention MIPRO (IEEE) pp 1811-1815.
[7] Bogo F, Romero J, Loper M, et al. 2014 FAUST: Dataset and evaluation for 3D mesh registration Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition pp 3794-3801.
[8] Zhang H, Sheffer A, Cohen-Or D, et al. 2008 Deformation-driven shape correspondence Computer Graphics Forum (Oxford, UK: Blackwell Publishing Ltd.) pp 1431-1439.
[9] Dubrovin A and Kimmel R 2011 Approximately isometric shape correspondence by matching pointwise spectral features and global geodesic structures Advances in Adaptive Data Analysis 3 (01n02) 203-228.
[10] Rodola E, Bronstein A M, Albarelli A, et al. 2012 A game-theoretic approach to deformable shape matching 2012 IEEE Conference on Computer Vision and Pattern Recognition pp 182-189.
[11] Aubry M, Schlickewei U and Cremers D 2011 The wave kernel signature: A quantum mechanical approach to shape analysis 2011 IEEE International Conference on Computer Vision Workshops (ICCV Workshops) pp 1626-1633.
[12] Litman R and Bronstein A M 2013 Learning spectral descriptors for deformable shape correspondence IEEE Transactions on Pattern Analysis and Machine Intelligence 36 (1) 171-180.
[13] Windheuser T, Vestner M, Rodolà E, et al. 2014 Optimal intrinsic descriptors for non-rigid shape analysis Proceedings of the British Machine Vision Conference.
[14] Kazhdan M and Hoppe H 2013 Screened poisson surface reconstruction ACM Transactions on Graphics (ToG) 32 (3) 29.
[15] Wainwright M J, Jaakkola T S and Willsky A S 2005 MAP estimation via agreement on trees: Message-passing and linear programming *IEEE Transactions on Information Theory* **51** (11) 3697-3717.

[16] Kolmogorov V 2006 Convergent tree-reweighted message passing for energy minimization *IEEE Transactions on Pattern Analysis and Machine Intelligence* **28** (10) 1568-1583.

[17] Agamennoni G, Fontana S, Siegwart R Y, et al. 2016 Point clouds registration with probabilistic data association 2016 *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)* pp 4092-4098.

[18] Gao X, Zhang T, Liu Y and Yan Q 2017 14 *Lectures on Visual SLAM: From Theory to Practice* (Publishing House of Electronics Industry).

[19] Pomerleau F, Colas F and Siegwart R 2015 A review of point cloud registration algorithms for mobile robotics *Foundations and Trends® in Robotics* **4** (1) 1-104.

[20] Gonzalez T F 1985 Clustering to minimize the maximum intercluster distance *Theoretical Computer Science* **38** 293-306.

[21] Bogo F, Romero J, Loper M, et al. 2014 FAUST: Dataset and evaluation for 3D mesh registration *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition* pp 3794-3801.