On integrability of transverse Lie-Poisson structures to nilpotent elements

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Abstract

We apply the argument shift method to transverse Poisson structures to nilpotent elements of Lie-Poisson structures in simple Lie algebras. Examples show that this method always leads to a simple construction of polynomial completely integrable system. We provide a uniform construction of these integrable systems for some distinguished nilpotent orbits of semisimple type which include nilpotent elements corresponding to the partition \([2m + 1, 2m - 1, 1]\) and \([2m + 1, 2m - 1]\) in the Lie algebras \(so_{4m+1}\) and \(so_{4m}\), respectively.

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1 Introduction

In a prominent article [19], Miscenko and Fomenko proved that for any simple Lie algebra there exists a set of polynomial functions which forms Liouville completely integrable system under the Lie-Poisson bracket. Their method could be applied to bihamiltonian manifolds and it is known as the argument shift method [2]. It works as follows: Let \(M\) be a bihamiltonian manifold with compatible Poisson structures \(B'_1\) and \(B'_2\), where \(B'_1\) is in general position. Then consider the family of functions

\[
F := \cup_{\lambda \in \mathbb{C}} \{ F_\lambda : F_\lambda \text{ is a Casimir of } B'_1 + \lambda B'_2 \}. \tag{1.1}
\]

This family commutes pairwise with respect to both Poisson brackets. The main task then is to show that \(F\) contains a family of functions which is completely integrable under \(B'_1\), i.e. it has the right number of linearly independent functions. Bolsinov [2] proved that this is the case under certain condition on dimensions of singular sets of the Poisson pencils \(B'_\lambda = B'_1 + \lambda B'_2, \lambda \in \overline{\mathbb{C}}\). The argument shift method for
a complex simple Lie algebra \( g \) are carried out by considering with Lie-Poisson structure \( B_1 \), a compatible Poisson structure \( B_2 \) which is obtained from a co-cycle defined by a regular semisimple element in \( g \) [19]. A question posted in recent review paper by Bolsinov et al. [4] on open problems and challenges on finite dimensional integrable systems is whether there exists a polynomial nonlinear Poisson structure which admits a polynomial completely integrable system. In this paper we will give a infinite number of examples of such completely integrable systems which happen to be on the transverse Poisson structure of Lie-Poisson structure.

The Lie-Poisson structure \( B_1 \) of a simple Lie algebra \( g \) is irregular and of corank equals the rank of \( g \). It is symplectic leaves coincide with the orbits of the adjoint action. Using Weinstein splitting theorem [1] for any irregular point \( s \), there is a neighborhood isomorphic to a product of a symplectic submanifold and a nontrivial Poisson submanifold of rank 0 at \( s \). The later defines the transverse Poisson structure of \( B_1 \) at \( s \). We are interested on constructing completely integrable system for the transverse Poisson structures of the Lie-Poisson bracket at nilpotent elements of \( g \).

Let \( A := \{e,h,f\} \subset g \) be a \( sl_2 \)-triples and consider the so called Slodowy slice \( Q := e + g^f \) where \( g^f \) is the centralizers of \( f \) in \( g \). Then \( Q \) is a transverse subspace to the adjoint orbit of \( e \) and it inherits the transverse Poisson structure \( B_1^Q \) of the Lie-Poisson structure \( B_1 \). It turns out that \( B_1^Q \) is a polynomial Poisson bstructure [9] (see also [12] where an alternative proof is given by using the notion of bihamiltonian reduction and finite dimensional version of Drinfeld-Sokolov reduction). The rank of \( B_1^Q \) in case \( e \) is regular or subregular nilpotent element is 0 and 2, respectively. Hence, integrability is trivial in those cases. However, for other types of nilpotent elements the rank is greater than 2. For example, when \( e \) is a nilpotent elements of type \( D_{2m}(a_{m-1}) \) in a Lie algebra of type \( D_{2m} \), the rank of \( B_1^Q \) is \( 2m - 2 \) while \( \dim Q = 4m - 2 \). This nilpotent element correspond to partition \([2m + 1, 2m - 1]\) when the Lie algebra is \( so_{2m} \). Thus, it is natural to ask about existence of functions which are completely integrable for these large family of polynomial Poisson structures. For this task, we will apply the argument shift method.

Consider a bihamiltonian structure on \( g \) formed by compatible Poisson structures \( B_1 \) and \( B_2 \), where in our case, \( B_2 \) is defined by means of a co-cycle introduced by a nilpotent element \( b \) satisfying certain conditions extracted from the properties of the nilpotent element \( e \). Then one can perform bihamiltonian reduction to obtain a bihamiltonian structure \( B_1^Q \) and \( B_2^Q \) on \( Q \) [12]. Here, the collection \( F \) of Casimirs of Poisson pencils \( B_1^Q := B_1^Q + \lambda B_2^Q \) needed to preform argument shift method is easy to describe. Let \( P_1, \ldots, P_r \) be a complete set of generators of the ring of invariant polynomials on \( g \). Then the restrictions \( P_i^Q \) of \( P_i(x + \lambda b) \) to \( Q \) are a complete set of independent Casimirs of the Poisson pencil \( B_1^Q \). Hence, it remains to investigate whether \( F \) contains enough number of linearly independent functions. Examples show that this is always the case. However, we will provide a proof only for some distinguished nilpotent elements of semisimple type as we rely on the notion of opposite Cartan subalgebras and the structure of the weights of the restriction of the adjoint representation to \( A \). This includes the family of nilpotent elements of type \( D_{2m}(a_{m-1}) \) mentioned above. Precisely, we proved the following

**Theorem 1.1.** Let \( A = \{e,h,f\} \) be an \( sl_2 \)-triples in a simple Lie algebra \( g \) of rank \( r \) where \( e \) is one of the following distinguished nilpotent orbits of semisimple type: \( D_{2m}(a_{m-1}) \), \( B_{2m}(a_m) \), \( F_4(a_1) \), \( F_4(a_2) \), \( E_6(a_1) \), \( E_6(a_3) \), \( E_7(a_1) \), \( E_8(a_1) \), \( E_8(a_2) \) and \( E_8(a_4) \).

Let \( Q := e + g^f \) be the Slodowy slice and consider the transverse Poisson structure \( B_1^Q \) of the Lie-Poisson structure on \( g \). Let \( P_1, \ldots, P_r \) be a complete set of homogenous generators of the invariant ring under the adjoint group action. Let \( b \in g \) be an eigenvector of \( \text{ad}_h \) of the minimal weight such that \( e + b \) is regular semisimple. Define the family of functions \( H_i^Q \) on \( Q \) by the formula

\[
P_i(x + \lambda a) := \sum_{j=1}^{\mu_i+1} \lambda^{j-1} \lambda_i(i)(x), \quad \forall x \in Q, \quad i = 1, \ldots, r
\]  

(1.2)
Then the set of all non constant functions $H^J_i$ are linearly independent and form a completely integrable system under $B^Q_i$.

The paper is organized as follows. In section 2 we review the construction of bihamiltonian structure on Slodowy slice and we give example of using argument shift method for arbitrary nilpotent element to construct an integrable system. We give a review about distinguished nilpotent orbits of semisimple type in in section 3. In section 4, we prove our theorem for some class of these orbits. In the last section, we comment about the extension of the results of section 4 to all remaining distinguished nilpotent orbits of semisimple type.

## 2 Argument shift method for transverse Poisson structure

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $r$ with the Lie bracket $[\cdot, \cdot]$. Define the adjoint representation $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $\text{ad}_x(y) := [x, y]$. For $x \in \mathfrak{g}$, denote $g^x$ the centralizer of $x$ in $\mathfrak{g}$, i.e. $g^x := \ker \text{ad}_x$. Let $e \in \mathfrak{g}$ be a nilpotent element. Then by Jacobson-Morozov theorem, there exist a nilpotent element $f$ and a semisimple element $h'$ such that $A = \{e, h', f\} \subseteq \mathfrak{g}$ is a $\mathfrak{sl}_2$-triples satisfying

\[
[h', e] = 2e, \quad [h', f] = -2f, \quad [e, f] = h'.
\] (2.1)

Let us fix a good grading of $\mathfrak{g}$ compatible with $A$ [14], i.e.

\[
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i; \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j},
\] (2.2)

where $e \in \mathfrak{g}_2$, $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$; and the map $\text{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$.

Let $\langle \cdot, \cdot \rangle$ be a normalization of the Killing form on $\mathfrak{g}$ such that $\langle e|f \rangle = 1$. We fix an isotropic subspace $l \subset \mathfrak{g}_{-1}$ under the symplectic bilinear form on $\mathfrak{g}_{-1}$ defined by $(x, y) \mapsto \langle e|[x, y]\rangle$. Let $l'$ denote the symplectic complement of $l$ and introduce the following nilpotent subalgebras

\[
m := l \oplus \bigoplus_{i \leq -2} \mathfrak{g}_i; \quad n := l' \oplus \bigoplus_{i \leq -2} \mathfrak{g}_i.
\] (2.3)

Let $\mathfrak{b}$ denote the orthogonal complement of $\mathfrak{n}$ under $\langle \cdot, \cdot \rangle$. We choose an element $k \in \mathfrak{g}$ which centralizes the subalgebra $\mathfrak{n}$, i.e.

\[
n \subset \mathfrak{g}^k.
\] (2.4)

Such an element always exists. For example, we can take $a$ to be a homogeneous element of minimal grading.

We fix a bihamiltonian structure on $\mathfrak{g}$ as follows. We define the gradient $\nabla F : \mathfrak{g} \to \mathfrak{g}$ for a function $F$ on $\mathfrak{g}$ by the formula

\[
\frac{d}{dt}|_{t=0} F(x+tq) = \langle \nabla F(x)|q\rangle, \quad \forall x, q \in \mathfrak{g}.
\] (2.5)

Then for any two functions $F$ and $G$, the Lie-Poisson bracket on $\mathfrak{g}$ is defined by

\[
\{F, G\}_1(x) = \langle x|[\nabla G(x), \nabla F(x)]\rangle.
\] (2.6)

Which is compatible with the Poisson bracket defined by

\[
\{F, G\}_2(x) = \langle k|[\nabla G(x), \nabla F(x)]\rangle.
\] (2.7)
The corresponding Poisson structures (tensors) are given, respectively, by the formulas

\[ B_1(v) = [q, v] \]
\[ B_2(v) = [k, v], \]  

for every \( v \in T_q^* \mathfrak{g} \cong \mathfrak{g}. \)

We fix a complete set of generators \( P_1, \ldots, P_r \) of the ring of invariant polynomials under the adjoint group action. It is well known that these functions form complete family of global independent Casimirs of Lie-Poisson bracket. Moreover, the functions \( P_i(x + \lambda k) \) are complete set of independent Casimirs of the Poisson pencil \( B_\lambda := B_1 + \lambda B_2 \) [3].

Define Slodowy slice \( Q := e + \mathfrak{g}^f \). This affine subspace is transverse to the adjoint orbit of \( e \). It turns out that \( Q \) inherits a bihamiltonain structure \( B_Q^1, B_Q^2 \) from \( B_1, B_2 \), respectively. Where \( B_Q^i \) is the transverse Poisson structure of \( B_1 \) at \( e \). This bihamiltonian structure is independent of the choice of good grading and isotropic subspace \( l \). It can be obtained equivalently [12] by using the bihamiltonian reduction [7] with Poisson tensor procedure [21], Dirac reduction and a finite dimensional version of the generalized Drinfeld-Sokolov reduction [13], [5]. This equivalence is obtained as a consequence of results appeared in many articles. For example the connection between Dirac and Drinfeld-Sokolov reductions was first mentioned in [16]. For the purpose of this article we summarize Poisson tensor procedure, also called the method of transverse subspace.

Let us apply Poisson tensor procedure to construct \( B_Q^\lambda := B_Q^1 + \lambda B_Q^2, \lambda \in \mathbb{C} \). Let \( z \in Q \) and \( w \in T_z^* Q \). We identify \( T_z^* Q \) with \( \mathfrak{g}^e \) using the bilinear form \( \langle \cdot, \cdot \rangle \) and we consider \( w \) as an element of \( \mathfrak{g}^e \). Then we extend \( w \) to a covector \( v \in T_z^* \mathfrak{g} \) by requiring that

1. The projection \( v_e \) of \( v \) to \( \mathfrak{g}^e \) equals \( w \), and
2. \( B_\lambda(v) \in \mathfrak{g}^f \simeq T_z Q. \)

It turns out that this extension \( v \) is unique and it is found by solving recursive equations. Then the value of the reduced Poisson structure is given by the formula

\[ B_\lambda^Q(w) = B_\lambda(v) = [z + \lambda k, v]. \]  

Note that this is a Lax representation of any Hamiltonian vector field in \( Q \) under \( B_\lambda^Q \). Thus, the following standard result is valid and the proof could be found in [1] (page 68).

**Proposition 2.1.** Let \( \phi \) be any faithful matrix representation of \( \mathfrak{g} \) and \( z \in Q \). Then the coefficients of the characteristic polynomial \( \det(\phi(z + \lambda a) - \mu I) \) are Casimirs of the Poisson pencil \( B_\lambda \). In particular, the restrictions \( P_i^Q \) of the invariant polynomials \( P_i \) to \( Q + \lambda a \) are Casimirs of the Poisson pencil \( B_\lambda^Q \).

Using argument shift method we consider the expansion

\[ P_i^Q(z + \lambda k) = \sum_{j=1}^{\gamma_i+1} \lambda^{j-1} P_i^j(z) \]  

Then the functions \( P_i^1 \) are Casimirs of \( B_1^Q \), \( P_i^{\gamma_i} \) are Casimirs of \( B_2^Q \) and the functions \( P_i^j \) are in involution with respect to both Poisson structures [19]. It remains to show that the family \( P_i^j \) contains a subset which is a completely integrable under \( B_1^Q \). This is indeed the case for all examples we calculated. In other words, using the notations of this section we can state the following:
Conjecture 2.2. For any nilpotent element \( e \) is a simple Lie algebra \( \mathfrak{g} \) and for arbitrary element \( k \in \mathfrak{g} \) satisfying (2.4) the non constant functions of the expansion (2.10) are linearly independent over \( Q \) and form a completely integrable system for \( B^1_1 \).

Example 2.3. Consider the Lie algebra \( \mathfrak{sl}_5 \) and a nilpotent element \( e \) corresponding to the partition \( [3, 2] \). In contrary to the treatment in next sections, \( e \) is not of semisimple type [15]. Using standard procedure to obtain the \( \mathfrak{sl}_2 \)-triples [6], we set

\[
e = \zeta_{1,2} + \zeta_{2,3} + \zeta_{4,5}, \quad h = 2\zeta_{1,1} - 2\zeta_{3,3} + \zeta_{4,4} - \zeta_{5,5}, \quad f = 2\zeta_{2,1} + 2\zeta_{3,2} + \zeta_{5,4}
\]

where \( \zeta_{i,j} \) denote the standard basis of \( \mathfrak{gl}_5(\mathbb{C}) \). Then points of Slodowy slice \( Q \) will take the form

\[
\left( \begin{array}{cccccc}
2u_6 & 1 & 0 & 0 & 0 \\
2u_1 - \frac{2}{3}u_5 & 2u_6 & 1 & u_8 & 0 \\
u_4 & 2u_1 - \frac{2}{3}u_5 & 2u_6 & u_7 & 2u_8 \\
u_2 & 0 & 0 & -3u_6 & 1 \\
u_3 & 0 & u_1 + \frac{1}{3}u_5 & -3u_6 \\
\end{array} \right)
\]

(2.11)

\( \zeta_{3,1} \) has the minimal weight and thus the shift will be along the variable \( u_4 \). Using the same notations given in (2.10), we give the following completely integrable system

\[
P_1^1 = u_1 + 3u_6^2, \quad P_3^2 = u_6, \quad P_4^2 = u_5 - \frac{45}{4}u_6^2 + \frac{5}{4}u_1,
\]

\[
P_2^1 = u_4 - 10u_6^3 + 10u_1u_6 - 8u_5u_6 + 5u_3u_8,
\]

\[
P_3^1 = u_4u_6 - 10u_6^4 + 2u_5u_6^2 - \frac{2}{3}u_1^2 + \frac{8}{15}u_5^2 - \frac{2}{5}u_1u_5 + \frac{1}{2}u_3u_7 + \frac{1}{2}u_2u_8,
\]

\[
P_4^1 = u_4u_5 - 90u_5^5 + 100u_1u_6^3 - 10u_5u_6^4 - \frac{45}{4}u_4u_6^2 + 25u_3u_8u_6^2 - 10u_1^2u_6 + \frac{8}{5}u_5^2u_6
\]

\[
-6u_1u_5u_6 - 5u_3u_7u_6 - 5u_2u_8u_6 + \frac{5}{4}u_1u_4 - \frac{5}{4}u_2u_7 - 5u_1u_3u_8 - 4u_3u_5u_8.
\]

We make the following change of coordinates using the Casimirs of \( B^2_2 \) which has the property that the inverse map is also a polynomial map

\[
(w_1, w_2, \ldots, w_8) := (P_1^1, P_2^1, P_3^2, P_4^2, u_2, u_3, u_7, u_8).
\]

(2.13)

Then the nonzero terms of the transverse Poisson bracket \( B^1_1 \) are

\[
\{w_3, w_5\} = \frac{5}{6}w_3, \quad \{w_3, w_6\} = \frac{5}{6}w_6, \quad \{w_3, w_7\} = -\frac{5}{6}w_7, \quad \{w_3, w_8\} = -\frac{5}{6}w_8,
\]

\[
\{w_4, w_5\} = -\frac{5}{6} (140w_5w_6^2 + 35w_5w_3 + 18w_4w_6), \quad \{w_4, w_6\} = -\frac{25}{12} (w_5 + 4w_3w_6),
\]

\[
\{w_4, w_7\} = \frac{5}{6} (140w_8w_3^2 + 35w_7w_3 + 18w_4w_8), \quad \{w_4, w_8\} = \frac{25}{12} (w_7 + 4w_3w_8), \quad \{w_7, w_8\} = \frac{10}{3}w_2^2,
\]

\[
\{w_5, w_6\} = -\frac{10}{3}w_6^2, \quad \{w_5, w_8\} = \frac{5}{9} (1080w_3^3 - 110w_1w_3 + 64w_4w_3 + 3w_2 - 21w_6w_8),
\]

\[
\{w_6, w_7\} = \frac{5}{9} (1080w_3^3 - 110w_1w_3 + 64w_4w_3 + 3w_2 - 21w_6w_8), \quad \{w_6, w_8\} = \frac{1}{9} (-540w_3^3 + 25w_1 - 8w_4),
\]

\[
\{w_5, w_7\} = \frac{2}{45} (59400w_3^3 - 10250w_1w_3^2 - 2840w_4w_3^2 + 375w_2w_3^2 - 3375w_6w_8w_3 - 144w_7^2)
\]

\[
+ \frac{2}{45}(450w_1w_4 + 75w_4w_7 + 75w_5w_8).
\]

In particular the vector field

\[
\chi_4 = \sum_{i=5}^{8} \{w_4, w_i\} \frac{\partial}{\partial w_i}
\]

(2.15)

is an integrable Hamiltonian vector field on \( Q \).
3  Distinguished nilpotent elements of semisimple type

Let us recall some definitions and notations from the reference [6]. The nilpotent orbit of a nilpotent element $e$ in a simple Lie algebra $\mathfrak{g}$ is called distinguished, and hence also $e$, if it has no representative in a proper Levi subalgebra of $\mathfrak{g}$. Distinguished nilpotent orbits, along with other nilpotent orbits, are classified by using weighted Dynking diagram. Distinguished nilpotent orbits are listed in the form $Z_r(a_i)$ where $Z$ is the type of the simple Lie algebra $\mathfrak{g}$, $r$ is its rank and $i$ is the number of vertices of weight 0 in the corresponding weighted Dynkin diagram. If there is another orbit of the same number $i$ of 0’s then the notation $Z_r(b_i)$ is used. When the nilpotent element $e$ is distinguished, then the eigenvalues of the semisimple element $h'$ of a choice of a $sl_2$-triples satisfying equation (2.1) are all even integers. Hence we can use the semisimple element $h := h'/2$ instead of $h'$ for better follow of arguments (see equation (3.1)).

The nilpotent element $e$ is said to be of semisimple type [15], and so its orbit, if there exists an element $g$ of the minimal eigenvalue of $ad_{h'}$ such that $e + g$ is semisimple. In this case $e + g$ is called a cyclic element. In case $e$ is also distinguished, the element $e + g$ will be regular semisimple. The list of distinguished nilpotent elements of semisimple types can be found in [15] and [11]. It consists of

1. All regular nilpotent orbits in simple Lie algebras and subregular nilpotent orbits $F_4(a_1)$, $E_6(a_1)$, $E_7(a_1)$ and $E_8(a_1)$.
2. Nilpotent orbits of type $B_{2m}(a_m)$, $D_{2m}(a_{m-1})$, $F_4(a_2)$, $F_4(a_3)$, $E_6(a_3)$, $E_7(a_5)$, $E_8(a_2)$, $E_8(a_4)$, $E_8(a_6)$ and $E_8(a_7)$.

In this article $B_{2m}(a_m)$ denote the nilpotent orbit corresponding to the partition $[2m + 1, 2m - 1, 1]$ when the Lie algebra $\mathfrak{g}$ is $so_{4m+1}$ (type $B_{2m}$).

In this section we fix a distinguished nilpotent element $L_1$ of type $Z_r(a_{r-s})$ where $Z_r(a_{r-s})$ is one of the nilpotent orbits

$$B_{2m}(a_m), D_{2m}(a_{m-1}), F_4(a_1), F_4(a_2), E_6(a_1), E_6(a_3), E_7(a_1), E_8(a_1), E_8(a_2), \text{ and } E_8(a_4).$$

Hence $r$ is always denote the rank of the corresponding Lie algebra $\mathfrak{g}$ and $r - s$ is the number of 0’s on the weighted Dynkin diagram of $L_1$. The number $s$ is introduced in this form in order to give universal statements to all nilpotent orbits under consideration. Note that the nilpotent orbit of $L_1$ is of codimension $n := r + 2(r - s) = 3r - 2s$.

We fix a semisimple element $h$ and a nilpotent element $f$ in $\mathfrak{g}$ such that the set $A = \{L_1, h, f\}$ forms a $sl_2$-triple with relations

$$[h, L_1] = L_1, \quad [h, f] = -f, \quad [L_1, f] = 2h. \quad (3.1)$$

Note that these relations are different from the ones given in (2.1). The eigenvalues of $ad_h$ are all integers. Let $\kappa$ denote the maximal eigenvalue. The Dynkin grading associated to $L_1$ takes the form:

$$\mathfrak{g} = \bigoplus_{-\kappa} \mathfrak{g}_i; \quad \mathfrak{g}_i := \{g \in \mathfrak{g} : ad_h g = ig\} \quad (3.2)$$

which is a good grading for $A$. Note that $L_1$ is a distinguished nilpotent element of semisimple type [15]. Hence, we fix an element $K_1 \in \mathfrak{g}_{-\kappa}$ such that the cyclic element $Y_1 := L_1 + K_1$ is regular semisimple. In what follows we give a general setup associated to cyclic elements following the work of Kostant for the case of cyclic elements associated to regular nilpotent elements [17].

6
The following give a bijective map between the sets $E(g)$ and $E(e)$

$$
\nu_i = \begin{cases} 
\eta_i, & i \leq s; \\
\eta_i + \kappa + 1, & i > s.
\end{cases}
$$

Let $Y_1, Y_2, \ldots, Y_r$ be a basis of $\mathfrak{h}'$ of eigenvectors of $w$ such that $w(Y_i) = e^{\nu_i} Y_i$. Then the elements $Y_i$ will have the form

$$
Y_i = L_i + K_i; \quad L_i \in \mathfrak{g}_{\eta_i}, \quad K_i \in \mathfrak{g}_{\eta_i + (\kappa + 1)}, \quad i = 1, \ldots, r.
$$

Let $\mathfrak{h}' := \mathfrak{g}^Y_1$ be the Cartan subalgebra containing $Y_1$. It is known as the opposite Cartan Subalgebra. Then the adjoint group element $w$ defined by

$$
w := \exp \frac{2\pi i}{\kappa + 1} \text{ad}_h
$$

acts on $\mathfrak{h}'$ as a representative of a regular conjugacy class $[w]$ in the Weyl group $W(\mathfrak{g})$ of $\mathfrak{g}$ of order $\kappa + 1$.

The element $Y_1$ is an eigenvector of $w$ of eigenvalue $\epsilon$ where $\epsilon$ is the primitive $(\kappa + 1)$th root of unity. We also define the multiset $E(L_1)$ which consists of natural numbers $\eta_i$, $i = 1, \ldots, r$, such that $\epsilon^\nu$’s are the eigenvalues of the action of $w$ on $\mathfrak{h}'$. We call $E(L_1)$ the exponents of the nilpotent element $L_1$. Our justification of the name exponents for $E(L_1)$ is that, in this context, the exponents $E(g)$ of the Lie algebra equal the exponents of the regular nilpotent element [17] (a nilpotent element of type $Z_r(a_0)$).

In table 1, we list in the first 2 columns the elements of multiset $E(L_1)$ and the exponents of the regular nilpotent elements, hence of $E(g)$. We will denote the elements $E(g)$ by $\nu_i$ and we assume

$$
\nu_1 \leq \nu_2 \leq \ldots \leq \nu_r.
$$

Note that we give $E(L_1)$ is special order such that the following important relation between $E(L_1)$ and $E(g)$ is simple to state.

**Lemma 3.1.** The following give a bijective map between the sets $E(g)$ and $E(e)$

$$
\nu_i = \begin{cases} 
\eta_i, & i \leq s; \\
\eta_i + \kappa + 1, & i > s.
\end{cases}
$$

| Orbit | $Z_r(a_{r-s})$ | $B_{2m}(a_0)$ | $B_{2m}(a_m)$ | $D_{2m}(a_0)$ | $D_{2m}(a_{m-1})$ | $F_4(a_0)$ | $F_4(a_1)$ | $F_4(a_2)$ | $E_6(a_0)$ | $E_6(a_1)$ | $E_6(a_3)$ | $E_7(a_0)$ | $E_7(a_1)$ | $E_7(a_2)$ | $E_7(a_3)$ | $E_8(a_0)$ | $E_8(a_1)$ | $E_8(a_2)$ | $E_8(a_3)$ | $E_8(a_4)$ |
|-------|----------------|---------------|---------------|---------------|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $W(L_1)$ | $\eta_1, \eta_2, \ldots, \eta_r$ | $\eta_{r+1}, \ldots, \eta_r$ | $\eta_{r+1}, \ldots, \eta_r$ | $1, 3, \ldots, 4m - 1$ | $1, 3, \ldots, 2m - 1$ | $1, 3, \ldots, 2m - 1$ | $1, 3, \ldots, 4m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ | $1, 3, \ldots, 2m - 3$ |
| Table 1: Weights and Exponents of some distinguished nilpotent elements under consideration |
The commutators \([Y_i, Y_j] = 0\) imply that the set \(\{L_1, \ldots, L_r\}\) generates a commutative subalgebra of \(g^{L_1}\).

Hence upon considering the restriction of the adjoint representation of the \(sl_2\)-subalgebra \(\mathcal{A}\) generated by \(\{L_1, h, f\}\), the vectors \(L_i\) are maximal weight vectors of irreducible \(\mathcal{A}\)-submodules of dimensions \(2\eta_i + 1\). We observe that the total number of irreducible \(\mathcal{A}\)-submodules is \(n := 3r - 2s\). The numbers

\[
\eta_{r+1}, \ldots, \eta_n \label{3.6}
\]

are given in the third column of table 1. Let us fix a decomposition of \(g\) into irreducible \(\mathcal{A}\)-submodules, i.e.

\[
g = \bigoplus_{j=1}^n \mathcal{V}_j \label{3.7}
\]

where \(\dim \mathcal{V}_j = 2\eta_j + 1\) and \(L_i\) is maximal weight vector of \(\mathcal{V}_i\) for \(i = 1, \ldots, r\). We found it more convenient to denote also the maximum vectors of the remaining spaces \(\mathcal{V}_j\) by \(L_j\). The numbers \(\eta_1, \ldots, \eta_n\) are known in the literature as the weights of the nilpotent element \(L_1\) and will be denoted \(W(L_1)\).

Recall that we fix the invariant nondegenerate bilinear form \(\langle \cdot, \cdot \rangle\) on \(g\) such that \(\langle L_1| f \rangle = 1\). Let us define the matrix of its restriction to \(h'\)

\[
A_{ij} = \langle Y_i|Y_j \rangle \label{3.8}
\]

**Proposition 3.2.** The matrix \(A_{ij}\) is nondegenerate and antidiagonal in the sense that

\[
A_{ij} = 0, \text{ if } \eta_i + \eta_j \neq \kappa + 1. \label{3.9}
\]

**Proof.** It follows from the properties of Cartan subalgebras that the matrix \(A_{ij}\) is nondegenerate. We will use the fact that the matrix \(\langle \cdot, \cdot \rangle\) is a nondegenerate invariant bilinear form on \(h'\). Hence for any element \(Y_i\) there exists an element \(Y_j\) such that \(\langle Y_i | Y_j \rangle \neq 0\). But for the Weyl group element \(w\) we have the equality

\[
\langle Y_i|Y_j \rangle = \langle wY_i|wY_j \rangle = \exp \frac{2(\eta_i + \eta_j)\pi i}{\kappa + 1} \langle Y_i|Y_j \rangle
\]

forces \(\eta_i + \eta_j = \kappa + 1\) in case \(\langle Y_i|Y_j \rangle \neq 0\). \(\square\)

### 4 Argument shift method and adjoint quotient map

We keep the notations and definition from the last section but we assume the nilpotent element \(L_1\) is of type \(B_{2m}(a_m), D_{2m}(a_{m-1}), F_4(a_2)\) or \(E_6(a_3)\). Then the rank \(r\) is even and we set \(m := r/2\).

**Lemma 4.1.** The elements \(Y_i, i > 1\) can be normalized such that the only nonzero values of \(\langle \cdot, \cdot \rangle\) on the basis \(Y_i\) are given as follows

\[
\langle Y_i|Y_{m+i} \rangle = \kappa + 1, \quad \langle Y_{m+i}|Y_{2m-i+1} \rangle = \kappa + 1; \quad i = 1, \ldots, m, \label{4.1}
\]

i.e.

\[
\langle Y_i|Y_j \rangle = \kappa + 1, \quad \text{if} \quad i + j = m + 1 \text{ or } i + j = 3m + 1 \label{4.2}
\]

**Proof.** Form the last lemma the elements \(Y_1, \ldots, Y_r\) can be grouped to subsets of 4 or 2 elements where the restriction of \(\langle \cdot, \cdot \rangle\) will be nondegenerate and has the form

\[
\begin{pmatrix}
0 & 0 & * & * \\
0 & 0 & * & * \\
* & * & 0 & 0 \\
* & * & 0 & 0
\end{pmatrix}
\text{ or }
\begin{pmatrix}
0 & * \\
* & 0 \\
0 & 0
\end{pmatrix}, \label{4.3}
\]

respectively. Using simple linear changes, they can be transform to blocks of anti-diagonal matrices without losing the fact that their are eigenvectors of the action of \(w\) on \(h'\). \(\square\)
We will assume that the basis $Y_i$ of $h'$ are normalized and satisfy the hypothesis of the previous lemma. Then we get the following identities

**Lemma 4.2.**

\[
\langle L_i | K_j \rangle = \eta_j (\delta_{i+j,m+1} + \delta_{i+j,3n+1}) \tag{4.4}
\]

**Proof.** Recall that

\[
Y_i = L_i + K_i; \quad L_i \in \mathfrak{g}_h, \quad K_i \in \mathfrak{g}_{h-(\kappa+1)}, \quad i = 1, \ldots, r. \tag{4.5}
\]

Using the identity $0 = [Y_i, Y_j] = [L_i, K_j] + [K_i, L_j]$ with the invariant bilinear form yields

\[
0 = \langle h|[L_i, K_j] + [K_i, L_j] \rangle = (\kappa - \eta_j + 1)\langle L_i | K_j \rangle + \eta_j \langle K_i | L_j \rangle. \tag{4.6}
\]

This equation with the normalization $\langle Y_i | Y_j \rangle = \langle L_i | K_j \rangle + \langle K_i | K_j \rangle = (\kappa + 1)(\delta_{i+j,m+1} + \delta_{i+j,3m+1})$ yield the required identity. \hfill \square

Recall that the Hamiltonian vector fields under Lie-Poisson structure of a function $F$ at a point $x \in \mathfrak{g}$ takes the form

\[
\chi_F(x) := \text{ad}_{\nabla F(x)} x = [\nabla F(x), x]. \tag{4.7}
\]

Also, the symplectic leave through $x$ coincides with the adjoint orbit of $x$. Let $G$ be the adjoint group of $\mathfrak{g}$. Using Chevalley’s theorem, we fix a complete system of homogeneous generators $P_1, \ldots, P_r$ of the algebra $S(\mathfrak{g}^*)^G$ of invariant polynomials under the adjoint group action. We assume that degree $P_i$ equals $\nu_i + 1$. It is known that these generators give a complete set of global Casimir functions of the Lie-Poisson bracket. In particular

\[
\nabla P_i(x) \in \mathfrak{g}^x, \quad \forall x \in \mathfrak{g}, \quad i = 1, \ldots, r \tag{4.8}
\]

Consider the adjoint quotient map

\[
\Psi : \mathfrak{g} \to \mathbb{C}, \quad \Psi(x) = (P_1(x), \ldots, P_r(x)). \tag{4.9}
\]

From the work of Kostant in [18] the rank of $\Psi$ at $x$ equals $r$ if and only if $x$ is regular element in $\mathfrak{g}$, i.e. $\dim \mathfrak{g}^x = r$. Later, Slodowy showed that the rank of $\Psi$ is $r - 1$ at subregular nilpotent elements [22]. Finally, the rank of $\Psi$ at distinguished nilpotent elements in $\mathfrak{g}$ was stated without proof by Richardson [20] except the orbit $E_k(a_2)$. However, the ideas in his work inspire many of the results in this article. Later, we give alternative way to find the rank of $\Psi$ for the nilpotent orbits under consideration which includes $E_k(a_2)$ (see corollary 4.7).

It is obvious that for any $x \in \mathfrak{g}$ the rank of $\Psi$ at $x$ equals the dimension of the vector space generated by $\nabla P_i(x)$ Since $Y_1$ is regular, the rank of $\Psi$ at $Y_1$ equals $r$. Thus the gradients $\nabla P_i(Y_1)$ are linearly independent and in fact a basis of $h'$. We use this remark in the following proposition.

Under the normalization given in lemma 4.2, we fix a basis $e_0, e_1, e_2, \ldots$ for $\mathfrak{g}$ such that:

1. The first $n + r$ are given in the following order

\[
K_m, L_m, L_1, L_2, \ldots, L_{m-1}; L_{m+1}, \ldots, L_n, K_1, K_2, \ldots, K_{m-1}; K_{m+1}, \ldots, K_{2m}, \tag{4.10}
\]

2. $\langle e_i | Y_1 \rangle \neq 0$ if and only if $i = 0$ or $i = 1$.

Then we define the linear coordinates

\[
x_i(g) := \langle e_i | g \rangle, \quad i = 0, 1, 2, \ldots \tag{4.11}
\]

In what follows we will trace the dependence of the invariant polynomials on the coordinates $x_0$ and $x_m$ since they are dual to $Y_1$. Note that $\nabla F$ of a function $F$ on $\mathfrak{g}$ will be given by the formula $\nabla F = \sum \frac{\partial F}{\partial x_i} e_i$. 

9
Lemma 4.3. The matrix with entries $\frac{\partial P_i}{\partial x_j}(Y_1)$, $i,j = 1,...,r$, is non-degenerate. Moreover, $P_i$ have the following form

$$P_i = \sum_{s=0}^{i-1} \sum_{j=2}^{r} c_{i,j,s}x_1^s x_0^{i-s}(x_j + x_{j+n-1}) + c_{i,1,s}x_1^s x_0^{i-s}(x_0 + x_1)H_i(x),$$

(4.12)

$$c_{i,j,s} \in \mathbb{C}, \frac{\partial H_i}{\partial x_k}(Y_1) = 0, \forall k.$$  

(4.13)

Proof. Let $P$ be homogeneous invariant polynomial of degree $\xi$. Note that $\xi \geq 2$. Then $\nabla P(Y_1) \in \mathfrak{g}^{Y_1} = \mathfrak{h}'$. Since $\mathfrak{h}'$ has basis $Y_i = L_i + K_i$, we get

$$\nabla P(Y_1) = \sum_{i=1}^{r} c_i (L_i + K_i) = \sum_{i=1}^{r} c_i Y_i$$  

(4.14)

where

$$c_i = \begin{cases} 
\frac{\partial P}{\partial x_{i+1}}(Y_1) = \frac{\partial P}{\partial x_{i+n-1}}(Y_1), & i < m; \\
\frac{\partial P}{\partial x_{i}}(Y_1) = \frac{\partial P}{\partial x_{0}}(Y_1), & i = m; \\
\frac{\partial P}{\partial x_{i}}(Y_1) = \frac{\partial P}{\partial x_{i+n-1}}(Y_1), & i > m.
\end{cases}$$

(4.15)

By construction, the values of the coordinates at $Y_i$ are all zero except $x_m(Y_1)$ and $x_0(Y_1)$. We conclude that for $\frac{\partial P}{\partial x_i}(Y_1) \neq 0$, the polynomial $\frac{\partial P_i}{\partial x_i}$ must contains a term of the form

$$\sum_{s=0}^{\xi-1} c_{i,s}x_1^s x_0^{\xi-s-1}, \quad c_{i,s} \in \mathbb{C}$$

(4.16)

This gives (4.12). The non-degeneracy condition follows from the fact that $\nabla P_i(Y_1)$ are a basis for $\mathfrak{h}'$. □

We consider again the Slodowy slice $Q = L_1 + \mathfrak{g}^l$. A good coordinates of Slodowy slice are $x_1,...,x_n$ as defined above since $\mathfrak{g}^{l-1}$ is orthogonal to $\langle \cdot, \cdot \rangle$. Note that from lemma 4.2, $x_0(q) = \langle K_m | L_1 \rangle = \eta_m$ is constant for every $q \in Q$. We will use the following result due to Slodowy.

Theorem 4.4. [22] The restriction of an invariant polynomial $P$ of degree $\mu$ to $Q$ is quasi-homogeneous polynomial of degree $\mu$ when degree $x_i$ equals eigenvalue of $e_i$, under $ad_h$ for $i = 1,\ldots, r$.

This leads to the following refinement of the last lemma

Proposition 4.5. The invariant polynomials $P_i$ can be normalized such that their restrictions $P_i^Q$ to $Q$ will have the from

$$P_i^Q(x_1,\ldots,x_n) := H_i^Q(x) + \begin{cases} 
\sum_{\deg x_j = \deg P_i} c_{j,x_j}, & i = 1,\ldots,s-1; \\
x_1, & i = s, \\
x_1 x_i, & i = s+1,\ldots,r
\end{cases}$$

(4.17)

where $\frac{\partial H_i^Q}{\partial x_k} |_{x_i = 0} = 0, \forall k$. Moreover, the square matrix $\frac{\partial P_i^Q}{\partial x_j}$ evaluated at $x_k = \delta_{1k}$ is nondegenerate.

Proof. The restriction $P_i^Q$ of $P_i$ to is obtained by setting $x_i = 0$ for $i > n$ and $x_0 = \eta_m$. Hence from the quasihomogeneity and the formula (4.12) we get
\[ P_i^Q(x_1, \ldots, x_n) = \sum_{s=0}^{\nu_i-1} \sum_{\deg P_i - \deg x_j = s(k+1)} \bar{c}_{i,j,s} x_i^s x_j + H_i^Q(x), \quad \bar{c}_{i,j,s} \in \mathbb{C} \]  

(4.18)

where \( H_i^Q(x) \) is the restriction of \( H_i \). Then it is clear that \( \frac{\partial H_i^Q}{\partial x_k} |_{x_k = \delta_{i,1}} = 0 \). Then from the relation between \( E(L_1) \) and \( E(\mathfrak{g}) \) observed in lemma 3.1, \( P_i^Q \) will get the form

\[ P_i^Q(x_1, \ldots, x_n) = \sum_{\deg P_i - \deg x_j = s(k+1)} \bar{c}_{i,j,s} x_i^s x_j + H_i^Q(x) \]  

(4.19)

where \( s = 0 \) for \( i \leq s \) and \( s = 1 \) otherwise. For the nondegeneracy condition, observe the variable \( x_1 \) appears with the same power in each \( P_i^Q \). Hence the value of the square matrix \( \frac{\partial P_i^Q}{\partial x_j} \) when \( x_k = \delta_{1,k} \) is nonzero constant multiplication of the nondegenerate square matrix \( \frac{\partial P_j^Q}{\partial x_j}(Y_1) \). Finally, the normalization for \( P_i^Q, i \geq s \) comes from normalizing first \( P_s^Q \) which is possible from the structure of the set \( E(L_1) \). Then use \( P_s^Q P_i^Q, i < s \) to eliminate unwanted variables repeatedly appears in \( P_i^Q \) with \( \deg P_j^Q = \deg P_s + \deg P_i \).

We assume from now on that the invariant polynomials satisfies the hypothesis of the previous proposition. The following two corollaries follows from this normalization.

**Corollary 4.6.** The following define a coordinates on \( Q \)

\[ t_i := \begin{cases} P_i^Q, & i = 1, \ldots, s; \\ \frac{\partial P_i^Q}{\partial x_i}, & i = s+1, \ldots, r; \\ x_i, & i = r+1, \ldots, n \end{cases} \]  

(4.20)

Where the Jacobian equals \( \pm 1 \), i.e. the inverse map is a quasihomogeneous polynomials of the same degrees.

Note that \( t_i = P_i^Q \) for \( i = 1, \ldots, s \) where the nilpotent elements is of type \( Z_r(a_{r-s}) \). Hence, in this coordinates the restriction of the quotient map \( \Psi \) to \( Q \) takes the form

\[ \Psi^Q(t_1, t_2, \ldots, t_n) = (t_1, \ldots, t_{r-s}, P_{r-s+1}^Q, \ldots, P_r^Q). \]  

(4.21)

**Corollary 4.7.** The rank of the quotient map \( \Psi \) at \( L_1 \) equals \( s \).

**Proof.** From quasi-homogeneity, the rank of \( \Psi \) at \( L_1 \) is the same as the rank of \( \Psi^Q \) at the origin which equals \( r-s \). \( \square \)

We fix the bihamiltonain structure \((2.8)\) in \( \mathfrak{g} \) with \( k = K_1 \). Let \( B_1^Q \) and \( B_2^Q \) be their reductions to \( Q \). Recall that the dimension of \( Q \) is \( n = r + 2(r-s) \) and the rank of the transverse Poisson pencil max\(_{\lambda \in \mathbb{C}} B_\lambda^Q = n - r = 2(r-s) \). Define the polynomials \( P_i^j \) from the formula

\[ P_i^Q(x + \lambda a) = P_i^1(x) + \lambda P_i^2(x) \]  

(4.22)

Here \( P_i^2 = 0 \) for \( i = 1, \ldots, s \). This gives \( \dim Q - (r-s) = 2r-s \) functions in involution for both Poisson brackets where \( P_i^1 \) are Casimirs of \( B_1^Q \). Hence, to get completely integrable systems for \( B_1^Q \), we need to show that this family of functions are independent at some point of \( Q \).
Theorem 4.8. The functions $P^i_j$ are linearly independent on some open dense subset. In particular, they form completely integrable system under $B^Q_1$.

Proof. We consider what is called the momentum map in the coordinates developed in the last section

$$
\Phi(t_1, \ldots, t_n) := (P^1_1, \ldots, P^1_s, P^2_s, P^1_{s+1}, \ldots, P^1_r),
$$

(4.23)

$$
:= (t_1, \ldots, t_r, P^1_{s+1}, \ldots, P^1_r).
$$

Observe that the functions are independent if and only if the map $\Phi$ is regular. This means the matrix $\frac{\partial P^i_{s+i}}{\partial t_{r+j}}$; $1 \leq i \leq r - s$ and $1 \leq j \leq n - r$ is of maximal rank. which is equivalent to the statement that the quotient map $\Psi^Q$ has a regular points in the subvariety of $Q$ defined as the zero of the determinant of the matrix $\frac{\partial P^i_{s+i}}{\partial t_{s+j}}$; $1 \leq i, j \leq r - s$. The later follows from the fact that the set of regular points of the quotient map is dense open subset in $g$, and hence in $Q$. Note that the determinant is nonzero since it must contains, up to constant, the monomial $t_1^{r-s}$.

5 Other distinguished nilpotent elements

We tried to give a uniform statements which covers all nilpotent orbits given in table 1. But we found that we are introducing more notations and definition than necessary to cover a finite number of orbits. We hope after the last section, it is clear for the reader how to extend the statements and the proofs in the case the nilpotent element $L_1$ is of type $F_4(a_1)$, $E_6(a_1)$, $E_7(a_1)$, $E_8(a_2)$ or $E_8(a_4)$. One has only to introduce a normalization of the basis of $h'$ as obtained in lemma 4.1 and modify the statements accordingly to the relation between exponents $E(L_1)$ and $E(\mathfrak{g})$ given in lemma 3.1. This leads to the prove of our main theorem 1.1.

There are 5 non-regular distinguished nilpotent orbits of semisimple type which are not included in table 1. In these cases, argument shift method produce the right number of functions but it is hard to prove, even by direct computations, that they are linearly independent (the proof of theorem 4.8 is invalid in these cases). The reason is that the dimension of the nilpotent orbit will be too small so that the argument shift methods produce set of functions which are neither Casimirs nor can be included as coordinates on Slodowy slice. Hence it is not possible to relate regularity of the momentum map to the regularity of the adjoint quotient map as we did in proving theorem 4.8.

In future publications we will generalize the results of this article to all distinguished nilpotent elements in simple Lie algebras.

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