A parametrized family of Tversky metrics connecting the Jaccard distance to an analogue of the normalized information distance

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Abstract. Jiménez, Becerra, and Gelbukh (2013) defined a family of “symmetric Tversky ratio models” $S_{\alpha, \beta}$, $0 \leq \alpha \leq 1$, $\beta > 0$. Each function $D_{\alpha, \beta} = 1 - S_{\alpha, \beta}$ is a semimetric on the powerset of a given finite set. We show that $D_{\alpha, \beta}$ is a metric if and only if $0 \leq \alpha \leq \frac{1}{2}$ and $\beta \geq \frac{1}{1-\alpha}$. This result is formally verified in the Lean proof assistant. The extreme points of this parametrized space of metrics are $J_1 = D_{1/2, 2}$, the Jaccard distance, and $J_\infty = D_{0, 1}$, an analogue of the normalized information distance of M. Li, Chen, X. Li, Ma, and Vitányi (2004).

Keywords: Jaccard distance · Normalized information distance · metric space · proof assistant.

1 Introduction

Distance measures (metrics), are used in a wide variety of scientific contexts. In bioinformatics, M. Li, Badger, Chen, Kwong, and Kearney [12] introduced an information-based sequence distance. In an information-theoretical setting, M. Li, Chen, X. Li, Ma and Vitányi [13] rejected the distance of [12] in favor of a normalized information distance (NID). The Encyclopedia of Distances [3] describes the NID on page 205 out of 583, as

$$\frac{\max\{K(x \mid y^*), K(y \mid x^*)\}}{\max\{K(x), K(y)\}}$$

where $K(x \mid y^*)$ is the Kolmogorov complexity of $x$ given a shortest program $y^*$ to compute $y$. It is equivalent to be given $y$ itself in hard-coded form:

$$\frac{\max\{K(x \mid y), K(y \mid x)\}}{\max\{K(x), K(y)\}}$$

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Another formulation (see \[13\] page 8) is
\[
\frac{K(x, y) - \min\{K(x), K(y)\}}{\max\{K(x), K(y)\}}.
\]

The fact that the NID is in a sense a normalized metric is proved in \[13\]. Then in 2017, while studying malware detection, Raff and Nicholas \[14\] suggested Lempel–Ziv Jaccard distance (LZJD) as a practical alternative to NID. As we shall see, this is a metric. In a way this constitutes a full circle: the distance in \[12\] is itself essentially a Jaccard distance, and the LZJD is related to it as Lempel–Ziv complexity is to Kolmogorov complexity. In the present paper we aim to shed light on this back-and-forth by showing that the NID and Jaccard distances constitute the endpoints of a parametrized family of metrics.

For comparison, the Jaccard distance between two sets \(X\) and \(Y\), and our analogue of the NID, are as follows:
\[
J_1(X, Y) = \frac{|X \setminus Y| + |Y \setminus X|}{|X \cup Y|} = 1 - \frac{|X \cap Y|}{|X \cup Y|} \tag{1}
\]
\[
J_{\infty}(X, Y) = \frac{\max\{|X \setminus Y|, |Y \setminus X|\}}{\max\{|X|, |Y|\}} \tag{2}
\]

Our main result Theorem 11 shows which interpolations between these two are metrics.

Incidentally, the names of \(J_1\) and \(J_{\infty}\) come from the observation that they are special cases of \(J_p\) given by
\[
J_p(A, B) = \left(2 \cdot \frac{|B \setminus A|^p + |A \setminus B|^p}{|A|^p + |B|^p + |B \setminus A|^p + |A \setminus B|^p}\right)^{1/p} = \begin{cases} J_1(A, B) & p = 1 \\ J_{\infty} & p \to \infty \end{cases}
\]

We conjecture that \(J_p\) is a metric for each \(p\), but shall not attempt to prove it here.

The way we arrived at Section 1 as an analogue of NID is via Lempel–Ziv complexity. While there are several variants \[11,18,19\], the LZ 1978 complexity \[19\] of a sequence is the cardinality of a certain set, the dictionary.

**Definition 1** Let \(\text{LZSet}(A)\) be the Lempel–Ziv dictionary for a sequence \(A\). We define LZ–Jaccard distance LZJD by
\[
\text{LZJD}(A, B) = 1 - \frac{\text{LZSet}(A) \cap \text{LZSet}(B)}{\text{LZSet}(A) \cup \text{LZSet}(B)}.
\]

It is shown in \[12\] Theorem 1] that the triangle inequality holds for a function which they call an information-based sequence distance. Later papers give it the notation \(d_s\) in \[13\] Definition V.1], and call their normalized information distance \(d\). Raff and Nicholas \[14\] introduced the LZJD and did not discuss the appearance of \(d_s\) in \[13\] Definition V.1], even though they do cite \[13\] (but not \[12\]).
Kraskov et al. [10,9] use $D$ and $D'$ for continuous analogues of $d_s$ and $d$ in [13] (which they cite). The Encyclopedia calls it the normalized information metric,

$$\frac{H(X | Y) + H(X | Y)}{H(X,Y)} = 1 - \frac{I(X;Y)}{H(X,Y)}$$

or Rajski distance [15].

This $d_s$ was called $d$ by [12] — see Table 1. Conversely, [13, near Definition Reference Jaccard notation NID notation

| Reference | Jaccard notation | NID notation |
|-----------|-----------------|--------------|
| [12]      | $d_s$           | $d$          |
| [9]       | $D$             | $D'$         |
| [14]      | LZJD            | NCD          |

Table 1: Overview of notation used in the literature. (It seems that authors use simple names for their favored notions.)

Remark 2 Ridgway [4] observed that the entropy-based distance $D$ is essentially a Jaccard distance. No explanation was given, but we attempt one as follows. Suppose $X_1, X_2, X_3, X_4$ are iid Bernoulli($p = 1/2$) random variables, $Y$ is the random vector $(X_1, X_2, X_3)$ and $Z$ is $(X_2, X_3, X_4)$. Then $Y$ and $Z$ have two bits of mutual information $I(Y,Z) = 2$. They have an entropy $H(Y) = H(Z) = 3$ of three bits. Thus the relationship $H(Y,Z) = H(Y) + H(Z) - I(Y,Z)$ becomes a Venn diagram relationship $|\{X_1, X_2, X_3, X_4\}| = |\{X_1, X_2, X_3\}| + |\{X_2, X_3, X_4\}| - |\{X_2, X_3\}|$. The relationship to Jaccard distance may not have been well known, as it is not mentioned in [9,2,12,1].

A more general setting is that of STRM (Symmetric Tversky Ratio Models), Theorem 10. These are variants of the Tversky index (Theorem 4) proposed in [7].

Definition 3 A semimetric on $X$ is a function $d : X \times X \to \mathbb{R}$ that satisfies the first three axioms of a metric space, but not necessarily the triangle inequality: $d(x,y) \geq 0$, $d(x,y) = 0$ if and only if $x = y$, and $d(x,y) = d(y,x)$ for all $x,y \in X$.

Definition 4 ([17]) For sets $X$ and $Y$ the Tversky index with parameters $\alpha, \beta \geq 0$ is a number between 0 and 1 given by

$$S(X,Y) = \frac{|X \cap Y|}{|X \cap Y| + \alpha |X \setminus Y| + \beta |Y \setminus X|}.$$ We also define the corresponding Tversky dissimilarity $d^{T}_{\alpha, \beta}$ by

$$d^{T}_{\alpha, \beta}(X,Y) = \begin{cases} 1 - S(X,Y) & \text{if } X \cup Y \neq \emptyset; \\ 0 & \text{if } X = Y = \emptyset. \end{cases}$$
To motivate Theorem 3, we include the following lemma without proof.

**Lemma 5** Suppose \( d \) is a metric on a collection of nonempty sets \( \mathcal{X} \), with \( d(X,Y) \leq 2 \) for all \( X,Y \in \mathcal{X} \). Let \( \hat{\mathcal{X}} = \mathcal{X} \cup \{ \emptyset \} \) and define \( \hat{d} : \hat{\mathcal{X}} \times \hat{\mathcal{X}} \to \mathbb{R} \) by stipulating that for \( X,Y \in \mathcal{X} \),

\[
\hat{d}(X,Y) = d(X,Y); \quad d(X,\emptyset) = 1 = d(\emptyset,X); \quad d(\emptyset,\emptyset) = 0.
\]

Then \( \hat{d} \) is a metric on \( \hat{\mathcal{X}} \).

**Theorem 6** (Gragera and Suppakitpaisarn [5,6]) The optimal constant \( \rho \) such that

\[
d^T_{\alpha,\beta}(X,Y) \leq \rho(d^T_{\alpha,\beta}(X,Y) + d^T_{\alpha,\beta}(Y,Z))
\]

for all \( X,Y,Z \) is

\[
\frac{1}{2} \left( 1 + \sqrt{\frac{1}{\alpha \beta}} \right).
\]

**Corollary 7** \( d^T_{\alpha,\beta} \) is a metric only if \( \alpha = \beta \geq 1 \).

**Proof.** Clearly, \( \alpha = \beta \) is necessary to ensure \( d^T_{\alpha,\beta}(X,Y) = d^T_{\alpha,\beta}(Y,X) \). Moreover \( \rho \leq 1 \) is necessary, so Theorem 6 gives \( \alpha \beta \geq 1 \).

**Definition 8** The Szymkiewicz—Simpson coefficient is defined by

\[
\text{overlap}(X,Y) = \frac{|X \cap Y|}{\min(|X|,|Y|)}
\]

We may note that \( \text{overlap}(X,Y) = 1 \) whenever \( X \subseteq Y \) or \( Y \subseteq X \), so that \( 1 - \text{overlap} \) is not a metric.

**Definition 9** The Sørensen–Dice coefficient is defined by

\[
\frac{2|X \cap Y|}{|X| + |Y|}.
\]

**Definition 10** (Section 2) Let \( \mathcal{X} \) be a collection of finite sets. We define \( S : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) as follows. For sets \( X,Y \in \mathcal{X} \) we define \( m(X,Y) = \min\{|X \setminus Y|,|Y \setminus X|\} \) and \( M(X,Y) = \max\{|X \setminus Y|,|Y \setminus X|\} \). The symmetric TRM is defined by

\[
S(X,Y) = \frac{|X \cap Y| + \text{bias}}{|X \cap Y| + \text{bias} + \beta(m + (1-\alpha)M)}
\]

The unbiased symmetric TRM is the case where \( \text{bias} = 0 \), which is the case we shall assume we are in for the rest of this paper. The Tversky semimetric \( D_{\alpha,\beta} \) is defined by \( D_{\alpha,\beta}(X,Y) = 1 - S(X,Y) \), or more precisely

\[
D_{\alpha,\beta} = \begin{cases} 
\beta \frac{am+(1-\alpha)M}{|X \cap Y| + \beta(\alpha m + (1-\alpha)M)}, & \text{if } X \cup Y \neq \emptyset; \\
0, & \text{if } X = Y = \emptyset.
\end{cases}
\]
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Fig. 1: A Tversky semimetric $D_{\alpha,\beta}$ is a metric if and only if $(\alpha, \beta)$ belongs to the green region. The parameter values corresponding to the Jaccard distance $J_1$, the analogue of normalized information distance analogue $J_\infty$, the Sørensen–Dice semimetric, and the Szymkiewicz—Simpson semimetric are indicated.

Note that for $\alpha = 1/2$, $\beta = 1$, the STRM is equivalent to the Sørensen–Dice coefficient. Similarly, for $\alpha = 1/2$, $\beta = 2$, it is equivalent to Jaccard’s coefficient.

Our main result is (see Figure 1):

**Theorem 11** Let $0 \leq \alpha \leq 1$ and $\beta > 0$. Then $D_{\alpha,\beta}$ is a metric if and only if $0 \leq \alpha \leq 1/2$ and $\beta \geq 1/(1 - \alpha)$.

Theorem 11 gives the converse to the Gragera and Suppakitpaisarn inspired

**Corollary 12** The Tversky dissimilarity $d_{\alpha,\beta}^T$ is a metric iff $\alpha = \beta \geq 1$.

**Proof.** Suppose the Tversky dissimilarity $d_{\alpha,\beta}^T$ is a semimetric. Let $X, Y$ be sets with $|X \cap Y| = |X \setminus Y| = 1$ and $|Y \setminus X| = 0$. Then

$$1 - \frac{1}{1 + \beta} = d_{\alpha,\beta}^T(Y, X) = d_{\alpha,\beta}^T(X, Y) = 1 - \frac{1}{1 + \alpha},$$

hence $\alpha = \beta$. Let $\gamma = \alpha = \beta$.

Now, $d_{\alpha,\beta}^T = D_{\alpha_0,\beta_0}$ where $\alpha_0 = 1/2$ and $\beta_0 = 2\gamma$. Indeed, let $m = \min\{|X \setminus Y|, |Y \setminus X|\}$ and $M = \max\{|X \setminus Y|, |Y \setminus X|\}$. Since

$$D_{\alpha_0,\beta_0} = \beta_0 \frac{\alpha_0 m + (1 - \alpha_0) M}{|X \cap Y| + \beta_0 [\alpha_0 m + (1 - \alpha_0) M]},$$

$$D_{\frac{1}{2}, 2\gamma} = 2\gamma \frac{\frac{1}{2} m + (1 - \frac{1}{2}) M}{|X \cap Y| + 2\gamma [\frac{1}{2} m + (1 - \frac{1}{2}) M]}$$
\[
\gamma |X \setminus Y| + |Y \setminus X| = 1 - |X \cap Y| + |X \setminus Y| + |Y \setminus X| = d_{\gamma,\gamma}^T.
\]

By Theorem 11, \(d_{\gamma,\gamma}^T\) is a metric if and only if \(\beta_0 \geq \frac{1}{1 - \alpha_0}\). This is equivalent to \(2\gamma \geq 2\), i.e., \(\gamma \geq 1\).

The truth or falsity of Theorem 12 does not arise in Gragera and Suppakitpaisarn’s work, as they require \(\alpha, \beta \leq 1\) in their definition of Tversky index. We note that Tversky [17] only required \(\alpha, \beta \geq 0\).

### 2 Results

**Lemma 13** Let \(u, v, w, \epsilon > 0\). Then

\[
\frac{1}{u} \leq \frac{1}{v + \frac{1}{w}} \implies \frac{1}{u + \epsilon} \leq \frac{1}{v + \epsilon} + \frac{1}{w + \epsilon}.
\]

**Proof.** It is of course equivalent to show

\[
vw \leq uv + uw \implies (v + \epsilon)(w + \epsilon) \leq (u + \epsilon)(w + \epsilon) + (u + \epsilon)(v + \epsilon),
\]

which reduces to

\[(v + w)\epsilon \leq (u + w)\epsilon + (u + v)\epsilon + \epsilon^2,
\]

which is clearly the case.

**Lemma 14** Suppose \(a(x, y) = a_{xy}\) and \(b(x, y) = b_{xy}\) are functions. Suppose the function \(d\) given by \(d(x, y) = a_{xy}/b_{xy}\) is a metric, and \(\epsilon > 0\) is a real number. Let \(\tilde{d}(x, y) = \frac{a_{xy}}{b_{xy} + \epsilon a_{xy}}\). Then \(\tilde{d}\) is also a metric.

**Proof.** The only nontrivial task is to verify the triangle inequality. Define further functions \(u, v, w\) by

\[u = b_{xy}/a_{xy}, \quad v = b_{xz}/a_{xz}, \quad w = b_{zy}/a_{zy}.
\]

Since \(d\) is a metric we have

\[
\frac{a_{xy}}{b_{xy}} \leq \frac{a_{xz}}{b_{xz}} + \frac{a_{zy}}{b_{zy}}
\]

and hence \(\frac{1}{u} \leq \frac{1}{v} + \frac{1}{w}\). We proceed by forward reasoning: we need the truth of the following equivalent conditions:

\[
\frac{a_{xy}}{b_{xy} + \epsilon a_{xy}} \leq \frac{a_{xz}}{b_{xz} + \epsilon a_{xz}} + \frac{a_{zy}}{b_{zy} + \epsilon a_{zy}},
\]

\[\frac{1}{u + \epsilon} \leq \frac{1}{v + \epsilon} + \frac{1}{w + \epsilon}.
\]

By Theorem 13 we are done.
Theorem 15 For each $\alpha$, the set of $\beta$ for which $D_{\alpha, \beta}$ is a metric is upward closed.

Proof. Suppose $D_{\alpha, \beta_0}$ is a metric and $\epsilon = \beta - \beta_0 \geq 0$. Let $a_{XY} := \alpha m(X, Y) + (1 - \alpha) M(X, Y)$. Since

$$D_{\alpha, \beta}(X, Y) = \frac{\alpha a_{XY}}{|X \cap Y| + \beta a_{XY}}$$

and since the upfront factor of $\beta$ may be removed without loss of generality, the question reduces to Theorem 14.

Some convenient notation to be used below includes $\overline{\alpha} = 1 - \alpha; \gamma := \beta \alpha \leq 1$ with $\beta = 1/\overline{\alpha}; x \cdot y = |X \cap Y|; x = |X| \text{ etc.};$

$$\begin{align*}
x_y &= |X \setminus Y|, x_{zy} = |X \setminus (Z \cup Y)| = |(X \setminus Z) \setminus Y|, \\
x_{000} &= |X \cap Y \cap Z|, x_{001} = |X \cap Y \cap Z|, x_{010} = |X \cap Y \cap Z|, x_{011} = |X \cap Y \cap Z|, \\
x_{100} &= |X \cap Y \cap Z|, x_{101} = |X \cap Y \cap Z|, x_{110} = |X \cap Y \cap Z|, x_{111} = |X \cap Y \cap Z|.
\end{align*}$$

Theorem 16 $\delta := \alpha m + \overline{\alpha} M$ satisfies the triangle inequality if and only if $\alpha \leq 1/2$.

Proof. We first show the only if direction. Let $X = \{0\}, Y = \{1\}, Z = \{0, 1\}$. Then

$$\begin{align*}
\alpha m(X, Y) + \overline{\alpha} M(X, Y) &= 1, \\
\alpha m(X, Z) + \overline{\alpha} M(X, Z) &= \alpha m(Y, Z) + \overline{\alpha} M(Y, Z) = 0 + \overline{\alpha}.
\end{align*}$$

The triangle inequality then is equivalent to $1 \leq 2\overline{\alpha}$, i.e., $\alpha \leq 1/2$.

Now let us show the if direction. The triangle inequality says

$$\alpha \min\{xy, yz\} + \overline{\alpha} \max\{yz, xy\} \leq \alpha \min\{xz, xz\} + \overline{\alpha} \max\{xz, xz\} + \alpha \min\{yz, yz\}$$

By symmetry between $x$ and $y$, we may assume that $y \leq x$. Hence either $y \leq z \leq x$, $y \leq x \leq z$, or $z \leq y \leq x$. Thus our proof splits into three Cases, I, II, and III.

Case I: $y \leq z \leq x$; we must show that $\alpha y_z + \overline{\alpha} x_y \leq \alpha x_z + \overline{\alpha} x_z + \alpha y_z + \overline{\alpha} z_y$. Since $y_z \leq y_z + z_y$ and $x_y \leq x_z + z_y$, this holds for all $\alpha$.

Case II: $y \leq x \leq z$; We want to show that $\alpha y_z + \overline{\alpha} x_y \leq \alpha x_z + \overline{\alpha} x_z + \alpha y_z + \overline{\alpha} z_y$. In terms of $\gamma = \alpha/\overline{\alpha}$ this says

$$0 \leq (y_z + x_z - y_x)\gamma + z_x + z_y - x_y = C\gamma + D.$$
elements that belong to exactly two of $X, Y, Z$ once each. Since $x \leq z$, it follows that

$$C = y_z + x_z - y_x \leq z_x + z_y - x_y = D.$$ 

Subcase II.1: $C \geq 0$. Then $C + D = 2C \geq 0$, as desired.

Subcase II.2: $C < 0$. In order to show $C \gamma + D \geq 0$ for all $0 \leq \gamma \leq 1$ it suffices that $C + D \geq 0$, since then $C \gamma + D = D - |C| \gamma \geq D - |C| \geq 0$.

We have $C + D = (x_z + z_y - x_y) + (y_z + z_x - y_x) \geq 0$.

Case III: $z \leq y \leq x$: We now need $\alpha y_x + \alpha z_x \leq y_z + \alpha z_y + \alpha y_z$,

$$0 \leq \gamma(z_x + z_y - y_x) + (x_z + y_z - x_y) = C \gamma + D.$$ 

The statement $C \leq D$ says $z_y + (z_x + x_y) \leq y_z + (y_x + x_z)$, which holds by the reasoning from Case II using now $z \leq y$. And now

$$C + D = (x_z + y_z - x_y) + (x_z + z_y - x_y) \geq 0.$$ 

**Theorem 17** The function $D_{\alpha, \beta}$ is a metric only if $\beta \geq 1/(1 - \alpha)$.

**Proof.** Consider the same example as in Theorem 16. Ignoring the upfront factor of $\beta$, we have

$$D = \frac{\delta}{|X \cap Y| + \beta \delta}.$$ 

In our example,

$$D(X, Y) = \frac{1}{0 + \beta \cdot 1} = \frac{1}{\beta},$$

$$D(X, Z) = D(Y, Z) = \frac{\pi}{1 + \beta \cdot \pi} = \frac{\pi}{1 + \beta \pi}.$$ 

The triangle inequality is then equivalent to:

$$\frac{1}{\beta} \leq 2 \frac{\pi}{1 + \beta \pi} \iff \beta \geq \frac{1 + \beta \pi}{2 \pi} \iff \beta \geq 1/(1 - \alpha).$$

**Theorem 18** The function $D_{\alpha, \beta}$ is a metric on all finite power sets only if $\alpha \leq 1/2$.

**Proof.** Suppose $\alpha > 1/2$. Let $Z_n = \{-(n-1), -(n-2), \ldots, 0\}$, a set of cardinality $n$ disjoint from $\{1, 2\}$, and let $Y_n = Z_n \cup \{1\}$, $X_n = Z_n \cup \{2\}$. The triangle inequality says

$$\frac{\beta}{n + \beta \cdot 1} = D(X_n, Y_n) \leq D(X_n, Z_n) + D(Z_n, Y_n) = 2\beta \frac{\pi}{n + \beta \pi}$$

$$n + \beta \pi \leq 2\pi(n + \beta)$$

$$n(1 - 2\pi) \leq \beta \pi.$$ 

Since $\alpha > 1/2$, we have $2\pi < 1$. Let $n > \frac{\beta \pi}{1 - 2\pi}$. Then the triangle inequality does not hold, so $D_{\alpha, \beta}$ is not a metric on the power set of $\{-(n-1), -(n-2), \ldots, 0, 1, 2\}$. 
Proof (Proof of Theorem 11). We saw in Theorem 16 that $\delta$ is a metric for $0 \leq \gamma \leq 1$. (Recall that $\beta = 1/(1 - \alpha)$, so that $\gamma = \alpha/\beta$.) In general if $d$ is a metric and $a$ is a function, we may hope that $d/(a + d)$ is a metric. We shall use the observation, mentioned by [16], that in order to show $d_{xy} \leq d_{xz} + d_{yz}$, it suffices to show the following pair of inequalities:

$$\frac{d_{xy}}{a_{xy} + d_{xy}} \leq \frac{d_{xz}}{a_{xz} + d_{xz}} + \frac{d_{yz}}{a_{yz} + d_{yz}},$$

it suffices to show the following pair of inequalities:

$$\frac{d_{xy}}{a_{xy} + d_{xy}} \leq \frac{d_{xz} + d_{yz}}{a_{xz} + d_{xz} + d_{yz}},$$

$$\frac{d_{xz} + d_{yz}}{a_{xz} + d_{xz} + d_{yz}} \leq \frac{d_{xz}}{a_{xz} + d_{xz}} + \frac{d_{yz}}{a_{yz} + d_{yz}}.$$

Here (3) follows from $d$ being a metric, i.e., $d_{xy} \leq d_{xz} + d_{yz}$, since $c \geq 0 < a \leq b$ $\Rightarrow \frac{a}{b + c} \leq \frac{b}{b + c}$.

Next, (4) would follow from $a_{xy} + d_{yz} \geq a_{xz}$ and $a_{xy} + d_{xz} \geq a_{yz}$. By symmetry between $x$ and $y$ and since $a_{xy} = a_{yx}$ in our case, it suffices to prove the first of these, $a_{xy} + d_{yz} \geq a_{xz}$. This is equivalent to

$$x \cap y + \gamma \min\{z_y, y_z\} + \max\{z_y, y_z\} \geq x \cap z,$$

which holds for all $0 \leq \gamma \leq 1$ if and only if $x \cap y + \max\{z_y, y_z\} \geq x \cap z$. There are now two cases.

Case $z \geq y$: We have

$$x \cap y + z_y \geq x \cap z$$

since any element of $X \cap Z$ is either in $Y$ or not.

Case $y \geq z$:

$$x \cap y + z_z \geq x \cap z$$

$$x_{110} + x_{111} + x_{110} + x_{010} \geq x_{101} + x_{111}$$

$$x_{110} + x_{111} + x_{010} \geq x_{101}$$

This is true since $z_y \geq x \cap z_y$.

3 Application to phylogeny

The mutations of spike glycoproteins of coronaviruses are of great concern with the new SARS-CoV-2 virus causing the disease CoViD-19. We calculate several distance measures between peptide sequences for such proteins. The distance

$$Z_{2, \alpha}(x_0, x_1) = \alpha \min(|A_1|, |A_2|) + \overline{\alpha} \max(|A_1|, |A_2|)$$

where $A_i$ is the set of subwords of length 2 in $x_i$ but not in $x_{1-i}$, counts how many subwords of length 2 appear in one sequence and not the other.
We used the Ward linkage criterion for producing Newick trees using the `hclust` package for the Go programming language. The calculated phylogenetic trees were based on the metric $Z_{2,\alpha}$.

We found one tree isomorphism class each for $0 \leq \alpha \leq 0.21$, $0.22 \leq \alpha \leq 0.36$, and $0.37 \leq \alpha \leq 0.5$, respectively (Figure 2, Figure 3). In Figure 3 we are also including the tree produced using the Levenshtein edit distance in place of $Z_{2,\alpha}$.

We see that the various intervals for $\alpha$ can correspond to “better” or “worse” agreement with other distance measures. Thus, we propose that rather than focusing on $\alpha = 0$ and $\alpha = 1/2$ exclusively, future work may consider the whole interval $[0, 1/2]$.

4 Conclusion

Many researchers have considered metrics based on sums or maxima, but we have shown that these need not be considered in “isolation” in the sense that they form the endpoints of a family of metrics.
A parametrized family of Tversky metrics

More general set-theoretic metrics can be envisioned. The Steinhaus transform of $\delta$ with $\beta = 1/\alpha$ is:

$$\delta'(X, Y) = \frac{2\delta(X, Y)}{\delta(X, \emptyset) + \delta(Y, \emptyset) + \delta(X, Y)}$$

$$= 2 \frac{\gamma \min\{x_y, y_x\} + \max\{x_y, y_x\}}{(x + y) + \gamma \min\{y_x, x_y\} + \max\{y_x, x_y\}}$$

A question for future research is whether this Steinhaus transform is more or less useful than what Jiménez et al. [7] considered. We can consider a general setting for potential metrics that contains both the Steinhaus transform of $\delta$ and the STRM metrics. In terms of $m(X, Y) = \min\{x_y, y_x\}$ and $M(X, Y) = \max\{x_y, y_x\}$, we can consider $\Delta_{\gamma, s} := \frac{\gamma m + M}{x + y + s(x + y) + (\gamma m + M)}$. When $s = 0$ this is our STRM metric. When $s = 1$ it is the Steinhaus transform, ignoring constant upfront factors.

**Correctness of results.** We have formally proved Theorem 11 in the Lean theorem prover, with a more streamlined proof than that presented here. The Github repository can be found at [8].
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