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Regular Articles

Mean field game for modeling of COVID-19 spread✩

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The paper presents one of the possible approaches to pandemic spread modeling. The proposed model is based on the mean-field control inside separate groups of population, namely, susceptible (S), infected (I), removed (R) and cross-immune (C) ones. The numerical algorithm to solve this problem ensures conservation of the total population mass during timeline. The numerical experiments demonstrate modeling results for COVID-19 spread in Novosibirsk (Russia) for two 100-day periods.

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1. Introduction

The COVID-19 pandemic has not only caused hundreds of thousands of deaths around the world, but also had a huge impact on the global economy, and everyone’s daily life. As a rule, SIR-type compartmental differential models are used to describe the dynamics of transmission of infectious diseases in a population. However, a SIR model stops working if it is necessary to account for population heterogeneity or some stochastic phenomena that are significant for small populations at the initial phase of disease spread. Since the spread of COVID-19 has a significant spatial characteristic, various restrictive measures have been used to slow down disease propagation, exerting an additional influence on the dynamics of population behavior (by “spatial” we mean certain local features characteristic of a particular population, determined, for example, by location, introduced policies, mass-media coverage, etc.). Thus, to study an infectious-disease spread, it is also necessary to take into account the stochastic parameters of the system and its heterogeneity, which, in the classical sense, leads to systems with a large number of differential equations, even if it

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is possible to cluster the population somehow. Computationally, the approach to describing population dynamics can be simplified by using the mean-field game (MFG) theory, which enables one to describe the behavioral dynamics of a system with a small number of partial differential equations, taking into account population heterogeneity.

The paper presents a SIRC model for COVID-19 spread, based on a system of ODEs using the MFG approach for which optimality conditions for mean-field system have been obtained (see 2). In Section 3, the optimal control problem with external influence has been solved. In Section 4, a modified computational scheme for the MFG system has been proposed and the results of numerical experiments for COVID-19 spread in the Novosibirsk Region have been discussed.

2. Mathematical model

There are a lot of mathematical models of infectious disease spread that are based on the mass balance law and described by systems of nonlinear ordinary differential equations [2]. The basic compartmental model to describe transmission of infectious diseases was proposed by Kermack and McKendrick in 1927 [7] where the population was divided into three groups: susceptible, infected and removed. Various disease outbreaks such as the SARS epidemic of 2002-2003, possible H5N1 influenza epidemic in 2005, H1N1 influenza pandemic of 2009, Ebola outbreak of 2014 and the COVID-19 pandemic in 2019 have reignited the interest in epidemic models and led to reformulation of the Kermack-McKendrick model. In the next section we present one of the variations of the compartmental model where the population is divided into four groups and that accounts for the cross-immune cases taking place during the pandemic.

2.1. SIRC model

In this subsection, we will focus on the macroscopic description of the spread without taking into account the individual effect of an agent on the whole population. For these purposes epidemiological models with different characteristics are generally used. Most of these models are based on the susceptible-infected-removed (SIR) separation. In Casagrandet et al. [3], a SIRC model was presented for describing the dynamic behavior of influenza A by introducing a new component, namely, cross-immunity (C) of a population for people who has recovered from infection with different strains of the same virus subtype at previous time intervals. In [5,16,22] this idea was developed and it was shown in [12] that this model can be used to account for an infected but asymptomatic part of the population.

Now, let us consider the dynamics of an epidemic following a simplified SIRC model. Assume that the epidemic occurs in a short time period if compared to the population dynamics (birth and death), so the last one can be neglected. The flow diagram of the simplified SIRC model is presented in Fig. 1. From the point of modeling, the population at any time moment is divided into four compartments: susceptible individuals $S(t)$, i.e., those who do not have specific immune defenses against the infection; infected individuals $I(t)$; and two compartments of those who are totally or partially immune i.e., recovered $R(t)$ and cross-immune $C(t)$.

\[
\begin{align*}
\text{S} & \quad \beta \quad \gamma \quad \delta \quad \mu \\
\rightarrow & \quad I \quad \rightarrow \quad R \quad \rightarrow \quad C
\end{align*}
\]

Fig. 1. General flow diagram of the SIRC model.
Formally, in accordance with the diagram shown in Fig. 1 and the mass balance law, the SIRC model can be represented by the following system of four ordinary differential equations

\[
\begin{align*}
\frac{dS(t)}{dt} &= -\beta S(t) I(t) + \mu C(t), \\
\frac{dI(t)}{dt} &= \beta S(t) I(t) + \varepsilon \beta C(t) I(t) - \gamma I(t), \\
\frac{dR(t)}{dt} &= (1 - \varepsilon) \beta C(t) I(t) + \gamma I(t) - \delta R(t), \\
\frac{dC(t)}{dt} &= \delta R(t) - \beta C(t) I(t) - \mu C(t)
\end{align*}
\]  

(1)

with initial conditions: \(S(0) = S_0;\ I(0) = I_0;\ R(0) = R_0;\ C(0) = C_0,\) where \(S_0,\ I_0,\ R_0,\ C_0 \in \mathbb{R}^+.\) Note, that \(S(t) + I(t) + R(t) + C(t) = 1\) represents the total population. Here the following parameters are used: \(\mu\) is the rate at which the cross-immune population becomes susceptible again; \(\beta\) is the contact/infection transmission rate; \(\varepsilon\) is the average reinfection probability of a cross-immune individual; \(\gamma\) is the recovery rate of the infected population; \(\delta\) is the rate at which the recovered population becomes cross-immune and moves from total to partial immunity. Parameter \(\mu\) is connected with the average number of days neutralizing antibodies disappear, and \(\delta\) is the estimated number of days when CD4+ and CD8+ T-cells decrease [4]. Note that in the absence of cross-immunity, i.e. \((1 - \varepsilon) = 0,\) the SIRC model becomes similar to the SIRS model for the fractions \(S\) and \(C\) become immunologically indistinguishable [16].

2.2. Mean field approach to epidemic control

In fact, greater consistency with some real-world phenomena can be achieved if stochastic changes in a system are taken into account. SIR-based epidemic models assume that observed dynamics are determined by the deterministic cases assuming the population is homogenic and neglecting the characteristics of individual agents. At the same time, accounting for the rationality of an individual who is a part of the population, i.e., for their ability to be in a state \(X(t)\) and be able to change it, leads to management problems due to a large number of participants in the system. Taking into account the stochasticity of the process, we assume that the dynamics of a single individual is subject to the Što diarrhoal equation

\[
\begin{align*}
\frac{dX^N(t)}{dt} &= b(t, X^N(t), \theta^N(t), \alpha_i(t))dt + \sigma(t, X^N(t), \theta^N(t))dW^N(t),
\end{align*}
\]  

(2)

where \(i \in 1, \ldots, N;\ W^N \) is an independent standard Wiener processes; \(\alpha_i(t)\) is the \(i\)-th agent’s strategy, and \(\theta^N(t)\) is the empirical measure of distribution of agents (individuals, players) in the system at time \(t\) [6]. Let function \(b,\ \sigma\) be continuous in time and same for all players. In [6] it was shown that when the number of agents in system rises extremely (\(N \to \infty\)), the mass of single individuals can be substituted by a representative agent whose state is determined by the following control equation:

\[
\begin{align*}
\frac{dX(t)}{dt} &= b(t, X(t), m(t), \alpha(t))dt + \sigma(t, X(t), m(t))dW(t).
\end{align*}
\]  

(3)

Here, \(X(t) : [0, T] \to \Omega; m(t) : \theta^N(t) \overset{N \to \infty}{\longrightarrow} m(t)\) is agents distribution over state space \(\Omega\) at time \(t,\) and \(\alpha(t)\) is the control process (or, in other words, strategy the representative agent’s strategy) to ensure the Nash equilibrium of a system of interacting agents and minimize the function:

\[
J(\alpha) = \mathbb{E} \left[ \int_0^T f(s, X(s), m(t), \alpha(t))ds + G(X(T), m(T)) \right],
\]  

(4)
where \( f \) and \( G \) are Lipschitz functions. This approach to controlling a population with a large number of interacting agents is known as Mean Field Game (MFG). For the more detailed description, refer to original works \([1, 6, 10]\).

For our goals, a population of “atomized” agents is characterized by their states (or positions) \( x \in \Omega \) at each time moment \( t \in [0, T] \). The term “atomized” means that each agent of an infinite set has no influence on the situation (because of its zero-measured support) and chooses a rational strategy \( \alpha(t, x) \) accounting for the agent’s own position \( m(t, x) : [0, T] \times \Omega \rightarrow R \) relative to other agents. In \([1]\), it was shown when \( \sigma \) in \((2), (3)\) characterized by the stochastic nature of agents interaction equilibrium is constant, the distribution of agents \( m(t, x) \) obeys the Kolmogorov (Fokker-Planck) equation

\[
\partial m/\partial t - \Delta m\sigma^2/2 + \nabla (ma) = 0 \text{ in } [0, T] \times \Omega
\]

with initial condition

\[m(0, x) = m_0(x) \text{ on } \Omega \]

and the Neumann boundary condition

\[
\partial m/\partial x = 0 \text{ for } t \text{ and } x \in \Gamma_\Omega.
\]

Here, the boundary condition (7) prevents the loss of density \( m(t, x) \) with time.

So, on the one hand, we have a differential SIRC model (1) that describes a population at a macroscopic level, divide it into several parts, but provides no individual description. On the other hand, the individuality can be introduced and controlled using the MFG model (4)–(7). Following the ideas proposed in \([11]\), the spatial SIRC model was combined with the MFG one and a management problem was introduced to control virus spread in several groups of population.

Instead of \( S, I, R, C \) denoting the proportions of groups in a population, the distribution density of agents within these groups \( m_i(t, x) : [0, T] \times [0, 1] \rightarrow R \) was introduced, where \( i \in \{S, I, R, C\} \). If the state variable \( x \) varies within \( \Omega = [0, 1] \) and denotes the population’s propensity to comply with quarantine measures such as physical distancing and self-isolation, 0 means that an agent is loyal to the introduced restriction measures and inclined to comply with them, and 1 means the opposite. Introduced functions \( \alpha_i(t, x) : [0, T] \times [0, 1] \rightarrow R, i \in \{S, I, R, C\} \) denote the compliance strategy the representative agent of each group in the population. Due to (1), (5)–(7) the functions \( m_i(t, x) \) are subject to the following PDE system:

\[
\begin{align*}
\partial m_S + \nabla (m_S \alpha_S) + \beta m_S m_I - \mu m_C - \sigma_S^2 \Delta m_S/2 &= 0, \\
\partial m_I + \nabla (m_I \alpha_I) - \beta m_S m_I - \varepsilon \beta m_C m_I + \gamma m_I - \sigma_I^2 \Delta m_I/2 &= 0, \\
\partial m_R + \nabla (m_R \alpha_R) - (1 - \varepsilon) \beta m_C m_I - \gamma m_I + \delta m_R - \sigma_R^2 \Delta m_R/2 &= 0, \\
\partial m_C + \nabla (m_C \alpha_C) - \delta m_R + \beta m_C m_I + \mu m_C - \sigma_C^2 \Delta m_C/2 &= 0.
\end{align*}
\]

Here \( \sigma_i, i \in \{S, I, R, C\} \) are non-negative parameters to, as before, characterize the stochastic processes within the population. If the initial value of \( m_i \) is given:

\[m_i(0, x) = m_{i0}(x) \text{ for } x \in [0, 1].\]

and, similar to (7), it meets zero Neumann boundary conditions that no mass can flow in or out of \( \Omega \), one obtains:

\[
\partial m_i/\partial x = 0 \text{ for } t \in [0, T] \text{ and } x = 0, 1.
\]
Note that
\[
\int_0^1 (m_S(t, x) + m_I(t, x) + m_R(t, x) + m_C(t, x)) \, dx = \int_0^1 m(t, x) \, dx, \quad \forall t \in [0, T].
\]

It is noteworthy that summing up the equations in system (8), one obtains the equation (5) with respect to the total population mass \(m(t, x)\), which leads to the following requirement:
\[
\frac{\partial}{\partial t} \int_0^1 (m_S(t, x) + m_I(t, x) + m_R(t, x) + m_C(t, x)) \, dx = 0,
\]
i.e., the total mass of the four groups will be conserved \(\forall t\).

Now, following the MFG principles, an assumption that agents are rational and tend to choose a strategy to maximize their own benefit in accordance with the cost functions (4) produces the management problem that minimizes the value functional in respect to \((m_i, \alpha_i) \forall i \in \{S, I, R, C\}\)
\[
J(m_{SIRC}, \alpha_{SIRC}) = \int_0^T \int_0^1 \sum_{i \in \{S,I,R,C\}} (F_i(\alpha_i, t, x) m_i + g_i(t, x, m_i)) \, dx dt + \int_0^1 m_i^2(T, x) / 2 \, dx,
\]
where the index \(SIRC\) denotes the set of all possible values from \(\{S, I, R, C\}\). Here \(F_i\) function is the running cost to implement strategy \(\alpha_i\); \(g_i\) is payment for current position or state and \(\int_0^1 m_i(T, x) \, dx\) is the terminal cost of epidemic spread.

As the function of strategy implementation cost \(F_i(\bar{\alpha}_i, t, x)\) (an agent’s payment for state changing) a piecewise continuous function was considered:
\[
F_i(\bar{\alpha}_{sirc}, t, x) = \begin{cases} \bar{\alpha}_i^2(100 + \bar{\alpha}_i^2) / 2 & \text{if } \bar{\alpha}_i \leq 0, \\ \bar{\alpha}_i^2(20 + \bar{\alpha}_i^2) / 2 & \text{otherwise} \end{cases}
\] (12)

for all admissible values of \(\bar{\alpha}_i \in \mathbb{R}\) and \(\forall i \in \{S, I, R, C\}\). Note that function has the same form for any part of the population. This can help to account for infected, but asymptotic population masses. If an individual is an infection carrier but does not know about it, in fact, they continue their everyday life for it is more beneficial for them. The piecewise continuous form of (12) means that the transition to self-isolation for an agent (when \(\bar{\alpha}_i \leq 0\)) is more expensive than maintaining close contact with other people. It is noteworthy that for (12) the following properties satisfy \(\forall (t, x) \in [0, T] \times [0, 1]\):
\[
1. \quad \frac{\partial F_i}{\partial \bar{\alpha}_i} (\bar{\alpha}_i, t, x) \text{ is continuous for all admissible } \bar{\alpha}_i;
2. \quad F_i(0, t, x) = \frac{\partial F_i}{\partial \alpha}(0, t, x) = 0;
3. \quad \frac{\partial F_i}{\partial \bar{\alpha}_i} (\bar{\alpha}_i, t, x) \text{ is strictly monotonous for } \bar{\alpha}_i \in (-\infty, +\infty).
\] (13)

These properties ensure the unique solvability of the equation
\[
\frac{\partial F_i}{\partial \bar{\alpha}_i} (\bar{\alpha}_i, t, x) = z \text{ for any } z \in \mathbb{R} \text{ and } (t, x) \in [0, T] \times [0, 1].
\]
For the $g_i(t,x,m_i)$ function:

\[
g_i(t,x,m_i) = \begin{cases} 
  c_i(1-x)m_i & \text{if } i \in \{S, R, C\}, \\
  c_1m_i & \text{if } i = I.
\end{cases}
\] (14)

Here $c_{1i}, c_{2i} \in [0,1]$ are positive constants. Thus, the function $g_i$ together with the function $F_i$ determines the costs paid by an agent to comply with quarantine restrictions and social distancing. The difference is, from a mathematical point of view, that the function $g_i$ doesn’t explicitly depend on the player’s strategy $\alpha_i$ and determines the so-called “implicit” costs. The choice of function $g_i$ was based on the concept that if a person is not infected and the number of non-isolated people in a population is rising, a person’s profit decreases in an attempt to comply with the restrictions (non-linearity with respect to $m_i$). This effect is commonly known as “economies of scale” [13]. On the contrary, if a person is infected, then they are inclined to comply with the restrictions.

Thus, the MFG approach leads to the optimization problem that requires you to find the optimal strategy $\alpha_i$ by minimizing the function (11) with the restriction in the form of PDE system (8) with initial (9) and boundary (10) conditions.

### 2.3. Optimality conditions

To ensure the strategy chosen by a person is optimal the Lagrange multiplier method [1] was used. For that the first equation in (8) was multiplied by an arbitrary smooth function $v_S(t,x) \in C^\infty([0,T]\times[0,1])$ and the resulting expression was integrated by parts with respect to $t$ and $x$:

\[
L_S := -\int_0^T \int_0^1 \left( \partial v_S/\partial t + \sigma_S^2 \Delta v_S/2 + \alpha_S \cdot \partial v_S/\partial x \right) m_S dx dt + \\
+ \int_0^T \int_0^1 (\beta m_S m_I v_S - \mu m_C v_S) dx dt + \int_0^1 (v_S(T,x)m_S(T,x) - v_S(0,x)m_S0(x)) dx = 0.
\] (15)

Do the same with other equations in (8) for smooth function $v_i(t,x) \in C^\infty([0,T]\times[0,1])$ where $i \in \{I, R, C\}$:

\[
L_I := -\int_0^T \int_0^1 \left( \partial v_I/\partial t + \sigma_I^2 \Delta v_I/2 + \alpha_I \cdot \partial v_I/\partial x \right) m_I dx dt + \\
+ \int_0^T \int_0^1 (-\beta m_S m_I v_I - \varepsilon \beta m_C m_I v_I + \gamma m_I v_I) dx dt + \int_0^1 (v_I(T,x)m_I(T,x) - v_I(0,x)m_{I0}(x)) dx = 0.
\] (16)

\[
L_R := -\int_0^T \int_0^1 \left( \partial v_R/\partial t + \sigma_R^2 \Delta v_R/2 + \alpha_R \cdot \partial v_R/\partial x \right) m_R dx dt + \\
+ \int_0^T \int_0^1 (-1 - \varepsilon) \beta m_C m_R v_R - \gamma m_R v_R + \delta m_R v_R) dx dt + \int_0^1 (v_R(T,x)m_R(T,x) - v_R(0,x)m_{R0}(x)) dx = 0.
\] (17)
\begin{equation}
L_C := -\int_0^T \int_0^1 \left( \frac{\partial v_C}{\partial t} + \sigma_C^2 \Delta v_C/2 + \alpha_C \cdot \partial v_C/\partial x \right) m_C \, dx \, dt + \\
+ \int_0^T \int_0^1 \left( -\delta m_R v_C + \beta m_C m_I v_C + \mu m_C v_C \right) \, dx \, dt + \int_0^1 \left( v_C(T, x)m_C(T, x) - v_C(0, x) m_C(0, x) \right) \, dx = 0.
\end{equation}

Note that expressions (15)–(18) are valid when the following boundary conditions are satisfied

\begin{equation}
\partial v_i/\partial x = 0 \quad \forall t \in [0, T] \quad \text{and} \quad x = 0, 1 \quad \forall i \in \{S, I, R, C\}
\end{equation}

and

\begin{equation}
\alpha_i(t, 0) = \alpha_i(t, 1) = 0 \quad \forall t \in [0, T] \quad \forall i \in \{S, I, R, C\}.
\end{equation}

Now, the Lagrange function corresponding to the optimization problem under consideration can be written down as:

\begin{equation}
\mathcal{G}(m_{SIRC}, \alpha_{SIRC}, v_{SIRC}) := J(m_{SIRC}, \alpha_{SIRC}) - L_S - L_I - L_R - L_C.
\end{equation}

As the result, the minimization of (11) together with the (8)–(10) can be represented [1] as the problem of finding a saddle point:

\begin{equation}
\inf_{(m_i, \alpha_i)} \sup_{v_i} \mathcal{G}(m_{SIRC}, \alpha_{SIRC}, v_{SIRC}) \forall i \in \{S, I, R, C\}.
\end{equation}

The variation (15)–(18) with respect to the components \( m_i \forall (t, x) \in [0, T] \times [0, 1] \) to find a stationary point produces a system analogous to the Hamilton-Jacobi-Bellman equation:

\begin{equation}
\begin{cases}
\partial v_S/\partial t + \sigma_S^2 \Delta v_S/2 + \alpha_S \cdot \partial v_S/\partial x + \beta m_I(v_I - v_S) = -F_S - \partial g_S/\partial m_S , \\
\partial v_I/\partial t + \sigma_I^2 \Delta v_I/2 + \alpha_I \cdot \partial v_I/\partial x + \beta m_S(v_S - v_I) + \beta m_C(v_R - v_C) + \\
+ (\varepsilon \beta m_C - \gamma)(v_I - v_R) = -F_I - \partial g_I/\partial m_I - \delta(T - t)m_I, \\
\partial v_R/\partial t + \sigma_R^2 \Delta v_R/2 + \alpha_R \cdot \partial v_R/\partial x + \delta(v_C - v_R) = -F_R - \partial g_R/\partial m_R, \\
\partial v_C/\partial t + \sigma_C^2 \Delta v_C/2 + \alpha_C \cdot \partial v_C/\partial x + \mu(v_S - v_C) + \\
+ \varepsilon \beta m_I(v_I - v_R) + \beta m_I(v_R - v_C) = -F_C - \partial g_C/\partial m_C ,
\end{cases}
\end{equation}

where \( \delta(T - t) \) is the delta function and zero initial condition:

\begin{equation}
v_i(t, x) = 0 \quad \forall x \in [0, 1].
\end{equation}

The variation (15)–(18) with respect to the components \( \alpha_i \forall (t, x) \in [0, T] \times [0, 1] \) gives the following optimality conditions for \( \alpha_i \in \mathbb{R} \) in addition to system (23), (24), (19)

\begin{equation}
\frac{\partial F_i}{\partial \alpha_i}(\alpha_i, t, x) + \frac{\partial v_i}{\partial x}(t, x) = 0
\end{equation}

\( \forall i \in \{S, I, R, C\} \forall (t, x) \in [0, T] \times [0, 1] \). The properties (13) ensure that (25) has a unique solution that satisfies (20).

Thus, two coupled PDE systems with initial and boundary conditions (8)–(10) and (19), (23), (24) together with (25) give the necessary conditions for the minimization of (11).
3. Optimal control problem with external influence

In our case, agents are considered as “rational” but “selfish” i.e., inclined to choose a strategy to maximize their own benefits based on other agents’ strategies, which can lead to a faster spread of the infection. In reality, however, the situation is controlled by some external influence such as introduction of strict quarantine and other restrictions or allocation of social benefits, etc. These quarantine regulations not only determine agents’ behavior but, as a rule, are strict requirements and therefore cannot be accounted for in the cost functional where their non-fulfillment becomes possible. In this way, an assumption can be made that the total control of a system’s behavior can be determined by two processes:

\[ \alpha_i = \hat{\alpha}_i (t, x) + \rho_i (\hat{\alpha}_i (t, x), t, x). \]

(26)

Here \( \hat{\alpha}_i \), as before, is a person’s strategy and \( \rho_i (\hat{\alpha}_i, t, x) \) is the external (corrective) control which adjusts the player’s strategy depending on the fulfillment of a certain condition. For example, we can put the following expression:

\[ \rho_i (\hat{\alpha}_i, t, x) = \begin{cases} 0 & \text{if } \hat{\alpha}_i \leq 0, \\ -c_{3i} \hat{\alpha}_i & \text{otherwise}, \end{cases} \]

(27)

where \( c_{3i} \in [1, 2] \) are positive constants and (27) does not violate the boundary condition (20) for \( \alpha_i \). From physical point of view, (27) means that if agents are inclined to follow the quarantine measures, the government doesn’t impose additional restrictions and takes some restrictive actions otherwise. As before, the dynamics of the agent’s mass was defined as the solution of the system of equations (8) taking into account the representation (26). Instead of (11) the same function \( J(m_{SIRC}, \hat{\alpha}_{SIRC}) \) was minimized for a set of \( \hat{\alpha}_i \) \( \forall i \in \{S, I, R, C\} \)

\[ J(m_{SIRC}, \hat{\alpha}_{SIRC}) = \int_0^T \int_0^1 \sum_{i \in \{S, I, R, C\}} (F_i (\hat{\alpha}_i, t, x) m_i + g_i (t, x, m_i))dxdt + \int_0^1 m_i^2 (T, x)/2 dx. \]

(28)

so the cost of implementing external control was neglected as being not important for a single individual. This assumption gave us an optimality conditions in form of the system (23) with (26) where (25) was replaced by

\[ \frac{\partial F_i}{\partial \hat{\alpha}_i} + \left(1 + \frac{\partial \rho_i (\hat{\alpha}_i, t, x)}{\partial \hat{\alpha}_i}\right) \frac{\partial v_i}{\partial x} = 0 \text{ in } [0, T] \times [0, 1]. \]

(29)

in a way the optimization problem with external control can be formulated as finding the minima of the function (28) in accordance with (26) with the restrictions (8)–(10), (19), (23), (24) and (29).

4. Numerical experiments

A series of numerical experiments was performed to demonstrate the difference between the standard differential SIRC model (1) and its MFG extension.

4.1. Numerical scheme for FPK equations

To search for a numerical implementation of the differential problem mentioned above, discrete uniform grids were introduced in time
\[ t_k = k\tau, \quad k = 0, \ldots, M, \quad \tau = T/M; \]

and in space

\[ x_{i+1/2} = (i + 1/2)h, \quad i = -1, \ldots, N, \quad h = 1/N. \]

The chosen solution was as a piecewise linear functions \( m^{i,h}(t, x) \) at each time level \( t_k \), which are continuous on \([0,1]\) and linear on each segment \( \omega_j = [x_{j-1/2}, x_{j+1/2}] \) \( \forall j = 1, \ldots, N - 1 \). In addition, \( m^{i,h}(t_k, x) \) was assumed to be constant on the intervals \( \omega_0 = [0, x_{1/2}] \) and \( \omega_N = [x_{N-1/2}, 1] \) to satisfy boundary condition (10). Thus, at each time level \( t_k \), the function \( m^{i,h}(t_k, x) \) is completely determined by its discrete values \( m^{i,h}_{k,j+1/2} := m^{i,h}(t_k, x_{j+1/2}) \):

\[ m^{i,h}(t_k, x) = m^{i,h}_{k,j-1/2}(x_{j+1/2} - x)/h + m^{i,h}_{k,j1/2}(x - x_{j-1/2})/h \quad \forall x \in \omega_i. \]

To satisfy the boundary condition (10), it was put that

\[ m^{i,h}_{k,-1/2} = m^{i,h}_{k,1/2} \quad \text{and} \quad m^{i,h}_{k,N+1/2} = m^{i,h}_{k,N-1/2}. \]

To construct the computational scheme for equations (8), the idea firstly proposed by Prof. V.V. Shaydurov [19] was modified for the numerical solution of FPK equation (5) and applied to various MFG statements in \([17,18,20]\) written in collaboration with V. Petrakova (formerly Kornienko), one of the authors of this article. The idea was to split the approximation of the FPK equation (5) into two parts: advection and diffusion ones. Note, that the advection part can be written in (8) as:

\[ \frac{\partial m_i}{\partial t} + \frac{\partial (m_i \alpha_i)}{\partial x} = f^i_1(t, x), \quad (31) \]

\[ \forall i \in \{S, I, R, C\} \] where \( f^i_1(t, x) \) are

\[ \left\{ \begin{array}{l}
\quad f^S_1 = -\beta m_S m_I + \mu m_C + \sigma^2_S \Delta m_S/2 , \\
\quad f^I_1 = \beta m_S m_I + \varepsilon \beta m_C m_I - \gamma m_I + \sigma^2_I \Delta m_I/2 , \\
\quad f^R_1 = (1 - \varepsilon) \beta m_C m_I + \gamma m_I - \delta m_R + \sigma^2_R \Delta m_R/2 , \\
\quad f^C_1 = \delta m_R - \beta m_C m_I - \mu m_C + \sigma^2_C \Delta m_C/2 = 0 .
\end{array} \right. \]

(32)

For diffusion part the same property can be used in (8):

\[ \left\{ \begin{array}{l}
\quad \beta m_S m_I - \mu m_C - \sigma^2_S \Delta m_S/2 = f^S_2(t, x) , \\
\quad -\beta m_S m_I - \varepsilon \beta m_C m_I + \gamma m_I - \sigma^2_I \Delta m_I/2 = f^I_2(t, x) , \\
\quad - (1 - \varepsilon) \beta m_C m_I - \gamma m_I + \delta m_R - \sigma^2_R \Delta m_R/2 = f^R_2(t, x) , \\
\quad - \delta m_R + \beta m_C m_I + \mu m_C - \sigma^2_C \Delta m_C/2 = f^C_2(t, x) ,
\end{array} \right. \]

(33)

where \( f^i_2 = -\partial_t m_i - \nabla (m_i \alpha_i) \). Following the idea proposed in \([17–20]\) (31) was approximated at in the point \((t_k, x_{i+1/2})\) as

\[ \frac{1}{8\tau} m^{i,h}_{k,j-1/2} + \frac{3}{4\tau} m^{i,h}_{k,j+1/2} + \frac{1}{8\tau} m^{i,h}_{k,j+3/2} - \gamma^{i,1}_{k,j+1/2} m^{i,h}_{k-1,j-1/2} - \gamma^{i,2}_{k,j+1/2} m^{i,h}_{k-1,j+1/2} \]

\[ - \gamma^{i,3}_{k,j+1/2} m^{i,h}_{k-1,j+3/2} - \gamma^{i,1}_{k,j+1/2} m^{i,h}_{k-1,j-1/2} - \gamma^{i,2}_{k,j+1/2} m^{i,h}_{k-1,j+1/2} - \gamma^{i,3}_{k,j+1/2} m^{i,h}_{k-1,j+3/2} = f^i_1(t_k, x_{j+1/2}) \]

(34)
\[ \forall i \in \{S, I, R, C\} \text{ and } \forall j = 0, \ldots, N - 1 \forall k = 1, \ldots, M \text{ with coefficients} \]
\[
\begin{align*}
\gamma_{k,j+1/2}^{i,1} &= \frac{1}{8\tau} \left( 1 + \frac{4\tau}{h} \alpha_{k,j}^i \right), \\
\gamma_{k,j+1/2}^{i,2} &= \frac{1}{8\tau} \left( 3 + \frac{4\tau}{h} \alpha_{k,j}^i \right) + \frac{1}{8\tau} \left( 3 - \frac{4\tau}{h} \alpha_{k,j+1}^i \right), \\
\gamma_{k,j+1/2}^{i,3} &= \frac{1}{8\tau} \left( 1 - \frac{4\tau}{h} \alpha_{k,j+1}^i \right).
\end{align*}
\] (35)

The following notation for the approximation of the differential expression \( \sigma_i^2 \Delta m_i / 2 \forall i \in \{S, I, R, C\} \) was introduced:
\[
\Delta_{k,j+1/2}^{i,h} = \sigma_i^2 m_{k,j-1/2}^{i,h} - 2m_{k,j+1/2}^{i,h} + m_{k,j+3/2}^{i,h}.
\] (36)

For approximation of (33), the following finite-difference scheme was applied:
\[
\begin{align*}
\left\{ \begin{array}{l}
\beta m_{k-1,j+1/2}^{I,h} m_{k-1,j+1/2}^{I,h} - \mu m_{k-1,j+1/2}^{C,h} - \Delta_{k,j+1/2}^{S,h} \approx f_k^S(t_k, x_{j+1/2}), \\
-\beta m_{k-1,j+1/2}^{S,h} m_{k-1,j+1/2}^{I,h} - \varepsilon \beta m_{k-1,j+1/2}^{C,h} m_{k-1,j+1/2}^{I,h} + \gamma m_{k-1,j+1/2}^{I,h} - \Delta_{k,j+1/2}^{S,h} \approx f_k^I(t_k, x_{j+1/2}), \\
-(1 - \varepsilon) \beta m_{k-1,j+1/2}^{C,h} m_{k-1,j+1/2}^{I,h} - \gamma m_{k-1,j+1/2}^{C,h} + \delta m_{k-1,j+1/2}^{R,h} - \Delta_{k,j+1/2}^{C,h} \approx f_k^C(t_k, x_{j+1/2}), \\
-\delta m_{k-1,j+1/2}^{R,h} + \beta m_{k-1,j+1/2}^{C,h} m_{k-1,j+1/2}^{I,h} + \mu m_{k-1,j+1/2}^{C,h} - \Delta_{k,j+1/2}^{C,h} \approx f_k^R(t_k, x_{j+1/2}).
\end{array} \right.
\end{align*}
\] (37)

Note that
\[
\forall k = 1, \ldots, M, \ j = 1, \ldots, N - 1 \forall i \in \{S, I, R, C\}. \text{ It leads to the following finite-difference approximation of equations (8)}
\]
\[
D_{k,j+1/2}^{i,h} = f_{k-1,j+1/2}^{i,h} + \Gamma_{k-1,j+1/2}^{i,h},
\] (38)

where
\[
D_{k,j+1/2}^{i,h} = \left( \frac{1}{8\tau} - \frac{\sigma_i^2}{2h^2} \right) m_{k,j-1/2}^{i,h} + \left( \frac{3}{4\tau} + \frac{\sigma_i^2}{h^2} \right) m_{k,j+1/2}^{i,h} + \left( \frac{1}{8\tau} - \frac{\sigma_i^2}{2h^2} \right) m_{k,j+3/2}^{i,h},
\] (39)
\[
\Gamma_{k-1,j+1/2}^{i,h} = \gamma_{k,j+1/2}^{i,1} m_{k-1,j-1/2}^{i,h} + \gamma_{k,j+1/2}^{i,2} m_{k-1,j+1/2}^{i,h} + \gamma_{k,j+1/2}^{i,3} m_{k-1,j+3/2}^{i,h}
\]
and
\[
\begin{align*}
\left\{ \begin{array}{l}
f_{k-1,j+1/2}^{S,h} = -\beta m_{k-1,j+1/2}^{S,h} m_{k-1,j+1/2}^{I,h} + \mu m_{k-1,j+1/2}^{C,h}, \\
f_{k-1,j+1/2}^{I,h} = \beta m_{k-1,j+1/2}^{S,h} m_{k-1,j+1/2}^{I,h} + \varepsilon \beta m_{k-1,j+1/2}^{C,h} m_{k-1,j+1/2}^{I,h} - \gamma m_{k-1,j+1/2}^{I,h}, \\
f_{k-1,j+1/2}^{C,h} = (1 - \varepsilon) \beta m_{k-1,j+1/2}^{C,h} m_{k-1,j+1/2}^{I,h} + \gamma m_{k-1,j+1/2}^{C,h} - \delta m_{k-1,j+1/2}^{R,h}, \\
f_{k-1,j+1/2}^{R,h} = \delta m_{k-1,j+1/2}^{R,h} - \beta m_{k-1,j+1/2}^{C,h} m_{k-1,j+1/2}^{I,h} - \mu m_{k-1,j+1/2}^{C,h}.
\end{array} \right.
\end{align*}
\] (40)

and initial conditions corresponding to (9)
\[
m_{0,j+1/2}^{i,h} = m_0^i(x_{j+1/2}) \forall j = 0, \ldots, N - 1.
\] (41)
Thus, instead of differential equations (8) \( \forall i \in \{S, I, R, C\} \) the following systems of algebraic equations were obtained:

\[
\begin{bmatrix}
A & -B_2 & A & \ldots & -B_M & A
\end{bmatrix}
\begin{bmatrix}
m^i_{1,1} \\
m^i_{2,1} \\
\vdots \\
m^i_{M,1}
\end{bmatrix}
= 
\begin{bmatrix}
B^i_1 m_{0,1}^i + f_{0,1}^i \\
f_{1,1}^i \\
\vdots \\
f_{M-1,1}^i
\end{bmatrix} = \bar{S}^i m_{0,1}^i. \tag{42}
\]

Hereinafter, the subscript points indicate acceptable values in a corresponding position of the mesh function. \( A \) and \( B_k^i \) indicate the following matrices: \( A \) and \( B_k^i \) are the matrices of the form

\[
B_k^i = 
\begin{bmatrix}
\gamma_{k,-1/2}^i + \gamma_{k,1/2}^i & \gamma_{k,3/2}^i & \gamma_{k,5/2}^i \\
\gamma_{k,1/2}^i & \gamma_{k,3/2}^i & \gamma_{k,5/2}^i \\
\gamma_{k,3/2}^i & \gamma_{k,5/2}^i & \gamma_{k,7/2}^i \\
\gamma_{k,5/2}^i & \gamma_{k,7/2}^i & \gamma_{k,9/2}^i \\
\vdots & \vdots & \vdots \\
\gamma_{k,N-3/2}^i & \gamma_{k,N-1/2}^i + \gamma_{k,N+1/2}^i & \gamma_{k,N+3/2}^i \\
\gamma_{k,N-1/2}^i & \gamma_{k,N+1/2}^i & \gamma_{k,N+3/2}^i
\end{bmatrix},
\]

\[
A = 
\begin{bmatrix}
\frac{7}{8 \tau} + \frac{\sigma^2}{2 h^2} & \frac{1}{8 \tau} - \frac{\sigma^2}{2 h^2} & \frac{1}{8 \tau} - \frac{\sigma^2}{2 h^2} & \frac{1}{8 \tau} - \frac{\sigma^2}{2 h^2} & \frac{1}{8 \tau} - \frac{\sigma^2}{2 h^2}
\end{bmatrix}.
\]

Impose the conditions

\[
h^2 \leq 4 \tau \sigma^2 \quad \text{and} \quad \tau |\alpha^i_{k,j}| \leq h/4. \tag{43}
\]

**Remark 1.** Condition (43) guarantees that all \( \gamma_{k,j+1/2}^i \) are positive, and matrices \( A \) and \( \bar{A}^i \) are M-matrices [14].

**Proposition 1.** Components \( m_{k,j+1/2}^i \) are non-negative \( \forall i \in \{S, I, R, C\}, \forall k = 1, \ldots, M, \forall j = 1, \ldots, N - 1 \) when \( m_{0}^i(x_{j+1/2}) \geq 0 \) \( \forall j = 1, \ldots, N - 1 \) and (43) is performed.

**Proof.** For non-negative components of \( f_{k,j}^i \), the non-negativity of \( m_{k,j}^i \), is provided by the monotonously property of the M-matrix [14]. Put that \( \exists i_0, k_0, j_0 \) for which \( f_{k_0 - 1,j_0}^i \) is negative and \( m_{k_0,j_0}^i \) change its value to negative one. From the non-negativity of \( m_{k_0,j_0}^i(x_{j+1/2}) \) \( \forall j = 1, \ldots, N - 1 \) follows that \( \exists \zeta \in [(k_0 - 1) \tau, k_0 \tau] \) that \( m_{k_0,j_0}^{i-\delta}(\zeta, x_{j_0+1/2}) = 0 \). Then \( m_{k_0,j_0}^{i-\delta} = O(\zeta - (k_0 - 1) \tau) \) and from (40) it follows that \( f_{k_0 - 1,j_0}^{i-\delta} \) is non-negative for non-negative parameters \( \beta, \mu, \varepsilon, \delta \). The obtained contradiction proves the proposition.

**Remark 2.** Sum expressions in (38) over \( j = 0, \ldots, N - 1 \) and multiply obtained expression by \( \tau h \). As a result, for non-negative \( m_{k-1,j}^{i,h}(t_k, x) \) the following equality is satisfied:

\[
\int_0^1 \sum_{i=S,I,R,C} m_{k-1,j}^{i,h}(t_k, x) \, dx = \int_0^1 m_{k-1,j}^{i,h}(t_k, x) \, dx = \int_0^1 \sum_{i=S,I,R,C} m_{k-1,j}^{i,h}(t_k, x) \, dx = \int_0^1 m_{k-1,j}^{i,h}(t_k, x) \, dx \tag{44}
\]

since the following property is performed \( \forall i \in \{S, I, R, C\} \).
Having\(\gamma_{k,j+1/2}^{i,1} + \gamma_{k,j+1/2}^{i,2} + \gamma_{k,j+1/2}^{i,3} = 1/\tau.\) (45)

Here \(m^h\) is grid value of total mass of population and expression (44) for non-negative \(m^{i,h}\) is the discrete analogue of the conservation law for total mass of agents.

**Proposition 2.** For (38)–(40) with initial (41) and boundary (30) conditions the following assessment is performed

\[
\max_{0 \leq k \leq M} \|m^{i,h}(t_k, \cdot)\|_{1,h} \leq \|m^i_0(\cdot)\|_{1,h} + T \max_{0 \leq k \leq M} \|f^{i,h}(t_k, \cdot)\|_{1,h},
\]

where \(\|m^{i,h}(t_k, \cdot)\|_{1,h}\) is discrete analogue of \(L_1(0,1)\)-norm for a grid function

\[
\int_0^1 m^h(t_k, x) \, dx = \|m^h(t_k, \cdot)\|_{1,h} := \sum_{j=0}^{N-1} |m_{k,j+1/2}^{i,h}| h.
\]

**Proof.** Use the key property (45) of the approximation scheme (38). Put that \(\tilde{m}^{i,h}(t_k, \cdot)\) is the solution of (38) for all non-negative \(f_{k-1,j+1/2}^{i,h}\). The components \(\tilde{m}_{k,j+1/2}^{i,h}\) will be non-negative due to M-matrices properties [14]. Multiply (38) by \(\tau\) and \(h\) and sum up over \(j = 0, ..., N - 1\) for non-negative \(f_{k-1,j+1/2}^{i,h}\).

Taking into account (45), the following equality is obtained:

\[
\|\tilde{m}^{i,h}(t_k, \cdot)\|_{1,h} = \|\tilde{m}^{i,h}(t_k, \cdot)\|_{1,h} + \|f^{i,h}(t_k, \cdot)\|_{1,h}.
\]

\(\forall i \in \{S, I, R, C\}, \forall k = 1, ..., M.\) Mathematical induction on \(k\) leads to

\[
\|\tilde{m}^{i,h}(t_k, \cdot)\|_{1,h} = \|\tilde{m}^i_0(\cdot)\|_{1,h} + \tau \sum_{s=1}^{k} \|f^{i,h}(t_s, \cdot)\|_{1,h}.
\]

Now put \(\tilde{m}^{i,h}(t_k, \cdot)\) into the solution of system (38), where \(f_{k-1,j+1/2}^{i,h}\) can be of any sign. Note that

\[
\tilde{m}^{i,h}(t_k, x_{j+1/2}) - m_{k,j+1/2}^{i,h} \geq 0,
\]

which is due to the monotone property of the M-matrix [14] that occurs after the expression \(\tilde{m}^{i,h}(t_k, x_{j+1/2}) - m_{k,j+1/2}^{i,h}\) is substituted by (38). The non-negativity of components \(\tilde{m}_{k,j+1/2}^{i,h}\) and \(m_{k,j+1/2}^{i,h}\) constitute:

\[
\|m^{i,h}(t_k, \cdot)\|_{1,h} \leq \|\tilde{m}^{i,h}(t_k, \cdot)\|_{1,h} = \|\tilde{m}^i_0(\cdot)\|_{1,h} + \tau \sum_{s=1}^{k} \|f^{i,h}(t_s, \cdot)\|_{1,h},
\]

(46)

Having taken the maximum of both parts of (46) leads to the required estimate.

**4.2. Discrete optimal control problem**

Replace the integral cost function in (11) by the discrete one:

\[
J^h(m_{IRC}^h, \alpha_{IRC}^h) = \sum_{i} \sum_{k=0}^{M-1} \sum_{j=0}^{N-1} \left( r_{k,j+1/2}^{i,h} m_{k,j+1/2}^{i,h} + g_{k,j+1/2}^{i,h} \right) \tau h + \sum_{j=0}^{N-1} h(m_{M,j+1/2}^i)^2 / 2.
\]

(47)

Here \(r_{k,i+1/2}^{i,h}\) is carried out for \(F(\alpha, t, x)\) in the following way:
\[
\tau^{i,h}_{k,j+1/2} = F^i \left( \alpha^h_{k,j}, t_k, x_j \right)/2 + F^i \left( \alpha^h_{k,j+1}, t_k, x_{j+1} \right)/2
\]

with \( \alpha^{i,h}_{k,j} := \alpha^{i,h}(t_k, x_j) \), so the differential minimization problem is replaced by the grid one:

\[
\begin{aligned}
\inf_{\alpha_{SIRC}} & \ J^h(\mathbf{m}^h_{SIRC}, \alpha^h_{SIRC}), \\
\mathfrak{A}^h \mathbf{m}^h_{\cdot \cdot} = \mathfrak{A}^h \mathbf{m}^0_{\cdot \cdot}.
\end{aligned}
\]  

**Remark 3.** Decomposition into Taylor’s series shows that the finite difference problem (49) approximates the initial differential formulation by the order of \( O(\tau + h^2) \).

Also, introduce the following:

\[
(a, b) = h \sum_{j=0}^{N-1} a_{j+1/2} b_{j+1/2}
\]  

for any vectors \( a = \{a_{j+1/2}\}_{j=0,\ldots,N-1} \) and \( b = \{b_{j+1/2}\}_{j=0,\ldots,N-1} \). To formulate a discrete optimal control problem, introduce the grid-function set: \( v^{i,h}_{\cdot \cdot} = \{v^{i,h}_{k,j+1/2}\}_{k=0,\ldots,M} \). Multiply the \( k \)-th component of \( i \)-th equation from (42) by \( \tau v^{i,h}_{k,\cdot} \) and sum over \( k \) taking into account the notation (50):

\[
L^{i,h} := \tau \sum_{k=1}^{M} \left( \left( \mathfrak{A}^i m^{i,h}_{k,\cdot}, v^{i,h}_{k,\cdot} \right) - \left( \mathbf{B}^i_k m^{i,h}_{k-1,\cdot}, v^{i,h}_{k-1,\cdot} \right) - \langle v^{i,h}_{k,\cdot}, f^{i,h}_{k-1,\cdot} \rangle - \langle m^{i,h}_{0,\cdot}, \mathfrak{A}^i v^{i,h}_{0,\cdot} \rangle \right) \tau + \langle m^{i,h}_{M,\cdot}, \mathfrak{A}^i v^{i,h}_{M,\cdot} \rangle \tau = \tau \sum_{k=0}^{M-1} \left( \langle m^{i,h}_{k,\cdot}, \mathfrak{A}^i v^{i,h}_{k,\cdot} \rangle - \langle m^{i,h}_{k+1,\cdot}, (\mathbf{B}^i_{k+1})^* v^{i,h}_{k+1,\cdot} \rangle - \langle v^{i,h}_{k+1,\cdot}, f^{i,h}_{k+1,\cdot} \rangle \right).
\]  

Now write down the Lagrangian for the grid optimization problem (49)

\[
\mathfrak{A}^h(\mathbf{m}^h_{SIRC}, \alpha^h_{SIRC}, v^h_{SIRC}) := J^h(\mathbf{m}^h_{SIRC}, \alpha^h_{SIRC}) - \sum_{i \in \{S,I,R,C\}} L^{i,h}.
\]  

Here \( (\mathbf{B}^i_k)^* = (\mathbf{B}^i_k)^T \) means the matrix conjugates to \( \mathbf{B}^i_k \). Then the problem of finding a saddle point (22) in the grid case is written as:

\[
\inf_{(\mathbf{m}^h_{SIRC}, \alpha^h_{SIRC})} \sup_{v^h_{SIRC}} \mathfrak{A}^h(\mathbf{m}^h_{SIRC}, \alpha^h_{SIRC}, v^h_{SIRC}).
\]  

Differentiate the Lagrangian (52) with respect to the individual components to obtain the following system of algebraic equations:

\[
\mathfrak{A}^i v^{i,h}_{k,\cdot} = (\mathbf{B}^i_{k+1})^* v^{i,h}_{k+1,\cdot} + z^{i,h}_{k,\cdot},
\]  

\( \forall i \in \{S,I,R,C\}, \forall k = M - 1, M - 2, \ldots, 0 \). Here

\[
z^{i,h}_{k,j+1/2} = \partial g^i / \partial m^i \left( t_k, x_{j+1/2}, m^h_{k,j+1/2} \right) + z^{i,h}_{k,j+1/2} + \sum_{l \in \{S,I,R,C\}} v^{l,h}_{k+1} \partial f^{l,h}_{k} / \partial m^{l,h}_{k}.
\]  

and
\[ v_{k,-1/2}^h = v_{k,1/2}^h, \quad v_{k,N+1/2}^h = v_{k,N-1/2}^h \]

\( \forall k = M - 1, \ldots, 0 \forall i = 0, \ldots, N - 1. \) The initial conditions for (54) after variation of the Lagrangian (52) can be written in the following form

\[
\begin{align*}
\begin{cases}
v_{M,i+1/2}^{i,h} &= 0, & \text{if } i \in \{S, R, C\}, \\
\mathcal{A}v_{M,i+1/2}^{i,h} &= m_{M,i+1/2}^{i,h} & \text{if } i = I.
\end{cases}
\end{align*}
\]

Proposition 3. For (54)–(57) under the restrictions (43) the following assessments are performed

\[
\max_{0 \leq k \leq M} \|v^{i,h}(t_k, \cdot)\|_{\infty,h} \leq T \max_{0 \leq k \leq M} \|z^{i,h}(t_k, \cdot)\|_{\infty,h},
\]

for \( i \in \{S, R, C\} \) and for \( i = I \):

\[
\max_{0 \leq k \leq M} \|v^{i,h}(t_k, \cdot)\|_{\infty,h} \leq \|m^{I}(t_M, \cdot)\|_{\infty,h} + T \max_{0 \leq k \leq M} \|z^{i,h}(t_k, \cdot)\|_{\infty,h},
\]

where \( \|m^{i,h}(t_k, \cdot)\|_{\infty,h} \) is discrete analogue of \( L_\infty(0,1) \)–norm for grid function

\[
\max_{0 \leq x \leq 1} |v^i(t_k, x)| = \|v^i(t_k, \cdot)\|_{\infty,h} := \max_{0 \leq i \leq N-1} |v_{k,i+1/2}^i|.
\]

Proof. Put \( |\tilde{v}^{i,h}(t_k, x_{j+1/2})| \) as the component reaching its maximum absolute value on the layer \( t_k \) so that \( |\tilde{v}^{i,h}(t_k, x_{j+1/2})| = \|v^{i,h}(t_k, \cdot)\|_{\infty,h} \). Use again the key property of the coefficients (45) to obtain the inequality

\[
\|v^{i,h}(t_k, \cdot)\|_{\infty,h} = |\tilde{v}^{i,h}(t_k, x_{j+1/2})| \leq \|v^{i,h}(t_{k+1}, \cdot)\|_{\infty,h} + \tau \|z^{h,i}(t_k, \cdot)\|_{\infty,h}.
\]

Mathematical induction on \( k \) leads to

\[
\|v^{i,h}(t_k, \cdot)\|_{\infty,h} \leq \|v^{h,i}(t_M, \cdot)\|_{\infty,h} + (M-k)\tau \|z^{h,i}(t_k, \cdot)\|_{\infty,h}.
\]

For \( i \in \{S, R, C\} \) with zero “initial” conditions for (54) the required estimate can be obtained after taking a maximum over \( k \) in (58). For \( i = I \) from (57) follows that

\[
\|v^{h,i}(t_M, \cdot)\|_{\infty,h} \leq \|A^{-1}\|_{\infty,h}\|m^{h,I}(t_M, \cdot)\|_{\infty,h}.
\]

It is known that for an M-matrix with strict diagonal dominance the following statement holds (Theorem 2 from [21]):

\[
\|A^{-1}\|_{\infty,h} = 1/R,
\]

where \( R \) is the amount of diagonal dominance, which is the same for each row of the matrix \( A \) and equaled to \( 1/\tau \). Then for \( i = I \) (58) can be rewritten as

\[
\|v^{h,I}(t_k, \cdot)\|_{\infty,h} \leq \tau \|m^{h,I}(t_M, \cdot)\|_{\infty,h} + (M-k)\tau \|z^{h,I}(t_k, \cdot)\|_{\infty,h}.
\]

Taking a maximum over \( k \) in (59) one obtains the required estimate for \( i = I \).

Also (53) gives the following grid analogue for optimality conditions (29) at the point \( (t_k, x_j) \):

\[
\frac{\partial F_i}{\partial \alpha_i}(a_{i,k,j}, t_k, x_j) + \frac{\tau_i h_{j+1/2}^i - \tau_i h_{j-1/2}^i}{h} = 0
\]

(60)
\( \forall k = 1, \ldots, M \ \forall j = 1, \ldots, N - 1. \)

Thus, the solution to the discrete optimization problem (49) can be found by the following iterative algorithm:

1. Put the initial value of grid functions \( \hat{\alpha}_{k,j} \) equal to zero.
2. For the zero functions \( \alpha^i_{k,j} \) obtain the initial value of functions \( m^i_{k,j} \) using (38)–(42) and \( J^h(m^i_{k,j}, \alpha^i_{k,j}) \) using (47), (48).
3. Calculate the value of \( v^i_{k,j} \) functions solving the system (54)–(57).
4. Obtain a new value for the functions \( \alpha^i_{k,j} \) by solving (60).
5. Obtain a new value for the functions \( m^i_{k,j} \) by solving (38)–(43).
6. Obtain a new value for the functions by solving \( J^h(m^i_{k,j}, \alpha^i_{k,j}) \) by (47), (48).
7. If the cost function \( J^h(m^i_{k,j}, \alpha^i_{k,j}) \) reaches its minima with the given accuracy then choose functions \( m^i_{k,j}, \alpha^i_{k,j} \) obtained on the previous steps as a solution of (49). Otherwise go to step 3 for new iteration.

4.3. Discrete MFG with corrective control

For the discrete statement of the optimal control problem with external influence on agent’s strategy, the schemes (38)–(43) and (54)–(57) introducing into consideration the following representation can be used:

\[
\hat{\alpha}^i_{k,j} = \alpha^i_{k,j} + \rho^h (\hat{\alpha}^i_{k,j})
\]

(61)

\( \forall i \in \{S, I, R, C\}, \ \forall k = 1, \ldots, M \ \forall j = 1, \ldots, N - 1. \) Instead of the integral function (28) use the discrete one:

\[
J^h(m^h_{SIRC}, \hat{\alpha}^h_{SIRC}) = \sum_i \sum_{k=0}^{M-1} \sum_{j=0}^{N-1} \left( i^h_{k,j+1/2} m^i_{k,j+1/2} + g^i_{k,j+1/2} \right) \tau^h + \sum_{j=0}^{N-1} h(m^f_{M,j+1/2})^2 / 2,
\]

(62)

where \( i^h_{k,j+1/2} \) is carried out for \( F(\hat{\alpha}, t, x) \) in the following way:

\[
i^h_{k,j+1/2} = F^i(\hat{\alpha}^h_{k,j}, t_k, x_j) / 2 + F^i(\hat{\alpha}^h_{k,j+1}, t_k, x_{j+1}) / 2.
\]

(63)

Then, the grid optimization problem can be rewritten as:

\[
\begin{align*}
\inf_{\hat{\alpha}^h_{SIRC}} J^h(m^h_{SIRC}, \hat{\alpha}^h_{SIRC}), \\
\exists^i m^i_{j,.} = \hat{\exists}^i m^i_{0,.}.
\end{align*}
\]

(64)

The solution of (64) can be found by the iterative algorithm described in the previous section taking into account the representation (61).

4.4. Initial dataset

For numerical implementation of the algorithm the parameters presented below were applied. Table 1 describes the epidemiological constants obtained by solving the inverse problem for a statistical data set describing COVID-19 incidence in the Novosibirsk region [15].

For the MFG model the parameters presented in Table 2 were applied. Note that the combination of constants \( c_{11} \) and \( c_{21} \) is quite close in its physical meaning, the so-called “index of self-isolation” introduced by the Yandex company (https://yandex.ru/company/researches/2020/podomam). The current values of constants correspond to the low self-isolation level (about 0.5 – 1) and describe the situation in the period
under consideration. The values of the coefficients \( c_{3i} \) were chosen because during these time periods the quarantine measures introduced by the government were not so restrictive.

The initial distributions of agents mass were written as:

\[
m_{0i} = \frac{A_i}{B_i} \left( \exp \left( \frac{-(x - x_i^c)^2}{2 \sigma_i^c} \right) \right) / \sigma_i^c \sqrt{2\pi} + a_i x^2 + b_i (1 - x)^2 ,
\]

(65)

where \( A_i \) is proportion of current fraction in relation to the total population at the initial time moment; \( B_i \) is normalization factor equaled to integral over \([0,1]\) from expression in brackets; \( a_i = \exp \left( -\left(1 - x_i^c\right)^2 / 2 \sigma_i^c \right) \left(1 - x_i^c\right) / 2 \sigma_i^c \sqrt{2\pi} \) and \( b_i = \exp \left( -(x_i^c)^2 / 2 \sigma_i^c \right) (x_i^c) / 2 \sigma_i^c \sqrt{2\pi} \) ensure boundary conditions (7) for \( m_{0i} \). Note that \( A_i, \forall i \in \{S, I, R, C\} \) coincides with the corresponding initial value \( S(0), I(0), R(0), C(0) \) for the differential SIRC model, so the following values for \( x_i^c \) and \( \sigma_i^c \) were chosen:

\[
\begin{align*}
x_S^c &= 0.8; & \sigma_S^c &= 0.1; & x_I^c &= 0.2; & \sigma_I^c &= 0.1; \\
x_R^c &= 0.7; & \sigma_R^c &= 0.2; & x_C^c &= 0.3; & \sigma_C^c &= 0.2.
\end{align*}
\]

(66)

The physical meaning of constants (66) is that non-infected (S) people do not seek to comply with restrictions and self-isolation for economic reasons while infected (I) ones, in general, comply with quarantine measures. The recovered part of population (R) has full immunity and often does not comply with social measures since they are confident in their immunity; and cross-immune ones (C) are afraid of re-infection after a while.

For the \( A_i \) constants, the following parameters presented in Table 3 were chosen for different time periods. Here parameters \( A_I \) are also a solution for the inverse problem.
Table 3
Initial data for population distribution.

| Symbol | Time period | 
|--------|-------------|
|        | from 05/01/20 to 06/30/20 | from 06/30/20 to 08/08/20 | from 12/01/20 to 03/10/21 |
| $A_S = S(0)$ | 0.999757 | Data obtained at | 0.991199 |
| $A_I = I(0)$ | 0.000204 | horizon time $T$ from previous time period | 0.001505 |
| $A_R = R(0)$ | 0.000039 | | 0.006937 |
| $A_C = C(0)$ | 0.000000 | | 0.000359 |

Fig. 2. Comparison of SIRC and MFG-SIRC models for number of daily diagnoses in Novosibirsk region.

4.5. Numerical results of modeling COVID-19 spread in the Novosibirsk region

First, the difference between using the standard SIRC differential model and the MFG model based on it was considered. The comparison was made using official statistical data from a coronavirus report [15] for a period of 100 days from May 1, 2020 in Novosibirsk city, Russia. The collected statistic data, as well the solutions of differential SIRC and grid MFG-SIRC models can be seen in Fig. 2.

The figure shows the statistical data has two areas describing incidence rise and fall. Since SIR-type models are bad for long-term prediction, the two curves with different epidemiological parameters were put together to approximate the epidemiological situation. The model’s epidemiological parameters were restored based on the statistical data and the optimization method presented in the work [8] for two intervals: from May 1 to June 30 and from June 30 to August 8, 2020. The obtained parameters are presented in Table 1. These parameters were used for numerical implementation of the two models in question. The spatial and stochastic constants for the MFG implementation are given in Section 4.4. As can be seen from the figure, MFG gives a more accurate approximation to statistical data, but like the parent SIRC model, it does not work well for long-term forecasts.

Fig. 2 also shows that introducing spatial distribution into consideration has a significant impact on population behavior. Fig. 3 demonstrates the dependence of the obtained solution on the selected initial spatial distribution. The comparison was made on based on a coronavirus report for the city of Novosibirsk from December 1, 2020 when a decline in the incidence was observed. For this time period, the restored epidemiological parameters are given in Table 1 while the stochastic parameter of the system $\sigma^2$ was chosen equal to 0.5, and the following curves were used as initial distributions for different population groups:

1. initial distributions $m_{0i}$ determined by (65) with parameters (66);
2. initial distributions $m_{0i}$ determined by (65) with parameters $x^c_i = 0.2; \sigma^c_i = 0.1 \forall i \in \{S, I, R, C\}$;
3. initial distributions $m_{0i}$ determined by (65) with parameters $x^c_i = 0.5; \sigma^c_i = 0.2 \forall i \in \{S, I, R, C\}$.
In physical sense, it means that at the initial time of the second case, agents are clearly positive to quarantine measures and physical distancing, and in third case, they are not so patient to conform with them. The motivation for the first case is presented in Section 4.4.

For the third case (3), external corrective control as a solution of the system (64) was introduced. Comparison of the obtained curves is shown in Fig. 4, which demonstrates that taking into account external restrictive measures that are not the agent’s choice has a significant impact on population dynamics.

The Fig. 4 shows that taking into account external restrictive measures that are not the agent’s choice has a significant impact on population dynamics.

5. Conclusion and discussions

The article describes a way to apply the well-known economic mean field models for forecasting a pandemic spread of epidemics, in particular that of COVID-19. This approach has been opted because traditional epidemiological SIR-based models do not take into account population heterogeneity and therefore cannot be used for long-term forecasts. Another well-known approach to a virus spread, the so-called agent-based models, allow taking into account non-epidemiological factors, but lead to computationally complex systems
Numerical experiments into modeling the dynamics of COVID-19 spread in Novosibirsk, Russia have shown that despite the fact that the dynamics of the population is determined by the differential SIRC model, taking into account the spatial agent characteristics has a huge impact on the final result. This what gives this model an advantage over SIR-type models but also generates a huge class of problems for determining the parameters of a system, related to the fields of statistical analysis and inverse problem solving. In addition, there are several more ways to modify the presented model. The greatest interest in this respect presents considering the epidemiological parameters $\beta, \gamma$ and others depending on the position $x$ of the agent in space or even on its strategy $\alpha(t,x)$. This computational approach is applicable for any SIR-type model.

The numerical algorithm considered in this paper is based on the ideas proposed in the works [17–20] and modified for application to epidemiology problems. It gives a direct and simple rule for minimizing the cost functional, ensures the fulfillment of the law of conservation of the entire agents mass, and allows to take into account more complex control functions $F_i(\alpha, t, x)$ instead of those that are quadratically dependent on $\alpha$ [1].

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