One Instanton Predictions of a Seiberg-Witten curve from M-theory: the Symmetric Representation of SU(N)

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Abstract

We consider N=2 supersymmetric Yang-Mills theories in four dimensions with gauge group SU(N) for N larger than two. Using the cubic curve for a matter hypermultiplet transforming in the symmetric representation, obtained from M-theory by Landsteiner and Lopez, we calculate the prepotential up to the one instanton correction. We treat the curve to be approximately hyperelliptic and perform a perturbation expansion for the Seiberg-Witten differential to get the one instanton contribution. We find that it reproduces the correct result for one-loop, and we obtain the prediction for that curve for the one instanton correction term.

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1 Introduction

Over the last few years enormous progress has been made in four-dimensional N=2 supersymmetric gauge theory. The seminal work was that of Seiberg and Witten (SW) for SU(2) pure Yang-Mills [1], and also with matter transforming in the defining representation [2]. The main idea is that all the relevant information about the low energy limit of the theory can be obtained from an algebraic variety (in most cases an algebraic curve), which is taken to be a fibre over the moduli space of vacua. In particular, the low energy effective action is completely determined from a holomorphic function, the prepotential. This prepotential can be separated into three parts, a classical piece, a one-loop piece, and a sum of instanton contributions, and is calculable from the curve.

This program was generalized by a large number of people to the classical groups without matter or with matter transforming in the defining representation [3]. In general, these papers do not make direct predictions for the prepotential, but rather they present a curve which may be used to obtain the prepotential, with suitable methods, if available. The conjectured curves were checked for consistency using double scaling limits, monodromy properties, decoupling limits, etc.

The curves for the classical groups, with the above mentioned matter content, can all be represented by a double covering of the Riemann sphere, i.e. they are of hyperelliptic type. (The exceptional groups have curves of non-hyperelliptic type [4]; see also [5] for an interesting account for SU(N) pure Yang-Mills with non-hyperelliptic curves.) A general approach to calculate instanton corrections to the prepotential for hyperelliptic curves was developed by D’Hoker, Krichever and Phong (DKP) [6, 7]. These authors give explicit forms for the one and two instanton corrections for classical groups with matter in the
defining representation.

More recently, the curves of SW theory have been rederived from M-theory [8] and also using geometric engineering [3]. Working in the context of M-theory, Landsteiner and Lopez [10] suggested curves for SW theory for SU($N$) with matter transforming in the symmetric as well as in the antisymmetric representations. No predictions for the prepotential were presented which could be compared with results obtained from N=2 field theory, and hence a thorough test is desirable. Furthermore, the suggested curves are of non-hyperelliptic type. Although several consistency checks were made by Landsteiner-Lopez, there was no indication given as how the prepotential might be computed.

It is the purpose of this paper to derive the prepotential from the curve given in [10] for matter transforming in the symmetric representation up to the one instanton contribution. We only consider $N \geq 3$, since for SU(2) the symmetric representation coincides with the adjoint which is a scale invariant theory with four supersymmetries. This example is not within the scope of this paper.

Our result should be compared with microscopic instanton calculations, when they become available. The importance of those tests is obvious. The SW method, however elegant and powerful, must still be tested on a case by case basis. One must of course check if the model can make predictions for the prepotential, and if it reproduces the same results as a calculation entirely in the context of field theory. By giving a prediction for the first instanton correction to the prepotential we provide a target for such calculations.

The basic idea is to treat the Landsteiner-Lopez curve, which is of cubic form

$$y^3 + fy^2 + g\bar{\Lambda}^2 y + \epsilon = 0,$$

to be approximately hyperelliptic, i.e. $y^2 + fy + g\bar{\Lambda}^2 = 0$. This will be a good approximation in the part of moduli space where $\epsilon = \bar{\Lambda}^6 x^6$ is small, which corresponds to the weak coupling region. We then perform a perturbation expansion in
for the SW differential to sufficient order to obtain the one instanton result. As we will see, first order corrections will be sufficient. To determine the prepotential we follow the work by DKP closely. Our result for the one instanton correction will, in fact, depend on residue functions $S_k(a_k)$ in the same way as for SU($N$) with matter in the defining representation. The difference will be in the detailed form of the residue functions.

Our calculation is subjected to several consistency checks. We may take the mass of the hypermultiplet to infinity. In this limit, one should recover the result for pure Yang-Mills, and we show that this is indeed the case. There are also tests of the formal mathematical correctness of the result. First of all, for technical reasons, terms proportional to $\bar{\Lambda}$ will appear at various stages of the calculation, however, we find that they cancel in the final results, as they must. Also the dual periods, according to the SW ansatz, must be integrable. Since we start with a Riemann surface the integrability is guaranteed, and this provides a non-trivial test of the final result for the dual periods as well as for the method itself.

One interesting fact that appears in our calculation is that the one-loop part of the prepotential follows from the hyperelliptic part of the curve by itself. Obtaining the one-loop part of the prepotential is thus not conclusive evidence for a particular ansatz. In our calculation, the evidence in favor of the M-theory curve (as opposed to a possible hyperelliptic candidate similar to the truncated M-theory curve) comes from the one instanton correction term to the prepotential.
2 General Background

We consider $\mathbb{N}=2$ supersymmetric Yang-Mills theory in four dimensions with gauge group $\text{SU}(N)$, for $N$ larger than two, and with matter multiplets transforming in the symmetric rank two tensor representation.

The model contains a vector multiplet corresponding to a pure gauge multiplet transforming in the adjoint representation, and matter hypermultiplets transforming in the symmetric representation. Asymptotic freedom is ensured if the beta function is negative. The beta function is proportional to the difference of the Dynkin index of the adjoint representation and the sum of the Dynkin indices of the matter multiplet representations. The adjoint representation has index $2N$, while the symmetric representation has index $N+2$, and hence one flavor is the only asymptotically free case.

From general arguments, the low energy effective Lagrangian is determined completely by a holomorphic function, the prepotential $F$, and is given by

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left( \int d^4\theta \frac{\partial F(A)}{\partial A_i} \tilde{A}_i + \frac{1}{2} \int d^2\theta \frac{\partial^2 F(A)}{\partial A_i \partial A_j} W^\alpha_i W^\alpha_j \right). \quad (1)$$

It is known that the prepotential for $\mathbb{N}=2$ supersymmetric theories consists of three parts; the classical contribution, the one-loop contribution (which corresponds to the only perturbative corrections by virtue of non-renormalization theorems), and non-perturbative instanton corrections [1]. Furthermore, the prepotential must be of the form

$$F = \frac{1}{2} \tau_0 A^2 + \frac{i}{4\pi} \sum_{\alpha \in \Delta_+} (A \cdot \alpha)^2 \log \left( \frac{A \cdot \alpha}{\Lambda} \right)^2$$

$$- \frac{i}{8\pi} \sum_{w \in W_G} \sum_{j=1}^{N_f} (A \cdot w + m_j)^2 \log \left( \frac{A \cdot w + m_j}{\Lambda} \right)^2 + \sum_{d=1}^{\infty} F_d(A) \Lambda^{2N-I_D} d. \quad (2)$$

In this expression, $\Delta_+$ denotes the set of positive roots, $W_G$ is the set of weights, $F_d$ are instanton corrections, $I_D$ denotes the sum of the Dynkin indices of the matter hyper-
multiplet representations, $\Lambda$ is the dynamically generated scale of the theory, $N_f$ is the number of matter hypermultiplets, and $m_j$ are the respective masses. Furthermore, $A_i$ is an $N=1$ chiral superfield whose scalar components $e_i$ parametrize the flat directions of the potential

$$V(\phi) = Tr[\phi, \phi^\dagger]^2.$$  

Here $\phi$ is the scalar field of the $N=2$ vector supermultiplet transforming in the adjoint representation.

The SW ansatz postulates that the prepotential $F(a)$ can be obtained from a fibration of algebraic varieties over the moduli space of vacua, where $a_i$ is the quantum corrected $e_i$. Associated with each fiber there is a preferred differential $\lambda$, the SW differential. Using a canonical set of one cycles $(A_i, B_i)$ that form a basis for the first homology group, the periods $a_k$ and their dual partners $a_{D,k}$ are obtained from the curve and $\lambda$ by means of the period integrals

$$2\pi i a_k = \oint_{A_k} \lambda, \quad 2\pi i a_{D,k} = \oint_{B_k} \lambda.$$  

The prepotential is related to the periods by

$$a_{D,k} = \frac{\partial F}{\partial a_k}.$$  

To verify the validity of the ansatz, one must perform microscopic instanton calculations similar to those in ref. [12]. Those calculations will provide the correct (i.e. obtained from the underlying field theory) instanton coefficients $F_d(a)$. The purpose of this paper is to derive the one instanton contributions to the prepotential, from the curve deduced by
Landsteiner and Lopez from M-theory for the symmetric representation of $SU(N)$. This will provide a prediction that can be checked against microscopic instanton calculations, when they become available.

3 The Curve

The curve proposed from M-theory considerations by Landsteiner and Lopez for matter transforming in the symmetric representation takes the form\(^1\)

$$y^3 + f(x)y^2 + g(x)\Lambda^2 y + \epsilon(x) = 0,$$

(6)

where

$$f(x) = \prod_{i=1}^{N}(x - e_i) \quad g(x) = (-1)^N x^2 \prod_{i=1}^{N}(x + e_i) \quad \epsilon(x) = \bar{\Lambda}^6 x^6.$$ (7)

Here $\bar{\Lambda}^2 = \Lambda^{N-2}$, and $e_i$ are the classical moduli which parametrize the weights of the defining representation of $SU(N)$. Asymptotic freedom requires the number of flavors to be at most one, and the classical order parameters satisfy

$$\sum_{i=1}^{N} e_i = 0,$$

(8)

for the massless hypermultiplet.

It can be checked that the curve (6) is invariant under the involution

$$y \rightarrow \frac{\Lambda^4 x^4}{y} \quad x \rightarrow -x.$$ (9)

The existence of an involution in this case reflects the fact that the genus of the algebraic variety corresponding to the curve (6) is larger than the dimension of the moduli space.
Fig.1: The sheet structure for the cubic curve.

The curve (6) can be regarded as a three-sheeted branched covering of the Riemann sphere parametrized by the complex coordinate $x$. It describes a Riemann surface of genus $2N - 2$. If we change the variable $y$ to $w = y + f(x)/3$, the curve becomes $w^3 + a(x)w + b(x) = 0$. To find the positions of the ramification points of the curve we look for the points where two sheets coincide, i.e. when $w_i = w_j$ for $i \neq j$, which implies that the discriminant $s(x) = 0$, where

$$s(x) = \frac{a^3(x)}{27} + \frac{b^2(x)}{4} = -\frac{\Lambda^4}{108} \left(f^2g^2 + O(\Lambda^2)\right).$$

(10)

To zeroth order in $\Lambda$, the solutions to the equation $s(x) = 0$ correspond to the set $\{0, \pm e_i\}$. The corrections to these solutions will be such that the ramification points $\pm e_i$ split into pairs for non-vanishing $\Lambda$. In the perturbative regime, i.e. for $\Lambda \ll e_i$, $i = 1, ..., N$, one
has square root cuts over $\pm e_i$, while at $x = 0$ there are no cuts at all. A careful analysis of how the phase changes as one moves between the ramification points shows that sheets two and three are connected over $-e_i$, while sheets one and two are connected over $e_i$. This structure is depicted in fig.1. The dashed line over $x = 0$ indicates that sheets two and three touch here, but there are no branch cuts at $x = 0$. The other vertical lines indicate the location of the square root cuts, which we choose to run between $x_i^\pm$. (Thus $x_i^\pm \to e_i$ as $\bar{\Lambda} \to 0$.)

The structure of the Riemann surface may also be obtained in a more intuitive way from the involution of the curve. Since the curve (9) is invariant under the involution (3), the set of roots is at most permuted by the involution. In fact, one finds that the involution exchanges sheets one and three, and maps sheet two onto itself. This is easy to verify by using the action of the involution on the roots, and computing to the lowest order in $\bar{\Lambda}$, where the roots are given by

$$
y_1 = -f + \ldots \quad y_2 = -\bar{\Lambda}^2 g + \ldots \quad y_3 = -\bar{\Lambda}^4 x^6 + \ldots
$$

(11)

Since the involution includes the discrete map $x \to -x$, the sheet structure follows.

The strategy of this paper is to make a systematic expansion of the solutions of the curve around the hyperelliptic approximation, which induces a perturbation series for the SW differential. We will keep terms to sufficient accuracy to obtain the one instanton contribution to the prepotential. The approximation scheme means that only a two-sheeted structure will appear in the calculation. The zeroth approximation to the curve (3) is the hyperelliptic equation

$$y^2 + yf(x) + \bar{\Lambda}^2 g(x) = 0,$$

(12)
with roots

\[ y_\pm = -\frac{f(x)}{2} \pm \left[ \frac{(f(x))^2}{4} - \bar{\Lambda}^2 g(x) \right]^{1/2} \equiv -\frac{f(x)}{2} \pm r(x). \quad (13) \]

There are square root branch cuts over \( e_i \) connecting sheets one and two. When computing the periods and the dual periods, we only need to consider cycles related to those branchings, since the approximation only involves these two sheets. (Even when we include perturbations in \( \epsilon \), only the same cycles are required. This means that this is indeed a projection onto a subvariety.) Notice that the hyperelliptic approximation breaks down near \( x = 0 \). However, this does not change the dual periods.

To find the correction terms to the roots of the cubic equation (6) as well as to the SW differential

\[ \lambda = x \frac{dy}{y}, \quad (14) \]

we make a perturbation expansion around the last term of the cubic equation, that is, around \( \epsilon(x) \). Expand solutions to (6) in powers of \( \epsilon \),

\[ y_i(x) = \sum_{n=0}^{\infty} \alpha_{n}^{(i)} \epsilon^n, \quad (i = 1, 2, 3) \quad (15) \]

and insert \((13)\) into \( \prod_{i=1}^{3} (y - y_i) = 0 \). One may solve for the coefficients \( \alpha_{n}^{(i)} \) iteratively order by order in the perturbation theory by comparing this expression with \((6)\). We find, correct to first order in \( \epsilon \), the roots \footnote{See ref. \cite{13}, Appendix A for a more explicit account of the perturbation expansion.}

\[ y_1(x) = -\left( \frac{f}{2} + r \right) \left[ 1 + \epsilon \left( \frac{1}{2\Lambda^2 gr} + \frac{r - f/2}{\Lambda^2 g^2} \right) \right] + ... \]

\[ y_2(x) = -\left( \frac{f}{2} - r \right) \left[ 1 + \epsilon \left( \frac{-1}{2\Lambda^2 gr} - \frac{r + f/2}{\Lambda^2 g^2} \right) \right] + ... \]

\[ y_3(x) = -\frac{\epsilon}{\Lambda^2 g} + ... \quad (16) \]
The perturbation expansion enables us to express the SW differential for sheet one to first order in $\epsilon(x) = \bar{\Lambda}^6 x^6$ as

$$\lambda_1 = \frac{x}{r + f/2} \left[ r + \frac{f}{2} \right] + x d \left[ \epsilon \left( \frac{1}{2\bar{\Lambda}^2 g r} + \frac{r - f/2}{\bar{\Lambda}^4 g^2} \right) \right]. \quad (17)$$

The first term corresponds to the differential for the hyperelliptic approximation. Although this term has the standard form, notice that the function $g(x)$ in our case is moduli dependent, which is different than SW theory for SU($N$) with no matter or matter in the defining representation. We will find that the hyperelliptic approximation correctly predicts the one-loop part of the prepotential. However, the $O(\epsilon)$ correction to the hyperelliptic approximation will be important for our calculation of the one instanton contribution to the prepotential.

For later reference, note that the weights of the symmetric representation are

$$e_i + e_j \quad i \leq j \quad i, j = 1, \ldots, N. \quad (18)$$

4 The Periods

The purpose of this paper is to compute the explicit form of the prepotential up to the one instanton contribution for the $N = 2$ supersymmetric Yang-Mills theory with matter transforming in the symmetric representation from the Landsteiner Lopez curve. In order to do that, we must calculate the explicit expression of the periods $a_k$ and the dual periods $a_{D,k}$ up to order $\bar{\Lambda}^2$.

As the curve (6), proposed from M-theory for the symmetric representation, is not hyperelliptic, we develop a systematic procedure which generalizes previous methods. This involves a systematic expansion of the solution to (6) about the hyperelliptic approxima-
tion to this curve, which then induces similar corrections to the SW differential $\lambda$. We will show that the corrections to $O(\epsilon)$ in $\lambda$ do not contribute to the result of the periods $a_k$, although they are important in the computation of the dual periods $a_{D,k}$.

For the computation of the periods, we will work on the first sheet and consider the SW differential given in eq. (17). We arrange the branch cuts to extend from $x_k^-$ to $x_k^+$, where $x_k^\pm \to e_k$ as $\bar{\Lambda} \to 0$. We describe the cycles after projecting onto the sub-variety i.e. that of the hyperelliptic approximation. The $A_k$ cycles are then closed contours encircling the branch cuts with centers around $e_k$ (see fig.1). The $B_k$ cycles consist of curves which go from $x_1^-$ to $x_k^-$ on sheet one, pass through the branch cut between $x_k^-$ and $x_k^+$, and return to $x_1^-$ on sheet two. It is clear that the intersection number of the cycles $A_i$ and $B_j$ will be proportional to $\delta_{ij}$.

First of all, we will evaluate the portion of the periods that comes from the hyperelliptic approximation. For this purpose, define

$$\lambda = \lambda_I + \lambda_{II} + ..., \quad (19)$$

where the first term refers to the hyperelliptic approximation, and the second corresponds to the first-order corrections in $\epsilon$. In order to do so, express the $\epsilon = 0$ part of the SW differential on sheet one, eq. (17), as

$$\left(\lambda_1\right)_I = \frac{x dx}{2r} \left(f' - \frac{g'}{2g} f\right) + \frac{x dx}{2g} g', \quad (20)$$

where $f'$ and $g'$ denote $\partial f/\partial x$ and $\partial g/\partial x$, respectively. The last term in (20) can be neglected in the computation of the periods $a_k$, since it does not have any residues within the contour $A_k$. Following DKP (3), the integral we need to evaluate is
\[ 2\pi i(a_k)_I \equiv \oint_{A_k} dx x \frac{f' - g'}{2g} \sqrt{1 - 4\Lambda^2 g/f^2} \]
\[ = \oint_{A_k} dx \frac{f'}{f} + \sum_{m=1}^{\infty} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1)\Gamma(1/2)} (4\bar{\Lambda}^2)^m \oint_{A_k} dx x \left( \frac{f'}{f} - \frac{g'}{2g} \right) \frac{g^m}{f^{2m}}, \quad (21) \]

where we have expanded the denominator in a convergent power series in \( \bar{\Lambda}^2 \). The first integral in \((21)\) can be easily evaluated, with the result

\[ \oint_{A_k} dx \frac{f'}{f} = 2\pi i e_k. \quad (22) \]

For the remaining part of the integral \((21)\), we use the following identity \[6\]
\[ x \left( \frac{f'}{f} - \frac{g'}{2g} \right) \frac{g^m}{f^{2m}} = -\frac{d}{dx} \left( \frac{xg^m}{2mf^{2m}} \right) + \frac{1}{2m} \frac{g^m}{f^{2m}}. \quad (23) \]

The total derivative in \((23)\) drops out, and for the remainder we only need the contribution with \( m = 1 \), since we are only interested in \( \bar{\Lambda}^2 \) corrections. Introduce the residue functions \( S_k(x) \) defined by

\[ \frac{4g}{f^2} \equiv \frac{S_k(x)}{(x - e_k)^2}, \quad (24) \]

where

\[ S_k(x) = \frac{4(-1)^N x^2 \prod_{i=1}^{N} (x + e_i)}{\prod_{i\neq k} (x - e_i)^2}. \quad (25) \]

Using the definition of \( S_k(x) \) and that \( \frac{\Gamma(3/2)}{\Gamma(2)\Gamma(1/2)} = 1/2 \), we can conclude that the final expression for the periods \( a_k \), up to one-instanton order is

\[ a_k = e_k + \bar{\Lambda}^2 \frac{\partial S_k}{\partial x}(e_k). \quad (26) \]

Equation \((26)\) is not changed to \( O(\bar{\Lambda}^2) \) by the \( O(\epsilon) \) corrections to the SW differential, as can be checked directly from \((17)\). Indeed, the first term \( \epsilon/(2\Lambda^2 r) \) is of higher order,
while the remaining terms do not have residues within the $A_k$ contours. In order to see that $\epsilon/(2\Lambda^2 r)$ is higher order than $\Lambda^2$, we notice that its lowest order contribution is $2\Lambda^4 x^6/f(x)$. Although we can expand this factor by using partial fractions, we will obviously never get contributions to the one instanton term.

5 The Dual Periods

One evaluates the SW differential for the $B_k$ cycles by means of a contour that goes from $x^-_1$ to $x^-_k$ on sheet one, crosses the branch cut at $e_k$ to sheet two, runs back from $x^-_k$ to $x^-_1$ on sheet two, and passes to sheet one through the branch cut at $e_1$. The SW differential on sheet two is obtained by taking $r \rightarrow -r$ in $\lambda_1$. The $B_k$ cycles require the difference $\lambda_1 - \lambda_2$ in the integral therefore only terms with odd powers of $r$ will contribute to the dual periods.

The dual periods $a_{D,k}$ have two different contributions; the first corresponds to the hyperelliptic approximation, while the second comes from the correction to the hyperelliptic approximation. They can be expressed as

$$a_{D,k} = (a_{D,k})_I + (a_{D,k})_{II}. \quad (27)$$

In order to find the contribution coming from the hyperelliptic approximation, we need to introduce a regularization parameter $\xi$, which allows us to expand $r^{-1}$ in the integral for the dual periods, as was argued by DKP [3]. In what follows, it is understood that the limit $\xi \rightarrow 1$ is taken in the final results. Then the hyperelliptic approximation for the dual periods is given by

$$2\pi i (a_{D,k})_I = 2 \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)}{\Gamma(m+1)\Gamma(1/2)} \xi^{2m}(4\Lambda^2)^m \int_{x^-_1}^{x^-_k} dx x \left( f' - \frac{g'}{2g} \right) \frac{g^m}{f^{2m}}. \quad (28)$$
For notational convenience we will suppress the dependence on $x_1$, which we will restore at the end.

In order to truncate the series at the desired order in $\bar{\Lambda}$, one must know $x_k$ to sufficiently high order. By definition, $x_k$ satisfies $s(x_k) = 0$. The explicit form of the discriminant, defined in eq. (10), in terms of the functions $f$ and $g$, is

$$s(x) = -\frac{\bar{\Lambda}^4}{108} \left[ f^2 g^2 - 4\bar{\Lambda}^2 (f^3 x^6 + g^3) + 18\bar{\Lambda}^4 x^6 f g - 27\bar{\Lambda}^8 x^{12} \right].$$ \hspace{1cm} (29)

To order $\bar{\Lambda}^2$, this reduces to the identity $f^2(x_k) - 4\bar{\Lambda}^2 g(x_k) = 0$, which is appropriate to the hyperelliptic approximation, i.e. $r(x_k) = 0$ in (13). This is derived in an alternate way in ref. [13]. After a Taylor expansion, we get to one instanton accuracy

$$x_k = \epsilon_k - \bar{\Lambda} S_k^{1/2}(\epsilon_k) + \frac{\bar{\Lambda}^2}{2} \frac{\partial S_k}{\partial \epsilon}(\epsilon_k) + ...$$ \hspace{1cm} (30)

Since some integrations in eq. (28) produce inverse powers of $\bar{\Lambda}$, we need to examine all orders of $m$ in eq. (28). For future reference, introduce the sum

$$H(\omega, n) \equiv \sum_{m=n}^{\infty} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1)\Gamma(1/2)} \omega(m).$$ \hspace{1cm} (31)

The relevant functions $\omega(m)$ and values of $H(\omega, n)$ for our calculation are presented in Table 1, which we have extracted from DKP [3].

| $\omega(m)$ | $n$ | $H$       |
|-------------|-----|----------|
| $1/(2m)$    | 1   | $\log 2$ |
| $1/[2(m-1)]$| 2   | $(1/2)\log 2 + 1/4$ |
| $1/[2m(2m-1)]$ | 1 | $-\log 2 + 1$ |

Table 1
First, consider the \( m = 0 \) contribution to the dual period \((28)\)

\[
2 \int_{x_k^-}^{x_k^+} dx \left( \frac{f'}{f} - \frac{g'}{2g} \right) = 2 \int_{x_k^-}^{x_k^+} dx \left[ \sum_{i=1}^{N} \frac{1}{x - e_i} - \frac{1}{x} - \sum_{i=1}^{N} \frac{1}{2(x + e_i)} \right]
\]

\[
= (N - 2)x_k^- + 2 \sum_{i=1}^{N} e_i \log(x_k^- - e_i) + \sum_{i=1}^{N} e_i \log(x_k^- + e_i).
\]

(32)

By using eq. \((23)\), the integral \((28)\) for all \( m \geq 1 \) can be expressed in a more convenient form. The total derivative can be evaluated by recalling that \( 4g(x_k^-)/f^2(x_k^-) = 1 \) to our order of accuracy, and the prefactors of this term sum to \( H(1/m, 1) = 2 \log 2 \), according to Table 1. Hence, the contribution coming from the \( m \neq 0 \) terms in eq. \((32)\) becomes

\[
2 \sum_{m=1}^{\infty} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1)\Gamma(1/2)} \xi^{2m}(4\bar{\Lambda}^2)^m \int_{x_k^-}^{x_k^+} dx \left( \frac{f'}{f} - \frac{g'}{2g} \right) \frac{g^m}{f^{2m}}
\]

\[
= -2x_k^- \log 2 + 2 \sum_{m=1}^{\infty} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1)\Gamma(1/2)} \xi^{2m}(4\bar{\Lambda}^2)^m \frac{2m}{2m} \int_{x_k^-}^{x_k^+} dx \frac{g^m}{f^{2m}}.
\]

(33)

To evaluate the remaining integral we need to be careful to keep all terms contributing to order \( \bar{\Lambda}^2 \), as the integration may produce inverse powers of \( \bar{\Lambda} \). Because of this, we will have to deal with an infinite power series in \( m \). In order to carry out these integrations, we use partial fractions for the integrand, i.e.

\[
\frac{4^m g^m}{f^{2m}} \equiv \sum_{i=1}^{N} \sum_{p=1}^{2m} Q_{i,p}^{(2m)} (x - e_i)^p
\]

(34)

for some suitable constants \( Q_{i,p}^{(2m)} \). To study the relevant contributions of eq. \((34)\) to \((33)\) we need consider three separate cases. The first possibility appears when \( i = k \). In this case, since \( (x_k^- - e_k) \) is of order \( \bar{\Lambda} \), only \( p = 2m \) and \( p = 2m - 1 \) will contribute to order \( \bar{\Lambda}^2 \). However, all \( m \) will contribute to the sum in \((33)\) for these values of \( p \). For example, for \( p = 2m \), we obtain \( \bar{\Lambda}^{2m}/(x_k^- - e_k)^{2m-1} \) after performing the integration, which provides contributions of order \( \bar{\Lambda} \) for all \( m \geq 1 \).
The second possibility to consider is \( i \neq k \). In this case, the factor \( x_k^- - e_i \) is of zeroth order in \( \Lambda \), so \( m = 1 \) will be the only relevant contribution to order \( \Lambda^2 \) in (33).

Finally the value \( p = 1 \) requires special attention in our discussion, since the contribution (33) in this case will generate logarithmic terms, both for \( i = k \) and for \( i \neq k \).

First, let us turn our attention to \( p = 1 \). Notice from (34) that \( Q^{(2m)}_{i,1} \) is nothing but the residue of the function \((4g)^m / f^{2m}\). Hence

\[
Q^{(2m)}_{i,1} = \frac{2m}{2\pi i} \oint_{A_k} dx \left( \frac{f'}{f} - \frac{g'}{2g} \right) \frac{4^m g^m}{f^{2m}}. \tag{35}\]

Also

\[
\frac{1}{2\pi i} \sum_{m=1}^{\infty} 4^m \Lambda^{2m} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1) \Gamma(1/2)} \oint_{A_k} dx \left( \frac{f'}{f} - \frac{g'}{2g} \right) \frac{g^m}{f^{2m}} \\
= \frac{1}{2\pi i} \oint_{A_k} dx \left( \frac{f'}{f} - \frac{g'}{2g} \right) \left( \frac{1}{\sqrt{1 - 4\Lambda^2 g/f^2}} - 1 \right) \\
= a_k - e_k. \tag{36}\]

Taking into account that the integrand for \( p = 1 \) is \( 1/(x - e_i) \), the contribution of this term to (33) is

\[
2 \sum_{m=1}^{\infty} \sum_{i=1}^{N} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1) \Gamma(1/2)} \xi^{2m} \Lambda^{2m} \int_{x_k^-}^{x_k^+} dx \frac{Q^{(2m)}_{i,1}}{x - e_i} \\
= 2 \sum_{i=1}^{N} (a_i - e_i) \log(x_k^- - e_i). \tag{37}\]

As we have argued, for \( i \neq k \), only \( m = 1 \) will be relevant to order \( \Lambda^2 \). Its contribution to (33) becomes

\[
2 \sum_{i \neq k} \sum_{m=1}^{\infty} \sum_{p=2}^{2m} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1) \Gamma(1/2)} \xi^{2m} \Lambda^{2m} \int_{x_k^-}^{x_k^+} dx \frac{Q^{(2m)}_{i,p}}{(x - e_i)^p} \\
= -\frac{1}{2} \Lambda^2 \sum_{i \neq k} \frac{Q^{(2)}_{i,2}}{x_k^- - e_i} + \mathcal{O}(\Lambda^3). \tag{38}\]
For \( m \neq 0 \), our integral thus becomes

\[
2 \sum_{m=1}^{\infty} \frac{\Gamma(m+1/2)}{\Gamma(m+1)\Gamma(1/2)} \frac{\xi^{2m}}{2m} (4\bar{\Lambda}^2)^m \int_{x^-}^x dx \frac{g^m}{f^{2m}}
\]

(39)

\[
= 2 \sum_{i=1}^{N} (a_i - e_i) \log(x_k^- - e_i) - \frac{1}{2} \bar{\Lambda}^2 \sum_{i \neq k} Q^{(2)}_{i,2} x_k^- - e_i
\]

\[
-2 \sum_{m=1}^{\infty} \frac{\Gamma(m+1/2)}{\Gamma(m+1)\Gamma(1/2)} \frac{(\bar{\Lambda} \xi)^{2m}}{2m} \left[ \frac{1}{(2m-1)} \frac{Q^{(2m)}_{k,2m}}{(x_k^- - e_k)^{2m-1}} + \frac{\theta_{m-2}}{(2m-2)} \frac{Q^{(2m)}_{k,2m-1}}{(x_k^- - e_k)^{2m-2}} \right]
\]

to one instanton order, and where \( \theta_s = 1 \) for \( s \geq 0 \), and \( \theta_s = 0 \) for \( s < 0 \).

In order to obtain the final expression for the integral, we must know the partial fraction coefficients \( Q^{(2m)}_{i,p} \). These terms can be obtained from the evaluation of \( (4g)^m / f^{2m} \) near \( x = e_i \). We find

\[
\frac{4^m g^m}{f^{2m}} = \left( \frac{S_k(x)}{(x - e_k)^2} \right)^m \xrightarrow{x \to e_k} \frac{S_k^m(e_k)}{(x - e_k)^2} + \frac{m S_k^{m-1}(e_k) \partial S_k(x)}{(x - e_k)^{2m-1}} + \ldots
\]

(40)

which, by identification, gives \( Q^{(2m)}_{k,2m} = S_k^m \) and \( Q^{(2m)}_{k,2m-1} = m S_k^{m-1} \partial S_k(x) \). Using this result, Table 1 for the \( m \) summation, and a Taylor expansion of \( 1/(x_k^- - e_k) \) with the explicit form of \( x_k^- \) c.f. eq. (30), the total contribution for \( p = 2m \) and \( p = 2m+1 \) from eq. (30) when \( i = k \) becomes

\[
-2 \sum_{m=1}^{\infty} \frac{\Gamma(m+1/2)}{\Gamma(m+1)\Gamma(1/2)} \frac{(\bar{\Lambda} \xi)^{2m}}{2m} \left[ \frac{1}{(2m-1)} \frac{Q^{(2m)}_{k,2m}}{(x_k^- - e_k)^{2m-1}} + \frac{\theta_{m-2}}{(2m-2)} \frac{Q^{(2m)}_{k,2m-1}}{(x_k^- - e_k)^{2m-2}} \right]
\]

\[
= 2\bar{\Lambda}(1 - \log 2) S_k^{1/2}(e_k) + \bar{\Lambda}^2 \left( \frac{1}{2} \log 2 - \frac{1}{4} \right) \partial S_k(x) + O(\bar{\Lambda}^3).
\]

(41)

Equation (32) and the relevant parts of (33), (37), (38), and (31) add to

\[
2\pi i (a_{D,k})_l = (N - 2 - 2\log 2) x_k^- + 2 \sum_{i=1}^{N} a_i \log(x_k^- - e_i) + \sum_{i=1}^{N} e_i \log(x_k^- + e_i)
\]

\[
+2\bar{\Lambda}(1 - \log 2) S_k^{1/2}(e_k) + \bar{\Lambda}^2 \left[ \frac{1}{2} \sum_{i \neq k} S_i(e_i) x_k^- - e_i + \left( \frac{1}{4} \log 2 - \frac{1}{4} \right) \partial S_k(x) \right] + O(\bar{\Lambda}^3).
\]

(42)
This is not yet the expression for the dual periods in the hyperelliptic approximation. We must express $x_k^-$ and $e_i$ in terms of $a_k$ and $a_i$ respectively. Also notice that we have contributions proportional to $\Lambda$ in (42). This is, of course, unphysical since the one instanton contribution corresponds to terms of order $\bar{\Lambda}^2$. We have verified that the next term in perturbation theory in $\epsilon(x)$ will contribute only to one instanton order and higher, hence the terms of order $\bar{\Lambda}$ must cancel in the hyperelliptic approximation.

The calculation, being tedious and relatively straightforward, can, however, be facilitated by a few useful rearrangements, which we quote in what follows. We then state the result for the dual periods obtained from the zeroth order approximation in $\epsilon$ in the perturbation theory. Additional intermediate results are given in Appendix A.

First of all, notice that from the expression of the prepotential, we expect the dual periods at order $(\bar{\Lambda})^0$ to contain terms of the type $(a_k - a_i) \log(a_k - a_i)$ as well as $(a_k + a_i) \log(a_k + a_i)$. With this in mind, we use the following identity that is valid to order $\bar{\Lambda}^2$

$$f(x_k^-) = -2\bar{\Lambda}g^{1/2}(x_k^-) \iff \prod_{i=1}^{N}(x_k^- - e_i) = -2\bar{\Lambda}x_k^- (-1)^{\frac{N}{2}} \prod_{i=1}^{N}(x_k^- + e_i)^{1/2}. \quad (43)$$

Taking the logarithm yields

$$0 = \log(-\bar{\Lambda}) + \log 2 + \frac{1}{2} \log(-1)^N - \sum_{i=1}^{n} \log(x_k^- - e_i) + \frac{1}{2} \sum_{i=1}^{n} \log(x_k^- + e_i) + \log x_k^- \cdot (44)$$

Multiplying (44) by $2x_k^-$ and adding it to $2\pi i (a_{D,k})_I$, we obtain
\[2\pi i(a_{D,k})_I\]

\[= [N - 2 + 2 \log(-\bar{\Lambda}) + \log(-1)^N] x_k^- - 2 \sum_{i=1}^N (x_k^- - a_i) \log(x_k^- - e_i)\]

\[+ \sum_{i=1}^N (x_k^- + e_i) \log(x_k^- + e_i) + 2x_k^- \log x_k^- + 2\bar{\Lambda}(1 - \log 2) S_k^{1/2}(e_k)\]

\[+ \bar{\Lambda}^2 \left[-\frac{1}{2} \sum_{i \neq k} \frac{S_i(e_i)}{x_k^- - e_i} + \left(\frac{1}{2} \log 2 - \frac{1}{4} \right) \frac{\partial S_k}{\partial x}(e_k)\right].\]  

(45)

To evaluate the terms of the type \((x_k^- - a_i) \log(x_k^- - e_i)\) for \(i \neq k\), we use the following expansion

\[\log(x_k^- - e_i) \approx \log(e_k - e_i) + \frac{x_k^- - e_i}{e_k - e_i} - \frac{1}{2} \frac{(x_k^- - e_i)^2}{(e_k - e_i)^2}.\]  

(46)

Taking into account eqs. (44), (45), and the results in Appendix A, we obtain the classical and one-loop contribution to the dual periods in the form

\[2\pi i [(a_{k,D})_{cl} + (a_{k,D})_{1-loop}]_I = [N - 2 + 2 \log(-\bar{\Lambda}) + \log(-1)^N] a_k\]

\[-2 \sum_{i \neq k} (a_k - a_i) \log(a_k - a_i) + \sum_{i=1}^N (a_k + a_i) \log(a_k + a_i) + 2a_k \log a_k.\]  

(47)

To evaluate the one instanton terms, we must use

\[\frac{\partial S_k}{\partial x}(e_k) = S_k(a_k) \left(\sum_{i=1}^N \frac{1}{a_k + a_i} - 2 \sum_{i \neq k} \frac{1}{a_k - a_i} + \frac{2}{a_k}\right),\]  

(48)

and eqs. (8) and (26), which imply

\[\sum_{i=1}^N a_i + \mathcal{O}(\bar{\Lambda}^4) = \frac{\bar{\Lambda}^2}{4} \sum_{i=1}^N \frac{\partial S_i}{\partial x}(a_i),\]  

(49)

for a massless hypermultiplet. Eq. (49) vanishes as we will verify. To see this, introduce the function

\[K(x) = \frac{4(-1)^N x^2 \prod_{i=1}^N (x + e_i)}{\prod_{i=1}^N (x - e_i)^2}.\]  

(50)
Notice that the residues of $K(x)$ are given by
\[
\text{Res } K(x) \big|_{x=e_k} = \frac{\partial S_k}{\partial x}(e_k),
\]
and since $K(x)$ does not have any poles at infinity its residues must sum to zero. This gives the result
\[
\sum_{i=1}^{N} a_i = 0,
\]
to this order.

Making use of these identities, it can be checked that the $\bar{\Lambda}$ contributions cancel as promised, and the contribution from the hyperelliptic approximation to the one instanton term takes the form
\[
2\pi i \left[ (a_{D,k})_{1\text{-inst}} \right]_I = \bar{\Lambda}^2 \left( -\frac{1}{2} \sum_{i \neq k} \frac{S_i(a_i)}{a_k - a_i} - \frac{1}{4} \sum_{i=1}^{N} \frac{\partial S_i}{\partial x}(a_i) \log(a_k + a_i) + \frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) \right). \quad (53)
\]

From the SW ansatz we know that this result, if complete as it stands, must be integrable order by order in $\bar{\Lambda}^2$. The zero instanton result is exactly what one expects from the prepotential. However, the logarithmic term at the one-instanton level is spurious, and indicates that we need additional terms in the expansion beyond the hyperelliptic approximation. The corrections that we need are given by the second term in eq. (17).

By arguments similar to the ones used in the evaluation of the dual period contribution from the hyperelliptic approximation, one can convince oneself that the $\epsilon/(2g^2\bar{\Lambda}^2)$ term in (17) will not contribute to one instanton effects. What remains is $xd(\epsilon r/(g^2\bar{\Lambda}^4))$ since only odd terms in $r$ will survive. Hence we wish to evaluate
\[
2\pi i \left( a_{D,k} \right)_{II} = 2\bar{\Lambda}^2 \int^{x_{\bar{\Lambda}}} x d \left( \frac{x^6 r}{g^2} \right) = 2\bar{\Lambda}^2 \int^{x_{\bar{\Lambda}}} dx \left[ \frac{\partial}{\partial x} \left( \frac{x^7 r}{g^2} \right) - \frac{x^6 r}{g^2} \right]. \quad (54)
\]
To this order we may take \( r = f/2 \). It is also easy to convince oneself that the total derivative terms are of order \( \bar{\Lambda}^3 \). This leaves us with

\[
-\bar{\Lambda}^2 \int_{x_k}^{x_k} dx \frac{x^2 \prod_{i=1}^{N} (x - e_i)}{\prod_{i=1}^{N} (x + e_i)^2} = -\bar{\Lambda}^2 \sum_{i=1}^{N} \left( -\frac{\bar{Q}_{i,2}}{x_k + e_i} + \bar{Q}_{i,1} \log(x_k + e_i) \right),
\]

(55)

where we have introduced the partial fraction coefficients \( \bar{Q}_{i,p} \) for \( p = 1, 2 \), which are defined by the relation

\[
\frac{x^2 \prod_{i=1}^{N} (x - e_i)}{\prod_{i=1}^{N} (x + e_i)^2} \equiv \sum_{i=1}^{N} \sum_{p=1}^{2} \frac{\bar{Q}_{i,p}}{(x + e_i)^p}.
\]

(56)

Evaluating the right hand side near \( x = -e_i \), one finds that

\[
\bar{Q}_{i,1} = -\frac{1}{4} \frac{\partial S_i}{\partial x}(e_i) \quad \text{and} \quad \bar{Q}_{i,2} = \frac{1}{4} S_i(e_i).
\]

(57)

The total correction term from eq. (55) then is

\[
2\pi i (a_{D,k})_{II} = \bar{\Lambda}^2 \sum_{i=1}^{N} \left( \frac{S_i(a_i)}{4(a_k + a_i)} + \frac{1}{4} \frac{\partial S_i}{\partial x}(a_i) \log(a_k + a_i) \right).
\]

(58)

Adding this to the one instanton contribution obtained in the hyperelliptic approximation, we at last get the complete one-instanton contribution to the dual period

\[
2\pi i (a_{D,k})_{1\text{-inst}} = \bar{\Lambda}^2 \left( -\frac{1}{2} \sum_{i \neq k} \frac{S_i(a_i)}{a_k - a_i} + \frac{1}{4} \sum_{i=1}^{N} \frac{S_i(a_i)}{a_k + a_i} + \frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) \right) - (k \to 1),
\]

(59)

where we have restored the dependence on the lower integration limit which was suppressed throughout the calculation.

The final result for the dual period derived from the curve (3) up to one instanton order takes the form
2\pi i a_{D,k} = 2\pi i [(a_{D,k})_{cl} + (a_{D,k})_{1-loop} + (a_{D,k})_{1-inst}]

= [N - 2 + 2\log(-\Lambda) + \log(-1)^N] a_k

-2 \sum_{i \neq k} (a_k - a_i) \log(a_k - a_i) + \sum_{i=1}^N (a_k + a_i) \log(a_k + a_i) + 2a_k \log a_k

+ \tilde{\Lambda}^2 \left( -\frac{1}{2} \sum_{i \neq k} \frac{S_i(a_i)}{a_k - a_i} + \frac{1}{4} \sum_{i=1}^N \frac{S_i(a_i)}{a_k + a_i} + \frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) \right) - (k \to 1). \hspace{1cm} (60)

6 The Prepotential

Finally we must use the dual periods to find the prepotential, i.e. to find the holomorphic function whose derivative is $a_{D,k}$. This will provide us with the one instanton correction to the prepotential for SU($N$) with matter transforming in the symmetric representation. The validity of this result can be tested by performing microscopic instanton calculations along the lines of previous calculations testing similar results for matter in the defining representation.

We must show that the dual periods $a_{D,k}$ can be expressed as a derivative of some function with respect to $a_k$, as in eq. (5). We can express the prepotential as:

$$F = F_{cl} + F_{1-loop} + \sum_{d=1}^\infty \tilde{\Lambda}^{2d} F_d.$$ \hspace{1cm} (61)

The one-loop part of the prepotential takes the explicit form

$$F_{1-loop} = \frac{i}{8\pi} \left[ \sum_{i,j=1}^N (a_i - a_j)^2 \log \left( \frac{a_i - a_j}{\Lambda} \right)^2 - \sum_{i \leq j} (a_i + a_j)^2 \log \left( \frac{a_i + a_j}{\Lambda} \right)^2 \right], \hspace{1cm} (62)$$

for the symmetric representation. The derivative of this expression with respect to $a_k$ reproduces the functional behaviour of the zero instanton term of the dual periods $a_{D,k}$ c.f. eq. (51).
To check the one instanton term, first notice that we are interested in derivatives with respect to $a_k$, but in (60) all the derivatives are considered with respect to $x$. However, we have the identity

$$\frac{\partial S_k(a_k)}{\partial a_k} = \frac{\partial S_k}{\partial x}(a_k) + \frac{S_k(a_k)}{2a_k}. \quad (63)$$

Furthermore

$$\sum_{i \neq k} S_i(a_i) - 2 \sum_{i \neq k} S_i(a_i) = \frac{\partial}{\partial a_k} \sum_{i \neq k} S_i(a_i). \quad (64)$$

Thus, the one instanton terms of the dual periods $a_{D,k}$ can be then expressed as

$$2\pi i (a_{D,k})_{1-inst} = \bar{\Lambda}^2 \left[ -\frac{1}{2} \sum_{i \neq k} S_i(a_i) + \frac{1}{4} \sum_{i=1}^{N} S_i(a_i) + \frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) \right] = \bar{\Lambda}^2 \frac{\partial}{\partial a_k} \sum_{i=1}^{N} S_i(a_i). \quad (65)$$

This demonstrates the integrability, and gives the one instanton correction to the prepotential, which is

$$\mathcal{F}_1 = \frac{1}{8\pi i} \sum_{i=1}^{N} S_i(a_i). \quad (66)$$

It is straightforward to see that we can continue our calculation along the same lines to arbitrary instanton number. However, the number of terms will increase quickly and the calculations will become increasingly more difficult.

Our result for the one instanton correction $\mathcal{F}_1$ must reproduce the result for pure Yang-Mills as we take the mass of the hypermultiplet to infinity, because in this limit we “integrate out” the massive degrees of freedom and we are left with the pure gauge case only. In order to restore the mass dependence, we shift

$$a_i \rightarrow a_i + \frac{m}{2}. \quad (67)$$
which is consistent with the mass dependence in Landsteiner and Lopez [10]. In the limit $m \to \infty$, the one instanton term takes the form (by using the explicit form for $S_i(a_i)$, c.f. (25)),

$$
\sum_{i=1}^{N} \frac{4(a_i + m/2)^2 \prod_{j=1}^{N} (a_i + a_j + m)}{\prod_{j \neq i} (a_i - a_j)^2} \to \sum_{i=1}^{N} \frac{m^{N+2}}{\prod_{j \neq i} (a_i - a_j)^2}.
$$

(68)

This coincides with the result for pure Yang-Mills theory [6], with appropriate rescalings of the respective scales $\Lambda$.

Notice that our expression for the one instanton correction $\mathcal{F}_1$ to the prepotential is formally identical to the one derived for matter transforming in the defining representation [4], with the difference in the theories just expressed by the different definition of the residue functions $S_k$. This is also true for the antisymmetric representation with the minor addition of a similar term (a residue at $x = 0$) [13].

7 Conclusions

In this paper, we have calculated the prepotential up to one instanton order for the N=2 supersymmetric Yang-Mills theories in four dimensions with gauge group SU($N$) and matter transforming in the symmetric representation. The calculation has been performed with the SW ansatz as the starting point, together with a curve derived from M-theory. The curve, being cubic, is not as straightforward to deal with as the quadratic curves found for matter in the defining representation.

It was shown that the curve reproduces the correct result for the one-loop part of the prepotential, which although a non-trivial test of the curve, is not a conclusive one. First of all, the curve has already been tested for self consistency when derived from M-theory.
These tests might already be accurate enough to guarantee the one-loop result. Furthermore, as we have seen in this paper, the hyperelliptic approximation is sufficient to get the correct one-loop result, which tells us that there is more than one curve with the same one-loop prepotential. We also have unpublished results in which we find entirely different curves for SU(3) and SU(4) which also reproduce the correct one-loop prepotential, but fail to be integrable to one-instanton order. (This approach might be extended to SU(N) although we have not done so.) One thus must go beyond one-loop to obtain non-trivial tests of any proposed curve. In fact, the one instanton term will distinguish between various approaches that reproduce the one-loop part of the prepotential. Therefore a convincing test of the M-theory curve requires at least the one-instanton term.

We also gave the one instanton correction to the prepotential. It was found to be of the same general form as the one obtained from matter transforming in the defining representation. This is also the case for the antisymmetric representation, which is discussed in a separate publication [13].

The one instanton term is also important from the physical point of view. No matter how powerful the SW approach is, and no matter how successful it has proven in previous examples, it is still a conjecture whose results must be tested in detailed field theoretic calculations. This is something which can be done in microscopic instanton calculations, and here we provide a target for those calculations.

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Appendix A

Some partial results are listed below which were used to obtain the dual periods for the hyperelliptic approximation. The left-hand sides are taken from eq. (45), and we have made extensive use of eq. (46) and analogous expressions.

\[-2(x_k^- - a_k)\log(x_k^- - e_k)\]  \hspace{1cm} (A.1)

\[= 2\bar{\Lambda} \left[ \log(-\bar{\Lambda}) + \log S_k^{1/2}(a_k) \right] S_k^{1/2}(a_k) - \bar{\Lambda}^2 \left[ 1 + \frac{1}{2} \log(-\bar{\Lambda}) + \frac{1}{2} \log S_k^{1/2}(a_k) \right] \frac{\partial S_k}{\partial x}(a_k).\]

\[-2 \sum_{i \neq k} (x_k^- - a_i)\log(x_k^- - e_i)\]  \hspace{1cm} (A.2)

\[= -2 \sum_{i \neq k} (a_k - a_i)\log(a_k - a_i) + 2\bar{\Lambda} \left[ N - 1 + \sum_{i \neq k} \log(a_k - a_i) \right] S_k^{1/2}(a_k)
\]

\[-\bar{\Lambda}^2 \left[ \frac{N - 1}{2} \frac{\partial S_k}{\partial x}(a_k) + \frac{1}{2} \frac{\partial S_k}{\partial x}(a_k) \sum_{i \neq k} \log(a_k - a_i) + \frac{1}{2} \sum_{i \neq k} \frac{\partial S_i}{\partial x}(a_i) + S_k(a_k) \sum_{i \neq k} \frac{1}{a_k - a_i} \right].\]

\[\sum_{i=1}^{N} (x_k^- + e_i)\log(x_k^- + e_i)\]  \hspace{1cm} (A.3)

\[= \sum_{i=1}^{N} (a_k + a_i)\log(a_k + a_i) - \bar{\Lambda} \left[ N + \sum_{i=1}^{N} \log(a_k + a_i) \right] S_k^{1/2}(a_k)
\]

\[+\bar{\Lambda}^2 \left[ \frac{N}{4} \frac{\partial S_k}{\partial x}(a_k) + \frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) \sum_{i=1}^{N} \log(a_k + a_i) - \frac{1}{4} \sum_{i=1}^{N} \frac{\partial S_i}{\partial x}(a_i) \log(a_k + a_i)
\]

\[-\frac{1}{4} \sum_{i=1}^{N} \frac{\partial S_i}{\partial x}(a_i) + \frac{1}{2} S_k(a_k) \sum_{i=1}^{N} \frac{1}{a_k + a_i} \right].\]

\[2x_k^-\log x_k^-\]  \hspace{1cm} (A.4)

\[= 2a_k\log a_k - 2\bar{\Lambda} S_k^{1/2}(a_k)(1 + \log a_k) + \bar{\Lambda}^2 \left[ \frac{1}{2} \frac{\partial S_k}{\partial x}(a_k)(1 + \log a_k) + \frac{S_k(a_k)}{a_k} \right].\]
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