BIHOLOMORPHISMS BETWEEN HARTOGS DOMAINS OVER HOMOGENEOUS SIEGEL DOMAINS

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Abstract. In this paper, we characterize the Hartogs domains over homogeneous Siegel domains of type II and explicitly describe their automorphism groups. Moreover we prove that any proper holomorphic map between Hartogs domains over homogeneous Siegel domains over type II is a biholomorphism.

1. Introduction

Let $D$ be a domain in the complex Euclidean space $\mathbb{C}^r$. Let $K_D: D \times \overline{D} \to \mathbb{C}$ be the Bergman kernel of $D$, i.e., $K_D$ is the reproducing kernel of the Hilbert space of holomorphic $L^2$ functions on $D$. Suppose that $K_D(z, z)$ is positive for any $z \in D$. For a positive real number $s \geq 1$ and a positive integer $N$, let $D_{N,s}$ be the Hartogs domain over $D$ defined by

$$\{(z, \zeta) \in D \times \mathbb{C}^N : ||\zeta||^2 < K_D(z, z)^{-s}\},$$

(1.1)

where $|| \cdot ||$ denotes the standard Hermitian norm on $\mathbb{C}^N$.

The purpose of this paper is to prove the following two theorems:

Theorem 1.1. Let $D$ and $D'$ be homogeneous Siegel domains of type II. Suppose that there exists a biholomorphism $f: D_{N,s} \to D'_{N',s'}$. Then $D$ and $D'$ are biholomorphic, $N = N'$, $s = s'$ and $f$ is of the form (1.2) with

$$f_1(z, \zeta) = \phi(z), \quad f_2(z, \zeta) = U(\zeta) \left(J\phi(z)\right)^s$$

for some biholomorphism $\phi$ from $D$ to $D'$ and some unitary transformation $U$.

Corollary 1.2. Let $D$ be a homogeneous Siegel domain of type II. Then the set of all biholomorphic self-maps $\text{Aut}(D_{N,s})$ consists of the maps of the form (1.2) with any $\phi \in \text{Aut}(D)$ and any unitary transformation $U$ in $\mathbb{C}^n$.

Theorem 1.3. Let $f: D_{N,s} \to D'_{N',s'}$ be a proper holomorphic map between equidimensional Hartogs domains over homogeneous Siegel domains of type II which are not biholomorphic to the unit ball. Then $f$ is a biholomorphism.

The motivation of this paper is to generalize the work of Ahn-Byun-Park in [1] and Tu-Wong in [13].

Theorem 1.4 (Ahn, Byun and Park [1]). Let $D$ be a bounded symmetric domain of classical type which is not biholomorphic to the unit ball. Then the set of all biholomorphic self-maps $\text{Aut}(D_{N,s})$ consists of biholomorphic automorphisms of the form $\Phi = (\Phi_1, \Phi_2): D_{N,s} \to D_{N,s}$ with

$$\Phi_1(z, \zeta) = \phi(z), \quad \Phi_2(z, \zeta) = U(\zeta) \left(J\phi(z)\right)^s$$

(1.3)

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for any $\phi \in Aut(D)$ and unitary transformation $U$.

In their original work, Tu and Wang studied Hua domains over bounded symmetric domains of classical type and hence their result is more general. However for simplicity of exposition, we only state their theorems in case of Hartogs domains.

**Theorem 1.5** (Tu and Wang [13]).

(1) Let $D$ and $D'$ be irreducible bounded symmetric domains of classical type. Let $f: D_{N,s} \rightarrow D'_{N',s'}$ be a biholomorphism. Then $N = N'$, $s = s'$ and there exist an automorphism $\Phi$ of $D'_{N',s'}$, a complex linear isomorphism $A$ of $\mathbb{C}^{\dim D}$ and a unitary transformation $U$ of $\mathbb{C}^N$ such that

\[
\Phi \circ f(z, \zeta) = (Az, U\zeta).
\]

In particular any automorphism of $D_{N,s}$ is of the form (1.2).

(2) Let $f: D_{N,s} \rightarrow D'_{N's'}$ be a proper holomorphic map between equidimensional Hartogs domains over bounded symmetric domains of classical type. Then $f$ is a biholomorphism.

When $D$ is the unit ball, $D_{N,s}$ is biholomorphic to a generalized ellipsoid. In this case Naruki [10] showed that every biholomorphism has the form (1.4) and Kodama [8] proved that every automorphism has the form (1.2).

The automorphism group of Hartogs domains over minimal homogeneous domain in $\mathbb{C}^n$ with center 0 was described in [15]. Note that every bounded homogeneous domain can be realized as a homogeneous Siegel domain of type II. One of the difficulties in proving Corollary 1.2 for general bounded homogeneous domains is linked with fact that $D_{N,s}$ might not be circular if $D$ is not a bounded symmetric domain. Here one says that a domain is circular if $(z_1, z_2, \ldots, z_n) \in D$ implies that also $(e^{i\theta}z_1, e^{i\theta}z_2, \ldots, e^{i\theta}z_n)$ belongs to the domain for every $\theta \in \mathbb{R}$. By a work of Bell [3], if the bounded domain is circular, every proper holomorphic map between equidimensional domains can be extended holomorphically over the boundary. Hence we can use the boundary structure of the domains. Moreover, if $D$ is a circular domain containing 0, then the automorphism of $D$ fixing 0 is a linear map (cf. [5] Corollary 1.3.2.).

Recently, in [6], the Bergman kernel of the domain (1.1) over the bounded homogeneous domain was explicitly obtained by Ishi-Park-Yamamori. With this explicitly expressed Bergman kernel, we can exploit the Tumanov’s method in [14]. In this paper he proved the following:

**Theorem 1.6** (Tumanov [14]). Let $D$ and $D'$ be equidimensional Siegel domains of type II. Then any proper holomorphic map $f: D \rightarrow D'$ is biholomorphic and rational.
2. PRELIMINARIES

2.1. Homogeneous Siegel domains of type II and their properties. In this section, we recall the definition and properties of homogeneous Siegel domains of type II that we need later. For more detail, see [11].

A domain \( V \) in \( \mathbb{R}^n \) is called a convex cone if the following three conditions are satisfied:

1. For any \( x \in V \) and for any \( \lambda > 0 \), \( \lambda x \in V \);
2. If \( x, y \in V \), then \( x + y \in V \);
3. \( V \) contains no entire straight lines.

A convex cone \( V \) is called homogeneous if the linear automorphism group, \( G(V) = \{ A \in GL(\mathbb{R}^n) : AV = V \} \), acts transitively on \( V \). A map \( F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{R}^n \) is called a \( V \)-Hermitian form if the following conditions are satisfied:

1. \( F(u, v) \) is \( \mathbb{C} \)-linear in \( u \),
2. \( F(u, v) = F(v, u) \),
3. \( F(u, u) \in V \) where \( V \) is the closure of \( V \) in \( \mathbb{R}^n \),
4. \( F(u, u) = 0 \) implies \( u = 0 \).

The Siegel domain of type II is defined by

\[ D(V, F) = \{ (w, u) \in \mathbb{C}^n \times \mathbb{C}^m : \text{Im} w - F(u, u) \in V \} \]

When \( \text{Aut}(D(V, F)) \) acts transitively on \( D(V, F) \), it is called homogeneous and when the affine linear group of \( \mathbb{C}^n \times \mathbb{C}^m \) which preserves \( D(V, F) \) acts transitively on \( D(V, F) \), it is called affine homogeneous.

**Theorem 2.1** (Pyateskii-Shapiro [11]).

1. Every homogeneous bounded domain in the complex Euclidean space is holomorphically equivalent to a homogeneous Siegel domain of type II.
2. The followings are equivalent: (i) \( D(V, F) \) is homogeneous, (ii) \( D(V, F) \) is affine homogeneous, (iii) \( V \) is homogeneous.
3. Let \( G_a \) be the affine automorphism groups of a homogeneous Siegel domain of type II \( D(V, F) \). Let \( H \) be the isotropy subgroup of \( G_a \) at \( (0, 0) \in \mathbb{C}^n \times \mathbb{C}^m \) and \( K_a \) the isotropy subgroup of \( G_a \) for some fixed \( (ir, 0) \in \mathbb{C}^n \times \mathbb{C}^m \) with \( r \in \mathbb{R}^n \). Let \( \mathfrak{g}_a, \mathfrak{t}_a \) be the Lie algebras of \( G_a, K_a \) respectively. Then one has \( G_a = H \cdot \mathbb{R}^n \mathbb{C}^m, K_a \subset H \) and

\[ \mathfrak{g}_a = \mathfrak{t}_a + j \mathbb{R}^n + \mathbb{R}^n + \mathbb{C}^n \]

where \( j \) is a linear endomorphism of \( \mathfrak{g}_a \) induced from the complex structure of \( D(V, F) \).
4. Define \( \mathfrak{g} = j \mathbb{R}^n + \mathbb{R}^n + \mathbb{C}^n \) which is a normal \( j \)-algebra corresponding to \( D(V, F) \) (for more detail, see [11, 17]). Then there exist an 1-dimensional ideal \( \mathfrak{t} \), a subalgebra \( \mathfrak{z} \) and a normal \( j \)-subalgebra \( \mathfrak{g}_1 \subset \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{t} + j \mathfrak{t} + \mathfrak{z} + \mathfrak{g}_1 \) satisfying

\[ [\mathfrak{t} + j \mathfrak{t}, \mathfrak{g}_1] = 0, \quad [\mathfrak{z}, \mathfrak{g}_1] \subset \mathfrak{z} \]

**Remark** that the normal \( j \)-subalgebra \( \mathfrak{t} + j \mathfrak{t} + \mathfrak{z} \) corresponds to the Siegel domain of type II which is biholomorphic to the unit ball. Besides \( \mathfrak{t} + j \mathfrak{t} \) corresponds to the upper half plane in \( \mathbb{C} \).

**Lemma 2.2.** Let \( D \) be a homogeneous Siegel domain of type II which is not biholomorphic to the unit ball. Then there is a complex manifold \( M \) such that \( \Delta \times M \) is holomorphically and properly embedded in \( D \).

**Proof.** Let \( \mathfrak{g} \) be a normal \( j \)-algebra corresponding to \( D \) and \( G \) its Lie group. Then there exist a \( j \)-subalgebra \( \mathfrak{t} + j \mathfrak{t} + \mathfrak{z} \) and a normal \( j \)-subalgebra \( \mathfrak{g}_1 \) such that \( \mathfrak{g} = \mathfrak{t} + j \mathfrak{t} + \mathfrak{z} + \mathfrak{g}_1 \) and
\(|r + j\varpi, g_1| = 0\). Let \(R\) and \(G_1\) be corresponding connected Lie subgroups of \(\varpi\) and \(g_1\) respectively in \(G\). Then the \(R\)-orbit of the base point of \(D\) is biholomorphic to the unit disc and \(G_1\)-orbit is a complex submanifold \(M\) such that the product of the \(R\)-orbit and \(M\) can be properly and holomorphically embedded in \(D\).

For a Siegel domain of type II, \(D = \{(w, u) \in \mathbb{C}^n \times \mathbb{C}^m : \text{Im}(w) = F(u, u) \in V\}\), the Shilov boundary of \(D\) is given by

\[
S = \{(w, u) \in \mathbb{C}^n \times \mathbb{C}^m : \text{Im}(w) = F(u, u)\}.
\]

Let \(D' = \{(w, u) \in \mathbb{C}^n \times \mathbb{C}^m : \text{Im}(w) = F'(u, u) \in V'\}\) be another Siegel domains of type II and \(S'\) its Shilov boundary.

**Lemma 2.3.** Let \(f : D \to D'\) be a biholomorphism. Then \(f\) maps the Shilov boundary of \(D\) into that of \(D'\).

**Proof.** Suppose that \(f(S) \not\subset S'\). Then there is \(p \in S\) and \(q \in \partial D' \setminus S'\) such that \(f(p) = q\). Let \(\mu\) be a holomorphic function on some open neighborhood of \(D\) which attains \(\sup_{\partial D} |\mu|\) only at \(p\). Then \(G \circ f^{-1}\) attains the supremum at \(q\). This contradicts to the definition of the Shilov boundary. \(\square\)

For a given homogeneous Siegel domain \(D(V, F)\) of type II, the Bergman kernel has the form

\[
K_{D(V, F)}((w, u), (w', u')) = c_{V, F} \left( \frac{u - w'}{2\sqrt{-1}} - F(w, w') \right)^{2d-q}
\]

\[
= b \left( \frac{u - w'}{2\sqrt{-1}} - F(w, w') \right)
\]

where \(c_{V, F}\) is a constant independent of \((z, \zeta)\) and \((z', \zeta')\) and multi-indices \(d = (d_1, \ldots, d_R)\), \(q = (q_1, \ldots, q_R)\) are given by the Lie algebra structure of affine automorphism group of \(D(V, F)\).

In the second line, \(b\) is a holomorphic function on the tube domain \(\{u \in \mathbb{C}^m : \text{Im} u \in V\}\) which is a product of rational functions. Note that every \(d_j\) is semi-negative integer and every \(q_k\) is semi-positive integer. Moreover \(2(d_1 + \cdots + d_R) - (q_1 + \cdots + q_R) < -2\). For the detail, see [4].

### 2.2. The Bergman kernel of \(D_{N,s}\)

For a homogeneous Siegel domain \(D\), the Bergman kernel \(K_{N,s} \) of \(D_{N,s}\) is given by the following [6 Theorem 4.3]:

\[
K_{N,s}((z, \zeta), (z', \zeta')) = \frac{N!}{\pi^N} K_{D(V, F)}(z, z')^{N+1} b \left( \frac{t d}{dt} + N \right) \left[ (1 - t)^{-m-1} \right]_{t = K_{D(V, F)}(z, z')^{-1}(\zeta, \zeta')}
\]

where \(b\) is a polynomial. For more detail, see [6 Section 4].

For a bounded domain \(D\) in the Euclidean space, we say that the Bergman kernel \(K_D\) of \(D\) extends over the boundary if for every compact subset \(E\) of \(D\), there is an open neighborhood \(U\) of \(D\) such that \(\overline{E} \subset U\) and \(K(\cdot, p)\) is holomorphic on \(U\) for every \(p \in E\). By the equations (2.5) and (2.6), we can obtain the following:

**Lemma 2.4.** Let \(D\) be a Siegel domain of type II. Then the Bergman kernels of \(D\) and \(D_{N,s}\) extend over the boundary.
3. Hartogs domains over general domains

3.1. Automorphisms of \( D_{N,s} \). Let \( D \) be a domain in \( \mathbb{C}^n \) such that \( K_D(z,z) \neq 0 \) for any \( z \in D \). For an automorphism \( \phi \in \text{Aut}(D) \), let \( \Phi \) be a holomorphic mapping defined by \( \Phi = (\Phi_1, \Phi_2): D_{N,s} \to D_{N,s} \) with
\[
\Phi_1(z,\zeta) = \phi(z), \quad \Phi_2(z,\zeta) = U(\zeta) \left( J\phi(z) \right)^s.
\]
Here \( J\phi \) denotes the Jacobian determinant of \( \phi \) and \( U \) is a unitary transformation. Since
\[
\|\Phi_2(z,\zeta)\|^2 = \|\zeta\|^2 |J\phi(z)|^{2s}
\]
and
\[
K_D(\Phi_1(z,\zeta),\Phi_1(z,\zeta))^{-s} = \left( K_D(z,z) |J\phi(z)|^{-2} \right)^{-s},
\]
it follows that \( \Phi \) is well-defined and the holomorphic map \( \Phi \) is an automorphism of \( D_{N,s} \).

**Remark 3.1.** Let \( \phi: D \to D' \) be a biholomorphism between domains \( D \) and \( D' \). Then \( D_{N,s} \) and \( D_{N,s}' \) are biholomorphic by the map
\[
(z,\zeta) \mapsto (\phi(z), J\phi(z)^s \zeta)
\]
where \( J\phi \) denotes the Jacobian determinant of \( \phi \). By the transformation formula of the Bergman kernel by biholomorphisms, it is easy to show that \( \Phi \) is a biholomorphism.

Let \( G \) be the set of automorphisms induced by all \( \phi \in \text{Aut}(D) \).

**Lemma 3.2.** \( G \) is a subgroup of \( \text{Aut}(D_{N,s}) \).

**Proof.** Let \( \phi, \psi \in \text{Aut}(D) \) and \( \Phi, \Psi \in \text{Aut}(D_{N,s}) \) induced by \( \phi, \psi \) respectively. Then
\[
\Psi \circ \Phi(z,\zeta) = \Psi(\Phi_1(z,\zeta), \Phi_2(z,\zeta)) = \Psi(\phi(z), U(\zeta)(J\phi(z))^s)
\]
\[
= (\psi \circ \phi(z), V(U(\zeta)(J\phi(z))^s)(J\psi)^s \circ (\phi(z)))
\]
\[
= (\psi \circ \phi(z), VU(\zeta)(J\psi \circ \phi(z))^2)
\]
and hence the lemma is proved. \( \square \)

3.2. Strongly pseudoconvex boundary points. Decompose the boundary of \( D_{N,s} \) by
\[
\partial D_{N,s} = \partial_0 D_{N,s} \cup (\partial D \times \{0\})
\]
with
\[
\partial_0 D_{N,s} := \{(z,\zeta) \in D \times \mathbb{C}^N : ||\zeta||^2 = K_D(z,z)^{-s}\}.
\]

**Lemma 3.3.** Suppose that the Bergman metric of \( D \) is well-defined. Then any point in \( \partial_0 D_{N,s} \) is strongly pseudoconvex.

**Proof.** For the notational convention, we will write \( K_D(z,z) \) by \( K \). Denote \( \rho(z) = ||\zeta||^2 - K_D(z,z)^{-s} \). Note that
\[
\partial \overline{\partial} \rho(z) = \left( \begin{array}{cc}
-\partial \overline{\partial} K^{-s} & 0 \\
0 & I_{N \times N}
\end{array} \right),
\]
\[
-\partial \overline{\partial} K^{-s} = sK^{-s-1} \partial \overline{\partial} K - s(s+1)K^{-s-2} \partial K \wedge \overline{\partial} K,
\]
\[
\partial \overline{\partial} \log K = \frac{\partial \overline{\partial} K}{K} = \frac{\partial K \wedge \overline{\partial} K}{K^2}.
\]
This implies that
\[
-\partial \overline{\partial} K^{-s} = sK^{-s} \partial \overline{\partial} \log K - s^2 K^{-s-2} \partial K \wedge \overline{\partial} K,
\]
Let $V = X + Y$ with $X = \sum_{j=1}^{r} X_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{j=1}^{N} Y_j \frac{\partial}{\partial x_j}$ be a complex tangent vector on the boundary, i.e., $\partial \rho(V) = \sum_{j=1}^{N} \overline{\zeta_j} Y_j + sK^{-s-1}\partial K(X) = 0$. Hence we obtain

$$|\partial K(X)|^2 = s^{-2}K^{2s+2}\left|\sum_{j=1}^{N} \overline{\zeta_j} Y_j\right|^2. \tag{3.10}$$

Since $\partial \partial \rho(V, \overline{V}) = ||Y||^2 - \partial \partial K^{-s}(X, \overline{X})$,

$$\partial \partial \rho(V, \overline{V}) = ||Y||^2 + sK^{-s}\partial \partial \log K(X, \overline{X}) - s^2K^{-s-2}\left(s^{-2}K^{2s+2}\left|\sum_{j=1}^{N} \overline{\zeta_j} Y_j\right|^2\right)$$

$$= sK^{-s}\partial \partial \log K(X, \overline{X}) + ||Y||^2 - K^s\left|\sum_{j=1}^{N} \overline{\zeta_j} Y_j\right|^2$$

$$= sK^{-s}\partial \partial \log K(X, \overline{X}) + ||\zeta||^{-1}\left(||Y||^2||\zeta||^2 - \left|\sum_{j=1}^{N} \overline{\zeta_j} Y_j\right|^2\right) > 0 \tag{3.11}$$

where the third equality is due to $||\zeta||^2 = K^{-s}$. \hfill \Box

**Remark 3.4.** Lemma 3.3 holds Hartogs domain over a domain $D$ of the form $\{1\}$ with $s > 0$.

## 4. Proof of Theorems

The first step that we need to consider is that the maps in the theorems send $D \times \{0\}$ to $D' \times \{0\}$. When the map is an automorphism of the bounded symmetric domains of classical type, it is proved in [14], and when the map is a proper holomorphic map between bounded symmetric domains of classical type, it is proved in [13]. In our case, we use the following lemma:

**Lemma 4.1.** [9, Proposition 2.2.] Suppose that $f$ is a bounded holomorphic function defined on $\Delta \times W$ where $W$ is a bounded domain in $\mathbb{C}^k$. Then there exists a holomorphic function $f_0$ on $W$ satisfying $\lim_{r \to 1} f(re^{i\theta}, w) = f_0(w)$ for almost every $\theta \in \partial \Delta$. Moreover, if $f_0$ is constant for a measurable subset of $\partial \Delta$, then $f$ is independent of variables of $W$.

**Lemma 4.2.** Let $D$ and $D'$ be homogeneous Siegel domains of type II and $f: D_{N,s} \to D'_{N',s'}$ a proper holomorphic map. Suppose that $D$ and $D'$ are not biholomorphic to the unit ball. Then $f$ maps $D \times \{0\}$ to $D' \times \{0\}$.

**Proof.** Suppose that $f(D \times \{0\}) \not\subseteq D' \times \{0\}$. Choose $z_0 \in D$ such that $f(z_0, 0) \in D'_{N',s'} \setminus D' \times \{0\}$. Let $\nu: \Delta \times M \hookrightarrow D$ be a proper holomorphic embedding such that $\nu(0, y_0) = z_0$ for some $y_0 \in M$, i.e.,

$$f \circ \nu(0, y_0) \in D'_{N',s'} \setminus D' \times \{0\}. \tag{4.1}$$

Choose $\theta \in [0, 2\pi)$ such that $f \circ \nu(e^{i\theta}, \cdot): M \to \mathbb{C}^{r+N'+r'}$ is a well-defined holomorphic map and $f \circ \nu(e^{i\theta}, M) \subset \partial D'_{N',s'}$. Because of (4.1), there exists $y \in M$ such that $f \circ \nu(e^{i\theta}, y) \in \partial \partial D'_{N',s'}$. Since $\partial \partial D'_{N',s'}$ is strongly pseudoconvex, $f \circ \nu(e^{i\nu}, \cdot)$ is a constant map for each $\nu$ in some neighborhood of $\theta$. Hence $f \circ \nu$ is independent of $M$ and it is a contradiction to the fact that $f$ is proper. Therefore $f(D \times \{0\}) \subseteq D' \times \{0\}$. \hfill \Box

**Proposition 4.3.** Let $U$ be an open neighborhood of $0 \in \mathbb{C}^{r+N}$ and $f: U \to \mathbb{C}^{r+N}$ a holomorphic map such that
(1) \( f(D_{N,s} \cap U) \subset D_{N,s'} \) and \( f(\partial D_{N,s} \cap U) \subset \partial D_{N,s'} \) with \( s \leq s' \),
(2) \( f(U \cap (D \times \{0\})) \subset D \times \{0\} \) and \( f|_{U \cap (D \times \{0\})} \equiv \text{id} \),
(3) \( df|_{(0,0)} \neq 0 \).

Then \( s = s' \) and the Jacobian matrix of \( f \) at 0 is \( \begin{pmatrix} Id & b \\ 0 & c \end{pmatrix} \) for some unitary \( N \) by \( N \) matrix \( c \) and \( r \) by \( N \) matrix \( b \).

**Proof.** Denote \( f = (f_1, f_2) \) where \( f_1 : D_{N,s} \rightarrow D, f_2 : D_{N,s} \rightarrow \mathbb{C}^N \) are projections and \( \rho_1(z, \zeta) = |\zeta|^2 - K_D(z, z)^{-s} \). Fix a point \((z_0, \zeta_0) \in \partial D_{N,s} \cap U \) such that \( \zeta_0 \neq 0 \). Note that \( d\rho_1|_{(z_0, \zeta_0)} \neq 0 \).

Since \(|f_2(z, \zeta)|^2 - K_D(f_1(z, \zeta), f_1(z, \zeta))^{-s} = 0 \) whenever \( z \in U \) and \(|\zeta|^2 - K_D(z, z)^{-s} = 0 \), there exists a real analytic function \( \mu(z, \zeta) \) on a sufficiently small neighborhood of \((z_0, \zeta_0) \) such that
\[
|f_2(z, \zeta)|^2 - K_D(f_1(z, \zeta), f_1(z, \zeta))^{-s'} = \mu((z, \zeta), (\overline{z}, \overline{\zeta})) \left(|\zeta|^2 - K_D(z, z)^{-s}\right).
\]

By polarization, we obtain the holomorphic equation,
\[
f_2(z, \zeta) \cdot \overline{\mathcal{J}_2(w, \xi)} - K_D(f_1(z, \zeta), f_1(\overline{w}, \overline{\xi}))^{-s'} = \mu((z, \zeta), (w, \xi)) \left(\zeta \cdot \xi - K_D(z, \overline{w})^{-s}\right).
\]

By the identity property of holomorphic functions, it follows that
\[
f_2(z, \zeta) \cdot \overline{\mathcal{J}_2(w, \xi)} - K_D(f_1(z, \zeta), f_1(\overline{w}, \overline{\xi}))^{-s'} = 0
\]
whenever \( \zeta \cdot \xi - K_D(z, \overline{w})^{-s} = 0 \) and \((z, \zeta), (w, \xi) \in U \times U \). This implies that \([4.2]\) holds for all \((z, \zeta) \in U \) by substituting \((w, \xi) = (\overline{z}, \overline{\zeta})\).

Substitute \( \zeta = 0 \) on \([4.2]\). Then
\[
\mu(z, 0) = K(z, z)^{s-s'}. \]

If we expand \( \mu \) in terms of \( \zeta \) variable, we can express \( \mu \) as
\[
\mu(z, \zeta) = K(z, z)^{s-s'} + \sum_{j=1}^{N} \zeta_j g_j(z, \overline{z}) + \sum_{j=1}^{N} \overline{\zeta}_j g_j(z, \overline{z}) + O(\zeta^2)
\]
with some functions \( g_j \) where \( O(\zeta^2) \) denotes the higher order terms with respect to \( \zeta \). Note that \( K(0,0)^{s-s'} = 0 \) if \( s \neq s' \). Let
\[
f_1(z, \zeta) = z + b\zeta + O^2
\]
\[
f_2(z, \zeta) = az + c\zeta + O^2
\]
with linear transformations \( a, b, c \) where \( O^2 \) denotes the higher order terms in \( \zeta \) and \( z \). Then the equation \([4.2]\) can be expressed by
\[
|az + c\zeta + O^2|^2 - c(\text{Im}(\pi_1(z + b\zeta + O^2))) - F(\pi_2(z + b\zeta + O^2), \pi_2(z + b\zeta + O^2))^{-(2d-q)s'}
\]
\[
= (K(z, z)^{s-s'}) + \sum_{j=1}^{N} \zeta_j g_j(z, \overline{z}) + \sum_{j=1}^{N} \overline{\zeta}_j g_j(z, \overline{z}) + O(\zeta^2) \left(|\zeta|^2 - K(z, z)^{-s}\right)
\]
where \( \pi_1 : D \rightarrow \mathbb{C}^n \) and \( \pi_2 : D \rightarrow \mathbb{C}^m \) are projections. In the first line of \([4.7]\), the second order term in \( \zeta \) variable is \(|c\zeta|^2\) which does not vanish by the condition \([3]\). On the other hand in the second line of \([4.7]\), the second order term in \( \zeta \) variable exists only when \( K(z, z)^{s-s'}|\zeta|^2 = |\zeta|^2 \). That is \( s = s' \). Moreover \(|cc|| = ||\zeta||\) and hence we obtain that \( c \) is a unitary transformation. Furthermore, if we consider the \(|z|^2\) term, it follows that \( a = 0 \).

**Remark 4.4.** In the proof of Theorem \([1.1]\) we used the condition \( s \geq 1 \) in the definition of the Hartogs domains.
Lemma 4.5. [14] Lemma 2.1 Let $f: D \to D'$ be a proper holomorphic map of bounded domains in $\mathbb{C}^n$ each of whose kernel functions $K_D$ and $K_{D'}$ extends to the boundary. Assume that there is a sequence of points $z_k \in D$, $k = 1, 2, \ldots$, satisfying the following conditions:

1. $z \to a \in \partial D$ and $f(z_k) \to b \in \partial D'$ as $k \to \infty$.
2. $|Jf(z_k)| > \epsilon$ for some $\epsilon > 0$.
3. There exists a point $p \in D'$ such that $K'(b, p) \neq 0$ and $R'(b, p) \neq 0$, where

$$R'(z, \zeta) = \det \left( \frac{\partial^2}{\partial z_i \partial \zeta_j} \log K_{D'}(z, \zeta) \right).$$

Then $f$ extends holomorphically to a neighborhood of $a \in \partial D$.

Proof of Theorem 1.1. Suppose that $D$ is the unit ball. Then $D_{N,s}$ is an ellipsoid. Suppose that there is a biholomorphic map $f: D_{N,s} \to D'_{N',s'}$. If $D'$ is not biholomorphic to the unit ball, then there is an embedding from $\Delta \times M$ for some complex submanifold $M$ into $D'$ and hence there should be a complex submanifold of dimension greater than zero in $\partial D_{N,s}$ by the same reason of Lemma 4.2. This contradiction implies that $D'$ is also the unit ball. Hence $D'_{N',s'}$ is an ellipsoid. In this case the theorem is proved by Naruki in [10].

Now suppose that $D$ and $D'$ are not the unit balls. By Lemma 4.2, it follows that $f|_{D \times \{0\}}: D \times \{0\} \to D' \times \{0\}$ is a biholomorphism and by Theorem 4.5, it is rational. Hence $N = N'$ and there exists a sequence $z_k \in D \times \{0\}$ such that $z_k \to S \times \{0\}$ and $f(z_k) \to b$ for some $\epsilon > 0$. Say $a := \lim_{k \to \infty} z_k$. Here $S$ denotes the Shilov boundary of $D$. By Lemma 4.5, $f$ extends holomorphically to a neighborhood of $a$.

By Lemma 2.3, it follows that $b := \lim_{k \to \infty} f(z_k)$ is also contained in the Shilov boundary of $D'$. Therefore there exist $\phi \in \text{Aut}(D)$ and $\psi \in \text{Aut}(D')$ such that $\phi(a) = 0$ and $\psi(b) = 0$. Denote $\Phi \in \text{Aut}(D_{N,s})$ and $\Psi \in \text{Aut}(D'_{N',s'})$ induced by $\phi$ and $\psi$ according to (1.2) respectively. Then $\Psi \circ f \circ \Phi^{-1}$ is a biholomorphism from $D_{N,s}$ to $D'_{N',s'}$ which extends holomorphically to a neighborhood of $(0,0)$.

Let $\sigma := (\Psi \circ f \circ \Phi^{-1})^{-1}|_{D' \times \{0\}}$. Note that it is a biholomorphism from $D'$ to $D$. Let $\Sigma$ be a biholomorphism from $D'_{N',s'}$ onto $D_{N,s}$ defined by (3.1). Then $\Sigma \circ \Psi \circ f \circ \Phi^{-1}: D_{N,s} \to D_{N,s}$ is a biholomorphism satisfying the conditions in Proposition 1.1 and hence $s = s'$ and hence $\Sigma \circ \Psi \circ f \circ \Phi^{-1}$ is an automorphism of $D_{N,s}$.

Note that $\Sigma \circ \Psi \circ f \circ \Phi^{-1}(z,0) = (z,0)$ for any $z \in D$. By Proposition 4.3, it follows that

$$d(\Sigma \circ \Psi \circ f \circ \Phi^{-1})|_{(0,0)} = \begin{pmatrix} I_r & b \\ 0 & c \end{pmatrix}$$

where $c$ is a $m \times m$ unitary matrix. Define $C(z, \zeta) = (z, c^{-1} \zeta)$ and $F := C \circ \Sigma \circ \Psi \circ f \circ \Phi^{-1}$. Then

$$dF|_{(0,0)} = \begin{pmatrix} I_r & b \\ 0 & I_N \end{pmatrix}$$

On the other hand $k$-th composition of $F$, say $F^k$, is also an automorphism and there is an automorphism $\tilde{F}$ of $D_{N,s}$ such that $F^k$ converges uniformly in compact-open topology since $F|_{D \times \{0\}}$ is the identity map. By Lemma 4.5 $\tilde{F}$ extends holomorphically over the boundary and hence $d\tilde{F}|_{(0,0)}$ should be finite. This implies that $b = 0$.

Expanding $F$ in a power series at $(0,0)$ yields

$$F(\xi) = \xi + P_k(\xi) + O(|\xi|^{k+1}),$$
where $P_k$ is the first nonvanishing homogeneous polynomial of degree $k$ of order exceeding 1 in the Taylor expansion. Then direct computation gives that

$$F^2(\xi) = \xi + 2P_k(\xi) + O(|\xi|^{k+1})$$

(4.10)

$$F^j(\xi) = \xi + jP_k(\xi) + O(|\xi|^{k+1})$$

where $\xi = (z, \zeta)$. But this gives a contradiction to that $F$ extends to a neighborhood of $(0, 0)$. This implies that $F = id$ on $D_{N,s}$ and hence the theorem is proved. \qed

**Proof of Theorem**. Because of Lemma 2.2, it follows that $f$ maps $D \times \{0\}$ into $D' \times \{0\}$. Let $H = \{z \in D_{N,s} : Jf(z) = 0\}$. Suppose that there is a sequence $z_k$ in $H$ such that $z_k \to \partial_0 D_{N,s}$ and $f(z_k) \to \partial_0 D_{N',s'}$ as $k \to \infty$. Since $\partial_0 D_{N,s}$ and $\partial_0 D_{N',s'}$ are strongly pseudoconvex by Lemma 3.3 we may take scaling sequences $s_k : D_{N,s} \to C^d$ and $S_k : D_{N',s'} \to C^d$ where $d = \text{dim} D_{N,s} = \text{dim} D_{N',s'}$ such that $s_k$ and $S_k$ are biholomorphisms onto their images and $s_k(D_{N,s})$, $S_k(D_{N',s'})$ converge to the unit ball in $C^d$ (For more detail, see [5, 9.2 Higher Dimensional Scaling and the Wong-Rosay Theorem]). Then $S_k \circ f \circ s_k^{-1}$ converges a proper holomorphic map between the unit ball in $C^d$, say $F$. Since every proper holomorphic self-map of the unit ball is an automorphism ([2]), $F$ is an automorphism of the unit ball and hence extends to the closure of the unit ball as a diffeomorphism. This contradicts to that $z_k \in H$ and hence $f(H) \subset D' \times \{0\}$ by the maximum principle. Since the codimension of $f(H)$ is 1, there is an open subset in $D$ such that $Jf$ vanishes. It is a contradiction to the fact that $f$ is a proper holomorphic map. Hence $H = \emptyset$, i.e., $f$ is unbranched. Since $D_{N,s}$ is simply connected, $f$ is a biholomorphism. \qed

**Remark 4.6.** When $D$ and $D'$ are the unit balls, that is, when $D_{N,s}$ and $D'_{N',s'}$ are complex ellipsoids, there is a proper holomorphic map which is not a biholomorphism.

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