Complex Hadamard matrices from Sylvester inverse orthogonal matrices

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Abstract
A novel method to obtain parameterizations of complex inverse orthogonal matrices is provided. These matrices are natural generalizations of complex Hadamard matrices which depend on non zero complex parameters. The method we use is via doubling the size of inverse complex conference matrices. When the free parameters take values on the unit circle the inverse orthogonal matrices transform into complex Hadamard matrices, and in this way we find new parameterizations of Hadamard matrices for dimensions \( n = 8, 10, 12 \).

1 Introduction

In the seminal paper \cite{14} Sylvester defined the notion of a self-reciprocal matrix as a square array of elements of which each is proportional to its first minor. If the sum of the squares of the terms in each row or in each column is equal to unity the matrix becomes strictly orthogonal. When the strictness condition was withdrawn he defined a more general class of matrices by a homographic relation between each element and its first minor. However in this paper we shall make use only of the particular class of inverse orthogonal matrices \( A = (a_{ij}) \) that are those matrices whose inverse is given by \( A^{-1} = (1/a_{ij})^t = (1/a_{ji}) \), where \( t \) means transpose, and their entries \( 0 \neq a_{ij} \in \mathbb{C} \) satisfy the relation

\[
AA^{-1} = nI_n
\]  

(1.1)

In the above relation \( I_n \) is the \( n \)-dimensional identity matrix. When the entries \( a_{ij} \) take values on the unit circle \( A^{-1} \) coincides with the Hermitian conjugate \( A^* \) of \( A \), and in this case (1.1) is the definition of complex Hadamard matrices. Complex Hadamard matrices have applications in quantum information theory, several branches of combinatorics, digital signal processing, image analysis, coding theory, cryptology, etc.

Complex orthogonal matrices also appeared in the description of topological invariants of knots and links, see e.g. Ref. \cite{5}. They were called two-weight spin models being related to symmetric statistical Potts models. These matrices have been generalized to two-weight spin models, \cite{6}, also called generalized spin models, by removing the symmetry condition. After that these models have been extended to four-weight spin models, \cite{11}. These last matrices are also known under the name of type II matrices, see Ref. \cite{11}. Particular cases of type II matrices also appeared as generalized Hadamard transform for processing multiphase or multilevel signals, see \cite{7} and \cite{4}, which includes the classical Fourier, Walsh-Hadamard and Reverse Jacket transforms. They are defined for \( 2n \times 2n \)-dimensional matrices and, until now, only for a particular class given in terms of \( p^{th} \) roots of unity and/or one complex non-zero
The aim of the paper is to provide an analytic method for the construction of inverse orthogonal matrices, and, as a byproduct, to get from them nonequivalent complex Hadamard matrices. It is well known, see [11], that a complete classification of inverse orthogonal matrices was given only for dimensions, \( n \leq 5 \). For \( n \geq 6 \) the problem is still open. Even the “simpler” problem of classification of nonequivalent complex Hadamard matrices is not yet solved, and new families frequently appear, see in this respect the papers [9], [2], [10], [12] and [13]. The method we use in the following to construct complex Hadamard matrices is via doubling the size of inverse complex conference matrices. The first non-trivial case is \( n = 4 \), and here we give a few examples for dimensions \( m = 2n \) when \( n = 4, 5, 6 \).

### 2 Inverse Orthogonal Matrices

The complex \( n \times n \) conference matrices, \( C_n \), are matrices with \( a_{ii} = 0 \), \( i = 1, \ldots, n \) and \( |a_{ij}| = 1 \), \( i \neq j \) that satisfy

\[
C_n C_n^* = (n - 1)I_n
\]

where \( C_n^* \) is the Hermitian conjugate of \( C_n \). Conference matrices, \( C_n \), are important because by construction the matrix

\[
H_{2n} = \begin{pmatrix} C_n + I_n & C_n^* - I_n \\ C_n - I_n & -C_n^* - I_n \end{pmatrix}
\]

is complex Hadamard of size \( 2n \times 2n \), as one can easily verify. The above formula can be slightly generalized by taking the entries of the first column in (2.2) as \( C_n \pm I_n e^{ia} \) and those from the second column as \( \pm C_n^* - I_n e^{-ia} \), with \( a \in \mathbb{R} \).

Let us now suppose that we have a complex conference matrix \( C_n \) which depends on a few arbitrary phases \( e^{i\alpha_j} \), \( j = 1, \ldots, k \), then it can be transformed into a complex inverse orthogonal conference matrix by the change \( e^{i\alpha_j} \rightarrow a_j \) with \( 0 \neq a_j \in \mathbb{C} \) complex non-zero numbers. Of course one can look directly for inverse orthogonal conference matrices, but our method provided us quite easy formulas for the cases \( n = 4, 5, 6 \). Thus the complex inverse orthogonal conference matrices are those matrices with \( a_{ij} \), \( i \neq j \), complex non-zero numbers, and \( a_{ii} = 0 \), defined by a similar relation to the relation (2.1). It is well known that for complex Hadamard and conference matrices the Hermitian conjugate coincides with the inverse matrix. Hence to extend the above construction (2.2) to complex inverse orthogonal conference matrices we have to provide a recipe for a proper treating of the zero entries on the main diagonal. Our formula for the inverse is given by

\[
C_n^{-1} = (1/(C_n + I_n) - I_n)^t
\]

where \( 1/A \) is the matrix whose elements are \( 1/a_{ij} \). We remark that the above relation makes sense since all the entries of \( C_n + I_n \) are complex non-null numbers. In the case of complex orthogonal matrices the formula (2.2) takes the form

\[
O_{2n} = \begin{pmatrix} C_n + a I_n & C_n^{-1} - I_n/a \\ C_n - a I_n & -C_n^{-1} - I_n/a \end{pmatrix}
\]

with \( a \in \mathbb{C}^* \) any complex non-zero number. From the equations (2.3) and (2.4) it results that the matrix \( O_{2n} \) nonlinearly depends on the parameter \( a \) on all the parameters entering \( C_n \).

Now we show how to proceed in this case to obtain complex Hadamard matrices from Sylvester orthogonal matrices. First we consider the case \( n = 4 \) and start with the complex conference matrix given
In [3], p. 5373, by changing its arbitrary phase $e^{it}$ into the complex number $b \neq 0$

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -b & b \\ 1 & b & 0 & -b \\ 1 & -b & b & 0 \end{pmatrix}$$  \hspace{1cm} (2.5)

We remark that the transposed matrix $C_4^T$, which can be obtained by a sign change $b \rightarrow -b$ in relation (2.5), provides an other inequivalent complex orthogonal conference matrix. The second remark is that in both the doubling formulas (2.2) and (2.4) the diagonal matrix $I_n$ acts as a symmetry breaking and by consequence we can multiply the columns, or the rows, of $C_n$ by arbitrary non-null complex numbers. Thus in the above case and those similar to we multiply (2.5) at right by a diagonal matrix. When $n = 4$ the diagonal of this diagonal matrix has the form $d_1 = (A_1, A_2, A_3, A_4)$, where $A_i$ are non-zero complex numbers, and by taking into account the nonlinear dependence of the formula (2.4) on $C_n$ entries we will obtain complex orthogonal matrices depending on more free parameters. By using the formula (2.4) one gets

$$O_8 = \begin{pmatrix} a & A_2 & A_3 & A_4 & -\frac{1}{a} & \frac{1}{A_1} & \frac{1}{A_1} & \frac{1}{A_1} \\ A_1 & a & -A_3b & A_4b & \frac{1}{A_2} & -\frac{1}{a} & \frac{1}{A_2b} & -\frac{1}{A_2b} \\ A_1 & A_2b & a & -A_4b & \frac{1}{A_3} & -\frac{1}{A_3b} & \frac{1}{a} & -\frac{1}{A_3b} \\ A_1 & -A_2b & A_3b & a & \frac{1}{A_4} & -\frac{1}{A_4b} & \frac{1}{a} & -\frac{1}{A_4b} \\ -a & A_2 & A_3 & A_4 & \frac{1}{a} & -\frac{1}{a} & \frac{1}{A_1} & -\frac{1}{A_1} \\ A_1 & -a & -A_3b & a & \frac{1}{A_2} & -\frac{1}{a} & \frac{1}{A_2b} & -\frac{1}{A_2b} \\ A_1 & A_2b & -a & -A_4b & \frac{1}{A_3} & -\frac{1}{A_3b} & \frac{1}{a} & -\frac{1}{A_3b} \\ A_1 & -A_2b & A_3b & -a & \frac{1}{A_4} & -\frac{1}{A_4b} & \frac{1}{a} & -\frac{1}{A_4b} \end{pmatrix}$$  \hspace{1cm} (2.6)

In this form $O_8$ also depends non linearly on the parameters $A_i$. Now we put $O_8$ in the standard form by multiplying (2.6) at right and, respectively, at left by diagonal matrices generated by the inverse of the first row, respectively, of the first column. After that we interchange, through a transposition, the fifth and eight rows. Finally one finds after the substitutions

$$A_1 = 1/c, \quad A_2 = 1/d, \quad A_3 = 1/e, \quad A_4 = 1/f$$  \hspace{1cm} (2.7)

the following standard form of the $O_8$ matrix

$$O_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a^2cd & -abc & abc & -a^2cd & -1 & \frac{ad}{b} & -\frac{ad}{b} \\ 1 & abc & a^2ce & -abc & -a^2ce & -\frac{ae}{b} & -1 & \frac{ae}{b} \\ 1 & -abc & abc & a^2cf & -a^2cf & \frac{af}{b} & -\frac{af}{b} & -1 \\ 1 & -abc & abc & -a^2cf & a^2cf & \frac{af}{b} & -\frac{af}{b} & -1 \\ 1 & -a^2cd & -abc & abc & a^2cd & -1 & -\frac{ad}{b} & \frac{ad}{b} \\ 1 & abc & -a^2ce & -abc & a^2ce & \frac{ae}{b} & -1 & \frac{ae}{b} \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}$$  \hspace{1cm} (2.8)

matrix which depends on six complex arbitrary non-zero parameters. Please remark the still nonlinear dependence on the parameters $a$, and $b$, but the linear dependence on parameters $1/A_i$, i.e. $c$, $d$, $e$, and $f$. This property is generic for all the cases discussed in the paper.
When the parameters entering the $O_8$ matrix take values on the 6-dimensional torus $\mathbb{T}^6$, i.e. one makes the substitutions $a \to e^{ib}$, etc, $O_8$ gets a complex Hadamard matrix. It can be written in the form

$$D^{(6)}_{8(1)}(a, b, c, d, e, f) = H^{(1)}_8 \circ \text{EXP} \left( i \cdot R^{(6)}_8(a, b, c, d, e, f) \right)$$

(2.9)

The above relation is the standard form introduced in [15] to present complex Hadamard matrices, where now all the six parameters, $a, b, c, d, e, f \in \mathbb{R}$, are real numbers, $\circ$ denotes the Hadamard product, $i = \sqrt{-1}$ and $H^{(1)}_8$ is obtained from (2.8) by taking all the six parameters equal to unity, i.e.

$$H^{(1)}_8 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1
\end{bmatrix}$$

(2.10)

and

$$R^{(6)}_8 = \begin{bmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & 2a + c + d & a + b + c & a + b + c & 2a + c + d & a - b + d & a - b + d & \bullet \\
\bullet & a + b + c & 2a + c + e & a + b + c & 2a + c + e & a - b + e & \bullet & \bullet \\
\bullet & a + b + c & a + b + c & 2a + c + f & 2a + c + f & a - b + f & a - b + f & \bullet \\
\bullet & a + b + c & a + b + c & 2a + c + f & 2a + c + f & a - b + f & a - b + f & \bullet \\
\bullet & 2a + c + d & a + b + c & a + b + c & 2a + c + d & a - b + d & a - b + d & \bullet \\
\bullet & a + b + c & 2a + c + e & a + b + c & 2a + c + e & a - b + e & a - b + e & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{bmatrix}$$

(2.11)

where $\bullet$ means zero. It is well known that matrices under Hadamard product and usual addition generate a commutative algebra such that all the usual functions, like exponential entering (2.9), are well defined.

In our opinion the matrix (2.9) is the most general affine form of a 8-dimensional Hadamard matrix, although we have no formal proof. The previous known results are the matrices $F_8^{(5)}$, see [15], and $S_8^{(4)}$, see [9], that depend on five, respectively, four free parameters.

As we said before the transposed matrix $C_4^T$ leads to another inequivalent matrix which has the form

$$D^{(6)}_{8(2)}(a, b, c, d, e, f) = H^{(2)}_8 \circ \text{EXP} \left( i \cdot R^{(6)}_8(a, b, c, d, e, f) \right)$$

(2.12)

where

$$H^{(2)}_8 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & 1
\end{bmatrix}$$

(2.13)
If in relation (2.8) one makes the replacements $abc = i$, $a^2cd = i$, $ad = b$, $a^2ce = i$, $b = af$, and finally $f = e$, one gets the matrix

$$D_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & i & -i & i & -i & -1 & 1 & -1 \\
1 & i & i & -i & -i & -1 & 1 & -1 \\
1 & -i & i & -i & i & -i & 1 & 1 \\
1 & -i & i & i & -i & i & 1 & -1 \\
1 & i & -i & i & -i & i & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1
\end{pmatrix}$$  \hspace{0.5cm} (2.14)

Similar matrices are the $K_4$ matrix obtained by Horadam, \[4\], from a back-circulant matrix at its turn derived from a quadriphase perfect sequence, the $S_8$ matrix obtained by Matolcsi et al, \[9\], via tiling abelian groups, and the $J_8$ matrix obtained by Lee and Vavrek, \[8\], from a modified Paley construction. All these four matrices are inequivalent. $D_8$ has the form of a jacket matrix, i.e. its entries from the first row and column are 1, while the entries from the last row and column are $\pm 1$.

If now in (2.8) one makes the substitutions $d = e = b/a$, followed by $f = bv/a$, and finally $abc = u$, one gets an orthogonal jacket matrix which depends on two arbitrary non-zero complex parameters

$$d_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & u & -u & u & -u & 1 & 1 & -1 \\
1 & u & u & -u & -u & -1 & 1 & 1 \\
1 & -u & u & uv & -uv & v & v & -v \\
1 & -u & u & -uv & uv & v & v & -v \\
1 & -u & u & u & u & -1 & -1 & 1 \\
1 & u & -u & -u & u & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1
\end{pmatrix}$$  \hspace{0.5cm} (2.15)

The choice $u = i$ and $v = 1$ in relation (2.15) leads to the matrix $D_8$, and $u = v = i$ leads to an other inequivalent matrix.

By using the doubling formula

$$C = \begin{pmatrix} A & DB \\ A & -DB \end{pmatrix}$$  \hspace{0.5cm} (2.16)

see \[3\], Eq. (8), where $A$ and $B$ are matrices of the form $D_8^{(6)}$, each one depending on six different parameters, and $D \in \mathbb{T}^8$ is a diagonal matrix with its first entry 1, and the other entries seven arbitrary phases, one gets a matrix $D_8^{(19)}$ that depends on 19 arbitrary phases. The only known results for this dimension are the matrices $F_1^{(17)}$ from \[15\], and $R_1^{(11)}$ from \[9\], which depend on 17, respectively 11, free parameters.

When $n = 5$ we start with the complex orthogonal conference matrix

$$C_5 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & b & \omega b & \omega^2 b \\
1 & b & 0 & \omega b & \omega^2 b \\
1 & \omega b & \omega^2 b & 0 & b \\
1 & \omega^2 b & \omega b & b & 0
\end{pmatrix}$$  \hspace{0.5cm} (2.17)

which depends on two complex parameters, a discrete parameter $\omega$ that is solution of the equation $\omega^2 + \omega + 1 = 0$, and a free parameter $b \in \mathbb{C}^*$. By using the same procedure as before one finds
which depends on eight complex parameters: \( \omega, a, b, c, d, e, f, g \), one of them, \( \omega \), taking two discrete values.

Similar to the case \( n = 8 \) we use the same notation for the complex parameters from (2.18), and the real parameters entering (2.19). Thus if into the formula (2.18) one takes \( a, b, c, d, e, f, g \in \mathbb{F}^7 \) one gets the Hadamard matrix

\[
D^{(8)}_{10}(\omega, a, b, c, d, e, f, g) = H^{(\omega)}_{10} \circ \text{EXP} \left( i \cdot R^{(7)}_{10}(a, b, c, d, e, f, g) \right)
\]

where now in (2.19) and (2.20) \( a, b, c, d, e, f, g \in \mathbb{R} \)

\[
R^{(7)}_{10}(a, b, c, d, e, f, g) =
\]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & a^2cd & abc & abc\omega & abc\omega^2 & -a^2cd & -1 & ad/b & ad/b\omega & ad/b\omega^2 \\
1 & abc & a^2ce & abc\omega^2 & abc\omega & -a^2ce & ae/b & -1 & ae/b\omega^2 & ae/b\omega \\
1 & abc\omega & abc\omega^2 & a^2cf & abc & -a^2cf & af/b\omega & -1 & af/b \omega & -a^2cf \\
1 & abc\omega^2 & abc\omega & abc & a^2cg & -a^2cg & ag/b\omega^2 & ag/b\omega & -ag/b\omega & -ag/b \\
1 & -a^2cd & abc & abc\omega^2 & a^2cd & -1 & -ad/b & -ad/b\omega & -ad/b\omega^2 \\
1 & abc & -a^2ce & abc\omega^2 & abc\omega & a^2ce & -ae/b & -1 & -ae/b\omega & -ae/b\omega \\
1 & abc\omega^2 & abc\omega & abc & a^2cf & abc & -af/b\omega & -af/b\omega & -af/b \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
and

\[
H^{(ω)}_{10} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & ω & ω^2 & -1 & -1 & 1 & ω^2 & ω \\
1 & 1 & 1 & ω & -1 & 1 & -1 & ω & ω^2 & -ω \\
1 & ω & ω^2 & 1 & 1 & -1 & ω & ω & -1 & 1 \\
1 & ω^2 & ω & 1 & -1 & 1 & -ω & ω & -1 & -1 \\
1 & ω^2 & ω & 1 & 1 & -1 & ω & ω^2 & 1 & -1 \\
1 & -1 & 1 & ω & ω^2 & 1 & -1 & -1 & -ω & -ω \\
1 & 1 & -1 & ω^2 & ω & 1 & -1 & -1 & -ω & -ω \\
1 & ω & ω^2 & -1 & 1 & 1 & -ω & -ω & -ω & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

Within the array (2.20) the horizontal lines were introduced to separate the rows. Excluding the first and the last rows, the entries on the other rows have been written on two levels. Thus the second entry from the second row is \(2a + c + d\), etc. In this case we obtained two inequivalent matrices corresponding to the two complex values for \(ω\). Other known results in this dimension are those found from 10-dimensional Fourier matrix by Tadej and Życzkowski [13], which depends on four parameters, and that found by Szöllősi [12] which depends on three parameters.

By using the doubling formula (2.10) with \(A\) and \(B\) of the form \(D^{(9)}_{10}\) and \(D \in ℤ_{10}\) one gets a formula for \(D^{(23)}_{20}\) which depends on 23 arbitrary phases, and either \(ω = e^{2πi/3}\), or \(ω = e^{4πi/3}\).

From formula (2.18) one finds an orthogonal jacket matrix which depends on three complex parameters \(A\), \(B\) and \(ω\)

\[
D^{(3)}_{10}(ω, A, B) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & Aω^2 & Aω & Aω^2 & -Aω^2 & -1 & ω^2 & ω & 1 & 1 \\
1 & A & Aω & Aω^2 & -Aω & -A & ω & -ω & -ω & -ω \\
1 & Aω & Aω^2 & A & A & -A & ω & ω^2 & ω & -1 \\
1 & Aω^2 & AW & A & AB & -AB & Bω & Bω^2 & B & -1 \\
1 & Aω^2 & Aω & A & AB & -AB & Bω & Bω^2 & B & -1 \\
1 & -Aω^2 & -Aω & Aω^2 & -Aω & -A & ω & ω^2 & ω & -1 \\
1 & A & -Aω & Aω^2 & Aω & -Aω & -ω & -ω & -ω & -ω \\
1 & Aω & Aω^2 & -A & A & A & -ω^2 & -ω & -ω & -ω \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

From it, by taking for example \(A = B = 1\), one gets a generalized Butson matrix

\[
d^{(ω)}_{10} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & ω^2 & 1 & ω & ω^2 & -ω^2 & -1 & ω^2 & ω & 1 \\
1 & 1 & ω & ω^2 & ω & -ω & ω & -ω & -ω & -ω \\
1 & ω & ω^2 & 1 & 1 & -1 & ω & ω^2 & ω & -1 \\
1 & ω^2 & ω & 1 & -1 & 1 & -ω & ω & -ω & -ω \\
1 & ω^2 & ω & 1 & 1 & -1 & ω & ω^2 & 1 & -1 \\
1 & -ω^2 & 1 & ω & ω^2 & ω & 1 & -ω & ω & -ω \\
1 & 1 & -ω & ω^2 & ω & -ω & -ω & ω & -ω & -ω \\
1 & ω & ω^2 & -1 & 1 & 1 & -ω^2 & -ω & -ω & -ω \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

which is different from \(H^{(ω)}_{10}\), and so on.
For \( n = 6 \) there are three different inverse complex conference matrices

\[
C_{6A} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & b & b & -b & -b \\
1 & b & 0 & -b & b/c & -b/c \\
1 & b & -b & 0 & -b/c & b/c \\
1 & -b & bc & -bc & 0 & b \\
1 & -b & -bc & bc & b & 0
\end{bmatrix}
\]  

(2.23)

\[
C_{6G} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & b & b & -b & -b \\
1 & b & 0 & -b & c & -c \\
1 & b & -b & 0 & -c & c \\
1 & -b & b^2/c & b^2/c & -b^2/c & b^2/c \\
1 & -b & -b^2/c & -b^2/c & 0 & b
\end{bmatrix}
\]  

(2.24)

and

\[
C_{6M} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & bc & bc & -bc & -bc \\
1 & bc & 0 & -bc & b^2 & -b^2 \\
1 & bc & -bc & 0 & -b^2 & b^2 \\
1 & -bc & b & -b & 0 & bc \\
1 & -bc & -b & b & bc & 0
\end{bmatrix}
\]  

(2.25)

Starting with (2.23) one gets another five inverse orthogonal conference matrices. The first is obtained from the transposed matrix \( C_{6B} = C_{6A}^t \) and the other four from the matrices \( C_{6C} \) and \( C_{6E} \) which are obtained from (2.23) by the substitutions \( b \to -b \), and, respectively, \( c \to -c \), and theirs transposes, \( C_{6D} \), and \( C_{6F} \). Similar results are obtained from (2.24) and (2.25). In conclusion the total number of inequivalent complex Hadamard matrices will be 18. Similar to the cases \( n = 4 \), and \( n = 5 \), there will be four matrices, \( H_{12}^{(i)} \), \( i = 1, 2, 3, 4 \), which all are equivalent, but since we preserve the standard form (2.26) the obtained Hadamard matrices will be inequivalent matrices.

One conclusion of the above results is that for \( n \geq 6 \), in general, there are many inequivalent inverse orthogonal conference matrices which lead to many inequivalent complex Hadamard matrices. For a general dimension \( n \) this number depends on the intrinsic local entries symmetry of the corresponding inverse orthogonal conference matrix. It is easily seen that the above construction can be generalised such that the following proposition holds:

**Proposition 1.** Let us suppose that the \( n \times n \) complex inverse orthogonal conference matrix \( C_n \) has the standard form, i.e. all the entries on the first row and the first column are equal to unity, excepting that on the main diagonal which is zero. If \( C_n \) depends on \( p \) complex parameters, then the complex inverse orthogonal matrix \( O_{2n} \) obtained by using formula (2.4) depends on \( n + p + 1 \) complex parameters. The number of inequivalent complex Hadamard matrices is at least two, obtained from the inverse conference matrix and its transpose, and, in general, this number depends on the intrinsic local entries symmetry of the inverse conference matrix which one starts with.

Now one remark concerning the difference between the cases \( n = 4 \) and \( n = 5 \). In the first case the second solution appeared from the local symmetry \( b \to -b \), or from the transposed matrix, \( C_{4}^t \), while
in the second case the two solutions are a consequence of the two discrete values taken by $\omega$. Thus for $n \geq 7$ and odd, the number of orthogonal conference matrices could depend on the complex solutions of the equation $\omega^{n-2} = 1$, and also on the local entry symmetries.

The matrix $(2.23)$ leads to the inverse orthogonal matrix $(2.26)$ which depends on nine complex non-zero parameters, and to complex Hadamard matrices of the form $(2.19)$.

$$O_{12A} =$$

$$D_{12A}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{1} \circ \exp \left( i \cdot R_{12A}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12B}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(1)} \circ \exp \left( i \cdot R_{12B}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12C}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(2)} \circ \exp \left( i \cdot R_{12C}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12D}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(2)} \circ \exp \left( i \cdot R_{12D}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12E}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(3)} \circ \exp \left( i \cdot R_{12E}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12F}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(3)} \circ \exp \left( i \cdot R_{12F}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12G}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(1)} \circ \exp \left( i \cdot R_{12G}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12H}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(1)} \circ \exp \left( i \cdot R_{12H}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12I}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(4)} \circ \exp \left( i \cdot R_{12I}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12J}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(4)} \circ \exp \left( i \cdot R_{12J}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$

$$D_{12K}^{(9)}(a, b, c, d, e, f, g, h, i) = H_{12}^{(3)} \circ \exp \left( i \cdot R_{12K}^{(9)}(a, b, c, d, e, f, g, h, i) \right)$$
\[ D^{(9)}_{12L}(a, b, c, d, e, f, g, h, i) = H^{(3)}_{12} \circ \text{EXP} \left( i \cdot R^{(9)}_{12L}(a, b, c, d, e, f, g, h, i) \right) \quad (2.40) \]
\[ D^{(9)}_{12M}(a, b, c, d, e, f, g, h, i) = H^{(1)}_{12} \circ \text{EXP} \left( i \cdot R^{(9)}_{12M}(a, b, c, d, e, f, g, h, i) \right) \quad (2.41) \]
\[ D^{(9)}_{12N}(a, b, c, d, e, f, g, h, i) = H^{(1)}_{12} \circ \text{EXP} \left( i \cdot R^{(9)}_{12N}(a, b, c, d, e, f, g, h, i) \right) \quad (2.42) \]
\[ D^{(9)}_{12O}(a, b, c, d, e, f, g, h, i) = H^{(2)}_{12} \circ \text{EXP} \left( i \cdot R^{(9)}_{12O}(a, b, c, d, e, f, g, h, i) \right) \quad (2.43) \]
\[ D^{(9)}_{12P}(a, b, c, d, e, f, g, h, i) = H^{(2)}_{12} \circ \text{EXP} \left( i \cdot R^{(9)}_{12P}(a, b, c, d, e, f, g, h, i) \right) \quad (2.44) \]
\[ D^{(9)}_{12Q}(a, b, c, d, e, f, g, h, i) = H^{(4)}_{12} \circ \text{EXP} \left( i \cdot R^{(9)}_{12Q}(a, b, c, d, e, f, g, h, i) \right) \quad (2.45) \]
\[ D^{(9)}_{12R}(a, b, c, d, e, f, g, h, i) = H^{(3)}_{12} \circ \text{EXP} \left( i \cdot R^{(9)}_{12R}(a, b, c, d, e, f, g, h, i) \right) \quad (2.46) \]

where

\[
H^{(1)}_{12} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\quad (2.47)
\]

\[
H^{(2)}_{12} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
\end{bmatrix}
\quad (2.48)
\]
\[
H_{12}^{(3)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(2.49)

\[
H_{12}^{(4)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(2.50)

\[
R_{12A}(a, b, c, d, e, f, g, h, i) = \begin{bmatrix}
\end{bmatrix}
\]

(2.51)
\[
R_{12B}^{(9)}(a, b, c, d, e, f, g, h, i)) = \]
\[
\begin{bmatrix}
2a + d & a + b & a + b & a + b & a + b & 2a + d & a - b & a - b & a - b & a - b \\
+e & +d & +d & +d & +d & +e & +e & +e & +e & +e \\
\end{bmatrix}
\]

\[
R_{12C}^{(9)}(a, b, c, d, e, f, g, h, i)) = 
\]
\[
\begin{bmatrix}
2a + d & a + b & a + b & a + b & a + b & 2a + d & a - b & a - b & a - b & a - b \\
+e & +d & +d & +d & +d & +e & +e & +e & +e & +e \\
\end{bmatrix}
\]

\[ R_{12}^{(9)}(a, b, c, d, e, f, g, h, i) = \] (2.54)

\[
\begin{array}{cccccccccccc}
2a + d & a + b & a + b & a + b & a + b & a + b & 2a + d & \bullet & a - b & a - b & a - b & a - b \\
+e & +d & +d & +d & +d & +d & +e & +e & +e & +e & +e \\
a + b & 2a + d & a + b & a + 2b & a + 2b & 2a + d & a - b & \bullet & a - b & a - c & a - c \\
+d & +f & +d & -c + d & -c + d & +f & +f & +f & +f & +f & +f \\
a + b & a + c & a + c & 2a + d & a + b & 2a + d & a - b & a - 2b & a - 2b & a - b \\
+d & +d & +d & h & d & h & h & +c + h & +c + h & +h & +h \\
a + b & a + c & a + c & a + b & 2a + d & 2a + d & a - b & a - 2b & a - 2b & a - b \\
+d & +d & +d & +d & +i & +i & +i & +i & +c + i & +c + i & +i \\
a + b & a + c & a + c & a + b & 2a + d & 2a + d & a - b & a - 2b & a - 2b & a - b \\
+d & +d & +d & +d & +d & +d & +e & +e & +e & +e & +e \\
a + b & 2a + d & a + b & a + 2b & a + 2b & 2a + d & a - b & a - c & a - c \\
+d & +f & +d & -c + d & -c + d & +f & +f & +f & +f & +f & +f \\
a + b & a + b & a + d & a + 2b & a + 2b & 2a + d & a - b & a - b & a - c & a - c \\
+d & +d & +d & -c + d & -c + d & +f & +f & +f & +f & +f & +f \\
a + b & a + c & a + c & 2a + d & a + b & 2a + d & a - b & a - 2b & a - 2b & a - b \\
+d & +d & +d & +d & +h & +d & +h & +h & +c + h & +c + h & +h \\
\end{array}
\]

\[ R_{12}^{(9)}(a, b, c, d, e, f, g, h, i) = \] (2.55)

\[
\begin{array}{cccccccccccc}
2a + d & a + b & a + b & a + b & a + b & a + b & 2a + d & \bullet & a - b & a - b & a - b & a - b \\
+e & +d & +d & +d & +d & +d & +e & +e & +e & +e & +e \\
a + b & 2a + d & a + b & a + 2b & a + 2b & 2a + d & a - b & \bullet & a - b & a - c & a - c \\
+d & +f & +d & -c + d & -c + d & +f & +f & +f & +f & +f & +f \\
a + b & a + b & a + b & 2a + d & a + b & 2a + d & a - b & a - b & a - c & a - c \\
+d & +d & +d & -c + d & -c + d & +f & +f & +f & +f & +f & +f \\
a + b & a + c & a + c & a + b & 2a + d & 2a + d & a - b & a - 2b & a - 2b & a - b \\
+d & +d & +d & +d & +i & +i & +i & +i & +c + i & +c + i & +i \\
a + b & a + c & a + c & a + b & 2a + d & 2a + d & a - b & a - 2b & a - 2b & a - b \\
+d & +d & +d & +d & +d & +i & +i & +i & +i & +c + i & +c + i & +i \\
2a + d & a + b & a + b & a + b & a + b & a + b & 2a + d & \bullet & a - b & a - b & a - b & a - b \\
+e & +d & +d & +d & +d & +d & +e & +e & +e & +e & +e \\
a + b & 2a + d & a + b & a + 2b & a + 2b & 2a + d & a - b & \bullet & a - b & a - c & a - c \\
+d & +f & +d & -c + d & -c + d & +f & +f & +f & +f & +f & +f \\
a + b & a + b & 2a + d & a + b & a + 2b & a + 2b & 2a + d & a - b & a - b & a - c & a - c \\
+d & +d & +d & -c + d & -c + d & +f & +f & +f & +f & +f & +f \\
a + b & a + c & a + c & 2a + d & a + b & 2a + d & a - b & a - 2b & a - 2b & a - b \\
+d & +d & +d & +d & +h & +d & +h & +h & +c + h & +c + h & +h \\
\end{array}
\]
\[ R_{12M}^{(9)}(a, b, c, d, e, f, g, h, i) = \]
\[
\begin{array}{cccccccccccccccc}
2a + d & a + b & a + b & a + b & a + b & 2a + d & \cdot & a - b & a - b & a - b & a - b \\
+e & +c + d & +c + d & +c + d & +c + d & +e & -c + e & -c + e & -c + e & -c + e \\
\hline
a + b & 2a + d & a + b & a + b & a + b & 2a + d & a - b & a - b & a - b & a - b \\
+e & +c + d & +f & +c + d & 2c + d & 2c + d & +f & -c + f & -c + f & +f & +f \\
\hline
\end{array}
\]

\[ R_{12N}^{(9)}(a, b, c, d, e, f, g, h, i) = \]
\[
\begin{array}{cccccccccccccccc}
2a + d & a + b & a + b & a + b & a + b & 2a + d & \cdot & a - b & a - b & a - b & a - b \\
+e & +c + d & +c + d & +c + d & +c + d & +e & -c + e & -c + e & -c + e & -c + e \\
\hline
a + b & 2a + d & a + b & a + b & a + b & 2a + d & a - b & a - b & a - b & a - b \\
+e & +c + d & +f & +c + d & 2c + d & 2c + d & +f & -c + f & -c + f & +f & +f \\
\hline
\end{array}
\]
\[
R_{12O}^{(9)}(a,b,c,d,e,f,g,h,i) =
\]
\[
\begin{array}{cccccccccccc}
2a + d & a + b & a + b & a + b & a + b & 2a + d & \bullet & a - b & a - b & a - b & a - b \\
+e & +c + d & +c + d & +c + d & +c + d & +e & -c + e & -c + e & -c + e & -c + e \\
\end{array}
\]

\[
R_{12R}^{(9)}(a,b,c,d,e,f,g,h,i) =
\]
\[
\begin{array}{cccccccccccc}
2a + d & a + b & a + b & a + b & a + b & 2a + d & \bullet & a - b & a - b & a - b & a - b \\
+e & +c & +c & +c & +c & +e & +e & +e & +e & +e \\
\end{array}
\]
Similar to the preceding cases from \((2.26)\) one gets an orthogonal jacket matrix that depends on four complex non-zero parameters

\[
D^{(4)}_{12}(u,v,w,x) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & u & u & u & -u & -u & -u & w & -1 & -w & 1 & -1 \\
1 & u & uw & -u & u/w & -u/w & u/w & w & -w & 1 & -1 & 1 \\
1 & u & -u & uw & -u & u/w & -uw & w & -w & -1 & -1 & 1 \\
1 & -u & uw & -uw & u & u & -u & 1 & w & -w & -1 & 1 \\
1 & -u & -uw & uw & u & u^2 & -u^2 & -u^2 & -w^2 & -uw & w & 1 \\
1 & -u & -uw & uw & u & -u^2 & -u^2 & u^2 & w & uw & -uw & -w & -1 \\
1 & -u & u & u & -u & u & -w & 1 & -1 & -1 & 1 & 1 \\
1 & u & -uw & -u & u/w & -u/w & uw & w & -1 & 1 & -1 & 1 \\
1 & -u & uw & -uw & -u & u & w & -1 & -w & w & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\] (2.60)

and from it by taking \(u = v = x = i\) and \(w = 1\) one gets the Hadamard matrix

\[
d_{12} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & i & i & i & -i & -i & -i & 1 & 1 & -1 & -1 & 1 \\
1 & i & -i & i & -i & i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & i & -i & i & -i & i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -i & i & -i & i & i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -i & i & i & i & i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -i & i & i & -i & -i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -i & -i & i & -i & i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -i & -i & i & i & i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -i & i & -i & i & i & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\] (2.61)

which, for this dimension, is a novel matrix in literature. By other parameter choices one finds other inequivalent matrices.

Using the same doubling formula \((2.16)\) and the matrices \(D^{(9)}_{12}, (2.27), (2.46)\), one find eighteen matrices \(D^{(29)}_{24}\) that depends on 29 arbitrary phases.

## 3 Conclusion

In this paper we provided a procedure to find parametrizations of complex inverse orthogonal matrices by doubling the size of complex inverse orthogonal conference matrices. Due to the nonlinear dependence of the doubling formula \((2.3)\) on complex entering parameters, the new found complex Hadamard matrices depend on more independent phases in all the analysed cases, than those found in the published literature on the subject.

The number of inequivalent complex Hadamard matrices depend on the number of inequivalent conference matrices one starts with, but also on their local entries symmetries, such that for \(n = 12\), new eighteen solutions were found.
Our approach can be used to obtain \( n \times n \) complex weighing matrices, \( W_{n,k} \), which are similar to orthogonal conference matrices \( [21] \), being complex matrices that have on each row and column exactly \( k \) zero entries and satisfy the relation

\[
W_{n,k} W_{n,k}^* = (n - k)I_n
\]  

(3.1)

where \( k \) is called weight. When \( k = 1 \) one gets a conference matrix.

It is easily seen that such a matrix can be brought to the form \( (W_{n,k})_{ii} = 0 \), such that the doubling formula \( [21] \) can be used if the \( W_{n,k}^{-1} \) entries are given by the following formula

\[
(W_{n,k}^{-1})_{ij} = \begin{cases} 
1/(W_{n,k})_{ji} & \text{for } (W_{n,k})_{ji} \neq 0 \\
0 & \text{for } (W_{n,k})_{ji} = 0 
\end{cases}
\]  

(3.2)

The obtained matrix will be again a weighing matrix whose weight is now \( 2(k - 1) \); for \( k = 1 \) one gets Hadamard matrices, for \( k = 2 \) one gets weighing matrices with the same number of zero entries but with a double size, and for \( k \geq 3 \) one finds sparse matrices of the form \( W_{2n,2(k-1)} \). These matrices are important for applications in quantum information theory, cryptography, signal processing and orthogonal code design, see for example the paper \[16\].

The weighing matrices for \( k \geq 2 \) can be obtained very easily, and here we give an example of a complex orthogonal \( W_{4,2} \)

\[
W_{4,2} = \begin{bmatrix}
0 & a & 0 & b \\
c & 0 & d & 0 \\
0 & e & 0 & -\frac{be}{a} \\
f & 0 & -\frac{df}{c} & 0 
\end{bmatrix}
\]  

(3.3)

which depends on six arbitrary non-zero parameters. Applications of weighing matrices will be given elsewhere.

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**References**

[1] Ei. Bannai and Et. Bannai, Generalized generalized spin models (four-weight spin models), *Pacific J.Math.*, 170, 1-16 (1995)

[2] K. Beauchamp and R. Nicoara, Orthogonal maximal abelian \( \ast \)-subalgebras of \( 6 \times 6 \) matrices, *Lin.Alg.Appl.* 428, 1833-1853 (2008)

[3] P. Diţă, Some results on the parametrization of complex Hadamard matrices, *J.Phys.A: Math. Gen.*, 37, 5355-5374 (2004)

[4] K.J. Horadam, A generalized Hadamard transform, [arXiv:cs/0508104v1]

[5] V.F.R. Jones, On knot invariants related to statistical mechanical models, *Pacific J.Math.*, 137, 311-334 (1989)

[6] K. Kawagoe, A. Munemasa, and Y. Watatani, Generalized spin models, *J. Knot Theory Ramificat.* 3, 465-475 (1994)
[7] M. H. Lee, S. Rajan, and J.Y. Park, A generalized reverse jacket transform, *IEEE Trans. Circuits Syst. II*, 48, 684-690 (2001)

[8] M. H. Lee and V.Vl.Vavrek, Eleventh International Workshop on Algebraic and Combinatorial Coding Theory, Jacket conference matrices and Paley transformation, Pamporovo, Bulgaria, 181-185 (2008)

[9] M. Matolcsi, J. Réfy and F. Szöllösi, Constructions of complex Hadamard matrices via tiling abelian groups, *Open Syst.InfDyn.*, 14, 247-263 (2007)

[10] M. Matolcsi and F. Szöllösi, Towards a classification of 6 complex Hadamard matrices, *Open Syst.InfDyn.*, 15, 93-108 (2008)

[11] K. Nomura, Type II matrices of size five, *Graphs and Combinatorics*, 15, 79-92 (1999)

[12] F. Szöllösi, Parametrizing complex Hadamard matrices, *Eur.J.Comb.* 29, 1219-1234 (2008)

[13] F. Szöllösi, A two parameter family of complex Hadamard matrices of order 6 induced by a hypocycloid, [arXiv:0811.3930](https://arxiv.org/abs/0811.3930)

[14] J.J. Sylvester, Thoughts on inverse orthogonal matrices, simultaneous sign-successions, and tesselated pavements in two or more colors, with applications to Newton’s rule, ornamental tile-work, and the theory of numbers, *Phil.Mag.* 34, 461-475 (1867)

[15] W. Tadej and K. Życzkowsky, A concise guide to complex Hadamard matrices, *Open Syst.Inf.Dyn.*, 13, 133-177 (2006)

[16] G. Zeng and M. H. Lee, A generalized reverse block jacket transform, *IEEE Trans. Circuits Syst. I*, 55, 1589-1600 (2008)