Amplitude modulated Bloch oscillations of photon probability distribution in a cavity–atom system

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Abstract
We study the dynamics of the Rabi Hamiltonian in the medium coupling regime with $|g/\omega| \sim 0.07$, where $g$ is the atom-field coupling constant and $\omega$ is the field frequency, for the quantum state with average photon number $\bar{n} \sim 10^4$. We map the original Hamiltonian to an effective one, which describes a tight-binding chain subjected to a staggered linear potential. It is shown that the photon probability distribution of a Gaussian-type state exhibits amplitude-modulated Bloch oscillations (BOs), which are a superposition of two conventional BOs with a half-BO-period delay between them and are essentially another type of Bloch–Zener oscillation. The probability transition between the two BOs can be controlled and suppressed by the ratio $g\sqrt{\bar{n}}/\omega$, as well as the in-phase resonant oscillating atomic frequency $\Omega(t)$, leading to multiple zero-transition points.

Keywords: cavity quantum electrodynamics, tunneling, quantum optics

(Some figures may appear in colour only in the online journal)

1. Introduction

The interaction of matter and light is one of the fundamental processes occurring in nature. Its elemental constituent is the coupling between single atoms and photons. The quantum Rabi model considers a two-level atom coupled to a quantized field, describing the simplest interaction between quantum light and matter. It was used to describe the interaction between a rapidly varying, weak magnetic field and nuclear spin [1, 2]. Now it applies to a variety of physical systems with the two-level atom, including trapped ions, quantum dots and superconducting qubits. Such systems can be exploited as a building block for quantum information processing and other potential applications to future quantum technologies [3].

Although the quantum Rabi model seems relatively simple and was declared to be solved recently [4, 5], the dynamics are actually quite complicated, depending on the system parameters and the initial state [6]. The rotating wave approximation (RWA) is justified in the strong coupling regime (although the average photon number cannot be too large) and results in the Jaynes–Cummings (JC) model [7], which predicts non-classical phenomena, such as revivals of the initial excited state of the atom [8–10]. It is well known that an atom-cavity system undergoes Rabi oscillations. In the case of the single cavity mode being initially prepared in a coherent state with a large average photon number, although the oscillations experience the collapse and revival, the photon probability distribution remains unchanged in the framework of the JC model. The question is whether it is true if the original Rabi model is considered, and what will be observed in experiments. With strong couplings in the solid state cavity QED [11], the failure of RWA has reignited interest in the quantum Rabi model [4, 6, 12–19].

In this paper, we study the dynamics of the Rabi Hamiltonian in the medium coupling regime with $|g/\omega| \sim 0.07$, where $g$ is the atom-field coupling constant and $\omega$ is the field frequency, for the quantum state with
average photon number $\bar{n} \sim 10^4$. The effective coupling constant becomes $[g\sqrt{\bar{n}}] \sim 7\omega$, which goes beyond the RWA. Within this parameter regime, the original Hamiltonian is mapped to an effective one, which describes a uniform tight-binding chain subjected to a staggered linear potential. An approximate solution suggests that the photon number distribution of a Gaussian-type state exhibits amplitude-modulated Bloch oscillations (BOs), which are a superposition of two conventional BOs with a half-BO-period delay between them. This phenomenon essentially belongs to Bloch–Zener oscillation in the strong tunneling limit [20–25]. We find that the probability transition between the two BOs can be controlled and suppressed by the ratio $g\sqrt{\bar{n}}/\omega$, as well as the resonant oscillating atomic frequency $\Omega(t)$, leading to multiple zero-transition points. A numerical simulation of dynamics in the Rabi model confirms this prediction.

This paper is organized as follows. In section 2, we propose the effective Hamiltonian in the concerned parameter regime. In section 3, we present the approximate solution of the effective Hamiltonian, based on which the dynamics of a local state are investigated. Section 4 is dedicated to a numerical simulation in the original Rabi model, focusing on the control of the probability transition between two BOs by the atomic transition frequency. Finally, we give a summary and discussion in section 5.

2. Model and equivalent Hamiltonian

We start by considering the single-mode atom-cavity model whose Hamiltonian can be written as

$$H = \omega a^\dagger a + \frac{\Omega}{2} \hat{\sigma}_z + g \hat{\sigma}_z (a^\dagger + a),$$  \hspace{1cm} (1)

where $\omega$ and $\Omega$ are the field and atomic transition frequencies, respectively, and $g$ is the coupling constant. $a^\dagger (a)$ is the creation (annihilation) operator of the light field, while $\hat{\sigma}_z = |e\rangle \langle e| - |g\rangle \langle g|$. $\hat{\sigma}_z = |e\rangle \langle g| + |g\rangle \langle e|$ are atomic operators, where $|e\rangle$ and $|g\rangle$ denote the ground and excited atomic states, respectively. There is an approximation which has been developed, the RWA [26, 27] under the condition $g\sqrt{\bar{n}} \ll \omega$.

In this paper, we consider the case with the medium coupling regime with $|g/\omega| \sim 0.07$ and the average photon number $\bar{n} \sim 10^4$. Such a parameter regime is accessible in experiments [28–30]. The aim of this paper is to investigate the dynamics of a particular initial state with a Gaussian-type photon number distribution, which allows us to obtain the approximate analytical result from the Hamiltonian (1).

Since the Hamiltonian (1) is parity-conserving for the excitation number

$$\mathcal{N} = a^\dagger a + \frac{1}{2}(\hat{\sigma}_z + 1),$$  \hspace{1cm} (2)

it can be written in two independent equivalent Hamiltonians $H^e_{\text{eq}}$ ($H^o_{\text{eq}}$) in the bases with even (odd) excitation numbers:

$$H^\lambda_{\text{eq}} = g \sum_{i=0}^{\infty} \sqrt{i + 1} |i\rangle_\lambda \langle i + 1 + \text{H. c.}| + \frac{\Omega + i\omega}{2} |i\rangle_\lambda \langle i|, \hspace{1cm} (3)$$

where $\lambda = e, o$ and $\gamma = 1, \gamma = 0$. This is schematically illustrated in figure 1. The equivalence between $H$ and $H^\lambda_{\text{eq}}$ is based on the mapping of the corresponding basis: for the even invariant subspace we have $\{ |g, 0\rangle, |e, 1\rangle, |g, 2\rangle, |e, 3\rangle, |g, 4\rangle, \ldots \} \longrightarrow \{ |i\rangle_e \}$, where $i = 0, 1, 2, \ldots$, while for the odd invariant subspace we have $\{ |e, 0\rangle, |g, 1\rangle, |e, 2\rangle, |g, 3\rangle, |e, 4\rangle, \ldots \} \longrightarrow \{ |i\rangle_o \}$. Note that $H^\lambda_{\text{eq}}$ is a standard tight-binding chain with coordinate-dependent nearest-neighbor (NN) hopping strength and on-site potential. It is easy to find out that, within the region of large excitation-number limit $i \gg 1$, when the initial state is local near $n \sim 10^4$, we can take truncated approximation to $H^\lambda_{\text{eq}}$. In this case, the expansion of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The atom-cavity level structure used for the mapping to a tight-binding chain, and the implementation of Bloch oscillations in the photon-number space. Atom-cavity level diagram showing the lower-energy states for a two-level atom of transition frequency $\Omega$ coupled (with single-photon Rabi frequency $g$) to a single-mode cavity field of frequency $\omega$, with $\Omega = \omega$. (a) and (b) correspond to even- and odd-parity cases, respectively. The transition strengths are in units of $g$. (c) For large photon region $n \gg 1$, both even- and odd-parity cases can be mapped to a tight-binding chain with uniform NN hopping strength ($\sqrt{n + 1} \approx \sqrt{n}$) and linear potential (as represented by the gray shadow) approximately, which allows the occurrence of Bloch oscillations.}
\end{figure}
the coupling strength in equation (3) is 
\[ \sqrt{\Omega} = \sqrt{\hbar} + \frac{1 - \delta}{2\sqrt{\hbar}} + \ldots \]
We take the first term only, then \( \sqrt{\Omega} \approx \sqrt{\hbar} \); therefore, \( H_{eq}^0 \) can be approximately equivalent to a tight-binding chain with uniform NN hopping strength and staggered linear potential. In the following, we neglect the label \( \lambda \) in the effective Hamiltonian for simplicity and demonstrate that such a discrete system admits the existence of a robust BO [31–33].

3. Approximate formalism

In this section, we will introduce an approximate formalism for the approximate solution of the effective Hamiltonian and investigate the dynamical behavior as an application.

3.1. Effective Hamiltonian

We rewrite the effective Hamiltonian in the form
\[ H_{eff} = H_0 + H_1 \]  
\[ H_0 = g \sqrt{\hbar} \sum_{i=0}^{N} \left( |i\rangle \langle i+1| + H. c. \right) \]
\[ + \omega \sum_{i=0}^{N} \left( |i - N/2\rangle \langle i| \right) \]
\[ H_1 = \frac{\Omega}{2} \sum_{i=0}^{N} (-1)^i \langle i \rangle \langle i| \]
where \( \bar{n} \) is the average number of photons and can be regarded as a constant in the context of the problem we are concerned with in this paper. This Hamiltonian is equivalent to (3) for the dynamics of a local state within the region \( i \approx 1 \). Note that the Hamiltonian \( H_{eq}^0 \) is nothing but the tight-binding Hamiltonian to describe a single particle subjected to a staggered linear potential, which has been well studied in previous literature [20]. It is demonstrated that the Bloch–Zener oscillations with a single period occur in the parameter regime \( g/\omega \sim 0.2 \) and \( \Omega = 6.734\omega \). In this paper, we revisit the same model within the parameter regime \( g/\omega \sim 0.07 \) and \( \Omega = \omega \), which is accessible in experiments [28–30]. We will show that the dynamics undergo Bloch–Zener oscillations characterized by two periods.

In the absence of term \( H_1 \), it is a standard model which admits the existence of Wannier–Stark localization and Bloch oscillations, and has been studied extensively [34–37]. Now the question is: what happens to the dynamics of the model in equation (4). The basic idea is to consider the term \( H_1 \) as a perturbation. According to the theory of Wannier–Stark localization [31, 32], the solution of \( H_0 \) has the form
\[ H_0 \psi_m = E_m^0 \psi_m, \]
with
\[ E_m^0 = c m - \frac{N}{2} \omega, \]
and
\[ \psi_m = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp \left[ -i \left( mk + \frac{L}{2} \sin k \right) \right] |k\rangle \]  
\[ |k\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{isk} |i\rangle, m = 0, \pm 1, \pm 2, \ldots \] In real space, it reads
\[ \psi_m = \sum_{i=1}^{2N} J_{m-i} \left( \frac{L}{2} \right) |i\rangle. \]
where \( L = -4g\sqrt{\hbar}/\omega \) is the spatial extent of a single Bloch wave packet oscillation [32], and \( g \) is negative.

In order to examine the effect of \( H_1 \) on the dynamics of the system, it is convenient to work in the interaction picture. The propagator of the whole Hamiltonian \( H \) in the interaction picture is obtained by the unitary transformation
\[ H_1(t) = e^{iH_0 t}H_1 e^{-iH_0 t}. \]
The propagator is
\[ U^{i}(t, 0) = T \exp \left[ -i \int_0^t H_1(t') dt' \right]. \]
where \( T \) is the time order operator. In the basis of states \( \{ \psi_m \} \), one obtains
\[ \langle \psi_n | H_1(t) | \psi_m \rangle \approx e^{i(m-n)\omega t} \frac{\Omega}{2} (-1)^m J_{n-m}(L). \]
We will reduce it by the following two steps. Firstly, we can neglect rapidly oscillating terms in \( e^{i\omega nt} \) for \( n \gg 2 \). Secondly, we note that \( |J_1(x)| \ll |J_0(x)| \) when \( x \) is around the zeros of \( J_1(x) \), \( J_1(x_0) = 0 \). Then when the parameter \( L \approx x_0, \) with \( x_0 = 2.41, 5.53, \ldots \), one can drop the term \( \langle \psi_m | H_1(t) | \psi_{m+1} \rangle \).

Therefore the approximation and the feature of the Bessel function lead to the following approximate expression
\[ \langle \psi_m | H_1(t) | \psi_m \rangle \approx \frac{\Omega}{2} (-1)^m J_0(L) \delta_{mn}, \]
i.e., state \( \psi_m \) is the simultaneous eigenstate of the time-dependent Hamiltonian \( H_1(t) \). Then in the basis of states \( \{ \psi_m \} \), the propagator in equation (13) is reduced to
\[ U^{i}_{mn}(t, 0) \approx \delta_{mn} \exp \left[ -i \int_0^t \langle \psi_n | H_1(t') | \psi_m \rangle dt' \right]. \]
The corresponding propagator \( U(t, 0) \) in Schrödinger picture is in the form
\[ U_{mn}(t, 0) \approx \delta_{mn} \exp \left[ -i \left[ m\omega + (-1)^m \gamma \right] t \right], \]
where
\[ \gamma = \frac{\Omega}{2} J_0(L). \]
3.2. Dynamics

To clarify the feature of the dynamics, we consider the time evolution of an arbitrary state. We note that any state can be decomposed into even- and odd-parity portions, which are spanned by the state $|\psi_n\rangle$ with even or odd $n$, respectively. It shows that the parity is conserved during the time evolution under an arbitrary (either odd or even) $H$ within the approximate framework. We see that the even- and odd-parity parts of the quantum state evolve independently. Furthermore, the appearance of the term $H$ only contributes an overall phase on each parts. It is presumable that the probability of the evolved state is determined by the factor $\gamma$ in a simple manner. It plays a central role in the emerging interference phenomena of two subwaves with different parity.

Actually, for a given initial state
\[ |\psi(0)\rangle = \sum_n f_n |\psi_n\rangle, \tag{19} \]
we have
\[ |\psi(t)\rangle = U(t,0) |\psi(0)\rangle = e^{-iHt} \sum_n f_n |\psi_n\rangle. \tag{20} \]
In order to demonstrate the physical picture of state $|\psi(t)\rangle$, we consider the time evolution of the same initial state under the free Hamiltonian $H_0$, which can be written as
\[ |\psi_0(t)\rangle = e^{-iH_0t} |\psi(0)\rangle = \sum_n e^{-i\omega_nt} f_n |\psi_n\rangle. \tag{21} \]
It is revealed that $|\psi_0(t)\rangle$ exhibits standard Bloch oscillation with period $T_B = 2\pi/\omega$ for a local initial state. Furthermore, a straightforward derivation shows that
\[ |\psi(t)\rangle = \cos(\gamma t) |\psi_0(t)\rangle - \sin(\gamma t) |\psi_0(t + T_B/2)\rangle. \tag{22} \]
It shows that the state $|\psi(t)\rangle$ is regarded as the superposition of two evolved states under the Hamiltonian $H_0$. In order to get a physical picture of the phenomenon, we introduce an operator $\hat{B}_x$ which relates two states $|\psi_0(t)\rangle$ and $|\psi_0(t + T_B/2)\rangle$ in the following way
\[ \hat{B}_x |\psi_0(t)\rangle = |\psi_0(t + T_B/2)\rangle. \tag{23} \]
Obviously, $\hat{B}_x$ can be expressed in terms of $H_0$ as
\[ \hat{B}_x = e^{-iH_0T_B/2}. \tag{24} \]
The physical meaning of operator $B_x$ becomes clear if we apply it to the state $|k\rangle$. A direct derivation shows that
\[ \hat{B}_x |k\rangle = \exp[iL \sin k] |k + \pi\rangle. \tag{25} \]
It indicates that the operator $\hat{B}_x$ is nothing but the $\pi$-boost operator that shifts all momentum states by $\pi$ with an extra phase factor. Then if we consider the initial state as a Gaussian wave packet $|\phi(n_0, k_0)\rangle$ with momentum $k_0$ and the center position $n_0$, approximately, it evolves as
\[ |\psi(t)\rangle = \cos(\gamma t) e^{-iH_0t} |\phi(n_0, k_0)\rangle - \sin(\gamma t) e^{-iH_0t} \exp[iL \sin k_0] \times |\phi(n_0 + n_x, k_0 + \pi)\rangle \tag{26} \]
where $n_x = -L \cos k_0$. Here the Gaussian wave packet has the form
\[ |\phi(n_0, k_0)\rangle = \frac{1}{\sqrt{R}} \sum_{n=0}^{\infty} e^{-\frac{1}{2}(n-n_0)^2} e^{i\alpha n} |n\rangle, \tag{27} \]
where $R$ is the normalization factor and $\alpha$ determines the half-width of the wave packet, $\Delta = 2\sqrt{2} \ln 2/\alpha$ in the case of $\alpha \ll 1$. We can see that the evolved state represents the superposition of Bloch oscillations of two wave packets with an $n_x$-shift in position and a $\pi$-shift in momentum. The amplitude of each oscillation is modulated with sinusoidal time dependence.

Particularly, we see that in the case of the parameters satisfying the equation
\[ \gamma n T_B = \frac{\pi}{2}, \quad (n \in \mathbb{N}) \tag{28} \]
the wave packet $|\phi(n_0, k_0)\rangle$ is separated as two equal-probability ones, $|\phi(n_0, k_0)\rangle$ and its counterpart $|\phi(n_0 + n_x, k_0 + \pi)\rangle$, at instant $t = \frac{T_B}{2}$. On the other hand, we see that $|\psi(t)\rangle$ and $|\psi_0(t)\rangle$ coalesce with each other at instants $t = t_n$, where
\[ t_n = \frac{n\pi}{\gamma}, \quad (n \in \mathbb{N}). \tag{29} \]
To exemplify these features, we give the center positions $n_a(t)$, $n_b(t)$ and corresponding probabilities $P_a(t)$, $P_b(t)$ of two wave packets as a function of time for the initial state $|\psi(0)\rangle = |\phi(n_0, 0)\rangle$ as the form
\[ n_a(t) = n_b + L \cos(\alpha t) = n_0 - L \sin^2(\alpha t/2) \tag{30} \]
and
\[ P_a(t) = 1 - P_b(t) = \cos^2(\gamma t) \tag{31} \]
which is evolved in the Hamiltonian $H_{\text{eff}}$ in equation (4) with constant $\Omega$. However, it is worthy to point out that the above conclusions are obtained within the approximate framework, which we will see from the following analysis. One can consider the mechanism of the modulated BOs in an alternative way. For a wave packet $|\phi(n_0, k_0)\rangle$ at a certain location, the action of $H_0$ is to drive it to evolve as a Bloch oscillation, while the action of $H_1$ is to make a transition from $|\phi(n_0, k_0)\rangle$ to its counterpart, i.e.,
\[ H_1 |\phi(n_0, k_0)\rangle \rightarrow |\phi(n_0, k_0 + \pi)\rangle. \tag{32} \]
Meanwhile, the energy difference between two wave packets

\[
\Delta E = \frac{\Omega}{2} \left\{ \phi(n_0, k_0) \right\} H_0 \left\{ \phi(n_0, k_0) \right\} - \frac{\Omega}{2} \left\{ \phi(n_0, k_0 + \pi) \right\} H_0 \left\{ \phi(n_0, k_0 + \pi) \right\} = 2\Omega g \sqrt{\Delta} \cos k_0
\]

suppresses this process. It indicates that only under the condition \( k_0 \approx \pm \pi/2 \), such a transition has a higher probability to occur due to the fact that \( \Delta E=0 \). We note that the condition \( k_0 \approx \pm \pi/2 \) is satisfied when \( e^{-iH_{\text{eff}}t} \left\{ \phi(n_0, k_0) \right\} \) and \( e^{-iH_{\text{eff}}t} \left\{ \phi(n_0, k_0 + \pi) \right\} \) overlap with each other in real space. The underlying mechanism of this phenomenon is that \( H_1 \) is a local interaction operator. To demonstrate this point, numerical simulation is performed by exact diagonalization for the truncated matrix of the Hamiltonian \( H_{\text{eq}} \) rather than \( H_{\text{eff}} \). The probabilities \( P_a(t) \) and \( P_b(t) \) are computed by \( \left\{ \psi(0) e^{iH_{\text{eq}}t} e^{-iH_{\text{eff}}t} \left\{ \psi(0) \right\} \right\}^2 \) and \( \left\{ \psi(0) e^{iH_{\text{eq}}(t+T/2)} e^{-iH_{\text{eff}}t} \left\{ \psi(0) \right\} \right\}^2 \), respectively. In Figure 2 we plot the center positions and the probability distributions of two evolved wave packets \( e^{-iH_{\text{eff}}t} \left\{ \psi(0) \right\} \) and \( e^{-iH_{\text{eff}}(t+T/2)} \left\{ \psi(0) \right\} \), obtained by analytical expression in equations (30) and (31), and the numerical simulation.

It shows that the probability distributions are step-like as functions of time and the probability exchange happens when two wave packets meet together. When they meet each other for the first time, the splitting of a Gaussian wave packet can be observed. It also indicates that our analytical results, equations (30) and (31), are the time-averaged approximation for the probability transition.

A Gaussian wave packet with the center position \( \bar{n} \) and the half-width \( \Delta = 2\sqrt{2} \ln 2/\alpha = 23.5 \), considering the evolution which has been discussed above, can be regarded as

\[ J_0(x) \]

being local within a spatial extent, \( D = L + 6\alpha \), where \( L = -4 g \sqrt{n}/\omega \). When \( n \sim 10^4 \), \( L \sim 28 \), \( D < 100 \), now we let \( D = 100 \); in this case, if we want to satisfy \( \left| \sqrt{i} - \sqrt{n} \right| / \sqrt{n} < 5\% \) for \( i \in D \), \( n \sim 10^3 \) is mandatory at least.

In the next section, the validity of the prediction will be investigated in the Rabi system by numerical simulation. We will quantitatively evaluate the extent of approximation of the above analysis.

4. Controllable dynamics of cavity-atom system

Now we apply the obtained results to a concrete case and then demonstrate the dynamic property of the system. We investigate the time evolution of the wave packet in the single-mode atom-cavity model. As mentioned above, the dynamics of the local state in a large excitation number region are equivalent to those of the Hamiltonian in equation (4). We will show that the modulation of the Bloch oscillations can be controlled by the coupling constant and the energy level of the atom, which can be adjusted via the external field.

As an application of the obtained result, let us take a simple case as an example. Without loss of generality, we assume that the initial state is in the form

\[ |\psi(0)\rangle = \frac{1}{\sqrt{2}} \left( |g\rangle + \sum_n e^{-\frac{x^2}{\alpha^2}} |n\rangle \right). \]

Transforming the basis \{ |g, n\rangle, |e, n\rangle \} to \{ |\bar{r}\rangle \} and \{ |\bar{l}\rangle \}, we have

\[ |\psi(0)\rangle = \frac{1}{\sqrt{2}} \sum_{\bar{r}} \sum_{\bar{l}=0} e^{-\frac{x^2}{\alpha^2}} |\bar{l}\rangle |\bar{r}\rangle, \]

which corresponds to the superposition of two independent stationary wave packets in two chains \( H_{\text{eff}} \) with \( \lambda = e, o \), respectively. According to the above analysis, this state
evolves to
\[\psi(\gamma) = \sum_{\gamma} \psi(\theta) \cos(\gamma t) + \sin(\gamma t) \in (\mu, n) \right\}
+ \left[ \psi(\theta) - i \psi(\theta) \sin(\gamma t) \right] \in (e, n) \right\}, \quad (36)
\]
where
\[\theta = \exp \left[ -i \sigma \left( \frac{\alpha^2}{2} (n - n_t) \right)^2 \right]. \quad (37)
\]
\[\mu = -\frac{L}{2} \sin(\omega t), \quad (38)
\]
\[n_t = n - L \sin^2 \left( \frac{\omega t}{2} \right). \quad (39)
\]

We focus on the evaluation of the photon number distribution \(P(n, t)\), characterizing the dynamics of the state. A straightforward derivation shows that
\[
P(n, t) = \left| \langle g, n | \psi(t) \rangle \right|^2 + \left| \langle e, n | \psi(t) \rangle \right|^2
= \frac{1}{R} \left| \theta_i \right|^2 \cos^2(\gamma t) + \left| \theta_i + \gamma \theta_i \right|^2 \sin^2(\gamma t) \right\} \quad (40)
\]
where \(\theta_i\) has the explicit form
\[\left| \theta_i \right|^2 = \exp \left[ -a^2 (n - n_t)^2 \right]. \quad (41)
\]
which represents a Gaussian distribution with time-dependent center \(n_t\). It is shown that the photon number distribution \(P(n, t)\) exhibits amplitude-modulated BOs.

We note that \(P(n, t)\) is directly determined by the factor \(\gamma\) in equation (18), which depends on the ratio \(g \sqrt{n}/\omega\). The numerical method is employed to simulate the time evolution process. It is performed by exact diagonalization of the Hamiltonian with truncated approximation. We are interested in the suppression and process of the probability transition. In the following, we will investigate the cases with two types of parameters: (i) adjusting the ratio \(g \sqrt{n}/\omega\) for the zero-transition point, and (ii) in-phase resonant oscillating \(\Omega(t)\).

4.1. Constant \(\Omega\)
According to the approximate analysis, when we take \(g \sqrt{n}/\omega\) satisfying \(J_0(L) = 0\), the transition between two BOs can be frozen. However, in practice a deviation may occur with respect to the approximate solution. We compute the time evolution and search the zero-transition point for the initial state in the form of equation (34). We consider four typical cases with \(L = 24.31, 25.73, 27.50\) and 28.89. From the plot of the function \(J_0(x)\) in figure (3), we can see that two of the four points are located in the vicinity of the zeros of the Bessel function, while the others are located at the stationary points. In figure 4, the photon number distributions \(P(n, t)\) for these parameters are plotted. The corresponding analytical results \(n_t\) and \(n_t + \gamma \theta_i\) in equation (39) are also added on the plots of \(P(n, t)\) as a comparison and guide.

4.2. In-phase resonant oscillating \(\Omega(t)\)
We consider the resonant \(\Omega(t)\), which has the half-period as the Bloch oscillations. To clarify the action of \(\Omega(t)\), we take a
rectangular wave as

\[ \Omega(t) = \begin{cases} \omega, & \frac{1}{2} (nT - T_i) < |t - \varphi_0| \leq \frac{1}{2} (nT + T_i), \\ 0, & (n = 1, 2, 3 \ldots) \\ \varphi_0, & \text{otherwise} \end{cases} \]  

(42)

with \( T = T_B, T_i = 0.1 \) T. Here \( \varphi_0 \) controls the phase between \( \Omega(t) \) and the Bloch oscillations. We consider two typical cases with \( \varphi_0 = 0 \) and \( T/4 \), which are in phase and out of phase with the Bloch oscillations, respectively.

Similarly, we can also take a sinusoidal wave as

\[ \Omega(t) = 0.5 \left( 1 + \cos \left( 2\omega \left( t + \varphi_0 \right) \right) \right) \omega, \]  

(43)

with \( \varphi_0 = 0 \) and \( T/4 \), respectively. Here we use a uniform mesh in the time discretization for a time-dependent Hamiltonian, i.e.,

\[ \psi(t) = \exp \left[ -i \int_0^t H(t')dt' \right] \psi(0). \]  

(44)

The profiles of the evolutions with different types of \( \Omega(t) \) are plotted in figure 5. The numerical result accords with our prediction: in the case of in-phase \( \Omega(t) \), the wave dynamics are similar to the case with constant \( \Omega \), while in the case of out-of-phase \( \Omega(t) \), they are similar to the case with zero \( \Omega \).

5. Summary

In summary, we have studied the dynamics of the Rabi Hamiltonian, identifying a parameter regime corresponding to amplitude-modulated BOs, another type of Bloch–Zener oscillations. A periodic phenomenon for this well-
known model when the coupling constant goes beyond the RWA is presented for the first time. It also reveals a significant effect on the field: Bloch–Zener oscillations of the photon probability distribution with a distinct amplitude. It is remarkable that such a macroscopic phenomenon is induced by a single atom. Moreover, the probability transition between the two BOs can be controlled and suppressed by the ratio $\frac{\omega_{\text{gn}}}{\bar{\omega}}$, as well as the in-phase resonant oscillating atomic frequency $\Omega(t)$, leading to multiple zero-transition points. The numerical simulation of dynamics in the Rabi model confirms this prediction. However, the experimental observation of this prediction requires a coherent time scale of $\frac{\omega_{\text{gn}}}{\bar{\omega}}$, which is still a challenge so far. At the same time, with such a high number of photons in a quantum resonator, dissipation may also have a role.

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