MODEL THEORY OF VALUED FIELDS

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Abstract. We give a proposal for future development of the model theory of valued fields. We also summarize recent results on $p$-adic numbers.

1. Valued fields

Let $K$ be a valued field with a valuation map $v : K \to G \cup \{\infty\}$ to an ordered group $G$; this is a map satisfying

(i) $v(x) = \infty$ if and only if $x = 0$;
(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in K$;
(iii) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$.

We write $R$ for the valuation ring $\{x \in K \mid v(x) \geq 0\}$ of $K$, $M$ for its unique maximal ideal $\{x \in K \mid v(x) > 0\}$ and we write $k$ for the residue field $R/M$ and $\bar{\cdot} : R \to k : x \mapsto \bar{x}$ for the natural projection. We call $K$ a Henselian valued field if $R$ is a Henselian valuation ring.

A valued field often carries an angular component map modulo $M$, or angular component map for short; it is a group homomorphism $ac : K \times \to k \times$, extended by putting $ac(0) = 0$, and satisfying $ac(x) = \bar{x}$ for all $x$ with $v(x) = 0$ (see [18]).

2. Different languages

The model theory of valued fields can be studied at different levels of complexity. One can use the most basic language to study fields, the language $L_{\text{ring}}$ of rings, and for some valued fields (like the $p$-adic numbers) the relation $v(x) \leq v(y)$ is already definable in this language. In general, this relation is not automatically definable (like in algebraically closed valued fields). It is very natural to add a divisibility predicate, or even more conveniently, a restricted division function $D$ as follows

$$D : K^2 \to K^2 : (x, y) \mapsto \begin{cases} x/y & \text{if } v(x) \geq v(y), y \neq 0; \\ 0 & \text{else.} \end{cases}$$

Let us call $L_D$ the language $L_{\text{ring}}$ together with $D$. Algebraically closed valued fields have quantifier elimination in $L_D$, see [19]. It is also convenient to add unary predicates $P_n$ to $L_{\text{ring}}$, corresponding to the set of $n$-th powers in $K^\times = K \setminus \{0\}$; one thus obtains the language of Macintyre $L_{\text{Mac}}$. Macintyre [15] proved that $p$-adic fields have quantifier elimination$^2$ in $L_{\text{Mac}}$. It is less known that there are many other valued fields which have quantifier elimination in $L_{\text{Mac}}$, like for example some fields of (formal) Laurent series, $\mathbb{R}((t))$, $\mathbb{C}((t))$, $\mathbb{Q}_p((t))$, and fields of

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$^1$Here, an ordered group is a totally ordered non-trivial abelian group $G$ such that $x < y$ implies $x + z < y + z$ for all $x, y, z \in G$.

$^2$we always allow parameters in formula’s; thus by quantifier elimination we mean that every formula is equivalent to a quantifier free formula possibly containing parameters.
iterated Laurent series like \( \mathbb{R}((t_1))((t_2)) \), and so on. This follows from quantifier elimination results of F. Delon. Delon adds more predicates (variants of the \( P_n \) predicates) and obtains quantifier elimination for a big class of valued fields; this language \( \mathcal{L}_{\text{Delon}} \) reduces to \( \mathcal{L}_{\text{Mac}} \) for the above mentioned fields. Let us denote \( \mathcal{L}_{\text{Mac},D} \) for the language \( \mathcal{L}_{\text{Mac}} \) together with restricted division \( D \).

One can also add extra sorts to study valued fields, like the value group, and the residue field. This way, one obtains languages of Denef - Pas. More precisely, let \( \mathcal{L}_k \) be an arbitrary expansion of \( \mathcal{L}_{\text{ring}} \) and let \( \mathcal{L}_G \) be an arbitrary expansion of the language of ordered groups with infinity \( (+, -, 0, \infty, \leq) \). A language of Denef - Pas is a three-sorted language of the form \( (\mathcal{L}_k, \mathcal{L}_{\text{ring}}, \mathcal{L}_G, v, ac) \) with as sorts:

(i) a residue field-sort, with the language \( \mathcal{L}_k \),
(ii) a valued field-sort, with the language \( \mathcal{L}_{\text{ring}} \), and
(iii) a value group-sort, with the language \( \mathcal{L}_G \).

The function symbol \( v \) stands for the valuation map and \( ac \) stands for an angular component map.

Pas obtains quantifier elimination for the valued-field-variables for a very general class of fields, namely, for all Henselian valued fields \( K \) with an angular component map and with \( \text{char } k = 0 \) (and also in other cases).

The idea of these different languages is to be able to study more and more complex kinds of valued fields, but of course also the geometry of the definable sets gets more complex. In the following sections we will define several notions of minimality with respect to different languages.

### 3. Basic open problems

In this section, we let \( K \) be a valued field which is not real closed, nor algebraically closed. For the model theory of real closed fields with a valuation, we refer to [11].

**Open Problem 1.** Suppose that \( K \) has quantifier elimination in \( \mathcal{L}_{\text{Mac},D} \), does it follow that \( R \) is Henselian? If not, under which extra conditions does this follow?

Similar question can also be posed for languages of Denef - Pas instead of \( \mathcal{L}_{\text{Mac},D} \). One can put it even more challenging: does Henselianess of \( R \) follow if we only know that the definable subsets of \( K \) are quantifier free definable in \( \mathcal{L}_{\text{Mac},D} \)? Question [11] is a generalization of an open problem mentioned in [10], where the analogue question is asked when \( K \) is a \( p \)-valued field.

**Open Problem 2** (Same assumptions as in problem [11].) Does it follow that the indices \( [K^x : P_n] \) are finite for \( n > 1 \)?

**Open Problem 3** (Same assumptions as in problem [11].) Does it follow that \( R \) is already \( \mathcal{L}_{\text{Mac}} \)-definable, and hence, that \( \mathcal{L}_{\text{Mac},D} \) is an expansion by definition of \( \mathcal{L}_{\text{Mac}} \)?

- Under which conditions does it follow that \( G \) is a \( \mathbb{Z} \)-group\(^3\), or that \( G \) is elementary equivalent to \( \mathbb{Z}^n \) with lexicographical order\(^4\)?
- Does it follow that \( K \) has characteristic zero?

\(^3\)By this we mean that \( G \) is elementary equivalent to \( \mathbb{Z} \)
\(^4\)This is the case for iterated Laurent series fields.
Also under the extra condition that the value group $G$ is a $\mathbb{Z}$-group (or elementary equivalent to $\mathbb{Z}^n$ with lexicographical order) the above questions are worth looking at. An example of a valued field with finite residue field and value group $\mathbb{Z}^2$ with lexicographical order is $\mathbb{Q}_p((t))$.

4. A GENERAL NOTION OF P-MINIMAL FIELDS

In view of the above questions, it is natural to give a definition of the following kind for $K$ a valued field which is not algebraically closed, nor real closed.

Definition 1. Let $\mathcal{L}$ be an expansion of $\mathcal{L}_{\text{Mac,D}}$ and suppose that each $\mathcal{L}$-definable subset of $K$ is already quantifier free definable in $\mathcal{L}_{\text{Mac}}$. Then we call $(K, \mathcal{L})$ general $p$-minimal (where the $p$ of $p$-minimal comes from the powersets $P_n$). A theory of valued fields in the language $\mathcal{L}$ is general $p$-minimal if each of its models is.

If $K$ is a $p$-adically closed field, this corresponds to the notion of $p$-minimality by Haskell and Macpherson in [12]. We can, of course, define analogously a notion of $\text{Pas}$-minimality:

Definition 2. Let $\mathcal{L}_{\text{Pas}}$ be a language of Denef - Pas and let $\mathcal{L}$ be an expansion of $\mathcal{L}_{\text{Pas}}$ and suppose that each $\mathcal{L}$-definable subset of $K$ is already quantifier free definable in $\mathcal{L}_{\text{Pas}}$, then we call $(K, \mathcal{L})$ $\text{Pas}$-minimal.

On the $p$-adic numbers with either semialgebraic or subanalytic structure, a cell decomposition theorem holds (see theorem 1 below, and also [7] and [2]). In a weaker formulation, Mourgues [17] proves a cell decomposition theorem for any $p$-minimal structure on $\mathbb{Q}_p$ having definable Skolem functions. Mourgues’ decomposition is weaker in the sense that it only gives a partition of definable sets into cells, and not a preparation of definable functions (see theorem 1 below for preparations of definable functions). To obtain cell decomposition à la Mourgues, definable Skolem functions are really needed. Not all general $p$-minimal fields have definable Skolem functions, like for example $\mathbb{C}((t))$; for $p$-minimal structures on $\mathbb{Q}_p$ this is an open question. Languages of Denef - Pas are very robust even if Skolem functions are not definable; the residue field sort often replaces the need of definable Skolem functions.

It is natural to ask the following questions:

Open Problem 4. • Can one put on any general $p$-minimal field a $p$-minimal analytic structure (as in section 6 for $\mathbb{Q}_p$) yielding subanalytic sets?  

• Can one obtain cell decomposition (with preparation of definable functions) for subanalytic sets on a general $p$-minimal field with definable Skolem functions?  

• Do general $p$-minimal fields with definable Skolem functions have cell decomposition (with or without preparation)?  

Similar questions can be put for $\text{Pas}$-minimal fields and other variants.

5This is currently under development in [3].
5. TAME P-MINIMAL FIELDS

**Definition 3** (Temporary definition in view of open problems 1, 2, and 3). Let $K$ be a valued field of characteristic zero such that $G$ is elementary equivalent to $\mathbb{Z}^n$ with lexicographical order. Let $\mathcal{L}$ be an expansion of $\mathcal{L}_{\text{Mac}, P}$. Suppose that $(K, \mathcal{L})$ is a general p-minimal field, that $R$ is Henselian and that the indices $[K^\times : P_n]$ are finite for each $n > 1$. Then we call $K$ a tame p-minimal field. We call a theory in the language $\mathcal{L}$ tame p-minimal if each of its models is.

**Remark 1.** If the above problems 1, 2, and 3 can be solved positively, any general p-minimal field would be tame p-minimal.

**Example 1.** An easy example of a tame p-minimal field is $\mathbb{R}((t))$, the field of Laurent series over $\mathbb{R}$. If we fix a parameter corresponding to $t$, then this field has definable Skolem functions, because an ordering can be defined on $\mathbb{R}((t))$ using cosets of the squares. Similar results hold for $\mathbb{R}((t_1))((t_2))$ and so on. Other basic examples are $\mathbb{Q}_p$, $\mathbb{C}((t))$, and $\mathbb{Q}_p((t))$, and fields of iterated (formal) Laurent series over $\mathbb{C}$, over $\mathbb{Q}_p$, and over finite field extensions of $\mathbb{Q}_p$. This follows from the quantifier elimination results of Delon.

In [3] and [4], criterions are given for the existence of a $\mathcal{L}_{\text{ring}}$-definable bijection $K \to K^\times$ for a tame p-minimal field $K$; it is a sufficient condition that the residue field $k$ is finite (like for $\mathbb{Q}_p((t))$, $\mathbb{F}_p((t))(s)$ and so on).

I think it would be very interesting to develop the model theory of tame p-minimal fields and general p-minimal fields. For example, one can look for a generalization of the notion of a $p$-adic cell (see section 5), which can be used for all tame p-minimal fields to obtain cell decomposition. The notion of $p$-adic cells itself might be too strict: in general, the residue field is not finite and therefore one might need cells of a more general kind. Something in the style of the following subsets of $K$ might be needed as cells:

\begin{equation}
\{ x \in K \mid \wedge_{i,j,r} v(a_{ijr}) \triangleleft v(x - c_i) < v(x - c_j), \ x - c_j \in \lambda_j P_{n_j} \},
\end{equation}

where $a_{ijr} \in K^\times$, $c_i, c_j, \lambda_j \in K$, and $\triangleleft$ and $<_{ij}$ are either $<$, $>$, $=$, or no condition, and the conjunction is finite.

**Open Problem 5.** Let $K$ be a tame p-minimal field. Does it follow that $K$ has only finitely many algebraic field extensions of a given finite degree?

6. FIELDS OF $p$-ADIC NUMBERS

In this section we let $K$ be a finite field extension of $\mathbb{Q}_p$ with valuation ring $R$ and $| \cdot |$ is the $p$-adic norm. We put on $K$ the structure of subanalytic sets as follows. For $X = (X_1, \ldots, X_m)$ let $R(X)$ be the ring of restricted power series over $R$ in the variables $X$; it is the ring of power series $\sum a_i X^i$ in $R[[X]]$ such that $|a_i|$ tends to $0$ as $|i| \to \infty$. (Here, we use the multi-index notation where $i = (i_1, \ldots, i_m)$, $|i| = i_1 + \ldots + i_m$ and $X^i = X_1^{i_1} \ldots X_m^{i_m}$.) For $x \in R^m$ and $f = \sum a_i X^i$ in $R(X)$ the series $\sum a_i x^i$ converges to a limit in $K$, thus, one can associate to $f$ a **restricted analytic function** given by

\[ f : K^m \to K : x \mapsto \begin{cases} \sum_i a_i x^i & \text{if } x \in R^m, \\ 0 & \text{else.} \end{cases} \]

We let $\mathcal{L}_{\text{sub}}$ be the language $\mathcal{L}_{\text{Mac}}$ together with a function symbol $f$ for each $f \in R(X)$, with the interpretation as restricted analytic function. A set $X \subset K^m$
is called (globally) subanalytic if it is \( L_{\text{sub}} \)-definable. We call a function \( g : A \to B \) subanalytic if its graph is a subanalytic set. We refer to \cite{10, 12, 13} and \cite{2} for the theory of subanalytic \( p \)-adic geometry and to \cite{14} for the theory of rigid subanalytic sets.

\( P \)-adic cell decomposition makes use of basic sets called cells, which we define inductively. For \( m, l > 0 \) we write \( \pi_m : K^{m+l} \to K^m \) for the linear projection on the first \( m \) variables, and, for \( A \subset K^{m+l} \) and \( x \in \pi_m(A) \), we write \( A_x \) for the fiber \( \{ t \in K^l \mid (x, t) \in A \} \).

Let \( \mathcal{L} \) be an expansion of the language \( L_{\text{Mac}} \), then we define \( \mathcal{L} \)-cells.

**Definition 4.** A cell \( A \subset K \) is a (nonempty) set of the form

\[
\{ t \in K \mid |a| \square_1 |t - \gamma| \square_2 |\beta|, \ t - \gamma \in \lambda \mathcal{P}_n \},
\]

with constants \( n > 0, \gamma, \lambda \in K, \alpha, \beta \in K^\times \), and \( \square_i \) either \(<\) or no condition. If \( \lambda = 0 \) we call \( A \) a \((0)\)-cell and if \( \lambda \neq 0 \), we call \( A \) a \((1)\)-cell.

A \( \mathcal{L} \)-cell \( A \subset K^{m+1} \) is a set of the form

\[
\{ (x, t) \in K^{m+1} \mid x \in D, \, |a(x)| \square_1 |t - \gamma(x)| \square_2 |\beta(x)|, \ t - \gamma(x) \in \lambda \mathcal{P}_n \},
\]

with \( (x, t) = (x_1, \ldots, x_m, t) \), \( n > 0, \lambda \in K, D = \pi_m(A) \) a cell, \( \mathcal{L} \)-definable functions \( \alpha, \beta : K^m \to K^\times \) and \( \gamma : K^m \to K \), and \( \square_i \) either \(<\) or no condition. If \( D \) is a \((i_1, \ldots, i_m)\)-cell and \( \lambda = 0 \) we call \( A \) a \((i_1, \ldots, i_m, 0)\)-cell, and if \( \lambda \neq 0 \) we call \( A \) a \((i_1, \ldots, i_m, 1)\)-cell. Further, we call \( \gamma \) the center of the cell \( A \) and we call \( \lambda \mathcal{P}_n \) the coset of \( A \).

**Remark 2.**

- If \( \mathcal{L} = L_{\text{Mac}} \), we speak of semialgebraic cells; if \( \mathcal{L} = L_{\text{sub}} \), we speak of subanalytic cells. By induction we can call \( A \) an analytic cell if all functions \( \alpha, \beta, \gamma \) are analytic on \( D \) and \( D \) is an analytic cell.
- An analytic \((i_1, \ldots, i_m)\)-cell is a \((\sum_j i_j)\)-dimensional \( p \)-adic manifold.
- A cell \( A \subset K^{m+1} \) is either the graph of a subanalytic function defined on \( \pi_m(A) \) (if \( \lambda = 0 \)), or, for each \( x \in \pi_m(A) \), the fiber \( A_x \subset K \) contains a nonempty open.

Theorem 1 below gives a cell decomposition for subanalytic sets and for semi-algebraic sets into analytic cells, and at the same time it gives a preparation of definable functions. In \cite{9}, an overview is given of applications of the semialgebraic cell decomposition.

**Theorem 1** \((p\text{-adic cell decomposition})\). Let \( \mathcal{L} \) be either the language \( L_{\text{Mac}} \) or \( L_{\text{sub}} \). Let \( X \subset K^{m+1} \) be a \( \mathcal{L} \)-definable set, \( m \geq 0 \), and \( f_j : X \to K \) \( \mathcal{L} \)-definable functions for \( j = 1, \ldots, r \). Then there exists a finite partition of \( X \) into analytic \( \mathcal{L} \)-cells \( A \) with center \( \gamma : K^m \to K \) and with coset \( \lambda \mathcal{P}_n \) such that for each \( (x, t) \in A \)

\[
|f_j(x, t)| = |h_j(x)| \cdot |(t - \gamma(x))^{a_j} \lambda^{-a_j}|^{\frac{1}{p}},
\]

for each \( j = 1, \ldots, r \), with \( (x, t) = (x_1, \ldots, x_m, t) \), integers \( a_j \), and \( h_j : K^m \to K \) \( \mathcal{L} \)-definable functions, analytic on \( \pi_m(A) \). If \( \lambda = 0 \), we put \( a_j = 0 \), and we use the convention \( 0^0 = 1 \).

The semialgebraic case is due to Denef \cite{17} and is stated in this form in \cite{11}. The subanalytic case is recently proven by the author in \cite{2}. The fact that we can take the cells and the functions \( h_j \) to be analytic follows from the fact that all semialgebraic and subanalytic functions are piecewise analytic, see \cite{10}. Namely,
this fact can be used to partition a given cell $A$ further into analytic cells with the same center on which the restrictions of $h_j$ are analytic.

The proof of Thm. $L$ when $L = \mathcal{L}_{\text{sub}}$ uses compactness arguments and recent results of $[13]$. Once one knows this cell decomposition, it follows that $(K, \mathcal{L}_{\text{Mac}})$ is p-minimal (although the subanalytic cell decomposition actually relies on this fact, proven in $[13]$). Recently, it has been proven by D. Haskell and the author (unpublished notes) that if one adds a nontrivial entire analytic function to $\mathcal{L}\text{Mac}$, the obtained structure on $\mathbb{Q}_p$ is not p-minimal anymore.

An important use of p-adic cell decomposition is to calculate p-adic integrals, see e.g. $[8]$. We explain this into some detail.

**Definition 5.** Let $L$ be either the language $\mathcal{L}\text{Mac}$ or $\mathcal{L}_{\text{sub}}$. The algebra $\mathcal{C}_L(K^m)$ of ($L$-definable) constructible functions on $K^m$ is the $\mathbb{Q}$-algebra generated by the functions $K^m \to \mathbb{Q}$ of the form $x \mapsto v(h(x))$ and $x \mapsto |h'(x)|$ where $h : K^m \to K^\times$ and $h' : K^m \to K$ are $L$-definable.

To any function $f \in \mathcal{C}_L(K^{m+n})$, $m, n \geq 0$, we associate a function $I_m(f) : K^m \to \mathbb{Q}$ by putting

$$I_m(f)(\lambda) = \int_{K^n} f(\lambda, x) |dx|$$

if the function $x \mapsto f(\lambda, x)$ is absolutely integrable for all $\lambda \in K^m$, and by putting $I_m(f)(\lambda) = 0$ otherwise. Of course, $|dx|$ stands here for a Haar measure.

Constructible functions often appear naturally in number theory, for example, local singular series, introduced by A. Weil, are constructible (see $[9]$ and $[2]$).

**Theorem 2** (Basic Theorem on p-adic Analytic Integrals). For any function $f \in \mathcal{C}_L(K^{m+n})$, the function $I_m(f)$ is in $\mathcal{C}_L(K^m)$.

The semialgebraic case of Thm. 2 is proven in $[9]$, the subanalytic case in $[2]$. The following questions are natural to ask.

**Open Problem 6.**

- Does any p-minimal structure on $\mathbb{Q}_p$ have definable Skolem functions?
- Does any p-minimal structure on $\mathbb{Q}_p$ allow a cell decomposition with preparation of definable functions?
- Apart from the semialgebraic and subanalytic structures on $\mathbb{Q}_p$, can one find other p-minimal structures on $\mathbb{Q}_p$?

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