Regularity of foliations and Lyapunov exponents of partially hyperbolic dynamics on 3-torus

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Abstract
In this work we study the relation between the regularity of invariant foliations and the Lyapunov exponents of partially hyperbolic diffeomorphisms. We suggest a new regularity condition for foliations in terms of disintegration of the Lebesgue measure which can be considered to be a criterium for the rigidity of Lyapunov exponents.

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1. Introduction
In this paper we address the regularity of invariant foliations of partially hyperbolic dynamics and its relations to Lyapunov exponents and rigidity. We suggest a new regularity condition (uniform bounded density property) for foliations, which is defined in terms of the disintegration of the Lebesgue measure along the leaves of the foliation. In principle it can be compared with the absolute continuity of foliations, however for (un)stable foliations of partially hyperbolic diffeomorphisms the works of Pesin–Sinai [11] and Ledrappier [9] shed light on the subject, suggesting that for these foliations our condition imposes a kind of regularity much stronger than absolute continuity. However, we believe that exploiting this regularity condition is a geometric measure theoretical criterium for the rigidity of partially hyperbolic dynamics.

From now on, we shall consider a smooth measure \( m \) (the Lebesgue measure) on \( \mathbb{T}^3 \) and a \( C^2 \) diffeomorphism \( f: M \to M \) preserving \( m \). \( f \) is called (absolute) partially hyperbolic if there exists a \( Df \)-invariant splitting of the tangent bundle \( TM = E^s_f \oplus E^c_f \oplus E^u_f \) and constants \( \nu_- \leq \nu_+ \leq 1 \leq \mu_+ \leq \lambda_- \leq \lambda_+ \) and \( C > 0 \), satisfying

\[
\frac{1}{C} \nu^\alpha ||v|| \leq ||Df^\alpha(x)v|| \leq C \nu^\alpha ||v||, \quad \forall v \in E^f_f(x),
\]
We show that, if we assume Lebesgue measure if and only if for almost every $x \in M$, $\phi$ is leafwise absolutely continuous, if it satisfies the following: a measurable set $\phi(B)$ are an equivalent class of measures up to scaling. In a foliation chart $U \subset M$; write $m_{\mathcal{F}_{\mathcal{F}}}$; the probability measure which comes from the Rokhlin disintegration of $m$ restricted to $U$. In what follows we use the unique notation $m_{\mathcal{F}_{\mathcal{F}},(B)}$ to denote the disintegration of the plaque inside foliated box $B$, which is a probability measure.

**Definition 1.1 (Leafwise absolute continuity).** Let $\mathcal{F}$ be a foliation on $M$. We say that $\mathcal{F}$ is leafwise absolutely continuous, if it satisfies the following: a measurable set $Z$ has zero Lebesgue measure if and only if for almost every $p \in M$, the leaf $\mathcal{F}_p$ meets $Z$ in a $\lambda_{\mathcal{F}_p}$ zero measure set, that is, $\lambda_{\mathcal{F}_p}(Z) = 0$, for almost everywhere $p \in M$.

Locally, it is equivalent to

$$\lambda_{\mathcal{F}_p} \sim m_{\mathcal{F}_p}, \ m\text{-almost everywhere } p \in U.$$  

In a general setting it is not easy to understand the disintegration $[[m_{\mathcal{F}_p}]]$ of $m$. In the case of leafwise absolutely continuous foliations, the Radon–Nikodym derivative $dm_{\mathcal{F}_p}/d\lambda_{\mathcal{F}_p}$ is an interesting object for study. This motivated us to introduce a new regularity condition. We show that, if we assume $m_{\mathcal{F}_p}$ is 'universally proportional' to $\lambda_{\mathcal{F}_p}$, for almost everywhere $x \in M$, independent of the size of $\mathcal{F}_p \cap U$ then many rigidity results hold. To begin, we need to work with long foliated boxes.

$$\frac{1}{C} |\mu_u^n||v|| \leq ||Df^n(x)v|| \leq C\mu_u^n||v||, \quad \forall v \in E^u_f(x),$$

$$\frac{1}{C} |\lambda_c^n||v|| \leq ||Df^n(x)v|| \leq C\lambda_c^n||v||, \quad \forall v \in E^c_f(x).$$

It is possible to choose a Riemannian metric in $M$ that makes $C = 1$ in the above definition. In this paper all partially hyperbolic diffeomorphisms are defined on $\mathbb{T}^3$. For simplicity we denote $Df(x)|E^s_f(x)$ by $J^s f(x)$, $\sigma \in \{s, c, u\}$. The distributions $E^s_f$ and $E^u_f$, respectively, a stable and an unstable bundle, are uniquely integrable to foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ (see [8]). In the general case, $E^c$ is not integrable. However for absolute partially hyperbolic diffeomorphisms on $\mathbb{T}^3$, the centre bundle is integrable [2]. The foliation tangent to $E^c$ is denoted by $\mathcal{F}^c$. One special example is the case of Anosov diffeomorphisms on $\mathbb{T}^3$ with splitting $TM = E^{uu} \oplus E^u \oplus E^s$. In this case, the weak unstable bundle $E^u$ can be considered as the central bundle.

### 1.1. Regularity of foliations

Roughly speaking, a foliation is an equivalence relation on a manifold, such that the equivalence classes (the leaves) are connected via immersed submanifolds. For dynamic invariant foliations the leaves are not stacked up in a smooth fashion, although they typically enjoy a high degree of regularity. To define the different regularity conditions we need foliated charts. For instance a codimension-$k$ foliation is $C^r$ if there exists a covering of the manifold by $C^r$ charts $\phi : U \to \mathbb{R}^n \times \mathbb{R}^k$ such that each plaque is sent into the hyperplane $\mathbb{R}^n \times \{\phi(p)\}$.

For a $C^r$-partially hyperbolic diffeomorphism the invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are typically at most Hölder continuous with $C^r$ leaves. An important feature of stable and unstable foliations of partially hyperbolic diffeomorphism is their ‘absolute continuity’ property. In smooth ergodic theory, absolute continuity of foliations has been used by Anosov to prove the ergodicity of Anosov diffeomorphisms. One of the weakest definitions (leafwise absolute continuity) is sufficient to prove the ergodicity of Anosov diffeomorphisms. See [13] for other definitions and state of the art of absolute continuity of foliations.

Consider $\mathcal{F}$ a foliation over $M$. Denote by $m$ the Riemannian measure over $M$; and $\lambda_{\mathcal{F}_p}$; the Riemannian measure over $\mathcal{F}_p$; the leaf through $x \in M$. There is a unique disintegration $[[m_{\mathcal{F}_p}]]$ of $m$ along the leaves of the foliation. $[[m_{\mathcal{F}_p}]]$ are an equivalent class of measures up to scaling. In a foliation chart $U \subset M$; write $m_{\mathcal{F}_p}$; the probability measure which comes from the Rokhlin disintegration of $m$ restricted to $U$. In what follows we use the unique notation $m_{\mathcal{F}_{\mathcal{F}},(B)}$ to denote the disintegration of the plaque inside foliated box $B$, which is a probability measure.
Definition 1.2 (Long foliated box). Let $\mathcal{F}$ be a one dimensional foliation of $M^n$. A set $B \subset M$ is called a foliated box by $\mathcal{F}$ of size greater than or equal to $R > 0$, if:

1. $B$ is homeomorphic to $D^{n-1} \times (0, 1)$ where $D^{n-1}$ is $(n-1)$-dimensional ball representing transversal to the plaques in $B$;
2. for each $x \in B$, the length of the connected component of $\mathcal{F}_x \cap B$ containing $x$ is greater than or equal to $R > 0$ in the intrinsic Riemannian metric of $\mathcal{F}_x$.

For any foliated box $B$ we denote by $m|_B$ the normalized Lebesgue measure of $B$ and for any plaque $\mathcal{F}_x(B)$ the probability induced Lebesgue measure on the plaque is denoted by $\text{Leb}_{\mathcal{F}_x}(B)$. In the cases where the box is fixed, we write $\text{Leb}_{\mathcal{F}_x}$.

Definition 1.3 (Uniform bounded density). Let $\mathcal{F}$ be a one dimensional foliation on $M$. We say that $\mathcal{F}$ has uniform bounded density (UBD) properties, if there is $K > 1$ such that for every long foliated box of $\mathcal{F}$ in $M$ we have

$$\frac{1}{K} < \frac{\text{d}m_{\mathcal{F}_x}}{\text{d}\text{Leb}_{\mathcal{F}_x}} < K$$

independent of the size of the foliated box and $x$.

For example if $A$ is a linear partially hyperbolic automorphism of the torus then the invariant foliations have UBD properties. In fact this is the case for any $f$ close to $A$ and $C^1$ conjugated to it.

Another example of foliations with UBD properties is the case of central foliation of ergodic partially hyperbolic diffeomorphisms on $M^3$ whenever it is absolutely continuous and the leaves are circles. Indeed, as the lengths of the central leaves are uniformly bounded (see [5, 6] for general statements) the UBD property is equivalent to leafwise absolute continuity. A recent result of Avila–Viana–Wilkinson [1] establishes that absolute continuity of central foliation in this setting implies $C^\infty$ regularity. We hope that UBD properties of central foliations in general may imply its differentiability.

Lyapunov exponents are important constants and measure the asymptotic behaviour of dynamics in the tangent space level. Let $f: M \to M$ be a measure preserving $C^1$-diffeomorphism. Then by Oseledets' theorem, for almost every $x \in M$ and any $v \in T_x(M)$

the following limit exists:

$$\lim_{n \to \infty} \frac{1}{n} \log \|Df^n(x)v\|$$

and is equal to one of the Lyapunov exponents of the orbit of $x$. For a conservative partially hyperbolic diffeomorphism of $\mathbb{T}^3$, which is the main object of the study in this paper, we get a full Lebesgue measure subset $\mathcal{R}$ such that for each $x \in \mathcal{R}$ :

$$\lim_{n \to \infty} \frac{1}{n} \log \|Df^n(x)v^\sigma\| = \lambda^\sigma(f, x)$$

where $\sigma \in \{s, c, u\}$ and $v^\sigma \in E^\sigma$.

Every diffeomorphism of the torus $f: \mathbb{T}^n \to \mathbb{T}^n$ induces an automorphism of the fundamental group and there exists a unique linear diffeomorphism $f_*$ which induces the same automorphism on $\pi_1(\mathbb{T}^n)$. $f_*$ is called the linearization of $f$ and in this paper we study the relations between Lyapunov exponents of $f$ and its linearization in the partially hyperbolic setting.

3 This condition is similar to bounded jacobian or global absolute continuity in [1]. However, we do not have any evidence that they are equivalent.
2. Statement of results and questions

First we prove that the uniform bounded density is a criterium for the rigidity of Lyapunov exponents in the context of partially hyperbolic diffeomorphisms of $\mathbb{T}^3$.

**Theorem 2.1.** Let $f : \mathbb{T}^3 \to \mathbb{T}^3$, be a conservative partially hyperbolic diffeomorphism. Denote by $A = f_*$ and suppose that stable and unstable foliations have the uniform bounded density property, then $\lambda^\sigma(f, \cdot) = \lambda^\sigma_A, \sigma \in \{s, c, u\}$ for almost every $x \in \mathbb{T}^3$.

**Remark 2.2.** In the above theorem if we just assume the UBD property of one of the foliations $\mathcal{F}^s$ or $\mathcal{F}^u$, we conclude the rigidity of the corresponding Lyapunov exponent. In the above theorem the rigidity of the central Lyapunov exponent is just a corollary of the volume preserving property of $f$. However, the same rigidity for the central exponent also holds if we just assume $\mathcal{F}^c$ has UBD properties. As we do not have a good description for the disintegration along the central leaves, the proof for the central exponent rigidity is different from the stable and unstable foliation cases and it appears in the proof of theorem 2.5.

The above result shows that UBD properties impose restrictions on the dynamics in the level of Lyapunov exponents.

The above theorem assumes the UBD properties and concludes some rigidity of Lyapunov exponents. We should mention that even leafwise absolute continuity imposes some restrictions on the Lyapunov exponents, as we see in the following theorem. Recall that stable and unstable foliation of any $C^2$-partially hyperbolic diffeomorphism are leafwise absolutely continuous ([3]).

**Theorem 2.3.** Let $f$ be a $C^2$ conservative partially hyperbolic diffeomorphism on the 3-torus and $A$ its linearization then

$$\lambda_s(f, x) \leq \lambda^s(A) \quad \text{and} \quad \lambda^s(f, x) \geq \lambda^s(A) \quad \text{for almost everywhere} \quad x \in \mathbb{T}^3.$$ 

Similar to the statement above, the theorem appears in [14] and is proven in [15] for $f$ $C^1$-close to $A$. In [15], the authors need unique homological data for the strong unstable foliation and they prove that it is the case when $f$ is closed to its linearization.

**Corollary 2.4.** Any conservative linear partially hyperbolic diffeomorphism is a local maximum point for

$$f \mapsto \int \lambda^u(f) \, dm.$$ 

Analogously any conservative linear partially hyperbolic diffeomorphism is a local minimum point for

$$f \mapsto \int \lambda^s(f) \, dm.$$ 

**Problem 1.** Classifying the local maximum points of the unstable Lyapunov exponent. Are these diffeomorphisms $C^1$ conjugated to the linear?

**Problem 2.** In the context of the above theorem, suppose that $\lambda^c(f) > 0$ and $\mathcal{F}^c$ is upper leafwise absolutely continuous then $\lambda^c(f) \leq \lambda^c(A)$.

Another interesting issue in the setting of partially hyperbolic diffeomorphisms is the characterization of the topological type of central leaves. It is clear that for a general partially hyperbolic diffeomorphism (general 3-manifolds) with one dimensional central bundle, the leaves of central foliation may be circles, lines or both (consider the suspension of an Anosov diffeomorphism of $\mathbb{T}^2$). However by Hammerlindl’s result [7], central leaves of a partially hyperbolic diffeomorphism on $\mathbb{T}^3$ are homeomorphic to central leaves of its linearization and consequently all the leaves have the same topological type. A very natural question is:
**Question 1.** Suppose $f$ is volume preserving (absolute) partially hyperbolic on $\mathbb{T}^3$ and the central Lyapunov exponent vanishes almost everywhere. Is it true that all centre leaves are compact?

In a general setting, this question has been answered negatively in [12]. We would like to mention a recent result of Hammerlindl and Ures, a non-ergodic derived from Anosov diffeomorphism on $\mathbb{T}^3$. If this exists, it will have zero central Lyapunov exponent and non-compact central leaves. It is interesting to know whether this example of such partially hyperbolic non-ergodic diffeomorphisms on the torus exists.

Assuming UBD properties of central foliation we get the following theorem, which gives an affirmative answer to the above question.

**Theorem 2.5.** Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a conservative partially hyperbolic diffeomorphism. Suppose that $\mathcal{F}^\sigma$ has the uniform bounded density property, then $\lambda^\sigma(f, x) = \lambda^\sigma(f_\ast)$, almost everywhere. In particular if $\lambda^\sigma(f, x) = 0$ for a.e. $x \in \mathbb{T}^3$, then all centre leaves are circles.

We do not know whether one can prove that all the central leaves are homeomorphic to $S^1$ merely under the assumption of absolute continuity of central foliation and vanishing central Lyapunov exponent.

### 3. Preliminaries

In this section we review some definitions and known results about partially hyperbolic diffeomorphisms on $\mathbb{T}^3$.

#### 3.1. Partially hyperbolic diffeomorphisms on $\mathbb{T}^3$

In the rest of the preliminary section we will recall some topological properties of invariant foliations of partially hyperbolic diffeomorphisms on a 3-torus. One of the key properties of the invariant foliations of partially hyperbolic diffeomorphisms in the 3-torus is their quasi-isometric property. Quasi-isometric foliation $W$ of $\mathbb{R}^d$ means that the leaves do not fold back on themselves much.

**Definition 3.1.** A foliation $W$ is quasi-isometric if there exists a positive constant $Q$ such that for all $x, y$ in a common leaf of $W$ we have

$$d_W(x, y) \leq Q^{-1}||x - y||.$$

Here $d_W$ denotes the Riemannian metric on $W$ and $||x - y||$ is the distance on the ambient manifold of the foliation.

In the partially hyperbolic case, we denote by $d^\sigma(\cdot, \cdot)$, the Riemannian metric on $\mathcal{F}^\sigma, \sigma \in \{s, c, u\}$. We define $d^\sigma(\cdot, \cdot)$ in the dynamical coherent case. The foliation $\mathcal{F}^\sigma$ is called quasi-isometric if its lift to the universal covering ($\mathbb{R}^3$) is quasi-isometric.

Let us recall some recent results concerning the geometric and topological properties of the invariant foliations of partially hyperbolic diffeomorphisms on $\mathbb{T}^3$.

**Theorem 3.2 ([4, 7]).** If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is a partially hyperbolic diffeomorphism, then $\mathcal{F}^\sigma, \sigma \in \{s, c, u\}$ are quasi-isometric foliations.

**Proposition 3.3 ([7]).** Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism and $A : \mathbb{T}^3 \to \mathbb{T}^3$ the linearization of $f$. Then

$$\lim_{||y - x|| \to +\infty} \frac{y - x}{||y - x||} = E^\sigma_A, \quad y \in \mathcal{F}^\sigma_x, \sigma \in \{s, c, u\}$$

and the convergence is uniform.
Proposition 3.4 ([7]). Let $f : T^3 \to T^3$ be a partially hyperbolic diffeomorphism and $A : T^3 \to T^3$ the linearization of $f$ then for each $k \in \mathbb{Z}$ and $C > 1$ there is an $M > 0$ such that for $x, y$,

$$\|x - y\| > M \Rightarrow \frac{1}{C} < \frac{\|f^k(x) - f^k(y)\|}{\|A^k(x) - A^k(y)\|} < C.$$ 

More generally, for each $k \in \mathbb{Z}$, $C > 1$, and linear map $\pi : \mathbb{R}^d \to \mathbb{R}^d$ there is an $M > 0$ such that for $x, y \in \mathbb{R}^d$,

$$\|\pi(x - y)\| > M \Rightarrow \frac{1}{C} < \frac{\|\pi(f^k(x) - f^k(y))\|}{\|\pi(A^k(x) - A^k(y))\|} < C.$$ 

Theorem 3.5 ([7]). Every partially hyperbolic diffeomorphism of the 3-torus is leaf conjugated to its linearization by a homeomorphism $h$. Furthermore, $h$ restricted to each centre leaf is bi-Lipschitz and denoting by $\tilde{h}$ a lift of $h$ in $\mathbb{R}^3$ one has that

$$\|\tilde{h} - 1d_{\mathbb{R}^3}\|$$ is bounded.

The above theorem and propositions have the following corollaries which are useful in the rest of the paper.

Lemma 3.6 ([10]). Let $f : T^3 \to T^3$ be a partially hyperbolic diffeomorphism and $A : T^3 \to T^3$ the linearization of $f$. For all $n \in \mathbb{Z}$ and $\epsilon > 0$ there exists $M$ such that for $x, y$ with $y \in F_\sigma f(x)$ and $\|x - y\| > M$ we have

$$(1 - \epsilon)e^{\lambda^\sigma N} \|y - x\| \leq \|A^N(x) - A^N(y)\| \leq (1 + \epsilon)e^{\lambda^\sigma N} \|y - x\|$$

where $\lambda^\sigma$ is the Lyapunov exponent of $A$ corresponding to $E^\sigma$ and $\sigma \in \{s, c, u\}$.

Proof. Let us fix $\sigma$ and denote by $E^\sigma_A$ the eigenspace corresponding to $\lambda^\sigma_A$, $\mu := e^{\lambda^\sigma}$. Let $N \in \mathbb{Z}$ and choose $x, y \in F^\sigma_A(x)$, such that $\|x - y\| > M$. By proposition 3.3, we have

$$\frac{x - y}{\|x - y\|} = v + e_M,$$

where the vector $v = e_{E_A}$ is a unit eigenvector of $A$, in the $E_A$ direction and $e_M$ is a correction vector that converges to zero uniformly as $M$ goes to infinity.

So, considering $\mu$, the eigenvalue of $A$ in the $E_A$ direction

$$A^N\left(\frac{x - y}{\|x - y\|}\right) = \mu^N v + A^N e_M = \mu^N \left(\frac{x - y}{\|x - y\|}\right) - \mu^N e_M + A^N e_M$$

implies that

$$\|x - y\|\|(\mu^N - \mu^N)\|e_M\| - \|A\|^N\|e_M\|\| \leq \|A^N(x - y)\| \leq \|x - y\|\|(\mu^N + \mu^N)\|e_M\| + \|A\|^N\|e_M\|\|.$$

Since $N$ is fixed, we can choose $M > 0$, such that

$$\mu^N\|e_M\| + \|A\|^N\|e_M\| \leq \epsilon \mu^N,$$

and the lemma is proved. \qed
4. Technical rigidity results and proof of theorem 2.1

In this section we prove some technical rigidity results for Lyapunov exponents which will be used in the proofs of the main theorems. In particular we prove theorem 2.1.

Let us concentrate on the volume preserving partially hyperbolic diffeomorphisms of $T^3$. One important result which appears in the works of Pesin–Sinai and Ledrappier (see [9, 11]) is the exact formula for the disintegration of the Lebesgue measure along unstable manifolds (even in the Pesin theory setting): take $\xi$ to be a measurable partition subordinated to the unstable foliation. For $y \in \xi(x)$ define

$$
\Delta_1(x, y) := \prod_{i=1}^{\infty} \frac{J_u(f^{-i}(x))}{J_u(f^{-i}(y))}.
$$

(4.1)

After normalizing $\rho(y) := \Delta_1(x, y)/L(x)$ where $L(x) = \int_{\xi(x)} \Delta_1(x, y) d\text{Leb}_x$ and $\rho(\cdot)$ is the Radon–Nikodym derivative $dm_{\xi}/d\text{Leb}_x$. We emphasize that such a clear formula for the disintegration along a general leafwise absolutely continuous foliation (for instance for central foliation whenever it is absolutely continuous) is not available. We use this formula in the proof of theorem 2.1. Here we observe some elementary properties of $\Delta_1$. First of all note that the $C^2$-regularity of $f$ and the Hölder continuous dependence of $E^u$ with the base point gives us:

**Lemma 4.1.** For any $\epsilon > 1$ there exists $\delta > 0$ such that if $y \in W^u_{h, x} \subset F^u_x$ then

$$
1 - \epsilon \leq \Delta_1(x, y) \leq 1 + \epsilon.
$$

**Proof.** Taking a logarithm as a $\alpha$-Hölder continuity of the unstable bundle and $J^u f$ implies that

$$
\log \Delta_1(x, y) = \sum_{i=1}^{\infty} \log J^u f(f^{-i}(x)) - \log J^u f(f^{-i}(y)) \\
\leq \sum_{i=1}^{\infty} C_1 \lambda^{-i} d^2(x, y) \leq \left( C_1 \sum_{i=1}^{\infty} \lambda^{-i} \right) d^2(x, y)
$$

where $\lambda^{-}$ comes from the definition of partial hyperbolicity. This completes the proof of the lemma.

**Lemma 4.2.** Suppose that $F^u$ has bounded density properties. There exists $K > 1$ such that for almost every $x \in T^3$ and every $y_1, y_2 \in F^u_x$:

$$
K^{-1} \leq \Delta_1(y_1, y_2) \leq K.
$$

(4.2)

Moreover, for any $n \in \mathbb{N}$:

$$
K^{-2} \leq \prod_{i=0}^{n-1} J^u f(f^i(y_1)) J^u f(f^i(y_2)) \leq K^2.
$$

(4.3)

**Proof.** By definition of uniform bounded density (1.3) it comes out that $(\rho(y_2)/\rho(y_1)) \in [K^{-2}, K^2]$. Abusing the notations for simplicity, we substitute $K^2$ by $K$ and conclude the first claim of the lemma.

We can suppose that the points $x$ satisfying (4.2) belong to an invariant set. So changing $x$ to $f^n(x)$ we have

$$
K^{-1} \leq \Delta_1(f^n(y_1), f^n(y_2)) \leq K.
$$

(4.4)
Dividing equation (4.4) by (4.2) we conclude the proof of the second claim of the lemma.

In the stable case, we take $f^{-1}$ and apply (4.1) in the $E^u_f = E^u_{f^{-1}}$ direction. Similarly for $y \in F_x^u$ we define

$$\Delta^s(x, y) := \prod_{i=0}^{\infty} \frac{J^s f(f^i(x))}{J^s f(f^i(y))}.$$  \hfill (4.5)

From now on we use the notation $\Delta^s(f^n(y_1), f^n(y_2)) = O(1)$ to denote that $\Delta^s(f^n(y_1), f^n(y_2))$ is bounded from below and above by constants just depending on $f$.

Now we state a technical proposition which guarantees the constancy of unstable periodic data and rigidity of Lyapunov exponents.

**Proposition 4.3.** Let $f$ be a partially hyperbolic diffeomorphism of $T^3$. Suppose that for any $x$ in an invariant full measure set and any $y_1, y_2 \in F_x^\sigma$:

$$\Delta^\sigma(f^n(y_1), f^n(y_2)) = O(1),$$

then $\lambda^\sigma(f, x) = \lambda^\sigma(A)$ where $A$ is the linearization of $f$ and $\sigma \in \{s, u\}$.

### 4.1. Proof of theorem 2.1

As we mentioned above (lemma 4.2) the UBD property of unstable foliation implies the desired boundedness condition and using proposition 4.3 we conclude that $\lambda^u(f, x) = \lambda^u_A$.

Similarly, taking the inverse $f^{-1}$, the UBD property implies that $\lambda^u(f^{-1}, x) = \lambda^u_{A^{-1}}$, it means $\lambda^s(f, x) = \lambda^s_A$ and as $f$ is conservative $\lambda^c(f, x) = \lambda^c_A$.

### 4.2. Proof of proposition 4.3

Take any $\sigma \in \{s, u\}$ and suppose that $Z = \{x \in T^3 | \lambda^\sigma(f, x) > \lambda^\sigma_A\}$ has a positive volume. Let $\varepsilon > 0$ be a small number and define

$$A_n = \{x \in Z | ||J^\sigma f^m(x)|| > e^{m\lambda^\sigma(x) + \varepsilon} \text{ for all } m \geq n\}.$$

Take $n$ large enough such that

$$m(A_n) > 0 \quad \text{and} \quad \frac{Q e^{m\lambda^\sigma + \varepsilon}}{2K^{2m\lambda^\sigma_A}} > 2,$$

where $Q$ is as in the definition (3.1) of quasi-isometric foliations (we know that stable, unstable and central foliations of partially hyperbolic diffeomorphisms in $T^3$ are quasi-isometric) and the constant $K$ is such that

$$K^{-1} \leq \Delta^\sigma(f^n(y_1), f^n(y_2)) \leq K.$$

Similar to (4.3) we get

$$K^{-2} \leq \prod_{i=0}^{n-1} \frac{J^\sigma f(f^i(y_1))}{J^\sigma f(f^i(y_2))} \leq K^2.$$  \hfill (4.6)

for any $n \in \mathbb{N}$.

Using proposition 3.4 and lemma 3.6, choose $M > 0$ such that for any $y \in F_x^c$ and $d^c(x, y) \geq M$

$$\frac{1}{2} < \frac{||f^n x - f^n y||}{||A^n x - A^n y||} < 2.$$  \hfill (4.7)
Take any regular point \( x \in A_\sigma \). By definition we have \( J_\sigma f^n(x) > e^{n(\lambda_\sigma A + \epsilon)} \) and by (4.6) we get
\[
J_\sigma f^n(y) \geq \frac{1}{K^2} e^{n(\lambda_\sigma A + \epsilon)}
\]
for any \( y \in F_\sigma^x \). Now
\[
\frac{||f^n x - f^n y||}{||A^\sigma x - A^\sigma y||} \geq \frac{Q}{K^2 e^{n\lambda_\sigma A}} \frac{||D^n f^n|||d\lambda_{F^n}}{||A^\sigma x - A^\sigma y||} \geq \frac{Q e^{n(\lambda_\sigma A + \epsilon)||x - y||}}{2K^2 e^{n\lambda_\sigma A}} > 2.
\]
which gives a contradiction. Thus \( \{ x \in T^3 | \lambda^{\sigma}(f, x) > \lambda^{\sigma}_A \} \) has zero volume. In the same way, considering \( f^{-1} \), it comes out that
\[
m(\{ x \in T^3 | \lambda^{\sigma}(f^{-1}, x) > \lambda^{\sigma}_A - 1 \}) = m(\{ x \in T^3 | \lambda^{\sigma}(f, x) < \lambda^{\sigma}_A \}) = 0.
\]

5. Local maxima for Lyapunov exponents

5.1. Proof of theorem 2.3

Proof. We prove the statement on \( \lambda^{\sigma}(f, \cdot) \). Suppose by contradiction that there is a positive volume set \( Z \subset T^3 \), such that, for every \( x \in Z \) we have \( \lambda^{\sigma}(f, x) > \lambda^{\sigma}_A \). Since \( f \) is \( C^2 \), the unstable foliation \( F^u \) for \( f \) is upper absolutely continuous, in particular there is a positive volume set \( B \) such that for every point \( x \in B \) we have
\[
\lambda^{F^u}_x (F^u_x \cap Z) > 0.
\]
Choose a \( p \in B \) satisfying (5.1) and \( \epsilon > 0 \) a small number. Now consider a segment \( [x, y]_B \subset F^u_p \) satisfying \( \lambda^{F^u}_x ([x, y]_B \cap Z) > 0 \) such that the length of \( [x, y]_B \) is bigger than \( M \) as required in lemma 3.6 and proposition 3.4. We can choose \( M \) such that
\[
||Ax - Ay|| < (1 + \epsilon) e^{\lambda_\sigma A} ||y - x||
\]
and
\[
||fx - fy|| \leq 1 + \epsilon.
\]
whenever \( d^u(x, y) \geq M \). The above equation implies that
\[
||fx - fy|| < (1 + \epsilon)^2 e^{\lambda_\sigma A} ||y - x||.
\]
Inductively, we assume that for \( n \geq 1 \) we have
\[
||f^n x - f^n y|| < (1 + \epsilon)^{2n} e^{n\lambda_\sigma A} ||y - x||.
\]
Since \( f \) expands uniformly on the \( u \)-direction we have \( d^u(f^n x, f^n y) > M \), it leads
\[
||f(f^n x) - f(f^n y)|| < (1 + \epsilon)|A(f^n x) - A(f^n y)|| < (1 + \epsilon)^2 e^{\lambda_\sigma A} ||f^n x - f^n y|| < (1 + \epsilon)^{2(n+1)} e^{(n+1)\lambda_\sigma A}.\]
For each $n > 0$, let $A_n \subset Z$ be the following set

$$A_n = \{ x \in Z : \| D^m f^k x \| > (1 + 2\varepsilon)^{2k} e^{k\lambda_A} \text{ for any } k \geq n \}.$$  

We have $m(Z) > 0$ and $A_n \uparrow Z$. Consider a big $n$ and $a_n > 0$ such that $\text{Leb}_{\mathcal{F}}([-x, y] \cap A_n) = a_n \text{Leb}_{\mathcal{F}}([-x, y] \cap A)$. Note that when $n$ increases to infinity the proportion $a_n$ converges to $\text{Leb}_{\mathcal{F}}([-x, y] \cap Z)$.

We can assume with lost generality $a_n > a_0 > 0$ for any $n > 1$. Then

$$\| f^m x - f^m y \| > Q \int_{[x, y] \cap A_n} |Df^m(z)| \, d\lambda_{\mathcal{F}}(z) \quad (5.3)$$
$$> Q(1 + 2\varepsilon)^{2m} e^{m\lambda_\mathcal{F}}([-x, y] \cap A_n) \quad (5.4)$$
$$> a_0 Q(1 + 2\varepsilon)^{2m} e^{m\lambda_\mathcal{F}} \| x - y \|. \quad (5.5)$$

The inequalities (5.2) and (5.5) give a contradiction. We conclude that $\lambda_\mathcal{F}(f, x) \leq \lambda_\mathcal{A}$, for almost everywhere $x \in T^3$. Considering the inverse $f^{-1}$ we conclude that $\lambda_\mathcal{F}(f, x) \leq \lambda_\mathcal{A}$ for almost every $x \in T^3$. □

It would be interesting to prove a result similar to theorem 2.3 for the central Lyapunov exponent of partially hyperbolic diffeomorphisms (see problem 2). However, the above arguments can be used to prove the same result for the special case of Anosov diffeomorphisms. In what follows we consider Anosov diffeomorphisms as partially hyperbolic systems and by the central bundle we refer to the weak unstable bundle of the invariant decomposition.

**Theorem 5.1.** Consider $f : T^3 \to T^3$ a $C^2$ volume preserving Anosov diffeomorphism with decomposition $E^{uu} \oplus E^s \oplus E^c$. Let $A : T^3 \to T^3$, the linearization of $f$. Suppose that $\mathcal{F}$ is a quasi isometric, (upper) absolutely continuous foliation then $\lambda^c(f, \cdot) \leq \lambda^c_A$, a.e. $x \in T^3$.

**Proof.** By [7] we have that $A$ is partially hyperbolic and $\dim E^c_f = \dim E^c_A$. Since $f$ is Anosov with a partially hyperbolic structure, then by propositions 3.3 and 3.4, we have that $A$ is Anosov and $\lambda_A > 0$, for a.e. Furthermore there is $\mu > 1$, such that $\| Df^m(x) \| > \mu$, for any $x \in T^3$. In the other words, $f$ has uniform expansion in the central direction. Since $\mathcal{F}$ is quasi-isometric, we can apply the same argument of the previous theorem, and we conclude $\lambda^c(f, \cdot) \leq \lambda^c_A$, a.e. $x \in T^3$. □

6. Topology of central leaves

To prove theorem 2.5 we show that the UBD properties of the central foliation imply that for almost every $x$ we have $\lambda^c(x) = \lambda^c_A$. Consequently under the assumption of $\lambda^c(f, x) = 0$, a.e., $A$ has compact central leaves and by leaf conjugacy between $f$ and $A$, the same is true for $f$.

The idea of the proof is similar to that of theorem 2.1. However a main difficulty here is that we do not have a formula for the density of the disintegration along the central foliation. Let

$$Z = \{ x \in T^3 | \lambda^c(f, x) > \lambda^c_A \}$$

and $\varepsilon > 0$ be a small number. Define

$$A_n = \{ x \in Z | \| Df^m(x) E^c \| > e^{m(\lambda_A + \varepsilon)} \text{ for all } m \geq n \}.$$  

Take $n$ large enough such that

$$m(A_n) > \alpha \quad \text{and} \quad \frac{Q\alpha e^{n(\lambda_A + \varepsilon)}}{4Ke^{n\lambda_A}} > 2. \quad (6.1)$$
for some positive $\alpha > 0$ which will be fixed. We choose $M$ satisfying 4.7 and 4.8 for $\sigma = c$.

Now the idea is to find a central plaque $F^s_x$ of size $M$ such that $\text{Leb}_x(A_n) \geq \alpha/2K$. Of course, if we could provide a measurable partition of $M$ into plaques of size $M$ by Rokhlin decomposition we would get a plaque such that $\text{m}_z(\pi^{-1}(A_n)) \geq \alpha$ and by definition $\text{Leb}_x(A_n) \geq \alpha/K$. As we ignore the existence of such partition we construct a partition covering a large measurable subset of $T^3$ in the next subsection.

Let us complete the proof of the theorem assuming the existence of such plaque. The idea is to get the same contradiction as in the proof of theorem 2.1. More precisely, we get, similar to (5.5),

$$\frac{||f^n x - f^n y||}{||A^n x - A^n y||} \geq \frac{Q \int_{F^s_1(B)} ||D f^n| |E'||d\lambda_{F^s_1}}{||A^n x - A^n y||} \geq \frac{Q \alpha e^{n(\lambda_c + \varepsilon)}}{4K e^{n\lambda_c}} ||x - y|| \geq \frac{Q \alpha}{4K} > 2. \quad (6.2)$$

To find such a plaque we need to construct a measurable partition by plaques as follows.

### 6.0.1. Measurable partition by long plaques.

It is more convenient to work in the universal covering $\pi : \mathbb{R}^3 \to T^3$. We lift the foliations to $\mathbb{R}^3$ and use the same notations $F^s_x$ for the leaf passing through $x$ in $\mathbb{R}^3$. First we recall a property of central foliation proved by Hammerlind [7]:

**Proposition 6.1.** There is a constant $R_c$ such that for all $x \in \mathbb{R}^3$, $F^s_x \subset U_{R_c}(A^*_n)$ where $U_{R_c}(A^*_n)$ denotes the neighbourhood of radius $R_c$ around the central leaf of $A$ through $x$.

For $M$ large enough, we take $D$ as a ball centered at $O \in \mathbb{R}^3$ of radius $M$, transverse to $F^s_x$ and in the $su$-leaf of $A$. Now saturate $D$ by central plaques of size $M$ and let $\hat{D} := \bigcup_{z \in D} F^s_{z,M}$.

**Lemma 6.2.** If $M$ is large enough there exists a plaque $F^s_{z,M}$ such that

$$\text{Leb}_z(\pi^{-1}(A_n)) \geq \alpha/2K.$$

**Proof.** Recall that $m(A_n) \geq \alpha$. As $M$ is large, $\hat{D}$ will include a large number $\tilde{N}(M)$ of fundamental domains (cubes) where $\pi$ is invertible. That is $C_i \subset \hat{D}$ where $C_i$ are unitary cubes for $i = 1, \ldots, \tilde{N}(M)$. However $\hat{D}$ may partially intersect $\tilde{N}(M)$ other fundamental cubes, i.e $\hat{D} \cap C_i \neq \emptyset$ but $C_i \not\subset \hat{D}$ for $i = 1, \ldots, \tilde{N}(M)$. By the above proposition we claim that

$$\lim_{M \to \infty} \frac{\tilde{N}(M)}{\tilde{N}(M)} = 0.$$

So for large enough $M$ we have

$$m(\pi^{-1}(A_n) \cap \hat{D}) \geq \frac{\alpha \tilde{N}(M)}{\tilde{N}(M) + \tilde{N}(M)} \geq \alpha/2.$$

Now we disintegrate along the plaques in $\hat{D}$ by Rokhlin, gaining plaques such that $m_z(\pi^{-1}(A_n)) \geq \alpha/2$, and by definition 1.3 of uniform bounded density property, it yields that $\text{Leb}_x(\pi^{-1}(A_n)) \geq \alpha/2K$. □
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