Generalized uncertainty principles associated with the quaternion offset linear canonical transform

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ABSTRACT
The quaternionic offset linear canonical transform (QOLCT) can be defined as a generalization of the quaternionic linear canonical transform (QLCT). In this paper, we define the QOLCT, we derive the relationship between the QOLCT and the quaternion Fourier transform (QFT). Based on this fact, we prove the Rayleigh formula and some properties related to the QOLCT. Then, we generalize some different uncertainty principles (UPs), including Heisenberg-Weyl’s UP, Hardy’s UP, Beurling’s UP, and logarithmic UP to the QOLCT domain.

1. Introduction
The QFT plays a relevant role in the representation of signals. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency-domain signal. In [1], the authors provided Heisenberg’s inequality [2] and Hardy’s UP for the two-sided QFT. Thereafter, in the paper [3], they generalized Beurling’s UP [4, 5] to the QFT domain. It is well-known that the LCT provides a more general framework for all remarkable linear integral transforms in signal processing and optics, such as the Fourier transform FT, the fractional FT, the Fresnel transform, and the Lorentz transform.

The LCT was extended to Clifford analysis by Kou et al. [6] in 2013, to study the generalized prolate spheroidal wave functions and their connection with energy concentration problems.

In [7], the authors introduced the quaternion linear canonical transform (QLCT), which is a generalization of the LCT in the framework of quaternion algebra. Several properties, such as Parseval’s formula and the UP associated with the QLCT, are established. In the
light of being a generalization of the LCT, the OLCT has wide applications in signal processing and optics, and naturally one is motivated to also extend the OLCT to the quaternionic algebra framework.

To the best of our knowledge, the generalization of the OLCT to quaternionic algebra, and the study of the properties and UPs associated with this generalization have not been carried out yet. Therefore, the results in this paper are new in the literature.

The main objective of the present study is to develop further technical methods in the theory of partial differential equations [8]. In the present work, we study the QOLCT that transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency-domain signal. Some important properties of the two-sided QOLCT are established. Well-known UPs for the two-sided QOLCT are generalized.

The rest of the paper is organized as follows: Section 2 gives a brief introduction to some general definitions and basic properties of quaternionic analysis and contains a reminder of the definition and some results for the two-sided QFT useful in the sequel. The QOLCT of a 2D quaternionic signal is introduced and studied in Section 3. Some important properties such as Rayleigh’s theorem are obtained, we also give the QOLCT of Gaussian quaternionic functions (Gabor filters) to be indeed the only functions that minimize Heisenberg-Weyl’s UP associated with the QOLCT, which has been proven in Section 4. In this section, we generalize the corresponding results of Hardy’s UP, Beurling’s UP, and the logarithmic UP to the QOLCT domain. In Section 5, we conclude the paper.

2. Preliminaries

The quaternion algebra

In the present section, we collect some basic facts about quaternions, which will be needed throughout the paper.

\[ \mathbb{H} = \{ q = q_0 + iq_1 + jq_2 + kq_3; \ q_0, q_1, q_2, q_3 \in \mathbb{R} \}, \]

which is an associative non-commutative four-dimensional algebra.

The elements \( i, j, k \) satisfy Hamilton’s multiplication rules:

\[ ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j; \quad i^2 = j^2 = k^2 = -1. \]

In this way, quaternion algebra can be seen as an extension of the complex field \( \mathbb{C} \).

Quaternions are isomorphic to the Clifford algebra \( Cl_{(0,2)} \) of \( \mathbb{R}^{(0,2)} \):

\[ \mathbb{H} \cong Cl_{(0,2)}. \]

The scalar part of a quaternion \( q \in \mathbb{H} \) is \( q_0 \) denoted by \( Sc(q) \), the non-scalar part (or pure quaternion) of \( q \) is \( iq_1 + jq_2 + kq_3 \) denoted by \( Vec(q) \).

The quaternion conjugate of \( q \in \mathbb{H} \) defined by

\[ \overline{q} = q_0 - iq_1 - jq_2 - kq_3, \]

is an anti-involution, namely,

\[ \overline{qp} = \overline{p} \overline{q}, \quad \overline{p + q} = \overline{p} + \overline{q}, \quad \overline{\overline{p}} = p. \]
The norm or modulus of \( q \in \mathbb{H} \) is defined by
\[
|q|_Q = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2},
\] (6)
then, we have
\[
|pq|_Q = |p|_Q |q|_Q.
\] (7)
In particular, when \( q = q_0 \) is a real number, the modulus \( |q|_Q \) reduces to the ordinary Euclidean modulus \( |q| = \sqrt{q_0^2} \).

It is easy to verify that \( 0 \neq q \in \mathbb{H} \) implies:
\[
q^{-1} = \frac{\bar{q}}{|q|^2_Q}.
\] (8)
Any quaternion \( q \) can be written as \( q = |q|Qe^{i\theta} \), where \( e^{i\theta} = \cos(\theta) + i \sin(\theta) \), the generalized Euler formula for the quaternions, with
\[
\theta = \arctan \frac{|\text{Vec}(q)|_Q}{|q|_Q}, \quad 0 \leq \theta \leq \pi \text{ and } u \overset{\text{def}}{=} \frac{\text{Vec}(q)}{|\text{Vec}(q)|_Q} \text{ verifying } u^2 = -1.
\]
In this paper, we will study the quaternion-valued signal \( f : \mathbb{R}^2 \to \mathbb{H} \), which can be expressed as \( f = f_0 + if_1 + jf_2 + kf_3 \), with \( f_m : \mathbb{R}^2 \to \mathbb{R} \) for \( m = 0, 1, 2, 3 \).

Let us introduce the left quaternionic inner product for quaternion valued functions \( f, g : \mathbb{R}^2 \to \mathbb{H} \), as follows:
\[
\langle f, g \rangle = \int_{\mathbb{R}^2} f(t) \bar{g}(t) \, dt.
\] (9)
Hence, the natural norm is given by
\[
|f|_{2,Q} = \sqrt{\langle f, f \rangle} = \left( \int_{\mathbb{R}^2} |f(t)|^2_Q \, dt \right)^{\frac{1}{2}},
\] (10)
and the left quaternionic Hilbert space \( L^2(\mathbb{R}^2, \mathbb{H}) \), is given by
\[
L^2(\mathbb{R}^2, \mathbb{H}) = \left\{ f : \mathbb{R}^2 \to \mathbb{H}, \ |f|_{2,Q} < \infty \right\}.
\] (11)
We denote by \( \mathcal{S}(\mathbb{R}^2, \mathbb{H}) \), the quaternion Schwartz space of \( C^\infty \)-functions \( f \), from \( \mathbb{R}^2 \) to \( \mathbb{H} \), such that for all \( m, n \in \mathbb{N} \)
\[
\sup_{t \in \mathbb{R}^2, \alpha_1 + \alpha_2 \leq n} \left( 1 + |t|^m \right) \left| \frac{\partial^{\alpha_1 + \alpha_2}}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2}} f(t) \right|_Q < \infty, \quad \text{where } (\alpha_1, \alpha_1) \in \mathbb{N}^2.
\] (12)
Throughout this paper, we use the following real vector notation:
\( t = (t_1, t_2) \in \mathbb{R}^2, \ f(t) = f(t_1, t_2), \ dt = dt_1 dt_2, \) and so on.
2.1. The general two-sided QFT

The QFT which has been defined by Ell [9] is a generalization of the classical Fourier transform (CFT) using the quaternionic algebra framework. Several known and useful properties and theorems of this extended transform are generalizations of the corresponding ones of the CFT with some modifications (e.g. [1,3,9–13]). The QFT belongs to the family of Clifford Fourier transformations because of (3). There are three different types of QFT, the left-sided QFT, the right-sided QFT, [14] and two-sided QFT [15].

Let us define the two-sided QFT and provide some properties used in the sequel.

Definition 2.1 (Two-sided QFT with respect to two pure unit quaternions \( \lambda; \mu \) [16]): Let \( \lambda, \mu \in \mathbb{H}, \lambda^2 = \mu^2 = -1 \), be any two pure unit quaternions.

For \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), the two-sided QFT with respect to \( \lambda; \mu \) is

\[
F_{\lambda, \mu} \{ f \}(u) = \int_{\mathbb{R}^2} e^{-\lambda u_1 t_1} f(t) e^{-\mu u_2 t_2} \, dt, \quad \text{where} \ t, \ u \in \mathbb{R}^2.
\]  

(13)

We define a new modulus of \( F_{\lambda, \mu} \{ f \} \) as follows:

\[
\| F_{\lambda, \mu} \{ f \} \|_Q \overset{\text{def}}{=} \sqrt{\sum_{m=0}^{m=3} \| F_{\lambda, \mu} \{ f_m \} \|^2_Q}.
\]  

(14)

Furthermore, we define a new \( L^2 \)-norm of \( F \{ f \} \) by

\[
\| F_{\lambda, \mu} \{ f \} \|_{2,Q} \overset{\text{def}}{=} \sqrt{\int_{\mathbb{R}^2} \| F_{\lambda, \mu} \{ f \}(y) \|^2_Q \, dy}.
\]  

(15)

Notice that \( \| . \|_Q \), which was introduced at first in [11], is a norm indeed (for the proof see [3, Lemma 3.3]) and is used throughout this paper only to state Theorem 4.6.

Lemma 2.2 ((Dilation property), see example 2 on page 50 [10]): Let \( k_1, k_2 \) be a positive scalar constants, we have

\[
F_{\lambda, \mu} \{ f(t_1, t_2) \} \left( \frac{u_1}{k_1}, \frac{u_2}{k_1} \right) = k_1 k_2 F_{\lambda, \mu} \{ f(k_1 t_1, k_2 t_2) \} (u_1, u_2).
\]  

(16)

The following lemma is known as Rayleigh Energy Theorem (Parseval’s Theorem), it states that the \( L^2 \)-norm of a signal is invariant under the quaternion Fourier transform.

Lemma 2.3 (QFT Rayleigh): Let \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), then

\[
\int_{\mathbb{R}^2} \| F_{\lambda, \mu} \{ f \}(u) \|^2_Q \, du = 4\pi^2 \int_{\mathbb{R}^2} |f(t)|^2_Q \, dt.
\]  

(17)

Proof: See [[10], Thm. 2.7].

Lemma 2.4: If \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), \( \frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} f \) exist and are in \( L^2(\mathbb{R}^2, \mathbb{H}) \) for \( m, n \in \mathbb{N}_0 \), then

\[
F_{\lambda, \mu} \left\{ \frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} f \right\}(u) = (\lambda u_1)^m \, F_{\lambda, \mu} \{ f \}(u) \, (\mu u_2)^n.
\]  

(18)
Lemma 2.5 (Inverse QFT [17]): If \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), and \( \mathcal{F}^{\lambda,\mu}(f) \in L^1(\mathbb{R}^2, \mathbb{H}) \), then the two-sided QFT is an invertible transform and its inverse is given by
\[
f(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\lambda u_1 t_1} \mathcal{F}^{\lambda,\mu}(f)(u)e^{\mu u_2 t_2} \, du.
\] (19)

3. The offset quaternionic linear canonical transform

In [6], Kou et al. introduced the quaternionic linear canonical transform (QLCT) by considering a pair of unit determinant two-by-two matrices
\[
A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \mathbb{R}^{2 \times 2},
\]
with unit determinant, that is \( a_1d_1 - b_1c_1 = 1 \), \( a_2d_2 - b_2c_2 = 1 \).

The third author [18] generalized the definitions of [6] to be:

The two-sided QLCT of signals \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), is defined as
\[
L^{\lambda,\mu}_{A_1;A_2}(f)(u) = \int_{\mathbb{R}^2} K^{\lambda}_{A_1}(t_1,u_1)f(t)K^{\mu}_{A_2}(t_2,u_2) \, dt,
\] (20)
with \( \lambda, \mu \in \mathbb{H} \), two pure unit quaternions, \( \lambda^2 = \mu^2 = -1 \), including the cases \( \lambda = \pm \mu \),
\[
K^{\lambda}_{A_1}(t_1,u_1) = \frac{1}{\sqrt{2\pi}} e^{\lambda(a_1t_1^2 - 2t_1u_1 + d_1u_1^2)/2b_1},
\]
\[
K^{\mu}_{A_2}(t_2,u_2) = \frac{1}{\sqrt{2\pi}} e^{\mu(a_2t_2^2 - 2t_2u_2 + d_2u_2^2)/2b_2}.
\]

In [6], for \( \lambda = i \) and \( \mu = j \), the right-sided QLCT and its properties, including an UP are studied in some detail.

We now generalize the definitions of [16,17] as follows:

Definition 3.1: Let \( A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \), parameters \( a_i, b_i, c_i, d_i, \tau_i, \eta_i \in \mathbb{R} \) such as \( a_id_i - b_ic_i = 1 \), for \( i = 1,2 \).

The two-sided quaternionic offset linear canonical transform (QOLCT) of a signal \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \), is given by \( \mathcal{O}^{\lambda,\mu}_{A_1;A_2}(f(t))(u) \)
\[
= \begin{cases} 
\int_{\mathbb{R}^2} K^{\lambda}_{A_1}(t_1,u_1)f(t)K^{\mu}_{A_2}(t_2,u_2) \, dt, & \text{if } b_1, b_2 \neq 0, \\
\sqrt{d_1}e^{\frac{\lambda}{2}(u_1 - \tau_1)^2 + u_1\tau_1}f((d_1(u_1 - \tau_1), t_2)K^{\mu}_{A_2}(t_2,u_2), & \text{if } b_1 = 0, b_2 \neq 0, \\
\sqrt{d_2}K^{\lambda}_{A_1}(t_1,u_1)f(t_1,d_2(u_2 - \tau_2))e^{\mu\left(\frac{c_2d_2}{2}((u_2 - \tau_2)^2 + u_2\tau_2)\right)}, & \text{if } b_1 \neq 0, b_2 = 0, \\
\sqrt{d_1d_2}f((d_1(u_1 - \tau_1), d_2(u_2 - \tau_2))e^{\lambda\left(\frac{c_1d_1}{2}(u_1 - \tau_1)^2 + u_1\tau_1\right)}), & \text{if } b_1 = b_2 = 0, \\
\mu\left(\frac{c_2d_2}{2}((u_2 - \tau_2)^2 + u_2\tau_2)\right), & \text{if } b_1 = b_2 = 0,
\end{cases}
\] (21)
with \( K_{A_1}^\lambda (t_1, u_1) = \frac{1}{\sqrt{2\pi b_1}} e^{\lambda(a_1 t_1^2 - 2t_1(u_1 - \tau_1) - 2u_1(t_1 - b_1 \eta_1) + d_1(u_1^2 + \tau_1^2)) + \frac{1}{2\lambda t_1}} \), for \( b_1 \neq 0 \), and
\[ K_{A_2}^\mu (t_2, u_2) = \frac{1}{\sqrt{2\pi b_2}} e^{\mu(a_2 t_2^2 - 2t_2(u_2 - \tau_2) - 2u_2(t_2 - b_2 \eta_2) + d_2(u_2^2 + \tau_2^2)) + \frac{1}{2\mu t_2}} , \]
for \( b_2 \neq 0 \), with \( \frac{1}{\sqrt{\lambda}} = e^{-\frac{\lambda}{\tau_1}}, \frac{1}{\sqrt{\mu}} = e^{-\frac{\mu}{\tau_2}}. \)

The left-sided and right-sided QOLCTs can be defined correspondingly by placing the two kernel factors both on the left or on the right, respectively. We note that when \( \tau_1 = \tau_2 = \eta_1 = \eta_2 = 0 \), the two-sided QOLCT reduces to the QLCT. Also, when \( A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), the conventional two-sided QFT is recovered. Namely,
\[
O^\lambda_{A_1, A_2} \{ f(t) \}(u) = \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}^2} e^{-\lambda t_1 u_1} f(t) e^{-\mu t_2 u_2} \, dt \right) \frac{1}{\sqrt{2\pi} \lambda b_1},
\]
where \( \mathcal{F}^\lambda_{A_1, A_2}(f) \) is the QFT of \( f \) given by (13).

For simplicity’s sake, in this paper, we restrict our attention to the two-sided QLCTs of 2D quaternion-valued signals. Note that when \( b_1 b_2 = 0 \) or \( b_1 = b_2 = 0 \), the QOLCT of a function is essentially a chirp multiplication and is of no particular interest in our objective interests. Hence, we deal with only the case when \( b_1 b_2 \neq 0 \) in this paper, without loss of generality, we set \( b_1 > 0 \) \( (l = 1, 2) \).

The following lemma gives the relationships of two-sided QOLCTs and two-sided QFTs of 2D quaternion-valued signals. It will prove useful for our analysis of the QOLCT.

**Lemma 3.2:** The QOLCT of a signal \( f \in L^1(\mathbb{R}^2, \mathbb{H}) \) can be reduced to the QFT
\[
O^\lambda_{A_1, A_2} \{ f \} (u_1, u_2) = e^{\lambda \left[ -\frac{1}{b_1} u_1(d_1 t_1 - b_1 \eta_1) + \frac{d_1}{2b_1}(u_1^2 + \tau_1^2) \right]} \mathcal{F}^\mu_{A_1, A_2} \{ h \} \left( \frac{u_1}{b_1} \right) \left( \frac{u_2}{b_2} \right),
\]
where
\[
h(t) = \frac{1}{\sqrt{2\pi \lambda b_1}} e^{\lambda \left( \frac{d_1}{2b_1} t_1 + \frac{d_1}{2b_1} \tau_1^2 \right)} f(t) e^{\mu \left( \frac{d_2}{2b_2} t_2 + \frac{d_2}{2b_2} \tau_2^2 \right)} \frac{1}{\sqrt{2\pi} \mu b_2}.
\]

**Proof:** From the definition of the QOLCT, we have
\[
O^\lambda_{A_1, A_2} \{ f \} (u_1, u_2) = \int_{\mathbb{R}^2} K_{A_1}^\lambda (t_1, u_1) f(t) K_{A_2}^\mu (t_2, u_2) \, dt
\]
\[
= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi \lambda b_1}} e^{\lambda \left[ \frac{d_1}{2b_1} t_1 - \frac{1}{b_1} t_1 (u_1 - \tau_1) - \frac{1}{b_1} u_1(d_1 t_1 - b_1 \eta_1) + \frac{d_1}{2b_1}(u_1^2 + \tau_1^2) \right]} f(t) \frac{1}{\sqrt{2\pi \mu b_2}}
\]
\[
\times e^{\mu \left[ \frac{d_2}{2b_2} t_2 - \frac{1}{b_2} t_2 (u_2 - \tau_2) - \frac{1}{b_2} u_2(d_2 t_2 - b_2 \eta_2) + \frac{d_2}{2b_2}(u_2^2 + \tau_2^2) \right]} \, dt
\]
\[
= e^{\lambda \left[ -\frac{1}{b_1} u_1(d_1 t_1 - b_1 \eta_1) + \frac{d_1}{2b_1}(u_1^2 + \tau_1^2) \right]} \int_{\mathbb{R}^2} e^{-\lambda \frac{1}{b_1} t_1 u_1} \frac{1}{\sqrt{2\pi \lambda b_1}} e^{\lambda \left[ \frac{d_1}{2b_1} t_1 + \frac{d_1}{2b_1} \tau_1^2 \right]} f(t)
\]
\[
= e^{\lambda \left[ -\frac{1}{b_1} u_1(d_1 t_1 - b_1 \eta_1) + \frac{d_1}{2b_1}(u_1^2 + \tau_1^2) \right]} \int_{\mathbb{R}^2} e^{-\lambda \frac{1}{b_1} t_1 u_1} \frac{1}{\sqrt{2\pi \lambda b_1}} e^{\lambda \left[ \frac{d_1}{2b_1} t_1 + \frac{d_1}{2b_1} \tau_1^2 \right]} f(t)
\]
\[
= e^{\lambda \left[ -\frac{1}{b_1} u_1(d_1 t_1 - b_1 \eta_1) + \frac{d_1}{2b_1}(u_1^2 + \tau_1^2) \right]} \mathcal{F}^\mu_{A_1, A_2} \{ h \} \left( \frac{u_1}{b_1} \right) \left( \frac{u_2}{b_2} \right).
\]
Theorem 3.4: \( \text{We derive shift and modulation properties for the QOLCT.} \)

\[ QO L CT \text{ can be derived from the QFT.} \]

Theorem 3.5: \( \text{Let } f \in L^1(\mathbb{R}^2, \mathbb{H}). \text{ Then its QOLCT satisfies:} \)

- **The map** \( f \to \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f) \) **is real linear. That is, for** \( \alpha, \beta \in \mathbb{R}, \text{ we have} \)
  \[
  \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(\alpha f + \beta g) = \alpha \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f) + \beta \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(g). \tag{24}
  \]

- **\( \lim_{|u| \to \infty} |\mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f)(u)|_Q = 0. \)**

- **\( \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f) \) is uniformly continuous on** \( \mathbb{R}^2. \)

Following the proofs of [19, Thm. 11, Thm. 12], and by straightforward computation we derive shift and modulation properties for the QOLCT.

Theorem 3.4: \( \text{Let } f \in L^1(\mathbb{R}^2, \mathbb{H}), \text{ with } t, u \in \mathbb{R}^2, \text{ constants } \xi = (\xi_1, \xi_2), k = (k_1, k_2) \in \mathbb{R}^2. \text{ We have:} \)

- **Shift property**
  \[
  \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f(t - k))(u)
  = e^{\lambda \left( -\frac{a_1^2}{2} + k_1u_1c_1 + k_1(t_1c_1 - a_1) \right)} \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f(t))(u - k_1a_1, u_2 - k_2a_2)
  \times e^{\mu \left( -\frac{a_2^2}{2} + k_2u_2c_2 - k_2(t_2c_2 - a_2) \right)}. \tag{25}
  \]

- **Modulation property**
  \[
  \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(e^{\lambda t_1 \xi_1} f(t) e^{\mu t_2 \xi_2})(u)
  = e^{-\lambda \left[ \frac{d_1}{2} (b_1 \xi_1^2 - 2 \xi_1 u_1) + \xi_1 (d_1 t_1 - b_1) \right]} \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f(t))(u_1 - b_1 \xi_1, u_2 - b_2 \xi_2)
  \times e^{-\mu \left[ \frac{d_2}{2} (b_2 \xi_2^2 - 2 \xi_2 u_2) + \xi_2 (d_2 t_2 - b_2) \right]}. \tag{27}
  \]

Theorem 3.5: \( \text{If } f \text{ and } \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f) \text{ are in } L^1(\mathbb{R}^2, \mathbb{H}), \text{ then the inverse transform of the QOLCT can be derived from that of the QFT.} \)
Proof: Indeed, let
\[ g(t) = e^{\frac{1}{\sqrt{2\pi}} t_1 + \frac{\lambda}{\sqrt{2\pi}} t_2} f(t) e^{\frac{\mu}{\sqrt{2\pi}} t_2 t_2 + \frac{\alpha}{\sqrt{2\pi}} t_2^2}. \] (28)

We have
\[
\mathcal{O}^{\lambda,\mu}_{A_1,A_2} \{ f \} (u_1, u_2) = \frac{1}{\sqrt{2\pi \lambda b_1}} e^{-\frac{1}{\sqrt{2\pi}} u_1 (d_1 t_1 - b_1 \eta_1) + \frac{\lambda}{\sqrt{2\pi}} u_1^2 + \frac{\tau_1^2}{2}} \mathcal{F}^{\lambda,\mu} \{ g \} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) 
\times e^{-\frac{1}{\sqrt{2\pi}} u_2 (d_2 t_2 - b_2 \eta_2) + \frac{\mu}{\sqrt{2\pi}} u_2^2 + \frac{\tau_2^2}{2}} \frac{1}{\sqrt{2\pi \mu b_2}}.
\]

This implies
\[
\mathcal{F}^{\lambda,\mu} \{ g \} (u_1, u_2) = \sqrt{2\pi \lambda b_1} e^{\frac{\lambda u_1 (d_1 t_1 - b_1 \eta_1) - \lambda d_1}{\sqrt{2\pi}} (b_1^2 u_1^2 + \tau_1^2)} \mathcal{O}^{\lambda,\mu}_{A_1,A_2} \{ f \} (b_1 u_1, b_2 u_2) 
\times e^{\mu u_2 (d_2 t_2 - b_2 \eta_2) - \mu d_2}{\sqrt{2\pi \mu b_2}} (b_2^2 u_2^2 + \tau_2^2) \frac{1}{\sqrt{2\pi \mu b_2}}.
\]

From Lemma 2.5, it follows that
\[
g(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\lambda t_1 u_1} \mathcal{F}^{\lambda,\mu} \{ g \} (u) e^{\mu t_2 u_2} du,
\]
or, equivalently
\[
e^{\frac{1}{\sqrt{2\pi}} t_1 + \frac{\lambda}{\sqrt{2\pi}} t_2 + \frac{\mu}{\sqrt{2\pi}} t_2^2} f(t) e^{\frac{\mu}{\sqrt{2\pi}} t_2 t_2 + \frac{\alpha}{\sqrt{2\pi}} t_2^2} 
\times \mathcal{O}^{\lambda,\mu}_{A_1,A_2} \{ f \} (b_1 u_1, b_2 u_2) e^{\mu u_2 (d_2 t_2 - b_2 \eta_2) - \mu d_2}{\sqrt{2\pi \mu b_2}} (b_2^2 u_2^2 + \tau_2^2) e^{\mu t_2 u_2} \sqrt{2\pi \mu b_2} du.
\]

It means that
\[
f(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \sqrt{2\pi \lambda b_1} e^{-\frac{1}{\sqrt{2\pi}} t_1 + \frac{\lambda}{\sqrt{2\pi}} t_2 + \frac{\mu}{\sqrt{2\pi}} t_2^2} \mathcal{F}^{\lambda,\mu} \{ g \} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) 
\times \mathcal{O}^{\lambda,\mu}_{A_1,A_2} \{ f \} (b_1 u_1, b_2 u_2) e^{\mu u_2 (d_2 t_2 - b_2 \eta_2) - \mu b_2}{\sqrt{2\pi \mu b_2}} (b_2^2 u_2^2 + \tau_2^2) e^{\mu t_2 u_2} \frac{1}{\sqrt{2\pi \mu b_2}} 
\times \sqrt{2\pi \mu b_2} K^{\mu}_{A_2} (t_2, b_2 u_2) du
\]
\[
= b_1 b_2 \int_{\mathbb{R}^2} K^{\lambda}_{A_1} (t_1, b_1 u_1) \mathcal{O}^{\lambda,\mu}_{A_1,A_2} \{ f \} (b_1 u_1, b_2 u_2) \frac{K^{\mu}_{A_2} (t_2, b_2 u_2)}{\sqrt{2\pi \mu b_2}} du
\]
\[
= \int_{\mathbb{R}^2} K^{\lambda}_{A_1} (t_1, u_1) \mathcal{O}^{\lambda,\mu}_{A_1,A_2} \{ f \} (u_1, u_2) \frac{K^{\mu}_{A_2} (t_2, u_2)}{\sqrt{2\pi \mu b_2}} du,
\]
which is the inverse transform of the QOLCT. This proves the theorem. \(\blacksquare\)
Theorem 3.6 (Rayleigh’s theorem of the QOLCT): Every 2D quaternion-valued signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) and its QOLCT are related to the Rayleigh identity in the following way:

\[
\left| \mathcal{O}^{\lambda, \mu}_{A_1, A_2} \{ f \} \right|_{Q, 2} = \left| f \right|_{Q, 2}.
\]  

Proof: Let \( h \) be rewritten in the form of (23).

By the definition of the norm \( \left| \cdot \right|_{Q, 2} \), Lemmas 2.2 and 3.2, we have

\[
\left| \mathcal{O}^{\lambda, \mu}_{A_1, A_2} \{ f \} \right|_{Q, 2}^2 = \int_{\mathbb{R}^2} \left| e^{\lambda \left\{ \int \frac{1}{b_1} u_1 (d_1 t_1 - b_1 \eta_1) + \frac{d_1}{b_1^2} (u_1^2 + \tau_1^2) \right\} f} \} \{ h \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) \right|^2 Q du \\
\times e^{\mu \left\{ \int \frac{1}{b_2} u_2 (d_2 t_2 - b_2 \eta_2) + \frac{d_2}{b_2^2} (u_2^2 + \tau_2^2) \right\}^2 Q du}
\]

\[
= \int_{\mathbb{R}^2} \left| \mathcal{F}^{\lambda, \mu} \{ h(t) \} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) \right|^2 Q du
\]

\[
= \int_{\mathbb{R}^2} \left| b_1 b_2 \mathcal{F}^{\lambda, \mu} \{ h(b_1 t_1, b_2 t_2) \} (u_1, u_2) \right|^2 Q du
\]

\[
= b_1^2 b_2^2 \int_{\mathbb{R}^2} \left| \mathcal{F}^{\lambda, \mu} \{ h(b_1 t_1, b_2 t_2) \} (u_1, u_2) \right|^2 Q du.
\]

From Lemma 2.3 we get

\[
\int_{\mathbb{R}^2} \left| \mathcal{F}^{\lambda, \mu} \{ h(b_1 t_1, b_2 t_2) \} (u_1, u_2) \right|^2 Q du = 4\pi^2 \int_{\mathbb{R}^2} \left| h(b_1 t_1, b_2 t_2) \right|_Q^2 dt.
\]

Let \( s_l = b_l t_l \), for \( l = 1, 2 \), we have

\[
\int_{\mathbb{R}^2} \left| h(b_1 t_1, b_2 t_2) \right|_Q^2 dt = \frac{1}{b_1 b_2} \int_{\mathbb{R}^2} \left| h(s_1, s_2) \right|_Q^2 ds
\]

\[
= \frac{1}{4\pi^2 b_1^2 b_2^2} \int_{\mathbb{R}^2} \left| f(s_1, s_2) \right|_Q^2 ds.
\]

The last statement follows from \( |h(t)|_Q = \frac{1}{2\pi \sqrt{b_1 b_2}} |f(t)|_Q \), therefore, we get

\[
\left| \mathcal{O}^{\lambda, \mu}_{A_1, A_2} \{ f \} \right|_{Q, 2}^2 = \int_{\mathbb{R}^2} \left| f(s) \right|_Q^2 ds = \left| f \right|_{Q, 2}^2.
\]

This ends the proof. \( \blacksquare \)

Lemma 3.7: If \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), \( \frac{\partial}{\partial t_l} f \) exist and are in \( L^2(\mathbb{R}^2, \mathbb{H}) \) for \( l = 1, 2 \), then

\[
(1) \int_{\mathbb{R}^2} u_1^2 \left| \mathcal{O}^{\lambda, \mu}_{A_1, A_2} \{ f(u) \} \right|^2 Q du = b_1^2 \int_{\mathbb{R}^2} \left| \lambda \left( \frac{a_1}{b_1} t_1 + \frac{\tau_1}{b_1} \right) f(t) + \frac{\partial}{\partial t_1} f(t) \right|^2 Q dt.
\]

\[
(2) \int_{\mathbb{R}^2} u_2^2 \left| \mathcal{O}^{\lambda, \mu}_{A_1, A_2} \{ f(u) \} \right|^2 Q du = b_2^2 \int_{\mathbb{R}^2} \left| \left( \frac{a_2}{b_2} t_2 + \frac{\tau_2}{b_2} \right) f(t) \mu + \frac{\partial}{\partial t_2} f(t) \right|^2 Q dt.
\]
Proof: Let $h$ be rewritten in the form of (23). For the first statement, using Lemma 2.4 shows that

$$F^{\lambda,\mu} \left\{ \frac{\partial}{\partial t_1} f \right\} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) = \frac{\partial}{\partial t_1} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right).$$

Then, using Lemma 3.2, (14), Lemma 2.4 and the above equality we get

$$\int_{\mathbb{R}^2} u_1^2 \left| \mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{ f \} (u) \right|^2_Q \, du = \int_{\mathbb{R}^2} \left| \frac{1}{\lambda} b_1 F^{\lambda,\mu} \left\{ \frac{\partial}{\partial t_1} h \right\} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) \right|^2_Q \, du
= \int_{\mathbb{R}^2} \frac{1}{\lambda^2} b_1^2 \left| F^{\lambda,\mu} \left\{ \frac{\partial}{\partial t_1} h \right\} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) \right|^2_Q \, du
= b_1^2 \int_{\mathbb{R}^2} \left| F^{\lambda,\mu} \left\{ \frac{\partial}{\partial t_1} h \right\} \left( \frac{u_1}{b_1}, \frac{u_2}{b_2} \right) \right|^2_Q \, du
= b_1^2 \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial t_1} h(t) \right|^2_Q \, dt,$$

where the last equation is the consequence of using Lemma 2.3. Moreover, using

$$\left| \frac{\partial}{\partial t_1} h(t) \right|^2_Q = \left| \frac{\partial}{\partial t_1} \left( \frac{1}{\sqrt{2\pi \lambda b_1}} e^{\frac{\lambda}{\sqrt{2\pi \lambda b_1}^2} t_1^2 + \frac{\lambda^2}{2\pi \mu b_2} t_2^2} f(t) e^{\mu \left( \frac{a_1^2 t_1^2 + \tau_1^2}{2b_2} \right)} \frac{1}{\sqrt{2\pi \mu b_2}} \right) \right|^2_Q
= \frac{1}{2\pi \sqrt{b_1 b_2}} \left| e^{\lambda \left( \frac{a_1^2 t_1^2 + \tau_1^2}{2b_2} \right)} \left[ \lambda \left( \frac{a_1}{b_1} t_1 + \frac{\tau_1}{b_1} \right) f(t) + \frac{\partial}{\partial t_1} f(t) \right] \right|^2_Q
= \frac{1}{2\pi \sqrt{b_1 b_2}} \left| \lambda \left( \frac{a_1}{b_1} t_1 + \frac{\tau_1}{b_1} \right) f(t) + \frac{\partial}{\partial t_1} f(t) \right|^2_Q,$$

we further get

$$\int_{\mathbb{R}^2} u_1^2 \left| \mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{ f \} (u) \right|^2_Q \, du = b_1^2 \int_{\mathbb{R}^2} \left| \lambda \left( \frac{a_1}{b_1} t_1 + \frac{\tau_1}{b_1} \right) f(t) + \frac{\partial}{\partial t_1} f(t) \right|^2_Q \, dt.$$

To prove the statement 2, we argue in the same way as in the previous proof.
Applying Lemma 3.2, (14), Lemmas 2.4 and 2.3 shows that

\[ \int_{\mathbb{R}^2} u_2^2 |O_{A_1 A_2}^{\lambda, \mu} \{ f \} (u) |^2 \, du = \int_{\mathbb{R}^2} u_2^2 |F_{\lambda, \mu}^{A_1, A_2} \{ h(t) \} (u_1/b_1, u_2/b_2) |^2 \, du \]

\[ = \int_{\mathbb{R}^2} b_2^2 \int_{\mathbb{R}^2} |F_{\lambda, \mu}^{A_1, A_2} \{ \frac{\partial}{\partial t_2} h \} (u_1/b_1, u_2/b_2) |^2 \, du \]

\[ = b_2^2 \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial t_2} h \right|^2 \, dt. \]

Since \( \left| \frac{\partial}{\partial t_2} h(t) \right|_{Q} = \frac{1}{2 \pi \sqrt{b_1 b_2}} |f(t) \mu (\frac{\partial}{\partial t_2} t_2 + \frac{\lambda}{b_2}) + \frac{\partial}{\partial t_2} f(t) |_{Q}, \) it follows that

\[ \int_{\mathbb{R}^2} u_2^2 |O_{A_1 A_2}^{\lambda, \mu} \{ f \} (u) |^2 \, du = b_2^2 \int_{\mathbb{R}^2} |(\frac{\partial}{\partial t_2} t_2 + \frac{\lambda}{b_2}) f(t) \mu + \frac{\partial}{\partial t_2} f(t) |_{Q}^2 \, dt. \]
Table 1. Properties of the quaternionic offset linear canonical transform (QOLCT).

| Property                        | Function                                                                 | QOLCT                                                                 |
|---------------------------------|--------------------------------------------------------------------------|-----------------------------------------------------------------------|
| Real linearity                  | $a f + \beta g$                                                          | $\alpha \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f) + \beta \mathcal{O}_{A_1, A_2}^{\lambda, \mu}(g)$ |
| Rayleigh's identity             | $|f|_{Q, 2}$                                                             | $|\mathcal{O}_{A_1, A_2}^{\lambda, \mu}(f)|_{Q, 2}$ |

Note: Let $f$ and $g \in L^1(\mathbb{R}^2, \mathbb{H})$, the constants $\alpha$ and $\beta \in \mathbb{R}$, $u \in \mathbb{R}^2$, $A_1 = \left[\begin{smallmatrix} a_1 & b_1 & \eta_1 \\ c_1 & d_1 & \eta_1 \end{smallmatrix}\right]$, parameters $a_i, b_i, c_i, d_i, \eta_1, \eta_2 \in \mathbb{R}$ such that $a_i d_i - b_i c_i = 1$, for $i = 1, 2$.

$$
\int_{\mathbb{R}} K_{A_1}^{\lambda}(t_1, u_1) e^{-\alpha_1 t_1^2} dt_1 = \frac{1}{\sqrt{2b_1 \alpha_1 + a_1}} e^{-\frac{a_1 (u_1 - \tau_1)^2}{4 \alpha_1^2 t_1^2 + a_1^2}} \left(2u_1 (d_1 t_1 - b_1 \eta_1) + d_1 (u_1^2 + \tau_1^2) - \frac{a_1 (u_1 - \tau_1)^2}{4 \alpha_1^2 t_1^2 + a_1^2}\right) \frac{1}{2b_1}.
$$

Consequently,

$$
\mathcal{O}_{A_1, A_2}^{\lambda, \mu}\{f\}(u_1, u_2) = e^{\frac{a_1 (u_1 - \tau_1)^2}{4 \alpha_1^2 t_1^2 + a_1^2}} e^{\frac{a_2 (u_2 - \tau_2)^2}{4 \alpha_2^2 t_2^2 + a_2^2}} \frac{1}{\sqrt{2b_1 \alpha_1 + a_1 \lambda}} \beta_1 \frac{1}{\sqrt{2b_2 \alpha_2 + a_2 \mu}} 
$$

$$
\times e^{\lambda \left(-2u_1 (d_1 t_1 - b_1 \eta_1) + d_1 (u_1^2 + \tau_1^2) - \frac{a_1 (u_1 - \tau_1)^2}{4 \alpha_1^2 t_1^2 + a_1^2}\right) \frac{1}{2b_1}} \times e^{\mu \left(-2u_2 (d_2 t_2 - b_2 \eta_2) + d_2 (u_2^2 + \tau_2^2) - \frac{a_2 (u_2 - \tau_2)^2}{4 \alpha_2^2 t_2^2 + a_2^2}\right) \frac{1}{2b_2}} \beta_2.
$$

Some properties of the QOLCT are summarized in Table 1.

4. Uncertainty principles for the offset quaternionic linear canonical transform

In harmonic analysis, roughly speaking, the UP states that a non-trivial function and its FT cannot both be sharply localized. The UP plays an important role in signal processing and quantum mechanics. In quantum mechanics, the UP asserts that one cannot measure the position and momentum of a particle at the same time, i.e. increasing the knowledge of the
position decreases the knowledge of the momentum and vice versa. There are many different forms of UPs in the time-frequency plane, such as Heisenberg-Weyl’s UP, Hardy’s UP, Beurling’s UP, and the logarithmic UP, and so on in terms of different notations of ‘localization’. As far as we know, in 2013, Kit-Ian Kou et al. [7] extended the Heisenberg-type UP to the QLCT. Recently Huo [20] generalized some different UPs for the OLCT of a signal \( f \in L^2(\mathbb{R}) \). Bahri and Ashino [21] established the logarithmic UP associated with the QLCT. Considering that the QOLCT is a generalized version of the QLCT and the quaternionic Fourier, transform and so of the QFT, it is natural and interesting to study the simultaneous localization of a function and its QOLCT by further extending the aforementioned UPs to the QOLCT domain. Therefore, in this section, we prove and generalize Heisenberg-Weyl’s UP, Hardy’s UP, Beurling’s UP, and the logarithmic UP to 2D quaternion-valued signals using the two-sided QOLCT.

4.1. Heisenberg–Weyl’s uncertainty principle

The following proposition is Heisenberg–Weyl uncertainty principle associated with the quaternion Fourier transform provided in terms of absolute covariance. It has been established in [1, Thm. 4.1] for the norm \( \| \cdot \|_{2,Q} \). However, the same proof can be applied using the norm \( \| \cdot \|_{2,Q} \).

**Proposition 4.1:** Let \( f(t) = |f(t)|_{Q} e^{u(t)\theta(t)} \). If \( f, \frac{\partial}{\partial t} f, t_k f \in L^2(\mathbb{R}^2, \mathbb{H}) \) for \( k = 1, 2 \), then

\[
|t_k f(t)|_{2,Q}^2 \left| \frac{\xi_k}{2\pi b_k} \mathcal{O}_1^\mu_{A_1 A_2} \{ f \} (\xi) \right|^2_{2,Q} \geq \frac{1}{16\pi^2} |f(t)|_{2,Q}^4 + \text{COV}_k^2,
\]

where \( \text{COV}_k \) is defined as

\[
\text{COV}_k \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{R}^2} |f(t)|_{Q}^2 |t_k(\frac{\partial}{\partial t} e^{u(t)\theta(t)})|_{Q} \, dt.
\]

The equation holds if and only if \( f(t) = \text{De}^{-a_k t_k^2} e^{u(t)\theta(t)} \) and \( \frac{\partial}{\partial t_k} e^{u(t)\theta(t)} = \delta_k t_k \), where \( a_1, a_2 > 0, D \in \mathbb{R}^+ \) and \( \delta_1, \delta_2 \) are pure quaternions.

**Theorem 4.2:** Suppose that \( f, \frac{\partial}{\partial t} f, t_k f \in L^2(\mathbb{R}^2, \mathbb{H}) \) for \( k = 1, 2 \), then

\[
|t_k f(t)|_{2,Q}^2 \left| \frac{\xi_k}{2\pi b_k} \mathcal{O}_1^\mu_{A_1 A_2} \{ f \} (\xi) \right|^2_{2,Q} \geq \frac{1}{16\pi^2} |f(t)|_{2,Q}^4 + \text{COV}_k^2,
\]

where \( \text{COV}_k \) is defined as

\[
\text{COV}_k \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{R}^2} |f(t)|_{Q}^2 |t_k(\frac{\partial}{\partial t} e^{u(t)\theta(t)})|_{Q} \, dt,
\]

and

\[
e^{u(t)\theta(t)} = \frac{1}{|f(t)|_{Q}^2} \frac{1}{\sqrt{\lambda}} e^{\frac{1}{b_1} t_1 t_1 + \frac{1}{b_2} t_2 t_2 + \mu} f(t) e^{\mu} \frac{1}{b_1} t_1^2 + \frac{1}{b_2} t_2^2 + \frac{1}{\sqrt{\mu}}.
\]

The equation holds if and only if \( f(t) = \text{De}^{-a_k t_k^2} e^{u(t)\theta(t)} \) and \( \frac{\partial}{\partial t_k} e^{u(t)\theta(t)} = \delta_k t_k \) where \( a_1, a_2 > 0, D \in \mathbb{R}^+ \) and \( \delta_1, \delta_2 \) are pure quaternions.

**Proof:** Let \( h \) be rewritten as (23).

Since \( \frac{\partial}{\partial t} f, t_k f \in L^2(\mathbb{R}^2, \mathbb{H}) \), and \( |h(t)|_{Q} = \frac{1}{2\pi b_1 b_2} |f(t)|_{Q} \), we get

\[
|t_k h(t)|_{2,Q}^2 = \int_{\mathbb{R}^2} t_k^2 |h(t)|_{Q}^2 \, dt = \frac{1}{4\pi^2 b_1 b_2} |t_k f(t)|_{2,Q}^2.
\]

and

\[
|h|_{2,Q}^4 = \frac{1}{16\pi^4 b_1^2 b_2^2} |f|_{2,Q}^4.
\]
According to Lemma 3.2, we have

\[ |\xi_k \mathcal{F}^{\lambda,\mu} \{ h \} (2\pi \xi) |^2_{2,Q} = |\xi_k \mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{ f \} (2\pi b_1 \xi_1, 2\pi b_2 \xi_2) |^2_{2,Q} \]

\[ = \frac{1}{4\pi^2 b_1 b_2} |\xi_k \mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{ f \} (\xi) |^2_{2,Q}. \]

Hence, it follows from Proposition 4.1 that

\[ (|t_k h(t)|^2_{2,Q}) (\xi_k^2 |\mathcal{F}^{\lambda,\mu} \{ h \} (2\pi \xi) |^2_{2,Q}) \geq \frac{1}{16\pi^2} |h|^4_{2,Q} + \text{COV}^2_{t_k}, \tag{32} \]

where \( \text{COV}_{t_k} = \frac{1}{2\pi} \int_{\mathbb{R}^2} |h(t)|^2_Q |t_k (\frac{\partial}{\partial t_k} e^{u(t)\theta(t)})|_Q \, dt \), and

\[ e^{u(t)\theta(t)} = \frac{1}{|h(t)|_Q} h(t) \]

\[ = \frac{2\pi \sqrt{b_1 b_2}}{|f(t)|_Q} \frac{1}{\sqrt{2\pi \lambda b_1}} e^{\frac{\lambda}{b_1^2} t_1 \tau_1 + \frac{\lambda}{2b_1^2} \tau_1^2} f(t) e^{\mu \frac{1}{b_2^2} t_2 \tau_2 + \mu \frac{\mu}{2b_2^2} \tau_2^2} \frac{1}{\sqrt{2\pi \mu b_2}} \]

\[ = \frac{1}{|f(t)|_Q} \frac{1}{\sqrt{\lambda}} e^{\frac{\lambda}{b_1^2} t_1 \tau_1 + \frac{\lambda}{2b_1^2} \tau_1^2} f(t) e^{\mu \frac{1}{b_2^2} t_2 \tau_2 + \mu \frac{\mu}{2b_2^2} \tau_2^2} \frac{1}{\sqrt{\mu}}. \]

(32) yields

\[ |t_k f(t)|^2_{2,Q} \frac{\xi_k}{2\pi b_k} \mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{ f \} (\xi) \geq \frac{1}{16\pi^2} |f(t)|^4_{2,Q} + 16\pi^4 b_1^2 b_2^2 \text{COV}^2_{t_k}. \]

After straightforward calculation one obtains

\[ 16\pi^4 b_1^2 b_2^2 \text{COV}^2_{t_k} = \frac{1}{4\pi^2} \left[ \int_{\mathbb{R}^2} \left( |f(t)|_Q \right)^2 \left| t_k \left( \frac{\partial}{\partial t_k} e^{u(t)\theta(t)} \right) \right|_Q \, dt \right]^2. \]

By means of Proposition 4.1, equality holds in (32) if and only if \( \frac{\partial}{\partial t_k} e^{u(t)\theta(t)} = \delta_k t_k \), and

\[ |h(t)|_Q = Ce^{-a_k t_k^2} \]

that is \( |f(t)|_Q = 2\pi \sqrt{b_1 b_2} Ce^{-a_k t_k^2} \), where \( a_1, a_2 > 0, C, D \in \mathbb{R}^+ \) and \( \delta_1, \delta_2 \) are pure quaternions. This proves the theorem.

\[ \square \]

4.2. Hardy’s uncertainty principle

Hardy’s theorem [22] is a qualitative UP, it asserts that it is impossible for a function and its Fourier transform to decrease rapidly simultaneously. The following proposition is Hardy’s UP for the two-sided QFT.
Proposition 4.3 ([1, Thm. 5.3]): Let \(\alpha\) and \(\beta\) be positive constants. Suppose \(f \in L^1(\mathbb{R}^2, \mathbb{H})\) with

\[
|f(t)|_Q \leq Ce^{-\alpha|t|^2}, \quad t \in \mathbb{R}^2,
\]

\[
|\mathcal{F}^{\lambda,\mu}\{f\}(u)|_Q \leq C'e^{-\beta|u|^2}, \quad u \in \mathbb{R}^2,
\]

for some positive constants \(C, C'\). Then, three cases can occur:

(i) If \(\alpha\beta > \frac{1}{4}\), then \(f = 0\).

(ii) If \(\alpha\beta = \frac{1}{4}\), then \(f(t) = Ae^{-\alpha|t|^2}\), with \(A\) a quaternion constant.

(iii) If \(\alpha\beta < \frac{1}{4}\), then there are infinitely many such functions \(f\).

Based on Proposition 4.3, we now state Hardy’s UP associated with the QOLCT, which shows that it is impossible for \(f\) and its two-sided QOLCT to both decrease very rapidly.

Theorem 4.4: Let \(\alpha\) and \(\beta\) be positive constants. Suppose \(f \in L^1(\mathbb{R}^2, \mathbb{H})\) with

\[
|f(t)|_Q \leq Ce^{-\alpha|t|^2}, \quad t \in \mathbb{R}^2,
\]

\[
|\mathcal{O}_{A_1,A_2}^{\lambda,\mu}(b_1u_1, b_2u_2)|_Q \leq C'e^{-\beta|u|^2}, \quad u \in \mathbb{R}^2,
\]

for some positive constants \(C, C'\). Then, three cases can occur:

(i) If \(\alpha\beta > \frac{1}{4}\), then \(f = 0\).

(ii) If \(\alpha\beta = \frac{1}{4}\), then \(f(t) = e^{-\alpha|t|^2}e^{-\lambda \frac{a_1}{2b_1}t_1^2 - \lambda \frac{1}{b_1}t_1 + \lambda \frac{1}{b_1}t_1 \tau_1}Ae^{-\mu \frac{a_2}{2b_2}t_2^2 - \mu \frac{1}{b_2}t_2 + \mu \frac{1}{b_2}t_2 \tau_2}\), where \(A\) is a quaternion constant.

(iii) If \(\alpha\beta < \frac{1}{4}\), then there are infinitely many \(f\).

Proof: The proof is similar in spirit to that of [20]. Let \(g\) be rewritten in the form of (28), then we have

\[
\mathcal{O}_{A_1,A_2}^{\lambda,\mu}\{f\}(u_1, u_2) = \frac{1}{\sqrt{2\pi \lambda b_1}}e^{-\lambda \frac{1}{b_1}u_1(d_1 t_1 - b_1 n_1) + \lambda \frac{d_1}{2b_1}(u_1^2 + \tau_1^2)}\mathcal{F}^{\lambda,\mu}\{g\}(\frac{u_1}{b_1}, \frac{u_2}{b_2})
\times e^{-\mu \frac{a_2}{b_2}u_2(d_2 t_2 - b_2 n_2) + \mu \frac{d_2}{b_2}(u_2^2 + \tau_2^2)} \frac{1}{\sqrt{2\pi \mu b_2}}.
\]

Since

\[
|g(t)|_Q = |f(t)|_Q,
\]

we get \(g \in L^1(\mathbb{R}^2, \mathbb{H})\) and \(|g(t)|_Q \leq Ce^{-\alpha|t|^2}\).

On the other hand, combining (35) and (34) we obtain

\[
|\mathcal{F}^{\lambda,\mu}\{g\}(u_1, u_2)|_Q = 2\pi \sqrt{b_1b_2} |\mathcal{O}_{A_1,A_2}^{\lambda,\mu}\{f\}(b_1u_1, b_2u_2)|_Q
\leq 2\pi \sqrt{b_1b_2} C'e^{-\beta|u|^2}.
\]
Therefore, it follows from Proposition 4.3 that, if $\alpha \beta = \frac{1}{4}$ then $g(t) = Ae^{-\alpha|t|^2}$, for some constant $A \in \mathbb{H}$.

Hence

$$f(t) = e^{-\alpha|t|^2} e^{-\lambda \frac{1}{\beta_1} t_1 \tau_1 - \alpha \frac{1}{\beta_1} t_1^2} e^{-\mu \frac{1}{\beta_2} t_2 \tau_2 - \mu \frac{1}{\beta_2} t_2^2}.$$

If $\alpha \beta > \frac{1}{4}$ then $g = 0$, so $f = 0$.

If $\alpha \beta < \frac{1}{4}$, then there are infinitely many such functions $f$, that verify (33) and (34). This completes the proof.

\[\square\]

### 4.3. Beurling’s uncertainty principle

The following proposition is Beurling’s UP for the two-sided QFT.

**Proposition 4.5 ([3, Thm. 4.2]):** Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $d \geq 0$ satisfy

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|f(x)|_Q \| \mathcal{F} \{ f \} (y) \|_{Q_2}}{(1 + |x| + |y|)^d} e^{2\pi |x||y|} \, dx \, dy < \infty.$$ 

Then $f(x) = P(x)e^{-a|x|^2}$, where $a > 0$ and $P$ is a polynomial of degree $< \frac{d-2}{2}$. In particular, $f$ is identically 0 when $d \leq 2$.

Let us define the operator $G_{A_1A_2}^{\lambda,\mu}$ on $L^2(\mathbb{R}^2, \mathbb{H})$ by

$$G_{A}^{\lambda,\mu} \{ f \} (u) = e^{\lambda u_1 (d_1 \tau_1 - b_1 \eta_1) - \lambda \frac{d_1}{2b_1} (b_1^2 u_1^2 + \tau_1^2)} e^{\mu u_2 (d_2 \tau_2 - b_2 \eta_2) - \mu \frac{d_2}{2b_2} (b_2^2 u_2^2 + \tau_2^2)},$$

$f \in L^2(\mathbb{R}^2, \mathbb{H})$. On the basis of Proposition 4.5, we give Beurings’ UP in the QOLCT domain.

**Theorem 6:** Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $d \geq 0$ satisfy

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|f(t)|_Q \| G_{A_1A_2}^{\lambda,\mu} \{ G_{A_1A_2}^{\lambda,\mu} \{ f \} \} (u) \|_{Q_2}}{(1 + |t| + |u|)^d} e^{2\pi |t||u|} \, dt \, du < \infty,$$

then

$$f(t) = e^{-a|t|^2} e^{-\lambda \frac{1}{\beta_1} t_1 \tau_1 - \alpha \frac{1}{\beta_1} t_1^2} e^{-\mu \frac{1}{\beta_2} t_2 \tau_2 - \mu \frac{1}{\beta_2} t_2^2} P(t)e^{-a|t|^2},$$

where $a > 0$ and $P$ is a quaternion polynomial of degree $< \frac{d-2}{2}$.

In particular, $f = 0$ when $d \leq 2$.

**Proof:** Let $h$ be rewritten in the form of (23), and we have $h \in L^2(\mathbb{R}^2, \mathbb{H})$.

It follows from Lemma 3.2, and $|h(t)|_Q = \frac{1}{2\pi \sqrt{b_1 b_2}} |f(t)|_Q$ that,

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|h(t)|_Q \| \mathcal{F}^{\lambda,\mu} \{ h \} (u) \|_{Q_2}}{(1 + |t| + |u|)^d} e^{2\pi |t||u|} \, dt \, du$$

...
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| f(t) \right| Q \left\| G^\lambda,\mu_{A_1,A_2} \{ \mathcal{O}^\lambda,\mu_{A_1,A_2} \{ f \} \} (u) \right\|_{Q} e^{||t||u} \, dt \, du < \infty.
\]

Then by Proposition 4.5, we get \( h(t) = P(t)e^{-a|t|^2} \) where \( a > 0 \) and \( P \) is a quaternion polynomial of degree \( < \frac{d-2}{2} \), i.e.

\[
f(t) = e^{-a|t|^2} e^{-\frac{1}{b_1} t_1 t_1 - \frac{a_1}{b_1} t_1^2} P(t)e^{-\frac{1}{b_2} t_2 t_2 - \frac{a_2}{b_2} t_2^2}.
\]

As a consequence of Theorem 4.6, and using the fact that \( \|F^\lambda,\mu \{ f \} \|_Q \) is equivalent to \( \|F^\lambda,\mu \{ f \} \|_Q \) when \( f \) is real valued, we obtain the following corollary.

**Corollary 4.7:** Let \( f \in L^2(\mathbb{R}^2, \mathbb{R}) \) be such that

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| f(t) \right| Q \left\| \mathcal{O}^\lambda,\mu_{A_1,A_2} \{ f \} (u) \right\|_{Q} e^{||t||u} \, dt \, du < \infty,
\]

for some \( d \geq 0 \).

Then

\[
f(t) = P(t)e^{-a|t|^2}, \quad a.e.,
\]

for some real polynomial \( P \) of degree \( < \frac{d-2}{2} \) and \( a > 0 \).

In particular, \( f = 0 \) when \( d \leq 2 \).

**Remark 4.8:** It is worth pointing out that Beurling’s UP, Theorem 4.6, is a variant of Hardy’s UP, Theorem 4.4, and one can derive immediately from it the UPs of Gelfand-Shilov and Cowling-Price in the QOLT domain [3].

### 4.4. Logarithmic uncertainty principle

The logarithmic UP [23], the more general form of Heisenberg type UP, is derived by using Pitt’s inequality. Its localization is measured in terms of entropy.

**Lemma 4.9** (Pitt’s inequality for the two-sided QFT): For \( f \in S(\mathbb{R}^2, \mathbb{H}) \), and \( 0 \leq \alpha < 2 \),

\[
\int_{\mathbb{R}^2} |u|^{-\alpha} |F^\lambda,\mu \{ f \} (u_1, u_2)|^2 \, du_2 \leq C_\alpha \int_{\mathbb{R}^2} |t|^\alpha |f(t)|^2 \, dt,
\]

with \( C_\alpha \overset{\text{def}}{=} \frac{4\pi^2}{2\alpha} \left[ \Gamma\left(\frac{2-\alpha}{4}\right) / \Gamma\left(\frac{2+\alpha}{4}\right) \right]^2 \), and \( \Gamma(.) \) is the Gamma function.

**Proof:** The lemma has been proved in [11, Thm. 4.1] for \( F^{ij} \), the same proof remains valid for any two pure unit quaternions \( \lambda, \mu \) as well.

**Theorem 4.10:** Under the assumptions of Lemma 4.9, one has

\[
\int_{\mathbb{R}^2} \left| \left( \begin{array}{c} z_1 \\ b_1 \\ z_2 \\ b_2 \end{array} \right) \right|^{-\alpha} \left\| \mathcal{O}^\lambda,\mu_{A_1,A_2} \{ f \} (z_1, z_2) \right\|_Q^2 \, dz \leq C_\alpha \frac{4\pi^2}{2\alpha} \int_{\mathbb{R}^2} |t|^\alpha |f(t)|^2 \, dt.
\]
**Proof:** Let $h$ be rewritten in the form of (23).
It is clear that $h \in S(\mathbb{R}^2, \mathbb{H})$, and $|h(t)|_Q = \frac{1}{2\pi \sqrt{b_1 b_2}} |f(t)|_Q$.

Let $\mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{ f \}$ be rewritten as (21), by Lemma 4.9, we obtain

$$
\int_{\mathbb{R}^2} |u|^{-\alpha} \left| \mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{ f \} (b_1 u_1, b_2 u_2) \right|^2 \, du = \int_{\mathbb{R}^2} |u|^{-\alpha} \left| \mathcal{F}_{\lambda, \mu} \{ h(t) \} (u_1, u_2) \right|^2 \, du \\
\leq C_\alpha \int_{\mathbb{R}^2} |t|^{\alpha} |h(t)|_Q^2 \, dt \\
= \frac{C_\alpha}{4\pi^2 b_1 b_2} \int_{\mathbb{R}^2} |t|^{\alpha} |f(t)|_Q^2 \, dt.
$$

Let $z_1 = b_1 u_1$ and $z_2 = b_2 u_2$, then we have

$$
\frac{1}{b_1 b_2} \int_{\mathbb{R}^2} \left| \begin{pmatrix} z_1 \\ z_2 \\ b_1 \\ b_2 \end{pmatrix} \right|^{-\alpha} \left| \mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{ f \} (z) \right|^2 \, dz \leq \frac{C_\alpha}{4\pi^2 b_1 b_2} \int_{\mathbb{R}^2} |t|^{\alpha} |f(t)|_Q^2 \, dt,
$$
i.e.

$$
\int_{\mathbb{R}^2} \left| \begin{pmatrix} z_1 \\ z_2 \\ b_1 \\ b_2 \end{pmatrix} \right|^{-\alpha} \left| \mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{ f \} (z) \right|^2 \, dz \leq \frac{C_\alpha}{4\pi^2} \int_{\mathbb{R}^2} |t|^{\alpha} |f(t)|_Q^2 \, dt.
$$

![Proof](image)

**Theorem 4.11 (Logarithmic UP for the QOLCT):** Let $f \in S(\mathbb{R}^2, \mathbb{H})$, then

$$
\int_{\mathbb{R}^2} \ln \left| \begin{pmatrix} z_1 \\ z_2 \\ b_1 \\ b_2 \end{pmatrix} \right| \left| \mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{ f \} (z) \right|^2 \, dz + \int_{\mathbb{R}^2} \ln (|t|) |f(t)|_Q^2 \, dt \geq A \int_{\mathbb{R}^2} |f(t)|_Q^2 \, dt,
$$
with $A = \ln(2) + \Gamma'(\frac{1}{2})/\Gamma(\frac{1}{2})$.

**Proof:** Let $f \in S(\mathbb{R}^2, \mathbb{H})$, $0 \leq \alpha < 4$, $D_\alpha = \frac{C_\alpha}{4\pi^2} = \frac{1}{\pi^2} \left[ \Gamma(\frac{2-\alpha}{4})/\Gamma(\frac{2+\alpha}{4}) \right]^2$, and $\Phi(\alpha) = \int_{\mathbb{R}^2} \left| \begin{pmatrix} z_1 \\ z_2 \\ b_1 \\ b_2 \end{pmatrix} \right|^{-\alpha} \left| \mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{ f \} (z) \right|^2 \, dz - D_\alpha \int_{\mathbb{R}^2} |t|^{\alpha} |f(t)|_Q^2 \, dt$.

By differentiating $\Phi(\alpha)$, we have

$$
\Phi'(\alpha) = - \int_{\mathbb{R}^2} \ln \left( \begin{pmatrix} z_1 \\ z_2 \\ b_1 \\ b_2 \end{pmatrix} \right) \left| \begin{pmatrix} z_1 \\ z_2 \\ b_1 \\ b_2 \end{pmatrix} \right|^{-\alpha} \left| \mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{ f \} (z) \right|^2 \, dz \\
- D_\alpha' \int_{\mathbb{R}^2} |t|^{\alpha} |f(t)|_Q^2 \, dt - D_\alpha \int_{\mathbb{R}^2} \ln (|t|) |t|^{\alpha} |f(t)|_Q^2 \, dt,
$$
with

$$
D_\alpha' = - \ln(2) 2^{-\alpha} \left[ \Gamma \left( \frac{2-\alpha}{4} \right)/\Gamma \left( \frac{2+\alpha}{4} \right) \right]^2 \\
+ 2^{-\alpha} \left[ \frac{1}{2} \Gamma \left( \frac{2-\alpha}{4} \right) \Gamma' \left( \frac{2-\alpha}{4} \right) \Gamma^2 \left( \frac{2+\alpha}{4} \right) \\
- \frac{1}{2} \Gamma^2 \left( \frac{2-\alpha}{4} \right) \Gamma \left( \frac{2+\alpha}{4} \right) \Gamma' \left( \frac{2+\alpha}{4} \right) \right]/\Gamma^4 \left( \frac{2+\alpha}{4} \right).
$$

We have $D_0 = 1$ and $D_0' = -\ln(2) - \Gamma'(\frac{1}{2})/\Gamma(\frac{1}{2})$. 

![Proof](image)
Because of (38), we see that $\Phi(\alpha) \leq 0$ for $0 \leq \alpha < 2$, also by Theorem 3.6 we have $\Phi(0) = 0$. Then $\Phi'(0^+) = \lim_{\alpha \to 0^+} \frac{\Phi(\alpha) - \Phi(0)}{\alpha} \leq 0$.

Therefore,

$$
\left( \ln(2) + \Gamma \left( \frac{1}{2} \right) / \Gamma \left( \frac{1}{2} \right) \right) \int_{\mathbb{R}^2} |f(t)|_Q^2 \, dt \leq \int_{\mathbb{R}^2} \ln \left( \left| \frac{z_1}{b_1}, \frac{z_2}{b_2} \right| \right) |\mathcal{O}_{A_1, A_2}^\lambda \mu (f)(z)|_Q^2 \, dz
$$

$$
+ \int_{\mathbb{R}^2} \ln(|t|) |f(t)|_Q^2 \, dt.
$$

From which the theorem follows.

\textbf{Remark 4.12:} Applying Jensen’s inequality to (39), we can show that the logarithmic UP implies Heisenberg–Weyl’s inequality Theorem 4.2.

5. Conclusion

In this paper, we first presented a new generalization of the QLCT and so of the QFT, namely the QOLCT. Second, we established some properties of the QOLCT including the Rayleigh’s formula. Then, we derived three UPs in the QOLCT domain: Heisenberg-Weyl’s UP, Hardy’s UP, and its variant-Beurling’s UP. These three UPs assert that it is impossible for a non-zero function and its QOLCT to both decrease very rapidly. Finally, we generalized Pitt’s inequality to the QOLCT domain, and then obtained a logarithmic UP associated with QOLCT. In future work, we will consider these UPs for the offset linear canonical transform in Clifford analysis.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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