Rational method for calculating statically indeterminate symmetric systems

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Abstract. Nowadays, the process of calculating structures is more and more entrusted to automatic calculation systems or artificial intelligence. This increases the accuracy and reliability of the calculations, but does not contribute to the understanding of the real work of the structures by the design engineer or architect. The main difficulty in carrying out such calculations is associated with the multifactorial nature of the solutions, and, as a rule, a significant number of unknowns and variables included in the calculation. The calculation is always carried out with the introduction of a certain degree of accuracy, which, in combination with a large number of variables, leads to the appearance of the effect of a cumulative decrease in the degree of reliability of the calculations carried out and the accumulation of statistical error. In turn, this can develop into a critical state. The technique presented in the article allows one to reduce the number of variables when solving problems of calculating statically indeterminate structures without changing the final result of the calculation.

1. Introduction

In real practice, there are simply no statically definable tasks; they serve as teaching material for indefinable ones. At the beginning of the course, when solving computational and graphic works, students receive a specific task to solve a problem by the method of forces or by the method of displacement, and when solving dynamic problems, they must independently choose a solution method. The choice of the method for solving the problem is often not so unambiguous. For example, it may be necessary to additionally group unknowns in the case of symmetrical frames, or use a combination of both methods. There are cases that during the writing of a diploma, former students come with questions that none of the methods gives a simple solution. Of course, in modern conditions of development of information technologies and software systems, it is possible to solve a complex problem using a computer, but this solution does not instil engineering thinking skills and does not give an understanding of the essence of the processes in the future. It is also not always possible to suggest the literature - in modern textbooks there is no information about the mixed solution method at all.

All that was found was textbooks on "Structural Mechanics" [4, 5, 7-9], while more or less new [8] is actually a reprint [7]. Similar educational publications are available outside the Russian Federation, but they also provide only basic examples of performing calculations [10-12]. In the textbook Dykhovichny A.I. [4] it is said that the theory of methods for calculating statically indeterminate
systems was developed by AA Gvozdev and presented in the book "General method for calculating statically indeterminate systems" (1927), which simply could not be found. In the aforementioned textbook [4] for the mixed method, the following is formulated: “A. A. Gvozdev established the equality in absolute value of the conjugate coefficients of the equations of displacement, of which one is an effort, and the other is a displacement”. The authors of this article came to the conclusion that it is necessary to fill this gap and consider a full-fledged solution of a complex problem, where the mixed method gives a clear rational solution, and also explain the physical essence of this method.

2. Materials and methods
Solving problems of statically indeterminate structures sometimes results in a system with many unknowns, depending on each other. It is well known that the more unknowns are included in the task and the more interrelationships that arise between them, the more likely it is that minor errors and assumptions appear that, due to their accumulation, can lead to critical errors. The method described below for solving this kind of problems eliminates this problem, allowing us to reduce both the number of unknowns in the problem and to concentrate their core of interaction.

3. Results
Consider a complex symmetrical frame with no external load (Fig. 1). At the same time, remember that symmetry should be not only in the support fixtures and outlines of the axis, but also in the rigidity of the elements. Let's calculate the number of unknowns of both methods and show their main systems.

Let us determine the amount of unknown forces or “extra connections” in the frame of Fig. 1 using the Chebyshev formula [1-9]:

$$W = 3 \cdot L - 2 \cdot J - G_0 = 3 \cdot 1 - 2 \cdot 1 - 12 = -11.$$ 

Number of unknown movement methods: $$n = n_u + n_l,$$

where:

- $n_u = 4$ - rigid node not connected to the supports, if you show a conditionally hinge scheme, then the upper crossbar can be displaced not only horizontally, but also vertically at the hinge location, since is an instantly changeable system.

Therefore: $$n_l = 2 \Rightarrow n = n_u + n_l = 4 + 2 = 6.$$ 

![Figure 1. The preset scheme (an example is made for this article as an illustration of the described method).](image)

The main frame of the force method, taking into account the grouping of unknowns (Figs. 2a and 2b), as described in [1-12]. Because the solution of the general canonical system splits into two, then it is better to depict the main systems for symmetric and reverse symmetric diagrams separately at once, so as not to clutter up the diagram.

The canonical system splits into two, in which one has 5 unknowns, and the other has 6.
When solving the problem by the displacement method, taking into account the grouping of unknowns [1-3], the canonical system will have three symmetric unknowns and three reverse symmetric ones (Fig. 3b, Fig. 3a).

a) Inversely symmetric main frame

b) Symmetrical main frame

*Figure 2. Main frame of the method of forces taking into account the symmetry of the frame.*

Therefore, for this task, it is advisable to use both methods simultaneously. In this case, the main system (Figures 5a and 5b) is formed according to two principles: in its less rigid part, containing broken outlines (a-b-c-d) Fig. 4, the frame is cut along the hinge (c). In the stiffer part: on the lower transom, which includes rigid nodes (a-f), initially not connected to the supports, rigid terminations are added.

*Figure 4. Main frame of mixed method with node notation.*

We immediately group the unknown displacement methods, as described in [1-3], and depict the symmetric (Fig. 5b) and inversely symmetric (Fig. 5a) main systems separately. Moreover, regardless of the external load, the general system of canonical equations splits into two independent groups. One group contains only reverse symmetric unknowns, while the other contains symmetric ones.
Symmetrical unknowns discarded in the force method or added in the displacement method create symmetric moment diagrams (Fig. 6).

For reverse symmetric diagrams, the diagrams are the same in size, but with a different sign, i.e. located on the other side of the rods (Fig. 7).

a) Inversely symmetric main frame

b) Symmetrical main frame

Figure 5. Main frame of the mixed method taking into account the symmetry of the frame.

Figure 6. Symmetrical unit plot from \( x_3 = 1 \).

Figure 7. Inversely symmetric unit plot from \( x_1 = 1 \).

The general system of canonical circuit equations for the model shown in Figures 5a and 5b is as follows:
When solving the general canonical system, the multiplication of the diagrams $\overline{M}_1$ and $\overline{M}_3$ gives zero, since multiplying the left side of the frame of Figure 6 by the left of Figure 7 is a positive number, and the right one is the same in magnitude, but negative. The situation is similar for the displacement method, therefore, system (1) splits into two independent ones:

$$
\begin{align}
\delta'_{11} \cdot x_1 + \delta'_{12} \cdot z_2 + \delta_{13} \cdot x_3 + \delta_{14} \cdot z_4 + \Delta_{1p} &= 0 \\
\delta_{31} \cdot x_1 + \delta'_{32} \cdot z_2 + \delta_{33} \cdot x_3 + \delta'_{34} \cdot z_4 + \Delta_{3p} &= 0 \\
\delta_{41} \cdot x_1 + \delta_{42} \cdot z_2 + \delta_{43} \cdot x_3 + \delta_{44} \cdot z_4 + R_{4p} &= 0
\end{align}
$$

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\delta_{31} \cdot x_1 + \delta_{33} \cdot x_3 + \delta_{34} \cdot z_4 + \Delta_{3p} &= 0 \\
\delta_{41} \cdot x_1 + \delta_{43} \cdot x_3 + \delta_{44} \cdot z_4 + R_{4p} &= 0
\end{align}
$$

Canonical systems (2) and (3) containing terms of both the method of forces and the method of displacement, and having basic systems of schemes (Figs. 5a and 5b) are called the mixed method. For mixed terms in the canonical system of this method, the following rules apply:

$$
\delta_{km} = \delta_{mk}, \ r_{km} = r_{mk}, \ \delta'_{km} = -\delta_{mk}, \ \delta'_{km} = -\delta_{mk},
$$

the last two equalities follow from the statics axioms about the equality of action and reaction. This is due to the physical meaning of the coefficients of the system.

Consider system (2) and give a physical explanation of the coefficients of the system. The first line of the system is related to displacements:

$\delta'_{11} \cdot x_1$ - moving point c (Fig. 7) of the main system (Fig. 5a) in the direction $x_1$ from the action of the force $x_1$;

$\delta_{12}' \cdot z_2$ - displacement of the section at point c (Fig. 4) of the main system (Fig. 5a) in the direction $x_1$ from the rotation of the seal 2 (nodes a and f) by $z_2$. Let us explain this statement using the example of the deformation pattern (Fig. 8).

$\delta_{12}' = \delta_{12}' z_2 = 1$

$\delta'_{12} = \delta'_{12} z_2 = 1$

Figure 8. Deformation of the frame when turning unknown 2, i.e., nodes a and f simultaneously clockwise.
Δ₁₁₂ - moving point c (Fig. 4) of the main system (Fig. 5a) in the direction x₁ from the action of the external load.

Second line of system (2):

r₂₁·x₁ - the reaction arising in the termination 2 (nodes A and F of Fig. 7) of the main system (Fig. 5a) from the movement of the point C the left and right parts of the frame in the direction x₁ for the left side of the frame downwards, right - upwards (Fig. 7);

r₂₂·z₂ - the reaction arising in the seal 2 (nodes A and F in Fig. 4) of the main system (Fig. 5a) from the rotation of the seal 2 (nodes A and F) by an amount z₂ (Fig. 9), while on the broken upper sections A-B-C and C-D-F (Fig. 9) diagrams do not appear, they just rotate (Fig. 8);

R₂₂ - the reaction arising in the seal 2 (nodes a and f in Fig. 4) of the main system (Fig. 5a) from the action of an external load.

![Diagram of moments from rotation z₂ = 1.](image)

Let us find the coefficients for the unknowns, we will accept \( \frac{EJ_1}{EJ_2} = \alpha \Rightarrow EJ_1 = \alpha \cdot EJ_2 \) (the multiplication of the diagrams is carried out by the Mohr method, using the trapezoid formulas):

\[
\delta_{11} = \int \frac{M_1 \cdot M_1}{EJ} \, dz = 2 \left[ \frac{\ell_2}{6 \cdot \alpha E \cdot J_2} \cdot \left(2 \cdot \ell_2^2\right) + \frac{h_1}{6E \cdot J_2} \cdot \left(6 \cdot \ell_2^2\right) \right] = \\
= \frac{2 \cdot \ell_2^2}{6E \cdot J_2} \cdot \left[ 4 \cdot \ell_2^2 + 6 \cdot h_1 \right] \frac{\alpha}{\alpha}
\]

\( \delta'_{12} \) cannot be found by multiplying the diagrams \( \overline{M}_1 \) (Fig. 7) and \( \overline{M}_2 \) (Fig. 9), because the diagrams are located in different areas. In this case, it is clearly seen from the deformation pattern (Fig. 8) that there is a displacement \( \delta'_{12} \) in the direction \( x_1 \). Therefore, we will determine the coefficient \( r'_{21} \) (Fig. 10), from the diagram \( \overline{M}_1 \) (Fig. 7).
Figure 10. Finding the coefficient $r'_{21}$ of the canonical system (2).

$$r'_{21} = r''_{21} + r''_{21} = -2 \cdot \ell_2$$
and therefore $a_{12} = -r'_{21} = +2 \cdot \ell_2$.

The last coefficient for unknowns $r_{22}$ (Fig. 11).

Figure 11. Finding the coefficient $r_{22}$ of the canonical system (2).

$$r_{22} = r'_{22} + r'_{22} = 2 \cdot EJ_2 \cdot \left(\frac{4\alpha}{l_1} + \frac{4}{h_2} + \frac{6\alpha}{2l_2}\right).$$

The weight terms of the canonical system depend on the external load. We know what $\Delta_{1P}$ contains multiplier $\frac{1}{EJ_2}$ and the resulting reaction $z_2$ value too, therefore, in the first equality of system (2), this common factor can be canceled immediately. In the second equality of the same system, only the value $r_{22}$ - has a multiplier $EJ_2$, and as said earlier $z_2$, it has the value - as a multiplier $\frac{1}{EJ_2}$, therefore, all terms of 2 rows of system (2) have the same dimension. Therefore, the system itself must be solved without $EJ$ initially, taking into account only the ratio between the stiffness of the rods: $EJ_1 = \alpha \cdot EJ_2$.

If the preset external load is not arbitrary, but also symmetric or back-symmetric, then the entire solution of the problem is reduced to the solution of only one of the systems. For a symmetric external load, the main frame (Fig. 5b) and canonical (3), for an inversely symmetric main frame (Fig. 5a) and canonical (2).

4. Summary
Application of this calculation method for complex symmetrical frames showed the following results:

1. To simplify the calculation, it is necessary to maintain symmetry for the main system of any method (force method, displacement method and mixed method).
2. To group the unknowns so that the main system breaks up into an inversely symmetric and symmetric part, which makes it possible to split one canonical system into two independent ones.

3. In the case when one part of the frame is very rigid - the lower part (Fig. 1), then additional ties are superimposed on it. And the other part of the frame (on the upper floor) is less rigid, then "extra connections" are eliminated. Thus, a basic mixed method system is obtained. Moreover, in the canonical systems (2) and (3), part of the unknown forces is \( x_1 \) and \( x_3 \), and part of the displacement is \( z_2 \) and \( z_4 \). Accordingly, the coefficients for them are displacements \( \delta_{km} \) or reactions \( r_{km} \). This technique allows reducing the system of 11 unknown methods of forces or a system of 6 unknown methods of displacement into two simple ones (2) and (3) with two unknowns in each.

Thus, the application of this technique greatly simplifies the entire calculation. In turn, this leads not only to a decrease in the critical mass of errors, increasing its accuracy, but also allows you to perform calculations manually (without the use of electronic computing systems and software systems), which gives a significant positive effect in methodological terms when teaching students of higher educational institutions.

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