An Accelerated Stochastic Algorithm for Solving the Optimal Transport Problem

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Abstract

A primal-dual accelerated stochastic gradient descent with variance reduction algorithm (PDASGD) is proposed to solve linear-constrained optimization problems. PDASGD could be applied to solve the discrete optimal transport (OT) problem and enjoys the best-known computational complexity—$\tilde{O}(n^2/\epsilon)$, where $n$ is the number of atoms, and $\epsilon > 0$ is the accuracy. In the literature, some primal-dual accelerated first-order algorithms, e.g., APDAGD, have been proposed and have the order of $\tilde{O}(n^{2.5}/\epsilon)$ for solving the OT problem. To understand why our proposed algorithm could improve the rate by a factor of $\tilde{O}(\sqrt{n})$, the conditions under which our stochastic algorithm has a lower order of computational complexity for solving linear-constrained optimization problems are discussed. It is demonstrated that the OT problem could satisfy the aforementioned conditions. Numerical experiments demonstrate superior practical performances of the proposed PDASGD algorithm for solving the OT problem.

1 Introduction

Optimal transport (OT) [16, 24, 29] has attracted intensive research efforts in the past few years and could be applied to numerous areas. For instance, since OT could provide an optimal transportation plan between histograms and preserve the geometric data structure well, we could apply OT to the data analytics task of domain adaptation [11] and image processing task of color transfer [10]. Considering Wasserstein distance (a special case of OT) is a powerful tool to quantify the distance
between probability distributions, OT could be applied to statistical tests [30] and distributionally robust optimization [23] as well.

A central question is how to solve the OT problem efficiently. We will focus on solving the OT problem between two discrete probability distributions in this paper. It is well known that solving the discrete OT problem exactly is computationally costly since the discrete OT problem is a linear programming problem with \( n^2 \) variables, where \( n \) is the number of atoms of the distributions. In this regard, we develop an approximation algorithm — *primal-dual accelerated stochastic gradient descent with variance reduction algorithm* (PDASGD)—to solve the discrete OT problem.

Approximating the OT problem by PDASGD could be divided into two steps. Firstly, we apply PDASGD to solve the entropy-regularized OT problem [6]. Secondly, we round the solution to the constraint set of the OT problem, as suggested by [3], to obtain an \( \epsilon \)-approximation solution to the OT problem. The total computational complexity of applying the proposed method to give an \( \epsilon \)-approximation solution to the OT problem is \( \widetilde{O} \left( \frac{n^2}{\epsilon} \right) \), which matches the best-known complexity for OT approximation algorithms [15, 19, 22]. Notably, the computational complexity of PDASGD is lower than the non-stochastic accelerated primal-dual first-order algorithms [8, 13, 14, 20], which have the order of \( \widetilde{O} \left( \frac{n^{2.5}}{\epsilon} \right) \), by a factor of \( \widetilde{O} \left( \sqrt{n} \right) \), and is also lower than the Sinkhorn-based algorithms [1, 3, 6], which are of the order \( \widetilde{O} \left( \frac{n^2}{\epsilon^2} \right) \), by a factor of \( \widetilde{O} \left( \frac{1}{\epsilon} \right) \).

To facilitate a deeper understanding and broader applications of PDASGD for solving the generic linear-constrained optimization problems, we will discuss in detail later under which conditions our accelerated primal-dual stochastic algorithm helps relax the total complexity dependence on problem size \( n \) over other primal-dual counterparts [8, 13, 14, 20]. Informally, if the \( L \)-smooth parameter w.r.t. \( l_2 \) norm, the average \( L \)-smooth parameter w.r.t. \( l_2 \) norm and \( L \)-smooth parameter w.r.t. \( l_\infty \) norm of the dual objective function are in the same order, our proposed algorithm will outperform. More details are provided in Section 4. In particular, the aforementioned conditions could be satisfied by the semi-dual of the entropy-regularized OT problem.

Finally, numerical experiments are carried out to show the favorable empirical efficiency of PDASGD. More specifically, we make a fairly complete comparison with the existing approximation algorithms
for solving the OT problem. The experimental results show that PDASGD enjoys numerical stability and low practical computational complexity.

1.1 Related Work

In this subsection, we review the algorithms for solving the OT problem.

The OT problem could be reformulated as the minimum-cost flow problem and solved by the network simplex algorithm, interior algorithm, and other combinatorial algorithms [27]. The exact algorithms have large computational complexities and may have lower complexities in special cases [31]. In this sense, developing fast approximations algorithms for the OT problem is of much interest.

We make a comprehensive review of the existing approximation algorithms [1, 3, 4, 5, 6, 8, 13, 14, 15, 18, 19, 20, 21, 22, 28] for solving the OT problem by summarizing them into three categories:

(1) The first category takes advantage of the Bregman projection, including Sinkhorn [6], and Greenkhorn [3], stochastic Sinkhorn [1], accelerated Sinkhorn [21].

(2) The second category includes accelerated first-order optimization algorithms, including primal-dual accelerated gradient descent (APDAGD) [8], primal-dual accelerated mirror descent (APDAMD) [20], accelerated alternating minimization (AAM) [13], primal-dual accelerated randomized coordinate descent (APDRCD) [14], accelerated hybrid primal-dual (HPD) [5], and primal-dual accelerated stochastic proximal mirror descent (PDASMD) [22].

(3) The third category applies other advanced optimization techniques, including dual extrapolation [15], box-constrained Newton method [4], packing LP [28], extragradient [19], and graph-based search algorithm [18].

We compare the computational complexities of different algorithms in Table 1 and provide more details as follows:

(1) The best provable computational complexity of the first category is $\tilde{O}(n^{7/3}/\epsilon^{4/3})$ (accelerated Sinkhorn [21]) and $\tilde{O}(n^2/\epsilon^2)$ (others). Our algorithm has a lower order of $\epsilon$. 

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(2) Among the second category, the best complexity of the non-stochastic algorithms [5, 8, 13, 14, 20] is $\widetilde{O}(n^{2.5}/\epsilon)$, which has a higher order of $n$ than our algorithm. Notably, PDASMD [22] in the second category is an accelerated stochastic algorithm based on the proximal mirror descent and has an order of $\widetilde{O}(n^2/\epsilon)$, which is theoretically comparable to our algorithm. We discuss the difference and the improvement of our algorithm compared with PDASMD as follows. PDASMD utilizes the mirror descent to update the variables, while PDASGD simply utilizes the gradient descent. It is demonstrated in [22] that the adoption of mirror descent helps improve the rate from $\mathcal{O}(n^{2.5}/\epsilon)$ to $\mathcal{O}(n^2/\epsilon)$, while the adoption of gradient descent still has the complexity of order $\mathcal{O}(n^{2.5}/\epsilon)$. However, in this paper, we show that simple adoption of gradient descent could achieve the state-of-the-art computational complexity—$\mathcal{O}(n^2/\epsilon)$. This implies that the reduced complexity of the stochastic algorithm may not come from which mirror map is chosen but from the nature of the design of the stochastic algorithm and the problem structure. To illustrate this perspective, we will discuss when our proposed stochastic algorithm PDASGD could outperform other primal-dual first-order algorithms. More importantly, we find out PDASGD could improve the practical performance after parameter tuning. Our implementation of PDASGD compares favorably to the existing state-of-the-art OT solvers, which is not achieved by [22].

(3) In the third category, the complexity bound is $\widetilde{O}(n^2/\epsilon + n/\epsilon^2)$ (graph-based algorithm [18]) and $\widetilde{O}(n^2/\epsilon)$ (others). Our algorithm is easier to understand and implement than the algorithms in this category. In addition to solving the OT problem, we also provide the convergence rate of applying our algorithm to the generic linear-constrained optimization problems.

1.2 Notations and Definitions

Some notations and definitions are introduced in this subsection. They will be needed in the rest of this paper.

The $m$-dimensional column of all ones is denoted by $\mathbf{1}_m$. The $i$th basis of the $m$-dimensional Euclidean space is denoted by $\mathbf{e}_i$. The $l_p$ norm of the vector is denoted by $\|\cdot\|_p$. $X \otimes Y$ denotes the standard Kronecker product of matrix $X$ and matrix $Y$. The entropy of matrix $X$ is defined as $H(X) = -\sum X_{ij} \ln(X_{ij})$. We adopt the element-wise exponential and logarithm operator $\exp(X)$ and $\ln(X)$.
Table 1: Computational complexities of OT algorithms.

| Algorithm                  | Computational Complexity |
|----------------------------|--------------------------|
| Sinkhorn [6]               | $\frac{n^2}{\epsilon^2}$ |
| Greenkhorn [3]             | $\frac{n^2}{\epsilon^2}$ |
| Stochastic Sinkhorn [1]    | $\frac{n^2}{\epsilon^2}$ |
| Accelerated Sinkhorn [21]  | $\frac{n^{7/3}}{\epsilon^{4/3}}$ |
| APDAGD [8]                 | $\frac{n^{2.5}}{\epsilon}$ |
| APDAMD [20]                | $\frac{n^{2.5}}{\epsilon}$ |
| APDRCD [14]                | $\frac{n^{2.5}}{\epsilon}$ |
| AAM [13]                   | $\frac{n^{2.5}}{\epsilon}$ |
| Hybrid Primal-Dual [5]     | $\frac{n^{2.5}}{\epsilon}$ |
| PDASMD [22]                | $\frac{n^2}{\epsilon}$ |
| Packing LP [4, 28]         | $\frac{n^2}{\epsilon}$ |
| Box Constrained Newton [4] | $\frac{n^2}{\epsilon}$ |
| Dual Extrapolation [15]    | $\frac{n^2}{\epsilon}$ |
| Dijkstra’s search + DFS [18]| $\frac{n^2}{\epsilon} + \frac{n}{\epsilon^2}$ |
| Extragradient [19]         | $\frac{n^2}{\epsilon}$ |
| PDASGD (This paper)        | $\frac{n^2}{\epsilon}$ |

for matrix $X$. A concatenation $\text{Vec}(X)$ is defined as $\text{Vec}(X) = (X_{11}, \cdots, X_{n1}, \cdots, X_{1m}, \cdots, X_{nm})^\top$, where $X \in \mathbb{R}^{n \times m}$. The matrix norm induced by two arbitrary vector norms $\| \cdot \|_H$ and $\| \cdot \|_E$ is denoted by $\|X\|_{E \rightarrow H} = \max_{x: \|a\|_E \leq 1} \|Xa\|_H$.

**Definition 1.** For a continuously differentiable function $f : Q \rightarrow \mathbb{R}$ is $L$-smooth or strongly convex if the following conditions are satisfied:

- **$f$ is $L$-smooth w.r.t. $\| \cdot \|_2$** if $\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2, \forall x, y \in Q$. Or equivalently, $f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + L\|x - y\|_2^2/2, \forall x, y \in Q$.

- **$f$ is $L$-smooth w.r.t. $\| \cdot \|_\infty$** if $\|\nabla f(x) - \nabla f(y)\|_1 \leq L\|x - y\|_\infty, \forall x, y \in Q$. Or equivalently, $f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + L\|x - y\|_\infty^2/2, \forall x, y \in Q$.

- **$f$ is $\sigma$-strongly convex w.r.t. $\| \cdot \|_E$** if $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \sigma\|x - y\|_E^2/2, \forall x, y \in Q$.

1.3 Organization

The remainder of this paper is organized as follows. In Section 2, we give the formulation of the OT problem, the entropy-regularized OT problem, and the $\epsilon$-approximation framework for the OT problem. In Section 3, we introduce our proposed algorithm PDASGD to solve linear-constrained optimization problems and give the associated convergence rate. In Section 4, we make a careful
comparison with existing accelerated primal-dual first-order methods. In Section 5, we apply the proposed algorithm PDASGD to approximate the OT problem and calculate the resulting computational complexity. In Section 6, we carry out numerical experiments to show the efficiency of PDASGD. We discuss some future work in Section 7. The proofs are relegated to Appendix or Supplementary Materials whenever possible.

2 Problem Formulation

We present the formulations of the OT problem and describe the \( \epsilon \)-approximation framework.

2.1 Optimal Transport and Entropic Optimal Transport

In this paper, we study the optimal transport (OT) problem between two discrete probability distributions \( \alpha \) and \( \beta \) as follows,

\[
\min_{X \in \mathcal{U}(\alpha, \beta)} \langle C, X \rangle,
\]

\[
\mathcal{U}(\alpha, \beta) = \left\{ X \in \mathbb{R}_{+}^{n \times n} \left| X1_n = \alpha, X^T1_n = \beta \right. \right\},
\]

where \( \alpha, \beta \geq 0, \alpha^T1_n = \beta^T1_n = 1 \). \( X \) and \( C \) denote the transportation plan and the cost matrix, respectively. The matrix inner product is defined as \( \langle C, X \rangle = \sum_{i,j=1}^{n} C_{ij}X_{ij} \).

To enable faster computing and obtain smooth solutions, the entropy-regularized OT problem [6] is proposed by adding an entropy regularizer to the objective function. The formulation of this approach is as follows,

\[
\min_{X \in \mathcal{U}(\alpha, \beta)} \langle C, X \rangle - \eta H(X),
\]

where \( \eta \) is the penalty parameter and \( H(X) \) is the entropy.

In this paper, we first solve the entropy-regularized OT problem (2) and then round the solution to the constraint set \( \mathcal{U}(\alpha, \beta) \) to obtain an approximation solution to the OT problem (1). Detailed elaboration on this process will be provided in Section 5.
2.2  $\epsilon$-approximation

A prevalent measure of the numerical efficiency of the approximation algorithms for solving the OT problem is the computational complexity of obtaining an $\epsilon$-approximation solution $\hat{X} \in \mathcal{U}(\alpha, \beta)$ to the OT problem (1), such that

$$\langle \hat{X}, C \rangle - \langle X^*, C \rangle \leq \epsilon,$$

where $X^*$ is the solution to problem (1). For the stochastic algorithms, we formulate the inequality as

$$\mathbb{E}[\langle \hat{X}, C \rangle] - \langle X^*, C \rangle \leq \epsilon$$

to accommodate the random-variable outputs. To assess the performance of PDASGD, we will use the second formulation.

3  The Algorithm: PDASGD

The entropy-regularized OT problem (2) is an optimization problem with the linear constraints $X \in \mathcal{U}(\alpha, \beta)$. Accordingly, we describe a linear-constrained optimization problem, which could cover the entropy regularized OT as a special case, in Section 3.1. We adopt an accelerated stochastic algorithm to solve this optimization problem in Section 3.2.

3.1  Linear-constrained Optimization Problem

In this subsection, we present the general optimization problem with linear constraints, develop its dual, and clarify some necessary assumptions.

Consider the optimization problem as follows,

$$\min_{x \in Q \subset \mathbb{R}^q} f(x)$$

s.t. $Ax = b \in \mathbb{R}^d$, \hfill (3)

where $A \in \mathbb{R}^{d \times q}$. The associated Lagrange dual problem is stated as follows,

$$\min_{\lambda \in \mathbb{R}^d} \left\{ \phi(\lambda) = -\langle b, \lambda \rangle + \max_{x \in Q \subset \mathbb{R}^q} (-f(x) + \langle Ax, \lambda \rangle) \right\}.$$ \hfill (4)
Denote
\[ x(\lambda) = \arg \max_{x \in Q \subset \mathbb{R}^q} (-f(x) + \langle Ax, \lambda \rangle), \tag{5} \]
then the dual problem (4) could be expressed as:
\[ \min_{\lambda \in \mathbb{R}^d} \{ \phi(\lambda) = -\langle b, \lambda \rangle - f(x(\lambda)) + \langle Ax(\lambda), \lambda \rangle \}. \]

Equipped with the problem formulation above, we further consider the optimization problems with particular properties introduced in the following assumption.

**Assumption 1.** In the optimization problem (3) and its dual problem (4), we assume that

- the following equation involving the (sub)gradient of \( \phi(\lambda) \) and \( x(\lambda) \) holds:
  \[ \nabla \phi(\lambda) = Ax(\lambda) - b. \tag{6} \]
- \( \phi(\lambda) \) could be written as a finite sum of functions:
  \[ \phi(\lambda) = \frac{1}{h} \sum_{i=1}^{h} \phi_i(\lambda). \]
- \( \phi_i(\lambda) \) is convex and \( L_i \)-smooth w.r.t. \( \| \cdot \|_2 \). The associated average smooth parameter is denoted by
  \[ L = \frac{1}{h} \sum_{i=1}^{h} L_i. \]

The first part in Assumption 1 depends on the primal-dual structure. The second part is to let the function meet the requirement of applying the stochastic gradient descent algorithm. And the third part is to promise the convergence of the algorithm.

In the following remark, we discuss Assumption 1 when \( f(x) \) is strongly convex.

**Remark 1.** If \( f(x) \) is \( \sigma \)-strongly convex w.r.t. \( \| \cdot \|_E \) on \( Q \), Danskin’s theorem implies the equation (6) holds and \( \phi(\lambda) \) is convex. It follows from [26] that \( \phi(\lambda) \) is \( L \)-smooth w.r.t. \( \| \cdot \|_2 \), and the associated smooth parameter could be specified by
  \[ L \leq \| A \|_{E \rightarrow 2}/\sigma. \]
Further, as long as \( \phi(\lambda) \) is
finite-sum, $\phi_i(\lambda)$ is $hL$-smooth w.r.t. $\| \cdot \|_2$ and the associated average smooth parameter for $\phi(\lambda)$ could be specified by $\overline{L} \leq h\|A\|_{E^{2}\rightarrow2}/\sigma$ [2]. In conclusion, if $f(x)$ is strongly convex, one only needs to check whether the dual objective function $\phi(\lambda)$ is finite-sum and each element function is convex.

Notably, it is well-known that the primal objective function of the entropy-regularized OT problem is strongly convex w.r.t. $\| \cdot \|_1$ [9, 13, 20], which indicates the potential of applying the proposed algorithm for solving the OT problem.

3.2 Our Algorithm

In this subsection, we present our primal-dual accelerated stochastic gradient descent (PDASGD) algorithm for solving problem (3), and then analyze the convergence rate of PDASGD.

The pseudo-code of PDASGD is presented in Algorithm 1. It is a primal-dual algorithm motivated by [2]. It updates the dual variable sequences $\lambda, z, y$ in the inner loop (Step 7-11), and transforms to the primal variable by (5) in the outer loop (Step 14-16). In the inner loop, Step 8 utilizes the Katyusha momentum [2] to accelerate the algorithm; Steps 10 and 11 use the variance-reduced gradient calculated in Step 9. As for the outer loop, Step 5 is to compute the full gradient used to calculate the variance-reduced gradient. Step 14-16 is to obtain the output by taking the weighted average of the primal variable historic values. This iterative updating helps avoid storing all past values of $x$ and $\tau$. Note that Step 14 indicates that one calculates the primal variable once per outer iteration via a dual variable that is sampled randomly from the previous $m$-step inner loop.

3.3 Convergence Analysis

This subsection discusses the convergence rate of PDASGD, which is presented in Algorithm 1.

We summarize the theoretical convergence rate of PDASGD in the following theorem.

**Theorem 1.** Under Assumption 1, if one applies PDASGD to problem (3) and (4), the output $x^S$ of Algorithm 1 satisfies

$$f(\mathbb{E}(x^S)) - f(x^*) \lesssim \frac{\phi(0) - \phi(\lambda^*)}{S^2} + \frac{\overline{L}\|\lambda^*\|_2^2}{mS^2},$$
Algorithm 1 PDASGD

1: **Input:** dual objective function \( \phi(\lambda) \) and the associated smooth parameter w.r.t. \( \| \cdot \|_2: L_i, \bar{L} \);
number of inner iterations: \( m \); number of outer iterations: \( S \).

2: \( \tau_2 \leftarrow 1/2; y_0 = z_0 = \lambda_0 = \lambda^* = 0 = C_0 = D_0 = 0 \leftarrow 0 \).

3: for all \( s = 0, \cdots, S - 1 \) do

4: \( \tau_{1,s} \leftarrow 2/(s + 4); \gamma_s \leftarrow 1/(9\tau_{1,s} \bar{L}) \).

5: \( u^s \leftarrow \nabla \phi(\lambda^s) \).

6: for all \( j = 0 \) to \( m - 1 \) do

7: \( k \leftarrow sm + j \).

8: \( \lambda_{k+1} \leftarrow \tau_{1,s} z_k + \tau_2 \lambda^* + (1 - \tau_{1,s} - \tau_2) y_k \).

9: \( \nabla_{k+1} \leftarrow u^s + (\nabla \phi_i(\lambda_{k+1}) - \nabla \phi_i(\lambda^*))/h p_i \),
where \( i \) is randomly chosen from \( \{1, 2, \cdots, h\} \), each with probability \( p_i = L_i/h \bar{L} \).

10: \( z_{k+1} \leftarrow z_k - \gamma_s \nabla_{k+1}/2 \).

11: \( y_{k+1} \leftarrow \lambda_{k+1} - \nabla_{k+1}/9 \bar{L} \).

end for

12: \( \lambda^{s+1} \leftarrow \sum_{j=1}^m y_{sm+j}/m \).

13: \( D_s \leftarrow D_s + x(\hat{\lambda}_s)/\tau_{1,s} \), where \( \hat{\lambda}_s \) is randomly chosen from \( \{\lambda_{sm+1}, \cdots, \lambda_{sm+m}\} \).

14: \( C_s \leftarrow C_s + 1/\tau_{1,s} \).

15: \( x^s \leftarrow D_s/C_s \).

16: \( s \leftarrow s + 1 \).

end for

19: **Output:** \( x^S \).

\[
\| \mathbb{E}[Ax^S - b] \|_2 \lesssim \phi(0) - \phi(\lambda^*) - \frac{L\|\lambda^*\|_2}{S^2}\|\lambda^*\|_2 + \frac{\bar{L}\|\lambda^*\|_2}{mS^2},
\]

where \( x^* \) is the solution to problem (3) and \( \lambda^* \) is the solution to problem (4).

Then, we suppose the dual function \( \phi(\lambda) \) is \( L' \)-smooth w.r.t. \( \| \cdot \|_\infty \). The convergence rate of PDASGD could be further simplified. We present the results in the following corollary. This will help us understand under which conditions we can obtain a better bound in Section 4.

**Corollary 1.** Under Assumption 1, if one applies PDASGD to problem (3) and (4) and supposes that \( \phi(\lambda) \) is \( L' \)-smooth w.r.t. \( \| \cdot \|_\infty \), the output \( x^S \) of Algorithm 1 satisfies

\[
f(\mathbb{E}(x^S)) - f(x^*) \lesssim \frac{L'\|\lambda^*\|_\infty^2}{S^2\|\lambda^*\|_2} + \frac{\bar{L}\|\lambda^*\|_2^2}{mS^2},
\]

\[
\| \mathbb{E}[Ax^S - b] \|_2 \lesssim \frac{L'\|\lambda^*\|_\infty^2}{S^2\|\lambda^*\|_2} + \frac{\bar{L}\|\lambda^*\|_2^2}{mS^2},
\]

where \( x^* \) is the solution to problem (3) and \( \lambda^* \) is the solution to problem (4).
Remark 2. If we assume that the number of the inner loop \( m \) equals \( d \), it follows from the inequality \( \| \lambda^* \|_2 \leq \sqrt{d} \| \lambda^* \|_\infty \) that
\[
f(\mathbb{E}(x^S)) - f(x^*) \leq \frac{(L' + \overline{L})\| \lambda^* \|_\infty^2}{S^2}.
\]

In this way, the convergence rate depends on the orders of parameters \( L' \) and \( \overline{L} \). It will be proven that \( L' \) and \( \overline{L} \) are constants for the semi-dual of the entropy-regularized OT problem, which enables us to improve the theoretical bound of computational complexity for solving the OT problem. More details will be discussed in Section 4 and Section 5.

4 PDASGD Improves the Complexity

In this section, we discuss when our proposed accelerated stochastic algorithm, compared with non-stochastic primal-dual first-order algorithms, could reduce the computational complexity.

We revisit the primal problem and the associated dual problem introduced in Section 3.1 as follows.

The primal problem:
\[
\min_{x \in Q \subseteq \mathbb{R}^q} f(x) \quad \text{s.t.} \quad Ax = b \in \mathbb{R}^d. \tag{7}
\]

The associated dual problem:
\[
\min_{\lambda \in \mathbb{R}^d} \left\{ \phi(\lambda) = -\langle b, \lambda \rangle + \max_{x \in Q \subseteq \mathbb{R}^q} (-f(x) + \langle Ax, \lambda \rangle) \right\}, \tag{8}
\]

where we suppose \( d = \Theta(n) \) and \( \phi(\lambda) \) could be written as a finite sum of functions: \( \phi(\lambda) = \sum_{i=1}^{n} \phi_i(\lambda)/n. \)

Some assumptions should be further made before we present our observations.

Assumption 2. In problem (8), with a little abuse of notation, we assume that

- \( \phi(\lambda) \) is \( \overline{L} \)-smooth w.r.t. \( \| \cdot \|_2 \)
- \( \phi_i(\lambda) \) is \( L_i \)-smooth w.r.t. \( \| \cdot \|_2 \). The associated average smooth parameter is denoted by
\[ \overline{L} = \sum_{i=1}^{n} L_i/n. \]

- \( \phi(\lambda) \) is \( L' \)-smooth w.r.t. \( \| \cdot \|_\infty \).
- The smooth parameters \( \overline{L}, \overline{L}, L' \) are in the same order w.r.t. \( n \).

We restate the convergence rate of APDAGD [9] in the following theorem for the convenience of comparison.

**Theorem 2** (Theorem 3 in [9]). Suppose \( f(x) \) in problem (7) is strongly convex. If one applies APDAGD to problem (7) and (8), the output \( x^k \) of the \( k \)th iteration generated by APDAGD satisfies

\[
 f(x^k) - f(x^*) \lesssim \frac{\overline{L}\|\lambda^*\|_2^2}{k^2}.
\]

Equipped with the convergence rate of the APDAGD, we make a comparison in the following corollary.

**Corollary 2.** Suppose that the computational complexity of each iteration is \( O(K) \) for PDASGD and \( O(nK) \) for APDAGD, and \( f(x) \) is strongly convex. Under Assumption 1 and Assumption 2, if one applies PDASGD (let the number of inner loops be \( m = n \)) and APDAGD to problem (7) and (8), the total computational complexity of obtaining an \( \epsilon \)-approximation solution \( \widehat{x} \), i.e., \( f(\widehat{x}) - f(x^*) \leq \epsilon \) (or \( f(E[\widehat{x}]) - f(x^*) \leq \epsilon \) for the random-variable output \( \widehat{x} \)), is

\[
 O(n^{3/2}K\sqrt{\frac{L}{\epsilon} \|\lambda^*\|_\infty})
\]

for APDAGD and

\[
 O(nK\sqrt{\frac{L'}{\epsilon} \|\lambda^*\|_\infty})
\]

for PDASGD.

**Remark 3.** Notably, PDASGD computes the gradient of only one element function \( \phi_i(\lambda) \) in each iteration, and there are \( n \) element functions in total. Hence, it is reasonable to impose assumptions regarding the computational complexity of each iteration. For clarity, we additionally assume that \( f(x) \) is strongly convex, ensuring the smoothness of the dual objective function \( \phi(\lambda) \) as discussed in
Remark 1. This assumption can be met by entropy-regularized OT.

Corollary 2 implies that the proposed PDASGD could reduce the theoretical complexity of giving the $\epsilon$-approximation solution over APDAGD by $O(\sqrt{n})$. Other primal-dual accelerated first-order algorithms have a similar convergence rate or computational complexity as APDAGD, e.g., APDAMD [20], APDRCD [14], AAM [13]. We omit these comparisons due to the page limit.

In conclusion, Corollary 2 helps us understand the conditions under which the total complexity of applying the proposed PDASGD to solve the OT problem is lower than that of other first-order counterparts which are not based on stochastic gradient descent. The key component is that the smooth parameters $\bar{L}, \bar{L}, L'$ in Assumption 2 are in the same order. The following is an example where $\bar{L}, \bar{L}, L'$ are not in the same order.

**Example 1.** If the dual objective function $\phi(\lambda)$ has the following expression

$$\phi(\lambda) = \sum_{i=1}^{n} i \langle e_i, \lambda \rangle^2,$$

$\phi(\lambda)$ is $2n$-smooth w.r.t. $\| \cdot \|_2$ and $2n^2$-smooth w.r.t. $\| \cdot \|_{\infty}$.

Then, we rewrite $\phi(\lambda)$ in the finite-sum formulation as follows,

$$\phi(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(\lambda), \quad \phi_i(\lambda) = n i \langle e_i, \lambda \rangle^2.$$

We could derive that $\phi_i(\lambda)$ is $2ni$-smooth w.r.t. $\| \cdot \|_2$, and the associated average smooth parameter is $n(n+1)$.

In Example 1, we have $\bar{L} = 2n, \bar{L} = n(n+1), L' = 2n^2$, which are of different orders. However, it will be demonstrated in Section 5 that the smooth parameters are in the same order for the semi-dual of the entropy-regularized OT problem.
5 Apply PDASGD to Solve the OT Problem

In this section, we apply PDASGD to the OT problem and provide a rigorous computational complexity analysis.

There are two steps for solving the OT problem. The first step is to apply PDASGD to solve the entropy-regularized OT problem, and the second step is to round the output to the feasible area $U(\alpha, \beta)$, where $\alpha$ and $\beta$ are given marginal distributions.

5.1 Apply PDASGD to the Entropy Regularized OT

Recall the entropy-regularized OT problem [6]:

$$\min_{X \in U(\alpha, \beta)} \langle C, X \rangle - \eta H(X),$$  \hspace{1cm} (9)

where $\eta$ is the penalty parameter and $H(X)$ is the entropy. It could be rewritten in the formulation discussed in Section 3.1:

$$\min_{x \in \mathbb{R}^n_+} \left\{ f(x) = c^\top x + \eta x^\top \ln x \right\}$$

$$\text{s.t. } Ax = b,$$  \hspace{1cm} (10)

where $c = \text{Vec}(C)$, $x = \text{Vec}(X)$, $b = (\alpha^\top, \beta^\top)^\top$, and $A$ is the linear operator such that $Ax = A\text{Vec}(X) = \begin{pmatrix} X1_n \\ X^\top 1_n \end{pmatrix}$.

To apply the primal-dual algorithm, we first derive the dual of problem (10) as follows,

$$\min_{\lambda \in \mathbb{R}^{2n}} \left\{ -(b, \lambda) + \eta \exp\left( \frac{A^\top \lambda - c - \eta 1_n}{\eta} \right) 1_n^\top 1_n \right\}.$$  \hspace{1cm} (11)

The adoption of PDASGD requires that the dual objective function is finite-sum. To achieve this structure, we follow [12] to derive the semi-dual from the dual problem (11). We start with splitting $\lambda$ into two parts $\lambda = (u^\top, v^\top)^\top$, where $u, v \in \mathbb{R}^n$. In this way, the dual problem (11) can be written.
as follows,

\[
\min_{u \in \mathbb{R}^n, v \in \mathbb{R}^n} -\langle \alpha, u \rangle - \langle \beta, v \rangle + \eta \sum_{i=1, j=1}^n \exp \left( \frac{u_i + v_j - c_{ij} - \eta}{\eta} \right).
\] (12)

Then, for a fixed \( v \), we solve \( u \) by the optimality condition of minimizing (12) and the result \( u(v) \) is shown in equation (14). Finally, we plug \( u(v) \) into problem (12). The resulting optimization problem is as follows,

\[
\min_{v \in \mathbb{R}^n} \left\{ G(v) = \frac{1}{n} \sum_{i=1}^n g_i(v) = \frac{1}{n} \sum_{i=1}^n n\alpha_i h_i(v) \right\},
\] (13)

where

\[
h_i(v) = \eta \ln \sum_{j=1}^n \exp \left( \frac{v_j - c_{ij} - \eta}{\eta} \right) - \sum_{j=1}^n \beta_j v_j - \eta \ln \alpha_i + \eta.
\]

Problem (13) is referred to as the semi-dual of the entropy regularized OT problem. We further derive the expression of the primal variable \( x \) in terms of the semi-dual variable \( v \):

\[
x(v) = \exp \left( \frac{A^\top (u(v)^\top, v^\top)^\top - c - \eta 1_n^2}{\eta} \right),
\]

where

\[
u_i(v) = \eta \ln(\alpha_i) - \eta \ln \sum_{j=1}^n \exp \left( \frac{v_j - c_{ij} - \eta}{\eta} \right).
\] (14)

In this way, the closed form of \( x(v) \) could be given by

\[
[x(v)]_{i+n(j-1)} = \exp \left( \frac{u_i + v_j - c_{ij} - \eta}{\eta} \right)
\]

\[
= \exp \left( \eta \ln(\alpha_i) - \eta \ln \sum_{j=1}^n \exp \left( \frac{v_j - c_{ij} - \eta}{\eta} \right) + v_j - c_{ij} - \eta \right)
\]

\[
= \frac{\alpha_i \exp \left( \frac{v_j - c_{ij} - \eta}{\eta} \right)}{\sum_{l=1}^n \exp \left( \frac{u_l - c_{il} - \eta}{\eta} \right)}.
\] (15)

Before applying PDASGD, we should check that Assumption 1 holds. For the first part of Assumption
1, we rewrite equation (6) to accommodate the semi-dual formulation, and check it in Proposition 1.

**Proposition 1.** Following the notations that have been introduced above, we have that

\[ [Ax(v) - b]_{i=1,\ldots,n} = 0, \]

\[ \nabla G(v) = [Ax(v) - b]_{i=n+1,\ldots,2n}, \]

where \([\cdot]_{i=1,\ldots,n}\) denotes the 1st row to the nth row of the matrix and \([\cdot]_{i=n+1,\ldots,2n}\) denotes the \((n+1)\)st row to the \((2n)\)th row of the matrix.

For the second and the third part of Assumption 1, we check that \(g\)'s in (13) are convex and smooth, for which we specify the smoothness parameters. See details in the following proposition.

**Proposition 2.** In problem (13),

- \(g_i(v)\)'s are convex.
- \(g_i(v)\) is \(n\alpha_i/\eta\)-smooth w.r.t. \(\|\cdot\|_2\) for all \(1 \leq i \leq n\). The average \(L\)-smooth parameter w.r.t. \(\|\cdot\|_2\) of \(G(v)\) is \(1/\eta\).
- \(G(v)\) is \(5/\eta\)-smooth w.r.t. \(\|\cdot\|_\infty\).

**Remark 4.** Proposition 2 implies that the smooth parameters of the semi-dual objective function of the entropy-regularized OT are constants independent of \(n\), enabling us to apply our proposed PDASGD algorithm to improve the computational complexity as stated in Section 4.

We have verified that the entropy-regularized OT could satisfy the formulations and assumptions stated in Section 3, so we could apply our proposed algorithm PDASGD to the entropy-regularized OT (10) and its semi-dual (13).

### 5.2 Round to Feasible Area

After performing PDASGD, we employ the rounding algorithm in [3] to round the PDASGD output to the feasible area \(\mathcal{U}(\alpha, \beta)\). We restate the associated rounding algorithm in Algorithm 2.

We combine the aforementioned two subroutines, i.e., solving the entropy-regularized OT and
**Algorithm 2** Round to $U(\alpha, \beta)$ (Algorithm 2 in [3]).

1: **Input**: $F$.
2: $X \leftarrow D(x)$ with $x_i = \min\{\alpha_i/r_i(F), 1\}$, where $r_i(F)$ denotes the $i$th row sum of $F$. For a vector $x \in \mathbb{R}^n$, matrix $D(x) \in \mathbb{R}^{n \times n}$ is a diagonal matrix with entries $[D(x)]_{ii} = x_i$.
3: $F' \leftarrow XF$.
4: $Y \leftarrow D(y)$ with $y_j = \min\{\beta_j/c_j(F'), 1\}$, where $c_j(F')$ denotes the $j$th column sum of $F'$.
5: $F'' \leftarrow F'Y$.
6: $\text{err}_r \leftarrow \alpha - r(F'')$, $\text{err}_c \leftarrow \beta - c(F'')$.
7: **Output**: $E \leftarrow F'' + \text{err}_r \text{err}_c^T/\|\text{err}_r\|_1$.

Rounding to the feasible area, in Algorithm 3. The output gives an $\epsilon$-approximation solution to the OT problem.

**Algorithm 3** Approximating OT by PDASGD

1: **Input**: Accuracy $\epsilon > 0$, $\eta = \epsilon/(8 \ln n)$ and $\epsilon' = \epsilon/(6\|C\|_{\infty})$.
2: Let $\tilde{\alpha} \in \Delta_n$ and $\tilde{\beta} \in \Delta_n$ be defined by
   $$\tilde{b} = \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} 1 - \epsilon' \\ \frac{\epsilon'}{8n} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
   where $\Delta_n = \{p : \sum_{i=1}^n p_i = 1\}$.
3: Apply PDASGD (Algorithm 1) to approximate the solution to the entropy-regularized OT problem (9) with marginal $\tilde{b} = (\tilde{\alpha}^T, \tilde{\beta}^T)^T$ and penalty parameter $\eta$. Run PDASGD until the output $\tilde{X}$ satisfies $f(\mathbb{E}[\tilde{X}]) - f(x^*) \leq \epsilon/4$ and $\|\mathbb{A}E[\tilde{X}] - \tilde{b}\|_1 \leq \epsilon'/2$, where $\mathbb{E}[\tilde{X}] = \text{Vec}(\mathbb{E}[\tilde{X}])$, $x^* = \text{Vec}(X^*)$, and $X^*$ is the solution to the entropy-regularized OT problem.
4: Round $\tilde{X}$ to $\hat{X}$ by Algorithm 2 such that $\mathbb{E}[\hat{X}]1_n = \alpha$, $\mathbb{E}[\hat{X}]^T1_n = \beta$.
5: **Output**: $\hat{X}$.

**Remark 5.** Notice that the output of PDASGD is a random variable. When we analyze the convergence rates of PDASGD (Algorithm 1) and Algorithm 3, we focus on the values of expectation, i.e., $\mathbb{E}[\hat{X}]$ and $\mathbb{E}[\tilde{X}]$.

The total computational complexity of Algorithm 3 for giving an $\epsilon$-approximation solution to the OT problem is $\tilde{O}(n^2/\epsilon)$ as shown in the following theorem.

**Theorem 3.** PDASGD gives an $\epsilon$-approximation solution to the OT problem (1), i.e., $\hat{X} \in U(\alpha, \beta)$, $\mathbb{E}[\langle C, \hat{X} \rangle] - \langle C, x^* \rangle \leq \epsilon$, in a total number of

$$O\left(\frac{n^2\|C\|_{\infty}\sqrt{\ln n}}{\epsilon}\right)$$
Thanks to the structure of the semi-dual of the entropy-regularized OT problem, PDASGD achieves the state-of-the-art computational complexity $O(n^2/\epsilon)$ for computing the OT problem. Notably, the primal-dual first-order methods, including APDAGD, APDAMD, AAM, and APDRCD, focus on the dual problem of OT problem. The associated smooth parameter of the dual problem w.r.t. $\| \cdot \|_2$ is $2/\eta$ [8, 14, 20], which is in the same order as the smooth parameters of the semi-dual problem, seeing Proposition 2. This relationship explains why our PDASGD can be faster by a factor of $O(\sqrt{n})$, as stated in Corollary 2.

6 Numerical Experiments

We conduct numerical experiments in this section to compare our algorithm PDASGD with the existing approximation algorithms, including Sinkhorn, stochastic Sinkhorn, APDAGD, AAM, and PDASMD, for solving the OT problem.\footnote{The code is available at \url{https://github.com/YilingXie27/PDASGD}.}

6.1 Dataset

We use MNIST\footnote{http://yann.lecun.com/exdb/mnist/} as our real dataset. We also generate synthetic grey-scale images by the approach introduced in [3]. Each image is produced by randomly positioning a foreground square in an otherwise black background. The foreground occupies 20%, and the foreground and background intensities follow the uniform distribution on $[0,10]$ and $[0,1]$, respectively.

6.2 Experiment Setting

We set the number of the inner loops of PDASGD $m$ as $2\sqrt{n}$, where $n$ is the dimension of marginals. It is observed that PDASGD converges faster when the step size in Step 10 in Algorithm 1 increases, so we set the step size as $15\gamma_s$ for better practical performance.

We utilize the distance between the output $\hat{X}$ and the transportation polytope to measure the accuracy following [9, 20], i.e., $d(\hat{X}) = \| r(\hat{X}) - r \|_1 + \| c(\hat{X}) - c \|_1$, where $r$ and $c$ are given marginal distributions, $r_i(\hat{X})$ denotes the $i$th row sum of $\hat{X}$, and $c_j(\hat{X})$ denotes the $j$th column sum of $\hat{X}$.
We assess the efficiency of different algorithms by the total number of arithmetic operations. The desired accuracy is chosen from the set \{0.005, 0.01, 0.015, 0.02\}.

We run the algorithms on five pairs of randomly selected MNIST images or synthetic images. The error bar is plotted for each algorithm in different dimension size and accuracy choices.

### 6.3 Experiment Results

The numerical results are shown in Figure 1 to 8. The figures demonstrate that the practical computational complexity of our algorithm PDASGD is less than AAM, APDAGD, and PDASMD. Regarding Sinkhorn and Stochastic Sinkhorn, the average performance of PDASGD is comparable to theirs. However, Sinkhorn and stochastic Sinkhorn have a much larger variance than PDASGD. It implies that our proposed algorithm is more stable than the Sinkhorn algorithm and stochastic Sinkhorn algorithm.

In conclusion, considering its low practical complexity and stable performance, the proposed algorithm PDASGD is a desirable choice for solving the OT problem in practice.

![Figure 1: Synthetic Data, accuracy = 0.005](image1.png)  
![Figure 2: MNIST Dataset, accuracy = 0.005](image2.png)

### 7 Discussions

This paper proposes a primal-dual accelerated stochastic gradient descent method for solving the OT problem. One interesting extension of the OT problem is the computation for the Wasserstein Barycenter (WB). In the literature, people have been applying the primal-dual first-order algo-
Figure 3: Synthetic Data, accuracy = 0.01

Figure 4: MNIST Dataset, accuracy = 0.01

Figure 5: Synthetic Data, accuracy = 0.015

Figure 6: MNIST Dataset, accuracy = 0.015

Figure 7: Synthetic Data, accuracy = 0.02

Figure 8: MNIST Dataset, accuracy = 0.02
rithms to solve the distributed fixed-support WB problem over networks [17, 14] and achieve the computational complexity of the order $\tilde{O}(mn^{2.5}/\epsilon)$ [17], where $m$ denotes the number of distributions. Exploring the application of PDASGD for computing the Wasserstein Barycenter (WB) and achieving the state-of-the-art convergence rate of $\tilde{O}(mn^2/\epsilon)$ [7] could be considered as a promising future direction. In addition, the adoption of PDASGD to other formulations of the OT problem, e.g., unbalanced OT problem, multi-marginal OT problem, and partial OT problem, is also very interesting.

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A Proof of Theorem 1

A.1 A Lemma

The proof of Theorem 1 is based on the following lemma, and the associated proof is relegated to the Supplementary Materials.

Lemma 1.\
\[
\frac{\tau_2 m}{\tau_1, S} (E[\phi(\tilde{\lambda}^S)] - \phi(\lambda^*)) + \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) (f(E[x^S]) - f(x(\lambda^*))) + \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} (f(\tilde{\lambda}^0) - \phi(\lambda^*)) \\
\leq \frac{\tau_2 m}{\tau_{1,0}} (E[x^S] - b, \lambda) + \frac{\tau_2 m}{\tau_{1,0}} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) \\
+ \frac{1 - \tau_{1,0}}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 9T \|z_0 - \lambda\|_2^2.
\]

(16) holds for all \( \lambda \in \mathbb{R}^d \).

Proof. See G. □

A.2 Proof of Theorem 1

Proof. We first derive the upper bound of \( E[\phi(\tilde{\lambda}^S)] + f(E(x^S)) \).

We minimize both sides of (16) on \( B(2\|\lambda^*\|_2) \), where \( B(r) \) is defined as \( B(r) = \{ \|\lambda\|_2 \leq r \} \), as
follows,
\[
\frac{\tau_2 m}{\tau_{1,0}^2} (\mathbb{E}[\phi(\tilde{\lambda}^S)] - \phi(\lambda^*)) + \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) (f(\mathbb{E}[x^S]) - f(x(\lambda^*)))
\]
\[
\leq \min_{\lambda \in B(2\|\lambda^*\|_2)} \left\{ \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) (\mathbb{E}[x^S] - b, \lambda) + \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + \frac{1 - \tau_1,0 - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 36\mathcal{L}\|\lambda^*\|_2^2 \right\}
\]
\[
\leq \min_{\lambda \in B(2\|\lambda^*\|_2)} \left\{ \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) (\mathbb{E}[x^S] - b, \lambda) \right\}
\]
\[
+ \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + \frac{1 - \tau_1,0 - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 36\mathcal{L}\|\lambda^*\|_2^2
\]
\[
= -2 \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) \|\lambda^*\|_2 \|\mathbb{E}[Ax^S - b]\|_2 + \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + \frac{1 - \tau_1,0 - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 36\mathcal{L}\|\lambda^*\|_2^2
\]

where (a) uses the Hölder inequality.

Rearrange (17):
\[
\frac{\tau_2 m}{\tau_{1,0}^2} (\mathbb{E}[\phi(\tilde{\lambda}^S)] - \phi(\lambda^*)) \leq \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) (-f(\mathbb{E}[x^S]) + f(x(\lambda^*)) - \|\lambda^*\|_2 \|\mathbb{E}[Ax^S - b]\|_2)
\]
\[
+ \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + \frac{1 - \tau_1,0 - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 36\mathcal{L}\|\lambda^*\|_2^2.
\]

Notice that
\[
\phi(\lambda^*) = -f(x(\lambda^*)) + \langle \lambda^*, \nabla \phi(\lambda^*) \rangle = f(x(\lambda^*)),
\]
and
\[
f(\mathbb{E}[x^S]) - f(x(\lambda^*)) = f(\mathbb{E}[x^S]) + \phi(\lambda^*)
\]
\[
= f(\mathbb{E}[x^S]) - \langle \lambda^*, b \rangle + \max_x \{-f(x) + \langle Ax, \lambda^* \rangle\}
\geq f(\mathbb{E}[x^S]) - \langle \lambda^*, b \rangle - f(\mathbb{E}[x^S]) + \langle AE[x^S], \lambda^* \rangle
= (\lambda^*, AE[x^S] - b).
\]

Then, it follows from Hölder inequality that
\[
-f(\mathbb{E}[x^S]) + f(x(\lambda^*)) - \|\lambda^*\|_2\|\mathbb{E}[Ax^S - b]\|_2 \leq 0. \tag{19}
\]

We assume \(S \geq 2\) and could derive that
\[
\sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \geq \frac{S^2 + 7S}{4} \geq \frac{S^2 + 8S + 16}{8} = \frac{\tau_2}{\tau_{1,S}}. \tag{20}
\]

Combining (18), (19) and (20), we could obtain that
\[
\frac{\tau_2}{\tau_{1,S}}(\mathbb{E}[\phi(\tilde{\lambda}^S)] - \phi(\lambda^*)) \leq \frac{\tau_2}{\tau_{1,S}}(-f(\mathbb{E}[x^S]) + f(x(\lambda^*)) - \|\lambda^*\|_2\|\mathbb{E}[Ax^S - b]\|_2)
\]
\[
+ \frac{\tau_2}{\tau_{1,0}} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + \frac{1 - \tau_{1,0}}{\tau_{1,0}} (\phi(y_0) - \phi(\lambda^*)) + 36\bar{L}\|\lambda^*\|_2^2,
\]
which is equivalent to
\[
\frac{\tau_2}{\tau_{1,S}}(\mathbb{E}[\phi(\tilde{\lambda}^S)] + f(\mathbb{E}[x^S])) \leq \frac{\tau_2}{\tau_{1,S}}(\phi(\lambda^*) + f(x(\lambda^*)) - \|\lambda^*\|_2\|\mathbb{E}[Ax^S - b]\|_2)
\]
\[
+ \frac{\tau_2}{\tau_{1,0}} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + \frac{1 - \tau_{1,0}}{\tau_{1,0}} (\phi(y_0) - \phi(\lambda^*)) + 36\bar{L}\|\lambda^*\|_2^2.
\]

Considering \(\phi(\lambda^*) = -f(x(\lambda^*))\) holds, we have that
\[
\frac{\tau_2}{\tau_{1,S}}(\mathbb{E}[\phi(\tilde{\lambda}^S)] + f(\mathbb{E}[x^S])) \leq \frac{\tau_2}{\tau_{1,0}} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*))
\]
\[
+ \frac{1 - \tau_{1,0}}{\tau_{1,0}} (\phi(y_0) - \phi(\lambda^*)) + 36\bar{L}\|\lambda^*\|_2^2. \tag{21}
\]

\[24\]
Since
\[ \frac{\tau_2}{\tau_{1,S}} = \frac{S^2 + 8S + 16}{8} \geq \frac{S^2}{8} \]
holds, we could derive from (21) that
\[
\mathbb{E}[\phi(\tilde{\lambda}^S)] + f(\mathbb{E}(x^S)) \leq 8 \left( \frac{2(\phi(\lambda^0) - \phi(\lambda^*))}{S^2} + \frac{36\mathcal{L}\|\lambda^*\|^2}{mS^2} \right). \tag{22}
\]

Then, we derive the upper bound of \( \|\mathbb{E}[Ax^S - b]\|_2 \).

One may note that (16) is equivalent to
\[
\mathbb{E}[\phi(\tilde{\lambda}^S)] + f(\mathbb{E}(x^S)) \geq \phi(\lambda^*) + f(\mathbb{E}(x^S)) = f(\mathbb{E}(x^S)) - f(\mathbb{x^S}) + \|\lambda^*\|_2\|\mathbb{E}[Ax^S - b]\|_2
\]
\[
\leq \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \|\lambda^*\|_2\|\mathbb{E}[Ax^S - b]\|_2 + \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \mathcal{L}\|z_0 - \lambda\|^2.
\]

Similarly, we could obtain that
\[
\mathbb{E}[\phi(\tilde{\lambda}^S)] + f(\mathbb{E}(x^S)) \leq \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \|\lambda^*\|_2\|\mathbb{E}[Ax^S - b]\|_2 + \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \mathcal{L}\|z_0 - \lambda\|^2.
\]

Further, notice that we have the following inequality:
\[
\mathbb{E}[\phi(\tilde{\lambda}^S)] + f(\mathbb{E}(x^S)) \geq \phi(\lambda^*) + f(\mathbb{E}(x^S)) = f(\mathbb{E}(x^S)) - f(\mathbb{x^S}) \geq -\|\lambda^*\|_2\|\mathbb{E}[Ax^S - b]\|_2, \tag{24}
\]
where (a) uses (19).

Plugging (24) into (23) and minimizing both sides on \( B(2\|\lambda^*\|_2) \), where \( B(r) \) is defined as \( B(r) = \)
\[ \{\|\lambda\|_2 \leq r\}, \text{ we could have that} \]

\[
- \frac{\tau_2 m}{\tau_{1,s}^2} \|\lambda^*\|_2 \|\mathbb{E}[Ax^S - b]\|_2 \\
\leq \min_{\lambda \in B(2\|\lambda^*\|_2)} \left\{ \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) \langle A\mathbb{E}[x^S] - b, \lambda \rangle + \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) \right\} \\
+ \frac{1 - \tau_{1,0} - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 9L\|z_0 - \lambda\|_2^2 \\
+ \frac{\tau_2 m}{\tau_{1,0}^2} \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) \|\lambda^*\|_2 \|\mathbb{E}[Ax^S - b]\|_2 \\
\leq \min_{\lambda \in B(2\|\lambda^*\|_2)} \left\{ \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) \langle A\mathbb{E}[x^S] - b, \lambda \rangle \right\} \\
+ \frac{\tau_2 m}{\tau_{1,0}^2} \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) \|\lambda^*\|_2 \|\mathbb{E}[Ax^S - b]\|_2 \\
\overset{(a)}{=} - \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) \|\lambda^*\|_2 \|\mathbb{E}[Ax^S - b]\|_2 + \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) \\
+ \frac{1 - \tau_{1,0} - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 36L\|\lambda^*\|_2^2,
\]

where (a) uses the Hölder inequality.

Then, we rearrange the terms as follows,

\[
m \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} - \frac{\tau_2}{\tau_{1,s}^2} \right) \|\lambda^*\|_2 \|\mathbb{E}[Ax^S - b]\|_2 \\
\leq \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + \frac{1 - \tau_{1,0} - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 36L\|\lambda^*\|_2^2 \\
= 2m (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + 36L\|\lambda^*\|_2^2.
\]

We assume \( S \geq 3 \), then we have

\[
\left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} - \frac{\tau_2}{\tau_{1,s}^2} \right) \geq S^2/8.
\]
In this way, we could conclude that

$$
\|E[Ax^S - b]\|_2 \leq 8 \left( \frac{2}{S^2} \left( \frac{\phi(\tilde{\lambda}^0) - \phi(\lambda^*)}{\|\lambda^*\|_2} + \frac{36L\|\lambda^*\|_2}{mS^2} \right) \right). \tag{26}
$$

Notice that $E[\phi(\tilde{\lambda}^S)] \geq -f(x^*)$ and $\tilde{\lambda}^0 = 0$ holds, thereby we could get the following from (22) and (26):

$$
f(E(x^S)) - f(x^*) \lesssim \frac{\phi(0) - \phi(\lambda^*)}{S^2\|\lambda^*\|_2} + \frac{T\|\lambda^*\|_2}{mS^2},
$$

$$
\|E[Ax^S - b]\|_2 \lesssim \frac{\phi(0) - \phi(\lambda^*)}{S^2\|\lambda^*\|_2} + \frac{T\|\lambda^*\|_2}{mS^2}.
$$

\(\Box\)

### B  Proof of Corollary 1

**Proof.** It follows from $L'$-smoothness of $\phi$ w.r.t. $\|\cdot\|_\infty$ that

$$
\phi(0) - \phi(\lambda^*) \leq \langle \nabla \phi(\lambda^*), 0 - \lambda^* \rangle + \frac{L'}{2} \|0 - \lambda^*\|_\infty^2 = \frac{L'}{2} \|\lambda^*\|_\infty^2.
$$

Thus, we could obtain the following immediately from Theorem 1 that

$$
f(E(x^S)) - f(x^*) \lesssim \frac{L'\|\lambda^*\|_\infty^2}{S^2} + \frac{T\|\lambda^*\|_2}{mS^2},
$$

$$
\|E[Ax^S - b]\|_2 \lesssim \frac{L'\|\lambda^*\|_\infty^2}{S^2\|\lambda^*\|_2} + \frac{T\|\lambda^*\|_2}{mS^2}.
$$

\(\Box\)

### C  Proof of Corollary 2

**Proof.** For the output $x^S$ of PDASGD, we have that

$$
f(E(x^S)) - f(x^*) \lesssim \frac{L'\|\lambda^*\|_\infty^2}{S^2} + \frac{T\|\lambda^*\|_2}{nS^2} \leq \frac{(L' + T)\|\lambda^*\|_\infty^2}{S^2}.
$$
Considering that $L'$ and $\overline{L}$ have the same order, we have that

$$f(\mathbb{E}(x^S)) - f(x^*) \lesssim \frac{L'\|\lambda^*\|_2^2}{S^2}.$$

To let $f(\mathbb{E}(x^S)) - f(x^*) \leq \epsilon$ hold, it follows from $k = nS$ that

$$k = \mathcal{O} \left( n\sqrt{\frac{L'}{\epsilon} \|\lambda^*\|_{\infty}} \right),$$

where $k$ denotes the number of total iterations.

For the output $x^k$ of APDAGD, we have that

$$f(x^k) - f(x^*) \lesssim \frac{\tilde{L}\|\lambda^*\|_2^2}{k^2} \leq \frac{n\tilde{L}\|\lambda^*\|_2^2}{k^2}.$$

Considering that $L'$ and $\overline{L}$ have the same order, we have that

$$f(x^k) - f(x^*) \lesssim \frac{nL'\|\lambda^*\|_2^2}{k^2}.$$

To let $f(x^k) - f(x^*) \leq \epsilon$ hold, we have that

$$k = \mathcal{O} \left( \sqrt{n} \sqrt{\frac{L'}{\epsilon} \|\lambda^*\|_{\infty}} \right),$$

where $k$ denotes the number of total iterations.

If the computational complexity of each iteration is $\mathcal{O}(K)$ in PDASGD and $\mathcal{O}(nK)$ in APDAGD, the associated computational complexities for APDAGD and PDASGD are $\mathcal{O}(n^{3/2}K\sqrt{L'/\epsilon}\|\lambda^*\|_{\infty})$ and $\mathcal{O}(nK\sqrt{L'/\epsilon}\|\lambda^*\|_{\infty})$, respectively. □
D Proof of Proposition 1

Proof. We calculate the gradient of $G(v)$ as follows.

$$[\nabla G(v)]_j = -\beta_j + \sum_{i=1}^{n} \alpha_i \exp\left(\frac{v_j - c_{ij} - \eta}{\eta}\right) \sum_{l=1}^{n} \exp\left(\frac{v_l - c_{il} - \eta}{\eta}\right).$$

It follows from equation (15) that

$$[x(v)]_{i+n(j-1)} = \alpha_i \exp\left(\frac{v_j - c_{ij} - \eta}{\eta}\right) \sum_{l=1}^{n} \exp\left(\frac{v_l - c_{il} - \eta}{\eta}\right),$$

and

$$[Ax(v)]_{i=n+1,...,2n} = [A]_{i=n+1,...,2n} x(v) = \sum_{i=1}^{n} \alpha_i \exp\left(\frac{v_j - c_{ij} - \eta}{\eta}\right) \sum_{l=1}^{n} \exp\left(\frac{v_l - c_{il} - \eta}{\eta}\right).$$

It is obvious that $\nabla G(v) = [Ax(v) - b]_{i=n+1,...,2n}$ holds.

For the first $n$ rows, we have that

$$[Ax(v)]_{i=1,...,n} = [A]_{i=1,...,n} x(v) = \sum_{j=1}^{n} \alpha_i \exp\left(\frac{v_j - c_{ij} - \eta}{\eta}\right) \sum_{l=1}^{n} \exp\left(\frac{v_l - c_{il} - \eta}{\eta}\right) = \alpha_i,$$

which means that $[Ax(v) - b]_{i=1,...,n} = 0$.

E Proof of Proposition 2

Proof. We begin with proving $g_i(v)$'s are convex.

We compute the first order derivative of $g_i(v)$ as follows,

$$\frac{\partial g_i(v)}{\partial v_j} = -n\alpha_i \left( \beta_j - \frac{\exp(\frac{v_j - c_{ij} - \eta}{\eta})}{\sum_{l=1}^{n} \exp(\frac{v_l - c_{il} - \eta}{\eta})} \right),$$
and the second order derivative of $g_i(v)$ as follows,

$$\frac{\partial^2 g_i(v)}{\partial v_j^2} = n \frac{\alpha_i}{\eta} \left( \frac{\exp \left( \frac{v_j - c_{ij} - \eta}{\eta} \right)}{\sum_{l=1}^n \exp \left( \frac{v_l - c_{il} - \eta}{\eta} \right)} - \frac{(\exp \left( \frac{v_j - c_{ij} - \eta}{\eta} \right))^2}{\left( \sum_{l=1}^n \exp \left( \frac{v_l - c_{il} - \eta}{\eta} \right) \right)^2} \right),$$

$$\frac{\partial^2 g_i(v)}{\partial v_j \partial v_k} = -n \frac{\alpha_i}{\eta} \frac{\exp \left( \frac{v_j - c_{ij} - \eta}{\eta} \right) \exp \left( \frac{v_k - c_{ik} - \eta}{\eta} \right)}{\left( \sum_{l=1}^n \exp \left( \frac{v_l - c_{il} - \eta}{\eta} \right) \right)^2}.$$

Then, we denote the corresponding Hessian matrix of $G(v)$ by $H$ and let $K_j = \exp((v_j - c_{ij} - \eta)/\eta)$. In this way, we could obtain that

$$H = n \frac{\alpha_i}{\eta} \left( \text{diag} \left( \frac{K_j}{\sum_l K_l} \right) - \frac{K_i K_j}{\left( \sum_l K_l \right)^2} \right).$$

For any nonzero vector $y$, we have that

$$y^\top H y = n \frac{\alpha_i}{\eta} \sum_{i,j} y_i H_{ij} y_j = n \frac{\alpha_i}{\eta} \left( \frac{\sum_j y_j^2 K_j}{\sum_l K_l} - \frac{(\sum_i y_i K_i)^2}{(\sum_l K_l)^2} \right).$$

(27)

It follows from the Cauchy Schwartz inequality that

$$\left( \sum_i y_i K_i \right)^2 = \left( \sum_i y_i \sqrt{K_i} \sqrt{K_i} \right)^2 \leq \sum_l K_l \sum_j y_j^2 K_j.$$

Plugging the inequality above into equation (27), we could obtain that

$$y^\top H y \geq 0.$$

Thus, the Hessian matrix is positive semi-definite, which implies that $g_i$’s are convex.

Secondly, we prove $g_i(v)$ is $L_i$-smooth and specify the smoothness parameter.
Notice the equation (27) implies the following inequality:

\[ y^\top H y \leq n \frac{\alpha_i}{\eta} \sum_j y_j^2 K_j \]

which is equivalent to

\[ \frac{y^\top H y}{y^\top y} \leq n \frac{\alpha_i}{\eta}. \]

By Rayleigh quotient theory, the maximum eigenvalue of \( H \) is less than or equal to \( n\alpha_i/\eta \). According to Lemma 1.2.2 in [25], \( g_i(v) \) is \( L_i \)-smooth, where \( L_i = n\alpha_i/\eta \). We could compute the average \( L \)-smooth parameter of \( G(v) \) immediately:

\[
L = \frac{1}{n} \sum_{i=1}^{n} L_i = \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha_i}{\eta} = \frac{1}{\eta}.
\]

**Regarding the \( L \)-smoothness w.r.t. \( \| \cdot \|_\infty \),** Lemma 1 in [22] shows that \( G(v) \) is \( 5/\eta \)-smooth w.r.t. \( \| \cdot \|_\infty \).

## F Proof of Theorem 3

*Proof.* Firstly, we prove that the output \( \tilde{X} \) of Algorithm 3 is an \( \epsilon \)-approximation to the OT problem, i.e., \( E[\langle C, \tilde{X} \rangle] - \langle C, X^* \rangle \leq \epsilon. \)

We use the notations \( \alpha' = r(E[\tilde{X}]) \) and \( \beta' = c(E[\tilde{X}]) \), where \( r_i(\cdot) \) denotes the \( i \)th row sum of the matrix and \( c_j(\cdot) \) denotes the \( j \)th column sum of the matrix. By Lemma 7 in [3], given \( \alpha', \beta' \) and \( X^* \), there exists a matrix \( X' \in \mathcal{U}(\alpha', \beta') \) such that

\[
\| X' - X^* \|_1 \leq 2 \left[ \| r(X^*) - \alpha' \|_1 + \| c(X^*) - \beta' \|_1 \right]
\]

\[
= 2 \left[ \| \tilde{\alpha} - \alpha' \|_1 + \| \tilde{\beta} - \beta' \|_1 \right].
\]

It follows from \( f(E[\tilde{X}]) - f(x^*) \leq \epsilon/4 \) and \( f(x^*) - f(x') \leq 0 \) that

\[ f(E[\tilde{X}]) - f(x') \leq \epsilon/4, \]

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where \( \tilde{x} = \text{Vec}(\tilde{X}) \), \( x^* = \text{Vec}(X^*) \) and \( x' = \text{Vec}(X') \).

Recall that \( f \) denotes the primal function of the entropy-regularized OT problem, so could obtain that

\[
\mathbb{E}[\langle \tilde{X}, C \rangle] - \eta H(\mathbb{E}[\tilde{X}]) \leq \langle X', C \rangle - \eta H(X') + \frac{\epsilon}{4}.
\]  

(29)

Then, we have the following inequalities:

\[
\mathbb{E}[\langle \tilde{X}, C \rangle] - \langle X^*, C \rangle = \mathbb{E}[\langle \tilde{X}, C \rangle] - \langle X', C \rangle + \langle X', C \rangle - \langle X^*, C \rangle \\
\quad \leq (a) \eta(H(\mathbb{E}[\tilde{X}]) - H(X')) + \frac{\epsilon}{4} + \langle X' - X^*, C \rangle \\
\quad \leq (b) \eta(H(\mathbb{E}[\tilde{X}]) - H(X')) + \frac{\epsilon}{4} + \|X' - X^*\|_1 \|C\|_\infty \\
\quad \leq (c) \eta(H(\mathbb{E}[\tilde{X}]) - H(X')) + \frac{\epsilon}{4} + 2 \left[ \|\tilde{\alpha} - \alpha'\|_1 + \|\tilde{\beta} - \beta'\|_1 \right] \|C\|_\infty \\
\quad \leq (d) 2\eta \ln n + \frac{\epsilon}{4} + 2 \left[ \|\tilde{\alpha} - \alpha'\|_1 + \|\tilde{\beta} - \beta'\|_1 \right] \|C\|_\infty,
\]

(30)

where (a) uses (29), (b) uses the Hölder inequality, (c) uses (28) and (d) comes from \( 0 \leq H(\mathbb{E}[\tilde{X}]), H(X') \leq 2\ln n \).

It follows from Lemma 7 in [3] that \( \mathbb{E}[\tilde{X}] \) and \( \mathbb{E}[\tilde{X}] \) satisfies

\[
\|\mathbb{E}[\tilde{X}] - \mathbb{E}[\tilde{X}]\|_1 \leq 2 \left[ \|\alpha' - \alpha\|_1 + \|\alpha' - \beta\|_1 \right].
\]

(31)

We further have that

\[
\mathbb{E}[\langle \tilde{X}, C \rangle] - \langle X^*, C \rangle \\
= \mathbb{E}[\langle \tilde{X}, C \rangle] - \langle X^*, C \rangle + \mathbb{E}[\langle \tilde{X}, C \rangle] - \mathbb{E}[\langle \tilde{X}, C \rangle] \\
\quad \leq (a) \mathbb{E}[\langle \tilde{X}, C \rangle] - \langle X^*, C \rangle + \|\mathbb{E}[\tilde{X}] - \mathbb{E}[\tilde{X}]\|_1 \|C\|_\infty \\
\quad \leq (b) \mathbb{E}[\langle \tilde{X}, C \rangle] - \langle X^*, C \rangle + 2 \left[ \|\alpha' - \alpha\|_1 + \|\alpha' - \beta\|_1 \right] \|C\|_\infty \\
\quad \leq (c) 2\eta \ln n + \frac{\epsilon}{4} + 2 \left[ \|\alpha' - \alpha\|_1 + \|\beta' - \beta\|_1 \right] \|C\|_\infty + 2 \left[ \|\tilde{\alpha} - \alpha'\|_1 + \|\tilde{\beta} - \beta'\|_1 \right] \|C\|_\infty \\
\quad \leq (d) \frac{\epsilon}{4} + \frac{\epsilon}{4} + 2 \left[ \|\alpha' - \alpha\|_1 + \|\beta' - \beta\|_1 \right] \|C\|_\infty + 2 \left[ \|\tilde{\alpha} - \alpha'\|_1 + \|\tilde{\beta} - \beta'\|_1 \right] \|C\|_\infty
\]

(32)
where (a) uses Hölder inequality, (b) uses (31), (c) uses (30), and (d) comes from $\eta = \epsilon/8 \ln n$.

Notice we could get that
\[
\|\bar{\alpha} - \alpha'\|_1 + \|\bar{\beta} - \beta'\|_1 = \|\mathbb{E}[A\bar{x} - \bar{b}]\|_1 \leq \frac{\epsilon'}{2}
\] (33)
from the stopping criteria, and
\[
\|\alpha' - \alpha\|_1 + \|\beta' - \beta\|_1 \leq \|\alpha' - \bar{\alpha}\|_1 + \|\beta' - \bar{\beta}\|_1 + \|\alpha - \bar{\alpha}\|_1 + \|\beta - \bar{\beta}\|_1
\]
\[
= \|\mathbb{E}[A\bar{x} - \bar{b}]\|_1 + \|\alpha - \bar{\alpha}\|_1 + \|\beta - \bar{\beta}\|_1
\]
\[
\leq (a) \frac{\epsilon'}{2} + \frac{\epsilon'}{2} = \epsilon',
\] (34)
where (a) comes from the definition of $\bar{b}$ in Step 2 in Algorithm 3.

Since $\epsilon' = \epsilon/6\|C\|_\infty$, it follows from (32), (33) and (34) that
\[
\mathbb{E}[\langle \bar{X}, C \rangle] - \langle X^*, C \rangle \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + 3\epsilon'\|C\|_\infty = \epsilon.
\]

It remains to compute the computational complexity. We set the number of inner loops $m$ in PDASGD as $n$. (One could check that the choice of the value of $m$ does not affect the convergence results.) The proof of this part is based on the following lemma, and the associated proof is relegated to the Supplementary Materials.

**Lemma 2.** The output $x^S$ of Step 2 in Algorithm 3 satisfies
\[
f(\mathbb{E}[x^S]) - f(x^*) \lesssim \frac{R^2}{\eta S^2} + \frac{R^2}{\eta m S^2},
\]
\[
\|\mathbb{E}[Ax^S - \bar{b}]\|_2 \lesssim \frac{R'^2}{\eta S^2 R} + \frac{R}{\eta m S^2},
\]
where $x^*$ is the solution to (10), $\lambda^*$ is the solution to (11), $R$ is the upper bound of $\|\lambda^*\|_2$, and $R'$ is the upper bound of $\|\lambda^*\|_\infty$. Note that the marginals of the entropy-regularized OT problem are $\bar{\alpha}$ and $\bar{\beta}$. 

Proof. See G.3.

Notably, Lemma 3.2 in [20] proves that

\[ R = \eta \sqrt{n(\tilde{R} + 1/2)}, \quad R' = \eta (\tilde{R} + 1/2), \]

where

\[ \tilde{R} = \frac{\|C\|_{\infty}}{\eta} + \ln n - 2 \ln(\min_{1 \leq i, j \leq n} \{\tilde{\alpha}_i, \tilde{\beta}_j\}). \]

If we let \( N = nS \) be the number of total iterations, then we could have that

\[
f(\mathbb{E}[x^N]) - f(x^*) \leq \frac{n^2 R^2}{\eta N^2} + \frac{n R^2}{\eta N^2} \leq \frac{n^2 R^2 (\tilde{R} + 1/2)^2}{N^2},
\]

\[
\|\mathbb{E}[Ax^N - b]\|_1 \leq \sqrt{n} \|\mathbb{E}[Ax^N - b]\|_2 \leq \sqrt{n} \left( \frac{n^2 R^2}{\eta N^2 R} + \frac{n R}{\eta N^2} \right) \leq \frac{n^2 R' R}{\eta N^2} = \frac{n^2 (\tilde{R} + 1/2)}{N^2}.
\]

It follows from the definition of \( \tilde{b} \) in Step 2 in Algorithm 3 that

\[
\min_{1 \leq i, j \leq n} \{\tilde{\alpha}_i, \tilde{\beta}_j\} \geq \frac{\epsilon'}{8n}.
\]

Accordingly, it follows from \( \epsilon' = \epsilon/(6\|C\|_{\infty}) \) and \( \eta = \epsilon/(8 \ln n) \) that

\[
\tilde{R} \leq 8 \frac{\|C\|_{\infty}}{\epsilon} \ln n + \ln n - 2 \ln \left( \frac{\epsilon}{48n \|C\|_{\infty}} \right) = O \left( \frac{\|C\|_{\infty} \ln n}{\epsilon} \right).
\]

To make \( f(\mathbb{E}[x^N]) - f(x^*) \leq \epsilon/4 \) hold, we have that

\[
N = O \left( n \tilde{R} \sqrt{\frac{\eta}{\epsilon}} \right) = O \left( \frac{n \|C\|_{\infty} \sqrt{\ln n}}{\epsilon} \right).
\]
Consequently, to let $\|E[Ax^N - \bar{b}]\|_1 \leq \epsilon'/2$ hold, we have that

$$N = \mathcal{O}\left(n\sqrt{\frac{R}{\epsilon'}}\right) = \mathcal{O}\left(n\sqrt{\frac{6\|C\|_\infty \|C\|_\infty \ln n}{\epsilon'}}\right) = \mathcal{O}\left(n\|C\|_\infty \sqrt{\ln n}\right).$$

Each iteration of PDASGD requires $\mathcal{O}(n)$ operations on average. $\tilde{\alpha}$ and $\tilde{\beta}$ in Step 1 in Algorithm 2 could be found in $\mathcal{O}(n)$ operations. Step 3 requires $\mathcal{O}(n^2)$ operations. Therefore, the total number of operations is $\mathcal{O}\left(n^2\|C\|_\infty \sqrt{\ln n}/\epsilon\right).$ 

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Supplementary Materials

G Proof of Lemma 1

G.1 Two Lemmas

We first present two lemmas that will be used to prove Lemma 1.

Notably, our algorithm is motivated by Algorithm 5 (Katyusha\textsuperscript{2}ns) in [2]. Similar to the analysis in [2], we analyze our algorithm for a fixed $k$. Hence, $y_k$, $z_k$ and $\lambda_{k+1}$ are fixed, and the randomness only comes from a choice of $i$. We rewrite the single iteration as:

$$\lambda_{k+1} = \tau_1 z_k + \tau_2 \bar{\lambda} + (1 - \tau_1 - \tau_2)y_k,$$

$$\bar{\nabla}_{k+1} = u + (\nabla \phi_i(\lambda_{k+1}) - \nabla \phi_i(\bar{\lambda}))/hp_i,$$

$$z_{k+1} = z_k - \gamma \bar{\nabla}_{k+1}/2,$$

$$y_{k+1} = \lambda_{k+1} - \bar{\nabla}_{k+1}/g\mathcal{L}.$$

Lemma 3.

$$\gamma \langle \nabla \phi(\lambda_{k+1}), z_k - \lambda \rangle$$

$$\leq \frac{\gamma}{\tau_1} \left( \phi(\lambda_{k+1}) - \mathbb{E}[\phi(y_{k+1})] + \tau_2 \phi(\bar{\lambda}) - \tau_2 \phi(\lambda_{k+1}) - \tau_2 \langle \nabla \phi(\lambda_{k+1}), \bar{\lambda} - \lambda_{k+1} \rangle \right)$$

$$+ \|z_k - \lambda\|_2^2 - \mathbb{E}\|z_{k+1} - \lambda\|_2^2$$

holds for all $\lambda \in \mathbb{R}^d$.

Proof. Set $\psi = 0$ and $V_2(u) = \|z - u\|_2^2$ in Lemma E.4 of [2]. □
Lemma 4.

\[ E[\phi(y_{k+1})] \leq \tau_1 (\phi(\lambda_{k+1}) + \langle \nabla \phi(\lambda_{k+1}), \lambda - \lambda_{k+1} \rangle) + \tau_2 \phi(\bar{\lambda}) + (1 - \tau_1 - \tau_2) \phi(y_k) \]
\[ + \frac{\tau_1}{\gamma} \|z_k - \lambda\|^2 \quad \text{for all } \lambda \in \mathbb{R}^d. \]

Proof. Following the proof logic of Lemma 2.7 in [2], we have that

\[ \gamma \langle \nabla \phi(\lambda_{k+1}), \lambda_{k+1} - \lambda \rangle \]
\[ = \gamma \langle \nabla \phi(\lambda_{k+1}), \lambda_{k+1} - z_k \rangle + \gamma \langle \nabla \phi(\lambda_{k+1}), z_k - \lambda \rangle \]
\[ \leq \frac{\gamma \tau_1}{\tau_1} \langle \nabla \phi(\lambda_{k+1}), \bar{\lambda} - \lambda_{k+1} \rangle + \frac{\gamma (1 - \tau_1 - \tau_2)}{\tau_1} \langle \nabla \phi(\lambda_{k+1}), y_k - \lambda_{k+1} \rangle + \gamma \langle \nabla \phi(\lambda_{k+1}), z_k - \lambda \rangle \quad (36) \]

where (a) uses (35), (b) uses the convexity of \( \phi \).

Then, we apply Lemma 3 to (36), we obtain that

\[ \gamma \langle \nabla \phi(\lambda_{k+1}), \lambda_{k+1} - \lambda \rangle \]
\[ \leq \frac{\gamma \tau_2}{\tau_1} \phi(y_k) + \frac{\gamma (1 - \tau_1 - \tau_2)}{\tau_1} \phi(\lambda_{k+1}) + \frac{\gamma \tau_2}{\tau_1} \phi(\bar{\lambda}) + \|z_k - \lambda\|^2 - E[\|z_{k+1} - \lambda\|^2]. \]

After rearranging, we could get that

\[ E[\phi(y_{k+1})] \leq \tau_1 (\phi(\lambda_{k+1}) + \langle \nabla \phi(\lambda_{k+1}), \lambda - \lambda_{k+1} \rangle) + \tau_2 \phi(\bar{\lambda}) + (1 - \tau_1 - \tau_2) \phi(y_k) \]
\[ + \frac{\tau_1}{\gamma} \|z_k - \lambda\|^2 - \frac{\tau_1}{\gamma} E[\|z_{k+1} - \lambda\|^2]. \]
G.2 Proof of Lemma 1

Equipped with Lemma 3 and Lemma 4, we begin to prove Lemma 1.

Proof. We first sum up both sides of the inequality in Lemma 4 for \( k = sm, \ldots, sm + m - 1 \) and could get that

\[
E\left[ \sum_{j=1}^{m} \phi(y_{sm+j}) \right] \\
\leq \tau_{1,s} \sum_{j=1}^{m} (\phi(\lambda_{sm+j}) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle) + \tau_{2} m \phi(\lambda) + (1 - \tau_{1,s} - \tau_{2}) \sum_{j=0}^{m-1} \phi(y_{sm+j}) \quad (37)
\]

where \( \tau_{1,s} = 2/(s + 4), \gamma_{s} = 1/9\tau_{1,s}L \) and \( \tau_{2} = 1/2. \)

Step 13 in PDASGD implies \( \bar{\lambda} = \sum_{j=1}^{m} y_{(s-1)m+j}/m. \) Plugging this expression into (37), together with the convexity of \( \phi \), we could obtain that

\[
E\left[ \sum_{j=1}^{m} \phi(y_{sm+j}) \right] \leq \tau_{1,s} \sum_{j=1}^{m} (\phi(\lambda_{sm+j}) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle) \\
+ \tau_{2} \sum_{j=1}^{m} \phi(y_{(s-1)m+j}) + (1 - \tau_{1,s} - \tau_{2}) \sum_{j=0}^{m-1} \phi(y_{sm+j}) \\
+ \frac{\tau_{1,s}}{\gamma_{s}} \sum_{j=0}^{m-1} ||z_{sm+j} - \lambda||_{2}^{2} - \frac{\tau_{1,s}}{\gamma_{s}} \sum_{j=1}^{m} E[||z_{sm+j} - \lambda||_{2}^{2}].
\]

By taking the expectation of both sides, we could have that

\[
E\left[ \sum_{j=1}^{m} \phi(y_{sm+j}) \right] \leq \tau_{1,s} \sum_{j=1}^{m} E[(\phi(\lambda_{sm+j}) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle)] + \tau_{2} E[\sum_{j=1}^{m} \phi(y_{(s-1)m+j})] \\
+ (1 - \tau_{1,s} - \tau_{2}) E[\sum_{j=0}^{m-1} \phi(y_{sm+j})] + \frac{\tau_{1,s}}{\gamma_{s}} E[||z_{sm} - \lambda||_{2}^{2}] - \frac{\tau_{1,s}}{\gamma_{s}} E[||z_{sm+m} - \lambda||_{2}^{2}].
\]

Divide both sides by \( \tau_{1,s}^{2} \), rewrite \( \sum_{j=0}^{m-1} \phi(y_{sm+j}) \) as \( \sum_{j=1}^{m} \phi(s_{sm+j}) + \phi(y_{sm}) - \phi(y_{sm+m}) \) on the
right side, and rearrange the terms:

\[
\begin{align*}
\mathbb{E}[\frac{\tau_1, s + \tau_2}{\tau_1, s} \sum_{j=1}^{m} \phi(y_{sm+j}) + \frac{1 - \tau_1, s - \tau_2}{\tau_1, s} \phi(y_{sm+m})] & \\
\leq \frac{1}{\tau_1, s} \sum_{j=1}^{m} \mathbb{E}[(\phi(\lambda_{sm+j}) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle)] + \frac{\tau_2}{\tau_1, s} \mathbb{E}[\sum_{j=1}^{m} \phi(y_{s-1}m+j)] & \\
+ \frac{1 - \tau_1, s - \tau_2}{\tau_1, s} \mathbb{E}[\phi(y_{sm})] + 9\mathbb{I}\mathbb{E}[\|z_{sm} - \lambda\|_2^2] - 9\mathbb{I}\mathbb{E}[\|z_{sm+m} - \lambda\|_2^2].
\end{align*}
\]

Then, we split the first term on the left side as \(\sum_{j=1}^{m} \phi(y_{sm+j}) = \sum_{j=1}^{m-1} \phi(y_{sm+j}) + \phi(y_{sm+m})\) and the second term on the right side as \(\sum_{j=1}^{m-1} \phi(y_{s-1}m+j) = \sum_{j=1}^{m-1} \phi(y_{s-1}m+j) + \phi(y_{sm})\):

\[
\begin{align*}
\mathbb{E}[\frac{\tau_1, s + \tau_2}{\tau_1, s} \sum_{j=1}^{m-1} \phi(y_{sm+j})] & + \mathbb{E}[\frac{1}{\tau_1, s} \phi(y_{sm+m})] & \\
\leq \frac{1}{\tau_1, s} \sum_{j=1}^{m} \mathbb{E}[(\phi(\lambda_{sm+j}) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle)] & + \frac{\tau_2}{\tau_1, s} \mathbb{E}[\sum_{j=1}^{m} \phi(y_{s-1}m+j)] & \\
+ \frac{1 - \tau_1, s - \tau_2}{\tau_1, s} \mathbb{E}[\phi(y_{sm})] & + 9\mathbb{I}\mathbb{E}[\|z_{sm} - \lambda\|_2^2] & - 9\mathbb{I}\mathbb{E}[\|z_{sm+m} - \lambda\|_2^2].
\end{align*}
\]

For analysis convenience, we introduce the optimal value term \(\phi(\lambda^*)\) to both sides and make the inequality still holds:

\[
\begin{align*}
\mathbb{E}[\frac{\tau_1, s + \tau_2}{\tau_1, s} \sum_{j=1}^{m-1} (\phi(y_{sm+j}) - \phi(\lambda^*))] & + \mathbb{E}[\frac{1}{\tau_1, s} (\phi(y_{sm+m}) - \phi(\lambda^*))] & \\
\leq \frac{1}{\tau_1, s} \sum_{j=1}^{m} \mathbb{E}[(\phi(\lambda_{sm+j}) - \phi(\lambda^*) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle)] & & \\
+ \frac{\tau_2}{\tau_1, s} \mathbb{E}[\sum_{j=1}^{m-1} (\phi(y_{s-1}m+j) - \phi(\lambda^*))] & + \frac{1 - \tau_1, s - \tau_2}{\tau_1, s} \mathbb{E}[(\phi(y_{sm}) - \phi(\lambda^*))] & + 9\mathbb{I}\mathbb{E}[\|z_{sm} - \lambda\|_2^2] & - 9\mathbb{I}\mathbb{E}[\|z_{sm+m} - \lambda\|_2^2].
\end{align*}
\]

(38)
From the setting $\tau_1,s = 2/(s + 4)$ and $\tau_2 = 1/2$, we get the following inequalities:

$$\frac{1}{\tau_1^2} \geq \frac{1 - \tau_1,s+1}{\tau_1^2}, \quad \frac{\tau_1,s + \tau_2}{\tau_1^2} \geq \frac{\tau_2}{\tau_1^2,s+1}.$$  

We use the inequalities above, rearrange the terms of (38), and telescope for $s = 0, \cdots, S - 1$:

$$\frac{\tau_2}{\tau_1,S} \mathbb{E}\left[\sum_{j=1}^{m-1} \left(\phi(y(s-1)m+j) - \phi(\lambda^*)\right)\right] + \frac{1 - \tau_1,S}{\tau_1,S} \mathbb{E}[\phi(y_{Sm}) - \phi(\lambda^*)]$$

$$\leq \sum_{s=0}^{S-1} \frac{1}{\tau_1,s} \sum_{j=1}^{m} \mathbb{E}[\left(\phi(\lambda_{sm+j}) - \phi(\lambda^*) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle\right)]$$

$$+ \frac{1 - \tau_1,0 - \tau_2}{\tau_1,0} (\phi(y_0) - \phi(\lambda^*)) + \frac{\tau_2 m}{\tau_1,0} (\phi(\tilde{\lambda}^0) - \phi(\lambda^*)) + 9L\|z_0 - \lambda\|_2^2 - 9L\mathbb{E}[\|z_{Sm} - \lambda\|_2^2].$$  

(39)

Now we analyze the expression of $\phi(\lambda)$:

$$\phi(\lambda) = -f(x(\lambda)) + \langle \lambda, A x(\lambda) - b \rangle \overset{(a)}{=} -f(x(\lambda)) + \langle \lambda, \nabla \phi(\lambda) \rangle,$$  

(40)

where (a) uses Assumption 1 ($\nabla \phi(\lambda) = A x(\lambda) - b$).

We could also derive the following equation from (40) and Assumption 1 ($\nabla \phi(\lambda) = A x(\lambda) - b$):

$$\phi(\lambda_{sm+j}) + \langle \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j} \rangle = -f(x(\lambda_{sm+j})) + \langle A x(\lambda_{sm+j}) - b, \lambda \rangle.$$  

(41)

Next, we introduce an ancillary variable

$$x_a = \sum_{s=0}^{S-1} \frac{1}{\tau_1,s} \sum_{j=1}^{m} x(\lambda_{sm+j}) \bigg/ m \sum_{s=0}^{S-1} \frac{1}{\tau_1,s}.$$  

Plugging this ancillary variable and (41) into the first term on the right side of (39), and we could
obtain that
\[
\sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \sum_{j=1}^{m} \mathbb{E}[(\phi(\lambda_{sm+j}) - \phi(\lambda^* + \nabla \phi(\lambda_{sm+j}), \lambda - \lambda_{sm+j}))]
\]
\[
= \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \sum_{j=1}^{m} (\mathbb{E}[-f(x(\lambda_{sm+j})) + f(x(\lambda^*)) + \langle Ax(\lambda_{sm+j}) - b, \lambda \rangle])
\]
\[
\leq \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \left( -(f(\mathbb{E}[x^S_a]) - f(x(\lambda^*))) + \langle A\mathbb{E}[x^S_a] - b, \lambda \rangle \right),
\]
where (a) uses Jensen’s inequality and the definition of \( x^S_a \).

By the choice of \( \tau_{1,s} = 2/(s + 4) \) and \( \tau_2 = 1/2 \), we have that
\[
\frac{\tau_2}{\tau_{1,s}} \leq \frac{1 - \tau_{1,s}}{\tau_{1,s}}.
\]

Thus, the following inequality involves the left side of (39) holds:
\[
\frac{\tau_2}{\tau_{1,s}} \mathbb{E}\left[ \sum_{j=1}^{m-1} (\phi(y(S-1)m+j) - \phi(\lambda^*)) \right] + \frac{1 - \tau_{1,s}}{\tau_{1,s}} \mathbb{E}[\phi(yS_m) - \phi(\lambda^*)]
\]
\[
\geq \frac{\tau_2}{\tau_{1,s}} \mathbb{E}\left[ \sum_{j=1}^{m} (\phi(y(S-1)m+j) - \phi(\lambda^*)) \right].
\]

Plugging (42) and (43) into (39), we could obtain that
\[
\frac{\tau_2}{\tau_{1,s}} \mathbb{E}\left[ \sum_{j=1}^{m} (\phi(y(S-1)m+j) - \phi(\lambda^*)) \right]
\]
\[
\leq \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \left( -(f(\mathbb{E}[x^S_a]) - f(x(\lambda^*))) + \langle A\mathbb{E}[x^S_a] - b, \lambda \rangle \right)
\]
\[
+ \frac{\tau_2 m}{\tau_{1,s}} \left( \phi(\lambda^0) - \phi(\lambda^*) \right) + \frac{1 - \tau_{1,0} - \tau_2}{\tau_{1,0}} \langle \phi(y_0) - \phi(\lambda^*) \rangle + 9\mathbb{E}\|z_0 - \lambda\|^2 - 9\mathbb{E}\|z_{S_m} - \lambda\|^2.
\]
By definition of $\tilde{\lambda}^S$ and convexity of $\phi$, we could have that

$$\frac{\tau_2 m}{\tau_{1,s}^2} E[\phi(\tilde{\lambda}^S) - \phi(\lambda^*)] \leq \frac{\tau_2}{\tau_{1,s}} E\left[ \sum_{j=1}^{m} (\phi(y_{(s-1)m+j}) - \phi(\lambda^*)) \right]$$

$$\leq \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \left( - (f(E[x^S_0]) - f(x(\lambda^*))) \right) + \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \langle A E[x^S_s] - b, \lambda \rangle$$

$$+ \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\lambda^0) - \phi(\lambda^*)) + \frac{1 - \tau_{1,0} - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 9L\|z_0 - \lambda\|_2^2 - 9L E[\|z_{Sm} - \lambda\|_2^2].$$

(44)

Notice that

$$x^S = \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} x(\tilde{\lambda}_s) / \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}}, \quad E[x^S] = E[x^S_s]$$

In this way, we rearrange (44) and get that

$$\frac{\tau_2 m}{\tau_{1,s}^2} (E[\phi(\tilde{\lambda}^S)] - \phi(\lambda^*)) + \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) (f(E[x^S]) - f(x(\lambda^*)))$$

$$\leq \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} m \right) \langle A E[x^S] - b, \lambda \rangle + \frac{\tau_2 m}{\tau_{1,0}^2} (\phi(\lambda^0) - \phi(\lambda^*))$$

$$+ \frac{1 - \tau_{1,0} - \tau_2}{\tau_{1,0}^2} (\phi(y_0) - \phi(\lambda^*)) + 9L\|z_0 - \lambda\|_2^2.$$
proof happen from (40) to (41) as follows,

\[
G(v) = -\langle (u^T, v^T)^\top, b \rangle + \max_x (-f(x) + \langle (u^T, v^T)^\top, Ax \rangle)
\]

\[
= -\langle (u^T, v^T)^\top, b \rangle - f(x(v)) + \langle (u^T, v^T)^\top, Ax(v) \rangle
\]

\[
= -f(x(v)) + \langle (u^T, v^T)^\top, Ax(v) - b \rangle
\]

\[
= -f(x(v)) + \langle u, [Ax(v) - b]_{i=n+1,\ldots,2n} \rangle + \langle v, [Ax(v) - b]_{i=1,\ldots,n} \rangle.
\]

Proposition 1 implies that \( \nabla G(v) = [Ax(v) - b]_{i=n+1,\ldots,2n} \) and \([Ax(v) - b]_{i=1,\ldots,n} = 0 \). In this way, \( G \) has the following expression:

\[
G(v) = -f(x(v)) + \langle v, \nabla G(v) \rangle.
\]

Similarly,

\[
G(v^*) = -f(x(v^*)) + \langle v^*, \nabla G(v^*) \rangle = -f(x(v^*)).
\]

Thus, we could get that

\[
G(v_{sm+j}) + \langle \nabla G(v_{sm+j}), v - v_{sm+j} \rangle
\]

\[
= -f(x(v_{sm+j})) + \langle \nabla G(v_{sm+j}), v_{sm+j} \rangle + \langle \nabla G(v_{sm+j}), v - v_{sm+j} \rangle
\]

\[
= -f(x(v_{sm+j})) + \langle \nabla G(v_{sm+j}), v \rangle
\]

\[
= -f(x(v_{sm+j})) + \langle [Ax(v_{sm+j}) - b]_{i=n+1,\ldots,2n}, v \rangle
\]

\[
= -f(x(v_{sm+j})) + \langle [Ax(v_{sm+j}) - b]_{i=n+1,\ldots,2n}, v \rangle + \langle [Ax(v_{sm+j}) - b]_{i=1,\ldots,n}, u \rangle
\]

\[
= -f(x(v_{sm+j})) + \langle Ax(v_{sm+j}) - b, (u^T, v^T)^\top \rangle
\]

\[
= -f(x(v_{sm+j})) + \langle Ax(v_{sm+j}) - b, \lambda \rangle
\]
In this way, we could get that

\[ \frac{\tau_2}{\tau_{1,0}} \left( \mathbb{E}[G(v^*)] - G(v^*) \right) + \left( m \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) (f(\mathbb{E}[x^S]) - f(x^*)) \]

\[ \leq \left( \sum_{s=0}^{S-1} \frac{1}{\tau_{1,s}} \right) \langle A \mathbb{E}[x^S] - b, \lambda \rangle + \tau_2 \frac{\mathbb{E}[x^S] - \mathbb{E}[x^0]}{\tau_{1,0}} (G(v^0) - G(v^*)) \]

\[ + \frac{1 - \tau_{1,0}}{\tau_{1,0}} (G(y_0) - G(v^*)) + 9L\|z_0 - v\|_2^2. \]

We could follow the same logic of proof for Theorem 1 in Section A.2. For analysis convenience, we minimize on \( B(2R) \) instead of \( B(2\|\lambda^*\|_2) \) in (17) and (25), where \( R \) is the upper bound of \( \|\lambda^*\|_2 \).

In this way, we could have that

\[ f(\mathbb{E}(x^S)) - f(x^*) \lesssim \frac{L'\|v^*\|_2^2}{S^2} + \frac{\mathcal{L}\|v^*\|_2^2}{nS^2} \leq \frac{L'R^2}{S^2} + \frac{LR^2}{nS^2} \]

\[ \|\mathbb{E}[Ax^S - b]\|_2 \lesssim \frac{L'|v^*\|_\infty^2}{S^2R} + \frac{\mathcal{L}|v^*\|_\infty^2}{nS^2R} \leq \frac{L'R^2}{S^2R} + \frac{LR}{nS^2} \]

where \( x^* \) is the solution to problem (10), \( v^* \) is the solution to problem (13), \( R \) is the upper bound of \( \|\lambda^*\|_2 \), and \( R' \) is the upper bound of \( \|\lambda^*\|_\infty \).

Since Proposition 2 illustrates that

\[ L' = \frac{5}{\eta}, \quad \mathcal{L} = \frac{1}{\eta}, \]

we obtain that

\[ f(\mathbb{E}(x^S)) - f(x^*) \lesssim \frac{R^2}{\eta S^2} + \frac{R^2}{\eta nS^2} \]

\[ \|\mathbb{E}[Ax^S - b]\|_2 \lesssim \frac{R^2}{\eta S^2R} + \frac{R}{\eta nS^2}. \]

\[ \square \]