Polynilpotent Multipliers of Finitely Generated Abelian Groups *

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Abstract
In this paper, we present an explicit formula for the Baer invariant of a finitely generated abelian group with respect to the variety of polynilpotent groups of class row 
\((c_1, \ldots, c_t), N_{c_1,\ldots,c_t}\). In particular, one can obtain an explicit structure of the \(\ell\)-solvable multiplier (the Baer invariant with respect to the variety of solvable groups of length at most \(\ell \geq 1\), \(S_{\ell}\)) of a finitely generated abelian group.

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1. Introduction and preliminaries

I. Schur [13], in 1907, found a formula for the Schur multiplier of a direct product of two finite groups as follows:

$$M(A \times B) \cong M(A) \oplus M(B) \oplus A_{ab} \otimes B_{ab}.$$  

One of the important corollaries of the above fact is an explicit formula for the Schur multiplier of a finite abelian group $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots, \mathbb{Z}_{n_k}$, where $n_{i+1} | n_i$ for all $1 \leq i \leq k - 1$, as follows:

$$M(G) \cong \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_3}^{(2)} \oplus \ldots \oplus \mathbb{Z}_{n_k}^{(k-1)},$$

where $\mathbb{Z}_{n}^{(m)}$ denotes the direct sum of $m$ copies of the cyclic group $\mathbb{Z}_n$ (see [10]).

In 1997, the first author, in a joint paper [11], succeeded to generalize the above formula for the Baer invariant of a finite abelian group $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots, \mathbb{Z}_{n_k}$, where $n_{i+1} | n_i$ for all $1 \leq i \leq k - 1$, with respect to the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, as follows:

$$\mathcal{N}_c M(G) \cong \mathbb{Z}_{n_2}^{(b_2)} \oplus \mathbb{Z}_{n_3}^{(b_3-b_2)} \oplus \ldots \oplus \mathbb{Z}_{n_k}^{(b_k-b_{k-1})},$$

where $b_i$ is the number of basic commutators of weight $c+1$ on $i$ letters (see [4]).

$\mathcal{N}_c M(G)$ is also called the $c$-nilpotent multiplier of $G$ (see [3]). Note that, by a similar method of the paper [11], we can obtain the structure of the $c$-nilpotent multiplier of a finitely generated abelian group as the following theorem.

**Theorem 1.1.** Let $G \cong \mathbb{Z}^{(m)} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \mathbb{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} | n_i$ for all $1 \leq i \leq k - 1$, then

$$\mathcal{N}_c M(G) \cong \mathbb{Z}^{(b_m)} \oplus \mathbb{Z}_{n_1}^{(b_{m+1}-b_m)} \oplus \mathbb{Z}_{n_2}^{(b_{m+2}-b_{m+1})} \oplus \ldots \oplus \mathbb{Z}_{n_k}^{(b_{m+k}-b_{m+k-1})},$$
where $b_i$ is the number of basic commutators of weight $c + 1$ on $i$ letters and $b_0 = b_1 = 0$.

Now, in this paper, we intend to generalize the above theorem to obtain an explicit formula for $N_{c_1, \ldots, c_t}M(G)$, the Baer invariant of $G$ with respect to the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $N_{c_1, \ldots, c_t}$, where $G$ is a finitely generated abelian group. We also call $N_{c_1, \ldots, c_t}M(G)$, a polynilpotent multiplier of $G$. As an immediate consequence, one can obtain an explicit formula for the $\ell$-solvable multiplier of $G$, $S_\ell M(G)$.

**Definition 1.2.** Let $G$ be any group with a free presentation $G \cong F/R$, where $F$ is a free group. Then, after R. Baer [1], the Baer invariant of $G$ with respect to a variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$
\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},
$$

where $V$ is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of $F$ with respect to $\mathcal{V}$ and

$$
[RV^*F] = \langle v(f_1, \ldots, f_{i-1}, f_ir, f_{i+1}, \ldots, f_n)v(f_1, \ldots, f_i, \ldots, f_n)^{-1} | r \in R, 1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbb{N} \rangle.
$$

In special case, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of $G$ will be

$$
\frac{R \cap F'}{[R, F]},
$$

which, following Hopf [7], is isomorphic to the second cohomology group of $G$, $H_2(G, C^*)$, in finite case and also is isomorphic to the well-known notion the Schur multiplier of $G$, denoted by $M(G)$. The multiplier $M(G)$ arose in Schur’s work [12] of 1904 on projective representation of a group, and has subsequently found a variety of other applications. The survey article of
Wiegold [14] and the books of Beyl and Tappe [2] and Karpilovsky [10] form a fairly comprehensive account of $M(G)$.

If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer invariant of $G$ with respect to $\mathcal{N}_c$ will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]},$$

where $\gamma_{c+1}(F)$ is the $(c + 1)$-st term of the lower central series of $F$ and $[R, 1F] = [R, F]$, $[R, cF] = [[R, c_{-1}F], F]$, inductively.

If $\mathcal{V}$ is the variety of solvable groups of length at most $\ell \geq 1$, $\mathcal{S}_\ell$, then the Baer invariant of $G$ with respect to $\mathcal{S}_\ell$ will be

$$\mathcal{S}_\ell M(G) = \frac{R \cap \delta^i(F)}{[R, F, \delta^1(F), \ldots, \delta^{l-1}(F)]},$$

where $\delta^i(F)$ is the $i$-th derived subgroup of $F$. See [8, corollary 2.10] for the equality $[R S^*_\ell F] = [R, F, \delta^1(F), \ldots, \delta^{l-1}(F)]$.

In a very more general case, let $\mathcal{V}$ be the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1,\ldots,c_t}$, then the Baer invariant of a group $G$ with respect to this variety is as follows:

$$\mathcal{N}_{c_1,\ldots,c_t} M(G) \cong \frac{R \cap \gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F)}{[R, c_1 F, c_2 \gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)]},$$

where $\gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots))$ are the terms of iterated lower central series of $F$. See [6, Corollary 6.14] for the equality

$$[RN_{c_1,\ldots,c_t}^* F] = [R, c_1 F, c_2 \gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)].$$

In the following, we are going to mention some definitions and notations of T.C. Hurley and M.A. Ward [9], which are vital in our investigation.
**Definition and Notation 1.3.** Commutators are written $[a, b] = a^{-1}b^{-1}ab$ and the usual convention for left-normed commutators is used, $[a, b, c] = [[a, b], c]$, $[a, b, c, d] = [[[a, b], c], d]$ and so on, including the trivial case $[a] = a$.

**Basic commutators** are defined in the usual way. If $X$ is a fully ordered independent subset of a free group, the basic commutators on $X$ are defined inductively over their weight as follows:

(i) All the members of $X$ are basic commutators on $X$ of weight one on $X$.

(ii) Assuming that $n > 1$ and that the basic commutators of weight less than $n$ on $X$ have been defined and ordered.

(iii) A commutator $[a, b]$ is a basic commutator of weight $n$ on $X$ if $wt(a) + wt(b) = n$, $a < b$, and if $b = [b_1, b_2]$, then $b_2 \leq a$. The ordering of basic commutators is then extended to include those of weight $n$ in any way such that those of weight less than $n$ precede those of weight $n$. The natural way to define the order on basic commutators of the same weight is lexicographically, $[b_1, a_1] < [b_2, a_2]$ if $b_1 < b_2$ or if $b_1 = b_2$ and $a_1 < a_2$.

A word of the form $[c, a_1, a_2, \ldots, a_p, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$ is a “standard invertator” will be meant to imply that the $\beta_i$’s are $\pm 1$, $c > a_1 \leq a_2 \leq \ldots \leq a_p \leq b_1 \leq b_2 \leq \ldots \leq b_q$ and if $b_i = b_j$ then $\beta_i = \beta_j$ for all $i, j$.

Whenever this terminology is used it will be accomplished by a statement of what set $X$, the $a_i$ and the $b_j$ are chosen from and this will be always be a set which is known to be fully ordered in some way. Restrictions on the values of $p$ and $q$ will be given, the value $p = 0$ and $q = 0$ being permissible so that we may, when we wish, specify standard invertators of the forms $[c, a_1, \ldots, a_p]$ or $[c, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$.

Let $F$ be a free group on alphabet $X$ and $m$ and $n$ be integers. Then

(i) $A_{m,n}$ denotes the set of all basic commutators on $X$ of weight exactly $n$ and of the form $[c, a_1, \ldots, a_p]$, where $b$ and the $a_i$ are all basic commutators
on $X$ of weight less than $m$.

(ii) $B_{m,n}$ denotes the set of all standard invertators on $X$ of the form

$$
[b, a_1, a_2, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}, \ldots, a_q^{\alpha_q}],
$$

where $0 \leq p < q$, $b$ and the $a_i$ are basic commutators on $X$ of weight less than $m$,

$$
wt([b, a_1, a_2, \ldots, a_p]) < n \leq wt([b, a_1, a_2, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}])
$$

and $b = [b_1, b_2]$ implies $b_2 \leq a_1$. Note that $[b, a_1, a_2, \ldots, a_p] \in A_{m,r}$, where $r$ is the weight of this commutator and $r < n$. Also, observe that $A_{m,m}$ is just the set of all basic commutators of weight $m$ on $X$.

**Theorem 1.4** (P.Hall [4,5]). Let $F = \langle x_1, x_2, \ldots, x_d \rangle$ be a free group, then

$$
\frac{\gamma_n(F)}{\gamma_{n+i}(F)}, \quad 1 \leq i \leq n
$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, \ldots, x_d\}$.

**Theorem 1.5** (Witt Formula [4]). The number of basic commutators of weight $n$ on $d$ generators is given by the following formula:

$$
\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m)d^{n/m}
$$

where $\mu(m)$ is the *Mobious function*, and defined to be

$$
\mu(m) = \begin{cases} 
1 & \text{if } m = 1, \\
0 & \text{if } m = p_1^{\alpha_1} \ldots p_s^{\alpha_s} \exists \alpha_i > 1, \\
(-1)^s & \text{if } m = p_1 \ldots p_s,
\end{cases}
$$

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The following important theorem presents interesting free generating sets for the terms of the lower central series of a free group which will be used several times in this paper.

**Theorem 1.6** (T.C.Hurley and M.A.Ward 1981). Let $F$ be a free group, freely generated by some fully ordered set $X$, and let $m$ and $n$ be integers satisfying $2 \leq m \leq n$. Then the members of the set

$$A_{m,m} \cup A_{m,m+1} \cup \ldots \cup A_{m,n-1} \cup B_{m,n}$$

are distinct as written, so that in particular this is a disjoint union, and the set freely generates $\gamma_m(F)$.

**Proof.** See [9, Theorem 2.2].

**Corollary 1.7.** Let $F$ be a free group freely generated by some fully ordered set $X$. Then $\gamma_{c_2+1}(\gamma_{c_1+1}(F))$ is freely generated by

$$\hat{A}_{c_2+1,c_2+1} \cup \hat{B}_{c_2+1,c_2+2},$$

where $\hat{A}_{c_2+1,c_2+1}$ is the set of all basic commutators of weight $c_2 + 1$ on the set

$$Y = A_{c_1+1,c_1+1} \cup B_{c_1+1,c_1+2},$$

and $\hat{B}_{c_2+1,c_2+2}$ is the set of all standard invertators on $Y$ of the form

$$[b, a_1, a_2, \ldots, a_p, a_{p+1}^{\alpha_p+1}, \ldots, a_q^{\alpha_q}],$$

where $0 \leq p < q$, $b$ and the $a_i$ are basic commutators on $Y$ of weight less than $c_2 + 1$,

$$wt([b, a_1, a_2, \ldots, a_p]) < c_2 + 2 \leq wt([b, a_1, a_2, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}])$$
and $b = [b_1, b_2]$ implies $b_2 \leq a_1$.

**Proof.** Using Theorem 1.6, $\gamma_{c_1+1}(F)$ is freely generated by $A_{c_1+1,c_1+1} \cup B_{c_1+1,c_1+2}$, when putting $m = c_1 + 1, n = c_1 + 2$. Now we can suppose $\mathcal{F} = \gamma_{c_1+1}(F)$ is a free group, freely generated by fully ordered set $Y = A_{c_1+1,c_1+1} \cup B_{c_1+1,c_1+2}$. Applying Theorem 1.6 again for $\gamma_{c_2+1}(\mathcal{F})$ and $m = c_2 + 1, n = c_2 + 2$, the result holds. \(\square\)

As an immediate consequence we have the following corollary.

**Corollary 1.8.** Let $F$ be a free group freely generated by some fully ordered set $X$. Then the second derived subgroup of $F$, $\delta^2(F) = F''$, is freely generated by

$$\hat{A}_{2,2} \cup \hat{B}_{2,3},$$

where $\hat{A}_{2,2}$ is the set of all basic commutators of weight 2 on the set $A_{2,2} \cup B_{2,3}$, and $\hat{B}_{2,3}$ is the set of all standard invertators on $A_{2,2} \cup B_{2,3}$ of the form $[b, a_1, a_2^{a_2}, \ldots, a_q^{a_q}]$, where $b, a_i \in A_{2,2} \cup B_{2,3}$.

### 2. The Main Results

In this section, first, we concentrate on the calculation of the Baer invariant of a finitely generated abelian group with respect to the variety of metabelian groups, i.e. solvable groups of length 2, $S_2$.

Let $Z_{r_i} = \langle x_i \mid x_i^{r_i} \rangle, 1 \leq i \leq t$, be cyclic groups of order $r_i \geq 0$, and let

$$0 \rightarrow R_i = \langle x_i^{r_i} \rangle \rightarrow F_i = \langle x_i \rangle \rightarrow Z_{r_i} \rightarrow 0,$$

be a free presentation of $Z_{r_i}$. Also, suppose $G \cong \oplus \sum_{i=1}^t Z_{r_i}$ is the direct sum of the cyclic groups $Z_{r_i}$. Then

$$0 \rightarrow R \rightarrow F \rightarrow G \rightarrow 0$$

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is a free presentation of \( G \), where \( F = \prod_{i=1}^{t'} F_i = \langle x_1, \ldots, x_t \rangle \) is the free product of \( F_i \)'s, and \( R = \prod_{i=1}^{t} R_i \gamma_2(F) \). Therefore, the metabelian multiplier of \( G \) is as follows:

\[
S_2 M(G) \cong \frac{R \cap \delta^2(F)}{[R, F, \delta^1(F)]} = \frac{F''}{[R, F, F']} \quad (\text{since} \quad F' \leq R).
\]

Now, the following theorem presents an explicit structure for the metabelian multiplier of a finitely generated abelian group.

**Theorem 2.1.** With the above notation and assumption, let \( G \cong \mathbb{Z}^{(m)} \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k} \) be a finitely generated abelian group, where \( n_{i+1} | n_i \) for all \( 1 \leq i \leq k - 1 \). Then the following isomorphism holds:

\[
S_2 M(G) \cong \mathbb{Z}^{(d_m)} \oplus \mathbb{Z}_{n_1}^{(d_{m+1} - d_m)} \oplus \ldots \oplus \mathbb{Z}_{n_k}^{(d_{m+k} - d_{m+k-1})},
\]

where \( d_i = \chi_2(\chi_2(i)) \), and \( \chi_2(i) \) is the number of basic commutators of weight 2 on \( i \) letters.

**Proof.** With the previous notation, put \( t = m + k \), \( r_1 = r_2 = \ldots = r_m = 0, \ r_{m+j} = n_j \), \( 1 \leq j \leq k \). Then \( \mathbb{Z}_{r_1} \cong \ldots \cong \mathbb{Z}_{r_m} \cong \mathbb{Z}, \ \mathbb{Z}_{r_{m+j}} \cong \mathbb{Z}_{n_j}, \ G \cong \oplus \sum_{i=1}^{m+k} \mathbb{Z}_{r_i} \), and

\[
S_2 M(G) \cong \frac{F''}{[R, F, F']},
\]

where \( F \) is the free group on the set \( X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\} \).

By corollary 1.8 \( F'' \) is a free group with the basis \( \hat{A}_{2,2} \cup \hat{B}_{2,3} \). Put \( L \) the normal closure of those elements of the basis \( F'' \), \( \hat{A}_{2,2} \cup \hat{B}_{2,3} \), of weight, as commutators on the set \( X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\} \), greater than 4 in \( F'' \). In other words

\[
L = \langle w \in \hat{A}_{2,2} \cup \hat{B}_{2,3} \mid w \notin \{u \in \hat{A}_{2,2} \cup \hat{B}_{2,3} \mid u \text{ is of the form } [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]] \rangle^{F''}.
\]
It is easy to see that $F''/L$ is a free group freely generated by the following set

$$Y = \{wL \mid w \in \hat{A}_{2,2} \cup \hat{B}_{2,3} \text{ and } w \text{ is of the form } [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]]\}.$$ 

Therefore

$$\frac{F''/L}{(F''/L)'} \cong \frac{F''}{LF''}$$

is a free abelian group with the basis $\overline{Y} = \{wLF'' \mid wL \in Y\}$. Since $S_2M(G) \cong F''/[R, F, F']$ is abelian, so $F''' \leq [R, F, F']$. Thus, we have

$$S_2M(G) \cong \frac{F''/LF'''}{[R, F, F']/LF'''}.$$ 

Now we are going to describe explicitly the bases of the free abelian group $F''/LF'''$ and its subgroup $[R, F, F']/LF'''$ in order to find the structure of the metabelian multiplier of $G$, $S_2M(G)$. According to the basis $\overline{Y}$ of the free abelian group $F''/LF'''$, it is easy to see that

$$\overline{Y} = C_0 \cup C_1 \cup \ldots \cup C_k,$$

where

$$C_0 = \{wLF''' \in \overline{Y} \mid w = [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]], 1 \leq i_1, i_2, i_3, i_4 \leq m\},$$

and for all $1 \leq \lambda \leq k$

$$C_\lambda = \{wLF''' \in \overline{Y} \mid w = [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]], 1 \leq i_1, i_2, i_3, i_4 \leq m + \lambda$$

$$, \exists 1 \leq j \leq 4, \text{ s.t. } i_j = m + \lambda\}.$$ 

In order to find an appropriate basis for the free abelian group $[R, F, F']/LF'''$, first we claim that $\gamma_5(F) \cap F'' \leq LF'''$ (*), since, let $u \in F''$, using the basis $\overline{Y}$ of the free abelian group $F''/LF'''$, we have

$$uLF''' = w_{i_1}^{\epsilon_1} \ldots w_{i_t}^{\epsilon_t} LF'''$$,
where \( w_{i1}LF''', \ldots, w_{it}LF''' \in \mathcal{A} \), and \( \epsilon_1, \ldots, \epsilon_t \in \mathbb{Z} \). Clearly \( LF''' \leq \gamma_5(F) \), so, if \( u \in \gamma_5(F) \), then we have \( w_{i1}^{\epsilon_1} \ldots w_{it}^{\epsilon_t} \in \gamma_5(F) \). It is easy to see that \( w_{i1}^{\epsilon_1}, \ldots, w_{it}^{\epsilon_t} \) are basic commutators of weight 4 on \( X \). By Theorem 1.4 \( \gamma_4(F)/\gamma_5(F) \) is the free abelian group with basis of all basic commutators of weight 4 on \( X \). Thus we have \( \epsilon_1 = \ldots = \epsilon_t = 0 \), and hence \( u \in LF''' \). As an immediate consequence we have \( [F', F, F'] \leq LF''' \). Note that \( R = (\prod_{i=1}^{m+k} R_i)F' \), where \( R_i = \langle x_i^0 \rangle = 1 \), for all \( 1 \leq i \leq m \), and \( R_{m+j} = \langle x_{m+j}^j \rangle \), for all \( 1 \leq j \leq k \), so

\[
\frac{[R, F, F']}{LF'''} = \prod_{j=1}^{k} [R_{m+j}, F, F'] \frac{LF'''}{LF'''}.
\]

Using the above equality and the congruence

\[
[[x_{i1}^{\alpha_1}, x_{i2}^{\alpha_2}], [x_{i3}^{\alpha_3}, x_{i4}^{\alpha_4}]] \equiv [[x_{i1}, x_{i2}], [x_{i3}, x_{i4}]]^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \pmod{LF'''}
\]

for all \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} \), (By (*)), it is routine to check that the free abelian group \( [R, F, F']/LF''' \) has the following basis

\[
D_1 \cup D_2 \cup \ldots \cup D_k,
\]

where \( D_\lambda = \{ w^{n_\lambda}LF''' \mid wLF''' \in C_\lambda, 1 \leq \lambda \leq k \} \).

Using the form of the elements \( C_\lambda \) and the number of basic commutators of weight 2 on \( i \) letters, \( \chi_2(i) \), one can easily see that \( |C_0| = \chi_2(\chi_2(m)) \), and \( |C_\lambda| = \chi_2(\chi_2(m + \lambda)) - \chi_2(\chi_2(m + \lambda - 1)) \). Hence the result holds. \( \square \)

Now, trying to generalize the proof of the above theorem, which is the basic idea of the paper, we are going to present an explicit formula for the polynilpotent multiplier of a finitely generated abelian group with respect to the variety \( \mathcal{N}_{c_1, \ldots, c_t} \). Because of applying an iterative method and avoiding complicacy for the reader, first, we state and prove the beginning step of the
method for the variety $\mathcal{N}_{c_1,c_2}$ in the following theorem.

**Theorem 2.2.** Let $\mathcal{N}_{c_1,c_2}$ be the polynilpotent variety of class row $(c_1, c_2)$ and $G \cong \mathbb{Z}^{(m)} \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k - 1$. Then the following isomorphism holds:

$$\mathcal{N}_{c_1,c_2}M(G) \cong \mathbb{Z}^{(e_m)} \oplus \mathbb{Z}^{(e_{m+1} - e_m)} \oplus \ldots \oplus \mathbb{Z}^{(e_{m+k} - e_{m+k-1})},$$

where $e_i = \chi_{c_2+1}(\chi_{c_1+1}(i))$ for all $m \leq i \leq m+k$.

**Proof.** By the notation of the Theorem 2.1 we have

$$\mathcal{N}_{c_1,c_2}M(G) = \frac{\gamma_{c_2+1}(\gamma_{c_2+1}(F))}{[R, c_1F, c_2\gamma_{c_1+1}(F)]},$$

where $F$ is the free group on the set $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\}$. By considering the basis of the free group $\gamma_{c_2+1}(\gamma_{c_2+1}(F))$ presented in corollary 1.7, we put

$$L = \langle w \in \hat{A}_{c_2+1,c_2+1} \cup \hat{B}_{c_2+1,c_2+2} \mid w \notin E \rangle,$$

where $E$ is the set of all basic commutators of weight exactly $c_2 + 1$ on the set of all basic commutators of weight exactly $c_1 + 1$ on the set $X$.

Clearly $\gamma_{c_2+1}(\gamma_{c_1+1}(F))/L$ is free on the set

$$Y = \{wL \mid w \in \hat{A}_{c_2+1,c_2+1} \cup \hat{B}_{c_2+1,c_2+2} \text{ and } w \in E\}$$

and $\gamma_{c_2+1}(\gamma_{c_1+1}(F))/L \gamma_{c_2+1}(\gamma_{c_1+1}(F))$ is free abelian with the basis $\bar{Y} = \{wL \gamma_{c_2+1}(\gamma_{c_1+1}(F)) \mid wL \in Y\}$. Considering the form of the elements of $L$ and noticing to the abelian group $\mathcal{N}_{c_1,c_2}M(G)$, we have

$$L \gamma_{c_2+1}(\gamma_{c_1+1}(F)) \leq [R, c_1F, c_2 \gamma_{c_1+1}(F)].$$
Thus the following isomorphism holds:

\[ N_{c_1,c_2} M(G) \cong \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F)) \langle R, c_1 F, c_2 \gamma_{c_1+1}(F) \rangle}{L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F)))}. \]

By Theorem 1.4 \( \gamma_{c_1+c_2+c_1c_2+1}(F) \langle c_1 F, c_2 \gamma_{c_1+1}(F) \rangle / L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \) is the free abelian group with the basis of all basic commutators of weight \( c_1 + c_2 + c_1c_2 + 1 \) on \( X \).

Using the above fact and the basis \( \bar{Y} \) of the free abelian group \( \gamma_{c_2+1}(\gamma_{c_1+1}(F)) \langle c_1 F, c_2 \gamma_{c_1+1}(F) \rangle / L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \) we can conclude the following inclusion:

\[ \gamma_{c_1+c_2+c_1c_2+2}(F) \cap \gamma_{c_2+1}(\gamma_{c_1+1}(F)) \leq L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))). \]

Now, it is easy to see that \( \bar{Y} = C_0 \cup C_1 \cup \ldots \cup C_k \) is a basis for the free abelian group \( \gamma_{c_2+1}(\gamma_{c_1+1}(F)) \langle c_1 F, c_2 \gamma_{c_1+1}(F) \rangle / L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \) and \( D_1 \cup D_2 \cup \ldots \cup D_k \) is a basis for the free abelian group

\[ \frac{[R, c_1 F, c_2 \gamma_{c_1+1}(F)]}{L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F)))}, \]

where

\[ C_0 = \{ wL\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in \bar{Y} \mid w \in E \}
\]

and \( w \) is a commutator on letters \( x_1, \ldots, x_m \),

and for \( 1 \leq \lambda \leq k \):

\[ C_\lambda = \{ wL\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in \bar{Y} \mid w \in E \text{ and } w \text{ is a commutator on letters } x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+\lambda} \text{ such that the letter } x_{m+\lambda} \text{ does appear in } w \}, \]

\[ D_\lambda = \{ w^\lambda L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in \bar{Y} \mid w L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in C_\lambda \}. \]

Note that using the form of the elements of \( C_\lambda \) and the number of basic commutators, we can conclude that \( | C_0 | = \chi_{c_2+1}(\chi_{c_1+1}(m)) \) and \( | C_\lambda | = \chi_{c_2+1}(\chi_{c_1+1}(m+\lambda)) - \chi_{c_2+1}(\chi_{c_1+1}(m+\lambda-1)) \). Hence the result holds. □

Now, we are ready to state and prove the main result of the paper in general case.
Theorem 2.3. Let $\mathcal{N}_{c_1, c_2, \ldots, c_t}$ be the polynilpotent variety of class row $(c_1, c_2, \ldots, c_t)$ and $G \cong \mathbb{Z}^{(m)} \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} | n_i$ for all $1 \leq i \leq k - 1$. Then an explicit structure of the polynilpotent multiplier of $G$ is as follows.

$$\mathcal{N}_{c_1, c_2, \ldots, c_t} M(G) \cong \mathbb{Z}^{(f_m)} \oplus \mathbb{Z}^{(f_{m+1} - f_{m+1})} \oplus \ldots \oplus \mathbb{Z}^{(f_{m+k} - f_{m+k-1})},$$

where $f_i = \gamma_{c_{i+1}}(\chi_{c_{i-1}+1}(\cdots (\chi_{c_{i+1}}(i)) \ldots))$ for all $m \leq i \leq m + k$.

Proof. Let $F$ be the free group on the set $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\}$. Then by previous notation, we have

$$\mathcal{N}_{c_1, c_2, \ldots, c_t} M(G) = \frac{\gamma_{c_{i+1}}(\gamma_{c_{i-1}+1}(\cdots (\gamma_{c_{i+1}}(F)) \ldots))}{[R, c_1 F, c_2 \gamma_{c_{i+1}}(F), \ldots, c_t \gamma_{c_{i-1}+1}(\cdots (\gamma_{c_{i+1}}(F)) \ldots))].}$$

We define $\rho_t(F), E_t, X_t$ inductively on $t$ as follows:

$\rho_1(F) = \gamma_{c_1+1}(F)$, $\rho_t(F) = \gamma_{c_{i+1}}(\rho_{i-1}(F))$;

$E_1 = X$, $E_i =$ the set of all basic commutators of weight $c_i + 1$ on the set $E_{i-1}$;

$X_1 = A_{c_1+1, c_1+1} \cup B_{c_1+1, c_1+2}$. $X_i = \hat{A}_{c_1+1, c_i+1} \cup \hat{B}_{c_{i+1}, c_{i+2}}$, where $\hat{A}_{c_1+1, c_{i+1}}$ is the set of all basic commutators of weight $c_{i+1}$ on the set $X_{i-1}$, and $\hat{B}_{c_{i+1}, c_{i+2}}$ is the set of all standard invertors on $X_{t_i}$ of the form

$$[b, a_1, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}, \ldots, a_q^{\alpha_q}],$$

where $0 \leq p < q$, $b$ and the $a_i$ are basic commutators on $X_{i-1}$ of weight less than $c_{i+1}$, $wt([b, a_1, \ldots, a_p]) < c_i + 2 \leq wt([b, a_1, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}])$ and $b = [b_1, b_2]$ implies $b_2 \leq a_1$.

Using Theorem 1.6 and induction on $t$, it is easy to see that $\gamma_{c_{i+1}}(\gamma_{c_{i-1}+1}(\cdots (\gamma_{c_{i+1}}(F)) \ldots)) = \rho_t(F)$ is freely generated by $X_t$. Now, putting

$$L_t = \langle w \in X_t \mid w \notin E_t >^{\rho_t(F)} \rangle,$$

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one can easily see that \( \rho_t(F)/L_t \) is free on the set \( Y_t = \{ wL_t \mid w \in E_t \} \) and \( \rho_t(F)/L_t \gamma_2(\rho_t(F)) \) is free abelian with the basis \( \tilde{Y}_t = \{ wL_t \gamma_2(\rho_t(F)) \mid w \in E_t \} \). By considering the abelian group \( N_{c_1,c_2,\ldots,c_t}M(G) \) and the form of the elements of \( L_t \), we have \( L_t \gamma_2(\rho_t(F)) \leq [R, c_1 F, c_2 \rho_1(F), \ldots, c_t \rho_{t-1}(F)] \), and the following isomorphism

\[
N_{c_1,c_2,\ldots,c_t}M(G) \cong \frac{\rho_t(F)/L_t \gamma_2(\rho_t(F))}{[R, c_1 F, c_2 \rho_1(F), \ldots, c_t \rho_{t-1}(F)]/L_t \gamma_2(\rho_t(F))}.
\]

Clearly \( \gamma_\pi(F)/\gamma_{\pi+1}(F) \) is the free abelian group with the basis of all basic commutators of weight \( \pi \) on \( X \), where \( \pi = \prod_{i=1}^t (c_i + 1) \). Using the above fact and \( \tilde{Y}_t \), the basis of the free abelian group \( \rho_t(F)/L_t \gamma_2(\rho_t(F)) \), one can obtain the following inclusion:

\[
\gamma_{\pi+1}(F) \cap \rho_t(F) \leq L_t \gamma_2(\rho_t(F)).
\]

Therefore, it is clear that \( \tilde{Y}_t = C_{0,t} \cup C_{1,t} \cup \ldots \cup C_{k,t} \) is a basis for the free abelian group \( \rho_t(F)/L_t \gamma_2(\rho_t(F)) \) and \( D_{0,t} \cup D_{1,t} \cup \ldots \cup D_{k,t} \) is a basis for the free abelian group \( [R, c_1 F, c_2 \rho_1(F), \ldots, c_t \rho_{t-1}(F)]/L_t \gamma_2(\rho_t(F)) \), where

\[
C_{0,t} = \{ wL_t \gamma_2(\rho_t(F)) \mid w \in E_t \text{ and } w \text{ is a commutator on letters } x_1, \ldots, x_m \};
\]

\[
C_{\lambda,t} = \{ wL_t \gamma_2(\rho_t(F)) \mid w \in E_t \text{ and } w \text{ is a commutator on letters } x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+\lambda} \text{ such that the letter } x_{m+\lambda} \text{ does appear in } w \};
\]

\[
D_{\lambda,t} = \{ w^{n_\lambda}L_t \gamma_2(\rho_t(F)) \mid wL_t \gamma_2(\rho_t(F)) \in C_{\lambda,t} \};
\]

for all \( 1 \leq \lambda \leq k \).

Note that \( |C_{0,t}| = \chi_{c_1+1}(\ldots (\chi_{c_1+1}(m)) \ldots) \) and

\[
|C_{\lambda,t}| = \chi_{c_1+1}(\ldots (\chi_{c_1+1}(m+\lambda)) \ldots) - \chi_{c_1+1}(\ldots (\chi_{c_1+1}(m + \lambda - 1)) \ldots).
\]

Hence the result holds. \( \square \)

Now we can state the following interesting corollary.
Corollary 2.4. Let $\mathcal{S}_\ell$ be the variety of solvable groups of length at most $\ell$ and $G \cong \mathbb{Z}^{(m)} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots, \mathbb{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} | n_i$ for all $1 \leq i \leq k - 1$. Then the following isomorphism holds:

$$\mathcal{S}_\ell M(G) \cong \mathbb{Z}^{(h_m)} \oplus \mathbb{Z}^{(h_{m+1}-h_m)} \oplus \ldots \oplus \mathbb{Z}^{(h_{m+k}-h_{m+k-1})}$$

where $h_i = \chi_2 (\ldots (\chi_2 (i)) \ldots)$ for all $m \leq i \leq m+k$.

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