Holographic dark energy with varying gravitational constant

Mubasher Jamil,† Emmanuel N. Saridakis,‡ and M. R. Setare§

1Center for Advanced Mathematics and Physics, National University of Sciences and Technology, Rawalpindi, 46000, Pakistan
2Department of Physics, University of Athens, GR-15771 Athens, Greece
3Department of Science, Payame Noor University, Bijar, Iran

We investigate the holographic dark energy scenario with a varying gravitational constant, in flat and non-flat background geometry. We extract the exact differential equations determining the evolution of the dark energy density-parameter, which include $G$-variation correction terms. Performing a low-redshift expansion of the dark energy equation of state, we provide the involved parameters as functions of the current density parameters, of the holographic dark energy constant and of the $G$-variation.

PACS numbers: 95.36.+x, 98.80.-k

I. INTRODUCTION

Recent cosmological observations obtained by SNe Ia [1], WMAP [2], SDSS [3] and X-ray [4] indicate that the universe experiences an accelerated expansion. Although the simplest way to explain this behavior is the consideration of a cosmological constant $\Lambda$, the two relevant problems (namely the “fine-tuning” and the “coincidence” one) led to the dark energy paradigm. The dynamical nature of dark energy, at least in an effective level, can originate from various fields, such as a canonical scalar field (quintessence) [6], a phantom field, that is a scalar field with a negative sign of the kinetic term [7], or the combination of quintessence and phantom in a unified model named quintom [8].

Although going beyond the above effective description requires a deeper understanding of the underlying theory of quantum gravity [9], unknown at present, physicists can still make some attempts to probe the nature of dark energy according to some basic quantum gravitational principles. An example of such a paradigm is the holographic dark energy scenario, constructed in the light of the holographic principle [10,11,12,13] (although the recent developments in Horawa gravity could offer a dark energy candidate with perhaps better quantum gravitational foundations [14]). Its framework is the black hole thermodynamics [15] and the connection (known from AdS/CFT correspondence) of the UV cut-off of a quantum field theory, which gives rise to the vacuum energy, with the largest distance of the theory [10]. Thus, determining an appropriate quantity $L$ to serve as an IR cut-off, imposing the constraint that the total vacuum energy in the corresponding maximum volume must not be greater than the mass of a black hole of the same size, and saturating the inequality, one identifies the acquired vacuum energy as holographic dark energy:

$$\rho_{\Lambda} = \frac{3c^2}{8\pi GL^2},$$

with $G$ the Newton’s gravitational constant and $c$ a constant. The holographic dark energy scenario has been tested and constrained by various astronomical observations [16,17,18,19,20] and it has been extended to various frameworks [21,22,23].

Until now, in all the investigated holographic dark energy models a constant Newton’s “constant” $G$ has been considered. However, there are significant indications that $G$ can by varying, being a function of time or equivalently of the scale factor [24]. In particular, observations of Hulse-Taylor binary pulsar [22,24], helio-seismological data [27], Type Ia supernova observations [1] and asteroseismological data from the pulsating white dwarf star G117-B15A [29] lead to $|\dot{G}/G| \lesssim 4.10 \times 10^{-11}\text{yr}^{-1}$, for $z \lesssim 3.5$ [30]. Additionally, a varying $G$ has some theoretical advantages too, alleviating the dark matter problem [31], the cosmic coincidence problem [32] and the discrepancies in Hubble parameter value [33].

There have been many proposals in the literature attempting to theoretically justified a varying gravitational constant, despite the lack of a full, underlying quantum gravity theory. Starting with the simple but pioneering work of Dirac [34], the varying behavior in Kaluza-Klein theory was associated with a scalar field appearing in the metric component corresponding to the 5-th dimension [35] and its size variation [36]. An alternative approach arises from Brans-Dicke framework [37], where the gravitational constant is replaced by a scalar field coupling to gravity through a new parameter, and it has been generalized to various forms of scalar-tensor theories [38], leading to a considerably broader range of variable-$G$ theories. In addition, justification of a varying Newton’s constant has been established with the use of conformal invariance and its induced local transformations [39]. Finally, a varying $G$ can arise perturbatively through a semiclassical treatment of Hilbert-Einstein action [40], non-perturbatively through quantum-gravitational ap-

---

†Electronic address: mjamil@camp.edu.pk
‡Electronic address: msaridak@phys.uoa.gr
§Electronic address: rezakord@ipm.ir
proaches within the “Hilbert-Einstein truncation” [41], or through gravitational holography [42, 43].

In this work we are interested in investigating the holographic dark energy paradigm allowing for a varying gravitational constant, and extracting the corresponding corrections to the dark energy equation-of-state parameter. In order to remain general and explore the pure varying-$G$ effects in a model-independent way, we do not use explicitly any additional, geometrical or quintessence-like, scalar field, considering just the Hilbert-Einstein action in an affective level, as it arises from gravitational holography [42, 43]. In other words, we effectively focus on the dark energy and dark matter sectors without examining explicitly the mechanism of G-variation, which value is considered as an input fixed by observations. Additionally, generality requires to perform our study in flat and non-flat FRW universe.

The plan of the work is as follows: In section II we construct the holographic dark energy scenario with a varying Newton’s constant and we extract the differential equations that determine the evolution of dark energy density-parameter. In section III we use these expressions in order to calculate the corrections to the dark energy equation-of-state parameter at low redshifts. Finally, in section IV we summarize our results.

II. HOLOGRAPHIC DARK ENERGY WITH VARYING GRAVITATIONAL CONSTANT

A. Flat FRW geometry

Let us construct holographic dark energy scenario allowing for a varying Newton’s constant $G$. The spacetime geometry will be a flat Robertson-Walker:

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2d\Omega^2),$$

with $a(t)$ the scale factor and $t$ the comoving time. As usual, the first Friedmann equation reads:

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_A\right),$$

with $H$ the Hubble parameter, $\rho_m = \frac{m}{a^3}$, where $\rho_m$ and $\rho_A$ stand respectively for matter and dark energy densities and the index 0 marks the present value of a quantity. Furthermore, we will use the density parameter $\Omega_\Lambda = \frac{8\pi G}{3H^2}\rho_A$, which, imposing explicitly the holographic nature of dark energy according to relation (1), becomes

$$\Omega_\Lambda = \frac{c^2}{H^2L^2}.$$  

Finally, in the case of a flat universe, the best choice for the definition of $L$ is to identify it with the future event horizon [12, 13, 42, 44, 45], that is $L \equiv R_h(a)$ with

$$R_h(a) = a \int_0^\infty \frac{dt'}{a(t')} = a \int_a^\infty \frac{da'}{Ha'^2}.$$  

In the following we will use $\ln a$ as an independent variable. Thus, denoting by dot the time-derivative and by prime the derivative with respect to $\ln a$, for every quantity $F$ we acquire $\dot{F} = F'\dot{a}$.

Differentiating (1) using (3), and observing that $\dot{R}_h = H R_h - 1$, we obtain:

$$\frac{\dot{\Omega}_\Lambda}{\Omega_\Lambda} = 2\frac{H}{\Omega_\Lambda} \left[1 - \frac{\dot{H}}{H^2} + \frac{\sqrt{\Omega_\Lambda}}{c}\right].$$

Until now, the varying behavior of $G$ has not become manifested. However, the next step is to eliminate $\dot{H}$. This can be obtained by differentiating Friedman equation (3), leading to

$$2\frac{\dot{H}}{H^2} = -3 + \Omega_\Lambda \left(1 + 2\frac{\sqrt{\Omega_\Lambda}}{c}\right) + \frac{G'}{G} (1 - \Omega_\Lambda),$$

where $G$ is considered to be a function of $\ln a$. In the extraction of this relation we have additionally used the auxiliary expression

$$\rho'_\Lambda = \rho_\Lambda \left(-\frac{G'}{G} - 2 + \frac{2\sqrt{\Omega_\Lambda}}{c}\right),$$

which arises from differentiation of (1). Therefore, substituting (7) into (9) we finally obtain:

$$\Omega_\Lambda = \frac{\Omega_\Lambda(1 - \Omega_\Lambda)}{1 + 2\frac{\sqrt{\Omega_\Lambda}}{c}} - \Omega_\Lambda(1 - \Omega_\Lambda)\frac{G'}{G}.$$  

The first term is the usual holographic dark energy differential equation [13]. The second term is the correction arising from the varying nature of $G$. Note that $G'/G$ is a pure number as expected.

Finally, for completeness, we present the general solution for arbitrary $c$ and $G'/G \equiv \Delta_G$, which in an implicit form reads

$$\frac{\ln a}{c} + x_0 = \frac{\ln \Omega_\Lambda}{c(1 - \Delta_G)} - \frac{\ln(1 - \sqrt{\Omega_\Lambda})}{2 + c(1 - \Delta_G)} + \frac{\ln(1 + \sqrt{\Omega_\Lambda})}{2 + c(\Delta_G - 1)} - \frac{8\ln|c(1 - \Delta_G) + 2\sqrt{\Omega_\Lambda}|}{c(\Delta_G - 1)(c^2(\Delta_G - 1)^2 - 4)}. \quad (10)$$

The constant $x_0$ can be straightforwardly calculated if we determine $a_0$ and $\Omega_{\Lambda 0}$ today (for example choosing $a_0 = 1$ $x_0$ is equal to the left hand side with $\Omega_\Lambda$ replaced by $\Omega_{\Lambda 0}$). Clearly, for $\Delta_G = 0$ and $c = 1$, expression (10) coincides with that of [13].

B. Non-flat FRW geometry

In this subsection we generalize the aforementioned analysis in the case of a general FRW universe with line element

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2d\Omega^2\right) \quad (11)$$

in comoving coordinates $(t, r, \theta, \phi)$, where $k$ denotes the spacial curvature with $k = -1, 0, 1$ corresponding to
open, flat and closed universe respectively. In this case, the first Friedmann equation writes:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} (\rho_m + \rho_{\Lambda}).$$  \hfill (12)

According to the formulation of holographic dark energy in non-flat geometry, the cosmological length \( L \) in [4] is considered to be [21:

$$L = \frac{a(t)}{\sqrt{|k|}} \sin \left( \frac{\sqrt{|k|} R_h}{a(t)} \right),$$  \hfill (13)

where

$$\frac{1}{\sqrt{|k|}} \sin \left( \sqrt{|k|} y \right) = \begin{cases} \sin y & k = +1, \\
y & k = 0, \\
\sinh y & k = -1. \end{cases}$$  \hfill (14)

A straightforward calculation leads to

$$\dot{L} = HL - \cos \left( \frac{\sqrt{|k|} R_h}{a} \right),$$  \hfill (15)

where

$$\cos \left( \sqrt{|k|} y \right) = \begin{cases} \cos y & k = +1, \\
1 & k = 0, \\
\cosh y & k = -1. \end{cases}$$  \hfill (16)

Repeating the procedure of the previous sub-section and differentiating [4] using [13] and [15] we obtain:

$$\frac{\Omega'_\Lambda}{\Omega_\Lambda} = \frac{2}{\Omega_\Lambda} \left( -1 - \frac{\dot{H}}{H^2} + \frac{\Omega_\Lambda}{c} \cos(\sqrt{|k|} y) \right).$$  \hfill (17)

On the other hand, differentiating Friedmann equation [12] we finally obtain

$$2 \frac{\dot{H}}{H^2} = -3 - \Omega_k + \Omega_\Lambda + 2 \frac{\Omega^3/2}{c} \cos \left( \frac{\sqrt{|k|} R_h}{a} \right) +$$

$$+ \left( 1 + \Omega_k - \Omega_\Lambda \right) \frac{G'}{G},$$  \hfill (18)

where we have introduced the curvature density parameter \( \Omega_k \equiv \frac{k}{a^2 H^2} \). Therefore, substituting [18] into [17] we result to

$$\Omega'_\Lambda = \Omega_\Lambda \left[ 1 + \Omega_k - \Omega_\Lambda + \frac{2 \Omega_\Lambda}{c} \cos \left( \frac{\sqrt{|k|} R_h}{a} \right) \left( 1 - \Omega_\Lambda \right) \right] -$$

$$- \Omega_\Lambda \left( 1 + \Omega_k - \Omega_\Lambda \right) \frac{G'}{G}. \hfill (19)$$

Expression [19] provides the correction to holographic dark energy differential equation in non-flat universe, due to the varying nature of \( G \). Clearly, for \( k = 0 \) (and thus \( \Omega_k = 0 \)) it leads to [9].

### III. COSMOLOGICAL IMPLICATIONS

Since we have extracted the expressions for \( \Omega'_\Lambda \), we can calculate \( w(z) \) for small redshifts \( z \), performing the standard expansions of the literature. In particular, since \( \rho_\Lambda \sim a^{-3(1+w)} \) we acquire

$$\ln \rho_\Lambda = \ln \rho_\Lambda^0 + \frac{d \ln \rho_\Lambda}{d \ln a} \ln a + \frac{1}{2} \frac{d^2 \ln \rho_\Lambda}{d (\ln a)^2} (\ln a)^2 + \ldots,$$  \hfill (20)

where the derivatives are taken at the present time \( a_0 = 1 \) (and thus at \( \Omega_\Lambda = \Omega_\Lambda^0 \)). Then, \( w(\ln a) \) is given as

$$w(\ln a) = -1 - \frac{1}{3} \frac{d \ln \rho_\Lambda}{d \ln a} + \frac{1}{6} \frac{d^2 \ln \rho_\Lambda}{d (\ln a)^2} \ln a,$$  \hfill (21)

up to second order. Since \( \rho_\Lambda = 3H^2 \Omega_\Lambda/(8\pi G) = \Omega_\Lambda \rho_m/\Omega_m = \rho_m / (1 + \Omega_k - \Omega_\Lambda) a^{-3} \), the derivatives are easily computed using the obtained expressions for \( \Omega'_\Lambda \). In addition, we can straightforwardly calculate \( w(z) \), replacing \( \ln a = - \ln(1+z) \simeq -z \), which is valid for small redshifts, defining

$$w(z) = -1 - \frac{1}{3} \frac{d \ln \rho_\Lambda}{d \ln a} + \frac{1}{6} \frac{d^2 \ln \rho_\Lambda}{d (\ln a)^2} z \equiv w_0 + w_1 z.$$  \hfill (22)

The role of \( G \)-variation will be expressed through the pure number \( G'/G \equiv \Delta_G \), which will be extracted from observations. In particular, observations of Hulse-Taylor binary pulsar B1913+16 lead to the estimation \( G'/G \sim 2 \pm 4 \times 10^{-12} \text{yr}^{-1} \) [25, 26], while helio-seismological data provide the bound \( -1.6 \times 10^{-12} \text{yr}^{-1} < G'/G < 0 \) [27]. Similarly, Type Ia supernova observations [1] give the best upper bound of the variation of \( G \) as \( -10^{-11} \text{yr}^{-1} \leq \frac{\Delta G}{G} < 0 \) at redshifts \( z \simeq 0.5 \) [28], while asteroeoseismological data from the pulsating white dwarf star G117-B15A lead to \( \left| \frac{\Delta G}{G} \right| \leq 4.10 \times 10^{-11} \text{yr}^{-1} \) [29]. See also [30] for various bounds on \( G'/G \) from observational data, noting that all these measurements are valid at relatively low redshifts, i.e. \( z \lesssim 3.5 \).

Since the limits in \( G \)-variation are given for \( \dot{G}/G \) in units \( \text{yr}^{-1} \), and since \( \dot{G}/G = H G'/G \), we can estimate \( \Delta_G \) substituting the value of \( H \) in \( \text{yr}^{-1} \). In the following we will use \( \left| \frac{\Delta G}{G} \right| \lesssim 4.10 \times 10^{-11} \text{yr}^{-1} \). Thus, inserting an average estimation for the Hubble parameter \( H \approx \langle H \rangle \approx 6 \times 10^{-11} \text{yr}^{-1} \) [45], we obtain that \( 0 < \left| \Delta_G \right| \lesssim 0.07 \). Clearly, this estimation is valid at low redshifts, since only in this range the measurements of \( G'/G \) and the estimation of the average \( \langle H \rangle \) are valid. However, the restriction to this range is consistent with the \( z \)-expansion of \( w \) considered above.
A. Flat FRW geometry

In this case $\Omega_0'$ is given by (19), and the aforementioned procedure leads to

$$w_0 = -\frac{1}{3} - \frac{2}{3c} \sqrt{\Omega_0^0} + \frac{\Delta_G}{3}$$

$$w_1 = \frac{1}{6c} \sqrt{\Omega_0^0 (1 - \Omega_0^0)} \left( 1 + \frac{2}{c} \sqrt{\Omega_0^0} - (1 - \Omega_0^0) \sqrt{\Omega_0^0} \Delta_G \right).$$

(23)

These expressions provide $w_0$ and $w_1$, for the holographic dark energy with varying $G$, in a flat universe. Obviously, when $\Delta_G = 0$, they coincide with those of (13).

In general, apart from the relevant uncertainty in $\Omega_0^0$ measurements, we face the problem of the uncertainty in the constant $c$. In particular, observational data from type la supernovae give the best-fit value $c = 0.21$ within 1-$\sigma$ error range [16], while those from the X-ray mass fraction of galaxy clusters lead to $c = 0.61$ within 1-$\sigma$ [17]. Similarly, combining data from type la supernovae, Cosmic Microwave Background radiation and large scale structure give the best-fit value $c = 0.91$ within 1-$\sigma$ [18], while combining data from type Ia supernovae, X-ray gas and Baryon Acoustic Oscillation lead to $c = 0.73$ as a best-fit value within 1-$\sigma$ [19]. However, expressions [23, 24] provide the pure change due to the variation of gravitational constant for given $c$ and $\Omega_0^0$. For example, and in order to compare with the corresponding result of [13], imposing $\Omega_0^0 = 0.73$ and $c = 1$, and using $0 < |\Delta_G| < 0.07$ we obtain:

$$w_0 = -0.903^{+0.023}_{-0.023}$$

$$w_1 = 0.1041^{+0.0025}_{-0.0025}.$$  

(25)

where we have neglected uncertainties other than $G$-variation. Finally, note that the $w_0$-variation due to $\Delta_G$ is absolute, that is it does not depend on $c$ and $\Omega_0^0$, while that of $w_1$ does depend on these parameters. However, the relative variations of $w_0, w_1$ do depend on the $c$-value, and they are smaller for smaller $c$.

B. Non-flat FRW geometry

In this case $\Omega_0'$ is given by (19), and the aforementioned procedure leads to

$$w_0 = \frac{1}{3} - \frac{2}{3c} \sqrt{\Omega_0^0} \cos \frac{\sqrt{|k|} R_{h0}}{a_0} + \frac{\Delta_G}{3}$$

$$w_1 = \frac{\sqrt{\Omega_0^0}}{6c} \left[ 1 + \Omega_0^0 - \Omega_0^0 + \frac{2}{c} \sqrt{\Omega_0^0} \left( \frac{\sqrt{|k|} R_{h0}}{a_0} \right) (1 - \Omega_0^0) \right] \cos \left( \sqrt{\frac{|k|}{} R_{h0}} \right) + \frac{\Omega_0^0}{3c^2} q a_0 \left( \frac{\sqrt{|k|} R_{h0}}{a_0} \right) - \frac{\sqrt{\Omega_0^0}}{6c} (1 + \Omega_0^0 - \Omega_0^0) \cos \left( \frac{\sqrt{|k|} R_{h0}}{a_0} \right) \Delta_G.$$  

(26)

(27)

In these expressions, $\Omega_0^0$ is the present day value of the curvature density parameter, and we have defined

$$q(\sqrt{|k|}) = \begin{cases} \sin^2 k & k = +1, \\ 0 & k = 0, \\ -\sinh^2 k & k = -1. \end{cases}$$

(28)

Finally, $R_{h0}$ and $a_0$ are the present values of the corresponding quantities. Clearly, for $k = 0$, that is for a flat geometry, (26), (27) coincide with (23), (24) respectively.

As we observe, expressions (26), (27), apart from the present values of the parameters $\Omega_0^0, \Omega_0^0$ contain $a_0$ and the value of $R_{h0}$ at present. This last term is present in a non-flat universe, and it is a “non-local” quantity which has to be calculated by an integration (see relations (13) and (5)). However, making use of the holographic nature of dark energy, we can overcome this difficulty. Indeed, from (21) we obtain that $L_0 = c/(H_0 \sqrt{\Omega_0^0})$, with $H_0$ the present value of the Hubble parameter. On the other hand, from (13) we acquire $R_{h0}/a_0 = \frac{1}{\sqrt{|k|}} \sin^{-1}(\sqrt{|k|} L_0/a_0)$. Therefore, we conclude that

$$\frac{R_{h0}}{a_0} = \frac{1}{\sqrt{|k|}} \sin^{-1} \left( \frac{c \sqrt{|k|}}{a_0 H_0 \sqrt{\Omega_0^0}} \right) = \frac{1}{\sqrt{|k|}} \sin^{-1} \left( \frac{c \sqrt{\Omega_0^0}}{\sqrt{\Omega_0^0}} \right).$$

(29)

a relation which proves very useful. Substituting into (26), (27) we finally obtain the simple expressions:
Finally, we mention that in general, the possible un-
certainty of the constant c can have a larger effect on
\( w(z) \) than that of \( G \)-variation. In the above investiga-
tion we have just provided the complete expressions, in-
cluding the correction terms due to the variation of the
gravitational constant. One could proceed to a combined
observational constraint analysis, allowing for variations
and uncertainties in all parameters, as it was partially
performed in the specific Brans-Dicke framework in [47].
This extended examination, with not-guaranteed results
due to complexity, is under current investigation and it
is left for a future publication.

\[
\begin{align*}
    w_0 &= -\frac{1}{3} - \frac{2}{3c} \sqrt{\Omega^0_\Lambda - c^2 \Omega^0_k} + \frac{\Delta G}{3} \\
    w_1 &= \frac{\Omega^0_k}{3} + \frac{1}{6c} \sqrt{\Omega^0_\Lambda - c^2 \Omega^0_k} \left[ 1 + \Omega^0_k - \Omega^0_\Lambda + \frac{2}{c} (1 - \Omega_\Lambda) \sqrt{\Omega^0_\Lambda - c^2 \Omega^0_k} \right] - \frac{1}{6c} \sqrt{\Omega^0_\Lambda - c^2 \Omega^0_k} \left( 1 + \Omega^0_k - \Omega^0_\Lambda \right) \Delta G. 
\end{align*}
\]

(30)

Note that \( w_0, w_1 \) depend eventually only on \( \Omega^0_\Lambda, \Omega^0_k, c \) and
of course \( \Delta G \). Similarly to the previous subsection, in
order to give a representative estimation and neglecting
uncertainties of other quantities apart from \( G \)-variation,
we use \( c = 1, \Omega^0_\Lambda \approx 0.73, \Omega^0_k \approx 0.02, 0 < |\Delta G| < 0.07, \)
obtaining:

\[
\begin{align*}
    w_0 &= -0.895^{+0.023}_{-0.023} \\
    w_1 &= 0.111^{+0.003}_{-0.003}. 
\end{align*}
\]

(32)

Finally, we mention that the relative variations of \( w_0, w_1 \)
depend on the \( c \)-value, and they are smaller for smaller
\( c \).

IV. CONCLUSIONS

In this work we have investigated the holographic dark
energy scenario with a varying gravitational constant, go-
ing beyond the simple scenarios of [40]. Imposing flat
and non-flat background geometry we have extracted the
exact differential equations that determine the evo-
lation of the dark energy density-parameter, where the
\( G \)-variation appears as a coefficient in additional terms.
Thus, performing a low-redshift expansion of the dark
energy equation-of-state parameter \( w(z) \approx w_0 + w_1 z \), we
provide \( w_0, w_1 \) as functions of \( \Omega^0_\Lambda, \Omega^0_k \), of the holographic
dark energy constant \( c \), and of the \( G \)-variation \( \Delta G \) (ex-
pressions (30), (31)). As expected, the variation of the
gravitational constant increases the variation of \( w(z) \).

In the aforementioned analysis, the \( G \)-variation has
been considered as a constant quantity at the cosmo-
logical epoch of interest, that is at low redshifts, as it is
measured in observations with satisfactory accuracy
[23, 24, 25, 26, 28, 29, 30]. A step forward would be to
consider possible \( G(z) \)-parametrizations [17, 48] and ex-
tract their effect on \( w(z) \). However, such parametriza-
tions have a significant amount of arbitrariness, since the
present observational data do not allow for such a reso-
lution, and thus we have not performed this extension in
the present work.

Finally, we mention that in general, the possible un-
certainty of the constant \( c \) can have a larger effect on
\( w(z) \) than that of \( G \)-variation. In the above investiga-
tion we have just provided the complete expressions, in-
cluding the correction terms due to the variation of the
gravitational constant. One could proceed to a combined
observational constraint analysis, allowing for variations
and uncertainties in all parameters, as it was partially
performed in the specific Brans-Dicke framework in [47].
This extended examination, with not-guaranteed results
due to complexity, is under current investigation and it
is left for a future publication.

[1] A. G. Riess et al. [Supernova Search Team Collabora-
tion], Astron. J. 116, 1009 (1998); S. Perlmutter et
al. [Supernova Cosmology Project Collaboration], Astra-
phys. J. 517, 565 (1999).

[2] C. L. Bennett et al., Astrophys. J. Suppl. 148, 1 (2003).

[3] M. tegmark et al. [SDSS Collaboration], Phys. Rev. D
69, 103501 (2004).

[4] S. W. Allen, et al., Mon. Not. Roy. Astron. Soc. 353, 457
(2004).

[5] V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D 9, 373
(2000); P. J. Peebles and B. Ratra, Rev. Mod. Phys. 75,
559 (2003).

[6] B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406
(1988); C. Wetterich, Nucl. Phys. B 302, 668 (1988); A.
R. Liddle and R. J. Scherrer, Phys. Rev. D 59, 023500
(1999); I. Zlatev, L. M. Wang and P. J. Steinhardt,
Phys. Rev. Lett. 82, 896 (1999); Z. K. Guo, N. Ohta
and Y. Z. Zhang, Mod. Phys. Lett. A 22, 883 (2007);
S. Dutta, E. N. Saridakis and R. J. Scherrer, Phys. Rev.
D 79, 103005 (2009) [arXiv:0903.3412 [astro-ph.CO]].

[7] R. R. Caldwell, Phys. Lett. B 545, 23 (2002); R. R. Cald-
well, M. Kamionkowski and N. N. Weinberg, Phys. Rev.
Lett. 91, 071301 (2003); S. Nojiri and S. D. Odintsov,
Phys. Lett. B 562, 147 (2003); V. K. Onemli and R.
P. Woodard, Phys. Rev. D 70, 107301 (2004); M. R.
Setare, J. Sadeghi, A. R. Amani, Phys. Lett. B 666,
288, (2008); X. m. Chen, Y. g. Gong and E. N. Saridakis,
JCAP 0904, 001 (2009); E. N. Saridakis, Nucl. Phys. B
819, 116 (2009).

[8] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B
607, 35 (2005); Z. K. Guo, et al., Phys. Lett. B 608,
177 (2005); M.-Z Li, B. Feng, X.-M Zhang, JCAP, 0512,
002 (2005); B. Feng, M. Li, Y.-S. Piao and X. Zhang,
Phys. Lett. B 634, 101 (2006); M. R. Setare, Phys. Lett. B
641, 130 (2006); W. Zhao and Y. Zhang, Phys. Rev. D
73, 123509 (2006); M. R. Setare, J. Sadeghi, and A. R.
Amani, Phys. Lett. B 660, 299 (2008); J. Sadeghi, M. R.
Setare, A. Banijamali and F. Milani, Phys. Lett. B 662,
92 (2008); M. R Setare and E. N. Saridakis, Phys. Lett. B
688, 177 (2008); M. R. Setare and E. N. Saridakis, JCAP
