The Ising $M$-$p$-spin mean-field model for the structural glass: continuous vs. discontinuous transition

F. Caltagirone$^{1}$, U. Ferrari$^{1}$, L. Leuzzi$^{1,2}$, G. Parisi$^{1,2,3}$ and T. Rizzo$^{11}$

$^{1}$ Dip. Fisica, Università “Sapienza”, Piazzale A. Moro 2, I-00185, Rome, Italy
$^{2}$ IPCF-CNR, UOS Rome, Università “Sapienza”, Piazzale A. Moro 2, I-00185, Rome, Italy
$^{3}$ INFN, Piazzale A. Moro 2, 00185, Rome, Italy

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The critical behavior of a family of fully connected mean-field models with quenched disorder, the $M-p$ Ising spin glass, is analyzed, displaying a crossover between a continuous and a random first order phase transition as a control parameter is tuned. Due to its microscopic properties the model is straightforwardly extendable to finite dimensions in any geometry.

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I. INTRODUCTION

Since the work of Kirkpatrick, Thirumalai and Wolynes$^{1-6}$ a certain set of mean-field spin-glass models have been shown to own the salient properties of the behavior of structural glasses. In particular, these models display dynamic equations that are equivalent to those predicted by the Mode Coupling Theory (MCT)$^{7-9}$ above the so-called mode coupling temperature $T_{mc}$ where ergodicity breaking occurs in that theory. Moreover, two kinds of transition are predicted: a dynamic one at $T_d = T_{mc}$ and a thermodynamic phase transition at a lower $T$, often referred to as Kauzmann transition. Mean-field models exhibiting structural glass features are characterized by multi-body microscopic interactions and their thermodynamics is self-consistently described by implementing a discontinuous Replica Symmetry Breaking (RSB) Ansatz (usually one step: 1RSB).

The dynamic transition is due to the presence of a large number of metastable excited glassy states, represented as local minima in the free energy landscape in the configurations space, growing exponentially with the size $N$ of the system. In the mean-field approximation, barriers between minima grow with the size, so that, in the thermodynamic limit, the relaxing dynamics to equilibrium of the system at $T \leq T_d$ remains stuck forever inside the first “meta”-stable state where it ends up in. In real glassy systems, however, there is a slow dynamics occurring through activated processes and this dynamic arrest is an artefact due to the mean-field approximation. In finite dimensions the glass transition occurs because at some glass temperature $T_g$ the time-scales of observation are shorter than the characteristic time-scales of the slowest structural processes taking place in the glass-former sample. Metastable states really have a finite lifetime, even though (much) longer than the experimental time of observation. The effect of activated processes starting from spin-glass 1RSB mean-field models has been analyzed, e.g., by working at finite $N$ in the fully connected Random Orthogonal Model (ROM)$^{10-15}$ and finding a glass behavior, similar to the one observed in computer glasses, cf., e.g., Refs.$^{14,15}$

As we mentioned, another property occurring in the glass-like mean-field models (see also Refs.$^{16-18}$) is a thermodynamic transition between the supercooled liquid (below $T_d$) and a thermodynamically stable glass. This occurs with a jump in the order parameter, but without discontinuity in the internal energy (no latent heat is exchanged). This mixture of first order and continuous phase transition in presence of disorder has been termed random first order (RFOT).$^6$

One of the most accredited theories, the Adam-Gibbs-Di Marzio entropic theory$^{19,20}$ predicts the existence of a thermodynamic transition to an ideal glass phase, the so-called Kauzmann transition. The Kauzmann temperature is generally associated with the asymptote of the Vogel-Fulcher law$^{21,22}$ of the relaxation time and, thus, related to the transition one might have in an infinitely slow cooling of a never-crystallizing glass-former. Because of the impossibility of experimental measurements of glass relaxation in those conditions, the very existence of the Kauzmann point and the nature of that transition is still a matter of debate.

Attempts to follow the properties envisaged in mean-field models in realistic systems have faced the problem of finding a proper way of embedding the model microscopic features into a given finite dimensional geometry (e.g., 3D cubic lattice) with-
out altering the discontinuous nature of the transition. So that one can actually falsify the hypothesis of RFOT in finite dimensional systems. Indeed, in Ref. [23] a generalization of the \( p = 3 \)-spin model with \( M = 2 \) Ising spins on each site was numerically studied on a \( D = 4 \) hypercubic lattice finding evidence for a continuous phase transition. The same continuous behavior was recently found, already in the mean-field regime, in the same \( p = 3 \), \( M = 2 \) model in a \( D = 1 \) chain on a “Levy lattice” [24].

Starting from this observation, that the RFOT becomes continuous in finite \( D \), the work of Moore, Drossel and Yeo [25–27] shows that this is equivalent to the critical behavior of the Edwards-Anderson model in a field, where the transition line is called de Almeida-Thouless (dAT) line. Applying droplet theory (that rules out the existence of a dAT line outside the limit of validity of mean-field theory) it is, thus, inferred that no thermodynamic random first order transition can occur in real structural glasses. The issue of the existence of a dAT line in finite dimensional spin-glasses will not be addressed here. For recent bibliography on that subject see Refs. [28–33] and references therein.

In the present work we will focus on deriving a mean-field class of models, to whom Ising \( p \)-spins\(^{16,34}\) belong, whose critical behavior shifts from continuous to discontinuous in a controlled way. The aim is to clarify why the finite dimension extensions of mean-field glasses studied so far do not display RFOT and to devise mean-field models whose discontinuous critical nature can be conserved also beyond the limit of validity of the mean-field approximation. The model consists of \( N \) sites, each one containing \( M \) spins interacting with spins on other sites in \( p \)-uples. We will see how, changing \( p \) and the number \( M \) of spins living on a single site, it is possible to move from systems displaying a second order phase transition to systems displaying a random first order transition, that is, yielding both a dynamic and a Kauzmann-like phase transition. The finite dimension counterpart of the model under probe can be easily achieved since the \( p \)-spin interaction is always exchanged between two sites, e.g., nearest neighbors on a \( d \)-dimensional (hyper)cubic lattice.

We mention that moving from mean-field to finite dimensions, also standard Ising \( p \)-spin and Potts models might conserve the random first order nature of the transition and keep reproducing basic features of structural glasses. Even though it is not straightforward to conceive a short-range finite dimensional Ising \( p \)-spin, the Potts model can be easily defined on a hyper-cubic lattice. Nevertheless, no numerical evidence has been collected so far for a discontinuous RFOT in disordered Potts models with number of states \( p_{\text{Potts}} = 5, 6, 10^{35–37} \) and, actually, we found no argument to infer that in the finite dimensional lattice case the \( p_{\text{Potts}} \rightarrow \infty \) limit can be kept under control.

On the other hand, the model considered in the present work has the advantage to reduce to an exact mean-field model for the RFOT as \( M \rightarrow \infty \) even in finite dimension (and finite size), for any values of \( p \). Moreover, we can work out a sufficient criterion to determine the smallest value of \( M \) above which continuous transitions cannot occur.

The manuscript is organized as follows: in Sec. II we will study the statistical mechanics of the model; in Sec. III we show that the large \( M \) limit corresponds to standard \( p \)-spin and in Sec. IV, expanding near criticality, we build the corresponding field-theory, compute the coupling constants and study the relevance of terms competing for continuous/discontinuous transition. In Sec. V we present our conclusions.

## II. THE MODEL

The model consists on \( N \) sites, each one hosting a set of \( M \) spins. Two sites interact through a \( p \)-body interaction involving spins belonging to the two sets of \( M \) spins. The Hamiltonian reads

\[
\mathcal{H} = -\sum_{\langle x, y \rangle} \sum_{g(x,y)} J_g \prod_{\mu \in g} s_\mu
\]

(1)

where \( \langle x, y \rangle \) indicates the sum over all couples of sites and \( g(x, y) \) are all the possible \( p \)-uples among the \( 2M \) spins, with an exception if \( p \leq M \): those \( p \)-uples completely pertaining to a single site are excluded. This choice actually defines our model when \( p \leq M \), as we will discuss in the following.

The disordered interactions are Gaussian i.i.d. variables, with distribution:

\[
P(J_g) = \frac{1}{\sqrt{2\pi \sigma_{J_g}^2}} e^{-\frac{J_g^2}{2\sigma_{J_g}^2}}
\]

(2)

where, to provide the right thermodynamic convergence of the free energy, the variance scales like

\[
\sigma_{J_g}^2 = \frac{1}{NM^{p-1}}
\]

(3)
A. Free energy and order parameters

Replicating \( n \) times the system we compute the average over quenched disorder of the replicated partition function:

\[
\overline{Z^n} = \int \prod_{(x,y)} P(J_y) \, dJ_y
\]

\[
\times \text{Tr}[a] \exp \left[ \frac{\beta^2}{4NM} \sum_{a=1}^{1,N} \sum_{x,y} \sum_{g(x,y)} \sum_{\mu=\varepsilon} s^a_{\mu} \right]
\]

yielding

\[
\overline{Z^n} = \text{Tr}[a] \exp \left[ \frac{\beta^2}{4NM^{p-1}} \sum_{a,b} \sum_{x\neq y} \sum_{a,b} \sum_{k} s^a_i(x) s^b_{i_k}(x) \cdots s^a_i(x) s^b_{i_k}(x) \right. \\
\left. \sum_{i_{k+1} \cdots i_p} s^a_{i_{k+1}}(y) s^b_{i_{k+1}}(y) \cdots s^a_{i_{p}}(y) s^b_{i_{p}}(y) \right]
\]

(5)

Explicitly separating those spins belonging to site \( x \) from those on site \( y \) one can obtain a general expression for the partition function valid both for \( p > M \) and \( p \leq M \):

\[
\overline{Z^n} = \text{Tr}_{\{s(x),s(y)\}} \exp \left[ \frac{\beta^2}{4NM^{p-1}} \right]
\]

\[
\times \sum_{a,b} \sum_{x \neq y} \sum_{k} \sum_{i_1 < \cdots < i_k} s^a_i(x) s^b_{i_k}(x) \cdots s^a_i(x) s^b_{i_k}(x) \\
\sum_{i_{k+1} \cdots i_p} s^a_{i_{k+1}}(y) s^b_{i_{k+1}}(y) \cdots s^a_{i_{p}}(y) s^b_{i_{p}}(y)
\]

(6)

For \( p > M \) the sum over \( k \) runs from \( p - M \) to \( M \); in the case \( p \leq M \) the sum over \( k \) runs from \( 1 \) to \( p - 1 \). In principle, it might be possible to include an extra term due to self-interaction: \( p \) out of \( M \) spins interact on a single site (“a single site standard \( p \)-spin”). As already mentioned, in the present work we will consider a model \textit{without} site self-interaction.

We now introduce a set of multi-overlaps between \( k \) spins on the same site \( x \) in two replicas:

\[
Q^{(k)}_{ab} = \frac{1}{NM^k} \sum_{x=1}^{N} \sum_{i_1 < \cdots < i_k} s^a_i(x) s^b_{i_k}(x) \cdots s^a_i(x) s^b_{i_k}(x)
\]

(7)

By means of multi-overlaps we can write the replicated partition function Eq. (7) as

\[
\overline{Z^n} = e^{NC} \int DQ \text{Tr}_{\{s(x),s(y)\}} \exp \left[ \frac{\beta^2 NM}{4} \sum_{a,b} \sum_{k} Q^{(k)}_{ab} Q^{(p-k)}_{ab} \right] \times
\]

\[
\times \prod_{k} \prod_{a \neq b} \delta \left( NM Q^{(k)}_{ab} \right)
\]

where the parameter \( C \), proportional to minus the paramagnetic free energy, reads

\[
\frac{C}{n} = \frac{\beta^2}{4M^{p-1}} \left[ \sum_{k} \left( \frac{M}{k} \right) \left( \frac{M}{p-k} \right) \right]
\]

(9)

Introducing the integral representation for the delta functions in Eq. (8) one obtains:

\[
\overline{Z^n} = e^{NC} \int DQ D\Delta \exp \left[ -NG(Q,\Delta) \right]
\]

(10)

\[
G(Q,\Delta) = -\frac{\beta^2 M}{4} \sum_{a \neq b} Q^{(k)}_{ab} Q^{(p-k)}_{ab} + \frac{M}{2} \sum_{k} \sum_{a \neq b} \Lambda^{(k)}_{ab} Q^{(k)}_{ab} - \log Z(\Delta)
\]

(11)

\[
Z(\Delta) = \text{Tr}e^{S(\Delta)}
\]

(12)

\[
S(\Delta) = \frac{1}{2} \sum_{k} \sum_{a \neq b} \frac{\Lambda^{(k)}_{ab}}{M^{k-1}} \sum_{i_1 < \cdots < i_k} s^a_i s^b_{i_k} \cdots s^a_{i_k} s^b_{i_k}
\]

\[
DQ = \prod_{k} \prod_{a \neq b} dQ^{(k)}_{ab}
\]

\[
D\Delta = \prod_{k} \prod_{a \neq b} d\Lambda^{(k)}_{ab}
\]

(13)
The stationarity equations in $\Lambda$ and $Q$ are

$$Q_{ab}^{(k)} = \frac{1}{Z(\Delta)} \text{Tr}[s^a] \frac{1}{M^k} \Lambda_{ab}^{(k)}$$

$$(14)$$

$$\Lambda_{ab}^{(k)} = \beta^2 Q_{ab}^{(p-k)}$$

$$(15)$$

Substituting the saddle point value for $\Lambda$ in the effective action we obtain

$$G(Q) = \frac{\beta^2 M}{4} \sum_{k \neq b} Q_{ab}^{(p-k)}$$

$$- \log \text{Tr}[s^a] e^{S(Q)}$$

$$S(Q) = \frac{\beta^2}{2} \sum_{k \neq b} Q_{ab}^{(p-k)} M^{k-1} \sum_{i_1 < i_2 < \cdots < i_k} s_i^a s_i^b \ldots s_i^a s_i^b$$

$$(16)$$

The physical meaning of the overlap matrix at saddle point value is the usual one and, more precisely

$$Q_{ab}^{(k)} = \frac{1}{NM^k} \sum_{x=1}^{N} \sum_{i_1 < i_2 < \cdots < i_k} \langle s_i^a(x) \ldots s_i^b(x) \rangle^2$$

$$= \lim_{n \to 0} \frac{2}{n(n-1)} \sum_{a < b} Q_{ab}^{(k)} S_P$$

$$(17)$$

### III. LARGE $M$ LIMIT: STANDARD $p$-SPIN

For large $M$, neglecting diagonal terms in the sum over $i_1, \ldots, i_k$, in Eq. (16), the log Tr term can be rewritten as

$$S(Q) = M^{2} \sum_{k=1}^{p-1} \sum_{k \neq b} Q_{ab}^{(p-k)} \left( \frac{1}{M} \sum_{i=1}^{M} s_i^a s_i^b \right)^k$$

$$(18)$$

Performing the saddle point for large $M$, rather than $N$, and introducing the auxiliary parameter

$$q_{ab} = \frac{1}{M} \sum_{i=1}^{M} s_i^a s_i^b$$

$$(19)$$

we obtain, for the free energy Eq. (16)

$$G(Q) = M \left[ \frac{\beta^2}{4} \sum_{k=1}^{p-1} \sum_{a \neq b} Q_{ab}^{(p-k)} Q_{ab}^{(p-k)} - \frac{\beta^2}{2} \sum_{k=1}^{p-1} \frac{1}{k!} \sum_{a \neq b} Q_{ab}^{(p-k)} q_k + \lambda_{ab} q_{ab} \right]$$

$$- \log \text{Tr}[s^a] \exp \left\{ \sum_{a \neq b} \lambda_{ab} s^a s^b \right\}$$

$$(20)$$

The saddle point self-consistency equation w.r.t. $Q^{(p-k)}$ yields

$$Q_{ab}^{(k)} = \frac{1}{k!} q_{ab}$$

$$(21)$$

Substituting Eq. (21) in Eq. (20), we obtain the expression

$$G(q, \lambda)/M = -\frac{\beta^2}{4} \sum_{a \neq b} \sum_{k=1}^{p-1} \frac{1}{k!(p-k)!} q_{ab}^p + \lambda_{ab} q_{ab}$$

$$- \log \text{Tr}[s^a] \exp \left\{ \sum_{a \neq b} \lambda_{ab} s^a s^b \right\}$$

$$(22)$$

that is, the standard formal free energy of the fully connected Ising $p$-spin model:

$$G(q, \lambda)/M = -\frac{\beta^2}{4} \sum_{a \neq b} \frac{2p-2}{p!} q_{ab}^p + \lambda_{ab} q_{ab}$$

$$- \log \text{Tr}[s^a] \exp \left\{ \sum_{a \neq b} \lambda_{ab} s^a s^b \right\}$$

$$(23)$$

with

$$q_{ab} = \langle s^a s^b \rangle$$

$$\lambda_{ab} = \frac{p \beta^2}{2} q_{ab}^{p-1}.$$  

$$(24)$$

$$(25)$$

### IV. ANALYSIS OF THE CRITICAL POINT

Our aim is to find the transition point and to study its thermodynamic nature as $M$ and $p$ are changed. In particular, we will verify that, at given $p$ (vice-versa $M$) there are threshold values of $M$ (resp. $p$) beyond which the transition switches from continuous to discontinuous.
First, to identify the critical point we expand the stationarity equation (14) to first order in $Q^{(k)}_{ab}$, obtaining:

$$Q^{(k)}_{ab} = \frac{\beta^2}{M^k} \frac{Q^{(p-k)}_{ab}}{M^{k-1}} (M) \tag{26}$$

There are “multi”- critical temperatures for the “multi” - overlaps, whose expressions read

$$\beta_c(k) = \frac{M^{\frac{k-1}{2}}}{(k)^{\frac{1}{2}} (p-k)^{\frac{1}{2}}} \tag{27}$$

The largest critical temperature is obtained for $k = p/2$ if $p$ is even, and for $k = (p+1)/2$, $(p-1)/2$ if $p$ is odd. The overlap corresponding to the smallest $\beta_c$ (slightly above $\beta_c$) is non-zero and of order $\tau \propto (T_c - T)/T_c$, while the others are at least of order $\tau^2$.

Proceeding to the second order expansion of Eq. (14) we have

$$Q^{(k)}_{ab} = \frac{\beta^2}{M^k} \frac{Q^{(p-k)}_{ab}}{M^{k-1}} (M)$$

$$+ \text{Tr}[\sigma^z] \frac{1}{M^k} \sum_{i_1 < \ldots < i_k} s_{i_1} s_{i_2} \ldots s_{i_k} s_{i_k}$$

$$\times \frac{\beta^4}{4 \times 2!} \sum_{l,m \neq d,e \neq f} Q^{(p-l)}_{cd} \frac{Q^{(p-m)}_{ef}}{M^{l-1} M^{m-1}}$$

we will focus only on the equations for the overlaps corresponding to the largest critical temperature, cf. Eq. (27), that is, on the terms of the type

$$Q^{(p/2)}_{ab} Q^{(p/2)}_{ab}, \quad \text{for even } p$$

else

$$Q^{(p+1)}_{ab} Q^{(p+1)}_{ab}, \quad \text{for odd } p.$$  

More specifically, we are interested in the terms of the series at the r.h.s of Eq. (28) with $k = l = m = p/2$, if $p$ is even, or with $k, l, m = \frac{p+1}{2}$, if $p$ is odd.

It is interesting to notice that we would have the same physics considering a model in which $p/2$-uples on each site interact with $p/2$ on another site (p even) or $(p+1)/2$-uples on a site interact with $(p-1)/2$ on another site (p odd).

In Eq. (28) each spin in each replica has to be matched with another one in another replica in order to get a non-zero result from the trace. At second order we are, thus, left with only two kinds of possible matching, yielding terms:

$$\sum_c Q_{ac}(x) Q_{cb}(x) = \left( Q_{(x)}^{(c)} \right)^2 \quad \text{and} \quad \left( Q_{ab}^{(x)} \right)^2.$$

We will see how, depending on the parity of $p$ even the multiplicity of such terms will change, leading to different expressions of their coefficients as functions of $p$ and $M$.

Using the above results, Eq. (16), approximated to the second order in $Q$, can then be written as

$$G(Q) = \frac{T}{2} \sum_{a,b} Q^2_{ab} + \frac{w_1}{6} \text{Tr} Q^3 + \frac{w_2}{6} \sum_{a,b} Q^3_{ab} \tag{29}$$

where $Q_{ab}$ stays for $Q^{(x)}_{ab}$.

As already noticed by Gross, Kanter and Sompolinsky in the Potts model (threshold was $p_{\text{Potts}} = 4$ colors) and in Ref. [38], it can be shown (see App. A) that if the ratio between coupling constants on the nonlinear terms is larger than one the phase transition cannot be continuous. We will now proceed to the computation of the coupling constants for the $M$-$p$ Ising spin model. Since, as already mentioned, the computation of the third order coefficients will yield different functional expressions depending on the values of $p$, we have to distinguish between four cases:

$$p = \begin{cases} 
4a & \text{A} \\
4a - 2 & \text{B} \\
4a - 1 & \text{C} \\
4a - 3 & \text{D} 
\end{cases} \quad a \in \mathbb{N}^+ \quad (30)$$

and we will analyze them separately.

**A. Even $p$ and $p/2$, $p = 4a$**

The only surviving term in the sum over $l$ and $m$ in the r.h.s. of Eq. (28) is for $l = m = k = p/2$. The trace term turns out to be

$$w_1 \sum_{c=1}^{n} Q_{ac}^{(p/2)} Q_{cb}^{(p/2)}$$

$$= \frac{\beta^4}{M^{p/2 - 2}} \left( \frac{M}{2} \right)^n \sum_{c=1}^{n} Q_{ac}^{(p/2)} Q_{cb}^{(p/2)} \tag{31}$$
The squared term is
\[ w_2 \left( Q_{ab}^{(p/2)} \right)^2 \]
\[ = \frac{1}{2M^2 p - 2} \left( \frac{M}{p/2} \right)^{p/2} \left( \frac{M - p/2}{p/2} \right)^{p/2} \left( Q_{ab}^{(p/2)} \right)^2 \]
and the ratio
\[ \frac{w_2}{w_1} = \frac{1}{2} \left( \frac{p/2}{p/4} \right) \left( \frac{M - p/2}{p/2} \right) \] (33)

**B. Even \( p, \) odd \( p/2, p = 4a - 2 \)**

In the r.h.s. of Eq. (29) only the coefficient in front of the \( \text{Tr} Q_{ab}^3 \) term survives, whereas \( w_2 = 0 \) always. The ratio is
\[ \frac{w_2}{w_1} = 0 \] (34)
According to the small \( Q \) expansion, Eqs. (28), (29), when \( p \) is even and \( p/2 \) is odd the transition at the largest critical temperature \( 1/\beta_c(p/2) \), cf. Eq. (27), turns out can to be continuous, no matter how many spins \( M \) stay on each site. This might appear at contrast with the large \( M \) limit leading to Eq. (23) that is equivalent to an Ising \( p \)-spin mean-field model for any \( p > 2 \). There, however, no perturbative expansion was carried out, while Eq. (34) is the outcome of an expansion for small overlap values that cannot help in identifying discontinuous transitions with large jumps in \( Q \). Indeed, the condition expressed by Eq. (A7) is sufficient but not necessary to rule out a continuous transition.

**C. Odd \( p, \) even \( (p + 1)/2, p = 4a - 1 \)**

When \( p \) is odd we have to deal with two relevant overlaps \( Q^{(p+1)/2} \) and \( Q^{(p+1)/2} \) and one critical temperature. In order to determine the coupling constants of the cubic terms one, thus, has to diagonalize a \( 2 \times 2 \) matrix. In App. B we report the details of the computation leading to:
\[ w_1(p, M) = \frac{\sqrt{2M - p + 1} + \sqrt{p + 1}}{4M^{p-3/2} \sqrt{p + 1}} \sqrt{\left( \frac{M}{p-1} \right)} \] (35)
for the coefficient of the cubic trace term in the action, Eq. (29). The expression for the coupling constant depends, further, on \( (p + 1)/2 \) being even or odd. For \( p = 4a - 1 \) we obtain
\[ w_2(p, M) = \frac{1}{8M^{p-3/2}} \frac{5M - 3p + 2}{2M - p + 1} \]
\[ \times \sqrt{\left( \frac{M}{p-1} \right)} \left( \frac{M}{p + 1} \right) \left( \frac{M - p - 1}{p + 1} \right) \] (36)

and, eventually, the ratio is:
\[ \frac{w_2}{w_1} = \frac{2 + 5M - 3p}{2 + 4M - 2p} \frac{\sqrt{p + 1}}{\sqrt{2M - p + 1}} \times \left( \frac{M - p - 1}{p + 1} \right) \left( \frac{M}{p + 1} \right) \] (37)

**D. Odd \( p, \) even \( (p - 1)/2, p = 4a - 3 \)**

In this last case the coupling of the trace cubic term is still given by Eq. (37) and the second nonlinear coupling constant is expressed as:
\[ w_2(p, M) = \frac{1}{4M^{p-3/2}} \frac{6 - 5p^2 + 9M + 7pM}{(p + 3)(2M - p + 1)} \]
\[ \times \sqrt{\left( \frac{M}{p - 1} \right)} \left( \frac{M - p - 1}{p - 1} \right) \left( \frac{p - 1}{p - 1} \right) \] (38)
yielding the ratio:
\[ \frac{w_2}{w_1} = \frac{6 - 5p^2 + 9M + 7pM}{(p + 3)(2M - p + 1)} \]
\[ \times \frac{\sqrt{2M - p + 1}}{\sqrt{p + 1} + \sqrt{2M - p + 1}} \left( \frac{M - p - 1}{p - 1} \right) \left( \frac{p - 1}{p - 1} \right) \] (39)
We now have a complete description of the critical behavior of the \( M-p \) system. Already at the mean-field level, in order to have a discontinuous transition a \( p > 2 \) interaction between spins in not enough. We find that for each given \( p \) one needs a minimal number of spins \( M_{\text{disc}} \) on each site in order to have a random first order phase transition, corresponding to the lowest integer \( M \) for which \( w_2/w_1 > 1 \); cf. Eq. (33), (37), or (39) depending on the parity of \( p \) and \((p + 1)/2\).

In table Tab. I we report some values of the ratios for systems with small \( p \) and \( M \). In Fig. 1 we plot the \( M_{\text{disc}}(p) \) behavior.

**V. CONCLUSIONS**

In the present work we have performed an analytic computation of the critical behavior of a mean-field
TABLE I: Ratio values for small p and M around the threshold 1

| p | M       | \(w_2/w_1\)      |
|---|---------|------------------|
| 3 | 2       | 3(1 - 1/√2) = 0.87868 |
| 3 | 3       | 2                |
| 4 | 2       | 0                |
| 4 | 3       | 1                |
| 4 | 4       | 2                |
| 5 | 3       | (√3 − 1)/2 = 0.366025 |
| 5 | 4       | 13(√3/2 − 1) = 2.92168 |
| 6 | any     | 0                |

FIG. 1: Lowest integer values of \(M\) at given \(p\), for which a discontinuous transition is certainly expected (sufficient condition to have RFOT is \(M \geq M_{\text{disc}}\)).

The particular case studied in Ref. [23], \((M = 2, p = 3)\), yielded numerical evidence for a continuous phase transition in dimension four. This is consistent with the value of the \(w_2/w_1 = 3[1 − 1/√2] = 0.87868\) as computed in the mean-field theory, cf. Tab. I and Ref. [39]. The same applies to the model recently studied in Ref. [24], a one-dimensional \((M, p) = (2, 3)\) model on a Levy lattice.

The continuous-discontinuous cross-over is like the one found in Potts\(^{17}\) varying the number of colors, and in the spherical \(p\)-spin varying an external magnetic field\(^{18}\). The advantage of the present model is that it can be easily represented in finite dimensions on lattices of given geometry, e.g., on a cubic lattice with short-range interactions, and that it always displays a RFOT in the \(M \to \infty\) limit for all lattices, both finite dimensional and mean-field like.

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Appendix A: Threshold value for \(w_2/w_1\)

Starting from the self-consistency equation for small \(Q\)'s (implying a continuous transition in \(Q\)), cf. Eq. (29) where also the quartic term is considered,

\[
\tau Q_{ab} + w_1(Q^2)_{ab} + w_2 Q^2_{ab} + y Q^3_{ab} = 0 \quad (A1)
\]

we have, in the RSB Ansatz,

\[
\tau q(x) - w_1 \left[ 2 q(x) \int_0^1 q(s) \, ds \right] + \int_0^x [q(x) - q(s)]^2 \, ds \right] + w_2 q(x)^2 + y q(x)^3 = 0 \quad (A2)
\]

Deriving once Eq. (A2) w.r.t. \(x\) one has

\[
q'(x) \left\{ \tau - 2 w_1 \left[ \int_x^1 q(s) \, ds + x q(x) \right] + 2 w_2 q(x) + 3 y q(x)^2 \right\} = 0 \quad (A3)
\]

If \(q'(x) \neq 0\), deriving a second time w.r.t. \(x\), one finds

\[
q'(x) [-w_1 x + w_2 + 3 y q(x)] = 0 \quad (A4)
\]

If \(y > 0\), then the overlap function around criticality can be written as:

\[
q(x) = \begin{cases} 
0 & x < x_1 \\
\frac{1}{3y} (w_1 x - w_2) & x_1 < x < x_2 \\
q(1) & x > x_2 
\end{cases} \quad (A5)
\]
where, for continuity, \( x_1 = w_2/w_1 \) and

\[
q(1) = \frac{\tau}{p} \frac{2}{1 + \sqrt{1 - 6yt/p^2}}, \quad p \equiv w_1(1 - x_1) \quad (A6)
\]

As a consequence, in order to have a continuous transition it must be

\[
\frac{w_2}{w_1} \leq 1. \quad (A7)
\]

The argument for the threshold value of \( x \) still works also if \( y \leq 0 \) in Eq. (A1). In that case, rather than a continuous function, we simply have a 1RSB step function for \( q(x) = \theta(x - x_1)q \), with

\[
q = \frac{\tau}{p} \frac{2}{1 + \sqrt{1 - 10yt/p^2}}, \quad p \equiv w_1 \left(1 - \frac{w_2}{w_1}\right) \quad (A8)
\]

\[
x_1 = \frac{w_2}{w_1} + 3\sqrt{q} \quad (A9)
\]

**Appendix B: Coupling constants with odd \( p \)**

When \( p \) is odd we have to deal with two relevant overlaps and one critical temperature. The second order equation, Eq. (28), has the structure:

\[
\mathcal{A} \mathbf{Q}_{ab} = \mathbf{F}(\{\mathbf{Q}\}) \quad (B1)
\]

where \( \mathbf{Q}_{ab} = \{Q_{ab}^{(p-M)}, \ldots, Q_{ab}^{(M)}\} \). Diagonalizing \( \mathcal{A} \rightarrow \mathcal{D}_A = \mathcal{P}^{-1} \mathcal{A} \mathcal{P} \) one obtains:

\[
\mathcal{P}^{-1} \mathcal{A} \mathcal{P} \mathcal{P}^{-1} \mathbf{Q}_{ab} = \mathcal{P}^{-1} \mathbf{F}(\{\mathbf{Q}_{ab}\}) \quad (B2)
\]

Introducing new variables \( \Theta_{ab} \), linear combinations of \( \mathbf{Q}_{ab} \), the above expression can be rewritten as

\[
\mathcal{D}_A \Theta_{ab} = \mathcal{P}^{-1} \mathbf{F}(\{\mathbf{P} \Theta_{ab}\}) \quad (B3)
\]

Rearranging the entries in a proper way, \( \mathcal{A} \) can be written as a block matrix of \( 2 \times 2 \) elements per block, and each block can be diagonalized separately, with eigenvalues

\[
\lambda^{(k)} = 1 \pm \beta^2 \sqrt{f(k)f(p-k)} \quad (B4)
\]

and eigenvectors:

\[
v^{k} = \begin{bmatrix} \frac{2\sqrt{f(p-k)}}{w} \\pm \frac{1}{\sqrt{2\sqrt{f(k)}}} \end{bmatrix}. \quad (B5)
\]

For each block of the matrix, labeled by \( k \), the eigenvector matrix and its inverse, thus, are

\[
\mathcal{P} = \begin{bmatrix} \frac{1}{2\sqrt{f(p-k)}} \\pm \frac{1}{\sqrt{2\sqrt{f(k)}}} \\frac{2\sqrt{f(p-k)}}{w} \\pm \frac{1}{\sqrt{2\sqrt{f(k)}}} \end{bmatrix} \quad (B6)
\]

\[
\mathcal{P}^{-1} = \begin{bmatrix} \sqrt{f(p-k)} \\pm \frac{1}{\sqrt{2\sqrt{f(k)}}} \\frac{2\sqrt{f(p-k)}}{w} \\pm \frac{1}{\sqrt{2\sqrt{f(k)}}} \\frac{2\sqrt{f(p-k)}}{w} \\pm \frac{1}{\sqrt{2\sqrt{f(k)}}} \end{bmatrix} \quad (B7)
\]

with

\[
f(k) = \frac{1}{M2^{k-1}} \left(\begin{array}{c} M \\kappa \end{array}\right) \quad (B8)
\]

and

\[
\Theta^{(k^+)} = \sqrt{f(p-k)}Q_{ab}^{(k)} - \sqrt{f(k)}Q_{ab}^{(p-k)} \quad (B9)
\]

\[
\Theta^{(k^-)} = \sqrt{f(p-k)}Q_{ab}^{(k)} + \sqrt{f(k)}Q_{ab}^{(p-k)} \quad (B10)
\]

In the present case, since \( p \) is odd, the only overlaps we need to consider are \( Q_{ab}^{(\pm 1)} \) and \( Q_{ab}^{(\pm 2)} \). Their self-consistency equation can be written in the form:

\[
\mathcal{A} \begin{bmatrix} Q_{ab}^{(\pm 1)} \\ Q_{ab}^{(\pm 2)} \end{bmatrix} = \begin{bmatrix} F_{\pm 1}(Q) \\ F_{\pm 2}(Q) \end{bmatrix} \quad (B11)
\]

with

\[
\mathcal{A} = \begin{bmatrix} 1 & -\beta^2 f(\frac{\pm 1}{2}) \\ -\beta^2 f(\frac{\pm 1}{2}) & 1 \end{bmatrix} \quad (B12)
\]

The functions \( F_{(p+1)/2} \) are two polynomials of degree two in all the \( Q^{(k)} \)'s. However, as mentioned above, in order to study the nature of the critical behavior (continuous or discontinuous) we only need the terms relevant at the highest critical temperature, cf. Eq. (27), and we, thus, set to zero all the overlap matrices except for \( Q_{ab}^{(\pm 1)} \) and \( Q_{ab}^{(\pm 2)} \).

Depending on the parity of \((p + 1)/2\) the relevant terms contributing to the nonlinear couplings in the action Eq. (29) differ. We now consider the two cases separately.

1. \((p + 1)/2\) even

If \( p = 4a - 1 \) with \( a \in \mathbb{N} \) the functions on the r.h.s. of Eq. (B11) read:

\[
F_{p+1}(Q) = \frac{\beta^4}{M2^{p-1}} \sum_{c=1}^{n} \left( \begin{array}{c} M \\frac{p+1}{2} \end{array}\right) Q_{ac}^{(\pm 1)} Q_{cb}^{(\pm 1)}
\]

\[
\quad + \frac{\beta^4}{M2^{p-1}} \left( \begin{array}{c} M \\frac{p+1}{2} \end{array}\right) \left( \begin{array}{c} M \\frac{p+1}{2} \end{array}\right) Q_{ab}^{(\pm 1)} Q_{ab}^{(\pm 1)}
\]

\[(B13)\]
\[ F_{\pm 1}(Q) = \frac{\beta^4}{M^{\pm p-\frac{1}{2}}} \left( M \right) \sum_{c=1}^{n} Q_{ac}^{(\pm 1)} Q_{cb}^{(\pm 1)} \]
\[ + \frac{\beta^4}{2M^{\pm p-\frac{1}{2}}} \left( M \right) \left( M - \frac{p+1}{2} \right) \left( M - \frac{p-1}{2} \right) \left( Q_{ab}^{(\pm 1)} \right)^2 \]
\[ + \frac{\beta^4}{2M^{\pm p-\frac{1}{2}}} \left( M \right) \left( M - \frac{p+1}{2} \right) \left( M - \frac{p-1}{2} \right) \left( Q_{ab}^{(\pm 1)} \right)^2 \]
\[ (B14) \]

b. \( (p-1)/2 \) even

If, otherwise, \( p = 4a + 1 \) with \( a \in \mathbb{N} \), one obtains

\[ F_{\pm 1} = \frac{\beta^4}{M^{\pm p-\frac{1}{2}}} \left( M \right) \sum_{c} Q_{ac}^{(\pm 1)} Q_{cb}^{(\pm 1)} \]
\[ + \frac{\beta^4}{2M^{\pm p-\frac{1}{2}}} \left( M \right) \left( M - \frac{p+1}{2} \right) \left( M - \frac{p-1}{2} \right) \left( Q_{ab}^{(\pm 1)} \right)^2 \]
\[ + \frac{\beta^4}{2M^{\pm p-\frac{1}{2}}} \left( M \right) \left( M - \frac{p+1}{2} \right) \left( M - \frac{p-1}{2} \right) \left( Q_{ab}^{(\pm 1)} \right)^2 \]
\[ (B15) \]

Computation of the coupling constants

In order to decouple Eqs. (B11)-(B12) we specify the two new variables, Eqs. (B9)-(B10), for \( k = (p-1)/2 \):

\[ \Theta_{ab}^{(+)} = \sqrt{f \left( \frac{p+1}{2} \right)} Q_{ab}^{(\pm 1)} - \sqrt{f \left( \frac{p-1}{2} \right)} Q_{ab}^{(\pm 1)} \]
\[ (B17) \]
\[ \Theta_{ab}^{(-)} = \sqrt{f \left( \frac{p+1}{2} \right)} Q_{ab}^{(\pm 1)} + \sqrt{f \left( \frac{p-1}{2} \right)} Q_{ab}^{(\pm 1)} \]
\[ (B18) \]

Applying the diagonalization transformation described above, cf. Eqs. (B1)-(B3), one finds

\[ \lambda^{(+)\Theta_{ab}^{(+)} = \sqrt{f \left( \frac{p+1}{2} \right)} F_{\pm 1} - \sqrt{f \left( \frac{p-1}{2} \right)} F_{\pm 1} \]
\[ (B19) \]
\[ \lambda^{(-)\Theta_{ab}^{(-)} = \sqrt{f \left( \frac{p+1}{2} \right)} F_{\pm 1} + \sqrt{f \left( \frac{p-1}{2} \right)} F_{\pm 1} \]
\[ (B20) \]

where the eigenvalues, cf. Eq. (B4), are

\[ \lambda^{(\pm)} = 1 \pm \beta^2 \sqrt{f((p-1)/2)f((p+1)/2)} \]

Since the \( F \)'s depend on the \( Q \)'s, we have to apply the inverse transformation to get equations in terms of the \( \Theta \)'s. The eigenvalue \( \lambda^{(+)} \) is always positive, so that \( \Theta^{(+)} \) plays the same role of the “non critical” overlaps and can be put to zero. The inverse transformation, thus, reduces to

\[ Q_{ab}^{(\pm)} = \frac{\Theta_{ab}^{(-)}}{2 \sqrt{f \left( \frac{p-1}{2} \right)}} \]
\[ (B21) \]
\[ Q_{ab}^{(\pm)} = \frac{\Theta_{ab}^{(-)}}{2 \sqrt{f \left( \frac{p-1}{2} \right)}} \] so that Eq. (B20) decouples in

\[ \lambda^{(-)\Theta_{ab}^{(-)} = w_1(p, M) \sum_{c} \Theta_{ac}^{(-)} \Theta_{cb}^{(-)} + w_2(p, M) (\Theta_{ab}^{(-)})^2 . \]
\[ (B22) \]

The constants \( w_1 \) and \( w_2 \) depend on \( p \) and \( M \). The expression for \( w_1 \) is

\[ w_1(p, M) = \frac{1}{4M^{p-3/2}} \left( \sqrt{\left( \frac{M}{p-1} \right)} + \sqrt{\left( \frac{M}{p+1} \right)} \right) \]
\[ (B23) \]

The formula for \( w_2 \) changes depending on the parity of \((p+1)/2\). For even \((p+1)/2\):

\[ w_2(p, M) = \frac{1}{8M^{p-3/2}} \sqrt{\left( \frac{M}{p+1} \right)} \]
\[ \times \left[ \left( \frac{p+1}{p+1} \right) \left( M - \frac{p+1}{2} \right) + \left( \frac{p+1}{p+1} \right) \left( M - \frac{p-1}{2} \right) + 2M - p + 1 \right] \]
If \((p + 1)/2\) is odd it reads:

\[
\begin{align*}
    w_2(p, M) &= \frac{1}{8M^{p-\frac{1}{2}}} \left( \frac{M}{p+1} \right)^{\left( \frac{p-1}{2} \right)} \\
    &\times \left[ \left( \frac{p-1}{4} \right) \left( \frac{M - p-1}{p-1} \right) \\
    &+ \frac{1 + p}{2M - p + 1} \left( \frac{p-1}{2} \right) \left( \frac{M - p-1}{p+1} \right) \\
    &+ 2 \left( \frac{p+1}{2} \right) \left( \frac{M - p-1}{p+1} \right) \right]
\end{align*}
\]

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