A NOTE ON THE NONEXISTENCE OF QUASI-HARMONIC SPHERES

JIAYU LI AND LINLIN SUN

ABSTRACT. In this paper we study the properties of quasi-harmonic spheres from $\mathbb{R}^m, m > 2$. We show that if the universal covering $\tilde{N}$ of $N$ admits a nonnegative strictly convex function $\rho$ with the exponential growth condition $\rho(y) \leq C \exp\left(\frac{1}{4}d(y)^{2/m}\right)$ where $d(y)$ is the distance function on $\tilde{N}$, then $N$ does not admit a quasi-harmonic sphere, which generalize Li-Zhu’s result [8]. We also show that if $u$ is a quasi-harmonic sphere, then the property that $u$ is of finite energy $\left(\int_{\mathbb{R}^m} e(u)e^{-|x|^2/4} \, dx < \infty\right)$ is equivalent to the property that $u$ satisfies the large energy condition $\left(\lim_{R \to \infty} R^m e^{-R^2/4} \int_{B_R(0)} e(u)e^{-|x|^2/4} \, dx = 0\right)$.

1. Introduction

Let $M^m, N^n$ be two compact Riemannian manifolds of dimension $m$ and $n$ respectively. Let $u \in W^{1,2}(M, N)$, the energy of $u$ is defined by

$$E(u) = \frac{1}{2} \int_M |du|^2 \, d\text{Vol}_M.$$ 

The critical points of the energy functional are called harmonic maps. Eells and Sampson [4] introduce the heat flow and prove that, the heat flow has a global solution which subconverges strongly to a harmonic map at infinity if the sectional curvature of the target manifold is non-positive. This result was generalized by Ding and Lin [3] to the case that the universal covering of $N$ admits a nonnegative strictly convex function with quadratic growth.

However, in general, the heat flow may produce singularities at a finite time (e.g. [1,2]). Struwe divided singularities of the heat flow into two different types. One of this type is associated to quasi-harmonic spheres (c.f. [9]).

Definition 1.1. A quasi-harmonic sphere is a harmonic map from $\big(\mathbb{R}^m, \exp(-x^2/2(m-2))g_0\big)$ to a Riemannian manifold, where $g_0$ is the Euclidean metric in $\mathbb{R}^m$ ($m > 2$), i.e.,

(1.1) \hspace*{1cm} \tau(u) = \frac{1}{2} x \cdot du,

with finite energy

(1.2) \hspace*{1cm} \int_{\mathbb{R}^m} e(u)e^{-|x|^2/4} \, dx < \infty,$
where
\[ e(u) = \frac{1}{2} |du|^2. \]

Based on the work of Lin and Wang\(^9\), we know that Liouville theorems for harmonic spheres
(harmonic maps from spheres) and quasi-harmonic spheres imply the global existence of the heat
flows. Li and Wang\(^6\) proved that there are no non-constant quasi-harmonic spheres with images
in a regular ball. Li and Wang\(^6\) proved that there are no non-constant quasi-harmonic spheres with images
in a regular ball. Li and Zhu\(^8\) proved that, if the heat flow has a global solution and there is
no harmonic map from \(S^l\) to \(N\) for \(2 \leq l \leq m - 1\), then this flow subconverges in \(C^2\) norm to
a smooth harmonic map at infinity. Moreover, in the same paper, they also proved that the heat
flow exists globally provided that the universal covering \(\tilde{N}\) of \(N\) admits a strictly convex positive
function \(\rho\) with polynomial growth, i.e.,
\[ \tilde{\nabla}^2 \rho > 0, \quad 0 < \rho(y) < C(1 + \tilde{d}(y, y_0))^p, \quad \forall y \in \tilde{N}, \]
for some \(y_0 \in \tilde{N}\) and some positive constants \(C, P\). Here \(\tilde{d}\) is the distance function on \(\tilde{N}\). Li
and Yang\(^7\) generalized these results to the case of “quasi-harmonic sphere with large energy
condition” under the same assumption on \(\rho\). The large energy condition is defined by
\[
\lim_{R \to \infty} R^m e^{-R^2/4} \int_{B_R(0)} e(u) e^{-|x|^2/4} \, dx = 0.
\]

Our first main result is as follows.

**Theorem 1.1.** Suppose \(u\) satisfies (1.1), then the following three conditions are equivalent to
each other.

(1) The large energy condition holds, i.e., (1.3) holds.

(2) \[ \int_{\mathbb{R}^m} |u_r|^2 |x|^{4-m} \, dx < \infty. \]

(3) The total energy is finite, i.e., (1.2) holds.

**Remark 1.1.** Li and Zhu\(^8\) stated the following estimate for quasi-harmonic sphere,
\[
\int_{B_R(0)} |du|^2 \, dx \leq CR^{m-2}, \quad \forall R > 0,
\]
where \(C\) is a constant independent of \(R\). As a consequence, this condition \((1.4)\) is equivalent to
\((1.2)\) and is also equivalent to the following condition
\[
\int_{\mathbb{R}^m} |du|^2 |x|^{2-m-\delta} \, dx < \infty
\]
for some or every \(\delta > 0\). In fact, one can get more, see Corollary 2.5.

Our second main result is that, Li-Zhu’s result holds, if the universal covering \(\tilde{N}\) of \(N\) admits
a nonnegative strictly convex function \(\rho\) with the following exponential growth condition: for
some constant \(C\),
\[
\rho(y) \leq C \exp \left( \frac{1}{4} \tilde{d}(y)^2/m \right), \quad \forall y \in \tilde{N}.
\]

\(^1\) We thank ZHU Xiangrong for pointing out this equivalent condition.
Here \( \dd(y) = \dd(y, y_0) \) is the distance function on \( \tilde{N} \) from some fixed point \( y_0 \in \tilde{N} \). It is easy to check that this assumption is weaker than the one in \([8]\).

**Theorem 1.2.** Suppose \( m \geq 3 \) and there is a nonnegative strictly convex function \( \rho \) on the universal covering of the target manifold \( N \) such that (1.5) holds. Then there is no non-constant quasi-harmonic sphere \( u \) from \( \mathbb{R}^m \) to \( N \).

2. Proof of Theorem 1.1

In this section, we derive some estimates and prove Theorem 1.1. Introduce

\[
H(r) := \int_{S^{m-1}} (|u_r|^2 - e(u)) \, d\theta, \quad \forall r > 0.
\]

We begin with the following Lemma.

**Lemma 2.1.** Suppose \( u \) satisfies (1.1). Then

1. either

\[
-R^2(m - 2) \int_{B_{\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx \leq H(R) \leq 0, \quad \forall R > 0,
\]

2. or there exists \( R_0 \geq \sqrt{2(m - 2)} \) such that

\[
H(R) \geq R^{2-2m} e^{R^2/2} R_0^{2m-2} e^{-R_0^2/2} H(R_0) > 0, \quad \forall R > R_0.
\]

Here \( S^{m-1} \) stands for the unit sphere in \( \mathbb{R}^m \) centering at 0 and \( B_R = B_R(0) \).

**Proof.** A direct computation gives (c.f. Lemma 3.3 in \([8]\))

\[
(2.3) \quad \frac{d}{dr} \int_{S^{m-1}} (|u_r|^2 - e(u)) \, d\theta - \int_{S^{m-1}} \left( \frac{2}{r} e(u) + \left( \frac{r}{2} - \frac{m}{r} \right) |u_r|^2 \right) d\theta = 0, \quad \forall r > 0.
\]

According to this identity, we get

\[
\frac{d}{dr} \int_{S^{m-1}} (|u_r|^2 - e(u)) \, d\theta + \frac{2}{r} \int_{S^{m-1}} (|u_r|^2 - e(u)) \, d\theta = \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 \, d\theta.
\]

From this formula, we know

\[
(2.4) \quad \frac{d}{dr} (r^2 H(r)) = r^2 \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 \, d\theta.
\]

Thus, \( r^2 H(r) \) is increase from \( \sqrt{2(m - 2)} \) to infinity, and is decrease from 0 to \( \sqrt{2(m - 2)} \). Setting \( C_0 := \sqrt{2(m - 2)} \), we get

\[
r^2 H(r) \geq C_0^2 H(C_0), \quad \forall r > 0.
\]

Again according to (2.3) to obtain

\[
\frac{d}{dr} \int_{S^{m-1}} (|u_r|^2 - e(u)) \, d\theta + \left( \frac{2(m-1)}{r} - r \right) \int_{S^{m-1}} (|u_r|^2 - e(u)) \, d\theta = \left( r - \frac{2(m-2)}{r} \right) \int_{S^{m-1}} \left( e(u) - \frac{1}{2} |u_r|^2 \right) \, d\theta,
\]
which implies

\[(2.5) \quad \frac{d}{dr} \left( r^{2m-2} e^{-r^2/2} H(r) \right) = r^{2m-2} e^{-r^2/2} \left( r - \frac{2m - 4}{r} \right) \int_{S^{m-1}} \left( e(u) - \frac{1}{2} |u_r|^2 \right) \, d\theta. \]

Hence, \( r^{2m-2} e^{-r^2/2} H(r) \) is increase from \( \sqrt{2(m-2)} \) to infinity, and is decrease from 0 to \( \sqrt{2(m-2)} \). It is obvious that

\[ r^{2m-2} e^{-r^2/2} \int_{S^{m-1}} \left( |u_r|^2 - e(u) \right) \, d\theta \to 0, \quad \text{as } r \to 0. \]

Moreover,

\[ \frac{d}{dr} \left( r^2 H(r) \right) \geq -(m-2) r \int_{S^{m-1}} |u_r|^2 \, d\theta, \]

which yields

\[ R^2 H(R) \geq -(m-2) \int_{B_R} r^{2-m} |u_r|^2 \, dx, \quad \forall R > 0. \]

Here we have used the fact

\[ \lim_{r \to 0} r^2 H(r) = 0. \]

Therefore,

\[ r^2 H(r) \geq C_0^2 H(C_0) \geq -(m-2) \int_{B_{C_0}} r^{2-m} |u_r|^2 \, dx, \quad \forall r > 0. \]

Now we can finish the proof of this Lemma. If we do not have (2.1), then there exists \( R_0 \geq \sqrt{2(m-2)} \), such that

\[ \int_{(R_0) \times S^{m-1}} \left( |u_r|^2 - e(u) \right) \, d\theta > 0, \]

then for every \( r > R_0 \),

\[ r^{2m-2} e^{-r^2/2} H(r) \geq R_0^{2m-2} e^{-R_0^2/2} H(R_0) > 0, \]

which means that (2.2) holds. \( \square \)

**Remark 2.1.** Suppose \( u \) satisfies (1.1), then

\[ -R^2 H(R) \leq (m-2) \int_{B_{\sqrt{2m-2}}} r^{2-m} |u_r|^2 \, dx, \]

\[ -R^{2m-2} e^{-R^2/2} H(R) \leq (m-2) \int_{B_{\sqrt{2m-2}}} r^{m-2} e^{-r^2/2} \frac{|u_r|^2}{r^2} \, dx, \]

\[ -R^m e^{-R^2/4} H(R) \leq (m-2) \int_{B_{\sqrt{2m-2}}} e^{-r^2/4} e(u) \, dx, \]

holds for all \( R > 0. \)
Proof. The proof of (2.6) and (2.7) can be found in the proof of Lemma 2.1. The proof of (2.8) can be proved similarly since (2.3) implies the following formula
\[
\frac{d}{dr} \left( r^m e^{-r^2/4} H(r) \right) = \left( \frac{r}{2} - \frac{m-2}{r} \right) r^m e^{-r^2/4} \int_{S^{m-1}} e(u) d\theta, \quad \forall r \in (0, \infty).
\]
\[\square\]

Lemma 2.2. Suppose \( u \) satisfies (1.1) and
\[
\liminf_{R \to \infty} R^{2m-2} e^{-R^2/2} \int_{[R] \times S^{m-1}} (|u_r|^2 - e(u)) d\theta > 0,
\]
then
\[
\liminf_{R \to \infty} R^m e^{-R^2/4} \int_{B_R} (|u_r|^2 - e(u)) e^{-r^2/4} dx > 0.
\]

Proof. A direct computation. \[\square\]

Next, we prove the following energy estimate.

Proposition 2.3. Suppose \( u \) satisfies (1.1), then there is a constant \( C_1 \) depending only on \( m \) such that for every \( 0 \leq \delta \leq 2 \), we have
\[
\int_{B_R} r^{4-m-\delta} |u_r|^2 \ dx \leq C_1 \int_{B_{2 \sqrt{m-2}}} r^{2-m} |u_r|^2 \ dx + 4R^2 H(R)^+, \quad \forall R > 0.
\]

Here \( f^+ = \max \{f, 0\} \).

Proof. We only consider the case \( R > 2 \sqrt{m-2} \) and start with the formula (2.4), i.e.,
\[
\frac{d}{dr} \left( r^2 H(r) \right) = r^2 \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 \ d\theta.
\]
For every \( 0 < \rho < R \), we have
\[
R^2 H(R) - \rho^2 H(\rho) = \int_{\rho}^{R} r^2 \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 \ d\theta \ dr
\]
\[
= \int_{B_R \setminus B_{\rho}} \left( \frac{r}{2} - \frac{m-2}{r} \right) r^{3-m} |u_r|^2 \ dx.
\]
For \( \sqrt{4(m-2)} \leq \rho < R \), we have
\[
\int_{B_R \setminus B_{\rho}} r^{4-m} |u_r|^2 \ dx \leq 4R^2 H(R)^+ - 4\rho^2 H(\rho),
\]
which implies
\[
\int_{B_R \setminus B_{2 \sqrt{m-2}}} r^{4-m} |u_r|^2 \ dx \leq 4R^2 H(R)^+ - 4 \left( 2 \sqrt{m-2} \right)^2 H \left( 2 \sqrt{m-2} \right)
\]
\[
\leq 4R^2 H(R)^+ + 4(m-2) \int_{B_{2 \sqrt{m-2}}} r^{2-m} |u_r|^2 \ dx.
\]
Here we have used (2.6). In particular, we get the desired estimate for $\delta = 0$. In general $0 \leq \delta \leq 2$, 
\[
\int_{B_R} r^{4-m-\delta} |u_r|^2 \, dx = \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{4-m-\delta} |u_r|^2 \, dx + \int_{B_{2\sqrt{m-2}}} r^{4-m-\delta} |u_r|^2 \, dx \\
\leq \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{4-m} |u_r|^2 \, dx + (2 \sqrt{m-2})^{2-\delta} \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx \\
\leq 8(m-2) \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+.
\]

As a consequence,

**Corollary 2.4.** Suppose $u$ satisfies (1.1). Then there is a constant $C_2$ such that for every $0 < \delta < 1$,
\[
\delta R^{-\delta} \int_{B_R} r^{2-m+\delta} e(u) \, dx \leq C_2 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+, \quad \forall R > 0.
\]

In particular,

(2.9) \[
R^{2-m} \int_{B_R} e(u) \, dx \leq C_2 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+, \quad \forall R > 0.
\]

**Proof.** Since
\[
\int_{B_R} r^{2-m+\delta} e(u) \, dx = -\int_0^R r^{1+\delta} H(r) \, dr + \int_{B_R} r^{2-m+\delta} |u_r|^2 \, dx \\
\leq \sup_{0 < r < R} (-r^2 H(r)) \times \int_0^R r^{\delta-1} \, dr + R^\delta \int_{B_R} r^{2-m} |u_r|^2 \, dx \\
= \sup_{0 < r < R} (-r^2 H(r)) \times \frac{R^\delta}{\delta} + R^\delta \int_{B_R} r^{2-m} |u_r|^2 \, dx.
\]

Now applying Lemma 2.1 and Proposition 2.3, there exists a constant $C_2$ depending only on $m$ such that
\[
\delta R^{-\delta} \int_{B_R} r^{2-m+\delta} e(u) \, dx \leq C_2 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+.
\]

Also, we can prove the following

**Corollary 2.5.** Suppose $u$ satisfies (1.1), then there is a constant $C_3$ depending only on $m$ such that for every $0 < \delta < 1$,
\[
\delta \int_{B_R} r^{2-m-\delta} e(u) \, dx \leq C_3 \int_{B_{2\sqrt{m-2}}} r^{1-m} e(u) \, dx + 4R^2 H(R)^+, \quad \forall R > 0.
\]
Proof. Similar to the proof of Corollary 2.4, for $0 < \delta < 1$ and $R > 2 \sqrt{m-2}$,

$$
\int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m-\delta} e(u) \, dx = - \int_{2\sqrt{m-2}}^{R} r^{1-\delta} H(r) \, dr + \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m-\delta} |u_r|^2 \, dx
\leq \sup_{2 \sqrt{m-2} < r < R} (-r^2 H(r)) \times \int_{2 \sqrt{m-2}}^{R} r^{\beta-1} \, dr + \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx
\leq \sup_{2 \sqrt{m-2} < r < R} (-r^2 H(r)) \times \frac{2 \sqrt{m-2}}{\delta} + \int_{B_R} r^{2-m} |u_r|^2 \, dx.
$$

Then Lemma 2.1 and Proposition 2.3 gives the desired estimate. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Suppose the large energy condition holds, i.e., the claim (1) is true. Then according to Lemma 2.1 and Lemma 2.2 (or c.f. [7]), we know that $H(r) \leq 0$ for every $r > 0$. Now the claim (2) follows from Proposition 2.3.

From the claim (2) to the claim (3), we need only to prove that

$$
\int_{\mathbb{R}^m} r^{2-m-\delta} |du|^2 \, dx < \infty.
$$

holds for some $\delta > 0$. According to Corollary 2.5, we need only to claim that $\liminf_{R \to \infty} R^2 H(R)^+ \leq 0$. This is true because

$$
\liminf_{R \to \infty} R^2 H(R)^+ \leq \liminf_{R \to \infty} \int_{[R] \times S^{m-1}} |u_r|^2 \, d\theta
$$

and the claim (2) implies the righthand is zero.

From the claim (3) to the claim (1) is obvious. □

3. Proof of Theorem 1.2

The following Lemma is proved in [8]. Here we provide another proof which is simpler for $m > 2$.

Lemma 3.1. Suppose $f$ is a non-constant nonnegative smooth function satisfying

$$
\Delta f \geq \frac{1}{2} r f_r,
$$

then there exists a constant $C > 0$ such that for $r$ large enough,

$$
\int_{S^{m-1}} f(r, \theta) \, d\theta > C r^{-m} e^{r^2/4}.
$$

Proof. Let

$$
v(r) = \int_{S^{m-1}} f(r, \theta) \, d\theta,
$$

then a direct computation yields

$$
\frac{d}{dr} \left( r^{m-1} e^{-r^2/4} \frac{d}{dr} v \right) \geq 0.
$$
Since $\frac{d v}{d r} = O\left(\frac{1}{r}\right)$ as $r \to 0$, we obtain
\[
\lim_{r \to 0} r^{m-1} e^{-r^2/4} \frac{d}{d r} v = 0,
\]
since $m > 2$. In particular,
\[
r^{m-1} e^{-r^2/4} \frac{d}{d r} v \geq 0.
\]
Since $f$ is not a constant, there exists $a > 0$ such that $\frac{d v}{d r} |_{a} > 0$. The rest of the proof is simple (c.f. [8]).

Let $d(x) = \text{dist}(u(x), u(0))$, then we have the following

**Lemma 3.2** (Refine energy estimate). Suppose $u$ is a quasi-harmonic sphere, then there is a constant $C_m$ depending only on $m$ such that for all $R > 0$,
\[
\int_{B_R} d^2 \, dx \leq C_m R^m \int_{B_{2R}} e^{-r^2/4} \, dx,
\]
\[
\int_{B_R} |\nabla d|^2 \, dx \leq C_m R^{m-2} \int_{B_{2R}} e^{-r^2/4} \, dx.
\]

**Remark 3.1.** (1) Denoted $E_R(u)$ by the energy of $u$ on $B_R$, i.e.,
\[
E_R(u) = \frac{1}{2} \int_{B_R} |d u|^2 e^{-x^2/4} \, dx.
\]
Then apply Corollary 2.5 to this Lemma to obtain the following estimate
\[
\int_{B_R} d^2 \, dx \leq C_m R^m E_R(u),
\]
\[
\int_{B_R} |\nabla d|^2 \, dx \leq C_m R^{m-2} E_R(u).
\]

(2) Li and Zhu (c.f. Lemma 3.2 in [8]) obtained a similar result with constant $C_{m,u}$ depending only on $m$ and the total energy of $u$ such that
\[
\int_{B_R} d^2 \, dx \leq C_{m,u} R^m,
\]
\[
\int_{B_R} |\nabla d|^2 \, dx \leq C_{m,u} R^{m-2}.
\]

**Proof of Lemma 3.2**  It is clear that
\[
d(r, \theta) \leq \int_{0}^{r} |u(x, \theta)| \, ds, \quad |\nabla d| \leq |d u|.
\]

Since the total energy of $u$ is finite, by Lemma 2.2 we have
\[
\int_{S^{m-1}} \left( |u| - e(u) \right) \, d\theta \leq 0, \quad r > 0.
\]
Applying (2.9), we obtain
\[ \int_{B_R} |\nabla d|^2 \leq 2C_2R^{m-2} \int_{B_2 \setminus B_1} r^{2-m} |u_r|^2 \, dx, \quad R > 0. \]

Next, we show
\[ \int_{S^{m-1}} \left( \int_0^r |u_s(s, \theta)| \, ds \right)^2 \, d\theta \leq C_m \int_{B_2 \setminus B_1} r^{1-m} |u_r|^2 \, dx, \quad \forall r > 0. \]

Then the first part of the this Lemma follows from this inequality. Without loss of generality, assume \( r > 1 \). Applying Proposition 2.3 and taking \( \delta = 1/2 \), we get
\[ \int_{B_R} r^{7/2-m} |u_r|^2 \, dx \leq C_1 \int_{B_2 \setminus B_1} r^{2-m} |u_r|^2 \, dx, \quad R > 0. \]

Using Minkowski’s inequality, we get
\[
\left( \int_{S^{m-1}} \left( \int_0^r |u_s(s, \theta)| \, ds \right)^2 \, d\theta \right)^{1/2} \leq \int_0^1 \left( \int_{S^{m-1}} |u_s(s, \theta)|^2 \, d\theta \right)^{1/2} \, ds \\
\leq \left( \int_0^1 \left( \int_{S^{m-1}} |u_s(s, \theta)|^2 \, d\theta \right)^{1/2} \, ds \right) + \left( \int_1^r \left( \int_{S^{m-1}} |u_s(s, \theta)|^2 \, d\theta \right)^{1/2} \, ds \right) \\
\leq \left( \int_0^1 \left( \int_{S^{m-1}} |u_s(s, \theta)|^2 \, d\theta \right)^{1/2} \, ds \right)^{1/2} \\
+ \left( \int_1^r s^{5/2} \int_{S^{m-1}} |u_s|^2 \, d\theta \, ds \right)^{1/2} \left( \int_1^r s^{-5/2} \, ds \right)^{1/2} \\
\leq C_m \left( \int_{B_2 \setminus B_1} r^{1-m} |u_r|^2 \, dx \right)^{1/2}.
\]

Lemma 3.3. Suppose \( u \) is a quasi-harmonic sphere, then there is a constant \( C_m \) depending only on \( m \) such that
\[ \int_{B_r} \exp \left( C_m^{-1} E_r(u)^{-1/2} r^{2-m} d \right) \, dx \leq C_m, \quad \forall r > 1. \]

Proof. By the energy estimate Corollary 2.5, using an argument similar to the one used in the proof of Lemma 3.5 in [5], we can prove that the BMO subnorm \([d]_{s,B_2} \) of \( d \) over \( B_2r \) satisfies
\[ [d]_{s,B_2} \triangleq \sup_{x \in B_2 \setminus B_1} \int_B |d(y) - d_0| \, dy \leq C_m \sqrt{E_2(u)} (1 + r)^{m-2}, \]

where the supermum is taken over all cubes \( x \in B_1 \subseteq B_2r \). The John-Nirenberg theorem (c.f. Lemma 1 in [5]) claims that there is two constants \( C_5, C_6 \) depends only on \( m \) such that for all cubes \( Q \subseteq B_2r \),
\[ \left| \left\{ x \in Q : |d(x) - d_0| > s \right\} \right| \leq C_5 \exp \left( -\frac{C_6 s}{[d]_{s,B_2}} \right) |Q|, \]
which implies
\[ \int_{B_r} \exp \left( C_6 \frac{|d - d_{B_r}|}{2|d|_{B_r}} \right) \, dx \leq C_5, \quad \forall r > 0. \]
Since we have the estimate (3.1), as a consequence, there is a constant \( C_7 \) which depends only on \( m \) such that
\[ \int_{B_r} \exp \left( C_7^{-1} E_r(u)^{-1/2} r^{2-m} |d - d_{B_r}| \right) \, dx \leq C_7, \quad \forall r > 1. \]
Finally, according to Lemma 3.2 we can find a constant \( C_8 \) depending only on \( m \) such that
\[ d_{B_r} := \int_{B_r} d \, dx \leq C_8 E_r(u)^{1/2}. \]
Therefore, we get the desired estimate. \( \square \)

**Remark 3.2.** Checking the proof of Lemma 3.5 in [8] step by step, and using the argument mentioned above, one can prove the following refine estimate,
\[ \int_{B_r} \exp \left( C_9^{-1} \tilde{E}_r(u)^{-1/2} r^{2-m} \right) \, dx \leq C_9, \quad \forall r > 1. \]
Here
\[ \tilde{E}_r(u) = \int_{B_r} r^{1-m} |u_r|^2 \, dx. \]
In fact, checking the proof (c.f. page 455 in [8]), the constants come from either Lemma 3.2 or \( \tilde{E}_{3m}(u) \) which can be controlled by \( \tilde{E}_{2 \sqrt{m-2}}(u) \) thanks to Corollary 2.5. Hence one can prove the required refine BMO estimate (3.1).

Now we give a proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \tilde{N} \) be the universal covering of \( N \). Let \( \tilde{u} : \mathbb{R}^m \to \tilde{N} \) be a lift of \( u \) with \( \tilde{u} = u \circ \pi \) where \( \pi : \tilde{N} \to N \) is the covering map. It is easy to see that
\[ \int_{\mathbb{R}^m} e(\tilde{u}) e^{-|x|^2/4} \, dx < \infty. \]
Set \( f = \rho \circ \tilde{u} \), then
\[ \Delta f - \frac{1}{2} r \partial_r f = \tilde{V}^2 \rho(\tilde{u})(d\tilde{u}, d\tilde{u}) > 0. \]
Fixed \( p > 0 \). Notice that there is a constant \( C > 0 \) such that
\[ \int_{B_{2R}} f^p \, dx = \int_{B_{2R}} (\rho \circ \tilde{u})^p \, dx \leq C^p \int_{B_{2R}} e^{\frac{\tilde{E}_{2\sqrt{m-2}}}{4}} \, dx, \quad R > 0. \]
Applying Young’s inequality,
\[ A + B \geq (PA)^{1/p} (QB)^{1/Q}, \quad A, B > 0, \quad P, Q \geq 1, \quad 1/P + 1/Q = 1, \]
we obtain that for $\tilde{\delta} = p/(2m)$,

$$
\tilde{\delta} r^{2-m} \tilde{d} + \left(\frac{p}{4} - \tilde{\delta}\right) r^2 = \frac{p}{2m} r^{2-m} \tilde{d} + \left(\frac{p}{4} - \frac{p}{2m}\right) r^2 \\
= \frac{p}{4} \left(\frac{2}{m} r^{2-m} \tilde{d} + \frac{m-2}{m} r^2\right) \\
\geq \frac{p}{4} \left(r^{2-m} \tilde{d}\right)^{2/m} \left(r^2\right)^{(m-2)/m} \\
= \frac{p}{4} d^{2/m}.
$$

Therefore, according to (3.2), for $R > 0$, we have

$$(3.3) \quad \int_{B_{2R}} f^p \, dx \leq C_p \int_{B_{2R}} e^{\tilde{\delta} r^{2-m} \tilde{d}(\tilde{u},y_0)} e^{(p/4-\tilde{\delta})R^2} \, dx = C_p \int_{B_{2R}} e^{2m^{-2} \tilde{d}(2R)^{-m} \tilde{d}(\tilde{u},y_0) e^{(p/4-\tilde{\delta})R^2}} \, dx.$$

We can choose $p > 0$ sufficiently small so that

$$2^m \tilde{\delta} = 2^m \frac{1}{m} p \leq C_m E^{-1/2},$$

which is equivalent to

$$E \leq \frac{m^2}{4^{m-3}C_m^2 p^2}.$$

According to Lemma 3.3 and (3.3), we can see that

$$\int_{B_{2R}} f^p \, dx \leq C_p e^{(p/4-\tilde{\delta})R^2} \int_{B_{2R}} \exp \left(C_m^{-1/2} (2R)^{-m} \tilde{d}(\tilde{u},y_0)\right) \, dx \leq C_p C_m (2R)^m e^{(p/4-p/(2m))R^2}$$

holds for $R$ large enough.

If $f$ is not a constant, applying Lemma 3.1 we obtain that for $R$ large enough,

$$\int_{B_R} f \, dx \geq C_u R^{-2} e^{R^2/4}.$$

Here $C_u > 0$ is a constant which is independent of $R$. Since $f \geq 0$ satisfies

$$\text{div} \left( e^{-|x|^2/4} \nabla f \right) \geq 0,$$

applying Moser’s iteration (c.f. page 167 in [2]), for every $p > 0$, there is a constant $C_p > 0$ depending only on $p, m$ such that

$$\int_{B_R} f \, dx \leq C_p R^{m/p} \left( \int_{B_{2R}} f^p \, dx \right)^{1/p}$$

holds for $R$ large enough. Consequently, for $R$ large enough

$$(3.4) \quad \int_{B_{2R}} f^p \, dx \geq C_p R^{-p} C_u R^{-(m+2)p-m} e^{p R^2/4}.$$

Together with (3.3) and (3.4), we know that

$$0 < C_p^{-1} C_u \leq C_p^{-1} C_m 2^m R^{2m+(m+2)p} e^{-p R^2/(2m)} \to 0, \quad \text{as} \ R \to \infty.$$

This contradiction means that $f$ is a constant. Moreover, since $\rho$ is a strictly convex function, we get that $d\tilde{u} = 0$, i.e., $\tilde{u}$ is a constant. As a consequence, $u$ is a constant.
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School of Mathematics Sciences, University of Science and Technology of China, 230026 Hefei, Anhui, China
E-mail address: jiayuli@ustc.edu.cn

School of Mathematics Sciences, University of Science and Technology of China, 230026 Hefei, Anhui, China
E-mail address: sunll@ustc.edu.cn