Some Reflections on the Status of Conventional Quantum Theory when Applied to Quantum Gravity

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All current approaches to quantum gravity employ essentially standard quantum theory including, in particular, continuum quantities such as the real or complex numbers. However, I wish to argue that this may be fundamentally wrong in so far as the use of these continuum quantities in standard quantum theory can be traced back to certain a priori assumptions about the nature of space and time: assumptions that may be incompatible with the view of space and time adopted by a quantum gravity theory. My conjecture is that in, some yet to be determined sense, to each type of space-time there is associated a corresponding type of quantum theory in which continuum quantities do not necessarily appear, being replaced with structures that are appropriate to the specific space-time.

Topos theory then arises as a possible tool for ‘gluing’ together these different theories associated with the different space-times. As a concrete example of the use of topos ideas, I summarise recent work applying presheaf theory to the Kochen-Specher theorem and the assignment of values to physical quantities in a quantum theory.

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1 Introduction

The period in the late 1960’s when I was a postgraduate student at Imperial College saw rapid changes in theoretical physics in response to data streaming from the world’s particle accelerators. One consequence was that the subject matter of a student’s PhD thesis sometimes changed uncomfortably rapidly during the course of his or her studies. As a result, there was a tendency for supervisors to assign a provisional thesis title like “Topics in elementary particle physics”—a practice that was understandable, but which was finally blocked by the University of London some years ago!

When asked to speak at this Symposium in honour of Stephen Hawking, I adopted a similar tactic by choosing as the provisional title “Prima facie questions in quantum gravity”, on the grounds that this would give maximum flexibility when it came to actually write the talk. However, in the event, I have chosen to focus on one single issue, and the title of my lecture has been readjusted accordingly.

The question I wish to address is the extent to which ideas of standard quantum theory are adequate for the formulation of a quantum theory of gravity: in particular, in regard to (i) the use of continuum quantities in the mathematical foundations of quantum theory; and (ii) possible roles for topos theory. In this context it should be emphasised that all the current mainstream approaches to quantum gravity use standard quantum theory in one form or another.

Of course, it is well understood that, at a conceptual level, the standard interpretation of quantum theory is inadequate when, for example, applied to quantum cosmology. Specifically, the lack of any external observer of the universe ‘as a whole’ throws into doubt the instrumentalism of the Copenhagen interpretation; as does any attempt to construct a quantum gravity theory with no background space-time in which an ‘observer’ could be placed. The extent to which such reservations apply to quantum gravity away from the cosmological regime is still debated, but in practice most work on quantum gravity pays only lip service to these conceptual issues.

However, what I have in mind are not conceptual issues per se but rather certain mathematical ingredients in the formalism of quantum theory that are invariably taken for granted and yet which, I claim, implicitly assume certain properties of space and time that may be fundamentally incompatible with the spatio-temporal concepts needed for a successful quantum gravity theory.
An example of particular interest is the use of the *continuum* (via the real or complex numbers).

More generally, one can question the almost universal assumption that spatio-temporal concepts are to be implemented mathematically using standard point-set theory: this notwithstanding the frequently-voiced objection that the literal idea of a space, or time, point is physically meaningless. In fact, there exists something—namely topos theory—that can replace set theory as the foundation of mathematics, and which could arguably be a more appropriate way of modelling spatio-temporal concepts in physical regimes where quantum gravity effects are paramount. As we shall see, topos theory is also relevant to questions concerning the status of the continuum. These considerations have motivated my focussing the lecture on two main areas: (i) the *a priori* status of spatio-temporal concepts in quantum theory, particularly in regard to the use of continua; and (ii) certain possible roles for topos theory in theoretical physics.

Considerations of this type are part of the general question of the role of standard spatio-temporal concepts in a theory of quantum gravity. In the current major quantum gravity programmes, most of these concepts are inserted by hand as part of the overall background structure of the theory. On the other hand, there is a school of thought that maintains that the standard ideas of space and time should ‘emerge’ from the theory only in some appropriate limit or physical regime; in which case, a crucial question is whether the theory contains *any* fundamental concepts/structures that can be be broadly identified as ‘spatio-temporal’, or if all such concepts or structures are emergent in some way. One of the attractions of the consistent-histories approach to quantum theory (of which more later) is that it allows for the idea of emergent structures in a natural way via the process of coarse-graining: analogous to how thermodynamical concepts arise from statistical physics when microscopic details of the system are ignored.

*En passent*, one might ask what else could arise from the fundamental theory in some appropriate physical limit. This might include the entire mathematical formalism of standard quantum theory, including its use of Hilbert spaces defined over $\mathbb{R}$ or $\mathbb{C}$. Perhaps the conceptual structure of standard quantum theory is also an emergent structure: in particular, the special role for measurement, and the use of probabilities that lie in the closed interval $[0,1]$ of the real numbers. Certainly, there is no compelling logical reason why whatever plays the role of standard quantum theory at
the Planck length—if, indeed, there is any such theory—should possess all the features of the theory that is known to work empirically only at atomic and nuclear scales.

However, one of the key questions of interest in the present paper is not how the standard ideas of space and time (and probability) might emerge from a different formalism; but rather how one might proceed to construct a quantum theory \textit{ab initio} in which whatever fundamental spatio-temporal concepts are present are definitely not the familiar continuum ones: for example, if one is given a finite causal set as a background structure. The first step, and the only one taken in the present paper, is to sound a cautionary note by emphasising how strongly the continuum ideas of space and time are implicitly embedded in the standard formulation of quantum theory.

The plan of the paper is as follows. In Section 2 there is a discussion of the role of continuum concepts in the formulation of quantum theory in the presence of a non-standard background (such as a causal set \cite{2}, \cite{3}). The conclusion of the discussion is that, in some appropriate sense, there may be a different \textit{type} of quantum theory for each type of background space-time.

If this is indeed the case, the question then arises as to how these different theories are to be ‘patched’ together in a true quantum gravity theory in which these backgrounds are themselves subject to quantum effects. One possibility is the use of \textit{topos} theory. In Section 3 we discuss other ways in which the standard notion of space-time may change and, again, find a possible role for topos theory.

Since topos theory is an important mathematical ingredient in our considerations, one part of the subject—the theory of presheaves—is introduced in Section 4. It is then shown in Section 5 how this can be used in a natural way to illustrate certain key features (specifically the Kochen-Specker theorem) of standard quantum theory. The main physical idea here is a role for contextual, multi-valued logic—an idea that in itself has many possible fruitful applications in theoretical physics.

2 The Danger of \textit{A Priori} Assumptions
2.1 The use of the real and complex numbers in quantum theory

The use of the real and complex numbers is a basic feature of all approaches to quantum theory: Hilbert spaces of states, $C^*$-algebras of observables, quantum logic of propositions, functional integral methods, etc., etc. These number systems have a variety of relevant mathematical properties, but the one of particular interest here is that they are continua, by which—in the present context—is meant not only that $\mathbb{R}$ and $\mathbb{C}$ have the appropriate cardinality, but also that they come equipped with the familiar topology and differential structure that makes them manifolds of real dimension one and two respectively.

My concern is that the use of these numbers may be problematic in the context of a quantum gravity theory whose underlying notion of space and time is different from that of a smooth manifold. The danger is that by imposing a continuum structure in the quantum theory a priori, one may be creating a theoretical system that is fundamentally unsuitable for the incorporation of spatio-temporal concepts of a non-continuum nature: this would be the theoretical-physics analogue of what a philosopher might call a ‘category error’. For this reason, it is important to consider carefully the origin, and role, in standard quantum theory of this particular facet of the real (and complex) numbers.

In general terms, the real numbers arise in three ways in physical theories: (i) as the values of physical quantities; (ii) to model space and time; and (iii) as the values of probabilities. Our present task is to consider more precisely the use real numbers in quantum theory in these terms.

As a first step, consider the simple example with which most undergraduate courses on quantum theory begin: a non-relativistic point particle moving in one dimension. The state of the system at a time $t$ is represented by a wave function $\psi_t(x)$, and we see at once that continuum quantities are involved in three ways: (i) as the argument $x$ in the wave function; (ii) as the value of the wave function; and (iii) as the time parameter $t$. Let us consider these in turn.
2.1.1 The $x$ in $\psi(x)$

From one perspective, the $x$ in $\psi(x)$ arises because we are starting with a classical theory and then ‘quantising’ it. In the present example, the classical configuration space $Q$ is identified with the real line because the system is a point particle moving in (one-dimensional) physical space, and the latter is modelled by the real numbers.

In general, the configuration space (if there is one) $Q$ for a classical system is modelled mathematically by a differentiable manifold, and the classical state space is the co-tangent bundle $T^*Q$. The physical motivation for using a manifold to represent $Q$ again reduces to the fact that we represent physical space with a manifold. This is clearly so for configurations that correspond to the position of the centre-of-mass of an object in space, or its overall orientation in space, but it also applies to internal degrees of freedom of relative positions of constituent entities.

Thus, in assuming that the state space of a classical system is a manifold of the form $T^*Q$ we are importing into the classical theory a powerful a priori picture of physical space: namely, that it is a differentiable manifold. This then carries across to the corresponding quantum theory. For example, if ‘quantisation’ is construed to mean defining the quantum states to be cross-sections of some flat vector bundle over $Q$, then the domain of these state functions is the continuum space $Q$.

However, for this argument to have any force we need to consider why quantisation is so defined, and this takes us to the issue of the space in which the wave function has its values.

2.1.2 The value of the function $\psi(x)$—the role of classical physical quantities.

In the example of the quantum theory of a particle moving in one-dimension, the value of the state function $\psi(x)$ is a complex number: so, once again, there may be cases where $S$ is a symplectic manifold that is not a cotangent bundle; for example, $S := S^2$. However, I would argue that the reason $S$ is assumed to be a manifold is still ultimately grounded in an a priori assumption about the nature of physical space (and time).

The bundle is chosen to be flat so that a covariant derivative of sections can be defined without the need to introduce extra local ‘connection’ variables into the theory.
a continuum concept arises. This particular one comes from two different sources.

On the one hand, the operator \( \hat{x} \) that represents the position of the particle acts on the state function as

\[
(\hat{x}\psi)(x) := x\psi(x).
\]  

(1)

More generally, for a system with a configuration manifold \( Q \), a classical physical configuration quantity corresponds to a real-valued function \( f : Q \to \mathbb{R} \), and this function is represented in the quantum theory by the operator \( (\hat{f}\psi)(q) := f(q)\psi(q) \).

(2)

on sections of the appropriate vector bundle.

Although this equation does not prove that \( \psi(q) \) is \( \mathbb{C} \)-valued, it does show that, for each \( q \in Q \), the space in which \( \psi(q) \) takes its values must be such that it admits a multiplication operation by real numbers.

We see that this particular sources of the real numbers in quantum theory comes from the assumption that classical physical quantities are real-valued, which is then translated into an analogous requirement on the quantum variables.

A related feature is that in any quantum system the eigenvector equation for a physical quantity \( A \) is of the form \( \hat{A}|a\rangle = a|a\rangle \), where \( a \) is a real number. So the state space has to be such that its elements can be multiplied by real numbers. Note that this applies even to quantum physical quantities that have no classical analogue: we still assume that their eigenvalues are real numbers. Of course, for many quantities the set of all eigenvalues will be a discrete subset of \( \mathbb{R} \), but that does not detract from the point being made here.

It is thus pertinent to ask why physical quantities—classical or quantum—are taken to be real-valued. Many will doubtless say that the answer is obvious, or that it is even part of the definition of a physical quantity, but I would challenge these assertions as being over-hasty.

One reason why the values of physical quantities are assumed to be real numbers is undoubtedly the operational one that—at least, in the pre-digital age—physical quantities are ultimately measured with rulers and pointers, and so it is the assumed continuum nature of physical space that comes into play.
However, it is by no means obvious that physical quantities should necessarily be real-valued in, for example, a quantum gravity theory in which it is not appropriate to think of space as a smooth manifold, and where, therefore, there is no place for operational considerations that presuppose a continuum nature for space and/or time.

Of course, it is a totally open question as to what should replace $\mathbb{R}$ as the value space of a physical quantity in these circumstances—it could be something as obvious as a finite number field, but it could also be something far more radical. In any event, a key role in deciding this issue should be played by any underlying spatio-temporal concepts (albeit, non-standard) that are present.

### 2.1.3 The value of the function $\psi(x)$ — the role of probability.

A different source of the $\mathbb{C}$-valued nature of the wave-function is its probabilistic interpretation. Of course, this extends outside simple wave mechanics, with the general quantum-theory result that if $\hat{E}(A \in \Delta)$ is the spectral projector onto the eigenspace of $\hat{A}$ with eigenvalues in the (Borel) set $\Delta \subset \mathbb{R}$ then, if the (normalised) state is $\psi$, the probability that the proposition “The physical quantity $A$ lies in $\Delta$” is true is

$$\text{Prob}(A \in \Delta; \psi) = \langle \psi, \hat{E}(A \in \Delta) \rangle.$$  \hfill (3)

From our present perspective, the key point is that the assumption that probabilities should lie in the interval $[0, 1]$ of the real numbers requires the field over which the Hilbert state space is defined to be such as to accommodate this assumption via the right hand side of Eq. (3). So this is yet another source of the use of continuum quantities in the mathematical formulation of quantum theory.

In the context of standard physics, it is clear why probabilities are required to lie in the interval $[0, 1]$. As physicists, we most commonly employ a relative-frequency interpretation of probability in which an experiment is repeated a large number, say $N$, times, and the probability associated with a particular result is then defined to be the ratio $N_i/N$, where $N_i$ is the number of experiments in which that result was obtained. The rational numbers

\[4\] Of course, in the standard Copenhagen interpretation of quantum theory it would be more appropriate to say that the proposition represented by the spectral projector $\hat{E}(A \in \Delta)$ is “If a measurement is made of $A$, then the value will be found to lie in $\Delta$.  

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\(N_i/N\) necessarily lie between 0 and 1, and if we take the limit as \(N \to \infty\), as is appropriate for a hypothetical ‘infinite ensemble’, we get real numbers in the closed interval \([0, 1]\).

Although the relative-frequency interpretation of probability may seem natural in standard physics, it is not meaningful in situations where there is no classical spatio-temporal background in which observations could be made; or, if there is a background, it is such that there is no meaningful analogue of the relative-frequencies interpretation adapted to that background.

Under such circumstances it might be more natural to follow Aristotle, Heisenberg and Popper in adopting a \textit{propensity} interpretation of probability, perhaps within the context of a ‘post-Everett’ form of quantum theory, such as consistent-histories theory.

However, if probability is viewed in this more realist way, there is no overwhelming reason for assigning its values to be real numbers lying in the interval \([0, 1]\). The minimal requirement is presumably only that the value space should be a partially ordered set \((\mathcal{V}, \leq)\) so that it makes sense to say that certain events are more, or less, probable (in the sense of the partial-ordering operation \(\leq\)) than others.

Note that this allows for the possibility of pairs of events whose propensities are incomparable: \textit{i.e.}, the probability value-space \(\mathcal{V}\) may not be a \textit{totally}-ordered set. We would, however, expect there to be a unit element \(1 \in \mathcal{V}\), corresponding to the probability of an event that is certain to happen (or the proposition that is identically true), and with \(p \leq 1\) for all \(p \in \mathcal{V}\). Similarly, there should be a null element \(0 \in \mathcal{V}\), corresponding to the probability of an event that is certain not to happen (or the proposition that is identically false), and with \(0 \leq p\) for all \(p \in \mathcal{V}\).

It also seems natural to require that \(\mathcal{V}\) has some ‘semi-additive’ structure so that the probability of two disjoint events is the ‘sum’ of the probabilities of the individual events. At the very least, if \(P\) is any proposition and \(\neg P\) is its negation, we would expect the probability of \(P \lor \neg P\) to be the unit element \(1 \in \mathcal{P}\), and equal to the ‘sum’ of the probabilities of \(P\) and \(\neg P\).

Of course, it is an open question as to what precise mathematical structure should be used as the value-space for probabilities in the absence of any classical spatio-temporal background; or, indeed, what it should be in the

\[\text{This would not be so if for some reason the quantum propositions obeyed an intuitionistic logic (see later) where the principle of excluded middle does not necessarily apply.}\]
presence of a non-standard background such as a causal set. But the key point is that there is no fundamental reason why this value-space has to involve the real numbers; and the form of quantum theory in such a situation should reflect this fact.

### 2.1.4 The t in $\psi_t(x)$

The time-parameter $t$ in the wave-function is taken directly from the corresponding parameter in classical, non-relativistic physics. It is Newtonian time, and as such it is part of the background structure of standard Newtonian physics. It is represented by a real number: indeed, the full manifold structure of $\mathbb{R}$ (and of the classical state space) is invoked when defining the differential equations of motion of classical physics.

In relativistic physics, space and time are placed on a more equal footing, with a background space-time manifold rather just a background time. In special relativity, this background manifold has the topological and differential structure of $\mathbb{R}^4$, and is equipped with the, fixed, Minkowskian metric tensor.

Things change considerably when we come to the space and time of general relativity: indeed in the context of quantum gravity, time is a difficult concept—in particular, there is the well-known ‘problem of time’ that affects all approaches to quantum gravity in one way or another.

This problem was first explicitly encountered in the context of the canonical approach to quantum gravity, whose central feature is the constraint equations on the state vector $\Psi$

\[
\hat{H}_i(x)\Psi = 0 \\
\hat{H}_\perp(x)\Psi = 0
\]

where $\mathcal{H}_i$ and $\mathcal{H}_\perp$ are constructed from the metric tensor $g$ (and its conjugate variable) on an underlying 3-manifold $\mathbb{M}$. $\mathbb{M}$.

Equation (4) simply asserts the invariance of $\Psi$ under (small) spatial diffeomorphisms. However, equation (5) is more problematic. In the representation in which $\Psi$ appears as a functional $\Psi[g]$, eq. (5) is the Wheeler-DeWitt equation, and—in one approach to the ‘problem of time’—is interpreted as a dynamical equation with respect to an ‘internal’ time variable that has to be constructed from the metric tensor and its conjugate. It is always assumed
that this variable will be represented by a real number: indeed, the internal time is usually sought from the perspective of classical canonical general relativity—which is bound to lead to a real quantity. So once again we see how a priori assumptions about the nature of time can be placed into the quantum theory from the outset. (Of course, the canonical approach already comes with an explicit background spatial manifold.)

2.2 Space-time dependent quantum theory

The main conclusion I wish to draw from the discussion above is that a number of a priori assumptions about the nature of space and time are present in the mathematical formalism of standard quantum theory, and it may therefore be necessary to seek a major restructuring of this formalism in situations where the underlying spatio-temporal concepts (if there are any at all) are different from the standard ones which are represented mathematically with the aid of differential geometry.

A good example would be to consider from scratch how to construct a quantum theory when space-time is a finite causal set: either a single such— which then forms a fixed, but non-standard, spatio-temporal background—or else a collection of such sets in the context of a type of quantum gravity theory. In the case of a fixed background, this new quantum formalism should be adapted to the precise structure of the background, and can be expected to involve a substantial departure from the standard formalism: particularly in regard to the use of real numbers as the values of physical quantities and probabilities.

The fundamental emphasis in a causal set is on a space-time structure as a single unit, rather than separate space and time structures, and this suggests strongly that it would be better to start ab initio with a history theory rather than one in which some type of ‘temporal slicing’ is introduced. It should be emphasised that the path-integral approach to standard quantum theory is not a history theory in the way the phrase is being used here. Indeed, a path integral generates transition amplitudes between canonical states, which implicitly requires some type of time slicing. In fact, the only genuine ‘history’ theory I know that can handle space-time structures as integral entities is the consistent-histories formalism of Griffiths [8], Omnes [9] and Gell-Mann and Hartle [10].

Thus an instructive research programme would be to develop a version
of consistent-history quantum theory that is appropriate for a background causal set. In this context, a particularly useful approach could be the Gell-Mann and Hartle method as axiomatised in the language of quantum temporal logic by Isham [9], and Isham and Linden [10], together with the completely new perspective on the role of time introduced by Savvidou [16]. Here one has an orthoalgebra $\mathcal{U}P$ of propositions about the history of the system, and a space $\mathcal{D}$ of ‘decoherence functions’ that are maps $d : \mathcal{U}P \times \mathcal{U}P \rightarrow \mathbb{C}$ and which encode both the dynamics and the initial conditions. From a physical perspective, if a proposition $\alpha \in \mathcal{U}P$ belongs to a consistent set, then $d(\alpha, \alpha)$ is interpreted as the probability that $\alpha$ is true in the context of that consistent set.

It follows from the discussion above, that if there is a background causal set the quantum history formalism should be such that this structure is reflected in (i) the choice of the space $\mathcal{U}P$ of propositions about the ‘universe’; and (ii) the choice of the space in which decoherence functions take their values, with an associated change in the mathematical representation of probability.

Finally, if it is indeed the case that, in some sense, to each background space-time there is associated a corresponding type of quantum theory, then the question arises as to how these different theories can be ‘patched’ together to give a quantum space-time theory in which the different backgrounds are themselves the subject of quantum effects. One possibility is the use of topos theory: in particular, the theory of presheaves which provides a powerful way of handling situations where there is a space of ‘contexts’ with respect to which individual structures are associated. For example, a context could be a causal set.

Topos theory is of potential interest in theoretical physics in a number of ways, and it will recur in much of what follows. For this reason, an introduction to some of the basic ideas is given in Section 4.

3 Alternative Conceptions of Spacetime

3.1 Points or Regions?

Doubts about the use of the continuum in present-day physical theories prompts one to consider more general alternative conceptions of space and
time. We turn now to briefly sketch two such, both of which involve topos theory, and which raise the even more iconoclastic idea that the use of set theory itself may be inappropriate for modelling space and time in the context of quantum gravity.

3.1.1 From points to regions

In standard general relativity—and, indeed, in all classical physics—space (and similarly time) is modelled by a set, and the elements of that set correspond to points in space. However, it is often claimed that the notion of a spatial (or temporal) point has no real physical meaning, and this motivates trying to construct a theory in which ‘regions’ are the primary concept. In such a theory, ‘points’—if they exist at all—would play a secondary role in which they are determined in some way by the regions (rather than regions being collections of points, as in standard set theory).

In fact, there are axiom systems for regions, some of whose models do not contain anything corresponding to points of which the regions are composed. As an example, consider a topological space $X$. The family of all open sets has the algebraic operations of conjunction, disjunction and negation defined by $O_1 \land O_2 := O_1 \cap O_2$; $O_1 \lor O_2 := O_1 \cup O_2$; and $\neg O := \text{int}(X - O)$ respectively; and with these operations, the open sets form a complete Heyting algebra, also known as a locale. Here, a Heyting algebra $H$ is defined to be a distributive lattice, with null and unit elements, that is relatively complemented, which means that to any pair $S_1, S_2$ in $H$, there exists an element $S_1 \Rightarrow S_2$ of $H$ with the property that, for all $S \in H$, we have $S \leq (S_1 \Rightarrow S_2)$ if and only if $S \land S_1 \leq S_2$.

Heyting algebras are thus a generalization of Boolean algebras. In particular, they need not obey the law of excluded middle, and so provide natural algebraic structures for intuitionistic logic. A Heyting algebra is said to be complete if every family of elements has a least upper bound. Thus, when partially ordered by set-inclusion, the open sets of any topological space form a Heyting algebra. This algebra is complete since arbitrary unions of open sets are open, and the disjunction of an arbitrary family of open sets can be defined as the interior of their intersection.

However, it transpires that not every locale is isomorphic to the Heyting algebra of open sets of some topological space; and in this sense, the theory of regions given by the definition of a locale is a generalisation of the idea of
a topological space that allows regions that are not composed of underlying points. This might be an interesting alternative to standard topology for modelling space-time in the context of quantum gravity.

A far-reaching generalisation of this idea is given by topos theory. As we shall see in Section 4, in any topos the idea of a ‘subobject’ is the analogue of the set-theoretic notion of a subset of a given set; and for any object $X$ in a topos, the family of subobjects of $X$ is a Heyting algebra, and hence another possible model for the regions of space-time.

### 3.2 Synthetic Differential Geometry

Recent decades have seen a revival of the idea of infinitesimals: nilpotent real numbers $d$ such that $d^2 = 0$. At first sight this seems nonsensical (apart from the trivial case $d = 0$) but it turns out that sense can be made of this, and in two different ways.

In the first approach, called ‘non-standard analysis’, every infinitesimal has a reciprocal, so that there are different infinite numbers corresponding to the different infinitesimals. There were attempts in the 1970s to apply this idea to quantum field theory: in particular, it was shown how the different orders of ultra-violet divergences correspond to different types of infinite number in the sense of non-standard analysis [11].

In the second approach, there are infinitesimals but without the corresponding infinite numbers. This is possible provided we work within the context of a topos rather than normal set theory: for example, a careful study of the proof that the only real number $d$ such that $d^2 = 0$ is 0, shows that it involves the principle of excluded middle, which in general does not hold in the intuitionistic logic of a topos [12].

This approach is known as ‘synthetic differential geometry’ (SDG), and it is intriguing to see if our familiar physical theories can be rewritten using this structure. For example, Fearns has recently shown how some of the features of standard quantum theory can be expressed in this way [13] (see also [14]).

Of even greater importance, however, is the possibility that there may be regimes in physics, in particular involving quantum space-time structures, where SDG is more appropriate than the standard approach.

One such possibility is suggested by the ‘History Projection Operator’ approach to consistent histories in which there are copies of the standard canonical commutation relations at each moment of time. For example, for
a particle moving in one dimension we have the history algebra \[ 15 \]

\[
\begin{align*}
[x_t, x_{t'}] &= 0 \quad (6) \\
[p_t, p_{t'}] &= 0 \quad (7) \\
[x_t, p_{t'}] &= i\hbar(t' - t) \quad (8)
\end{align*}
\]

where the label \( t \) on the (Schrödinger picture) operators \( \hat{x}_t \) and \( \hat{p}_t \) refers to the time at which propositions about the system are asserted—the time of ‘temporal logic’.

A major advance in the HPO formalism took place when time was introduced by Savvidou in a completely new way \[ 16 \] \[ 17 \]. It was realised that it is natural to consider time in a two-fold manner: the ‘time of being’—the time at which events ‘happen’ (the time label \( t \) in Eqs. (6)–(8) can be regarded as such), and the ‘time of becoming’—the time of dynamical change, represented by a time label \( s \). This second time appears in the history analogue \( \hat{x}_t(s) \) of the Heisenberg picture, which is defined as

\[
\hat{x}_t(s) := e^{is\hat{H}/\hbar}\hat{x}_t e^{-is\hat{H}/\hbar}
\]

where \( \hat{H} := \int dt \hat{H}_t \) is the history quantity that represents the time average of the energy of the system. The notion of time evolution is now recovered for the time-averaged physical quantities, for example

\[
\hat{x}_f(s) := e^{is\hat{H}/\hbar}\hat{x}_fe^{-is\hat{H}/\hbar}
\]

where \( f(t) \) is a smearing function.

Associated with these two manifestations of the concept of time are two types of time transformation: the ‘external’ translation \( \hat{x}_t(s) \mapsto \hat{x}_{t+t'}(s) \); and the ‘internal’ translation \( \hat{x}_t(s) \mapsto \hat{x}_t(s+s') \). The external time translation is generated by the ‘Louiville’ operator \[ 14 \]

\[
\hat{V} := \int dt \hat{p}_t \frac{d\hat{x}_t}{dt}
\]

whereas the internal time translation is generated by the time-averaged energy operator \( \hat{H} \).

More importantly, it was shown in \[ 16 \] that the generator of time translation in the HPO theory is the ‘action’ operator \( \hat{S} \) defined as

\[
\hat{S} := \int dt \hat{p}_t \frac{d\hat{x}_t}{dt} - \hat{H} = \hat{V} - \hat{H}.
\]

Hence the action operator is the generator of both types of time translation: \( \hat{x}_t(s) \mapsto \hat{x}_{t+t'}(s+s') \). It is a striking result that in the HPO theory the quantum analogue of the classical action functional is an actual operator in the formalism, and is the generator of time translations.

In the context of SDG, it is the view of Savvidou (with which I agree) that the infinitesimals of SDG are particularly well adapted to describe transformations in the external time-parameter.
If true, this has significant implications for the construction of a history theory of quantum gravity. In particular, in the context of general relativity, Savvidou has shown that the analogue of the Liouville transformations is the full space-time diffeomorphism group \(^{18}\). Thus the intriguing possibility arises that, in a history version of general relativity, there may be a natural role for SDG, and hence for topos theory, in implementing the actions of this fundamental group.

4 Presheaves and Related Notions from Topos Theory

From now on we shall concentrate on topos theory itself, culminating in a particular application to standard quantum theory.

There are various approaches to the notion of a topos but the focus here will be on one that emphasises the underlying logical structure. To keep the discussion simple, we will not develop the full definition of a topos but will concentrate on the role of a ‘subobject classifier’. This involves a generalization of the set-theoretic idea of a characteristic function, and has a particularly interesting logical structure in the kind of topos to which the discussion in Section 5 is confined: namely, a topos of presheaves \(^{19}\) \(^{20}\).

A topos is a type of category that behaves much like the category of sets Set.\(^6\) In the category Set, the objects are sets and the arrows/morphisms are ordinary functions between them (set-maps). In many other categories, the objects are sets equipped with some type of additional structure, and the arrows are functions that preserve this structure. An example is the category of groups, where an object is a group, and an arrow \(f : G_1 \rightarrow G_2\) is a group homomorphism from \(G_1\) to \(G_2\). However, a category need not have ‘structured sets’ as its objects. An example is given by any partially-ordered

\(^6\)Recall that a category consists of a collection of objects and a collection of arrows (or morphisms), with the following three properties: (1) Each arrow \(f\) is associated with a pair of objects, known as its domain (\(\text{dom} f\)) and the codomain (\(\text{cod} f\)), and is written in the form \(f : B \rightarrow A\) where \(B = \text{dom} f\) and \(A = \text{cod} f\); (2) Given two arrows \(f : B \rightarrow A\) and \(g : C \rightarrow B\) (so that the codomain of \(g\) is equal to the domain of \(f\)), there is a composite arrow \(f \circ g : C \rightarrow A\), and this composition of arrows obeys the associative law; and (3) Each object \(A\) has an identity arrow, \(\text{id}_A : A \rightarrow A\), with the properties that for all \(f : B \rightarrow A\) and all \(g : A \rightarrow C\), \(\text{id}_A \circ f = f\) and \(g \circ \text{id}_A = g\).
It can be regarded as a category in which (i) the objects are the elements of $\mathcal{P}$; and (ii) if $p, q \in \mathcal{P}$, an arrow from $p$ to $q$ is defined to exist if, and only if, $p \leq q$ in the poset structure. Thus, in a poset regarded as a category, there is at most one arrow between any pair of objects $p, q \in \mathcal{P}$.

In any category, an object $T$ is called a terminal (resp. initial) object if for every object $A$ there is exactly one arrow $f : A \to T$ (resp. $f : T \to A$). Any two terminal (resp. initial) objects are isomorphic. So we can fix on one such object and write ‘the’ terminal (resp. initial) object as $1$ (resp. $0$). An arrow $1 \to A$ is called a point, or global element, of $A$. For example, applying these definitions to the category of sets, we see that (i) each singleton set is a terminal object; (ii) the empty set $\emptyset$ is initial; and (iii) the points of $A$ are in one-to-one correspondence with the elements of $A$ (in the usual sense of the word ‘element’ of a set).

### 4.1 Toposes and Subobject Classifiers

We turn now to the very special kind of category called a ‘topos’, concentrating on the requirement that a topos contains a generalization of the set-theoretic concept of a characteristic function.

Recall that for any set $X$, and any subset $A \subseteq X$, there is a characteristic function $\chi_A : X \to \{0, 1\}$, with $\chi_A(x) = 1$ or $0$ according as $x \in A$ or $x \notin A$. One can think of $\{0, 1\}$ as truth-values, with $\chi_A$ classifying the various $x \in X$ in response to question “Is $x$ an element of $A$?” Furthermore, $\{0, 1\}$ is itself a set—i.e. an object in the category Set—and for each $A, X$ with $A \subseteq X$, $\chi_A$ is an arrow from $X$ to $\{0, 1\}$.

These concepts extend to a general category as follows.

1. A ‘subobject’ is the analogue of the set-theoretic idea of a subset. More precisely, one generalizes the idea that a subset $A$ of $X$ has a preferred injective (i.e., one-to-one) map $A \to X$ sending $x \in A$ to $x \in X$. The categorial analogue of an injective map is called a ‘monic arrow’, and a subobject of any object $X$ in a category is defined to be a monic arrow with codomain $X$.

2. Any topos is required to have an analogue, written $\Omega$, of the set $\{0, 1\}$ of truth-values; in particular, $\Omega$ is an object in the topos. Furthermore, $\{0, 1\}$ is itself a set. Two objects $A$ and $B$ in a category are said to be isomorphic if there exists arrows $f : A \to B$ and $g : B \to A$ such that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$. 

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7Two objects $A$ and $B$ in a category are said to be isomorphic if there exists arrows $f : A \to B$ and $g : B \to A$ such that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$. 

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there is a one-to-one correspondence between subobjects of an object $X$, and arrows from $X$ to $\Omega$.

3. In a topos, $\Omega$ acts as an object of generalized truth-values, just as \{0, 1\} does in set-theory (though $\Omega$ typically has more than two global elements). Moreover, $\Omega$ has a natural logical structure. More precisely, $\Omega$ has the internal structure of a Heyting algebra object: the algebraic structure appropriate for intuitionistic logic, mentioned in Section 3.1.1.

In addition, the collection of subobjects of any given object $X$ in a topos is a complete Heyting algebra.

4.2 Toposes of Presheaves

In preparation for the application to quantum theory discussed in Section 5, we turn now to the theory of presheaves.

First recall that a ‘functor’ between a pair of categories $\mathcal{C}$ and $\mathcal{D}$ is an arrow-preserving function from one category to the other. More precisely, a covariant functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a function that assigns (i) to each $\mathcal{C}$-object $A$, a $\mathcal{D}$-object $F(A)$; and (ii) to each $\mathcal{C}$-arrow $f : B \to A$, a $\mathcal{D}$-arrow $F(f) : F(B) \to F(A)$ such that $F(\text{id}_A) = \text{id}_{F(A)}$.

These assignments are such that if $g : C \to B$, and $f : B \to A$ then $F(f \circ g) = F(f) \circ F(g)$.

A presheaf (or varying set) on the category $\mathcal{C}$ is defined to be a covariant functor $X$ from the category $\mathcal{C}$ to the category of sets. We want to make the collection of presheaves on $\mathcal{C}$ into a category, and so it is necessary to define what is meant by an ‘arrow’ between two presheaves $X$ and $Y$. This is defined to be a natural transformation $N : X \to Y$, which is a family of maps (the components of $N$) $N_A : X(A) \to Y(A)$, where $A$ an object in $\mathcal{C}$, such that if $f : A \to B$ is an arrow in $\mathcal{C}$, then the composite map $X(A) \overset{N_A}{\to} Y(A) \overset{Y(f)}{\to} Y(B)$ is equal to $X(A) \overset{X(f)}{\to} X(B) \overset{N_B}{\to} Y(B)$, as shown.

\footnote{More precisely, the theory of presheaves on an arbitrary ‘small’ category $\mathcal{C}$ (the qualification ‘small’ means that the collection of objects in $\mathcal{C}$ is a genuine set, as is the collection of all arrows in $\mathcal{C}$).}
in the commutative diagram

\[
\begin{array}{ccc}
X(A) & \xrightarrow{X(f)} & X(B) \\
\downarrow^{N_A} & & \downarrow^{N_B} \\
Y(A) & \xrightarrow{Y(f)} & Y(B)
\end{array}
\] (10)

An object \(K\) is said to be a subobject of \(X\) if there is an arrow in the category of presheaves \(i : K \to X\) with the property that, for each \(A\), the component map \(i_A : K(A) \to X(A)\) is a subset embedding, i.e., \(K(A) \subseteq X(A)\). Thus, if \(f : A \to B\) is any arrow in \(C\), we get the commutative diagram

\[
\begin{array}{ccc}
K(A) & \xrightarrow{K(f)} & K(B) \\
\downarrow & & \downarrow \\
X(A) & \xrightarrow{X(f)} & X(B)
\end{array}
\] (11)

where the vertical arrows are subset inclusions.

The category of presheaves on \(C\), \(\text{Set}^C\), forms a topos. We turn now to discussing the subobject classifier of this particular topos.

4.2.1 Sieves and the Subobject Classifier in a Topos of Presheaves.

A key concept in presheaf theory—and something of particular importance for the quantum theory application discussed later—is that of a ‘sieve’, which plays a central role in the construction of the subobject classifier in the topos \(\text{Set}^C\) of presheaves on a category \(C\).

A sieve on an object \(A\) in \(C\) is defined to be a collection \(S\) of arrows \(f : A \to B\) in \(C\) with the property that if \(f : A \to B\) belongs to \(S\), and if \(g : B \to C\) is any arrow, then \(g \circ f : A \to C\) also belongs to \(S\). In the simple case where \(C\) is a poset, a sieve on \(p \in C\) is any subset \(S\) of \(C\) such that if \(r \in S\) then (i) \(p \leq r\), and (ii) \(r' \in S\) for all \(r \leq r'\). Thus a sieve is just an upper set in the poset.

The presheaf \(\Omega : C \to \text{Set}\) is now defined as follows. If \(A\) is an object in \(C\), then \(\Omega(A)\) is defined to be the set of all sieves on \(A\); and if \(f : A \to B\), then \(\Omega(f) : \Omega(A) \to \Omega(B)\) is defined as

\[
\Omega(f)(S) := \{h : B \to C \mid h \circ f \in S\}
\] (12)

for all \(S \in \Omega(A)\). Note that if \(S\) is a sieve on \(A\), and if \(f : A \to B\) belongs to \(S\), then from the defining property of a sieve

\[
\Omega(f)(S) := \{h : B \to C \mid h \circ f \in S\} = \{h : B \to C\} =: \uparrow B
\] (13)
where \( \uparrow B \) denotes the principal sieve on \( B \), defined to be the set of all arrows in \( C \) whose domain is \( B \).

A crucial property of sieves is that the set \( \Omega(A) \) of sieves on \( A \) has the structure of a Heyting algebra where the unit element \( 1_{\Omega(A)} \) in \( \Omega(A) \) is the principal sieve \( \uparrow A \), and the null element \( 0_{\Omega(A)} \) is the empty sieve \( \emptyset \). The partial ordering in \( \Omega(A) \) is defined by \( S_1 \leq S_2 \) if and only if \( S_1 \subseteq S_2 \); and the logical connectives are defined as:

\[
S_1 \land S_2 := S_1 \cap S_2 \quad (14)
\]
\[
S_1 \lor S_2 := S_1 \cup S_2 \quad (15)
\]
\[
S_1 \Rightarrow S_2 := \{ f : A \to B \mid \forall g : B \to C \text{ if } g \circ f \in S_1 \text{ then } g \circ f \in S_2 \} \quad (16)
\]

As in any Heyting algebra, the negation of an element \( S \) (called the pseudo-complement of \( S \)) is defined as \( \neg S := S \Rightarrow 0 \); so that

\[
\neg S := \{ f : A \to B \mid \text{for all } g : B \to C, g \circ f \notin S \}. \quad (17)
\]

As remarked earlier, the main distinction between a Heyting algebra and a Boolean algebra is that, in the former, the negation operation does not necessarily obey the law of excluded middle: instead, all that be can said is that, for any element \( S \),

\[
S \lor \neg S \leq 1. \quad (18)
\]

It can be shown that the presheaf \( \Omega \) is a subobject classifier for the topos \( \text{Set}^C \). Thus subobjects of any object \( X \) in this topos (i.e., any presheaf on \( C \)) are in one-to-one correspondence with arrows \( \chi : X \to \Omega \). This works as follows. Let \( K \) be a subobject of \( X \). Then there is an associated characteristic arrow \( \chi^K : X \to \Omega \), whose component \( \chi^K_A : X(A) \to \Omega(A) \) at each ‘stage of truth’ \( A \) in \( C \) is defined as

\[
\chi^K_A(x) := \{ f : A \to B \mid X(f)(x) \in K(B) \} \quad (19)
\]

for all \( x \in X(A) \). That the right hand side of Eq. (19) actually is a sieve on \( A \) follows from the defining properties of a subobject.\[9\]

\[^9\text{There is a converse to Eq. (19): namely, each arrow } \chi : X \to \Omega \text{ (i.e., a natural transformation between the presheaves } X \text{ and } \Omega \text{) defines a subobject } K^\chi \text{ of } X \text{ via}
\]
\[
K^\chi(A) := \chi_A^{-1}\{1_{\Omega(A)}\}. \quad (20)
\]

at each stage of truth \( A \).
Thus, in each ‘branch’ of the category $\mathcal{C}$ going ‘upstream’ from the stage $A$, $\chi^K_A(x)$ picks out the first member $B$ in that branch for which $X(f)(x)$ lies in the subset $K(B)$, and the commutative diagram Eq. (11) then guarantees that $X(h \circ f)(x)$ will lie in $K(C)$ for all $h : B \to C$.

Thus each ‘stage of truth’ $A$ in $\mathcal{C}$ serves as a possible context for an assignment to each $x \in X(A)$ of a generalised truth-value, which is a sieve belonging to the Heyting algebra $\Omega(A)$. This is the sense in which contextual, generalised truth-values arise naturally in a topos of presheaves.

4.2.2 Global Sections of a Presheaf

For the category of presheaves on $\mathcal{C}$, a terminal object $1 : \mathcal{C} \to \text{Set}$ can be defined by $1(A) := \{\ast\}$ (a singleton set) at all stages $A$ in $\mathcal{C}$; if $f : A \to B$ is an arrow in $\mathcal{C}$ then $1(f) : \{\ast\} \to \{\ast\}$ is defined to be the map $\ast \mapsto \ast$. This is indeed a terminal object since, for any presheaf $X$, we can define a unique natural transformation $N : X \to 1$ whose components $N_A : X(A) \to 1(A) = \{\ast\}$ are the constant maps $x \mapsto \ast$ for all $x \in X(A)$.

A global element (or point) of a presheaf $X$ is also called a global section. As an arrow $\gamma : 1 \to X$ in the topos $\text{Set}^\mathcal{C}$, a global section corresponds to a choice of an element $\gamma_A \in X(A)$ for each stage of truth $A$ in $\mathcal{C}$, such that, if $f : A \to B$, the ‘matching condition’

$$X(f)(\gamma_A) = \gamma_B$$

is satisfied. As we shall see, the Kochen-Specker theorem can be read as asserting the non-existence of any global sections of certain presheaves that arises naturally in quantum theory.

5 Presheaves of Propositions, and Valuations in Quantum Theory

The contextual, multi-valued logic that arises naturally in a topos of presheaves has some very interesting potential applications in theoretical physics. Here, however, I shall briefly present just one particular example that has been developed in detail elsewhere [21, 22, 23]. This is the proposal to retain a ‘realist flavour’ in the assignment of values to quantum-theoretic quantities by using the non-Boolean logical structure of a particular topos of presheaves.
Before stating the proposal precisely, recall the Kochen-Specker theorem which asserts the impossibility of associating real values $V(\hat{A})$ to all physical quantities in a quantum theory (if $\text{dim} \mathcal{H} > 2$) whilst preserving the ‘$\text{FUNC}$’ rule that $V(f(\hat{A}) = f(V(\hat{A}))$—i.e., the value of a function $f$ of a physical quantity $A$ is equal to the function of the value of the quantity. Equivalently, it is not possible to assign true-false values to all the propositions in a quantum theory in a way that respects the structure of the associated lattice of projection operators. As we shall see, our topos-theoretic proposal is such that the truth value ascribed to a proposition about the value of a physical quantity need not be just ‘true’ or ‘false’.

Thus consider the proposition “$A \in \Delta$”, which asserts that the value of the quantity $A$ lies in a Borel set $\Delta \subseteq \mathbb{R}$. Roughly speaking, our proposal is that any such proposition should be ascribed as a truth-value a set of coarse-grainings, $f(\hat{A})$, of the operator $\hat{A}$ that represents $A$. Exactly which coarse-grainings are in the truth-value depends in a precise way on $\Delta$ and the quantum state $\psi$: specifically, $f(\hat{A})$ is in the truth-value if and only if $\psi$ is in the range of the spectral projector $\hat{E}[f(\Delta)]$. Note the contrast with the conventional eigenstate-eigenvalue link: our requirement is not that $\psi$ be in the range of $\hat{E}[A \in \Delta]$, but a weaker one since, generally, $\hat{E}[f(\Delta)]$ is a larger spectral projector than $\hat{E}[f(\Delta)]$; i.e., in the lattice $\mathcal{L}(\mathcal{H})$ of projectors on the Hilbert space $\mathcal{H}$, we have $\hat{E}[A \in \Delta] \leq \hat{E}[f(\Delta)]$.

So the intuitive idea is that the new proposed truth-value of “$A \in \Delta$” is given by the set of weaker propositions “$f(\Delta)$” that are true in the old (i.e., eigenstate-eigenvalue link) sense. More precisely, the truth-value of “$A \in \Delta$” is the set of quantities $f(\Delta)$ for which the corresponding weaker proposition “$f(\Delta)$” is true in the old sense. Thus the truth-value of a proposition in the new sense is given by the set of its consequences that are true in the old sense.

The first step in stating the proposal precisely is to introduce the set $\mathcal{O}$ of all bounded self-adjoint operators on the Hilbert space $\mathcal{H}$ of a quantum system. The set $\mathcal{O}$ is turned into a category by defining the objects to be the elements of $\mathcal{O}$, and saying that there is an arrow from $\hat{A}$ to $\hat{B}$ if there exists a real-valued function $f$ on the spectrum $\sigma(\hat{A}) \subseteq \mathbb{R}$ of $\hat{A}$, such that $\hat{B} = f(\hat{A})$. If $\hat{B} = f(\hat{A})$, for some $f : \sigma(\hat{A}) \to \mathbb{R}$, then the corresponding arrow in the category $\mathcal{O}$ will be denoted $f_\hat{A} : \hat{A} \to \hat{B}$.

The next step is define two presheaves on the category $\mathcal{O}$, called the dual presheaf and the coarse-graining presheaf respectively. The former affords an
elegant formulation of the Kochen-Specker theorem, namely as the statement that the dual presheaf does not have global sections. The latter is at the basis of the proposed generalised truth-value assignments.

The dual presheaf on \( \mathcal{O} \) is the covariant functor \( D : \mathcal{O} \to \text{Set} \) defined as follows:

1. On objects: \( D(\hat{A}) \) is the dual of \( W_A \), where \( W_A \) is the spectral algebra of the operator \( \hat{A} \) (i.e. \( W_A \) is the collection of all projectors onto the subspaces of \( \mathcal{H} \) associated with Borel subsets of \( \sigma(\hat{A}) \)). Thus \( D(\hat{A}) \) is defined to be the set of all homomorphisms from the Boolean algebra \( W_A \) to the Boolean algebra \{0, 1\}.

2. On arrows: If \( f_\mathcal{O} : \hat{A} \to \hat{B} \), so that \( \hat{B} = f(\hat{A}) \) and \( W_B \subseteq W_A \), then \( D(f_\mathcal{O})(\chi) := \chi|_{W_f(\hat{A})} \) where \( \chi|_{W_f(\hat{A})} \) is the restriction of \( \chi \in D(W_A) \) to the subalgebra \( W_f(\hat{A}) \subseteq W_A \).

A global element (global section) of the functor \( D : \mathcal{O} \to \text{Set} \) is then a function \( \gamma \) that associates to each \( \hat{A} \in \mathcal{O} \) an element \( \gamma_A \) of the dual of \( W_A \) such that if \( f_\mathcal{O} : \hat{A} \to \hat{B} \) (so \( \hat{B} = f(\hat{A}) \) and \( W_B \subseteq W_A \)), then \( \gamma_A|_{W_B} = \gamma_B \).

Thus, for all projectors \( \hat{\alpha} \in W_B \subseteq W_A \), we have \( \gamma_B(\hat{\alpha}) = \gamma_A(\hat{\alpha}) \).

Since each \( \hat{\alpha} \) in the lattice \( \mathcal{L}(\mathcal{H}) \) of projection operators on \( \mathcal{H} \) belongs to at least one such spectral algebra \( W_A \) (for example, the algebra \{0, 1, \hat{\alpha}, 1-\hat{\alpha}\}) it follows that a global section of \( D \) associates to each projection operator \( \hat{\alpha} \in \mathcal{L}(\mathcal{H}) \) a number \( V(\hat{\alpha}) \) which is either 0 or 1, and is such that if \( \hat{\alpha} \) and \( \hat{\beta} \) are disjoint propositions then \( V(\hat{\alpha} \lor \hat{\beta}) = V(\hat{\alpha}) + V(\hat{\beta}) \). A global section \( \gamma \) of the presheaf \( D \) would correspond to an assignment of truth-values \{0, 1\} to all propositions of the form “\( A \in \Delta \)”, which obeyed the \textit{FUNC} condition \( \gamma_A|_{W_B} = \gamma_B \). But these are precisely the types of valuation prohibited by the Kochen-Specker theorem provided that \( \dim \mathcal{H} > 2 \! \). So an alternative way of expressing the Kochen-Specker theorem is the statement that (if \( \dim \mathcal{H} > 2 \)) the dual presheaf \( D \) has no global sections.

However, we can use the subobject classifier \( \Omega \) in the topos \( \text{Set}^\mathcal{O} \) of all presheaves on \( \mathcal{O} \) to assign \textit{generalized} truth-values to the propositions “\( A \in \Delta \)”. These truth-values will be sieves—as defined in Section \( \text{[4.2.2]} \)—and since they will be assigned relative to each ‘context’ or ‘stage of truth’ \( \hat{A} \) in \( \mathcal{O} \), these truth-values will be contextual as well as generalized. Note that because in any topos the subobject classifier \( \Omega \) is unique up to isomorphism the traditional objection to multi-valued logics in quantum theory—
that their structure often seems arbitrary—does not apply to these particular
generalized, contextual truth-values.

The first step is to define the appropriate presheaf of propositions. The
*coarse-graining presheaf* over \( \mathcal{O} \) is the covariant functor \( G : \mathcal{O} \to \text{Set} \) defined
as follows.

1. On objects in \( \mathcal{O} \): \( G(\hat{A}) := W_A \), the spectral algebra of \( \hat{A} \).

2. On arrows in \( \mathcal{O} \): If \( f : \hat{A} \to \hat{B} \) (i.e., \( \hat{B} = f(\hat{A}) \)), then \( G(f) : W_A \to W_B \) is defined as\(^{10}\)

   \[
   G(f)(\hat{E}[A \in \Delta]) := \hat{E}[f(A) \in f(\Delta)] \tag{22}
   \]

A function \( \nu \) that assigns to each object \( \hat{A} \) in \( \mathcal{O} \) and each Borel set
\( \Delta \subseteq \sigma(\hat{A}) \), a sieve of arrows in \( \mathcal{O} \) on \( \hat{A} \) (i.e., a sieve of arrows with \( \hat{A} \) as
domain), will be called a *sieve-valued valuation* on \( G \). We write the values
of this function as \( \nu(A \in \Delta) \).

From the logical point of view, a natural requirement for any kind of
valuation on a presheaf of propositions such as \( G \) is that the valuation should
specify a subobject of \( G \). But subobjects are in one-one correspondence with
arrows, i.e., natural transformations, \( N : G \to \Omega \). So it is natural to require
a sieve-valued valuation \( \nu \) to define such a natural transformation by the
equation \( N\nu(\hat{E}[A \in \Delta]) := \nu(A \in \Delta) \) for all stages-contexts \( \hat{A} \).

This requirement leads directly to the analogue for presheaves of the func-
tional composition condition of the Kochen-Specker theorem, called *FUNC*
above. Indeed, it transpires that a sieve-valued valuation defines a natural
transformation if and only if it obeys (the presheaf version of) *FUNC*.

To spell this out, first recall that sieves are ‘pushed forward’ by the sub-
object classifier \( \Omega \) according to Eq. (12). For the category \( \mathcal{O} \): if \( f : \hat{A} \to \hat{B} \), then \( \Omega(f) : \Omega(\hat{A}) \to \Omega(\hat{B}) \) is defined by

\[
\Omega(f)(S) := \{ h : B \to C \mid h \circ f \in S \} \tag{23}
\]
for all sieves \( S \in \Omega(\hat{A}) \).

\(^{10}\)If \( f(\Delta) \) is not Borel, the right hand side is to be understood in the sense of Theorem
4.1 of [21]—a measure-theoretic nicety that we shall not discuss here.
Accordingly, we say that a sieve-valued valuation $\nu$ on $G$ satisfies *generalized functional composition*—for short, $\text{FUNC}$—if for all $\hat{A}, \hat{B}$ and $f_\mathcal{O} : \hat{A} \to \hat{B}$ and all $\hat{E}[A \in \Delta] \in G(\hat{A})$, we have

$$\nu(B \in G(f_\mathcal{O})(\hat{E}[A \in \Delta])) \equiv \nu(f(A) \in f(\Delta)) = \Omega(f_\mathcal{O})(\nu(A \in \Delta)). \quad (24)$$

It can readily be checked that $\text{FUNC}$ is exactly the condition a sieve-valued valuation must obey in order to define a natural transformation—i.e., a subobject of $G$—by the equation $N^\nu(\hat{E}[A \in \Delta]) := \nu(A \in \Delta)$. That is, a sieve-valued valuation $\nu$ on $G$ obeys $\text{FUNC}$ if and only if the functions at each context $\hat{A}$

$$N^\nu(\hat{E}[A \in \Delta]) := \nu(A \in \Delta) \quad (25)$$

define a natural transformation $N^\nu$ from $G$ to $\Omega$.

It turns out that with any quantum state there is associated such a sieve-valued valuation obeying $\text{FUNC}$. Furthermore, this valuation gives the natural generalization of the eigenvalue-eigenstate link described earlier. That is, a quantum state $\psi$ induces a sieve on each $\hat{A}$ in $\mathcal{O}$ by the requirement that an arrow $f_\mathcal{O} : \hat{A} \to \hat{B}$ is in the sieve if and only if $\psi$ is in the range of the spectral projector $\hat{E}[B \in f(\Delta)]$. To be precise, we define for any $\psi$, and any Borel subset $\Delta$ of the spectrum $\sigma(\hat{A})$ of $\hat{A}$,

$$\nu^\psi(A \in \Delta) := \{f_\mathcal{O} : \hat{A} \to \hat{B} \mid \hat{E}[B \in f(\Delta)] \psi = \psi\} = \{f_\mathcal{O} : \hat{A} \to \hat{B} \mid \text{Prob}(B \in f(\Delta); \psi) = 1\} \quad (26)$$

where $\text{Prob}(B \in f(\Delta); \psi)$ is the usual Born-rule probability that the result of a measurement of $B$ will lie in $f(\Delta)$, given the state $\psi$.

One can check that the definition satisfies $\text{FUNC}$, and also has other properties that it is natural to require of a valuation (discussed in [21, 22, 23]). Thus, by using topos theory we are able to assign generalised truth values to all propositions whilst preserving the appropriate analogue of the $\text{FUNC}$ condition.

The key feature of these truth assignments is that they involve the contextual, multi-valued logic that is an intrinsic feature of a topos of presheaves. My expectation is that a similar topos structure could serve to patch together the different types of quantum theory that, as discussed earlier, I anticipate should be associated with different background space-time structures.
6 Conclusions

In general, the real numbers enter physical theories in three ways: as the values of physical quantities; as coordinates on a manifold model for space and time; and as the values of probabilities. The main thrust of the present paper is to argue that all three uses may become problematic in physical regimes that characterise strong quantum gravity effects.

In particular, I have argued that the assignment of real numbers as values of physical quantities and probabilities is to some extent motivated by certain \textit{a priori} ideas about the continuum nature of space and time. Thus it may be fundamentally wrong to attempt to construct a quantum theory of gravity whilst using a quantum formalism in which these \textit{a priori} continuum ideas are present from the beginning. My contention is that there should be a different type of quantum structure for each ‘type’ of background space-time: in particular, the mathematical spaces in which physical quantities and probabilities take their values should reflect the structure of this background.

If this is correct, the question then arises as to how to patch together a collection of such theories in the situation where the ‘background’ space-times are themselves the subject of quantum effects. I have suggested that the appropriate mathematical tool for doing this is topos theory; in particular the theory of presheaves with its intrinsic contextual, multi-valued logic. As an example of the use of this theory I have briefly reviewed an application of presheaf theory to the Kocken-Specher theorem in standard quantum theory.

Topos theory is also an essential ingredient in synthetic differential geometry, and this too may have important applications in theoretical physics; particularly perhaps in the context of the two-pronged way in which time arises in the consistent history theory.

What is sketched in the first half of this paper is only a collection of ideas. It remains an outstanding challenge to implement some of these general thoughts in the context, say, of a specific non-standard spatio-temporal background, such as a causal set. This could give valuable insight into what is perhaps the hardest task of all: to construct a quantum formalism for use in situations where there are no \textit{prima facie} spatio-temporal concepts at all—a situation that could well arise in a quantum gravity theory in which all of what we might want to call “spatio-temporal concepts” emerge from the basic formalism only in some limiting sense.
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