The Existence of Dyon Solutions for Generalized Weinberg-Salam Model

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Abstract

The generalized Weinberg-Salam model which is presented in a recent study of Kimm, Yoon and Cho [17], is arising in electroweak theory. In this paper, we prove the existence and asymptotic behaviors at infinity of static and radially symmetric dyon solutions to the boundary-value problem of this model. Moreover, as a by product, the qualitative properties of dyon solutions are also obtained. The methods used here are the extremum principle, the Schauder fixed point theory and the shooting approach depending on one shooting parameter. We provide an effective framework for constructing the dyon solutions in general dimensions and develop the existing results.

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1 Introduction and main results

This paper is concerned with the existence and asymptotic behaviors at infinity of static and radially symmetric dyon solutions to the boundary-value problem of the generalized Weinberg-Salam model (see [17])

\[ \mathcal{L} = -|D_\mu \phi|^2 - \frac{\lambda}{2} \left( \phi^\dagger \phi - \frac{\mu^2}{\lambda} \right)^2 - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} G_{\mu\nu}^2, \]  

(1.1)

where \( D_\mu \) describes the covariant derivative of the SU(2) subgroup only, \( \dagger \) represents conjugate transpose, \( F_{\mu\nu} \) and \( G_{\mu\nu} \) are the gauge field strengths of \( SU(2) \) and \( U(1)_Y \) with the potentials \( \overline{A}_\mu \) and \( B_\mu \) and the corresponding coupling constants \( g, g', \lambda, \mu \) are parameter, \( \phi, D_\mu \phi \) is represented as

\[ \phi = \frac{1}{\sqrt{2}} \rho \xi, \quad (\xi^\dagger \xi = 1), \]  

(1.2)

\[ D_\mu \phi = \left( \partial_\mu - ig \frac{g'}{2} \overline{A}_\mu - ig' B_\mu \right) \phi \triangleq \left( D_\mu - ig' B_\mu \right) \phi, \]  

(1.3)

where \( \rho, \xi \) are the Higgs field and unit doublet, respectively.

It is well known that theoretical physics and field theory, in particular, provides a rich and challenging topic of study for mathematics. The study of these problems not only contributes to a deeper understanding of physical concepts, theories, and the relationships among them, but also provides new theories, methods, and techniques for the development of mathematics. In 1931, Dirac has generalized the Maxwell’s theory with his magnetic monopole [9]. Since then, magnetic monopoles have been the subject of extensive research [18]. Dirac’s monopole lives in the classical Maxwell field theory for electromagnetism and carries infinite energy [34]. By further studying Maxwell’s equation, Schwinger extended Dirac’s idea of magnetic monopole [12], and discovered and defined a new kind of particle like solution that carries both electric charge and magnetic charge, which is called dyon [27]. Dyon has a lot of important applications in high-temperature superconductivity, quantum Hall effect and superfluids [8]. Generalized Yang-Mills theory has a covariant derivative which contains both vector and scalar gauge bosons [17]. Based on this theory, some people construct an SU(3) unified model [12] of weak and electromagnetic interactions. By using the
NJL mechanism, the symmetry breaking can be realized dynamically [13]. The masses of $W$, $Z$ [5] are obtained and interactions between various particles are the same as that of Weinberg-Salam(WS) model. As for monopoles, 't Hooft [12] and Polyakov [23] have shown the existence of finite-energy solutions in arrSO(3) Higgs model. However, the more relevant model for electromagnetic and weak interactions is the SU(2) x U(1) model of Weinberg [31] and Salam. The Weinberg-Salam electroweak model has important theoretical significance and research value in classical field theory [10, 11, 16, 20, 26].

From a mathematical point of sight, the proof of the existence of a magnetic monopole or dyon is a complex subject. Their existences rely either on explicit constructions in the self-dual limit [1, 2, 24] or nonlinear functional analysis [4, 15, 19, 25, 30], as well as numerical simulation [3, 16]. Ever since Dirac [9] predicted the existence of the monopole, the monopole has been an obsession. The Abelian monopole has been generalized to the non-Abelian gauge theory by Wu and Yang [7, 32] who showed a non-Abelian monopole solution in the pure SU(2) gauge theory, and by 't Hooft [12] and Polyakov [23] who have shown that the SU(2) gauge theory allows a finite energy monopole solution in Georgi-Glashow model as a topological soliton in the presence of a triplet scalar source. Moreover, the monopole in grand unification has been constructed by Dokos and Tomaras [10]. The discovery of vortex solutions in the Weinberg-Salam model of the electroweak interactions raises the possibility that such solutions may exist in a wider class of field theories. Indeed some time ago Cho and Maison [8] have established that Weinberg-Salam model and Georgi-Glashow model have exactly the same topological structure, and demonstrated the existence of a new type of monopole and dyon solutions in the standard Weinberg-Salam model. Originally the solutions of Cho and Maison were obtained by a numerical integration [17]. But a mathematically rigorous existence proof has been established which endorses the numerical results. And the solutions are now referred to as Cho-Maison monopole and dyon [28, 34, 35]. Up to now, many experimental results have proved that Weinberg-Salam (WS) model [31] is correct in the current energy range.

In the spherically symmetric case, the two-point boundary value problem to (1.1) is consisting of six nonlinear ordinary differential equations. Namely, this is a highly nonlinear and strongly coupled nonlinear ordinary differential system. Therefore it is very difficult to handle. The purpose of our paper is to establish an existence theorem of the dyon solutions for the generalized Weinberg-Salam model (1.1). In fact, such a study was carried out in the earlier paper of Mcleod and the existence of the Weinberg-Salam dyon was rigorously established by the method of calculus of variations in the article of Yang [35]. In a recent interesting work by Kimm, Yoon and Cho [17], three different ways is discussed to estimate the mass of the electroweak monopole and the differential equations governing the static radially symmetric the Cho-Maison monopole and dyon solutions are constructed in the generalized Weinberg-Salam model.

For the generators of $SU(2)$, we use the conventional Pauli matrices $\tau^\alpha (\alpha = 1, 2, 3)$. 
Then, (1.1) is reduced to the following equations of motion

\[
\begin{align*}
\partial^2 \rho &= |D_\mu \xi|^2 \rho + \frac{1}{8} (\rho^2 - 2\mu^2) \rho, \\
D^2 \xi &= -2 \frac{\partial \mu \rho}{\rho} D_\mu \xi + [\xi^\dagger D^2 \xi + 2 \frac{\partial \mu \rho}{\rho} (\xi^\dagger D_\mu \xi)] \xi, \\
D_\mu F_{\mu \nu} &= i \frac{2}{\sqrt{2}} \rho^2 [\xi^\dagger \bar{\tau} (D_\nu \xi) - (D_\nu \xi)^\dagger \bar{\tau} \xi], \\
\partial_\mu G_{\mu \nu} &= i \frac{g}{\sqrt{2}} \rho^2 [\xi^\dagger (D_\nu \xi) - (D_\nu \xi)^\dagger \xi].
\end{align*}
\]

(1.4)

By the Abelian decomposition [17]

\[
\Phi = \rho \hat{n}, \quad \hat{A}_\mu + \hat{W}_\mu = \bar{A}_\mu,
\]

we have

\[
\begin{align*}
\mathcal{L} &= -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\rho^2}{2} |D_\mu \xi|^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} \hat{F}_{\mu \nu}^2 - \frac{1}{4} G_{\mu \nu}^2 - \frac{g}{2} \hat{F}_{\mu \nu} \cdot (\hat{W}_\mu \times \hat{W}_\nu) \\
&\quad - \frac{1}{4} (\hat{D}_\mu \hat{W}_\nu - \hat{D}_\nu \hat{W}_\mu)^2 - \frac{g^2}{8} \rho^2 (\hat{W}_\mu)^2 - \frac{g^2}{4} (\hat{W}_\mu \hat{W}_\nu)^2,
\end{align*}
\]

(1.6)

where

\[
\hat{D}_\mu = \partial_\mu - \frac{g}{2} \bar{\tau} \cdot \hat{A}_\mu - i \frac{g'}{2} B_\mu.
\]

To construct the desired solutions we enlarge $U(1)_Y$ and embed it to another $SU(2)$. And then, we introduce a hypercharged vector field $X_\mu$ and a Higgs field $\sigma$, and generalize the Lagrangian (1.6) adding the following Lagrangian

\[
\Delta \mathcal{L} = -\frac{1}{2} |\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 + i g' G_{\mu \nu} X_\mu^* X_\nu + \frac{1}{4} g'^2 (X_\mu^* X_\nu - X_\mu X_\nu)^2 \\
- \frac{1}{2} (\partial_\mu \sigma)^2 - g'^2 \sigma^2 |X_\mu|^2 - \kappa \left( \sigma^2 - \frac{m^2}{\kappa} \right)^2,
\]

(1.7)

where

\[
\hat{D}_\mu = \partial_\mu + i g' B_\mu.
\]

Therefore, we can get the generalized Weinberg-Salam model in electroweak theory as follows

\[
\begin{align*}
\mathcal{L} &= -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\rho^2}{2} |D_\mu \xi|^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} \hat{F}_{\mu \nu}^2 - \frac{1}{4} G_{\mu \nu}^2 - \frac{g}{2} \hat{F}_{\mu \nu} \cdot (\hat{W}_\mu \times \hat{W}_\nu) - \frac{g^2}{8} \rho^2 (\hat{W}_\mu)^2 \\
&\quad - \frac{1}{4} (\hat{D}_\mu \hat{W}_\nu - \hat{D}_\nu \hat{W}_\mu)^2 - \frac{g^2}{4} (\hat{W}_\mu \times \hat{W}_\nu)^2 - \frac{1}{2} |\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu|^2 + i g' G_{\mu \nu} X_\mu^* X_\nu \\
&\quad + \frac{1}{4} g'^2 (X_\mu^* X_\nu - X_\mu X_\nu)^2 - \frac{1}{2} (\partial_\mu \sigma)^2 - g'^2 \sigma^2 |X_\mu|^2 - \kappa \left( \sigma^2 - \frac{m^2}{\kappa} \right)^2.
\end{align*}
\]

(1.8)
In order to pursue a static radially symmetric dyon solution, we follow Kimm, Yoon and Cho to use the following general ansatz \[ \rho = \rho(r), \quad \sigma = \sigma(r), \]
\[ \xi = i \left( \sin \left( \frac{\theta}{2} \right) e^{-i\varphi} - \cos \left( \frac{\theta}{2} \right) \right), \]
\[ X_\mu = \frac{i}{g} \frac{h(r)}{r^2} e^{i\varphi} (\partial_\mu \theta + i \sin \theta \partial_\mu \varphi), \]
\[ B_\mu = \frac{1}{g} B(r) \partial_\mu t - \frac{1}{g} \frac{1}{(1 - \cos \theta)} \partial_\mu \varphi, \]
\[ \vec{A}_\mu = \frac{1}{g} A(r) \partial_\mu \hat{r} + \frac{1}{g} (f(r) - 1) \hat{r} \times \partial_\mu \hat{r}, \]

where \( \xi^\dagger \vec{\tau} \xi = -\hat{r}, (t, r, \theta, \varphi) \) are the spherically symmetric coordinates.

With the spherically symmetric ansatz the equations of motion for functions \( f(r), \rho(r), A(r), B(r), h(r), \sigma(r), 0 < r < \infty \) are reduced to

\[
\frac{f''}{r^2} - \frac{f^2 - 1}{r^2} f = \left( \frac{g^2}{4} \rho^2 - A^2 \right) f, \quad (1.9)
\]
\[
\rho'' + \frac{2}{r} \rho' - \frac{f^2}{2r^2} \rho = -\frac{1}{4} (A - B)^2 \rho + \frac{\lambda}{2} \left( \rho^2 - \frac{2\rho^2}{\lambda} \right) \rho, \quad (1.10)
\]
\[
A'' + \frac{2}{r} A' - \frac{2f^2}{r^2} A = \frac{1}{4} g^2 \rho^2 (A - B), \quad (1.11)
\]
\[
B'' + \frac{2}{r} B' - \frac{2h^2}{r^2} B = \frac{1}{4} g' \rho^2 (B - A), \quad (1.12)
\]
\[
h'' - \frac{h^2}{r^2} - \frac{1}{r^2} h = \left( g^2 \sigma^2 - B^2 \right) h, \quad (1.13)
\]
\[
\sigma'' + \frac{2}{r} \sigma' - \frac{2h^2}{r^2} \sigma = \kappa \left( \sigma^2 - \frac{m^2}{\kappa} \right) \sigma. \quad (1.14)
\]

The boundary conditions for a regular field configuration can be chosen as

\[
f(0) = h(0) = 1, A(0) = B(0) = \rho(0) = \sigma(0) = 0, \quad (1.15)
\]
\[
f(\infty) = h(\infty) = 0, A(\infty) = A_0, B(\infty) = B_0, \rho(\infty) = \rho_0, \sigma(\infty) = \sigma_0, \quad (1.16)
\]

where \( \sigma_0 = \sqrt{\frac{m^2}{\kappa}}, \rho_0 = \mu \sqrt{\frac{2}{\kappa}}, \lambda, \mu, g, g', \kappa, m \) are parameters, \( A_0, B_0 \) are given positive constants.

For the above nonlinear ordinary differential equation with two-point boundary value problem, inspired by the literature [6,14,21,22,29,33], we develop the methods and techniques in which we prove the existence of the solution and study the related properties of the solution.

To do so, we require parameters to satisfy the following two assumptions

\[
(H1) \quad \frac{1}{4} g \rho_0^2 > A_0^2, \quad g' \sigma_0^2 > B_0^2,
\]
\[
(H2) \quad B_0 = A_0.
\]
Then the main results of this paper are stated as follows.

**Theorem 1.1 (Existence of solutions to the boundary-value problem)** Under the assumption (H1) and (H2), the Weinberg-Salam dyon equations (1.9) – (1.14) have a family of finite energy smooth solutions which satisfy the radial symmetry properties. The obtained solution configuration functions \(f, \rho, A, B, h, \sigma\) have the properties that

1. The function \(f, A, B, h \in C^1([0, +\infty))\) and \(f'(0) = h'(0) = 0;\)
2. \(0 \leq f(r), h(r) \leq 1, \rho^2(r) \leq \rho_0^2 + \frac{A_0^2}{2}, A(r) \leq A_0, B(r) \leq A_0, B(r) \geq A(r)\) for all \(r \geq 0;\)
3. \(r^2 \rho(r), rA(r), rB(r), r\sigma(r)\) are increasing, \(f(r), h(r), r^{-1}B(r), r^{-1}A(r)\) are decreasing;
4. \(r^{-k}\rho(r)\) is decreasing as long as \(\rho \leq \rho_0\), where \(k = \frac{1}{2}(\sqrt{3} - 1)\); \(r^{-2}\sigma(r)\) is decreasing as long as \(\sigma \leq 0;\)
5. \(r^{-1}B(r), r^{-1}A(r)\) is bounded as \(r \to 0\).

About the asymptotics of the solutions as \(r \to \infty\), we have the following result.

**Theorem 1.2 (The asymptotic exponential decay)** As \(r \to \infty\), there hold the sharp asymptotic estimates

\[
\begin{align*}
f(r) &= O(e^{-\kappa(1-\varepsilon)r}), \quad (1.17) \\
\rho(r) &= \rho_0 + O(r^{-1}e^{-\sqrt{2}\mu_0(1-\varepsilon)r}), \quad (1.18) \\
A(r) &= A_0 + O(r^{-1}), \quad (1.19) \\
h(r) &= O(e^{-\varsigma(1-\varepsilon)r}), \quad (1.20) \\
B(r) - A(r) &= O(r^{-1}e^{-\nu_0(1-\varepsilon)r}), \quad (1.21) \\
\sigma(r) &= \sigma_0 + O(r^{-1}e^{-\sqrt{2}\nu(1-\varepsilon)r}), \quad (1.22)
\end{align*}
\]

where \(\varepsilon > 0\) can be taken to be arbitrarily small, \(\rho_0 = \mu \sqrt{\frac{2}{\kappa}}\), \(\nu = \frac{\rho_0}{2} \sqrt{g^2 + g'^2}\) and the decay exponents are defined by the expressions

\[
\begin{align*}
\kappa &= \sqrt{\frac{1}{4}g^2 \rho_0^2 - A_0^2}, \quad \mu_0 = \min\{\mu, \sqrt{2}\kappa, \frac{\nu}{\sqrt{2}}\}, \\
\varsigma &= \sqrt{g^2 \sigma_0^2 - A_0^2}, \quad \nu_0 = \min\{2\kappa, \nu\}, \quad \xi = \sqrt{2}\sigma_0.
\end{align*}
\]

**Remark:** In the proof of Theorem 1.1, we will give asymptotic estimates for the solution of the above problem when \(r \to 0\), and we omit them here.

As the end of this section, we state the arrangement of this paper as follows. In Section 2, we first give a series of lemmas as the primary works for proving our main results. And then, by using the shooting method and the Sturm comparison principle, we prove the existence of the dyon solution of each second-order nonlinear ordinary differential equation and establish the qualitative properties of solutions. In Section 3, by using the Schauder fixed point theory, we study the two-point boundary value problem (1.9)-(1.16) and obtain the asymptotic behaviors of the solutions at infinity.
2 Primaries

The proof of the theorem depends on a fixed-point argument, and we outline this in the statement of a series of lemmas, which will be proved in sequence.

**Lemma 2.1** Given a pair of functions \((\rho(r), A(r), h(r)) \in C([0, +\infty))\) such that

1. \(h'(0) = h(\infty) = 0, \ h(0) = 1, \ \rho(\infty) = \rho_0, \ A(\infty) = A_0;\)
2. When \(r \to 0, \ r^{-k}\rho(r), \ r^{-1}A(r), \ r^{-1}(h(r) - 1)\) exist finite limits;
3. \(h(r)\) is decreasing, \(r\rho(r), rA(r)\) are increasing;
4. \(r^{-k}\rho(r), \ r^{-1}A(r)\) is decreasing so long as \(r^\alpha \leq \frac{\rho_0}{R^*} \leq 1,\) that is \(\rho(r) \leq \rho_0;\)
5. \(\rho^2(r) \leq \rho_0^2 + \frac{A^2}{2r}, \ A(r) \leq A_0;\)
6. \(0 < \alpha < k = \frac{1}{2}(\sqrt{5} - 1), \ R^* = R^*(\rho_0, \lambda, \mu, A_0, g, g')\) is a positive constant (without loss of generality, we could assume \(R^* \geq \rho_0\)), then we can find a unique continuously differentiable function \(f\) satisfying equation (1.9) and the conditions as follow
   1. \(f(0) = 1, \ f(\infty) = 0, \ f'(0) = 0, \ f(r)\) is decreasing;
   2. \(|\rho^{-2}\psi(r)| \leq N \left( R^* \frac{2^2}{1+\alpha} + r^{2\alpha} R^* \right)^2 \ (\forall r \leq 1), \) where \(\psi(r) = f(r) - 1, \ N = N(\rho_0, \lambda, \mu, A_0, g, g')\) is a positive constant.

**Lemma 2.2** Given the functions \((\rho(r), A(r), h(r))\) as in Lemma 2.1, and the associated function \(f(r)\), then we can find a unique function \(B(r) \in C^1([0, +\infty))\) satisfying the equation (1.12) and the conditions

\[ B(0) = 0, \ B(\infty) = B_0, \ A(r) \leq B(r) \leq A_0, \ |r^{-1}B(r)| \leq R^* \ (\forall r \leq 1). \]

**Lemma 2.3** Given the functions \((\rho(r), A(r), h(r))\) as in Lemma 2.1, and the associated function \(f(r)\), then we can find a unique function \(\sigma(r) \in C^1([0, +\infty))\) satisfying the equation (1.14) and the conditions

\[ \sigma(0) = 0, \ \sigma(\infty) = \sigma_0, \ r\sigma(r)\text{ is increasing}, \]

\[ r^{-1}\sigma(r)\text{ is decreasing so long as } \sigma \leq \sigma_0, \ |r^{-1}\sigma(r)| \leq R^* \ (\forall r \leq 1). \]

**Lemma 2.4** Given the function \((\rho(r), A(r), h(r))\) as in Lemma 2.1, and the associated function \(B(r), \sigma(r)\) from Lemmas 2.2, 2.3, then we can find a unique function \(\tilde{h}(r) \in C^1([0, +\infty))\) satisfying the equation

\[ \tilde{h}'' - \frac{\tilde{h}^2 - 1}{r^2} \tilde{h} = (g^2\sigma^2 - B^2) \tilde{h} \]  \hspace{1cm} (2.1)  

and the conditions \(\tilde{h}(r)\) is decreasing,

\[ \tilde{h}(0) = 0, \ \tilde{h}(\infty) = 1, \ \tilde{h}'(0) = 0, \ |r^{-2}(\tilde{h}(r) - 1)| \leq N \left( R^* \frac{2^2}{1+\alpha} + r^{2\alpha} R^* \right)^2 \ (\forall r \leq 1). \]

**Lemma 2.5** Given the function \((\rho(r), A(r), h(r))\) as in Lemma 2.1, and the associated function \(f(r), B(r)\) from Lemmas 2.1, 2.2, then we can find a unique function \(\tilde{\rho}(r) \in C^1([0, +\infty))\) satisfying the equation

\[ (r\tilde{\rho})'' - \frac{f^2}{2r} \tilde{\rho} = -\frac{1}{4}r \ (A - B)^2 \tilde{\rho} + \frac{\lambda}{2} r \tilde{\rho} \left( \tilde{\rho}^2 - \frac{2\mu^2}{\lambda} \right) \]  \hspace{1cm} (2.2)
and the conditions
\[ \dot{\rho}(0) = 0, \quad \dot{\rho}(\infty) = \rho_0, \quad \ddot{\rho}(r) \leq \rho_0^2 + \frac{A^2}{2r}, \quad r\dot{\rho}(r) \text{ is increasing,} \]
\[ r^{-k}\dot{\rho}(r) \text{ is decreasing so long as } \dot{\rho}(r) \leq \rho_0, \quad |r^{-k}\dot{\rho}(r)| \leq R_2^* (\forall r \leq 1), \] where \( k = \frac{1}{2}(\sqrt{3} - 1) \).

**Lemma 2.6** Given the function \((\rho(r), A(r), h(r))\) as in Lemma 2.1, and the associated function \(f(r), B(r)\) from Lemmas 2.1, 2.2, then we can find a unique function \(\tilde{A}(r) \in C^1([0, +\infty))\) satisfying the equation
\[ (r\tilde{A})'' - \frac{2}{r} f^2 \tilde{A} = -\frac{1}{4} g^2 \rho^2 r \left( B - \tilde{A} \right) \] (2.3)
and the conditions
\[ \tilde{A}(0) = 0, \quad \tilde{A}(\infty) = A_0, \quad r\tilde{A}(r) \text{ is increasing,} \]
\[ r^{-1}\tilde{A}(r) \text{ is decreasing,} \quad \tilde{A}(r) \leq B(r), \quad r^{-1}\tilde{A}(r) \leq R_3^* (\forall r \leq 1). \]

The final theorem is then just a matter of constructing a mapping and showing it has a fixed point, and this is proved in the final section and so does the asymptotics of the solutions.

### 2.1 Proof of Lemma 2.1 (Existence and uniqueness of \(f(r)\))

In this subsection, we prove in two steps. To solve the two-point boundary value problem, we use a single parameter shooting method. When we do this, we need to consider the initial value problem
\[ f'' - \frac{f^2 - 1}{r^2} f = \left( \frac{g^2}{4} \rho^2 - A^2 \right) f, \quad f(0) = 1. \] (2.4)

Firstly, we prove the existence of local solutions to the initial value problem (2.4).

**Lemma 2.7** There exists a locally continuous solution of the initial value problem (2.4) near \(x = 0\).

**Proof** Using the idea from the literature [6] and [29], this lemma will be proved. Firstly, we introduce a new variable \(w = f'\), and rewrite the equation (2.4) as the first-order ordinary differential equation system
\[ f' = w, \] (2.5)
\[ (r^2 w)' = 2rw + \left[ (f^2 - 1) + r \left( \frac{g^2}{4} \rho^2 - A^2 \right) \right] f. \] (2.6)

Then consider the space \(X\) as follows
\[ X = \{ (f(r), w(r)) \in C([0, x]) | \|f(r) - 1\| \leq 1, \|w(r)\| \leq 1, \forall r \in [0, x] \} \]
where \(\|h\| = \sup \{|h(r)| : 0 \leq r \leq x\}\). It is clear that \(X\) is a complete normed linear space if we take as metric the maximum of the two components. We define a map \(T : (f, w) \rightarrow (T_1, T_2)\) on \(X\) where
\[ T_1 = 1 + \int_0^r wds, \] (2.7)
\[ T_2 = \frac{1}{r^2} \int_0^r 2sw + \left[ (f^2 - 1) + s \left( \frac{g^2}{4} \rho^2 - A^2 \right) \right] fds. \] (2.8)
One verifies easily that $T$ does in fact take $X$ to $X$ and that $T$ is a contracting map if $r$ is sufficiently small, and that a fixed point of $T$ is a solution to our equation. Therefore the lemma follows.

Setting $\psi = f - 1$, then we may rewrite equation (2.4) as

$$\psi'' = \frac{2\psi}{r^2} + \left(\frac{1}{4}g'\rho^2 - A^2\right)(\psi + 1) + \frac{3\psi^2 + \psi^3}{r^2}. \quad (2.9)$$

Using the basic theory of ordinary differential equations and initial value condition, the differential equation can be transformed into the following integral equation

$$\psi(r) = Cr^2 + \frac{1}{3} \int_0^r (r^2 s^{-1} - r^{-1} s^2) \left\{ \left(\frac{1}{4} g'\rho^2 - A^2\right)(\psi + 1) + \frac{3\psi^2 + \psi^3}{s^2} \right\} ds, \quad (2.10)$$

where $C$ is an arbitrary constant. Equation can be solved by lemma 2.7, at least for $r$ sufficiently small. And according to (2.10), we obtain a solution with

$$\psi(r) = Cr^2 + O(r^{2+2\alpha}) \ (r \to 0^+). \quad (2.11)$$

Applying the extension theorem of the solution of the ordinary differential equation, the solution can be extended to its maximum existence interval $[0, R_C)$, where either $R_C = \infty$ or $\lim_{r \to R_C} \psi(r) = \infty$. By the continuous dependence of the solution on the parameters theorem we obtain that the solution $\psi$ depends continuously on the parameter $C$.

Next we show the existence of the global solution to the boundary value problem, then we consider $C > 0$ and define the sets $S^f_1, S^f_2, S^f_3$ as follows

$$S^f_1 = \{ C < 0 : \psi'(r; C) \text{ becomes positive before } \psi(r; C) \text{ reaches } -1 \},$$

$$S^f_2 = \{ C < 0 : \psi(r; C) \text{ crosses } -1 \text{ before } \psi'(r; C) \text{ becomes 0} \},$$

$$S^f_3 = \{ C < 0 : \forall r > 0, \psi'(r; C) \leq 0, -1 < \psi(r; C) < 0 \}.$$

Obviously, it follows from the construction of the set

$$S^f_1 \cup S^f_2 \cup S^f_3 = S^f, \quad S^f_1 \cap S^f_2 = S^f_2 \cap S^f_3 = S^f_3 \cap S^f_1 = \emptyset.$$

**Lemma 2.8** The sets $S^f_1, S^f_2$ are both open and nonempty.

**Proof** Firstly, $S^f_1$ contains $C$ small. Inserting $C = 0$ into the equation (2.10), we obtain the equations as follows

$$\psi(r) = \frac{1}{3} \int_0^r (r^2 s^{-1} - r^{-1} s^2) \left\{ \left(\frac{1}{4} g'\rho^2 - A^2\right)(\psi + 1) + \frac{3\psi^2 + \psi^3}{s^2} \right\} ds, \quad (2.12)$$

$$\psi'(r) = \frac{1}{3} \int_0^r (2rs^{-1} + r^{-2} s^2) \left\{ \left(\frac{1}{4} g'\rho^2 - A^2\right)(\psi + 1) + \frac{3\psi^2 + \psi^3}{s^2} \right\} ds. \quad (2.13)$$

When $r > 0$ is sufficiently small, it is obvious that $\psi' > 0$. According to (2.4), we obtain that $\frac{1}{4}g\rho^2 - A^2 > 0$. In addition, $\psi + 1 > 0$, $\frac{3\psi^2 + \psi^3}{s^2} > 0$ are bounded. Hence $\psi > 0$ when
\( r > 0 \) is sufficiently small. By the continuous dependence of \( \psi \) and \( \psi' \) on the parameter \( C \), we obtain that \( \psi > 0, \psi' > 0 \) when \( C < 0, r > 0 \) are both sufficiently small. Therefore \( C \in S^f_1 \) and the nonemptiness of \( S^f_1 \) is established. It is evident that \( S^f_1 \) is open because the continuous dependence of \( \psi \) and \( \psi' \) on the parameter \( C \).

Secondly, \( S^f_2 \) contains \( C \) large. Here we introduce a transformation to consider a modified variable \( t = |C|^{2r} \) in (2.10) so that it becomes

\[
\psi(t) = -t^2 + \frac{1}{3} \int_0^t (t^2 \tau - 1 - t^{-1} \tau^2) \left\{ |C|^{-1} \left( \frac{1}{4} g^2 \rho^2 - A^2 \right) (\psi + 1) + \frac{3\psi^2 + \psi'^3}{\tau^2} \right\} d\tau, \quad (2.14)
\]

\[
\psi'(t) = -2t + \frac{1}{3} \int_0^t (2t \tau - 1 - t^{-2} \tau^2) \left\{ |C|^{-1} \left( \frac{1}{4} g^2 \rho^2 - A^2 \right) (\psi + 1) + \frac{3\psi^2 + \psi'^3}{\tau^2} \right\} d\tau. \quad (2.15)
\]

The differential equation corresponding to (2.14) is

\[
\psi_{tt} - \frac{\psi(\psi + 1)(\psi + 2)}{t^2} = \left( \frac{1}{4} g^2 \rho^2 - A^2 \right) (\psi + 1)|C|^{-1}. \quad (2.16)
\]

Since \( r^{-\alpha} \rho, r^{-\alpha} A \leq R^*, \forall r \leq 1 \), we obtain that

\[
\rho \leq |C|^{-1} \tau^\alpha R^*, \quad A \leq |C|^{-1} \tau^\alpha R^*, \quad \forall t \leq |C|^{\frac{1}{2}}.
\]

From the continuity of the function \( \psi(t) \) on \([0, R]\), then for all \( t \leq |C|^{\frac{1}{2}} \), there is an \( M > 0 \) such that \( |\psi(t)| \leq M \). Hence we obtain the estimate

\[
|C|^{-1} \left( \frac{1}{4} g^2 \rho^2 - A^2 \right) (\psi(t) + 1) \leq |C|^{-1-\alpha} \tau^{2\alpha} R^* M \left( \frac{1}{4} g^2 + 1 \right).
\]

According to

\[
\left( \frac{1}{4} g^2 \rho^2 - A^2 \right) (\psi + 1)|C|^{-1} \to 0 \text{ as } C \to -\infty,
\]

we obtain that (2.14) is equivalent to the differential equation

\[
\frac{d^2 \psi}{dt^2} = \frac{\psi(\psi + 1)(\psi + 2)}{t^2}, \quad \forall t \in [0, R],
\]

where \( R \) is a positive constant. It is clear that the solution of (2.16) as follows

\[
\psi \sim -t^2, \quad (r \to 0, \ C \to -\infty).
\]

Thus the solution \( \psi \) could cross \(-1\) at \( t_0 = 1 + \varepsilon_0 (\varepsilon_0 > 0) \). Noting that \( \psi(t) < 0, \forall t \in (0, t_0) \), then \( \psi(r; C) \) crosses \(-1\) before \( \psi'(r; C) \) becomes 0. Therefore there exists some positive constant \( N = N(\rho_0, \lambda, \mu, A_0, g, g') \) independent of the choice of \( \rho, A, \) and \( R^* \) such that \( C \in S^f_2 \) when \( R^* |C|^{-1-\alpha} < \frac{1}{N} \). In other words, \( S^f_2 \) is nonempty. The fact that \( S^f_2 \) is open is self-evident.

Since the connected set \( C < 0 \) cannot consist of two open disjoint non-empty sets, there must be some value of \( C \) in neither \( S^f_1 \) nor \( S^f_2 \). For this value of \( C \), say \( C_0 \), we have a solution with \( \psi'(r; C) \leq 0, \ -1 < \psi(r; C) < 0 \).
Lemma 2.9 The solution corresponding to the parameter $C_0$ in $S_0^I$ satisfies $\lim_{r \to \infty} f(r) = 0$.

Proof Since $-1 < \psi(r; C_0) < 0$ and $\psi'(r; C_0) \leq 0$, $\forall r > 0$, we obtain that $\lim_{r \to \infty} f(r; C_0) \triangleq L \geq 0$. It is obvious that $f(r) = O(e^{-\sqrt{\kappa r}})(r \to \infty)$ because $f'' \sim (\frac{g^2}{4} \rho^2 - A^2)f$ as $r \to \infty$. Noting that

$$
\lim_{r \to \infty} \kappa(r) = \lim_{r \to \infty} \left(\frac{g^2}{4} \rho^2 - A^2\right) = \frac{g^2}{4} \rho_0^2 - A_0^2 > 0,
$$
we have $L = \lim_{r \to \infty} f(r) = 0$.

Finally, we want to prove that the solution $f$ is the only solution satisfying the conditions that

$$
f'(0) = 0, \quad |r^{-2} \psi(r)| \leq N \left(R^\frac{2}{1+\alpha} + r^{2\alpha} R^2\right) \quad (\forall r \leq 1).
$$

Lemma 2.10 The solution for the given parameter $C_0$ is unique.

Proof Suppose otherwise that there are two solutions $f_1, f_2$, and set $\Psi(r) = f_2(r) - f_1(r)$. Then the function $\Psi(r)$ satisfies the boundary condition $\Psi(0) = \Psi(\infty) = 0$ and the equation

$$
r^2 \Psi''(r) = \left[(f_1''(r) - 1) + \left(\frac{1}{4} g^2 \rho^2(r - A^2(r)) r^2 + (f_2''(r) + f_1(r) f_2(r))\right)\right] \Psi(r)
\triangleq Q(r) \Psi(r), \quad 0 < r < +\infty. \quad (2.18)
$$

Without loss of generality, we assume $\Psi > 0$ when $r > 0$ is sufficiently small. According to (2.4), we obtain that

$$
r^2 f''_i(r) = \left[(f_i''(r) - 1) + \left(\frac{1}{4} g^2 \rho^2(r - A^2(r)) r^2 + f_i(r) f_i(r)\right)\right] f_i(r)
\triangleq q_i(r) f_i(r), \quad 0 < r < +\infty, \quad (i = 1, 2). \quad (2.19)
$$

Since $Q(r) - q_1(r) > 0$, Applying the Sturm-Picone comparison theorem to (2.18) – (2.19), we conclude that $f_1$ have more zero points than $\Psi$. Note that $f_1(r) \neq 0$ as $r \in [0, +\infty)$, then we have $\Psi(r) \neq 0$ for all $r \in [0, +\infty)$, which contradicts $\Psi(0) = 0$.

Lemma 2.11 If $C_0 \in S_0^I$, for $r \leq 1$, we can find $N = N(\rho_0, \lambda, \mu, A_0, g, g') > 0$ such that

$$
|r^{-2} \psi(r)| \leq N \left(R^\frac{2}{1+\alpha} + r^{2\alpha} R^2\right),
$$
where $\psi = f - 1$. Furthermore, we have $f'(0) = 0$.

Proof According to (2.10), $A(s) \leq s^\alpha R^s, \rho(s) \leq s^\alpha R^s, \forall s \leq 1$, and $|C_0|^{1+\alpha} \leq N R^2$ (because $C_0 \in S_0^I$), we arrive at

$$
\left|\frac{\psi}{r^2}\right| \leq |C_0| + \frac{1}{3} \int_0^r s^{-1} \left(\frac{1}{4} g^2 + 1\right) s^{2\alpha} R^2 ds
\leq \left(N R^2\right)^{\frac{2}{1+\alpha}} + \left(\frac{\frac{1}{4} g^2 + 1}{3}\right) \int_0^r s^{2\alpha-1} R^2 ds
\leq N' \left(R^\frac{2}{1+\alpha} + r^{2\alpha} R^2\right) \leq N'' \left(R^\frac{2}{1+\alpha} + R^2\right) \triangleq R^*, \quad \forall r \leq 1, \quad (2.20)
$$
where $N' = \max\{\frac{1}{12}, \frac{1}{60}, 1\}$. Using the above result, we have $\psi(r) = O(r^2)(r \to 0^+)$. By the definition of the derivative of the function $\psi(r)$ at $r = 0$, we obtain that

$$
\lim_{r \to 0^+} \frac{\psi(r) - \psi(0)}{r} = \lim_{r \to 0^+} \frac{O(r^2)}{r} = 0.
$$

Namely, $f'(0) = 0$. This completes the proof of the Lemma 2.1.

\[ \blacksquare \]

### 2.2 Proof of Lemma 2.2 (Existence and uniqueness of $B(r)$)

By noting that we can rewrite (1.12) as

$$
r^2B'' + 2rB' - 2B = 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2s^2(B - A). \tag{2.21}
$$

Similarly, differential equation (2.21) can be transformed into the integral equation form

$$
B(r) = br + \frac{1}{3} \int_0^r (r - r^{-2}s^3) \left[ 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2s^2(B - A) \right] ds, \tag{2.22}
$$

where $b$ is an arbitrary constant, which can be solved by Picard iteration to give a locally continuous solution of the initial value problem that exists at least for $r$ sufficiently small, and we obtain a solution with $B(r) = br + O(r^{2+2\alpha})$, and the solution continuously depends on the parameter $b$.

Same as previous section, we introduce three sets

$$
S_1^B = \{ b > 0 : B(r; b) < A \text{ before } B(r; b) \text{ reaches } A_0 \},
$$

$$
S_2^B = \{ b > 0 : B(r; b) \text{ crosses } A_0 \text{ before } B(r; b) = A \},
$$

$$
S_3^B = \{ b > 0 : \forall r > 0, B(r; b) \leq A_0, B(r; b) \geq A \}.
$$

Obviously, $S_1^B \cup S_2^B \cup S_3^B = S^B$, $S_1^B \cap S_2^B = S_2^B \cap S_3^B = S_3^B \cap S_1^B = \emptyset$.

**Lemma 2.12** The set $S_1^B, S_2^B$ are both open and nonempty.

**Proof** According to the integral equation (2.22), if $b = 0$, since

$$
B'(r) = \frac{1}{3} \int_0^r \left[ 1 + 2r^{-3}s^3 \right] \left[ 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2s^2(B - A) \right] ds, \tag{2.23}
$$

by iteration we obtain that $B' < 0$ when $r > 0$ is sufficiently small. If $r > 0$ is near zero, it is clear that $B - A < 0$ because $0 < A < 1$ and $B(0) = 0$. By the continuous dependence of $B$ on the parameter $b$ we obtain that $B - A < 0$ for the sufficiently small $b, r > 0$. Let $r = r_0$ be the first value satisfying $B(r) = A(r)$, then

$$
B'(r) \bigg|_{r=r_0} = \left( b + \frac{1}{3} \int_0^r \left[ 1 + 2r^{-3}s^3 \right] (h^2 - 1) A \right) ds \bigg|_{r=r_0} > 0.
$$

Obviously,

$$
0 < b < \left\{ \frac{1}{3} \int_0^r \left[ 1 + 2r^{-3}s^3 \right] (h^2 - 1) A \right) ds \right\|_{r=r_0}.
$$

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for all $b \in (0, A_0r_0)$, then $B'|_{r=r_0} < 0$, which contradicts $B'|_{r=r_0} > 0$. Hence, $B < A$ when $0 < b \leq A_0r_0$. The following proves that the above $b$ can be taken. Since

\[
\begin{align*}
\frac{b}{r^2} &< \left| \frac{1}{3} \left\{ \int_0^t [(1 + 2r^{-3}s^3)(h^2 - 1)]ds \right\} \right|_{r=r_0} \\
&\leq \frac{2}{3} \left\{ \int_0^t [(1 + 2r^{-3}s^3) |h^2 - 1| |A|]ds \right\} \right|_{r=r_0} \\
&\leq \frac{2}{3} A_0 \left\{ \int_0^t (1 + 2r^{-3}s^3) ds \right\} \right|_{r=r_0} = A_0r_0,
\end{align*}
\]

then $S^B_1$ is nonempty.

On the other hand, $S^B_2$ contains big $b$. Here we introduce a transformation $t = br$ in (2.22) to consider

\[
\begin{align*}
B(t) &= t + \frac{1}{3} \int_0^t \frac{\tau^2}{b^2} (t\tau - 2 - t^2\tau) \left[ 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2b^{-2}\tau^2(B - A) \right] d\tau, \quad (2.24) \\
B'(t) &= 1 + \frac{1}{3} \int_0^t \frac{\tau^2}{b^2} (t\tau - 2 + t^2\tau) \left[ 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2b^{-2}\tau^2(B - A) \right] d\tau. \quad (2.25)
\end{align*}
\]

In view of

\[
\left| \frac{h - 1}{r^2} \right| = \left| \frac{\Phi}{r^2} \right| \leq N \left( r^{2\alpha}R^{\tau^2} + \frac{R}{r^{1+\alpha}} \right), \quad \forall r \leq 1,
\]

we obtain that

\[
\left| (h^2 - 1)(\tau) \right| \leq N \left( R^{\frac{2\alpha}{1+\alpha}} + b^{-2\alpha}R^{\tau^2} \right) b^{-2}\tau^2, \quad \forall \tau \leq t \leq b.
\]

By the continuous of $A(r), B(r), \rho(r)$ for all $r \in [0, R]$, there exist $M_1, M_2, M_3 > 0$ such that $|B(t)| \leq M_1, |A(t)| \leq M_2, |\rho(t)| \leq M_3$ for all $t \leq b$, then we have

\[
\frac{1}{b^2} \left[ 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2b^{-2}\tau^2(B - A) \right] \leq \frac{1}{b^2} \left[ 2N(R^{\frac{2\alpha}{1+\alpha}} + R^{\tau^2})M_1 + \frac{1}{4}g^2M_3^2(M_1 + M_2) \right].
\]

If $b \to \infty$, according to the integral value theorem we get

\[
\frac{1}{3} \int_0^t \frac{\tau^2}{b^2} (t\tau - 2 - t^2\tau) \left[ 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2b^{-2}\tau^2(B - A) \right] d\tau \to 0.
\]

Substituting it into (2.24), there exists a constant $R$ such that $B$ crosses $A_0$ at $t_0 = A_0 + 1$ for all $t \in [0, R]$ when $(R^{\frac{2\alpha}{1+\alpha}} + b^{-2\alpha}R^{\tau^2})b^{-2}$ is sufficiently small. In addition, when $b > 0$ is sufficiently large, for any $t \in (0, t_0]$, we have $B'(t) > 0$. In particular, $B'(0) = b > 0$ is sufficiently large at $r = 0$. The continuity can ensure that two sets are open.

Since the connected set $b > 0$ cannot consist of two open disjoint nonempty sets, there must be some value of $b$ in neither $S^B_1$ nor $S^B_2$. For this value of $b$, say $b_0$, we have a solution with $0 \leq A(r) \leq B(r; b_0) \leq A_0, \forall r > 0.$
Lemma 2.13 The solution corresponding to the parameter $b_0$ in $S_3^B$ satisfies $\lim_{r \to \infty} B(r) = A_0$.

Proof Since $0 \leq A(r) \leq B(r) \leq A_0$ and $(rB(r))'' \geq 0$, $\forall r > 0$, then $\lim_{r \to \infty} (rB)' \equiv L \leq +\infty$. If $L = +\infty$, for convenience, we assume $G = 4A_0$. When $r \geq r_0$, integrating over $(r_0, r)$ for $(rB)' > G$, we see easily that

$$B(r) > \frac{r_0}{r}B(r_0) + (1 - \frac{r_0}{r})G > 2A_0,$$

which contradicts $\lim_{r \to \infty} B(r) = A_0$. Using the L'Hopital's rule, we have

$$\lim_{r \to \infty} B(r) = \lim_{r \to \infty} \frac{rB(r)}{r} = \lim_{r \to \infty} (rB(r))' = L.$$

On the one hand, $L \leq A_0$ because $B \leq A_0$. On the other hand, applying $B \geq A$ and $\lim_{r \to \infty} A(r) = A_0$, we arrive at $L \geq A_0$. Consequently, $\lim_{r \to \infty} B(r) = L = A_0$. \hfill $\square$

Now we have obtained that the solution corresponding to the parameter $b_0$ is the solution of the boundary value problem consisting of (1.12), (1.15) and (1.16). The following will prove that the solution is unique. To this end, we have the following lemma.

Lemma 2.14 The solution for the given parameter $b_0$ is unique.

Proof Assume that there are two solutions $B_1(r), B_2(r)$, and set $\Psi(r) = B_2(r) - B_1(r)$. Then it satisfies the boundary condition $\Psi(0) = \Psi(\infty) = 0$ and the equation

$$(r\Psi(r))'' = \frac{2}{r}h^2(r)\Psi(r) + \frac{1}{4}g^2\rho^2(r) r\Psi(r), \ 0 < r < +\infty. \quad (2.26)$$

When $r > 0$, we assume $\Psi(r) > 0$ (Similarly, it can be seen that $\Psi(r) < 0$), therefore $\Psi'(r) = 0, \Psi''(r) \leq 0$ at $r = r_0$. Applying the maximum principle to (2.26), we conclude that $\Psi(r) \equiv 0$, which contradicts the assumption that $B_1(r) \neq B_2(r)$. \hfill $\square$

Lemma 2.15 If $b_0 \in S_3^B$, for $r \leq 1$, we can find a suitably large constant $R^*$ such that $|r^{-1}B(r)| \leq R^*$.

Proof According to the equation (2.22), since $|h^2 - 1| \leq N \left( r^{2\alpha} R^{*2} + R^{*\frac{2}{1+\alpha}} \right) r^2$, $B - A \leq A_0$, $\rho \leq \rho_0$, $B \leq A_0$ for $r \leq 1$, we have

$$|B - b_0 r| \leq \frac{1}{3} \int_0^r \left[ 2(h^2 - 1)B + \frac{1}{4}g^2\rho^2s^2(B - A) \right] ds \leq \frac{1}{3} \int_0^r \left[ 2r^3N \left( R^{*2}r^{2\alpha} + R^{*\frac{2}{1+\alpha}} \right) A_0 + \frac{r}{4}s^2g^2\rho_0^2 A_0 \right] ds \leq N_1 \left( R^{*2}r^{2\alpha+4} + R^{*\frac{2}{1+\alpha}r^4} + r^4 \right),$$

where $N_1 = \max\{ \frac{2}{3}NA_0, \frac{1}{36}g^2\rho_0^2 A_0 \}$. 

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Suppose \( r = 2b_0^{-1}A_0 \), there is a \( N_2 = \max\{2^{2\alpha+4}A_0^{2\alpha+4}N_1, 2^4A_0^4N_1\} \) such that
\[
|B(r) - 2A_0| \leq N_2 \left[ R^{\ast 2}b_0^{-2\alpha+4} + \left( R^{\ast 2\alpha+2} + 1 \right) b_0^{-4} \right].
\]

Since the left-hand side is bounded, we conclude that there exist a \( C > 0 \) such that
\[
N_2[R^{\ast 2}b_0^{-2\alpha+4} + (R^{\ast 2\alpha+2} + 1)b_0^{-4}] \geq C.
\]

That is, we have \( b_0 \leq N_3R^{\ast 2\alpha+2} \triangleq R^{\ast} \), where \( N_3 = \max\{(C^{-1}N_2)^{2\alpha+2}, (C^{-1}N_2)^{\frac{3}{2}}\} \). In view of \( B(r) = b_0r + O(r^{2+2\alpha}) (r \to 0^+) \) for all \( r \leq 1 \), we obtain that
\[
|r^{-1}B(r)| \leq |b_0| + |O(r^{1+2\alpha})| \leq (R^{\ast} + 1) \triangleq R^{\ast}. \tag{2.27}
\]

Thus, for \( r \leq 1 \), we can find a suitably large constant \( R^{\ast} \) such that \( |r^{-1}B(r)| \leq R^{\ast} \). The proof of Lemma 2.2 is complete. \( \Box \)

### 2.3 Proof of Lemma 2.3 (Existence and uniqueness of \( \sigma(r) \))

In order to study the solution of the boundary value problem be related to the equation (1.14) subject to the boundary conditions \( \sigma(0) = 0 \) and \( \sigma(\infty) = \sigma_0 \), we can rewrite (1.14) as
\[
(r\sigma)^{''} - \frac{2}{r^2}(r\sigma) = \kappa \left( \alpha^2 - \frac{m^2}{\kappa} \right) (r\sigma) + \frac{2}{r^2}(h^2 - 1)(r\sigma). \tag{2.28}
\]

Let \( H = r\sigma \), we can convert (2.28) into the integral equation
\[
H(r) = Dr^2 + \frac{1}{3} \int_0^r \left( s^{-1}r^2 - r^{-1}s^2 \right) H(s) \left[ \kappa \left( \frac{H^2}{s^2} - \frac{m^2}{\kappa} \right) + \frac{2}{s^2}(h^2 - 1) \right] ds, \tag{2.29}
\]
where \( D \) is an arbitrary constant.

Similar to the idea in Section 3, it follows that there exists a locally continuous solution \( \rho(r) = Dr + O(r^2) (r \to 0^+) \) of the initial value problem consisting of (1.14) and (1.15). We are interested in \( D > 0 \), and now we define three sets as follows
\[
S_1^\sigma = \{ D > 0 : \exists r_0 > 0 \text{ such that } (r\sigma(r))' \big|_{r=r_0} < 0 \text{ before } H(r; D) \text{ becomes infinite} \},
\]
\[
S_2^\sigma = \{ D > 0 : r\sigma(r) \text{ becomes infinite before } (r\sigma(r))' \text{ becomes zero} \},
\]
\[
S_3^\sigma = \{ D > 0 : \forall r > 0, \sigma(r; D) \text{ is finite, } r\sigma(r; D) \geq 0 \}.
\]

It is easy to see that \( S_1^\sigma \cup S_2^\sigma \cup S_3^\sigma = S^\sigma \), \( S_1^\sigma \cap S_2^\sigma = S_2^\sigma \cap S_3^\sigma = S_3^\sigma \cap S_1^\sigma = \emptyset \).

**Lemma 2.16** The set \( S_1^\sigma, S_2^\sigma \) are both open and nonempty.

**Proof** When \( D > 0 \) is sufficiently small, we could assume \( \sigma^2 - \frac{m^2}{\kappa} = -\frac{m^2}{2\kappa} \) for any bounded range of \( r \), then
\[
(r\sigma(r))'' = \left( \frac{2}{r^2}h^2 - \frac{m^2}{2} \right) (r\sigma(r)). \tag{2.30}
\]

It is clear that \( (r\sigma(r))' > 0, r\sigma(r) > 0 \) for all small \( r \). If \( r > \frac{2}{m} \), (2.30) is an oscillatory equation in the above bounded range of \( r \). Hence, for \( r > \frac{2}{m} \), there exists a bounded range...
of \( r \) such that \((r\sigma)' < 0\). Meanwhile, \( r\sigma \) is finite, which means \( \sigma \) is finite. Then there exists a \( r = r_0 > 0 \) such that \((r\sigma)'|_{r=r_0} < 0\) before \( H(r;D) \) becomes infinite. Hence \( S_1^\sigma \) is nonempty.

In order to proof \( S_2^\sigma \) is nonempty, we use the variable \( t \) to replace \( r : r = D^{-\frac{1}{2}}t \). Thus (2.29) becomes

\[
H(t) = t^2 + \frac{1}{3} \int_0^t (\tau^{-1} - t^2 \tau^{-1}) H(\tau) \left[ \frac{H^2}{\tau^2} - \frac{m^2}{D} - \frac{2}{\tau^2} (h^2 - 1) \right] d\tau. \tag{2.31}
\]

Then we get

\[
H'(t) = 2t + \frac{1}{3} \int_0^t (2 \tau^{-1} t + \tau^2 \tau^{-3}) H(\tau) \left[ \frac{H^2}{\tau^2} - \frac{m^2}{D} - \frac{2}{\tau^2} (h^2 - 1) \right] d\tau, \tag{2.32}
\]

\[
H''(t) = 2 + \frac{1}{3} \int_0^t 2 \tau^{-1} (1 - \tau^3 \tau^{-3}) H(\tau) \left[ \frac{H^2}{\tau^2} - \frac{m^2}{D} - \frac{2}{\tau^2} (h^2 - 1) \right] d\tau
\]

\[+ H(\tau) \left[ \frac{H^2}{\tau^2} - \frac{m^2}{D} - \frac{2}{\tau^2} (h^2 - 1) \right]. \tag{2.33}\]

According to

\[
\left| \frac{\Phi}{\tau^2} \right| \leq N \left( r^{2\alpha} R^2 + R^{3 \alpha + \frac{2}{\alpha}} \right), \forall r \leq 1,
\]

we arrive at

\[
\left| \frac{(h^2 - 1)}{\tau^2} \right| \leq N \left( R^{3 \alpha + \frac{2}{\alpha}} + D^{-\alpha} \tau^{2\alpha} R^2 \right) D^{-1}, \forall \tau \leq D^{\frac{1}{2}}.
\]

If \( D \to \infty \) in above inequation, then

\[
\frac{m^2}{D} + \frac{2}{\tau^2} (h^2 - 1) \leq \frac{m^2}{D} + N \frac{R^{3 \alpha + \frac{2}{\alpha}} + R^2}{D} \to 0.
\]

By using (2.31)-(2.33) and the above estimate, we obtain that \( H(t) > 0, H'(t) > 0, H''(t) > 0 \). Therefore \( S_2^\sigma \) is nonempty. Continuity can ensure that two sets are open. \( \square \)

Since the connected set \( D > 0 \) cannot consist of two open disjoint non-empty sets, there must be some value of \( D \) in neither \( S_1^\sigma \) nor \( S_2^\sigma \). For this value of \( D \), say \( D_0 \), we have a solution with \( \sigma(r;D) \) is finite and \( r\sigma(r;D) \geq 0, \forall r > 0 \).

**Lemma 2.17** The solution corresponding to the parameter \( D_0 \) in \( S_3^\sigma \) satisfies \( \lim_{r \to \infty} \sigma(r) = \sigma_0 \).

**Proof** The fact that \( \sigma(r;D_0) \) is bounded at \([0, +\infty)\) follows immediately from the fact \( D_0 \in S_3^\sigma \). To see that

\[
\lim_{r \to \infty} \sigma(r) = m \sqrt{\frac{1}{\kappa}} \triangleq \sigma_0,
\]

we have the following two steps.

On the one hand, we have \( \sigma \leq \sigma_0 \). If not, we assume \( \sigma > \sigma_0 \). Next we claim \( \lim_{r \to \infty} \sigma'(r) = 0 \). We could assume \( \lim_{r \to \infty} \sigma'(r) = \beta > 0 \), then \( \sigma' > \frac{\beta}{2} > 0 \) when \( r \) is sufficiently small. And then

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\(\sigma(r) > \frac{2}{3}r + C\), which contradicts the finiteness of \(\sigma(r)\). Hence \(\lim_{r \to \infty} \sigma'(r) = 0\). Applying (2.28), we observe that \(\sigma'' > 0\) when \(r\) is sufficiently large, which contradicts \(\lim_{r \to \infty} \sigma'(r) = 0\).

On the other hand, we have \(\sigma \geq \sigma_0 - \epsilon, \forall \epsilon > 0\). If not, we assume \(\sigma < \sigma_0 - \epsilon\). Similarly, \(\lim_{r \to \infty} \sigma'(r) = 0\) is valid. According to (2.28), we obtain that \((r\sigma)'' < 0\) when \(r > 0\) is sufficiently large, that is, there exist a positive constant \(C\) such that \(H'' < -CH\). So \(D_0 \in S^q\), which contradicts \(D_0 \in S^q\).

With the above analysis, we must have \(\sigma_0 - \epsilon \leq \sigma \leq \sigma_0\) as \(r \to +\infty\), in other words, \(\lim_{r \to \infty} \sigma(r) = \sigma_0\).

Finally, we will prove the uniqueness and some properties of the solution \(\sigma\). To this end, we have the following lemmas.

**Lemma 2.18** The solution for the given parameter \(D_0\) is unique.

**Proof** Suppose otherwise that there are two solutions \(\sigma_1, \sigma_2\), and set \(\Psi(r) = \sigma_2(r) - \sigma_1(r)\). Then the function \(\Psi(r)\) satisfies the boundary condition \(\Psi(0) = \Psi(\infty) = 0\) and the equation

\[
(r \Psi(r))'' = \left[ \kappa \left( \frac{\sigma_2'(r) - \frac{m^2}{\kappa}}{\sigma_2'} \right) + \frac{2}{r^2} h^2(r) + \kappa(\sigma_2^2(r) + \sigma_1(r)\sigma_2(r)) \right] (r \Psi(r))
\]

\[
\triangleq Q_2(r)(r \Psi(r)), \quad 0 < r < +\infty.
\] (2.34)

Without loss of generality, we assume \(\Psi(r) > 0\) when \(r > 0\) is sufficiently small. In view of (2.28), we obtain that

\[
(r \sigma_i(r))'' = \left[ \kappa \left( \frac{\sigma_i^2(r) - \frac{m^2}{\kappa}}{\sigma_i'} \right) + \frac{2}{r^2} h^2(r) \right] (r \sigma_i(r))
\]

\[
\triangleq q_i(r)(r \sigma_i(r)), \quad 0 < r < +\infty, \quad (i = 1, 2).
\] (2.35)

Since \(Q_2(r) - q_1(r) > 0\), applying the Sturm-Picone comparison theorem to (2.34)–(2.35), we conclude that \(r \sigma_2(r)\) have more zero points than \(r \Psi(r)\) for all \(r \in (0, +\infty)\). Note that \(r \sigma_2(r) \neq 0\) for all \(r \in (0, +\infty)\), then we have \(\Psi(r) \neq 0\) at a finite internal of \(r\). Multiplying the equation (2.34) by \(r \sigma_2(r)\), and the equation (2.35) (take \(i = 2\)) by \(r \Psi(r)\), and then subtracting, we get

\[
(r \Psi(r))''(r \sigma_2(r)) - (r \sigma_2(r))''(r \Psi(r)) = \kappa(\sigma_2^2(r) + \sigma_1(r)\sigma_2(r))(r \Psi(r))(r \sigma_2(r)) > 0.
\]

Then \(\left[(r \Psi(r))''(r \sigma_2(r)) - (r \sigma_2(r))''(r \Psi(r))\right]' > 0\), that is, \(r \Psi(r))''(r \sigma_2(r)) - (r \sigma_2(r))''(r \Psi(r))\) is monotonically increasing. According to

\[
\left[(r \Psi(r))'(r \sigma_2(r)) - (r \sigma_2(r))'(r \Psi(r))\right]' \bigg|_{r=0} = \left[r^2(\Psi'(r)\sigma_2(r) - \Psi(r)\sigma_2'(r))\right]' \bigg|_{r=0} = 0,
\]

we observe that \(r \Psi(r))'(r \sigma_2(r)) - (r \sigma_2(r))'(r \Psi(r)) > 0\).

In view of

\[
((\sigma_2^{-1}(r))\Psi'(r))'(\sigma_2^{-2}(r))(\Psi'(r)\sigma_2(r) - \sigma_2'(r)) > 0
\]
and \((\sigma_2^{-1}(r))\Psi(r) > 0\) at \(r = 0 + \varepsilon\), we easily obtain \((\sigma_2^{-1}(r))\Psi(r) > 0\) as \(r \to \infty\), which contradicts

\[
\lim_{r \to \infty} \left( \frac{\Psi(r)}{\sigma_2(r)} \right) = \lim_{r \to \infty} \left( \frac{\sigma_2(r) - \sigma_1(r)}{\sigma_2(r)} \right) = \lim_{r \to \infty} \left( 1 - \frac{\sigma_1(r)}{\sigma_2(r)} \right) = 0.
\]

Thus the Lemma 2.18 follows.

**Lemma 2.19** If \(D_0 \in D_3^*\), for \(r \leq 1\), we can find a suitably large constant \(R^*\) such that \(|r^{-1}\sigma(r)| \leq R^*\). Moreover, if \(\sigma(r) \leq \sigma_0\), then \(r^{-2}H(r)\) is decreasing.

**Proof** Since the equation (2.29) and \(H(r) > 0, \sigma(r) \leq \sigma_0, 0 \leq h(r) \leq 1\) for all \(r > 0\), we must have \((r^{-2}H(r))' \leq 0\). In other words, \(r^{-2}H(r)\) is decreasing. To prove the other part, for all \(r \leq 1\), we arrive at

\[
|h(r) - 1| \leq N \left( r^{2\alpha}R^* + r^{1+\alpha} \right)^2, \quad \forall r \leq 1 \quad (0 < \alpha < k = \frac{\sqrt{3} - 1}{2}),
\]

where \(N_1 = \frac{2N_0\sigma_0}{3}, N_2 = \max\{\frac{N_1}{2\alpha + 1}, N_1\}\).

Suppose \(r = 2\sigma_0D_0^{-1}\), there exists \(N' = \max\{2^{2+2\alpha}N_2, 4\sigma_0^2N_2\}\) such that

\[
|\sigma(r) - 2\sigma_0| \leq N' \left( D_0^{-2-2\alpha}R^* + R^* \frac{\sigma_0^2}{1+\alpha} D_0^{-2} \right).
\]

Since the left-hand side of the above inequality is bounded, we conclude that there exist a \(C > 0\) such that

\[
N'[R^*D_0^{-2\alpha - 2} + R^* \frac{\sigma_0^2}{1+\alpha} D_0^{-2}] \geq C.
\]

Thus, we obtain \(D_0 \leq N_3R^* \frac{1}{1+\alpha}\), where \(N_3 = \max\{(C^{-1}N')^{\frac{1}{2+\alpha}}, (C^{-1}N')^{\frac{1}{2}}\}\). Notice that the only positive term in the integrand of (2.29) is \(\frac{H^2}{s^7}\) and \(\frac{H^2}{s^7} \leq \sigma_0^2\) is bounded. Then for all \(\forall r \leq 1\), we get

\[
\sigma(r) \leq D_0r + \frac{1}{3} \int_0^r \left[ s^{-1}r(1 - s^3r^{-3}) \times s\sigma_0 \times \kappa \frac{H^2}{s^7} \right] ds \leq D_0r + \frac{1}{3} \int_0^r \left[ s^{-1}r \times s\sigma_0 \times \kappa \sigma_0^2 \right] ds
\]

\[
= D_0r + \frac{\kappa \sigma_0^3 r^2}{3} \leq N_3R^* \frac{1}{1+\alpha} \sigma_0 + \frac{\kappa \sigma_0^3}{3} r^2 \leq N \left( R^*r + r^2 \right) \leq 2NR^*r \triangleq R^*r,
\]

where \(R^*\) is a suitably large constant, \(N = \max\{N_3, \frac{1}{3}\kappa \sigma_0^3\}\), \(0 < \frac{1}{1+\alpha} < 1\). This completes the proof of the Lemma 2.3. \(\square\)
2.4 Proof of Lemma 2.4 (Existence and uniqueness of $\tilde{h}(r)$)

The proof of this subsection is similar to that of subsection 2.2, and only the conclusion and part of the proof are given here. Setting $\Phi = \tilde{h} - 1$, then we may rewrite equation (2.1) as

$$\Phi'' = \frac{2\Phi}{r^2} + (g'^2\sigma^2 - B^2) (\Phi + 1) + \frac{3\Phi^2 + \Phi^3}{r^2}.$$  (2.37)

From (2.37) and $\Phi(0) = 0$ we can exhibit (2.37) alternatively as the integral equation

$$\Phi(r) = Er^2 + \frac{1}{3} \int_0^r (r^2 s^{-1} - r^{-1} s^2) \left\{ (g'^2\sigma^2 - B^2) (\Phi + 1) + \frac{3\Phi^2 + \Phi^3}{s^2} \right\} ds,$$  (2.38)

where $E$ is an arbitrary constant.

Then it follows that there exists a locally continuous solution $\Phi(r) = Er^2 + O(r^{2+2\alpha}) (r \to 0^+)$ of the initial value problem. We are interested in $E < 0$, and now we define three sets as follows

$$S_{\tilde{h}}^1 = \{ E < 0 : \Phi'(r; E) \text{ becomes positive before } \Phi(r; E) \text{ reaches } -1 \},$$

$$S_{\tilde{h}}^2 = \{ E < 0 : \Phi(r; E) \text{ crosses } -1 \text{ before } \Phi'(r; E) \text{ becomes } 0 \},$$

$$S_{\tilde{h}}^3 = \{ E < 0 : \forall r > 0, \Phi'(r; E) \leq 0, -1 < \Phi(r; E) < 0 \}.$$

It is clear that $S_{\tilde{h}}^1 \cup S_{\tilde{h}}^2 \cup S_{\tilde{h}}^3 = S_{\tilde{h}}, S_{\tilde{h}}^1 \cap S_{\tilde{h}}^2 = S_{\tilde{h}}^3 \cap S_{\tilde{h}}^3 = \emptyset$.

The following two lemmas are used to prove the existence of the solution of two-point boundary value problem about (2.37), and the procedure is similar to Lemma 2.8-Lemma 2.9. Thus the proofs are omitted here.

**Lemma 2.20** The set $S_{\tilde{h}}^1, S_{\tilde{h}}^2$ are both open and nonempty.

Since the connected set $E < 0$ cannot consist of two open disjoint non-empty sets, there must be some value of $E$ in neither $S_{\tilde{h}}^1$ nor $S_{\tilde{h}}^2$. For this value of $E$, say $E_0$, we have a solution with $\Phi'(r; E) \leq 0, -1 < \Phi(r; E) < 0$.

**Lemma 2.21** The solution corresponding to the parameter $E_0$ in $S_{\tilde{h}}^2$ satisfies $\lim_{r \to \infty} \tilde{h}(r) = 0$.

**Lemma 2.22** The solution for the given parameter $E_0$ is unique.

The conclusion can be obtained by the Sturm-Picone comparison theorem, and the proofs are also omitted here.

**Lemma 2.23** If $E_0 \in S_{\tilde{h}}^3$, for $r \leq 1$, we can find $N = N(\sigma_0, \kappa, \mu, B_0, m, g') > 0$ such that

$$|r^{-2}\Phi(r)| \leq N \left( R^* \frac{r^2}{1 + r^{2\alpha}} + r^{2\alpha} R^{*2} \right),$$

where $\Phi(r) = \tilde{h}(r) - 1$. Furthermore, we have $\tilde{h}'(0) = 0$. 

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Proof. According to (2.38), \(B(s) \leq s^\alpha R_s^s\), \(\sigma(s) \leq s^\alpha R_s^s\), \(\forall s \leq 1\), and \(|E_0|^{1+\alpha} \leq NR^s^2\) (because \(E_0 \in S_2^1\)), we arrive at
\[
\left|\frac{\Phi}{r^2}\right| \leq |E_0| + \frac{1}{3} \int_0^r s^{-1} \left(g^2 + 1\right) s^{2\alpha R_s^2} ds
\]
\[
\leq \left(NR^s^2\right)^\frac{1+\alpha}{\alpha} + \frac{\left(g^2 + 1\right)}{3} \int_0^r s^{2\alpha - 1 R_s^2} ds
\]
\[
\leq N' \left(R^s_1 + r^{2\alpha R_s^2}\right) \leq N' \left(R^s_1 + R_s^2\right) \triangleq R^*_1, \quad \forall r \leq 1, \quad (2.39)
\]
where \(N' = \max\{N^{\frac{1+\alpha}{\alpha}}, \frac{\left(g^2 + 1\right)}{6\alpha}\}\). Using the above result, we have \(\Phi(r) = O(r^2)(r \to 0^+)\). By the definition of the derivative of the function \(\Phi(r)\) at \(r = 0\) we obtain that
\[
\Phi'(0) = \lim_{r \to 0^+} \frac{\Phi(r) - \Phi(0)}{r} = \lim_{r \to 0^+} \frac{O(r^2)}{r} = 0,
\]
In other words, \(h'(0) = 0\). The proof of Lemma 2.4 is complete. \(\square\)

2.5 Proof of Lemma 2.5 (Existence and uniqueness of \(\tilde{\rho}(r)\))

The proof of this subsection is similar to that of subsection 2.4, and only the conclusion and part of the proof are given here. In order to study the solution of the boundary value problem be related to the equation (2.2) subject to the boundary conditions \(\tilde{\rho}(0) = 0\) and \(\tilde{\rho}(\infty) = \rho_0\), we can rewrite (2.2) as
\[
(r\tilde{\rho})'' - \frac{1}{2r^2} (r\tilde{\rho}) = -\frac{1}{4} (A - B)^2 (r\tilde{\rho}) + \frac{\lambda}{2} (r\tilde{\rho}) (\tilde{\rho}^2 - \rho_0^2) + \frac{1}{2r^2} (f^2 - 1) (r\tilde{\rho}). \quad (2.40)
\]
Let \(Q = r\tilde{\rho}\), we can convert (2.40) into the integral equation
\[
Q(r) = F r^{k+1} + \frac{1}{\sqrt{3}} \int_0^r \left(s^{-k} r^{k+1} - r^{-k} s^{k+1}\right) Q(s) T(s) ds, \quad (2.41)
\]
where \(F\) is an arbitrary constant,
\[
T(s) = -\frac{1}{4} (A - B)^2 + \frac{\lambda}{2} \left(\frac{Q^2}{s^2} - \rho_0^2\right) + \frac{1}{2s^2} (f^2 - 1).
\]

Then it follows that there exists a locally continuous solution \(Q(r) = F r^{k+1} + O(r^{2+k})\) \((r \to 0^+)\) of the initial value problem consisting of (2.2) and \(\tilde{\rho}(0) = 0\). We are interested in \(F > 0\), and now we define three sets as follows
\[
S_1^\rho = \left\{ F > 0 : \exists r_0 > 0 \text{ such that } (r\tilde{\rho}(r))'|_{r=r_0} < 0 \text{ before } \tilde{\rho}(r; F) \text{ becomes infinite} \right\},
\]
\[
S_2^\rho = \left\{ F > 0 : r\tilde{\rho}(r) \text{ becomes infinite before } (r\tilde{\rho}(r))' \text{ becomes zero} \right\},
\]
\[
S_3^\rho = \left\{ F > 0 : \forall r > 0, \tilde{\rho}(r; F) \text{ is finite}, r\tilde{\rho}(r; F) \geq 0 \right\}.
\]

It is obvious that \(S_1^\rho \cup S_2^\rho \cup S_3^\rho = S^\rho, S_1^\rho \cap S_2^\rho = S_2^\rho \cap S_3^\rho = S_3^\rho \cap S_1^\rho = \emptyset\).

The following lemmas are used to prove the existence of the solution of two-point boundary value problem about (2.40), and the procedure is similar to Lemma 2.16-Lemma 2.17. Thus the proof is omitted here.
Lemma 2.24 The set \( S_1^\rho, S_2^\rho \) are both open and nonempty.

Since the connected set \( F > 0 \) cannot consist of two open disjoint non-empty sets, there must be some value of \( F \) in neither \( S_1^\rho \) nor \( S_2^\rho \). For this value of \( F \), say \( F_0 \), we have a solution with \( \rho(r; F) \) is finite and \( r\rho(r; F) \geq 0, \forall r > 0. \)

Lemma 2.25 If \( F_0 \in S_3^\rho \), then \( \rho^2(r; F_0) \leq \rho_0^2 + \frac{A^2}{2A} \) as \( r \to \infty \).

**Proof** If not, we assume that \( r = r_0 \) is the first point such that \( \rho^2(r) = \rho_0^2 + \frac{A^2}{2A} \). Since \( \rho(0) = 0 \), we get \( \rho'(r_0) \geq 0 \). If \( \rho'(r_0) = 0 \), in view of \( -\frac{1}{4}(B - A)^2 > -\frac{A^2}{2A} \), then \( \rho''(r_0) > 0 \). That is, \( r_0 \) is the minimum point of \( \rho \), which contradicts \( r_0 \) is the maximum point of \( \rho \). If \( \rho''(r_0) > 0 \), it is clear that

\[
(r\rho)''' > \left( \rho_0^2 + \frac{2\lambda}{A^2} \right) \left[ \frac{A^2}{4} - \frac{1}{4}(B(r) - A(r))^2 \right] r \triangleq Nr, (N > 0),
\]

because \( \rho(\infty) > \{\rho_0^2 + \frac{A^2}{2A}\}^\frac{1}{2} \). Then \( \rho \) becomes infinite at a finite point, in other words, \( F_0 \in S_2^\rho \), which contradicts \( F_0 \in S_3^\rho \). To sum up, we have \( \rho^2(r; F_0) \leq \rho_0^2 + \frac{A^2}{2A} \) as \( r \to \infty \). \(

Lemma 2.26 The solution corresponding to the parameter \( F_0 \) in \( S_3^\rho \) satisfies \( \lim_{r \to \infty} \rho'(\infty) = \rho_0 \).

Lemma 2.27 If \( F_0 \in S_3^\rho, \forall \varepsilon > 0 \), there exist a \( R(\varepsilon), \forall r > R(\varepsilon) \), such that

\[
\rho^2(r) > \rho_0^2 - \varepsilon,
\]

where \( R(\varepsilon) \) is independent of the choice of \( \rho, A, R^* \) in Lemma 2.1.

**Proof** When \( r^2 \geq 2(\varepsilon \lambda)^{-1} \triangleq R_1^2(\varepsilon) \), we assume \( \rho^2(r) \leq \rho_0^2 - \varepsilon \). Inserting this result into (2.40), we obtain that

\[
Q''(r) \leq -\frac{\varepsilon}{2}\lambda Q(r) + \frac{1}{4\varepsilon}\lambda Q(r) = -\frac{1}{4}\varepsilon\lambda Q(r).
\]

It is easy to see that \( Q(r) \neq 0, \forall r \in (0, +\infty) \) because \( Q(0) = 0 \) and \( Q'(r) > 0 \). On the other hand, if \( r \geq R_1 \), applying the Sturm-Picone comparison theorem to \( Q(r)' = -\frac{1}{4}\varepsilon\lambda Q_1(r) \) and (2.43), we obtain that \( Q(r) \) have more zero points than \( Q_1(r) \). Then in view of the above equation with zeros \( 2k\pi(\lambda\varepsilon)^{-\frac{1}{2}} \) (where \( k \) is an integer), there exists at least one zero point of \( Q(r) \) in the interval \( (R_1, R(\varepsilon)) \) (where \( R(\varepsilon) \triangleq R_1 + 4\pi(\lambda\varepsilon)^{-\frac{1}{2}} \)), which contradicts \( Q(r) \neq 0, \forall r \in (0, +\infty) \). Consequently, there is a \( r_0 \in (R_1, R(\varepsilon)) \) such that \( \rho^2(r) > \rho_0^2 - \varepsilon \). According to the existence of solutions of two point boundary value problem of ordinary differential equation (2.2) and \( Q'(r) > 0 \), we can get \( \rho^2(r) > \rho_0^2 - \varepsilon, \forall r \in (R(\varepsilon), +\infty) \). \(

Lemma 2.28 The solution for the given parameter \( F_0 \) is unique.

Following the proof of Lemma 2.18, we obtain

Lemma 2.29 If \( F_0 \in S_3^\rho \), for \( r \leq 1 \), then there exists a suitably large constant \( R_2^* \) such that \( |r^{-k}\rho(r)| \leq R_2^* \). Moreover, if \( \rho(r) \leq \rho_0 \), then \( r^{-k-1}Q(r) \) is decreasing.
**Proof** Imitating the proof of Lemma 2.19, it can be concluded that \( r^{-k-1}Q(r) \) is decreasing. To prove the other part, for all \( r \leq 1 \), we arrive at

\[
|\hat{\rho}(r) - F_0 r^k| \leq N_1 \int_0^r r^k s^{1-k} \left(1 + s^{2\alpha} R^s + R^{\frac{s}{1+\alpha}}\right) ds \\
\leq 2N_1 \left(r^{2+2\alpha} R^s + r^2 R^{\frac{s}{1+\alpha}}\right),
\]

in which we have used \( 0 \leq f(r) \leq 1 \), \( Q(r) \leq r \left(\rho_0^2 + \frac{A_2}{2\lambda}\right)^{\frac{1}{2}} \) and

\[
|f^2(r) - 1| \leq N \left(r^{2\alpha} R^s + R^{\frac{s}{1+\alpha}}\right) r^2, \forall r \leq 1 \ (0 < \alpha < k = \frac{\sqrt{3} - 1}{2}).
\]

Here \( N_1 = \max\{\frac{A_2^2}{4\sqrt{3}}(\rho_0^2 + \frac{A_2^2}{2\lambda})^{\frac{1}{2}}, \frac{N_2}{2\sqrt{3}}(\rho_0^2 + \frac{A_2^2}{2\lambda})^{\frac{1}{2}}\} \).

Suppose \( r^k = 2(\rho_0^2 + \frac{A_2^2}{2\lambda})^{\frac{1}{2}} F_0^{-1} \), there exists

\[
N_2 = 2N_1 \max\{2^{\frac{2(1+\alpha)}{k}}(\rho_0^2 + \frac{A_2^2}{2\lambda})^{\frac{1+\alpha}{k}}, 2^{\frac{2\alpha}{k}}(\rho_0^2 + \frac{A_2^2}{2\lambda})^{\frac{1}{2}}\}
\]

such that

\[
|\hat{\rho}(r) - 2 \left(\rho_0^2 + \frac{A_2^2}{2\lambda}\right)^{\frac{1}{2}}| \leq N_2 \left(F_0^{-\frac{2(1+\alpha)}{k}} R^s + F_0^{-\frac{2\alpha}{k}} R^{\frac{s}{1+\alpha}}\right).
\]

Since the left-hand side of the above inequality is bounded, we conclude that there exist a \( C > 0 \) such that

\[
N_2(F_0^{-\frac{2(1+\alpha)}{k}} R^s + F_0^{-\frac{2\alpha}{k}} R^{\frac{s}{1+\alpha}}) \geq C.
\]

That is, we have \( F_0 \leq N_3 R^{\frac{k}{1+\alpha}} \), where \( N_3 = \max\{(N_2 C^{-1})^{\frac{k}{1+\alpha}}, (N_2 C^{-1})^{\frac{k}{2}}\} \). Notice that the only positive term in the integrand of (2.40) is \( \frac{Q^2}{s^2} \) and \( \frac{Q^2}{s^2} \leq \rho_0^2 + \frac{A_2^2}{2\lambda} \) is bounded. Then for all \( \forall r \leq 1 \), we get

\[
\hat{\rho}(r) \leq F_0 r^k + \frac{1}{\sqrt{3}} \int_0^r \left[s^{-k-1} \left(1 - s^{2k+1} r^{-2k-1}\right) \times s \left(\rho_0^2 + \frac{A_2^2}{2\lambda}\right)^{\frac{1}{2}} \times \frac{\lambda Q^2}{2s^2}\right] ds \\
\leq F_0 r^k + \frac{1}{\sqrt{3}} \int_0^r r^k s^{1-k} \left(\rho_0^2 + \frac{A_2^2}{2\lambda}\right)^{\frac{1}{2}} ds \leq N_3 R^{\frac{k}{1+\alpha}} r^k + \frac{\lambda r^2}{2\sqrt{3}} \left(\rho_0^2 + \frac{A_2^2}{2\lambda}\right)^{\frac{1}{2}} \\
\leq N_4 \left(R^{\frac{1-\alpha}{1+\alpha}} r^k + r^2\right) \leq 2N_4 R^{r^k} \triangleq R^*_2 r^k,
\]

where \( R^*_2 \) is a suitably large constant, \( N_4 = \max\{N_3, \frac{\lambda}{2\sqrt{3}}(\rho_0^2 + \frac{A_2^2}{2\lambda})^{\frac{3}{2}}\}, 0 < \frac{1}{1+\alpha} < 1 \). That is, \( \forall r \leq 1 \), we arrive at \( |r^{-k} \hat{\rho}| \leq R^*_2 \). The proof of Lemma 2.5 is finished. \( \square \)
2.6 Proof of Lemma 2.6 (Existence and uniqueness of \( \tilde{A}(r) \))

The proof of this subsection is similar to that of subsection 2.3, and only the conclusion and part of the proof are given here. Firstly, we rewrite (2.3) as

\[
r^2 \ddot{A} + 2r\dot{A} - 2\tilde{A} = 2 \left( f^2 - 1 \right) \tilde{A} - \frac{1}{4} g^2 \rho^2 r^2 \left( B - \tilde{A} \right).
\]

(2.45)

Then the differential equation (2.45) can be transformed into the integral equation form

\[
\tilde{A}(r) = ar + \frac{1}{3} \int_0^r s^2 (rs^{-2} - r^{-2}s) \left[ 2 \left( f^2 - 1 \right) \tilde{A} - \frac{1}{4} g^2 \rho^2 s^2 \left( B - \tilde{A} \right) \right] ds,
\]

(2.46)

for arbitrary constant \( a \), which can be determined by Picard iteration. There exists a locally continuous solution to the initial value problem for \( r \) sufficiently small. Then we obtain solution \( \tilde{A}(r) = ar + O(r^2) \) which continuously depends on the parameter \( a \).

Next, we introduce three sets

\[
S_1^\tilde{A} = \left\{ a > 0 : (r\tilde{A}(r))' \text{ becomes negative before } \tilde{A}(r; a) = B(r) \right\},
\]

\[
S_2^\tilde{A} = \left\{ a > 0 : \tilde{A}(r; a) \text{ crosses } B(r) \text{ before } (r\tilde{A}(r))' = 0 \right\},
\]

\[
S_3^\tilde{A} = \left\{ a > 0 : \forall r > 0, B(r; b) \leq A_0, B(r; b) \geq A(r) \right\}.
\]

**Case (i)** \( (r\tilde{A}(r))' = 0, \tilde{A}(r) = B(r) \) at \( r = r_0 \), while \( (r\tilde{A}(r))' \geq 0, \tilde{A}(r) \leq B(r) \) for all \( r < r_0 \). We claim that case (i) is not valid. Inserting \( \tilde{A}(r_0) = B(r_0) \) into (2.3), then we easily have \( (r\tilde{A}(r))''|_{r=r_0} > 0 \). In addition, it is clear that \( (r\tilde{A}(r))' < 0, \forall r \in (r_0 - \delta, r_0) \) because \( (r\tilde{A}(r))'|_{r=r_0} = 0 \), which contradicts \( (r\tilde{A}(r))' \geq 0 \) for all \( r < r_0 \).

Obviously, \( S_1^\tilde{A} \cup S_2^\tilde{A} \cup S_3^\tilde{A} = \mathcal{S}^\tilde{A}, S_1^\tilde{A} \cap S_2^\tilde{A} = S_2^\tilde{A} \cap S_3^\tilde{A} = S_3^\tilde{A} \cap S_1^\tilde{A} = \emptyset \).

**Lemma 2.30** The set \( S_1^\tilde{A}, S_2^\tilde{A} \) are both open and nonempty.

**Proof** Firstly, \( S_1^\tilde{A} \) contains small \( a \). Inserting \( a = 0 \) into the equation (2.46), we obtain the equations as follows

\[
\tilde{A}(r) = \frac{1}{3} \int_0^r s^2 (rs^{-2} - r^{-2}s) \left[ 2 \left( f^2 - 1 \right) \tilde{A} - \frac{1}{4} g^2 \rho^2 s^2 (B - \tilde{A}) \right] ds.
\]

(2.47)

In fact, when \( r > 0 \) is sufficiently small, \( \tilde{A}(r) < 0 \) and \( r\tilde{A}(r) \) is decreasing because of

\[
(r\tilde{A}(r))' = \int_0^r \left[ \frac{2}{s} f^2 \tilde{A} - \frac{1}{4} g^2 \rho^2 s \left( B - \tilde{A} \right) \right] ds < 0.
\]

When \( a > 0, r > 0 \) is sufficiently small, since \( r\tilde{A}(r) = ar^2 + O(r^3)(r \to 0^+) \), we have \( r\tilde{A}(r) > 0 \). As \( r \) increases, when \( r > a \) is sufficiently small, the dominant term becomes \( O(r^3) \), and by the negative coefficient of \( r^3 \) we conclude that \( r\tilde{A}(r) < 0 \). Therefore, \( (r\tilde{A}(r))' < 0 \) when \( r \) is sufficiently small. Meanwhile, \( B(r) > \tilde{A}(r) \). So \( S_1^\tilde{A} \) is nonempty.

The proof that the set \( S_2^\tilde{A} \) is a nonempty set is similar to Lemma 2.12 and will be omitted here. The continuity can ensure that two sets are open. \( \square \)
Since the connected set $a > 0$ cannot consist of two open disjoint nonempty sets, there must be some value of $a$ in neither $S^A_1$ nor $S^A_2$. For this value of $a$, say $a_0$, we have a solution with $B(r; b) \leq A_0$, $B(r; b) \geq A(r), \forall r > 0$.

**Lemma 2.31** The solution corresponding to the parameter $a_0$ in $S^A_3$ satisfies $\lim_{r \to \infty} \tilde{A}(r) = A_0$.

**Proof** Since $a_0 \in S^A_3$, then $(r \tilde{A}(r))' > 0$, $\tilde{A}(r) \leq B(r) \leq A_0$ for all $r > 0$. In view of $(r \tilde{A}(r))|_{r=0} = 0$ and $(r \tilde{A}(r))' > 0$, we obtain that $r \tilde{A} \geq 0$ for all $r$. Hence $0 \leq \tilde{A} \leq B \leq A_0$ for all $r$, that is, $\tilde{A}(r)$ is bounded on $[0, +\infty)$.

We claim that $B(r) \leq \tilde{A}(r)$ when $r > 0$ is sufficiently large. If not, inserting $B(r) > \tilde{A}(r)$ into (2.3), we obtain that

$$(r \tilde{A}(r))'' = \left[ \frac{2}{r^2} f^2(r)\tilde{A}(r) - \frac{1}{4} g^2 \rho^2(r)r \left( B(r) - \tilde{A}(r) \right) \right] r < -Kr.$$

That is, $(r \tilde{A}(r))'' \sim -Kr$, $(r \tilde{A}(r))' \sim -Kr^2(r \to \infty)$, which contradicts $(r \tilde{A}(r))' > 0$. Combining $B(r) \leq \tilde{A}(r)$ and $\tilde{A}(r) \leq B(r)$ as $r \to +\infty$, we arrive at

$$\lim_{r \to \infty} \tilde{A}(r) = \lim_{r \to \infty} B(r) = A_0.$$

\[\square\]

**Lemma 2.32** The solution for the given parameter $a_0$ is unique.

The conclusion can be obtained by using the extremum principle, and the proof will be omitted here.

**Lemma 2.33** If $a_0 \in S^A_3$, for $r \leq 1$, we can find a suitably large constant $R_3^*$ such that $|r^{-1} \tilde{A}(r)| \leq R_3^*$. Moreover, $r^{-1} \tilde{A}(r)$ is decreasing.

**Proof** Imitating the proof of Lemma 2.19, it can be concluded that $r^{-1} \tilde{A}(r)$ is decreasing. To prove the other part, for all $r \leq 1$, we arrive at

$$\left| \tilde{A}(r) - a_0 r \right| \leq \frac{1}{3} \int_0^r \left| 2r(f^2 - 1)\tilde{A} + \frac{r}{4} g^2 \rho^2 s^2 \left( B - \tilde{A} \right) \right| ds$$

$$\leq \frac{1}{3} \int_0^r r \left[ 2N_{a_0} s^3 \left( R^{s^2} s^{2\alpha} + R^{s^{2\alpha}} \right) + \frac{1}{2} g^2 s^{2k+2} R^{s^2} A_0 \right] ds$$

$$\leq N_1 \left[ R^{s^2} \left( a_0 r^{2\alpha+5} + A_0 r^{2k+4} + R^{s^{2\alpha}} a_0 r^5 \right) \right]$$

because $|B(r) - \tilde{A}(r)| \leq 2A_0$, $\rho(r) \leq \rho_0$, $\tilde{A}(r) \leq a_0 r$ and

$$|f^2(r) - 1| \leq N \left( r^{2\alpha} R^{s^2} + R^{s^{2\alpha}} \right) r^2, \forall r \leq 1,$$

where $N_1 = \max\{\frac{2}{3}N, \frac{1}{4}g^2\}$.

Suppose $r = 2a_0^{-1} A_0$, there is a

$$N_2 = \max\{2^{2\alpha+5} A_0^{2\alpha+5} N_1, 2^{2k+4} A_0^{2k+5} N_1, 2^5 A_0^5 N_1\}$$

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such that

\[ \|\tilde{A}(r) - 2A_0\| \leq N_2 \left[ R^2 \left( a_0^{-2\alpha - 4} + a_0^{-2k-4} \right) + R^{\frac{2}{1+\alpha}} a_0^{-5} \right]. \]

Since the left-hand side of the above inequality is bounded, we conclude that there exist a \( a > 0 \) such that

\[ N_2[R^2(a_0^{-2\alpha - 4} + a_0^{-2k-4}) + R^{\frac{2}{1+\alpha}} a_0^{-5}] \geq C. \]

That is, we have \( a_0 \leq N_3 R^{\frac{1}{2+\alpha}} \varepsilon \leq R_3^* \), where \( N_3 = \max\{ (C^{-1} N_2)^{\frac{1}{2+\alpha}}, (C^{-1} N_2)^{\frac{1}{2+\alpha}}, (C^{-1} N_2)^{\frac{1}{2+\alpha}} \} \).

In view of \( r^{-1} \tilde{A} \leq a_0 \) for all \( r \leq 1 \), there exist a suitably large constant \( R_3^* \) such that

\[ r^{-1} \tilde{A}(r) \leq a_0 \leq R_3^*. \quad (2.48) \]

The proof of Lemma 2.6 is complete. \( \square \)

### 3 Proof of Theorem 1.1 and 1.2

In this section, we complete the proof of Theorems 1.1-1.2 by the Schauder fixed point theorem and the extremum principle.

Firstly, we define the space \( \mathcal{B} \) as follows

\[ \mathcal{B} = \{ (\rho, A, \Phi) | \rho, A, \Phi \in C([0, +\infty)), r^{-\alpha} (1 + r^\alpha) \rho, r^{-\alpha} (1 + r^\alpha) A, r^{-\alpha} (1 + r^\alpha) \Phi \text{ are bounded} \} \]

with

\[ \|(\rho, A, \Phi)\|_{\mathcal{B}} = \sup_{r \in [0, +\infty)} \{ |r^{-\alpha} (1 + r^\alpha) \rho| + |r^{-\alpha} (1 + r^\alpha) A| + |r^{-\alpha} (1 + r^\alpha) \Phi| \}. \]

Next, we define the mapping \( F: (\rho, A, \Phi) \to (\tilde{\rho}, \tilde{A}, \tilde{\Phi}) \) on \( \mathcal{B} \). In the following, we will demonstrate that it is continuous.

**Lemma 3.1** \( F: (\rho, A, \Phi) \to (\tilde{\rho}, \tilde{A}, \tilde{\Phi}) \) is continuous on \( \mathcal{B} \).

**Proof** In order to obtain this conclusion, we shall show that if \( (\rho_1, A_1, \Phi_1), (\rho_2, A_2, \Phi_2) \in \mathcal{B} \), then when \( \|(\rho_1 - \rho_2, A_1 - A_2, \Phi_1 - \Phi_2)\|_{\mathcal{B}} \to 0 \), we have

\[ \|F(\rho_1, A_1, \Phi_1) - F(\rho_2, A_2, \Phi_2)\|_{\mathcal{B}} \to 0. \]

Since \( F(\rho_1(\infty)) = F(\rho_2(\infty)) = \rho_0 \), then for any \( \varepsilon > 0 \) there is \( R_0 > 0 \) such that

\[ \sup_{r \in [R_0, +\infty)} \left| r^{-\alpha} (1 + r^\alpha) [F(\rho_1(\infty)) - F(\rho_2(\infty))] \right| < \varepsilon. \quad (3.1) \]

Likewise, we have

\[ \sup_{r \in [R_0, +\infty)} \left| r^{-\alpha} (1 + r^\alpha) [F(A_1(\infty)) - F(A_2(\infty))] \right| < \varepsilon, \quad (3.2) \]

\[ \sup_{r \in [R_0, +\infty)} \left| r^{-\alpha} (1 + r^\alpha) [F(\Phi_1(\infty)) - F(\Phi_2(\infty))] \right| < \varepsilon. \quad (3.3) \]
For the $\varepsilon$ given above, there is $\delta > 0$ such that
\[
\sup_{r \in [0, \delta]} |r^{-\alpha} (1 + r^\alpha) (\rho_1 - \rho_2)| \to 0, \ (n \to \infty)
\]
and $F(\rho_1(0)) = F(\rho_2(0)) = 0$, then
\[
\sup_{r \in [0, \delta]} |r^{-\alpha} (1 + r^\alpha) [F(\rho_1(0)) - F(\rho_2(0))]| < \varepsilon. \tag{3.4}
\]
Similarly, we get
\[
\sup_{r \in [0, \delta]} |r^{-\alpha} (1 + r^\alpha) [F(A_1(0)) - F(A_2(0))]| < \varepsilon, \tag{3.5}
\]
\[
\sup_{r \in [0, \delta]} |r^{-\alpha} (1 + r^\alpha) [F(\Phi_1(0)) - F(\Phi_2(0))]| < \varepsilon. \tag{3.6}
\]
In view of the continuity of $F$ at $[\delta, R_0]$, thus $F$ is continuous on $\mathcal{B}$.

After that, we define the nonempty bounded closed convex subset $\mathcal{S}$ of $\mathcal{B}$ by
\[
\mathcal{S} = \{(\rho, A, \Phi) \in \mathcal{B} \mid (1) \text{ for } r \leq 1, \text{ we have } |r^{-k} \rho| \leq R^*, |r^{-1} A| \leq R^*, |r^{-2} \Phi| \leq R^*;
\]
\[
(2) \ r \rho(r), r A(r) \text{ is increasing, } r^{-1} \Phi(r) \text{ is decreasing};
\]
\[
\text{if } r \leq \frac{\rho_0}{R^*}, \text{ then } r^{-k} \rho(r) \text{ is decreasing};
\]
\[
(3) \ \rho^2 \leq \rho_0^2 + \frac{A^2_0}{2\lambda}, \frac{1}{4} g^2 \rho^2 \geq A^2, g^2 \sigma^2 \geq B^2, \ A \leq A_0, |\Phi| \leq 1,
\]
\[
\text{if } r \geq R(\varepsilon, F_0), \text{ then } \frac{1}{4} g^2 \rho^2 \geq \frac{1}{2} \left( \frac{1}{4} g_0^2 \rho_0^2 + A_0^2 \right);
\]
\[
(4) \ A(\infty) = A_0, \rho(\infty) = \rho_0, \Phi(\infty) = -1 \},
\]
where $R^* = \max\{R_1^*, R_2^*, R_3^*\}$, $R(\varepsilon, F_0)$ is $R(\varepsilon)$ in Lemma 2.27. It is straightforward that the set $\mathcal{S}$ is indeed nonempty, bounded, closed and convex.

Finally, we show that the mapping $F$ is compact which maps $\mathcal{S}$ into itself.

**Lemma 3.2** For the mapping $F: (\rho, A, \Phi) \to (\tilde{\rho}, \tilde{A}, \tilde{\Phi})$,

(1) The mapping $F$ takes $\mathcal{S}$ into itself;

(2) $F$ is compact.

**Proof** Part (1) is already established in Lemmas 2.1-2.6 or their proofs. Next we concentrate on the proof of Part (2). Firstly, from the fact that the mapping $F$ is continuous on $\mathcal{B}$ and $\mathcal{S}$ is the subset of $\mathcal{B}$, it follows that $F$ is continuous on $\mathcal{S}$. Secondly, to show $F$ is compact, we will demonstrate that the mapping $F$ maps an arbitrary bounded set in $\mathcal{S}$ to a column-compact set. That is if $\{\rho_n(r), A_n(r), \Phi_n(r)\}$ is the arbitrary bounded sequences in $\mathcal{S}$, then we must prove that $\{\tilde{\rho}_n(r), \tilde{A}_n(r), \tilde{\Phi}_n(r)\}$ have convergent sub-sequence in $\mathcal{S}$. Take $\{\tilde{A}_n(r)\}$ as an example.

Using Lemma 2.6 and $\{\tilde{A}_n(r)\} \in \mathcal{S}$, it is straightforward to show
\[
\begin{align*}
\tilde{A}_n'(r) &= a + \frac{1}{3} \int_0^r s^2 (s^{-2} + 2r^{-3}s) \left\{ 2 (f_n^2 - 1) \tilde{A}_n - \frac{1}{4} g^2 \tilde{\rho}_n^{-2} (B_n - \tilde{A}_n) \right\} ds
\end{align*}
\]
is bounded on any inner closed subinterval of \((0, +\infty)\). In other words, there is \(L_1 > 0\) such that \(\|\tilde{A}_n'(r)\|_\infty \leq L_1\) on any inner closed subinterval of \((0, +\infty)\), where \(L_1\) have no connection with \(n\). Applying the mean value theorem, for any \(\varepsilon > 0\), \(r_1, r_2 \in [\delta, R] \subset (0, +\infty)\) there exist \(\delta = \frac{\varepsilon}{L_1 + 1} > 0\) such that \(|\tilde{A}_n(r_1) - \tilde{A}_n(r_2)| = |\tilde{A}_n'(|\xi|)| r_1 - r_2| < \varepsilon\) when \(|r_1 - r_2| < \delta\), where \(\xi\) is between \(r_1\) and \(r_2\). Hence the equicontinuity of \(\{\tilde{A}_n(r)\}\) is apparently established. It is clear that \(\{A_n(r)\}\) is uniformly bounded because \(\{\tilde{A}_n(r)\} \in S\). According to the Arzela-Ascoli theorem, there is a subsequence of \(\{\tilde{A}_n(r)\}\) (denoted as \(\{\tilde{A}_n\}\)) and the continuous function \(\tilde{A}(r)\), which uniformly converges to \(\tilde{A}(r)\) in any compact subinterval of \((0, +\infty)\) (denoted as \([\delta, R]\)), therefore

\[
\sup_{r \in [\delta, R]} |r^{-\alpha}(1 + r^\alpha)(\tilde{A}_n(r) - \tilde{A}(r))| \leq (1 + \sigma^{-\alpha}) \sup_{r \in [\delta, R]} |\tilde{A}_n(r) - \tilde{A}(r)| \to 0 (n \to \infty). \tag{3.7}
\]

We only need to prove that when \(r \to 0\) and \(r \to +\infty\)(that is in interval \((0, \delta)\) and \((R, +\infty)\)), such that if \(n \to +\infty\)

\[
\|\tilde{A}_n(r) - \tilde{A}(r)\|_\infty \to 0.
\]

As \(r \to 0\) (that is in interval \((0, \delta)\)), we can get \(\tilde{A}_n(r) \leq rR^\alpha, \tilde{A}(r) \leq rR^\alpha\) for all \(r \leq 1\) because of \(\{\tilde{A}_n(r)\}, \tilde{A}(r) \in S\). Then, for the \(\varepsilon > 0\) given above, we can choose a sufficiently small \(\delta = (\frac{\varepsilon}{4R^\alpha})^{1-\alpha}\) such that

\[
\sup_{r \in (0, \delta)} |r^{-\alpha}(1 + r^\alpha)(\tilde{A}_n(r) - \tilde{A}(r))| < 2 \sup_{r \in (0, \delta)} |r^{-\alpha}(1 + r^\alpha)(\tilde{A}_n(r) - \tilde{A}(r))| < 4\delta^{1-\alpha} R^\alpha \leq \varepsilon.
\]

With \(\delta > 0\) fixed, we can then find \(n\) sufficiently large that

\[
\sup_{r \in (0, \delta)} |r^{-\alpha}(1 + r^\alpha)(\tilde{A}_n(r) - \tilde{A}(r))| \to 0 (n \to \infty). \tag{3.8}
\]

As \(r \to +\infty\) (that is in interval \((R, +\infty)\)), we want to show that

\[
\sup_{r \in (R, \infty)} |r^{-\alpha}(1 + r^\alpha)(\tilde{A}_n(r) - \tilde{A}(r))| < \varepsilon
\]

for \(\forall \varepsilon > 0\). It is only necessary to prove that \(|\tilde{A}_n(r) - \tilde{A}(r)| < \varepsilon\). From (2.3), we have

\[
(rA_\tilde{n}(r))'' = \frac{2}{r}f_n^2(r)\tilde{A}_n(r) - \frac{1}{4}g^2\rho_n^2(r)(B_n(r) - \tilde{A}_n(r)) \leq \frac{2}{r}f_n^2(r)\tilde{A}_n(r).
\]

Then, integrating the above inequality over \((r, \infty)\), we have

\[
(s\tilde{A}_n)'(\infty) - (s\tilde{A}_n)'(r) \leq \int_r^{+\infty} \frac{2}{s}f_n^2(s\tilde{A}_n)ds.
\]

In view of \(\tilde{A}_n(\infty) = A_0, \tilde{A}_n'(\infty) = O(r^{-2})\), we obtain that

\[
|A_0 - \tilde{A}_n(r)| \leq r\tilde{A}_n'(r) + \int_r^{+\infty} \frac{2}{s}f_n^2(s\tilde{A}_n)ds.
\]
Since \( f_0(r) = O(e^{-\kappa(1-\varepsilon)r}) \), \( \tilde{A}_n(r) = A_0 + O(r^{-1}) \), \( \tilde{A}_n'(r) = O(r^{-2}) \), therefore we can get 
\[ |A_0 - \tilde{A}_n(r)| \to 0(n \to \infty) \] as \( r \to \infty \), that is, 
\[ |\tilde{A}(r) - \tilde{A}_n(r)| \to 0(n \to \infty) \]. Thus

\[
\sup_{r \in (R, +\infty)} |r^{-\alpha}(1 + r^\alpha)(\tilde{A}_n(r) - \tilde{A}(r))| \to 0(n \to \infty).
\] (3.9)

In summary, we have proved for any \( r \), if \( n \to \infty \), then \( \|\tilde{A}_n(r) - \tilde{A}(r)\|_{\mathcal{B}} \to 0 \).

Evidenced by the same token, it is easy to show that \( \|\tilde{\rho}_n(r) - \tilde{\rho}(r)\|_{\mathcal{B}} \to 0(n \to \infty) \), \( \|\tilde{\Phi}_n(r) - \tilde{\Phi}(r)\|_{\mathcal{B}} \to 0(n \to \infty) \) as well.

In conclusion, according to Lemma 3.1-3.2, the mapping \( F \) satisfies the conditions of the Schauder fixed point theorem. Applying the Schauder fixed point theorem, we obtain that \( (f(r), \rho(r), A(r), B(r), h(r), \sigma(r)) \) is the solution of the two-point boundary value problem (1.9)-(1.16) for the system of nonlinear ordinary differential equations. Thus the Theorem 1.1 is proved.

Next we show some properties of the solutions obtained above.

**Proof of Theorem 1.2**

Define the comparison function \( \eta(r) = Ce^{-\zeta(1-\varepsilon)r} \), where \( \zeta = \sqrt{g^2\sigma^2 - A_0^2} \), \( C > 0 \) is a constant to be chosen later, \( \varepsilon > 0 \) is sufficiently small. From the equation (1.13) and the property \( h(r) > 0 \) we obtain that for any \( \varepsilon > 0 \), there is a sufficiently large \( r_\varepsilon > 0 \) so that

\[
(h - \eta)^{''} = (g^2\sigma^2 - B^2) h + \frac{1}{r^2} h (h^2 - 1) - \zeta^2 (1 - \varepsilon)^2 \eta
\]

\[
= \zeta^2 (1 - \varepsilon)^2 (h - \eta) + \zeta^2 \left[ 1 - (1 - \varepsilon)^2 \right] h + \frac{1}{r^2} h (h^2 - 1)
\]

\[
\geq \zeta^2 (1 - \varepsilon)^2 (h - \eta), \; r > r_\varepsilon.
\] (3.10)

Taking \( C > 0 \) be large enough to make \((h - \eta)(r_\varepsilon) \leq 0\). Thus, in view of this and the boundary condition \((h - \eta)(r) \to 0(r \to \infty)\), we obtain by applying the maximum principle theorem in (3.10) the result \( 0 < h \leq \eta = Ce^{-\zeta(1-\varepsilon)r}, \; r > r_\varepsilon \) as expected in (1.21).

As \( r \to +\infty \), the asymptotic estimate of the function \( h \) can be proved by imitating \( h \). In other words, the estimate for \( f \) in (1.18) is established.

Next, set \( \tau(r) = r(B - A) \), then \( \tau > 0 \). By virtue of the equations (1.11) and (1.12), we have

\[
\tau^{''} = \frac{1}{4} (g^2 + g^2) \rho^2 \tau + \frac{2}{r} (h^2 - f^2 A),
\]

then \( \tau^{''} = \frac{1}{4} (g^2 + g^2) \rho^2 \tau \) as \( r \to \infty \). Now let \( \eta(r) = Ce^{-\nu_0(1-\varepsilon)r} \), where \( C > 0 \) is a constant to be chosen later, \( \varepsilon > 0 \) is sufficiently small, \( \kappa = \sqrt{\frac{1}{4} g^2 \rho^2 - A_0^2} \), \( \nu = \frac{1}{2} \rho_0 \sqrt{g^2 + g^2} \), \( \nu_0 = \frac{1}{2} \rho \sqrt{g^2 + g^2} \).
\[ (\tau - \eta)'' = \frac{1}{4} (g^2 + f^2) \rho^2 \eta + \frac{2}{r} (h^2 B - f^2 A) - \nu_0^2 (1 - \varepsilon)^2 \eta \]
\[ \geq \nu_0^2 \left( 1 - \frac{\varepsilon}{2} \right)^2 \tau - \nu_0^2 (1 - \varepsilon)^2 \eta + \frac{2}{r} (h^2 B - f^2 A) \]
\[ \geq \nu_0^2 \left( 1 - \frac{\varepsilon}{2} \right)^2 (\tau - \eta) + I_1 \]
\[ \geq \nu_0^2 \left( 1 - \frac{\varepsilon}{2} \right)^2 (\tau - \eta), \quad r > r_\varepsilon. \quad (3.11) \]

Taking \( C > 0 \) be large enough to make

\[ I_1 = \nu_0^2 \left[ \left( 1 - \frac{\varepsilon}{2} \right)^2 - (1 - \varepsilon)^2 \right] \eta + \frac{2}{r} \left( C_1^2 e^{-2\kappa(1-\varepsilon)\tau} A_0 - C_2^2 e^{-2\kappa(1-\varepsilon)\tau} A_0 \right) \]
\[ = \nu_0^2 \left[ \left( 1 - \frac{\varepsilon}{2} \right)^2 - (1 - \varepsilon)^2 \right] C e^{-\nu_0(1-\varepsilon)r} + \frac{2}{r} \left( C_1^2 e^{-2\kappa(1-\varepsilon)\tau} A_0 - C_2^2 e^{-2\kappa(1-\varepsilon)\tau} A_0 \right) > 0 \]

and \((\tau - \eta)(r_\varepsilon) = \{ r_\varepsilon [B (r_\varepsilon) - A (r_\varepsilon)] - C e^{-\nu_0(1-\varepsilon)r_\varepsilon} \} \leq 0\), where \( h(r) = C_1 e^{-\zeta(1-\varepsilon)r}, \ C_1 \) is an arbitrary constant, \( f(r) = C_2 e^{-\kappa(1-\varepsilon)r}, \ C_2 \) is also an arbitrary constant. Therefore, according to this and the boundary condition \((\tau - \eta)(r) \to 0 \ (r \to \infty)\), we obtain by applying the maximum principle theorem in (3.11) the result \( 0 < \tau \leq \eta = C_3 e^{-\nu_0(1-\varepsilon)r}, \ r > r_\varepsilon \) as expected in (1.22).

Then we consider the estimate for \( \rho \). For the new function \( T(r) = r (\rho - \rho_0) \), the equation (1.10) gives us

\[ T'' = \frac{\lambda}{2} (\rho + \rho_0) \rho T + \frac{1}{2r} \left[ f^2 - \frac{1}{2} r^2 (A - B)^2 \right] \rho. \]

It is seen that \( T'' = \frac{\lambda}{2} (\rho_0 + \rho_0) \rho T = \lambda \rho_0^2 T \) as \( r \to \infty \). Set \( \eta(r) = C_3 e^{-\sqrt{2\mu_0}(1-\varepsilon)r} \) with \( \varepsilon > 0, \ C > 0, \ \mu = \sqrt{\frac{\lambda_{\min}}{2}}, \ \mu_{\min} = \min\{\sqrt{2\mu}, 2\kappa, 2\nu_0\}. \) Hence for any \( \varepsilon > 0 \), there exist a sufficiently large \( r_\varepsilon > 0 \) so that

\[ (T - \eta)'' = \frac{\lambda}{2} (\rho + \rho_0) \rho T + \frac{1}{2r} \left[ f^2 - \frac{1}{2} r^2 (A - B)^2 \right] \rho - 2 \mu^2 (1 - \varepsilon)^2 \eta \]
\[ \geq \frac{\lambda}{2} (\rho + \rho_0) \rho (T - \eta) + I_2 \geq \frac{\lambda}{2} (\rho + \rho_0) \rho (T - \eta), \quad r > r_\varepsilon, \quad (3.12) \]

where \( r_\varepsilon > 0 \) is sufficiently large. Then we can choose \( C > 0 \) large enough to make

\[ I_2 = \left[ 2 \mu^2 \left( 1 - \frac{\varepsilon}{2} \right)^2 - 2 \mu^2 (1 - \varepsilon)^2 \right] \eta + \frac{1}{2r} \left[ f^2 - \frac{1}{2} r^2 (A - B)^2 \right] \rho \]
\[ = 2 \mu^2 \varepsilon \left( 1 - \frac{3}{4} \varepsilon \right) C e^{-\sqrt{2\mu_0}(1-\varepsilon)r} + \frac{1}{2r} \left[ C_1^2 e^{-2\kappa(1-\varepsilon)r} - \frac{1}{2} C_3^2 e^{-2\nu_0(1-\varepsilon)r} \right] \rho_0 > 0 \]

and \((T - \eta)(r_\varepsilon) = \{ r_\varepsilon [\rho(r_\varepsilon) - \rho_0] - C e^{-\sqrt{2\mu_0}(1-\varepsilon)r_\varepsilon} \} \leq 0\) where \( f(r) = C_2 e^{-\kappa(1-\varepsilon)r}, \ C_2 \) is an arbitrary constant, \( B(r) - A(r) = C_3 r^{-1} e^{-\nu_0(1-\varepsilon)r} \), \( C_3 \) is also an arbitrary constant. Since
the boundary condition \((T - \eta)(r) \to 0 \ (r \to \infty)\), using the maximum principle theorem in (3.12), then the decay estimate for \(\rho\) near infinity stated in (1.19) is established.

Then we study the asymptotic estimates (1.22). Setting \(S(r) = r(\sigma - \sigma_0)\), for (1.14), we obtain that \(S'' = 2\kappa \sigma^2 S\) as \(r \to \infty\). To get the estimate for \(\sigma(r)\) in (1.22), we use the comparison function \(\eta(r) = Ce^{-\sqrt{2}\xi(1-\varepsilon)r}\), where \(C > 0\) is a constant to be chosen later, \(\varepsilon > 0\) is sufficiently small, \(\xi = \sqrt{k\sigma_0}\). Then for any \(\varepsilon > 0\), there is a sufficiently large \(r_\varepsilon > 0\) such that

\[
(S - \eta)'' = \kappa (\sigma_0 + \sigma) \sigma S + \frac{2}{r} h^2 \sigma - 2\xi^2 (1-\varepsilon)^2 \eta \\
\geq \kappa (\sigma_0 + \sigma) (S - \eta) + \left[2\xi^2 \left(1 - \frac{\varepsilon}{2}\right)^2 - 2\xi^2 (1-\varepsilon)^2\right] \eta \\
\geq \kappa (\sigma_0 + \sigma) (S - \eta) , \quad r > r_\varepsilon.
\]

Let \(C > 0\) be large enough to make \((S - \eta)(r_\varepsilon) = \{r_\varepsilon[\sigma(r_\varepsilon) - \sigma_0] - Ce^{-\sqrt{2}\xi(1-\varepsilon)r_\varepsilon}\} \leq 0\). Thus, in view of this and the boundary condition \((S - \eta)(r) \to 0 \ (r \to \infty)\), we obtain by applying the maximum principle theorem in (3.13) the result \(0 < S \leq \eta = Ce^{-\sqrt{2}\xi(1-\varepsilon)r} , \quad r > r_\varepsilon\) as expected in (1.22).

Then the estimate for \(A\) in (1.20) follows. The proof of the Theorem 1.2 is completed. □

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