ARC DIAGRAMS AND 2-TERM SIMPLE-MINDED COLLECTIONS OF PREPROJECTIVE ALGEBRAS OF TYPE A

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Abstract. We study an explicit description of semibricks and 2-term simple-minded collections over preprojective algebras of type $A$ via arc diagrams. We provide a bijection between the set of noncrossing arc diagrams (resp. the set of double arc diagrams), which is in bijective correspondence with elements of the symmetric group, and the set of semibricks (resp. the set of 2-term simple-minded collections) over the algebra. Moreover we define a mutation and a partial order on the set of double arc diagrams. In particular, we obtain a poset isomorphism between the symmetric group and the set of 2-term simple-minded collections. As an application of our results, we study semibricks of some quotient algebras of the preprojective algebras of type $A$ and we reprove some important results shown by the other authors.

Contents

1. Introduction
   1.1. Background
   1.2. Main results
2. Preliminaries
   2.1. Symmetric groups and canonical join representations
   2.2. Arc diagrams
   2.3. Torsion classes and semibricks
   2.4. Simple-minded collections
3. Double arc diagrams and 2-term SMCs
   3.1. Bricks and arc modules
   3.2. Hom-spaces of arc modules
   3.3. A partial order and mutation
4. Quotient algebras of the preprojective algebra
References

1. Introduction

1.1. Background. Preprojective algebras are one of the important classes of algebras not only in representation theory of algebras but also in many areas of mathematics. One of the remarkable properties of this class is that they can unify path algebras of all orientations of the given quiver. It has recently turned out this property provides a close connection between tilting theory of preprojective algebras and the corresponding Weyl groups [BIRS, IR, M1]. This strong link leads to a lot of fruitful consequences...
to analyze categorical structures using combinatorics of Weyl groups, for example \[\text{AM, AIRT, As2, BIRS, GLS, IRRT, IRTT, IZ, M1, M2, MT}\]. In this recent development, a theory of semibricks and $\tau$-tilting modules have been playing an crucial role \[\text{AIR, As1}\]. This theory allows us to give a systematic way to study several key categories such as torsion classes and wide subcategories.

By the result of \[\text{M1, As1}\], we have established a one-to-one correspondence between semibricks over preprojective algebras of Dynkin type and the corresponding Weyl group through $\tau$-tilting theory. One of the main aims of this paper is to give an explicit description of semibricks for type $A$ via arc diagrams, and provide a direct and simple proof of the bijection.

1.2. Main results. Fix a positive integer $n$. Let $\Pi$ be the preprojective algebra of type $A_n$, $\text{mod}\Pi$ the category of finitely generated right $\Pi$-modules and $D^b(\text{mod}\Pi)$ the bounded derived category of $\text{mod}\Pi$. We denote by $\text{sbrick}\Pi$ the set of semibricks (Definition 2.13) of $\text{mod}\Pi$ and by $2\text{-smc}\Pi$ the set of the 2-term simple-minded collections (SMCs for short) on $D^b(\text{mod}\Pi)$ (Definition 2.16).

Our fundamental tool is the notion of arc diagrams introduced by Reading \[\text{Re3}\]. The set of noncrossing arc diagrams of $n+1$ points (Definition 2.2, NAD for short), which is bijective correspondence with elements of the symmetric group of degree $n+1$, provides a combinatorial model for canonical join representations \[\text{Re3}\]. We give a map from each arc to a module, which is a brick, and we will show that this map can be extended to a bijective map from NAD to $\text{sbrick}\Pi$ (Theorem 3.13). Moreover, we attach two gradings on arcs, which we call green and red, and introduce the notion of the set of double arc diagrams (Definition 2.4, DAD for short), which is a set of bigraded arc diagrams enhanced from NAD. Then we give an interpretation of the gradings as shift functors of the derived category of $D^b(\text{mod}\Pi)$ and extend the above map from DAD to $2\text{-smc}\Pi$, which turned out to be bijection. For this purpose, we define a mutation on DAD (Definition 3.16). The mutation is compatible with the action of a simple generator on the Weyl group (Proposition 3.19), and this fact induces a partial order on DAD (Corollary 3.20).

The following picture shows the mutation behavior of NAD of 4 points.
Furthermore, we show that mutation on $DAD$ is also compatible with mutation on $2$-smc $\Pi$ (Proposition 3.24). Our main results are summarized as follows.

**Theorem 1.1.** (see Proposition 2.9, Theorems 3.13, 3.25) We have the following commutative diagrams, and all maps are poset isomorphisms.

\[ W \xrightarrow{\mathcal{D}} DAD \xrightarrow{\Phi} 2\text{-smc } \Pi \xrightarrow{H^0} \text{sbrick } \Pi, \]

\[ W \xrightarrow{\mathcal{G}} NAD \xrightarrow{\Psi} \text{sbrick } \Pi. \]

We remark that (semi)bricks over the preprojective algebras also have been studied by [As2 DIRRT] in a complete different way: Asai gave a classification of (semi)bricks in terms of Young diagram-like notation [As2] and Demonet-Iyama-Reading-Reiten-Thomas applied their reduction technique to the algebras and reduced this problem to gentle algebras [DIRRT]. On the other hand, Barnard-Carroll-Zhu [BCZ] studied torsion classes for a quotient of the preprojective algebras via arc diagrams. An explicit relationship between this quotient algebra and the preprojective algebra can be explained by using results of [AIR As1 M1 DIRRT]. In this paper, we will establish the above result without depending on these results.

As an application, we study semibricks over quotient algebras of preprojective algebras of type $A_n$. Let $I$ be a two-sided ideal of $\Pi$. Using the above map, we define a subposet of $NAD$ as follows

\[ NAD_I := \{ \mathcal{G} \in NAD \mid \Phi(\mathcal{G}) \subset \mod(\Pi/I) \}. \]

Then we have the following corollary, which recovers some part of results shown by the other authors such as [DIRRT BCZ IT Ao AMN].

**Corollary 1.2.** Let $I$ be a two-sided ideal of $\Pi$. We have a poset isomorphism

\[ NAD_I \rightarrow \text{sbrick}(\Pi/I). \]

In particular, we have the following results.

(i) Let $I_{\text{cyc}}$ be the ideal of $\Pi$ generated by all 2-cycles. Then we have a poset isomorphism

\[ NAD \rightarrow \text{sbrick}(\Pi/I_{\text{cyc}}). \]

(ii) Let $\vec{Q}$ be a linear quiver of type $A_n$ and $\text{RNAD}$ the set of right noncrossing arc diagrams (Definition 4.5). Then we have a poset isomorphism

\[ \text{RNAD} \rightarrow \text{sbrick } K\vec{Q}. \]

(iii) Let $\text{ANAD}$ be the set of alternating noncrossing arc diagrams (Definition 4.9). Then we have a poset isomorphism

\[ \text{ANAD} \rightarrow \text{sbrick}(\Pi/\text{rad}^2(\Pi)). \]

Finally we expect that the techniques of this paper shed a new light on the study of SMCs and can be widely applied to several classes of algebras formulated by geometric models such as gentle algebras.

2. Preliminaries

**Notation** Fix a natural number $n \geq 1$. For a finite dimensional algebra $A$ over an algebraically closed field $K$, we denote by $\mod A$ the category of finitely generated right $A$-modules and by $D^b(\mod A)$ the bounded derived category of $\mod A$. 

2.1. Symmetric groups and canonical join representations. We recall basic definitions and terminologies about the symmetric group.

Definition 2.1. Let $L$ be a finite lattice.

(1) An element of $x \in L$ is called **join-irreducible** if it is not the minimum element of $L$ and if $x = y \vee z$ for some $y, z \in L$, then $y = x$ or $z = x$, where $\vee$ denotes the join of $y$ and $z$, or equivalently, the join-irreducible elements are those which cover precisely one element.

(2) We call $C \subseteq L$ a **canonical join representation** if
   (i) $x = \bigvee_{c \in C} c$.
   (ii) For any proper subset $C' \subset C$, the join $\bigvee_{c \in C'} c$ never coincides with $x$.
   (iii) If $U \subseteq L$ satisfies the properties (i) and (ii), then, for every $c \in C$, there exists $u \in U$ such that $c \leq u$.

In this case, we also call $x = \bigvee_{c \in C} c$ a canonical join representation.

Note that if $x = \bigvee_{c \in C} c$ is a canonical join representation, then it is unique and each element $c \in C$ is join-irreducible. As a finite lattice $L$, we consider the symmetric group in this paper.

Fix a natural number $n \geq 1$ and let $[n + 1] := \{1, 2, \ldots, n + 1\}$. A permutation $w$ of $[n + 1]$ is a sequence $w = w_1w_2\ldots w_{n+1}$ such that $\{w_1, \ldots, w_{n+1}\} = [n]$. The weak order on permutations is a partial order whose cover relations are $w_1 \cdots w_{n+1} < v_1 \cdots v_{n+1}$ whenever there exists $i$ such that $w_i = v_{i+1} < v_i = w_{i+1}$ and such that $w_j = v_j$ for $j \notin \{i, i + 1\}$. This is the Weyl group of type $A_n$ and it is generated by the transpositions $s_i = (i \ i + 1) \ (1 \leq i \leq n)$, which satisfy the relations $s_i^2 = 1$, $s_i s_j = s_j s_i \ (|i - j| \geq 2)$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. We denote by $W = W_n$ for the set of permutations of $[n + 1]$ and we regard $W$ as a partially ordered set defined by the weak order.

In this paper, for $w = w_1w_2\ldots w_{n+1}$, we define the left action of $s_i$ by

$$s_i w = \begin{cases} w_{s_i(1)}w_{s_i(2)}\cdots w_{s_i(n+1)} & \text{if } i \neq w_i \text{ and } i+1 \neq w_i+1 \\ w_1w_2\cdots w_{i-1}w_i w_{i+1}w_{i+2}\cdots w_{n+1}. & \text{if } i = w_i \text{ or } i+1 = w_i+1 \end{cases}$$

We say that a **descent** of a permutation is a pair $w_i$ and $w_{i+1}$ of adjacent entries such that $w_i > w_{i+1}$. A permutation is join-irreducible if and only if it has exactly one descent (see Re3 for more detail).

2.2. Arc diagrams. Next we introduce arc diagrams, which is the main subject of this paper. We slightly modify the original definition of Re3 for our convenience. In particular, we arrange points of an arc diagram horizontally.

Definition 2.2. Put $n + 1$ distinct points $1, 2, \ldots, n + 1$ on a horizontal line from left to right.

(1) An **arc** is a curve which connects a point $p \in [n + 1]$ to a strictly higher point $q \in [n + 1]$, moving monotone upwards from $p$ to $q$ and passing either to the left (=above) or to the right (=below) of each point between $p$ and $q$.

(2) An **arc diagram** consists of some (or no) arcs connecting the points. We identify an arc (or arcs) with an arc diagram.

(3) An arc diagram is called **noncrossing** if it satisfies the following two conditions:
   (nc1) No two arcs intersect, except possibly at their endpoints.
   (nc2) No two arcs share the same right endpoint or the same left endpoint.
In this paper, by arcs intersect, we always mean that arcs cross in their interiors. We also treat arcs and arc diagrams up to isotopies, or equivalently, an arc diagram is determined by which pairs of points are joined by an arc and which points are above and below of each arc. We denote the set of arc diagrams (resp. noncrossing arc diagrams) by $AD = AD_n$ (resp. $NAD = NAD_n$).

**Example 2.3.** This is an example of a noncrossing arc diagram consisting of three arcs.

The following arc diagrams does not satisfy (nc1) and (nc2), respectively.

Next we relate arc diagrams with elements of $W$ following [Re3]. We first define some set of lines.

**Definition 2.4.** Given a permutation $w = w_1 \ldots w_{n+1}$, write each entry $w_i$ at the point $(i, w_i)$ in the plane $\{(a, b) \mid 1 \leq a, b \leq n + 1\}$.

1. We draw **green lines** between $(i, w_i)$ and $(i + 1, w_{i+1})$ if $w_i > w_{i+1}$. We denote the set of green lines by $G(w)$.
2. We draw **red lines** between $(i, w_i)$ and $(i + 1, w_{i+1})$ if $w_i < w_{i+1}$. We denote the set of red lines by $R(w)$.
3. We let $D(w) := G(w) \cup R(w)$.

For $w \in W$, define $G(w)$ as above, and move all of the points into a single vertical line, allowing the lines to curve but not to pass through any of the points. Then rotate it 90 degrees clockwise. These lines become the arcs in an arc diagram. By abuse of notation, we also write this map by $G : W \to AD_n$. Similarly, we define $R : W \to AD_n$ and $D : W \to AD_n$. We call $G(w)$ (resp. $R(w)$, $D(w)$) a **green arc diagram** (resp. a red arc diagram, a double arc diagram) of $w$. We denote the set of double arc diagrams by $DAD = DAD_n$, that is, $DAD = \{D(w) \mid w \in W\}$.

**Remark 2.5.** The name of green and red arcs comes from maximal green sequences in the sense of Keller (we refer to [K]). This name will be justified later (subsection 3.3).

**Example 2.6.** Let $w = 53271468$. The left figure shows the construction of green lines and the right one is the corresponding green arc diagram $G(w)$. 
Moreover, the red arc diagram $\mathcal{R}(w)$ and the double arc diagram $\mathcal{D}(w)$ are, respectively, illustrated as follows.

![Diagram showing red and double arc diagrams]

Here we write green arcs as solid lines and red arcs as dashed lines.

The following lemma follows immediately from its construction.

**Lemma 2.7.**
(1) An element $w \in W$ has exactly one decent (or equivalently, join-irreducible) if and only if $\mathcal{G}(w)$ consists of one arc.
(2) A double arc diagram always consists of $n$ arcs and any two arcs never intersect.

Moreover we recall the following important result due to Reading.

**Theorem 2.8.** [Re3] The map $\mathcal{G}: W \to \mathcal{AD}$ and $\mathcal{R}: W \to \mathcal{AD}$ gives a bijection $W \to \mathcal{NAD}$.

Moreover $\mathcal{G}(w)$ gives a canonical join representation of $w$ by identifying the arcs with join-irreducible elements.

**Proof.** [Re3, Theorems 2.4 and 3.1] shows that the map $\mathcal{G}: W \to \mathcal{NAD}$, where $\mathcal{G}$ is denoted by $\delta$ in [Re3], gives a bijection and $\mathcal{G}(w)$ gives a canonical join representation of $w$. Moreover, it is easy to check a bijection $\mathcal{R}: W \to \mathcal{NAD}$ by the similar argument of [Re3, Theorem 3.1]. $\square$

The following lemma also follows immediately from its construction and Theorem 2.8.

**Proposition 2.9.** Define the maps

$\mathcal{G}: \mathcal{DAD} \to \mathcal{NAD}$, $\mathcal{D}(w) \mapsto \mathcal{G}(w)$,

$\mathcal{R}: \mathcal{DAD} \to \mathcal{NAD}$, $\mathcal{D}(w) \mapsto \mathcal{R}(w)$

by the restriction of arcs of $\mathcal{D}(w)$ to green arcs and red arcs, respectively. Then the following diagram commutes, and all maps are bijections.

![Diagram showing bijections]

**Remark 2.10.** In [Re3], an explicit bijective map $\mathcal{NAD} \to W$ is explained. It implies that, for a given $\mathcal{G}(w), \mathcal{R}(w)$ or $\mathcal{D}(w)$, it is possible to calculate one of the others.

**Example 2.11.** We illustrate the correspondence $W$ and the set of arc diagrams. Here we write $W$ by the Hasse quiver. The corresponding quiver consisting of $\mathcal{NAD}$ and $\mathcal{DAD}$ is justified from the view point of mutation and partial orders, which will be explained in section 3.3.
(1) We give the correspondence $\mathcal{G} : W \to \text{NAD}$ for $n = 2$.

Moreover, the correspondence $\mathcal{D} : W \to \text{DAD}$ is illustrated as follows.

Here we write green arcs as normal lines and red arcs as dotted lines.

(2) We give the correspondence $\mathcal{D} : W \to \text{DAD}$ for $n = 3$. 
The aim of this paper is to give a natural interpretation of these arc diagrams in the representation theory of preprojective algebras.

2.3. Torsion classes and semibricks. In this subsection, we recall the notion of torsion classes and semibricks. Let $A$ be a basic finite dimensional algebra. We denote by $|A|$ the number of isoclasses of indecomposable direct summands of $A$ and we assume that $|A| = n$.

First we recall torsion classes and torsion-free classes.

**Definition 2.12.** A full subcategory $\mathcal{T} \subset \text{mod} A$ is called a torsion class if $\mathcal{T}$ is closed under taking extensions and factor modules. A torsion class $\mathcal{T} \subset \text{mod} A$ is called functorially finite if there exists $M \in \text{mod} A$ such that $\mathcal{T} = \text{Fac} M$, where $\text{Fac} M$ is the full subcategory of $\text{mod} A$ consisting of factor modules of finite direct sums of copies of $M$. Dually we can define a full subcategory $\mathcal{F} \subset \text{mod} A$ torsion-free class and functorially finite torsion-free class. We denote $f_{\text{tors}} A$ by the set of functorially finite torsion classes in $\text{mod} A$ and $f_{\text{torf}} A$ by the set of functorially finite torsion-free classes in $\text{mod} A$.

Next we recall the notion of bricks and semibricks.

**Definition 2.13.**

1. A module $S$ in $\text{mod} A$ is called a brick if $\text{End}_A(S)$ is a division $K$-algebra. We denote by $\text{brick} A$ the set of isoclasses of bricks in $\text{mod} A$.
2. A subset $\mathcal{S} \subset \text{brick} A$ is called a semibrick if $\text{Hom}_A(S_i, S_j) = 0$ holds for any $S_i \neq S_j \in \mathcal{S}$. We denote by $\text{sbrick} A$ the set of semibricks in $\text{mod} A$.
3. We say that a semibrick $\mathcal{S}$ is left finite (resp. right finite) if the smallest torsion class (resp. torsion-free class) $\mathcal{T}(\mathcal{S}) \subset \text{mod} A$ (resp. $\mathcal{F}(\mathcal{S}) \subset \text{mod} A$) containing $\mathcal{S}$ is functorially finite. We denote by $f_{\text{L}-\text{sbrick}} A$ (resp. $f_{\text{R}-\text{sbrick}} A$) the set of left finite (resp. right finite) semibricks in $\text{mod} A$.

Then we briefly recall some important results about torsion classes and semibricks.

**Definition-Theorem 2.14.** [As1, Proposition 1.9] [Ri] We have bijections

$$\mathcal{T} : f_{\text{L}-\text{sbrick}} A \to f_{\text{tors}} A, \quad \mathcal{F} : f_{\text{R}-\text{sbrick}} A \to f_{\text{torf}} A.$$
For \( S, S' \in fL\text{-sbrick} A \), we write \( S \leq S' \) if \( T(S) \subset T(S') \), and define a partial order on \( fL\text{-sbrick} A \) by \( S \leq S' \). A partial order on \( fR\text{-sbrick} A \) is similarly defined.

Finally, we recall the following \( \tau \)-tilting finite property in terms of bricks.

**Theorem 2.15.** [DII] [As1] Let \( A \) be a finite dimensional algebra. If \( \text{brick} A \) is finite, then we have
\[
\text{brick} A = fL\text{-sbrick} A = fR\text{-sbrick} A.
\]

2.4. **Simple-minded collections.** We next recall the definition of simple-minded collections in \( D^b(\text{mod} A) \). We keep the same notation as the previous subsection.

**Definition 2.16.** A set \( \mathcal{X} := \{X_1, X_2, \ldots, X_r\} \) of isoclasses of objects in \( D^b(\text{mod} A) \) is called a simple-minded collection (SMC for short) if it satisfies the following conditions.

(sm1) For any \( i \in \mathcal{X} \), the endomorphism ring \( \text{End}_{D^b(\text{mod} A)}(X_i) \) is a division \( K \)-algebra.

(sm2) For any \( X_i \neq X_j \in \mathcal{X} \), we have \( \text{Hom}_{D^b(\text{mod} A)}(X_i, X_j) = 0 \).

(sm3) For any \( X_i, X_j \in \mathcal{X} \) and \( m < 0 \), we have \( \text{Hom}_{D^b(\text{mod} A)}(X_i, X_j[m]) = 0 \).

(sm4) The smallest thick subcategory of \( D^b(\text{mod} A) \) containing \( \mathcal{X} \) is \( D^b(\text{mod} A) \).

In this case, we always have \( r = n \) [KY, Corollary 5.5]. We denote by \( \text{smc} A \) for the set of SMCs in \( D^b(\text{mod} A) \).

Moreover we recall a partial order on \( \text{smc} A \).

**Proposition 2.17.** [KY, Proposition 7.9] For \( \mathcal{X}, \mathcal{X}' \in \text{smc} A \), we write \( \mathcal{X} \leq \mathcal{X}' \) if
\[
\text{Hom}_{D^b(\text{mod} A)}(X, X'[m]) = 0
\]
for any \( m < 0 \), \( X \in \mathcal{X} \) and \( X' \in \mathcal{X}' \). Then \( \leq \) gives a partial order on \( \text{smc} A \).

Moreover, we call a complex \( X \in D^b(\text{mod} A) \) 2-term if the \( i \)-th cohomology \( H^i(X) \) is 0 for any \( i \neq -1, 0 \). A SMC \( \mathcal{X} \) in \( D^b(\text{mod} A) \) is called 2-term if any \( X \in \mathcal{X} \) is 2-term. We denote by \( 2\text{-smc} A \) for the set of 2-term SMCs in \( D^b(\text{mod} A) \).

The following simple remark is also quite important.

**Remark 2.18.** [BY, Remark 4.11] If \( \mathcal{X} \) is a 2-term SMC, then every \( X \in \mathcal{X} \) belongs to either \( \text{mod} A \) or \( (\text{mod} A)[1] \) in \( D^b(\text{mod} A) \).

Finally we recall an important relationship between semibricks and SMCs.

**Theorem 2.19.** [BY, KY, As1] There exists a poset isomorphism and anti-isomorphism
\[
H^0: 2\text{-smc} A \rightarrow fL\text{-sbrick} A, \quad H^{-1}: 2\text{-smc} A \rightarrow fR\text{-sbrick} A
\]
given by \( \mathcal{X} \mapsto H^0(\mathcal{X}) = \mathcal{X} \cap \text{mod} A \) and \( \mathcal{X} \mapsto H^{-1}(\mathcal{X}) = \mathcal{X}[-1] \cap \text{mod} A \), respectively.

3. **Double arc diagrams and 2-term SMCs**

In this section, we establish a direct relationship between the set of noncrossing arc diagrams (resp. double arc diagrams) and the set of semibricks (resp. 2-term SMCs) of a preprojective algebra of type \( A_n \). In particular, we establish a bijection between \( W \) and \( 2\text{-smc} A \), which gives another proof of [As2]. For this purpose, we relate each arc (with a color) to a brick (with a shift functor). Moreover, we study hom-spaces between two bricks in terms of combinatorics of arcs.
From now on, fix an integer \( n \geq 1 \). As before, we denote by \( W = \Gamma \) for the set of permutations of \( \{1, 2, \ldots, n+1\} \). Let \( Q \) be the following quiver

\[
Q = ( v_1 \xrightarrow{a_1^-} v_2 \xrightarrow{a_2^-} \cdots \xrightarrow{a_n^-} v_{n-1} \xrightarrow{a_{n-1}^-} v_n ).
\]

We denote by \( Q_0 \) the set of vertices of \( Q \), that is, \( Q_0 = \{v_1, \ldots, v_n\} \), and \( Q_1 \) the set of arrows of \( Q \), that is, \( Q_1 = \{a_1, \ldots, a_{n-1}, a_1^+, \ldots, a_n^-\} \). For the convenience, we write \( a_i = a_i^+ \) and hence we can write any arrow of \( Q \) as \( a_i^{\epsilon_i} \) for some \( \epsilon_i \in \{+, -\} \) and \( 1 \leq i \leq n-1 \). Let \( \Pi = \Pi_n \) be the preprojective algebra of type \( A_n \), that is, \( \Pi = KQ/(\sum_{i=1}^{n-1} (a_i^- a_i - a_i a_i^-)) \).

### 3.1. Bricks and arc modules

In this subsection, we give a correspondence between arcs and bricks, which is also explained in \[1\] Appendix.

**Definition 3.1.** For an arc \( \alpha \), we define a \( \Pi \)-module \( S(\alpha) \) by the following steps.

1. Draw \( n \) vertical dashed lines between adjacent points in the arc diagram, and name these lines as \( v_1, v_2, \ldots, v_n \) from left to right.
2. Define a subquiver \( Q_\alpha \) of \( Q \) as follows.
   - The vertex set of \( Q_\alpha \) consists of \( v_i \in Q_0 \) such that \( \alpha \) and the line \( v_i \) intersect.
   - Suppose that we have \( v_i, v_{i+1} \in (Q_\alpha)_0 \). If the segment of \( \alpha \) cut by the lines \( v_i \) and \( v_{i+1} \) is below the point \( i+1 \), then we put an arrow \( a_i : v_i \to v_{i+1} \), and put \( a_i^- : v_{i+1} \to v_i \) if the segment is above the point.
3. Construct \( S(\alpha) \) as a representation of \( Q \) as follows.
   - To each \( v_i \in Q_0 \), we assign \( K \) if \( i \in (Q_\alpha)_0 \), and 0 otherwise.
   - To each arrow \( v_i \to v_j \in Q \), we assign the identity map if \( v_i \to v_j \) belongs to \( Q_\alpha \), and 0 otherwise.

By this correspondence, we define a map

\[
S : \{\text{arcs}\} \to \text{mod} \Pi, \alpha \mapsto S(\alpha),
\]

and we call \( S(\alpha) \) an **arc module**.

Moreover, we can give another formulation of arc modules as follows.

The conditions (1) and (2) imply that if an arc \( \alpha \) has the left endpoint \( i \) and the right endpoint \( j+1 \), then we can define the sequence \( a_\epsilon a_{i+1}^{\epsilon_{i+1}} a_{i+2}^{\epsilon_{i+2}} \cdots a_{j-1}^{\epsilon_{j-1}} \) of arrows (possibly an idempotent) of \( Q \). More precisely, if \( i < j \), let \( k(\alpha)_{k+1} \) be the central segment of \( \alpha \) cut by the lines \( v_k \) and \( v_{k+1} \) \( (i \leq k < j) \). Then \( k(\alpha)_{k+1} \) corresponds to the arrow \( a_k : v_k \to v_{k+1} \) if the segment is below the point, and \( a_k^- : v_{k+1} \to v_k \) if the segment is above the point. We repeat this procedure from \( i \) to \( j+1 \) (left to right) and we have a unique sequence of arrows. If \( i = j \), then we associate \( \alpha \) with the idempotent \( e_j \).

By this correspondence, we associate an arc with the sequence of arrows and we call it the **arrow sequence** of \( \alpha \) (which is not necessary a path) and write \( \alpha = a_\epsilon a_{i+1}^{\epsilon_{i+1}} \cdots a_{j-1}^{\epsilon_{j-1}} \).

Then we define \( S(a_\epsilon a_{i+1}^{\epsilon_{i+1}} \cdots a_{j-1}^{\epsilon_{j-1}}) \) by assigning \( K \) to the vertices \( v_i, v_{i+1}, \ldots, v_j \) and assigning the identity maps for the arrows \( a_\epsilon, a_{i+1}^{\epsilon_{i+1}}, \ldots, a_{j-1}^{\epsilon_{j-1}} \). Then we have \( S(\alpha) = S(a_\epsilon a_{i+1}^{\epsilon_{i+1}} \cdots a_{j-1}^{\epsilon_{j-1}}) \).

We remark that arc modules can be defined by **non-revisiting walks** as \[2\] and by a **Young diagram-like notation** as \[3\]. It is also defined by using an arc in \[4\] for a quotient algebra of the preprojective algebra.

Now we give an example.
Example 3.2. Let $\alpha$ be the following arc.

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (v2) at (1,0) {$v_2$};
  \node (v3) at (2,0) {$v_3$};
  \node (v4) at (3,0) {$v_4$};
  \node (v5) at (4,0) {$v_5$};
  \node (v6) at (5,0) {$v_6$};
  \node (v7) at (6,0) {$v_7$};

  \draw[dashed] (v1) -- (v2);
  \draw[dashed] (v2) -- (v3);
  \draw[dashed] (v3) -- (v4);
  \draw[dashed] (v4) -- (v5);
  \draw[dashed] (v5) -- (v6);
  \draw[dashed] (v6) -- (v7);

  \foreach \i in {1,2,3,4,5,6,7} {
    \fill (\i,0) circle (2pt);
  }
\end{tikzpicture}
\end{center}

Then $a_1a_2a_3^{-1}a_4a_5^{-1}$ is the arrow sequence of $\alpha$. Then the corresponding arc module $S(\alpha)$ is defined as follows

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (v2) at (1,0) {$v_2$};
  \node (v3) at (2,0) {$v_3$};
  \node (v4) at (3,0) {$v_4$};
  \node (v5) at (4,0) {$v_5$};
  \node (v6) at (5,0) {$v_6$};

  \draw[->] (v1) -- (v2);
  \draw[->] (v2) -- (v3);
  \draw[->] (v3) -- (v4);
  \draw[->] (v4) -- (v5);
  \draw[->] (v5) -- (v6);

  \fill (v1) circle (2pt);
  \fill (v2) circle (2pt);
  \fill (v3) circle (2pt);
  \fill (v4) circle (2pt);
  \fill (v5) circle (2pt);
  \fill (v6) circle (2pt);
\end{tikzpicture}
\end{center}

where $v_i$ shows a $K$-vector space $K$ lying on the vertex $v_i$, and each arrow is the identity map.

Remark 3.3. We can also interpret the above construction in terms of surface models as follows. Let $S$ be a disk with $n + 1$ punctures with a set of distinguished points $1, 2, 3, \cdots, n + 1$ from left to right in the interior of $S$. Assume that the boundary has counterclockwise orientation.

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (v2) at (1,0) {$v_2$};
  \node (v3) at (2,0) {$v_3$};
  \node (v4) at (3,0) {$v_4$};
  \node (v5) at (4,0) {$v_5$};
  \node (v6) at (5,0) {$v_6$};

  \draw[->] (v1) to[bend right=90] (v2);
  \draw[->] (v2) to[bend right=90] (v3);
  \draw[->] (v3) to[bend right=90] (v4);
  \draw[->] (v4) to[bend right=90] (v5);
  \draw[->] (v5) to[bend right=90] (v6);

  \fill (v1) circle (2pt);
  \fill (v2) circle (2pt);
  \fill (v3) circle (2pt);
  \fill (v4) circle (2pt);
  \fill (v5) circle (2pt);
  \fill (v6) circle (2pt);
\end{tikzpicture}
\end{center}

Then each segment $k|\alpha|k+1$ is isotopy to the boundary of $S$ and can be oriented uniquely as same as the boundary of $S$. This orientation determines an arrow sequence of given arc.

From now on, for given arcs $\alpha$ and $\beta$, we will study hom-space $\text{Hom}_\Pi(S(\alpha), S(\beta))$. Since arc modules are nothing but representations of some path algebra of type $A_n$, it is not difficult to calculate it directly. Here we will formulate its calculation using a similar technique of gentle algebras by [CB1, S, BDMTY].

For an arc $\alpha$, consider a factorization of the arrow sequence. We denote the set of all factorizations of the arrow sequence $\alpha = a_{i_1}^t a_{i_1+1}^{s_1} \cdots a_{i_j-1}^{s_{j-1}}$ by

$$\mathcal{P}(\alpha) := \{(b, c, d) \mid a_{i_1}^t a_{i_1+1}^{s_1} \cdots a_{i_j-1}^{s_{j-1}} = bcd\}.$$ 

We call a triple $(b, c, d) \in \mathcal{P}(\alpha)$ a quotient factorization of $\alpha$ if the following conditions are satisfied

- $b$ is empty or $b = b'a_{s}^{-1}$ for some $i \leq s \leq j - 1$.
- $d$ is empty or $d = a_t d'$ for some $i \leq t \leq j - 1$.

It is easy to see that a quotient factorization $(b, c, d) \in \mathcal{P}(\alpha)$ induces a surjective map $S(\alpha) \twoheadrightarrow S(c)$. We denote by $\mathcal{F}(\alpha)$ the set of all quotient factorizations of $\alpha$.

Dually, we can define a submodule factorization of $\alpha$ and denote by $\mathcal{S}(\alpha)$ the set of all submodule factorizations of $\alpha$. Similarly, we have an injective map $S(c) \hookrightarrow S(\alpha)$ for a submodule factorization $(b, c, d) \in \mathcal{S}(\alpha)$. 
Let \((b, c, d) \in \mathcal{F}(\alpha)\) and \((b', c, d') \in \mathcal{S}(\beta)\). For a pair \(T := ((b, c, d), (b', c, d'))\), we have a natural homomorphism
\[ f_T : \mathcal{S}(\alpha) \to \mathcal{S}(\beta), \]
defined as the composition
\[ \mathcal{S}(\alpha) \xrightarrow{\mathcal{S}(\alpha)} \mathcal{S}(\beta). \]

We call these homomorphisms \(f_T\) graph maps following [BDMTY].

Note that a preprojective algebra is not necessary a gentle (string) algebra in general, but we can easily check that the similar result works for arc modules since \(\mathcal{S}(\alpha) \in \mod \Pi/I_{\text{cyc}}\), where \(I_{\text{cyc}}\) is the ideal generated by all 2-cycles and \(\Pi/I_{\text{cyc}}\) is gentle. The following result is due to [CB1].

**Theorem 3.4.** [CB1] The set of graph maps from \(\mathcal{S}(\alpha)\) to \(\mathcal{S}(\beta)\) is a basis for \(\text{Hom}_\Pi(\mathcal{S}(\alpha), \mathcal{S}(\beta))\).

From this theorem, we can formulate \(\text{Hom}_\Pi(\mathcal{S}(\alpha), \mathcal{S}(\beta))\) by a simple combinatorics of \(\alpha\) and \(\beta\).

We start with the following easy lemma about arc modules.

**Lemma 3.5.** For an arc \(\alpha\), \(\mathcal{S}(\alpha)\) is a brick (i.e. arc modules are brick).

**Proof.** This follows from Theorem 3.4 (or it is easy to check it directly since \(\mathcal{S}(\alpha)\) is an indecomposable \(KQ_\alpha\)-module whose entries of the dimension vector consists of 1).

Next we will show that any brick is an arc module. For this purpose, we recall some basic result of preprojective algebras, which comes from 2-Calabi-Yau property.

Let \((-,-)\) be the symmetric bilinear form on \(\mathbb{Z}^n\) defined by
\[ (x, y) := \sum_{i \in Q_0} 2x_i y_i - \sum_{a : i \rightarrow j \in Q_1} x_i y_j, \]
where \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\). In particular, we have the associated quadratic form
\[ q(x) := (x, x) = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_{n-1} - x_n)^2 + x_n^2. \]

**Proposition 3.6.** [CB2] Let \(\dim X\) (resp. \(\dim Y\)) be the dimension vector of \(X \in \mod \Pi\) (resp. \(Y \in \mod \Pi\)). Then we have
\[ (\dim X, \dim Y) = \dim \text{Hom}_\Pi(X, Y) + \dim \text{Hom}_\Pi(Y, X) - \dim \text{Ext}^1_\Pi(X, Y). \]

Using this result, we determine all bricks in terms of arcs.

**Proposition 3.7.**
1. The map
\[ S : \{\text{arcs}\} \to \text{brick} \Pi \]
is bijection.
2. Any brick does not admit non-trivial self-extension.
3. We have
\[ \text{sbrick} \Pi = f_L^{-1}\text{sbrick} \Pi = f_R^{-1}\text{sbrick} \Pi. \]

**Proof.** By Lemma 3.5, the map is well-defined and clearly injective. We will show the surjectivity. Let \(X \neq 0 \in \mod \Pi\) be a brick, which is clearly indecomposable, and \(\dim X = (x_1, \ldots, x_n)\) the dimension vector of \(X\). Then, Proposition 3.6 implies
\[ q(\dim X) \leq 2 \dim \text{Hom}_\Pi(X, X) = 2. \]
Therefore we have
\[ x_1^2 + (x_1 - x_2)^2 + \cdots + (x_{n-1} - x_n)^2 + x_n^2 \leq 2. \]

We can easily check that neither \( q(\dim X) = 0 \) nor \( q(\dim X) = 1 \) occur, and hence we have \( q(\dim X) = 2 \). Thus we get \( \text{Ext}^1_{\Pi}(X, X) = 0 \) and (2) follows. Moreover, by \( x_1^2 + (x_1 - x_2)^2 + \cdots + (x_{n-1} - x_n)^2 + x_n^2 = 2 \), \( x_1 \) and \( x_n \) are both 0 or 1. Then it is easy to check that \( x_i \) is also 0 or 1 for any \( 2 \leq i \leq n - 1 \) and hence \( \dim X \) has entries only 0 or 1. Since \( X \) is indecomposable, any non-zero entry is adjacent to each other, that is, \( \dim X \) has the form \((0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)\) possibly without zeroes. Therefore it is isomorphic to \( S(\alpha) \) for some arc \( \alpha \). Finally (3) follows from (1) and Theorem 2.15. \( \square \)

**Remark 3.8.** A classification of \( \text{sbrick} \Pi \) is also formulated in \( \text{As2} \) \( \text{DIRRT} \) in a different way.

3.2. **Hom-spaces of arc modules.** In this subsection, we study hom-spaces of arc modules in terms of combinatorics of arcs. Moreover, we establish a bijection between the set of noncrossing arc diagrams and the set of semibricks.

**Lemma 3.9.** Let \( \alpha \) and \( \beta \) be arcs. Assume that \( \alpha \) and \( \beta \) do not intersect.

1. Assume that \( \alpha \) and \( \beta \) share the same left endpoint but not the right endpoint. Then we have \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 0 \) or \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 1 \). Moreover, \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 0 \) (resp. \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 1 \)) if and only if \( \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)) = 1 \) (resp. \( \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)) = 0 \)).

2. Assume that \( \alpha \) and \( \beta \) share the same right endpoint but not the left endpoint. Then we have \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 0 \) or \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 1 \). Moreover, \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 0 \) (resp. \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 1 \)) if and only if \( \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)) = 1 \) (resp. \( \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)) = 0 \)).

3. Assume that the right endpoint of \( \alpha \) coincides with the left endpoint of \( \beta \), or left endpoint of \( \alpha \) coincides with the right endpoint of \( \beta \). Then we have
   \[ \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 0 = \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)). \]

4. Assume that \( \alpha \) and \( \beta \) share both the right endpoint and the left endpoint. Then we have
   \[ \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) \neq 0 \text{ and } \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) \neq 0. \]

**Proof.** We will show (1); (2) is similar.

We assume that \( \alpha \) and \( \beta \) share the left endpoint \( i \). Since \( \alpha \) and \( \beta \) do not intersect, there exists \( i < j \) such that \( \alpha \) and \( \beta \) are isotopy between the interval \( v_i \) and \( v_{j-1} \) satisfying one of the following situations

\[
\begin{align*}
\text{(Case 1). Consider the following case.} & \\
\end{align*}
\]

\[
\begin{align*}
\text{or} & \\
\end{align*}
\]
Then, by Theorem 3.4 we have an inclusion $S(\beta) \to S(\alpha)$ and there is no graph map from $S(\alpha)$ to $S(\beta)$. Thus we have $\dim \text{Hom}_\Pi(S(\alpha), S(\beta)) = 0$ and $\dim \text{Hom}_\Pi(S(\beta), S(\alpha)) = 1$. Clearly, if we replace $\alpha$ and $\beta$, then we have $\dim \text{Hom}_\Pi(S(\beta), S(\alpha)) = 0$ and $\dim \text{Hom}_\Pi(S(\alpha), S(\beta)) = 1$.

(Case 2). Consider the following case.

Then, by Theorem 3.4 we have a surjection $S(\beta) \to S(\alpha)$ and there is no graph map from $S(\alpha)$ to $S(\beta)$. Thus we have $\dim \text{Hom}_\Pi(S(\alpha), S(\beta)) = 0$ and $\dim \text{Hom}_\Pi(S(\beta), S(\alpha)) = 1$. Clearly, if we replace $\alpha$ and $\beta$, then we have $\dim \text{Hom}_\Pi(S(\beta), S(\alpha)) = 0$ and $\dim \text{Hom}_\Pi(S(\alpha), S(\beta)) = 1$.

(Case 3). Consider the following case.

Then, for the arrow sequence of $\alpha$ and $\beta$, we have factorizations $\alpha = ba_{j-1}d$ and $\beta = ba_{j-1}d'$. Then we have

$$S(\beta) \longrightarrow S(b) \longleftarrow S(\alpha).$$

Moreover, because $\alpha$ and $\beta$ do not share the right endpoint, the right endpoint of $\alpha$ is bigger or smaller than the right endpoint of $\beta$. Therefore, there exists $k > j$ such that

Then we can check that the above map $S(\beta) \to S(b) \hookrightarrow S(\alpha)$ is a unique graph map and there is no graph map from $S(\alpha)$ to $S(\beta)$. Thus we have $\dim \text{Hom}_\Pi(S(\beta), S(\alpha)) = 1$ and $\dim \text{Hom}_\Pi(S(\alpha), S(\beta)) = 0$ by Theorem 3.4. Clearly, if we replace $\alpha$ and $\beta$, then we have $\dim \text{Hom}_\Pi(S(\beta), S(\alpha)) = 0$ and $\dim \text{Hom}_\Pi(S(\alpha), S(\beta)) = 1$.

Using Theorem 3.4 it is easy to check (3) and (4) by the same argument. □

Next we give the following simple observation.

**Proposition 3.10.** Two arcs $\alpha$ and $\beta$ are noncrossing (i.e. $\alpha$ and $\beta$ satisfy (nc1),(nc2)) if and only if $\dim \text{Hom}_\Pi(S(\alpha), S(\beta)) = 0 = \dim \text{Hom}_\Pi(S(\beta), S(\alpha))$.

**Proof.** The only if part follows from Lemma 3.9 (3). We will show the if part. Assume that $\alpha$ and $\beta$ are crossing.

(nc1). First we assume that $\alpha$ and $\beta$ intersect between the points $i$ and $i + 1$. Then it is easy to check that the simple module $S_{v_i}$ associated to the vertex $v_i$ belongs to one of the top of $S(\alpha)$ and $S(\beta)$. We assume that it belongs to the top of $S(\alpha)$. Then since $\alpha$ and
\[ \beta \text{ intersect, } S_{v_i} \text{ belongs to the socle of } S(\beta). \] Therefore have \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) \neq 0. \)

Clearly, if we replace \( \alpha \) and \( \beta \), then we have \( \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)) \neq 0. \)

\textbf{(nc2).} Next we assume that \( \alpha \) and \( \beta \) share either the same right endpoint or the same left endpoint, or both. Then Lemma 3.9 implies that \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) \neq 0 \) or \( \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)) \neq 0. \)

\textbf{Remark 3.11.} We remark that Proposition 3.10 is essentially identified with [BCZ, Lemma 4.2.8], where the authors studied some quotient algebra \( \Pi/I \) of \( \Pi. \) Since there exists a natural fully faithful functor from \( \text{mod}(\Pi/I) \) to \( \text{mod} \Pi, \) we can apply this result to our case.

For \( G(w) \in \text{NAD}, \) we define the map \( \Phi \) as follows

\[ \Phi(G(w)) = \{ S(\alpha) \mid \alpha \in G(w) \} \subset \text{mod} \Pi. \]

\textbf{Example 3.12.} Let \( w = 53271468. \) Then, by Example 2.6, we have the following green arc diagram \( G(w) \)

Then \( \Phi(G(w)) \) consist of three indecomposable \( \Pi \)-modules

\[ v_1 \oplus v_2 \oplus v_3 \oplus v_5 \oplus v_6 \oplus v_7. \]

Using the above notion, we have the following consequence, which provides a bijection between \( W \) and \( \text{sbrick} \Pi. \)

\textbf{Theorem 3.13.} We have a bijection

\[ \Phi : \text{NAD} \rightarrow \text{sbrick} \Pi. \]

In particular, there exists a bijection \( W \) and \( \text{sbrick} \Pi. \)

\textbf{Proof.} From Propositions 3.7 and 3.10 the map is well-defined and surjective, and it is clearly injective. The second statement follows from Theorem 2.8 \( \square \)

We prepare the following lemma for later use.

\textbf{Lemma 3.14.} Assume that two arcs \( \alpha \) and \( \beta \) do not intersect nor share any endpoint.

Then we have

\[ \dim \text{Ext}^1_{\Pi}(S(\alpha), S(\beta)) = 0 = \dim \text{Ext}^1_{\Pi}(S(\beta), S(\alpha)). \]

\textbf{Proof.} Since the dimension vectors of \( S(\alpha) \) and \( S(\beta) \) have entries only 0 or 1, we have \( (\dim S(\alpha), \dim S(\beta)) = 0 \) by the assumption. On the other hand, by Lemma 3.9 (3), we have \( \dim \text{Hom}_{\Pi}(S(\alpha), S(\beta)) = 0 = \dim \text{Hom}_{\Pi}(S(\beta), S(\alpha)). \) Therefore, by Proposition 3.6 we have

\[ 0 = (\dim S(\alpha), \dim S(\beta)) = -\dim \text{Ext}^1_{\Pi}(S(\alpha), S(\beta)). \]

Thus, we get \( \dim \text{Ext}^1_{\Pi}(S(\alpha), S(\beta)) = 0 \) and, similarly, \( \dim \text{Ext}^1_{\Pi}(S(\beta), S(\alpha)) = 0. \) \( \square \)
3.3. A partial order and mutation. In this subsection, we discuss mutation and partial orders of arc diagrams. We define mutation of DAD such that the left action of a transposition on \( w \) is compatible with the mutation of the corresponding double arc diagrams. Moreover we will show the compatibility of the mutation of double arc diagrams and 2-term SMCs. Using this result, we show a poset isomorphism \( W \to 2\text{-smc}\Pi \).

For our purpose, we give a more precise correspondence \( \mathcal{D} : W \to \text{DAD} \). Let \( w = (w_1 w_2 \cdots w_{n+1}) \in W \). By the map \( \mathcal{D} : W \to \text{DAD} \), for a pair \( (w_i, w_{i+1}) \), we have an arc whose endpoints are \( w_i \) and \( w_{i+1} \). We denote this arc by \( \alpha_{w_i w_{i+1}} \) (note that the arc \( \alpha_{w_i w_{i+1}} \) depends not only on \( (w_i, w_{i+1}) \) but also on \( (w_1 w_2 \cdots w_{n+1}) \)).

Recall that \( \alpha_{w_i w_{i+1}} \) is green (resp. red) if \( w_i > w_{i+1} \) (resp. \( w_i < w_{i+1} \)). For simplicity, we write \( \alpha_{w_i w_{i+1}}(0) \) (resp. \( \alpha_{w_i w_{i+1}}(1) \)) if \( \alpha_{w_i w_{i+1}} \) is green (resp. red). We denote by \( c_i \in \{0, 1\} \) the color of \( \alpha_{w_i w_{i+1}} \). In this notation, we can write

\[
\mathcal{D}(w) = \{ \alpha_{w_1 w_2}(c_1), \alpha_{w_2 w_3}(c_2), \ldots, \alpha_{w_n w_{n+1}}(c_n) \},
\]

and we define \( \Psi : \text{DAD} \to \{ \text{the set of 2-term complexes of } \mathcal{D}^\Pi(\text{mod } \Pi) \} \)

\[
\Psi(\mathcal{D}(w)) = \Psi(\{ \alpha_{w_i w_{i+1}}(c_i) \}) = \{ S(\alpha_{w_i w_{i+1}}[c_i]) \} \quad (1 \leq i \leq n).
\]

It is also written as \( \Psi(\mathcal{D}(w)) = \Phi(\mathcal{G}(w)) \sqcup \Phi(\mathcal{R}(w))[1] \). By abuse of notation, for an arc \( \alpha_{w_i w_{i+1}}(c_i) \) of \( \mathcal{D}(w) \), we denote by

\[
\Psi(\alpha_{w_i w_{i+1}}(c_i)) = S(\alpha_{w_i w_{i+1}}[c_i])
\]

the corresponding 2-term indecomposable complex obtained by \( \Psi \). We will show that \( \Psi \) gives a bijection between DAD and 2-smc\Pi.

First, we will define a mutation on DAD. This is defined by applying a half-twist to a diagram. To avoid an confusion, we give another formulation here.

We introduce the following terminology. For an arc \( \alpha \), we denote by \( L(\alpha) \in \{ 1, 2, \ldots, n \} \) the left endpoint of \( \alpha \) and \( R(\alpha) \in \{ 2, 3, \ldots, n + 1 \} \) the right endpoint of \( \alpha \). Recall that, for an arc \( \alpha \) and \( 1 \leq i \leq n \), we denote by \( \alpha|_{v_i} \) (resp. \( v_i | \alpha \), \( \alpha|_{v_{i+1}} \)) the left segment (resp. the right segment, the central segment) of \( \alpha \) cut by the line \( v_i \) (resp. \( v_i, v_i \) and \( v_{i+1} \)). Then we define the following transformation of arcs.

**Definition 3.15.** Fix \( w \in W \). Let \( \alpha, \beta \) (\( \alpha \neq \beta \)) be arcs of \( \mathcal{D}(w) \). Assume that \( \alpha \) and \( \beta \) share one of the endpoint \( i \). Then, by transforming \( \beta \), we define a new arc \( \gamma = \sigma(\beta; \alpha) \) as follows.

(1-i) Assume that \( R(\alpha) = L(\beta) = i \). Then we define \( \gamma \), where \( \gamma|_{v_{i-1}} \) is isotopy to \( \alpha|_{v_{i-1}} \), \( v_i | \gamma \) is isotopy to \( v_i | \beta \) and \( v_{i+1} | \gamma|_{v_i} \) is the arc above to the point \( i \). This is illustrated by the following example.

![Diagram](image)

We remark that we have a natural interpretation as an exact sequence. For example, if \( \alpha \) and \( \beta \) are green, then we have the exact sequence

\[
0 \longrightarrow S(\alpha) \longrightarrow S(\gamma) \longrightarrow S(\beta) \longrightarrow 0.
\]
(1-ii) If \( L(\alpha) > L(\beta) \), then we define \( \gamma \) such that \( \gamma_{v_{i-1}} \) is isotopy to \( \beta_{v_{i-1}} \) and \( i \) is the right endpoint of \( \gamma \).

(1-iii) If \( L(\alpha) < L(\beta) \), then we define \( \gamma \), where \( \gamma_{v_{i-1}} \) is isotopy to \( \alpha_{v_{i-1}} \) and \( i \) is the right endpoint of \( \gamma \).

(2-i) Assume that \( L(\alpha) = R(\beta) = i \). Then we define \( \gamma \), where \( \gamma_{v_{i-1}} \) is isotopy to \( \beta_{v_{i-1}} \), \( v_{i}|\gamma \) is isotopy to \( v_{i}|\alpha \) and \( v_{i-1}|\gamma|v_{i} \) is the arc below to the point \( i \).

(2-ii) If \( R(\alpha) > R(\beta) \), then we define \( \gamma \) such that \( v_{i}|\gamma \) is isotopy to \( v_{i}|\alpha \) and \( i \) is the left endpoint of \( \gamma \).

(2-iii) If \( R(\alpha) < R(\beta) \), then we define \( \gamma \) such that \( v_{i}|\gamma \) is isotopy to \( v_{i}|\beta \) and \( i \) is the left endpoint of \( \gamma \).

Using the above transformation, we define a mutation of DAD as follows.

**Definition 3.16.** Let \( w = (w_1w_2 \cdots w_{n+1}) \in W \) and \( D(w) = \{ \alpha_{w_1w_2}(c_1), \alpha_{w_2w_3}(c_2), \ldots, \alpha_{w_nw_{n+1}}(c_n) \} \). Fix \( 1 \leq i \leq n \) and assume that \( w_i > w_{i+1} \), or equivalently, \( \alpha_{w_iw_{i+1}} \) is green. We define left mutation

\[ \mu_i^- (D(w)) := \{ \mu_i^- (\alpha_{w_1w_2}(c_1)), \mu_i^- (\alpha_{w_2w_3}(c_2)), \ldots, \mu_i^- (\alpha_{w_nw_{n+1}}(c_n)) \} \]

as follows.

(1) For \( k \neq i - 1, i, i + 1 \), we do not change \( \alpha_{w_kw_{k+1}} \) including its color. Equivalently, we define

\[ \mu_i^- (\alpha_{w_kw_{k+1}}(c_k)) = \alpha_{w_kw_{k+1}}(c_k) \]

(2) We only change the color of \( \alpha_{w_iw_{i+1}} \) from green to red. Equivalently, we define

\[ \mu_i^- (\alpha_{w_iw_{i+1}}(1)) = \alpha_{w_iw_{i+1}}(0) \]

(3) We define \( \mu_i^- (\alpha_{w_{i-1}w_i}(c_{i-1})) = \sigma(\alpha_{w_{i-1}w_i} \alpha_{w_{i}w_{i+1}})(c_{i-1}') \), where

\[ c_{i-1}' = \begin{cases} 1 & \text{if } w_{i-1} < w_{i+1}, \\ 0 & \text{if } w_{i-1} > w_{i} \text{ or } w_{i} > w_{i-1} > w_{i+1}. \end{cases} \]
Similarly, we define $\mu_i(\alpha_{w_{i+1}w_{i+2}}(c_{i+1})) = \sigma(\alpha_{w_{i+1}w_{i+2}}; \alpha_{w_{i+2}w_{i+1}})(c'_{i+1})$, where

$$c'_{i+1} = \begin{cases} 1 & \text{if } w_i < w_{i+2}, \\ 0 & \text{if } w_{i+1} > w_{i+2} \text{ or } w_{i+1} < w_{i+2} < w_i. \end{cases}$$

If $w_i < w_{i+1}$, or equivalently, $\alpha_{w_iw_{i+1}}$ is red, then we can dually define right mutation $\mu_i^+(D(w))$. We denote by $\mu_i$ left or right mutation.

**Remark 3.17.** We can formulate the above mutation in terms of half-twists. Namely, the new diagram is defined by applying a half-twist [KT 1.6.2 Half-twists] to $\alpha_{w_iw_{i+1}}$ and hence mutation is nothing but graded half-twists.

**Example 3.18.** Consider $n = 3$ and the set of double noncrossing arc diagrams $\text{DAD}_3$. The following graph is a mutation graph of $\text{DAD}_3$, where the arrows denote left mutations.

The following lemma is straightforward.

**Proposition 3.19.** Let $s_i = (i \ i+1)$ be the transposition of $W$. Then the left action of $s_i$ on $w$ is compatible with mutation $\mu_i$, that is, we have the following commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{D} & \text{DAD} \\
\downarrow{s_i} & & \downarrow{\mu_i} \\
W & \xrightarrow{D} & \text{DAD}
\end{array}
$$

In particular, we have $\mu_i(D(w)) \in \text{DAD}$ and $\mu_i^- \circ \mu_i^+ = \text{id} = \mu_i^+ \circ \mu_i^-$. 

**Proof.** Let $w = w_1 \cdots w_{n+1}$ and 

$$D(w) = \{\alpha_{w_1w_2}(c_1), \alpha_{w_2w_3}(c_2), \cdots, \alpha_{w_nw_{n+1}}(c_n)\}.$$ 

Let $u := s_iw = u_1 \cdots u_{n+1}$ and 

$$D(u) = \{\alpha_{u_1u_2}(d_1), \alpha_{u_2u_3}(d_2), \cdots, \alpha_{u_nu_{n+1}}(d_n)\}.$$
We will show that
\[ \{\alpha_{uk_{k+1}}(d_k) \mid 1 \leq k \leq n\} = \mu_{i}(\{\alpha_{uk_{k+1}}(c_k) \mid 1 \leq k \leq n\}). \]
It is also enough to show that \( \alpha_{uk_{k+1}}(d_k) = \mu_{i}(\alpha_{uk_{k+1}}(c_k)) \) for any \( 1 \leq k \leq n \).
Since \( u_1 \cdots u_{n+1} = w_1 w_2 \cdots w_{i-1} w_{i+1} w_{i+2} \cdots w_{n+1} \), we have
\[ \alpha_{uk_{k+1}}(d_k) = \alpha_{uk_{k+1}}(c_k) = \mu_{i}(\alpha_{uk_{k+1}}(c_k)) \]
for any \( k \neq i-1, i, i+1 \). By case-by-case analysis, we can easily check the case of \( k = i-1, k = i \) and \( k = i+1 \).

By Propositions 2.19 and 3.19, we obtain the following corollary.

**Corollary 3.20.** Let \( D, D' \in \text{DAD} \). We write \( D \leq D' \) if \( D \) is obtained from \( D' \) by a sequence of left mutation. Then it gives a partial order on \( \text{DAD} \) and it is isomorphic to \( W \) as posets.

Moreover we define a partial order on \( \text{NAD} \) as follows.

**Definition 3.21.** Let \( G, G' \in \text{NAD} \). We define a partial order on \( \text{NAD} \) by \( G \leq G' \) if \( G^{-1}(G) \leq G^{-1}(G') \). In particular, \( \text{NAD} \cong \text{DAD} \cong W \).

Next we will show the compatibility of mutation of \( \text{DAD} \) and \( 2\text{-smc} \). For this purpose, we recall mutation of simple minded collections based on [KY] [BY].

**Definition 3.22.** [KY] Let \( A \) be a finite dimensional algebra such that \(|A| = n\). Let \( X := \{X_1, X_2, \cdots, X_n\} \) be a SMC of \( \text{D}^b(\text{mod} \ A) \). Fix \( 1 \leq i \leq n \). We define a left mutation of SMCs
\[ \mu_i^{-}(X) := \{ \mu_i^{-}(X_1), \mu_i^{-}(X_2), \cdots, \mu_i^{-}(X_n) \} \subset \text{D}^b(\text{mod} \ A) \]
as follows. Let \( E := \text{Filt} X_i \) be the extension closure of \( X_i \) of \( \text{D}^b(\text{mod} \ A) \). Take a minimal left \( E \)-approximation of \( X_k[-1] \)
\[ X_k[-1] \xrightarrow{f_k} E_k. \]
Then we define \( \mu_i^{-}(X_k) = \text{cone}(f_k) \), where \( \text{cone}(f_k) \) is the mapping cone of \( f_k \). Dually we can define the right mutation \( \mu_i^{+} \) and we have \( \mu_i^{+} \circ \mu_i^{-} = \text{id} \) and \( \mu_i^{-} \circ \mu_i^{+} = \text{id} \). We denote by \( \mu_i \) a left or right mutation.

Even if \( X \) is 2-term, \( \mu_i^{-}(X) \) is not necessary 2-term in general. With regard to this fact, we have the following nice characterization of 2-term SMCs [BY] Subsection 3.7.

**Lemma 3.23.** [BY] Let \( X := \{X_1, X_2, \cdots, X_n\} \) be a 2-term SMCs of \( \text{D}^b(\text{mod} \ A) \).

1. We have \( \mu_i^{-}(X) \in 2\text{-smc} \ A \) if and only if we have \( X_i \in \text{mod} \ A \).
2. If the above equivalent condition (1) holds, then \( \mu_i^{-}(X) \) has the following description.
   (i) We have
   \[ \mu_i^{-}(X_i) = X_i[1]. \]
   (ii) Assume \( X_k \in \text{mod} \ A \ (k \neq i) \). Then we have \( \mu_i^{-}(X_k) = \text{cone}(f_k) \in \text{mod} \ A \) and we have an exact sequence \( 0 \to E_k \to \text{cone}(f_k) \to X_k \to 0 \) in \( \text{mod} \ A \).
   (iii) Assume \( X_k \in \text{mod} \ A[1] \). Then we have one of the following two cases:
   (Case1). We have an exact sequence in \( \text{mod} \ A \)
   \[ 0 \longrightarrow X_k[-1] \xrightarrow{f_k} E_k \longrightarrow \text{Coker}(f_k) \longrightarrow 0. \]
In this case, we have \( \mu_i^- (X_k) = \text{Coker}(f_k) \).

(Case 2). We have an exact sequence in \( \mod A \)

\[
0 \longrightarrow \ker(f_k) \longrightarrow X_k[-1] \xrightarrow{f_k} E_k \longrightarrow 0.
\]

In this case, we have \( \mu_i^- (X_k) = \ker(f_k)[1] \).

The following proposition is a key result of this section.

**Proposition 3.24.** The map \( \Psi : \text{DAD} \rightarrow 2\text{-smc II} \) is well-defined and the mutation of \( \text{DAD} \) and \( 2\text{-smc II} \) are compatible, that is, we have the following commutative diagram

\[
\begin{array}{ccc}
\text{DAD} & \xrightarrow{\Psi} & 2\text{-smc II} \\
\mu_i & \downarrow & \downarrow \mu_i \\
\text{DAD} & \xrightarrow{\Psi} & 2\text{-smc II}
\end{array}
\]

**Proof.** Let \( \mathcal{D}(w) = \{ \alpha_{w_1w_2}(c_1), \alpha_{w_2w_3}(c_2), \ldots, \alpha_{w_nw_{n+1}}(c_n) \} \). Fix \( 1 \leq i \leq n \). We will show the compatibility of left mutation. We assume that \( w_i > w_{i+1} \), or equivalently, \( c_i = 0 \) and we will show that

\[
\Psi(\mu_i^{-} (\{ \alpha_{w_kw_{k+1}}(c_k) \}_{1 \leq k \leq n})) = \mu_i^{-} (\Psi(\{ \alpha_{w_kw_{k+1}}(c_k) \}_{1 \leq k \leq n})).
\]

In our notation, it is enough to show that \( \Psi(\mu_i^{-} (\alpha_{w_kw_{k+1}}(c_k))) = \mu_i^{-} (\Psi(\alpha_{w_kw_{k+1}}(c_k))) \) for any \( 1 \leq k \leq n \). Then, because of \( \Psi(\mathcal{D}(\text{id})) = \{ S_1, S_2, \ldots, S_n \} \in 2\text{-smc II} \) and Lemma 3.23, it also implies that \( \Psi : \text{DAD} \rightarrow 2\text{-smc II} \) is well-defined.

For simplicity, we write \( S(\alpha_{w_kw_{k+1}}) \) by \( S(w_kw_{k+1}) \). Then we have

\[
\Psi(\mathcal{D}(w)) = \{ S(w_1w_2)c_1, S(w_2w_3)c_2, \ldots, S(w_nw_{n+1})c_n \}.
\]

Let \( \mathcal{E} := \text{Filt}\{ S(w_kw_{k+1}) \} \) be the extension closure of \( S(w_kw_{k+1}) \). Then, by Proposition 3.7 (2), we have \( \mathcal{E} = \text{add}\{ S(w_kw_{k+1}) \} \).

(1) Fix \( 1 \leq k \leq n \) such that \( k \neq i - 1, i, i + 1 \). Then the two arcs \( \alpha_{w_kw_{k+1}} \) and \( \alpha_{w_{i+1}w_{i+1}} \) are noncrossing. If \( c_k = 0 \), then Lemma 3.14 implies that

\[
\text{Hom}_{D^b(\text{mod II}))(S(w_kw_{k+1})[-1], S(w_kw_{k+1}))} = \text{Ext}_{\text{III}}^1(S(w_kw_{k+1}), S(w_kw_{k+1})) = 0.
\]

If \( c_k = 1 \), then Lemma 3.9 implies that

\[
\text{Hom}_{D^b(\text{mod II}))(S(w_kw_{k+1}), S(w_kw_{k+1}))} = 0.
\]

Therefore, in both cases,

\[
S(w_kw_{k+1})[c_k - 1] \xrightarrow{f_k} 0,
\]

is a left minimal \( \mathcal{E} \)-approximation and hence we have

\[
\mu_i^{-} (S(w_kw_{k+1})[c_k]) = \text{cone}(f_k) = S(w_kw_{k+1})[c_k] = \Psi(\mu_i^{-} (\alpha_{w_kw_{k+1}}(c_k))).
\]

Thus we get

\[
\mu_i^{-} (\Psi(\alpha_{w_kw_{k+1}}(c_k))) = \Psi(\mu_i^{-} (\alpha_{w_kw_{k+1}}(c_k))).
\]

(2) Since \( c_i = 0 \), we have

\[
\mu_i^{-} (S(w_kw_{i+1})[0]) = S(w_kw_{i+1})[1].
\]

Thus we get \( \mu_i^{-} (\Psi(\alpha_{w_kw_{i+1}}(c_i))) = \Psi(\mu_i^{-} (\alpha_{w_kw_{i+1}}(c_i))). \)
(3) We will calculate \( \mu_i^-(S(w_{i-1}w_i)\{c_{i-1}\}) \).

(i) First assume \( w_{i-1} > w_i \), or equivalently, \( c_{i-1} = 0 \). Since the dimension vectors of \( S(w_{i-1}w_i) \) and \( S(w_iw_{i+1}) \) do not share any non-zero entry and they are adjacent, we have \( (\dim S(w_{i-1}w_i), \dim S(w_iw_{i+1})) = -1 \). On the other hand, we have \( \text{Hom}_{\Pi}(S(w_{i-1}w_i), S(w_iw_{i+1})) = 0 = \text{Hom}_{\Pi}(S(w_iw_{i+1}), S(w_{i-1}w_i)) \) by Proposition 3.10. Thus, Proposition 3.6 implies

\[
\dim \text{Ext}^1_{\Pi}(S(w_{i-1}w_i), S(w_iw_{i+1})) = 1.
\]

Therefore there exists a unique (up to scalar) non-zero map

\[
S(w_{i-1}w_i)[-1] \xrightarrow{f_{i-1}} S(w_iw_{i+1}),
\]

which is a left minimal \( E \)-approximation. Then, by Lemma 3.23 (ii), we have \( \text{cone}(f_{i-1}) \in \text{mod} \Pi \) which is given by the following non-split exact sequence

\[
0 \to S(w_{i-1}w_i) \to \text{cone}(f_{i-1}) \to S(w_{i-1}w_i) \to 0.
\]

Thus we get \( \text{cone}(f_{i-1}) \cong S(\sigma(\alpha_{w_{i-1}w_i}; \alpha_{w_iw_{i+1}})) = \Psi(\mu_i^- (\alpha_{w_{i-1}w_i}(c_{i-1}))) \).

Therefore we obtain \( \mu_i^- (\Psi(\alpha_{w_{i-1}w_i}(c_{i-1}))) = \Psi(\mu_i^- (\alpha_{w_{i-1}w_i}(c_{i-1}))) \).

(ii) Next assume \( w_{i-1} < w_i \), or equivalently, \( c_{i-1} = 1 \). As same as Lemma 3.9 (2), if \( w_{i-1} > w_{i+1} \), then we have a unique inclusion from \( S(w_{i-1}w_i) \) to \( S(w_iw_{i+1}) \) and if \( w_{i-1} < w_{i+1} \), then we have a unique surjection from \( S(w_{i-1}w_i) \) to \( S(w_iw_{i+1}) \). Thus, we have

\[
\dim \text{Hom}_{\Pi}(S(w_{i-1}w_i), S(w_iw_{i+1})) = 1.
\]

Therefore there exists a unique non-zero map

\[
S(w_{i-1}w_i)[1][-1] \cong S(w_{i-1}w_i) \xrightarrow{f_{i-1}} S(w_iw_{i+1}),
\]

which is a left minimal \( E \)-approximation.

(Case 1). Assume \( w_{i-1} > w_{i+1} \). Then we have an exact sequence

\[
0 \longrightarrow S(w_{i-1}w_i) \xrightarrow{f_{i-1}} S(w_iw_{i+1}) \longrightarrow \text{Coker}(f_{i-1}) \longrightarrow 0.
\]

By Lemma 3.23 (iii), we have

\[
\mu_i^- (S(w_{i-1}w_i)[1]) = \text{Coker}(f_{i-1}) = S(\sigma(\alpha_{w_{i-1}w_i}; \alpha_{w_iw_{i+1}}))[c_{i-1}].
\]

(Case 2). Assume \( w_{i-1} < w_{i+1} \). Then we have an exact sequence

\[
0 \longrightarrow \text{Ker}(f_{i-1}) \longrightarrow S(w_{i-1}w_i) \xrightarrow{f_{i-1}} S(w_iw_{i+1}) \longrightarrow 0.
\]

By Lemma 3.23 (iii), we have

\[
\mu_i^- (S(w_{i-1}w_i)[1]) = \text{ker}(f_{i-1})[1] = S(\sigma(\alpha_{w_{i-1}w_i}; \alpha_{w_iw_{i+1}}))[c_{i-1}].
\]

Therefore, we obtain \( \mu_i^- (\Psi(\alpha_{w_{i-1}w_i}(c_{i-1}))) = \Psi(\mu_i^- (\alpha_{w_{i-1}w_i}(c_{i-1}))) \).

(4) We can calculate \( \mu_i^- (S(w_{i+1}w_{i+2})[c_{i+1}]) \) as same as (3) and we get

\[
\mu_i^- (\Psi(\alpha_{w_{i+1}w_{i+2}}(c_{i+1}))) = \Psi(\mu_i^- (\alpha_{w_{i+1}w_{i+2}}(c_{i+1}))).
\]

We leave to the reader the calculation.

By (1), (2), (3) and (4), we get the conclusion. \( \square \)

As a consequence, we have the following result.
**Theorem 3.25.**  
(1) We have a poset isomorphism 
\[ \Psi : DAD \rightarrow 2\text{-smc} \Pi. \]
(2) We have a poset isomorphism 
\[ \Phi : NAD \rightarrow \text{sbrick} \Pi. \]

Proof. (1) First we will show that the map is injective. Let \( D, D' \in DAD \) and 
\[ D = \{ \alpha_1(c_1), \alpha_2(c_2), \ldots, \alpha_n(c_n) \}, \quad D' = \{ \alpha'_1(c'_1), \alpha'_2(c'_2), \ldots, \alpha'_n(c'_n) \}. \]
Assume that \( \Psi(D) = \Psi(D') \). It is equivalent to saying that 
\[ \{ S(\alpha_i)[c_i] \} = \{ S(\alpha'_i)[c'_i] \}. \quad (1 \leq i \leq n) \]
Then, Proposition 3.7 implies \( D = D' \). Thus the map is injective.

Next we will show that the map is an order embedding. Let \( D, D' \in DAD \) and assume that \( D' \succ D \), or equivalently, \( D \) is obtained by left mutation from \( D' \). By Proposition 3.24, it is also equivalent to saying that \( \Psi(D) \) is obtained by left mutation from \( \Psi(D') \). Equivalently, we have \( \Psi(D') \succ \Psi(D) \) by [AI] Theorem 2.35 and [KY] Theorem 7.12. Therefore, the map \( W \rightarrow 2\text{-smc} \Pi \) is an order embedding.

Finally we will show that the map is surjective, which follows from the same argument as [AIR] Corollary 2.38. From the above argument, \( \{ \Psi(D) \mid D \in DAD \} \) consists of the connected component in the Hasse quiver of \( 2\text{-smc} \Pi \), which is isomorphic to \( W \). It is enough to show that any element \( X \in 2\text{-smc} \Pi \) belongs to this component. Then, since \( X \in 2\text{-smc} \Pi \), we have \( \Psi(D(\text{id})) \geq X \). Then [AI] Proposition 2.35 and [KY] Theorem 7.12 implies that there exists a sequence \( \Psi(D(\text{id})) \succ X_1 \succ X_2 \succ \cdots \) of left mutation of \( 2\text{-smc} \Pi \) such that \( X_i \geq X \) for any \( i \). Since the component \( \{ \Psi(D) \mid D \in DAD \} \) is finite and \( \Psi(D(\text{id})) \in \{ \Psi(D) \mid D \in DAD \} \), this sequence must be finite. Thus we have \( X = X_j \) for some \( j \) and hence \( X \) belongs to \( \{ \Psi(D) \mid D \in DAD \} \).

(2) We have the following commutative diagram and all maps are bijections by Proposition 2.9 Theorems 2.10 and 3.13 

![Diagram](https://via.placeholder.com/150)

Moreover, since the maps \( G, \Psi, H^0 \) are poset isomorphisms, we obtain the conclusion.

\[ \square \]

The next corollary allows us to compare the orders of arc diagrams in a simple way.

**Corollary 3.26.** \( D(w) \leq D(v) \) (or equivalently, \( G(w) \leq G(v) \)) if and only if there exists no graph map from any \( M \in \Phi(G(w)) \) to any \( N \in \Phi(R(v)) \).

Proof. By Theorem 3.25, \( D(w) \leq D(v) \) if and only if \( \Psi(D(w)) \leq \Psi(D(v)) \).

Since any object of \( \Psi(D(w)) \) and \( \Psi(D(v)) \) is 2-term, we have \( \Psi(D(w)) \leq \Psi(D(v)) \) if and only if
\[ \text{Hom}_{D(\text{mod II})}(X, Y[-1]) = 0 \]
for any \( X \in \Psi(D(w)) \) and \( Y \in \Psi(D(v)) \). Because
\[ \Psi(D(w)) = \Phi(G(w)) \sqcup \Phi(R(w))[1], \]

it is also equivalent to saying that $\text{Hom}_\Pi(M, N) = 0$ for any $M \in \Phi(G(w))$ and $N \in \Phi(R(v))$. Then the conclusion follows from Theorem 3.4. \qed

At the end of this section, we will discuss a consequence of the above results.

Let $A$ be a finite dimensional algebra. Even though we know that there exists a bijection between $f_L\text{-sbrick} A$ and $2\text{-smc} A$, it is not known that a calculatable map from $f_L\text{-sbrick} A$ to $2\text{-smc} A$ in general.

Theoretically, for a given $S \in f_L\text{-sbrick} A$, we obtain the corresponding SMC $X_S$ by the following steps (see [As1, Figure 1] for more details).

(i) From $S$, we calculate the smallest torsion class $T(S)$ containing $S$.

(ii) From the torsion class $T(S)$, we calculate a Ext-projective of $T(S)$, which we denote by $T$. It turns out that $T$ is a support $\tau$-tilting module and it is identified with a support $\tau$-tilting pair $(T, P)$ for some projective module $P$ [AIR, Theorem 2.7].

(iii) From $(T, P)$, we calculate $D(T, P) := (\tau T \oplus \nu P, \nu T_{pr})$, where $\nu$ is the Nakayama functor and $T_{pr}$ is a maximal projective direct summand of $T$. It turns out to be a support $\tau$-tilting pair [AIR, Theorem 2.14].

(iv) From $U := \tau T \oplus \nu P$, we calculate the set $S' = \{X_1, \ldots, X_i\}$ of isoclasses of indecomposable direct summands of $\text{soc}\text{End}_A(U)(U)$.

Then, we conclude $S \sqcup S'[1]$ is a 2-term SMC, and this correspondence gives a bijection between $f_L\text{-sbrick} A$ and $2\text{-smc} A$ [As1].

However, it is very hard to calculate all modules of $T(S)$ and the Ext-projective of $T(S)$. From this viewpoint, we pose the following question.

**Question 3.27.** For a given left finite semibricks, how can we calculate an explicit description of the corresponding SMC ?

In our situation, the map $f_L\text{-sbrick} \Pi = \text{sbrick} \Pi \rightarrow 2\text{-smc} \Pi$ is given by simple combinatorics of arc diagrams, that is, all we have to do is to calculate $D(w)$ from $G(w)$, which is done in [Re3] as pointed out in Remark 2.10.

4. Quotient algebras of the preprojective algebra

In this section, we discuss semibricks for several quotient algebras of preprojective algebras. As an application, we study semibricks of some important classes of algebras and recover some known results.

As before, fix an integer $n \geq 1$ and let $\Pi = \Pi_n$ be the preprojective algebra of type $A_n$ and $W = W_n$, the symmetric group of degree $n + 1$.

Let $I$ be a two-sided ideal of $\Pi$. Then we define

$$\text{NAD}_I := \{G \in \text{NAD} \mid S(\alpha) \in \text{mod}(\Pi/I), \forall \alpha \in G\}.$$ 

Then we naturally regard $\text{NAD}_I$ as an induced subposet of $\text{NAD}$, that is, for $G, G' \in \text{NAD}_I$, we have $G \leq G'$ on $\text{NAD}_I$ if and only if $G \leq G'$ on $\text{NAD}$.

As before, for $G \in \text{NAD}_I$, we can define

$$\Phi(G) = \{S(\alpha) \mid \alpha \in G\} \subset \text{mod}(\Pi/I).$$

Moreover, for $C \subset \text{mod}(\Pi/I)$, we denote $T_{\Pi}(C)$ (resp. $T_{\Pi/I}(C)$) by the smallest torsion class containing $C$ in $\text{mod} \Pi$ (resp. in $\text{mod}(\Pi/I)$). Note that $T_{\Pi}(C) = \text{Filt}_H(\text{Fac} C)$ (see [MS, Lemma 3.1]). We let

$$\text{sbrick}(\Pi) \cap \text{mod}(\Pi/I) := \{S \in \text{sbrick} \Pi \mid S \in \text{mod}(\Pi/I), \forall S \in S\}.$$
Lemma 4.1. We have a poset isomorphism

\[ \text{brick}(\Pi) \cap \text{mod}(\Pi/I) \to \text{brick}(\Pi/I). \]

Proof. Because \( \text{mod}(\Pi/I) \) is a full subcategory of \( \text{mod} \Pi \), we can identify \( \text{brick}(\Pi/I) \) with \( \text{brick}(\Pi) \cap \text{mod}(\Pi/I) \).

Take \( S, S' \in \text{brick}(\Pi) \cap \text{mod}(\Pi/I) \) and assume that \( S \leq S' \), that is, \( T_\Pi(S) \subset T_\Pi(S') \).

Since \( T_\Pi(C) = \text{Filt}_\Pi(\text{Fac}C) \), we have \( T_\Pi(S) \cap \text{mod}(\Pi/I) = T_{\Pi/I}(S) \). Thus, we have \( T_{\Pi/I}(S) \subset T_{\Pi/I}(S') \).

Conversely, take \( S, S' \in \text{brick}(\Pi/I) \) and assume that \( S \leq S' \), that is, \( T_{\Pi/I}(S) \subset T_{\Pi/I}(S') \). Then since \( T_\Pi(T_{\Pi/I}(S)) = T_\Pi(S) \), we have \( T_\Pi(S) \subset T_\Pi(S') \). Thus the map is an isomorphism. □

Theorem 4.2. Let \( I \) be a two-sided ideal of \( \Pi \). Then we have a poset isomorphism

\[ \Phi : \text{NAD} \to \text{brick}(\Pi/I). \]

Proof. By Theorem 3.25, we have a poset isomorphism

\[ \text{NAD} \to \text{brick}(\Pi) \cap \text{mod}(\Pi/I). \]

Moreover, Lemma 4.1 gives a poset isomorphism \( \text{brick}(\Pi) \cap \text{mod}(\Pi/I) \to \text{brick}(\Pi/I) \). □

As a first application of Theorem 4.2, we consider the case that semibricks of the algebras do not change.

Corollary 4.3. Let \( I_{\text{cyc}} \) be the ideal of \( \Pi \) generated by all 2-cycles. Then we have a poset isomorphism

\[ \Phi : \text{NAD} \to \text{brick}(\Pi/I_{\text{cyc}}). \]

Proof. Since we have \( \text{NAD} = \text{NAD}_{I_{\text{cyc}}} \), we get \( \text{NAD} \cong \text{NAD}_{I_{\text{cyc}}} \cong \text{brick}(\Pi/I_{\text{cyc}}) \) by Theorem 4.2. □

Remark 4.4. Corollary 4.3 was also shown in [BCZ, Lemma 4.2.8] and [DIRRT, Proposition 6.7].

As a second application of Theorem 4.2, we study semibricks of a path algebra in terms of a special class of arc diagrams.

Definition 4.5. [Re3] A right arc is an arc that does not pass to the left (=above) of any point, and we call a noncrossing arc diagram having only right arcs a right noncrossing arc diagram. We denote the set of right noncrossing arc diagrams by \( \text{RNAD} = \text{RNAD}_n \) and regard it as a subposet of \( \text{NAD} \).

Remark 4.6. (i) The right noncrossing arc diagrams are nothing but noncrossing partitions, and its enumeration is well-known: The number of right noncrossing arc diagrams on \( n \) points is the Catalan number \( \frac{1}{n+1} \binom{2n}{n} \).

(ii) The name of right arc follows from [Re3], which makes sense if we arrange an arc diagram vertically.

As a corollary of Theorem 4.2, we have the following result.

Corollary 4.7. Let \( \vec{Q} \) be a linear quiver of type \( A_n \). Then we have a poset isomorphism

\[ \Phi : \text{RNAD} \to \text{brick} K \vec{Q}. \]
Proof. For \( I := \langle a_1^-, a_2^-, \ldots, a_{n-1}^- \rangle \), we have \( \Pi/I \cong K \tilde{Q} \). On the other hand, it is easy to check
\[ \text{NAD}_I = \text{RNAD}. \]
Thus Theorem 4.2 implies the assertion. \( \Box \)

Remark 4.8. Corollary 4.7 can be easily generalized to an arbitrary quiver of type \( A \) by the notion of \( c \)-sortable arcs (\( c \)-sortable permutations) (see [BaR, Re1, Re2] for definitions and further background). More precisely, let \( Q \) be a quiver of type \( A \) and \( c \)-\( \text{NAD} \) the set of noncrossing arc diagrams consisting of \( c \)-sortable arcs corresponding \( Q \). Then we have a poset isomorphism
\[ \Phi : c\text{-NAD} \rightarrow s\text{brick } KQ. \]
This result together with Theorem 2.14 recovers one of the main results by Ingalls-Thomas [IT], which relate \( c \)-sortable elements and torsion classes.

To give a last application, we introduce the notion of alternating arc diagrams.

Definition 4.9. [BaR] A right-even alternating arc is an arc that passes to the right (=below) of even points and to the left (=above) of odd points. A left-even alternating arc is an arc that passes to the left of even points and to the right of odd points.

An alternating arc is an arc that is either right-even alternating or left-even alternating or both. We call a noncrossing arc diagram consisting of alternating arcs an alternating noncrossing arc diagram. We denote the set of alternating noncrossing arc diagrams by \( \text{ANAD} = \text{ANAD}_n \).

Let \( \text{rad}(\Pi) \) be a radical of \( \Pi \) and consider the quotient algebra \( \Pi/\text{rad}^2(\Pi) \), where \( \text{rad}^2(\Pi) \) is the radical square. It is equivalent to saying that \( \text{rad}^2(\Pi) \) is the ideal of \( \Pi \) generated by all paths of length two.

Then we have the following result.

Corollary 4.10. We have a poset isomorphism
\[ \text{ANAD} \rightarrow s\text{brick}(\Pi/\text{rad}^2(\Pi)). \]

Proof. It is easy to check
\[ \text{NAD}_{\text{rad}^2(\Pi)} = \text{ANAD}. \]
Thus Theorem 4.2 implies the assertion. \( \Box \)

Remark 4.11. Let \( B \) be a Brauer line algebra with the multiplicity 1 (see, for example, [RZ, section 4] and [AMN, section 2] for the definition). Then we have a poset isomorphism
\[ s\text{brick } B \cong s\text{brick}(\Pi/\text{rad}^2(\Pi)) \]
(for example, this follows from [AsI, section 1.4] or [Ad, Proposition 4.3]). Thus, Corollary 4.10 gives a classification of semibricks of a Brauer line algebra in terms of alternating arc diagrams. Moreover the result of [BaR] gives a bijection a between ANAD and the quotient lattice \( W/\theta_{bcC} \), where \( \theta_{bcC} \) denotes by the biCambrian congruence. Since there exists a bijection between the set of torsion classes, the set of 2-term silting complexes and the set of semibricks [AIR, AsI], the above result also implies results of [DIRRT, Theorem 7.10], [Ag] and [AMN, Theorem 5.2].

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