Relations between M-brane and D-brane quantum geometries

Josie Huddleston*  
28 June 2010

Abstract

This paper investigates M-brane quantum geometry (and its representation by equations involving the 3-bracket) by looking at the compactification of an M-brane system to a D-brane system. Particularly of interest is the system where coincident M2-branes end at an angle on an M5-brane, and its reduction to D1- or F1-branes ending on a D3-brane. Since equations for the quantum geometries of both of these systems are known, the paper will attempt to directly relate them via a compactification.

*Funded by an EPSRC studentship. With thanks to my supervisor, Douglas Smith, and to everyone I’ve met within the Durham bubble. Particular thanks to Sam and Eimear, who have both (in their turn) put up with me during the worst moments, on my way through this stuff!
1 Introduction

The M-branes of M-theory are not understood nearly so well as their D-brane counterparts. Undoubtedly this has something to do with the age of M-theory compared to that of string theory, but it is also true to say that M-theory, as a higher dimensional theory and a generalisation, has presented new challenges in areas that are now straightforward for string theorists. One of these problems involves the quantisation of N coincident D-branes, which in string theory has a low energy limit based on a non-Abelian Yang-Mills Lie algebra. In M-theory, the analogues are coincident M2-branes, which up until recently had no simple description for their quantisation. Work by Bagger and Lambert [2][3][4], and later Gustavsson [5] shed light on this, using a 3-algebra in the same sort of way as the multiple-F1 system used the Lie algebra, and hence constructing an action (the Bagger-Lambert-Gustavsson action) for coincident M2-branes.

However, the 3-algebra used by Bagger, Lambert and Gustavsson is itself not well understood. When looking at two coincident M2s, the only compatible 3-algebra found at the time had the simple 3-bracket \[ [T^i, T^j, T^k] = \varepsilon^{ijk} T^l \] (although new examples are now coming to light [6]). The difficulty in finding new solutions (as well as the requirement for an invariant metric) is making the 3-bracket satisfy the Fundamental Identity, the generalisation of the Jacobi Identity for a 3-algebra.

In this paper, we would like to shed some more light on the 3-bracket’s role in M-brane geometry. This task is aided in part by the work of Chu and Smith, whose paper [10] uses Basu-Harvey fuzzy funnels in order to probe the \( C \)-field on the M5-brane with multiple M2-branes to deduce properties of the M5 theory, similarly to how multiple D1-branes can probe a \( B \)-field on the D3-brane. In the latter case, this NS \( B \)-field translates via a Seiberg-Witten map into the known non-commutative geometry on the D-brane represented by \[ [X^i, X^j] = \Theta_{ij} \] (for some \( B \)-field-related constants \( \Theta_{ij} \)). For the 3-algebra, in contrast, the \( C \)-field leads to an equation \[ [X^i, X^j, X^k] = \Theta^{ijk} \], which represents some novel quantum geometry for the M5-brane.

To get more information about the 3-bracket (both its form and its meaning within the theory), it would be useful to know how it relates to the commutator - i.e. to reduce the M5 action to, say, a D3 action and hence (by comparison with known D3 actions) relate the constants \( \Theta^{ijk} \) from the M-brane geometry to the constants \( \Theta_{ij} \). This would begin to give us some insight into how the quantum geometry of the M5-brane works.

The results of two papers are useful here. The previously mentioned paper [10], of course, provides the link between the 3-bracket equation and the \( C \)-field. The second paper, by Pasti, Sorokin and Tonin [8] provides an action for an M5-brane with a \( C \)-field, and a reduction as far as a D4-brane for a simplified case.

The full reduction using a \( C \)-field with all components switched on will be beyond the scope of this paper, but instead we will focus on the two-component case considered in [10] where the coincident M2-branes end on an M5-brane at an angle \( \alpha \). This will involve following the process of [8] but without setting \( A_{\mu5} \)
to zero. A key question is whether the results of this compactification will tally with known results about coincident strings ending on D3-branes at an angle, and in the final part of the paper a comparison with the non-commutative geometry known for D-branes will be attempted to assess this, using results from [13].

2 The compactification

We consider an M2-brane with coordinates 0 and 2 in common with the M5-brane that it connects with:

| M2: | 0 | 2 | 6 |
|-----|---|---|---|
| M5: | 0 | 1 | 2 | 3 | 4 | 5 |

2.1 Dualising the M5 action

We begin with the known M5 action, from [7].

\[ S_{\text{M5}} = \int d^6x \left( \sqrt{-\det (g_{mn} + iH_{mn})} + \frac{\sqrt{-g}}{4} \partial_m a(\ast H)^{mnl} H_{nlp} \partial^p a \right) + \int \left( C^{(6)} + \frac{1}{2} F^{(3)} \wedge C^{(3)} \right) \quad (1) \]

where \( H = F - C \) (\( F = dA \) the worldvolume field strength), \( \ast H \) is the Hodge star of \( H \), and \( H_{mn} = \frac{1}{\sqrt{(d\alpha)^2}}(\ast H)_{mn} \partial^\alpha a \). The field \( a \) is an auxiliary field introduced to ensure \( d=6 \) covariance of the action [7], since the self-dual field-strength \( H_{(3)} \) on its own would prevent us from writing a kinetic term (since \( H \wedge \ast H = 0 \)). In order for \( S_{\text{M5}} \) to reduce to a standard D4 action, it is also necessary to dualise it. [8] contains a reduction/dualisation of the M5 action, but they make one assumption that we don’t want to make here, namely that \( A_{\mu r} = 0 \) (where \( r \) is the direction of reduction). We retain the assumption of a gauge choice such that the auxiliary field \( a \) satisfies \( \partial_\mu a = \delta_\mu^r \), and the assumption of a metric whose determinant is unchanged after reduction, specifically in directions 2 and 5.

The removal of this assumption causes terms involving \( F^{(3)}_{mn5} = (dA)_{mn5} \) to become non-zero. This means that any direction of reduction will result in non-zero fields \( F^{(2)} \) as well as the \( F^{(3)} \) fields found in [8]. This in turn makes dualisation of the fields rather more complicated - before, the \( F \)-fields could be dualised by replacement with their Hodge star (give or take a constant factor) but this is only true for \( A_{\mu r} = 0 \) and compactification in the \( r \) direction.

To do the dualisation from scratch, we begin by expanding some of the \( Hs \), and splitting \( F \) and \( C \) into their 2- and 3-form parts:
The next step is to add two different Lagrange multiplier terms to the action in order to impose the $F = dA$ constraint in both the 2- and 3-form cases [9]:

$$\hat{S} \left[ F^{(2)}, F^{(3)} \right] = S \left[ F^{(2)}, F^{(3)} \right] + \int \epsilon^{\alpha\beta\gamma\delta\epsilon} \left( \frac{1}{2} i \left( \star \hat{F} \right)^{(3)}_{\alpha\gamma} \left( F^{(2)}_{\delta\epsilon} - 2\partial_\delta A_\epsilon \right) + \frac{1}{6} i \left( \star \hat{F} \right)^{(2)}_{\alpha\beta} \left( F^{(3)}_{\gamma\delta\epsilon} - 2\partial_\gamma A_\delta \epsilon \right) \right)$$

The two Lagrange multipliers are $\frac{1}{2} i \left( \star \hat{F} \right)^{(3)}_{\alpha\gamma}$ and $\frac{1}{6} i \left( \star \hat{F} \right)^{(2)}_{\alpha\beta}$, with their constants chosen for convenience. Note that despite the factors of $i$, these terms will turn out to be real.

From here, taking equations of motion in each of $F^{(3)}$ and $F^{(2)}$ in turn, results in two independent equations that determine the new $F$s in terms of the old:

$$\hat{F}^{(2)} = \frac{i}{2} \hat{C}^{(2)} - \frac{i}{4} \hat{F}^{(2)}$$

$$\hat{F}^{(3)} = \frac{i}{6} \hat{C}^{(3)} - \frac{i}{12} \hat{F}^{(3)}$$

where tildes indicate Hodge stars. Finally, in order to keep the Born-Infeld part of the new action in the same form as the old one, we define the new $\hat{C}$-fields by saying that $\hat{H}^{(2)} = k \left( i \star H^{(3)} \right)$ (where $\hat{H}^{(2)} = \hat{F}^{(2)} - \hat{C}^{(2)}$ analogous to $H = F - C$). This results in:

$$\hat{C}^{(2)} = \frac{i}{12} \star C^{(3)}$$

$$\hat{C}^{(3)} = i \star C^{(2)}$$

Substituting into (1), the result is the dualised-reduced-M5 action:

$$\hat{S} = \int d^5 x \sqrt{\det \left( g_{\alpha\beta} + 12i \hat{H}_{\alpha\beta} \right)} + \epsilon^{\alpha\beta\gamma\delta\epsilon} \left( \hat{C}^{(5)}_{\alpha\beta\gamma\delta\epsilon} + 2 \hat{F}^{(3)} \hat{F}^{(2)}_{\alpha\beta\gamma\delta\epsilon} \right)$$

where we have absorbed the $\hat{C}^{(2)} \hat{C}^{(3)}$ term along with $C^{(5)}$ into the new $\hat{C}^{(5)}$.

Note that this is a D4 action, but we haven’t chosen which numbers the indices run over yet. We do so in the next section.

### 2.2 The D2-on-D4 and F1-on-D4 actions

In order to simplify things a bit for the purposes of more reduction, we’ll start with the above (dualised-reduced-)M5 action with a specific $C$-field which has two non-zero parts: one including the direction of the M2-brane and one not.
Specifically, we’ll look at the choice found in [10] which represents an M2-brane at an angle of \( \alpha \) to the M5-brane, i.e. \( \tilde{C}_{012}^{(3)} = \frac{1}{4} \sin \alpha, \tilde{C}_{345}^{(3)} = -\frac{1}{4} \tan \alpha \) (with all other \( \tilde{C}^{(3)} \) and also \( \tilde{C}^{(5)} \) zero). We then have two choices of how to compactify down to a D4-brane in IIA - by compactifying a direction perpendicular to the M2-brane (say 5), or a dimension parallel to the M2-brane (say 2). These produce a D2 on a D4, and an F1 on a D4 respectively, with actions as shown here.

\[
S_{(D4,D2)} = \int d^5x \sqrt{\det \left( g_{\alpha\beta} + \tilde{F}_{\alpha\beta}^{(2)} \right) - \sin \alpha + \frac{i}{24} \epsilon^{\alpha\beta\gamma\delta\epsilon} \tilde{F}_{\alpha\beta\gamma}^{(3)} \tilde{F}_{\delta}\epsilon} \\
- \frac{i}{12} \tilde{F}_{34}^{(2)} \tan \alpha - \frac{i}{12} \tilde{F}_{012}^{(3)} \sin \alpha + \frac{i}{6} \sin \alpha \tan \alpha
\] (3)

\[
S_{(D4,F1)} = \int d^5x \sqrt{\det \left( g_{\alpha\beta} + \tilde{F}_{\alpha\beta}^{(2)} \right) - \tan \alpha + \frac{i}{24} \epsilon^{\alpha\beta\gamma\delta\epsilon} \tilde{F}_{\alpha\beta\gamma}^{(3)} \tilde{F}_{\delta}\epsilon} \\
- \frac{i}{12} \tilde{F}_{01}^{(2)} \sin \alpha - \frac{i}{12} \tilde{F}_{345}^{(3)} \tan \alpha + \frac{i}{6} \sin \alpha \tan \alpha
\] (4)

Note also that the original two C-field components come through this unchanged; to get a result for a more general two-component C-field is simply a matter of replacing \( \tan \alpha \) and \( \sin \alpha \) above respectively with \( \tilde{C}_{012}^{(3)} \) and \( \tilde{C}_{345}^{(3)} \) (D4-D2 case) or \( \tilde{C}_{01}^{(2)} \) and \( \tilde{C}_{34}^{(2)} \) (D4-F1 case).

### 2.3 The D1-on-D3 and F1-on-D3 actions

To turn the D4 actions into their related D3 actions we must T-dualise, in directions 2 and 5 respectively [11]. Since all the WZ terms have a component of these directions (recall that \( \sin \alpha \) and \( \tan \alpha \) are C-fields in the directions perpendicular to the \( F_s \)) this simply has the effect of removing all \( 2's \) and \( 5's \) and reducing the dimensions by 1. (Note that the Born-Infeld part of each action remains the same.)

\[
S_{(D3,D1)} = \int d^4x \sqrt{\det \left( g_{\alpha\beta} + \tilde{F}_{\alpha\beta}^{(2)} \right) - \sin \alpha + \frac{i}{24} \epsilon^{\alpha\beta\gamma\delta} \left( \left( \tilde{F}_{\alpha\beta\gamma}^{(3)} \tilde{F}_{\delta}\epsilon^{(1)} + \tilde{F}_{\alpha\beta}^{(2)} \tilde{F}_{\delta}\epsilon \right) \right) \\
- \frac{i}{12} \tilde{F}_{34}^{(2)} \tan \alpha - \frac{i}{12} \tilde{F}_{01}^{(2)} \sin \alpha + \frac{i}{6} \sin \alpha \tan \alpha
\] (5)

\[
S_{(D3,F1)} = \int d^4x \sqrt{\det \left( g_{\alpha\beta} + \tilde{F}_{\alpha\beta}^{(2)} \right) - \tan \alpha + \frac{i}{24} \epsilon^{\alpha\beta\gamma\delta} \left( \left( \tilde{F}_{\alpha\beta\gamma}^{(3)} \tilde{F}_{\delta}\epsilon^{(1)} + \tilde{F}_{\alpha\beta}^{(2)} \tilde{F}_{\delta}\epsilon \right) \right) \\
- \frac{i}{12} \tilde{F}_{01}^{(2)} \sin \alpha - \frac{i}{12} \tilde{F}_{34}^{(2)} \tan \alpha + \frac{i}{6} \sin \alpha \tan \alpha
\] (6)
Again, the general two-component $C$-field result follows by directly substituting for $\sin$ and $\tan$ alpha, crossing out the 2s and 5s from the general $C$-fields given at the end of the last section. From this, and the standard D3 action found in \[1\], we can see that in each case one of the two parts of the $C$-field has become a RR $C^{(2)}$-field and one part has become a NS $B^{(2)}$-field. Specifically, we get $C_{01}^{(2)} = \tan \alpha$, $B_{34}^{(2)} = C_{34}^{(2)} = \sin \alpha$ for the D3-D1 case, and $C_{34}^{(2)} = \sin \alpha$, $B_{01} = C_{01}^{(2)} = \tan \alpha$ for the D3-F1 case.

2.4 A word about geometry

At this point, it is worth taking a moment to consider how all this relates back to the geometry of the D- and M-brane systems. We are looking particularly at set-ups where a lower dimensional brane ends on a higher dimensional brane at an angle, and have established that it’s possible to reduce and dualise in such a way as to turn the M5-M2 system into either a D3-D1 system or a D3-F1 system. In terms of the D-brane geometry, we know that adding a constant NS $B$-field to the D-brane worldvolume causes the worldvolume coordinates $X^i$ to obey a commutation relation

$$\Theta^{ij} = i \left[X^i, X^j\right]$$

(7)

with components of $\Theta$ dependent on the $B$-field. For small $B$, this can be written as

$$(2\pi \alpha')^2 F^{ij} = i \left[X^i, X^j\right]$$

(8)

where $F = F + B$ and $F$ is the worldvolume field strength on the brane \[10\]. Note that since the geometry depends only on total field $F$, it is indifferent to which parts of the field come from the field strength $F$ and which from the $B$-field.

In the M-brane case we expect an analogous situation with the quantum geometry represented by the 3-bracket - that is, upon adding a $C^{(3)}$-field it takes the form

$$\Theta^{ijk} \sim i \left[X^i, X^j, X^k\right]$$

(9)

and if $C$ is sufficiently small we can approximate this by

$$F^{ijk} \sim i \left[X^i, X^j, X^k\right]$$

(10)

with $F = F + C^{(3)}$ \[10\]. In the reduction, as we’ve seen, this $C^{(3)}$-field splits into two pieces, with one part becoming a RR $C^{(2)}$-field and one part becoming a NS $B^{(2)}$-field. Now in equation (8), we have $F = F_0 + B + C^{(2)}$ (with $F_0$ the original worldvolume field strength). Hence it is clear that in theory the same D-brane geometry could be produced by various sets of fields, so long as they come together to the same $F$.

From a purely geometric standpoint, we can see that reducing and dualising an angled-M2-on-M5 system should produce an angled-D1/F1-on-D3 system, with the same angle $\alpha$ involved throughout. Thus, we might expect that the
geometry as specified by (9) should reduce to the geometry as specified by (7). The question is, can we explicitly check this for the particular case we’re looking at? From [10], we know that
\[
[X, X, X] = \begin{cases} 
\frac{i\varepsilon_{\mu\nu\lambda} \sin \alpha}{K \cos^2 \alpha} & \mu, \nu, \lambda \in \{0, 1, 2\} \\
\frac{i\varepsilon_{\mu\nu\lambda} \tan \alpha}{K \sec^2 \alpha} & \mu, \nu, \lambda \in \{3, 4, 5\} \\
0 & \text{otherwise}
\end{cases}
\tag{11}
\]
(where \(K\) is a dimensional proportionality constant) is the relevant equation for the angled-M2-on-M5 (with \(C\)-fields as specified in 2.2), while
\[
[X, X] = \begin{cases} 
\frac{iz^{\mu\nu}(2\pi\alpha') \tan \alpha}{1+\tan^2 \alpha} & \mu, \nu \in \{3, 4\} \\
0 & \text{otherwise}
\end{cases}
\tag{12}
\]
is the equation for an angled-D1-on-D3, arrived at by including a \(B\)-field \(B_{34} = \tan \alpha\) \[11\]. Thus we make the “Geometry Reduction conjecture” that the system with geometry specified by (11) should reduce to one with geometry specified by (12).

3 Checking the Geometry Reduction conjecture

So far, we have found the explicit reduction of \(C^{(3)}\) yields a \(B\) and \(C^{(2)}\) combination. For example, in the D3-D1 case \[9\], we get \(C_{01}^{(2)} \sim \tan \alpha\) plus \(B_{34} = C_{34}^{(2)} \sim \sin \alpha\). This clearly is not the same as the standard example of an angled D3-D1 system found in the literature, which has only a \(B\)-field \(B_{34}^{(2)} \sim \tan \alpha\). However, as we showed in \[22\], it is entirely possible that two such systems could indeed have the same non-commutative geometry, and indeed there is good reason to suggest that they should. To check this, though, it is necessary to construct the total field strength from the various fields obtained in the reduction.

In \[13\], Cornalba et al. give a method for constructing a field strength \(F\) from a background \(U(1)\) gauge field strength \(F_0\) with infinitesimal \(\delta B\)- and \(\delta C\)-fields added to it using
\[
\delta F = \frac{1}{2} \left(1 + F_0 \frac{1}{g}\right) \Omega \left(1 - \frac{1}{g} F_0\right)
\tag{13}
\]
where
\[
\Omega = \frac{1}{1 + F_0 \frac{1}{g}} \left(\delta B - F_0 \frac{1}{g} \delta B - F_0\right) \frac{1}{1 - \frac{1}{g} F_0}
\]
for the \(B\)-field, and
\[
\Omega_{mn} \gamma^m n = e^{-\omega \gamma} \delta C \gamma^4 \ldots \gamma^9 - \delta C \gamma^4 \ldots \gamma^9 e^{\omega \gamma}|_{2\text{-form}}
\]

\[7\]
for the $C$-field, with $\omega \cdot \gamma = \omega_{mn} \gamma^m \gamma^n$ and $\omega$ a specific 2-form that depends on the initial $F_0$.

It would be useful to know whether this method of deriving $\mathcal{F}$ is still applicable outside the infinitesimal domain, i.e. for general $B$ and $C$. Certainly there is an obvious way to proceed, by calculating $\delta \mathcal{F}$ in terms of $\delta B$ or $\delta C$ and then solving the resulting differential equation(s) to find the new $\mathcal{F}$ in terms of fields $F_0, B$ and $C$. In a footnote of [13] this is done for a finite $B$-field with $C$ and $F_0$ set to zero, and the answer is $\mathcal{F} = g \tanh \left( \frac{1}{2} B \right)$.

\[ e^{\pm \omega \cdot \gamma} = \frac{1}{\sqrt{1 + F^2}} \pm \frac{F}{\sqrt{1 + F^2}} \gamma^0 \gamma^1 \]

\[ \delta F = \frac{1}{2} \delta C \sqrt{F^2 + 1} \quad \implies \quad F_1 = F_0 + \frac{1}{2} \delta C \sqrt{F_0^2 + 1} + O ( (\delta C)^2 ) \quad (14) \]

Figure 1: Finite $B$-field, infinitesimal $\delta C$-field

In order for the method to make sense for finite order though, it is necessary for it to be path independent. That is to say, it shouldn’t matter in what order the finite $B$ and $C$ are added (or even if they are added piece by piece) - the resulting $\mathcal{F}$ should be the same. A first step towards this is to look at putting together a finite $B$-field with an infinitesimal $\delta C$-field, such as shown in Figure 1. There are two paths on this diagram - one that takes the simple route from $F_0$ to $F_1$ by the addition of an infinitesimal $\delta C$, and one that takes the long way round to $F'_1$ by adding a finite $B$, then the $\delta C$, then $-B$. Note that the process of putting a new field into $\mathcal{F}$ is not really as simple as addition, so it is a nontrivial problem to ask whether $F_1$ and $F'_1$ are the same, or to what order in $\delta C$ equality holds. (If equality holds for sufficiently high order, it should be possible to “sum up” (i.e. integrate) a block of such paths to prove path-independence for finite $B$ and $C$.)

Checking this for general $B$ and $\delta C$ turns out to be rather difficult. When all components of the two fields are switched on the result is a matrix of interlinked differential equations with no easy solutions. However, with the results of Section 2 in mind it would be instructive to look at the case where $\delta C = \delta C_{01}$, $B = B_{34}$. With only one component in each field, equation (13) can be solved repeatedly to find the new field. First to find $F_1$:

\[ \delta F = \frac{1}{2} \delta C \sqrt{F^2 + 1} \quad \implies \quad F_1 = F_0 + \frac{1}{2} \delta C \sqrt{F_0^2 + 1} + O ( (\delta C)^2 ) \quad (14) \]
Then $\tilde{F}$:
\[
\delta F = \frac{1}{2} \delta B \left( 1 + F^2 \right) \implies \frac{1}{2} B = \int_{\tilde{F}_0}^{\tilde{F}} \frac{dF}{F^2 + 1} \implies \tilde{F} = \frac{\tan \frac{1}{2} B + F_0}{1 - F_0 \tan \frac{1}{2} B}
\]

Then $\hat{F}$:
\[
\hat{F} = \tilde{F} + \frac{1}{2} \delta C \sqrt{\tilde{F}^2 + 1} + O \left( (\delta C)^2 \right)
\]

And finally $F'_1$:
\[
-\frac{1}{2} B = \int_{\tilde{F}}^{F'_1} \frac{dF}{F^2 + 1} \implies F'_1 = -\tan \frac{1}{2} B + \hat{F} \\
= \frac{-\tan \frac{1}{2} B + \hat{F}}{1 + \hat{F} \tan \frac{1}{2} B}
\]

After substituting and simplifying:
\[
F'_1 = F_0 + \frac{1}{2} \delta C \sqrt{1 + F_0^2} \left( \cos \frac{1}{2} B - F_0 \sin \frac{1}{2} B \right) + O \left( (\delta C)^2 \right)
\]

This agrees to constant term, but no further, with the $F_1$ result in (13). It also agrees to first order and beyond in $\delta C$ if $B$ and $F_0$ are infinitesimal, in line with the results of [13] which says that $F_1$ and $F'_1$ should be equal in this case.

This unfortunately suggests that the $F$-construction method will not work for general $B$ and $C$. Hence it cannot reliably be used to check the Geometry Reduction conjecture made at the end of Section 2.

4 Conclusions

We have seen that it is possible to generalise the results of [8] to the case where $A_{ur} \neq 0$. The continued assumption of gauge choice such that $\partial_\mu a = \delta_\mu^r$ seems reasonable, while it might be interesting (if rather trickier) in future work to look at breaking the final assumption: i.e. to look at some specific cases where the determinant of the metric contributes extra terms to the action when reduced in directions 2 and/or 5.

In this paper, the generalisation has led to some new D4 and D3 actions representing various possible compactifications of an M5 with background C-field. In particular, by comparison with well-known actions for the D3-brane, we have seen that under these conditions a $C^{(3)}$-field with two orthogonal components reduces to two 2-forms, $B$ and $C^{(2)}$, with directly comparable components to those of the original field. These have suggested that the non-commutative geometry of the D3-brane could be arrived at using a combination of $B$- and $C$-fields, as well as purely from a $B$-field as we are accustomed to. If proved fully, this illumination could work both ways: the method could be useful in Bagger-Lambert theory, if the poorly-understood 3-bracket used to describe the quantum geometry of the M5-brane could be in some way directly compared to the simpler commutator used for D3-branes. Alternatively, finding that this particular $B$
and $C$ combination does in fact result in a different D-brane non-commutative geometry (i.e. that the conjecture is false) would also be an interesting result, though perhaps it is less likely given what we know from the geometrical picture.

In the last section of the paper, we saw that the method for constructing a single field $\mathcal{F}$ out of background fields (found in [13]) was path-dependent, and hence badly-defined, for non-infinitesimal combinations of $B$ and $C$ fields. Any future work modifying this method for finite fields should be able to use the results of this paper to check the Geometry Reduction conjecture, i.e. to see for sure whether the reduction of an M-brane quantum geometry is (at least in this specific case) equivalent to the known D-brane non-commutative geometry. If this is indeed the case, the next step would be to try and get some idea of what happens to the 3-bracket during the reduction process, as this might lead to a method to convert any known $\Theta^{jk}(\alpha)$ (i.e. any specific instance of (9)), into the $\Theta^j(\alpha)$ that specifies the non-commutative geometry of the reduced system.

References

[1] C. V. Johnson, “D-branes” (2003)

[2] J. Bagger and N. Lambert, “Modeling multiple M2’s” Phys. Rev. D75 045020 (2007) arXiv:hep-th/0611108

[3] J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes” Phys. Rev. D77 065008 (2008) [arXiv:0711.0955 [hep-th]]

[4] J. Bagger and N. Lambert, “Comments On Multiple M2-branes” JHEP 0802, 105 (2008) [arXiv:0712.3738 [hep-th]]

[5] A. Gustavsson, “Algebraic structures on parallel M2-branes” arXiv:0709.1260 [hep-th].

[6] P. M. Ho, R. C. Hou, Y. Matsuo, “Lie 3-Algebra and Multiple M2-branes” arXiv:0804.2110v2 [hep-th]

[7] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, “Covariant Action for the Super–Five–Brane of M–Theory” arXiv:hep-th/9701149v2

[8] P. Pasti, D. Sorokin and M. Tonin “Covariant Action for a D=11 Five–Brane with the Chiral Field” arXiv:hep-th/9701037v3

[9] A. A. Tseytlin, Nucl. Phys. B469 (1996) 51-67

[10] C. S. Chu and D. J. Smith, “Towards the Quantum Geometry of the M5-brane in a Constant C-Field from Multiple Membranes” arXiv:0901.1847v1 [hep-th]

[11] J. L. Karczmarek and and C. G. Callan, Jr. “Tilting the Noncommutative Bion” arXiv:hep-th/0111133v2
[12] G.W. Gibbons and D.A. Rasheed, “SL(2,R) Invariance of Non–Linear Electrodynamics Coupled to an Axion and a Dilaton” Phys. Lett. B365 46 (1996) [arXiv:hep-th/9509141]

[13] L. Cornalba, M. S. Costa and R. Schiappa, “D–Brane Dynamics in Constant Ramond–Ramond Potentials, S–Duality and Noncommutative Geometry” [arXiv:hep-th/0209164v5]