Learning with Proper Partial Labels

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Abstract

Partial-label learning is a kind of weakly-supervised learning with inexact labels, where for each training example, we are given a set of candidate labels instead of only one true label. Recently, various approaches on partial-label learning have been proposed under different generation models of candidate label sets. However, these methods require relatively strong distributional assumptions on the generation models. When the assumptions do not hold, the performance of the methods is not guaranteed theoretically. In this paper, we propose the notion of \textit{properness} on partial labels. We show that this proper partial-label learning framework includes many previous partial-label learning settings as special cases. We then derive a unified unbiased estimator of the classification risk. We prove that our estimator is risk-consistent by obtaining its estimation error bound. Finally, we validate the effectiveness of our algorithm through experiments.

1 Introduction

Partial-label learning (PL) \cite{Cour2011, Zeng2013} is a kind of weakly-supervised learning \cite{Sugiyama2018, Zhou2018} where for each training example, we only have access to a set of candidate labels. Recently, the framework of PL has been successfully applied to various real-world tasks such as web mining \cite{Luo2010}, face recognition \cite{Zeng2013}, birdsong classification \cite{Liu2012}, and multimedia content analysis \cite{Cour2011}.

Research on PL can date back to the work of Jin and Ghahramani \cite{Jin2002}. In the paper, the authors first introduced the problem setting of PL and then proposed an Expectation-Maximization model within the maximum-likelihood framework. Along this line, Liu and Dietterich \cite{Liu2012} proposed the \textit{Logistic Stick-Breaking Conditional Multinomial Model} (CMM) where they introduced a mapping component and a coding component to better capture the underlying structure of the data under weak supervision. Another line of research focused more on analyzing the theoretical part of PL. Cour et al. \cite{Cour2011} proposed a learning formulation based on a convex loss for partial labels within the \textit{empirical risk minimization} (ERM) framework \cite{Vapnik2013}. Then, they proposed the concept of the \textit{ambiguity degree} to describe the hardness of PL problems and studied the statistical consistency of their algorithms. Later, Liu and Dietterich \cite{Liu2014} further studied the ambiguity degree and obtained conditions for the learnability of ERM minimizers.

Within the framework of ERM, although the statistical consistency and the learnability of the classifier has been well-studied based on the notion of the ambiguity degree \cite{Cour2011}, the formulation of data distribution remains unexplored. To achieve this goal, various partial label generation models have been studied recently \cite{Ishida2019, Feng2020a, Feng2020b}. However,
distributional assumptions made in these papers are too strong. As a result, these assumptions can hold only in a limited number of real-world cases and may not be very practical. Motivated by the above observations, in this paper, we study the PL problem under a weak distributional assumption. Specifically, we have the following contributions:

- We propose the notion of *properness* for PL problems. We show that our Proper Partial-Label Learning (PPL) framework includes much recent work as special cases, such as Learning with Complementary Labels (CL) [Ishida et al., 2019], Learning with Multiple Complementary Labels (MCL) [Feng et al., 2020a], and Provably Consistent Partial-Label Learning (PCPL) [Feng et al., 2020b].

- We derive a unified risk-consistent method for PPL problems, which is both model-independent and optimizer-independent. Theoretically, we establish an estimation error bound for our method. Experimentally, We demonstrate the effectiveness of the proposed method on various benchmark datasets.

## 2 Preliminaries

In this section, we first introduce the formulations of ordinary multi-class classification and PL. Then, we review related work on PL problems.

### 2.1 Ordinary Multi-class Classification

In multi-class classification, we let $K \geq 3$ denote the number of total classes. We let $\mathcal{X} \in \mathbb{R}^d$ denote the instance space and $\mathcal{Y} := \{1, 2, \ldots, K\}$ denote the label space where $d$ is the dimension of the instance space. We let $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{Y}$ be the input and output random variables. We assume that the input-output pair $(\mathbf{x}, \mathbf{y})$ is sampled independently from an unknown probability distribution with joint density denoted as $p(\mathbf{x}, \mathbf{y})$. Let $f : \mathcal{X} \rightarrow \mathbb{R}^K$ be a multi-class classifier, with which we determine a class $\hat{y}$ for input $\mathbf{x}$ by $\hat{y} = \arg \max_{y \in \mathcal{Y}} f_y(\mathbf{x})$, where $f_y$ is the $y$-th element of $f$. Our goal of multi-class classification is to learn a multi-class classifier $f : \mathcal{X} \rightarrow \mathbb{R}^K$ that minimizes the following classification risk:

$$ R(f; \mathcal{L}) = \mathbb{E}_{p(\mathbf{x}, \mathbf{y})} [\mathcal{L}(f(\mathbf{x}), \mathbf{y})], $$

where $\mathbb{E}_{p(\mathbf{x}, \mathbf{y})}[:]$ denotes the expectation over the joint probability density $p(\mathbf{x}, \mathbf{y})$ and $\mathcal{L}$ is the loss function. However, in general, the classification risk can not be computed directly because the joint probability density $p(\mathbf{x}, \mathbf{y})$ is often unknown. In ordinary multi-class classification, we assume that we have access to a training dataset $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=1}^n$ where each example is independently drawn from $p(\mathbf{x}, \mathbf{y})$. Following the ERM principle, we minimize the empirical risk instead:

$$ \hat{R}_n(f; \mathcal{L}) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}(f(\mathbf{x}_i), \mathbf{y}_i) $$

### 2.2 Partial-Label Learning

In PL, we do not have direct access to the true labels $\mathbf{y}_i$. Each $\mathbf{x}_i$ is only equipped with a partial label (a set of candidate labels). More specifically, the partial label space is defined as $\mathcal{S} := 2^\mathcal{Y} \setminus \{\emptyset, \mathcal{Y}\}$,
where $2^\mathcal{Y}$ denotes the power set of $\mathcal{Y}$. Let $s \in \mathcal{S}$ be the partial label of $x$. With a little abuse of notation, we assume that $(x, s)$ is drawn from an unknown probability distribution with joint probability density denoted as $p(x, s)$. We let $p(x, y, s)$ denote the joint probability density of the whole triplet $(x, y, s)$. The key assumption of PL is that the partial label $s$ always includes the true label $y$ [Cour et al., 2011, Liu and Dietterich, 2012, 2014]:

$$P(y \in s | x, s) = 1,$$

where $P$ denotes the probability. We denote the PL training dataset by $D_{\text{PL}} = \{(x_i, s_i)\}_{i=1}^n$, where each example is assumed to be independently drawn from the joint probability density $p(x, s)$. The empirical risk $\hat{R}_n(f; \mathcal{L})$ is not accessible from $D_{\text{PL}}$. This is the common difficulty for applying ERM in weakly supervised classification. One popular solution is to rewrite the classification risk in the form whose direct unbiased estimator can be computed using weak labels [Sugiyama et al., in press, Bao et al., 2018, Charoenphakdee et al., 2019, Feng et al., 2021, Cao et al., 2021b]. The risk rewriting method is model-independent and optimizer-independent. This key property enables the algorithms to be highly flexible and scalable to large-scale datasets.

### 2.3 Related Work

Here, we introduce some previous work related to our proposed method: Learning with Complementary Labels (CL) [Ishida et al., 2019], Learning with Multiple Complementary Labels (MCL) [Feng et al., 2020a], and Provably Consistent Partial-Label Learning (PCPL) [Feng et al., 2020b].

**Learning with Complementary Labels.** A complementary label [Ishida et al., 2017, 2019, Yu et al., 2018] is a kind of weak labels indicating an incorrect class of an instance. We let $\bar{y}$ denote the complementary label. By taking $s = \mathcal{Y} \setminus \{\bar{y}\}$, the complementary label can be equivalently expressed as a partial label. CL is an extreme case of PL since the size of the partial label is fixed to its maximum, $K - 1$: $|s| = K - 1$. Ishida et al. [2017, 2019] assumed that the training dataset $D_{\text{CL}} = \{(x_i, \bar{y}_i)\}_{i=1}^n$ is independently sampled from the joint probability density

$$\bar{p}(x, \bar{y}) = \frac{1}{K-1} \sum_{y \neq \bar{y}} p(x, y).$$

This assumption implies that

$$\bar{p}(\bar{y}|x, y) = \frac{1}{K-1} \mathbf{1}\{ \bar{y} \neq y \},$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. Equivalently, this can be expressed as

$$p(s|x, y) = \frac{1}{K-1} \mathbf{1}\{ y \in s \},$$

which means that all labels except the correct label $y$ are chosen uniformly to be the complementary label. Based on this assumption, the authors derived the following unbiased risk estimator:

$$\hat{R}_{\text{CL}}(f; \mathcal{L}) = \frac{1}{n} \sum_{i=1}^n \left[ \sum_{k \neq \bar{y}_i} \mathcal{L}(f(x_i), k) - (K - 2)\mathcal{L}(f(x_i), \bar{y}_i) \right].$$
Learning with Multiple Complementary Labels. Feng et al. [2020a] generalized the problem to deal with multiple complementary labels. We let $\bar{s}$ denote the set of complementary labels. Similarly, by taking $s = \mathcal{Y} \setminus \bar{s}$, the multiple complementary label can be equivalently expressed as a partial label. MCL assumes that we have access to a dataset $D_{\text{MCL}} = \{(x_i, \bar{s}_i)\}_{i=1}^n$, where each example is independently drawn from the joint probability density $\bar{p}(x, \bar{s})$ given as

$$\bar{p}(x, \bar{s}) = \sum_{i=1}^{K-1} \bar{Q}_i \bar{p}(x, \bar{s} \mid |\bar{s}| = i),$$

where $|\bar{s}|$ denotes the size of $\bar{s}$, $\bar{Q}_i := P(|\bar{s}| = i)$, and

$$\bar{p}(x, \bar{s} \mid |\bar{s}| = i) := \begin{cases} \frac{1}{(K-i)} \sum_{y \notin \bar{s}} p(x, y) & \text{if } |\bar{s}| = i, \\ 0 & \text{otherwise}. \end{cases}$$

This assumption implies that

$$p(\bar{s} | x, y) = \frac{\bar{Q}_{|\bar{s}|}}{(K-1)} 1\{y \notin \bar{s}\},$$

and equivalently, that

$$p(s | x, y) = \frac{Q_{|s|}}{(K-1)} 1\{y \in s\},$$

where $Q_i = P(|s| = i)$. Based on this assumption, the authors derived the following unbiased risk estimator:

$$\hat{R}_{\text{MCL}}(f; \mathcal{L}) = \frac{1}{n} \sum_{i=1}^n \left( \sum_{y \notin \bar{s}_i} \mathcal{L}(f(x_i), y) - \frac{K-1-|\bar{s}_i|}{|\bar{s}_i|} \sum_{y \in \bar{s}_i} \mathcal{L}(f(x_i), y) \right).$$

Provably Consistent Partial-Label Learning. In PCPL [Feng et al., 2020b], the authors assumed that

$$p(s | x, y) = \frac{1}{2K-1} 1\{y \in s\}.$$

This assumption implies that given the true label $y$, the remaining part of the partial label $s$ is uniformly sampled from $2^\mathcal{Y}\setminus\{y\} \setminus \{\emptyset, \mathcal{Y}\setminus\{y\}\}$. With this assumption, the authors derived the following unbiased risk estimator:

$$\hat{R}_{\text{PCPL}}(f; \mathcal{L}) = \frac{1}{2n} \sum_{i=1}^n \sum_{j=1}^K \frac{p(y = j | x_i)}{\sum_{k \in s} p(y = k | x_i)} \mathcal{L}(f(x_i), j).$$

The posterior probability $p(y = k | x)$ is not accessible. Therefore, the softmax function was utilized to approximate $p(y = k | x)$:

$$p(y = k | x) \approx \frac{e^{f_k(x)}}{\sum_{i=1}^K e^{f_i(x)}}.$$
Table 1: Examples of special PPL settings.

| \(C(x, s)\) | CMM | CL | MCL | PCPL |
|-------------|-----|----|-----|------|
| \(C(s)\)   | \(\frac{1}{K-1}\) | \(\frac{Q_{|s|}}{K-1}\) | \(\frac{1}{2^{K-1}-1}\) |

3 Method

In this section, we first propose the notion of properness for PL. Then, we propose a general PL framework called the proper partial-label learning (PPL) framework. Within this framework, we derive a unified risk-consistent algorithm for PL problems. Finally, we establish an estimation error bound for the proposed method.

3.1 Proper Partial-Label Learning

Here, we introduce the concept of properness in PL:

**Definition 1.** We say that a PL problem is proper if there exists a function \(C : \mathcal{X} \times \mathcal{S} \rightarrow \mathbb{R}\) such that the following condition holds:

\[
p(s|x, y) = C(x, s)1\{y \in s\}. \tag{7}
\]

The direct interpretation for this definition is twofold. First, an example \((x, y)\) can not generate a partial label \(s\) where \(s\) does not include \(y\). This coincides with the basic property (2) of PL. Second, when \((x, y)\) is given, the probability of generating \(s\) does not depend on \(y\) as long as \(y \in s\).

The properness assumption is general in the sense that it includes many previous problem settings as special cases. In the Logistic Stick-Breaking Conditional Multinomial Model (CMM) [Liu and Dietterich, 2012], the authors treated the partial label \(s\) as label noise and assumed that \(s\) and \(x\) are conditionally independent when the true label \(y\) is given:

\[
p(s|x, y) = p(s|y). \tag{8}
\]

This conditional independence assumption is common in modelling label noise [Han et al., 2018]. Further, the authors proposed the following assumption to model the noise distribution:

\[
p(s|y) = C(s)1\{y \in s\}. \tag{9}
\]

Here, the function \(C\) only depends on \(s\). In the following proposition, we obtain an equivalent expression for (8) and (9):

**Proposition 1.** (8) and (9) hold if and only if there exists a function \(C : \mathcal{S} \rightarrow \mathbb{R}\) such that the following condition holds:

\[
p(s|x, y) = C(s)1\{y \in s\}. \tag{10}
\]

All the proofs are given in Appendix A. By the above proposition, we can see that the CMM assumption (10) is stronger than the properness assumption (7). Within the PL framework, we can reproduce the CL, MCL, and PCPL models introduced in Section 2 with different formulations of \(C(x, s)\) (Table 1). We notice that the function \(C\) of CL, MCL, and PCPL does not depend on \(x\).
This shows that the distributional assumptions of CL, MCL, and PCPL are stronger than CMM. For CL classification, we have
\[ Q_{|s|} = \begin{cases} 
1 & \text{if } |s| = K - 1, \\
0 & \text{otherwise.} 
\end{cases} \]

This shows that the CL setting is stricter than the MCL setting. The following proposition shows that the PCPL assumption is stronger than the MCL assumption:

**Proposition 2.** If the PCPL assumption \( p(s|x, y) = \frac{1}{2^{K-1}-1} \{y \in s\} \) holds, then the MCL assumption \( p(s|x, y) = \frac{Q_{|s|}}{(|s|-1)} \{y \in s\} \) also holds. But the opposite is not true.

We summarize the relations between different special PPL settings in Figure 1. The area \( \text{AND}(\text{CL, PCPL}) \) represents MCL problems where \( K = 2 \). The partial label generation process adopted in Section 4 lies in the area \( \text{MCL} \setminus \text{OR}(\text{CL, PCPL}) \). The area \( \text{PPL} \setminus \text{CMM} \) represents PPL problems where the function \( C \) also depends on \( x \). The area outside of PPL represents PL problems where \( p(s|x, y) \) depends on the choice of \( y \) given \( y \in s \).

### 3.2 Risk Consistent Algorithm

Here, we describe our proposed algorithm for PPL.

**Candidate Label Confidence.** Various weakly supervised learning settings based on confidence data have been studied recently, including positive-confidence data [Ishida et al., 2018], similarity-confidence data [Cao et al., 2021b], confidence data for instance-dependent label-noise [Berthon et al., 2021] and single-class confidence [Cao et al., 2021a]. Here, we propose the following candidate label confidence:

**Definition 2.** The candidate label confidence \( r_y(x, s) \) is defined as
\[ r_y(x, s) = p(y|x, s). \]

The following theorem of the candidate label confidence provides an equivalent definition for properness in PPL:

**Theorem 1.** The properness assumption (7) holds if and only if
\[ r_y(x, s) = \frac{p(y|x)}{\sum_{k \in s} p(y = k|x)} \{y \in s\}. \]

(11)
The theorem shows that in PPL problems, the candidate label confidence can be expressed in terms of confidence scores (i.e., the class-posterior probabilities). Specifically, the properness assumption requires that the confidence for a candidate label $y$ in the candidate label set $s$ is proportional to its confidence score. The advantage of this property is that we can compute $r_y(x, s)$ without specifying $C(x, s)$, which enables our derivation for a risk-consistent algorithm.

Unbiased Risk Estimator. Here, we will use the PPL training dataset $D_{PPL} = \{(x_i, s_i)\}_{i=1}^n$ to construct an unbiased estimator of the objective risk (1).

**Theorem 2.** The classification risk (1) can be equivalently expressed as

$$R(f; \mathcal{L}) = \mathbb{E}_{p(x, s)} \left[ \sum_{j \in s} r_j(x, s) \mathcal{L}(f(x), j) \right].$$  (12)

Based on Theorem 1 and Theorem 2, we can have the following theorem:

**Theorem 3.** For PPL, the classification risk (1) can be equivalently expressed as

$$R(f; \mathcal{L}) = \mathbb{E}_{p(x, s)} \left[ \sum_{j \in s} \sum_{k \in s} \frac{p(y = j|x)}{p(y = k|x)} \mathcal{L}(f(x), j) \right].$$  (13)

Finally, an unbiased risk estimator is given by the following corollary:

**Corollary 1.** When the properness assumption (7) holds, given the dataset $D_{PPL} = \{(x_i, s_i)\}_{i=1}^n$, the following is an unbiased estimator of the classification risk:

$$\hat{R}_{PPL}(f; \mathcal{L}) = \frac{1}{n} \sum_{i=1}^n \sum_{j \in s_i} \frac{p(y = j|x_i)}{p(y = k|x_i)} \mathcal{L}(f(x_i), j).$$  (14)

Risk-Consistent Algorithm. We approximate the confidence score $p(y = j|x)$ by the softmax output:

$$p_f(y|x) = \frac{e^{f_j(x)}}{\sum_{i=1}^K e^{f_i(x)}}.$$

Then we can compute the empirical risk (14) with $D_{PPL}$ and (15). The pseudo-code and the implementation of the proposed method are presented in Algorithm 1 and Figure 2. We can see that the proposed method is model-independent, optimizer-independent, and loss-independent. Note that our algorithm coincides with that of the progressive identification (PRODEN) algorithm [Lv et al., 2020] and the PCPL algorithm [Feng et al., 2020b]. However, for PRODEN, it was derived without any theoretical guarantees in totally different manners. For PCPL, the algorithm was derived from a completely different risk estimator (6) under a very strong distributional assumption.

Next, we establish an estimation error bound to show the consistency of the proposed method.

For simplicity, we omit $\mathcal{L}$ and write the risk $R$ as $R(f)$. Let $\mathcal{F}$ be the hypothesis class and $\mathcal{F}_i = \{f_i : f \in \mathcal{F}\}$ be the function class generated by $\mathcal{F}$, where $f_i$ denotes the $i$-th element of $f$. Let $f^* = \arg \min_{f \in \mathcal{F}} R(f)$ be the optimal classifier and $\hat{f}_{PPL} = \arg \min_{f \in \mathcal{F}} \hat{R}_{PPL}(f)$ be the minimizer of the empirical risk. We have the following theorem:
Algorithm 1 PPL

**Input:** Model $f$, number of epochs $T$, number of mini-batches $B$, partially labeled training dataset $\mathcal{D}_{\text{PPL}} = \{(x_i, s_i)\}_{i=1}^n$

1: **Initialize** $\hat{p}(y = j|x_i) = 1/K$;
2: Let $\mathcal{A}$ be an stochastic optimization algorithm;
3: for $t = 1, 2, \ldots, T$ do
4:   Shuffle $\mathcal{D}_{\text{PPL}}$;
5:   for $l = 1, 2, \ldots, B$ do
6:     Fetch mini-batch $\mathcal{D}_l$ from $\mathcal{D}_{\text{PPL}}$;
7:     Compute loss from Eq. (14) using $\hat{p}$;
8:     Update $f$ by $\mathcal{A}$;
9:     Update $\hat{p}$ by Eq. (15);
10: end for
11: end for

**Output:** $f$

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Figure 2: Implementation diagram of PPL.

**Theorem 4.** Suppose that $M := \sup_{x \in \mathcal{X}, y \in \mathcal{Y}, f \in \mathcal{F}} \mathcal{L}(f(x), y) < \infty$ and that the loss $\mathcal{L}(f(x), s)$ is $\rho$-Lipschitz with respect to $f(x)$ for all $y \in \mathcal{Y}$. Let $\mathcal{R}_n$ denote the Rademacher complexity. Then, for any $\delta > 0$, the following holds with probability at least $1 - \delta$: 

$$ R(\hat{f}_{\text{PPL}}) - R(f^*) \leq 4\sqrt{2}\rho \sum_{i=1}^K \mathcal{R}_n(\mathcal{F}_i) + 2M \sqrt{\frac{\log \frac{2}{\delta}}{2n}}. $$

In general, we can further prove that the Rademacher complexity $\mathcal{R}_n(\mathcal{F}_y)$ is bounded by $O(1/\sqrt{n})$ [Golowich et al., 2018]. Hence, the above theorem demonstrates that $R(\hat{f}_{\text{PPL}})$ converges in probability to $R(f^*)$ as the sample size $n \to \infty$.

4 Experiments

In this section, we demonstrate the effectiveness of the proposed method through experiments. Experiment details are provided in Appendix B.
Datasets. Our experiments were conducted on four widely-used benchmark datasets in image classification, namely MNIST [LeCun et al., 1998], Fashion-MNIST [Xiao et al., 2017], Kuzushiji-MNIST [Clanuwat et al., 2018], and CIFAR-10 [Krizhevsky, 2009].

Model and Loss Function. We performed the experiments with various base models, including a simple linear model (d-10), a 5-layer MLP (d-300-300-300-300-10), 32-layer ResNet [He et al., 2016], and 22-layer DenseNet [Huang et al., 2017]. We applied batch normalization [Ioffe and Szegedy, 2015] to each hidden layer. We used $l_2$-regularization and the stochastic gradient descent optimizer [Robbins and Monro, 1951] with momentum 0.9. We chose the popular cross-entropy loss as our loss function.

Baseline Methods. To evaluate the performance of the general PPL framework, we compared it with the MCL approach using partially labeled datasets generated in the area MCL\OR(CL, PCPL) (Figure 1). For the MCL approach, we used the surrogate loss $r_{LOG}$ which achieved the overall best performance in Feng et al. [2020a].

Partial Label Generation Process. In the experiment of PCPL [Feng et al., 2020b], the partial label $s$ for each instance was uniformly sampled from $S$ given $y \in s$. The resulting distribution of $|s|$ is symmetric. The same generation process was also adopted in the experiment of MCL [Feng et al., 2020a] to generate the complementary label set $\overline{s}$. However, in most real-world cases, smaller complementary label sets are easier to collect. Motivated by this observation, we consider the $\alpha$-skewed distribution: $\overline{Q}_{k+1} = \alpha \overline{Q}_k, 0 \leq k \leq K - 1$. Here, $\overline{Q}_k := P(|\overline{s}| = k)$. Figure 3 compares the PCPL distribution with the $\alpha$-skewed distribution.

Table 2: Average partial label size for different distributions.

|       | PCPL  | 0.9-skewed | 0.8-skewed | 0.7-skewed |
|-------|-------|------------|------------|------------|
| $E[|s|]$ | 5.0   | 5.69       | 6.40       | 7.05       |
Table 3: Test performance (mean±standard error) of classification accuracy over five trials on benchmark datasets.

| Dataset         | α    | Linear       | MCL       | PPL       | MLP       | MCL       | PPL       |
|-----------------|------|--------------|-----------|-----------|-----------|-----------|-----------|
|                 |      | MCL          | PPL       |           | MCL       | PPL       |           |
| MNIST           | 0.9  | 91.90±0.05%  | 92.18±0.06% | 98.12±0.54% | 98.40±0.09% | 92.18±0.06% | 98.40±0.09% |
|                 | 0.8  | 91.84±0.04%  | 92.14±0.02% | 98.13±0.28% | 98.28±0.04% | 92.14±0.02% | 98.28±0.04% |
|                 | 0.7  | 91.76±0.04%  | 92.02±0.06% | 97.70±0.10% | 98.22±0.04% | 92.02±0.06% | 98.22±0.04% |
| Fashion-MNIST   | 0.9  | 83.63±0.08%  | 84.11±0.07% | 87.97±0.53% | 88.95±0.13% | 87.97±0.53% | 88.95±0.13% |
|                 | 0.8  | 83.54±0.10%  | 83.88±0.08% | 87.87±0.40% | 88.59±0.13% | 87.87±0.40% | 88.59±0.13% |
|                 | 0.7  | 83.18±0.03%  | 83.71±0.06% | 87.73±0.16% | 88.23±0.16% | 87.73±0.16% | 88.23±0.16% |
| Kuzushiji-MNIST | 0.9  | 68.49±0.06%  | 69.25±0.08% | 90.19±1.06% | 91.82±0.20% | 90.19±1.06% | 91.82±0.20% |
|                 | 0.8  | 67.98±0.10%  | 68.68±0.19% | 89.88±0.67% | 90.72±0.32% | 89.88±0.67% | 90.72±0.32% |
|                 | 0.7  | 66.42±0.19%  | 67.37±0.14% | 87.81±1.24% | 90.24±0.26% | 87.81±1.24% | 90.24±0.26% |

Next, we define the average partial label size as $E[|s|]$, which can measure the ambiguity and the hardness of PL problems. We summarize the average partial label size for different distributions in Table 2. We can see that the ambiguity of PL problems increases as $\alpha$ decreases. Based on the $\alpha$-skewed distribution, we consider the following partial label generation process:

1. Instantiate $\bar{Q}_k$ according to $\alpha$;

2. For each instance $x$, we first sample the size $|s|$ from $\bar{Q}$. Then, we uniformly sample a complementary label set $\bar{s}$ with size $|\bar{s}|$. Finally, a partial label $s$ is generated by $s = Y \setminus \bar{s}$.

**Experimental Results** We randomly sampled 10% of the training set to construct a validation set. We selected the learning rate and weight decay from the set $\{10^{-6}, 10^{-5}, \ldots, 10^{-1}\}$ to achieve the best validation score. The mini-batch size was set to 256 and the epoch number was set to 250. We compared MCL and PPL on different choices of $\alpha$. The test performance was evaluated based on 5 trials on the four benchmark datasets. We performed the paired $t$-test at 5% significance level and highlighted MCL or PPL in boldface if one performed significantly better than the other. The experimental results are summarized in Table 3. From the table, we can see that PPL always performed better than MCL. We also notice that the test accuracy decreased as we decreased $\alpha$, which accords with our conjecture that the average partial label size can be used to measure the hardness of PL problems. More detailed visualized results are provided in Appendix B.

We further checked the loss variance of the MCL and PPL learners. We highlighted the results with less variance in boldface. The experimental results are reported in Table 4. We see that the loss
Table 4: Loss (mean±standard error) of classification accuracy over five trials on benchmark datasets.

| Dataset      | α  | Linear MCL | Linear PPL | MLP MCL | MLP PPL |
|--------------|----|------------|------------|---------|---------|
| MNIST        | 0.9| 0.234±0.012| 0.256±0.007 | 0.073±0.005 | 0.069±0.010 |
|              | 0.8| 0.231±0.012| 0.250±0.006 | 0.085±0.028 | 0.109±0.004 |
|              | 0.7| 0.223±0.016| 0.247±0.005 | 0.097±0.015 | 0.135±0.005 |
| Fashion-MNIST| 0.9| 0.368±0.007| 0.409±0.003 | 0.533±0.048 | 0.244±0.011 |
|              | 0.8| 0.354±0.009| 0.404±0.005 | 0.481±0.057 | 0.232±0.004 |
|              | 0.7| 0.356±0.018| 0.412±0.005 | 0.549±0.059 | 0.232±0.004 |
| Kuzushiji-MNIST| 0.9| 0.447±0.010| 0.539±0.003 | 0.111±0.013 | 0.128±0.017 |
|              | 0.8| 0.432±0.009| 0.530±0.005 | 0.130±0.011 | 0.139±0.004 |
|              | 0.7| 0.431±0.011| 0.528±0.003 | 0.171±0.052 | 0.167±0.005 |


data variance of PPL was always less than MCL. This helps explain why PPL had better performance in previous experiments.

5 Conclusions

We presented a unified framework for learning from proper partial labels which accommodates many previous problems settings as special cases. We derived an unbiased estimator of the classification risk for proper partial label learning problems and theoretically established an estimation error bound for the proposed method. We also demonstrated the effectiveness of the proposed method through experiments on benchmark datasets.
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Appendix

A Proofs

A.1 Proof of Proposition 1

If (8) and (9) hold, we can easily see that (10) holds. Conversely, if (10) holds, we have

\[ p(s|y) = \int_{\mathcal{X}} p(x, s|y)dx = \int_{\mathcal{X}} p(s|x, y)p(x|y)dx \]

\[ = \int_{\mathcal{X}} C(s)1\{y \in s\}p(x|y)dx = C(s)1\{y \in s\} \cdot \int_{\mathcal{X}} p(x|y)dx \]

\[ = p(s|x, y) \int_{\mathcal{X}} p(x|y)dx. \]

By noticing that \( \int_{\mathcal{X}} p(x|y)dx = 1 \), we have

\[ p(s|y) = p(s|x, y) \int_{\mathcal{X}} p(x|y)dx = p(s|x, y). \]

Hence, (8) holds and we have

\[ p(s|y) = p(s|x, y) = C(s)1\{y \in s\}, \]

which concludes the proof.

A.2 Proof of Proposition 2

First, CL is a special case of MCL and the CL assumption is different from the PCPL assumption when \( K \geq 3 \). Hence, we know the opposite is not true. Next, it suffices to prove that

\[ Q_k = \frac{(K-1)}{2^{K-1} - 1}. \] (16)

By definition, we have

\[ Q_k = P(|s| = k) = \sum_{|s|=k} p(s) = \sum_{|s|=k} \int_{\mathcal{X}} \sum_{i=1}^{K} p(x, y = i, s)dx \]

\[ = \sum_{|s|=k} \sum_{i=1}^{K} \int_{\mathcal{X}} p(s|x, y = i)p(x, y = i)dx \]

\[ = \sum_{|s|=k} \sum_{i=1}^{K} \int_{\mathcal{X}} \frac{1}{2^{K-1} - 1}1\{i \in s\}p(x, y = i)dx \]

\[ = \frac{1}{2^{K-1} - 1} \sum_{|s|=k} \sum_{i=1}^{K} 1\{i \in s\}p(y = i) \]

\[ = \frac{1}{2^{K-1} - 1} \sum_{i=1}^{K} \sum_{|s|=k} 1\{i \in s\}p(y = i). \]
When \(i\) is fixed in \(s\), there are \(\binom{K-1}{k-1}\) different choices of \(s\) with size \(k\). By noticing that 
\[
\sum_{i=1}^{K} p(y = i) = 1,
\]
we have
\[
Q_k = \frac{1}{2^{K-1}-1} \sum_{i=1}^{K} \binom{K-1}{k-1} p(y = i) = \frac{\binom{K-1}{k-1}}{2^{K-1}-1},
\]
which completes the proof.

A.3 Proof of Theorem 1

If (7) holds, i.e., 
\[
p(s|x, y) = C(x,s) \mathbf{1}\{ y \in s \},
\]
then we have
\[
r_y(x, s) = \frac{p(y, s|x)}{p(s|x)} = \frac{p(y, s|x)}{\sum_{k \in s} p(y = k, s|x)}
\]
\[
= \frac{p(s|x, y)p(y|x)}{\sum_{k \in s} p(s|x, y = k)p(y = k|x)}
\]
\[
= \frac{C(x, s) \mathbf{1}\{ y \in s \} p(y|x)}{\sum_{k \in s} C(x, s) \mathbf{1}\{ k \in s \} p(y = k|x)}
\]
\[
= \frac{p(y|x)}{\sum_{k \in s} p(y = k|x) \mathbf{1}\{ y \in s \}}.
\]
Conversely, if (11) holds, we have
\[
p(s|x, y) = \frac{p(y, s|x)}{p(y|x)} = \frac{p(y|x, s)p(s|x)}{p(y|x)} = \frac{p(s|x)}{\sum_{k \in s} p(y = k|x) \mathbf{1}\{ y \in s \}}.
\]
Hence, there exists a function
\[
C(x, s) = \frac{p(s|x)}{\sum_{k \in s} p(y = k|x)}
\]
such that
\[
p(s|x, y) = C(x, s) \mathbf{1}\{ y \in s \}.
\]
A.4 Proof of Theorem 2

By definition, we have
\[
R(f; \mathcal{L}) = \int_X \sum_{j=1}^K \mathcal{L}(f(x), j)p(x, y = j)dx = \int_X \sum_{j=1}^K \mathcal{L}(f(x), j)p(x, y = j, s)dx
\]
\[
= \int_X \sum_{j=1}^K \sum_{s \in S} r_j(x, s) \mathcal{L}(f(x), j)p(x, s)dx
\]
\[
= \mathbb{E}_{p(x, s)} \left[ \sum_{j \in s} r_j(x, s) \mathcal{L}(f(x), j) \right].
\]

A.5 Proof of Theorem 3

By Theorem 1, (11) holds. By Theorem 2, we have
\[
R(f; \mathcal{L}) = \mathbb{E}_{p(x, s)} \left[ \sum_{j \in s} r_j(x, s) \mathcal{L}(f(x), j) \right]
\]
\[
= \mathbb{E}_{p(x, s)} \left[ \sum_{j \in s} \sum_{k \in s} p(y = j|x) \mathcal{L}(f(x), j) \right],
\]
which completes the proof.

A.6 Proof of Theorem 4

We first introduce the following lemma:

Lemma 1. The following inequality holds:
\[
0 \leq R(\hat{f}_{PPL}) - R(f^*) \leq 2 \sup_{f \in \mathcal{F}} |R(f) - \hat{R}_{PPL}(f)|.
\]

Proof. By definition, \( R(\hat{f}_{PPL}) - R(f^*) \geq 0 \). Also, we have
\[
R(\hat{f}_{PPL}) - R(f^*) = \left( R(\hat{f}_{PPL}) - \hat{R}_{PPL}(\hat{f}_{PPL}) \right)
+ \left( \hat{R}_{PPL}(\hat{f}_{PPL}) - \hat{R}_{PPL}(f^*) \right)
+ \left( \hat{R}_{PPL}(f^*) - R(f^*) \right)
\leq \left( R(\hat{f}_{PPL}) - \hat{R}_{PPL}(\hat{f}_{PPL}) \right) + \left( \hat{R}_{PPL}(f^*) - R(f^*) \right)
\leq 2 \sup_{f \in \mathcal{F}} |R(f) - \hat{R}_{PPL}(f)|.
\]

\qed
The Rademacher complexity Mohri et al. [2018] is defined as follows:

**Definition 3 (Empirical Rademacher Complexity).** Let $\mathcal{G}$ be a class of functions mapping $\mathbb{Z}$ to $\mathbb{R}$ and $S = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ a fixed sample of size $n$. Then, the empirical Rademacher complexity of $\mathcal{G}$ with respect to the sample $S$ is defined as

$$\hat{R}_S(\mathcal{G}) = \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i g(z_i) \right],$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$, with $\sigma_i$'s independent uniform random variables taking values in $\{-1, +1\}$.

**Definition 4 (Rademacher Complexity).** Suppose the sample $S$ of size $n$ is drawn independently from the distribution denoted by a probability density function $p$. The Rademacher complexity of $\mathcal{G}$ with respect to $p$ is defined as

$$R_n(\mathcal{G}) = \mathbb{E}_{z_i \sim p} \left[ \hat{R}_S(\mathcal{G}) \right].$$

We introduce a class of functions defined on $X \times S$ according to Eq. (13):

$$\mathcal{G} = \{ (x, s) \rightarrow \sum_{j \in s} \sum_{k \in s} p(y = j | x) p(y = k | x) L(f(x), j) : f \in F \}.$$

Then, the Rademacher complexity of $\mathcal{G}$ with respect to $p(x, s)$ is given as:

$$R_n(\mathcal{G}) = \mathbb{E}_{(x_i, s_i) \sim p} \left[ \mathbb{E}_\sigma \left[ \sup_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i g(x_i, s_i) \right] \right].$$

We have the following lemma:

**Lemma 2.** Suppose $M := \sup_{x \in X, y \in Y, f \in F} L(f(x), y) < \infty$, then, for any $\delta > 0$, the following holds with probability at least $1 - \delta$:

$$\sup_{f \in F} |R(f) - \hat{R}_{\text{PPL}}(f)| \leq 2R_n(\mathcal{G}) + M \sqrt{\frac{\log \frac{2}{\delta}}{2n}}.$$ 

**Proof.** For a sample $S$, we define $\phi(S) = \sup_{f \in F} \{ R(f) - \hat{R}_{\text{PPL}}(f) \}$. Suppose we replace an example $(x_i, s_i)$ in the sample $S$ with another example $(x_i', s_i')$, the change of $\phi(S)$ is no greater than

$$\sup_{g \in \mathcal{G}} \frac{g(x_i, s_i) - g(x_i', s_i')}{n} \leq \frac{M}{n},$$

since $L$ is bounded by $M$. Then, by McDiarmid’s inequality McDiarmid [1989], for any $\delta > 0$, with probability at least $1 - \delta/2$, the following holds:

$$\phi(S) \leq \mathbb{E}_{(x_i, s_i) \sim p} [\phi(S)] + M \sqrt{\frac{\log \frac{2}{\delta}}{2n}}.$$
It is a routine work Mohri et al. [2018] to show $E_{(x_i, s_i) \sim p} [\phi(S)] \leq 2R_n(G)$. Hence, the following holds with probability at least $1 - \delta/2$:

$$\sup_{f \in F} \left\{ R(f) - \hat{R}_{PPL}(f) \right\} \leq 2R_n(G) + M \sqrt{\frac{\log \frac{2}{\delta}}{2n}}.$$  \hspace{1cm} (20)

Similarly, we can prove that the following holds with probability at least $1 - \delta/2$:

$$\sup_{f \in F} \left\{ \hat{R}_{PPL}(f) - R(f) \right\} \leq 2R_n(G) + M \sqrt{\frac{\log \frac{2}{\delta}}{2n}}.$$  \hspace{1cm} (21)

We complete the proof by combining (20) and (21).

Next, we bound the Rademacher complexity $R_n(G)$ by the following lemma Feng et al. [2020b]:

**Lemma 3.** Suppose that the loss $L(f(x), s)$ is $\rho$-Lipschitz with respect to $f(x)$ for all $y \in Y$. Then, the following inequality holds:

$$R_n(G) \leq \sqrt{2\rho} \sum_{i=1}^{K} R_n(F_i).$$

**Proof.** Let $\Pi = \{(x, y) \rightarrow L(f(x), y) : f \in F\}$. Notice that the candidate label confidence $r_y(x, s)$ satisfies that $0 \leq r_y(x, s) \leq 1$ and that $\sum_{k \in s} r_k(x, s) = 1$. In this way, we can obtain $R_n(G) \leq R_n(\Pi)$. Since $L$ is $\rho$-Lipschitz with respect to $f(x)$, following the Rademacher vector contraction inequality Maurer [2016], we have $R_n(\Pi) \leq \sqrt{2\rho} \sum_{i=1}^{K} R_n(F_i)$, which concludes the proof.

Finally, the proof of Theorem 3 is completed by combining Lemma 1, Lemma 2 and Lemma 3.

### B Experiment Details

**B.1 Datasets and Models**

We list here the details of the four benchmark datasets used in our experiments:

- **MNIST:** A 10-class dataset containing handwritten digits from 0 to 9. MNIST has in total 60,000 training images and 10,000 test images. Each instance is a $28 \times 28$ grayscale image.

- **Fashion-MNIST:** A 10-class dataset of fashion items. Fashion-MNIST has in total 60,000 training images and 10,000 test images. Each instance is a $28 \times 28$ grayscale image.

- **Kuzushiji-MNIST:** A 10-class dataset of cursive Japanese (Kuzushiji) characters. Kuzushiji-MNIST has in total 60,000 training images and 10,000 test images. Each instance is a $28 \times 28$ grayscale image.
**Table 5: Information of datasets and corresponding models.**

| Dataset            | #Train | #Test | Dimension | Model         |
|--------------------|--------|-------|-----------|---------------|
| MNIST              | 60,000 | 10,000| 784       | Linear, MLP   |
| Fashion-MNIST      | 60,000 | 10,000| 784       | Linear, MLP   |
| Kuzushiji-MNIST    | 60,000 | 10,000| 784       | Linear, MLP   |
| CIFAR-10           | 50,000 | 10,000| 3,072     | ResNet, DenseNet |

**CIFAR-10:** A 10-class dataset of colored images (airplane, automobile, bird, cat, deer, dog, frog, horse, ship, and truck). CIFAR-10 has in total 50,000 training images and 10,000 test images. Each instance is a $32 \times 32 \times 3$ colored image.

The information of each dataset and its corresponding models is summarized in Table 5.

**C Experimental Results**

For more detailed visualized results, we recorded the test accuracy at each training epoch and plotted the test accuracy curve on the four benchmark datasets in Figure 4, Figure 5, Figure 6 and Figure 7. As can be seen from the figures, the classification task grew harder as $\alpha$ decreased. We also observe that PPL had more stable performance than MCL when more complex models were used.
Figure 4: Visualized experimental results for MNIST and Fashion-MNIST. Linear model is trained on the two datasets. Dark colors represent the mean accuracy of 5 trials and light colors represent the standard error.
Figure 5: Visualized experimental results for Kuzushiji-MNIST and CIFAR-10. Linear model is trained on Kuzushiji-MNIST, and ResNet is trained on CIFAR-10. Dark colors represent the mean accuracy of 5 trials and light colors represent the standard error.
Figure 6: Visualized experimental results for MNIST and Fashion-MNIST. MLP is trained on the two datasets. Dark colors represent the mean accuracy of 5 trials and light colors represent the standard error.
Figure 7: Visualized experimental results for Kuzushiji-MNIST and CIFAR-10. MLP is trained on Kuzushiji-MNIST, and DenseNet is trained on CIFAR-10. Dark colors represent the mean accuracy of 5 trials and light colors represent the standard error.