Implementation of Kalman filter algorithm on models reduced using singular perturbation approximation method and its application to measurement of water level

Vimala Rachmawati 1, Didik Khusnul Arif 2 and Dieky Adzkiya 3
Department of Mathematics, Faculty of Mathematics, Computing, and Data Sciences, Institut Teknologi Sepuluh Nopember, Kampus ITS Sukolilo-Surabaya 60111, Indonesia

Email: 1 vimalarachmawati@gmail.com, 2 didik@matematika.its.ac.id, 3 dieky@matematika.its.ac.id

Abstract. The systems contained in the universe often have a large order. Thus, the mathematical model has many state variables that affect the computation time. In addition, generally not all variables are known, so estimations are needed to measure the magnitude of the system that cannot be measured directly. In this paper, we discuss the model reduction and estimation of state variables in the river system to measure the water level. The model reduction of a system is an approximation method of a system with a lower order without significant errors but has a dynamic behaviour that is similar to the original system. The Singular Perturbation Approximation method is one of the model reduction methods where all state variables of the equilibrium system are partitioned into fast and slow modes. Then, The Kalman filter algorithm is used to estimate state variables of stochastic dynamic systems where estimations are computed by predicting state variables based on system dynamics and measurement data. Kalman filters are used to estimate state variables in the original system and reduced system. Then, we compare the estimation results of the state and computational time between the original and reduced system.

1. Introduction
River is one source of water that holds and drain the flow of water. River can be viewed as a system which is a combination of several components that work together to achieve a certain objective. In this case, the river level system is used to keep the flow of river water flow in the desired condition. The system from this river level can be represented in the form of mathematical models.

The systems contained in the universe often have a large order. Thus, the mathematical model has many state variables that affect the computation time. Therefore, we need a simplification of large order systems into smaller orders without significant errors. Simplification of this system is called model reduction [1]. Currently, there are many developed methods of model reduction such as balanced truncation methods [2,3,7,8,9,11] and singular perturbation approximation [10]. In [10], we have developed the algorithm of singular perturbation approximation. In this paper, we apply the Singular Perturbation Approximation (SPA) method to shallow water equations.
In large systems, generally not all variables are known, so a useful estimate is needed to measure the magnitude of the system that cannot be measured directly. Kalman filter is a recursive algorithm for estimating state variables of a stochastic dynamic system. Estimation using this method is done by predicting state variables based on system dynamics and measurement data [4].

In [1], Kalman filter algorithm has been developed in the reduced model and applied to the heat conduction distribution problem. The simulation results show that the Kalman filter estimation on the reduced system is more accurate and faster compared to Kalman filter on the original system. In this paper, we estimate the model of shallow water equations that has been reduced using Singular Perturbation Approximation (SPA) method by implementing Kalman Filter algorithm to measure water level of the river.

2. Shallow Water Equations
In this section, we discuss the shallow water equation that describe the flow of water in the river [5]:

\[
\begin{align*}
\frac{\partial h}{\partial t} + D \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} + C_f u &= 0
\end{align*}
\]  

with initial conditions:

\[
h(x, 0) = 2 + \sin(2\pi x), \quad u(x, 0) = 0
\]

and boundary conditions:

\[
\begin{align*}
h(0, t) &= h(x - 1, t), \quad h(L, t) = h(2, t) \\
u(0, t) &= u(x - 1, t), \quad u(L, t) = u(2, t)
\end{align*}
\]

where \( h(x, t) \) is the water level above the reference plane at position \( x \) and time \( t \), \( u(x, t) \) is the average current velocity at position \( x \) and time \( t \), \( t \) is the time variable, \( x \) is the position along the river, \( D \) is the water depth above the reference point, \( g \) is the gravitational acceleration and \( C_f \) is a friction constant.

3. Discretization
In this section, we discretization the Shallow Water Equations with Lax Wendroff Scheme so that we can obtain a discrete-time system that is suitable for model reduction. First, from equation (1) is discretized with FTCS (Forward Time Centered Space) Scheme as follows:

\[
\begin{align*}
h_i^{k+1} &= h_i^k - \frac{\Delta t}{2\Delta x} (u_{i+1}^k - u_{i-1}^k) \\
u_i^{k+1} &= (1 - C_f \Delta t)u_i^k - \frac{g\Delta t}{2\Delta x} (h_{i+1}^k - h_{i-1}^k)
\end{align*}
\]

The Lax Wendroff Scheme is a combination of Lax Friedrichs Scheme and Leapfrog Scheme. The Lax Friedrichs Scheme works as follows replace \( h_i^k \) and \( u_i^k \) in equation (4) respectively with \( \frac{1}{2}(h_i^{k+1} + h_i^{k-1}) \) and \( \frac{1}{2}(u_{i+1}^k + u_{i-1}^k) \). So we obtain:

\[
\begin{align*}
h_i^{k+1} &= \frac{1}{2}(h_i^{k+1} + h_i^{k-1}) - \frac{\Delta t}{2\Delta x} (u_{i+1}^k - u_{i-1}^k) \\
u_i^{k+1} &= \left(\frac{1-C_f\Delta t}{2}\right)(u_{i+1}^k + u_{i-1}^k) - \frac{g\Delta t}{2\Delta x} (h_i^k - h_{i-1}^k)
\end{align*}
\]
The Leapfrog Scheme works as follows replace $\Delta t$ with $2\Delta t$ in equation (4) with the aim that $g\Delta t$ or $D\Delta t$ has a smaller value than $\Delta x$ to achieve the desired accuracy. So we obtain :

$$h_i^{k+1} = h_i^k - \frac{D\Delta t}{\Delta x} (u_{i+1}^k - u_{i-1}^k)$$

$$u_i^{k+1} = (1 - 2C_f\Delta t)u_i^k - \frac{g\Delta t}{\Delta x} (h_{i+1}^k - h_{i-1}^k)$$

(6)

Substitution $u_i^{k+1}$ in equation (5) to $h_i^{k+1}$ in equation (6) and $h_i^{k+1}$ in equation (5) to $u_i^{k+1}$ in equation (6). Then we obtain :

$$h_i^{k+1} = h_i^k - a(u_{i+1}^k - u_{i-1}^k) + c(h_{i+1}^k + h_{i-1}^k)$$

$$u_i^{k+1} = du_i^k - b(h_{i+1}^k - h_{i-1}^k) + c(u_{i+1}^k - 2u_i^k + u_{i-1}^k)$$

(7)

with

$$a = \frac{D\Delta t}{\Delta x} (1 - C_f\Delta t), b = \frac{g\Delta t}{\Delta x}, c = \frac{Dg\Delta t^2}{2\Delta x^2}, d = (1 - 2C_f\Delta t)$$

Generate terms (7) for $i = 0,1,2,\ldots,N-1$. So if equations (7) is written in the form of state space system, we obtain

$$x(k+1) = Ax(k) + Bu(k)$$

(8)

where

$$x(k+1) = \begin{bmatrix} h_1^k \\ u_1^k \\ h_2^k \\ u_2^k \\ h_3^k \\ \vdots \\ h_{N-1}^k \\ u_{N-1}^k \end{bmatrix}, x(k) = \begin{bmatrix} h_1^k \\ u_1^k \\ h_2^k \\ u_2^k \\ h_3^k \\ \vdots \\ h_{N-1}^k \\ u_{N-1}^k \end{bmatrix}, u(k) = \begin{bmatrix} u_0^k \\ \vdots \\ u_{N-1}^k \end{bmatrix}$$

Then, we construct a measurement equation at time $k$ that is

$$y(k) =Cx(k) + Du(k)$$

(9)

Furthermore, a simulation is done to determine the water level of the river by using the parameter values as follows

$$D = 10 \ m, g = 9.8 \ \frac{m}{s^2}, C_f = 0.0002, x = 1, t = 0.2$$

So we obtained the matrix for the original system $(A, B, C, D)$ with the order $n = 20$ as follows
The requirement for model reduction is that the system must be asymptotically stable, controllable and observable. So it is necessary to analyze the original system \((A, B, C, D)\) above whether it is asymptotically stable, controllable and observable.

4. Model Reduction

Model Reduction is an attempt to replace large models or systems with simpler models without significant errors. Model reduction can be done with several methods. One of them is the method of Singular Perturbation Approximation. Model Reduction is done by establishing a balanced system first.

4.1. Balanced Systems

The balanced system \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) is a new system derived from the original system \((A, B, C, D)\) with controllability gramian \(\tilde{W}\) and observability gramian \(\tilde{M}\) which is the unique solution of the Lyapunov equations. Thus it satisfies \(\tilde{W} = \tilde{M} = \Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_n), \sigma_1 \geq \cdots \geq \sigma_r \geq \cdots \geq \sigma_n > 0\). With \(\sigma_i\) is the Hankel singular value of the system \((A, B, C, D)\) which can be defined as \(\sigma_i = \sqrt{\lambda_i(WM)}\), \(i = 1, ..., n\), where \(\lambda_i\) is the eigenvalues of \(WM\). The balanced system is obtained by transforming the original system of equations (8) and (9) to the matrix of transformation \(T\) which satisfies [6]:

\[
x_k = T\tilde{x}_k
\]  

\[
A = \begin{bmatrix}
0.02 & 0.49 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.49 & 0.49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.49 & 0.5 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.49 & 0.49 & 0 & 0.02 & -0.49 & 0.49 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.49 & 0.5 & 0.02 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.49 & 0.49 & 0 & 0.02 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.02 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.49 & 0.49 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.49 & 0.49 & 0.02 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.49 & 0.49 & 0.02 & 0.02
\end{bmatrix}_{20 \times 20}
\]

\[
B = \begin{bmatrix}
0.49 & 0.49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 0.49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.49 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.49 & 0.49
\end{bmatrix}_{4 \times 20}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}_{10 \times 20}
\]

\[
D = [0]
\]
Thus we obtained the matrix for the balanced system \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) by order \(n = 20\) with 
\[
\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = C, \quad \tilde{D} = D
\]
that is asymptotically stable, controllable and observable as follows
\[
\tilde{A} = \begin{bmatrix}
0.7942 & 2.0188e-15 & \cdots & \cdots & 5.7600e-15 & 0.1254 \\
-1.5623e-14 & 0.2265 & \cdots & \cdots & -0.0482 & 3.0281e-15 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
-9.1001e-16 & 0.0226 & \cdots & \cdots & -0.1770 & -5.0922e-15 \\
-0.0357 & -1.1540e-15 & \cdots & \cdots & -6.7856e-15 & -0.2743
\end{bmatrix}_{20 \times 20}
\]
\[
\tilde{B} = \begin{bmatrix}
0.4565 & -0.6739 & -0.4899 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0.0465 & 0.0508 \\
0.4616 & -0.6815 & -0.4955 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0.0428 & 0.0473 \\
0.4565 & 0.6739 & -0.4899 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -0.0465 & 0.0508 \\
-0.4616 & -0.6815 & 0.4955 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0.0428 & -0.0473
\end{bmatrix}^T_{4 \times 20}
\]
\[
\tilde{C} = \begin{bmatrix}
0.2084 & -0.3456 & \cdots & \cdots & -0.4641 & -0.4376 \\
0.2040 & -0.0857 & \cdots & \cdots & 0.0842 & 0.0738 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0.2040 & 0.0857 & \cdots & \cdots & -0.0842 & 0.0738 \\
0.2084 & 0.3456 & \cdots & \cdots & 0.4641 & -0.4376
\end{bmatrix}_{10 \times 20}
\]
\[
\tilde{D} = [0]
\]

4.2. Singular Perturbation Approximation Methods

Having obtained the balanced system \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) with the same controllability gramian \(\tilde{W}\) and observability gramian \(\tilde{M}\) that equals the diagonal matrix \(\Sigma\). Then the system \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) is partitioned according to \(\Sigma = diag(\Sigma_1, \Sigma_2)\), where \(\Sigma_1 = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)\) and \(\Sigma_2 = diag(\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n)\). Thus, the system realization \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) can be written as follows:

\[
\begin{bmatrix}
\tilde{x}_1(k+1) \\
\tilde{x}_2(k+1)
\end{bmatrix} =
\begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} \\
\tilde{A}_{21} & \tilde{A}_{22}
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1(k) \\
\tilde{x}_2(k)
\end{bmatrix} +
\begin{bmatrix}
\tilde{B}_1 \\
\tilde{B}_2
\end{bmatrix} u(k)
\]

\[
\begin{bmatrix}
\tilde{y}(k)
\end{bmatrix} =
\begin{bmatrix}
\tilde{C}_1 & \tilde{C}_2
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1(k) \\
\tilde{x}_2(k)
\end{bmatrix} + \tilde{D} u(k)
\]

with \(\tilde{x}_1(k) \in \mathbb{R}^r\) and \(\tilde{A}_{11} \in \mathbb{R}^{rxr}\) corresponds to gramian \(\Sigma_1\), and \(\tilde{x}_2(k) \in \mathbb{R}^{n-r}\) corresponds to \(\Sigma_2\).

The next step, we take \(\tilde{x}_2(k+1) = 0\) and we assume \(\tilde{A}_{22}\) is a nonsingular matrix. So that the reduced system of \(r\) corresponding to gramian \(\Sigma_1\) is as follows:

\[
\begin{bmatrix}
\tilde{x}_1(k+1) \\
\tilde{y}(k)
\end{bmatrix} =
\begin{bmatrix}
\tilde{A}_{11} & \tilde{B}_1 \\
\tilde{C}_1 & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1(k) \\
u(k)
\end{bmatrix}
\]

for \(k = 0, 1, 2, \cdots\), with \(\tilde{x}_1(k) \in \mathbb{R}^r\), \(u(k) \in \mathbb{R}^s\) and \(\tilde{y}(k) \in \mathbb{R}^t\) then we obtained
Thus we obtained a reduced system \((\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r)\) of size \(r\) and expressed in the form of:

\[
\begin{align*}
\tilde{x}_{r+1} &= \tilde{A}_r \tilde{x}_r + \tilde{B}_r \tilde{u}_k, \\
\tilde{y}_r &= \tilde{C}_r \tilde{x}_r + \tilde{D}_r \tilde{u}_k
\end{align*}
\]

From model reduction with the Singular Perturbation Approximation (SPA) method on the asymptotically stable, controllable and observable, \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\) systems produced by the reduced system \((\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r)\) with \(r < n\) order that is asymptotically stable.

For example, if we reduced the original system by order \(n = 20\) to order \(r = 8\), then we get the matrix for reduced system \((\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r)\) as follows:

\[
\begin{bmatrix}
-0.2848 & -1.4130e - 11 & \cdots & 1.1471e - 12 & 0.5900 \\
-1.3615e - 11 & 0.5768 & \cdots & -0.0568 & 1.0024e - 11 \\
0.9040 & 1.5027e - 11 & \cdots & 7.4797e - 14 & 0.1567 \\
1.2735e - 11 & -0.7229 & \cdots & -0.1769 & 1.2704e - 13 \\
2.6574e - 12 & -0.2175 & \cdots & -0.2041 & 6.1157e - 13 \\
-0.1784 & -2.2115e - 12 & \cdots & -2.7623e - 15 & -0.0259 \\
3.1440e - 13 & -0.0653 & \cdots & -0.3026 & 3.1992e - 15 \\
-0.1299 & -2.1658e - 12 & \cdots & -5.9488e - 15 & -0.2961
\end{bmatrix}_{8 \times 8}
\]

\[
\begin{bmatrix}
0.4471 & -0.6071 & 0.1357 & -0.3188 & -0.0694 & -0.0156 & -0.1725 & -0.0939 \\
0.4528 & -0.6135 & 0.1365 & -0.3232 & -0.0700 & -0.0159 & -0.1749 & -0.0952 \\
0.4528 & 0.6135 & 0.1365 & 0.3232 & 0.0700 & -0.0159 & 0.1749 & 0.0952 \\
0.3242 & -0.3205 & 0.1012 & -0.1717 & 0.0204 & -0.0271 & 0.1363 & -0.0615 \\
-0.0938 & -0.2004 & 0.3330 & 0.2593 & -0.0084 & 0.0198 & 0.2535 & -0.0407 \\
0.0355 & -0.0797 & -0.0269 & 0.1329 & 0.2710 & 0.3324 & 0.1249 & -0.0177 \\
-0.0719 & -0.0759 & -0.0179 & -0.0629 & 0.3043 & -0.0233 & 0.2433 & -0.2477 \\
0.0489 & -0.0775 & -0.0586 & -0.0250 & 0.1606 & -0.0013 & 0.5051 & -0.1272 \\
-0.0489 & -0.0775 & -0.0586 & -0.0250 & 0.1606 & -0.0013 & 0.5051 & 0.1272 \\
0.0719 & -0.0759 & -0.0179 & -0.0629 & 0.3043 & 0.0233 & 0.2433 & 0.2477 \\
-0.0355 & -0.0797 & -0.0269 & 0.1329 & 0.2710 & -0.3324 & 0.1249 & 0.0177 \\
0.0938 & -0.2004 & -0.3330 & 0.2593 & -0.0084 & -0.0198 & 0.2535 & 0.0407 \\
-0.3242 & -0.3025 & -0.1012 & -0.1717 & 0.0204 & 0.0271 & 0.1363 & 0.0615
\end{bmatrix}_{10 \times 8}
\]

\[
\begin{bmatrix}
-0.1468 & -0.1281 & -0.1010 & \cdots & 0.0544 & 0.0438 & 0.0460 \\
-0.1437 & -0.1293 & -0.1019 & \cdots & 0.0550 & 0.0443 & 0.0466 \\
0.0460 & 0.0438 & 0.0544 & \cdots & -0.1010 & -0.1281 & -0.1468 \\
-0.0466 & -0.0443 & -0.0550 & \cdots & 0.1019 & 0.1293 & 0.1437
\end{bmatrix}_{4 \times 10}
\]
5. Estimation

In this section, we discuss about estimation of the original system and reduced system. Then, we compare the estimation results of the state and computational time between the original system and reduced system.

5.1 Kalman Filter Algorithm

Kalman filter is one method to estimate state variable from stochastic dynamic system first introduced by Rudolf E. Kalman in 1960. Estimation using this method is done by predicting state variable based on system dynamics and measurement data [4]. In system modelling, there is no mathematical model of a perfect system. This can be caused by the noise factor that affects the system. Therefore, it is necessary to add stochastic factors to the deterministic system of equations (8) and (9) in the form of noise system and noise measurement, thus becoming the stochastic dynamic system as follows:

\[ x_{k+1} = A_kx_k + B_ku_k + G_kw_k \]
\[ z_k = H_kx_k + v_k \]  

Kalman Filter algorithm consists of 4 parts. The first and second sections provide the model system and measurement model as well as the initial value (initialization), while the third and fourth sections are the prediction and the correction phase.

a. System models and measurement models

\[ x_{k+1} = A_kx_k + B_ku_k + G_kw_k \]
\[ z_k = H_kx_k + v_k \]
\[ x_0 \sim (\bar{x}_0, P_{x_0}), \quad w_k = (0, Q_k), \quad v_k = (0, R_k) \]

b. Initialization

\[ \bar{x}_0 = \bar{x}_0, \quad P_0 = P_{x_0} \]

c. Time Update

Covariance error: \( P_{k+1}^- = A_kP_kA^T + G_kQ_kG^T \)
Estimation: \( \hat{x}_{k+1}^- = A_k\hat{x}_k + B_ku_k \)

d. Measurement Update

Kalman Gain: \( K_{k+1} = P_{k+1}^-H_{k+1}^{-1} (H_{k+1}P_{k+1}^-H_{k+1}H_{k+1}^T R_{k+1}^{-1})^{-1} \)
Covariance error: \( P_{k+1} = (I - K_{k+1}H_{k+1})P_{k+1}^- \)
Estimation: \( \hat{x}_{k+1} = \hat{x}_{k+1}^- + K_{k+1}(z_{k+1} - H_{k+1}\hat{x}_{k+1}^-) \)

5.2 Simulation

In this section, we will do the model reduction on the original system \((A, B, C, D)\) beginning with a balanced system \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\). Next, we focus on model reduction using the Singular Perturbation Approximation method to obtain a reduced system \((A_r, B_r, C_r, D_r)\). Based on the reduction procedure in the original system, it can be concluded that the system can be reduced from order 1 to 19 which satisfies \( \|G_s - G_r\|_{\infty} \leq 2(\sigma_{r+1} + \cdots + \sigma_n) \), with \( G_s \) and \( G_r \) respectively is a system transfer function \((A, B, C, D)\) and its reduced system.

Furthermore, estimation is done with the implementation of the kalman filter algorithm on the original system and reduced system from the order 8 to 19 using MATLAB software. From the simulation results, we get the graphic estimations in Figure 1 and 2 and the comparison of error values in Table 1.
Based on Figure 1, the graphic between the original system and the estimated value is almost the same so as to produce a small error. While in Figure 2, the graphic shown between the reduced system at the order of 8, 11, 14 and 16 with estimation result have slightly different value at some point so that it yield big error value. For more details about error values in each system can be seen from the following table:

**Table 1.** Error value by using Kalman Filter

| Experiment | Original | 8th Order | 9th Order | 10th Order | 11th Order | 12th Order | 13th Order | 14th Order | 15th Order | 16th Order | 17th Order | 18th Order | 19th Order |
|------------|----------|-----------|-----------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 1          | 1.63E-04 | 2.0945    | 1.8717    | 0.0199     | 1.0363     | 3.2058     | 0.0605     | 0.4596     | 7.37E-04   | 1.07E-05   | 0.5243     | 0.0154     | 0.1357     |
| 2          | 6.26E-06 | 23.7105   | 0.0183    | 0.0356     | 0.9372     | 0.2381     | 0.5023     | 0.3160     | 0.0095     | 0.0011     | 1.2355     | 0.0092     | 0.0610     |
| 3          | 1.99E-06 | 0.6903    | 0.0999    | 0.0452     | 0.0457     | 0.0048     | 0.6105     | 0.0242     | 0.1725     | 0.0401     | 0.9412     | 0.0014     | 0.0046     |
| 4          | 4.39E-06 | 10.3773   | 0.6125    | 0.0864     | 0.1522     | 0.0023     | 0.0651     | 0.0576     | 0.0737     | 0.0065     | 0.6803     | 0.0135     | 0.0058     |
| 5          | 2.86E-06 | 0.0335    | 0.0496    | 0.0192     | 0.3528     | 0.5858     | 0.0075     | 0.0933     | 0.2205     | 0.0010     | 1.0084     | 0.0519     | 0.0108     |
| **Mean Value** | **3.58E-05** | **7.3812** | **0.5304** | **0.0413** | **0.5048** | **0.8074** | **0.2492** | **0.1901** | **0.0954** | **0.0097** | **0.8779** | **0.0183** | **0.0436** |
Based on Table 1, the original system has the smallest average error compared to the others. Whereas for comparison in the reduced system, the order 16 has the smallest error value. For computational time comparison is shown in Table 2 as follows:

| Order of reduced systems | Computational time (Seconds) |
|--------------------------|------------------------------|
| Original                 | 0.01015                      |
| 8th order                | 0.01913                      |
| 9th order                | 0.00685                      |
| 10th order               | 0.00724                      |
| 11th order               | 0.00711                      |
| 12th order               | 0.00655                      |
| 13th order               | 0.00679                      |
| 14th order               | 0.00665                      |
| 15th order               | 0.00652                      |
| 16th order               | 0.00645                      |
| 17th order               | 0.00645                      |
| 18th order               | 0.00668                      |
| 19th order               | 0.00717                      |

According to Table 2, the reduced system has faster computation time than the original system and those are reduced system order 16 and 17.

6. Conclusions
In this work, we have studied estimation of the river flow models that have been reduced by the singular perturbation approximation method. Our approach is as follows. First, we discretized the differential equation using Lax Wendroff Scheme to obtain a discrete-time linear-time-invariant system for the original system. Then, we construct the balanced system. After that, we apply model reduction using singular perturbation approximation method. Next, we estimate the original system and reduced system using Kalman filter algorithm. Finally, we compare the results of the state and computational time between the original system and reduced system. According to our simulation results, the original system has the smallest estimation error value and the reduced system has the fastest computation time.

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