HIBI ALGEBRAS AND REPRESENTATION THEORY

SANGJIB KIM AND VICTOR PROTSAK

Abstract. This paper gives a survey on the relation between Hibi algebras and representation theory. The notion of Hodge algebras or algebras with straightening laws has been proved to be very useful to describe the structure of many important algebras in classical invariant theory and representation theory \cite{2,5,10,33}. In particular, a special type of such algebras introduced by Hibi \cite{12} provides a nice bridge between combinatorics and representation theory of classical groups. We will examine certain poset structures of Young tableaux and affine monoids, Hibi algebras in toric degenerations of flag varieties, and their relations to polynomial representations of the complex general linear group.

1. Young tableaux and Gelfand-Tsetlin poset

In this section, we will define some partially ordered sets and investigate their properties.

1.1. Poset of column tableaux. A Young diagram $\lambda$ is a collection of boxes arranged in left-justified rows with the row lengths in non-increasing order.

$$
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
$$

Writing $\lambda_i$ for the length of the $i$th row of $\lambda$ counting from top to bottom, we will identify $\lambda$ with a non-increasing sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \ldots) \text{ such that } \lambda_1 \geq \lambda_2 \geq \cdots \geq 0.$$ 

The transpose (or conjugate) $\lambda'$ of a Young diagram $\lambda$ is the Young diagram $(d_1, d_2, \ldots)$ where $d_j$ is the number of boxes in the $j$th column of $\lambda$ counting from left to right. The depth of $\lambda$ is the number of non-empty rows in $\lambda$ and will be denoted by $d(\lambda)$.

A Young tableau is a filling of the boxes of a Young diagram with positive integers. The content of a Young tableau $T$ is a sequence $(c_1, c_2, \ldots)$ where $c_i$ is the number of boxes containing $i$ in $T$. A Young tableau is called semistandard if its entries in each row weakly increase from left to right and its entries in each column strictly increase from top to bottom.

From now on, we fix a positive integer $n$ and then consider Young diagrams whose depths are not more than $n$ and Young tableaux whose entries are from $\{1, 2, \ldots, n\}$. For
\[(1^k) = (k)^t = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \end{array} \quad \text{and} \quad [i_1, i_2, \ldots, i_k] = \begin{array}{c} i_1 \\ i_2 \\ \vdots \\ i_k \\ \end{array} \]

Figure 1. A Young diagram with a single column having \(k\) boxes and a column tableau of depth \(k\).

Example, when \(n = 6\), the following is a semistandard Young tableau

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 3 & 3 & 5 & 6 & 6 \\
2 & 2 & 2 & 3 & 3 & 5 & 5 & 6 & 6 \\
3 & 3 & 3 & 5 & 5 & 6 & 6 \\
5 & 5 & 6 & 6 \\
\end{array}
\]

on a Young diagram \((12, 10, 6, 4, 0, 0)\) with content \((4, 5, 9, 0, 8, 6)\).

For notational convenience, for \(1 \leq k \leq n\), we let \((1^k)\) or \((k)^t\) denote a Young diagram having \(k\) boxes in a single column, and write \([i_1, i_2, ..., i_k]\) for a semistandard tableau on a Young diagram \((1^k)\) whose \(j\)th entry counting from top to bottom is \(i_j\) for \(1 \leq j \leq k\). We will call \([i_1, i_2, ..., i_k]\) a column tableau of depth \(k\). See Figure 1.

**Definition 1.1.** The poset of column tableaux is the set of column tableaux

\[L_n = \bigcup_{1 \leq k \leq n} \{[i_1, i_2, ..., i_k] : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}\]

with the following partial order. For \(I, J \in L_n\)
\nwe let \(I \geq_{\text{tab}} J\) if \(a \leq b\) and \(i_\ell \geq j_\ell\) for all \(1 \leq \ell \leq a\).

It is straightforward to check that the poset \((L_n, \geq_{\text{tab}})\) forms a distributive lattice \([10]\). For \(I = [i_1, i_2, ..., i_a]\) and \(J = [j_1, j_2, ..., j_b]\) in \(L_n\) with \(a \leq b\), their join and meet are

\[I \vee J = [x_1, ..., x_a]\quad \text{and}\quad I \wedge J = [y_1, ..., y_b]\]

respectively where

\[x_\ell = \max(i_\ell, j_\ell)\quad \text{for} \quad 1 \leq \ell \leq a,\quad y_\ell = \begin{cases} \min(i_\ell, j_\ell) & \text{for} \quad 1 \leq \ell \leq a, \\ j_\ell & \text{for} \quad a < \ell \leq b. \end{cases}\]

We will call \(L_n\) a distributive lattice of column tableaux. See Figure 2.

We remark that every multichain (i.e., linearly ordered multiset) of \(L_n\) can be identified with a semistandard Young tableau. For every multichain, by concatenating its elements in weakly increasing order, we obtain a semistandard tableau. Conversely, from the definition of semistandard Young tableaux, the columns of every semistandard Young tableau form a multichain of \(L_n\) with respect to the order \(\geq_{\text{tab}}\). See Figure 3.
1.2. GT poset and indicator functions. Usually, a Gelfand-Tsetlin (GT) pattern is defined as a triangular array of integers satisfying certain inequalities \cite{8, 31}. Here, we want to define it using a poset.

**Definition 1.2.**

1. The GT poset $\Gamma_n$ is the set
   \[ \Gamma_n = \left\{ z^{(i)}_j : 1 \leq j \leq i \leq n \right\} \]
   with the partial order $z^{(i)}_j \geq z^{(i)}_{j+1}$ for all $1 \leq j \leq i \leq n - 1$.

2. A GT pattern is an order-preserving map from $\Gamma_n$ to non-negative integers, that is, a map $f : \Gamma_n \rightarrow \mathbb{Z}_{\geq 0}$ such that
   \[ z^{(a)}_b \geq z^{(c)}_d \text{ in } \Gamma_n \text{ implies that } f(z^{(a)}_b) \geq f(z^{(c)}_d). \]

3. We let $S_n$ denote the set of all GT patterns.

We shall draw $\Gamma_n$ in the form of an inverted pyramid as in Figure 3. Then, for $f \in S_n$, by placing its values $f(z^{(i)}_j)$ at the positions of $z^{(i)}_j$ in $\Gamma_n$, we can identify $f$ with a triangular array of integers as GT patterns are usually defined. We can also identify $f$ with an integral point $(f(z^{(i)}_j))_{1 \leq j \leq i \leq n}$ in $\mathbb{R}^{n(n+1)/2}$ and then $S_n$ can be considered an integral lattice cone in $\mathbb{R}^{n(n+1)/2}$ \cite{16}.

Now let us focus on GT patterns $f \in S_n$ whose images are contained in $\{0, 1\}$, or equivalently, the points in the lattice cone $S_n \subseteq \mathbb{R}^{n(n+1)/2}$ whose coordinates are either 1 or 0. Since $f$ is order-preserving, its support

\[ \text{Supp}(f) = \{ x \in \Gamma_n : f(x) \neq 0 \} \]

is an order-increasing subset of $\Gamma_n$, i.e., for $x, y \in \Gamma_n$, if $x \in \text{Supp}(f)$ and $y \geq x$, then $y \in \text{Supp}(f)$. In fact, for every order-increasing subset $A$ of $\Gamma_n$, its indicator function

\[ \rho_{\Gamma_n}(A) = \sum_{x \in A} 1_{x} \]

is a GT pattern with $f(z^{(i)}_j) = \rho_{\Gamma_n}(\{z^{(i)}_j\})$.

**Figure 2.** The Hasse diagram of $L_4$. The elements decrease along the lines from left to right.

\[
\begin{array}{cccc}
[1] & [1, 2] & [1, 2, 3] \\
[4] & [2] & [1, 4] & [1, 2] & [1, 2, 3] \\
& [3] & [2, 4] & [1, 3] & [1, 2, 4] \\
& & [3, 4] & [2, 3] & [1, 3, 4] \\
& & & [2, 3, 4] & \\
\end{array}
\]

**Figure 3.** A multichain of $L_4$ and a semistandard Young tableau.
Figure 4. The Hasse diagram of the GT poset $\Gamma_4$. The elements decrease along the lines from left to right.

1. $A : \Gamma_n \to \{0, 1\}$ belongs to $S_n$ where
\[
A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \notin A.
\end{cases}
\]

Definition 1.3. The poset of indicator functions is the set
\[
\Lambda_n = \{ A \in S_n : A \text{ is a non-empty order-increasing subset of } \Gamma_n \}
\]
with the reverse inclusion order on the order-increasing subsets of $\Gamma_n$, that is,
\[
A \geq_{\text{ind}} B \text{ if and only if } A \subseteq B.
\]

Then, with the following join and meet
\[
1_A \lor 1_B = 1_{A \cup B} \quad \text{and} \quad 1_A \land 1_B = 1_{A \cap B}
\]
respectively, the poset $(\Lambda_n, \geq_{\text{ind}})$ is a distributive lattice.

Theorem 1.4. The poset $(L_n, \geq_{\text{tab}})$ of column tableaux is order-isomorphic to the poset $(\Lambda_n, \geq_{\text{ind}})$ of indicator functions.

Proof. Let $I = [i_1, i_2, \ldots, i_k] \in L_n$. For each $1 \leq a \leq n$, we let $\ell_a$ be the number of the entries $i_j$ in $I$ which are not more than $a$. Then, we consider an order-preserving map $f_I : \Gamma_n \to \{0, 1\}$ such that for each $a$, the number of $z_b^{(a)}$ for $1 \leq b \leq a$ such that $f_I(z_b^{(a)}) = 1$ is $\ell_a$. Since we know that
\[
f_I(z_1^{(a)}) \geq f_I(z_2^{(a)}) \geq \cdots \geq f_I(z_{a-1}^{(a)}) \geq f_I(z_a^{(a)})
\]
for $1 \leq a \leq n$, the numbers $\ell_a$ can completely determine $f_I$. Then, it is straightforward to check that the map sending $I$ to $f_I$ gives an order-isomorphism from $L_n$ to $\Lambda_n$. See [21 §3.3].

For $I \in L_n$, if we write $A$ for the support of the corresponding map $f_I \in \Lambda_n$, then $f_I$ is the indicator function of the order-increasing subset $A$ of $\Gamma_n$
\[
f_I = 1_A \quad \text{where } A = \text{Supp}(f_I).
\]

With the dual relation between order-decreasing subsets (also called order ideals) and order-increasing subsets, Theorem 1.4 is basically Birkhoff’s representation theorem for distributive lattices (also known as the fundamental theorem for finite distributive lattices [34 §3.4]) applied to the distributive lattice $L_n$. Recall that an element in a lattice is join-irreducible, if it is neither the least element of the lattice nor the join of any two smaller elements. Then, Theorem 1.4 tells us that the GT poset $\Gamma_n$ can be identified
with the set \( J(L_n) \) of join-irreducible elements of \( L_n \) with an additional greatest element \( z_1^{(n)} \). See Figure 5 and compare it with Figure 4.

With Birkhoff’s theorem, the greatest column tableau \([n]\) in \( L_n \) corresponds to the largest order ideal of \( J(L_n) \), which is itself. Then, by taking its complement in \( J(L_n) \), the order-increasing subset of \( J(L_n) \) corresponding to \([n]\) is the empty set. For us, the column tableau \([n]\) corresponds to the indicator function of the singleton set \( \{z_1^{(n)}\} = \Gamma_n \setminus J(L_n) \). By considering the GT poset \( \Gamma_n \) rather than \( J(L_n) \), we can reserve the indicator function of the empty set defined on \( \Gamma_n \) for the identity in the monoid \( S_n \) we will study in the following section.

**Example 1.5.** The following gives an order isomorphism between \( L_3 \) and \( \Lambda_3 \).

```
1 0 0  1 0 0  1 0 0
1 0 0  1 0 0  0 0 0

1 2 3  1 0 0  1 1 0
1 1 0  1 1 0  1 1 0
1 1 1  1 1 0  0 0 0
```

2. **Affine monoid of GT patterns and Hibi algebra**

In this section, we study the monoid structure of \( S_n \), the Hibi algebra on \( L_n \), and their properties.

2.1. **Affine monoid of GT patterns.** The sum of any two order-preserving maps is again order-preserving, and therefore the set \( S_n \) of all GT patterns can be considered an affine semigroup with respect to the usual addition of functions. The zero map is its identity.

**Definition 2.1.** The affine monoid of GT patterns is the set of all GT patterns

\[
S_n = \{ f : \Gamma_n \to \mathbb{Z}_{\geq 0} \mid f \text{ is order-preserving} \}.
\]
with the usual addition of functions.

For a semistandard Young tableau $T$, let $I_j$ be its $j$th column counting from left to right. Then they form a multichain of $L_n$ as we remarked at the end of §1.1

$$I_1 \leq_{tab} I_2 \leq_{tab} \cdots \leq_{tab} I_k.$$  

If we let $1_{A_j} = f_{I_j}$ be the indicator functions corresponding to the column tableaux $I_j$ given in Theorem 1.4, then they form a multichain of $\Lambda_n$ and their sum

$$1_{A_1} + 1_{A_2} + \cdots + 1_{A_k} \quad \text{with } A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k$$

is again an order-preserving map. This is an extension of the bijection in Theorem 1.4, and it gives a correspondence between the multichains of $L_n$ and the elements in $S_n$.

**Proposition 2.2.** There is a bijection between semistandard Young tableaux and GT patterns.

**Proof.** With the above discussion, it is enough to show that every GT pattern can be expressed as a sum of linearly ordered elements in $\Lambda_n$. Let $f \in S_n$ and its image be $\{i_1, \ldots, i_m\}$ with $0 \leq i_1 < i_2 < \cdots < i_m$. If we write $A_k$ for the inverse image of $\{y \in \mathbb{Z} : y \geq i_k\}$ under $f$, then they are order-increasing subsets of $\Gamma_n$ and satisfy $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_m$. Therefore, their indicator functions $1_{A_k}$ form a multichain of $\Lambda_n$. Now $f$ can be expressed as

$$(2.1) \quad f = c_1 1_{A_1} + c_2 1_{A_2} + \cdots + c_m 1_{A_m}$$

where $c_1 = i_1$ and $c_k = i_k - i_{k-1}$ for $2 \leq k \leq m$. See Example 2.3. \qed

We remark that there is a well-known bijection between semistandard Young tableaux and GT patterns which does not refer to these poset structures. For a semistandard Young tableau $T$, we define $f_T : \Gamma_n \to \mathbb{Z}$ by

$$(2.2) \quad f_T(z_j^{(i)}) = \text{the number of entries in the } j\text{th row of } T \text{ which are less than or equal to } i$$

for all $1 \leq j \leq i \leq n$. Then, the correspondence $T \mapsto f_T$ gives a bijection between semistandard Young tableaux and GT patterns. This bijection is the same as the one given in Proposition 2.2 in terms of multichains in the posets $L_n$ and $\Lambda_n$.

**Example 2.3.** Using the formula (2.1), we can express

\[
\begin{array}{cccccccccccc}
10 & 7 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 \\
7 & 7 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \\
7 & 3 & 2 & 2 & 1 & 1 & 1 & 0 \\
3 & 2 & 2 & 2 & 1 & 1 & 0 \\
3 & 2 & 2 & 2 & 1 & 1 & 0 \\
\end{array}
\]
Then, this GT pattern corresponds to the following multichain of $L_4$

\[
1 \leq_{\text{tab}} 2 \leq_{\text{tab}} 3 \leq_{\text{tab}} 4 \leq_{\text{tab}} 2 \leq_{\text{tab}} 3 \leq_{\text{tab}} 2 \leq_{\text{tab}} 4 \leq_{\text{tab}} 4
\]

or equivalently the semistandard Young tableau

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 \\
2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\
3 & 3 & 4 & & & & & \\
4 & 4 & & & & & & \\
\end{array}
\]

2.2. Hibi algebra and affine monoid algebra. In [12], Hibi introduced an algebra $H_L$ attached to a finite lattice $L$, now called the Hibi algebra on $L$. It is the quotient of the polynomial ring with variables $x_{\alpha}$ indexed by $\alpha \in L$ by the ideal $I_L$ generated by the binomials $x_{\alpha}x_{\beta} - x_{\alpha \wedge \beta}x_{\alpha \vee \beta}$ for all incomparable pairs $(\alpha, \beta)$ in $L$.

Among many others, it is shown that if $L$ is a distributive lattice then $H_L$ is an algebra with straightening laws on $L$ and therefore all the monomials which are not divisible by $x_{\alpha}x_{\beta}$ for any incomparable pairs $(\alpha, \beta)$ in $L$ form a $C$-basis for $H_L$. We would like to study the Hibi algebra on the distributive lattice $L_n$ of column tableaux

$$H_n = \mathbb{C}[x_I : I \in L_n]/I_{L_n}.$$  

On the other hand, we have the affine monoid algebra $\mathbb{C}[S_n]$ of the monoid $S_n$ of GT patterns. Note that the following identities hold

$$1_A + 1_B = 1_{A \cup B} + 1_{A \cap B}$$

for all pairs $(A, B)$ of incomparable order-increasing subsets of $\Gamma_n$, and therefore, with $(1.1)$, the map

$$\psi : H_n \rightarrow \mathbb{C}[S_n]$$

sending $x_I$ to $f_I$ for all $I \in L_n$ is well-defined. Indeed it gives an algebra isomorphism. See [12, §2]. See also [16, §3.3] and [21, §2.3].

**Proposition 2.4.** The affine monoid algebra $\mathbb{C}[S_n]$ of $S_n$ is isomorphic to the Hibi algebra on $L_n$.

For a Young diagram $\lambda = (\lambda_1, ..., \lambda_n)$, let $\mathbb{C}[S_n]_{\lambda}$ denote the set of all formal linear combinations of GT patterns $f$ such that

$$f(z_j^{(n)}) = \lambda_j \quad \text{for all} \ 1 \leq j \leq n.$$  

Then, the affine monoid algebra $\mathbb{C}[S_n]$ is multigraded by Young diagrams

$$\mathbb{C}[S_n] = \bigoplus_{\lambda} \mathbb{C}[S_n]_{\lambda}$$

with $\mathbb{C}[S_n]_{\lambda} \cdot \mathbb{C}[S_n]_{\mu} \subseteq \mathbb{C}[S_n]_{\lambda + \mu}$. Similarly, $H_n$ is multigraded by Young diagrams

$$H_n = \bigoplus_{\lambda} (H_n)_{\lambda}$$

and in this case, once monomials $\prod_{j=1}^{r} x_{I_j}$ are identified with multisubsets $\{I_1, ..., I_r\}$ of $L_n$, by Proposition 2.2, the space $(H_n)_{\lambda}$ consists of all formal linear combinations of semistandard Young tableaux on the Young diagram $\lambda$. 
We remark that for each Young diagram \( \lambda \), there is a finite dimensional irreducible representation \( V_\lambda^n \) of the general linear group \( GL_n(\mathbb{C}) \). Such a representation has a \( \mathbb{C} \)-basis which can be labeled by GT patterns satisfying (2.3), or equivalently, by semistandard Young tableaux on the Young diagram \( \lambda \). Therefore, we can think of \( \mathbb{C}[S_n]_\lambda \) and \( (H_n)_\lambda \) as combinatorial models of the representation space \( V_\lambda^n \). We will be more precise about it in the next section.

3. Flag algebra and representations of \( GL_n \)

Let us consider the complex general linear group \( GL_n = GL_n(\mathbb{C}) \), that is, the group of complex \( n \times n \) invertible matrices with matrix multiplication. We will construct algebras carrying polynomial representations of \( GL_n \).

3.1. Representations of \( GL_n \). Let us begin with some basic concepts of representation theory. For more details, we refer the reader to [11] especially, §1.5, §2.1, and §3.2.

A representation of a group \( G \) on a vector space \( V \) (over \( \mathbb{C} \) in this paper) is a group homomorphism \( \phi \) from \( G \) to the group of all automorphisms of \( V \). Then, \( G \) acts on \( V \) by

\[
g \cdot v = \phi(g)v \quad \text{for } g \in G \text{ and } v \in V.
\]

When such an action is understood, we often say \( V \) is a representation of \( G \). A representation \( V \) is irreducible if it is a nonzero representation that has no proper subrepresentation closed under the action of \( G \). We will focus on the polynomial representation of \( GL_n \), which means that the matrix coefficients of \( \phi(g) \) of a typical element \( g = (g_{ij}) \in GL_n \) are polynomials generated by \( g_{ij} \).

We let \( A_n \) be the maximal torus of \( GL_n \) consisting of all invertible diagonal matrices. A character of \( A_n \) is a regular homomorphism

\[
\psi_n^\kappa : A_n \rightarrow \mathbb{C}^\times \text{ defined by } \psi_n^\kappa(a) = a_1^{\kappa_1}a_2^{\kappa_2}\cdots a_n^{\kappa_n}
\]

for some \( \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{Z}^n \). Here, \( a = \text{diag}(a_{11}, \ldots, a_{nn}) \in A_n \) is the diagonal matrix with diagonal entries \( a_{11}, a_{22}, \ldots, a_{nn} \). We call \( \psi_n^\kappa \) a polynomial dominant character, if \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \geq 0 \). Note that the set of all polynomial dominant characters form a semigroup

\[
\hat{A}_n^+ = \{ \psi_n^\kappa : \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \geq 0 \} \quad \text{with} \quad \psi_n^\alpha \cdot \psi_n^\beta = \psi_n^{\alpha + \beta}.
\]

Every polynomial representation of \( GL_n \) has a \( \mathbb{C} \)-basis, called weight basis, consisting of vectors \( v \) satisfying

\[
a \cdot v = \psi_n^\kappa(a)v \quad \text{for all } a \in A_n
\]

for some \( \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{Z}^n \) such that \( \kappa_i \geq 0 \) for all \( i \). Such a vector \( v \) is called a weight vector of weight \( \psi_n^\kappa \).

Now we let \( U_n \) be the maximal unipotent subgroup of \( GL_n \) consisting of all upper triangular matrices with 1’s on the diagonal. Let us write \( V^{U_n} \) for the subspace of a polynomial representation \( V \) of \( GL_n \) consisting of all vectors invariant under the action of \( U_n \). Since \( A_n \) normalizes \( U_n \), the action of \( A_n \) will leave \( V^{U_n} \) invariant. Moreover, if \( V \) is irreducible, then Theorem of the Highest Weight shows that \( V^{U_n} \) is a one-dimensional subspace of \( V \) spanned by a weight vector of weight \( \psi_n^\kappa \in \hat{A}_n^+ \) and determines the representation \( V \) up to equivalence. In this case, we call a vector \( v \in V^{U_n} \) a highest weight vector for \( V \) and \( \psi_n^\kappa \) the highest weight of \( V \). Characters of \( A_n \) occurring as the highest
weights of irreducible polynomial representations of $GL_n$ are exactly polynomial dominant characters.

Therefore, we can associate each irreducible polynomial representation of $GL_n$ with a polynomial dominant character $\psi^\kappa_n$ or equivalently a sequence $\kappa$ of non-increasing non-negative integers. We can further identify such a sequence $\kappa$ with a Young diagram as given in §11 and this establishes a one-to-one correspondence between irreducible polynomial representations of $GL_n$ and Young diagrams with not more than $n$ rows. From now on, for a Young diagram $\lambda$ with $d(\lambda) \leq n$, we let $V^\lambda_n$ denote the irreducible polynomial representation with highest weight $\psi^\kappa_n$.

3.2. Weight basis and GT patterns. There is a nice labeling system for weight basis elements for $V^\lambda_n$. First, we recall a simple branching rule. See [11 §8].

Lemma 3.1. For a Young diagrams $\mu = (\mu_1, ..., \mu_k)$, the irreducible representation $V^\mu_k$ of $GL_k$, under the restriction of $GL_k$ down to its block diagonal subgroup $GL_{k-1} \times GL_1$, decomposes in a multiplicity-free fashion

$$V^\mu_k = \bigoplus_\nu V^\nu_{k-1} \otimes V^{(r)}_1$$

where the sum is over Young diagrams $\nu = (\nu_1, ..., \nu_{k-1})$ interlacing $\mu$, i.e.

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \cdots \geq \nu_{k-1} \geq \mu_k$$

and $r = \sum_{j=1}^k \mu_j - \sum_{j=1}^{k-1} \nu_j$.

For each GT pattern $f \in S_n$, let us write

$$\lambda[k] = (f(z_1^{(k)}), f(z_2^{(k)}), ..., f(z_i^{(k)}))$$

for $1 \leq k \leq n$ and $\lambda = \lambda[n]$. Then, $f$ can encode the successive applications of Lemma 3.1 for $k = n, n-1, ..., 2$. Since every irreducible polynomial representation of $GL_1$ is one-dimensional, we can find a vector $v_f$ in the chain of spaces

$$V^{\lambda[n]}_n \supset \left( V^{\lambda[n-1]}_{n-1} \otimes V^{(\kappa_1)}_1 \right) \supset \left( V^{\lambda[n-2]}_{n-2} \otimes V^{(\kappa_2)}_1 \otimes V^{(\kappa_1)}_1 \right) \supset \cdots \supset \left( V^{(\kappa_1)}_1 \otimes \cdots \otimes V^{(\kappa_1)}_1 \right)$$

where

$$\kappa_1 = \sum_{j=1}^i f(z_j^{(i)}) - \sum_{j=1}^{i-1} f(z_j^{(i-1)})$$

for $2 \leq i \leq n$ and $\kappa_1 = f(z_1^{(1)})$.

and therefore $(\kappa_1) = \lambda[1]$. The vector $v_f$ is stable under the action of $A_n \cong GL_1 \times \cdots \times GL_1$ ($n$ times) with weight $\psi^\kappa_n$ where $\kappa = (\kappa_1, ..., \kappa_n)$.

This gives a one-to-one correspondence between the set of GT patterns satisfying (2.4) and a weight basis for $V^\lambda_n$. See [8 §31] for more details. With the correspondence (2.2), we can also label weight basis elements with semistandard Young tableaux. In this case, semistandard Young tableaux on a Young diagram $\lambda$ with content $\kappa$ corresponds to weight basis elements in $V^\lambda_n$ with weight $\psi^\kappa_n$. We refer the reader to [11 §8.1].
3.3. Flag algebra for $GL_n$. To construct an algebra carrying irreducible representations of $GL_n$, we recall the $GL_n$-$GL_m$ duality (see [11, §9.2] and [15]). Let us write $\mathbb{C}[M_{n,m}]$ for the ring of polynomials on the space $M_{n,m}$ of $n \times m$ complex matrices. We use the coordinates $x_{ab}$ to write a typical element $X \in M_{n,m}$

$$X = \begin{bmatrix}
    x_{11} & x_{12} & \cdots & x_{1m} \\
    x_{21} & x_{22} & \cdots & x_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n1} & x_{n2} & \cdots & x_{nm}
\end{bmatrix}$$

and let the group $GL_n \times GL_m$ act on $h \in \mathbb{C}[M_{n,m}]$ by

$$( (g_1, g_2) \cdot h)(X) = h( g_1^t X g_2)$$

for $(g_1, g_2) \in GL_n \times GL_m$ and $X \in M_{n,m}$. Then, as a $GL_n \times GL_m$ representation, the algebra $\mathbb{C}[M_{n,m}]$ decomposes as

$$\mathbb{C}[M_{n,m}] \cong \bigoplus_{\lambda} V^\lambda_n \otimes V^\lambda_m$$

where the summation runs over Young diagrams $\lambda$ with not more than $\min(n, m)$ rows.

Now we let $n \geq m$ and consider the subring $\mathcal{R}_{n,m}$ of $\mathbb{C}[M_{n,m}]$ consisting of all polynomials invariant under the action of $1 \times U_m$,

$$\mathcal{R}_{n,m} = \{ h \in \mathbb{C}[M_{n,m}] : h(Xu) = h(X) \text{ for all } u \in U_m \} \cong \bigoplus_{d(\lambda) \leq m} V^\lambda_n \otimes (V^\lambda_m)^{U_m}.$$

We will call $\mathcal{R}_{n,m}$ the *flag algebra* for $GL_n$. Since $V^\lambda_m$ is irreducible, $(V^\lambda_m)^{U_m}$ is one-dimensional. Therefore, $\mathcal{R}_{n,m}$ contains exactly one copy of every polynomial representation $V^\lambda_n$ of $GL_n$ for $\lambda$ with $d(\lambda) \leq m$.

Let us write $\mathcal{R}^\lambda_{n,m}$ for the space of weight vectors in $\mathcal{R}_{n,m}$ with weight $\psi^\lambda_n$ under the right action of $A_m$. Then we obtain the graded algebra structure of $\mathcal{R}_{n,m}$ with respect to the semigroup $\hat{A}_m^+$

$$\mathcal{R}_{n,m} = \bigoplus_{\psi_n^\lambda \in \hat{A}_m^+} \mathcal{R}^\lambda_{n,m}$$

where $\mathcal{R}^\lambda_{n,m} \cong V^\lambda_n$ as a representation of $GL_n$ and $\mathcal{R}^\lambda_{n,m} \cdot \mathcal{R}^\mu_{n,m} \subseteq \mathcal{R}^{\lambda+\mu}_{n,m}$.

3.4. Standard monomial basis. Let us review a presentation of the flag algebra $\mathcal{R}_{n,m}$. We are particularly interested in weight bases for the individual homogeneous spaces $\mathcal{R}^\lambda_{n,m}$ of $\mathcal{R}_{n,m}$ under the left action of $A_n$. Let us consider the subposet $L_{n,m}$ of $L_n$ consisting of all column tableaux of depth at most $m$

$$L_{n,m} = \{ I \in L_n : \text{the depth of } I \text{ is not more than } m \}.$$

For a column tableau $I = [i_1, i_2, \ldots, i_k] \in L_{n,m}$, we define a function $\delta_I$ on $M_{n,m}$ by the determinant of the submatrix of $X = (x_{ab}) \in M_{n,m}$ obtained by selecting the rows
$i_1, i_2, \ldots, i_k$ and columns $1, 2, \ldots, k$ of $X$

\[
\delta_I(X) = \det \begin{bmatrix}
x_{i_11} & x_{i_12} & \cdots & x_{i_1k} \\
x_{i_21} & x_{i_22} & \cdots & x_{i_2k} \\
\vdots & \vdots & \ddots & \vdots \\
x_{i_k1} & x_{i_k2} & \cdots & x_{i_kk}
\end{bmatrix}.
\] (3.1)

It is easy to check that $\delta_I \in \mathbb{C}[M_{n,m}]$ are invariant under the action of $1 \times U_m$ and therefore the product $\prod_j \delta_{I_j}$ of any finite number of such elements belong to $R_{n,m}$.

**Definition 3.2.** Let $\{I_1, \ldots, I_r\}$ be a multisubset of $L_{n,m}$ such that the depth of $I_j$ is $d_j$ and $d_1 \geq d_2 \geq \cdots \geq d_r$. Then, the initial monomial in $\lambda$ $\Delta$ in $R_{n,m}$ in lexicographic order $\geq$ it is easy to check that

\[
(3.1)
\]

Let $\Delta = \prod_j \delta_{I_j}$ is called a standard monomial if it is not divisible by $\delta_I \delta_J$ for any incomparable pairs $(I, J)$ in $L_{n,m}$. Therefore, its indices form a multi-chain of the poset $L_{n,m}$ and we write $\Delta_T$ for $\prod_{j=1}^r \delta_{I_j}$ where $T = (I_1 \leq_{tab} I_2 \leq_{tab} \cdots \leq_{tab} I_r)$.

Note that every product $\Delta = \prod_{j=1}^r \delta_{I_j}$ is a weight vector under the right action of $A_m$. The weight of $\Delta$ is $\psi_{\lambda}^m$ (therefore $\Delta \in R^\lambda_{n,m}$) if and only if the shape of $\Delta$ is $\lambda$. This shows that the space $R^\lambda_{n,m}$ is spanned by the products $\Delta$ whose shapes are $\lambda$.

**Theorem 3.3.** For each Young diagram $\lambda$ with $d(\lambda) \leq m$, standard monomials of shape $\lambda$ form a $\mathbb{C}$-basis for the homogeneous component $R^\lambda_{n,m}$ of $R_{n,m}$.

**Proof.** Let us give a sketch of a proof. For more details or different proofs, we refer the reader to, for example, [2,5,7,10,30,33]. We can begin with a determinantal identity: for each incomparable pair $(I, J)$ in $L_{n,m}$, we have

\[
\delta_I \delta_J = \delta_{I \wedge J} \delta_{I \vee J} + \sum_{E,F} c_{E,F} \delta_E \delta_F
\]

where $E \leq_{tab} (I \wedge J) \leq_{tab} (I \vee J) \leq_{tab} F$. By applying it to a non-standard monomial $\Delta$ in $R^\lambda_{n,m}$ as many as possible, we can express $\Delta$ as a linear combination of standard monomials of shape $\lambda$. Therefore, standard monomials of shape $\lambda$ span the space $R^\lambda_{n,m}$.

To show that they are linearly independent, we can use a monomial order on the set of all monomials $\prod_{ij} x_{ij}^{r_{ij}}$ in the polynomial ring $\mathbb{C}[M_{n,m}]$. Let us consider the graded lexicographic order $\geq_{glex}$ with respect to the following order on the variables

\[
x_{ab} > x_{cd} \quad \text{if } b < d; \text{ or } b = d \text{ and } a < c.
\]

Then, the initial monomial $in(\delta_I)$ of $\delta_I$, that is, the monomial appearing in the polynomial $\delta_I$ which is greater than all other monomials appearing in $\delta_I$ with respect to $\geq_{glex}$, is the product of the diagonal entries in $\delta_I$(3.1)

\[
in(\delta_I) = x_{i_11} x_{i_22} \cdots x_{i_kk}.
\]

From $in(\prod_j \delta_{I_j}) = \prod_j in(\delta_{I_j})$, one can easily compute the initial monomials of standard monomials and show that standard monomials of shape $\lambda$ have distinct initial monomials with respect to $\geq_{glex}$. Therefore, they are linearly independent. $\square$
We note that standard monomials are stable under the left action of $A_n$, and therefore standard monomials of shape $\lambda$ form a weight basis for $\mathcal{R}_{\lambda}^{n,m} \cong V_{\lambda}^n$. By identifying multichains in $L_{n,m}$ with semistandard Young tableaux, we obtain the following result.

**Corollary 3.4.** For a Young diagram $\lambda$ with $d(\lambda) \leq n$, the dimension of the representation $V_{\lambda}^n$ is equal to the number of semistandard Young tableaux on the Young diagram $\lambda$ with entries from $\{1, 2, \ldots, n\}$.

3.5. **Initial algebra and toric degeneration.** Now let us consider the initial algebra of the flag algebra with respect to the monomial order $\geq_{glex}$ in $\mathcal{R}_{n,m} = \{f \in \mathcal{R}_{n,m} : f \in \mathcal{R}_{n,m}\}$.

**Theorem 3.5.** There is a flat one-parameter family of algebras whose general fiber is isomorphic to $\mathcal{R}_{n,m}$ and special fiber is isomorphic to the initial algebra $\mathcal{R}_{n,m}$.

**Proof.** From Theorem 3.3, every element $f \in \mathcal{R}_{n,m}$ can be uniquely expressed as a linear combination of standard monomials

$$f = c_1 \Delta_1 + c_2 \Delta_2 + \cdots + c_k \Delta_k.$$  

Since standard monomials have distinct initial monomials, $in(f) = in(\Delta_i)$ for some $i$. Also, $\Delta_i$ is the product of some $\delta_{I_j}$ and therefore its initial monomial is the product of $in(\delta_{I_j})$. This shows that the initial algebra $in(\mathcal{R}_{n,m})$ is generated by $in(\delta_{I_j})$ for $I_j \in L_{n,m}$ and that the set $\{\delta_{I_j} \in \mathcal{R}_{n,m} : I_j \in L_{n,m}\}$ forms a finite SAGBI basis for the algebra $\mathcal{R}_{n,m}$. This guarantees that there is a flat degeneration from the flag algebra $\mathcal{R}_{n,m}$ to its initial algebra $in(\mathcal{R}_{n,m})$. See [3, 9, 29, 30].

To investigate the structure of the initial algebra $in(\mathcal{R}_{n,m})$, we restrict the bijection from $L_n$ to $\Lambda_n$ given in Theorem 1.4 to $L_{n,m}$. For $I \in L_{n,m}$, since the depth of $I$ is not more than $m$, we have $f_I(z_{j}^{(n)}) = 0$ for all $m + 1 \leq j \leq n$. By the order structure of $\Gamma_n$, this condition forces $f_I(z_{j}^{(i)}) = 0$ for all $j \geq m + 1$. We define the smallest subposet of $\Gamma_n$ containing $Supp(f_I)$ for all $I \in L_{n,m}$

$$\Gamma_{n,m} = \{z_{j}^{(i)} \in \Gamma_n : j \leq m \}.$$  

See Figure 6. We write $S_{n,m}$ for the submonoid of $S_n$ consisting of all order-preserving maps in $S_n$ whose supports are in $\Gamma_{n,m}$ and let $\Lambda_{n,m} = \Lambda_n \cap S_{n,m}$.

![Figure 6. The Hasse diagram of the poset $\Gamma_{5,3}$. The elements decrease along the lines from left to right.](image-url)
Proposition 3.6. The initial algebra $in(R_{n,m})$ of the flag algebra $R_{n,m}$ is isomorphic to the affine monoid algebra $\mathbb{C}[S_{n,m}]$ of $S_{n,m}$.

Proof. With the correspondence between $L_{n,m}$ and $\Lambda_{n,m}$, we define a map $\phi: in(R_{n,m}) \to \mathbb{C}[S_{n,m}]$ sending $in(\delta_I)$ to $f_I$ for $I \in L_{n,m}$. For $I = [i_1, ..., i_a]$ and $J = [j_1, ..., j_b]$ with $a \leq b \leq m$, we have

$$in(\delta_I)in(\delta_J) = \prod_{k=1}^{a} (x_{i_k,k}x_{j_k,k}) \times \prod_{k=a+1}^{b} x_{i_k,k} = in(\delta_{I\cup J})in(\delta_{I\wedge J}).$$

With (2.3), $\phi$ is well-defined and it can be extended to the initial monomials of standard monomials to give a semigroup isomorphism between the semigroup of the initial monomials of all $h \in R_{n,m}$ and $S_{n,m}$. □

With Proposition 2.4, this shows that the initial algebra $in(R_{n,m})$ has the structure of the Hibi algebra on $L_{n,m}$. We also remark that its spectrum $Spec(in(R_{n,m}))$ can be understood as an affine toric variety associated with the lattice cone $S_{n,m}$ of GT patterns defined on $\Gamma_{n,m}$.

4. More subposets of $L_n$ and $\Gamma_n$

There are some subposets of $L_{n,m}$ and $\Gamma_{n,m}$ whose associated Hibi algebras are closely related to important questions in invariant theory and representation theory of classical groups. In this section, we list some of them and relevant works.

4.1. Grassmannians. For $m \leq n$, let us consider the subposet $G_{n,m}$ of $L_{n,m}$ consisting of all column tableaux of depth $m$

$$G_{n,m} = \{ I \in L_{n,m} : \text{the depth of } I \text{ is } m \}.$$  

Using the argument in §1.2 we can find its associated GT poset. See [21, §3] and Figure 7. The multichains of $G_{n,m}$, the corresponding GT patterns, and the Hibi algebra attached to them can be used to describe the Grassmannian variety of $m$ dimensional subspaces of $\mathbb{C}^n$, a ring of polynomials in $\mathbb{C}[M_{n,m}]$ invariant under the right action of the special linear group $SL_m(\mathbb{C})$, and finite dimensional representations of the general linear group $GL_n(\mathbb{C})$ labeled by rectangular Young diagrams of depth $m$. See, for example, [10, 11, 36]. This poset also has an interesting connection with double tableaux or pairs of Young tableaux. See [22].

4.2. Symplectic groups. For $n = 2m$, let us consider the subposet of $L_{n,m}$

$$P_n = \{ I \in L_{n,m} : I \succeq_{tab} [1, 3, 5, ..., 2m - 1] \}.$$  

We can find its associated GT poset using the argument in §1.2. See [21] and Figure 8. The multichains of $P_n$ and the GT patterns corresponding to them can be used to label weight basis elements for the rational representations of the symplectic group $Sp_n(\mathbb{C})$. See, for example, [1, 4, 21, 28, 31, 32].
4.3. Branching rules. For $m \leq n$ and $k < n$, let us consider the subposet $B_{n,m,k}$ of $L_{n,m}$ consisting of all column tableaux of the forms

$$\begin{align*}
&[1, 2, ..., p], \quad [i_1, i_2, ..., i_q], \quad [1, 2, ..., r, j_1, j_2, ..., j_s] \\
&\text{where } 1 \leq p, r \leq \min(k, m), \quad 1 \leq q, s \leq \min(n - k, m), \quad 1 \leq r + s \leq m, \quad \text{and } k + 1 \leq i_c, j_d \leq n. 
\end{align*}$$

The GT poset associated with $B_{n,m,k}$ can be computed as in §1.2. See Figure 9, Figure 10, and [23].

For each semistandard tableau $T$ obtained from a multichain of $B_{n,m,k}$, by erasing the entry $i$ in the $i$th row of $T$ for $1 \leq i \leq k$ and replacing the entry $j$ in $T$ with $j - k$ for all $j \geq k + 1$, we can realize $T$ as a semistandard tableau on a skew Young diagram $\lambda/\mu$ with content $\nu = (\nu_1, ..., \nu_{n-k})$. Here, the inner diagram is $\mu = (\mu_1, \mu_2, ...)$ where $\mu_i$ is the number of all boxes in the $i$th row of $T$ containing $i$ for $1 \leq i \leq k$, and $\nu_j$ is the number of boxes in $T$ containing $j + k$ for $1 \leq j \leq n - k$.

For example, with $n = 10$, $m = 5$, and $k = 4$, a semistandard Young tableau $T$ on a skew Young diagram $(12, 10, 6, 4, 0)/(8, 5, 3, 0)$ with content $(5, 2, 3, 2, 0)$ where

$$T = \begin{array} {cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 5 & 5 \\
2 & 2 & 2 & 2 & 5 & 7 & 8 & 9 \\
3 & 3 & 3 & 6 & 7 & 9 \\
5 & 5 & 6 & 8 \\
\end{array} \quad \leftrightarrow \quad T' = \begin{array} {cccc}
1 & 1 & 3 & 4 \\
1 & 3 & 4 & 5 \\
2 & 3 & 5 \\
1 & 1 & 2 & 4 \\
\end{array}.$$  

Then, the multichains of $B_{n,m,k}$, the corresponding GT patterns, and the Hibi algebra attached to them can be used to describe branching rules for some pairs $(G, H)$ of classical groups, that is, how a representation of $G$ decomposes into irreducible representations of a subgroup $H$ of $G$. See [23, 26, 31, 32].

4.4. Tensor product of representations. The tensor product decomposition problem to determine how tensor products of group representations decomposes is an important problem in representation theory with many applications. Recently, Howe and his collaborators have shown that answers to many of these questions can be given nicely in terms of the Hibi algebras associated with some subposets of $\Gamma_{n,m}$ and their variations. We refer the interested reader to [17, 20, 24, 25, 27, 37].


Figure 9. The subposet of $\Gamma_{8,3}$ associated with $B_{8,3,5}$.

Figure 10. The subposet of $\Gamma_{5,3}$ associated with $B_{5,3,2}$.

Acknowledgment. Parts of this article were presented at The Prospects for Commutative Algebra, Osaka, Japan, July 2017. We express our sincere gratitude to the organizers for the wonderful and stimulating conference.

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E-mail address: sk23@korea.ac.kr

Department of Mathematics, Korea University, Seoul 02841, South Korea

E-mail address: vp35@cornell.edu

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA