Abstract

$\mathcal{N}$-fold supersymmetry is an extension of the ordinary supersymmetry in one-dimensional quantum mechanics. One of its major property is quasi-solvability, which means that energy eigenvalues can be obtained for a portion of the spectra. We show that recently found Type A $\mathcal{N}$-fold supersymmetry can be constructed by using $\mathfrak{sl}(2)$ algebra, which provides a basis for the quasi-solvability. By this construction we find a condition for the Type A $\mathcal{N}$-fold supersymmetry which is less restrictive than the condition known previously. Several explicitly known models are also examined in the light of this construction.
1 Introduction

One novel feature of supersymmetry is its nonrenormalization theorem [1, 2]. The same is true [3] for higher derivative extensions of supersymmetry [4]-[17]. In fact, this observation was crucial for identification of a simple form of \( \mathcal{N} \)-fold supersymmetry, whose supercharges of \( \mathcal{N} \)-fold supersymmetry are \( \mathcal{N} \)-th order polynomials of the momentum. Namely, vanishing of the leading Borel singularity of the perturbative series in a quartic potential model was found [18] by the use of the valley method [19]-[26] and this motivated a search for supersymmetry or its extension, which lead to “\( \mathcal{N} \)-fold supersymmetry” in Ref.[3]. Later it was extended to a periodic and exponential potentials in Ref.[27] and to a sextic potential in Ref.[28].

The (extended) nonrenormalization theorem, which applies to all of these models, states that the perturbative corrections to a part of the spectrum are either nonexistent, or are obtained in a closed form. When \( \mathcal{N} \)-fold supersymmetry is not spontaneously broken, there is no nonperturbative corrections, and thus that part of the spectra is exactly solved. This property has also been known in literatures, quite independently from the supersymmetry considerations, as “quasi-exact solvability” [29]-[31]. When \( \mathcal{N} \)-fold supersymmetry is spontaneously broken, in general there are non-perturbative corrections to energy eigenvalues and we only know of the perturbative part. We dubbed this property “quasi-perturbative solvability” [32].

Quasi-exact solvability is known to be connected with the property that the Hamiltonian can be written in terms of \( sl(2) \) generators [29]. In Ref.[3], we made the same connection for the quartic potential, which is only quasi-perturbatively solvable. Similar connections were made for the sextic potential in Ref.[33, 34] and for exponential potential in Ref.[35]. Through these works it became apparent that all of the models, which can be classified as Type A [28], may be expressed in terms of the \( sl(2) \) generators. The purpose of this letter then is to make the general connection between the Type A \( \mathcal{N} \)-fold supersymmetry and the \( sl(2) \) representations.

In the following, we first review \( \mathcal{N} \)-fold supersymmetry in general and the definition of the Type A \( \mathcal{N} \)-fold supersymmetry in Section 2. The method to construct \( \mathcal{N} \)-fold supersymmetric models from quasi-solvability is briefly reviewed in Section 3. Based on these knowledge, we show that quasi-solvability naturally leads to the \( sl(2) \) representations and show that this leads to Type A \( \mathcal{N} \)-fold supersymmetry in Section 4. Type A \( \mathcal{N} \)-fold supersymmetric models which can be written down explicitly are examined in Section 5. A summary is given in Section 6.

2 \( \mathcal{N} \)-fold supersymmetry and its Type A subclass

We first present a concise definition of \( \mathcal{N} \)-fold supersymmetry and Type A \( \mathcal{N} \)-fold supersymmetry. One dimensional quantum mechanical model with \( \mathcal{N} \)-fold supersymmetry has one ordinary (bosonic) coordinate \( q \) and fermionic coordinates \( \psi \) and \( \psi^\dagger \), which satisfy the following:

\[
\{ \psi, \psi \} = \{ \psi^\dagger, \psi^\dagger \} = 0, \quad \{ \psi, \psi^\dagger \} = 1.
\]
The Hamiltonian $H_N$ is given as follows,

$$H_N = H_N^- \psi \psi^\dagger + H_N^+ \psi^\dagger \psi,$$

where $H_N^\pm$ are ordinary Hamiltonians,

$$H_N^\pm = \frac{1}{2} p^2 + V_N^\pm(q),$$

with $p = -i d/dq$. The $\mathcal{N}$-fold supercharges are generically defined as

$$Q_N = P_N^\dagger \psi, \quad Q_N^\dagger = P_N \psi^\dagger,$$

where $P_N$ is an $\mathcal{N}$-th order polynomial of $p$,

$$P_N = w_N(q) p^N + w_{N-1}(q) p^{N-1} + \cdots + w_1(q) p + w_0(q).$$

The $\mathcal{N}$-fold supersymmetry algebra is defined as follows:

$$\{Q_N, Q_N\} = \{Q_N^\dagger, Q_N^\dagger\} = 0,$$

$$[Q_N, H_N] = [Q_N^\dagger, H_N] = 0.$$  \hspace{1cm} (2.6)

(2.7)

Among the above, the nilpotency (2.6) is guaranteed by the property of the fermionic coordinates (2.1), while the latter leads to

$$P_N H_N^- - H_N^+ P_N = 0,$$

and its conjugate, which are essentially $\mathcal{N} + 2$ differential equations for the potentials $V_N^\pm(q)$ and the coefficient functions $w_{\mathcal{N},...,0}(q)$. Since there are $\mathcal{N} + 3$ functions to be determined, one function remains arbitrary. One may, for example, choose it one of the potentials. Therefore it may be said that any ordinary (purely bosonic, nonsupersymmetric) one-dimensional quantum model may be extended so that it is a part of an $\mathcal{N}$-fold supersymmetric system. This does not mean that any one-dimensional system can be even partially solved by means of nonrenormalization theorem(s) of $\mathcal{N}$-fold supersymmetry: The differential equations that are obtained from Eq.(2.8) are just as difficult to solve as Schrödinger equations.

The explicitly known models noted in the introduction are, in contrast, solvable: The $\mathcal{N}$-fold supersymmetric algebra (2.8) can be solved explicitly, and the part of the spectra are solved. All these models belong to Type A $\mathcal{N}$-fold supersymmetry, which is defined to have the following form of the supercharge [28]:

$$P_N = (D + i(\mathcal{N} - 1)E(q))(D + i(\mathcal{N} - 2)E(q)) \cdots (D + iE(q))D,$$

where $D \equiv p - iW(q)$. In Refs.[28, 32], we showed, by induction in $\mathcal{N}$, that the algebra (2.8) is satisfied if the following conditions are met:

$$V_N^\pm = \frac{1}{2} \left( W^2 \pm W' \right)$$

$$+ \frac{\mathcal{N} - 1}{2} \left[ -E W + \frac{2\mathcal{N} - 1}{6} E^2 - \frac{\mathcal{N} + 1}{6} E' \pm \left( W' - \frac{\mathcal{N}}{2} E' \right) \right],$$

$$(\mathcal{W} + E\mathcal{W})'' - E(\mathcal{W} + E\mathcal{W})' = 0 \quad \text{(for } \mathcal{N} \geq 2),$$

$$(E' + E^2)'' - E(E' + E^2)' = 0 \quad \text{(for } \mathcal{N} \geq 3),$$

(2.10) (2.11) (2.12)
where a prime denotes a derivative with respect to $q$ and
\[ \tilde{W}(q) \equiv W(q) - \frac{N-1}{2} E(q). \] (2.13)

3 Quasi-solvability and $\mathcal{N}$-fold supersymmetry

$\mathcal{N}$-fold supersymmetry allows alternative construction based on quasi-solvability \cite{32}. Since this will be a key for the $sl(2)$ construction, we will briefly review this aspect.

We first note that the $p^Nq^{N+2}$-terms in the $\mathcal{N}$-fold supersymmetry algebra (2.8) trivially vanish. The $p^Nq^{N+1}$-terms yield $w_N(q)' = 0$. Therefore we choose that $w_N(q) = 1$, (3.1)

hereafter without losing generality. Then, the $p^N$-terms yield the following relation:
\[ V^+_N(q) = V^-_N(q) + iw_{N-1}(q)'. \] (3.2)

Let us assume that an operator $P(p, q)$, whose highest order of $p$ (when all $p$’s are moved to right of all $q$’s) is $\mathcal{N}$, has a nontrivial kernel $\mathcal{V} \equiv \{ \phi(q) | P(p, q)\phi(q) = 0 \}$ of dimension $\mathcal{N}$. Further let us assume that there is a Hamiltonian $H$ of the form,
\[ H = \frac{1}{2} p^2 + V(q) \] (3.3)

that maps the kernel $\mathcal{V}$ into itself:
\[ H\phi(q) \in \mathcal{V} \text{ for any } \phi \in \mathcal{V}. \] (3.4)

Such a system is “quasi-solvable” \cite{29}–\cite{31}, in the sense that the energy eigenvalues are obtained in a closed form for the part of the spectrum that is spanned by the kernel.\footnote{We do not require that the equation $P(p, q)\phi(q) = 0$ is algebraically solvable for the system to be "quasi-solvable". This definition does not conflict with, e.g., the definition of "quasi-exact solvability" in Ref.\cite{31}.} This is because of the following: Let $\phi_n(q) (n = 1, 2, \cdots, \mathcal{N})$ be a basis of the kernel $\mathcal{V}$. Then, the above property (3.4) means that $H\phi_n(q)$ is given by a linear combination of $\phi_1,...,\phi_N(q)$:
\[ H\phi_n(q) = \sum_{m=1}^{\mathcal{N}} S_{n,m} \phi_m(q). \] (3.5)

Therefore, by diagonalizing the $\mathcal{N} \times \mathcal{N}$ matrix $S$, we obtain the energy eigenvalues of $\mathcal{N}$-states in a closed form.\footnote{It should be noted that we did not require normalizability for $\phi_i(q)$ in the above. Therefore, the normalizability of the resulting eigenfunctions should be separately examined, especially in connection with the perturbation theory \cite{32}.}

It is straightforward to show that existence of such a Hamiltonian $H$ that satisfy the property (3.4) implies that the system is $\mathcal{N}$-fold supersymmetric. This is because of the following. Let us denote $P(p, q)$ as follows:
\[ P(p, q) = p^N + c_{N-1}(q)p^{N-1} + \cdots + c_0(q), \] (3.6)
where we have chosen the coefficient function of $p^N$ in $P(p,q)$ equal to one, since it is irrelevant for the definition of the kernel $V.$ (Its form is also motivated by the allowed choice \( 1 \).) We introduce another Hamiltonian $K$ as follows,

$$K = \frac{1}{2} p^2 + Y(q),$$

$$Y(q) = V(q) + ic_{N-1}(q)'.$$  

(3.7)

(3.8)

It may be noted that the latter form is motivated by Eq. (3.2). Then the operator $G(p,q) \equiv P(p,q)H - KP(p,q)$ contains only up to $(N-1)$-powers of $p$, by the same reason that lead to Eq. (3.2). It also satisfies the following:

$$G(p,q)\phi_n = 0 \text{ for } n = 1, 2, \ldots, N.$$  

(3.9)

Since $G(p,q)$ is an $(N-1)$-th order differential operator, it can not non-trivially annihilate $N$ independent functions $\phi_1, \ldots, \phi_N(q)$. Therefore, the operator $G(p,q)$ is identically zero:

$$P(p,q)H - KP(p,q) = 0.$$  

(3.10)

If we identify $P_N = P$, $H_N = H$, and $H_N^\dagger = K$, the above relation is equivalent to the $N$-fold supersymmetry algebra (2.8).

The above argument shows that if we can construct a Hamiltonian $H$ and an operator $P(p,q)$ that satisfy the above quasi-solvability condition we have a $N$-fold supersymmetric system. In the following, we will carry out this construction for Type A $N$-fold supersymmetry.

### 4. sl(2) Construction

We first derive a simpler expression for $P_N$ of Type A $N$-fold supersymmetry (2.9). Let us first introduce

$$U(q) \equiv e^{\int^q W(q')dq'},$$

(4.1)

and transform $P_N$ as follows:

$$U P_N U^{-1} \equiv (-i)^N \tilde{P}_N.$$  

(4.2)

This leads to the following expression:

$$\tilde{P}_N = \left( \frac{d}{dq} - (N-1)E(q) \right) \left( \frac{d}{dq} - (N-2)E(q) \right) \cdots \left( \frac{d}{dq} - E(q) \right) \frac{d}{dq}.$$  

(4.3)

Next we introduce a function $h(q)$ defined by the following;

$$h(q) = c_1 \int^q dq_1 e^{\int^q_0 E(q_2)dq_2} + c_2,$$  

(4.4)

where $c_{1,2}$ are constants. The above is a general solution of the following differential equation:

$$h''(q) - E(q)h'(q) = 0.$$  

(4.5)
We then find that $\tilde{P}_N$ may be written as follows:

$$\tilde{P}_N = (h')^N \left( \frac{d}{dh} \right)^N. \quad (4.6)$$

We therefore arrive at the following simple expression:

$$P_N = (-i)^NU^{-1}(h')^N \left( \frac{d}{dh} \right)^NU. \quad (4.7)$$

The form (4.7) of $P_N$ allows straightforward identification of the kernel $V$: The equations for its basis $\{\phi_1, \phi_2, \cdots, \phi_N\}$;

$$P_N\phi_n = 0, \quad (4.8)$$

can be simply solved as follows:

$$\phi_n = h^{n-1}U^{-1} \quad (n = 1, 2, \cdots, N). \quad (4.9)$$

Next we need to find a Hamiltonian $H_N$ that satisfy;

$$P_NH_N\phi_n = 0, \quad (4.10)$$

for $n = 1, 2, \cdots, N$. By transforming the Hamiltonian $H_N$ by $U$ as

$$H_N = U^{-1}\tilde{H}_NU, \quad (4.11)$$

we find that the condition (4.10) may be written as follows:

$$\left( \frac{d}{dh} \right)^N\tilde{H}_N h^{n-1} = 0, \quad (4.12)$$

for $n = 1, 2, \cdots, N$. In the following, we will obtain $\tilde{H}_N$ that satisfy the above as a function of $h$ and $d/dh$, noting that since $\tilde{H}_N$ contains second derivatives with respect to $q$ it contains second derivatives with respect to $h$ as well.

Evidently, arbitrary constants are allowed in $\tilde{H}_N$ in Eq.(4.12). The operators with first derivative with respect to $h$ allowed in $\tilde{H}_N$ are given by the following:

$$\frac{d}{dh}, \quad h\frac{d}{dh}, \quad h^2\frac{d}{dh} - (N-1)h. \quad (4.13)$$

It should be noted that in case of $N = 1$, the above is not the complete list of such operators, since any operator of the form $g(h)d/dh$ with an arbitrary function $g(h)$ is allowed. Therefore, the following construction applies only for $N \geq 2$.

All of the operators in the list (4.13), when combined with constants, can be written in terms of $sl(2)$ generators, whose representation on an $N$-dimensional space spanned by the basis $(1, h, h^2, \cdots, h^{N-1})$ are the following:

$$J^+ \equiv h^2\frac{d}{dh} - (N-1)h, \quad J^0 \equiv h\frac{d}{dh} - \frac{N-1}{2}, \quad J^- \equiv \frac{d}{dh}. \quad (4.14)$$
These generators satisfy the algebra,
\[ [J^+, J^-] = -2J^0, \quad [J^\pm, J^0] = \mp J^\pm, \]  
and form the following Casimir operator:
\[ \frac{1}{2} (J^+ J^- + J^- J^+) - (J^0)^2 = -\frac{1}{4}(N^2 - 1). \]  

Operators that contain second derivatives with respect to \( h \) that satisfy Eq. (4.12) are only the following:
\[ \frac{d^2}{dh^2} = (J^-)^2, \]  
\[ h \frac{d^2}{dh^2} = J^0 J^- + \frac{N - 1}{2} J^- , \]  
\[ h^2 \frac{d^2}{dh^2} = J^+ J^- + (N - 1) J^0 + \frac{(N - 1)^2}{2}, \]  
\[ h^3 \frac{d^2}{dh^2} - (N - 1)(N - 2) h = J^+ J^0 - \frac{5 - 3N}{2} J^+ , \]  
\[ h^4 \frac{d^2}{dh^2} - 2(N - 2) h^3 \frac{d}{dh} + (N - 1)(N - 2) h^2 = (J^+)^2 . \]  

This time, the above list of the operators are complete only for \( N \geq 3 \), for which we restrict our construction. From Eq. (4.14)–(4.21), we find that the Hamiltonian \( \tilde{H}_N \) that satisfy the \( N \)-fold supersymmetric condition (4.12) can be always written in terms of the \( sl(2) \) generators \( J^{+,0,-} \) as follows:
\[ \tilde{H}_N = - \sum_{i,j=+,.0,-} a_{ij} J^i J^j + \sum_{i=+,.0,-} b_i J^i + C. \]  

where \( a_{ij}, b_i \) and \( C \) are constants.

The Hamiltonian \( \tilde{H}_N \) (4.22) is written in terms of \( h \) as follows:
\[ \tilde{H}_N = - P_4(h) \frac{d^2}{dh^2} + P_3(h) \frac{d}{dh} + P_2(h), \]  
where the coefficient functions \( P_{4,3,2}(h) \) are;
\[ P_4(h) = a_{++} h^4 + a_{+,0} h^3 + (a_{+-} + a_{00}) h^2 + a_{0-} h + a_{--} , \]  
\[ P_3(h) = 2(N - 2) a_{++} h^3 + \left( \frac{3N - 5}{2} a_{+,0} + b_{+} \right) h^2 \]  
\[ + ((N - 1) a_{+-} + (N - 2) a_{00} + b_{0}) h + \frac{N - 1}{2} a_{0-} + b_{-} , \]  
\[ P_2(h) = -(N - 1)(N - 2) a_{++} h^2 - (N - 1) \left( \frac{N - 1}{2} a_{+,0} + b_{+} \right) h \]  
\[ - \left( \frac{N - 1}{2} \right) a_{00} - \frac{N - 1}{2} a_{0-} + b_{-} + C. \]
Transforming the Hamiltonian $\tilde{H}_N$ in Eq.(1.23) back to the original Hamiltonian $H_N$ by the $U$-transformation (1.11) and writing it in terms of the coordinate $q$, we find the following:

$$H_N = -\frac{P_4(h)}{(h')^2} \frac{d^2}{dq^2} + \left[ \frac{P_4(h)}{(h')^2} \left( -2W + \frac{h''}{h'} \right) + \frac{P_3(h)}{h'} \right] \frac{d}{dq} \frac{d}{dq} - \frac{P_4(h)}{(h')^2} (W' + W^2) + \frac{P_4(h)h''}{(h')^3} W + \frac{P_3(h)}{h'} W + P_2(h). \quad (4.27)$$

Comparing this with the regular form of the Hamiltonian (2.3), we find the following requirement for $h(q)$ from the $d^2/dq^2$ term:

$$P_4(h) = \frac{1}{2} (h')^2, \quad (4.28)$$

which is also a requirement for $E(q)$ through (1.4). Similarly, from the $d/dq$ term:

$$P_3(h) = h' \left( W - \frac{E}{2} \right). \quad (4.29)$$

Finally, by the use of Eq.(4.28)–(4.29), the potential $V_N^-(q)$ is obtained as follows:

$$V_N^-(q) = \frac{1}{2} (W(q)^2 - W(q)') + P_2(h(q)). \quad (4.30)$$

The other potential $V_N^+(q)$ may be obtained from Eq.(3.3), or equivalently Eq.(3.8): Since Eq.(4.7) induces

$$P_N = p^N - iN\tilde{W}(q)p^{N-1} + \text{[terms with less powers of } p], \quad (4.31)$$

we find that

$$V_N^+(q) = V_N^-(q) + N\tilde{W}(q)' \quad (4.32)$$

We will now compare the $sl(2)$ construction explained above with the previous results on Type A $\mathcal{N}$-fold supersymmetry [28, 32]. From Eq.(1.24) we find the following identity:

$$\frac{d^5 P_4}{dh^5} = 0. \quad (4.33)$$

On the other hand, using Eq.(1.5) repeatedly on the expression of $P_4(h)$ in Eq.(1.28), we find that

$$0 = \frac{d^5 P_4}{dh^5} = \frac{1}{h'} \left( \frac{d}{dq} - 2E \right) \left[ (E' + E^2)' - E(E' + E^2)' \right], \quad (4.34)$$

which is a generalization of one of the Type A conditions (2.12). Eq.(4.34) maybe rewritten as,

$$\frac{((E' + E^2)'' - E(E' + E^2)'')'}{(E' + E^2)'' - E(E' + E^2)'} = 2E = \frac{h''}{h'}, \quad (4.35)$$
4. sl(2) Construction

which is integrated to yield that

\[(E' + E^2)'' - E(E' + E^2)' = \beta_1 (h')^2,\]

(4.36)

where \(\beta_1\) is an integration constant. The above is further integrated to yield the following:

\[E' + E^2 = \frac{1}{2} \beta_1 h^2 + \beta_2 h + \beta_3,\]

(4.37)

where \(\beta_{2,3}\) are integration constants. Since we have

\[\frac{d^2 P_4}{dh^2} = E' + E^2,\]

(4.38)

from Eq.\(\text{(4.28)}\), we find the following identification of the constants:

\[a_{++} = \frac{\beta_1}{4!}, \quad a_{+0} = \frac{\beta_2}{3!}.\]

(4.39)

Similarly to Eq.\(\text{(4.34)}\), we can obtain the following:

\[0 = -\frac{\mathcal{N} - 2}{2} \frac{d^4 P_3}{dh^4} + \frac{d^3 P_3}{dh^3} = \frac{1}{h^2} \left[ (\tilde{W}' + E\tilde{W})'' - E(\tilde{W}' + E\tilde{W})' \right],\]

(4.40)

which is exactly one of the Type A conditions \(\text{(2.11)}\). This equation is integrated to give the following:

\[\tilde{W}' + E\tilde{W} = \beta_4 h + \beta_5,\]

(4.41)

where \(\beta_{4,5}\) are integration constants. By comparing the above and

\[\frac{dP_3}{dh} = (\tilde{W}' + E\tilde{W}) + \frac{\mathcal{N} - 2}{2}(E' + E^2),\]

(4.42)

we find the following identification:

\[b_+ = \frac{1}{2} \beta_4 - \frac{1}{12} \beta_2.\]

(4.43)

Using Eqs.\(\text{(4.39)}\) and \(\text{(4.43)}\), we can obtain the following expression of \(P_2(h)\) in terms of \(E(q)\) and \(W(q)\):

\[P_2(h) = -\frac{\mathcal{N} - 1}{2} \left[ \mathcal{N} - 2 \left( E' + E^2 \right) + (\tilde{W}' + E\tilde{W}) \right] + \text{constants},\]

(4.44)

which, together with Eqs.\(\text{(4.30)}\) and \(\text{(4.32)}\), reproduces the potentials \(V_N^\pm(q)\) in Eq.\(\text{(2.10)}\).

In summary, we have shown that sl(2) construction yields Type A \(\mathcal{N}\)-fold supersymmetry, which is defined by the form of the supercharge \(\text{(2.3)}\). We find that following set of conditions is sufficient for \(\mathcal{N} = 1, 2\) and is necessary and sufficient for \(\mathcal{N} \geq 3\):

\[V_N^\pm = \frac{1}{2} \left( W^2 \pm W' \right) + \frac{\mathcal{N} - 1}{2} \left[ -EW + \frac{2\mathcal{N} - 1}{6} E^2 - \frac{\mathcal{N} + 1}{6} E' \pm \left( W' - \frac{\mathcal{N}}{2} E' \right) \right],\]

(4.45)

\[(\tilde{W}' + E\tilde{W})'' - E(\tilde{W}' + E\tilde{W})' = 0,\]

(4.46)

\[\left( \frac{d}{dq} - 2E \right) \left[ (E' + E^2)'' - E(E' + E^2)' \right] = 0,\]

(4.47)
in place of Eqs. (2.10)–(2.12). We note that one may conversely derive the \(\text{sl}(2)\) form from the above, by solving these equations and defining \(P_{4,3,2}(q)\) as in Eqs. (4.28), (4.29) and (4.44), respectively.

We have found above that \(\text{sl}(2)\) construction contains the original conditions Eqs. (2.10)–(2.12), but gives a less-restrictive condition (4.47). This is explained by the fact that original conditions were obtained by induction in \(N\). In such an induction, we implicitly assumed that \(E(q)\) and \(W(q)\) are independent from \(N\). In our \(\text{sl}(2)\) construction, \(N\)-independence of \(E(q)\) implies \(N\)-independence of \(h(q)\) (assuming that the coefficients \(c_1\) and \(c_2\) in Eq. (4.4) are \(N\)-independent as well) and thus of \(P_4(q)\) through (1.28), which in turn means \(N\)-independence of all the \(a\)-coefficients in Eq. (4.24). Further, \(N\)-independence of \(W(q)\) implies \(N\)-independence of all four coefficients of \(h^{n-1}\) terms in \(P_3(q)\) in Eq. (1.25). The \(N\)-independence of coefficient of \(h^3\) term implies that

\[
a_{++} = 0, \tag{4.48}
\]

while the latter three coefficients may be made \(N\)-independent by appropriate choice of the \(b\)-coefficients. Using our identification (4.39), we find that the above corresponds to \(\beta_1 = 0\), which, as seen in Eq. (4.36), reproduces the the Type A condition (2.12).}

5 Specific Examples

In this section, we illustrate the correspondence between some specific examples of the type A potentials, some of which appear in Ref. [28], and some of the \(\text{sl}(2)\) models mentioned in Ref. [29].

In Ref. [29], for a quasi-solvable potential \(V(q)\) and the Hamiltonian,

\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} V(q), \tag{5.1}
\]

the wave function can be written in the following form:

\[
\psi(q) = \varphi(z(q)) \ e^{-g(q)}. \tag{5.2}
\]

Therefore, by comparing (4.9) with (5.2), we suppose the general relations among the two formalisms are

\[
h(q) = z(q), \tag{5.3}
\]

\[
W(q) = \frac{d}{dq} g(q). \tag{5.4}
\]

We note that in the following we will write down only the potential \(V_N(q)\). The other potential \(V_N^+(q)\) can be constructed easily using the relation (4.32). Throughout this analysis, we neglect constant terms in the potentials.

ootnote{Recently we have proven the conditions (4.45)–(4.47) by a direct calculation. This proof will be published in near future.}
5.1 Quadratic Type

First, we consider the case of $E = 0$, which is a trivial solution of Eq.(4.47). In this case, the following quadratic $W(q)$ and quartic potential $V_N^-(q)$ are obtained from Eqs. (4.45) and (4.46):

\[ W(q) = C_1q^2 + C_2q + C_3, \]  
\[ V_N^-(q) = \frac{C_1^2}{2}q^4 + C_1C_2q^3 + \left( \frac{C_2^2}{2} + C_1C_3 \right)q^2 + (C_2C_3 - NC_1)q. \]

This type of quartic potential is not mentioned in Ref.[29]. However, we see easily that this example is constructed from $sl(2)$ generators. By transforming $H_N$ by as (4.11) and setting $h(q) = q$, which is consistent with $E = 0$ and Eq.(4.5), we get

\[ \tilde{H}_N = -\frac{1}{2} \left( J^- \right)^2 + C_1J^+ + C_2J^0 + C_3J^- . \]

This can be put into the bilinear and linear form of the $sl(2)$ generators;

\[ \tilde{H}_N = -\frac{1}{2}(J^-)^2 + C_1J^+ + C_2J^0 + C_3J^- . \]

This reproduces the form noted in Footnote 12 in Ref.[3] with suitable choice of $C_i$.

5.2 Exponential and Periodic Type

Next, we take $E = E_0$ (a non-zero constant), which also trivially satisfies Eq.(4.47). In this case, we find the following from Eqs. (4.45) and (4.46):

\[ W(q) = C_1e^{E_0q} + C_2e^{-E_0q} + C_3, \]  
\[ V_N^-(q) = \frac{C_1^2}{2}e^{2E_0q} + \frac{1}{2}C_1\{2C_3 - E_0(2N - 1)\}e^{E_0q} + \frac{1}{2}C_2(2C_3 + E_0)e^{-E_0q} + \frac{C_2^2}{2}e^{-2E_0q}. \]

When $E_0$ is chosen to be real, this exponential potential corresponds to the potential (I) in Ref.[29]. Furthermore, when $E_0$ is chosen to be complex and coefficients are appropriately chosen so that potential is real, the above potential reproduces the potential (X) in Ref.[29]. The potential and the wave function there are as follows:

\[ V(q) = a^2e^{-2\alpha q} - a\{2b + \alpha(2N - 1)\}e^{-\alpha q} + c(2b - \alpha)e^{\alpha q} + c^2e^{2\alpha q}, \]  
\[ \varphi(q) = A_0e^{-\alpha(N-1)q} + A_1e^{-\alpha(N-2)q} + \ldots + A_{N-1}, \]  
\[ g(q) = a\alpha e^{-\alpha q} + bq + c\alpha e^{\alpha q}, \]  
\[ z(q) = e^{-\alpha q}. \]

\[ ^4 \text{This may be because this potential makes sense only perturbatively: The wavefunction is normalizable at any finite order of the perturbation theory, but not at full order. Therefore, the obtained energy eigenvalues represent only the perturbative part [13, 22].} \]
Therefore, from Eq. (5.4), (5.3) and (5.14), we obtain the relation;

\[ E_0 = -\alpha. \]  

(5.15)

In addition, the relations of the other parameters are determined from Eqs. (5.4), (5.9) and (5.13);

\[ C_1 = -a, \quad C_2 = c, \quad C_3 = b, \]  

(5.16)

and then the potential (5.11) equals to (5.11).

### 5.3 Cubic Type

Furthermore, we consider a solution of (4.47), \( E = 1/q \). In this case, we get the following \( W(q) \) and the potential \( V_N^{-}(q) \) in the same way as above:

\[ W(q) = C_1 q^3 + C_2 q + C_3 q^{-1}, \]  

(5.17)

\[ V_N^{-}(q) = \frac{C_2}{2} q^6 + C_1 C_2 q^4 + \frac{1}{2} \{ C_2^2 - (4N - 1 - 2C_3)C_1 \} q^2 + \frac{C_3(C_3 + 1)}{2} q^{-2}. \]  

(5.18)

This sextic potential is equivalent to the potentials (VI) and (VII) in Ref. [29]. The potentials and the wave functions\(^5\) are given by the following:

\[ V(q) = a^2 q^6 + 2abq^4 + \{ b^2 - (4N - 1 - 2c)a \} q^2 + c(c + 1) \frac{1}{q^2}, \]  

(5.19)

\[ \psi(q) = A_0(q^2)^{N-1} + A_1(q^2)^{N-2} + \ldots + A_{N-1}, \]  

(5.20)

\[ g(q) = \frac{a}{4} q^4 + \frac{b}{2} q^2 + c \ln q, \]  

(5.21)

\[ z(q) = q^2. \]  

(5.22)

Therefore, from (5.4), (5.17) and (5.21), the relations of the parameters are determined as follows:

\[ C_1 = a, \quad C_2 = b, \quad C_3 = c. \]  

(5.23)

Under these relations, the potential (5.18) equals to the potential (5.14). Note that \( E = 1/q \) is consistent with Eqs. (4.5), (5.3) and (5.22).

\(^5\)The potential and the wave function are originally given as an example of the spherically symmetric quasi-solvable models. As we are discussing one-dimensional potentials here, we set \( d = 1 \) and \( l = 0 \).
5.4 Hyperbolic Type

At the end of this section, we consider another solution of Eq. (4.47),

\[ E = \alpha \left( \frac{1}{\cosh \alpha q \sinh \alpha q} - 2 \tanh \alpha q \right). \]  

(5.24)

In this case, the following \( W(q) \) and the potential \( V_{\mathcal{N}}(q) \) are obtained:

\[ W(q) = \frac{C_1}{\cosh \alpha q \sinh \alpha q} + C_2 \cosh \alpha q \sinh \alpha q + C_3 \tanh \alpha q, \]  

(5.25)

\[ V_{\mathcal{N}}(q) = \frac{C_2^2}{2} \cosh^4 \alpha q - C_2 \left( \frac{C_2}{2} - C_3 + \alpha \right) \cosh^2 \alpha q \]

\[ - \left[ \frac{C_2^2}{2} - C_1 C_3 - \alpha (2 \mathcal{N} - 1) \{ C_1 - (\mathcal{N} - 1) \alpha \} + \frac{C_3}{2} \alpha (4 \mathcal{N} - 3) \right] \frac{1}{\cosh^2 \alpha q} \]

\[ + \frac{C_1}{2} (C_1 + \alpha) \frac{1}{\cosh^2 \alpha q \sinh^2 \alpha q}. \]  

(5.26)

This type of hyperbolic potential corresponds to the potential (IV) in Ref. [29]. The potential and wave function are

\[ V(q) = c^2 \cosh^4 \alpha q - c (c + 2 \alpha - 2 a) \cosh^2 \alpha q - \{ a (a + \alpha) + \alpha k (\alpha k + \alpha + 2 a) \} \cosh^{-2} \alpha q, \]  

(5.27)

\[ \varphi(q) = A_k \tanh^k \alpha q + A_{k-1} \tanh^{k-1} \alpha q + \ldots + A_0, \]

(5.28)

\[ g(q) = \frac{c}{4 \alpha} \cosh 2 \alpha q + \frac{a}{\alpha} \ln \cosh \alpha q, \]

(5.29)

\[ z(q) = \cosh^{-2} \alpha q, \]  

(5.30)

where \( \mathcal{N} = \lfloor k/2 \rfloor + 1 \). Also, \( A_{i=\text{odd}} = 0 \) for even \( k \) and \( A_{i=\text{even}} = 0 \) for odd \( k \). The solution (5.24) is consistent with Eqs. (4.5), (5.3) and (5.30), and we get the following relations from Eqs. (5.4), (5.25) and (5.29) for even \( k \):

\[ C_1 = 0, \quad C_2 = c, \quad C_3 = a. \]  

(5.31)

When these relations hold, the potential (5.26) equals to the potential (5.27). For odd \( k \), we modify Eq. (5.4) as,

\[ W(q) = \frac{d}{dq} g(q) - \frac{\alpha}{\cosh \alpha q \sinh \alpha q}, \]  

(5.32)

to find that the relations,

\[ C_1 = -\alpha, \quad C_2 = c, \quad C_3 = a, \]  

(5.33)

reproduce the potential and the wave function.

We note that some other models are also known, notably ones with \( E = -3/q \) and \( E = \alpha / \tan (\alpha q) \). All of those models and some other new models we have found recently will be discussed in a separate literature.
6 Summary

In this paper we have shown that by quasi-solvability considerations \( sl(2) \) emerges naturally and \( \mathcal{N} \)-fold supersymmetry can be constructed by the use of \( sl(2) \) generators. This is done by first constructing the kernel space of the supercharge \( P_N \) of the Type A form (2.9) and then showing that the Hamiltonian that leave the kernel invariant is always written in terms of \( sl(2) \) generators as Eq.(4.22). By requiring that this Hamiltonian induces the canonical form (2.3), we have obtained a condition (4.47) on \( E(q) \), a relation (4.46) between \( E(q) \) and \( W(q) \), and have found a expression (4.45) of the potentials \( V_{N}^{\pm}(q) \) in terms of \( E(q) \) and \( W(q) \). These equations are more general than the previously obtained equations (2.10)–(2.12) and are necessary and sufficient for Type A \( \mathcal{N} \)-fold supersymmetry. We have also examined explicitly known models in the light of the \( sl(2) \) construction.

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References

[1] E. Witten, Nucl. Phys. B188 (1981) 513.
[2] E. Witten, Nucl. Phys. B202 (1982) 253.
[3] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato and S. Wada, Nucl. Phys. B553 (1999) 644.
[4] A. A. Andrianov, M. V. Ioffe and V. P. Spiridonov, Phys. Lett. A174 (1993) 273.
[5] A. A. Andrianov, M. V. Ioffe, F. Cannata and J.-P. Dedonder, Int. J. Mod. Phys. A10 (1995) 2683.
[6] A. A. Andrianov, M. V. Ioffe and D. N. Nishnianidze, Phys. Lett. A201 (1995) 103.
[7] A. A. Andrianov, M. V. Ioffe and D. N. Nishnianidze, Theor. Math. Phys. 104 (1995) 1129.
[8] V. G. Bagrov and B. F. Samsonov, Theor. Math. Phys. 104 (1995) 1051.
[9] B. F. Samsonov, Mod. Phys. Lett. A11 (1996) 1563.
[10] V. G. Bagrov and B. F. Samsonov, Phys. Part. Nucl. 28 (1997) 374.
[11] B. F. Samsonov, Phys. Lett. A263 (1999) 274.
[12] M. Plyushchay, Int. J. Mod. Phys. A15 (2000) 3679.
[13] S. Klishevich and M. Plyushchay, Mod. Phys. Lett. A14 (1999) 2739.
[14] J. O. Rosas-Ortiz, *J. Phys.* **A31** (1998) 10163.

[15] A. Khare and U. Sukhatme, *J. Math. Phys.* **40** (1999) 5473.

[16] D. J. Fernández C. and V. Hussin, *J. Phys.* **A32** (1999) 3630.

[17] D. J. Fernández C., J. Negro and L. M. Nieto, *Phys. Lett.* **A275** (2000) 338.

[18] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato and S. Wada, *Phys. Lett.* **B424** (1998) 93.

[19] D. J. Rowe and A. Ryman, *J. Math. Phys.* **23** (1982) 732.

[20] I. I. Balitsky and A. V. Yung, *Phys. Lett.* **B168** (1986) 13.

[21] P. G. Silvetrov, *Sov. J. Nucl. Phys.* **51** (1990) 1121.

[22] H. Aoyama and H. Kikuchi, *Nucl. Phys.* **B369** (1992) 219.

[23] H. Aoyama and S. Wada, *Phys. Lett.* **B349** (1995) 279.

[24] T. Harano and M. Sato, *hep-ph/9703457*.

[25] H. Aoyama, H. Kikuchi, T. Harano, M. Sato and S. Wada, *Phys. Rev. Lett.* **79** (1997) 4052.

[26] H. Aoyama, H. Kikuchi, T. Harano, I. Okouchi, M. Sato and S. Wada, *Prog. Theor. Phys. Supplement* **127** (1997) 1.

[27] H. Aoyama, M. Sato, T. Tanaka and M. Yamamoto, *Phys. Lett.* **B498** (2001) 117.

[28] H. Aoyama, M. Sato and T. Tanaka, *Phys. Lett.* **B503** (2001) 423.

[29] A. V. Turbiner, *Commun. Math. Phys.* **118** (1988) 467.

[30] M. A. Shifman, *Int. J. Mod. Phys.* **A12** (1989) 2897.

[31] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics*, (IOP Publishing, Bristol, 1994), and references cited therein.

[32] H. Aoyama, M. Sato and T. Tanaka, *quant-ph/00106037*.

[33] S. M. Klishevich and M. S. Plyushchay, *hep-th/0105135*.

[34] P. Dorey, C. Dunning and R. Tateo, *hep-th/0103051*.

[35] S. M. Klishevich and M. S. Plyushchay, *Nucl. Phys.* **B606** (2001) 583 (*hep-th/0012023*).