Thermal Field Theory in Small Systems

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Abstract. We compute the finite size corrections to the partition function in a Cartesian space of finite extent in \( M \) directions and of infinite extent in \( D - M \) directions for a massless, non-interacting scalar field theory. We then use this partition function to compute numerically the energy density, pressure, entropy density, and speed of sound for this theory for \( M = 1, 2, \) and 3 for \( D = 3 \) total spatial dimensions. The finite size corrections for the speed of sound are \( \sim 600\% \), which indicates the need to consider these corrections in hydrodynamic simulations of small collision systems in high energy nuclear physics.

1. Introduction
The goal of heavy ion phenomenology is to understand the non-trivial, emergent, moderate-body physics of the strong nuclear force. This is an immodest goal as there isn’t even a first principles calculation of the phase diagram of many of the most important many-body systems in QED, including water.

Approximately a microsecond after the Big Bang, the universe had cooled to a trillion degrees, or a few hundred MeV in natural units, a hundred thousand times hotter than the center of the sun [1]. We are precisely interested in these conditions, when, for instance, lattice simulations suggest a phase transition from normal nuclear matter to what has become known as the quark-gluon plasma (QGP) [2, 3].

At asymptotically high temperatures, asymptotic freedom demands that the relevant degrees of freedom are (at most slightly thermally modified) quarks and gluons. But at these moderate temperatures of a few hundred MeV, the picture is not so clear. Lattice data shows a \( \sim 15\% \) deviation from the Stefan-Boltzmann ideal gas limit of QCD for quantities such as the energy density and pressure of nuclear matter [2, 3]. These slight but non-zero differences may be rather equally well attributed to the dynamics of nearly zero coupling [4] or of nearly infinite coupling [5].

Amazingly, we are able to experimentally probe these theoretical ideas at colliders such as the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory in Upton, New York, USA and at the Large Hadron Collider (LHC) in Geneva, Switzerland. The enormous wealth of data generated by these massive machines requires thousands of physicists to analyze, to put into a usable form for comparison to theoretical calculations.

A “standard model” of the collisions of large nuclei such as \(^{197}\text{Au}\) and \(^{208}\text{Pb}\) in heavy ion collisions is emerging in which the initial state physics of the color glass condensate (CGC) rapidly hydrodynamizes into a system for which nearly inviscid relativistic hydrodynamics with an equation of state given by lattice QCD provides an excellent description of the dynamics. The
system cools and hadronizes, with the final state nuclear matter rescattering until it is so dilute that it propagates freely until detection. This picture of the evolution of a heavy ion collision provides an excellent description of the distribution of the 99% of particles measured in heavy ion collisions, those particles with momenta in the direction transverse to the beam below ∼few GeV/c [6]. Moreover, energy loss calculations—jet tomography, in which very high momentum particles created at the first instant of the collision that then subsequently propagate through the soft medium particles—appear to provide a very good qualitatively consistent picture of the distribution of particles above ∼15 GeV/c in transverse momentum [7].

A recent surprise is that this same “standard model” picture appears to also give a good description of the distribution of the low momentum particles in collisions of small systems such as a proton on a large nucleus [8]. This shocking conclusion raises two important questions: how are the thermodynamics changed in such a small system (previous calculations assumed a system in the thermodynamic limit) and how do particles lose energy in such a small system (previous calculations assumed a large system size)?

We will attempt to provide the start of an answer to the first question. The beginnings of an answer to the second question were discussed in [9].

2. Thermal Field Theory
Since relativistic hydrodynamics provides such a good description of the measured low momentum particles, and because the distribution of the low momentum particles show correlations amongst up to eight particles simultaneously [10], we conclude that the systems created in these heavy ion collisions behave collectively. We thus assume, presumably safely, that thermal quantum field theory [11, 12] provides a good description of the physics in these systems. In particular, we will use the technology of the canonical ensemble to compute various quantities such as the energy density, pressure, and speed of sound relevant for understanding the dynamics of quark-gluon plasma, even in systems far from the thermodynamic limit. Therefore we will concern ourselves with computing the partition function

$$Z = \sum_i e^{-E_i/T},$$

where the sum over $i$ is a sum over all possible states the system might take, $E_i$ is the energy associated with the $i^{th}$ state, and $T$ is the fixed temperature of the system.

The various quantities of interest may be computed from $Z$ through

$$e = \frac{T^2}{V} \frac{\partial \ln Z}{\partial T},$$

$$p = T \frac{\partial \ln Z}{\partial V},$$

$$S = \ln Z + \frac{E}{T},$$

$$c_s^2 = \frac{\partial p}{\partial e}.$$

In order to make the connection between the usual partition function of statistical mechanics and quantum field theory, we exploit the relation

$$Z = \sum_i e^{-E_i/T} = \text{tr} e^{-\beta H} = \int D\phi e^{-S_E};$$

the sum over states is equal to the trace over the Hamiltonian operator weighted by $\beta \equiv 1/T$ which is, in turn, equal to the path integral over the Euclideanized action $S_E$. With the path integral in hand, one may use the methods of usual perturbative field theory, e.g. Feynman diagrams, even for systems for which $T \neq 0$. 
3. Results for Systems of Infinite Size

For simple enough Lagrangians, one may analytically solve for the partition function \( Z \) and thus also for the various thermodynamic quantities of interest. For example, one may find \( E, p, \) etc. for a non-interacting scalar field

\[
L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2.  \tag{7}
\]

One method for finding \( Z \) for the above Lagrangian is to place the system in a box with sides of length \( L \) and expand \( \phi(x) \) in the Fourier modes supported by the box. The calculation is simpler if we impose Neumann boundary conditions; then we may take

\[
\phi(\tau, \vec{x}) = \sqrt{\frac{\beta}{V}} \sum_{n, \vec{p}} \tilde{\phi}(\omega_n, \vec{p}) e^{i(\vec{p} \cdot \vec{x}) + \omega_n \tau}.  \tag{8}
\]

For a massless theory in \( D \) spatial dimensions, one finds after the \( T = 0 \) vacuum subtraction that

\[
\ln Z(\beta, D, m) = -\frac{V_D \beta^{-D}}{(4\pi)^{D/2}} \sum_{\ell=1}^{\infty} \left( \frac{2m\beta}{\ell} \right)^{D+1} K_{D+1}^2 (m\beta\ell)  \tag{9}
\]

By expanding the modified Bessel function for small argument, \( K_\nu(x) = 2^{\nu-1} \Gamma(\nu)x^{-\nu} + O(x) \), one finds

\[
\ln Z\bigg|_{\beta,D,m=0} = \frac{V_d}{\pi(D+1)/2} \zeta(D+1) \Gamma(D+1/2) \beta^{-D}  \tag{10}
\]

\[
\rightarrow \frac{V_3 \pi^2}{90} T^3, \text{ for } D = 3.  \tag{11}
\]

Therefore one immediately finds that

\[
p(D,T) = \frac{\partial T \ln Z}{\partial V} = \frac{1}{\pi(D+1)/2} \zeta(D+1) \Gamma(D+1/2) T^{D+1} \rightarrow \frac{\pi^2}{90} T^4, \text{ for } D = 3,
\]

\[
e(D,T) = \frac{T^2 \partial \ln Z}{V \partial T} = \frac{d}{\pi(D+1)/2} \zeta(D+1) \Gamma(D+1/2) T^{D+1} \rightarrow \frac{\pi^2}{30} T^4, \text{ for } D = 3,
\]

\[
c_s(D,T) = \sqrt{\frac{\partial p}{\partial e}} = \frac{1}{\sqrt{D}} \rightarrow \frac{1}{\sqrt{3}}, \text{ for } D = 3.  \tag{12}
\]

4. Results for Systems of Finite Size

In order to model QGP-like systems of finite size, we choose to replace the Neumann boundary conditions with Dirichlet boundary conditions in \( M \) of the \( D \) spatial directions. The idea is that these Dirichlet boundary conditions model how the QGP created in heavy ion collisions is of a finite extent inside a larger vacuum.

In the above derivation of the partition function Eq. (9) we took

\[
\sum_{\vec{p}} \rightarrow V_D \int \frac{d^D p}{(2\pi)^D}.  \tag{13}
\]
With these $M$ compactified directions we take
\[ \sum_{\vec{p}} \sum_{\ell=(...,-1,...)} V_{D-M} \int \frac{d^{D-M}p}{(2\pi)^{D-M}}. \]  

(14)

The logarithm of the partition function then becomes more complicated:
\[
\ln Z = V_{D-M} \left( \frac{\beta}{(2\pi)^{D+1}} \sum_{i=0}^{M} (-1)^i \frac{\pi^i}{i!(M-i)!} \sum_{\sigma} (L_{\sigma(i+1)} \cdots L_{\sigma(M)}) \right) \\
\times \left[ 1 - \delta_{M,i} \right] \sum_{\vec{\ell} \in \mathbb{Z}^{M-i}\{} 0 \{
\frac{1}{2} \Gamma \left( \frac{D-i+1}{2} \right) \left[ \frac{1}{\pi} \sum_{j=1}^{M-1} k_j^2 L_{\sigma(i+j)}^2 \right]^{-\frac{D+1}{2}} \right]
\]
\[ + 2 \frac{\beta^{-(D-M)}}{(2\pi)^{D-M+1}} \sum_{\vec{\ell}=(...,-1,...)} \sum_{\ell=1}^{\infty} \left( \frac{\tilde{\omega}_\ell}{\ell} \right) \frac{D-M+1}{2} K_{D-M+1} \left( \tilde{\omega}_\ell \beta \ell \right), \]  

(15)

where
\[ \tilde{\omega}_\ell \equiv \left( \sum_{i=1}^{M} \left( \frac{\pi \ell_i}{L_i} \right)^2 \right)^{1/2}. \]  

(16)

In Eq. (15), the $L_i$ are the finite extents of the space in each of the $M$ compactified directions. The sum over $\sigma$ is the sum over all permutations of the compactified directions, thus guaranteeing that Eq. (15) is symmetric under the interchange of any of the $L_i$.

One may compute analytically the result for, e.g., $D=3$ and $M=1$ from Eq. (15) to find that
\[
\ln Z \bigg|_{T_L,D=3,M=1,m=0} = \frac{V_2}{L^2} \left[ \frac{\pi^2}{1440} \frac{1}{T_L} + \frac{(TL)^2}{2\pi} \sum_{\ell=1}^{\infty} \left( \frac{\pi \ell}{TL} L_2(e^{-\frac{\pi \ell}{T_L}} + L_3(e^{-\frac{\pi \ell}{T_L}}) \right) \right], \]  

(17)

where $\text{Li}_n(z)$ is the usual polylogarithmic function. One can show that Eq. (17) has the correct $TL \to \infty$ behavior:
\[
\ln Z \bigg|_{T_L,D=3,M=1,m=0} \to \frac{V_2 \pi^2(TL)^3}{L^2} = V_3 \frac{\pi^2 T^3}{90}. \]  

(18)

One may see the numerical results for various thermodynamic relations given the above partition function Eq. (15) in Fig. (1).

5. Conclusions and Outlook

As one can see from Fig. (1), the small system size corrections to the usual thermodynamic properties of an ideal gas can be very large. In particular, while the energy density, pressure, and entropy density differences are relatively small, $\sim 20\%$, the speed of sound increases by up to a factor of 6! It is difficult to understand 1) why the speed of sound is so much more sensitive to the small size corrections than the other thermodynamic quantities, 2) how to interpret a speed of sound that is larger than the speed of light in vacuum, and 3) at what length scale the derivation becomes invalid. These issues are currently under investigation. We are also examining the effect of a non-zero mass for the field.

Future work includes improving the calculation of the finite system size corrections to energy loss. Previous work [9] found the finite distance corrections to the usual DGLV energy loss formalism while keeping the infinite size GW model for the medium scattering centers. It will be very interesting to compute the finite size corrections to, e.g., the Debye screening mass.
Figure 1: (Color online) Ratios of thermodynamic quantities in a finite size system to that of an infinite size system as a function of temperature for a massless, non-interacting scalar field theory. The large figures have a finite length of $L = 10 \text{ fm}$ while the insets have $L = 2 \text{ fm}$. The three curves are for $M = 1$ (plates), $M = 2$ (tube), and $M = 3$ (box) compact dimensions in a $D = 3$ spatial dimensions universe. Clockwise from top left are: energy density, pressure, speed of sound, and entropy density.

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