Necessary Optimality Conditions for a Dead Oil Isotherm Optimal Control Problem*

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Abstract

We study a system of nonlinear partial differential equations resulting from the traditional modelling of oil engineering within the framework of the mechanics of a continuous medium. Recent results on the problem provide existence, uniqueness and regularity of the optimal solution. Here we obtain the first necessary optimality conditions.

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Key Words: optimal control of distributed systems; dead oil isotherm problem; necessary optimality conditions.

1 Introduction

We are interested in the optimal control of the dead oil isotherm problem:

\[
\begin{align*}
\partial_t u - \Delta \varphi(u) &= \text{div} (g(u)\nabla p) \quad \text{in } Q_T = \Omega \times (0, T), \\
\partial_t p - \text{div} (d(u)\nabla p) &= f \quad \text{in } Q_T = \Omega \times (0, T), \\
u|_{\partial \Omega} &= 0, \quad u|_{t=0} = u_0, \\
p|_{\partial \Omega} &= 0, \quad p|_{t=0} = p_0,
\end{align*}
\]

where $\Omega$ is an open bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary. Equations (1) serve as a model for an incompressible biphasic flow in a porous medium, with applications to the industry of exploitation of hydrocarbons. To understand the optimal control problem we will consider here, some words about the recovery of hydrocarbons are in order. At the time of the first

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run of a layer, the flow of the crude oil towards the surface is due to the energy stored in the gases under pressure in the natural hydraulic system. To mitigate the consecutive decline of production and the decomposition of the site, water injections are carried out, well before the normal exhaustion of the layer. The water is injected through wells with high pressure, by pumps specially drilled to this end. The pumps allow the displacement of the crude oil towards the wells of production. The wells must be judiciously distributed, which gives rise to a difficult problem of optimal control: how to choose the best installation sites of the production wells? This is precisely the question we address in this work: our main goal is to present a method to carry out the optimal control of (1) with respect to all the important parameters that intervene in the process.

More precisely, we seek necessary conditions for the admissible parameters $u$, $p$ and $f$ to minimize the functional

$$J(u, p, f) = \frac{1}{2} \| u - U \|_{2,Q_T}^2 + \frac{1}{2} \| p - P \|_{2,Q_T}^2 + \frac{\beta_1}{2} \| f \|_{2,q_0,Q_T}^{2q_0} + \frac{\beta_2}{2} \| \partial_t f \|_{2,Q_T}^2$$

(2)

where $q_0 > 1$ and $\beta_1 > 0$ and $\beta_2 > 0$ are two coefficients of penalization. The first two terms in (2) make possible to minimize the difference between the reduced saturation of oil $u$, the global pressure $p$ and the given data $U$ and $P$.

Existence and uniqueness to the system (1), for the case when the term $\partial_t p$ is missing but for more general boundary conditions, is established in [1]. In [2] we obtain conditions which provide existence and regularity of the optimal solutions to the problem of minimizing (2) subject to (1). Here we are interested to obtain necessary optimality conditions which permit to find the solutions predicted by the results in [2]. This is, to the best of our knowledge, an important open question.

Several techniques for deriving optimality conditions are available in the literature of optimal control systems governed by partial differential equations [3, 4, 5]. We obtain the optimality conditions by making use of a Lagrangian approach recently used with success by O. Bodart, A. V. Boureau and R. Touzani for an optimal control problem of the induction heating [6], and by H.-C. Lee and T. Shilkin for the thermistor problem [7].

2 Preliminaries

Our main objective is to obtain necessary conditions for a triple $(\bar{u}, \bar{p}, \bar{f})$ to minimize (2) among all the functions $(u, p, f)$ verifying (1). The intended necessary optimality conditions are proved in §3 under adequate hypotheses on the data of the problem, which provide regularity of the optimal solution.

2.1 Notation and Functional Spaces

In the sequel we assume that $\varphi$, $g$ and $d$ are real valued $C^1$-functions satisfying:

$$(H1) \ 0 < c_1 \leq d(r), \ \varphi(r) \leq c_2; \ |d'(r)|, \ |\varphi'(r)|, \ |\varphi''(r)| \leq c_3 \ \forall r \in \mathbb{R}.$$
(H2) $u_0, p_0 \in C^2(\bar{\Omega})$, $U, P \in L^2(\Omega_T)$, where $u_0, p_0 : \Omega \to \mathbb{R}$, $U, P : \Omega_T \to \mathbb{R}$, and $u_0|_{\partial\Omega} = p_0|_{\partial\Omega} = 0$.

We consider the following spaces:

$W^{1,0}_p(\Omega_T) := L^p(0, T, W^{1,0}_p(\Omega)) = \{ u \in L^p(\Omega_T), \nabla u \in L^p(\Omega_T) \},$

endowed with the norm $\| u \|_{W^{1,0}_p(\Omega_T)} = \| u \|_{p, \Omega_T} + \| \nabla u \|_{p, \Omega_T}$;

$W^{2,1}_p(\Omega_T) := \{ u \in W^{1,0}_p(\Omega_T), \nabla^2 u, \partial_t u \in L^p(\Omega_T) \},$

with the norm $\| u \|_{W^{2,1}_p(\Omega_T)} = \| u \|_{W^{1,0}_p(\Omega_T)} + \| \nabla^2 u \|_{p, \Omega_T} + \| \partial_t u \|_{p, \Omega_T}$;

$V := \left\{ u \in W^{1,0}_2(\Omega_T), \partial_t u \in L^2(0, T, W^{-1}_2(\Omega)) \right\};$

$W := \left\{ u \in W^{2,1}_2(\Omega_T), u|_{\partial\Omega \times (0, T)} = 0 \right\};$

$\mathcal{T} := \left\{ f \in L^q(\Omega_T), \partial_t f \in L^2(\Omega_T) \right\};$

$H := L^q(\Omega_T) \times W^{2-\frac{2}{q}}_{\frac{q}{2}}(\Omega).$

### 2.2 Coercive Estimate

The following lemma provides a coercive estimate to linear parabolic systems that is useful for our purposes.

**Lemma 2.1 (S).** Let $\Omega$ be a bounded domain with a $C^2$-boundary, and assume that

$A_{ijkl} \in C(\bar{\Omega}_T), \quad b_{ijk} \in L^r(\Omega_T), \quad c_{ij} \in L^r(\Omega_T)$

with $r > n + 2$ and $A$ satisfying the strong ellipticity condition:

$\exists \gamma_0 > 0 : A_{ijkl}(u)B_{ijkl} \geq \gamma_0 |B|^2 \quad \forall B \in \mathbb{M}^{n \times n},$

$A_{ijkl}A_{ijkl} = A_{ijkl} = A_{ijkl}.$

Then, for any $s \in (1, r), \ s \neq \frac{3}{2}$, and for arbitrary functions $f \in L^q(\Omega_T, \mathbb{R}^n)$ and $u_0 \in W^{2-\frac{2}{q}}_{\frac{q}{2}}(\Omega, \mathbb{R}^n)$ there exists a unique solution $u \in W^{2,1}_s(\Omega_T, \mathbb{R}^n)$ of the problem

$\partial_t u - A_{ijkl}(x)u_{ijkl} + b_{ijk}(x)u_{jk} + c_{ij}(x)u_j = f_i(x),$

$u|_{t=0} = u_0, \quad u|_{\partial\Omega} = 0.$

Moreover, the estimate

$\| u \|_{W^{2,1}_s(\Omega_T)} \leq c \left( \| f \|_{s, \Omega_T} + \| u_0 \|_{W^{2-\frac{2}{q}}_{\frac{q}{2}}(\Omega)} \right)$

holds for some constant $c$ depending only on $n, \Omega, \sigma_0, \gamma_0$ and the norms of the coefficients.
2.3 Existence of Optimal Solution

The following existence theorem is proved in [2] using a technical lemma found in [9], Young’s inequality and Aubin’s Lemma, together with the theorem of Lebesgue and some compactness arguments of J. L. Lions [3]. The conclusion follows from the fact that J is lower semicontinuous with respect to the weak convergence.

Theorem 2.2 ([2]). Under the hypotheses (H1) and (H2) there exists a \( q > 1 \), depending on the data of the problem, such that the problem of minimizing (2) subject to (1) has an optimal solution \( (\bar{u}, \bar{p}, \bar{f}) \) satisfying \( \bar{u} \in W^{2,1}_q(Q_T) \), \( \bar{p} \in C ([0,T]; L^2(\Omega)) \cap W^{1,0}_2(Q_T) \), \( \partial_t \bar{p} \in L^2 (0,T, W^{-1}_2(\Omega)) \), \( \bar{f} \in L^{2q_0}(Q_T) \), \( \partial_t \bar{f} \in L^2(Q_T) \).

2.4 Regularity of Solutions

Regularity of the solutions given by Theorem 2.2 is also proved in [2]. Theorem 2.3 is obtained using Young’s and Holder’s inequalities, Gronwall Lemma, De Giorgi-Nash-Ladyzhenskaya-Uraltseva theorem, an estimate from [10], and some technical lemmas found in [11].

Theorem 2.3 ([2]). Let \( (\bar{u}, \bar{p}, \bar{f}) \) be an optimal solution to the problem of minimizing (2) subject to (1). Suppose that (H1) and (H2) are satisfied. Then, there exist \( \alpha > 0 \) such that the following regularity conditions hold:

\[
\bar{u} \in C^{\alpha, \frac{\alpha}{2}} (Q_T), \quad \bar{p} \in W^{1,0}_q(Q_T), \quad \bar{p}_0 \in W^{2,1}_q(Q_T), \quad \partial_t \bar{p} \in L^\infty (0,T; L^2(\Omega)) \cap W^{1,0}_2(Q_T),
\]

\[
\bar{u} \in C^{\frac{\alpha}{4}}(Q_T), \quad \bar{u} \in W^{2,1}_q(Q_T), \quad \bar{p} \in W^{2,1}_q(Q_T).
\]

3 Main Results

We define the following nonlinear operator corresponding to (1):

\[
F : W \times W \times \Upsilon \rightarrow H \times H (u, p, f) \rightarrow F(u, p, f) = 0
\]

where

\[
F(u, p, f) = \left( \begin{array}{l}
\partial_t u - \Delta \varphi(u) - \operatorname{div}(g(u)\nabla p), \\
\partial_t p - \operatorname{div}(d(u)\nabla p) - f
\end{array} \right),
\]

\( \gamma_0 u = u|_{t=0} \). Owing to the estimate

\[
\|v\|_{W^{1,0}_{\frac{q}{q-1}}(Q_T)} \leq c \|v\|_{W^{2,1}_{2q}(Q_T)}, \quad \forall v \in W^{2,1}_q(Q_T), \quad 1 < q < 2
\]

(see [11]) and hypothesis (H1), we have

\[
\varphi''(u) |\nabla u|^2, \quad g'(u) |\nabla u| \nabla p, \quad d(u) |\nabla u| \nabla p \in L^{\frac{2q}{q-1}}(Q_T) \subset L^{2q}(Q_T).
\]

Thus, it follows that \( F \) is well defined.
3.1 Gâteaux differentiability

Theorem 3.1. In addition to the hypotheses (H1) and (H2), let us suppose that
\[(H3) \quad |\varphi'''| \leq c.\]

Then, the operator $F$ is Gâteaux differentiable and its derivative is given by
\[
\delta F(u,p,f)(e,w,h) = \frac{d}{ds} F(u + se, p + sw, f + sh) \bigg|_{s=0} = (\delta F_1, \delta F_2)
\]
\[
= \left( \frac{\partial e - \text{div} (\varphi'(u)\nabla e)}{\partial t} - \text{div} (\varphi''(u)e\nabla e) - \text{div} (g(u)\nabla w) - \text{div} (g'(u)e\nabla p), \quad \gamma_0 e \right),
\]
for all $(e, w, h) \in W \times W \times \mathcal{Y}$.

Furthermore, for any optimal solution $(\bar{u}, \bar{p}, \bar{f})$ of the problem of minimizing $F$ among all the functions $(u, p, f)$ satisfying $F$, the image of $\delta F (\bar{u}, \bar{p}, \bar{f})$ is equal to $H \times H$.

To prove Theorem 3.1 we make use of the following lemma.

Lemma 3.2. The operator $\delta F(u,p,f) : W \times W \times \mathcal{Y} \rightarrow H \times H$ is linear and bounded.

Proof. of Lemma 3.2 We have for all $(e, w, h) \in W \times W \times \mathcal{Y}$
\[
\delta_p F_2(u,p,f)(e,w,h) = \partial w - \text{div} (d(u)\nabla w) - \text{div} (d'(u)e\nabla p) - h
\]
\[
= \partial w - d(u)\nabla w - d'(u)e\nabla w - d'(u)e\nabla p - h,
\]
with $\delta_p F$ the Gâteaux derivative of $F$ with respect to $p$. Then, using hypothesis (H1), we obtain that
\[
\|\delta_p F_2(u,p,f)(e,w,h)\|_{2q,Q_T} \leq \|\partial w\|_{2q,Q_T} + \|\nabla w\|_{2q,Q_T} + c\|\nabla w\|_{2q,Q_T} + c\|\nabla e\|_{2q,Q_T} + c\|\nabla e\|_{2q,Q_T} + c\|\nabla e\|_{2q,Q_T} + c\|\nabla e\|_{2q,Q_T} + c\|\nabla e\|_{2q,Q_T}.
\]
In what follows we consider the term $\|e\nabla u, \nabla p\|_{2q,Q_T}$. Similar arguments apply to the remaining terms of (3). We have
\[
\|e\nabla u, \nabla p\|_{2q,Q_T} \leq \|e\|_{\infty,Q_T} \|\nabla u, \nabla p\|_{2q,Q_T}
\leq \|e\|_{\infty,Q_T} \|\nabla u\|_{4q,4Q_T} \|\nabla p\|_{4q,4Q_T}
\leq c\|u\|_{W} \|p\|_{W} \|e\|_{W}.
\]
Then,
\[
\|\delta_p F_2(u,p,f)(e,w,h)\|_{2q,Q_T} \leq c (\|u\|_{W}, \|p\|_{W}, \|f\|_{\mathcal{Y}}) (\|e\|_{W} + \|w\|_{W} + \|h\|_{\mathcal{Y}}).
\]
On the other hand,

\[
\delta_u F_1(u, p, f)(e, w, h) = \partial_t e - \operatorname{div} (\varphi'(u)\nabla e) - \operatorname{div} (\varphi''(u)e\nabla u) \\
- \operatorname{div} (g(u)\nabla w) - \operatorname{div} (g'(u)e\nabla p) \\
= \partial_t e - \varphi'(u)\triangle e - \varphi''(u)\nabla u.\nabla e \\
- \varphi''(u)e\triangle u - \varphi''(u)\nabla u \cdot \nabla e - \varphi'''(u)e\nabla u^2 \\
- g(u)\triangle w - g'(u)e\nabla u.\nabla w \\
- g'(u)e\triangle p - g'(u)e\nabla p \cdot \nabla e - g''(u)e\nabla u \cdot \nabla p,
\]

where \(\delta_u F\) is the Gâteaux derivative of \(F\) with respect to \(u\). The same argument as above give that

\[
\|\delta_u F_1(u, p, f)(e, w, h)\|_{2Q_T} \leq c(\|u\|_W, \|p\|_W, \|f\|_T)(\|e\|_W + \|w\|_W + \|h\|_T).
\]

Consequently, by (4) and (5) we can write

\[
\|\delta F(u, p, f)(e, w, h)\|_{H \times H \times \mathcal{T}} \leq c(\|u\|_W, \|p\|_W, \|f\|_T)(\|e\|_W + \|w\|_W + \|h\|_T).
\]

\(\Box\)

\textit{Proof.} of Theorem 3.1. In order to show that the image of \(\delta F(\pi, \bar{p}, \tilde{f})\) is equal to \(H \times H\), we need to prove that there exists a \((w, e, h) \in W \times W \times \mathcal{T}\) such that

\[
\begin{align*}
\partial_t e - \operatorname{div} (\varphi'(\pi)\nabla e) - \operatorname{div} (\varphi''(\pi)e\nabla \pi) - \operatorname{div} (g(\pi)\nabla w) - \operatorname{div} (g'(\pi)e\nabla p) &= \alpha, \\
\partial_t w - \operatorname{div} (d(\pi)\nabla w) - \operatorname{div} (d'(\pi)e\nabla \pi - \nabla e) &= \beta,
\end{align*}
\]

for any \((\alpha, a)\) and \((\beta, b)\) \in \(H\). Writing the system (6) for \(h = 0\) as

\[
\begin{align*}
\partial_t e - \varphi'(\pi)\triangle e - 2\varphi''(\pi)\nabla \pi \cdot \nabla e - \varphi''(\pi)e\triangle \pi - \varphi'''(\pi)e\nabla \pi^2, \\
- g(\pi)\triangle w - g'(\pi)e\nabla \pi.\nabla w - g'(\pi)e\triangle \pi - g''(\pi)e\nabla \pi.\nabla e - g'''(\pi)e\nabla \pi &= \alpha, \\
\partial_t w - d(\pi)\triangle w - d'(\pi)e\nabla \pi.\nabla w - d'(\pi)e\triangle \pi - d''(\pi)e\nabla \pi.\nabla e - d'''(\pi)e\nabla \pi &= \beta,
\end{align*}
\]

it follows from the regularity of the optimal solution (Theorem 2.3) that

\[
\varphi''(\pi)e\nabla \pi^2, \varphi''(\pi)e\nabla \pi^2, g'(\pi)e\nabla \pi.\nabla e, g''(\pi)e\nabla \pi.\nabla e, d'(\pi)e\nabla \pi.\nabla e, d''(\pi)e\nabla \pi.\nabla e, d'''(\pi)e\nabla \pi \in L^{2q_0}(Q_T),
\]

\[
\varphi''(\pi)e\nabla \pi, g'(\pi)e\nabla \pi, g''(\pi)e\nabla \pi, d'(\pi)e\nabla \pi, d''(\pi)e\nabla \pi, d'''(\pi)e\nabla \pi \in L^{4q_0}(Q_T).
\]

By Lemma 2.1 there exists a unique solution of the system (7), hence there exists \((e, w, 0)\) verifying (6). We conclude that the image of \(\delta F\) is equal to \(H \times H\). \(\Box\)
3.2 Necessary Optimality Condition

We consider the cost functional $J : W \times W \times Y \to \mathbb{R}$ and the Lagrangian $\mathcal{L}$ defined by

$$\mathcal{L}(u, p, f, p_1, e_1, a, b) = J(u, p, f) + \left\langle F(u, p, f), \left( \begin{array}{c} p_1 \\ e_1 \\ a \\ b \end{array} \right) \right\rangle,$$

where the bracket $\langle \cdot, \cdot \rangle$ denote the duality between $H$ and $H'$.

**Theorem 3.3.** Under hypotheses (H1)–(H3), if $(\pi, \overline{\pi}, \overline{f})$ is an optimal solution to the problem of minimizing (2) subject to (1), then there exist Lagrange multipliers $(\pi_1, \overline{\pi}_1) \in W_2^2(Q_T) \times W_2^1(Q_T)$ satisfying the following conditions:

$$\partial_t \pi_1 + \text{div} (\varphi'(\pi) \nabla e_1) - d'(\pi) \nabla \pi \cdot \nabla \overline{\pi} - \varphi''(\pi) \nabla \pi \cdot \nabla \overline{\pi} - g'(\pi) \nabla \pi \cdot \nabla \overline{\pi} = \pi - U,$$

$$\text{s.t.} \quad \overline{\pi}_1|_{\partial \Omega} = 0, \quad \overline{\pi}_1|_{t=T} = 0,$$

$$\partial_t \overline{\pi}_1 + \text{div} (d(\pi) \nabla \overline{\pi}_1) + d(\pi) \nabla \pi \cdot \nabla \overline{\pi}_1 = \overline{\pi} - P,$$

$$\text{s.t.} \quad \overline{\pi}_1|_{\partial \Omega} = 0, \quad \overline{\pi}_1|_{t=T} = 0,$$

$$- \beta_2 \frac{\partial^2 \overline{f}}{\partial t^2} + 2q_0 \beta_1 |\overline{f}|^{2q_0-2} \overline{f} = \overline{p}_1, \quad \left. \frac{\partial \overline{f}}{\partial t} \right|_{t=0} = \left. \frac{\partial \overline{f}}{\partial t} \right|_{t=T} = 0. \quad (8)$$

**Proof.** Let $(\pi, \overline{\pi}, \overline{f})$ be an optimal solution to the problem of minimizing (2) subject to (1). It is well known (cf. e.g. [12]) that there exist Lagrange multipliers $(\overline{\pi}_1, \pi_1, (\pi_1, \overline{\pi}_1, \overline{f})) \in H' \times H'$ verifying

$$\delta_{(u, p, f)} \mathcal{L}(\pi, \overline{\pi}, \overline{f}, \pi_1, \overline{\pi}_1, \overline{f}) (e, w, h) = 0 \quad \forall (e, w, h) \in W \times W \times Y,$$

with $\delta_{(u, p, f)} \mathcal{L}$ the Gâteaux derivative of $\mathcal{L}$ with respect to $(u, p, f)$. We then obtain

$$\int_{Q_T} ((\pi - U)e + (\overline{\pi} - P)w + 2q_0 \beta_1 |\overline{f}|^{2q_0-2} \overline{f} + \beta_2 \partial_t \overline{f} \partial_t h) \, dx dt$$

$$- \int_{Q_T} (\partial_t e - \text{div} (\varphi'(\pi) \nabla e) - d'(\pi) \nabla \pi \cdot \nabla \overline{\pi} - \varphi''(\pi) \nabla \pi \cdot \nabla \overline{\pi} - g'(\pi) \nabla \pi \cdot \nabla \overline{\pi} - \text{div} (g(\pi) \nabla w) - \text{div} (g'(\pi) \nabla \overline{\pi})) \, dx dt$$

$$- \int_{Q_T} (\partial_t w - \text{div} (d(\pi) \nabla w) - \text{div} (d'(\pi) \nabla \overline{\pi} - h) \, dx dt$$

$$- (\gamma_0 e, \pi) + (-\gamma_0 w, \overline{f}) = 0 \quad \forall (e, w, h) \in W \times W \times Y.$$

This last system is equivalent to the following one:

$$\int_{Q_T} ((\pi - U)e - d'(\pi) \nabla \pi \cdot \nabla \overline{\pi} + \partial_t e \overline{\pi} - \text{div} (\varphi'(\pi) \nabla e) \overline{\pi}$$

$$- \text{div} (\varphi''(\pi) \nabla \pi \cdot \nabla \overline{\pi} - g'(\pi) \nabla \pi \cdot \nabla \overline{\pi}) \, dx dt$$

$$+ \int_{Q_T} ((\overline{\pi} - P)w + \partial_t w \overline{\pi} - \text{div} (d(\pi) \nabla w) \overline{\pi} - \text{div} (g(\pi) \nabla w) \overline{\pi}) \, dx dt$$

$$+ \int_{Q_T} (2q_0 \beta_1 |\overline{f}|^{2q_0-2} \overline{f} + \beta_2 \partial_t \overline{f} \partial_t h - \overline{p}_1 h) \, dx dt$$

$$+ (\gamma_0 e, \pi) + (-\gamma_0 w, \overline{f}) = 0 \quad \forall (e, w, h) \in W \times W \times Y.$$


In others words, we have

\[
\int_{Q_T} (\pi - U) + d'(u)\nabla p, \nabla \pi \nabla - \partial_t \pi - \text{div} (\varphi'(\pi) \nabla \pi) + \varphi''(\pi) \nabla \pi, \nabla \pi + g'(u) \nabla \pi, \nabla \pi) e \, dx \, dt
\]

+ \int_{Q_T} (\langle \pi - P \rangle + \partial_t P - \text{div} (d(\pi) \nabla \pi) - \text{div} (g(\pi) \nabla \pi)) w \, dx \, dt

+ \int_{Q_T} (2q_0 \beta_1 |T|^{2u_0 - 2T}h + \beta_2 \partial_t \partial_t h - \partial_t h) \, dx \, dt

+ (\gamma_0 e, \overline{\pi}) + (\gamma_0 w, \overline{h}) = 0 \forall (e, w, h) \in W \times W \times \Omega.

(9)

Consider now the system

\[
\begin{align*}
\partial_t e_1 + \text{div} (\varphi'(\pi) \nabla e_1) - d'(\pi) \nabla p_1, \nabla e_1 - \varphi''(\pi) \nabla \pi, \nabla e_1 - g'(u) \nabla \pi, \nabla e_1 &= \pi - U, \\
\partial_t p_1 + \text{div} (d(\pi) \nabla p_1) + \text{div} (g(\pi) \nabla e_1) &= p - P, \\
e_1|_{\partial \Omega} &= p_1|_{\partial \Omega} = 0, \quad e_1|_{t=T} = p_1|_{t=T} = 0.
\end{align*}
\]

(10)

It follows by Lemma 2.1 that (10) has a unique solution $(e_1, p_1) \in W^{2,1}_2(Q_T) \times W^{2,1}_2(Q_T)$. Since the problem of finding $(e, w)$ in $W \times W$ satisfying

\[
\begin{align*}
\partial_t e - \text{div} (\varphi'(\pi) \nabla e) - \text{div} (\varphi''(\pi)e \nabla \pi) - \text{div} (g(\pi) \nabla w) - \text{div} (g'(\pi)e \nabla \pi) &= \text{sign}(e_1 - \overline{\pi}), \\
\partial_t w - \text{div} (d(\pi) \nabla w) - \text{div} (d'(\pi)e \nabla \pi) &= \text{sign}(p_1 - \overline{p_1}),
\end{align*}
\]

\[
\gamma_0 e = \gamma_0 w = 0
\]

is uniquely solvable on $W^{2,1}_2 \times W^{2,1}_2$ by Lemma 2.1 by choosing $h = 0$ in (9), multiplying (10) by $(e, w)$, integrating by parts, and making the difference with (9), we obtain

\[
\begin{align*}
\int_{Q_T} (\partial_t e - \text{div} (\varphi'(\pi) \nabla e) - \text{div} (\varphi''(\pi)e \nabla \pi) - \text{div} (g(\pi) \nabla w) - \text{div} (g'(\pi)e \nabla \pi))(e_1 - \overline{\pi}) \, dx \, dt
\]

+ \int_{Q_T} (\partial_t w - \text{div} (d(\pi) \nabla w) - \text{div} (d'(\pi)e \nabla \pi))(p_1 - \overline{p_1}) \, dx \, dt

+ (\gamma_0 e, \gamma_0 e_1 - \overline{\pi}) + (\gamma_0 w, \gamma_0 p_1 - \overline{h}) = 0 \quad \forall (e, w) \in W \times W.
\]

(11)

Choosing $(e, w)$ in (12) as the solution of the system (11), we have

\[
\int_{Q_T} \text{sign}(e_1 - \overline{\pi})(e_1 - \overline{\pi}) \, dx \, dt + \int_{Q_T} \text{sign}(p_1 - \overline{p_1})(p_1 - \overline{p_1}) \, dx \, dt = 0.
\]

It follows that $e_1 = \overline{\pi}$ and $p_1 = \overline{p_1}$. Coming back to (12), we obtain $\gamma_0 e_1 = \pi$ and $\gamma_0 p_1 = h$. On the other hand, choosing $(e, w) = (0, 0)$ in (9) it follows (8), which conclude the proof of Theorem 3.3.
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