RACE: Sub-Linear Memory Sketches for Approximate Near-Neighbor Search on Streaming Data

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Abstract

We present the first sublinear memory sketch which can be queried to find the ν nearest neighbors in a dataset. Our online sketching algorithm can compress an N-element dataset to a sketch of size $O(N^b \log^3 N)$ in $O(N^{b+1} \log^3 N)$ time, where $b < 1$ when the query satisfies a data-dependent near-neighbor stability condition.

We achieve data-dependent sublinear space by combining recent advances in locality sensitive hashing (LSH)-based estimators with compressed sensing. Our results shed new light on the memory-accuracy tradeoff for near-neighbor search. The techniques presented reveal a deep connection between the fundamental compressed sensing (or heavy hitters) recovery problem and near-neighbor search, leading to new insight for geometric search problems and implications for sketching algorithms.
1 Introduction

Approximate near-neighbor search (ANNS) is a fundamental problem with widespread applications in databases, learning, computer vision, and much more [1]. Furthermore, ANNS is the first stage of several data processing and machine learning pipelines and is a popular baseline data analysis method. Informally, the problem is as follows. Given a dataset \( \mathcal{D} = x_1, x_2, \ldots, x_N \), observed in a one pass sequence, build a data structure \( \mathcal{S} \) that can efficiently identify a small number of data points \( x_i \in \mathcal{D} \) that have high similarity to any dynamically generated query \( q \).

In this paper, we focus on low-memory ANNS in settings where it is prohibitive to store the complete data in any form. Such restrictions naturally arise in extremely large databases, computer networks, and internet-of-things systems [2]. We want to compress the dataset \( \mathcal{D} \) into a sketch \( \mathcal{S} \) that is as small as possible while still retaining the ability to find near-neighbors for any query. Furthermore, the algorithm should be one pass as the second pass is prohibitive when we cannot store the data in any form. It is common wisdom that the size of \( \mathcal{S} \) must scale linearly (\( \geq O(N) \)). In this work, we challenge that wisdom by constructing a sketch of size \( O(N^b \log^3 N) \) bits in \( O(N^{b+1} \log^3 N) \) time. Our sketch can identify near-neighbors for stable queries with high probability in \( O(N^{b+1} \log^3 N) \) time. The value of \( b \) is data-dependent and can be significantly less than 1 for many applications of practical importance.

1.1 Main Results

Our main contribution is a one-pass algorithm that produces a sketch \( \mathcal{S} \) which solves the exact \( v \)-nearest neighbor problem in sub-linear memory with high probability. A formal problem statement for the \( v \)-nearest neighbor problem is available in Section 2.1. We obtain the following theorem.

Theorem 1. Main Theorem

Given a query \( q \), a dataset \( \mathcal{D} \) in a metric space \( (\mathcal{M}, d(\cdot, \cdot)) \), and a \( (d(q, x_v), d(q, x_{v+1}), p_1, p_2) \)-sensitive hash function with a finite range \( r \), we can construct a sketch \( \mathcal{S} \) that solves the exact \( v \)-nearest neighbor problem with probability \( 1 - \delta \) using

\[
O \left( (v + 1)^3 N^b \log^3 \left( \frac{N}{\delta} \right) \right)
\]

bits, where

\[
b = \frac{6|\log p_1| + 2 \log r}{\log \frac{1}{\Delta}}
\]

\( x_v \) is the \( v \)th nearest neighbor of \( q \) in \( \mathcal{D} \), \( x_{v+1} \) is the \( (v + 1) \)th nearest neighbor of \( q \) in \( \mathcal{D} \), and \( \Delta = \frac{p_2}{p_1} \leq 1 \).

Our algorithm requires \( O(N^{b+1} \log^3 N) \) time to construct \( \mathcal{S} \) and the same time to return the \( v \) nearest neighbors for a query. In our theorem, \( \Delta \) is a data-dependent value that characterizes the difficulty of the nearest neighbor problem for a given query. Our guarantees are general and work for any query and any value of \( \Delta \). However, the sketch is only sub-linear when \( b < 1 \). Our approach supports a large class of data-dependent assumptions; with stronger assumptions we can obtain smaller sketches. Such assumptions are frequently valid in practice - many common cases satisfy our requirements. We support our theoretical findings with real experiments on real social-network datasets.

It should be noted that the exact \( v \)-nearest neighbor problem is a stronger near-neighbor problem formulation than the \((R, cR)\)-approximate near-neighbor problem that is popular in the literature. By a corollary to our main theorem, our sketch also provides a data-dependent sub-linear solution for this problem. Our techniques have two main implications for sketching and data-dependent geometric problems.

First, our analysis reveals a deep link between compressed sensing and near-neighbor search. Surprisingly, the hardness of near-neighbor is directly related to the notions of sparsity, or signal-to-noise-ratio (SNR), in compressed sensing [3, 4]. This connection allows us to analyze data assumptions in the context of the very well-studied compressed sensing framework. Data-dependent assumptions are gaining popularity because they can provide strong guarantees. For instance, the first improvements over the seminal near-neighbor
| Method                        | Sketch Size (bits) | Sketch Time | Comments                      |
|-------------------------------|--------------------|-------------|-------------------------------|
| No compression                | \(dN \log N\)     | N/A         | -                             |
| Random projections \[7\]      | \(N \log^2 N\)    | N log \(N\) | Widely used in practice       |
| Compressed clustering tree \[8\] | \(N \log N\)   | \(Nd \log^{O(1)} N\) | Multiple passes              |
| Coresets \[9\]               | \(d\epsilon^{-(d-1)}\) | \(N + \epsilon^{-(d-1)}\) | Multiple Passes and ANNS only |
| This work                     | \(N^b \log^3 N\)   | \(N^{b+1} \log^3 N\) | Data-dependent \(b < 1\)     |

Table 1: Summary of related work on limited memory near-neighbor search. The results are shown for a \(d\)-dimensional dataset of \(N\) points. Existing methods \[7, 8, 9\] can estimate distances to all points in the dataset with a \(1 \pm \epsilon\) multiplicative error. The full dependence on \(\epsilon\) is not shown here. Note that core-sets perform poorly in high dimensions, even if one uses \[7\] to reduce \(d\) to \(O(\log N)\). Our method estimates similarities with a \(\pm \epsilon\) additive error, where \(b\) depends on the properties of the dataset.

search results of \[5\] were obtained using data-dependent hashing \[4\]. We use data-dependent methods for a similar purpose. In general, the communication complexity of near neighbor problems is \(O(N)\). Our method breaks the linear memory bound because of our data assumptions.

Second, our use of compressed sensing provides a way to construct sketches that can process queries not seen during sketching. The sketching literature has primarily considered problems involving data that has already been seen by the sketch. Examples of such problems include estimating frequency counts, norms, and statistics of a dataset. Our sketch encodes the relationship between the dataset and any number of arbitrary, unknown queries. To the best of our knowledge, our techniques yield the first sub-linear sketch that can provably represent important structural relationships between the dataset and any query.

### 1.2 Related Work

The problem of finding near-neighbors in sub-linear time is a very well-studied problem with several solutions \[5\]. However, the memory requirement for near-neighbor search has only recently started receiving attention \[8, 10\]. The best theoretical result in this direction requires \(O(N \log N)\) memory and therefore does not break the linear memory bound \[8\]. Table 1 contains a summary of existing work in the area. To the best of our knowledge, the algorithm described in this paper is the first to perform near-neighbor search using sub-linear memory.

**Coresets or Clustering Based Approaches:** A reasonable approach for compressing data in the context of near-neighbor search is to construct a coreset or to represent the dataset as a set of clusters. Such methods can obtain a subset \(P\) of \(D\) that is a good representation of the dataset. More formally, given \(\epsilon > 0\), for every \(x \in D\) we can guarantee the existence of a point \(p \in P\) such that \(d(p, x) < \epsilon\). However, our procedure is superior in the following two regards. First, our approach works in the one-pass streaming setting. Coresets and cluster-based compression methods require access to the entire dataset at once to determine which points to retain in the sketch. Therefore, it is impossible to stream queries to the sketch efficiently. One pass online is a hard requirement for many high-speed similarity search applications.

Second, cluster or coreset approximation of the data cannot solve the exact \(v\)-nearest neighbor problem because the sketching process removes points from the dataset. Coresets and cluster-based compression methods guarantee the existence of a point within a \(\epsilon\) distance to the query. However, there may be any number of near-neighbors within \(\epsilon\) of the query that have been discarded during sketching. In summary, these methods cannot solve the \(v\)-nearest neighbor problem with high probability, as they throw away data instances entirely, akin to sampling. On the contrary, our sketch can report any point from the dataset at query time and hence can report the true top-\(v\) near neighbors with high probability. To the best of our knowledge, no non-trivial algorithm in the literature can report exact top-\(v\) near-neighbors with high probability. This new possibility could be of independent interest in itself.
Perhaps most importantly, our method requires weaker assumptions about the dataset. Cluster-based methods assume that the dataset has a clustered structure and store a small collection of central points for each cluster. Coreset methods also require a clustering assumption to achieve high compression ratios. Our method is valid even when there is no efficient cluster representation.

1.3 Techniques

We obtain our results by combining recent advances in locality-sensitive hashing (LSH)-based estimation with standard compressed sensing techniques.

LSH-based estimators: Recent work has shown that LSH can be used for efficient unbiased statistical estimation \cite{11, 12, 13}. In particular, the array-of-counts estimator (ACE) in \cite{13} provides an unbiased estimate for sums of all the LSH collision probabilities in a dataset. The first step of our approach is to extend ACE to estimate arbitrary linear combinations of collision probabilities. We repeat the ACE method (RACE) to get sharp estimates of these linear combinations. Under data-dependent assumptions, we only require $O(\epsilon^{-2} \log(1/\delta))$ ACE repetitions to obtain estimates with an $\epsilon$ additive accuracy. Once we have sharp estimates, we apply standard compressed sensing techniques.

Compressed sensing: A central result of compressed sensing is that a $v$ sparse vector of length $N$ can be recovered from $O(v \log N/v)$ linear combinations of its elements. The coefficients of the linear combination are defined by the sensing or measurement matrix, which is a very well-studied topic in compressed sensing. We choose the RACE linear combination coefficients so that they describe a valid measurement matrix. In this context, our measurements are of the vector $s(q) \in \mathbb{R}^N$, where the $i^{th}$ component of $s(q)$ is the LSH collision probability of $q$ and $x_i \in \mathcal{D}$. That is, $(s(q))_{i} = p(x_i, q)$. Therefore, we require $O(v \log N/v)$ RACEs to estimate all of the measurements, where $v$ is the sparsity of $s(q)$. By performing compressed sensing recovery for $s(q)$, we can obtain the LSH collision probabilities between $q$ and each element in the dataset. Elements with large collision probabilities are the near-neighbors of $q$.

Data-dependent guarantees: To identify near-neighbors for a query $q$ from our estimated measurements, we recover $s(q)$ and report the indices with the largest values (the heavy hitters) as the identities of near-neighbors. For successful compressed sensing recovery, $s(q)$ must obey a sparsity condition. This is where our main data-dependent assumptions enter the picture. To identify the heavy-hitting indices, we need $s(q)$ to contain only a few large elements. The geometric interpretation of this condition is that most elements in the dataset are not near-neighbors of $q$. If the dataset already satisfies our sparsity condition, then we may proceed directly to recovery.

If not, then we can force $s(q)$ to be sparse using standard methods for amplifying a LSH family. Specifically, we can construct a hash function from $K$ independent realizations of a LSH family with collision probability $p(x, q)$ to get a new LSH family with a larger range and a collision probability $p(x, q)^K$. Therefore, by amplifying the LSH family used to construct the sketch, we also amplify any sparsity that is already present in $s(q)$. We show that well-established notions of near-neighbor stability \cite{14} are equivalent to weak sparsity conditions on $s(q)$, allowing us to express our guarantees in terms of near neighbor stability. This result establishes a deep link between compressed sensing sparsity and the difficulty of the near-neighbor search problem. We can analyze a large class of common data-dependent geometric assumptions by interpreting them as sparsity conditions.

Reduce near-neighbor to compressed sensing recovery: Using compressed sensing, we can estimate the components of $s(q)$ within an $\epsilon$ additive tolerance. To solve the near-neighbor problem, we make $\epsilon$ small enough to distinguish between near-neighbors and the rest of the dataset. The value of $\epsilon$ depends on $K$; increasing $K$ makes $s(q)$ sparse but also decreases the magnitude of the components of $s(q)$. Balancing the sparsity requirement with $\epsilon$ introduces a data-dependent multiplicative $O(N^\epsilon)$ term. This term is sub-linear ($b < 1$) when $s(q)$ is sufficiently sparse or, equivalently, when $q$ is a stable query. In general, our sketch requires $O(N^b \log^3(N))$ bits, where $b$ depends on the query stability.
2 Preliminary

2.1 Formal Problem Statements

In this paper, we solve the exact $v$-nearest neighbor problem. The $v$-nearest neighbor problem is to identify all of the $v$ closest points to a query with high probability. When $v = 1$ we have the well-known nearest neighbor problem.

Definition 1. Exact $v$-nearest neighbor

Given a set $D$ of points in a $d$-dimensional space and a parameter $v$, construct a data structure which, given any query point $q$, reports a set of $v$ points in $D$ with the following property: Each of the $v$ nearest neighbors to $q$ is in the set with probability $1 - \delta$.

The well-known $(R, c)$-approximate near-neighbor problem is a relaxation of the $v$-nearest neighbor problem. We use the standard definition of this problem.

Definition 2. Randomized $(R, c)$-approximate near-neighbor

Given a set $D$ of points in a $d$-dimensional space and parameters $R > 0$, $c > 1$ and $\delta > 0$, construct a data structure which, given any query point $q$, does the following with probability $1 - \delta$: If there is an $R$-near neighbor of $q$ in $D$, it reports a $cR$-near neighbor of $q$ in $D$.

Let $x_v$ denote the $v^{th}$ nearest neighbor and note that any algorithm which solves the $v$-nearest neighbor problem also solves the $(R, c)$-approximate near-neighbor problem for any $R \geq d(q, x_v)$ and any value of $c > 1$. We apply our algorithm to the $v$-nearest neighbor problem because it provides better intuition for the query and data-dependent nature of our guarantees. Unlike the $(R, c)$-approximate near-neighbor problem, the difficulty of the $v$-nearest neighbor problem is data-dependent. We motivate our data-dependent conditions using the notion of near-neighbor stability from the seminal paper [14].

Definition 3. Unstable near-neighbor search

A nearest neighbor query is unstable for a given $\epsilon$ if the distance from the query point to most data points is $\leq (1 + \epsilon)$ times the distance from the query point to its nearest neighbor.

Finally, we formally describe our compressed searching framework for streaming data. A popular setting where we are not allowed to remember the complete dataset is the one pass online setting [15]. Given a dataset $D$ with $N$ elements, we read each element $x_i \in D = \{x_1, x_2, x_3, \ldots, x_N\}$ in an online fashion, observing each $x_i$ one at a time. The goal is to construct a sketch or summary $S$ of $D$. $S$ must support incremental addition and deletion of elements. After observing $x_i$, we update $S$ by adding the vector $x_i$ to the sketch. Then, we discard $x_i$ and observe vector $x_{i+1}$. We are not allowed a second pass through the data since we do not store the complete dataset anywhere. We want the size of $S$, which determines the communication complexity and storage cost, to scale less than $O(N)$. We will show that our RACE sketch solves the $v$-nearest neighbor problem under these constraints.

2.2 Locality-Sensitive Hashing

LSH [5] is a popular technique for efficient approximate nearest-neighbor search. A LSH family is a family of functions with the following property: Under the hash mapping, similar points have a high probability of having the same hash value. The formal definition of a LSH family is as follows [5].

Definition 4. $(R, cR, p_1, p_2)$-sensitive hash family

A family $\mathcal{H}$ is called $(R, cR, p_1, p_2)$-sensitive with respect to a metric space $(\mathcal{M}, d(\cdot, \cdot))$ if for any two points $p, q \in \mathcal{M}$ the following properties hold

- If $d(p, q) \leq R$ then $\Pr_{H}[h(p) = h(q)] \geq p_1$
- If $d(p, q) \geq cR$ then $\Pr_{H}[h(p) = h(q)] \leq p_2$

We say that a collision occurs whenever the hash values for two points are equal, i.e. $h(p) = h(q)$. The probability $\Pr_{H}[h(p) = h(q)]$ is known as the collision probability of $p$ and $q$. In this paper we will use the
notation \( p(p, q) \) to denote the collision probability of \( p \) and \( q \). For our arguments, we will assume a slightly stronger notion of LSH than the one specified by Definition 1. We will suppose that the collision probability is a monotonic function of the similarity between \( p \) and \( q \). That is

\[
p(p, q) \propto f(\text{sim}(p, q))
\]

where \( \text{sim}(p, q) \) is a similarity function and \( f(\cdot) \) is monotone increasing. LSH is a very well-studied topic with a number of well-known LSH families in the literature [1]. Most LSH families satisfy this assumption. Finally, it should be noted that for any positive integer \( K \), if there exists an LSH function \( h(\cdot) \) with collision probability \( p(x, y) \), then the same \( h(\cdot) \) can be independently concatenated \( K \) times to obtain a new \( h(\cdot) \) with collision probability \( p(x, y)^K \). If \( h(\cdot) \) had a finite range \([1, R]\), then the new LSH function has the range \([1, R^K]\).

### 2.3 Array-of-Counts Estimator (ACE)

Recent work has shown that LSH can be used for efficient unbiased statistical estimation [11, 12, 13]. The ACE algorithm capitalizes on this view of LSH [13]. Given a LSH function \( l(\cdot) \) with finite range \([1, R]\), the ACE algorithm compresses a dataset \( D \) into an \( R \times 1 \) array of counters. The algorithm itself is simple. First, the hash value \( l(x) \) of each element \( x \in D \) is found. Then, the array of counters \( A \) is incremented by 1 at the index corresponding to this hash value. Here, we re-state the main theorem from [13] using simpler notation.

**Theorem 2. ACE Estimator [13]**

Given a dataset \( D \), a LSH function \( l(\cdot) \mapsto [1, R] \) and a parameter \( K \), construct a LSH function \( h(\cdot) \mapsto [1, R^K] \) by concatenating \( K \) independent \( l(\cdot) \) hashes. Let \( A \in \mathbb{R}^{R^K} \) be an array where the \( i^{th} \) component is

\[
A[i] = \sum_{x \in D} \mathbb{I}_{h(x) = i}
\]

Then for any query \( q \),

\[
\mathbb{E}[A[h(q)]] = \sum_{x \in D} p(x, q)^K
\]

We will heavily leverage the observation that \( A[h(q)] \) is an unbiased estimate of the summation of collision probabilities. In this work, we extend the theory of [13]. Experimental results from [13] show that \( A[h(q)] \) has low variance, but the original paper did not attempt to bound the variance of \( A[l(q)] \). Therefore, we prove a data-dependent upper bound on the variance of this estimator. We modify the estimator to accommodate linear combinations of \( p(x_i, q)^K \) and apply standard techniques to our variance bound to obtain sharp estimates.

Finally, we note the space required by an ACE estimator. For a dataset \( D \) containing \( N \) elements, it is straightforward to see that \( A[i] \leq N \). Therefore, each cell of \( A \) requires \( O(\log N) \) bits, meaning that the size of \( A \) is \( O(R^K \log N) \) bits.

### 2.4 Compressed Sensing and the Count Min Sketch

Compressed sensing is the area in signal processing which solves the following problem.

**Definition 5. The compressed sensing problem**

Given a vector \( \mathbf{x} \in \mathbb{R}^N \) and a matrix \( \Psi \) of basis vectors for \( \mathbb{R}^N \), construct a measurement matrix \( \Phi \in \mathbb{R}^{M \times N} \) and a recovery algorithm that can accurately recover \( \mathbf{x} \) from \( \mathbf{y} \in \mathbb{R}^M \), where \( \mathbf{y} = \Phi \mathbf{x} \) is a set of \( M << N \) measurements of \( \mathbf{x} \).

The problem is tractable when \( \mathbf{x} \) has a \( v \)-sparse representation in \( \Psi \), i.e. when \( \mathbf{x} \) is a linear combination of at most \( v \) basis vectors from \( \Psi \in \mathbb{R}^{N \times N} \). If this is true, we say that \( \mathbf{x} \) is a compressible vector. The fundamental result in compressed sensing is the existence of measurement matrices and recovery algorithms that can exactly recover \( \mathbf{x} \) from \( \mathbf{y} \) using only \( M = O(v \log N/v) \) measurements. For a detailed description of the compressed sensing problem, see [16].
In this paper, all of our vectors are sparse with respect to the identity matrix \( \Psi = I \). That is, we are concerned with vectors that contain only a few nonzero elements. In the streaming literature, these elements are often referred to as heavy hitters. The problem of identifying heavy hitters in a data stream is equivalent to the compressed sensing problem when \( \Psi = I \) [17].

The count-min sketch (CMS) is a data stream summary that is popularly used to solve the online heavy hitter problem. The CMS is a \( d \times w \) array of counts that are incremented in a randomized fashion. Given a vector \( s \), for every element \( s_i \) in \( s \), we apply \( d \) hash functions \( h_1(\cdot), \ldots, h_d(\cdot) \) to \( s_i \) to obtain a set of \( d \) indices. Then, we increment the CMS cells at these indices. In terms of compressed sensing, the update procedure of the CMS defines the measurement matrix \( \Phi \). Here, \( \Phi \) is a random sparse matrix, with non-zeros at locations defined by the \( d \) hash functions.

One particularly important result is that when all elements of \( s \) are non-negative, we have a point-wise bound on the estimated \( s_i \) values returned by the CMS [18].

**Theorem 3.** Given a CMS sketch of the non-negative vector \( s \in \mathbb{R}_+^N \) with \( d = O \left( \log \left( \frac{N}{\epsilon} \right) \right) \) rows and \( w = O \left( \frac{1}{\epsilon^2} \right) \) columns, we can recover a vector \( s_{CMS} \) such that we have the following point-wise recovery guarantee for each recovered element \( s_i^{CMS} \):

\[
s_i \leq s_i^{CMS} \leq s_i + \epsilon |s|_1
\]

with probability \( 1 - \delta \).

For the sake of simplicity, we only consider the CMS when presenting our results. However, RACE supports arbitrary linear combinations with bounded coefficients. Therefore, we can accommodate nearly any measurement matrix and recovery procedure. There are an enormous number of valid measurement matrices in the literature; for details see [19]. Other compressed sensing methods can improve our bounds, but we defer this discussion to Section 7.3.

### 2.5 Notation

For the rest of the paper, let \((\mathcal{M}, d(\cdot, \cdot))\) be a metric space and let \( l(\cdot) \) be a LSH function on \( \mathcal{M} \). We let \( \mathcal{D} \) be a dataset in \( \mathcal{M} \) with \( N \) elements \( \mathcal{D} = \{ x_1, x_2, \ldots, x_N \} \), and we let \( q \) be a near-neighbor query on \( \mathcal{D} \).

Throughout the paper, we suppose that \( l(\cdot) \) has been concatenated \( K \) times as described in Section 2.2 to form a new hash function \( L(\cdot) \). We use \( p(x_i, q) \) to denote the collision probability of \( x_i \) and \( q \) under \( l(\cdot) \) and we note that the collision probability under \( L(\cdot) \) is \( p(x_i, q)^K \). We will often abbreviate \( p(x_i, q) \) as \( p_i \) when the context specifies the query and the dataset. The event of a hash collision under \( L(\cdot) \) between \( q \) and the \( i^{th} \) element \( x_i \) is denoted by the indicator function \( 1_{L(x_i) = L(q)} \). When the context is obvious, we abbreviate \( 1_{L(x_i) = L(q)} \) as \( 1_i \). Finally, we let \( s(q) \) be the vector of collision probabilities for every element in \( \mathcal{D} \) under \( L(\cdot) \). That is, \( s_i(q) = p(x_i, q)^K \). We suppose for convenience and simplicity of notation that \( s(q) \) is sorted in decreasing order. As before, we will occasionally refer to \( s(q) \) as \( s \).

\( A \) is an ACE estimator and we use the notation \( A[i] \) to refer to the array cell at index \( i \) in \( A \). \( \Phi \in \mathbb{R}^{M \times N} \) is a compressed sensing measurement matrix and \( y \in \mathbb{R}^M \) is a vector of \( M \) compressed sensing measurements. The compressed sensing measurements \( y \) will always be of the vector \( s(q) \), \( |s|_1 \) is the 1-norm of \( s \), and \( |s|_2 \) is the 2-norm of \( s \). When discussing compressed sensing measurements \( y \) that correspond to a CMS, we use \( CMS_{i,j} \) to refer to cell \((i, j)\) in the CMS. When we discuss the CMS or the count sketch, we use \( d \) for the number of rows (the depth) and \( w \) for the number of columns (the width). A CMS with \( d \) rows and \( w \) columns has \( M = dw \) measurements.

### 3 Estimation of Compressed Sensing Measurements

In this section, our goal is to prove that the ACE algorithm can estimate the compressed sensing measurements of \( s(q) \). We begin by constructing a modified version of ACE that can estimate any linear combination of \( s(q) \) components. Then, we prove a data-dependent variance bound on this estimator and apply the median of means technique.

We can estimate the linear combination by incrementing the ACE array using the linear combination coefficients. Suppose we are given a sequence of linear combination coefficients \( \{r_i\}_{i=1}^N \). The original ACE...
estimator simply increments \( A \) at index \( L(x_i) \) by 1, resulting in an estimator where \( A[L(q)] = \sum_{x_i \in D} 1_i \). In our case, we increment \( A[L(x_i)] \) and therefore we have \( A[L(q)] = \sum_{x_i \in D} r_i 1_i \). Using the same argument as \[3\], \( A[L(q)] \) is an unbiased estimator of the linear combination of collision probabilities.

**Theorem 4.** Given a dataset \( D \), \( K \) independent LSH functions \( l(\cdot) \) from the same LSH family and any choice of constants \( r_i \in \mathbb{R} \), we can design an ACE array \( A \) such that for any query, \( A[L(q)] \) is an unbiased estimator of a linear combination of \( s_i(q) = p(x_i, q)^K \).

\[
A[L(q)] = \sum_{x_i \in D} r_i 1_i
\]  
(3)

\[
E[A[L(q)]] = \sum_{x_i \in D} r_i p(x_i, q)^K
\]  
(4)

where \( L(\cdot) \) is formed by concatenating the \( K \) copies of \( l(\cdot) \).

Now consider the set of \( M \) compressed sensing measurements of \( s(q) \) (in the \( I \) basis) that we obtain from the matrix multiplication \( y(q) = \Phi s(q) \). Measurement \( y_j(q) \) can be expressed as \( \sum_i \Phi_{j,i} s_i(q) \). Therefore, a direct corollary of Theorem \[1\] is that we can obtain an unbiased estimator of \( y_j(q) \) by choosing \( r_i = \Phi_{j,i} \). The main problem is to bound the variance of this estimator. Unfortunately, the variance is highly data-dependent due to the appearance of correlated cross-terms \( E[1_i, 1_j] \). First, we restrict our attention to linear combinations where the coefficients \( r_i \) are bounded to the interval \([-1, 1]\). Since we may always normalize the entries of \( \Phi \), this is not a limiting assumption. Then, we apply the Cauchy-Schwarz inequality to the \( E[1_i, 1_j] \) terms and use the fact that \( E[1_i^2] = E[1_i] \). This yields the following bound on the variance.

**Theorem 5.** Given a query \( q \), the variance of the ACE estimator \( A[l(q)] \) from Theorem \[4\] with linear combination coefficients \( r_i \in [-1, 1] \) is bounded by \( |\tilde{s}(q)|^2 \), where \( \tilde{s}(q) = \sqrt{s_i(q)} = p(x_i, q)^{\frac{1}{2}} \).

\[
\text{var}(A[l(q)]) \leq |\tilde{s}(q)|^2
\]  
(5)

Theorem \[5\] holds for any linear combination with bounded coefficients. Our bound can be improved by forcing \( \Phi \) to obey more limiting restrictions, but we defer this discussion to Section \[7.1\].

Using Theorem \[5\] and the median-of-means (MoM) technique, we can obtain an arbitrarily close estimate of each compressed sensing measurement \( y_i(q) \). Suppose we independently repeat the ACE estimator and compute the MoM estimate from the repetitions. Let \( \hat{y}_i(q) \) be the MoM estimate of \( y_i(q) \) computed from a set of independent ACE repetitions of \( A[l(q)] \). Then we have a pointwise bound on the error for each \( y_i(q) \).

**Theorem 6.** For any \( \epsilon > 0 \) and given

\[
O\left( \frac{|\tilde{s}(q)|^2 \log \left( \frac{1}{\delta} \right)}{\epsilon^2} \right)
\]  

independent ACE repetitions, we have the following bound for the MoM estimator

\[
y_i(q) - \epsilon \leq \hat{y}_i(q) \leq y_i(q) + \epsilon
\]  
(6)

with probability \( 1 - \delta \) for any query \( q \).

To ensure that all \( M \) measurements obey this bound with probability \( 1 - \delta \), we apply the probability union bound to get the following result.

**Theorem 7.** Let \( \hat{y}(q) \in \mathbb{R}^M \) be the vector of measurement estimates \( \hat{y}_i \). Then for any \( \epsilon > 0 \) and given

\[
O\left( \frac{M|\tilde{s}(q)|^2 \log \left( \frac{M}{\delta} \right)}{\epsilon^2} \right)
\]  

independent ACE repetitions, for all \( M \) measurements \( \hat{y}_i \) we have that

\[
y(q)_i - \epsilon \leq \hat{y}(q)_i \leq y(q)_i + \epsilon
\]  
(7)

with probability \( 1 - \delta \) for any query \( q \).

Therefore, by repeating ACE estimators (RACE), we can obtain low-variance estimates of the compressed sensing measurements of \( s(q) \). The exact number of measurements \( M \) depends on both \( \Phi \) and the dataset, but \( M < O(N) \).
4 Data-Dependent Sparsity Conditions

Before we can discuss compressed recovery of \( s(q) \), we need to introduce some data-dependent conditions. In this section, we introduce sparsity and stability conditions for the near-neighbor query. We analyze these conditions in the context of compressed sensing and computational geometry. For our compressed sensing measurements to be useful, \( s(q) \) needs to be sparse. That is, only a few elements of \( s(q) \) are allowed to be large; all others must be small. This is equivalent to specifying a bound for \(|s(q)|_1\) [3]. We also need a bound on \(|\hat{s}(q)|_1\) to avoid a memory blow-up in Theorem 7.

Our signal \( s(q) \) has three fortunate properties. First, the collision probabilities are bounded: \( p(x_i, q) \in [0, 1] \). Second, increasing \( K \) causes each element of \( s(q) \) to decrease, since \( s(q)_i = p(x_i, q)^K \). Finally, we control \( K \). Therefore, we can force \(|s(q)|_1\) to be arbitrarily small by choosing \( K \) sufficiently large. However, recall from Section 2.3 that the ACE estimator requires \( O(v^K \log N) \) memory. Therefore, we want \( K \) to be just large enough.

Here, we provide values of \( K \) that will bound \(|s(q)|_1\) and \(|\hat{s}(q)|_1\). Here, our guarantees are for the worst-case sparsity situation when \(|s|_1 = O(N)\). We will refer to the worst-case sparsity assumption as the equidistant assumption, since it implies that all points in the dataset other than the \( v \) nearest neighbors are equidistant to the query. In Section 2.3 we present additional results for stronger assumptions that are common in practical applications. Stronger assumptions require smaller \( K \) and therefore less space. To choose \( K \) appropriately, we need to characterize the sparsity of \( s(q) \) under the equidistant assumption. We begin by defining two data-dependent values \( \Delta \) and \( B \).

For the \( v \)-nearest neighbor problem, let \( x_v \) and \( x_{v+1} \) be the \( v \)th and \((v+1)\)th nearest neighbors, respectively. Using the same notation as before, let \( \Delta \) be defined as

\[
\Delta = \frac{p(x_{v+1}, q)}{p(x_v, q)}
\]

(8)

\( \Delta \) governs the stability of the nearest neighbor query. It is a measure of the gap between the near-neighbors and the rest of the dataset. Our definition of \( \Delta \) is motivated by the definition of an \( \epsilon \)-unstable query (Definition 3). Note that Definition 3 is in terms of distances, while \( \Delta \) is in terms of collision probabilities. Without specifying the LSH and similarity function, it is impossible to express our notion of stability in terms of Definition 3. However, \( \Delta \) is always in the interval \([0, 1]\) because \( p_{v+1} \leq p_v \). If \( \Delta \approx 1 \), then \( x_v \) and \( x_{v+1} \) are both highly similar to the query and, according to Definition 3, the query is unstable.

We also require a characterization of \(|s(q)|_1\). To recover the \( v \)-nearest neighbors of \( q \), the nonzero components of \( s(q) \) should correspond to the largest \( p(x_v, q)^K \) values. The tail of \( s(q) \), which contains the rest of the dataset, should be small. Again let \( x_v \) denote the \( v \)th nearest neighbor and define a constant \( B \) as

\[
B = \sum_{i=v+1}^{N} \frac{\hat{s}_i}{s_{v+1}} = \sum_{i=v+1}^{N} \sqrt{\frac{p(x_i, q)^K}{p(x_{v+1}, q)^K}}
\]

(9)

\( B \) is a query-dependent value that measures the sparsity of \( s \). A bound on \( B \) implies a bound on \(|s|_1\) and \(|\hat{s}|_1\) because \(|\hat{s}|_1 \leq v + B \sqrt{\frac{N}{v+1}}\) and \(|s|_1 \leq |\hat{s}|_1\). Under the equidistant assumption, \( B = O(N) \) because \( s \) is not sparse. We are now ready to present our results for \( K \) in terms of \( B \) and \( \Delta \).

Theorem 8. Given a query \( q \) and query-dependent parameters \( B \) and \( \Delta \), if

\[
K = \left\lceil \frac{2 \log B}{\log \frac{1}{\Delta}} \right\rceil
\]

then

\[
p(x_v, q)^K \geq \sum_{i=v+1}^{N} p(x_i, q)^K
\]

and we have the bounds

\[
|s(q)|_1 \leq v + 1
\]

\[
|\hat{s}(q)|_1 \leq v + 1
\]
Figure 1: Geometric interpretation of compressed sensing sparsity. If $|\tilde{s}|_1$ is not bounded, then all points in the dataset are equidistant to the query (center). If $s$ is sparse, then far fewer points in the dataset are near the query (right).

If $K$ is chosen according to Theorem 8, then $s(q)$ becomes sparse, and the top $v$ components of $s(q)$ are heavy hitters. $K$ can be viewed as a corrective step to induce sparsity in $s$. The stronger our sparsity assumption, the smaller $K$ can be. When all non-neighbors have the same collision probability (i.e. when $p(x_i, q) = p(x_{v+1}, q) \forall i > v + 1$), $K = \left\lceil \frac{2 \log N}{\log \frac{1}{\Delta}} \right\rceil$. Under these conditions, the sparsity of $s$ depends entirely on $\Delta$. If $\Delta$ is small, then the query is stable and $s$ is at least somewhat sparse. Figure 1 provides intuition for this relationship. It should be noted that the situation where all points in the dataset are equidistant from the query has been identified as a particularly difficult case \[14\]. In Section 5, we will see that under the equidistant assumption, the memory required by our sketch depends on $p_v$ and $\Delta$.

5 Reduce Near-Neighbor to Compressed Recovery

In Section 3, we provided an estimator for compressed sensing measurements. In Section 4, we provided a way to ensure that $s(q)$ is sparse for the equidistant assumption, which can characterize near-neighbor queries that are arbitrarily difficult to answer. In this section, we will use these results to find near neighbors using compressed sensing. For simplicity, we restrict our attention to the CMS. The main challenge is to ensure that the collision probabilities recovered by our algorithm are within $\epsilon$ of the actual collision probabilities. First, we use Theorem 7 to estimate the CMS cells. This results in an approximated CMS, which we call $\hat{CMS}$. We want to recover an estimate $\hat{s}$ of $s$ from $CMS$.

First, we estimate the CMS cells using Theorem 7, and we suppose that there are $M = O\left(\frac{1}{\epsilon} \log \frac{N}{\delta}\right)$ cells. According to the CMS pointwise recovery guarantee (Theorem 3), we can recover the vector that produced $\hat{CMS}$ with $\epsilon_C$ pointwise error. However, each cell of $\hat{CMS}$ has its own estimation error $\epsilon_E$. Our compressed sensing measurements themselves are noisy.

This situation is known as measurement noise. Fortunately, the CMS has a particularly simple relationship between measurement noise and recovery error. A measurement error of $\epsilon_E$ propagates as-is during recovery to produce a recovery error of $\epsilon_E$. This occurs because the CMS recovery procedure returns one of the cell values as its estimate for each component of $\hat{s}$. If the cell values in $\hat{CMS}$ deviate from the true CMS values by $\leq \epsilon_C$, then the output of $\hat{CMS}$ deviates from the output of CMS by $\leq \epsilon_E$. It should be noted that other compressed sensing methods can be more or less sensitive to measurement noise. In general, the recovery procedure solves a nonlinear optimization problem, producing a complex relationship between $\epsilon_E$ and the recovery error. In Section 7.1, we discuss how to modify our analysis to handle other recovery schemes. However, for the CMS, we have a simple pointwise error for the recovered $\hat{s}$.

$$s_i(q) - \epsilon_E \leq \hat{s}_i(q) \leq s_i(q) + \epsilon_E + \epsilon_C|s(q)|_1 \quad (10)$$
By choosing appropriate values for $\epsilon_C$ and $\epsilon_E$, we obtain a concise statement for the pointwise recovery guarantee of our estimated CMS.

**Theorem 9.** We require

$$O\left(\frac{\|\hat{s}(q)\|_1^2\|s(q)\|_1}{\epsilon^3} \log \left(\frac{\|s(q)\|_1}{\epsilon \delta} \log \left(\frac{N}{\delta}\right)\right) \log \left(\frac{N}{\delta}\right)\right)$$

ACE estimates to recover $\hat{s}(q)$ such that

$$s_i(q) - \frac{\epsilon}{2} \leq \hat{s}_i(q) \leq s_i(q) + \frac{\epsilon}{2}$$

with probability $1 - \delta$.

If $s$ is sparse, then this result can be used to identify the top $v$ elements of $s$ by setting $\epsilon = s_v - s_{v+1}$. Suppose that $K$ has been chosen so that $s(q)$ is sparse, and set $\epsilon = s_v - s_{v+1}$ in Theorem 9. Then it is easy to see that we can identify near-neighbors by finding the heavy hitting indices of $\hat{s}(q)$, which correspond to the largest collision probabilities. We formalize this observation in Corollary 9.1.

**Corollary 9.1.** Suppose $K$ is chosen so that $|\hat{s}(q)|_1 \leq C$ and $|s(q)|_1 \leq C$. Then we can identify the $v$ nearest neighbors with probability $1 - \delta$ using

$$O\left(\frac{C^3}{p_v^3(1 - \Delta K)^3} \log \left(\frac{C}{p_v^3(1 - \Delta K)\delta} \log \left(\frac{N}{\delta}\right)\right) \log \left(\frac{N}{\delta}\right)\right)$$

ACE estimates.

### 5.1 Near-Neighbor Sketch Size

Here, we will use Corollary 9.1 to find the size of the sketch for the equidistant assumption. Under these conditions, $K = \lceil 2\log N / \log \frac{N}{\Delta} \rceil$ (Theorem 8). To obtain useful guarantees that are easy to interpret, we restrict $\Delta$ to the range where $2 \log N / \log \frac{N}{\Delta} > 1$. Then the following expression holds for $p_v$ and $p_{v+1}$.

$$\epsilon = p_v^K - p_{v+1}^K = O\left((1 - N^{-2})N^{2\log p_v \Delta / \Delta} \right) = O\left(N^{2\log p_v \Delta / \Delta} \right)$$

It is important to note that this does not hold when $\Delta \leq N^{-2}$. In this case, $|s|_1$ is already sparse and we may choose $K = 1$. We explore these corner cases in Section 7.3. Also, recall that each ACE estimator requires $O(r^K \log N)$ bits. The final sketch size is the number of ACE estimators multiplied by $O(r^K \log N)$ bits.

**Theorem 10.** Given a query $q$, a dataset $D$ in a metric space $(M, d(\cdot, \cdot))$, and a LSH function with a range $r$, we can construct a sketch that solves the exact $v$-nearest neighbor problem with probability $1 - \delta$ in size

$$O\left((v + 1)^3 N^{b_1} \log \left(\frac{(v + 1)}{\delta} \right) N^{b_2} \log \left(\frac{N}{\delta}\right) \log N\right)$$

bits, where

$$b_1 = \frac{6\|p_v\| + 2\log r}{\log \frac{1}{\Delta}}$$

$$b_2 = \frac{2\|p_v\|}{\log \frac{1}{\Delta}}$$

$x_v$ is the $v$th nearest neighbor of $q$ in $D$, $x_{v+1}$ is the $v + 1$th nearest neighbor of $q$ in $D$, and $\Delta = \frac{p_{v+1}}{p_v} \geq N^{-2}$.

Our main theorem is a substantially simplified version of Theorem 10. To get the simplified result, we expand the logarithms and keep the $O(N^b \log^3 N)$ term, which dominates the asymptotic behavior in $N$. By corollary to our main theorem, we also have a solution to the $(R, cR)$-approximate near-neighbor problem.
Corollary 10.1. Given a query \( q \), a dataset \( D \) in a metric space \((M, d(\cdot, \cdot))\), and a \((R, cR, p_1, p_2)\)-sensitive hash function with a range \( r \), we can construct a sketch \( S \) that solves the \((R, cR)\)-approximate near-neighbor problem with probability \( 1 - \delta \) in size

\[
O \left( (v + 1)^3 N^{b_1} \log \left( \frac{(v + 1)}{\delta} N^{b_2} \log \frac{N}{\delta} \right) \log \left( \frac{N}{\delta} \log N \right) \right)
\]

bits, where

\[
b_1 = \frac{6|\log p_1| + 2\log r}{\log \frac{1}{\Delta}}
\]

\[
b_2 = \frac{2|\log p_1|}{\log \frac{1}{\Delta}}
\]

\( v \) is the number of points in the ball \( B(q, cR) \), and \( \Delta \leq \frac{p_2}{p_1} \leq 1 \).

Our sketch is sublinear when \( b_1 < 1 \). A necessary condition for \( b_1 < 1 \) is that \( p_{c+1} < \frac{1}{r} \). This restricts the LSH families that we can use to solve the near neighbor problem using RACE. For instance, we cannot create a LSH family by using a universal hashing to re-hash \( K \) LSH values to a finite range \( r \). This introduces random collisions which occur with probability \( \frac{1}{r} \), forcing \( p_2 \geq \frac{1}{r} \). Also, we prefer families for which \( r \) is small. If the range \( r \) is large, the result is not useful since \( b < 1 \) only for trivial \( \Delta \). In Section 6, we provide an example of a LSH family that does satisfy the sub-linearity condition in a useful way.

5.2 Streaming Near-Neighbors

We propose the following online sketching and querying algorithms to implement the sketch from Theorem 10.

**Algorithm 1** One-Pass Online Sketching Algorithm

**Require:** \( D \)

**Ensure:** \( d \times w \times R \) ACE arrays indexed as \( A_{i,j,o} \)

**Initialize:** \( k \times d \times w \times R \) independent LSH family (denoted by \( L(\cdot) \)) and \( d \) independent 2-universal hash functions \( h_i(\cdot), i \in [1 - d] \), each taking values in range \([1 - w]\).

**while** not at end of data \( D \) do

\( \text{read current } x_j; \)

\( \text{for } o \text{ in } 1 \text{ to } r \text{ do} \)

\( \text{for } i \text{ in } 1 \text{ to } d \text{ do} \)

\( A_{i,h_i(j),o}[L[x_j]]++; \)

\( \text{end for} \)

\( \text{end for} \)

\( \text{end while} \)

In Algorithm 1, the \( h(\cdot) \) hash functions are the internal hash functions used by the CMS. These are used to determine the randomized measurement matrix \( \Phi \). Each of the \( d \times w \times R \) ACE estimators internally has a range \( r^K \), where \( r \) is the range of the LSH function \( l(\cdot) \). It should be noted that given an element \( x_j \) and its identifier \( j \), we can add/remove \( x_j \) to/from \( S \) in the streaming setting.
Algorithm 2 Querying Algorithm

Require: Sketch from Algorithm 1, query \( q \)
Ensure: Identities of Top-\( v \) neighbors of \( q \)

We already have: \( k \times d \times w \times R \) independent LSH family (denoted by \( L(\cdot) \)) and \( d \) independent 2-universal hash functions \( h_i(\cdot), i \in [1-d] \), each taking values in range \([1-w]\) from Algorithm 1.

for \( i \) in 1 to \( d \) do
  for \( j \) in 1 to \( w \) do
    \( CMS(i,j) = \text{MoM}(A_{i,j,o}[L(q)]) \)
  end for
end for
for \( j \) in 1 to \( n \) do
  \( \hat{s}_j = \hat{p}(q,x_j)^K = \min_i CMS(i,h_i(j)) \)
end for
Report top-\( v \) indices of \( \hat{s} \) as the neighbors.

Algorithm\( 2 \) begins by computing the \( CMS \) for a given query. We estimate \( E[A[L(q)]] \) by performing median of means as in Theorem 7. This yields \( CMS(i,j) \), to which we apply the standard CMS recovery procedure.

5.3 Near-Neighbor Sketch and Query Time

For each update to the sketch, we compute \( K \) hash values for each ACE estimator. Corollary 9.1 gives an expression for the number of ACE estimators in terms of \( K \) and \( C \). This gives the time complexity for each update in the dataset. To query the sketch, we must compute the same hash functions. Then, we perform median-of-means and the CMS recovery. Provided \( b < 1 \), the querying algorithm is dominated by the \( O(N \log N) \) compressed recovery. Formally, we have the following theorem for RACE when applied under the equidistant assumption.

Theorem 11. Provided \( b < 1 \), the sketch from Theorem 10 can be constructed in

\[
O \left( (v+1)^3 N^{b+1} \log \left( \frac{(v+1)}{\delta} \right) N^{b} \log N \right) \log \left( \frac{N}{\delta} \right) \log \frac{1}{\Delta}
\]

time and queried in

\[
O \left( N \log \left( \frac{N}{\delta} \right) \right)
\]
time.

6 A Sub-Linear Near-Neighbor Sketch for the Angular Distance

The analysis in Theorem 10 assumes the existence of a LSH function with finite range \( r \) that can satisfy \( p_2 < \frac{1}{r} \) for a useful range of \( \Delta \). In this section we show that the popular signed random projection LSH family satisfies our sub-linearity conditions.

Definition 6. Signed random projections

The signed random projection LSH family is a family of functions \( l(\cdot) : \mathbb{R}^d \to \{-1,1\} \) such that

\[ l_w(x) = \text{sign}(w^T x) \]

where \( w \in \mathbb{R}^d \) is an independent standard Gaussian vector: \( w \sim \mathcal{N}(0, I) \)
Figure 2: Relationship between $b$ and $(p_v, \Delta)$ for the signed random projection LSH family (left) and the geometric interpretation of the equidistant assumption for the angular distance (right). Here, we solve the $v = 4$ nearest neighbor problem, where $\theta_v$ is the distance between the query and the 4th nearest neighbor. Gap is inversely proportional to $\Delta$.

From [20], the collision probability under $l(\cdot)$ obeys

$$\Pr[l(q) = l(x)] = 1 - \frac{\theta(q, x)}{\pi}$$

(13)

where

$$\theta(q, x) = \cos^{-1}\left(\frac{\langle x, y \rangle}{||x||||y||}\right)$$

Here, $\theta(q, x)$ is the angle between $q$ and $x$. We will refer to $\theta(q, x)$ as the angular distance, and we note that $\theta(q, x)$ is a metric on the unit sphere in $\mathbb{R}^d$. Note that the range of $l(\cdot)$ is $r = 2$. This fully specifies the sub-linearity condition for this problem. In Figure 2 we show the value of $b$ for a range of $p_v$ and $\Delta$ values. We also show the geometric interpretation of the worst-case sparsity condition using the angular distance. Our sketch is clearly sub-linear for reasonable range of $(p_v, \Delta)$. These results may be improved further using the cross-polytope LSH [21]. For a given angular distance $d(q, x_{v+1})$, the collision probability under cross-polytope LSH yields a lower $\Delta$.

7 Tightening the Bound

In this section we discuss ways to reduce the space required by the sketch. First, we discuss the use of compressed sensing matrices with special structural properties. This improves the ACE variance bound, ultimately attaining a smaller value of $K$. Since these compressed sensing matrices employ different recovery procedures than the CMS, we also show how to adapt our analysis for most other compressed recovery methods. We also describe properties that are desirable for the LSH functions used to construct our sketch. Finally, we repeat our analysis under stronger sparsity assumptions and show that stronger assumptions require smaller $K$.

7.1 Alternative Compressed Sensing Matrices

In the previous sections, we exclusively considered the CMS as our compressed sensing method. The CMS has a randomized measurement matrix $\Phi$ with a simple, computationally efficient recovery procedure. If we entertain the notion of structured $\Phi$ coupled with more complex recovery procedures, we can improve the variance bound in Theorem 8. This is desirable because a smaller variance bound usually allows us to choose a smaller $K$, which improves the sketch size. To see this, observe that Theorem 8 bounds two quantities under the equidistant assumption: $|s|_1$ and $|\tilde{s}|_1$. We cannot avoid the sparsity requirement for $|s|_1$, but the
value of $K$ is actually determined by $|\tilde{s}|_1$, which comes from the RACE variance. If this requirement can be relaxed, $K$ can be up to two times smaller. This is a $O(\sqrt{N})$ improvement; the exponent $b$ improves from $b = \frac{6 \log p_v + 2 \log r}{\log \frac{1}{\Delta}}$ to $b = \frac{3 \log p_v + \log r}{\log \frac{1}{\Delta}}$.

**Balanced $\Phi$ Matrices** Consider the $M \times N$ measurement matrix $\Phi$ formed by independently selecting $M$ rows of the Hadamard matrix, excluding the first row. It is easy to see that the sum of linear combination coefficients for each row $\sum r_i = 0$. More generally, we will say that a matrix $\Phi$ is balanced if the summation over the rows of $\Phi$ is zero. Equivalently, $\sum r_i = 0$ for every measurement. If we use RACE to estimate a balanced linear combination, some of the cross-correlation terms may cancel, resulting in a lower variance bound. The RACE variance for a balanced measurement is

$$E[A[L(q)]^2] \leq \left( \sum_{i=1}^{N} \Phi(m,i) \sqrt{p_i^K} \right)^2 = \left( \sum_{i=1}^{N} \Phi(m,i) \tilde{s}_i \right)^2$$

Since exactly half of the $\Phi(m,i) < 0$, the summation over $\tilde{s}$ is smaller than the summation over the largest $\frac{N}{2}$ elements of $\tilde{s}$. Using a similar procedure to Theorem 8, we have that

$$K = \left\lceil \frac{\log \frac{N}{2}}{\log \frac{1}{\Delta}} \right\rceil$$

**Sparse $\Phi$ Matrices** Consider a measurement matrix where each row contains at most $S$ nonzero elements. These measurement matrices are referred to as sparse matrices [22]; examples include the count-min sketch and count sketch. Such matrices are of interest due to the substantial computational speedup of the recovery procedure due to the presence of the zero entries. We also note that $S$ may scale sub-linearly. There exist matrices with $S = \text{polylog}(N)$ while retaining a sub-linear number of measurements [23]. The second moment of the ACE estimator for measurement $y_m$ is

$$E[A[L(q)]^2] = \sum_{i=1}^{N} \sum_{j=1}^{N} \Phi(m,i) \Phi(m,j) E[1_{i,j}]$$

When $\Phi$ is a sparse matrix with $S$ nonzeros in each row, we get that variance is bounded by the $S$ largest elements.

$$E[A[l(q)]^2] \leq \sum_{i=1}^{S} p(x_i, q) \tilde{\Phi}$$

When $\log N > 2 \log S$, the bound on $|s|_1$ dominates in Theorem 8 and $K$ becomes twice as small, as desired.

**Generalized Pointwise Result** Theorem 9 uses the CMS recovery guarantee. A similar pointwise guarantee exists for the count sketch, which has sparse updates and approximately balanced coefficients. Unfortunately, the count sketch does not provably satisfy the conditions for the improvement in $K$. For $\Phi$ that do meet the requirements, the compressed sensing literature provides recovery guarantees in the following (inconvenient) form.

$$||\hat{s} - s||_2 \leq \frac{C}{\sqrt{v}} ||s - \bar{s}||_1$$

where $s \in \mathbb{R}^N$ is the signal, $\hat{s}$ is the recovered signal, $\bar{s}$ is the best $v$-sparse approximation to $s$, and $C$ is a constant. In the context of our problem, $\hat{s}$ is the vector that contains the collision probabilities of the $v$-nearest neighbors of the query. To adapt Theorem 9 to a general compressed recovery scheme, observe that the pointwise error $(\hat{s}_i - s_i)^2 \leq ||\hat{s} - s||_2$. Then, we simply use $\Phi$-dependent recovery guarantee to select $\epsilon$ and proceed as before.
7.2 Desirable LSH Properties

For good performance, we require \( p_v \) to be large and \( \Delta \) to be small. Most hashing schemes in the literature are compared on the basis of \( \rho = \log p_1 / \log p_2 \). There has been a tremendous research effort to find hashing methods that reduce \( \rho \) as far as possible. This benefits our method because reducing \( \rho \) also reduces \( \Delta \). Therefore, existing results from data-dependent LSH can improve our method.

Our sketch also benefits from techniques to boost the collision probability of similar elements. As an example, consider the covering LSH family from [23]. Covering LSH is a family \( \mathcal{H} \) of hash functions for the Hamming space that guarantee hash collisions for some LSH function \( h(\cdot) \in \mathcal{H} \). Specifically, for two points \( x \) and \( q \) with \( d(x, q) < R \), there exists a hash function \( h(\cdot) \in \mathcal{H} \) that has \( p(x, q) = 1 \). Suppose we maintain one RACE for each \( h(\cdot) \in \mathcal{H} \) and set \( \epsilon \) sufficiently small. Such a scheme ensures that \( p_v = 1 \) for at least one RACE.

7.3 Strong Sparsity Assumptions

**Bounded \( |s|_1 \) assumption:** The most natural sparsity assumption is to suppose that for \( K = 1 \), \( |s|_1 \) and \( |s|_1 \leq C \). Here, \( C \) is a constant or a value that weakly depends on \( N \), i.e. \( C < O(N) \). Under these conditions, we can perform sub-linear CMS recovery directly from Corollary 9.1 without needing to bound \( |s|_1 \) with Theorem 8. Specifically, we take \( \epsilon = p_v - p_{v+1} \) and use \( C \).

**Power Law assumption:** Suppose that \( s \) decreases according to a power law distribution when \( K = 1 \). More specifically, suppose that for \( i > v \), \( p(x_i, q) \) decays with \( C \alpha^i \) where \( C \) is a constant and \( \alpha < 1 \).

\[
p(x_i, q) \leq C \alpha^i \quad \forall i \geq v
\]

Then \( |s|_1 \) is bounded by the geometric series sum \( \frac{1}{1 - \alpha} \). In particular, note that if we use \( K \) hashes, the bound becomes \( |s|_1 \leq \frac{1}{1 - \alpha^K} \). Using a similar procedure to the one employed in Theorem 8, we bound \( |s|_1 \leq 2 \) by choosing \( K = \lceil \frac{1}{\log \alpha} \rceil \).

**Arbitrarily small \( \Delta \):** In our main analysis, we considered the case when \( 2 \log N / \log N_\alpha > 1 \). This is by far the most common and useful situation, but there are corner cases for arbitrarily small \( \Delta \). Theorem 10 is valid for any \( p_v \), provided that \( \Delta > N^{-2} \). However, when \( \Delta \) is sufficiently small, \( s \) is already sparse. In particular, if \( \Delta \leq N^{-2} \) then \( 2 \log N / \log N_\alpha < 1 \). Therefore, we choose \( K = 1 \) in this case. Under these conditions, we directly apply Corollary 9.1 to solve the \( v \)-nearest neighbor problem. For any pair of \( (p_v, p_{v+1}) \) which cause \( K = 1 \), the sketch requires

\[
O \left( (v + 1)^3 \frac{1}{p_v^3} \log \left( \frac{v + 1}{p_v} \log \left( \frac{N}{\delta} \right) \right) \log \left( \frac{N}{\delta} \right) \right)
\]

(18) bits. It should be noted that the behavior with regard to \( p_v \) is the same as in Theorem 10. As \( p_v \) becomes arbitrarily small, the memory requirement always increases without bound regardless of \( \Delta \). This occurs because we are forced to choose \( \epsilon \) arbitrarily small.

8 Experimental Evaluations

In this section, we rigorously evaluate our RACE data structures on a friend recommendation task on the Google plus graph, similar to the ones described in [25]. Our goal is to compare and contrast the practical compression-accuracy tradeoff of RACE with baselines. We find that RACE provides orders of magnitude better performance for similarity search in limited-memory conditions than the dimensionality reduction based alternative. We compare RACE with random projections and random sampling, which are the only two practical baseline methods that have provable tradeoffs with memory usage. We investigate the empirical memory-accuracy tradeoff and show that our method provides superior compression while still recovering the near-neighbors with high recall.
| Nodes | Nonzeros | Mean Edges | Mean Similarity |
|-------|----------|------------|-----------------|
| 107614| 13673453 | 127        | 0.002           |

Table 2: Google Plus Dataset Statistics

**Figure 3:** Implementation of sketching (Algorithm 1) and querying (Algorithm 2) using RACE data structures. During sketching, we compute \( d \times w \times R \times k \) hash values for each \( x \in D \) and update the RACEs selected using \( h(\cdot) \). During querying, we compute the hash values of the query \( q \) and estimate the CMS measurements.

### 8.1 Friend Recommendation Task

We motivate our experiments with a recommendation systems task. However, we anticipate that several other applications will directly benefit from our compressed searching framework. RACE is a natural fit for low-memory, communication constrained applications.

In recommendation systems, we represent relationships, such as friendships or co-purchases, between users (or items) as graphs. Given \( N \) users, we represent each user as an \( N \)-dimensional sparse vector, where non-zero entries correspond to edges or connections. For recommendations, we are interested in finding similar elements - usually, users with similar connections or sets of items that are frequently purchased together.

The core problem in finding similar elements is the task of identifying vectors similar to the given user’s \( N \)-dimensional sparse vector representation [26]. This problem reduces to a near-neighbor search problem on the sparse vector representation, where the objective is to recommend nodes in the network with high similarity. Unfortunately, direct similarity search is not feasible for large \( N \), and online graphs are usually large. For example, it is not difficult to find graphs with billions of nodes (\( N = 10^9 \)) and over a trillion edges [27].

Graphs of this size are prohibitively expensive to store and transmit. Methodologies capable of compressing the network (or vectors) into small sketches while still allowing for near-neighbor queries are invaluable in large-scale recommendations since we only need to communicate the sketch.

We use the Google Plus social network dataset, obtained from [28], to evaluate our algorithm. Google Plus is a directed graph of 107,614 Google Plus users, where each element in the dataset is an adjacency list of connections to other users. The uncompressed dataset size is 121 MB when stored in a sparse format as the smallest possible unsigned integral type. Additional statistics are displayed in Table 2. These characteristics are typical for large scale graphs, where the data is high dimensional and sparse. Note that the low mean similarity between elements indirectly implies that \( s(q) \) is sparse.

### 8.2 Implementation

We use the CMS version of RACE that was presented in Section 5. However, we deviate from the algorithm described in Algorithm 1 in our implementation. In particular, we rehash the \( K \) LSH hash values to a range \( r \) using a universal hashing, meaning that \( p_2 \geq \frac{1}{r} \). Therefore, our LSH function violates the sub-linearity condition for the equidistant assumption. However, this is not of practical consequence due to the sparsity
naturally present in our dataset.

We implemented RACE in C++ with the following considerations. Our algorithm is characterized by the hyperparameters \( k, d, w, R \) and \( r \) and by the hash functions \( l(\cdot) \) and \( h(\cdot) \). Here, \( l(\cdot) \) is MinHash, a LSH function for the Jaccard distance. \( h(\cdot) \) is the universal hash function used in the construction of the CMS, which we implement using MurmurHash. For all experiments, we fix \( K = 2 \). The hyperparameters \( d, w \) and \( R \) control the size of the count array: \( d \) and \( w \) are the number of rows and columns in the CMS, and \( R \) is the number of ACE repetitions. In the experiments, we vary these hyperparameters to evaluate the algorithm when different dataset compression ratios are used. An implementation diagram is shown in Figure 3.

An important implementation detail is that the counters of the ACE arrays do not need to be stored as full 32 bit integers because the count array increments are distributed across \( d \times w \times r \) total buckets. Empirically, we observed that the counts were less than 32 for all experiments. This occurs because the CMS only assigns each data point to \( d \) cells out of \( dw \) total cells. Therefore, we implemented the count arrays using unsigned 8 bit integers, which have a maximum count value of 255.

The hash function computations and ACE data structure updates are trivially parallel, since the algorithm simply runs \( d \times w \) parallel ACE structures. Therefore, our C++ implementation uses multicore parallelism to update the array of ACEs and to compute the \( k \times d \times w \times R \) independent hash signatures. For the Google Plus dataset, the sketching and batch querying algorithms can run on a 2015 MacBook Pro in under a minute (batch size = 200).

8.3 Baselines

We compare our method with dimensionality reduction and random sampling followed by exact near-neighbor search. Our methodology operates in the strict one-pass streaming environment, which is required in many high-speed applications. There are no other methods in the literature that can operate in such a setting.

For comparisons, we reduce the size of the dataset using dimensionality reduction techniques or by sampling the dataset until the desired compression ratio is achieved. Then, given a set of queries, we perform exact near-neighbor search using the Euclidean distance metric.

Random Projections: We use sparse random projections [29] to project the dataset. This approach uses the Johnson-Lindenstrauss lemma to reduce the dimensionality of the dataset, and it is the best known method that is practical to implement. The compression ratio is reported as the minimum cost to store the projected dataset. We then perform exact nearest neighbor search using the projected queries.

Random Sampling: With random sampling, we reduce the original dataset to the desired size by selecting a random set of elements of the dataset. Given a query, we perform exact nearest neighbor search on the random samples.

8.4 Experimental Setup

We computed the ground truth Jaccard similarities and nearest neighbors for each element of the Google Plus dataset. We are primarily interested in queries for which high similarity neighbors exist in the dataset due to the constraints of the friend recommendation problem. This is also consistent with the near-neighbor problem statements in Section 2.1 which assume the existence of a near-neighbor. We used two sets of queries. The first set is a collection of Google Plus nodes, where each query has at least one neighbor with a Jaccard similarity measure greater than 0.8 (\( p_v = 0.8 \)). For this set of queries, the average \(|s(q)|_1 = 188\). The second set of queries is a set of nodes which have a neighbor with similarity greater than 0.9. For this set of queries, the average \(|s(q)|_1 = 204\). Note that since \(|s(q)|_1 << N\), this dataset satisfies the sparsity conditions.

We evaluate our algorithm and our baseline methods for a range of dataset compression ratios and report the recall of all ground truth near neighbors with similarity greater than 0.8 and 0.9, for the first and second query sets respectively. To show that our algorithm supports out-of-sample queries, we remove the query from the dataset before creating the sketch. Thus, we confirm that the algorithm is not simply returning the query.
Figure 4: Average recall vs compressed dataset size. The dataset size is expressed as the inverse compression ratio, or the ratio of the compressed size to the uncompressed size. Recall is reported as the average recall of neighbors with Jaccard similarity \( \text{sim}(x, q) \geq 0.8 \) (left) and 0.9 (right) over the set of queries. Higher is better.

8.4.1 Hyperparameter Settings

Our nearest-neighbor algorithm requires hyperparameters \( k, r, d, w, R \). We fixed \( k = 2 \) and ran our method for values of \( d \) between 4 and 6, \( r \) between 8 and 22, \( w = 600 \) and the hash function range \( R = 450 \). We varied \( r \) and \( d \) to show the compression performance tradeoff at a range of compressed memory sizes.

For random projections, we performed a sweep of the number of random projections from 5 to 500. Random sampling was performed by decimating the dataset (without replacement) so that the re-sampled dataset had the desired size.

8.5 Evaluation

We evaluated the algorithms based on their recall of ground truth nearest neighbors. Since we are primarily interested in high similarity results, we query the algorithm for all near-neighbors that have similarity greater than 0.8 and 0.9. We report the recall of ground truth near-neighbors with similarity greater than 0.8 and 0.9 for each set of queries. This corresponds to the \( v \)-nearest neighbor problem, where \( v \) is the number of nodes with similarity greater than our threshold.

8.6 Results

Figure 4 shows the mean recall of the most similar ground truth neighbors on both sets of queries for the RACE algorithm, random projections, and random sampling. We obtained good recall (> 0.85) on the set of queries that have very similar neighbors (> 0.9) even for an extreme 20x compression ratio. It is evident that RACE is better for high similarity. This is due to increased sparsity of \( s(q) \) (any two random users are unlikely to share a friend and hence have similarity zero) and higher \( p(x_v, q) \). In the recommender system setting, we wish to recommend nodes with very high similarity. If we require the algorithm to recover neighbors with similarity measure greater than 0.9 with an expected recall of 80% or higher, our algorithm requires only 5% of the space of the original dataset (6 MB) while random projections require 60 MB (50%) and random sampling requires nearly the entire dataset. For neighbors with lower similarity (0.8), our method requires roughly one quarter of the memory needed by random projections.

9 Conclusion

We have presented RACE, the first sub-linear memory algorithm for near-neighbor search. Our analysis connects the stability of a near-neighbor search problem with the memory required to provide an accurate solution. Additionally, our core idea of using LSH to estimate compressed sensing measurements creates a sketch that can encode structural information and can process data not seen during the sketching process.
We supported our theoretical findings with experimental results. In a practical test setting with a reasonably large dataset (100,000 elements), RACE outperformed existing methods for low-memory near-neighbor search by a factor of 10. We expect that RACE will enable large-scale similarity search for a variety of applications and will find utility in situations where memory and communication are limiting factors.

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