Improved bounds for the extremal number of subdivisions

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Abstract

Let $H_t$ be the subdivision of $K_t$. Very recently, Conlon and Lee have proved that for any integer $t \geq 3$, there exists a constant $C$ such that $\text{ex}(n, H_t) \leq C n^{3/2 - 1/6}$\textsuperscript{t}. In this paper, we prove that there exists a constant $C'$ such that $\text{ex}(n, H_t) \leq C' n^{3/2 - 1/4}$\textsuperscript{t}.

1 Introduction

For a graph $H$, the extremal function $\text{ex}(n, H)$ is defined to be the maximal number of edges in an $H$-free graph on $n$ vertices. This function is well understood for graphs $H$ with chromatic number at least three by the Erdős-Stone-Simonovits theorem. However, for bipartite graphs $H$, much less is known. For a survey on the subject, see [7]. One of the few general results, proved by Füredi [6], and reproved by Alon, Krivelevich and Sudakov [1] is the following.

Theorem 1 (Füredi, Alon-Krivelevich-Sudakov). Let $H$ be a bipartite graph such that in one of the parts all the degrees are at most $r$. Then there exists a constant $C$ such that $\text{ex}(n, H) \leq C n^{2-1/r}$.

Conlon and Lee [2] have conjectured that the only case when this is tight up to the implied constant is when $H$ contains a $K_{r,r}$ (it is conjectured [3] that $\text{ex}(n, K_{r,r}) = \Omega(n^{2-1/r})$), and that for other graphs $H$ there exists some $\delta > 0$ such that $\text{ex}(n, H) = O(n^{2-1/r-\delta})$.

The subdivision of a graph $L$ is the bipartite graph with parts $V(L)$ and $E(L)$ (the vertex set and the edge set of the graph $L$, respectively) where $v \in V(L)$ is joined to $e \in E(L)$ if $v$ is an endpoint of $e$. It is easy to see that any $C_4$-free bipartite graph in which every vertex in one part has degree at most two is a subgraph of $H_t$ for some positive integer $t$, where $H_t$ is the subdivision of $K_t$. Conlon and Lee have verified their conjecture in the $r = 2$ case by proving the following result.

Theorem 2 (Conlon and Lee [2, Theorem 5.1]). For any integer $t \geq 3$, there exists a constant $C_t$ such that $\text{ex}(n, H_t) \leq C_t n^{3/2-1/6}$.

They have observed the lower bound $\text{ex}(n, H_t) \geq c n^{3/2 - 1/2}$ coming from the probabilistic deletion method, and have asked for an upper bound of the form $\text{ex}(n, H) \leq C_t n^{3/2 - \delta_t}$, where $1/\delta_t$ is bounded by a polynomial in $t$. We can prove such a bound even for a linear $\delta_t$.  

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Theorem 3. For any integer \( t \geq 3 \), there exists a constant \( C_t \) such that \( \text{ex}(n, H_t) \leq C_t n^{1+\frac{1}{2t-3}} = C_t n^{3/2-\frac{t}{t-3}}. \)

It would be very interesting to know whether or not this bound is tight up to the implied constant. It certainly is tight for \( t = 3 \) as \( \text{ex}(n, C_6) = \Theta(n^{4/3}) \).

We can in fact prove a slightly stronger result. For integers \( s \geq 1 \) and \( t \geq 3 \), let \( L_{s,t} \) be the graph which is a \( K_{s+t-1} \) with the edges of a \( K_s \) removed. That is, the vertex set of \( L_{s,t} \) is \( S \cup T \) where \( S \cap T = \emptyset, |S| = s \) and \( |T| = t - 1 \), and \( xy \) is an edge if and only if \( x \in T \) or \( y \in T \). Let \( L_{s,t}' \) be the subdivision of \( L_{s,t} \).

Theorem 4. For any two integers \( s \geq 1 \) and \( t \geq 3 \), there exists a constant \( C_{s,t} \) such that \( \text{ex}(n, L_{s,t}') \leq C_{s,t} n^{3/2-\frac{1}{s+t-3}}. \)

This result certainly implies Theorem 3 as \( L_{1,t} = K_t \). Moreover, we can apply Theorem 4 to obtain good bounds on the extremal number of the subdivision of the complete bipartite graph \( K_{a,b} \) as well. Let us write \( H_{a,b} \) for the subdivision of \( K_{a,b} \). Conlon and Lee [2, Theorem 4.2] have proved that for any \( 2 \leq a \leq b \) there exists a constant \( C \) such that \( \text{ex}(n, H_{a,b}) \leq C_n^{3/2-\frac{1}{2a}}. \) They have also observed the lower bound \( \text{ex}(n, H_{a,b}) = \Omega_{a,b}(n^{3/2-\frac{1}{2a}+\epsilon}) \) (which follows from the probabilistic deletion method). Hence their upper bound is reasonably close to best possible when \( a = b \), but is weak when \( b \) is much larger then \( a \).

Since \( K_{a,b} \) is a subgraph of \( L_{b,a+1} \), Theorem 4 implies the following result, by taking \( s = b \) and \( t = a + 1 \).

Corollary 5. For any two integers \( 2 \leq a \leq b \), there exists a constant \( C_{a,b} \), such \( \text{ex}(n, H_{a,b}) \leq C_{a,b} n^{3/2-\frac{1}{2a}+\epsilon}. \)

2 Proof of Theorem 4

We shall use the following lemma of Conlon and Lee [2, Lemma 2.3], which is a slight modification of a result of Erdős and Simonovits [4]. Let us say that a graph \( G \) is \( K\)-almost-regular if \( \max_{v \in V(G)} \deg(v) \leq K \min_{v \in V(G)} \deg(v) \). Moreover, following Conlon and Lee, we say that a bipartite graph \( G \) with a bipartition \( A \cup B \) is balanced if \( \frac{1}{2} |B| \leq |A| \leq 2 |B| \).

Lemma 6. For any positive constant \( \alpha < 1 \), there exists \( n_0 \) such that if \( n \geq n_0 \), \( C \geq 1 \) and \( G \) is an \( n \)-vertex graph with at least \( Cn^{1+\alpha} \) edges, then \( G \) has a \( K\)-almost-regular balanced bipartite subgraph \( G' \) with \( m \) vertices such that \( m \leq n^{\frac{1+\alpha}{1+2\alpha}}, |E(G')| \geq C m^{1+\alpha} \) and \( K = 60 \cdot 2^{1+1/\alpha^2} \).

This reduces Theorem 4 to the following.

Theorem 7. For every \( K \geq 1 \), and positive integers \( s \geq 1, t \geq 2 \), there exists a constant \( c = c(s,t,K) \) with the following property. Let \( n \) be sufficiently large and let \( G \) be a balanced bipartite graph with bipartition \( A \cup B \), \( |B| = n \) such that the degree of every vertex of \( G \) is between \( \delta \) and \( K\delta \), for some \( \delta \geq c n^{1/2-1/3} \). Then \( G \) contains a copy of \( L_{s,t}' \).
Given a bipartite graph $G$ with bipartition $A \cup B$, the neighbourhood graph is the weighted graph $W_G$ on vertex set $A$ where the weight of the pair $uv$ is $d_G(u,v) = |N_G(u) \cap N_G(v)|$. Here and below $N_G(v)$ denotes the neighbourhood of the vertex $v$ in the graph $G$. For a subset $U \subset A$, we write $W(U)$ for the total weight in $U$, i.e., $W(U) = \sum_{u \in U} d_G(u,v)$.

We shall use the following simple lemma of Conlon and Lee [2, Lemma 2.4].

**Lemma 8.** Let $G$ be a bipartite graph with bipartition $A \cup B$, $|B| = n$, and minimum degree at least $\delta$ on the vertices in $A$. Then for any subset $U \subset A$ with $\delta|U| \geq 2n$,

$$\sum_{u \in U} d_G(u,v) \geq \frac{\delta^2}{2n} \binom{|U|}{2}$$

In other words, the conclusion of Lemma 8 is that $W(U) \geq \frac{\delta^2}{2n} \binom{|U|}{2}$.

In the next definition, and in the rest of this paper, for a weighted graph $W$ on vertex set $A$, if $u, v \in A$, then $W(u,v)$ stands for the weight of $uv$. Moreover, we shall tacitly assume throughout the paper that $s \geq 1$ and $t \geq 3$ are fixed integers.

**Definition 9.** Let $W$ be a weighted graph on vertex set $A$ and let $u, v \in A$ be distinct. We say that $uv$ is a light edge if $1 \leq W(u,v) < \binom{s+t-1}{2}$ and that it is a heavy edge if $W(u,v) \geq \binom{s+t-1}{2}$.

Note that if there is a $K_{s+t-1}$ in $W_G$ formed by heavy edges, then clearly there is an $L_{s,t}$ in $W_G$ formed by heavy edges, therefore there is an $L'_{s,t}$ in $G$.

The next lemma is one of our key observations.

**Lemma 10.** Let $G$ be an $L'_{s,t}$-free bipartite graph with bipartition $A \cup B$, $|B| = n$ and suppose that $W(A) \geq 8(s+t)^2n$. Then the number of light edges in $W_G$ is at least $\frac{W(A)}{4(s+t)^2}$.

**Proof.** Let $B = \{b_1, \ldots, b_n\}$. Let $k_i = |N_G(b_i)|$ and suppose that $k_i \geq 2(s+t-2)$ for some $i$. As $G$ is $L'_{s,t}$-free, there is no $K_{s+t-1}$ in $W[N_G(b_i)]$ formed by heavy edges.

Thus, by Turán’s theorem, the number of light edges in $N_G(b_i)$ is at least

$$(s+t-2) \binom{k_i}{2} = \frac{1}{2} k_i \left( \frac{k_i}{s+t-2} - 1 \right) \geq \frac{k_i^2}{4(s+t-2)}.
$$

But

$$\sum_{i : k_i < 2(s+t-2)} \binom{k_i}{2} < 4(s+t)^2n \leq \frac{W(A)}{2},$$

so

$$\sum_{i : k_i \geq 2(s+t-2)} \binom{k_i}{2} \geq \frac{W(A)}{2}.$$ 

Since every light edge is present in at most $\binom{s+t-1}{2}$ of the sets $N_G(b_i)$, it follows that the total number of light edges is at least
\[
\frac{1}{\binom{s+t-1}{2}} \sum_{i,k \geq 2(s+t-2)} \frac{k^2_i}{4(s+t-2)} \geq \frac{W(A)}{4(s+t)^3}.
\]

\[\square\]

**Corollary 11.** Let \( G \) be an \( L_{s,t} \)-free bipartite graph with bipartition \( A \cup B \), \( |B| = n \), and minimum degree at least \( \delta \) on the vertices in \( A \). Then for any subset \( U \subset A \) with \( |U| \geq \frac{8(s+t)n}{\delta} \) and \( |U| \geq 2 \), the number of light edges in \( W_G[U] \) is at least \( \frac{\delta^2}{8(s+t)^3}|U|^2 \).

**Proof.** By Lemma 8, we have \( W(U) \geq \frac{\delta^2 |U|}{2n} \geq \frac{\delta^2}{8n}|U|^2 \geq 8(s+t)^2n \). Now the result follows by applying Lemma 10 to the graph \( G[U \cup B] \).

We are now in a position to complete the proof of Theorem 7.

**Proof of Theorem 7** Let \( c \) be specified later and suppose that \( n \) is sufficiently large. Assume, for contradiction, that \( G \) is \( L_{s,t} \)-free. We shall find distinct \( u_1, \ldots, u_{t-1} \in A \) with the following properties.

(i) Each \( u_i u_j \) is a light edge in \( W_G \).

(ii) If \( i, j, k \) are distinct, then \( N_G(u_i) \cap N_G(u_j) \cap N_G(u_k) = \emptyset \).

(iii) For each \( 1 \leq i \leq t - 1 \), the number of \( v \in A \) with the property that for every \( j \leq i \), \( u_j v \) is a light edge is at least \( \left( \frac{\delta^2}{32(s+t)^3} \right)^i |A| \).

As \( n \) is sufficiently large, we have \( |A| \geq n/2 \geq \frac{8(s+t)n}{\delta} \), therefore by Corollary 11 there are at least \( \frac{\delta^2}{8(s+t)^3} |A| \) light edges in \( A \), so we may choose \( u_1 \in A \) such that the number of light edges \( u_1 v \) is at least \( \frac{\delta^2}{32(s+t)^3} |A| \).

Now suppose that \( 2 \leq i \leq t - 1 \), and that \( u_1, \ldots, u_{i-1} \) have been constructed satisfying (i),(ii) and (iii). Let \( U_0 \) be the set of vertices \( v \in A \) with the property that \( u_j v \) is a light edge for every \( j \leq i - 1 \). By (iii), we have \( |U_0| \geq \left( \frac{\delta^2}{32(s+t)^3} \right)^{i-1} |A| \).

Now let \( U \) consist of those \( v \in U_0 \) for which \( N_G(u_j) \cap N_G(u_k) \cap N_G(v) = \emptyset \) holds for all \( 1 \leq j < k \leq i - 1 \). Since \( u_j u_k \) is a light edge for any \( 1 \leq j < k \leq i - 1 \), we have that \( d_G(u_j, u_k) < (s+t-1) \). But the degree of every \( b \in B \) is at most \( K\delta \), therefore the number of \( v \in A \) for which \( N_G(u_j) \cap N_G(u_k) \cap N_G(v) \neq \emptyset \) is at most \( (s+t-1)^2 K \delta \), so \( |U_0 \setminus U| \leq \left( \frac{i-1}{i-1} \right)^{(s+t-1)^2 K \delta} \). But note that for sufficiently large \( n \), we have \( \left( \frac{\delta^2}{32(s+t)^3} n \right)^{i-1} |A| \geq 2 \left( \frac{s+t}{s+t} \right)^{(s+t-1)^2 K \delta} \) because \( \delta = o((\delta^2 n)^{t-2}) \) and \( \delta = o((\delta^2/n)n) \).

Thus,

\[|U| \geq \frac{1}{2} |U_0| \geq \frac{1}{2} \left( \frac{\delta^2}{32(s+t)^3} \right)^{i-1} |A|.
\]

But for sufficiently large \( c = c(s, t, K) \), we have \( \frac{1}{2} \left( \frac{\delta^2}{32(s+t)^3} \right)^{i-1} |A| \geq \frac{8(s+t)n}{\delta} \). Indeed, this is obvious when \( \delta^2 \geq 32(s+t)^3 n \), and otherwise, using \( \delta \geq cn^{\frac{t-2}{t}} \), we have

\[\frac{1}{2} \left( \frac{\delta^2}{32(s+t)^3} \right)^{i-1} |A| \geq \frac{1}{2} \left( \frac{\delta^2}{32(s+t)^3} \right)^{i-2} |A| \geq \frac{1}{4(32(s+t)^3)} \frac{\delta^{2i-4}}{n^{t-3}} \geq \frac{8(s+t)n}{\delta}.
\]

Thus, by Corollary 11 there exists some \( u_i \in U \) with at least \( \frac{\delta^2}{8(s+t)^3} |U| - 1 \) \( \left( \frac{\delta^2}{32(s+t)^3} \right)^i |A| \) light edges adjacent to it in \( U \). This completes the recursive construction of the vertices \( \{u_j\}_{1 \leq j \leq t-1} \).
By (iii) for \( i = t - 1 \), there is a set \( V \subseteq A \) consisting of at least \( (\frac{\delta^2}{32(s+t)n})^{t-1}|A| \) vertices \( v \) such that for every \( j \leq t - 1 \), \( u_jv \) is a light edge. We shall now prove that there exist distinct \( v_1, \ldots, v_s \in V \) such that \( N_G(u_i) \cap N_G(u_j) \cap N_G(v_k) \neq \emptyset \) for all \( i \neq j \), and \( N_G(u_i) \cap N_G(v_j) \cap N_G(v_k) \neq \emptyset \) for all \( j \neq k \). It is easy to see that this suffices since then there is a copy of \( L'_{s,t} \) in \( G \), which is a subdivision of the copy of \( L_{s,t} \) in \( W_G \) whose vertices are \( v_1, \ldots, v_s, u_1, \ldots, u_{t-1} \).

We shall now choose \( v_1, \ldots, v_s \) one by one. Since every \( u_iu_j \) is a light edge, the number of those \( v \in A \) with \( N_G(u_i) \cap N_G(u_j) \cap N_G(v) \neq \emptyset \) for some \( i \neq j \) is at most \((t-1)(s+t-1)K\delta\). Moreover, given any choices for \( v_1, \ldots, v_{k-1} \in V \), as each \( u_iv_j \) is a light edge, the number of those \( v \in A \) with \( N_G(u_i) \cap N_G(v_j) \cap N_G(v) \neq \emptyset \) for some \( i, j \) is at most \((t-1)(k-1)(s+t-1)K\delta\). Therefore as long as \( |V| > (t-1)(s+t-1)K\delta + (t-1)(s-1)(s+t-1)K\delta \), suitable choices for \( v_1, \ldots, v_s \) can be made. Since \( |V| \geq (\frac{\delta^2}{32(s+t)n})^{t-1}|A| \), this last inequality holds for large enough \( c = c(s, t, K) \). \( \Box \)

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