A HILBERT-KUNZ CRITERION FOR SOLID CLOSURE IN DIMENSION TWO (CHARACTERISTIC ZERO)

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Abstract. Let $I$ denote a homogeneous $R_+$-primary ideal in a two-dimensional normal standard-graded domain over an algebraically closed field of characteristic zero. We show that a homogeneous element $f$ belongs to the solid closure $I^*$ if and only if $e_{HK}(I) = e_{HK}((I, f))$, where $e_{HK}$ denotes the (characteristic zero) Hilbert-Kunz multiplicity of an ideal. This provides a version in characteristic zero of the well-known Hilbert-Kunz criterion for tight closure in positive characteristic.

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INTRODUCTION

Let $(R, \mathfrak{m})$ denote a local Noetherian ring or an $\mathbb{N}$-graded algebra of dimension $d$ of positive characteristic $p$. Let $I$ denote an $\mathfrak{m}$-primary ideal, and set $I^{[q]} = (f^q : f \in I)$ for a prime power $q = p^e$. Then the Hilbert-Kunz function of $I$ is given by

$$e \mapsto \lambda(R/I^{[p^e]}),$$

where $\lambda$ denotes the length. The Hilbert-Kunz multiplicity of $I$ is defined as the limit

$$e_{HK}(I) = \lim_{e \to \infty} \lambda(R/I^{[p^e]})/p^{ed}.$$  

This limit exists as a positive real number, as shown by Monsky in [9]. It is an open question whether this number is always rational.

The Hilbert-Kunz multiplicity is related to the theory of tight closure. Recall that the tight closure of an ideal $I$ in a Noetherian ring of characteristic $p$ is by definition the ideal

$$I^* = \{ f \in R : \exists c \text{ not in any minimal prime : } cf^q \in I^{[q]} \text{ for almost all } q = p^e \}.$$  

For an analytically unramified and formally equidimensional local ring $R$ the equation $e_{HK}(I) = e_{HK}(J)$ holds if and only if $I^* = J^*$ holds true for ideals $I \subseteq J$ (see [6, Theorem 5.4]). Hence $f \in I^*$ if and only if $e_{HK}(I) = e_{HK}((I, f))$. This is the Hilbert-Kunz criterion for tight closure in positive characteristic.

The aim of this paper is to give a characteristic zero version of this relationship between Hilbert-Kunz multiplicity and tight closure for $R_+$-primary homogeneous ideals in a normal two-dimensional graded domain $R$. There
are several notions for tight closure in characteristic zero, defined either by reduction to positive characteristic or directly. We will work with the notion of solid closure (see [3]). In dimension two, the containment in the solid closure $f \in (f_1, \ldots, f_n)^*$ means that the open subset $D(\mathfrak{m}) \subset \text{Spec } A$ is not an affine scheme, where $A = \mathbb{R}[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ is the so-called forcing algebra, see [1] Proposition 1.3.

The definition of the Hilbert-Kunz multiplicity in positive characteristic does not suggest at first sight an analogous notion in characteristic zero. However, a bridge is provided by the following result of [2], which gives an explicit formula for the Hilbert-Kunz multiplicity and proves its rationality in dimension two (the rationality of the Hilbert-Kunz multiplicity for the maximal ideal was also obtained independently in [10]).

**Theorem 1.** Let $R$ denote a two-dimensional standard-graded normal domain over an algebraically closed field of positive characteristic, $Y = \text{Proj } R$. Let $I = (f_1, \ldots, f_n)$ denote a homogeneous $R_+$-primary ideal generated by homogeneous elements $f_i$ of degree $d_i$, $i = 1, \ldots, n$. Then the Hilbert-Kunz multiplicity of the ideal $I$ equals

$$e_{HK}(I) = \frac{\deg(Y)}{2} \left( \sum_{k=1}^t r_k \nu_k^2 - \sum_{i=1}^n d_i^2 \right).$$

Here the numbers $r_k$ and $\nu_k$ come from the strong Harder-Narasimhan filtration of the syzygy bundle $\text{Syz}(f_1^q, \ldots, f_n^q)(0)$ given by the short exact sequence

$$0 \rightarrow \text{Syz}(f_1^q, \ldots, f_n^q)(0) \rightarrow \bigoplus_{i=1}^t \mathcal{O}(-qd_i) \xrightarrow{f_1^q, \ldots, f_n^q} \mathcal{O}_Y \rightarrow 0.$$

This syzygy bundle is a locally free sheaf on the smooth projective curve $Y = \text{Proj } R$, and its strong Harder-Narasimhan filtration is a filtration $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_t = \text{Syz}(f_1^q, \ldots, f_n^q)(0)$ such that the quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are strongly semistable, meaning that every Frobenius pull-back is semistable. Such a filtration exists for $q$ big enough by a theorem of Langer, [5] Theorem 2.7.

Then we set $r_k = \text{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $\nu_k = -\mu(\mathcal{S}_k/\mathcal{S}_{k-1})/q \deg(Y)$, where $\mu$ denotes the slope.

To define the Hilbert-Kunz multiplicity in characteristic zero we now take the right hand side of the above formula as our model.

**Definition 1.** Let $R$ denote a two-dimensional normal standard-graded $K$-domain over an algebraically closed field $K$ of characteristic zero. Let $I = (f_1, \ldots, f_n)$ be a homogeneous $R_+$-primary ideal given by homogeneous ideal generators $f_i$ of degree $d_i$. Let $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_t = \text{Syz}(f_1, \ldots, f_n)(0)$ denote the Harder-Narasimhan filtration of the syzygy bundle on $Y = \text{Proj } R$, set $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $r_k = \text{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$. Then the Hilbert-Kunz multiplicity of $I$ is by definition

$$e_{HK}(I) = \frac{\deg(Y)}{2} \left( \sum_{k=1}^t r_k \frac{\mu_k}{\deg(Y)^2} - \sum_{i=1}^n d_i^2 \right) = \frac{\sum_{k=1}^t r_k \mu_k^2}{2 \deg(Y)} - \frac{\deg(Y)^2 \sum_{i=1}^n d_i^2}{2 \deg(Y)}.$$
Hilbert-Kunz Criterion

It is easy to show that this definition does not depend on the chosen ideal generators and is therefore an invariant of the ideal, see [2, Proposition 4.9]. With this invariant we can in fact give the following Hilbert-Kunz criterion for solid closure in characteristic zero in dimension two (see Theorem 3.3):

**Theorem 2.** Let \( K \) denote an algebraically closed field of characteristic zero, let \( R \) denote a standard-graded two-dimensional normal \( K \)-domain. Let \( I \) be a homogeneous \( R_+ \)-primary ideal and let \( f \) denote a homogeneous element. Then \( f \) is contained in the solid closure, \( f \in I^* \), if and only if \( e_{HK}(I) = e_{HK}((I, f)) \).

To prove this theorem it is convenient to consider more generally for a locally free sheaf \( S \) on a smooth projective curve \( Y \) the expression

\[
\mu_{HK}(S) = \sum_{k=1}^{t} r_k \mu_k^2,
\]

where \( r_k \) and \( \mu_k \) are the ranks and the slopes of the semistable quotient sheaves in the Harder-Narasimhan filtration of \( S \). We call this number the Hilbert-Kunz slope of \( S \). With this notion the Hilbert-Kunz multiplicity of an ideal \( I = (f_1, \ldots, f_n) \) is related to the Hilbert-Kunz slope of the syzygy bundle by

\[
e_{HK}((f_1, \ldots, f_n)) = \frac{1}{2 \deg(Y)} \left( \mu_{HK}(\text{Syz}(f_1, \ldots, f_n)(0)) - \mu_{HK}(\bigoplus_{i=1}^{n} \mathcal{O}(-d_i)) \right).
\]

With this notion we will in fact prove the following theorem, which implies Theorem 2 (see Theorem 2.6).

**Theorem 3.** Let \( Y \) denote a smooth projective curve over an algebraically closed field of characteristic 0. Let \( S \) denote a locally free sheaf on \( Y \) and let \( c \in H^1(Y, S) \) denote a cohomology class given rise to the extension \( 0 \to S \to S' \to \mathcal{O}_Y \to 0 \) and the affine-linear torsor \( \mathbb{P}(S') - \mathbb{P}(S) \). Then \( \mathbb{P}(S') - \mathbb{P}(S) \) is an affine scheme if and only if \( \mu_{HK}(S') < \mu_{HK}(S) \).

1. **The Hilbert-Kunz Slope of a Vector Bundle**

We recall briefly some notions for locally free sheaves (or vector bundles), see [2] or [3]. Let \( Y \) denote a smooth projective curve over an algebraically closed field and let \( S \) denote a locally free sheaf of rank \( r \). Then \( \text{deg}(S) = \text{deg}(\bigwedge^r S) \) is called the degree of \( S \) and \( \mu(S) = \text{deg}(S)/r \) is called the slope of \( S \). If \( \mu(T) \leq \mu(S) \) holds for every locally free subsheaf \( T \subseteq S \), then \( S \) is called semistable. In general there exists the so-called Harder-Narasimhan filtration. This is a filtration of locally free subsheaves \( S_1 \subset \ldots \subset S_t = S \) such that the quotient sheaves \( S_k/S_{k-1} \) are semistable locally free sheaves with decreasing slopes \( \mu_1 > \ldots > \mu_t \). The Harder-Narasimhan filtration is uniquely determined by these properties. \( S_1 \) is called the maximal destabilizing subsheaf, \( \mu_1 = \mu_{\max}(S) \) is called the maximal slope of \( S \) and \( \mu_t = \mu_{\min}(S) \) is
called the minimal slope of $S$. If $S ightarrow T$ is a non-trivial sheaf homomorphism, then $\mu_{\min}(S) \leq \mu_{\max}(T)$.

We begin with the definition of the Hilbert-Kunz slope of $S$.

**Definition 1.1.** Let $S$ denote a locally free sheaf on a smooth projective curve over an algebraically closed field of characteristic 0. Let $S_1 \subset \ldots \subset S_t = S$ denote the Harder-Narasimhan filtration of $S$, set $r_k = \text{rk}(S_k/S_{k-1})$ and $\mu_k = \mu(S_k/S_{k-1})$. We define the Hilbert-Kunz slope of $S$ by

$$\mu_{HK}(S) = \sum_{k=1}^{t} r_k \mu_k^2 = \sum_{k=1}^{t} \frac{\text{deg}(S_k/S_{k-1})^2}{r_k}.$$ 

The only justification for considering this number is Theorem 3.3 below. We gather together some properties of this notion in the following proposition.

**Proposition 1.2.** Let $S$ denote a locally free sheaf on a smooth projective curve over an algebraically closed field of characteristic 0. Then the following hold true.

1. If $S$ is semistable, then $\mu_{HK}(S) = \text{deg}(S)^2 / \text{rk}(S)$.
2. Let $T \subset S$ denote a locally free subsheaf occurring in the Harder-Narasimhan filtration of $S$. Then $\mu_{HK}(S) = \mu_{HK}(T) + \mu_{HK}(S/T)$.
3. We have $\mu_{HK}(S \oplus T) = \mu_{HK}(S) + \mu_{HK}(T)$.
4. $\mu_{HK}(S) = \mu_{HK}(S^\vee)$.
5. Let $\mathcal{L}$ denote an invertible sheaf. Then

$$\mu_{HK}(S \otimes \mathcal{L}) = \mu_{HK}(S) + 2 \text{deg}(S) \text{deg}(\mathcal{L}) + \text{rk}(S) \text{deg}(\mathcal{L})^2.$$ 

6. Let $\varphi : Y' \rightarrow Y$ denote a finite morphism between smooth projective curves of degree $n$. Then $\mu_{HK}(\varphi^*(S)) = n^2 \mu_{HK}(S)$.

**Proof.** (i) and (ii) are clear from the definition. (iii). The maximal destabilizing subsheaf of $S \oplus T$ is either $S_1 \oplus 0$, $0 \oplus T_1$ or $S_1 \oplus T_1$. Hence the result follows from (ii) by induction on the rank of $S \oplus T$.

(iv). Let $0 = S_0 \subset S_1 \subset \ldots \subset S_t = S$ denote the Harder-Narasimhan filtration of $S$. Set $Q_k = S_k/S_k$. This gives a filtration $0 \subset Q_{t-1}^\vee \subset \ldots \subset Q_1^\vee \subset Q_0^\vee = S^\vee$. From $0 \rightarrow Q_k^\vee \rightarrow Q_{k-1}^\vee \rightarrow Q_k^\vee \rightarrow Q_{k-1}^\vee$ we get $0 \rightarrow Q_k^\vee \rightarrow Q_{k-1}^\vee \rightarrow Q_{k-1}^\vee/ Q_k^\vee \cong (S_k/S_{k-1})^\vee \rightarrow 0$. Hence the filtration is the Harder-Narasimhan filtration of $S^\vee$ and the result follows from $\mu(Q_{k-1}^\vee/ Q_k^\vee) = -\mu(S_k/S_{k-1})$.

(v). The Harder-Narasimhan filtration of $S \otimes \mathcal{L}$ is $S_1 \otimes \mathcal{L} \subset \ldots \subset S_t \otimes \mathcal{L}$ and $\mu(S_k \otimes \mathcal{L}/S_{k-1} \otimes \mathcal{L}) = \mu((S_k/S_{k-1}) \otimes \mathcal{L}) = \mu(S_k/S_{k-1}) + \mu(\mathcal{L})$. Therefore

$$\mu_{HK}(S \otimes \mathcal{L}) = \sum_{k=1}^{t} r_k \mu_k(S \otimes \mathcal{L})^2 = \sum_{k=1}^{t} r_k (\mu_k + \text{deg}(\mathcal{L}))^2.$$
\[
\sum_{k=1}^{t} r_k (\mu_k^2 + 2\mu_k \deg(L) + \deg(L)^2) = \mu_{HK}(S) + 2 \deg(L) \sum_{k=1}^{t} r_k \mu_k + \deg(L)^2 \sum_{k=1}^{t} r_k .
\]

This is the stated result, since \(\deg(S) = \sum_{k=1}^{t} r_k \mu_k\) and \(\text{rk}(S) = \sum_{k=1}^{t} r_k\).

(vi). The pull-back of a semistable sheaf under a separable morphism is again semistable, and the pull-back of the Harder-Narasimhan filtration is the Harder-Narasimhan filtration of \(\varphi^*(S)\). Hence the result follows from \(\deg(\varphi^*(S)) = n \deg(S)\). \(\square\)

**Lemma 1.3.** The Hilbert-Kunz multiplicity of a locally free sheaf \(S\) has the property that \(\mu_{HK}(S) \geq \deg(S)^2 / \text{rk}(S)\), and equality holds if and only if \(S\) is semistable.

**Proof.** We have to show that
\[
\sum_{k=1}^{t} r_k \mu_k^2 \geq \deg(S)^2 / \text{rk}(S) = (r_1 \mu_1 + \ldots + r_t \mu_t)^2 / (r_1 + \ldots + r_t)
\]
or equivalently that
\[
(r_1 + \ldots + r_t) (\sum_{k=1}^{t} r_k \mu_k^2) \geq (r_1 \mu_1 + \ldots + r_t \mu_t)^2 .
\]
The left hand side is \(\sum_{k=1}^{t} r_k^2 \mu_k^2 + \sum_{i \neq k} r_i r_k \mu_i \mu_k\) (we sum over ordered pairs), and the right hand side is \(\sum_{k=1}^{t} r_k^2 \mu_k^2 + \sum_{i \neq k} r_i r_k \mu_i \mu_k\). Hence left hand minus right hand is
\[
\sum_{i \neq k} r_i r_k \mu_i \mu_k - \sum_{i \neq k} r_i r_k \mu_i \mu_k
\]
So this follows from \(0 \leq (\mu_i - \mu_k)^2 = \mu_i^2 + \mu_k^2 - 2\mu_i \mu_k\) for all pairs \(i \neq k\). Equality holds if and only if \(\mu_i = \mu_k\), but then \(t = 1\) and \(S\) is semistable. \(\square\)

**Remark 1.4.** Lemma \([1.3]\) implies that the number \(\mu_{HK}(S) - \deg(S)^2 / \text{rk}(S)\) \(\geq 0\), and \(= 0\) holds exactly in the semistable case. It follows from Proposition \([1.2]\) (v) that this number is invariant under tensoring with an invertible sheaf.

**Proposition 1.5.** Let \(S\) and \(T\) denote two locally free sheaves on \(Y\). Then
\[
\mu_{HK}(S \otimes T) = \text{rk}(T) \mu_{HK}(S) + \text{rk}(S) \mu_{HK}(T) + 2 \deg(S) \deg(T) .
\]

**Proof.** Let \(r_i, \mu_i, i \in I,\) and \(r_j, \mu_j, j \in J, (I\) and \(J\) disjoint\) denote the ranks and slopes occurring in the Harder-Narasimhan filtration of \(S\) and \(T\) respectively. It is a non-trivial fact (in characteristic zero!) that the tensor product of two semistable bundle is again semistable, see \([7\] Theorem 3.1.4]. From this it follows that the semistable quotients of the Harder-Narasimhan
filtration of $S \otimes T$ are given as $(S_i/S_{i-1}) \otimes (T_j/T_{j-1})$ of rank $r_i \cdot r_j$ and slope $\mu_i + \mu_j$. Therefore the Hilbert-Kunz slope is

$$\mu_{HK}(S \otimes T) = \sum_{i,j} r_i r_j (\mu_i + \mu_j)^2$$

$$= \sum_{i,j} r_i r_j \mu_i^2 + \sum_{i,j} r_i r_j \mu_j^2 + 2 \sum_{i,j} r_i r_j \mu_i \mu_j$$

$$= (\sum_j r_j)(\sum_i r_i \mu_i^2) + (\sum_i r_i)(\sum_j r_j \mu_j^2) + 2(\sum_i r_i \mu_i)(\sum_j r_j \mu_j)$$

$$= \text{rk}(T) \mu_{HK}(S) + \text{rk}(S) \mu_{HK}(T) + 2 \deg(S) \deg(T)$$

\[\blacksquare\]

2. A Hilbert-Kunz criterion for affine torsors

In this section we consider a locally free sheaf $S$ on a smooth projective curve $Y$ together with a cohomology class $c \in H^1(Y, S) \cong \text{Ext}(O_Y, S)$. Such a class gives rise to an extension $0 \to S \to S' \to O_Y \to 0$. Of course $\deg(S') = \deg(S)$ and $\text{rk}(S') = \text{rk}(S) + 1$. We shall investigate the relationship between $\mu_{HK}(S)$ and $\mu_{HK}(S')$.

Lemma 2.1. Let $Y$ denote a smooth projective curve over an algebraically closed field. Let $S$, $T$ and $Q$ denote locally free sheaves on $Y$. Then the following hold.

(i) Let $\varphi : T \to S$ denote a sheaf homomorphism, $c \in H^1(Y, T)$ with corresponding extension $T'$, let $S'$ denote the extension of $S$ corresponding to $\varphi(c) \in H^1(Y, S)$. Then there is a sheaf homomorphism $\varphi' : T' \to S'$ extending $\varphi$.

(ii) Suppose that $0 \to T \to S \to Q \to 0$ is a short exact sequence, and $c \in H^1(Y, T)$. Then $T' \subseteq S'$ and $S'/T' \cong S/T$.

(iii) Suppose that $0 \to T \to S \to Q \to 0$ is a short exact sequence, and $c \in H^1(Y, S)$. Then $S' \to Q' \to 0$ and $Q' \cong S'/T'$.

(iv) If $S$ is semistable of degree 0 and $c \in H^1(Y, S)$, then also $S'$ is semistable.

Proof. The cohomology class $c$ is represented by the Čech cocycle $\check{c} \in H^0(U_1 \cap U_2, S)$, where $Y = U_1 \cup U_2$ is an affine covering. Then $S'$ arises from $S'_1 = S|_{U_1} \oplus O$ and $S'_2 = S|_{U_2} \oplus O$ by glueing $S'_1|U_1 \cap U_2 \cong S'_2|U_1 \cap U_2$ via $(s + t, t)$ $\mapsto (s + t, t)$. The natural mappings $T'_i \to S'_i$, $i = 1, 2$, glue together to a morphism $T' \to S'$. The injectivity and surjectivity transfer from $\varphi$ to $\varphi'$, since these are local properties. (ii) and (iii) then follow from suitable diagrams.

(iv). Suppose that $F \subseteq S'$ is a semistable subsheaf of positive slope. Then the induced mapping $F \to O$ is trivial and therefore $F \subseteq S$, which contradicts the semistability of $S$. \[\blacksquare\]
Let $S_1 \subset \ldots \subset S_t = S$ denote the Harder-Narasimhan filtration of $S$ and $c \in H^1(Y, S)$. If the image of $c$ in $H^1(Y, S/S_{k-1})$ is zero, then $c$ stems from a class $c_{i-1} \in H^1(Y, S_{i-1})$. So we find inductively a class $c_n \in H^1(Y, S_n)$ mapping to $c$ and such that the image in $H^1(Y, S_n/S_{n-1})$ is not zero (or $c$ itself is 0). This yields extensions $S'_k$ of $S_k$ for $k \geq n$. It is crucial for the behavior of $S'$ whether $\mu(S_n/S_{n-1}) \geq 0$ or $< 0$. The following Proposition deals with the case $\mu(S_n/S_{n-1}) \geq 0$.

**Proposition 2.2.** Let $S_1 \subset \ldots \subset S_t = S$ be the Harder-Narasimhan filtration of $S$ and let $c \in H^1(Y, S)$. Let $n$ be such that the image of $c$ in $H^1(Y, S_n/S_{n-1})$ is 0 for $k > n$ but such that the image in $H^1(Y, S_n/S_{n-1})$ is $\neq 0$. Suppose that $\mu(S_n/S_{n-1}) \geq 0$. Let $i$ be the biggest number such that $\mu(S_i/S_{i-1}) \geq 0$ (hence $n \leq i$).

(i) Suppose that $\mu_i > 0$. Then the Harder-Narasimhan filtration of $S'$ is

$$S_1 \subset \ldots \subset S_i \subset S'_i \subset S'_{i+1} \subset \ldots \subset S'.$$

(ii) Suppose that $\mu_i = 0$. Then the Harder-Narasimhan filtration of $S'$ is

$$S_1 \subset \ldots \subset S_{i-1} \subset S'_i \subset S'_{i+1} \subset \ldots \subset S'.$$

**Proof.** (i). The quotients of the filtration are $S_k/S_{k-1}$, $k \leq i$, which have positive slope, $S'_i/S_i \cong O_Y$, and $S'_k/S'_{k-1} \cong S_k/S_{k-1}$ (Lemma 2.1(ii)) for $k > i$, which have negative slope. These quotients are all semistable and the slope numbers are decreasing.

(ii). The quotients $S_k/S_{k-1}$ are semistable with decreasing positive slopes for $k = 1, \ldots, i-1$. The quotients $S'_k/S'_{k-1} \cong S_k/S_{k-1}$ are semistable with decreasing negative slopes for $k = i+1, \ldots, t$. The quotient $S'_i/S_{i-1}$ is isomorphic to $(S_i/S_{i-1})'$ by Lemma 2.1(iii), hence semistable of degree 0 by Lemma 2.1(iv).

In the rest of this section we study the remaining case, that $\mu(S_n/S_{n-1}) < 0$. In this case it is not possible to describe the Harder-Narasimhan filtration of $S'$ explicitly. However we shall see that in this case the Hilbert-Kunz slope of $S'$ is smaller than the Hilbert-Kunz slope of $S$. We need the following two lemmata.

**Lemma 2.3.** Let $T$ denote a locally free sheaf on $Y$ with Harder-Narasimhan filtration $T_k$, $\mu_k = \mu(T_k/T_{k-1})$ and $r_k = \text{rk}(T_k/T_{k-1})$. Let

$$(\tau_i) = (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2, \mu_3, \ldots, \mu_{t-1}, \mu_t, \ldots, \mu_t)$$

denote the slopes where each $\mu_k$ occurs $r_k$-times. Let $S \subseteq T$ denote a locally free subsheaf of rank $r$ and let $\sigma_i$, $i = 1, \ldots, r$ denote the corresponding numbers for $S$. Then $\sigma_i \leq \tau_i$ for $i = 1, \ldots, r$.

Moreover, if $S$ is saturated (meaning that the quotient sheaf is locally free) and if no subsheaf $S_j$ of the Harder-Narasimhan filtration of $S$ occurs in the Harder-Narasimhan filtration of $T$, then $\sigma_i \leq \tau_{i+1}$ for $i = 1, \ldots, r$. 


Proof. Let \( i, i = 1, \ldots, r \) be given and let \( j \) be such that \( \text{rk}(S_{j-1}) < i \leq \text{rk}(S_j) \), hence \( \sigma_i = \mu_j(S) = \mu(S_j/S_{j-1}) \). We may assume that \( i = \text{rk}(S_j) \). Let \( k \) be such that \( \text{rk}(T_{k-1}) < i \leq \text{rk}(T_k) \). Therefore \( S_j \not\subseteq T_{k-1} \), and the induced morphism \( S_j \to T/T_{k-1} \) is not trivial. Hence \( \sigma_i = \mu_j(S) = \mu(S_j) \leq \mu_{\min}(S_j) \leq \mu_{\max}(T/T_{k-1}) = \mu_k(T) = \tau_i \).

Now suppose that \( \sigma_i > \tau_{i+1} \). Then necessarily \( \sigma_i > \tau_{i+1} \) and \( \tau_i > \tau_{i+1} \) by what we have already proven. Therefore \( i = \text{rk}(S_j) = \text{rk}(T_k) \). If \( S_j \subseteq T_k \), then they are equal, since both sheaves are saturated of the same rank, but this is excluded by the assumptions. Hence \( S_j \not\subseteq T_k \) and \( S_j \to T/T_k \) is non-trivial. Therefore \( \sigma_i = \mu_{\min}(S_j) \leq \mu_{\max}(T/T_k) \). 

Remark 2.4. If the numbers \( \tau_i \) are given as in the previous lemma, then \( \deg(T) = \sum_i \tau_i \) and \( \mu_{HK}(T) = \sum_i \tau_i^2 \).

Lemma 2.5. Let \( \alpha_1 \leq \ldots \leq \alpha_r \) and \( \beta_1 \leq \ldots \leq \beta_{r+1} \) denote positive real numbers such that \( \alpha_i \geq \beta_{i+1} \) for \( i = 1, \ldots, r \) and \( \sum_{i=1}^r \alpha_i = \sum_{i=1}^{r+1} \beta_i \). Then \( \sum_{i=1}^r \beta_i^2 \leq \sum_{i=1}^{r+1} \alpha_i^2 \) and equality holds if and only if \( \alpha_i = \beta_i \).

Proof. Let \( \alpha_i = \beta_i + \delta_i, \delta_i \geq 0 \). From \( \sum_{i=1}^r \alpha_i = \sum_{i=1}^r \delta_i + \sum_{i=1}^r \beta_i = \sum_{i=1}^{r+1} \beta_i \) we get \( \beta_1 = \sum_{i=1}^r \delta_i \). The quadratic sums are

\[
\sum_{i=1}^r \alpha_i^2 = \sum_{i=2}^{r+1} \beta_i^2 + \sum_{i=1}^r \delta_i^2 + 2 \sum_{i=1}^r \delta_i \beta_i,
\]

and

\[
\sum_{i=1}^{r+1} \beta_i^2 = (\sum_{i=1}^r \delta_i)^2 + \sum_{i=2}^{r+1} \beta_i^2 = 2 \sum_{i<j} \delta_i \delta_j + \sum_{i=1}^r \delta_i^2 + \sum_{i=2}^{r+1} \beta_i^2.
\]

So we have to show that \( \sum_{i<j} \delta_i \delta_j \leq \sum_{j=1}^r \delta_i \beta_j \). But this is clear from \( \sum_{i<j} \delta_i \delta_j \leq \sum_{j=1}^r \delta_j \leq \beta_2 \leq \beta_{i+1} \) for all \( i = 1, \ldots, r \). Equality holds if and only if \( \delta_i = 0 \).

A cohomology class \( H^1(Y,S) \) corresponds to a geometric \( S \)-torsor \( T \to Y \). This is an affine-linear bundle on which \( S \) acts transitively. A geometric realization is given as \( T = \mathbb{P}(S^\vee) - \mathbb{P}(S^\vee) \). The global cohomological properties of this torsor are related to the Hilbert-Kunz slope in the following way.

Theorem 2.6. Let \( Y \) denote a smooth projective curve over an algebraically closed field of characteristic 0. Let \( S \) denote a locally free sheaf on \( Y \) and let \( c \in H^1(Y,S) \) denote a cohomology class given rise to the extension \( 0 \to S \to S' \to \mathcal{O}_Y \to 0 \) and the affine-linear torsor \( \mathbb{P}(S^\vee) - \mathbb{P}(S^\vee) \). Then the following are equivalent.

(i) There exists a locally free quotient \( \varphi : S \to Q \to 0 \) such that \( \mu_{\max}(Q) < 0 \) and the image \( \varphi(c) \in H^1(Y,Q) \) is non-trivial.
(ii) The torsor \( \mathbb{P}(S^\vee) - \mathbb{P}(S^\vee) \) is an affine scheme.
(iii) The Hilbert-Kunz slope drops, that is \( \mu_{HK}(S') < \mu_{HK}(S) \).
Proof. The equivalence (i) ⇔ (ii) was shown in [3, Theorem 2.3]. The implication (iii) ⇒ (i) follows from Proposition 2.2 for if (i) does not hold, then we are in the situation of Proposition 2.2 that \( \mu(S_n/S_{n-1}) \geq 0 \). The explicit description of the Harder-Narasimhan filtration of \( S' \) gives in both cases that \( \mu_{HK}(S') = \mu_{HK}(S) \).

So suppose that (i) holds. This means that there exists a subsheaf \( S_n \subseteq S \) occurring in the Harder-Narasimhan filtration of \( S \) such that \( c_n \in H^1(Y, S_n) \) and such that its image in \( H^1(Y, S/S_{n-1}) \) is non-trivial with \( \mu_{\text{max}}(S/S_{n-1}) = \mu(S_n/S_{n-1}) = \mu_n < 0 \).

Let \( T_1 \subset \ldots \subset T_i = S' \) denote the Harder-Narasimhan filtration of \( S' \) with slopes \( \mu_k = \mu(T_k/T_{k-1}) \) and ranks \( r_k = \text{rk}(T_k/T_{k-1}) \). Suppose that the maximal slope \( \mu(T_1) \) is positive. Then the induced mapping \( T_1 \to S'/S \cong O_Y \) is trivial, and \( T_1 \subseteq S \). This is then also the maximal destabilizing subsheaf of \( S \), since \( \mu_{\text{max}}(S) \leq \mu_{\text{max}}(S') = \mu(T_1) \). Therefore \( \mu_{HK}(S) = \mu_{HK}(T_1) + \mu_{HK}(S/T_1) \) and \( \mu_{HK}(S'/T_1) = \mu_{HK}(T_1) + \mu_{HK}(S'/T_1) \) by Proposition 1.2(ii).

Since \( S'/T_1 \) is the extension of \( S/T_1 \) defined by the image of the cohomology class in \( H^1(Y, S/T_1) \) (Lemma 2.1(iii)), we may mod out \( T_1 \). Note that this does not change the condition in (i). Hence we may assume inductively that \( \mu_{\text{max}}(S) \leq 0 \) and \( \mu_{\text{max}}(S') \leq 0 \).

Now suppose that \( T_1 \) has degree 0. Again, if \( T_1 \subseteq S \), then this is also the maximal destabilizing subsheaf of \( S \), and we can mod out \( T_1 \) as before. So suppose that \( T_1 \to O_Y \) is non-trivial. Then this mapping is surjective, let \( K \subseteq S \) denote the kernel. This means that the extension defined by \( c \in H^1(Y, S) \) comes from the extension given by \( 0 \to K \to T_1 \to O_Y \to 0 \), and \( \tilde{c} \in H^1(Y, K) \). \( K \) is semistable, since its degree is 0 and \( \mu_{\text{max}}(S) \leq 0 \). But then the image of \( c \) is 0 in every quotient sheaf of \( S \) with negative maximal slope, which contradicts the assumptions. Therefore we may assume that \( \mu_{\text{max}}(S') < 0 \).

We want to apply Lemma 2.3 to \( S \subset S' = T \). Assume that \( S \) and \( S' \) have a common subsheaf occurring in both Harder-Narasimhan filtrations. Then they have the same maximal destabilizing subsheaf \( F = S_1 = T_1 \), which has negative degree. If \( c \) comes from \( \tilde{c} \in H^1(Y, F) \), then \( F \subset F' \subset S' \) and \( \mu(F) = \deg(F)/\text{rk}(F) < \deg(F)/(\text{rk}(F) + 1) = \mu(F') \), which contradicts the maximality of \( F \). Hence the image of \( c \) in \( H^1(Y, S/F) \) is not zero and we can mod out \( F \) as before.

Therefore we may assume that \( S \) and \( S' \) do not have any common subsheaf in their Harder-Narasimhan filtrations. Then Lemma 2.3 yields that \( \sigma_i \leq \tau_{i+1} \), and all these numbers are \( \leq 0 \) and moreover \( \tau_i < 0 \). Lemma 2.3 applied to \( \alpha_i = -\sigma_i \) and \( \beta_i = -\tau_i \) yields that \( \sum_{i=1}^{r} \sigma_i^2 \geq \sum_{i=1}^{r+1} \tau_i^2 \), and \( > 0 \) holds since \( \tau_1 \neq 0 \).

\[ \square \]

Remark 2.7. Suppose that \( S \) is a semistable locally free sheaf of negative degree, and let \( c \in H^1(Y, S) \) with corresponding extension \( S' \). Then Theorem
Lemma 2.6 together with Lemma 1.3 yield the inequalities
\[ \frac{\deg(S)^2}{r + 1} \leq \mu_{HK}(S') \leq \frac{\deg(S)^2}{r}. \]
If $S'$ is also semistable, then we have equality on the left.

3. A Hilbert-Kunz criterion for solid closure

We come now back to our original setting of interest, that of a two-dimensional normal standard-graded domain $R$ over an algebraically closed field $K$. A homogeneous $R_+$-primary ideal $I = (f_1, \ldots, f_n)$ gives rise to the syzygy bundle $\text{Syz}(f_1, \ldots, f_n)(0)$ on $Y = \text{Proj} \ R$ defined by the presenting sequence

\[ 0 \rightarrow \text{Syz}(f_1, \ldots, f_n)(m) \rightarrow \bigoplus_{i=1}^n O_Y(m - d_i) f_1^{1 \ldots f_n} O_Y(m) \rightarrow 0. \]

Another homogeneous element $f$ of degree $m$ yields an extension

\[ 0 \rightarrow \text{Syz}(f_1, \ldots, f_n)(m) \rightarrow \text{Syz}(f_1, \ldots, f_n, f)(m) \rightarrow O_Y \rightarrow 0 \]

which corresponds to the cohomology class $\delta(f) \in H^1(Y, \text{Syz}(f_1, \ldots, f_n)(m))$ coming from the presenting sequence via the connecting homomorphism

\[ \delta : H^0(Y, O_Y(m)) = R_m \rightarrow H^1(Y, \text{Syz}(f_1, \ldots, f_n)(m)). \]

The Hilbert-Kunz multiplicities of the ideals and the Hilbert-Kunz slopes of the syzygy bundles are related in the following way.

Lemma 3.1. Let $K$ denote an algebraically closed field of characteristic $0$. Let $R$ denote a standard-graded two-dimensional normal $K$-domain, $Y = \text{Proj} \ R$. Let $I$ be a homogeneous $R_+$-primary ideal and let $f$ denote a homogeneous element of degree $m$. Then the Hilbert-Kunz multiplicities $e_{HK}(I) = e_{HK}((I, f))$ are equal if and only if the Hilbert-Kunz slopes of the corresponding syzygies bundles $\mu_{HK}(\text{Syz}(f_1, \ldots, f_n)(m)) = \mu_{HK}(\text{Syz}(f_1, \ldots, f_n, f)(m))$ are equal.

Proof. Let $\mu_k$ and $r_k$ ($\tilde{\mu}_k$ and $\tilde{r}_k$) denote the ranks and the slopes in the Harder-Narasimhan filtration of $\text{Syz}(f_1, \ldots, f_n)(0)$ (of $\text{Syz}(f_1, \ldots, f_n, f)(0)$ respectively). For the Hilbert-Kunz multiplicities of the ideals $(f_1, \ldots, f_n)$ and $(f_1, \ldots, f_n, f)$ we have to compare

\[ e_{HK}(I) = \frac{1}{2 \deg(Y)} \left( \sum_{k=1}^i \frac{r_k \mu_k^2 - \deg(Y)^2 \sum_{i=1}^n d_i^2}{2} \right) \]

and

\[ e_{HK}((I, f)) = \frac{1}{2 \deg(Y)} \left( \sum_{k=1}^i \frac{\tilde{r}_k \tilde{\mu}_k^2 - \deg(Y)^2 (m^2 + \sum_{i=1}^n d_i^2)}{2} \right). \]

The extension defined by $c = \delta(f) \in H^1(Y, \text{Syz}(f_1, \ldots, f_n)(m))$ is

\[ 0 \rightarrow S = \text{Syz}(f_1, \ldots, f_n)(m) \rightarrow S' = \text{Syz}(f_1, \ldots, f_n, f)(m) \rightarrow O_Y \rightarrow 0 \]
and the Hilbert-Kunz slopes of these sheaves are due to Proposition 1.2(v) (since \(\deg(\text{Syz}(f_1, \ldots, f_n)(0)) = -\deg(Y) \sum_{i=1}^n d_i\))

\[
\mu_{HK}(S) = \sum_{k=1}^t r_k \mu_k^2 + 2(-\sum_{i=1}^n d_i \deg(Y))m \deg(Y) + (n-1)m^2 \deg(Y)^2
\]

and \(\mu_{HK}(S') = \)

\[
= \sum_{k=1}^i \tilde{r}_k \tilde{\mu}_k^2 + 2(-\sum_{i=1}^n d_i + m) \deg(Y))m \deg(Y) + nm^2 \deg(Y)^2
\]

\[
= \sum_{k=1}^i \tilde{r}_k \tilde{\mu}_k^2 - 2(\sum_{i=1}^t d_i)m \deg(Y)^2 \quad \text{for} \quad n \geq 1.
\]

So the difference is in both cases (up to the factor \(1/2 \deg(Y)\))

\[
\sum_{k=1}^i \tilde{r}_k \tilde{\mu}_k^2 - \sum_{k=1}^t r_k \mu_k^2 - m^2 \deg(Y)^2.
\]

Therefore \(e_{HK}(I) = e_{HK}((I, f))\) if and only if

\[
\mu_{HK}(\text{Syz}(f_1, \ldots, f_n)(m)) = \mu_{HK}(\text{Syz}(f_1, \ldots, f_n, f)(m)).
\]

\[\square\]

**Remark 3.2.** Let \(0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{Q} \to 0\) denote a short exact sequence of locally free sheaves. Then the alternating sum of the Hilbert-Kunz slopes, that is \(\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q})\) does not change when we tensor the sequence with an invertible sheaf. This follows from Proposition 1.2(v) and we suspect that this is true in general. From the short exact sequence \(0 \to \text{Syz}(f_1, \ldots, f_n)(0) \to \mathcal{O}_Y \to \mathcal{O}_Y \to 0\) this number is \(\leq 0\) by Theorem 2.6 and we expect this to be true in general. From the short exact sequence \(0 \to \text{Syz}(f_1, \ldots, f_n)(0) \to \mathcal{O}_Y \to \mathcal{O}_Y \to 0\) it follows via \(e_{HK}(I) = \frac{1}{2\deg(Y)}(\mu_{HK}(\text{Syz}(f_1, \ldots, f_n)(0)) - \mu_{HK}(\mathcal{O}_Y))\) that the Hilbert-Kunz multiplicity of an ideal is always nonnegative. In fact \(I = R\) is the only ideal with \(e_{HK}(I) = 0\). This follows from Theorem 3.3 below, since \(1 \notin I^*\) for \(I \neq R\).

We come now to the main result of this paper. Recall that the solid closure of an \(m\)-primary ideal \(I = (f_1, \ldots, f_n)\) in a two-dimensional normal excellent domain \(R\) is given by the condition that \(f \in (f_1, \ldots, f_n)^*\) if and only if \(D(m) \subset \text{Spec } R[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n + f)\) is not an affine scheme. In positive characteristic this is the same as tight closure, see \([5, \text{Theorem } 8.6]\). In the case of an \(R_+\)-primary homogeneous ideal in a standard-graded normal \(K\)-domain this is equivalent to the property that the torsor \(\mathbb{P}((S')^* - \mathbb{P}(S'))\) over the corresponding curve \(Y = \text{Proj } R\) is not affine (see \([1, \text{Proposition } 3.9]\)). This relates solid closure to the setting of the previous section.

**Theorem 3.3.** Let \(K\) denote an algebraically closed field. Let \(R\) denote a standard-graded two-dimensional normal \(K\)-domain. Let \(I\) be a homogeneous \(R_+\)-primary ideal and let \(f\) denote a homogeneous element. Then \(f \in I^*\) if and only if \(e_{HK}(I) = e_{HK}((I, f))\).
Proof. If the characteristic is positive then this is a standard result from tight closure theory as mentioned in the introduction. So suppose that the characteristic is 0. Let $I = (f_1, \ldots, f_n)$ be generated by homogeneous elements, and set $m = \deg(f)$. The containment in the solid closure, $f \in (f_1, \ldots, f_n)^*$, is equivalent with the non-affineness of the torsor $\mathbb{P}(S'\vee) - \mathbb{P}(S'\vee)$ [1, Proposition 3.9], where $S = \text{Syz}(f_1, \ldots, f_n)(m)$ and $S'$ is the extension given by the cohomology class $\delta(f)$. Hence the result follows from Theorem 2.6 and Lemma 3.1.

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