Chiral Skyrmionic matter
in non-centrosymmetric magnets

Ulrich K. Rößler, Andrei A. Leonov, Alexei N. Bogdanov
IFW Dresden, Postfach 270116, D-01171 Dresden, Germany
E-mail: u.roessler@ifw-dresden.de

Abstract. Axisymmetric magnetic strings with a fixed sense of rotation and nanometer sizes (chiral magnetic vortices or Skyrmions) have been predicted to exist in a large group of non-centrosymmetric crystals more than two decades ago. Recently these extraordinary magnetic states have been directly observed in thin layers of cubic helimagnet (Fe,Co)Si. In this report we apply our earlier theoretical findings to review main properties of chiral Skyrmions, to elucidate their physical nature, and to analyse these recent experimental results on magnetic-field-driven evolution of Skyrmions and helicoids in chiral helimagnets.

1. Introduction
In non-centrosymmetric magnetic systems, antisymmetric Dzyaloshinskii-Moriya (DM) exchange causes particular magnetic couplings and inhomogeneous states [1]. These chiral interactions stabilize two- and three- dimensional localized structures [2, 3, 4]. Phenomenologically the inhomogeneous Dzyaloshinskii energy contributions [1, 5] are described by Lifshitz invariants, antisymmetric differential forms linear in first spatial derivatives of the magnetization \( M \)

\[
\Lambda^{(k)}_{ij} = M_i \frac{\partial M_j}{\partial x_k} - M_j \frac{\partial M_i}{\partial x_k}
\]

where \( x_k \) are Cartesian components of the spatial variable \( r \). Solutions for chiral magnetic Skyrmions as static states localized in two dimensions have been derived first in 1989 [2]. It was shown that these topological solitonic field configurations exist in magnetic systems for all non-centrosymmetric crystallographic classes that allow Lifshitz invariants in their magnetic free energy [2]. The unique character of these textures stems from the general instability of multidimensional solitonic states in field theories [3, 6]. Nonlinear continuum models for condensed matter systems do not contain solutions for static and smooth multidimensional localized states. Such states appear only as dynamic excitations, while static configurations are generally unstable and collapse spontaneously into topological singularities.

For a long time investigations of chiral Skyrmions have been restricted to theoretical studies [3, 7, 8, 9, 10]. The solutions for two-dimensional localized and bound states (isolated Skyrmions and Skyrmion lattices) have been studied in non-centrosymmetric ferromagnets [7, 8] and antiferromagnets [5], in cubic helimagnets [3, 8, 9], and in confined centrosymmetric magnetic systems with surface/interface-induced chiral interactions (e.g. nanolayers of magnetic metals) [10]. In [3], we formulated the idea of Skyrmionic matter in non-centrosymmetric crystals.
predicting the existence of mesophases, composed of Skyrmions as ‘molecular units’, similar to vortex matter in type-II superconductors [11]. Various effects observed in MnSi and other cubic helimagnets with B20 structure [12] indicate multidimensionally modulated magnetic states conforming with our theoretical predictions of Skyrmions and their properties [3, 7]. The theoretical ideas of Skyrmions in chiral magnets have triggered various experimental efforts to find evidence for these twisted textures [13, 14]. These experiments collected an impressive range of data that suggest complex magnetic order phenomena. But, mainly using diffraction or indirect evidence by transport measurements, these experimental results remained essentially inconclusive and have been contested, see, e.g., [15]. Moreover, the interpretation of the experimental data has been based on approximate solutions [14, 16] to the Dzyaloshinskii model by using variational approaches in terms of a mode instability. The corresponding results do not describe the properties of Skyrmions and the phase transition behavior of chiral magnets, which is governed by the nucleation of a localized mesoscale entity [1, 7, 17]. Our theoretical developments [8, 9] show that multiply modulated chiral states in non-centrosymmetric magnets are composed of localized solitonic states with particle-like behavior. The thermodynamic stability of condensed Skyrmionic phases has been shown [3, 8, 9] for the standard and modified Dzyaloshinskii models devised for cubic helimagnets [1, 18]. Meanwhile, experimental efforts culminated in the direct microscopic observations of chiral Skyrmions in thin layers of (Co,Fe)Si [19]. This breakthrough is the first clear experimental evidence for the existence of Skyrmions as axisymmetric chiral localized states that are stabilized by a complex interplay of nonlinear and chiral effects, as predicted earlier [2, 3].

Here, we address the problem of multidimensional solitonic states in nonlinear systems lacking inversion symmetry. This emerging field of nonlinear physics is based on solutions of nonlinear partial differential equations [3, 7, 9], and intricate mathematical methods of micromagnetics [20], and physics of solitons [21]. Concentrating on the physical side of the problem rather than on mathematical details we give an elementary introduction into the properties of chiral Skyrmions in magnetism.

2. Dzyaloshinskii theory for cubic helimagnets

2.1. Phenomenological energy and equations

Within the phenomenological theory introduced by Dzyaloshinskii [1] the magnetic energy density of a cubic non-centrosymmetric ferromagnet with the magnetization \( \mathbf{M} \) can be written as [1, 18]

\[
w = A (\text{grad} \mathbf{M})^2 - D \mathbf{M} \cdot \mathbf{rot} \mathbf{M} - \mathbf{M} \cdot \mathbf{H} - \sum_{i=1}^{3} \left[ B (\partial M_i / \partial x_i)^2 + K_c M_i^4 \right] - f(M)
\]

including the exchange stiffness with constant \( A \). The inhomogeneous DM exchange with constant \( D \) is a combination of Lifshitz invariants (1), \( w_D = D (\Lambda_{xy} + \Lambda_{yz} + \Lambda_{zx}) = D \mathbf{M} \cdot \mathbf{rot} \mathbf{M} \). These isotropic interactions together with the Zeeman energy are the essential couplings to stabilize chiral modulations (\( w_0 \)). The sum in Eq. (2) includes exchange (\( K \)) and cubic (\( K_c \)) magnetic anisotropy contributions; \( f(M) \) comprises magnetic interactions imposed by variation of the magnetization modulus \( M \equiv |M| \). In a broad temperature range the magnetization vector practically does not change its length, and nonuniform magnetic states display only rotation of \( M \). The low temperature properties, thus, can be described by the model (2) with a fixed magnetization modulus \( M = \text{const} \).

2.2. One-dimensional chiral modulations: cones and helicoids

In non-centrosymmetric magnets, chiral couplings of type (1) favour rotation of the magnetization vector. They destabilize the homogeneous magnetic structure and induce long-
Figure 1. One-dimensional chiral modulations in cubic helimagnets. In a helical "array" (a) the magnetization rotates in the plane spanned by the orthogonal unity vectors $n_1$ and $n_2$ and the rotation sense is determined by the sign of Dzyaloshinskii constant $D$. Chiral helices (b) are composed of arrays (a) with the same propagation direction. Under the influence of the applied field the helix (b) transforms into longitudinal distorted cones (c) or into transversally distorted helicoids (d).

range modulations of the magnetization [1]. At zero field helical modulations

$$M = M [n_1 \cos (k \cdot r) + n_2 \sin (k \cdot r)], \quad |k| = 2\pi/L_D, \quad L_D = 4\pi A/|D|$$

with period $L_D$ and wave vector $k$ correspond to the absolute minimum of the isotropic functional $w_0$ (Fig. 1 a, b). In Eq. (3) $n_1, n_2$ are orthogonal unit vectors in the plane of the magnetization rotation (Fig. 1). The modulations (3) have a fixed rotation sense determined by the sign of constant $D$ and are continuously degenerate with respect to propagation directions in the space. An applied magnetic field lifts the degeneracy of the helices (3) and stabilizes a cone solution with the propagation direction along the magnetic field

$$\cos \theta = |H|/H_D, \quad \psi = 2\pi z/L_D, \quad H_D = D^2 M/(2A),$$

where $M = M(\sin \theta \cos \psi; \sin \theta \sin \psi; \cos \theta)$ is written in spherical coordinates. In such a helix the magnetization component along the applied field has a fixed value $M_\perp = M \cos \theta = M(H/H_D)$, and the magnetization vector $M$ rotates within a cone surface. The critical value $H_D$ marks the saturation field of the cone phase. The helix periods ($L_D$) and critical fields ($H_D$) for some non-centrosymmetric cubic ferromagnets (helimagnets) are presented in Table 1.

| Compound      | $T_N$ [K] | $L_D$ [nm] | $H_D$ [T] |
|---------------|-----------|------------|-----------|
| MnSi          | 29.5      | 18.0       | 0.62      |
| FeGe          | 278.7     | 68.3-70.0  | 0.2       |
| Fe0.3Co0.7Si  | 8.8       | 230        | (6.0 ± 1.5)·10^{-3} |
| Fe0.5Co0.5Si  | 43.5      | 90.0       | (4.0 ± 0.5)·10^{-2} |
| Fe0.8Co0.2Si  | 32.2      | 29.5       | 0.18      |

In real non-centrosymmetric magnets anisotropic forces fix the propagation of chiral modulations along certain easy axes directions imposed by the crystal symmetry. A magnetic field perpendicular to the propagation direction violates the uniform rotation of the magnetization ($\psi \propto z$, Eq. (3)). In a sufficiently high magnetic field $H = H_F$ distorted helicoids infinitely expand their period into a system of isolated $2\pi$ domain walls separating domains with the magnetization along the applied field (Figs. 3 (d), 5 (a)) [1, 7].
Figure 2. Isolated Skyrmions in a nanolayer (a) and a bulk sample (b) of a cubic helimagnet. The magnetization profiles $\theta(\rho)$ (c) and corresponding energy densities $\varepsilon(\rho)$ (d) for typical solutions of Eq. (5). A cross-section through an isolated Skyrmion shows axisymmetric distribution of the magnetization (shaded area indicates the core with the diameter $D_0$ ((a) and Inset in (c)). Phase trajectories of Skyrmions correspond to separatrix curves in the phase space (curve $b-o$)(e). Typical eigen functions for radial excitations (f).

3. Chiral flux-lines: the building blocks for Skyrmionic textures

3.1. Chiral Skyrmions in cubic helimagnets

The equations minimizing energy (2) include axisymmetric localized solutions $\psi = \phi + \pi/2$, $\theta = \theta(\rho)$, $M = \text{const}$ (Fig. 2, (a), (b)) where the spatial variable $r = (\rho \cos \phi; \rho \sin \phi; z)$ is written in cylindrical coordinates and the magnetization in spherical coordinates (4). The Euler equation for $\theta(\rho)$ to determine the equilibrium structure of isolated Skyrmions [2, 7] is

$$A\left(\frac{d^2 \theta}{d\rho^2} + \frac{1}{\rho} \frac{d \theta}{d \rho} - \frac{1}{\rho^2} \sin \theta \cos \theta\right) - \frac{H}{2M} \sin \theta - \frac{D}{\rho} \sin^2 \theta = 0$$ (5)

with boundary conditions $\theta(0) = \pi$, $\theta(\infty) = 0$. In non-centrosymmetric uniaxial magnets functionals $w_D$ have anisotropic forms depending on crystallographic class and may include several terms [2]. The solutions $\psi(\phi)$ have different form depending on the crystal symmetry, however, the Euler equation for $\theta(\rho)$ has the same functional form for all non-centrosymmetric uniaxial ferromagnets [2].

3.2. Solutions for Skyrmions

Typical solutions of Eq. (5), $\theta(\rho)$, consist of arrow-like cores $(\pi - \theta \propto \rho$ for $\rho \leq L_D)$ and exponential ”tails”, $\theta \propto \exp(-\rho)$ for $\rho \gg L_D$ [2, 7] (Fig. 2). In phase space $(\theta, d\theta/d\rho)$ these localized solutions correspond to separatrix trajectories (e.g. curve $b-o$ in Fig. 2(e)). A Skyrmion core diameter $D_0$ can be defined as two times the value of $R_0$, which is the coordinate of the point where the tangent at the inflection point $(\rho_0, \theta_0)$ intersects the $\rho$-axis (Fig. 2(d)), in analogy to definitions for domain wall width [20]. Skyrmion energy per unit length and diameter are

$$E = 2\pi \int_0^\infty \varepsilon(\theta, \rho) d\rho,$$

$$D_0 = 2L_0[\rho_0 - \theta_0(d\theta/d\rho)^{-1}]_{\rho=\rho_0}$$ (6)

where $\varepsilon(\theta, \rho)$ is the ”linear” energy density [7]. The energy density distributions $\varepsilon(\rho)$ (Fig. 2(f)) reveal two distinct regions: positive energy ”bags” are concentrated in the Skyrmion center and are surrounded by extended areas with negative energy density, where the DM exchange
Figure 3. Skyrmion lattices: (a) unit cells have axisymmetric magnetization structure near the center; (b) a thin layer with a hexagonal lattice. The magnetization profiles $\theta(\rho/L_D)$ of a Skyrmion cell (solid blue lines) and of an isolated Skyrmion (dashed red lines) for different values of applied magnetic field (c). Equilibrium sizes of the cell core ($D_0$) and lattice period compared to helicoid periods (d). Inset shows the transformation of the hexagonal lattice into a set of isolated Skyrmions in increasing magnetic field. Snapshots correspond to the field values indicated in the diagram for period ($L/L_D$) vs. applied field ($H/H_D$) by hollow circles (d).

4. Skyrmion lattices: where does ”double-twist” become beneficial?

4.1. Solutions for bound Skyrmions

The equilibrium parameters of a Skyrmion lattice are derived from a system of differential equations for $\theta(x,y)$, $\psi(x,y)$ minimizing the system energy. For Skyrmions at low temperatures the circular cell approximation provides a very elegant and reliable approximation [7]. In this method the lattice cell is replaced by a circle of equal area. The equilibrium parameters of Skyrmion lattices are derived by integration of Eq. (5) with boundary conditions $\theta(0) = \pi$, $\theta(R) = 0$ and a subsequent minimization of the lattice energy density $W_S = (2/R^2) \int_0^R w(\rho) \rho \, d\rho$ with respect to the cell radius $R$ [7]. The solutions for magnetization profiles $\theta(\rho)$ and the lattice periods $L$ are plotted in Fig. 3. The reduced perpendicular magnetization of a Skyrmion lattice averaged over the Skyrmion cell $m_S = (2/R^2) \int_0^R \cos(\theta) \rho \, d\rho$ is plotted together with the magnetization of the helicoid, $m_H$ (Fig. 5). Contrary to the helicoid the Skyrmion lattice has a finite magnetization at zero field, $m_S(0) = 0.124$.

Table 2. Critical fields and characteristic parameters of the hexagonal Skyrmion lattice: $H_1$ transition field between the helicoid and Skyrmion lattice; $H_S$ saturation field of the Skyrmion lattice; last column for isolated Skyrmions as excitations of the saturated state.

| Reduced magnetic field, $H/H_D$ | $H_1$ | $H_S$ | $L/L_D$ | $D_0/L_D$ | $m_S$ |
|---------------------------------|-------|-------|---------|-----------|-------|
| 0                               | 0.216 | 0.801 | 1.376   | 1.362     | 0.124 |
| $L/L_D$                         | 1.4   |       |         |           |       |
| Core diameter, $D_0/L_D$        | 1.270 | $\infty$ | 0.920   | 0.461     |       |
| Averaged magnetization, $m_S$   | 0.278 | 1     |         |           |       |
4.2. Skyrmions compete with helicoids. Transition field $H_1$

In the Skyrmion lattices rotation of the magnetization in two directions leads to a larger reduction in the Dzyaloshinskii-Moriya energy than single-direction rotation in helical phases. On the other hand, such double-twist modulations increase the exchange energy. The equilibrium energy of the Skyrmion cell at zero field $\bar{w}_S(\zeta) = \left(\frac{2}{\zeta^2}\int_0^\zeta w(\rho)\rho d\rho\right)$ plotted as a function of the distance from the center $\zeta$ (Fig. 4 (a)) shows that an energy excess near the border outweighs the energy gain at the Skyrmion center (for details see [3]). At higher magnetic fields, however, the Skyrmion lattice has lower energy than the helicoid. The first order transition between these two modulated states occurs at $H_1 = 0.2168H_D$ [7].

Figure 4. Local energies $\bar{w}(\zeta)$ of the Skyrmion lattice and helicoid at zero field (a). Below the critical field $H_S$ the negative internal energies of Skyrmions overcome the repulsive Skyrmion-Skyrmion interaction and a dense packed (hexagonal) lattice is created (b). Under the influence of induced uniaxial anisotropies ($E_u$) the Skyrmion lattice becomes globally stable in a broad range of the applied fields, adopted from [8], $K_0$ is the critical value for uniaxial distortions suppressing the cones in zero field (c).

4.3. "Receding" Skyrmions. Critical field $H_S$

Properties of the Skyrmion lattice solutions are collected in Figs. 3, 5 and in Table 2. With increasing magnetic field, a gradual localization of the Skyrmion core $D_0$ is accompanied by the expansion of the lattice period. The lattice transforms into the homogeneous state by infinite expansion of the period at the critical field $H_S = 0.80132H_D$. Remarkably, the Skyrmion core retains a finite size, $D_0(H_S) = 0.920L_D$ and the lattice releases a set of repulsive isolated Skyrmions at the transition field $H_S$, owing to their topological stability. These free Skyrmions can exist far above $H_S$. On decreasing the field again below $H_S$, they can re-condense into a Skyrmion lattice (Fig. 4 (b)). A similar type of sublimation and resublimation of particle-like textures occurs in helicoids at the critical field $H = H_H/H_D = \pi^2/8 = 0.6168$: the period infinitely expands and the helicoid splits into a set of isolated $2\pi$ domain walls or kinks [1, 7].

This peculiar transformation of chiral modulations into homogeneous states constitutes a nucleation type of continuous phase transition, according to a classification introduced by De Gennes [17]. In contrast to the common instability type of continuous (2nd order) transitions that is described by the amplitude of a sole fundamental mode acting as the order parameter, the nucleation transitions retain a full spectrum of lattice modes at the transition. Magnetic-field-driven transitions of multidomain states into the homogeneous phase also belong to the nucleation type of phase transformations [20].
4.4. Stabilization effects of magnetic anisotropy
For isotropic model $w_0$ (2) the cone phase (4) is the global minimum in the whole range of the applied fields where the modulated states exist ($0 < H < H_D$). The helicoids and Skyrmion lattices can exist only as metastable states. Under the influence of anisotropies (2), however, the Skyrmion lattices may gain thermodynamic stability in a certain interval of applied fields, for details see [8, 9]. E.g., uniaxial anisotropies can be induced in bulk helimagnets by uniaxial stresses or they arise due to surface effects in thin layers. These anisotropies are described by $E_u = K (\mathbf{M} \cdot \mathbf{n})^2$, where $K$ is a constant of uniaxial anisotropy, $\mathbf{n}$ is a unity vector along the distortion axis. Such uniaxial distortions suppress the cone phase in cubic helimagnets and establish global stability of the Skyrmion lattice in broad ranges of the thermodynamic parameters (Fig. 4 (c)) [8].

5. On the observation of Skyrmionic and helical textures in Fe$_{0.5}$Co$_{0.5}$Si nanolayers
Real-space images of Skyrmion states in a thin layer of cubic helimagnet Fe$_{0.5}$Co$_{0.5}$Si have recently been obtained by using Lorentz transmission electron microscopy [19]. This is the first clear experimental manifestation of chiral Skyrmion states. The first-order transition of a helicoid into a Skyrmion lattice and its subsequent transformation into a system of isolated Skyrmions observed in bias magnetic fields (Figs. 1, d-f, 2, 3 (a-d) in [19]) are in excellent agreement with the theoretical predictions on the behavior of Skyrmions and the field-driven transitions into densely packed Skyrmion lattices according to the magnetic phase diagrams calculated earlier [3, 7] (Fig. 5).

In the experiments, the thickness 20 nm of the magnetic layer Fe$_{0.5}$ Co$_{0.5}$Si is much smaller than the helix period $L_D = 90$ nm [19]. But, even for such a small thickness, the conical state propagating only for a fraction of a period perpendicularly through the layer has lower energy than a Skyrmion lattice, absent additional effects that stabilize it in applied fields. Usually in magnetic nanolayers strong perpendicular uniaxial anisotropy arises, either as a result of surface effects [22] or of lattice strains. Thus, induced anisotropies give a possible explanation for the experimental observation of the Skyrmions in these Fe$_{0.5}$Co$_{0.5}$Si layers, in accordance with the phase diagram for cubic helimagnets with uniaxial distortions (Fig. 4 (c)) [8]. Fig. 5

**Figure 5.** The ideal magnetization curves for a bulk sample (based on results of [7]) (a) and for a thin layer (b) of a cubic helimagnet with suppressed cone phases. Solid lines indicate the thermodynamically stable states; dashed lines in Fig. (a), metastable configurations. The 1st order transition line at $H_1$ is expanded into a region of multidomain states in the thin layer. Fragments of experimentally observed images [19] demonstrate excellent agreements with theoretically calculated magnetization curves. The patterns marked by “(!)” display isolated chiral Skyrmions.
presents the magnetization curve for a bulk isotropic helimagnet (a) (based on results of [7], Fig. 12) and the corresponding magnetization curve for a thin layer involving demagnetization effects [20] (b). Compared to theoretically calculated values in a bulk material ($H_S$, $H_H$) the corresponding critical fields in a thin layer are shifted, and their values can be estimated as $\tilde{H}_S(H) = H_S + 4\pi M$. Due to demagnetization effects multidomain states can be stabilized in the vicinity of the transition field $H_1$ [7]. The boundaries of these regions with coexisting phases can be estimated as $\tilde{H}_1S = H_1 + 4\pi Mm_H(H_1)$, $\tilde{H}_1L = H_1 + 4\pi Mm_H(H_1)$. The magnetizations of the competing phases at the transition field equal $m_H(H_1) = 0.111$ and $m_S(H_1) = 0.278$. The jump of the magnetization during the transition equals $\Delta M = [m_S(H_1) - m_N(H_1)]M = 0.167M$, i.e., it reaches about 17 % of the saturation value.

The magnetization curves in Fig. 5 are constructed for ideally soft magnetic material under the condition that only the equilibrium states are realized in the magnetic sample. In real materials the formation of the equilibrium states is often hindered (especially during the phase transitions), and evolution of metastable states and hysteresis effects play an important role in the magnetization processes. Particularly, the formation of the Skyrmion lattice below $H_S$ can be suppressed. Then isolated Skyrmions exist below this critical field. At a critical field $H_{El}$ the Skyrmions become unstable with respect to elliptical deformations and “strip-out” into isolated 2$\pi$ domain walls. In a bulk material $H_{EI} = 0.534H_D$ (indicated in Fig. 5 with a red arrow). In a thin layer, one estimates $\tilde{H}_{EI} = H_{EI} + 4\pi M$. As discussed earlier [2, 3, 7] the evolution of chiral Skyrmions in magnetic fields has many features in common with that of bubble domains in perpendicular magnetized films [20], and with Abrikosov vortices in superconductors [11].

The fragments of images from Ref. [19] (Fig.5 (b)) reflect in details the theoretically predicted evolution of the chiral modulations in the applied magnetic field: the helicoid phase is realized at low fields (region (I)); at higher field this transforms into the Skyrmion lattice (region (II)) via an intermediate state ($\tilde{H}_{1S} < H < \tilde{H}_{1L}$); finally the Skyrmion lattice by extension of the period transforms into the homogeneous phase where isolated Skyrmions still exist as topologically stable 2D solitons.

Two patterns indicated in Fig. 5 (b) with exclamation mark manifest the main result of Ref. [19]: the first images of static two-dimensional localized states aka chiral Skyrmions! In Ref. [19] this result has been overlooked and misinterpreted as a coexisting ferromagnetic and Skyrmion lattice phases. As it was expounded in the previous section, the transition of the Skyrmion lattice into the homogeneous state is a continuous transition, but of the particular nucleation type. Such transitions exclude the formation of coexisting states.

The condensed Skyrmion phases in the micrograph of Ref. [19] also appear as heavily distorted densely packed two-dimensional lattice configurations. This is expected for Skyrmionic matter. As these mesophases are composed from elastically coupled radial strings, dense Skyrmion configurations generally do not form ideal crystalline lattices but various kinds of partially ordered states, e.g. hexatic ordering implying only orientational order of bonds without positional long-range order, or other glassy arrangements following standard arguments put forth for the similar vortex matter in type-II superconductors [11]. The observation derives from the particle-like (or string-like) nature of Skyrmions and suggests that Skyrmionic mesophases may display rich phase diagrams.

6. Chiral modulations near the ordering temperature

6.1. Solutions for high-temperature Skyrmions. Confinement temperature $T_p$

Near the ordering temperature the magnetization amplitude varies under the influence of the applied field and temperature. Basic properties of chiral modulations in this region can be derived by minimization of the isotropic model $w = w_0 - f(M)$ derived from Eq. (2) with respect to all components of the magnetization vector $(M, \theta, \psi)$. Within this model axisymmetric isolated structures are described by equations $\psi = \phi + \pi/2$ and the solutions of the Euler
Eqs. (8) describe the structure of isolated Skyrmions. The magnetization of the homogeneous region of "precursor state" ($T < T_L$) separates the main part of phase diagram with regular chiral modulations ($T < T < T_N$). The confined Skyrmion textures include square (half-Skyrmion) (c) and hexagonal (d) lattices.

Typical solutions for magnetization profiles $\theta(\rho)$, $M(\rho)$ (Eqs. 8) (a). Magnetic phase diagram of an isotropic helimagnet near the ordering temperature includes areas with repulsive (I), attractive (II) Skyrmions, and confined Skyrmion states (III) (b). Confinement temperature $T_c$ separates the main part of phase diagram with regular chiral modulations ($T < T_c$) from the region of "precursor state" ($T_L < T < T_N$). The confined Skyrmion textures include square (half-Skyrmion) (c) and hexagonal (d) lattices.

\[
A \left( \frac{d^2\theta}{d\rho^2} + \frac{1}{\rho} \frac{d\theta}{d\rho} - \frac{\sin \theta \cos \theta}{\rho^2} \right) - D \frac{2}{\rho} \sin^2 \theta - H \sin \theta + A \frac{2}{M} \frac{dM}{d\rho} \left( \frac{d\theta}{d\rho} - 1 \right) = 0, \tag{7}
\]

With $f(M) = a_1 M^2 + a_2 M^4$ ($a_1 = J(T_c - T)$, $a_2 > 0$) which is commonly used to describe magnetization processes in cubic helimagnets at high temperatures the solutions of Eqs. (8) have been investigated in [3, 9]. With boundary conditions $\theta(0) = \pi$, $\theta(\infty) = 0$, $M(\infty) = M_0$, Eqs. (8) describe the structure of isolated Skyrmions. The magnetization of the homogeneous phase $M_0$ is derived from equation $2a_1 M + 4a_2 M^3 - H = 0$. It is convenient to introduce new parameters, an effective temperature $a = (T_c - T)/T_D$ and $H_0 = (D^2/2A)^{3/2}/\sqrt{a_2}$ where "chiral shift", $T_D = D^2/(8A)$, is a characteristic temperature equal to the difference between the ordering temperature of the helimagnet ($T_N = T_c + T_D$) and the ferromagnet with $D = 0$ ($T_c$).

Typical solutions $\theta(\rho)$, $M(\rho)$ of Eqs. (8) demonstrate an inhomogeneity of $M$ associated with the Skyrmion cores, Fig. 6(a). Detailed analysis of the solutions for isolated Skyrmions has revealed a number of remarkable phenomena imposed by the interplay between the angular ($\theta$) and modulus ($M$) order parameters [9].

(i) **Collapse of Skyrmions at high fields.** The solutions of Eqs. (8) exist only below a critical collapse line (Fig. 6). As the applied field approaches this line the magnetization in the Skyrmion center $M(0)$ gradually shrinks, and as $M(0)$ becomes zero the Skyrmion collapses. This is in contrast with low-temperature Skyrmions, which exist without collapse even at very large magnetic fields [7] because the stiffness of the magnetization modulus maintains topological stability of Skyrmions. At high temperatures the softening of the magnetization allows the order parameter to pass through zero and to unwind the Skyrmion core.

(ii) **Crossover of Skyrmion-Skyrmion interactions.** The coupling of Skyrmions is repulsive in a broad temperature range, but it starts to oscillate at high temperatures, Fig. 6 (b). The equation to describe this crossover in the inter-Skyrmion coupling is $\eta H^* = \sqrt{2 + \sqrt{3 + 4a}} (a + 1 \pm \sqrt{3 + 4a}/2)$. For $a < -0.5$ this equation yields a crossover line in the phase diagram, Fig. 6(b).
This is ended in turning point (-0.5, 0). Another turning point (-0.75, \(\sqrt{2}\eta/4\)) indicates the lowest temperature, i.e., a confinement temperature \(T_L = T_N - 4T_D\), where Skyrmions begin to attract each other at certain distances, owing to the oscillatory character of their interactions. This results in energetic confinement of Skyrmions, as they can form clusters to lower their energy.

(iii) **Confinement.** For \(-0.5 < a < 0.25\) the crossover line delimits a small pocket (III) in the vicinity of the ordering temperature. Within this region Skyrmions can exist only as bound states. In the confinement region Skyrmionic states drastically differ from those in the main part of the phase diagram. Confined Skyrmion textures (hexagonal and square half-Skyrmion lattices, Fig. 6 (c)) arise from the disordered state through a rare case of an instability-type nucleation transition, and their field-driven transformation evolves by distortions of the modulus profiles \(M(\rho)\) while the equilibrium periods of the lattices do not change strongly with increasing applied field (for details see [9]).

6.2. **Confinement phenomenon and precursor states**

The magnetic phase diagram of isotropic helimagnet (Fig. 6 (b)) includes two distinct temperature intervals: (i) in the main part (0 < \(T < T_L\)) the rotation of the local magnetization vector determines the chiral modulation, while the magnetization amplitude remains constant. (ii) At high temperatures (\(T_L < T < T_N\)) spatial variation of the magnetization modulus becomes a sizeable effect, and strong interplay between \(M\) and angular variables is the main factor in the formation and peculiar behaviour of chiral modulations in this region. The confinement temperature \(T_L\) [9] provides the scale that delineates the border between these two regimes in the phase diagram. The characteristic temperature \(T_L\) is of fundamental importance for chiral magnets. It is of the same order of magnitude as the temperature interval \((T_N - T_c) \propto D^2/A\), where chiral couplings cause inhomogeneous precursor states around the magnetic order temperature. Due to the relativistic origin and corresponding weakness of the DM exchange the shift \(\Delta T = T_N - T_L\) is small (for MnSi this is estimated to be about 2 K [9]).

Since discovery of chiral modulations in MnSi family of B20 compounds numerous magnetic anomalies have been observed near the ordering temperature of these helimagnets [12]. However, the nature of the chiral spin textures in this region of the phase diagram is largely unresolved in experiment. During last years these precursor effects have become a subject of intensive investigations [13, 14] motivated by the expectation to identify Skyrmionic states [3].

The rigorous solutions for helical and Skyrmionic modulations recently derived within the basic model \(w = w_0 + f(M)\) (2) and the theoretical description of novel phenomena attributed to this region [9] give a first explanation of these anomalies. The stabilization of Skyrmionic textures and other chiral modulations near magnetic ordering involve the confinement of localized state, but also a strong influence of minor energy contributions. E.g., the phase diagram of cubic helimagnets contains defected half-Skyrmion lattices at small applied fields and densely packed full-Skyrmion lattices. However, the conformation of these mesophases and their relative stability will depend on various small additional effects, such as dipolar couplings, thermal fluctuations, quenched defects etc. It would be naive to expect these phase diagrams to be simple and to be determined by one dominating mechanism able to stabilize Skyrmions over helicoids. Owing to the hierarchy of interactions, the properties and stability of the Skyrmion cores will always be provided by direct exchange and Dzyaloshinskii-Moriya couplings via the double-twist mechanism. But the mesophase formation will be ruled by much weaker couplings, owing to the localized and frustrated nonlinear character of these solitonic entities.

7. **Topogical solitons, vortices, Skyrmions...**

7.1. **How the chiral Skyrmion got its name**

Localized solutions of Eq. (5) have been initially introduced under the name **magnetic vortices** [2] because, to a certain extent, they are similar to 2D topogical defects investigated in magnetism.
and known as "two-dimensional topological solitons" or "vortices" (e.g. well-known Belavin-Polyakov solutions for magnetic vortices [23]). On the other hand, the term Skyrmion has been conceived in a field rather distant from condensed-matter physics and initially was related to the localized solutions derived by Skyrme within his model for low-energy dynamics of mesons and baryons [24]. In fact, the Skyrme model [24] comprises three spatial dimensions, and the name "baby Skyrmion" was used by some field theorists to distinguish two-dimensional localized states from “mature” three-dimensional solutions in the original Skyrme model [24], both types of them being topological static solitons. During the last decades the "Skyrmion" has progressively won currency in general physics to designate any non-singular localized and topologically stable field configuration. Complying with this trend, in 2002 we have renamed "chiral magnetic vortices" into "chiral Skyrmions" [5]. Ironically, the fate of the localized states near ordering transition, as they can decay by longitudinal magnetization processes, betrays that these “chiral Skyrmions" are not topologically stable, at high enough temperature.

7.2. 'What exactly is a chiral Skyrmion’?

Thus, the term Skyrmion is an umbrella title for smooth localized structures to distinguish them from singular localized states, e.g., disclinations in liquid crystal textures [25]. This convention provides only a formal label for a large variety of very different solitonic states from many fields of physics [26]. With respect to the subject of this paper a "Skyrmion" designates well-defined solutions of Eq. (5) which are (i) localized, (ii) axisymmetric, and have (iii) fixed rotation sense. Examples of chiral Skyrmions have been presented in Fig. 2. The axisymmetric structure of the Skyrmion core and its localized character are retained in bound states as Skyrmion lattices [3, 4]. This reflects the particle-like character of chiral Skyrmions and the most general features of their energetics (Fig. 4 (a), and Refs. [3, 4]). Skyrmions as countable entities can be arranged in various ways to create dense magnetic textures. This is the essence of Skyrmionic matter and entails the possibility to form a variety of mesophases with crystalline, but also with liquid-like or glassy configurations in chiral magnetism.

Alternative approaches construct Skyrmionic textures from crossing plane waves (helices) as fundamental modes of so-called spin-spiral crystals [14, 16]. In [14] a Skyrmion lattice is composed of three helices superimposed under an angle of 120 degree. According to [14] such a "triple-Q anti-Skyrmion lattice” ansatz even reaches the global free energy minimum in the A phase of MnSi. This construction is inconsistent with the properties of Skyrmions. Chiral Skyrmionic textures incorporate isolated or embedded axisymmetric lines. The notions of a "spin-spiral crystal" [16] or "multi-Q Skyrmions" [14] are misconceptions because they blend mutually exclusive ideas of particle-like localized Skyrmions from one side and delocalized plane waves from the other. The radial structure of the chiral Skyrmion cannot be reduced to a superposition of harmonic helical waves. Calculations based on such multi-Q ansaetze predict incorrect phase diagrams. E.g., the hexagonal Skyrmion lattice is stable in remanent state at \( H = 0 \) [7], but in [14] an instability at a finite field is presented for the variational hexagonal spin-spiral solution. The idea of a spin-spiral crystal ruled by one harmonic mode, where contributions of higher-order modes are small in some sense [14, 16], also is at variance with the nucleation type of transition actually observed in the chiral Skyrmion textures. This idea wrongly places the Dzyaloshinskii theory in a class of models for modulated states with instability type of transitions.

8. Conclusions: What makes chiral Skyrmions interesting?

In non-centrosymmetric magnets chiral magnetic Skyrmions arise as a result of the specific stabilization mechanism imposed by the handedness of the underlying crystal structure [1, 2, 3]. In centrosymmetric magnets such solutions are radially unstable and collapse spontaneously under the influence of the applied magnetic field or intrinsic short-range interactions. In
nonlinear field models Skyrmion states can be stabilized by higher order spatial derivatives (often this is referred to as Skyrme mechanism). In condensed-matter systems there are no physical interactions providing such energy contributions. Chiral interactions present a unique mechanism to stabilize Skyrmion states in ordered condensed-matter systems. This singles out chiral systems (including non-centrosymmetric magnets, multiferroics, liquid crystals, and metallic nanostructures with induced chiral interactions) into a particular class of materials with Skyrmionic states.

Acknowledgment. We thank S. Blügel, S.V. Grigoriev, G. Güntherodt, R. Schäfer, W. Selke, and H. Wilhelm, for discussions. Support by DFG project RO 2238/9-1 is gratefully acknowledged.

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