Geometric properties of critical collapse

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(Dated: 12th October 2018)

Critical collapse of a massless scalar field in spherical symmetry is systematically studied. Regarding the methodology, we combine numerical simulations and asymptotic analysis, and synthesize critical collapse, spacetime singularities, and complex science. The first approximate analytic expression for the metric components near the center is obtained. The spacetime in a small fraction of the discrete self-similar region is nearly conformally flat, which is one fundamental geometric feature. The slopes for the metric components and the scalar field versus the logarithm of the proper time are at the same order of magnitude as those in black hole formation, respectively. In this sense, one rough analytic explanation to the values of these slopes is given for the first time. The results show that the dynamics is not described by the Kasner solution. It is speculated that the scalar field in critical collapse may be a special standing wave.

I. INTRODUCTION

Complex systems widely exist in Nature. The interactions between the parts, alternatively the nonlinearities, in a complex system make the system more than the sum of its parts, causing the emergence of the intriguing critical phenomena and discrete scale invariance in some circumstances [1, 2]. The critical phenomena in gravitational collapse were originally discovered by Choptuik via numerical simulations in 1992, and the results were published in 1993 [3]. A naked singularity is formed in critical collapse [4], implying potential connections between critical collapse and the dynamics in the vicinities of singularities.

Choptuik simulated critical collapse of a massless scalar field in spherical symmetry in general relativity. When the scalar field is weak, the scalar field will collapse, get reflected at the center, and disperse eventually, leaving a flat spacetime behind; while when the scalar field is strong enough, a black hole (BH) will be formed. By fine tuning one parameter of the scalar field, a critical solution which is on the threshold of BH formation can be obtained. The collapse in which a BH just can (or cannot) be formed is called super-critical (or sub-critical) collapse. The results for critical collapse include discrete self-similarity (DSS), universality, and mass scaling law. The DSS states that the solution to the scalar field oscillates periodically at ever-decreasing time and length scales related by a factor of $e^\Delta$ with $\Delta \approx 3.44$. Regarding the universality, all the families of the near-critical evolutions approach the same solution. The mass scaling law describes that, in the super-critical case, the mass of the small BH has a scaling relation with the parameter of the scalar field, $M_{BH} \approx B|p - p^*|^\gamma$, where $B$ is a family-dependent parameter, $p$ is one parameter for the scalar field, describing the strength of the scalar field, $p^*$ is the critical value for $p$, and $\gamma$ is a universal scaling exponent $\gamma \approx 0.37$.

After Choptuik’s discovery, simulations of critical collapse in other gravitational theories, symmetries, and matter fields have been implemented [5–12]. Critical collapse of perfect fluids with continuous self-similarity was explored in Refs. [13, 14]. It was proposed that the scaling exponent $\gamma$ can be obtained via a linear stability analysis of the critical solution in Ref. [13], and such a proposal was implemented successfully in Ref. [14, 17].

The scaling exponent was also obtained by evolving near-critical initial data numerically in Ref. [15], confirming the results from the linear perturbation theory [14].

Considering that the results on critical collapse by Choptuik are numerical, it is natural to study this issue analytically. In Refs. [19, 21], under the requirements of discrete scale invariance, analyticity, and an additional reflection-type symmetry, the critical collapse is reduced to an eigenvalue problem. The rescaling factor $\Delta$ becomes an eigenvalue and is solved numerically by a relaxation algorithm with high precision. Motivated by understanding the nature of the echoing of the scalar field, Price and Pullin found that although the oscillatory behavior of the scalar field seems to come from the nonlinearities of general relativity, it can be approximated by a scalar field solution in flat spacetime [22]. In addition, analytic models for the continuous self-similar collapse in spherical symmetry (Roberts solution) [23, 25] and cylindrical symmetry [26] were presented. In Ref. [27], with analytic perturbation methods, it was found that a generic perturbation departs from the Roberts solution in a universal way. For reviews on critical collapse, see Refs. [28, 29].

Despite the above efforts on analytic studies, due to the complexity of Einstein equations, ever since the publication of Choptuik’s numerical results in 1993, the analytic solution remains unknown, and the nature of critical collapse is far from being well understood. In this paper, we explore the dynamics of critical collapse. In order to obtain some analytic information, we consider multiple methods and examine critical collapse from multiple
angles. We combine numerical simulations and asymptotic analysis. Moreover, considering that critical collapse ends up with a naked singularity, we connect critical collapse to the results on the dynamics near spacetime singularities that have been obtained before, including the Belinskii, Khalatnikov, and Lifshitz (BKL) conjecture and BH formation. With such efforts, the following results are obtained: (i) The first approximate analytic expression for the metric components near the center is obtained. The slopes of the metric components and the scalar field versus the logarithm of the proper time are at the same order of magnitude as those in BH formation, respectively. Noting that the values of the slopes in BH formation can be obtained analytically, in this sense, one rough analytic explanation to the values of the slopes in critical collapse is given for the first time.

(ii) It is found that the spacetime in critical collapse is nearly conformally flat, which is one fundamental geometric feature. (iii) The dynamics near the center is not described by the Kasner solution. (iv) It is speculated that the scalar field in critical collapse may be a special self-similarity pattern and seek the full analytic solution to critical collapse.

The paper is organized as follows. In Sec. II we describe the framework. The numerical results on the evolution are presented in Sec. III. In Sec. IV approximate analytic information on critical collapse is reported. Section V describes the nearly conformal flatness of the spacetime in critical collapse. We compare critical collapse with the BKL conjecture and BH formation in Sec. VI. In Sec. VII the connection between the scalar field in critical collapse and standing waves is argued. In Sec. VIII the results are summarized and discussed. Throughout the paper, we set $G = c = \hbar = 1$.

II. FRAMEWORK

In this section, we present the framework for numerical simulations of critical collapse, including the coordinates, the equations of motion, initial conditions, and boundary conditions.

Critical collapse of a massless scalar field in spherical symmetry is simulated in double-null coordinates,

$$ds^2 = -4e^{-2\sigma(u,v)} \, du \, dv + r^2(u,v) \, d\Omega^2 = e^{-2\sigma(t,x)} (-dt^2 + dx^2) + r^2(t,x) \, d\Omega^2,$$

where $u = (t - x)/2$ and $v = (t + x)/2$. In these coordinates, the two-manifold metric, $d\gamma^2 = e^{-2\sigma}(-dt^2 + dx^2)$, is conformally flat, such that one can know the speed of information propagation everywhere directly. Moreover, the evolution penetrates the horizon of the small BH formed in super-critical collapse.

Consider a massless scalar field $\psi$ with the energy-momentum tensor $T_{\mu \nu} = \psi,_{\mu} \psi,_{\nu} - (1/2)g_{\mu \nu}g^{\alpha \beta} \psi,_{\alpha} \psi,_{\beta}$. Then the equations of motion can be written as

$$r(-r,_{tt} + r,_{xx}) - r,_{tt}^2 + r,_{xx}^2 = e^{-2\sigma},$$

$$-\sigma,_{tt} + \sigma,_{xx} + \frac{r,_{tt} - r,_{xx}}{r} + 4\pi(\psi,^2_t - \psi,^2_x) = 0,$$

$$-\psi,_{tt} + \psi,_{xx} + \frac{2}{r}(-r,_{t}\psi,_{t} + r,_{x}\psi,_{x}) = 0.$$

The constraint equations are

$$r,_{tx} + r,_{t}\sigma,_{x} + r,_{x}\sigma,_{t} + 4\pi r\psi,_{t}\psi,_{x} = 0,$$

$$r,_{tt} + r,_{xx} + 2r,_{t}\sigma,_{t} + 2r,_{x}\sigma,_{x} + 4\pi r(\psi,^2_t + \psi,^2_x) = 0.$$

In this paper, $r,_{t}$ is defined as $r,_{t} \equiv \partial r(t,x)/\partial t$, and other quantities, e.g., $r,_{x}$, $r,_{tt}$, etc, are defined analogously.

For numerical stability concern, the Misner-Sharp mass $m$ is used as an auxiliary variable in the numerical set-up as has been successfully implemented in Ref. [32],

$$g^{\mu \nu} r,_{\mu} r,_{\nu} = e^{2\sigma}(-r,_{tt}^2 + r,_{xx}^2) + 1 - \frac{2m}{r}.$$  

Then Eqs. (2) and (3) can be rewritten as

$$-r,_{tt} + r,_{xx} - e^{-2\sigma} \cdot \frac{2m}{r^2} = 0,$$

$$-\sigma,_{tt} + \sigma,_{xx} - e^{-2\sigma} \cdot \frac{2m}{r^2} + 4\pi(\psi,^2_t - \psi,^2_x) = 0.$$

The dynamics of $m$ is described by

$$m,_{t} = 4\pi r^2 \cdot e^{2\sigma} \left[ -\frac{1}{2} r,_{t}(\psi,^2_t + \psi,^2_x) + r,_{x}\psi,_{t}\psi,_{x} \right].$$

The constraint equation for $m$ is

$$m,_{x} = 4\pi r^2 \cdot e^{2\sigma} \left[ \frac{1}{2} r,_{x}(\psi,^2_t + \psi,^2_x) - r,_{t}\psi,_{t}\psi,_{x} \right].$$

The derivations of Eqs. (10) and (11) are given in the Appendix. In the simulations, Eqs. (4), (5), (9), and (10) are integrated numerically with finite difference method and leapfrog integration scheme.

We impose $r,_{tt} = \sigma,_{t} = \psi,_{t} = 0$ at $t = 0$. The initial profile of $\psi$ is set as

$$\psi(t = 0, x) = a \cdot \exp \left[ -\frac{(x - x_0)^2}{b} \right],$$

with $a$ being tuned as $a = 0.0908379861$, $b = 0.01$, and $x_0 = 0.25$. We require $r = \sigma = m = 0$ at the origin $(x = 0, t = 0)$. The expressions for $r,_{xx}$, $\sigma,_{x}$, and $m,_{x}$ can be obtained from Eqs. (3), (9), and (11), respectively. We obtain the quantities $r$, $\sigma$, and $m$ on the initial slice of $t = 0$ by numerically integrating such expressions.
from $x = 0$ to $x = 2$ with the fourth-order Runge-Kutta method. From Eqs. (8), (9), and (4), the values of $r$, $\sigma$, and $\psi$ at $t = \Delta t$ can be determined via a second-order Taylor expansion. Take the first-order time derivative of Eq. (10), one can obtain $m_{,tt}$. With $m_{,t}$ and $m_{,tt}$, $m$ at $t = \Delta t$ is obtained via a second-order Taylor series expansion.

Regarding the boundary conditions, we always set $r = m = 0$ at $x = 0$. Then there are always $r_{,t} = r_{,tt} = 0$ at $x = 0$. From Eqs. (4) and (8), one obtains respectively $\psi_x \equiv 0$ and $\sigma_{,xx} \equiv 0$ at $x = 0$. Then with Eq. (6), there is $r_{,x} \sigma_{,x} \equiv 0$ at $x = 0$. Since $r_{,x}$ is usually not zero, one obtains $\sigma_{,x} \equiv 0$ at $x = 0$.

The simulations of critical collapse need to be highly accurate. In order to achieve this objective, adaptive or fixed mesh refinement techniques are usually used. While in this paper, we take an even simpler approach. From the beginning, we use very small spatial and temporal grid spacings $\Delta x = \Delta t = 2.5 \times 10^{-2}$, not making any mesh refinement throughout the whole simulations. It turns out that the numerical results by this approach can show the basic features of critical collapse and are adequate for us to study the dynamics of critical collapse. The code is second-order convergent.

III. RESULTS ON EVOLUTION

The numerical results for the evolution of $r$, $\sigma$, $m$, and $\psi$ are shown in Fig. 1. Near the center, $r$ approaches zero, and $\sigma$ goes to $+\infty$. The scalar field $\psi$ oscillates, and at the same time, moves toward the center under gravity.

In Choptuik’s original work [3], the simulations were run in the polar coordinates $ds^2 = -\alpha^2(t, r)dt^2 + \beta^2(t, r)dr^2 + r^2d\Omega^2$. It was found that, in the scales of $(ln r, ln \tau)$, the scalar field repeats itself with a period $\Delta \approx 3.44$, where $\tau$ is the proper time of a central observer. After Choptuik’s work, similar results have been reported by others. We run the simulations in double-null coordinates, and find that, as shown in Figs. 2(a) and 2(b), the periods on $ln r$ and $ln \tau$ are about $3.47$ and $3.59$, respectively. Currently, our attention is not focused on the discrepancies of the periods in the two sets of coordinates. In Ref. 3, it was emphasized that the DSS only occurs in a neighborhood of the center. This is also shown in Fig. 2(a) of this paper.

It is noticeable that there are at least two basic parameters for a wave: period and amplitude. Physically, the amplitude is also significant, which relates the energy of the wave. However, the amplitude of the scalar field has not been paid as much attention as the period was in the literature in the past. As shown in Figs. 1(e) and 2(b), the amplitude is about 0.6. The amplitude of the scalar field seems to be dependent on the coordinates. In Choptuik’s original work, where the polar coordinates were used, the amplitude is about 0.45 [3]. We call attention to the amplitude not only because the amplitude of the oscillations is one basic parameter, but also because that one may obtain additional hints on the nature of and possible analytic solution to critical collapse by pondering the amplitude and period together.

IV. APPROXIMATE ANALYTIC EXPRESSIONS

It is highly challenging to obtain the full analytic solution to critical collapse. In this section, we present partial analytic information obtained via combination of numerical simulations and asymptotic analysis.

A. $\sigma(t, x)$ and $r(t, x)$

We investigate the contribution of each term in the equation of motion for $r$ on the slice of $x = 4.75 \times 10^{-4}$. Remarkably, as shown in Fig. 3(a), near the center, ever since the beginning of the collapse, there is always $e^{-\sigma} \approx r_{,x}$. (13)

We plot $r_{,x}$ vs. $r$ on the same slice, and find that, as shown in Fig. 3(b),

$$r_{,x} \approx \frac{r}{x}.$$ (14)

Then there is

$$r(t, x) \approx D(t)x,$$ (15)

where $D(t)$ is a function of $t$. Combine Eqs. (13)–(15),

$$e^{-\sigma} \approx r_{,x} \approx \frac{r}{x} \approx D(t).$$ (16)

Some comments on Eq. (16) are made as follows:

(i) Equation (16) implies that $\sigma$ and $r$ have strong coherence and/or resonances.

(ii) As shown in Figs. 2(a) and 2(b), Eq. (16) is valid only in a small fraction ($\sim 5\%$) of the DSS region.

(iii) We also simulate collapse toward BH formation and flat spacetime (collapse-and-dispersion), and find that Eq. (16) also exist in these two cases. Details are skipped.

B. $\sigma$ vs. $ln \tau$

In Fig. 4(a), we plot $-\sigma$ vs. $-ln \tau$ on the slice of $x = 4.75 \times 10^{-4}$, where $\tau$ is the proper time on this slice, $\tau = \int_0^\xi e^{-\sigma} d\xi$, $\xi \equiv t - t_0$, and $t_0 = 0.6095$ is the coordinate time where $r$ goes to zero. As shown in Fig. 4(a), $-\sigma$ is a superposition of a linear function and a periodic function of $-ln \tau$. In fact, in many discrete scale invariance systems, the log-periodic oscillations exist in the time
dependence of the energy release as the impending rupture is approached. The typical time-to-failure formula is [1][2]

\[ E \sim (t_r - t)^n \left[ 1 + \beta \cos \left( 2\pi \frac{\ln(t_r - t)}{\ln \lambda} + \varphi \right) \right], \quad (17) \]
Figure 2. (color online). The scalar field $\psi$ in critical collapse. (a) Discrete self-similarity for $\psi$. Near the center, the profile of $\psi$ vs. $r$ on the slice of $t = 0.4730$ and the profile of $\psi$ vs. $32r$ on the slice of $t = 0.5945$ are similar. The difference of $\ln \tau$ between the two slices is $\Delta(\ln \tau) \approx 3.59$. (b) The approximate analytic expression, $e^{-\sigma} \approx r_x \approx r/x$, is valid only in a small fraction of the DSS region. (c) $\psi$ vs. $x$ on the slice of $t = 0.4730$. The profile of $\psi$ vs. $x$ is fit according to $f(x) = ae^{-bx} \cos(cx + d) + f$. The fitting results are the following: $a = 0.6623 \pm 0.0025$, $b = 10.37 \pm 0.08$, $c = 13.96 \pm 0.08$, $d = -0.4068 \pm 0.0095$, and $f = 0.0005 \pm 0.0002$.

Figure 3. (color online). Contributions of all the terms in the equation of motion for $r$ \[^{2}\]. (a) $e^{-\sigma} \approx r_x$. (b) $r_x \approx r/x$, which yields $r(t,x) \approx D(t)x$. Slice: $x = 4.75 \times 10^{-4}$.

where $E$ is the energy released or some other variable describing the on-going damage, $t_r$ is the rupture time, $n$ is a critical exponent, $\lambda$ is a preferred scaling factor,
Figure 4. (color online). $(-\sigma, \psi)$ vs. $-\ln \tau$. The numerical results are fit for $4.96 \leq -\ln \tau \leq 8.55$, correspondingly for $0.4730 \leq t \leq 0.5945$. (a) The numerical results on $-\sigma$ vs. $-\ln \tau$ are fit according to $f(\zeta) = a\zeta + \ln[1 + b\cos(\omega\zeta + c)] + d$. The period for $\sigma$ is $T_\sigma \approx 1.795$, and $\omega$ is fixed to be $\omega = 2\pi/T_\sigma = 3.50$. The fitting results are $a = -0.5028 \pm 0.0015$, $b = 0.2287 \pm 0.0018$, $c = -1.482 \pm 0.009$, and $d = 0.1757 \pm 0.0096$. (b) The numerical results on $\psi$ vs. $-\ln \tau$ are fit according to $f(\zeta) = a_0 + \sum_{n=1}^4 a_n \cos(n\omega\zeta) + b_0 \sin(n\omega\zeta)$. The period for $\psi$ is $T_\psi \approx 3.59$, and $\omega$ is fixed to be $\omega = 2\pi/T_\psi = 1.75$. The fitting results are $a_0 = 0.0106 \pm 0.0004$, $a_1 = -0.3451 \pm 0.0006$, $b_1 = 0.3927 \pm 0.0007$, $a_2 = 0.0110 \pm 0.0006$, $b_2 = -0.0001 \pm 0.0006$, $a_3 = 0.0546 \pm 0.0006$, $b_3 = 0.0508 \pm 0.0006$, $a_4 = 0.0007 \pm 0.0006$, $b_4 = 0.00035 \pm 0.00059$. Slice: $x = 4.75 \times 10^{-4}$.

Figure 5. (color online). Upper panel: $\xi$ vs. $\tau$ by Eq. (19) and the approximate inverse function $\tau$ vs. $\xi$ by Eq. (21). For both curves in the upper panel, the horizontal axis is for $\tau$, and the vertical one is for $\xi$. Lower panel: The relative error for the two curves in the upper panel.

and $\varphi$ is a phase. The critical collapse system is also a discrete scale invariance system. So we fit the numerical results of $-\sigma$ vs. $-\ln \tau$ according to the logarithm of Eq. (17),

$$f(\zeta) = a\zeta + \ln[1 + b\cos(\omega\zeta + c)] + d. \quad (18)$$

Noting that the period for $\sigma$ in Fig. 4(a) is $T_\sigma \approx 1.795$, we fix the angular frequency $\omega$ in Eq. (18) as $\omega = 2\pi/T_\sigma = 3.50$. As shown in Fig. 5(a), the numerical results can be well fit by this formula. The fitting results are the following:

$$a = -0.5028 \pm 0.0015,$$

$$b = 0.2287 \pm 0.0018,$$

$$c = -1.482 \pm 0.009,$$

$$d = 0.1757 \pm 0.0096.$$ The close agreement between the numerical results and the fitting formula strongly implies that critical collapse is indeed one more subfield of complex science.

### C. $(\tau, \sigma, r)$ vs. $\xi$

In this subsection, we seek the approximate analytic expressions for $(\tau, \sigma, r)$ vs. $\xi$. Note that $\tau = \int_0^\xi e^{-\sigma} d\xi$ and the approximate analytic expression for $\sigma = \sigma(\tau)$ is given by Eq. (18), the approximate analytic expression for $\xi = \xi(\tau)$ can be obtained by integrating $\xi = \int_0^\tau e^\sigma d\tau$. The fitting result for $a$ in Eq. (18) is $-0.5028 \pm 0.0015$. For simplicity, we set $a = -1/2$. In addition, let $y = \omega \ln \tau - c$ and use Taylor series expansion, there is

$$\xi \approx \frac{e^{-d+c/(2\omega)}}{\omega} \int_{-\infty}^{y} \frac{e^{y/(2\omega)}}{1 + b \cos y} dy \approx \frac{e^{-d+c/(2\omega)}}{\omega} \int_{-\infty}^{y} \frac{e^{y/(2\omega)}}{(1 - b \cos y + b^2 \cos^2 y) dy} \approx e^{-d(2 + b^2)^{1/2}} [1 - H \sin(\omega \ln \tau - c + \delta)],$$

(19)
where \( H = \frac{2b}{(2 + b^2)\sqrt{1 + 4\omega^2}} \approx 0.03 \), and \( \delta = \arccos\frac{2\omega}{\sqrt{1 + 4\omega^2}} \approx 0.14 \) radians.

Noting that \( H \approx 0.03 \ll 1 \), the major part for the inverse function \( \tau = \tau(\xi) \) is

\[
\tau_m = \left( \frac{e^{d\xi}}{2 + b^2} \right)^2. \tag{20}
\]

Substitution of Eq. (20) into the second term in the square brackets in Eq. (19) yields an approximate analytic expression for \( \tau = \tau(\xi) \)

\[
\tau^{1/2} \approx \frac{e^{d\xi}}{2 + b^2} [1 + H \sin(\omega \ln \tau_m - c + \delta)]. \tag{21}
\]

The relative error for Eq. (21) compared to Eq. (19) is less than 1%. See Fig. 3.

Combining Eqs. (16), (18), and (21), one obtains

\[
e^{-\sigma} \approx \frac{c^2}{x} \\
= e^{2d\xi} \left[ 1 + \frac{b^2}{(2 + b^2)(1 + 4\omega^2)} \right. \\
+ b \cos(\omega \ln \tau_m - c) \\
+ \frac{2b}{(2 + b^2)(1 + 4\omega^2)} \sin(\omega \ln \tau_m - c + \delta) \\
\left. + \frac{b^2}{(2 + b^2)(1 + 4\omega^2)} \sin(2\omega \ln \tau_m - 2c + \delta) \right]. \tag{22}
\]

Noting that \( \xi \approx t_0 - t \), Eq. (22) is the first approximate analytic expression for the metric components in terms of the coordinate time near the center in critical collapse. As shown in Fig. 6, Eq. (22) is well consistent with the numerical results for \( \xi < 0.15 \), which corresponds to formal critical collapse. There is a large deviation between the analytic expression (22) and the numerical results for \( 0.15 < \xi < 0.6 \), which should be because that this region is not a formal critical collapse region, but an entrance to critical collapse. Also see Figs. 3 and 4.

D. \( \psi \) vs. \( \ln \tau \)

The numerical results on \( \psi \) vs. \( -\ln \tau \) are plotted in Fig. 3(b). Roughly speaking, \( \psi \) is a linear function of \( \ln \tau \) and keeps switching between \( \alpha \ln \tau \) and \( -\alpha \ln \tau \) with \( \alpha \approx 0.6 \sim 0.8 \). It is noticeable that the slope \( \alpha \) describes the ratio between the amplitude and period of \( \psi \):

\[
\frac{\text{Amplitude}}{\text{Period}} \approx \frac{\alpha}{4} \approx 0.15 \sim 0.2. \tag{23}
\]

Equation (23) may be another meaningful expression in critical collapse. In seeking analytic solution to critical collapse, it may be instructive to take Eq. (23) into account and ponder where the values of the amplitude and period come from together.

The numerical results on \( \psi \) vs. \( -\ln \tau \) are fit according to fourth-order Fourier sine and cosine series. The fitting results are shown in the caption of Fig. 4. In addition, for reference consideration, we fit the numerical results on \( \psi \) vs. \( x \) on the slice of \( t = 0.4730 \) according to \( f(x) = ae^{-bx} \cos(cx + d) + f \). The fitting results are given in the caption of Fig. 2(c). Further studies are needed.

In Sec. VI we will give one preliminary analytic explanation to the slopes of \( (\sigma, \psi) \) vs. \( -\ln \tau \).

V. THE SPACETIME NEAR THE CENTER IS NEARLY CONFORMALLY FLAT

Substitution of Eq. (16) into Eq. (1) yields

\[
d\bar{s}^2 \approx D^2(t) \left( -dt^2 + dx^2 + x^2d\Omega^2 \right). \tag{24}
\]**
So the spacetime near the center is nearly conformally flat (NCF), which is one fundamental geometric feature. In conformally flat spacetime, the Weyl tensor \( C_{\alpha \beta \mu \nu} \) and the scalar \( C_W \equiv \sqrt{C_{\alpha \beta \mu \nu}C^{\alpha \beta \mu \nu}} \) are zero. Then, with Eq. (2), for the metric (1), there is

\[
C_W = \sqrt{\frac{4}{3}e^{2\sigma}} \left[ -\sigma,tt + \sigma,xx + \frac{-r,tt + r,xx}{r} \right. \\
+ \left. \frac{r,tt - r,xx + e^{-2\sigma}}{r^2} \right]
\]

(25)

\[
= \sqrt{\frac{4}{3}e^{2\sigma}} \left[ -\sigma,tt + \sigma,xx + \frac{2(-r,tt + r,xx)}{r} \right] \\
\approx 0.
\]

Taking into account Eq. (3), one obtains

\[
|-\sigma,tt + \sigma,xx| : \frac{|r,tt - r,xx|}{r} : 4\pi|\psi_t^2 - \psi_{,tt}^2| \approx 2 : 1 : 3.
\]

(26)

Equation (26) shows that the metric components (\( \sigma, r \)) and the scalar field (\( \psi \)) have strong coherence and/or resonances. The complete analytic solution to critical collapse has not been obtained, and any possible analytic solution near the center should satisfy this relation. So this relation is one guiding star for seeking the final analytic solution.

Using \( \sigma \approx -\ln (r/x) \) and (\( r(t, x) \approx D(t)x \)), one obtains \( \sigma,tt + r,tt/r - r,xx/r^2 \approx 0 \) and \( \sigma,xx + r,xx/r \approx 0 \). Then with \( e^{-\sigma} \approx r/x \), the first line of Eq. (25) leads to \( C_W \approx 0 \), verifying that the spacetime in which \( e^{-\sigma} \approx r/x \) is indeed nearly conformally flat.

With the comments on Eq. (16) made in Sec. IV.A, one obtains that the NCF region is a small fraction of the DSS region, and the NCF region also exists near the center in BH formation and collapse-and-dispersion cases.

We also compute the Weyl tensor in the Roberts solution, which is one type of continuously self-similar critical collapse of a scalar field, and obtain nonzero results. The solution can be written as below [24]:

\[
ds^2 = -dudv + r^2 d\Omega^2, \\
r^2 = \frac{1}{4}|(1 - k^2)v^2 - 2uv + u^2|, \\
\psi = \pm \frac{1}{2} \log \frac{(1 - k)v + u}{(1 + k)v - u},
\]

(27)

(28)

(29)

where \( k \) is a parameter. For this solution,

\[
C_{\alpha \beta \mu \nu}C^{\alpha \beta \mu \nu} = \frac{256u^2v^2k^4}{3(-v^2 + v^2k^2 + 2vu - u^2)^4},
\]

(30)

which is nonzero for non-Minkowskian spacetime.

VI. CRITICAL COLLAPSE VS. DYNAMICS NEAR SPACETIME SINGULARITIES

A singularity is formed in critical collapse. Therefore, it is meaningful to connect critical collapse to the results on the dynamics near spacetime singularities that have been obtained before, including the BKL conjecture and BH formation. We will do so in this section.

A. Scalar collapse toward BH formation

The BKL conjecture is one of the guiding principles in the studies of the dynamics in the vicinities of spacetime singularities. According to the BKL conjecture, near the singularities, the temporal derivatives dominate the matter field ones, and the behaviors of the spacetime and the matter field are described by the Kasner solution [39–41]. The four-dimensional homogeneous but anisotropic Kasner solution with a massless scalar field \( \psi \) minimally coupled to gravity can be described as follows [39–40]:

\[
ds^2 = -dt^2 + \sum_{i=1}^{3} r^{2p_i} dx_i^2, \\
p_1 + p_2 + p_3 = 1, \\
p_1^2 + p_2^2 + p_3^2 = 1 - q^2, \\
\sqrt{8\pi}\psi = q \ln \tau.
\]

The parameter \( q \) is constrained by Eq. (31) as \( q^2 \leq 2/3 \). For reviews on the BKL conjecture, see Refs. [39–41].

In Ref. [31], scalar collapse toward BH formation in \( f(R) \) gravity was simulated with a formalism essentially the same as the one used in this paper. The dynamics in the vicinity of the singularity of the formed BH was studied, and approximate analytic solution was obtained. As the singularity is approached, the conformal scalar field \( \phi \equiv \sqrt{3/2 \ln |df(R)/dR|} \) is dominant, and the physical scalar field \( \psi \) is negligible. The solution on slices of \( x = \text{Constant} \) is the following:

\[
\tau \approx A\xi^\frac{2}{3} \approx A \left( \frac{3 + 2C^2}{4} \right)^{\frac{2}{3 + 2C^2}},
\]

(32)

\[
e^{-\sigma} \approx \xi^{-\frac{1+2C^2}{2}} \approx \left( \frac{3 + 2C^2}{4} \right)^{-\frac{1+2C^2}{2}},
\]

(33)

\[
\sqrt{8\pi}\phi \approx C \ln \xi \approx \frac{4C}{3 + 2C^2} \ln \tau,
\]

(34)

where

\[
\tau \equiv \int_0^\xi e^{-\sigma} d\xi \approx \frac{4}{3 + 2C^2} \xi^{\frac{1+2C^2}{4}},
\]

(35)
Figure 7. (color online). Some numerical results on BH formation presented in Ref. [31]. (a) Apparent horizon and singularity curve, \( r = 0 \), of the formed BH. (b) \( A \) for Eq. (32), \( r \approx A\xi^3 \), along the singularity curve. For \( x < 1 \), \( A \approx x \). (c) The Kasner exponents \( p_i \) and the parameter \( q \) which are described by Eqs. (36)-(38), respectively. (d) \( C \) and \( q \) for Eq. (34), \( \sqrt{8\pi\phi} \approx C \ln \xi \approx q \ln \tau \). At \( x = 0 \) along the curve of \( r = 0 \), \( C = \pm\sqrt{3}/2 \). This corresponds to a special configuration: (i) A black hole including a central singularity is just formed. (ii) \( p_1 = p_2 = p_3 = 1 \), and the metric is asymptotically conformally flat: \( ds^2 = -d\tau^2 + \tau^{2/3} \sum_{i=1}^{3} dx_i^2 \). (iii) \( |q| \) approaches its maximum \( |q| = \sqrt{2}/3 \).

In critical collapse, \( e^{-\sigma} \approx r/x \). We check whether this is also true in BH formation. Enforcing \( e^{-\sigma} \sim r \), one obtains from Eqs. (32) and (33)

\[
C = \pm \sqrt{3}/2. \tag{39}
\]

The result of \( C = \pm\sqrt{3}/2 \) corresponds to a special configuration: (i) As shown in Fig. 7(d), at \( x = 0 \) along the singularity curve, there is \( C = \pm\sqrt{3}/2 \). (ii) A black hole including a central singularity is just formed, see Fig. 7(a). (iii) As shown in Fig. 7(c), \( p_1 = p_2 = p_3 = 1 \), and the metric is asymptotically conformally flat:

\[
ds^2 = -d\tau^2 + \tau^{2/3} \sum_{i=1}^{3} dx_i^2. \tag{39}
\]

| \( p_1 \) | \( p_2 = p_3 = 1/3 \) | \( q = \sqrt{2}/3 \) |
|---|---|---|

These parameters satisfy Eq. (31): \( p_1 + p_2 + p_3 = 1 \) and \( p_1^2 + p_2^2 + p_3^2 = 1 - q^2 \). So the solution is the Kasner solution, and the BKL conjecture is valid in the vicinity of the singularity of the BH formed in collapse.

\[\xi \equiv t_0 - t, \text{ and } t_0 \text{ is the coordinate time on the singularity curve } r = 0 \text{ shown in Fig. 7(a). Comparing Eqs. (32) to (31), we extract}
\]

\[
p_1 = \frac{-1 + 2C^2}{3 + 2C^2}, \tag{36}
\]

\[
p_2 = p_3 = \frac{2}{3 + 2C^2}, \tag{37}
\]

\[
q = \frac{4C}{3 + 2C^2}. \tag{38}
\]

These parameters satisfy Eq. (31): \( p_1 + p_2 + p_3 = 1 \) and \( p_1^2 + p_2^2 + p_3^2 = 1 - q^2 \). So the solution is the Kasner solution, and the BKL conjecture is valid in the vicinity of the singularity of the BH formed in collapse.
Actually, as shown in Fig. 7(b), for $x < 1$, the 'coefficient' $A$ in Eq. (32) has a rough relation $A \approx x$. Therefore, similar to critical collapse, in BH formation, for $x$ close to zero along the singularity curve, there is also $e^{-\sigma} \approx r/x$.

With Eqs. (33) and (34), one obtains

$$
\sigma \approx -\frac{1}{3} \ln \tau,
$$

$$
\sqrt{8\pi} \phi \approx q \ln \tau \approx \pm \sqrt{\frac{2}{3}} \ln \tau.
$$

As discussed in Ref. [31], near the singularity, the ratios between the spatial and temporal derivatives of $r$, $\sigma$, and $\psi$ are determined by the slope of the singularity curve. Then replacing $\psi$ by $\phi$, Eq. (3) is reduced to

$$
- \sigma_{,tt} + \frac{r_{,tt}}{r} + 4\pi \phi_{,t}^2 = 0.
$$

With Eqs. (32)-(34), in the case of $C = \pm \sqrt{3}/2$, there is

$$
|\sigma_{,tt}| : \left|\frac{r_{,tt}}{r}\right| : 4\pi \phi_{,t}^2 = 2 : 1 : 3.
$$

In the metric (31), the Kreschmann scalar is [42]

$$
K = \frac{12H}{(3 + 2C^2)^\frac{3}{2}} \tau^{-4} \approx \frac{3H}{A}\left(\frac{r}{A}\right)^{-2(3+2C^2)}.
$$

where $H = 16 - 40C^2 + 99C^4 - 26C^6 + 3C^8$. In the case of $C = \pm \sqrt{3}/2$, $K \propto r^{-12}$.

**B. Critical collapse vs. BH formation**

We compare the dynamics of critical collapse in the NCF region and that at $x = 0$ along the singularity curve in BH formation.

(i) In collapse toward BH formation, at $x = 0$, from the beginning of the collapse to the formation of the central singularity, there is $e^{-\sigma} \approx r/x$. In critical collapse, in the NCF region, there is always $e^{-\sigma} \approx r/x$.

(ii) In BH formation, the scalar field and gravity are strong. Then, the scalar field is simply absorbed into the central singularity without reflections, $\psi = \alpha \ln \tau$ or $\psi = -\alpha \ln \tau$ with $\alpha \approx 0.16$, and $\sigma = -(1/3) \ln \tau$. In critical collapse, the scalar field and gravity are weak, and there is a balance between gravity and reflections at the center. Consequently, $\psi$ keeps switching between $\alpha \ln \tau$ and $-\alpha \ln \tau$ with $\alpha \approx 0.6 \sim 0.8$, and $\sigma$ is a superposition of a linear function and a periodic function of $\ln \tau$ with the slope for the linear part being about $-0.50$.

The slopes of $(\sigma, \psi)$ vs. $\ln \tau$ in critical collapse are at the same order of magnitude as those in BH formation, respectively. However, as shown in Eqs. (10) and (11), the values of the slopes in BH formation, $-1/3$ and $\pm 0.16$, can be obtained analytically. In this sense, for the first time, we give one rough analytic explanation to the slopes of $(\sigma, \psi)$ vs. $\ln \tau$ in critical collapse.

(iii) In BH formation, the spatial and temporal derivatives are decoupled. Then, the ratios between the three temporal derivative terms in the equation of motion for $\sigma$ (3) are described by Eq. (43). We also check the equations of motion in critical collapse, and find that the spatial and temporal derivatives are mixed. As an example, the behaviors of all the terms in Eq. (3) are plotted in Fig. 8(a). The ratios between the three temporal derivatives in Eq. (3) do not have the simple values as in BH formation. We check the ratios of the sums of the spatial and temporal derivatives, and find that, as shown in Fig. 8(b), such ratios have a simpler form as described by Eq. (26). Interestingly, as discussed in Sec. V, this result can also be derived from the NCF feature of the spacetime.
In critical collapse, the scalar field moves toward the center under gravity. At the same time, the scalar field is reflected at the center. The resulting scalar field is discretely self-similar. This picture makes us to recall the concept of standing waves. The scalar field in critical collapse and a standing wave share at least the following two features: (i) superposition of incoming and outgoing waves, (ii) DSS. So in this paper, we speculate that the scalar field in critical collapse may be a special standing wave. The word ‘special’ is due to the fact that, in critical collapse, the scalar wave keeps shrinking toward the center, which is different from usual standing waves. We think that this special standing wave picture helps to interpret the DSS feature and seek the full analytic solution to critical collapse, noting that the DSS feature describes that the scalar field repeats itself at ever-decreasing time and length scales by a factor of $e^\Delta$.

In fact, standing waves are quite common in astrophysics: e.g., (i) The 5-minute solar oscillation in local surface velocities discovered by Leighton [43, 44]. (ii) The density wave theory on the spiral structure of disk galaxies by Lin and Shu [45–49]. (iii) The microwave cavity hypothesis on ball lighting by Kapitsa [50, 51].

VIII. SUMMARY AND DISCUSSIONS

Now we summarize the methodology and the physical results, and discuss the perspectives on studies of critical collapse.

A. Summary on methodology

(i) Numerical simulations vs. asymptotic analysis. The numerical simulations and asymptotic analysis are complementary. In order to obtain analytic information on critical collapse, we combined these two approaches together. We firstly simulated critical collapse, then examined the behaviors of all the terms in the equations of motion, and studied the connections between different quantities. With this strategy, we obtained some approximate analytic information on critical collapse.

(ii) Critical collapse vs. dynamics near spacetime singularities. A naked singularity is formed in critical collapse. So we investigated the connection between critical collapse and the results on the dynamics near spacetime singularities that have been obtained before, including the BKL conjecture and BH formation.

(iii) Critical collapse vs. complex science. Critical phenomena are an important subject in complex science. So it is meaningful to investigate critical collapse from the point of view of complex science. In this paper, we fit $\sigma$ vs. $\ln \tau$ according to one typical log-periodic formula in discrete scale invariance systems in complex science.

(iv) Reflections on the boundary. The dynamics of a system is determined by the field equations, initial condi-

---

Figure 9. (color online). $\ln K$ vs. $\ln r$, where $K$ is the Kreschmann scalar. The relation of $\ln K$ vs. $\ln r$ is fit according to $f(x) = ax + b$ with $a = -5.30 \pm 0.04$ and $b = -27.16 \pm 0.38$. Slice: $t = 0.3$.

(iv) As shown in Eq. (44), in BH formation, in the case of $C = \pm \sqrt{3}/2$, the Kreschmann scalar $K \propto r^{-12}$. We plot the Kreschmann scalar in critical collapse in Fig. 9. Roughly speaking, $K \propto r^{-5.30}$, much weaker than the BH formation case, and also weaker than the Schwarzschild BH case where $K \propto r^{-6}$.

The above comparisons are also listed in Table 1. Now we would like to give some comments. Critical collapse and BH formation share some common features: after all they both are on the dynamics in the vicinities of singularities. On the other hand, they have some remarkable differences: the scalar field and gravity are weaker in critical collapse than in BH formation.

C. Critical collapse vs. the BKL conjecture

One important statement in the BKL conjecture is that the dynamics near the singularities is described by the Kasner solution. In the vicinity of the singularity of the BH formed by scalar collapse, the expressions of $r$, $\sigma$, and $\psi$ are the Kasner solution. However, the singularity formed in critical collapse is much weaker than those in BHs. Then the corresponding expressions in critical collapse are different from those in BH formation, and are not the Kasner solution.

The complete analytic solution to critical collapse has not been obtained yet, and the confrontation of the BKL conjecture in critical collapse needs to be explored further.

VII. SCALAR FIELD IN CRITICAL COLLAPSE VS. STANDING WAVES

In critical collapse, the scalar field moves toward the center under gravity. At the same time, the scalar field...
Table I. BH formation vs. critical collapse

| x = 0 on the singularity curve in BH formation | DSS region in critical collapse |
|-----------------------------------------------|--------------------------------|
| From beginning of collapse to formation of singularity: | | |
| $e^{-\sigma} \approx \frac{r}{x}$ | $e^{-\sigma} \approx \frac{r}{x}$ |
| $\sigma \approx -\frac{1}{3} \ln \tau$ | $\sigma \approx -\frac{1}{3} \ln \tau - \ln[1 + 0.23 \cos(3.50 \ln \tau + 1.48)] - 0.18$ |
| $\psi$ is absorbed into the central singularity, and $\psi = \alpha \ln \tau$ or $-\alpha \ln \tau$ with $\alpha \approx 0.16$. | $\psi$ is reflected at $x = 0$, and keeps switching between $\alpha \ln \tau$ and $-\alpha \ln \tau$ with $\alpha \approx 0.6 \sim 0.8$. |
| Spatial and temporal derivatives are separated. | Spatial and temporal derivatives are mixed. |
| $|\sigma,tt| : \frac{r,tt}{r} : 4\pi \psi_{,t}^2 = 2 : 1 : 3$ | $| - \sigma,tt + \sigma,xx | : \frac{r,tt-r,xx}{r} : 4\pi \psi_{,t}^2 - \psi_{,x}^2 | \approx 2 : 1 : 3$ |
| The spacetime is asymptotically conformally flat. | The spacetime is nearly conformally flat. |
| $K \propto r^{-12}$ | $K \propto r^{-5.30}$ |

| x = 0 on the singularity curve in BH formation | DSS region in critical collapse |
|-----------------------------------------------|--------------------------------|
| From beginning of collapse to formation of singularity: | | |
| $e^{-\sigma} \approx \frac{r}{x}$ | $e^{-\sigma} \approx \frac{r}{x}$ |
| $\sigma \approx -\frac{1}{3} \ln \tau$ | $\sigma \approx -\frac{1}{3} \ln \tau - \ln[1 + 0.23 \cos(3.50 \ln \tau + 1.48)] - 0.18$ |
| $\psi$ is absorbed into the central singularity, and $\psi = \alpha \ln \tau$ or $-\alpha \ln \tau$ with $\alpha \approx 0.16$. | $\psi$ is reflected at $x = 0$, and keeps switching between $\alpha \ln \tau$ and $-\alpha \ln \tau$ with $\alpha \approx 0.6 \sim 0.8$. |
| Spatial and temporal derivatives are separated. | Spatial and temporal derivatives are mixed. |
| $|\sigma,tt| : \frac{r,tt}{r} : 4\pi \psi_{,t}^2 = 2 : 1 : 3$ | $| - \sigma,tt + \sigma,xx | : \frac{r,tt-r,xx}{r} : 4\pi \psi_{,t}^2 - \psi_{,x}^2 | \approx 2 : 1 : 3$ |
| The spacetime is asymptotically conformally flat. | The spacetime is nearly conformally flat. |
| $K \propto r^{-12}$ | $K \propto r^{-5.30}$ |

B. Summary on results

(i) Small DSS region vs. small NCF region. In Chop- tui's original paper [3], it was emphasized that the DSS feature “pertains only to the strong-field dynamics of critical collapse, which always occurs in a neighborhood of $r = 0$.” In this paper, we found that the NCF region is a small fraction of the DSS region. All the results obtained in this paper are on this small NCF region.

(ii) One missing parameter: the amplitude of the scalar field. As one of the basic parameters for a wave, the amplitude of the scalar field has not been paid as much attention as the period was in the past. We pointed out that, for critical collapse in double-null coordinates, the amplitude is about $0.04$.

(iii) Approximate analytic results and NCF. Near the center, $e^{-\sigma} \approx \frac{r}{x}$, $\sigma \approx \frac{r}{x} \approx D(t)$. As a consequence, the spacetime around the center is nearly conformally flat. This is one fundamental geometric feature of spacetime. The relation of $e^{-\sigma} \approx \frac{r}{x}$ and the ratios of $2 : 1 : 3$ between $| - \sigma,tt + \sigma,xx |$, $| r,tt - r,xx | / r$, and $4\pi \psi_{,t}^2 - \psi_{,x}^2$ show that there are strong coherence/resonances between some quantities in critical collapse. Interestingly, we found that such results also exist in BH formation and collapse-and-dispersion cases.

The metric component $\sigma$ can be well fit by a typical log-periodic formula in discrete scale invariance systems in complex science. Starting from this result, we obtain the first approximate analytic expression for the metric components in terms of the coordinate time near the center in critical collapse.

The slopes of $(\sigma, \psi)$ vs. $\ln \tau$ are at the same order of magnitude as those in BH formation, respectively. Considering that the values of the slopes in BH formation can be obtained analytically, we give one rough analytic explanation to these slopes in critical collapse for the first time.

(iv) Critical collapse vs. BH formation. These two types of collapse share common features, since they are both on the dynamics near the singularities. On the other hand, there are some differences between them: the scalar field and gravity are weaker in critical collapse than in BH formation.

(v) The confrontation of the BKL conjecture in critical collapse. The singularity formed in critical collapse is much weaker than those in BHs. As a consequence, the metric components and the scalar field in critical collapse are described by a new solution, rather than the Kasner solution. So the BKL conjecture may not apply to critical collapse.

(vi) Scalar field in critical collapse vs. standing waves. In critical collapse, the scalar field is reflected at the center and has DSS pattern. With this picture in mind, we speculated that the scalar field in critical collapse may be a special standing wave.

C. Discussions

(i) Further studies on the interplay between critical collapse and complex science. The interplay between critical collapse and complex science deserves to be explored further, which is meaningful for both sides.

In critical collapse, gravity and reflections at the center compete and compromise. Similar mechanisms widely exist in other branches of complex science [52].

(ii) Further studies on the BKL conjecture in critical collapse. The BKL conjecture may still be a guiding principle to achieve deeper understanding on critical collapse.

(iii) More attention to mathematics. The dynamics of critical collapse is largely determined by the equations of motion. Therefore, it may be instructive to examine the structure of the partial differential equations.
Critical collapse is an interdisciplinary subject, connecting gravitation (including BH physics and spacetime singularities), complex science, and partial differential equations, etc. Studies in critical collapse can enrich each of these, and deserve further efforts.

ACKNOWLEDGMENTS

The authors are grateful to Zhoujian Cao, David Hilditch, Li-Wei Ji, Prashant Kocherlakota, Yun-Kau Lau, Junbin Li, Daoyan Wang, and Xuefeng Zhang for useful discussions, and Beijing Normal University and Sun Yat-sen University for hospitality. This work is supported in part by the National Natural Science Foundation of China (NSFC) under grant No.11575083, and Shandong Province Natural Science Foundation under grant No.ZR201709220395.

Appendix: Derivations of $m_t$ and $m_x$

Taking the first-order temporal derivative on Eq. (7), one obtains

$$-\frac{m_t}{r} + \frac{mr_t}{r^2} = e^{2\sigma}[-r_t(r, t \sigma_{,t} + r, \tau t) + r_x (r, x \sigma_{,x} + r, x_t)].$$  \hspace{1cm} \text{(A.1)}

Subtraction of Eq. (8) from Eq. (6) yields

$$r, t \sigma_{,t} + r, \tau t = -2\pi r (\psi^2_{,t} + \psi^2_{,x}) - e^{-2\sigma} \frac{m}{r^2} - r, x \sigma_{,x}.$$  \hspace{1cm} \text{(A.2)}

Rewrite the constraint equation (5) as

$$r, x \sigma_{,x} + r, x_t = -4\pi r \psi_{,t} \psi_{,x} - r, t \sigma_{,x}.$$  \hspace{1cm} \text{(A.3)}

Substitutions of Eqs. (A.2) and (A.3) into (A.1) yield Eq. (10):

$$m_t = 4\pi r^2 (r, t T^t_t - r, x T^x_t)$$
$$= 4\pi r^2 e^{2\sigma} \left[ -\frac{1}{2} r, t (\psi^2_{,t} + \psi^2_{,x}) + r, x \psi_{,t} \psi_{,x} \right].$$  \hspace{1cm} \text{(A.4)}

Taking the first-order spatial derivative on Eq. (7), one obtains

$$-\frac{m_x}{r} + \frac{mr_x}{r^2} = e^{2\sigma}[-r, t (r, t \sigma_{,x} + r, x t) + r, x (r, x \sigma_{,t} + r, x_x)].$$  \hspace{1cm} \text{(A.5)}

Addition of Eqs. (6) and (8) yields

$$r, x \sigma_{,x} + r, x_x = -2\pi r (\psi^2_{,t} + \psi^2_{,x}) + e^{-2\sigma} \frac{m}{r^2} - r, t \sigma_{,t}.$$  \hspace{1cm} \text{(A.6)}

Rewrite the constraint equation (5) as

$$r, t \sigma_{,x} + r, x_t = -4\pi r \psi_{,t} \psi_{,x} - r, t \sigma_{,x}.$$  \hspace{1cm} \text{(A.7)}

Substitutions of Eqs. (A.6) and (A.7) into (A.5) yield Eq. (11):

$$m_x = 4\pi r^2 (r, x T^x_x - r, t T^t_t)$$
$$= 4\pi r^2 e^{2\sigma} \left[ \frac{1}{2} r, x (\psi^2_{,t} + \psi^2_{,x}) - r, t \psi_{,t} \psi_{,x} \right].$$  \hspace{1cm} \text{(A.8)}

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