AXIOMATICATION OF IF-THEN-ELSE OVER MONOIDS OF POSSIBLY NON-HALTING PROGRAMS AND TESTS

GAYATRI PANICKER, K. V. KRISHNA, AND PURANDAR BHADURI

ABSTRACT. In order to study the axiomatization of the if-then-else construct over possibly non-halting programs and tests, the notion of \( C \)-sets was introduced in the literature by considering the tests from an abstract \( C \)-algebra. This paper extends the notion of \( C \)-sets to \( C \)-monoids which include the composition of programs as well as composition of programs with tests. For the class of \( C \)-monoids where the \( C \)-algebras are adas a canonical representation in terms of functional \( C \)-monoids is obtained.

INTRODUCTION

The algebraic properties of the program construct if-then-else have been studied in great detail under various contexts. For example, in \[8,17,21\], the authors investigated on axiom schema for determination of the semantic equivalence between the conditional expressions. The authors in \[2,5,18\] studied complete proof systems for various versions of if-then-else. While a transformational characterization of if-then-else was given in \[16\], an axiomatization of equality test algebras was considered in \[9,20\]. In \[11,22\], if-then-else was studied as an action of Boolean algebra on a set. Due to their close relation with program features, functions have been canonical models for studies on algebraic semantics of programs.

In \[12\] Kennison defined comparison algebras as those equipped with a quaternary operation \( C(s,t,u,v) \) satisfying certain identities modelling the equality test. He also showed that such algebras are simple if and only if \( C \) is the direct comparison operation \( C_0 \) given by \( C_0(s,t,u,v) \) taking value \( u \) if \( s = t \) and \( v \) otherwise. This was extended by Stokes in \[23\] to semigroups and monoids. He showed that every comparison semigroup (monoid) is embeddable in the comparison semigroup (monoid) \( T(X) \) of all total functions \( X \rightarrow X \), for some set \( X \). He also obtained a similar result in terms of partial functions \( X \rightarrow X \). In \[10\] Jackson and Stokes gave a complete axiomatization of if-then-else over halting programs and tests. They also modelled composition of functions and of functions with predicates and called this object a \( B \)-monoid and further showed that the more natural setting of only considering composition of functions would not admit a finite axiomatization. They proved that every \( B \)-monoid is embeddable in a functional \( B \)-monoid comprising total functions and halting tests and thus achieved a Cayley-type theorem for the class of \( B \)-monoids. The work listed above predominantly considered the case where the tests are halting and drawn from a Boolean algebra. A natural interest is to study non-halting tests and programs.

2010 Mathematics Subject Classification. 08A70, 03G25 and 68N15.
Key words and phrases. Axiomatization, if-then-else, non-halting programs, \( C \)-algebra.
There are multiple studies (e.g., see [3, 7, 13, 14]) on extending two-valued Boolean logic to three-valued logic. However McCarthy’s logic (cf. [17]) is distinct in that it models the short-circuit evaluation exhibited by programming languages that evaluate expressions in sequential order, from left to right. In [6], Guzmán and Squier gave a complete axiomatization of McCarthy’s three-valued logic and called the corresponding algebra a $C$-algebra, or the algebra of conditional logic. While studying if-then-else algebras in [15], Manes defined an ada (Algebra of Disjoint Alternatives) which is essentially a $C$-algebra equipped with an oracle for the halting problem.

Jackson and Stokes in [11] studied the algebraic theory of computable functions, which can be viewed as possibly non-halting programs, together with composition, if-then-else and while-do. In this work they assumed that the tests form a Boolean algebra. Further, they demonstrated how an algebra of non-halting tests could be constructed from Boolean tests in their setting. Jackson and Stokes proposed an alternative approach by considering an abstract collection of non-halting tests and posed the following problem:

Characterize the algebras of computable functions associated with an abstract $C$-algebra of non-halting tests.

The authors in [19] have approached the problem by adopting the approach of Jackson and Stokes in [10]. The notion of a $C$-set was introduced through which a complete axiomatization for if-then-else over a class of possibly non-halting programs and tests, where tests are drawn from an ada, was provided.

In this paper, following the approach of Jackson and Stokes in [10], we extend the notion of $C$-sets to include composition of possibly non-halting programs and of these programs with possibly non-halting tests. This object is termed a $C$-monoid and we show that every $C$-monoid where the tests are drawn from an ada is embeddable in a canonical model of $C$-monoids, viz., functional $C$-monoids. The organisation of the paper is as follows. In Section 1 we provide necessary background material including the notion of $C$-sets and their properties. We introduce the notion of a $C$-monoid in Section 2 and give various examples thereof. In Section 3 we delineate the procedure to achieve a Cayley-type theorem and embed every $C$-monoid where the tests are drawn from an ada into a functional $C$-monoid. We conclude the paper in Section 4.

1. Preliminaries

In this section we present the necessary background material. First we recall the concept of $B$-sets. The notion of a $B$-set was introduced by Bergman in [4] and elucidated by Jackson and Stokes in [10] to study the theory of halting programs equipped with the operation of if-then-else.

**Definition 1.1.** Let $\langle Q, \lor, \land, \neg, T, F \rangle$ be a Boolean algebra and $S$ be a set. A $B$-set is a pair $(S, Q)$, equipped with a function $\eta : Q \times S \times S \rightarrow S$, called $B$-action, where $\eta(\alpha, a, b)$ is denoted by $\alpha[a, b]$, read “if $\alpha$ then $a$ else $b$”, that satisfies the following axioms for all $\alpha, \beta \in Q$ and $a, b, c \in S$:
In [10] Jackson and Stokes also considered the case of modelling if-then-else over a collection of programs with composition by including an operation to capture the composition of programs with tests.

**Definition 1.2.** Let \((S, \cdot)\) be a monoid with identity element 1 and \((S, Q)\) be a \(B\)-set. The pair \((S, Q)\) equipped with a function \(\circ: S \times Q \to Q\) is said to be a \(B\)-monoid if it satisfies the following axioms for all \(a, b, c \in S\) and \(\alpha, \beta \in Q\):

\[
\begin{align*}
(1) & \quad \alpha[a, a] = a \\
(2) & \quad \alpha[\alpha[a, b], c] = \alpha[a, c] \\
(3) & \quad \alpha[a, \alpha[b, c]] = \alpha[a, c] \\
(4) & \quad F[a, b] = b \\
(5) & \quad -\alpha[a, b] = \alpha[b, a] \\
(6) & \quad (\alpha \land \beta)[a, b] = \alpha[\beta[a, b], b]
\end{align*}
\]

Since functions on sets model programs, standard examples of \(B\)-sets and \(B\)-monoids come from functions on sets. Let \(X\) and \(Y\) be two sets. The set of all functions \(X \to Y\) will be denoted by \(Y^X\) while the set of all functions \(X \to X\) will be denoted by \(T(X)\).

Let \(2\) be the two element Boolean algebra. For any set \(X\) the pair \((T(X), 2^X)\) is a \(B\)-set with the following action for all \(f \in T(X)\) and \(\alpha \in 2^X\):

\[
\alpha[g, h](x) = \begin{cases} 
  g(x), & \text{if } \alpha(x) = T; \\
  h(x), & \text{if } \alpha(x) = F.
\end{cases}
\]

Note that \(T(X)\) is a monoid with respect to usual composition of mappings. The \(B\)-set \((T(X), 2^X)\) equipped with the operation \(\circ\) defined by

\[
(f \circ \alpha) = \{x \in X : f(x) \in \alpha\}
\]

for all \(f \in T(X)\) and \(\alpha \in 2^X\) is a \(B\)-monoid. In fact, Jackson and Stokes obtained the following theorem.

**Theorem 1.3 ([10]).** Every \(B\)-monoid \((S, Q)\) is embeddable as a two-sorted algebra into the \(B\)-monoid \((T(X), 2^X)\) for some set \(X\). Furthermore, if \(S\) and \(Q\) are finite, then \(X\) is finite.

In [6] Guzmán and Squier introduced the notion of a \(C\)-algebra as the algebra corresponding to McCarthy’s three-valued logic (cf. [17]).
Definition 1.4. A $C$-algebra is an algebra $\langle M, \lor, \land, \neg \rangle$ of type $(2, 2, 1)$, which satisfies the following axioms for all $\alpha, \beta, \gamma \in M$:

\begin{align*}
\neg \neg \alpha &= \alpha \\
\neg(\alpha \land \beta) &= \neg \alpha \lor \neg \beta \\
(\alpha \land \beta) \land \gamma &= \alpha \land (\beta \land \gamma) \\
\alpha \land (\beta \lor \gamma) &= (\alpha \land \beta) \lor (\alpha \land \gamma) \\
(\alpha \lor \beta) \land \gamma &= (\alpha \land \gamma) \lor (\neg \alpha \land \beta \land \gamma) \\
\alpha \lor (\alpha \land \beta) &= \alpha
\end{align*}

It is easy to see that every Boolean algebra is a $C$-algebra. In particular, $\mathbb{2}$ is a $C$-algebra. Let $\mathbb{3}$ denote the $C$-algebra with the universe $\{T, F, U\}$ and the following operations.

| $\neg$ | $\land$ | $\lor$ |
|--------|--------|--------|
| T      | T, F   | T, T   |
| F      | F, T   | F, F   |
| U      | U, U   | U, U   |

In fact, the $C$-algebra $\mathbb{3}$ is the McCarthy’s three-valued logic.

In view of the fact that the class of $C$-algebras is a variety, for any set $X$, $\mathbb{3}^X$ is a $C$-algebra with the operations defined pointwise. Guzmán and Squier in [6] showed that elements of $\mathbb{3}^X$ along with the $C$-algebra operations may be viewed in terms of pairs of sets. This is a pair $(A, B)$ where $A, B \subseteq X$ and $A \cap B = \emptyset$. Akin to the well-known correlation between $2^X$ and the power set $\mathcal{P}(X)$ of $X$, for any element $\alpha \in \mathbb{3}^X$, associate the pair of sets $(\alpha^{-1}(T), \alpha^{-1}(F))$. Conversely, for any pair of sets $(A, B)$ where $A, B \subseteq X$ and $A \cap B = \emptyset$ associate the function $\alpha$ where $\alpha(x) = T$ if $x \in A$, $\alpha(x) = F$ if $x \in B$ and $\alpha(x) = U$ otherwise. With this correlation, the operations can be expressed as follows:

\begin{align*}
\neg(A_1, A_2) &= (A_2, A_1) \\
(A_1, A_2) \land (B_1, B_2) &= (A_1 \cap B_1, A_2 \cup (A_1 \cap B_2)) \\
(A_1, A_2) \lor (B_1, B_2) &= ((A_1 \cup (A_2 \cap B_1), A_2 \cap B_2)
\end{align*}

Notation 1.5. We use $M$ to denote an arbitrary $C$-algebra. By a $C$-algebra with $T, F, U$ we mean a $C$-algebra with nullary operations $T, F, U$, where $T$ is the (unique) left-identity (and right-identity) for $\land$, $F$ is the (unique) left-identity (and right-identity) for $\lor$ and $U$ is the (unique) fixed point for $\neg$. Note that $U$ is also a left-zero for both $\land$ and $\lor$ while $\neg$ is a left-zero for $\land$.

In [15] Manes introduced the notion of ada (algebra of disjoint alternatives) which is a $C$-algebra equipped with an oracle for the halting problem.
Definition 1.6. An ada is a C-algebra $M$ with $T,F,U$ equipped with an additional unary operation $\downarrow$ subject to the following equations for all $\alpha, \beta \in M$:

\begin{align*}
F \downarrow &= F \\
U \downarrow &= F \\
T \downarrow &= T \\
\alpha \land \beta \downarrow &= \alpha \land (\alpha \land \beta) \downarrow \\
\alpha \downarrow \lor \neg(\alpha \downarrow) &= T \\
\alpha &= \alpha \downarrow \lor \alpha
\end{align*}

The C-algebra $\mathbb{3}$ with the unary operation $(\ )\downarrow$ defined by (22), (23) and (24) forms an ada. This ada will also be denoted by $\mathbb{3}$. One may easily resolve the notation overloading – whether $\mathbb{3}$ is a C-algebra or an ada – depending on the context. In [15] Manes showed that the ada $\mathbb{3}$ is the only subdirectly irreducible ada. For any set $X$, $\mathbb{3}^X$ is an ada with operations defined pointwise. Note that the ada $\mathbb{3}$ is also simple.

We use the following notations related to sets and equivalence relations.

Notation 1.7.

1. Let $X$ be a set and $\bot \notin X$. The pointed set $X \cup \{\bot\}$ with base point $\bot$ is denoted by $X_\bot$.

2. The set of all functions on $X_\bot$ which fix $\bot$ is denoted by $T\sigma(X_\bot)$, i.e.,

$$T\sigma(X_\bot) = \{f \in T(X_\bot) : f(\bot) = \bot\}.$$ 

3. Under an equivalence relation $\sigma$ on a set $A$, the equivalence class of an element $p \in A$ will be denoted by $p^\sigma$. Within a given context, if there is no ambiguity, we may simply denote the equivalence class by $p$.

In order to axiomatize if-then-else over possibly non-halting programmes and tests, in [19], Panicker et al. considered the tests from a C-algebra and introduced the notion of C-sets. We now recall the notion of a C-set.

Definition 1.8. Let $S_\bot$ be a pointed set with base point $\bot$ and $M$ be a C-algebra with $T,F,U$. The pair $(S_\bot, M)$ equipped with an action

$$\llbracket \cdot, \cdot \rrbracket : M \times S_\bot \times S_\bot \to S_\bot$$

is called a C-set if it satisfies the following axioms for all $\alpha, \beta \in M$ and $s, t, u, v \in S_\bot$:

\begin{align*}
(28) \quad U[s,t] &= \bot & (U\text{-axiom}) \\
(29) \quad F[s,t] &= t & (F\text{-axiom}) \\
(30) \quad (\neg \alpha)[s,t] &= \alpha[t,s] & (\neg\text{-axiom}) \\
(31) \quad \alpha[\alpha[s,t],u] &= \alpha[s,u] & \text{(positive redundancy)} \\
(32) \quad \alpha[s,\alpha[t,u]] &= \alpha[s,u] & \text{(negative redundancy)} \\
(33) \quad (\alpha \land \beta)[s,t] &= \alpha[\beta[s,t],t] & (\land\text{-axiom}) \\
(34) \quad \alpha[\beta[s,t],\beta[u,v]] &= \beta[\alpha[s,u],\alpha[t,v]] & \text{(premise interchange)} \\
(35) \quad \alpha[s,t] = \alpha[t,t] \Rightarrow (\alpha \land \beta)[s,t] = (\alpha \land \beta)[t,t] & (\land\text{-compatibility})
\end{align*}
Let $M$ be a $C$-algebra with $T,F,U$ treated as a pointed set with base point $U$. The pair $(M,M)$ is a $C$-set under the following action for all $\alpha,\beta,\gamma \in M$:

$$\alpha[\beta,\gamma] = (\alpha \land \beta) \lor (\neg \alpha \land \gamma).$$

We denote the action of the $C$-set $(M,M)$ by $\cdot$. In [19], Panicker et al. showed that the axiomatization is complete for the class of $C$-sets $(S_\perp,M)$ when $M$ is an ada. In that connection, they obtained some properties of $C$-sets. Amongst, in Proposition 1.9 below, we list certain properties related to congruences which are useful in the present work. Viewing $C$-sets as two-sorted algebras, a congruence of a $C$-set is a pair $(\sigma,\tau)$, where $\sigma$ is an equivalence relation on $S_\perp$ and $\tau$ is a congruence on the ada $M$ such that $(s,t),(u,v) \in \sigma$ and $(\alpha,\beta) \in \tau$ imply that $(\alpha[s,u],\beta[t,v]) \in \sigma$.

**Proposition 1.9** (19). Let $(S_\perp,M)$ be a $C$-set where $M$ is an ada. For each maximal congruence $\theta$ on $M$, let $E_{\theta}$ be the relation on $S_\perp$ given by

$$E_{\theta} = \{(s,t) \in S_\perp \times S_\perp : \beta[s,t] = \beta[t,t] \text{ for some } \beta \in T^\theta\}.$$ 

Then we have the following properties:

(i) For $\alpha \in M$ and $s,t \in S_\perp$, if $(\alpha,\beta) \in \theta$ then according to $\beta = T,F$ or $U$, we have $(\alpha[s,t],s) \in E_{\theta}$, $(\alpha[s,t],t) \in E_{\theta}$ or $(\alpha[s,t],\perp) \in E_{\theta}$, respectively.

(ii) The pair $(E_{\theta},\theta)$ is a $C$-set congruence.

(iii) For the $C$-set $(M,M)$ the equivalence $E_{\theta}$ on $M$, denoted by $E_{\theta,M}$, is a subset of $\theta$.

(iv) $\bigcap_{\theta} E_{\theta} = \Delta_{S_\perp}$, where $\theta$ ranges over all maximal congruences on $M$.

(v) The intersection of all maximal congruences on $M$ is trivial, that is $\bigcap_{\theta} = \Delta_{M}$ where $\theta$ ranges over all maximal congruences on $M$.

For more details on $C$-sets one may refer to [19].

2. $C$-MONOIDS

We now include the case where the composition of two elements of the base set and of an element with a predicate is allowed. Our motivating example is $(T_o(X_\perp),3^X)$, where $T_o(X_\perp)$ is considered to be a monoid with zero by equipping it with composition of functions. The composition will be written from left to right, i.e., $(f \cdot g)(x) = g(f(x))$. The monoid identity in $T_o(X_\perp)$ is the identity function $id_{X_\perp}$ and the zero element is $\perp_{\perp}$, the constant function taking the value $\perp$. We also include composition of functions with predicates via the natural interpretation given by the following for all $f \in T_o(X_\perp)$ and $\alpha \in 3^X$:

$$\alpha \cdot (f \circ \alpha)(x) = \begin{cases} T, & \text{if } \alpha(f(x)) = T; \\ F, & \text{if } \alpha(f(x)) = F; \\ U, & \text{otherwise.} \end{cases}$$

(36)

Note that if the composition takes value $T$ or $F$ at some point $x \in X_\perp$ then as $\alpha \in 3^X$ this implies that $f(x) \neq \perp$.

With this example in mind we define a $C$-monoid as follows.
**Definition 2.1.** Let \((S_\bot, \cdot)\) be a monoid with identity element 1 and zero element \(\bot\) where \(\bot \cdot s = \bot = s \cdot \bot\). Let \(M\) be a \(C\)-algebra and \((S_\bot, M)\) be a \(C\)-set with \(\bot\) as the base point of the pointed set \(S_\bot\). The pair \((S_\bot, M)\) equipped with a function
\[
\circ : S_\bot \times M \rightarrow M
\]
is said to be a **\(C\)-monoid** if it satisfies the following axioms for all \(s, t, r, u \in S_\bot\) and \(\alpha, \beta \in M\):

\begin{align*}
(37) \quad &\bot \circ \alpha = U \\
(38) \quad &t \circ U = U \\
(39) \quad &1 \circ \alpha = \alpha \\
(40) \quad &s \circ (\neg \alpha) = \neg (s \circ \alpha) \\
(41) \quad &s \circ (\alpha \land \beta) = (s \circ \alpha) \land (s \circ \beta) \\
(42) \quad &\alpha[s, t] \cdot u = \alpha[s \cdot u, t \cdot u] \\
(43) \quad &r \circ \alpha[s, t] = (r \circ \alpha)[r \cdot s, r \cdot t] \\
(44) \quad &\alpha[s, t] \circ \beta = \alpha[s \circ \beta, t \circ \beta]
\end{align*}

(\(\bot\)-o-axiom)

(\(U\)-o-axiom)

(1-o-axiom)

(\(\neg\)-o-axiom)

(\(\land\)-o-axiom)

(semigroup action)

(right composition)

(left composition)

(\(\circ\)-interchange)

The following are examples of \(C\)-monoids.

**Example 2.2.** Recall from [19] that the pair \((T_0(X_\bot), 3^X)\) equipped with the action \((40)\) for all \(f, g \in T_0(X_\bot)\) and \(\alpha \in 3^X\) is a \(C\)-set. Note that \(T_0(X_\bot)\) is treated as a pointed set with base point \(\zeta_\bot\).

\[
\alpha[f, g](x) = \begin{cases} f(x), & \text{if } \alpha(x) = T; \\ g(x), & \text{if } \alpha(x) = F; \\ \bot, & \text{otherwise}. \end{cases}
\]

The \(C\)-set \((T_0(X_\bot), 3^X)\) equipped with the operation \(\circ\) given in \((40)\) and with \(T_0(X_\bot)\) treated as a monoid with zero is in fact a \(C\)-monoid. For verification of axioms \((37) - (45)\) refer to Appendix A.1. Such \(C\)-monoids will be called **functional \(C\)-monoids**.

**Example 2.3.** Let \(S_\bot\) be a non-trivial monoid with identity 1 and zero \(\bot\) and no non-zero zero-divisors, i.e., \(s \cdot t = \bot \Rightarrow s = \bot = t = \bot\). Then \(S_\bot^X\) is also a monoid with zero for any set \(X\) with operations defined pointwise. For \(f, g \in S_\bot^X\) define \((f \cdot g)(x) = f(x) \cdot g(x)\). The identity of \(S_\bot^X\) is the constant function \(\zeta_\bot\) taking the value \(\bot\). The zero and base point of \(S_\bot^X\) is the constant function \(\zeta_\bot\) taking the value \(\bot\). Recall from [19] that the pair \((S_\bot^X, 3^X)\) is a \(C\)-set under action \((40)\). In fact it is also a \(C\)-monoid with \(\circ\) defined as follows for all \(f \in S_\bot^X\) and \(\alpha \in 3^X\):

\[
(f \circ \alpha)(x) = \begin{cases} \alpha(x), & \text{if } f(x) \neq \bot; \\ U, & \text{otherwise}. \end{cases}
\]

For verification of axioms \((37) - (45)\) refer to Appendix A.2.

**Example 2.4.** Let \(S_\bot\) be a non-trivial monoid with zero and no non-zero zero-divisors, i.e., \(s \cdot t = \bot \Rightarrow s = \bot = t = \bot\). In [19] the authors showed that for any
pointed set \( S_\perp \) with base point \( \perp \), the pair \((S_\perp, 3)\) is a (basic) \( C \)-set with respect to the following action for all \( a, b \in S_\perp \) and \( \alpha \in 3 \):

\[
\alpha[a, b] = \begin{cases} 
a, & \text{if } \alpha = T; 
b, & \text{if } \alpha = F; 
\perp, & \text{if } \alpha = U. 
\end{cases}
\]

This basic \( C \)-set \((S_\perp, 3)\) equipped with \( \circ : S_\perp \times 3 \to 3 \) defined below for all \( s \in S_\perp \) and \( \alpha \in 3 \) is a \( C \)-monoid.

\[
s \circ \alpha = \begin{cases} 
\alpha, & \text{if } s \neq \perp; 
U, & \text{if } s = \perp.
\end{cases}
\]

For verification of axioms (37) – (45) refer to Appendix A.3.

### 3. Representation of a class of \( C \)-monoids

In this section we obtain a Cayley-type theorem for a class of \( C \)-monoids as stated in the following main theorem.

**Theorem 3.1.** Every \( C \)-monoid \((S_\perp, M)\) where \( M \) is an ada is embeddable in the \( C \)-monoid \((\text{T}_0(X_\perp), 3^X)\) for some set \( X \). Moreover, if both \( S_\perp \) and \( M \) are finite then so is \( X \).

**Sketch of the proof.** For each maximal congruence \( \theta \) of \( M \), we consider the \( C \)-set congruence \((E_\theta, \theta)\) of \((S_\perp, M)\). Corresponding to each such congruence, we construct a homomorphism of \( C \)-monoids from \((S_\perp, M)\) to the functional \( C \)-monoid over the set \( S_\perp/E_\theta \). This collection of homomorphisms has the property that every distinct pair of elements from each component of the \( C \)-monoid will be separated by some homomorphism from this collection. We then set \( X \) to be the disjoint union of \( S_\perp/E_\theta \)'s excluding the equivalence class \( \perp/E_\theta \). We complete the proof by constructing a monomorphism – by pasting together each of the individual homomorphisms from the collection defined earlier – from the \( C \)-monoid \((S_\perp, M)\) to the functional \( C \)-monoid over the pointed set \( X_\perp \) with a new base point \( \perp \).

The proof of Theorem 3.1 will be developed through various sub sections. First in Subsection \( 3.1 \) we study some properties of maximal congruences of adas. We then present a collection of homomorphisms which separate every distinct pair of elements from each component of \((S_\perp, M)\) in Subsection \( 3.2 \). In Subsection \( 3.3 \) we construct the required functional \( C \)-monoid and establish an embedding from \((S_\perp, M)\). Finally, we consolidate the proof in Subsection \( 3.4 \).

In what follows \((S_\perp, M)\) is a \( C \)-monoid with \( M \) as an ada. Let \( \theta \) be a maximal congruence on \( M \) and \( E_\theta \) be the equivalence on \( S_\perp \) as defined in Proposition 1.9 so that the pair \((E_\theta, \theta)\) is a congruence on \((S_\perp, M)\). We denote the quotient set \( S_\perp/E_\theta \) by \( S_\theta \) and use \( \overline{S_\theta} \) to denote the set \( S_\theta \setminus \{\perp/E_\theta\} \). Further, we use \( q, s, t, u, v \) to denote elements of \( S_\perp \) and \( \alpha, \beta, \gamma \) to denote elements of the ada \( M \).

#### 3.1. Properties of maximal congruences

The following properties are useful in proving the main theorem.

**Proposition 3.2.** No two elements of \( \{T, F, U\} \) are related under \( \theta \). That is, \((T, F) \notin \theta, (T, U) \notin \theta \) and \((F, U) \notin \theta \).
Proof. If \((T, F) \in \theta\) then we show that \(\theta = M \times M\); contradicting the maximality of \(\theta\). Suppose \((T, F) \in \theta\) and let \(\alpha, \beta \in M\). Then \((T, F), (\alpha, \alpha) \in \theta \Rightarrow (T \wedge \alpha, F \wedge \alpha) \in \theta\) that is \((\alpha, F) \in \theta\). Similarly \((\beta, F) \in \theta\) and so using the symmetry and transitivity of \(\theta\) we have \((\alpha, \beta) \in \theta\) and consequently \(\theta = M \times M\). The proof of \((T, U) \notin \theta\) follows along similar lines. Finally since \((F, U) \in \theta \iff (T, U) \in \theta\), the result follows. \(\square\)

Proposition 3.3. For each \(q \in S_{\bot}\), we have

(i) \((q \circ T)[q, \bot] = q\).

(ii) \((q \circ T, F) \notin \theta\).

(iii) \((q \circ T, U) \in \theta \iff (q, \bot) \in E_{\theta}\).

(iv) \((q \circ T, T) \in \theta \iff (q \circ F, F) \in \theta \iff (q, \bot) \notin E_{\theta}\).

(v) \((s, t) \in E_{\theta} \Rightarrow (s \circ \alpha, t \circ \alpha) \in \theta\) for all \(\alpha \in M\).

(vi) \((1, \bot) \notin E_{\theta}\).

Proof.

(i) Using (11) we have \(q = q \cdot 1 = q \cdot T[1, \bot] = (q \circ T)[q \cdot 1, q \cdot \bot] = (q \circ T)[q, \bot]\).

(ii) We prove the result by contradiction. Suppose \((q \circ T, F) \in \theta\). Using the fact that \(\theta\) is a congruence on \(M\) and (11) we have \((q \circ T, F) \in \theta \Rightarrow (\neg(q \circ T), \neg F) \in \theta \Rightarrow (q \circ F, T) \in \theta\). Similarly using the fact that \(\theta\) is a congruence, (11) and (10) we have \(((q \circ F) \lor (q \circ T), (T \lor F)) \in \theta \Rightarrow (q \circ T, T) \in \theta\) and \((q \circ T, T) \in \theta\). From the symmetry and transitivity of \(\theta\) it follows that \((T, F) \in \theta\), a contradiction by Proposition 3.2. The result follows.

(iii) \((\Rightarrow):\) Let \((q \circ T, U) \in \theta\). Using Proposition 1.9(i) we can say that for any choice of \(s, t \in S_{\bot}\) we have \(((q \circ T)[s, t], \bot) \in E_{\theta}\). On choosing \(s = q, t = \bot\) and using Proposition 3.3(i) we have \(((q \circ T)[q, \bot], \bot) \in E_{\theta}\) that is \((q, \bot) \in E_{\theta}\) as desired.

(\(\Leftarrow\)) First note that, for \(\alpha \in M\),

\[(7)\]

(cf. 19 Proposition 2.8(1))]. Now assume that \((q, \bot) \in E_{\theta}\). Then there exists \(\beta \in T\) such that \(\beta[q, \bot] = \beta[\bot, \bot]\). However, by (17), we have \(\beta[q, \bot] = \bot\). Thus, \(\beta[q, \bot] \circ T = \bot \circ T\) so that \(\beta[q \circ T, \bot] \circ T = \bot\) (using (7) and (15)). Consequently, using (17) on \((M, M), we have \(\beta[q \circ T, U] = \beta[U, U]\). Hence \((q \circ T, U) \in E_{\theta,M}\) and so from Proposition 1.9(iii), \((q \circ T, U) \in \theta\).

Thus \((q \circ T, U) \in \theta \iff (q, \bot) \notin E_{\theta}\).

(iv) We first show that \((q \circ T, T) \in \theta \Rightarrow (q \circ F, F) \in \theta\) by making use of the substitution property of the congruence \(\theta\) with respect to \(\neg\), the fact that \(\neg\) is an involution and (10). Thus \((q \circ T, T) \in \theta \Rightarrow (\neg(q \circ T), \neg T) \in \theta \Rightarrow (q \circ (\neg T), \neg T) \in \theta \Rightarrow (q \circ F, F) \in \theta\). Using Proposition 3.2 Proposition 3.3(ii) and Proposition 3.3(iii) we show the equivalence \((q \circ T, T) \in \theta \iff (q, \bot) \notin E_{\theta}\). We have \((q \circ T, T) \in \theta \Rightarrow (q \circ T, U) \notin \theta \Rightarrow (q, \bot) \notin E_{\theta}\). Conversely \((q, \bot) \notin E_{\theta} \Rightarrow (q \circ T, U) \notin \theta\). Using Proposition 3.3(ii) it follows that \((q \circ T, F) \notin \theta\). Since \(\theta\) is a maximal congruence the only remaining possibility is that \((q \circ T, T) \in \theta\) which completes the proof.

(v) Consider \((s, t) \in E_{\theta} \text{ and } \alpha \in M\). Then there exists \(\beta \in T\) such that \(\beta[s, t] = \beta[t, t]\). Thus \(\beta[s, t] \circ \alpha = \beta[t, t] \circ \alpha\). Using (15) we have \(\beta[s \circ \alpha, t \circ \alpha] = \beta[t \circ \alpha, t \circ \alpha]\) from which it follows that \((s \circ \alpha, t \circ \alpha) \in E_{\theta,M} \subseteq \theta\) by Proposition 1.9(iii).
(vi) Suppose that \((1, \perp) \in E_\theta\). Using Proposition 3.3(v), 3.9, 3.7, we have \((1, \perp) \in E_\theta \Rightarrow (1 \circ T, \perp \circ T) \in \theta \) and so \((T, U) \in \theta \) a contradiction by Proposition 3.2.

\[\square\]

3.2. A class of homomorphisms separating pairs of elements. For each maximal congruence \(\theta\) on \(M\), in this subsection, we present homomorphisms \(\phi_\theta : S_\perp \to T_\theta(S_{\perp \theta})\) and \(\rho_\theta : M \to 3^{S_\perp}\). Then we establish that \((\phi_\theta, \rho_\theta)\) is a homomorphism from \((S_\perp, M)\) to the functional \(C\)-monoid \((T_\theta(S_{\perp \theta}), 3^{S_\perp})\). Further, we ascertain that every pair of elements in \(S_\perp\) (or \(M\)) are separated by some \(\phi_\theta\) (or \(\rho_\theta\)).

Proposition 3.4. The function \(\phi_\theta : S_\perp \to T_\theta(S_{\perp \theta})\) given by \(\phi_\theta(s) = \psi_\theta(s)\), where \(\psi_\theta(s) = t \cdot \overline{s^o}\), is a monoid homomorphism that maps the zero (and base point) of \(S_\perp\) to that of \(T_\theta(S_{\perp \theta})\), that is \(\perp \mapsto \bar{\perp}\).

Proof. Claim: \(\phi_\theta\) is well-defined. It suffices to show that \(\psi_\theta(s)\) is well-defined and that \(\psi_\theta(s) \in T_\theta(S_{\perp \theta})\), that is \(\psi_\theta(\perp) = \bar{\perp}\). In order to show the well-definedness of \(\psi_\theta\), consider \(\bar{t} = \bar{\theta}\) that is \((u, t) \in E_\theta\). Then there exists \(\beta \in T\) such that \(\beta[u, t] = \beta[t, t]\). Consequently

\[
\beta[u \cdot s, t \cdot s] = \beta[u, t] \cdot s = \beta[t, t] \cdot s = \beta[t \cdot s, t \cdot s]
\]

Thus \((u \cdot s, t \cdot s) \in E_\theta\) and so \(\psi_\theta(\bar{u}) = \psi_\theta(\bar{t})\). Also \(\psi_\theta(\bar{\perp}) = \bar{\perp} \cdot \bar{s} = \bar{\perp}\). Thus \(\psi_\theta(\bar{\perp}) = \bar{\perp}\).

Claim: \(\phi_\theta(\perp) = \bar{\perp}\). We have \(\phi_\theta(\perp) = \psi_\theta(\bar{\perp})\) where \(\psi_\theta(\bar{\perp}) = \bar{\perp} \cdot \bar{\perp} = \bar{\perp}\). Thus \(\phi_\theta(\perp) = \bar{\perp}\).

Claim: \(\phi_\theta(1) = id_{S_{\perp \theta}}\). We have \(\phi_\theta(1) = \psi_\theta(1)\) where \(\psi_\theta(1) = \bar{1} \cdot \bar{1} = \bar{1}\).

Claim: \(\phi_\theta\) is a semigroup homomorphism. Consider \(\phi_\theta(s \cdot t) = \psi_\theta(s \cdot t)\) where \(\psi_\theta(s \cdot t) = u \cdot \overline{s \cdot t} = (u \cdot s) \cdot \bar{t} = \psi_\theta(u) \cdot \psi_\theta(s) = \psi_\theta(\psi_\theta(s)) = (\psi_\theta(s) \cdot \psi_\theta(s))\). Thus \(\phi_\theta(s \cdot t) = \phi_\theta(s) \cdot \phi_\theta(t)\). \[\square\]

Proposition 3.5. The function \(\rho_\theta : M \to 3^{S_\perp}\) given by

\[
\rho_\theta(\alpha) = \begin{cases} 
(S_\theta, \emptyset), & \text{if } \alpha = T; \\
(\emptyset, S_\theta), & \text{if } \alpha = F; \\
(A_\theta, B_\theta^\emptyset), & \text{otherwise}
\end{cases}
\]

where \(A_\theta = \{T^o : t \circ \alpha \in T^o\}\) and \(B_\theta^\emptyset = \{T^o : t \circ \alpha \in T^o\}\), is a homomorphism of \(C\)-algebras with \(T, F, U\).

Proof. Claim: \(\rho_\theta\) is well-defined. If \(\alpha \in \{T, F\}\) then the proof is obvious. If \(\alpha \notin \{T, F\}\) then we show that \(A_\theta \cap B_\theta^\emptyset = \emptyset\) and that \(A_\theta^\emptyset, B_\theta^\emptyset \subseteq S_\theta\), that is, \(\perp \notin A_\theta \cup B_\theta^\emptyset\). Let \(\perp \in A_\theta^\emptyset \cup B_\theta^\emptyset\). Then \(t \circ \alpha \in T^o\) and \(t \circ \alpha \in T^o\) and so \((T, F) \notin \theta\) which is a contradiction to Proposition 3.2. Using (37) we have \(\perp \circ \alpha = U \) and so if \(\perp \in A_\theta^\emptyset \cup B_\theta^\emptyset\) we have \(\perp \circ \alpha = U \in \{T^o, F^o\}\), a contradiction to Proposition 3.2. Finally we show that the image under \(\rho_\theta\) is independent of the representative of the equivalence class chosen. Using Proposition 3.3(v) we have \(\bar{\alpha} = \bar{\alpha} \Rightarrow (s \circ \alpha, t \circ \alpha) \in \theta\). The result follows.
Claim: \( \rho_0 \) preserves the constants \( T, F, U \). It is clear that \( \rho_0(T) = (S_0, \emptyset) \), \( \rho_0(F) = (0, S_0) \) and, using \([\text{I}3]\) and Proposition \([\text{I}2]\) that \( \rho_0(U) = (A_0^T, B_0^U) = (\emptyset, \emptyset) \) from which the result follows.

Claim: \( \rho_0 \) is a C-algebra homomorphism. We show that \( \rho_0(\neg) = \neg(\rho_0(\alpha)) \).

If \( \alpha \in \{ T, F \} \) the proof is obvious. Suppose that \( \alpha \notin \{ T, F \} \). Then we have the following.

\[
\rho_0(\neg) = (A_0^{\neg\alpha}, B_0^{\neg\alpha})
\]

\[
= (\{ \overline{t} : t \circ (\neg) \in \overline{T}\}, \{ \overline{t} : (t \circ (\neg)) \in \overline{T} \})
\]

\[
= (\{ \overline{t} : \neg(t \circ \alpha) \in \overline{T}\}, \{ \overline{t} : (\neg(t \circ \alpha)) \in \overline{T} \})
\]

\[
= (\{ \overline{t} : t \circ \alpha \in \overline{T}\}, \{ \overline{t} : t \circ \alpha \in \overline{T} \})
\]

Finally we show that \( \rho_0(\alpha \land \beta) = \rho_0(\alpha) \land \rho_0(\beta) \). Note that the proof of \( \rho_0(\alpha \lor \beta) = \rho_0(\alpha) \lor \rho_0(\beta) \) follows using the double negation and De Morgan’s laws, viz., \([\text{I}5]\) and \([\text{I}6]\) respectively in conjunction with the fact that \( \rho_0 \) preserves \( \neg \) and \( \land \). In order to prove that \( \rho_0(\alpha \land \beta) = \rho_0(\alpha) \land \rho_0(\beta) \) we proceed by considering the following cases.

**Case I:** \( \alpha, \beta \notin \{ T, F \} \). We have the following subcases:

**Subcase 1:** \( \alpha, \beta \notin \{ T, F \} \). Then \( \rho_0(\alpha \land \beta) = (A_0^\alpha \land \beta, B_0^{\alpha \land \beta}), \rho_0(\alpha) = (A_0^\alpha, B_0^\alpha) \) and \( \rho_0(\beta) = (A_0^\beta, B_0^\beta) \). Now \( (A_0^\alpha, B_0^\alpha) \land (A_0^\beta, B_0^\beta) = (A_0^\alpha \cap A_0^\beta, B_0^\alpha \cup (A_0^\alpha \cap B_0^\beta)) \). Thus we have to show that \( (A_0^{\alpha \land \beta}, B_0^{\alpha \land \beta}) = (A_0^\alpha \cap A_0^\beta, B_0^\alpha \cup (A_0^\alpha \cap B_0^\beta)) \).

We show that the pairs of sets are equal componentwise.

Let \( \overline{t} \in A_0^{\alpha \land \beta} \). Then \( q \circ (\alpha \land \beta) \in \overline{T} \),

\[
\Rightarrow (q \circ (\alpha \land \beta), T) \in \emptyset \quad \text{(using } [\text{I}11]\)
\]

\[
\Rightarrow ((q \circ (\alpha \land \beta)), (q \circ (\alpha \land \beta)) \land T) \in \emptyset \quad \text{(since } \theta \text{ is a congruence)}
\]

\[
\Rightarrow ((q \circ (\alpha \land \beta)), q \circ \alpha, q \circ \beta) \in \emptyset \quad \text{(using the properties of } \land)\]

\[
\Rightarrow (q \circ (\alpha \land \beta), T) \in \emptyset \quad \text{(by transitivity of } \theta)\]

so that \( \overline{t} \in A_0^\alpha \). Along similar lines one can observe that

\[
(((q \circ (\alpha \land \beta)), q \circ \beta), T \land (q \circ \beta)) \in \emptyset.
\]

Consequently \( q \circ (\alpha \land \beta), T \in \theta \) so that \( \overline{t} \in A_0^\beta \). Hence \( A_0^{\alpha \land \beta} \subseteq A_0^\alpha \cap A_0^\beta \).

For reverse inclusion let \( \overline{t} \in A_0^\alpha \cap A_0^\beta \). Then \( q \circ (\alpha, T), (q \circ (\alpha \land \beta), T) \in \theta \). Since \( \theta \) is a congruence we have \( ((q \circ (\alpha \land \beta)), T \land (q \circ \beta)) = (q \circ (\alpha \land \beta), T) = ((q \circ (\alpha \land \beta), T) \in \emptyset \text{ and so } \overline{t} \in A_0^{\alpha \land \beta} \text{. Hence } A_0^{\alpha \land \beta} = A_0^\alpha \cap A_0^\beta \).

In order to show that \( B_0^{\alpha \land \beta} \subseteq B_0^\alpha \cup (A_0^\alpha \cap B_0^\beta) \) consider \( \overline{t} \in B_0^{\alpha \land \beta} \) that is \( q \circ (\alpha \land \beta), F \in \emptyset \). Since \( \theta \) is a maximal congruence consider the following three possibilities:

\( q \circ (\alpha, F) \in \emptyset \): Then clearly \( \overline{t} \in B_0^\alpha \) and so \( \overline{t} \in B_0^\alpha \cup (A_0^\alpha \cap B_0^\beta) \).

\( q \circ (\alpha, T) \in \emptyset \): Then we have \( \overline{t} \in A_0^\alpha \). We show that \( q \circ (\alpha, F) \in \emptyset \). If this is not the case then either \( q \circ (\alpha, T) \in \emptyset \) or \( q \circ (\alpha, U) \in \emptyset \). If \( q \circ (\alpha, T) \in \emptyset \) then
since \((q \circ \alpha, T) \in \theta\) we have \((q \circ (\alpha \land \beta), T \land T) = (q \circ (\alpha \land \beta), T) \in \theta\) using (11) and the fact that \(\theta\) is a congruence. However since \((q \circ (\alpha \land \beta), F) \in \theta\) we obtain a contradiction that \((T, F) \in \theta\) (cf. Proposition 5.2). Along similar lines if \((q \circ \beta, U) \in \theta\) then as \((q \circ \alpha, T) \in \theta\) we have \((q \circ (\alpha \land \beta), U) \in \theta\) and so \((F, U) \in \theta\) a contradiction to Proposition 5.2. Hence \((q \circ \beta, F) \in \theta\) so that \(\overline{q} \in (A^\alpha_\theta \cap B^\beta_\theta) \subseteq B^\alpha_\theta \cup (A^\alpha_\theta \cap B^\beta_\theta)\).

\((q \circ \alpha, U) \in \theta\): Since \((q \circ \beta, q \circ \beta) \in \theta\) we have \((q \circ (\alpha \land \beta), U \land q \circ \beta) = (q \circ (\alpha \land \beta), U) \in \theta\) using (11) and the fact that \(\theta\) is a congruence. However since \((q \circ (\alpha \land \beta), F) \in \theta\) we have \((F, U) \in \theta\) a contradiction to Proposition 5.2. Thus this case cannot occur.

To show the reverse inclusion let \(\overline{q} \in B^\alpha_\theta \cup (A^\alpha_\theta \cap B^\beta_\theta)\) that is \(\overline{q} \in B^\alpha_\theta\) or \(\overline{q} \in A^\alpha_\theta \cap B^\beta_\theta\). If \(\overline{q} \in B^\alpha_\theta\) then \((q \circ \alpha, F) \in \theta\)

\[\Rightarrow ((q \circ \alpha) \land (q \circ \beta), F \land (q \circ \beta)) \in \theta \quad \text{(since \(\theta\) is a congruence)}\]

\[\Rightarrow (q \circ (\alpha \land \beta), F \land (q \circ \beta)) \in \theta \quad \text{ (using 11)}\]

\[\Rightarrow (q \circ (\alpha \land \beta), F) \in \theta \quad \text{(since \(F\) is a left-zero for \(\land\))}\]

from which it follows that \(\overline{q} \in B^\alpha_\theta \land \beta\). In the case where \(\overline{q} \in A^\alpha_\theta \cap B^\beta_\theta\) that is \((q \circ \alpha, T), (q \circ \beta, F) \in \theta\), along similar lines it follows that \((q \circ (\alpha \land \beta), T \land F) = (q \circ (\alpha \land \beta), F) \in \theta\) and so \(\overline{q} \in B^\alpha_\theta \land \beta\). Therefore \(B^\alpha_\theta \land \beta\) is a contradiction to Proposition 5.3. Hence \(A^\alpha_\theta \cap A^\beta_\theta = \emptyset\).

Subcase 2: \(\alpha \land \beta \in \{T, F\}\). Using the fact that \(M \leq 3^X\) for some set \(X\) it is easy to see that if \(\alpha, \beta \notin \{T, F\}\) then \(\alpha \land \beta \neq T\). It follows that the only possibility in this case is that \(\alpha \land \beta = F\). Therefore \(\rho_\theta(\alpha \land \beta) = \rho_\theta(F) = (\emptyset, S_\theta)\) and \(\rho_\theta(\alpha) \land \rho_\theta(\beta) = (A^\alpha_\theta \cap A^\beta_\theta, B^\alpha_\theta \cup (A^\alpha_\theta \cap B^\beta_\theta))\) so and we have to show that

\((\emptyset, S_\theta) = (A^\alpha_\theta \cap A^\beta_\theta, B^\alpha_\theta \cup (A^\alpha_\theta \cap B^\beta_\theta))\).

We first show that \(A^\alpha_\theta \cap A^\beta_\theta \neq \emptyset\). If \(A^\alpha_\theta \cap A^\beta_\theta \neq \emptyset\) then let \(\overline{q} \in A^\alpha_\theta \cap A^\beta_\theta\) so that \((q \circ \alpha, T, T) \in \theta\).

\[\Rightarrow ((q \circ \alpha) \land (q \circ \beta), T \land T) = ((q \circ \alpha) \land (q \circ \beta), T) \in \theta \quad \text{(since \(\theta\) is a congruence)}\]

\[\Rightarrow (q \circ (\alpha \land \beta), T) \in \theta \quad \text{ (using 11)}\]

\[\Rightarrow (q \circ F, T) \in \theta \quad \text{(since \(\alpha \land \beta = F\))}\]

\[\Rightarrow (q \circ \neg F, T) = (q \circ \neg F, F) \in \theta \quad \text{(since \(\theta\) is a congruence)}\]

\[\Rightarrow (q \circ \neg F, F) = (q \circ T, F) \in \theta \quad \text{ (using 10)}\]

which is a contradiction to Proposition 5.3 (ii). Hence \(A^\alpha_\theta \cap A^\beta_\theta = \emptyset\).

In order to show that \(B^\alpha_\theta \cup (A^\alpha_\theta \cap B^\beta_\theta) = S_\theta\) consider \(\overline{q} \in S_\theta\) that is \(\overline{q} \neq \neg \overline{q}\) which gives \((q, \bot) \notin E_\theta\). We proceed by considering the following three cases:

\((q \circ \alpha, T) \in \theta\): Then it is clear that \(\overline{q} \in B^\alpha_\theta \subseteq B^\alpha_\theta \cup (A^\alpha_\theta \cap B^\beta_\theta)\).

\((q \circ \alpha, F) \in \theta\): Then we have \(\overline{q} \in A^\alpha_\theta\). We show that \((q \circ \beta, F) \in \theta\). Suppose that this is not the case. Since \(\theta\) is a maximal congruence it implies that either \((q \circ \beta, T) \in \theta\) or \((q \circ \beta, U) \in \theta\). If \((q \circ \beta, T) \in \theta\) then since \((q \circ \alpha, T) \in \theta\) it follows that \((q \circ (\alpha \land \beta), T \land T) = (q \circ F, T) \in \theta\) so that \((q \circ T, F) \in \theta\). This is a contradiction to Proposition 5.3 (ii). In the case that \((q \circ \beta, U) \in \theta\) proceeding as earlier we have \((q \circ (\alpha \land \beta), T \land U) = (q \circ F, U) \in \theta\) so that \((q \circ T, U) \in \theta\). It follows from Proposition 5.3 (ii) that \((q, \bot) \in E_\theta\) which is a contradiction to the assumption that \(\overline{q} \in S_\theta\). Consequently it must be
the case that \((q \circ \beta, F) \in \theta\) so that \(\overline{q} \in A^\theta \cap B^\theta \subseteq B^\theta \cup (A^\theta \cap B^\theta)\).

\((q \circ \alpha, U) \in \theta\): Since \(\theta\) is a congruence we have \((q \circ \beta, q \circ \beta) \in \theta\)

\[\Rightarrow (\underbrace{(q \circ \alpha) \land (q \circ \beta)}_{\text{(since } \theta \text{ is a congruence)}}, U \land (q \circ \beta)) \in \theta\]

\[\Rightarrow ((q \circ (\alpha \land \beta), U) = (q \circ F, U) \in \theta\] (since \(U\) is a left-zero for \(\land\) and using (\ref{eq:5}))

\[\Rightarrow (\neg (q \circ F), \neg U) \in \theta\] (since \(\theta\) is a congruence)

\[\Rightarrow (q \circ \neg F, \neg U) = (q \circ T, U) \in \theta\] (using (\ref{eq:4}))

Thus using Proposition \ref{prop:3.3}(iii) we have \((q, \bot) \in E^\theta\) which is a contradiction to the assumption that \(\overline{q} \in S^\theta\). Hence this case cannot occur.

Thus \(B^\theta \cup (A^\theta \cap B^\theta) = S^\theta\) which completes the proof in the case where \(\alpha, \beta \notin \{T, F\}\).

Case II: \(\alpha \in \{T, F\}\). The verification is straightforward by considering \(\alpha = T\) and \(\alpha = F\) casewise.

Subcase 1: \(\alpha = T\). Then \(\rho_\theta(\alpha \land \beta) = \rho_\theta(T \land \beta) = \rho_\theta(\beta) = (S^\theta, \emptyset) \land \rho_\theta(\beta) =\)

\[\Rightarrow \rho_\theta(T) \land \rho_\theta(\beta) = \rho_\theta(\alpha) \land \rho_\theta(\beta).\]

Subcase 2: \(\alpha = F\). Then \(\rho_\theta(\alpha \land \beta) = \rho_\theta(F \land \beta) = \rho_\theta(F) = (\emptyset, S^\theta) = (\emptyset, S^\theta) \land \rho_\theta(\beta) =\)

\[\Rightarrow \rho_\theta(\alpha) \land \rho_\theta(\beta) = \rho_\theta(\alpha) \land \rho_\theta(\beta).\]

Case III: \(\beta \in \{T, F\}\). We have the following subcases:

Subcase 1: \(\beta = T\). The proof follows along the same lines as Case II above since \(T\) is the left and right-identity for \(\land\). Thus \(\rho_\theta(\alpha \land \beta) = \rho_\theta(\alpha \land T) = \rho_\theta(\alpha) =\)

\[\Rightarrow \rho_\theta(\alpha) \land (S^\theta, \emptyset) = \rho_\theta(\alpha) \land \rho_\theta(T) = \rho_\theta(\alpha) \land \rho_\theta(\beta).\]

Subcase 2: \(\beta = F\). If \(\alpha \in \{T, F\}\) then this reduces to Case II proved above and consequently we have \(\rho_\theta(\alpha \land \beta) = \rho_\theta(\alpha) \land \rho_\theta(\beta)\) in this case. Thus it remains to consider the case where \(\alpha \notin \{T, F\}\). We then have the following subcases depending on \(\alpha \land \beta\):

\(\alpha \land \beta \notin \{T, F\}\): Then \(\rho_\theta(\alpha \land \beta) = \rho_\theta(\alpha \land F) = (A^\theta \cap F, B^\theta \cap F)\) while \(\rho_\theta(\alpha) = (A^\theta, B^\theta)\) and \(\rho_\theta(\beta) = \rho_\theta(F) = (\emptyset, S^\theta)\). Thus \(\rho_\theta(\alpha) \land \rho_\theta(\beta) = (\emptyset, A^\theta \cup B^\theta)\).

We show that \(A^\theta \cap F = \emptyset\) as earlier by proving that the pairs of sets are equal componentwise.

We show that \(A^\theta \cap F = \emptyset\) by contradiction. If \(A^\theta \cap F \neq \emptyset\) then consider \(\overline{q} \in A^\theta \cap F\). It follows that \((q \circ (\alpha \land F), T) \in \theta\)

\[\Rightarrow (\underbrace{(q \circ (\alpha \land F)) \land (q \circ F)}_{\text{(since } \theta \text{ is a congruence)}}, T \land q \circ F) \in \theta\]

\[\Rightarrow ((q \circ (\alpha \land F)) \land (q \circ F), q \circ F) \in \theta\] (since \(T\) is a left-identity for \(\land\))

\[\Rightarrow ((q \circ (\alpha \land F)) \land (q \circ F), q \circ F) \in \theta\] (using (\ref{eq:5}))

\[\Rightarrow (q \circ (\alpha \land F), q \circ F) \in \theta\] (using the properties of \(\land\))

\[\Rightarrow (q \circ F, T) \in \theta\] (since \(\theta\) is a congruence)

\[\Rightarrow (q \circ T, F) \in \theta\] (from (\ref{eq:4}) and since \(q \circ \beta \in \theta\) is a congruence)

which is a contradiction to Proposition \ref{prop:3.3}(ii). Hence \(A^\theta \cap F = \emptyset\).
We show that $B_θ^{α ∧ F} = A_θ^α \cup B_θ^α$ using standard set theoretic arguments. Let $\overline{q} \in B_θ^{α ∧ F}$ and so $\overline{(q \circ (α ∧ F), F)} \in θ$ so that $((q \circ (α ∧ F), F), F) \in θ$. In view of the maximality of $θ$ it suffices to consider three cases. If either $(q \circ α, T) \in θ$ or $(q \circ α, F) \in θ$ then $\overline{q} \in A_θ^α \cup B_θ^α$. If $(q \circ α, U) \in θ$ then $((q \circ α) ∧ \{(q \circ α) ∧ (q \circ F), U ∧ F\}) = ((q \circ α) ∧ (q \circ F), U) \in θ$. Thus $(F, U) \in θ$ which is a contradiction to Proposition 3.2. Hence this case cannot occur and so $B_θ^{α ∧ F} \subseteq A_θ^α \cup B_θ^α$.

For the reverse inclusion consider $\overline{q} \in A_θ^α \cup B_θ^α$ so that $\overline{q} \in A_θ^α$ or $\overline{q} \in B_θ^α$. If $\overline{q} \in A_θ^α$ then $(q \circ α, T) \in θ$. Since $\overline{q} \in A_θ^α \subseteq S_θ$ using Proposition 3.3(iv) we have $((q, \perp) \notin E_θ \Rightarrow (q \circ F, F) \in θ$. Consequently $(q \circ (α ∧ F), (T ∧ F)) = (q \circ (α ∧ F), F) \in θ$ and so $\overline{q} \in B_θ^{α ∧ F}$. Along similar lines if $\overline{q} \in B_θ^α$ we have $(q \circ (α ∧ F), F) \in θ$ so that $\overline{q} \in B_θ^{α ∧ F}$. Hence $(A_θ^{α ∧ F}, B_θ^{α ∧ F}) = (\emptyset, A_θ^α \cup B_θ^α)$.

Let $α \land β ∈ \{T, F\}$: Using the fact that $M ≤ 3^X$ for some set $X$ we have $α \land F \neq T$ from which it follows that the only case is $α \land β = α \land F = F$. Thus $ρ_θ(α ∧ F) = ρ_θ(F) = (\emptyset, S_θ)$ while $ρ_θ(α) ∧ ρ_θ(F) = (\emptyset, A_θ^α \cup B_θ^α)$. We show that $(\emptyset, S_θ) = (\emptyset, A_θ^α \cup B_θ^α)$.

In order to show that $A_θ^α \cup B_θ^α = S_θ$ consider $\overline{q} \in S_θ$. If $(q \circ α, T) \in θ$ or $(q \circ α, F) \in θ$ then the proof is complete. If $(q \circ α, U) \in θ$ then since $\overline{q} \neq \overline{T}$ that is $(q, \perp) \notin E_θ$ by Proposition 3.3(iv) we have $(q \circ F, F) \in θ$. Thus $(q \circ (α ∧ F), U ∧ F) = (q \circ F, U) \in θ$. Consequently from the transitivity of $θ$ it follows that $(F, U) \in θ$ which is a contradiction to Proposition 3.2. Hence $(\emptyset, S_θ) = (\emptyset, A_θ^α \cup B_θ^α)$.

Thus $ρ_θ$ is a homomorphism of $C$-algebras with $T, F, U$. □

**Lemma 3.6.** The pair $(φ_θ, ρ_θ)$ is a $C$-monoid homomorphism from $(S_θ, M)$ to the functional $C$-monoid $(T_θ(S_θ), 3^{S_θ})$.

**Proof.** In view of Proposition 3.4 and Proposition 3.5 it suffices to show that $φ_θ(α[α, s]) = ρ_θ(α) [φ_θ(s), φ_θ(t)]$ and $ρ_θ(s \circ α) = φ_θ(s) \circ ρ_θ(α)$ hold. In order to show that $φ_θ(α[α, s]) = ρ_θ(α) [φ_θ(s), φ_θ(t)]$ we proceed casewise depending on the value of $α$ as per the following:

**Case I:** $α ∈ \{T, F\}$. If $α = T$ then $φ_θ(α[α, s]) = φ_θ(T[α, s]) = φ_θ(s) = (S_θ, \emptyset) [φ_θ(s), φ_θ(t)] = ρ_θ(T) [φ_θ(s), φ_θ(t)] = ρ_θ(α) [φ_θ(s), φ_θ(t)]$. Along similar lines if $α = F$ then $φ_θ(α[α, s]) = φ_θ(F[α, s]) = φ_θ(t) = (\emptyset, S_θ) [φ_θ(s), φ_θ(t)] = ρ_θ(F) [φ_θ(s), φ_θ(t)] = ρ_θ(α) [φ_θ(s), φ_θ(t)]$.

**Case II:** $α \notin \{T, F\}$. If $α \notin \{T, F\}$ then using (44) we have $φ_θ(α[α, s]) = ψ_θ^α[α, s, t]$ where $ψ_θ^α[α, s, t](\overline{q}) = v \cdot (α[α, s]) = (v \circ α)[v \cdot s, v \cdot t]$. Consider $ρ_θ(α) [φ_θ(s), φ_θ(t)] = (A_θ^α, B_θ^α)(ψ_θ^α, ψ_θ^t)! (\overline{q}) = \begin{cases} v \cdot s, & \text{if } \overline{q} \in A_θ^α, \text{ that is } (v \circ α) \in T_θ^α; \\ v \cdot t, & \text{if } \overline{q} \in B_θ^α, \text{ that is } (v \circ α) \in F_θ^α; \\ 1, & \text{otherwise.} \end{cases}$
It suffices to consider the following three cases:

Subcase 1: \((v \circ \alpha) \in \overline{T}_\theta^\theta\). Using Proposition 1.2(i) we have \((v \circ \alpha)[v \cdot s, v \cdot t], v \cdot s) \in E_\theta\). Consequently \(v \circ \alpha)|v \cdot s, v \cdot t| = v \cdot s\).

Subcase 2: \((v \circ \alpha) \in \overline{F}_\theta^\theta\). Along similar lines if \((v \circ \alpha) \in \overline{F}_\theta^\theta\) then \((v \circ \alpha)[v \cdot s, v \cdot t], v \cdot t) \in E_\theta\), by Proposition 1.2(i) and so \((v \circ \alpha)|v \cdot s, v \cdot t| = v \cdot t\).

Subcase 3: \((v \circ \alpha) \in \overline{U}_\theta^\theta\). Then \((v \circ \alpha)[v \cdot s, v \cdot t], \bot) \in E_\theta\), by Proposition 1.2(i) which gives \((v \circ \alpha)|v \cdot s, v \cdot t| = \bot\).

Thus we have \(\psi_\alpha^{|s,t|}(\overline{v}) = (A^\alpha, B^\alpha)\{\psi^s, \psi^t\}v(\overline{v})\) for every \(v \in S_\theta\) and so \(\phi_\theta(\alpha[s,t]) = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]\).

We show that \(\rho_\theta(s \circ \alpha) = \phi_\theta(s) \circ \rho_\theta(\alpha)\) by proceeding casewise depending on the value of \(\alpha\) and \(s \circ \alpha\).

Case 1: \(\alpha \notin \{T, F\}, s \circ \alpha \notin \{T, F\}\). Then \(\rho_\theta(s \circ \alpha) = (A^{s \circ \alpha}, B^{s \circ \alpha})\) and \(\rho_\theta(\alpha) = (A^\alpha, B^\alpha)\). Then \(\phi_\theta(s) \circ (A^\alpha, B^\alpha) = \psi^s \circ (A^\alpha, B^\alpha) = (C, D)\), where \(C = \{\overline{v} \in S_\theta : \psi^s(\overline{v}) \in A^\alpha\}\) and \(D = \{\overline{v} \in S_\theta : \psi^s(\overline{v}) \in B^\alpha\}\). We have to show that \((A^{s \circ \alpha}, B^{s \circ \alpha}) = (C, D)\).

It is clear that \(\overline{v} \in C\)

\[\Leftrightarrow \psi^s(\overline{v}) \in A^\alpha\]
\[\Leftrightarrow \overline{v} \cdot S \in A^\alpha\]
\[\Leftrightarrow ((q \cdot s) \circ \alpha, T) \in \theta\]
\[\Leftrightarrow (q \circ (s \circ \alpha), T) \in \theta\] (using Proposition 1.2(i))
\[\Leftrightarrow \overline{v} \in A^{s \circ \alpha}\]

Along similar lines we have \(\overline{v} \in D \Leftrightarrow \overline{v} \in B^{s \circ \alpha}\).

Case II: \(\alpha \in \{T, F\}, s \circ \alpha \notin \{T, F\}\). If \(\alpha = T\) then \(\rho_\theta(s \circ \alpha) = \rho_\theta(s \circ T) = (A^{s \circ T}, B^{s \circ T})\). On the other hand \(\phi_\theta(s) \circ \rho_\theta(\alpha) = \phi_\theta(s) \circ \rho_\theta(T) = \psi^s \circ (S_\theta, \emptyset) = (C, D)\) where \(C = \{\overline{v} \in S_\theta : \psi^s(\overline{v}) \in S_\theta\}\) and \(D = \emptyset\). We have to show that \((A^{s \circ T}, B^{s \circ T}) = (C, \emptyset)\).

We show that \(B^{s \circ T} = \emptyset\) by contradiction. If \(B^{s \circ T} \neq \emptyset\) then let \(\overline{v} \in B^{s \circ T}\)

\[\Rightarrow (q \circ (s \circ T), F) \in \theta\]
\[\Rightarrow ((q \cdot s) \circ T, F) \in \theta\] (using Proposition 1.2(i))

which is a contradiction to Proposition 3.3(ii). Thus \(B^{s \circ T} = \emptyset\).

We now show that \(A^{s \circ T} = C\). It is clear that \(\overline{v} \in C\)

\[\Leftrightarrow \psi^s(\overline{v}) \in S_\theta\]
\[\Leftrightarrow \overline{v} \in S_\theta\]
\[\Leftrightarrow ((q \cdot s, \bot) \notin E_\theta\]
\[\Leftrightarrow ((q \cdot s) \circ T, T) \in \theta\] (using Proposition 3.3(iv))
\[\Leftrightarrow (q \circ (s \circ T), T) \in \theta\] (using Proposition 1.2(i))
\[\Rightarrow \overline{v} \in A^{s \circ T}\]
In the case where \( \alpha = F \) the proof follows along similar lines.

**Case III:** \( \alpha \notin \{T, F\}, s \circ \alpha \in \{T, F\} \). We have the following subcases:

**Subcase 1:** \( s \circ \alpha = T \). Then \( \rho_0(s \circ \alpha) = \rho_0(T) = (S_\theta, \emptyset) \). On the other hand \( \phi_0(s) \circ \rho_0(\alpha) = \psi_0^s \circ (A^\alpha_\emptyset, B^\alpha_\emptyset) = (C, D) \) where \( C = \{ \overline{q} \in S_\theta : \psi_0^s(\overline{q}) \in A^\alpha_\emptyset \} \) and \( D = \{ \overline{q} \in S_\theta : \psi_0^s(\overline{q}) \in B^\alpha_\emptyset \} \). We have to show that

\[
(C, D) = (S_\theta, \emptyset).
\]

We first show by contradiction that \( D = \emptyset \). If \( D \neq \emptyset \) consider \( \overline{q} \in D \)

\[
\Rightarrow \quad \psi_0^s(\overline{q}) \in B^\alpha_\emptyset \\
\Rightarrow \quad \overline{q} \cdot s \in A^\alpha_\emptyset \\
\Rightarrow \quad ((q \cdot s) \circ \alpha, T) \in \theta \\
\Rightarrow \quad (q \circ (s \circ \alpha), F) \in \theta \quad \text{(using (42))} \\
\Rightarrow \quad (q \circ T, F) \in \theta
\]

which is a contradiction to Proposition 3.3(ii).

In order to show that \( C = S_\theta \) consider \( \overline{q} \in S_\theta \) that is \( (q, \perp) \notin E_\emptyset \)

\[
\Rightarrow \quad (q \circ T, T) \in \theta \quad \text{(using Proposition 3.3(iv))} \\
\Rightarrow \quad (q \circ (s \circ \alpha), T) \in \theta \\
\Rightarrow \quad ((q \cdot s) \circ \alpha, T) \in \theta \\
\Rightarrow \quad \overline{q} \cdot s \in A^\alpha_\emptyset \\
\Rightarrow \quad \psi_0^s(\overline{q}) \in A^\alpha_\emptyset \\
\Rightarrow \quad \overline{q} \in C.
\]

**Subcase 2:** \( s \circ \alpha = F \). Then \( \rho_0(s \circ \alpha) = \rho_0(F) = (\emptyset, S_\theta) \) while \( \phi_0(s) \circ \rho_0(\alpha) = \psi_0^s \circ (A^\alpha_\emptyset, B^\alpha_\emptyset) = (C, D) \) where \( C = \{ \overline{q} \in S_\theta : \psi_0^s(\overline{q}) \in A^\alpha_\emptyset \} \) and \( D = \{ \overline{q} \in S_\theta : \psi_0^s(\overline{q}) \in B^\alpha_\emptyset \} \). We have to show that

\[
(C, D) = (\emptyset, S_\theta).
\]

We first show \( C = \emptyset \) by contradiction. If \( C \neq \emptyset \) consider \( \overline{q} \in C \)

\[
\Rightarrow \quad \psi_0^s(\overline{q}) \in A^\alpha_\emptyset \\
\Rightarrow \quad \overline{q} \cdot s \in A^\alpha_\emptyset \\
\Rightarrow \quad ((q \cdot s) \circ \alpha, T) \in \theta \\
\Rightarrow \quad (q \circ (s \circ \alpha), T) \in \theta \quad \text{(using (42))} \\
\Rightarrow \quad (q \circ T, F) \in \theta \\
\Rightarrow \quad (q \circ T, F) \in \theta
\]

which is a contradiction to Proposition 3.3(ii).

In order to show that \( D = S_\theta \) consider \( \overline{q} \in S_\theta \) that is \( (q, \perp) \notin E_\emptyset \)

\[
\Rightarrow \quad (q \circ F, F) \in \theta \quad \text{(using Proposition 3.3(iv))} \\
\Rightarrow \quad (q \circ (s \circ \alpha), F) \in \theta \\
\Rightarrow \quad ((q \cdot s) \circ \alpha, F) \in \theta \quad \text{(using (42))} \\
\Rightarrow \quad \overline{q} \cdot s \in B^\alpha_\emptyset \\
\Rightarrow \quad \psi_0^s(\overline{q}) \in B^\alpha_\emptyset \\
\Rightarrow \quad \overline{q} \in D
\]

which completes the proof for the case where \( \alpha \notin \{T, F\} \) and \( s \circ \alpha \in \{T, F\} \).

**Case IV:** \( \alpha \in \{T, F\}, s \circ \alpha \in \{T, F\} \). Note that \( s \circ T \neq F \) as a consequence of Proposition 3.3(ii). If \( s \circ T = F \) then as \( \theta \) is a congruence, \( (F, F) \in \theta \Rightarrow (s \circ T, F) \in \theta \),
a contradiction to Proposition 3.3(ii). Similarly we have $s \circ F \neq T$. In view of the above it suffices to consider the following cases:

**Subcase 1**: $\alpha = T, s \circ \alpha = T$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(T) = (S_\theta, \emptyset)$ and $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi^*_\theta \circ (S_\theta, \emptyset) = (C, D)$ where $C = \{ \overline{q} \in S_\theta : \psi^*_\theta(\overline{q}) \in S_\theta \}$ and $D = \emptyset$. Thus it suffices to show that $C = S_\theta$. Let $\overline{q} \in S_\theta$ that is $(q, \bot) \notin E_\theta$

1. $(q \circ T, T) \in \theta$ (using Proposition 3.3(iv))
2. $(q \circ (s \circ T), T) \in \theta$
3. $(q \circ s, T) \in \theta$ (using Prop. 3.3(iv))
4. $(q \circ s, \bot) \notin E_\theta$
5. $\overline{q} \cdot \overline{s} \in S_\theta$
6. $\psi^*_\theta(\overline{q}) \in S_\theta$
7. $\overline{q} \in C$.

Thus $C = S_\theta$.

**Subcase 2**: $\alpha = F, s \circ \alpha = F$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(F) = (\emptyset, S_\theta)$ and $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi^*_\theta \circ (\emptyset, S_\theta) = (C, D)$ where $C = \emptyset$ and $D = \{ \overline{q} \in S_\theta : \psi^*_\theta(\overline{q}) \in S_\theta \}$. The proof follows along similar lines as above. In order to show that $D = S_\theta$ consider $\overline{q} \in S_\theta$ that is $(q, \bot) \notin E_\theta$

1. $(q \circ F, F) \in \theta$ (using Proposition 3.3(iv))
2. $(q \circ (s \circ F), F) \in \theta$
3. $(q \circ s, F) \in \theta$ (using Prop. 3.3(iv))
4. $(q \circ s, \bot) \notin E_\theta$
5. $\overline{q} \cdot \overline{s} \in S_\theta$
6. $\psi^*_\theta(\overline{q}) \in S_\theta$
7. $\overline{q} \in D$.

Hence $D = S_\theta$ which completes the proof.

Thus $(\phi_\theta, \rho_\theta)$ is a homomorphism of $C$-monoids. \hfill \qed

**Proposition 3.7.** For all $\alpha \in M$ the following statements hold:

(i) $\rho_\theta(\alpha) = (S_\theta, \emptyset) \Rightarrow (\alpha, T) \in \theta$.

(ii) $\rho_\theta(\alpha) = (\emptyset, S_\theta) \Rightarrow (\alpha, F) \in \theta$.

Proof.

(i) If $\alpha = T$ then the result is obvious. Suppose that $\alpha \neq T$ and $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha) = (S_\theta, \emptyset)$. It follows that $(t \circ \alpha, T) \in \theta$ for all $\overline{t} \in S_\theta$. Using Proposition 3.3(vi) and (39) we have $\overline{t} \in S_\theta$ and so $(1 \circ \alpha, T) = (\alpha, T) \in \theta$.

(ii) Along similar lines if $\alpha \neq F$ then $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha) = (\emptyset, S_\theta)$ gives $(t \circ \alpha, F) \in \theta$ for all $\overline{t} \in S_\theta$. Using Proposition 3.3(vi) and (39) we have $\overline{t} \in S_\theta$ and so $(1 \circ \alpha, F) = (\alpha, F) \in \theta$.

\hfill \qed

**Lemma 3.8.** For every $s, t \in S^\bot$ where $s \neq t$ there exists a maximal congruence $\theta$ on $M$ such that $\phi_\theta(s) \neq \phi_\theta(t)$.

Proof. Using Proposition 1.9(iv) we have $\bigcap E_\theta = \Delta S^\bot$ and so since $s \neq t$ there exists a maximal congruence $\theta$ on $M$ such that $(s, t) \notin E_\theta$, i.e., $\overline{s} \neq \overline{t}$. For this $\theta$, consider $\phi_\theta : S^\bot \to T_\theta(S_{\theta^\bot})$. Then $\phi_\theta(s) = \psi^*_\theta, \phi_\theta(t) = \psi^*_\theta$. For $\overline{t} \in S_{\theta^\bot}$
we have $\psi_\alpha(T) = 1 \cdot s = s$ while $\psi_\beta(T) = 1 \cdot t = t$. Since $s \neq t$ it follows that $\phi_\beta(s) \neq \phi_\beta(t)$. \qed

**Lemma 3.9.** For every $\alpha, \beta \in M$ where $\alpha \neq \beta$ there exists a maximal congruence $\theta$ on $M$ such that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$.

**Proof.** Using Proposition 1.9(v) since $\alpha \neq \beta$ there exists a maximal congruence $\theta$ on $M$ such that $(\alpha, \beta) \notin \theta$. We show that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$. If $\alpha$ or $\beta$ is in $\{T, F\}$ but $\rho_\theta(\alpha) = \rho_\theta(\beta)$ then using Proposition 3.4 we have $(\alpha, \beta) \in \theta$, a contradiction. In the case where $\alpha, \beta \notin \{T, F\}$ we show that

$$(A_\theta^\alpha, B_\theta^\alpha) \neq (A_\theta^\beta, B_\theta^\beta)$$

by showing that either $A_\theta^\alpha \neq A_\theta^\beta$ or that $B_\theta^\alpha \neq B_\theta^\beta$. Owing to Proposition 3.2 it suffices to consider the following three cases:

**Case I:** $\langle \alpha, T \rangle \in \theta$. Note that Proposition 3.3(vi) gives $\overline{T} \in S_\theta$. Thus we have $\overline{T} \in S_\theta$ for which $1 \circ \alpha = \alpha \in \overline{T}^\theta$ and so $\overline{T} \in A_\theta^\alpha$. However $\overline{T} \notin A_\theta^\beta$ since $(\alpha, \beta) \notin \theta$.

**Case II:** $\langle \alpha, F \rangle \in \theta$. Along similar lines for $\overline{T} \in S_\theta$ we have $1 \circ \alpha = \alpha \in \overline{T}^\theta$ and so $\overline{T} \in B_\theta^\alpha$. It is clear that $\overline{T} \notin B_\theta^\beta$ since $(\alpha, \beta) \notin \theta$.

**Case III:** $\langle \alpha, U \rangle \in \theta$. In view of Proposition 3.2 it suffices to consider the following cases:

**Subcase 1:** $\langle \beta, T \rangle \in \theta$. As earlier we have $\overline{T} \in A_\theta^\beta \setminus A_\theta^\alpha$.

**Subcase 2:** $\langle \beta, F \rangle \in \theta$. It is clear that $\overline{T} \in B_\theta^\beta \setminus B_\theta^\alpha$.

Thus $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$ which completes the proof. \qed

### 3.3. Embedding into a functional $C$-monoid.

Let $\{\theta\}$ be the collection of all maximal congruences of $M$. Define the set $X$ to be the disjoint union of $S_\theta$ taken over all maximal congruences of $M$, written

$$X = \bigsqcup_{\theta} S_\theta$$

(48)

Set $X_\bot = X \cup \{\bot\}$ with base point $\bot \notin X$. For notational convenience we use the same symbol $\bot$ in $X_\bot$ as well as in $S_\bot$. Which $\bot$ we are referring to will be clear from the context of the statement.

In this subsection we obtain monomorphisms $\phi : S_\bot \rightarrow T_\alpha(X_\bot)$ and $\rho : M \rightarrow 3^X$, using which we establish that $(S_\bot, M)$ can be embedded into the functional $C$-monoid $(T_\alpha(X_\bot), 3^X)$.

**Remark 3.10.**

(i) Let $q \in S$ be fixed. For different $\theta$’s the representation of classes $\overline{T}^\theta$’s are different in the disjoint union $X$ of $S_\theta$’s.

(ii) Let $\{A_\lambda\}, \{B_\lambda\}$ be two families of sets indexed over $\Lambda$. Then

$$\left( \bigsqcup_\lambda A_\lambda \right) \cap \left( \bigsqcup_\lambda B_\lambda \right) = \left( \bigsqcup_\lambda \left( A_\lambda \cap B_\lambda \right) \right) \text{ and } \left( \bigsqcup_\lambda A_\lambda \right) \cup \left( \bigsqcup_\lambda B_\lambda \right) = \left( \bigsqcup_\lambda A_\lambda \right) \cup \left( \bigsqcup_\lambda B_\lambda \right).$$

**Notation 3.11.**

(i) For the pair of sets $(A, B)$, we denote by $\pi_1(A, B)$ the first component $A$, and by $\pi_2(A, B)$ the second component $B$.
(ii) For a family of pairs of sets \((A_\lambda, B_\lambda)\) where \(\lambda \in \Lambda\) we denote by \(\bigcup_\lambda (A_\lambda, B_\lambda)\) the pair of sets \(\bigcup_\lambda A_\lambda, \bigcup_\lambda B_\lambda\).

**Lemma 3.12.** Consider \(\phi : S_\bot \rightarrow \mathcal{T}_o(X_\bot)\) given by

\[
(\phi(s))(x) = \begin{cases} 
(\phi_\theta(s))(\overline{q}_\theta), & \text{if } x = \overline{q}_\theta \in S_\theta \text{ and } (\phi_\theta(s))(\overline{q}_\theta) \neq \bot; \\
\bot, & \text{otherwise}. 
\end{cases}
\]

Then \(\phi\) is a monoid monomorphism that maps the zero (and base point) of \(S_\bot\) to that of \(\mathcal{T}_o(X_\bot)\), that is \(\bot \mapsto \zeta_\bot\).

**Proof.** It is clear that \(\phi\) is well-defined and that \(\phi(s) \in \mathcal{T}_o(X_\bot)\) since \((\phi(s))(\bot) = \bot\).

**Claim:** \(\phi\) is injective. Let \(s \neq t \in S_\bot\). Using Lemma 3.8 there exists a maximal congruence \(\theta\) on \(M\) such that \(\phi_\theta(s) \neq \phi_\theta(t)\). Hence there exists a \(\overline{q}_\theta \neq \bot\) such that \((\phi_\theta(s))(\overline{q}_\theta) \neq (\phi_\theta(t))(\overline{q}_\theta)\). By extrapolation it follows that \((\phi(s))(\overline{q}_\theta) \neq (\phi(t))(\overline{q}_\theta)\) and so \(\phi(s) \neq \phi(t)\).

**Claim:** \(\phi(\bot) = \zeta_\bot\). Using Proposition 3.4 we have \(\phi_\theta(\bot) = \zeta_\overline{q}_\theta\) for all \(\theta\) and so by definition \((\phi(\bot))(x) = \bot\) for all \(x \in X_\bot\).

**Claim:** \(\phi(s \cdot t) = \phi(s) \cdot \phi(t)\). Clearly \((\phi(s \cdot t))(\bot) = \bot = (\phi(s) \cdot \phi(t))(\bot)\). Let \(\overline{q} \in X\) that is \(\overline{q}_\theta \in S_\theta\) for some \(\theta\). Then by Proposition 3.4 we have \((\phi(s))(\overline{q}_\theta)) = (\phi_\theta(1))(\overline{q}_\theta) = \overline{q}_\theta\) and hence \(\phi(1) = id_{X_\bot}\).

**Claim:** \(\phi(s \cdot t) = \phi(s) \cdot \phi(t)\). Clearly \((\phi(s \cdot t))(\bot) = \bot = (\phi(s) \cdot \phi(t))(\bot)\). Let \(\overline{q} \in X\) that is \(\overline{q}_\theta \in S_\theta\) for some \(\theta\). Suppose that \((\phi(s \cdot t))(\overline{q}) = \bot\) so that \((\phi_\theta(s \cdot t))(\overline{q}) = \bot\)

\[
\Rightarrow (\phi_\theta(s) \cdot \phi_\theta(t))(\overline{q}) = \bot \\
\Rightarrow \phi_\theta(t)(\phi_\theta(s)(\overline{q})) = \bot \\
\Rightarrow \phi(t)(\phi_\theta(s)(\overline{q})) = \bot
\]

Noting that there are only two possibilities for \(\phi(s)(\overline{q})\) we see that if \(\phi(s)(\overline{q}) = \phi_\theta(s)(\overline{q})\) then we are through. On the other hand if \(\phi(s)(\overline{q}) = \bot\) that is \(\phi_\theta(s)(\overline{q}) = \bot\) then we have \((\phi(s \cdot t))(\overline{q}) = \bot = (\phi(s) \cdot \phi(t))(\overline{q})\) which completes the proof in this case.

Consider the case where \((\phi(s \cdot t))(\overline{q}) \neq \bot\). Using Proposition 3.4 it follows that \((\phi(s \cdot t))(\overline{q}) = (\phi(s \cdot t))(\overline{q}) = (\phi(s) \cdot \phi(t))(\overline{q}) = \phi_\theta(t)(\phi_\theta(s)(\overline{q}))\) and so \((\phi_\theta(s))(\overline{q}) \neq \bot\). Consequently \(\phi(t)(\phi(s)(\overline{q})) = \phi_\theta(t)(\phi_\theta(s)(\overline{q}))\) since \(\phi_\theta(s)(\overline{q}) \neq \bot\). It follows that \((\phi(s \cdot t))(\overline{q}) = (\phi(s) \cdot \phi(t))(\overline{q})\) which completes the proof.

**Lemma 3.13.** The function \(\rho : M \rightarrow 3^X\) defined by

\[
\rho(\alpha) = \sqcap_\theta \rho_\theta(\alpha)
\]

is a monomorphism of \(C\)-algebras with \(T, F, U\).

**Proof.** **Claim:** \(\rho\) is well defined. Let \(\alpha \in M\). Using Remark 3.10(i) we have \(\pi_1(\rho(\alpha)) \cap \pi_2(\rho(\alpha)) = \emptyset\) due to the distinct representation of equivalence classes. Also by Proposition 3.5 we have \(\pi_1(\rho_\theta(\alpha)) \subseteq S_\theta\) and so \(\bot \notin \pi_1(\rho(\alpha)) \cup \pi_2(\rho(\alpha))\) that is \(\rho(\alpha)\) is can be identified with a pair of sets over \(X\).
Claim: \( \rho \) is injective. Let \( \alpha \neq \beta \in M \). By Lemma 3.10 there exists a \( \theta \) such that \( \rho_\theta(\alpha) \neq \rho_\theta(\beta) \). Without loss of generality we infer that there exists a \( \overline{\theta^s} \in \pi_1(\rho_\theta(\alpha)) \setminus \pi_1(\rho_\theta(\beta)) \). Since \( \rho(\alpha) \) is formed by taking the disjoint union of the individual images under \( \rho_\theta(\alpha) \), using Remark 3.11(i) we can say that \( \overline{\theta} \in \pi_1(\rho(\alpha)) \setminus \pi_1(\rho(\beta)) \) that is \( \rho(\alpha) \neq \rho(\beta) \).

Claim: \( \rho \) preserves the constants \( T, F, U \). It follows easily from Proposition 3.26 that \( \rho(T) = (X, \emptyset), \rho(F) = (\emptyset, X) \) and \( \rho(U) = (\emptyset, \emptyset) \).

Claim: \( \rho(\neg \alpha) = \neg(\rho(\alpha)) \). If \( \alpha \in \{T, F\} \) then the result is obvious. If \( \alpha \notin \{T, F\} \) then \( \neg \alpha \notin \{T, F\} \). Using Proposition 3.26 we have \( \rho(\neg \alpha) = (\biguplus A_\theta^{-\alpha}, \biguplus B_\theta^{-\alpha}) = (\biguplus B_\theta^{\neg \alpha}, \biguplus A_\theta^{\neg \alpha}) \). Thus \( \rho(\neg \alpha) = (\biguplus B_\theta^{\neg \alpha}, \biguplus A_\theta^{\neg \alpha}) = \neg(\rho(\alpha)) \).

Claim: \( \rho(\alpha \land \beta) = \rho(\alpha) \land \rho(\beta) \). In view of Remark 3.11(ii) we have \( \biguplus ((A_\lambda, B_\lambda) \land (C_\lambda, D_\lambda)) = (\biguplus A_\lambda, \biguplus B_\lambda) \land (\biguplus C_\lambda, \biguplus D_\lambda) \) for the family of pairs of sets \( (A_\lambda, B_\lambda), (C_\lambda, D_\lambda) \) where \( \lambda \in \Lambda \) over \( X \). In view of the above and Proposition 3.26 we have \( \biguplus \rho(\alpha \land \beta) = \biguplus (\rho(\alpha) \land \rho(\beta)) = (\biguplus \rho(\alpha)) \land (\biguplus \rho(\beta)) = \rho(\alpha) \land \rho(\beta) \) which completes the proof. \( \square \)

Lemma 3.14. The pair \( (\phi, \rho) \) is a C-monoid monomorphism from \( (S_\bot, M) \) to the functional C-monoid \( (T_\oplus(X_\bot), 3^X) \).

Proof. In view of Lemma 3.12 and Lemma 3.13 it suffices to show \( \phi(\alpha[s, t]) = (\rho(\alpha))[\phi(s), \phi(t)] \) and \( \rho(s \circ \alpha) = \phi(s) \circ \rho(\alpha) \).

In order to show that \( \phi(\alpha[s, t]) = (\rho(\alpha))[\phi(s), \phi(t)] \) we show that \( \phi(\alpha[s, t])(x) = (\rho(\alpha))[\phi(s), \phi(t)](x) \) for all \( x \in X_\bot \). Thus we have the following cases:

Case I: \( x = \bot \). It is clear that \( \phi(\alpha[s, t])(\bot) = \bot = (\rho(\alpha))[\phi(s), \phi(t)](\bot) \) since \( \pi_1(\rho(\alpha)), \pi_2(\rho(\alpha)) \subseteq X \) and \( \bot \notin X \).

Case II: \( x \in X \). Consider \( \overline{\theta} \in X \) that is \( \overline{\theta^s} \in S_\theta \) for some \( \theta \). We have the following subcases:

Subcase 1: \( \phi(\alpha[s, t])(\overline{\theta}) = \bot \). then \( \phi_\theta(\alpha[s, t])(\overline{\theta}) = \bot \) and so using Lemma 3.10 we have \( \phi_\theta(\alpha[s, t])(\overline{\theta}) = \bot = (\rho(\alpha))[\phi_\theta(s), \phi_\theta(t)](\overline{\theta}) \). It follows that either \( \overline{\theta} \notin \pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha)) \) or that \( \overline{\theta} \in \pi_1(\rho_\theta(\alpha)) \) and \( \phi_\theta(s)(\overline{\theta}) = \bot \) or, similarly, that \( \overline{\theta} \in \pi_2(\rho_\theta(\alpha)) \) and \( \phi_\theta(t)(\overline{\theta}) = \bot \). Thus we have the following:

\[ \overline{\theta} \notin \pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha)) : \] In view of Remark 3.10(i) it follows that \( \overline{\theta} \notin \pi_1(\rho(\alpha)) \cup \pi_2(\rho(\alpha)) \) and so \( (\rho(\alpha))[\phi(s), \phi(t)](\overline{\theta}) = \bot \).

\[ \overline{\theta} \in \pi_1(\rho_\theta(\alpha)) \text{ and } \phi_\theta(s)(\overline{\theta}) = \bot : \] Then \( \overline{\theta} \in \pi_1(\rho(\alpha)) \) and \( \phi(s)(\overline{\theta}) = \bot \) and so \( (\rho(\alpha))[\phi(s), \phi(t)](\overline{\theta}) = \bot \).

\[ \overline{\theta} \in \pi_2(\rho_\theta(\alpha)) \text{ and } \phi_\theta(t)(\overline{\theta}) = \bot : \] Along similar lines we have \( (\rho(\alpha))[\phi(s), \phi(t)](\overline{\theta}) = \bot \).
Subcase 2: \( \phi(\alpha[s, t]) (\overline{q}) \neq \bot \). Then \( \phi(\alpha[s, t]) (\overline{q}) = \phi_\theta(\alpha[s, t]) (\overline{q}) \) and so using Lemma 3.6 we have \( \phi(\alpha[s, t]) (\overline{q}) = (\rho_\theta(\alpha)) [\phi_\theta(s), \phi_\theta(t)] (\overline{q}) \). It follows that

\[
\phi(\alpha[s, t]) (\overline{q}) = (\rho_\theta(\alpha)) [\phi_\theta(s), \phi_\theta(t)] (\overline{q}) = \begin{cases} 
(\phi_\theta(s)) (\overline{q}), & \text{if } \overline{q} \in \pi_1(\rho_\theta(\alpha)); \\
(\phi_\theta(t)) (\overline{q}), & \text{if } \overline{q} \in \pi_2(\rho_\theta(\alpha)); \\
\bot, & \text{otherwise.}
\end{cases}
\]

If \( \overline{q} \in \pi_1(\rho_\theta(\alpha)) \): It follows that \( \overline{q} \in \pi_1(\rho(\alpha)) \) and so \( (\rho(\alpha)) [\phi(s), \phi(t)] (\overline{q}) = (\phi(s)) (\overline{q}) \). Note that \( \phi_\theta(s) (\overline{q}) \neq \bot \) else \( \phi(\alpha[s, t]) (\overline{q}) = \bot, \) a contradiction.

Thus \( \phi(s) (\overline{q}) = \phi_\theta(s) (\overline{q}) \) so that \( \phi(\alpha[s, t]) (\overline{q}) = (\rho(\alpha)) [\phi(s), \phi(t)] (\overline{q}) \).

If \( \overline{q} \notin (\pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha))) \): This case cannot occur since we assumed that \( \phi(\alpha[s, t]) (\overline{q}) \neq \bot \).

Thus \( \phi(\alpha[s, t]) = (\rho(\alpha)) [\phi(s), \phi(t)] \).

We now show that \( \rho(s \circ \alpha) = \phi(s) \circ \rho(\alpha) \). In order to prove this we proceed by showing that

\[ \pi_i(\rho(s \circ \alpha)) = \pi_i(\phi(s) \circ \rho(\alpha)) \]

for \( i \in \{1, 2\} \).

Let \( \overline{q} \in \pi_1(\rho(s \circ \alpha)) = \sqcup \pi_1(\rho_\theta(s) \circ \alpha)) \). Then \( \overline{q}_\theta \in S_\theta \) for some \( \theta \) and \( \overline{q}_\theta \in \pi_1(\rho_\theta(s) \circ \alpha)) \)

\[
\Rightarrow \overline{q}_\theta \in \pi_1(\phi_\theta(s) \circ \rho_\theta(\alpha)) \quad \text{(using Lemma 3.6)}
\]

\[
\Rightarrow \phi_\theta(s)(\overline{q}_\theta) \in \pi_1(\rho_\theta(\alpha)) \subseteq S_\theta
\]

\[
\Rightarrow \phi_\theta(s)(\overline{q}_\theta) \neq \bot
\]

\[
\Rightarrow \phi_\theta(s)(\overline{q}_\theta) = \phi_\theta(s)(\overline{q}_\theta)(\neq \bot)
\]

\[
\Rightarrow \phi_\theta(s)(\overline{q}_\theta) \in \pi_1(\rho_\theta(\alpha)) \quad \text{(using Remark 3.10(i))}
\]

\[
\Rightarrow \overline{q}_\theta \in \pi_1(\phi(s) \circ \rho(\alpha)) \quad \text{(using Lemma 3.14)}
\]

and so \( \pi_1(\rho(s \circ \alpha)) \subseteq \pi_1(\phi(s) \circ \rho(\alpha)) \).

For the reverse inclusion assume that \( \overline{q} \in \pi_1(\phi(s) \circ \rho(\alpha)) \). Consequently we have \( \overline{q}_\theta \in S_\theta \) for some \( \theta \) and \( \phi(s)(\overline{q}_\theta) \in \pi_1(\rho(\alpha)) \subseteq X \)

\[
\Rightarrow \phi(s)(\overline{q}_\theta) \neq \bot
\]

\[
\Rightarrow \phi(s)(\overline{q}_\theta) = \phi(s)(\overline{q}_\theta)(\neq \bot)
\]

\[
\Rightarrow \phi(s)(\overline{q}_\theta) \in \pi_1(\rho(\alpha)) \quad \text{(using Remark 3.10(i))}
\]

\[
\Rightarrow \overline{q}_\theta \in \pi_1(\phi(s) \circ \rho(\alpha)) \quad \text{(using Lemma 3.14)}
\]

from which it follows that \( \pi_1(\phi(s) \circ \rho(\alpha)) \subseteq \pi_1(\rho(s \circ \alpha)) \). Proceeding along exactly the same lines we can show that \( \pi_2(\rho(s \circ \alpha)) = \pi_2(\phi(s) \circ \rho(\alpha)) \) which completes the proof.

3.4. Proof of Theorem 3.1 Let \( \{\theta\} \) be the collection of all maximal congruences of \( M \). Consider the set \( X \) as in (48). The functions \( \phi : S_\perp \rightarrow T_\theta(X_\perp) \) and \( \rho : M \rightarrow 3^X \) as defined in Lemma 3.12 and Lemma 3.13 respectively, are monomorphisms. Further, by Lemma 3.14 the pair \( (\phi, \rho) \) is a monomorphism from \( (S_\perp, M) \) to the functional C-monoid \( (T_\theta(X_\perp), 3^X) \). From the construction of \( X \) it is also evident that if \( M \) and \( S_\perp \) are finite then there are only finitely many maximal congruences \( \theta \) on \( M \) and finitely many equivalence classes \( E_\theta \) on \( S_\perp \) and so \( X \) must be finite.
Corollary 3.15. An identity is satisfied in every $C$-monoid $(S_\bot, M)$ where $M$ is an ada if and only if it is satisfied in all functional $C$-monoids.

In view of Corollary 3.15 and (36), we have the following result.

Corollary 3.16. In every $C$-monoid $(S_\bot, M)$ where $M$ is an ada we have $(f \circ T)[f, f] = f$.

4. Conclusion

The notion of $C$-sets axiomatize the program construct if-then-else considered over possibly non-halting programs and non-halting tests. In this work, we extended the axiomatization to $C$-monoids which include the composition of programs as well as composition of programs with tests. For the class of $C$-monoids where the $C$-algebra is an ada we obtain a Cayley-type theorem which exhibits the embedding of such $C$-monoids into functional $C$-monoids. Using this, we obtain a mechanism to determine the equivalence of programs through functional $C$-monoids. It is desirable to achieve such a representation for the general class of $C$-monoids with no restriction on the $C$-algebra, which can be considered as future work. Note that the term $f \circ T$ in the standard functional model of a $C$-monoid represents the aspect of the domain of the function, as used in [4, 11]. It is interesting to study the relation between these two concepts in the current set up.

References

[1] G. M. Bergman. Actions of Boolean rings on sets. *Algebra Universalis*, 28:153–187, 1991.
[2] S. L. Bloom and R. Tindell. Varieties of “if-then-else”. *SIAM J. Comput.*, 12:677–707, 1983.
[3] D. A. Bochvar. Ob odnom tróhznacnom isčisenii i égo priménennii k analizu paradoksov klassičeskogo rassirennogo funkcional’nego isčisenii (in Russian). *Matematicheskij sbornik*, 4: 287–308, 1939. Translated to English by M. Bergmann “On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus”. *History and Philosophy of Logic*, 2:87–112, 1981.
[4] J. Desharnais, P. Jipsen, and G. Struth. Domain and antidomain semigroups. In *Relations and Kleene algebra in computer science*, volume 5827 of *Lecture Notes in Comput. Sci.*
[5] I. Guessarian and J. Meseguer. On the axiomatization of “if-then-else”. *SIAM J. Comput.*, 16:332–357, 1987.
[6] F. Guzmán and C. C. Squier. The algebra of conditional logic. *Algebra Universalis*, 27:88–110, 1990.
[7] A. Heyting. Die formalen regeln der intuitionistischen logik, sitzungsberichte der preussischen akademie der wissenschaften, physikalischmathematische klasse,(1930), 42:57–71 158–169 in three parts. *Sitzungsber. preuss. Akad. Wiss*, 42:158–169, 1934.
[8] S. Igarashi. Semantics of ALGOL-like statements. In *Symposium on Semantics of Algorithmic Languages*, pages 117–177. Springer, 1971.
[9] M. Jackson and T. Stokes. Agreeable semigroups. *J. Algebra*, 266(2):393–417, 2003.
[10] M. Jackson and T. Stokes. Semigroups with if-then-else and halting programs. *Int. J. Algebra Comput.*, 19:937–961, 2009.
[11] M. Jackson and T. Stokes. Monoids with tests and the algebra of possibly non-halting programs. *J. Log. Algebr. Methods Program.*, 84:259–275, 2015.
[12] J. F. Kennison. Triples and compact sheaf representation. *J. Pure Appl. Algebra*, 20:13–38, 1981.
[13] S. Kleene. On notation for ordinal numbers. *The Journal of Symbolic Logic*, 3:150–155, 1938.
[14] J. Łukasiewicz. On three-valued logic. Ruch Filozoficzny, 5,(1920), English translation in Borkowski, L.(ed.) 1970. Jan Łukasiewicz: Selected Works, 1920.
[15] E. Manes. Adas and the equational theory of if-then-else. *Algebra Universalis*, 30:373–394, 1993.
In other words $\circ$ where $\alpha$

Axiom (40): Let $

\alpha

\begin{align*}

\alpha

\end{align*}

Axiom (41): Let $

\alpha, \beta

\begin{align*}

\alpha, \beta

\end{align*}

Appendix A. Proofs

A.1. Verification of Example 2.2

We use the pairs of sets representation given by Guzmán and Squier in [6] and identify $\alpha \in 3^X$ with a pair of sets $(A, B)$ of $X$ where $A = \alpha^{-1}(T)$ and $B = \alpha^{-1}(F)$. In this representation $T = (X, \emptyset), F = (\emptyset, X)$ and $U = (\emptyset, \emptyset)$. Thus the operation $\circ$ is given as follows:

$(f \circ \alpha)(x) = \begin{cases} 
T, & \text{if } f(x) \in A; \\
F, & \text{if } f(x) \in B; \\
U, & \text{otherwise}.
\end{cases}$

In other words $f \circ \alpha$ can be identified with the pair of sets $(C, D)$ where $C = \{x \in X : f(x) \in A\}$ and $D = \{x \in X : f(x) \in B\}$.

Axiom (37): Let $\alpha$ be identified with the pair of sets $(A, B)$. Then $\zeta \circ \alpha = (\emptyset, \emptyset) = U$ as $\zeta(x) = \bot / \notin (A \cup B)$.

Axiom (38): Consider $U = (\emptyset, \emptyset)$. Then $f \circ U = (\emptyset, \emptyset) = U$.

Axiom (39):

$(1 \circ \alpha)(x) = \begin{cases} 
T, & \text{if } id_{\bot}(x) \in A; \\
F, & \text{if } id_{\bot}(x) \in B; \\
U, & \text{otherwise}
\end{cases}
\begin{align*}
\beta
\end{align*}

= \alpha(x).

Thus $1 \circ \alpha = \alpha$.

Axiom (40): Let $\alpha$ be identified with the pair of sets $(A, B)$. Then $f \circ \alpha = (C, D)$ where $C = \{x \in X : f(x) \in A\}$ and $D = \{x \in X : f(x) \in B\}$. Thus $\neg(f \circ \alpha) = (D, C)$. Also $f \circ (\neg \alpha) = f \circ (B, A) = (E, F)$ where $E = \{x \in X : f(x) \in B\}$ and $F = \{x \in X : f(x) \in A\}$. It follows that $(E, F) = (D, C)$.

Axiom (41): Let $\alpha, \beta$ be represented by the pairs of sets $(A_1, A_2)$ and $(B_1, B_2)$ respectively. Then $\alpha \wedge \beta = (A_1 \cap B_1, A_2 \cup (A_1 \cap B_2))$. Also let $f \circ \alpha = (C_1, C_2)$.
where $C_1 = \{x \in X : f(x) \in A_1\}$ and $C_2 = \{x \in X : f(x) \in A_2\}$, and

$f \circ \beta = (D_1, D_2)$ where $D_1 = \{x \in X : f(x) \in B_1\}$ and $D_2 = \{x \in X : f(x) \in B_2\}$. Then $(C_1, C_2) \wedge (D_1, D_2) = (C_1 \cap D_1, C_2 \cup (C_1 \cap D_2))$. Thus

$C_1 \cap D_1 = \{x \in X : f(x) \in A_1 \cap B_1\}$ and

$C_2 \cup (C_1 \cap D_2) = \{x \in X : f(x) \in A_2 \cup (A_1 \cap B_2)\}$. Hence

$f \circ (\alpha \wedge \beta) = (f \circ \alpha) \wedge (f \circ \beta)$.

Axiom (12): Consider $f, g \in T_o(X\perp)$ and $\alpha \in 3^X$ represented by the pair of sets $(A, B)$.

$$((f \cdot g) \circ \alpha)(x) = \begin{cases} T, & \text{if } g(f(x)) \in A; \\
F, & \text{if } g(f(x)) \in B; \\
U, & \text{otherwise.} \end{cases}$$

Let $g \circ \alpha = (C, D)$ where $C = \{x \in X : g(x) \in A\}$ and $D = \{x \in X : g(x) \in B\}$.

$$f \circ (g \circ \alpha))(x) = \begin{cases} T, & \text{if } f(x) \in C; \\
F, & \text{if } f(x) \in D; \\
U, & \text{otherwise.} \end{cases}$$

We may consider the following three cases.

Case I: $x \in X$ such that $g(f(x)) \in A$: Then $((f \cdot g) \circ \alpha)(x) = T$. Also $f(x) \in C$ as $g(f(x)) \in A$. Thus $(f \circ (g \circ \alpha))(x) = T$.

Case II: $x \in X$ such that $g(f(x)) \in B$: Then $((f \cdot g) \circ \alpha)(x) = F$. Similarly $g(f(x)) \in B$ means that $f(x) \in D$. Thus $(f \circ (g \circ \alpha))(x) = F$.

Case III: $x \in X$ such that $g(f(x)) \notin (A \cup B)$: Then $((f \cdot g) \circ \alpha)(x) = U$. Since $f(x)$ is in neither $C$ nor $D$ it follows that $(f \circ (g \circ \alpha))(x) = U$.

Axiom (13): Consider $\alpha \in 3^X$ represented by the pair of sets $(A, B)$.

$$(\alpha[f, g] \cdot h)(x) = h(\alpha[f, g](x)) = \begin{cases} h(f(x)), & \text{if } x \in A; \\
h(g(x)), & \text{if } x \in B; \\
\bot, & \text{otherwise.} \end{cases}$$

Hence $\alpha[f, g] \cdot h = \alpha[f \cdot h, g \cdot h]$.

Axiom (14): Let $\alpha \in 3^X$ be represented by the pair of sets $(A, B)$.

$$(h \cdot \alpha[f, g])(x) = \alpha[f, g](h(x)) = \begin{cases} f(h(x)), & \text{if } h(x) \in A; \\
g(h(x)), & \text{if } h(x) \in B; \\
\bot, & \text{otherwise.} \end{cases}$$

Let $h \circ \alpha$ be represented by the pair of sets $(C, D)$ where $C = \{x \in X : h(x) \in A\}$ and $D = \{x \in X : h(x) \in B\}$.

$$(h \circ \alpha)[h \cdot f, h \cdot g](x) = \begin{cases} (h \cdot f)(x), & \text{if } x \in C; \\
(h \cdot g)(x), & \text{if } x \in D; \\
\bot, & \text{otherwise} \end{cases} = \begin{cases} f(h(x)), & \text{if } h(x) \in A; \\
g(h(x)), & \text{if } h(x) \in B; \\
\bot, & \text{otherwise.} \end{cases}$$
Thus \( h \cdot \alpha[f, g] = (h \circ \alpha)[h \cdot f, h \cdot g] \).

**Axiom**: Let \( \alpha, \beta \in S^X \) be represented by the pairs of sets \((A_1, A_2)\) and \((B_1, B_2)\) respectively. For \( f, g \in \mathcal{T}_T(X) \) we have the following:

\[
    h(x) = \alpha[f, g](x) = \begin{cases} 
        f(x), & \text{if } x \in A_1; \\
        g(x), & \text{if } x \in A_2; \\
        \bot, & \text{otherwise.}
    \end{cases}
\]

Also \( h \circ \beta = (C_1, C_2) \) where \( C_1 = \{x \in X : h(x) \in B_1\} \) and \( C_2 = \{x \in X : h(x) \in B_2\} \). Similarly \( f \circ \beta = (D_1, D_2) \) where \( D_1 = \{x \in X : f(x) \in B_1\} \) and \( D_2 = \{x \in X : f(x) \in B_2\} \). Let \( g \circ \beta = (E_1, E_2) \) where \( E_1 = \{x \in X : g(x) \in B_1\} \) and \( E_2 = \{x \in X : g(x) \in B_2\} \). Thus \( \alpha[f \circ \beta, g \circ \beta] = ((A_1, A_2) \cap (D_1, D_2)) \lor ((A_1, A_2) \cap (E_1, E_2)) \).

This evaluates to

\[
    \alpha[f \circ \beta, g \circ \beta] = (A_1 \cap D_1, A_2 \cup (A_1 \cap D_2)) \lor (A_2 \cap E_1, A_1 \cup (A_2 \cap E_2))
\]

\[
    = ((A_1 \cap D_1) \cup ((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)),
\]

\[
    (A_2 \cup (A_1 \cap D_2)) \cap (A_1 \cup (A_2 \cap E_2))
\]

\[
    = (S_1, S_2) \text{ (say)}
\]

We show that \((C_1, C_2) = (S_1, S_2)\) by standard set theoretic arguments.

First we prove that \( C_1 \subseteq S_1 \). Let \( x \in C_1 \). Then \( h(x) \in B_1 \). Consider the following cases:

**Case I**: \( x \in A_1 \): Then \( h(x) = f(x) \in B_1 \) hence \( x \in D_1 \). Therefore \( x \in A_1 \cap D_1 \) and so \( x \in S_1 \).

**Case II**: \( x \in A_2 \): Then \( h(x) = g(x) \in B_1 \) hence \( x \in E_1 \). Hence \( x \in A_2 \cap E_1 \subseteq A_2 \) we have \( x \in S_1 \).

**Case III**: \( x \notin (A_1 \cup A_2) \): Then \( h(x) = \bot \notin B_1 \) a contradiction to our assumption that \( h(x) \in B_1 \). It follows that this case cannot occur.

We show that \( S_1 \subseteq C_1 \). Let \( x \in S_1 \). Thus \( x \in A_1 \cap D_1 \) or \( x \in ((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)) \). If \( x \in A_1 \cap D_1 \) then \( h(x) = f(x) \) as \( x \in A_1 \) and \( f(x) \in B_1 \) as \( x \in D_1 \). Thus \( h(x) \in B_1 \) and so \( x \in C_1 \). If \( x \in ((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)) \), then \( x \in (A_2 \cap E_1) \). Thus \( h(x) = g(x) \) as \( x \in A_2 \) and \( g(x) \in B_1 \) as \( x \in E_1 \). Hence \( h(x) \in B_1 \) thus \( x \in C_1 \).

We show that \( C_2 \subseteq S_2 \). Let \( x \in C_2 \) hence \( h(x) \in B_2 \). Consider the following cases:

**Case I**: \( x \in A_1 \): Then \( h(x) = f(x) \in B_2 \), therefore \( x \in D_2 \). Hence \( x \in A_1 \cap D_2 \subseteq A_1 \) and so \( x \in S_2 \).

**Case II**: \( x \in A_2 \): Then \( h(x) = g(x) \in B_2 \) therefore \( x \in E_2 \). Thus \( x \in A_2 \cap E_2 \subseteq A_2 \) and so \( x \in S_2 \).

**Case III**: \( x \notin (A_1 \cup A_2) \): Then \( h(x) = \bot \notin B_2 \) which is a contradiction. It follows that this case cannot occur.

Finally we show that \( S_2 \subseteq C_2 \). Since \( A_1 \cap A_2 = \emptyset \) it follows that \( x \in A_1 \cap D_2 \) or \( x \in A_2 \cap E_2 \). If \( x \in A_1 \cap D_2 \) then \( h(x) = f(x) \in B_2 \) and hence \( x \in C_2 \). If \( x \in A_2 \cap E_2 \) then \( h(x) = g(x) \in B_2 \) hence \( x \in C_2 \).

Thus \( \alpha[f, g] \circ \beta = \alpha[f \circ \beta, g \circ \beta] \).
A.2. Verification of Example 2.3

Let \( f, g, h \in S^X_\perp \) and \( \alpha, \beta \in 3^X \).

Axiom (37): It is easy to see that \((\zeta_\perp \circ \alpha)(x) = U\) for all \( x \in X \).

Axiom (38): It is clear that \((f \circ U)(x) = U\).

Axiom (39): Since \( S^X_\perp \) is non-trivial we must have \( 1 \neq \perp \). If not then for \( a \in S^X_\perp \setminus \{\perp\} \) we have \( a = a \cdot 1 = a \cdot \perp = \perp \) a contradiction. It follows that \( \zeta_1 \neq \zeta_\perp \). Hence \((\zeta_1 \circ \alpha)(x) = \alpha(x)\) as \( \zeta_1(x) = 1 \neq \perp \).

Axiom (40): We have

\[
(f \circ (\neg \alpha))(x) = \begin{cases} 
(\neg \alpha)(x), & \text{if } f(x) \neq \perp; \\
U, & \text{otherwise}
\end{cases}
= \begin{cases} 
(\alpha(x)), & \text{if } f(x) \neq \perp; \\
U, & \text{otherwise}
\end{cases}
= \neg(f \circ \alpha)(x).
\]

Thus \( f \circ (\neg \alpha) = \neg(f \circ \alpha) \).

Axiom (41): We have

\[
(f \circ (\alpha \land \beta))(x) = \begin{cases} 
(\alpha \land \beta)(x), & \text{if } f(x) \neq \perp; \\
U, & \text{otherwise}
\end{cases}
= \begin{cases} 
\alpha(x) \land \beta(x), & \text{if } f(x) \neq \perp; \\
U \land U, & \text{otherwise}
\end{cases}
= (f \circ \alpha)(x) \land (f \circ \beta)(x).
\]

Thus \( f \circ (\alpha \land \beta) = (f \circ \alpha) \land (f \circ \beta) \).

Axiom (42): Since \( S^X_\perp \) has no zero-divisors we have

\( f(x) \cdot g(x) = \perp \iff f(x) = \perp \) or \( g(x) = \perp \). Consequently

\[
((f \cdot g) \circ \alpha)(x) = \begin{cases} 
\alpha(x), & \text{if } (f \cdot g)(x) \neq \perp; \\
U, & \text{otherwise}
\end{cases}
= \begin{cases} 
\alpha(x), & \text{if } f(x) \cdot g(x) \neq \perp; \\
U, & \text{otherwise}
\end{cases}
= \begin{cases} 
\alpha(x), & \text{if } f(x) \neq \perp \text{ and } g(x) \neq \perp; \\
U, & \text{otherwise}
\end{cases}
= (f \circ (g \circ \alpha))(x).
\]

Thus \( (f \cdot g) \circ \alpha = f \circ (g \circ \alpha) \).
Axiom \textbf{[13]}: We have
\[
(\alpha[f, g] \cdot h)(x) = \alpha[f, g](x) \cdot h(x) = \begin{cases} f(x) \cdot h(x), & \text{if } \alpha(x) = T; \\ g(x) \cdot h(x), & \text{if } \alpha(x) = F; \\ \bot, & \text{otherwise} \end{cases}
\]
Thus \( \alpha[f, g] \cdot h = \alpha[f \cdot h, g \cdot h] \).

Axiom \textbf{[14]}: Consider
\[
h \cdot \alpha[f, g](x) = h(x) \cdot \alpha[f, g](x) = \begin{cases} h(x) \cdot f(x), & \text{if } \alpha(x) = T; \\ h(x) \cdot g(x), & \text{if } \alpha(x) = F; \\ \bot, & \text{otherwise}. \end{cases}
\]
On the other hand
\[
(h \circ \alpha)[h \cdot f, h \cdot g](x) = \begin{cases} h(x) \cdot f(x), & \text{if } (h \circ \alpha)(x) = T; \\ h(x) \cdot g(x), & \text{if } (h \circ \alpha)(x) = F; \\ \bot, & \text{otherwise}. \end{cases}
\]
Note that if \( h(x) = \bot \) then \( h \cdot \alpha[f, g](x) = \bot = (h \circ \alpha)[h \cdot f, h \cdot g](x) \). Suppose that \( h(x) \neq \bot \) then \( (h \circ \alpha)(x) = \alpha(x) \). It is clear that in this case as well \( h \cdot \alpha[f, g](x) = (h \circ \alpha)[h \cdot f, h \cdot g](x) \) holds. Thus \( h \cdot \alpha[f, g] = (h \circ \alpha)[h \cdot f, h \cdot g] \).

Axiom \textbf{[15]}: Consider
\[
(\alpha[f, g] \circ \beta)(x) = \begin{cases} \beta(x), & \text{if } \alpha[f, g](x) \neq \bot; \\ U, & \text{otherwise} \end{cases}
\]
We have \((\alpha[f \circ \beta, g \circ \beta])(x) = (\alpha(x) \land (f \circ \beta)(x)) \lor (\neg \alpha(x) \land (g \circ \beta)(x)).\)
If \( f(x) \neq \bot \) and \( \alpha(x) = T \) we have
\[
(\alpha[f \circ \beta, g \circ \beta])(x) = (T \land (\beta(x)) \lor (F \land (g \circ \beta)(x)) = \beta(x) \lor F = \beta(x) = (\alpha[f, g] \circ \beta)(x).
\]
If \( g(x) \neq \bot \) and \( \alpha(x) = F \) we have
\[
(\alpha[f \circ \beta, g \circ \beta])(x) = (F \land (f \circ \beta)(x)) \lor (T \land (\beta(x)) = F \lor \beta(x) = \beta(x) = (\alpha[f, g] \circ \beta)(x).
\]
In all other cases it can be easily ascertained that
\[
(\alpha[f \circ \beta, g \circ \beta])(x) = U = (\alpha[f, g] \circ \beta)(x). \]
Thus \( \alpha[f, g] \circ \beta = \alpha[f \circ \beta, g \circ \beta] \).

A.3. Verification of Example \textbf{2.4}

Axiom \textbf{[27]}: It is clear that \( \bot \circ \alpha = U \).

Axiom \textbf{[28]}: It is obvious that \( t \circ U = U \).

Axiom \textbf{[29]}: Since \( S_\bot \) is non-trivial it follows that \( 1 \neq \bot \). Consequently \( 1 \circ \alpha = \alpha \).

Axiom \textbf{[30]}: If \( s = \bot \) then \( s \circ (\neg \alpha) = U = \neg (s \circ \alpha) \). If \( s \neq \bot \) then
\[
s \circ (\neg \alpha) = \neg \alpha = \neg (s \circ \alpha). \]
Thus \( s \circ (\neg \alpha) = \neg (s \circ \alpha) \).
Axiom (11): If $s = \perp$ then $s \circ (\alpha \land \beta) = U$ and $(s \circ \alpha) \land (s \circ \beta) = U \land U = U$. If $s \neq \perp$ then $s \circ (\alpha \land \beta) = \alpha \land \beta = (s \circ \alpha) \land (s \circ \beta)$. Thus $s \circ (\alpha \land \beta) = (s \circ \alpha) \land (s \circ \beta)$.

Axiom (12): Consider $s,t \in S_\perp$ such that $s \cdot t = \perp$. Then $(s \cdot t) \circ \alpha = \perp \circ \alpha = U$. Since $S_\perp$ has no non-zero zero-divisors we have $s = \perp$ or $t = \perp$ and so $s \circ (t \circ \alpha) = U$ in either case. If $s \cdot t \neq \perp$ then $(s \cdot t) \circ \alpha = \alpha$ and $s \circ (t \circ \alpha) = t \circ \alpha = \alpha$ as neither $s$ nor $t$ are $\perp$. Thus $(s \cdot t) \circ \alpha = s \circ (t \circ \alpha)$.

Axiom (13): As $\alpha \in \{T,F,U\}$ we consider the following three cases:

Case I: $\alpha = T$: Then $\alpha[s,t] \cdot u = T[s,t] \cdot u = s \cdot u = T[s \cdot u,t \cdot u]$.

Case II: $\alpha = F$: Then $\alpha[s,t] \cdot u = F[s,t] \cdot u = t \cdot u = F[s \cdot u,t \cdot u]$.

Case III: $\alpha = U$: Then $\alpha[s,t] \cdot u = U[s,t] \cdot u = \perp \cdot u = \perp = U[s \cdot u,t \cdot u]$.

Thus $\alpha[s,t] \cdot u = \alpha[s \cdot u,t \cdot u]$.

Axiom (14): Consider the following cases:

Case I: $r = \perp$: Then $r \cdot \alpha[s,t] = \perp \cdot \alpha[s,t] = \perp = U[r \cdot s, r \cdot t] = (\perp \circ \alpha)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Case II: $r \neq \perp$: We again consider the following three cases:

Case i: $\alpha = T$:
$r \cdot \alpha[s,t] = r \cdot T[s,t] = r \cdot s = T[r \cdot s, r \cdot t] = (r \circ T)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Case ii: $\alpha = F$:
$r \cdot \alpha[s,t] = r \cdot F[s,t] = r \cdot t = F[r \cdot s, r \cdot t] = (r \circ F)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Case iii: $\alpha = U$:
$r \cdot \alpha[s,t] = r \cdot U[s,t] = r \cdot \perp = \perp = U[r \cdot s, r \cdot t] = (r \circ U)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Thus $r \cdot \alpha[s,t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Axiom (15): Consider the following three cases:

Case I: $\alpha = T$: $\alpha[s,t] \circ \beta = T[s,t] \circ \beta = (s \circ \beta) = T[s \circ \beta,t \circ \beta]$.

Case II: $\alpha = F$: $\alpha[s,t] \circ \beta = F[s,t] \circ \beta = t \circ \beta = F[s \circ \beta,t \circ \beta]$.

Case III: $\alpha = U$: $\alpha[s,t] \circ \beta = U[s,t] \circ \beta = \perp \circ \beta = U[s \circ \beta,t \circ \beta]$.

Thus $\alpha[s,t] \circ \beta = \alpha[s \circ \beta,t \circ \beta]$.

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, India

E-mail address: p.gayatri@iitg.ac.in

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, India

E-mail address: kvk@iitg.ac.in

Department of Computer Science and Engineering, Indian Institute of Technology Guwahati, Guwahati, India

E-mail address: pbhaduri@iitg.ac.in