Entanglement production by evolution operator

V.I. Yukalov\textsuperscript{1} and E.P. Yukalova\textsuperscript{2}

\textsuperscript{1}Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia
\textsuperscript{2}Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna 141980, Russia
E-mail: yukalov@theor.jinr.ru

Abstract. Entanglement production, generated by an evolution operator, is considered. The measure of entanglement production, introduced earlier by one of the authors, is employed. As an illustration, a two-qubit register is studied and the corresponding measure of evolutional entanglement production is calculated. Such two-qubit registers can be realized by atomic systems in a double well or by trapped atoms with two coherent modes.

1. Entangled and disentangled states

Entanglement is a pivotal quantum property that can give a principal advantage for the use of quantum processes, as compared to classical ones, in such fields as quantum information processing, quantum computing, and the functioning of artificial quantum intelligence \cite{1-7}.

The states, represented by wave functions or state vectors, are distinguished onto non-entangled and entangled, as explained below.

One considers a composite system characterized by a Hilbert space

\[ \mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i, \tag{1} \]

which is a tensor product of partial Hilbert spaces

\[ \mathcal{H}_i = \text{span}\{|n_i\}\] \tag{2}

that are closed linear envelopes over the related bases. So that the total Hilbert space can be written as

\[ \mathcal{H} = \text{span}\left\{ \bigotimes_{i=1}^{N} |n_i\rangle \right\}. \tag{3} \]

Disentangled states have the form of the products

\[ \varphi_{\text{dis}} = \bigotimes_{j=1}^{N} \varphi_j \tag{4} \]
of the partial states
\[ \varphi_j = \sum_{n_j} c_{n_j} |n_j\rangle \in \mathcal{H}_j \]
(5)
pertaining to the partial Hilbert spaces. Respectively, a disentangled state is of the form
\[ \varphi_{\text{dis}} = \bigotimes_{j=1}^N \sum_{n_j} c_{n_j} |n_j\rangle . \]
(6)
All disentangled states of the space \( \mathcal{H} \) compose a disentangled set
\[ \mathcal{D} \equiv \{ \forall \varphi_{\text{dis}} \in \mathcal{H} \} . \]
(7)
Entangled states of the space \( \mathcal{H} \) can be written as
\[ \varphi_{\text{ent}} = \sum_{\{n_i\}} c_{n_1n_2\ldots n_N} |n_1n_2\ldots n_N\rangle , \]
(8)
where at least one of the coefficients cannot be represented as a product of \( c_{n_i} \), so that at least for one coefficient
\[ c_{n_1n_2\ldots n_N} \neq \prod_{i=1}^N c_{n_i} . \]
(9)
The collection of all entangled states forms an entangled set
\[ \mathcal{H} \setminus \mathcal{D} \equiv \{ \varphi_{\text{ent}} \in \mathcal{H}, \varphi_{\text{ent}} \notin \mathcal{D} \} . \]
(10)

Thus the total Hilbert space \( \mathcal{H} \) consists of two sets, a disentangled set \( \mathcal{D} \) and an entangled set \( \mathcal{H} \setminus \mathcal{D} \).

2. Measure of entanglement production

One should distinguish entangled states and entanglement production by quantum operations. Entanglement of states can be induced by operators acting on the states of a Hilbert space. So, entangled states characterize a static feature of a considered set, while entanglement production describes the operational feature of an operator.

Let \( \hat{A} \) be a non-singular trace-class operator on \( \mathcal{H} \), such that
\[ 0 \neq |\text{Tr}_\mathcal{H} \hat{A}| < \infty . \]
(11)
The action of the operator on a disentangled state \( \varphi_{\text{dis}} \in \mathcal{D} \) can result either in a different disentangled state \( \varphi_{\text{dis}}' \in \mathcal{D} \) or in an entangled state \( \varphi_{\text{ent}} \in \mathcal{H} \setminus \mathcal{D} \). How would it be possible to quantify the entangling ability of an operator?

For two given states \( \varphi_{\text{dis}} \in \mathcal{D} \) and \( \varphi_{\text{ent}} \in \mathcal{H} \setminus \mathcal{D} \), one could define a kind of the entanglement probability
\[ p_{\text{ent}} \left( \hat{A} \varphi_{\text{dis}} \rightarrow \varphi_{\text{ent}} \right) \propto |\langle \varphi_{\text{ent}} | \hat{A} \varphi_{\text{dis}} \rangle|^2 , \]
(12)
being appropriately normalized. However, this would be not a general property of the operator on the whole Hilbert space, but merely its property with regard to the two given particular states.

The global ability of an operator of producing entanglement in the total Hilbert space \( \mathcal{H} \), when
\[ \hat{A} \mathcal{D} \rightarrow \mathcal{H} \setminus \mathcal{D} , \]
(13)
can be quantified as follows.

The entangling ability of the operator \( \hat{A} \) can be understood by comparing its action on the given Hilbert space with the action of its non-entangling counterpart \( \hat{A}^\otimes \) that does not entangle the states,

\[
\hat{A}^\otimes \mathcal{D} \rightarrow \mathcal{D} ,
\]

being defined as

\[
\hat{A}^\otimes \equiv \bigotimes_{i=1}^{N} \hat{A}_i / (\text{Tr}_\mathcal{H} \hat{A})^{N-1} ,
\]

where the partially traced operators

\[
\hat{A}_i \equiv \text{Tr}_{\mathcal{H}/\mathcal{H}_i} \hat{A}
\]

are obtained by tracing out the operator \( \hat{A} \) over all partial Hilbert spaces, composing \( \mathcal{H} \), except one space \( \mathcal{H}_i \). The denominator in the non-entangling counterpart is such that the normalization

\[
\text{Tr}_\mathcal{H} \hat{A}^\otimes = \text{Tr}_\mathcal{H} \hat{A}
\]

be preserved.

The measure of entanglement production, for an operator \( \hat{A} \), is defined \([8, 9]\) as

\[
\varepsilon(\hat{A}) \equiv \log \frac{||\hat{A}||_p}{||\hat{A}^\otimes||_p} .
\]

The logarithm can be taken with respect to any base. The norm here can be accepted as a Schatten \( p \)-norm

\[
||\hat{A}||_p \equiv \left( \text{Tr}_\mathcal{H} | \hat{A} |^p \right)^{1/p} ,
\]

in which

\[
p \in [1, \infty) , \quad | \hat{A} | \equiv \sqrt{\hat{A}^+ \hat{A}} .
\]

The Schatten norms are isometrically invariant, and therefore unitary invariant, such that

\[
||\hat{A}_1||_p = ||U_1 \hat{A} U_2||_p
\]

for any linear isometries, hence, for any unitary transformations \( U_1 \) and \( U_2 \). Thence, the Schatten norms are independent of the basis used when taking the trace operation.

It is convenient to employ the Hilbert-Schmidt, or Frobenius, norm

\[
||\hat{A}||_2 = \sqrt{\text{Tr}_\mathcal{H} (\hat{A}^+ \hat{A})} ,
\]

which is the Schatten 2-norm.

The standard operator norm corresponds to the Schatten \( \infty \)-norm

\[
||\hat{A}||_\infty \equiv \sup_{\varphi \in \mathcal{H}} \frac{||\hat{A} \varphi||}{||\varphi||} (\varphi \neq 0) ,
\]

denoted in what follows as

\[
||\hat{A}|| \equiv ||\hat{A}||_\infty .
\]
Keeping in mind an orthonormal basis \( \{ e_\alpha \} \) in \( \mathcal{H} \), we can write
\[
\text{Tr}_{\mathcal{H}}(\hat{A}^+ \hat{A}) = \sum_\alpha (e_\alpha, \hat{A}^+ \hat{A} e_\alpha) = \sum_\alpha (\hat{A} e_\alpha, \hat{A} e_\alpha) = \sum_\alpha ||\hat{A} e_\alpha||^2.
\]
Therefore the Hilbert-Schmidt norm can be defined as
\[
||\hat{A}||_2^2 = \sum_\alpha ||\hat{A} e_\alpha||^2.
\]

Quantity (18) enjoys all necessary properties required for being a measure. Thus, it is semi-positive. Because of the importance of this property, we prove it below.

**Theorem.** For trace-class operators \( \hat{A} : \mathcal{H} \to \mathcal{H} \) and \( \hat{A}^\otimes : \mathcal{D} \to \mathcal{D} \), defined in (15), the measure of entanglement production (18) is semi-positive,
\[
\varepsilon(\hat{A}) \geq 0.
\] (21)

**Proof.** The operators, enjoying finite norm (19), are Hilbert-Schmidt operators. Note that the trace-class operators are also Hilbert-Schmidt operators. A family of the operators \( \hat{A} : \mathcal{H} \to \mathcal{H} \), supplemented by norm (19), forms a Banach space
\[
B(\mathcal{H}) \equiv \{ \hat{A} : \mathcal{H} \to \mathcal{H}, ||\hat{A}||_2 \},
\] (22)
that is a normed, complete, linear space. Respectively, a family of the operators \( \hat{A}^\otimes : \mathcal{D} \to \mathcal{D} \), complimented by norm (19), forms a Banach space
\[
B^{\otimes}(\mathcal{D}) \equiv \{ \hat{A}^\otimes : \mathcal{D} \to \mathcal{D}, ||\hat{A}^\otimes||_2 \},
\] (23)
which is a subspace of space (22),
\[
B^{\otimes}(\mathcal{D}) \subset B(\mathcal{H}).
\]
It is possible to introduce a projection \( P^{\otimes} \) in the space of bounded linear operators [10], such that
\[
P^{\otimes}B(\mathcal{H}) = B^{\otimes}(\mathcal{D}),
\] (24)
with the standard properties of idempotence and self-adjointness,
\[
P^{\otimes 2} = P^{\otimes}, \quad P^{\otimes +} = P^{\otimes}.
\]
The operator norm of the projector reads as
\[
||P^{\otimes}|| = \sup_{\hat{A} \in B(\mathcal{H})} \frac{||P^{\otimes} \hat{A}||}{||\hat{A}||} \quad (\hat{A} \neq 0),
\]
where in the right-hand side the operator norms are given by (20). Then
\[
||P^{\otimes}|| = \sup_{\hat{A} \in B(\mathcal{H})} \sup_{\varphi \in \mathcal{H}} \frac{||P^{\otimes} \hat{A} \varphi||}{||\hat{A} \varphi||} \quad (\hat{A} \neq 0, \varphi \neq 0),
\]
expressed through the vector norms. Since \( \hat{A} \varphi \in \mathcal{H} \), the projector norm takes the form
\[
||P^{\otimes}|| = \sup_{\hat{A} \varphi \in \mathcal{H}} \frac{||P^{\otimes} \hat{A} \varphi||}{||\hat{A} \varphi||} \quad (\hat{A} \varphi \neq 0).
\]
The non-entangling counterpart (15) is obtained from the initial operator \( \hat{A} \) by means of the transformation

\[
P \otimes \hat{A} = \hat{A} \otimes \in \mathcal{B}(\mathcal{D}) \subset \mathcal{B}(\mathcal{H})
\]

preserving normalization (17). Therefore

\[
||P|| = 1.
\]

Keeping in mind that

\[
||\hat{A}||^2 = ||P \otimes \hat{A}||^2 = \sum_\alpha ||P \otimes \hat{A}_\alpha||^2 ,
\]

we have

\[
||P \otimes \hat{A}_\alpha|| \leq ||P|| ||\hat{A}_\alpha|| .
\]

Hence

\[
\sum_\alpha ||P \otimes \hat{A}_\alpha||^2 \leq \sum_\alpha ||\hat{A}_\alpha||^2 = ||\hat{A}||^2 ,
\]

from where

\[
||\hat{A}||_2 \leq ||\hat{A}||_2 ,
\]

which implies that measure (15) is semi-positive, satisfying (21). □

Employing the properties of the Hilbert-Schmidt norm [11,12], it is straightforward to prove that measure (15) satisfies all other conditions required for being a measure [9]. Thus, it is continuous with respect to the norm convergence, it is zero for non-entangling operators, it is additive with respect to copies, and invariant under local unitary operations.

3. **Evolutional entanglement production**

Entanglement production has been considered for density matrices of two-mode Bose systems of trapped atoms and multimode Bose systems in optical lattices [13–20]. Also, it has been studied for pseudospin entanglement production of radiating systems [21]. Here we investigate the entanglement production by the evolution operators [22].

The evolution operator of a closed quantum system,

\[
\hat{U}(t) = e^{-iHt} ,
\]

characterizes the evolution of quantum states in time, generated by a Hamiltonian \( H \). Even when an initial state \( \varphi(0) \) would be not entangled, it can become entangled with time yielding an entangled state

\[
\varphi(t) = \hat{U}(t)\varphi(0) .
\]

The entangling properties of the evolution operator are described by the entanglement-production measure

\[
\varepsilon(\hat{U}(t)) = \log \frac{||\hat{U}(t)||_2}{||\hat{U}^\otimes(t)||_2} \equiv \varepsilon(t) .
\]

Since evolution operators are unitary, so that

\[
\hat{U}^+(t)\hat{U}(t) = \hat{1}_\mathcal{H} ,
\]

with the right-hand side being the identity operator in \( \mathcal{H} \), the related norm equals the dimensionality of the Hilbert space \( \mathcal{H} \),

\[
||\hat{U}(t)||^2_2 = \text{Tr}_{\mathcal{H}} \hat{1}_\mathcal{H} = \text{dim}\mathcal{H} .
\]
The calculation of the norm for the nonentangling counterpart $||\hat{U}\otimes||_2$ is more involved. In the limit of short times, we have

$$||\hat{U}(t)||_2^2 \simeq \dim\mathcal{H} - \mu t^2,$$

with a constant $\mu$ depending on the system Hamiltonian. As a result, the short-time behavior of the entanglement-production measure is

$$\varepsilon(t) \simeq \frac{1}{2} \mu t^2 \quad (t \to 0). \quad (30)$$

4. Entanglement for two-qubit registers

Two-qubit registers are widely used for quantum information processing [1–7]. Such registers can be realized with the help of cold atoms, trapped ions, spin systems, and photons in cavities [19,23–30]. Below we show how the entanglement-production measure for arbitrary time can be explicitly calculated for two-qubit registers.

In the case of a two-qubit register, a disentangled state is a product

$$\varphi_{\text{dis}} = \varphi_1 \otimes \varphi_2,$$

in which

$$\varphi_1 = a_1 |\uparrow\rangle + a_2 |\downarrow\rangle, \quad \varphi_2 = b_1 |\uparrow\rangle + b_2 |\downarrow\rangle.$$

And an entangled state reads as

$$\varphi_{\text{ent}} = c_1 |\uparrow\uparrow\rangle + c_2 |\uparrow\downarrow\rangle + c_3 |\downarrow\uparrow\rangle + c_4 |\downarrow\downarrow\rangle,$$

where at least one of the coefficients $c_j$ is not a product of the type $a_j b_j$.

The two-qubit Hamiltonian has the form

$$H = H_0 + H_{\text{int}}, \quad (31)$$

consisting of a free term

$$H_0 = -h \left( S_z^1 \otimes \hat{I}_2 + \hat{I}_1 \otimes S_z^2 \right), \quad (32)$$

where $S_j^z$ are spin operators, while $h$ is an external field, and the interaction term is

$$H_{\text{int}} = 2JS_1^x \otimes S_2^x. \quad (33)$$

The Hamiltonian is of a two-site Ising type with the interaction strength $J$.

Calculating the entanglement-production measure, as explained in the previous sections, we obtain

$$\varepsilon(t) = \frac{1}{2} \log \frac{1 + \cos^2(ht) + 2\cos(ht)\cos(Jt)}{[1 + \cos(ht)\cos(Jt)]^2}. \quad (34)$$

At the initial time, as it should be, no entanglement is yet produced by the evolution operator,

$$\lim_{t \to 0} \varepsilon(t) = \varepsilon(0) = 0.$$

Also, notice the importance of interactions, without which no entanglement could be produced,

$$\lim_{J \to 0} \varepsilon(t) = 0 \quad (t \geq 0).$$
The measure is invariant with respect to the inversion of the signs in $h$ and $J$. At short times, the measure behaves as

$$\varepsilon(t) \simeq \frac{J^2}{8} t^2 + \frac{J^2(J^2 - 12h^2)}{192} t^4,$$

in agreement with formula (30).

The temporal behavior of the measure is either periodic, when the ratio $h/J$ is rational, or quasi-periodic, if this ratio is irrational. This is illustrated by Figs. 1 and 2, where time is measured in units of $1/J$.

5. Conclusion and possible extensions

There exist two types of entanglement measures that characterize either the entanglement of given states or the amount of entanglement produced by quantum operations. Here the latter notion is considered quantified by a measure of entanglement production. The theorem on the semi-positiveness of this measure is proved. The entanglement produced by evolution operators is analyzed. Detailed calculations are accomplished for a two-qubit register modeled by an Ising-type Hamiltonian. The measure of entanglement production for such a case is either periodic or quasi-periodic in time, depending on the relation between the values of an external field and pseudospin interactions.

It is straightforward to notice that in the same way, as has been done in the present paper, one can define thermal entanglement production by statistical operators of equilibrium systems. Let $\hat{\rho}$ be a statistical operator. Since this operator is trace-normalized to one, its non-entangling, or distilled, counterpart reads as

$$\hat{\rho}_\odot = \bigotimes_{i=1}^{N} \hat{\rho}_i,$$

with the partial statistical operators

$$\hat{\rho}_i \equiv \text{Tr}_{\mathcal{H}_i} \hat{\rho}.$$

The measure of entanglement production by the statistical operator is

$$\varepsilon(\hat{\rho}) = \log \frac{||\hat{\rho}||_2}{||\hat{\rho}_\odot||_2}.$$

In equilibrium, the statistical operator of a system with a Hamiltonian $\mathcal{H}$ has the form

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \mathcal{H}}, \quad Z = \text{Tr}_\mathcal{H} e^{-\beta \mathcal{H}},$$

where $\beta = 1/T$ is inverse temperature. It easy to notice that the exponential here can be represented through the evolution operator in imaginary time as

$$e^{-\beta \mathcal{H}} = \hat{U}(-i\beta).$$

Hence we have

$$\hat{\rho} = \frac{1}{Z} \hat{U}(-i\beta), \quad Z = \text{Tr}_\mathcal{H} \hat{U}(-i\beta).$$

With the partial term

$$\hat{\rho}_i = \frac{1}{Z} \text{Tr}_{\mathcal{H}_i} \hat{U}(-i\beta).$$
the distilled operator becomes

\[ \hat{\rho}_\otimes = \frac{1}{Z^N} \bigotimes_{i=1}^{N} \text{Tr}_{\mathcal{H}_i} \hat{U}(-i\beta) \).

Thus we come to the measure

\[ \varepsilon(\hat{\rho}) = \frac{1}{2} \log \frac{Z^{2N-2} \text{Tr}_\mathcal{H} \hat{U}(-2i\beta)}{\prod_{i=1}^{N} \text{Tr}_{\mathcal{H}_i} [\text{Tr}_{\mathcal{H}/\mathcal{H}_i} \hat{U}(-i\beta)]^2}.
\]

The calculational procedure for this measure, as is seen, reduces to that for the evolution operator. A detailed investigation of the thermal entanglement production, based on the above measure, will be treated in a separate paper.

Acknowledgement
Financial support from the Russian Foundation for Basic Research (grant # 14-02-00723) is appreciated.

Figure 1. The entanglement-production measure, for the case of periodic evolution, as a function of time measured in units of $1/J$, for different fields: (a) $h/J = 1$ (the period is $\pi$); (b) $h/J = 5/7$ (the period is $7\pi$); (c) $h/J = 7$ (the period is $\pi$); (d) $h/J = 8$ (the period is $2\pi$).
Figure 2. The measure of evolitional entanglement production, illustrating quasi-periodic behavior, for different fields: (a) $h/J = \sqrt{2}$; (b) $h/J = \sqrt{3}/2$; (c) $h/J = \sqrt{5}$; (d) $h/J = \sqrt{7}$.

[1] Williams C P and Clearwater S H 1998, Explorations in Quantum Computing (New York: Springer)
[2] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (New York: Cambridge University)
[3] Vedral V 2002 Rev. Mod. Phys. 74 197
[4] Keyl M 2002 Phys. Rep. 369 431
[5] Yukalov V I and Sornette D 2009 Entropy 11 1073
[6] Wilde M 2013, Quantum Information Theory (Cambridge: Cambridge University)
[7] Yukalov V I and Sornette D 2013 Laser Phys. 23 105502
[8] Yukalov V I 2003 Phys. Rev. Lett. 90 167905
[9] Yukalov V I 2003 Phys. Rev. A 68 022109
[10] Kuo T H 1974 Pacif. J. Math. 52 475
[11] Reed M and Simon B 1972 Methods of Modern Mathematical Physics Vol 1 (New York: Academic)
[12] Horn R A and Johnson C R 1991 Topics in Matrix Analysis (New York, Cambridge University)
[13] Yukalov V I 2003 Mod. Phys. Lett. B 17 95
[14] Yukalov V I, Yukalova E P, and Bagnato V S 2003 Laser Phys. 13 551
[15] Yukalov V I, Yukalova E P, and Bagnato V S 2003 Laser Phys. 13 861
[16] Yukalov V I and Yukalova E P 2006 Laser Phys. 16 354
[17] Yukalov V I and Yukalova E P 2006 Phys. Rev. A 73 022335
[18] Yukalov V I and Yukalova E P 2008 J. Phys. Conf. Ser. 104 012003
[19] Yukalov V I 2009 Laser Phys. 19 1
[20] Yukalov V I, Yukalova E P, and Sornette D 2014 J. Phys. Conf. Ser. 497 012034
[21] Yukalov V I 2004 Laser Phys. 14 1403
[22] Yukalov V I and Yukalova E P 2015 Phys. Rev. A 92 052121
[23] Gallagher T F 1994 Rydberg Atoms (Cambridge: Cambridge University)
\[24\] DiVincenzo D P 2000 \textit{Fortschr. Phys.} \textbf{48} 771
\[25\] Raithel G and Morrow N 2006 \textit{Adv. At. Mol. Opt. Phys.} \textbf{53} 187
\[26\] Choi J H, Knuffman B, Leibisch T C, Reinhard A, and Raithel G 2007 \textit{Adv. At. Mol. Opt. Phys.} \textbf{54} 131
\[27\] Gallagher T F and Pillet P 2008 \textit{Adv. At. Mol. Opt. Phys.} \textbf{56} 161
\[28\] Saffman M, Walker T G, and Mølmer K 2010 \textit{Rev. Mod. Phys.} \textbf{82} 2313
\[29\] Buluta I, Ashhab S, and Nori F. 2011 \textit{Rep. Prog. Phys.} \textbf{74} 104401
\[30\] Murray C and Pohl T 2016 \textit{Adv. At. Mol. Opt. Phys.} \textbf{65} 321