On the variety of the inflection points of plane cubic curves

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Abstract. In this paper we study properties of the nine-dimensional variety of the inflection points of plane cubics. We describe the local monodromy groups of the set of inflection points near singular cubic curves and give a detailed description of the normalizations of the surfaces of the inflection points of plane cubic curves belonging to general two-dimensional linear systems of cubics. We also prove the vanishing of the irregularity of a smooth manifold birationally isomorphic to the variety of the inflection points of plane cubics.

Keywords: plane cubic curves, inflection points, monodromy.

Introduction

The study of the properties of inflection points of plane non-singular cubic curves has a rich and long history. In the middle of the 19th century Hesse ([1], [2]) proved that the nine inflection points of a non-singular plane cubic form a projectively rigid configuration of points, that is, the 9-tuples of the inflection points of non-singular plane cubic curves constitute an orbit with respect to the action of the group $\text{PGL}(3, \mathbb{C})$ on the set of 9-tuples of points of the projective plane. Jordan [3] defined a subgroup $\text{Hes}$ of order 216 in $\text{PGL}(3, \mathbb{C})$ leaving invariant the set of inflection points of the Fermat cubic, which is given in $\mathbb{P}^2$ by the equation $z_1^3 + z_2^3 + z_3^3 = 0$, and called it the Hesse group. Invariants of the group $\text{Hes}$ were described by Maschke in [4]. A brief historical overview of the results concerning the inflection points of plane cubics can be found in [5].

Let $F(\bar{a}, \bar{z}) = \sum_{0 \leq i+j \leq 3} a_{i,j}z_1^i z_2^j z_3^{3-i-j}$ be a homogeneous polynomial of degree three in the variables $z_1, z_2, z_3$ and of degree one in the variables $a_{i,j}$, $0 \leq i + j \leq 3$. We write $\mathcal{C} \subset \mathbb{P}^9 \times \mathbb{P}^2$ for the complete family of plane cubic curves given by the equation $F(\bar{a}, \bar{z}) = 0$. Let $\kappa: \mathcal{C} \to \mathbb{P}^9$ and $h: \mathcal{I} = \mathcal{C} \cap \mathcal{H} \to \mathbb{P}^9$ be the restrictions of the projection $\text{pr}_1: \mathbb{P}^9 \times \mathbb{P}^2 \to \mathbb{P}^9$ to $\mathcal{C}$ and $\mathcal{I}$, where

$$\mathcal{H} = \left\{ (\bar{a}, \bar{z}) \in \mathbb{P}^9 \times \mathbb{P}^2 \mid \det\left( \frac{\partial^2 F(\bar{a}, \bar{z})}{\partial z_i \partial z_j} \right) = 0 \right\}.$$
It is well known (see, for example, [6]) that for a generic point \( \overline{a}_0 \in \mathbb{P}^9 \) the intersection of the curve \( C_{\overline{a}_0} = \kappa^{-1}_1(\overline{a}_0) \) and its Hesse curve \( H_{C_{\overline{a}_0}} \), given by
\[
\det \left( \frac{\partial^2 F(\overline{a}_0, \overline{z})}{\partial z_i \partial z_j} \right) = 0,
\]
consists of the nine inflection points of \( C_{\overline{a}_0} \). Therefore, \( \deg h = 9 \).

Let \( \mathcal{B} \subset \mathbb{P}^9 \) be the discriminant hypersurface consisting of all points \( \overline{a} \) such that the curve \( C_{\overline{a}} \) is singular. Then \( h : \mathcal{I} \setminus h^{-1}(\mathcal{B}) \to \mathbb{P}^9 \setminus \mathcal{B} \) is an unbranched covering of degree nine and, therefore, this covering determines a homomorphism \( h_* : \pi_1(\mathbb{P}^9 \setminus \mathcal{B}, \overline{a}_0) \to \mathbb{S}_9 \) (here \( \mathbb{S}_9 \) is the symmetric group acting on the set \( I_{\overline{a}} = C_{\overline{a}_0} \cap \mathcal{I} \)). The group \( \mathcal{G} = \text{Im} h_* \) is called the monodromy group of the inflection points of plane cubic curves. The following theorem was proved in [7].

**Theorem 1.** The group \( \mathcal{G} \) is a group of order 216. It is isomorphic to the group \( \text{Hes} \) of projective transformations of the plane leaving invariant the set of inflection points of the Fermat curve of degree three.

In this paper we continue the study of the properties of the varieties of inflection points of plane curves begun in [8]. In §1 we recall known facts about the fundamental groups of the complements of subvarieties of codimension 1 and about monodromy groups of dominant morphisms. In §2 we describe the local monodromy groups of the set of inflection points of plane cubic curves near the points \( \overline{a} \in \mathbb{P}^9 \) parametrizing singular cubics (Propositions 3–5). In §§3 and 4 we give a detailed description of the normalizations of the surfaces of inflection points of plane cubic curves belonging to generic two-dimensional linear systems of cubic curves, and also prove the main result of this paper (Theorem 4), namely, we prove the vanishing of the irregularity of smooth varieties birationally isomorphic to \( \mathcal{I} \).

**§1. The covering monodromy**

**1.1. On the fundamental groups of complements of subvarieties of codimension 1.** Let \( B \subset Z \) be a reduced closed subvariety of codimension 1 of a simply connected smooth variety \( Z \), \( \dim Z = k \). It is well known that the fundamental group \( \pi_1(Z \setminus B, p) \) is generated by the so-called geometric generators (or bypasses around \( B \)) \( \gamma_q \), that is, elements represented by loops \( \Gamma_q \), \( q \in B \setminus \text{Sing} B \), of the following form. Let \( L \subset Z \) be the germ of a smooth curve crossing \( B \) transversally at a point \( q \in B \setminus \text{Sing} B \), and let \( S_1 \subset L \) be a circle of small radius centred at \( q \). The right orientation on \( Z \) (defined by the complex structure) determines an orientation on \( S_1 \) and, in this case, \( \Gamma_q \) is a loop consisting of a path \( l \) lying in \( Z \setminus B \) and connecting the point \( p \) with some point \( q_1 \in S_1 \), the loop \( S_1 \) (with right orientation) starting and ending at \( q_1 \), and the return to \( p \) along \( l \). (Of course, the geometric generators \( \gamma_q \) depend not only on the choice of the point \( q \), but also on the choice of the path \( l \). If we choose different paths and different points, then the geometric generators \( \gamma_q \) are conjugate to each other in \( \pi_1(Z \setminus B, p) \) for points \( q \) belonging to the same irreducible component of \( B \).)

When \( B \) is a hypersurface in \( \mathbb{P}^k \) and \( \mathbb{P}^n \) is an \( n \)-plane in \( \mathbb{P}^k \) in general position with respect to \( B \), the Zariski–van Kampen theorem asserts that the homomorphism
\[
i_* : \pi_1(\mathbb{P}^n \setminus B) \to \pi_1(\mathbb{P}^k \setminus B)
\]
induced by an embedding \( i: \mathbb{P}^n \hookrightarrow \mathbb{P}^k \) is an isomorphism when \( n \geq 2 \) and an epimorphism when \( n = 1 \). In the case \( n = 1 \), the group \( \pi_1(\mathbb{P}^1 \setminus B, p) \) is generated by \( d = \deg B \) bypasses \( \gamma_1, \ldots, \gamma_d \) in \( \mathbb{P}^1 \setminus B \) around the points of the set \( \{q_1, \ldots, q_d\} = B \cap \mathbb{P}^1 \). An ordered set \( \{i_*(\gamma_1), \ldots, i_*(\gamma_d)\} \) is called a good geometric base of the group \( \pi_1(\mathbb{P}^k \setminus B, p) \) if the product \( \gamma_1 \cdots \gamma_d \) is equal to the identity of \( \pi_1(\mathbb{P}^1 \setminus B, p) \).

Let \( m = m_o(B) \) be the multiplicity of singularity of \( B \) at a point \( o \) and let \( z_1, \ldots, z_k \) be local analytic coordinates in an analytic neighbourhood \( V \subset Z \) of the point \( o = (0, \ldots, 0) \in B \) chosen in such a way that the intersection number \( (L_0, B)_o \) of the ‘line’ \( L_0 \subset V \) given by the equations \( z_1 = \cdots = z_{k-1} = 0 \) and the hypersurface \( B \) at the point \( o \) is equal to \( m \). Let

\[
\Delta_{\delta}^k = \{(z_1, \ldots, z_k) \in V \mid |z_i| < \delta_i \text{ for } i = 1, \ldots, k \} \subset Z
\]

be the polydisc of multiradius \( \delta = (\delta_1, \ldots, \delta_k) > 0 \) centred at \( o \). We always assume that the multiradius \( \delta = (\delta_1, \ldots, \delta_k) \) is chosen in such a way that the restriction of the projection

\[
\pr: (z_1, \ldots, z_{k-1}, z_k) \mapsto (z_1, \ldots, z_{k-1})
\]

to \( \Delta_{\delta}^k \cap B \) is a proper finite map of degree \( m \). Suppose that the ‘line’ \( L \subset \Delta_{\delta}^k \) given by the equations \( z_1 = c_1 < \delta_1, \ldots, z_{k-1} = c_{k-1} < \delta_{k-1} \) intersects \( B \) at \( m \) distinct points \( q_1, \ldots, q_m \) and \( p = (c_1, \ldots, c_{k-1}, c_k) \in L_1 \setminus B \). The fundamental group \( \pi_1(L \setminus B, p) \) is a free group of rank \( m \) generated by \( m \) bypasses \( \gamma_1, \ldots, \gamma_m \) around the points \( q_1, \ldots, q_m \). An ordered set \( \{\gamma_1, \ldots, \gamma_m\} \) is called a good geometric base of \( \pi_1(L \setminus B, p) \) if the product \( \gamma_1 \cdots \gamma_m \) is equal to the element of \( \pi_1(L \setminus B, p) \) represented by the circuit along the circle \( \Delta = \{|z_k| = \delta_k\} \). The ordered set \( \{i_*(\gamma_1), \ldots, i_*(\gamma_m)\} \) is also called a good geometric base of \( \pi_1(\Delta_{\delta}^k \setminus B, p) \). It is easy to show that the elements of a good geometric base generate the group \( \pi_1(\Delta_{\delta}^k \setminus B, p) \).

For multiradii \( \delta_1 \) and \( \delta_2 \) with \( \delta_1 \leq \delta_2 \), the embedding \( \Delta_{\delta_1}^k \subset \Delta_{\delta_2}^k \) induces a homomorphism \( i_*: \pi_1(\Delta_{\delta_1}^k \setminus B) \to \pi_1(\Delta_{\delta_2}^k \setminus B) \) of fundamental groups. The following theorem is well known (see, for example, [9]).

**Theorem 2.** There is a multiradius \( \overline{\delta} > 0 \) such that for every multiradius \( \delta \leq \overline{\delta} \), the homomorphism \( i_*: \pi_1(\Delta_{\delta}^k \setminus B) \to \pi_1(\Delta_{\delta}^k \setminus B) \) induced by the embedding of polydiscs is an isomorphism.

The group \( \pi_1^{\text{loc}}(B, o) := \pi_1(\Delta_{\delta}^k \setminus B) \) is called the local fundamental group of \( B \) at \( o \).

Let \( \Pi \) be a linear subspace of \( \Delta_{\delta}^k \), \( \dim \Pi = n \), passing through \( o \) and in general position with respect to \( B \) at \( o \). The following theorem is a direct consequence of results in [10].

**Theorem 3.** The homomorphism \( i_*: \pi_1^{\text{loc}}(B \cap \Pi, o) \to \pi_1^{\text{loc}}(B, o) \) induced by the embedding \( i: \Pi \hookrightarrow \Delta_{\delta}^k \) is an isomorphism when \( n \geq 3 \) and an epimorphism when \( n = 2 \).

The following assertions are well known.
Assertion 1. Let $B \subset \Delta_r^2$ be the curve given by the equation $z_1^3 + z_2^3 = 0$. Hence $B$ is a singularity of type $A_2$ at $o$. Then the group $\pi_1^{\text{loc}}(B, o)$ has the following presentation:

$$\pi_1^{\text{loc}}(B, o) = \langle \gamma_1, \gamma_2 : \gamma_1\gamma_2\gamma_1 = \gamma_1\gamma_2\gamma_1 \rangle,$$

where $\{\gamma_1, \gamma_2\}$ is a good geometric base of $\pi_1(\Delta_r^2 \setminus B)$.

Assertion 2. Let $B \subset \Delta_r^k$ be given by the equation $z_1 \cdots z_n = 0$, $n \leq k$. Hence $B$ is a divisor with normal crossings at $o$. Then the group $\pi_1^{\text{loc}}(B, o) \simeq \mathbb{Z}^m$ is a free Abelian group generated by the bypasses $\gamma_i$, $i = 1, \ldots, m$, around the hypersurfaces $\{z_i = 0\}$.

Lemma 1 (see [11]). Let $(B, o)$ be the germ of a curve in $\Delta_r^2$, $\sigma : \tilde{\Delta}_r^2 \to \Delta_r^2$ the $\sigma$-process centred at $o$, and $E = \sigma^{-1}(o)$ its exceptional divisor. Then the conjugacy class of the bypass $\gamma$ around $E$ in the group $\pi_1(\Delta_r^2 \setminus \sigma^{-1}(B), \sigma^{-1}(p)) = \pi_1(\Delta_r^2 \setminus B, p)$ contains the element $\Delta = \gamma_1 \cdots \gamma_m$, where $\{\gamma_1, \ldots, \gamma_m\}$ is a good geometric base of $\pi_1(\Delta_r^2 \setminus B, p)$.

1.2. On the monodromy groups of dominant morphisms. Let $f : Y \to Z$ be a dominant proper holomorphic map of a reduced complex-analytic variety $Y$ onto a smooth variety $Z$, $\dim Y = \dim Z = k$. Then there is a subvariety $B \subset Z$ of codimension 1 (which we call the discriminant of $f$) such that $f : Y \setminus f^{-1}(B) \to Z \setminus B$ is a finite unramified covering. Note that $Y \setminus f^{-1}(B)$ is a smooth variety.

Let $n$ be the degree of $f : Y \setminus f^{-1}(B) \to Z \setminus B$. The map $f$ induces a homomorphism $f_* : \pi_1(Z \setminus B, p) \to \mathbb{S}_n$ (called the monodromy of $f$) whose image $G_f := f_*(\pi_1(Z \setminus B, p))$ is called the monodromy group of $f$. It is a subgroup of the symmetric group $\mathbb{S}_n$ acting on the fibre $f^{-1}(p) = \{p_1, \ldots, p_n\}$ in the following way. A loop $\Gamma \subset Z \setminus B$ representing an element $\gamma \in \pi_1(Z \setminus B, p)$ can be lifted to $Y$ and, as a result, we get $n$ paths $\Gamma_1, \ldots, \Gamma_n \subset Y \setminus f^{-1}(B)$ starting and ending at points of $f^{-1}(p)$. Therefore this lift determines an action of $f_*\gamma$ on $f^{-1}(p)$ sending the initial point $p_i$ of the path $\Gamma_i$ to its end point for every $i = 1, \ldots, n$.

By the Grauert–Remmert–Riemann–Stein theorem ([12]), every epimorphism $\varphi_* : \pi_1(Z \setminus B) \to G \subset \mathbb{S}_n$ uniquely determines a finite covering $\varphi : X \to Z$ (which we call the Stein covering associated with $\varphi_*$) of degree $\deg \varphi = n$, unramified over $Z \setminus B$, where $X$ is a normal variety and the group $G \subset \mathbb{S}_n$ is the monodromy of $\varphi$, and if $\varphi_* = f_*$ and $Y$ is a normal variety, then there is a dominant map $\psi : Y \to X$ (the factorization map contracting the connected components of the fibres of $f$ to points of $X$) such that $f = \varphi \circ \psi$. In the case when $n = |G|$ and the embedding $G \subset \mathbb{S}_{|G|}$ is the Cayley embedding, that is, $G$ acts on itself by right multiplication, the group $G$ acts on $X$ in such a way that $X/G = Z$ and in this case $f$ is a Galois covering. The following proposition holds.

Proposition 1. If $G \subset \mathbb{S}_n$ is the monodromy group of a Stein covering $\varphi : X \to Z$ and $\mathbb{S}_n$ acts on the set $\{q_1, \ldots, q_n\}$, then for the Galois covering $\tilde{\varphi} : \tilde{X} \to X$ with Galois group $G^1 = \{g \in G \mid g(q_1) = q_1\}$ such that $\tilde{\varphi} = \varphi \circ \varphi$.

In the notation of §1.1, let $o$ be a point of the hypersurface $B$. The choice of a path connecting the base points of the fundamental groups induces a homomorphism $i_* : \pi_1^{\text{loc}}(B, o) \to \pi_1(Z \setminus B)$ and a homomorphism $f_{*, \text{loc}} = f_* \circ i_* : \pi_1^{\text{loc}}(B, o) \to G$. 
section 2.1. On the inflection points of plane cubic curves.

Consider a cubic curve $C_{\pi_0} \subset \mathbb{P}^2$ given by the equation $F(\bar{a}_0, \bar{z}) = 0$, and let $H_{C_{\pi_0}} \subset \mathbb{P}^2$ be its Hesse curve given by the equation

$$\det \left( \frac{\partial^2 F(\bar{a}_0, \bar{z})}{\partial z_i \partial z_j} \right) = 0.$$
By definition, the *inflection points* of $C_{\pi_0}$ are the points belonging to the intersection $C_{\pi_0} \cap H_{C_{\pi_0}}$.

**Assertion 5.** *The singular points of a cubic curve $C_{\pi_0}$ are its inflection points.*

**Proof.** If $s = (0, 0, 1)$ is a singular point of $C_{\pi_0}$, then $C_{\pi_0}$ is given by an equation of the form

$$z_3 \sum_{i+j=2} c_{i,j} z_1^i z_2^j + \sum_{i+j=3} c_{i,j} z_1^i z_2^j = 0.$$

It is easy to see that the elements of the third column of the Hesse matrix on the left-hand side of this equation vanish at $s$. □

**Assertion 6.** *If a line $L$ is a component of the cubic curve $C_{\pi_0}$, then the points of $L$ are inflection points of $C_{\pi_0}$.*

**Proof.** Suppose that the line $L$ is given by the equation $z_1 = 0$. Then the cubic curve $C_{\pi_0}$ is given by an equation of the form

$$z_1 \sum_{0 \leq i+j \leq 2} c_{i,j} z_1^i z_2^j z_3^{2-i-j} = 0.$$

It is easy to see that only the elements of the first column and first row of the Hesse matrix on the left-hand side of this equation can take non-zero values at the points of $L$. Hence the Hessian vanishes at the points of $L$. □

### 2.2. Equisingular stratification of the variety of singular cubic curves.

Write

$$\mathcal{S} := \{(\bar{a}, \bar{z}) \in \mathbb{P}^9 \times \mathbb{P}^2 \mid \bar{z} \in \text{Sing } C_{\pi}\}.$$

We have $\text{pr}_1(\mathcal{S}) = \mathcal{B}$. The discriminant $\mathcal{B} \subset \mathbb{P}^9$ is irreducible [13] and has a natural stratification

\[\begin{array}{c}
\mathcal{B}_7 \\ \rightarrow \\
\mathcal{B}_5 \\
\rightarrow 
\mathcal{B}_4 \\
\rightarrow \\ \mathcal{B}_{3,1} \\
\rightarrow \\
\mathcal{B}_{2,1} \\
\rightarrow \\
\mathcal{B}_1 \subset \mathcal{B}, \\
\mathcal{B}_{3,2} \\
\rightarrow \\
\mathcal{B}_{2,2}
\end{array}\]

where $\mathcal{B}_7$ is the variety parametrizing the triple lines; $\mathcal{B}_5$ is the variety parametrizing the cubic curves consisting of two lines one of which is included in the cubic with multiplicity two; $\mathcal{B}_4$ is the variety parametrizing the cubic curves consisting of three lines with a common point; $\mathcal{B}_{3,1}$ is the variety parametrizing the cubic curves consisting of three lines in general position; $\mathcal{B}_{3,2}$ is the variety parametrizing the cubic curves consisting of smooth conics and lines touching each other; $\mathcal{B}_{2,1}$ is the variety parametrizing the cubic curves consisting of transversal conics and lines; $\mathcal{B}_{2,2}$ is the variety parametrizing the cuspidal rational cubic curves; $\mathcal{B}_1$ is the variety parametrizing the nodal rational cubic curves. The arrows denote the adjacency of strata and the subscript $i$ in the notation for $\mathcal{B}_i$ and $\mathcal{B}_{i,j}$ is equal to the codimension of these strata in $\mathbb{P}^9$. All strata are smooth manifolds and, moreover, they are orbits of the action of the group $\text{PGL}(3, \mathbb{C})$ on the space of plane cubic curves. Since $\mathcal{B}$ is irreducible, all the geometric generators of the group $\pi_1(\mathbb{P}^9 \setminus \mathcal{B})$ belong to the same conjugacy class of elements of this group and generate it. We also have the following assertion.
**Assertion 7.** (i) For any points $\bar{a}_1$ and $\bar{a}_2$ in the same stratum $\mathcal{B}'$ of the equisingular stratification of $\mathcal{B}$, the local monodromy groups $G_{\bar{a}_1}$ and $G_{\bar{a}_2}$ are conjugate in $G$.

(ii) If a stratum $\mathcal{B}'$ is adjacent to a stratum $\mathcal{B}''$, then the local monodromy group $G_{\bar{a}_1}$ at a point $\bar{a}_1 \in \mathcal{B}'$ contains a subgroup conjugate in $G$ to the local monodromy groups $G_{\bar{a}_2}$ at the points $\bar{a}_2 \in \mathcal{B}''$.

**Proof.** The group $\text{PGL}(3, \mathbb{C})$ is a connected complex manifold. This yields part (i) since each stratum $\mathcal{B}'$ is an orbit under the action of $\text{PGL}(3, \mathbb{C})$ on the variety of plane cubic curves.

Part (ii) follows since for every sufficiently small neighbourhood $V_1 \subset \mathbb{P}^9$ of a point $\bar{a}_1 \in \mathcal{B}'$ one can find a point $\bar{a} \in \mathcal{B}''$ such that some neighbourhood $V$ of $\bar{a}$ lies in $V_1$. □

The space $\mathbb{P}^9$ has a natural covering by ten affine spaces $\mathbb{C}_{i,j}^9 = \{ \bar{a} \in \mathbb{P}^9 \mid a_{i,j} \neq 0 \}$, $0 \leq i + j \leq 3$. The coordinates of the points $\bar{a}$ in each space $\mathbb{C}_{i_0,j_0}^9$ are the coefficients of the equations

$$
\sum_{0 \leq i+j \leq 3} a_{i,j} z_1^i z_2^j z_3^{3-i-j} = 0
$$

of plane cubic curves normalized by the condition $a_{i_0,j_0} = 1$.

Properties of the discriminant $\mathcal{B}$ at the points corresponding to nodal cubics were investigated in [14].

**Proposition 2** [14]. Let $C_{\bar{a}_0}$ be a nodal cubic curve. Then $\mathcal{B}$ is a divisor with normal crossings in some analytic neighbourhood of $\bar{a}_0$.

To calculate the local monodromy groups of the morphism $h$ at the points corresponding to nodal cubics, we need a more detailed description of the discriminant $\mathcal{B}$ at these points. Therefore we consider a nodal cubic curve $C_{\bar{a}_0}$ given in the non-homogeneous coordinates $x = z_1/z_3$, $y = z_2/z_3$ on $\mathbb{C}^2 \subset \mathbb{P}^2$ by the equation

$$
xy + \sum_{i+j=3} c_{i,j} x^i y^j = 0.
$$

The point $\bar{a}_0$ lies in the affine space $\mathbb{C}_{1,1}^9$ and its coordinates $a_{0,0}$, $a_{1,0}$ and $a_{0,1}$ vanish. The variety $\mathcal{S} \cap (\mathbb{C}_{1,1}^9 \times \mathbb{C}^2)$ is clearly given by the equations

$$
\begin{align*}
a_{0,0} + a_{1,0} x + a_{0,1} y + xy + a_{2,0} x^2 + a_{0,2} y^2 + \sum_{i+j=3} a_{i,j} x^i y^j &= 0, \\
a_{1,0} + y + 2a_{2,0} x + \sum_{i+j=3} ia_{i,j} x^{i-1} y^j &= 0, \\
a_{0,1} + x + 2a_{0,2} y + \sum_{i+j=3} ja_{i,j} x^i y^{j-1} &= 0.
\end{align*}
$$

(2)

It follows from the specific form of the equations (2) that the eight-dimensional variety $\mathcal{S}$ is smooth at the point $(\bar{a}_0, z_0)$, where $z_0 = (0, 0, 1)$ and the functions $a_{1,0}$, $a_{0,1}$ and $a_{i,j} - c_{i,j}$ with $2 \leq i + j \leq 3$, $(i,j) \neq (1,1)$, are local parameters on $\mathcal{S}$ at $(\bar{a}_0, \bar{z}_0)$. Hence the image $\text{pr}_1(\mathcal{S} \cap V) \subset \mathbb{C}_{1,1}^9$ is a non-singular hypersurface at
the point \(\bar{a}_0\), where \(V\) is a sufficiently small analytic neighbourhood of the point \((\bar{a}_0, \bar{z}_0)\) in \(\mathbb{C}^9_1 \times \mathbb{C}^2\). Moreover, we easily see that \(a_{0,0} = 0\) is the equation of the tangent space to \(\text{pr}_1(S \cap V)\) at \(\bar{a}_0\).

The strata \(B_1, B_{2,1}\) and \(B_{3,1}\) of the discriminant hypersurface \(B\) parametrize the nodal cubic curves. Since these strata are orbits in \(\mathbb{P}^9\) under the action of \(\text{PGL}(3, \mathbb{C})\), we can assume without loss of generality that the nodal cubic \(C_{\pi_0}\) is given by one of the following three equations:

\[
\begin{align*}
1) \quad & z_1z_2z_3 + z_1^3 + z_2^3 = 0 \quad \text{if } \bar{a}_0 \in B_1, \\
2) \quad & z_1z_2z_3 + z_1^3 = 0 \quad \text{if } \bar{a}_0 \in B_{2,1}, \\
3) \quad & z_1z_2z_3 = 0 \quad \text{if } \bar{a}_0 \in B_{3,1}.
\end{align*}
\]

In all three cases, the point \(\bar{a}_0\) belongs to the affine space \(\mathbb{C}^9_1\). In case 1), \(B\) is a non-singular hypersurface at \(\bar{a}_0\). In case 2), \(B\) is locally a union of two hypersurfaces which are non-singular at \(\bar{a}_0\) and intersect each other transversally at this point (the tangent spaces to these hypersurfaces are given by the equations \(a_{0,0} = 0\) and \(a_{0,3} = 0\)). In case 3), \(B\) is locally a union of three hypersurfaces which are non-singular at \(\bar{a}_0\) and intersect each other transversally at this point (the tangent spaces to these hypersurfaces are given by the equations \(a_{0,0} = 0\), \(a_{0,3} = 0\) and \(a_{3,0} = 0\)).

### 2.3. Properties of the monodromy group \(\mathcal{G} = \text{Hes}\) of the morphism \(h\). The group \(\text{Hes}\) was originally defined as the subgroup of the group \(\text{PGL}(3, \mathbb{C})\) of linear transformations of \(\mathbb{P}^2\) that leaves invariant the Hesse pencil, that is, the one-dimensional linear system of plane cubic curves given by the equation

\[
C_{(t_1, t_2)} : \quad t_1(z_1^3 + z_2^3 + z_3^3) + t_2z_1z_2z_3 = 0, \quad (t_1, t_2) \in \mathbb{P}^1. \tag{3}
\]

The Hesse pencil has nine fixed points

\[
\begin{align*}
q_1 &= (0, 1, -1), & q_4 &= (0, 1, -\omega), & q_7 &= (0, 1, -\omega^2), \\
q_2 &= (1, 0, -1), & q_5 &= (1, 0, -\omega^2), & q_8 &= (1, 0, -\omega), \\
q_3 &= (1, -1, 0), & q_6 &= (1, -\omega, 0), & q_9 &= (1, -\omega^2, 0),
\end{align*}
\]

where \(\omega = e^{2\pi i/3}\) is a primitive cube root of unity. A direct verification shows that these nine points are inflection points of every non-singular curve in the Hesse pencil. Therefore, the group \(\text{Hes} \subset \text{PGL}(3, \mathbb{C})\) can also be defined as the subgroup of projective transformations leaving invariant the set of inflection points of the Fermat cubic \(F = C_{(1,0)}\).

The properties of the group \(\text{Hes}\) are well known (see, for example, [15]). Its order is equal to 216 and its action on the nine inflection points of \(F\) determines an embedding \(\text{Hes} \subset S_9\) such that \(\text{Hes}\) is a 2-transitive subgroup of the symmetric group \(S_9\) generated by the permutations

\[
g_0 = (1, 2, 4)(5, 6, 8)(3, 9, 7) \quad \text{and} \quad g_1 = (4, 5, 6)(7, 9, 8).
\]

The subgroups

\[
\text{Hes}^i = \{g \in \text{Hes} \mid g(q_i) = q_i\}, \quad i = 1, \ldots, 9,
\]
which consist of the elements of Hes fixing the point \(q_i\), are conjugate to each other in Hes and isomorphic to \(SL(2, \mathbb{Z}_3)\). Their order is 24. As a subgroup of \(S_9\), the group Hes\(^1\) is generated by the permutations \(g_1\) and \(g_2 = g_0g_1g_0^{-1} = (2, 8, 5)(3, 6, 9)\).

As an abstract group, it has the following presentation:

\[
\text{Hes}^1 = \{g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2, g_1^3 = 1\}.
\]

We note for further use that the class of elements conjugate to \(g_1\) in Hes\(^1\) consists of four elements: \(g_1, g_2, g_1g_2, g_1^{-1}, g_2g_1g_2^{-1}\), and every pair of these four elements generates the group Hes\(^1\) and satisfies the relation \(xyz = yxy\). Moreover (see [16]), the group Hes\(^1\) has a faithful linear representation in \(GL(2, \mathbb{C})\) (group no. 4 in [16]; see also [17]) such that the images of \(g_1\) and \(g_2\) are reflections.

The group Hes\(^1\) is naturally embedded in the symmetric group \(S_8\) acting on \(\{q_2, \ldots, q_8\}\). We enumerate the (left) cosets of the subgroup \(\langle g_1 \rangle\) generated by \(g_1\) in Hes\(^1\) as follows:

\[
\begin{align*}
c_2 &= \langle g_1 \rangle, & c_3 &= g_2^2g_1g_2^2\langle g_1 \rangle, & c_4 &= g_1^2g_2^2\langle g_1 \rangle, & c_5 &= g_2^2\langle g_1 \rangle, \\
c_6 &= g_1g_2^2\langle g_1 \rangle, & c_7 &= g_1g_2\langle g_1 \rangle, & c_8 &= g_2\langle g_1 \rangle, & c_9 &= g_1^2g_2\langle g_1 \rangle.
\end{align*}
\]

The group Hes\(^1\) acts on the set \(\{c_2, \ldots, c_9\}\).

Put Hes\(^1,2\) = \(\{g \in \text{Hes}^1 \subset S_8 \mid g(q_2) = q_2\}\).

**Assertion 8.** We have Hes\(^1,2\) = \(\langle g_1 \rangle\) and the action of the group

\[
\text{Hes}^1 = \langle g_1 = (4, 5, 6)(7, 9, 8), g_2 = (2, 8, 5)(3, 6, 9) \rangle \subset S_8
\]

on the set \(\{c_2, \ldots, c_9\}\) coincides with the action on the set \(\{q_2, \ldots, q_9\}\).

**Proof.** Write down all 24 permutations belonging to the group Hes\(^1\) \(\subset S_8\). Then we easily see that the assertion holds. \(\square\)

**2.4. On the local monodromy groups of the morphism \(h\) at points corresponding to nodal cubic curves.**

**Proposition 3.** (i) The local monodromy groups of \(h\) at the points \(\overline{a}_1 \in B_1\) are Abelian groups isomorphic to \(\mathbb{Z}_3\) and conjugate in Hes to the subgroup \(G_1 \subset S_9\) generated by the permutation \(g_2 = (2, 8, 5)(3, 6, 9)\).

(ii) The local monodromy groups of \(h\) at the points \(\overline{a}_{2,1} \in B_{2,1}\) are Abelian groups isomorphic to \(\mathbb{Z}_2^3\) and conjugate in Hes to the subgroup \(G_{2,1} \subset S_9\) generated by the permutations \(g_2 = (2, 8, 5)(3, 6, 9)\) and \(g_3 = (1, 4, 7)(3, 9, 6)\).

(iii) The local monodromy groups of \(h\) at the points \(\overline{a}_{3,1} \in B_{3,1}\) are Abelian groups isomorphic to \(\mathbb{Z}_2^3\) and conjugate in Hes to the subgroup \(G_{3,1} \subset S_9\) generated by the permutations \(g_2 = (2, 8, 5)(3, 6, 9)\) and \(g_3 = (1, 4, 7)(3, 9, 6)\). Hence these groups coincide with \(G_{2,1}\).

**Proof.** We first prove part (iii). There is no loss of generality in assuming that the nodal curve \(C_{\overline{a}_{3,1}}\) is given by the equation \(z_1z_2z_3 = 0\).

Consider a three-dimensional family \(C_{\overline{a}_{1}}\) of cubic curves defined by the equation

\[
z_1z_2z_3 + a_{3,0}z_1^3 + a_{0,3}z_2^3 + a_{0,0}z_3^3 = 0
\]

\[\tag{4}\]
and its projection to the affine linear subspace \( \text{pr}_1(C_{\Pi_1}) = \Pi_1 \simeq \mathbb{C}^3 \) of \( \mathbb{C}^9_{1,1} \subset \mathbb{P}^9 \). Note that \((a_{3,0}, a_{0,3}, a_{0,0})\) are affine coordinates in \( \Pi_1 \) and it can easily be checked that the Hessian \( H_{C_{\Pi_1}} \) of this family is given by the equation

\[
(b^3 a_{3,0} a_{0,3} a_{0,0} + 2)z_1 z_2 z_3 + 6(a_{3,0} z_1^3 + a_{0,3} z_2^3 + a_{0,0} z_3^3) = 0.
\]

Therefore \( h^{-1}(\Pi_1) \) is given in \( \Pi_1 \times \mathbb{P}^2 \) by the equations

\[
z_1 z_2 z_3 = a_{3,0} z_1^3 + a_{0,3} z_2^3 + a_{0,0} z_3^3 = 0.
\]

Write \( p = (\delta, \delta, \delta) \in \Pi_1 \), where \( 0 < \delta \ll 1 \), and put \( C_p = \kappa^{-1}(p) \). Note that the cubic curve \( C_p \) has the same inflection points as the Fermat curve (see § 2.3).

It follows from the results in § 2.2 that, locally at the point \( \overline{a}_{3,1} \) with coordinates \( a_{3,0} = a_{0,3} = a_{0,0} = 0 \), the space \( \Pi_1 \) has transversal intersection with irreducible branches of the hypersurface \( B \) and, in a neighbourhood of this point, the surface \( B \cap \Pi_1 \) is given in \( \Pi_1 \) by the equation \( a_{3,0} a_{0,3} a_{0,0} = 0 \). By Theorem 3, the group \( \pi_1^{\text{loc}}(B, \overline{a}_{3,1}) \) is generated by the elements \( \gamma_j \in \pi_1(\Pi_1 \setminus B, p), j = 1, 2, 3 \), represented by the following loops (see Assertion 2):

\[
c_1(t) = \{ (\delta e^{2\pi it}, \delta, \delta) \in \Pi_1 \mid t \in [0, 1] \},
\]

\[
c_2(t) = \{ (\delta, \delta e^{2\pi it}, \delta) \in \Pi_1 \mid t \in [0, 1] \},
\]

\[
c_3(t) = \{ (\delta, \delta, \delta e^{2\pi it}) \in \Pi_1 \mid t \in [0, 1] \}.
\]

The pre-image \( h^{-1}(c_1) \) consists of nine paths

\[
q_1(t) = (c_1(t), (0, 1, -1)),
\]

\[
q_2(t) = (c_1(t), (1, 0, -\lambda(t))),
\]

\[
q_3(t) = (c_1(t), (1, -\lambda(t), 0)),
\]

\[
q_4(t) = (c_1(t), (0, 1, -\omega)),
\]

\[
q_5(t) = (c_1(t), (1, 0, -\lambda(t)\omega^2)),
\]

\[
q_6(t) = (c_1(t), (1, -\lambda(t)\omega, 0)),
\]

\[
q_7(t) = (c_1(t), (0, 1, -\omega^2)),
\]

\[
q_8(t) = (c_1(t), (1, 0, -\lambda(t)\omega)),
\]

\[
q_9(t) = (c_1(t), (1, -\lambda(t)\omega^2, 0)),
\]

where \( \lambda(t) = e^{2\pi it/3} \) and \( \omega = e^{2\pi i/3} \) and, therefore, \( h_*(\gamma_1) = (2, 8, 5)(3, 6, 9) = g_2 \in S_9 \).

The pre-image \( h^{-1}(c_2) \) consists of nine paths

\[
q_1(t) = (c_2(t), (0, 1, -\lambda(t))),
\]

\[
q_2(t) = (c_2(t), (1, 0, -1)),
\]

\[
q_3(t) = (c_2(t), (1, -\lambda^2(t), 0)),
\]

\[
q_4(t) = (c_2(t), (0, 1, -\lambda(t)\omega)),
\]

\[
q_5(t) = (c_2(t), (1, 0, -\omega^2)),
\]

\[
q_6(t) = (c_2(t), (1, -\lambda^2(t)\omega, 0)),
\]

\[
q_7(t) = (c_2(t), (0, 1, -\lambda(t)\omega^2)),
\]

\[
q_8(t) = (c_2(t), (1, 0, -\omega)),
\]

\[
q_9(t) = (c_2(t), (1, -\lambda^2(t)\omega^2, 0))
\]

and, therefore, \( h_*(\gamma_2) = (1, 4, 7)(3, 9, 6) = g_3 \).

Similarly, the pre-image \( h^{-1}(c_3) \) consists of nine paths

\[
q_1(t) = (c_3(t), (0, 1, -\lambda^2(t))),
\]

\[
q_2(t) = (c_3(t), (1, 0, -\lambda^2(t))),
\]

\[
q_3(t) = (c_3(t), (1, -1, 0)),
\]

\[
q_4(t) = (c_3(t), (0, 1, -\lambda^2(t)\omega)),
\]

\[
q_5(t) = (c_3(t), (1, 0, -\lambda^2(t)\omega^2)),
\]

\[
q_6(t) = (c_2(t), (1, -\omega, 0)),
\]

\[
q_7(t) = (c_3(t), (0, 1, -\lambda^2(t)\omega^2)),
\]

\[
q_8(t) = (c_3(t), (1, 0, -\lambda^2(t)\omega)),
\]

\[
q_9(t) = (c_2(t), (1, -\omega^2, 0))
\]
and, therefore, \( h_*(\gamma_3) = (1, 7, 4)(2, 5, 8) := g_4 \). It is easy to see that \( g_2g_3 = g_4^{-1} \).

Therefore the local monodromy group of the morphism \( h \) at \( a_{3,1} \) is the group \( \mathcal{G}_{3,1} \simeq \mathbb{Z}_3^3 \) generated in \( \mathcal{G} = \text{Hes} \) by the permutations \( g_2 \) and \( g_3 \).

To prove part (ii), we may assume that the point \( \overline{a}_{2,1} \in \Pi_1 \) has coordinates \( a_{3,0} = a_{0,3} = 0 \) and \( a_{0,0} = \varepsilon \), where \( 0 < \varepsilon \ll \delta \), that is, the nodal cubic \( C_{\overline{a}_{2,1}} \) is given by the equation \( z_1z_2z_3 + \varepsilon z_3^3 = 0 \). Then it is easy to see that the local fundamental group of the morphism \( h \) at \( \overline{a}_{2,1} \) is generated by \( \gamma_2 \) and \( \gamma_3 \). And in the case (i) we can assume that the point \( \overline{a}_1 \in \Pi_1 \) has coordinates \( a_{3,0} = 0 \) and \( a_{0,0} = a_{0,3} = \varepsilon \), where \( 0 < \varepsilon \ll \delta \), that is, the nodal cubic \( C_{\overline{a}_{2,1}} \) is given by the equation \( z_1z_2z_3 + \varepsilon z_3^3 + \varepsilon z_3^3 = 0 \). Then it is easy to see that the local fundamental group of the morphism \( h \) at \( \overline{a}_1 \) is generated by \( \gamma_2 \).

\[ \square \]

2.5. On the local monodromy groups of the morphism \( h \) at points corresponding to cuspidal cubic curves.

**Proposition 4.** The local monodromy groups \( \mathcal{G}_a \) of the morphism \( h \) at points \( \overline{a} \in \mathcal{B}_{2,2} \) are conjugate to the group \( \text{Hes}^1 \).

**Proof.** We can assume without loss of generality that the cuspidal cubic curve \( C_{\overline{a}_0} \subset \mathbb{P}^2 \) is given by the equation \( z_1^3 + z_2^2z_3 = 0 \). Its singular point \( s \) has coordinates \((0, 0, 1)\), and the point \( p = (0, 1, 0) \) is its unique non-singular inflection point.

The fibre \( h^{-1}(\overline{a}_0) \) of \( h \) is the intersection of the Hessian variety \( \mathcal{H} \) and the fibre \( C_{\overline{a}_0} \) of the morphism \( \kappa: \mathcal{C} \to \mathbb{P}^0 \). Calculating the Hessian of the polynomial \( z_1^3 + z_2^2z_3 \), we find that the Hesse curve \( H_{C_{\overline{a}_0}} \subset \mathbb{P}^2 \) is given by the equation \( z_1z_2^2 = 0 \). It is easy to see that the intersection numbers of the curves \( C_{\overline{a}_0} \) and \( H_{C_{\overline{a}_0}} \) at the points \( p \) and \( s \) are equal to \((C_{\overline{a}_0}, H_{C_{\overline{a}_0}}) = 1 \) and \((C_{\overline{a}_0}, H_{C_{\overline{a}_0}}) = 8 \). It follows that the variety \( \mathcal{I} \) is non-singular in some analytic neighbourhood \( V \subset \mathcal{I} \) of the point \( (\overline{a}_0, p) \) and the restriction of \( h \) to \( V \) is a biholomorphic isomorphism of \( V \) onto its image \( h(V) \). Therefore at least one orbit of the action of the local monodromy group \( \mathcal{G}_{\overline{a}_0} \) of the morphism \( h \) on the set \( \{q_1, \ldots, q_9\} \) consists of one point and, therefore, \( \mathcal{G}_{\overline{a}_0} \) is contained in a group conjugate to \( \text{Hes}^1 \).

It will be shown in § 3.2 that the common point \( \overline{a}_0 \in \mathcal{B} \cap \Pi \) of \( \mathcal{B} \) and a generic projective plane \( \Pi \subset \mathbb{P}^9 \) through \( \overline{a}_0 \) is an ordinary cusp of the curve \( \mathcal{B} \cap \Pi \). The group \( \pi^c_1(\mathcal{B} \cap \Pi, \overline{a}_0) \) (see Assertion 1) is generated by the geometric generators \( \gamma_1 \) and \( \gamma_2 \) satisfying the relation \( \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 \), and the elements \( h_*(\gamma_1) \) and \( h_*(\gamma_2) \) generate the group \( \mathcal{G}_{\overline{a}_0} \). It follows from the properties of the group \( \text{Hes}^1 \) (see § 2.3) that if \( h_*(\gamma_1) \neq h_*(\gamma_2) \), then the group \( \mathcal{G}_{\overline{a}_0} \) is conjugate in \( \mathcal{G} \) to the group \( \text{Hes}^1 \), but if \( h_*(\gamma_1) = h_*(\gamma_2) \), then \( \mathcal{G}_{\overline{a}_0} \) is a cyclic group of order 3.

We claim that the second case does not occur. Indeed, consider the family of cubic curves \( C_{\overline{a}_\tau} \) parametrized by the points of the circle \( S_1 = \{\overline{a}_\tau \in \mathbb{P}^9\}, \tau = \delta e^{2\pi i t} \), and given by the equation

\[ C_{\overline{a}_\tau}: \quad z_1^3 + z_2^2z_3 + \tau z_3^3 = 0, \]  

where \( t \in [0, 1] \) and \( \delta \) is a positive real number. It is easy to see that the Hessian variety \( H_{C_{\overline{a}_\tau}} \) of this family is given by the equation

\[ z_1(3\tau z_3^2 - z_2^2) = 0. \]
It follows from (6) and (7) that the pre-image $h^{-1}(S_1) \subset \mathcal{I}$ of the circle $S_1$ consists of nine paths. Three of them are

\begin{align*}
l_1(t) &= \{ (\overline{a}(t), \overline{z}(t)) \in \mathcal{I} \mid \overline{a}(t) = \overline{a}_\tau(t), \overline{z}(t) = (0, 1, 0) \}, \\
l_j(t) &= \{ (\overline{a}(t), \overline{z}(t)) \in \mathcal{I} \mid \overline{a}(t) = \overline{a}_\tau(t), \overline{z}(t) = (0, \sqrt{-\delta} e^{\pi i(t+j)}, 1) \}, & j &= 2, 3,
\end{align*}
and the other six are

\begin{align*}
l_{j,k}(t) &= \{ (\overline{a}(t), \overline{z}(t)) \in \mathcal{I} \mid \overline{a}(t) = \overline{a}_\tau(t), \overline{z}(t) = \left(-\sqrt{4\delta} e^{2\pi i(t+j)/3}, \sqrt{3\delta} e^{\pi i(t+k)}, 1 \right) \},
\end{align*}
where $j = 0, 1, 2$ and $k = 0, 1$. If $\delta \ll 1$, then $S_1$ (up to conjugation) represents an element $\gamma \in \pi^1_{\text{loc}}(B, \overline{a}_0)$ and it is easy to see that the cyclic permutation type of $h_*(\gamma) \in \mathcal{G}_{\pi_0}$ is $(6, 2, 1)$. □

2.6. On the local monodromy groups of the morphism $h$ at points belonging to strata of $\mathcal{B}$ of codimension $\geq 3$. Consider a point $\pi \in B_{3,2}$. In every neighbourhood of this point there are points $\overline{a}_1 \in B_{2,1}$ and $\overline{a}_2 \in B_{2,2}$. Therefore, by Assertion 7, the groups $\mathcal{G}_{\overline{a}_1}$ and $\mathcal{G}_{\overline{a}_2} \simeq \text{Hes}^1$ are subgroups of the local monodromy group $\mathcal{G}_{\pi} \subset \text{S}_9$. The action of $\mathcal{G}_{\pi}$ on the set $\{1, \ldots, 9\}$ has no fixed elements, and the group $\text{Hes}^1$ acts transitively on the set $\{2, \ldots, 9\}$. Hence, $\mathcal{G}_{\pi}$ contains all the groups $\text{Hes}^i$, $i = 1, \ldots, 9$.

Since $\mathcal{B}$ is an irreducible variety, it follows from Proposition 3 that the monodromy group $\mathcal{G}$ of the morphism $h$ is generated by permutations of cyclic type $(3,3,1,1,1)$. Each of these generators is contained in some group $\text{Hes}^i$. This yields the following proposition.

Proposition 5. The local monodromy group $\mathcal{G}_{\pi}$ of the morphism $h$ at a point $\pi \in B_{3,2}$ and the local monodromy groups $\mathcal{G}_{\pi}$ at points $\overline{a}$ belonging to the strata $\mathcal{B}'$ of codimension $\geq 4$ in $\mathcal{B}$ coincide with the whole group $\mathcal{G} = \text{Hes}$.

§ 3. Surfaces of inflection points of generic two-dimensional linear systems of plane cubic curves

3.1. Two-dimensional linear systems of plane cubic curves. Let $\overline{z} = (z_1, z_2, z_3)$ and $\overline{t} = (t_1, t_2, t_3)$ be homogeneous coordinates in $\mathbb{P}^2$ and in a projective plane $\Pi \subset \mathbb{P}^9$ respectively. Consider a (projectively) two-dimensional linear system $C_\Pi$ of plane cubic curves given by the equation $F(\overline{t}, \overline{z}) = 0$, where

$$F(\overline{t}, \overline{z}) = t_1 F(\overline{a}_1, \overline{z}) + t_2 F(\overline{a}_2, \overline{z}) + t_3 F(\overline{a}_3, \overline{z}).$$

Let $H(\overline{t}, \overline{z}) = \det(\partial^2 F(\overline{t}, \overline{z})/\partial z_i \partial z_j)$ be the Hessian of the polynomial $F(\overline{t}, \overline{z})$. The equations

$$F(\overline{t}, \overline{z}) = 0, \quad H(\overline{t}, \overline{z}) = 0 \quad (8)$$
on $\Pi \times \mathbb{P}^2$ determine a surface $\mathcal{I}_\Pi = \mathcal{I} \cap \text{pr}^{-1}(\Pi)$ of inflection points of cubic curves belonging to the linear system $C_\Pi$.

Put $x = z_1/z_3, y = z_2/z_3$ and $\alpha = t_2/t_1, \beta = t_3/t_1$. In what follows, without special mention, we will choose the basis elements $F_1(z) := F(\overline{a}_1, \overline{z})$ of the linear system $C_\Pi$ and the homogeneous coordinates $(z_1, z_2, z_3)$ in $\mathbb{P}^2$ in such a way that the equations (8) in the non-homogeneous coordinates $(\alpha, \beta, x, y)$ of $\mathbb{C}^2 \times \mathbb{C}^2 \subset \Pi \times \mathbb{P}^2$ take the most convenient form for the study of the properties of the variety $\mathcal{I}_\Pi$. 
3.2. Generic projections to the plane. The complete linear system of cubic curves parametrized by the points of \( \mathbb{P}^9 \) determines the so-called Veronese-3 embedding \( \varphi_3 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^9, \varphi_3(\mathbb{P}^2) = \mathcal{V}_3 \), of the projective plane \( \mathbb{P}^2 \) in the projective space dual to the space \( \mathbb{P}^9 \) of parameters. Two-dimensional linear subspaces \( \Pi \subset \mathbb{P}^9 \) determine linear projections \( \text{pr}_{\Pi} : \mathbb{P}^9 \to \mathbb{P}^9 \) onto the projective planes \( \mathbb{P}^9 \) that are dual to the projective planes \( \Pi \). It is well known (see, for example, [18]) that for all planes \( \Pi \) in a certain Zariski-open and dense subset \( \mathcal{W} \) of the Grassmannian \( \text{Gr}(3, 10) \) of two-dimensional planes in \( \mathbb{P}^9 \), the composites \( \xi := \text{pr}_{\Pi} \circ \varphi_3 : \mathbb{P}^2 \to \mathbb{P}^9 \) are generic coverings of the projective plane, that is, they possess the following properties:

(i) \( \xi \) is a finite morphism of degree nine;

(ii) the ramification curve \( \widetilde{R} \subset \mathbb{P}^2 \) of \( \xi \) is non-singular and the ramification index along \( \widetilde{R} \) is equal to 2;

(iii) the discriminant curve \( \hat{B} = \xi(\widetilde{R}) \) has no singular points except for ordinary nodes and cusps, and the restriction of \( \xi \) to \( \widetilde{R} \) is a birational morphism.

It follows from the proof of Theorem 4 in [19] that the degree of the curve \( \hat{B} \) is equal to 18, \( \hat{B} \) has 42 ordinary cusps and 84 ordinary nodes, and the geometric genus \( g(\hat{B}) \) of the curve \( \hat{B} \) is equal to 10. The curve \( \widetilde{R} \subset \mathbb{P}^2 \) is a smooth plane curve of degree six.

The cubic curves \( C_{\pi} = \xi^{-1}(L_{\pi}) \) parametrized by the points of a plane \( \Pi \in \mathcal{W} \) are inverse images of the lines \( L_{\pi} \) in \( \mathbb{P}^9 \) dual to the points \( \pi \in \Pi \). A cubic curve \( C_{\pi} \) is singular if and only if the line \( L_{\pi} \) is tangent to \( \hat{B} \). Therefore, the curve \( \hat{B} \) is dual to the curve \( B := B \cap \Pi \), and when the cubic curve \( C_{\pi} \) is singular, its singular points belong to \( C_{\pi} \cap \widetilde{R} \). For every point \( q \in L_{\pi} \cap \hat{B} \), only one point of \( \xi^{-1}(q) \) may be a singular point of \( C_{\pi} \). Moreover, the restriction of the projection \( \text{pr}_2 : \Pi \times \mathbb{P}^2 \to \mathbb{P}^2 \) to \( S = \mathcal{S} \cap (\Pi \times \mathbb{P}^2) \) is a one-to-one map onto the curve \( \widetilde{R} \).

It follows from the description of the equisingular stratification of singular cubic curves (see § 2.2) that \( \hat{B} \) may have only bitangent and three-tangent lines \( L_{\pi} \), (in this case, the cubic curves \( C_{\pi} \), are unions either of transversally intersecting lines and quadrics, that is, \( \pi_i \in \mathcal{B}_{2,1} \), or of three lines, that is, \( \pi_i \in \mathcal{B}_{3,1} \)). Moreover, if \( L_{\pi} \) is the tangent line to \( \hat{B} \) at an inflection point, then \( C_{\pi} \) is a cuspidal cubic, that is, \( \pi_i \in \mathcal{B}_{2,2} \). Therefore, by Plicker’s formulae (see, for example, [20]), the degree of \( B \) is equal to 12 and the singular points of \( B \) are 24 ordinary cusps, \( n_1 \) ordinary nodes and \( n_2 \) ordinary triple points (singularities of type \( D_4 \)), where \( n_1 + 3n_2 = 21 \).

Remark 1. If the plane \( \Pi \in \mathcal{W} \) is sufficiently general, that is, \( \Pi \cap \mathcal{B}_{3,1} = \emptyset \), then the curves \( B \subset \Pi \) and \( \hat{B} \subset \mathbb{P}^9 \) give an example of dual cuspidal curves such that the fundamental groups \( \pi_1(\Pi \setminus B) \) and \( \pi_1(\hat{B} \setminus \hat{B}) \) are not Abelian.

We write

\[ \mathcal{B}_{\widetilde{R},1} = \{ \zeta \in \widetilde{R} \mid \xi(\zeta) \text{ is an inflection point of } \hat{B} \}, \]

\[ \mathcal{B}_{\widetilde{R},2} = \{ \zeta \in \widetilde{R} \mid \xi(\zeta) \text{ is a cusp of } \hat{B} \}. \]

The set \( \mathcal{B}_{\widetilde{R},1} \) consists of 24 points and \( \mathcal{B}_{\widetilde{R},2} \) consists of 42 points.

In what follows we assume that \( \Pi \in \mathcal{W} \).
3.3. Properties of the Stein coverings associated with the morphisms $h_{\Pi}$.
A two-dimensional linear system $C_{\Pi}$ of cubic curves determines a generic covering of the plane $\xi: \mathbb{P}^2 \to \overset{\circ}{\Pi}$ and it follows from the Zariski–van Kampen theorem that the monodromy group of the covering $h_{\Pi}: \mathcal{I}_{\Pi} \to \Pi$ is the group $\mathcal{G} = \text{Hes} \subset S_9$.

Consider the Stein covering $\varphi: Y \to \Pi$, $\deg \varphi = 9$, associated with the morphism $h_{\Pi}$. The covering $\varphi$ is unramified over $\Pi \setminus B$. The following description of the properties of $\varphi: \varphi^{-1}(V) \to V$, where $V \subset \Pi$ is a sufficiently small complex-analytic neighbourhood of a point $o \in B$, is based on the results in §§2 and 1.2.

If $o \in B_1$, then the local monodromy group of $\varphi$ at $o$ is the cyclic group generated by a permutation of cycle type $(1,1,1,3,3)$. Therefore, $\varphi^{-1}(V)$ is a disjoint union of five irreducible open subsets of $Y$, $\varphi^{-1}(V) = \bigcup_{i=1}^{5} U_i$. Each of these subsets is non-singular and, up to a relabelling, the coverings $\varphi_i: U_i \to V$ with $i = 1, 2, 3$ are biholomorphic maps while the coverings $\varphi: U_i \to V$ with $i = 4, 5$ are three-sheeted coverings branched along $B \cap V$.

If $o \in B_2$, and $B \cap V = B_1 \cup B_2$, where $B_1'$ and $B_2'$ are branches of the curve $B$ that intersect each other transversally at $o$, then the local monodromy group $\mathcal{G}_{2,1} \subset S_9$ of the covering $\varphi$ at $o$ is the Abelian group generated by the permutations $g_3 = (2,8,5)(3,6,9)$ (the image of the bypass around $B_1'$) and $g_4 = (1,4,7)(3,9,6)$ (the image of the bypass around $B_2'$). The set $\{1, \ldots, 9\}$ splits into three orbits $\{1,4,7\}$, $\{2,8,5\}$ and $\{3,9,7\}$ of the action of $\mathcal{G}_{2,1}$. Therefore, $\varphi^{-1}(V)$ is a disjoint union of three irreducible open subsets of $Y$, $\varphi^{-1}(V) = \bigcup_{i=1}^{3} \bigcup_{o_i} U_i$. Up to relabelling, the sets $U_1$ and $U_2$ are non-singular and $U_3$ has a singularity at the point $o' = U_3 \cap \varphi^{-1}(o)$. The coverings $\varphi_i: U_i \to V$, $i = 1, 2, 3$, are three-sheeted. The covering $\varphi: U_1 \to V$ is ramified over $B_1'$, the covering $\varphi: U_2 \to V$ is ramified over $B_2'$, and the covering $\varphi: U_3 \to V$ is ramified over $B_1' \cup B_2'$. Note that if $\psi: W \to U_3$ is the minimal desingularization of the singular point $o'$, then $E = \psi^{-1}(o')$ is a smooth rational curve and $(E^2)_W = -3$ (see the proof of Assertion 4).

If $o \in B_3$, and $B \cap V = B_1 \cup B_2 \cup B_3'$, where $B_1'$, $B_2'$ and $B_3'$ are three branches of the curve $B$ with pairwise-transversal intersection at $o$, then the local monodromy group $\mathcal{G}_{3,1} \subset S_9$ of the covering $\varphi$ at $o$ is the Abelian group generated by the permutations $g_3 = (2,8,5)(3,6,9)$ (the image of the bypass around $B_1'$), $g_4 = (1,4,7)(3,9,6)$ (the image of the bypass around $B_2'$) and $g_5 = (2,5,8)(1,7,4)$ (the image of the bypass around $B_3'$). The set $\{1, \ldots, 9\}$ splits into two orbits $\{1,4,7\}$, $\{2,8,5\}$ and $\{3,9,7\}$ of the action of $\mathcal{G}_{3,1}$. Therefore, $\varphi^{-1}(V)$ is a disjoint union of three irreducible open subsets of $Y$, $\varphi^{-1}(V) = \bigcup_{i=1}^{3} U_i$. Up to relabelling, the sets $U_1$, $U_2$ and $U_3$ have singularities at the points $o_i = U_i \cap \varphi^{-1}(o)$. The coverings $\varphi_i: U_i \to V$, $i = 1, 2, 3$, are three-sheeted. The covering $\varphi: U_1 \to V$ is ramified over $B_1' \cup B_2'$, the covering $\varphi: U_2 \to V$ is ramified over $B_2' \cup B_3'$, and the covering $\varphi: U_3 \to V$ is ramified over $B_1' \cup B_3'$. As above, we note that if $\psi: W_i \to U_i$ are minimal desingularizations of the singular points $o_i$ and $E_i = \psi^{-1}(o_i)$, then the $E_i$ are smooth rational curves and $(E^2)_W = -3$.

If $o \in B_2$, then the local monodromy group $\mathcal{G}_{2,2} \subset S_9$ of the covering $\varphi$ at $o$ is the group $\mathcal{G}_{2,2} \simeq \text{Hes}^1$ generated by two permutations, $g_1 = (4,5,6)(7,9,8)$ and $g_2 = (2,8,5)(3,6,9)$. The set $\{1, \ldots, 9\}$ splits into two orbits $\{1\}$ and $\{2, \ldots, 9\}$ of the action of $\mathcal{G}_{2,2}$. Therefore, $\varphi^{-1}(V)$ is a disjoint union of two irreducible open subsets of $Y$, $\varphi^{-1}(V) = U_1 \cup U_2$. The covering $\varphi: U_1 \to V$ is a biholomorphic map and, therefore, $U_1$ is non-singular. The degree of the covering $\varphi: U_2 \to V$ is equal to eight. This covering is branched over the curve $B \cap V = B'$, which has
a singularity of type $A_2$ at $o$. The group $\text{Hes}^1 \subset S_8$ is the monodromy group of the covering $\varphi: U_2 \to V$ ($S_8$ acts on $\{2, \ldots, 9\}$).

We claim that $U_2$ is non-singular, the ramification curve $R' \subset U_2$ of the covering $\varphi$ is also non-singular and $\varphi: R' \to B'$ is a two-sheeted map ramified at the point $o$. Indeed, recall that the group $\text{Hes}^1 = \langle g_1, g_2 \rangle$ can be embedded in $\text{GL}(2, \mathbb{C})$ (group no. 4 in [16]; see also [17]) in such a way that the elements $g_1$ and $g_2$ are 3-reflections:

$$g_1 = -\frac{\omega}{\sqrt{2}} \begin{pmatrix} \varepsilon & \sqrt{3} \\ \varepsilon^3 & \varepsilon \end{pmatrix}, \quad g_2 = \frac{\omega}{\sqrt{2}} \begin{pmatrix} \varepsilon^3 & \varepsilon^5 \\ \varepsilon^7 & \varepsilon \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$ and $\varepsilon = e^{2\pi i/8}$. Since the group $\text{Hes}^1 \subset \text{GL}(2, \mathbb{C})$ is generated by reflections, the quotient space $\mathbb{C}^2/\text{Hes}^1$ is $\mathbb{C}^2$. Write $\tilde{\psi}: \mathbb{C}^2 \to \mathbb{C}^2$ for the factorization morphism. We have $\deg \tilde{\psi} = 24$ and it is known [17] that $\tilde{\psi}$ is given by the functions

$$\Psi = u^4 - 2\sqrt{-3}u^2v^2 + v^4, \quad \Theta = uv(u^4 - v^4).$$

Moreover, $\tilde{\psi}$ is ramified with multiplicity three along a quadruple of lines $\bigcup_{i=1}^4 L_i$ given by the equation $\Phi = 0$, where

$$\Phi = u^4 + 2\sqrt{-3}u^2v^2 + v^4,$$

and the branch curve $B$ is given by the equation

$$\Psi^3 + 12\sqrt{-3}\Theta^2 = 0.$$

The degree of the restriction of $\tilde{\psi}$ to each line $L_i$, $\tilde{\psi}: L_i \to B$, is equal to 2. The group $\langle g_1 \rangle$ permutes the lines $L_2$, $L_3$ and $L_4$ and leaves fixed the line $L_1$ given by the equation

$$(\sqrt{2} + \omega \varepsilon)u + \omega \varepsilon^3 v = 0.$$ 

The covering $\tilde{\psi}$ is the compose $\tilde{\psi} = \psi \circ \overline{\psi}$, where $\overline{\psi}: \mathbb{C}^2 \to \mathbb{C}^2/\langle g_1 \rangle \simeq \mathbb{C}^2$ is the three-sheeted factorization map defined by the action of the group $\langle g_1 \rangle$. This map is ramified along $L_1$ and $\psi: \mathbb{C}^2/\langle g_1 \rangle \to \mathbb{C}^2$ is an eight-sheeted covering ramified with multiplicity three along the non-singular curve $R = \overline{\psi}(L_2 \cup L_3 \cup L_4)$. The monodromy group of this covering is $\text{Hes}^1 \subset S_8$ acting on the eight left cosets of the group $\langle g_1 \rangle$. The covering $\psi: R \to B$ is a two-sheeted map ramified at the origin. We now show that $U_2$ is non-singular, the ramification curve $R' \subset U_2$ of $\varphi$ is also non-singular and $\varphi: R' \to B'$ is a two-sheeted covering branched at $o$. To do this, it suffices to embed the pair $(V, B')$ in $(\mathbb{C}^2, B)$ and apply the Grauert–Remmert–Riemann–Stein theorem, Proposition 1 and Assertion 8 to $\psi: \psi^{-1}(V) \to V$ and $\varphi: U_2 \to V$.

We denote the ramification curve of $\varphi$ by $\mathcal{R} \subset Y$. It follows from what was said above that $\varphi: \mathcal{R} \to B$ is a two-sheeted covering branched at 24 points (the cusps of $B$). Therefore, by the Hurwitz formula, the geometric genus $g(\mathcal{R})$ of $\mathcal{R}$ is equal to

$$g(\mathcal{R}) = \frac{1}{2} [2(2g(B) - 2) + 24] + 1 = 31.$$

The number of singular points of the surface $Y$ is equal to $n_1 + 3n_2 = 21$. 
Proposition 6. Let $\sigma: X \to Y$ be the minimal resolution of singularities of $Y$. Then the square $K_X^2$ of the canonical class of the surface $X$ is given by $K_X^2 = 18$ and the Euler characteristic $\chi(O_X) = \sum_{i=0}^{2} (-1)^i \dim H^i(X, O_X)$ of the structure sheaf $O_X$ of $X$ is given by $\chi(O_X) = 1 - q + p_g = 9$.

Proof. Put $\psi = \varphi \circ \sigma$ and write $R \subset X$ for the proper pre-image of the curve $\mathcal{R}$. By Assertion 4, $\sigma^{-1}(\text{Sing} \, Y)$ is a disjoint union of 21 rational curves, $E := \sigma^{-1}(\text{Sing} \, Y) = \bigsqcup_{i=1}^{21} E_i$. Moreover, by the same assertion, $(E_i^2)_X = -3$ and $(R, E_i)_X = 2$. We have $K_X = \psi^*(K_{\Pi}) + 2R + E$ since $\psi$ is ramified along $R$ with multiplicity three, $(E_i, \psi^*(K_{\Pi}))_X = 0$ and $(K_X + E_i, E_i)_X = -2$. In addition, we also have $(\psi^*(K_{\Pi}), \psi^*(K_{\Pi}))_X = 81$ since $\deg \psi = K_{\Pi}^2 = 9$, and $(R, \psi^*(K_{\Pi}))_X = -72$ since the degree of the covering $\psi: R \to B$ is equal to 2 and $\deg B = 12$. Therefore the equalities

$$(R^2)_X = (R, K_X)_X = 30$$

follow from the equalities $(K_X + R, R)_X = (\psi^*(K_{\Pi}) + 3R + E, R)_X = 60$. As a result, we have

$$K_X^2 = (\psi^*(K_{\Pi}) + 2R + E, \psi^*(K_{\Pi}) + 2R + E)_X = 18.$$ 

Write $e(M)$ for the Euler characteristic of a topological space $M$. We have

$$e(X) = 9e(\Pi \setminus B) + 5e(B \setminus \text{Sing} \, B) + 4n_1 + 6n_2 + 2c$$

since $\deg \psi = 9$, the number of pre-images of every non-singular point of the curve $B$ is equal to 5, the pre-image of each node of $B$ is the disjoint union of a projective line ($e(\mathbb{P}^1) = 2$) and two points, the pre-image of each triple point of $B$ is the disjoint union of three projective lines, and the pre-image of each cusp of $B$ consists of two points. Since $g(B) = 10$, $n_1 + 3n_2 = 21$, $c = 24$ and $e(\Pi) = 3$, we have

$$e(B) = -2(g(B) - 1) - n_1 - 2n_2 = n_2 - 39, \quad e(\Pi \setminus B) = 42 - n_2,$$

$$e(B \setminus \text{Sing} \, B) = -2(g(B) - 1) - 2n_1 - 3n_2 - c = 3n_2 - 84.$$ 

Therefore $e(X) = 90$ and, applying the Noether formula, we obtain that

$$\chi(O_X) = \frac{K_X^2 + e(X)}{12} = 9. \quad \square$$

3.4. Resolution of singularities of the surface of inflection points for a generic two-dimensional linear system of plane cubic curves. We denote the normalization of the variety $\mathcal{I}_{\Pi}$ by $\nu: I := I_{\Pi} \to \mathcal{I}_{\Pi}$ and put $p_1 = h_{\Pi} \circ \nu: I \to \Pi$ and $S = S \cap \mathcal{C}_{\Pi}$. By Assertion 5, the curve $S$ is contained in $\mathcal{I}_{\Pi}$.

Proposition 7. The surface $I$ is non-singular and the morphism $p_1: I \to \Pi$ coincides with the morphism $\psi = \varphi \circ \sigma: X \to \Pi$, where $\varphi: Y \to \Pi$ is the Stein covering associated with $h_{\Pi}$ and $\sigma: X \to Y$ is the minimal resolution of singularities of the surface $Y$.

Proof. Clearly, the surface $\mathcal{I}_{\Pi}$ is non-singular at the points belonging to $h_{\Pi}^{-1}(\Pi \setminus B)$.

Assertion 9. The surface $\mathcal{I}_{\Pi}$ is non-singular at the points belonging to $\mathcal{I}_{\Pi} \setminus S$. 

Proof. An irreducible nodal cubic \( C_{\pi_1}, \pi_1 \in B_1 \cap \Pi \), has three smooth inflection points (and a cuspidal cubic \( C_{\pi_1}, \pi_1 \in B_{2,2} \cap \Pi \), has one smooth inflection point). It was shown in [20] that the intersection number of \( C_{\pi_1} \) and its Hesse curve \( H_{\pi_1} \) at a singular point \( s \in C_{\pi_1} \) is equal to \((C_{\pi_1}, H_{\pi_1})_s = 6 \) (resp., \((C_{\pi_1}, H_{\pi_1})_s = 8 \)). Therefore, by Assertion 3, the surface \( \mathcal{I}_\Pi \) is non-singular at the points belonging to \( \mathcal{I}_\Pi \setminus (S \cup h^{-1}(B_{2,1} \cup B_{3,1})) \). To complete the proof of Assertion 9, it suffices to show that \( \mathcal{I}_\Pi \) is non-singular at the points of \( h^{-1}(\pi_1) \setminus S \) when \( \pi_1 \in (B_{2,1} \cup B_{3,1}) \cap \Pi \).

There is no loss of generality in assuming that \( z_1z_2z_3 + a_{3,0}z_1^3 = 0 \) is the equation of \( C_{\pi_1} \) (if \( \pi_1 \in B_{3,1} \), then \( a_{3,0} = 0 \) and \( h^{-1}(\pi_1) \) consists of three lines in \( \{\pi_1\} \times \mathbb{P}^2 \) given by the equations \( z_i = 0, \ i = 1, 2, 3 \), but if \( \pi_1 \in B_{2,1} \), then \( a_{3,0} \neq 0 \) and \( h^{-1}(\pi_1) \) is a line in \( \{\pi_1\} \times \mathbb{P}^2 \) given by the equation \( z_1 = 0 \).

We claim that the points of the line \( E = \{\pi_1\} \times \{z_1 = 0\} \) are non-singular points of \( \mathcal{I}_\Pi \) if \( z_2z_3 \neq 0 \) (when \( \pi_1 \in B_{3,1} \), one can similarly prove that \( \mathcal{I}_\Pi \) is non-singular at all points of the lines \( \{z_2 = 0\} \) and \( \{z_3 = 0\} \) which are not singular points of the curve \( C_{\pi_1} \); we omit this proof). To show this, we choose basis elements \( F_1(\pi), F_2(\pi), F_3(\pi) \) of the linear system \( \mathcal{C}_\Pi \) in such a way that the linear system \( \mathcal{C}_\Pi \) is given in the non-homogeneous coordinates \((\alpha, \beta, x, y)\) by the equation

\[
(xy + a_{3,0}x^3) + \alpha \left( \sum_{1 \leq i+j \leq 3} b_{i,j}x^iy^j \right) + \beta \left( 1 + \sum_{1 \leq i+j \leq 3} c_{i,j}x^iy^j \right) = 0.
\] (10)

Let \( \mathfrak{m} \) be the ideal in the polynomial ring \( \mathbb{C}[\alpha, \beta, x, y] \) generated by the monomials \( \alpha, \beta \) and \( x \). We rewrite (10) in the form

\[
xy + \alpha \left( \sum_{1 \leq j \leq 3} b_{0,j}y^j \right) + \beta \left( 1 + \sum_{1 \leq j \leq 3} c_{0,j}y^j \right) + R_1(\alpha, \beta, x) = 0,
\] (11)

where \( R_1(\alpha, \beta, x) \in \mathfrak{m}^2. \) Simple calculations (which we omit) show that the Hessian of the linear system \( \mathcal{C}_\Pi \) in the non-homogeneous coordinates \((\alpha, \beta, x, y)\) is of the form

\[
2 \left[ xy + \alpha \left( \sum_{1 \leq j \leq 2} b_{0,j}y^j - 3b_{0,3}y^3 \right) + \beta \left( -3 + \sum_{1 \leq j \leq 2} c_{0,j}y^j - 3c_{0,3}y^3 \right) \right] + R_2(\alpha, \beta, x) = 0,
\] (12)

where \( R_2(\alpha, \beta, x) \in \mathfrak{m}^2. \)

The surface \( \mathcal{I}_\Pi \cap (\mathbb{C}^2 \times \mathbb{C}^2) \) is given by the equations (11) and (12) in \( \mathbb{C}^2 \times \mathbb{C}^2, \) and the line \( E \) is given by the equations \( \alpha = \beta = x = 0. \) Therefore a necessary and sufficient condition for a point \( p = (0,0,0,y_0) \) with \( y_0 \neq 0 \) to be a singular point of \( \mathcal{I}_\Pi \) is the proportionality of the linear forms (in the variables \( \alpha, \beta \) and \( x \)) occurring in the left-hand sides of (11) and (12), where \( y \) takes a concrete value, namely, \( y_0. \)

It is easy to see that the vanishing of \( b_{0,3} \) is a necessary condition. In this case, the cubic curve \( \mathcal{C}_{\pi_2} \) given by the equation \( F_2(\pi) = 0 \) contains not only the singular point \( q_1 = (0,0,1) \) of the curve \( \mathcal{C}_{\pi_1} \) but also the second singular point \( q_2 = (0,1,0) \) of \( \mathcal{C}_{\pi_1}. \) In our case this is impossible since the linear system \( \mathcal{C}_\Pi \) determines a generic covering \( \xi: \mathbb{P}^2 \to \check{\Pi} \) of the projective plane. \( \square \)

Assertion 10. The curve \( S \) is non-singular at \( s \in S \subset \mathcal{I}_\Pi \) if \( h_\Pi(s) \in B_1 \cup B_{2,1} \cup B_{3,1}. \)
Proof. As above, we can assume that the point $s$ lies in the neighbourhood $\mathbb{C}^2 \times \mathbb{C}^2 \subset \Pi \times \mathbb{P}^2$ with coordinates $(\alpha, \beta, x, y)$ and has coordinates $(0, 0, 0, 0)$. Choose the basis polynomials $F_1(x, y), F_2(x, y)$ and $F_3(x, y)$ of the linear system $\mathcal{C}_{\Pi}$ as follows:

$$F_1(x, y) = xy + a_{3,0}x^3 + a_{0,3}y^3, \quad F_2(x, y) = \sum_{1 \leq i+j \leq 3} b_{i,j}x^iy^j,$$

$$F_3(x, y) = 1 + \sum_{1 \leq i+j \leq 3} c_{i,j}x^iy^j.$$ 

Hence the linear system $\mathcal{C}_{\Pi}$ is given by the equation

$$(xy + a_{3,0}x^3 + a_{0,3}y^3) + \alpha \left( \sum_{1 \leq i+j \leq 3} b_{i,j}x^iy^j \right) + \beta \left( 1 + \sum_{1 \leq i+j \leq 3} c_{i,j}x^iy^j \right) = 0. \quad (13)$$

We can assume that $b_{1,1} = c_{1,1} = 0$ (correcting $F_2(x, y)$ and $F_3(x, y)$ using $F_1(x, y)$ if necessary) and $b_{1,0} = 1, c_{1,0} = 0$. Let $\mathfrak{m}$ be the ideal in the power-series ring $\mathbb{C}[[\alpha, \beta, x, y]]$ generated by the monomials $\alpha, \beta, x$ and $y$, and let $\mathfrak{m}_\beta$ be the ideal in the power-series ring $\mathbb{C}[[\alpha, x, y]]$ generated by the monomials $\alpha, x$ and $y$. We also denote the left-hand side of $(13)$ by $w$ and put $w_1 = \partial w / \partial x$ and $w_2 = \partial w / \partial y$. We have $w = \beta + R_0(\alpha, \beta, x, y)$ and

$$w_1 = y + \alpha + R_1(\alpha, \beta, x, y), \quad w_2 = x + b_{0,1}\alpha + c_{0,1}\beta + R_2(\alpha, \beta, x, y), \quad (14)$$

where $R_i(\alpha, x, y) \in \mathfrak{m}^2$ for $i = 0, 1, 2$. It is easy to see from $(14)$ that the functions $w, w_1, w_2$ and $\alpha$ are local coordinates in some neighbourhood $V$ of $s$, the variety $U = \mathcal{C}_{\Pi} \cap V$ is given in $V$ by the equation $w = 0$, and the curve $S \cap V$ is given in $U$ by the equations $w_1 = w_2 = 0$. \hfill $\square$

Assertion 11. If $h_{\Pi}(s) \in \mathcal{B}_1 \cup \mathcal{B}_{2,1} \cup \mathcal{B}_{3,1}$ for some $s \in S \subset \mathcal{I}_{\Pi}$, then the point $s$ has an analytic neighbourhood where the surface $\mathcal{I}_{\Pi}$ is the union of two non-singular surfaces intersecting transversally at $s$.

Proof. Use the notation introduced in the proof of Assertion 10. Simple calculations (which we omit) show that the equation of the Hesse surface $H_{\mathcal{C}_{\Pi}}$ in the non-homogeneous coordinates $(\alpha, \beta, x, y)$ is of the form

$$2[-3\beta + xy + \alpha(x + b_{0,1}y) + \beta(-3 + c_{0,1}y) + 4b_{0,1}\alpha^2 + 4c_{0,1}\alpha\beta] + R_3(\alpha, \beta, x, y) = 0, \quad (15)$$

where $R_3(\alpha, \beta, x, y) \in \mathfrak{m}^3$.

It follows from $(13)$ that the restriction of $\beta$ to $V$ takes the form

$$\beta = -[xy + \alpha(x + b_{0,1}y)] + R_4(\alpha, x, y), \quad (16)$$

where $R_4(\alpha, x, y) \in \mathfrak{m}^3_\beta$. Therefore the surface $\mathcal{I}_{\Pi} \cap U$ is given by the equation $(15)$ with $\beta$ replaced by the right-hand side of $(16)$. After this substitution and simple transformations, $(15)$ takes the form

$$4(y + \alpha)(x + b_{0,1}\alpha) + R_5(\alpha, x, y) = 0, \quad (17)$$

where $R_5(\alpha, x, y) \in \mathfrak{m}^3_\beta$. Therefore $s$ is a singular point of $\mathcal{I}_{\Pi}$ of multiplicity two. This also shows that $S = \text{Sing}\mathcal{I}_{\Pi}$ since $s$ is a generic point belonging to $S$. 

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It follows from (17) that $s$ is an ordinary node of the curve $D = \{\alpha = 0\} \cap \mathcal{I}_\Pi$. Let $\sigma: \tilde{U} \to U$ be the monoidal transformation centred at the curve $S \cap U$, $\sigma^{-1}(S \cap U) = E$. It is easy to see that the proper pre-image $\sigma^{-1}(D)$ of $D$ intersects $E$ transversally at two points. Since $s$ is a generic point of the curve $S \cap U$, the intersection $\sigma^{-1}(U \cap \mathcal{I}_\Pi) \cap E$ of the proper pre-image of the surface $U \cap \mathcal{I}_\Pi$ and $E$ is a disjoint union of two non-singular curves if the neighbourhood $V$ is chosen small enough. It follows that the surface $\mathcal{I}_\Pi \cap U$ is a union of two non-singular surfaces meeting transversally since the multiplicity of the singularity of the surface $\mathcal{I}_\Pi \cap U$ at every point $s \in S \cap U$ is equal to 2. □

**Assertion 12.** If $h_\Pi(s) \in \mathcal{B}_{2,2}$ when $s \in S \subset \mathcal{I}_\Pi$, then the surface $I$ is non-singular at the points of $\nu^{-1}(s)$.

**Proof.** Consider a sufficiently small complex-analytic neighbourhood $V \subset \Pi$ of the point $p_1(s)$. The morphism $p_1: p_1^{-1}(V) \to V$ is a finite covering and $p_1^{-1}(V)$ is a normal surface (consisting of two components). Therefore Assertion 12 follows from the Grauert–Remmert–Riemann–Stein theorem and the results in §3.3. □

To complete the proof of Proposition 7, it suffices to apply Assertions 9–12, 4 and the Grauert–Remmert–Riemann–Stein theorem. □

§ 4. Calculation of the irregularity

### 4.1. An additional condition of genericity

In what follows we use the notation introduced in §3. To compute the irregularity $q = \dim H^1(I, \mathcal{O}_I)$ of the surface $I = p_1^{-1}(\Pi)$, it is convenient to introduce the following additional genericiy condition for two-dimensional linear systems $\mathcal{C}_\Pi$ of plane cubic curves. We say that a linear system $\mathcal{C}_\Pi$ of cubic curves parametrized by points of the projective plane $\Pi \in \mathcal{W} \subset \text{Gr}(3, 10)$ satisfies the additional condition of genericity if, for every point $\bar{x} \in \mathbb{P}^2$, only finitely many cubics in $\mathcal{C}_\Pi$ passing through $\bar{x}$ have an inflection point at $\bar{x}$.

Write $J_{1,o}$ for the subset of planes $\Pi$ in $\mathcal{W} \subset \text{Gr}(3, 10)$ such that the point $o \in \mathbb{P}^2 \setminus \tilde{R}$ is an inflection point for every cubic in $\mathcal{C}_\Pi$ containing $o$.

**Assertion 13.** We have $\dim J_{1,o} \leq 17$.

**Proof.** Since $o \in \mathbb{P}^2 \setminus \tilde{R}$, the generic covering $\xi: \mathbb{P}^2 \to \tilde{\Pi}$ is a biholomorphic isomorphism in some neighbourhood of $o$. Therefore basis elements $F_1(x, y), F_2(x, y)$ and $F_3(x, y)$ of the linear system $\mathcal{C}_\Pi$ of the form

$$F_1(x, y) = x + \sum_{2 \leq i+j \leq 3} b_{i,j} x^i y^j, \quad F_2(x, y) = y + \sum_{2 \leq i+j \leq 3} c_{i,j} x^i y^j, \quad F_3(x, y) = 1 + \sum_{2 \leq i+j \leq 3} d_{i,j} x^i y^j$$

are uniquely determined by the point $o \in \mathbb{P}^2 \setminus \tilde{R}$ for every plane $\Pi \in \mathcal{W}$. Clearly, $\Pi$ belongs to $J_{1,o}$ if and only if the quadratic form

$$(t_1 b_{2,0} + t_2 c_{2,0})x^2 + (t_1 b_{1,1} + t_2 c_{1,1})xy + (t_1 b_{0,2} + t_2 c_{0,2})y^2$$
is divisible by the linear form $t_1x + t_2y$ for every point $(t_1,t_2) \in \mathbb{P}^1$. Putting $x = t_2 = 1$, we easily see that this condition is equivalent to the equalities

$$b_{2,0} = c_{1,1}, \quad b_{1,1} = c_{0,2}, \quad b_{0,2} = c_{2,0} = 0.$$  

Hence the dimension of the variety $J_{1,0}$ does not exceed the total number (equal to 17) of the indeterminate coefficients $b_{i,j}, c_{i,j}$ and $d_{i,j}$ of the monomials in $F_1(x, y), F_2(x, y)$ and $F_3(x, y)$. □

Write $J_{2,0}$ for the subset of planes $\Pi \subset \text{Gr}(3, 10)$ such that the point $o$ belongs to $\mathfrak{B}_{\tilde{R},2}$ and is an inflection point for every cubic in $\mathcal{C}_\Pi$ containing $o$.

**Assertion 14.** We have $\dim J_{2,0} \leq 17$.

*Proof.* In a neighbourhood of $o$, the covering $\xi$ is equivalent to the restriction of the projection $pr: (z, a, b) \mapsto (a, b)$ to the surface with equation $z^3 + az + b = 0$ in $\mathbb{C}^3$ with coordinates $(z, a, b)$. The properties of this projection are well studied (see, for example, [19]). In particular, it is easy to show that in our case the pencil of cubic curves with base point $o$ has a unique element singular at $o$ (we write $C_1$ for this cubic curve given by the equation $F_1(x, y) = 0$; the point $o$ is an ordinary node of $C_1$) and the other elements of the pencil are non-singular at $o$ and touch one of the branches of $C_1$ at that point. Hence there are basis elements $F_1(x, y), F_2(x, y)$ and $F_3(x, y)$ in $\mathcal{C}_\Pi$ of the form

$$F_1(x, y) = xy + \sum_{i+j=3} b_{i,j}x^i y^j, \quad F_2(x, y) = x + \sum_{2 \leq i+j \leq 3} c_{i,j}x^i y^j,$$

$$F_3(x, y) = 1 + d_{0, 1}y + \sum_{2 \leq i+j \leq 3} d_{i,j}x^i y^j, \quad c_{1,1} = d_{1,1} = 0.$$  

It is easy to see that $\Pi$ belongs to $J_{2,0}$ if and only if the coefficient $c_{0,2}$ vanishes. Clearly, $\dim J_{2,0}$ does not exceed the total number (equal to 17) of the indeterminate coefficients $b_{i,j}, c_{i,j}$ and $d_{i,j}$ of the monomials in $F_1(x, y), F_2(x, y)$ and $F_3(x, y)$. □

**Remark 2.** The proof of Assertion 14 shows that if $\Pi \not\subset J_{2,0}$ (that is, $c_{0,2} \neq 0$), then for all non-singular cubic curves belonging to the linear system $\mathcal{C}_\Pi$ and passing through a point $o \in \mathfrak{B}_{\tilde{R},2}$, the point $o$ is not an inflection point.

We write $J_{3,0}$ for the subset of planes $\Pi \subset \text{Gr}(3, 10)$ such that the point $o$ belongs to $\tilde{R} \setminus \mathfrak{B}_{\tilde{R},2}$ and is an inflection point for every cubic curve in $\mathcal{C}_\Pi$ passing through $o$.

**Assertion 15.** The set $J_{3,0}$ is empty.

*Proof.* In a neighbourhood of $o \in \tilde{R}$, the covering $\xi$ is ramified along $\tilde{R}$ with multiplicity two. Therefore, for any cubic curves $C_1$ and $C_2$ belonging to $\mathcal{C}_\Pi$ and passing through $o$, the intersection number $(C_1, C_2)_o$ of the curves $C_1$ and $C_2$ at $o$ is equal to 2. In addition, as mentioned above, the pencil of cubic curves with base point $o$ has a unique element singular at $o$, to be denoted by $C_1$ (it is equal to the pre-image of the tangent line to one of the branches of $\tilde{B}$ at $o'$), and all other elements of the pencil are non-singular at $o$. In all cases, $o$ is an ordinary node of $C_1$, except when $o'$ is an inflection point of $\tilde{B}$. In the last case, $o$ is an ordinary cusp of $C_1$.  


Let \( F_1(x, y) = 0 \) be the equation of \( C_1 \) and let \( t_1F_1(x, y) + t_2F_2(x, y) = 0 \) be the equation of the pencil of cubic curves passing through \( o \). We can assume without loss of generality that

\[
F_2(x, y) = x + \sum_{2 \leq i+j \leq 3} b_{i,j}x^iy^j,
\]

and if \( o \) is an ordinary node of \( C_1 \), then

\[
F_1(x, y) = \sum_{2 \leq i+j \leq 3} c_{i,j}x^iy^j,
\]

where \( c_{0,2} = 1 \) since \((C_1, C_2)_o = 2\) for the curve \( C_2 \) given by the equation \( F_2(x, y) = 0 \).

In the case when \( o \) is an ordinary cusp of \( C_1 \), we can assume that

\[
F_1(x, y) = y^2 + \sum_{i+j = 3} c_{i,j}x^iy^j.
\]

In all these cases, it is easy to see that \( o \) can be an inflection point for all elements of the pencil only when the equality \( t_1 + t_2b_{0,2} = 0 \) holds for all \((t_1, t_2) \in \mathbb{P}^1\), which is impossible. \( \square \)

**Remark 3.** The proof of Assertion 15 shows that if \( o \) is an ordinary cusp of \( C_1 \), then the pencil of cubic curves passing through \( o \) has exactly one other cubic curve (corresponding to \( t_1 = -b_{0,2} \) and \( t_2 = 1 \)) with an inflection point at \( o \).

**Assertion 16.** There is a non-empty Zariski-open subset \( \mathcal{W}_1 \) of \( \mathcal{W} \) such that all two-dimensional linear systems \( \mathcal{C}_\Pi \) of plane cubic curves with \( \Pi \in \mathcal{W}_1 \) satisfy the additional genericity condition.

**Proof.** We define \( J = J_1 \cup J_2 \), where \( J_i = \{(\Pi, z) \in \mathcal{W} \times \mathbb{P}^2 \ : \ \Pi \in J_i, z\} \), \( i = 1, 2 \), and put \( \mathcal{W}_1 := \mathcal{W} \setminus \text{pr}_1(J) \), where the bar means the closure of a set. It follows from Assertions 13 and 14 that \( \dim J \leq 19 \). Thus \( \mathcal{W}_1 \) is a non-empty Zariski-open set because \( \dim \mathcal{W} = \dim \text{Gr}(3, 10) = 21 \). Furthermore, it also follows from these assertions and Assertion 15 that the linear systems \( \mathcal{C}_\Pi \) of cubic curves with \( \Pi \in \mathcal{W}_1 \) satisfy the additional genericity condition. \( \square \)

### 4.2. Investigation of properties of the second projection.

We write \( p_2 = \text{pr}_2 \circ \nu \colon I \to \mathbb{P}^2 \), where \( \nu \colon I \to \mathcal{I}_\Pi \) is the normalization morphism and \( \text{pr}_2 \colon \Pi \times \mathbb{P}^2 \to \mathbb{P}^2 \) is the projection to \( \mathbb{P}^2 \).

By Proposition 7, we can identify \( I \) with the minimal resolution \( X \) of singularities of the Stein covering associated with the morphism \( h_\Pi \). Therefore, in what follows we use the notation introduced in the proof of Proposition 6. By Proposition 6, the surface \( I \) has the following invariants:

\[
K_I^2 = 18 \quad \text{and} \quad \chi(O_I) = \sum_{i=0}^{2} (-1)^i \dim H^i(I, O_I) = 1 - q + p_g = 9.
\]

The curve \( R \subset X = I \) is non-singular and \((R^2)_I = (R, K_I)_I = 30\) by (9). Moreover, \( I \) contains 21 rational curves \( E_i, i = 1, \ldots, 21 \), with \((E_i^2)_I = -3\) and \((E_i, K_I)_I = 1\), \( p_1(E_i) \) are points in \((\mathcal{B}_{2,1} \cup \mathcal{B}_{3,1}) \cap \Pi \), and \( p_2(E_i) \) are lines in \( \mathbb{P}^2 \).

Unless otherwise stated, we shall assume that \( \Pi \in \mathcal{W}_1 \).
Assertion 17. The morphism \( p_2: I \to \mathbb{P}^2 \) is a finite three-sheeted covering.

Proof. Let \( L \) and \( M \) be lines in \( \mathbb{P}^2 \) and \( \Pi \) respectively. The Picard group \( \text{Pic}(\Pi \times \mathbb{P}^2) \) is the free Abelian group generated by the divisors \( \Pi \times L \) and \( M \times \mathbb{P}^2 \). The surface \( \mathcal{I}_\Pi \) is the complete intersection of two subvarieties \( C_\Pi \) and \( H_{C_\Pi} \) of codimension 1. We have \( C_\Pi = \Pi \times L + M \times \mathbb{P}^2 \) and \( H_{C_\Pi} = 3(\Pi \times L + M \times \mathbb{P}^2) \) as elements of \( \text{Pic}(\Pi \times \mathbb{P}^2) \). The fibre of the projection \( \mathcal{I}_\Pi \to \mathbb{P}^2 \) is also the intersection of two subvarieties \( \Pi \times L_1 \) and \( \Pi \times L_2 \) (where \( L_1 \) and \( L_2 \) are lines in \( \mathbb{P}^2 \)). It follows that the degree of the restriction of \( \mathcal{I}_\Pi \) to \( \mathcal{I}_\Pi \to \mathbb{P}^2 \) is equal to the intersection number

\[
(3(\Pi \times L + M \times \mathbb{P}^2), \Pi \times L + M \times \mathbb{P}^2, \Pi \times L, \Pi \times L)_{\Pi \times \mathbb{P}^2} = 3.
\]

To study the properties of the discriminant curve \( \mathcal{B} \subset \mathbb{P}^2 \) and the ramification curve \( \mathcal{R} \subset I \) of the covering \( p_2: I \to \mathbb{P}^2 \), we need the following lemma.

Lemma 2. Let \((U,o')\) and \((V,o)\) be germs of non-singular surfaces at points \(o'\) and \(o\), and let \(\varphi: (U,o') \to (V,o)\) be a two-sheeted finite covering branched along the germ \((B,o)\) of a curve with \(\varphi^*((B,o)) = 2(R,o')\). Then the germs \((B,o)\) and \((R,o')\) are non-singular at the points \(o\) and \(o'\), and if \((C,o') \subset (U,o')\) and \((\varphi(C),o) \subset (V,o)\) are non-singular germs of curves, then either \(\varphi: (C,o') \to (\varphi(C),o)\) is a two-sheeted covering branched at \(o'\) and the germs \((\varphi(C),o)\) and \((B,o)\) intersect each other transversally at \(o\), or \(\varphi: (C,o') \to (\varphi(C),o)\) is a biholomorphic map and the intersection numbers \((\varphi(C),B)_o\) and \((C,R)_{o'}\) of germs at \(o\) and \(o'\) are connected by the equality \((\varphi(C),B)_o = 2(C,R)_{o'}\).

Proof. Let \(\sum_{i+j=1}^{\infty} c_{i,j}x^iy^j = 0\) be the equation of \((B,o)\) in \((V,o)\). Then the germ \((U,o')\) is biholomorphic to the germ in \((C^3,o')\) given by the equation \(z^2 = \sum_{i+j=1}^{\infty} c_{i,j}x^iy^j\). The linear part \(c_{1,0}x + c_{0,1}y\) of this equation is non-degenerate since \((U,o')\) is the germ of a non-singular surface. Hence we can always assume (after a change of coordinates) that the germ \((B,o)\) is given by the equation \(x = 0\) and the covering \(\varphi\) is given by the equation \(z^2 = x\).

If the germs \((\varphi(C),o)\) and \((B,o)\) intersect each other transversally at \(o\), we can assume that the equation of \((\varphi(C),o)\) is \(y = 0\). Then, clearly, \(\varphi: (C,o') \to (\varphi(C),o)\) is a two-sheeted covering branched at \(o'\) and the germs \((R,o')\) and \((C,o')\) are given by the equations \(z = 0\) and \(y = 0\).

If the germs \((\varphi(C),o)\) and \((B,o)\) are tangent to each other with multiplicity \(r > 1\) at \(o\), then we can assume after an analytic change of coordinates that \(x = 0\) is the equation of \((B,o)\) and \(x - y^r = 0\) is the equation of \((\varphi(C),o)\). Hence the full pre-image of \((\varphi(C),o)\) is given by the equation \(z^2 - y^r = 0\). If \(r\) is odd, this pre-image is irreducible and it is a germ of a singular curve. If \(r = 2k\) is even, the pre-image of \((\varphi(C),o)\) splits into two germs given by the equations \(z - y^k = 0\) and \(z + y^k = 0\).

Assertion 18. The covering \( p_2 \) is ramified along \( \mathcal{R} \) with multiplicity two. The degree of the curve \( \mathcal{B} \) is \( \deg \mathcal{B} = 18 \). The curve \( \mathcal{B} \) intersects \( \mathcal{R} \) transversally at 24 points belonging to the set \( \mathcal{B}_{\mathcal{R},1} \) and, moreover, \( \mathcal{B} \) and \( \mathcal{R} \) have 42 common points of simple tangency belonging to the set \( \mathcal{B}_{\mathcal{R},2} \).

Proof. We have \( p_2(R) = \widetilde{R} \) and, by the results in § 3.3, \( p_2: R \to \widetilde{R} \) is a double covering branched at 24 points of the set \( \mathcal{B}_{\mathcal{R},1} \). Moreover, it follows from Remark 3
that the full pre-image $p_2^{-1}(\bar{z})$ of any point $\bar{z} \in \bar{R}$ consists of at least two points. Therefore $\mathcal{B}$ has no irreducible components along which the covering $p_2$ is ramified with multiplicity three because $\deg p_2 = 3$ and the plane curve $\bar{R}$ has a non-empty intersection with each irreducible component of $\mathcal{B}$. Moreover, the morphism $p_2$ is a two-sheeted covering locally, at every point $q \in \mathcal{R} \cap \bar{R}$. Hence, by Lemma 2, the curves $\mathcal{R}$ and $\mathcal{B}$ are non-singular at their intersection points with $\bar{R}$ and $\bar{R}$ respectively.

By the adjunction formula we have $K_I = p_2^*(K_{\mathcal{B}2}) + \mathcal{R}$. Combining this with the facts that $p_2: \bar{R} \to R$ is a two-sheeted covering, $\deg \bar{R} = 6$ and $(K_I, \bar{R})_I = 30$, we have

$$(\mathcal{R}, \bar{R})_I = 30 - 2 \deg K_{\mathcal{B}2} \cdot \deg \bar{R} = 66. \quad (18)$$

Since $p_2: \bar{R} \to \bar{R}$ is ramified at 24 points belonging to $\mathcal{B}_{\bar{R},1}$, we have $\mathcal{B}_{\bar{R},1} \subset \mathcal{B}$. It follows from Remark 2 that the curves $\bar{R}$ and $\mathcal{B}$ also intersect each other at 42 points in $\mathcal{B}_{\bar{R},2}$, and these curves touch each other at these points by Lemma 2. Hence (applying Lemma 2 again) we obtain that $(\mathcal{R}, \bar{R})_I \geq |\mathcal{B}_{\bar{R},1}| + |\mathcal{B}_{\bar{R},2}| = 66$. Therefore, to complete the proof of Assertion 18, it suffices to compare this inequality with (18), apply Lemma 2 again and obtain the equalities $6 \deg \mathcal{B} = (\bar{R}, \mathcal{B})_{\mathcal{B}2} = |\mathcal{B}_{\bar{R},1}| + 2|\mathcal{B}_{\bar{R},2}| = 108$. $\square$

**Proposition 8.** The irregularity $q = \dim H^1(I, \mathcal{O}_I)$ of the surface $I$ vanishes.

**Proof.** We put $D = p_2^*(L)$, where $L$ is a line in $\mathbb{P}^2$. The divisor $D$ is ample, the complete linear system $|D|$ is free from fixed components and base points, and a general element of this system is a non-singular curve since $p_2$ is a finite morphism on $\mathbb{P}^2$. Hence, $\dim H^0(I, \mathcal{O}_I(D)) = 3$ because $(D^2)_I = 3$ and if $\dim H^0(I, \mathcal{O}_I(D)) = r > 3$, then the complete linear system $|D|$ determines a birational map onto a surface of degree three in $\mathbb{P}^{r-1}$, which is birationally isomorphic to a ruled surface. But this is impossible since $\chi(\mathcal{O}_I) = 9$ and the Euler characteristic of the structure sheaf of a ruled surface is less than 2. Moreover, we have

$$(D, K_I)_I = (D, p_2^*(K_{\mathcal{B}2}) + \mathcal{R})_I = 3(L, K_{\mathcal{B}2})_{\mathcal{B}2} + (L, \mathcal{B})_{\mathcal{B}2} = 9.$$ 

Hence, a generic curve in the linear system $|D|$ has geometric genus $g(D) = 7$. We also observe from the definitions of the divisor $D$ and the exceptional curves $E_i$ that $(D, E_i)_I = (K_I, E_i)_I = 1$ for all $i = 1, \ldots, 21$.

Write $\bar{D} = K_I - D$ and let $\dim H^1(I, \mathcal{O}_I(D)) = q(D)$. We have $(\bar{D}^2)_I = 3$ and $(D, \bar{D})_I = 6$. It follows from the Riemann–Roch theorem and Serre duality that

$$\dim H^0(I, \mathcal{O}_I(D)) + \dim H^0(I, \mathcal{O}_I(\bar{D})) = \frac{(D - K_I, D)_I}{2} + p_a(I) + q(D).$$

Hence, $\dim H^0(I, \mathcal{O}_I(\bar{D})) = 3 + q(D)$. We also note that the linear system $|\bar{D} - D|$ is empty. Indeed, $(\bar{D} - D, E_i)_I = -1$ for all $i = 1, \ldots, 21$ and the assumption that $|\bar{D} - D|$ is non-empty would mean that all the curves $E_i$ are its fixed components. On the other hand, $(\bar{D} - D, D)_I = 3$ and $D$ has a positive intersection with any effective divisor. We get a contradiction.

We claim that $q(D) = 0$. Assume the opposite and consider the map $\varphi_{|\bar{D}|}: I \to \mathbb{P}^{2+q(D)}$ (possibly rational) determined by the linear system $|\bar{D}|$. Let $N + M \in |\bar{D}|$,
where $N$ is the (non-empty) fixed part of the linear system $|\widehat{D}|$. Then $\varphi_{|\widehat{D}|}$ maps curves of genus $g(D) = 7$ in the linear system $|D|$ either birationally onto curves of degree $m = (D,M)_T \leq 5$ in $\mathbb{P}^{2+g(D)}$, or in a two-sheeted manner (when $m = 4$) onto curves of degree two, or in an $m$-sheeted manner onto lines. In all cases, the images do not lie in proper linear subspaces of $\mathbb{P}^{2+g(D)}$ since the linear system $|\widehat{D} - D|$ is empty. Therefore the last two cases are impossible. The first case is also impossible since the geometric genus of a curve of degree $m \leq 5$ does not exceed 6 (to see this, it suffices to project the image of $D$ birationally to the plane). Hence, the linear system $|\widehat{D}|$ has no fixed components.

Note that if $p \in I$ is a base point of the linear system $|\widehat{D}|$, then all the curves in this linear system (except perhaps for one) are non-singular at $q$ since $(\widehat{D}^2)_I = 3$ and the linear system $|\widehat{D}|$ has no fixed components. Hence the linear system $|\widehat{D}|$ cannot be composed from a pencil and, therefore, $\varphi_{|\widehat{D}|}$ maps the surface $I$ to a surface not lying in a proper linear subspace of $\mathbb{P}^{2+g(D)}$. It follows that $2 \leq \deg \varphi_{|\widehat{D}|}(I) \leq 3$ and $\varphi_{|\widehat{D}|} : I \to \varphi_{|\widehat{D}|}(I)$ is a birational map. This is impossible since $\chi(\mathcal{O}_I) = 9$. Hence, $q(D) = 0$.

We claim that $\dim H^0(D_0, \mathcal{O}_{D_0}(D)) = 2$ for a non-singular curve $D_0 \in |D|$. Indeed, if $\dim H^0(D_0, \mathcal{O}_{D_0}(D)) > 2$, then the regular sections of the sheaf $\mathcal{O}_{D_0}(D)$ determine a birational map of the curve $D_0$ onto a curve of degree three. This is impossible since $g(D_0) = 7$. Therefore Proposition 8 follows from the exact sequence

$$0 \to H^0(I, \mathcal{O}_I) \to H^0(I, \mathcal{O}_I(D)) \to H^0(D_0, \mathcal{O}_{D_0}(D)) \to H^1(I, \mathcal{O}_I) \to 0$$

combined with the equalities $\dim H^0(I, \mathcal{O}_I) = 1$, $\dim H^0(I, \mathcal{O}_I(D)) = 3$ and $\dim H^0(D_0, \mathcal{O}_{D_0}(D)) = 2$, which were established above. $\square$

4.3. Calculation of the irregularity of the variety $\mathcal{I}$. Let $\sigma : \mathcal{J} \to \mathcal{I}$ be a resolution of singularities of $\mathcal{I}$ such that $\sigma : \mathcal{J} \setminus \sigma^{-1}(\mathcal{S}) \to \mathcal{J} \setminus \mathcal{S}$ is an isomorphism.

Theorem 4. The irregularity $q(\mathcal{J}) = \dim H^1(\mathcal{J}, \mathcal{O}_\mathcal{J})$ is equal to zero.

Proof. This is proved by applying Proposition 8 and the following theorem to the morphism $h \circ \sigma : \mathcal{J} \to \mathbb{P}^9$. $\square$

Theorem 5. Let $f : M \to \mathbb{P}^k$ be a dominant morphism of a smooth projective variety $M$, $\dim M = k \geq 3$. If the irregularity of the surfaces $f^{-1}(\Pi)$ vanishes for all projective planes $\Pi \subset \mathbb{P}^k$ belonging to an open dense subset $W$ of $\text{Gr}(3,k+1)$, then the irregularity $q(M) = \dim H^1(M, \mathcal{O}_M)$ is equal to zero.

Proof. Consider a projective variety $N \subset \mathbb{P}^n$ along with a Zariski-open subset $N_0$ of $N$ contained in $N \setminus \text{Sing} \ N$. We introduce the following notation:

$\text{Gr}_2(V)$ is the Grassmannian variety of all two-dimensional vector subspaces of a vector space $V$;

$\mathfrak{G}_2(N_0) = \{(p,V) \mid p \in N_0, V \in \text{Gr}_2(T_{N_0, p})\}$, where $T_{N_0, p}$ is the tangent space to $N_0$ at the point $p$;

$\mathfrak{P}_2(N_0) = \{(p,\Pi) \in N_0 \times \text{Gr}(3,n+1) \mid p \in \Pi \subset N\}$, where the $\Pi$ are projective planes in $\mathbb{P}^n$.

The map $(p,\Pi) \mapsto (p,T_{\Pi,p})$ clearly determines an embedding $i_N : \mathfrak{P}_2(N) \hookrightarrow \mathfrak{G}_2(N)$. When $N = \mathbb{P}^k$, the map $i_{\mathbb{P}^k} : \mathfrak{P}_2(\mathbb{P}^k) \hookrightarrow \mathfrak{G}_2(\mathbb{P}^k)$ is an isomorphism.
The projection $(p, V) \mapsto p$ determines a morphism $\text{pr}_{N,1}: \mathfrak{G}_2(N_0) \to N_0$ whose fibres over the points $p \in N_0$ are of dimension $\dim \text{pr}_{N,1}^{-1}(p) = 2(\dim N - 2)$. Therefore,

$$\dim \mathfrak{G}_2(N) = 3 \dim N - 4.$$  \hspace{1cm} (19)

Assume that $q(M) > 0$. Then there is a non-zero holomorphic 1-form $\nu$ on $M$. We put $K = \{ p \in M \mid \ker \nu(p) = T_{M,p} \}$, where the linear map $\nu(p): T_{M,p} \to \mathbb{C}$ is defined by the pairing $\langle T_{M,p} \times T_{M,p}^*, \mathbb{C} \rangle$. Let $B$ be a hypersurface in $\mathbb{P}^k$ such that $f: M \setminus f^{-1}(B) \to \mathbb{P}^k \setminus B$ is a finite unramified covering. We write $M_0$ for the open dense subset $M \setminus (f^{-1}(D))$ of $M$, where $D = f(K) \cup B \subset \mathbb{P}^k$. The morphism $f: M_0 \to \mathbb{P}^k \setminus D$ clearly induces a morphism $f_*: \mathfrak{G}_2(M_0) \to \mathfrak{G}_2(\mathbb{P}^k \setminus D) \simeq \mathfrak{P}_2(\mathbb{P}^k \setminus D)$.

Consider a subset

$$\Upsilon = \{ (p, V) \in \mathfrak{G}_2(M_0) \mid V \subset \ker \nu(p) \}$$

of $\mathfrak{G}_2(M_0)$ along with its image $f_*(\Upsilon) \subset \mathfrak{G}_2(\mathbb{P}^k \setminus D)$. We have

$$\dim \Upsilon = \dim M + \dim \text{Gr}(2, \dim M - 1) = 3k - 6.$$  

Therefore,

$$\dim f_*(\Upsilon) \leq 3k - 6.$$  

On the other hand, let $\text{pr}_2: \mathfrak{P}_2(\mathbb{P}^k) \to \text{Gr}(3, k + 1)$ be the projection by the formula $(p, \Pi) \mapsto \Pi$. The variety $\mathfrak{G}_2(\mathbb{P}^k \setminus D) \cap \text{pr}_2^{-1}(W)$ is open and dense in $\mathfrak{G}_2(\mathbb{P}^k) \simeq \mathfrak{P}_2(\mathbb{P}^k)$. By (19), we have

$$\dim \mathfrak{G}_2(\mathbb{P}^k \setminus D) \cap \text{pr}_2^{-1}(W) = 3k - 4.$$  

Hence there is a point $(p, \Pi) \in \text{pr}_2^{-1}(W)$ that does not belong to the set $f_*(\Upsilon)$ and, therefore, the restriction of $\nu$ to the surface $f^{-1}(\Pi)$ does not vanish at points of $f^{-1}(p)$. Thus the irregularity of the surface $f^{-1}(\Pi)$ must be positive, contrary to the assumption $q(f^{-1}(\Pi)) = 0$. \hfill \square

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