AN INJECTIVITY RADIUS ESTIMATE IN TERMS OF METRIC SPHERE

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Abstract. In this paper we prove that if a point $p$ in a complete Riemannian manifold is not a cut point of any point whose distance to $p$ is $r$, then the injectivity radius of $p$ is strictly large than $r$. As a corollary we give a positive answer to a problem raised by Z. Sun and J. Wan.

This paper is to answer a question asked by Z. Sun and J. Wan in [2]. Let $M$ be a complete noncompact Riemannian manifold, and let $i_p$ denote the injectivity radius at $p$ of $M$. Let

$$i(p, r) = \min\{i_x : \forall x \in M \text{ s.t. } d(x, p) = r\},$$

where $d(x, p)$ is the distance between two points $x$ and $p$. According to [2], they defined a number $\alpha(M)$ to be

$$\alpha(M) = \liminf_{r \to \infty} \frac{i(p, r)}{r},$$

which is called the injectivity radius growth of $M$. Because in the definition of $\alpha(M)$ $r$ goes to infinity and the distance from $p$ to any other fixed point is a definite finite number, it can be seen directly (see also a proof in [2]) that $\alpha(M)$ is not depending on $p$. One of their questions in [2] is the following

Question 1 ([2]). For a complete noncompact manifold $M$, can one prove that every geodesic $\gamma : (-\infty, +\infty) \to M$ is a line as long as $\alpha(M) > 1$?

In other words, they asked that whether the injectivity radius of every point in $M$ is infinity when $\alpha(M) > 1$? A positive answer of Question 1 directly follows from Proposition 2 below.

Proposition 2. Let $M$ be a complete Riemannian manifold and $p \in M$. If for some $r > 0$, $p$ is not a cut point of any point $x$ such that $d(x, p) = r$, then the injectivity radius $i_p$ at $p > r$.

Remark 3. The point in proving Proposition 2 is to show that the minimal geodesics for $p$ to points in the metric sphere $S_r(p) = \{x \in M : d(p, x) = r\}$ covers the whole ball $B_r(p) = \{x \in M : d(p, x) \leq r\}$. Thous the conclusion

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of Proposition 2 may be already known by some experts, it seems that it is still not well-known and there is no proof can be found in the earlier literature. That is the reason why I decided to write down a proof.

**Remark 4.** It can be proved that for \( p \in M \) and \( r > 0 \), if the minimizing geodesic from \( p \) to each point \( x \) such that \( d(x, p) = r \) is unique, then the injectivity radius of \( p \geq r \). However, the proof is more complicated than that of Proposition 2. So we will not go into that case here.

**Proof of Proposition 2.** Let \( T^1_p M \) denote the set of all unit vectors at \( p \) in \( M \). Let us denote

\[
A(p, r) = \{ X \in T^1_p M : \exp_p(tX) \text{ is minimal on } [0, r'] \text{ for some } r' > r \}.
\]

It suffices to show that \( A(p, r) \) is open and closed in \( T^1_p M \). Firstly, it is well-known that the function \( \sigma : T^1_p M \to \mathbb{R}^+ \),

\[
\sigma(X) = \sup \{ t : \exp(tX) \text{ is minimal on } [0, t] \},
\]

is continuous (see 2.1.5 Lemma in [1]). Hence by definition \( A(p, r) \) is open.

Now let us show that \( A(p, r) \) is closed in \( T^1_p M \). Assume a sequence of unit vectors \( X_i \in A(p, r) \) converges to a unit vector \( X \in T^1_p M \), then the geodesic \( \exp(tX_i) \) converges to \( \exp(tX) \) point-wise. Because all geodesic \( \exp(tX_i) \) is minimal on \( [0, r] \), the limit \( \exp(tX) \) is also a minimal geodesic on \( [0, r] \), and thus \( d(\exp(rX), p) = r \). Moreover, by the assumption of Proposition 2, \( \exp(rX) \) is not a cut point of \( p \). Hence, there is \( \epsilon > 0 \) such that \( \exp(tX) \) is also minimal on \( [0, r + \epsilon] \). Thus \( X \in A(p, r) \) and \( A(p, r) \) is closed.

Because \( A(p, r) \) is open and closed, it coincides with \( T^1_p M \). Therefore the injectivity radius at \( p \) is \( r \). \( \square \)

The following corollaries directly follows from Proposition 2. Recall that \( p \) is called a pole if the injectivity radius of \( p \) is infinity. In particular, \( M \) is diffeomorphic to \( \mathbb{R}^n \) by the exponential map \( \exp_p : T_p M \to M \) at a pole.

**Corollary 5.** Let \( M \) be a complete non-compact manifold. \( M \) possesses a pole at \( p \) if (and only if) there is a sequence \( r_k \to \infty \) such that \( p \) is not a cut point of any point in \( S(p, r_k) \).

**Corollary 6.**

\[
\limsup_{r \to \infty} \frac{i(p, r)}{r} > 1,
\]

implies that every point in \( M \) is a pole. Hence either \( \limsup_{r \to \infty} \frac{i(p, r)}{r} \in [0, 1] \), or \( \limsup_{r \to \infty} \frac{i(p, r)}{r} = \infty \).

Because \( \alpha(M) \leq \limsup_{r \to \infty} \frac{i(p, r)}{r} \), Corollary 6 not only answers Question 1, but also strength the homeomorphism result of Theorem 1.2 in [2] to diffeomorphism in the case of dimension 4.
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