Equivalent boundedness of Marcinkiewicz integrals on non-homogeneous metric measure spaces

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Received July 27, 2012; accepted April 03, 2013

Abstract Let \((X, d, \mu)\) be a metric measure space satisfying the upper doubling condition and the geometrically doubling condition in the sense of T. Hytönen. In this paper, the authors prove that the \(L^p(\mu)\) boundedness with \(p \in (1, \infty)\) of the Marcinkiewicz integral is equivalent to either of it's boundedness from \(L^1(\mu)\) into \(L^{1, \infty}(\mu)\) or from the atomic Hardy space \(H^1(\mu)\) into \(L^1(\mu)\). Moreover, the authors show that, if the Marcinkiewicz integral is bounded from \(H^1(\mu)\) into \(L^1(\mu)\), then it is also bounded from \(L^\infty(\mu)\) into the space \(RBLO(\mu)\) (the regularized BLO), which is a proper subset of \(RBMO(\mu)\) (the regularized BMO) and, conversely, if the Marcinkiewicz integral is bounded from \(L^\infty(\mu)\) (the set of all \(L^\infty(\mu)\) functions with bounded support) into the space \(RBMO(\mu)\), then it is also bounded from the finite atomic Hardy space \(H^{1, \infty}_{\text{fin}}(\mu)\) into \(L^1(\mu)\). These results essentially improve the known results even for non-doubling measures.

Keywords upper doubling, geometrically doubling, Marcinkiewicz integral, atomic Hardy space, \(RBMO(\mu)\)

MSC(2010) 42B20, 42B25, 42B35, 30L99

Citation: Lin H, Yang D. Equivalent boundedness of Marcinkiewicz integrals on non-homogeneous metric measure spaces. Sci China Math, 2012, 55, doi: 10.1007/s11425-000-0000-0

1 Introduction

It is well known that the Littlewood-Paley \(g\)-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley \(g\)-function. In 1938, as an analogy of the classical Littlewood-Paley \(g\)-function without going into the interior of the unit disk, Marcinkiewicz [28] introduced the integral on one-dimensional Euclidean space \(\mathbb{R}\), which is today called the Marcinkiewicz integral, and conjectured that it is bounded on \(L^p([0, 2\pi])\) for any \(p \in (1, \infty)\). In 1944, by using a complex variable method, Zygmund [38] proved the Marcinkiewicz conjecture. The higher-dimensional Marcinkiewicz integral was introduced by Stein [30] in 1958. Let \(\Omega\) be homogeneous of degree zero in \(\mathbb{R}^n\) for \(n \geq 2\), integrable and have mean value zero on the unit sphere \(S^{n-1}\). The higher-dimensional Marcinkiewicz integral \(M_\Omega\) is then defined by

\[
M_\Omega(f)(x) := \left[ \int_0^\infty \left[ \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \, dy \right]^2 \frac{dt}{t^3} \right]^{1/2}, \quad x \in \mathbb{R}^n.
\]

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Stein [30] proved that, if $\Omega \in \text{Lip}_\alpha(S^{n-1})$ for some $\alpha \in (0, 1]$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for any $p \in (1, 2]$ and also bounded from $L^1(\mathbb{R}^n)$ to $L^{1, \infty}(\mathbb{R}^n)$. Since then, many papers focus on the boundedness of this operator on various function spaces. We refer the reader to see [7–9, 13, 14, 27, 33, 34, 36] for its developments and applications.

On the other hand, many results from real analysis and harmonic analysis on the classical Euclidean spaces have been extended to the space of homogeneous type introduced by Coifman and Weiss [5]. Recall that a metric space $(\mathcal{X}, d)$ equipped with a Borel measure $\mu$ is called a space of homogeneous type, if $(\mathcal{X}, d, \mu)$ satisfies the following measure doubling condition that there exists a positive constant $C_\mu$ such that, for all balls $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$
\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)).
$$

Moreover, it is known that many results concerning the theory of Calderón-Zygmund operators and function spaces remain valid even for non-doubling measures; see, for example [2, 3, 12, 21, 29, 31, 32, 37]. In particular, let $\mu$ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the polynomial growth condition that there exist positive constants $C_\mu$ and $\kappa \in (0, n]$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$
\mu(B(x, r)) \leq C_\mu r^n,
$$

where $B(x, r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. Such a measure $\mu$ need not satisfy the doubling condition (1.1). The analysis with non-doubling measures plays a striking role in solving the long-standing open Painlevé problem by Tolsa in [32]. In [12], the authors introduced the Marcinkiewicz integral on $\mathbb{R}^n$ with a measure as in (1.2). Moreover, under the assumption that the Marcinkiewicz integral is bounded on $L^2(\mu)$, the authors then obtained its boundedness, respectively, from $L^1(\mu)$ into $L^{1, \infty}(\mu)$, from the atomic Hardy space $H^1(\mu)$ into $L^1(\mu)$ or from $L^\infty(\mu)$ into the space $\text{RBLO}(\mu)$.

However, as pointed out by Hytönen in [17] that the measures satisfying the polynomial growth condition are different from, not general than, the doubling measures. Hytönen [17] introduced a new class of metric measure spaces which satisfy the so-called upper doubling condition and the geometrically doubling condition (see also, respectively, Definitions 1.1 and 1.3 below). This new class of metric measure spaces is called the non-homogeneous space, which includes both the spaces of homogeneous type and metric spaces with the measures satisfying (1.2) as special cases.

From now on, we always assume that $(\mathcal{X}, d, \mu)$ is a non-homogeneous space in the sense of Hytönen [17]. In this setting, Hytönen [17] introduced the space $\text{RBMO}(\mu)$, the space of the regularized BMO, and Hytönen and Martikainen [19] further established a version of Tb theorem. Later, Hytönen, Da. Yang and Do. Yang [20] studied the atomic Hardy space $H^1(\mu)$ and proved that the dual space of $H^1(\mu)$ is just the space $\text{RBMO}(\mu)$. Some of results in [20] were also independently obtained by Bui and Duong [1] via different approaches. Moreover, Lin and Yang [23] introduced the space $\text{RBLO}(\mu)$ (the space of the regularized BLO) and applied this space to the boundedness of the maximal Calderón-Zygmund operators. Several equivalent characterizations for the boundedness of the Calderón-Zygmund operators and the maximal Calderón-Zygmund operators were established in [16, 18, 25, 26]. Some weighted norm inequalities for the multilinear Calderón-Zygmund operators were presented by Hu, Meng and Yang in [15]. Very recently, Fu, Yang and Yuan [11] established the boundedness of multilinear commutators of Calderón-Zygmund operators with $\text{RBMO}(\mu)$ functions on Orlicz spaces. Moreover, by a method different from the classical one, Lin and Yang [24] proved an interpolation result that a sublinear operator, which is bounded from $H^1(\mu)$ to $L^{1, \infty}(\mu)$ and from $L^\infty(\mu)$ to $\text{RBMO}(\mu)$, is also bounded on $L^p(\mu)$ for all $p \in (1, \infty)$. More developments on harmonic analysis in this setting can be found in the monograph [37].

The main purpose of this paper is to generalize and improve the corresponding results in [12] for $\mathcal{X} := \mathbb{R}^n$ with a measure $\mu$ as in (1.2) to the present setting $(\mathcal{X}, d, \mu)$. Precisely, we prove that the $L^p(\mu)$ boundedness with $p \in (1, \infty)$ of the Marcinkiewicz integral is equivalent to either of its boundedness from $L^1(\mu)$ into $L^{1, \infty}(\mu)$ or from the atomic Hardy space $H^1(\mu)$ into $L^1(\mu)$. As for the endpoint case of $p = \infty$, we show that, if the Marcinkiewicz integral is bounded from $H^1(\mu)$ into $L^1(\mu)$, then it is bounded from $L^\infty(\mu)$ into the space $\text{RBLO}(\mu)$, which is a proper subset of $\text{RBMO}(\mu)$. Moreover, we prove that, if
the Marcinkiewicz integral is bounded from $L^\infty_b(\mu)$ (the set of all $L^\infty(\mu)$ functions with bounded support) into the space $\text{RBMO}(\mu)$, then it is also bounded from the finite atomic Hardy space $H^{1,\infty}_\text{fin}(\mu)$ into $L^1(\mu)$. These results essentially improve the existing results.

To state our main results, we first recall some necessary notions and notation. We start with the notion of the upper doubling and geometrically doubling metric measure space introduced in [17].

**Definition 1.1.** A metric measure space $(X, d, \mu)$ is called upper doubling, if $\mu$ is a Borel measure on $X$ and there exist a dominating function $\lambda : X \times (0, \infty) \to (0, \infty)$ and a positive constant $C_\lambda$ such that, for each $x \in X$, $r \to \lambda(x, r)$ is non-decreasing and, for all $x \in X$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2). \quad (1.3)$$

**Remark 1.2.** (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function $\lambda(x, r) := \mu(B(x, r))$. Moreover, let $\mu$ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the polynomial growth condition. By taking $\lambda(x, r) := Cr^\kappa$, we see that $(\mathbb{R}^n, |\cdot|, \mu)$ is also an upper doubling measure space.

(ii) It was proved in [20] that there exists a dominating function $\tilde{\lambda}$ related to $\lambda$ satisfying the property that there exists a positive constant $C_\tilde{\lambda}$ such that $\tilde{\lambda} \leq \lambda$, $C_\tilde{\lambda} \leq C_\lambda$ and, for all $x, y \in X$ with $d(x, y) \leq r$,

$$\tilde{\lambda}(x, r) \leq C_\tilde{\lambda} \tilde{\lambda}(y, r). \quad (1.4)$$

Based on this, in this paper, we always assume that the dominating function $\lambda$ also satisfies (1.4).

We now recall the notion of the geometrically doubling space (see, for example, [17]).

**Definition 1.3.** A metric space $(X, d)$ is said to be geometrically doubling, if there exists some $N_0 \in \mathbb{N} := \{1, 2, \ldots\}$ such that, for any ball $B(x, r) \subset X$, there exists a finite ball covering $\{B(x_i, r/2)\}_{i}$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0$.

**Remark 1.4.** Let $(X, d)$ be a metric space. In [17, Lemma 2.3], Hytönen showed that the following statements are mutually equivalent:

(i) $(X, d)$ is geometrically doubling.

(ii) For any $\epsilon \in (0, 1)$ and ball $B(x, r) \subset X$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_{i}$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n}$, here and in what follows, $N_0$ is as in Definition 1.3 and $n := \log_2 N_0$.

(iii) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset X$ can contain at most $N_0 \epsilon^{-n}$ centers $\{x_i\}_i$ of disjoint balls with radius $\epsilon r$.

(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset X$ can contain at most $M$ centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

The following coefficients $\delta(B, S)$ for all balls $B$ and $S$ were introduced in [17] as analogues of Tolsa’s numbers $K_{Q, R}$ in [31]; see also [20].

**Definition 1.5.** For all balls $B \subset S$, let

$$\delta(B, S) := \int_{(2S)\setminus B} \frac{d\mu(x)}{\lambda(c_B, d(x, c_B))},$$

where above and in that follows, for a ball $B := B(c_B, r_B)$ and $\rho \in (0, \infty)$, $\rho B := B(c_B, \rho r_B)$.

The following atomic Hardy space was introduced in [20] and a slight different equivalent variant was independently introduced in [1]. In what follows, $L^1_{\text{loc}}(\mu)$ denotes the space of all $\mu$-locally integrable functions.

**Definition 1.6.** Let $\rho \in (1, \infty)$ and $p \in (1, \infty]$. A function $b \in L^1_{\text{loc}}(\mu)$ is called a $(p, 1)$-atomic block, if
(i) there exists some ball $B$ such that $\text{supp}(b) \subset B$;

(ii) $\int_{X} b(x) \, d\mu(x) = 0$;

(iii) for any $j \in \{1, 2\}$, there exist a function $a_j$ supported on a ball $B_j \subset B$ and $\kappa_j \in \mathbb{C}$ such that

$$b = \kappa_1 a_1 + \kappa_2 a_2$$

and

$$\|a_j\|_{L^p(\mu)} \leq [\mu(\rho B_j)]^{1/p-1}[1 + \delta(B_j, B)]^{-1}.$$  

Moreover, let

$$|b|_{H^1_{\text{atb}}(\mu)} := |\kappa_1| + |\kappa_2|.$$  

**Definition 1.7.** Let $p \in (1, \infty]$.  

(1) The space $H^1_{\text{fin}}(\mu)$ is defined to be the set of all finite linear combinations of $(p, 1)_\lambda$-atomic blocks.  

The norm of $f$ in $H^1_{\text{fin}}(\mu)$ is defined by

$$\|f\|_{H^1_{\text{fin}}(\mu)} := \inf \left\{ \sum_{j=1}^{N} |b_j|_{H^1_{\text{atb}}(\mu)} : f = \sum_{j=1}^{N} b_j, b_j \text{ is a } (p, 1)_\lambda \text{-atomic block, } N \in \mathbb{N} \right\}.$$  

(2) A function $f \in L^1(\mu)$ is said to belong to the atomic Hardy space $H^1_{\text{atb}}(\mu)$, if there exist $(p, 1)_\lambda$-atomic blocks $\{b_j\}_{j \in \mathbb{N}}$ such that $f = \sum_{j=1}^{\infty} b_j$ and $\sum_{j=1}^{\infty} |b_j|_{H^1_{\text{atb}}(\mu)} < \infty$.  

The norm of $f$ in $H^1_{\text{atb}}(\mu)$ is defined by

$$\|f\|_{H^1_{\text{atb}}(\mu)} := \inf \left\{ \sum_{j} |b_j|_{H^1_{\text{atb}}(\mu)} \right\},$$

where the infimum is taken over all the possible decompositions of $f$ as above.

**Remark 1.8.** (1) It was proved in [20] that, for each $p \in (1, \infty]$, the atomic Hardy space $H^1_{\text{atb}}(\mu)$ is independent of the choice of $\rho$ and that, for all $p \in (1, \infty)$, the spaces $H^1_{\text{atb}}(\mu)$ and $H^1_{\text{atb}}(\mu)$ coincide with equivalent norms. Thus, in the following, we denote $H^1_{\text{atb}}(\mu)$ simply by $H^1(\mu)$.

(2) When $\mu(X) < \infty$, as in the space of homogeneous type, the constant function having value $[\mu(X)]^{-1}$ is also regarded as a $(p, 1)_\lambda$-atomic block (see [6, p. 591]) and, moreover, $||[\mu(X)]^{-1}||_{H^1(\mu)} \leq 1$.

We now recall the definition of the space $\text{RBMO(\mu)}$ introduced in [17].

**Definition 1.9.** Let $\rho \in (1, \infty)$.  

A function $f \in L^1_{\text{loc}}(\mu)$ is said to be in the space $\text{RBMO}(\mu)$, if there exist a positive constant $C$ and a number $f_B$ for any ball $B$ such that, for all balls $B$,

$$\frac{1}{\mu(\rho B)} \int_{B} |f(y) - f_B| \, d\mu(y) \leq C$$

and, for balls $B \subset S$,

$$|f_B - f_S| \leq C[1 + \delta(B, S)].$$

Moreover, the norm of $f$ in $\text{RBMO}(\mu)$ is defined to be the minimal constant $C$ as above and denoted by $\|f\|_{\text{RBMO}(\mu)}$.

It was proved in [17, Lemma 4.6] that the space $\text{RBMO}(\mu)$ is independent of the choice of $\rho$.

Let $K$ be a locally integrable function on $(X \times X) \setminus \{(x, x) : x \in X\}$.  

Assume that there exists a positive constant $C$ such that, for all $x, y \in X$ with $x \neq y$,

$$|K(x, y)| \leq C \frac{d(x, y)}{\lambda(x, d(x, y))}.$$  

(1.5)

and, for all $y, y' \in X$,

$$\int_{d(x, y) > 2d(y, y')} \left[ |K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \right] \frac{1}{d(x, y)} \, d\mu(x) \leq C.$$  

(1.6)
The Marcinkiewicz integral \( \mathcal{M}(f) \) associated to the above kernel \( K \) is defined by setting, for all \( x \in \mathcal{X} \),

\[
\mathcal{M}(f)(x) := \left( \int_0^\infty \left( \int_{d(x, y) < t} K(x, y) f(y) \, d\mu(y) \right)^2 \frac{dt}{t^3} \right)^{1/2}.
\]  

(1.7)

Obviously, by taking \( \lambda(x, r) := C r^n \), we see that, in the classical Euclidean space \( \mathbb{R}^n \), if

\[
K(x, y) := \frac{1}{|x - y|^{n-1}} \Omega(x - y)
\]

with \( \Omega \) being homogeneous of degree zero and \( \Omega \in \text{Lip}_\alpha(S^{n-1}) \) for some \( \alpha \in (0, 1] \), then \( K \) satisfies (1.5) and (1.6), and \( \mathcal{M} \) in (1.7) is just the Marcinkiewicz integral \( \mathcal{M}_\Omega \) introduced by Stein in [30]. Thus, \( \mathcal{M} \) in (1.7) is a natural generalization of the classical Marcinkiewicz integral in the present setting.

One of main results of this article is as follows.

**Theorem 1.10.** Let \( K \) satisfy (1.5) and (1.6), and \( \mathcal{M} \) be as in (1.7). Then the following four statements are equivalent:

(i) \( \mathcal{M} \) is bounded on \( L^{p_0}(\mu) \) for some \( p_0 \in (1, \infty) \);

(ii) \( \mathcal{M} \) is bounded from \( L^1(\mu) \) into \( L^{1, \infty}(\mu) \);

(iii) \( \mathcal{M} \) is bounded on \( L^p(\mu) \) for all \( p \in (1, \infty) \);

(iv) \( \mathcal{M} \) is bounded from \( H^1(\mu) \) into \( L^1(\mu) \).

Comparing with the corresponding result in [12], Theorem 1.10 makes an essential improvement.

As for the endpoint case of \( p = \infty \), we obtain the following result. Recall that \( L_0^\infty(\mu) \) denotes the set of all \( L^\infty(\mu) \) functions with bounded support and \( \text{RBLO}(\mu) \) the regularized BLO space introduced in [23] (see also Definition 2.2 below).

**Theorem 1.11.** Let \( K \) satisfy (1.5) and (1.6), and \( \mathcal{M} \) be as in (1.7).

(i) If \( \mathcal{M} \) is bounded from \( H^1(\mu) \) into \( L^1(\mu) \), then, for \( f \in L^\infty(\mu) \), \( \mathcal{M}(f) \) is either infinite everywhere or finite \( \mu \)-almost everywhere, more precisely, if \( \mathcal{M}(f) \) is finite at some point \( x_0 \in \mathcal{X} \), then \( \mathcal{M}(f) \) is finite \( \mu \)-almost everywhere and

\[
\|\mathcal{M}(f)\|_{\text{RBLO}(\mu)} \leq C \|f\|_{L^\infty(\mu)},
\]

where the positive constant \( C \) is independent of \( f \).

(ii) If there exists a positive constant \( C \) such that, for all \( f \in L_0^\infty(\mu) \),

\[
\|\mathcal{M}(f)\|_{\text{RBMO}(\mu)} \leq C \|f\|_{L^\infty(\mu)},
\]

then \( \mathcal{M} \) is bounded from \( H_0^1, \infty(\mu) \) into \( L^1(\mu) \).

**Remark.** (i) Recall that it was proved in [23] that \( \text{RBLO}(\mu) \) is a proper subset of \( \text{RBMO}(\mu) \), which, together with Theorem 1.11(i), further implies that, if \( \mathcal{M} \) is bounded from \( H^1(\mu) \) into \( L^1(\mu) \), then, for any \( f \in L^\infty(\mu) \), \( \mathcal{M}(f) \) is either infinite everywhere or

\[
\|\mathcal{M}(f)\|_{\text{RBMO}(\mu)} \lesssim \|f\|_{L^\infty(\mu)}.
\]

This is the known result for the Marcinkiewicz integral over the classical Euclidean space \( \mathbb{R}^n \). Moreover, Lin et al. [22] constructed a nonnegative function belonging to \( \text{BMO}(\mathbb{R}^n) \) but not to \( \text{BLO}(\mathbb{R}^n) \), which further shows that our result indeed improves the known corresponding result even on the classical Euclidean space \( \mathbb{R}^n \).
(ii) From Theorem 1.10, we deduce that, if $\mathcal{M}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$, then it is also bounded on $L^p(\mu)$ for all $p \in (1, \infty)$. Since $L^\infty_0(\mu) \subset L^p(\mu)$ for all $p \in (1, \infty)$, we then see that, if $\mathcal{M}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$, then, for any $f \in L^\infty_0(\mu)$, $\mathcal{M}(f)$ is finite at some point $x_0 \in \mathcal{X}$, which, together with Theorem 1.11(i), implies that it is bounded from $L^\infty_0(\mu)$ into RBLO($\mu$) and hence, by (i) of this remark, it is also bounded from $L^\infty_0(\mu)$ into RBMO($\mu$).

(iii) In the present setting, it is still unclear whether the boundedness of sublinear operators on the atomic Hardy space can be deduced only from their behaviors on atoms. More precisely, it is unclear whether the uniform boundedness in some Banach space $\mathcal{B}$ of a sublinear operator $T$ on all $(\infty, 1)$-atoms can guarantee the boundedness of $T$ from $H^1(\mu)$ to $\mathcal{B}$ or not. Thus, under the assumption of Theorem 1.11, it is unclear whether the Marcinkiewicz integral $\mathcal{M}$ can extends boundedly from $H^1(\mu)$ to $L^1(\mu)$ or not.

This paper is organized as follows. In Section 2, under the assumption that the Marcinkiewicz integral is bounded on $L^{p_0}(\mu)$ for some $p_0 \in (1, \infty)$, we then obtain its boundedness, respectively, from $L^1(\mu)$ to $L^{1, \infty}(\mu)$, from $H^1(\mu)$ to $L^1(\mu)$, from $L^\infty(\mu)$ to the space RBLO($\mu$) and on $L^p(\mu)$ for all $p \in (1, \infty)$; see Theorem 2.3 below. From this, we deduce that Theorem 1.10(i) implies (ii), (iii) and (iv) of Theorem 1.10, which slightly improves the corresponding result in [12] by relaxing the assumption that the Marcinkiewicz integral is bounded on $H^1(\mu)$ to $\mathcal{B}$ or not. In this section, under the assumption that the Marcinkiewicz integral is bounded on $L^{p_0}(\mu)$ for some $p_0 \in (1, \infty)$, we then obtain its boundedness, respectively, from $L^1(\mu)$ to $L^{1, \infty}(\mu)$, from $H^1(\mu)$ to $L^1(\mu)$, from $L^\infty(\mu)$ to the space RBLO($\mu$) and on $L^p(\mu)$ for all $p \in (1, \infty)$.

In Section 3, we prove Theorem 1.10. Indeed, by Theorem 2.3 and an obvious fact that Theorem 1.10(iii) implies Theorem 1.10(i), to prove Theorem 1.10, we only need to prove that Theorem 1.10(ii) implies Theorem 1.10(iii) and Theorem 1.10(iv) implies Theorem 1.10(iii). To this end, we need some fine estimates on the sharp maximal function $M^2_f$ (see (3.1) below) and the non-centered doubling Hardy-Littlewood maximal function $N_r$ (see (3.2) below); for example, see the technical Lemmas 3.1 and 3.2 concerning with the operators $M^2_f$ and $N_r$ from [24] and the estimate for $M^2_f(\mathcal{M}(f))$ in Lemma 3.4. We also need to consider the decomposition of the function $f$; for example, in the proof that Theorem 1.10(ii) implies Theorem 1.10(iii), for any fixed $\ell \in (0, \infty)$, we split $f$ into $f_1$ and $f_2$ with $f_1 := f \chi_{\{|y| \leq \ell\}}$ and $f_2 := f \chi_{\{|y| > \ell\}}$, while in the proof that Theorem 1.10(iv) implies Theorem 1.10(iii), we use the Calderón-Zygmund decomposition from [1, Theorem 6.3] (see also Lemma 2.5 below). Here and in what follows, for any $\mu$-measurable set $E$, $\chi_E$ denotes its characteristic function. Based on these facts, by some argument similar to that used in the proof of [24, Theorem 1.1], we then complete the proof of Theorem 1.10.

Section 4 is devoted to the proof of Theorem 1.11. Indeed, Theorem 1.11(i) can be deduced directly from Theorems 1.10 and 2.3. By using a technical estimate for $\mathcal{M}$ (see (4.3) below) and some argument used in the proof of Theorem 2.3, we then obtain the desired conclusion of Theorem 1.11(ii).

We finally make some conventions on notation. Throughout this paper, we denote by $C$ a positive constant which is independent of the main parameters involved, but may vary from line to line. Positive constants with subscripts, such as $C_1$, do not change in different occurrences. The subscripts of a constant indicate the parameters it depends on. The symbol $Y \lesssim Z$ means that there exists a positive constant $C$ such that $Y \leq CZ$. The symbol $A \sim B$ means that $A \lesssim B \lesssim A$. For any ball $B \subset \mathcal{X}$, we denote its center and radius, respectively, by $c_B$ and $r_B$ and, moreover, for any $\rho \in (1, \infty)$, the ball $B(c_B, \rho r_B)$ by $\rho B$. Given any $q \in (1, \infty)$, let $q := q/(q - 1)$ denote its conjugate index. Also, let $\mathbb{N} := \{1, 2, \ldots\}$.

## 2 Boundedness of Marcinkiewicz integrals

In this section, under the assumption that the Marcinkiewicz integral is bounded on $L^{p_0}(\mu)$ for some $p_0 \in (1, \infty)$, we then obtain its boundedness on Lebesgue spaces and Hardy spaces. We first recall the notions of $(\alpha, \beta)$-doubling and the space RBLO($\mu$).

**Definition 2.1.** Let $\alpha, \beta \in (1, \infty)$. A ball $B := B(x, r) \subset \mathcal{X}$ is called $(\alpha, \beta)$-doubling, if $\mu(\alpha B) \lesssim \beta \mu(B)$.

It was proved in [17] that, if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\beta > C^\log_2 \alpha =: \alpha'$, then, for every ball $B(x, r) \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ such that $\alpha^j B$ is $(\alpha, \beta)$-doubling.
Moreover, let \((\mathcal{X}, d)\) be geometrically doubling, \(\beta > \alpha^n\) with \(n := \log_2 N_0\) and \(\mu\) be a Borel measure on \(\mathcal{X}\) which is finite on bounded sets. Hytönen [17] also showed that, for \(\mu\)-almost every \(x \in \mathcal{X}\), there exist arbitrarily small \((\alpha, \beta)\)-doubling balls centered at \(x\). Furthermore, the radius of these balls may be chosen to be of the form \(\alpha^{-3}\) for \(j \in \mathbb{N}\) and any preassigned number \(r \in (0, \infty)\). Throughout this paper, for any \(\alpha \in (1, \infty)\) and ball \(B, \tilde{B}^\alpha\) always denotes the smallest \((\alpha, \beta_\alpha)\)-doubling ball of the form \(\alpha^j B\) with \(j \in \mathbb{N}\), where

\[
\beta_\alpha := \max \{\alpha^{3n}, \alpha^{3\nu}\} + 30^n + 30^\nu = \alpha^{3(\max\{n, \nu\})} + 30^n + 30^\nu. \tag{2.1}
\]

If \(\alpha = 6\), we denote the ball \(\tilde{B}^\alpha\) simply by \(\tilde{B}\).

The following space \(\text{RBLO}(\mu)\) was introduced in [23]. Recall that the classical space \(\text{BLO}(\mathbb{R}^n)\) was introduced by Coifman and Rochberg [4] and, in the setting of \((\mathbb{R}^n, | \cdot |, \mu)\) with \(\mu\) only satisfying the polynomial growth condition, the space \(\text{RBLO}(\mu)\) was first introduced by Jiang [21].

**Definition 2.2.** Let \(\eta, \rho \in (1, \infty)\), and \(\beta_\rho\) be as in (2.1). A real-valued function \(f \in L^1_{\text{loc}}(\mu)\) is said to be in the space \(\text{RBLO}(\mu)\), if there exists a non-negative constant \(C\) such that, for all balls \(B,\)

\[
\frac{1}{\mu(\eta B)} \int_B \left[ f(y) - \left. \inf \right|_{\rho} f \right] \, d\mu(y) \leq C
\]

and, for all \((\rho, \beta_\rho)\)-doubling balls \(B \subset S,\)

\[
\inf \left. \inf \right|_{B \rho} f - \inf \left. \inf \right|_{S} f \leq C[1 + \delta(B, S)].
\]

Moreover, the \(\text{RBLO}(\mu)\) norm of \(f\) is defined to be the minimal constant \(C\) as above and denoted by \(\|f\|_{\text{RBLO}(\mu)}\).

It was proved in [23] that \(\text{RBLO}(\mu) \subset \text{RBMO}(\mu)\) and the definition of \(\text{RBLO}(\mu)\) is independent of the choice of the constants \(\eta, \rho \in (1, \infty)\).

**Theorem 2.3.** Let \(K\) satisfy (1.5) and (1.6), and \(M\) be as in (1.7). Suppose that \(M\) is bounded on \(L^{p_0}(\mu)\) for some \(p_0 \in (1, \infty)\). Then,

(i) \(M\) is bounded from \(L^1(\mu)\) into \(L^{1, \infty}(\mu)\);

(ii) \(M\) is bounded from \(H^1(\mu)\) into \(L^1(\mu)\);

(iii) for \(f \in L^{\infty}(\mu), M(f)\) is either infinite everywhere or finite \(\mu\)-almost everywhere; more precisely, if \(M(f)\) is finite at some point \(x_0 \in \mathcal{X}\), then \(M(f)\) is finite \(\mu\)-almost everywhere and

\[
\|M(f)\|_{\text{RBLO}(\mu)} \leq C\|f\|_{L^{\infty}(\mu)},
\]

where the positive constant \(C\) is independent of \(f\);

(iv) \(M\) is bounded on \(L^p(\mu)\) for all \(p \in (1, \infty)\).

To prove Theorem 2.3, we first recall some necessary technical lemmas. The following useful properties of \(\delta\) were proved in [20].

**Lemma 2.4.** (i) For all balls \(B \subset R \subset S\), it holds true that \(\delta(B, R) \leq \delta(B, S)\).

(ii) For any \(\rho \in [1, \infty)\), there exists a positive constant \(C\), depending on \(\rho\), such that, for all balls \(B \subset S\) with \(r_S \leq \rho r_B\), \(\delta(B, S) \leq C\).

(iii) For any \(\alpha \in (1, \infty)\), there exists a positive constant \(\tilde{C}\), depending on \(\alpha\), such that, for all balls \(B, \delta(B, \tilde{B}^\alpha) \leq \tilde{C}\).

(iv) There exists a positive constant \(c\) such that, for all balls \(B \subset R \subset S, \delta(B, S) \leq \delta(B, R) + c\delta(R, S)\). In particular, if \(B\) and \(R\) are concentric, then \(c = 1\).

(v) There exists a positive constant \(\tilde{c}\) such that, for all balls \(B \subset R \subset S, \delta(R, S) \leq \tilde{c}[1 + \delta(B, S)];\) moreover, if \(B\) and \(R\) are concentric, then \(\delta(R, S) \leq \delta(B, S)\).
Now we recall the Calderón-Zygmund decomposition from [1, Theorem 6.3].

**Lemma 2.5.** Let $p \in [1, \infty)$, $f \in L^p(\mu)$ and $\ell \in (0, \infty)$ ($\ell > \ell_0 := \frac{2}{\gamma_0} [\mu(X)]^{-\frac{1}{p}} \|f\|_{L^p(\mu)}$ if $\mu(X) < \infty$, where $\gamma_0$ is any fixed positive constant satisfying that $\gamma_0 > \max\{C_\lambda \log_\delta 6, 6^{3n}\}$, $C_\lambda$ is as in (1.3) and $n := \log_2 N_0$). Then,

(i) there exists an almost disjoint family $\{6B_j\}_j$ of balls such that $\{B_j\}_j$ is pairwise disjoint,

\[
\frac{1}{\mu(6^2B_j)} \int_{B_j} |f(x)|^p \, d\mu(x) > \frac{\ell^p}{\gamma_0} \quad \text{for all } j,
\]

\[
\frac{1}{\mu(6^2\eta B_j)} \int_{\eta B_j} |f(x)|^p \, d\mu(x) \leq \frac{\ell^p}{\gamma_0} \quad \text{for all } j \text{ and all } \eta \in (2, \infty)
\]

and

\[
|f(x)| \leq \ell \quad \text{for } \mu - \text{almost every } x \in X \setminus (\cup_j 6B_j);
\]

(ii) for each $j$, let $S_j$ be a $(3 \times 6^2, C_\lambda \log_\delta (3 \times 6^2)^{+1})$-doubling ball of the family $\{(3 \times 6^2)^{k} B_j\}_{k \in \mathbb{N}}$ and $\omega_j := \chi_{6B_j}/(\sum_k \chi_{6B_k})$. Then, there exists a family $\{\varphi_j\}_j$ of functions such that, for each $j$, $\text{supp} (\varphi_j) \subset S_j$, $\varphi_j$ has a constant sign on $S_j$,

\[
\int_X \varphi_j(x) \, d\mu(x) = \int_{6B_j} f(x) \omega_j(x) \, d\mu(x),
\]

\[
\sum_j |\varphi_j(x)| \leq \gamma \ell \quad \text{for } \mu - \text{almost every } x \in X,
\]

where $\gamma$ is some positive constant, depending only on $(X, \mu)$, and there exists a positive constant $C$, independent of $f$, $\ell$ and $j$, such that, when $p = 1$, it holds true that

\[
\|\varphi_j\|_{L^\infty(\mu)(S_j)} \leq C \int_X |f(x)\omega_j(x)| \, d\mu(x)
\]

and, when $p \in (1, \infty)$, it holds true that

\[
\left[ \left( \int_{S_j} |\varphi_j(x)|^p \, d\mu(x) \right)^{1/p} \right]^{1/p} \mu(S_j)^{1/p'} \leq \frac{C}{\ell^{p-1}} \int_X |f(x)\omega_j(x)|^p \, d\mu(x);
\]

(iii) for $p \in (1, \infty)$, if, for any $j$, choosing $S_j$ in (ii) to be the smallest $(3 \times 6^2, C_\lambda \log_\delta (3 \times 6^2)^{+1})$-doubling ball of the family $\{(3 \times 6^2)^{k} B_j\}_{k \in \mathbb{N}}$, then $h := \sum_j (f \omega_j - \varphi_j) \in H^{1,p}_{\text{atb}}(\mu)$ and there exists a positive constant $C$, independent of $f$ and $\ell$, such that

\[
\|h\|_{H^{1,p}_{\text{atb}}(\mu)} \leq \frac{C}{\ell^{p-1}} \|f\|_{L^p(\mu)}
\]

The following characterization of the space RBLO($\mu$) was proved in [23].

**Lemma 2.6.** Let $\rho \in (1, \infty)$ and $\beta_\rho$ be as in (2.1). If $f \in \text{RBLO}(\mu)$, then there exists a non-negative constant $C_1$ satisfying that, for all $(\rho, \beta_\rho)$-doubling balls $B$,

\[
\frac{1}{\mu(B)} \int_B \left[ f(y) - \text{ess inf}_B f \right] \, d\mu(y) \leq C_1
\]

and, for all $(\rho, \beta_\rho)$-doubling balls $B \subset S$,

\[
m_B(f) - m_S(f) \leq C_1 \left[ 1 + \delta(B, S) \right],
\]

where above and in what follows, $m_B(f)$ denotes the mean of $f$ over $B$, namely,

\[
m_B(f) := \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y).
\]

Moreover, the minimal constant $C_1$ is equivalent to $\|f\|_{\text{RBLO}(\mu)}$. 

To prove Theorem 1.10, we also need the following two interpolation results.

**Lemma 2.7.** Let \( p \in (1, \infty) \) and \( T \) be a sublinear operator bounded from \( L^1(\mu) \) to \( L^{1,\infty}(\mu) \). If there exists a positive constant \( C \) such that, for all \((p, 1)_\lambda\)– atomic blocks \( b \),

\[
\|Tb\|_{L^1(\mu)} \leq C|b|_{H^1_{\lambda, \infty}^{(p)}(\mu)},
\]

then \( T \) extends to be a bounded sublinear operator from \( H^1(\mu) \) to \( L^1(\mu) \).

The proof of Lemma 2.7 is similar to that of \cite[Theorem 1.13]{24}, the details being omitted. The following lemma is just \cite[Theorem 1.1]{24}.

**Lemma 2.8.** Suppose that \( T \) is a sublinear operator bounded from \( L^\infty(\mu) \) to \( \text{RBMO}(\mu) \) and from \( H^1(\mu) \) to \( L^{1,\infty}(\mu) \). Then \( T \) extends boundedly to \( L^p(\mu) \) for every \( p \in (1, \infty) \).

Based on the above lemmas, we now turn to the proof of Theorem 2.3.

**Proof of Theorem 2.3.** We first show (i). Let \( f \in L^1(\mu) \) and \( \ell \in (0, \infty) \). To prove (i), it suffices to show that

\[
\mu\left(\{x \in \mathcal{X} : \mathcal{M}(f)(x) > \ell\}\right) \lesssim \epsilon^{-1} \|f\|_{L^1(\mu)}.
\]

By applying Lemma 2.5 and its notation, we see that \( f = g + h \), where \( h := \sum_j (f \omega_j - \varphi_j) =: \sum_j h_j \).

Obviously, \( \|g\|_{L^\infty(\mu)} \lesssim \epsilon \) and \( \|g\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)} \). This, together with the \( L^{p_0}(\mu) \) boundedness of \( \mathcal{M} \), implies that

\[
\mu\left(\{x \in \mathcal{X} : \mathcal{M}(g)(x) > \ell\}\right) \lesssim \epsilon^{-p_0} \|g\|_{L^{p_0}(\mu)} \lesssim \epsilon^{-1} \|f\|_{L^1(\mu)}.
\]

By Lemma 2.5(i), to prove (2.3), we only need to prove that

\[
\mu\left(\left\{x \in \mathcal{X} \setminus \left(\bigcup_j 6^2 B_j\right) : \mathcal{M}(h)(x) > \ell\right\}\right) \lesssim \epsilon^{-1} \int_{\mathcal{X}} |f(x)| \, d\mu(x).
\]

To this end, for each fixed \( j \), let \( S_j \) be as in Lemma 2.5(iii) with \( c_{S_j} \) and \( r_{S_j} \) being, respectively, its center and radius, and write

\[
\int_{\mathcal{X} \setminus 2S_j} \mathcal{M}(h_j)(x) \, d\mu(x)
\]

\[
\lesssim \int_{\mathcal{X} \setminus 2S_j} \left[ \int_0^{d(x, c_{S_j}) + r_{S_j}} \left( \int_{d(x, y) < \ell} K(x, y) h_j(y) \, d\mu(y) \right)^2 \frac{dt}{t^3} \right]^{1/2} \, d\mu(x)
\]

\[
+ \int_{\mathcal{X} \setminus 2S_j} \left[ \int_{d(x, c_{S_j}) + r_{S_j}}^\infty \ldots \right]^{1/2} \, d\mu(x) =: I_1 + I_2,
\]

From (1.3) and (1.4), we deduce that, for any ball \( B \) with the center \( c_B, x \notin kB \) with \( k \in (1, \infty) \) and \( y \in B \),

\[
\lambda(c_B, d(x, c_B)) \sim \lambda(x, d(x, c_B)) \sim \lambda(x, d(x, y)),
\]

which, together with the Minkowski inequality, (1.3) and (1.5), shows that

\[
I_1 \lesssim \int_{\mathcal{X} \setminus 2S_j} \left[ \int_{d(x, y)}^{d(x, c_{S_j}) + r_{S_j}} \frac{dt}{t^3} \left| h_j(y) \frac{d(x, y)}{\lambda(x, d(x, y))} \right| d\mu(y) \right]^{1/2} \, d\mu(x)
\]

\[
\lesssim \frac{1}{S_j} \left( \int_{\mathcal{X}} |h_j(y)| \, d\mu(y) \right) \int_{\mathcal{X} \setminus 2S_j} \frac{1}{d(x, c_{S_j})^{1/2} \lambda(x, d(x, c_{S_j}))} \, d\mu(x) \lesssim \|h_j\|_{L^1(\mu)}.
\]

For \( x \in \mathcal{X} \setminus 2S_j \) and \( y \in S_j \), it holds true that \( d(x, y) < d(x, c_{S_j}) + r_{S_j} \). Thus, by the vanishing moment of \( h_j \) and (1.6), we obtain

\[
I_2 \lesssim \int_{\mathcal{X} \setminus 2S_j} \left[ \int_{\mathcal{X}} \left| K(x, y) - K(x, c_{S_j}) h_j(y) \right| \, d\mu(y) \right]^{1/2} \, d\mu(x)
\]

\[
\lesssim \frac{1}{S_j} \left( \int_{\mathcal{X}} |K(x, y) - K(x, c_{S_j}) h_j(y)| \, d\mu(y) \right) \int_{\mathcal{X} \setminus 2S_j} \frac{1}{d(x, c_{S_j}) + r_{S_j}} \, d\mu(x)
\]

\[
\lesssim \frac{1}{S_j} \left( \int_{\mathcal{X}} |K(x, y) - K(x, c_{S_j}) h_j(y)| \, d\mu(y) \right) \int_{\mathcal{X} \setminus 2S_j} \frac{1}{d(x, c_{S_j}) + r_{S_j}} \, d\mu(x)
\]
\[
\begin{align*}
\lesssim & \int_X |h_j(y)| \int_{X \setminus 2B_j} |K(x, y) - K(x, c_{S_j})| \frac{1}{d(x, c_{S_j})} \, d\mu(x) \, d\mu(y) \\
& \lesssim \|h_j\|_{L^1(\mu)}.
\end{align*}
\]

Notice that supp \((f \omega_j) \subset 6B_j\) and \(|\omega_j| \leq 1\). From this, the Minkowski inequality, (1.5), (2.5) and Lemma 2.4, it follows that

\[
\int_{(2S_j) \setminus 6^2B_j} M(f \omega_j)(x) \, d\mu(x) \\
\lesssim \int_{(2S_j) \setminus 6^2B_j} \int_X \left[ \int_{d(x, y)}^\infty \frac{1}{t^3} \frac{|f(y)\omega_j(y)|}{(x, d(x, y))} \, d\mu(y) \right] \, d\mu(x) \\
\lesssim \int_{(2S_j) \setminus 6^2B_j} \frac{1}{\lambda(c_{B_j}, d(x, c_{B_j}))} \, d\mu(x) \int_{6B_j} |f(y)| \, d\mu(y) \\
\lesssim \delta(B_j, S_j) \int_{6B_j} |f(y)| \, d\mu(y) \lesssim \int_{6B_j} |f(y)| \, d\mu(y). \tag{2.6}
\]

On the other hand, by the H"older inequality, the \(L^{p_0}(\mu)\)-boundedness of \(M\) and Lemma 2.5(ii), we conclude that

\[
\int_{2S_j} M(\varphi_j)(x) \, d\mu(x) \lesssim \left\{ \int_{2S_j} |M(\varphi_j)(x)|^{p_0} \, d\mu(x) \right\}^{1/p_0} \left[ \mu(2S_j) \right]^{1/p_0'} \lesssim \left[ \int_{S_j} |\varphi_j(x)| \, d\mu(x) \right]^{1/p_0} \left[ \mu(S_j) \right]^{1/p_0'} \lesssim \int_{6B_j} |f(y)| \, d\mu(y),
\]

where \(1/p_0 + 1/p_0' = 1\). The above two estimates, together with the estimates for \(I_1\) and \(I_2\) and Lemma 2.5, show that

\[
\mu \left( \left\{ x \in X \setminus \left( \bigcup_j 6^2B_j \right) : M(h)(x) > \ell \right\} \right) \\
\leq \ell^{-1} \left[ \sum_j \int_{X \setminus 2B_j} M(h_j)(x) \, d\mu(x) + \sum_j \int_{(2S_j) \setminus 6^2B_j} \cdots \right] \\
\lesssim \ell^{-1} \left[ \sum_j \|h_j\|_{L^1(\mu)} + \sum_j \int_{6B_j} |f(y)| \, d\mu(y) \right] \lesssim \ell^{-1} \int_X |f(x)| \, d\mu(x),
\]

which implies (2.4) and hence completes the proof of (i).

To prove (ii), as pointed out in Remark 1.8, since the definition of \(H^1(\mu)\) is independent of the choice of the constant \(\rho \in (1, \infty)\), without loss of generality, we may assume that \(\rho = 2\) in Definition 1.6. It follows, from (i), that \(M\) is bounded from \(L^1(\mu)\) to \(L^{1, \infty}(\mu)\). Thus, by Lemma 2.7, to show (ii), it suffices to prove that, for all \((p_0, 1)\lambda\)-atomic blocks \(b\),

\[
\|M(b)\|_{L^1(\mu)} \lesssim |b|_{H^{1, p_0}_{\text{at}}(\mu)}. \tag{2.7}
\]

Let \(b := \sum_{j=1}^2 \kappa_j a_j\) be a \((p_0, 1)\lambda\)-atomic block, where, for any \(j \in \{1, 2\}\), supp \((a_j) \subset B_j \subset B\) for some \(B_j\) and \(B\) as in Definition 1.6. Write

\[
\int_X M(b)(x) \, d\mu(x) \\
= \int_{X \setminus 2B} M(b)(x) \, d\mu(x) + \int_{2B} \cdots \\
\leq \int_{X \setminus 2B} M(b)(x) \, d\mu(x) + \sum_{j=1}^2 \left| \int_{2B_j} M(a_j)(x) \, d\mu(x) + \int_{(2B) \setminus 2B_j} \cdots \right|
\]

\[
\|M(b)\|_{L^1(\mu)} \lesssim |b|_{H^{1, p_0}_{\text{at}}(\mu)}.
\]
\[ J_2 \leq |k_1| + |k_2| \sim |b|_{B^{1,p_0}}. \]

which, together with (2.8), implies (2.7) and hence completes the proof of (ii).

By Definition 1.6 and an argument similar to that used in the estimates for \( I_1 \) and \( I_2 \), we see that

\[ J_1 \lesssim \|b\|_{L^1(\mu)} \lesssim \|b\|_{H^{1,p_0}(\mu)}. \]  

(2.8)

From the Hölder inequality, the \( L^{p_0}(\mu) \) boundedness of \( M \) and Definition 1.6(iii), it follows that, for each fixed \( j \),

\[ \int_{2B_j} M(a_j)(x)\,d\mu(x) \lesssim \| M(a_j) \|_{L^{p_0}(\mu)} [\mu(2B_j)]^{1/p_0} \lesssim \| a_j \|_{L^{p_0}(\mu)} [\mu(2B_j)]^{1/p_0} \lesssim 1. \]  

(2.9)

Similar to the estimate for (2.6), by Definition 1.6(iii), we have

\[ \int_{(2S)\setminus B_j} M(a_j)(x)\,d\mu(x) \lesssim \int_{(2S)\setminus B_j} \frac{1}{\lambda(c_{B_j}, d(x, c_{B_j}))}\,d\mu(x) \|a_j\|_{L^1(\mu)} \lesssim \delta(B_j,S) \|a_j\|_{L^1(\mu)} \lesssim 1. \]

Combining the above estimates, we see that

\[ J_2 \lesssim |k_1| + |k_2| \sim |b|_{B^{1,p_0}}. \]

which, together with (2.8), implies (2.7) and hence completes the proof of (ii).

We now prove (iii). First, we claim that there exists a positive constant \( C \) such that, for any \( f \in L^\infty(\mu) \) and \( (6, \beta_6) \)-doubling ball \( B \),

\[ \frac{1}{\mu(B)} \int_B M(f)(y)\,d\mu(y) \leq C \|f\|_{L^\infty(\mu)} + \inf_{y \in B} M(f)(y). \]  

(2.10)

To prove this, we decompose \( f \) as

\[ f(x) = f \chi_{5B} + f \chi_{X \setminus 5B} =: f_1 + f_2. \]

By the Hölder inequality and \( L^{p_0}(\mu) \) boundedness of \( M \), we have

\[ \frac{1}{\mu(B)} \int_B M(f_1)(y)\,d\mu(y) \leq \frac{1}{[\mu(B)]^{1/p_0}} \left\{ \int_X [M(f_1)(y)]^{p_0}\,d\mu(y) \right\}^{1/p_0} \lesssim \frac{[\mu(5B)]^{1/p_0}}{[\mu(B)]^{1/p_0}} \|f\|_{L^\infty(\mu)} \lesssim \|f\|_{L^\infty(\mu)}. \]  

(2.11)

Noticing that, for \( y \in B \) and \( z \in X \setminus 5B \), it holds true that \( d(y,z) > r_B \). By the Minkowski inequality, (1.3) and (1.5), we conclude that, for any \( y \in B \),

\[ M(f_2)(y) = \left[ \int_{r_B}^{\infty} \left( \int_{d(y,z)<t} K(y,z)f_2(z)\,d\mu(z) \right)^2 \frac{dt}{t^3} \right]^{1/2} \]

\[ \leq \left[ \int_{r_B}^{\infty} \left( \int_{d(y,z)<t} K(y,z)f(z)\,d\mu(z) \right)^2 \frac{dt}{t^3} \right]^{1/2} \]

\[ + \left[ \int_{r_B}^{\infty} \left( \int_{d(y,z)<t} K(y,z)f_1(z)\,d\mu(z) \right)^2 \frac{dt}{t^3} \right]^{1/2} \]

\[ \leq M(f)(y) + \left[ \int_{r_B}^{\infty} \left( \int_{d(y,z)<6r_B} K(y,z)f_1(z)\,d\mu(z) \right)^2 \frac{dt}{t^3} \right]^{1/2} \]
\[
\leq M(f)(y) + C\|f\|_{L^\infty(\mu)} r_B^{-1} \int_{d(y, z) < 6r_B} \frac{d(y, z)}{\lambda(y, d(y, z))} d\mu(z)
\leq M(f)(y) + C\|f\|_{L^\infty(\mu)},
\]  

(2.12)

where \(C\) is a positive constant independent of \(f\) and \(y\). Thus, the proof of the estimate (2.10) can be reduced to proving that, for all \(x, y \in B\),

\[
|M(f_2)(x) - M(f_2)(y)| \lesssim \|f\|_{L^\infty(\mu)}.
\]  

(2.13)

To this end, write

\[
|\mathcal{M}(f_2)(x) - \mathcal{M}(f_2)(y)|
\leq \left[ \int_0^\infty \left( \int_{d(y, z) < t} K(x, z) f_2(z) \, d\mu(z) - \int_{d(y, z) < t} K(y, z) f_2(z) \, d\mu(z) \right)^2 \frac{dt}{t^3} \right]^{1/2}
\leq \left\{ \int_0^\infty \left[ \int_{d(y, z) < t \leq d(x, z)} |K(y, z)| |f_2(z)| \, d\mu(z) \right]^2 \frac{dt}{t^3} \right\}^{1/2}
+ \left\{ \int_0^\infty \left[ \int_{d(x, z) < t \leq d(y, z)} |K(x, z)| |f_2(z)| \, d\mu(z) \right]^2 \frac{dt}{t^3} \right\}^{1/2}
+ \left\{ \int_0^\infty \left[ \int_{d(x, z) < t \leq d(x, y)} |K(y, z) - K(x, z)| |f_2(z)| \, d\mu(z) \right]^2 \frac{dt}{t^3} \right\}^{1/2}
=: M_1 + M_2 + M_3.
\]

Applying the Minkowski inequality, (1.3), (1.5) and (2.5), we conclude that, for all \(x, y \in B\),

\[
M_1 \lesssim \int_X \frac{|f_2(z)| d(y, z)}{\lambda(y, d(y, z))} \left[ \int_{d(y, z) < t \leq d(x, z)} \frac{dt}{t^3} \right]^{1/2} d\mu(z)
\lesssim \|f\|_{L^\infty(\mu)} \int_{X \setminus 5B} \frac{r_B^{1/2}}{d(z, c_B)^{1/2} \lambda(c_B, d(z, c_B))} d\mu(z) \lesssim \|f\|_{L^\infty(\mu)}.
\]

Similarly, \(M_2 \lesssim \|f\|_{L^\infty(\mu)}\). Another application of the Minkowski inequality and (1.6) shows that

\[
M_3 \lesssim \int_X \frac{|K(y, z) - K(x, z)| |f_2(z)|}{d(z, c_B)} \left[ \int_{d(x, z) < t \leq d(x, y)} \frac{dt}{t^3} \right]^{1/2} d\mu(z)
\lesssim \|f\|_{L^\infty(\mu)} \int_{X \setminus 5B} \frac{|K(y, z) - K(x, z)|}{d(z, c_B)} d\mu(z) \lesssim \|f\|_{L^\infty(\mu)}.
\]

Combining the estimates for \(M_1\), \(M_2\) and \(M_3\), we obtain (2.13). Thus, (2.10) holds true.

By (2.10), for \(f \in L^\infty(\mu)\), if \(M(f)(x_0) < \infty\) for some point \(x_0 \in X\), then \(M(f)\) is finite \(\mu\)-almost everywhere and, in this case,

\[
\frac{1}{\mu(B)} \int_B \left[ M(f)(x) - \text{ess inf}_{x \in B} M(f)(x) \right] d\mu(x) \lesssim \|f\|_{L^\infty(\mu)},
\]

provided that \(B\) is a \((6, \beta_0)\)-doubling ball. To prove that \(M(f) \in \text{RBLO}(\mu)\), by Lemma 2.6, we still need to prove that \(M(f)\) satisfies (2.2). Let \(B \subset S\) be any two \((6, \beta_0)\)-doubling balls. For any \(x \in B\) and \(y \in S\), we write

\[
M(f)(x) \lesssim M(f \chi_{5B})(x) + M(f \chi_{5S \setminus 5B})(x).
\]
Moreover, for all \( r \in S \), we have
\[
\mathcal{M}(f \chi_{S \setminus \{S\}}(y)) \leq \mathcal{M}(f(y)) + C\|f\|_{L^\infty(\mu)},
\]
where \( C \) is a positive constant independent of \( f \) and \( y \). On the other hand, by the estimate same as that of (2.13), for all \( x, y \in S \), we see that
\[
|\mathcal{M}(f \chi_{S \setminus \{S\}}(y)) - \mathcal{M}(f \chi_{S \setminus \{S\}}(y))| \lesssim \|f\|_{L^\infty(\mu)}.
\]
For all \( x \in B \), by the Minkowski inequality, (1.5), (2.5) and Lemma 2.4, we obtain
\[
\mathcal{M}(f \chi_{S \setminus \{S\}}(x)) \lesssim \mathcal{M}(f \chi_{S \setminus \{S\}}(x)) + \mathcal{M}(f(x)) + [1 + \delta(B, S)]\|f\|_{L^\infty(\mu)}.
\]
Therefore, for any \( x \in B \) and \( y \in S \), we find that
\[
\mathcal{M}(f(x)) \lesssim \mathcal{M}(f \chi_{S \setminus \{S\}}(x)) + \mathcal{M}(f)(y) + [1 + \delta(B, S)]\|f\|_{L^\infty(\mu)}.
\]
Taking mean value over \( B \) for \( x \) and over \( S \) for \( y \), we conclude that
\[
m_B(\mathcal{M}(f)) - m_S(\mathcal{M}(f)) \lesssim [1 + \delta(B, S)]\|f\|_{L^\infty(\mu)},
\]
where we used (2.11). This finishes the proof of Theorem 2.3.(iii).

Notice that \( \text{RBLO}(\mu) \subset \text{RBMO}(\mu) \). It then follows, from (ii), (iii) and Lemma 2.8, that \( \mathcal{M} \) is bounded on \( L^p(\mu) \) for all \( p \in (1, \infty) \), which implies (iv) and hence completes the proof of Theorem 2.3.

3 Proof of Theorem 1.10

To prove Theorem 1.10, we need some maximal functions in [1,17] as follows. Let \( f \in L^1_{\text{loc}}(\mu) \) and \( x \in \mathcal{X} \). The non-centered doubling Hardy-Littlewood maximal function \( N(f)(x) \) and the sharp maximal function \( M^2(f)(x) \) are, respectively, defined by setting,
\[
N(f)(x) := \sup_{B \ni x \atop B \text{ \( \delta \)-doubling}} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y)
\]
and
\[
M^2(f)(x) := \sup_{B \ni x \atop \mu(5B)} \frac{1}{\mu(5B)} \int_B |f(y) - m_B(f)| \, d\mu(y)
= \sup_{B \ni x \atop \mu(5B)} \frac{|m_B(f) - m_S(f)|}{1 + \delta(B, S)}.
\]
Moreover, for all \( r \in (0, \infty) \), the operators \( M^2_r \) and \( N_r \) are defined, respectively, by setting, for all \( f \in L^1_{\text{loc}}(\mu) \) and \( x \in \mathcal{X} \),
\[
M^2_r(f)(x) := \{M^2(|f|^r)(x)\}^{1/r}
\]
and
\[
N_r(f) := [N(|f|^r)]^{1/r}.
\]
By the Lebesgue differentiation theorem, we see that, for \(\mu\)-almost every \(x \in \mathcal{X}\),
\[
|f(x)| \leq N(f)(x);
\]
see \cite[Corollary 3.6]{17}. Moreover, it follows, from \cite[Proposition 3.5]{17}, that, for any \(p \in [1, \infty]\), \(Nf\) is bounded from \(L^p(\mu)\) to \(L^{p, \infty}(\mu)\).

The following two technical lemmas were, respectively, \cite[Lemmas 3.2 and 3.3]{24}.

**Lemma 3.1.** Let \(p \in [1, \infty)\) and \(f \in L^1_{\text{loc}}(\mu)\) such that \(\int_{\mathcal{X}} f(x) \, d\mu(x) = 0\) if \(\mu(\mathcal{X}) < \infty\). If, for any \(R \in (0, \infty)\),
\[
\sup_{\ell \in (0, R)} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) < \infty,
\]
then there exists a positive constant \(C\), independent of \(f\), such that
\[
\sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) \leq C \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in \mathcal{X} : M^2(f)(x) > \ell\}).
\]

**Lemma 3.2.** Let \(r \in (0, 1)\) and \(N_r(f)\) be as in (3.2). Then, for any \(p \in [1, \infty)\), there exists a positive constant \(C\), depending on \(r\), such that, for any suitable function \(f\) and \(\ell \in (0, \infty)\),
\[
\mu(\{x \in \mathcal{X} : N_r(f)(x) > \ell\}) \leq C \ell^{-p} \sup_{\tau \in (\ell, \infty)} \tau^p \mu(\{x \in \mathcal{X} : |f(x)| > \tau\}).
\]

**Lemma 3.3.** Let \(r \in (0, 1)\), \(K\) satisfy (1.5) and (1.6), and \(\mathcal{M}\) be as in (1.7). If \(\mathcal{M}\) is bounded from \(H^1(\mu)\) into \(L^1(\mu)\), then there exists a positive constant \(C\), depending on \(r\), such that, for any \(\rho \in (1, \infty)\), ball \(B\) and function \(a \in L^{\rho \infty}(\mu)\) supported on \(B\),
\[
\frac{1}{\mu(\rho B)} \int_B |\mathcal{M}(a)(x)|^\rho \, d\mu(x) \leq C\|a\|_{L^\rho \infty(\mu)}^\rho.
\]

Proof. Without loss of generality, we may assume that \(\rho = 2\). For any given ball \(B := B(c_B, r_B)\), we consider the following two cases on \(r_B\).

**Case (i) \(r_B \leq \text{diam}(\text{supp } \mu)/40\).** We use the same notation as in the proof of \cite[Lemma 3.1]{26}. Let \(S\) be the smallest ball of the form \(6^k B\) such that \(\mu(6^j B \setminus 2B) > 0\) with \(j \in \mathbb{N}\). Thus, \(\mu(6^{-1} S \setminus 2B) = 0\) and \(\mu(S \setminus 2B) > 0\). This leads to \(\mu(S \setminus (6^{-1} S \cup 2B)) > 0\) and \(B \subset S\). By this and \cite[Lemma 3.3]{17}, we choose \(x_0 \in \mathcal{X} \setminus (6^{-1} S \cup 2B)\) such that the ball center at \(x_0\) with the radius \(6^{-k} r_S\) for some integer \(k \geq 2\) is \((6, \beta_0)\)-doubling. Let \(B_0\) be the largest ball of this form. Then, it is easy to show that \(B_0 \subset 2S\) and \(d(B_0, B) \geq r_B/2\). It was proved, in the proof of \cite[Lemma 3.1]{26}, that \(d(B, 2S) \leq 1\) and \(d(B_0, 2S) \leq 1\), which imply that \(d(B, 2S) \leq 1\) and \(d(B_0, 2S) \leq 1\).

For any \(a \in L^{\infty}(\mu)\) supported on \(B\), set
\[
C_{B_0} := -\frac{1}{\mu(B_0)} \int_{\mathcal{X}} a(x) \, d\mu(x) \quad \text{and} \quad b := a + C_{B_0} \chi_{B_0}.
\]

It is easy to see that \(b\) is an \((\infty, 1)\)-atomic block with \(\text{supp } (b) \subset 2S\) and \(\int_{\mathcal{X}} b(x) \, d\mu(x) = 0\). Moreover, by the choice of \(C_{B_0}\), the doubling property of \(B_0\) and the assumption of \(a\), we have
\[
|C_{B_0}| \mu(2B_0) \lesssim |C_{B_0}| \mu(B_0) \lesssim \|a\|_{L^1(\mu)} \lesssim \|a\|_{L^{\infty}(\mu)} \mu(2B),
\]
which further shows that
\[
\|b\|_{H^1_{\text{ad}, \infty}(\mu)} \leq [1 + \delta(B, 2S)] \|a\|_{L^{\infty}(\mu)} \mu(2B) + [1 + \delta(B_0, 2S)] |C_{B_0}| \mu(2B_0) \lesssim \|a\|_{L^{\infty}(\mu)} \mu(2B).
\]

Notice that, for any \(x \in B\) and \(y \in B_0\), it holds true that \(d(x, y) \geq r_B/2\). It then follows, from the Minkowski inequality, (1.5), (1.3), (1.4) and (3.5), that, for any \(x \in B\),
\[
\mathcal{M}(C_{B_0} \chi_{B_0})(x) = \left[ \int_0^\infty \left| \int_{d(x, y) < t} K(x, y) C_{B_0} \chi_{B_0}(y) \, d\mu(y) \right|^2 \frac{dt}{t^3} \right]^{1/2}
\]
\[
\begin{align*}
\lesssim |C_{B_0}| & \int_{B_0} \left[ \int_{d(x,y)}^\infty \frac{dt}{t^4} \right]^{1/2} \frac{d(x, y)}{\lambda(x, d(x, y))} d\mu(y) \\
\lesssim |C_{B_0}| & \frac{\mu(B_0)}{\lambda(x, r_B)} \lesssim \frac{\|a\|_{L^\infty(\mu)} \mu(2B)}{\lambda(\mu(B), r_B)} \lesssim \|a\|_{L^\infty(\mu)}.
\end{align*}
\]

On the other hand, by the boundedness from \(H^1(\mu)\) to \(L^1(\mu)\) of \(\mathcal{M}\), together with the Hölder inequality, we see that, for any \(r \in (0, 1)\), there exists a positive constant \(C_r\), depending on \(r\), such that, for all \(b \in H^1(\mu)\) and balls \(B\),

\[
\int_B |\mathcal{M}(b)(x)|^r d\mu(x) \leq C_r \frac{\|b\|_{H^1(\mu)}}{|\mu(B)|^{r-1}}.
\]

Combining the above estimates, we see that

\[
\begin{align*}
\int_B |\mathcal{M}(a)(x)|^r d\mu(x) & \leq \int_B |\mathcal{M}(b)(x)|^r d\mu(x) + \int_B |\mathcal{M}(C_{B_0} \chi_{B_0})|^r d\mu(x) \\
& \lesssim \frac{\|b\|_{H^1(\mu)}}{|\mu(B)|^{r-1}} + \mu(B) \|a\|_{L^\infty(\mu)} \lesssim \mu(2B) \|a\|_{L^\infty(\mu)}.
\end{align*}
\]  

(3.6)

Case (ii) \(r_B > \text{diam}(\text{supp} \mu)/40\). In this case, without loss of generality, we may assume that \(r_B \leq 8 \text{diam}(\text{supp} \mu)\). Then Remark 1.4(ii) tells us that \(B \cap \text{supp} \mu\) is covered by finite number balls \(\{B_j\}_{j=1}^N\) with radius \(r_B/800\), where \(N \in \mathbb{N}\). For \(j \in \{1, \ldots, N\}\) and \(a\) as Lemma 3.3, we define \(a_j := \sum_{k \in j} \chi_{B_k} - a\).

By the argument used in Case (i), we see that (3.6) also holds true, if we replace \(B\) and \(a\) by \(2B_j\) and \(a_j\), respectively. It then follows that

\[
\begin{align*}
\int_B |\mathcal{M}(a)(x)|^r d\mu(x) & \leq \sum_{j=1}^N \int_{2B_j} |\mathcal{M}(a_j)(x)|^r d\mu(x) \\
& \lesssim \sum_{j=1}^N \mu(4B_j) \|a_j\|_{L^\infty(\mu)} \lesssim \mu(2B) \|a\|_{L^\infty(\mu)},
\end{align*}
\]

which, combined with (3.6), completes the proof of Lemma 3.3. \(\square\)

**Lemma 3.4.** Let \(r \in (0, 1)\), \(K\) satisfy (1.5) and (1.6), and \(\mathcal{M}\) be as in (1.7). Suppose that \(\mathcal{M}\) is bounded from \(H^1(\mu)\) into \(L^1(\mu)\), or from \(L^1(\mu)\) into \(L^{1, \infty}(\mu)\). Then, there exists a positive constant \(C_r\), depending on \(r\), such that, for all \(f \in L^\infty(\mu)\),

\[
\|\mathcal{M}^r(\mathcal{M}(f))\|_{L^\infty(\mu)} \leq C_r \|f\|_{L^\infty(\mu)}.
\]  

(3.7)

**Proof.** For any ball \(B \subset \mathcal{X}\) and \(r \in (0, 1)\), set

\[
h_{B, r} := m_B([\mathcal{M}(f \chi_{\mathcal{X} \setminus 2B})]^r).
\]

Observe that, for any ball \(B \subset \mathcal{X}\),

\[
\frac{1}{\mu(5B)} \int_B |\mathcal{M}(f)(x)|^r - m_B([\mathcal{M}(f)]^r)| d\mu(x) \leq \frac{1}{\mu(5B)} \int_B |\mathcal{M}(f)(x)|^r - h_{B, r, |}d\mu(x) + |h_{B, r} - h_{B, r, |}|
\]

and, for two doubling balls \(B \subset S\),

\[
|m_B([\mathcal{M}(f)]^r) - m_S([\mathcal{M}(f)]^r)| \leq |m_B([\mathcal{M}(f)]^r) - h_{B, r} + |h_{B, r} - h_{S, r}| + |h_{S, r} - m_S([\mathcal{M}(f)]^r)|.
\]
Therefore, to show (3.7), it suffices to prove that, for all balls $B \subset X$,

$$D_1 := \frac{1}{\mu(5B)} \int_B |[\mathcal{M}(f)(x)]^r - h_{B,r}| \, d\mu(x) \lesssim \|f\|_{L^\infty(\mu)}^r \tag{3.8}$$

and, for all balls $B \subset S \subset X$ with $S$ being $(6, \beta_6)$-doubling ball,

$$D_2 := |h_{B,r} - h_{S,r}| \lesssim [1 + \delta(B, S)]^{r} \|f\|_{L^\infty(\mu)}^r \tag{3.9}$$

To prove (3.8), from the trivial inequality, $\|a^r - b^r\| \lesssim |a - b|^r$ for all $a, b \in \mathbb{C}$ and $r \in (0, 1)$, and the fact that $\mathcal{M}$ is sublinear, we deduce that

$$D_1 \leq \frac{1}{\mu(5B)} \int_B |[\mathcal{M}(f)(x)]^r - [\mathcal{M}(f\chi_{X\setminus B})(x)]^r| \, d\mu(x)$$

$$+ \frac{1}{\mu(5B)} \int_B |[\mathcal{M}(f\chi_{X\setminus B})(x)]^r - h_{B,r}| \, d\mu(x)$$

$$\leq \frac{1}{\mu(5B)} \int_B |[\mathcal{M}(f\chi_{X\setminus B})(x)]^r \, d\mu(x)$$

$$+ \frac{1}{\mu(B)} \frac{1}{\mu(5B)} \int_B |\mathcal{M}(f\chi_{X\setminus B})(x) - \mathcal{M}(f\chi_{X\setminus B})(y)| \, d\mu(y) \, d\mu(x)$$

$$=: D_{1,1} + D_{1,2}.$$ 

For the term $D_{1,1}$, we consider the following two cases.

Case (i) $\mathcal{M}$ is bounded from $H^1(\mu)$ into $L^1(\mu)$. By Lemma 3.3, we have

$$D_{1,1} \lesssim \frac{1}{\mu(5B)} \int_{2B} |[\mathcal{M}(f\chi_{2B})(x)]^r \, d\mu(x) \leq \|f\chi_{2B}\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\mu)}.$$ 

Case (ii) $\mathcal{M}$ is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. By the Kolmogorov inequality (see [10, p. 102]), we conclude that

$$D_{1,1} \leq \frac{\mu(2B)^{1-r}}{\mu(5B)} \|f\chi_{2B}\|_{L^1(\mu)} \lesssim \|f\|_{L^\infty(\mu)}.$$ 

Therefore, $D_{1,1} \lesssim \|f\|_{L^\infty(\mu)}.$

For the term $D_{1,2}$, by an argument used in the estimate for (2.13), we see that, for all $x, y \in B$,

$$|\mathcal{M}(f\chi_{X\setminus B})(x) - \mathcal{M}(f\chi_{X\setminus B})(y)| \lesssim \|f\|_{L^\infty(\mu)},$$

which implies that $D_{1,2} \lesssim \|f\|_{L^\infty(\mu)}.$ 

Combining the estimates for $D_{1,1}$ and $D_{1,2}$, we obtain the desired estimate (3.8).

Now we prove (3.9). Write

$$|h_{B,r} - h_{S,r}| = |m_B([\mathcal{M}(f\chi_{X\setminus B})]^r) - m_S([\mathcal{M}(f\chi_{X\setminus S})]^r)|$$

$$\leq |m_B([\mathcal{M}(f\chi_{4S\setminus 2B})]^r)| + |m_S([\mathcal{M}(f\chi_{4S\setminus 2S})]^r)|$$

$$+ |m_B([\mathcal{M}(f\chi_{X\setminus 4S})]^r) - m_S([\mathcal{M}(f\chi_{X\setminus 4S})]^r)|$$

$$=: D_{2,1} + D_{2,2} + D_{2,3}.$$ 

Similar to the estimate for (2.14), we see that, for all $x \in B$, $\mathcal{M}(f\chi_{4S\setminus 2B})(x) \lesssim [1 + \delta(B, S)]\|f\|_{L^\infty(\mu)}$, which further implies that $D_{2,1} \lesssim [1 + \delta(B, S)]^{r} \|f\|_{L^\infty(\mu)}.$

To estimate $D_{2,2}$, notice that $S$ is a $(6, \beta_6)$-doubling ball. Then, similar to the estimate for $D_{1,1}$, we have

$$D_{2,2} \lesssim \frac{1}{\mu(6S)} \int_{4S} |\mathcal{M}(f\chi_{4S\setminus 2S})(x)| \, d\mu(x) \lesssim \|f\|_{L^\infty(\mu)}.$$ 

Similar to the estimate for $D_{1,2}$, it is easy to see that $D_{2,3} \lesssim \|f\|_{L^\infty(\mu)}^r$, which, together with the estimates for $D_{2,1}$ and $D_{2,2}$, implies (3.9) and hence completes the proof of Lemma 3.4. □
Proof of Theorem 1.10. By Theorem 2.3, we have already known that (i) implies (ii), (iii) and (iv). Obviously, (iii) implies (i). Therefore, to prove Theorem 1.10, it suffices to prove that (ii) implies (iii) and (iv) implies (iii).

To prove (ii) implies (iii), by the Marcinkiewicz interpolation theorem, we only need to prove that, for all $f \in L^p(\mu)$ with $p \in (1, \infty)$ and $\ell \in (0, \infty)$,

$$
\mu\{x \in X : \mathcal{M}(f)(x) > \ell\} \lesssim \ell^{-p} \|f\|^p_{L^p(\mu)}.
$$

(3.10)

Let $r \in (0, 1)$ and $N_r$ be as in (3.2). Notice that $\mathcal{M}(f) \leq N_r(\mathcal{M}(f))$ $\mu$-almost everywhere on $X$ and $L^\infty_b(\mu)$ is dense in $L^p(\mu)$ for all $p \in (1, \infty)$. Then, by a standard density argument, to prove (3.10), it suffices to prove that, for all $f \in L^\infty_b(\mu)$ and $p \in (1, \infty)$,

$$
\sup_{\ell \in (0, \infty)} \ell^p \mu\{x \in X : N_r(\mathcal{M}(f))(x) > \ell\} \lesssim \|f\|^p_{L^p(\mu)}.
$$

(3.11)

To this end, we consider the following two cases for $\mu(X)$.

Case (i) $\mu(X) = \infty$. Fix $\ell \in (0, \infty)$. For any $f \in L^\infty_b(\mu)$, we split $f$ into $f_1$ and $f_2$ with $f_1 := f \chi_{\{y \in X : |f(y)| > \ell\}}$ and $f_2 := f \chi_{\{y \in X : |f(y)| \leq \ell\}}$. It is easy to see that

$$
\|f_1\|_{L^\infty(\mu)} \leq \ell^{1-p} \|f\|_{L^p(\mu)}, \quad \|f_2\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\mu)} \quad \text{and} \quad \|f_2\|_{L^\infty(\mu)} \leq \ell.
$$

(3.12)

For each $r \in (0, 1)$, let $M^2_r$ be as in (3.1). From Lemma 3.4 and (3.12), it follows that

$$
\|M^2_r(\mathcal{M}(f_2))\|_{L^\infty(\mu)} \lesssim \|f_2\|_{L^\infty(\mu)} \lesssim \ell.
$$

Hence, if $c_0$ is a sufficiently large constant, we have

$$
\mu\{x \in X : M^2_r(\mathcal{M}(f_2))(x) > c_0 \ell\} = 0.
$$

(3.13)

On the other hand, by (3.12), together with the boundedness from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ of $\mathcal{M}$ and Lemma 3.2, we see that, for any $p \in (1, \infty)$ and $R \in (0, \infty)$,

$$
\sup_{\ell \in (0, R)} \ell^p \mu\{x \in X : N_r(\mathcal{M}(f_2))(x) > \ell\}
\leq \sup_{\ell \in (0, R)} \ell^{p-1} \sup_{\tau \in [\ell, \infty)} \mu\{x \in X : \mathcal{M}(f_2)(x) > \tau\} < \infty.
$$

It then follows, from the fact that $N_r \circ \mathcal{M}$ is quasi-linear, Lemma 3.1 and (3.13), that there exists a positive constant $C$ such that

$$
\sup_{\ell \in (0, \infty)} \ell^p \mu\{x \in X : N_r(\mathcal{M}(f))(x) > Cc_0 \ell\}
\leq \sup_{\ell \in (0, \infty)} \ell^p \mu\{x \in X : N_r(\mathcal{M}(f_2))(x) > c_0 \ell\}
+ \sup_{\ell \in (0, \infty)} \ell^p \mu\{x \in X : N_r(\mathcal{M}(f_1))(x) > c_0 \ell\}
\lesssim \sup_{\ell \in (0, \infty)} \ell^p \mu\{x \in X : M^2_r(\mathcal{M}(f_2))(x) > c_0 \ell\}
+ \sup_{\ell \in (0, \infty)} \ell^p \mu\{x \in X : N_r(\mathcal{M}(f_1))(x) > c_0 \ell\}
\sim \sup_{\ell \in (0, \infty)} \ell^p \mu\{x \in X : N_r(\mathcal{M}(f_1))(x) > \ell\}.
$$

(3.14)

By the boundedness from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ of $N$, the boundedness from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ of $\mathcal{M}$ and (3.12), we conclude that

$$
\mu\{x \in X : N_r(\mathcal{M}(f_1))(x) > \ell\}
$$
\[ \mu \left( \left\{ x \in X : N \left( \mathcal{M}(f_1)_{\{y \in X : \mathcal{M}(f_1)(y) > \ell/2^k\}}(x) > \frac{\ell^r}{2} \right) \right\} \right) \]
\[ \lesssim \ell^{-r} \int_X \mathcal{M}(f_1)(x) \chi_{\{y \in X : \mathcal{M}(f_1)(y) > \ell/2^k\}}(x) \, d\mu(x) \]
\[ \lesssim \ell^{-r} \mu \left( \left\{ x \in X : \mathcal{M}(f_1)(x) > \ell/2^k \right\} \right) \int_0^{\ell/2^k} s^{r-1} \, ds + \ell^{-r} \int_{\ell/2^k}^{\infty} s^{r-1} \mu(\{x \in X : \mathcal{M}(f_1)(x) > s\}) \, ds \]
\[ \lesssim \mu \left( \left\{ x \in X : \mathcal{M}(f_1)(x) > \ell/2^k \right\} \right) + \frac{1}{\ell} \sup_{s > \ell/2^k} s \mu(\{x \in X : \mathcal{M}(f_1)(x) > s\}) \]
\[ \lesssim \frac{\|f_1\|_{L^1(\mu)}}{\ell} \lesssim \ell^{-p}\|f\|_{L^p(\mu)}^p, \tag{3.15} \]

which, together with (3.14), implies (3.11).

**Case (ii) \( \mu(X) < \infty \).** In this case, we assume that \( f \in L^\infty(\mu) \). For each fixed \( \ell \in (0, \infty) \), with the same notation \( f_1 \) and \( f_2 \) as in Case (i), we have \( f := f_1 + f_2 \). Let \( r \in (0, 1) \). We claim that
\[ F := \frac{1}{\mu(X)} \int_X [\mathcal{M}(f_2)(x)]^r \, d\mu(x) \lesssim \ell^r. \tag{3.16} \]

Indeed, by the boundedness from \( L^1(\mu) \) into \( L^{1, \infty}(\mu) \) of \( \mathcal{M} \), we have
\[ \int_X [\mathcal{M}(f_2)(x)]^r \, d\mu(x) = r \int_0^\infty \|f_2\|_{L^1(\mu)/\mu(x)} \, t^{r-1} \mu(\{x \in X : \mathcal{M}(f_2)(x) > t\}) \, dt + r \int_0^\infty \|f_2\|_{L^1(\mu)/\mu(x)} \, \cdots \]
\[ \lesssim \mu(X) \int_0^\|f_2\|_{L^1(\mu)/\mu(x)} \, t^{r-1} \, dt + \|f_2\|_{L^1(\mu)} \int_0^\infty \|f_2\|_{L^1(\mu)/\mu(x)} \, t^{r-2} \, dt \]
\[ \lesssim [\mu(X)]^{1-r} \|f_2\|_{L^1(\mu)}^r \lesssim \mu(X) \ell^r, \]

which implies (3.16). Observe that \( \int_X (\mathcal{M}(f_2)(x))^r - F \, d\mu(x) = 0 \) and, for any \( R \in (0, \infty) \),
\[ \sup_{\ell \in (0, R)} \ell^p \mu(\{x \in X : N(\mathcal{M}(f_2))^r(x) > \ell\}) \lesssim R^p \mu(X) < \infty. \]

It then follows, from Lemma 3.1, \( M_\ell^p(F) = 0 \), (3.13) and (3.15), that there exists a positive constant \( \tilde{c} \) such that
\[ \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in X : N(\mathcal{M}(f))^r(x) > \tilde{c} \ell\}) \]
\[ \lesssim \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in X : N(\mathcal{M}(f_2))^r(x) > (\ell \ell)^r\}) \]
\[ \lesssim \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in X : M_\ell^r(\mathcal{M}(f_2))(x) > \ell \ell\}) \]
\[ \lesssim \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in X : N(\mathcal{M}(f_1))(x) > c_0 \ell\}) \]
\[ \lesssim \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in X : N(\mathcal{M}(f_1))(x) > \ell \}) \lesssim \|f\|_{L^p(\mu)}^p, \]

where, in the first inequality, we chose \( c_0 \) large enough such that \( F \leq (\ell \ell)^r \). This finishes the proof that (ii) implies (iii).
Now we prove that (iv) implies (iii). To this end, we consider the following two cases for $\mu(\mathcal{X})$.

**Case (I) $\mu(\mathcal{X}) = \infty$.** In this case, let $L^\infty_b(\mu) := \{ f \in L^\infty_b(\mu) : \int_{\mathcal{X}} f(x) \, d\mu(x) = 0 \}$. Then, $L^\infty_b(\mu)$ is dense in $L^p(\mu)$ for all $p \in (1, \infty)$. Therefore, it suffices to prove that (3.11) holds true for all $f \in L^\infty_b(\mu)$ and $p \in (1, \infty)$.

For each fixed $\ell \in (0, \infty)$, applying Lemma 2.5, we conclude that $f = g + h$, where $h$ is as in Lemma 2.5 and $g := f - h$, such that
\[ \|g\|_{L^\infty(\mu)} \lesssim \ell, \] (3.17)
and
\[ h \in H^1(\mu), \quad \|h\|_{H^1(\mu)} \lesssim \ell^{1-p}\|f\|_{L^p(\mu)}^p. \] (3.18)
For each $r \in (0, 1)$, let $M^r$ be as in (3.1). Similar to (3.13), if $\tilde{c}_0$ is a sufficiently large constant, we then have
\[ \mu(\{ x \in \mathcal{X} : M^r(\mathcal{M}(g))(x) > \tilde{c}_0 \ell \}) = 0. \] (3.19)
On the other hand, since both $f$ and $h$ belong to $H^1(\mu)$, we see that $g \in H^1(\mu)$ and
\[ \|g\|_{H^1(\mu)} \leq \|f\|_{H^1(\mu)} + \|h\|_{H^1(\mu)} \lesssim \|f\|_{H^1(\mu)} + \ell^{1-p}\|f\|_{L^p(\mu)}, \]
which, together with the boundedness from $H^1(\mu)$ into $L^1(\mu)$ of $\mathcal{M}$ and Lemma 3.2, implies that, for any $p \in (1, \infty)$ and $R \in (0, \infty)$,
\[ \sup_{\ell \in (0, R)} \ell^p \mu(\{ x \in \mathcal{X} : N_r(\mathcal{M}(g))(x) > \ell \}) \lesssim \sup_{\ell \in (0, R)} \ell^{p-1} \sup_{\tau \in [\ell, \infty)} \tau \mu(\{ x \in \mathcal{X} : \mathcal{M}(g)(x) > \tau \}) < \infty. \]
By some estimates similar to those of (3.14) and (3.15), via the boundedness of $\mathcal{M}$ from $H^1(\mu)$ into $L^1(\mu)$ and (3.18), we conclude that there exists a positive constant $C$ such that
\[ \sup_{\ell \in (0, \infty)} \ell^p \mu(\{ x \in \mathcal{X} : N_r(\mathcal{M}(f))(x) > C\tilde{c}_0 \ell \}) \lesssim \sup_{\ell \in (0, \infty)} \ell^p \mu(\{ x \in \mathcal{X} : N_r(\mathcal{M}(h))(x) > \ell \}) \lesssim \sup_{\ell \in (0, \infty)} \ell^p \|h\|_{H^1(\mu)} \lesssim \|f\|_{L^p(\mu)}, \]
which implies that (3.11) holds true for all $f \in L^\infty_b(\mu)$ and $p \in (1, \infty)$.

**Case (II) $\mu(\mathcal{X}) < \infty$.** In this case, we assume that $f \in L^\infty_b(\mu)$. Notice that, if $\ell \in (0, \ell_0]$, where $\ell_0$ is as in Lemma 2.5, then (3.10) holds true trivially. Thus, we only need to consider the case when $\ell \in (\ell_0, \infty)$. For each fixed $\ell \in (\ell_0, \infty)$, applying Lemma 2.5, we see that $f = g + h$ with $g$ and $h$ satisfying (3.17) and (3.18), respectively. Let $r \in (0, 1)$. We claim that
\[ G := \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} [\mathcal{M}(g)(x)]^r \, d\mu(x) \lesssim \ell^r. \] (3.20)
Indeed, since $\mu(\mathcal{X}) < \infty$, we regard $\mathcal{X}$ as a ball and the constant function having value $[\mu(\mathcal{X})]^{-1}$ as a $(p, 1)$-atomic block, respectively. By (3.17), we have
\[ g_0 := g - \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} g(x) \, d\mu(x) \in H^1(\mu) \quad \text{and} \quad \|g_0\|_{H^1(\mu)} \lesssim \ell. \]
It then follows, from the Hölder inequality, the boundedness from $H^1(\mu)$ into $L^1(\mu)$ of $\mathcal{M}$ and (3.17), that
\[ \int_{\mathcal{X}} [\mathcal{M}(g)(x)]^r \, d\mu(x) \leq [\mu(\mathcal{X})]^{1-r} \left[ \int_{\mathcal{X}} \mathcal{M}(g)(x) \, d\mu(x) \right]^r. \]
Proof of Theorem 1.11.

From Theorems 1.10 and 2.3, we deduce Theorem 1.11(i) immediately. To prove Lemma 4.1.

To prove Theorem 1.11, we need the following lemma, which is a corollary of [23, Lemma 3.2].

\[ \|g_0\|_{H^\ast(\mu)} + \ell\|\mu(\mathcal{X})^{-1}\|_{H^\ast(\mu)} \leq \ell^r, \]

which implies (3.20).

Notice that \( \int_{\mathcal{X}}[|\mathcal{M}(g)(x)|^r - G] \, d\mu(x) = 0 \) and, for any \( R \in (0, \infty), \)

\[ \sup_{\ell \in (0, R)} \ell^p \mu(\{x \in \mathcal{X} : N(|\mathcal{M}(g)|^r - G)(x) > \ell \}) \leq R^p \mu(\mathcal{X}) < \infty. \]

Therefore, from an argument similar to that used in Case (ii), together with Lemma 3.1, \( M_5^B(G) = 0 \) and (3.19), we deduce that

\[ \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in \mathcal{X} : N_r(\mathcal{M}(f))(x) > \ell \}) \leq \sup_{\ell \in (0, \infty)} \ell^p \mu(\{x \in \mathcal{X} : N_r(\mathcal{M}(h))(x) > \ell \}) \leq \|f\|_{L^p(\mu)}, \]

which completes the proof that (iv) implies (iii) and hence the proof of Theorem 1.10. \( \square \)

4 Proof of Theorem 1.11

To prove Theorem 1.11, we need the following lemma, which is a corollary of [23, Lemma 3.2].

Lemma 4.1. Let \( \eta \in (1, \infty) \) and \( \beta_6 \) be as in (2.1). Then, there exists a positive constant \( C \) such that, for all \( f \in \text{RBMO}(\mu) \) and balls \( B, \)

\[ \frac{1}{\mu(\eta B)} \int_B |f(y) - m_B(f)| \, d\mu(y) \leq C\|f\|_{\text{RBMO}(\mu)} \tag{4.1} \]

and, for all \((6, \beta_6)\)-doubling balls \( B \subset S, \)

\[ |m_B(f) - m_S(f)| \leq C[1 + \delta(B, S)]\|f\|_{\text{RBMO}(\mu)}. \tag{4.2} \]

Proof of Theorem 1.11. From Theorems 1.10 and 2.3, we deduce Theorem 1.11(i) immediately. To prove Theorem 1.11(ii), we first claim that, for all \( f \in L_5^\infty(\mu) \) with \( \text{supp} \, f \subset B, \)

\[ \int_B \mathcal{M}(f)(x) \, d\mu(x) \leq \mu(2B)\|f\|_{L^\infty(\mu)}. \tag{4.3} \]

We consider the following two cases for \( r_B. \)

Case (i) \( r_B \leq \text{diam} \, (\text{supp} \, \mu)/40. \) In this case, choose \( \eta = 2 \) in Lemma 4.1. It then follows, from Lemma 4.1 and (1.8), that

\[ \int_B |\mathcal{M}(f)(x) - m_B(\mathcal{M}(f))| \, d\mu(x) \leq \mu(2B)\|\mathcal{M}(f)\|_{\text{RBMO}(\mu)} \leq \mu(2B)\|f\|_{L^\infty(\mu)}. \]

Therefore, the proof of (4.3) is reduced to showing

\[ |m_B(\mathcal{M}(f))| \leq \|f\|_{L^\infty(\mu)}. \tag{4.4} \]

Let \( S, B_0 \) be the same notation as in the proof of Lemma 3.3. Recall that \( \delta(B, 2S) \leq 1, \delta(B_0, 2S) \leq 1, \delta(B, 2S) \leq 1 \) and \( \delta(B_0, 2S) \leq 1. \) By this, together with Lemmas 2.4 and 4.1, we see that

\[ |m_{B_0}(\mathcal{M}(f)) - m_{\hat{B}}(\mathcal{M}(f))| \leq |m_{B_0}(\mathcal{M}(f) - m_{2\hat{S}}(\mathcal{M}(f))| + |m_{2\hat{S}}(\mathcal{M}(f)) - m_{\hat{B}}(\mathcal{M}(f))| \]

\[ \leq \left[ 2 + \delta(B_0, 2\hat{S}) + \delta(\hat{B}, 2\hat{S}) \right]\|\mathcal{M}(f)\|_{\text{RBMO}(\mu)} \leq \|f\|_{L^\infty(\mu)}. \]
which further implies that, to prove (4.4), it suffices to prove that
\[ |m_{B_0}(\mathcal{M}(f))| \lesssim \|f\|_{L^\infty(\mu)}. \] (4.5)

Notice that, for all \( y \in B_0 \) and \( z \in B \), it holds true that \( d(y, z) \geq r_B/2 \) and hence \( d(c_B, y) \leq d(c_B, z) + d(z, y) \leq d(y, z) \). By the Minkowski inequality, (1.5), (1.3), (1.4) and the fact that \( \text{supp} \, f \subset B \), we conclude that, for all \( y \in B_0 \),
\[ \mathcal{M}(f)(y) = \left[ \int_0^\infty \left( \int_{d(y, z) < t} K(y, z) f(z) \, d\mu(z) \right)^2 \frac{dt}{t^3} \right]^{1/2} \lesssim \|f\|_{L^\infty(\mu)} \int_B \left( \int_{d(y, z) < t} \frac{dt}{t^3} \right)^{1/2} \frac{d(y, z)}{\lambda(y, d(y, z))} \, d\mu(z) \lesssim \|f\|_{L^\infty(\mu)} \mu(B) \int_B \frac{d(y, z)}{\lambda(c_B, d(y, z))} \, d\mu(z) \lesssim \|f\|_{L^\infty(\mu)}, \]
which implies (4.5). Hence, (4.3) holds true in this case.

Case (ii) \( r_B > \text{diam} \left( \text{supp} \, f \right)/40 \). In this case, the argument is almost the same as the one of Case (ii) in the proof of Lemma 3.3. We omit the details, which shows that the claim (4.3) also holds true in this case.

Now based on the claim (4.3), we prove Theorem 1.11(ii). Take \( \rho = 4 \) and \( p = \infty \) in Definition 1.6. By the definition of \( H^{1, \infty}_{\mathrm{lin}}(\mu) \), it suffices to show that, for any \( (\infty, 1)_{\lambda} \)-atomic block \( b \),
\[ \|\mathcal{M}(b)\|_{L^1(\mu)} \lesssim \|b\|_{H^{1, \infty}_{\mathrm{lin}}(\mu)}. \] (4.6)
By the argument used in the estimate for (2.7), we see that (4.6) holds true if we replace any \( (p_0, 1)_{\lambda} \)-atomic block and (2.9) by an \( (\infty, 1)_{\lambda} \)-atomic block and (4.3), respectively. We omit the details, which completes the proof of Theorem 1.11.

Acknowledgements The first author is supported by the Mathematical Tianyuan Youth Fund of the National Natural Science Foundation of China (Grant No. 11026120) and Chinese Universities Scientific Fund (Grant No. 20120003110003). The second author is supported by the National Natural Science Foundation of China (Grant No. 11171027) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003). The authors would like to thank the referees for their careful reading and many valuable remarks which made this article more readable.

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