Complex Random Matrices have no Real Eigenvalues

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Abstract

Let $\zeta = \xi + i\xi'$ where $\xi, \xi'$ are iid copies of a mean zero, variance one, subgaussian random variable. Let $N_n$ be a $n \times n$ random matrix with iid entries $\zeta_{ij} = \zeta$. We prove that there exists a $c \in (0, 1)$ such that the probability that $N_n$ has any real eigenvalues is less than $c^n$. The bound is optimal up to the value of the constant $c$. The principal component of the proof is an optimal tail bound on the least singular value of matrices of the form $M_n := M + N_n$ where $M$ is a deterministic complex matrix with $\|M\| \leq Kn^{1/2}$ for some constant $K$. For this class of random variables, this result improves on the results of Pan and Zhou [17]. In the proof of the tail bound, we develop an optimal small-ball probability bound for complex random variables that generalizes the Littlewood-Offord theory developed by Tao-Vu [25, 30] and Rudelson-Vershynin [19, 20].

1 Introduction

The study of eigenvalues is a foundational aspect of random matrix theory. For non-symmetric random matrices, real eigenvalues are of particular interest ([10, 4, 3, 14, 12, 7]). Yet, very little is known about their behavior for general classes of random variables [16, Ch. 15.3]. Even the existence of real eigenvalues is only known for the real gaussian case, where it has been proven that there are roughly $\sqrt{2n \pi}$ real eigenvalues [4]. We demonstrate that for a general class of complex random variables, whose real and imaginary components are independent, there are unlikely to be any real eigenvalues.

A key element of the proof, which is of independent interest, is an optimal result on the tail probability of the least singular value of random complex matrices. Let $M$ be a $n \times n$ matrix and $s_1(M) \geq \cdots \geq s_n(M)$ its singular values. Of great interest to numerical analysts is the condition number of the matrix $M$, defined to be

$$\kappa(M) := s_1(M)/s_n(M) = \|M\|/\|M^{-1}\|$$

Attempts to understand the typical behavior of this parameter were instigated by von Neumann and Goldstine [31] in their seminal work on numerical matrix inversion. The condition number is also

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intimately tied to the efficiency of algorithms [22] and the difficulty of problems in numerical analysis [1]. Spielman and Teng were motivated by these concerns when they introduced the paradigm of smoothed analysis [23, 24]. Their goal was to understand the behavior of algorithms on fixed inputs that had been perturbed by random noise. Since the operator norm of a random matrix is well-understood, the difficulty in the analysis of the condition number reduces to understanding the least singular value. In our setting, we examine the least singular value of a fixed matrix $M$ plus a random matrix $N_n$. The real case with Gaussian noise was addressed in [21] and more general models were featured in [27]. There is a compelling practical motivation for understanding these more general models because discrete models, in particular, are more accurate representations of noise and error in digital settings.

2 Previous Results

2.1 Real Eigenvalues

Edelman, Kostlan, and Shub were able to find precise asymptotics for the expected number of real eigenvalues, $E_n$, for a random matrix with iid $\mathcal{N}(0,1)$ entries.

**Theorem 2.1** ([4]).

$$\lim_{n \to \infty} \frac{E_n}{\sqrt{n}} = \sqrt{\frac{2}{\pi}}$$

Later, Edelman [3] derived exact formulas for the probability that the random real gaussian matrix has exactly $k$ real eigenvalues and expressed the joint densities of those eigenvalues explicitly. The following theorem is a consequence of evaluating the formula for $k = n$.

**Theorem 2.2** ([3]). *The probability that the random real gaussian matrix has all real eigenvalues is $2^{-n(n-1)/4}$.***

The techniques used in the proof of the above results are specialized for gaussian random variables. Much less is known for general distributions. In fact, the following toy problem was posed in the Van Vu’s talk at the 2014 ICM in Seoul and remains unresolved.

**Problem 2.3.** *Prove that a random $\pm 1$ random matrix has at least two real eigenvalues with high probability.*

2.2 Least Singular Value

In contrast to real eigenvalues, much is known about the universality of the least singular value in the real case. One can deduce from a result of Edelman [2] that
**Theorem 2.4.** There exists a constant $C > 0$ such that for $N_n$, a random matrix populated with iid $\mathcal{N}(0,1)$ random variables, then for any $\varepsilon > 0$,

$$P(s_n(N_n) \leq \varepsilon n^{-1/2}) \leq \varepsilon$$

Sankar, Teng, and Spielman [21] were able to prove an analogous result for the smoothed analysis model.

**Theorem 2.5.** There exists a constant $C > 0$ such that for $M$, a deterministic matrix, and $N_n$ a random matrix populated with iid $\mathcal{N}(0,1)$ random variables, then for any $\varepsilon > 0$ and $M_n := M + N_n$,

$$P(s_n(M_n) \leq \varepsilon n^{-1/2}) \leq C\varepsilon$$

They further conjectured that

**Conjecture 2.6.** Let $\xi$ be a mean zero, variance at least 1, subgaussian random variable. Let $N_n$ be a $n \times n$ random matrix with iid entries $x$. Then for every $\varepsilon \geq 0$

$$P(s_n(N_n) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + c^n$$

**Definition 2.7.** A random variable $\xi$ is subgaussian if there exists a $B > 0$ such that

$$P(|\xi| > t) \leq 2\exp(-t^2/B^2) \text{ for all } t > 0$$

The minimal $B$ in the inequality is known as the subgaussian moment of $\xi$.

For a very general class of random variables, Tao and Vu [27] showed that

**Theorem 2.8.** Let $\xi$ be a random variable with mean zero and bounded second moment, and let $\gamma \geq 1/2$, $A \geq 0$ be constants. Then there is a constant $C$ depending on $\xi, \gamma, A$ such that the following holds. Let $N_n$ be the random matrix of size $n$ whose entries are iid copies of $\xi$. Let $M$ be a deterministic matrix satisfying $\|M\| \leq n^\gamma$ and let $M_n := M + N_n$. Then

$$P(s_n(M_n) \leq n^{-(2A+1)\gamma}) \leq C \left(n^{-A+o(1)} + P(\|N_n\| \geq n^\gamma)\right)$$

Furthermore, they showed that unlike the gaussian case, the bound necessarily requires conditions on $M$. In [19], Rudelson and Vershynin obtained the optimal rate for subgaussian random variables and $M = 0$.

**Theorem 2.9.** Let $\xi$ be a mean zero, variance at least 1, subgaussian random variable. Let $N_n$ be a $n \times n$ random matrix with iid entries $x$. Then there exists constants $C, c$ such that for every $\varepsilon \geq 0$

$$P(s_n(N_n) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + c^n$$

3
For the complex case, Edelman’s work [2] implies the following.

**Theorem 2.10.** For \( \zeta \) a complex gaussian and \( N_n \) a random \( n \times n \) matrix populated with iid entries \( \xi \), then

\[
\Pr(s_n(N_n) \leq \varepsilon n^{-1/2}) \leq \varepsilon^2
\]

For more general complex random variables, Pan and Zhou [17] showed

**Theorem 2.11.** Let \( \zeta \) be a complex random variables with mean zero, \( \mathbb{E}|\zeta|^2 = 1 \), and \( \mathbb{E}|\zeta|^3 < B \). Let \( M \) be a fixed complex matrix and \( N_n \) be a random matrix with iid entries \( \zeta \) and define \( M_n := M + N_n \). There exists a \( C > 0 \) and \( c \in (0, 1) \) such that for \( K \geq 1 \) and every \( \varepsilon > 0 \),

\[
\Pr(s_n(M_n) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + cn + \Pr(\|M_n\| > K\sqrt{n})
\]

Our work will improve this rate for the case when the real and imaginary components of the random variable are independent.

## 3 Main Results

We address the question of the existence of real eigenvalues for a general class of complex random variables whose real and imaginary parts are independent.

**Theorem 3.1.** Let \( \zeta = \xi + i\xi' \) be a complex random variable where \( \xi \) and \( \xi' \) are iid, mean zero, variance one, subgaussian with moment \( B \), real random variables. Let \( N_n \) be a \( n \times n \) random matrix populated with independent copies of \( \zeta \). There exists a \( c \in (0, 1) \) such that

\[
\Pr(N_n \text{ has a real eigenvalue}) \leq c^n
\]

**Remark 3.2.** This is best possible up to the value of the constant \( c \). For example, for \( \pm 1 \pm i \) random variables, the probability of having zero as an eigenvalue is lower bounded by the probability that there exists two rows or columns with the same entries. The probability of the latter is \((1 + o(1))n^24^{-n}\).

The crucial ingredient in the proof is a new result on the smoothed analysis of the least singular value for such complex matrices.

**Theorem 3.3.** Let \( \zeta \) and \( N_n \) be as in Theorem 3.1. There exists constants \( K, C, c > 0 \) such that for \( M \), a fixed complex matrix with \( \|M\| \leq K\sqrt{n} \), and \( M_n := M + N_n \)

\[
\Pr(s_n(M_n) \leq \varepsilon n^{-1/2}) \leq C\varepsilon^2 + c^n
\]

**Remark 3.4.** Edelman’s result [2] shows that \( C\varepsilon^2 \) is optimal up to the constant \( C \). Setting \( \varepsilon = 0 \) and considering \( \pm 1 \) random variables, we recover the complex analogue of Kahn, Komlós, and Szemerédi [13] that \( \pm 1 \) random matrices are singular with exponentially small probability. Thus, the \( c^n \) term is optimal for random sign matrices.
4 Notation

It will often be convenient to convert a problem from the complex setting to the real one. For this purpose, we introduce the following notation. For $v = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$ (all vectors are assumed to be column vectors), we let $\hat{v} := (\Re(v_1), \ldots, \Re(v_n), \Im(v_1), \ldots, \Im(v_n))^T \in \mathbb{R}^{2n}$ where $\Re(v_j)$ and $\Im(v_j)$ are respectively the real and imaginary parts of the complex number $v_j$. We will also need to convert $v$ into matrix form. Let $[v] \in \mathbb{R}^{2 \times 2n}$ be defined as

$[v] := \begin{pmatrix} \Re(v) & -\Im(v) \\ \Im(v) & \Re(v) \end{pmatrix}$

where $\Re(v)$ indicates the vector whose entries are the real parts of the corresponding entries in $v$. $\Im(v)$ is similarly defined (See Figure 4).

An important property is that for $a \in \mathbb{C}^n$,

$|\sum_{j=1}^{n} a_j v_j| = |v^T a| = \|\hat{a}\|_2$

We use $S_{2n}^{n-1}$ and $S_{C}^{n-1}$ to denote the unit sphere in $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively. For a $N \times N'$ real or complex matrix $M$, we denote the $\ell_2$ operator norm by $\|M\|$. For $J \in [N]$ we denote by $M_J$ the $|J| \times N'$ matrix composed of the rows of $M$ indexed by $J$. For two vectors $v, v'$, let $v \cdot v'$ represent the standard dot product of the two. $i$ will always mean $\sqrt{-1}$. $\zeta$ will typically denote a complex random variable and $\xi$ a real one. $C, c$ will be universal constants whose values may change from line to line.
5 Proof of Theorem 3.3

The argument will be a modification of that used by Rudelson and Vershynin [19]. We begin with a decomposition of the complex unit sphere.

5.1 Decomposition of $S_{\mathbb{C}^n}^{n-1}$

**Definition 5.1.** Let $\delta, \rho \in (0, 1)$ be two constants. A vector $v \in \mathbb{C}^n$ is called sparse if $|\text{supp}(v)| \leq 2\delta n$. A vector $v \in S_{\mathbb{C}^n}^{n-1}$ is compressible if it is within Euclidean distance $\rho$ from the set of all sparse vectors. A vector $v \in S_{\mathbb{C}^n}^{n-1}$ is called incompressible if it is not compressible.

The least singular value problem can thus divided into two subproblems.

$$
\mathbb{P}(s_n(M_n) \leq \varepsilon n^{-1/2}) \leq \mathbb{P}(\inf_{v \in \text{Comp}}\|M_nv\|_2 \leq \varepsilon n^{-1/2}) + \mathbb{P}(\inf_{x \in \text{Incomp}}\|M_nx\|_2 \leq \varepsilon n^{-1/2})
$$

We exploit the different properties of compressible and incompressible vectors to solve the problem for each set in a distinct way.

5.2 Compressible Vectors

For compressible vectors, the bound is much stronger than we need and the argument is essentially the same as [19, 17].

**Lemma 5.2.** For $M_n$ as in Theorem 3.3, there exists $\delta, \rho, c_1, c_2 > 0$ such that

$$
\mathbb{P}(s_n(M_n) \leq \varepsilon n^{-1/2}) \leq \mathbb{P}(\inf_{v \in \text{Comp}}\|M_nv\|_2 \leq c_1 n^{1/2}) \leq \exp(-c_2 n) \quad (1)
$$

**Proof.** See [17, Section 2.2].

5.3 Incompressible Vectors

For the remainder of the proof we fix a $\delta$ and $\rho$ satisfying inequality (1). For incompressible vectors, we leverage the fact that they have many coordinates of roughly the same size.

**Lemma 5.3.** [19, Lemma 3.4] Let $z \in \text{Incomp}(\delta, \rho)$. Then there exists a set $\sigma \subseteq \{1, \ldots, 2n\}$ of cardinality $|\sigma| \geq \nu_1 n$ and such that

$$
\frac{\nu_2}{\sqrt{n}} \leq |z_k| \leq \frac{\nu_3}{\sqrt{n}}
$$

for all $k \in \sigma$

where $0 < \nu_1, \nu_2, \nu_3$ are constants depending only on $\delta$ and $\rho$. $\sigma$ is known as the spread part of the vector $v$. 
5.3.1 Invertibility of Incompressible Vectors Via Distance

**Lemma 5.4 (Invertibility via Distance).** [18, Lemma 5.6] Let $M_n$ be a complex random matrix. Let $X_1, \ldots, X_n$ denote the column vectors of $M$, and let $H_k$ denote the span of all the column vectors except the $k$-th. Then for every $\delta, \rho \in (0, 1)$ and every $\varepsilon > 0$ one has

$$
P\left( \inf_{z \in \text{Incomp}(\delta, \rho)} \|M_n z\|_2 < \varepsilon \nu_2 n^{-1/2} \right) \leq \frac{1}{\nu_1} P(\text{dist}(X_n, H_n) < \varepsilon)$$

where $\nu_1, \nu_2$ are as in Lemma 5.3.

This lemma reduces the invertibility issue into a distance problem. In fact, conditioning on $H_n$, we can choose a unit normal vector independent of $X_n$. The distance is then simply the dot product of this unit normal vector with an independent random vector, so the question becomes one of small ball probability.

5.3.2 Small Ball Probability

For intuition and motivation, we briefly revert back to the real case. Consider the linear combination,

$$S = \sum_{k=1}^{n} a_k \xi_k.$$ 

**Definition 5.5.** The Lévy concentration function of $S$ is defined as

$$\mathcal{L}(S, \varepsilon) := \sup_{v \in \mathbb{R}} P(|S - v| \leq \varepsilon)$$

Clearly, the vector $a = (a_1, \ldots, a_n)$ has a strong influence on the Lévy concentration. For example, if

$$a = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \ldots, 0\right)$$

and $\xi$ are independent Rademacher random variables ($\pm 1$), then $P(S = 0) = \frac{1}{2}$. On the other hand, if

$$a = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$$

then it is known that $P(S = 0) \sim n^{-1/2}$ [5]. Littlewood and Offord [15] initiated the study of the dependence of the Lévy function on the arithmetic structure of $a$. Recently, Tao and Vu [30] proposed that a large small-ball probability implies a strong additive structure. Results in the classification of this additive structure are now called Inverse Littlewood-Offord theorems [30, 25, 28, 26, 19, 20, 8]. We now introduce a two-dimensional small ball probability bound which corresponds to a bound on the Levy concentration for complex sums with complex coefficients. Rudelson and Vershynin [19] proposed a measure for the additive structure of a vector $v \in \mathbb{R}^n$. They coined the term Essential Least Common Denominator (lcd).

$$\text{lcd}_\alpha(v) := \inf \left\{ \hat{\theta} > 0 : \text{dist}(\hat{\theta} v, \mathbb{Z}^n) < \min(\gamma \|\hat{\theta} v\|_2, \alpha) \right\}$$
We generalize this definition to handle complex vectors and our matrix construction \([v]\).

**Definition 5.6.** Fix a \(\gamma \in (0,1)\) and a parameter \(\alpha > 0\). We define the Essential Least Common Denominator of \(v \in \mathbb{C}^n\) to be

\[
LC\alpha(v) := \inf \left\{ \|\theta\|_2 : \theta \in \mathbb{R}^2, \text{dist}([v]^T \theta, \mathbb{Z}^{2n}) < \min(\gamma \|\theta\|_2, \alpha) \right\}
\]

We use this measure of structure to control the small-ball probability.

**Theorem 5.7** (Small Ball Probability via LCD). Consider a random vector \(\xi = (\xi_1, \ldots, \xi_{2n})\) and a \(v \in \mathbb{S}^{n-1}\). For every \(\alpha > 0\) and for \(\varepsilon \geq 4 \text{LCD}_\alpha(v)\) we have

\[
\sup_{v \in \mathbb{R}^2} \mathbb{P}(\|v\|\xi - v| \leq \varepsilon) \leq C \varepsilon^2 + C \exp(-c\alpha^2)
\]

**Remark 5.8.** In our application, we will set \(\alpha\) to be \(\beta \sqrt{n}\) for some small constant \(\beta\), so the \(\exp(-c\alpha^2)\) term is negligible.

**Proof.** For ease of exposition, we will prove the when \(\xi_k\) are the \(\pm 1\) random variables. The extension to general random variables is the same as in [19, Sec 4]. We begin with Esséen’s inequality ([29, Lemma 7.17], [6, 11]) which states that for a random variable \(X \in \mathbb{R}^2\)

\[
\mathcal{L}(X, \sqrt{2}) \leq C \int_{B(0,\sqrt{2})} |\phi_X(\theta)|d\theta
\]

where \(\phi_X(\theta) = \mathbb{E} \exp(2\pi i (\theta \cdot X))\) and \(\theta \cdot X\) denotes the usual dot product and \(\phi_X(\theta)\) is the characteristic function of a random vector [9, Ch. 5]. Let \(X = \sqrt{2}\varepsilon^{-1}[v]\xi\) in Esséen’s inequality. Let \([v]_j\) denote the \(j\)-th column of \([v]\), so we have

\[
(\theta \cdot \sqrt{2}\varepsilon^{-1}[v]\xi) = \sqrt{2} \sum_{k=1}^{2n} \varepsilon^{-1}(\theta \cdot [v]_k)\xi_k
\]

By independence,

\[
\phi_X(\theta) = \prod_{k=1}^{2n} \phi_k(\sqrt{2}\varepsilon^{-1}(\theta \cdot [v]_k))
\]

where \(\phi_k(t) = \mathbb{E} \exp(2\pi it\xi_k)\). Thus, we have

\[
\mathcal{L}([v]\xi, \varepsilon) \leq C \int_{B(0,2)} \prod_{k=1}^{2n} |\phi_k(\varepsilon^{-1}(\theta \cdot [v]_k))|d\theta
\]

8
Thus, for all \( k \),

\[
|\phi_k(t)| = E \exp(2\pi it\xi_k) = \cos(2\pi t)
\]

Utilizing the inequality \(|x| \leq \exp(-\frac{1}{2}(1 - x^2))\) that is valid for all \( x \), we get

\[
|\phi_k(t)| \leq \exp(-\frac{1}{2}(1 - \cos 2\pi t)) = \exp(-\sin^2(\pi t)) \leq \exp(-\min_{q \in \mathbb{Z}} |2t - q|^2)
\]

Putting all this together gives

\[
\mathcal{L}([v]\xi, \varepsilon) \leq C \int_{B(0,2)} \exp \left(-\sum_{k=1}^{2n} \min_{q_k \in \mathbb{Z}} |2\varepsilon^{-1}(\theta \cdot [v]_k) - q_k|^2 \right) d\theta \leq C \int_{B(0,2)} \exp(-h^2(\theta)/10) d\theta \quad (2)
\]

where

\[
h^2(\theta) := \text{dist}(2\varepsilon^{-1}[v]^T\theta, \mathbb{Z}^2)\]

Now we use the \( LCD_\alpha(v) \) to estimate the distance that appears in \( h(\theta) \). By the assumption that

\[
\varepsilon \geq \frac{4}{LCD_\alpha(v)}
\]

Then for any \( \theta \in B(0,2) \), we have

\[
\|2\varepsilon^{-1}\theta\|_2 \leq LCD_\alpha(v)
\]

Thus, by the definition of \( LCD_\alpha(v) \),

\[
\text{dist}(\varepsilon^{-1}[v]^T\theta, \mathbb{Z}^2) \geq \min\{\gamma \varepsilon^{-1}\|\theta\|_2, \alpha\}
\]

Backsubstituting into \( h(\theta) \) yields

\[
h(\theta)^2 \geq \min\{\gamma^2 \varepsilon^{-2}\|\theta\|_2^2, \alpha^2\}
\]

Plugging this into inequality \((2)\) gives

\[
\mathcal{L}([v]\xi, \varepsilon) \leq C \int_{B(0,2)} \exp\left(-\frac{1}{10} \gamma^2 \varepsilon^{-2}\|\theta\|_2^2 \right) + C \exp(-\alpha^2)
\]

\[
\leq C \varepsilon^2 \int_0^\infty r \exp(-cr^2) dr + C \exp(-\alpha^2)
\]

\[
\leq C \varepsilon^2 + C \exp(-\alpha^2)
\]
By the tensorization lemma [19, Lemma 2.2], we get the following bound for a single vector.

**Lemma 5.9** (Invertibility for a Single Vector). Let \( M' \) be a \( m \times n \) complex random matrix with entries of the form \( m_{ij} + \xi_{ij} + \xi'_{ij} \) where \( m_{ij} \) is a deterministic complex number and \( \xi_{ij} \) and \( \xi'_{ij} \) are iid, mean zero, variance 1, and subgaussian with moment \( B \). Then for any \( \alpha > 0 \) and for every vector \( v \in S_{C}^{n-1} \), and for every \( \varepsilon > 0 \), satisfying

\[
\varepsilon \geq \max \left( \frac{4}{\text{LCD}_\alpha(v)}, \exp(-c\alpha^2) \right)
\]

we have

\[
P(\|M_n v\|_2 < \varepsilon n^{1/2}) \leq (C\varepsilon)^{2m}
\]

### 5.4 Random Normal Vectors have Large LCD

We now show that it is unlikely that a random normal vector will have small LCD by an \( \varepsilon \)-net argument. We first prove a lower bound on the \( \text{LCD} \) for incompressible vectors that will be of use in the proof of Lemma 5.13.

**Lemma 5.10** (Lower Bound on LCD). There exists constants \( \gamma > 0 \) and \( \lambda > 0 \) only depending on \( \delta, \rho \) such that for any incompressible vector \( v \in S_{C}^{n-1} \) we have \( \text{LCD}_\alpha(v) \geq \lambda n^{1/2} \).

**Proof.** Assume to the contrary that \( \text{LCD}_\alpha(v) < \lambda n^{1/2} \). By definition of the \( \text{LCD} \) there exists \( \theta \in \mathbb{R}^2 \) and \( p \in \mathbb{Z}^{2n} \) such that

\[
\|[v]^T \theta - p\|_2 < \gamma \|\theta\|_2 < \gamma \lambda n^{1/2}.
\]

Recall the definition of the spread part of the vector \( v \) from Lemma 5.3. Let \( \sigma(v) \subseteq [2n] \) denote the spread part of the vector \( v \). Assume without loss of generality that half of the spread coordinates are real, i.e. \( |\sigma(v) \cap [n]| > \frac{\nu_1}{2} n \). Fix a constant \( k \) such that

\[
1/k^2 < \nu_1/4.
\]

Then there exists a set \( I(v) \in [n] \) of size at least \((1 - 1/k^2)n\) such that for \( j \in I(v), |\Im(v_j)| < k/\sqrt{n} \). Thus,

\[
|\sigma(v) \cap [n] \cap I(v)| \geq \frac{\nu_1}{4} n.
\]

Now let

\[
J(v) := \{ j : |([v]^T \theta)_j - p_j| < \frac{2\sqrt{2} \gamma \lambda}{\sqrt{\nu_1}} \}.
\]

We finally define

\[
L(v) := \sigma(v) \cap [n] \cap I(v) \cap J(v).
\]
By equation (3), we have that \(|J(v)| \geq (2 - \nu_1/8)n\) and combining this bound with equation (4) yields

\[ |L(v)| \geq \frac{\nu_1}{8} n. \]

Due to the symmetry of \([v]^T\) (See Figure 1), if

\[ \|\[v]^T \theta - p\|_2 = \left\| [v]^T \left( \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right) - \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) \right\|_2 \]

where \(p_1, p_2 \in \mathbb{Z}^n\) then

\[ \|\[v]^T \theta - p\|_2 = \|\[v]^T \theta' - p'\|_2 := \left\| [v]^T \left( \begin{array}{c} -\theta_2 \\ \theta_1 \end{array} \right) - \left( \begin{array}{c} -p_2 \\ p_1 \end{array} \right) \right\|_2 \]

Clearly, \(\|\theta\|_2 = \|\theta'\|_2\). This symmetry can be exploited by defining

\[ L'(v) := \{ j : (j - n) \in L(v) \}. \]

For any \(j \in L(v)\), by the definition of \(J(v)\),

\[ |p_j| < |([v]^T \theta)_j| + \frac{2\sqrt{2}\gamma \lambda}{\sqrt{\nu_1}} \leq |\theta_1||\Re(v_j)| + |\theta_2||\Im(v_j)| + \frac{2\sqrt{2}\gamma \lambda}{\sqrt{\nu_1}} < (\nu_3 + k + \frac{2\sqrt{2}\gamma}{\sqrt{\nu_1}}) \lambda < 1 \]

for small enough \(\lambda\) and assuming \(\gamma < 1\). Similarly, by applying the above argument with \(\theta'\) and \(p'\), we find that for \(j \in L'(v)\)

\[ |p_j| < 1 \]

Since \(p_j\) must be an integer, this implies \(p_j = 0\) for \(j \in L(v) \cup L'(v)\). Thus,

\[ \|([v]^T \theta)_L - p_L\|_2 = \|([v]^T \theta)_L\|_2 \]

and

\[ \|([v]^T \theta)_L' - p_L'\|_2 = \|([v]^T \theta)_L'\|_2. \]

We now lower bound \(\|\[v]^T \theta - p\|\) by

\[ \|([v]^T \theta)_L\|_2 + \|([v]^T \theta)_L'\|_2 = \|([v]^T \theta')_L\|_2 + \|([v]^T \theta')_L'\|_2. \]
Note that by the definition of $I(v)$,
\[ \| \Im(v) \|_2 < \frac{k\sqrt{\nu_1}}{2\sqrt{2}} \]
and by the definition of $\sigma(v)$,
\[ \| \Re(v) \|_2 \geq \frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}} \]
Now we have two cases to consider. Let $c' > 0$ be a constant such that
\[ \sqrt{1 - c'^2\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}}} - c'\frac{k\sqrt{\nu_1}}{2\sqrt{2}} > 0 \]
1. Assume that $|\theta_1| \geq c'||\theta||_2$ and $|\theta_2| \geq c'||\theta||_2$. In this case, and adding the condition that
\[ \theta_1 \Re(v) \cdot \theta_2 \Im(v) \geq 0 \]
we find that
\[ \|(v^T\theta)_L\|_2 = \|\theta_1 \Re(v) + \theta_2 \Im(v)\|_2 \]
\[ \geq c'\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}} ||\theta||_2 \]
If $\theta_1 \Re(v) \cdot \theta_2 \Im(v) < 0$ then
\[ \| - \theta_2 \Re(v) + \theta_1 \Im(v)\|_2 \geq |\theta_2||\Re(v)||_2 \]
so
\[ \|(v^T\theta')_L\|_2 \geq \| - \theta_2 \Re(v) + \theta_1 \Im(v)\|_2 \]
\[ \geq c'\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}} ||\theta||_2 \]
2. If we assume that $|\theta_2| < c'||\theta||_2$ then $|\theta_1| > \sqrt{1 - c'^2||\theta||_2}$. Therefore,
\[ \|(v^T\theta)_L\|_2 \geq \| \theta_1 ||\Re(v)||_2 - |\theta_2||\Im(v)||_2 \] \[ \geq (\sqrt{1 - c'^2\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}}} - c'\frac{k\sqrt{\nu_1}}{2\sqrt{2}})||\theta||_2 \]
A similar argument works for $L'$ if $|\theta_2| \geq c'||\theta||_2$.

We have shown that
\[ \|(v^T\theta - p)\|_2 \geq \min \left\{ c'\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}}, \sqrt{1 - c'^2\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}}} - c'\frac{k\sqrt{\nu_1}}{2\sqrt{2}} \right\} ||\theta||_2 \]
Setting $\gamma < \min \left\{ c'\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}}, \sqrt{1 - c'^2\frac{\nu_2\sqrt{\nu_1}}{2\sqrt{2}}} - c'\frac{k\sqrt{\nu_1}}{2\sqrt{2}} \right\}$ yields the desired contradiction.
For the remainder of the proof, we fix $\lambda$ and $\gamma$ from Lemma 5.10. We divide the set of potential normal vectors into classes of similar $\text{LCD}$. 

**Definition 5.11.** Define $S_D = \{v \in S^{n-1} : D \leq \text{LCD}_\alpha(x) \leq 2D\}$.

**Lemma 5.12 (Nets for Level Sets for LCD).** There exists a $2\alpha/D$-net of $S_D$ of cardinality at most $C(\alpha + 2D)\left(\frac{10D}{\alpha^2}\right)^{2n}$.

**Proof.** For a parameter $r$ to be chosen later, we create a $r$-net, $A_{D,r}$ of the annulus, $A_D$ in $\mathbb{R}^2$ defined by $A_D := \{\theta : D \leq ||\theta||_2 \leq 2D\}$.

For every $v \in S_D$, there exists $\theta \in A_D$ and $p \in \mathbb{R}^{2n}$ such that $\| [v]^T \theta - p \|_2 < \alpha$.

Let $\theta' \in A_{D,\beta}$ be within $\beta$ of $\theta$. For every $\theta'$ there is a unique $v' \in \mathbb{C}^n$ such that $[v']^T \theta' = p$. This can be seen by examining the $k$-th and $n+k$-th coordinates for all $1 \leq k \leq n$. This reduces to the following set of linear equations.

$$
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
-\theta_1
\end{pmatrix}
\begin{pmatrix}
\Re(v_k) \\
\Im(v_k)
\end{pmatrix} =
\begin{pmatrix}
p_k \\
p_{n+k}
\end{pmatrix}
$$

For any $\theta \neq 0$ the matrix

$$
\begin{pmatrix}
\theta_1 & \theta_2 \\
-\theta_1 & \theta_2
\end{pmatrix}
$$

is invertible so the system has a unique solution $v'$. The norm of $v'$ cannot be too large as

$$
\begin{pmatrix}
\Re(v_k) \\
\Im(v_k)
\end{pmatrix}
\|
= \left\|
\begin{pmatrix}
\theta_1 & \theta_2 \\
\theta_2 & -\theta_1
\end{pmatrix}^{-1}
\begin{pmatrix}
p_k \\
p_{n+k}
\end{pmatrix}
\right\|_2
\leq \frac{1}{||\theta||_2}
\begin{pmatrix}
p_k \\
p_{n+k}
\end{pmatrix}
\right\|_2
$$

so

$$
\|v'\|_2 \leq \frac{1}{D} \|p\|_2 \leq \frac{1}{D} (\alpha + \|v|^T \theta\|_2) \leq \frac{\alpha + 2D}{D}
$$

Also,

$$
\|v - v'\|_2 = \frac{1}{2} \|v - [v']^T\|_2
= \frac{1}{2||\theta||_2} \||v|^T - [v']^T\theta\|_2
\leq \frac{1}{2||\theta||_2} (\|\|v|^T \theta - p\|_2 + \|\|v'|^T \theta - p\|_2 + \|\|v|^T\||\theta - \theta'|\|_2)
\leq \frac{1}{2D} (\alpha + 2\alpha + 2D)
\leq \frac{\alpha}{D}
$$
The second inequality follows from the observation that the rows of \([v] - [v']\) are orthogonal and of the same length. The last inequality is achieved by letting \(r := \frac{D\alpha}{2(\alpha + 2D)}\). Let
\[
\mathcal{N} = \{v \in \mathbb{C}^n : \exists \theta' \in A_{D,\beta} \text{ and } p \in \mathbb{Z}^{2n} \cap B(0, \alpha + 2D) \text{ such that } [v]^T \theta' = p\}.
\]
We have shown that \(\mathcal{N}\) is an \(\alpha/D\)-net of \(S_D\). Now we bound the cardinality of \(\mathcal{N}\).

\[
|\mathcal{N}| \leq |A_{D,\beta}| \left(1 + \frac{3(\alpha + 2D)}{\sqrt{2n}}\right)^{2n} \leq C \frac{(\alpha + 2D)^2}{\alpha^2} \left(\frac{10D}{n^{1/2}}\right)^{2n}
\]
These bounds follow from the well-known result on the number of lattice points in a high-dimensional sphere and a simple covering argument in the plane for the annulus.

**Lemma 5.13** (Random Normal has Large LCD). Let \(Z_1, \ldots, Z_{n-1}\) be complex vectors. Consider a vector \(v\) orthogonal to all the \(Z_j\). Then there exist constants \(c, c' > 0\) such that
\[
\mathbb{P}(\text{LCD}_\alpha(v) < \exp(cn)) \leq \exp(-c'n)
\]

**Proof.**
\[
\mathbb{P}(\exists v \in S_{\mathbb{C}}^{n-1}, \text{LCD}_\alpha(v) < e^{cn} \text{ and } A'v = 0) \leq \mathbb{P}(\exists v \in \text{Comp}, A'v = 0) + \mathbb{P}(\exists v \in \text{Incomp}, \text{LCD}_\alpha(v) < e^{cn} \text{ and } A'v = 0)
\]
Lemma 5.2 handles the first summand. Note that \(Z_j v = 0\) is equivalent to \([v](Z_j)^T \tilde{v} = 0\). It suffices to show that the event
\[
\mathcal{E} := \{\exists v \in S_D : A'v = 0 \text{ and } \|A'\| < C\sqrt{n}\}
\]
holds with probability at most \(\exp(-cn)\).

Assume that the event \(\mathcal{E}\) holds. Let \(\mathcal{N}\) be the \(\alpha/D\)-net. Choose \(y \in \mathcal{N}\) such that \(\|x - y\|_2 < \alpha/D\).

By the triangle inequality,
\[
\|A'y\|_2 \leq \|A'\|\|x - y\|_2 \leq Cn^{1/2}\frac{\alpha}{D} = \frac{C\beta n}{D}
\]
recalling that \(\alpha = \beta n^{1/2}\). We can safely assume that \(\alpha \leq D\) by Lemma 5.10. Set \(\varepsilon = C\beta n^{1/2}/D\).

Finally, applying the union bound, we find
\[
\mathbb{P}(\mathcal{E}) \leq \mathbb{P}(\exists y \in \mathcal{N} : \|A'y\|_2 \leq \varepsilon \sqrt{n})
\]
\[
\leq |\mathcal{N}|(C\varepsilon)^{2n-2} \leq C \frac{(3D)^2}{n} \left(\frac{10D}{n^{1/2}}\right)^{2n} \left(\frac{C\beta n^{1/2}}{D}\right)^{2n+2} \leq \exp(-cn)
\]
for a suitably small \(\beta\).

At this point, we have all the necessary elements to complete the proof of Theorem 3.3.
5.5 Proof of Theorem 3.3

Proof. Recall that
\[ P(s_n(M_n) \leq \varepsilon n^{-1/2}) \leq P(\inf_{x \in \text{Comp}} \|M_n x\|_2 \leq \varepsilon n^{-1/2}) + P(\inf_{x \in \text{Incomp}} \|M_n x\|_2 \leq \varepsilon n^{-1/2}) \]

By Lemma 5.2, the first term on the right is exponentially small. By Lemma 5.4, the second term is upper bounded by
\[ \frac{1}{\nu_1} P(\text{dist}(X_n, H_n) < \varepsilon) = C P(|X_n \cdot Z_n| < \varepsilon) \]

where \( Z_n \) is a vector normal to \( H_n \). Let \( E_D \) be the event that \( LCD_\alpha(Z) > \exp(cn) \), then
\[ P(|X_n \cdot Z_n| < \varepsilon) \leq P(|X_n \cdot Z_n| < \varepsilon|E_D) + P(E_D^c) \]

By Theorem 5.7, the first term is less than \( C\varepsilon^2 + \exp(-cn) \) and the second term is less than \( \exp(-cn) \) by Lemma 5.13.

Finally, the proof of Theorem 3.1 is a consequence of Theorem 3.3.

6 Proof of Theorem 3.1

Proof. We can choose \( K \) large enough such that the probability that \( N_n \) has an eigenvalue of absolute value greater than \( K \sqrt{n} \) is less than \( e^{-cn} \). We can choose an \( \varepsilon n^{-1/2} \) net of the interval \([-K \sqrt{n}, K \sqrt{n}]\) of size at most \( 2K n/\varepsilon \). A real eigenvalue, \( \lambda \), in the interval \([-K \sqrt{n}, K \sqrt{n}]\) would imply that \( s_n(N_n - \lambda_0) \leq \varepsilon n^{-1/2} \) for some \( \lambda_0 \) in the net. By Theorem 3.3, this happens with probability at most \( C\varepsilon^2 + c^{-n} \). Thus, by the union bound, the probability that there exists a real eigenvalue is bounded by \( (2K n/\varepsilon)(C\varepsilon^2 + c^{-n}) \). Letting \( \varepsilon = c_1^{-n} \) with \( c_1 < c \) yields the result.

Acknowledgements

The author would like to thank Van Vu for his support and helpful discussions. The author also thanks Oanh Nguyen for her careful reading of the preliminary draft and many helpful comments.

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