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Investigation of the inerter-based dynamic vibration absorber for machining chatter suppression

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Abstract. This study proposes a novel approach to increase the chatter stability in machining operations. It shows the potential performance improvement when an inerter-based dynamic vibration absorber is employed in a machining operation. Tuned inerter based devices have been employed to decrease the magnitude of the vibrations in applications such as civil engineering structures and vehicle suspension systems but the nature of chatter in machining is different from these applications. Therefore, it requires a different tuning methodology to obtain the optimal design parameters. In this study, the machining operation is modelled as an undamped single degree of freedom system and different configurations of an inerter, a damper and two springs are used to ensure a stable region of operation. Strategies for the tuning parameters are developed both analytically and numerically. Using these techniques the performance improvement in the chatter stability provided by using inerter based devices instead of a traditional dynamic vibration absorber is demonstrated.

1. Introduction
Regenerative chatter can be seen as one of the biggest issues in machining processes, and can be defined as undesired vibrations between the workpiece and cutting tool. It causes poor surface finish and reduces productivity. Also, chatter leads to fast tool wear (or even tool breakage) which shortens the life cycle of the tool. Therefore, it is important to have a chatter-free operation to increase productivity.

Dynamic vibration absorbers have been widely used to increase the chatter performance of machining operations due to their simplicity, low cost and reliability. Tarng et al. [1] demonstrated that a piezoelectric vibration absorber can improve the chatter stability by matching the natural frequency of the absorber with the structure’s target mode. Moradi et al. [2] presented chatter stability improvement by investigating the position of a tunable vibration absorber along a boring bar in a turning operation. More sophisticated designs such as nonlinear, two-degree-of-freedom and multiple tuned mass dampers have been proposed to suppress chatter vibrations [3–6]. Also, active vibration absorbers have been investigated to improve chatter performance [7, 8]. Although active vibration absorbers tend to offer more performance improvement, they are more complex, expensive and difficult to implement than passive vibration absorbers.

The use of a relative-acceleration-dependent inertial mechanism, alongside the stiffness and the damping components, was already proposed by Kuroda et al. [9] and Saito et al. [10] in order to increase the limited performance of an absorber system. Similarly, the inerter, first defined by
Smith [11], provides an inertial force proportional to the relative acceleration at its terminals. Recently, inerter based devices have been investigated to improve performances of the vibratory systems. Wang et al. [12] studied the building suspension design with inerters and Lazar et al. [13] employed an inerter based device for structural vibration suppression. Inerter based devices were also tested for railway vehicles [14] and vehicle suspension systems [15,16]. Recently, Wang and Lee [17] proposed to utilize an inerter to increase vibration suppression of milling machine tools. However, they focused on the amplitude of the displacement of the vibration rather than the chatter stability. Furthermore, the suspension network that was proposed in the study did not operate with a proof-mass but instead required a mechanical connection to a rigid body, which limits its practical use.

Tuning parameters are crucial to obtain the best performance from traditional or inerter-based absorbers. For traditional dynamic vibration absorbers, it is common to use Den Hartog’s fixed-points method [18] to set the parameters of the components of the absorber. However, this method does not give the optimum parameters for a machining operations since machining chatter stability is related to the real part of the frequency response function (FRF). Therefore, an analytical tuning strategy, which focuses on the real part of the FRF, was developed by Sims [19]. Both analytical strategies developed by Den Hartog and Sims are valid if the primary system of interest is undamped. Numerical optimisation methods can be used for optimisation of damped systems [6,20,21]. $H_2$ and $H_\infty$ performances of six different configurations of inerter-based dynamic vibration absorbers were evaluated by Hu et al. [22] using a numerical optimisation method. Also, closed-form solutions for the optimal parameters of inerter-based dynamic vibration absorbers have been recently obtained by Barredo et al. [23]. Still, the focus of this study was the amplitude of the displacement of the vibration instead of the real part of the FRF, which is more related to machining chatter stability.

This study will investigate four different layouts of the inerter-based dynamic vibration absorber to improve machining chatter stability. The tuning strategies will be evaluated both analytically and numerically. After finding the optimal design parameters for the configurations, the performance evaluation will be made and a conclusion will be drawn.

2. Machining dynamics

For machining, it is crucial to understand the regenerative effect underlying chatter. In that way, the inverse proportional relationship between the limiting depth of cut and the real part of the FRF can be demonstrated and a tuning strategy can be developed. We consider a machining process that can be described as a turning operation in the following way. Due to undulations on the surface of the workpiece, the instantaneous chip thickness can be written as

$$h(t) = h_{\text{m}} + y(t - T) - y(t)$$

where $h_{\text{m}}$ is the intended chip thickness, $y(t - T)$ is the displacement of the vibration due to the previous cut and $y(t)$ is the displacement of the vibration due to the current cut as seen in Figure 1. $T$ is the time delay because of the workpiece rotation.

The cutting force can be derived by considering only the $y$-direction

$$F(t) = b_{\text{cut}} K_s h(t)$$

where $b_{\text{cut}}$ and $K_s$ are the depth of cut and specific cutting force coefficient, respectively. The multiplication of $b_{\text{cut}}$ and $h(t)$ gives the chip area removed from the workpiece and $K_s$ defines the interaction between the cutting tool and the workpiece. Taking the Laplace Transform of Equation 1 and Equation 2 gives

$$F(s) = b_{\text{cut}} K_s h(s)$$
Figure 1. The undulations due to the current and previous cuts

\[ h(s) = h_m(s) + y(s)(e^{-sT} - 1) \]  (4)

and the output can be written as the product of the transfer function and the input,

\[ y(s) = G(s)F(s) = G(s)b_{cut}K_s h(s) \]  (5)

where \( G(s) \) is the transfer function of the cutting tool-workpiece system. Substituting Equation 4 into Equation 3 yields

\[ \frac{h(s)}{h_m(s)} = \frac{1}{1 + (1 - e^{-sT})K_s b_{cut} G(s)} \]  (6)

The stability analysis of Equation 6 gives

\[ 1 + (1 - e^{-sT})K_s b_{lim} G(s) = 0 \]  (7)

as the condition for the limit of stability. For the frequency domain, \( s = j\omega_c \) and \( e^{-j\omega_c T} = \cos\omega_c T - j \sin\omega_c T \) can be written. Therefore,

\[
egin{align*}
1 + K_s b_{lim} (\Re\{G(s)\} - \Re\{G(s)\} \cos(\omega_c T) - \Im\{G(s)\} \sin(\omega_c T)) \\
+j K_s b_{lim} (\Im\{G(s)\} + \Re\{G(s)\} \sin(\omega_c T) - \Im\{G(s)\} \cos(\omega_c T)) &= 0
\end{align*}
\]  (8)

When the real part and imaginary part of Equation 8 are evaluated separately, these expressions can be found:

\[ \omega_c T = 3\pi + 2\psi \]  (9)

\[ b_{lim} = -\frac{1}{2K_s \Re\{G(\omega_c)\}} \]  (10)

where \( b_{lim} \) is the limit of depth of cut and defines the stability boundary. \( \omega_c \) is chatter frequency at which the workpiece oscillates, \( \psi = \tan^{-1}\left(\frac{\Im\{G(\omega_c)\}}{\Re\{G(\omega_c)\}}\right) \) is the phase angle of the structure and \( \omega_c T = \epsilon \) is the phase shift between inner and outer modulations (waviness at current and previous cuts). The number of vibration waves left on the workpiece is given by

\[ \frac{\omega_c T}{2\pi} = k + \frac{\epsilon}{2\pi} \]  (11)

where \( k \) is the integer number of waves and \( \frac{\epsilon}{2\pi} \) is the fraction of the redundant wave. In the case of \( \epsilon = 0 \), there is no fractional wave and two consecutive cuts are in phase.
Equation 10 demonstrates that the limiting depth of cut is inversely proportional to the negative real part of the FRF of the system. This means that a smaller absolute value of the most negative real value indicates an improvement in the chatter performance. Hence, the tuning strategy must focus on the absolute value of the most negative real value to make it as small as possible. In the following section, the mathematical model and the displacement transfer function will be presented in order to conduct simulations.

3. Mathematical model of the system

Six different configurations of inerter-based dynamic vibration absorbers were investigated for $H_\infty$ and $H_2$ by Hu et al. [22]. Their study has already shown that there is no benefit to add an inerter to traditional dynamic vibration absorber (e.g. configurations C1 and C2 in [22]). Also, it has been demonstrated that an inerter and an additional spring added to a traditional dynamic vibration absorber (e.g. configurations C3-C6 in [22]) improves the $H_\infty$ and $H_2$ performances of the absorbers. The question that is considered in the present study is whether these four configurations can increase the chatter stability in a machining operation. The four configurations and their transfer function representations are first defined in this section.

A machining system with an inerter-based dynamic vibration absorber can be modelled as seen in Figure 2. The displacement transfer function can be derived as [22]

$$H(s) = \frac{x}{x_s} = \frac{1}{\frac{x_s}{\omega_n^2} + \frac{k}{k}R(s) + 1} \quad (12)$$

where

$$R(s) = \frac{(k + sY(s))ms^2}{k + ms^2 + sY(s)} \quad (13)$$

and where $x_s = \frac{F}{k}$ and $\omega_n = \sqrt{\frac{k}{M}}$ are the static displacement and natural frequency of the machining system. The displacement transfer function is $H(s) = KG(s)$ so the transfer function of the cutting tool-workpiece system can be obtained easily. $Y(s)$ is the impedance of the inerter-based device and varies depending on the configurations which are given in Figure 3. The impedance for each configuration is also presented in Table 1.

![Figure 2. Inerter-based dynamic vibration absorber](image)

![Figure 3. Four configurations investigated in this study](image)

The displacement frequency response function can be defined by replacing $s$ with $j\omega$ in Equation 12 in the form of

$$H_i(j\lambda) = \frac{R_{ni} + jI_{ni}}{R_{di} + jI_{di}}, \quad i = 1, \ldots, 4 \quad (14)$$

by using the following non-dimensional terms
Table 1. Impedance for each configuration in Figure 3

\[
Y_1(s) = \frac{1}{s^{2} + \frac{k_1}{c} + \frac{1}{s}} \quad Y_2(s) = \frac{1}{s^{2} + \frac{1}{k_1} + \frac{1}{s}} \\
Y_3(s) = \frac{1}{s^{2} + \frac{b}{s} + \frac{1}{c}} \quad Y_4(s) = \frac{1}{s^{2} + \frac{b}{s} + \frac{1}{c}}
\]

mass ratio \( \mu = \frac{m}{M} \)

inertance-to-mass ratio \( \delta = \frac{b}{m} \)

damping ratio \( \zeta = \frac{c}{2\sqrt{mk}} \)

corner frequency ratio \( \eta = \frac{\omega_b}{\omega_m} \)

natural frequency ratio \( \gamma = \frac{\omega_m}{\omega_n} \)

forced frequency ratio \( \lambda = \frac{\omega}{\omega_n} \)

where \( \omega_m = \sqrt{\frac{k}{m}}, \omega_b = \sqrt{\frac{k_1}{b}}, \omega_n = \sqrt{\frac{K}{M}} \). The full expression of Equation 14 for each configuration, is given in Appendix A. The next section will present the methods to find the optimal design parameters to obtain the best performances from the configurations.

4. Methodology for tuning parameters

Self-adaptive Differential Evolution algorithm (SaDE) [24] is an quick and effective optimisation method for nonlinear functions and thus, it can be used as a numerical optimization method in this problem. The aim of using a numerical optimisation method is to find the optimum design parameters of the configurations. Hence, maximisation of the most negative real part of the FRF will be sought. However, numerical methods are not as easy as analytical methods to implement. For this reason, even though the closed-form solutions in [23] focus on the amplitude of the displacement of the vibration, they are also employed in this study to test whether any improvement in terms of chatter stability can be provided or not. The closed-form solutions are applied for only the configuration which gives the best improvement.

4.1. Numerical optimisation

The objective of the numerical optimisation is to maximise the minimum negative real part of the FRF. For a specified mass ratio, the optimisation problem can be expressed as

\[
\text{max}_{\gamma, \zeta, \delta, \eta} \left( \text{min}_{\lambda_n} \{ \Re \{ H_i(j\lambda_n) \} \} \right), \ i = 1, \ldots, 4
\]

The problem is solved by using SaDE [24] and Matlab is employed to apply the optimisation algorithm. SaDE solves the optimisation problems by generating a parameter candidate pool for each generation. It is an effective method since the choice of the learning strategy and the two parameters which have an effect on the performance of the optimisation result are adjusted adaptively in SaDE [24]. For mass ratio \( \mu = 0.1 \), the optimum design parameters that are obtained via SaDE are given in Table 2.
Table 2. Optimal design parameters for the configurations from SaDE

| Configurations/Design parameters | $\gamma$  | $\zeta$  | $\delta$  | $\eta$  |
|----------------------------------|----------|---------|----------|----------|
| C1                               | 1.0680   | 0.1657  | 0.2190   | 1.0476   |
| C2                               | 1.1082   | 0.0487  | 0.1890   | 0.9066   |
| C3                               | 0.9753   | 0.2093  | 1.4925   | 1.5231   |
| C4                               | 1.0134   | 0.0624  | 0.1579   | 1.2469   |

4.2. Closed-form solutions
As it will be seen in Section 5, configuration C1 provides the best improvement. Thus, the closed-form solutions which have been found in [23] are also tested. It must be noted that these expressions have not been derived for the real part of the FRF. In spite of this, there is a possibility that an inerter-based dynamic vibration absorber tuned by this method can still provide a better chatter suppression performance than traditional dynamic vibration absorber. Therefore, it might be possible to obtain a better chatter suppression performance, even without the computational effort of the numerical optimisation method, by using the closed-form expression. The main reason for this part of this study is to examine whether this is possible or not. The expressions which give the optimal design parameters for configuration C1 are [23]

$$
\gamma_{opt, C1} = \frac{1}{1 + \mu} \\
\delta_{opt, C1} = \frac{2\mu}{1 + \mu} \\
\eta_{opt, C1} = \sqrt{1 + \mu} \\
\zeta_{opt, C1} = \sqrt{\frac{11\mu}{9a_1a_2}}
$$

where

$$
a_1 = (\mu^{3/2} + 7\mu^{7/2} + 16\mu^{5/2} + 14\mu^{3/2} + 4\sqrt{\mu})\sqrt{\mu + 4 + \mu^5 + 9\mu^4 + 28\mu^3 + 36\mu^2 + 18\mu + 2}
$$

$$
a_2 = (-\mu^{7/2} - 6\mu^{5/2} - 10\mu^{3/2} - 4\sqrt{\mu})\sqrt{\mu + 4 + ma^4 + 8\mu^3 + 20\mu^2 + 16\mu + 2}
$$

5. Results
The simulations were conducted for the four configurations and the design parameters were chosen as presented in Table 2. A simulation was also conducted for configuration C1 with the optimal design parameters which were found by using closed-form solutions in Section 4.2. The primary mass, the mass ratio $\mu$, the natural frequency of the primary system $f_n$ and the specific cutting coefficient $K_s$ were taken as 1 kg, 0.1, 100 Hz and 1000 N/mm², respectively. The result of a machining operation with a traditional dynamic vibration absorber whose design parameters were tuned by Sims’ methodology was taken the benchmark analysis.

The improvements in terms of the limiting depth of cut ($b_{lim}$) which were provided from the different configurations are given in Table 3. Figure 4 presents the negative real part of the FRF of the systems for the four configurations and a traditional dynamic vibration absorber as the interest is the negative real part of the FRF in machining chatter stability. The optimal parameters were obtained from the numerical optimisation method for the four configurations and Sims’ methodology for the traditional dynamic vibration absorber. Figure 5 demonstrates the real part of the FRF for two different design parameters which were obtained by the numerical optimisation method and the closed-form solutions in [23] for configuration C1 with the benchmark.

The results show that the best performance is obtained by configuration C1 with the parameters which were found by the numerical optimisation method. Hence, the stability lobe diagram of this case is given in Figure 6 with a comparison with the benchmark.
Table 3. Improvements for each configuration

|                  | DVA    | C1       | C2       | C3       | C4       |
|------------------|--------|----------|----------|----------|----------|
| $\Re_{\text{min}} \{G(j\lambda)\}$ $[N/m]$ | $-4.3573\times10^{-6}$ | $-3.3217\times10^{-6}$ | $-3.3165\times10^{-6}$ | $-3.9780\times10^{-6}$ | $-3.3255\times10^{-6}$ |
| $b_{\text{lim}}$ for $\Re_{\text{min}} \{G(j\lambda)\}$ $[mm]$ | 0.1097 | 0.1505 | 0.1422 | 0.1257 | 0.1504 |
| Improvement (%)  | -37.19 | 29.63    | 14.59    | 37.10    | 29.63    |

Figure 4. Optimal design parameters results obtained by the numerical optimisation method for C1, C2, C3 and C4, and traditional dynamic vibration absorber result as the benchmark

Figure 5. Comparison of the real part of the FRF for different design parameters obtained by the numerical optimisation method and the closed-form solution for configuration C1

Figure 6. Stability lobe diagram for configuration C1 and traditional dynamic vibration absorber
6. Discussion
The results have shown that the configurations of the inerter-based dynamic vibration absorber studied in this paper can be used to improve machining chatter stability. The best performances have been obtained from configuration C1 (red solid line in Figure 4) and configuration C4 (magenta dashed line in Figure 4) with 37% improvement.

The results have also demonstrated that tuning parameters have a significant effect on the performance of the absorbers as can be seen in Figure 5. The design parameters obtained by using the closed-form solutions in [23] gave a worse performance (blue dashed line in Figure 5) than traditional dynamic vibration absorber (black dashed line in Figure 5) while the design parameters obtained by using the numerical optimisation gave a better performance (red solid line in Figure 5). This is because the closed-form solutions used in the previous study are derived for the magnitude of the displacement of the vibration whereas the performance of the chatter stability is related to the negative real part of the FRF as shown in Equation 10. Thus, the objective function of the numerical optimisation was the real part of the FRF and the design parameters for that case provided significant improvements, especially for configurations C1 and C4.

Although the primary system was assumed to be an undamped system in this study, the numerical optimisation method can also be used to find the optimal design parameters for a damped system, which would be more accurate model for a real machining operation. However, as we are seeking to prove the concept, an undamped model is sufficient for the sake of simplicity. Moreover, the use of an undamped model as a primary system in this study provided the observation of the difference between two tuning methodologies that are adjusted for the amplitude of the displacement of the vibration and the negative real part of the FRF in an analysis of the machining chatter stability. It has been seen that only employment of the inerter is not sufficient to obtain an improvement. The design parameters have a key role in the performance so they must be tuned with the right method.

7. Conclusion
This study has shown the benefit of the employment of an inerter-based dynamic vibration absorber in a machining operation in terms of the machining chatter stability. An undamped system and four different configurations were employed for the simulations. The design parameters of the configurations were obtained via Self-adaptive Differential Evolution algorithm. It has been seen that the objective function of the optimisation must be the negative real part of the FRF to achieve the optimal parameters which give the best performance. Almost 37% improvement was achieved in a comparison to a traditional dynamic vibration absorber that is similarly tuned to maximise the chatter stability.

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Appendix A. Expressions for \( R_{ni} \), \( R_{di} \), \( I_{ni} \) and \( I_{di} \)

The full expression of the terms in Equation 14:

For \( i = 1, \ldots, 4 \),

\[
\begin{align*}
R_{n1} & = -2\eta^2\gamma^2\lambda^2\zeta + 2\eta^2\gamma^4\zeta - 2\eta^2\gamma^2\lambda^2\zeta - 2\gamma^2\lambda^2\zeta + 2\lambda^4 \\
I_{n1} & = -\delta\eta^2\gamma^3\lambda - \delta\eta^2\gamma\lambda^3 \\
R_{d1} & = 2\eta^2\gamma^2\lambda^4\mu\zeta + 2\delta\eta^2\gamma\lambda^4\zeta - 2\eta^2\gamma^4\lambda^2\mu\zeta - 2\eta^2\gamma^2\lambda^4\zeta + 2\eta^2\gamma^2\lambda^2\zeta - 2\delta\eta^2\gamma^2\lambda^2\zeta \\
& + 2\gamma^2\lambda^4\mu\zeta + 2\delta\eta^2\gamma^4\lambda^2\zeta - 2\delta\eta^2\gamma^2\lambda^2\zeta + 2\gamma^2\lambda^4\zeta - 2\delta\lambda^6 - 2\gamma^2\lambda^2 - 2\lambda^4 \\
I_{d1} & = -\delta\eta^2\gamma^3\lambda^3 - \delta\eta^2\gamma^3\lambda + \delta\eta^2\gamma\lambda^5 + \delta\eta^2\gamma\lambda^3 - \delta\eta^2\gamma\lambda^3 \\
R_{n2} & = -\delta^2\eta^2\gamma^2\lambda^2 + \delta^2\eta^2\gamma^2\lambda^2 - \delta\gamma^2\lambda^2 + \delta\lambda^4 \\
I_{n2} & = -2\delta\gamma\lambda^3\zeta + 2\gamma^3\lambda\zeta - 2\gamma\lambda^3\zeta \\
R_{d2} & = \delta^2\eta^2\gamma^2\lambda^4\mu + \delta^2\eta^2\gamma\lambda^4\zeta - \delta\eta^2\gamma^4\lambda^2\mu - \delta\eta^2\gamma^2\lambda^4\mu + \delta\eta^2\gamma^2\lambda^2\mu - \delta^2\eta^2\gamma^2\lambda^2\mu \\
& + \delta\gamma^2\lambda^4\mu + \delta\eta^2\gamma^4 - \delta\eta^2\gamma^2\lambda^2\mu + \delta\gamma^2\lambda^4 - \delta\lambda^6 - \delta\gamma^2\lambda^2 + \delta\lambda^4 \\
I_{d2} & = 2\delta\gamma\lambda^5\mu\zeta + 2\delta\gamma\lambda^3\lambda\zeta - 2\gamma^3\lambda^3\mu\zeta - 2\gamma^3\lambda^3\zeta + 2\gamma^5\lambda^5 - 2\gamma^3\lambda^3\zeta + 2\gamma^3\lambda^3 - 2\gamma\lambda^3\zeta \\
R_{n3} & = -\delta\eta^2\gamma^4 - \delta\eta^2\gamma^2\lambda^2 - \delta\gamma^2\lambda^2 + \delta\lambda^4 \\
I_{n3} & = 2\delta\eta^2\gamma^3\lambda\zeta - 2\delta\gamma\lambda^3\zeta + 2\gamma^3\lambda\zeta - 2\gamma\lambda^3\zeta \\
R_{d3} & = \delta\eta^2\gamma^4\lambda^2\mu - \delta\eta^2\gamma^2\lambda^2\mu + \delta\gamma^4\lambda^2\mu + \delta\eta^2\gamma^2\lambda^4\mu + \delta\eta^2\gamma^4\lambda^2 - \delta\eta^2\gamma^2\lambda^2 + \delta\gamma^2\lambda^4 - \delta\lambda^6 - \delta\gamma^2\lambda^2 + \delta\lambda^4 \\
& - 2\gamma^3\lambda^3\mu\zeta + 2\delta\gamma\lambda^5\mu\zeta + 2\delta\eta^2\gamma^3\lambda^3\mu\zeta + 2\delta\gamma\lambda^5\lambda\zeta - 2\gamma^3\lambda^3\mu\zeta \\
I_{d3} & = -2\delta\gamma\lambda^5\mu\zeta + 2\delta\gamma\lambda^3\lambda\zeta - 2\gamma\lambda^3\zeta + 2\gamma\lambda^3\zeta \\
R_{n4} & = -\delta^2\eta^2\gamma^2\lambda^2 + \delta^2\eta^2\gamma^4 - \delta\eta^2\gamma^2\lambda^2 - \delta\gamma^2\lambda^2 + \delta\lambda^4 \\
I_{n4} & = 2\delta^2\eta^2\gamma^3\lambda\zeta + 2\gamma^3\lambda\zeta - 2\gamma\lambda^3\zeta \\
R_{d4} & = \delta^2\eta^2\gamma^2\lambda^4\mu + \delta^2\eta^2\gamma\lambda^4\zeta - \delta^2\eta^2\gamma^4\lambda^2\mu - \delta^2\eta^2\gamma^2\lambda^4\mu + \delta^2\eta^2\gamma^2\lambda^2\mu + \delta^2\eta^2\gamma^2\lambda^2\mu \\&+ \delta\gamma^2\lambda^4 + \delta\eta^2\gamma^4 - \delta\eta^2\gamma^2\lambda^2 + \delta\gamma^2\lambda^4 - \delta\lambda^6 - \delta\gamma^2\lambda^2 + \delta\lambda^4 \\
I_{d4} & = -2\delta^2\eta^2\gamma^3\lambda^3\mu\zeta - 2\delta^2\eta^2\gamma^3\lambda^3\mu\zeta + 2\delta^2\eta^2\gamma^3\lambda^3\mu\zeta - 2\gamma^3\lambda^3\mu\zeta - 2\gamma^3\lambda^3\zeta + 2\gamma\lambda^5\zeta \\
& + 2\gamma^3\lambda\zeta - 2\gamma\lambda^3\zeta
\end{align*}
\]
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