Hilbert evolution algebras and its connection with discrete-time Markov chains

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Received: 21 December 2021 / Accepted: 4 August 2022 / Published online: 12 August 2022 © The Indian National Science Academy 2022

Abstract Evolution algebras are non-associative algebras. In this work we provide an extension of this class of algebras, in a framework of Hilbert spaces, and illustrate the applicability of our approach by discussing a connection with discrete-time Markov chains with infinite countable state space. Specifically, if we associate to each possible state of such a Markov process a generator of the Hilbert evolution algebra structure, then the whole dynamics of the process can be described through consecutive applications of the evolution operator, provided certain boundedness conditions on the transition probabilities hold.

Keywords Genetic Algebra · Evolution Algebra · Hilbert Space · Markov Chain

Mathematics Subject Classification 17D92 · 46C99 · 60J10

1 Introduction

In this paper we contribute with the Theory of Evolution Algebras, which is developed around a special class of genetic algebras. At the beginning, the notion of evolution algebra was formulated in [18] as an algebraic way to mimic the self-reproduction of alleles in non-Mendelian genetics. Fortunately, like many objects in Mathematics, this concept proved to be very flexible for comparison with concepts from different fields. The best reference to start studying the subject is the seminal work of Tian, [17], where the author, after the formulation of basic properties for these algebras, explores an interesting correspondence between them and the theory of discrete-time Markov chains. In the same reference the reader may find a summary of possible connections with other fields like graph theory, group theory, statistical physics, and others. An evolution algebra is defined as follows.

Communicated by Rahul Roy.

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**Definition 1** Let $\mathbb{K}$ be a field and let $\mathcal{A} := (\mathcal{A}, \cdot)$ be a $\mathbb{K}$-algebra. We say that $\mathcal{A}$ is an evolution algebra if it admits a basis $S := \{e_i\}_{i \in \Lambda}$, such that

$$
e_i \cdot e_i = \sum_{k \in \Lambda} c_{ki} e_k, \quad \text{for } i \in \Lambda, \quad (1)$$

$$e_i \cdot e_j = 0, \quad \text{for } i, j \in \Lambda \text{ such that } i \neq j. \quad (2)$$

The scalars $c_{ki} \in \mathbb{K}$ are called the structure constants of $\mathcal{A}$ relative to $S$. A basis $S$ satisfying (2) is called a natural basis of $\mathcal{A}$. We emphasize that in the definition above basis means Hamel basis; i.e., a maximal linearly independent subset. It implies that for a fixed $i \in \Lambda$ only a finite number of constants $c_{ki}$ are non-zero.

Currently, there is a wide literature about this issue and its consequences. Here we mention some of the recent works, and we refer the reader to the references therein for a deeper study of the theory. In [3–7] the reader may find a survey of properties and results for general evolution algebras; the works in [1,2,8,15] are devoted to the connection between evolution algebras and graphs together with some related properties; and in [10,13] one may see a good review of results with relevance in genetics and other applications.

We are interested in providing a generalization of Definition 1 which is able to deal with infinite-dimensional spaces and, at the same time, to include an application not covered by it. Let us start with our motivation. If $\mathcal{A}$ is an evolution algebra such that $c_{ki} \in [0, 1]$, for any $i, k \in \Lambda$, and $\sum_{i \in \Lambda} c_{ki} = 1$, for any $i \in \Lambda$, then $\mathcal{A}$ is called a Markov evolution algebra. The name is due to a correspondence between evolution algebras and discrete-time Markov chains given in [17]. To see the connection, let us remember some basic notation for Markov chains. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; i.e., $\Omega$ is an arbitrary non-empty set, $\mathcal{F}$ is a $\sigma$-field of subsets of $\Omega$ and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. A sequence of random variables $\{X_n\}_{n \geq 0}$ living in this probability space and taking values in $\mathcal{X} := \{x_i\}_{i \in \Lambda}$, where $\Lambda$ is a countable set of indices, is called a discrete-time Markov chain if it satisfies the Markov’s property; namely, if

$$\mathbb{P}(X_{n+1} = x_j \mid X_0 = x_{i_0}, X_1 = x_{i_1}, \ldots, X_{n-1} = x_{i_{n-1}}, X_n = x_i)$$

is equal to $\mathbb{P}(X_{n+1} = x_j \mid X_n = x_i) =: p_{i j}$, for any $n \geq 1$, and for any subset $\{x_{i_0}, x_{i_1}, \ldots, x_{i_{n-1}}, x_i, x_j\} \subset \mathcal{X}$ with $\{i_0, i_1, \ldots, i_{n-1}, i, j\} \subset \Lambda$. The values $p_{i j}$ are called the transition probabilities of the Markov chain and do not depend on $n$; i.e., $\{X_n\}_{n \geq 0}$ is an homogeneous Markov chain. In [17, Chapter 4] it is defined an evolution algebra $\mathcal{A}$ with a natural basis $S := \{e_i\}_{i \in \Lambda}$ in such a way that each state of the Markov chain is in correspondence with each generator of $S$, and $c_{ki} = p_{ik}$, for any $i, k \in \Lambda$.

As far as we know [17, Chapter 4] was the first in proposing the interplay between evolution algebras and Markov chains. In such a work many well-known results coming from Markov chains were stated in the language of Markov evolution algebras. We point out that this is an interesting connection which deserves to be explored because it represents a new framework to describe random phenomena; i.e. through techniques of non-associative algebras. However, we have to take care when dealing with the connection of these mathematical objects. Although [17, Theorem 16, page 54] claims, using the correspondence mentioned above, that for any homogeneous Markov chain there is an evolution algebra whose structure constants are transition probabilities, and whose generator set is the state space of the Markov chain, this is not totally true whether the state space has infinitely many elements.

**Example 1** A Discrete-time Markov chain that does not determine an evolution algebra according to Definition 1

Let $\{X_n\}_{n \geq 0}$ be a Markov chain with state space given by $\mathcal{X} = \mathbb{N} \cup \{0\}$ and transition probabilities given by $p_{0i} = p_i > 0$, for any $i \in \mathbb{N}$, where $\sum_{i=1}^{\infty} p_i = 1$, and $p_{i(0)} = 1$ for any $i \in \mathbb{N}$. See Figure 1 for an illustration of the transitions of this Markov chain. In words, from any state $i \neq 0$ the process “jumps” to state $i − 1$ with probability 1, and as soon as the process hits state 0, it jumps to state 0 with probability $p_0$. Notice that from state 0 we can go to infinitely many states with positive probability. If we assume that there exists an evolution algebra whose generator set is in correspondence with the state space of this Markov chain, namely $\mathcal{S} = \{e_i\}_{i \in \mathbb{N} \cup \{0\}}$, then, taking $c_{ki} = p_{ik}$, it should be for $i \neq 0$, $e_i^2 = e_{i-1}$, while

$$e_0^2 = \sum_{i \in \mathbb{N}} p_i e_i,$$

with $p_i > 0$ for any $i \in \mathbb{N}$. But this is a contradiction because according to Definition 1, since $\mathcal{S}$ is a Hamel basis, the numbers $c_{ki}$ can be non-zero only for a finite number of $j$’s.
The previous example suggests that although many connections with other fields have been suggested in literature, still some gaps exist whether one considers applications involving infinite-dimensional spaces. This is because in the original definition of Tian [17] the basis is implicitly assumed to be a Hamel basis. Thus, the sum (1) can have only a finite number of nonzero terms. Following this work, in [5] the authors consider infinite-dimensional evolution algebras but still only with finite sums. In order to allow an infinite number of nonzero terms in the series, we need to consider other structures to give a meaning to the sum (1). The usual way to do this is through Functional Analysis, by introducing topologies and different notions of convergence. One approach to do this was taken in [14]. The novelty of such a work is the concept of Banach evolution algebras capable of dealing with infinite-dimensional algebras whose natural basis are uncountable. However, the case of evolution algebras with an infinite countable natural basis is not covered by their definition. With the motivation of fulfill this gap we propose a different approach and we work with Hilbert spaces, which leads us to consider other kind of basis; namely, Schauder basis. In other words, we propose an extension of Definition 1 by providing an evolution algebra structure in a given Hilbert space. We call that new structure a Hilbert evolution algebra, and after stating some basic properties, we illustrate its applicability to the connection with discrete-time Markov chains.

The rest of the paper is subdivided into two sections. In Section 2 we introduce the concept of Hilbert evolution algebra, we define its associated evolution operator, and we discuss a condition under which this operator is continuous. In Section 3 we include our application to the connection between these objects and discrete-time Markov chains.

2 Hilbert Evolution Algebras

We start with some definitions and notation. Let $V$ be a vector space over $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, with inner product $\langle \cdot, \cdot \rangle$. A subset $\{e_k\}_{k \in \Lambda} \subset V$, where $\Lambda$ is a countable set, is a Schauder Basis of $V$ if any $v \in V$ has a unique representation

$$v = \sum_{k \in \Lambda} v_k e_k, \quad \text{where } v_k \in \mathbb{K}.$$ 

We say that $V$ is a Hilbert space if it is also a complete metric space with respect to the distance function induced by the inner product. A subset $\{e_k\}_{k \in \Lambda} \subset V$ is an orthonormal basis if every $v \in V$ can be expressed as

$$v = \sum_{k \in \Lambda} \langle v, e_k \rangle e_k.$$ 

On the other hand, we say that $V$ is separable if it has a countable dense subset. In this case, any orthonormal basis is countable. The Gram-Schmidt orthonormalization process proves that every separable Hilbert space has an orthonormal basis. We highlight that if $V$ is finite-dimensional, the notion of Schauder basis coincides with that of Hamel basis.

We shall define an evolution algebra structure in a Hilbert space $A$. In order to do it two questions should be considered. The first one is that we would like to define a product in $A$ satisfying relations (1) and (2). The problem with that is the convergence of the series involved in the definition of such a new product; that is, if...
$v, w \in \mathcal{A}$ then $v \cdot w$ may not be in $\mathcal{A}$. Specifically, if we write $v = \sum_{k \in \Lambda} v_k e_k$ and $w = \sum_{k \in \Lambda} w_k e_k$ where \{\(e_k\)\}_{k \in \Lambda}$ is an orthonormal basis then, using \((1)\) and \((2)\) and extending by linearity, we must have

$$v \cdot w = \sum_{k \in \Lambda} \left( \sum_{i \in \Lambda} v_i w_i c_{ki} \right) e_k.$$  

However, the series can be non convergent in $\mathcal{A}$. To solve this problem we will work with separable Hilbert spaces and appeal to a well-known result of Hilbert spaces theory.

\textbf{Proposition 1} [12, Theorem 8.3.1] \textit{Let $\mathcal{A}$ be a Hilbert space and let \{\(e_i\)\}_{i \in \mathbb{N}} be an orthonormal subset. The series $\sum_{k=1}^{\infty} c_k e_k$ is convergent if, and only if, the numerical series $\sum_{k=1}^{\infty} |c_k|^2$ is convergent.}

Thus, given a separable Hilbert space $\mathcal{A}$, we want to define the product algebra for elements $v = \sum_{k=1}^{\infty} v_k e_k$ and $w = \sum_{k=1}^{\infty} w_k e_k$ satisfying

$$\sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i w_i c_{ki} \right)^2 < \infty,$$

for an orthonormal basis \{\(e_k\)\}_{k \in \mathbb{N}}. In this case, the product $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ can be defined in the basis \{\(e_k\)\}_{k \in \mathbb{N}} and extended by linearity.

The second issue to consider for a general definition of evolution algebra in a Hilbert space is the compatibility between the involved structures. Note that under the considerations described above we can introduce the left multiplication operator

$$L_v : \mathcal{A} \rightarrow \mathcal{A} \quad w \mapsto L_v(w) := v \cdot w,$$

for any $v \in \mathcal{A}$. So we shall require for the continuity of left multiplication operators whenever it is possible to define the product algebra. After the previous considerations we are able to introduce our definition.

\textbf{Definition 2} \textit{Let $\mathcal{A} = (\mathcal{A}, \langle \cdot, \cdot \rangle)$ be a real or complex separable Hilbert space which is provided with an algebra structure by the product $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. We say that $\mathcal{A} := (\mathcal{A}, \langle \cdot, \cdot \rangle, \cdot)$ is a separable Hilbert evolution algebra if it satisfies the following conditions:

(i) There exists an orthonormal basis \{\(e_i\)\}_{i \in \mathbb{N}} and scalars \{\(c_{ki}\)\}_{i, k \in \mathbb{N}}\), such that

$$e_i \cdot e_i = \sum_{k=1}^{\infty} c_{ki} e_k$$

and

$$e_i \cdot e_j = 0, \text{ if } i \neq j,$$

for any $i, j \in \mathbb{N}$.

(ii) For any $v \in \mathcal{A}$, the left multiplications $L_v$ defined by \((4)\) are continuous in the metric topology induced by the inner product; i.e., there exists constants $M_v > 0$ such that

$$\|L_v(w)\| \leq M_v \|w\|, \text{ for all } w \in \mathcal{A}. $$

A basis satisfying condition (i) will be called \textit{orthonormal natural basis}. In the sequel we will work only with separable Hilbert spaces, so we omit the word separable and talk about Hilbert evolution algebras. As the evolution algebras in the sense of Definition 1, the Hilbert evolution algebras are commutative and are, in general, non associative and without an unitary element. Also, it is not difficult to see that for any finite-dimensional evolution algebra it is possible to define a norm such that the algebra becomes a Hilbert evolution algebra. For more details see [17, Section 3.3]. Let us also point out that while checking \((7)\), we are also checking that the product algebra is well defined; that is, if \((7)\) holds then $\|L_v(w)\| = \|v \cdot w\| < \infty$, which written explicitly in any orthonormal natural basis is equivalent to \((3)\).

Note that if $V$ is a finite-dimensional vector space and \{\(e_i\)\}_{i \in \Lambda} is a basis for $V$, then it is always possible to give an evolution algebra structure to $V$ by defining a product satisfying the equations \((1)\) and \((2)\), for any finite subset of scalars \{\(c_{ki}\)\}_{i, k \in \Lambda}. However, as we prove in the next result, in the infinite-dimensional Hilbert case the sequence of scalars must satisfy an additional condition.
Proof  Let \( v = \sum_{i=1}^{\infty} v_i e_i \in \mathcal{A} \), we define a Hilbert evolution algebra structure in \( \mathcal{A} \).

Indeed, it is possible to choose numbers \( \{c_{ki}\}_{i,k\in\mathbb{N}} \) such that

\[
\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |v_i c_{ki}|^2 < \infty,
\]

for any \( v = \sum_{i=1}^{\infty} v_i e_i \in \mathcal{A} \), defines a Hilbert evolution algebra structure in \( \mathcal{A} \).

**Proof** Let \( v = \sum_{i=1}^{\infty} v_i e_i \in \mathcal{A} \) be an arbitrary vector and note that there are always numbers satisfying (8). Indeed, it is possible to choose numbers \( \{c_{ki}\}_{i,k\in\mathbb{N}} \) such that

\[
\sup \left\{ \sum_{k=1}^{\infty} |c_{ki}|^2 : i \in \mathbb{N} \right\} < \infty,
\]

from which (8) follows immediately, because we know that \( \|v\|^2 = \sum_{i=1}^{\infty} |v_i|^2 < +\infty \). Now, let \( w = \sum_{i=1}^{\infty} w_i e_i \) and consider the formal series

\[
L_v(w) = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i w_i c_{ki} \right) e_k.
\]

Thus, to prove the proposition we must analyze the convergence of

\[
\|L_v(w)\|^2 = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i w_i c_{ki} \right)^2.
\]

By equation (8) we have that \( \sum_{i=1}^{\infty} |v_i c_{ki}|^2 < \infty \), for any \( k \in \mathbb{N} \). On the other hand \( w \in \mathcal{A} \) implies that \( \|w\|^2 = \sum_{i=1}^{\infty} |w_i|^2 < \infty \). Then we can use the Cauchy-Schwartz inequality to obtain

\[
\left| \sum_{i=1}^{\infty} w_i v_i c_{ki} \right|^2 \leq \left( \sum_{i=1}^{\infty} |w_i|^2 \right) \left( \sum_{i=1}^{\infty} |v_i c_{ki}|^2 \right) = \|w\|^2 \sum_{i=1}^{\infty} |v_i c_{ki}|^2,
\]

for every \( k \in \mathbb{N} \). Hence

\[
\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |w_i v_i c_{ki}|^2 \leq \|w\|^2 \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |v_i c_{ki}|^2 = M_v^2 \|w\|^2,
\]

where by hypothesis we can define

\[
M_v := \left( \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |v_i c_{ki}|^2 \right)^{1/2}.
\]

It follows that \( \|L_v(w)\| \leq M_v \|w\| \), for any \( w \in \mathcal{A} \); i.e., the operators \( L_v \) are well defined and are continuous for \( v \in \mathcal{A} \). That is, if we define the product in the basis \( \{e_i\}_{i\in\mathbb{N}} \) by the equations (5) and (6) then, it is possible to extend the product by linearity to \( v \cdot w \) for all \( v, w \in \mathcal{A} \) using the equation (9), and in this framework the operators \( L_v \) are continuous. Therefore we have an Hilbert evolution algebra structure defined in \( \mathcal{A} \).

Based on the previous proof we see that there is an important special case to guarantee the existence of Hilbert evolution algebras.

**Corollary 1** Let \( \mathcal{A} \) be a separable Hilbert space. Consider an orthonormal basis \( \{e_i\}_{i\in\mathbb{N}} \) and suppose that the sequences \( \{c_{ki}\}_{i,k\in\mathbb{N}} \) satisfy

\[
K := \sup \left\{ \sum_{k=1}^{\infty} |c_{ki}|^2 : i \in \mathbb{N} \right\} < \infty.
\]

Then \( \mathcal{A} \) admits an Hilbert evolution algebra structure where the \( c_{ki} \) are the structure constants.
In analogy to the theory of [17, Section 3.2] in the finite dimensional case, the next step is to introduce the evolution operator induced by the Hilbert evolution algebra. We define the evolution operator as the linear operator
\[
C : D(C) \rightarrow A
\]
given by its values in a natural orthonormal basis,
\[
C(e_i) := e_i^2 = \sum_{k=1}^{\infty} c_{ki} e_k,
\]
and with domain \(D(C) \subset A\) defined as
\[
D(C) := \left\{ v = \sum_{i=1}^{\infty} v_i e_i \in A : \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} v_i c_{ki} \right| < \infty \right\}
\]
(11)

It is worth noting that \(D(C)\) is a vector space. This follows from the Minkowski’s inequality, which implies
\[
\left( \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} (\alpha v_i + w_i) c_{ki} \right)^2 \right)^{1/2} \leq \left( \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} \alpha v_i c_{ki} \right)^2 \right)^{1/2} + \left( \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} w_i c_{ki} \right)^2 \right)^{1/2} < \infty,
\]
for \(\alpha \in \mathbb{R}, v = \sum_{i=1}^{\infty} v_i e_i, w = \sum_{i=1}^{\infty} w_i e_i \in D(C).\) With this we can write
\[
C(v) := \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} v_i c_{ki} \right) e_k,
\]
(12)

for any \(v = \sum_{i=1}^{\infty} v_i e_i \in D(C).\) In the general case the operator \(C\) will be unbounded, thus it is important to find conditions on the structure constants to know when \(C\) is closable or closed. This matter is left for future work.

Next we show some cases where \(C\) is bounded.

**Proposition 3** Let \(A\) be a Hilbert evolution algebra with structure constants satisfying one of the following conditions:

(i) \(\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |c_{ki}|^2 < \infty.\)

(ii) There exists \(\alpha_k, \beta_i > 0, i, k \in \mathbb{N}\) and \(M_1, M_2 > 0\) such that

\[
\sum_{k=1}^{\infty} |c_{ki}| \alpha_k \leq M_1 \beta_i, \quad \text{for all } i \in \mathbb{N},
\]
\[
\sum_{i=1}^{\infty} |c_{ki}| \beta_i \leq M_2 \alpha_k, \quad \text{for all } k \in \mathbb{N}.
\]

(13)

Then \(D(C) = A\) and the evolution operator \(C : A \rightarrow A\) is bounded with \(\|C\| \leq (M_1 M_2)^{1/2}.\)

**Proof** Suppose the first condition. Let \(v = \sum_{i=1}^{\infty} v_i e_i \in A\) and note that
\[
\|C(v)\|^2 = \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} v_i c_{ki} \right|^2 \leq \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} |v_i|^2 \right) \left( \sum_{i=1}^{\infty} |c_{ki}|^2 \right) = \|v\|^2 \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |c_{ki}|^2 < \infty,
\]
where we use the Cauchy-Schwartz inequality. Therefore \(D(C) = A\) and the linear operator is bounded in this case. Now let us assume that the second condition (13) is satisfied. By a similar argument used to prove the Schur
Test [11, Section 45], we have that
\[
\|C(v)\|^2 = \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} v_i c_{ki} \right|^2
\]
\[
\leq \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} v_i \right|^2 |c_{ki}|^2
\]
\[
= \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} |c_{ki}| |v_i| \right)^2 \frac{\sqrt{|c_{ki}|}}{\sqrt{\beta_i}} \frac{\sqrt{|v_i|}}{\sqrt{\beta_i}}
\]
\[
\leq M_2 \sum_{k=1}^{\infty} \frac{|v_i|^2}{\beta_i} \left( \sum_{i=1}^{\infty} |c_{ki}| |\alpha_k| \right)
\]
\[
\leq M_1 M_2 \sum_{i=1}^{\infty} |v_i|^2 = M_1 M_2 \|v\|^2.
\]
That is, \( D(C) = \mathcal{A} \) and \( C \) is a bounded linear operator, with \( \|C\| \leq (M_1 M_2)^{1/2} \).

\[\qed\]

3 The connection with discrete-time Markov chains

In order to illustrate the applicability of Definition 2 we recover our motivation, the connection with discrete-time Markov chains. In what follows, the structure constants will be interpreted as probabilities so we assume \( K = \mathbb{R} \).

**Theorem 4** Let \( \{X_n\}_{n \geq 0} \) be an homogeneous discrete-time Markov chain with state space \( \mathcal{X} = \{x_i\}_{i \in \mathbb{N}} \) and transition probabilities given by \( p_{ik} \), for \( i, k \in \mathbb{N} \). If \( \mathcal{A}_\mathcal{X} \) is a separable Hilbert space with an orthonormal basis \( \{e_i\}_{i \in \mathbb{N}} \), then the structural constants \( \{c_{ki}\}_{i,k \in \mathbb{N}} \), such that \( c_{ki} := p_{ik} \) for any \( i, k \in \mathbb{N} \), define a Hilbert evolution algebra structure in \( \mathcal{A}_\mathcal{X} \), called a Markov Hilbert evolution algebra. Moreover, suppose there exists \( \alpha_k, \beta_i > 0 \), \( i, k \in \mathbb{N} \) and \( M_1, M_2 > 0 \) such that

\[
\sum_{k=1}^{\infty} p_{ik} \alpha_k \leq M_1 \beta_i, \quad \text{for all } i \in \mathbb{N},
\]
\[
\sum_{i=1}^{\infty} p_{ik} \beta_i \leq M_2 \alpha_k, \quad \text{for all } k \in \mathbb{N}.
\]

Then the evolution operator \( C : \mathcal{A}_\mathcal{X} \to \mathcal{A}_\mathcal{X} \) is a bounded linear operator and satisfies

\[
C^n(e_i) = \sum_{k=1}^{\infty} p_{ik}^{(n)} e_k,
\]

where

\[
p_{ik}^{(n)} := \mathbb{P}(X_n = x_k | X_0 = x_i).
\]

**Proof** The first part of the proof is a direct consequence of Corollary 1. Indeed, consider a separable Hilbert space \( \mathcal{A}_\mathcal{X} \) for which we can identify an orthonormal basis \( \{e_i\}_{i \in \mathbb{N}} \). Then, if we consider the constants \( \{c_{ki}\}_{i,k \in \mathbb{N}} \) as the transition probabilities; i.e., \( c_{ki} = p_{ik} \), then condition (10) holds because for any \( i \in \mathbb{N} \) we have \( \sum_{k=1}^{\infty} |c_{ki}|^2 \leq \sum_{k=1}^{\infty} c_{ki} = 1 \).
Let \( C : D(C) \to A_X \) the operator defined by (12). Note that the conditions (14) are just (13) written for \( p_{ik} = c_{ki} \). Thus, we have \( D(C) = A_X \), the continuity of \( C \) and \( \| C \| \leq M^{1/2} \). Now, (15) can be proved by induction in \( n \), by noting that

\[
C \left(C^{n-1}(e_i)\right) = C \left( \sum_{k=1}^{\infty} p_{ik}^{(n-1)} e_k \right) = \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} p_{ik}^{(n-1)} e_j \right) e_j = \sum_{k=1}^{n} p_{ij}^{(n)} e_j,
\]

where, since \( p_{ik}^{(n-1)} c_{jk} = p_{ik}^{(n-1)} p_{kj} \), the last equality of (16) is due to the Chapman-Kolmogorov Equations (see for example [16, Section 4.2]), which guarantee that

\[
p_{ij}^{(n_1+n_2)} = \sum_{k=1}^{\infty} p_{ik}^{(n_1)} p_{kj}^{(n_2)},
\]

for any \( i, j, n_1, n_2 \in \mathbb{N} \). \( \square \)

Note that the previous theorem ensures that each Markov chain with state space \( X = \{x_i\}_{i \in \mathbb{N}} \) induces a Markov Hilbert evolution algebra on every Hilbert space \( A_X \) associated to it. On the other hand, the condition (14) for the continuity of the evolution operator is difficult to check, thus we present a particular case, which is simpler to verify.

**Corollary 2** Let \( A_X \) be a Markov Hilbert evolution algebra and \( M > 0 \) such that

\[
\sum_{i=1}^{\infty} p_{ik} \leq M \quad \text{for all } k \in \mathbb{N},
\]

then the evolution operator \( C : A_X \to A_X \) is bounded and satisfies (15).

**Proof** We want to check that the equations (14) are satisfied. Note that, since \( \sum_{k=1}^{\infty} p_{ik} = 1 \), condition (17), allow us to use \( \alpha_i = \beta_k = 1 \) for all \( i, k \in \mathbb{N} \) and \( M_1 = M_2 = 1 \). Thus, we can apply Theorem 4. \( \square \)

**Example 2** Let us consider the Example 1 again. Let \( X = \{x_i\}_{i \in \mathbb{N}\cup\{0\}} \) be the state space and let the transition probabilities given by \( p_{0i} = p_i > 0 \), for any \( i \in \mathbb{N} \), where \( \sum_{i=1}^{\infty} p_i = 1 \) and \( p_{i(i-1)} = 1 \) for any \( i \in \mathbb{N} \). By the Theorem 4, we have an induced Markov Hilbert evolution algebra \( A_X \) by taking \( c_{ki} = p_{ik} \), for \( i, k \in X \). Moreover, note that

\[
\sum_{i=1}^{\infty} p_{0i} = p_{10} = 1,
\]

and, for any \( k \in \mathbb{N} \), we have

\[
\sum_{i=1}^{\infty} p_{ik} = p_{(k+1)k} + p_{0k} = 1 + p_{0k} \leq 2.
\]

Hence the equation (17) is satisfied, implying that the evolution operator \( C \) is bounded with \( D(C) = A_X \) and \( \| C \| \leq 2 \).

**Example 3** Consider a branching process with offspring distribution given by \( (p_i)_{i \geq 0} \), with \( p_0 \in (0, 1) \). That is, consider the discrete-time Markov chain \( (Z_n)_{n \geq 0} \) such that \( Z_0 = 1 \), and

\[
Z_{n+1} = \sum_{i=1}^{Z_n} X_i, \tag{18}
\]

where \( X_1, X_2, \ldots \) are independent and identically distributed random variables with a common law \( \mathbb{P}(X_1 = i) = p_i \), for \( i \in \mathbb{N} \cup \{0\} \). This is another example of stochastic process such that, depending of the offspring law, does not determine an evolution algebra according to Definition 1. However, by the Theorem 4, we have an
induced Markov Hilbert evolution algebra $\mathcal{A}$ by taking $c_{ki} = p_{ik}$, for $i, k \in \mathbb{N} \cup \{0\}$. Moreover, we can check equation (17). Let us consider first $k = 0$. Note that,

$$
\sum_{i=1}^{\infty} p_{0i} = \sum_{i=1}^{\infty} \mathbb{P}(Z_{n+1} = 0|Z_n = i)
$$

$$
= \sum_{i=1}^{\infty} \mathbb{P} \left( \sum_{\ell=1}^{i} X_\ell = 0 \right)
$$

$$
= \sum_{i=1}^{\infty} \mathbb{P} \left( \bigcap_{\ell=1}^{i} \{X_\ell = 0\} \right)
$$

$$
= \sum_{i=1}^{\infty} p_{0i}^i
$$

$$
= p_0 \frac{1}{1 - p_0}.
$$

(19)

The second line in (19) is due to (18) and the independence between the $X_\ell$’s and $Z_n$. The fourth line is a consequence of the independence of the $X_\ell$’s. Now, for any $k \in \mathbb{N}$ note that, similarly to the first steps in (19), we have:

$$
\sum_{i=1}^{\infty} p_{ik} = \sum_{i=1}^{\infty} \mathbb{P} \left( \sum_{\ell=1}^{i} X_\ell = k \right).
$$

(20)

Moreover, $\{ \sum_{\ell=1}^{i} X_\ell = k \} \subset \{ \sum_{\ell=1}^{i} X_\ell \geq 1 \}$ and, if we consider $s \in (0, 1)$, note that the event $\{ \sum_{\ell=1}^{i} X_\ell \geq 1 \}$ is equivalent to the event $\{ s \sum_{\ell=1}^{i} X_\ell \geq s \}$. Thus, (20) and the previous comments imply, by Markov’s inequality:

$$
\sum_{i=1}^{\infty} p_{ik} \leq \sum_{i=1}^{\infty} s^{-1} \mathbb{E} \left( s \sum_{\ell=1}^{i} X_\ell \right).
$$

Since the $X_\ell$’s are independent and identically distributed random variables, if we denote by $\varphi(s)$ the common probability generating function, we have

$$
\mathbb{E} \left( s \sum_{\ell=1}^{i} X_\ell \right) = \mathbb{E} \left( s X_1 \right)^i = \varphi(s)^i.
$$

Therefore,

$$
\sum_{i=1}^{\infty} p_{ik} \leq s^{-1} \sum_{i=1}^{\infty} \varphi(s)^i,
$$

(21)

where $\varphi(s) \in [p_0, 1)$ provided $s \in (0, 1)$. Since $s$ can be arbitrarily chosen in $(0, 1)$, take $s = 1/2$, and note that from (19) and (21) we conclude

$$
\sum_{i=1}^{\infty} p_{ik} \leq \max \left\{ \frac{p_0}{1 - p_0}, \frac{2 \varphi(1/2)}{1 - \varphi(1/2)} \right\},
$$

for all $k \in \mathbb{N} \cup \{0\}$. Hence the equation (17) is satisfied, implying that the evolution operator $C : \mathcal{A}_X \rightarrow \mathcal{A}_X$ is bounded.

Theorem 4 gains in interest if we realize that it includes as corollaries [17, Theorem 16] and [17, Lemma 4]. In fact, our result covers all the discrete-time Markov chains with a finite state space, and a wide range of discrete-time Markov chains with infinite, but countable, state space. It is worth pointing out that still there exist some Markov chains for which (14) does not hold, as we illustrate in the following example.
Example 4 The house-of-cards Markov chain

Let \( \{X_n\}_{n \geq 0} \) be the Markov chain with state space given by \( \mathcal{X} = \mathbb{N} \cup \{0\} \) and transition probabilities given by \( p_{i0} = p_i > 0 \) and \( p_{i(i+1)} = 1 - p_i \), for any \( i \in \mathbb{N} \), and \( p_{00} = p_0 = 1 - p_0 \). See Figure 2 for an illustration of the transitions of such a Markov chain. In other words, from any state \( i \neq 0 \) the process jumps to 0 with probability \( p_i \) or it jumps to state \( i + 1 \) with probability \( 1 - p_i \). This model is known as the house-of-cards Markov chain. Notice that to state 0 we can go from infinitely many states with positive probability. For this chain we can find examples for which (14) holds, or not.

i. If we let \( \sum_{i=0}^{\infty} p_i = 1 \), then \( \alpha_i = \beta_i = 1 \) for any \( i \), and \( M_1 = M_2 = 1 \) is enough to satisfy (14). In fact, we would have for any \( i \in \mathbb{N} \cup \{0\} \)

\[
\sum_{k=0}^{\infty} p_{ik} \alpha_k = p_{i0} \alpha_0 + p_{i(i+1)} \alpha_{i+1} = p_i + (1 - p_i) = 1.
\]

Moreover, for any \( k \in \mathbb{N} \), we would have

\[
\sum_{i=0}^{\infty} p_{ik} \beta_i = p_{(k-1)k} \beta_{k-1} = (1 - p_{k-1}) < 1.
\]

while for \( k = 0 \) we would have

\[
\sum_{i=0}^{\infty} p_{i0} \beta_i = \sum_{i=0}^{\infty} p_i = 1,
\]

which completes the verification of (14).

ii. Let \( p_i = p \) for all \( i \in \mathbb{N} \cup \{0\} \), and suppose that (14) holds. Then, for any \( i \in \mathbb{N} \cup \{0\} \) we have

\[
\sum_{k=0}^{\infty} p_{ik} \alpha_k = p_{i0} \alpha_0 + p_{i(i+1)} \alpha_{i+1} = p \alpha_0 + (1 - p) \alpha_{i+1},
\]

which implies that

\[
p \alpha_0 + (1 - p) \alpha_{i+1} \leq M_1 \beta_i
\]

for all \( i \in \mathbb{N} \cup \{0\} \). On the other hand,

\[
\sum_{i=0}^{\infty} p_{i0} \beta_i = p \sum_{i=0}^{\infty} \beta_i,
\]

which implies by (14) that

\[
\sum_{i=0}^{\infty} \beta_i \leq M_2 \alpha_0 < \infty.
\]

Then \( \lim_{i \to \infty} \beta_i = 0 \), and this in turns implies from (22) that \( p \alpha_0 = 0 \), which is a contradiction.

As pointed out in [17, Lemma 4 of Chapter 4], and now extended to the cases where (14) holds, the evolution operator can be used in the context of evolution algebras as the transition matrix, whose entries are transition probabilities, is used in the context of Markov chains. This is the spirit of (15), which can be easily extended to any \( v \in \mathcal{A}_\mathcal{X} \) such that \( v = \sum_{i=1}^{\infty} \alpha_i e_i \), with \( \{\alpha_i\}_{i \in \mathbb{N}} \) being a probability distribution on \( \mathcal{X} \); i.e., \( \alpha_i \in [0, 1] \) for any \( i \in \mathbb{N} \) and \( \sum_{i \in \mathbb{N}} \alpha_i = 1 \). In other words, \( v \) is a (possibly infinite) convex combination of points of the orthonormal basis \( \{e_i\}_{i \in \mathbb{N}} \). Let us denote by \( \text{conv}(A) \) the closed convex hull of the set \( A \), that is, the closure of the convex hull \( \text{conv}(A) \).

Corollary 3 Consider a Markov Hilbert evolution algebra \( \mathcal{A}_\mathcal{X} \) satisfying the equations (14) and let \( v \in \text{conv}(\{e_i\}_{i \in \mathbb{N}}) \) such that \( v = \sum_{i=1}^{\infty} \alpha_i e_i \). Then

\[
\mathcal{C}^n(v) = \sum_{i=1}^{\infty} \alpha_i^n(v) e_i,
\]

where \( \alpha_i^n(v) = P(X_n = x_i) \) provided \( P(X_0 = x_k) = \alpha_k \), for \( k \in \mathbb{N} \).
Proof. Let \( v \in \text{conv}\left(\left\{ e_i \right\}_{i \in \mathbb{N}}\right) \) such that \( v = \sum_{i=1}^{\infty} \alpha_i e_i \), and assume that \( \mathbb{P}(X_0 = x_k) = \alpha_k \), for \( k \in \mathbb{N} \). The law of total probability implies

\[
\mathbb{P}(X_n = x_i) = \sum_{k=1}^{\infty} \mathbb{P}(X_n = x_i | X_0 = x_k) \mathbb{P}(X_0 = x_k) = \sum_{k=1}^{\infty} p_{ki}^{(n)} \alpha_k.
\]

The proof is finished if we let \( \alpha_i^n(v) := \mathbb{P}(X_n = x_i) \) and we realize that (15) implies

\[
C^n(v) = \sum_{k=1}^{\infty} \alpha_k C^n(e_k) = \sum_{k=1}^{\infty} \alpha_k \left\{ \sum_{i=1}^{\infty} p_{ki}^{(n)} e_i \right\} = \sum_{i=1}^{\infty} \left\{ \sum_{k=1}^{\infty} \alpha_k p_{ki}^{(n)} \right\} e_i.
\]

Let us finish with a comment about this connection with Markov chains. The previous results claim that it is possible to model some random phenomena with an approach of Hilbert evolution algebras. When one uses Markov chains the first step is to identify the state space of the process, the second one is to determine the transition probabilities. The conclusion of this section is that if we associate to each possible state of the process a generator of the algebraic structure, then the whole evolution of the process can be observed through consecutive applications of the evolution operator, provided (14), or (17) holds. Indeed, Corollary 3 can be applied by assuming that the application of \( C \) to \( v \in \text{conv}\left(\left\{ e_i \right\}_{i \in \mathbb{N}}\right) \) represents that the process starts from the state represented by \( e_i \) with probability \( \alpha_i \). Then, the application of the evolution operator \( n \) times allows us to discover with which probability the process will be in a given state at time \( n \). We point out that our extension of the concept of evolution algebra allows us to advance in the analysis proposed in [17, Chapter 4], extending it to a wide class of infinite-dimensional Markov chains.

Acknowledgements. Part of this work was carried out during a visit of P.C. at the Universidade Federal de Pernambuco (UFPE); and visits of P.M.R. at the Universidade Federal do ABC (UFABC) and at the Universidad Nacional de la Patagonia “San Juan Bosco” (UNPSJB). The visit at UNPSJB was during the realization of the School EMALCA 2019. The authors are grateful with these institutions, and with the organizers of the School, for their hospitality and support. Part of this work has been supported by Fundação de Amparo à Pesquisa do Estado de São Paulo-FAPESP (Grant 2017/10555-0).

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