Computing $A^1$-Euler numbers with Macaulay2

Sabrina Pauli

Abstract

We use Macaulay2 for several enriched counts in $GW(k)$. First, we compute the count of lines on a general cubic surface using Macaulay2 over $\mathbb{F}_p$ in $GW(\mathbb{P}^3)$ for $p$ a prime number and over $\mathbb{Q}$ in $GW(\mathbb{Q})$.

1 Introduction

In [KW17] Kass and Wickelgren count the lines on a smooth cubic surface as an element of the Grothendieck-Witt ring $GW(k)$ of a field $k$ by computing the $A^1$-Euler number of the vector bundle $E := \text{Sym}^3 \mathcal{S}^* \to \text{Gr}(2, 4)$ which is by definition the sum of the local indices, that is the local $A^1$-degrees, of the zeros of a general section. Here, $\text{Gr}(2, 4)$ denotes the Grassmannian of lines in $\mathbb{P}^3$ and $\mathcal{S} \to \text{Gr}(2, 4)$ its tautological bundle.

For a field $L$, denote by $\mathcal{E}_L$ the base change of $\mathcal{E}$ to $L$. Let $F \in \mathbb{F}_p[x_0, x_1, x_2, x_3]$ be a random homogeneous degree 3 polynomial in 4 variables. Then $F$ defines a general cubic surface $X = \{F = 0\} \subset \mathbb{P}^3_{\mathbb{F}_p}$ and a section $\sigma_F$ of $\mathcal{E}_{\mathbb{F}_p}$ by restriction. The zeros of $\sigma_F$ are the lines on $X$.

Let $A^4_{\mathbb{F}_p} = \text{Spec}(\mathbb{F}_p[x_1, x_2, x_3, x_4]) \subset \text{Gr}(2, 4)$ be the open affine subset of the Grassmannian consisting of the lines spanned by $x_1 e_1 + x_3 e_2 + e_3$ and $x_2 e_1 + x_4 e_2 + e_4$ where $(e_1, e_2, e_3, e_4)$ is the standard basis for $\mathbb{F}_p^4$. For the general cubic surface $X$, all lines on $X$ are elements of this open affine subset of $\text{Gr}(2, 4)$ and hence the $A^1$-Euler number $e_{A^4}(\mathcal{E}_{\mathbb{F}_p}) \in GW(\mathbb{F}_p)$ (or the count of lines on the cubic surface $X$) can be computed as the sum of local $A^1$-degrees of the zeros of $\sigma_F|_{A^4} = (f_1, f_2, f_3, f_4) : A^4 \to A^4$ by [KW19].

The $\mathbb{F}_p$-algebra $\mathbb{F}_p[x_1, x_2, x_3, x_4]/(f_1, f_2, f_3, f_4)$ is 0 dimensional and thus there are finitely many lines on $X$. Call these lines $l_1, \ldots, l_n$. By [KW17] Corollary 51 the lines on a general and thus smooth cubic surface are simple. This means that the lines $l_1, \ldots, l_n$ are simple zeros of $(f_1, f_2, f_3, f_4) : A^4_{\mathbb{F}_p} \to A^4_{\mathbb{F}_p}$. It follows that $\mathbb{F}_p[x_1, x_2, x_3, x_4]/I$ is isomorphic to the product of fields $F_1 \times \cdots \times F_n$ where $F_j = \mathbb{F}_p[x_1, x_2, x_3, x_4]/m_j$ is the field of definition of $l_j$ (that is residue field of the point in $\text{Gr}(2, 4)$ corresponding to $l_j$) for $j = 1, \ldots, n$. By [KW19] Lemma 9 the local index at $l_j$ is equal to $\langle J_{F_j} \rangle \in GW(F_j)$ where $J_{F_j}$ is the image of the jacobian element $J := \text{det} \frac{\partial f_i}{\partial x_i}$ in $F_j = \mathbb{F}_p[x, y, z, w]/m_j$ and it follows that the $A^1$-Euler number of $\text{Sym}^3 \mathcal{S}^* \to \text{Gr}(2, 4)$ is given by

$$e_{A^4}(\mathcal{E}_{\mathbb{F}_p}) = \sum_{j=1}^n \text{Tr}_{F_j/\mathbb{F}_p}(\langle J_{F_j} \rangle) \in GW(\mathbb{F}_p).$$

(1)
We use Macaulay2 to compute the rank and discriminant of $\mathbb{1}$ when $p = 32003$. The computation gives an element in $\text{GW}(\mathbb{F}_p)$ of rank 27 and discriminant $1 \in \mathbb{F}_p^*/(\mathbb{F}_p^*)^2$. Two elements in $\text{GW}(\mathbb{F}_p)$ are equal if and only if they have the same rank and discriminant, so this determines the count of lines on a cubic surface in $\text{GW}(\mathbb{F}_p)$ completely.

Similarly, we use Macaulay2 to get the Gram matrix of a form representing $e^{A^1}(E_Q) \in \text{GW}(\mathbb{Q})$ over the rational numbers $\mathbb{Q}$. We view this form as a bilinear form over the real numbers $\mathbb{R}$ and compute its signature which is equal to 3.

By Theorem 5.8 in [BW20], $e^{A^1}(E_Q) = e^{A^1}(\text{Sym}^3 S^*)$ is equal to either

$$\frac{n_C + n_R}{2} (1) + \frac{n_C - n_R}{2} (-1) \in \text{GW}(k)$$

or

$$\frac{n_C + n_R}{2} (1) + \frac{n_C - n_R}{2} (-1) + \langle 2 \rangle - \langle 1 \rangle \in \text{GW}(k)$$

for $n_C, n_R \in \mathbb{Z}$ and a field $k$. By [BW20, Remark 5.7] $n_C$ and $n_R$ are the Euler numbers of the real and complex bundle, respectively. The complex count $n_C$ is equal to the rank of our form which is $n_C = 27$, and the real count is equal to the signature, so $n_R = 3$. This has already been known for a long time. The complex count $n_C$ is the classical result by Cayley and Salmon that there are 27 complex lines on a smooth cubic surface [Cay09]. Segre studied the real lines on a smooth cubic surface in [Seg42]. See also [FK15] and [OT14] for the real count.

Since 2 is not a square for every prime $p$, in particular not for our chosen prime 32003, we can rule out (3) for the count of lines on a cubic surface and hence we have a new proof of the fact that

$$e^{A^1}(\text{Sym}^3 S^*) = 15(1) + 12(-1) \in \text{GW}(k)$$

which is the main result in [KW17].

In [LEV19, §8] and [BW20, Corollary] it is shown that the $A^1$-Euler number of direct sums of symmetric powers of the dual tautological bundle on a Grassmannian is always of form (2) when defined, using the theory of Witt-valued characteristic classes. The proof here is independent of this theory and we may also apply it to bundles which are not of this form.

Similarly, we get an enriched count of lines meeting 4 general lines in $\mathbb{P}^3$ (this has already been computed in [SW18]) and of lines on a quadratic surface meeting one general line by computing the $A^1$-Euler numbers $e^{A^1}(\bigoplus_{i=1}^4 \wedge^2 S^* \to \text{Gr}(2, 4))$ and $e^{A^1}(\wedge^2 S^* \oplus \text{Sym}^2 S^* \to \text{Gr}(2, 4))$, respectively. Note, that neither of these vector bundles is a direct sum of symmetric powers of the dual tautological bundle and we cannot use [LEV19, §8] and [BW20, Corollary] to rule out (3). However, we already know that the $A^1$-Euler number of both of these bundles will be a multiple of the hyperbolic form $H = \langle 1 \rangle + \langle -1 \rangle$ since they have direct summands of odd rank [SW18, Proposition 12].

Furthermore, we count singular elements on a pencil of degree $d$ surfaces as the $A^1$-Euler number of $\bigoplus_{i=1}^4 \pi_2^*(d - 1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2}(1) \to \mathbb{P}^3 \times \mathbb{P}^1$.

Finally, we provide code for computing the EKL-form (see [KW19]) which computes the local $A^1$-degree for non-simple zeros.

In the appendices we compute the $A^1$-Milnor numbers of several Fuchsian singularities and provide one
explicit example of the Gram matrix of a form representing $e^{2\mathbb{A}}(\mathcal{E}_{11}) \in \text{GW}(\mathbb{F}_{11})$.

2.1 Definition of the $\mathbb{A}^1$-Euler number

Let $k$ be a field. We recall the definition of the $\mathbb{A}^1$-Euler number from [KW17] §4. Let $\pi : E \to X$ be a vector bundle of rank $r$ over a smooth scheme $X$ over $k$. Recall that a (weak) orientation of $E$ is an isomorphism $\phi : \det E \cong L \otimes 2$ where $L \to X$ is a line bundle.

Definition 2.1. A relative orientation of $E$ is an orientation of the line bundle $\text{Hom}(\det TX, \det E)$, that is, an isomorphism $\phi : \text{Hom}(\det TX, \det E) \cong (\det TX)^{-1} \otimes \det E$. However, $\pi : E \to X$ can still be relatively orientable even though $E$ and $TX$ are not orientable.

Let $\pi : E \to X$ be a vector bundle over a smooth and proper scheme $X$ over $k$ equipped with a relative orientation $\phi$ and assume that $\dim X = rk E = r$. In our examples, $X$ is either a Grassmannian of lines or projective space which both have standard coverings by open affine spaces $U \cong \mathbb{A}^r$. An open affine subset $U \cong \mathbb{A}^r$ of $X$ defines local coordinates, that is, a trivialization of $TX|_U$.

Remark 2.2. If both the tangent bundle of $X$ and $E$ are orientable, then $E$ is relatively orientable since $\text{Hom}(\det TX, \det E) \cong (\det TX)^{-1} \otimes \det E$. However, $\pi : E \to X$ can still be relatively orientable even though $E$ and $TX$ are not orientable.

Let $\pi : E \to X$ be a vector bundle over a smooth and proper scheme $X$ over $k$ equipped with a relative orientation $\phi$ and assume that $\dim X = rk E = r$. In our examples, $X$ is either a Grassmannian of lines or projective space which both have standard coverings by open affine spaces $U \cong \mathbb{A}^r$. An open affine subset $U \cong \mathbb{A}^r$ of $X$ defines local coordinates, that is, a trivialization of $TX|_U$.

Definition 2.4. A trivialization of $E|_U$ with $U \cong \mathbb{A}^r$ is compatible with the relative orientation $\phi$ and the local coordinates on $X$ if the element of $\text{Hom}(\det TX|_U, \det E|_U)$ sending the distinguished basis element of $\det TX|_U$ to the distinguished element of $\det E|_U$, is sent to a square by $\phi$.

Let $\sigma : X \to E$ be a section of $E$ with an isolated zero $x \in X$. Choose a neighborhood $x \in U \cong \mathbb{A}^r$ and a trivialization $E|_U \cong \mathbb{A}^r \times \mathbb{A}^r$ compatible with $\phi$ and the local coordinates defined by $U \cong \mathbb{A}^r$. Then the local index $\text{ind}_x \sigma$ at $x$ is the local $\mathbb{A}^1$-degree $\deg_{x}^{A^1}(f_1, \ldots, f_r)$ of $(f_1, \ldots, f_r) : \mathbb{A}^r \to \mathbb{A}^r$ at $x$ where

$$(f_1, \ldots, f_r) : \mathbb{A}^r \xrightarrow{\sigma_1 \circ \cdots \circ \sigma_r} (\text{id} \circ (f_1, \ldots, f_r)) : \mathbb{A}^r \times \mathbb{A}^r \xrightarrow{\pi_2} \mathbb{A}^r.$$ 

The local $\mathbb{A}^1$-degree is the analog of the local degree in $\mathbb{A}^1$-homotopy theory. We do not use its formal definition and refer the interested reader to [KW19] §2 for the definition and to [WW19] for an introduction to $\mathbb{A}^1$-homotopy theory.

Let $L/k$ be a finite separable field extension and let $\beta : V \times V \to L$ be a non-degenerate symmetric bilinear form over $L$. Then the trace form $\text{Tr}_{L/k}(\beta)$ is the form

$$V \times V \xrightarrow{\beta} L \xrightarrow{\text{Tr}_{L/k}} k \quad (5)$$
where Tr_{L/k} denotes the field trace. Assume \( x \in X \) is simple zero, that is the jacobian element \( J(x) := \frac{\partial F}{\partial x}(x) \) at \( x \) is non-zero. If \( x \) is a rational point, its local degree is equal to \( \langle J(x) \rangle \in GW(k) \). When \( x \) is not rational, its local \( A^1 \)-degree can be computed as the trace form \( \text{Tr}_{k(x)/k}(\langle J(x) \rangle) \in GW(k) \) of \( \langle J(x) \rangle \in GW(k(x)) \) for finite separable field extensions \( k(x)/k \) by [KW19].

**Remark 2.5.** When \( x \in X \) is a non-simple zero, its local \( A^1 \)-degree can be computed as the EKL-form (see section 3).

We define the \( A^1 \)-Euler number \( e^{A^1}(E, \sigma) \) with respect to a section \( \sigma : X \to E \) with only isolated zeros to be sum of local indices of the zeros of \( \sigma \). It turns out that \( e^{A^1}(E, \sigma) \) does not depend on the chosen section [BW20] Theorem 1.1 and we can define the \( A^1 \)-Euler number independently of \( \sigma \).

**Definition 2.6.** Let \( \pi : E \to X \) be a vector bundle of rank \( r \) equal to the dimension of the smooth, proper scheme \( X \) over a field \( k \) equipped with a relative orientation, then the \( A^1 \)-Euler number is defined by \( e^{A^1}(E) := e^{A^1}(E, \sigma) \) for a section \( \sigma \) with only isolated zeros.

### 2.2 Cubic Surfaces

We compute the rank and discriminant of the \( A^1 \)-Euler number of \( E = \text{Sym}^3 S^* \to \text{Gr}(2,4) \) over \( \mathbb{F}_p \) with \( p = 32003 \).

\[
\begin{align*}
P &= 32003 \\
\mathbb{F} &= \mathbb{Z}/P \\
\end{align*}
\]

We generate a random homogeneous degree 3 polynomial \( F \) in 4 variables \( X_0, X_1, X_2 \) and \( X_3 \).

\[
\begin{align*}
R &= \mathbb{F}[X_0, X_1, X_2, X_3] \\
F &= \text{random}(3, R) \\
\end{align*}
\]

We replace \( X_0, X_1, X_2 \) and \( X_3 \) by \( x_1 s + x_2, x_3 s + x_4, s \) and 1, respectively, and define \( I \) to be the ideal in \( C = \mathbb{F}_p[x_1, x_2, x_3, x_4] \) generated by the coefficients \( s^3, s^2, s \) and 1 of \( F(x_1 s + x_2, x_3 s + x_4, s, 1) \). That means, we let \( \text{Spec} \ C = \text{Spec}(\mathbb{F}_p[x_1, x_2, x_3, x_4]) \subset \text{Gr}(2,4) \) be the open affine subset consisting of the lines spanned by \( x_1 e_1 + x_2 e_2 + e_3 \) and \( x_2 e_1 + x_1 e_2 + e_4 \) for the standard basis \( (e_1, e_2, e_3, e_4) \) of \( \mathbb{F}_p^4 \). The monomials \( s^3, s^2, s \) and 1 define a basis of \( E|_{\text{Spec} C} \) and thus a trivialization. This means \( I \) is the ideal generated by \( f_1, f_2, f_3, f_4 \) where

\[
(f_1, f_2, f_3, f_4) : A^4 \xrightarrow{\sigma|_{A^4}} A^4 \times A^4 \xrightarrow{\pi_2} A^4,
\]

is equal to the restriction of the section \( \sigma_F \) of \( E \) defined by \( F \) to the chosen open affine subset \( \text{Spec} \ C = \text{Spec}(\mathbb{F}_p[x_1, x_2, x_3, x_4]) \) in the chosen trivialization of \( E|_{\text{Spec} C} \).

**Remark 2.7.** By [KW17] Corollary 45] the vector bundle \( E \) is relatively orientable and the local coordinates defined by the subset \( \text{Spec} \ C \subset \text{Gr}(2,4) \) and the trivialization \( E|_{U} \) defined above, are compatible with this relative orientation.

\[
\begin{align*}
C &= \mathbb{F}[x_1, x_2, x_3, x_4] \\
S &= C[s] \\
g &= \{x_1 s + x_2, x_3 s + x_4, s, 1\}
\end{align*}
\]
\[ m = \text{map}(S, R, g) \]
\[ I = \text{sub(ideal flatten entries last coefficients m F, C)} \]

We compute the dimension and degree \( \text{rk} \) of \( C/I = \mathbb{F}_p[x_1, x_2, x_3, x_4]/I \).

\[
\text{dim } I \\
\text{rk} = \text{degree } I
\]

Since there are in general finitely many lines on a cubic surface, the expected dimension of \( C/I \) is 0. The degree gives the rank \( \text{rk} \) of (1) which turns out to be 27 as expected.

\[ i10 : \text{dim } I \]
\[ o10 = 0 \]

\[ i11 : \text{rk} = \text{degree } I \]
\[ o11 = 27 \]

So \( C/I \) is a finite \( \mathbb{F}_p \)-algebra of rank 27. We know that the finitely many lines \( l_1, \ldots, l_n \) on \( \{ F = 0 \} \subset \mathbb{P}^3 \) are simple zeros of \( (f_1, f_2, f_3, f_4) : \mathbb{A}^4_{\mathbb{F}_p} \rightarrow \mathbb{A}^4_{\mathbb{F}_p} \) [KW17 Corollary 51].

This implies that \( C/I = \mathbb{F}_p[x_1, x_2, x_3, x_4]/I \) is isomorphic to the product of fields

\[
\mathbb{F}_p[x_1, x_2, x_3, x_4]/\mathfrak{m}_1 \times \cdots \times \mathbb{F}_p[x_1, x_2, x_3, x_4]/\mathfrak{m}_n = F_1 \times \cdots \times F_n
\]

where \( \mathfrak{m}_i \) is maximal ideal defining \( l_i \) as point in \( \text{Gr}(2, 4) \) and \( F_i \) is the field of definition of \( l_i \), i.e., the residue field of \( l_i \) in \( \text{Gr}(2, 4) \), for \( i = 1, \ldots, n \). We use a primary decomposition of \( I \) to find the \( \mathfrak{m}_i \).

\[ L = \text{primaryDecomposition } I \]
\[ n = \text{length } L \]

**Remark 2.8.** Since the ideals \( \mathfrak{m}_i \) are actually primes, the primary ideals in the primary decomposition are the (unique) minimal primes and we can let Macaulay2 compute the minimal primes instead of the primary decomposition of \( I \). This is much more time efficient. However, for a non-smooth cubic surface, \( (f_1, f_2, f_3, f_4) : \mathbb{A}^4_{\mathbb{F}_p} \rightarrow \mathbb{A}^4_{\mathbb{F}_p} \) could have non-simple zeros and we would need to use the primary decomposition and the EKL-form (see section 3) for the non-reduced factors of \( C/I \) to find the count of lines on the cubic surface.

The contribution of the line \( l_i \) to (1) is \( \text{Tr}_{F_i/\mathbb{F}_p}((J_{F_i})) \) where \( J_{F_i} \) is the image of the jacobian element \( J = \det \frac{\partial f_i}{\partial x_j} \) of \( I \) in \( F_i = C/\mathfrak{m}_i \). The discriminant of (1) is the product of the discriminants of the forms \( \text{Tr}_{F_i/\mathbb{F}_p}((J_{F_i})) \). By [KW17 Lemma 58] the discriminant of \( \text{Tr}_{F_i/\mathbb{F}_p}((J_{F_i})) \) is a square in \( \mathbb{F}_p \) if \( J_{F_i} \) is a square in \( F_i = \mathbb{F}_p[x_1, x_2, x_3, x_4]/\mathfrak{m}_i \) when the degree \( [F_i : \mathbb{F}_p] \) is odd and if \( J_{F_i} \) is a non-square in \( F_i = \mathbb{F}_p[x_1, x_2, x_3, x_4]/\mathfrak{m}_i \) when \( [F_i : \mathbb{F}_p] \) is even. Since the units \( \mathbb{F}_q^* \) of a finite field \( \mathbb{F}_q \) with \( q \) elements form the cyclic group of order \( q - 1 \), \( \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). By Fermat’s little theorem \( b^{q-1} \equiv 1 \mod q \) for
If \( b \in \mathbb{F}_q^* \) and \( b \) is a square if and only if \( b^{\frac{q-1}{2}} \equiv 1 \mod q \). So to find the discriminant of \( \text{1} \) we compute the product

\[
\text{disc(1)} = \prod_{i=1}^{n} \epsilon_i J_{F_i}^{-1}
\]

where \( \epsilon = -1 \) when \( [F_i : \mathbb{F}_p] \) is even and \( \epsilon = 1 \) when \( [F_i : \mathbb{F}_p] \) is odd.

\[
J = \text{determinant jacobian I}
\]

\[
disc = 1_{\text{FF}}; i=0; \text{while } i < n \text{ do}
\]

(if even degree \( L_i \)

then

\[
disc = \text{disc*lift}(J_{(C/L_i)}^{-1}(\langle P^\text{(degree L_i)} - 1 \rangle/2), \text{FF})*(-1)_{\text{FF}}
\]

else

\[
disc = \text{disc*lift}(J_{(C/L_i)}^{-1}(\langle P^\text{(degree L_i)} - 1 \rangle/2), \text{FF}); i=i+1
\]

The discriminant of \( \text{1} \) is a square.

\[
i17 : \text{disc}
\]

\[
o17 = 1
\]

### 2.3 The trace form

The trace form \([5]\) can also be defined when \( L \) is a finite étale \( k \)-algebra, i.e., a product \( L \cong L_1 \times \cdots \times L_s \) of finitely many finite separable field extensions \( L_1, \ldots, L_s \) of \( k \). For example \( C/I = \mathbb{F}_p[x_1, x_2, x_3, x_4]_{(f_1, f_2, f_3, f_4)} \) is a finite étale algebra isomorphic to \( F_1 \times \cdots \times F_n \) and the trace form \( \text{Tr}_C/I/F_p(\langle J_{C/I} \rangle) \) is equal to \( \sum_{i=1}^{n} \text{Tr}_{F_i/F_p}(J_{F_i}) \) with the notation from subsection \([22]\) and where \( J_{C/I} \) is the image of the jacobian element \( J \) in \( C/I \).

The following code computes the trace form \( \text{Tr}_{L/k}(\langle J_{C/I} \rangle) \) where \( FF \) is a field and \( I \) an ideal in a polynomial ring \( C \) over \( FF \) such that \( C/I \) is a finite étale algebra over \( FF \) and \( J \in C \).

\[
\text{traceForm} = (C,I,J,FF) \rightarrow ( \text{B:=basis(C/I);} \text{r:=degree I;} \text{Q:=(J(C/I))*transpose B)*B;} \text{toVector := q \rightarrow last coefficients(q,Monomials=>B);} \text{fieldTrace := q \rightarrow (M:=toVector(q*B_(0,0)));i=1;while i<r do (M=M|(toVector (q*B_(0,i))); i=i+1); trace M);} \text{matrix applyTable(entries Q, q->lift(fieldTrace q,FF))})
\]

#### 2.3.1 Lines meeting four general lines in \( \mathbb{P}^3 \)

As an example we compute the count of lines meeting 4 general lines in \( \mathbb{P}^3 \), i.e., we compute the \( \mathbb{A}^1 \)-Euler number of the bundle \( \mathcal{E}_2 := \wedge^2 S^* \oplus \wedge^2 S^* \oplus \wedge^2 S^* \oplus \wedge^2 S^* \rightarrow \text{Gr}(2, 4) \). We know from \([SW18]\) that this is equal to the hyperbolic form \( \mathbb{H} := (1) + (-1) \).
Clearly, \( \det(\wedge^2 S^* \oplus \wedge^2 S^* \oplus \wedge^2 S^* \oplus \wedge^2 S^*) \cong (\wedge^2 S^*)^\otimes 4 \) and thus the vector bundle \( \mathcal{E}_2 \) is orientable. The Grassmannian \( \text{Gr}(2, 4) \) is orientable as well (i.e., its tangent bundle \( T\text{Gr}(2, 4) \cong S^* \otimes Q \) is orientable). Those two orientations yield a canonical relative orientation \( \phi : \text{Hom}(T\text{Gr}(2, 4), \mathcal{E}_2) \cong L^\otimes 2 \) of \( \mathcal{E}_2 \). Over the open affine subset \( \text{Spec}(\mathbb{F}_p[x_1, x_2, x_3, x_4]) \subset \text{Gr}(2, 4) \) from subsection 2.2, the dual tautological bundle \( S^* \to \text{Gr}(2, 4) \) has basis the two monomials \( s \) and 1 (where \( s \) is again the variable on the line). This basis induces a trivialization of the restriction of \( \mathcal{E}_2 \) to \( \text{Spec}(\mathbb{F}_p[x_1, x_2, x_3, x_4]) \). By [SW18, Lemma 4] the relative orientation \( \phi \) is compatible with the chosen local coordinates and trivialization of \( \mathcal{E} \).

Let \( l_1, \ldots, l_4 \) be 4 general lines in \( \mathbb{P}^3 \) and let \( a_i, b_i \) be two independent linear forms cutting out \( l_i \) for \( i = 1, \ldots, 4 \).

\[
\begin{align*}
a_1 &= \text{random}(1, R) \\
b_1 &= \text{random}(1, R) \\
a_2 &= \text{random}(1, R) \\
b_2 &= \text{random}(1, R) \\
a_3 &= \text{random}(1, R) \\
b_3 &= \text{random}(1, R) \\
a_4 &= \text{random}(1, R) \\
b_4 &= \text{random}(1, R)
\end{align*}
\]

The linear forms \( a_i \) and \( b_i \) define a section \( s_i := a_i \wedge b_i \) of \( \wedge^2 S^* \). A line \( l \) in \( \mathbb{P}^3 \) meets the line \( l_i \) if and only if \( s_i(l) = 0 \) by [SW18, Lemma 5].

\[
\begin{align*}
s_1 &= \text{lift}((\text{last coefficients } m a_1)_{(0,0)}(\text{last coefficients } m b_1)_{(1,0)} \\
&- (\text{last coefficients } m a_1)_{(1,0)}(\text{last coefficients } m b_1)_{(0,0)}, C) \\
s_2 &= \text{lift}((\text{last coefficients } m a_2)_{(0,0)}(\text{last coefficients } m b_2)_{(1,0)} \\
&- (\text{last coefficients } m a_2)_{(1,0)}(\text{last coefficients } m b_2)_{(0,0)}, C) \\
s_3 &= \text{lift}((\text{last coefficients } m a_3)_{(0,0)}(\text{last coefficients } m b_3)_{(1,0)} \\
&- (\text{last coefficients } m a_3)_{(1,0)}(\text{last coefficients } m b_3)_{(0,0)}, C) \\
s_4 &= \text{lift}((\text{last coefficients } m a_4)_{(0,0)}(\text{last coefficients } m b_4)_{(1,0)} \\
&- (\text{last coefficients } m a_4)_{(1,0)}(\text{last coefficients } m b_4)_{(0,0)}, C)
\end{align*}
\]

\[
I_2 = \text{ideal}(s_1, s_2, s_3, s_4) \\
J_2 = \text{determinant jacobian } I_2 \\
\text{traceForm}(C, I_2, J_2, \mathbb{F}^p)
\]

Let \( I_2 \) be the ideal generated by the sections \( s_1, \ldots, s_4 \) and \( J_2 := \text{det} \frac{\partial s_i}{\partial x_j} \). We compute the trace form \( \text{Tr}(C/\mathbb{F}^p)(J_2) \) (where \( C \) is still \( \mathbb{F}_p[x_1, x_2, x_3, x_4] \)) and get a form of rank 2 and discriminant \( -1 \in \mathbb{F}_p^*/(\mathbb{F}_p)^2 \) as expected.

### 2.3.2 Lines on a degree 2 hypersurface in \( \mathbb{P}^3 \) meeting 1 general line

We compute the count of lines on a quadratic surface meeting a general line as the \( \mathbb{A}^1 \)-Euler number of \( \mathcal{E}_2 := \text{Sym}^2 S^* \oplus \wedge^2 S^* \to \text{Gr}(2, 4) \).

There is a canonical isomorphism \( \det(\text{Sym}^2 S^* \oplus \wedge^2 S^*) \cong (\det S^*)^\otimes 4 \). So \( \mathcal{E}_3 \) is orientable. Since \( \text{Gr}(2, 4) \) is orientable, too (there are canonical isomorphisms \( \det T\text{Gr}(2, 4) \cong \det(S^* \otimes Q) \cong (\det S^*)^\otimes 2 \otimes (\det Q)^\otimes 2 \),
we get a canonical relative orientation on $E_3$. We choose the same local coordinates on $\text{Gr}(2, 4)$ as above and find a trivialization of $E_3|_{\text{Spec } \mathbb{C}}$: As in [KW17, Definition 39] we define a basis $\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = x_1e_1 + x_3e_2 + e_3$ and $\tilde{e}_4 = x_2e_1 + x_4e_2 + e_4$ of $\mathbb{F}_p[x_1, x_2, x_3, x_4]^4$, and let $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\tilde{\phi}_3$ and $\tilde{\phi}_4$ be its dual basis. Here $e_1, e_2, e_3, e_4$ is a basis of $\mathbb{F}_4^4$. Then the open affine subset of lines spanned by $x_1e_1 + x_3e_2 + e_3$ and $x_2e_1 + x_4e_2 + e_4$, $U = \text{Spec } \mathbb{F}_p[x_1, x_2, x_3, x_4] \subset \text{Gr}(2, 4)$, yields a basis

$$\tilde{\phi}_3 \otimes \tilde{e}_1, \tilde{\phi}_4 \otimes \tilde{e}_1, \tilde{\phi}_3 \otimes \tilde{e}_2, \tilde{\phi}_4 \otimes \tilde{e}_2$$

(6)

of $T\text{Gr}(2, 4)|_U$ and a basis

$$(\tilde{\phi}_3^2, 0), (\tilde{\phi}_3\tilde{\phi}_4, 0), (\tilde{\phi}_4^2, 0), (0, \tilde{\phi}_3 \wedge \tilde{\phi}_4)$$

(7)

of $E_2|_U$.

**Lemma 2.9.** The coordinates defined by the basis (6) (which are equal to the coordinates chosen in subsection 2.2) and the trivialization of $E_3|_U$ defined by the basis (7) are compatible with the canonical relative orientation of $E_3$ described above.

**Proof.** The wedge product of (6) is

$$(\tilde{\phi}_3 \wedge \tilde{\phi}_4 \otimes \tilde{e}_1 \wedge \tilde{e}_2)^{\otimes 2} \in (\det S^* \otimes \det Q)^{\otimes 2}|_U$$

and the wedge product of (7) is

$$(\tilde{\phi}_3 \wedge \tilde{\phi}_4)^{\otimes 4} \in (\det S^*)^{\otimes 4}|_U$$

which are both squares. It follows that

$$(\tilde{\phi}_3 \wedge \tilde{\phi}_4 \otimes \tilde{e}_1 \wedge \tilde{e}_2)^{\otimes -2} \otimes (\tilde{\phi}_3 \wedge \tilde{\phi}_4)^{\otimes 4} \in ((\det S^* \otimes \det Q)^{\otimes -2} \otimes (\det S^*)^{\otimes 4})|_U$$

is a square, too. \hfill \Box

We compute $e^{\mathbb{H}}(E_3)$.

F2 = random(2,R)

a5 = random(1,R)

b5 = random(1,R)

s5 = lift((last coefficients m a5)_<(0,0)*

(last coefficients m b5)_<(1,0)

-(last coefficients m a5)_<(1,0)*

(last coefficients m b5)_<(0,0),C)

Q = sub(ideal flatten entries last coefficients m F2, C)

I3 = Q+ideal(s5)

J3 = determinant jacobian I3

traceForm(C,I3,J3,FF)

It is a rank 4 form of discriminant $1 \in \mathbb{F}_p^*/(\mathbb{F}_p^*)^2$. When we compute the form over the real numbers $\mathbb{R}$ (this can be done similarly as in subsubsection 2.3.3), we get a form of signature 0. Hence, we can use [BW20, Theorem 5.8] to conclude that $e^{\mathbb{H}}(E_3) = 2\mathbb{H}$. 8
**Remark 2.10.** We know already from [SW18, Proposition 12] that we get a multiple of $H = (1) + (-1)$ for the $A_1$-Euler number $e^{A_1}(E \oplus E')$ when the rank of $E$ or $E'$ is odd.

### 2.3.3 Signature of $e^{A_1}(E) = e^{A_1}(\text{Sym}^3 S_\ast)$

Let $G$ be a random degree 3 homogeneous polynomial in 4 variables with coefficients in $\mathbb{Q}$. We now compute $e^{A_1}(E_Q, \sigma_G) \in GW(\mathbb{Q})$ where $\sigma_G$ is the section of $E$ defined by $G$. Base change yields a form over $\mathbb{R}$ of which we compute the signature as the number of positive eigenvalues minus the negative eigenvalues.

```plaintext
R2 = QQ[Y0,Y1,Y2,Y3]
G=random(3,R2)
```

Exactly as before, we restrict $\sigma_G : \text{Gr}(2,4) \to \text{Sym}^3 S^\ast$ to

```plaintext
Spec C2 := Spec(Q[y1, y2, y3, y4]) \subset \text{Gr}(2,4)
```

and get $(g_1, g_2, g_3, g_4) : A_4^4 \to A_4^4$ and let $I_4 = (g_1, g_2, g_3, g_4)$.

```plaintext
C2 = QQ[y1, y2, y3, y4]
S2 = C2[r]
g2 = {y1*r+y2, y3*r+y4, r, 1}
m2 = map(S2, R2, g2)
I4 = sub(ideal flatten entries last coefficients m2 G, C2)
J4 = determinant jacobian I4
```

We compute the trace form $\text{Tr}(C2/I4)/\mathbb{Q}$ where $J_4$ is the jacobian element of $I_4$, and get a $27 \times 27$-matrix with values in $\mathbb{Q}$. Viewing it as a form over $\mathbb{R}$, its signature is equal to the number of positive eigenvalues minus the number of negative eigenvalues because any real symmetric matrix can be diagonalized orthogonally.

```plaintext
Sol=traceForm(C2,I4,J4,QQ)
E=eigenvalues Sol
sgn=0; i=0;
while i<rk do(if E_i<0 then sgn=sgn-1 else sgn=sgn+1; i=i+1)
sgn
```

The signature is 3.

```plaintext
i53 : sgn
```

```plaintext
o53 = 3
```

So we know that the signature of $e^{A_1}(E)$ is $n_\mathbb{R} = 3$ and its rank $n_\mathbb{C} = 27$. Since the discriminant of $e^{A_1}(E_p) \in GW(F_p)$ with $p = 32003$ is a square (and 2 is not a square in $F_p$), we can conclude that

$$e^{A_1}(E) = e^{A_1}(\text{Sym}^3 S^\ast) = 15(1) + 12(-1)$$

applying [BW20, Theorem 5.8].
2.3.4 Singular elements on a pencil of degree $d$ hypersurfaces in $\mathbb{P}^3$

Let $\{F_t = t_0 F_0 + t_1 F_1 = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^1$ be a pencil of degree $d$ surfaces in $\mathbb{P}^3$. A surface in the pencil is singular if there is a point on the surface on which all 4 partial derivatives vanish simultaneously. Consider the vector bundle $F := \bigoplus_{i=1}^4 \pi_1^*(\mathcal{O}_{\mathbb{P}^3}(d-1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^3 \times \mathbb{P}^1$ where $\pi_1 : \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^3$ and $\pi_2 : \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1$ are the projections to the first and second factor, respectively. A pencil $X_t = \{F_t = t_0 F_0 + t_1 F_1 = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^1$ defines a section $\sigma = (\frac{\partial F_t}{\partial X_0}, \ldots, \frac{\partial F_t}{\partial X_3})$ of this bundle where $X_0, \ldots, X_3$ are the coordinates on $\mathbb{P}^3$. A general singular hypersurface of degree $d$ has a unique singularity which is an ordinary double point by [EH16, Proposition 7.1 (b)] and, whence, the zeros of $\sigma$ are simple and count the singular elements on the pencil $X_t$. The bundle $F$ is relatively orientable since $\mathbb{P}^3 \times \mathbb{P}^1$ are orientable, and we can enrich the count of singular elements on the pencil over $\text{GW}(k)$.

Let $\mathbb{A}^3 \cong U_0 \subset \mathbb{P}^3$ and $\mathbb{A}^1 \cong V_0 \subset \mathbb{P}^1$ be the open affine subsets where the first variable does not vanish and let $\mathbb{A}^4 \cong U := U_0 \times V_0 \subset \mathbb{P}^3 \times \mathbb{P}^1$. One can show that $U$ and the evident trivialization of $F|_U$ are compatible with the relative orientation of $F$ in the same manner as in [McK20, Lemma 3.10].

Example 2.11. We provide the code for $d = 2$ over the field $\mathbb{F}_p$.

```
F0 = random(2,R)
F1 = random(2,R)
T = R[t]
Ft = F0+t*F1
D0 = diff(X0, Ft)
D1 = diff(X1, Ft)
D2 = diff(X2, Ft)
D3 = diff(X3, Ft)
C3 = FF[x1,x2,x3,t]
m3 = map(C3,T,{t,1,x1,x2,x3})
I5 = ideal(m3 D0,m3 D1,m3 D2,m3 D3)
J5 = determinant jacobian I5
traceForm(C3,I5,J5,FF)
```

For the enriched count of singular elements on a pencil of degree 2 surfaces in $\mathbb{P}^3$ we get a form of rank 4, discriminant $1 \in \mathbb{F}_p^*/(\mathbb{F}_p^*)^2$ and signature 0, that is the form $2\mathbb{H}$. For $d = 3$, i.e., the enriched count of singular elements on a pencil of cubic surfaces, we get $16\mathbb{H}$ and for $d = 4$, $54\mathbb{H}$.

Remark 2.12. Again we know by [SW18, Proposition 12] that we get a multiple of the hyperbolic form $\mathbb{H} = \langle 1 \rangle + \langle -1 \rangle$.

Remark 2.13. Proposition 7.4 in [EH16] computes the number of singular elements on a pencil of degree $d$ hypersurfaces in $\mathbb{P}^n$ over the complex numbers to be $(n+1)(d-1)^n$. Whenever $n$ is odd, the corresponding bundle is relatively orientable, and the count can be enriched in $\text{GW}(k)$ to the form $\frac{(n+1)(d-1)^n}{2}\mathbb{H}$ by [SW18, Proposition 12]. One checks that this coincides with our count for $n = 3$ and $d = 2, 3, 4$.

Remark 2.14. Levine finds a formula [Lev17, Corollary 12.4] to count singular elements as the sum of $A^1$-Milnor numbers (see subsection 3.2) of the singularities in a more general setting. It would be interesting to find an interpretation for the local indices in our count and then compare our result to Levine’s count.
3 EKL-class

EKL is short for Eisenbud-Khimshiashvili-Levine who computed the local degree of non-simple, isolated zeros as the signature of a certain non-degenerate symmetric bilinear form (a representative of the EKL-class) over \( \mathbb{R} \) in [EL77] and [Khi]. Eisenbud asked whether the class represented by the EKL-form which is defined in purely algebraic terms, had a meaningful interpretation over an arbitrary field \( k \). His question was answered affirmatively in [KW19] where it is shown that the EKL-class is equal to the local \( A^1 \)-degree.

We recall the definition of the EKL-class from [KW19]. Let \( k \) be a field. Assume that \( f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k \) has an isolated zero at the origin and let \( Q := k[x_1, \ldots, x_n]_{f_1, \ldots, f_n} \). Define \( E := \det \{a_{ij}\} \) where the \( a_{ij} \in k[x_1, \ldots, x_n] \) are chosen such that \( f_i = \sum_{j=1}^n a_{ij}x_j \). We call \( E \) the distinguished socle element since it generates the socle of \( Q \) (that is the sum of the minimal nonzero ideals) when \( f \) has an isolated zero at the origin [KW19, Lemma 4].

Remark 3.1. Let \( J = \det \frac{\partial f_i}{\partial x_j} \) be the jacobian element. By [SS75, Korollar 4.7] \( J = \text{rank}_k Q \cdot E \).

Let \( \phi : Q \to k \) be a \( k \)-linear functional which sends \( E \) to 1.

Definition 3.2. The EKL-class of \( f \) is the class of \( \beta_\phi : Q \times Q \to k \) defined by \( \beta_\phi(a, b) = \phi(ab) \) in \( GW(k) \).

Remark 3.3. By [KW19, Lemma] the EKL-class is well-defined, i.e., it does not depend on the choice of \( \phi \) and \( \beta_\phi \) is non-degenerate. One can for example choose a \( k \)-basis \( b_1, \ldots, b_{n-1}, E \) for \( Q \) and choose \( \phi \) such that \( \phi(b_i) = 0 \) for \( i = 1, \ldots, n - 1 \) and \( \phi(E) = 1 \).

3.1 EKL-code

The following code computes the EKL-form of \( f : \mathbb{A}^n \to \mathbb{A}^n \) with one isolated zero at the origin when \( \text{char } k \) does not divide \( \text{rank}_k Q \). The input is a triple \((C, I, FF)\) where \( k = FF \) is a field and the ideal \( I = (f_1, \ldots, f_n) \subset C = FF[x_1, \ldots, x_n] \) is a complete intersection in the polynomial ring \( C \).

EKL=(C,I,FF)->(r=degree I; B = basis(C/I); B2=mutableMatrix B; J=determinant jacobian I; toVector = q -> last coefficients(q, Monomials=>B); E=J_(C/I)/r; p=0;j=0; while j<r do (if (toVector E)_(j,0)!=0 then p=j;j=j+1); B2_(0,p)=E; B2=matrix(B2); Q = transpose B2 * B2; T=mutableIdentity(C/I,r); i=0;while i<r do (T_(i,p)=(toVector E)_(i,0); i=i+1); T=matrix T; T1=T^(-1); linear = v -> v_(p,0); M=matrix applyTable(entries Q,q->lift(linear(T1*(toVector q)),FF)); M)
Table 1: Du Val singularities

| Singularity | Equation $f$ | $\mu^{A_1}(f) = \text{EKL-class of } \text{grad}(f) \in \text{GW}(\mathbb{Q})$ |
|-------------|--------------|------------------------------------------------|
| $A_n$, $n$ odd | $x^2 + y^2 + z^{n+1}$ | $\frac{n-1}{2} \mathbb{H} + \langle n + 1 \rangle$ |
| $A_n$, $n$ even | $x^2 + y^2 + z^{n+1}$ | $\frac{n}{2} \mathbb{H}$ |
| $D_n$, $n > 1$ odd | $x^2 + y^2 z + z^{n-1}$ | $\frac{n-1}{2} \mathbb{H} + \langle -1 \rangle$ |
| $D_n$, $n$ even | $x^2 + y^2 z + z^{n-1}$ | $\frac{n}{2} \mathbb{H} + \langle -1 \rangle + \langle n - 1 \rangle$ |
| $E_6$ | $x^2 + y^3 + z^4$ | $3 \mathbb{H}$ |
| $E_7$ | $x^2 + y^3 + yz^3$ | $3 \mathbb{H} + \langle -6 \rangle$ |
| $E_8$ | $x^2 + y^3 + z^5$ | $4 \mathbb{H}$ |

3.2 $A^1$-Milnor numbers

Kass and Wickelgren define and compute several $A^1$-Milnor numbers as an application of the EKL-form in [KW19]. Let $0 \in X = \{ f = 0 \} \subset \mathbb{A}^n$ be a hypersurface with an isolated singularity at the origin. Then the $A^1$-Milnor number of $X$ is

$$\mu^{A_1}(f) := \deg_0^{A_1}(\text{grad}(f)).$$

Kass and Wickelgren show that the $A^1$-Milnor number is an invariant of the singularity. When $n$ is even $\mu^{A_1}(f)$ counts the nodes to which $X$ bifurcates (see [KW19] for more details). They compute the $A^1$-Milnor numbers of $ADE$ singularities.

3.2.1 Du Val Singularities

We compute the EKL class of Du Val singularities, that is simple singularities in 3 variables, in Table 1.

Example 3.4. As an example we provide the computation for $E_6$.

```plaintext
C4=QQ[x,y,z]
f=x^2+y^3+z^3*y
I6=ideal(diff(x,f),diff(y,f),diff(z,f))
EKL(C4,I6,QQ)
```

We get the following EKL-form.

```
i74 : EKL(C4,I6,QQ)
o74 = |
| 0 0 0 1 0 0 0 |
| 0 0 0 0 0 1/18 0 |
| 0 0 0 0 1/18 0 |
| 1 0 0 0 0 0 0 |
| 0 1/18 0 0 0 0 |
| 0 0 1/18 0 0 0 |
| 0 0 0 0 0 0 -1/6 |

7 7
o74 : Matrix QQ <--- QQ
```

12
It is easy to see that this is $3\mathbb{H} + \langle -6 \rangle$.

### A  An example of lines on a cubic

As an example, we provide the Gram matrix of $\psi^A (\mathcal{E}_{F_{11}}, \sigma_H)$ for

$$H = Z_0^3 - Z_0^2 Z_1 - Z_1^2 Z_2 + Z_0 Z_2^2 - 2 Z_1 Z_2^2 - 2 Z_0^2 Z_3 - Z_0 Z_1 Z_3 - Z_1^2 Z_3 + Z_1 Z_2 Z_3 + Z_1 Z_3^2 + 2 Z_2 Z_3^2,$$

that is, the count of lines on the cubic surface $\{ H = 0 \} \subset \mathbb{P}^3_{F_{11}}$.

P2 = 11
FF2 = ZZ/P2
R3 = FF2[Z0,Z1,Z2,Z3]
H = Z0^5-Z0^2*Z1-Z1^2*Z2+Z0*Z2^2-2*Z1*Z2^2-
2*Z0^2*Z3-Z0*Z1*Z3-Z1^2*Z3+Z1*Z2*Z3+Z1*Z3^2+2*Z2*Z3^2
C5 = FF2(z1,z2,z3,z4]
S3 = C5[u]
g3 = {z1*u+z2,z3*u+z4,u,1}
m4 = map(S3,R3,g3)
I7 = sub(ideal flatten entries last coefficients m4 H, C5)
L2=minimalPrimes I7
n2=length L2
J7=determinant jacobian I7

There are 5 lines on $X$.

i85 : n2=length L2

o85 = 5

Let $F_1, \ldots, F_5$ be the fields of definitions of the 5 lines. We compute the trace forms of $\text{Tr}_{F_j/F_{11}} (J_{F_j})$ for $j = 1, \ldots, 5$ and sum them up to get $\psi^A (\mathcal{E}_{F_{11}}, \sigma_F) \in \text{GW}(F_{11})$.

Sol2 = traceForm(C5,L2_0,J7,FF2); j=1;
while j<n2 do (Sol2=Sol2++traceForm(C5,L2_j,J7,FF2);j=j+1);
Sol2
The sizes of the blocks are the degrees \( [F_j : F_{11}] \) of the field extension \( F_j/F_{11} \) for \( j = 1, \ldots, 5 \). So there is one rational line on \( X \), one defined over a field extension of degree 2 and 3 lines defined over a field extension of degree 8 on \( X \).

B  More \( \mathbb{A}^1 \)-Milnor numbers

We provide \( \mathbb{A}^1 \)-Milnor numbers of some Fuchsian singularities (see [Ebe03]) in Table 2.
Table 2: Fuchsian singularities

| Singularity | Equation $f$ | $\mu^A(f) = \text{EKL-class of grad}(f) \in \text{GW}(\mathbb{Q})$ |
|-------------|--------------|---------------------------------------------------------------|
| $E_{12}$    | $x^7 + y^3 + z^2$ | $6\mathbb{H}$ |
| $Z_{11}$    | $x^5 + xy^3 + z^2$ | $5\mathbb{H} + (-6)$ |
| $Q_{10}$    | $x^4 + y^3 + xz^2$ | $5\mathbb{H}$ |
| $E_{13}$    | $x^5 y + y^3 + z^2$ | $6\mathbb{H} + (-10)$ |
| $Z_{12}$    | $x^4 y + xy^3 + z^2$ | $5\mathbb{H} + (-22) + (-66)$ |
| $Q_{11}$    | $x^3 y + y^3 + xz^2$ | $5\mathbb{H} + (2)$ |
| $W_{12}$    | $x^5 + y^4 + z^2$ | $6\mathbb{H}$ |
| $S_{11}$    | $x^4 + y^2 z + xz^2$ | $5\mathbb{H} + (-2)$ |
| $E_{14}$    | $x^8 + y^3 + z^2$ | $7\mathbb{H}$ |
| $Z_{13}$    | $x^6 + xy^3 + z^2$ | $6\mathbb{H} + (-6)$ |
| $Q_{12}$    | $x^5 + y^3 + xz^2$ | $6\mathbb{H}$ |
| $W_{13}$    | $x^4 y + y^4 + z^2$ | $6\mathbb{H} + (-2)$ |
| $S_{12}$    | $x^3 y + y^2 z + xz^2$ | $6\mathbb{H}$ |
| $U_{12}$    | $x^4 + y^3 + z^3$ | $6\mathbb{H}$ |
| $J_{0,3}$   | $x^9 + y^3 + z^2$ | $8\mathbb{H}$ |
| $Z_{1,0}$   | $x^7 + xy^3 + z^2$ | $7\mathbb{H} + (-6)$ |
| $Q_{2,0}$   | $x^6 + y^3 + xz^2$ | $7\mathbb{H}$ |
| $W_{1,0}$   | $x^6 + y^4 + z^2$ | $7\mathbb{H} + (3)$ |
| $S_{1,0}$   | $x^5 + y^3 + xz^2$ | $7\mathbb{H}$ |
| $U_{1,0}$   | $x^3 y + y^3 + z^3$ | $7\mathbb{H}$ |
| $W_{12}$    | $x^5 + y^4 + z^2$ | $6\mathbb{H}$ |
| $NA_{0,0}$  | $x^5 + y^5 + z^2$ | $8\mathbb{H}$ |
| $VNA_{0,0}$ | $x^4 + y^4 + y^2 z$ | $7\mathbb{H} + (-2)$ |
| $J_{3,0}$   | $x^{12} + y^3 + z^2$ | $11\mathbb{H}$ |
| $Z_{2,0}$   | $x^{10} + xy^3 + z^2$ | $10\mathbb{H} + (-6)$ |
| $Q_{3,0}$   | $x^9 + y^3 + xz^2$ | $10\mathbb{H}$ |
| $X_{2,0}$   | $x^8 + y^4 + z^2$ | $10\mathbb{H} + (1)$ |
| $S_{2,0}$   | $x^7 + y^2 z + xz^2$ | $10\mathbb{H}$ |
| $U_{2,0}$   | $x^6 + y^3 + z^3$ | $10\mathbb{H}$ |
| $x^6 + y^6 + z^2$ | $12\mathbb{H} + (2)$ |
| $x^5 + y^5 + xz^2$ | $12\mathbb{H}$ |
| $x^4 + y^4 + z^4$ | $10\mathbb{H} + (1)$ |
Acknowledgements

I would like to thank Kirsten Wickelgren for introducing me to the topic and her excellent guidance and feedback on this project. I am also very grateful to Anton Leykin for introducing me to Macaulay2. I gratefully acknowledge support by the RCN Frontier ResearchGroup Project no. 250399 “Motivic Hopf Equations.”

References

[BW20] Tom Bachmann and Kirsten Wickelgren. $A^1$-euler classes: six functors formalisms, dualities, integrality and linear subspaces of complete intersections, 2020.

[Cay09] Arthur Cayley. On the Triple Tangent Planes of Surfaces of the Third Order, volume 1 of Cambridge Library Collection - Mathematics, page 445–456. Cambridge University Press, 2009.

[Ebe03] Wolfgang Ebeling. The Poincaré series of some special quasihomogeneous surface singularities. *Publ. Res. Inst. Math. Sci.*, 39(2):393–413, 2003.

[EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[EL77] David Eisenbud and Harold I. Levine. An algebraic formula for the degree of a $C^\infty$ map germ. *Ann. of Math. (2)*, 106(1):19–44, 1977.

[FK15] Sergey Finashin and Viatcheslav M. Kharlamov. Abundance of 3-planes on real projective hypersurfaces. *Arnold Math. J.*, 1(2):171–199, 2015.

[Khi] G. N. Khimshiashvili. The local degree of a smooth mapping. *Sakharth. SSR Mecn. Akad. Moambe*, 85(2).

[KW17] Jesse Leo Kass and Kirsten Wickelgren. An Arithmetic Count of the Lines on a Smooth Cubic Surface. *arXiv e-prints*, page arXiv:1708.01175, Aug 2017.

[KW19] Jesse Leo Kass and Kirsten Wickelgren. The class of Eisenbud-Khimshiashvili-Levine is the local $A^1$-Brouwer degree. *Duke Math. J.*, 168(3):429–469, 2019.

[Lev17] Marc Levine. Toward an enumerative geometry with quadratic forms, 2017.

[LEV19] MARC LEVINE. Motivic euler characteristics and witth-valued characteristic classes. *Nagoya Mathematical Journal*, page 1–60, Mar 2019.

[McK20] Stephen McKean. An arithmetic enrichment of b´ezout’s theorem, 2020.

[OT14] Christian Okonek and Andrei Teleman. Intrinsic signs and lower bounds in real algebraic geometry. *J. Reine Angew. Math.*, 688:219–241, 2014.

[Seg42] Beniamino Segre. *The Non-singular Cubic Surfaces*. Oxford University Press, Oxford, 1942.
[SS75] Günter Scheja and Uwe Storch. Über Spurfunktionen bei vollständigen Durchschnitten. *J. Reine Angew. Math.*, 278(279):174–190, 1975.

[SW18] Padmavathi Srinivasan and Kirsten Wickelgren. An arithmetic count of the lines meeting four lines in p3. 2018.

[WW19] Kirsten Wickelgren and Ben Williams. Unstable Motivic Homotopy Theory. *arXiv e-prints*, page arXiv:1902.08857, Feb 2019.