Abstract

We present formulae for the characters of coset conformal field theories and apply these to specific examples to determine the integer shift of the conformal weights of primary fields. We also present an example of a coset conformal field theory which cannot be described by the identification current method.
1. Introduction

The coset construction [1] of conformal field theories (CFTs) [2,3] has proved to be a practical method for constructing rational conformal field theories. Indeed it may be that all rational CFTs have a coset realisation. Thus the properties of coset conformal field theories are important in understanding the general structure of rational CFTs. In this paper we investigate various aspects of coset models. In particular, the spectrum of a conformal field theory is determined by the characters [4,5], and much information can be gleaned by their examination. We provide character formulae for a wide class of coset theories and examine these formulae in specific examples. We also use the character formulae to deduce formulae for the “integer shifts” of the conformal weights of primary fields.

In coset CFTs \( \hat{g} / \hat{h} \), where \( \hat{g} \) and \( \hat{h} \) are affine algebras [6,7], the primary fields are labeled by \( \phi_{\Lambda}^{\lambda} \), where \( \Lambda \) and \( \lambda \) are weights of the Lie algebras \( g \) and \( h \) respectively. The conformal weight of \( \phi_{\Lambda}^{\lambda} \) is given by \( h = h_{\Lambda} - h_{\lambda} + n \) where \( n \) is an integer known as the integer shift. The evaluation of the integer shifts is crucial in determining the spectrum of CFTs, which is important, for example, in obtaining the physical spectrum in four dimensional superstring models [8]. Not all pairs of labels \( \Lambda \) and \( \lambda \) give genuine and distinct primary fields but some combinations do not correspond to fields present in the coset, and some combinations of labels are equivalent [9]. In examining the spectrum of primary fields in a coset CFT, the procedure, due to Schellekens and Yankielowicz [10], for specifying which fields are non-zero and distinct, by the introduction of an “identification” current, has proved extremely useful. However, by examining the specific example \( \hat{su}(3)_2 / \hat{su}(2)_8 \) we show that this procedure is not always applicable, contrary to various suppositions, and for this “maverick” coset the spectrum of valid inequivalent fields is not determined solely by an identification current.

The plan of this paper is as follows: In section 2 we review the coset formulation of Goddard, Kent and Olive of rational conformal fields theories [1] and also the Schellekens and Yankielowicz procedure [10] for determining the spectrum of primary fields in such models. In section 3 and the appendix we derive explicit character formulae for these theories. In section 4 we use these formulae to obtain expressions for the integer shifts in the case \( \hat{su}(2)_{k_1} \times \hat{su}(2)_{k_2} / \hat{su}(2)_{k_1+k_2} \). In sections 5 and 6 we use these formulae to obtain information on the conformal field theory for specific examples namely, \( \hat{g} / \hat{su}(3)_k \) and \( \hat{su}(3)_k / \hat{su}(2)_4k \). The case \( \hat{g} / \hat{su}(3)_2 \) gives an example of how the coset CFT may have a larger symmetry than the Virasoro algebra. In this case the coset provides a realisation of the \( W_3 \) algebra [11]. The case \( \hat{su}(3)_2 / \hat{su}(2)_8 \) is very interesting because is provides an example of a coset CFT which cannot be described in terms of an identification current. This provides a counter example to several propositions regarding coset CFTs.
2. Review of the GKO construction

Here we briefly review the Goddard, Kent and Olive (GKO) [1] construction for rational conformal field theories and the Schellekens-Yankielowicz [10] method for describing primary fields of coset models by the use of an identification current.

Consider a Kac-Moody algebra $\hat{g}$ [6], associated with the Lie algebra $g$. The corresponding current algebra has fields $J^a(z)$ which satisfy the operator product expansion (O.P.E.)

$$J^a(z)J^b(w) = \frac{-k\delta^{ab}}{(z-w)^2} + \frac{f^{ab}_cJ^c(w)}{(z-w)} + \text{n.s.t.} \quad (2.1)$$

where the integer $k$ is the level of the algebra, $f^{ab}_c$ are the structure constants of the Lie algebra and ‘n.s.t.’ denotes terms which are non-singular as $z \to w$. The Laurent coefficients of the currents $J^a(z) = \sum_n J^a_n z^{-n-1}$, satisfy

$$[J^a_m, J^b_n] = f^{ab}_cJ^c_{m+n} + km\delta^{ab}\delta_{m+n,0} \quad (2.2)$$

As is well known, Kac-Moody algebras contain Virasoro algebras. The stress-energy tensor is formed using the Sugawara construction [7]

$$T_g(z) = \frac{-1}{2(k+g)} \sum_a :J^a J^a(z): \quad (2.3)$$

which satisfies

$$T_g(z)T_g(w) = \frac{c_g/2}{(z-w)^4} + \frac{T_g(w)}{(z-w)^2} + \frac{\partial T_g(w)}{z-w} + \text{n.s.t.} \quad (2.4)$$

as $z \to w$. The central charge is related to the level of the Kac-Moody algebra by

$$c_g = \frac{k \dim g}{k + g} \quad (2.5)$$

where $g$ is the dual Coxeter number of the Lie algebra $g$.

Suppose $g$ contains a subalgebra $h$. Then $\hat{g}$ will contain a subalgebra $\hat{h}$. We can then form the stress-energy tensor [1]

$$T_{g/h} = T_g - T_h \quad (2.6)$$

which satisfies the O.P.E. for an energy momentum tensor with central charge

$$c_{g/h} = c_g - c_h \quad (2.7)$$

This construction of the energy momentum tensor is the well known GKO construction for coset conformal field theories. An important feature for this construction is that the O.P.E. of the currents of the subalgebra $J^a(z)$ with $T_{g/h}(w)$ is non-singular

$$J^a(z)T_{g/h}(w) = \text{n.s.t.} \quad (2.8)$$
or, equivalently,
\[
[J_m^a, L_n^{g/h}] = 0 \quad (2.9)
\]
Using this construction it is possible to construct many CFTs with relatively small central charge, and it may be that all rational CFTs have a coset realisation. Representations of \( \hat{g} \) at level \( k \) are labeled by the highest weights of \( g \), \( \Lambda \), satisfying \( \psi \cdot \Lambda \leq k \) where \( \psi \) is the highest root of \( g \). The conformal weight of the associated primary field is given by
\[
h_\Lambda = \frac{\Lambda^2 + 2 \Lambda \cdot \rho_g}{2(k + g)} \quad (2.10)
\]
where \( \rho_g \) is half the sum of the positive roots of \( g \).

If we consider \( \hat{g} \) as the algebra generated by \( J_n^a \), then this contains, by definition, the algebra, \( \hat{h} \) and the algebra generated by \( L_n^{g/h} \). However the full algebra of the coset will be the algebra \( A_{g/h} \), which we define to be the largest algebra such that
\[
A_{g/h} \otimes \hat{h} \subset \hat{g} \quad (2.11)
\]
In general, a representation of \( \hat{g} \) will decompose into a sum of representations of the smaller algebra \( A_{g/h} \otimes \hat{h} \),
\[
\Lambda = \sum_\lambda R_\lambda^A \otimes \lambda \quad (2.12)
\]
where \( R_\lambda^A \) is a representation of \( A_{g/h} \).

For a representation \( \Lambda \) of the Kac-Moody algebra the character is defined as
\[
ch_\Lambda(\tau, z^i) \equiv \text{tr}(exp(\{2\pi i \tau L_0 + c/24 - h_\Lambda + \sum z_i H_0^i\}))
\]
\[
= \text{tr}(q^{L_0 + c/24 - h_\Lambda} \prod w_i^{H_0^i}) \quad (2.13)
\]
where \( H^i \) are the elements of the Cartan subalgebra of \( g \), \( q \equiv \exp(2\pi i \tau) \) and \( w_i \equiv \exp(2\pi iz_i) \). The restricted characters \( \chi_\Lambda(\tau) \) are defined by
\[
\chi_\Lambda(\tau) \equiv q^{h_\Lambda - c/24} ch_\Lambda(\tau, 0) \quad (2.14)
\]
Assuming, for the moment, that the Cartan subalgebra has the same dimension for both \( g \) and \( h \), i.e. they are of the same rank, then we will have \( z_i H_0^i = z_i H_0^i \) and, since \( L_0^g = L_0^{g/h} + L_0^h \)
\[
ch_\Lambda(\tau, z_i) = \sum_\lambda b_\lambda^A(\tau) ch_\lambda(\tau, z_i) \quad (2.15)
\]
and for the restricted characters
\[
\chi_\Lambda(\tau) = \sum_\lambda \chi_\lambda^A(\tau) \chi_\lambda(\tau) \quad (2.16)
\]
From this we see that

\[ h^\Lambda = h_\Lambda - h_\lambda + n \]  

(2.17)

where \( n \) is the order of the leading term in the power series expansion of \( b^\Lambda \)(\( q \)). There are many cases where the rank of \( g \) and \( h \) are different (in particular the important example \( \hat{g} \times \hat{g} / \hat{g} \)), but these formulae still stand, provided we choose the \( z_i(z_i) \) such that \( z_iH_0^1 \) can be written as a linear combination of the \( z_iH_0 \), which is equivalent to specifying the embedding. The functions \( b^\Lambda (\tau) \) are the “branching functions” of the coset CFT. Not all combinations of labels \( \Lambda \) and \( \lambda \) give rise to non-zero branching functions and not all distinct labels give rise to distinct branching functions [9]. Which possibilities are non-zero and are distinct has been resolved by the introduction [10] of the concept of an “identification current” in the coset CFT.

The Schellekens-Yankielowicz mechanism [10] for deciding which fields are non-zero and inequivalent is to use an “identification current” which is defined in terms of simple currents of the factors \( \hat{g} \) and \( \hat{h} \). Before discussing the identification current we define simple currents and the relation to non-diagonal modular invariants. A simple current of a general CFT, \( J \), is a primary field with the simple fusion rules [2,12]

\[ J \cdot \phi = \phi' \]  

(2.18)

That is, for a primary field the operator product of \( J \) with the field only yields fields from a single conformal family. If we consider the fusion of \( J \) with itself we obtain a field \( J^2 \) etc. For a rational CFT with a finite number of primary fields there must be an integer \( N \) such that \( J^N = 1 \). In general the action of \( J \) upon a primary field \( \phi \) will yield fields \( \{ J^r \phi, r = 0, 1, \ldots, N\phi - 1 \} \), where \( J^{N\phi} \phi = \phi \).

The integer \( N\phi \) must be a divisor of \( N \). If a field satisfies \( J \cdot \phi = \phi \) then this field is said to be a fixed point w.r.t \( J \). The existence of fixed points (and fields with \( N\phi < N \)) does a great deal to complicate the analysis. For the most part the difficulties associated with fixed points have been resolved in ref.[10] and we will not linger on their resolution. The case of most interest to us will be when \( J \) has integer conformal weight, that is \( h(J) \) is integer. (In general it can be shown that \( h(J) = r/N, \mod(1) \)). In this case, the CFT has a non-diagonal modular invariant (NDMI) [13,14]

\[
Z = \sum_{\phi: h(J \cdot \phi) - h(\phi) \in Z} \frac{N}{N\phi} |\chi_\phi + \chi_{J \cdot \phi} + \cdots + \chi_{J^{N\phi - 1} \cdot \phi}|^2
\]

(2.19)

This form has been suggested to be the diagonal modular invariant of an extended algebra [9,13,14,15] whose characters are \( \chi_\phi + \chi_{J \cdot \phi} + \cdots \). Most Kac-Moody algebras contain simple currents, some examples of which are given in following...
sections. For cases where $N_\phi < N$ the extended algebra appears to have a multiplicity of primary fields with identical characters.

The relationship between the characters of the coset algebra and the branching functions has been the source of some confusion, but has been elegantly resolved by Schellekens and Yankielowcz [10]. This relies on the observation that the diagonal combination of characters,

$$Z = \sum_a \chi_a \chi_a^*$$

where the summation runs over all genuine characters of the coset CFT, must be modular invariant. Since the $\chi_a$ can be expressed as a sum of the branching functions this can be rewritten as

$$Z = \sum_a \left| \sum_{\Lambda,\lambda} n_{\Lambda,\lambda}^a b_{\lambda}^\Lambda \right|^2$$

Hence one should look for modular invariant combinations of the branching functions of this form (which can be recognised as that generated by a simple current.) To look for such modular invariants, note that the branching functions of $\hat{g}/\hat{h}$ transform as the characters of $\hat{g} \times \hat{h}^*$. Hence if one can find a suitable modular invariant for $\hat{g} \times \hat{h}^*$ then the corresponding object for $\hat{g}/\hat{h}$ will be modular invariant and a candidate for the diagonal modular invariant of the coset.

The critical observation is that it is possible to find such a current which generates a modular invariant corresponding to the characters of $\hat{g}/\hat{h}$. This current, denoted $J_I$, is called the identification current. After determining $J_I = \phi_{J_{I2}}^{J_{I1}}$, the non-zero branching functions are those which have

$$h(J_I \cdot \phi^\Lambda_\lambda) - h(\phi^\Lambda_\lambda) = 0 \mod(1)$$

and we have the following equivalence

$$\phi^\Lambda_\lambda \equiv \phi^{J_{I1},\Lambda}_{J_{I2},\lambda}$$

The details of the identification current, for a variety of cosets is given in [10]. As a simple example, for $\text{su}(2)_k$ the simple current is $(k)$ which satisfies $(k) \cdot (l) = (k-l)$. For $k$ odd there are no fixed points. For $\text{su}(2)_k \times \text{su}(2)_1/\text{su}(2)_{k+1}$ (a realisation of the minimal models) the fields are $\phi_{l_3,l_2}^{l_1}$. The condition for the branching function to be non-zero reduces to $l_1 + l_2 - l_3 = 0 \mod(2)$, and we have the equivalence

$$\phi_{l_3,l_2}^{l_1} \equiv \phi_{k+l_1-1,l_2}^{k-1,l_1-1}$$

which we see can be rearranged as the standard labelling for the fields of the minimal models.
The identification current method is elegant and applies to many cosets, however, as we shall show in this paper, there do exist cosets which can not be described in terms of such an identification current.

3. Character Formulae

In this section we explore and determine the branching functions for coset CFTs. The branching functions are determined by equ. (2.15). In general, one cannot set \( z^i = 0 \) to solve these equations because there is then not enough information due to the multiplicity of terms on the R.H.S. The solution to the special case \( \hat{su}(2)_{k_1} \times \hat{su}(2)_{k_2} / \hat{su}(2)_{k_1+k_2} \) has been presented in ref.[16]. Throughout this section we follow the notation of ref [5].

The Weyl-Kac formula for the characters of a Kac-Moody algebra is

\[
ch^g_{\Lambda,k}(\tau, z_i) = \frac{N^g_{\Lambda,k}(\tau, z_i)}{D_g(\tau, z_i)}
\]

where

\[
N^g_{\Lambda,k}(\tau, z_i) = \sum_{w \in W_g} \epsilon(w) \Theta_{w(\Lambda+\rho),k+g}(\tau, z_i)
\]

and

\[
D_g(\tau, z_i) = \prod_{\alpha > 0} (1 - e^{-\alpha})^{\text{mult}(\alpha)}(\tau, z_i)
\]

The \( w \) are elements of the Weyl group of the Lie algebra, \( W_g \), and \( \epsilon(w) = \pm 1 \) is the parity of \( w \). The \( \Theta_{w(\Lambda+\rho),k+g} \) are the generalised \( \Theta \)-functions

\[
\Theta_{\Lambda,k}(\tau, z_i) = e^{2\pi i (\rho \cdot z_i e_i)} \sum_{\gamma \in M} q^{\frac{1}{2} \gamma \cdot \gamma + \gamma \cdot \Lambda} \exp\left\{-2\pi i (k \cdot (z_i e_i) + \Lambda \cdot (z_i e_i))\right\}
\]

where \( \{e_i\} \) form a basis for the weight lattice of \( g \) and \( M \) is the long root lattice of \( g \) (which is the root lattice whenever \( g \) is simply laced.)

The denominator in eqn.(3.3) is defined in terms of \( \alpha \) where \( \alpha \) are all the positive roots of the Kac-Moody algebra. The denominator can be rewritten (or is defined as)

\[
D_g(\tau, z_i) = \\
\left( \prod_{n=1}^{\infty} \left(1 - q^n\right) \right)^{\text{rank}(g)} \prod_{\bar{\alpha} \in g^+} \left(1 - e^{2\pi i \bar{\alpha} \cdot z_i e_i}\right) \left( \prod_{n=1}^{\infty} \left(1 - q^n e^{2\pi i \bar{\alpha} \cdot z_i e_i}\right) \left(1 - q^n e^{-2\pi i \bar{\alpha} \cdot z_i e_i}\right) \right)
\]

where \( \bar{\alpha} \) are the roots of the Lie algebra, and \( g^+ \) denotes the set of positive roots of \( g \).

In this section, for clarity, we restrict ourselves to an illustrative case, leaving the rather complex general case to the appendix. Suppose we have a pair of Lie
algebras $h$ and $g$, of the same rank such that the root lattice of $h$ is contained in that of $g$. Writing eq. (2.15) using eq. (3.1),

$$\frac{N^g_\Lambda(q, z)}{D_g} = \sum_\mu b^g_\mu(q) \frac{N^h_\mu(q, z)}{D_h}. \tag{3.6}$$

The denominators are defined in terms of a product over the roots. For the simple case examined here (since the roots of $h$ are a subset of the roots of $g$), it is easy to construct $D_{g/h} \equiv D_g D_h^{-1}$:

$$D_{g/h} = \prod_{\alpha \in S} (1 - e^{-\alpha})^{\text{mult} \alpha} \tag{3.7}$$

where $S$ consists of positive roots of $\hat{g}$ which are not roots of $\hat{h}$. Then

$$\frac{N^g_\Lambda(q, z)}{D_{g/h}(q, z)} = \sum_\mu b^h_\mu(q) N^h_\mu(q, z) \tag{3.8}$$

Consider the coefficient, on the r.h.s., of

$$\exp(-2\pi i \lambda \cdot z) \tag{3.9}$$

where $\lambda$ labels some highest weight irreducible representation of $\hat{h}$. From (3.2) and (3.4), there is a unique term of this form, i.e. that which has $\gamma = 0, \omega = 1, \mu = \lambda$, and moreover its coefficient is the branching function. Thus by comparing coefficients on both sides of eq. (3.8)

$$b^\lambda_\Lambda = \exp(2\pi i \lambda \cdot z) \frac{N^g_\Lambda(q, z)}{D_{g/h}(q, z)} \bigg|_{\text{coeff of } \zeta^i} \tag{3.10}$$

where $\zeta_i \equiv \exp(2\pi iz_i), i = 1 \ldots \text{rank } g$. The branching function can then be obtained by expanding the r.h.s. as a Laurent series in $\{\zeta_i\}$. The numerator $N^g_\Lambda$ is already in this form and we can expand the denominator $D_{g/h}$ using the identity \[17\]:

$$\frac{1}{(1-t) \prod_{n=1}^{\infty} (1-q^n t)(1-q^n t^{-1})} = \phi(q)^2 \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} \frac{1}{(1-q^n t)^2}$$

$$= -\phi(q)^2 \sum_{n=-\infty}^{1} (-1)^n q^{n(n+1)/2} \sum_{r=0}^{\infty} q^{-rn-r-t-r-1}$$

$$+ \phi(q)^2 \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sum_{r=0}^{\infty} q^{rn} t^r$$

$$= \phi(q)^2 \sum_{n=-\infty}^{\infty} (-1)^n \delta_n q^{n(n+1)/2} \sum_{r=st=tn}^{\infty} q^{rn} t^r \delta_n. \tag{3.11}$$
where \( \delta_n \) and \( st_n \) are defined by
\[
\begin{align*}
  & n \geq 0 : \delta_n = 1, \quad st_n = 0 \\
  & n < 0 : \delta_n = -1, \quad st_n = 1
\end{align*}
\]
and where
\[
\phi(q) \equiv \prod_{n > 0} (1 - q^n)
\]
Thus we can rewrite \( D_{g/h}^{-1} \) as
\[
D_{g/h}^{-1}(q, \{ z_i \}) =
\phi(q)^{|h| - |g|} \prod_{\alpha \in S} \left( \sum_{n_{\alpha} = -\infty}^{\infty} (-1)^{n_{\alpha}} \delta_{n_{\alpha}} q^{n_{\alpha}(n_{\alpha} + 1)/2} \sum_{r_{\alpha} = st_{n_{\alpha}}} q^{r_{\alpha} n_{\alpha} \delta_{\alpha}} \exp(2\pi i \delta_{n_{\alpha}} r_{\alpha} \cdot z_i) \right)
\]
and obtain the branching function,
\[
b^\lambda_\Lambda(q) = \exp(2\pi i \lambda \cdot z) \phi(q)^{|h| - |g|} \sum_{w \in W} \epsilon(w) e^{2\pi i p \cdot z}
\]

The relevant terms which contribute to \( b^\lambda_\Lambda \) are those which satisfy:
\[
\lambda + \rho - (k + g) \gamma - \omega(\Lambda + \rho) + \sum_{\alpha \in S \setminus S_0} \alpha \delta_{\alpha} r_{\alpha} = 0 .
\]
To solve these constraints choose a basis of the lattice spanned by the elements of \( S \), say \( S_0 \subset S \). Let the dual be \( S^* = \{ e_\alpha \} \), such that \( e_\alpha \cdot \beta = \delta_{\alpha, \beta}, \ \forall \alpha, \beta \in S_0 \).
If we now define
\[
V_\beta \equiv e_\beta \cdot \left\{ (k + g) \gamma + \omega(\Lambda + \rho) - \rho - \lambda - \sum_{\alpha \in S \setminus S_0} \alpha \delta_{\alpha} r_{\alpha} \right\}
\]
then the constraints (3.15) have solution
\[
r_{\alpha} = \delta_{\alpha} V_{\alpha} \quad \forall \alpha \in S_0
\]
which allows elimination of rank(\( g \)) of the summations over the \( r_{\alpha} \). Since \( r_{\alpha} \geq 0 \) we have \( \text{sign}(V_{\alpha}) = \text{sign}(\delta_{\alpha}) = \text{sign}(n_{\alpha}), \ \forall \alpha \in S_0 \).

The final form of the branching function is then
\[
b^\lambda_\Lambda = \phi(q)^{|h| - |g|} \sum_{w \in W} \epsilon(w) F_w(\Lambda + \rho) (\tau)
\]
where
\[
F_{\Lambda}(\tau) = \sum_{\gamma \in M} \sum_{\alpha \in S \setminus S_0} \sum_{\alpha \in S \setminus S_0} \sigma q^N
\] (3.20)

and
\[
\sigma = \left( \prod_{\alpha \in S} \delta_{n_{\alpha}} \right) (-1)^{\sum_{\alpha \in S} n_{\alpha}}
\]
\[
N = \frac{1}{2}(k + g)\gamma^2 + \gamma \cdot \Lambda + \frac{1}{2} \sum_{\alpha \in S} n_{\alpha}(n_{\alpha} + 1) + \sum_{\alpha \in S \setminus S_0} n_{\alpha} r_{\alpha} \delta_{n_{\alpha}} + \sum_{\alpha \in S_0} V_{\alpha} n_{\alpha}
\] (3.21)

and
\[
V_{\alpha} \equiv e_{\alpha} \cdot \left\{ (k + g)\gamma + \Lambda - \lambda - \sum_{\alpha \in S \setminus S_0} \alpha r_{\alpha} \delta_{n_{\alpha}} \right\}.
\] (3.22)

A cursory examination of this formula shows that it involves summation over a lattice of dimension \(|g| - |h|\). This formula provides an explicit form for the branching functions and hence for the characters of the theory. It is however not always the most efficient possible and in special cases, such as \(\hat{su}(n)_k \times \hat{su}(n)_{k+1}\), formulae be obtained which are considerably more efficient. In the following sections we shall apply (3.17), evaluated by computer, to examine the properties of the character. For large cosets, the computer time taken to evaluate characters can become considerable.

4. Integer Shifts

In coset models, as in any conformal field theory, the values of the conformal weights of the primary fields is essential information. For a primary field of the coset, \(\phi_{\Lambda}^{\lambda}\), the \(h\)-value is related to that of \(\Lambda\) and \(\lambda\) by
\[
h_{\lambda}^{\Lambda} = h_{\lambda}^{\Lambda} - h_{\Lambda} + n
\] (4.1)

where \(n\) is a non-negative integer. The value of the integer shift \(n\) is not in general known, although its value is important. (As for example, in the construction of four dimensional string theories based upon CFTs [8].) In principle, \(n\) can be extracted from the character formulae of the previous section and, as we noted in section two, \(n\) is simply the degree of the first term in the branching function: \(b_{\lambda}^{\Lambda} = q^n(n_0 + n_1 q + \ldots)\). For illustration, consider the coset
\[
\hat{su}(2)_{k_1} \times \hat{su}(2)_{k_2} / \hat{su}(2)_{k_1 + k_2}.
\] (4.2)

For this coset the primary fields are \(\phi_{\Lambda_3}^{\lambda_1 \lambda_2}\) where \(\lambda_i = 0, 1, \ldots, k_i\). The identification current is \(\phi_{k_3}^{k_1 k_2}\), and the non-zero fields are those where \(\lambda_1 + \lambda_2 - \lambda_3\) is even,
and we have the equivalence $\phi_{\lambda_3}^{\lambda_1,\lambda_2} \equiv \phi_{\lambda_3-\lambda_1-\lambda_2}^{k_1-k_2}$. We determine $n$ as a function of the parameters $\lambda_1, \lambda_2, \lambda_3, k_1, k_2$. By evaluating the branching functions for a large and representative range of these parameters, we determine a formula for $n$ by inspection. The form of $n$ is sufficiently simple that this procedure is amenable, and we obtain, for $k_1 \geq k_2$,

$$n = k_2 x^2 + (2s + \lambda_2) x + s + a + b$$  \hspace{1cm} (4.3)$$

where

$$s = \left(\frac{\lambda_3 - \lambda_1 - \lambda_2}{2}\right) \mod k_2$$

$$x = \left(\frac{\lambda_3 - \lambda_1 - \lambda_2}{2}\right) \div k_2$$

$$a = (s + \lambda_2 - k_2) \theta(s + \lambda_2 - k_2)$$

$$b = \left(\frac{\lambda_2 - \lambda_1 - \lambda_3}{2}\right) \theta(\lambda_2 - \lambda_1 - \lambda_3)$$  \hspace{1cm} (4.4)$$

For the case $k_1 = k_2$ the formula is valid provided we choose $\lambda_1 \leq \lambda_2$. It is worthwhile checking that if we have:

$$|\lambda_2 - \lambda_1| \leq \lambda_3 \leq \lambda_1 + \lambda_2$$  \hspace{1cm} (4.5)$$

then the integer shift vanishes. The case $\lambda_3 = \lambda_2 + \lambda_1$ is immediate. Consider the case $\lambda_1 > \lambda_2$. Then setting $\lambda_3 = \lambda_1 + \lambda_2 + 2\alpha$, where $0 \leq \alpha < \lambda_2 \leq k_2$ it follows $s = k_2 + \alpha - \lambda_2$, $x = -1$, and hence $a = \alpha$, and $b = 0$. Then (4.3) yields zero. The case $\lambda_2 \geq \lambda_1$ is equally trivial. We can similarly demonstrate the invariance of

$$h_{\lambda_3}^{\lambda_1,\lambda_2} = n + h_{\lambda_1} + h_{\lambda_2} - h_{\lambda_3}$$  \hspace{1cm} (4.6)$$

under the mapping $\lambda_i \rightarrow k_i - \lambda_i$. This particular coset has been studied previously by Date et al, [16] and we reproduce their result. In the following sections we will apply this technique to obtain formulae for the integer shift in new examples.

5. Special Case 1: $(\hat{g}_2)_k/\hat{su}(3)_k$

In this section, the techniques of chapter three are applied to the particular case of $\hat{g}_2/\hat{su}_3$. This coset is fairly simple in structure, as $su(3)$ is the algebra associated with the long root lattice of $g_2$. We use the usual notation for $g_2$: the simple roots are $\alpha_1$ and $\alpha_2$ where $\alpha_1^2 = 2$ and $\alpha_2^2 = \frac{2}{3}$. Then the $su(3)$ has positive roots $\alpha_1, \alpha_1 + 3\alpha_2$ and $2\alpha_1 + 3\alpha_2$. For this coset, unusually, there are no identification currents. If we label the fields $\phi_\Lambda^\lambda$, where $\Lambda$ and $\lambda$ label irreducible representations of $g_2$ and $su(3)$ respectively, then all pairs $(\Lambda, \lambda)$ are allowed and inequivalent. In evaluating the branching function formula, (3.19), we find (3.15) becomes

$$\lambda + \sum_{a \in S} \alpha \delta_{a\alpha} r_\alpha + \rho - (k + g) \gamma - w(\Lambda + \rho) = 0$$  \hspace{1cm} (5.1)$$
The summation over $S$ is simply the summation over $\mathfrak{g}_2^+ \backslash \mathfrak{su}(3)^+$, the short positive roots of $\mathfrak{g}_2$, which is $\{ \alpha_2, \alpha_{12}, \alpha_{122} \}$, where $\alpha_{12} \equiv \alpha_1 + 2\alpha_2$. We use these constraints to eliminate two of the summations over $r_\alpha$ from (3.19). Letting $r_2 \equiv r_{\alpha_2}$ etc., we find

\[
\delta_2 r_2 = -\delta_{12}^2 r_{12}^2 - \alpha_{12}^2 \cdot V,
\]

\[
\delta_{12} r_{12} = -\delta_{12}^2 r_{12}^2 - \alpha_{122}^2 \cdot V,
\]

where

\[
V \equiv \lambda + \rho - w(\Lambda + \rho) - (k + g)\gamma.
\]

Thus, given $n_{12}^2, r_{12}^2, w, \gamma$, we can fix $r_2, r_{12}$ and the sign of $n_2$ and $n_{12}$. The explicit formula for the branching functions is then,

\[
b^{\lambda}(q) = \phi(q)^{-6} \sum_{w \in W_{\mathfrak{g}_2}} \sum_{\gamma \in M} \sum_{n_{12} \in \mathbb{Z}} \sum_{r_{12} = n_{12}} \sum_{\nu = 0}^{\infty} \sum_{\nu_2 \geq 0, \nu_2 \geq 0} \sum_{\nu_2 \geq 0, \nu_2 \geq 0} \sigma q^N (5.4)
\]

where

\[
\sigma = \epsilon(w).(-1)^{n_2 + n_{12} + n_{122}} \delta_{n_2} \delta_{n_{12}} \delta_{n_{122}}
\]

\[
N = r_2 n_2 \delta_2 + r_{12} n_{12} \delta_{12} + r_{122} n_{122} \delta_{122} + \frac{1}{2} (k + g)\gamma^2 + \gamma \cdot w(\Lambda + \rho)
\]

\[
+ n_2 (n_2 + 1)/2 + n_{12} (n_{12} + 1)/2 + n_{122} (n_{122} + 1)/2
\]

Consider the theory at level 1. The central charge is $c = 4/5$ and the non-zero weight labels are $(0, 1)$ and $(1, 0)$ for $\mathfrak{su}_3$, both at spin 1/3, and $(0, 1)$ for $\mathfrak{g}_2$, at spin 2/5. We therefore have six fields and no identification currents. The weights of the fields are $h_\lambda^\mu = h_\lambda - h_\mu + n$, where $n$ is the degree of the first term in the corresponding branching function. The computed branching functions are:

\[
b^{0,0}_{(0,0)} = 1 + q^2 + 2q^3 + 3q^4 + 4q^5 + 7q^6 + \ldots
\]

\[
b^{0,1}_{(0,0)} = q + q^2 + 2q^3 + 2q^4 + 4q^5 + 5q^6 + \ldots
\]

\[
= b^{1,0}_{(0,0)}
\]

\[
b^{0,1}_{(0,0)} = 1 + 2q + 2q^2 + 4q^3 + 5q^4 + 8q^6 + \ldots
\]

\[
b^{0,1}_{(0,1)} = 1 + q + 2q^2 + 3q^4 + 5q^5 + 7q^6 + \ldots
\]

\[
= b^{1,0}_{(0,1)}
\]

Hence we obtain fields with weights: $h = 0, \frac{2}{5}, (\frac{3}{7})^2, (\frac{1}{15})^2$. The weights correspond to the NDMI of the $c = \frac{4}{5}$ minimal model [18]. This well known example, corresponds to the first element of the $W_3$ algebras. Thus we have the equivalence

\[
\frac{\mathfrak{g}_2}{\mathfrak{su}(3)} \equiv \frac{\mathfrak{su}(3)_1 \times \mathfrak{su}(3)_1}{\mathfrak{su}(3)_2} \equiv \frac{\mathfrak{su}(2)_2 \times \mathfrak{su}(2)_1}{\mathfrak{su}(2)_4} \big|_{NDMI}.
\]
Examining the identity character, one can observe an additional state at level three corresponding to the spin-3 field which generates the $W_3$ symmetry. (For a pure Virasoro algebra the representation with $h = 0$ generally has only a single state at level 3, $L_{-3}|0 >$, since the state $L_{-1}|0 >$ is null.) For a coset theory, the algebra is generally an extension of the Virasoro algebra [20]. Examining the zero character is an easy way to discover the spin of the lowest extra field in the symmetry algebra.

We present a formula for the integer shifts using the same technique as for the diagonal $\widehat{su}(2)$ case. As in that case we only have five parameters, and hence the solution is not hard to obtain. We find for a primary field $\phi^\Lambda_\mu$ the integer shift is

$$n = \kappa_1 \theta(\kappa_1) \theta(\kappa_2) + \kappa_2 \theta(\kappa_2) + (\lambda_1 - \mu_1 - \mu_2) \theta(\lambda_1 - \mu_1 - \mu_2) + (\kappa_1 + \kappa_2) \theta(\kappa_1 + \kappa_2) \theta(\lambda_1 + \lambda_2 - 1 - \mu_2)$$

(5.8)

where

$$\Lambda = \lambda_i w_i \quad \mu = \mu_i \nu_1 \quad \mu - \Lambda = \kappa_i \nu_i \quad \mu_1 \leq \mu_2$$

(5.9)

having chosen $w_i$ and $\nu_i$ as the standard fundamental weights of $g_2$ and $su(3)$ respectively. (The $\theta$-function is the usual step function with $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geq 0$.) Note that when $\mu - \Lambda$ has positive Dynkin components in the $su(3)$ basis then the formula reduces to $(\mu - \Lambda) \cdot \psi$, where $\psi$ is the highest root.

6. Special Case 2: $\widehat{su}(3)_k/\widehat{su}(2)_{4k}$

This very interesting example of a coset CFT provides a counter example to some of the preconceptions relating to coset spaces. For $k = 1$ the coset is the trivial theory. This is related to the conformal embedding of $\widehat{su}(2)_4$ within $\widehat{su}(3)_1$ [19,20]. For $k > 2$ the coset is a perfectly normal example of a theory where the non-zero characters and equivalence between characters is determined by an identification current. This current is given by

$$J_I = \phi^{(0,0)}_{4k}$$

(6.1)

and the action of $J_I$ upon a field is

$$J_I \cdot \phi^\Lambda_\mu = \phi^\Lambda_{4k-l}$$

(6.2)

Using this, the fields $\phi^\Lambda_\mu$ are non-zero provided $l$ is even and we have the equivalence $\phi^\Lambda_{l} \equiv \phi^\Lambda_{4k-l}$. The non-zero condition is completely equivalent to demanding that $P\Lambda - l$ lies within the root lattice of $su(2)$, where $P\Lambda$ is the $su(2)$ weight $\Lambda$ is mapped to by the embedding. For $k = 3$, $c = 10/7$ and it can be shown that this coset is equivalent to the third member of the $W_3$ minimal series. This provides a check that the identification current does entirely specify the spectrum for $k > 2$. 
For $k = 2$ however we find we have a “maverick” coset theory. For this coset $c = 4/5$. Hence this coset must correspond to the minimal model with this $c$-value. There are two possibilities — either it corresponds to the diagonal modular invariant or it corresponds to the non-diagonal modular invariant [18] (which is equivalent to the first element of the $W_3$ minimal series). However, if we consider the potential field $\phi_2^{(0,0)}$ say, then this has $h = 4/5$ which does not lie within the spectrum of the $c = 4/5$ minimal model. To investigate this example further we must give a formula for the characters of the spectrum of the conventions the embedding of the potential field.

\[ \phi = \frac{1}{4} \text{equiv to the first element of the } W_3 \text{ minimal series}. \]

In section 3, we can deduce a formula for a character $\chi^A$, 

\[ \chi^A = \phi(\tau)^{-5} \sum_{w \in W_{SU(3)}} \sum_{\gamma \in M} \sum_{n_1 = -\infty}^{\infty} \sum_{r_1 = \text{st}(n_1)}^{\infty} (-1)^{n_1 + n_2 - 1} \delta_{n_1} \delta_{n_2} q^{N(\gamma, n_1, r_1, V)} \]  

where

\[ V = (w(\Lambda + \rho) - \rho) \cdot \alpha_1 - \frac{3}{2} - 2r_1 \delta_1 - 1 + (k + g) \gamma \cdot \alpha_1 \]

\[ N = \frac{1}{2} (k + g) \gamma^2 + \gamma \cdot w(\Lambda + \rho) + \frac{1}{2} n_1 (n_1 + 1) + \delta_1 n_1 r_1 + \frac{1}{2} n_2 (n_2 + 1) + \delta_2 n_2 V \]

If we evaluate this character for $k = 2$ then we find more characters are zero than we would expect from the identification current and there exist more equivalences than one would expect. The equivalences of the characters are given in table 1. From this, and comparing with the minimal model, we find there are more zero characters, and also there are more equivalences than expected. Roughly, speaking there being one extra equivalence so that $\chi^{00}_0$ is equivalent to $\chi^{41}_{11}$ in addition to the expected $\chi^{00}_8$. Taking these into account we find the spectrum of $h$-values matches that of the NDMI of the $c = 4/5$ minimal model ($h = 0, 2/5, (2/3)^2, (1/5)^2$). This is also the first element of the $W_3$ minimal series so this coset has an extended algebra containing a spin-3 field as in the $\hat{g}_2$ case considered earlier. This maverick coset space is related to the NDMI which exists for $\hat{su}(3)_2 \times \hat{su}(2)_8$, 

\[ Z = \left| \chi^{(00)}_0 + \chi^{(00)}_8 + \chi^{(11)}_4 \right|^2 + \left| \chi^{(11)}_2 + \chi^{(11)}_6 + \chi^{(00)}_4 \right|^2 + \left| \chi^{(10)}_2 + \chi^{(10)}_6 + \chi^{(02)}_4 \right|^2 + \left| \chi^{(01)}_2 + \chi^{(01)}_6 + \chi^{(20)}_4 \right|^2 + \left| \chi^{(20)}_0 + \chi^{(20)}_6 + \chi^{(01)}_4 \right|^2 + \left| \chi^{(02)}_0 + \chi^{(02)}_8 + \chi^{(10)}_4 \right|^2 \]  

As we can see this NDMI is generating the characters of the coset theory in an analogous manner to that in which the identification current works normally.
However this case is not generated by any simple current, as can be seen, for
example, by looking at the component $\chi_{00}^{00} + \chi_{8}^{00} + \chi_{4}^{11}$. The last term is not
related to the others by any simple current.

We feel this single example to be very instructive. Although the technology of
Schellekens and Yankielowicz provides a means of determining the primary fields
of a coset theory in most cases this is an example where it does not work. This
method relies on several assumptions, clearly stated as e.g. in ref.[10], which may
not hold. In particular, this theory is a counterexample to the proposition that
characters will be non-zero provided that $PA - \lambda$ lies within the lattice $PM_g, (P$
denoting the embedding of the weight space), as can be seen by examining the
character $\chi_{2}^{00}$ which is zero for $k = 2$. The equivalence of characters is also clearly
greater in this case than that expected by considering the automorphisms of the
extended dynkin diagrams of $\hat{g}$ and $\hat{h}$. Maverick coset spaces will be dealt with in
a future work [21]. Coset theories provide a rich framework in which to construct
rational CFTs which is not yet fully understood.

7. Conclusions Coset conformal field theories have proved a valuable tool
in the study of rational conformal field theories. In this paper we have studied
the characters of a coset conformal field theory and produced some quite general
formulae for these characters. These characters determine the spectrum of the
coset theories and as such are crucial in determining the physical spectrum in
string theories. In particular the spectrum of the $h$–values is determined by the
characters, including the integer shifts. We obtained a formula for the integer
shift agreeing with calculations done previously for specific cases. The characters
can also be used to study the existence of extended symmetries in coset CFTs.
We have also used these character formulae to provide an example of a coset
theory which cannot be described by the identification current method. This
coset $su(3)_2/su(2)_8$, is a “maverick” theory which has more zero characters and
 equivalences than specified by the identification current method. We expect the
study of this and other maverick theories, via the branching function formulae or
other methods to provide much insight into the study of coset CFTs.

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Appendix A. Generalising the Character formulae

Suppose we have a coset $\hat{g}/\hat{h}$ which decomposes into the form:

$$\frac{\hat{g}_k}{\hat{h}_l} = \frac{(\hat{g}_1)_{k_1} \times \cdots \times (\hat{g}_n)_{k_n}}{(\hat{h}_1)_{l_1} \times \cdots \times (\hat{h}_m)_{l_m}} \quad (A.1)$$
where \(\{G_i\}\) and \(\{H_j\}\) are semisimple compact Lie groups other than \(U(1)\). Note that this means that for each \(j\), \(H_j \subset G_{i(j)}\), for some \(i(j)\), which will be an aid to the calculation in what follows. In this appendix we are using barred indices to refer to the subalgebras \(\hat{h}\) and unbarred indices for the algebras \(\hat{g}\). The embedding of \(h\) in \(g\) is specified by

\[
(H^h)^j = P_{kj}^j (H^g)_l \quad E^\alpha_i = M_{\alpha_k}^\alpha E_{\alpha_k}
\]

(A.2)

where \((H^g)_l\) denotes an element of the CSA of \(g_k\) and \(\alpha_k\) denotes a root of \(g_k\). Associated with each Cartan generator \((H^g)_l\) there is a basis vector of the root space \(e^k_l\). Denoting by \(M\) the root lattice and \(W\) the weight lattice, let elements of \(M_g\) be \(\alpha \equiv (\alpha_1, \ldots, \alpha_n)\) we have the mapping:

\[
P : M_g \rightarrow W_h
\]

\[
\alpha \rightarrow \bar{\lambda} = P_{ij}^l (\alpha_k \cdot e^k_l) e^j_i
\]

(A.3)

An arbitrary vector in the weight space of \(\hat{h}\) may be written

\[
v \equiv -2\pi i \sum_{i=1}^m (\sum_{j=1}^{\text{rank}(h_i)} z^i_i e^i_j + \tau(\Lambda_0)_i + u \bar{\delta}_i)
\]

(A.4)

Again writing \(\Lambda = (\Lambda_1, \ldots, \Lambda_n)\), where \(\Lambda_i\) is a highest weight label for \(\hat{g}_i\), the branching function equation becomes

\[
\text{ch}_{\Lambda}^g (v) = \sum_{\mu} b_{\Lambda}^\mu \text{ch}_\mu (v)
\]

(A.5)

where \(\mu\) are weight labels for \(\hat{h}\). In evaluating the lhs we can use (3.1),(3.2),(3.3) directly, provided we set

\[
z^i_k = P_{ki}^{ij} z^j_i
\]

(A.6)

When evaluating \(D_g\) using (3.2) we can use the following relation

\[
\alpha_k \cdot z_k = \alpha_i \cdot z_i \quad \forall \alpha_k, \alpha_i \text{ st. } M_{\alpha_k}^{\alpha_i} \neq 0
\]

(A.7)

It is therefore easy to decompose \(D_g\):

\[
D_{\hat{g}_{j(i)}} = D_{\hat{h}_{i_1}} \cdots D_{\hat{h}_{i_p}} \prod_{\alpha \in S} (1 - e^{-\alpha})^{\text{mult} \alpha}
\]

(A.8)

where \(H_{i_1} \times \cdots \times H_{i_p} \subset G_j\), and \(S\) is the complement of the set of positive roots of the \(h_i\) in those of \(Pg_{j(i)}\). It is important to note (A.8) is meaningful only when
evaluated on the weight space of $h$ using (A.6). Having thus removed $D_h$ and writing $D_{\hat{g}} = D_{g/h}D_h$ we have

\[
\prod_{i=1}^{n} \frac{N^\delta_{\Lambda_i}(q, z_i)}{D_{g/h}(q, \{z_j\})} = \sum_{\mu} b^{\mu_1 \ldots \mu_m}_{\Lambda_1 \ldots \Lambda_n} \prod_{j=1}^{m} N^{h_j}_{\mu_j}(q, z_j) \tag{A.9}
\]

It is convenient to define the variables

\[
y_{ij} \equiv \exp(2\pi i z_j) \quad i = 1 \ldots m, \quad j = 1 \ldots \text{rank}(h_i) \tag{A.10}
\]

Thus, as in section 3, we obtain

\[
b^{\lambda_1 \ldots \lambda_m}_{\Lambda_1 \ldots \Lambda_n}(q) = \exp\left(2\pi i \left(\sum_{i=1}^{m} \lambda_i \cdot z_i\right)\right) \prod_{i=1}^{n} \frac{N^\delta_{\Lambda_i}(q, z_i)}{D_{g/h}(q, \{z_j\})} \left|_{\text{coeff of } (y_i)^0} \right. \tag{A.11}
\]

Again, $\delta_n$ and $s_{tn}$ are defined by

\[
\begin{align*}
&n \geq 0 : \delta_n = 1, s_{tn} = 0 \\
&n < 0 : \delta_n = -1, s_{tn} = 1
\end{align*} \tag{A.12}
\]

Thus (3.2), (3.4), (3.13), (A.11) together give us our general result for the branching function

\[
b^{\lambda_1 \ldots \lambda_m}_{\Lambda_1 \ldots \Lambda_n}(q) = \exp(2\pi i \left(\sum_{i} \lambda_i \cdot z_i\right) ) \phi(q)^{|h|-|g|} \prod_{i=1}^{n} \left( \sum_{w_i \in W_{g_i}} \epsilon(w_i) e^{2\pi i p_i \cdot z_i} \right) \times \sum_{\gamma_i \in M^L_i} q^{\frac{1}{2}(k_i + g_i)\gamma_i^2 + \gamma_i \cdot w_i(\Lambda_i + \rho_i)} \exp\left(-2\pi i (k_i \gamma_i \cdot z_i + w_i(\Lambda_i + \rho_i) \cdot z_i)\right) \\
&\times \prod_{\tilde{\alpha}_i \in S} \left( \sum_{n_{\tilde{\alpha}_i} = -\infty}^{\infty} (-1)^{n_{\tilde{\alpha}_i}} \delta_{n_{\tilde{\alpha}}} q^{n_{\alpha}(n_{\alpha} + 1)/2 - r_{\alpha} \cdot \delta_{\alpha}} \exp\left(2\pi i n_{\alpha} \cdot z_i\right) \right) \left|_{\text{coeff of } (y_i)^0} \right. \tag{A.13}
\]

The relevant terms satisfy

\[
\lambda + P \sum_{\alpha \in S} \alpha \delta_{n_{\alpha}} r_{\alpha} + P \rho = P \left\{ \sum_{i=1}^{n} (k_i + g_i) \gamma_i + w_i(\Lambda_i + \rho_i) \right\} \tag{A.14}
\]

Note this this equation implies the well known condition $\lambda - PA \in PM_g$. Thus using (A.13) directly we may compute any branching function. In order to make such a computation efficient it is necessary to solve the constraints (A.14) for some of the $r_{\tilde{\alpha}_{ik}}$ so as to eliminate them from (A.13). To do this define $S_0 \subset PS$ form a basis for $PM_g$, and let $S_0^* = \{w_{\alpha}\}$ be its dual, such that $w_{\alpha} \cdot P \beta = \delta_{\alpha,\beta}$. Define

\[
V = P \left( \sum_{k} (k_k + g_k)\gamma_k + w_k(\Lambda_k + \rho_k) - \rho_k \right) - \sum_{\alpha \in S \setminus S_0} \alpha \delta_{n_{\alpha}} r_{\alpha} - \lambda \tag{A.15}
\]

$V_{\beta} \equiv w_{\beta} \cdot V$
Thus the constraints (A.14) become

\[ V_\beta = r_\beta \delta n_\beta \quad (A.16) \]

We can then insert this into (A.13) to obtain the general result:

\[ b_{\lambda_1 \ldots \lambda_m}(q) = \phi(q)|g|^{-|h|} \prod_{i=1}^{n} \sum_{w_i \in W_{\gamma_i}} \epsilon(w_i) F_w,\Lambda(q) \quad (A.17) \]

where

\[ F_w,\Lambda(q) = \sum_{\gamma_i \in M_i} \sum_{n_\alpha = -\infty}^{\infty} \sum_{r_\alpha = s_\alpha} \sum_{\alpha \in P \setminus S_0} \sigma q^N \quad (A.18) \]

where

\[ \sigma = \left( \prod_{\alpha \in S} \delta n_\alpha \right) (-1)^{\sum_{\alpha \in S} n_\alpha} \]

\[ N = \frac{1}{2} \sum_i (k_i + g_i) \gamma_i^2 + w_i (\Lambda_i + \rho_i) \cdot \gamma_i \]

\[ + \frac{1}{2} \sum_{\alpha \in S} n_\alpha (n_\alpha + 1) + \sum_{\alpha \in P \setminus S_0} n_\alpha r_\alpha \delta n_\alpha + \sum_{\alpha \in S_0} V_\alpha n_\alpha \quad (A.19) \]
### Table 1.

For the coset theory $\hat{su}(3)_2/\hat{su}(2)_8$, we show the extra equivalences and vanishing of characters beyond that expected by the identification current method. Using the identification current method all characters shown are expected to be non-zero.

| Equivalences of Character | Character Value |
|---------------------------|-----------------|
| $\chi_0^{(00)} \equiv \chi_8^{(00)} \equiv \chi_4^{(11)}$ | $1 + q^2 + 2q^3 + 3q^4 + 4q^5 \cdots$ |
| $\chi_{(11)} \equiv \chi_6^{(11)} \equiv \chi_4^{(00)}$ | $1 + 2q + 2q^2 + 4q^3 + 5q^4 + 8q^5 \cdots$ |
| $\chi_2^{(10)} \equiv \chi_6^{(10)} \equiv \chi_4^{(02)}$ | $1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 \cdots$ |
| $\chi_2^{(01)} \equiv \chi_6^{(01)} \equiv \chi_4^{(20)}$ | $1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 \cdots$ |
| $\chi_2^{(20)} \equiv \chi_8^{(20)} \equiv \chi_4^{(01)}$ | $1 + q + 2q^2 + 2q^3 + 4q^4 + 5q^5 \cdots$ |
| $\chi_0^{(02)} \equiv \chi_8^{(02)} \equiv \chi_4^{(10)}$ | $1 + q + 2q^2 + 2q^3 + 4q^4 + 5q^5 \cdots$ |
| $\chi_0^{(00)} \equiv \chi_8^{(00)} \equiv \chi_4^{(11)}$ | $0$ |
| $\chi_2^{(20)} \equiv \chi_6^{(20)}$ | $0$ |
| $\chi_2^{(02)} \equiv \chi_6^{(02)}$ | $0$ |
| $\chi_2^{(10)} \equiv \chi_6^{(10)}$ | $0$ |
| $\chi_0^{(01)} \equiv \chi_8^{(01)}$ | $0$ |
| $\chi_0^{(11)} \equiv \chi_8^{(11)}$ | $0$ |
| $\chi_0^{(10)} \equiv \chi_8^{(10)}$ | $0$ |
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