XOR games with \(d\) outcomes and the task of non-local computation

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A natural generalization of the binary XOR games to the class of XOR-\(d\) games with \(d > 2\) outcomes is studied. We propose an algebraic bound to the quantum value of these games and use it to derive several interesting properties of these games. As an example, we re-derive in a simple manner a recently discovered bound on the quantum value of the CHSH-\(d\) game for prime \(d\). It is shown that no total function XOR-\(d\) game with uniform inputs can be a pseudo-telepathy game, there exists a quantum strategy to win the game only when there is a classical strategy also. We then study the principle of lack of quantum advantage in the distributed non-local computation of binary functions which is a well-known information-theoretic principle designed to pick out quantum correlations from amongst general no-signaling ones. We prove a large-alphabet generalization of this principle, showing that quantum theory provides no advantage in the task of non-local distributed computation of a restricted class of functions with \(d\) outcomes for prime \(d\), while general no-signaling boxes do. Finally, we consider the question whether there exist two-party tight Bell inequalities with no quantum advantage, and show that the binary non-local computation game inequalities for the restricted class of functions are not facet defining for any number of inputs.

Introduction. Non-local correlations are one of the most intriguing aspects of Nature, evidenced in the violation of Bell inequalities. Besides their foundational interest, these correlations have also proven to be useful in information processing tasks such as secure device-independent randomness amplification and expansion [1], cryptographic secure key generation [2] and reduction of communication complexity [3]. Concerning such applications, it is typically of interest to compute the classical and quantum value of the Bell expression, the classical value being the maximum over local realistic assignments of outcomes while the quantum value is the maximum attained using measurements on entangled quantum states. Computing the classical value is in general a hard problem, being an instance of an integer program [4, 5]. It has also been shown [6] that computing the quantum value is hard, although in some instances it is possible to compute it efficiently or to find a good approximation. A hierarchy of semi-definite programs from [7] is typically used to get (upper) bounds on the quantum value, although the quality of approximation achieved by these bounds remains unknown.

An important class of Bell inequalities for which the quantum value can be computed exactly is the class known as two-party binary XOR games or equivalently as bipartite two-outcome correlation inequalities. In a binary XOR game, the two parties Alice and Bob receive inputs \(x \in [m_x], y \in [m_y]\) (where \([m_A] = \{1, \ldots, m_A\}\)) and respond with outputs \(a, b \in \{0, 1\}\). The winning constraint for each pair of inputs \((x, y)\) only depends on the XOR modulo 2 of the parties’ answers, i.e., the Bell expression in the binary XOR game only involves probabilities \(P(a \oplus_2 b = k|x, y)\) for \(k \in \{0, 1\}\). The fact that these are equivalent to correlation inequalities for binary outcomes is simply seen by noting that in this case the correlators \(\mathcal{E}_{x,y}\) are given by \(\mathcal{E}_{x,y} = \sum_{k=0,1}(-1)^kP(a \oplus_2 b = k|x, y)\). For these games, it was shown in [8, 9] based upon a theorem by Tsirelson [10] that the quantum value can be computed efficiently by means of a semi-definite program, although computing the classical value is known to be a hard problem even for this class of games [5]. Besides binary XOR games, few general results are known regarding the maximum quantum violation of classes of Bell inequalities, although computations have been performed in some specific instances [11].

A natural generalization of the binary XOR games is to the class of XOR-\(d\) games, where the outputs of the two parties are not restricted to be binary, although the winning constraint still depends upon the XOR modulo \(d\), with \(d\) being the number of outcomes. The generalization can also be extended to the class known as L\(E\)AR games [5], where the parties output answers that are elements of a finite Abelian group and the winning constraint depends upon the group operation acting on the outputs. These games have been studied [5, 12] in the context of hardness of approximation of several important optimization problems, in attempts to identify the existence of polynomial time algorithms to approximate the optimum solution of the problem to within a constant factor. In the context of Bell inequalities, these were first studied in [13] where a large alphabet generalization of the CHSH inequality called CHSH-d was considered, which has since been investigated in [14–17], [34]. While an efficient algorithm has been shown in [19] for approximating the quantum value of any lin-
ear game up to a constant factor, this is more useful for proving certain interesting results such as the entangled parallel repetition theorem and for approximating the quantum value in instances where this is close to the (maximum possible value of) unity, while being less useful for instances such as CHSH-d. An important property of the XOR-d games concerns their relationship with communication complexity, following [20, 21] it is seen that correlations (boxes) winning a non-trivial total function XOR-d game can result in a trivialization of communication complexity. A related information-theoretic principle called absence of non-local computation (no-NLC) has also been suggested in [22]; this proposes that quantum correlations are those that do not provide any advantage over classical correlations in the task of distributed non-local computation of arbitrary binary functions, while general no-signaling correlations do.

In this paper, we present a linear algebraic bound to the quantum value of linear games and use it to derive several interesting properties of these games, specifically those of the XOR-d games. We illustrate the bound with the example of the CHSH-d game for prime and prime power \(d\), comparing it with recently derived results using alternative methods. We show that for uniformly chosen inputs, no non-trivial total function XOR-d game can be won with a quantum strategy (in other words there is no pseudo-telepathy game [11] within this class) and consequently that these no-signaling boxes that trivialize communication complexity cannot be realized within quantum theory. We prove a certain large alphabet generalization of the no-NLC principle, showing that quantum theory provides no advantage in the task of non-local distributed computation of a restricted class of functions with \(d\) outcomes for prime \(d\). As it has been of interest recently [23–25] to compute a non-trivial boundary (facet) of the quantum correlation set, we also consider the question whether the non-local computation inequalities are facet-defining and answer the question in the negative for the restricted class of functions with binary outputs, while the general question remains open.

A bound on the quantum value of linear games. Linear games are a generalization of binary XOR games to an arbitrary output alphabet size and are defined as follows:

**Definition 1.** A two-player linear game \( (g^l, q) \) is one where two players Alice and Bob receive questions \( u, v \) from sets \( Q_A, Q_B \) respectively, chosen from a probability distribution \( q(u, v) \) by a referee. They reply with respective answers \( a, b \in (G, +) \) where \( G \) is a finite Abelian group with associated operation \(+\). The game is defined by a winning constraint \( a + b = f(u, v) \) for some function \( f : Q_A \times Q_B \rightarrow G \).

XOR-d games, denoted \( g^\oplus \) are linear games with the associated group being the cyclic group \( \mathbb{Z}_d \), the integers with operation \( \oplus_d \) addition modulo \( d \). The value of the game is given by the expression

\[
\omega(g) = \sum_{u \in Q_A} \sum_{v \in Q_B} q(u, v) P(a + b = f(u, v) | u, v). \tag{1}
\]

The maximum classical value of the game (the maximum over all deterministic assignments of \( a, b \) or their convex combinations) is denoted \( \omega_c(g) \), the value of the game achieved by a quantum strategy (POVM measurements on a shared entangled state) is denoted \( \omega_q(g) \), while the value achieved by no-signaling strategies (where neither party can signal their choice of input using the correlations) is denoted \( \omega_{ns}(g) \). Linear games belong to the class of unique games [19] and for every game in this class, a no-signaling box exists that wins the game, i.e., \( \omega_{ns}(g) = 1 \). Such a box is simply defined by the entries \( P(a, b | u, v) = 1/|G| \) if \( a + b = f(u, v) \) and 0 otherwise for all input pairs \((u, v)\), this strategy clearly wins the game, and is no-signaling since the output distribution seen by each party is fully random, i.e., \( P(a | u) = P(b | v) = 1/|G| \).

As in the case of Boolean functions [26, 27], it can be seen that the classical value \( \omega_c(g^l) \) for any linear game is strictly greater than the pure random guess value \( 1/|G| \).

**Lemma 1.** For any linear game \( g^l \) corresponding to a function \( f(u, v) \) with \( u \in Q_A, v \in Q_B \) and for an arbitrary probability distribution \( q(u, v) \), we have

\[
\omega_c(g^l) = \min \left\{ |Q_A|, |Q_B| \right\}, \tag{2}
\]

where \( m = \min \{ |Q_A|, |Q_B| \} \).

We are now ready to present a bound on the quantum value of a linear game using a set of norms of its game matrices defined using the characters, generalizing the bound on binary XOR games from [22] which was rediscovered in [28].

**Theorem 2.** The quantum value of a linear game \( g^l \) with input sets \( Q_A, Q_B \) can be bounded as

\[
\omega_q(g^l) \leq \frac{1}{|G|} \left[ 1 + \sqrt{|Q_A| |Q_B|} \sum_{x \in G \setminus \{e\}} \| \Phi_x \| \right], \tag{3}
\]

where \( \Phi_x = \sum_{(u,v) \in Q_A \times Q_B} q(u,v) \chi_x(f(u,v)) |u\rangle \langle v| \) are the game matrices, \( \chi_x \) are the characters of the group \( G \) and \( \| \cdot \| \) denotes the spectral norm. In particular, for an XOR-d game with \( m_A \) and \( m_B \) inputs for the two parties, the quantum value can be bounded as

\[
\omega_q(g^\oplus) \leq \frac{1}{d} \left[ 1 + \sqrt{m_A m_B} d^{-d} \sum_{k=1}^{d-1} \| \Phi_k \| \right], \tag{4}
\]

with \( \Phi_k = \sum_{u \in [m_A]} \sum_{v \in [m_B]} q(u, v) \zeta^k \chi_{f(u,v)}(u) \langle v| \) and \( \zeta = \exp(2\pi i/d) \).
It should be noted that as shown in [19], the quantum value of a linear game can be efficiently approximated, to be precise for any $g' \in G$ with $\omega_q(g') = 1 - \delta'$, there exists an efficient algorithm to approximate this value using a semi-definite program and a rounding procedure that gives an entangled strategy achieving $\omega_q^{app}(g') = 1 - \delta''$, where $\delta/4 \leq \delta'' \leq \delta$. While this is highly significant and useful for proving results such as the entangled parallel repetition theorem for such games [19], it would appear to be good for approximating the quantum value only when the quantum value is close to unity, which is not the case for simple examples like the CHSH-d game.

For uniform probability inputs $q(u,v) = 1/|Q_A||Q_B|$ or when the input distribution possesses certain symmetries, as we shall see the simple linear algebraic bound above supplants this result and proves to be very useful to derive other interesting properties of these games.

We first illustrate the applicability of the bound by considering the flagship scenario of the CHSH-d game which is a natural generalization of the CHSH inequality to a higher dimensional output. In this game, Alice and Bob are asked questions $u, v$ chosen uniformly at random from a finite field $F_d$ of size $d$ so that $q(u,v) = 1/d^2$, where $d$ is a prime, or a prime power. They return answers $a, b \in F_d$ with an aim to satisfy $a \oplus b = u \cdot v$ where the arithmetic operations are taken modulo $d$. In [16], an intensive study of this game was performed, with two significant results obtained on the asymptotic classical and quantum values of the game. We now apply Theorem 2 to re-derive in a simple manner the upper bound for the quantum value of CHSH-d for prime $d$.

**Lemma 3** (see also [16]). The quantum value of the CHSH-d game for prime $d$ can be bounded as

$$\omega_q(CHSH - d) \leq \frac{1}{d} + \frac{d - 1}{d \sqrt{d}}, \quad (5)$$

For prime power $d = p^r$ where $p$ is prime and $r > 1$ is an arbitrary integer, the quantum value is bounded by

$$\omega_q(CHSH - d) \leq \frac{1}{d} + \frac{1}{\sqrt{d}} \sum_{q=0}^{r-1} \phi(p^{r-q}) \frac{\sqrt{p^q}}{d \sqrt{d}} \leq \frac{1}{d} + \frac{d - 1}{d \sqrt{d}}, \quad (6)$$

where $\phi(\cdot)$ is Euler’s totient function.

Comparing with the numerical results of [14, 15] indicates that while the above bound may be achieved for some small $d$, it may not be tight in general, also note that the optimum value of the game for Pauli measurements was recently derived in [17].

Let us now show that no non-trivial game for a total function $f(u,v)$ (a total function is one which is defined for all input pairs $(u,v)$) within the class of XOR-d games $g^\ominus$ with uniformly chosen inputs can be won by a quantum strategy, meaning that there is no pseudo-telepathy game [11] within this class. This extends a similar statement proven for binary games in [8] (beyond XOR games, binary games are all games with outputs in $\{0,1\}$).

**Lemma 4.** For XOR-d games $g^\ominus$ corresponding to total functions with $m$ questions per player, when the input distribution is uniform $q(u,v) = 1/m^2$, $\omega_q(g^\ominus) = 1$ iff $\omega_q(g^\ominus) = 1$, i.e., when rank($\Phi_1$) = 1.

Following the results of [20, 21], any non-trivial total function XOR-d game for prime $d$ and any number of dits as input $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n)$ is won by a no-signaling box that trivializes communication complexity, and Lemma 4 shows that these boxes cannot be realized within quantum theory. To elaborate, it can be seen that any no-signaling box that wins a non-trivial total function XOR-d game for prime $d$ must contain as a sub-box, one of the functional boxes of the form $P(a \oplus b = f(x_i, y_i) | u, v) = 1/d$ for $a,b,u,v \in \{0,\ldots,d-1\}$; having $d^p$ copies of the box and addressing this sub-box in each, Alice and Bob can compute any function of $d$ outputs with a single bit of communication, resulting in a trivialization of communication complexity. It was recently shown that all the extremal points of the no-signaling polytope for any number of inputs and outputs cannot be realized within quantum theory [30]. It remains an open question whether all such vertices lead to a trivialization of communication complexity (at least in a probabilistic setting), if so this would be a compelling reason for their exclusion from correlations that can be realized in Nature.

**XOR-d games with no quantum advantage: the task of non-local computation.** Even though the quantum non-local correlations cannot be used to transmit information, they enable the performance of several tasks impossible in a classical world, such as the expansion and amplification of intrinsic randomness, device-independent secure key generation, etc. An unexpected limitation of quantum correlations however is the fact that they do not provide any advantage over classical correlations in the performance of an important information-theoretic task, namely the non-local distributed computation of Boolean functions [22], even though certain super-quantum no-signaling correlations do.

Consider a Boolean function $f(z_1, \ldots, z_n)$ from $n$ bits to 1 bit. A non-local (distributed) computation of the function is defined as follows. Two parties, Alice and Bob, are given inputs $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ obeying $x_i \oplus y_i = z_i$ each bit $x_i, y_i$ being 0 or 1 with equal probability. This ensures that neither party has access to any input $z_i$ on their own. To perform the non-local computation, Alice and Bob must output bits $a$ and $b$ respectively such that $a \oplus b = f(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$. Their goal is thus to maximize the probability of success in this task for some given input distribution $p(z_1, \ldots, z_n) = p(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$. In [22], it was shown that surprisingly for any input distribution
The games \( \text{N LC} \), Alice and Bob sharing quantum resources cannot do any better than classical resources (both give rise to only a linear approximation of the computation), while they could successfully perform the task if the resources they shared were limited by the no-signaling principle alone. This lack of advantage in non-local computation was so striking that it was postulated as an information-theoretic principle that picks out quantum theory from among general no-signaling theories, in relation to the correlations that the theory gives rise to [22]. It is of interest to find whether any of the inequalities defined by these games define facets of the classical polytope (note that a facet of a polytope is a face with dimension one less than that of the polytope).

The above consideration of functions with a single-bit output is important since these encapsulate all decision problems, a natural class of problems used to define computational complexity classes. In the program of characterizing quantum correlations however, we must consider functions with multi-bit outputs as well as functions with higher input and output alphabets. We now use the bound (12) to construct a generalized non-local computation task for functions with higher input output alphabet. Consider the following generalization of the non-local computation task to \( \text{XOR-d} \) games, namely the computation of the function \( g(z_1, \ldots, z_n) \) with \( z_i \in \{0, \ldots, d-1\} \) where \( d \) is a prime. In these games which we label \( \text{N LC}_d \), Alice and Bob receive \( n \) dits \( x_n = (x_1, \ldots, x_n) \) and \( y_n = (y_1, \ldots, y_n) \) which obey \( x_i \oplus_d y_i = z_i \). Their task is to output dits \( a, b \) respectively such that

\[
a \oplus_d b = g(x_n \oplus_d y_n) \cdot (x_n \oplus_d y_n),
\]

(7)

where \( x_n \oplus_d y_n \) is the dit-wise XOR of the \( n \) \(-d \) dits, i.e., \( \{ x_1 \oplus_d y_1, \ldots, x_n \oplus_d y_n \} \) and \( g \) is an arbitrary function from \( n-1 \) dits to 1 dit. The inputs are chosen according to

\[
\frac{1}{d^{n+1}} p(x_{n-1} \oplus_d y_{n-1})
\]

(8)

for \( p(x_{n-1} \oplus_d y_{n-1}) \) being an arbitrary probability distribution.

We now show that the games \( \text{N LC}_d \) as defined above exhibit no quantum advantage. The idea behind the proof is to show that the game matrices \( \Phi_k^1 \Phi_k \) for these games are diagonal in a basis composed of tensor products of the Fourier vectors corresponding to dimension \( d \). We then present a classical strategy which achieves the quantum value, which is essentially given by the maximum singular vectors of \( \Phi_k \).

**Theorem 5.** The games \( \text{N LC}_d \) for arbitrary prime \( d \) and for input distribution satisfying (8) have no quantum advantage, i.e., \( \omega_c(\text{N LC}_d) = \omega_q(\text{N LC}_d) \).

As mentioned previously, all unique games including the \( \text{XOR-d} \) games have no-signaling value of unity, so that in general \( 1 = \omega_{nd}(\text{N LC}_d) > \omega_q(\text{N LC}_d) \). Note that the slight restriction in Eq. (7) (a fixed dependence on \( x_n \oplus_d y_n \)), means that the games do not cover the entire class of functions considered in [22], it remains open whether there is no quantum advantage for the remaining functions in this class as well.

We now proceed to consider whether any of the games within the non-local computation class is a tight Bell inequality, i.e., whether any of them define a facet of the local polytope. Recall that a facet of a polytope of dimension \( D \) is a face of dimension \( D - 1 \). The facets are the simplest possible description of the Bell polytope, for any box outside the polytope, there exists a facet-defining inequality violated by the box. There has been considerable interest in identifying a portion of the quantum boundary which is of maximal dimension, such a task has been achieved in the case of three or more parties [23] and has given rise to the interesting information-theoretic principle known as local orthogonality [25]. The question whether the \( \text{NLC}_d \) games give rise to facet-defining inequalities was left open in [22], a partial answer was provided in [23] where it was shown that the \( \text{NLC}_d \) games are not facets for inputs of 2 and 3 bits. Here we answer this question in the negative for the binary \( \text{NLC}_d \) games for the class of functions in Eq. (7) with \( d = 2 \) for arbitrary input size and present as open the corresponding question for the \( \text{NLC}_d \) games and for more general bipartite inequalities.

**Theorem 6.** The non-local computation game inequalities for functions of the form in Eq. (7) for \( d = 2 \) do not define facets of the local polytope for any input size \( 2^n \).

The proof of this statement relies on a decomposition of the non-local computation inequalities into multiple face-defining inequalities, indicating that they cannot be facets. However, the proof does not seem to extend in a straightforward manner to the \( \text{NLC}_d \) games. The question of whether there exist tight two-party Bell inequalities with no quantum violation thus remains open.

**Conclusions.** In this paper, we have presented an easily computable bound on the quantum value of linear games, with particular emphasis on \( \text{XOR-d} \) games for prime \( d \). We have used this bound to rule out from the quantum set a class of no-signaling boxes that result in a trivialization of communication complexity. To do this, we have shown that no uniform input total function \( \text{XOR-d} \) game can be a pseudo-telepathy game. We have also shown how the recently discovered bound on the CHSH-d game in [16] can be derived in a simple manner for prime \( d \). Finally, we have extended the principle of no-NLC to general prime dimensional output, showing that quantum theory provides no advantage over classical theories in the distributed non-local computation of functions with prime dimensional output. We have also considered the question whether there exist tight two-party Bell inequalities with no quantum ad-
vantage, showing that while the NLC inequalities (for the restricted class of functions) do not define facets, the question remains open for NLC$_d$ and for general bipartite Bell inequalities.

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Supplemental Material. Here, we present the proofs of the propositions stated in the main text.

Lemma 1. For any linear game $g'$ corresponding to a function $f(u,v)$ with $u \in Q_A, v \in Q_B$ and for an arbitrary probability distribution $q(u,v)$, we have

$$\omega_d(g') \geq \frac{1}{|G|} \left(1 + \frac{|G|-1}{m}\right), \quad (9)$$

where $m = \min\{|Q_A|, |Q_B|\}$.

Proof. Let $d = |G|$, Alice and Bob receive as inputs $u,v$ of $\log_d |Q_A|$ and $\log_d |Q_B|$ bits respectively. Suppose without loss of generality that $|Q_A| \leq |Q_B|$ ($m = |Q_A|$), and let the two parties Alice and Bob share a uniformly distributed random variable $\omega$ of $\log_d |Q_A|$ bits. The following classical strategy achieves the lower bound in Eq.(9). Bob outputs $b = f(w,v)$, while Alice checks if $u = w$ and outputs $a = e$ if this is the case; if not she outputs a uniformly distributed $a \in G$. In the case when $u \neq w$ which happens with probability $1/m$, we see that $a + b = e + f(w,v) = f(u,v)$ and the strategy succeeds. When $u = w$, we have that $a + f(w,v)$ is uniformly random since $a$ is uniform, and the strategy succeeds with probability $1/d$. The value achieved by this strategy is therefore $1/m + (1-1/m)/d$ which gives the lower bound in Eq. (9). □

In what follows, we will use the notion of the characters of a finite Abelian group, defined in a standard manner as follows.

Definition 2. Let $G$ be a finite Abelian group with $|G|$ elements, with operation $+$ and identity element $e$. A character
of $G$ denoted $\chi$ is a homomorphism from $G$ to the multiplicative group of complex roots of unity:

$$\chi(a + b) = \chi(a)\chi(b) \quad (a, b \in G)$$

(10)

The characters of $G$ form a finite group denoted $\hat{G}$ under elementwise multiplication. The identity element of $G$ is denoted $\chi(e)$ and satisfies $\chi(e) = 1$ for all $g \in G$.

A useful property of the characters is that for any $\chi \neq e \in \hat{G}$, we have $\sum_{x \in G} \chi(x) = 0$ and that for any $e \neq g \in G$, we have $\sum_{x \in G} \chi'(g) = 0$. Note that the dual group $\hat{G}$ and $G$ are in fact isomorphic to each other. For each $x \in G$, let us denote by $\chi_x$ the image of $x$ under a fixed isomorphism of $G$ with $\hat{G}$.

Theorem 2. The quantum value of a linear game $g^l$ with input sets $Q_A, Q_B$ can be bounded as

$$\omega_q(g^l) \leq \frac{1}{|G|^2} \left[ 1 + \sqrt{|Q_A||Q_B|} \sum_{x \in G} \|\Phi_x\| \right],$$

(11)

where $\Phi_x = \sum_{(u,v) \in Q_A \times Q_B} q(u,v)\chi_x(f(u,v))|u⟩⟨v|$ are the game matrices, $\chi_x$ are the characters of the group $G$ and $\|\cdot\|$ denotes the spectral norm. In particular, for an XOR-game with $m_A$ and $m_B$ inputs for the two parties, the quantum value can be bounded as

$$\omega_q(g^{\oplus}) \leq \frac{1}{d} \left[ 1 + \sqrt{m_A m_B} \sum_{k=1}^{d-1} \|\Phi_k\| \right],$$

(12)

with $\Phi_k = \sum_{u \in [m_A]} \sum_{v \in [m_B]} q(u,v)\zeta^k f(u,v) |u⟩⟨v|$ and $\zeta = \exp(2\pi i/d)$.

Proof. To derive a bound on the quantum value of a linear game $\omega_q(g^l)$, we make use of the generalized Fourier transform on finite Abelian groups [31]. Let us first note that by the fundamental theorem of finite Abelian groups, any finite Abelian group $G$ can be seen as a direct product of cyclic groups as $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ for some integers $n_1, \ldots, n_k$, where $\times$ denotes the direct product and $\mathbb{Z}_n$ denotes the cyclic group of order $n$. Every element $x \in G$ can thus be seen as a $k$-tuple $(x_1, \ldots, x_k)$ with $x_i \in \mathbb{Z}_{n_i}$. Denoting by $\chi_a$ the characters of the Abelian group $G$, we see that these can be written as $\chi_a(x) = \prod_{j=1}^k \zeta_{a_j}^{x_j}$, where $\zeta_j = \exp(2\pi i/n_j)$ is the $n_j$-th root of unity, and $a_j \in \mathbb{Z}_{n_j}$ for $j \in [k]$. The above relation gives a total of $\prod_{j=1}^k n_j = |G|$ (orthogonal) characters and consequently accounts for all the characters of $G$. Note that $\bar{\chi_a}(x) = \chi_a(-x)$, where $\bar{\chi_a}$ denotes the conjugate character, and $\chi_a(x) = \chi_x(a)$. We now introduce the generalized correlators $\langle A^{x}_u B^{y}_v \rangle$ via the Fourier transform of probabilities $P(a, b|u, v)$ on the group, defined as

$$\langle A^{x}_u B^{y}_v \rangle = \sum_{a,b \in G} \bar{\chi}_x(a) \bar{\chi}_y(b) P(a, b|u, v).$$

(13)

The probabilities are then given by the inversion formula

$$P(a, b|u, v) = \frac{1}{|G|^2} \sum_{x,y \in G} \chi_a(x) \chi_b(y) \langle A^{x}_u B^{y}_v \rangle.$$

(14)

The marginals $\langle A^{x}_u \rangle$ are given by

$$\langle A^{x}_u \rangle = \sum_{a,b \in G} \bar{\chi}_x(a) \bar{\chi}_y(b) P(a, b|u, v) = \sum_{a \in G} \chi_x(a) P(a|u),$$

(15)

where $e$ denotes the identity element of the group with $\chi_e$ being the trivial character ($\chi_e(a) = 1 \forall a \in G$) and we have used the no-signaling condition $\sum_{a \in G} P(a, b|u, v) = P(a|u)$; an analogous expression holds for $\langle B^{y}_v \rangle = \sum_{b \in G} \chi_y(b) P(b|v)$.

The normalization constraint is written as $\langle A^{x}_u B^{y}_v \rangle = 1 \forall (u, v) \in Q_A \times Q_B$. The probabilities $P(a, b|u, v)$ that enter the game expression can therefore be evaluated as

$$P(a + b = f(u,v)|u,v) = \frac{1}{|G|^2} \sum_{x,y \in G} \chi_x(x) \chi_y(y) \langle A^{x}_u B^{y}_v \rangle.$$

(16)

Using the orthogonality of the characters $\chi_x(x) \chi_y(y) = |G|\delta_{x,y}$ where $\delta_{x,y}$ denotes the Kronecker delta, and the property of the characters that $\chi_x(a+b) = \chi_x(a)\chi_x(b)$ we get that

$$P(a + b = f(u,v)|u,v) = \frac{1}{|G|^2} \sum_{x \in G} \chi_x(x) f(u,v) \langle A^{x}_u B^{y}_v \rangle = \frac{1}{|G|^2} \sum_{x \in G} \chi_x(x) f(u,v) \langle A^{x}_u B^{y}_v \rangle.$$

(17)

Now, since we do not restrict the dimension of the shared entangled states, the probabilities $P(a, b|u, v)$ are given by projective measurements $\{\Pi^a_u \}, \{\Sigma^b_v \}$ on a pure state $|\Psi⟩ \in \mathcal{C}^{D\times D}$ as $P(a, b|u, v) = ⟨\Psi| \Pi^a_u \otimes \Sigma^b_v |\Psi⟩$ the correlators can be written as the expectation value of observables $A^{x}_u B^{y}_v$ as $\langle A^{x}_u B^{y}_v \rangle = ⟨\Psi| A^{x}_u B^{y}_v |\Psi⟩$ with observables defined by

$$A^{x}_u = \sum_{a \in G} \bar{\chi}_x(a) \Pi^a_u$$

(18)

and $B^{y}_v = \sum_{b \in G} \bar{\chi}_y(b) \Sigma^b_v$. The game expression $\sum_{(u,v) \in Q_A \times Q_B} q(u,v) P(a + b = f(u,v)|u,v)$ can therefore be rewritten using Eq.(17) and the above observables as $(1/|G|) \sum_{x \in G} |a_x⟩⟨1 \otimes \Phi_x |β_y⟩$ with vectors $|a_x⟩, |β_y⟩$ and the linear game matrices $\Phi_x$.
defined as
\[
|\alpha_x\rangle = \sum_{u \in Q_A} (A_{u}^x \otimes \mathbf{1}) |\Psi\rangle \otimes |u\rangle,
\]
\[
|\beta_y\rangle = \sum_{v \in Q_B} (1 \otimes B_{v}^y) |\Psi\rangle \otimes |v\rangle,
\]
\[
\Phi_x = \sum_{(u,v) \in Q_A \times Q_B} q(u,v)\chi_x(f(u,v))|u\rangle \langle v|.
\]

The normalization of the input probability distribution \(\sum_{u,v} q(u,v) = 1\) translates to \(\langle \alpha_x| \mathbf{1} \otimes \Phi_x |\beta_x\rangle = 1\). The quantum value \(\omega_q(g^1)\) of the linear game can therefore be bounded as
\[
\omega_q(g^1) = \frac{1}{|G|} \sum_{x \in G} \langle \alpha_x| \mathbf{1} \otimes \Phi_x |\beta_x\rangle \\
\leq \frac{1}{|G|} \left[ 1 + \sqrt{|Q_A||Q_B|} \sum_{x \in G} \|\Phi_x\| \right],
\]
where \(\|\Phi_x\|\) denotes the norm of the game matrices \(\Phi_x\).

For games where the winning constraint only depends upon the XOR of the outcomes, i.e., \(V(a,b|u,v) = 1\) iff \(a \oplus d \oplus b = f(u,v)\) for \(u \in [m_A], v \in [m_B]\) and \(f(u,v) \in \{0,\ldots,d-1\}\), the above reduces to
\[
\omega_q(g^{\oplus}) = \frac{1}{d} \sum_{k=0}^{d-1} \langle \alpha_k| \mathbf{1} \otimes \Phi_k |\beta_k\rangle \\
\leq \frac{1}{d} \left[ 1 + \sqrt{m_A m_B} \sum_{k=1}^{d-1} \|\Phi_k\| \right].
\]

Lemma 3. For \(d\) outcome XOR games \(g^{\oplus}\) with \(m\) questions per player, when the input distribution is uniform \(q(u,v) = 1/m^2\), \(\omega_q(g^{\oplus}) = 1\) iff \(\omega_c(g^{\oplus}) = 1\), i.e., when rank(\(\Phi_1\)) = 1.

Proof: The constraint that the input distributions of questions to the players are uniform, \(q(u,v) = 1/m^2\) for all \(u,v\), is equivalent to \(\|\Phi_k\| \leq 1/m\) since both the maximum (absolute value) column sum and row sum matrix norms are equal to \(1/m\). Now \(\omega_c(g^{\oplus}) = 1\) requires from the bound in Eq.(21) that \(\|\Phi_k\| = 1/m\) for all \(k \in \{1,\ldots,d-1\}\). Consider the matrix \(\Phi_1 = \Phi_{k+1}\) which has entries \((\Phi_1^+\Phi_k)_{u,v} = m \sum_{w=1}^{m} q(w,u)q(w,v)\zeta^{-f(w,u)+f(w,v)}\), where \(\zeta = \exp(2\pi i/d)\) is the \(d\)-th root of unity. Let \(\lambda_j\) be the maximum eigenvector corresponding to eigenvalue \(1/m^2\) of \(\Phi_1^+\Phi_k\), with complex entries \(\lambda_j = |\lambda_j|e^{\phi_j}\).

Let the entries of the eigenvector be ordered by absolute value, \(|\lambda_1| \geq \cdots \geq |\lambda_m|\) and consider the eigenvalue equation corresponding to \(\lambda_1\), we have
\[
\sum_{v,w=1}^{m} |\lambda_1| \zeta^{-f(w,v)+f(w,v)+\theta_0} = m^2 |\lambda_1| e^{\phi_1}.
\]

Clearly the above equation can only be satisfied when \(\lambda_1 = |\lambda_j| \quad \forall j, j'\) and when the phases add, i.e., when \(f(w,v) - f(w',v') + \theta_0 = f(w',v') - f(w',1) + \theta_0 \quad \forall v, w, v', w'\), in particular \(f(w,v) - f(w,v') = \theta_0 - \theta v, \forall w, v, v'\). Now, with all \(|\lambda_j|\) equal, the rest of the eigenvalue equations (for \(u \neq 1\)) can only be satisfied with \(f(w,v) - f(u,w) + \theta_0 = f(w',v') - f(u,w') + \theta_0; \forall w, v, w', v'\). From these, we deduce that \(\omega_q(g^{\oplus}) = 1\) only when \(f(w,v) - f(u,v') = \theta_0 - \theta v\), in other words, when the columns of the game matrix \(\Phi_1\) are proportional to each other, the proportionality factor between columns \(k,l\) being \(\zeta^{k-l}\).

In this case (with \(\text{rank}(\Phi_1) = 1\)), a classical winning strategy which always exists for the first column of the game matrix \(\Phi_1\) can be straightforwardly extended to a classical winning strategy for the entire game, meaning \(\omega_c(g^{\oplus}) = 1\) also.

\[\square\]

Lemma 4. [see also [16]]. The quantum value of the CHSH-\(d\) game for prime \(d\) can be bounded as
\[
\omega_q(\text{CHSH} - d) \leq \frac{1}{d} + \frac{d-1}{d\sqrt{d}}.
\]

For prime power \(d = p^r\) where \(p\) is prime and \(r > 1\) is an arbitrary integer, the quantum value is bounded by
\[
\omega_q(\text{CHSH} - d) \leq \frac{1}{d} + \sum_{q=0}^{r-1} \phi(p^{r-q}) \sqrt{p^{r-q}} \leq \frac{1}{d} + \frac{d-1}{d\sqrt{p^r}}.
\]

where \(\phi(\cdot)\) is Euler’s totient function.

Proof. The entries of the game matrix \(\Phi_k\) are by definition \(\Phi_k(u,v) = q(u,v)\zeta^{k(u,v)}\) where \(\zeta = \exp(2\pi i/d)\) and \(u,v \in \{0,\ldots,d-1\}\), and we consider uniform probability inputs \(q(u,v) = 1/d^2\). It is readily seen that for prime \(d\), the game matrices \(\Phi_k\) for \(k \in \{1,\ldots,d-1\}\) are equal to each other up to a permutation of rows (or columns). Moreover, a direct calculation using \(\sum_{j=0}^{d-1} \zeta^j = 0\) yields that \(\Phi_1^2 = \mathbf{1}/d^2\), so that \(\|\Phi_1\| = 1/d\sqrt{d}\) for all \(k \in \{0,\ldots,d-1\}\). Substituting into Eq.(11) with \(m_A = m_B = d\) yields the bound in Eq.(23) for prime \(d\).

For prime power \(d = p^r\) where \(p\) is prime and \(r \geq 2\) is an arbitrary integer, the norms \(\|\Phi_k\|\) are not all equal. The matrices \(\Phi_1^k\Phi_k\) take the form \((1/d^2)^k|j\rangle \langle j|\) and \(q|p\rangle \langle p-q|\) for \(k/p^r\) being an integer relatively prime to \(d/p^r\), where \(J\) is the all-ones matrix and \(q\) is an integer with \(0 < q < r - 1\). Therefore, the norms are given by \(\sqrt{p^{r-q}/d}\) with respective degeneracy given by \(\phi(p^{r-q})\) where \(\phi(p^r) = p^r - p^{r-1}\) is the Euler totient function. Evidently, the largest norm is attained at \(k = p^{r-1}\), in which case we get \(\|\Phi_{p^{r-1}}\| = \sqrt{p^{r-1}/d^2} = 1/d\sqrt{p}\). Substituting into Eq.(11) with \(m_A = m_B = d\) gives Eq.(24) for prime power \(d\).

\[\square\]
Theorem 5. The games $NLC_d$ for arbitrary prime $d$ and for input distribution satisfying (8) have no quantum advantage, i.e., $\omega_q(NLC_d) = \omega_c(NLC_d)$.

Proof. We first consider the case of uniformly chosen inputs. The games $NLC_d$ consider functions of the following form (all arithmetic operations being performed modulo $d$)

$$a \oplus_d b = g(x_1 \oplus_d y_1, \ldots, x_{n-1} \oplus_d y_{n-1}) \cdot (x_n \oplus_d y_n),$$

with $g$ being an arbitrary function. Such a game is therefore composed of “building-block games” $G(t)$ which are of the form

$$G(t) := \{a \oplus_d b = t \cdot (x \oplus_d y)\},$$

with $t \in \{0, \ldots, d-1\}$, i.e., $f(x, y) = t \cdot (x \oplus_d y)$. There are $d$ different games $G(t)$, each with single dit input for each party (which we will take to be $x_i$ and $y_i$), and these games all have classical value $\omega_c(G(t)) = 1$ for each $t$. Explicitly, the classical strategy $a = t \cdot x$ and $b = t \cdot y$ wins the game $G(t)$. We can write the corresponding (non-normalized) game matrices $\Phi^{(1)}_k(t)$ for games $G(t)$ and they take the form

$$\Phi^{(1)}_k(t) := \sum_{x,y \in \{0, \ldots, d-1\}} \zeta^{kt(x \oplus_d y)}|x\rangle \langle y|,$$

with $\zeta = \exp\left(2\pi i / d\right)$. Here the $t$ in the superscript denotes that these matrices correspond to the $NLC_d$ game matrices for $n = 1$. Let us analyze some properties of the $\Phi^{(1)}_k(t)$. Firstly, we see that $\Phi^{(1)}_k(t) \Phi^{(1)}_k(t'^*)$ for any $k, t$ is diagonal in the Fourier basis defined by the Fourier vectors $|f_j\rangle$ with

$$|f_j\rangle = \left(1, \zeta^j, \zeta^{2j}, \ldots, \zeta^{(d-1)j}\right)^T$$

with $j \in \{0, \ldots, d-1\}$. Moreover, we also see that each $\Phi^{(1)}_k(t) \Phi^{(1)}_k(t)$ has only one eigenvalue ($=d^2$) different from zero and this corresponds to the eigenvector $|f_{d-k,1}\rangle$. This gives the orthogonality $\Phi^{(1)}_k(t) \Phi^{(1)}_k(t'^*) = 0$ for $k \cdot t \neq k' \cdot t'$. Since, we will be concerned with finding the maximum singular vectors corresponding to a fixed $k$, we can encapsulate the above properties by the equation

$$\left[\Phi^{(1)}_k(t) \Phi^{(1)}_k(t')\right]_{f_j} = d^2 \delta_{t,t'} \delta_{j,d-k}|f_j|$$

We shall use these properties of the $\Phi^{(1)}_k(t)$ as we proceed to analyze the game matrices $\Phi^{(n)}_k$ for the general $n$ dit input $NLC_d$ games themselves. Consider the games $NLC_d$ for prime $d$ and arbitrary number $n$ of input dits for each party. Denote the total number of inputs for each party by $m = d^n$, and the corresponding game matrices by $\Phi^{(n)}_k$. Due to the structure of the function in Eq. (25), namely the fact that the games only depend on the dit-wise XOR of the $n$ dits, we see that $\Phi^{(n)}_k \Phi^{(n)}_k$ acquires a block circulant structure (for $1 \leq i \leq n$ the corresponding matrices $\Phi^{(i)}_k \Phi^{(i)}_k$ for each $k$ are block-wise circulant matrices). For example, a possible (unnormalized) game matrix $\Phi^{(n)}_{ex}$ for $n = 2, d = 3$ of the form

$$\begin{bmatrix}
\Phi^{(1)}_{(0)}(0) & \Phi^{(1)}_{(1)}(0) & \Phi^{(1)}_{(1)}(2) \\
\Phi^{(1)}_{(1)}(1) & \Phi^{(1)}_{(2)}(0) & \Phi^{(1)}_{(1)}(0) \\
\Phi^{(1)}_{(2)}(2) & \Phi^{(1)}_{(0)}(0) & \Phi^{(1)}_{(1)}(1)
\end{bmatrix}
$$

with the $\Phi^{(1)}_{(i)}(t)$ defined as in Eq. (27) would have $\Phi^{(n)}_{ex} \Phi^{(n)}_{ex}$ equal to

$$\sum_{i=0}^{d-1} \Phi^{(1)}_{(i)} \Phi^{(1)}_{(i)} \Phi^{(1)}_{(i+1)}$$

which is a block-wise circulant matrix. In general, the entries of $\Phi^{(n)}_k \Phi^{(n)}_k$ are explicitly given by

$$\left[\Phi^{(n)}_k \Phi^{(n)}_k\right]_{ij} = \sum_{u_1, \ldots, u_{n-1} = 0}^{d-1} k_{ij}(x_{i-1} \oplus_d y_{i-1}) k_{ij}(u_{n-1} \oplus_d y_{n-1})$$

where as before $x_{n-1} = (x_1, \ldots, x_{n-1})$ and $y_{n-1} = (y_1, \ldots, y_{n-1})$ are strings of $n - 1$ dits, and we have omitted the normalization factor (of $1/d^{4n}$) for clarity. Due to this block circulant structure, we have that $\Phi^{(n)}_k \Phi^{(n)}_k$ for any $n, k$ is diagonal in the basis formed by the tensor products of the Fourier vectors $\{|f_{i_1}\rangle \otimes \ldots |f_{i_n}\rangle\}$ with $i_1, \ldots, i_n \in \{0, \ldots, d-1\}$.

We now proceed to calculate the maximum eigenvector of $\Phi^{(n)}_k \Phi^{(n)}_k$ among the basis formed by $\{|f_{i_1}\rangle \otimes \ldots |f_{i_n}\rangle\}$. To do this, let us consider the case of fixed $i_n$ varying $i_1, \ldots, i_{n-1}$. Using the properties of the game matrices $\Phi^{(1)}_k(t)$ recapitulated by Eq. (29), we see that for any fixed $i_n$, the eigenvalue corresponding to $|f_{i_1}\rangle \otimes \ldots |f_{i_{n-1}}\rangle \otimes |f_{i_n}\rangle$ cannot be smaller than that corresponding to any other $|f_{i_1}\rangle \otimes \ldots |f_{i_{n-1}}\rangle$. This is due to the fact that the other eigenvectors contribute only phases $\zeta^j$ to the eigenvalue expression corresponding to $|f_{i_1}\rangle \otimes \ldots |f_{i_{n-1}}\rangle \otimes |f_{i_n}\rangle$ and the properties stated above. It therefore follows that the maximum eigenvector is among the $|f_{i_1}\rangle \otimes \ldots |f_{i_{n-1}}\rangle \otimes |f_{i_n}\rangle$.

Let us compute the eigenvalues corresponding to $|f_{i_1}\rangle \otimes \ldots |f_{i_n}\rangle$ for $i_n \in \{0, \ldots, d-1\}$. To do this, fix an input string $x_{n-1}$ (to say $(0, \ldots, 0)$) and vary over $y_{n-1}$. In other words we consider the first row block of $\Phi^{(n)}_k$ corresponding to the game blocks $\Phi^{(1)}_{k,g}(x_{n-1} \oplus_d y_{n-1})$ of size $d \times d$. Denote by $\lambda_{x_{n-1}}(i_n, k)$ the number of times
the game $G(d−k⋅i_n)$ appears for this choice of $x_{n−1}$ in matrix $Φ_k^{(n)}$. Due to the symmetry of the game constraint, $λ^{x_{n−1}}(i_n,k)$ is independent of the choice of row $x_{n−1}$ so we may drop the superscript. Moreover, since $Φ_k^{(n)}$ is a symmetric matrix, we also have $λ^{x_{n−1}}(i_n,k) = λ^{y_{n−1}}(i_n,k)$ for an analogously defined $λ^{y_{n−1}}(i_n,k)$. Let us define $Λ(k) := \max_i λ(i_n,k)$ and let $μ \in \{0, \ldots, d−1\}$ denote the value of $i_n$ for which the maximum of $λ(i_n,k)$ is achieved. Again using Eq. (29), we have that

$$[Φ_k^{(n)+} Φ_k^{(n)}] |f_0⟩⊗_{n−1} ⊗ |f_{i_n}⟩ = d^2Λ^2(i_n,k) |f_0⟩⊗_{n−1} ⊗ |f_{i_n}⟩.$$  \hfill (33)

We therefore obtain that $∥Φ_k^{(n)}∥ = dΛ(k)$.

For prime $d$, we see that $Λ(k) = Λ$, constant and independent of $k$. This follows from the fact that the number of generators of the additive group $\mathbb{Z}_d$ for prime $d$ is simply equal to $d−1$ (all numbers less than prime $d$ are relatively prime to it). Therefore, for prime $d$, we obtain the following bound on the quantum value in the uniform case

$$ω_q(NLC_d) ≤ \frac{1}{d} \left(1 + \frac{(d−1)Λ}{d^{n−1}}\right).$$  \hfill (34)

We now consider the classical deterministic strategy where Alice outputs $a = μx_n$ independently of her input $x_{n−1}$ and Bob outputs $b = μy_n$ independently of his input $y_{n−1}$. Note that for the $d × d$ blocks described by $G(μ)$ all the $d^2$ constraints will be satisfied. On the other hand, for the blocks described by $G(t)$ for $t ≠ μ$, only $d$ constraints are satisfied with the use of this strategy. The score achieved by this strategy is therefore given by

$$ω_c(NLC_d) = \frac{d^{n−1}}{d^2} \left[Λd^2 + (d^{n−1} − Λ)d\right],$$  \hfill (35)

which equals the upper bound on the quantum value in Eq. (34); this completes the proof for uniformly chosen inputs.

Having solved the problem for uniformly distributed inputs, we can generalize to the case of probability distributions

$$\frac{1}{d^m+1} p(x_{n−1} ⊕_d y_{n−1})$$  \hfill (36)

For this input distribution, the matrix $Φ_k^{(n)}$ is still composed of the elementary games $Φ_k^{(1)}(t)$ that can be classically saturated. The difference is that a weight $p(x_{n−1} ⊕_d y_{n−1})/d^m+1$ is now attributed to each element of the $d × d$ block

$$[Φ_k^{(n)}]_{x_{n−1}y_{n−1}} = \frac{1}{d^m+1} p(x_{n−1} ⊕_d y_{n−1}) Φ_k^{(1)}(x_{n−1} ⊕_d y_{n−1}).$$  \hfill (37)

This preserves the block-wise circulant structure of $Φ_k^{(n)+} Φ_k^{(n)}$ ensuring that these matrices are still diagonal in the basis formed by the tensor products of Fourier vectors. As in the case of uniformly distributed inputs, the properties of $Φ_k^{(1)}(t)$ in Eq. (29) imply that the maximum eigenvector corresponds to one choice of $i_n ≡ \{0, \ldots, d−1\}$ within the $|f_0⟩⊗_{n−1} ⊗ |f_{i_n}⟩$. To compute the eigenvalues corresponding to $|v_0⟩⊗_{n−1} ⊗ |v_{i_n}⟩$, we have to take into account the number of times a game $G(d−k⋅i_n)$ appear in a given row block as well as the respective weights. Denote by $λ(i_n,k)$ the weighted sum of the times the game $G(d−k⋅i_n)$ appears in a row block, i.e.,

$$λ(i_n,k) = \sum_{y_{n−1} s.t. g(i_n,k) = i_n} \frac{1}{d^2} p(0_n−1 ⊕_d y_{n−1}).$$  \hfill (38)

As before, let us define $Λ(k) := \max_i λ(i_n,k)$ and let $μ$ denote the $i_n$ for which the maximum is reached. For the weighted matrix we have

$$[Φ_k^{(n)+} Φ_k^{(n)}] |f_0⟩⊗_{n−1} ⊗ |f_{i_n}⟩ = d^2λ(i_n,k) |f_0⟩⊗_{n−1} ⊗ |f_{i_n}⟩.$$  \hfill (39)

We therefore obtain that $∥Φ_k^{(n)}∥ = dΛ(k)$.

Again, for prime $d$, the maximum of this sum is independent of $k$. Therefore, for prime $d$, we obtain the following bound on the quantum value for a general $NLC_d$ game

$$ω_q(NLC_d) ≤ \frac{1}{d} \left[1 + d^{n+1}(d−1)Λ\right].$$  \hfill (40)

Consider the classical deterministic strategy where Alice outputs $a = μx_n$ independently of $x_{n−1}$ and Bob outputs $b = μy_n$ independently of $y_{n−1}$. For the $d × d$ blocks described by $G(μ)$ all the $d^2$ constraints will be satisfied. On the other hand, for the blocks described by $G(t) ≠ μ$, only $d$ constraints are satisfied with the use of this strategy. The score achieved by this strategy is therefore given by

$$ω_c(NLC_d) = d^{n−1} \left[Λd^2 + \left(\frac{1}{d^{n+1}} − Λ\right)d\right],$$  \hfill (41)

which equals the upper bound on the quantum value in Eq. (40); this completes the proof that quantum strategies cannot outperform classical ones in the $NLC_d$ game. \hfill □

**Theorem 6.** The non-local computation game inequalities for functions of the form in Eq. (7) for $d = 2$ do not define facets of the local polytope for any input size $2^n$.

**Proof.** A facet of a polytope of dimension $D$ is a face of the polytope of dimension $D−1$, i.e., facets are $D−1$ faces. The dimension of a convex polyhedron $P$ is the dimension of its affine hull, it is the smallest dimension of an Euclidean space containing a congruent copy of $P$. A general Bell inequality is of the form $c⋅p ≤ b$ where
c is the vector of coefficients for the inequality acting on the box of probabilities \{P[a\|u]\} written as a vector p and \(b \in \mathbb{R}\) is the classical (local realistic) bound. By definition, a way to check that a Bell inequality is tight, i.e., defines a facet of the local polytope \(P_L\) of dimension D is to show that the classical deterministic boxes \(\{P_{det}[a\|u]\}\) that achieve value \(b\) for the inequality span an affine subspace of dimension \(D - 1\). Let us recall the following formal definitions of the above facts [32, 33].

A polytope \(P \subseteq \mathbb{R}^n\) is the convex hull of a finite number of points in \(\mathbb{R}^n\), alternatively \(P\) is a bounded polyhedron, where \(Q\) is a polyhedron if there is a system of finitely many inequalities \(C \cdot q \leq b\) such that \(Q = \{q \in \mathbb{R}^n \mid C \cdot q \leq b\}\). Let \(P \subseteq \mathbb{R}^n, c \in \mathbb{R}^n\) and \(b \in \mathbb{R}\). An inequality \(c \cdot p \leq b\) is valid for \(P\) if it holds for all \(p \in P\).

The hyperplane \(H_{(c; b)}\) given as

\[
H_{(c; b)} = \{p \in \mathbb{R}^n \mid c \cdot p = b\}
\]

is said to be a supporting hyperplane of the polytope \(P\) if \(c \cdot p \leq b\) is valid for \(P\). A facet of \(P\) is a face of \(P\) if \(P = \bigcap F\) and \(F = P \cap H\) for some supporting hyperplane \(H\) of \(P\). A set \(P = \{p_1, \ldots, p_k\} \subseteq \mathbb{R}^n\) is affine dependent iff there exist \(\lambda_1, \ldots, \lambda_k \in \mathbb{R}\) not all zero such that \(\sum_{i=1}^{k} \lambda_i p_i = 0\) and \(\sum_{i=1}^{k} \lambda_i = 0\). The dimension of \(P\) is \(\dim(P) := \max\{\dim Q \mid Q \subseteq P, Q\textit{ affine independent}\}\). A facet of \(P\) if \(P\) is a face and \(\dim(F) = \dim(P) - 1\). An inequality \(c \cdot p \leq b\) is said to be facet-defining (or essential) for \(P\) if \(H_{(c; b)}\) is a facet of \(P\). Let \(C \cdot p \leq b\) be a system of valid inequalities for polytope \(P\) such that for each facet \(F\) of \(P\), there is a row \(c_i \cdot p \leq b_i\) of \(C \cdot p \leq b\) such that \(F = P \cap H_{(c; b)}\). Then \(P = \{p \in \mathbb{R}^n \mid C \cdot p \leq b\}\).

One approach to show that the games NLC do not define facets of the local polytope is thus to prove that there are fewer than \(D\) affinely independent classical deterministic boxes saturating the corresponding NLC inequality. Here, we use an alternative approach based on the following straightforward Lemma.

**Lemma 7.** If \(P\) is a polytope, then the intersection of two faces of \(P\) is a face of \(P\). A facet of \(P\) cannot be obtained as the intersection of two or more different faces of \(P\).

**Proof.** Suppose \(F\) and \(G\) are two faces of \(P\), so there are corresponding supporting hyperplanes \(H_{(c_F, b_F)}\) and \(H_{(c_G, b_G)}\) given as

\[
H_{(c_F, b_F)} := \{p \mid c_F \cdot p = b_F\}
\]

\[
H_{(c_G, b_G)} := \{p \mid c_G \cdot p = b_G\}\]

such that \(F = P \cap H_{(c_F, b_F)}\) and \(G = P \cap H_{(c_G, b_G)}\). The half-space

\[
\{p \mid (c_F + c_G) \cdot p = b_F + b_G\}
\]

contains \(P\) and for any \(p \in P\), we have that \((c_F + c_G) \cdot p = b_F + b_G\) only when both \(c_F \cdot p = b_F\) and \(c_G \cdot p = b_G\). Hence the intersection of \(F\) and \(G\) is the intersection of \(P\) with the hyperplane \(H_{(c_F + c_G, b_F + b_G)}\), and so \(F \cap G\) is a face of \(P\). If \(F \cap G\) is a facet of \(P\), then by the above argument, we have that \(F\) and \(G\) must also be facets of \(P\). The affinely independent boxes \(p\) that define \(F \cap G\) also define \(F\) and \(G\) and so \(F = G\). This shows that a facet cannot be obtained as the intersection of two (or more) differing faces.

We will now prove that the non-local computation games NLC for functions of the form in Eq. (7), i.e. of the form \(f(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = g(x_1 \oplus y_1, \ldots, x_{n-1} \oplus y_{n-1}) \cdot (x_n \oplus y_n)\) do not define facets by exhibiting a decomposition of the corresponding inequalities \(c_{NLC} \cdot p \leq b_{NLC}\) into multiple faces of the corresponding polytope. To do this, we first prove the following interesting result concerning the classical strategies for these games. Consider the relationship between the optimal classical strategy for the NLC game corresponding to a function \(f(x_1 \oplus y_1, \ldots, x_n \oplus y_n)\) and its sub-game NLCs corresponding to a function \(f(\tilde{x}_1 \oplus y_1, \ldots, \tilde{x}_{n-1} \oplus y_{n-1}, x_n \oplus y_n)\). Here, \(\tilde{x}_i \in \{0, 1\}\) are fixed inputs, i.e., in the sub-game NLC, Alice receives a single bit input \(x_0\) while Bob receives \(n\) bits \(y_1, \ldots, y_n\) with probability \((1/2^{n-1})p(\tilde{x}_1 \oplus y_1, \ldots, \tilde{x}_{n-1} \oplus y_{n-1}, x_n \oplus y_n)\). Notice that the probabilities in the sub-game do not sum to 1, the maximum value of the sub-game is 1/2^{n-1}.

**Lemma 8.** The optimal classical strategy for any NLC game corresponding to a function \(f(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = g(x_1 \oplus y_1, \ldots, x_{n-1} \oplus y_{n-1}) \cdot (x_n \oplus y_n)\) is also the optimal classical strategy for the sub-game NLCs corresponding to a function \(f(\tilde{x}_1 \oplus y_1, \ldots, \tilde{x}_{n-1} \oplus y_{n-1}, x_n \oplus y_n)\), where \(\tilde{x}_i \in \{0, 1\}\) for \(i \in \{1, \ldots, n-1\}\) are fixed inputs.

**Proof.** Let us first construct the optimal classical strategy that achieves \(\omega_c\) for the NLC game \(f(x_1 \oplus y_1, \ldots, x_n \oplus y_n)\). A standard convexity argument shows that it is enough to consider deterministic strategies. Note that arithmetic operations in this proof are taken modulo 2 since we are considering binary XOR games. As we have seen in the proof of Theorem (5), the classical optimal strategy is given by the maximum singular (in this case eigen) vector of the game matrix \(\Phi\) (note that for \(d = 2\) we have only one game matrix, which we can denote as \(\Phi\) instead of \(\Phi_1\)). For \(d = 2\), the NLC game is composed of two types of games \(G(0)\) and \(G(1)\) each with classical winning strategies, which appear as \(2 \times 2\) sub-matrices of \(\Phi\). Explicitly, \(G(0)\) is defined by the winning constraint \(a \oplus b = 0\) for \(x, y \in \{0, 1\}\) and \(G(1)\) is defined by the winning constraint \(a \oplus b = x \oplus y\) for \(x, y \in \{0, 1\}\). Clearly, there is a classical strategy that wins game \(G(j)\) given by \(a = j \cdot x\) and \(b = j \cdot y\), such a strategy satisfies the \(d^2 = 4\) constraints in the game, and moreover playing the optimal strategy for \(G(j)\) for the game \(G(j \oplus 2 1)\) only satisfies \(d = 2\) constraints of \(G(j \oplus 2 1)\). Further-
more, any deterministic strategy for $G(j)$ necessarily satisfies $2t$ constraints of $G(j)$ for $0 \leq t \leq 2$.

Now, any classical deterministic strategy for the game $NLC^6$ corresponding to the function $f(\bar{x}_1 \oplus_2 y_1, \ldots, \bar{x}_{n-1} \oplus_2 y_{n-1}, \bar{x}_n \oplus_2 y_n)$ picks one of two possible output strategies for Alice, namely Alice may output $a = 0$ or $a = x$ up to a relabeling of 0 and 1; given Alice’s strategy, for each $2 \times 2$ block $G(j)$, Bob optimizes over his deterministic strategies. Let $\lambda_j$ denote the weighted sum of times the block $G(j)$ appears in the game, i.e.,

$$
\lambda_j = \sum_{y_{n-1}} [p(y_{n-1}, 0) + p(y_{n-1}, 1)],
$$

where as usual $y_{n-1} = (y_1, \ldots, y_{n-1})$. Let $\Lambda := \max \{\lambda_j\}$ and let $\mu \in \{0, 1\}$ be the choice of $j$ that achieves this maximum. Then, from the properties of $G(j)$ outlined above, namely that any deterministic strategy satisfies either 0, 2 or 4 constraints in each $G(j)$, it is clear that the optimum strategy is for Alice to output $a = \mu \cdot x_n$ and for Bob to output $b = \mu \cdot y_n$ independently of $y_{n-1}$. Such a strategy achieves value $(1 + \Lambda) / 2^n$ for the sub-game $NLC^6$. Now, the classical strategy for the $NLC$ game itself is given by the maximum eigenvector of $\Phi$. As in Eq. (40) in the proof of Theorem (5), we see that this eigenvector defines exactly the strategy outlined above and gives the value $(1 + \Lambda) / 2^n$ achieved by playing the strategy for $NLC^6$ a total of $2^{n-1}$ times for different choices of $(\bar{x}_1, \ldots, \bar{x}_n)$. This completes the proof.

Note that the sub-games $NLC^6$ themselves could have quantum strategies that outperform classical ones, the Lemma (8) only points out using the symmetry of the $NLC$ game that the optimal classical strategy of $NLC$ satisfies exactly the same fraction of constraints in each sub-game $NLC^6$. This implies that the hyper-plane $H_{NLC^6}$ defined by the $NLC^6$ inequality given as $c_{f_{NLC}} \cdot p = b_{f_{NLC}}$ is a supporting hyperplane of the local Bell polytope and therefore $NLC^6$ gives a face $F_{NLC}$ of the local Bell polytope. We also see from Lemma (8) that every $NLC$ inequality can be written as the sum of $2^{n-1}$ inequalities corresponding to $NLC^6$, i.e., the face $F_{NLC}$ defined by the $NLC$ game inequality is obtained as the intersection of the $2^{n-1}$ faces $F_{NLC^6}$ ($b_{NLC} = \sum b_{NLC^6}$ and $c_{NLC} = \sum c_{NLC^6}$) so that

$$
F_{NLC} = \bigcap_{s=1}^{2^{n-1}} F_{NLC^s}. \tag{46}
$$

Therefore, since for $n \geq 2$ the different $F_{NLC^s}$ have differing $c_{f_{NLC}}$ for the corresponding hyperplanes, by Lemma (7) $F_{NLC}$ cannot be a facet of the local polytope. As we have seen, for $n = 1$ any $NLC$ game is won by a classical strategy.

□