A Sharp Tail Bound for the Expander Random Sampler

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Abstract

Consider an expander graph in which a $\mu$ fraction of the vertices are marked. A random walk starts at a uniform vertex and at each step continues to a random neighbor. Gillman showed in 1993 that the number of marked vertices seen in a random walk of length $n$ is concentrated around its expectation, $\Phi := \mu n$, independent of the size of the graph. Here we provide a new and sharp tail bound, improving on the existing bounds whenever $\mu$ is not too large.

1 Introduction

Let $G = (V, E)$ be a finite regular undirected graph, and let $A$ be its normalized adjacency matrix. Let

$$
\lambda(G) = \|A - J\|
$$

be the second largest absolute value of an eigenvalue of $A$, where $\| \cdot \|$ denotes the operator norm, and $J$ is the matrix whose entries are all $1/|V|$. There exist families of $d$-regular graphs $G$ for some constant $d$ so that $\lambda(G)$ is bounded above by a constant less than 1 [LPS88, Mar88, Fri08]. Such graphs are known as expander graphs.

An expander sampler samples vertices of an expander graph by performing a simple random walk on the graph. Note that when $G$ is the complete graph with self-loops, we have $A = J$ and $\lambda(G) = 0$, and this corresponds to sampling vertices uniformly and independently. One remarkable property of expander samplers is that the sampled vertices behave in various ways like vertices chosen uniformly and independently. This is even though the sampled vertices are not independent—knowing one vertex in the random walk narrows down the number of choices of the next vertex in the walk to just $d$. More precisely, fix a graph $G$, an arbitrary $n \geq 1$, and functions $f_1, \ldots, f_n : V \to [0, 1]$ (where often $f_1 = \cdots = f_n = f$). Let $(Y_1, \ldots, Y_n)$ be the simple random walk on $G$ of length $n$, i.e., $Y_1$ is chosen uniformly at random from $V$, and each subsequent $Y_i$ is chosen uniformly from the set of neighbors of $Y_{i-1}$. We will be interested in the random variable $S_n = f_1(Y_1) + \cdots + f_n(Y_n)$.

When the vertices are sampled uniformly and independently, the behavior of $S_n$ is described by the classical central limit theorem. In particular, the distribution of $S_n$ tends towards a normal distribution. Refinements include the Berry-Esseen theorem, which describes the rate at which this occurs [Ber41, Ess42], and various large deviation bounds. In all cases, analogous results hold

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†This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-1342536.
‡Supported by the Simons Collaboration on Algorithms and Geometry and by the National Science Foundation (NSF) under Grant No. CCF-1320188. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.
for expander samplers, with the appropriate dependence on $\lambda(G)$. For a more complete listing of these results, see Section 1.3.2 of the thesis of Mann [Man96].

Here we are interested in bounding $E[\alpha S_n]$, the moment generating function of $S_n$. Such bounds easily imply tail bounds on $S_n$ by Markov’s inequality. When vertices are sampled uniformly and independently, letting $\mu_i = E[f_i(v)]$ and $\Phi := E[S_n] = \mu_1 + \cdots + \mu_n$, a simple calculation shows that the moment generating function is bounded above by

$$E[\alpha S_n] \leq \exp((\alpha - 1) \cdot \Phi).$$

In this note we prove an analogous bound for the case of expander walks.

**Theorem 1.1.** Let $G = (V, E)$ be a regular undirected graph and let $\lambda = \lambda(G)$. Let $f_1, \ldots, f_n : V \to [0, 1]$, and let $\mu_i = E[f_i(v)]$. For a random walk $(Y_1, \ldots, Y_n)$, let $S_n = f_1(Y_1) + \cdots + f_n(Y_n)$ and $\Phi := E[S_n] = \mu_1 + \cdots + \mu_n$. Then for $1 < \alpha < 1/\lambda$,

$$E[\alpha S_n] \leq \exp((\alpha - 1) \cdot \Phi \cdot \left(\frac{1 - \lambda}{1 - \alpha \lambda}\right)).$$ (1)

As stated previously, bounds on the moment generating function immediately imply tail bounds. In particular, by plugging in $\alpha = \lambda^{-1} - (1 - \lambda)/(t^{1/2} \lambda^{3/2})$ in Theorem 1.1 we obtain the following.

**Corollary 1.2.** In the setting of Theorem 1.1, for all $t > 1/\lambda$,

$$\Pr[S_n \geq t\Phi] \leq \left(\frac{1}{\lambda} - \frac{1 - \lambda}{t^{1/2} \lambda^{3/2}}\right)^{-t\Phi} \exp(\Phi \cdot (1 - \lambda)(\sqrt{t\lambda} - 1)/\lambda).$$ (2)

For instance, for $\lambda = 1/2$, we obtain

$$\Pr[S_n \geq t\Phi] \leq \left(2 - \frac{2}{t}\right)^{-t\Phi} \exp(\Phi \cdot (\sqrt{2/2} - 1)).$$

We also note that for large $t$, the bound in (2) is roughly $\lambda^{t\Phi}$, which is again close to tight by the example in Section 1.2.

**Related Work.** Theorem 1.1 and Corollary 1.2 are part of a long line of work on proving tail bounds for expander walks starting from Gillman [Gil93, Gil98], and further developed in [Din95, Kah97, Lez98, Wag08, LP04, Hea08, CLLM12, HH15, Pau15, NRR17]. Closest to our work are the results of Lezaud [Lez98] and León and Perron [LP04]. Lezaud proved a bound of a form similar to (1) (see his Remark 2) with a large constant in the exponent. León and Perron [LP04] later improved on his result, proving a sharp bound on the moment generating function with corresponding tail bounds. Although their results are sharp, their bounds appear to be somewhat unwieldy, and they do not explicitly include convenient tail bounds as in our Corollary 1.2. Another difference is that both results [Lez98, LP04] assume $f_1 = \cdots = f_n$, although with some work, one can probably extend them to the case of general $f_1, \ldots, f_n$ as we consider here (and which is required for the application mentioned below). A final difference between our work and past work is in terms of proof techniques. Most work in this area including [Lez98, LP04] use perturbation theory, while some more recent work uses direct linear algebra arguments. Our proof is of the latter type, and arose from our joint work with Naor [NRR17]. We also feel that our proof is somewhat cleaner than some previous proofs.
1.1 Application to low-randomness samplers

One motivation for our work comes from a recent paper of Meka [Mek17], as explained next. For a set $V$, define a sampler from $V$ of length $n$ as a function whose range is $V^n$. The seed length of a sampler is defined as the logarithm of the cardinality of the function’s domain, and can be seen as the number of random bits necessary to sample from the function. With this terminology, an expander sampler for a $d$-regular graph $G = (V, E)$ has seed length $\log_2 |V| + (n-1) \log_2 d$. In particular, if $G$ is a constant degree expander, the seed length of $H$ is $\log_2 |V| + O(n)$.

In [Mek17], Meka constructs a (non-expander) sampler satisfying a bound similar to that in Eq. (1) using the worse seed length of $O(n + \log |V| + n(\log \log |V| + \log(1/(\mu_1 + \cdots + \mu_n))))/\log n$. Our results show that the sampler of [Mek17] can be replaced with an expander sampler, providing an improved seed length and a simpler construction. Notice however that Meka does not require optimal constants, and instead of using our bounds, one could also use some of the earlier work, such as [Lez98, Wag08] (after extending them to deal with $f_1, \ldots, f_n$ that are not necessarily all equal; see also [CLLM12, Section 3.2]).

1.2 Sharpness

We now sketch an argument showing that Theorem 1.1 is sharp in the following sense. Fix arbitrary $\lambda \in [0,1]$ and $\Phi > 0$, and consider the matrix $A = \lambda I + (1-\lambda)J$ of dimensions $|V| \times |V|$. This corresponds to the walk where at each step we either stay in place with probability $\lambda$, or choose a uniform vertex with probability $1-\lambda$. (Strictly speaking, $A$ does not correspond to a regular unweighted graph; it is straightforward to modify the example to this case.) Let $f_1 = \cdots = f_n = f$ be the function that assigns 1 to a $\mu = \Phi/n$ fraction of “marked” vertices and 0 to the remaining vertices (where we assume for simplicity that $\Phi|V|/n$ is integer). Equivalently, one can consider a Markov chain with two states, one marked and one unmarked; a step in the chain stays in the same state with probability $\lambda$ and otherwise chooses from the stationary distribution, which assigns mass $\mu$ to the marked state and $1-\mu$ to the unmarked state.

Then we claim that as $n$ goes to infinity, the left-hand side of Eq. (1) converges to the right-hand side. To see that, we say that a step of the walk is a “hit” if (1) the walk chooses a uniform vertex (which happens with probability $1-\lambda$), and (2) that chosen vertex is marked. Then observe that the random variable counting the number of hits during the walk converges to a Poisson distribution with expectation $(1-\lambda)\Phi$ (since it is the sum of $n$ independent Bernoulli random variables, each with probability $(1-\lambda)\mu$ of being 1). Moreover, each time a hit occurs, we stay in that vertex a number of steps that is distributed like a geometric distribution with success probability $1-\lambda$. (We are ignoring here lower order effects, such as reaching the end of the walk.) Therefore, using the probability mass function of the Poisson distribution and the moment generating function of the geometric distribution, we see that for any $\alpha < 1/\lambda$, as $n$ goes to infinity, $E[\alpha^S_n]$ converges to

$$\sum_{k=0}^{\infty} \frac{\exp(-(1-\lambda)\Phi)((1-\lambda)\Phi)^k}{k!} \left(\frac{(1-\lambda)\alpha}{1-\lambda\alpha}\right)^k = \exp\left(-(1-\lambda)\Phi + (1-\lambda)\Phi \frac{(1-\lambda)\alpha}{1-\lambda\alpha}\right)$$

$$= \exp\left(\Phi \cdot \left(\frac{(1-\lambda)(\alpha - 1)}{1 - \alpha\lambda}\right)\right),$$

as desired.
2 Preliminaries

For $p \geq 1$, let the $p$-norm of an $N$-dimensional vector $v$ be defined as

$$\|v\|_p = \left(\frac{|v_1|^p + |v_2|^p + \cdots + |v_N|^p}{N}\right)^{1/p}.\$$

Note that under this definition, $\|v\|_p \leq \|v\|_q$ if $p \leq q$. Additionally, we let the operator norm of a matrix $A \in \mathbb{R}^{N \times N}$ be defined as

$$\|A\| = \max_{v: \|v\|_2 = 1} \|Av\|_2.\$$

For a vector $v$, we let $\text{diag}(v)$ be the diagonal matrix where $\text{diag}(v)_{i,i} = v_i$.

We denote by $\mathbf{1}$ the all-$1$ vector, and by $J$ the matrix whose entries are all $1/N$. Given a regular undirected graph $G = (V,E)$, let $A$ be its normalized adjacency matrix. We let $\lambda(G) = \|A - J\|$ (where $J$ is the matrix whose entries are all $1/|V|$) be the second largest eigenvalue in absolute value of $A$.

3 Bounding monomials

In this section we prove Lemma 3.3 bounding the expectation of monomials in the $f_i(Y_i)$. We start with two simple claims.

**Claim 3.1.** For all $k \geq 1$ and matrices $R_1, \ldots, R_k \in \mathbb{R}^{N \times N}$,

$$\|R_1 J R_2 J \cdots J R_k \mathbf{1}\|_1 \leq \prod_{i=1}^k \|R_i\|_1.\$$

**Proof.** Notice that for any vector $v$, $Jv = \alpha \mathbf{1}$ where $\alpha$ is the average of the coordinates of $v$ and hence satisfies $|\alpha| \leq \|v\|_1$. The claim then follows by induction. \(\square\)

**Claim 3.2.** For all $k \geq 1$, vectors $u_1, \ldots, u_k \in [0,1]^N$, $U_i = \text{diag}(u_i)$ for all $i$, and matrices $T_1, \ldots, T_{k-1} \in \mathbb{R}^{N \times N}$,

$$\|U_1 T_1 U_2 T_2 \cdots T_{k-1} U_k \mathbf{1}\|_1 \leq \sqrt{\|u_1\|_1 \|u_k\|_1 \prod_{i=1}^{k-1} \|T_i\|}.\$$

**Proof.** The case $k = 1$ is immediate since $U_1 \mathbf{1} = u_1$. For $k > 1$,

$$\|U_1 T_1 U_2 T_2 \cdots T_{k-1} U_k \mathbf{1}\|_1 \leq \|u_1\|_2 \|T_1 U_2 T_2 \cdots T_{k-1} U_k\|_2 \leq \|u_1\|_2 \|u_k\|_2 \prod_{i=1}^{k-1} \|T_i\|.$$

where the first inequality is the Cauchy-Schwarz inequality, and the second inequality is by the definition of operator norm and the observation that $\|U_i\| = \|u_i\|_\infty \leq 1$. We complete the proof by noting that $\|u_i\|_2^2 \leq \|u_i\|_1$ since the entries of $u_i$ are in $[0,1]$. \(\square\)

We can now prove the main result of this section.
Lemma 3.3. Let \( G = (V, E) \) be a regular undirected graph and let \( \lambda = \lambda(G) \). Let \( f_1, \ldots, f_n : V \to [0,1] \), and let \( \mu_i = E[f_i(v)] \). For a random walk \( (Y_1, \ldots, Y_n) \), let \( Z_i = f_i(Y_i) \) for all \( i \). Then for all \( k \geq 1 \) and \( w \in [n]^k \) such that \( w_1 \leq w_2 \leq \cdots \leq w_k \),

\[
E[Z_{w_1} Z_{w_2} \cdots Z_{w_k}] \leq \sum_{s \in \{0,1\}^{k-1}} \sqrt{\mu_{w_1}} \mu_{w_k} \left( \prod_{i:s_i=0} (1 - \lambda^{w_{i+1}-w_i}) \sqrt{\mu_{w_i} \mu_{w_{i+1}}} \right) \left( \prod_{i:s_i=1} \lambda^{w_{i+1}-w_i} \right).
\]

Proof. Let \( d_i = w_{i+1} - w_i \) for all \( i \), and let \( A \) be the normalized adjacency matrix of \( G \). Let \( u_i \) be given by \( (u_i)_v = f_{w_i}(v) \) and let \( U_i = \text{diag}(u_i) \). Then

\[
E[Z_{w_1} Z_{w_2} \cdots Z_{w_k}] = \| U_1 A^{d_1} U_2 A^{d_2} \cdots A^{d_{k-1}} U_k 1 \|_1.
\]

Let \( T_{i,0} = (1 - \lambda^{d_i}) I \) and \( T_{i,1} = A^{d_i} - (1 - \lambda^{d_i}) I \) and notice that \( \| T_{i,1} \|_2 = \lambda^{d_i} \). Using the triangle inequality, we can bound the right-hand side of Eq. (4) from above by

\[
\sum_{s \in \{0,1\}^{k-1}} \| U_1 T_{1,s_1} U_2 T_{2,s_2} \cdots T_{k-1,s_{k-1}} U_k 1 \|_1.
\]

The main claim is that for each \( s \), the term corresponding to \( s \) satisfies

\[
\| U_1 T_{1,s_1} U_2 T_{2,s_2} \cdots T_{k-1,s_{k-1}} U_k 1 \|_1 \leq \sqrt{\mu_{w_1}} \mu_{w_k} \left( \prod_{i:s_i=0} (1 - \lambda^{d_i}) \sqrt{\mu_{w_i} \mu_{w_{i+1}}} \right) \left( \prod_{i:s_i=1} \lambda^{d_i} \right).
\]

Notice that this claim immediately implies the lemma by summing over \( s \in \{0,1\}^{k-1} \). To see why the claim is true, let \( 1 \leq r_1 < \cdots < r_t \leq k - 1 \) be the indices where \( s \) is zero. Then by Claim 3.1, the left-hand side of Eq. (5) is at most

\[
\left( \prod_{i:s_i=0} (1 - \lambda^{d_i}) \right) \| U_1 T_{1,1} U_2 T_{2,1} \cdots T_{r_1-1,1} U_{r_1} 1 \|_1 \cdot \| U_{r_1+1} T_{r_1+1,1} U_{r_1+2} T_{r_1+2,1} \cdots T_{r_2-1,1} U_{r_2} 1 \|_1 \cdots \| U_{r_t+1} T_{r_t+1,1} U_{r_t+2} T_{r_t+2,1} \cdots T_{k-1,1} U_k 1 \|_1.
\]

The claim now follows by applying Claim 3.2.

When \( \mu_1 = \cdots = \mu_n = \mu \), the bound in Eq. (3) simplifies to

\[
E[Z_{w_1} Z_{w_2} \cdots Z_{w_k}] \leq \mu \prod_{i=1}^{k-1} ((1 - \lambda^{w_{i+1}-w_i}) \mu + \lambda^{w_{i+1}-w_i}).
\]

Observe that for the two-state Markov chain described in Section 1.2 for every \( n \geq 1 \) and every \( 1 \leq w_1 \leq \cdots \leq w_k \leq n \), this inequality is actually an equality. Indeed, the left-hand side is the probability that we are in the marked state at all the steps \( w_1, \ldots, w_k \). The probability of being in the marked state at step \( w_i \) is \( \mu \) (as we are in the stationary distribution); and the probability of being in the marked state at step \( w_{i+1} \) conditioned on being there at step \( w_i \) is \( (1 - \lambda^{w_{i+1}-w_i}) \mu + \lambda^{w_{i+1}-w_i} \).

This observation implies that the moment generating function \( E[e^{S_n}] \) of an arbitrary graph and arbitrary \( f_1, \ldots, f_n \) with all \( E[f_i] \) equal can be bounded by the moment generating function of the corresponding two-state Markov chain (as can be seen from the Taylor expansion; see Eq. (8) below). This can be used to give an alternative (and perhaps more intuitive) proof of Theorem 1.1. We do not include this proof here since it is not clear how to extend it to the case of general \( \mu_i \).
4 Proof of Theorem 1.1

In this section we complete the proof of the main theorem using the bound in Lemma 3.3. We start with the following easy corollary of Cauchy-Schwarz.

Claim 4.1. Let \( P \in \mathbb{R}[X_1, \ldots, X_n] \) be a multivariate polynomial with non-negative coefficients. Then for \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}, \)
\[
P(x_1 y_1, x_2 y_2, \ldots, x_n y_n) \leq \max \{ P(x_1^2, x_2^2, \ldots, x_n^2), P(y_1^2, y_2^2, \ldots, y_n^2) \}.
\]

Proof. Let
\[
P(X_1, \ldots, X_n) = \sum_{m \in \mathbb{N}^n} a_m X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}
\]
for some \( a_m \geq 0. \) Then
\[
P(x_1 y_1, x_2 y_2, \ldots, x_n y_n) = \sum_{m \in \mathbb{N}^n} a_m (x_1 y_1)^{m_1} (x_2 y_2)^{m_2} \cdots (x_n y_n)^{m_n}
\]
\[= \sum_{m \in \mathbb{N}^n} \left( \sqrt{a_m x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}} \right) \left( \sqrt{a_m y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}} \right)
\leq \left( \sum_{m \in \mathbb{N}^n} a_m x_1^{2m_1} x_2^{2m_2} \cdots x_n^{2m_n} \right)^{1/2} \left( \sum_{m \in \mathbb{N}^n} a_m y_1^{2m_1} y_2^{2m_2} \cdots y_n^{2m_n} \right)^{1/2}
\leq \max \{ P(x_1^2, x_2^2, \ldots, x_n^2), P(y_1^2, y_2^2, \ldots, y_n^2) \},
\]
where the first inequality follows from Cauchy-Schwarz. \( \square \)

Lemma 4.2. Let \( G = (V, E) \) be a regular undirected graph, let \( \lambda = \lambda(G) \), and let \( f_1, \ldots, f_n : V \to [0, 1]. \) For a random walk \( (Y_1, \ldots, Y_n) \), let \( Z_i = f_i(Y_i) \) for all \( i \), and let \( \mu_i = E[f_i(v)] \), \( \Phi = \mu_1 + \cdots + \mu_n. \) For all \( k \in [n] \), let \( W_k \subseteq [n]^k \) be the set of all \( w \) such that \( w_1 < w_2 < \cdots < w_k. \) Then
\[
E \left[ \sum_{w \in W_k} Z_{w_1} Z_{w_2} \cdots Z_{w_k} \right] \leq \sum_{i=0}^{k-1} \binom{k-1}{i} \Phi^{i+1} \lambda^{k-i-1} \frac{1}{(i+1)!(1-\lambda)^{k-i-1}}.
\]

Proof. By Lemma 3.3 and Claim 4.1,
\[
E \left[ \sum_{w \in W_k} Z_{w_1} Z_{w_2} \cdots Z_{w_k} \right] \leq \sum_{w \in W_k} \sum_{s \in \{0,1\}^{k-1}} \sqrt{\mu_{w_1} \mu_{w_k}} \left( \prod_{i:s_i=0} \sqrt{\mu_{w_i}} \right) \left( \prod_{i:s_i=1} \lambda^{w_{i+1} - w_i} \right)
\]
\[\leq \max \left\{ \sum_{w \in W_k} \sum_{s \in \{0,1\}^{k-1}} \mu_{w_1} \prod_{i:s_i=0} \mu_{w_i+1} \prod_{i:s_i=1} \lambda^{w_{i+1} - w_i}, \right. \]
\[\left. \sum_{w \in W_k} \sum_{s \in \{0,1\}^{k-1}} \mu_{w_k} \prod_{i:s_i=0} \mu_{w_i} \prod_{i:s_i=1} \lambda^{w_{i+1} - w_i} \right\}.
\]

We assume for the remainder of the proof that the second term is the maximum. A similar proof holds under the assumption that the first term is the maximum.

We will show that for each \( s \in \{0,1\}^{k-1}, \)
\[
\sum_{w \in W_k} \mu_{w_k} \left( \prod_{i:s_i=0} \mu_{w_i} \right) \left( \prod_{i:s_i=1} \lambda^{w_{i+1} - w_i} \right) \leq \frac{(\mu_1 + \cdots + \mu_n)^k |s| \lambda^{|s|}}{(k-|s|)! (1-\lambda)^{|s|}}, \tag{6}
\]
For a random walk \( (\Phi) \) the lemma then follows from Lemma 4.2.

**Lemma 4.3.** Let \( G = (V, E) \) be a regular undirected graph, let \( \lambda = \lambda(G) \), and let \( f_1, \ldots, f_n : V \to [0, 1] \). For a random walk \( (Y_1, \ldots, Y_n) \), let \( Z_i = f_i(Y_i) \) for all \( i \), and let \( S_n = Z_1 + \cdots + Z_n \), and let \( \mu_i = E[Z_i] \), \( \Phi = \mu_1 + \cdots + \mu_n \). Then for all positive integers \( q \),

\[
E[S_n^q] \leq \sum_{k=1}^q \frac{q!}{k!} \sum_{i=0}^{k-1} \binom{k-1}{i} \Phi^{i+1} \lambda^{k-i-1} \frac{(i+1)!(1-\lambda)^{k-i-1}}{(i+1)(1-\lambda)^{k-i-1}}.
\]

where \( \{\} \) denotes the Stirling number of the second kind.

**Proof.** Consider the subset \( D_k \subseteq [n]^q \) of vectors with exactly \( k \) distinct coordinates. Note that

\[
E[S_n^q] = \sum_{w \in [n]^q} E \left[ \prod_{j=1}^q Z_{w_j} \right] = \sum_{k=1}^q \sum_{w \in D_k} E \left[ \prod_{j=1}^q Z_{w_j} \right].
\]

We will upper bound each term on the right-hand side separately.

Fix a \( k \), and let \( W_k \subseteq [n]^k \) be the set of vectors \( w \) so that \( w_1 < w_2 < \cdots < w_k \). Let \( \psi : D_k \to W_k \) be the function mapping each \( w \in D_k \) to the vector whose coordinates are exactly those in \( w \) in sorted order and without repetition. Then because \( Z_i \in [0, 1] \) for all \( i \),

\[
\sum_{w \in D_k} E \left[ \prod_{j=1}^q Z_{w_j} \right] \leq \sum_{w \in D_k} E \left[ \prod_{j=1}^k Z_{\psi(w)_j} \right]. \tag{7}
\]

Moreover, for all \( w \in W_k \) we have \( |\psi^{-1}(w)| = \binom{q}{k} \) (as this is the number of ways to partition \( q \) labeled balls into \( k \) nonempty labeled boxes), and thus Eq. (7) is equal to

\[
\binom{q}{k} k! \sum_{w \in W_k} E \left[ \prod_{j=1}^k Z_{w_j} \right].
\]

The lemma then follows from Lemma 4.2. \( \square \)
Finally we can insert the upper bounds from Lemma 4.3 in the Taylor expansion of $\alpha S_n$ to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 4.3,

$$E[\alpha S_n] = \sum_{q=0}^{\infty} \frac{\log(\alpha)^q E[S_q^2]}{q!} \leq 1 + \sum_{q=1}^{\infty} \frac{\log(\alpha)^q}{q!} \sum_{k=1}^{q} \frac{k!}{k(k-1)} \sum_{i=1}^{k} \frac{\Phi^i \lambda^{k-i}}{i!(1-\lambda)^{k-i}}.$$  

Rearranging the sums yields

$$1 + \sum_{i=1}^{\infty} \frac{\Phi^i}{i!} \sum_{k=i}^{\infty} \frac{k!}{k(k-1)} \sum_{q=k}^{\infty} \frac{q^q}{q!} \left( \sum_{i=1}^{q} \frac{\log(\alpha)^q}{q!} \right).$$ (9)

Using the following identity \[\text{Sta12}, \text{Eq. 1.94(b)},\]

$$\sum_{q=k}^{\infty} \frac{q^q}{q!} = \frac{(\alpha - 1)^k}{k!},$$

(which can be seen by writing $\alpha - 1 = e^{\log(\alpha)} - 1 = \log(\alpha) + \frac{1}{2!} \log(\alpha)^2 + \frac{1}{3!} \log(\alpha)^3 + \cdots$) we can rewrite Eq. (9) as

$$1 + \sum_{i=1}^{\infty} \frac{\Phi^i}{i!} \sum_{k=i}^{\infty} \frac{k!}{k(k-1)} \sum_{q=k}^{\infty} \frac{q \log(\alpha)^q}{q!} \left( \frac{\lambda^{k-i}}{1 - \lambda} \right)^{k-i}.$$

Using the following identity for $0 \leq x < 1$,

$$\sum_{j=i}^{\infty} \frac{(j-1)^i}{i-1} x^j = (1-x)^{-i},$$

(which follows from differentiating $\sum_{j=0}^{\infty} x^j = (1-x)^{-1}$ a total of $i-1$ times) we can rewrite Eq. (10) as

$$1 + \sum_{i=1}^{\infty} \frac{\Phi^i (\alpha - 1)^i}{i!} \left( 1 - \frac{\lambda(\alpha - 1)}{1 - \lambda} \right)^{-i} = \exp \left( \Phi \cdot \left( \frac{(1-\lambda)(\alpha - 1)}{1-\alpha \lambda} \right) \right).$$

\[\square\]

**Acknowledgements** We thank Assaf Naor for many useful discussions, and Michael Forbes for referring us to prior work. We also thank the Oberwolfach Research Institute for Mathematics and the organizers of the “Complexity Theory” workshop there in November 2015 where this work was initiated.
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