The complexity of soundness in workflow nets

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Abstract

Workflow nets are a popular variant of Petri nets that allow for the algorithmic formal analysis of business processes. The central decision problems concerning workflow nets deal with soundness, where the initial and final configurations are specified. Intuitively, soundness states that from every reachable configuration one can reach the final configuration. We settle the widely open complexity of the three main variants of soundness: classical, structural and generalised soundness. The first two are EXPSPACE-complete, and, surprisingly, the latter is PSPACE-complete, thus computationally simpler.

CCS Concepts  • Software and its engineering → Petri nets; • Theory of computation → Computational complexity and cryptography;

Keywords Workflow nets, Petri nets, soundness, generalised soundness, structural soundness, complexity

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1 Introduction

Workflow nets are a formalism that allows for the modeling of business processes. Specifically, they allow to formally represent workflow procedures in Workflow Management Systems (WFMSs) (see e.g. [23, Section 4], where Figure 6 shows a workflow net for the processing of complaints; and [22, Section 3] for details on modeling procedures). Such a mathematical representation enables the algorithmic formal analysis of their behaviour. This is particularly relevant for large organisations that seek to manage the workflow of complex business processes. Such challenges have received, and continue to receive, intense academic attention, e.g. through the foundations track of the Business Process Management Conference (BPM), and via a discipline coined as process mining and pioneered prolifically by Wil van der Aalst.1

In particular, many tools, such as those integrated in the ProM framework [26], can extract events from logs, e.g. of enterprise resource planning (ERP) systems, from which they synthesize workflow nets (and other models) to be formally analyzed (see [24] for a book on the topic).

More formally, workflow nets form a subset of (standard) Petri nets. They consist of places that can contain resources (called tokens) which can be consumed and produced via transitions in a nondeterministic and concurrent fashion. Two designated places, namely the initial place i and the final place f, respectively model the initialisation and termination of a business process. No token can be produced in the initial place, and no token can be consumed from the final place.

A central property studied since the inception of workflow nets is 1-soundness [22, 23]. Informally, quoting [22], it states that “For any case, the procedure will terminate eventually [...]”. More formally, from the configuration with a single token in the initial place i, every reachable configuration can reach the configuration with a single token in the final place f. For readers familiar with computation temporal logic (CTL), 1-soundness can be loosely rephrased as i |= VG EF f. More generally, k-soundness states the same but for k tokens, i.e. (k · i) |= VG EF (k · f).

Classical soundness. Several variants of soundness have been considered in the literature (see [25] for a survey). The best-known is classical soundness. It states that a workflow net is 1-sound and that each transition is meaningful, i.e. each transition can be fired in at least one execution (often called quasi-liveness). It is well-known that deciding classical soundness amounts to checking boundedness and liveness of a slightly modified net. In particular, this means that classical soundness is decidable since boundedness and liveness are decidable problems. However, to the best of our knowledge, the (exact) complexity of classical soundness remains widely open. It has been suggested that classical

1See http://www.processmining.org.
soundness is EXPSPACE-hard. For example, the author of [9] mentions that "IO-soundness is decidable but also EXPSPACE-hard ([21])", yet [21] merely states the following:

[...] [I]t may be intractable to decide soundness.
(For arbitrary [workflow]-nets liveness and boundedness are decidable but also EXPSPACE-hard [...].)

Further, [23, p. 38] suggests that intractability follows from the fact that "deciding liveness and boundedness is EXPSPACE-hard", which is attributed to [5]. However, the latter only mentions liveness to be EXPSPACE-hard (which was known prior to [5]).

The confusion arises from the fact that boundedness and liveness are independently EXPSPACE-hard problems, which suggests that classical soundness must naturally be at least as hard. However, this needs not be the case. For example, for a well-studied subclass of Petri nets, called free-choice nets, testing simultaneously boundedness and liveness has lower complexity than testing both properties independently\(^5\) [10].

Moreover, since liveness is equivalent to the Petri net reachability problem [13], the only known upper bound is not even primitive recursive [15]. As a first contribution, we show that classical soundness and \(k\)-soundness are in fact both EXPSPACE-hard and in EXPSPACE, and hence EXPSPACE-complete. The upper bound is derived with a fortiori surprisingly little effort by invoking known results on coverability and so-called cyclicity. The hardness result is obtained by a careful reduction from the reachability problem for reversible Petri nets [4, 17]. There, we exploit subtle known results in a technically challenging way.

**Generalised and structural soundness.** Among the variants of soundness catalogued by the survey of van der Aalst et al. [23], generalised soundness [28, Def. 3] is the only fundamentally distinct property (in particular, see [25, Fig. 7]). It asks whether a given workflow net is \(k\)-sound for all \(k \geq 1\).

Generalised soundness, unlike classical soundness, preserves nice properties like composition [28]. The existential counterpart of generalised soundness, where "for all" is replaced by "for some", is known as structural soundness [1].

It is of priori not clear whether generalised and structural soundness are decidable, as the approach for deciding other types of soundness reasons about \(k\)-soundness for a given or fixed number \(k\). Nonetheless, both problems have been shown decidable [7, 29]. The two algorithms, and a subsequent one [27], rely on Petri net reachability, which has very recently been shown Ackermann-complete [8, 14, 15].

As for classical soundness, the computational complexity of generalised and structural soundness remains open. In fact, we are not aware of any complexity result. In this work, we prove that generalised and structural soundness have much lower complexity than Petri net reachability: they are respectively PSPACE-complete and EXPSPACE-complete. In particular, the fact that generalised soundness is simpler than classical soundness is arguably surprising: positive instances of both problems require the given workflow net to be bounded, but for generalised soundness, one can avoid explicitly checking this EXPSPACE-complete property.

To derive the PSPACE membership, we introduce the notion of strong soundness which is defined in terms of a relaxed reachability relation (sometimes known as \(Z\)-reachability or pseudo-reachability, e.g., see [2]). Through results on integer linear programming and exploiting Steinitz Lemma on reordering vectors [20], we prove that \(k\)-unsoundness of a workflow net must occur for a “small” number \(k\). Furthermore, we show that it suffices to witness such a \(k\) for so-called \(Z\)-bounded nonredundant nets, with \(Z\)-boundedness being a more restrictive property than (standard) boundedness. By building upon these results, we establish the EXPSPACE membership of structural soundness, and, in fact, effectively characterise the set of sound numbers of workflow nets, which settles the open problem of [7].

The hardness results for PSPACE and EXPSPACE are respectively obtained via reductions from the reachability problem for conservative Petri nets [18], and from 1-soundness.

**Contribution and organisation.** In summary, we settle, after around two decades, the exact computational complexity of the central decision problems for workflow nets. This is achieved in the rest of this work, organised as follows. In Section 2, we introduce general notation, Petri nets, workflow nets and soundness. In Section 3, we prove that classical soundness is EXPSPACE-complete. In Section 4, we provide bounds on vector reachability, which in turn allows us to prove PSPACE-completeness of generalised soundness (Section 5), and EXPSPACE-completeness of structural soundness (Section 6). In Section 7, we leverage the previous results to give a characterisation of numbers \(k\) for which a workflow net is \(k\)-sound. Finally, we conclude in Section 8. Due to space constraints, some proofs are deferred to an appendix.

**2 Preliminaries**

We denote naturals and integers with the usual font: \(n \in \mathbb{N}\) and \(z \in \mathbb{Z}\). Given \(i, j \in \mathbb{Z}\), we write \([i..j]\) for \(\{i, i + 1, \ldots, j\}\). We use the bold font for vectors and matrices, e.g. \(\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n\) and \(A \in \mathbb{Z}^{m \times n}\). Given \(n \in \mathbb{N}\), we write \(n^d = (n, \ldots, n) \in \mathbb{N}^d\). We omit the dimension \(d\) when it is clear from the context. e.g. \(0\) denotes the null vector. We write \(\mathbf{a}[i] = a_i\) and \(A[i,j]\) for matrix entries where \(i \in [1..m]\) and \(j \in [1..n]\). We write \(x \leq y\) if \(x[i] \leq y[i]\) holds for all \(i \in [1..n]\). We write \(x < y\) if at least one inequality is strict. Given a vector \(\mathbf{a} \in \mathbb{Z}^n\) or a matrix \(A \in \mathbb{Z}^{m \times n}\), we define the norms \(|\mathbf{a}| := \max_{1 \leq i \leq n} |a[i]|\) and \(\|A\| := \max_{1 \leq j \leq m, 1 \leq i \leq n} |A[j, i]|\).
2.1 Petri nets

A Petri net is a triple $\mathcal{N} = (P, T, F)$ such that:

- $P$ and $T$ are disjoint finite sets whose elements are respectively called places and transitions,
- $F : (P \times T) \cup (T \times P) \to \mathbb{N}$ is the flow function.

A marking is a vector $m : P \to \mathbb{N}$ where $m[p]$ indicates how many tokens are contained in place $p$. Informally, $F[p, t]$ and $F[t, p]$ respectively denote the amount of tokens to be consumed and produced in place $p$. Let $*t, *t^+ \in \mathbb{N}^P$ respectively denote the vectors such that $*t[p] := F[p, t]$ and $*t^+[p] := F[t, p]$. Let $\Delta(t) := *t^+ - *t$ denote the effect of $t$. We say that a transition $t \in T$ is enabled in $m$ if $m \geq *t$. If $t$ is enabled in $m$, then $t$ may be fired, which leads to the marking $m' := m + \Delta(t)$. The latter is denoted by $m \xrightarrow{t} m'$, or simply by $m \rightarrow m'$ whenever we do not care about the transition that led to $m'$. We use a standard notation for markings, listing only nonzero values, e.g. if $P = \{p_1, p_2\}$, $m[p_1] = 2$ and $m[p_2] = 0$, then $m = (p_1 : 2)$.

A run is a sequence of transitions $\rho = t_1 \cdots t_n \in T^*$. A run is enabled in a marking $m_0$ if there is a sequence of markings $m_0, \ldots, m_n$ such that $m_i \xrightarrow{t_i} m_{i+1}$ for all $0 \leq i < n$. If it is the case, then we denote this by $m_0 \rightarrow_* m_n$, or $m_0 \rightarrow^* m_n$ if $\rho$ is not important. Given $\ell \in \mathbb{N}$, we say that $\rho$ is $\ell$-bounded if $\|m_i\| \leq \ell$ for all $0 \leq i \leq n$. The support of a run is the set of transitions occurring in it, denoted $\text{supp}(\rho) := \{t_1, \ldots, t_n\}$.

We introduce a semantics where transitions can always be fired, and hence where markings may become negative. Formally, a $\mathbb{Z}$-marking is a vector $m : P \to \mathbb{Z}$. We write $m \rightarrow_\mathbb{Z} m'$ (or simply $m \rightarrow m'$) if $m' = m + \Delta(t)$. Given a run $\rho$, we define in the obvious way $\rightarrow_\mathbb{Z}$ and $\rightarrow^*_\mathbb{Z}$. Note that markings are $\mathbb{Z}$-markings (with the domain restricted to $\mathbb{N}$). The definition of $\mathbb{Z}$-markings is mostly needed to use $\rightarrow_\mathbb{Z}$.

We define the absolute value and norm of a Petri net $\mathcal{N} = (P, T, F)$ by $|\mathcal{N}| := |P| + |T|$ and $||\mathcal{N}|| := ||F|| + 1$, where $F$ is seen as a vector over $(P \times T) \cup (T \times P)$. The size of a Petri net is defined as $\text{size}(\mathcal{N}) := |\mathcal{N}|(1 + \log(|\mathcal{N}||))$. For some complexity problems, we will be given a Petri net and some markings, e.g. $m$ and $m'$. By the size of the input, we understand $\text{size}(\mathcal{N}, m, m') := \text{size}(\mathcal{N}) + \log(\|m\| + 1) + \log(\|m'\| + 1)$.

A transition $t$ is said to be quasi-live from marking $m$ if there exists a marking $m'$ such that $m \rightarrow^* m'$ and $t$ is enabled in $m'$. A transition $t$ is said to be live from $m$ if $t$ is quasi-live from all $m'$ such that $m \rightarrow^* m'$. We say that a Petri net $\mathcal{N}$ is quasi-live (resp. live) from $m$ if each transition $t$ of $\mathcal{N}$ is quasi-live (resp. live) from $m$. Informally, quasi-liveness states that no transition is useless, and liveness states that transitions can always eventually be fired.

Example 2.1. Consider the Petri net $\mathcal{N}_{\text{middle}} = (P, T, F)$ illustrated in the middle of Figure 1. Places $P = \{i, q_1, q_2, f\}$ and transitions $T = \{t_1, t_2, t_3, t_4\}$ are depicted respectively as circles and squares. The flow function $F$ is depicted by arcs, where weight 1 is omitted and arcs with weight 0 are not drawn, e.g. $F(i, t_1) = 1$, $F(t_1, i) = 0$, $F(t_4, f) = 2$ and $F(f, t_4) = 0$. In particular, transitions $t_1$, $t_2$ and $t_3$ are quasi-live from marking $\{i : 1\}$ since

\[
\{i : 1\} \xrightarrow{t_1} \{q_1 : 1\} \xrightarrow{t_2} \{q_2 : 1\} \xrightarrow{t_3} \{q_1 : 1\}.
\]

However, as no other marking is reachable, transition $t_4$ is not quasi-live. Note that $t_2$ and $t_3$ are both live from $\{i : 1\}$, while $t_1$ is not live since it can only be fired once.

2.2 Workflow nets and soundness

A workflow net $\mathcal{N}$ is a Petri net that satisfies the following:

- there is a dedicated initial place $i$ with $*i[i] = 0$ for every transition $t$ (cannot produce tokens in $i$);  
- there is a dedicated final place $f \neq i$ with $*f[f] = 0$ for every transition $t$ (cannot consume tokens from $f$);
- each place and transition lies on at least one path from $i$ to $f$ in the underlying graph of $\mathcal{N}$, i.e. the graph $(V, E)$ where $V := P \cup T$ and $(u, v) \in E$ iff $F[u, v] > 0$.

Given $k \in \mathbb{N}$, we say that $\mathcal{N}$ is $k$-sound iff for all $m$, $\{i : k\} \xrightarrow{t} m$ implies $m \xrightarrow{t} \{f : k\}$, i.e. starting from $k$ tokens in the initial place, it is always possible to move the $k$ tokens into the final place. We say that $\mathcal{N}$ is:

- classically sound iff $\mathcal{N}$ is 1-sound and quasi-live from $\{i : 1\}$;
- generalised sound iff $\mathcal{N}$ is $k$-sound for all $k \geq 0$;
- structurally sound iff $\mathcal{N}$ is $k$-sound for some $k > 0$.

Example 2.2. Consider the workflow nets $\mathcal{N}_{\text{left}}, \mathcal{N}_{\text{middle}}$ and $\mathcal{N}_{\text{right}}$ depicted respectively in Figure 1.

Workflow nets $\mathcal{N}_{\text{left}}$ and $\mathcal{N}_{\text{middle}}$ are not 1-sound since their only transition that can mark place $f$ is not quasi-live from $\{i : 1\}$, namely $s_2$ and $t_4$. In particular, this means that both workflow nets are neither classically sound, nor generalized sound. Workflow net $\mathcal{N}_{\text{right}}$ is 1-sound, and in fact classically sound, as shown by the reachability graph of Figure 2.

In particular, this means that $\mathcal{N}_{\text{right}}$ is structurally sound. Workflow net $\mathcal{N}_{\text{left}}$ is not structurally sound as no matter the marking $\{i : k\}$ from which it starts, there is no way to empty place $p_1$ once it is marked. Workflow net $\mathcal{N}_{\text{middle}}$ is 2-sound, and hence structurally sound. Indeed, from $\{i : 2\}$, the two tokens must enter $\{q_1, q_2\}$ from which they can escape via $\{q_1 : 1, q_2 : 1\}$ by firing $t_4$, reaching marking $\{f : 2\}$.

Workflow net $\mathcal{N}_{\text{right}}$ is not 2-sound, and hence not generalized sound. Indeed, we have $\{i : 2\} \xrightarrow{k \in \mathbb{N}} \{q_2 : 2, f : 1\}$ and no transition is enabled in the latter marking.

To gain some intuition on why soundness in workflow nets is in general easier than reachability in Petri nets, let us end this section by proving a simple property which lets us conclude $k$-unsoundness from strict coverability of the final marking $\{f : k\}$.

Lemma 2.3. Let $\mathcal{N} = (P, T, F)$ be a workflow net and let $k \in \mathbb{N}$. If $\{i : k\} \xrightarrow{t} \{f : k\} + m$ for some marking $m > 0$, then $\mathcal{N}$ is not $k$-sound.
As mentioned in the introduction, classical soundness is characterized by boundedness and liveness:

\[ \text{Proposition 3.1 (Lemma 8). A workflow net } \mathcal{N} \text{ is classically sound iff } \mathcal{N}_{\text{rc}} \text{ is live and bounded from } \{i: 1\}. \]

In the proof of Lemma 2.3 we rely on two properties of workflow nets: there are no outgoing edges from \( f \); and that from a nonzero marking one cannot reach a zero marking (since all nodes are on a path from \( i \) to \( f \)).

**Proof of Lemma 2.3.** By definition, each transition lies on a path from \( i \) to \( f \). In particular, this means \( r^* \neq 0 \) for all \( t \in T \). So \( \{f: k\} + m \not\rightarrow^* \{f: k\} \), and thus \( \mathcal{N} \) is not \( k \)-sound. \( \square \)

### 3 Classical soundness

As mentioned in the introduction, classical soundness is decidable, but its complexity has not yet been established. Let us recall why decidability holds. We say that a Petri net \( \mathcal{N} \) is **bound**ed from marking \( m \) if there exists \( b \in \mathbb{N} \) such that \( m \rightarrow^* m' \) implies \( m' \leq b \). Otherwise, \( \mathcal{N} \) is unbound**ed from \( m \). It is well-known that unboundedness holds iff there exist markings \( m' < m'' \) such that \( m \rightarrow^* m' \rightarrow^* m'' \). The short-circuit net \( \mathcal{N}_{\text{sc}} \) of a workflow net \( \mathcal{N} \) is \( \mathcal{N} \) extended with a transition \( t_{\text{sc}} \) such that \( F[f, t_{\text{sc}}] = F[t_{\text{sc}}, i] = 1 \) (and 0 for other entries relating to \( t_{\text{sc}} \)). Informally, the short-circuit net allows restoring the system upon completion, i.e. by moving a token from \( f \) to \( i \).

For example, the left side of Figure 3 illustrates a short-circuit net \( \mathcal{N}_{\text{sc}} \). By inspecting the graph of markings reachable from \( \{i: 1\} \) in \( \mathcal{N}_{\text{sc}} \), we see that \( \mathcal{N}_{\text{sc}} \) is live and bounded, i.e. it is always possible to (re)fire any transition, and each place is bounded by \( b = 1 \) token. It turns out that liveness and boundedness characterize classical soundness:

**Proposition 3.1 (Lemma 8). A workflow net \( \mathcal{N} \) is classically sound iff \( \mathcal{N}_{\text{sc}} \) is live and bounded from \( \{i: 1\} \).**

Decidability of classical soundness follows from Theorem 3.1. Indeed, boundedness can be tested in EXPSPACE [19], and liveness is decidable since it reduces to reachability [13, Thm 5.1] which is decidable [16]. However, the liveness problem is hard for the reachability problem [13, Thm 5.2], which was recently shown Ackermann-complete [8, 14, 15]. In this section, we first give a slightly different characterization involving liveness which yields EXPSPACE membership. Then, we show that classical soundness is EXPSPACE-hard, and hence EXPSPACE-complete, via a reduction from the reachability problem for so-called reversible Petri nets.

#### 3.1 EXPSPACE membership

Let us reformulate the characterization of Theorem 3.1 so that it deals with another property than liveness, namely “cyclicity”. We say that a Petri net is **cyclic** from a marking \( m \) if for any marking \( m', m \rightarrow^* m' \) implies \( m' \rightarrow^* m \), i.e. it is always possible to go back to \( m \). For example, the short-circuit net \( \mathcal{N}_{\text{sc}} \), illustrated on the left of Figure 3, is cyclic since each marking reachable from \( \{i: 1\} \) can reach \( \{f: 1\} \), which in turn can reach \( \{i: 1\} \).

Rather than directly considering classical soundness, we first consider 1-soundness. The characterization of Theorem 3.1 can be adapted to this problem as follows:

**Proposition 3.2.** A workflow net \( \mathcal{N} \) is 1-sound iff \( \mathcal{N}_{\text{sc}} \) is bounded and transition \( t_{\text{sc}} \) is live from \( \{i: 1\} \).
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Theorem 3.5. Both 1-soundness and classical soundness are in EXPSPACE.

Proof. Checking whether a transition \( t \) satisfies \( \pi \tau = \{i : 1\} \) can be carried in polynomial time. The other properties of Theorem 3.3 for 1-soundness, namely boundedness and cyclicity, belong to EXPSPACE \([3, 19]\).

For quasi-liveness, we proceed as follows. The coverability problem asks whether given a Petri net and two markings \( m, m' \), there exists a marking \( m'' \geq m' \) such that \( m \rightarrow m'' \). This problem belongs to EXPSPACE \([19]\). Recall that quasi-liveness asks whether for each transition \( t \in T \cup \{t_{sc}\} \), it is the case that \( \{i : 1\} \rightarrow m \) for some some marking \( m \) that enables \( t \), i.e. such that \( m \geq \tau t \). The latter is a coverability question. Hence, quasi-liveness amounts to \( |T| + 1 \) coverability queries, which can be checked in EXPSPACE.

We further show that the previous result can be extended to \( k \)-soundness through the following lemma.

Lemma 3.6. Given a workflow net \( N \) and \( k > 0 \), one can compute, in polynomial time, a workflow net \( N' \) with \( ||N'|| = ||N|| + \log(k) \) such that, for all \( c > 0 \), \( N \) is \( c \)-sound iff \( N' \) is \( c \)-sound.

Proof. Let \( N = (P, T, F) \). We define \( N' \equiv (P', T', F') \) that rescales everything by \( k \). Formally, we add two new places that are the new initial and final places \( P' : = P \cup \{i', f'\} \). We denote by \( i' \) and \( f' \) the previous initial and final places. We add two new transitions \( t_i \) and \( t_f \) defined by:

- \( t_i[i'] = 1 \) and \( t_i[p] = 0 \) for \( p \neq i' \),
- \( t_i'[i] = k \) and \( t_i'[p] = 0 \) for \( p \neq i' \),
- \( t_f[i] = k \) and \( t_f[p] = 0 \) for \( p \neq f' \),
- \( t_f'[f'] = 1 \) and \( t_f'[p] = 0 \) for \( p \neq f' \).

It is straightforward that \( N' \) satisfies the lemma.

Corollary 3.7. The \( k \)-soundness problem is in EXPSPACE.

Proof. It suffices to invoke Theorem 3.6 with \( c = 1 \), and test 1-soundness of the resulting workflow net via Theorem 3.5.

3.2 EXPSPACE-hardness

Let us now establish EXPSPACE-hardness of classical soundness. We will need the forthcoming lemma that essentially states that so-called reversible Petri nets can count up to (or down from) a doubly exponential number. Formally, we say that a Petri net \( N = (P, T, F) \) is reversible if each transition of \( N \) has an inverse, i.e. for every \( t \in T \), there exists \( t^{-1} \in T \).
such that \((t\rightarrow r) = t\) and \((t\rightarrow r) = t\). Note that for reversible Petri nets, it is the case that \(m \rightarrow m'\) if and only if \(m' \rightarrow m\). To emphasise this, we will sometimes write \(m \leftrightarrow m'\).

**Lemma 3.8** ([17, Lemma 3]). Let \(N\) be a reversible Petri net and let \(m\) and \(m'\) be two markings. Let \(n := \text{size}(N, m, m')\). There exists \(c_n \in 2^{O(n)}\) such that if \(m \rightarrow m'\) then \(m \rightarrow m'\) for a \(c_n\)-bounded run \(p\).

**Lemma 3.9** ([17, reformulation of Lemma 6 and Lemma 8]). Let \(n \in \mathbb{N}\) and \(c_n \in 2^{O(n)}\). There exists a reversible Petri net \(N = (P, T, F, c_n)\) with four distinguished places \(s, c, f, b \in P\). Let \(m_n := \{s: 1, c: 1\}\) and \(m'_n := \{f: 1, c: 1, b: c_n\}\). The following holds for all \(m\):

1. \(m_n \rightarrow m'_n\);
2. \(m_n \rightarrow m\) and \(m[f] > 0\) implies \(m = m'_n\);
3. \(m \rightarrow m'_n\) and \(m[s] > 0\) implies \(m = m_n\);
4. if \(m < m'_n\) and \(m[f] = 0\) then no transition can be fired from \(m\);
5. for all \(p \in P\) there exists \(m_n \rightarrow m\) s.t. \(m[p] > 0\).

Furthermore, \(N_n\) is: of polynomial size in \(n\); constructible in polynomial time in \(n\); and quasi-live both from \(m_n\) and \(m'_n\).

**Theorem 3.10.** The classical soundness and 1-soundness problems are EXSPACE-hard.

**Proof.** We give a reduction from the reachability problem for reversible Petri nets. This problem is known to be EXSPACE-complete [4, 17]. Let \(N = (P, T, F)\) be a reversible Petri net, and let \(m, m'\) be two markings for which we would like to know whether \(m \rightarrow m'\) in \(N\).

Let \(n := \text{size}(N, m, m')\), let \(c_n\) be the value given by Theorem 3.8 for \(n\), and let \(N_n = (P_n, T_n, F_n)\) be the Petri net given by Theorem 3.9 for \(c_n\).

We construct a workflow net \(N' = (P', T', F')\) such that \(N'\) is classical sound if and only if \(m \rightarrow m'\) in \(N\). To avoid any confusion, we will denote markings in \(N'\) by \(n, n', \ldots\), etc.

The construction will ensure that \(m \rightarrow m'\) in \(N\) iff \(N'\) is classically sound. (1)

Moreover, 1-soundness of \(N'\) will imply \(m \rightarrow m'\), which will prove that both classical soundness and 1-soundness are EXSPACE-hard.

Informally, we wish for \(N'\) to convert \(\{i: 1\}\) into \(m\), simulate \(N\), and convert \(m'\) into \(\{f: 1\}\). Here, the reversibility of \(N\) is crucial to ensure soundness: “erroneous runs” should still be able to reach \(f\). This approach is however idealised since \(N'\) has no way to test whether \(N\) has reached \(m'\).

By Theorem 3.8, we know that \(m \rightarrow m'\) is witnessed by a \(c_n\)-bounded run. Hence, \(c_n\) tokens per place suffice. Thus, we add a dual place \(\overline{p}\) for each place \(p\) of \(N\) such that, in marking \(n, \overline{p}\) contains \(c_n - n[p]\) tokens. This allows to implement a form of equality test for \(m'\). Yet, this is again oversimplified as it must be implemented with great care. Indeed, if \(N\) has reached \(m'' > m'\), then the gadget for equality test will consume some tokens, but not all \(c_n\) tokens from the dual places (recall that producing and consuming \(c_n\) cannot be achieved atomically, but rather via \(N_n\)). Thus, a mechanism is needed to restore \(m''\) and the budget, as \(m'' \rightarrow m'\) could hold.

Formally, the set of places \(P'\) consists of: \(P_1\) its disjoint copy \(\overline{P} := \{\overline{p} \mid p \in P\}\); seven extra places

\[
\{i, f, \rho_{\text{start}}, \rho_{\text{inProgress}}, \rho_{\text{cover}}, \rho_{\text{simple}}, \rho_{\text{canFire}}\}
\]

two disjoint copies of \(P_n\) (from Theorem 3.9), with one copy of \(b\) removed. One of the copies will be marked with \(\vee\) to avoid any confusion, thus we write \(e.g. p_0^\vee \in P_n^\vee\). The two places \(b\) and \(b^\vee\) are merged into a single place denoted \(b\).

Before presenting the transitions, we would like to emphasise that, intuitively, place \(\overline{p} \in \overline{P}\) will contain a “budget” of tokens that is an upper bound on how many more tokens can be present in \(p\). Most of the time, for every marking \(n\) and place \(p \in P\), we will keep \(n[p] + n[\overline{p}] = c_n\) as an invariant.

In Figure 4, we present the most relevant parts of \(N'\). Formally, the set of transitions is divided into four subsets \(T' = T_1 \cup T_2 \cup T_3 \cup T_4\). Transitions will be defined by \(\{t'\}[p]\) and \(\{t\}[p]\). The values are zero on unmentioned places.

First, for every transition \(t \in T_1\), we define \(t' \in T_1\) by:

\[
\begin{align*}
&t'[p] := t[p] \text{ and } t'[p] := t'[p] \text{ for all } p \in P; \\
&t'[\overline{p}] := t[p] \text{ and } t'[\overline{p}] := t'[p] \text{ for all } p \in P; \\
&t'[\rho_{\text{canFire}}] := t[\rho_{\text{canFire}}] := 1.
\end{align*}
\]

It is easy to see that since \(N\) is a reversible Petri net, for every transition in \(T_1\), its reverse is also in \(T_1\). We will say that \(T_1\) is reversible. Notice that, for all \(t' \in T_1\) and \(p \in P\), the sum of tokens in \(p\) and \(\overline{p}\) does not change under \(t'\).

Second, for every \(t \in T_2\), we add \(t' \in T_2\) such that:

\[
\begin{align*}
&t'[p] := t[p] \text{ and } t'[p] := t'[p] \text{ for all } p \in P; \\
&t'[\overline{p}] := t[p] \text{ and } t'[\overline{p}] := t'[p] \text{ for all } p \in P.
\end{align*}
\]

Intuitively, places in \(\overline{P}\) behave as \(b\) to initialise the budget of \(c_n\) tokens. Similarly, for every \(t' \in T_3\), we add \(t' \in T_3\) such that:

\[
\begin{align*}
&t'[\rho] := t[p] \text{ and } t'[\rho'] := t'[p] \text{ for all } p \in P; \\
&t'[\overline{p}] := t[b] \text{ and } t'[\overline{p}] := t'[b] \text{ for all } \overline{p} \in \overline{P}.
\end{align*}
\]

Note that since \(N_n\) is reversible, both \(T_2\) and \(T_3\) are reversible.

The set \(T_4\) consists of the ten remaining transitions

\[
\{f_{\text{hard}}, f_{\text{start}}, f_m, f_m', f_{\text{in}}, f_{\text{empt}}, f_{\text{reach}}, f_{\text{reach}}, f_{\text{simple}}, f_{\text{simple2}}\}.
\]

Intuitively, the first two are needed to initialise places in \(\overline{P}\) with \(c_n\) tokens; the next three transitions respectively add \(m, -m'\) and \(m'\) to \(P\); the next three transitions transfer tokens towards the final places; and the last two transitions are needed for quasi-liveness. Formally,

\[
\begin{align*}
&t'[f_{\text{hard}}] = t'[f_{\text{hard}}] = t'[f_{\text{hard}}] := 1; \\
&t'[f_{\text{start}}] = t'[f_{\text{start}}] = t'[f_{\text{start}}] := 1; \\
&t'_m[p] := t'_m[p] := t_m[p] \text{ for all } p \in P; \text{ and } t'_m[p_{\text{start}}] = t'_m[p_{\text{canFire}}] = t'_m[p_{\text{canFire}}] = 1; \\
&t'_m[p] := t'_m[p] := t_m[p] \text{ for all } p \in P; \text{ and } t'_m[p_{\text{canFire}}] = t'_m[p_{\text{canFire}}] := 1; \text{ and } t_m^{-1} \text{ is its reverse transition.}
\end{align*}
\]
Figure 4. A workflow net $N'$ which is classically sound iff $m \rightarrow^* m'$ in the reversible Petri net $N = (P, T, F)$. In the example, $P = \{p_1, p_2\}$, $m = (1, 0)$ and $m' = (0, 1)$. The original places are blue, their copies are green, and other new places are red. We omit the transitions in $T_1$ that originated from $T$ (recall that these transitions are modified to consume and produce tokens also in green places), and we place $p_{\text{canFire}}$ (used only to allow transitions in $T_1$ to fire). We only sketch transitions in $T_2$ and $T_3$ (and some other transitions), by writing the intuitive meaning of the gadgets that add/remove $c_t$ tokens (arcs of these “transitions” are marked with a different color). The transition $t_{\text{hard}}$ initiates the bottom part of $N'$ (by filling the green places with $c_t$ tokens) that checks $m \rightarrow^* m'$. The transition $t_{\text{simple}}$ initiates the top part of $N'$. We denote arcs in the top part with dotted gray color. This part is rather trivial and its only purpose is to ensure quasi-liveness of transitions in $T_1$ (by filling blue and green places with $\parallel N \parallel$ tokens).

Let us denote the last marking by $n$. Notice that

$$n \rightarrow^{t_{\text{start}} m} \{p_{\text{canFire}} : 1, b : c_n\} + \sum_{p \in P} \{p : c_n - m[p]\}.$$  

We denote the latter marking by $n'$. Since $N'$ is 1-sound, $n' \rightarrow^{\rho'} \{f : 1\}$ for some run $\rho$. This is possible if $t_{\text{reach}}$ was fired at least once in $\rho$. Let $n_1 \rightarrow^{t_{\text{reach}} n_2}$ be the last time $t_{\text{reach}}$ was fired in $\rho$. We claim that $n_2 = \{f^c : 1, c^c : 1, b : c_n\} + \sum_{p \in P} \{p : c_n\}$. Indeed, it has to be that

$$n_2 \rightarrow^{\rho''} \{s^c : 1, c^c : 1\} \rightarrow^{t_{\text{end}}} \{f : 1\},$$

where $\rho''$ uses transition only from $T_3$. By Theorem 3.9 (4), this is possible only if $n_2$ is as claimed. Let $\rho''$ be the prefix of the run $\rho$ from $n'$ such that it ends in $n_1$. Finally, $\rho''$, when restricted to $P$, witnesses reachability for $m \rightarrow^* m'$.

$(\Rightarrow)$ Suppose that $m \rightarrow^* m'$. The proof of 1-soundness is very technical and can be found in the appendix. In a nutshell, recall that $T_1$, $T_2$ and $T_3$ are reversible, and for $t_{\text{m}}'$, $t_{\text{reach}}$, in $T_4$ we include their reverse transitions. This allows us to revert any configuration to a configuration from which it is easy to define a run to $\{f : 1\}$.

To conclude this implication, we need to prove that $N'$ is quasi-live. Indeed, from the proof of 1-soundness it is easy...
to see that $m \rightarrow m'$ implies that all transitions are fireable, with the possible exception of transitions from $T_i$. However, 
\[
\{ i: 1 \} \rightarrow^{\text{simple}} \{ p_{\text{simple}} : 1 \} + \sum_{p \in P} \{ p : \| N \|, \bar{p} : \| N \| \}.
\]
From the latter configuration, any transition of $T_i$ is fireable.

Finally, observe that $N'$ is a workflow net. Indeed, by taking $t_{\text{simple}}$ we put tokens in $P$ and $\bar{P}$. Each place from copies in $N_n$ is on a path from $i$ to $f$ by Theorem 3.9 (3). The remaining places are clearly on such a path by definition (see Figure 4).

\[\square\]

4 Bounds on vector reachability

In this section, we present technical results that will be helpful to establish complexity bounds in the forthcoming sections. It is well-known that Petri nets are complex due to their nonnegativity constraints. Namely, markings are over $\mathbb{N}$ (not $\mathbb{Z}$), which blocks transitions from being fired whenever the amount of tokens would drop below zero. By lifting this restriction, i.e. allowing markings over $\mathbb{Z}$, transitions cannot be blocked and we obtain a provably simpler model (e.g. see [12]). We recall known results that provide bounds on reachability problems for vectors over $\mathbb{Z}$. Based on these results, we will derive useful bounds for the next sections.

4.1 Integer linear programs

Given positive natural numbers $n, m > 0$, let $A \in \mathbb{Z}^{m \times n}$ be an integer matrix, $b \in \mathbb{Z}^m$ an integer vector and $x = (x_1, \ldots, x_n)^T$ a vector of variables. We say that $G := A \cdot x \geq b$ is an $(m \times n)$-ILP, that is, an integer linear program (ILP) with $m$ inequalities and $n$ variables. The set of solutions of $G$ is 
\[
\| G \| := \{ \mu \in \mathbb{Z}^n | A \cdot \mu \geq b \};
\]
and the set of natural solutions is 
\[
\| G \|_\geq := \| G \| \cap \mathbb{N}^n.
\]

We will only be interested in the natural solutions $\| G \|_\geq$, but sometimes we will need to refer to $\| G \|$. We shall assume that these sets are equal, by implicitly adding a new inequality for each variable specifying that it is greater or equal to 0.

Often it is convenient to write an equality constraint, e.g., $x - y = 0$. This can be simulated by two inequalities, so we will allow to define $G$ both with equalities and inequalities.

We introduce some notation about semi-linear sets from [6] to obtain bounds on the sizes of solutions to ILPs. A set of vectors is called linear if it is of the form $L(b, P) = \{ b + \lambda_1 p_1 + \ldots + \lambda_k p_k | \lambda_1, \ldots, \lambda_k \in \mathbb{N} \}$, where $b \in \mathbb{Z}^n$ is a vector and $P = \{ p_1, \ldots, p_k \} \subseteq \mathbb{N}^n$ is a finite set of vectors. A set is called hybrid linear if it is of the form $L(b, P) = \bigcup_{b \in \mathbb{N}^n} L(b, P)$ for a finite set of vectors $B = \{ b_1, \ldots, b_r \} \subseteq \mathbb{N}^n$.

The size of a finite set of vectors $B$ and of an $(m \times n)$-ILP $G$ are defined respectively as $|B| := \max_{b \in \mathbb{B}} |b|$ and $|G| := |A| + |b| + m + n$.

**Lemma 4.1** ([30], presented adapted from [6, Prop. 3]). Let $G$ be an $(m \times n)$-ILP. It is the case that $|G| = \bigcup_{i \in I} L(B_i, P_i)$, where $\max_{i \in I} |B_i| \leq |G|^{O(n \log n)}$.

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For the forthcoming lemmas, recall that $c = (c, \ldots, c)$.

**Lemma 4.2.** Let $G$ be an $(m \times n)$-ILP. There exists a number $c \leq \| G \|^{O(n \log n)}$ such that for all $\mu \in \| G \|_\geq$, there is some $\mu' \in \| G \|_\geq$ such that $\mu' \leq \mu$ and $\mu' \leq c$.

**Proof.** Recall that we can assume $\| G \| = \| G \|_\geq$. By Theorem 4.1, $\| G \| = \bigcup_{i \in I} L(B_i, P_i)$. We set $c := \max_{i \in I} |B_i|$. Let $\mu \in \| G \|_\geq$. There exist $i \in I$ and $b \in B_i$ such that $\mu \in L(b, P_i)$. Note that $p \geq 0$ for all $p \in P$. Hence, we have $b \in \| G \|_\geq$, $b \leq \mu$ and $b \leq c$. Thus, we can set $\mu' := b$.

\[\square\]

**Lemma 4.3.** Let $G = A \cdot x \geq b$ be an $(m \times n)$-ILP, where $b \geq 0$. There exists $c \leq \| G \|^{O((m+n) \log(m+n))}$ such that the following holds. For every $\mu \in \| G \|_\geq$, there exists $\nu \in \| G \|_\geq$ such that $\nu \leq \mu$, $\nu \leq c$, and $A \cdot \nu \leq A \cdot \mu$.

**Proof.** Let $x_1, \ldots, x_n$ be the variables of $G$. We define a $(3m \times (m+n))$-ILP $G'$ by slightly modifying $G$. For every inequality in the original ILP $G$, we add one fresh variable. We denote them $y_1, \ldots, y_m$. Now, recall that the inequalities in $G$ are of the form: $\sum_{i=1}^n A[j, i] \cdot x_i \geq b[j]$ for $j \in [1..m]$. The ILP $G'$ is defined with the same inequalities, plus $m$ new equalities (recall that this requires $2m$ inequalities): $\sum_{i=1}^n A[j, i] \cdot x_i = y_j = 0$ for $j \in [1..m]$.

Notice that, in solutions for $G'$, the variables $y_j$ are uniquely determined by the valuation of $x_1, \ldots, x_n$. For convenience, we will write $\mu[x_i], \mu'[y_j]$ when referring to the components of solutions. For every $\mu \in \| G \|_\geq$, there is a unique $\mu' \in \| G' \|$ such that $\mu'[x_i] = \mu[x_i]$ for all $i \in [1..n]$. Thus, since $b \geq 0$, we have $\| G' \|_\geq = \{ \mu' | \mu \in \| G \| \}$. We define $c$ as the constant from Theorem 4.2 for $G'$. Now, let $\mu \in \| G \|_\geq$ and let $\mu' \in \| G' \|_\geq$ be its corresponding solution. By Theorem 4.2, there exists $\nu' \in \| G' \|_\geq$ such that $\nu' \leq \mu'$ and $\nu' \leq c$. We define $\nu \in \| G \|_\geq$ as the solution corresponding to $\nu'$. It is clear that $\nu \leq \mu$ and $\nu \leq c$. For the remaining part, fix $j \in [1..m]$. Recall that $\nu'[y_j] = \sum_{i=1}^n A[j, i] \cdot \nu'[x_i]$ and $\mu'[y_j] = \sum_{i=1}^n A[j, i] \cdot \mu'[x_i]$. Thus,

\[
\sum_{i=1}^n A[j, i] \cdot \nu[x_i] \leq \sum_{i=1}^n A[j, i] \cdot \mu[x_i],
\]
which concludes the proof.

\[\square\]

4.2 Steinitz Lemma

Let us recall the Steinitz Lemma [20] based on the presentation of [11].

**Lemma 4.4.** Let $x_1, \ldots, x_n \in \mathbb{R}^d$ be such that $\sum_{i=1}^n x_i = 0$ and $\| x_i \| \leq 1$ for all $i$. There exists a permutation $\pi$ on $[1..n]$ such that

\[
\sum_{j=1}^i x_{\pi(j)} \leq d \quad \text{for all } i \in [1..n].
\]

The following formulation of the lemma, which is depicted graphically in Figure 5, will be more convenient for us.
The complexity of soundness in workflow nets

Theorem 5.1. Generalised soundness is in PSPACE.

5 Generalised soundness

A Petri net \( N \) is \( \mathbb{Z} \)-bounded from a marking \( m \) if there exists \( b \in \mathbb{N} \) such that \( m = \sum_{j=0}^{n} x_j \). There exists a permutation \( \pi \) of \([0..n]\) such that: \( \pi(0) = 0 \); and there exist \( 0 \leq c_0 \leq c_1 \leq \ldots \leq c_n \leq 1 \), where

\[
\left| \sum_{j=0}^{i} x_{\pi(j)} - c_i \cdot z \right| \leq b(d + 2) \quad \text{for all } i \in [0..n].
\]

Lemma 4.5. Let \( x_0, x_1, \ldots, x_n \in \mathbb{Z}^d, b := \max_{j=0}^{n} \|x_j\|, \) and \( z := \sum_{j=0}^{n} x_j \). There exists a permutation \( \pi \) of \([0..n]\) such that: \( \pi(0) = 0 \); and there exist \( 0 \leq c_0 \leq c_1 \leq \ldots \leq c_n \leq 1 \), where

\[
\left| \sum_{j=0}^{i} x_{\pi(j)} - c_i \cdot z \right| \leq b(d + 2) \quad \text{for all } i \in [0..n].
\]

5.1 Nonredundant workflow nets

Fix a workflow net \( N = (P, T, F) \). We say that a transition \( t \) is \( k \)-sound if there exists \( k \in \mathbb{N} \) such that \( \{i: k \to m\} \). Note that every strongly \( k \)-sound net is also \( k \)-sound.

The aim of the next three subsections is to prove the following theorem.

Theorem 5.1. Generalised soundness is in PSPACE.

The proof has two parts. First, we prove that if there is a \( k \) for which the net is not \( k \)-sound, then there is also such a \( k \) bounded exponentially. Second, we prove that \( k \)-soundness for exponentially bounded \( k \) can be verified in PSPACE.

that this requires coverability checks for every place. However, since the number of initial tokens is arbitrary, finding redundant places amounts to a simple polynomial-time saturation procedure. More details can be found in [29, Thm. 8, Def. 10, Sect. 3.2] (and in the appendix). We will call workflow nets without redundant places nonredundant workflow nets. To summarise we conclude the following.

Proposition 5.2. Given a workflow net \( N \), one can identify and remove all redundant places from it in polynomial time. The resulting workflow net \( N' \) is nonredundant. Moreover, \( N \) is \( k \)-sound if and only if \( N' \) is \( k \)-sound for all \( k \in \mathbb{N} \).

In the following lemma, intuitively, we show that the initial budget is small for nonredundant workflow nets.

Lemma 5.3. Let \( N = (P, T, F) \) be a nonredundant workflow net and let \( p \in P \) be a place. There exists \( k < (\|T\| + 2)^{\|T\|} \) such that \( \{i: k \to m \} \) and \( m[p] > 0 \).

Proof. A transition \( t \) increases a place \( p' \) if \( \Delta(t)[p'] > 0 \). We say that a run \( \rho \) increases \( p' \) if there exists \( \gamma \in \text{supp}(\rho) \) such that \( \gamma \). For the proof of the lemma, we assume that \( p \neq i \), as otherwise it suffices to define \( k = 1 \). We prove that for all run \( \{i: k' \to m'\} \), there is a run \( \pi \) such that: \( \text{supp}(\pi) = \text{supp}(\rho) \), and \( \{i: k \to m \} \) for some \( k < (\|T\| + 2)^{\|T\|} \) and \( m \), where \( m[p'] \geq 1 \) for all places \( p' \) increased by \( \rho \). Note that, since \( N \) is a nonredundant workflow net, if we exhibit such a run then we are done as there exists \( p \) that increases \( p \).

Let \( \{i: k' \to m'\} \). The proof is by induction on \( n \), where \( \text{supp}(\rho) = \{t_1, \ldots, t_n\} \). Assume \( n = 1 \). The only transition used by \( \rho \) is \( t_1 \), which increases \( \rho \). Recall that \( \|T\| \) is the maximal number occurring on any arc of \( N \). Since workflow nets start with tokens only in place \( i \), we must have \( \{i: \|T\| \geq n\} \). It suffices to define \( \pi := t_1 \) and \( k := \|T\| < (\|T\| + 2) \).

For the induction step, assume \( n > 1 \) and that the lemma holds for \( n - 1 \). Let \( \rho_{n-1} \) be the longest prefix of \( \rho \) such that \( \text{supp}(\rho_{n-1}) = \{t_1, \ldots, t_{n-1}\} \). The induction hypothesis for

\( \|T\| \) holds for exponentially bounded \( k \) can be verified in PSPACE.
We recall a result by van Hee et al. that establishes a connection.

Thus, to put

\[
\frac{1}{p} \times \text{the run under } T_{\text{net}}. \text{ Let } n := \{k : \pi \leq (\|T\| + 1) + (\|T\| + 1) \cdot m_{n-1} \}. 
\]

By definition of \( m_{n-1} \), it holds that \( n[p'] \geq \|T\| + 1 \) for all \( p' \in \pi^* \). Furthermore, we can fire \( t_n \) from \( n \). Let \( n \rightarrow^{*} m \). To conclude, consider a place \( p' \) increased by \( p \). If it is increased by one of the transitions \( t_1, \ldots, t_{n-1} \), then after firing \( t_n \) at least one token was left in \( p' \). Otherwise, \( p' \) is increased by \( t_n \). In both cases, we have \( m[p] \geq 1 \). It remains to observe that \( k = (k_{n-1} + 1) \cdot (\|T\| + 1) < (\|T\| + 2)^n \).

5.2 Relating soundness and strong soundness

We recall a result by van Hee et al. that establishes a connection between the reachability relations \( \rightarrow^* \) and \( \rightarrow \).

Lemma 5.4 (adaptation of [29, Lemma 12]). Let \( N \) be a non-redundant, generalised sound workflow net, and let \( m \) be a marking for which there exists \( k \geq 0 \) satisfying \( \{i : k \rightarrow^\pi \} \).

There exists \( \ell \geq 0 \) such that \( \{i : k + \ell \rightarrow^* \} \).

Note that Theorem 5.4 is an easy consequence of the definition of non-redundancy. Namely, it suffices to put “enough budget” in each place so that the run under \( \rightarrow^\pi \) becomes a run under \( \rightarrow^\ast \). We restate the result to give a bond on \( \ell \), and so that it does not need to assume \( N \) is generalised sound.

Lemma 5.5. Let \( N = (P, T, F) \) be a non-redundant workflow net. Let \( k \in \mathbb{N}^P \) be such that \( \{i : k \rightarrow^z \} \).

There exists \( \ell \leq (\|T\| + 2) \cdot \max(\|T\|, k) \cdot |P| + 2 \) such that \( \{i : \ell \rightarrow^* \} \).

Proof. Let \( p = t_1 t_2 \cdots t_n \) be such that \( \{i : k \rightarrow^\pi \} \).

Let us define \( x_0 := \{i : k \} \) and \( x_j := \Delta(t_j) \) for all \( j \in [1..n] \). By Theorem 4.5, we can assume that the transitions \( t_j \) are ordered so that there exist \( c_0, \ldots, c_n \geq 0 \) where

\[
\{i : k + \sum_{j=1}^n A(t_j) - c_j m \} \leq \max(\|T\|, k) \cdot |P| + 2,
\]

for all \( i \in [0..n] \). Since \( m \geq 0 \), we get for all \( p \in P \):

\[
\{i : k + \sum_{j=1}^n A(t_j) \cdot p \} \geq -\max(\|T\|, k) \cdot |P| + 2.
\]

By Theorem 5.3, there exists \( \ell_p \leq (\|T\| + 2)^{|T|} \) such that for every place \( p \) there is a run \( \{i : \ell_p \rightarrow^\pi m_p \} \) with \( m_p[p] > 0 \). Thus, to put \( \max(\|T\|, k) \cdot |P| + 2 \) tokens in all places, it suffices to repeat \( \max(\|T\|, k) \cdot |P| + 2 \) times the run \( \pi_{\ell_p} \) for every \( p \in P \). This requires \( \ell \leq (\|T\| + 2)^{|T|} \cdot \max(\|T\|, k) \cdot |P| |P| + 2 \).

Let \( m' \) be the marking obtained afterwards. By (2), \( m' \) allows to fire \( \rho \). Thus, we obtain \( \{i : \ell \rightarrow^* m' \} \).

This lemma allows us to focus on \( \rightarrow^z \) instead of \( \rightarrow^\ast \).

Lemma 5.6. Let \( N = (P, T, F) \) be a non-redundant workflow net. It is the case that \( N \) is generalised sound iff it is strongly k-sound for all \( k \geq 0 \). Moreover, if \( N \) is not strongly k-sound, then there exists \( k' \leq k + 2(\|T\| + 2)^{|T|} \cdot \max(\|T\|, k) \cdot |P| |P| + 2 \) such that \( N \) is not k'-sound.

Proof. The “if” implication is trivial. Indeed, if \( N \) is not k-sound then it cannot be strongly k-sound.

To prove the “only if” implication, assume that \( N \) is not strongly k-sound. We show that there exists \( k' \) such that \( N \) is not k'-sound. We will also prove the promised bound on \( k' \). Since \( N \) is not strongly k-sound, there must be some \( m^* \in \mathbb{N}^P \) and \( \pi' \) such that \( \{i : k \rightarrow^\pi m \} \).

By Theorem 5.3, there exists \( \ell \leq (\|T\| + 2)^{|T|} \cdot \max(\|T\|, k) \cdot |P| |P| + 2 \) such that \( \{i : \ell \rightarrow^* m' \} \).

Thus, \( m + \{i : \ell \rightarrow^* \} \).

We are done since this means that \( N \) is not (\( k + k \))-sound.

5.3 Strong unsoundness occurs for small numbers

In this section, we will show that if there exists some \( k \) such that \( N \) is not strongly k-sound, then \( k \) is at most exponential in \( |N| \). We define an ILP which is closely related to the markings reachable from at least one initial number of tokens in \( N \). Essentially, the ILP will encode that there exists \( k > 0 \) and \( m \geq 0 \) such that \( \{i : k \rightarrow^z m \} \).

This can be done since only “ firing counts” matter, i.e. \( m \rightarrow^z m' \) implies \( m \rightarrow^z m' \) for any permutation \( \pi' \) of \( \pi \).

Let \( N = (P, T, F) \) be a workflow net. We define ILP \( N := \text{A} \cdot x \geq 0 \) as an ILP with \( |P| + |T| + 1 \) inequalities and \( |T| + 1 \) variables. The variables of ILP \( N = x := (\kappa, t_1, \ldots, t_n) \). For ease of notation, we write \( t = (t_1, \ldots, t_n) \). We assume an implicit bijection between \( T \) and \( [1..|T|] \), i.e. for every \( t \in T \) there is a unique \( i \) such that \( t[i] = t_i \). The matrix \( A \) is defined by the following inequalities:

1. \( \kappa + \sum_{t \in T} t[i] \cdot \Delta(t)[i] \geq 0 \),
2. \( \kappa \geq 1 \),
3. \( \sum_{t \in T} t[i] \cdot \Delta(t)[i] \geq 0 \) for all \( p \in P \) \( \{i \} \),
4. \( t_i \geq 0 \) for all \( i \in [1..|T|] \).

The first two inequalities concern the initial “budget” \( k \) of tokens in \( i \) which is represented by \( \kappa \). Intuitively, \( \kappa \geq 1 \) has to be at least as much as \( t \) consumes from the initial place. The last two inequalities guarantee that we obtain a marking over \( \mathbb{N}^P \) and that the “ firing count” is over \( \mathbb{N}^T \).

Let \( \mu : x \rightarrow \mathbb{N} \) be a solution to ILP \( N \). We define

\[
\text{marking}(\mu) := \{i : \mu(k) \} + \sum_{i \in \{1..|T|\}} \mu(t_i) \cdot \Delta(t_i).
\]

The following claim follows by definition of ILP \( N \) and \( \rightarrow^* \).

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Claim 5.7. Let $m \in \mathbb{N}^P$ and $k > 0$. It holds that \( \{i: k \} \rightarrow^* \mu \) if and only if there exists a solution $\mu$ to ILP$_N$ such that $\text{marking}(\mu) = m$ and $\mu[k] = k$.

We conclude this part with the following bound.

Lemma 5.8. Let $N$ be a nonredundant workflow net. If $\mathcal{N}$ is strongly $i$-sound for all $1 \leq i < k$, and not strongly $k$-sound, then $k \leq c$, where $c$ is the bound from Theorem 4.3 for ILP$_N$.

Proof. For the sake of contradiction, assume that $k > c$ as is the statement. Since $N$ is not strongly $k$-sound, there exists a marking $m \in \mathbb{N}^P$ such that $\{i: k \} \rightarrow^* m$ and $\mu \not\rightarrow^* \{f: k\}$. By Theorem 5.7, there exists a solution $\mu$ to ILP$_N$ such that $\text{marking}(\mu) = m$ and $\mu[k] = k$. By Theorem 4.3, there exists a solution $\mu' \leq \mu$ to ILP$_N$ such that $\mu'[k] \leq c < k = \mu[k]$ and $A\mu' \leq A\mu$, where $A$ is the underlying matrix of ILP$_N$. The latter inequality implies $\text{marking}(\mu') \leq \text{marking}(\mu)$.

Consider the vector $\pi := \mu - \mu'$. We prove that $\pi$ is a solution to ILP$_N$. Since $\mu' \leq \mu$ we know that $\pi$ is nonnegative. The inequalities of $A$ are satisfied since $A\pi \geq 0 \equiv A\mu \geq A\mu'$ and $\mu'[k] \leq c < k = \mu[k]$. Thus, $\pi$ is a solution to ILP$_N$.

By Theorem 5.7, $\{i: \mu'[k]\} \rightarrow^* \text{marking}(\mu')$ and $\{i: \pi[k]\} \rightarrow^* \text{marking}(\pi)$. Recall that $\mu'[k], \pi[k] < \mu[k] = k$. By assumption, $N$ is strongly $\mu'[k]$-sound and strongly $\pi[k]$-sound. Therefore, $\text{marking}(\mu') \rightarrow^* \{f: \mu'[k]\}$ and $\text{marking}(\pi) \rightarrow^* \{f: \pi[k]\}$. Since the function marking(·) is linear, we get

$$m = \text{marking}(\mu) = \text{marking}(\mu') + \text{marking}(\pi).$$

This implies $m \not\rightarrow^* \{f: \mu'[k]\} + \{f: \pi[k]\} = \{f: k\}$, which is a contradiction. \qed

5.4 Reachability in $\mathbb{Z}$-bounded nets is in PSPACE

Note that $\{i: 0\} = \{f: 0\} = 0$. We will use these notations interchangeably depending on the emphasis.

Lemma 5.9. Let $N = (P, T, F)$ be a nonredundant workflow net and $k > 0$. If $\mathcal{N}$ is $\mathbb{Z}$-unbounded from $\{i: k\}$, then $N$ is not generalised sound.

Proof. Since $N$ is $\mathbb{Z}$-unbounded from $\{i: k\}$, there exist $m, m'$ and $\pi$ such that $m < m'$ and $\{i: k\} \rightarrow^* \mu \rightarrow^* \{f: k\}$. Thus, $\{i: 0\} \rightarrow^* m' > 0$. For the sake of contradiction, assume that $N$ is generalised sound. Since $N$ is defined, $m$ is strongly $k$-sound in particular for $k = 0$ by Theorem 5.6, so we have $m' > m \rightarrow^* \{f: k\}$, which contradicts the fact that $t^* \neq 0$ for all $t \in T$. \qed

Lemma 5.10. Let $N = (P, T, F)$ be a workflow net. Let $m \in \mathbb{N}^P$ be a marking such that $||m|| \geq \max(||T||, k)^2 \cdot (|P| + 2) \cdot |P|$. If $\{i: k\} \rightarrow^* m$ then $\mathcal{N}$ is $\mathbb{Z}$-unbounded.

Proof. Let $\{i: k\} \rightarrow^* m$ for some $\sigma = t_1 t_2 \ldots t_n$. We use the notation $\cdot^*$ for multisets, e.g., $\{a, a, b\}$ contains two occurrences of $a$ and one of $b$. Without loss of generality, assume that no submultiset of $\{t_1, t_2, \ldots, t_n\}$ sums to 0. Otherwise, we can shorten $\sigma$ by removing such a submultiset. Further observe that since $||m|| \geq \max(||T||, k)^2 \cdot (|P| + 2) \cdot |P|$, we know that $n > \max(||T||, k) \cdot (|P| + 2) \cdot |P|$. By Theorem 4.5, we can assume that $t_1, t_2, \ldots, t_n$ are ordered so that there exist $0 \leq c_0 \leq c_1 \leq \ldots \leq c_n,$ where

$$\left| \{i: k\} + \sum_{j=1}^{i} \Delta(t_j) - c_i m \right| \leq \max(||T||, k) \cdot (|P| + 2),$$

for all $i \in [0..n]$. By the pigeonhole principle, there must exist $0 \leq i_1 < i_2 \leq n$ such that

$$\{i: k\} + \sum_{j=1}^{i_2} \Delta(t_j) - c_{i_2} m = \{i: k\} + \sum_{j=1}^{i_1} \Delta(t_j) - c_{i_1} m.$$

This is equivalent to

$$\sum_{j=i_1+1}^{i_2} \Delta(t_j) = \{c_{i_2} - c_{i_1}\} m.$$

We have $(c_{i_2} - c_{i_1}) m \geq 0$ and, since no subset of $\{t_1, t_2, \ldots, t_n\}$ sums to 0, we have a strict inequality. Let $z := \sum_{j=i_1+1}^{i_2} \Delta(t_j)$. We proved that $\{i: 0\} \rightarrow^* z > 0$, so $\mathcal{N}$ is $\mathbb{Z}$-unbounded. \qed

We are ready to prove the PSPACE membership of generalised soundness.

Proof of Theorem 5.1. Consider a workflow net $\mathcal{N} = (P, T, F)$. By Theorem 5.2, we can assume that $\mathcal{N}$ is a nonredundant workflow net. By Theorem 5.6 and Theorem 5.8, to prove generalised soundness it suffices to prove that it is $k$-sound for all $k \leq ||N||^{\text{poly}(|N|)}$.

By Theorem 5.9 and Theorem 5.10, if $\{i: k\} \rightarrow^* m$ and thus $\{i: k\} \rightarrow^* \mu \rightarrow^* m$ and $||m|| \geq C_k$ for some $C_k = ||N|| + k^{\text{poly}(|N|)}$, then the net is unsound. Since we need to consider only $k \leq ||N||^{\text{poly}(|N|)}$, all constants $C_k$ are bounded exponentially and can be written in polynomial space.

Thus, to verify $k$-soundness we proceed as follows. First, we check if a configuration $m$ such that $||m|| \geq C_k$ can be reached. This can be easily performed in NPSPACE = PSPACE as such a run would be witnessed by a sequence of configurations, such that each configuration can be stored in polynomial space. If such a configuration can be reached, then the algorithm outputs no. Otherwise, for every $m \in \mathbb{N}^P$ such that $||m|| < C_k$ one needs to verify whether $\{i: k\} \rightarrow^* m$ implies $m \rightarrow^* \{f: k\}$. This can be done in coNPSPACE = coPSPACE = PSPACE. \qed

5.5 PSPACE-hardness

A conservative Petri net is a Petri net $\mathcal{N} = (P, T, F)$ such that transitions preserve the number of tokens. That is, for all $m, m' \in \mathbb{N}^P$, it is the case that $m \rightarrow m'$ implies $\sum_{p \in P} m[p] = \sum_{p \in P} m'[p]$. The reachability problem for conservative Petri nets asks whether $m \rightarrow m'$, given $\mathcal{N}$, a source marking $m$ and a target marking $m'$.

Theorem 5.11. Generalised soundness is PSPACE-hard.

Proof. We give a reduction from reachability in conservative Petri nets, which is known to be PSPACE-complete [18].
Let \( N = (P, T, F) \) be a conservative Petri net, and let \( m, m' \) be the source and target markings. We define the constant \( c := \sum_{p \in P} m[p] = \sum_{p \in P} m'[p] \).

We construct a workflow net \( N' = (P', T', F') \) such that \( N' \) is generalised sound if and only if \( m \rightarrow^* m' \) in \( N \). To do so, we extend \( N \) with three new places \( P' := P \cup \{i, f, r\} \). Places \( i \) and \( f \) serve as dedicated initial and final places, respectively. Place \( r \) will be used to reset configurations. It could be merged with \( i \), if not for the restriction that, in a workflow net, place \( i \) cannot have any incoming arc.

We define \( T' \supseteq T \) by keeping the existing transitions and adding \( 3 + |P| \) new transitions. Namely:

1. transition \( t_i \) defined by \( t_i^* := \{i: 1\} \), and \( t_i^* := \{f: c\} \),
2. transition \( t_m \) defined by \( t_m^* := \{r: c\} \), and \( t_m^* := \{f\} \),
3. transition \( t_m' \) defined by \( t_m' := \{f\} \), and \( t_m' := \{f: 1\} \),
4. transition \( t_p \) defined by \( t_p^* := \{p: 1\} \), and \( t_p^* := \{f: 1\} \).

The first two transitions move a token from \( i \) and create the marking \( m \). The third transition consumes \( m' \) and puts a token into \( f \). Transitions from the fourth group allow to move tokens from any place in the original Petri net \( P \) to \( r \). See Figure 6 for a graphical presentation.

Figure 6. A workflow net \( N' \) which is generalised sound iff \( m \rightarrow^* m' \) in the conservative Petri net \( N = (P, T, F) \). Here, \( P = \{p_1, p_2, p_3\} \), \( m = \{p_1: 1, p_2: 1\} \), \( m' = \{p_2: 1, p_3: 1\} \) and \( c = 2 \). The original places are blue and the new places are red. We omit the original transitions (from \( T \)) in the picture.

It remains to show that \( N' \) is correct. Suppose \( N' \) is generalised sound. It must also be 1-sound and in particular \( \{i: 1\} \rightarrow^* \{f: 1\} \). Since \( N \) is conservative, it is easy to see that \( t_m \) can be fired only if there are no tokens in \( P \). Moreover, a token can be transferred to \( f \) only using \( t_m' \), which consumes \( m' \). Thus, we have \( m \rightarrow^* m' \) in \( N \).

For the converse implication, suppose that \( m \rightarrow^* m' \). Fix some \( k \) and suppose \( \{i: k\} \rightarrow^* v \). Notice that the transitions are defined in such a way that for every reachable configuration \( v \), the invariant \( ck = v[i] \cdot c + \sum_{p \in P \cup \{r\}} v[p] \cdot c + v[f] \cdot c \) holds. Thus, by repeatedly firing transitions \( t_i \) and \( t_p \), all tokens but those in \( f \) can be moved to \( r \), i.e.

\[ v \rightarrow^* \{r: (k - v[f]) \cdot c\} + \{f: v[f]\} \]

From there, to reach \( \{f: k\} \), it suffices to repeat \( (k - v[f]) \times \) times the following: fire \( t_m \); fire the run that witnesses \( m \rightarrow^* m' \); and fire \( t_m' \).

\[ \square \]

6 Structural soundness

In this section, we establish the EXPSpace-completeness of structural soundness. Recall that the latter asks whether, given a workflow net, \( k \)-soundness holds for some \( k \geq 1 \).

6.1 EXPSpace membership

Theorem 6.1. Structural soundness is in EXPSpace.

Let \( N = (P, T, F) \) be a workflow net. We define an \( (|T| + 2|P| + 1) \times (|T| + 1) \)-ILP, called \( ILP^*_N \). The variables are the same as for \( ILP_N \) in subsection 5.2: \( (k, t_1, \ldots, t_n) \), with the intuition that \( k \) denotes the number of initial tokens and \( t_i \) the number of times the transitions are used. We will keep the notation \( v = (t_1, \ldots, t_n) \) and the notation \( v[T] \) for \( t \in T \).

The inequalities are defined as follows:

1. \( \{i: k\} + \sum_{t \in T} v[T] \cdot \Delta(t) = \{f: k\} \) (expressed with \( 2|P| \) inequalities);
2. \( v \geq 0 \) \((|T|) \) inequalities;
3. and \( k > 0 \).

The first set of inequalities expresses that the effect of the transitions yields the final marking. The second type ensures that each transition is fired a nonnegative number of times. Finally the last one ensures that the initial marking has at least one token. The following is immediate.

Claim 6.2. There exists \( k > 0 \) such that \( \{i: k\} \rightarrow^* \{f: k\} \) if and only if there exists a solution \( v \) to \( ILP^*_N \) such that \( v[k] = k \).

Lemma 6.3. Let \( N = (P, T, F) \) be a nonredundant workflow net that is \( k \)-sound, and \( i \)-unsound for all \( 1 \leq i < k \). It is the case that \( c \leq c + (|T| + 2|P| + 1) \cdot \max(|T|, c) \cdot |P|(|P| + 2) \), where \( c \) is the bound given by Theorem 4.3 for \( ILP^*_N \).

Proof. Towards a contradiction, suppose that \( k > c + (|T| + 2|P| + 1) \cdot \max(|T|, c) \cdot |P|(|P| + 2) \). Consider \( ILP^*_N \). Since \( N \) is \( k \)-sound, there is a run \( \{i: k\} \rightarrow^* \{f: k\} \) and thus \( ILP^*_N \) has a solution \( \mu \). By Theorem 4.3, we can assume that \( \mu \leq c \).

By Theorem 6.2, \( (\{\mu[k]\} \rightarrow^* \{\mu[k]\}) \). By Theorem 5.5, there exist \( \ell \leq (|T| + 2|P| + 1) \cdot \max(|T|, c) \cdot |P|(|P| + 2) + \) and \( m \in \mathbb{N}^P \) such that \( \{\ell: \mu[m]\} \rightarrow^* m + \{f: \mu[m]\} \). Note that \( \mu[m] < k \). Let \( g := k - (\ell + \mu[m]) > 0 \). We have \( \{i: k\} = \{\ell + \mu[m] + g\} \rightarrow^* \{i: g\} + \mu + \{f: \mu[m]\} \).

Since \( N \) is \( k \)-sound, we have

\( \{i: g\} + \mu + \{f: \mu[m]\} \rightarrow^* \{\ell + \mu[m] + g\} \).

Thus, since \( v[f] = 0 \) for all \( t \in T \), we have \( \{i: g\} \rightarrow^* \{f: \ell + \mu[m] + g\} \).

Altogether, obtain

\( \{\ell + \mu[m] + g\} \rightarrow^* \{\ell + \mu[m] + g\} + \mu \rightarrow^* \{\ell + \mu[m]\} + \{f: \ell + g\} \).

Therefore, since \( N \) is \( k \)-sound, it must be \( \mu[m] \)-sound (recall that tokens in \( f \) are never consumed). This contradicts the fact that \( N \) is \( i \)-unsound for all \( 1 \leq i < k \).

\( \square \)

We may now prove Theorem 6.1.
Proof of Theorem 6.1. By Theorem 5.2, we can assume that the input \( \mathcal{N} \) is a non-redundant workflow net. By Theorem 6.3, it suffices to check if \( \mathcal{N} \) is \( k \)-sound for some value \( k \) bounded by \( \|\mathcal{N}\|_{\text{poly}} \). First, we guess \( k \), which can be written with polynomially many bits. Then, we test \( k \)-soundness in EXPSPACE via Theorem 3.7.

6.2 EXPSPACE-hardness

Theorem 6.4. Structural soundness is EXPSPACE-hard.

Proof: Let \( \mathcal{N} \) be a workflow net. We construct a workflow net \( \mathcal{N}' \) which is structurally sound if \( \mathcal{N} \) is 1-sound. We simply add a single new transition \( t \) to \( \mathcal{N} \) with \( t^* := \{i: 2\} \) and \( t^* := \{f: 1\} \). We show that \( \mathcal{N}' \) is \( k \)-unsound for every \( k \geq 2 \). Towards a contradiction, suppose it is \( k \)-sound for some \( k \geq 2 \).

Notice that \( k \) cannot be even because \( \{i: k\} \rightarrow t^{k/2} \{f: k/2\} \) and \( f \) has no outgoing arcs, and hence \( \{f: k/2\} \nrightarrow \{f: k\} \). Thus, it is the case that \( k \geq 3 \) is odd and \( \{i: k\} \rightarrow t^* \{i: 1\} + \{f: \lfloor k/2 \rfloor\} \). Since \( \mathcal{N}' \) is \( k \)-sound, \( \{i: 1\} \rightarrow t^* \{f: \lfloor k/2 \rfloor\} \). But that implies \( \{i: k\} \rightarrow f^* \{f: k \cdot \lfloor k/2 \rfloor\} \). Note that \( k \cdot \lfloor k/2 \rfloor > k \) as \( k \geq 3 \), which yields a contradiction since \( f \) has no outgoing arcs to get rid of the extra tokens.

To conclude, we observe that if the initial configuration in \( \mathcal{N}' \) is \( \{i: 1\} \), then it behaves like \( \mathcal{N} \) would, since \( t \) will never be enabled, i.e., it is not quasi-live. Thus, \( \mathcal{N}' \) is structurally sound if and only if \( \mathcal{N} \) is 1-sound, and EXPSPACE-hardness follows from Theorem 3.10.

7 Characterizing the set of sound numbers

Given a workflow net \( \mathcal{N} \), we define the set \( \text{Sound}(\mathcal{N}) := \{k \geq 1 \mid \mathcal{N} \text{ is } k\text{-sound}\} \). That is, \( \text{Sound}(\mathcal{N}) \) contains all the numbers for which \( \mathcal{N} \) is sound (except 0 which is trivial as any workflow net is 0-sound). This section is dedicated to providing and computing a representation of \( \text{Sound}(\mathcal{N}) \).

First, let us state a simple fact about \( \text{Sound}(\mathcal{N}) \).

Lemma 7.1. The set \( \text{Sound}(\mathcal{N}) \) is closed under subtraction with positive results.

Proof: Let \( g, k \in \text{Sound}(\mathcal{N}) \) be such that \( g > k \). We show that \( g - k \in \text{Sound}(\mathcal{N}) \). Since \( k \in \text{Sound}(\mathcal{N}) \), we have \( \{i: g\} = \{i: k + (g - k)\} \rightarrow \{f: k\} + \{i: g - k\} \). Since \( \mathcal{N} \) is \( g \)-sound, it must also be \( (g - k) \)-sound. So, \( g - k \in \text{Sound}(\mathcal{N}) \).

Corollary 7.2. There exist \( p > 0 \) and \( k \in \mathbb{N} \cup \{\infty\} \) such that \( \text{Sound}(\mathcal{N}) = \{i \cdot p \mid 1 \leq i \leq k\} \).

By the above, \( \text{Sound}(\mathcal{N}) \) is characterized by \( p \) and \( k \). We thus say that a net is \((k, p)\)-sound if and only if \( \text{Sound}(\mathcal{N}) = \{i \cdot p \mid 1 \leq i \leq k\} \). Note that \( k = 0 \) implies \( \text{Sound}(\mathcal{N}) = \emptyset \). Further, \( k = \infty \) if and only if \( \text{Sound}(\mathcal{N}) \) is infinite. Finally, a workflow net is generalised sound if it is \((1, +\infty)\)-sound; and it is structurally sound if there exist \( p, k \geq 1 \) such that

\[ (k, p) \text{-sound. We show that } k \text{ and } p \text{ can be computed. This will rely on insights from the prior sections about the smallest numbers for which a net is unsound or sound.}

Theorem 7.3. Given a workflow net \( \mathcal{N} \), the numbers \( p \) and \( k \) that characterize \( \text{Sound}(\mathcal{N}) \) are bounded by \( \|\mathcal{N}\|_{\text{poly}} \), and hence can be represented with polynomially many bits.

Given \( \mathcal{N} \), \( p' \) and \( k' \), the problem of deciding whether \( \mathcal{N} \) is \((k', p')\)-sound is EXPSPACE. Moreover, the algorithm computes \( p \) and \( k \) such that \( \mathcal{N} \) is \((k, p)\)-sound.

Proof: Consider a workflow net \( \mathcal{N} \). By Theorem 5.2, we can assume that \( \mathcal{N} \) is non-redundant. We will compute for which \( p \) and \( k \) the net \( \mathcal{N} \) is \((p, k)\)-sound. By Theorem 6.3, if \( \text{Sound}(\mathcal{N}) \neq \emptyset \), then there exists \( G \leq \|\mathcal{N}\|_{\text{poly}} \) such that \( \mathcal{N} \) is \( k \)-sound for some \( k \leq G \). By Theorem 3.7, it is possible to check \( 1 \)-soundness, \( 2 \)-soundness, \( \ldots \), \( G \)-soundness in EXPSPACE. Thus, in EXPSPACE, we can identify the smallest \( p \) such that \( \mathcal{N} \) is \( p \)-sound.

It remains to compute \( k \). Using Theorem 3.6, we construct a net \( \mathcal{N}' \) which is \( c \)-sound if and only if \( \mathcal{N} \) is \( c \)-sound for all \( c > 0 \). Thus, the smallest number \( c \) for which \( \mathcal{N}' \) is \( c \)-sound is the smallest \( c \) such that \( \mathcal{N} \) is \( c \)-sound.

By Theorem 5.8, if \( \text{Sound}(\mathcal{N}') \neq \emptyset \setminus \{0\} \) then there exists \( G' \leq \|\mathcal{N}\|_{\text{poly}} \) such that \( \mathcal{N}' \) is \( k \)-sound for some \( k \leq G' \). Thus, it suffices to check \( 1 \)-soundness, \( 2 \)-soundness, \( \ldots \), \( G' \)-soundness to identify whether \( k = +\infty \), or to compute the largest \( k \in \mathbb{N} \) such that \( \mathcal{N} \) is \( pk \)-sound. By Theorem 3.7, \( k \) can be computed in exponential space.

8 Conclusion

In this work, we settled, after around two decades, the complexity of the main decision problems concerning workflow nets: \( k \)-soundness, classical soundness, structural soundness, and generalised soundness. The first three are EXPSPACE-complete, while the latter is PSPACE-complete and hence surprisingly simpler. We have further characterised the set of sound numbers of workflow nets: they have a specific shape that can be computed with exponential space.

As further work, we intend to study extensions of these problems in the context of Petri nets. For example, a natural extension of generalised soundness asks, given markings \( m \) and \( m' \), whether for every \( k \in \mathbb{N} \), every marking reachable from \( k \cdot m \) can reach \( k \cdot m' \). Contrary to workflow nets, a Petri net that satisfies this property needs not to be bounded.

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