Heat transfer in a nonequilibrium spin-boson model: A perturbative approach

Wei Wu

Center of Interdisciplinary Studies, Lanzhou University, Lanzhou 730000, People’s Republic of China
Beijing Computational Science Research Center, Beijing 100193, People’s Republic of China

We investigate the heat transport in a nonequilibrium spin-boson model, where a two level system bridging two harmonic reservoirs at different temperatures, by employing a unitary transformation along with a resolvent operator expansion technique. Analytical expressions of the heat current and the thermal conductance of this model are obtained. Compared with the performances of other methods, namely, the nonequilibrium Green’s function method and the equation of motion formulation, our approach provides a reasonable description of heat transfer properties of the nonequilibrium spin-boson model for the weak-coupling region at low temperature.

PACS numbers: 03.65.Yz, 05.60.Gg, 44.10.+i

I. INTRODUCTION

A nonequilibrium quantum dissipative system is generally comprised of a small quantum system immersed in two (or more) dissipative reservoirs of different chemical potentials or temperatures. Due to the system-reservoir interaction, different reservoirs exchange their particles, spins and energy via the central quantum system in the form of a matter [1, 2], a spin [3, 4] and a heat current [5–10]. Usually, the small quantum system is modeled by a few-level system [3, 4, 8–10], a spin-chain [11–13], or an electromagnetic resonator [6, 7]. While, the constructions of reservoirs can be quite different and strongly influence statistical properties of the whole system. Roughly speaking, categorizing by their constituents of environmental modes, there are three kinds of nonequilibrium quantum dissipative systems: bosonic bath model [6–13], spin-bath model [3, 4] and fermionic bath model [5, 14–17]. The nonequilibrium spin-boson (NESB) model is the most intensively used toy model to simulate the phenomenology of heat transfer in molecular junctions [18, 19] and the Kondo problem for magnetic impurities [8].

On the other hand, studies of heat transfer are of great interest in chemical and biophysics disciplines [10, 20]. Thanks to the rapid development of nanotechnologies in experiments, nowadays, researchers can observe the heat flows in certain mesoscopic and microcosmic systems, say, aromatic molecules in Refs. [21, 22] and peptide helices of Ref. [23]. To explain and analyze these experimental results, it is highly desirable to formulate a systematic framework, which should contain a rigorous definition as well as a computable scheme of heat current (and other observables of interest), to describe heat transfer characteristics in these systems.

Several definitions of heat current are employed in previous literatures [5–8, 14, 24–27]. For example, in Refs. [5–8], the heat current operator is defined as the time derivative of reservoir’s Hamiltonian, which is the most simple but straightforward definition. In a few studies [14, 24], the heat current is defined as the energy change of reservoir plus system-reservoir interaction, where the proportion of system-reservoir part can be adjusted in different situations. For a weak-coupled nonequilibrium quantum dissipative system driven by a time-dependent periodic force or pump, its evolution equation can be approximately described by a standard Lindblad-Gorini-Kossakowski-Sudarshan type quantum master equation. Under this circumstance, the heat current is commonly evaluated via the Liouville-dissipation generators in the master equation [25–27], which is different from the first definition from the point of view of calculation. At all events, a good definition should be consistent with quantum thermodynamics, in which some important theorems or central requirements are satisfied [26–28]. It has been showed that the second definition, i.e., heat current is defined by the energy change of reservoir plus any nonzero fraction of the system-reservoir interaction, is not an exact differential when evaluated along reversible isothermal transformations [28]. In this scenario, the second one may not the most suitable definition of heat current.

So far, many analytical and numerical schemes have been developed to investigate the heat transport problem in the NESB model, such as, the nonequilibrium Green’s function (NEGF) [3–6, 8, 28–31], various versions of equation of motion (EOM) [2, 7, 10, 17–19, 32–36] including the Markovian quantum master as well as the reduced evolution equation based on a noninteracting-blip approximation, the multilayer multiconfiguration time-dependent Hartree method [37], the quasi-adiabatic path-integral approach [10, 36, 38] and the hierarchical equations of motion [39–42]. Each method has its own merit and corresponding weakness. Among these formulations, the NEGF and the

*Electronic address: wuw@lzu.edu.cn
EOM have advantages of obtaining analytical, but yet simple expressions of steady-state heat current and other thermodynamical quantities. The multilayer multiconfiguration time-dependent Hartree method and the hierarchical equations of motion approach allow us to simulate not only the steady-state properties, but also the real-time evolution of heat current beyond the weak-coupling approximation. Nevertheless, compared with a purely numerical method, an analytical scheme can make it possible for us to build a more clear physical picture. Moreover, we also believe that any alternative analytical approach can be important for a better understanding of the heat transport properties in a nonequilibrium quantum dissipative system.

In this paper, by combining a generalized polaron transformation [43–47] and a resolvent operator expansion technique [48–51], we present a perturbative approach to investigate the heat transport problem in nonequilibrium quantum dissipative systems. Due to the fact that both polaron transformation and resolvent operator expansion can be applied to bosonic bath systems [43, 44, 47–49, 51], spin-bath systems [45, 46] as well as metallic systems [50], we believe our approach may be a unified scheme for various nonequilibrium quantum dissipative systems. In this paper, we take the NESB model as an example to illustrate our formulation. To verify the validity of our approach, a careful comparison between our approach with other methods, the NEGF method and the EOM formula, is presented. We demonstrate our results agree with some well-accepted conclusions obtained by other methods in the valid region of our approach.

The paper is organized as follows. In Sec. II, we introduce the NESB model and the observables of interest (heat current and thermal conductance) concentrated in our study. In Sec. III, we present our formulation in detail, which is the main part of this paper. To manifest the validity of our approach, we display our results and results obtained by other methods as counterparts in Sec. IV. Main conclusions of this paper are drawn in Sec. V. Throughout the paper, we set $\hbar = k_B = e = 1$ for the sake of convenience, and all the other units are dimensionless as well.

II. MODEL AND OBSERVABLE OF INTEREST

In this section, we first give a brief outline of the NESB model considered in this paper, which is a paradigmatic example for quantum heat transport problem in a noisy environment. To accurately describe the heat transfer characteristics in the NESB model, we need to present rigorous definitions of heat current and thermal conductance. In our paper, we adopt the definition of steady-state heat current employed in literatures [3–6, 8–10, 29–32], which has a thermodynamically valid description of energy exchanges. A different definition of heat current is presented in Refs. [14, 24], unfortunately, it suffers from an inherent problem of inconsistency with quantum thermodynamics [28]. On the other hand, the heat current can be Taylor expand in powers of the temperature difference of two reservoirs, which is beyond the scope of this paper, need a more sophisticated technique.

### A. NESB Model

We consider a two level system (TLS) linearly coupled to two non-interacting harmonic reservoirs at different temperatures. In this paper, we use the notation $\nu$, where $\nu = R$ and $L$, as the subscript to distinguish the two different reservoirs. The whole Hamiltonian of the TLS plus its surrounding reservoirs is described by

$$H = H_s + H_b + H_i,$$

where $H_s = -\frac{1}{2}\Delta \sigma_x$ is the Hamiltonian of the TLS, and can be viewed as a bridge linking the two reservoirs. From a heat engine’s perspective, it plays the same role as the working fluid in classical thermodynamics. The parameter $\Delta$ denotes the tunneling between the two levels of TLS. The Hamiltonian of the environmental part is given by

$$H_b = \sum_{\nu} H_{b\nu} \text{ with } H_{b\nu} = \sum_k \omega_{\nu k} b_{\nu k}^\dagger b_{\nu k},$$

where $b_{\nu k}$ and $b_{\nu k}^\dagger$ are the bosonic annihilation and creation operators of the $(\nu, k)$-th environmental mode with frequency $\omega_{\nu k}$, respectively. We assume the interaction Hamiltonian has the form of $H_i = \sum_{\nu} H_{i\nu}$ with $H_{i\nu} = \frac{1}{2} \sum_k g_{\nu k} (b_{\nu k}^\dagger + b_{\nu k})$, where the parameter $g_{\nu k}$ quantifies the coupling strength between the TLS and the $k$th mode in the $\nu$th reservoir.

Generally, it is very convenient to encode the frequency dependence of the interaction strengths in the bath density spectral function, which is defined by $I_\nu(\omega) \equiv \sum_k g_{\nu k}^2 \delta(\omega - \omega_{\nu k})$ with $\delta(x)$ being the famous Dirac-$\delta$ function. For convenience, we also introduce a modified bath density spectral function $\tilde{I}_\nu(\omega)$ as follows [8]

$$I_\nu(\omega) = 2\alpha_\nu \tilde{I}_\nu(\omega),$$

where $\alpha_\nu$ characterizes the coupling strength between the TLS and the $\nu$th reservoir.
where $\alpha_{\nu}$ is a dimensionless coupling strength between the TLS and the $\nu$th reservoir. In our study, we assume an Ohmic-class modified density spectral function for both reservoirs, i.e.,

$$\tilde{I}_{\nu}(\omega) = \omega^2 e^{-s_{\nu c}^2} \Theta(\omega_{\nu c} - \omega),$$  

where $\omega_{\nu c}$ is the cutoff frequency of the $\nu$th reservoir. Without losing generality, we choose the same cutoff frequency for the two reservoirs, i.e., $\omega_{R c} = \omega_{L c} = \omega_{c}$, which means $\tilde{I}_{R}(\omega) = \tilde{I}_{L}(\omega) = \tilde{I}(\omega)$. The function $\Theta(x)$ is the usual Heaviside step function and parameter $s$ implies the so-called Ohmicity parameter. Depending on the value of the Ohmicity parameter, the bath density spectrum can be classified into three different categories: sub-Ohmic reservoirs $0 < s < 1$, Ohmic reservoir $s = 1$, and super-Ohmic reservoirs $s > 1$. It is necessary to point out that the control of the Ohmicity parameter $s$ is possible when simulating the dephasing process of trapped ultracold atoms, as demonstrated in Ref. [53]. In what follows, we focus on the Ohmic dissipation case, but our results can be also extended to non-Ohmic cases without difficulty.

### B. Heat current and thermal conductance

In the NESB model, a heat flow rises because the Hamiltonian of reservoirs does not commute with the total Hamiltonian $H$, i.e., $[H_{br}, H] \neq 0$. Similar to many previous literatures [3–6, 8–10, 29–32], we utilize the following definition for the stationary heat current from the $\nu$th harmonic bath to the TLS:

$$J_{\nu} \equiv -\frac{\partial}{\partial t} \langle H_{br} \rangle,$$

where the symbol $\langle \ldots \rangle$ implies taking an average at a nonequilibrium steady-state of the whole system, i.e., $\langle \mathcal{O} \rangle \equiv \text{Tr}_{br}(\rho_{br} \mathcal{O})$. The choice of $\rho_{br}$ is a very important question in our study [54]. For an equilibrium problem, $\rho_{br}$ is just the Gibbs thermal equilibrium state, but in the nonequilibrium situation considered in this paper, it is generally unknown and must be computed in some way [54]. The expression of $\rho_{br}$ adopted in this paper will be specified more precisely in Subsec. III B.

By making use of the Keldysh Green’s function approach, an exact expression of the steady-state heat current can be obtained [5, 6, 8, 9, 29, 30]

$$J_{\nu} = \frac{1}{2} \int_{0}^{\infty} d\omega \omega \left[ \chi''(\omega) I_{\nu}(\omega) n_{\nu}(\omega) - i \frac{1}{2} \tilde{G}^{<}(\omega) I_{\nu}(\omega) \right],$$

where $\tilde{f}(\omega)$ is defined as the Fourier transform of an arbitrary time-dependent function $f(t)$, i.e.,

$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt.$$

Throughout this paper, we adopt a check sign upon the function to symbolize the Fourier transform. The function $\chi''(t, t')$ is the imaginary part of the dynamical susceptibility $\chi(t, t')$, namely, $\chi''(t, t') \equiv -\text{Im} \chi(t, t')$ with $\chi(t, t') \equiv -i\Theta(t-t')\langle [\sigma_z(t), \sigma_z(t')] \rangle$. The function $G^{<}(t, t') \equiv -i\langle [\sigma_z(t'), \sigma_z(t)] \rangle$ denotes the lesser Green function and $n_{\nu}(\omega) \equiv [\exp(\beta_{\nu}\omega) - 1]^{-1}$ is the Bose-Einstein distribution of temperature $T_{\nu} = \beta_{\nu}^{-1}$. In the steady-state situation, the conservation law of current holds, i.e., $J = J_{L} = -J_{R}$. Then, the expression of the heat current in Eq. 5 can be further simplified [5, 6, 8, 9, 29, 30]

$$J = \frac{\alpha_{L} \alpha_{R}}{\alpha_{L} + \alpha_{R}} \int_{0}^{\infty} d\omega \omega \chi''(\omega) \tilde{I}(\omega) [n_{L}(\omega) - n_{R}(\omega)].$$

The other important physical quantity to describe the heat transport characteristics is heat conductance $\kappa$. In the linear response region, $J \simeq \kappa(T_{L} - T_{R})$, the energy conductance can be extracted as follows [8, 9, 29, 30]

$$\kappa = \frac{\alpha_{L} \alpha_{R}}{\alpha_{L} + \alpha_{R}} \int_{0}^{\infty} d\omega \omega \chi''_{eq}(\omega) \tilde{I}(\omega) \left[ \frac{\beta\omega/2}{\sinh(\beta\omega/2)} \right]^{-2},$$

where $\chi''_{eq}(\omega) \equiv \chi''(\omega)|_{T_{L}=T_{R}=T}$. From Eq. 6 and Eq. 7, we can see that the imaginary part of the dynamical susceptibility $\chi''(\omega)$ fully determines the heat transfer properties in the NESB model. Thus, the heat transport problem is reduced to finding the dynamical susceptibility. In the next section, we would like to show how to derive a quite general, but yet simple expression of $\chi''(\omega)$ by employing a perturbative method.
III. METHOD

In this section, we present a perturbative approach to study the heat transport problem in a NESB model, which is the main part of this paper. Basically, this perturbative approach is a combination of two common tricks, which are widely used in previous studies of quantum dissipative dynamics, a (generalized) polaron transformation [43–47] and a resolvent operator expansion technique [48–51]. Our approach allows an analytical expression of observable of interest, and we believe it can offer a more clear physical picture, compared with other purely numerical methods. At the end of the section, a brief discussion about the validity region of our approach is drawn.

A. Unitary transformation

We first present a unitary transformation, which can be viewed as a generalized polaron transformation and has been widely used in many previous studies. Similar to Refs. [43–45, 47], a unitary transformation is applied to the original Hamiltonian as $\hat{H} = e^SHe^{-S}$, where the generator $S$ is defined by

$$S \equiv \sigma_z \sum_{\nu,k} \frac{g_{\nu k}}{2\omega_{\nu k}} \xi_{\nu k}(b_{\nu k}^\dagger - b_{\nu k}).$$

(8)

Here $\xi_{\nu k}$ is a undetermined parameter, by choosing a suitable $\xi_{\nu k}$, the original Hamiltonian $H$ can be transformed to a better representation in which the exact solvable part contains the most important physical message of the whole system. If one choose $\xi_{\nu k} = 1$, then $S$ reduces to the generator of the well-known polaron transformation [55]. In this paper, we choose [43–45, 47]

$$\xi_{\nu k} = \frac{\omega_{\nu k}}{\omega_{\nu k} + \eta \Delta}.$$  

(9)

with $\eta$ being a renormalization factor

$$\eta = \exp \left[ - \sum_{\nu,k} \frac{g_{\nu k}^2}{2\omega_{\nu k}} \xi_{\nu k}^2 \coth \left( \frac{\beta \nu \omega_{\nu k}}{2} \right) \right].$$

(10)

By doing so, the transformed Hamiltonian $\hat{H}$ can be exactly expressed as $\hat{H} = \hat{H}_s + \hat{H}_b + \hat{H}_1 + \delta \hat{H}_i$. The first part $\hat{H}_s \equiv -\frac{1}{2}\eta \Delta \sigma_x$, where we have dropped some inconsequential constant terms. And expressions of $\hat{H}_i$ and $\delta \hat{H}_i$ are given by

$$\hat{H}_i = \sum_{\nu,k} \tilde{g}_{\nu k}(b_{\nu k}^\dagger \sigma_- + b_{\nu k} \sigma_+),$$

(11)

$$\delta \hat{H}_i = -\frac{\Delta}{2} \sigma_z \left\{ \cosh \left[ \sum_{\nu,k} \frac{g_{\nu k}}{\omega_{\nu k}} \xi_{\nu k}(b_{\nu k}^\dagger - b_{\nu k}) \right] - \eta \right\} - \frac{i}{2} \sigma_y \left\{ \sinh \left[ \sum_{\nu,k} \frac{g_{\nu k}}{\omega_{\nu k}} \xi_{\nu k}(b_{\nu k}^\dagger - b_{\nu k}) \right] - \eta \sum_{\nu,k} \frac{g_{\nu k}}{\omega_{\nu k}} \xi_{\nu k}(b_{\nu k}^\dagger - b_{\nu k}) \right\},$$

(12)

where $\tilde{g}_{\nu k} \equiv g_{\nu k} \eta \Delta/(\omega_{\nu k} + \eta \Delta)$ is the renormalized coupling strength and $\sigma_{\pm} \equiv \frac{1}{2}(\sigma_x \pm i\sigma_y)$. Because of the construction of $\eta$ in Eq. 10, one can easily check that $\langle \delta \hat{H}_i \rangle_b = 0$, where the symbol $\langle ... \rangle_b$ implies the thermodynamic average value with respect to the thermal equilibrium states of two reservoirs. In this sense, we can safely drop $\delta \hat{H}_i$, which is acceptable in a perturbative treatment [43–45, 47]. In fact, terms in $\hat{H}_2$ are related to multiboson nondiagonal transitions, say $b_{\nu k} b_{\nu k'}$, and $b_{\nu k}^\dagger b_{\nu k'}^\dagger$. The contributions of these nondiagonal terms to physical quantities are of order $O(g_{\nu k}^4)$, which is negligible in a perturbative scheme. It is necessary to point out that the drop of $\delta \hat{H}_i$ does not mean our calculation is valid only up to the second order in $g_{\nu k}$. As discussion in Refs. [43, 44, 47, 56], the strong-coupling effect is included in the construction of the renormalization factor $\eta$, which characterizes a coherent tunneling induced by the multiboson nondiagonal transitions. Therefore, the unitary transformation proposed by Zheng et al. [43–45, 47, 56] allows a investigation of a quantum dissipative system in a wide range of coupling strengths, but still keeps a simple mathematical structure. Therefore, in what follows, the transformed Hamiltonian is approximated as $\hat{H} \simeq \hat{H}_s + \hat{H}_b + \hat{H}_1$. 


Due to the fact that \( \sigma_z \) commutates with the generator \( S \), one can easily find that

\[
\chi(t) = -i[\sigma_z(t)|\sigma_z(0)\rangle - \langle \sigma_z(0)|\sigma_z(t)\rangle] = -i[\langle \sigma_z(t)|\sigma_z(0)\rangle - \langle \sigma_z(0)|\sigma_z(t)\rangle],
\]

where \( \sigma_z(t) \equiv e^{i\hat{H}t}\sigma_z e^{-i\hat{H}t} \). For the sake of convenience, we introduce the Liouvillian operator \( \hat{L} \), which is defined by \( \hat{L} \rho = [\hat{L}, \rho] \) for an arbitrary operator \( \rho \). Then the notation \( \sigma_z(t) \) can be simplified further by making use of the Liouvillian operator \( \hat{L} \), i.e., \( \sigma_z(t) = e^{i\hat{L}t}\sigma_z(0) \). And the expression of \( \chi(t) \) can be rewritten as follows

\[
\chi(t) = -i \left\{ \text{Tr}_{sb} \left[ \sigma_z \hat{\rho}_{sb} e^{i\hat{L}t}\sigma_z \right] - \text{Tr}_{sb} \left[ \hat{\rho}_{sb} e^{i\hat{L}t}\sigma_z \right] \right\} = -i \text{Tr}_{sb} \left[ (\sigma_z \hat{\rho}_{sb} - \hat{\rho}_{sb} \sigma_z) e^{i\hat{L}t}\sigma_z \right],
\]

where \( \hat{\rho}_{sb} \) denotes the density matrix of the transformed Hamiltonian \( \hat{H} \) in a steady-state.

To obtain the expression of \( \chi(t) \), we first have to specify \( \hat{\rho}_{sb} \), or equivalently, \( \rho_{sb} \). Following the adiabatic switch-on idea proposed in Ref. [54], we imagine that at \( t = -\infty \) the whole system has three decoupled parts, a TLS and two independent reservoirs, the interaction between the TLS and the two reservoirs is switch-off. The initial state at \( t = -\infty \) is assumed to be a factorising state \( \hat{\rho}_{sb}(-\infty) = \hat{\rho}_s(-\infty) \otimes \rho_{b} \), where \( \rho_s \equiv \bigotimes_{\nu} \rho_{\nu} \) with \( \rho_{\nu} \equiv e^{-\beta_{\nu} H_{\nu}}/\text{Tr}_{\nu} (e^{-\beta_{\nu} H_{\nu}}) \) being the thermal equilibrium state of the \( \nu \)th reservoir. The choice of \( \rho_s \) is the same with the widely used Keldysh formulation [5, 6, 8, 29, 30], while \( \hat{\rho}_{sb}(-\infty) \) can be an arbitrary state. At time \( t = 0 \), we switch on the system-reservoir interactions. As the time evolves, a nonequilibrium steady-state of the whole system is established correspondingly. In the long-time limit, we hypothesize the final steady-state \( \hat{\rho}_{sb} = \hat{\rho}_{sb}(+\infty) \) still has a factorising form of

\[
\hat{\rho}_{sb} \simeq \hat{\rho}_s \otimes \rho_{b},
\]

where \( \hat{\rho}_s = \hat{\rho}_s(+\infty) \) is the nonequilibrium steady-state of the TLS. In this paper, we assume \( \hat{\rho}_s \) has a canonical distribution

\[
\hat{\rho}_s = \frac{e^{-\beta_{\nu} H_{\nu}}}{\text{Tr}_s (e^{-\beta_{\nu} H_{\nu}})} = \frac{1}{2} (1 - \bar{\sigma}_z \sigma_z),
\]

with \( \bar{\sigma}_z = \tanh (\frac{1}{2} \beta_{\nu} \Delta) \) being the thermal expectation value of \( \sigma_z \). The parameter \( \beta_{\nu} \) in Eq. 16 means the final steady-state temperature of the TLS. Physically speaking, the steady-state temperature depends on a number of factors, say, \( \beta_{\nu} \), the coupling strengths and coupling types between the TLS and its surrounding reservoirs. It is very difficult to derive a general expression of \( \beta_{\nu} \) for an arbitrary nonequilibrium quantum dissipative system. In our treatment, consider the fact that the modified density spectral functions are the same for both reservoirs, we assume \( \beta \) can be partitioned as

\[
\beta = \frac{\alpha_L}{\alpha_L + \alpha_R} \beta_L + \frac{\alpha_R}{\alpha_L + \alpha_R} \beta_R,
\]

The above equation means \( \beta \) is the algebraic average of \( \beta_L \) and \( \beta_R \), and the partition coefficients reflect the contribution of the \( \nu \)th reservoir to the steady-state temperature \( \beta \). It can be physically understood as follows: a larger value of the coupling strength \( \alpha_{\nu} \) means a stronger influence of \( \nu \)th reservoir on the TLS, which leads to the final steady-state temperature is closer to \( \beta_{\nu} \). Moreover, in the weak-coupling limit, one can see \( \eta \to 1 \), and \( \bar{\sigma}_z \) reduces to the correct result at thermal equilibrium \( \beta_L = \beta_R = \beta \). In this case, \( \bar{\sigma}_z \) only depends on \( \beta \) and the tunneling parameter \( \Delta \), it does not carry information about the system-reservoir coupling. In contrast, for the nonequilibrium situation \( \beta_L \neq \beta_R \), except \( \Delta, \bar{\sigma}_z \) is now determined by \( \beta \), which contains message of the spectral functions of two reservoirs. This result can be regraded as a signature of the TLS away from equilibrium and is also reported in Ref.[10]. Anyway, Eq. 16 and Eq. 17 can be viewed as an ansatz, and its validity should be checked by a direct comparison with other approaches, which will be presented in Sec. IV.

We would like to give some concerned discussions about Eq. 15 and Eq. 16: (i) One may notice that the assumption adopted in Eq. 15 can be regarded as the common Born approximation [57]. The Born approximation has been widely
used in several previous studies of the single-reservoir spin-boson model [43–45, 47, 48]. As discussed in Ref. [48], when the system-reservoir coupling is weak, the result obtained by Born approximation is reliable in the entire range of temperatures. In this sense, Eq. 15 is also feasible in the perturbative framework adopted in this paper. The Born approximation erase the contributions from system-reservoir interactions, but naturally, $\tilde{\rho}_{s}$ is not a temperature in the standard thermodynamic sense, but it can recover the authentic temperature in the weak-coupling limit for a single-reservoir open system [59]. Very interestingly, within the EOM approach, Segal et al. [17, 63] found that $\tilde{\rho}_{s}$ is physical reasonable in our treatment. (iii) In the Keldysh formalism, the explicit expression of $\tilde{\beta}$ is physical reasonable in our treatment. (iii) In the Keldysh formalism, the explicit expression of $\tilde{\rho}_{s}$ is physical reasonable in our treatment. (iii) In the Keldysh formalism, the explicit expression of $\tilde{\rho}_{s}$ is physical reasonable in our treatment. However, how to construct $\tilde{\sigma}_{s}$ in this case is still an open question. (ii) Generally speaking, the nonequilibrium steady-state of the TLS is not a canonical distribution, when system-reservoir coupling is strong or the decoherence mechanism is strongly non-Markovian [59–62]. However, in our perturbative treatment, the higher-order corrections of system-reservoir couplings are dropped, in this case, the canonical distribution assumption is acceptable. Furthermore, several studies showed that [17, 59, 63], the steady-state of a small quantum system, whose Hamiltonian is $\mathcal{H}_{s}$, interacting with dissipative reservoirs can be formally written as a generalized canonical distribution with an effective temperature $\beta_{\text{eff}}$, i.e., $\rho_{s}(+\infty) \propto e^{-\beta_{\text{eff}}\mathcal{H}_{s}}$. The effective temperature $\beta_{\text{eff}}$ is not a temperature in the standard thermodynamic sense, but it can recover the authentic temperature in the weak-coupling limit for a single-reservoir open system [59]. Very interestingly, within the EOM approach, Segal et al. [17, 63] found that $\beta_{\text{eff}}$ reduces to the expression of Eq. 17 in the classic limit, which suggests the expression of $\tilde{\beta}$ is physical reasonable in our treatment. (iii) In the Keldysh formalism, the explicit expression of $\tilde{\rho}_{s}$ is 2 important, because this condition of the TLS is washed out in the Keldysh equation [29–31]. However, for any EOM approach, one need to specify $\tilde{\rho}_{s}$ exactly. In our study, we assume $\tilde{\rho}_{s}$ is a nonequilibrium steady-state of the TLS. A similar choice is also used in serval previous studies [10, 17–19, 63], in which the stationary heat current is explored by making use of various EOM methods.

With the help of Eq. 15, the expression of $\chi(t)$ reduces to

$$\chi(t) = -i \text{Tr}_{b} \left[ (\sigma_{z} \tilde{\rho}_{s} - \tilde{\rho}_{s} \sigma_{z}) \text{Tr}_{b} \left( \tilde{\rho}_{s} e^{iLt} \sigma_{z} \right) \right]$$

$$= -i \tilde{\sigma}_{z} \text{Tr}_{b} \left[ i \sigma_{y} Z(t) \right]$$

$$= -i \tilde{\sigma}_{z} [Z_{+}(t) - Z_{-}(t)],$$

where $Z_{\mu\mu'}(t) \equiv \langle \mu | (e^{iLt})_{b} \sigma_{z} | \mu' \rangle$, and $| \mu = \pm \rangle$ are the eigenstates of $\sigma_{x}$, i.e., $\sigma_{x}|\pm\rangle = \pm|\pm\rangle$. The remaining task is quite clear: we turn to evaluate the bath-averaged super-operator $Z(t)$. In the next subsection, we would like to show how to derive an analytical expression of $Z(t)$ by making use of a resolvent operator expansion technique.

C. Resolvent operator expansion technique

In this subsection, we use the strategy proposed by Dattagupta et al. [48, 49] to derive the expression of $Z(t)$. First, we transform it into its Laplace space or its resolvent $\hat{Z}(\lambda)$. Throughout the paper, we refer $\hat{f}(\lambda)$ as the Laplace transformation of an arbitrary time-dependent function $f(t)$, i.e.,

$$\hat{f}(\lambda) \equiv \int_{0}^{\infty} dt f(t) e^{-\lambda t},$$
where we have used a hat sign upon the function to denote the Laplace transformation. In the Laplace space, one can expand \( \hat{Z}(\lambda) \) in powers of \( \hat{L}_i \) as follows

\[
\hat{Z}(\lambda) = \left\langle \frac{1}{\lambda - iL_0} \right\rangle \sigma_z \\
= \left\langle \frac{1}{\lambda - iL_0} \right\rangle \sigma_z \\
= \left\langle \frac{1}{\lambda - iL_0} + i \frac{1}{\lambda - iL_0} \tilde{L}_i \frac{1}{\lambda - iL_0} - \frac{1}{\lambda - iL_0} \tilde{L}_i \frac{1}{\lambda - iL_0} + \ldots \right\rangle \sigma_z \\
= \left\langle \frac{1}{\lambda + iL_s} + i \frac{1}{\lambda - iL_s} \left\langle \tilde{L}_i \right\rangle \sigma_z \\
\right\rangle \sigma_z + \ldots \right\rangle \sigma_z \\
\approx \frac{1}{\lambda - iL_s + \Sigma(\lambda)} \sigma_z,
\]

where \( \tilde{L}_0 \equiv \tilde{L}_s + L_h \), and the super-operator \( \Sigma(t) \) is defined by

\[
\Sigma(t) \equiv \left\langle \tilde{L}_i \right\rangle_{\psi_0(t)}.
\]

In the derivation of Eq. 19, we have used the formula \( X^{-1} = Y^{-1} + Y^{-1}(Y - X)X^{-1} \) and truncated the series at the order \( L_i^2 \). This technique was originally proposed in Refs. [48, 49] and is called resolvent operator expansion method. Here, we want to stress that Eq. 19 still contains contributions from the higher order terms \( \hat{g}_{ik} \), owing to the construction of the renormalized coupling constants \( \hat{g}_{ik} \), see the discussions in Subsec. III A.

Transforming Eq. 19 back to the time space, one can find \( Z(t) \) obeys the following integro-differential equation

\[
\frac{d}{dt} Z(t) = i \tilde{L}_t Z(t) - \int_0^t d\tau \Sigma(t - \tau) Z(\tau).
\]

The above equation is the same the celebrated Nakajima-Zwanzig equation [64, 65] of a quantum dissipative system, simply replacing \( \hat{Z}(\tau) \) by a reduced density operator the quantum subsystem. Together with initial conditions \( Z_{-+}(0) = Z_{-+}(0) = 1 \), and after a intricate but straightforward calculation, we obtain

\[
\hat{Z}_{-+}(\lambda) = \left[ \lambda + i\eta \Delta + \sum_{\nu, k} \frac{\hat{g}_{ik}^2(2n_{\nu k} + 1)}{\lambda + i\omega_{\nu k}} \right]^{-1},
\]

\[
\hat{Z}_{-+}(\lambda) = \left[ \lambda - i\eta \Delta + \sum_{\nu, k} \frac{\hat{g}_{ik}^2(2n_{\nu k} + 1)}{\lambda - i\omega_{\nu k}} \right]^{-1}.
\]

Substituting the above expressions into Eq. 18, one can obtain an analytical expression of \( \hat{\chi}(\lambda) \). And due to the fact that \( \hat{\chi}(\omega) \) is related to \( \hat{\chi}(\lambda) \) by a simple analytic continuation [66], the spin susceptibility \( \hat{\chi}(\omega) \) can be evaluated explicitly as follows

\[
\hat{\chi}(\omega) = -i \hat{\sigma}_z \left[ \hat{Z}_{-+}(\lambda) - \hat{Z}_{-+}(\lambda) \right] |_{\lambda = -i(\omega + i\theta^+)} \\
= \hat{\sigma}_z \left\{ \left[ \frac{1}{\omega - \eta \Delta \sum_{\nu, k} \frac{\hat{g}_{ik}^2(2n_{\nu k} + 1)}{\omega - \omega_{\nu k} + i0^+}} \right]^{-1} - \left[ \frac{1}{\omega + \eta \Delta \sum_{\nu, k} \frac{\hat{g}_{ik}^2(2n_{\nu k} + 1)}{\omega + \omega_{\nu k} + i0^+}} \right]^{-1} \right\}.
\]

For the single-reservoir spin-boson model case, our result recovers that of Ref. [43] at \( T_\nu = 0 \). In Appendix VII, we demonstrate Eq. 23 can be retrieved by making use of an alternative technique: The chain equations of Green’s function approach.

The imaginary part of \( \hat{\chi}(\omega) \) can be obtained by making use of the Sokhotski-Plemelj theorem

\[
\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = \frac{1}{x} \mp i\delta(x).
\]
where P stands for the Cauchy principal value. Finally, one can find
\[
\chi''(\omega) = -\text{Im}[\chi(\omega)] = \delta_x \left\{ \frac{\Gamma(\omega)\Theta(\omega)}{[\omega - \eta \Delta - \Lambda(\omega)]^2 + \Gamma^2(\omega)} - \frac{\Gamma(-\omega)\Theta(-\omega)}{[\omega + \eta \Delta + \Lambda(-\omega)]^2 + \Gamma^2(-\omega)} \right\},
\]
(24)
where
\[
\Lambda(\omega) = P \sum_{\nu, k} \frac{\tilde{g}^2_{\nu k}(2n_{\nu k} + 1)}{\omega - \omega_{\nu k}}
= (\eta \Delta)^2 \sum_{\nu} \int_0^\infty d\varpi \frac{I_{\nu}(\varpi)}{(\omega - \varpi)(\eta \Delta + \varpi)^2} \coth \left( \frac{\beta_{\nu} \varpi}{2} \right).
\]
(25)
\[
\Gamma(\omega) = \pi \sum_{\nu, k} \frac{\tilde{g}^2_{\nu k}(2n_{\nu k} + 1)}{\omega - \omega_{\nu k}}
= \pi (\eta \Delta)^2 \sum_{\nu} \frac{I_{\nu}(\omega)}{(\eta \Delta + \omega)^2} \coth \left( \frac{\beta_{\nu} \omega}{2} \right).
\]
(26)
In a practical numerical calculation, the integral in Eq. 25 dramatically oscillates for the parameter regions considered in our study. Fortunately, in the low temperature limit, one can Taylor expand the trigonometric function in powers of $\beta_{\nu}^{-1}$ to obtain the stable dominant term. For the lowest-order approximation, $\coth(\frac{\beta_{\nu} \varpi}{2}) \simeq 1$, the integral in Eq. 25 can be exactly worked out [43]
\[
\Lambda(\omega) \simeq -2(\eta \Delta)^2 \sum_{\nu} \frac{\alpha_{\nu}}{\eta \Delta + \omega} \left\{ \frac{\omega_c}{\eta \Delta + \omega_c} - \frac{\omega}{\eta \Delta + \omega} \ln \left[ \frac{\omega_c (\omega_c + \eta \Delta)}{\eta \Delta (\omega_c - \omega)} \right] \right\}.
\]

In order to verify the validity of our approach, we make a comparison between our result of Eq. 24 with the result obtained by the NEGF method in Refs. [29, 30]. As shown in Fig. 1, our result is in agreement with that of the NEGF method. We notice that, in the weak-coupling region, the imaginary part of dynamical susceptibility $\chi''(\omega)$ exhibits two distinct peaks near $\omega = \pm \Delta$, corresponding to tunneling processes between the two levels of the TLS. This result is physically reasonable and convinces us that our approach truly captures the behavior of $\chi''(\omega)$.

Before moving on to the next section, we would like to make some remarks on the application sphere of our method. Basically, two approximations are employed in our method: the first one is the second order approximation to the system-reservoir coupling in obtaining the transformed Hamiltonian $\tilde{H}$. The neglected higher order terms are responsible for multiboson nondiagonal transitions. As discussed in Refs. [43, 44, 56], these multiboson excitation processes often happen in strong-coupling regions or at high temperature, thus, the first approximation is physical reliable for the low temperature region in the weak-coupling case. The second approximation is the extended Born approximation used in Eq. 15, which is acceptable in the weak-coupling region for the entire range of temperatures. In brief, our perturbative method can give convincing results in weak-coupling region at low temperature. In the next section, we would like to discuss results from our approach in the above valid region.

IV. RESULT

In order to compare with the result obtained by the NEGF approach [29, 30], below, we principally discuss our results in the so-called nonadiabatic region: $\Delta/\omega_c \ll 1$ and $\beta_{\nu} \omega_c \gg 1$. As demonstrated in Ref. [31], in the nonadiabatic limit, the result of NEGF method coincides with the result obtained by the noninteracting-blip approximation [67] for an Ohmic dissipative reservoir. In this sense, the NEGF method serves a good description of the heat transport in a NESM model. If our result agrees with that of the NEGF method, it would convince us that our approach truly captures the heat transfer property.

We begin with the symmetrical coupling case, i.e., $\alpha_L = \alpha_R = \alpha$. In Fig. 2(a), we display the results of our approach as well as predictions of the NEGF method in Refs. [29, 30]. No distinct difference is found between these two different methods in weak-coupling regions. While, with the increase of the coupling strength $\alpha$, the deviation of the two methods becomes more and more apparent. We also observe that, regardless of our result or that of the NEGF method, the heat current $J$ grows linearly with the parameter $\alpha$ in the weak-coupling limit, i.e., $J \propto \alpha$ when $\alpha \to 0$. As the coupling parameter $\alpha$ becomes larger, the heat current $J$ deviates from the above linearity, but it
The heat conductance for an Ohmic dissipation case is given by \[ \kappa = \frac{\alpha L G R}{\alpha L + \alpha R} \pi \Delta + 1 \left[ \frac{\beta \Delta / 2}{\sinh(\beta \Delta / 2)} \right]^{-2}. \] (27)

The above expression has been checked by a numerically exact Monte Carlo simulation [8], and EOM based on the noninteracting blip approximation [52], which implies it can accurately depict the heat conductance in the weak-coupling limit.

FIG. 2: (a) The heat current \( J \) is plotted as a function of the coupling strength parameter \( \alpha_L = \alpha_R = \alpha \). Red circles are our results. Blue squares denote the results obtained by the NEGF method reported in Ref. [29]. The purple dashed lines are linear fittings of the form \( J = a\alpha + b \). (b) The same with (a), but \( \alpha_R = \alpha, \alpha_L = 1.5\alpha_R \). Other parameters are chosen as follows: \( \Delta = 0.1, \beta_L = 20, \beta_R = 22, \omega_c = 10\Delta \).

FIG. 3: (a) Thermal conductance \( \kappa \) is displayed as a function of the coupling parameter \( \alpha_L = \alpha_R = \alpha \) with \( \beta_L = \beta_R = 50 \). (b) The same with (a), but \( \beta_L = \beta_R = 80 \). (c) Thermal conductance \( \kappa \) is displayed as a function of the coupling parameter \( \alpha_L = 1.5\alpha_R \) and \( \alpha_R = \alpha \) with \( \beta_L = \beta_R = 50 \). (d) The same with (c), but \( \beta_L = \beta_R = 80 \). Red circles are the results obtained by our approach. Blue squares denote the results obtained by the NEGF method reported in Ref. [29]. Purple dashed lines are the results from Eq. 27. Other parameters are chosen as follows: \( \Delta = 0.1 \) and \( \omega_c = 10\Delta \).

Still grows monotonically with \( \alpha \) in the entire range \( 0 \leq \alpha \leq 0.1 \). This conclusion coincides with previous results reported in Refs. [10, 20, 38, 52]. A similar result is also found in the asymmetrical coupling case, namely \( \alpha_L \neq \alpha_R \), see Fig. 2(b).

Next, we discuss the behavior of the heat conductance \( \kappa \) using our approach. For comparison, we also present the formula derived via the EOM approach by Segal et al. [10, 18–20, 52]. In the weak coupling limit \( \alpha_\nu \to 0 \), it is found that the heat conductance for an Ohmic dissipation case is given by [8, 52]

\[
\lim_{\alpha_\nu \to 0} \kappa = \frac{\alpha_L G R}{\alpha_L + \alpha_R} \pi \Delta + 1 \left[ \frac{\beta \Delta / 2}{\sinh(\beta \Delta / 2)} \right]^{-2}.
\] (27)

The above expression has been checked by a numerically exact Monte Carlo simulation [8], and EOM based on the noninteracting blip approximation [52], which implies it can accurately depict the heat conductance in the weak-coupling limit.

FIG. 4: (a) Thermal conductance \( \kappa \) is depicted as a function of the temperature parameter \( \beta_L = \beta_R = \beta \) with \( \alpha_L = \alpha_R = 0.01 \). (b) The same with (a), but \( \alpha_L = \alpha_R = 0.03 \). (c) Thermal conductance \( \kappa \) is depicted as a function of the temperature parameter \( \beta_L = \beta_R = \beta \) with \( \alpha_L = 1.5\alpha_R \) and \( \alpha_R = 0.01 \). (d) The same with (c), but \( \alpha_R = 0.03 \). Red circles are the results obtained by our approach. Blue squares denote the results obtained by the NEGF method reported in Ref. [29]. Purple dashed lines denote numerical fittings of the form \( \kappa = a\beta^{-1.3} + b \). Other parameters are chosen as follows: \( \Delta = 0.1 \) and \( \omega_c = 10\Delta \).
In Fig. 3, we display our results, NEGF results and the predictions from Eq. 27. It is clear to see they are in good agreement when the coupling parameter $\alpha$ approaches to zero. We notice that, in the weak coupling limit $\alpha \to 0$, $\kappa \propto \alpha$ whether the symmetrical coupling case or the asymmetrical coupling case. The heat conductance deviates this linear response relation as the coupling parameter $\alpha$ becomes larger. The same phenomenon is reported in Refs. [8, 52] as well. Moreover, we explore the relation between the heat conductance $\kappa$ and the temperature $\beta_L = \beta_R = \beta$. The thermal conductance $\kappa$ as a function of the temperature $\beta$ in low temperature regions is plotted in Fig. 4. We find that $\kappa \propto \beta^{-3}$ when $\beta \to \infty$, regardless of the symmetrical case or the asymmetrical coupling case. This $T^3$-law is also observed in Ref. [8] by means of a Monte Carlo simulation. As high temperature increases, this $T^3$-law breaks down, which agrees with the observations in Refs. [8, 52].

V. CONCLUSION

In summary, we investigate the heat transport problem in a NESB model by employing a perturbative approach. This approach is a combination of a unitary transformation and a resolvent operator expansion technique, which are well-established frameworks and originally developed for describing the dynamics of a quantum dissipative system. As discussed in Subsec. III C, our approach works in weak-coupling region at low temperature, and may be applied to other nonequilibrium quantum dissipative systems as well. Analytical expressions of the heat current and the thermal conductance of this NESB model are obtained. Using these results, we reexamine the heat exchange properties in the range of validity of our approach. It is found that our results agree with the predictions of the NEGF and the EOM methods. We believe our formulation provides an alternative way to describe the heat transfer properties in the NESB model with an analytical, but yet simple expression. Finally, due to the generality of nonequilibrium quantum dissipative systems, we expect our result to be of interest for a wide range of applications in quantum chemistry and biophysics.

VI. ACKNOWLEDGMENTS

W. Wu wishes to thank Wen-Li Zhu, Jun-Hong An, Hong-Guang Luo and Hai-Qing Lin for numerous discussions. This project was supported by the China Postdoctoral Science Foundation (Grant Nos.2017M610753 and 2018T110038), the NSFC (Grant No.11704025) and the NSAF (Grant No. U1530401).

VII. APPENDIX

In this appendix, we shall use an alternative method to reproduce Eq. 23. Let us define the retarded Green’s functions as follows

$$G_{AB}^\text{r}(t,t') \equiv -i\Theta(t-t')\langle\langle A(t), B(t')\rangle\rangle.$$  

For the sake of convenience, we use the notation $\langle\langle A, B\rangle\rangle_\omega$ to denote its Fourier transformation, i.e., $\langle\langle A, B\rangle\rangle_\omega \equiv \tilde{G}_{AB}^\text{r}(\omega)$. The chain equations of the retarded Green’s functions is given by [68]

$$\omega\langle\langle A, B\rangle\rangle_\omega = \langle\langle A, B\rangle\rangle + \langle\langle [A, H], B\rangle\rangle_\omega.$$  

Generally, there are infinite coupled equations in Eq. 29, which means one need to make a truncation to obtain a closed set of equations [68]. Following the cutoff scheme in Refs. [44, 45], we truncate the the chain equations at the order of $\tilde{g}_k^2$ and make use of the approximation $(b_{\nu k}^\dagger + b_{\nu k})(b_{\nu k}^\dagger + b_{\nu k}) \simeq 2n_{\nu k} + 1$, then Eq. 29 will be self-closed, and we find

$$\omega\langle\langle \sigma_x, \sigma_z \rangle\rangle_\omega = -\eta\Delta\langle\langle i\sigma_y, \sigma_z \rangle\rangle_\omega - \sum_{\nu,k} \tilde{g}_{\nu k}\langle\langle \sigma_x(b_{\nu k}^\dagger - b_{\nu k}), \sigma_z \rangle\rangle_\omega,$$  

$$\omega\langle\langle i\sigma_y, \sigma_z \rangle\rangle_\omega = -2\langle\langle \sigma_x \rangle\rangle - \eta\Delta\langle\langle \sigma_z, \sigma_z \rangle\rangle_\omega - \sum_{\nu,k} \tilde{g}_{\nu k}\langle\langle \sigma_x(b_{\nu k}^\dagger + b_{\nu k}), \sigma_z \rangle\rangle_\omega,$$  

$$\omega\langle\langle \sigma_z(b_{\nu k}^\dagger - b_{\nu k}), \sigma_z \rangle\rangle_\omega = -\omega_{\nu k}\langle\langle \sigma_x(b_{\nu k}^\dagger + b_{\nu k}), \sigma_z \rangle\rangle_\omega - \tilde{g}_{\nu k}(2n_{\nu k} + 1)\langle\langle \sigma_z, \sigma_z \rangle\rangle_\omega.$$
\[ \omega \langle \sigma_z (b^\dagger_{\nu k} + b_{\nu k}), \sigma_z \rangle = -\omega_{\nu k} \langle \sigma_z (b^\dagger_{\nu k} - b_{\nu k}), \sigma_z \rangle - \tilde{g}_{\nu k} (2n_{\nu k} + 1) \langle \langle i\sigma_y, \sigma_z \rangle \rangle. \]  

(33)

The solution for \( \langle \sigma_z, \sigma_z \rangle \) is

\[ \langle \sigma_z, \sigma_z \rangle = \langle \sigma_z \rangle \left\{ \left[ \omega - \eta \Delta - \sum_{\nu,k} \frac{\tilde{g}_{\nu k}^2 (2n_{\nu k} + 1)}{\omega - \omega_{\nu k}} \right]^{-1} - \left[ \omega + \eta \Delta - \sum_{\nu,k} \frac{\tilde{g}_{\nu k}^2 (2n_{\nu k} + 1)}{\omega + \omega_{\nu k}} \right]^{-1} \right\}, \]  

(34)

In our paper, \( \tilde{\rho}_{ab} \simeq \tilde{\rho}_s \otimes \rho_\sigma \) which results in \( \sigma_z \simeq \hat{\sigma}_z \). Then, one can obtain the dynamical susceptibility \( \chi(\omega) = \langle \sigma_z, \sigma_z \rangle + \omega \), which recovers the Eq. 23 in the main text. Although this approach is quite distinct from the resolvent operator expansion in methodology, such an agreement is not a surprise, because the level of approximations invoked in these two methods are the same, namely, both theories have utilized the second order truncation in perturbation expansions and the extended Born approximation.

[1] S. R. de Groot, and P. Mazur, *Non-equilibrium Thermodynamics* (Dover Press, New York, 1984).
[2] M. Esposito, U. Harbola and S. Mukamel, Rev. Mod. Phys. 81, 1665 (2009).
[3] M. Matsuo, Y. Ohnuma, T. Kato, and S. Maekawa, Phys. Rev. Lett. 120, 037201 (2018).
[4] Y. Ohnuma, M. Matsuo, and S. Maekawa, Phys. Rev. B 96, 134412 (2017).
[5] Y. Meir and N. S. Wingreen, Phys. Rev. Lett. 68, 2512 (1992).
[6] T. Ojanen, and A.-P. Jauho, Phys. Rev. Lett. 100, 155902 (2008).
[7] E. A. Martinez and J. P. Paz, Phys. Rev. Lett. 110, 130406 (2013).
[8] K. Saito and T. Kato, Phys. Rev. Lett. 111, 214301 (2013).
[9] J. Taylor and D. Segal, Phys. Rev. Lett. 114, 220401 (2015).
[10] N. Boudjada, and D. Segal, J. Phys. Chem. A 118, 11323 (2014).
[11] H. Wichterich, M. J. Henrich, H.-P. Breuer, J. Gemmer, and M. Michel, Phys. Rev. E 76, 031115 (2007).
[12] D. Michel, and O. Hess, Phys. Rev. B 77, 104303 (2008).
[13] L.-A. Wu, and D. Segal, Phys. Rev. A 84, 012319 (2011).
[14] M. F. Ludovico, J. S. Lim, M. Moskalets, L. Arrachea, and D. Sanchez, Phys. Rev. B 89, 161306(R) (2014).
[15] M. Galperin, A. Nitzan, and M. A. Ratner, Phys. Rev. B 75, 155312 (2007).
[16] D. Segal, and D. R. Reichman, Phys. Rev. B 76, 195316 (2007).
[17] L.-A. Wu, C. X. Yu, and D. Segal, Phys. Rev. E 80, 041103 (2009).
[18] D. Segal, A. Nitzan, J. Chem. Phys. 122, 194704 (2015).
[19] D. Segal, A. Nitzan, Phys. Rev. Lett. 94, 034301 (2005).
[20] D. Segal, B. K. Agarwalla, Annu. Rev. Phys. Chem. 65, 365 (2014).
[21] V. M. Kasyanenko, S. L. Tesar, G. I. Rubtsov, A. L. Burin, and I. V. Rubtsov, J. Phys. Chem. B 115, 11063 (2011).
[22] B. C. Pein and D. D. Drett, J. Phys. Chem. A 118, 965 (2014).
[23] V. Botan, E. H. G. Backus, R. Pfister, A. Moretto, M. Crisma, C. Toniolo, P. H. Nguyen, G. Stock, and P. Hamm, Proc. Natl. Acad. Sci. U.S.A. 104, 12749 (2007).
[24] M. F. Ludovico, J. S. Lim, M. Moskalets, L. Arrachea, and D. Sanchez, Phys. Rev. B 89, 161306(R) (2014).
[25] M. Kilgour and D. Segal, Phys. Rev. E 98, 012117 (2018).
[26] R. Kosloff and A. Levy, Annu. Rev. Phys. Chem. 65, 365 (2014).
[27] D. Gelbwaser-Klimovsky, and A. Aspuru-Guzik, J. Phys. Chem. Lett. 6, 3477 (2015).
[28] M. Esposito, M. A. Ochoa and M. Galperin, Phys. Rev. B 92, 235440 (2015).
[29] J. Liu, H. Xu, B. Li, and C. Wu, Phys. Rev. E 96, 012135 (2017).
[30] Y. Yang and C. Wu, Europhys. Lett. 107, 30003 (2014).
[31] J. Liu, H. Xu, C.-Q. Wu, Chem. Phys. 418, 42 (2016).
[32] J. Liu, C.-Y. Hsieh, C. Wu, and J. Cao, J. Chem. Phys. 148, 234104 (2018).
[33] C. Wang, J. Ren, and J. Cao, Phys. Rev. A 95, 023610 (2017).
[34] T. Ren, X.-B. Wang, and J. Ren, Phys. Rev. B 87, 144303 (2013).
[35] J. Ren, P. Hänggi, and B. Li, Phys. Rev. Lett. 104, 170601 (2010).
[36] D. Segal, Phys. Rev. B 87, 195436 (2013).
[37] R. A. Veližhanin, H. Wang, and M. Thoss, Chem. Phys. Lett. 460, 325 (2008).
[38] M. Kilgour, B. K. Agarwalla, and D. Segal, J. Chem. Phys. 150, 084111 (2019).
[39] A. Kato and Y. Tanimura, J. Chem. Phys. 143, 064107 (2015).
[40] L. Song and Q. Shi, Phys. Rev. B 95, 064308 (2017).
[41] J. Cerrillo, M, Buser, and T, Brandes, Phys. Rev. Lett. B 94, 214308 (2016).
[42] C. Schinabeck, R. Härtle, and M. Thoss, Phys. Rev. B 97, 235429 (2018).
[43] H. Zheng, Eur. Phys. J. B 38, 559 (2004).
[44] P. Huang, H. Zheng, Chem. Phys. Lett. 500, 256 (2010).
[45] Z. Lv and H. Zheng, J. Chem. Phys. 131, 134503 (2009).
[46] J. Shao and P. Hänggi, Phys. Rev. Lett. 81, 5710 (1998).
[47] W. Wu, and J.-B. Xu, Annals of Physics 377, 48 (2017).
[48] S. Dattagupta, H. Grabert, and R. Jung, J. Phys.: Condens. Matter 1, 1405 (1989).
[49] S. Dattagupta, Relaxation Phenomena in Condensed Matter Physics (Academic Press, New York, 1987).
[50] T. Qureshi, Phys. Rev. B 52, 7976 (1995).
[51] T. Qureshi, Phys. Rev. B 53, 3183 (1996).
[52] D. Segal, Phys. Rev. E 90, 012148 (2014).
[53] P. Haikka, S. McEndoo, G. De Chiara, G.M. Palma, S. Maniscalco, Phys. Rev. A 84, 031602 (2011).
[54] J.-S. Wang, N. Zeng, J. Wang, and C. K. Gan, Phys. Rev. E 75, 061128 (2007).
[55] R. Silbey, R. A. Harris, J. Chem. Phys. 80, 2615 (1984).
[56] H. Zheng, K. H. Bennemann, Phys. Lett. A 352, 99 (2006).
[57] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2002).
[58] C. K. Lee, J. Moix, and J. Cao, J. Chem. Phys. 136, 204120 (2012).
[59] C.-J. Yang, J.-H. An, H.-G. Luo, Y. Li, and C. H. Oh, Phys. Rev. E 90, 022122 (2014).
[60] C. K. Lee, J. Cao, and J. Gong, Phys. Rev. E 86, 021109 (2012).
[61] C. Y. Cai, L.-P. Yang, and C. P. Sun, Phys. Rev. A 89, 012128 (2014).
[62] H.-N. Xiong, P.-Y. Lo, W.-M. Zhang, F. Nori, and D. H. Feng, Sci. Rep. 5, 13353 (2015).
[63] D. Segal, Phys. Rev. B 73, 205415 (2006).
[64] S. Nakajima, Prog. Theor. Phys. 20, 948 (1958).
[65] R. Zwanzig, J. Chem. Phys. 33, 1338 (1960).
[66] U. Weiss, Quantum Dissipative Systems (World Scientific, Singapore, 2008).
[67] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg A and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987).
[68] G. D. Mahan, Many-Particle Physics (Plenum Press, New York, 1990).