Abstract

Motivated by their role for integrality and integrability in topological string theory, we introduce the general mathematical notion of “s-functions” as integral linear combinations of poly-logarithms. 2-functions arise as disk amplitudes in Calabi-Yau D-brane backgrounds and form the simplest and most important special class. We describe s-functions in terms of the action of the Frobenius endomorphism on formal power series and use this description to characterize 2-functions in terms of algebraic K-theory of the completed power series ring. This characterization leads to a general proof of integrality of the framing transformation, via a certain orthogonality relation in K-theory. We comment on a variety of possible applications. We here consider only power series with rational coefficients; the general situation when the coefficients belong to an arbitrary algebraic number field is treated in a companion paper.

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1 Introduction

A remarkable aspect of exact calculations in supersymmetric and topological quantum theories is the blending of discrete and analytic information. What we mean is that while, on the one hand, the microscopic Lagrangian formulation of some given supersymmetric observable makes manifest the holomorphic dependence on the parameters, and can, in some cases, be used to derive the behavior under certain duality transformations (or under analytic continuation), the typical answers localize to finite sums (or at the most, finite-dimensional integrals) over configurations of classical supersymmetric solutions (BPS states).

In other words, the expansion coefficients of the supersymmetric amplitude around the appropriate limit often admit an a priori perhaps unexpected interpretation as counting dimensions of certain vector spaces of BPS states (or, more generally, an index of some operator on such spaces). That, conversely, the generating functions for these dimensions have interesting analytic and modular properties, is a remarkable fact that can be understood, at least in part, as a consequence of the underlying duality symmetries of the microscopic formulation.

In these respects, supersymmetric partition functions are reminiscent of functions of interest in analytic number theory, and in fact there are many cases in which the two are very closely related. This has led to a number of results on, most notably, modular forms finding applications in diverse areas of mathematical physics related to supersymmetric field and string theories. This interest has also led to a number of new mathematical results. The ways in which the functions of interest are related to some geometric situation in both physics and number theory are also often similar. The rampant speculation about the deeper meaning of such coincidences is best restrained by pointing out that there remain large classes of very deep number theoretic functions (such as L-functions, ζ-functions) whose relevance for supersymmetric quantum theory are much less clear.

In this paper, we study elementary algebraic properties of a certain class of functions that we call “s-functions” (where, at least for now, s is a positive integer). We extract this notion from the appearance of s-function in perturbative computations in topological string theory, where they are building blocks of supersymmetric generating
functions. We define an $s$-function as an integral linear combination of $s$-logarithms. We give later an equivalent definition in terms of Frobenius map on (formal) power series with rational coefficients. That second definition can easily be generalized to the case when we allow coefficients to lie in arbitrary local or global number fields, see [5].

One of our main results concerns the most important special case, $s = 2$, and is an integrality statement of a certain algebraic transformation of 2-functions (viewed as formal power series) that we call "framing". We point out that while special cases of this framing transformation are known in the context of open topological string theory (where we borrowed the name, see [2]), the generality in which it applies has not been pointed out in the literature to our knowledge (Although it might be known to experts. We made our initial observations after reading [3].) Secondly, we will give a mathematical proof of this framing property, using an interpretation of the notion of 2-function in algebraic K-theory. (M. Kontsevich informed us that he also obtained this interpretation and used it to prove some integrality theorems.)

Finally, we point out a few further generalizations of our setup and constructions. One of these generalizations involves extending the field of definition of the coefficients from $\mathbb{Q}$ to a more general number field. The relevance of such extensions was first observed in [4], and we will elaborate on them in a companion paper [5].

At the moment the only immediate applications of our results that we are aware of come from open topological string theory and mirror symmetry. We suspect however that the concepts we introduce might play a role in other contexts as well. As a particular example, refs. [6, 3] lead us to expect certain connections with the theory of Mahler measures. In a different way, the characterization of 2-functions in algebraic K-theory is reminiscent of a certain integrability condition recently explored by Gukov and Sulkowski [7].

In mathematical terms, integrality of framing is the following statement. Say

$$W(z) = \sum_{d=1}^{\infty} n_d \text{Li}_2(z^d)$$  \hspace{1cm} (1.1)

is an integral linear combination of standard di-logarithms,

$$\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$  \hspace{1cm} (1.2)

Namely, the coefficients $n_d$ in (1.1) are integers, and we may as well assume that the
series (1.1) is convergent. We introduce the power series
\[ Y(z) = \exp\left(-z \frac{d}{dz} W(z)\right), \tag{1.3} \]
with constant term 1 around \( z = 0 \). Then the relation
\[ \tilde{z} = -zY(z) \tag{1.4} \]
may be inverted (formally, and as a convergent power series),
\[ z = -\tilde{z}\tilde{Y}(\tilde{z}) \tag{1.5} \]
to yield another power series \( \tilde{Y}(\tilde{z}) \) around \( \tilde{z} = 0 \) with constant term 1. Expanding
\[ \tilde{W}(\tilde{z}) = \int \frac{d\tilde{z}}{\tilde{z}}(-\log \tilde{Y}(\tilde{z})) = \sum_{d=1}^{\infty} \tilde{n}_d \text{Li}_2(\tilde{z}^d) \tag{1.6} \]
defines coefficients \( \tilde{n}_d \) which are \textit{a priori} rational numbers. It is elementary to see that \( \tilde{Y}(\tilde{z}) \) has integer coefficients, and as a consequence that \( d\tilde{n}_d \in \mathbb{Z} \). We claim that, in fact,
\[ \tilde{n}_d \in \mathbb{Z} \tag{1.7} \]
(whenever \( n_d \in \mathbb{Z} \)). See section 3 for the sketch of a proof of this statement.

In physical terms, we think of \( W(z) \) as a contribution to the space-time superpotential from D-branes wrapping supersymmetric cycles in some Calabi-Yau compactification of string theory preserving \( \mathcal{N} = 1 \) supersymmetry in 4 dimensions. In that context, \( z \) is a chiral superfield whose vacuum expectation value parametrizes the moduli space of open/closed string vacua, and corresponds to a geometric modulus of the configuration. It was shown long time ago by Ooguri and Vafa \[8\], generalizing work by Gopakumar and Vafa \[9\], that, for an appropriate choice of parametrization, one expects the coefficients \( n_d \) in the expansion (1.1) to count dimensions of spaces of appropriate BPS states, hence the integrality.

One of the features of the setup of Ooguri and Vafa is the dependence of the superpotential (and the BPS invariants) on an integer parameter, \( f \), known as “the framing”. Algebraically, the framing results from an ambiguity in the identification of the open string modulus \[2\]. Namely, framing by \( f \) amounts to replacing
\[ \log z \to \log z_f = \log z + f z \partial_z W(z), \quad f \in \mathbb{Z} \tag{1.8} \]
Although it might not be immediately obvious, the transformations (1.8) are very closely related to (namely, generated by) (1.4), (1.5). We will explain this connection, and also review the general setup in somewhat more detail, in section 2.

The dependence on $f$ of open topological string amplitudes in the Ooguri-Vafa setup is explained through the (large-$N$) duality with Chern-Simons theory and knot invariants. (The framing of a knot is the choice of non-vanishing section of the normal bundle of a knot in a three-manifold, and gets identified with $f$ under this duality.) For this duality to operate, it is important that the underlying Calabi-Yau manifold be non-compact. Mathematically, the framing dependence of the enumerative (open Gromov-Witten) invariants is only well understood when the manifold is toric [11].

In a series of works, from [10] to [4], it was shown that the expansion (1.1) continues to be valid in principle when the underlying Calabi-Yau manifold is compact, and non-toric, but requires a number of modifications in practice. Most prominently, the parameter $z$ should be a closed string modulus that remains massless at tree-level. Secondly, the standard di-logarithm (1.2) has to be “twisted” in general to take into account the symmetries of the open string vacuum structure. On the other hand, neither the geometric setup (in the A-model) nor the actual calculation (in the B-model) seem to involve any ambiguity that could be identified as “framing”. Most recently, it was pointed out in [4] that the generic B-model setup will predict invariants $n_d$ that are irrational numbers valued in some algebraic number field (a finite extension of $\mathbb{Q}$, fixed for each geometric situation). The symmetry of the space of vacua and the associated twist of the di-logarithm was related to the Galois group of that finite field extension. It is possible to prove some integrality statements also in this case (see [5] for detail).

The statement that D-brane superpotentials given by geometric formulas of [12, 2, 10] indeed admit a decomposition of the form (1.1) into integral pieces was proven mathematically in ref. [15], using and extending earlier work by the same authors [13, 14]. The central aspect of that series of works was to relate the BPS numbers $n_d$ to the action of the Frobenius automorphism on $p$-adic cohomology of the Calabi-Yau (together with an algebraic cycle).

The proofs of [15] show integrality of open topological string amplitudes, separately for any value of the framing, in situations in which this concept is well-defined. One of the main messages of the present paper is that the integrality of the framing transformation is more general, and in fact not tied to a particular geometric situation.
But the methods for proving the integrality statements developed in [15] continue to apply. This means in particular that we can define a “framed” superpotential and enumerative invariants even when we do not know a geometric interpretation for the integer ambiguity \( f \). We interpret this fact, together with the observation that instanton numbers are related to Mahler measures by a framing transformation [6, 3], as a hint that framing is an important intrinsic property of the di-logarithm.

As further support, we mention that the integrality of the framing transformation is naturally expressed as a certain torsion/orthogonality condition in algebraic K-theory, see section 3, and can also be given a Hodge theoretic interpretation. Framing can also be generalized to the multi-variable situation, in which it depends in an interesting way on the additional data of a symmetric bilinear form. Finally, while in this paper we are concerned mainly with the situation in which the coefficients are actually rational numbers, the generalization to arbitrary number fields is rather straightforward. We will consider it in a separate paper [5], where further mathematical details may also be found.

## 2 Dilogarithm, \( s \)-functions, and topological strings

(The mathematically inclined reader may gain from skipping the odd (numbered) subsections which provide some physics motivation for our definitions.)

### 2.1 A-model

From the point of view of the A-model, the origin of the formula (1.1) is, intuitively, easy to understand. Consider a Calabi-Yau threefold \( X \) and a Lagrangian submanifold \( L \subset X \). We view \( L \) as the support of a topological D-brane in the A-model, which we may want to use as an ingredient in a superstring theory construction. As is well-known, the classical deformation space of \( L \) modulo Hamiltonian isotopy (or, preserving the “special Lagrangian” condition, if one exists) is unobstructed and of dimension equal to \( b_1(L) \). Worldsheet instanton corrections however induce a space-time superpotential that schematically takes the form

\[
W = \ldots + \sum_{w:(D,\partial D)\to (X,L)} e^{\int_D u^* \omega} \Tr [Pe^{\int_{\partial D} u^* A}]
\]

and depends (via the symplectic form \( \omega \)) on the Kähler moduli of \( X \) and (via the (unitary) connection \( A \)) on the choice of a flat bundle over \( L \). Here, \( \ldots \) denotes certain
(subtle) classical terms that we will neglect in this paper, so the sum is over all (non-
constant) holomorphic maps

\[ u : (D, \partial D) \to (X, L) \]  

(2.2)

from the disk \( D \) to \( X \) mapping the boundary \( \partial D \) to \( L \).

It is well known that the expected (virtual) dimension of the space of such holomorphic maps is zero for any class \( \beta = u_*([D, \partial D]) \in H_2(X, L) \), and hence one expects to write\(^1\)

\[ W(Q) = \sum_{\beta \in H_2(X, L)} m_\beta q^\beta \]  

(2.3)

where \( \log q \) is the appropriate combination of moduli of \((X, L)\), and \( m_\beta \) is the “number” of holomorphic maps in a fixed class \( \beta \) (open Gromov-Witten invariants). While the general definition of \( m_\beta \) is plagued with difficulty, it is in any case clear that the moduli space \( \mathcal{M}(\beta) \) of such maps will contain components of positive dimension, if \( \beta \) is not primitive. Namely, say \( \beta = k\beta' \) with \( \beta' \) integral and \( k > 1 \). Then any \( u' \in \mathcal{M}(\beta') \) may be composed with a degree \( k \) covering map \( c : (D, \partial D) \to (D, \partial D) \) to give a map \( u = u' \circ c \in \mathcal{M}(\beta) \). Since the maps \( c \) come in families (of dimension \( 2k - 2 \)), so \( \mathcal{M}(\beta) \) will contain components of positive dimension.

The formula (1.1) is a reflection of these multi-covers. (Even though it might not be strictly true that all holomorphic maps can be factorized in this way, see [16], the success of the formula suggests that this is effectively the case.) The general statement is that a BPS state corresponding (intuitively) to an embedded disk with boundary on \( L \) in the class \( \beta \), together with all its multi-covers, makes a contribution to \( W \) of the form

\[ W_\beta(q) = \sum \frac{1}{k^2} q^{k\beta} = \text{Li}_2(q^\beta), \]  

(2.4)

so that if \( n_\beta \) is the (integer!) degeneracy of BPS states of charge \( \beta \), the total superpotential is

\[ W(q) = \sum_{\beta} n_\beta W_\beta(q) = \sum_{\beta} n_\beta \text{Li}_2(q^\beta) \]  

(2.5)

Eq. (1.1) is recovered when \( H_2(X, L) \) has rank one, and \( q = z \).

The prototypical example of a multi-cover formula like (2.4) is known from the beginning of mirror symmetry [17] as the Aspinwall-Morrison formula [18]. It states

\(^1\)Note that the \( \text{Tr} \) in (2.1) really depends on the homotopy class of \( u \) in \( \pi_2(X, L) \). The formula (2.3) makes sense if the fundamental group of \( L \) is abelian, so that \( \pi_2(X, L) = H_2(X, L) \).
that the large volume (A-model) expansion of the $\mathcal{N} = 2$ prepotential (i.e., the genus 0 Gromov-Witten potential) takes the form

$$F^{(0)} = \sum M_d q^d = \sum N_d \text{Li}_3(q^d) \tag{2.6}$$

with integer $N_d$. The $M_d$ are rational numbers, which is obvious from both the definition in Gromov-Witten theory, as well as from the B-model formulas (involving differential equations with rational coefficients). The integrality of the $N_d$ however is harder to see. It was proven mathematically in [13, 14, 15, 19]. A physical explanation was given in [9] by relating the $N_d$ to the degeneracy of BPS states. The generalization of the Aspinwall-Morrison formula to arbitrary genus $g$ was shown to involve the poly-logarithm $\text{Li}_{3-2g}$.

### 2.2 $s$-functions

The central importance of these multi-cover formula in relating the perturbative topological string amplitudes to the degeneracy of BPS states motivates us to introduce the following notion: If $s$ is a positive integer, we call a power series

$$V(z) = \sum_{d=1}^{\infty} m_d z^d \in \mathbb{Q}[[z]] \tag{2.7}$$

with rational coefficients $m_d$ an $s$-function if it can be written as an integral linear combination of $s$-logarithms.

$$V(z) = \sum_{d=1}^{\infty} n_d \text{Li}_s(z^d), \quad \text{Li}_s(z) := \sum_{k=1}^{\infty} k^{-s} z^k \tag{2.8}$$

with $n_d \in \mathbb{Z}$. It is convenient to define the logarithmic derivative,

$$\delta_z = \frac{d}{d \ln z} \tag{2.9}$$

So $\delta_z \text{Li}_s(z) = \text{Li}_{s-1}(z)$, and if $V(z)$ is an $s$-function, $\delta_z V(z)$ is an $(s - 1)$-function. For the topological string, the relevant values are $s = 3$ for genus 0 (closed string tree-level) invariants, $s = 2$ for disk invariants (open string tree-level), and $s = 1$ for all one-loop amplitudes (open or closed).

Sometimes it is convenient to consider $s$-functions with respect to prime number $p$ requiring that the denominators of the coefficients $m_d$ are not divisible by $p$ (the coefficients are $p$-integral).
2.3 B-model

As mentioned in the introduction, framing originally entered topological string theory through the relation between local toric manifolds and Chern-Simons gauge theory and knot invariants. Framing has also been explained in (toric) A-model [11] as a choice of linearization of the torus action required to make the localization calculation of open Gromov-Witten invariants well-defined. The operation itself is however most straightforward to explain geometrically in the B-model.

The B-model mirror of a general toric Calabi-Yau threefold has the form

$$\{uv = H(x, y)\} \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \ni (u, v, x, y) \quad (2.10)$$

where we do not need to write explicitly the dependence on complex structure parameters. Geometrically, (2.10) is a conic bundle over $\mathbb{C}^* \times \mathbb{C}^*$ with discriminant locus given by the (non-compact, i.e., punctured) curve

$$\mathcal{C} = \{H(x, y) = 0\} \subset \mathbb{C}^* \times \mathbb{C}^* \quad (2.11)$$

Aganagic and Vafa [12] study B-type D-branes in this geometry wrapped on one component of a reducible fiber, say $u = 0$, varying over $\mathcal{C}$. They identify certain “semiclassical” regimes of these branes as punctures of the curve, and show that the superpotential expanded near such a point is given by an Abel-Jacobi computation on $\mathcal{C}$. For simplicity, let us say the interest is in a puncture at $y = 1$. Then the superpotential is given by the formula

$$\delta z W(z) = -\log y(z) \quad (2.12)$$

where $z$ (the open string modulus) is a local coordinate on the curve such that $z = 0$ corresponds to the puncture. This choice is made such that the superpotential is critical, i.e., $y = 1$, at $z = 0$. In some simple cases, $z$ coincides with $x$ in (2.10). As pointed out in [2], however, this prescription is ambiguous: If $z$ is such a good coordinate, then so is any combination

$$z_f = z(-1)^f y^f \quad (2.13)$$

with integer $f$. Up to the sign, this is equation (1.8) from the introduction.$^2$

Let us pause briefly here to explain the relevance of the mirror map: $W$ as defined by (2.12) is a 2-function in the sense of the previous subsection when expanded, not

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$^2$The sign is thrown in to preserve integrality also at $p = 2$, see below.
only around $z = 0$, but also in the appropriate flat closed string coordinates around a degeneration of the curve. Assuming that the family of curves is defined over $\mathbb{Q}$, the 2-function property of $W$ (after the mirror map) was shown in general in [15]. However, as pointed out in general in [2], the mirror map is in fact independent of the open string coordinate itself, and as a consequence framing commutes with the mirror map. This observation also explains why we are using the traditional B-model notation $z$ interchangeably with the A-model $q$ for the argument of our $s$-functions.

Our main point now is to abstract the ambiguity (2.13) to the following general “framing transformation”, parameterized by an integer $f$.

### 2.4 The group of framing transformations

Say $V(z)$ is an $s$-function with $s \geq 1$. Define

$$Y = \exp\left(- (\delta_z)^{s-1} V\right) \in \mathbb{Z}[[z]] \quad (2.14)$$

(The integrality of the power series follows from $\delta_z^{s-1} \text{Li}_s(z) = \text{Li}_1(z) = -\ln(1-z)$.)

When $s = 2$ and $V = W$ is the superpotential of (the mirror of) a toric D-brane configuration, then the corresponding $Y = \exp(-\delta_z W) = y(z)$ will by the above construction satisfy an algebraic equation

$$H(z, Y) = 0 \quad (2.15)$$

while the “framed superpotential” $\delta_{z_f} W_f = -\log Y_f$ can be identified with a solution of the equation

$$H_f(z_f, Y_f) = H(z_f (-Y_f)^{-f}, Y_f) = 0 \quad (2.16)$$

Even though the equations are algebraic, it is natural (for the purposes of mirror symmetry, for example) to think of solutions $Y(z)$ and $Y_f(z_f)$ as local power series around $z = 0$, $z_f = 0$, respectively. By the construction, both $Y(0) = 1$, and $Y_f(0) = 1$. The relation between the two is then simply $Y_f(z_f) = Y(z)$, $z_f = (-Y)^f z$. Eliminating $z$, and renaming $z_f$ as $z$, this means that we can obtain $Y_f(z)$ as the solution to the equation in formal power series,

$$Y_f = Y(z (-Y_f)^f) \quad (2.17)$$

The “framed 2-function” is then the power series

$$W_f(z) = \int \frac{dz}{z}(- \log Y_f(z)) \quad (2.18)$$
and, as we prove below, in fact is also a 2-function. Since the relations (2.17), (2.18) make sense independent of the existence of an algebraic equation of the type (2.15), we may take this as the general definition of framing, even for cases in which no such equation is known to exist.

We note a few elementary properties of this definition. First of all, framing defines a group action \( \mathbb{Z} \ni f : W \to W_f \). Indeed, using the definitions, we find

\[
(Y_f)_{f'} = Y_f(z(-(Y_f)_{f'})^{f'}) \\
= Y(z(-(Y_f)_{f'})^{f'}(-(Y_f)_{f'})^f) \\
= Y(z(-(Y_f)_{f'})^{f+f'})
\]

Thus, the equation for \((Y_f)_{f'}\) is exactly the defining equation for \(Y_{f+f'}\).

Secondly, to make contact with the introduction, we write eqs. (1.4), (1.5) in the form

\[
z = -\tilde{z} \tilde{Y} = zY(-\tilde{z} \tilde{Y}) \tilde{Y}
\]

which is equivalent to

\[
Y(-\tilde{z} \tilde{Y}) = \tilde{Y}^{-1}
\]

Comparison with (2.17) shows that

\[
\tilde{Y} = (Y_{-1})^{-1}
\]

By substituting \(Y_1\) for \(Y\) into this equation, we learn that \(\tilde{Y}_1 = Y^{-1}\), and since \(Y \mapsto \tilde{Y}\) is obviously involutive, this implies

\[
Y_1 = \tilde{Y}^{-1}
\]

We conclude that framing transformations in the sense of (2.17) in fact are generated by the single transformation (1.4), (1.5), together with the operation \(Y \mapsto Y^{-1}\) (which in terms of the 2-function, corresponds simply to \(W \mapsto -W\)). Hence, we will refer to (1.4), (1.5) also as “framing”.

3 K-theoretical description of 2-functions. Integrality of framing

The purpose of this section is to give a description of 2-functions in terms of K-theory and to derive from this description the integrality of the framing transformation:
Theorem 1. If $W(z) \in \mathbb{Q}[z]$ is a 2-function, then its image under framing, $\tilde{W}(z)$ is also a 2-function.

To prove this theorem it is sufficient to check for all prime numbers $p$ that the fact that $W(z)$ is a 2-function with respect to $p$ implies that $\tilde{W}(z)$ is also a 2-function with respect to $p$.

A complete proof of this statement will be given in [5]. The $K$-theoretic proof that we will sketch in this section works only for odd primes.

We begin with a lightning review of algebraic $K$-theory.

3.1 Algebraic $K$-theory. Orthogonality relation

For a ring $A$, the group $K_1(A)$ is defined as the abelianization of the infinite linear group $GL(A)$:

$$K_1(A) = GL(A)/[GL(A), GL(A)].$$

If $A$ is a Euclidean domain (in particular, it is a commutative ring), then the group $K_1(A)$ is isomorphic to the group $A^\times$ of invertible elements of $A$. (For arbitrary commutative rings we have an embedding $A^\times \to K_1(A)$ induced by the embedding $A^\times = GL_1(A) \hookrightarrow GL(A)$, and a map $K_1(A) \to A^\times$ induced by the determinant map $\det : GL(A) \to A^\times$.)

Notice that $K_1(A)$ is usually regarded as an additive group, but in our situation it is isomorphic to a multiplicative group $A^\times$, and so the multiplicative notation is more convenient.

The group $K_2(A)$, which we will write additively, is defined for an arbitrary ring via the universal central extension of the commutator subgroup $E(A) = [GL(A), GL(A)]$. Thus it fits into the sequence

$$K_2(A) \to St(A) \to E(A) \to GL(A) \to K_1(A)$$

where $St(A)$ is the Steinberg group (when $A$ is Euclidean, we may think of the “universal cover” of $E(A) = SL(A) = \text{Ker}(\det)$).

For an arbitrary ring $A$ there exists a pairing $K_1(A) \otimes \mathbb{Z} K_1(A) \to K_2(A)$. Via the embedding $A^\times \to K_1(A)$, this pairing induces a (skew) pairing of invertible elements of $A$ in $K_2(A)$ (an antisymmetric bilinear map $\phi : A^\times \otimes A^\times \to K_2(A)$).

An important property of $\phi$ is that the pairing of two invertible elements $f, g$ vanishes if $f + g = 1$. Let us denote by $J$ the subgroup of $A^\times \otimes A^\times$ generated by elements
of the form \( f \otimes (1 - f) \); the above statement means that \( J \subset \text{Ker} \phi \). If \( A \) is a field, then \( J = \text{Ker} \phi \) (by Matsumoto’s theorem).

Using the notation

\[
K^0_2(A) = A^\times \otimes A^\times / J
\]

we can consider the pairing \( \phi \) as a composition of maps \( A^\times \otimes A^\times \to K^0_2(A) \) and \( K^0_2(A) \to K_2(A) \). We will work with the first map considered as a pairing on \( A^\times \); we denote this pairing by \( \{ f, g \} \). (It is possible to work also with \( \phi \), but this makes the proof more complicated.)

We will characterize 2-functions in terms of this pairing.

Let us take two invertible elements \( f, g \in A^\times \). By definition \( f \) and \( g \) are orthogonal if the element \( 2 \{ f, g \} = \{ f^2, g \} = \{ f, g^2 \} \) vanishes. We have used the bilinearity of the pairing \( \phi \) expressed by formulas \( \{ f_1 f_2, g \} = \{ f_1, g \} + \{ f_2, g \}, \{ f, g_1 g_2 \} = \{ f, g_1 \} + \{ f, g_2 \} \). These unusual formulas come from the fact that the operation in \( A^\times \) is written as multiplication, while the operations in \( K_2(A) \) and \( K^0_2(A) \) are written additively. \(^3\) Notice that it follows from bilinearity that an invertible element \( f \) that is orthogonal to elements \( g \in A^\times \) and \( h \in A^\times \) is orthogonal to their product \( gh \). It is also obvious that automorphisms of the ring \( A \) preserve orthogonality.

The relevant ring for us is \( A = \mathbb{Z}((q)) \), the ring of formal Laurent series in one variable \( q \), with integer coefficients. The ring \( A \) has a natural topology, that induces a topology in \( A^\times \). This allows us to modify the notion of orthogonality: we will say that \( f, g \) are orthogonal in the new sense if there exists a sequence of pairs \( (f_n, g_n) \) such that \( f_n \) tends to \( f \), \( g_n \) tends to \( g \) and \( f_n \) is orthogonal to \( g_n \) in the old sense.

Notice that starting with a 2-function \( W(q) = \sum_{d=1} n_d \text{Li}_2(q^d) \) represented as a sum of di-logarithms we construct an invertible element of \( A \) by the formula

\[
Y(q) = \exp(-\delta_q W(q)) = \prod (1 - q^d)^{dn_q}.
\]

If the sum of di-logarithms is finite the element \( Y(q) \) is orthogonal to \( q \). It is sufficient to check this statement for every factor. The fact that \( q \) is orthogonal to \( (1 - q^d)^d \) can be derived from the following chain of identities:

\[
\{ q, (1 - q^d)^d \} = \{ q^d, 1 - q^d \} = 0. \tag{3.3}
\]

\(^3\)Some other useful properties include \( \{ f, -f \} = 0, \{ f, 1 \} = 0 \), and anti-symmetry \( \{ f, g \} = -\{ g, f \} \).
If the sum of di-logarithms is infinite then $Y(q)$ is orthogonal to $q$ in the new sense (infinite product is a limit of finite products). In what follows we will understand the orthogonality in the new sense.

Let us notice that $q^m$ is orthogonal to $q$. It is sufficient to check this for $m = 1$. This follows from the identity

$$\{q, (-q)\} = \{q, (1 - q)\} + \{q^{-1}, (1 - q^{-1})\} = 0,$$

which in turn follows from $-q = \frac{1-q}{1-q^2}$. We see that $2\{q, q\} = \{q, (-q)^2\} = 2\{q, (-q)\} = 0$.

We obtain that $q$ is orthogonal to an expression of the form $q^m \prod (1 - q^d)^{dn_d}$. One can prove (Sec 5 and [5]) that the inverse statement is also correct:

**Theorem 2.** If $Y(q) = \exp(-\delta_q W(q))$ is orthogonal to $q$ then the series $W(q) \in \mathbb{Q}[\![q]\!]$ is a 2-function for all odd primes.

In other words $q$ is orthogonal to $Y$ where $Y$ behaves like $q^m$ as $q$ tends to 0 if and only if $\int \log(Y/q^m) d \log q$ is a 2-function for all odd primes.

### 3.2 Integrality of framing (Proof of Theorem 1)

Notice that any change of variables $\bar{q} = q + a_2 q^2 + a_3 q^3 + \cdots$ with $a_i \in \mathbb{Z}$ induces an automorphism of the algebra $A = \mathbb{Z}(\!(q)\!)$. This automorphism preserves the orthogonality relation.

Using this fact we can describe all orthogonal pairs $(f, g)$. It is sufficient to consider only the case when $f$ behaves like $q$ as $q$ tends to zero. Then we can take $f$ as a new variable; in terms of this variable $g$ has an expression of the form $f^m \prod (1 - f^d)^{dn_d}$ and integrates to a 2-function $\sum n_d \text{Li}_2(f^d)$. This follows from Theorem 2 above, applied to the ring $\mathbb{Z}(\!(f)\!)$, which is isomorphic to $A$ as remarked above.

Let us now consider all orthogonal pairs $(f, g)$ where both $f$ and $g$ behave like $q$ as $q \to 0$. The orthogonality relation is symmetric, hence $f$ and $g$ are on equal footing. Therefore, we can construct two different 2-functions (one from expression of $g$ in terms of $f$, another from expression of $f$ in terms of $g$.) These two 2-functions are related by framing transformation (1.4), (1.5). Thus we see that Theorem 1 is a simple consequence of the K-theoretical description of 2-functions and of the symmetry of the orthogonality relation.
Notice that in the proof we used orthogonal pairs where both \( f \) and \( g \) behave like \( q \) as \( q \to 0 \) (this is important for symmetry). In the orthogonal pair \((q,Y(q))\) the function \( Y(q) \) does not satisfy this condition, therefore in the construction of the framing transformation it should replaced by \( qY(q) \).

4 Frobenius automorphism and local \( s \)-functions

The purpose of this section is to reformulate the definition of an \( s \)-function in terms of the action of the Frobenius endomorphism acting on (formal) power series. Such a reformulation was crucial in the proofs of integrality theorems in [13, 14, 15]. We will apply it here to sketch a proof of the description of 2-functions in terms of algebraic K-theory (Theorem 2) that was used in the derivation of integrality of framing (see [5] for more detail).

First of all we will check that \( V(z) \in \mathbb{Q}[[z]] \) is an \( s \)-function if and only if for any prime number \( p \) the formal series

\[
\frac{1}{p^s}(V(z^p)) - V(z)
\]  

is \( p \)-integral (the denominators of the coefficients are not divisible by \( p \)). The proof is an easy consequence of Möbius inversion formula (and a trivial generalization of the special statements for \( s = 2, 3 \)).

Recall that if \((a_d)\) and \((b_d)\) are two sequences such that

\[
a_d = \sum_{k\mid d} b_k \tag{4.2}
\]

(where the sum is over all divisors of \( d \)), then

\[
b_d = \sum_{k\mid d} \mu\left(\frac{d}{k}\right) a_k = \sum_{k\mid d} \mu(k) a_{k/d} \tag{4.3}
\]

where \( \mu \) is the Möbius function: \( \mu(k) = 0 \) if \( k \) is not squarefree, \( \mu(k) = (-1)^r \) if \( k = p_1 \cdots p_r \) is the product of \( r \) distinct prime factors. The important property of \( \mu \) is that

\[
\sum_{k\mid d} \mu(k) = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{if } d > 1 \end{cases} \tag{4.4}
\]

which itself follows from the fact that if \( r > 0 \)

\[
\sum_{l\mid k} \mu(l) = \sum_{l\mid p_2 \cdots p_r} \mu(l) + \sum_{l\mid p_1 l} \mu(p_1 l) \tag{4.5}
\]
Then (4.3) follows from the computation

$$
\sum_{k|d} b_k = \sum_{l|d} \mu\left(\frac{d}{l}\right) a_l = \sum_{l|d} a_l \sum_{k|d} \mu\left(\frac{d}{k}\right) = \sum_{l|d} a_l \sum_{k|d} \mu(k) = a_d \tag{4.6}
$$

Returning to $s$-functions, we compare coefficients of $z^d$ in

$$
V(z) = \sum_{d=1}^{\infty} m_d z^d = \sum_{d,k=1}^{\infty} \frac{n_d}{k^s} z^d \tag{4.7}
$$

to conclude that

$$
d^s m_d = \sum_{k|d} k^s n_k \tag{4.8}
$$

where the sum is over all divisors of $d$. Applying Möbius inversion, we find

$$
d^s n_d = \sum_{k|d} \mu\left(\frac{d}{k}\right) k^s m_k \tag{4.9}
$$

On the other hand, the statement that (4.1) be $p$-integral for all primes $p$ is equivalent to the condition that

$$
m_d - \frac{1}{p^s} m_{d/p} \text{ be } p\text{-integral for all } p, d \tag{4.10}
$$

it being understood that $m_{d/p} = 0$ if $p \nmid d$. To see that the integrality of the $n_d$ implies this condition, we note that (4.8) implies

$$
m_d - \frac{1}{p^s} m_{d/p} = \frac{1}{d^s} \sum_{k|d} k^s n_k \tag{4.11}
$$

The sum is restricted to those $k$ divisible by as many powers of $p$ as $d$, and therefore the right hand side is $p$-integral if the $n_k \in \mathbb{Z}$.

Conversely, since $\mu(k) = 0$ if $k$ is divisible by $p^2$, and $\mu(pk) = -\mu(k)$ if $p \nmid k$, we see that the formula (4.9) may be rewritten as

$$
n_d = \sum_{k|d} \mu(k) \frac{m_{d/k}}{k^s} = \sum_{k|d} \frac{\mu(k)}{k^2} \left( m_{d/k} - \frac{1}{p^s} m_{d/(pk)} \right) \tag{4.12}
$$

with the same understanding that $m_{d/(pk)} = 0$ if $p \nmid d$. We see that if (4.10) holds, the $n_d$ are $p$-integral for any $p$, hence integral.
We can reformulate the above statement in $p$-adic terms. Let us denote by $V_p(z)$ the $p$-adic reduction of $V(z)$, i.e., the series obtained from $V(z)$ by viewing all coefficients as $p$-adic numbers. (Really, this is the same series.) We also denote by $\text{Fr}_p : \mathbb{Q}[[z]] \to \mathbb{Q}[[z]]$ the Frobenius endomorphism fixing $\mathbb{Q}$ and sending $z$ to $z^p$. Then the characterization (4.1) of $s$-functions is equivalent to the statement that for any $p$,

$$M_p(z) = \frac{1}{p^s} \text{Fr}_p V_p(z) - V_p(z) \in \mathbb{Z}_p[[z]]$$

is a series with $p$-adic integral coefficients. We may call a function that satisfies (4.13) (only) for some fixed prime $p$ a local $s$-function.

One can express the coefficients $n_d$ (or, better to say, their $p$-adic reductions) in terms coefficients of the series $m_p(z)$, see Lemma 3 of [15]. It follows immediately from this formula that $p$-adic integrality of coefficients of $M_p(z)$ for all $p$ guarantees the integrality of $n_p$.

5 Regulators. K-theoretic characterization of 2-functions

Fix an odd prime number $p$.

Let us define a map

$$(f, g)_p : \mathbb{Z}_p((z))^\times \otimes \mathbb{Z}_p((z))^\times \to \mathbb{Q}_p((z))/\mathbb{Z}_p((z))$$

(5.1)

to be a unique bilinear skew-symmetric pairing such that for every $g \in \mathbb{Z}_p[[z]]^\times$ and every $f \in \mathbb{Z}_p((z))^\times$, one has

$$(f, g)_p = \int \left( \frac{1}{p^2} \text{Fr}^* (\log g d \log f) - \log g d \log f \right) - \frac{1}{p} \text{Fr}^* (\log g) \left( \frac{1}{p} \text{Fr}^* (\log f) - \log f \right).$$

(5.2)

Here $\text{Fr}^*$ is the “Frobenius lifting”:

$$\text{Fr}^*(h(z)) = h(z^p).$$

The key property of the pairing $(f, g)_p$ is that it is invariant under any change of variables of the form $z \to h(z) = a_1 z + a_2 z^2 + \cdots$, $a_1 \in \mathbb{Z}_p^\times$. In particular, if $(f, g)_p = 0$ (i.e., $(f(z), g(z))_p$ is $p$-adically integral) then the same is true for $(f(h(z)), g(h(z)))_p$. This can be derived from a more general fact: the right-hand side of formula (5.2) viewed as an element of the quotient group $\mathbb{Q}_p((z))/\mathbb{Z}_p((z))$ does not get changed if one replaces $\text{Fr}^*$ by an arbitrary Frobenius lifting of the form

$$\text{F}^*(h(z)) = h(z^p(1 + pr(z)),$$
for some \( r(z) \) ∈ \( \mathbb{Z}_p[[z]] \).

One can derive from this fact (or prove directly) that \( (1 - f, f)_p = 0 \) for every \( f \in \mathbb{Z}_p((z)) \times \) such that \( 1 - f \) is also in \( \mathbb{Z}_p((z)) \times \). Thus, (5.1) factors through a homomorphism

\[
K_2^0(\mathbb{Z}_p((z))) \rightarrow \mathbb{Q}_p((z))/\mathbb{Z}_p((z)).
\] (5.3)

Finally, one can easily check that \((f, g)_p\) is continuous in each variable with respect to “\(z\)-adic” topology on \( \mathbb{Z}_p((z)) \).

The above properties are sufficient to prove Theorem 2. Indeed, if \( Y(z) \) is orthogonal to \( z \) then \((z, Y(z))_p = 0\) for every odd prime \( p \). On the other hand, we have that

\[
(z, Y(z))_p = \frac{1}{p^2} \text{Fr}^*W(z) - W(z).
\]

Therefore \( W(z) \) is a 2-function.

**Remark:** One can check that (5.3) factors uniquely through a homomorphism

\[
K_2(\mathbb{Z}_p((z))) \rightarrow \mathbb{Q}_p((z))/\mathbb{Z}_p((z)).
\]

6 Generalizations

One of our central contentions in writing this note is that \( s \)-functions are interesting algebraic objects in their own right, independent of relations to physics of topological strings. We also claim that the value \( s = 2 \) is special. We give further evidence in this section by pointing out some very natural generalizations of our discussion so far. One possible application that we will not (p)review in any detail here is to the theory of Mahler measures. Indeed, the reader will find it easy to verify that the relation between the so-called Modular Mahler Measures of [6] and the instanton expansion of certain “exceptional non-critical strings” [20], which was pointed out by Stienstra [3], is nothing but an elementary framing transformation (1.4), (1.5). Given our results, it is clear that the relation will be valid in much greater generality than the examples presented in [3]. It would be interesting to explore this further.

6.1 Arithmetic twists

The generalization that motivated our initial observations concerning 2-functions is related to the results of [4]. In that work, it was pointed out that the A-model instanton
expansion of the superpotential associated with a general D-brane on a compact Calabi-Yau three-fold is not rational (let alone integral in the usual sense). Instead, the coefficients were found to be contained in the algebraic number field $K$ over which the curve representing the D-brane in the B-model was defined. However, it was also observed that with an appropriate modification of the Ooguri-Vafa multi-cover formula, (2.5), at least integrality in the algebraic sense could be preserved. The expansion proposed in [4] was of the form

$$W(q) = \sum \tilde{n}_d q^d = \sum n_d \text{Li}_D^2(q^d)$$

(6.1)

where $n_d$ are algebraic integers$^4$ and $\text{Li}_D^2$, dubbed the “D-logarithm” is a certain (formal) power series with coefficients in $K$ that depend $n_d$, and its images under the Galois group of the extension $K/\mathbb{Q}$. More precisely, the definition of the D-logarithm given in [4] depended on the lifting mod $p^2$ of the Frobenius automorphism at each unramified prime $p$, so in that sense the coefficients $n_d$ (though integer for any choice of lifting) depend on several infinities of choices, and would not appear as true geometric invariants. The noteworthy exception is provided by abelian extensions, where $\text{Li}_D^2$ could be taken to be the di-logarithm twisted by a Dirichlet character $\chi$, of the form

$$\sum \chi(k) \frac{q^k}{k^2}$$

(6.2)

As will be clear, this can be rewritten as an integral linear combination of ordinary di-logarithms evaluated at appropriate roots of unity, which can therefore be viewed as a canonical basis in which to decompose the superpotential. It remains rather unclear at this point whether such a basis exists also for general extensions with non-abelian Galois group.

On the other hand, however, one may formulate the integrality statement of [4] without explicitly referring to any “D-logarithm”. Moreover, the proofs of [15] can rather straightforwardly be adapted to prove that integrality statement as well. We will explain this in the forthcoming paper [5].

Finally, and this is most relevant given with respect to the present note, it turns out that the framing transformation can also be defined, and preserves integrality (in the algebraic sense), for 2-functions with coefficients in an arbitrary number field [5].

$^4$Recall that an algebraic number $x$ can be identified with a root of a polynomial $P(x) \in \mathbb{Q}[x]$. If $P$ has coefficients in $\mathbb{Z}$, leading coefficient 1, and is irreducible, then $x$ is an algebraic integer.
6.2 Multi-variable case

As we have pointed out before, if $V$ is an $s$-function (for $s > 2$), then $W_V = \delta^{s-2}V$ is a 2-function, and its framed version $\tilde{W}_V = \delta^{s-2}\tilde{V}$ is also a 2-function. It is a natural question to ask whether this 2-function also comes from an $s$-function, namely whether there exists an $s$-function $\tilde{V}$ such that $\tilde{W}_V = \delta^{s-2}\tilde{V}$. It is not hard to see that really this is not the case (for instance, framing $\text{Li}_s$ for $s > 3$ in this way returns at most a 3-function). Thus, among $s$-functions for other values of $s$, 2-functions are distinguished by the integrality of framing.

Perhaps the most direct way to see that framing naturally only makes sense for 2-functions is to consider the generalization to the multi-variable situation. With rational coefficients, we say, as before, that a formal power series $W \in \mathbb{Q}[[z_1, \ldots, z_r]]$ is a 2-function if it can be written as an integral linear combination of di-logarithms,

$$W(z_1, \ldots, z_r) = \sum_{d_1, \ldots, d_r} n_{d_1, \ldots, d_r} \text{Li}_2(z_1^{d_1} \cdot \cdots \cdot z_r^{d_r})$$

(6.3)

Defining $\delta_i \equiv \frac{d}{d \ln z_i}$, and following eq. (1.3), we introduce

$$Y_i = \exp(\delta_i W)$$

(6.4)

Since the $\delta_i W$ are 1-functions, the $Y_i$ naturally have integer coefficients. An interesting distinction from the one-variable case is that we find an additional degree of freedom when identifying the “framed” variables $\tilde{z}_i$ with $Y_i$, in analogy to (1.4). Namely, say $(\kappa^{ij})_{i,j=1,\ldots,r}$ is a symmetric matrix with integer coefficients. Define $\sigma_i = (-1)^{\kappa_{ii}}$, and

$$\tilde{z}_i = \sigma_i z_i \prod_{j=1}^r Y_i^{\kappa_{ij}} = \sigma_i z_i \exp(\kappa^{ij} \delta_j W)$$

(6.5)

We may invert this relation as before, and upon writing

$$z_i = \sigma_i \tilde{z}_i \exp(\kappa^{ij} \delta_j \tilde{W})$$

(6.6)

we find that $\tilde{W} \in \mathbb{Q}[[\tilde{z}_1, \ldots, \tilde{z}_r]]$ is also a 2-function. This assertion can be proved rather straightforwardly by realizing the multi-dimensional operation as a combination of elementary one-dimensional framing, leading to the identification of the group of framing transformations with the additive group of symmetric integral matrices.

Now it is clear that if we had started from a multi-variable $s$-function with $s > 2$, we would in (6.4) have obtained more $Y$’s from multi-derivatives than variables, so the identification would not be one-to-one. Thus, again, $s = 2$ is special.
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