Complexity of MLDP

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Abstract

We carry out an explicit examination of the NP-hardness of a bi-objective optimization problem to minimize distance and latency of a single-vehicle route designed to serve a set of client requests. In addition to being a Hamiltonian cycle the route is to minimize the traveled distance of the vehicle as well as the total waiting time of the clients along the route.

1 Introduction

MLDP consists in finding a route to visit all clients, starting from and returning to a fixed depot, while minimizing the total latency at the clients and the total traveled distance. In this work, we limit to a single vehicle with no capacity restrictions that visits all of the clients in a single tour. The problem is a combination of the Minimum Latency Problem (MLP) (Blum et al., 1994; Chaudhuri et al., 2003; García et al., 2002; Lucena, 1990; Nagarajan and Ravi, 2008) and the Traveling Salesman Problem (TSP) (Applegate et al., 2007; Gutin and Punnen, 2006; Hoffman et al., 2013; Laporte, 1992).

In this work, we establish the NP-hardness of the problem, so as not to further contribute to the prevailing trend in related literature of omitting this step entirely and assuming that anything that resembles something NP-hard is also NP-hard, without any need for formal examination.

2 Formulation of MLDP

Consider a set of vertices $V = \{v_0, v_1, v_2, \ldots, v_n\}$, where $v_0$ is the depot and the other $n$ are clients. A matrix $T$ contains $\forall(i, j)$ the travel time from $i$ to $j$, $t_{i,j} \geq 0$, whereas $t_{i,i}$ represent the service time at client $i$; we set $t_{0,0} = 0$ and allow $t_{i,j} \neq t_{j,i}$ A cost matrix $C$ is given by

$$c_{i,j} = t_{i,i} + t_{i,j}. \quad (1)$$
Inspired by Picard and Queyranne (1978), we use variables $x_{i,k}$ to indicate that vertex $i$ is the $k$th vertex to visit along the route; we refer to the set of these binary variables as $X$. The travel distance from the depot to the first client is

$$T_0 = \sum_{i=1}^{n} c_{0,i}x_{i,1}, \quad (2)$$

the travel distance from the last client to the depot is

$$T_n = \sum_{i=1}^{n} c_{i,0}x_{i,n}, \quad (3)$$

for which the total travel distance is

$$T = T_0 + T_n + \sum_{k=1}^{n-1} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j}f_{i,j}(k). \quad (4)$$

The clients wait for the agent to reach and serve all the previously visited clients

$$W_0 = n \sum_{i=1}^{n} c_{0,i}x_{i,1}, \quad (5)$$

so the total waiting time (i.e., latency) is

$$W = W_0 + \sum_{k=1}^{n-1} \sum_{i=1}^{n} \sum_{j=1}^{n} (n - k)c_{i,j}f_{i,j}(k). \quad (6)$$

Returning from the last client to the depot is not considered part of the total latency of the clients, although it is included in Equation (4) for the total travel time.

Solving MLDP requires finding an assignment to $X$ that minimizes the objectives (Equations (1) and (5)) with equal importance and satisfies the restrictions of visiting each client exactly once and visiting a client at each step of the route (i.e., that the $x_{i,k}$ form a valid permutation of the vertex set).

## 3 NP-hardness of MLDP

The goal of an optimization problem $O_\sigma$ is to find the best solution according to a set of constraints that describes the environment of the problem. Each optimization problem is associated to a decision problem $D_\sigma$ which differs from $O_\sigma$ in the sense that the objective is treated as a constraint. Given a bound $B$ for the constraint representing the objective of $O_\sigma$, the answer to the decision problem is whether or not an assignment exists that satisfies both the original constraints and the new constraint with bound $B$. Hence the decision problem $D_\sigma$ has only two outcomes: yes or no. If the decision problem $D_\sigma$
is proven NP-complete, then the optimization problem $O$ is proven NP-hard (Garey and Johnson, 1990; Papadimitriou, 1994).

A complexity class is defined by several parameters such as non-determinism and restrictions on the amount of computational resources available (namely time and space) (Papadimitriou, 1994). NP stands for non-deterministic polynomial time meaning that the class is defined by bounding the execution time polynomially under non-deterministic computation: given an input for a problem $D$, an “oracle” can guess a correct solution in polynomial time. To prove the inclusion of any problem $D$ in the class NP it suffices to demonstrate that any given solution of $O$ can be verified as an actual solution of $O$ in polynomial time. Both of the single-objective optimization problems that are merged into MLDP are NP-hard (Afrati et al., 1986; Papadimitriou, 1994). In this section we demonstrate that also MLDP itself is NP-hard.

We first define a decision problem $D_{MLDP}$ associated to MLDP, then show that $D_{MLDP}$ belongs to NP, and finally establish that an efficient reduction from a known NP-complete problem to $D_{MLDP}$ exists (Garey and Johnson, 1990).

The corresponding decision problem is the following: given the costs (Equation (1)), an upper bound $T \leq \Theta$ to Equation (4), and an upper bound $W \leq \Omega$ to Equation (6), is there an assignment $X$ that satisfies all the original constraints as well as the upper bounds set on the objectives?

When service times and travel times are both constant, the problem is trivial, as any visit order will be optimal. When service times are a constant but travel times are arbitrary non-negative values, the problem is NP-hard, as it is now simply a TSP instance. The case of constant travel times and arbitrary non-negative service times is an MLP instance. In this section, we detail the case that both the service and the travel times are arbitrary.

The general MLDP consists in finding a route which visits all clients, leaves from a established depot and returns to it. All this while minimizing the total latency of all the clients and the total traveled distance of an uncapacitated vehicle. We assume this agent takes an arbitrary time $t_{ij}$ to reach every client and that all service times $s_i$ are also arbitrary.

We first establish that the feasibility of a given $X$ can be verified in polynomial-time: each $X$ captures a permutation that starts at the depot and the permutation can be efficiently recovered from the assignment matrix $X$ as a visit sequence $v^{(0)}, v^{(1)}, \ldots, v^{(n)}$ where $v^{(0)}$ is always the depot and the visits from $v^1$ to $v^{(n)}$ correspond to the clients.

We need three accumulator variables, one to measure time along the route, one to count the total travel time, and the third to count the total latency. We also need an array of $n$ binary decision variables, one per each client, to verify that each one is properly visited. We proceed in the visiting order, denoting the source by $i$ and the destination by $j$ adding the value of $t_{i,j}$ to both the time accumulator and the travel-time accumulator, then adding the service time $s_i$ to the time accumulator, and then the current value of the time accumulator to the latency accumulator, making the client $j$ as visited. If at the end of the permutation, all $n$ binary variables are one, the travel-time accumulator respects its upper bound $\Theta$, and the latency accumulator respects its upper bound $\Omega$,
the solution is feasible. As the verification is a polynomial procedure, $D_{\text{MLDP}} \in \text{NP}$.

To establish NP-completeness, we reduce the traveling salesman problem (TSP) in its decision version to $D_{\text{MLDP}}$, as illustrated in Figure 1. To prove the complexity of MLDP, a reduction algorithm must transform a TSP entry into a MLDP one. This MLDP input is then solved and receives a yes or no answer. As a whole, the process takes a TSP instance, reduces it into an MLDP instance, solves the reduced instance with an algorithm for MLDP, and responds correctly “yes” or “no” to the original instance.

In the decision version of TSP, the input is a cost matrix $C$ for the travel costs between $n$ clients (with the diagonal elements being zero) together with an upper bound to the total cost of the tour, $C$, and the question is whether a permutation over the set of clients exists that produces a tour with the sum of costs of the segments not exceeding $C$.

In order to transform the input for the decision version of the TSP into an input for $D_{\text{MLDP}}$, we will use the elements of $C$ as $\mathcal{T}$, set $\Theta = C$, and only need to compute efficiently an adequate value for the latency bound $\Omega$. We do this in terms of the worst-case costs, computing from the cost matrix $C$ the largest cost per each row (that is, as if the agent at each client chose the most expensive segment to continue the tour) in $O(n^2)$ time, and then sort these worst-case segments from largest to smallest in $O(n \log n)$ time to construct an upper bound to the worst-case latency (the service times in TSP are the diagonal elements of the cost matrix that are all zero): the worst-case total latency is the sum of the cumulative sums of the sorted list of costs and we use this as $\Omega$.

Hence, $D_{\text{MLDP}}$ responds “yes” to the transformed input if and only if the decision version of TSP would respond “yes” for the original input.

By reducing TSP to MLDP, we show that MLDP is at least as difficult as TSP. Thus, $D_{\text{MLDP}}$ is NP-complete and therefore MLDP is NP-hard.
4 Conclusions

We proved the NP-completeness of a decision problem corresponding to a bi-objective problem that combines the Minimum Latency Problem (MLP) and the Traveling Salesman Problem (TSP).

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