FIREBALLS, FLARES, AND FlickERING: A SEMIANALYTIC, LTE, EXPLOSIVE MODEL FROM ACCRETION DISKS TO supernovae

K. J. Pearson
Department of Physics and Astronomy, Louisiana State University, Nicholson Hall, Baton Rouge, LA 70803-4001

Keith Horne
School of Physics and Astronomy, University of St. Andrews, North Haugh, St. Andrews, Fife KY16 9SS, UK

and

Warren Skidmore
California Institute of Technology, Mail Code 105-24, Pasadena, CA 91125-24

Received 2004 July 14; accepted 2004 September 22

ABSTRACT

We derive simple analytic expressions for the continuum light curves and spectra of flaring and flickering events that occur over a wide range of astrophysical systems. We compare these results to data taken from the cataclysmic variable SS Cyg and also from SN 1987A, deriving physical parameters for the material involved. Fits to the data indicate a nearly time-independent photospheric temperature arising from the strong temperature dependence of opacity when hydrogen is partially ionized.

Subject headings: accretion, accretion disks — binaries: close — novae, cataclysmic variables — radiative transfer — stars: individual (SS Cygni) — supernovae: individual (SN 1987A)

1. INTRODUCTION

In a recent paper (Pearson et al. 2003) we explained the unusual flaring activity of the cataclysmic variable (CV) system AE Aqr in terms of the aftermath of the collision between two gas clouds. We modeled the resulting fireball numerically, comparing the results to analytic approximations for the optical light curves and continuum spectra and to observed light curves and spectra. We set out here an improved analytic calculation for an expanding fireball with an LTE ionization structure and compare our results to more detailed numerical simulations. We fit our results to multiwavelength light curves of two very different systems, deriving values for the physical conditions in them.

The process of flickering and flaring occurs across the whole range of astrophysical systems from stars to active galaxies. It appears to be a recurrent feature of accreting systems and, in particular, those where an accretion disk is present. This flickering process may well be associated with the anomalously high viscosity present in these systems and represent the effect of magnetic reconnections such as that from the viscosity mechanisms of Hawley & Balbus (1991). Such a sudden localized deposition of energy would raise the gas temperature and the consequent overpressure would give rise to a local expansion of the disk material. With material cooling adiabatically, this expansion would rapidly become supersonic.

Observations by Bruch (2000) suggest that the flickering in CVs is not uniformly distributed across the disk but instead is associated with the stream-disk impact point and the innermost boundary layer region of the disk. Similarly, Patterson (1981) found that the flickering in HT Cas was associated with the inner part of the accretion disk. These findings, however, are contradicted by Welsh & Wood (1995) for HT Cas, by Baptista & Bortoletto (2004) for V2051 Oph, and by Baptista et al. (2002) for the low-mass X-ray binary (LMXB) X1822−371. All these studies show flickering to arise from a range of disk radii.

The terms “flickering” and “flaring” are often used interchangeably when describing the stochastic variability of accreting sources and, even when defined in a particular field, are not necessarily used consistently. Warner (1995) and Baptista & Bortoletto (2004) describe flickering in CVs as the continuous, random brightness fluctuations of 0.01–1 mag on timescales of seconds to dozens of minutes, although the exact numerical ranges differ between the two. In contrast, the unusual CV AE Aqr is universally described as a “flaring” source since it has rarer but brighter events that often occur in batches. These events can raise the total optical luminosity of the system by factors of 2–3, contrasting with the 5%–20% typical of CV flickering (Bruch 1992) but on a similar timescale. Consequently, we adopt the convention of describing the small-amplitude, continuous variations as “flickering” and reserve the term “flaring” for larger scale events.

The underlying model used to reproduce the AE Aqr light curves was based on consideration of the flux emitted by a hot, spherically symmetric ball of gas as it expanded and cooled. While we saw that the “expanding fireball” situation in AE Aqr could arise from collisional heating of gas blobs, the same situation might arise from the very different causes outlined above. As a result, these fireball models may have a much wider range of applicability than simply the unusual behavior of AE Aqr. Accordingly, in this paper we compare analytic expressions derived for the light curves of such expansions to observations of SS Cygni and SN 1987A.

SS Cyg is a member of the dwarf nova subclass of CVs. It consists of a 1.2 $M_\odot$ white dwarf accreting material through an accretion disk from a 0.71 $M_\odot$ K5 V secondary that loses material via Roche lobe overflow (Ritter & Kolb 2003). SS Cyg was the second member of this subclass identified (Wells 1896) and has had a near continuously monitored light curve for over a century (Warner 1995). Like all dwarf novae, SS Cyg exhibits optical variability over a large range of timescales, e.g., outbursts ($\Delta m \approx 3.5$ mag, $t \approx 40$ days), orbital modulations ($\Delta m \approx 0.5$ mag, $P = 0.275130$ days), flickering ($\Delta m = 0.01–0.2$ mag,
radioactive decay of unstable nuclei in the ejecta. It was a Type II supernova with an identified progenitor (Sk 69°202) that was a blue supergiant. It is believed that it lasted \( \sim 10^7 \) yr as a 20–30 \( M_\odot \) main-sequence blue supergiant star, before becoming a red supergiant for \( \sim 10^6 \) yr, during which phase it lost 3–6 \( M_\odot \), and finally becoming a blue supergiant again for a few thousand years (McCray 1993). The rebrightening “bump” phase arises from a photosphere moving out with expelled material and eventually reversing as the material becomes optically thin again. It is interesting to note that the photosphere maintained a roughly constant temperature \( \sim 6000 \) K close to the recombination temperature for hydrogen at the expected densities in the expanding shell. The models of AE Aqr flares in Pearson et al. (2003) showed a similar isothermal behavior.

2. MODEL ASSUMPTIONS

We briefly recap here the dynamical model developed in Pearson et al. (2003), but the interested reader is referred to that paper for a more detailed derivation. Let us assume a spherically symmetric expansion of a Gaussian density profile with radial velocity proportional to the distance from the center of the expansion. We define

\[
\eta \equiv \frac{r}{a} \tag{1}
\]

as a dimensionless measure of the radius \( r \) in terms of the current length scale of the Gaussian \( a \). The expansion factor \( \beta \) acts as a dimensionless time, being the constant of proportionality between the current and fiducial scale length \( a_0 \). Hence,

\[
\beta \equiv \frac{a}{a_0} \tag{2}
\]

consistent with our above assumptions regarding velocity and implying

\[
\beta = 1 + H(t - t_0), \tag{3}
\]

where \( t_0 \) is the time at which all the fiducial values are determined and \( H \) is an “expansion constant” setting the speed of the expansion. For simplicity, we consider only the case of uniform three-dimensional expansion, avoiding factors such as the angle of the observer to lines of symmetry. We also restrict ourselves to a spatially uniform temperature distribution since, for example, a power-law distribution gives a result for the later integration in equation (22) in terms of the hypergeometric function \( _2F_1 \). We assume a power law with index \( \Gamma \) for the temporal dependence of temperature. Here \( \Gamma = 0 \) corresponds to an isothermal expansion and \( \Gamma = 2 \) to the adiabatic case.

In summary then, we have

\[
T = T_0 \beta^{-\Gamma}, \tag{4}
\]

\[
\rho = \rho_0 \beta^{-3} e^{-\eta^2}, \tag{5}
\]

where

\[
\rho_0 = \frac{M}{\left(\pi a_0^2\right)^{3/2}} \tag{6}
\]

and \( M \) is the total mass involved in the expansion.

Using \( v(r) = H/r = H/r^{1/2} \), we can integrate \( \frac{1}{2} \rho v^2 \) over all space to derive the total kinetic energy of the expansion

\[
E_{\text{kin}} = \frac{2}{\sqrt{\pi}} M_0^2 H^2 \int_0^\infty \eta^2 e^{-\eta^2} d\eta \tag{7}
\]

\[
= \frac{3}{4} M_0^2 H^2 \tag{8}
\]

\[
= \frac{3}{4} M v_0^2, \tag{9}
\]

where \( v_0 = H_0 \) is the speed of expansion at \( r = a \).

3. THEORETICAL LIGHT CURVES AND SPECTRAL DISTRIBUTIONS

The radiative transfer equation has a formal solution under conditions of LTE

\[
I = \int_0^\infty B e^{-\tau} d\tau, \tag{10}
\]

where \( I \) is the intensity of the emerging radiation, \( B \) is the Planck function, and \( \tau \) is the optical depth measured along the line of sight from the observer. For cases in which the source function is everywhere the same, this becomes

\[
I = B(1 - e^{-\tau}). \tag{11}
\]

We define \( x \) as the distance from the fireball center toward the observer and \( y \) as the distance perpendicular to the line of sight. The above line integral (eq. [11]) gives the intensity \( I(y) \) for lines of sight with different impact parameters \( y \). The fireball flux, obtained by summing intensities weighted by the solid angles of annuli on the sky, is then

\[
f(\lambda) = \int_0^\infty I(y) \frac{2\pi y}{d^2} dy, \tag{12}
\]

where \( d \) is the source distance.

We can calculate the evolution of the continuum light curve using expressions for the linear absorption coefficient. The free-free absorption coefficient (per unit distance) can be written as

\[
\kappa_{\text{ff}} = \kappa_0 \left(1 - e^{-(h\nu/kT)}\right)^{-1/2} \nu^{-3} n_e n_i, \tag{13}
\]

where

\[
\kappa_0 = 3.692 \times 10^{-2} Q^2 g_{\text{ff}}(\text{SI}) \tag{14}
\]

(Keady & Kilcrease 2000), \( g_{\text{ff}} \) is the free-free Gaunt factor (Gaunt 1930), and \( Q \) is the atomic charge. We wish to retain the explicit frequency dependence but otherwise follow a parallel derivation as for Kramer’s opacity. For a mixed elemental composition, then, we sum over all species (assumed fully ionized)

\[
\sum_{\text{all ions}} Q_i^2 n_i g_{\text{ff}} \approx \frac{\rho}{m_1} (1 - Z) g_{\text{ff}} \tag{15}
\]
Here \( \bar{g}_f \) is a mean Gaunt factor (close to unity) and \( Z \) is the metal mass fraction. We also have

\[
n_e = \frac{\rho}{\mu m_\text{H}} = \frac{\rho(1 + X)}{2m_\text{H}}
\]  

(Bowers & Deeming 1984), where \( X \) is the hydrogen mass fraction. Combining and writing the correction for stimulated emission as

\[
\epsilon = 1 - e^{-(h\nu/kT)}
\]

(17)
gives

\[
\kappa_\text{eff} = \kappa_1 \epsilon \rho^2 T^{-1/2} \nu^{-3},
\]

(18)
where we have defined

\[
\kappa_1 = 6.695 \times 10^{51}(1 - Z)(1 + X)\bar{g}_f (\text{SI}).
\]

Inserting our density profile (eq. [5]), we arrive at the result

\[
\kappa_\text{eff} = \frac{\kappa_1 \epsilon}{T^{1/2} \nu^3} \frac{M^2}{\pi^2 \sigma^6} e^{-2\eta^2}.
\]

(20)

On dimensional grounds if on no other, a similar expression to equation (18) and hence equation (20) must also exist for bound-free opacity when it dominates (when most species are fully recombined). Textbook derivations of Kramers’s opacity (e.g., Bowers & Deeming 1984) show us

\[
\kappa_{\text{bf}} \propto Z(1 + X)\bar{g}_{\text{bf}},
\]

(21)
where \( \bar{g}_{\text{bf}} \) is the bound-free Gaunt factor (Gaunt 1930). Useful tables for \( \bar{g}_{\text{bf}} \) have been calculated by Glasco & Zirin (1964). The constant of proportionality, however, is more difficult to determine than for free-free, since it must account for the ionization edges in the absorption and, in particular, the change of energy level populations with temperature for each ion. We assume that a relation analogous to equation (18) also holds for the situation of mixed ionized and recombined species where both forms of opacity contribute.

The optical depth parallel to the observer’s line of sight is

\[
\tau(y) = -\int_{-\infty}^{y} \kappa \, dx
\]

\[= \left( \frac{\kappa_1 \epsilon}{T^{1/2} \nu^3} \frac{M^2}{\pi^2 \sigma^6} \right) e^{-2(y/a)^2} \int_{-\infty}^{\infty} e^{-2(y/a)^2} d \left( \frac{X}{a} \right)
\]

\[= \tau_0 e^{-2(y/a)^2},
\]

(24)
where the optical depth on the line of sight through the center of the fireball is

\[
\tau_0 = \left( \frac{\beta_e}{\beta} \right)^{10 \to -1/2}
\]

(25)
and the time at which the fireball becomes optically thin along this line of sight is

\[
\beta_c \equiv \left( \frac{\kappa_1 \epsilon}{T^{1/2} \nu^3} \frac{M^2}{\pi^2 \sigma^6} \right)^{1/10 \to -1/2}
\]

(26)

It should be noted that in pulling \( \kappa_1(\mu, \mu_0) \) out of the integral given by equation (23) we have implicitly assumed that the spatial variation of the ionization fraction has negligible impact on the behavior. This is an assumption that we return to later.

In Pearson et al. (2003) we approximated the flux received from the fireball by splitting it into two components as viewed on the plane of the sky: an optically thick central region bounded by \( y = y_m \) (see eq. [44]) and an optically thin surrounding. Here, however, we calculate the integral exactly. Combining equations (11), (12), and (24) with a change of integration variable to \( u = \tau(y) \), we have

\[
f = \frac{\pi a^2 B}{2d^2} \int_{0}^{\tau_0} \frac{1 - e^{-u}}{u} \, du.
\]

(27)
Expressing the exponential in its series form, we integrate to find

\[
f = \frac{\pi a^2 B}{2d^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tau_0^n}{n!} ,
\]

(28)
which can also be rewritten in the form of standard functions

\[
f = \frac{\pi a^2 B}{2d^2} \left[ E_1(\tau_0) + \gamma + \ln(\tau_0) \right],
\]

(29)
where \( E_1 \) is the first-order exponential integral and \( \gamma \approx 0.577216 \) is Euler’s gamma constant (Abramowitz & Stegun 1972; Jeffreys & Jeffreys 1956; Press et al. 1992). This has the form of

\[
f = \Omega BS(\tau_0),
\]

(30)
where \( \Omega = \pi a^2/2d^2 \) is the solid angle subtended by the current standard deviation \( (a/\sqrt{2}) \) of the Gaussian density profile and the “saturation function”

\[
S(\tau_0) = [E_1(\tau_0) + \gamma + \ln(\tau_0)]
\]

(31)
is plotted in Figure 1. We note the asymptotic limits

\[
S(\tau) \approx \begin{cases} \tau, & \tau \ll 1, \\ \gamma + \ln \tau, & \tau \gg 1. \end{cases}
\]

(32)
The intensity of emitted radiation as a function of impact parameter is plotted for different times in Figure 2. We can see how the flux saturates at the blackbody function for highly optically thick impact parameters. In comoving coordinates this region gradually shrinks and at later times the entire emission becomes optically thin.

The total flux given by equation (29) is plotted against time in Figure 3 for different values of \( \Gamma \), using \( \beta_c = 1 \). Figure 4 shows light curves at different wavelengths assuming \( \beta_c = 1 \) and \( T_0 = 15,000 \) K. The y-axis is again parameterized in multiples of \( f_0 = (\pi a_0/2d^2)B_0(5000 \text{ Å}, 15,000 \text{ K}) \). The optically thick contribution at 5000 Å is plotted by a dashed line and the optically thin contribution by a dot-dashed line.

Recalling that \( a = \beta a_0 \), for the special case of \( \Gamma = 0 \) (isothermal expansion) we can neglect the time variation of the Planck function and differentiate equation (29) with respect to \( \beta \) to find a condition for the maximum flux

\[
4[E_1(\tau_0) + \gamma + \ln(\tau_0)] - (10 - \Gamma)(1 - e^{-\tau_0}) = 0. \tag{33}
\]

This can be solved numerically to find \( \tau_{0,\text{pk}} = 6.8204 \). From equation (25) we then find \( \beta_{\text{pk}} = 0.6811 \beta_c \). We can also differentiate equation (29) with respect to \( \beta \) for \( \Gamma \neq 0 \) although the form is much more untidy and we must solve for \( \beta \) directly. We plot \( \beta_{\text{pk}} \) against \( \Gamma \) in Figure 5 continuing to neglect any time dependence of \( \epsilon \). We plot \( \beta_{\text{pk}} \) against wavelength in Figure 6 for several values of \( \Gamma \).

Bruch (1992) found a correlation between the amplitude of flickering in different wavebands for several CVs (see inter alia his Fig. 2). Such a correlation could arise from an isothermal evolution at a consistent temperature. This can be understood

\[
\text{Fig. 2.—Comparison of the intensity profiles at different times calculated using eqs. (24) and (11) for a } \Gamma = 0 \text{ (isothermal) evolution.}
\]

\[
\text{Fig. 3.—Temporal behavior of the total flux at 5000 Å for different cooling indices } \Gamma = -2, -1, 0, 1, \text{ and } 2, \text{ assuming } \beta_c = 1 \text{ and } T_0 = 15,000 \text{ K. The y-axis is parameterized in multiples of } f_0 = (\pi a_0/2d^2)B_0(5000 \text{ Å}, 15,000 \text{ K}).
\]

\[
\text{Fig. 4.—Temporal behavior of the total flux at several wavelengths assuming } \beta_c = (\lambda/5000 \text{ Å})^{\gamma/5}, T_0 = 15,000 \text{ K, and } \Gamma = 0. \text{ The y-axis is again parameterized in multiples of } f_0. \text{ The optically thick contribution at 5000 Å is plotted by a dashed line and the optically thin contribution by a dot-dashed line.}
\]

\[
\text{Fig. 5.—Time of peak flux at 5000 Å plotted against } \Gamma, \text{ assuming } \beta_c = 1 \text{ and } T_0 = 15,000 \text{ K.}
\]
from equation (29). Since the peak flux occurs at the same central optical depth independent of wavelength, then for two wavelengths $\lambda_1$ and $\lambda_2$ the peak fluxes are related by

$$ \frac{f_{pk,1}}{f_{pk,2}} = \left( \frac{\beta_{\lambda,1}}{\beta_{\lambda,2}} \right)^2 \frac{B_{\lambda,1}}{B_{\lambda,2}}. $$

(34)

$$ = \left( \frac{\beta_{\lambda,1}}{\beta_{\lambda,2}} \right)^2 \left( \frac{\lambda_1}{\lambda_2} \right)^{2/5} \left( \frac{e^{c_2/\lambda_1 T_0} - 1}{e^{c_2/\lambda_2 T_0} - 1} \right)^{2/5} \left( \frac{1 - e^{-c_2/\lambda_1 T_0}}{1 - e^{-c_2/\lambda_2 T_0}} \right)^2, $$

(35)

where $c_2 = hc/k$ and assuming no complications such as a Balmer edge between them. We plot the predicted values for different temperatures as stars, alongside the data taken from Table 4 of Bruch (1992), in a color-color diagram in Figure 7.

(note that we used $B_1$ rather than $B_\nu$ in the above derivation purely for comparison with this data set).

This figure shows that a simple application of our model enables us to reproduce the observed range of $V - B$ reasonably well using $T$ as a free parameter. The $U - B$ color, however, is generally underpredicted. This results from not making allowance for the increased opacity above the Balmer jump when evaluating $\beta_\nu$ in deriving equation (36). The size of the Balmer jump depends on a nontrivial combination of mass, length scale, and temperature to produce an effective value of $\kappa_1$. Any individual system may produce flickers with a range of physical parameters causing it to produce points scattered across this diagram.

Using numerical methods outlined later, we derived values for $\kappa_1$ for several temperatures over a range of densities. These are plotted in Figure 8. We can see how, for a large range of temperatures of interest and for densities ranging over several orders of magnitude, $\kappa_1$ is well represented by a constant value. At lower temperatures the free-free opacity transitions to bound-free opacity at lower densities than it does at higher temperatures. At these lower temperatures we could introduce significant errors if we estimate $\kappa_1$ from a position that has one form of opacity when the other is in fact dominant. It is very unlikely that this would be the case, however, if we estimate the ionization state from conditions at a suitable position and consider how $\kappa_1$ smoothly changes with the ionization. Qualitatively then, at the very least, we can be confident of the ability of the analysis to allow us to examine the behavior of the proposed mechanism and to predict fluxes to within factors of order unity.

4. ANALYSIS

Equation (29) gives us the complete temporal and frequency behavior for the emergent flux in terms of $T_0$. We must use the Planck function appropriate to the time-dependent behavior of the temperature from equation (4). Equation (25) gives us $\tau_\nu$ as a function of $\beta$ and $\beta_\nu$. The behavior of the flux thus rests on the character of the parameter $\beta_\nu$. We know functionally that

$$ \beta_\nu = \beta_\nu(\kappa_1, \epsilon, \nu^{-3}), $$

(37)

$$ \kappa_1 = \kappa_1(\epsilon, \nu, \beta), $$

(38)

$$ \epsilon = \epsilon(\nu, \beta), $$

(39)
and we examine each of these dependencies and appropriate levels of approximation in each case.

4.1. Global Behavior $\beta_\epsilon = \text{const}$

In examining the behavior of the flux as a function of time, it is instructive to restrict ourselves initially to a single wavelength and ignore spatial or time dependency in $\beta_\epsilon$.

We see from equations (25) and (28) that the late-time behavior ($kT \ll h\nu$) of the light curve is determined by the temperature index $\Gamma$ such that $f \sim \beta^{-5}$ or $\beta^{-3}$ for isothermal or adiabatic cooling, respectively.

4.2. Spatial Dependence $\beta_\epsilon = \beta_\epsilon(\eta)$

4.2.1. One-Zone Model

We know that $\beta_\epsilon$ depends on $\kappa_1$, which in turn depends on the ionization structure across the expansion. At the outset we acknowledge that a detailed solution accounting for the changing ionization fraction across the density profile would require numerical integration such as that reported in Pearson et al. (2003) for AE Aqr. We are looking for simpler, more easily calculable approximations that avoid such detailed methods. To this end, we do not include here any explicit allowance for the variation of ionization states across the profile; instead, we approximate the behavior by calculating the conditions at a suitable point within the expansion. With a Gaussian density profile we might expect the flux to be dominated by the region close to the photosphere. As a result, we could approximate the ionization structure across the profile; instead, we approximate approximations that avoid such detailed methods. To this end, we must iteratively solve the network of Saha equations for $\Delta E Aqr$. We are looking for simpler, more easily calculable numerical integration such as that reported in Pearson et al. (2003) ionization fraction across the density profile would require numerical integration such as that reported in Pearson et al. (2003) for AE Aqr. We are looking for simpler, more easily calculable approximations that avoid such detailed methods. To this end, we do not include here any explicit allowance for the variation of ionization states across the profile; instead, we approximate the behavior by calculating the conditions at a suitable point within the expansion. With a Gaussian density profile we might expect the flux to be dominated by the region close to the photosphere. As a result, we could approximate the ionization structure across the profile; instead, we approximate approximations that avoid such detailed methods. To this end, we must iteratively solve the network of Saha equations for $\Delta E Aqr$. We are looking for simpler, more easily calculable approximations that avoid such detailed methods. To this end, we do not include here any explicit allowance for the variation of ionization states across the profile; instead, we approximate the behavior by calculating the conditions at a suitable point within the expansion. With a Gaussian density profile we might expect the flux to be dominated by the region close to the photosphere. As a result, we could approximate the ionization structure across the profile; instead, we approximate approximations that avoid such detailed methods. To this end, we must iteratively solve the network of Saha equations for $\Delta E Aqr$. We are looking for simpler, more easily calculable approximations that avoid such detailed methods. To this end, we do not include here any explicit allowance for the variation of ionization states across the profile; instead, we approximate the behavior by calculating the conditions at a suitable point within the expansion. With a Gaussian density profile we might expect the flux to be dominated by the region close to the photosphere. As a result, we could approximate the ionization structure across the profile; instead, we approximate approximations that avoid such detailed methods. To this end, we must iteratively solve the network of Saha equations for $\Delta E Aqr$. We are looking for simpler, more easily calculable approximations that avoid such detailed methods.

4.2.2. Pure Hydrogen Case—Semianalytic Solution

For simplicity, let us consider a fireball with almost pure hydrogen composition (note that if $Z = 0$, then eq. [21] implies $\kappa_{1,\text{br}} = 0$), with an ionization fraction

$$
\iota \equiv \frac{n_i}{n_i + n_n} = \frac{n_i}{n},
$$

which, setting $\tau(y) = 1$ for the special case $x' = x_{ph}$, gives

$$
e^{2(y/a)^2} = \frac{1}{2} \tau_0 \text{erfc} \left( \frac{\sqrt{2} x_{ph}}{a} \right).
$$

This defines the locus of points on the photospheric surface. Since the largest contribution to the opacity integral comes from the region of highest density, we introduce least error by evaluating $\kappa_1$ there. This will clearly occur along the central line of sight $y = 0$; hence, we arrive at

$$
\text{erfc} \left( \sqrt{2} \eta_{\text{ph}} \right) = \frac{2}{\tau_0},
$$

which we must solve numerically. However, we must remember that it is possible for the photospheric surface to lie behind the density peak. This will just occur when $\tau(0) = 1$ at $\eta = 0$, implying $\beta = 2^{1/(10-\Gamma)} / \beta_\epsilon$. Ultimately then, we have

$$
\rho_{\text{eval}} = \begin{cases} 
\rho_0 \beta^{-3} e^{-\eta_{\text{ph}}^2}, & \beta < 2^{1/(10-\Gamma)} / \beta_\epsilon, \\
\rho_0 \beta^{-3}, & \text{otherwise}.
\end{cases}
$$

It must be emphasized that we are using this position solely to evaluate a typical ionization state and, hence, to find a suitable approximate value for $\kappa_1$. The effect of the density profile on the approximate accuracy. This value can then be used to calculate the light-curve behavior at a given wavelength.

4.2.3. Mixed Composition

For the more realistic case of mixed composition, we cannot calculate the ionization state at a given position analytically. Instead, we must iteratively solve the network of Saha equations...
under the prevailing physical conditions to enable us to determine $\kappa_1$. This solution can then be iterated with $\beta_c$ around the equations (26) and (43) and $\kappa_1$ loop in a similar way as above.

4.2.4. Three-Zone Model

The above method works well in situations in which either free-free or bound-free opacity is dominating. Unfortunately, as we noted at the end of § 3, in the situation of mixed opacity sources we can introduce significant errors if we use a value for $\kappa_1$ assuming the incorrect source. We can rescue much of the above formalism, however, if we split the spatial profile into three zones. We are almost bound to improve our calculation regardless of where we place the boundaries of the zones, but clearly it makes sense to try to ensure that we have free-free-dominated (outer regions), bound-free-dominated (inner regions), and mixed (intermediate regions) zones. Assuming that hydrogen species provide the dominant opacity source, we select the boundary between these regions by the hydrogen ionization fractions $\epsilon = 0.1$ and 0.9, which occur at $\eta_{0.1}$ and $\eta_{0.9}$.

Specifically then, we need to replace the integral in equation (23) with one over the three types of zones and rework equation (26) to redefine $\beta_c$. Calculating $x_{0.1}$ and $x_{0.9}$ using $y = 0$ or $y_m$ as desired, the integral in equation (23) thus becomes

$$\int_{-\infty}^{\infty} \kappa_1 e^{-2(x/a)^2} d\left(\frac{x}{a}\right) = \frac{2}{a} \int_0^{x_{0.1}} \kappa_{1,ff} e^{-2(x/a)^2} \, dx$$

$$+ \frac{2}{a} \int_{x_{0.1}}^{x_{0.9}} \kappa_{1,ml} e^{-2(x/a)^2} \, dx$$

$$+ \frac{2}{a} \int_{x_{0.9}}^{\infty} \kappa_{1,rr} e^{-2(x/a)^2} \, dx$$

$$= 2\kappa_{1,ff} \text{erfc}\left(\sqrt{2\frac{x_{0.1}}{a}}\right)$$

$$+ 2\kappa_{1,ml} \left[ \text{erfc}\left(\sqrt{2\frac{x_{0.1}}{a}}\right) - \text{erfc}\left(\sqrt{2\frac{x_{0.9}}{a}}\right) \right]$$

$$+ 2\kappa_{1,rr} \left[ 1 - \text{erfc}\left(\sqrt{2\frac{x_{0.1}}{a}}\right) \right],$$

(52)

which we use to replace factor $\kappa_1(\pi/2)^{1/2}$ in equation (26).

The question remains of how to rapidly find values for $\eta_{0.1}$ and $\eta_{0.9}$. Again, in the spirit of finding an easily calculable approximation and noting that even inaccurately determining these boundaries will still improve our integral calculation, let us consider a situation in which all species other than hydrogen remain fully ionized. Using standard methods, we derive

$$\mu_c = \frac{2}{1 + X(2\epsilon - 1)}. \quad (54)$$

Incorporating this into equation (48) along with our density profile, we can show

$$\rho_0 e^{-\eta^2} = \frac{2m_1}{1 + (2\epsilon - 1)X} \frac{1 - \epsilon}{\epsilon} A(T), \quad (55)$$

which we can rearrange and solve directly for $\eta$. We note the particularly simple form this reduces to for $\epsilon = 0.5$.

For similar reasons to the single-zone model, we evaluate $\kappa_{0.1}$ and $\kappa_{0.9}$ at the point of highest density in their region (e.g., at either $\eta_{0.9}$ or $\eta_{\text{max}}$ for free-free). Given the rapid density variation in the mixed zone, we evaluate $\kappa_m$ at $\eta_{0.5}$ or $\eta_{\text{max}}$ as appropriate. The possible integration schemes are illustrated schematically in Figure 9.

4.3. Frequency Dependence $\beta_c = \beta_c(\eta, \nu)$

The parameter $\beta_c$ has a direct frequency dependence from the $\nu^{-3}$ in opacity and also dependence through both $\kappa_1$ and $\epsilon$. For regions where the Gaunt factors are slowly changing functions of frequency, e.g., along the Paschen continuum in the 4000–8000 Å range or when free-free opacity is dominating, we ignore the small error introduced by neglecting the contribution of $\eta_0$ and $\eta_{\text{ff}}$ to $\kappa_1$. Instead, we need only correct for the direct and $\epsilon$ frequency dependence of $\beta_c$. Thus,

$$\beta_c(\nu) \approx \beta_c(\nu_0) \left(\frac{\nu_0}{\nu}\right)^{6/(10-\Gamma)} \left(\frac{\epsilon}{\epsilon_0}\right)^{2/(10-\Gamma)}. \quad (56)$$

With this correction we have only to calculate $\beta_c$ accurately at a single wavelength with the iterative method outlined above. The light-curve behavior as a function of both wavelength and time then follows immediately from the combination of equations (56) and (29).

4.4. Time Dependence $\beta_c = \beta_c(\eta, \nu, \beta)$

The dependence of $\beta_c$ on time again comes through both $\kappa_1$ and $\epsilon$ (if $\Gamma \neq 0$). In principle, we can use a similar expression to equation (56) also to correct for the $\epsilon$ time dependence.
However, the exponential nature of the expression renders the form of the light curves more complex and less instructive than that derived above. As a result, this correction is probably best included only in numerical solutions. More seriously, we cannot, in general, predict the future ionization structure for a mixed composition gas from its current state. Thus, the time dependence of $\kappa_1$ can, in general, only be included with numerical solution at a series of different times $\beta$. The exception to this rule is when we can be sure that the dominant opacity is and will remain free-free throughout the time considered. In this case $\kappa_1$ is a constant in time and we can use the same rapid approximation as the previous section.

4.5. Comparison of Integral Methods

We can lift the restriction to the purely free-free opacity case by iterative numerical solution of the Saha equations for a gas of mixed composition at each time in the light curve. From the ionization profile of each species we can in principle determine the opacity at any point.

We compared the results achieved by the two integration lines of sight $y = y_m$ or 0 and using one- or three-zone integration schemes. The total continuum opacity was calculated numerically using the methods of Gray (1976) with the exception of H$^-$ bound-free (Geltman 1962) and free-free (Stilley & Callaway 1970), He$^-$ (McDowell et al. 1966), and He$^+$ bound-free (Huang 1948). Calculating $\beta_\iota$ from this opacity, we can iterate to a consistent solution at each time. Altering the number of zones used yielded virtually no discernible effect on the predicted light curves. For temperatures around 16,000 K, the two lines of sight considered produced results differing by at most around 0.5%. However, at lower temperatures the predicted fluxes could differ more, reaching around 5% at 10,000 K (see Figs. 10 and 11).

5. COMPARISON TO OBSERVATION—DEDUCING FIREBALL PARAMETERS

The model was fitted to light curves for the flickering in the dwarf nova SS Cyg and the “bump phase” of SN 1987A. We employed a $\chi^2$ minimization amoeba code (Press et al. 1992) to derive the best-fit values for $M$, $a_0$, $T_0$, $H$, and $\Gamma$. We use the approximation that the ionization fractions for all the species are well represented by the conditions at $x = 0$, $y = y_m$ and use the single-zone integration approach to arrive at a self-consistent solution for $\beta_\iota$ and $\kappa_1$ there at each time.

5.1. SS Cygni

Rapid spectroscopy of SS Cyg was carried out using the Low Resolution Imaging Spectrograph (LRIS; Oke et al. 1995) on the 10 m Keck II telescope on Mauna Kea, Hawaii, between 09:38 and 09:56 UT on 1998 July 6, covering orbital phases 0.8276–0.8715. The instrumental set and data reduction were the same as described by Steeghs et al. (2001), O’Brien et al. (2001), and Skidmore et al. (2003). A total of 14,309 spectra were obtained using the rapid data acquisition system. The spectra covered 3259–7985 Å with 2.4 Å pixel$^{-1}$ dispersion and a mean integration time of 72.075 ms and no dead time between individual spectra. Further details of these observations are given in W. Skidmore et al. (2005, in preparation). A particular flickering event was isolated between 9:39:10 and 9:43:44 UT and flux from other sources removed by fitting, for each wavelength, a low-order polynomial to the fluxes both before and after the event. Light curves were formed from the
Fig. 11.—Same as Fig. 10, but with $T_0 = 10,000$ K. The one- and three-zone models are indistinguishable in either case. The $y = 0$ models produce slightly lower peak fluxes.

Fig. 12.—Analytic fits (solid curve) to the SS Cyg data points and light curves generated from numerically calculated spectra using the best-fit parameters (dashed curve). Parameter $\Gamma$ was allowed to be a fit parameter and the points were weighted according to the observational errors.
Fig. 13.—Same as Fig. 12, but $\Gamma$ was allowed to be a fit parameter and the points were weighted equally.

Fig. 14.—Same as Fig. 12, but $\Gamma = 0$ (isothermal) was fixed and the points were weighted according to the observational errors.
mean flux in the regions 3590–3650, 4165–4270, 4520–4620, 5100–5800, 5970–6500, and 7120–7550 Å at 2 s resolution.

Model light curves were computed using a distance of 166 pc (Harrison et al. 1999) and assuming solar composition. It appears that the disagreement is dominated by the systematic error of the oversimplicity of our model rather than observational errors. Accordingly, we carried out fits both weighting the data points according to their formal observational errors and with equal weighting. We plot the derived analytic light curves alongside the observations in Figures 12 and 13. From each set of derived parameters, we carried out a detailed numerical integration of the opacity calculated with an LTE ionization varying across the density profile as described in Pearson et al. (2003). These models allowed us to generate a time series of continuum spectra and “numerical” light curves for comparison to the data in each case. These numerical light curves are also plotted in the figures. The fitted value of $/C_0$ is close to zero in both cases and so we refitted the data fixing $/C_0 = 0$ exactly. These results are shown in Figures 14 and 15. The best-fit parameters for each case are summarized in Table 1.

Although we would expect the numerical light curves to more accurately represent reality, the analytic light curves generally fit the data better. This is unsurprising since the parameters used in both cases have been optimized for the analytic forms. The numerical light curves do seem closer to the data in the 3615 Å window in Figures 12 and 13. However, Figures 14 and 15 and our other unpublished light curves suggest that this is serendipitous. The difficulty in reproducing the Balmer jump may well reflect that we have five fitting points on the Paschen continuum (wavelengths longer than 3646 Å) and only one above the Balmer jump. Since the continuum level will have a close to $\nu^{-3}$ relationship between discontinuities, fixing the level at several points may well bias the parameters toward an accurate fit here rather than the more complex interplay of parameters required to accurately reproduce the Balmer jump. In ideal circumstances we would like several more points at shorter wavelengths to address this issue.

The derived masses, $M \sim 1.6 \times 10^{17}$ kg, are equivalent to $\sim 400$ s of the estimated mean mass transfer rate from the secondary of $\dot{M} \approx 4 \times 10^{14}$ kg s$^{-1}$ and $\sim 2 \times 10^{-6}$ of the total disk mass $M_{\text{disk}} \approx 7 \times 10^{20}$ kg (Schreiber & Gänsicke 2002). Similarly, the length scale of the expanding region, $a \sim 9.1 \times 10^8$ m, is $\sim 1.5\%$ of the estimated disk radius $R_{\text{disk}} \approx 6 \times 10^8$ m (Schreiber & Gänsicke 2002). The kinetic energy is much greater than the thermal energy $E_{\text{th}} \approx 4 \times 10^{22}(M/10^{17}\text{ kg})(T/20,000 \text{ K})$ J but comparable to that of a putative, moderately strong magnetic field in a sphere of radius $a_0$, $E_{\text{mag}} = 6 \times 10^{28}(a_0/10^7 \text{ m})^3(B/6 \text{ T})^2$ J.

We generated full optical spectra from the “six parameter weighted” set of values and compare them to the observed spectra at three different times in Figure 16. The model data have been convolved with the instrumental blurring of 9.8 Å FWHM. The results show how parameters derived from the analytic forms for the continuum light curves can subsequently be used to reproduce spectra. The agreement with observation at early times is remarkable. At the peak flux and later, however, the models predict more metal and stronger lines than are observed, particularly in the 3000–5000 Å range. Many of the relative line strengths, however, are still reproduced.

5.2. SN 1987A

Given the similarities in the mechanisms underlying the flickering light curves in our models and those of supernovae, we attempted to fit the $UBVRI$ photometric data for SN 1987A obtained from Catchpole et al. (1987). The data were converted...
### Table 1

**Derived and Auxiliary Parameters from the Analytic Fits to Flickering of SS Cyg and the Light Curve of SN 1987A**

| Model                           | \( M \)     | \( a_0 \)  | \( T_0 \)  | \( t_0 \)    | \( H \)  | \( r_0 \) | \( E_{\text{kin}} \) |
|--------------------------------|-------------|------------|------------|--------------|---------|----------|----------------|
| SS Cyg                         | (10^{17} kg) | (10^6 m)  | (10^4 K)  | (UT)         | (s^{-1})| (km s^{-1})| (10^{28} J)  |
| Six parameters weighted        | 1.6         | 9.8        | 1.91       | 9.6680       | 0.090   | -0.092   | 890          | 9.7          |
| Six parameters unweighted      | 1.6         | 8.1        | 2.07       | 9.6677       | 0.106   | -0.064   | 860          | 8.8          |
| Isothermal weighted            | 1.5         | 7.4        | 2.38       | 9.6679       | 0.115   | ...      | 860          | 8.3          |
| Isothermal unweighted          | 1.5         | 11.0       | 2.38       | 9.6691       | 0.078   | ...      | 860          | 8.4          |
| SN 1987A                       | (10^{11} kg) | (10^{13} m) | (10^3 K)  | (MJD)       | (10^{-8} s^{-1}) | (km s^{-1}) | (10^{43} J) |
| Five-band \( UBVRI \)          | 2.6         | 2.1        | 4.72       | 47018.7      | 7.2     | 0.066    | 1500         | 4.5          |
| Four-band \( BVRI \)           | 2.5         | 2.2        | 4.69       | 47025.3      | 7.0     | 0.071    | 1500         | 4.3          |

**Fig. 16.**—Comparison of three spectra at 9.6722 (rise), 9.6830 (peak), and 9.6916 UT (fall) calculated from numerical integration across our fireballs with the observed spectra of SS Cyg. The “six parameter weighted” values were used to generate the numerical spectra.
to fluxes using the standard values and effective wavelengths for the bands in Murdin (2001). The fits assumed an interstellar reddening $E(B-V) = 0.15$ (Hamuy et al. 1988; Fitzpatrick 1988) and assumed the same distance to the LMC of 50.1 kpc as Catchpole et al. (1987). Given the results of sophisticated non-LTE spectral and light-curve modeling (e.g., Mitchell et al. 2002), we should not expect an accurate match between our simple models and the SN 1987A observations. However, it will allow us to highlight the underlying similarities of these problems and to gauge the robustness of our derived parameters. The fits used only the data after JD 2,446,875.0 to eliminate the flash phase, and the data were all assigned equal weights. For the sake of continuity with the SS Cyg fit, we continued to allow six fit parameters rather than fixing $t_0$ and $H$ to give a launch time of JD 2,446,849.82 when the associated neutrino event was detected by Kamiokande II (Koshiba et al. 1987). Since we have not attempted to remove a “background” level, the fit may be contaminated by the early stages of the radioactive tail. However, the flux in this region is dominated by the expanding/collapsing photosphere effect and any heating should show up in the fitted value of $\Gamma$ differing from zero.

We see in Figure 17 that we can achieve a qualitative match between the analytic expression and the data. However, the $U$ band is particularly poorly reproduced, representing the effect of blanketing noted by Menzies et al. (1987). We refitted the data using just the $BVRI$ bands and show the results in Figure 18. The parameters for the fits are included in Table 1. We see that eliminating the $U$-band data has relatively little effect and derive a value for the mass of the material in the expansion of $M = 2.6 \times 10^{31}$ kg ($\approx 13 M_\odot$). This compares to an estimate of $7 M_\odot < M_{\text{env}} < 11 M_\odot$ by Saio et al. (1988). The typical expansion velocity of 1500 km s$^{-1}$ compares to a value derived from measurements of the blueshifts in P Cygni profiles of $\approx 2500$ km s$^{-1}$ at maximum light (McCray 1993). Typical estimates of $E_{\text{kin}} \approx 2 \times 10^{44}$ J (Bethe & Pizzochero 1990; Woosley 1988) are somewhat larger than our $E_{\text{kin}} \approx 4 \times 10^{43}$ J.

It is unsurprising that the light curves do not match in detail, since the Gaussian density profile we have assumed does not accurately represent supernova ejecta. In particular, the reversal of the fast rise and slow decline derived analytically can be understood from the way in which a photosphere marching inward through a shell (in comoving coordinates) evolves. In the shell case, the material will first become optically thin at low impact parameters once the photosphere reaches the inner surface of the shell and later at higher impact parameters with their greater path lengths through the shell material. While these cannot be considered in any sense good fits, we do retrieve values for the mass, size, and temperature (see Fig. 19) for this phase comparable with those given by Catchpole et al. (1987). We also see the same isothermal behavior from the light-curve analysis as those authors. The size of our photosphere is less than the radius they derive, which is consistent with our including both optically thick and thin emission regions whereas they treat all the emission as optically thick. Our light curves also sometimes

![Figure 17](image-url)
show an intriguing feature (unfortunately most apparent in the poorly fitted $U$ band) of a point of inflexion at around JD 2,446,960. This is reminiscent, in timing and in character, of the break in the light curves attributed to heating by radioactive decay. In our light curves, however, it appears to arise from the point at which the material becomes completely optically thin with no contribution from an optically thick core region. All these factors, showing reasonable results for SN 1987A despite the inherent simplicity of our model, reinforce our confidence in the robustness of the parameters derived for SS Cyg, which had much better agreement between data and theory.

6. SUMMARY

We have derived a formalism and analytic expressions applicable to a variety of systems that reproduce the evolution of their optical light curves. The method has been tested for two widely different cases, allowing us to derive reasonable values for the physical parameters involved in the expansion. There is encouraging agreement with a method using a full integration of the opacity across the expanding region to generate both light curves and spectra. We have seen how the flickering of close binary systems can be modeled as smaller, hotter analogs of the well-known rebrightening bump in supernova light curves. We hope to test this approach further by comparison to data from LMXBs and other systems in the future.

We thank the referee for the thoughtful and helpful suggestions in response to an earlier version of this paper. K. J. P. has been supported, in part, by the US National Science Foundation through grant AST 99-87344 and, in part, through LSU’s Center for Applied Information Technology and Learning. He also thanks Juhan Frank for stimulating discussions that improved this paper.
REFERENCES

Abramowitz, M., & Stegun, I. A. 1972, Handbook of Mathematical Functions (10th ed.; New York: Dover)
Baptista, R., & Bortoletto, A. 2004, AJ, 128, 411
Baptista, R., Bortoletto, A., & Harlaftis, E. T. 2002, MNRAS, 335, 665
Bethe, H. A., & Pizzochero, P. 1990, ApJ, 350, L33
Bowers, R. L., & Deeming, T. 1984, Astrophysics I: Stars (Boston: Jones and Bartlett)
Bruch, A. 1992, A&A, 266, 237
———. 2000, A&A, 359, 998
Catchpole, R. M., et al. 1987, MNRAS, 229, 15P
Fitzpatrick, E. L. 1988, ApJ, 335, 703
Gaunt, J. A. 1930, Philos. Trans. R. Soc. London, A229, 163
Geltman, S. 1962, ApJ, 136, 935
Glasco, H. P., & Zirin, H. 1964, ApJS, 9, 193
Gray, D. F. 1976, The Observation and Analysis of Stellar Photospheres (New York: Wiley)
Hamuy, M., Suntzeff, N. B., Gonzalez, R., & Gabriel, M. 1988, AJ, 95, 63
Harrison, T. E., McNamara, B. J., Szokdy, P., McArthur, B. E., Benedict, G. F., Klemola, A. R., & Gilliland, R. L. 1999, ApJ, 515, L93
Hawley, J. E., & Balbus, S. A. 1991, ApJ, 376, 223
Honey, W. B., et al. 1989, MNRAS, 236, 727
Huang, S. 1948, ApJ, 108, 354
Jeffreys, H., & Jeffreys, B. S. 1956, Methods of Mathematical Physics (3rd ed.; Cambridge: Cambridge Univ. Press)
Keady, J. J., & Kilcrease, D. P. 2000, in Allen’s Astrophysical Quantities, ed. A. N. Cox (4th ed.; New York: Springer), chap. 5
Koshiba, M., et al. 1987, IAU Circ., 4338, 1
Lang, K. R. 1980, Astrophysical Formulae (2nd ed.; New York: Springer)
Mauche, C. W., & Robinson, E. L. 2001, ApJ, 562, 508
McCray, R. 1993, ARA&A, 31, 175
McDowell, M. R. C., Williamson, J. H., & Myerscough, V. P. 1966, ApJ, 144, 827
Menzies, J. W., et al. 1987, MNRAS, 227, 39P
Mitchell, R. C., Baron, E., Branch, D., Hauschildt, P. H., Nugent, P. E., Lunqvist, P., Blinnikov, S., & Pun, C. S. J. 2002, ApJ, 574, 293
Murdin, P. 2001, Encyclopedia of Astronomy and Astrophysics (Bristol: IOP)
O’Brien, K., Horne, K., Boronson, B., Still, M., Gomer, R., Oke, J. B., Boyd, P., & Vitilek, S. D. 2001, MNRAS, 326, 1067
Oke, J. B., et al. 1995, PASP, 107, 375
Patterson, J. 1981, ApJS, 45, 517
Pearson, K. J., Horne, K. D., & Skidmore, W. 2003, MNRAS, 338, 1067
Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, Numerical Recipes in FORTRAN: The Art of Scientific Computing (2nd ed.; Cambridge: Cambridge Univ. Press)
Ritter, H., & Kolb, U. 2003, A&A, 404, 301
Saio, H., Nomoto, K., & Kato, M. 1988, Nature, 334, 508
Schreiber, M. R., & Gansicke, B. T. 2002, A&A, 382, 124
Skidmore, W., O’Brien, K., Horne, K., Gomer, R., Oke, J. B., & Pearson, K. J. 2003, MNRAS, 338, 1057
Steehgs, D., O’Brien, K., Gomer, R., & Oke, J. B. 2001, MNRAS, 323, 484
Stilley, J. L., & Callaway, J. 1970, ApJ, 160, 245
van Teeseling, A. 1997, A&A, 324, L73
Warner, B. 1995, Cataclysmic Variable Stars (Cambridge: Cambridge Univ. Press)
———. 2004, PASP, 116, 115
Wells, L. D. 1896, Harvard Coll. Obs. Circ., 12
Welsh, W. F., & Wood, J. H. 1995, in IAU Colloq. 151, Flares and Flashes, ed. J. Greiner, H. W. Duerbeck, & R. E. Gershberg (Berlin: Springer), 300
Woosley, S. E. 1988, ApJ, 330, 218