Excitation of magnetostatic spin waves in ferromagnetic films

Yu. Kuzovlev, N. Mezin and G. Yaros
d
A.A. Galkin Physics and Technology Institute of NASU, ul. R. Luxemburg 72, 83114 Donetsk, Ukraine
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The torque equation of nonlinear spin dynamics is considered in the magnetostatic approximation. In this framework, exact expressions for propagator of linear magnetostatic waves in ferromagnetic film between two antennas and corresponding mutual impedance of the antennas are derived, under conditions of uniform but arbitrarily oriented static magnetization and arbitrary anisotropy. The results imply also full description of spectrum of the waves.

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I. INTRODUCTION

The terms “magnetostatic oscillations” or “magnetostatic waves” (MSW) \[\text{(MSW)}\] mean such relatively long spin waves (SW) whose properties are dominated by the long-range quasi-static dipole interaction between “spins” and eventually by geometry of ferromagnetic sample (its shape and dimensions). In opposite, properties of relatively short SW (spin waves in narrow sense) \[\text{(SW)}\] come rather from the short-range exchange interaction. In ferrite films, MSW can be easy excited, with the help of microwave-frequency magnetic fields induced by wire or strip-like antennas \[\text{(MSW)}\], up to the level of strongly nonlinear interactions of MSW between themselves and with SW. In addition, small velocity of MSW ensures compactness of nonlinear transformations of electromagnetic signals. When exploring these possibilities, it is important to know characteristics of linear MSW modes and corresponding linear impedances of their antennas. Concerning MSW in flat homogeneous films, there are widely used exact results by Damon and Eshbach \[\text{(Damon and Eshbach)}\]. Other authors \[\text{(see also paper by Kalinikos and coworkers in \[\text{and references therein}\) developed approximate approaches to films which are non-homogeneous because of specific surface effects. At the same time, to the best of our knowledge, the homogeneous case remains incompletely investigated. In the present paper we find new solutions to this case believing that they will be useful for both practice and theoretical modeling of more complicated situations.

We confine our consideration by “magnetostatic approximation” which neglects SW (in narrow sense) at all, as if radius of the exchange interaction, \(r_e\), is equal to zero. This formal trick seems reasonable when cross-section dimensions of all the antennas or/and distances from them to film’s surface are much greater than \(r_e\). Indeed, under this condition magnetic field created by antennas is so smoothly distributed in film’s interior (with spatial scales much greater than \(r_e\)) that must only very poorly excite those SW whose length is comparable with \(r_e\) or smaller. We do not know strict proof of this statement, but it rather convincingly follows from results of the “magnetostatic approximation” itself (in particular, results presented below) which demonstrate, at mentioned condition, insignificance of “infinitely short” MSW in the sense of both their private excitations and summary contribution to film’s linear response to weak external field. Other deal is when in definite nonlinear processes even SW much shorter than \(r_e\) can be generated from long MSW \[\text{(by the way, all the nonlinear models collected in \[\text{agree with the above statement). However, here we are interested in linear (small-amplitude) response only, although examining the path to it from basic nonlinear equations.}

Besides, firstly, we will keep in mind that film under consideration is thick, in the sense that its thickness, \(D\), is large as compared with \(r_e\). This condition will make our results be better applicable to real films with surface-induced non-uniformity. Secondly, if a ground (static) state of film’s magnetization is divided into domains and hence strongly non-uniform then this state itself must be understood and described before its magnetostatic excitations. Of course, we avoid this extremely difficult problem, being satisfied by consideration of MSW on background of uniformly magnetized static state. Fortunately, for example, in yttrium-iron garnet films such state can be enforced by comparatively small bias field. At formulated assumptions, we will obtain the function (“propagator of MSW”) which describes mutual influence of two antennas by means of MSW propagating in a film with arbitrarily directed ground magnetization in presence of arbitrary magnetic anisotropy.

II. BASIC EQUATIONS

The classical semi-phenomenological dynamics of magnetization in solid ferromagnets and ferrites \[\text{(Landau-Lifshitz-Gilbert equation)}\] is based on the Landau-Lifshitz-Gilbert equation (the torque equation)

\[
\frac{dS}{dt} = F \times S + \gamma (F - S(S \cdot F))
\]  

(1)

Here \(S\) is unit vector in the direction of local magnetization (\(|S| = 1\)); \(F\) is thermodynamic force, or effective

*Electronic address: kuzovlev@kinetic.ac.donetsk.ua
internal magnetic field, expressed in units of the saturation magnetization, \( M_s \); the time is expressed in units of \( \tau_0 = (2\pi gM_s)^{-1} \) (\( g \approx 2.8 \text{ MHz/Oe} \) is the gyromagnetic ratio); \( \times \) and \( \cdot \) are symbols of vector and scalar products, respectively; \( \gamma \) is phenomenological friction (dissipation) parameter. For our purposes, the simplest model of dissipation is sufficient as those introduced by Eq.\[4, 5, 11\]

The internal field, \( \mathbf{F} \), is composed at least from \( \mathbf{H}_e \), (i) external bias field, \( \mathbf{H}_e \), (ii) magnetic anisotropy field, \( \mathbf{H}_a \), (iii) own magnetic field created by the magnetization (by “spins”), \( \mathbf{H}_a \), and (iv) exchange field, \( \mathbf{H}_{\text{exch}} \). We do not add to this list magnetic field induced by electric currents in the ferromagnetic sample since assume that it is good isolator. Usually small dimensions of real samples allow to neglect time delay of \( \mathbf{H}_a \), therefore \( \mathbf{H}_a \) can be found from quasi-static version of the Maxwell equations. Particularly, in absence of conductors and other ferromagnets in vicinity of the sample, these equations yield

\[
\mathbf{H}_a = -\tilde{G}\mathbf{S} \equiv \nabla \left( \nabla \cdot \int \frac{S(r)dr}{|r - r'|} \right) \quad (2)
\]

In most simple model of exchange interaction, \( \mathbf{H}_{\text{exch}} = \gamma^2 \nabla^2 \mathbf{S} \quad [4, 5, 11]. \) It should be noted that exchange interaction ensures local smoothness of the magnetization distribution, \( \mathbf{S} \), while the operator of dipole-dipole interaction, \( \tilde{G} \), is bounded when acts on smooth distributions: \( \|\tilde{G}\| \leq 4\pi \). If \( A(S) \) denotes density of energy of the anisotropy then

\[
\mathbf{H}_a = -A'(S) \equiv -\partial A(S)/\partial S
\]

In principle, all the consideration in this section and next section can be easily generalized to non-uniform anisotropy if suppose that \( A(S) \) is some function of space coordinates.

Let the subscript “0” be attribute of static magnetization state at constant bias field, \( \mathbf{H}_e = \mathbf{H}_0 = \text{const} \). According to Eq.\[4\] in any such state the vectors \( \mathbf{S}_0 \) and \( \mathbf{F}_0 \) are parallel one to another, that is \( \mathbf{F}_0 = W_0 \mathbf{S}_0 \), where scalar field \( W_0 \) (absolute value of static internal magnetic field) is defined by the requirement \( |\mathbf{S}| = 1 \).

Let additional time-varying magnetic field is switched on: \( \mathbf{H}_e = \mathbf{H}_0 + \mathbf{h} \), \( \mathbf{h} = \mathbf{h}(t) \). Corresponding deviation of magnetization from its static value, \( \mathbf{s} = \mathbf{S} - \mathbf{S}_0 \), can be represented as

\[
\mathbf{s} = \mathbf{S}_\perp + (S_\parallel - 1)\mathbf{S}_0 \quad , \quad \mathbf{S}_\perp = \mathbf{\hat{P}}\mathbf{S} \perp \mathbf{S}_0 \quad , \quad (3)
\]

\[
\mathbf{\hat{P}} \equiv 1 - \mathbf{S}_0 \otimes \mathbf{S}_0 \quad , \quad S_\parallel = \pm \sqrt{1 - |\mathbf{S}_\perp|^2} \ ,
\]

where \( \otimes \) means tensor product, and \( \mathbf{\hat{P}} \) is operator (matrix) which performs projection of vectors onto the plane perpendicular to \( \mathbf{S}_0 \). If none spins are overturned by the perturbation then sign of \( S_\parallel \) is everywhere positive. In terms of \( \mathbf{s} \) and \( \mathbf{S}_\perp \), Eq.\[1\] transforms into

\[
\frac{d\mathbf{S}_\perp}{dt} = \mathbf{S}_\parallel [\mathbf{F}_\perp \times \mathbf{s}] + \gamma (1 - \mathbf{S}_\perp \otimes \mathbf{S}_\perp) \mathbf{F}_\perp \quad , \quad (4)
\]

where \( \mathbf{F}_\perp = -\delta E_\perp/\delta \mathbf{S}_\perp \), and \( E_\perp \) is the excess energy (energy of excitation) implied by the perturbation,

\[
E_\perp = \int \left[ \frac{1}{2} W_0 s^2 + \tilde{A}(s) + \frac{1}{2} s \cdot \mathbf{G}s - \mathbf{h} \cdot s + C(\nabla s) \right] \, dr \quad (5)
\]

Here function \( \tilde{A} \) is defined by

\[
\tilde{A}(s) = A(\mathbf{S}_0 + s) - A(\mathbf{S}_0) - s \cdot A'(\mathbf{S}_0) \quad ,
\]

and function \( C \) represents exchange contribution to the excess energy. In the mentioned model, \( C(\nabla s) = \frac{1}{2} r^2 \sum_{\alpha\beta} (\nabla_\alpha s_\beta)^2 \). The functional derivative \( \mathbf{F}_\perp \) in Eq.\[4\] should be evaluated with taking into account full dependence of \( s \) on \( \mathbf{S}_\perp \) in accordance with Eq.\[3\].

Importantly, the frictionless version of Eq.\[4\] (that is at \( \gamma = 0 \)) follows from the variational principle

\[
\delta \int \left\{ \int \left( \mathbf{s} \cdot \left[ \frac{d\mathbf{S}_\perp}{dt} \times \mathbf{S}_\perp \right] \right) \frac{dr}{1 + S_\parallel} + E_\perp \right\} dt = 0 \quad (6)
\]

Of course, in general \( \mathbf{S}_0 \) is a complicated function of spatial coordinates, hence all the related values (\( W_0, \mathbf{\hat{P}}, \tilde{A}(s) \), and so on) are space dependent.

### III. LINEAR WAVES

If the functional \( \tilde{F} \) represents positively defined quadratic form then the static magnetization pattern \( \mathbf{S}_0 \) is stable with respect to any small perturbation, and therefore we can speak about linear eigenmodes of the excitation. In this case, let us introduce the spin precession operator, \( \mathbf{R} \), the anisotropy matrix, \( \tilde{A} \), and besides the exchange operator, \( \tilde{G} \), by the relations

\[
\mathbf{R}V \equiv \mathbf{S}_0 \times V \quad , \quad \tilde{C}V \equiv -r^2 \nabla^2 V \quad ,
\]

\[
\tilde{A}_{\alpha\beta} = \partial^2 A(S_0)/\partial S_{0\alpha} \partial S_{0\beta} \quad ,
\]

For the linear regime Eq.\[4\] yields

\[
\frac{d\mathbf{S}_\perp}{dt} = (\mathbf{R} - \gamma \mathbf{\hat{P}})(\mathbf{\tilde{W}} \mathbf{S}_\perp - \mathbf{h}) \quad , \quad (7)
\]

where we introduce the integral-differential operator

\[
\mathbf{\tilde{W}} \equiv W_0 + \tilde{A} + \tilde{G} + \tilde{C} \quad ,
\]

Rejecting from Eq.\[7\] both dissipation and external pump, we obtain equations for SW and MSW eigenmodes and eigenfrequencies:

\[
\mathbf{S}_\perp \equiv \mathbf{V} e^{-i\omega t} \quad , \quad -i\omega V = \mathbf{R}\mathbf{\tilde{W}}V \quad (8)
\]

Hereafter it is sufficient to consider positive frequencies only. Let the eigenmodes be enumerated by an index
Since in a stable state the operator $\hat{W}$ is positively defined, we can write

$$\omega_k \bar{V}_k = i\bar{W}^{1/2} \hat{R} \bar{W}^{1/2} \bar{V}_k , \quad \bar{V}_k \equiv \bar{W}^{1/2} V_k$$  \hspace{1cm} (9)

The operator on right-hand side in the left of these two equalities must be self-adjoint, hence, its eigenfunctions $\bar{V}_k$ can be made mutually orthogonal. From here the orthogonality rule for the eigenmodes does follow:

$$i \int S_0 \cdot [V_k \times V_k^*] \, dr = \delta_{mk}$$  \hspace{1cm} (10)

The same rule is specified by the variational principle. To get more general formulation of linear theory, we should return from the “ready” dipole interaction operator $\hat{G}$ to Maxwell equations:

$$\frac{dS_\perp}{dt} = (\hat{R} - \gamma \hat{1}) \{ (W_0 + \hat{A} + \hat{C}) S_\perp - \hat{h} - h_S \} ,$$  \hspace{1cm} (11)

$$\nabla \cdot (h_S + 4\pi S_\perp) = 0 , \quad \nabla \times h_S = 0 ,$$  \hspace{1cm} (12)

where we introduced new vector field, $h_S$, which represents time-varying part of magnetic field self-induced by magnetization (i.e. the same as the whole field induced by $s$). As before, it is assumed that the sample is non-conducting.

Applying Fourier transform to Eq. (11) in the frequency representation (frequency domain) from Eq. (11) and (12) we have

$$S_\perp = \tilde{\chi} \{ h + h_S \} , \quad \nabla \cdot \tilde{\mu} \{ h + h_S \} = 0 ,$$  \hspace{1cm} (13)

$$\tilde{\chi} = \{ i\omega + (\hat{R} - \gamma \hat{1})(W_0 + \hat{A} + \hat{C}) \}^{-1}(\hat{R} - \gamma \hat{1}) ,$$  \hspace{1cm} (14)

where $\tilde{\mu} \equiv 1 + 4\pi \tilde{\chi}$. Obviously, because of presence of the differential operator $\hat{C}$ in denominator of the polarizability matrix $\tilde{\chi}$ in Eq. (14), in fact $\tilde{\chi}$ is an integral operator.

At this point we go to the “magnetostatic approximation” formulated and discussed in Sec.1. Concretely, we reject the exchange operator $\hat{C}$ from denominator of $\tilde{\chi}$. Formally, this is equivalent to that the exchange radius $r_e$ turns into zero (of course, herewith we do not neglect exchange interaction since it remains responsible for the magnetization phenomenon itself). Strictly speaking, simultaneously the static magnetization, i.e. the patterns $S_0$ and $W_0$, also must be treated in this limit. But this does not matter in the case of uniform static magnetization which we will investigate below.

After that, $\tilde{\chi}$ turns into algebraic expression becoming literally matrix, and the problem reduces to purely differential equations for the field $h_S$. Direct analytical calculation gives very simple expression for the polarizability:

$$\tilde{\chi} = \frac{(W_0 + A_1 + A_2) \tilde{\Pi} - \tilde{A}_\perp - i\bar{\omega} \tilde{R}}{(W_0 + A_1)(W_0 + A_2) - \bar{\omega}^2} ,$$  \hspace{1cm} (15)

$$\tilde{A}_\perp = \tilde{\Pi} \tilde{A} \tilde{\Pi} , \quad W_0 \equiv W_0 - i\gamma \bar{\omega} , \quad \bar{\omega} \equiv \frac{\omega}{1 + \gamma^2}$$

Here $A_1$ and $A_2$ are those two eigenvalues of matrix $\tilde{A}_\perp$ which correspond to the pair of its eigenvectors perpendicular to $S_0$ and one to another: $\tilde{A}_\perp a_{1,2} = A_{1,2} a_{1,2}$. We enumerate them so that $S_0 \cdot [a_1 \times a_2] > 0$. At practically interesting ferrite samples $\gamma \lesssim 10^{-3}$, therefore $\gamma^2$ plays no role.

**IV. PROPAGATOR OF MAGNETOSTATIC WAVES IN FILMS**

Let the time-varying field $h$ is induced by some conductors which are placed outside the ferromagnetic sample and carry a.c. currents $I_n$ distributed with densities $I_n J_n \,(n = 1, 2, ...)$ Then we can write

$$h = \sum h_m I_m , \quad \nabla \cdot h_n = 0 , \quad \nabla \times h_n = \frac{4\pi}{c} J_n$$  \hspace{1cm} (16)

Here $h_n$ is magnetic field created by unit-value current in $n$-th conductor. The same function determines voltage (e.m.f.), $\varepsilon_n$, induced in the $n$-th conductor by time-varying magnetization of the sample:

$$\varepsilon_n = \int \left( h_n \cdot \frac{ds}{dt} \right) \, dr$$  \hspace{1cm} (17)

The fields $h_n$, as well as the self-induced field $h_S$ can be represented in the potential form.

In the linear regime, when $s \to S_\perp$, the response of the sample divides into sum of partial responses:

$$h_n = -\nabla U_n , \quad h_S = \sum \hat{h}_{s n} I_m , \quad \hat{h}_{s n} = -\nabla \hat{U}_{s n}$$

where $\hat{U}_{s n}$ is potential of the field induced by the sample in response to influence by $n$-th conductor. After obtaining $h_S$ in company with $S_\perp$ from Eq. (17) we will determine mutual impedances of the conductors, $\hat{Z}_{n m}$, caused by their interaction through the ferromagnetic:

$$\varepsilon_n = \sum \hat{Z}_{n m} I_m$$

Now concretize the sample as plate (film) whose in-plane dimensions much exceed its thickness, formally as infinite plate. At sufficiently large bias field, $H_0$, ferromagnetic plate allows for stable state of uniform magnetization. In real finite-size films, such the state is demagnetized at film’s edges only, in strip-like regions whose width is few of $D$ ($D$ is thickness). This justifies the theory of plane MSW in infinite uniformly magnetized film.

Let film is disposed in the region $-D/2 < z < D/2$. Naturally, make Fourier transform with respect to time and in-plane coordinates, $x$ and $y$, marking transformed functions by tilde. Introduce designations $k = \{k_x, k_y\}$, $\nabla = \{i k_x, i k_y, \nabla_z\}$. In film’s interior $\nabla^2 \hat{U}_n = 0$, therefore the potentials of conductors have the form

$$\hat{U}_n(k, z) = \Phi_n(k) \exp\{i \sigma_n z - D/2\} ,$$  \hspace{1cm} (18)
where $\sigma_n = 1 (-1)$, if $n$-th conductor is placed above (below) film, and form-factor $\Phi_n(k)$ describes distribution of $n$-th current. In combination with (12) and (17) the latter formula implies the relation between the impedances, from one hand, and values of the potentials taken at film’s surfaces, from another hand:

$$Z_{nm} = \frac{i \omega}{2\pi} \int |k| \tilde{U}_{sn} \left( \omega, k, \sigma_n \frac{D}{2} \right) \tilde{U}_n \left( -k, \sigma_n \frac{D}{2} \right) dk$$

(19)

Here $dk \equiv dk_x dk_y / (2\pi)^2$ (in contrast with $\tilde{U}_n$, potentials $\tilde{U}_{sn}$ are frequency dependent). Thus, the potentials are taken at the surface most close to receiving antenna ($n$-th conductor).

The Eq.13 or equivalently,

$$(\nabla \cdot \hat{n} \nabla)(\tilde{U}_{sn} + \tilde{U}_n) = 0,$$

(20)

should be solved under standard boundary conditions (4 11). To write the answer, introduce the unit-length vectors

$$\nu \equiv \{ k_x / |k|, k_y / |k|, 0 \}, \quad \Xi \equiv \{ 0, 0, 1 \},$$

and besides the matrix

$$M = \left[ \begin{array}{cc} \mu_{\nu\nu} & \mu_{\nu z} \\ \mu_{z\nu} & \mu_{zz} \end{array} \right] \equiv \left[ \begin{array}{cc} \nu \cdot \hat{n} \nu & \nu \cdot \hat{n} z \\ \hat{n} \nu \cdot \nu & \hat{n} \nu \cdot \Xi \end{array} \right]$$

(21)

As usually, the solution is composed by two exponent:

$$\tilde{U}_{sn} + \tilde{U}_n = \sum_{\pm} u_{n,\pm} \exp(q_{\pm} z), \quad q_{\pm} = \lambda_{\pm} |k|,$$

(22)

$$\lambda_{\pm} \equiv \lambda_0 \pm \Lambda, \quad \lambda_0 = \frac{\mu_{\nu z} + \mu_{z\nu}}{2i\mu_{zz}},$$

(23)

$$\Lambda = \sqrt{\left( \frac{\mu_{\nu\nu}}{\mu_{zz}} - \frac{\mu_{\nu z} + \mu_{z\nu}}{2i\mu_{zz}} \right)^2}.$$

(24)

It should be emphasized, however, that in general case (at arbitrary orientation of the vector $S_0$) the exponents $q_{\pm}$ are neither poorly imaginary nor poorly real but complex, that is MSW is not standing in $Z$-direction.

If we took into account finite exchange radius $r_e$ and dealt with the operator-valued polarizability matrix (14), then in place of (22) we would get a sum of at least six terms, where transverse wave numbers of order of $|k|$ (as $q_{\pm}$ in (22)) are more or less hybridized with real or imaginary wave numbers of order of $\pi / r_e$. From the point of view of our aims, such complication would be meaningless. But it can be necessary when considering short SW or small-scale details of long MSW on background of a nonuniform domain structure. Most natural approach to these tasks is direct analysis of the system of equations (11) and (12).

Consider the susceptibility matrix $M$. In standard spherical coordinates let $\theta$ be the angle between $Z$-axis and $S_0$. In the $XY$-plane (film’s plane), we introduce quantities $\nu_0$ and $\nu_\perp$ as cosine and sine, respectively, of the angle (counted clockwise) between projection of $S_0$ onto this plane and the above defined unit vector $\nu$ lying in it. Besides, consider the plane $a_1 a_2$ perpendicular to $S_0$ and, in this plane, define $\psi$ being the angle between plane $ZS_0$ and vector $a_1$ (definition of vectors $a_1$ and $a_2$ was done at the end of Sec.3). Further, introduce the quantities $A_{\pm} \equiv (A_2 \pm A_1) / 2$. At last, let $\Omega$ be nominator of the polarizability matrix (13),

$$\Omega = (\tilde{W}_0 + A_1 + A_2)\Pi - \tilde{A}_\perp + i\omega \tilde{R}.$$

In these designations

$$\Omega_{zz} = (\tilde{W}_0 + A_+ + A_- \cos 2\psi) \sin^2 \theta,$$

(25)

$$\Omega_{\nu\nu} = (\tilde{W}_0 + A_+) (\nu_0^2 + \nu_{\perp}^2 \cos^2 \theta) +$$

(26)

$$+ A_- \left( \nu_0^2 \cos^2 \theta - \nu_{\perp}^2 \right) \cos 2\psi - 2\nu_{\perp} \nu_\perp \sin 2\psi \cos \theta,$$

(27)

$$\Omega_{z\nu, \nu} = \Omega_{\times} \mp i\omega \nu_\perp \sin \theta,$$

(28)

$$\Omega_{\times} \equiv \sin \theta \left( A_- \nu_\perp \sin 2\psi -$$

(29)

$$- \nu_{\perp} (\tilde{W}_0 + A_+ + A_- \cos 2\psi) \cos \theta \right).$$

These formulas make it evident that effects of anisotropy are determined by $A_-$, while $A_+$ merely redefines the magnitude of static internal field, $\tilde{W}_0$.

Then, for a given in-plane orientation of the wave, $\nu$, introduce characteristic frequencies by the following expressions:

$$\omega_0^2 \equiv (\tilde{W}_0 + A_+)^2 - A_-^2,$$

(29)

$$\omega_u^2 \equiv \omega_0^2 + 4\pi \Omega_{zz},$$

(30)

$$\omega_{1,2}^2 \equiv \omega_0^2 + 2\pi (\Omega_{zz} + \Omega_{\nu\nu}) \mp$$

(31)

$$\mp 2\pi \sqrt{(\Omega_{zz} + \Omega_{\nu\nu})^2 - (2\nu_\perp \sin \theta)^2 \omega_0^2},$$

(32)

The frequency $\omega_u$, which is independent on the in-plane wave vector $k$, is the uniform precession frequency. In terms of these frequencies,

$$\mu_{zz} \equiv \frac{\omega_u^2 - \omega_0^2}{\omega_0^2 - \omega_u^2}, \quad \mu_{z\nu} - \mu_{\nu z} = \frac{-8\pi i\omega_\perp \sin \theta}{\omega_0^2 - \omega_u^2},$$

(33)

$$\lambda_0 = \frac{4\pi i \Omega_{\times}}{\omega_0^2 - \omega_u^2}, \quad \Lambda = \frac{\sqrt{(\omega_1^2 - \omega_u^2)(\omega_2^2 - \omega_u^2)}}{\omega_0^2 - \omega_u^2}.$$

(34)
Besides, below we will need in the determinant

$$
\Delta \equiv \det M = \frac{2\omega^2 - \omega_0^2 - \omega^2}{\omega_0^2 - \omega^2} \quad (35)
$$

It appears that $\Delta$ being quadratic function of the matrix elements of $M$ and $\chi$ nevertheless always has simple pole only.

$$
P(\omega, k) \equiv \frac{1 - \Delta - i(\mu_{z\nu} - \mu_{x\nu})}{1 + \Delta + 2\mu_{zz} \Lambda \coth(\Lambda|k|D)} \quad (36)
$$

where denominator in the latter expression is given by

$$
G(\omega, k) \equiv \omega^2 - \omega^2 + (\omega_0^2 - \omega^2) \Lambda \coth(\Lambda|k|D) \quad (38)
$$

For brevity, we do mark dependencies of the factors $\Delta$ and $\Lambda$ on $\omega$ and $k$ as well as dependencies of $\omega_{1,2,3}$ on $k$ (or, to be precise, on direction of the in-plane wave vector $k$). Combining these formulas and Eq.(19) for mutual impedance of two antennas located on one and the same side from film, we obtain:

$$
Z_{nm} = \frac{i\omega}{2\pi} \int |k|\Phi_n(-k)\Phi_m(k)P(\omega, k)dk \quad (39)
$$

The latter formulas present main results of the paper and, as far as we know, can give useful addition to results of Damon and Eshbach [2] and other authors (see Sec.1). Function $P(\omega, k)$ is the required propagator of linear (weak) magnetostatic excitations from one antenna to another. At the same time, it contains complete information about spectrum of MSW. The condition that its denominator turns into zero, $G(\omega, k) = 0$ (in absence of dissipation, at $\gamma = 0$), yields a set of dispersion laws for all possible types of MSW. This will be the subject of separate work.

V. MUTUAL IMPEDANCE OF WIRE ANTENNAS

To be more concrete, consider relatively simple but practically interesting case of straight-line wire antennas which have round cross-sections and are parallel one to another and to film’s surface. Besides, let they be oriented along $Y$-axis and located at $X$-positions $x_n$, on the same side from the film and at distances $\rho_n$ from its closest surface. In this situation

$$
\Phi_n(k) = (4\pi^2/ick_x) \exp(-|k_x|\rho_n - ik_x x_n)\delta(k_y)
$$

(here $c$ is speed of light), and Eq.(39) concretizes to

$$
Z_{nm} = \frac{4\pi i}{\omega} \int_0^\infty \exp\{-q(\rho_n + \rho_m)\} \left\{ \frac{(1 - \Delta) \cos(qx) + (\mu_{z\nu} - \mu_{x\nu}) \sin(qx)}{1 + \Delta + 2\mu_{zz} \Lambda \coth(\Lambda qD)} \right\} dq \quad (40)
$$

Here on the left $w$ is the film’s width (formally infinite) measured in centimeters along antennas (i.e in $Y$-direction), $f$ is the frequency expressed in GHz, while on the right-hand side $x \equiv x_n - x_m$, and the integral is taken over $q \equiv k_x$ at $k_y \to 0$. The latter means that matrix elements of the magnetic susceptibility matrix $M$ and the functions $\Lambda$ and $\Delta$ are calculated at $\nu = \{1, 0, 0\}$.

Note that the dimensionless circular frequency $\omega$ which enters all these functions is connected with actual frequency $f$ expressed in GHz by the relation (see Sec.2)

$$
\omega = f/f_0 \ , \ f_0 \equiv (2\pi \tau_0)^{-1} = gM_s
$$

We omit trivial but tremendous evaluation of the surface potentials which appear in Eq.(19). The result, for the surface closest to a given antenna, looks as

$$
\tilde{U}_{sn}(\omega, k, \sigma_n D/2) = \Phi_n(k)P(\omega, k) \quad , (36)
$$
It should be noted also that all the formulas \( \text{[25]-[35]} \) will serve for analytical calculations (e.g. in next our publication), but if using computer it is sufficient to numerically calculate matrix \( \text{[21]} \) and then factors \( \text{[24]} \) and \( \text{[35]} \) only (by this reason we expressed the impedance in terms of these quantities). In general, of course, first of all one must find the static magnetization vector, \( \mathbf{S}_0 \), but this is also not a hard task for computer.

For the case when two parallel wires are situated on the opposite parties from the film, evaluation of corresponding boundary potentials yields (in the same units):

\[
Z_{nm}^{\omega f} = 2\pi i \int_{-\infty}^{\infty} e^{-q_0(x+\rho_m)+iqx} \left\{ e^{-\phi|q|D} - \frac{2\mu_{zz}\exp(\lambda_0|q|D)}{(1+\Delta)\sinh(\lambda|q|D)+2\mu_{zz}\cosh(\lambda|q|D)} \right\} dq |q|
\]

Of course, here in the integrand \( \nu = \{ \text{sign}(q), 0, 0 \} \), and \( \lambda_0 \) is defined in \( \text{[35]} \).

For simple example, let us evaluate self-impedance, \( Z_{11} \), of straight wire antenna (to be precise, the contribution to full self-impedance stipulated by the film), in the special case when bias magnetic field vector, \( \mathbf{H}_0 \), lies in the film’s plane. For concreteness, let it be oriented along \( Y \)-axis. Besides, we assume that characteristic magnetic field of anisotropy is small in comparison with \( |\mathbf{H}_0|+4\pi M_s \), which allows to neglect effects of anisotropy. At last, if we direct the antenna in parallel to \( \mathbf{H}_0 \) then the impedance must be caused primarily by the so-called surface MSW discovered by Damon and Eshbach \( \text{[2]} \). Under above formulated conditions, in \( \mathbf{k} \)-plane these waves occupy the sector \( |k_y/k_x| < \sqrt{4\pi M_s/|\mathbf{H}_0|} \). But, naturally, sufficiently (infinitely) long antenna excites mainly the waves with \( |k_y/k_x| \rightarrow 0 \) which run perpendicularly to the field. Apparently, the latter case is the only case when the dispersion law of MSW (Damon-Eshbach waves with \( k \perp \mathbf{H}_0 \), or DE-waves) can be written in the evident analytical form \( \text{[2]} \):

\[
\omega_{DE}(k) = \sqrt{|H_0|(|H_0|+4\pi)+4\pi^2[1-\exp(-2D|k|)]}
\]

Here field and frequency are expressed in the dimensionless units introduced in Sec.2.

In this case there is a good analytical approximation for the integral \( \text{[40]} \) which yields

\[
\frac{R_{11} [\text{Ohm}]}{w [\text{cm}] f [\text{GHz}]} \approx \frac{4\pi \omega H}{1-X^2} \left( \frac{1-X}{1+X} \right)^{\nu/2D} \left[ \ln \frac{1+X}{1-X} \right]^{-1},
\]

\[
X = \frac{\omega^2-\omega_u^2}{4\pi^2}, \quad \omega = \frac{f}{f_0}, \quad \omega_u = \sqrt{|H_0|(|H_0|+4\pi)}, \quad \omega_t = |H_0|+2\pi
\]

In this formula, \( R_{11} = \text{Re} Z_{11} \) is the film-induced contribution to resistance of the antenna (\( Z = R-2\pi i f L \)); magnetic field is dimensionless, i.e. expressed in units of \( M_s \); the frequency belongs to the interval \( f_0\omega_u < f < f_0\omega_t ; \omega_u \) is dimensionless frequency of uniform precession and at the same time lower bound of spectrum of the Damon-Eshbach waves, while \( \omega_t \) is upper bound of this spectrum (and spectrum of MSW at all). Outside this frequency interval, there are no waves periodic to the antenna, and hence \( R_{11} \) turns into zero (or, to be more precise, becomes comparatively small).

Let us notice that \( X \rightarrow 1 \) when \( f \rightarrow f_0\omega_t \), therefore under condition \( \rho/2D < 1 \) the resistance \( \text{[43]} \) tends to infinity at upper edge of spectrum of the DE-waves. The matter is that here the group velocity of DE-waves, \( v_g \), turns into zero, hence density of states (DE-wave modes) tends to infinity.

Under the opposite condition, \( \rho/2D > 1 \), this effect is canceled by sufficient weakness of excitation of short DE-waves. The presence in \( \text{[43]} \) of the exponent which depends on geometric parameters of the system is eventually consequence of scale invariance of the dipole interaction.

What is interesting, in the lower part of spectrum of DE-waves the resistance \( R_{11} \) is almost independent on the film’s thickness, \( D \), although seemingly the e.m.f. and thus the resistance must be proportional to amount of magnetic moments (spins) under excitation and thus to \( D \). The matter is that the energy outflow from the antenna, \( p \), is proportional to the group velocity, \( v_g \), of
the DE-waves under excitation: \( p \propto D v_g |\mathbf{S}_\perp|^2 \) (where \( \mathbf{S}_\perp \) represents magnitude of spin precession). From the other hand, we can write \( p \propto \varepsilon_1^2 / R_{11} \), while \( \varepsilon_1 \propto D|\mathbf{S}_\perp| \).

Three above relations result in \( R_{11} \propto D/v_g \). But, as it follows from (42), group velocity of long DE-waves is proportional to the thickness, \( v_g \propto D \). This is the reason for the indifference of \( R_{11} (\omega \rightarrow \omega_u) \) with respect to \( D \).

A simple analytical estimate for the inductance, \( L_{11} = -\operatorname{Im} Z_{11} / (2\pi f) \), can be deduced at \( \rho/D \sim 1 \) only, and then

\[
L_{11} [\text{nH}] = \frac{-\rho}{\omega [\text{cm}]} \approx -(2 + |H_0|/\pi) \exp(-X) \text{Ei}(X)
\]  
(44)

(\text{Ei} is the integral exponent function). Clearly, \( L_{11} \) can be both positive and negative.

VI. CONCLUSION

In brief, we found (i) propagator \(\mathbf{G}(37)\) of magnetostatic waves (MSW) running in infinite ferromagnetic film from one antenna to another and (ii) linear (small-amplitude) mutual impedance of the antennas, under arbitrary orientation of uniform static magnetization of the film and arbitrary magnetic anisotropy.

Additionally, the equation \( G(\omega, k) = 0 \) with \( G \) being the denominator \(\mathbf{G}(38)\) in \(\mathbf{G}(37)\) determines dispersion laws for various linear eigenmodes of MSW thus allowing generalizations of classical results obtained in \(^2\).

To conclude, let us touch the applicability of formulas, obtained for continuous spectrum of MSW in infinite film, to real finite-size films where MSW spectrum is discrete. When in-plane dimensions of a film decrease then characteristic frequency separation between neighbouring MSW modes increases, but, at the same time, selection of the modes by any simple (for instance, straight-line wire) antenna becomes more and more worsened. As the result, the formulas derived for infinite film can give good estimate for impedances of antennas interacting with real films.

This expectation was confirmed by comparison between the analytical estimates and measurements of impedances induced by millimeter-size ferrite films as well as by results of their numeric simulations. Moreover, numeric simulations of the torque equation \(\mathbf{11}\) with dipole-dipole interactions between spins show that spatial-temporal patterns of spin precession even in rather small films (with length to thickness ratio \( \sim 30 \div 100 \)) and even at essentially non-linear regimes possess clear imprints of qualitative and quantitative characteristics inherent to linear MSW modes in infinite system.

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