The effect of random coupling coefficients on decoherence

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The aim of this letter is to analyze the effect on decoherence of the randomness of the coupling coefficients involved in the interaction Hamiltonian. By studying the spin-bath model with computer simulations, we show that such randomness greatly improves the “efficiency” of decoherence and, then, its physical meaning deserves to be considered.

**Introduction.** Environment induced decoherence (EID) is usually tested by means of computer simulations on models whose coefficients are, in general, taken as random with the idea of simulating a generic situation (see, as examples, [1], [2], [3], [4], [5], [6]). This widely spread strategy is reasonable for the coefficients of the particles’ states, but it is not natural for the coupling coefficients involved in the interaction Hamiltonian: there is no reason to suppose that particles of the same nature interact with each other with different strengths. The point would be irrelevant if the randomness of the coupling coefficients did not affect the overall phenomenon. The aim of this letter is to show that this is not the case: the randomness of the coupling coefficients greatly improves the “efficiency” of decoherence. This result has to be taken as a warning against the uncritical use of random coupling coefficients for drawing conclusions about decoherence. We will argue for this claim by means of the analysis of a well-known model.

**The spin-bath model.** This is a very simple model that has been exactly solved in previous papers (see [1]). We will study it from the general theoretical framework for decoherence presented in a previous work [7]. Let us consider a closed system $U = S + E$ where (i) the system $S$ is a spin-1/2 particle $P$ represented in the Hilbert space $\mathcal{H}_S$, and (ii) the environment $E$ is composed of $N$ spin-1/2 particles $P_i$, each one represented in its own Hilbert space $\mathcal{H}_i$. The complete Hilbert space of the composite system $U$ is $\mathcal{H} = \mathcal{H}_S \otimes \bigotimes_{i=1}^N \mathcal{H}_i$. In the particle $P$, the two eigenstates of the spin operator $S_{\vec{I}}$ in direction $\vec{v}$ are $|\uparrow\rangle$ and $|\downarrow\rangle$, such that $S_{\vec{I}} |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle$ and $S_{\vec{I}} |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle$. In each particle $P_i$, the two eigenstates of the corresponding spin operator $S_{\vec{I}_i}$ in direction $\vec{v}$ are $|\uparrow_i\rangle$ and $|\downarrow_i\rangle$, such that $S_{\vec{I}_i} |\uparrow_i\rangle = \frac{1}{2} |\uparrow_i\rangle$ and $S_{\vec{I}_i} |\downarrow_i\rangle = -\frac{1}{2} |\downarrow_i\rangle$. Therefore, a pure initial state of $U$ reads

$$|\psi_0\rangle = (a |\uparrow\rangle + b |\downarrow\rangle) \otimes \bigotimes_{i=1}^N (\alpha_i |\uparrow_i\rangle + \beta_i |\downarrow_i\rangle)$$ (1)

where the coefficients $a$, $b$, $\alpha_i$, $\beta_i$ are such that satisfy $|a|^2 + |b|^2 = 1$ and $|\alpha_i|^2 + |\beta_i|^2 = 1$. The self-Hamiltonians $H_S$ and $H_E$ of $S$ and $E$, respectively, are taken to be zero.

Then the total Hamiltonian $H = H_S + H_E + H_{SE}$ of the composite system $U$ results (see [1], [2])

$$H = H_{SE} = S_{\vec{I}} \otimes \sum_{i=1}^{N} 2g_i S_{\vec{I}_i} \otimes \bigotimes_{j \neq i} I_j$$ (2)

where $I_j$ is the identity operator on the subspace $\mathcal{H}_j$, $S_{\vec{I}} = \frac{1}{2} (|\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|)$ and $S_{\vec{I}_i} = \frac{1}{2} (|\uparrow_i\rangle \langle \uparrow_i| - |\downarrow_i\rangle \langle \downarrow_i|)$. Under the action of $H = H_{SE}$, the state $|\psi_0\rangle$ evolves as $|\psi(t)\rangle = a |\uparrow\rangle |\mathcal{E}_\uparrow(t)\rangle + b |\downarrow\rangle |\mathcal{E}_\downarrow(t)\rangle$ where $|\mathcal{E}_\uparrow(t)\rangle = |\mathcal{E}_\uparrow(-t)\rangle$ and

$$|\mathcal{E}_\uparrow(t)\rangle = \bigotimes_{i=1}^N (\alpha_i e^{-ig_1 t/2} |\uparrow_i\rangle + \beta_i e^{ig_1 t/2} |\downarrow_i\rangle)$$ (3)

An observable of $U$, $O \in \mathcal{O} = \mathcal{H} \otimes \mathcal{H}$, reads

$$O = \sum_{s,s'=\uparrow,\downarrow} s s' \langle s |s'| \rangle \otimes \bigotimes_{i=1}^N \left( \sum_{i=1}^{N} \frac{e^{i\epsilon_{i} t} |\uparrow_i\rangle \langle \uparrow_i| + e^{-i\epsilon_{i} t} |\downarrow_i\rangle \langle \downarrow_i|}{\epsilon_{i}^2/2} \right)$$ (4)

In the typical situation studied by the EID approach, the system of interest $S$ is simply the particle $P$. Therefore, the relevant observables $O_R$ are obtained from eq. (4) by making $\epsilon_{1}^t = \epsilon_{1}^1 = 1$ and $\epsilon_{1}^1 = 0$:

$$O_R = \sum_{s,s'\in\uparrow,\downarrow} s s' \langle s |s'| \rangle \otimes \bigotimes_{i=1}^N \left( \sum_{i=1}^{N} \frac{e^{i\epsilon_{i} t} |\uparrow_i\rangle \langle \uparrow_i| + e^{-i\epsilon_{i} t} |\downarrow_i\rangle \langle \downarrow_i|}{\epsilon_{i}^2/2} \right) = O_S \otimes I_E$$ (5)

The expectation value of these observables in the state $|\psi(t)\rangle$ is given by

$$\langle O_R |\psi(t)\rangle = |a|^2 s_{\uparrow\uparrow} + |b|^2 s_{\downarrow\downarrow} + 2 Re[ab^* s_{\uparrow\downarrow} r(t)]$$ (6)

where (see [2])

$$r(t) = \langle \mathcal{E}_\uparrow(t) |\mathcal{E}_\uparrow(t)\rangle = \sum_{i=1}^{N} |\alpha_i|^2 e^{-ig_1 t} + |\beta_i|^2 e^{ig_1 t}$$ (7)

It is clear that the time-behavior of $\langle O_R |\psi(t)\rangle$ depends on the behavior of $r(t)$. In order to know such a behavior,
where the $|\alpha_i|^2$ were obtained from a random-number generator, and the $|\beta_i|^2$ were computed as $|\beta_i|^2 = 1 - |\alpha_i|^2$. But before presenting the computer simulations, we will study the Poincaré time for this model.

**Analyzing the Poincaré time.** Each $f_i(t)$ of eq.(8) comes back to its initial value for the first time at a time $t_{P_i}$, such that $2g_it_{P_i} = 2\pi \Rightarrow t_{P_i} = \pi/g_i$. In turn, $|r(t)|^2$ comes back to its initial value when all the $f_i(t)$ do it. Therefore, the Poincaré time $t_P$ of this model is the time when all the $f_i(t)$ come back to their initial values for the first time. Let us consider three cases:

(a) All the $g_i$ have the same value: $g_i = g$, for all $i$. So, the mean value is $\overline{g_i} = g$. In this case, all the $f_i(t)$ come back to their initial values at the same time $t_{P_i} = \pi/g$. Therefore, the Poincaré time is $t_P = \pi/g = \pi/\overline{g_i}$, and it does not depend on the number of particles $N$.

(b) All the $g_i$ are such that $g_i = n_i g_{\text{min}}$, with $n_i \in \mathbb{N}$. In this case, $t_P$ is the largest $t_{P_i}$, corresponding to the smallest $g_i$, $g_{\text{min}}$: $t_P = \pi/g_{\text{min}}$. So, given a $g_{\text{min}}$, the Poincaré time $t_P$ does not depend on $N$. Since the mean value is $\overline{g_i} > g_{\text{min}}$, then $t_P > \pi/\overline{g_i}$.

(c) All the $g_i$ are random. Since in the computations the $g_i$ are rational numbers, we can express them as $g_i = p_i/q_i$, with $p_i, q_i \in \mathbb{N}$. If we make $t_P = \pi Q$, the number $Q$ has to be such that $Q = n_i q_i/p_i$ for all $i$, with $n_i \in \mathbb{N}$. Then, $n_i = Q p_i/q_i$. Since the $p_i$ and $q_i$ are random numbers, the least $Q$ that guarantees that $n_i$ is a natural number for all $i$ is $Q = \prod_{i=1}^N q_i$. Therefore,

$$t_P = \pi Q = \pi \prod_{i=1}^N q_i$$

### Case 1: Homogeneous environment.

Let us consider the case where all the particles of the environment are of the same kind. In this case, the reasonable assumption is that the particle $P$ interacts in the same way with all the environmental particles and, as a consequence, $g_i = g$ for all $i$ (Case (a)). The time-behavior of $|r(t)|^2$ of eq.(8), for $N = 100$ and $g_i = g = 0.5$, is plotted in Figure 1: we can see that the Poincaré time $t_P = \pi/g = \pi/0.5 \approx 6.28$ is not sufficiently larger than the decoherence time to consider the result as an effective decoherence that may lead to classicality.

Now let us consider the case that $P$ does not interact in the same way with all the environmental particles. In particular, we will suppose that the coupling coefficients are random in an interval $[\overline{g_i} - \Delta g, \overline{g_i} + \Delta g]$ around the mean value $\overline{g_i}$. The time-behavior of $|r(t)|^2$ of eq.(8), for $N = 100$ and $g_i = g = 0.5$, is plotted in Figure 2. In this case, the Poincaré time is much larger than $(\pi/\overline{g_i})^N = (6.28)^{100}$ (Case (c)), a value that can be considered infinite for all practical purposes. Therefore, as Figure 2 shows, it can be legitimately said that the particle $P$ decoheres and may become classical.

**Case 2: Non-homogeneous environment.** In this case we will consider an environment whose particles are not all of the same kind, and the particles of each kind $j$ interact with $P$ through their own coupling coefficient $g_j$: $N_1$ particles through $g_1$, $N_2$ particles through $g_2$, ... and $N_p$ particles through $g_p$, such that $\sum_{j=1}^p N_j = N$.

Then, the $|r(t)|^2$ of eq.(8) can be rewritten as:
The coupling coefficients “contaminated” environment does not decohere. Then, the Poincaré time can be easily computed as:

$$|r(t)|^2 = \prod_{i=1}^{N} f_i(t) = \left( \prod_{i=1}^{N_1} f_{i,g_1}(t) \right) \left( \prod_{i=1}^{N_2} f_{i,g_2}(t) \right) \cdots \left( \prod_{i=1}^{N_p} f_{i,g_p}(t) \right)$$

where each particular product is the contribution of each kind of particles. In particular, we will consider a situation where the environment is composed of $N$ particles that are almost all of the same kind, with the exception of a slight “contamination” of particles of different kinds. The time-behavior of $|r(t)|^2$ of eq. (10), for $N = 100$, $N_1 = 91$, $N_2 = N_3 = N_4 = 3$, $g_1 = 2.4$, $g_2 = 1.2$, $g_3 = 0.6$ and $g_4 = 0.3$, is plotted in Figure 3. Since all the $g_i$ are such that $g_i = n_i g_{\text{min}} = n_i g_4$, this situation corresponds to Case (b). Then, the Poincaré time can be easily computed as $t_P = \pi / g_{\text{min}} = \pi / 0.3 \simeq 10.43$. Therefore, in Figure 3, the peak in 10.43 is the Poincaré time, but the peaks around it are not due to the recurrence of $|r(t)|^2$. This means that the particle $P$ in interaction with this “contaminated” environment does not decohere.

Now we will consider that $P$ does not interact in the same way with the particles of a same kind $j$, but the coupling coefficients $g_{ji}$ are random in the intervals $[g_{ji} - \Delta g_j, g_{ji} + \Delta g_j]$ around the corresponding mean value $g_{ji}$. $N_j$ particles with $g_{ji} \in [g_{ji} - \Delta g_j, g_{ji} + \Delta g_j]$, $N_2$ particles with $g_{2i} \in [g_{2i} - \Delta g_2, g_{2i} + \Delta g_2]$, ... and $N_p$ particles with $g_{pi} \in [g_{pi} - \Delta g_p, g_{pi} + \Delta g_p]$, such that

$$\sum_{j=1}^{N} N_j = N.$$ Then, the $|r(t)|^2$ of eq. (5) can be rewritten as:

$$|r(t)|^2 = \prod_{i=1}^{N} f_i(t) = \left( \prod_{i=1}^{N_1} f_{i,g_1}(t) \right) \left( \prod_{i=1}^{N_2} f_{i,g_2}(t) \right) \cdots \left( \prod_{i=1}^{N_p} f_{i,g_p}(t) \right)$$

In particular, we will study a situation similar than the previous one with respect to the values of $N$ and of the $N_j$, and where the $g_j$ used there become here the mean values $\bar{g}_{ji}$. Moreover, in all the cases the $\Delta g_j$ were selected as approximately the 30% of the corresponding mean value $\bar{g}_{ji}$. The time-behavior of $|r(t)|^2$ of eq. (11), for $N = 100$, $N_1 = 91$, $N_2 = N_3 = N_4 = 3$, $\bar{g}_{1i} = 2.4$, $\bar{g}_{2i} = 1.2$, $\bar{g}_{3i} = 0.6$, $\bar{g}_{4i} = 0.3$, $\Delta g_1 = 0.8$, $\Delta g_2 = 0.4$, $\Delta g_3 = 0.2$ and $\Delta g_4 = 0.1$, is plotted in Figure 4, where we can see the drastic decoherence of the model. The same result is plotted in Figure 5 with a different time scale, in order to show the decoherence time.

**Conclusions.** By means of computer simulations we have proved the great influence that the randomness of the coupling coefficients exerts on decoherence. Such an influence appears under two forms: (i) the huge increasing of the Poincaré time, and (ii) the strong damping off the peaks that preclude decoherence. Then, in both cases the change from constant coefficients to random coefficients greatly improves the “efficiency” of decoherence. It is worth stressing that, whereas the first effect may be expected, the second effect is not foreseeable: it is not easy to explain a priori why the randomness of the coupling coefficients washes off the high peaks present in the non-random case.

On the other hand, a realistic environment is usually composed by a limited number of different kinds of particles, and it is reasonable to suppose that the particle of interest $P$ interacts with all the particles of the same kind with the same strength. So, the models with random coupling coefficients are not realistic. As we have pointed out in the Introduction, this would be not relevant if the...
randomness of the coefficients had no significant effect. But now we know that such randomness has a dramatic influence on decoherence, whether it is expectable or not. Therefore, the criterion for the selection of the coupling coefficients has to be carefully analyzed in each model, in order to avoid conclusions drawn from unphysical results.

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