Giant atom induced zero modes and localization in the nonreciprocal Su–Schrieffer–Heeger chain

J J Wang\textsuperscript{1}, Fude Li\textsuperscript{1} and X X Yi\textsuperscript{1,2,}\textsuperscript{*}

\textsuperscript{1} Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, People’s Republic of China
\textsuperscript{2} Center for Advanced Optoelectronic Functional Materials Research, and Key Laboratory for UV-Emitting Materials and Technology of Ministry of Education, Northeast Normal University, Changchun 130024, People’s Republic of China

E-mail: yixx@nenu.edu.cn

Received 25 May 2023; revised 18 August 2023
Accepted for publication 20 September 2023
Published 18 October 2023

Abstract
A notable feature of non-Hermitian systems with skin effects is the sensitivity of their spectra and eigenstates to the boundary conditions. In the literature, three types of boundary conditions--periodic boundary condition, open boundary condition (OBC) and a defect in the system as a boundary, are explored. In this work we introduce the other type of boundary condition provided by a giant atom. The giant atom couples to a nonreciprocal Su–Schrieffer–Heeger (SSH) chain at two points and plays the role of defects. We study the spectrum and localization of eigenstates of the system and find that the giant atom can induce asymmetric zero modes. A remarkable feature is that bulk states might localize at the left or the right chain-atom coupling sites in weak localization regimes. This bipolar localization leads to Bloch-like states, even though translational invariance is broken. Moreover, we find that the localization is obviously weaker than the case with two small atoms or OBCs even in strong coupling regimes. These intriguing results indicate that nonlocal coupling of the giant atom to a nonreciprocal SSH chain weakens the localization of the eigenstates. We also show that the Lyapunov exponent in the long-time dynamics in real space can act as a witness of the localized bulk states.

Keywords: localization, Su–Schrieffer–Heeger, giant atom, non-Hermitian topology

(Some figures may appear in colour only in the online journal)

\textsuperscript{*} Author to whom any correspondence should be addressed.
1. Introduction

In closed quantum systems, observables like Hamiltonians are represented by Hermitian operators. This is not the case for real systems that are inevitably coupled to environments. It is widely accepted that open quantum systems can be described effectively by non-Hermitian operators based on the quantum trajectory approach. Recently non-Hermitian systems have attracted a lot of attention, [1–7] especially in the field of topological physics. Interesting features are predicted and observed such as the topological insulator laser [8–10], new topological invariants in non-Hermitian systems [11–18], and the breakdown of the conventional bulk-boundary correspondence [19–41].

The systems with non-Hermitian skin effect (NHSE) are very sensitive with respect to boundary conditions. For example, the properties of both spectrum and eigenstates of non-Hermitian systems may change dramatically by changing the boundary conditions from periodic to open ones [21]. In between, a defect introduced into the system could also play the role of boundary [42–46]. Non-Hermitian impurity physics [42] was developed theoretically in a non-reciprocal lattice, where the authors found a new type of steady-state localization characterized by scale-free accumulation of eigenstates with localization lengths proportional to the system size. The scale-free localization can be interpreted by the generalized non-Bloch band theory [43, 44], and strong defect-system couplings might induce an effective boundary in a periodic system with properties very similar to systems with the open boundary condition (OBC) [45–48].

Giant atom was first physically realized by nonlocally coupling a transmon qubit to surface acoustic waves via an interdigitated transducer [49]. Because of the slow velocity of the acoustic wave, the size of the transmon is bigger than the wavelength, so the qubit is called the giant atom. On the other hand, giant atom such as superconducting qubit has played an important role in superconducting quantum circuits. It can be nonlocally coupled to a waveguide at multiple points [50–58] and act as an effective boundary for the waveguide. The giant atom might induce chiral bound states for the Hermitian systems of the waveguide [55, 56]. This begs the question that what properties of the system will change if a giant atom couples to a non-Hermitian topological chain and what new physical phenomena emerge. Furthermore, it is interesting to ask if there is a difference between the localization induced by a giant atom or other impurities and defects.

In this work, we will answer these questions and focus on spectrum structures and the localization of eigenstates in a system composed of a giant atom and a nonreciprocal Su–Schrieffer–Heeger (SSH) chain. We find that the giant atom can induce asymmetric zero modes. Bulk states may localize at the left or the right chain–atom coupling sites, so we call this ‘bipolar localization’. Further examination shows that strong atom–chain coupling cannot induce transitions from skin-free states to skin states in the bulk states. This suggests that the localization is weaker than that with OBC even in strong coupling regimes, and nonlocal couplings weaken localization of the eigenstates. We also show that the feature of localization of the bulk states can be captured by the dynamics in real space.

The paper is organized as follows. In section 2, we introduce a model to describe the system composed of a giant atom and a nonreciprocal SSH chain and calculate the spectrum of the system. In section 3, we derive an analytical expression for the zero modes induced by the giant atom. Two cases, one with two small atoms and the other with a giant atom, are compared and discussed in detail. In section 4, we mainly focus on the localization feature of the bulk eigenstates. In section 5, we explore the relation between the Lyapunov exponent and the localization of the bulk states. Finally, we summarize our results in section 6.
2. Model and spectrum

As depicted in figure 1(a), we consider a nonreciprocal SSH chain with the periodic boundary conditions (PBC) in real space described by

\[ H_{\text{SSH}} = \sum_{l=1}^{L} \left[ (t_1 + \gamma) \hat{C}_{A,l}^{\dagger} \hat{C}_{B,l} + (t_1 - \gamma) \hat{C}_{B,l}^{\dagger} \hat{C}_{A,l} + t_2 \hat{C}_{A,l}^{\dagger} \hat{C}_{B,l+1} + t_2 \hat{C}_{B,l+1}^{\dagger} \hat{C}_{A,l} \right], \]

where the chain consists of \( L \) unit cells, and the staggered nearest-neighbor hopping amplitudes are \( t_1 \pm \gamma \) and \( t_2 \), respectively. The asymmetry of hopping amplitudes \( (\gamma \neq 0) \) leads to the non-Hermiticity of the system. \( \hat{C}_{A,l}^{\dagger} \) and \( \hat{C}_{A,l} \) are the creation and annihilation operators for the sublattice site \( A \) at site \( l \). To have an effective boundary, we introduce a two level giant atom coupling at two points to a nonreciprocal SSH chain via \( A \rightarrow B \) (or \( A \rightarrow A \)) couplings, where nonlocal coupling points locate at sites \( n \) and \( m \) of lattice. The interaction Hamiltonian for such a system can be written as

\[ H_{I,AB} = g \sigma^+ \left( \hat{C}_{A,n} + \hat{C}_{B,m} \right) + \text{H.c.}, \]
\[ H_{I,AA} = g \sigma^+ \left( \hat{C}_{A,n} + \hat{C}_{A,m} \right) + \text{H.c.}, \]

where \( g \) denotes the atom–chain coupling strength, \( \sigma^+ = |e\rangle\langle g| \) is the usual pseudospin ladder operator of the giant atom, \( |g\rangle \) and \( |e\rangle \) are the ground state and the excited state of the giant atom, respectively. The Hamiltonians of atom–chain coupling read

\[ H_{AB} = H_{\text{SSH}} + H_{I,AB}, \] \[ H_{AA} = H_{\text{SSH}} + H_{I,AA}. \]

This system can be physically realized in electrical circuit. The nonreciprocal coupling might be achieved by connecting capacitor in series with a voltage follower, which is composed of an operational amplifier with a negative feedback network. In addition, two-level giant atom can be realized by using Josephson junctions. For the particularly experimental scheme, adding a constant imaginary shift to all sites corresponding to a passive setting with loss only [9], this correction does not affect the localization of eigenstates and existence of boundary modes.

First, we consider a system consisting of a giant atom coupled to a nonreciprocal SSH chain via \( A \rightarrow B \) coupling. The Hamiltonian of the system in the momentum space reads

\[ H_{AB}(k) = H_{\text{SSH}}(k) + H_{I,AB}(k) = \sum_{k} \left[ (t_1 + \gamma) \hat{C}_{A,k}^{\dagger} \hat{C}_{B,k} + (t_1 - \gamma) \hat{C}_{B,k}^{\dagger} \hat{C}_{A,k} + t_2 \hat{C}_{A,k}^{\dagger} \hat{C}_{B,k+1} + t_2 \hat{C}_{B,k+1}^{\dagger} \hat{C}_{A,k} \right] + g \sqrt{L} \sum_{k} \left[ \sigma^+ \left( \hat{C}_{A,k} e^{i2n} + \hat{C}_{B,k} e^{i2m} \right) + \text{H.c.} \right], \]
Figure 1. Sketch of the system. (a) A giant atom coupled to a nonreciprocal SSH chain. (b) Two small atoms coupled to a nonreciprocal SSH chain as a contrast. There are two cases for couplings in (a) and (b). Solid lines correspond to $A - B$ couplings, and dashed lines correspond to $A - A$ couplings.

where $\hat{C}_{\alpha(\beta),k}$ and $\hat{\tilde{C}}_{\alpha(\beta),k}$ are the creation and annihilation operators for the sublattice site $\alpha(\beta)$ at site $k$. Let us consider the system with $2L$ lattice sites and one site for giant atom, for which the matrix form of the Hamiltonian $H_{AB}(k)$ in the local site basis satisfies

$$
H_{AB}(k) = \begin{pmatrix}
0 & t_1 + \gamma + t_2 e^{-ik_1} & 0 & 0 & \ldots & 0 \\
t_1 - \gamma + t_2 e^{ik_1} & 0 & 0 & 0 & \ldots & \frac{g e^{-ik_1 n} \sqrt{L}}{\sqrt{K}} \\
0 & 0 & 0 & t_1 + \gamma + t_2 e^{-ik_1} & \ldots & \frac{g e^{-ik_2 m} \sqrt{L}}{\sqrt{K}} \\
0 & 0 & t_1 - \gamma + t_2 e^{ik_2} & 0 & \ldots & \frac{g e^{-ik_2 m} \sqrt{L}}{\sqrt{K}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{g e^{ik_1 n} \sqrt{L}}{\sqrt{K}} & \frac{g e^{ik_1 n} \sqrt{L}}{\sqrt{K}} & \frac{g e^{ik_2 m} \sqrt{L}}{\sqrt{K}} & \frac{g e^{ik_2 m} \sqrt{L}}{\sqrt{K}} & \ldots & 0
\end{pmatrix}.
$$

(5)

In general, a pure nonreciprocal SSH model has a chiral symmetry. But this model does not preserve chiral symmetry, because the giant atom couples to the SSH chain with $A - B$ couplings.

In order to derive a condition for the emergence of the zero mode, in the single-excitation subspace, the eigenstate of the Bloch Hamiltonian in momentum space can be expressed as

$$
|\psi\rangle = U_e |e, G\rangle + \sum_k \alpha_k \hat{C}_{\alpha,k}^\dagger |g, G\rangle + \sum_k \beta_k \hat{\tilde{C}}_{\beta,k}^\dagger |g, G\rangle,
$$

(6)

where $|G\rangle$ denotes the ground state of the SSH chain (vacuum state). $\alpha_k(\beta_k)$ denotes the amplitude in site $\alpha(\beta)$ of the $k$th unit cell, and $U_e$ denotes the amplitude in site of the giant atom.
Making use of the time-independent Schrödinger equation $\mathbb{H}_{AB}(k)|\psi\rangle = E|\psi\rangle$, we obtain the eigenvalues of the system with $A - B$ couplings

$$E = \frac{2g^2}{L} \sum_k \left( E + t_1 \cos[k(m-n)] - \gamma \sin[k(m-n)] ight)$$

$$+ t_2 \cos[k(m-n+1)]/(E^2 - \omega_k^2),$$

with

$$\omega_k = \sqrt{(t_1 + \gamma + t_2 e^{-ik})(t_1 - \gamma + t_2 e^{ik})}. \quad (8)$$

Setting $E = 0$, we can derive a condition for the emergence of the zero mode, i.e. $t_1 \in [-t_2 + \gamma, t_2 - \gamma]$. In the absence of giant atoms, the different topological phases for a pure nonreciprocal SSH model can be distinguished by the winding number $[16] \gamma = \frac{1}{\pi} \int_{-\pi}^{\pi} dk \langle \phi^e | i \partial_k | \phi^h \rangle$, where $\langle \phi^e \rangle$ and $\langle \phi^h \rangle$ are the left and the right eigenstates of a pure nonreciprocal SSH model. Assuming $t_2 > \gamma$, it satisfies

$$\gamma = \begin{cases} 
1, & \left( -t_2 + \gamma < t_1 < t_2 - \gamma \right), \\
\frac{1}{2}, & \left( -t_2 - \gamma < t_1 < -t_2 + \gamma \right) \\
\text{or} \ (t_2 - \gamma < t_1 < t_2 + \gamma), \\
0, & \left( t_1 < -t_2 - \gamma \right) \text{ or } \left( t_1 > t_2 + \gamma \right). 
\end{cases} \quad (9)$$

Note that the condition for the emergence of the zero mode $[t_1 \in ( -t_2 + \gamma, t_2 - \gamma )]$ is identical to the non-trivial phase boundary of a pure SSH chain without giant atoms $\gamma = 1$ (see appendix A for analytical results and figures 2(a)–(c) for numerical simulations). Except for the modes in the energy gap, there are other eigenvalues outside (lying above and below) the continuous bands as shown in figure 2(a). We call the corresponding eigenstates upper and lower bound states, respectively. Interestingly, eigenvalues form a close circle (red loop) in figure 2(c), because the system crosses two exceptional points (EPs) as the increase of parameter $t_1$. Combined with figures 2(a) and (b), it is clearly shown that complex conjugate pair of eigenvalues become purely real when crossing the left EP, and adjacent real eigenvalues merge into complex conjugate pairs when crossing the right EP.

Next, we consider a system consisting of a giant atom coupled to a nonreciprocal SSH chain via $A - A$ coupling. Similarly, matrix form of Hamiltonian (3b) of the system with $A - A$ couplings in the momentum space can be written as

$$H_{AA}(k) = \begin{pmatrix}
0 & t_1 + \gamma + t_2 e^{-ik_1} & 0 & 0 & \cdots & 0 & \frac{\sqrt{L}}{2} e^{-i(k_1 + \cdots)} \\
t_1 - \gamma + t_2 e^{ik_1} & 0 & 0 & 0 & \cdots & 0 & \frac{\sqrt{L}}{2} e^{i(k_1 + \cdots)} \\
0 & 0 & t_1 + \gamma + t_2 e^{-ik_1} & 0 & \cdots & 0 & \frac{\sqrt{L}}{2} e^{-i(k_1 + \cdots)} \\
\sqrt{L} e^{-i(k_1 + \cdots)} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\sqrt{L} e^{i(k_1 + \cdots)} & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
\sqrt{L} & 0 & 0 & 0 & \cdots & 0 & 0 \\
\sqrt{L} & 0 & 0 & 0 & \cdots & 0 & 0 
\end{pmatrix}. \quad (10)$$
Figure 2. Spectrum as a function of $t_1$ in (a)–(c) with $A - B$ and (d)–(f) with $A - A$ coupling, respectively. First (second) row show real (imaginary) part of the spectrum, while last row shows absolute value of the spectrum. The results are obtained by numerically solve the Schrödinger equation. The chosen parameters are $L = 50, m = 26, n = 25, g = 1, t_1 = 0.2, t_2 = 1$, and $\gamma = 0.5$.

Fortunately, this system still has a chiral symmetry $\Gamma^{-1} H_{AA}(k) \Gamma = -H_{AA}(k)$ with

$$
\Gamma = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & -1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & -1
\end{pmatrix}.
$$

The corresponding eigenvalue satisfies

$$
E = \frac{2g^2}{L} \sum_k \left( E \left(1 + \cos k(m-n)\right) \right).
$$

Obviously, $E = 0$ is always the solution of equation (12), which can be seen in the numerical spectra of Hamiltonian $H_{AA}$ (figures 2(d)–(f)). Except for two lattice sites $A$ coupld to the giant atom as effective boundaries, the other parts of the SSH chain are similar to chains with a $B$ site at both ends, and no longer a whole SSH chain. Hence, there is always a zero energy state. Of course, the condition for the emergence of the zero mode with $A - A$ couplings is no longer identical to the phase boundary of a pure SSH chain. Besides, in figure 2(d), we...
find that the upper and lower eigenvalues merge into the continual band as \( t_1 \) decreases, which indicates that the corresponding bound states vanished.

### 3. Zero mode and bound states

To begin with, we define population as modular square of the probability amplitude \( |A(B)|^2 \) in real space. The population for zero modes and upper bound states as a function of site \( N \) with different parameters are shown in figure 3 for \( A-B \) coupling and in figure 4 for \( A-A \) coupling. The population on the giant atom is set at \( N = 2L + 1 = 101 \). The bars represent the numerical results and the empty circles represent the analytical results. Simple algebra shows that the analytical results of the probability amplitudes of bound states with energy \( E \) take (see appendix A2 for details)

\[
\frac{A_l}{U_e} = \frac{(-1)^{l+1}}{\sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)}} \left( \frac{T_{l_1}^{[l-n]} + gT_{l_2}^{[l-m-1]} + Y_{l_2}^{[l-m]}}{t_2 - \gamma} \right),
\]

\[
\frac{B_l}{U_e} = \frac{(-1)^{l+1}}{\sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)}} \left( \frac{Y_1^{[l-n]} + gY_{l_1}^{[l-n+1]} + Y_{l_1}^{[l-n]}}{t_2 - \gamma} \right),
\]

for the \( A-B \) coupling, and

\[
\frac{A_l}{U_e} = \frac{(-1)^{l+1} T_{l_1}^{[l-n]} + Y_{l_2}^{[l-m]}}{\sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)}} + g \left( \frac{Y_1^{[l-n]} + Y_{l_1}^{[l-n+1]}}{t_2 - \gamma} \right),
\]

for the \( A-A \) coupling, where \( A_l(B_l) \) denotes the amplitude in site \( A(B) \) of the \( l \)th unit cell. Here, \( y = \theta(x) \) is the step function, and \( x = (E^2 - (t_1 + \gamma)(t_1 - \gamma) - t_2^2)/t_2 \). We also define \( T = gE/t_2 \), \( Y_1 = g(t_1 - \gamma)/t_2 \), \( Y_2 = g(t_1 + \gamma)/t_2 \). Besides, \( \tau_1 = a(b) \) for \( l \leq n \), \( \tau_2 = a(b) \) for \( l > m \), \( \tau_3 = a(b) \) for \( l \geq m \). In the end, \( a = (x + \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 + \gamma), b = (x - \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 + \gamma) \), \( a = (x + \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 - \gamma) \), \( b = (x - \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 - \gamma) \) for \( x < 2|t_1| \), \( a = (x - \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 + \gamma), b = (x + \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 - \gamma) \) for \( x > 2|t_1| \).

For the system with \( A-B \) coupling, the zero mode present in the bulk topological non-trivial phase of nonreciprocal SSH model with \(-t_2 + \gamma < t_1 < t_2 - \gamma\). Setting \( E = 0 \), the population of the zero mode can be obtained from equations (13a) and (13b),

\[
A_l/U_e = Y_3 \times \begin{cases} (t_1 - \gamma)_{l-n}, & (l > m), \\ 0, & (l \leq m), \end{cases}
\]

\[
B_l/U_e = Y_4 \times \begin{cases} (t_1 + \gamma)_{n-l}, & (l < n), \\ 0, & (l \geq n), \end{cases}
\]

where \( Y_3 = g/(t_1 - \gamma) \), and \( Y_4 = g/(t_1 + \gamma) \). We find that the probability amplitudes in the wave function satisfy the asymmetrical property \( |A_l| \neq |B_{m+n-l}| \). Assuming \( \gamma > 0 \), we have \( |(t_1 - \gamma)/t_2| < |(t_1 + \gamma)/t_2| \) for \( t_1 > 0 \), and \( |(t_1 + \gamma)/t_2| < |(t_1 - \gamma)/t_2| \) for \( t_1 < 0 \), which suggests that the system mainly occupies \( B \) sites on the left side of the giant atom when \( t_1 > 0 \),
Figure 3. Population of bound states for the system (3a) with $A - B$ coupling as a function of site $N$. (a) and (c) are plotted for the population of zero mode. (b) and (d) are plotted for the population of the upper bound state. Here, $t_1 = 0.2$ in (a) and (b), and $t_1 = -0.2$ in (c) and (d). The other parameters are chosen as $L = 50, m = 40, n = 20, g = 1, t_2 = 1$, and $\gamma = 0.5$. The site of giant atom is set at $N = 2L + 1 = 101$.

Figure 4. Population of the system (3b) in bound states with $A - A$ coupling as a function of site $N$. (a) and (c) are plotted for the population of zero mode, and corresponding wave functions do not distribute on $A$ sites. (b) and (d) are plotted for the population of the upper bound state. The system in the bulk topological non-trivial phase with $t_1 = 0.2$ in (a) and (b) and the bulk topological trivial phase with $t_1 = 1.6$ in (c) and (d). The other parameters are set as $L = 50, m = 40, n = 20, g = 1, t_2 = 1$, and $\gamma = 0.5$. The site of giant atom is set at $N = 2L + 1 = 101$.

and occupies the $A$ sites on the right side of the giant atom when $t_1 < 0$. Obviously, population of wave functions on the left and right sides of the giant atom are the same for the critical point with $t_1 = 0$. Besides, as the increase of $g$, the probability of the system on the giant atom are gradually suppressed, but the spatial symmetry of zero modes on the SSH chain cannot be changed.
In figures 3(a) and (c), we show the population of zero modes for $A - B$ coupling as a function of lattice site $N$ with $t_1 = 0.2$ and $t_1 = -0.2$, respectively. In this case, the zero modes mainly locate on the left (right) sides of the giant atom with $t_1 = 0.2$ ($t_1 = -0.2$), and satisfy the chiral symmetry [55]. It is clear that the analytical results (empty circles) given by equation (15) are in good agreement with the numerical results (bars). In figures 3(b) and (d), we show the corresponding population of the upper bound state with $t_1 = 0.2$ and $t_1 = -0.2$, respectively. Although the population of upper bound state has no obvious symmetry, asymmetry of hopping amplitudes ($\gamma \neq 0$) can induce the upper bound states to distribute on the left (right) of coupling point more than the right (left) of coupling point when $t_1 = 0.2$ ($t_1 = -0.2$). Similarly, analytical results given by equations (13a) and (13b) are in good agreement with the numerical results.

For the system with $A - A$ coupling, zero modes are always exist regardless of the value of $t_1$. According to equations (14a) and (14b), we can get the population of the zero mode, which gives rise to $A_1 = 0$, and

$$\frac{B_l}{U_e} = Y_4 \times \begin{cases} \left(-\frac{\gamma}{t_1}\right)^{(n-l)} + \left(-\frac{\gamma}{t_2}\right)^{(m-l)}, & (l < n), \\ \left(-\frac{\gamma}{t_1}\right)^{(m-l)}, & (n \leq l < m), \\ 0, & (m \leq l), \end{cases}$$

(16)

in the bulk topological non-trivial phase of the SSH model with $-t_2 + \gamma < t_1 < t_2 - \gamma$ and

$$\frac{B_l}{U_e} = -Y_4 \times \begin{cases} 0, & (l < n), \\ \left(-\frac{\gamma}{t_1+\gamma}\right)^{l-n}, & (n \leq l < m), \\ \left(-\frac{\gamma}{t_1+\gamma}\right)^{l-n} + \left(-\frac{\gamma}{t_2+\gamma}\right)^{l-m}, & (m \leq l), \end{cases}$$

(17)

in the bulk topological trivial phase of the SSH model with $t_1 > t_2 + \gamma$ or $t_1 < -t_2 - \gamma$. As for $-t_2 - \gamma < t_1 < -t_2 + \gamma$ and $-t_2 - \gamma < t_1 < -t_2 + \gamma$, the analytical results of zero model are not given. Note that the amplitudes of the wave functions only distribute on $B$ sites on the left sides of the giant atom or between two coupling points with $-t_2 + \gamma < t_1 < t_2 - \gamma$, and distribute on $B$ sites on the right sides of the giant atom or between two coupling points with $t_1 > t_2 + \gamma$ or $t_1 < -t_2 - \gamma$. Similarly, population of wave function can also be changed by tuning the parameters of the system. Taking $t_1 t_2$ and $\gamma$ to be positive leads to $\gamma/t_2 < \left| (t_1 + \gamma)/t_2 \right| < 1$ in the bulk topological non-trivial phase, and $0 < \left| t_2/(t_1 + \gamma) \right| < t_2/(t_2 + 2\gamma)$ in the bulk topological trivial phase. Taking $t_2 \leq 2\gamma$ yields $\gamma/t_2 \geq t_2/(t_2 + 2\gamma)$. This indicates that the decay of population in the bulk topological non-trivial phase are always slower than the bulk topological trivial phase.

In order to verify the above analysis, in figures 4(a) and (c), we show the population of zero modes for $A - A$ coupling as a function of lattice site $N$ with $t_1 = 0.2$ and $t_1 = 1.6$, respectively. The results in figures 4(a) and (c) are obtained by numerical simulations and they show that $|B_l| \approx |B_{l+m-n}|$ are satisfied. The decay of population with $t_1 = 0.2$ are slower than the population with $t_1 = 1.6$. In figures 3(b) and (d), we show the corresponding population of the upper bound state with $t_1 = 0.2$ and $t_1 = 1.6$, respectively. It is easy to find that they satisfy $|A(B)_l| \approx |A(B)_{l+m-n}|$, and the non-reciprocity makes upper bound states to distribute on the left of coupling point more than the right of coupling point.

In figures 3 and 4, we mainly present zero modes for the system composed of a giant atom and a nonreciprocal SSH chain. Note that zero modes localize at two effective boundaries. While for a nonreciprocal SSH chain with OBC, zero modes only localize at one of the two
Figure 5. Population of zero modes for the system that two small atoms couple to a nonreciprocal SSH chain as a function of site \( N \). (a) and (b) are plotted for one of two zero modes with \( A - B \) coupling, respectively. (c) and (d) are plotted for one of two zero modes with \( A - A \) coupling, respectively. Here, \( L = 50, m = 40, n = 20, g = 1, t_1 = 0.2, t_2 = 1, \) and \( \gamma = 0.5 \). The site of giant atom is set at \( N = 2L + 1 = 101 \).

These two cases are different, because giant atom not only induces two effective boundaries, but also couples them. In order to show the validity of the above conclusions, we considered the population of zero modes for the system that two small atoms couple to a nonreciprocal SSH chain (figure 1(b)), where coupling points locate at sites \( m \) and \( n \), respectively. The corresponding Hamiltonians can be expressed as

\[
H_{AB}^s = H_{ssh} + H_{I,AB}^s, \tag{18a}
\]

\[
H_{AA}^s = H_{ssh} + H_{I,AA}^s, \tag{18b}
\]

with

\[
H_{I,AB}^s = g \left( \sigma_1^+ \hat{C}_{A,m} + \sigma_2^+ \hat{C}_{B,m} \right) + \text{H.c.},
\]

\[
H_{I,AA}^s = g \left( \sigma_1^+ \hat{C}_{A,m} + \sigma_2^+ \hat{C}_{A,m} \right) + \text{H.c.}, \tag{19}
\]

where \( \sigma_i^+ = |e_i\rangle \langle g_1| \), and \( \sigma_i^+ = |e_2\rangle \langle g_2| \) are the usual pseudospin ladder operators of two small atoms, \( |g_1(2)\rangle \) and \( |e_1(2)\rangle \) are the ground state and the excited state of the small atoms, respectively. As shown in figure 5, there are two zero modes for small-atom case. Both modes localize at the right coupling point for \( A - A \) coupling, and one for each coupling point for \( A - B \) coupling with \( m = 40, n = 20, g = 1, t_1 = 0.2, t_2 = 1, \) and \( \gamma = 0.5 \). It is clearly shown that although two small atoms as defects induce two effective boundaries, zero modes localize at only one of the two boundaries. Note that this case is similarly to the zero mode with OBC. Hence, it can be seen that the zero modes for systems with giant atom are very unique, because nonlocal couplings weaken the localization.
4. Localization of eigenstates

Our aim is to study the localization of eigenstates, and we only interest in the wave functions living at the SSH chain, so population on the giant atom is not considered. To this aim, we calculate the modified averaged inverse participation ratio $\overline{\text{IPR}}$,

$$\overline{\text{IPR}} = \frac{1}{N} \sum_{q=1}^{N} \text{IPR}_q$$

with $\text{IPR}_q = \sum_{N=1}^{2L} |\psi_N^q|^4 / \left( \sum_{N=1}^{2L} |\psi_N^q|^2 \right)^2$ and $\psi_N^q$ is the probability amplitude at site $N$ of the $q$th right eigenstate of Hamiltonian in real space. $\overline{\text{IPR}}$ will enhance as the increase of the localization. In figure 6(a), we show $\overline{\text{IPR}}$ on the parameter space of $g_m$ and $g_n$ for the giant-atom case with $A - B$ coupling ($A - A$ coupling is similar), $g_m$ and $g_n$ are the atom–chain coupling strength, where $m$ and $n$ denote the coupling sites. Note that $\overline{\text{IPR}}$ is suppressed when $g_m = g_n$, and can be lifted as the increase of $\delta g = |g_m - g_n|$, which indicates that the localization will be suppressed for the same coupling strength setting. Particularly, for the case of $g_n \neq 0$ ($g_m = 0$) or $g_m \neq 0$ ($g_n = 0$), i.e. like a small-atom case, $\overline{\text{IPR}}$ will significant enhance as the increase of $g_n$ or $g_m$, which suggests that nonlocal couplings weaken localization of eigenstates for giant-atom case.

In the last section, we have shown the behavior of the bound states. Next, to analyze the localization of bulk states in detail, let us consider the real-space eigen-equation of the system. In the bulk of chain, it satisfies

$$\begin{aligned}
t_2 \psi_{B,J+1} + \left[ t_1 + \gamma \right] \psi_{B,J} &= E \psi_{A,J}, \\
\left[ t_1 - \gamma \right] \psi_{A,J} + t_2 \psi_{A,J+1} &= E \psi_{B,J}.
\end{aligned}$$

With $(\psi_{A,J}, \psi_{B,J}) = \beta'(\psi_A, \psi_B)$, it yields

$$\beta_{1,2}(E) = \frac{\Delta \pm \sqrt{\Delta^2 - 4t_1^2 (t_1^2 - \gamma^2)}}{2t_2(t_1 + \gamma)},$$

where $\Delta = E^2 + \gamma^2 - t_1^2 - t_2^2$, and $+$($-$) corresponds to $\beta_1$ ($\beta_2$). In figures 6(b)–(e), we show $|\beta_1|$ (red star) and $|\beta_2|$ (blue circle) as a function of $q$ with different coupling strengths $(g_m, g_n) = (0, 0), (1, 1), (13, 13)$, and $(13, 1)$. For the system under OBC without the giant atom, $|\beta_{1,2}| = \sqrt{[(t_1 - \gamma)/\left(t_1 + \gamma]\right]}$ [21] correspond to the black line in the nearly middle in figures 6(b)–(e) and the wave functions take the form of skin states. While for the system under PBC without the giant atom, $|\beta_1| = 1, |\beta_2| = [(t_1 - \gamma)/(t_1 + \gamma)]$ correspond to the black line in the above and below in figures 6(b)–(e), respectively, and the wave functions take the form of skin-free states. When coupling strengths $g_m = g_n = 0$, the bulk states obviously take the form of skin-free states (figure 6(b)). Counter-intuitively, $|\beta_{1,2}|$ cross the line in the above and below, respectively as $g_m = g_n = 1 (or 13)$ (figures 6(c) and (d)), which suggests that bulk states might localize at the left ($|\beta_1| > 1$) or the right ($|\beta_1| < 1$) chain-atom coupling sites as long as coupling strengths are the same. Note that eigenstates are in a relatively weak localization regime in this case (figure 6(a)). This bipolar localization inevitably leads to Bloch-like states albeit broken translational invariance, but the bipolar localization will disappear in a relatively strong localization regime $(g_m, g_n) = (13, 1)$ as shown in figure 6(e).

In order to further illustrate that nonlocal couplings have significant effect on the localization of eigenstates, we show $\overline{\text{IPR}}$ as a function of the coupling strengths $g_m = g_n = g$ for
Figure 6. (a) IPR on the parameter space of $g_m$ and $g_n$ for the system (3a) with $A-B$ coupling given by equation (20). (b)–(e) $|\beta_1|$ (red star) and $|\beta_2|$ (blue circle) as a function of $q$ given by equation (22) with $(g_m, g_n) = (0, 0), (1, 1), (13, 13)$, and $(13, 1)$, respectively. The three reference lines from top to bottom correspond to $|\beta| = 1$, $\sqrt{|(t_1 - \gamma)/(t_1 + \gamma)|}$, and $|(t_1 - \gamma)/(t_1 + \gamma)|$. The parameters shared by all of the figures are $L = 20, m = n = 10, t_1 = 0.2, t_2 = 1$, and $\gamma = 0.5$.

the system that a giant atom or two small atoms couples to a nonreciprocal SSH chain in figure 7(a). Note that IPR for giant-atom case reaches peak at $g = 1$ corresponding to magenta filled rhombus in figure 7(a), and even higher than the case with a bigger $g$. Comparing to small-atom case, it clear that nonlocal coupling weakens localization of eigenstates.

In figure 7(b), we show $|\beta_1|$ (red star) and $|\beta_2|$ (blue circle) as a function of $q$ with $g = 7$ for small-atom case, and corresponding to black filled square in figure 7(a). It can be seen that $|\beta_{1,2}| \approx \sqrt{|(t_1 - \gamma)/(t_1 + \gamma)|}$ are similar to the case with OBC, which implies that bulk states are like the form of skin states. On the contrary, it cannot be realized for giant-atom case even for a bigger $g = 13$ as shown in figure 6(d).
Figure 7. (a) IPR as a function of $g$ given by equation (20) for the system that a giant atom (magenta rhombus) or two small atoms (black square) couples to a nonreciprocal SSH chain. (b) $|\beta_1|$ (red star) and $|\beta_2|$ (blue circle) as a function of $q$ given by equation (22) with $g=7$ corresponds to black filled square in (a). The parameters are chosen as $L = 20, m = n = 10, t_1 = 0.2, t_2 = 1$, and $\gamma = 0.5$.

5. Lyapunov exponent

In order to predict the localization of bulk states, we calculate the Lyapunov exponent $\lambda(v)$ in the long-time dynamics in real space far from edges,

$$\lambda(v) = \lim_{t \to \infty} \frac{\log \|\psi(t)\|}{t},$$

where the amplitude $|\psi(t)| = A_{s_0+t}(t)$ with $s = vt$ [or similarly $|\psi(t)| = B_{s_0+t}(t)$]. When $t = 0$, $|\psi(0)| = A_{s_0}(0)$ [or $B_{s_0}(0)$] represents the amplitude in the initial time, where $s_0$ could be an arbitrary cell in lattice far away from the giant atom (effect boundary). Next, focussing on wave functions change along the space-time path $s = vt$, where $v$ is the drift velocity \[59–61\], and $s$ is the $s$th unit cell away from the cell $s_0$. When observation time $t_{ob} \to \infty$, i.e. $t_{ob}$ is large enough to compute $\lambda(v)$ with good accuracy, we can obtain the relationship between the Lyapunov exponent $\lambda$ and the drift velocity $v$

$$\lambda(v) = \frac{\log \|\psi(t_{ob})\|}{t_{ob}} = \frac{\log |A_{s_0+vt_{ob}}(t_{ob})|}{t_{ob}}.$$  \hspace{1cm} (24)

Recent research shows that if $\lambda(v)$ does not reach its largest value at zero drift velocity, then the non-Hermitian system exhibits the NHSE \[61\]. Clearly, Lyapunov exponent can act as a witness of defect-induced localized bulk states in the same way. For example, the existence of giant atom induced localized bulk states can be captured by the Lyapunov exponent exhibiting its largest value at a nonvanishing drift velocity.

The Lyapunov exponent in the long-time dynamics is numerically computed by solving the time-dependent Schrödinger equation. This equation can be solved using an accurate variable-step fourth-order Runge–Kutta method. In figure 8, model 1 with localized bulk states corresponds to the nonreciprocal system (3a) with $A - B$ coupling ($A - A$ coupling is similar). As a comparison, model 2 without localized bulk states is described by

$$H'_{AB} = H'_{SSH} + H_{I,AB},$$

\hspace{1cm} (25)
Figure 8. Lyapunov exponent $\lambda(v)$ as a function of $v$ given by equation (23) for the system (3a) (black squares) or the system (25) (red diamonds) with $A-B$ coupling, respectively. The parameters are set as $L = 401$, $m = n = 1$, $t_1 = 0.6$, $t_2 = 1$, $\gamma = 1$, and $\delta = 1$.

with

$$H'_{\text{SSH}} = \sum_{l=1}^{L} \left[ t_1 \hat{C}_{A,l}^\dagger \hat{C}_{B,l} + t_1 \hat{C}_{B,l}^\dagger \hat{C}_{A,l} + t_2 \hat{C}_{A,l+1}^\dagger \hat{C}_{B,l} + t_2 \hat{C}_{B,l+1}^\dagger \hat{C}_{A,l} + i \delta \hat{C}_{A,l}^\dagger \hat{C}_{A,l} - i \delta \hat{C}_{B,l}^\dagger \hat{C}_{B,l} \right],$$

(26)

where gain or loss ($\pm i\delta$) of the sublattice $A(B)$ leads to non-Hermiticity. The initial state is prepared in the bulk of the lattice, i.e. $A(L+1)/2 + s(0) = B(L+1)/2 + s(0) = 0$, and we chose a long lattice ($L = 401$) to avoid the wave packet reaching the effective boundary (giant atom) at the observation time $t_{ob} = 50$. This time is also sufficient to compute $\lambda(v)$ with good accuracy. It is clearly shown that Lyapunov exponent exhibits its largest value at a nonvanishing drift velocity ($v \approx -0.6$) for model 1 with localized bulk states, while it exhibits its largest value at a zero drift velocity ($v = 0$) for model 2 without localized bulk states. This result confirmed that Lyapunov exponent exhibit its largest value at a nonvanishing drift velocity can act as a witness of the existence of localized bulk states.

6. Summary

In summary, we have studied a giant atom coupled to two points of a nonreciprocal SSH chain. We show the spectrum structures of the system, and give a condition for the emergence of the zero mode. The interplay of nonreciprocal hopping and the nonlocal couplings can induce asymmetric zero modes. It is clear that the analytical results of zero modes are precise and almost identify with the numerical results. The features of zero modes are unique and obviously different from the case with OBC or two small atoms. Counter-intuitively, we uncover that bulk states might localize at the left or the right chain-atom coupling sites in weak localization regimes, and the localization is obviously weaker than the case with small-atom or OBC even in strong coupling regimes. The above results suggest that nonlocal coupling of giant atoms to a nonreciprocal SSH chain weakens localization of both zero modes and bulk states. We also show that Lyapunov exponent exhibiting its largest value at a nonvanishing drift velocity can act as a witness of the localized bulk states.
Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

The authors acknowledge Weijun Cheng for helpful comments. This work was supported by the National Natural Science Foundation of China (NSFC) under Grant Nos. 12175033, 12147206 and National Key R&D Program of China (No. 2021YFE0193500).

Appendix A. Energy equation

Consider a system consisting of a giant atom coupled to a nonreciprocal SSH chain via $A-B$ coupling, the Hamiltonian of the system in the momentum space reads,

$$
\mathcal{H}_{AB}(k) = \mathcal{H}_{SSH}(k) + \mathcal{H}_{I,AB}(k) = \sum_k \left[ ((t_1 + \gamma) + t_2 e^{-ik}) \hat{C}_{\alpha,k}^\dagger \hat{C}_{\beta,k} + ((t_1 - \gamma) + t_2 e^{ik}) \hat{C}_{\beta,k}^\dagger \hat{C}_{\alpha,k} \right] + \frac{g}{\sqrt{L}} \sum_k \left[ |e\rangle \langle g| \right] \left( \hat{C}_{\alpha,k} e^{ikm} + \hat{C}_{\beta,k} e^{ikn} \right) + \text{H.c.}. 
$$

(A1)

The time-independent Schrödinger equation \( \mathcal{H}_{AB}(k)|\psi\rangle = E|\psi\rangle \) together with equation (6) leads to,

$$
EU_e = \frac{g}{\sqrt{L}} \sum_k \left( \alpha_k e^{ikn} + \beta_k e^{ikm} \right),
E\alpha_k = (t_1 + \gamma + t_2 e^{-ik}) \beta_k + \frac{g}{\sqrt{L}} e^{-ikn} U_e, 
E\beta_k = (t_1 - \gamma + t_2 e^{ik}) \alpha_k + \frac{g}{\sqrt{L}} e^{-ikm} U_e. 
$$

(A2)

Eliminating \( \alpha_k, \beta_k \) and \( U_e \) in equation (A2), we obtain the equation for \( E \),

$$
E = \frac{2g^2}{L} \sum_k \left( E + t_1 \cos[k(m-n)] - \gamma \sin[k(m-n)] \right)
+ \frac{g^2}{\pi} \int_{-\pi}^{\pi} dk \left( E + t_1 \cos[k(m-n)] - \gamma \sin[k(m-n)] \right)
+ \frac{g^2}{\pi} \int_{-\pi}^{\pi} dk \left( E + t_1 \cos[k(m-n)] - \gamma \sin[k(m-n)] \right)
+ t_2 \cos[k(m-n+1)] \right)/(E^2 - \omega_k^2).
$$

(A3)

To obtain the zero-mode, we set \( E = 0 \) and write \( z_1 = e^{ik} \) and \( z_2 = e^{-ik} \), the right-hand side of the last equation can be simplified via residue theorem (although \( E = 0 \), we keep it in the equation for clarity of expression).
Similarly, for the system with $A-A$ coupling, the interaction Hamiltonian in the momentum space reads
\[
\mathcal{H}_{I,AA}(k) = \frac{g}{\sqrt{L}} \sum_{k} |e\rangle \langle g| \hat{C}_{\alpha,k} (e^{i\alpha_n} + e^{i\beta_m}) + \text{H.c.},
\]
and $\mathcal{H}_{AA}(k)|\psi\rangle = E|\psi\rangle$ leads to
\[
EU_e = \frac{g}{\sqrt{L}} \sum_{k} \alpha_k (e^{i\alpha_n} + e^{i\beta_m}),
\]
\[
E\alpha_k = (t_1 + \gamma + t_2 e^{-i\kappa}) \beta_k + \frac{g}{\sqrt{L}} (e^{-i\alpha_n} + e^{-i\beta_m}) U_e, \tag{A6}
\]
\[
E\beta_k = (t_1 - \gamma + t_2 e^{i\kappa}) \alpha_k.
\]
Some algebras shows that the energy $E$ satisfies
\[
E = 2\frac{g^2}{L} \sum_{\kappa} \left( E (1 + \cos[k(m-n)]) / E^2 - \omega_k^2 \right).
\tag{A7}
\]
Evidently, $E = 0$ is always the solution of the equation (A7).

**Appendix B. States outside the band**

For the system with $A-B$ couplings, in accordance to the equation (A2), the probability amplitudes $\alpha_k$ and $\beta_k$ satisfy,
\[
\frac{\alpha_k}{U_e} = \frac{g}{\sqrt{L}t_2} (E e^{-i\alpha_n} + (t_1 + \gamma + t_2 e^{-i\kappa}) e^{-i\beta_m}) f(k),
\]
\[
\frac{\beta_k}{U_e} = \frac{g}{\sqrt{L}t_2} (E e^{-i\beta_m} + (t_1 - \delta + t_2 e^{i\kappa}) e^{-i\alpha_n}) f(k), \tag{B1}
\]
where \( f(k) = 1/(x - (t_1 + \gamma)e^{ik} - (t_1 - \gamma)e^{-ik}) \) and \( x = (E^2 - (t_1 + \gamma)(t_1 - \gamma) - t_2^2)/t_2 \). By Fourier expansion for \( f(k) \),

\[
f(k) = a_0 + \sum_p \left[ \left( \frac{a_p - ib_p}{2} \right) e^{ikp} + \left( \frac{a_p + ib_p}{2} \right) e^{-ikp} \right],
\]

with

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(k) dk,
\]

\[
\frac{a_p + ib_p}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(k) e^{ikp} dk,
\]

\[
\frac{a_p - ib_p}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(k) e^{-ikp} dk,
\]

we derive \( f(k) \) as follows

\[
f(k) = \frac{(-1)^{y+1}}{\sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)}} \left[ 1 + \sum_{p=1}^{L} (e^{-ikp}a^p + e^{ikp}b^p) \right],
\]

where \( y = \theta(x) \) is the step function, and \( a = (x + \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 + \gamma), b = (x - \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 + \gamma) \) for \( x < -2|t_1| \) or \( a = (x - \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 + \gamma), b = (x - \sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)})/2(t_1 - \gamma) \) for \( x > 2|t_1| \). We calculate Fourier expansion of \( f(k) \) to obtain the amplitude of systems in real space easily. Substituting the above results into equation (B1), we will get the probability amplitude in real space by inverse Fourier transformation. The result is,

\[
A_l/U_c = \frac{1}{\sqrt{L}} \sum_k e^{i\Omega_k} / U_c = \frac{(-1)^{y+1}(T_1^{[l-n]} + g\tau_1^{[l-m]} + Y_1\tau_2^{[l-n]})}{\sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)}},
\]

\[
B_l/U_c = \frac{1}{\sqrt{L}} \sum_k e^{i\beta_k} / U_c = \frac{(-1)^{y+1}(T_2^{[l-n]} + g\tau_1^{[l-n+1]} + Y_1\tau_1^{[l-n]})}{\sqrt{x^2 - 4(t_1 + \gamma)(t_1 - \gamma)}},
\]

where \( A_l(B_l) \) denotes the amplitude in site \( A(B) \) of the \( l \)th unit cell, and \( U_c \) denotes the amplitude in site of the giant atom. \( T = gE/t_2, Y_1 = g(t_1 - \gamma)/t_2, Y_2 = g(t_1 + \gamma)/t_2. \) Besides, \( \tau_1 = a \) for \( l \geq n, \tau_1 = b \) for \( l < n. \tau_2 = a \) for \( l > m, \tau_2 = b \) for \( l \leq m. \)

For the zero modes \( E = 0 \), we would have \( x < -2|t_1|, \) thus \( a \) and \( b \) are reduced to \(-t_1 - \gamma)/t_2 \) and \(-t_1 + \gamma)/t_2 \) for \(-t_2 + \gamma < t_1 < t_2 - \gamma \), respectively. \( a \) and \( b \) are reduced to \(-t_2 + t_1 + \gamma) \) and \(-t_2 + t_1 - \gamma) \) for \( t_1 > t_2 + \gamma \) or \( t_1 < t_2 - \gamma \), respectively. Then equations (B5) and (B6) can be simplified as,

\[
A_l/U_c = Y_3 \times \begin{cases}
\left( \frac{t_2 - \gamma}{t_2} \right)^{(l-m)}, & (l > m), \\
0, & (l \leq m),
\end{cases}
\]

\[
B_l/U_c = Y_4 \times \begin{cases}
\left( \frac{t_2 + \gamma}{t_2} \right)^{(n-l)}, & (l < n), \\
0, & (l \geq n),
\end{cases}
\]

where \( Y_3 = g/(t_1 - \gamma), Y_4 = g/(t_1 + \gamma). \)
Similarly, for the system with $A - A$ coupling, equation (A6) yields
\[
\frac{\alpha_k}{U_e} = \frac{g E}{\sqrt{Ld_2}} \left( e^{-ikn} + e^{-ikm} \right) f(k),
\]
\[
\frac{\beta_k}{U_e} = \frac{g \left( t_1 - \gamma + t_2 e^{ik} \right)}{\sqrt{Ld_2}} \left( e^{-ikn} + e^{-ikm} \right) f(k).
\]

By the inverse Fourier transformation, we can obtain
\[
\frac{A_l}{U_e} = \frac{1}{\sqrt{L}} \sum_k e^{ikl} \frac{\alpha_k}{U_e} = \frac{1}{\sqrt{L}} \sum_k e^{ikl} \frac{\beta_k}{U_e} = \frac{1}{\sqrt{L}} \sum_k e^{ikl} \frac{\beta_k}{U_e}.
\]
\[
\frac{B_l}{U_e} = \frac{1}{\sqrt{L}} \sum_k e^{ikl} \frac{\beta_k}{U_e} = \frac{1}{\sqrt{L}} \sum_k e^{ikl} \frac{\beta_k}{U_e}.
\]

where $\tau_i = a$ for $l \geq m$, $\tau_i = b$ for $l < m$. Then setting $E = 0$, the equations (B10) and (B11) can be simplified as
\[
\frac{A_l}{U_e} = 0,
\]
\[
\frac{B_l}{U_e} = \begin{cases} 
0, & (l < n), \\
\left( \frac{t_2}{t_1 + \gamma} \right)^{l-n} + \left( \frac{-t_2}{t_1 + \gamma} \right)^{l-m}, & (n \leq l < m), \\
\left( \frac{-t_2}{t_1 + \gamma} \right)^{l-n}, & (m \leq l),
\end{cases}
\]
\[
\frac{B_l}{U_e} = \begin{cases} 
\left( \frac{-t_2}{t_1 + \gamma} \right)^{l-n} + \left( \frac{-t_2}{t_1 + \gamma} \right)^{l-m}, & (l < n), \\
\left( \frac{-t_2}{t_1 + \gamma} \right)^{l-n}, & (n \leq l < m), \\
0, & (m \leq l),
\end{cases}
\]

for $t_1 > t_2 + \gamma$ or $t_1 < -t_2 - \gamma$, and

for $-t_2 + \gamma < t_1 < t_2 - \gamma$.

ORCID iDs

J J Wang  
[https://orcid.org/0009-0005-2388-953X](https://orcid.org/0009-0005-2388-953X)

Fude Li  
[https://orcid.org/0000-0002-2959-6667](https://orcid.org/0000-0002-2959-6667)

References

[1] Majorana E 1931 Sulla formazione dello ione molecolare dielio *Nuovo Cimento* **8** 22

[2] Feshbach H 1958 Unified theory of nuclear reactions *Ann. Phys., NY* **5** 357

[3] Hatano N and Nelson D R 1996 Localization transitions in non-Hermitian quantum mechanics *Phys. Rev. Lett.* **77** 570

[4] Bender C M and Boettcher S 1998 Real spectra in non-Hermitian hamiltonians having PT symmetry *Phys. Rev. Lett.* **80** 5243
[5] Berry M V 2004 Physics of non-Hermitian degeneracies Czech. J. Phys. 54 1039
[6] Moiseyev N 2011 Non-Hermitian Quantum Mechanics 1st edn (Cambridge University Press)
[7] Brody D C 2013 Biorthogonal quantum mechanics J. Phys. A: Math. Theor. 47 035305
[8] Parto M, Wittek S, Hodaei H, Harari G, Bandres M A, Ren J, Rechtsman M C, Segev M, Christodoulides D N and Khajavikhan M 2018 Edge-mode lasing in 1D topological active arrays Phys. Rev. Lett. 120 113901
[9] Weimann S, Kremer M, Plotnik Y, Lumer Y, Makris K G, Segev M, Rechtsman M C and Szameit A 2017 Topologically protected bound states in photonic parity-time-symmetric crystal Nat. Mater. 16 433
[10] Bandres M A, Wittek S, Harari G, Parto M, Ren J, Segev M, Christodoulides D N and Khajavikhan M 2018 Topological insulator laser: experiments Science 359 eaar4005
[11] Liang S-D and Huang G-Y 2013 Topological invariance and global Berry phase in non-Hermitian systems Phys. Rev. A 87 012118
[12] Lee T E 2016 Anomalous edge state in a non-Hermitian lattice Phys. Rev. Lett. 116 133903
[13] Leykam D, Blokhk K Y, Huang C, Chong Y D and Nori F 2017 Edge modes, degeneracies and topological numbers in non-Hermitian systems Phys. Rev. Lett. 118 040401
[14] Gong Z, Ashida Y, Kawabata K, Takasaka K, Higashikawa S and Ueda M 2018 Topological phases of non-Hermitian systems Phys. Rev. X 8 031079
[15] Liu T, Zhang Y-R, Ai Q, Gong Z, Kawabata K, Ueda M and Nori F 2019 Second-order topological phases in non-Hermitian systems Phys. Rev. Lett. 122 076801
[16] Yin C, Jiang H, Li L, Li R and Chen S 2018 Geometrical meaning of winding number and its characterization of topological phases in one-dimensional chiral non-Hermitian systems Phys. Rev. A 97 052115
[17] Lieu S 2018 Topological phases in the non-Hermitian Su-Schrieffer-Heeger model Phys. Rev. B 97 045106
[18] Shen H, Zhen B and Fu L 2018 Topological band theory for non-Hermitian Hamiltonians Phys. Rev. Lett. 120 146402
[19] Xiong Y 2018 Why does bulk boundary correspondence fail in some non-Hermitian topological models J. Phys. Commun. 2 035043
[20] Kunst F K, Edvardsson E, Budich J C and Bergholtz E J 2018 Biorthogonal bulk-boundary correspondence in non-Hermitian systems Phys. Rev. Lett. 121 026808
[21] Yao S and Wang Z 2018 Edge states and topological invariants of non-Hermitian systems Phys. Rev. Lett. 121 086803
[22] Yao S, Song F and Wang Z 2018 Non-Hermitian Chern bands Phys. Rev. Lett. 121 136802
[23] Song F, Yao S and Wang Z 2019 Non-Hermitian skin effect and chiral damping in open quantum systems Phys. Rev. Lett. 123 170401
[24] Song F, Yao S and Wang Z 2019 Non-Hermitian topological invariants in real space Phys. Rev. Lett. 123 246801
[25] Yang Z, Zhang K, Fang C and Hu J 2020 Non-Hermitian bulk-boundary correspondence and auxiliary generalized Brillouin zone theory Phys. Rev. Lett. 125 226402
[26] Zhang K, Yang Z and Fang C 2020 Correspondence between winding numbers and skin modes in non-Hermitian systems Phys. Rev. Lett. 125 126402
[27] Yi Y and Yang Z 2020 Non-Hermitian skin modes induced by on-site dissipations and chiral tunneling effect Phys. Rev. Lett. 125 186802
[28] Okuma N, Kawabata K, Shiozaki K and Sato M 2020 Topological origin of non-Hermitian skin effects Phys. Rev. Lett. 124 086801
[29] Borgnia D S, Kruchkov A J and Slager R-J 2020 Non-Hermitian boundary modes and topology Phys. Rev. Lett. 124 056802
[30] Martinez Alvarez V M, Barrios Vargas J E and Foa Torres L E F 2018 Non-Hermitian robust edge states in one dimension: anomalous localization and eigenspace condensation at exceptional points Phys. Rev. B 97 121401(R)
[31] Lee C H and Thomale R 2019 Anatomy of skin modes and topology in non-Hermitian systems Phys. Rev. B 99 201105(R)
[32] Li L, Lee C H and Gong J 2020 Topological switch for non-Hermitian skin effect in cold-atom systems with loss Phys. Rev. Lett. 124 250402
[33] Lee C H, Li L and Gong J 2019 Hybrid higher-order skin-topological modes in nonreciprocal systems Phys. Rev. Lett. 123 016805
[34] Li L, Lee C H, Mu S and Gong J 2020 Critical non-Hermitian skin effect Nat. Commun. 11 5491
[35] Han Y Z, Liu J S and Liu C S 2021 The topological counterparts of non-Hermitian SSH models New J. Phys. 23 123029
[36] Banerjee A, Hegde S S, Agarwala A and Narayan A 2021 Chiral metals and entrapped insulators in a one-dimensional topological non-Hermitian system (arXiv:2111.02223)
[37] Liu S, Shao R, Ma S, Zhang L, You O, Wu H, Xiang Y J, Cui T J and Zhang S 2021 Non-Hermitian skin effect in a non-Hermitian electrical circuit Research 9 5608038
[38] Hofmann T et al 2020 Reciprocal skin effect and its realization in a topolectrical circuit Phys. Rev. Res. 2 023265
[39] Xiao L, Deng T, Wang K, Zhu G, Wang Z, Yi W and Xue P 2020 Non-Hermitian bulk-boundary correspondence in quantum dynamics Nat. Phys. 16 751–6
[40] Helbig T, Hofmann T, Imhof S, Abdelghany M, Kiessling T, Molenkamp L W, Lee C H, Szameit A, Greiter M and Thomale R 2020 Generalized bulk-boundary correspondence in non-Hermitian topolectrical circuits Nat. Phys. 16 747–50
[41] Brandenbourger M, Locsin X, Lerner E and Coulais C 2019 Non-reciprocal robotic metamaterials Nat. Commun. 10 4608
[42] Li L, Lee C H and Gong J 2021 Impurity induced scale-free localization Commun. Phys. 4 42
[43] Yokomizo K and Murakami S 2021 Scaling rule for the critical non-Hermitian skin effect Phys. Rev. B 104 165117
[44] Guo C-X, Liu C-H, Zhao X-M, Liu Y and Chen S 2021 Exact solution of non-Hermitian systems with generalized boundary conditions: size-dependent boundary effect and fragility of the skin effect Phys. Rev. Lett. 127 116801
[45] Liu Y and Chen S 2020 Diagnosis of bulk phase diagram of nonreciprocal topological lattices by impurity modes Phys. Rev. B 102 075404
[46] Roccati F 2021 Non-Hermitian skin effect as an impurity problem Phys. Rev. A 104 022215
[47] Lu J, Shan W-Y, Lu H-Z and Shen S-Q 2011 Non-magnetic impurities and in-gap bound states in topological insulators New J. Phys. 13 103016
[48] Slager R-J, Rademaker L, Zaanen J and Balents L 2015 Impurity-bound states and Green’s function zeros as local signatures of topology Phys. Rev. B 92 085126
[49] Gustafsson M V, Aref T, Kockum A F, Ekström M K, Johansson G and Delsing P 2014 Propagating phonons coupled to an artificial atom Science 346 207–11
[50] Roccati F, Lorenzo S, Calajo G, Palma G M, Carollo A and Ciccarello F 2022 Exotic interactions mediated by a non-Hermitian photonic bath Optica 9 565–71
[51] Kannan B et al 2020 Waveguide quantum electrodyamics with superconducting artificial giant atoms Nature 583 775
[52] Manenti R, Kockum A F, Patterson A, Behrle T, Rahamim T, Tancredi G, Nori F and Leck P J 2017 Circuit quantum acoustodynamics with surface acoustic waves Nat. Commun. 8 975
[53] Sletten L R, Moores B A, Viennot J J and Lehnert K W 2019 Resolving phonon Fock states in a multimode cavity with a double-slit qubit Phys. Rev. X 9 021056
[54] Wang X, Gao Z-M, Li J-Q, Zhu H-B and Li H-R 2022 Unconventional quantum electrodynamics with a Hofstadter-Ladder waveguide Phys. Rev. A 106 043703
[55] Wang X, Liu T, Kockum A F, Li H-R and Nori F 2021 Tunable chiral bound states with giant atoms Phys. Rev. Lett. 126 043602
[56] Cheng W, Wang Z and Liu Y-X 2022 Topology and retardation effect of a giant atom in a topological waveguide Phys. Rev. A 106 033522
[57] Santos A C and Bachelard R 2022 Generation of maximally-entangled long-lived states with giant atoms in a waveguide (arXiv:2207.04696)
[58] Andersson G, Ekström M K and Delsing P 2020 Electromagnetically induced acoustic transparency with a superconducting circuit Phys. Rev. Lett. 124 240402
[59] Dee G and Langer J S 1983 Propagating pattern selection Phys. Rev. Lett. 50 383
[60] Longhi S 2013 Convective and absolute PT-symmetry breaking in tight-binding lattices Phys. Rev. A 88 052102
[61] Longhi S 2019 Probing non-Hermitian skin effect and non-Bloch phase transition Phys. Rev. Res. 1 023013