STRONGLY BADLY APPROXIMABLE MATRICES IN FIELDS OF POWER SERIES

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Abstract. We study the notion of strongly badly approximable matrices in the field of power series over a field \( K \). We prove a transference principle in this setting, and show that such matrices exist when \( K \) is infinite.

1. Introduction

Let \( K \) be a field and let \( L = K((x^{-1})) \) be the field of formal Laurent series with coefficients in \( K \). That is, the nonzero elements of \( L \) consist of all formal Laurent series

\[
F(x) = \sum_{m=-\infty}^{M} a(m)x^m,
\]

where each coefficient \( a(m) \) is in \( K \), \( a(M) \neq 0_K \), and \( M \) is an arbitrary integer. We write \( 0_L \) for the identically zero Laurent series, which is also in \( L \). Addition and multiplication in \( L \) are defined in the obvious way. Then we define \( \| \| : L \to [0, \infty) \) by \( \|0_L\| = 0 \), and

\[
\|F\| = e^M,
\]

where \( F(x) \neq 0_L \) in \( L \) is given by \( 1 \). It follows that \( \| \| : L \to [0, \infty) \) is a discrete, non-archimedean absolute value on \( L \), and the resulting metric space \((L, \| \|)\) is complete. In this situation we define the subring of \( L \)-adic integers

\[
\mathcal{O}_L = \{ F \in L : \|F\| \leq 1 \},
\]

and its unique maximal ideal

\[
\mathcal{M}_L = \{ F \in \mathcal{O}_L : \|F\| < 1 \}.
\]

Clearly the residue class field \( \mathcal{O}_L/\mathcal{M}_L \) is isomorphic to \( K \). We recall that \( \mathcal{O}_L \) is compact if and only if the residue class field \( \mathcal{O}_L/\mathcal{M}_L \) is finite (see [1], Chapter 4, Corollary to Lemma 1.5.) Obviously \( \mathcal{M}_L \subseteq \mathcal{O}_L \) is the principal ideal generated by the element \( x^{-1} \).

It is clear that the polynomial ring \( K[x] \) can be embedded in \( L \) by simply regarding a polynomial

\[
P(x) = \sum_{m=0}^{M} \xi(m)x^m
\]
as a Laurent series with $\xi(m) = 0_K$ for integers $m \leq -1$. In what follows we will always identify $K[x]$ with its image in $L$. Let $\varphi : L \to \mathcal{M}_L$ be the map defined by $\varphi(0_L) = 0_L$, and defined on nonzero elements $F$ in $L$ by

$$\varphi(F) = \varphi\left(\sum_{m=-\infty}^{M} a(m)x^m\right) = \sum_{m=-\infty}^{-1} a(m)x^m.$$ 

It is easy to check that $\varphi$ is a surjective homomorphism of additive groups, and

$$\ker\{\varphi\} = K[x]. \tag{3}$$

We put $T = L/K[x]$. Then $\varphi$ induces an isomorphism of additive groups

$$\overline{\varphi} : T \to \mathcal{M}_L. \tag{4}$$

The subset $K[x] \subseteq L$ is clearly discrete with respect to the metric topology induced by the absolute value $|\cdot|$. In particular, if $P(x) \neq Q(x)$ are polynomials in $K[x]$, then we have

$$1 \leq |P - Q|. \tag{5}$$

Next we define $\| \| : T \to [0, 1)$ on the additive group $T$ by

$$\| F \| = \min \left\{|F - P| : P \in K[x]\right\}.$$ 

Alternatively, $\| F \|$ is the distance in $L$ from $F$ to the nearest polynomial. It is trivial to check that $\| \| : T \to [0, 1)$ is a norm on $T$ in the sense of Kaplansky [7, Appendix 1]. Therefore the map

$$(F, G) \mapsto \| F - G \|$$

defines a metric in $T$, and so induces a metric topology in this quotient group.

There is a natural $K[x]$-module structure on the additive group $T$. If $(P(x), F(x) + K[x])$ is an element of $K[x] \times T$ we define the product

$$P(x)(F(x) + K[x]) = P(x)F(x) + K[x]. \tag{6}$$

Again it is easy to check that (6) does give the additive group $T$ the structure of a $K[x]$-module.

We can formulate Diophantine approximation problems in this setting, with $K[x], K(x), L$ and $T$ playing analogous roles to $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and the unit interval $[0, 1)$, respectively. Indeed, Davenport and Lewis [5] studied the analog of Littlewood’s conjecture in $L$ and proved that this analog is actually false when $K$ is infinite. Explicit counterexamples were later given by Baker [2]. In the case where $K$ is finite, the analog of the Littlewood conjecture in $L$ is believed to be true, but still remains an open problem (see [1]).

In [8], motivated by inequalities regarding fractional parts of linear forms, we introduced the notion of strongly badly approximable matrices, which is a strengthening of the usual notion of badly approximable matrices. Let $A = (\alpha_{mn})$ be an $M \times N$ matrix with entries in $\mathbb{R}/\mathbb{Z}$. Recall that $A$ is badly approximable if there exists a constant $\beta(A) > 0$ such that we have

$$\beta(A) \leq \left( \max_{1 \leq m \leq M} \left\| \sum_{n=1}^{N} \alpha_{mn} \xi_n \right\| \right)^{M} \left( \max_{1 \leq n \leq N} |\xi_n| \right)^{N}. \tag{7}$$
for all \( \xi \in \mathbb{Z}^N \setminus \{0\} \). Here \( \|\alpha\| \) denotes the distance from \( \alpha \) to the nearest integer. We say that \( A \) is strongly badly approximable if there exists a constant \( \gamma(A) > 0 \) such that we have

\[
\gamma(A) \leq \left( \prod_{m=1}^{M} \left\| \sum_{n=1}^{N} \alpha_{mn} \xi_n \right\| \right) \left( \prod_{n=1}^{N} (|\xi_n| + 1) \right)
\]

(8)

for all \( \xi \in \mathbb{Z}^N \setminus \{0\} \). Clearly if \( A \) is strongly badly approximable, then it is badly approximable. If \( M = N = 1, A = (\alpha) \), then \( A \) is strongly badly approximable if \( \alpha \) is a badly approximable number. If \( M = 2, N = 1, A = (\alpha \beta) \) then \( A \) is strongly badly approximable if and only if there is a constant \( \gamma > 0 \) such that

\[
\gamma \leq \|k\alpha\||k\beta||(|k| + 1)
\]

for all \( k \in \mathbb{Z} \setminus \{0\} \). That is, \( (\alpha, \beta) \) is a counterexample to Littlewood’s conjecture. In [8], using the geometry of numbers, we obtained a transference principle for strongly badly approximable matrices, namely that \( A \) is strongly badly approximable if and only if its transpose \( A^t \) is.

In this note, we consider the notion of strongly badly approximable matrices in \( L \). We prove an explicit criterion for strongly badly approximable matrices in this setting, from which the transference principle readily follows. We also show that, when \( K \) is infinite, then strongly badly approximable matrices do exist.

Consider an \( M \times N \) matrix \( A = (\alpha_{mn}) \) with entries in \( \mathbb{T} \). We say that \( A \) is strongly badly approximable if there exists a constant \( C_1 = C_1(A) > 0 \) such that

\[
e^{-C_1} < \left( \prod_{m=1}^{M} \left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\| \right) \left( \prod_{n=1}^{N} \max(|\xi_n|, 1) \right)
\]

(9)

for all \( \xi = (\xi_1, \ldots, \xi_N) \in K[x]^N \setminus \{0\} \). Clearly, \( C_1 \) is necessarily greater than \( M \).

Let us denote the set of all elements \( \xi \in K(x) \) with \( |\xi| < e^N \) by \( G_N \). Given nonnegative integers \( U_1, \ldots, U_N \), and \( V_1, \ldots, V_M \), let us consider the set of all \( \xi = (\xi_1, \ldots, \xi_N) \in K[x]^N \) satisfying the conditions

\[
|\xi_n| < e^{U_n} \text{ for any } n = 1, \ldots, N,
\]

(10)

\[
\left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\| < e^{-V_m} \text{ for any } m = 1, \ldots, M.
\]

(11)

Clearly, the solutions to the systems (10) and (11) form a subvector space of

\[ G_{U_1} \times \cdots \times G_{U_N} \subset K[x]^N, \]

which we denote by \( V(U_1, \ldots, U_N; V_1, \ldots, V_M) \). Furthermore, the condition (11) determines \( V_1 + \cdots + V_M \) equations (whose unknowns are the coefficients of \( \xi_1, \ldots, \xi_N \)). Thus we immediately have the inequality

\[
\dim V(U_1, \ldots, U_N; V_1, \ldots, V_M) \geq \max(0, U_1 + \cdots + U_N - V_1 - \cdots - V_M).
\]

(12)

One may regard this as a generalization of Dirichlet’s theorem.

2. A First Characterization

We now prove our first characterization of strongly badly approximable matrices in \( T \).
**Proposition 1.** Let $A = (\alpha_{mn})$ be an $M \times N$ matrix with entries in $\mathbf{T}$, where $m = 1, 2, \ldots, M$ indexes rows and $n = 1, 2, \ldots, N$ indexes columns. Then $A$ is strongly badly approximable if and only if there exists a constant $C_2 \geq 0$ such that for any nonnegative integers $U_1, \ldots, U_N$ and $V_1, \ldots, V_M$, we have

$$\dim \mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M) \leq \max(0, U_1 + \cdots + U_N - V_1 - \cdots - V_M) + C_2. \quad (13)$$

In other words, $A$ is strongly badly approximable if and only if the inequality (12) is essentially best possible. One could regard Proposition 1 as an analog of [8, Lemma 4.2].

**Proof.** Suppose $A$ is strongly badly approximable, that is, there is a constant $C_1$ such that (13) holds for any $\xi \in K [x]^N \setminus \{0\}$. We first claim that

$$\mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M) = \{0\}$$

whenever

$$W_1 + \cdots + W_N \leq V_1 + \cdots + V_M + M - C_1.$$

Indeed, suppose there is $\xi \in \mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M), \xi \neq 0$. By definition of $\mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M)$, we have

$$\max (|\xi_n|, 1) \leq e^{W_n} \text{ for any } n = 1, \ldots, N$$

$$\left\| \sum_{n=1}^N \alpha_{mn} \xi_n \right\| \leq e^{-V_m - 1} \text{ for any } m = 1, \ldots, M. \quad (15)$$

Therefore,

$$\left( \prod_{n=1}^M \left\| \sum_{n=1}^N \alpha_{mn} \xi_n \right\| \right) \left( \prod_{n=1}^N \max (|\xi_n|, 1) \right) \leq e^{W_1 + \cdots + W_N - V_1 - \cdots - V_M - M} \quad (16)$$

This contradicts (9) if $W_1 + \cdots + W_N \leq V_1 + \cdots + V_M + M - C_1$.

We now show that (13) holds for

$$C_2 := C_1 - M \quad (17)$$

(note that $C_2 > 0$). If $U_1 + \cdots + U_N \leq V_1 + \cdots + V_M - C_2$, then (13) is automatically true by what we just proved. Suppose

$$U_1 + \cdots + U_N \geq V_1 + \cdots + V_M - C_2.$$

We can find integers $0 \leq W_n \leq U_n$ such that

$$W_1 + \cdots + W_N = \max(V_1 + \cdots + V_M - C_2, 0).$$

Then $G_{W_1} \times \cdots \times G_{W_N}$ and $\mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M)$ are two subvector spaces of $G_{U_1} \times \cdots \times G_{U_N}$, whose intersection is $\mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M) = \{0\}$. It follows that

$$\dim \mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M) \leq U_1 + \cdots + U_N - (W_1 + \cdots + W_N)$$

$$= U_1 + \cdots + U_N - \max(V_1 + \cdots + V_M - C_2, 0)$$

$$\leq U_1 + \cdots + U_N - V_1 - \cdots - V_M + C_2$$

$$\leq \max(0, U_1 + \cdots + U_N - V_1 - \cdots - V_M) + C_2$$

as desired.
Combining (20) and (21), we see that (9) holds for $\mathbf{1}; V$.

Upon observing that $\max(0, V)$ is sufficiently large.

For the reverse direction, let us assume that (13) holds for some constant $C_2$. Let $\xi$ be an arbitrary element in $K[x]^N \setminus \{0\}$. Our goal is to find a lower bound for

$$P = \left( \prod_{m=1}^{M} \left\| \sum_{n=1}^{N} \alpha_{mn} \xi_n \right\| \right) \left( \prod_{n=1}^{N} \max(|\xi_n|, 1) \right).$$

Put $\max(|\xi_n|, 1) = e^{U_n}$ for $n = 1, \ldots, N$. First we observe that $\left\| \sum_{n=1}^{N} \alpha_{mn} \xi_n \right\| \neq 0$ for any $m = 1, \ldots, M$. Indeed, suppose for a contradiction that $\left\| \sum_{n=1}^{N} \alpha_{1n} \xi_n \right\| = 0$. Let $W$ be any integer greater than $C_2$. Then for any $V \geq 0$, the vector space $\mathcal{V}(U_1 + W + 1, \ldots, U_N + W + 1; V, 0, \ldots, 0)$ has dimension at least $W$ (since it contains $f \xi$, for any $f \in G_W$). On the other hand, in view of (13), its dimension cannot exceed $C_2$ if $V$ is sufficiently large.

Let us put

$$\left\| \sum_{n=1}^{N} \alpha_{mn} \xi_n \right\| = e^{-V_m},$$

where $V_m$ are nonnegative integers. Then we want to find a lower bound for

$$\sum_{n=1}^{N} U_n - \sum_{m=1}^{M} V_m.$$ (18)

We can assume that (18) is negative, otherwise, we are already done. Let $W$ be the smallest integer such that

$$\sum_{n=1}^{N} (W + U_n) - \sum_{m=1}^{M} \max(0, V_m - W) \geq 0.$$ (19)

Clearly $W$ exists, and $W \geq 1$. From (19) we immediately have

$$\sum_{n=1}^{N} U_n - \sum_{m=1}^{M} V_m \geq - (M + N) W.$$ (20)

By minimality of $W$, we have

$$\sum_{n=1}^{N} (W - 1 + U_n) - \sum_{m=1}^{M} \max(0, V_m - W + 1) < 0.$$ (21)

Upon observing that $\max(0, V_m - W + 1) \leq \max(0, V_m - W) + 1$, we have

$$\sum_{n=1}^{N} (W + U_n) - \sum_{m=1}^{M} \max(0, V_m - W) < M + N.\]$$

Let us now consider the space

$$\mathcal{V}(W + U_1, \ldots, W + U_N; \max(0, V_1 - W), \ldots, \max(0, V_M - W)).$$

On the one hand, by hypothesis, its dimension is at most

$$\max \left( 0, \sum_{n=1}^{N} (W + U_n) - \sum_{m=1}^{M} \max(0, V_m - W) \right) + C_2 < (M + N) + C_2.$$

On the other hand, its dimension is at least $W$, since it contains $f \xi$ for any $f \in G_W$. Therefore,

$$W < (M + N) + C_2.$$ (21)

Combining (20) and (21), we see that (19) holds for

$$C_1 := (M + N)^2 + (M + N)C_2.$$ (22)
Remark 2.1. One can contrast the values of $C_1$ and $C_2$ given by (17) and (24). It is an interesting problem to determine if these values are tight.

3. A matrix interpretation

The pleasantness of working in $L$ is that we can write out any element of $L$ in terms of its coordinates and express Diophantine inequalities in terms of linear equations. For any element $\alpha = \sum_{i=1}^{\infty} a_i x^{-i} \in T$ and nonnegative integers $U,V$, let us denote by $M_{U,V}(\alpha)$ the matrix

$$
M_{U,V}(\alpha) = \begin{pmatrix}
a_1 & a_2 & \cdots & a_U \\
a_2 & a_3 & \cdots & a_U+1 \\
\vdots & \vdots & \ddots & \vdots \\
a_V & a_v+1 & \cdots & a_v+V-1 \\
\end{pmatrix}.
$$

(23)

In particular, when $U = 0$ or $V = 0$, $M_{U,V}(\alpha)$ is the empty matrix.

Given nonnegative integers $U_1, \ldots, U_N, V_1, \ldots, V_M$ and an $M \times N$ matrix $A = (\alpha_{mn})$ with entries in $T$, the conditions (10) and (11) represent a system of $V_1 + \cdots + V_M$ linear equations in $U_1 + \cdots + U_N$ variables, which can be written out explicitly as follows. Suppose that

$$
\alpha_{mn} = \sum_{i=1}^{\infty} a_i^{(mn)} x^{-i} \quad (1 \leq m \leq M, 1 \leq n \leq N).
$$

(24)

Let us write

$$
\xi_n = \sum_{j=0}^{U_n-1} t_j^{(n)} x^j \quad (1 \leq n \leq N)
$$

(25)

where we regard the $t_j^{(n)}$ as variables. Then the conditions (10), (11) amount to the system

$$
\sum_{n=1}^{N} \sum_{j=0}^{U_n-1} t_j^{(n)} a_{i+j}^{(mn)} = 0
$$

for any $i = 1, \ldots, V_m$ and $m = 1, \ldots, M$. It is straightforward to see that the matrix of this system is

$$
M_{U_1, \ldots, U_N; V_1, \ldots, V_M}(A) = \begin{pmatrix}
M_{U_1; V_1}(\alpha_{11}) & M_{U_1; V_1}(\alpha_{12}) & \cdots & M_{U_1; V_1}(\alpha_{1N}) \\
M_{U_1; V_2}(\alpha_{21}) & M_{U_1; V_2}(\alpha_{22}) & \cdots & M_{U_1; V_2}(\alpha_{2N}) \\
\vdots & \vdots & \ddots & \vdots \\
M_{U_1; V_M}(\alpha_{M1}) & M_{U_1; V_M}(\alpha_{M2}) & \cdots & M_{U_1; V_M}(\alpha_{MN})
\end{pmatrix}.
$$

(26)

We thus arrive at another characterization of strongly badly approximable matrices.

Proposition 2. The $M \times N$ matrix $A$ is strongly badly approximable if and only if there is a constant $C_2 \geq 0$ such that for any nonnegative integers $U_1, \ldots, U_N, V_1, \ldots, V_M$, the matrix $M_{U_1, \ldots, U_N; V_1, \ldots, V_M}(A)$ defined in (26) has rank at least

$$
\min \left( \sum_{n=1}^{N} U_n, \sum_{m=1}^{M} V_m \right) - C_2.
$$

(27)
Proof. This follows from Proposition 1 the rank-nullity theorem, and the fact that
\[
\sum_{n=1}^{N} U_n - \max \left( 0, \sum_{n=1}^{N} U_n - \sum_{m=1}^{M} V_m \right) - C_2 = \min \left( \sum_{n=1}^{N} U_n, \sum_{m=1}^{M} V_m \right) - C_2.
\] (28) □

It is easy to see that the transpose of \(M_{U_1, \ldots, U_N; V_1, \ldots, V_M}(A)\) is \(M_{V_1, \ldots, V_M; U_1, \ldots, U_N}(A^t)\). Thus from Proposition 2, we immediately have the following transference principle:

Theorem 1. A matrix \(A\) with entries in \(T\) is strongly badly approximable if and only if its transpose \(A^t\) is strongly badly approximable.

Remark 3.1. Propositions 1, 2 show that, if \(A\) is strongly badly approximable, we can choose \(C_1(A^t) = (M + N)^2 + (M + N)C_1(A)\), where \(C_1(A)\) is the constant defined in (9). One can compare this to the bound (8.30) in [8].

Given Proposition 2 we will establish the existence of strongly badly approximable matrices when \(K\) is infinite. This follows from the following stronger statement.

Theorem 2. Suppose \(K\) is infinite. Then for any \(M\) and \(N\), there exists an \(M \times N\) matrix \(A = (\alpha_{mn})\) with entries in \(T\) satisfying the following property.

\[ (*) \text{ For any nonnegative integers } U_1, \ldots, U_N, V_1, \ldots, V_M \text{ with } \sum_{n=1}^{N} U_n = \sum_{m=1}^{M} V_m, \text{ the square matrix } M_{U_1, \ldots, U_N; V_1, \ldots, V_M}(A) \text{ is non-singular.} \]

For \(1 \times N\) matrices, this is a result of Bumby [3]. Our argument is a generalization of his.

Proof. We prove Theorem 2 by induction on \(M + N\). When \(M = 0\) or \(N = 0\), the theorem is vacuously true. Suppose \(M, N \geq 0\) and we know already the existence of an \(M \times N\) matrix \(A^t\) satisfying (*)\(^\text{.}\) We will show how to add one column to \(A^t\) such that the new \(M \times (N + 1)\) matrix retains this property. By symmetry, we can also add one row to \(A^t\). This way, we can create \(M \times N\) matrices satisfying (*) for any \(M\) and \(N\). (Note that when \(M = N = 0\), then this process creates badly approximable elements in \(L\).)

Suppose the \(M \times N\) matrix \(A^t = (\alpha_{mn})\) satisfies (*). We will find \(\alpha_{m,N+1} \in T\) \((1 \leq m \leq M)\) such that the \(M \times (N + 1)\) matrix \(A = (\alpha_{mn})\) satisfies (*).

Suppose
\[
\alpha_{mn} = \sum_{i=1}^{\infty} a_i^{(mn)} x^{-i} \quad (1 \leq m \leq M, 1 \leq n \leq N + 1).
\]

Our goal is to construct \(M\) sequences
\[
\left( a_i^{(m,N+1)} \right)_{i=1}^{\infty}, \quad (1 \leq m \leq M),
\]
such that for integers \(U_1, \ldots, U_N, U_{N+1}\) and \(V_1, V_2, \ldots, V_M\), with
\[
U_1 + \cdots + U_N + U_{N+1} = V_1 + \cdots + V_M,
\]
the square matrix
\[
\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A)
\]

\[
= \begin{pmatrix}
a^{(11)}_1 & \cdots & a^{(11)}_{U_1} & \cdots & a^{(N+1,1)}_1 & \cdots & a^{(N+1,1)}_{U_{N+1}} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a^{(11)}_{V_1} & \cdots & a^{(11)}_{U_1+V_1-1} & \cdots & a^{(N+1,1)}_{V_1} & \cdots & a^{(N+1,1)}_{U_{N+1}+V_1-1} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a^{(M1)}_1 & \cdots & a^{(M1)}_{U_1} & \cdots & a^{(N+1,M)}_1 & \cdots & a^{(N+1,M)}_{U_{N+1}} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a^{(M1)}_{V_M} & \cdots & a^{(M1)}_{U_1+V_M-1} & \cdots & a^{(N+1,M)}_{V_M} & \cdots & a^{(N+1,M)}_{V_M+U_{N+1}-1}
\end{pmatrix}
\]
is non-singular.

To this end, we will construct the \(M\)-tuples \(a^{(m,N+1)}_L\) \(1 \leq m \leq M\) indexed by \(L\) recursively. Let us refer to \(\max_{1 \leq m \leq M}(U_{N+1} + V_m - 1)\) as the order of the matrix \(\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A)\).

Suppose all the tuples \(a^{(m,N+1)}_L\) \(1 \leq m \leq M\), with \(1 \leq L \leq L - 1\), are already determined in such a way that all matrices of order smaller than \(L\) are non-singular. We want to find \(a^{(m,N+1)}_L\) such that all the matrices \(\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A)\) satisfying

- \(U_1 + \cdots + U_N = V_1 + \cdots + V_M\)
- \(\max_{1 \leq m \leq M}(U_{N+1} + V_m - 1) = L\)

have non-zero determinant.

It is clear that the number of such matrices is finite. For any such matrix, by expanding along the last column, the non-zero determinant condition corresponds to an equation of the form

\[
\sum_{m=1}^{M} r_m a^{(m,N+1)}_L + r_0 \neq 0 \tag{29}
\]

where \(r_0, \ldots, r_M \in K\). For each \(1 \leq m \leq M\), \(r_m\) is either 0 or the determinant of a matrix of lower order (it is 0 if \(a^{(m,N+1)}_L\) is not present in \(\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A)\), but at least one of them is nonzero. The number of equations of type (29) is finite. Since \(K\) is infinite, a choice of \(a^{(m,N+1)}_L\) \(1 \leq m \leq M\) is certainly possible. \(\square\)

4. Explicit examples of strongly badly approximable matrices

In this section, we discuss about explicit examples of strongly badly approximable matrices. As mentioned earlier, Baker \(\cite{2}\) gave explicit counterexamples to Littlewood’s conjecture in \(K\) in the case where \(K\) has characteristic zero. He also pointed out that the same method can be used to show that any \(1 \times N\) matrix of the form

\[
A = \begin{pmatrix}
ed^\lambda_1/x & ed^\lambda_2/x & \cdots & ed^\lambda_N/x
\end{pmatrix},
\]
where \( \lambda_1, \ldots, \lambda_N \) are distinct elements of \( K \), is (in our language) strongly badly approximable. Here for any \( \lambda \in K \), \( e^{\lambda/x} \) is the formal power series

\[
e^{\lambda/x} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} x^{-i}.
\]

In a different context, Jager [6] studied the notion of perfect systems of power series (see the definition in [4, p. 196]). This notion was developed by Mahler [9] and inspired by Hermite’s proof of the transcendence of \( e \). By examining the underlying linear equations, it is easy to see that if the system \((f_1(x), \ldots, f_M(x))\) is perfect (where each \( f_i \) is a power series of the form \( f_i(x) = \sum_{j=0}^{\infty} a_j^{(i)} x^j \)), then the matrix

\[
A = \begin{pmatrix}
    f_1(x^{-1}) \\
    f_2(x^{-1}) \\
    \vdots \\
    f_M(x^{-1})
\end{pmatrix}
\]

is strongly badly approximable.

Jager then gave some examples of perfect systems. If \( \lambda_1, \ldots, \lambda_N \) are distinct elements of \( K \), then the system \((e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_N x})\) is perfect. Coupled with the transference principle, this recovers Baker’s result (with a simpler proof). He also showed that if \( w_1, \ldots, w_N \) are elements of \( K \), no two of which differ by an integer, then the system \(((1-x)^{w_1}, \ldots, (1-x)^{w_N})\) is also perfect. Here \((1-x)^w\) is the formal power series

\[
(1-x)^w = \sum_{i=0}^{\infty} (-1)^i \binom{w}{i} x^i.
\]

This gives another example of \( 1 \times N \) (hence \( N \times 1 \)) strongly badly approximable matrices. However, it seems to us that neither Baker nor Jager’s method can be extended to give explicit examples of \( M \times N \) for some \( M, N > 1 \). It is therefore an interesting problem to give explicit examples of strongly badly approximable matrices of arbitrary dimensions.

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