Blind Determination of the Number of Sources Using Distance Correlation

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Abstract—A novel blind estimate of the number of sources from noisy, linear mixtures is proposed. Based on Székely et al.’s distance correlation measure, we define the Sources’ Dependency Criterion (SDC), from which our estimate arises. Unlike most previously proposed estimates, the SDC estimate exploits the full independence of the sources and noise, as well as the non-Gaussianity of the sources (as opposed to the Gaussianity of the noise), via implicit use of high-order statistics. This leads to a more robust, resilient and stable estimate w.r.t. the mixing matrix and the noise covariance structure. Empirical simulation results demonstrate these virtues, on top of superior performance in comparison with current state of the art estimates.

Index Terms—Distance correlation, independent component analysis, number of sources, high-order statistics.

I. INTRODUCTION

The fundamental problem of determining the number of sources from noisy measurements of their linear mixtures has been ubiquitous in signal processing research for decades. This is mainly because correct determination of the model order is a necessary preliminary step in many classical problems in signal processing, such as direction-of-arrival estimation (e.g., [1]–[3]), blind source separation using Independent Component Analysis (ICA, e.g., [4]–[6]) and signal decoding in multiple-input multiple-output wireless systems (see [7] and references therein), to name but a few.

Many solutions to this problem from various approaches have been proposed in the literature so far, such as the well-known Akaike Information Criterion (AIC) and Minimum Description Length (MDL) [3], Random Matrix Theory (RMT)-based [9], Second Order moment Statistic of the Eigenvalues (SORTE) [10], [11], the recently proposed Bayesian information criterion variant [12], mean squared eigenvalue error [13], and many others [14]–[20]. However, all these solutions are heavily based on an assumption of spatial-whiteness of the additive noise, which essentially leads to a (matrix) rank estimation problem. Thus, to the best of our knowledge, previously proposed methods eventually make explicit use of the eigenvalues of the measurements’ empirical correlation matrix for the final estimation rule.

In this work, we address the problem of blind determination of the number of sources, where only few, basic assumptions are made, leaving the model general and suitable for a wider range of applications. In contrary to previously proposed methods, our estimate is not directly based on the empirical correlation matrix’ eigenvalues, and implicitly incorporates high-order statistics, relying on the Gaussianity of the noise vs. the non-Gaussianity of the sources. As a result, our estimate is indifferent to the spatial correlation of the noise, and is considerably more robust and resilient in comparison with other dominant, state-of-the-art estimates. Since our proposed solution is based on the (still) less known distance covariance measure, the following section is devoted to a presentation of its definition and some of its interesting, important properties.

II. DISTANCE COVARIANCE AND DISTANCE CORRELATION

Distance covariance (dCov), introduced by Székely et al. [21], is a measure which quantifies the dependence between two random vectors, not necessarily of the same dimension.

More formally, let \( x \in \mathbb{R}^{M \times 1} \) and \( y \in \mathbb{R}^{N \times 1} \) be two random vectors with finite first moments. The dCov between \( x \) and \( y \) is the nonnegative number \( \gamma(x, y) \) defined by

\[
\gamma^2(x, y) \triangleq \| \varphi_{x,y}(t, s) - \varphi_x(t)\varphi_y(s) \|^2 \\
\triangleq \int_{\mathbb{R}^{M+N}} |\varphi_{x,y}(t, s) - \varphi_x(t)\varphi_y(s)|^2 \frac{c_{CMCN}[t]}{|2|^{M+1}} |s|^{N+1} dt ds, \tag{1}
\]

where \( c_d \triangleq \frac{\pi^{(1+d)/2}}{\Gamma((1+d)/2)} \), \( \Gamma(\cdot) \) is the complete Gamma function (e.g., [22]), \( \varphi_x(t) \), \( \varphi_y(s) \) and \( \varphi_{x,y}(t, s) \) denote the characteristic functions of \( x \), \( y \) and \( (x, y) \), resp., \( \| \cdot \|_2 \) denotes the \( l^2 \) norm. Similarly, distance variance (dVar) is defined as the square root of

\[
\gamma^2(x) \triangleq \| \varphi_{x}(t, s) - \varphi_x(t)\varphi_x(s) \|^2 \tag{2}
\]

The distance correlation (dCor) between \( x \) and \( y \) is the nonnegative coefficient \( R(x, y) \) defined by

\[
R^2(x, y) \triangleq \begin{cases} 
\frac{\gamma^2(x,y)}{\sqrt{\gamma^2(x)\gamma^2(y)}}, & \gamma^2(x)\gamma^2(y) > 0, \\
0, & \gamma^2(x)\gamma^2(y) = 0. \end{cases} \tag{3}
\]

An important property of dCor is the following (e.g., [21]):

P1: \( 0 \leq R(x, y) \leq 1 \), and \( R(x, y) = 0 \) if and only if \( x \perp \perp y \) (which denotes \( x \) and \( y \) are statistically independent).

We stress that unlike the classical Pearson’s correlation coefficient (e.g., [23]), which may equal zero even if its arguments are statistically dependent, zero dCor necessarily implies statistical independence of its arguments.

Remarkably, very simple empirical estimates of the distance covariance exist, which do not require direct integration: For an observed random independent, identically distributed (i.i.d.) sample \( \{(x_t, y_t)\}_{t=1}^T = \{X \in \mathbb{R}^{M \times T}, Y \in \mathbb{R}^{N \times T}\} \) from the joint distribution of \( x \) and \( y \), define

\[
(\mathbf{Y}^{(x)})_{t,\tau} \triangleq \|x_t - x_\tau\|_2, \quad (\mathbf{Y}^{(y)})_{t,\tau} \triangleq \|y_t - y_\tau\|_2, \quad \forall t, \tau \in \{1, \ldots, T\}. \tag{4}
\]
The empirical dCov, then, is the nonnegative number \( \nu_T(X,Y) \), defined by

\[
\nu_T^2(X,Y) \equiv \frac{1}{T^2} \text{Tr} \left( PP^T \right),
\]

where \( P \equiv I_T - \frac{1}{T} 11^T \) is a projection matrix (\( I_T \) denoting the \( T \times T \) identity matrix and \( 1 \) denoting a \( T \times 1 \) all-ones vector). Accordingly, the empirical dVar \( \nu_T(X) \in \mathbb{R}^+ \) is defined by

\[
\nu_T^2(X) \equiv \nu_T^2(X,X) = \frac{1}{T^2} \text{Tr} \left( PP^T \right),
\]

and the empirical dCor \( \mathcal{R}_T(X,Y) \in [0, 1] \) is defined by

\[
\mathcal{R}_T^2(X,Y) \equiv \begin{cases} 
\frac{\nu_T^2(X,Y)}{\sqrt{\nu_T^2(X)\nu_T^2(Y)}}, & \nu_T^2(X,Y) > 0, \\
0, & \nu_T^2(X,Y) = 0.
\end{cases}
\]

Note that the statistic \( \mathcal{R}_T(X,Y) \) may be computed rather simply (in terms of arithmetic operations), which is important in our context for practical considerations. As shown in [24], we assume that all the signals are temporally i.i.d. As a consequence, the sources’ scales are non-identifiable in this model. Since the sources’ scales are non-identifiable in this model, we focus on the spatial covariance of the sources which is known, here as \( \Gamma \) and \( \mathcal{V} \).

Having established the foundations for our proposed estimate, we now turn to the problem in hand.

### III. Problem Formulation

Consider the linear, instantaneous noisy ICA model

\[
x[t] = As[t] + v[t] \in \mathbb{R}^{L \times 1}, \forall t \in \{1, \ldots, T\},
\]

which may be written conveniently in matrix form as \( X = AS + V \in \mathbb{R}^{L \times T} \), where \( S = [s_1 \ldots s_M]^T \in \mathbb{R}^{M \times T} \) denotes a matrix of \( M > 1 \) source signals of length \( T \), \( A \in \mathbb{R}^{L \times M} \) is a (deterministic) full rank mixing matrix, \( V = [v_1 \ldots v_L]^T \in \mathbb{R}^{L \times T} \) denotes a matrix of \( L \) additive noise signals (one for each sensor), where we assume \( L > M \), and \( X = [x_1 \ldots x_L]^T \in \mathbb{R}^{L \times T} \) is the matrix of the observed mixture signals. As in the standard ICA model, the sources \( s_1, \ldots, s_M \in \mathbb{R}^{T \times 1} \) (i.e., the rows of \( S \)) are assumed to be mutually statistically independent random processes, associated with unknown distributions, and the mixing matrix \( A \) is assumed to be unknown. However, unlike the common (not necessarily justified) assumption that the number of sources is known, here \( M \) is considered to be (deterministic) unknown.

For notational convenience only, we assume that the sources’ scales are non-Gaussian and that each source is temporally i.i.d. As a scaling convention we assume, without loss of generality, that the spatial covariance of the sources is \( \mathbb{E} [s[t]s[t]^T] = \Gamma_M \) since the sources’ scales are non-identifiable in this model. Furthermore, we assume that the noise \( v_1, \ldots, v_L \in \mathbb{R}^{T \times 1} \) from all the sensors (i.e., the rows of \( V \)) are temporally-white Gaussian noise processes, statistically independent from all the sources, with an unknown spatial covariance matrix \( \mathbb{E} [v[t]v[t]^T] \equiv R_v \in \mathbb{R}^{L \times L} \), where \( R_v \) can be any Positive-Definite (PD) matrix. This completes the definition of our model and the problem in question may be stated concisely as follows:

**Problem:** Given \( X \), determine the number of sources \( M \).

### IV. The Sources’ Dependency Criterion Estimate

Our proposed solution approach is based on the ability to *injectively* determine the empirical statistical independence of estimated sources using the empirical dCor. However, in order to put this powerful tool to work in the context of our problem, we first assume that we have at our disposal an ICA algorithm which can be applied to the \( L \) mixture signals using any hypothesized number of sources \( N \) (“N-hypothesis”) with \( 1 < N < L \), and provides consistent separation in the following sense: Let

\[
\mathcal{S}^{(N)} \equiv \hat{B}^{(N)}X = \hat{B}^{(N)}AS + \hat{B}^{(N)}V
\]

denote the output of the separation algorithm under the \( N \)-hypothesis, where \( \hat{B}^{(N)} \in \mathbb{R}^{N \times L} \) and \( \mathcal{G}^{(N)} = \hat{B}^{(N)}A \in \mathbb{R}^{N \times M} \) denote, resp., the estimated separating matrix and the resulting overall mixing-unmixing matrix, all under the same \( N \)-hypothesis. By “consistency” we mean that asymptotically (in both SNR and sample size together) perfect separation is achieved in the sense: Let

\[
\hat{M}^{\text{SDC}} \equiv \text{argmin}_{N \in \{2, \ldots, L-1\}} \text{SDC}(N)
\]

where the Sources’ Dependency Criterion / Sources’ empirical Distance Correlation (SDC) is defined (for \( 1 < N < L \)) as

\[
\text{SDC}(N) \equiv \max_{n \in \{1, \ldots, N\}} \mathcal{R}_T \left( \hat{s}^{N+1}_n, \mathcal{S}^{(N+1)} \right).
\]

Put simply, the SDC measures the maximal empirical dCor between each of the \( N \) estimated sources under the \( N \)-hypothesis and the “new” additional \((N+1)\)-th source under the \((N+1)\)-hypothesis (i.e., the \((N+1)\)-th row of \( \mathcal{S}^{(N+1)} \)). To formally justify and further explain the rationale of the proposed estimate, we shall present an asymptotic (qualitative) analysis of its operation. We start by defining a few necessary notions. First, we denote the Singular Value Decompositions (SVDs) \( A \equiv U_A D_A V_A^T \) and \( R_v \equiv U_v D_v U_v^T \), and we assume that the singular values are sorted in a decreasing order on the diagonals of \( D_A \) and \( D_v \). With this, we have

\[
R_x \equiv \mathbb{E} [x[t]x[t]^T] = U_A D_A^2 U_A^T + U_v D_v U_v^T \equiv U_x D_x U_x^T.
\]
From Weyl’s inequality (e.g., [28]), we have for all $1 \leq \ell \leq L$
\[
(D_A^2)_{\ell,\ell} + (D_v)_{\ell,L} \leq (D_A)_{\ell,\ell} + (D_v)_{1,1} \label{eq:weyl-ineq}.
\]
Since $R_v$ is PD, $(D_v)_{\ell,\ell} > 0$ for every $1 \leq \ell \leq L$. Therefore,
\[
\forall \ell \in \{1, \ldots, L\}, \exists \delta_\ell^2 \in \mathbb{R}^+: (D_x)_{\ell,\ell} \triangleq (D_A^2)_{\ell,\ell} + \delta_\ell^2, \label{eq:delta-ell}
\]
such that $(D_v)_{\ell,L} \leq \delta_\ell^2 \leq (D_v)_{1,1}$ for all $1 \leq \ell \leq L$. Notice that $(D_x)_{\ell,\ell} = \delta_\ell^2$ for $M + 1 \leq \ell \leq L$, since rank$(A) = M$.

With these notations, we assume
\[
\text{A1: } (D_v)_{1,1} \leq (D_A^2)_{M,M} \Rightarrow 1 \leq \ell \leq L : \delta_\ell^2 \leq (D_A^2)_{M,M} \text{, i.e., high SNR.}
\]

\[
\text{A2: The sample size } T \text{ is (finite but) “large enough” such that we may approximate } \frac{1}{T} X X^T \approx R_v, R_T(\cdot,\cdot) P^2 \approx R(\cdot,\cdot).
\]

Approximately “successful” operation of the separation algorithm for $N \geq M$ under A1 and A2:
\[
N = M : \hat{G}^{(M)} = \Gamma^{(M)} + \mathcal{E}^{(M)} A1 A2 \approx \Gamma^{(M)},
\]
\[
N > M : \hat{G}^{(N)} = \begin{bmatrix} \Gamma^{(N)} \\ \mathcal{E}^{(N)} A1 A2 \approx \Gamma^{(N)} \\ \mathcal{O} \end{bmatrix},
\]
where $\{\mathcal{E}^{(N)} \in \mathbb{R}^{N \times L}\}$ denote estimation error matrices.

\[
\text{A4: “Poor” operation of the separation algorithm for } N < M:
\]
When $N < M$ the resulting $\hat{G}^{(N)}$ is generally a “non-separating” matrix. At least, in particular,
\[
N < M : \exists i_1, i_2 : (\hat{G}^{(N)})_{i_1,N+1} (\hat{G}^{(N+1)})_{i_2,N+1} \neq 0.
\]

\[
\text{A5: Elements of the estimated } \hat{B}^{(N)} \text{ are generally non-zeros. In particular, for } N > M, \text{ the matrix } \Omega^{(N)} \triangleq \hat{B}^{(N)} R_v (\hat{B}^{(N+1)})^T \in \mathbb{R}^{N \times N} \text{ satisfies}
\]
\[
\exists n \in \{M+1, \ldots, N\} : (\Omega^{(N)})_{n,N+1} \neq 0.
\]

We shall now examine the three possible cases of the hypothesis test ([11]), which defines our proposed estimate.

\[ A. \text{ Case 1: } N\text{-hypothesis, } 1 < N < M \]
Assume the $N$-hypothesis, with $1 < N < M$. Therefore, in this case we have
\[
\text{SDC}(N) = \max_{n \in \{1, \ldots, N\}} R_T (\hat{s}^{(N)}_n, \hat{s}^{(N+1)}_{n+1}) \label{eq:sdcn}
\]
\[
A2 \approx \max_{n \in \{1, \ldots, N\}} R_T (\hat{s}^{(N)}_n[t], \hat{s}^{(N+1)}_{n+1}[t]) \triangleq \varrho_2^N, \label{eq:varrho-2n}
\]
where $\varrho_2^N > 0$, since $N + 1 \leq M$, hence the $(N + 1)$-th estimated source under the $(N + 1)$-hypothesis is (at least partially) linearly “contained” in one of the $N$ estimated sources under the $N$-hypothesis, by A4.

\[ B. \text{ Case 2: } M\text{-hypothesis} \]
Assume the $M$-hypothesis, i.e., the true number of sources. In this case, since the separation algorithm is assumed to be consistent, we have
\[
\text{SDC}(M) = \max_{m \in \{1, \ldots, M\}} R_T (\hat{s}^{(M)}_m, \hat{s}^{(M+1)}_{m+1}) \label{eq:sdcm}
\]
\[
A2 \approx \max_{m \in \{1, \ldots, M\}} R_T (\hat{s}^{(M)}_m[t], \hat{s}^{(M+1)}_{m+1}[t]) \triangleq 0 \label{eq:varrho-2m}
\]
as $M$ out of the $M + 1$ estimated sources under the $(M + 1)$-hypothesis must be (noisy versions of) the true sources (due to the consistency of the separation algorithm), and the $(M + 1)$-th estimated source is (approximately) a linear combination of noise components only, by A3. Thus, asymptotically, we approximately have $\varrho_2^N[t] \perp \varrho_2^{M+1}[t]$ for all $n \in \{1, \ldots, M\}$.

\[ C. \text{ Case 3: } N < M < L \]
Assume the $N$-hypothesis, with $M < N < L$. By A3, asymptotically, for every $M < N < L$, the “spurious” estimated sources $\hat{s}^{(N)}_n$ are generally non-separable in model ([8]) (see, e.g., [29] and references therein), we assert that A5 is highly likely to hold, hence the estimated spurious source $\hat{s}^{(N+1)}_{n+1}$ would be dependent on at least one estimated spurious source out of $\{\hat{s}^{(N)}_n\}_{n=M+1}^N$ a.s. Therefore, in this case we have
\[
\text{SDC}(N) \approx \left\{ \begin{array}{ll}
0, & N = M, \\
\varrho_2^N, & 1 \leq N \neq M \leq L, \\
M_{\text{SDC}} = M.
\end{array} \right.
\]

(19)

We stress that for any finite SNR and sample size $T$, SDC$(M) \neq 0$ a.s. However, asymptotically SDC$(M) \to 0$, thus the resulting error probability approaches zero as well, implying the consistency of the SDC estimate. The reasons for this are twofold: The estimate $\hat{B}$ approaches a perfect separating matrix (A3) and the empirical dCor approaches dCor (P2, A2). This assures consistently improving performance as the overall SNR and sample size grow, which is not necessarily true for other, previously proposed estimates in spatially non-white noise scenarios for any finite (even if large) SNR.

To summarize, the complete proposed solution algorithm to the problem of estimating the number of sources is as follows:

\[ \text{The Proposed Solution Algorithm: SDC Estimation} \]
1. Initialization: Obtain $\hat{S}^{(2)} \in \mathbb{R}^{2 \times T}$.
2. For every $N_{\text{cand}} = 2, \ldots, L - 1$ do:
   2.1. Obtain $\hat{s}^{(N_{\text{cand}}+1)} \in \mathbb{R}^{(N_{\text{cand}}+1) \times T}$ (e.g., via JADE);
   2.2. Compute SDC$(N_{\text{cand}})$ according to ([12]);
3. Determine $M_{\text{SDC}}$ according to ([11]).
V. Simulation Results

We demonstrate the performance of the proposed SDC estimate according to model (b) in simulation results of four different scenarios. In the last three, we compare it with the MDL, RMT and SORTE estimates which, currently being the leading methods, serve as an appropriate benchmark. All the empirical results are based on $10^3$ independent trials. Unless stated otherwise, the elements of $A$ were independently drawn at each trial from the standard Gaussian distribution.

First, we consider a scenario of $L = 7$ sensors and $M = 4$ zero-mean, unit variance Laplace distributed sources with white noise, i.e., $R_w = \sigma^2 I_L$. Fig. 1 presents the SDC cost function value for all the hypotheses, $N$, vs. $k$, an index determining the sample size and SNR such that $T = 500 - k$ and $\sigma^2 = -5 \cdot k$[dB]. In accordance with our asymptotic analysis, it is seen that the SDC cost function yields a consistent estimate.

Next, we consider a scenario of $L = 7$ sensors and $M = 3$ zero-mean, unit variance Laplace, Uniform and Rademacher (e.g., [30]) distributed sources. The noise is “approximately” white, i.e., $R_w$ is diagonal with $(R_w)_{k,\ell} = \sigma^2[\text{dB}] + \Delta_k[\text{dB}]$, where $\sigma^2$ is fixed and $\{\Delta_k \sim N(0, c^2)\}_{k=1}^7$ are mutually independent perturbations, with $\epsilon$ symbolizing the deviation from an “ideal” white-noise model. Fig. 2 presents the empirical error probabilities of the estimates vs. $\epsilon$ for $\sigma^2 = -15$[dB] and $T = 3000$. Evidently, MDL and RMT are sensitive to deviations from the white-noise model, while SDC and SORTE are more resilient to such deviations. And yet, recall that the SDC is blind, so (unlike SORTE) it does not exploit the (valuable) prior assumption of white noise.

In the last scenario we examine the performance in spatially correlated noise and “troublesome” mixing conditions. Specifically, $R_w$ has $\sigma^2$ on its diagonal, $0.1 \cdot \sigma^2$ on its sub- and super-diagonals, and zero elsewhere. This structure describes a “small” spatial correlation between two neighboring sensors (only). Further, after $A$ was drawn, we substitute (only) $\{D_A\}_{M,M} = \sqrt{0.1}$, which is mostly expressed in “difficult” second-order statistics conditions, and specifically challenges

\[1\text{with } \beta = 1 \text{ and } \alpha = 0.1\]
\[2\text{We do not consider AIC since it is an inconsistent estimate [8]}\]
\[3\text{For SORTE } 2 \leq M \leq 5, \text{ since it can estimate (only) up to } L - 3 \text{ sources.}\]

V. CONCLUSION

We presented an algorithm for blind determination of the number of (non-Gaussian) sources from noisy, linear mixtures. The proposed SDC estimate, which arises from the notion of dCor, was shown to be robust and resilient w.r.t. the mixing matrix and the noise spatial covariance matrix, which is not assumed to be of any particular structure. Accordingly, it exhibits more stable performance than other estimates when facing deviations from the ideal white-noise model assumption.
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