REMARK ON AN INEQUALITY FOR CLOSED
HYPERSURFACES IN COMPLETE MANIFOLDS WITH
NONNEGATIVE RICCI CURVATURE

XIAODONG WANG

Abstract. We give a simple proof of a recent result due to Agostiniani, Fogagnolo and Mazzieri [AFM].

The following result was proved by Agostiniani, Fogagnolo and Mazzieri [AFM].

**Theorem 1.** Let \((M^n, g) (n \geq 3)\) be a complete Riemannian manifold with nonnegative Ricci curvature and \(\Omega \subset M\) a bounded open set with smooth boundary. Then

\[
\int_{\partial \Omega} \left| \frac{H}{n-1} \right|^{n-1} \, d\sigma \geq \text{AVR} (g) |S^{n-1}|,
\]

where \(H\) is the mean curvature of \(\partial \Omega\) and \(\text{AVR} (g)\) is the asymptotic volume ratio of \(M\). Moreover, if \(\text{AVR} (g) > 0\), equality holds iff \(M \setminus \Omega\) is isometric to \((r_0, \infty) \times \partial \Omega, dr^2 + \left( \frac{r}{r_0} \right)^2 \tilde{g}_{\partial \Omega} \) with

\[
r_0 = \left( \frac{|\partial \Omega|}{\text{AVR} (g) |S^{n-1}|} \right)^{\frac{1}{n-1}}
\]

In particular, \(\partial \Omega\) is a connected totally umbilic submanifold with constant mean curvature.

The proof in [AFM] is highly nontrivial. It is based on the study of the solution of the following problem

\[
\begin{cases}
\Delta u = 0, & \text{on } M \setminus \Omega \\
u = 1 & \text{on } \partial \Omega \\
u(x) \to 0 & \text{as } x \to \infty,
\end{cases}
\]

which exists when \(\text{AVR} (g) > 0\). The key step consists of showing that, with \(\beta \geq (n - 2) / (n - 1)\)

\[
U_\beta (t) = t^{-\beta \left( \frac{n-1}{n-2} \right)} \int_{u=t} |\nabla u|^{\beta+1} \, d\sigma
\]

is monotone in \(t \in (0, 1]\). The geometric inequality (1) then follows by analyzing the asymptotic behavior of \(U_\beta (t)\) as \(t \to 0\). It is a beautiful argument.

In this short note, we show that this theorem can be proved by standard comparison methods in Riemannian geometry.

To prove the inequality (1), we assume, without loss of generality, that \(\Omega\) has no hole, i.e. \(M \setminus \Omega\) has no bounded component. In the following we write \(\Sigma = \partial \Omega\) and
let \( \nu \) be the outer unit normal along \( \Sigma \). For each \( p \in \Sigma \) let \( \gamma_p(t) = \exp_p t \nu(p) \) be
the normal geodesic with initial velocity \( \nu(p) \). We define
\[
\tau(p) = \sup \{ L > 0 : \gamma_p \text{ is minimizing on } [0, L] \in (0, \infty) \}.
\]
It is well known that \( \tau \) is a continuous function on \( \Sigma \) and the focus locus
\[
C(\Sigma) = \{ \exp_p \tau(p) \nu(p) : \tau(p) < \infty \}
\]
is a closed set of measure zero in \( M \). Moreover the map \( \Phi(r, p) = \exp_p r \nu(p) \) is a
diffeomorphism from
\[
E = \{ (r, p) \in \Sigma \times [0, \infty) : r < \tau(p) \}
\]
onoonto \( (M \setminus \Omega) \setminus C(\Sigma) \). And on \( E \) the pull back of the volume form takes the form
\[
d\mu = A(r, p) \, drd\sigma(p).
\]
We will also understand \( r \) as the distance function to \( \Sigma \) and it is smooth on \( M \setminus \Omega \) away from \( C(\Sigma) \). By the Bochner formula and nonnegative
Ricci curvature condition
\[
0 = \frac{1}{2} \Delta |\nabla r|^2 = |D^2 r|^2 + \langle \nabla r, \nabla \Delta r \rangle + \text{Ric}(\nabla r, \nabla r)
\geq \frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r} \Delta r.
\]
In view of the initial condition \( \Delta r|_{r=0} = H \), it is standard to deduce from the above
inequality \( \tau \leq \frac{n-1}{H} \) and
\[
\frac{A'}{A} = \Delta r \leq \frac{(n-1)H}{n-1 + Hr}
\]
This shows that the function
\[
\theta(r, p) = \frac{A(r, p)}{\left(1 + \frac{H(p)}{n-1} r\right)^{n-1}}
\]
is non-increasing in \( r \) on \( [0, \tau(p)] \). As \( \theta(0, p) = 1 \), we obtain
\[
A(r, p) \leq \left(1 + \frac{H(p)}{n-1} r\right)^{n-1}.
\]
The above analysis is standard in Riemannian geometry. We also remark that this
argument involving the Bochner formula can be replaced by an argument involving
the index form along each individual geodesic \( \gamma_p \). For more details, cf. \([P, S]\) or
other books on Riemannian geometry.

Therefore for any \( R > 0 \)
\[
\text{Vol}\{x \in M : d(x, \Omega) < R\} = |\Omega| + \int_{\Sigma} \int_{0}^{\min(R, \tau(p))} A(r, p) \, drd\sigma(p)
\leq |\Omega| + \int_{\Sigma} \int_{0}^{\min(R, \tau(p))} \left(1 + \frac{H(p)}{n-1} r\right)^{n-1} \, drd\sigma(p)
\leq |\Omega| + \int_{\Sigma} \int_{0}^{\min(R, \tau(p))} \left(1 + \frac{H^+(p)}{n-1} r\right)^{n-1} \, drd\sigma(p)
\leq |\Omega| + \int_{\Sigma} \int_{0}^{R} \left(1 + \frac{H^+(p)}{n-1} r\right)^{n-1} \, drd\sigma(p)
= |\Omega| + \frac{R^n}{n} \int_{\Sigma} \left(\frac{H^+(p)}{n-1}\right)^{n-1} d\sigma(p) + O\left(R^{n-1}\right).
\]
Dividing both sides by \(|B^n| R^n = |S^{n-1}| R^n/n\) and letting \(R \to \infty\) yields
\[
\text{AVR} (g) \leq \frac{1}{|S^{n-1}|} \int_{\Sigma} \left( \frac{H^+}{n-1} \right)^{n-1} d\sigma,
\]
which implies (1).

We now analyze the equality case. Suppose
\[
(2) \quad \text{AVR} (g) = \frac{1}{|S^{n-1}|} \int_{\Sigma} \left( \frac{H^+}{n-1} \right)^{n-1} d\sigma > 0.
\]
It is clear from the proof that \(\tau \equiv \infty\) on the open set \(\Sigma^+ = \{p \in \Sigma : H (p) > 0\}\). For any \(R' < R\) we have
\[
\text{Vol} \{ x \in M : d (x, \Omega) < R \} = |\Omega| + \int_{\Sigma^+} \int_0^R A (r, p) drd\sigma (p) + \int_{\Sigma^+} \int_0^{\min(R, \tau (p))} A (r, p) drd\sigma (p)
\leq |\Omega| + \int_{\Sigma^+} \int_0^R \theta (r, p) \left( 1 + \frac{H (p)}{n-1} r \right)^{n-1} drd\sigma (p) + \int_{\Sigma^+} \int_0^R drd\sigma (p)
\leq |\Omega| + \int_{\Sigma^+} \int_0^{R'} \theta (r, p) \left( 1 + \frac{H (p)}{n-1} r \right)^{n-1} drd\sigma (p) + O (R)
+ \int_{\Sigma^+} \int_0^{R'} \theta (r, p) \left( 1 + \frac{H (p)}{n-1} r \right)^{n-1} drd\sigma (p) + O (R).
\]
Dividing both sides by \(|B^n| R^n = |S^{n-1}| R^n/n\) and letting \(R \to \infty\) yields
\[
\text{AVR} (g) \leq \frac{1}{|S^{n-1}|} \int_{\Sigma^+} \left( \frac{H (p)}{n-1} \right)^{n-1} \theta (R', p) d\sigma (p).
\]
Letting \(R' \to \infty\) yields
\[
\text{AVR} (g) \leq \frac{1}{|S^{n-1}|} \int_{\Sigma^+} \left( \frac{H}{n-1} \right)^{n-1} \theta_\infty d\sigma,
\]
where \(\theta_\infty (p) = \lim_{r \to \infty} \theta (r, p) \leq 1\). As we have equality (2) we must have \(\theta_\infty (p) = 1\) for a.e. \(p \in \Sigma^+\). It follows that
\[
A (r, p) = \left( 1 + \frac{H (p)}{n-1} r \right)^{n-1} \text{ on } [0, \infty)
\]
for a.e. \(p \in \Sigma^+\). By continuity the above identity holds for all \(p \in \Sigma^+\).

Inspecting the comparison argument, we must have on \(\Phi ([0, \infty) \times \Sigma^+)\)
\[
D^2 r = \frac{\Delta r}{n-1} g = \frac{H}{n-1 + H r g},
Ric (\nabla r, \nabla r) = 0.
\]
As \(\text{Ric} \geq 0\), it follows that \(\text{Ric} (\nabla r, \cdot) = 0\). From the 1st equation above \(\Sigma^+\) is an umbilic hypersurface, i.e. the 2nd fundamental form \(\Pi = \frac{H}{n-1} \theta_{\Sigma^+}\). Working with
an orthonormal frame \( \{ e_0 = \nu, e_1, \cdots, e_{n-1} \} \) along \( \Sigma^+ \) we have by the Codazzi equation, with \( 1 \leq i, j, k \leq n - 1 \):

\[
R(e_k, e_j, e_i, \nu) = \Pi_{ij,k} - \Pi_{ik,j} = \frac{1}{n-1} (H_k \delta_{ij} - H_j \delta_{ik}).
\]

Taking trace over \( i \) and \( k \) yields

\[
-\frac{n-2}{n-1} H_j = Ric(e_j, \nu) = 0.
\]

As a result \( H \) is locally constant on \( \Sigma^+ \). Therefore \( \Sigma^+ \) must be the union of several components of \( \Sigma \). We know that \( \Phi \) is a diffeomorphism form \([0, \infty) \times \Sigma^+ \) onto its image and the pullback metric \( \Phi^* g \) takes the following form

\[
dr^2 + h_r,
\]

where \( h_r \) is a \( r \)-dependent family of metrics on \( \Sigma^+ \) and \( h_0 = g_{\Sigma^+} \). We have

\[
D^2 r = \frac{H}{n-1 + H} g.
\]

In terms of local coordinates \( \{ x_1, \cdots, x_{n-1} \} \) on \( \Sigma^+ \) the above equation implies

\[
\frac{1}{2} \frac{\partial}{\partial r} h_{ij} = \frac{H}{n-1 + H} h_{ij}.
\]

Therefore \( h_r = \left( 1 + \frac{H}{n-1} r \right)^2 g_{\Sigma^+} \). This proves that \( \Phi ([0, \infty) \times \Sigma^+) \) is isometric to \( \left( [r_0, \infty) \times \Sigma^+, dr^2 + \left( \frac{r}{r_0} \right)^2 g_{\Sigma^+} \right) \), where \( r_0 = \frac{n-1}{H} \).

Since \( M \) has nonnegative Ricci curvature and Euclidean volume growth, it has only one end by the Cheeger-Gromoll theorem. Therefore \( \Sigma^+ \) is connected and if \( \Sigma \) has other components besides \( \Sigma^+ \), they all bound bounded components of \( M \\setminus \Omega \).

If we have the stronger identity

\[
AVR(g) = \frac{1}{|\Omega|} \int_{\Sigma} \frac{H}{n-1} |^{n-1} d\sigma > 0,
\]

inspecting the proof of the inequality \([1]\) shows that we must have \( H \geq 0 \) on \( \Sigma \). Then \( \Sigma \) is compact Riemannian manifold with mean convex boundary. It is a classic fact that its boundary must be connected, see \([1] [K] \) or \([HW] \) for an analytic argument. Therefore \( \Sigma = \Sigma^+ \) is connected and \( M \setminus \Omega \) is isometric to \( \left( [r_0, \infty) \times \Sigma, dr^2 + \left( \frac{r}{r_0} \right)^2 g_{\Sigma} \right) \).

**Acknowledgement.** I would like to thank Fengbo Hang for helpful discussions.

**References**

[AFM] V. Agostiniani; M. Fogagnolo; L. Mazzieri. Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature. Invent. Math. 222 (2020), no. 3, 1033-1101.

[HW] F. Hang; X. Wang. Vanishing sectional curvature on the boundary and a conjecture of Schroeder and Strake, Pacific J. Math. 232 (2007), no. 2, 283-287.

[I] R. Ichida. Riemannian manifolds with compact boundary. Yokohama Math. J. 29 (1981), no. 2, 169-177.

[K] A. Kasue. Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary. J. Math. Soc. Japan 35 (1983), no. 1, 117-131.
[P] P. Petersen. Riemannian Geometry. Third edition. Graduate Texts in Mathematics, 171. Springer, Cham, 2016.

[S] T. Sakai. Riemannian geometry. Translations of Mathematical Monographs, 149. American Mathematical Society, Providence, RI, 1996.

Department of Mathematics, Michigan State University, East Lansing, MI 48824

Email address: xwang@math.msu.edu