Hölder continuity of solutions to the Monge-Ampère equations on compact Kähler manifolds

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ABSTRACT. We study Hölder continuity of solutions to the Monge-Ampère equations on compact Kähler manifolds. In [DNS] the authors have shown that the measure $\omega^n u$ is moderate if $u$ is Hölder continuous. We prove a theorem which is a partial converse to this result.

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1. Introduction

Let $X$ be a compact $n$-dimensional Kähler manifold equipped with a fundamental form $\omega$ satisfying $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi : X \rightarrow [-\infty, +\infty)$ is called $\omega$-plurisubharmonic ($\omega$-psh) if $\varphi \in L^1(X)$ and $\omega + dd^c \varphi \geq 0$. By PSH$(X, \omega)$ (resp. PSH$^-(X, \omega)$) we denote the set of $\omega$-psh (resp. negative $\omega$-psh) functions on $X$. The complex Monge-Ampère equation $\omega^n u = f \omega^n$ was solved for smooth positive $f$ in the fundamental work of S. T. Yau (see [Yau]). Later S. Kolodziej showed that there exists a continuous solution if $f \in L^p(\omega^n)$, $f \geq 0$, $p > 1$ (see [Ko2]). Recently in [Ko5] he proved that this solution is Hölder continuous in this case (see also [EGZ] for the case $X = \mathbb{C}P^n$).

In Corollary 1.2 in [DNS] the authors have shown that the measure $\omega^n u$ is moderate if $u$ is Hölder continuous. The main result is the following theorem which give a partial answer to the converse problem:

**Theorem A.** Let $\mu$ be a non-negative Radon measure on $X$ such that

$$
\mu(B(z, r)) \leq Ar^{2n-2+\alpha},
$$

for all $B(z, r) \subset X$ ($A, \alpha > 0$ are constants). Then for every $f \in L^p(d\mu)$ with $p > 1$, $\int_X f d\mu = 1$, there exists a Hölder continuous $\omega$-psh function $u$ such that $\omega^n u = f d\mu$.

The following results are simple applications of Theorem A:

**Corollary B.** Let $\varphi \in \text{PSH}(X, \omega)$ be a Hölder continuous function. Then for every $f \in L^p(\omega\varphi \wedge \omega^{n-1})$ with $p > 1$, $\int_X f \omega\varphi \wedge \omega^{n-1} = 1$, there exists a Hölder continuous $\omega$-psh function $u$ such that $\omega^n u = f \omega\varphi \wedge \omega^{n-1}$.

**Corollary C.** Let $S$ be a $C^1$ smooth real hypersurface in $X$ and $V_S$ be the volume measure on $S$. Then for every $f \in L^p(dV_S)$ with $p > 1$, $\int_X f dV_S = 1$, there exists a Hölder continuous $\omega$-psh function $u$ such that $\omega^n u = f dV_S$. 

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2. Preliminaries

First we recall some elements of pluripotential theory that will be used throughout the paper. Details can be found in [BT1-2], [Ce1-2], [CK], [CGZ], [De1-3], [Di1-3], [GZ1-2], [H], [Ko1-5], [KoTi], [Si1-2], [Ze1-2].

2.1. In [Ko2] Kołodziej introduced the capacity $C_X$ on $X$ by

$$C_X(E) := \sup \left\{ \int_E \omega^n_\varphi : \varphi \in \text{PSH}(X,\omega), -1 \leq \varphi \leq 0 \right\}$$

for all Borel sets $E \subset X$.

2.2. In [GZ1] Guedj and Zeriahi introduced the Alexander capacity $T_X$ on $X$ by

$$T_X(E) = e^{-\sup_{x} V_{E,x}^*}$$

for all Borel sets $E \subset X$. Here $V_{E,x}^*$ is the global extremal $\omega$-psh function for $E$ defined as the smallest upper semicontinuous majorant of $V_{E,x}$ i.e,

$$V_{E,x}(z) = \sup \{ \varphi(z) : \varphi \in \text{PSH}(X,\omega), \varphi \leq 0 \text{ on } E \}.$$

2.3. The following definition was introduced in [EGZ]: A probability measure $\mu$ on $X$ is said to satisfy the condition $\mathcal{H}(\alpha, A)$ ($\alpha, A > 0$) if

$$\mu(K) \leq AC_X(K)^{1+\alpha},$$

for any Borel subset $K$ of $X$.

A probability measure $\mu$ on $X$ is said to satisfy the condition $\mathcal{H}(\infty)$ if for any $\alpha > 0$ there exist $A(\alpha) > 0$ dependent on $\alpha$ such that

$$\mu(K) \leq A(\alpha)C_X(K)^{1+\alpha},$$

for any Borel subset $K$ of $X$.

2.4. The following definition was introduced in [DS]: A measure $\mu$ is said to be moderate if for any open set $U \subset X$, any compact set $K \subset U$ and any compact family $\mathcal{F}$ of plurisubharmonic functions on $U$, there are constants $\alpha > 0$ such that

$$\sup_K \{ \int e^{-\alpha \varphi} d\mu : \varphi \in \mathcal{F} \} < +\infty.$$
2.5. The following class of $\omega$-psh functions was investigated by Guedj and Zeriahi in [GZ2]:

$$\mathcal{E}(X, \omega) = \{ \varphi \in \text{PSH}(X, \omega) : \lim_{j \to \infty} \int_{\{\varphi > -j\}} \omega_{\max(\varphi, -j)}^n = \int_X \omega^n = 1 \}. $$

Let us also define

$$\mathcal{E}^-(X, \omega) = \mathcal{E}(X, \omega) \cap \text{PSH}^-(X, \omega).$$

We refer to [GZ2] for the properties of the class $\mathcal{E}(X, \omega)$.

2.6. $S$ is called a $C^1$ smooth real hypersurface in $X$ if for all $z \in X$ there exists a neighborhood $U$ of $z$ and $\chi \in C^1(U)$ such that $S \cap U = \{ z \in U : \chi(z) = 0 \}$ and $D\chi(z) \neq 0$ for all $z \in S \cap U$.

Next we state a well-known result needed for our work.

2.7. Proposition. Let $\mu$ be a non-negative Radon measure on $X$ such that $\mu(B(z, r)) \leq Ar^{2n-2+\alpha}$ for all $B(z, r) \subset X$ ($A, \alpha > 0$ are constants). Then $\mu \in \mathcal{H}(\infty)$.

Proof. By Theorem 7.2 in [Ze2] and Proposition 7.1 in [GZ1] we can find $\epsilon, C > 0$ such that

$$\mu(K) \leq Ah^{2n-2+\alpha}(K) \leq \frac{AC}{\alpha} T_X(K)^{\epsilon\alpha} \leq \frac{ACe^{\frac{\alpha}{c_X(K)^{\frac{\alpha}{\epsilon}}}}}{\alpha},$$

for all Borel subsets $K$ of $X$, where $h^{2n-2+\alpha}$ is the Hausdorff content of dimension $2n-2+\alpha$. This implies that $\mu \in \mathcal{H}(\infty)$.

3. Stability of the solutions

The stability estimate of solutions to the Monge-Ampère equations on compact Kähler manifolds was obtained by Kolodziej ([Ko2]). Recently, in [DZ] S. Dinew and Z. Zhang proved a stronger version of this estimate. We will show a generalization of the stability theorem by S. Kolodziej. As a first step we have the following proposition. This proof follows ideas of the proof of Theorem 2.5 in [DH]. We include a proof for the reader’s convenience.

3.1. Proposition. Let $\varphi, \psi \in \mathcal{E}^-(X, \omega)$ be such that $\omega^n_{\varphi} \in \mathcal{H}(\alpha, A)$. Then there exist constants $t \in \mathbb{R}$ and $C(\alpha, A) \geq 0$ such that

$$\int_{\{|\varphi - \psi - t| > a\}} (\omega^n_{\varphi} + \omega^n_{\psi}) \leq C(\alpha, A)a^{n+1},$$

here $a = \left( \int_X \|\omega^n_{\varphi} - \omega^n_{\psi}\| \right)^{\frac{1}{2n+3+\frac{\alpha}{1+\alpha}}}$.

Proof. Since $\int_{\{|\varphi - \psi - t| > a\}} (\omega^n_{\varphi} + \omega^n_{\psi}) \leq 2$, it suffices to consider the case when $a$ is small. Set

$$\epsilon = \frac{1}{2} \inf_{\{|\varphi - \psi - t| > a\}} \int \omega^n_{\varphi} : t \in \mathbb{R}$$
Hence
\[ \int_{\{\varphi - \psi - t \leq a\}} \omega_{\varphi}^n \leq 1 - 2\epsilon \]
for all \( t \in \mathbb{R} \). Set
\[ t_0 = \sup\{ t \in \mathbb{R} : \int_{\{\varphi < \psi + t\}} \omega_{\varphi}^n \leq 1 - \epsilon \} \]
Replacing \( \psi \) by \( \psi + t_0 \) we can assume that \( t_0 = 0 \). Then
\[ \int_{\{\varphi \leq \psi + a\}} \omega_{\varphi}^n \geq 1 - \epsilon. \]
Hence
\[ \int_{\{\varphi < \psi + a\}} \omega_{\varphi}^n = 1 - \int_{\{\varphi + a \leq \psi\}} \omega_{\varphi}^n = 1 - \int_{\{\varphi \leq \psi + a\}} \omega_{\varphi}^n \]
\[ + \int_{\{\psi - a < \varphi \leq \psi + a\}} \omega_{\varphi}^n \leq 1 - \epsilon. \]
Since \( \int_{\{\varphi - \psi \leq a\}} \omega_{\varphi}^n \leq 1 \) we can choose \( s \in [-a + a^{n+2}, a - a^{n+2}] \) satisfying
\[ \int_{\{\varphi - \psi - s \leq a^{n+2}\}} \omega_{\varphi}^n \leq 2a^{n+1}. \]
Replacing \( \psi \) by \( \psi + s \) we can assume that \( s = 0 \). One easily obtains the following inequalities
\[ (1) \int_{\{\varphi < \psi + a^{n+2}\}} \omega_{\varphi}^n \leq 1 - \epsilon, \int_{\{\psi < \varphi + a^{n+2}\}} \omega_{\varphi}^n \leq 1 - \epsilon, \int_{\{\varphi - \psi \leq a^{n+2}\}} \omega_{\varphi}^n \leq 2a^{n+1}. \]
By [GZ2] we can find \( \rho \in \mathcal{E}(X, \omega) \), such that
\[ (2) \omega_{\rho}^n = \frac{1}{1 - \epsilon} 1_{\{\varphi < \psi\}} \omega_{\varphi}^n + c 1_{\{\varphi \geq \psi\}} \omega_{\varphi}^n \text{ and } \sup_X \rho = 0, \]
\( (c \geq 0 \) is chosen so that the measure has total mass \( 1 \). \) For simplicity of notation we set \( \beta = \frac{n+1}{1 + \alpha} \). Set
\[ U = \{(1 - a^{n+2+\beta}) \varphi < (1 - a^{n+2+\beta}) \psi + a^{n+2+\beta} \rho \} \subset \{ \varphi < \psi \}. \]
From Theorem 2.1 in [Di3] and (2) we get
\[ (3) \omega_{\varphi}^{n-1} \wedge \omega_{(1-a^{n+2+\beta})\psi + a^{n+2+\beta} \rho} \geq (1 - a^{n+2+\beta}) \omega_{\varphi}^{n-1} \wedge \omega_{\psi} + \frac{a^{n+2+\beta}}{(1 - \epsilon)^{n}} \omega_{\varphi}^n, \]
on $U$. From Theorem 2.3 in [Di3], Lemma 2.6 in [DH] and (3) we obtain

\[
(1 - a^{n+2+\beta}) \int_U \varphi_{\psi}^{n-1} \wedge \omega_{\psi} + \frac{a^{n+2+\beta}}{(1 - \epsilon)^{\frac{n}{\pi}}} \int_U \omega_{\varphi}^n \\
\leq \int_U \omega_{(1-a^{n+2+\beta})\varphi + a^{n+2+\beta}\varphi} \wedge \omega_{\varphi}^{n-1} \\
\leq \int_U \omega_{(1-a^{n+2+\beta})\varphi} \wedge \omega_{\varphi}^{n-1} = (1 - a^{n+2+\beta}) \int_U \omega_{\varphi}^n + a^{n+2+\beta} \int_U \omega \wedge \omega_{\varphi}^{n-1} \\
\leq (1 - a^{n+2+\beta}) (\int_U \omega_{\varphi}^{n-1} \wedge \omega_{\psi} + 2a^{2n+3+\beta}) + a^{n+2+\beta} \int_U \omega \wedge \omega_{\varphi}^{n-1}.
\]

Hence

\[
\frac{1}{(1 - \epsilon)^{\frac{n}{\pi}}} \int_U \omega_{\varphi}^n \leq 2a^{n+1} + \int_U \omega \wedge \omega_{\varphi}^{n-1}.
\]

From Proposition 3.6 in [GZ1] and (4) we get

\[
\frac{1}{(1 - \epsilon)^{\frac{n}{\pi}}} \int_U \omega_{\varphi}^n \leq C_1(\alpha, A) a^{n+1},
\]

\[
\leq \frac{1}{(1 - \epsilon)^{\frac{n}{\pi}}} \int_U \omega_{\varphi}^n - A[C_\chi(\{\rho \leq -\frac{1}{2a^{\beta}}\})]^{1+\alpha} \\
\leq \frac{1}{(1 - \epsilon)^{\frac{n}{\pi}}} \int_U \omega_{\varphi}^n - \int_{\{\rho \leq -\frac{1}{2a^{\beta}}\}} \omega_{\varphi}^n \\
\leq \frac{1}{(1 - \epsilon)^{\frac{n}{\pi}}} \int_U \omega_{\varphi}^n \\
\leq 2a^{n+1} + \int_U \omega \wedge \omega_{\varphi}^{n-1} \\
\leq 2a^{n+1} + \int_{\{\varphi < \psi\}} \omega \wedge \omega_{\varphi}^{n-1},
\]

Similarly to $\rho$ we define $\vartheta \in \mathcal{E}(X, \omega)$, such that

\[
\omega_{\vartheta}^n = \frac{1}{1 - \epsilon} 1_{\{\varphi < \psi\}} \omega_{\varphi}^n + l 1_{\{\psi \geq \varphi\}} \omega_{\varphi}^n \text{ and } \sup_{X} \vartheta = 0,
\]

(l plays the same role as $c$ above). Set

\[
V = \{(1 - a^{n+2+\beta})\psi < (1 - a^{n+2+\beta})\varphi + a^{n+2+\beta}\vartheta \} \subset \{\psi < \varphi\}.
\]
We get

\[
\frac{1}{(1 - \epsilon)^n} \left[ \int_{\{\psi < \phi - a^{n+2}\}} \omega_\phi^n - C_1(\alpha, A)a^{n+1} \right] \leq 2a^{n+1} + \int_{\{\psi < \phi\}} \omega \wedge \omega_\phi^{n-1}.
\]

From (1), (5) and (6) we obtain

\[
\frac{1}{(1 - \epsilon)^n} \left[ 1 - 2a^{n+1} - 2C_1(\alpha, A)a^{n+1} \right] \leq \frac{1}{(1 - \epsilon)^n} \left[ \int_{\{\phi - \psi \geq a^{n+1}\}} \omega_\phi^n - 2C_1(\alpha, A)a^{1+\alpha} \right]
\]

\[
\leq 4a^{n+1} + 1.
\]

Hence

\[
\epsilon \leq 1 - \frac{1 - 2(C_1(\alpha, A) + 1)a^{n+1}}{4a^{n+1} + 1} \leq C_2(\alpha, A)a^{n+1}.
\]

This implies that there exists \( t \in \mathbb{R} \) satisfying

\[
\int_{\{\phi - \psi - t \geq a\}} \omega_\phi^n \leq 2C_2(\alpha, A)a^{n+1}.
\]

Finally we have

\[
\int_{\{\phi - \psi - t \geq a\}} (\omega_\phi^n + \omega_\psi^n) = 2 \int_{\{\phi - \psi - t \geq a\}} \omega_\phi^n + \int_{\{\phi - \psi - t \geq a\}} (\omega_\psi^n - \omega_\phi^n)
\]

\[
\leq 2C_2(\alpha, A)a^{n+1} + a^{2n+3+\beta} \leq C(\alpha, A)a^{n+1}.
\]

The second step in proving our stability theorem is the following

**3.2. Proposition.** Let \( \phi, \psi \in \mathcal{E}^{-}(X, \omega) \) be such that \( \omega_\phi^n, \omega_\psi^n \in \mathcal{H}(\alpha, A) \). Then there exist constants \( t \in \mathbb{R} \) and \( C(\alpha, A) \geq 0 \) such that

\[
C_X(\{|\phi - \psi - t| > a\}) \leq C(\alpha, A)a,
\]

here \( a = \left[ \int_X \|\omega_\phi^n - \omega_\psi^n\| \right]^{\frac{1}{2n+3+\beta}} \).

**Proof.** Since \( C_X(\{|\phi - \psi - t| > a\}) \leq C_X(X) = 1 \), it suffices to consider the case when \( a \) is small. Without loss of generality we can assume that \( \sup X \phi = \sup X \psi = 0 \). By Remark 2.5 in [EGZ] there exists \( M(\alpha, A) > 0 \) such that \( \|\phi\|_{L^\infty(X)} < M(\alpha, A), \|\psi\|_{L^\infty(X)} < M(\alpha, A) \). By Proposition 3.1 we can find \( t > 0 \) such that

\[
\int_{\{\phi - \psi - t \geq a\}} (\omega_\phi^n + \omega_\psi^n) \leq C_1(\alpha, A)a^{n+1}.
\]
We consider the case \( a < \min(1, \frac{1}{C_1(\alpha, A)}) \). Since
\[
\int_{\{|\varphi - \psi - t| > a\}} (\omega^n_\varphi + \omega^n_\psi) < 1
\]
we get \( \{ |\varphi - \psi - t| > a \} \neq X \). This implies that
\[
|t| \leq \sup_\times |\varphi - \psi| + 1 \leq M(\alpha, A) + 1.
\]
Replacing \( \psi \) by \( \psi + t \) we can assume that \( t = 0 \) and
\[
||\psi||_{L^\infty(X)} < 2M(\alpha, A) + 1.
\]
Using Lemma 2.3 in [EGZ] for \( s = \frac{a}{2}, t = \frac{a}{2(2M(\alpha, A) + 1)} \) we get
\[
C_X(\{|\varphi - \psi| < -a\}) \leq C_X(\{|\varphi - \psi| < -a - \frac{a}{2(2M(\alpha, A) + 1)}\})
\]
\[
\leq \frac{2^n(2M(\alpha, A) + 1)^n}{a^n} \int_{\{\varphi - \psi < -a\}} \omega^n_\psi
\]
\[
\leq 2^n(2M(\alpha, A) + 1)^n C_1(\alpha, A)a.
\]

Similarly we get
\[
C_X(\{|\psi - \varphi| < -a\}) \leq 2^n(2M(\alpha, A) + 1)^n C_1(\alpha, A)a.
\]

Combination of these inequalities yields
\[
C_X(\{||\varphi - \psi| > a\}) \leq C(\alpha, A)a.
\]

Now we prove the promised generalization of Kolodziej stability theorem (Theorem 1.1 in [Ko5]).

**3.3. Theorem.** Let \( \varphi, \psi \in \mathcal{E}^-(X, \omega) \) be such that \( \sup_\times \varphi = \sup_\times \psi = 0 \) and \( \omega^n_\varphi, \omega^n_\psi \in \mathcal{H}(\alpha, A) \). Then there exists \( C(\alpha, A) > 0 \) such that
\[
\sup_\times |\varphi - \psi| \leq C(\alpha, A)[\int_\times ||\omega^n_\varphi - \omega^n_\psi||]^{\frac{\min(1, \frac{1}{n})}{2n + 3 + \frac{3}{4n}}}.
\]

**Proof.** Set
\[
a = \left[ \int_\times ||\omega^n_\varphi - \omega^n_\psi||]^{\frac{1}{2n + 3 + \frac{3}{4n}}}.
\]

By Proposition 3.2 there exists \( C_1(\alpha, A) > 0 \) and \( t \in \mathbb{R} \) such that \( |t| \leq M(\alpha, A) + 1 \) and
\[
C_X(\{||\varphi - \psi - t| > a\}) \leq C_1(\alpha, A)a.
\]

Moreover, by Proposition 2.6 in [EGZ] there exists \( C_2(\alpha, A) > 0 \) such that
\[
\sup_\times |\varphi - \psi - t| \leq 2a + C_2(\alpha, A)[C_X(\{||\varphi - \psi - t| > a\})]^\frac{\alpha}{\alpha}
\]
\[
\leq 2a + C_2(\alpha, A)[C_1(\alpha, A)a]^\frac{\alpha}{\alpha}
\]
\[
\leq C_3(\alpha, A)a^{\min(1, \frac{\alpha}{\alpha})}.
\]
Moreover, since \( \sup_X \varphi = \sup_X \psi = 0 \) we obtain \( |t| \leq C_3(\alpha, A) a^{\min(1, \frac{\alpha}{n})} \). Combination of these inequalities yields

\[
\sup_X |\varphi - \psi| \leq \sup_X |\varphi - \psi - t| + |t| \leq 2C_3(\alpha, A) a^{\min(1, \frac{\alpha}{n})} = C(\alpha, A) \left[ \int_X |\omega_\varphi^n - \omega_\psi^n| \right]^{\frac{\min(1, \frac{\alpha}{n})}{2n + 3 + \frac{\alpha}{1 + \alpha}}}. 
\]

3.4. Corollary. Let \( \mu \) be a non-negative Radon measure on \( X \) such that \( \mu(B(z, r)) \leq A r^{2n-2+\alpha} \) for all \( B(z, r) \subset X \) (\( A, \alpha > 0 \) are constants). Given \( p > 1, M > 0, \epsilon > 0 \) and \( f, g \in L^p(d\mu) \) with \( ||f||_{L^p(d\mu)}, ||g||_{L^p(d\mu)} \leq M \) and \( \int_X f d\mu = \int_X g d\mu = 1 \). Assume that \( \varphi, \psi \in \mathcal{E}^-(X, \omega) \) satisfy \( \omega_\varphi^n = f d\mu, \omega_\psi^n = g d\mu \) and \( \sup_X \varphi = \sup_X \psi = 0 \). Then there exists \( C(\alpha, A, M, \epsilon) > 0 \) such that

\[
\sup_X |\varphi - \psi| \leq C(\alpha, A, M, \epsilon) \left[ \int_X |f - g| d\mu \right]^{\frac{1}{2n + 3 + \frac{\alpha}{1 + \alpha}}}. 
\]

Proof. By Hölder inequality we have

\[
\int_K f d\mu \leq ||f||_{L^p(d\mu)} [\mu(K)]^{1-\frac{1}{p}} \leq M [\mu(K)]^{1-\frac{1}{p}},
\]

\[
\int_K g d\mu \leq ||g||_{L^p(d\mu)} [\mu(K)]^{1-\frac{1}{p}} \leq M [\mu(K)]^{1-\frac{1}{p}},
\]

for any Borel subset \( K \) of \( X \). By Proposition 2.7 we get \( f d\mu, g d\mu \in \mathcal{H}(\infty) \). Using Theorem 3.3 we can find \( C(\alpha, A, M, \epsilon) > 0 \) such that

\[
\sup_X |\varphi - \psi| \leq C(\alpha, A, M, \epsilon) \left[ \int_X |f - g| d\mu \right]^{\frac{1}{2n + 3 + \frac{\alpha}{1 + \alpha}}}. 
\]

4. Local estimates in Potential theory

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\). By \( \text{SH}(\Omega) \) (resp \( \text{SH}^{-}(\Omega) \)) we denote the set of subharmonic (resp. negative subharmonic) functions on \( \Omega \). For each \( u \in \text{SH}(\Omega) \) and \( \delta > 0 \) we denote

\[
\tilde{u}_\delta(x) = \frac{1}{c_n \delta^n} \int_{B_\delta} u(x + y) dV_n(y),
\]

\[
u_\delta(x) = \sup_{y \in B_\delta} u(x + y),
\]

for \( x \in \Omega_\delta = \{ x \in \Omega : d(x, \partial \Omega) > \delta \} \). Here \( B_\delta = \{ x \in \mathbb{R}^n : |x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}} < \delta \} \) and \( c_n \) is the volume of the unit ball \( B_1 \). We state some results which will be used in our main theorems.
4.1. **Theorem.** Let $\mu$ be a non-negative Radon measure on $\Omega$ such that $\mu(B(z,r)) \leq Ar^{n-2+\alpha}$ for all $B(z,r) \subset D \subset \subset \Omega$ ($A, \alpha > 0$ are constants). Then for $K \subset \subset D$ and $\epsilon > 0$ there exists $C(\alpha, A, K, \epsilon)$ such that

$$\int_{K} [\tilde{u}_\delta - u] d\mu \leq C(\alpha, A, K, \epsilon) \int_{\bar{D}} \Delta u \delta^{\frac{n-\alpha}{1+\alpha}},$$

for all $u \in SH(\Omega)$, where $\Delta$ is the Laplace operator.

**Proof.** Since the change of radii of the balls does not affect the statement we can assume that $\Omega = B_4$, $D = B_3$, $K = B_1$ and $u$ is smooth on $B_4$. By [Hö] we have

$$u(x) = \int_{B_2} G(x, z) \Delta u(z) + h(x),$$

where $G(x, y)$ is the fundamental solution of Laplace equation and $h$ is harmonic in $B_2$. By Fubini theorem we have

$$\int_{B_1} [\tilde{u}_\delta(x) - u(x)] d\mu(x) = \int_{B_1} \frac{1}{c_n \delta^n} \int_{B_3} [u(x + y) - u(x)] dV_n(y) d\mu(x)$$

$$= \int_{B_2} \Delta u(z) \frac{1}{c_n \delta^n} \int_{B_3} dV_n(y) \int_{B_1} [G(x + y, z) - G(x, z)] d\mu(x).$$

Set

$$F(y, z) = \int_{B_1} [G(x + y, z) - G(x, z)] d\mu(x).$$

It is enough to prove that $F(y, z) \leq C(\alpha, A, \delta) \delta^{\frac{\alpha}{1+\alpha}}$ for all $y \in B_\delta, z \in B_2$. We consider two cases:
Case 1: \( n = 2 \). For \( y \in B_\delta, z \in B_2, \delta < \frac{1}{2} \), we have

\[
F(y, z) = \int_{B_1} |\ln |x + y - z| - \ln |x - z||d\mu(x)
\]

\[
= \int_{B_1 \cap \{x - z \geq |y|^{1 + \alpha}\}} \ln |1 + \frac{y}{x - z}|d\mu(x) + \int_{B_1 \cap \{|x - z| < |y|^{1 + \alpha}\}} \ln |1 + \frac{y}{x - z}|d\mu(x)
\]

\[
\leq \int_{B_1 \cap \{x - z \geq |y|^{1 + \alpha}\}} \ln |1 + |y|^{1 + \alpha}|d\mu(x) + \int_{B_1 \cap \{|x - z| < |y|^{1 + \alpha}\}} \ln |1 + |y|^{1 + \alpha}|d\mu(x)
\]

\[
+ \int_{B_1 \cap \{|x - z| < |y|^{1 + \alpha}\}} \ln \frac{1}{|x - z|}d\mu(x)
\]

\[
\leq |y|^{\frac{\alpha}{1 + \alpha}} \mu(B_1) + A|y|^{\frac{\alpha}{1 + \alpha}} \ln 4 + |y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \int_{\{|x - z| < |y|^{1 + \alpha}\}} \frac{1}{|x - z|^{\alpha - \epsilon}} \ln \frac{1}{|x - z|}d\mu(x)
\]

\[
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1 + \alpha}} + |y|^{\frac{\alpha - \epsilon}{1 + \alpha}} C_1(\alpha, \epsilon) \int_{\{|x - z| < 1\}} \frac{d\mu(x)}{|x - z|^{\alpha - \frac{\epsilon}{2}}}
\]

\[
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1 + \alpha}} + C_1(\alpha, \epsilon)|y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \sum_{j=0}^{\infty} \int_{\{2^{-j-1} \leq |x - z| < 2^{-j}\}} \frac{d\mu(x)}{|x - z|^{\alpha - \frac{\epsilon}{2}}}
\]

\[
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1 + \alpha}} + C_1(\alpha, \epsilon)|y|^{\frac{\alpha - \epsilon}{1 + \alpha}} A \sum_{j=0}^{\infty} 2^{(j+1)(\alpha - \frac{\epsilon}{2}) - j\alpha}
\]

\[
\leq C(\alpha, A, \epsilon)|y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \leq C(\alpha, A, \epsilon) \delta^{\frac{\alpha - \epsilon}{1 + \alpha}}.
\]

Case 2: \( n \geq 3 \). Similarly for \( y \in B_\delta, z \in B_2, \delta < \frac{1}{2} \), we have

\[
F(y, z) = \int_{B_1} \frac{1}{|x + y - z|^{n-2}} + \frac{1}{|x - z|^{n-2}}d\mu(x)
\]

\[
= \int_{B_1 \cap \{x - z \geq |y|^{1 + \alpha}\}} \frac{|x + y - z|^{n-2} - |x - z|^{n-2}}{|x + y - z|^{n-2} - |x - z|^{n-2}}d\mu(x) + \int_{B_1 \cap \{|x - z| < |y|^{1 + \alpha}\}} \frac{d\mu(x)}{|x - z|^{n-2 + \alpha - \epsilon}}
\]

\[
\leq C_2(\alpha)|y|^{\frac{\alpha}{1 + \alpha}} \int_{B_1 \cap \{x - z \geq |y|^{1 + \alpha}\}} d\mu(x) + |y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \int_{B_1 \cap \{|x - z| < |y|^{1 + \alpha}\}} \frac{d\mu(x)}{|x - z|^{n-2 + \alpha - \epsilon}}
\]

\[
\leq AC_2(\alpha)|y|^{\frac{\alpha}{1 + \alpha}} + |y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \int_{\{|x - z| < 1\}} \frac{d\mu(x)}{|x - z|^{n-2 + \alpha - \epsilon}}
\]

\[
\leq C(\alpha, A, \epsilon)|y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \leq C(\alpha, A, \epsilon) \delta^{\frac{\alpha - \epsilon}{1 + \alpha}}.
\]
4.2. **Theorem.** Let \( \mu \) be a non-negative Radon measure on \( \Omega \) such that \( \mu(B(z, r)) \leq Ar^{n-2+\alpha} \) for all \( B(z, r) \subset D \subset \subset \Omega \) \((A, \alpha > 0 \text{ are constants})\). Then for \( K \subset \subset D \) and \( \epsilon > 0 \) there exists \( C(\alpha, A, K, \epsilon) \) such that

\[
\int_K [u_\delta - u] d\mu \leq C(\alpha, A, K, \epsilon) ||u||_{L^\infty(\Omega)} \delta^{\frac{\alpha-\epsilon}{2(1+\alpha)}},
\]

for all \( u \in SH \cap L^\infty(\Omega) \).

We need a well-known lemma:

4.3. **Lemma.** Let \( u \in SH \cap L^\infty(\Omega) \). Then

\[
|\tilde{u}_\delta(x) - \tilde{u}_\delta(y)| \leq \frac{||u||_{L^\infty(\Omega)} |x - y|}{\delta},
\]

for all \( x, y \in \Omega_\delta \).

**Proof of Theorem 4.2.** By Lemma 4.3 we have

\[
u_\delta(x) = \sup_{y \in B_\delta} u(x + y) \leq \sup_{y \in B_\delta} \tilde{u}_\delta(x + y) \leq \tilde{u}_\delta(x) + \delta^{\frac{1}{2}} ||u||_{L^\infty(\Omega)}.
\]

By Theorem 4.1 we get

\[
\int_K [u_\delta - u] d\mu \leq \int_K [\tilde{u}_\delta(x) - u] d\mu + ||u||_{L^\infty(\Omega)} \mu(K) \delta^{\frac{1}{2}}
\]

\[
\leq C(\alpha, A, K, \epsilon) ||u||_{L^\infty(\Omega)} \delta^{\frac{\alpha-\epsilon}{2(1+\alpha)}}.
\]

Next we state a well-known result is a direct consequence of the Jensen formula (see [AG])

4.4. **Proposition.** Let \( u \in SH(B_2) \) be such that \( |u(x) - u(y)| \leq A|x - y|^{\alpha} \) for all \( x, y \in B_2 \). Then there exists \( C(\alpha, A) > 0 \) such that

\[
\int_{B(x, r)} \Delta u \leq C(\alpha, A) r^{n-2+\alpha},
\]

for all \( B(x, r) \subset B_1 \).

5. **Main results**

**Proof of Theorem A.** We use the same scheme as the proof of Theorem 2.1 in [Ko5]. From Corollary 3.4 and from Theorem 4.2 we can replace \( \omega^n \) by \( d\mu \). This implies that \( u \) is Hölder continuous with the Hölder exponent dependent on \( \alpha, A, p, X \) and \( ||f||_{L^p(d\mu)} \).

**Proof of Corollary B.** It follows from Proposition 4.4 and Theorem A.

**Proof of Corollary C.** Direct application of Theorem A.
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