GROUND STATE OF THE POLARON HYDROGENIC ATOM IN
A STRONG MAGNETIC FIELD

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Abstract. The ground-state electron density of a polaron bound to a Coulomb potential in a homogeneous magnetic field—the transverse coordinates integrated out—converges pointwise and weakly in the strong magnetic field limit to the square of a hyperbolic secant function.

1. Introduction

A non-relativistic Hydrogen atom in a strong magnetic field interacting with the quantized longitudinal optical modes of an ionic crystal is considered within the framework of Fröhlich’s 1950 polaron model [13]. Starting with Platzman’s variational treatment in 1962 the polaron Hydrogenic atom has been of interest for describing an electron bound to a donor impurity in a semiconductor [28]. Its first rigorous examination however came much later in 1988 from Löwen who disproved several longstanding claims about a self-trapping transition [27].

A study of the polaron Hydrogenic atom in strong magnetic fields was initiated by Larsen in 1968 for interpreting cyclotron resonance measurements in InSb [20]. The model has since been considered in formal analogy to the Hydrogen atom in a magnetic field though the latter was understood rigorously again much later in 1981 by Avron et al. who proved several properties including the non-degeneracy of the ground state [1]. Whether or not these Hydrogenic properties indeed persist when a coupling to a quantized field is turned on remains to be seen.

In any case polarons are the simplest Quantum Field Theory models, yet their most basic features such as the effective mass, ground-state energy and wave function cannot be evaluated explicitly. And while several successful theories have been proposed over the years to approximate the energy and effective mass of various polarons, they are built entirely on unjustified, even questionable, Ansätze for the wave function. The paper provides for the first time an explicit description of the ground-state wave function of a polaron in an asymptotic regime.

For the polaron Hydrogenic atom in a homogeneous magnetic field its ground-state electron density in the magnetic-field direction is shown to converge in the strong field limit to the square of a hyperbolic secant function: a sharp contrast to the paradigmatic Gaussian variational wave functions [35], [32] & ref. therein. The explicit limiting function is realized as a density of the minimizer of a one-dimensional problem with a delta-function potential describing the second leading-order term of the ground-state energy cf. [31], [31], [23].

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2. Model and Main Result

The Fröhlich model is defined by the Hamiltonian

$$H(B) := H_B - \partial_2^2 - \beta |x|^{-1} + N + \frac{\sqrt{\alpha}}{2\pi} \int_{\mathbb{R}^3} \left( \frac{a_k e^{ik_x}}{|k|} + \frac{a_k^* e^{-ik_x}}{|k|} \right) dk$$  \hspace{1cm} (2.1)

acting on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3) \otimes F$ where $F := \oplus_{n \geq 0} \otimes_n L^2(\mathbb{R}^3)$ is a symmetric phonon Fock space over $L^2(\mathbb{R}^3)$. The creation and annihilation operators for a phonon mode $a_k^*$ and $a_k$ act on $F$ and satisfy $[a_k, a_k^*] = \delta(k-k')$. The energy of the electron is described by the operator $H_B = \partial_2^2$ acting on $L^2(\mathbb{R}^3)$, where $H_B = \sum_{j=1,2} (-i\partial_j + A_j(x))^2$ is the two-dimensional Landau Hamiltonian with the magnetic vector potential $A(x_1, x_2, x_3) = B/2 (-x_2, x_1, 0)$ corresponding to a homogeneous magnetic field of strength $B \geq 0$ in the $x_3$-direction; the transverse coordinates are denoted by $x_\perp = (x_1, x_2)$. Furthermore inf spec $H_B = B$. The parameters $\alpha \geq 0$, $\beta > 0$ denote the strengths of the Coulombic electron-phonon interaction and the localization Coulomb potential; the coupling function $1/|k|$ is proportional to the square-root of the Fourier transform of the Coulomb interaction. The ground-state energy is

$$E_0(B) := \inf \left\{ \langle \Psi, H(B) \Psi \rangle : \|\Psi\| = 1, \Psi \in H^1_\Lambda(\mathbb{R}^3) \otimes \text{dom} \left( \sqrt{\mathcal{N}} \right) \right\},$$  \hspace{1cm} (2.2)

where $H^1_\Lambda(\mathbb{R}^3)$ is a magnetic Sobolev space of order one. A ground state exists since $-i\nabla - \beta |x|^{-1}$ has a negative energy bound state in $L^2(\mathbb{R}^3)$ [15].

Unlike previous treatments here the arguments remain valid for all values of the parameters $\alpha \geq 0$, $\beta > 0$. First the large $B$ asymptotics of the ground-state energy is derived; the main result is given as Theorem 2.2 below. Since the pioneering work of Larsen the model has been considered only in the perturbative regime $\alpha \ll \beta$, and the ground-state energy $E_0(B)$ has been approximated as the Hydrogenic energy

$$E_\mu(B) := \inf \text{spec } H_B - \partial_2^2 - \beta |x|^{-1}$$

with a supposedly small correction from the electron-phonon interaction. The large $B$ asymptotics of the Hydrogenic energy was derived rigorously in 1981 by Avron et al. [1] using ideas from [4], [33]:

$$E_\mu(B) = B - \frac{\beta^2}{4} (\ln B)^2 + \beta^2 \ln \ln B - \beta^2 (\frac{\gamma_E}{2} + \ln 2) \ln B - \beta^2 (\ln \ln B)^2 + 2\beta^2 (\frac{\gamma_E}{2} - 1 + \ln 2) \ln \ln B + O(1) \text{ as } B \to \infty$$ \hspace{1cm} (2.3)

with $\gamma_E$ the Euler-Mascheroni constant, and the expansion can be carried out to arbitrary order. The first three terms are understood heuristically: For large $B$ the electron is tightly bound in the transverse plane to the lowest Landau orbit while localized in the magnetic-field direction by a one-dimensional effective Coulomb potential that behaves to leading order like a delta well of strength $\beta \ln B / (\ln B)^2$ [7], [30], [31]; see (1.1) and Appendix B below. The electron motion is effectively one-dimensional cf. [6]. The second and third leading-order terms describe the dominant asymptotic behavior of the ground-state energy of this one-dimensional electron confined along the magnetic field. The pronounced anisotropy in the system is reflected by the characteristic length scales of the electron density in the transverse and the magnetic-field direction $1/\sqrt{B}$ and $1/\ln B$ respectively.
The above Hydrogenic heuristics still apply when a coupling to the phonon field is introduced i.e. $\alpha > 0$. For large $B$ the phonons cannot follow the electron’s rapid motion in the transverse plane and so resign themselves to dressing its entire Landau orbit: not only is the electron again localized in the magnetic-field direction by the one-dimensional effective Coulomb potential, but also the electron-phonon coupling function is now proportional to the square-root of the Fourier transform of the same effective Coulomb interaction cf. [19], [34] and property (k) in [30]; the system behaves as a one-dimensional strongly coupled polaron to leading order with interaction strength $\alpha \ln(B/\ln B)$ confined along the magnetic field by a delta well of strength $\beta \ln(B/\ln B)^2$, i.e. in the effective one-dimensional model the electron-phonon coupling is mediated by the magnetic field. The analogous large $B$ asymptotics of the polaron Hydrogenic energy is derived to second order:

**Theorem 2.1.** Let $E_0(B)$ be as defined in (2.2) above. Then

$$E_0(B) = B + \epsilon_0 (\ln B)^2 + \mathcal{O}((\ln B)^{3/2}) \quad \text{as } B \to \infty$$

$$\epsilon_0 := \inf \left\{ \int_R |\varphi'|^2 dx - \frac{\alpha}{2} \int_R |\varphi|^4 dx - \beta |\varphi(0)|^2 : \int_R |\varphi|^2 dx = 1 \right\}$$

$$= -\frac{1}{48} \left( \alpha^2 + 6\alpha\beta + 12\beta^2 \right).$$

Here the second leading-order term describes the dominant asymptotic behavior of the ground-state energy of the effective one-dimensional strongly coupled polaron confined along the magnetic field. It is evaluated explicitly by minimizing a non-linear functional. Furthermore the cross term in (2.6) indicates for large $B$ the effect of the electron-phonon interaction is not perturbative.

The large $B$ asymptotics for the polaron Hydrogenic energy is argued differently from Avron et al.’s proof of (2.3) and generalizes the result of Frank and Geisinger who proved (2.4)-(2.6) when $\beta = 0$ using upper and lower bounds to the ground-state energy [9]. Their upper bound is established with a trial wave function. Their lower bound is established by showing that the Hamiltonian when restricted to the lowest Landau level is bounded from below in the sense of quadratic forms by an essentially one-dimensional strong-coupling Hamiltonian; the strategy from [24] is then used to arrive at the nonlinear minimization problem for the second leading-order term along with lower order error terms cf. [14].

For proving Theorem 2.1 Frank et al.’s strategy in [9] applies mutatis mutandis. Here the argument for the upper bound is simplified using the effective Coulomb potential, given below in (4.1), which plays an essential role in Avron et al.’s proof of (2.3) but is conspicuously absent from [9]. Here the argument for the lower bound uses the bathtub principle [21] to extract a delta function from the Coulomb potential in the lowest Landau level: just like in [9] the error terms are bounded by $\mathcal{O}((\ln B)^{3/2})$, but—and this has been demonstrated recently in [11] for a strongly coupled polaron—the error terms in the lower bound should be much smaller. Moreover the upper bound suggests the third leading-order term again analogously to its Hydrogenic counterpart in (2.3) is $-4\epsilon_0 \ln B \ln \ln B$.

The expansion in (2.4) should be carried out to higher order. I conjecture the first six leading-order terms behave analogously to their Hydrogenic counterparts in (2.3): these should describe the leading asymptotics for the minimization of a non-linear functional arising naturally in the proof of the upper bound and given below.
in \([1,3]\), which is a classical approximation to the Fröhlich model of the strongly coupled one-dimensional polaron confined along the magnetic field: in this classical approximation the electron-phonon interaction is replaced with the one-dimensional effective Coulomb self-interaction of the electron. The second leading-order term above arises from this classical approximation by estimating the one-dimensional effective Coulomb interaction as a delta interaction of strength \(\ln B\). Furthermore, in the seventh leading-order term there should be an order-one quantum correction to the classical approximation cf. \([1,17]\). These higher-order asymptotics should be provable with better control of the error terms when arguing the lower bound, cf. \([11]\), and by making full use of the one-dimensional effective Coulomb potential: the electron-phonon interaction of the one-dimensional Hamiltonian that is derived both here and in Frank et al.’s proof \([9]\) of the lower bound is described using an artificial coupling function, given in \([5,9]\) below, when instead it should really be argued that the coupling is proportional to the square-root of the Fourier transform of the electron-phonon interaction of the one-dimensional Hamiltonian that is derived in (4.3), which is a classical approximation to the Fröhlich model of the strongly polarized hydrogenic atom in a strong magnetic field: in this classical approximation the electron-phonon interaction is replaced with the one-dimensional effective Coulomb self-interaction of the electron. The second leading-order term above arises from this classical approximation by estimating the one-dimensional effective Coulomb interaction as a delta interaction of strength \(\ln B\). Furthermore, in the seventh leading-order term there should be an order-one quantum correction to the classical approximation cf. \([11,17]\). These higher-order asymptotics should be provable with better control of the error terms when arguing the lower bound, cf. \([11]\), and by making full use of the one-dimensional effective Coulomb potential: the electron-phonon interaction of the one-dimensional Hamiltonian that is derived both here and in Frank et al.’s proof \([9]\) of the lower bound is described using an artificial coupling function, given in \([5,9]\) below, when instead it should really be argued that the coupling is proportional to the square-root of the Fourier transform of the effective Coulomb potential in \([4,11]\); then the strategy from \([24]\) can be used to arrive at the classical approximation in \([4,3]\) now as a lower bound.

The two-term asymptotics of the ground-state energy achieved in Theorem 2.1 suffices for arguing the main result:

**Theorem 2.2.** Let \(\Psi^{(B)} \in \mathcal{H}\) be any approximate ground-state wave function satisfying \(\langle \Psi^{(B)}, \mathbb{H}(B)\Psi^{(B)} \rangle = E_0(B) + \mathcal{O}(\ln B)^2\). The one-dimensional minimization problem in \((2.6)\) admits up to complex phase a unique minimizer

\[
\phi_0(x_3) = \alpha + 2\beta \sqrt{8\alpha \cosh\left(\frac{\alpha + 2\beta}{\alpha + 2\beta}x_3 + \tanh^{-1}\left(\frac{\alpha + 2\beta}{\alpha + 2\beta}\right)\right)}.
\]

and for \(W\) a sum of a bounded Borel measure on the real line and a \(L^\infty(\mathbb{R})\) function

\[
\lim_{B \to \infty} \frac{1}{(\ln B)^2} \int_{\mathbb{R}} W(x_3) \left(\int_{\mathbb{R}^2} \left| \Psi^{(B)}(x_3, \frac{x_3}{\ln B}) \right|^2 dx_\perp \right) dx_3 = \int_{\mathbb{R}} W(x_3) \phi_0(x_3)^2 dx_3. \tag{2.7}
\]

By choosing \(W\) as a delta-function potential pointwise convergence is obtained. When \(\alpha = 0\) the limiting density in \((2.7)\) is \(\sqrt{\beta/2} \exp(-\beta|x_3|/2)\) cf. \([12,29]\). The idea of the proof is to add to the Hamiltonian \(\epsilon\) times the one-dimensional potential \(W\) scaled appropriately in the magnetic-field direction cf. \([2,10,11,22,23,26]\).

For

\[
\mathbb{H}_\epsilon(B) := \mathbb{H}(B) - \epsilon(\ln B)^2 W((\ln B)x_3) \tag{2.8}
\]

and \(E_\epsilon(B)\) the corresponding ground-state energy it is argued vis-à-vis Theorem 2.1 that

\[
E_\epsilon(B) = B + \epsilon \ln B + \mathcal{O}\left((\ln B)^{3/2}\right) \quad \text{as } B \to \infty \tag{2.9}
\]

with

\[
\epsilon := \inf_{\varphi \neq 0} \left\{ \int_{\mathbb{R}} |\varphi|^2 dx - \frac{\alpha}{2} \int_{\mathbb{R}} |\varphi|^4 dx - \beta |\varphi(0)|^2 - \epsilon \int_{\mathbb{R}} W(x) |\varphi|^2 dx \right\}. \tag{2.10}
\]

By the variational principle \(E_\epsilon \leq \langle \Psi^{(B)}, \mathbb{H}_\epsilon(B)\Psi^{(B)} \rangle\), and the expectation value on the right-hand side evaluates to

\[
E_0(B) + \mathcal{O}(\ln B)^2 - \epsilon(\ln B) \int_{\mathbb{R}} W(x_3) \left(\int_{\mathbb{R}^2} \left| \Psi^{(B)}(x_3, \frac{x_3}{\ln B}) \right|^2 dx_\perp \right) dx_3.
\]
Then for \( \epsilon > 0 \)
\[
\frac{E_0(B) - E_0(B)}{\epsilon (\ln B)^2} \geq \frac{1}{\ln B} \int_W \left( \int_{\mathbb{R}^2} \left\| \frac{\Psi(B)}{x_3} \right\|_F \left( x_3, \frac{x_3}{\ln B} \right) dx \right) dx_3 + o(1),
\]
and taking the limit \( B \to \infty \) by Theorem 2.1 and (2.9)
\[
e_0 - e_\epsilon \geq \lim \sup_{B \to \infty} \frac{1}{\ln B} \int_W \left( \int_{\mathbb{R}^2} \left\| \frac{\Psi(B)}{x_3} \right\|_F \left( x_3, \frac{x_3}{\ln B} \right) dx \right) dx_3.
\]
For \( \epsilon < 0 \) the above inequality is reversed with “lim sup” replaced by “lim inf.”

Hence the theorem follows if the quotient on the left-hand side has a limit as \( \epsilon \to 0 \).

The map \( \epsilon \mapsto e_\epsilon \) is differentiable at \( \epsilon = 0 \) for all \( W \) if and only if the one-dimensional problem for the energy \( e_0 \) in (2.5) admits up to complex phase a unique minimizer:

Uniqueness is established by explicitly solving the corresponding Euler-Lagrange equation cf. [23], and by the variational principle and a compactness argument

\[
\lim_{\epsilon \to 0} \frac{e_0 - e_\epsilon}{\epsilon} = \int_W (x_3) \phi_0(x_3)^2 dx_3.
\]

In Section 3 the differentiation of the one-dimensional energy (2.11) is proved as Theorem 3.3. In Section 4 an upper bound to the ground-state energy is proved as Theorem 4.1. In Section 5 a lower bound to the ground-state energy is proved as Theorem 5.1, and Theorem 2.1 follows from Theorem 4.1 and Theorem 5.1. In Section 6 the main result Theorem 2.2 is proved.

### 3. Differentiating the One-Dimensional Energy

The one-dimensional problem for the energy \( e_0 \) in (2.5) shall be denoted

\[
e_0 := \inf_{\| \varphi \|_2 = 1} \mathcal{E}_0(\varphi)
\]

with

\[
\mathcal{E}_0(\varphi) := \int_W |\varphi'|^2 dx - \frac{\alpha}{2} \int_W |\varphi|^4 dx - \frac{\beta}{2} |\varphi(0)|^2.
\]

**Lemma 3.1.** The minimization problem in (2.7) for the energy \( e_0 \) admits up to complex phase a unique minimizer

\[
\phi_0(x) = \frac{\alpha + 2\beta}{\sqrt{8\alpha \cosh \left( \frac{\alpha + 2\beta}{4} |x| \right) + \tanh^{-1} \left( \frac{2\beta}{\alpha + 2\beta} \right)}}
\]

and

\[
e_0 = \frac{1}{48} \left( \alpha^2 + 6\alpha\beta + 12\beta^2 \right).
\]

**Proof.** The existence of a minimizer is shown in Appendix A. Any minimizer up to multiplication by a complex phase is a nonnegative, \( C^2(\mathbb{R} \setminus \{0\}) \) function in \( H^1(\mathbb{R}) \) solving the Euler-Lagrange equation

\[
-\psi'' - \alpha \psi^3 - \beta \delta(x) \psi = -\lambda \psi
\]

and

\[
-\lambda = e_0 - \frac{\alpha}{2} \int_{\mathbb{R}} \psi^4 dx < 0.
\]

Or equivalently it must solve

\[
-\psi'' - \alpha \psi^3 = -\lambda \psi \text{ for } |x| > 0
\]

and satisfy the boundary condition

\[
\lim_{\epsilon \to 0^+} [\psi'(-\epsilon) - \psi'(\epsilon) = \beta \psi(0)].
\]
The first integral of (3.4) is
\[ \psi'^2 = -\frac{\alpha}{2} \psi^4 + \lambda \psi^2 \text{ for } |x| > 0. \] (3.6)

Any nonnegative, \( C^2(\mathbb{R}\{\{0\}) \) solution of (3.6) in \( H^1(\mathbb{R}) \) satisfying the boundary condition in (3.5) must be of the form
\[ \psi = \sqrt{\frac{2\lambda}{\alpha}} \cosh \left( \frac{1}{\sqrt{\lambda}} (|x| - \tau) \right) \] (3.7)
for some \( \tau \) cf. \( \mathbb{R} \). The boundary condition and that \( \|\psi\|_2 = 1 \) require respectively
\[ \tanh \left( \tau \sqrt{\lambda} \right) = -\frac{\beta}{2\sqrt{\lambda}} \text{ and } \frac{\alpha}{4\sqrt{\lambda}} = 1 + \tanh \left( \tau \sqrt{\lambda} \right). \]

Any minimizer up to complex phase must therefore be of the form in (3.7) with
\[ \lambda = \left( \frac{\alpha + 2\beta}{4} \right)^2 \text{ and } \tau = -\frac{4}{\alpha + 2\beta} \tanh^{-1} \left( \frac{2\beta}{\alpha + 2\beta} \right). \]
The explicit calculation of \( \epsilon_0 \) now follows from (3.3). \( \square \)

**Lemma 3.2.** If \( \{\phi_n\}_{n=1}^{\infty} \) is a minimizing sequence for \( \epsilon_0 \), then \( |\phi_n| \) converges in \( H^1(\mathbb{R}) \) to the minimizer \( \phi_0 \) given in (3.2).

**Proof.** By Theorem 7.8 in [21] \( \{\|\phi_n\|_{L^\infty} \}_{n=1}^{\infty} \) is also a minimizing sequence for \( \epsilon_0 \). It is argued in Appendix A that every minimizing sequence for \( \epsilon_0 \) has a subsequence converging in \( H^1(\mathbb{R}) \) to some minimizer. Since \( \phi_0 \) is up to complex phase the unique minimizer, every subsequence of \( \{\|\phi_n\|_{L^\infty} \}_{n=1}^{\infty} \) must converge in \( H^1(\mathbb{R}) \) to \( \phi_0 \). \( \square \)

**Theorem 3.3.** Let \( W \) be a sum of a bounded Borel measure on the real line and a \( L^\infty(\mathbb{R}) \) function. For \( \epsilon \) a real parameter consider the one-dimensional energy
\[ \epsilon_\epsilon := \inf_{\|\phi\|_2 = 1} \mathcal{E}_\epsilon(\varphi), \] (3.8)
where
\[ \mathcal{E}_\epsilon(\varphi) := \mathcal{E}_\epsilon(\varphi) - \epsilon \int_{\mathbb{R}} W(x) |\varphi(x)|^2 \, dx \] (3.9)
with the functional \( \mathcal{E}_\epsilon \) as given in (3.7). Then the map \( \epsilon \mapsto \epsilon_\epsilon \) is differentiable at \( \epsilon = 0 \) and
\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \epsilon_\epsilon = -\int_{\mathbb{R}} W(x) \phi_0(x)^2 \, dx \]
with \( \phi_0 \) the minimizer given in (3.2) for the energy \( \epsilon_0 \).

**Proof.** Writing \( W = \mu + \omega \) where \( \mu \) is a signed, bounded measure on the real line and \( \omega \in L^\infty(\mathbb{R}) \), it follows from Hölder’s inequality, the Sobolev inequality and completion of the square that for \( \varphi \in H^1(\mathbb{R}) \)
\[ \mathcal{E}_\epsilon(\varphi) \geq \|\varphi\|^2_2 - \|\varphi\|^2_2 \left( \frac{\alpha}{2} \|\varphi\|^2_2 + |\epsilon| |\mu| L^1(\mathbb{R}) + \beta \right) - |\epsilon| \|\omega\|_{L^\infty} \|\varphi\|^2_2 \]
\[ \geq \frac{3}{4} \|\varphi\|^2_2 - \|\varphi\|^2_2 \left( \frac{\alpha}{2} \|\varphi\|^2_2 + |\epsilon| |\mu| L^1(\mathbb{R}) + \beta \right)^2 - |\epsilon| \|\omega\|_{L^\infty} \|\varphi\|^2_2. \] (3.10)
Hence \( \epsilon_\epsilon > -\infty \), and for each \( \epsilon \) there is a \( \phi_\epsilon \in H^1(\mathbb{R}) \) satisfying
\[ \mathcal{E}_\epsilon(\phi_\epsilon) \leq \epsilon_\epsilon + \epsilon^2 \text{ and } \|\phi_\epsilon\|_2 = 1. \] (3.11)
By the variational principle
\[ \epsilon_0 \leq E_0(\phi_n) = \mathcal{E}_\epsilon(\phi_n) + \epsilon \int_\mathbb{R} W(x) |\phi_n(x)|^2 \, dx \leq \epsilon_\star + \epsilon^2 + \epsilon \int_\mathbb{R} W(x) |\phi_n(x)|^2 \, dx, \]
and
\[ \epsilon_\star \leq \mathcal{E}_\epsilon(\phi_n) = \epsilon_0 - \epsilon \int_\mathbb{R} W(x) \phi_0(x)^2 \, dx. \]

Then for \( \epsilon > 0 \)
\[ - \int_\mathbb{R} W(x) |\phi_n(x)|^2 \, dx - \epsilon \leq \frac{\epsilon_0 - \epsilon_\star}{\epsilon} \leq - \int_\mathbb{R} W(x) \phi_0(x)^2 \, dx. \] (3.12)

For \( \epsilon < 0 \) the inequalities in (3.12) are reversed. It suffices therefore to show for any sequence \( \{\epsilon_n\}_{n=1}^\infty, |\epsilon_n| > 0 \) and \( \epsilon_n \to 0 \) that
\[ \lim_{n \to \infty} \int_\mathbb{R} W(x) |\phi_n(x)|^2 \, dx = \int_\mathbb{R} W(x) \phi_0(x)^2 \, dx. \] (3.13)

Since \( \epsilon_\star > -\infty \), the concave map \( \epsilon \mapsto \epsilon_\star \) is continuous. It follows from (3.10), the Sobolev inequality and (3.11) that \( \|\phi_n\|_\infty < C \). Thus
\[ \lim_{n \to \infty} \epsilon_n \int_\mathbb{R} W(x) |\phi_n(x)|^2 \, dx = 0, \]
and \( \{\phi_n\}_{n=1}^\infty \) is a minimizing sequence for \( \epsilon_\star \). By Lemma 3.2 \( |\phi_n| \) converges in \( H^1(\mathbb{R}) \) to \( \phi_0 \), and by Theorem 8.6 in [21] \( |\phi_n| \) converges also pointwise uniformly to \( \phi_0 \) on bounded sets. The convergence in (3.13) now follows. \( \square \)

4. Upper Bound to the Ground-State Energy

**Theorem 4.1.** There is a constant \( C > 0 \) such that for \( B > 1 \)
\[ E_0(B) \leq B + \epsilon_0 (\ln B)^2 - 4\epsilon_0 \ln B \ln B + C \ln B. \]

Theorem 4.1 will follow from Lemma 4.2 and Lemma 4.3.

**Lemma 4.2.** \( E_0(B) \leq E_0^\circ(B) \), where \( E_0^\circ(B) := \inf \{ P(\psi): \|\psi\|_2 = 1 \} \) is the classical Pekar energy with \( P \) denoting the three-dimensional magnetic Pekar functional
\[ \langle \psi, \left( H_B - \frac{\partial^2}{\partial^2} \right) \psi \rangle_{L^2} = -\frac{\alpha}{2} \int_{\mathbb{R}^3} \frac{|\psi(x)|^2 |\psi(y)|^2}{|x-y|} \, dx \, dy - \beta \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|} \, dx. \]

**Proof.** By the variational principle
\[ E_0(B) \leq \inf \{ \langle \Psi, H(B) \Psi \rangle: \|\Psi\| = 1, \Psi = \varphi(x)\Phi \text{ and } \Phi \in \mathcal{F} \} = E_0^\circ(B), \]
the equality following from completion of the square. \( \square \)

**Lemma 4.3.** There is a constant \( C > 0 \) such that for \( B > 1 \)
\[ E_0^\circ(B) \leq B + \epsilon_0 (\ln B)^2 - 4\epsilon_0 \ln B \ln B + C \ln B. \]
Proof. With the lowest Landau state in the zero angular momentum sector
\[ \gamma_B(x_\perp) := \sqrt{\frac{B}{2\pi}} \exp \left(-\frac{B}{4} |x_\perp|^2 \right) \] i.e. \( H_B \gamma_B = B \gamma_B \),
by an elementary calculation \[7\]
\[ V^B_U(x_3) := \int_{\mathbb{R}^2} \frac{|\gamma_B(x_\perp)|^2}{\sqrt{|x_\perp|^2 + x_3^2}} \, dx_\perp = \int_0^\infty \frac{e^{-u}}{\sqrt{x_3^2 + \frac{u}{B}}} \, du \quad (4.1) \]
and
\[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\gamma_B(x_\perp)|^2 |\gamma_B(y_\perp)|^2}{\sqrt{|x_\perp - y_\perp|^2 + (x_3 - y_3)^2}} \, dx_\perp \, dy_\perp = \frac{1}{\sqrt{2}} V^B_U \left( \frac{x_3 - y_3}{\sqrt{2}} \right). \quad (4.2) \]
For \( L > 0 \) and with \( \phi_0 \) the minimizer for the energy \( \epsilon_\circ, \mu(B) := \ln B - 2 \ln \ln B \) and \( f_B(x_3) := \sqrt{\mu(B)} \phi_0(\mu(B)x_3) \) by the variational principle, Lemma \[B.1\] and Corollary \[B.2\]
\[ E_\circ(B) \leq \mathcal{P} (\gamma_B(x_\perp) f_B(x_3)) \]
\[ = B + \int_{\mathbb{R}} |f'_B|^2 \, dx_3 - \frac{\alpha}{\sqrt{8}} \int_{\mathbb{R} \times \mathbb{R}} |f_B(x_3)|^2 V^B_U \left( \frac{x_3 - y_3}{\sqrt{2}} \right) f_B(y_3) \, dx_3 \, dy_3 \]
\[ - \beta \int_{\mathbb{R}} V^B_U(x_3) |f_B(x_3)|^2 \, dx_3 \]
\[ \leq B + \int_{\mathbb{R}} |f'_B|^2 \, dx_3 - (\alpha/2) \mu(B) \int_{\mathbb{R}} |f_B(x_3)|^4 \, dx_3 - \beta \mu(B) |f_B(0)|^2 \]
\[ + (\alpha/2 + \beta) \left( L^{-1} + 8 \sqrt{L} \| f'_B \|^3/2 \right) \| f_B \|_2^2 \]
where \( \mathcal{G} (B, L) := 2 \ln L + 2 \ln \ln B + 2 \int_0^\infty e^{-u} \ln \left( \frac{1}{u} \right. \frac{2}{B^{1/4}} + \frac{1}{u} \) \) \( du \) \( - \ln 2 \). Now choosing \( L = 1/\ln B \) it can be verified \( |\mathcal{G} (B, L/\sqrt{2})| < C \) for \( B > 1 \). Since \( \| f'_B \|_2 = \mu(B) \| \phi_0 \|_2 \) and
\[ \int_{\mathbb{R}} |f'_B|^2 \, dx_3 - (\alpha/2) \mu(B) \int |f_B|^4 \, dx_3 - \beta \mu(B) |f_B(0)|^2 = (\mu(B))^2 \epsilon_\circ, \quad (4.4) \]
the lemma follows. \( \square \)

Corollary 4.4. Let \( W \) be a sum of a bounded Borel measure on the real line and a \( L^\infty(\mathbb{R}) \) function. For \( \epsilon \) a real parameter, \( E_\epsilon(B) \) the ground-state energy of the Hamiltonian \( \mathbb{H}_\epsilon(B) \) in \[2.5\] and \( \epsilon_\circ \) the one-dimensional energy in \[2.10\] there is a constant \( C > 0 \) such that for \( B > 1 \)
\[ E_\epsilon(B) \leq B + \epsilon_\circ (\ln B)^2 + C \ln B |\ln \ln B| + C \ln B. \]
Proof. By the estimate in \[3.10\] \( \epsilon_\circ > -\infty \). Then with the functional \( \mathcal{E}_\epsilon \) as given in \[3.9\] for \( B > 1 \) there is a \( \phi_B \in H^1(\mathbb{R}), \| \phi_B \|_2 = 1 \) satisfying
\[ \mathcal{E}_\epsilon (\phi_B) < \epsilon_\circ + 1/\ln B \text{ and } \| \phi_B \|_2 < C. \quad (4.5) \]
For \( L > 0 \) and with \( g_B(x_3) := \sqrt{\ln B} \phi_B((\ln B) x_3) \) by trivial modifications to Lemma \[3.1\] and Corollary \[3.2\] the arguments in the proofs of Lemma 4.2 and
Lemma 5.1 apply mutatis mutandis, and $E_0 (B) \leq E^{'c} (B)$ with

$$E^{'c} (B) := \inf_{\|\Psi\|_2 = 1} \left\{ P (\Psi) - \epsilon (\ln B)^2 \int_{\mathbb{R}^3} W ((\ln B) x_3) |\psi (x_1, x_3)|^2 dx_1 dx_3 \right\}$$

$$\leq B + \int_{\mathbb{R}} |g_B'|^2 dx_3 - (\alpha / 2) \ln B \int_{\mathbb{R}} |g_B|^4 dx_3 - \beta \ln B |g_B(0)|^2$$

$$- \epsilon (\ln B)^2 \int_{\mathbb{R}} W ((\ln B) x_3) |g_B (x_3)|^2 dx_3$$

$$+ (\alpha / 2 + \beta) \left( L^{-1} + 8 \sqrt{L} \|g_B'|_2^3 / 2 + \| \hat{G} (B, L / \sqrt{2}) \|_2 \right) \|g_B'\|_2$$

where $\hat{G} (B, L) := 2 \ln L + 2 \int_0^\infty e^{-u} \ln \left( \sqrt{1 + \frac{2}{nL^2}} + \sqrt{\frac{1}{n}} \right) du - \ln 2$. Now choosing $L = 1 / \ln B$ it can be verified $|\hat{G} (B, L / \sqrt{2}) | < 2 \ln \ln B + C$ for $B > 1$. Since $\|g_B'|_2 = (\ln B) \|\phi'_B\|_2$, the corollary follows from (5.4) and scaling as in (5.4). \qed

5. LOWER BOUND TO THE GROUND-STATE ENERGY

Theorem 5.1. There is a constant $C > 0$ such that for $B \geq C$

$$E_0 (B) \geq B + \epsilon_0 (\ln B)^2 - C (\ln B)^{3/2}.$$  

The proof of Theorem 5.1 shall be provided at the end of Subsection 5.2

5.1. Delta-function potential. For $\Psi \in H_1^1 (\mathbb{R}^3) \otimes \mathcal{F}$ its electron density in the magnetic-field direction shall be denoted $\tilde{\Psi}^2 (x_3)$ i.e.

$$\tilde{\Psi} (x_3) := \left( \int_{\mathbb{R}^2} \|\tilde{\Psi}^2_x (x_1, x_3) dx_1 \right)^{1/2}.$$  

By Hölder’s inequality $\|\partial_3 \tilde{\Psi}\|_2 \leq \|\partial_1 \tilde{\Psi}\|$ and $\tilde{\Psi} \in H^1 (\mathbb{R})$. Furthermore the integral operator $P_B^0$ acting on $L^2 (\mathbb{R}^2)$ with kernel

$$P_B^0 (x_1, y_1) := \frac{B}{2\pi} e^{-\frac{B}{2\pi}|x_1 - y_1|^2} e^{iB(x_1 y_2 - x_2 y_1)} \quad (5.1)$$

is the projection onto the lowest Landau level i.e. the ground state of the Landau Hamiltonian $H_B$, and $P_B^0 := 1 - P_B^0$. Below the operators $P_B^0 \otimes 1$ and $P_B^0 \otimes 1 \otimes 1$ acting respectively on $L^2 (\mathbb{R}^2) \otimes L^2 (\mathbb{R})$ and $L^2 (\mathbb{R}^2) \otimes L^2 (\mathbb{R}) \otimes \mathcal{F}$ shall also be denoted $P_B^0$.

Lemma 5.2. Let $L > 0$. For $B > 1$ and $\Psi \in H_1^1 (\mathbb{R}^3) \otimes \mathcal{F}$

$$\int_{\mathbb{R}^3} \left\| P_B^0 \Psi \right\|_F^2 (x_1, x_3) \frac{dx_1 dx_3}{|x_1|^2 + x_3^2} (\ln B - 2 \ln \ln B) \left( \tilde{\Psi} (0) \right)^2$$

$$\leq L^{-1} \|\tilde{\Psi}\|_2^2 + 8 \sqrt{L} \|\partial_3 \tilde{\Psi}\|_2^3 / 2 \|\tilde{\Psi}\|_2^1 / 2 + |\mathcal{D} (B, L)| \|\partial_3 \tilde{\Psi}\|_2 \|\tilde{\Psi}\|_2;$$

$$\mathcal{D} (B, L) := 2 \ln L + 2 \ln B + \frac{2}{\sqrt{L^2 + 1}} + 2 \ln \left( \sqrt{1 + \frac{2}{BL^2}} + 1 \right) - \ln 2. \quad (5.2)$$

Proof. By Hölder’s inequality

$$\| P_B^0 \Psi \|_F^2 (x_1, x_3) \leq \left( \frac{B}{2\pi} \right) \left( \tilde{\Psi} (x_3) \right)^2, \quad (5.3)$$

$$\mathcal{D} (B, L) := 2 \ln L + 2 \ln B + \frac{2}{\sqrt{L^2 + 1}} + 2 \ln \left( \sqrt{1 + \frac{2}{BL^2}} + 1 \right) - \ln 2. \quad (5.2)$$

Proof. By Hölder’s inequality

$$\| P_B^0 \Psi \|_F^2 (x_1, x_3) \leq \left( \frac{B}{2\pi} \right) \left( \tilde{\Psi} (x_3) \right)^2, \quad (5.3)$$
The function $G_2$ in (5.2), under the assumption that Corollary 5.3.

and since $P^B_0$ is a projection

$$\int_{\mathbb{R}^3} \|P^B_0 \Psi\|^2 x (x_1, x_3) \, dx_1 \, dx_3 \leq \int_{\mathbb{R}} \left( \tilde{\Psi}(x_3) \right)^2 \, dx_3. \tag{5.4}$$

It follows from the bathtub principle [21], [5] that the maximum of the expression

$$\int_{\mathbb{R}^2} \frac{G(x_1, x_3)}{\sqrt{|x_1|^2 + x_3^2}} \, dx_1$$

over all functions $G$ satisfying the conditions (5.3) and (5.4) above is attained by the function

$$G_{\text{max}}(x_1, x_3) = \begin{cases} \left( \frac{R}{2\pi} \right) \left( \tilde{\Psi}(x_3) \right)^2 & \text{when } |x_1| \leq R \\ 0 & \text{when } |x_1| > R \end{cases}$$

where $R = \sqrt{2/B}$. Therefore

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{\|P^B_0 \Psi\|^2 x (x_1, x_3)}{\sqrt{|x_1|^2 + x_3^2}} \, dx_1 \, dx_3 \leq \int_{\mathbb{R}} \int_{|x_1| \leq \sqrt{2/B}} \frac{G_{\text{max}}(x_1, x_3)}{\sqrt{|x_1|^2 + x_3^2}} \, dx_1 \, dx_3$$

$$= \int_{\mathbb{R}} V^B_{\text{CL}}(x_3) \left( \tilde{\Psi}(x_3) \right)^2 \, dx_3$$

with

$$V^B_{\text{CL}}(x_3) := \frac{2}{\sqrt{\frac{1}{x_1^2} + x_3^2 + |x_3|^2}}. \tag{5.5}$$

The lemma follows from Corollary B.3 in the appendix.

**Corollary 5.3.** Let $\tau > 0$. There is a constant $C > 0$ such that for $B \geq C$ and $\Psi \in H^1_A(\mathbb{R}^3) \otimes \mathcal{F}$

$$\|\partial_3 \Psi\|^2 - \tau \left( \Psi, P^B_0 \left| x \right|^{-1} P^B_0 \Psi \right) \geq \left( \frac{\tau^2}{4} (\ln B)^2 + \tau^2 \ln \ln B - C (\ln B) \right) \|\Psi\|^2$$

**Proof.** Recall for $\epsilon > 0$, $\|\partial_3 \tilde{\Psi}\|_2 \|\tilde{\Psi}\|_2 \leq \epsilon \|\partial_3 \tilde{\Psi}\|^2 + \epsilon^{-1} \|\tilde{\Psi}\|^2$. With $D(B, L)$ as given in (5.2), under the assumption that $2\tau \epsilon \|D(B, L)\| < 1/2$ and denoting $\mu(B) = (\ln B - 2 \ln \ln B)$ it follows from Lemma 5.2 that

$$\|\partial_3 \Psi\|^2 - \tau \left( \Psi, P^B_0 \left| x \right|^{-1} P^B_0 \Psi \right)$$

$$\geq (1 - 2\tau \epsilon \|D(B, L)\|) \|\partial_3 \tilde{\Psi}\|^2 - \tau \mu(B) \left( \Psi(0) \right)^2 - \tau (L^{-1} + \epsilon^{-1} |D(B, L)|) \|\tilde{\Psi}\|^2$$

$$+ \tau \epsilon \|D(B, L)\| \|\partial_1 \tilde{\Psi}\|_2^2 - 8\tau \sqrt{L} \|\tilde{\Psi}\|_2^{1/2} \|\partial_1 \tilde{\Psi}\|_2^{3/2}$$

$$\geq \frac{-\tau^2 (\mu(B))^2}{4(1 - 2\tau \epsilon |D(B, L)|)} \|\tilde{\Psi}\|_2^2 - \tau \left( \frac{432 L^2}{|D(B, L)|^3} \epsilon^3 + L^{-1} + \epsilon^{-1} |D(B, L)| \right) \|\tilde{\Psi}\|^2$$

$$\geq \left( \frac{-\tau^2}{4} - \tau^3 \epsilon |D(B, L)| \right) (\mu(B))^2 - \frac{432 \tau L^2}{|D(B, L)|^3} \epsilon^3 - \frac{\tau}{L} - \tau \epsilon \|D(B, L)\| \|\tilde{\Psi}\|^2.$$
There exists a constant $\epsilon = 1 / \ln B$ and $L = 1 / \ln B$ the above assumption that $2 \epsilon \| \mathcal{D}(B, L) \| < 1 / 2$ can be verified to be true for $B \geq C$. The lemma follows.

5.2. Concentration in the Lowest Landau Level. First an ultraviolet cutoff is needed. With $\mathcal{K} > 0$ and $\Gamma_\mathcal{K} := \{ k \in \mathbb{R}^3 : \max(|k_\perp|, |k_3|) \leq \mathcal{K}\}$ the cutoff Hamiltonian shall be denoted

$$b_\mathcal{K}^\alpha := \left(1 - \frac{8\alpha}{\pi \mathcal{K}}\right) (H_B - \partial_3^2) + \frac{1}{2} \int_{\Gamma_\mathcal{K}} a_k^\dagger a_k dk + \frac{1}{2} \mathcal{N}^{\mathcal{K}} \int_{\Gamma_\mathcal{K}} \left(\frac{a_k e^{ik \cdot x}}{|k|} + \frac{a_k^\dagger e^{-ik \cdot x}}{|k|}\right) dk.$$

Lemma 5.4. If $\mathcal{K} > 8\alpha / \pi$, then $\mathbb{H}(B) \geq b_\mathcal{K}^\alpha - \beta |x|^{-1} - 1/4$.

Proof. The lemma follows from Lemma 5.1 in [9].

Lemma 5.5. There exists a constant $C > 0$ such that for $B \geq C$

$$\mathbb{H}(B) \geq \left(1 - \frac{8\alpha (\ln B)^2}{\pi B}\right) H_B + \left(\frac{1}{2} - \frac{8\alpha (\ln B)^2}{\pi B}\right) (-\partial_3^2) - \beta |x|^{-1} + \frac{1}{2} \mathcal{N} - C (\ln B)^2.$$

Proof. Let $\mathcal{K} = B / (\ln B)^2$ and $\mathcal{K}_3 = 16\alpha |\ln B| / \pi$. By completion of square

$$\int_{|k_3| \leq \mathcal{K}_3} \int_{|k_\perp| \leq \mathcal{K}} \left(\frac{1}{2} a_k^\dagger a_k + \frac{\sqrt{\alpha}}{2\pi} \frac{a_k}{|k|} e^{ik \cdot x} + \frac{\sqrt{\alpha}}{2\pi} \frac{a_k^\dagger}{|k|} e^{-ik \cdot x}\right) dk_\perp dk_3$$

$$\geq -\frac{\alpha}{2\pi^2} \int_{|k_3| \leq \mathcal{K}_3} \int_{|k_\perp| \leq \mathcal{K}} \frac{1}{|k|^2} dk_\perp dk_3$$

$$= -\frac{\alpha}{\pi} \int_0^{\mathcal{K}_3} \ln \left(\frac{K^2 + K_3^2}{K_3^2}\right) dk_3$$

$$= -\frac{\alpha}{\pi} \left(\mathcal{K}_3 \ln \left(\frac{K^2 + K_3^2}{K_3^2}\right) + 1\right) + 2K \arctan \left(\frac{K_3}{K}\right)$$

$$\geq -\frac{\alpha}{\pi} K_3 \left(\ln \left(\frac{K^2}{K_3^2}\right) + 1\right) + 2 + \frac{2K^2}{3K_3^2}$$

$$\geq -C (\ln B)^2,$$  \hspace{1cm} (5.6)

valid for some constant $C$ and $B \geq C$. The argument corrects a mistake in the proof of Lemma 5.2 in [9].

Denoting $\Lambda(B) = \{(k_\perp, k_3) \in \Gamma_\mathcal{K} : k_3 \leq |k_3| \leq \mathcal{K}$ and $|k_\perp| \leq \mathcal{K}\}$ it can be argued as in the proof of Lemma 5.2 in [9] that for $B$ large

$$\sqrt{\alpha} \int_{\Lambda(B)} \left(\frac{a_k}{|k|} e^{ik \cdot x} + \frac{a_k^\dagger}{|k|} e^{-ik \cdot x}\right) dk \geq \frac{1}{2} \partial_3^2 - \frac{1}{2} \left(\int_{\Lambda(B)} a_k^\dagger a_k dk + \frac{1}{2}\right).$$  \hspace{1cm} (5.7)

The lemma follows from Lemma 5.4 and the estimates (5.6) and (5.7).

Lemma 5.6. There exists a constant $C > 0$ such that for $B \geq C$ and all $\Psi \in H_0^1 (\mathbb{R}^3) \otimes \text{dom} \left(\sqrt{\mathcal{N}}\right)$ with $\|\Psi\| = 1$

$$\langle \Psi, \mathbb{H}(B) \Psi \rangle \geq B + \frac{B}{2} \| P^B \Psi \|^2 + \frac{1}{2} \langle \Psi, \mathcal{N} \Psi \rangle - C (\ln B)^2.$$
Proof. Recall by the diamagnetic inequality for $\tau > 0$
\[
\langle \Psi, (H_B - \partial_x^2) \Psi \rangle \geq \tau \langle \Psi, \beta |x|^{-1} \Psi \rangle - 4^{-1} \beta^2 \tau^2 \|\Psi\|^2.
\]
Then with $0 < \eta < 1$, $A > 1$ and $\theta := \left(\frac{1}{2} - \frac{8\alpha}{\pi B}\right)$ for $B$ large
\[
\theta \langle \Psi, (H_B - \partial_x^2) \Psi \rangle - \langle \Psi, \beta |x|^{-1} \Psi \rangle = 0 \langle P^B_0 \Psi, (H_B - \partial_x^2) P^B_0 \Psi \rangle - \langle P^B_0 \Psi, \beta |x|^{-1} P^B_0 \Psi \rangle
\]
\[+ \theta (1 - \eta) \langle P^B_0 \Psi, (H_B - \partial_x^2) P^B_0 \Psi \rangle + \theta \eta \langle P^B_0 \Psi, (H_B - \partial_x^2) P^B_0 \Psi \rangle
\]
\[+ \langle \partial_t \Psi, (H_B - \partial_x^2) \eta \rangle \langle x \rangle - \langle \partial_t \Psi, \beta |x|^{-1} \eta \rangle \langle x \rangle - \langle \partial_t \Psi, \beta |x|^{-1} \eta \rangle \langle x \rangle
\]
\[\geq \theta \langle P^B_0 \Psi, (H_B - \partial_x^2) P^B_0 \Psi \rangle - \langle P^B_0 \Psi, \beta |x|^{-1} P^B_0 \Psi \rangle
\]
\[+ \theta (1 - \eta) \langle P^B_0 \Psi, (H_B - \partial_x^2) P^B_0 \Psi \rangle + \theta \eta \langle P^B_0 \Psi, (H_B - \partial_x^2) P^B_0 \Psi \rangle
\]
\[+ \langle \partial_t \Psi, (H_B - \partial_x^2) \eta \rangle \langle x \rangle - \langle \partial_t \Psi, \beta |x|^{-1} \eta \rangle \langle x \rangle - \langle \partial_t \Psi, \beta |x|^{-1} \eta \rangle \langle x \rangle
\]
Choosing $\eta = 1/4$ and $A = \sqrt{B}\theta/2\beta$ for $B$ large $(A/(A - 1)) < 2$, and by Corollary 5.3 there exists some $C > 0$ such that for $B \geq C$
\[\langle \Psi, (H_B - \partial_x^2) P^B_0 \Psi \rangle - \langle \Psi, (A/(A - 1)) (P^B_0 \Psi, \beta |x|^{-1} P^B_0 \Psi) \rangle \geq -C (\ln B)^2.
\]
The lemma now follows from Lemma 5.5. \hfill \Box

The following observation is immediate from Theorem 4.1. For every $M > \varepsilon_0$ and $B$ large there exist wave functions $\Psi \in H^1_A(\mathbb{R}^3) \otimes \text{dom} \left(\sqrt{\mathbb{N}}\right)$ satisfying
\[
\langle \Psi, \mathbb{N}(B) \Psi \rangle \leq B + M (\ln B)^2 \quad \text{and} \quad \|\Psi\| = 1.
\]

Corollary 5.7. For every $M \in \mathbb{R}$ there exists a constant $C_M > 0$ such that for $B \geq C_M$ and all $\Psi \in H^1_A(\mathbb{R}^3) \otimes \text{dom} \left(\sqrt{\mathbb{N}}\right)$ satisfying (5.3)
\[
\|P^B_0 \Psi\|^2 \leq C_M (\ln B)^2 B^{-1} \quad \text{and} \quad \langle \Psi, \mathbb{N} \Psi \rangle \leq C_M (\ln B)^2.
\]

Proof. The corollary follows from Lemma 5.6. \hfill \Box

Lemma 5.8. Let $K > 8\alpha/\pi$ and $1 < A < \sqrt{B/\ln B}$. Denoting $\kappa = 1 - (8\alpha/\pi K)$ for every $M \in \mathbb{R}$ there exists a constant $C_M > 0$ such that for $B \geq C_M$ and all $\Psi \in H^1_A(\mathbb{R}^3) \otimes \text{dom} \left(\sqrt{\mathbb{N}}\right)$ satisfying (5.3)
\[
\langle \Psi, \mathbb{N}(B) \Psi \rangle \geq \langle P^B_0 \Psi, \left(\frac{\alpha K}{K} - \beta \left(1 - (A - 1)^{-1}\right) \right) |x|^{-1} \rangle \|P^B_0 \Psi\|^2
\]
\[\quad - C_M (\ln B)^2 (KB^{-1} + \sqrt{KB^{-1}}) - C_M \kappa^{-1} \ln B - 1/4.
\]
Proof. It can be argued as in the proof of Lemma 5.6 with $0 < \eta < 1$

\[
\kappa \langle \Psi, (H_B - \partial_3^2) \Psi \rangle - \langle \Psi, \beta|x|^{-1} \Psi \rangle \\
\geq \kappa \langle P_0^B \Psi, (H_B - \partial_3^2) P_0^B \Psi \rangle - (A/(A-1)) \langle P_0^B \Psi, \beta|x|^{-1} P_0^B \Psi \rangle \\
+ \kappa B \|P_0^B \Psi\|^2 + \left[ \kappa (2B - 3B\eta) - \mathcal{A}^2\beta^2(4\eta\kappa)^{-1} \right] \|P_0^B \Psi\|^2.
\]

It now follows from Lemma 5.4 that

\[
\langle \Psi, \mathbb{H}(B) \Psi \rangle \geq \left[ (6\kappa^2 - \beta(1/(1 - \mathcal{A}^{-1})) |x|^{-1}) P_0^B \Psi \right] + \kappa B \|P_0^B \Psi\|^2 \\
+ \left[ \kappa (2B - 3B\eta) - \mathcal{A}^2\beta^2(4\eta\kappa)^{-1} \right] \|P_0^B \Psi\|^2 - \frac{1}{4} \\
+ \left\langle P_0^B \Psi, \left( \int_{\Gamma_K} a_ka_k + \sqrt{\alpha} a_ka_k e^{ik \cdot x} + \sqrt{\alpha} a_ka_k e^{-ik \cdot x} d\kappa \right) P_0^B \Psi \right\rangle \\
+ \left\langle P_0^B \Psi, \left( \frac{\sqrt{\alpha}}{2\pi} \int_{\Gamma_K} a_ka_k e^{ik \cdot x} + \sqrt{\alpha} a_ka_k e^{-ik \cdot x} d\kappa \right) P_0^B \Psi \right\rangle \\
+ \left\langle P_0^B \Psi, \left( \frac{\sqrt{\alpha}}{2\pi} \int_{\Gamma_K} a_ka_k e^{ik \cdot x} + \sqrt{\alpha} a_ka_k e^{-ik \cdot x} d\kappa \right) P_0^B \Psi \right\rangle.
\]

By completion of square

\[
\int_{\Gamma_K} \left( a_ka_k + \sqrt{\alpha} a_ka_k e^{ik \cdot x} + \sqrt{\alpha} a_ka_k e^{-ik \cdot x} \right) d\kappa \\
\geq -\frac{\alpha}{4\pi} \int_{\Gamma_K} \frac{1}{|k|^2} d\kappa d\kappa \\
= -\frac{\alpha (2 \ln(2) + \pi)}{4\pi} \kappa,
\]

and by Corollary 5.7 for $B \geq C_M$

\[
\left\langle P_0^B \Psi, \left( \int_{\Gamma_K} a_ka_k + \sqrt{\alpha} a_ka_k e^{ik \cdot x} + \sqrt{\alpha} a_ka_k e^{-ik \cdot x} d\kappa \right) P_0^B \Psi \right\rangle \\
\geq -\frac{\alpha (2 \ln(2) + \pi)}{4\pi} \kappa \|P_0^B \Psi\|^2 \\
\geq -C_M \kappa B^{-1} (\ln B)^2.
\]

Furthermore it can be argued as in the proof of Lemma 5.4 in [9] and using Corollary 5.7 that for $B \geq C_M$

\[
\left| \left\langle P_0^B \Psi, \left( \frac{\sqrt{\alpha}}{2\pi} \int_{\Gamma_K} a_ka_k e^{ik \cdot x} d\kappa \right) P_0^B \Psi \right\rangle \right| \\
\leq C\sqrt{\kappa} \|P_0^B \Psi\| \|\sqrt{N} + 1P_0^B \Psi\| \\
\leq C_M \sqrt{\kappa} B^{-1} (\ln B)^2.
\]

The remaining interaction terms are estimated similarly. Choosing $\eta = 2/3$ the lemma follows from Corollary 5.7. □
Proposition 5.9. There exists a constant $C > 0$ such that for $B, \kappa$ and $K$ satisfying $B \geq C$, $C (\ln B)^{-1/2} \leq \kappa \leq C^{-1} \ln B$, $K \geq \sqrt{B}$ and some $1 < A < \sqrt{B/\ln B}$

$$P_0^B \left( \frac{\hbar \omega^2}{\kappa} - \beta \left( 1/ (1 - A^{-1}) \right) |x|^{-1} \right) P_0^B$$

$$\geq \left( \kappa B + \kappa^{-1} \left( \ln B \right)^2 \epsilon_0 - C \kappa^{-1/2} \left( \ln B \right)^{3/2} - C \left( 1 + \kappa^{-2} \right) \ln B \right) P_0^B .$$

The proof of Proposition 5.9 shall be provided in Subsection 5.3.

Proof of Theorem 5.1. Fix $M > 0$. For $B$ large by Theorem 4.1 there exist wave functions satisfying (5.8). It suffices to argue the desired lower bound on $\langle \Psi, \mathbb{H}(B) \Psi \rangle$ with those wave functions. By Lemma 5.8 and Proposition 5.9

$$\langle \Psi, \mathbb{H}(B) \Psi \rangle \geq \kappa B + \kappa^{-1} \left( \ln B \right)^2 \epsilon_0 - C \kappa^{-1/2} \left( \ln B \right)^{3/2} - C \left( 1 + \kappa^{-2} \right) \ln B$$

with $\kappa = 1 - (8 \alpha / \pi K)$. Choosing $K = B \left( \ln B \right)^{-4/3}$ and since $\| P_0^B \Psi \| \leq 1$

$$\langle \Psi, \mathbb{H}(B) \Psi \rangle \geq B + \epsilon_0 \left( \ln B \right)^2 - C \left( \ln B \right)^{3/2}$$

which is the claimed lower bound. \hfill \Box

5.3. Proof of Proposition 5.9

Reduction to one dimension. In [9] the authors consider a one-dimensional Hamiltonian with $0 < K_3 \leq K$ and $1 \leq K_\perp \leq K$

$$\hat{h}^{1d} :=$$

$$\kappa_1 (-\partial_x^2) + \int_{|k_3| \leq K_3} \hat{a}_{k_3} \hat{a}_{k_3} dk_3 + \sqrt{\alpha} \int_{|k_3| \leq K_3} \nu (k_3) \left( \hat{a}_{k_3} e^{ik_3 x_3} + \hat{a}_{k_3}^\dagger e^{-ik_3 x_3} \right) dk_3$$

acting on $L^2(\mathbb{R}) \otimes \mathcal{F}$ with $\kappa_1 := \kappa - (8 \alpha / \pi K_3) \int_{0}^{\infty} (1 + t)^{-1} \exp \left( -tK_3^2 / 2B \right) dt$ and $\kappa$ as in the statement of Proposition 5.9, and

$$\hat{a}_{k_3} := \frac{1}{\nu (k_3)} \int_{1 \leq |k_\perp| \leq K_\perp} \frac{a_k}{|k|} e^{ik_\perp \cdot x_\perp} dk_\perp$$

with

$$\nu (k_3) := \left( \int_{1 \leq |k_\perp| \leq K_\perp} |k|^{-2} dk_\perp \right)^{1/2} = \sqrt{\pi} \left( \ln \left( K_\perp^2 + k_3^2 \right) - \ln \left( 1 + k_3^2 \right) \right)$$

and satisfying $[\hat{a}_{k_3}, \hat{a}_{k_3}^\dagger] = \delta (k_3 - k_3')$ and $[\hat{a}_{k_3}, \hat{a}_{k_3'}] = [\hat{a}_{k_3}^\dagger, \hat{a}_{k_3'}^\dagger] = 0$ for $k_3, k_3' \in \mathbb{R}$.

Lemma 5.10. Denoting $\kappa_2 = \kappa - 2 \alpha \pi^{-1} K_3 K_\perp^{-2}$

$$P_0^B \left( \frac{\hbar \omega^2}{\kappa} - \beta \left( 1/ (1 - A^{-1}) \right) |x|^{-1} \right) P_0^B \geq \kappa_2 B P_0^B + P_0^B \left( \frac{\hbar^{1d}}{\kappa} - \beta \left( 1/ (1 - A^{-1}) \right) \right) P_0^B P_0^B - \left( 1 + \frac{\alpha}{2} \right) P_0^B .$$

Proof. The lemma follows from Lemma 6.1, Lemma 6.2, Lemma 6.3 and Lemma 6.4 in [9]. The proof of Lemma 6.3 in [9] uses an incorrect vector operator for cutting off high modes in the $k_\perp$-direction cf. [25], the mistake can be fixed, and the lemma stands true. \hfill \Box
Localization and decomposition. In [9] the authors decompose the mode space into $\mathcal{M}$ intervals, indexed with $b$, each of length $2K_3/M$ and consider for $u \in \mathbb{R}$ and $0 < \gamma < 1$ the block Hamiltonian

$$h_\gamma^{(u)} := \kappa_1 (1 - \mu_3^2) + \sum_b \left[ (1 - \gamma) A_b^{(u)*} A_b^{(u)} + \frac{\sqrt{\alpha}}{2\pi} V(b) \left( A_b^{(u)*} e^{ik_3x_3} + A_b^{(u)} e^{-ik_3x_3} \right) \right]$$

with $k_b$ a mode in block $b$ and the block creation and annihilation operators $A_b^{(u)*}$ and $A_b^{(u)}$ acting on $F(L^2(\mathbb{R}^3))$, where

$$A_b^{(u)} := \frac{1}{V(b)} \int_b \nu(k_3) e^{i(k_3-k_3)u} \hat{a}_{k_3} dk_3$$

with $V(b) := \left( \int_b \nu(k_3)^2 dk_3 \right)^{1/2}$. Furthermore

$$[A_b^{(u)}, A_{b'}^{(u)*}] = \delta_{bb'}, [A_b^{(u)}, A_{b'}^{(u)}] = [A_b^{(u)*}, A_{b'}^{(u)*}] = 0$$

for all blocks $b, b'$.

**Lemma 5.11.** For $\chi \in C^\infty_c(\mathbb{R}), \|\chi\|_2 = 1$ a nonnegative function supported on the interval $[-1/2, 1/2]$ and for $J > 0$, denoting $\chi^\prime_0(x_3) = J^{-1/2} \chi(J^{-1} (x_3 - u))$,

$$h^{1d} - \beta (1/ (1 - A^{-1})) P_0 B \|x\|^{-1} P_0 B$$

$$\geq \int_{\mathbb{R}} \chi^J_u \left[ h_\gamma^{(u)} - \beta (1/ (1 - A^{-1})) P_0 B \|x\|^{-1} P_0 B \right] \chi^J_u du - \frac{\alpha K_3^2 J^2}{4\pi^2 \gamma^2 M^2} R - \|\chi^\prime\|^2_{J^{-2}}$$

with

$$R := \int_{|k_3| \leq K_3} \nu(k_3)^2 dk_3 = \pi \int_{|k_3| \leq K_3} \left( \ln (K_\perp + k_3^2) - \ln (1 + k_3^2) \right) dk_3.$$

**Proof.** The lemma follows from Lemma 6.5 in [9].

**Error estimates.** Similarly as in [9], [14] and [24] representing the block creation and annihilation operators by coherent state integrals and completing the square it follows for a suitably chosen $k_b$ that

$$h_\gamma^{(u)} - \beta (1/ (1 - A^{-1})) P_0 B \|x\|^{-1} P_0 B \geq I - \mathcal{M}, \quad (5.10)$$

where

$$I := \inf_{\|\phi\|_2=1} \left[ \kappa_1 \|\partial_3 \phi\|_2^2 - \frac{\alpha}{4\pi^2 (1 - \gamma)} \int_{\mathbb{R}} \nu(k_3)^2 \left| \int_{\mathbb{R}^3} e^{ik_3x_3} |\phi(x, x_3)|^2 dx \right|^2 dk_3 \right.$$  

$$- \frac{\beta}{1 - A^{-1}} \int_{\mathbb{R}^3} \|x\|^{-1} \left| (P_0 B \phi)(x, x_3) \right|^2 dx \right].$$

Combining (5.10) with Lemma 5.11

$$h^{1d} - \beta (1/ (1 - A^{-1})) P_0 B \|x\|^{-1} P_0 B \geq I - \mathcal{M} - \frac{\alpha K_3^2 J^2}{4\pi^2 \gamma^2 M^2} R - \|\chi^\prime\|^2_{J^{-2}}.$$  

Now it follows from Lemma 5.10 that for some constant $C > 0$

$$P_0 B \left( h^{co}_K - \beta (1/ (1 - A^{-1})) \right) P_0 B$$

$$\geq \kappa B P_0 B + \kappa P_0 B - C \left( \frac{K_3 B}{K_\perp^2} + \mathcal{M} + \frac{K_3^2 J^2}{\gamma^2 M^2} R + \frac{1}{J^2} \right) P_0 B. \quad (5.11)$$
Lemma 5.12. For any $L > 0$ and $\epsilon > 0$ and with $\mathcal{D}(B, L)$ as given in (5.3) assuming

$$
\frac{2 \ln K_\perp}{1 - \gamma} = \frac{\mu(B)}{1 - A^{-1}} \quad \text{and} \quad \tilde{\kappa}_1 := \kappa_1 - \frac{4 \beta \epsilon |\mathcal{D}(B, L)||\ln K_\perp|}{\mu(B)(1 - \gamma)} > 0
$$

(5.12)

with $\mu(B) := \ln B - 2 \ln \ln B$,

$$
I \geq \frac{4 (\ln K_\perp)^2 \epsilon_0}{\tilde{\kappa}_1 (1 - \gamma)^2} - \frac{2 \beta \ln K_\perp}{\mu(B)(1 - \gamma)} \left( \frac{432 L^2}{|\mathcal{D}(B, L)|^3} e^3 + \frac{1}{L} + \frac{|\mathcal{D}(B, L)|}{\epsilon} \right).
$$

Proof. For $\phi \in L^2(\mathbb{R}^3)$, $\|\phi\|_2 = 1$ and $\tilde{\phi}(x_3) := \left( \int_{\mathbb{R}^2} |\phi(x_\perp, x_3)|^2 \, dx_\perp \right)^{1/2}$,

$$
\kappa_1 \|\partial_3 \phi\|^2 \geq \frac{2 \beta}{1 - A^{-1}} \int_{\mathbb{R}^3} \frac{\nu(k_3) \left| \int_{\mathbb{R}^3} e^{i k_3 x_3} \phi(x_\perp, x_3) \, dx \right|^2}{|x|} \, dk_3
$$

$$
\geq \kappa_1 \|\partial_3 \phi\|^2 - \frac{\alpha \ln K_\perp}{1 - \gamma} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^2} \phi(x_\perp, x_3) \, dx_\perp \right|^2 \, dx_3
$$

$$
\geq \left( \kappa_1 - \frac{2 \beta \epsilon |\mathcal{D}(B, L)|}{1 - A^{-1}} \right) \|\partial_3 \tilde{\phi}\|^2 + \frac{\alpha \ln K_\perp}{1 - \gamma} \|\tilde{\phi}\|^4_4 - \frac{\beta \mu(B)}{1 - A^{-1}} \left( \tilde{\phi}(0) \right)^2
$$

$$
\geq \frac{2 \beta \ln K_\perp}{\mu(B)(1 - \gamma)} \left( \frac{432 L^2}{|\mathcal{D}(B, L)|^3} e^3 + \frac{1}{L} + \frac{|\mathcal{D}(B, L)|}{\epsilon} \right).
$$

At the first inequality it is used that

$$
\nu(k_3)^2 = \pi \left( \ln (K_\perp^2 + k_3^2) - \ln (1 + k_3^2) \right) \leq 2 \pi \ln K_\perp
$$

(5.13)

along with Plancherel’s identity. At the second inequality the bathtub principle in Lemma 5.2 and the argument in the proof of Corollary 5.3 apply mutatis mutandis.

At the third inequality the assumptions in (5.12) are used. \hfill \square

From Lemma 5.12 and further assuming

$$
\kappa - \tilde{\kappa}_1 \leq \frac{\kappa}{2} \quad \text{and} \quad \gamma \leq \frac{1}{2}
$$

(5.14)

it can be seen there is a constant $C > 0$ such that

$$
I \geq 4 \kappa^{-1} \left( \ln K_\perp \right)^2 \epsilon_0
$$

$$
- 4 \left( \frac{\ln K_\perp}{\kappa} \left( \frac{\kappa - \tilde{\kappa}_1}{\kappa} + \gamma \right) + \frac{\ln K_\perp}{\mu(B)} \left( \frac{L^2}{|\mathcal{D}(B, L)|^3} e^3 + \frac{1}{L} + \frac{|\mathcal{D}(B, L)|}{\epsilon} \right) \right).
$$

With the above bound and (5.11) the argument in [9] applies mutatis mutandis and choosing $J^2 = \kappa^{1/5} K_\perp^{-3/5} (\ln K_\perp)^{-3/5}$, $\mathcal{M} = |J^{-2}|$ and $\gamma = \kappa^{4/5} K_\perp^{3/5} (\ln K_\perp)^{-7/5}$...
Corollary 5.13. Let $\epsilon < 0$ and for $2.1$ and (2.9) that for $\epsilon > 0$ as explained in the introduction it follows from the variational principle, Theorem 2.2 and (2.9) that for $\epsilon > 0$

$$\frac{\epsilon_0 - \epsilon}{\epsilon} \geq \limsup_{B \to \infty} \frac{1}{\ln B} \int_{\mathbb{R}} W(x_3) \left( \int_{\mathbb{R}^2} \left\| \Psi(B) \right\|^2_{x_\perp} \frac{x_3}{\ln B} dx_\perp \right) dx_3$$

and for $\epsilon < 0$

$$\frac{\epsilon_0 - \epsilon}{\epsilon} \leq \liminf_{B \to \infty} \frac{1}{\ln B} \int_{\mathbb{R}} W(x_3) \left( \int_{\mathbb{R}} \left\| \Psi(B) \right\|^2_{x_\perp} \frac{x_3}{\ln B} dx_\perp \right) dx_3.$$
APPENDIX A. COMPACTNESS OF MINIMIZING SEQUENCES

Theorem A.1. If a sequence \( \{ \psi_n \}_{n=1}^{\infty} \), \( \| \psi_n \|_2 = 1 \) satisfies \( \lim_{n \to \infty} E_o(\psi_n) = e_o \) with the functional \( E_o \) as given in (3.1), then there exists a subsequence \( \{ \psi_{n_k} \}_{k=1}^{\infty} \) and some \( \psi \in H^1(\mathbb{R}) \) such that \( \| \psi \|_2 = 1 \), \( E_o(\psi) = e_o \) and \( \| \psi_{n_k} - \psi \|_{H^1} \to 0 \) as \( k \to \infty \).

Proof. For \( \varphi \in H^1(\mathbb{R}) \) it can be argued as in the proof of Theorem 3.3 that
\[
E_o(\varphi) \geq \frac{3}{4} \| \varphi' \|_2^2 - \| \varphi \|_2^2 \left( \frac{\alpha}{2} \| \varphi \|_2^2 + \beta \right)^2.
\]
Furthermore since \( \psi_n \) is a minimizing sequence, \( E_o(\psi_n) < e_o + 1 \) for \( n \) large. Then \( \| \psi_n \|_{H^1} < C \) and there exists a subsequence \( \{ \psi_{n_k} \}_{k=1}^{\infty} \) and some \( \psi \in H^1(\mathbb{R}) \) such that \( \psi_{n_k} \) converges to \( \psi \) weakly in \( H^1(\mathbb{R}) \).

Step 1 (Compactness). It shall be argued that the subsequence \( \{ \psi_{n_k} \}_{k=1}^{\infty} \) satisfies
\[
\forall \delta > 0, \exists R < \infty \text{ s.t. } \| \psi_{n_k} \|_{L^2(|x|<R)}^2 > 1 - \delta. \tag{A.1}
\]

Essential to the argument is the binding inequality \( e_o < e_T \) where \( e_T := \inf \frac{1}{\| \varphi \|_2 = 1} E_T(\varphi) \) and \( E_T(\varphi) := \int_\mathbb{R} |\varphi'|^2 dx - (\alpha/2) \int_\mathbb{R} |\varphi|^4 dx \) is the translation-invariant problem admitting a symmetric decreasing minimizer \( \phi_T \). Indeed
\[
e_o \leq E_o(\phi_T) = E_T(\phi_T) - \beta \phi_T^2(0) = e_T - \beta \phi_T^2(0) < e_T.
\]
Moreover it should be noted
\[
E_o(\varphi) \geq e_o \| \varphi \|_2^2 \quad \text{and} \quad E_T(\varphi) \geq e_T \| \varphi \|_2^2 \quad \text{when} \quad \| \varphi \|_2 \leq 1. \tag{A.2}
\]

Also a quadratic partition of unity is chosen, \( \chi^2 + \tilde{\chi}^2 \equiv 1 \), where \( 0 \leq \chi \leq 1 \) is a smooth function with \( \chi(x) = 1 \) when \( |x| < 1/2 \) and \( \chi(x) = 0 \) when \( |x| > 1 \). Denoting \( \chi_R = \chi(R^{-1} \cdot) \) it follows from (A.2) that
\[
E_o(\psi_{n_k}) = E_o(\chi_R \psi_{n_k}) + E_T(\tilde{\chi}_R \psi_{n_k})
- 2 \int_\mathbb{R} \chi_R^2 \tilde{\chi}_R^2 |\psi_{n_k}|^4 dx - \int_\mathbb{R} |\psi_{n_k}|^2 \left( |\chi_R'|^2 + |\tilde{\chi}_R'|^2 \right) dx
greater than \( e_0 - e_T \) \| \chi_R \psi_{n_k} \|_2^2 + e_T \]

\[
- 2 \int_\mathbb{R} \chi_R^2 \tilde{\chi}_R^2 |\psi_{n_k}|^4 dx - \int_\mathbb{R} |\psi_{n_k}|^2 \left( |\chi_R'|^2 + |\tilde{\chi}_R'|^2 \right) dx \tag{A.3}
\]

Since \( \chi, \tilde{\chi} \) have bounded derivatives
\[
\int_\mathbb{R} |\psi_{n_k}|^2 \left( |\chi_R'|^2 + |\tilde{\chi}_R'|^2 \right) dx < CR^{-2}
\]
for some \( C > 0 \). Furthermore with \( D_R := \{ R/2 \leq |x| \leq R \} \)
\[
\int_\mathbb{R} \chi_R^2 \tilde{\chi}_R^2 |\psi_{n_k}|^4 dx \leq \| \psi_{n_k} \|_{L^4(D_R)}^4 \quad \text{and} \quad \| \psi_{n_k} \|_{L^4(D_R)}^2 \to \| \psi \|_{L^4(D_R)}^2
\]
by Rellich-Kondrashov, so the first term in (A.3) can also be made arbitrarily small with \( R \) chosen to be large enough uniformly in \( k \). Hence for any \( \delta > 0 \) there is some \( R \) such that for all \( k \)
\[
E_o(\psi_{n_k}) \geq \left( e_0 - e_T \right) \| \chi_R \psi_{n_k} \|_2^2 + e_T - \delta (e_T - e_o)/2. \tag{A.4}
\]

Since \( \{ \psi_{n_k} \}_{k=1}^{\infty} \) is a minimizing sequence for \( e_o \), for \( k \) large
\[
E_o(\psi_{n_k}) \leq e_o + \delta (e_T - e_o)/2. \tag{A.5}
\]
Compactness now follows from (A.4) and (A.5).

**Step 2** (Weak Limit is a Minimizer). By Rellich-Kondrashov and the compactness property in (A.1)

\[ \|\psi_{n_k} - \psi\|_2 \to 0 \quad \text{and} \quad \|\psi\|_2 = 1. \]  

(A.6)

Since \(\|\psi\|_{H^1} < C\), by Sobolev and Hölder's inequalities

\[ \int_R \left( |\psi_{n_k}|^4 - |\psi|^4 \right) dx \leq C \|\psi_{n_k} - \psi\|_2 \to 0. \]

Furthermore by Theorem 8.6 in [21] \(\psi_{n_k}(0) \to \psi(0)\), so

\[ \frac{\alpha}{2} \int_R |\psi_{n_k}|^4 dx + \beta |\psi_{n_k}(0)|^2 \to \frac{\alpha}{2} \int_R |\psi|^4 dx + \beta |\psi(0)|^2. \]  

(A.7)

Then since \(\lim \inf \|\psi_{n_k}\|_2 \geq \|\psi\|_2\), \(c_0 = \lim \inf_{k \to \infty} E_n(\psi_{n_k}) \geq E_n(\psi) \geq c_0\) and \(E_n(\psi) = c_0\).

**Step 3** (Convergence in \(H^1(\mathbb{R})\)). From (A.4)

\[ \lim_{k \to \infty} \|\psi_{n_k}\|_2^2 = \lim_{k \to \infty} \left( E_n(\psi_{n_k}) + \frac{\alpha}{2} \|\psi_{n_k}\|_2^4 + \beta |\psi_{n_k}(0)|^2 \right) \]

\[ = c_0 + \frac{\alpha}{2} \|\psi\|_2^4 + \beta |\psi(0)|^2 = \|\psi\|_2^4, \]

and since \(\psi_{n_k} \to \psi\) in \(H^1\), \(\|\psi_{n_k} - \psi\|_2 \to 0\). Strong convergence in \(H^1\) now follows from (A.6). \(\square\)

**Appendix B. Bound on the Effective Coulomb Potential**

Recalling the effective Coulomb potential \(V^B_U\) from (4.1),

**Lemma B.1.** For any \(L > 0\) and \(\phi \in H^1(\mathbb{R})\) one has for \(B > 1\)

\[ \left| \int_R V^B_U (x) |\phi(x)|^2 \, dx - (\ln B - 2 \ln \ln B) |\phi(0)|^2 \right| \]

\[ \leq L^{-1} |\phi|_2^2 + 8 \sqrt{L} |\phi'|_{3/2}^2 |\phi|_{2/2}^2 + |\mathcal{G} (B, L)| \|\phi\|_2 \|\phi\|_2; \]

\[ \mathcal{G} (B, L) := 2 \ln L + 2 \ln \ln B + 2 \int_0^\infty e^{-u} \ln \left( \sqrt{\frac{1}{u} + \frac{2}{BL^2}} + \sqrt{\frac{1}{u}} \right) \, du - \ln 2. \]

**Proof.** Writing \(\int_R V^B_U (x) |\phi(x)|^2 \) as

\[ |\phi(0)|^2 \int_{|x| \leq L} V^B_U (x) + \int_{|x| \leq L} V^B_U (x) \left( |\phi(x)|^2 - |\phi(0)|^2 \right) + \int_{|x| \geq L} V^B_U (x) |\phi(x)|^2 \]

it is possible to bound

\[ \int_{|x| \geq L} V^B_U (x) |\phi(x)|^2 \, dx \leq L^{-1} \int_{|x| \geq L} |\phi(x)|^2 \, dx, \]  

(B.1)

\[ \left| \int_{|x| \leq L} V^B_U (x) \left( |\phi(x)|^2 - |\phi(0)|^2 \right) \, dx \right| \leq 8 \sqrt{L} |\phi'|_{3/2}^2 |\phi|_{2/2}^2 \]  

(B.2)

and to evaluate the integral

\[ \int_{|x| \leq L} V^B_U (x) \, dx = \ln B - 2 \ln \ln B + \mathcal{G} (B, L). \]

To arrive at the bound in (B.2) the following inequalities are used

\[ |\phi(x) - \phi(0)| \leq \sqrt{|x|} |\phi'|_2 \text{ and } |\phi|^2_\infty \leq ||\phi'||_2 ||\phi||_2. \]  

(B.3)
The lemma now follows from (B.1), (B.2) and the rightmost inequality of (B.3).

**Corollary B.2.** For any \( L > 0 \) and \( \phi \in H^1(\mathbb{R}) \) one has for \( B > 1 \)
\[
\left| \int_{\mathbb{R} \times \mathbb{R}} |\phi(x)|^2 \frac{1}{\sqrt{2}} V_B^B \left( \frac{x-y}{\sqrt{2}} \right) |\phi(y)|^2 \, dx \, dy - (\ln B - 2 \ln \ln B) \| \phi \|_2^2 \right| \\
\leq L^{-1} \| \phi \|_4^4 + 8 \sqrt{T} \| \phi' \|_2^4 \| \phi \|_2^4 + \left| \mathcal{G} \left( B, \frac{L}{\sqrt{2}} \right) \right| \| \phi' \|_2 \| \phi \|_2^2
\]
with \( \mathcal{G} (B, L) \) as above.

**Proof.** The corollary follows from Lemma B.1.

Now recalling the potential \( V_B^B \) from (5.5),

**Corollary B.3.** For any \( L > 0 \) and \( \phi \in H^1(\mathbb{R}) \) one has for \( B > 1 \)
\[
\left| \int_{\mathbb{R}} V_B^B (x) |\phi(x)|^2 \, dx - (\ln B - 2 \ln \ln B) |\phi(0)|^2 \right| \\
\leq L^{-1} \| \phi \|_4^4 + 8 \sqrt{T} \| \phi' \|_2^4 \| \phi \|_2^4 + \left| \mathcal{D}(B, L) \right| \| \phi' \|_2 \| \phi \|_2^2
\]
\[
\mathcal{D}(B, L) := 2 \ln L + 2 \ln \ln B + \frac{2}{\sqrt{2BL^2} + 1 + 1} + 2 \ln \left( \frac{1 + 2}{BL^2} + 1 \right) - \ln 2.
\]

**Proof.** Evaluating the integral
\[
\int_{|x| < L} V_B^B (x) \, dx = \ln B - 2 \ln \ln B + \mathcal{D}(B, L)
\]
the argument follows the proof of Lemma B.1.

**References**

[1] Avron, J.E., I.W. Herbst and B. Simon, “Schrödinger Operators with Magnetic Fields. III. Atoms in Homogeneous Magnetic Field,” Communications in Mathematical Physics Vol.79 529-572 (1981).

[2] Baumgartner, B., “The Thomas-Fermi-Theory as Result of a Strong-Coupling-Limit,” Communications in Mathematical Physics Vol.47(3) 215-219 (1976).

[3] Baumgartner, B., J.-Ph. Solovej and J. Yngvason, “Atoms in Strong Magnetic Fields: The High Field Limit at Fixed Nuclear Charge,” Communications in Mathematical Physics Vol.212(3) 703-724 (2000).

[4] Blankenbecler, R., M.L. Goldberger and B. Simon, “The bound states of weakly coupled long-range one-dimensional quantum hamiltonians,” Annals of Physics Vol.108(1) 69-78 (1977).

[5] Bonetto, F. and M. Loss, “Entropic Chaoticity for the Steady State of a Current Carrying System,” Journal of Mathematical Physics Vol.54 103303 (2013).

[6] Brummelhuis, R. and P. Duclos, “Effective Hamiltonians for atoms in very strong magnetic fields,” Journal of Mathematical Physics Vol.47 032103 (2006).

[7] Brummelhuis, R. and M.B. Ruskai, “One-dimensional models for atoms in strong magnetic fields, II: Anti-Symmetry in the Landau Levels,” Journal of Statistical Physics, Vol.116(1) 547-570 (2004).

[8] Frank, R.L., *Ground States of Semi-linear PDEs*, Lecture notes from the “Summerschool on Current Topics in Mathematical Physics,” Marseille (2014).

[9] Frank, R.L. and L. Geisinger, “The Ground State Energy of a Polaron in a Strong Magnetic Field,” Communications in Mathematical Physics Vol.338 1-29 (2015).

[10] Frank, R.L., K. Merz, H. Siedentop and B. Simon, “Proof of the Strong Scott Conjecture for Chandrasekhar Atoms,” arXiv:1907.04894v2 (2019).

[11] Frank, R.L. and R. Seiringer, “Quantum Corrections to the Pekar Asymptotics of a Strongly Coupled Polaron,” arXiv:1902.02489v2 (2019).
[12] Froese, R. and R. Waxler, “The spectrum of a Hydrogen Atom in an intense Magnetic Field,” Reviews in Mathematical Physics Vol.6 699-832 (1995).
[13] Fröhlich, H., H. Pelzer and S. Zienau, “Properties of slow electrons in polar materials,” The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science Vol. 41 221-242 (1950).
[14] Ghanta, R., Exact ground state energy of the 1D strong-coupling polaron, Junior thesis, Princeton (2012).
[15] Griesemer, M., E.H. Lieb and M. Loss, “Ground states in non-relativistic quantum electrodynamics,” Inventiones Mathematicae Vol.145(3) 557-595 (2001).
[16] Griesemer, M. and D. Wellig, “The strong-coupling polaron in static electric and magnetic fields,” Journal of Physics A: Mathematical and Theoretical Vol.46 425202 (2013).
[17] Gross, E.P., “Strong coupling polaron theory and translational invariance,” Annals of Physics Vol.99(1) 1-29 (1976).
[18] Iatchenko, A., E.H. Lieb and H. Siedentop, “Proof of a Conjecture about Atomic and Molecular Cores Related to Scott’s Conjecture,” Journal für die reine und angewandte Mathematik Vol.472 177-195 (1996).
[19] Kochetov, E.A., H. Leschke and M.A. Smolyonyrev, “Diagrammatic weak-coupling expansion for the magneto-polaron energy,” Zeitschrift für Physik B Vol. 89 177-186 (1992).
[20] Larsen, D.M., “Shallow Donor Levels of InSb in a Magnetic Field,” Journal of Physics and Chemistry of Solids Vol.29 271-280 (1968).
[21] Lieb, E.H. and L.E. Thomas, “Exact Ground State Energy of the Strong-Coupling Polaron,” Communications in Mathematical Physics Vol.183(3) 511-519 (1997).
[22] Lieb, E.H. and K. Yamazaki, “Ground-state energy and the effective mass of the polaron,” Physical Review Vol.111(3) 728-733 (1958).
[23] Lieb, E.H. and H.T. Yau, “The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics,” Communications in Mathematical Physics Vol.112(1) 147-174 (1987).
[24] Löwen, H., “Analytical behavior of the ground-state energy and pinning transitions for a bound polaron,” Journal of Mathematical Physics Vol.29 1505-1513 (1988).
[25] Platzman, P., “Magnetopolyron effect for shallow donor states in GaAs,” Physical Review B Vol. 48(8) 5202 (1993).
[26] Simon, B., “The Bound State of Weakly Coupled Schrödinger Operators in One and Two Dimensions,” Annals of Physics Vol.97(2) 279-288 (1976).
[27] Smolyonyrev, M.A., E.A. Kochetov, G. Verbist, F.M. Peeters and J.T. Devreese, “Equivalence of 3D bipolarons in a strong magnetic field to 1D bipolarons,” Europhysics Letters Vol.19(6) 519-524 (1992).
[28] Zorkani, I., R. Belhissi and E. Kartheuser, “The Ground State Energy of a Bound Polaron in the Presence of a Magnetic Field,” Physica Status Solidi (b) Vol. 197 411-419 (1996).

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