The Age or Excess of the $M|G|\infty$ Queue Busy Cycle Mean Value

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Abstract

In this work, approximate and exact results concerning a performance measure of the $M|G|\infty$ system, the age or the excess of the busy cycle, are presented. It will be seen that it is also a measure of the busy period performance. Service distributions for which it is given for a simple expression and others for which this does not happen are considered. For this last case, bounds are deduced. A special emphasis is given to the exponential distribution and to those related with it, useful in reliability theory.

Keywords: age, excess, $M|G|\infty$, busy cycle, busy period

1. Introduction

In a queue system it is usual to call busy period a period that begins when a customer arrives at the system being it empty, ends when a customer abandons the system letting it empty, and during it there is always at least one customer being served.

So, in a queue system, there is a sequence of idle periods and busy periods.

Be then the $M|G|\infty$ system initially empty. The instants $0, t_1, t_2, ...$ at which the system enters in state 0, are a renewal process arrival instant, see (Hokstad, 1979). A cycle is complete whenever a renewal occurs, that is: an entrance at state 0. These cycles are called busy cycles and their length is a random variable designated $Z$.

In (Takács, 1962) it is showed that

\[ E[Z] = \frac{\rho}{\lambda} \quad \text{and} \quad E[Z^2] = \frac{2e^{\rho} \int_0^\infty (e^{-\lambda t} \int_0^t (1 - G(v)) dv - e^{-\rho}) dt}{\lambda} + \frac{2e^\rho}{\lambda^2} \]

where $\lambda$ is the Poisson process arrivals rate, $G(\cdot)$ is the service distribution function, $\alpha$ its service time and $\rho = \lambda \alpha$ the traffic intensity.

Consider now a renewal process which time length between consecutive arrivals is a random variable $Z$ and be $A(t)$ the time spent since the last renewal till $t$, or the time spent after $t$ till the next renewal. If the renewals represent old devices turning out of order and being replaced, $A(t)$ is the age of the device in use at instant $t$ or the remaining lifetime of a device in use at instant $t$ – excess of a device in use at instant $t$, respectively.

Being interested in that device age, or excess, mean value, that is $\lim_{s \to \infty} \int_0^s \frac{A(t)dt}{s}$, it can be computed through, see (Ross, 1983).

\[ \lim_{s \to \infty} \int_0^s \frac{A(t)dt}{s} = \frac{E[Z^2]}{2E[Z]} \]

Note that being the $M|G|\infty$ queue a system with no waiting and no losses, it is mandatory to present immediately an available server when every customer arrives, it will be interesting, for instance, in a given instant of a busy period to have an idea of how much more time it will last. So there will have the notion of for how much time it will be necessary to have the servers available. This time is precisely the busy cycle excess.

The results presented will allow answering to this question in mean value terms.

2. $M|G|\infty$ Queue Busy Cycle Age or Excess Mean Value

Calling the $M|G|\infty$ system busy cycle age, or excess, mean value $\beta_c$.

\[ \beta_c = \beta + \frac{1}{\lambda} \]

where

\[ \beta = \int_0^\alpha (e^{\rho - \lambda t} \int_0^t (1 - G(v)) dv - 1) dt \]
In (Sathe, 1985) it is showed that \( \beta = \frac{1}{2} \rho^2 (\gamma_2^2 + 1) E[e^{\lambda U(t)}] \) and \( 1 + 2 \rho^{-2} (1 + \gamma_2^2)^{-1} (e^{\rho} - 1 - \rho - \frac{\rho^2}{2}) \leq E[e^{\lambda U(t)}] \leq 2 \rho^{-2} (e^{\rho} - 1 - \rho) \) where \( \gamma_s \) is the service distribution coefficient variation. So

\[
\frac{\gamma_2^2 \rho^2}{\lambda^2} - \alpha \leq \beta_c - E[Z] \leq \frac{\gamma_2^2}{\lambda^2} \sum_{n=2}^{\infty} \frac{\rho^n}{n!} - \alpha
\]

(5)
The Expressions (3) and (4) show that \( \beta_c \) depends on the whole service distribution structure and so is highly sensible to its form. The bounds given by (5) possess the great advantage of being valid whichever the service distribution is and depend only on \( \rho, \lambda \) and \( \gamma_s \).

The next proposition, immediate consequence of (5), allows to compare \( \beta_c \) with \( E[Z] \), that is insensible to the service distribution, since \( \rho \) and \( \gamma_s \) are known.

Proposition 1

If \( \gamma_2^2 \leq \frac{\rho}{e^{\rho} - 1 - \rho} \), \( \beta_c \leq E[Z] \)

If \( \gamma_2^2 \geq \frac{2}{\rho} \), \( \beta_c \geq E[Z] \)

Observation:

\[
\frac{2}{\rho} \geq \frac{\rho}{e^{\rho} - 1 - \rho}
\]

But \( \lim_{\rho \to 0} \frac{2}{\rho} = \frac{2}{\lambda} \).

3. Values of \( \beta_c \) for Some Service Distributions

As it was emphasised in section 2, \( \beta_c \) depends on the whole service time distribution. Then the values of \( \beta_c \) for some service time distributions, obtained after (3), are presented:

Exponential

\[ \beta_c^{M} = \frac{1}{\lambda} + \alpha \sum_{n=1}^{\infty} \frac{\rho^n}{nn!} \]

Constant

\[ \beta_c^{D} = E[Z] - \alpha \]

\[ G(t) = \frac{e^{-\rho}}{e^{-\rho t} (1 - e^{-\rho})}, t \geq 0, \text{ see (Ferreira & Andrade, 2012)} \]

\[ \beta_c = E[Z] \]

\[ G(t) = 1 - \frac{1}{1 + e^{-\rho t} (e^{\rho} - 1)}, t \geq 0, \text{ see (Ferreira & Andrade, 2012)} \]

\[ \beta_c = \frac{e^{\rho} + e^{-\rho} - 1}{\lambda} \]

Power function with parameter \( c (\alpha = \frac{c}{c+1}) \)

\[ \beta_c^{P} = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{\rho^{n-k}}{(n-k)!} \sum_{j=0}^{k} \frac{(\rho)^j}{j! (k-j)! (k+j+c+1)} \]

Uniform in \([0,1]\) (\( c=1 \))

\[ \beta_c^{P} = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{2^{n-k} (n-k)!}{(n-k)!} \sum_{j=0}^{k} \frac{(-1)^j}{2^j (k-j)! (k+j+1)} \]

But note that the result for the constant service distribution may be derived easily from (5) making \( \gamma_s = 0 \).

For any service time distribution, after (5),

\[ \beta_c \geq E[Z] - \alpha + \frac{\rho^2}{2\lambda} \gamma_s^2 \]

(6)

So, for the \( M|G|\infty \) system, fixed \( \alpha \) and \( \lambda \), the least value of \( \beta_c \) happens in the case of constant service time.

The values of \( \beta_c \) are then computed for some values of \( \alpha \) and \( \lambda \) and presented in Tables 1 and 2.
If the service time distribution is IMRL

\[ t \geq 0 \]

\[ G(t) = \frac{1}{1 + e^{-\rho} \left( e^{\frac{\lambda}{1-\rho^2}} - 1 \right)} \]

\[ t \geq 0 \]

For the service time distributions exponential and power function the \( \beta_c \) values were obtained through (3) by numerical methods.

The values in Tables 1 and 2 evidence the dependence of \( \beta_c \) from the service time distribution structure, although for high values of \( \rho \) that dependence vanishes expressively.

After (5) and the expression showed in this section for \( \beta_c^M \) it is possible to obtain lower and upper bounds for this parameter, since the busy period mean value is

\[ E[B] = \frac{e^{\rho-1}}{\lambda} \]

for any service time distribution.

If the service time distribution is NBUE-New Better than Used in Expectation with mean \( \alpha \), \( \int_1^\infty [1 - G(v)] dv \leq \int_1^\infty e^{-\frac{v}{\alpha}} dv \), see (Ross, 1983), and the lower bound obtained for \( \beta_c^M \) is good for \( \beta_c^{NBUE} \).

If the service time distribution is NWUE-New Worse than Used in Expectation with mean \( \alpha \), \( \int_1^\infty [1 - G(v)] dv \geq \int_1^\infty e^{-\frac{v}{\alpha}} dv \), see (Ross, 1983), and the upper bound obtained for \( \beta_c^M \) is good for \( \beta_c^{NWUE} \).

If the service time distribution is DFR-Decreasing Failure Rate, \( 1 - G(t) \geq e^{t \frac{\lambda}{\alpha} - \frac{\lambda^2}{2}} \), see (Ross, 1983), and a lower bound for \( \beta_c^{DFR} \) may be obtained following a methodology analogous to the one that allowed to obtain the lower bound for \( \beta_c^M \). If the service time distribution is IMRL-Increasing Mean Residual Life, \( 1 - \frac{1 - G(v) dv}{\alpha} \geq e^{\frac{2\mu_r - 2\lambda}{3\mu_r^2}} \), being \( \mu_r \) the \( G(.) \) \( r^{th} \) order moment around the origin, see (Brown, 1981) and (Cox, 1962), and it is possible to find a lower bound for \( \beta_c^{IMRL} \) analogous to the one for \( \beta_c^M \). For the power function service distribution, as \( \gamma^2 = [c(c+2)]^{-1} \), a lower bound and an upper bound for \( \beta_c^P \) that, for \( c = 1 \), are also valid for the uniform in \([0,1]\) service distribution are easily obtained. So

- \( \beta_c \) lower bounds
a) M and NWUE
\[ E[Z] - \alpha + \frac{\alpha \rho}{2} \left( 1 + \frac{\rho}{6} \right) \]
b) DFR
\[ E[Z] - \alpha + \frac{\alpha \rho}{2} \left( 2e^{\frac{1+\gamma}{2}} - 1 + \rho \frac{3e^{1-\gamma} - 2}{6} \right) \]
c) IMRL
\[ E[Z] - \alpha + \frac{\lambda}{4} \left( \mu_2e^{\frac{2\alpha}{3\mu_2^2\mu_3}} - 2\alpha^2 + \rho \frac{\mu_2e^{2(1-\frac{2\alpha}{3\mu_2^2\mu_3})} - 4\alpha^2}{6} \right) \]
d) Power function with parameter \( c \)
\[ E[Z] + \frac{\rho - 2c(c + 2)}{2(c + 1)(c + 2)} \]
e) Power function with parameter \( c \)
\[ E[Z] + \frac{\rho - 2c(c + 2)}{2(c + 1)(c + 2)} \]
f) Uniform in \([0, 1]\) \((c = 1)\)
\[ E[Z] + \frac{\lambda}{2} - \frac{6}{12} \]

- \( \beta_c \) upper bounds
  a) M and NBUE
\[ \frac{1}{\lambda} + \min \left\{ 2(E[B] - \alpha), \frac{\rho}{2}(E[B] + \alpha) \right\} \]
  b) Power function with parameter \( c \)
\[ \frac{1}{\lambda} + \frac{(c + 1)^2}{c(c + 2)} E[B] - \frac{c + 1}{c + 2} \]
  c) Uniform in \([0, 1]\) \((c = 1)\)
\[ \frac{1}{\lambda} + \frac{4}{3} E[B] - \frac{2}{3} \]

Finally, the ratio of the difference between the upper and the lower bound over the real value, for the exponential and power function service distributions were computed taking \( \alpha = 0.5 \) and \( \lambda = 2, 10, 100 \), and the results are in Table 3.

Table 3. Ratio of the difference between the upper and the lower bound over the real value- \( \alpha = 0.5 \)

| Service Time Distribution | \( \lambda = 2 \) | \( \lambda = 10 \) | \( \lambda = 20 \) | \( \lambda = 100 \) |
|---------------------------|-----------------|-----------------|-----------------|-----------------|
| Exponential               | 0.024818024     | 0.62565866      | 0.87899084      | 0.87295261      |
| Power function with parameter \( c \); \( \alpha = \frac{c}{c+1} \) | 0.018536302     | 0.25071787      | 0.28865152      | 0.32992972      |

The best results (that is: the lowest) happen for the power service distribution and for the lowest traffic intensities.

4. Conclusions
It was already emphasized the interest of the $M|G|\infty$ system age or excess of the busy cycle, in the management of that queue, particularly of the availability of the servers.

Then this search was oriented to look for the properties of this parameter. Of course, important are the exact formulae to compute it for the various service time distributions, but some of it result quite complicate, involving infinite sums, making its applicability problematic.

So, the importance of the lower and upper bounds that it was possible to compute, mathematically much simpler, namely for service time distributions important in reliability theory such as: Exponential, NBUE, NWUE, DFR and IMRL.

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