ANOTHER PROOF OF THE NOWICKI CONJECTURE

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Abstract. Let $K[X_d, Y_d] = K[x_1, \ldots, x_d, y_1, \ldots, y_d]$ be the polynomial algebra in $2d$ variables over a field $K$ of characteristic 0 and let $\delta$ be the derivation of $K[X_d, Y_d]$ defined by $\delta(y_i) = x_i, \delta(x_i) = 0, i = 1, \ldots, d$. In 1994 Nowicki conjectured that the algebra $K[X_d, Y_d]^\delta$ of constants of $\delta$ is generated by $X_d$ and the determinants $u_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = x_i y_j - x_j y_i, 1 \leq i < j \leq d$. The affirmative answer was given by several authors using different ideas. In the present paper we give another proof of the conjecture based on representation theory of the general linear group $GL_2(K)$.

1. Introduction

The linear operator $\delta$ of an algebra $A$ over a field $K$ is a derivation if it satisfies the Leibniz rule
$$\delta(a_1 a_2) = \delta(a_1)a_2 + a_1 \delta(a_2), \quad a_1, a_2 \in A.$$ The kernel of $\delta$ is called the algebra of constants of $\delta$ and is denoted by $A^\delta$. In the sequel $K$ is a field of characteristic 0. When $A = K[Z_m] = K[z_1, \ldots, z_m]$ is the algebra of polynomials in $m$ variables the derivation $\delta$ is called Weitzenböck if it acts as a nilpotent linear operator on the vector space $KZ_m$ with basis $Z_m = \{z_1, \ldots, z_m\}$. The classical theorem of Weitzenböck [17] in 1932 states that in this case $K[Z_m]^\delta$ is finitely generated. The algebra $K[Z_m]^\delta$ coincides with the algebra of invariants $K[Z_m]^{(K,+)}$, where the additive group $(K,+)$ of the field $K$ is embedded as a subgroup into the unitriangular group $UT_m(K)$ acting as $\{\exp(\alpha \delta) | \alpha \in K\}$. Hence the finitely generation of $K[Z_m]^\delta$ is equivalent to a theorem of classical invariant theory. A modern geometric proof of the Weitzenböck theorem in this spirit is given by Seshadri [10]. A translation in an algebraic language of this proof is given by Tyc [16]. For more information on Weitzenböck derivations one can see the books by Nowicki [14, Section 6.2], Derksen and Kemper [3, Chapter 2], and Dolgachev [4, Section 4.2].

In the special case of the polynomial algebra $K[X_d, Y_d]$ in $2d$ variables $X_d = \{x_1, \ldots, x_d\}$ and $Y_d = \{y_1, \ldots, y_d\}$ when the Weitzenböck derivation $\delta$ acts by
$$\delta(x_i) = 0, \quad \delta(y_i) = x_i, \quad i = 1, \ldots, d,$$
Nowicki [14] conjectured in 1994 that $K[X_d, Y_d]^\delta$ is generated by $X_d$ and the determinants
$$u_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = x_i y_j - x_j y_i, \quad 1 \leq i < j \leq d.$$ There are several proofs based on different methods confirming the Nowicki conjecture: by Khoury [10, 11], Bedratyuk [1], the author and Makar-Limanov [6], Kuroda [12], and others.

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There are unpublished proofs by Derksen and Panyushev. As Kuoda mentions in his paper [12] Goto, Hayasaka, Kurano, and Nakamura [8] and Miyazaki [13] determined sets of generators for algebras of invariants with $K[X_d, Y_d]^\delta$ included in the list.

In the present paper we give a new proof using easy arguments from representation theory of the general linear group $GL_2(K)$. Our proof is inspired by our paper with Gupta [5] devoted to the noncommutative version of Weitzenböck derivations.

2. Preliminaries

Let $V$ be a vector space with basis \{x, y\} with the canonical action of the general linear group $GL_2(K)$:

\[(3)\quad g(x) = \gamma_1 x + \gamma_2 y, \quad g(y) = \gamma_1 x + \gamma_2 y, \quad \text{where} \quad g = \begin{pmatrix} \gamma_1 & \gamma_1 \\ \gamma_2 & \gamma_2 \end{pmatrix} \in GL_2(K).\]

For a background on representation theory of the general linear group $GL_m(K)$ see, e.g., the books by Weyl [18, Chapter 4] or James and Kerber [9, Chapter 8]. We shall summarize the necessary facts for $m = 2$ only. The polynomial representations of $GL_2(K)$ are completely reducible and their irreducible components $W(\lambda)$ are indexed by partitions $\lambda = (\lambda_1, \lambda_2)$. If $\lambda$ is a partition of $n$ (notation $\lambda \vdash n$), then $W(\lambda)$ can be realized as a $GL_2(K)$-submodule of the $n$-th tensor degree $V^\otimes n$ equipped with the diagonal action of $GL_2(K)$

\[g(v_1 \otimes \cdots \otimes v_n) = g(v_1) \otimes \cdots \otimes g(v_n), \quad v_i \in V, \quad g \in GL_2(K).\]

As a vector space $V^\otimes n$ is $\mathbb{N}_0^2$-graded and the homogeneous component of degree $(n_1, n_2)$, $n_1 + n_2 = n$, is spanned on the tensors $z_{i_1} \otimes \cdots \otimes z_{i_n}$, $z_i = x, y$, of degree $n_1$ and $n_2$ in $x$ and $y$, respectively. Then $W(\lambda)$ has a basis of homogeneous elements

\[(4)\quad \{w_0, w_1, \ldots, w_{\lambda_1-\lambda_2}\}, \quad \deg w_i = (\lambda_1 - i, \lambda_2 + i), \quad i = 0, 1, \ldots, \lambda_1 - \lambda_2.\]

The element $w_0 = w(\lambda) \in W(\lambda)$ which is homogeneous of degree $\lambda$ is called the highest weight vector of $W(\lambda)$. One typical element $w(\lambda)(1) \in W(\lambda)(1) \subset V^\otimes n$, $W(\lambda)(1) \cong W(\lambda)$, is

\[(5)\quad w(\lambda)(1) = (x \otimes y - y \otimes x) \otimes \cdots \otimes (x \otimes y - y \otimes x) \otimes x \otimes \cdots \otimes x,\]

Here the skew-symmetric sums $(x \otimes y - y \otimes x)$ appear in positions $(1, 2), (3, 4), \ldots, (2\lambda_2 - 1, 2\lambda_2)$. The symmetric group $S_n$ acts from the right on $V^\otimes n$ by place permutation

\[(v_1 \otimes \cdots \otimes v_n)\sigma^{-1} = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad v_i \in V, \quad \sigma \in S_n.\]

Then every highest weight vector $w(\lambda) \in W(\lambda) \subset V^\otimes n$ is of the form

\[(6)\quad w(\lambda) = w(\lambda)(1) \sum_{\sigma \in S_n} \alpha_\sigma \sigma^{-1}, \quad \alpha_\sigma \in K.\]

Clearly, the skew-symmetries in $w(\lambda)(1)\sigma$ are in positions

\[(\sigma(1), \sigma(2)), (\sigma(3), \sigma(4)), \ldots, (\sigma(2\lambda_2 - 1), \sigma(2\lambda_2)).\]

Remark 2.1. Since $W(\lambda)$, $\lambda \vdash n$, participates in $V^\otimes n$ with multiplicity equal to the number of standard tableaux of shape $[\lambda]$, by [7, Proposition 0.1] we may choose a basis of the vector space of highest weight vectors $w(\lambda) \in V^\otimes n$ consisting of all $w(\lambda)(1)\sigma^{-1}$ such that the tableau
is standard, i.e.,

\[
\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \ldots, \sigma(2\lambda_2 - 1) < \sigma(2\lambda_2),
\]

\[
\sigma(1) < \sigma(3) < \cdots < \sigma(2\lambda_2 - 1) < \sigma(2\lambda_2 + 1) < \sigma(2\lambda_2 + 2) < \cdots < \sigma(n),
\]

\[
\sigma(2) < \sigma(4) < \cdots < \sigma(2\lambda_2).
\]

The highest weight vector of \(W(\lambda) \subset V^{\otimes n}\) can be characterized in the following way, see [2] Lemma 1.1.

**Lemma 2.2.** Let \(\Delta\) be the derivation of the tensor algebra

\[
T(V) = \bigoplus_{n \geq 0} V^{\otimes n} = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots
\]

defined by \(\Delta(x) = 0, \Delta(y) = x\). Then the homogeneous element \(w \neq 0\) of degree \(\lambda \vdash n\) is a highest weight vector of some \(W(\lambda) \subset V^{\otimes n}\) if and only if \(\Delta(w) = 0\).

**Remark 2.3.** Up to a nonzero multiplicative constant the derivation \(\Delta\) sends the element \(w_i\) from (4) to \(w_{i-1}\), \(i = 1, \ldots, \lambda_1 - \lambda_2\), and \(\Delta(w_0) = 0\).

Let \(n = (n_1, \ldots, n_d)\) be a \(d\)-tuple of nonnegative integers and let \(|n| = n_1 + \cdots + n_d\). Consider the vector spaces \(V_1, \ldots, V_d\) with bases \(\{x_1, y_1\}, \ldots, \{x_d, y_d\}\), respectively, and the canonical action of \(GL_2(K)\) as in (3) on them. Clearly, the tensor products \(V^{(n)} = V_1^{\otimes n_1} \otimes \cdots \otimes V_d^{\otimes n_d}\) and \(V^{\otimes n}\) are isomorphic as \(GL_2(K)\)-modules. As in the case of \(V^{\otimes n} \subset T(V)\) we define an \(\mathbb{N}_0^d\)-grading on \(V^{(n)}\) which counts the number of entries of \(X_d\) and \(Y_d\), respectively. Let \(\lambda \vdash |n|\). If at the first \(\lambda_2\) couples of positions \((1, 2), (3, 4), \ldots, (2\lambda_2 - 1, 2\lambda_2)\) in the tensor product \(V^{(n)}\) we have \((V_{i_1}, V_{j_1}), (V_{i_2}, V_{j_2}), \ldots, (V_{i_{\lambda_2}}, V_{j_{\lambda_2}})\), and the positions left are \(k_1, \ldots, k_{\lambda_1 - \lambda_2}\), then the analogue of the equation (5) is

\[
(x_{i_1} \otimes y_{j_1} - y_{i_1} \otimes x_{j_1}) \otimes \cdots \otimes (x_{i_{\lambda_2}} \otimes y_{j_{\lambda_2}} - y_{i_{\lambda_2}} \otimes x_{j_{\lambda_2}}) \otimes x_{k_1} \otimes \cdots \otimes x_{k_{\lambda_1 - \lambda_2}}.
\]

The equation (6) also can be restated in a similar way. As a consequence of Lemma 2.2 we obtain:

**Corollary 2.4.** Let \(\delta\) be the derivation of the tensor algebra \(T(V_1 \oplus \cdots \oplus V_d)\) defined by (11). Then a homogeneous element \(w \in V^{(n)} = V_1^{\otimes n_1} \otimes \cdots \otimes V_d^{\otimes n_d}\) of degree \(\lambda \vdash |n|\) is a highest weight vector of a submodule \(W(\lambda)\) of \(V^{(n)}\) if and only if \(\delta(w) = 0\).

3. The main result

We are ready to present our proof of the Nowicki conjecture.
Theorem 3.1. Let $K$ be a field of characteristic $0$ and let $\delta$ be the derivation of the polynomial algebra $K[X_d, Y_d]$ defined by (1). Then the algebra of constants $K[X_d, Y_d]^\delta$ is generated by $X_d$ and the determinants $\delta^n$.

Proof. The algebra $K[X_d, Y_d]$ has a canonical $\mathbb{N}_0^d$-grading. The homogeneous component $K[X_d, Y_d]^\delta(n)$ of degree $n = (n_1, \ldots, n_d)$ is spanned by the monomials which are of degree $n_i$ in $x_i$ and $y_i$, $i = 1, \ldots, d$. It follows from the definition of $\delta$ that $\delta(K[X_d, Y_d]^\delta(n)) \subset K[X_d, Y_d]^\delta(n)$. Hence we shall prove the theorem if we show that each component $(K[X_d, Y_d]^\delta(n))$ is spanned on the products

$$X_d^p U_d^n = x_1^{p_1} \cdots x_d^{p_d} \prod_{1 \leq i < j \leq d} u_{ij}^{q_{ij}}$$

of degree $n$. As a $GL_d(K)$-module $K[X_d, Y_d]^\delta(n)$ is isomorphic to the symmetric tensor power of $n_1$ copies of $V_1$, $n_2$-copies of $V_2$, $n_d$ copies of $V_d$. Hence it is a homomorphic image of $V(n) = V_1^{n_1} \otimes \cdots \otimes V_d^{n_d}$. The action of $\delta$ on $V(n)$ induces the canonical action on $K[X_d, Y_d]^\delta(n)$. Therefore the vector space of the highest weight vectors of $K[X_d, Y_d]^\delta(n)$ is an image of the vector space of the highest weight vectors of $V(n)$ and by Lemma 2.2 and Remark 2.3 coincides with $(K[X_d, Y_d]^\delta(n))$. The highest weight vectors of $V(n)$ are linear combinations of the products (7) with the property that

$$\{i_a, j_a, k_b \mid a = 1, 2, \ldots, \lambda_2, b = 1, 2, \ldots, \lambda_1 - \lambda_2 \} = \{1, 2, \ldots, |n|\}.$$

Obviously the image of the element (7) in $K[X_d, Y_d]^\delta(n)$ is

$$u_{i_1 j_1} \cdots u_{i_{\lambda_2} j_{\lambda_2}} x_{k_1} \cdots x_{k_{\lambda_1 - \lambda_2}}$$

Replacing $u_{i_a j_a}$ with $u_{j_a i_a}$ if $i_a > j_a$, we obtain that $(K[X_d, Y_d]^\delta(n))$ is spanned on the products (3) which completes the proof. \qed

References

[1] L. Bedratyuk, A note about the Nowicki conjecture on Weitzenböck derivations, Serdica Math. J. 35 (2009), 311-316.

[2] F. Benanti, V. Drensky, Defining relations of minimal degree of the trace algebra of $3 \times 3$ matrices, J. Algebra 320 (2009), 756-782.

[3] H. Derksen, G. Kemper, Computational Invariant Theory, Encyclopaedia of Mathematical Sciences, Invariant Theory and Algebraic Transformation Groups 130, Springer-Verlag, Berlin, 2002.

[4] I. Dolgachev, Lectures on Invariant Theory, London Mathematical Society Lecture Note Series 296, Cambridge University Press, Cambridge, 2003.

[5] V. Drensky, C. K. Gupta, Constants of Weitzenböck derivations and invariants of unipotent transformations acting on relatively free algebras, J. Algebra 292 (2005), 393-428.

[6] V. Drensky, L. Makar-Limanov, The conjecture of Nowicki on Weitzenböck derivations of polynomial algebras, J. Algebra Appl. 8 (2009), 41-51.

[7] V. Drensky, Ts. G. Rashkova, Weak polynomial identities for the matrix algebras, Comm. Algebra 21 (1993), 3779-3795.

[8] S. Goto, F. Hayasaka, K. Kurano and Y. Nakamura, Rees algebra of the second syzygy module of the residue field of a regular local ring, Contemp. Math. 390 (2005), 97-108.

[9] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Reprint of the 1985 hardback ed. Encyclopedia of Mathematics and its Applications 16, Cambridge University Press, Cambridge, 2009.

[10] J. Khoury, Locally Nilpotent Derivations and Their Rings of Constants, Ph.D. Thesis, Univ. Ottawa, 2004.

[11] J. Khoury, A Groebner basis approach to solve a conjecture of Nowicki, J. Symbolic Comput. 43 (2008), 908-922.
[12] S. Kuroda, A simple proof of Nowicki’s conjecture on the kernel of an elementary derivation, Tokyo J. Math. 32 (2009), 247-251.
[13] M. Miyazaki, Invariants of the unipotent radical of a Borel subgroup, Proceedings of the 29th Symposium on Commutative Algebra in Japan, Nagoya, Japan, November 19-22, 2007, 43-50.
[14] A. Nowicki, Polynomial Derivations and Their Rings of Constants, Uniwersytet Mikolaja Kopernika, Torun, 1994. www-users.mat.umk.pl/~anow/ps-dvi/pol-der.pdf.
[15] C. S. Seshadri, On a theorem of Weitzenböck in invariant theory, J. Math. Kyoto Univ. 1 (1961), 403-409.
[16] A. Tyc, An elementary proof of the Weitzenböck theorem, Colloq. Math. 78 (1998), No. 1, 123-132.
[17] R. Weitzenböck, Über die Invarianten von linearen Gruppen, Acta Math. 58 (1932), 231-293.
[18] H. Weyl, The Classical Groups, Their Invariants and Representations, Reprint of the second edition (1946) of the 1939 original, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997.

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