Generalization of Agranovich-Toshich transformation and constraint free bosonic representation for systems of truncated oscillators.

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Abstract

The generalization of Agranovich-Toshich representation of paulion operators in terms of bosonic ones for the case of truncated oscillators of higher ranks is represented. We use this generalization to introduce a new constraint free bosonic description of truncated oscillator systems. The corresponding functional integral representations for thermodynamic quantities are given and the application to investigations of Long Rang Order in the system is discussed.

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1 Introduction

About three decades ago in the paper [1] Agranovich and Toshich proposed the bosonic expression for the creation and annihilation operators of paulions (i.e. particles obeyed the fermionic anticommutation relations on same sites and the bosonic commutation relations on different sites; alternatively, it is possible to realize them as lattice spins 1/2 or truncated oscillators of rank 2). Explicitly, the formula has the form:

\[ \hat{P}_i^+ = b_i^+ \sqrt{\sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} (b_i^+)^k b_i^k} \quad \hat{P}_i = (\hat{P}_i^+)^+ . \]  

(1)

Here \( \hat{P}_i^+, \hat{P}_i \) are the creation and annihilation operators of a paulion on site \( i \), which obey the paulionic commutation relations

\[ \hat{P}_i^+ \hat{P}_i + \hat{P}_i \hat{P}_i^+ = 1 \quad (\hat{P}_i^+)^2 = \hat{P}_i^2 = 0 \]

\[ [\hat{P}_i^+, \hat{P}_j] = [\hat{P}_i, \hat{P}_j] = 0 \quad \text{for} \quad i \neq j \]

(2)

and \( b_i^+, b_i \) are the bosonic operators on \( i \)-th site.

Particles with anticommutation relations (2) on site and the bosonic commutation relations on different sites widely arise in spin lattices, magnetics, models describing excitons in molecular crystals, defectons in quantum crystals and many others. In the original paper [1] the Frenkel excitons were considered in connection with the possibility of their Bose-condensation. As usually in the problem of the Bose-condensation, the definition of an auxiliary bosonic description of the system is a central point because then the standard theory of a nonideal Bose-gas can be applied.

On the other hand, there are several applications of high rank truncated oscillators. For example, it was recently shown that such operators can be used for the second quantization of particles with Haldane exclusion statistics [4]. Truncated oscillators also find applications in nonlinear optics, semiconductors, parasupersymmetric theories and other fields. That is why it seems to be interesting to generalize Agranovich-Toshich description of truncated oscillators of rank 2 to higher ranks and investigate the corresponding bosonic representation.

Similar to the approach of the sigma-model with Wess-Zumino term [4] we will treat the constraint on number of particles on site exactly. To do this we will use the mapping of the orthogonal sum of identical copies of a truncated oscillator space of states to the bosonic space of states. In such mapping the creation and annihilation operators of truncated bosons are represented in a form of power series on the usual bosonic creation and annihilation operators. This compels us to deal with infinite series of different vertices in the diagram technique. The choice of relevant contributions in such series should be dictated as usually by feathers of the concrete problem.

The paper is constructed as follows. In the next section we prove the generalization of the Agranovich-Toshich formula for the case of an arbitrary rank of truncated oscillators. We give both variants of the mapping – with and without the square root (which corresponds to the formula proposed by Chernyak [3] for the paulionic case; in practical use the latter is even more convenient). In section 3 we describe an associated bosonic system for the case of many degrees of freedom and give the functional integral representation of thermodynamic quantities of the system. The conclusion completes the letter with several remarks.
2 Generalization of Agranovich-Toshich representation for truncated oscillators

The goal of this section is to express the creation and annihilation operators of truncated bosons $B^+, B$ of rank $m$ with the algebra:

$$BB^+ - B^+ B = 1 - \frac{m}{(m-1)!}(B^+)^{m-1}B^{m-1}, \quad (B^+)^+ = B, \quad (B^+)^m = B^m = 0 \quad (3)$$

in terms of the standard bosonic creation and annihilation operators $b^+, b$. In this section only one degree of freedom is considered but the generalization on many degrees of freedom is straightforward and will be considered in the next section. First of all we will construct the number particle operator of the truncated bosons $\hat{N}$ using the following operator

$$1 + q^{(b^+b-k)} + q^{2(b^+b-k)} + \ldots + q^{(m-1)(b^+b-k)} ,$$

where $q = \exp \left( i \frac{2\pi}{m} \right)$ and $m$ is the rank of the truncated bosons. One can prove that this operator does not equal to zero only on states $|lm + k\rangle$, where $l = 0, 1, 2, \ldots$. So the operator $\hat{N} = B^+ B$ can be expressed as follows:

$$\hat{N} = 0 \cdot \left( 1 + q^{b^+b} + q^{2b^+b} + \ldots + q^{(m-1)b^+b} \right) \frac{1}{m} +$$

$$+ 1 \cdot \left( 1 + q^{(b^+b-1)} + q^{2(b^+b-1)} + \ldots + q^{(m-1)(b^+b-1)} \right) \frac{1}{m} + \ldots$$

$$+ (m-1) \left( 1 + q^{(b^+b-m+1)} + q^{2(b^+b-m+1)} + \ldots + q^{(m-1)(b^+b-m+1)} \right) \frac{1}{m} .$$

Summing terms of same orders one can obtain the following expression for operator $\hat{N}$:

$$\hat{N} = \sum_{k=1}^{m-1} \frac{q^k}{1-q^k} q^{kb^+b} + \frac{m-1}{2} . \quad (4)$$

Now to order bosonic operators we can use the formula for the normal ordered exponent

$$q^{kb^+b} = \exp \left( i \frac{2\pi}{m} kb^+b \right) =: \exp(q^k - 1)b^+b : ,$$

where $:$ : denotes normal ordering. Then expression (4) takes the form

$$\hat{N} = \sum_{l=1}^{m-1} \sum_{k=1}^{m-1} \frac{(-1)^l}{l!} q^k (1 - q^k)^{l-1} (b^+)^l b^l . \quad (5)$$

Now let us look for the creation (annihilation) operators of truncated bosons in the following form:

$$B^+ = b^+ \sqrt{\sum_{k=0}^{\infty} \alpha_k (b^+)^k b^k}, \quad B = (B^+)^+ .$$

and assume that $\alpha_k$ are real. Then, using expression (4) for the number particle operator $\hat{N} = B^+ B$, one can obtain the expression for the coefficients $\alpha_k$:
\[ \alpha_{l} = \sum_{k=1}^{m-1} \frac{(-1)^{l+1}}{(l+1)!} q^{k}(1-q^{k})^{l}. \]

For \( m = 2 \) (\( q = -1 \)) the coefficients take the form

\[ \alpha_{l} = \frac{(-2)^{l}}{(l+1)!} \]

which gives us Agranovich-Toshich representation (8) for the creation (annihilation) operators of truncated bosons of rank 2. So we have proved the generalization of Agranovich-Toshich formula for truncated oscillators of higher ranks:

\[ B^{+} = b^{+} \sqrt{\sum_{k=0}^{m-1} \frac{(-1)^{l+1}}{(l+1)!} q^{k}(1-q^{k})^{l}(b^{+})^{k}b^{k}}, \quad B = (B^{+})^{+}, \quad q = e^{i \frac{2\pi}{m}} \]  

(6)

Let us now "take square root" in formula (6). To do this we will follow the method proposed by Chernyak in Ref. [3]. The main point in the method is to use the projection operator on the vacuum state of the auxiliary boson system, i.e on the vector \(|0\rangle\). To express this projection operator \( \mathcal{P} \) in terms of \( b^{+}, b \) the coherent state representation is convenient

\[ |z\rangle = \exp(-\frac{1}{2} \bar{z} z) \exp(z b^{+})|0\rangle, \]

\[ <z'|z\rangle = \exp(-\frac{1}{2} \bar{z}' z' - \frac{1}{2} \bar{z} z + \bar{z}' z). \]

From last relations one easily derives

\[ <z'|\mathcal{P}|z\rangle = \exp(-\bar{z}' z) <z'|z\rangle. \]

On the other hand, for any \( k, l \) we have

\[ <z'| (b^{+})^{k}b^{l}|z\rangle = (\bar{z}')^{k} z^{l} <z'|z\rangle. \]

Then we can conclude that

\[ <z'|\mathcal{P}|z\rangle = \sum_{l=0}^{\infty} \frac{(-1)^{l}}{l!} <z'| (b^{+})^{l}b^{l}|z\rangle. \]

It means that the projection operator has the following expression in terms of the bosonic creation and annihilation operators:

\[ \mathcal{P} = \sum_{l=0}^{\infty} \frac{(-1)^{l}}{l!} (b^{+})^{l}b^{l} \equiv: \exp(-b^{+} b):. \]  

(7)

We now can use formula (7) to construct the creation and annihilation operators \( B^{+}, B \) which obey algebra (3). Indeed, it is easy to check from the matrix form that the following relations take place:

\[ B^{+} = \sum_{n=0}^{m-2} \sum_{k=0}^{m-k+1} (b^{+})^{mn+k+1} \mathcal{P}b^{mn+k} \sqrt{k+1} \sqrt{mn+k+1}, \quad B = (B^{+})^{+}, \quad q = e^{i \frac{2\pi}{m}} \]
\[ B = \sum_{n=0}^{\infty} \sum_{k=0}^{m-2} (b^+)^{mn+2k} P b^{mn+k+1} \frac{\sqrt{k+1}}{(mn+k)!\sqrt{mn+k+1}} \] 

It is obvious that relations (8) satisfies to algebra (3). On the other hand, the operators given by relations (6) satisfy the same algebra and have the same matrix form. Hence, we can realize formulae (8) as the "taking of square root" in eq.(6). For the particular case \( m = 2 \) our formulae are reduced to the formulae originally obtained by Chernyak [3] for the case of Paulionic operators.

### 3 Paulion-boson mapping and functional integral representation

In this section we will describe the mapping from the system of truncated oscillators to the auxiliary bosonic system which will be main object of the investigation in the section. The goal is to escape the introduction of a constraint. To do this we will embed an infinite number of copies of the finite dimensional space of states in the bosonic space of states and then proceed with the consideration of this new (auxiliary) bosonic space.

To explain this in details, let us first of all consider one degree of freedom (i.e a single site). Then the creation \( B^+ \) and annihilation \( B \) operators have the following matrix form in the \( m \)-dimensional Hilbert space of states \( \mathcal{H}_B \) (\( m \) is a rank of the truncated oscillator):

\[
B^+ = \begin{pmatrix}
0 & 0 & 0 & 0 & ... & 0 \\
1 & 0 & 0 & 0 & ... & 0 \\
0 & \sqrt{2} & 0 & 0 & ... & 0 \\
0 & 0 & \sqrt{3} & 0 & ... & 0 \\
& & & ... & & \\
0 & 0 & 0 & 0 & \sqrt{m-1} & 0 \\
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & 1 & 0 & 0 & ... & 0 \\
0 & 0 & \sqrt{2} & 0 & ... & 0 \\
0 & 0 & 0 & \sqrt{3} & ... & 0 \\
& & & ... & & \\
0 & 0 & 0 & 0 & 0 & \sqrt{m-1} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

with basis \( \{|0>, |1>, ..., |m-1>\} \) and the obvious notations. Now we introduce the infinite orthogonal sum \( \mathcal{H}_b = \bigoplus \mathcal{H}_{B,n} \) of such \( m+1 \)-dimensional Hilbert spaces \( \mathcal{H}_{B,n} \) with basis \( \{|0>, |1>, ..., |m-1>\}, ..., \{|nm+1>, |nm+2>, ..., |nm+m-1>\}, \ldots \). The extensions of the creation and annihilation operators \( B^+, B \) in this space have the form:

\[ \hat{B}^+ = \text{diag}(B^+, B^+, \ldots) \quad \hat{B} = \text{diag}(B, B, \ldots) \]

There is no problem to see that all thermodynamic quantities calculated with operators \( \hat{B}^+, \hat{B} \) are exactly the same as ones calculated with the original operators \( B^+, B \). Indeed, for example,

\[
< \hat{B}^+ \hat{B} > = \frac{Sp(\hat{B}^+ \hat{B} e^{-\beta(E-\mu)} B^+ B)}{Sp(e^{-\beta(E-\mu)B^+ B})}
\]
coincides with the same expressions but without hats due to the block structure of our operators (we should add that only the partition functions differ by an infinite numerical constant which does not affect observable physical quantities). The conclusion will be kept if we start with a lattice of truncated oscillators and then introduce hats for the operators.

In the previous section we have found the corresponding expressions for the creation and annihilation operators $\hat{B}^+, \hat{B}$ in terms of the bosonic creation and annihilation operators $b^+, b$ acting in the Hilbert space $\mathcal{H}_b$. They are given by formula (8) with the square root or by formula (8) without it (which we will use below).

The formulae considered above in this letter can be applied to construct the Hamiltonian of the auxiliary bosonic system. So if we start with the following Hamiltonian $H_t$ of truncated oscillators on a lattice:

$$H_t = \sum_i \Delta B_i^+ B_i + \sum_{i \neq j} M_{i,j} B_i^+ B_j + \sum_{i \neq j} (L_{i,j} B_i^+ B_j^+ + \text{h.c.}) + \sum_{i \neq j} J_{i,j} B_i^+ B_i B_j^+ B_j.$$ 

then the corresponding Hamiltonian of the auxiliary bosons has the form:

$$H = \sum_i \Delta \sum_{l=0}^{\infty} a(l)(b_i^+)^{l+1}b_i^{l+1} + \sum_{i \neq j} M_{i,j} b_i^+ S_{i,j} b_j + \sum_{i \neq j} (L_{i,j} b_i^+ b_j^+ S_{i,j} + \text{h.c.}) + \sum_{i \neq j} J_{i,j} \sum_{l,m=0}^{\infty} a(l)a(m)(b_i^+)^{l+1}(b_j^+)^{m+1}b_i^{l+1}b_j^{m+1}.$$ 

Here the following notations have been introduced:

$$S_{i,j} = \sum_{l,m=0}^{\infty} A(l)A(m)(b_i^+)^l(b_j^+)^m b_i^l b_j^m,$$

and coefficients $A(l)$ and $a(l)$ are defined as:

$$A(l) \equiv \sum_{k=0}^{\min(m-2,l)} \sum_{n=0}^{\left[ \frac{m}{2} \right]} (-1)^{l-mn-k} \frac{\sqrt{k+1}}{(l - mn - k)! (mn + k)! \sqrt{mn + k + 1}}$$

and

$$a(l) \equiv \frac{(-1)^{l+1} m-1}{(l+1)!} \sum_{k=1}^{m-1} q^k (1 - q^k)^l.$$ 

Let us note once more that the system with Hamiltonian $H$ is equivalent to the original Hamiltonian of truncated bosons $H_t$ and does not require any additional constraint.

Using the standard procedure, we can put down the functional integral representation of the partition function and correlators of the auxiliary bosonic system and the original system of truncated oscillators. For example, according to the definition and the formula (8), the following relations take place

$$Z \equiv Sp(e^{-\beta H}) = \int Db^+ (\tau) Db (\tau) e^S$$

$$< B_i^+ B_j > = \int Db^+ (\tau) Db (\tau) \sum_{l,m=0}^{\infty} A(l)A(m)(b_i^+ (\tau))^{l+1}(b_j^+ (\tau))^{m+1}b_i^l (\tau)b_j^{m+1}(\tau)e^S / Z,$$
where the action $S$ is defined by the form of the Hamiltonian $H$:

$$
S = \int_0^\beta \left( \sum_i \frac{\partial b_i^+(\tau)}{\partial \tau} b_i(\tau) - \sum_i \Delta \sum_{l=0}^\infty a(l)(b_i^+(\tau))^{l+1}b_i^{l+1}(\tau) + \sum_{i\neq j} M_{i,j}b_i^+(\tau)S_{ij}(\tau)b_j(\tau) + \sum_{i\neq j} (L_{i,j}b_i^+(\tau)b_j^+(\tau)S_{ij}(\tau) + h.c.) + \sum_{i\neq j} J_{i,j} \sum_{l,m=0}^\infty a(l)a(m)(b_i^+(\tau))^{l+1}(b_j^+(\tau))^{m+1}b_i^{l+1}(\tau)b_j^{m+1}(\tau) \right) d\tau .
$$

All other correlators can be obtain in the same manner and give us the bosonic functional integral representation which is free of constraints and limiting procedures (how it would be if we considered an analog of the hard-core interaction on site and took a limit). As usually, the functional integral form allows the simplest approach to the derivation of diagram technique rules which are standard ones for the problems in question. It is tempting to note that such technique is much less complicated and much more straightforward than spin operator one and is very natural for the consideration of problems concerning to the Bose-condensation (Long Range Order) in the system.

4 Conclusion

In this letter we considered the generalization of Agranovich-Toshich representation for the creation and annihilation operators of truncated oscillators in terms of auxiliary bosons. This allowed us to formulate the model of interacting bosons with the equivalent thermodynamic behaviour and express various correlators of a truncated oscillator system through series of correlators of interacting bosons. It is important that such description is free of constraints or limiting procedures which take place in other approaches.

Moreover, this technique can be applied to the high spin systems or Hubbard-like models using the obvious transformation of truncated oscillator operators to the corresponding spin operators or the Hubbard operators. This allows to escape the complicated operator technique and make use only standard one.

However, we have to note the difficulties which arise in this framework. Indeed, we have the infinite series of types of interactions which leads to the infinite series of the various vertices in diagrams. What contribution is relevant has to be determined by the concrete physical problem where several assumptions about the structure of a ground state and excitations have to be chosen. This common problem takes place for any perturbation theory. We think that the technique described above could be convenient in the consideration of questions concerning the existence of Long Range Order in a system which is equivalent to the Bose-condensation. Indeed, the interacting boson picture seems to be most natural for such investigation.

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