CONSTANT MEAN CURVATURE AND TOTALLY UMBILICAL BIHARMONIC SURFACES IN 3-DIMENSIONAL GEOMETRIES

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Abstract

We prove that a totally umbilical biharmonic surface in any 3-dimensional Riemannian manifold has constant mean curvature. We use this to show that a totally umbilical surface in Thurston’s 3-dimensional geometries is proper biharmonic if and only if it is a part of $S^2(1/\sqrt{2})$ in $S^3$. We also give complete classifications of constant mean curvature proper biharmonic surfaces in 3-dimensional geometries and in 3-dimensional Bianchi-Cartan-Vranceanu spaces, and a complete classifications of proper biharmonic Hopf cylinders in 3-dimensional Bianchi-Cartan-Vranceanu spaces.

1. Introduction and preliminaries

We assume that all manifolds, maps, tensor fields and other objects studied in this paper are smooth.

A map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is biharmonic map if $\varphi|\Omega$ is a critical point of the bienergy

$$E^2 (\varphi, \Omega) = \frac{1}{2} \int_{\Omega} |\tau(\varphi)|^2 \, dx$$

for every compact subset $\Omega$ of $M$, where $\tau(\varphi) = \text{Trace}_g \nabla d\varphi$ is the tension field of $\varphi$. Locally, biharmonic maps are solutions of the following system of 4th order PDEs:

$$\text{Trace}_g (\nabla^\varphi \nabla^\varphi - \nabla^\varphi_M) \tau(\varphi) - \text{Trace}_g R^N (d\varphi, \tau(\varphi)) d\varphi = 0,$$
where $R^N$ denotes the curvature operator of $(N, h)$ defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$ 

As any harmonic map (the one with $\tau(\varphi) = 0$) is biharmonic we use the name proper biharmonic for those biharmonic maps which are not harmonic.

A submanifold is a biharmonic submanifold if the isometric immersion that defines the submanifold is a biharmonic map. Biharmonic submanifolds include minimal submanifolds as a subset as it is well known that an isometric immersion is harmonic if and only if it is minimal. We use proper biharmonic submanifolds to name those biharmonic submanifolds which are not minimal.

Many recent works in the geometric study of biharmonic maps have been focused on the following two fundamental problems: (1) existence problem: given two model spaces (e.g., some “good” spaces such as spaces of constant sectional curvature or more general symmetric or homogeneous spaces), does there exist a proper biharmonic map mapping one space into another? (2) classification problem: classify all proper biharmonic maps between two model spaces where the existence is known. Some typical and challenging classification problems are the following

Chen’s conjecture [CH]: any biharmonic submanifold in a Euclidean space is minimal, and

The generalized Chen’s conjecture: any biharmonic submanifold of $(N, h)$ with $\text{Riem}^N \leq 0$ is minimal (see e.g., [CMO1], [MO], [BMO1], [BMO2], [BMO3], [Ba1], [Ba2], [Ou1], [Ou2], [IIU]).

For some recent progress of classifications of biharmonic submanifolds we refer the readers to [CMO1], [CMO2], [BMO1], [BMO2], [BMO3], [MO], and the references therein.

In the recent paper [Ou1], the first named author of this paper derived the equation for biharmonic hypersurfaces in a generic Riemannian manifold which can be stated in the following

**Theorem 1.1.** [Ou1] Let $\varphi : M^m \rightarrow N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\mathbf{\eta} = H\mathbf{\xi}$. Then $\varphi$ is biharmonic if
and only if:
\[
\begin{align*}
\Delta H - H|A|^2 + H\text{Ric}^N(\xi, \xi) &= 0, \\
2A(\text{grad } H) + \frac{m}{2}\text{grad } H^2 - 2H(\text{Ric}^N(\xi))\top &= 0,
\end{align*}
\]
where $\text{Ric}^N : T_qN \to T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \text{Ric}^N(Z), W \rangle = \text{Ric}^N(Z, W)$ and $A$ is the shape operator of the hypersurface with respect to the unit normal vector $\xi$.

An nice application of the above equation was made in [OT] where the equation was used to determine a conformally flat metric on $\mathbb{R}^5$ so that a foliation by the hyperplanes defined by the graphs of linear functions becomes a proper biharmonic foliation. Those proper biharmonic hyperplanes were eventually used to construct counter examples to prove that the generalized Chen’s conjecture is false (see [OT] for details).

In this paper, we will use equation (1) to study biharmonic surfaces in Thurston’s 3-dimensional geometries. We first show that a totally umbilical biharmonic surface in any 3-dimensional Riemannian manifold has constant mean curvature. We then use this to show that the only totally umbilical proper biharmonic surfaces in 3-dimensional geometries is a part of $S^2(1/\sqrt{2})$ in $S^3$. We also show that the only constant mean curvature proper biharmonic surface in 3-dimensional geometries are a part of $S^2(1/\sqrt{2})$ in $S^3$ or a part of $S^1(1/\sqrt{2}) \times \mathbb{R}$ in $S^2 \times \mathbb{R}$, and the only constant mean curvature proper biharmonic surfaces in a 3-dimensional Bianchi-Cartan-Vranceanu space is apart of $S^2(1/\sqrt{2})$ in $S^3$ or a part of a Hopf cylinder in $S^2(1/(2\sqrt{m})) \times \mathbb{R}$ or $SU(2)$ whose base curve is a circle with radius $R = 1/\sqrt{8m-1}$ in the base sphere $S^2(1/\sqrt{m})$ identified with $\left(\mathbb{R}^2, h = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2}\right)$.

2. Constant mean curvature biharmonic surfaces in 3-dimensional geometries

It is well known that Thurston’s eight models for 3-dimensional geometries consist of : 3-dimensional space forms $\mathbb{R}^3$, $S^3, H^3$, the product spaces: $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$ and $\tilde{SL}(2, \mathbb{R})$, Nil, Sol.

It is also known (see, e.g., [BDI], [CMOP]) that Bianchi-Cartan-Vranceanu 3-dimensional spaces :
\[
M_{m,l}^3 = \left(\mathbb{R}^3, g = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + [dz + \frac{l}{2} \frac{ydx - xdy}{1 + m(x^2 + y^2)}]^2\right)
\]
include six of Thurston’s eight 3-dimensional geometries in the family except for the hyperbolic space $H^3$ and Sol.

As biharmonic surfaces in 3-dimensional space forms have been completely classified ([Ji], [CI], [CMO1]), we can obtain the classification of constant mean curvature biharmonic surfaces in 3-dimensional geometries by classifying constant mean curvature biharmonic surfaces in Bianchi-Cartan-Vranceanu 3-dimensional spaces and in Sol.

### 2.1 Constant mean curvature biharmonic surfaces in Bianchi-Cartan-Vranceanu 3-spaces

We adopt the following notations and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,$$

and the Riemannian and the Ricci curvatures:

$$R(X, Y, Z, W) = g(R(Z, W)Y, X),$$
$$\text{Ric}(X, Y) = \text{Trace}_g R = \sum_{i=1}^m R(Y, e_i, X, e_i) = \sum_{i=1}^m \langle R(X, e_i)e_i, Y \rangle.$$

For a Bianchi-Cartan-Vranceanu 3-space given in (2), one can easily check that the vector fields

$$E_1 = F \frac{\partial}{\partial x} - \frac{ly}{2} \frac{\partial}{\partial z}, \quad E_2 = F \frac{\partial}{\partial y} + \frac{lx}{2} \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z},$$

where $F = 1 + m(x^2 + y^2)$, form an orthonormal frame.

A straightforward computation shows that (see also [CMOP])

$$[E_1, E_2] = 2mx E_2 - 2my E_1 + lE_3,$$

all other $[E_i, E_j] = 0$, $i, j = 1, 2, 3$.

Let $\nabla$ denote the Levi-Civita connection of a 3-dimensional Bianchi-Cartan-Vranceanu space, then we can check (see also [BDI] and [CMOP]) that

$$\nabla_{E_i} E_j = 0, \quad i, j = 1, 2, 3.$$
A further computation (see also [BDI] and [CMOP]) gives the possible nonzero components of the curvatures:

\[
\begin{align*}
R_{1212} &= g(R(E_1, E_2)E_2, E_1) = 4m - \frac{3l^2}{4}, \\
R_{1313} &= g(R(E_1, E_3)E_3, E_1) = \frac{l^2}{4}, \\
R_{2323} &= g(R(E_2, E_3)E_3, E_2) = \frac{l^2}{4},
\end{align*}
\]

and the Ricci curvature:

\[
\begin{align*}
\text{Ric} (E_1, E_1) = \text{Ric} (E_2, E_2) &= 4m - \frac{l^2}{2}, \\
\text{Ric} (E_3, E_3) &= \frac{l^2}{2}, \\
\text{all other } \text{Ric} (E_i, E_j) &= 0, \ i \neq j.
\end{align*}
\]

**Theorem 2.1.** A constant mean curvature surface in a 3-dimensional Bianchi-Cartan-Vranceanu space is proper biharmonic if and only if it is a part of:

1. \(S^2(1/\sqrt{2m})\) in \(S^3(1/\sqrt{m})\), or
2. \(S^1((1/(2\sqrt{2m})) \times \mathbb{R}\) in \(S^2(1/(2\sqrt{m}) \times \mathbb{R}\), or
3. a Hopf cylinder in \(SU(2)\) with \(4m - l^2 > 0\) over a circle of radius \(R = 1/\sqrt{8m - l^2}\) in the base sphere \(S^2(\frac{1}{2\sqrt{m}})\) identified with \(\left(\mathbb{R}^2, h = \frac{dx^2+dy^2}{1+m(x^2+y^2)}\right)\).

**Proof.** If the mean curvature \(H\) is constant, then the biharmonic equation reduces to

\[
\begin{align*}
-\text{H|A|^2 + HRic}^N(\xi, \xi) &= 0, \\
H (\text{Ric}(\xi))^\top &= 0.
\end{align*}
\]

Let \(\{e_i = a_i^\gamma E_\gamma, \ \xi = c^\gamma E_\gamma, \ i = 1, 2\}\) be an orthonormal frame on the ambient space adapted to the surface with \(\xi\) being the unit normal vector field of the
surface. Then, a straightforward computation using (6) gives

\begin{equation}
(\text{Ric} (\xi))^T = \sum_{i=1}^{2} \text{Ric}(c^iE_{\gamma}, a_i^\gamma E_{\gamma})e_i
= \sum_{i=1}^{2} \left( 4m - \frac{l^2}{2} \right) \sum_{\gamma=1}^{2} c^i a_i^\gamma + \frac{l^2}{2} c^3 a_3^i \right] e_i
= (l^2 - 4m)c^3 \sum_{i=1}^{2} a_i^3 e_i,
\end{equation}

\begin{equation}
\text{Ric}^N(\xi, \xi) = \text{Ric}^N(c^iE_{\gamma}, c^\gamma E_{\gamma}) = \sum_{i=1}^{2} \left( 4m - \frac{l^2}{2} \right) (c^i)^2 + \frac{l^2}{2} (c^3)^2
= (4m - \frac{l^2}{2}) + (l^2 - 4m)(c^3)^2.
\end{equation}

Substituting (8) and (9) into (7) we conclude that the constant mean curvature surface is biharmonic if and only if

\begin{equation}
\begin{cases}
-H[|A|^2 - (4m - \frac{l^2}{2}) - (l^2 - 4m)(c^3)^2] = 0, \\
(l^2 - 4m)c^3 a_1^3 H = 0, \\
(l^2 - 4m)c^3 a_2^3 H = 0,
\end{cases}
\end{equation}

which has solution $H = 0$ meaning that the surface is minimal, or

\begin{equation}
\begin{cases}
|A|^2 - (4m - \frac{l^2}{2}) - (l^2 - 4m)(c^3)^2 = 0, \\
(l^2 - 4m)c^3 a_1^3 = 0, \\
(l^2 - 4m)c^3 a_2^3 = 0.
\end{cases}
\end{equation}

We can solve Equation (10) by considering the following cases:

**Case I:** $l^2 - 4m = 0$. In this case, $|A|^2 = \frac{l^2}{2}$, and the corresponding Bianchi-Cartan-Vranceanu 3-space is locally either $\mathbb{R}^3$ or $S^3(1/\sqrt{m})$ and, by the classification of biharmonic surfaces in 3-dimensional space form (see [Ji], [CL], [CMO1]), we conclude that in these cases, the only proper biharmonic surface is a part of $S^2(1/\sqrt{2m})$ in $S^3(1/\sqrt{m})$.

**Case II:** $l^2 - 4m \neq 0$. In this case, by the last two equations of (10), we have either $c^3 = 0$ or $c^3 \neq 0$. 

For Case II-A: \( c^3 = 0 \), we use the first equation of (10) to conclude that

\[
|A|^2 = 4m - \frac{r^2}{2}.
\]

Noting that \( c^3 = 0 \) means that the normal vector field of the surface \( \Sigma \) is always orthogonal to \( E_3 = \frac{\partial}{\partial z} \) so we can take an another orthonormal frame \{\( e_1 = aE_1 + bE_2 \), \( e_2 = E_3 \), \( \xi = bE_1 - aE_2 \)\} adapted to the surface with \( a^2 + b^2 = 1 \) and \( \xi \) being the unit normal vector field. Using (4) we can compute (see also [Vd] Example 3.4.1)

\[
\begin{align*}
\nabla_{e_1}\xi &= \{ae_1(b) - be_1(a) + 2m(ay - bx)\}e_1 - \frac{l}{2}e_2, \\
\nabla_{e_2}\xi &= \{ae_2(b) - be_2(a) - \frac{l}{2}\}e_1.
\end{align*}
\]

A further computation gives the second fundamental form of the surface with respect to the chosen adapted orthonormal frame:

\[
\begin{align*}
\h(e_1, e_1) &= -(\nabla_{e_1}\xi, e_1) = -(ae_1(b) - be_1(a) + 2m(ay - bx)), \\
\h(e_1, e_2) &= -(\nabla_{e_1}\xi, e_2) = \frac{l}{2}, \\
\h(e_2, e_1) &= -(\nabla_{e_2}\xi, e_1) = \frac{l}{2} - ae_2(b) + be_2(a), \\
\h(e_2, e_2) &= 0,
\end{align*}
\]

It follows from (13), the symmetry \( \h(e_1, e_2) = \h(e_2, e_1) \), and \( 0 = \h(e_2(a^2 + b^2) = 2ae_2(a) + 2be_2(b) \) that

\[
\begin{align*}
\h(e_2(a) = \h(e_2(b) = 0,
\end{align*}
\]

which means the functions \( a, b \) are constant along the fibers of the Riemannian submersion

\[
\pi : M^3_{m,l} \longrightarrow \left( \mathbb{R}^2, h = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)} \right), \quad \pi(x, y, z) = (x, y).
\]

On the other hand, it is not difficult to see that the integral curves of \( e_2 \) are geodesics on the surface \( \Sigma \). It follows from a well-known fact in the differential geometry of surfaces that we can parametrize \( \Sigma \) by \( r = r(u, v) \) so that the \( u \) \(-\) \( curves \) are the integral curves of \( e_2 \) and the \( v \) \(-\) \( curves \) are the orthogonal trajectories of the \( u \) \(-\) \( curves \). Let \( \gamma : I \longrightarrow M^3_{m,l}, \gamma = \gamma(s) \) be a \( v \) \(-\) \( curve \) on the surface with arclength parameter, then it is horizontal with respect to the Riemannian submersion \( \pi \). Let \( \alpha(s) = \pi(\gamma(s)) \) be the curve in the base space of the Riemannian submersion, then the surface \( \Sigma \) can be viewed as \( \cup_{s \in I} \pi^{-1}(\alpha(s)) \), a Hopf cylinder over the curve \( \alpha(s) \subset \left( \mathbb{R}^2, h = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)} \right) \).

If we write \( \alpha(s) = (x(s), y(s)) \), then the surface \( \Sigma \) can be parametrized as \( r(s, t) = (x(s), y(s), t) \) since the fiber of \( \pi \) over a point \((x_0, y_0)\) is \( \pi^{-1}(x_0, y_0) = \{(x, y, t)|t \in \mathbb{R}\} \).
Let \( \kappa_g \) denote the geodesic curvature of the base curve. Then, we can use the Frenet formula to check (see also [Ve], Example 3.4.1) that \( \kappa_g = -(ae_1(b) - be_1(a) + 2m(ay - bx)) \). It follows from Equations (13) and (14) that
\[
|A|^2 = \kappa_g^2 + l^2/2, \quad H = \kappa_g/2.
\]
Combining (16) and (11) we have
\[
|A|^2 = \kappa_g^2 + l^2/2, \quad H = \kappa_g/2.
\]
To have a geometric characterization of the base curve we first notice that the proper biharmonicity \( (H^2 > 0) \) of the Hopf cylinder implies that \( m > 0 \). It follows that the potential BCV space has to be either \( S^2(1/(2\sqrt{m}) \times \mathbb{R} \) or \( SU(2) \). Using the well-known curvature relation (the O'Neill’s formula) of the Riemannian submersion \( 15 \) we conclude that the base space must have positive curvature \( 4m \) so it can be viewed as a sphere \( S^2(\frac{1}{2\sqrt{m}}) \). As the curve in the base sphere \( S^2(\frac{1}{2\sqrt{m}}) \) has constant geodesic curvature \( \kappa_g = \sqrt{4m - l^2} \) one can check that this curve, viewed as a curve in Euclidean 3-space of which \( S^2(\frac{1}{2\sqrt{m}}) \) is a subset, has curvature \( \kappa = \sqrt{\kappa^2_n + \kappa^2_g} = \sqrt{8m - l^2} \) and torsion \( \tau = -\frac{2\sqrt{n}\kappa'}{\kappa n\kappa_g} = 0. \)

From this we conclude that the base curve of the Hopf cylinder is a circle on \( S^2(\frac{1}{2\sqrt{m}}) \) with radius \( \frac{1}{\sqrt{8m - l^2}} \). In particular, when \( l = 0 \) we obtain the Hopf cylinder \( S^1((1/(2\sqrt{2m})) \times \mathbb{R} \) in \( S^2(1/(2\sqrt{m}) \times \mathbb{R} \), and when \( l = 0, m = 1/4 \) the proper Hopf cylinder \( S^1(1/\sqrt{2}) \times \mathbb{R} \) found in [Ou1].

For Case II-B: \( c^3 \neq 0 \), we use the last two equations of (10) to conclude that \( a_1^3 = a_2^3 = 0 \). It follows that \( \text{Span}\{e_1, e_2\} = \text{Span}\{E_1, E_2\} \). This means the distribution determined by \( \{E_1, E_2\} \) is integrable and hence (by Frobeniu’s theorem) is involutive. It follows from (3) that \( l = 0 \). It also follows that \( \xi = \pm E_3 \) and hence \( c^3 = \pm 1 \). Substituting \( l = 0 \) and \( c^3 = \pm 1 \) into the first equation of (10) we obtain \( |A|^2 = 0 \) which means that the surface is totally geodesic.

Combining the results proved above we obtain the theorem.

\[ \square \]

Remark 1. It is interesting to know that it is shown in [In] that there exists proper biharmonic Hopf cylinder in a Sasakian 3-manifold of constant holomorphic sectional curvature \( c = 4m - 3 > 1 \) which, according to [Ta], is isometric to a Bianchi-Cartan-Vranceanu space with \( l = 2 \), i.e., a \( SU(2) \). On the other hand, using a different method, the authors in [FO] gave an explicit equation of a Hopf cylinder in Bianchi-Cartan-Vranceanu space with \( l = 2 \) and \( c = 4m - 3 > 1 \).
Our results show that this is the only proper biharmonic surface in $SU(2)$ with constant mean curvature. Clearly, the proper biharmonic Hopf cylinder is not totally umbilical.

Now we are ready to prove the following theorem which gives a complete classification a proper biharmonic cylinder in Bianchi-Cartan-Vranceanu spaces.

**Theorem 2.2.** Let $\pi : M^3_{m,l} = \left( \mathbb{R}^3, g = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)} + dz + \frac{1}{2} \frac{ydy - xdx}{1 + m(x^2 + y^2)} \right) \rightarrow (\mathbb{R}^2, h = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)})$, $\pi(x, y, z) = (x, y)$ be a Riemannian submersion. Let $\alpha : I \rightarrow (\mathbb{R}^2, h = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)})$ be an immersed regular curve parametrized by arclength. Then the Hopf cylinder $\Sigma = \cup_{s \in I} \pi^{-1}(\alpha(s))$ is a proper biharmonic surface in Bianchi-Cartan-Vranceanu space if and only if it is a part of:

1. $S^1((1/(2\sqrt{m})) \times \mathbb{R}$ in $S^2(1/(2\sqrt{m}) \times \mathbb{R}$, or
2. a Hopf cylinder in $SU(2)$ with $4m - l^2 > 0$ over a circle of radius $R = 1/\sqrt{8m - l^2}$ in the base sphere $S^2(\frac{1}{2\sqrt{m}})$ identified with $(\mathbb{R}^2, h = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)})$.

**Proof.** Let $\alpha : I \rightarrow (\mathbb{R}^2, h = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)}) = S^2(\frac{\sqrt{m}}{2m})$ with $\alpha(s) = (x(s), y(s))$ be an immersed regular curve parametrized by arclength with the geodesic curvature $\kappa_g$. As in [On1], we can take the horizontal lifts of the tangent and the principal normal vectors of the curve $\alpha$: $X = \frac{\sqrt{m}}{F} E_1 + \frac{\sqrt{m}}{F} E_2$ and $\xi = \frac{\sqrt{m}}{F} E_1 - \frac{\sqrt{m}}{F} E_2$ (where $F = 1 + m(x^2 + y^2)$) together with $V = E_3$ to be an orthonormal frame adapted to the Hopf cylinder. A straightforward computation gives:

\[
\begin{align*}
\text{Ric}(\xi, \xi) &= (4m - \frac{l^2}{2})(\frac{x'^2 + y'^2}{F^2}) = 4m - \frac{l^2}{2}, \\
\text{Ric}(\xi, X) &= (4m - \frac{l^2}{2})(\frac{-x'y' + x'x'}{F^2}) = 0, \\
\text{Ric}(\xi, V) &= \text{Ric}(\frac{\sqrt{m}}{F} E_1 - \frac{\sqrt{m}}{F} E_2, E_3) = 0,
\end{align*}
\]

and the torsion of the lifting curve $\pi^{-1}(\alpha(s))$

\[
\tau_g = - (\nabla_X V, \xi) = - (\nabla_{\frac{\sqrt{m}}{F} E_1 + \frac{\sqrt{m}}{F} E_2} E_3, \frac{y'}{F} E_1 - \frac{x'}{F} E_2) = - \frac{l}{2},
\]

Substituting (18) and (19) into Equation (16) in [On1], we have

\[
\begin{align*}
\kappa_g'' - \kappa_g^3 + (4m - l^2)\kappa_g &= 0, \\
3\kappa_g \kappa_g' &= 0, \\
- \frac{l}{2} \kappa_g' &= 0.
\end{align*}
\]

Solving Equation (20) we have $\kappa_g = 0$ which gives the minimal surface $\Sigma = \cup_{s \in I} \pi^{-1}(\alpha(s))$, or $\alpha$ has constant geodesic curvature $\kappa_g^2 = 4m - l^2$. It follows from [On1] (page 229) that the mean curvature of the Hopf cylinder is given by
\[ H = \frac{\kappa_2}{2} \text{ and } |A|^2 = \kappa_2^2 + 2\tau^2 = 4m - \frac{\ell^2}{2} = \text{constant}. \]

From these we conclude that the Hopf cylinder \( \Sigma = \bigcup_{s \in I} \pi^{-1}(\alpha(s)) \) is proper biharmonic if only if

\[
\begin{align*}
H^2 &= \frac{4m-\ell^2}{4} > 0, \\
|A|^2 &= 4m - \frac{\ell^2}{2} > 0.
\end{align*}
\]

It follows from (21) that \( m > 0 \) and hence the potential Bianchi-Cartan-Vranceanu space is either \( S^2(1/(2\sqrt{m}) \times \mathbb{R} \) or \( SU(2) \) with \( m > 0 \). Applying our characterizations of Hopf cylinders in \( S^2(1/(2\sqrt{m}) \times \mathbb{R} \) or \( SU(2) \) given in Theorem 2.1 we obtain the Theorem. \( \square \)

2.2 Constant mean curvature biharmonic surfaces in Sol space

Let \((\mathbb{R}^3, g_{Sol})\) denote Sol space, where the metric can be written as \( g_{Sol} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2 \) with respect to the standard coordinates \((x, y, z)\) in \( \mathbb{R}^3 \). One can easily check that an orthonormal frame on Sol space can be chosen to be:

\[
E_1 = e^{-z} \frac{\partial}{\partial x}, \quad E_2 = e^z \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.
\]

With respect to this orthonormal frame, the Lie brackets and the Levi-Civita connection can be easily computed as:

\[
\begin{align*}
[E_1, E_2] &= 0, \quad [E_2, E_3] = -E_2, \quad [E_1, E_3] = E_1, \\
\nabla_{E_1} E_1 &= -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1, \\
\nabla_{E_2} E_1 &= 0, \quad \nabla_{E_2} E_2 = E_3, \quad \nabla_{E_2} E_3 = -E_2, \\
\nabla_{E_3} E_1 &= 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.
\end{align*}
\]

A further computation gives

\[
\begin{align*}
R(E_1, E_2)E_1 &= -E_2, \quad R(E_1, E_3)E_1 = E_3, \quad R(E_1, E_2)E_2 = E_1, \\
R(E_2, E_3)E_2 &= E_3, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_3)E_3 = -E_2,
\end{align*}
\]

and the possible nonzero components of the Riemannian curvature:

\[
\begin{align*}
R_{1212} &= g(R(E_1, E_2)E_2, E_1) = 1, \\
R_{1313} &= g(R(E_1, E_3)E_3, E_1) = -1, \\
R_{2323} &= g(R(E_2, E_3)E_3, E_2) = -1.
\end{align*}
\]

The Ricci curvature has components:

\[
Ric(E_3, E_3) = -2, \quad Ric(E_1, E_1) = Ric(E_2, E_2) = 0.
\]

Proposition 2.3. A constant mean curvature surface in Sol space is biharmonic if and only if it is minimal.
Proof. Let \( \{ e_1 = a^i E_i, \ e_2 = b^i E_i, \ \xi = c^i E_i \} \) be an adapted orthonormal frame with \( \xi \) being normal to the surface. Use the Ricci curvature (22) we have \( \text{Ric}(\xi,\xi) = -2(c^3)^2, \ \text{(Ric}(\xi))^{\top} = -2c^3a^3E_1 - 2c^3b^3E_2 \). From these together with biharmonic surface equation and the assumption that the mean curvature \( H \) is constant we conclude that the surface is biharmonic if and only if
\[
\begin{align*}
-H[|A|^2 + 2(c^3)^2] &= 0, \\
-2c^3a^3H &= 0, \\
-2c^3b^3H &= 0,
\end{align*}
\]
which has solution \( H = 0 \) meaning that the surface is minimal, or
\[
\begin{align*}
|A|^2 + 2(c^3)^2 &= 0, \\
c^3a^3 &= 0, \\
c^3b^3 &= 0.
\end{align*}
\]
Solving Equations (23) we have \( c^3 = 0 \) and \( |A|^2 = 0 \), which implies the surface is minimal. Thus, we obtain the proposition. \( \square \)

Corollary 2.4. The only constant mean curvature proper biharmonic surfaces in Thurston’s 3-dimensional geometries are a part of \( S^2(1/\sqrt{2}) \) in \( S^3 \), or a part of \( S^1(1/\sqrt{2}) \times \mathbb{R} \) in \( S^2 \times \mathbb{R} \).

Proof. By the classification results of \[ j, \ C1 \] and \[ CMO2 \], the only proper biharmonic surface in space forms \( \mathbb{R}^3, H^3 \) and \( S^3 \) is a part of \( S^2(1/\sqrt{2}) \) in \( S^3 \). It follows from Theorem 2.1 and Proposition 2.3 that the only constant mean curvature proper biharmonic surface in \( S^2 \times \mathbb{R}, \ H^2 \times \mathbb{R}, \ SL(2,\mathbb{R}), \text{Nil, and Sol spaces} \) is a part of \( S^1(1/\sqrt{2}) \times \mathbb{R} \) in \( S^2 \times \mathbb{R} \). Combining these we obtain the corollary. \( \square \)

3. Totally umbilical biharmonic surfaces in 3-dimensional geometries

In this section, we first prove that a totally umbilical biharmonic surface in any 3-dimensional Riemannian manifold must have constant mean curvature. We then use this theorem to show that the only totally umbilical proper biharmonic surface in 3-dimensional geometries is a part of \( S^2(1/\sqrt{2}) \) in \( S^3 \).

Theorem 3.1. A totally umbilical biharmonic surface in 3-dimensional Riemannian manifolds must have constant mean curvature.
Proof. Take an orthonormal frame \( \{e_1 = a^i E_i, e_2 = b^i E_i, \xi = c^i E_i \} \) of 3-dimensional Riemannian manifold adapted to the surface \( M \) such that \( Ae_i = \lambda_i e_i \), where \( A \) is the Weingarten map of the surface and \( \lambda_i \) is the principal curvature in the direction \( e_i \). Since \( M \) is supposed to be totally umbilical, i.e., all principal normal curvatures at any point of \( M \) are equal to the same number \( \lambda \). It follows that

\[
H = \frac{1}{2} \sum_{i=1}^{2} \langle Ae_i, e_i \rangle = \lambda, \tag{24}
\]

\[
A(\text{grad} H) = A(\sum_{i=1}^{2} (e_i \lambda) e_i) = \frac{1}{2} \text{grad} \lambda^2, \tag{25}
\]

\[
|A|^2 = 2\lambda^2. \tag{26}
\]

On the other hand, a straightforward computation gives

\[
\langle R(e_1, e_2) e_1, \xi \rangle = R(\xi, e_1, e_1, e_2) = -\text{Ric}(e_2, \xi), \tag{27}
\]

\[
\langle R(e_1, e_2) e_2, \xi \rangle = R(\xi, e_2, e_1, e_2) = \text{Ric}(e_1, \xi). \tag{28}
\]

Noting that \( e_1, e_2 \) are principal directions with principal curvature \( \lambda \) we can check that

\[
(\nabla_{e_1} h)(e_2, e_1) = e_1(h(e_2, e_1)) - h(\nabla_{e_1} e_2, e_1) - h(\nabla_{e_1} e_1, e_2) = -h(e_1, e_1) \langle \nabla_{e_1} e_2, e_1 \rangle - h(e_2, e_2) \langle \nabla_{e_1} e_1, e_2 \rangle = -\lambda(\langle \nabla_{e_1} e_2, e_1 \rangle + \langle \nabla_{e_1} e_1, e_2 \rangle) = 0, \tag{29}
\]

\[
(\nabla_{e_2} h)(e_1, e_1) = e_2(h(e_1, e_1)) - h(\nabla_{e_2} e_1, e_1) - h(\nabla_{e_2} e_1, e_1) = e_2(\lambda), \tag{30}
\]

\[
(\nabla_{e_1} h)(e_2, e_2) = e_1(h(e_2, e_2)) - h(\nabla_{e_1} e_2, e_2) - h(\nabla_{e_1} e_2, e_2) = e_1(\lambda), \tag{31}
\]

and

\[
(\nabla_{e_2} h)(e_1, e_2) = e_2(h(e_1, e_2)) - h(\nabla_{e_2} e_1, e_2) - h(\nabla_{e_2} e_2, e_1) = 0. \tag{32}
\]

Using (25), (26), (27), (28), (29), (30) and the Codazzi equation for a hypersurface:

\[
(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = (R^N(X, Y)Z) = (R^N(X, Y)Z, \xi), \tag{33}
\]

where the covariant derivative of the second fundamental form \( h \) is defined by

\[
(\nabla h)(X, Y, Z) = X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_Z Z) , \tag{34}
\]
we have

\[
\begin{align*}
&\{ e_1(\lambda) = \text{Ric}(e_1, \xi), \\
&e_2(\lambda) = \text{Ric}(e_2, \xi) \}.
\end{align*}
\]

On the other hand, using (24) and the second equation of (1) we have

\[2\lambda \text{grad} \lambda - \lambda (\text{Ric}(\xi, e_1)e_1 + \text{Ric}(\xi, e_2)e_2) = 0.\]

Substituting Equation (31) into (32) we have

\[\lambda \text{grad} \lambda = 0,\]

from which we conclude that \(\lambda\) is a constant. Thus, we obtain the theorem. \(\Box\)

Note that to classify totally umbilical biharmonic surfaces in 3-dimensional geometries it is enough to know totally umbilical biharmonic surfaces in Bianchi-Cartan-Vranceanu 3-spaces and in the hyperbolic 3-space and Sol space. As biharmonic surfaces in 3-dimensional space forms have been classified we need only to classify totally umbilical biharmonic surfaces in 3-dimensional Bianchi-Cartan-Vranceanu spaces and in Sol space. This is done by the following two corollaries.

**Corollary 3.2.** A totally umbilical biharmonic surface in Sol space is biharmonic if and only if it is minimal.

*Proof.* This is a consequence of Theorem 3.1 and Proposition 2.3. \(\Box\)

Remark 2. Note that there are many totally umbilical surfaces in Sol space (see [ST] for classifications of totally umbilical surfaces in Sol space and in a more general homogeneous 3-manifold).

**Corollary 3.3.** A totally umbilical surface in a 3-dimensional Bianchi-Cartan-Vranceanu space is proper biharmonic if and only if it is part of \(S^2(1/\sqrt{2m})\) in \(S^3(1/\sqrt{m})\).

*Proof.* By Theorem 3.1 a totally umbilical biharmonic surface in a 3-dimensional Bianchi-Cartan-Vranceanu space has constant mean curvature. This, together with Theorem 2.1 implies that the only potential totally umbilical proper biharmonic surface in these spaces are a part of \(S^2(1/\sqrt{2m})\) in \(S^3(1/\sqrt{m})\), or a part of a Hopf cylinder. As the latter surface is clearly not totally umbilical we conclude. \(\Box\)
We remark that totally umbilical surfaces in Bianchi-Cartan-Vranceanu spaces with four-dimensional isometry group has been classified in [Ve] whilst a classification of such surfaces in other three-dimensional homogeneous spaces has not yet appeared in the literatures (see [Ve]).

Now we can summarize our classification of totally umbilical biharmonic surfaces in Thurston’s 3-dimensional geometries in the following

**Theorem 3.4.** A totally umbilical surface in 3-dimensional geometries is proper biharmonic if and only if it is a part of $S^2(1/\sqrt{2})$ in $S^3$.

**Proof.** Recall that the eight 3-dimensional geometries are: $\mathbb{R}^3$, $S^3$, $H^3$, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, Sol, Nil, and $SL(2,\mathbb{R})$. It is well known (see [Ji], [CI], and [CMO2]) that there is no proper biharmonic surface in $\mathbb{R}^3$, $H^3$ and that (see [CMO1]) the only proper biharmonic surface in $S^3$ is a (part of) sphere $S^2(1/\sqrt{2})$. These, together with Corollaries 3.2 and 3.3, give the complete classification. □

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