Convolution-multiplication identities for Tutte polynomials of matroids

Joseph P. S. Kung
Department of Mathematics,
University of North Texas, Denton, TX 76203, U.S.A.
e-mail: kung@unt.edu

Abstract. We give a general multiplication-convolution identity for the multivariate
and bivariate rank generating polynomial of a matroid. The bivariate rank generating
polynomial is transformable to and from the Tutte polynomial by simple algebraic op-
erations. Several identities, almost all already known in some form, are specialization of
this identity. Combinatorial or probabilistic interpretations are given for the specialized
identities.

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1. A convolution-multiplication identity

This paper originated in an attempt to understand the relationship between Identities
7 and 8 in this paper. One way to do this is to find a hidden identity which contains
both identities as special cases. Identity 1 and its specialization, Identity 2, provide
candidates for such an identity. As the reader will see, the proofs of Identities 1 and 2,
and perhaps the identities themselves, are trivial. Identities have a way of receding to a
Zen state of triviality. In the other direction, we add meaning by giving combinatorial
or probabilistic interpretations for many identities derived from the hidden identity.

Let $M$ be a rank-$r$ matroid on the set $E$ with rank function $rk$ and $L(M)$ be its lattice
of closed sets or flats. If $M$ has no loops (that is, elements of rank zero), its characteristic
polynomial $\chi(M; \lambda)$ is the polynomial in the variable $\lambda$ defined by

$$
\chi(M; \lambda) = \sum_{X : X \in L(M)} \mu(\hat{0}, X)\lambda^{r-rk(X)},
$$

where $\mu$ is the Möbius function in the lattice $L(M)$. If $M$ has a loop, then $\chi(M; \lambda)$
is defined to be the zero polynomial. The size-corank polynomial $SC(M; x, \lambda)$ is the
polynomial in two variables, $x$ and $\lambda$ defined by

$$
SC(M; x, \lambda) = \sum_{A : A \subseteq E} x^{|A|}\lambda^{r-rk(A)}.
$$

The size-corank polynomial has a natural “polarized” multivariate generalization. Let
$\underline{x}$ be a labeled multiset $\{x_e : e \in E\}$ of variables or numbers, one for each element $e$
in the ground set $E$ of the matroid $M$. If $A \subseteq E$, define $\underline{x}^A$ to be the product $\prod_{e \in A} x_e$.
This notation is used analogously, so that, for example, $\underline{\hat{x}} = \{-x_e : e \in E\}$ and
$(-\underline{x})^A = \prod_{e \in A} (-x_e)$. In addition, if $\{y_e : e \in E\}$ is another multiset, then $\underline{x}y$ is the
set $\{x_e y_e : e \in E\}$ and $(\underline{x}y)^A = \prod_{e \in A} x_e y_e$. The subset-corank polynomial $SC(M; \underline{x}, \lambda)$
is defined by

$$\text{SC}(M; x, \lambda) = \sum_{A: A \subseteq E} x^A \lambda^{-\text{rk}(A)}.$$ 

The subset-corank polynomial specializes to the size-cornk polynomial when we set all the variables $x_e$ to the same variable $x$. Variants of the subset-corank polynomial, usually defined for graphs, have been rediscovered many times. The original discovery is by Fortuin and Kasteleyn in statistical mechanics. An almost complete list can be made by merging the lists in the surveys of Farr [4] and Sokal [14]. For matroids, the subset-corank polynomial is due to R.T. Tugger and it is sometimes named after her (see [11]).

The nullity-corank polynomial, usually known as the rank generating polynomial, $R(M; x, \lambda)$ is defined by

$$R(M; x, \lambda) = \sum_{A: A \subseteq E} x^{|A| - \text{rk}(A)} \lambda^{-\text{rk}(A)}.$$ 

The size-corank and nullity-corank polynomials are closely related: indeed,

$$x^{-r} \text{SC}(M; x, x \lambda) = R(M; x, \lambda)$$

and

$$\text{SC}(M; x, \lambda) = x^r R(M; x/x, \lambda).$$

Both polynomials specialize to the characteristic polynomial. Specifically,

$$\text{SC}(M; -1, \lambda) = \chi(M; \lambda), \quad R(M; -1, -\lambda) = (-1)^r \chi(M; \lambda).$$

The Tutte polynomial is defined to be the polynomial $R(M; x - 1, \lambda - 1)$. The three bivariate polynomials transform into each other by simple algebra. (Note, however, that we may need to divide; thus, problems may arise when we wish to set a variable to 0.) We shall choose the polynomial that gives the simplest or most general identities. Thus, despite the title, we shall work mostly with subset-corank or size-corank polynomials.

We begin with a general multiplication-convolution identity.

**Identity 1.** Let $M$ be a matroid on the set $E$, $x$ and $y$ be multisets of variables labeled by $E$, and $\lambda$ and $\xi$ be variables. Then

$$\text{SC}(M; xy, \lambda \xi) = \sum_{T: T \subseteq E} \lambda^{r-\text{rk}(T)} (-y)^T \text{SC}(M|T; x, \lambda) \text{SC}(M/T; y, \xi).$$

**Proof.** We use the fact that if $B$ and $A$ are subsets of $E$ such that $B \subseteq A$, then the sum

$$\sum_{T: B \subseteq T \subseteq A} (-1)^{|T|-|B|}$$

equals 0 except in the case $A = B$, when it equals 1. Then
\[
\text{SC}(M; xy, \lambda \xi) = \sum_{B, A : B, A \subseteq E} x^B \lambda^{r - \text{rk}(B)} y^A \xi^{r - \text{rk}(A)} \left[ \sum_{T : B \subseteq T \subseteq A} (-1)^{|T| - |B|} \right] \\
= \sum_{T : T \subseteq E} \lambda^{r - \text{rk}(T)} (-y)^T \left[ \sum_{B : B \subseteq T} (-x)^B \lambda^{r - \text{rk}(B)} \right] \left[ \sum_{A : T \subseteq A \subseteq E} y^A \xi^{(r - \text{rk}(T)) - (\text{rk}(A) - \text{rk}(T))} \right]
\]

\[
= \sum_{T : T \subseteq E} \lambda^{r - \text{rk}(T)} (-y)^T \text{SC}(M|T; -x, \lambda) \text{SC}(M/T; y, \xi).
\]

Setting \( x_e = x \), we obtain the following specialization of Identity 1.

**Identity 2.** Let \( x, y, \lambda, \xi \) be variables. Then

\[
\text{SC}(M; xy, \lambda \xi) = \sum_{T : T \subseteq E} \lambda^{r - \text{rk}(T)} (-y)^T \text{SC}(M|T; -x, \lambda) \text{SC}(M/T; y, \xi).
\]

Next we state, without proof, the analog of Identity 2 for the nullity-corank polynomial.

**Identity 3.**

\[
R(M; xy, \lambda \xi) = \sum_{T : T \subseteq E} \lambda^{r - \text{rk}(T)} (-y)^{|T| - \text{rk}(T)} R(M|T; -x, -\lambda) R(M/T; y, \xi).
\]

The algebraic operation dual to convolution is comultiplication. Thus, we can express the identities in this section in the language of coalgebras. However, we will wait until we have more than just a formal theory.

***Proof of Identity 3.***

2. Weighted sums of polynomials

Identities 1, 2, and 3 specialize to several known identities. We begin with identities expressing the size-corank or subset-corank polynomial as a weighted sum of other polynomials.

\footnote{This proof is included so that one can easily checked the identity. As indicated in the text, it will be removed in the final version.}
Identity 4.

\[ \text{SC}(M; xy, \xi) = \sum_{T \subseteq E} (x+1)^T y^T \text{SC}(M/T; -y, \xi). \]

Proof. Set \( y_e = -y_e, \lambda = 1, x_e = -x_e, \) leaving \( \xi \) unchanged, in Identity 1 to obtain

\[ \text{SC}(M; xy, \xi) = \sum_{T \subseteq E} y^T \text{SC}(M/T; -y, \xi) \]

In the last step in the derivation, we used a multivariate version of the binomial identity:

\[ \text{SC}(M/T; x, 1) = \sum_{B \subseteq T} \prod_{b \in B} x_b \]

\[ = \prod_{e \in T} (x_e + 1) = (x+1)^T. \]

Identity 4 specializes to more familiar identities.

Identity 5.

\[ \text{SC}(M; x, \xi) = \sum_{X \subseteq L(M)} (x+1)^X \chi(M/X; \xi), \]

\[ \text{SC}(M; x, \xi) = \sum_{X \subseteq L(M)} (x+1)^{|X|} \chi(M/X; \xi), \]

where the sums range over all closed sets \( X \) in the lattice \( L(M) \).

Proof. If \( T \) is not a closed set, then \( \text{SC}(M/T; -1, \xi) = 0 \) and if \( T \) is a closed set, then \( \text{SC}(M/T; -1, \xi) = \chi(M/T; \xi) \). Thus the range of the sums can be restricted to closed sets.

\[
R(M xy, \lambda \xi) \]

\[ = \sum_{B, A, B, A \subseteq E} x^{B \cap \{\text{rk}(B)\}} y^{A \cap \{\text{rk}(A)\}} \xi^{B \cap \{\text{rk}(A)\}} \left[ \sum_{T \subseteq E} (-1)^{|T|-|B|} \right]
\]

\[ = \sum_{T \subseteq E} x^{|B| \text{rk}(B)} y^{|A| \text{rk}(A)} \xi^{|A| \text{rk}(A)} \left[ \sum_{B, A \subseteq T} (-y)^{|B| \text{rk}(B)} (-\lambda)^{|A| \text{rk}(A)} \right]
\]

\[ = \sum_{T \subseteq E} \chi(x \text{rk}(T) - y \text{rk}(T)) R(M; x, -\lambda) R(M/T; y, \xi). \]
The bivariate form of Identity 5 is a fundamental identity of Tutte [15]. Tutte found it for graphs and Crapo [2] extended it to matroids. The multivariate form appeared in [11]. It has the following interpretation. The subset-corank polynomial encodes the rank function of the matroid $M$ in the sense that the rank of a set $A$ is $r - d$, where $d$ is the degree of $\lambda$ in the monomial $x^A \lambda^d$ in $SC(M; x, \lambda)$. On the other hand, $SC(M; x - 1, \xi)$ encodes the collection of closed sets of $M$ in the following way: a set $T$ is closed in $M$ if and only if the monomial $cx_T^{-1}$ occurs in $SC(M; x - 1, \xi)$ with a nonzero coefficient $c$. (The nonzero coefficient is $\chi(M/T; \lambda)$.) Thus, Identity 5 gives an “algebraic transformation” of the rank description of a matroid to its closed set description.

3. Random matroids

We next consider identities which give interpretations of size-corank or subset-corank polynomials as expected values of enumerative invariants of a random submatroid.

We begin with an identity which is an “order dual” of Identity 5. Setting $x_e = 1$, $\xi = 1$, $y_e = -y_e$, and leaving $\lambda$ unchanged in Identity 1, we obtain

$$SC(M; -y, \lambda) = \sum_{T: T \subseteq E} \lambda^{-rk(T)} y^T SC(M|T; -1, \lambda) SC(M/T; -y, 1).$$

Using the fact that $SC(M/T; -y, 1) = 1 - y^{E \setminus T}$, we obtain the following identity.

Identity 6.

$$SC(M; -y, \lambda) = \sum_{T: T \subseteq E} \lambda^{-rk(T)} \chi(M|T; \lambda) y^T (1 - y)^{E \setminus T}.$$  

The next identity, obtained by setting $x_e = -x_e$, $y_e = -y_e$, $\xi = 1$, leaving $\lambda$ unchanged, is a generalization of Identity 6.

Identity 7.

$$SC(M; xy, \lambda) = \sum_{T: T \subseteq E} \lambda^{-rk(T)} SC(M|T; x, \lambda) y^T (1 - y)^{E \setminus T}.$$  

Bivariate versions of Identities 6 and 7, stated for graphs and given interpretations in terms of random graphs, are known. See Welsh [17]. Specifically, Identity 6 is related to an identity in Vertigan’s Oxford thesis [16] and Welsh has given a proof using random subgraphs in [17]. A version of Identity 7 was found by Grimmett [7] (see also [17]).

The interpretations by random subgraphs (given in [17]) generalize easily to interpretations by random submatroids. Let $M$ be a matroid on the set $E$. We generate a random subset and hence, a random submatroid, of $M$ by deleting each element $e$ in $E$ independently and at random with probability $1 - p_e$. We need two somewhat artificial definitions, designed to make the interpretations work. Let $N$ be a rank-$s$ submatroid of the rank-$r$ matroid $M$. Then the normalized characteristic polynomial $\chi^N(N; \lambda)$ to be $\lambda^{r-s} \chi(N; \lambda)$. Similarly, the normalized size-corank polynomial $SC^N(N; x, \lambda)$ is defined to
be \( \lambda^r \cdot \text{SC}(N; x, \lambda) \). Since the probability that the subset \( T \) is chosen is \( p^T (1 - p)^{E \setminus T} \), it is clear that the expected value of the normalized characteristic (respectively, normalized size-corank polynomial) of a random submatroid of \( M \) equal \( \text{SC}(M; -p, \lambda) \) (respectively, \( \text{SC}(M; px, \lambda) \)).

In analogy with random graphs, one can develop a theory of random sets of vectors or points in a finite ambient space. This was done in Kelly and Oxley [9]. There are two choices for the ambient space: the (affine) vector space \( \text{GF}(q)^d \) or the projective space \( \text{PG}(d - 1, q) \) over \( \text{GF}(q) \). Here, \( q \) is a prime power and \( \text{GF}(q) \) is the finite field of order \( q \). As a matroid, the projective space \( \text{PG}(d - 1, q) \) is a simplification of \( \text{GF}(q)^d \).

In particular, the two ambient spaces have the same lattice \( L(d, q) \) of flats. The two ambient spaces give the same theory of random sets, more or less. Since probability theorists usually work with random matrices and vectors, we shall work with \( \text{GF}(q)^d \). We define a random set \( V(d, q, p) \) of vectors to be a random submatroid of \( \text{GF}(q)^d \) in which each element is chosen with the same probability \( p \).

**Lemma.**

\[
\text{SC}(\text{GF}(q)^d; x, \lambda) = \sum_{j=0}^{d} \binom{d}{j}_q (\lambda - 1)(\lambda - q) \cdots (\lambda - q^{d-1})(x + 1)^{q^j}
\]

\[
\sum_{d=0}^{\infty} \text{SC}(\text{GF}(q)^d; x, \lambda) \frac{z^d}{[d]!_q} = \left[ \prod_{d=0}^{\infty} \frac{1 + zq^d}{1 + \lambda zq^d} \right] \left[ \sum_{d=0}^{\infty} (x + 1)^{q^d} \frac{z^d}{[d]!_q} \right],
\]

where \( [d]_q = (1 - q)(1 - q^2) \cdots (1 - q^d) \) and \( \binom{d}{j}_q = \frac{[d]!_q}{[j]!_q [d - j]!_q} \).

**Proof.** The formula for \( \text{SC}(\text{GF}(q)^d; x, \lambda) \) follows easily from known counting formulas in finite vector spaces and Identity 5. It is also a special case of formulas in Mphako [13]. The generating function of \( \text{SC}(\text{GF}(q)^d; x, \lambda) \) factors into the product

\[
\left[ \prod_{d=0}^{\infty} (\lambda - 1)(\lambda - q) \cdots (\lambda - q^{d-1}) \frac{z^d}{[d]!_q} \right] \left[ \sum_{d=0}^{\infty} (x + 1)^{q^d} \frac{z^d}{[d]!_q} \right].
\]

The formula now follows from the \( q \)-binomial theorem, in the version stated in [5], Section 1.3. \( \square \)

Similar formulas exist for \( \text{SC}(\text{PG}(d - 1, q); x, \lambda) \). Simply replace the exponent \( q^j \) in \( x^{q^j} \) by \( q^{j-1} + q^{j-2} + \cdots + q + 1 \).

We next give two typical results about expected values of matroid invariants of random vectors. We shall need to assume some knowledge of critical problems (see, for example, [10]).

**Theorem.** (a) The expected number \( D(d, p, q, s) \) of \( s \)-tuples of linear functionals on \( \text{GF}(q)^d \) distinguishing a random set \( V(d, p, q) \) equals

\[
\sum_{k=0}^{s} \binom{d}{k}_q q^k (1 - p)^{q^r - k}
\]
(b) The expected number \( sp(d, p, q) \) of subsets spanning \( GF(q)^d \) in a random subset \( V(d, p, q) \) of vectors equals

\[
\sum_{k=0}^{d} \binom{d}{k} (-1)^k q^k (1 + p)^{q^{d-k}}.
\]

**Proof.** Since the number of \( s \)-tuples distinguishing a matroid \( M \) represented as a multiset of vectors in \( GF(q)^d \) is the normalized characteristic polynomial \( \chi^*(M; q^s) \) evaluated at \( q^s \),

\[
D(d, p, q, s) = SC(GF(q)^d; -p, \lambda).
\]

When \( \lambda = q^s \), the infinite product in the generating function telescopes into a finite product. Thus,

\[
\sum_{d=0}^{\infty} D(d, p, q, s) \frac{z^d}{[d]!} q^d = \left[ \prod_{d=0}^{s-1} (1 + zq^d) \right] \left[ \sum_{d=0}^{\infty} (1 - p)^{q^s} \frac{z^d}{[d]!} q^d \right].
\]

By an identity attributed to Euler or Cauchy (see, for example, [6], p. 254),

\[
\prod_{d=0}^{s-1} (1 - zq^d) = \sum_{k=0}^{s} (-1)^k \binom{d}{k} q^k z^k.
\]

The formula now follows from replacing the product with the sum and expanding.

For part (b), observe that \( SC(M|T; 1, 0) \) is the number of subsets \( A \) in \( T \) such that \( r - \text{rk}(A) = 0 \), that is, \( A \) spans \( M \). Setting \( \lambda = 0 \), \( y_e = p_e \), and \( x_e = 1 \) in Identity 7, we have

\[
sp(d, p, q) = SC(GF(q)^d; p, 0).
\]

From the generating function, in the form given in eqn (3), it follows that

\[
\sum_{d=0}^{\infty} sp(d, p, q) \frac{z^d}{[d]!} q^d = \left[ \sum_{d=0}^{\infty} (-1)^d q^d \frac{z^d}{[d]!} q^d \right] \left[ \sum_{d=0}^{\infty} (1 + p)^{q^s} \frac{z^d}{[d]!} q^d \right].
\]

Expanding the product yields the formula. This formula can also be obtained by Möbius inversion on the lattice \( L(d, q) \). \( \square \)

Identities 4 and 5 can also be given probabilistic interpretations. If \( M \) is a matroid on the set \( E \), we construct a random contraction \( H \) by choosing a random subset \( T \) by choosing each element \( e \) independently and at random with probability \( p_e \) and letting \( H = M/T \). A typical result is that the expected value of the characteristic polynomial \( \chi(H; \xi) \) of a random contraction is

\[
(1 - p)^E SC \left( M; \frac{p}{1-p} - 1; \xi \right).
\]

4. The motivating identities

We end by deriving the two identities which motivated this paper. We begin by setting \( x = -x, y = -1, \) and \( \lambda = 1 \), leaving \( \xi \) unchanged in Identity 3. Doing so, we obtain
the following convolutional identity, explicitly stated by Kook, Reiner, and Stanton in [8] and implicit in the combinatorial construction of Étienne and Las Vergnas in [3]:

\[ R(M; x, \xi) = \sum_{T: T \subseteq E} R(M|T; x, -1)R(M/T; -1, \xi). \] (4)

We can restrict the range of summation to flats for the same reason as in Identity 5. Because \( R(M/T; -1, \xi) \) is zero if \( M \) has isthmuses (or coloops), the range can be further restricted to cyclic flats, that is, flats with no isthmuses.

Next recall that

\[ R(M; x, -1) = R(M^\perp; -1, x) = (-1)^{|E| - r} \chi(M^\perp; -x), \]

where \( M^\perp \) is the orthogonal dual of \( M \). Thus, we obtain a version of the “KRSEV” identity which expresses the rank generating polynomial as a weighted-convolution of characteristic polynomials.

**Identity 8.**

\[ (-1)^r R(M; x, -\xi) = \sum_{X \in L^\circ(M)} (-1)^{|X|} \chi((M|X)^\perp; x)\chi(M/X; \xi), \]

where \( L^\circ(M) \) is the lattice of cyclic flats of \( M \).

Next, we set \( x = 1 \), \( y = -1 \), leaving \( \lambda \) and \( \xi \) unchanged in Identity 1, rederiving a multiplication identity in [12].

**Identity 9.**

\[ \chi(M; \lambda\xi) = \sum_{X \in L(M)} \lambda^{r - rk(X)} \chi(M|X; \lambda)\chi(M/X; \xi). \]

Kook, Reiner, and Stanton [8] have given an interpretation of Identity 8 for graphs using pairs of colorings and flows. One can easily adapt their interpretation using the critical problem. We will just give the result, referring the reader to [10], Section 4.7, for the necessary background.

Let \( M \) be a rank-\( r \) matroid on the set \( E \) and suppose that \( Q \) is an \( r \times |E| \) matrix over \( \text{GF}(s) \) representing \( M \) and \( R \) be an \( (|E| - r) \times |E| \) matrix over \( \text{GF}(|t|) \) representing the dual of \( M \). If \( T \subseteq E \), an \((s, t)\)-duet \((u, v)\) with support \( T \) is a pair of row vectors, such that \( u \) an \( |E|\) dimensional vector in the row space of \( Q \) with all coordinates in \( T \) equal to 0 and all coordinates in \( E \setminus T \) not equal to 0, and \( v \) is a \(|T|\) dimensional vector in the row space of the of \( R|T \), the submatrix of \( R \) consisting of all the columns labeled by \( T \), with all coordinates not equal to 0. Then

\[ R(M; t, s) = (-1)^r \sum_{(u, v)} (-1)^{|\text{support}(T)|}, \]

where the sum ranges over all \((s, t)\)-duets \((u, v)\).

Interpretations of Identity 9 for graphs and representable matroids can be found in [12].
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