Anomaly detection in static networks using egonets

Srijan Sengupta*

Abstract: Network data has rapidly emerged as an important and active area of statistical methodology. In this paper we consider the problem of anomaly detection in networks. Given a large background network, we seek to detect whether there is a small anomalous subgraph present in the network, and if such a subgraph is present, which nodes constitute the subgraph. We propose an inferential tool based on egonets to answer this question. The proposed method is computationally efficient and naturally amenable to parallel computing, and easily extends to a wide variety of network models. We demonstrate through simulation studies that the egonet method works well under a wide variety of network models. We obtain some interesting empirical results by applying the egonet method on several well-studied benchmark datasets.

Keywords: Egonet, Anomaly Detection, Subgraph Detection, Clique Detection, Statistical Network Analysis, Stochastic Blockmodel, Social Networks.

1 Introduction

We live in an increasingly interconnected world, where agents constantly interact with each other. This general agent-interaction framework describes many important systems, such as social interpersonal systems (Milgram 1967) and the World Wide Web (Huberman and Adamic 1999), to name a few. By denoting agents as nodes and their interconnections as edges, any such system can be represented as a network. Such networks provide a powerful and universal representation for analyzing a wide variety of systems spanning a remarkable range of scientific disciplines.

Fittingly, there has been a lot of recent emphasis in the literature towards developing methods for statistical analysis of network data. However, the problem of anomaly detection in networks has received relatively less attention from the statistics community.

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This paper is aimed at partially addressing this gap by formulating a statistical method for anomalous subgraph detection in static networks.

Hawkins (1980) defined an outlier as “an observation which deviates so much from other observations as to arouse suspicions that it was generated by a different mechanism.” This is an (intentionally, one would presume) open-ended definition where the keyword is “deviates”, and the deviant behavior of the outlier can manifest in many different ways. This is particularly true in the context of networks, where an anomalous subgraph of a large network can relate to many interesting graph properties, for example degrees, nodes that are too popular or unpopular, cliques, star subgraphs, etc. The process of anomaly detection depends on the specific kind of anomaly we are trying to detect. In this paper, we consider cliques as the network motif of interest. A clique is a (sub)graph where all pairs of nodes are connected. Note that cliques can form randomly in a network generated from a random graph model (Bollobás, 1998), and here we mean anomalous cliques that are unlikely to be formed in such fashion.

Anomalous clique detection is an important and relevant problem in many networked systems. In financial trading networks (Li et al., 2010; Pan et al., 2012), an anomalous clique can indicate a group of traders trading among each other to manipulate the stock market. Cliques can have important scientific interpretation in biological networks, e.g., in brain networks, neural cliques (Lin et al., 2006) are network-level memory coding units in the hippocampus. In social networks and online social media, an anomalous clique may be indicative of mechanisms of opinion manipulation or ideological echo chambers. Closer to home, identifying cliques in academic collaboration or citation networks can provide valuable insights into the research community.

We now describe the clique detection problem and provide a heuristic introduction to our proposed method with a simple example and visual illustration. Let $A$ be the symmetric adjacency matrix of a network where $A_{ij} = 1$ if nodes $i, j$ are connected, and $A_{ij} = 0$ otherwise. Consider the undirected Erdős-Rényi random graph model where for $i > j$, $A_{ij} \sim \text{Bernoulli}(p)$, $A_{ij} = A_{ji}$, and $A_{ii} = 0$. The top row of Figure 1 shows the heatmap of a network generated from this model with $n = 500$ nodes and $p = 0.05$ on the left, and the same network with an embedded 10-node clique on the right. We want a method that can perform the following tasks.

1. Detect whether a given network has an anomalous clique, and
2. If a clique is detected, identify the nodes forming the clique.

It would appear that these tasks are very similar to community detection (Fortunato, 2010).
The community detection problem has been very well studied in the recent statistics literature \cite{bickel2009consistent, rohe2011spectral, zhao2012consistency, jin2015spectral, bickel2015information} and an extensive set of statistical methods have been developed for community detection. However, community detection methods may not work well in detecting small anomalous cliques embedded in a large network.

To see this, let us consider only the second task of identifying the clique nodes when it has been detected that a clique is present. Note that the anomalous network (Fig. 1, top row, right) follows a stochastic blockmodel with $K=2$ communities, since the 10 clique nodes form a dense community of their own, and the remaining 490 nodes form the second, sparse community. Therefore, we should be able to identify the nodes in the two communities by using the spectral clustering method \cite{rohe2011spectral, qin2013spectral, sengupta2013two}.

The community detection results obtained by applying spectral clustering (both with and without regularization) on the anomalous network (Figure 1 top row, right panel) are given in Table 1. In both cases we obtained two large communities, and the 10 clique nodes cannot be identified within these communities. We interpret these results as a manifestation of the resolution limit phenomenon \cite{fortunato2007resolution} in community detection, the communities are too unbalanced (10 nodes vs 490 nodes) for spectral clustering to be accurate.

| Nodes  | With Regularization |  | Without Regularization |  |
|--------|---------------------|---|------------------------|---|
|        | Community 1 | Community 2 | Total | Community 1 | Community 2 | Total |
| Non-clique | 294     | 196     | 490 | 119     | 371     | 490 |
| Clique  | 0       | 10      | 10  | 10      | 0       | 10  |
| Total   | 294     | 206     | 500 | 129     | 371     | 500 |

Table 1: Spectral clustering with $K=2$ communities on the anomalous network, with regularization (left) and without regularization (right).

Note that the network heatmaps in Figure 1 are virtually identical, and the only reason the clique is at all visible in the right panel is that the clique consists of consecutively numbered nodes. It seems that it is hard to differentiate between the non-anomalous network and the anomalous network using network level metrics. Instead, such small anomalies might be detectable if we could “zoom in” to look at subgraph level metrics (e.g., degrees of all subgraphs). If we looked at all subgraphs, the clique would stand out on account of being much more densely connected than the rest of the network. However, it is computationally infeasible to compute all subgraph degrees as there are potentially
2^n sub-networks in a network with n nodes.

Instead, we propose looking at a special class of subgraphs, called egonets [Crossley et al., 2015]. The egonet of the i^{th} node is defined as the subgraph spanned by all neighbors of the i^{th} node. The adjacency matrix of the i^{th} egonet is given by \{A_{jk} : A_{ij} = 1, A_{ik} = 1\}. Since each node has exactly one egonet, there are n subgraphs to monitor, which is a relatively manageable number. For illustration, Fig. 1 (second row) shows the egonet heatmaps of an individual node from the non-anomalous network and the anomalous network. The difference between the two egonets is quite prominent, and one could formulate a decision rule to differentiate between them.

We propose a statistical decision rule for anomalous clique detection using p-values of egonet degrees. Here we briefly outline and illustrate the idea without getting into technical details. Let \(d_i = \sum_{j=1}^{n} A_{ij}\) be the degree of the i^{th} node, and \(e_i = \sum_{j,k:A_{ij}=1,A_{ik}=1} A_{jk}\) be the degree of the i^{th} egonet. Heuristically, clique-like subgraphs will have \(e_i\) disproportionately high compared to \(d_i\). If we fit a model to the whole network and compute the p-values of \(\{e_i\}_{i=1}^{n}\) with respect to this fitted model, then the p-values for egonets without anomalous clique would be approximately uniformly distributed between 0 and 1, whereas the p-values for egonets with an anomalous clique would be very low. We formulate a decision rule for clique detection based on p-values falling below a certain threshold.

The results of this method for the networks in Fig. 1 (top row) are plotted in the third and fourth rows of Fig. 1. From the third row, egonet degrees of the non-anomalous network (left) are found to be well within the detection threshold, whereas for the anomalous network (right), non-anomalous egonet degrees are well within the detection threshold, but egonet degrees for the anomalous egonets (represented as stars) breach the detection threshold. In the fourth row, the p-values are plotted in log scale, and we can see the results confirming the heuristics described above.

We now present a very brief overview of related work. Although the problem of anomaly detection in static networks has not received much attention in the statistics literature, it has been studied by the computer science community, and a survey of related work can be found in Akoglu et al. (2015). Most of these methods, however, are heuristic in nature without an explicitly formulated test statistic and decision rule. In particular, Akoglu et al. (2010) also used egonets for anomaly detection in networks, however their approach is based on a heuristic interpretation of plotted regression lines of \(\log(e_i)\) on \(\log(d_i)\), and
Figure 1: Top row: Adjacency heatmap of (left) an Erdős-Rényi network with $n = 500$ nodes and $p = 0.05$, and (right) same network with an embedded 10-node clique.

2nd row: Egonet from the non-anomalous network (left) and the anomalous network (right).

3rd row: Scatterplot of egonet degrees (Y-axis) vs node degrees (Y-axis), with the solid curve representing the detection threshold. Stars in the right plot represent nodes forming the embedded clique.

4th row: Plot of egonet p-values in log scale, with the solid horizontal line representing the detection threshold. Stars in the right plot represent nodes forming the embedded clique.
they do not have a test statistic or a statistical decision rule. The problem of finding a small hidden clique (the so-called “planted clique” problem) has been extensively studied in the theoretical computer science literature, some notable contributions being Alon et al. (1998), Kučera (1993), and Feige and Krauthgamer (2000), to name a few. The goal is similar to the second task that we listed, i.e., to identify the nodes constituting the hidden clique, rather than to detect whether a given network contains a hidden clique. A suite of network anomaly detection methods based on spectral properties was presented in Miller et al. (2015). Their method is formulated as a statistical decision rule to determine whether a given network contains a hidden clique, which makes their formulation directly comparable to ours. We used one of their methods for benchmarking the performance of our egonet method, as reported in Section 5.

Our main contribution in this paper is the formulation of a statistical inferential framework for anomalous clique detection in static networks. Using the egonet method, one can simultaneously perform two tasks — first, detect whether there is a small anomalous clique in a network, and second, if the presence of an anomalous clique is detected, then identify the nodes forming the clique. To the extent of our knowledge, this is one of the first methodological papers in the statistics literature to investigate the problem of anomaly detection in static networks. This is also one the first papers to study egonets, which are an important class of network substructures.

The rest of the article is organized as follows. In Section 2, we introduce the egonet method under the Erdős-Rényi model, and in Section 3 we present theoretical results pertaining to statistical guarantees. In Section 4, we extend the egonet method to several network models beyond the Erdős-Rényi model. In Section 5, we present simulation results for the egonet method and the benchmark $\chi^2$ method for synthetic networks generated from a wide variety of network models. In Section 6, we present case studies reporting the results of the egonet method applied on three well-known network datasets. We conclude the paper with a brief discussion in Section 7. In supplementary materials we provide technical proofs of the theoretical results, as well as R code for implementing the egonet method for a wide variety of network models.

2 The Egonet method

Consider an Erdős-Rényi random graph with $n$ nodes and probability $p$, then the adjacency matrix $A$ is a symmetric $n$-by-$n$ matrix with $A_{ij} \sim \text{Bernoulli}(p)$ for all $i > j$, and $A_{ii} = 0$ for $i = 1, \ldots, n$. Under this model, let $d_i$ be the degree of the $i^{th}$ node, and let $e_i$ be
the degree of the $i^{th}$ egonet. We define the $i^{th}$ egonet as the subgraph spanned by the $d_i$ neighbors of the $i^{th}$ node. It follows that

$$d_i \sim \text{Binomial}(n - 1, p), e_i|d_i \sim \text{Binomial} \left( \frac{d_i}{2}, p \right),$$

for $i = 1, \ldots, n$. The egonet p-value is defined as the probability of observing an egonet degree greater than or equal to the observed value, i.e.,

$$G \left( e_i; \frac{d_i}{2}, p \right) = P[B \geq e_i] \text{ where } B \sim \text{Binomial} \left( \frac{d_i}{2}, p \right).$$

From the definition of $G(\cdot)$, it follows that for any $\alpha \in (0, 1)$,

$$P \left[ G \left( e_i; \frac{d_i}{2}, p \right) < \alpha \right] \leq \alpha,$$

which means egonet p-values are unlikely to be very small when the network is indeed generated from an Erdős-Rényi model.

Now consider the anomalous case where there is a small clique embedded in the network. Let $m$ be the number of nodes in the clique, and $\mathcal{C} = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ be the set of nodes forming the clique. Then for $i > j$,

$$A_{ij} = 1 \text{ when } i, j \in \mathcal{C}, \text{ and } A_{ij} \sim \text{Bernoulli}(p) \text{ when } i \notin \mathcal{C} \text{ or } j \notin \mathcal{C}.$$

For $i \in \mathcal{C}$, the node degrees and egonet degrees are distributed as

$$d_i \sim (m - 1) + \text{Binomial}(n - m, p),$$

$$e_i|d_i \sim \left( \frac{m - 1}{2} \right) + \text{Binomial} \left( \frac{d_i}{2} - \left( \frac{m - 1}{2} \right), p \right).$$

When $\left( \frac{m - 1}{2} \right)$ is sufficiently large, $e_i$ is going to be large for anomalous nodes, and therefore the anomalous egonet p-values, $\{G \left( e_i; \frac{d_i}{2}, p \right) : i \in \mathcal{C}\}$ are likely to be very small, whereas the non-anomalous p-values, $\{G \left( e_i; \frac{d_i}{2}, p \right) : i \notin \mathcal{C}\}$ are unlikely to be very small.

In practice we do not know the value of the parameter $p$, but we can estimate it as

$$\hat{p} = \frac{\sum_{i > j} A_{ij}}{\binom{n}{2}}.$$
In the anomalous case this estimate is biased due to the presence of the clique, however the size of the clique is assumed to be too small for this bias to be significant. For illustration, in the networks from Figure 1, \( \hat{p} = 0.0491 \) for the non-anomalous network and \( \hat{p} = 0.0494 \) for the anomalous network, while the true value of the parameter is \( p = 0.05 \).

We now formally state the hypothesis testing problem. Given an \( n \times n \) simple, symmetric, binary network adjacency matrix \( A \), we want to determine whether to reject the null hypothesis

\[
H_0 : \exists \ p \in (0, 1) \text{ such that } A_{ij} \sim Bernoulli(p) \text{ for all } i > j
\]

against the alternate hypothesis

\[
H_1 : \exists C = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}, \exists \ p \in (0, 1) \text{ such that for all } i > j
A_{ij} = 1 \text{ when } i, j \in C, \text{ and } \ A_{ij} \sim Bernoulli(p) \text{ when } i \notin C \text{ or } j \notin C.
\]

Consider the test statistic

\[
T_n(A) = \min_{1 \leq i \leq n} G \left( e_i; \left( \frac{d_i}{2} \right), \hat{p} \right).
\]  

(4)

When \( H_0 \) is true, based on the preceding discussion, the values of \( G \left( e_i; \left( \frac{d_i}{2} \right), \hat{p} \right) \) are unlikely to be very small. Given that \( \hat{p} \) is sufficiently close to \( p \), the values of \( G \left( e_i; \left( \frac{d_i}{2} \right), \hat{p} \right) \) are, therefore, also unlikely to be very small, and therefore \( T_n \) is unlikely to be very small. Under \( H_1 \), the values of \( G \left( e_i; \left( \frac{d_i}{2} \right), \hat{p} \right) \) for \( i \in C \), closely matched by \( G \left( e_i; \left( \frac{d_i}{2} \right), \hat{p} \right) \) for \( i \in C \), are likely to be very small, and therefore the value of \( T_n \) is likely to be very small. Therefore we seek a critical region of the form \( T_n < \alpha_n \) for a suitably small value of \( \alpha_n \).

The choice of \( \alpha_n \) depends on the target significance level of the test, denoted by \( \alpha \). The egonet degrees \( e_i \) can be mutually dependent. To see this, consider the egonets for the \( i^{th} \) and \( j^{th} \) nodes, then if these nodes have shared neighbors, the \( i^{th} \) and \( j^{th} \) egonet degrees are going to be mutually dependent. This means the corresponding egonet p-values \( G \left( e_i; \left( \frac{d_i}{2} \right), \hat{p} \right) \) and \( G \left( e_j; \left( \frac{d_j}{2} \right), \hat{p} \right) \) may also be dependent, and the strength of their dependence is a function of shared neighbors between the \( i^{th} \) and the \( j^{th} \) node. Therefore we cannot model \( T_n \) as the minimum of \( n \) independent random variables. Instead, we use
a conservative bound and choose $\alpha_n = \frac{\alpha}{n}$. Let

$$
\tau_n = \min_{1 \leq i \leq n} G \left( e_i; \left( \frac{d_i}{2} \right), p \right)
$$

be the population version of $T_n$. Then $\tau_n$ can be interpreted as the minimum of the population p-values $G \left( e_i; \left( \frac{d_i}{2} \right), p \right)$, while the test statistic $T_n$ is the minimum of the estimated p-values $G \left( e_i; \left( \frac{d_i}{2} \right), \hat{p} \right)$. Under $H_0$, note that

$$
P \left[ \tau_n < \frac{\alpha}{n} \right] = P \left[ \left( \min_{1 \leq i \leq n} G \left( e_i; \left( \frac{d_i}{2} \right), p \right) \right) < \frac{\alpha}{n} \right]
$$

$$
\leq P \left[ \bigcup_{i=1}^{n} \{ G \left( e_i; \left( \frac{d_i}{2} \right), p \right) < \frac{\alpha}{n} \} \right]
$$

$$
\leq \sum_{i=1}^{n} P \left[ G \left( e_i; \left( \frac{d_i}{2} \right), p \right) < \frac{\alpha}{n} \right]
$$

$$
\leq \sum_{i=1}^{n} \frac{\alpha}{n} = \alpha
$$

from (2). Thus, the conservative choice of $\alpha_n = \frac{\alpha}{n}$ is justified in the sense that this choice ensures that under $H_0$, the rejection probability using the population test statistic $\tau_n$ is less than or equal to the target significance level. In Section 3, we establish that a similar guarantee holds for the sample test statistic $T_n$. If the null hypothesis is rejected, i.e., $T_n < \frac{\alpha}{n}$, we identify the nodes constituting the clique as

$$
\hat{C} = \left\{ i \in [1, n] : G \left( e_i; n_i, \hat{p} \right) \leq \frac{\alpha}{n} \right\}.
$$

(5)

We conclude this section with a statement of the algorithm and two remarks.

**Remark 2.1. Algorithmic scalability:** An advantage of this method is that the test statistic is based on degrees, which can be calculated very fast, compared to more complicated network metrics. Algorithm [1] outlines the steps involved in implementing the egonet method for clique detection under the Erdős-Rényi model. Step 2 of this algorithm is the main step in terms of computational expense, and this step can easily be parallelized to share the computation of $G \left( e_i; n_i, \hat{p} \right)$ for $i = 1, \ldots, n$ across a certain number of processors. This makes the algorithm computationally scalable and suitable for large networks.

**Remark 2.2. Dense subgraphs:** Under the egonet method, we reject the null hypothesis if the minimum egonet p-value is lower than the threshold value, i.e., if at least one egonet degree is disproportionately high compared to the corresponding node degree. While the presence of an anomalous clique can cause this to happen, this can also happen due to
Input: A symmetric, binary adjacency matrix $A_{n \times n}$ where $A_{ii} = 0 \ \forall i$

1. Estimate model parameter: $\hat{p} = \frac{\sum A_{ij}}{n(n-1)}$

2. For $i \leftarrow 1$ to $n$
   (a) Compute $d_i = \text{degree of } i^{th} \text{ node}$, $e_i = \text{degree of } i^{th} \text{ egonet}$
   (b) Compute $G(e_i; \left(\frac{d_i}{2}\right), \hat{p}) = P[B \geq e_i]$ where $B \sim Binomial\left(\left(\frac{d_i}{2}\right), \hat{p}\right)$.

3. Compute test statistic: $T_n = \min_{1 \leq i \leq n} G(e_i; n_i, \hat{p})$

4. Reject $H_0$ if $T_n < \frac{\alpha}{n}$, and in that case,

5. $\hat{C} = \{ i \in [1, n] : G(e_i; n_i, \hat{p}) < \frac{\alpha}{n} \}$

Algorithm 1: Egonet algorithm for the Erdős-Rényi model

presence of a small dense anomalous subgraph. In fact, a clique is a special (and interesting) case of such small dense subgraphs. It is worth noting that the problem of identifying a small, dense subgraph embedded in a large, sparse network is an active research area in itself [Montanari, 2015; Hajek et al., 2017; Komusiewicz et al., 2015; Zhang and Chen, 2015]. However, most of the work in this area is aimed at finding an anomalous subgraph assuming one exists, rather than determining whether there is an anomalous subgraph in the network.

3 Theory under the Erdős-Rényi model

In this section we establish statistical guarantees under the null hypothesis as well as the alternative hypothesis.

3.1 Statistical guarantees under $H_0$:

We know that when $H_0$ is true, the population p-values satisfy

$$P \left[ G \left( e_i; \left(\frac{d_i}{2}\right), \hat{p} \right) < \frac{\alpha}{n} \right] \leq \frac{\alpha}{n}.$$ 

\footnotetext{1}{To draw a heuristic analogy, suppose we have a sample from a $\text{Normal}(\mu, 1)$ distribution and we use a test of the form $\sqrt{n} \bar{X} > K$ for testing $H_0 : \mu = 0$ vs $H_1 : \mu = 1$. Then we are likely to reject when the actual data generating distribution has $\mu = 0.8$ or $\mu = 0.9$.}
We would like to establish a similar result for the estimated p-value, $G(e_i; \left(\frac{d_i}{2}\right), \hat{p})$. For notational convenience, we denote $n_i = \left(\frac{d_i}{2}\right)$.

The p-value function can be expressed as the incomplete regularized beta function,

$$G(k; n, p) = P[B(n, p) \geq k] = \frac{1}{\text{Beta}(k, n - k + 1)} \int_0^p t^{k-1}(1 - t)^{n-k} dt,$$  \hspace{1cm} (6)

where $\text{Beta}(\cdot, \cdot)$ is the Beta function. Keeping $e_i$ and $n_i$ fixed, clearly $G(e_i; n_i, \hat{p})$ is a non-decreasing function of $p$. When $\hat{p} > p$, $G(e_i; n_i, \hat{p}) \geq G(e_i; n_i, p)$ for all possible combinations of $e_i$ and $n_i$, which implies

$$P \left[ G(e_i; n_i, \hat{p}) < \frac{\alpha}{n} \right] \leq P \left[ G(e_i; n_i, p) < \frac{\alpha}{n} \right] \leq \frac{\alpha}{n}.$$  

Therefore, when $H_0$ is true and $\hat{p} > p$,

$$P \left[ T_n < \frac{\alpha}{n} \right] = P \left[ (\min_{1 \leq i \leq n} G(e_i; n_i, \hat{p})) < \frac{\alpha}{n} \right] \leq P \left[ \bigcup_{i=1}^p \{ G(e_i; n_i, \hat{p}) < \frac{\alpha}{n} \} \right] \leq \sum_{i=1}^n P \left[ G(e_i; n_i, \hat{p}) < \frac{\alpha}{n} \right] \leq \sum_{i=1}^n \frac{\alpha}{n} = \alpha \text{ from Eqn (2)}.$$  

Heuristically, when $\hat{p} > p$, a large value of $e_i$ is more likely under $\hat{p}$ than under $p$, and therefore it is less likely that the estimated p-value will be lower than the threshold $\frac{\alpha}{n}$. Therefore when $H_0$ is true, we only need to concern ourselves with the case when $\hat{p} < p$.

Next, consider the case when $H_0$ is true and $\hat{p} < p$. In this case, $G(e_i; n_i, \hat{p}) \leq G(e_i; n_i, p)$ for any $n_i, e_i$, and therefore $T_n \leq \tau_n$. To ensure that the type I error rate $P[T_n(A) < \frac{\alpha}{n}]$ is still asymptotically less than or equal to $\alpha$, we need a concentration result which shows that under some reasonable assumptions on $p$, the difference $(G(e_i; n_i, p) - G(e_i; n_i, \hat{p}))$ is asymptotically too small to affect the type I error rate substantially.

**Assumption 3.1.** As $n \to \infty$,

1. There exists some $\delta_1 \in (0, 0.5)$ such that

$$\log(n)p^{1-2\delta_1} \to 0.$$
2. There exists some $\delta_2 > 0$ such that 

$$\frac{(n - 1)p}{(1 + \delta_2) \log(n)} \to \infty.$$ 

**Remark 3.1.** As $n$ increases, the first part of the assumption ensures that the network does not become too dense, while the second part ensures that the network does not become too sparse, requiring the expected node degree $(n - 1)p$ to be larger than $\log(n^{1+\delta_2})$. Taken together, this assumption allows $p$ to take a wide range of values — for example, as $n$ goes to infinity, $p$ can take the value $p = Cn^{-1+\epsilon}$ for any $\epsilon \in (0, 1)$.

Consider the smallest value of $e_i$ that breaches the estimated p-value threshold, i.e.,

$$\hat{k}_i(n_i, \hat{p}) = \min\{j \in \mathbb{N} : G(j; n_i, \hat{p}) \leq \alpha/n\}.$$ 

For notational convenience we abbreviate $\hat{k}_i(n_i, \hat{p})$ as $\hat{k}_i$. We are now ready to state the first lemma.

**Lemma 3.1.** When $\hat{p} < p$, under assumption 3.1,

1. Let $B_n = \{\frac{p - \hat{p}}{\hat{p}} \leq \frac{1}{np^{1+\delta_1}}\}$, then $P[B_n^c] \to 0$.

2. Let $D_n = \{d_i \in np(c, e) \text{ for all } i = 1, \ldots, n\}$. Then $\exists c > 0$ such that $P[D_n^c] \to 0$.

3. When $B_n \cap D_n$ holds, $G(\hat{k}_i; n_i, \hat{p}) - G(\hat{k}_i; n_i, \hat{p}) = o\left(\frac{1}{n}\right)$.

**Remark 3.2.** Lemma 3.1 shows that with probability going to 1, the sample quantities $\hat{p}$ and $d_i$ are “well-behaved”, and as long as that happens, the estimated p-values are close enough to the population p-values. A proof of this lemma is in the supplementary material. The proof relies on a Chernoff bound to establish the first and second parts, and the incomplete regularized beta representation to prove the third part.

**Theorem 3.1.** Under Assumption 3.1, when $H_0$ is true,

$$P\left[T_n(A) < \frac{\alpha}{n}\right] < \alpha \text{ as } n \to \infty.$$ 

**Remark 3.3.** Theorem 3.1 formalizes the statistical guarantee for type I error rates under $H_0$. Note that under the null model, the true distribution of $e_i|n_i$ is $\text{Binomial}(n_i, p)$, but the reference distribution for estimated p-values is $\text{Binomial}(n_i, \hat{p})$. The estimated p-value threshold is breached by the $i^{th}$ egonet when $G(e_i; n_i, \hat{p})$ is less than $\frac{\alpha}{n}$, i.e., when
is greater than or equal to $\hat{k}_i$. The probability of this event is $G(\hat{k}_i; n_i, p)$, and from the definition of $G$ we know that $G(\hat{k}_i; n_i, \hat{p})$ is less than or equal to $\frac{\alpha}{n}$. A proof of the theorem, relying on the results in the lemma, is in the supplementary material.

### 3.2 Statistical guarantees under $H_1$

Under $H_1$, there is a set of nodes $\mathcal{C} = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ forming a clique. Usually one is interested in establishing the statistical guarantee that power of the test goes to 1 asymptotically, i.e., under $H_1$,

$$P\left[T_n < \frac{\alpha}{n}\right] \rightarrow 1 \text{ as } n \rightarrow \infty.$$

However, in the context of clique detection, in addition to being able to detect that there is a clique, we also want to identify the nodes that constitute the clique. To formalize this notion, we define the following metric for the set of anomalous nodes:

$$T_n^{(1)}(A) = \max_{i \in \mathcal{C}} G(e_i; n_i, \hat{p}). \tag{7}$$

With respect to this metric, we would like to prove that

$$P\left[T_n^{(1)} < \frac{\alpha}{n}\right] \rightarrow 1 \text{ as } n \rightarrow \infty,$$

which ensures that all anomalous nodes are detected with asymptotically high probability. Note that this is a stronger result than power going to 1 asymptotically, since

$$T_n(A) = \min_{1 \leq i \leq n} G(e_i; n_i, \hat{p}) \leq \max_{i \in \mathcal{C}} G(e_i; n_i, \hat{p}) = T_n^{(1)}(A).$$

To establish these guarantees, we need the following conditions relating to the size of the clique relative to the size of the network and the edge probability.

**Assumption 3.2.** Let $c_n = \binom{\frac{n}{2}}{\frac{m}{2}}$, then as $n \rightarrow \infty$,

1. There exists some $\delta_3 > 0$ such that $\frac{\delta_3 c_n}{p} \rightarrow 0$.

2. $\frac{c_n}{\delta_4} \rightarrow \infty$, where $\delta_4 = \max \left( p^3, \frac{\log(n)}{n^2} \right)$.

**Remark 3.4.** The first part of the assumption gives an upper bound on the size of the clique to ensure that the presence of the clique does not distort the rest of the network too
much. The second part gives a lower bound on the clique size to ensure that the presence of the clique makes egonet degrees of anomalous nodes large enough to be detected.

**Lemma 3.2.** Under assumptions 3.1 and 3.2, the following results hold.

1. Let $B_n = \{ \hat{p} \leq p(\frac{1}{\delta_3} + \epsilon) \}$, then $P[B_n^c] \to 0$.

2. Let $D_n = \{ d_i \leq m + ep \text{ for all } i \in C \}$. Then $P[D_n^c] \to 0$.

3. When $B_n \cap D_n$ holds, $G \left( \binom{m-1}{2}; n_i, \hat{p} \right) < \frac{\alpha}{n}$ for all $i \in C$.

**Remark 3.5.** Lemma 3.2 is analogous to Lemma 3.1 particularly with respect to the first two parts, the difference being that we now need results on anomalous nodes. A proof of the lemma is in the supplementary materials, and is based on similar ideas as the proof of Lemma 3.1.

**Theorem 3.2.** Under assumptions 3.1 and 3.2, when $H_1$ is true, $P \left[ T_n < \frac{\alpha}{n} \right] \to 1$, and $P \left[ T_n^{(1)} < \frac{\alpha}{n} \right] \to 1$ as $n \to \infty$.

**Remark 3.6.** Note that when $H_1$ is true, the neighborhood of any anomalous node includes the other $(m-1)$ anomalous nodes, which means the $i^{th}$ egonet contains at least $\binom{m-1}{2}$ edges connecting these nodes. The results in Lemma 3.2 are therefore sufficient to establish the statistical guarantees. A proof of the theorem, relying on the results in the lemma, is in the supplementary material.

## 4 Extension to other network models

An important advantage of the egonet method is that it easily extends to a wide variety of network models. As long as we have an adequately accurate estimator of the generative model, we can fit the model to the observed network, and obtain p-values of egonets with respect to this estimated model. In this section we describe the estimation methodology and formulate the egonet algorithm under four well-known network models beyond the simple Erdős-Rényi model. We do not include theoretical results under these models in the interest of space, and we plan to address this in future work.

Under the Erdős-Rényi model, egonet degrees are sums of independent and identically distributed Bernoulli random variables, and therefore one can easily compute egonet p-values from the Binomial distribution. However, when edges are modeled as Bernoulli
random variables in network models involving many parameters, the egonet degrees are sums of independent but not identically distributed Bernoulli random variables, i.e., egonet degrees have the Poisson Binomial Distribution (Ehm, 1991; Wang, 1993). Computing exact p-value for this distribution is a notoriously difficult problem (Hong, 2013), and most existing methods for p-value calculation depend on some kind of approximation. In our case, since the critical value of the test is based on the far right \((1 - \frac{2}{n})\)th quantile, such approximations can substantially distort the size and power of the test when \(n\) is large. Therefore, in this section, we model network edges as Poisson random variables, thereby allowing the edges to be integer-valued. This ensures that egonet degrees also follow the Poisson distribution, making p-value computation substantially easier compared to the Poisson Binomial Distribution.

Let \(A\) be the symmetric \(n\)-by-\(n\) adjacency matrix of a network with \(n\) nodes and no self-loops. Then under the general Poisson model,

\[
A_{ij} \sim \text{Poisson}(\lambda_{ij}) \quad \text{for } i > j,
\]

and \(A_{ii} = 0\). Under this general model, node degrees have the distribution

\[
d_i = \sum_j A_{ij} \sim \text{Poisson} \left( \sum_j \lambda_{ij} \right),
\]

and given the neighborhood \(N_i = \{j \leq n : A_{ij} > 0\}\) of the \(i\)th node, egonet degrees have the distribution

\[
e_i = \sum_{j,j' \in N_i, j > j'} A_{ij} \sim \text{Poisson} \left( \sum_{j,j' \in N_i, j > j'} \lambda_{jj'} \right).
\]

Suppose we have an estimator \(\hat{\lambda}_{ij}\) for each \(\lambda_{ij}\), then we can compute estimated p-values \(G(e_i; N_i, \{\hat{\lambda}_{ij}\}_{i,j=1}^n)\) as \(P[X \geq e_i] \text{ where } X \sim \text{Poisson} \left( \sum_{j,j' \in N_i, j > j'} \hat{\lambda}_{jj'} \right)\).

We now introduce four well-known network models and outline the egonet method for networks generated from these models. Since these are well-studied and well-documented models, we describe them very briefly in the interest of space. R functions for implementing the egonet method under these four models, as well as the Erdös-Rényi model, are available in supplementary materials.

- **Chung-Lu model:** Under the Chung-Lu model (Aiello et al., 2000),

\[
\lambda_{ij} = \theta_i \theta_j,
\]
where \( \{\theta_i\}_{i=1}^n \) are the degree parameters. We estimate the model parameters as

\[
\hat{\lambda}_{ij} = \frac{d_i d_j}{2M},
\]

where \( d_i \) is the degree of the \( i^{th} \) node and \( m = \sum_{i>j} A_{ij} \) is the degree of the network.

- **Stochastic blockmodel**: The stochastic blockmodel (Holland et al., 1983; Fienberg et al., 1985) is probably the most well-studied network model in the statistics literature. Under a stochastic blockmodel with \( K \) communities,

\[
\lambda_{ij} = \omega_{c_i} \omega_{c_j},
\]

where \( \omega \) is a \( K \)-by-\( K \) symmetric matrix of community-community interaction probabilities, and \( \{c_i\}_{i=1}^n \) are the communities of the nodes, with \( c_i \) taking its value in \( 1, \ldots, K \). We use the regularized spectral clustering method (Rohe et al., 2011; Qin and Rohe, 2013; Sengupta and Chen, 2015) for estimating the communities \( \{\hat{c}_i\}_{i=1}^n \), and estimate \( \omega_{rs} \) as

\[
\hat{\omega}_{rs} = \frac{\sum_{i,j:\hat{c}_i=\hat{c}_j=r} A_{ij}}{n_r n_s}
\]

where \( n_r \) is the size of the estimated \( r^{th} \) community. The Poisson parameters are estimated by plug-in estimators

\[
\hat{\lambda}_{ij} = \hat{\omega}_{c_i} \hat{\omega}_{c_j}.
\]

- **Degree-corrected stochastic blockmodel**: The degree-corrected stochastic blockmodel (Karrer and Newman, 2011) is a generalization of the stochastic blockmodel that allows for flexible degree distributions. Under this model with \( K \) communities,

\[
\lambda_{ij} = \theta_i \omega_{c_i} \omega_{c_j} \theta_j,
\]

where \( \omega \) is a \( K \)-by-\( K \) symmetric matrix of community-community interaction probabilities, and \( \{c_i\}_{i=1}^n \) are the communities of the nodes, and \( \{\theta_i\}_{i=1}^n \) are degree parameters. We use the regularized spectral clustering method with row-normalization (Rohe et al., 2011; Qin and Rohe, 2013; Sengupta and Chen, 2015) for estimating
the communities $\{c_i\}_{i=1}^n$, and estimate the remaining parameters as

$$\hat{\omega}_{rs} = \sum_{i,j:c_i,c_j=r} A_{ij}, \quad \hat{\theta}_i = \frac{d_i}{\delta_r}$$

where $\delta_r = \sum_{i:c_i=r} d_i$ is the degree of the estimated $r^{th}$ community. The Poisson parameters are estimated by plug-in estimators

$$\hat{\lambda}_{ij} = \hat{\theta}_i \hat{\omega}_{i^c} \hat{\omega}_{i^c} \hat{\theta}_j.$$

- **Popularity adjusted blockmodel**: The popularity adjusted blockmodel was proposed by [Sengupta and Chen (2018)](sengupta2018) for flexible modeling of node popularities in the presence of community structure. Under this model with $K$ communities,

$$\lambda_{ij} = \theta_{i_{c_i}} \theta_{j_{c_j}},$$

where $\theta_{i_{r}}$ represents the popularity of the $i^{th}$ node in the $r^{th}$ community, and $\{c_i\}_{i=1}^n$ are the node communities. We use the extreme points method of [Le et al. (2016)](le2016) to estimate communities, and estimate the popularity parameters as

$$\hat{\theta}_{ir} = \frac{\sum_{j:c_j=r} A_{ij}}{\sqrt{\sum_{i,j:c_i,c_j=r} A_{ij}}}.$$

The Poisson parameters are estimated by plug-in estimators

$$\hat{\lambda}_{ij} = \hat{\theta}_{i_{c_i}} \hat{\theta}_{j_{c_j}}.$$

5 Simulation results

In this section we present results for the egonet method for synthetic networks generated from the five network models discussed in Sections 2–4. From each model, we generated networks with $n = 500, 1000, 2000$ nodes. For an apples to apples comparison across network models, we configured the parameters under the various models in a manner such that the expected network density, defined as

$$\delta = \frac{1}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[A_{ij}],$$

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is fixed at $\delta = 0.05$ for all models and all values of $n$. For each network thus generated, we randomly selected a small set of $m$ nodes and embedded a clique consisting of the selected nodes. We used $m = 5, 10, 20$, to construct three different anomalous networks from each non-anomalous network. We deliberately included some scenarios where the egonet method may not be very successful as the clique is too small to detect.

The egonet method was implemented on both the original non-anomalous network and the networks with embedded clique, with significance levels $\alpha = 1\%, 2\%, 5\%$. We evaluated the performance of the method using the following metrics:

1. False alarm rate (size): The false alarm rate is the proportion of non-anomalous networks where the null hypothesis is rejected, i.e., the method wrongly detects the presence of a an anomalous clique although there isn’t one. This should be small, and less than or equal to the chosen significance level $\alpha$.

2. Anomaly detection rate (power): The detection rate is the proportion of anomalous networks where the null hypothesis is rejected, i.e., the method correctly detects the presence of a clique. This should be close to 1.

3. Node false alarm rate: the node false alarm rate is the proportion of non-anomalous nodes whose egonet p-value is lower than $\frac{\alpha}{n}$, i.e., nodes that are wrongly flagged by the method. This should be close to zero.

4. Node detection rate: in networks with embedded clique, the node detection rate is the proportion of anomalous nodes whose egonet p-value is lower than $\frac{\alpha}{n}$, i.e., nodes that are correctly flagged by the method. This should ideally be close to 1.

To illustrate the relation between the node level metrics (3–4) and the network level metrics (1–2), we consider an anomalous network with $n = 1000$ nodes with an embedded 10-node clique, i.e., the network contains 10 anomalous nodes and 990 non-anomalous nodes. Recall that the egonet test statistic is the minimum of the $n$ egonet p-values, and the null hypothesis is rejected if any of the $n$ egonet p-values falls below the $\frac{\alpha}{n}$ threshold. Suppose that the null hypothesis is indeed rejected (i.e., the minimum egonet p-value falls below the threshold), which implies that the correct decision was made at the network level. Further, suppose that at the node level, 9 of the 1000 egonet p-values fall below the threshold, of which 8 nodes belong to the 10-node clique and 1 node does not belong to the clique. In this example, the node false alarm rate is $1/990$ or $0.1\%$, and the node detection rate is $8/10$ or $80\%$. Note that detecting a single anomalous node is sufficient for detecting the presence of the anomaly at the network level. A high node detection
rate (together with a low node false alarm rate) ensures that in addition to detecting the presence of the clique, the method is also accurate in identifying the nodes that constitute the clique. In our simulation study, the node false alarm rate from the egonet method was below 0.01% under all models and all scenarios, and therefore we do not report this metric in the interest of space.

For benchmarking, we use the $\chi^2$ method of Miller et al. (2015), which is preferred over the other methods described in Miller et al. (2015) based on the statistical evaluation carried out by Komolafe et al. (2017). Given an observed network assumed to be generated under a given model, we first fit the model to the network and obtain an estimate of the expected adjacency matrix, $\hat{E}[A]$. Then we compute the residual matrix, defined as

$$R = A - \hat{E}[A],$$

and compute the first two principal components $X_1, X_2$ of $R$. The $n$ rows of the principal component matrix are then plotted on a notional Cartesian coordinate system, and the number of points falling in the four quadrants are tabulated as a 2-by-2 table. The test statistic is the $\chi^2$ test of independence statistic calculated from this table, maximized over all possible rotations of the Cartesian coordinate system. Since it is not feasible to look at all possible rotations, we use the approach of Komolafe et al. (2017) and take the maximum of 32 rotations over the grid $[\frac{\pi}{16}, \ldots, 2\pi]$. The null hypothesis is rejected if the maximized $\chi^2$ statistic exceeds the $(1 - \alpha)$ quantile of the $\chi^2_1$ distribution. The $\chi^2$ method does not automatically provide a method for identifying anomalous nodes. Therefore we do not calculate node false alarm rates and node detection rates for this method.

### 5.1 Erdős-Rényi model

Under the Erdős-Rényi model, we used $p = 0.05$ (which ensures $\delta = 0.05$) in conjunction with $n = 500, 1000, 2000$ to generate large, sparse networks. To obtain accurate estimates of performance metrics, we generated 10,000 networks from each parameter configuration. The results are tabulated in Tables 2–4. From Table 2, we note that other than the $n = 500, m = 5$ and $n = 2000, m = 10$ cases, the egonet method performs very well with respect to detection rates. To put this in context, the size of the (non-anomalous) maximal clique for an Erdős-Rényi graph is approximately 4 when $n = 500, p = 0.05$ and 5 when $n = 2000, p = 0.05$ (see Theorem 9.1 of Pinsky (2014)), and therefore an anomalous, embedded clique of size $m = 5$ (in the former case) or $m = 10$ (in the latter case) can be very hard to distinguish from a randomly formed clique. The false alarm rates are less
then and reasonably close to the specified $\alpha$. The egonet method performs substantially better than the $\chi^2$ method (Table 3), whose false alarm rates are much smaller than $\alpha$ and detection rates are generally lower than that of the egonet method. From Table 4, we observe that the general trend in node-level results for the egonet are in line with network level results. The egonet method is almost perfectly accurate with respect to node detection under the larger clique scenario for each value of $n$. Comparing Tables 2 and 4, it is interesting to observe that a node detection rate of 25% to 30% is adequate for clique detection at the network level.

| $n$   | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|-------|-------------|----------|-------------|-----------|-------------|-----------|
| 500   | 0.62%       | 1%       | 5           | 0.98%     | 10          | 100%      |
| 500   | 1.1%        | 2%       | 5           | 1.8%      | 10          | 100%      |
| 500   | 2.9%        | 5%       | 5           | 4.51%     | 10          | 100%      |
| 1000  | 0.68%       | 1%       | 10          | 93%       | 20          | 100%      |
| 1000  | 1.26%       | 2%       | 10          | 96.33%    | 20          | 100%      |
| 1000  | 3.6%        | 5%       | 10          | 98.93%    | 20          | 100%      |
| 2000  | 0.86%       | 1%       | 10          | 7.24%     | 20          | 100%      |
| 2000  | 1.66%       | 2%       | 10          | 10.99%    | 20          | 100%      |
| 2000  | 4.07%       | 5%       | 10          | 19.5%     | 20          | 100%      |

Table 2: Erdős-Rényi model: false alarm rates and detection rates for egonet method.

| $n$   | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|-------|-------------|----------|-------------|-----------|-------------|-----------|
| 500   | 0.04%       | 1%       | 5           | 0.09%     | 10          | 99.99%    |
| 500   | 0.18%       | 2%       | 5           | 0.25%     | 10          | 100%      |
| 500   | 1.48%       | 5%       | 5           | 1.6%      | 10          | 100%      |
| 1000  | 0%          | 1%       | 10          | 65.37%    | 20          | 100%      |
| 1000  | 0.07%       | 2%       | 10          | 77.22%    | 20          | 100%      |
| 1000  | 0.63%       | 5%       | 10          | 89.07%    | 20          | 100%      |
| 2000  | 0%          | 1%       | 10          | 0.05%     | 20          | 100%      |
| 2000  | 0.04%       | 2%       | 10          | 0.2%      | 20          | 100%      |
| 2000  | 0.48%       | 5%       | 10          | 1.05%     | 20          | 100%      |

Table 3: Erdős-Rényi model: false alarm rates and detection rates for $\chi^2$ method.
| $n$  | $\alpha$ | Clique size | node DR | Clique size | node DR |
|------|----------|-------------|---------|-------------|---------|
| 500  | 1%       | 5           | 0.07%   | 10          | 95.22%  |
| 500  | 2%       | 5           | 0.14%   | 10          | 96.64%  |
| 500  | 5%       | 5           | 0.32%   | 10          | 98.06%  |
| 1000 | 1%       | 10          | 24.22%  | 20          | 100%    |
| 1000 | 2%       | 10          | 29.64%  | 20          | 100%    |
| 1000 | 5%       | 10          | 37.79%  | 20          | 100%    |
| 2000 | 1%       | 10          | 0.66%   | 20          | 100%    |
| 2000 | 2%       | 10          | 1.00%   | 20          | 100%    |
| 2000 | 5%       | 10          | 1.77%   | 20          | 100%    |

Table 4: Erdös-Rényi model: node detection rates for egonet method.

5.2 Chung-Lu model

Under the Chung-Lu model, the $\theta_i$ parameters were sampled from the Beta(1,5) distribution to emulate power-law type behavior in node degrees, and the resultant matrix of $\lambda_{ij}$’s was then scaled to ensure that the expected network density is $\delta = 0.05$. As before, we generated 10,000 networks from each parameter configuration. The results are tabulated in Tables 5–7. From Table 5, we note that false alarm rates are much lower than the specified $\alpha$, implying that the decision rule is very conservative in this case. On the other hand, the detection rates are much higher than that under the Erdös-Rényi model, with close to perfect detection in all cases other than $n = 500, m = 5$. The $\chi^2$ method performs poorly under this model, with detection rates close to false alarm rates in most scenarios, and false alarm rates much higher than the specified $\alpha$. From Table 7, we observe that the general trend in node-level results for the egonet are in line with network level results. Comparing Tables 5 and 7, it is interesting to observe once again that a node detection rate of around 30% is adequate for clique detection at the network level.
| $n$  | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|------|-------------|----------|-------------|-----------|-------------|-----------|
| 500  | 0.12%       | 1%       | 5           | 17.18%    | 10          | 99.15%    |
| 500  | 0.27%       | 2%       | 5           | 20.35%    | 10          | 99.36%    |
| 500  | 0.77%       | 5%       | 5           | 25.17%    | 10          | 99.52%    |
| 1000 | 0.18%       | 1%       | 10          | 94.85%    | 20          | 100%      |
| 1000 | 0.32%       | 2%       | 10          | 95.5%     | 20          | 100%      |
| 1000 | 0.82%       | 5%       | 10          | 96.48%    | 20          | 100%      |
| 2000 | 0.18%       | 1%       | 10          | 81.7%     | 20          | 100%      |
| 2000 | 0.22%       | 2%       | 10          | 83.22%    | 20          | 100%      |
| 2000 | 0.64%       | 5%       | 10          | 85.6%     | 20          | 100%      |

Table 5: Chung-Lu model: false alarm rates and detection rates for egonet method.

| $n$  | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|------|-------------|----------|-------------|-----------|-------------|-----------|
| 500  | 1.26%       | 1%       | 5           | 1.12%     | 10          | 1.18%     |
| 500  | 3.15%       | 2%       | 5           | 2.9%      | 10          | 3.02%     |
| 500  | 10.06%      | 5%       | 5           | 10.18%    | 10          | 10.04%    |
| 1000 | 0.84%       | 1%       | 10          | 0.83%     | 20          | 97.58%    |
| 1000 | 2.17%       | 2%       | 10          | 2.29%     | 20          | 98.79%    |
| 1000 | 8.75%       | 5%       | 10          | 8.91%     | 20          | 99.62%    |
| 2000 | 0.54%       | 1%       | 10          | 0.62%     | 20          | 0.78%     |
| 2000 | 1.76%       | 2%       | 10          | 1.8%      | 20          | 1.78%     |
| 2000 | 7.34%       | 5%       | 10          | 7.24%     | 20          | 6.86%     |

Table 6: Chung-Lu model: false alarm rates and detection rates for $\chi^2$ method.
| $n$  | $\alpha$ | Clique size | node DR  | Clique size | node DR  |
|------|----------|-------------|----------|-------------|----------|
| 500  | 1%       | 5           | 4.15%    | 10          | 45.42%   |
| 500  | 2%       | 5           | 4.95%    | 10          | 47.26%   |
| 500  | 5%       | 5           | 6.16%    | 10          | 49.74%   |
| 1000 | 1%       | 10          | 28.55%   | 20          | 79.89%   |
| 1000 | 2%       | 10          | 29.89%   | 20          | 81.06%   |
| 1000 | 5%       | 10          | 31.78%   | 20          | 82.62%   |
| 2000 | 1%       | 10          | 16.6%    | 20          | 58.22%   |
| 2000 | 2%       | 10          | 17.32%   | 20          | 59.63%   |
| 2000 | 5%       | 10          | 18.52%   | 20          | 61.60%   |

Table 7: Chung-Lu model: node detection rates for egonet method.

5.3 Stochastic blockmodel

Under the stochastic blockmodel, we used $K = 2$ communities of equal size with

$$\omega \propto \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix},$$

and the resultant matrix of $\lambda_{ij}$’s was then scaled to ensure that the expected network density is $\delta = 0.05$. We generated 10,000 networks for $n = 500, 1000$ and 5000 networks for $n = 2000$. The results are tabulated in Tables 8–10. The performance of the egonet method is quite good and the results are similar to those under the Erdős-Rényi model with respect to network level metrics (Table 8 vs Table 2) as well as node level metrics (Table 10 vs Table 4). The $\chi^2$ method performs rather poorly (Table 9) in comparison.
| $n$  | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|------|-------------|----------|-------------|-----------|-------------|-----------|
| 500  | 0.7%        | 1%       | 5           | 1.04%     | 10          | 100%      |
| 500  | 1.35%       | 2%       | 5           | 1.97%     | 10          | 100%      |
| 500  | 3.3%        | 5%       | 5           | 4.49%     | 10          | 100%      |
| 1000 | 0.9%        | 1%       | 10          | 86.07%    | 20          | 100%      |
| 1000 | 1.69%       | 2%       | 10          | 92.3%     | 20          | 100%      |
| 1000 | 3.85%       | 5%       | 10          | 97.27%    | 20          | 100%      |
| 2000 | 0.78%       | 1%       | 10          | 5.28%     | 20          | 100%      |
| 2000 | 1.8%        | 2%       | 10          | 8.8%      | 20          | 100%      |
| 2000 | 4.24%       | 5%       | 10          | 16.08%    | 20          | 100%      |

Table 8: Stochastic blockmodel: false alarm rates and detection rates for egonet method.

| $n$  | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|------|-------------|----------|-------------|-----------|-------------|-----------|
| 500  | 1.01%       | 1%       | 5           | 0.88%     | 10          | 98.54%    |
| 500  | 1.99%       | 2%       | 5           | 2.13%     | 10          | 99.36%    |
| 500  | 6.47%       | 5%       | 5           | 6.38%     | 10          | 99.87%    |
| 1000 | 0.18%       | 1%       | 10          | 39.88%    | 20          | 100%      |
| 1000 | 0.65%       | 2%       | 10          | 52.53%    | 20          | 100%      |
| 1000 | 2.79%       | 5%       | 10          | 69.59%    | 20          | 100%      |
| 2000 | 0.02%       | 1%       | 10          | 0.04%     | 20          | 100%      |
| 2000 | 0.16%       | 2%       | 10          | 0.22%     | 20          | 100%      |
| 2000 | 1.16%       | 5%       | 10          | 1.48%     | 20          | 100%      |

Table 9: Stochastic blockmodel: false alarm rates and detection rates for $\chi^2$ method.
| $n$  | $\alpha$ | Clique size | node DR | Clique size | node DR |
|------|---------|-------------|---------|-------------|---------|
| 500  | 1%      | 5           | 0.07%   | 10          | 91.45%  |
| 500  | 2%      | 5           | 0.13%   | 10          | 93.68%  |
| 500  | 5%      | 5           | 0.26%   | 10          | 96.06%  |
| 1000 | 1%      | 10          | 18.07%  | 20          | 100%    |
| 1000 | 2%      | 10          | 22.78%  | 20          | 100%    |
| 1000 | 5%      | 10          | 30.16%  | 20          | 100%    |
| 2000 | 1%      | 10          | 0.47%   | 20          | 99.95%  |
| 2000 | 2%      | 10          | 0.734%  | 20          | 99.98%  |
| 2000 | 5%      | 10          | 1.31%   | 20          | 99.99%  |

Table 10: Stochastic blockmodel: node detection rates for egonet method.

5.4 Degree corrected blockmodel

Under the degree corrected blockmodel, the $\theta_i$ parameters were sampled from the Beta(1,5) distribution to emulate power-law type behavior in node degrees. We used $K = 3$ unbalanced communities with 25%, 25%, and 50% of the nodes, and

$$\omega \propto \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix},$$

and the resultant matrix of $\lambda_{ij}$’s was then scaled to ensure that the expected network density is $\delta = 0.05$. We generated 10,000 networks for $n = 500, 1000$ and 5000 networks for $n = 2000$. The results are tabulated in Tables 11–13.

The network level results are somewhat similar to those under the Chung-Lu model, with very low false alarm rates and high detection rates. The only scenario with low detection rates is $n = 500, m = 5$. In contrast, the $\chi^2$ method performs very poorly and erratically under the degree corrected blockmodel pretty much under all scenarios. The general trend in node-level results for the egonet are in line with network level results, although overall the node level detection rates are relatively low under this model.
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
n & False alarm & \alpha & Clique size & Detection & Clique size & Detection \\
\hline
500 & 0.1\% & 1\% & 5 & 16.3\% & 10 & 96.9\% \\
500 & 0.19\% & 2\% & 5 & 18.85\% & 10 & 97.64\% \\
500 & 0.41\% & 5\% & 5 & 23.14\% & 10 & 98.29\% \\
\hline
1000 & 0.08\% & 1\% & 10 & 94.13\% & 20 & 90.68\% \\
1000 & 0.21\% & 2\% & 10 & 94.96\% & 20 & 91.93\% \\
1000 & 0.49\% & 5\% & 10 & 95.83\% & 20 & 93.68\% \\
\hline
2000 & 0.06\% & 1\% & 10 & 79.14\% & 20 & 99.9\% \\
2000 & 0.12\% & 2\% & 10 & 80.62\% & 20 & 99.9\% \\
2000 & 0.6\% & 5\% & 10 & 83.18\% & 20 & 99.9\% \\
\hline
\end{array}
\]

Table 11: Degree corrected blockmodel: false alarm rates and detection rates for egonet method.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
n & False alarm & \alpha & Clique size & Detection & Clique size & Detection \\
\hline
500 & 6.35\% & 1\% & 5 & 6.62\% & 10 & 4.87\% \\
500 & 12.05\% & 2\% & 5 & 11.90\% & 10 & 9.63\% \\
500 & 25.28\% & 5\% & 5 & 25.1\% & 10 & 21.62\% \\
\hline
1000 & 4.65\% & 1\% & 10 & 4.64\% & 20 & 27.35\% \\
1000 & 9.23\% & 2\% & 10 & 9.34\% & 20 & 38.66\% \\
1000 & 20.92\% & 5\% & 10 & 21.05\% & 20 & 58.23\% \\
\hline
2000 & 9.44\% & 1\% & 10 & 9.34\% & 20 & 9.26\% \\
2000 & 15.92\% & 2\% & 10 & 15.64\% & 20 & 16.16\% \\
2000 & 30.66\% & 5\% & 10 & 30.44\% & 20 & 31.02\% \\
\hline
\end{array}
\]

Table 12: Degree corrected blockmodel: false alarm rates and detection rates for \( \chi^2 \) method.
| $n$ | $\alpha$ | Clique size | node DR | Clique size | node DR |
|-----|----------|-------------|---------|-------------|---------|
| 500 | 1%       | 5           | 3.91%   | 10          | 30.27%  |
| 500 | 2%       | 5           | 4.55%   | 10          | 32.45%  |
| 500 | 5%       | 5           | 5.62%   | 10          | 35.48%  |
| 1000| 1%       | 10          | 27.30%  | 20          | 40.75%  |
| 1000| 2%       | 10          | 28.66%  | 20          | 43.36%  |
| 1000| 5%       | 10          | 30.50%  | 20          | 47.25%  |
| 2000| 1%       | 10          | 15.64%  | 20          | 57.01%  |
| 2000| 2%       | 10          | 16.34%  | 20          | 58.35%  |
| 2000| 5%       | 10          | 17.41%  | 20          | 60.31%  |

Table 13: Degree corrected blockmodel: node detection rates for egonet method.

5.5 Popularity adjusted blockmodel

For the popularity adjusted blockmodel we used parameter configurations from the simulation study of Sengupta and Chen (2018). We consider networks with $K = 2$ equal size communities. Model parameters are set as $\lambda_{ir} = \alpha \sqrt{\frac{h}{1+h}}$ when $r = c_i$, and $\lambda_{ir} = \beta \sqrt{\frac{1}{1+h}}$ when $r \neq c_i$, where $h = 4$ is the homophily factor. In each community, we designate 50% of the nodes as category 1 and 50% of the nodes as category 2. We set $\alpha = 0.8$, $\beta = 0.2$ for category 1 nodes and $\alpha = 0.2$, $\beta = 0.8$ for category 2 nodes. Due to the high runtime for the extreme points method implemented on larger networks, we used a smaller number of networks for this model compared to other models, 10000 networks for $n = 500$, 5000 networks for $n = 1000$, and 500 networks for $n = 2000$.

The results are tabulated in Tables 14–16. The overall performance of the egonet method under this model is similar to that under the Chung-Lu model and the Degree corrected stochastic blockmodel, however the detection rates are slightly lower than these models. The false alarm rates are practically zero in most scenarios. In contrast, results from the $\chi^2$ method are very erratic, with very high false alarm rates.
| $n$ | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|-----|-------------|----------|-------------|-----------|-------------|-----------|
| 500 | 0.03%       | 1%       | 5           | 0.14%     | 10          | 99.58%    |
| 500 | 0.11%       | 2%       | 5           | 0.32%     | 10          | 99.73%    |
| 500 | 0.3%        | 5%       | 5           | 0.77%     | 10          | 99.85%    |
| 1000| 0.04%       | 1%       | 10          | 86.30%    | 20          | 100%      |
| 1000| 0.1%        | 2%       | 10          | 90.18%    | 20          | 100%      |
| 1000| 0.28%       | 5%       | 10          | 94.18%    | 20          | 100%      |
| 2000| 0%          | 1%       | 10          | 6.80%     | 20          | 100%      |
| 2000| 0%          | 2%       | 10          | 11%       | 20          | 100%      |
| 2000| 0.20%       | 5%       | 10          | 17.6%     | 20          | 100%      |

Table 14: Popularity adjusted blockmodel: false alarm rates and detection rates for egonet method.

| $n$ | False alarm | $\alpha$ | Clique size | Detection | Clique size | Detection |
|-----|-------------|----------|-------------|-----------|-------------|-----------|
| 500 | 21.2%       | 1%       | 5           | 21.24%    | 10          | 20.82%    |
| 500 | 28.23%      | 2%       | 5           | 27.99%    | 10          | 30.22%    |
| 500 | 40.41%      | 5%       | 5           | 39.60%    | 10          | 46.55%    |
| 1000| 27.42%      | 1%       | 10          | 26.9%     | 20          | 98.22%    |
| 1000| 34.56%      | 2%       | 10          | 33.78%    | 20          | 99.08%    |
| 1000| 46.01%      | 5%       | 10          | 45.50%    | 20          | 99.66%    |
| 2000| 39.6%       | 1%       | 10          | 38.2%     | 20          | 74.6%     |
| 2000| 45%         | 2%       | 10          | 44.2%     | 20          | 79%       |
| 2000| 54.6%       | 5%       | 10          | 54.2%     | 20          | 84.4%     |

Table 15: Popularity adjusted blockmodel: false alarm rates and detection rates for $\chi^2$ method.
### Table 16: Popularity adjusted blockmodel: node detection rates for egonet method.

| $n$  | $\alpha$ | Clique size | node DR | Clique size | node DR |
|------|----------|-------------|---------|-------------|---------|
| 500  | 1%       | 5           | 0.02%   | 10          | 47.49%  |
| 500  | 2%       | 5           | 0.04%   | 10          | 49.51%  |
| 500  | 5%       | 5           | 0.10%   | 10          | 52.32%  |
| 1000 | 1%       | 10          | 20.85%  | 20          | 98.19%  |
| 1000 | 2%       | 10          | 24.09%  | 20          | 98.64%  |
| 1000 | 5%       | 10          | 28.64%  | 20          | 99.12%  |
| 2000 | 1%       | 10          | 0.74%   | 20          | 57.68%  |
| 2000 | 2%       | 10          | 1.2%    | 20          | 59.86%  |
| 2000 | 5%       | 10          | 2.1%    | 20          | 63.48%  |

5.6 Overall comments from simulations

We make the following general observations from this simulation study.

**Remark 5.1. Performance of the egonet method:** Overall the egonet method performs well with respect to all four performance metrics. Recall that the node false alarm rate from the egonet method was below 0.01% under all models and all scenarios, and therefore we do not report this metric in the interest of space. It definitely performs better than the benchmark $\chi^2$ method by a substantial margin. The results from egonet follow statistical intuition, while the results from $\chi^2$ are very erratic and unreliable.

**Remark 5.2. Comparison across scenarios:** Across the different models, the $n = 500, m = 5$ scenario is the most difficult to detect. The $n = 2000, m = 10$ scenario is also hard to detect under the Erdős-Rényi model, the Stochastic blockmodel, and the Popularity adjusted blockmodel. Performance of the egonet method was very good under all other scenarios.

**Remark 5.3. Comparison across models:** While the egonet method works well under all models studied here, it seems to work particularly well under the Chung-Lu model and the Degree corrected blockmodel, followed closely by the Popularity adjusted blockmodel. Under these models, the egonet method has very low false alarm rates and high detection rates. A possible explanation is that the egonet method works best when there is a lot of variation in the edge probabilities $p_{ij}$.

On the other hand, the results from the Erdős-Rényi model and the stochastic blockmodel are quite similar. Note that, under the stochastic blockmodel each community is
distributed like an Erdős-Rényi random graph. Therefore, there seems to be interesting ways in which the similarity between network models translates to similar clique detection results using the egonet method. This is an intriguing phenomenon that we plan to investigate in greater detail in future work.

Remark 5.4. **Recommendation on significance level:** the power of the test (detection rate) does not seem to be very sensitive to the choice of the significance level $\alpha$. Therefore, we recommend the use of a small significance level, e.g., $\alpha = 1\%$.

6 Case studies with real-world network data

We implemented the egonet method on three well-known networks. In real-world applications, an important consideration is which null model to use for fitting the data and computing p-values. A practitioner can choose a null model based on heuristic considerations about the structure of the network, or take the cautious route of carrying out multiple versions of the test using different null models. In the latter case the inference can vary from one null model to another. Another option is to use the most general model (e.g., among the models used in this paper, the popularity adjusted blockmodel is the most general model and all other models are its special cases), although this can potentially lead to overfitting the data. In the following well-studied examples, we used null models from recent studies where a certain model was shown to work well for a certain network. Following the recommendation from Section 5, we used $\alpha = 1\%$ as the significance level.

Our first dataset is Zachary’s karate club (Zachary, 1977), a classical network dataset representing friendship relations between the 34 members of a karate club. Following Karrer and Newman (2011), we used the 2-community degree corrected blockmodel as the null model. The null hypothesis was not rejected, i.e., no anomalous clique was detected.

Our second dataset is the dolphin network (Lusseau, 2003), a well-studied network representing frequent associations between 62 bottlenose dolphins. Following Newman and Reinert (2016), we again used the 2-community degree corrected blockmodel as the null model. The null hypothesis was not rejected, i.e., no anomalous clique was detected.

Our third and final case study is on the political blogs network (Adamic and Glance, 2005), representing 16,714 hyperlinks between 1222 political weblogs before the 2004 US Presidential election. This network has been studied by many papers (Karrer and Newman, 2011; Zhao et al., 2012; Amini et al., 2013; Jin, 2015; Bickel and Sarkar, 2016; Le et al., 2016; Sengupta and Chen, 2018) in the statistics literature pertaining to community structure. A
prominent feature of this network is its strong community structure with two communities representing liberals and conservatives. Following Sengupta and Chen (2018), we used the 2-community popularity adjusted blockmodel as the null model. The null hypothesis was rejected, which indicates the presence of an anomalous clique.

To check the veracity of this inference as well as to gain empirical insight, we sought to recover the clique and investigate its constituent nodes. For this purpose, we constructed the subgraph spanned by nodes whose egonets breached the $\alpha_n$ threshold, and ran a clique detection algorithm from the igraph package in R on this small subgraph. We recovered a 10-node clique (Fig. 2), consisting of 2 liberal nodes and 8 conservative nodes, which confirms that the egonet method was correct in indicating the presence of a clique.

![Figure 2: Political blogs clique consisting of two liberal (blue) and 8 conservative (red) nodes.](image)

The political blogs network is a sparse network with a network density of only 2.24%, and therefore it is surprising to discover a fully connected 10-node clique in this network. The clique contains both liberal and conservative blogs, which is further intriguing, since in this network there is little interaction between the liberal and conservative communities — only 0.4% of the 372,696 liberal-conservative pairs are connected to each other. One would expect that if any clique was present it would either consist of only liberal nodes or only conservative nodes.

Therefore, we looked into the members of the clique to figure out, at least heuristically, the reason for their unusual inter-connection. For this, we focused on the two liberal blogs, namely buzzmachine.com and oxblog.blogspot.com. The first blog is run by Jeff Jarvis, a professor at the City University of New York, who is known to be a centrist liberal (Kurtz, 2005). The second blog also appears to espouse a centrist viewpoint, to the extent that five of the eight conservative blogs in the clique are listed on the oxblog website under “blogs we like”. Therefore, a possible explanation of the unusual clique is that centrists from both communities are likely to have dialogues among themselves, which makes them different
from other blogs that are much more likely to interact only within their own community.

7 Discussion

The goal of anomaly detection is to detect whether there is an anomaly, and if any anomaly is detected, to identify the part of the system that constitutes the anomaly. In this paper we have formulated a statistical inferential framework using egonets to address both these questions simultaneously in the context of clique detection in static networks. The egonet method is naturally amenable to parallel computing, making it computationally scalable. The flexible nature of the method makes it amenable for a wide variety of network models. We established theoretical guarantees under the Erdős-Rényi model and extended the method to four well-known network models. The egonet method performs well for synthetic networks from a wide variety of models, performing better than the benchmark $\chi^2$ method by a significant margin. We implemented the method on three well-studied network datasets and obtained some interesting empirical results.

The problem of anomaly detection in networks is an important and relevant problem which has not received much attention from the statistics community. We envisage this work as one of the first steps towards addressing this methodological gap.

Supplementary material to “Anomaly detection in static networks using egonets”

This supplementary document contains proofs of the theoretical results stated in the main manuscript. We first state an inequality for Binomial distributions from Hagerup and Rübl (1990) and Boucheron et al. (2013) (page 48) that will be used throughout these proofs: if $B \sim Binomial(n, p)$ and $a < p$,

$$P[B \leq an] \leq \left[ \left( \frac{p}{a} \right)^a e^{a-p} \right]^n, \quad (8)$$

and conversely, if $B \sim Binomial(n, p)$ and $a > p$,

$$P[B > an] \leq \left[ \left( \frac{p}{a} \right)^a e^{a-p} \right]^n. \quad (9)$$
7.1 Proof of Lemma 3.1

Lemma. When \( \hat{p} < p \), under Assumption 3.1,

1. Let \( B_n = \{ \frac{p - \hat{p}}{\hat{p}} \leq \frac{1}{np^{1+\delta_2}} \} \), then \( P[B_n^c] \to 0 \).

2. Let \( D_n = \{ d_i \in np(c, e) \text{ for all } i = 1, \ldots, n \} \). Then \( \exists c > 0 \) such that \( P[D_n^c] \to 0 \).

3. When \( B_n \cap D_n \) holds, \( G(\hat{k}_i; n_i, p) - G(\hat{k}_i; n_i, \hat{p}) = o \left( \frac{1}{n} \right) \).

Proof of 1: Let \( B = \sum_{i>j} A_{ij} \), then \( B \sim Binomial(\binom{n}{2}, p) \), note that \( \hat{p} = \frac{B}{\binom{n}{2}} \). Let \( \epsilon = \frac{1}{np^{1+\delta_2}} \), and \( a = \frac{p}{1+\epsilon} \). Note that \( p - \hat{p} \hat{p} \leq \epsilon \iff \hat{p} \leq \frac{p}{1+\epsilon} \). Then for \( n \) large enough,

\[
P \left[ \frac{p - \hat{p}}{\hat{p}} \leq \epsilon \right] = P \left[ \hat{p} \leq \frac{p}{1+\epsilon} \right]
\]

\[
= P \left[ B \leq \binom{n}{2} \frac{p}{1+\epsilon} \right]
\]

\[
\leq \left[ (1+\epsilon)^{\frac{p}{1+\epsilon}} e^{\frac{p}{1+\epsilon} - 1} \right]^{\binom{n}{2}} \text{ using (1)}
\]

\[
= \left[ e^{\log(1+\epsilon)^{\frac{p}{1+\epsilon}} e^{\frac{p}{1+\epsilon} - 1}} \right]^{\binom{n}{2}}
\]

\[
= \exp \left( -\binom{n}{2} p \left( 1 - \frac{1+\log(1+\epsilon)}{1+\epsilon} \right) \right)
\]

\[
\approx \exp \left( -\frac{n^2 \epsilon^2 p}{8} \right) \text{ using Taylor series approximation of } \log(1 + \epsilon)
\]

\[
\leq \exp \left( -\frac{n^2 \epsilon^2 p}{8} \right) = \exp \left( -\frac{1}{8p^{2\delta_2}} \right).
\]

As \( n \to \infty, p \to 0 \) from condition 1, hence \( P \left[ \frac{p - \hat{p}}{\hat{p}} \leq \epsilon \right] \leq \exp \left( -\frac{1}{8p^{2\delta_2}} \right) \to 0 \). □

Proof of 2: We know that \( d_i \sim Binomial(n-1, p) \). Then,

\[
P[d_i > epn]
\]

\[
\leq P[d_i > ep(n-1)]
\]

\[
\leq \left[ \left( \frac{p}{ep} \right)^e e^{-p} \right]^{n-1} \text{ using (2)}
\]

\[
= e^{-n e^{-p}}
\]

\[
\leq \frac{1}{n^{1+\delta_2}} \text{ from Assumption 3.1, part 2}
\]

Therefore,

\[
P[\bigcup_{i=1}^{n} \{ d_i > epn \}] \leq \sum_{i=1}^{n} P[d_i > epn] \leq \sum_{i=1}^{n} \frac{1}{n^{1+\delta_2}} = \frac{1}{n^{\delta_2}} \to 0.
\]
For the other direction, note that \( 1 - \frac{1+\log(x)}{x} \uparrow 1 \) as \( x \to \infty \), and therefore there is some \( c > 0 \) such that \( 1 - \frac{1+\log(c)}{c} > \frac{1+0.5\delta_2}{1+\delta_2} \). Therefore for large enough \( n \), with \( a = \frac{p}{c} \),

\[
P[d_i \leq \frac{np}{c}] \leq \sum_{i=1}^{n} P[d_i < \frac{np}{c}] \leq \sum_{i=1}^{n} \frac{1}{n^{1+0.5\delta_2}} = \frac{1}{n^{1+0.5\delta_2}} \to 0. \tag{11}
\]

Combining (3) and (4),

\[
P[D_n^c] \leq \sum_{i=1}^{n} P[d_i > epn] + P[\cup_{i=1}^{n} \{d_i < \frac{np}{c}\}] \to 0. \quad \Box
\]

Note that in the lemma, we denote \( c \) as \( 1/c \).

**Proof of 3:** We first prove two facts that will be useful in proving the result.

1. Let \( \hat{a} = \hat{k}_i/n_i \), then \( \hat{a}_i - \hat{p} \leq \sqrt{\frac{\log(n/\alpha)}{2n_i}} \).

   **Proof:** Let \( B \sim \text{Binomial}(n_i, \hat{p}) \). Using Hoeffding’s inequality,

   \[
P[B \geq n_i(\hat{p} + \epsilon)] \leq \exp(-2n_i\epsilon^2).
\]

   Put \( \hat{a}_i = \hat{p} + \epsilon \) where \( \epsilon = \frac{\log(n/\alpha)}{2n_i} \). Then

   \[
P[B \geq n_i\hat{a}_i] \leq \exp(-2n_i\frac{\log(n/\alpha)}{2n_i}) = \exp(-\log(n/\alpha)) = \alpha/n.
\]

   From the definition of \( \hat{k}_i \), we know that \( \hat{k}_i \) is the minimum value such that \( P[B \geq \hat{k}_i] \leq \alpha/n \). Therefore, \( \hat{k}_i/n_i \leq \hat{a} \leq \hat{p} + \epsilon \) where \( \epsilon = \sqrt{\frac{\log(n/\alpha)}{2n_i}} \).
2. Suppose $B \sim Binomial(n, p)$ and $k = na \geq np$, then for $n$ large enough $P[B = an] \leq \frac{2(a-p)}{a(1-p)} P[B \geq an]$.

Proof: This follows from Theorem 2 of [Arratia and Gordon (1989)] and discussions in their paper, in particular the comments just before equation (3) in pg 5 of their paper. They show that, asymptotically,

$$P[B = an] = (1 - r)P[B \geq an],$$

where $r = \frac{p(1-a)}{a(1-p)}$, which implies $(1 - r) = \frac{a-p}{a(1-p)}$. We skip a detailed proof and refer interested readers to their paper for more details.

Armed with these two facts, we are now ready to prove the third part of Lemma 3.1. Note that the following derivations are conditional on the event $B_n \cap D_n$. Using the regularized incomplete beta expression, for $n$ large enough,

$$G(\hat{k}_i; n_i, p) = \frac{1}{Beta(k_i, n_i-k_i+1)} \int_0^p t^{\hat{k}_i-1}(1-t)^{n_i-\hat{k}_i} dt,$$

$$G(\hat{k}_i; n_i, \hat{p}) = \frac{1}{Beta(k_i, n_i-k_i+1)} \int_0^{\hat{p}} t^{\hat{k}_i-1}(1-t)^{n_i-\hat{k}_i} dt,$$

$$\Rightarrow G(\hat{k}_i; n_i, p) - G(\hat{k}_i; n_i, \hat{p}) = \frac{1}{Beta(k_i, n_i-k_i+1)} \int_0^{\hat{p}} t^{\hat{k}_i-1}(1-t)^{n_i-\hat{k}_i} dt$$

$$= \frac{n_i!}{(k_i-1)!(n_i-k_i)!} \hat{p}^{\hat{k}_i}(1-\hat{p})^{n_i-\hat{k}_i} p \frac{(p-\hat{p})}{\hat{p}}$$

Taylor series approximation

$$\Rightarrow \frac{n_i!}{k_i!(n_i-k_i)!} \hat{p}^{\hat{k}_i}(1-\hat{p})^{n_i-\hat{k}_i} \frac{p}{\hat{p}}$$

expressing the first part as a Binomial pmf

$$\leq \frac{2(a-p)}{a(1-p)} \times \frac{\alpha}{n} \times \hat{a} n_i \frac{p-\hat{p}}{\hat{p}}$$

from fact 2 and definition of $\hat{k}_i$

$$= \frac{2(a-p)}{a(1-p)} \times \frac{\alpha}{n} \times n_i \frac{p-\hat{p}}{\hat{p}}$$

cancelling $\hat{a}$

$$\leq 2 \sqrt{\log(n/\alpha) \frac{2n_i}{(1-p)}} \times \frac{\alpha}{n} \times n_i \frac{p-\hat{p}}{\hat{p}}$$

from fact 1

$$\leq 4 \sqrt{\log(n/\alpha) \frac{2n_i}{(1-\hat{p})}} \times \frac{\alpha}{n} \times n_i \frac{p-\hat{p}}{\hat{p}}$$

since $\hat{p} < 1/2$ for $n$ large

$$= 2\sqrt{2} \sqrt{\log(n/\alpha) \times \frac{\alpha}{n} \times \sqrt{n_i} \frac{p-\hat{p}}{\hat{p}}}$$

under $D_n$, by Lemma 3.1 part 2

$$\leq 2\sqrt{2} \sqrt{\log(n/\alpha) \times \frac{\alpha}{n} \times n \frac{p-\hat{p}}{\hat{p}}}$$

under $B_n$, by Lemma 3.1 part 1

$$\leq 2\alpha e \sqrt{\log(n/\alpha) \frac{p}{\sqrt{2}} \frac{p}{\hat{p}}}$$

by Assumption 3.1 part 1. □
7.2 Proof of Lemma 3.2

Lemma. Under assumptions 3.1 and 3.2, the following results hold.

1. Let \( B_n = \{ \hat{p} \leq p(\frac{1}{\delta_3} + e) \} \), then \( P[B_n^c] \to 0 \).

2. Let \( D_n = \{ d_i \leq m + enp \text{ for all } i \in C \} \). Then \( P[D_n^c] \to 0 \).

3. When \( B_n \cap D_n \) holds, \( G \left( \binom{m-1}{2}; n, \hat{p} \right) \leq \frac{a}{n} \) for all \( i \in C \).

Proof of 1: We know that

\[
\hat{p} = \frac{1}{\binom{n}{2}} \sum_{A_{ij}} = \frac{1}{\binom{n}{2}} \left( \binom{m}{2} + B \right),
\]

where

\[
B \sim \text{Binomial} \left( \binom{n}{2} - \binom{m}{2}, p \right).
\]

Let

\[
B' = B + B_1, \text{ where } B_1 \sim \text{Binomial} \left( \binom{m}{2}, p \right),
\]

then clearly

\[
\hat{p} \leq \frac{1}{\binom{n}{2}} \left( \binom{m}{2} + B' \right), \text{ and } B' \sim \text{Binomial} \left( \binom{n}{2}, p \right).
\]

Applying inequality (2) on \( B' \),

\[
P \left[ B' > ep \binom{n}{2} \right] \leq \left( \frac{p}{ep} \right)^{ep} \binom{n}{2} = \left[ e^{-ep} e^{ep} \binom{n}{2} \right] = e^{-\binom{n}{2}p} \to 0 \text{ by assumption 3.1, part 2.}
\]

Note that for sufficiently large \( n \),

\[
B' \leq ep \binom{n}{2} \Rightarrow \hat{p} \leq p \left( \frac{cn}{p} + e \right) \leq p \left( \frac{1}{\delta_3} + e \right) \text{ by assumption 3.2 part 1}
\]

Therefore,

\[
P[B_n^c] \leq P \left[ B' > ep \binom{n}{2} \right] \to 0.
\]
Proof of 2: The proof is very similar to the first proof. Fix \( i \in \mathcal{C} \), then

\[
d_i = (m - 1) + \sum_{j \notin \mathcal{C}} A_{ij} \leq m + \sum_{j \notin \mathcal{C}} A_{ij} = m + B
\]

where

\[
B \sim \text{Binomial}(n - m, p).
\]

Let

\[
B' = B + B_1, \quad \text{where} \quad B_1 \sim \text{Binomial}(m, p),
\]

then clearly

\[
d_i \leq m + B', \quad \text{and} \quad B' \sim \text{Binomial}(n, p).
\]

Applying inequality (2) on \( B' \),

\[
P[B' > epn] \leq \left[ \left( \frac{p}{e} \right)^{ep} e^{ep-p} \right]^n = \left[ e^{-ep} e^{ep-p} \right]^n = e^{-np} \leq \frac{1}{n^{1+\delta_2}} \text{ by assumption 3.2, part 2}
\]

So for any \( i \in \mathcal{C} \),

\[
P[d_i > m + enp] \leq P[B' > epn] \leq \frac{1}{n^{1+\delta_2}},
\]

which implies that

\[
P[D^C_n] \leq \sum_{i \in \mathcal{C}} P[d_i > m + enp] \leq \sum_{i \in \mathcal{C}} \frac{1}{n^{1+\delta_2}} = \frac{m}{n^{1+\delta_2}} \to 0.
\]

Proof of 3: Fix \( i \in \mathcal{C} \), and define

\[
a_i = \frac{\binom{m-1}{2}}{n_i}.
\]

We first prove that under \( B_n \cap D_a \),

\[
a_i \geq e^2 \hat{p}.
\]

Under \( B_n \cap D_a \), we have

\[
\hat{p} \leq p \left( \frac{1}{\delta_3} + e \right), \quad \text{and} \quad d_i \leq m + enp, \implies n_i \hat{p} \leq \left( \frac{1}{\delta_3} + e \right)(m + enp)^2 p.
\]

Therefore it suffices to prove that

\[
e^2 \left( \frac{1}{\delta_3} + e \right)(m + enp)^2 p \leq \left( \frac{m - 1}{2} \right).
\]
Consider two cases.

1. When \( m \geq enp \),

\[
e^2 \left( \frac{1}{\delta_3} + e \right) (m + enp)^2 p \leq 4e^2 \left( \frac{1}{\delta_3} + e \right) m^2 p \leq \left( \frac{m-1}{2} \right)
\]

for sufficiently large \( n \), since \( p \to 0 \) as \( n \to \infty \) by assumption 3.1 part 1.

2. When \( m < enp \),

\[
e^2 \left( \frac{1}{\delta_3} + e \right) (m + enp)^2 p \leq 4e^4 \left( \frac{1}{\delta_3} + e \right) n^2 p^3 \leq \left( \frac{m-1}{2} \right)
\]

for sufficiently large \( n \), since \( \frac{c_n}{n^3} \to 0 \) as \( n \to \infty \) by assumption 3.2 part 2.

Now we are ready to prove the main result by using (1).

\[
G \left( \binom{m-1}{2}; n_i, \hat{p} \right) \leq \left[ \left( \frac{\hat{p}}{a_i} \right)^{a_i} e^{a_i - \hat{p}} \right]^{n_i} \text{ by (2)}
\]

\[
= \left[ \left( \frac{\hat{p}}{a_i} \right)^{a_i} e^{-\hat{p}} \right]^{n_i}
\]

\[
\leq \left( \frac{\hat{p}}{a_i} \right)^{a_i n_i} \text{ since } e^{-n_i \hat{p}} \leq 1
\]

\[
\leq e^{-a_i n_i} \text{ by (13)}
\]

\[
= e^{-\binom{m-1}{2}} \leq \frac{\alpha}{n}
\]

for sufficiently large \( n \), since \( \frac{\binom{m-1}{2}}{\log(n)} \to \infty \) by assumption 3.2 part 2.

**Proof of Theorem 3.1**

Note that under \( H_0 \), the true distribution of \( e_i \) is \( \text{Binomial}(n_i, p) \), but the reference distribution for estimated p-values is \( \text{Binomial}(n_i, \hat{p}) \). The estimated p-value threshold is breached by the \( i^{th} \) egonet when \( G(e_i; n_i, \hat{p}) \) is less than or equal to \( \frac{\alpha}{n} \), i.e., when \( e_i \) is greater than or equal to \( \hat{k}_i \). The probability of this event is therefore the same as the probability than a \( \text{Binomial}(n_i, p) \) random variable takes a value greater than or equal to \( \hat{k}_i \), and this probability is (from definition of \( G(\cdot) \)) precisely \( G(\hat{k}_i; n_i, p) \), i.e.,

\[
P[G(e_i; n_i, \hat{p}) \leq \frac{\alpha}{n}] = G(\hat{k}_i; n_i, p).
\]  

(13)
When $H_0$ is true, $\hat{p} < p$, and $B_n \cap D_n$ holds, the type I error rate can be expressed as

\[
P[T_n \leq \frac{\alpha}{n}] = P \left[ \left( \min_{1 \leq i \leq n} G(e_i; n_i, \hat{p}) \right) \leq \frac{\alpha}{n} \right] \leq P \left[ \bigcup_{i=1}^{n} \left\{ G(e_i; n_i, \hat{p}) \leq \frac{\alpha}{n} \right\} \right] \leq \sum_{i=1}^{n} P \left[ G(e_i; n_i, \hat{p}) \leq \frac{\alpha}{n} \right] = \sum_{i=1}^{n} G(\hat{k}_i; n_i, \hat{p}) \text{ from Eqn (6)} \leq \sum_{i=1}^{n} \left( G(\hat{k}_i; n_i, \hat{p}) + o \left( \frac{1}{n} \right) \right) \text{ from Lemma 1.3} \leq \sum_{i=1}^{n} \left( \frac{\alpha}{n} + o \left( \frac{1}{n} \right) \right) \text{ from definition of } \hat{k}_i = \alpha(1 + o(1)).
\]

Recall that when $H_0$ is true and $\hat{p} \geq p$, we have already proved that $P[T_n \leq \frac{\alpha}{n}] \leq \alpha$. Therefore, under $H_0$

\[
P[T_n \leq \frac{\alpha}{n}] = P[T_n \leq \frac{\alpha}{n} | \hat{p} < p]P[\hat{p} < p] + P[T_n \leq \frac{\alpha}{n} | \hat{p} > p]P[\hat{p} \geq p] + P[B_n^C] + P[D_n^C] \leq \alpha(1 + o(1))P[\hat{p} < p] + \alpha P[\hat{p} > p] + o(1) + + o(1) \rightarrow \alpha \text{ as } n \rightarrow \infty.
\]

**Proof of Theorem 3.2**

As observed before, since $T_n(A) \leq T_n^{(1)}(A)$,

\[
P \left[ T_n^{(1)} < \frac{\alpha}{n} \right] \rightarrow 1 \Rightarrow P \left[ T_n < \frac{\alpha}{n} \right] \rightarrow 1,
\]

and therefore we only need to prove the second result.

When $H_1$ is true, let $i \in \mathcal{C}$ be an anomalous node. Then $i$ is connected to all the other $(m - 1)$ anomalous nodes, which means the $i^{th}$ egonet contains the remaining $(m - 1)$ anomalous nodes, and the $(m - 1)$ edges between them. Therefore

\[
e_i \geq \binom{m-1}{2} \text{ for all } i \in \mathcal{C}.
\]

Recall that for any given $n_i$ and $p$, $G(x; n_i, \hat{p})$ is a non-increasing function of $x$. Therefore,
when $B_n \cap D_n$ holds,

$$G(e_i; n_i, \hat{p}) \leq G \left( \binom{n}{2}; n_i, \hat{p} \right) \quad \text{for all } i \in \mathcal{C}$$

$$\Rightarrow G(e_i; n_i, \hat{p}) < \alpha \frac{n}{n} \quad \text{for all } i \in \mathcal{C} \Rightarrow T_n^{(1)}(A) < \frac{\alpha}{n}.$$

Therefore,

$$P \left[ T_n^{(1)}(A) < \frac{\alpha}{n} \right] \leq P[B_n \cap D_n] \leq 1 - (P[B_n^C] + P[D_n^C]) \to 1 \quad \text{as } n \to \infty. \quad \Box$$

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