RANKIN-SELBERG METHOD FOR JACOBI FORMS OF INTEGRAL WEIGHT AND OF HALF-INTEGRAL WEIGHT ON SYMPLECTIC GROUPS.

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ABSTRACT. In this article we show analytic properties of certain Rankin-Selberg type Dirichlet series for holomorphic Jacobi cusp forms of integral weight and of half-integral weight. The numerators of these Dirichlet series are the inner products of Fourier-Jacobi coefficients of two Jacobi cusp forms. The denominators and the range of summation of these Dirichlet series are like the ones of the Koecher-Maass series. The meromorphic continuations and functional equations of these Dirichlet series are obtained. Moreover, an identity between the Petersson norms of Jacobi forms with respect to linear isomorphism between Jacobi forms of integral weight and half-integral weight is also obtained.

1. INTRODUCTION

1.1. The aim of this paper is to show analytic properties of certain Rankin-Selberg type Dirichlet series of Jacobi cusp forms. In [K-S 89] Kohnen and Skoruppa introduced the following Dirichlet series associated to Siegel cusp forms of degree two:

$$\sum_{m=1}^{\infty} \frac{\langle f_m, g_m \rangle}{m^s},$$

where $f_m$ and $g_m$ are $m$-th Fourier-Jacobi coefficients of two Siegel cusp forms of degree two $F$ and $G$, respectively, and where $\langle f_m, g_m \rangle$ is the Petersson inner product of Jacobi cusp forms $f_m$ and $g_m$. They showed a meromorphic continuation and the functional equation of this Dirichlet series. This result has been generalized to Siegel cusp forms of arbitral degree $n$ by Yamazaki [Ya 90]. It means that $F$ and $G$ can be replaced by Siegel cusp forms of arbitral degree $n$ and $f_m$ and $g_m$ are Fourier-Jacobi coefficients of $F$ and $G$ of not only integer index, but also matrix index $m$. If $m$ runs over matrices of a fixed size, the range of summation and the denominators of Yamazaki Dirichlet series are the same to the ones of the Koecher-Maass Dirichlet series.

In this paper we start with two Jacobi cusp forms $\phi_M$ and $\psi_M$ instead of two Siegel cusp forms $F$ and $G$. Here $\phi_M$ and $\psi_M$ are Jacobi cusp forms of index $\mathcal{M}$ and where $\mathcal{M}$ is a half-integral symmetric matrix. The Fourier-Jacobi coefficients $\phi_N$ and $\psi_N$ of

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\( \phi_M \) and \( \psi_M \), respectively, are also Jacobi cusp forms. We will show a meromorphic continuation and the functional equation of similar Dirichlet series as above.

The importance of this result is that one can apply it to obtain also similar result for Jacobi cusp forms (which includes Siegel cusp forms) of half-integral weight in a generalized plus space. This result is a generalization of the result for the Rankin-Selberg type Dirichlet series of Siegel cusp forms of half-integral weight obtained by Katsurada and Kawamura in [K-K 15, Corollary to Proposition 3.1]. It means that in [K-K 15] they treated Fourier coefficients of Siegel cusp forms of half-integral weight to construct the Rankin-Selberg type Dirichlet series. In this paper we shall treat Jacobi cusp forms of half-integral weight of arbitral degree instead of Siegel cusp forms of half-integral weight and we also shall treat Fourier-Jacobi coefficients instead of Fourier coefficients.

1.2. To be more precise, let \( L_n^+ \) be the set of all half-integral symmetric matrices of size \( n \) and \( L_n^{t+} \) be the subset of all positive-definite matrices in \( L_n^+ \).

We fix a matrix \( M \in L_n^{t+} \). We set

\[
L_{t,r}^+(M) := \left\{ N = \begin{pmatrix} N & \frac{1}{2} R \\ \frac{1}{2} R^T \end{pmatrix} \in L_{t+r}^+ \mid N \in L_t^+, R \in \mathbb{Z}^{(t,r)} \right\}
\]

and we put

\[
B_{t,r}(\mathbb{Z}) := \left\{ \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ 0 & 1_r \end{pmatrix} \in GL(t + r, \mathbb{Z}) \mid \gamma_1 \in GL(t, \mathbb{Z}), \gamma_2 \in \mathbb{Z}^{(t,r)} \right\}.
\]

The group \( B_{t,r}(\mathbb{Z}) \) acts on \( L_{t,r}^+(M) \) by \( \gamma \cdot N := N^{[t]} \gamma \) for \( \gamma \in B_{t,r}(\mathbb{Z}) \) and \( N \in L_{t,r}^+(M) \).

Here we put \( X[Y] := tYXY \) for matrices \( X \) and \( Y \) of suitable sizes.

We set

\[
\epsilon(M) := \{ \gamma \in GL(r, \mathbb{Z}) \mid \gamma \cdot M = M \}.
\]

For \( N \in L_{t,r}^+(M) \) we set

\[
\epsilon_{t,r}(N) := \{ \gamma \in B_{t,r}(\mathbb{Z}) \mid \gamma \cdot N = N \}.
\]

The symbol \( \text{Sp}(n, \mathbb{R}) \) denotes the real symplectic group of size \( 2n \). We put \( \Gamma_n := \text{Sp}(n, \mathbb{Z}) = \text{Sp}(n, \mathbb{R}) \cap \mathbb{Z}^{(2n,2n)} \). The symbol \( \mathcal{H}_n \) denotes the Siegel upper half space of size \( n \).

Let \( \phi_M \) and \( \psi_M \) be Jacobi cusp forms of weight \( k \) of index \( M \) on \( \Gamma_n \) (cf. [Z 89, Definition 1.3]). In the case \( r = 0 \) we regard \( \phi_M \) and \( \psi_M \) as Siegel cusp forms of weight \( k \) with respect to \( \Gamma_n \). Let \( 0 < t \leq n \) and we take Fourier-Jacobi expansions of \( \phi_M \) and \( \psi_M \):

\[
\phi_M(\tau, z)e(M\omega) = \sum_{N \in L_{t,r}^+(M)} \phi_{N'}(\tau', z')e(N\omega'),
\]

\[
\psi_M(\tau, z)e(M\omega) = \sum_{N \in L_{t,r}^+(M)} \psi_{N'}(\tau', z')e(N\omega'),
\]

(1.1)
where \( \left( \frac{\tau}{t}, \frac{z}{\omega} \right) = \left( \frac{\tau'}{t'}, \frac{z'}{\omega'} \right) \), \( \tau \in \mathfrak{H}_n \), \( \omega \in \mathfrak{H}_r \), \( z \in \mathbb{C}^{(n,r)} \), \( \tau' \in \mathfrak{H}_{n-t} \), \( \omega' \in \mathfrak{H}_{t+r} \) and \( z' \in \mathbb{C}^{(n-t,t+r)} \). We have the fact that \( \phi_N \) and \( \psi_N \) are Jacobi cusp forms of weight \( k \) of index \( N' \) on \( \Gamma_{n-t} \).

The aim of this article is to obtain analytic properties of the Dirichlet series

\[
D_t(\phi_M, \psi_M; s) := \sum_{N \in B_t, r(\mathbb{Z})/L_r(M)} \frac{\langle \phi_N, \psi_N \rangle}{|\epsilon_{t,r}(N)| \det(N)^s},
\]

where we denote by \( \langle \phi_N, \psi_N \rangle \) the Petersson inner product of \( \phi_N \) and \( \psi_N \) (see §2). This Dirichlet series converges for sufficiently large \( \text{Re}(s) \) (see Lemma 2.2). If \( t = n \), then \( \phi_N \) and \( \psi_N \) are Fourier coefficients of \( \phi_M \) and \( \psi_M \), respectively, and we set \( \langle \phi_N, \psi_N \rangle = \phi_N \overline{\psi_N} \) in this case. We remark \( D_1(\phi_M, \psi_M) = \frac{1}{2} \sum_{N \in \mathbb{Z}_{>0}} \frac{\langle \phi_N, \psi_N \rangle}{N^s} \) if \( t = 1 \) and \( r = 0 \).

Remark that if \( r = 0 \) and \( n \geq 2 \), then \( \phi_M \) and \( \psi_M \) are Siegel cusp forms. And analytic properties of the Dirichlet series \( D_t(\phi_M, \psi_M; s) \) in the case \( r = 0 \) have been shown by Maass [Ma 73] (for \( t = n = 2 \)), by Kohnen-Skoruppa [K-S 89, Theorem 1] (for \( t = 1 \) and \( n = 2 \)), by Kalinin [Kal 84] (for \( t = n \geq 2 \)) and by Yamazaki [Ya 90] (for \( t \geq 1 \) and \( n \geq 2 \)). Moreover, if \( r = 1 \), then meromorphic continuations, functional equations and residues of the above Dirichlet series have been shown in the papers by Kohnen-Zagier [K-Z 81] p.189-191 (for \( t = n = 1 \) and \( M = 1 \)), by Katsurada-Kawamura [K-K 15, Proposition 3.1] (for \( t = n \geq 2 \) and \( M = 1 \)) and by Imamo˘glu-Martin [I-M 03, Theorem 2(d) and Proposition 1(b)] (for \( t = n = 1 \) and for arbitral integer \( M \)).

The main result in the present article is that the Dirichlet series \( D_t(\phi_M, \psi_M; s) \) (for \( 1 \leq t \leq n \) and for arbitral index \( M \)) has a meromorphic continuation to the whole complex plane and has a functional equation (see Theorem 2.3). The residue at \( s = k - \frac{1}{2} \) is also determined (cf. Theorem 2.3). Such properties are shown by using Rankin-Selberg method. The properties of certain real analytic Siegel-Eisenstein series, which are necessarily to prove Theorem 2.3, have been shown by Kalinin [Kal 77] (for \( t = n \)) and by Yamazaki [Ya 90] (for \( 1 \leq t < n \)). Hence the main issue in this paper is to obtain an integral expression of the Dirichlet series \( D_t(\phi_M, \psi_M; s) \) by using such Siegel-Eisenstein series. To obtain the integral expression, we shall refine a method treated by Katsurada and Kawamura in [K-K 15, Proposition 3.1]. It means that we take vector valued modular forms through theta decompositions of two Jacobi forms and take the inner product of these two vector valued modular forms. By coupling this inner product with a certain Siegel-Eisenstein series we obtain the integral expression of the Dirichlet series \( D_t(\phi_M, \psi_M; s) \) (cf. Proposition 2.3). To show the integral expression of the Dirichlet series, we use the compatibility between the theta decomposition and the Fourier-Jacobi expansion of Jacobi forms.
1.3. As for half-integral weight case, we will show also analytic properties for similar Dirichlet series of certain Jacobi cusp forms (including certain Siegel cusp forms) of half-integral weight of certain indices. Such Jacobi forms of half-integral weight belong to the so-called plus-space, which is a generalization of Kohnen plus-space of elliptic modular forms of half-integral weight. A generalization of Kohnen plus-space for Jacobi forms has been introduced in [Ha 18]. Let \( M = \left( \begin{array}{c} N \\ \frac{1}{2} R \\ \frac{1}{2} R \\ 1 \end{array} \right) \) be a matrix in \( L^+_{r-1,1} \). We put \( M = 4N - R^t R \). We assume that \( k \) is an even integer. It is shown in [Ib 92] (for \( r = 1 \)) and in [Ha 18] (for \( r > 1 \)) that there exists a linear isomorphism between the space \( J_{k,\mathcal{M}}^{(n)} \) of Jacobi forms of weight \( k \) of index \( M \) on \( \Gamma_n \) and the space of Jacobi forms \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)} \), where \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)} \) is a certain subspace of Jacobi forms of weight \( k - \frac{1}{2} \) of index \( \mathfrak{M} \) on \( \Gamma_n \) (see §3). By this linear isomorphism the Fourier coefficients of forms in \( J_{k,\mathcal{M}}^{(n)} \) correspond each other.

We remark that if \( r = 1 \), then \( \mathfrak{M} = \emptyset \) and \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)} \) is the plus-space of Siegel modular forms of weight \( k - \frac{1}{2} \) introduced by Kohnen [Ko 80] (for \( n = 1 \)) and by Ibukiyama [Ib 92] (for \( n > 1 \)). Let \( \phi_{\mathfrak{M}} \) and \( \psi_{\mathfrak{M}} \) be Jacobi cusp forms in \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)} \). We construct a Dirichlet series \( D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) \) in the same manner as in the case of integral weight. Then, by using the linear isomorphism between \( J_{k,\mathcal{M}}^{(n)} \) and \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)} \), we obtain a meromorphic continuation, a functional equation and residues of \( D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) \) (see Theorem 3.3).

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2. Dirichlet series of Jacobi forms of integral weight

We denote by \( \mathcal{H}_n \) the Siegel upper half space of size \( n \). For any ring \( R \), we denote by \( R^{(l,m)} \) the set of all matrices of size \( l \times m \) with the entries in \( R \). The symbol \( 0^{(l,m)} \) denotes the zero matrix in \( \mathbb{C}^{(l,m)} \). We denote by \( \delta_{i,j} \) the Kronecker delta. It means that \( \delta_{i,j} = 1 \) if \( i = j \) and 0 otherwise. By abuse of language we put \( \det(\mathcal{M}) = \det(4\mathcal{M}) = 1 \), if the size of the matrix \( \mathcal{M} \) is 0.

Let \( k \) be an integer and let \( \mathcal{M} \in L^+_{r-1,1} \). We denote by \( J_{k,\mathcal{M}}^{(n)} \) (resp. \( J_{k,\mathcal{M}}^{(n)cusp} \)) the space of Jacobi forms (resp. Jacobi cusp forms) of weight \( k \) of index \( \mathcal{M} \) on \( \Gamma_n \). (See the definition [Zi 89, Definition 1.3]).

For \( \phi, \psi \in J_{k,\mathcal{M}}^{(n)cusp} \), the Petersson inner product is defined by

\[
\langle \phi, \psi \rangle := \int_{\mathcal{H}_n} \phi(\tau, z) \overline{\psi(\tau, z)} e^{-4\pi Tr(\mathcal{M}v^{-1}[y])} \det(v)^{k-n-r-1} \, dv \, dx \, dy,
\]
where $\mathcal{F}_{n,r} := \Gamma_{n,r}' \backslash (\mathcal{H}_n \times \mathbb{C}^{(n,r)})$, $\tau = u + iv$, $z = x + iy$, $du = \prod_{i \leq j} u_{i,j}$, $dv = \prod_{i \leq j} v_{i,j}$, $dx = \prod_{i,j} x_{i,j}$ and $dy = \prod_{i,j} y_{i,j}$. Here we put

$$\Gamma_{n,r}' := \left\{ \begin{pmatrix} A & 0 & B & * \\ * & 1_r & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & 1_r \end{pmatrix} \in \Gamma_{n+r} \right\} \Gamma_n,$$

and the group $\Gamma_{n,r}'$ acts on $\mathcal{H}_n \times \mathbb{C}^{(n,r)}$ in the usual manner (cf. [Zi 89, p.193]). For the sake of simplicity we put $J_{k,M}^{(0),cusp} := \mathbb{C}$ and for $\phi, \psi \in \mathbb{C}$, we set $\langle \phi, \psi \rangle := \phi \overline{\psi}$.

**Lemma 2.1.** Let $\phi_N$ be the $N$-th Fourier-Jacobi coefficient of $\phi_M$ defined in (1.1). Then, there exists a constant $C'$ which does not depend on the choice of $N$ such that

$$|\phi_N(\tau', z')e(iNv'^{-1}[y])(\det v')^{\frac{1}{2}}| < C' \det(N)^{\frac{1}{2}}$$

for any $(\tau', z') \in \mathcal{H}_{n-t} \times \mathbb{C}^{(n-t,t)}$, and where $v' = \text{Im}(\tau')$ and $y' = \text{Im}(z')$.

**Proof.** Since $\phi_M$ is a Jacobi cusp form, there exists a constant $C_\phi$ which depends only on $\phi_M$ such that

$$|\phi_M(\tau, z) \det(v)^{\frac{1}{2}} \exp(-2\pi \text{Tr}(Mv^{-1}[y]))| < C_\phi$$

for any $(\tau, z) \in \mathcal{H}_n \times \mathbb{C}^{(n,r)}$, and where $v = \text{Im}(\tau)$ and $y = \text{Im}(z)$. On the other hand, we have

$$\phi_N(\tau', z')e(iNT')$$

$$= \int_{\text{Sym}_n(\mathbb{Z}) \backslash \text{Sym}_n(\mathbb{R})} \int_{\mathbb{R}^{(t,r)}} \int_{\text{Sym}_n(\mathbb{Z}) \backslash \text{Sym}_n(\mathbb{R})} \phi_M \left( \tau + \begin{pmatrix} 0 & 0 \\ 0 & X'_1 \end{pmatrix}, z + \begin{pmatrix} 0 \\ X'_2 \end{pmatrix} \right)$$

$$\times e(M(\omega + X'_2))e(-N \begin{pmatrix} X'_1 & X'_2 \\ t_1 & t_2 \end{pmatrix}) dX'_1 dX'_2 dX'_2,$$

where $(\tau'_2 \ z'') = (\tau'_1 \ z'_1 \omega) \in \mathcal{H}_{n+r}$, $\tau \in \mathcal{H}_n$, $\omega \in \mathcal{H}_r$, $z \in M_{n,r}(\mathbb{C})$, $\tau' \in \mathcal{H}_{n-t}$, $\omega' \in \mathcal{H}_{t+r}$, and $z' \in M_{n-t,t+r}(\mathbb{C})$.

We decompose the matrix $(\tau'_2 \ z'')$ as

$$\begin{pmatrix} \tau & z \\ t & \omega \end{pmatrix} = \begin{pmatrix} \tau' & z'_1 \\ t_1 & \tau_2 \\ z'_2 & z_2 \end{pmatrix} = \begin{pmatrix} \tau' & z' \\ t & \omega' \end{pmatrix} \in \mathcal{H}_{n+r}$$

with

$$\tau = \begin{pmatrix} \tau'_1 \\ z'_1 \\ t \end{pmatrix}, z = \begin{pmatrix} z'_1 \\ z'_2 \\ \tau \end{pmatrix}, \omega = \begin{pmatrix} \tau_2 \ z_2 \\ t_2 \omega \end{pmatrix},$$

$$z'_1 \in \mathbb{C}^{(n-t,t)}, z'_2 \in \mathbb{C}^{(n-t,t)}, z_2 \in \mathbb{C}^{(t,t)}, \text{ and } \tau_2 \in \mathcal{H}_t.$$
We write
\[
\begin{pmatrix}
v' & y'_1 & y'_2 \\
\tau' & z'_1 & z'_2 \\
\tau & z_2 & \omega
\end{pmatrix} := \text{Im} \begin{pmatrix}
v & \tau' & z'_1 & z'_2 \\
\tau & z_2 & \omega \\
v & \tau & z_2 & \omega
\end{pmatrix},
\]
\[v' = \text{Im}(\tau'), \quad v_2 = \text{Im}(\tau), \quad T = \text{Im}(\omega) \quad \text{and} \quad T' := \text{Im}(\omega') = \begin{pmatrix} v_2 & y_2 \end{pmatrix}.
\]
Furthermore, we write \(y' := \text{Im}(z') = \text{Im} \begin{pmatrix} z'_1 & z'_2 \end{pmatrix} = \begin{pmatrix} y'_1 & y'_2 \end{pmatrix},\) then we have
\[
|\phi_N(\tau', z') e(iN v'^{-1}[y'])(\det v')^{\frac{1}{2}}| = \left| e(-iNT') \int_{\text{Sym}_n(\mathbb{Z}) \backslash \text{Sym}_n(\mathbb{R})} \int_{\mathbb{R}^{l(r)}} \int_{\text{Sym}_m(\mathbb{Z}) \backslash \text{Sym}_m(\mathbb{R})} \phi_M(\tau + (0 \ 0 \ x_1), \ z + (0 \ x_1)) \right.
\times e(\mathcal{M}(\omega + X'_2)) e(-N \left( x'_1 x'_3 \right)) dX'_1 dX'_2 dX'_2 e(iN v'^{-1}[y'])(\det v')^{\frac{1}{2}}
\[
= \left| e(-iNT') \int_{\text{Sym}_n(\mathbb{Z}) \backslash \text{Sym}_n(\mathbb{R})} \int_{\mathbb{R}^{l(r)}} \int_{\text{Sym}_m(\mathbb{Z}) \backslash \text{Sym}_m(\mathbb{R})} \phi_M(\tau + (0 \ 0 \ x_1), \ z + (0 \ x_1)) \right.
\times \det(\text{Im}(\tau + (0 \ x_1)))^\frac{1}{2} e(i\mathcal{M}(\text{Im}(\tau + (0 \ x_1))^{-1})(\det(\text{Im}(z + (x_1)))) \times e(\mathcal{M}(\omega + X'_2)) e(-N \left( x'_1 x'_3 \right)) dX'_1 dX'_2 dX'_2 e(iN v'^{-1}[y'])(\det v')^{\frac{1}{2}}
\times \det(\text{Im}(\tau + (0 \ x_1)))^\frac{1}{2} e(-i\mathcal{M} v^{-1}[y])
\[
< C^\phi e(-iNT') e(i\mathcal{M}T) e(iN v'^{-1}[y']) \det(v')^{\frac{1}{2}} \det(v)^{\frac{1}{2}} e(-i\mathcal{M} v^{-1}[y])
\[
= C^\phi e(-iN(T' - v'^{-1}[y']))) e(i\mathcal{M}(T - v^{-1}[y])) \det(v')^{\frac{1}{2}} \det(v)^{\frac{1}{2}}.
\]
We now write \(\mathcal{N} := \left( N_2 \ \frac{1}{2}R_2 \ \frac{1}{2}R_2 \ \mathcal{M} \right)\) and put
\[
\Delta := v_2 - v'^{-1}[y'], \quad \eta := y_2 - i' y'_1 v'^{-1} y'_2.
\]
Then, by a straightforward calculation we have
\[
\text{Tr}(\mathcal{N}(T' - v'^{-1}[y'])) - \text{Tr}(\mathcal{M}(T - v^{-1}[y])) = \text{Tr} \left( \mathcal{N} \left( \begin{pmatrix} \Delta & \eta \\ \iota \eta & \Delta^{-1}[\eta] \end{pmatrix} \right) \right).
\]
Since
\[
\mathcal{N} = \left( N_2 - \frac{1}{4} \mathcal{M}^{-1} \left[ \begin{array}{c} R_2 \\ 0 \end{array} \right] \begin{array}{c} 0 \\ \mathcal{M} \end{array} \right) \left[ \begin{pmatrix} 1_t & 0 \\ \frac{1}{2} \mathcal{M}^{-1} t^r R_2 & 1_r \end{pmatrix} \right]
\]
and since
\[
\left( \begin{array}{cc} \Delta & \eta \\ \iota \eta & \Delta^{-1}[\eta] \end{array} \right) = \left( \begin{array}{cc} \Delta & 0 \\ 0 & \Delta^{-1}[\eta] \end{array} \right) \left( \begin{pmatrix} 1_t & \Delta^{-1}[\eta] \\ 0 & 1_r \end{pmatrix} \right),
\]
there exists \((v_2, y_2) \in \text{Sym}_t^+ \times \mathbb{R}^{(t,r)}\) which satisfies \(N^\frac{\Delta}{\eta} \begin{pmatrix} \frac{\Delta}{\eta} & \eta \Delta^{-1} \cr t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\). By a straightforward calculation, such \((v_2, y_2)\) is given by

\[
\begin{align*}
v_2 &= v'^{-1}[y'_1] + \left(N_2 - \frac{1}{4}\mathcal{M}^{-1}[t R_2]\right)^{-1}, \\
y_2 &= -\frac{1}{2}(N_2 - \frac{1}{4}\mathcal{M}^{-1}[t R_2])^{-1} \mathcal{M}^{-1} R_2 + t y'_1 v'^{-1} y'_2.
\end{align*}
\]

With this \((v_2, y_2)\) we have

\[
|\phi_{\mathcal{N}}(\tau', z') e(i\mathcal{N} v'^{-1}[y'_1])(\det v')^{\frac{1}{2}}| < C e^{-it} \det(v')^{\frac{1}{2}} \det(v)^{-\frac{1}{2}}
\]

\[
= C e^{-it} \det(v_2 - v'^{-1}[y'_1])^{-\frac{1}{2}}
\]

\[
= C e^{-it} \det(N_2 - \frac{1}{4}\mathcal{M}^{-1}[t R_2])^{\frac{1}{2}}
\]

\[
= C e^{-it} \det(\mathcal{M})^{-\frac{1}{2}} \det(\mathcal{N})^{\frac{1}{2}}.
\]

Hence we conclude this lemma. \(\square\)

Let \(\phi_M\) and \(\psi_M\) be Jacobi cusp forms of weight \(k\) of index \(\mathcal{M}\) on \(\Gamma_n\). Let \(D_t(\phi_M, \psi_M; s)\) be the Dirichlet series defined in (1.2).

**Lemma 2.2.** The Dirichlet series \(D_t(\phi_M, \psi_M; s)\) converges absolutely with sufficient large \(Re(s)\).

**Proof.** By virtue of Lemma 2.1 we have

\[
\langle \phi_{\mathcal{N}}, \psi_{\mathcal{N}} \rangle < C \det(\mathcal{N})^k
\]

with a certain positive number \(C\) which does not depend on the choice of \(\mathcal{N}\). Hence, it is enough to show that the series

\[
\sum_{\mathcal{N} \in B_{t,r}(\mathbb{Z}) \setminus L_t^+ M} \frac{1}{|\epsilon_{t,r}(\mathcal{N})| \det(\mathcal{N})^s}
\]

converges absolutely for sufficiently large \(Re(s)\).

It is known that the series

\[
\sum_{\tau \in GL(t,\mathbb{Z}) \setminus L_t^+} \frac{1}{|\epsilon(T)| \det(T)^s}
\]

is absolutely convergent for \(Re(s) > \frac{t+1}{2}\) (cf. [Shi 75]).

There exists a natural map \(B_{t,r}(\mathbb{Z}) \setminus L_{t,r}^+(\mathcal{M}) \to GL(t,\mathbb{Z}) \setminus L_t^+\) given by

\[
\mathcal{N} = \begin{pmatrix} N_2 & \frac{1}{2} R_2 \\ \frac{1}{2} R_2 & \mathcal{M} \end{pmatrix} \to 4l N_2 - l\mathcal{M}^{-1}[t R_2],
\]
where $2l$ is the smallest positive integer which satisfies $2l(2\mathcal{M})^{-1}\mathbb{Z}^{(r,1)} \subset \mathbb{Z}^{(r,1)}$. This map may not be injective, but if two matrices $\mathcal{N}_i = \begin{pmatrix} N_{2,i} & \frac{1}{2}R_{2,i} \\ \frac{1}{2}R_{2,i} & \mathcal{M} \end{pmatrix} \in L_{t,r}^+(\mathcal{M})$ $(i = 1, 2)$ satisfy the conditions

$$4N_{2,1} - \mathcal{M}^{-1}[tR_{2,1}] = 4N_{2,2} - \mathcal{M}^{-1}[tR_{2,2}]$$

and $R_{2,1} - R_{2,2} \in \mathbb{Z}^{(t,r)}(2\mathcal{M})$, then $\mathcal{N}_1$ and $\mathcal{N}_2$ belong to the same equivalent class in $B_{t,r}(\mathbb{Z}) \setminus L_{t,r}^+(\mathcal{M})$. Therefore, for a fixed representative $\mathcal{T}$ in $GL(t,\mathbb{Z}) \setminus L_t^+$, there exist at most $\det(2\mathcal{M})^t$ representatives in $B_{t,r}(\mathbb{Z}) \setminus L_{t,r}^+(\mathcal{M})$ which map to $\mathcal{T}$.

For $\mathcal{N} = \begin{pmatrix} N_2 & \frac{1}{2}R_2 \\ \frac{1}{2}R_2 & \mathcal{M} \end{pmatrix}$, we remark the identity

$$\det(\mathcal{N}) = (4l)^{-t} \det(\mathcal{M}) \det(4lN_2 - l\mathcal{M}^{-1}[tR_2]).$$

For any real number $s$, we have

$$\frac{1}{|\epsilon_{t,r}(\mathcal{N})| \det(\mathcal{N})^s} \leq \frac{1}{\det(\mathcal{N})^s} = (4l)^{ts} \det(\mathcal{M})^{-s} \frac{|\epsilon(\mathcal{N}^r)|}{|\epsilon(\mathcal{N}^r)| \det(\mathcal{N})^s},$$

where we put $\mathcal{N}^r = 4lN_2 - l\mathcal{M}^{-1}[tR_2]$.

It is not difficult to show that there exists a constant $c$ such that $|\epsilon(\mathcal{N}^r)| < \det(\mathcal{N}^r)^c$ for any $\mathcal{N}^r \in L_t^+$. Therefore, for sufficiently large real number $s$, we have

$$\sum_{\mathcal{N} \in B_{t,r}(\mathbb{Z}) \setminus L_{t,r}^+(\mathcal{M})} \frac{1}{|\epsilon_{t,r}(\mathcal{N})| \det(\mathcal{N})^s} \leq (4l)^{ts} 2^s \det(\mathcal{M})^{-s+t} \sum_{\mathcal{N}^r \in GL(t,\mathbb{Z}) \setminus L_t^+} \frac{1}{|\epsilon(\mathcal{N}^r)| \det(\mathcal{N}^r)^{s-c}}.$$

Thus we conclude this lemma. \hfill \Box

For $0 \leq t \leq n$, we put

$$P_{n-t,t} := \left\{ \begin{pmatrix} * & * \\ 0(t,2n-t) & * \end{pmatrix} \in \Gamma_n \right\} = \left\{ \begin{pmatrix} * & 0^{(n-t,t)} \\ 0(t,n-t) & 0^{(t,t)} \end{pmatrix} \in \Gamma_n \right\}.$$

For $\tau \in \mathfrak{H}_n$ and for $s \in \mathbb{C}$, we set

$$E_t^{(n)}(s; \tau) := \sum_{\gamma \in P_{n-t,t} \setminus \Gamma_n} \left( \frac{\det(\text{Im}(\gamma \cdot \tau))}{\det(\text{Im}(\gamma \cdot \tau)_1)} \right)^s,$$

where $(\gamma \cdot \tau)_1$ is the left upper part of $\gamma \cdot \tau$ of size $(n-t) \times (n-t)$. The series $E_t^{(n)}(s; \tau)$ converges absolutely for $\text{Re}(s) > n - \frac{t-1}{2}$ (see [Ya 90, p.42-43]).

For $R \in \mathbb{Z}^{(n,r)}$ we put

$$\partial_{\mathcal{M},R}(\tau, z) := \sum_{p \equiv R \mod \mathbb{Z}^{(n,r)}(2\mathcal{M})} e \left( \frac{1}{4} p \mathcal{M}^{-1} p\tau + p^t z \right).$$

We remark that $\partial_{\mathcal{M},R}$ is defined for $R$ modulo $\mathbb{Z}^{(n,r)}(2\mathcal{M})$. 

\[2.1\]
We take the following decompositions with theta series:

\[
\phi_M(\tau, z) = \sum_{R \mod \mathbb{Z}^{(n,r)}(2M)} \varphi_R(\tau) \vartheta_{M,R}(\tau, z),
\]

(2.2)

\[
\psi_M(\tau, z) = \sum_{R \mod \mathbb{Z}^{(n,r)}(2M)} \psi_R(\tau) \vartheta_{M,R}(\tau, z).
\]

We call it theta decomposition. We also take the Fourier expansions of \(\phi_M\) and \(\psi_M\):

\[
\phi_M(\tau, z) = \sum_{N,R} C_{\phi}(N,R) e\left(N\tau + R^t z\right),
\]

\[
\psi_M(\tau, z) = \sum_{N,R} C_{\psi}(N,R) e\left(N\tau + R^t z\right),
\]

where in the above summations \(N \in L_n^+\) and \(R \in \mathbb{Z}^{(n,r)}\) run over matrices which satisfy \(4N - \mathcal{M}^{-1}[tR] > 0\). Then we have

\[
f_R(\tau) = \sum_{N} C_{\phi}(N,R) e\left(\frac{1}{4}(4N - \mathcal{M}^{-1}[tR])\tau\right),
\]

\[
g_R(\tau) = \sum_{N} C_{\psi}(N,R) e\left(\frac{1}{4}(4N - \mathcal{M}^{-1}[tR])\tau\right),
\]

where in the above summations \(N \in L_n^+\) runs over matrices which satisfy the condition \(4N - \mathcal{M}^{-1}[tR] > 0\).

Proposition 2.3. We have an integral expression of \(D_t(\phi_M, \psi_M; s)\) as follows. If \(\text{Re}(s)\) is sufficiently large, then we obtain

\[
(1 + \delta_{t,n})^{-1} \pi^{-\frac{1}{2}(t-1)+t(s+k-n+\frac{t-r-1}{2})} \prod_{j=1}^{t} \Gamma\left(s+k-n+\frac{t-r-j}{2}\right)^{-1}
\]

\[
\times \int_{\Gamma_n \backslash \mathfrak{H} \atop \text{mod } \mathbb{Z}^{(n,r)}(2M)} \sum_{R \mod \mathbb{Z}^{(n,r)}(2M)} f_R(\tau) g_R(\tau) (\text{det}(\text{Im}(\tau)))^{k-\frac{s}{2}} E_t^{(n)}(s; \tau) d\tau
\]

\[
= \text{det}(\mathcal{M}) \frac{\pi^{\frac{t}{2}+s+k-n+\frac{t-r-1}{2}}}{2^{(n-t)r-2(t+r)(s+k-n+\frac{t-r-1}{2})}}
\]

\[
\times D_t(\phi_M, \psi_M; s + k - n + (t - r - 1)/2). \]

We will show this proposition in \[\text{§4}\]

We put \(\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)\) and set

\[
\mathcal{E}_t^{(n)}(s; \tau) := \prod_{i=1}^{t} \xi(2s + 1 - i) \prod_{i=1}^{[t/2]} \xi(4s - 2n + 2t - 2i) E_t^{(n)}(s; \tau).
\]
The following theorem has been shown by Kalinin [Kal 77] for \( t = n \) and by Yamazaki [Ya 90] for \( 1 \leq t < n \).

**Theorem 2.4** ([Kal 77], [Ya 90]). The function \( E_t^{(n)}(s; \tau) \) has a meromorphic continuation to the whole complex plane as the function of \( s \) and holomorphic for \( \text{Re}(s) > (2n - t + 1)/2 \). Moreover, \( E_t^{(n)}(s; \tau) \) satisfies the functional equation

\[
E_t^{(n)}(s; \tau) = E_t^{(n)} \left( \frac{2n - t + 1}{2} - s; \tau \right).
\]

It has a simple pole at \( s = n - (t - 1)/2 \) with the residue

\[
\frac{1 + \delta_{t,n}}{2} \prod_{j=2}^{t} \xi(j) \prod_{j=1}^{[t/2]} \xi(2n - 2t + 2j + 1)
\]

when \( n > 1 \) and with the residue \( 1/2 \) when \( n = t = 1 \). (cf. [Kal 77] Theorem 2 for \( t = n > 1 \), [Ya 90] Theorem 2.2] for \( 1 \leq t < n \)).

Moreover, if \( t = n \), then the function \( \xi(2s) \prod_{i=1}^{t} \xi(4s - 2i) E_n^{(n)}(s; \tau) \) has a meromorphic continuation to the whole complex plane in \( s \) except the possible poles of finite order at \( s = j/4 \) for integers \( j \) (0 \( \leq j \leq 2n + 2 \)).

If \( t = 1 \), then \( E_1^{(n)}(s; \tau) \) has a meromorphic continuation to the whole complex plane in \( s \) except the poles at \( s = n \) and 0 with residues \( \frac{1}{2} \) and \( -\frac{1}{2} \), respectively.

It is remarked in [Ya 90] that if \( t \geq 2n - 2t + 2 \), then we can simplify the gamma factor of \( E_t^{(n)} \) by virtue of the cancellation of the above functional equation. It means that it is possible to take

\[
\prod_{i=1}^{2n - 2t + 1} \xi(2s + 1 - i) \prod_{i=1}^{[t/2]} \xi(4s - 2n + 2t - 2i) E_t^{(n)}(s; \tau)
\]

as the choice of the definition of \( E_t^{(n)} \) in this case. The residue of \( E_t^{(n)} \) in the theorem will be changed if we change the gamma factor.

We put

\[
D_t(\phi_M, \psi_M; s) := (4\pi)^{-ts}(\det M)^s \prod_{j=1}^{t} \left( \Gamma \left( s - \frac{j - 1}{2} \right) \xi(2s - 2k + 2n + r + 2 - t - j) \right)
\]

\[
\times \prod_{j=1}^{[t/2]} \xi(4s - 4k + 2n + 2r + 2 - 2j)
\]

\[
\times D_t(\phi_M, \psi_M; s).
\]

We remark that if \( r = 0 \), then we regard \( \det(M) \) as \( 1 \).
By virtue of Proposition 2.3 the function $D_t(\phi_M, \psi_M; s)$ equals to
\[
\int_{\Gamma \cup \delta_n R \mod \mathbb{Z}[n,r](2M)} f_R(\tau)g_R(\tau)(\det(\text{Im}\tau))^{k - \frac{t}{2}} \mathcal{E}^{(n)}_t (s - k + n - (t - r - 1)/2; \tau) \, d\tau
\]
times the constant $\pi^{-\frac{t}{4}(t-1)} \det(4M)^{-\frac{n-r}{2}(1 + \delta_{t,n})^{-1}}$.
Thus, due to Theorem 2.4 we have the following.

**Theorem 2.5.** The function $D_t(\phi_M, \psi_M; s)$ has a meromorphic continuation to the whole complex plane and holomorphic for $\text{Re}(s) > k - \frac{r}{2}$. It has a simple pole at $s = k - \frac{r}{2}$ with the residue
\[
(1 + \delta_{0,r})^{-1} \pi^{-\frac{t}{4}(t-1)} \det(4M)^{\frac{t}{2}} \langle \phi_M, \psi_M \rangle \prod_{j=2}^{t} \xi(j) \prod_{j=1}^{|t/2|} \xi(2n - 2t + 2j + 1)
\]
when $n > 1$ and with the residue $\frac{1}{2}(1 + \delta_{0,r})^{-1} \det(4M)^{\frac{t}{2}} \langle \phi_M, \psi_M \rangle$ when $n = t = 1$.
It satisfies a functional equation
\[
D_t(\phi_M, \psi_M; s) = D_t(\phi_M, \psi_M; 2k - n - r + \frac{t - 1}{2} - s).
\]
Moreover, if $t = 1$, then $D_1(\phi_M, \psi_M; s)$ has a meromorphic continuation to the whole complex plane and holomorphic except for simple poles at $s = k - \frac{r}{2}$ and $s = k - \frac{r}{2} - n$.
The residue at $s = k - \frac{r}{2}$ is
\[
(1 + \delta_{1,n})^{-1} (1 + \delta_{0,r})^{-1} \det(4M)^{\frac{1}{2}} \langle \phi_M, \psi_M \rangle.
\]
The case $r = 0$ has been shown in [Ya 90].

We remark that if $n = r = t = 1$, then the above residue coincides with Proposition 1 (a) in [I-M 03]. However, Proposition 1 (a) in [I-M 03] should read
\[
\text{Res}_{s_2 = k - 1/2} D_{F,G}(s_1, s_2) = \pi^{k + \frac{1}{2}} \Gamma \left( k - \frac{1}{2} \right)^{-1} \zeta(2)^{-1} L(F, G, s_1 + k - 1).
\]

After we shall explain some similar results of Theorem 2.5 for Jacobi cusp forms of half-integral weight in Section 3, we will prove Proposition 2.3 in Section 4.

3. Rankin-Selberg method for the plus space of Jacobi forms

In this section we shall explain the half-integral weight case. In this section we assume that $k$ is an even integer. We assume $r \geq 1$. Let $M = \begin{pmatrix} M_1 & \frac{1}{2}L \\ \frac{1}{2}L & 1 \end{pmatrix} \in L^{+}_r$ with $M_1 \in L^{+}_{r-1}$ and $L \in M_{r-1,1}(\mathbb{Z})$. If $r \geq 2$, we set
\[
\mathfrak{M} := 4M_1 - LL^t.
\]
If \( r = 1 \), then \( \mathcal{M} = 1 \) and we regard \( \mathfrak{M} = \emptyset \) as the empty set and we put \( \det(\mathfrak{M}) = 1 \) by abuse of notation.

We set \( \Gamma_0^{(n)}(4) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \in 4\mathbb{Z}^{(n,n)} \right\} \).

Let \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+} \) be the plus-space of Jacobi forms of weight \( k - \frac{1}{2} \) of index \( \mathfrak{M} \) on \( \Gamma_0^{(n)}(4) \) which is a generalization of generalized plus-space of Siegel modular forms of weight \( k - \frac{1}{2} \) to Jacobi forms. The space \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+} \) is defined as follows. Let \( \phi \) be a Jacobi form of weight \( k - \frac{1}{2} \) of index \( \mathfrak{M} \) on \( \Gamma_0^{(n)}(4) \). The reader is referred to [Ha 18] for the precise definition of Jacobi forms of half-integral weight. We take the Fourier expansion

\[
\phi(\tau, z) = \sum_{N', R'} C_\phi(N', R') e(N' \tau + R' z)
\]

for \( (\tau, z) \in \mathcal{H}_n \times \mathbb{C}^{(n,r-1)} \), where \( N' \) and \( R' \) run over \( L_n^* \) and \( \mathbb{Z}^{(n,r-1)} \), respectively, such that \( 4N' - R' \mathfrak{M}^{-1} R' \geq 0 \). Then \( \phi \) belongs to \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+} \) if and only if \( C_\phi(N', R') = 0 \) unless

\[
\left( \frac{N'}{2} R', \frac{1}{2} R' \mathfrak{M} \right) \equiv \lambda \lambda \mod 4
\]

with some \( \lambda \in \mathbb{Z}^{(n+r-1,1)} \).

If \( r = 1 \), then the space \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+} \) coincides with the generalized plus-space of Siegel modular forms. There exists a linear isomorphism map \( \iota_{\mathcal{M}} \) from \( J_{k,\mathcal{M}}^{(n)} \) to \( J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+} \) (cf. [EZ 85] (for \( r = n = 1 \)), [1b 92] (for \( r = 1, n > 1 \), [Ha 18] (for \( r > 1, n \geq 1 \))). This map \( \iota_{\mathcal{M}} \) is given as follows.

Let \( \phi_{\mathcal{M}} \in J_{k,\mathcal{M}}^{(n)} \) be a Jacobi form. We denote by \( C_{\phi_{\mathcal{M}}}(*, *) \) the Fourier coefficients of \( \phi_{\mathcal{M}} \). For \( \tau \in \mathcal{H}_n \) and for \( z = (z_1, z_2) \in \mathbb{C}^{(n,r)} \) \((z_1 \in \mathbb{C}^{(n,r-1)}, z_2 \in \mathbb{C}^{(n,1)})\), we take the theta decomposition

\[
\phi_{\mathcal{M}}(\tau, z) = \sum_{\substack{R \in \mathbb{Z}^{(n,1)} \\
R \mod 2\mathbb{Z}^{(n,1)}}} f_{R,\mathcal{M}_1}(\tau, z_1) \vartheta_{1,L,R}(\tau, z_1, z_2),
\]

where

\[
f_{R,\mathcal{M}_1}(\tau, z_1) = \sum_{N_1 \in L_n^*, N_3 \in \mathbb{Z}^{(n,r-1)}} C_{\phi_{\mathcal{M}}}(N_1, (N_3, R)) \times e(\left( N_1 - 1 \right) R' L \tau + \left( N_3 - 1 \right) R' L z_1)
\]

and the function \( \vartheta_{1,L,R} \) will be denoted in [13] (cf. [Ha 18] Lemma 4.1). We put

\[
\iota_{\mathcal{M}}(\phi_{\mathcal{M}})(\tau, z_1) = \sum_{R \in \mathbb{Z}^{(n,1)} / (2\mathbb{Z}^{(n,1)})} f_{R,\mathcal{M}_1}(4\tau, 4z_1).
\]
For the sake of simplicity we write $\phi_{nR} = \iota_M(\phi_M)$. Then $\phi_{nR}$ belongs to $J^{(n)\cup}_{k - \frac{1}{2}; nR}$ (cf. [Ha 18 Proposition 4.4]). If $\phi_M$ is a Jacobi cusp form, then $\phi_{nR}$ is also a Jacobi cusp form. If $r = 1$, then $\phi_{nR}$ is a Siegel modular form (cf. [E-Z 85], [Ib 92]).

Let $\phi$ and $\psi$ be Jacobi cusp forms of weight $k - \frac{3}{2}$ of index $S \in L_+^+$ on $\Gamma_0^{(n)}(4)$. The Petersson inner product is defined by

$$\langle \phi, \psi \rangle := \left[ \Gamma_n : \Gamma^{(n)}_0(4) \right]^{-1} \int_{F_{n,r}} \phi(\tau, z) \overline{\psi(\tau, z)} e^{-4\pi Tr(S\nu^{-1}\nu)} \det(\nu)^{k-n-r-\frac{3}{2}} \, du \, dv \, dx \, dy,$$

where $F_{n,r} := \Gamma_{n,r}^{J}(4) \L (\mathfrak{g}_n \times \mathbb{C}^{(n,r)})$, $\tau = u + iv$, $z = x + iy$; $du = \prod_{i \leq j} u_{i,j}$, $dv = \prod_{i \leq j} v_{i,j}$, $dx = \prod_{i,j} x_{i,j}$, $dy = \prod_{i,j} y_{i,j}$ and $\left[ \Gamma_n : \Gamma^{(n)}_0(4) \right]$ denotes the index of $\Gamma^{(n)}_0(4)$ in $\Gamma_n$. Here we put

$$\Gamma_{n,r}^{J}(4) := \left\{ \begin{pmatrix} A & 0 & B & * \\ * & 1_r & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & 1_r \end{pmatrix} \in \Gamma_{n+r} \bigg| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)}_0(4) \right\}.$$

Lemma 3.1. Let $\phi_M$ and $\psi_M$ be Jacobi cusp forms in $J^{(n)\cup}_{k_M}$, we put $\phi_{nR} = \iota_M(\phi_M)$ and $\psi_{nR} = \iota_M(\psi_M)$. As for the Petersson inner product we obtain the identity

$$\langle \phi_M, \psi_M \rangle = (1 + \delta_{1,r})^{-1} 2^{n(k-1)} \langle \phi_{nR}, \psi_{nR} \rangle.$$

We remark that, in the case of $r = 1$ and $n = 1$, this identity has been obtained by combining Kohnen and Zagier [K-Z 81, p.p. 189–191] and Eichler-Zagier [E-Z 85, Theorem 5.3]. Remark that the denominator of RHS of [E-Z 85, Theorem 5.3] is neither $2m$ nor $\sqrt{4m}$ but $4\sqrt{m}$. We remark that, in the case of $r = 1$ and $n > 1$, the above identity has been obtained by Katsurada and Kawamura [K-K 08, p. 2051 (6)]. Remark that $2^{(2k-2)(n-1)}$ in [K-K 08, p. 2051 (6)] should read $2^{(2k-2)(n-1)-1}$.

Proof. We recall the symbol $P_{0,n} = \{ (\gamma(n, r), *) \in \Gamma_n \}$. We set

$$E^{(n)}_{n,4}(s; \tau) := \sum_{\gamma \in P_{0,n} \backslash \Gamma^{(n)}_0(4)} \det(\text{Im}(\gamma \cdot \tau))^s.$$

We take the theta decompositions

$$\phi_{nR}(\tau, z_1) = \sum_{R \mod \mathbb{Z}^{(n,r-1)}(2n)} \tilde{f}_R(\tau) \vartheta_{nR}(\tau, z_1),$$

$$\psi_{nR}(\tau, z_1) = \sum_{R \mod \mathbb{Z}^{(n,r-1)}(2n)} \tilde{g}_R(\tau) \vartheta_{nR}(\tau, z_1).$$

We put

$$I_n(\phi_{nR}, \psi_{nR}; s) := \int_{\Gamma^{(n)}_0(4) \backslash \Delta_n} \sum_{R \mod \mathbb{Z}^{(n,r-1)}(2n)} \tilde{f}_R(\tau) \overline{\tilde{g}_R(\tau)} (\det(\text{Im}(\tau)))^{k-rac{3}{2}} \, E^{(n)}_{n,4}(s; \tau) \, d\tau.$$
where we put \( \tau = u + \sqrt{-1}v \) and \( d\tau := \det(v)^{-1}du \, dv \) and \( du = \prod_{l \leq m} u_{l,m} \, dv = \prod_{l \leq m} v_{l,m} \). Here \( u = (u_{l,m}) \) and \( v = (v_{l,m}) \). We put

\[
I_n(\phi_M, \psi_M; s) := \int_{\Gamma_n \setminus \mathcal{G}_n \mod \mathbb{Z}^{(n,r)}(2M)} \sum_{\tau \in \mathbb{R}_+} f_R(\tau) g_R(\tau) \det(\text{Im}(\tau))^{k-s} E^{(n)}(s; \tau) \, d\tau,
\]

where \( f_R \) and \( g_R \) are denoted in (2.2) through the decompositions of \( \phi_M \) and \( \psi_M \) with the theta series.

We will show this lemma by comparing the residue of \( I_n(\phi_{3R}, \psi_{3R}; s) \) with the one of \( I_n(\phi_M, \psi_M; s) \) at \( s = \frac{n+1}{2} \). Let

\[
\phi_{3R}(\tau, z_1) = \sum_{(N_2, R_2) \in L^{+}_{n,r-1}(\mathfrak{M})} C_{\phi_{3R}}(N_2', R_2') e(N_2 T + R_2 l z_1)
\]

and

\[
\phi_M(\tau, z) = \sum_{(N_2, R_2) \in L^{+}_{n,r-1}(\mathcal{M})} A_{\phi_M}(N_2, R_2) e(N_2 \tau + R_2 z)
\]

be the Fourier expansions of \( \phi_{3R} \) and \( \phi_M \), respectively. We similarly denote by \( C_{\psi_{3R}}(N_2', R_2') \) and \( A_{\psi_M}(N_2, R_2) \) the Fourier coefficients of \( \psi_{3R} \) and \( \psi_M \), respectively. We remark that if

\[
(3.1) \quad \left( \begin{array}{cc}
N_2' & \frac{1}{l} R_2' \\
\frac{1}{l} R_2' & 2\mathfrak{M}
\end{array} \right) = 4 \left( \begin{array}{cc}
N_2 & \frac{1}{2} R_{2,1} \\
\frac{1}{2} R_{2,1} & M_1
\end{array} \right) - \left( \begin{array}{cc}
R_{2,2} & L \\
L & R_{2,2}
\end{array} \right)
\]

and if

\[
(3.2) \quad \left( \begin{array}{cc}
N_2 & \frac{1}{2} R_2 \\
\frac{1}{2} R_2 & 2\mathcal{M}
\end{array} \right) = \left( \begin{array}{cc}
N_2 & \frac{1}{2} R_{2,1} \\
\frac{1}{2} R_{2,1} & M_1
\end{array} \right) \left( \begin{array}{cc}
\frac{1}{2} R_{2,2} & \frac{1}{2} L \\
\frac{1}{2} L & 1
\end{array} \right),
\]

then \( C_{\phi_{3R}}(N_2', R_2') = A_{\phi_M}(N_2, R_2) \) and \( C_{\psi_{3R}}(N_2', R_2') = A_{\psi_M}(N_2, R_2) \).

By the similar argument of the proof of the identity (4.2) in Proposition 4.1 which will be appeared in [41] we have

\[
I_n(\phi_{3R}, \psi_{3R}; s)
= 2\pi^{\frac{3}{4}} (n-1) \prod_{i=1}^{n} \Gamma(s + k - \frac{1}{2} - r - 1 - \frac{n+1}{2} - \frac{i-1}{2})
\times \sum_{\mathfrak{g} \in L^{+}_{n,r-1}(\mathfrak{M})/B_{n,r-1}(\mathbb{Z})} \frac{1}{|\epsilon_{n,r-1}(\mathfrak{M})|} C_{\phi_{3R}}(N_2', R_2') C_{\psi_{3R}}(N_2', R_2')
\times \det(\pi(4N_2' - 2\mathfrak{M}^{-1}[l R_2']))^{-k+s+\frac{1}{2}+\frac{1}{2}+\frac{n+1}{2}+\frac{1}{2}},
\]
where in the summation we set \( \mathfrak{N} = \left( \frac{N_2'}{\frac{1}{2}R_2'} \begin{array}{c} \frac{1}{2}R_2' \\ \mathfrak{M} \end{array} \right) \). We remark that if the identities (3.1) and (3.2) hold, then

\[
\det(\pi(4N_2' - \mathfrak{M}^{-1}|R_2')) = 2^{2n} \det(\pi(4N_2 - \mathfrak{M}^{-1}|R_2'))
\]

and \(|\epsilon_{n,r-1}(\mathfrak{M})| = |\epsilon_{n,r}(\mathfrak{N})|\), where we put \( \mathfrak{N} = \left( \frac{N_2}{\frac{1}{2}R_2} \begin{array}{c} \frac{1}{2}R_2 \\ \mathfrak{M} \end{array} \right) \). The map \( \left( \frac{N_2}{\frac{1}{2}R_2} \begin{array}{c} \frac{1}{2}R_2 \\ \mathfrak{M} \end{array} \right) \in L^+_{n,r}(\mathfrak{M})/B_{n,r}(\mathbb{Z}) \to \left( \frac{N_2'}{\frac{1}{2}R_2'} \begin{array}{c} \frac{1}{2}R_2' \\ \mathfrak{M} \end{array} \right) \in L^+_{n,r-1}(\mathfrak{M})/B_{n,r}(\mathbb{Z}) \) given by the identities (3.1) and (3.2) is bijective. Therefore, by using the identity (4.2) which will be appeared in §4, we have

\[
I_n(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) = 2^{2n} \pi^{\frac{1}{4}n(n-1)} \prod_{i=1}^{n} \Gamma(s + k - \frac{1}{2} - \frac{r-1}{2} - \frac{n+1}{2} - \frac{i-1}{2}) \times \sum_{\mathfrak{N} \in L^+_{n,r}(\mathfrak{M})/B_{n,r}(\mathbb{Z})} \frac{1}{|\epsilon_{n,r}(\mathfrak{N})|} A_{\phi_{\mathfrak{M}}}(N_2, R_2) A_{\psi_{\mathfrak{M}}}(N_2, R_2) \times 2^{2n(-k+s+n+1)} \det(\pi(4N_2 - \mathfrak{M}^{-1}|R_2'))^{-k+s+n+1} = 2^{2n(-k+s+n+1)} I_n(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s). \tag{3.3}
\]

We put

\[
\gamma_n(s) := \prod_{i=1}^{n} \xi(2s + 1 - i) \prod_{i=1}^{[n/2]} \xi(4s - 2i),
\]

where \( \xi(s) = \pi^{-\frac{1}{2}} \Gamma\left( \frac{s}{2} \right) \zeta(s) \) is the symbol denoted before Theorem 2.4, and we put

\[
\mathcal{E}_{n,4}^{(n)}(s; \tau) := \gamma_n(s) E_{n,4}^{(n)}(s; \tau).
\]

Then \( \mathcal{E}_{n,4}^{(n)}(s; \tau) \) has a meromorphic continuation to the whole complex plane in \( s \) (cf. [Kal 77, Theorem 1]). Moreover, \( \mathcal{E}_{n,4}^{(n)}(s; \tau) \) has a simple pole at \( s = (n+1)/2 \) with the residue

\[
\text{Res}_{s=\frac{n+1}{2}} \mathcal{E}_{n,4}^{(n)}(s; \tau) = \frac{1}{[\Gamma_n : \Gamma_0^{(n)}(4)]} \text{Res}_{s=\frac{n+1}{2}} \mathcal{E}_{n}^{(n)}(s; \tau) = \frac{1}{[\Gamma_n : \Gamma_0^{(n)}(4)]} \prod_{j=2}^{n} \xi(j) \prod_{j=1}^{[n/2]} \xi(2j + 1).
\]
Therefore the residue of $\gamma_n(s)I_n(\phi_{2\mathfrak{M}}, \psi_{2\mathfrak{M}}; s)$ at $s = \frac{n+1}{2}$ is

$$
\frac{1}{[\Gamma_n : \Gamma_0^{(n)}(4)]} \prod_{j=2}^{n} \xi(j) \prod_{j=1}^{[n/2]} \xi(2j + 1) \times \int_{\Gamma_0^{(n)}(4)\backslash \mathbb{H}} \sum_{\delta \in \mathbb{Z}^{(n,r-1)}(2\mathfrak{M})} f_R(\tau)g_R(\tau)(\det(\text{Im} \tau))^{k-\frac{1}{2} - \frac{n}{2}} d\tau
$$

$$
= \frac{2}{1 + \delta_{0,r-1}} \det(4\mathfrak{M})^\frac{n}{2} \prod_{j=2}^{n} \xi(j) \prod_{j=1}^{[n/2]} \xi(2j + 1) \langle \phi_{2\mathfrak{M}}, \psi_{2\mathfrak{M}} \rangle.
$$

On the other hand the residue of $\gamma_n(s)I_n(\phi_{\mathcal{M}}, \psi_{\mathcal{M}})$ at $s = \frac{n+1}{2}$ is

$$
2 \det(4\mathcal{M})^\frac{n}{2} \prod_{j=2}^{n} \xi(j) \prod_{j=1}^{[n/2]} \xi(2j + 1) \langle \phi_{\mathcal{M}}, \psi_{\mathcal{M}} \rangle.
$$

We remark the identity $\det(4\mathfrak{M}) = 2^{2r-4} \det(4\mathcal{M})$. Thus, by virtue of the identity (3.3), we have the lemma.

Let $\phi_{2\mathfrak{M}}, \psi_{2\mathfrak{M}} \in J_{k-\frac{1}{2},2\mathfrak{M}}^{(n)\text{cusp}}$ be Jacobi cusp forms of weight $k - \frac{1}{2}$ with the index $\mathfrak{M} \in L^+_{r-1}$ on $\Gamma_0^{(n)}(4)$. We remark that if $r = 1$, then $\phi_{2\mathfrak{M}}$ and $\psi_{2\mathfrak{M}}$ are Siegel cusp forms of weight $k - \frac{1}{2}$. For any natural number $t$ ($1 \leq t \leq n$), we take the Fourier-Jacobi expansions

$$
\phi_{2\mathfrak{M}}(\tau, z)e(\mathfrak{M} \omega) = \sum_{\mathfrak{m} \in L^+_{t,r-1}(2\mathfrak{M})} \phi_{\mathfrak{m}}(\tau', z')e(\mathfrak{m} \omega'),
$$

$$
\psi_{2\mathfrak{M}}(\tau, z)e(\mathfrak{M} \omega) = \sum_{\mathfrak{m} \in L^+_{t,r-1}(2\mathfrak{M})} \psi_{\mathfrak{m}}(\tau', z')e(\mathfrak{m} \omega').
$$

For complex number $s$ which real part is sufficient large, we set

$$
D_t(\phi_{2\mathfrak{M}}, \psi_{2\mathfrak{M}}; s) := \sum_{\mathfrak{m} \in B_{t,r-1}(\mathfrak{M}) \backslash L^+_{t,r-1}(2\mathfrak{M})} \langle \phi_{2\mathfrak{M}}, \psi_{2\mathfrak{M}} \rangle |e_{t,r-1}(\mathfrak{M})| \det(\mathfrak{M})^s.
$$

Lemma 3.2. We assume that $\phi_{2\mathfrak{M}}$ and $\psi_{2\mathfrak{M}}$ belong to the plus space $J_{k-\frac{1}{2},2\mathfrak{M}}^{(n)\text{cusp}}$. Let $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}} \in J_{k,\mathcal{M}}^{(n)\text{cusp}}$ be Jacobi cusp forms which satisfy $\phi_{2\mathfrak{M}} = \iota_{\mathcal{M}}(\phi_{\mathcal{M}})$ and $\psi_{2\mathfrak{M}} = \iota_{\mathcal{M}}(\psi_{\mathcal{M}})$. Then, for any $t$ ($1 \leq t \leq n$), we have

$$
D_t(\phi_{2\mathfrak{M}}, \psi_{2\mathfrak{M}}; s) = (1 + \delta_{1,r})2^{-2(k-1)(n-t)-2(r+t-1)s}D_t(\phi_{\mathcal{M}}, \psi_{\mathcal{M}}; s).
$$

In the case of $r = 1$, $\mathcal{M} = 1$ and $t = n$ this lemma has been shown in [K-K 15].

Proof. Assume $N$ is a matrix in $L^+_{t,r}(\mathcal{M})$. Let $\phi_N$ and $\psi_N$ be the $N$-th Fourier-Jacobi coefficients of $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$, respectively. We put $\phi_{\mathfrak{M}} = \iota_N(\phi_N)$ and $\psi_{\mathfrak{M}} = \iota_N(\psi_N)$. Then
\( \phi_{\mathfrak{M}} \) and \( \psi_{\mathfrak{M}} \) are \( \mathfrak{M} \)-th Fourier-Jacobi coefficients of \( \phi_{\mathfrak{M}} \) and \( \psi_{\mathfrak{M}} \), respectively. We remark that \( \phi_{\mathfrak{M}} \) and \( \psi_{\mathfrak{M}} \) belong to \( J_{k-\frac{1}{2}n}^{(n-t)+\text{cusp}} \) and remark that \( \phi_{\mathfrak{N}} \) and \( \psi_{\mathfrak{N}} \) belong to \( J_{k-\frac{1}{2}n}^{(n-t)+\text{cusp}} \).

By virtue of Lemma 3.3, we have \( \langle \phi_{\mathfrak{M}}, \psi_{\mathfrak{M}} \rangle = (1 + \delta_{1,r})2^{-2(n-t)(k-1)} \langle \phi_{\mathfrak{N}}, \psi_{\mathfrak{N}} \rangle \). We have also \( |\epsilon_{t,r-1}(\mathfrak{M})| = |\epsilon_{t,r}(\mathfrak{N})| \) and \( \det \mathfrak{M} = 2^{2(r+t-1)} \det \mathfrak{N} \). Thus we conclude the lemma.

We set
\[
D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) := \pi^{-ts} (\det \mathfrak{M})^s \prod_{j=1}^t \left( \Gamma \left(s - \frac{j}{2}\right) \xi(2s - 2k + 2n + r + 2 - t - j) \right)
\times \left( \prod_{j=1}^{[t/2]} \xi(4s - 4k + 2n + 2r + 2 - 2j) \right) D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s).
\]

Then
\[
D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) = (1 + \delta_{1,r})2^{-2(k-1)(n-t)} D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s).
\]

Due to Theorem 2.5 we have the followings.

**Theorem 3.3.** The function \( D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) \) has a meromorphic continuation to the whole complex plane and holomorphic for \( \text{Re}(s) > k - \frac{r}{2} \). It has a simple pole at \( s = k - \frac{r}{2} \) with the residue
\[
(1 + \delta_{1,r})^{-1} 2^{2k-t-\frac{1}{2}t(t-1)} (\det \mathfrak{M})^{\frac{t}{2}} \langle \phi_{\mathfrak{M}}, \psi_{\mathfrak{M}} \rangle \prod_{j=2}^t \xi(j) \prod_{j=1}^{[t/2]} \xi(2n - 2t + 2j + 1)
\]
when \( n > 1 \) and with the residue \( 2^{2k-3} \det(\mathfrak{M})^{\frac{t}{2}} \langle \phi_{\mathfrak{M}}, \psi_{\mathfrak{M}} \rangle \) when \( n = t = 1 \).

It satisfies the functional equation
\[
D_t(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) = D_t \left( \phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; 2k - n - r + \frac{t-1}{2} - s \right).
\]

Moreover, if \( t = 1 \), then \( D_1(\phi_{\mathfrak{M}}, \psi_{\mathfrak{M}}; s) \) has a meromorphic continuation to the whole complex plane and holomorphic except for simple poles at \( s = k - \frac{r}{2} \) and \( k - \frac{r}{2} - n \). The residue at \( s = k - \frac{r}{2} \) is
\[
(1 + \delta_{1,n})^{-1}(1 + \delta_{1,r})^{-1} 2^{2k-1} \det(\mathfrak{M})^{\frac{1}{2}} \langle \phi_{\mathfrak{M}}, \psi_{\mathfrak{M}} \rangle.
\]

In particular, if \( r = 1 \) and \( \mathcal{M} = 1 \), then the space \( J_{k,1}^{(n)} \) of Jacobi cusp forms of degree \( n \) is linearly isomorphic to the generalized plus-space \( S^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4)) \) as Hecke algebra modules. Here \( S^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4)) \) is a certain subspace of Siegel cusp forms of weight
$k - \frac{1}{2}$ of degree $n$ (see [1b 92] for the definition and the isomorphism). We have the following.

**Corollary 3.4.** Let $F, G \in S^+_{k - \frac{1}{2}}(\Gamma(G)^0(4))$. The function $D_t(F, G; s)$ has a meromorphic continuation to the whole complex plane and holomorphic for $\text{Re}(s) > k - \frac{1}{2}$. It has a simple pole at $s = k - \frac{1}{2}$ with the residue

$$2^{2k-t-1} \pi^{-\frac{t}{2}} t(t-1) \langle F, G \rangle \prod_{j=2}^{t} \xi(j) \prod_{j=1}^{[t/2]} \xi(2n - 2t + 2j + 1)$$

when $n > 1$ and with the residue $2^{2k-3} \langle F, G \rangle$ when $n = t = 1$.

It satisfies the functional equation

$$D_t(F, G; s) = D_t(F, G; 2k - n + \frac{t - 3}{2} - s).$$

Moreover, if $t = 1$, then $D_1(F, G; s)$ has a meromorphic continuation to the whole complex plane and holomorphic except for simple poles at $s = k - \frac{1}{2}$ and $k - \frac{1}{2} - n$. The residue at $s = k - \frac{1}{2}$ is

$$\text{Res}_{s=k-\frac{1}{2}} D_1(F, G; s) = \text{Res}_{s=k-\frac{1}{2}} (D_1(F, G; s) \pi^{-s} \Gamma(s) \xi(2s - 2k + 2n + 1))$$

$$= (1 + \delta_{1,n})^{-1} 2^{2(k-1)} \langle F, G \rangle.$$

We remark that the case $t = n$ in Corollary 3.4 has been shown in [K-Z 81] (for $n = 1$) and in [K-K 13] (for $n > 1$).

**4. Proof of Proposition 2.3**

In this section we shall prove Proposition 2.3. We use the same notation in §2. For $\tau \in H_n$, we decompose $\tau$ as $\tau = \left( \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right)$, $\tau_1 \in H_{n-t}$, $\tau_2 \in H_t$, $\tau'_1 \in \mathbb{C}^{(n-t,t)}$. We write

$$\tau = u + iv, \tau_j = u_j + iv_j \quad (j = 1, 2) \quad \text{and} \quad \tau'_j = x'_j + iy'_j \quad \text{with matrices} \quad u, v, u_j, v_j, x'_j, y'_j \quad \text{which entries are real numbers.}$$

For $\tau_1 \in H_{n-t}$, we fix a fundamental domain

$$D_t(\tau_1) := \mathbb{C}^{(n-t,t)}/(\tau_1 \mathbb{Z}^{(n-t,t)} + \mathbb{Z}^{(n-t,t)})$$

and put

$$\widehat{D_t(\tau_1)} := \left\{ (\tau'_1, \tau_2) \in D_t(\tau_1) \times H_t \mid \left( \begin{array}{c} \tau_1 \\ \tau'_1 \end{array} \right) \in H_n \right\}.$$
The group $P_{0,t} = \left\{ \begin{pmatrix} A & B \\ 0(t,t) & tA^{-1} \end{pmatrix} \in \Gamma_t \right\}$ acts on $\widetilde{D_t}(\tau_1)$ by

$$\left( \begin{array}{cc} A & B \\ 0(t,t) & tA^{-1} \end{array} \right) \cdot (z_1', \tau_2) := (z_1'A, \tau_2[tA] + B'tA)$$

for $\left( \begin{array}{cc} A & B \\ 0(t,t) & tA^{-1} \end{array} \right) \in P_{0,t}$ and for $(z_1', \tau_2) \in \widetilde{D_t}(\tau_1)$.

We put

$$I_t(\phi_{\mathcal{M}}, \psi_{\mathcal{M}}; s) := \int_{\Gamma_n \setminus \delta_n \mod \mathbb{Z}^{(n,r)(2\mathcal{M})}} f_R(\tau) g_R(\tau) \det(v)^{k-\frac{d}{2}+s} \det(v_1)^{-s} d\tau,$$

where $f_R$ and $g_R$ are denoted in (2.2) through the decompositions of $\phi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$ with the theta series, and where we put $d\tau := \det(v)^{-n-1} du \, dv$ and $du = \prod_{t \leq m} u_{t,m}$, $dv = \prod_{t \leq m} v_{t,m}$. Here $u = (u_{t,m})$ and $v = (v_{t,m})$.

We have

$$I_t(\phi_{\mathcal{M}}, \psi_{\mathcal{M}}; s) = \int_{P_{0,t} \setminus \Gamma_n \setminus \delta_n \mod \mathbb{Z}^{(n,r)(2\mathcal{M})}} f_R(\tau) g_R(\tau) \det(v)^{k-\frac{d}{2}+s} \det(v_1)^{-s} d\tau$$

$$= \frac{1 + \delta_{t,n}}{2} \int_{\Gamma_{n-t,t} \setminus \delta_{n-t} \mod \mathbb{Z}^{(n-t,r)(2\mathcal{M})}} \int_{P_{0,t} \setminus \Gamma_t \setminus \delta_{t} \mod \mathbb{Z}^{(t,r)(2\mathcal{M})}} \sum f_R(\tau) g_R(\tau) \det(v)^{k-\frac{d}{2}+s-(n+1)}$$

$$\times \det(v_1)^{-s} d\tau \cdot d\tau' \cdot du \cdot dv \cdot du_1 \cdot dv_1.$$
Proposition 4.1. We have the identity

\[
I_t(\phi_M, \psi_M; s) = \frac{1 + \delta t_n}{2} \prod_{i=1}^{t} \Gamma\left(s + k - \frac{r}{2} - (n + 1) + \frac{t + 1}{2} - i - 1\right) \times \int_{\Gamma_{t-1}\delta_{t-1}} \int_{D_t(\tau_1)} \sum_{\mathbb{Z}^{(n-t,r)}(2M)} \sum_{\mathbb{N} \in \mathcal{B}_{t,r}(\mathbb{Z})} \frac{1}{|\varepsilon_{t,r}(\mathcal{N})|} \times \det(\pi(4N_2 - \mathcal{M}^{-1}[tR_2]))^{-k + \frac{t}{2} - s + (n+1) - \frac{t}{2}} \times f_{R,N_2}(\tau_1, z_1') g_{R,N_2}(\tau_1, z_1') e\left(\frac{\sqrt{-1}}{2} (4N_2 - \mathcal{M}^{-1}[tR_2]) v_1^{-1}[y_1]\right) \det(v_1)^{k - \frac{t}{2} - (n+1)} \times dx_1' dy_1' du_1 dv_1.
\]

Proof. We obtain

\[
I_t(\phi_M, \psi_M; s) = \frac{1 + \delta t_n}{2} \int_{\Gamma_{t-1}\delta_{t-1}} \int_{D_t(\tau_1)} \sum_{\mathbb{Z}^{(n-t,r)}(2M)} \sum_{\mathbb{N} \in \mathcal{B}_{t,r}(\mathbb{Z})} f_{R,N_2}(\tau_1, z_1') g_{R,N_2}(\tau_1, z_1') e\left(\frac{\sqrt{-1}}{2} (4N_2 - \mathcal{M}^{-1}[tR_2]) v_2 \right) \det(v)^{k - \frac{t}{2} + s - (n+1)} \det(v_1)^{-s} \times dx_1' dy_1' du_2 dv_2 du_1 dv_1.
\]

Since \(\int_{\text{Sym}_t(\mathbb{R})/\text{Sym}_t(\mathbb{Z})} e(N_2 \tau_2 - N_2' \tau_2) du_2 = \delta_{N_2,N_2'}\) and since \(\det(v) = \det(v_1) \det(v_2 - v_1^{-1}[y_1])\), we have

\[
I_t(\phi_M, \psi_M; s) = \frac{1 + \delta t_n}{2} \int_{\Gamma_{t-1}\delta_{t-1}} \int_{\text{GL}(t,\mathbb{Z})/D_t(\tau_1)' \mod \mathbb{Z}^{(n-t,r)}(2M)} \sum_{\mathbb{N} \in \mathcal{B}_{t,r}(\mathbb{Z})} f_{R,N_2}(\tau_1, z_1') g_{R,N_2}(\tau_1, z_1') e\left(2\sqrt{-1}N_2 v_2 - \sqrt{-1}(4N_2 - \mathcal{M}^{-1}[tR_2]) v_2 \right) \det(v_2 - v_1^{-1}[y_1])^{k - \frac{t}{2} + s - (n+1)} \det(v_1)^{k - \frac{t}{2} - (n+1)} \times dv_2 dx_1' dy_1' du_1 dv_1,
\]

where we put \(D_t(\tau_1)' := \left\{(z_1', \sqrt{-1}v_2) \in \widetilde{D_t(\tau_1)}\right\}\) and where \(\text{GL}(t,\mathbb{Z})\) acts on \(D_t(\tau_1)'\) by \(A \cdot (z_1', \sqrt{-1}v_2) = (z_1't A, \sqrt{-1}v_2[tA])\) for \(A \in \text{GL}(t,\mathbb{Z})\) and for \((z_1', \sqrt{-1}v_2) \in \widetilde{D_t(\tau_1)}\).
We substitute $v_2$ by $v_2 + v_1^{-1}[y_1]$, then

$$I_t(\phi_M; \psi_M; s) = \frac{1 + \delta_{t,n}}{2} \int_{t \mod \Gamma_{n-t} \setminus \Gamma_{n-t}} \int GL(t, \mathbb{Z}) \setminus D_t(\tau_1) \sum_{R \mod \mathbb{Z}(n,r)} \sum_{N_2 \in L_t^+} f_{R,N_2}(\tau_1, z_1') g_{R,N_2}(\tau_1, z_1')$$

$$\times e\left(\frac{\sqrt{-1}}{2} \left(4N_2 - M^{-1}[t R_2]\right) v_2\right) e\left(\frac{\sqrt{-1}}{2} \left(4N_2 - M^{-1}[t R_2]\right) v_1^{-1}[y_1]\right)$$

$$\times \det(v_2)^{k-\frac{t}{2}+s-(n+1)} \det(v_1)^{k-\frac{t}{2}-(n+1)} dv_2 dx_2' dy_2' du_1 dv_1,$$

where we put $D_t(\tau_1)' = \{(z_1', \sqrt{-1}v_2) | (z_1', v_2) \in D_t(\tau_1) \times Sym_t^+(\mathbb{R})\}$ and where we denote by $Sym_t^+(\mathbb{R})$ the positive definite symmetric matrices of size $t$ which entries are real numbers.

Since $\phi_M$ is a Jacobi form, we obtain

$$C_{\phi}\left(\left(\frac{N_1}{2} \frac{1}{2} N_3, N_2[A]\right), \left(1_{n-t} t A\right) R\right) = C_{\phi}\left(\left(\frac{1}{2} A^{-1} t N_3, \frac{1}{2} N_2 A^{-1}\right), \left(1_{n-t} t A\right) R\right)$$

for any $A \in GL(t, \mathbb{Z})$. Thus, for a fixed $R = \left(\begin{array}{c} R_1 \\ R_2 \end{array}\right) \in \mathbb{Z}(n,r)$, we have

$$f_{R,N_2}(\tau_1, z_1') = \sum_{N_1 \in L_{n-t}^+ \cap \mathbb{Z}(n-t,t)} C_{\phi}\left(\left(\frac{N_1}{2} \frac{1}{2} N_3, N_2[A]\right), R\right)$$

$$\times e\left((N_1 - \frac{1}{4} M^{-1}[t R_1]) \tau_1 + (N_3 - \frac{1}{2} R_1, M^{-1}[t R_2], A) A t z_1'\right)$$

$$= \sum_{N_1 \in L_{n-t}^+ \cap \mathbb{Z}(n-t,t)} C_{\phi}\left(\left(\frac{1}{2} A^{-1} t N_3, \frac{1}{2} N_2 A^{-1}\right), R\right)$$

$$\times e\left((N_1 - \frac{1}{4} M^{-1}[t R_1]) \tau_1 + (N_3 - \frac{1}{2} R_1, M^{-1}[t R_2], z_1' A) A t z_1'\right)$$

$$= f_{R,N_2}(\tau_1, z_1') A.$$
We remark the isomorphism $B_{t,r}^\infty(Z) \backslash B_{t,s}(Z) \cong GL(t, Z)$. Therefore we have

\[
I_t(\phi_M, \psi_M; s) = \frac{1 + \delta_{t,n}}{2} \int_{\Gamma_{n-t} \backslash \Gamma_{n-t}} \int_{GL(t,Z)/D_t(\tau_1)^{\nu}} \sum_{R_1 \text{ mod } Z^{(n-t,r)}(2M) R_2 \text{ mod } Z^{(t,r)}(2M) N_2 \in L_{t,r}^+} \sum_{f_{R,N_2}(\tau_1, z_1')} \times g_{R,N_2}(\tau_1, z_1') e\left(\frac{-1}{2} (4N_2 - M^{-1}[t R_2]) v_2\right) e\left(\frac{-1}{2} (4N_2 - M^{-1}[t R_2]) v_1^{-1}[y_1']\right) \times \det(v_2)^{-\frac{s}{2}+s-(n+1)} \det(v_1)^{-\frac{s}{2}-(n+1)} \ dv_2 \ dx_1' \ dy_1' \ du_1 \ dv_1
\]

\[
= \frac{1 + \delta_{t,n}}{2} \int_{\Gamma_{n-t} \backslash \Gamma_{n-t}} \int_{GL(t,Z)/D_t(\tau_1)^{\nu}} \sum_{R_1 \text{ mod } Z^{(n-t,r)}(2M) N \in B_{t,r}^\infty(Z) \backslash L_{t,r}^+} \sum_{f_{R,N}(\tau_1, z_1')} \times g_{R,N}(\tau_1, z_1') e\left(\frac{-1}{2} (4N_2 - M^{-1}[t R_2]) v_2\right) e\left(\frac{-1}{2} (4N_2 - M^{-1}[t R_2]) v_1^{-1}[y_1']\right) \times \det(v_2)^{-\frac{s}{2}+s-(n+1)} \det(v_1)^{-\frac{s}{2}-(n+1)} \ dv_2 \ dx_1' \ dy_1' \ du_1 \ dv_1,
\]

where $\mathcal{N} = \left(\frac{N_2}{2} R_2 \frac{1}{2} R \mathcal{M}\right)$. We have

\[
I_t(\phi_M, \psi_M; s) = \frac{1 + \delta_{t,n}}{2} \int_{\Gamma_{n-t} \backslash \Gamma_{n-t}} \int_{GL(t,Z)/D_t(\tau_1)^{\nu}} \sum_{R_1 \text{ mod } Z^{(n-t,r)}(2M) N \in B_{t,r}(Z) \backslash L_{t,r}^+} \sum_{f_{R,N}(\tau_1, z_1')} \times g_{R,N}(\tau_1, z_1') e\left(\frac{-1}{2} (4N_2[A] - M^{-1}[t R_2 A]) v_2\right) e\left(\frac{-1}{2} (4N_2[A] - M^{-1}[t R_2 A]) v_1^{-1}[y_1']\right) \times \det(v_2)^{-\frac{s}{2}+s-(n+1)} \det(v_1)^{-\frac{s}{2}-(n+1)} \ dv_2 \ dx_1' \ dy_1' \ du_1 \ dv_1
\]

\[
= \frac{1 + \delta_{t,n}}{2} \int_{\Gamma_{n-t} \backslash \Gamma_{n-t}} \int_{GL(t,Z)/D_t(\tau_1)^{\nu}} \sum_{R_1 \text{ mod } Z^{(n-t,r)}(2M) N \in B_{t,r}(Z) \backslash L_{t,r}^+} \sum_{f_{R,N}(\tau_1, z_1')} \times g_{R,N}(\tau_1, z_1') e\left(\frac{-1}{2} (4N_2[A] - M^{-1}[t R_2 A]) v_2\right) e\left(\frac{-1}{2} (4N_2[A] - M^{-1}[t R_2 A]) v_1^{-1}[y_1'^t A]\right) \times \det(v_2)^{-\frac{s}{2}+s-(n+1)} \det(v_1)^{-\frac{s}{2}-(n+1)} \ dv_2 \ dx_1' \ dy_1' \ du_1 \ dv_1
\]
Here, in the last identity, we used the formula shown by Maass \cite[p.91, l.10-11]{Ma71}.

We denote by $C$ the set
\[\times\left(\frac{\sqrt{-1}}{2} (4N_2 - \mathcal{M}^{-1}[^t R_2]) \right) \det(v_2)^{k - \frac{s}{2} + s - (n+1)} dv_2 \, dx'_1 \, dy'_1 \, du_1 \, dv_1.\]

As for the last identity, we remark that the set
\[\{(z'_1, \sqrt{-1}v_2[^t A]) \mid (z'_1, \sqrt{-1}v_2) \in GL(t, \mathbb{Z}) \setminus D_t(\tau_1)'' \}, \quad A \in GL(t, \mathbb{Z})\]
covers $D_t(\tau_1)''$ twice if $t = n$ and once if $t \neq n$.

Here we have
\[
\int_{Sym^t_t(\mathbb{R})} e\left(\frac{\sqrt{-1}}{2} (4N_2 - \mathcal{M}^{-1}[^t R_2]) \right) dv_2 \, \det(v_2)^{k - \frac{s}{2} + s - (n+1)} dv_2 \\
= \int_{Sym^t_t(\mathbb{R})} \exp(-\pi Tr (4N_2 - \mathcal{M}^{-1}[^t R_2])) \, dv_2 \, \det(v_2)^{k - \frac{s}{2} + s - (n+1) + \frac{t+1}{2}} dv_2 \\
= \det(\pi (4N_2 - \mathcal{M}^{-1}[^t R_2]))^{-k + \frac{s}{2} - s + (n+1) - \frac{t+1}{2}} \\
\times \int_{Sym^t_t(\mathbb{R})} \exp(-Tr(v_2)) \, dv_2 \, \det(v_2)^{-k + \frac{s}{2} - s + (n+1) + \frac{t+1}{2}} dv_2 \\
= \det(\pi (4N_2 - \mathcal{M}^{-1}[^t R_2]))^{-k + \frac{s}{2} - s + (n+1) - \frac{t+1}{2}} \, \pi^{t-1} \\
\times \prod_{i=1}^t \Gamma\left(s + k - \frac{r}{2} - (n+1) + \frac{t+1}{2} - i - \frac{1}{2}\right).
\]

Here, in the last identity, we used the formula shown by Maass \cite[p.91, l.10-11]{Ma71}.

Thus we obtain the identity (4.2).

We denote by $D_{n,r}$ the complex domain $\mathfrak{H}_n \times \mathbb{C}^{(n,r)}$. Let $\phi_N$ and $\psi_N$ be Fourier-Jacobi coefficients of $\phi_\mathcal{M}$ and $\psi_\mathcal{M}$ denoted in (1.1).

We write $\mathcal{N} = \left(\frac{N_2}{\frac{1}{2}R_2} \mathcal{M}\right) \in L_{l,\mathcal{M}}^+(\mathcal{M})$. For $z' \in \mathbb{C}^{(n-t,t+r)}$ we write $z' = (z'_1, z'_2)$ with $z'_1 \in \mathbb{C}^{(n-t,t)}$ and $z'_2 \in \mathbb{C}^{(n-t,r)}$. We have theta decompositions of $\phi_N$ and $\psi_N$ as follows:

\[\phi_N(\tau_1, z'_1, z'_2) = \sum_{R_1 \mod \mathbb{Z}^{(n-t,t)}} f_{R,N_2}(\tau_1, z'_1) \vartheta_{\mathcal{M},R,2,R_1}(\tau_1, z'_1, z'_2),\]
\[\psi_N(\tau_1, z'_1, z'_2) = \sum_{R_1 \mod \mathbb{Z}^{(n-t,t)}} g_{R,N_2}(\tau_1, z'_1) \vartheta_{\mathcal{M},R,2,R_1}(\tau_1, z'_1, z'_2),\]

where we put
\[\vartheta_{\mathcal{M},R,2,R_1}(\tau_1, z'_1, z'_2) := \vartheta_{\mathcal{M},R}(\tau_1, \frac{1}{2}z'_1 R_2 \mathcal{M}^{-1} + z'_2),\]

and where $R = \left(\frac{R_1}{R_2}\right) \in \mathbb{Z}^{(n,r)}$, and where the function $\vartheta_{\mathcal{M},R}$ is denoted in (2.1). Here we remark that $f_{R,N_2}$ (resp. $g_{R,N_2}$) is appeared in (4.1) in the Fourier-Jacobi expansion.
of $f_R(\tau)$ (resp. $g_R(\tau)$). We omitted the detail of the proof of these decompositions, since the proof is similar to [Ha 18] Lemma 4.1. In other words, we can say that the Fourier-Jacobi expansions and the theta decompositions are compatible.

We write $\tau_1 = u_1 + iv_1$, $z_j^1 = x_j^1 + iy_j^1$ ($j = 1, 2$) with matrices $u_1, v_1 \in \text{Sym}_{n-t}(\mathbb{R})$, $x_1$, $y_1, y_2 \in \mathbb{R}^{n-t,t}$ and $x_2^1$, $y_2 \in \mathbb{R}^{n-t,r}$.

We need the following lemma to calculate $D_s(\phi_M, \psi_M; s)$.

**Lemma 4.2.** We have

$$\int_{D_s(\tau_1)} \partial_{\mathcal{M}, R_1, R_1}(\tau_1, z_1^1, z_2^1)\partial_{\mathcal{M}, R_2, R_1}(\tau_1, z_2^1, z_2^2)$$

$$\times e(2\sqrt{-1}\mathcal{M}^{-1}v_1^{-1}[\frac{1}{2}y_1R_2\mathcal{M}^{-1} + y_2]) \, dx_2 \, dy_2'$$

$$= \delta_{R_1, R_1}(\det v_1)^{\frac{s}{2}} \det(\mathcal{M})^{-\frac{n-t}{2}}.$$  

**Proof.** By a straightforward calculation we have

$$\int_{D_s(\tau_1)} \partial_{\mathcal{M}, R_2, R_1}(\tau_1, z_1^1, z_2^1)\partial_{\mathcal{M}, R_2, R_1}(\tau_1, z_2^1, z_2^2)$$

$$\times e(2\sqrt{-1}\mathcal{M}^{-1}v_1^{-1}[\frac{1}{2}y_1R_2\mathcal{M}^{-1} + y_2]) \, dx_2 \, dy_2'$$

$$= \int_{D_s(\tau_1)} \partial_{\mathcal{M}, R}(\tau_1, \frac{1}{2}z_1^1R_2\mathcal{M}^{-1} + z_2^1)\partial_{\mathcal{M}, R}(\tau_1, \frac{1}{2}z_2^1R_2\mathcal{M}^{-1} + z_2^2)$$

$$\times e(2\sqrt{-1}\mathcal{M}^{-1}v_1^{-1}[\frac{1}{2}y_1R_2\mathcal{M}^{-1} + y_2]) \, dx_2 \, dy_2'$$

$$= \int_{D_s(\tau_1)} \partial_{\mathcal{M}, R}(\tau_1, z_1^1)\partial_{\mathcal{M}, R}(\tau_1, z_2^1)e(2\sqrt{-1}\mathcal{M}^{-1}v_1^{-1}[y_2^1]) \, dx_2 \, dy_2'$$

$$= \delta_{R, R}(\det v_1)^{\frac{s}{2}} \det(\mathcal{M})^{-\frac{n-t}{2}}.$$  

Here, in the last identity, we used the formula in [Zi 89] p.211, 1.19-20.  

**Proposition 4.3.** We have

$$D_s(\phi_M, \psi_M; s + k - n + (t - r - 1)/2)$$

$$= \frac{1 + \delta_{t,n}}{2} \det(2\mathcal{M})^{-\frac{n-t}{2}} \det(\mathcal{M})^{-s-k+n-(t-r-1)/2} - \frac{r(s-k-n)(t-r-1)}{4}$$

$$\times \sum_{N \in \mathcal{B}_{r}(R_2) \setminus L_{t,r}(\mathcal{M})} \frac{1}{|e_{t,r}(N)| \det(4N_2 - \mathcal{M}^{-1}[t R_2])^{s+k-n+(t-r-1)/2}}$$

$$\times \int_{\Gamma_{n-t} \setminus B_{n-t}} \sum_{D_s(\tau_1)} \int_{D_s(\tau_1) \text{ mod } \mathbb{Z}^{n-t,r}} f_{R,N_2}(\tau_1, z_1^1)g_{R,N_2}(\tau_1, z_1^1)$$

$$\times \det(v_1)^{k-(n+\frac{s}{2})+1}e(2i \left( N_2 - \frac{1}{4}\mathcal{M}^{-1}[t R_2] \right) v_1^{-1}[y_1^1]) \, dx_1 \, dy_1 \, du_1 \, dv_1.$$
Proof. We put $D_r(\tau_1) = \mathcal{O}^{(n-t,r)}/(\tau_1 \mathbb{Z}^{(n-t,r)} + \mathbb{Z}^{(n-t,r)})$. We have

$$
\langle \phi_N, \psi_N \rangle = \int_{\Gamma_{n-t,t+r}\setminus \hat{D}_{n-t}^{(n-t+r)}} \phi_N^{\tau_1} \psi_N^{\tau_1} e(2iNv_1^{-1}[(y_1^t y_2^t)])
\times \det(v_1)^{k-(n+r+1)} du_1 dv_1 dx'_1 dy'_1 dx'_2 dy'_2
= \frac{1 + \delta_{tn}}{2} \int_{\Gamma_{n-t,r} \setminus \hat{D}_{n-t}^{(n-t+r)}} \phi_N^{\tau_1} \psi_N^{\tau_1} e(2iNv_1^{-1}[(y_1^t y_2^t)])
\times \det(v_1)^{k-(n+r+1)} dx'_2 dy'_2 dx'_1 dy'_1 du_1
= \frac{1 + \delta_{tn}}{2} \int_{\Gamma_{n-t,r} \setminus \hat{D}_{n-t}^{(n-t+r)}} \phi_N^{\tau_1} \psi_N^{\tau_1} e(2iNv_1^{-1}[(y_1^t y_2^t)])
\times \det(v_1)^{k-(n+r+1)} dx'_2 dy'_2 dx'_1 dy'_1 du_1,
$$

where we write $R = \left( \begin{array}{c} R_1 \\ R_2 \end{array} \right)$ and $\tilde{R} = \left( \begin{array}{c} \tilde{R}_1 \\ R_2 \end{array} \right)$. Since

$$
N = \left( \begin{array}{cc} N_2 & \frac{1}{2} R_2 \\ \frac{1}{2} R_2 & M \end{array} \right) = \left( \begin{array}{cc} N_2 - \frac{1}{4} M^{-1} [t^t R_2] & 0 \\ 0 & M \end{array} \right) \left[ \begin{array}{cc} 1_t & 1_r \\ -1 & 1_r \end{array} \right],
$$

we obtain

$$
\langle \phi_N, \psi_N \rangle = \frac{1 + \delta_{tn}}{2} \int_{\Gamma_{n-t,r} \setminus \hat{D}_{n-t}^{(n-t+r)}} \phi_N^{\tau_1} \psi_N^{\tau_1} e(2iNv_1^{-1}[(y_1^t y_2^t)])
\times \det(v_1)^{k-(n+r+1)} dx'_2 dy'_2 dx'_1 dy'_1 du_1.
$$

Due to Lemma 4.2, we obtain

$$
\langle \phi_N, \psi_N \rangle = \frac{1 + \delta_{tn}}{2} \int_{\Gamma_{n-t,r} \setminus \hat{D}_{n-t}^{(n-t+r)}} \phi_N^{\tau_1} \psi_N^{\tau_1} e(2iNv_1^{-1}[(y_1^t y_2^t)])
\times \det(v_1)^{k-(n+r+1)} \det(2M) \frac{n-t}{2} \frac{e(2iNv_1^{-1}[(y_1^t y_2^t)])}{(n-t)} dx'_1 dy'_1 du_1 dv_1.
$$
Therefore we have
\[
D_t(\phi_M, \psi_M; s + k - n + (t - r - 1)/2) = \frac{1 + \delta_{t,n}}{2} \det(2M)^{-\frac{n-1}{2} - \frac{r(n-t)}{2}} D_t(\psi_M, \phi_M; s - k + n - (t-r-1)/2) \times
\sum_{N \in B_{t,r}(\mathbb{Z}) \backslash L^+_{t,r}(M)} \frac{1}{\det(N) \mid \det(N)^{s+k-n+(t-r-1)/2}}
\int_{\Gamma_{n-t} \backslash \mathbb{H}} \int_{D_t(\tau_1) R_1 \mod \mathbb{Z}^{(n-t,r)}} \sum_{\mathbb{Z}^{(n-t,r)}} f_{R,N_2}^*(\tau_1, z'_1) g_{R,N_2}(\tau_1, z'_1)
\int_{\Gamma_{n-t} \backslash \mathbb{H}} \int_{D_t(\tau_1) R_1 \mod \mathbb{Z}^{(n-t,r)}} \sum_{\mathbb{Z}^{(n-t,r)}} f_{R,N_2}^*(\tau_1, z'_1) g_{R,N_2}(\tau_1, z'_1)
\times \det(v_1)^{k-(n+\frac{r}{2}+1)} e(2i \left( N_2 - \frac{1}{4} M^{-1} \left[ t R_2 \right] \right) v_1^{-1}[y_1]) dx_1 dy_1^t du_1 dv_1
\]

We conclude this proposition. \qed

By comparing the identity (4.2) with the identity (4.4), we obtain Proposition 2.3.

References

[E-Z 85] M. Eichler and D. Zagier: Theory of Jacobi Forms, Progress in Math. 55, Birkhäuser, Boston-Basel-Stuttgart, (1985).

[Ha 18] S. Hayashida: On Kohnen plus-space of Jacobi forms of half integral weight of matrix index, Osaka J. Math., 55, no.3 (2018), 499–522.

[Ib 92] T. Ibukiyama: On Jacobi forms and Siegel modular forms of half integral weights, Comment. Math. Univ. St. Paul. 41, no.2 (1992), 109–124.

[Ik 01] T. Ikeda: On the lifting of elliptic cusp forms to Siegel cusp forms of degree 2n, Ann. of Math. (2) 154, no.3 (2001), 641–681.

[I-M 03] Ö. Imamoğlu and Y. Martin: On a Rankin-Selberg convolution of two variables for Siegel modular forms, Forum Math. 15, no. 4 (2003), 565–589.

[Ka 77] V. L. Kalinin: Eisenstein series on the symplectic group, Mat. USSR Sb. 32 (1977), 449–476.

[Ka 84] V. L. Kalinin: Analytic properties of the convolution products of genus g, Math. USSR Sb. 48 (1984), 193–200.

[K-K 08] H. Katsurada and H. Kawamura: A certain Dirichlet series of Rankin-Selberg type associated with the Ikeda lifting, J. Number Theory 128, no. 7 (2008), 2025–2052.
[K-K 15] H. Katsurada and H. Kawamura: Ikeda’s conjecture on the period of the Duke-Imamoglu-Ikeda lift, *Proc. Lond. Math. Soc. (3)* **111**, no. 2 (2015), 445–483.

[Ko 80] W. Kohnen: Modular forms of half integral weight on $\Gamma_0(4)$, *Math. Ann.* **248** (1980), 249–266.

[K-S 89] W. Kohnen and N.-P. Skoruppa: A certain Dirichlet series attached to Siegel modular forms of degree two. *Invent. Math.* **95**, no.3 (1989), 541–558.

[K-Z 81] W. Kohnen and D. Zagier: Values of L-series of modular forms at the center of the critical strip. *Invent. Math.* **64**, no. 2 (1981), 175–198.

[Ma 71] H. Maass: Siegel’s modular forms and Dirichlet series. *Lecture Notes in Mathematics*, **216**. (1971) Springer-Verlag.

[Ma 73] H. Maass: Dirichletsche Reihen und Modulformen zweiten Grades. *Acta Arith.* **24** (1973), 225–238.

[Shi 75] T. Shintani: On zeta-functions associated with the vector space of quadratic forms, *J. Fac. Sci. Univ. Tokyo Sect. I A Math.* **22** (1975), 25–65.

[Ya 90] T. Yamazaki: Rankin-Selberg method for Siegel cusp forms, *Nagoya Math. J.* **120** (1990), 35–49.

[Zi 89] C. Ziegler: Jacobi forms of higher degree, *Abh. Math. Sem. Univ. Hamburg*. **59** (1989), 191–224.

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