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VECTOR INVARIANTS FOR THE TWO DIMENSIONAL MODULAR REPRESENTATION OF A CYCLIC GROUP OF PRIME ORDER

H E A CAMPBELL, R J SHANK, AND D L WEHLAU

Abstract. In this paper, we study the vector invariants of the 2-dimensional indecomposable representation $V_2$ of the cyclic group, $C_p$, of order $p$ over a field $F$ of characteristic $p$, $F[m V_2]^{C_p}$. This ring of invariants was first studied by David Richman [21] who showed that the ring required a generator of degree $m(p - 1)$, thus demonstrating that the result of Noether in characteristic 0 (that the ring of invariants of a finite group is always generated in degrees less than or equal to the order of the group) does not extend to the modular case. He also conjectured that a certain set of invariants was a generating set with a proof in the case $p = 2$. This conjecture was proved by Campbell and Hughes in [3]. Later, Shank and Wehlau in [24] determined which elements in Richman’s generating set were redundant thereby producing a minimal generating set.

We give a new proof of the result of Campbell and Hughes, Shank and Wehlau giving a minimal algebra generating set for the ring of invariants $F[m V_2]^{C_p}$. In fact, our proof does much more. We show that our minimal generating set is also a SAGBI basis for $F[m V_2]^{C_p}$. Further, our results provide a procedure for finding an explicit decomposition of $F[m V_2]$ into a direct sum of indecomposable $C_p$-modules. Finally, noting that our representation of $C_p$ on $V_2$ is as the $p$-Sylow subgroup of $SL_2(F_p)$, we describe a generating set for the ring of invariants $F[m V_2]^{SL_2(F_p)}$ and show that $(p + m - 2)(p - 1)$ is an upper bound for the Noether number, for $m > 2$.

Contents

1. Introduction 2
2. Preliminaries 5
2.2. Relations involving the $u_{ij}$ 6
3. Polarisation 8
4. Partial Dyck Paths 10
5. Lead Monomials 11

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1. Introduction

We suppose $G$ is a group represented on a vector space $V$ over a field $F$. If $\{x_1, x_2, \ldots, x_n\}$ is a basis for the hom-dual, $V^* = \text{hom}_F(V, F)$, of $V$, then we denote the symmetric algebra on $V^*$ by

$$F[V] = F[x_1, x_2, \ldots, x_n]$$

and we note that $G$ acts on $f \in F[V]$ by the rule

$$\sigma(f)(v) = f(\sigma^{-1}(v)).$$

As an aside, the notation $F[V]$ is often used in the literature to denote the ring of regular functions on $V$. Our notation coincides with the usual notion when the field $F$ is infinite. However, for example, if $F = F_p$, the prime field, then the functions $x_1$ and $x_p^1$ coincide on $V$.

The ring of functions left invariant by this action of $G$ is denoted $F[V]^G$. Invariant theorists often seek to relate algebraic properties of the invariant ring to properties of the representation. For example, when $G$ is finite of order $|G|$ and the characteristic $p$ of $F$ does not divide $|G|$ – the non-modular case – then $F[V]^G$ is a polynomial algebra if and only if $G$ is generated by reflections (group elements fixing a hyperplane of $V$). This is a famous result due to Coxeter [8], Shephard and Todd [26], Chevalley [6], and Serre[22]. For another example in the non-modular case, it is known by work of Noether [19] (when $p = 0$), Fogarty [12] and Fleischmann [13] (when $p > 0$), that $F[V]^G$ is generated in degrees less than or equal to $|G|$. And, in the non-modular case with $G$ finite, it is well-known that $F[V]^G$ is always Cohen-Macaulay.

The case when $p > 0, G$ is finite, $V$ is finite dimensional and $p$ does divide $|G|$ is that of modular invariant theory. Many results that are well understood in the non-modular case are not yet understood or even within reach in the modular case. For example, in the modular case it is known that if $F[V]^G$ is a polynomial algebra then $G$ must be generated by reflections, but this is far from sufficient. For another example, in the modular case $F[V]^G$ is “most often” not Cohen-Macaulay. Finally, in the modular case, there are examples where $F[V]^G$ requires generators of degrees (much) larger than $|G|$, see below: this paper re-examines the first known such example in considerable detail.
There are now several references for modular invariant theory, see Benson [1], Smith [27], Neusel and Smith [18], Derksen and Kemper [9], Campbell and Wehlau [3].

Invariant theorists also seek to determine generators for $F[V]^G$ and, if possible, relations among those generators. A famous example is the case of *vector invariants*, see Weyl [28]. Here we consider the vector space

$$m V = V \oplus V \oplus \cdots \oplus V$$

with $G$ acting diagonally. The invariants $F[m V]^G$ are called vector invariants, and in this case, we seek to describe, determine or give constructions for, the generators of this ring, a *first main theorem for* $(G, V)$. Once this is done a theorem determining the relations among the generators is referred to as a *second main theorem for* $(G, V)$.

The cyclic group $C_p$ has exactly $p$ inequivalent indecomposable representations over a field $F$ of characteristic $p$. There is one indecomposable $V_n$ of dimension $n$ for each $1 \leq n \leq p$. To see this choose a basis for $V_n$ with respect to which a fixed generator, $\sigma$, of $C_p$ is represented by a matrix in Jordan Normal form. Since $V_n$ is indecomposable this matrix has a single Jordan block and since $\sigma$ has order $p$ the common eigenvalue must be 1, the only $p^{th}$ root of unity in a field of characteristic $p$. Thus $\sigma$ is represented on $V_n$ by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & \ldots & 0 & 0 \\
: & : & : & \ddots & : & : \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 1
\end{pmatrix}
$$

In order that this matrix have order $p$ (or 1) we must have $n \leq p$. We call such a basis of $V_n$ for which $\sigma$ is in (lower triangular) Jordan Normal form a *triangular* basis.

Observe the following chain of inclusions:

$$V_1 \subset V_2 \subset \cdots \subset V_p.$$ 

If $V$ is any finite dimensional $C_p$-module then $V$ can be decomposed into a direct sum of indecomposable $C_p$-modules:

$$V \cong m_1 V_1 \oplus m_2 V_2 \cdots \oplus m_p V_p$$

where $m_i \in \mathbb{N}$ for all $i$. This decomposition is far from unique but does have the property that the *multiplicities* $m_i$ are unique.
We are interested in the representation $mV_2$ and the action of $C_p$ on $F[mV_2]$. The ring of invariants $F[mV_2]^{C_p}$ was first studied by David Richman [21]. He showed that this ring required a generator of degree $m(p - 1)$, showing that the result of Noether in characteristic 0 did not extend to the modular case. He also conjectured that a certain set of invariants was a generating set with a proof in the case $p = 2$. This conjecture was proved by Campbell and Hughes in [3]: the proof is long, complex, and counter-intuitive in some respects. Later, Shank and Wehlau in [24] determined which elements in Richman’s generating set were redundant thereby producing a minimal generating set.

We will show later (and the proof is not difficult), that $F[mV_2]^{C_p}$ is not Cohen-Macaulay, or see Ellingsrud and Skjelbred [11].

In this paper, we give a new proof of the result of Campbell and Hughes, Shank and Wehlau giving a minimal algebra generating set for the ring of invariants $F[mV_2]^{C_p}$. In fact, our proof does much more. We show that our minimal generating set is also a SAGBI basis for $F[mV_2]^{C_p}$. In our view, this result is extraordinary. Further, our techniques also yield a procedure for finding a decomposition of $F[mV_2]$ into a direct sum of indecomposable $C_p$-modules.

Our paper is organised as follows. In the second section of our paper, Preliminaries, we provide more details on the representation theory of $C_p$, our use of graded reverse lexicographical ordering on the monomials in $F[mV_2]^{C_p}$, and define the term SAGBI basis. In the next section, Polarisation, we define the polarisation map $F[V] \to F[mV]$, its (roughly speaking) inverse, known as restitution, and we note that these maps are $G$-equivariant, hence map $G$-invariants to $G$-invariants. These techniques allow us to focus our attention on multi-linear invariants. The next section, Partial Dyck Paths, describes a concept arising in the study of lattices in the plane, see, for example the book by Koshy [17, p. 151], and is followed by a section on Lead Monomials. Here we show that there is a bijection between the set of lead monomials of multi-linear invariants and certain collections of Partial Dyck Paths. This work requires us to count the number of indecomposable $C_p$ summands in

$$m \otimes V_2 = \underbrace{V_2 \otimes V_2 \otimes \cdots \otimes V_2}_m$$

and in fact we are able to determine a decomposition of $m \otimes V_2$ as a $C_p$-module, see Theorem 5.5. Following this, in section § 6, we prove that we have a generating set for our ring of invariants. The next section describes how our techniques provide a procedure for finding
a decomposition of $F[mV_2]$ as a $C_p$-module. In the final section, noting that our representation of $C_p$ on $V_2$ is as the $p$-Sylow subgroup of $SL_2(F_p)$, we are able to describe a generating set for the ring of invariants $F[mV_2]^{SL_2(F_p)}$.

We thank the referee for a thorough and careful reading of our paper.

2. Preliminaries

Suppose $\{e_1, e_2, \ldots, e_n\}$ is a triangular basis for $V_n$. Note that the $C_p$-module generated by $e_1$ is all of $V_n$. We also note that the indecomposable module $V_n^* = \text{hom}(V_n, F)$ is isomorphic to $V_n$ since $\dim(V_n^*) = \dim(V_n)$. Because of our interest in invariants we often focus on the $C_p$ action on $V_n^*$ rather than on $V_n$ itself. Therefore we will choose the dual basis $\{x_1, x_2, \ldots, x_n\}$ for $V^*$ to the basis $\{e_1, e_2, \ldots, e_n\}$. With this choice of basis the matrices representing $G$ are upper-triangular on $V^*$. We note that $\sigma^{-1}(x_1) = x_1$ and $\sigma^{-1}(x_i) = x_i + x_{i-1}$ for $2 \leq i \leq n$: for convenience, and since $\sigma^{-1}$ also generates $C_p$, we will change notation and write $\sigma$ instead of $\sigma^{-1}$ for the remainder of this paper. With this convention, we note that $(\sigma - 1)^r(x_n) = x_{n-r}$ for $r < n$ and $\dim(V_n) = n$ is the largest value of $r$ such that $x_1 \in (\sigma - 1)^{-1}(V_n^*)$. We say that the invariant $x_1$ has length $n$ in this case and write $\ell(x_1) = n$. We observe that the socle of $V_n$ is the line $V_n^{C_p}$ spanned by $\{e_n\}$. Similarly $(V_n^*)^{C_p}$ has basis $\{x_1\}$.

Note that the kernel of $\sigma - 1 : V_i \to V_i^{C_p}$ which is one dimensional for all $i$. Thus

$$\dim((\sigma - 1)^j(V_i)) = \begin{cases} 0 & \text{if } j - 1 \geq i; \\ i - j & \text{if } j - 1 < i. \end{cases}$$

For

$$V \cong m_1 V_1 \oplus m_2 V_2 \cdots \oplus m_p V_p$$

this gives $(p-j)m_p + (p-1-j)m_{p-1} + \cdots + (i-j)m_i = \dim((\sigma - 1)^j(V))$ for all $0 \leq j \leq p - 1$ and this system of equations uniquely determines the coefficients $m_1, m_2, \ldots, m_p$.

Each indecomposable $C_p$-module, $V_n$, satisfies $\dim(V_n)^{C_p} = 1$. Therefore the number of summands occurring in a decomposition of $V$ is given by $m_1 + m_2 + \cdots + m_p = \dim V^{C_p}$.

Consider $Tr := \sum_{\tau \in C_p} \tau$, an element of the group ring of $C_p$. If $W$ is any finite dimensional $C_p$-representation, we also use $Tr$ to denote the corresponding $F[W]^{C_p}$-module homomorphism,

$$Tr : F[W] \to F[W]^{C_p}.$$
Similarly we define
\[ N : F[W] \to F[W]^{C_p} \]
by \( N(w) = \prod_{\tau \in C_p} \tau(w) \).

Note that \((\sigma - 1)^{p-1} = \sum_{i=0}^{p-1} (-1)^i (p-1)^i \sigma^i = \sum_{i=0}^{p-1} \sigma^i = \text{Tr}.\) It follows that \(\text{Tr}(v) = 0\) if \(v \in V_n\) for \(n < p\), while \(\text{Tr}(x_p) = x_1 \in V_p\).

It is also the case that \(V_p \cong FC_p\) is the only free \(C_p\)-module and hence also the only projective.

The next theorem plays an important role in our decomposition of \(F[V]^{(d_1, d_2, \ldots, d_m)}\) as a \(C_p\)-module (modulo projectives). A proof in the case \(V = V_n\) may be found in Hughes and Kemper [14, section 2.3], and a proof of the version cited here is in Shank and Wehlau [25, section 2]

**Theorem 2.1** (Periodicity Theorem). Let \(V = V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_m}\). Let \(d_1, d_2, \ldots, d_m\) be non-negative integers and write \(d_i = q_i p + r_i\) where \(0 \leq r_i \leq p - 1\) for \(i = 1, 2, \ldots, m\). Then
\[ F[V]^{(d_1, d_2, \ldots, d_m)} = F[V]^{(r_1, r_2, \ldots, r_m)} \oplus t V_p \]
as \(C_p\)-modules for some non-negative integer \(t\).

Comparing dimensions shows that in the above theorem
\[ t = \left( \prod_{i=1}^{m} \binom{n_i + d_i - 1}{d_i} - \prod_{i=1}^{m} \binom{n_i + r_i - 1}{r_i} \right) / p . \]

In this paper, we are primarily interested in the case \(V = m V_2\). We denote the basis for the \(i^{th}\)-copy of \(V_2^*\) in this direct sum by \(\{x_i, y_i\}\) and we have \(\sigma(x_i) = x_i\) and \(\sigma(y_i) = y_i + x_i\).

For this representation of \(C_p\), there is another “obvious” family of invariants, namely the
\[ u_{ij} = x_i y_j - x_j y_i = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \]
for \(m \geq 2\).

**2.2. Relations involving the \(u_{ij}\).** We will consider now two important families of relations involving the invariants \(u_{ij} = x_i y_j - x_j y_i\). First we consider algebraic dependencies among the \(u_{ij}\). Suppose \(m \geq 4\) and let \(1 \leq i < j < k < \ell \leq m\). It is easy to verify that \(0 = u_{ij} u_{k\ell} - u_{ik} u_{j\ell} + u_{i\ell} u_{jk}\). It can be shown that these relations generate all the algebraic relations among the \(u_{st}\).

It is useful to represent products of the various \(u_{st}\) graphically as follows. We consider the vertices of a regular \(m\)-gon and label them clockwise by \(1, 2, \ldots, m\). We represent the factor \(u_{ij}\) by an arrow or directed edge from vertex \(i\) to vertex \(j\). Thus a product of various \(u_{st}\) is
represented by a number of directed edges joining the labelled vertices of the regular \( m \)-gon.

Returning to the relation \( u_{ij}u_{k\ell} - u_{ik}u_{j\ell} + u_{i\ell}u_{jk} \), we say that the middle product in this relation, \( u_{ik}u_{j\ell} \), is a crossing since the arrows representing the two factors \( u_{ik} \) and \( u_{j\ell} \) cross (intersect). Rewriting the relation as \( u_{ik}u_{j\ell} = u_{ij}u_{k\ell} + u_{i\ell}u_{jk} \), we see that we may replace a crossing with a sum of two non-crossings. As observed by Kempe [16], since the length of two (directed) diagonals representing \( u_{ik} \) and \( u_{j\ell} \) exceeds both the lengths represented by the sides \( u_{ij} \) and \( u_{k\ell} \) and the two sides \( u_{i\ell} \) and \( u_{jk} \), we may repeatedly use “uncrossing” relations to rewrite any product of \( u_{st} \)'s by a sum of such products without any crossings. Thus the space generated by the monomials in the \( u_{st} \) of degree \( d \) has a basis represented by planar directed graphs on \( m \) vertices having \( d \) directed edges. Here we allow multiple (or weighted) edges to represent powers such as \( u_{ij}^a \) for \( a \geq 2 \).

Now we consider another important class of relations, this time involving the \( u_{st} \) and the \( x_r \). Take \( m \geq 3 \), let \( 1 \leq i < j < k \leq m \) and consider the matrix

\[
\begin{pmatrix}
x_i & x_j & x_k \\
x_i & x_j & x_k \\
y_i & y_j & y_k
\end{pmatrix}.
\]

Computing the determinant by expanding along the first row we find

\[x_iu_{jk} - x_ju_{ik} + x_ku_{ij} = 0.\]

Since \( x_1, x_2, x_3 \) is a partial homogeneous system of parameters in \( \mathbf{F}[mV_2] \) consisting of invariants it is a partial homogeneous system of parameters in \( \mathbf{F}[mV_2]^{Cr} \). The relation

\[x_1u_{23} - x_2u_{13} + x_3u_{12} = 0\]

shows that the product \( x_3u_{12} \) represents 0 in the quotient ring \( \mathbf{F}[mV_2]^{Cr}/(x_1, x_2) \). Considering degrees, it is easy to see that \( u_{12} \) and \( x_3 \) do not lie in the ideal of \( \mathbf{F}[mV_2]^{Cr} \) generated by \( (x_1, x_2) \). Thus \( x_3 \) represents a zero divisor in the quotient ring \( \mathbf{F}[mV_2]^{Cr}/(x_1, x_2) \). This shows that the partial homogeneous system of parameters \( x_1, x_2, x_3 \) in \( \mathbf{F}[mV_2]^{Cr} \) does not form a regular sequence. Therefore \( \mathbf{F}[mV_2]^{Cr} \) is not a Cohen-Macaulay ring when \( m \geq 3 \). For \( m \leq 2 \) the ring of invariants \( \mathbf{F}[mV_2]^{Cr} \) is Cohen-Macaulay since \( \mathbf{F}[V_2]^{Cr} = \mathbf{F}[x_1, N(y_1)] \) is a polynomial ring and \( \mathbf{F}[2V_2]^{Cr} = \mathbf{F}[x_1, x_2, u_{12}, N(y_1), N(y_2)] \) is a hypersurface ring.

Throughout this paper we will use graded reverse lexicographic term orders. We write \( \text{LM}(f) \) for the lead monomial of \( f \) and \( \text{LT}(f) \) for the lead term of \( f \). We follow the convention that monomials are power products of variables and terms are scalar multiples of power products of variables. If \( S = \oplus_{d=0}^\infty S_d \) is a graded subalgebra of a polynomial ring then we say a set \( B \) is a SAGBI basis for \( S \) in degree \( d \) if for every \( f \in S_d \)
we can write $\text{LM}(f)$ as a product $\text{LM}(f) = \prod_{g \in B} \text{LM}(g)^{e_g}$ where each $e_g$ is a non-negative integer. If $B$ is a SAGBI basis for $S$ in degree $d$ for all $d$ then we say that $B$ is a SAGBI basis for $S$. If $B$ is a SAGBI basis for $S$ then $B$ is an algebra generating set for $S$. The word SAGBI is an acronym for “sub-algebra analogue of Gröbner bases for ideals”, and was introduced by Robbianno and Sweedler in [20] and (independently) by Kapur and Madlener in [15]. For a detailed discussion of term orders we direct the reader to Chapter 2 of Cox, Little and O’Shea [7]. For a discussion and application of SAGBI bases in modular invariant theory, we recommend Shank’s paper [23].

Given a sequence of variables $z_1, z_2, \ldots, z_m$ and an element $E = (e_1, e_2, \ldots, e_m)$ we write $z^E$ to denote the monomial $z_1^{e_1}z_2^{e_2}\cdots z_m^{e_m}$ and we denote the degree $e_1 + e_2 + \cdots + e_m$ of this monomial by $|E|$. The following well-known lemma is very useful for computing the lead term of a transfer.

**Lemma 2.3.** Let $t$ be a positive integer. Then

$$\sum_{i=0}^{p-1} t_i = \begin{cases} -1, & \text{if } p - 1 \text{ divides } t; \\ 0, & \text{if } p - 1 \text{ does not divide } t. \end{cases}$$

For a proof of this lemma see for example, [5, Lemma 9.4].

3. **Polarisation**

Let $V$ be a representation of a group $G$ and let $r \in \mathbb{N}$ with $r \geq 2$ and consider the map of $G$-representations

$$\nabla^*: rV \rightarrow (r-1)V$$
defined by $\nabla^*(v_1, v_2, \ldots, v_r) = (v_1, v_2, \ldots, v_{r-2}, v_{r-1} + v_r)$. We also consider the morphism

$$\Delta^*: (r-1)V \rightarrow rV$$
given by $\Delta^*(v_1, v_2, \ldots, v_{r-1}) = (v_1, v_2, \ldots, v_{r-2}, v_{r-1}, v_{r-1})$. Dual to these two maps we have the corresponding algebra homomorphisms:

$$\nabla : \mathbf{F}[(r-1)V] \rightarrow \mathbf{F}[rV]$$

and

$$\Delta : \mathbf{F}[rV] \rightarrow \mathbf{F}[(r-1)V]$$
given by $\nabla(f) = f \circ \nabla^*$ and $\Delta(F) = F \circ \Delta^*$. We also define $\nabla_r^* = (\nabla^*)^{r-1} : rV \rightarrow V$ and $\Delta_r^* = (\Delta^*)^{r-1} : V \rightarrow rV$.

Thus $\nabla_r : \mathbf{F}[V] \rightarrow \mathbf{F}[rV]$ is given by $(\nabla_r(f))(v_1, v_2, \ldots, v_r) = f(v_1 + v_2 + \cdots + v_r)$ and $\Delta_r : \mathbf{F}[rV] \rightarrow \mathbf{F}[V]$ is given by $(\Delta_r(F))(v) =$
For each multi-degree, \(\pi\) where \(P\) multi-linear function \(\in \pi\).

Again, for ease of notation, if \(f\) we will use the symbols \(F\) to denote \(\Delta\) homomorphism determined by \(\Delta\) since the action of \(F\) only if all its homogeneous components are invariant.

More generally, we may use isomorphisms of the form \(F[V \oplus W] \cong F[V] \otimes F[W]\) to define

\[
\nabla_{r_1, r_2, \ldots, r_m} = \nabla_{r_1} \otimes \nabla_{r_2} \otimes \cdots \otimes \nabla_{r_m} : F[\bigoplus_{i=1}^{m} W_i] \longrightarrow F[\bigoplus_{i=1}^{m} r_i W_i].
\]

Again, for ease of notation, if \(f \in F[\bigoplus_{i=1}^{m} W_i]_{(\lambda_1, \lambda_2, \ldots, \lambda_m)}\) we write \(\mathcal{P}(f) = \pi(1, 1, \ldots, 1)(\nabla_{\lambda_1, \lambda_2, \ldots, \lambda_m}(f)) \in F[\bigoplus_{i=1}^{m} \lambda_i W_i]_{(1, 1, \ldots, 1)}\). Here again we call the multi-linear function \(\mathcal{P}(f)\) the full polarisation of \(f\).

Similarly we define the restitution map

\[
\Delta_{r_1, r_2, \ldots, r_m} = \Delta_{r_1} \otimes \Delta_{r_2} \otimes \cdots \otimes \Delta_{r_m} : F[\bigoplus_{i=1}^{m} r_i W_i] \longrightarrow F[\bigoplus_{i=1}^{m} W_i].
\]

In this setting, if we have co-ordinate variables such as \(x_i, y_i, z_i\) for \(W_i\) we will use the symbols \(x_{ij}, y_{ij}, z_{ij}\) with \(1 \leq j \leq r_i\) to denote the co-ordinate variables for \(r_i W_i\). In this notation, restitution is the algebra homomorphism determined by \(\Delta_{r_1, r_2, \ldots, r_m}(x_{ij}) = x_{ij}, \Delta_{r_1, r_2, \ldots, r_m}(y_{ij}) = y_{ij}, \Delta_{r_1, r_2, \ldots, r_m}(z_{ij}) = z_{ij}\), etc. For this reason, restitution is sometimes referred to as erasing subscripts. For ease of notation, we will write \(R\) to denote the algebra homomorphism \(\Delta_{\lambda_1, \lambda_2, \ldots, \lambda_m}\) when restricted to \(F[\bigoplus_{i=1}^{m} \lambda_i W_i]_{(1, 1, \ldots, 1)}\) Then if \(F \in F[\bigoplus_{i=1}^{m} \lambda_i W_i]_{(1, 1, \ldots, 1)}\) then \(R(F) \in F[\bigoplus_{i=1}^{m} \lambda_i W_i]_{(1, 1, \ldots, 1)}\). (However, we will sometimes abuse notation and use \(R\) to denote \(\Delta_{\lambda_1, \lambda_2, \ldots, \lambda_m}\) when the indices \(\lambda_1, \lambda_2, \ldots, \lambda_m\) are clear from the context.)

It is a relatively straightforward exercise to verify that for any \(f \in F[\bigoplus_{i=1}^{m} W_i]_{(\lambda_1, \lambda_2, \ldots, \lambda_m)}\) we have \(R(\mathcal{P}(f)) = (\lambda_1!, \lambda_2!, \ldots, \lambda_m! f)\).

Since \(\nabla^+\) and \(\Delta^+\) are both \(G\)-equivariant, so are all the homomorphisms \(\nabla_{r_1, r_2, \ldots, r_m}\) and \(\Delta_{r_1, r_2, \ldots, r_m}\). In particular, if \(f\) is \(G\)-invariant then so is \(\mathcal{P}(f)\). Similarly, \(R(F)\) is \(G\)-invariant if \(F\) is. We also note that since the action of \(G\) preserves degree an element \(f\) is invariant if and only if all its homogeneous components are invariant.
4. Partial Dyck Paths

In this section we consider a generalization of Dyck paths (see the book by Koshy [17, p. 151] for an introduction to Dyck paths). For us, a lattice path is a finite sequence of steps in the first quadrant of the xy-plane starting from the origin. Each step in the path is given by either the vector (1,0) (an x-step) or the vector (0,1) (a y-step). The number of steps in the path is called its length. The path is said to have height $h$ if $h$ is the largest integer such that the path touches the line $y = x - h$. A lattice path has finishing height $h$ if the final step ends at a point on the line $y = x - h$.

Associated to each lattice path of length $d$ is a word of length $d$, i.e., an ordered sequence of $d$ symbols each either an $x$ or a $y$. This word encodes the path as follows: the $i^{th}$ symbol of the word is $x$ if the $i^{th}$ step of the path is an $x$-step and the $i^{th}$ symbol of the word is $y$ if the $i^{th}$ step is a $y$-step.

We will consider two types of lattice paths: (i) partial Dyck paths and (ii) initial Dyck paths of escape height $p - 1$.

**Definition 4.1.** A partial Dyck path is a lattice path which stays on or below the diagonal (the line with equation $y = x$). A partial Dyck path of finishing height 0, i.e., which finishes on the diagonal, is called a Dyck path.

**Definition 4.2.** An initial Dyck path of escape height $t$ is a lattice path of height at least $t$ and which if it crosses above the diagonal does so only after it touches the line $y = x - t$. Expressed another way, these are paths which consist of an partial Dyck path of finishing height $t$ followed by an entirely arbitrary sequence of $x$-steps and $y$-steps.

Clearly there are $2^d$ lattice paths of length $d$. We may associate these paths with the $2^d$ monomials in $F[dV_2](1,1,...,1) \cong \otimes^d V_2$. The lattice path $\gamma$ of length $d$ is associated to the word $\gamma_1\gamma_2\cdots \gamma_d$ and is associated to the multi-linear monomial $\Lambda(\gamma) = z_1z_2\cdots z_d$ where $\begin{cases} z_i = x_i, & \text{if } \gamma_i = x; \\ z_i = y_i, & \text{if } \gamma_i = y. \end{cases}$

We let $\text{PDP}^{d}_{\leq q}$ denote the set of all partial Dyck paths of length $d$ and height at most $q$. Furthermore, we will denote by $\text{PDP}^{d}_{\leq q}(h)$ the set of partial Dyck paths of length $d$, height at most $q$ and finishing height $h$. We write $\text{IDP}^{d}_{q}$ to denote the set of all initial Dyck paths of escape height $q$ and length $d$. 

5. Lead Monomials

We wish to consider the $C_ρ$-representation $F[dV_2]_{(1,1,\ldots,1)} \cong \otimes^d V_2$. We consider a decomposition of $\otimes^d V_2$ into a direct sum of indecomposable $C_ρ$-representations. Each summand $V_h$ has a one dimensional socle spanned by an element $f$ and we associate to this summand the monomial $LM(f)$. We say that the invariant $f$ has length $h$ and we write $\ell(f) = h$. In general a non-zero invariant has length $h$ if $h - 1$ is the maximal value of $r$ for which $f$ lies in the image of $(\sigma - 1)^r$.

In order to study $F[dV_2]_{(1,1,\ldots,1)} \cong (\otimes^d V_2)^C_ρ$ we use the graded reverse lexicographic order determined by $y_1 > x_1 > y_2 > x_2 \cdots > y_d > x_d$ and consider

$$M = \{LM(f) \mid f \in (\otimes^d V_2)^C_ρ\}.$$ 

We will show that the set map

$$\Lambda : PDP^d_{\leq p-2} \sqcup IDP^d_{p-1} \rightarrow M$$

is a bijection.

We begin by showing that the image of $\Lambda$ lies inside $M$. In fact we will show that if $γ \in PDP^d_{\leq p-2}(h)$ then $Λ(γ)$ is the lead monomial of an invariant of length $h + 1$. Furthermore if $γ \in IDP^d_{p-1}$ then $Λ(γ)$ is the lead monomial of an invariant of length $p$, i.e., an invariant lying in $Tr(\otimes^d V_2)$.

Consider a path $γ \in PDP^d_{\leq p-1}(h)$ and let $γ_1 γ_2 \cdots γ_d$ be the associated word. We wish to match each symbol $γ_j$ which is a $y$ with an earlier symbol $γ_{ρ(j)}$ which is an $x$. We do this recursively as follows. Choose the smallest value $j$ such that $γ_j = y$ and for which we have not yet found a matching $x$. Take $i$ to be maximal such that $i < j$, $γ_i = x$ and $i \neq ρ(s)$ for all $s < j$. Then we define $ρ(j) = i$. Let $I_1 = \{j \mid γ_j = y\}$, $I_2 = ρ(I_1)$ and $I_3 = \{1, 2, \ldots, d\} \setminus (I_1 \sqcup I_2)$. Then $|I_1| = |I_2| = (d - h)/2$, $|I_3| = h$ and $γ_i = x$ for all $i \in I_3$.

Define

$$θ_0(γ) = \left( \prod_{j \in I_1} u_{ρ(j),j} \right) \prod_{i \in I_3} x_i \quad \text{and} \quad θ'_0(γ) = \left( \prod_{j \in I_1} u_{ρ(j),j} \right) \prod_{i \in I_3} y_i.$$ 

Then $θ_0(γ) \in (\otimes^d V_2)^C_ρ$ and

$$LM(θ_0(γ)) = \prod_{j \in I_1} LM(u_{ρ(j),j}) \prod_{i \in I_3} x_i = \prod_{j \in I_1} x_{ρ(j)} y_j \prod_{i \in I_3} x_i = Λ(γ).$$

Lemma 5.1. $(σ - 1)^h(θ_0(γ)) = h! θ_0(γ)$ and thus $\ell(θ_0(γ)) \geq h + 1$.

Proof. We will prove a more general statement. We will show that

$$(σ - 1)^{|E|}(y^E) = |E|! x^E.$$
Note that this also implies that \((\sigma - 1)^{|E|+1}(y^E) = 0\). We proceed by induction on \(|E|\). The result is clear for \(|E| = 0, 1\). Assume, without loss of generality, that \(e_i \geq 1\) for all \(i\) and define \(Z_i \in \mathbb{N}^d\) by \(x_i = x^{Z_i}\). For \(|E| \geq 2\) we have

\[
(\sigma - 1)^{|E|}(y^E) = (\sigma - 1)^{|E|-1}(\sigma - 1)(y^E)
\]

\[
= (\sigma - 1)^{|E|-1} \left( \sum_i e_i x_i y^{E-Z_i} + \text{ terms divisible by some } x_k x_\ell \right)
\]

\[
= (\sigma - 1)^{|E|-1} \left( \sum_i e_i x_i y^{E-Z_i} \right)
\]

since the other terms lie in the kernel of \((\sigma - 1)^{|E|-1}\)

\[
= \sum_i e_i x_i (\sigma - 1)^{|E|-1}(y^{E-Z_i})
\]

\[
= \sum_i e_i x_i (|E| - 1)! x^{E-Z_i} \text{ by induction}
\]

\[
= \sum_i e_i (|E| - 1)! x^E = \left( \sum_i e_i \right) (|E| - 1)! x^E
\]

\[
= |E|(|E| - 1)! x^E = |E|! x^E
\]

\(\square\)

If \(\gamma \in \text{PDP}_d^{\leq p-2}\) then we define \(\theta(\gamma) = \theta_0(\gamma)\) and \(\theta'(\gamma) = \theta'_0(\gamma)\).

Suppose instead that \(\gamma \in \text{IDP}_d^{p-1}\) and let \(\gamma_1 \gamma_2 \cdots \gamma_d\) be the word associated to \(\gamma\). Take \(s\) minimal such that the path \(\gamma'\) associated to \(\gamma_1 \gamma_2 \cdots \gamma_s\) is a partial Dyck path of finishing height \(p - 1\). Since \(\gamma' \in \text{PDP}_{\leq p-1}(p - 1)\), from the above we have \(I_1 = \{ j \leq s \mid \gamma_j = y \}\), \(I_2 = \rho(I_1)\) and \(I_3 = \{ 1, 2, \ldots, s \} \setminus (I_1 \cup I_2)\) with \(|I_1| = |I_2| = (s-p+1)/2\), \(|I_3| = p - 1\) and \(\gamma_i = x\) for all \(i \in I_3\). We further define \(I_4 = \{ i > s \mid \gamma_i = y \}\) and \(I_5 = \{ i > s \mid \gamma_i = y \}\). Define

\[
\theta'(\gamma) = \theta'_0(\gamma') \prod_{i \in I_5} y_i \prod_{i \in I_4} x_i = \prod_{j \in I_1} u_{\rho(j),j} \prod_{i \in I_3} y_i \prod_{i \in I_4} x_i
\]

and

\[
\theta(\gamma) = \text{Tr} \left( \theta'_0(\gamma') \right) \prod_{i \in I_5} y_i \prod_{i \in I_4} x_i = \text{Tr} \left( \prod_{i \in I_3} y_i \right) \prod_{j \in I_1} u_{\rho(j),j} \prod_{i \in I_4} x_i
\]

Then \(\theta(\gamma) \in \text{Tr}(\otimes^d V_2) \subset (\otimes^d V_2)^C\) and \(\ell(\theta(\gamma)) = p\).
By Lemma 2.3

\[
\text{LM}(\theta(\gamma)) = \left( \prod_{i \in I_4} x_i \prod_{j \in I_1} \text{LM}(u_{p(j),j}) \right) \text{LM}(\text{Tr}(\prod_{i \in I_3 \cup I_5} y_i))
\]

\[
= \left( \prod_{i \in I_4} x_i \prod_{j \in I_1} x_{p(j)} y_j \right) \prod_{i \in I_3} x_i \prod_{i \in I_5} y_i = \Lambda(\gamma)
\]

In summary, if \( \gamma \in \text{PDP}^d_{\leq p-2}(h) \) then \( \theta(\gamma) \) is an invariant of length at least \( h + 1 \) and lead monomial \( \Lambda(\gamma) \). If \( \gamma \in \text{IDP}^d_{p-1} \) then \( \theta(\gamma) \) is an invariant of length \( p \) and with lead monomial \( \Lambda(\gamma) \). Note that since these lead monomials are all distinct, the maps \( \theta \) and \( \Lambda \) are injective.

It remains to show that \( \Lambda \) is onto \( M \) and to determine the exact length of the invariants \( \theta(\gamma) \) when \( \gamma \in \text{PDP}^d_{\leq p-2}(h) \). We will show that \( \Lambda \) is onto by showing \( |M| = |\text{PDP}^d_{\leq p-2} \cup \text{IDP}^d_{p-1}| \). To determine \( |M| \) we examine the number of indecomposable summands in the decomposition of \( \otimes^d V_2 \).

Define non-negative integers \( \mu^d_p(h) \) by the direct sum decomposition of the \( C_p \)-module \( \otimes^d V_2 \) over \( \mathbb{F} \):

\[
\otimes^d V_2 \cong \bigoplus_{h=1}^{p} \mu^d_p(h) V_h.
\]

Using the convention \( \otimes^0 V_2 = V_1 \), we have the following lemma.

**Lemma 5.2.** Let \( p \geq 3 \). Then

\[
\mu^0_p(h) = \delta^1_h \quad \text{and} \quad \mu^1_p(h) = \delta^2_h,
\]

and

\[
\mu^{d+1}_p(h) = \begin{cases} 
\mu^d_p(2), & \text{if } h = 1; \\
\mu^d_p(h - 1) + \mu^d_p(h + 1), & \text{if } 2 \leq h \leq p - 2; \\
\mu^d_p(p - 2), & \text{if } h = p - 1; \\
\mu^d_p(p - 1) + 2\mu^d_p(p), & \text{if } h = p;
\end{cases}
\]

for \( d \geq 1 \).

**Proof.** The initial conditions are clear. The recursive conditions follow immediately from the following three equations which may be found for example in Hughes and Kemper [14, Lemma 2.2]:

\[
V_1 \otimes V_2 \cong V_2
\]

\[
V_h \otimes V_2 \cong V_{h-1} \oplus V_{h+1} \quad \text{for all } 2 \leq h \leq p - 1
\]

\[
V_p \otimes V_2 \cong 2 V_p.
\]
Next we count lattice paths. Let $\nu_q^d(h) = |\text{PDP}_{\le q}(h)|$ for $1 \le h \le q$. We also define $\bar{\nu}_q^d = |\text{IDP}_q|$. With this notation we have the following lemma.

**Lemma 5.3.** Let $q \ge 2$. Then

$$
\nu_q^0(h) = \delta_h^0 \quad \text{and} \quad \nu_q^1(h) = \delta_h^1,
$$

and

$$
\nu_q^{d+1}(h) = \begin{cases} 
\nu_q^d(1), & \text{if } h = 0; \\
\nu_q^d(h - 1) + \nu_q^d(h + 1), & \text{if } 1 \le h \le q - 1; \\
\nu_q^d(q - 1), & \text{if } h = q;
\end{cases}
$$

and

$$
\bar{\nu}_q^{d+1} = \nu_{q-1}^d(q - 1) + 2\bar{\nu}_q^d
$$

for all $d \ge 1$.

**Proof.** All of these equations are easily seen to hold except perhaps the final one. Its left-hand term $\bar{\nu}_q^{d+1} = |\text{IDP}_{q+1}|$ is the number of initial Dyck paths of length $d + 1$ and escape height $q$. We divide such paths into two classes: those which first achieve height $q$ on their final step and those which achieve height $q$ sometime during the first $d$ steps. Paths in the first class are partial Dyck paths of length $d$, height at most $q - 1$ and finishing height $q - 1$ followed by an $x$-step for the $(d + 1)^{\text{st}}$ step. There are $\nu_{q-1}^d(q - 1) = |\text{PDP}_{q-1}(q - 1)|$ such paths. The second class consists of initial Dyck paths of escape height $q$ and length $d$ followed by a final step which may be either an $x$-step or a $y$-step. Clearly there are $2|\text{IDP}_q| = 2\bar{\nu}_q^d$ paths of this kind. \qed

**Corollary 5.4.** For all $d \in \mathbb{N}$, all primes $p$ and all $h = 1, 2, \ldots, p - 1$ we have

$$
\mu_p^d(h) = \nu_p^{d-2}(h - 1) \quad \text{and} \quad \mu_p^d(p) = \bar{\nu}_{p-1}^d.
$$

**Proof.** Comparing the recursive expressions and initial conditions for $\mu_p^d(h)$ and $\nu_p^{d-2}(h - 1)$ and for $\mu_p^d(p)$ and $\bar{\nu}_{p-1}^d$ given in the previous two lemmas makes the result clear for $p \ge 5$.

For $p = 2$ it is easy to see that $\mu_2^d(1) = \nu_0^d(0) = \delta_d^0$ for $d \ge 0$ and $\mu_2^d(2) = 2^{d-1} = \bar{\nu}_1^d$ for $d \ge 1$.

For $p = 3$ and $h = 1, 2$ we have

$$
\mu_3^d(h) = \nu_1^d(h - 1) = \begin{cases} 
1, & \text{if } h + d \text{ is odd}; \\
0, & \text{if } h + d \text{ is even}.
\end{cases}
$$
Hence \( \mu_d^3(3) = \left\lfloor \frac{2d-1}{3} \right\rfloor \) for \( d \geq 0 \). From the recursive relation \( \bar{\nu}_d^{d+1} = \nu_1^d(1) + 2\nu_2^d \) it is easy to see that \( \bar{\nu}_2^d = \left\lfloor \frac{2d-1}{3} \right\rfloor = \mu_d^3(3) \). \( \square \)

This corollary implies that the map \( \Lambda \) is a bijection. Furthermore for all \( d \), every element of \( \{ \text{LM}(f) \mid f \in (\otimes^d V_2)^{C_p} \} \) may be written as a product with factors from the set \( \{ \text{LM}(g) \mid g \in B \} \) where

\[
B = \{ x_i \mid 1 \leq i \leq d \} \cup \{ u_{ij} \mid 1 \leq i < j \leq d \} 
\cup \{ \text{Tr}(\prod_{i=1}^d y_i^{e_i}) \mid 0 \leq e_i \leq 1, \forall i = 1, 2, \ldots, d \}.
\]

We record and extend these results in the following theorem.

**Theorem 5.5.** Let \( p \) be a prime, let \( d \in \mathbb{N} \) and suppose \( 0 \leq h \leq p - 2 \). Let \( \gamma \in \text{PDP}^d_{\leq p-2} \cup \text{IDP}^d_{p-1} \). Then

1. \( \text{LM}(\theta(\gamma)) = \Lambda(\gamma) \).
2. If \( \gamma \in \text{PDP}^d_{\leq p-2}(h) \) then the invariant \( \theta(\gamma) \) lies in

\[
\mathbb{F}[d V_2]_{(1,1,\ldots,1)}^{C_p} \cong (\otimes^d V_2)^{C_p}
\]

and has length \( h + 1 \).
3. If \( \gamma \in \text{IDP}^d_{p-1} \) then the invariant \( \theta(\gamma) \) lies in

\[
\mathbb{F}[d V_2]_{(1,1,\ldots,1)}^{C_p} \cong (\otimes^d V_2)^{C_p}
\]

and has length \( p \).
4. \( B \) is a SAGBI basis in multi-degree \( (1,1,\ldots,1) \) for \( \mathbb{F}[d V_2]^{C_p} \).

Furthermore, we have the following decomposition of the \( C_p \) representation \( \otimes^d V_2 \) into indecomposable summands:

\[
\bigotimes_{\gamma \in \text{PDP}^d_{\leq p-2} \cup \text{IDP}^d_{p-1}} V_2 \cong \bigoplus_{\gamma \in \text{PDP}^d_{\leq p-2} \cup \text{IDP}^d_{p-1}} V(\gamma)
\]

where \( V(\gamma) \cong V_{h+1} \) is a \( C_p \)-module generated by \( \theta'(\gamma) \), with socle spanned by \( \theta(\gamma) \) and

\[
h = \ell(\theta(\gamma)) - 1 = \begin{cases} 
\text{the finishing height of } \gamma; & \text{if } \gamma \in \text{PDP}^d_{\leq p-2}(h); \\
p - 1 & \text{if } \gamma \in \text{IDP}^d_{p-1}.
\end{cases}
\]

**Proof.** The assertions (1) and (3) have already been proved.

To prove the other assertions we consider the \( C_p \)-module

\[
W = \sum_{\gamma \in \text{PDP}^d_{\leq p-2} \cup \text{IDP}^d_{p-1}} V(\gamma)
\]
generated by the set \( \{ \theta'(\gamma) \mid \gamma \in \text{PDP}_d^{\leq p-2} \cup \text{IDP}_d^{p-1} \} \). The set of vectors \( \{ \theta(\gamma) \mid \gamma \in \text{PDP}_d^{\leq p-2} \cup \text{IDP}_d^{p-1} \} \) spanning the socles of the \( V(\gamma) \) is linearly independent since these vectors have distinct lead monomials. This implies that the above sum is direct:

\[
W = \bigoplus_{\gamma \in \text{PDP}_d^{\leq p-2} \cup \text{IDP}_d^{p-1}} V(\gamma).
\]

Thus \( \dim W = (\sum_{h=0}^{p-2} (h + 1) \cdot \nu^d_p(h)) + p \cdot \bar{\nu}_p^d \). Applying Corollary 5.4, yields \( \dim W = \dim \otimes^d V_2 \). Since \( W \) is a submodule of \( \otimes^d V_2 \) we see that \( W = \otimes^d V_2 \). Furthermore, any set of (spanning vectors for the) socles in any direct sum decomposition of \( \otimes^d V_2 \) there will be exactly \( \nu^d_p(h) \) invariants of length \( h + 1 \) for \( 0 \leq h \leq p - 2 \) (and \( \bar{\nu}_p^d \) of length \( p \)).

Combining this fact with \( \ell(\gamma) \geq h + 1 \) for all \( \gamma \in \text{PDP}_d^{\leq p-2}(h) \), we get \( \ell(\gamma) = h + 1 \) for all \( \gamma \in \text{PDP}_d^{\leq p-2}(h) \), completing the proof of assertion (2) as well as the final assertion of the theorem. Assertion (4) also follows now since we have \( \{ \text{LM}(f) \mid f \in \otimes^d V_2 \}^{C_p} = \{ \text{LM}(\gamma) \mid \gamma \in \text{PDP}_d^{\leq p-2} \cup \text{IDP}_d^{p-1} \} \) and each of these lead monomials may be factored into a product of lead monomials of elements of \( B \).  

6. A Generating Set

Consider the set

\[
B = \{ x_i, N(y_i) \mid 1 \leq i \leq m \} \cup \{ u_{ij} \mid 1 \leq i < j \leq m \}
\cup \{ \text{Tr}(y^e) \mid 0 \leq e_i \leq p - 1 \}.
\]

We will show that \( B \) is a generating set, in fact a SAGBI basis for \( \mathbf{F}[m V_2]^{C_p} \). Let \( f \in \mathbf{F}[m V_2]^{C_p} \) be monic and multi-homogeneous, of multi-degree(\( \lambda_1, \lambda_2, \ldots, \lambda_m \)). Let \( A \) denote the subalgebra \( \mathbf{F}[B] \). We proceed by induction on the total degree \( d = \lambda_1 + \lambda_2 + \cdots + \lambda_m \) of \( f \).

Clearly if \( f \) has total degree 0 then \( f \) is constant, \( f \in A \) and \( \text{LM}(f) = 1 \) and there is nothing more to prove.

Suppose then that the total degree \( d \) of \( f \) is positive. First suppose that \( \lambda_i \geq p \) for some \( i \). We consider \( f \) as a polynomial in \( y_i \) and write \( f = \sum_{j=0}^{\lambda_i} f_j y_i^j \) where \( f_j \) is a polynomial which is homogeneous of degree \( \lambda_i - j \) in \( x_i \). Dividing \( f \) by \( N(y_i) \) in \( \mathbf{F}[m V_2] \) yields \( f = q N(y_i) + r \) where \( r \) is a polynomial whose degree in \( y_i \) is at most \( p - 1 \) and there is nothing more to prove.

Applying \( \sigma \) we have \( f = \sigma(f) = \sigma(q) N(y_i) + \sigma(r) \). Since applying \( \sigma \) cannot increase the degree in \( y_i \), we see that \( \sigma(r) \) also has degree at most \( p - 1 \). By the uniqueness of remainders and quotients we must have \( \sigma(r) = r \) and \( \sigma(q) = q \), i.e., \( q, r \in \mathbf{F}[m V_2]^{C_p} \). Since \( \lambda_i \geq p \), we see that \( x_i \) divides \( r \) and so we have \( f = q N(y_i) + x_i r' \) with
q, r′ ∈ F[m V2]Cp. By induction q, r′ ∈ A and thus f ∈ A. Also by induction we have that LM(q) and LM(r′), hence also LM(f) may be written as products with factors from LM(B).

Therefore, we may assume that \( \lambda_i < p \) for all \( i = 1, 2, \ldots, m \). Then \( \kappa = \lambda_1! \lambda_2! \cdots \lambda_m! \neq 0 \). Define

\[
F = \mathcal{P}(f) \in F[d V_2]^{C_p}(1, 1, \ldots, 1) = (\otimes_{i=1}^d V_2)^{C_p}.
\]

At this point we want to fix some notation. We will use \( \{x_{ij}, y_{ij} | 1 \leq i \leq m, 1 \leq j \leq \lambda_i\} \) as co-ordinate variables for \( \lambda_1 V_2 \oplus \lambda_2 V_2 \oplus \cdots \oplus \lambda_m V_2 \). We write \( u_{ij,k} = x_{ij}y_{kl} - y_{kl}y_{ij} \). We use a graded reverse lexicographic order on \( F[\oplus_{i=1}^m \lambda_i V_2] \) after ordering these variables such that the following conditions hold

- \( y_{ij} > x_{ij} \),
- if \( i < k \) then \( y_{ij} > y_{kl} \) and \( x_{ij} > x_{kl} \),
- if \( j < \ell \) then \( y_{ij} > y_{i\ell} \) and \( x_{ij} > x_{i\ell} \).

We will first show that \( \mathcal{B} \) generates \( F[m V_2]^{C_p} \) as an \( F \)-algebra and then show that it is a SAGBI basis. Of course, the former statement follows from the latter but we include a separate proof of the former since the proof is short and illustrates the main idea we will need for the latter proof.

By Theorem 5.5, we may write

\[
F = \sum_I \alpha_I \prod_{ij} x_{ij} \prod_{ij,k} u_{ij,k} \prod_{E} \text{Tr}(\prod_{ij} y_{ij}^{e_{ij}}).
\]

Let \( e_i = \sum_j e_{ij} \).

\[
f = \kappa^{-1} \mathcal{R}(\mathcal{P}(f)) = \kappa^{-1} \mathcal{R}(F)
= \kappa^{-1} \mathcal{R} \left( \sum_I \alpha_I \prod_{ij} x_{ij} \prod_{ij,k} u_{ij,k} \prod_{E} \text{Tr}(\prod_{ij} y_{ij}^{e_{ij}}) \right)
= \kappa^{-1} \sum_I \alpha_I \prod_{ij} \mathcal{R}(x_{ij}) \prod_{ij,k} \mathcal{R}(u_{ij,k}) \prod_{E} \mathcal{R}(\text{Tr}(\prod_{ij} y_{ij}^{e_{ij}}))
= \kappa^{-1} \sum_I \alpha_I \prod_{ij} x_{ij} \prod_{ij,k} u_{ik} \prod_{E} \text{Tr}(\prod_{i} y_i^{e_i}) \in A
\]

where the last equality follows from the following equalities

\[
\mathcal{R}(\text{Tr}(y^E)) = \mathcal{R}(\sum_{\tau \in C_p} \tau(y^E)) = \sum_{\tau \in C_p} \mathcal{R}(\tau(y^E)) = \sum_{\tau \in C_p} \tau(\mathcal{R}(y^E)) = \text{Tr}(\mathcal{R}(y^E)).
\]
This completes the proof that $B$ generates $F[m V_2]|_{C^F}$ as an $F$-algebra. We continue with the proof that $B$ is a SAGBI basis. First we prove a lemma relating our term orders and polarisation.

**Lemma 6.1.** Suppose $\gamma_1, \gamma_2$ are two monomials in $F[m V_2](\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\gamma_1 > \gamma_2$. Then $\text{LT}(\mathcal{P}(\gamma_1)) > \text{LT}(\mathcal{P}(\gamma_2))$.

**Proof.** Write $\gamma_1 = \prod_{i=1}^m x_i^{a_i} y_i^{\lambda_i - a_i}$ and $\gamma_2 = \prod_{i=1}^m x_i^{b_i} y_i^{\lambda_i - b_i}$. Choose $s$ such that $a_s \neq b_s$ but $a_{s+1} = b_{s+1}, \ldots, a_m = b_m$. Since $\gamma_1 > \gamma_2$ we must have $b_s > a_s$.

Now

$$\text{LT}(\mathcal{P}(\gamma_1)) = \prod_{i=1}^m \prod_{j=1}^{a_i} x_{ij} \prod_{j=a_i+1}^{a_s} y_{ij}$$

and

$$\text{LT}(\mathcal{P}(\gamma_2)) = \prod_{i=1}^m \prod_{j=1}^{b_i} x_{ij} \prod_{j=b_i+1}^{b_s} y_{ij}.$$

we have

$$\text{LT}(\mathcal{P}(\gamma_1)) = \Gamma_0 \Gamma_1 = \prod_{j=1}^{a_s} x_{sj} \prod_{j=a_s+1}^{a_i} y_{sj}$$

and

$$\text{LT}(\mathcal{P}(\gamma_2)) = \Gamma_0 \Gamma_2 = \prod_{j=1}^{b_s} x_{sj} \prod_{j=b_s+1}^{b_i} y_{sj}.$$

Since $a_s < b_s$ we see that $\text{LT}(\mathcal{P}(\gamma_1)) > \text{LT}(\mathcal{P}(\gamma_2))$. 

Write $f = \gamma_1 + \gamma_2 + \cdots + \gamma_s$ where each $\gamma_i$ is a term and $\text{LM}(f) = \text{LT}(f) = \gamma_1$ since $f$ was assumed to be monic. Define $F = \mathcal{P}(f)$. By Lemma 6.1, $\text{LM}(F) = \text{LM}(\mathcal{P}(\gamma_1))$. Furthermore, each monomial of $\mathcal{P}(\gamma_1)$ restitutes to $\gamma_1$. In particular, $\mathcal{R}(\Gamma_1) = \gamma_1$ where $\Gamma_1 = \text{LM}(F)$. By Proposition 5.5(4), we may write

$$\Gamma_1 = \text{LM}(F) = \text{LM}\left(\prod_{ij} x_{ij} \prod_{i,j,k,l} u_{ij,kl} \prod_{E} \text{Tr}(\prod_{i,j \in E} y_{ij}^{e_{ij}^{kl}})\right)$$

$$= \prod_{ij} x_{ij} \prod_{i,j,k,l} \text{LM}(u_{ij,kl}) \prod_{E} \text{LM}(\text{Tr}(\prod_{i,j \in E} y_{ij}^{e_{ij}^{kl}})).$$
Restituting we find
\[
\gamma_1 = R(\Gamma_1) = R\left( \prod_{ij} y_{ij} \prod_{ij,kl} \text{LM}(u_{ij,kl}) \prod_{E} \text{LM}(\text{Tr}(\prod_{ij} y_{ij}^{e_{ij}})) \right)
\]
\[
= \prod_{ij} R(x_{ij}) \prod_{ij,kl} R(\text{LM}(u_{ij,kl})) \prod_{E} R(\text{LM}(\text{Tr}(\prod_{ij} y_{ij}^{e_{ij}})))
\]
\[
= \prod_{ij} x_i \prod_{ij,kl} \text{LM}(u_{i,k}) \prod_{E} \text{LM}(\text{Tr}(\prod_{i} y_i^{\sum_{j} e_{ij}}))
\]
where the last equality follows using Lemma 6.2 below. Thus \( LM(f) \) may be written as a product of factors from \( LM(B) \). This shows that \( B \) is a SAGBI basis for \( F[m V_2]^{C_p} \).

**Lemma 6.2.** Let \( y^E = \prod_{i=1}^{m} \prod_{j=1}^{\lambda_i} y_{ij}^{e_{ij}} \) where \( e_{ij} \in \{0, 1\} \) for all \( i, j \).

Let \( e_i = \sum_{j=1}^{\lambda_i} e_{ij} \). If \( e_i < p \) for all \( i = 1, 2, \ldots, m \) then
\[
R\left( \text{LM}(\text{Tr}(y^E)) \right) = \text{LM}\left( \text{Tr}(R(y^E)) \right).
\]

**Proof.** Let \( s \) be minimal such that \( e_1 + e_2 + \cdots + e_s \geq p - 1 \). (If no such \( s \) exists then \( \text{Tr}(y^E) = 0 \) and \( \text{Tr}(R(y^E)) = 0 \).) Let \( r \) be minimal such that \( e_1 + e_2 + \cdots + e_s - 1 + e_{s+1} + e_{s+2} + \cdots + e_{s+r} = p - 1 \). By Lemma 2.3
\[
\text{LM}(\text{Tr}(y^E)) = \left( \prod_{i=1}^{s-1} x_i^{e_{ij}} \right) \prod_{j=1}^{r} x_j^{e_{ij}} \prod_{j=r+1}^{s} y_j^{e_{ij}} \prod_{i=s+1}^{m} \prod_{j=1}^{\lambda_i} y_{ij}^{e_{ij}}.
\]
Since \( R(y^E) = \prod_{i=1}^{m} y_i^{e_i} \), again using Lemma 2.3 we see that
\[
\text{LM}(\text{Tr}(R(y^E))) = \left( \prod_{i=1}^{s-1} x_i^{e_{ij}} \right) x_t y_s^{e_s-1} \left( \prod_{i=s+1}^{m} y_i^{e_i} \right)
\]
where \( t = (p - 1) - (e_1 + e_2 + \cdots + e_{s-1}) = \sum_{j=1}^{r} e_{ij} \). Thus
\[
R\left( \text{LM}(\text{Tr}(y^E)) \right) = \text{LM}\left( \text{Tr}(R(y^E)) \right)
\]
as required. \(\square\)

**Theorem 6.3.** The set
\[
B' = \{ x_i, N(y_i) \mid 1 \leq i \leq m \} \cup \{ u_{ij} \mid 1 \leq i < j \leq m \}
\]
\[
\cup \{ \text{Tr}(y^E) \mid 0 \leq e_i \leq p - 1, \ 2(p - 1) < |E| \}
\]
is both a minimal algebra generating set and a SAGBI basis for \( F[m V_2]^{C_p} \).

**Proof.** We start by showing \( B' \) is a SAGBI basis. We need to see why we do not need invariants of the form \( \text{Tr}(y^E) \) where \( |E| \leq 2(p - 1) \) as generators. To see this, consider such a transfer \( \text{Tr}(y^E) \). By Lemma 2.3...
its lead term is $x_i^{p-1-e_i} y_i^{t-p+1} \prod_{i=1}^{r-1} x_i^{e_i} \prod_{i=r+1}^{d} y_i^{e_i}$ where $r$ is minimal such that $t = \sum_{i=1}^{r} e_i \geq p - 1$. (We may assume that $r$ exists since if $|E| < p - 1$ then $\text{Tr}(y_E) = 0$.)

Write $\text{LM}(\text{Tr}(y_E)) = x_{i_1} x_{i_2} \cdots x_{i_{p-1}} y_{i_p} y_{i_{p+1}} \cdots y_{i_e}$ where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_e \leq m$. Consider $f = \prod_{j=1}^{2p-2-|E|} x_{i_j} \prod_{j=1}^{|E|-(p-1)} u_{i_{p-j},i_{p-1+j}}$. Then $\text{LM}(f) = \text{LM}(\text{Tr}(y_E))$. Thus $\{\text{LM}(f) \mid f \in \mathcal{B}'\}$ generates the same algebra as $\{\text{LM}(f) \mid f \in \mathcal{B}\}$ which shows that $\mathcal{B}'$ is a SAGBI basis (and hence a generating set) for $\mathbf{F}[m V_2]^C$.

Now we show that $\mathcal{B}'$ is a minimal generating set. It is clear that the elements $x_i$ and $u_{ij}$ cannot be written as polynomials in the other elements of $\mathcal{B}'$. Furthermore, since $\text{LM}(N(y_i)) = y_i^p$ is the only monomial occurring in any element of $\mathcal{B}'$ which is a pure power of $y_i$, we see that $N(y_i)$ is required as a generator. This leaves elements of the form $\text{Tr}(y_E^P)$ with $|E| > 2(p-1)$. We proceed similarly to the proof of [24, Lemma 4.3]. Assume by way of contradiction that $\text{Tr}(y_E^P) = \gamma_1 + \gamma_2 + \cdots + \gamma_r$ where each $\gamma_i$ is a scalar times a product of elements from $\mathcal{B}' \setminus \{\text{Tr}(y_E^P)\}$ and that $\text{LM}(\gamma_1) \geq \text{LM}(\gamma_2) \cdots \geq \text{LM}(\gamma_r)$. Then $\text{LM}(\text{Tr}(y_E^P)) \leq \text{LM}(\gamma_1)$. First we suppose that $\text{LM}(\gamma_1) = \text{LT}(\text{Tr}(y_E^P))$.

As above we have

$$\text{LM}(\gamma_1) = \text{LM}(\text{Tr}(y_E^P)) = x^A y^B = x_{i_1}^{p-1-e_1} y_{i_1}^{t-p+1} \prod_{i=1}^{r-1} x_i^{e_i} \prod_{i=r+1}^{d} y_i^{e_i}$$

where $r$ is minimal such that $t = \sum_{i=1}^{r} e_i \geq p - 1$.

Since each $e_i < p$ and $\text{LM}(N(y_i)) = y_i^p$ we see that $N(y_i)$ does not divide $\gamma_1$. But then since $|A| = p - 1$ we see that $|A| < |E| - |A| = |B|$ and thus there must be at least one transfer which divides $\gamma_1$. Conversely since $|A| = p - 1$ exactly one transfer (to the first power) may divide $\gamma_1$. But then the lead monomials of the other factors must divide $y^P$ and no element of $\mathcal{B}'$ has a lead monomial satisfying this constraint. This shows that for $|E| > 2(p-1)$, the monomial $\text{LM}(\text{Tr}(y_E^P))$ cannot be properly factored using lead monomials from $\mathcal{B}'$.

Therefore we must have $\text{LM}(\gamma_1) > \text{LM}(\text{Tr}(y_E^P))$ (and $\text{LM}(\gamma_1) = \text{LM}(\gamma_2)$). Since we may assume that each term of each $\gamma_i$ is homogeneous of degree $E$, we may write $\text{LM}(\gamma_1) = x^C y^D$ where $C + D = E$. But $\text{LM}(\text{Tr}(y_E^P)) = x^A y^B$ is the biggest monomial in degree $E$ which satisfies $|A| \geq p - 1$. Hence $\text{LM}(\gamma_1) > \text{LM}(\text{Tr}(y_E^P))$ implies that $|C| < p - 1$. Therefore $\gamma_1$ must be a product of elements of the form $x_i, u_{ij}$ and $N(y_i)$ from $\mathcal{B}'$. As above, since each $e_i < p$, no $N(y_i)$ can divide $\gamma_1$. But then $\text{LM}(\gamma_1)$ is a product of factors of the form $x_i$
and \(\text{LM}(u_{ij}) = x_iy_j\) and this forces \(|C| \geq |D| = |E| - |C|\). Therefore \(2(p - 1) > 2|C| \geq |E|\). This contradiction shows that we cannot express \(\text{Tr}(y^E)\) as a polynomial in the other elements of \(B'\) when \(|E| > 2(p - 1)\). \(\square\)

7. Decomposing \(F[m V_2]\) as a \(C_p\)-module

In this section we show that our techniques give a decomposition of the homogeneous component

\[ F[m V_2](d_1, d_2, \ldots, d_m) \]

as a \(C_p\)-module. We will describe \(F[m V_2](d_1, d_2, \ldots, d_m)\) modulo projectives, i.e., we compute the multiplicities of the indecomposable summands \(V_k\) of this component for which \(k < p\). Having done this, a simple dimension computation will give the complete decomposition.

By the Periodicity Theorem (Theorem 2.1), we may assume that each \(d_i < p\). Let \(d = d_1 + d_2 + \cdots + d_m\). The symmetric group on \(d\) letters, \(\Sigma_d\), acts on \(\otimes^d V_2\) by permuting the factors. This action commutes with the action of \(C_p\) (in fact with the action of all of \(GL(V_2)\)). The image of the polarization map consists of those tensors which are fixed by the Young subgroup \(Y = \Sigma_{d_1} \times \Sigma_{d_2} \times \cdots \times \Sigma_{d_m}\) of \(\Sigma_d\). Since each \(d_i < p\), we see that \(Y\) is a non-modular group. Maschke’s Theorem then implies that polarization embeds \(F[m V_2]_d\) into \(\otimes^d V_2\) as a \(C_p\)-summand. Therefore \(\ell(\mathcal{P}(f)) = \ell(f)\) for all \(f \in F[m V_2]^{C_p}(d_1, d_2, \ldots, d_m)\) and \(\ell(\mathcal{R}(F)) = \ell(F)\) for all \(F \in (\otimes^d V_2)^{C_p \times Y}\).

Using the relations given in Section 2.2, it is straightforward to write down a basis, consisting of products of \(u_{ij}\)'s and \(x_i\)'s, for the invariants in multi-degree \((d_1, d_2, \ldots, d_m)\) which lie in the subring generated by \(\{x_i \mid 1 \leq i \leq m\} \cup \{u_{i,j} \mid 1 \leq i < j \leq m\}\). Associated to the lead term of each invariant in this basis is an indecomposable summand of \(F[m V_2](d_1, d_2, \ldots, d_m)\). The dimension of this summand may be found using Theorem 5.5. More directly, consider a product of \(u_{ij}\)'s and \(x_i\)'s, say

\[ f := \prod_{i=1}^{m} x_i^{a_i} \cdot \prod_{1 \leq i < j \leq m} u_{i,j}^{b_{i,j}} \in F[m V_2]^{C_p}. \]

It is not too difficult to show that \(\text{LT}(f)\) is the lead term of an element of the transfer if and only if there exists \(r\) with \(1 \leq r \leq m\) such that

\[ \sum_{i=1}^{r} a_i + \sum_{i \leq r < j \leq m} b_{i,j} \geq p - 1. \]
If no such $r$ exists then $\ell(f) = 1 + \sum_{i=1}^{m} a_i$ gives the dimension of the associated summand.

Rather than working with the invariants lying in $F[mV_2]$ directly, one may instead use Theorem 5.5 to decompose $\otimes^d V_2$. It is then possible to perturb this decomposition so that it is a refinement of the splitting given by polarisation/restitution and thus gives a decomposition of $F[mV_2](a_1, \ldots, a_m)$.

**Example 7.1.** As an example we compute the decomposition of $F[4V_2](p+1,1,1,p+2)$. This space has dimension $(p + 2)(2)(2)(p + 3) = 4p^2 + 20p + 24$. By Theorem 2.1, we know $F[4V_2](p+1,1,1,p+2) \cong F[4V_2](1,1,1,2) \oplus (4p + 20)V_p$ and we need to compute the decomposition of $F[4V_2](1,1,1,2) = V_2 \otimes V_2 \otimes V_2 \otimes S^2(V_2)$.

We have available the invariants $x_1, x_2, x_3, x_4$ and $u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}$. Suppose now that $p \geq 7$. The products of these 10 invariants which lie in degree $(1, 1, 1, 2)$ are as follows (sorted by length):

- $\ell = 2$: $x_4u_{12}u_{34}$, $x_4u_{13}u_{24}$, $x_4u_{14}u_{23}$, $x_1u_{24}u_{34}$, $x_2u_{24}u_{34}$, $x_3u_{14}u_{24}$
- $\ell = 4$: $x_3x_4^2u_{12}$, $x_1x_2x_4^2u_{34}$, $x_1x_2x_4u_{23}$, $x_2x_1x_4u_{24}$, $x_2x_3x_4u_{14}$
- $\ell = 6$: $x_1x_2x_3x_4^2$

Consider the invariants of length 2. Among the available relations for those of length 2 we have:

0 = $x_4(u_{12}u_{34} - u_{13}u_{24} + u_{14}u_{23})$,
0 = $u_{34}(x_1u_{24} - x_2u_{14} + x_4u_{23})$, and
0 = $u_{14}(x_2u_{34} - x_3u_{24} + x_4u_{23})$.

Using these three relations we see that the three invariants

$x_4u_{13}u_{24}$, $x_2u_{14}u_{34}$, $x_3u_{14}u_{24}$

may be expressed in terms of the other three invariants

$x_4u_{12}u_{34}$, $x_4u_{14}u_{23}$, $x_1u_{24}u_{34}$.

Furthermore there are no relations involving only these latter three invariants and thus they represent the socles of 3 summands isomorphic to $V_2$. 
Among the available relations involving invariants of length 4 we have

\[ 0 = x_3^2(x_1u_{23} - x_2u_{13} + x_3u_{12}), \]
\[ 0 = x_1x_4(x_2u_{34} - x_3u_{24} + x_4u_{23}), \text{ and} \]
\[ 0 = x_3x_4(x_1u_{24} - x_2u_{14} + x_4u_{12}). \]

These allow us to express the three invariants

\[ x_2x_4^2u_{13}, \quad x_1x_3x_4u_{24}, \quad x_2x_3x_4u_{14} \]

using only

\[ x_3x_4^2u_{12}, \quad x_1x_4^2u_{23}, \quad x_1x_2x_4u_{34}. \]

Again these there are no relations involving only these latter 3 invariants and so they represent the socles of 3 summands isomorphic to \( V_4 \).

Since \( x_1x_2x_3x_4^2 \) spans the socle of a summand isomorphic to \( V_6 \) we conclude that

\[ F[4V_2]_{(1,1,1,2)} \cong 3V_2 \oplus 3V_4 \oplus V_6 \quad \text{for} \ p \geq 7. \]

For \( p = 5 \), the foregoing is all correct except that the lattice paths corresponding to \( x_1x_2x_3x_4^2 \) and \( x_1x_2x_3x_4y_4 = \text{LT}(x_1x_2u_{34}x_4) \) both attain height \( p - 1 = 4 \). Thus in this case these two invariants both represent a projective summand and we have the decomposition

\[ F[4V_2]_{(1,1,1,2)} \cong 3V_2 \oplus 2V_4 \oplus 2V_5 \quad \text{for} \ p = 5. \]

For \( p = 2, 3 \) all the relevant lattice paths attain height \( p - 1 \) and so the summand is projective. Thus

\[ F[4V_2]_{(1,1,1,2)} \cong 8V_3 \quad \text{for} \ p = 3, \text{ and} \]
\[ F[4V_2]_{(1,1,1,2)} \cong 12V_2 \quad \text{for} \ p = 2. \]

We will also illustrate how to use the decomposition of \( \otimes^5V_2 \) to find the decomposition of \( F[4V_2]_{(1,1,1,1,2)} \). By the results of Section 5, we have

\[ \otimes^5V_2 \cong 5V_2 \oplus 4V_4 \oplus V_6 \quad \text{for} \ p \geq 7. \]

Here the lead monomials are

\[ \ell = 2: \ x_1y_2x_3y_4x_5, \quad x_1x_2y_3x_4y_5, \quad x_1y_2x_3x_4y_5, \quad x_1x_2x_3y_4y_5 \]
\[ \ell = 4: \ x_1y_2x_3x_4x_5, \quad x_1x_2y_3x_4x_5, \quad x_1x_2x_3y_4x_5, \quad x_1x_2x_3x_4y_5 \]
\[ \ell = 6: \ x_1x_2x_3x_4x_5 \]

and the corresponding invariants are

\[ \ell = 2: \ x_3u_{12}u_{34}, \quad x_5u_{14}u_{23}, \quad x_3u_{12}u_{35}, \quad x_1u_{23}u_{45}, \quad x_1u_{25}u_{34} \]
\[ \ell = 4: \ x_3x_4x_5u_{12}, \quad x_1x_4x_5u_{23}, \quad x_1x_2x_3x_{34}, \quad x_1x_2x_3u_{45} \]
\[ \ell = 6: \ x_1x_2x_3x_4x_5 \]
The Young subgroup $Y := \Sigma_1 \times \Sigma_1 \times \Sigma_1 \times \Sigma_2$ acts by simultaneously interchanging $x_4$ with $x_5$ and $y_4$ with $y_5$. Clearly the action preserves length. The $C_p \times Y$ invariants are

$\ell = 2$: $x_5 u_1 u_2 u_{34} + x_4 u_1 u_2 u_{35}, \ x_5 u_1 u_4 u_{23} + x_4 u_1 u_4 u_{23}, \ x_5 u_1 u_2 u_{34} + x_5 u_1 u_2 u_{34}, \ x_1 u_2 u_{45} + x_1 u_2 u_{45} = 0, \ x_1 u_2 u_{34} + x_1 u_2 u_{35}$

$\ell = 4$: $x_3 x_4 x_5 u_{12}, \ x_1 x_4 x_5 u_{23}, \ x_1 x_2 x_5 u_{34} + x_1 x_2 x_4 u_{35}, \ x_1 x_2 x_3 u_{45} + x_1 x_2 x_3 u_{54} = 0$

$\ell = 6$: $x_1 x_2 x_3 x_4 x_5$

We now restate these $C_p \times Y$ invariants to $F[4 V_2]^{C_p}_{(1,1,1,2)}$. We find

$$\mathcal{R}(x_5 u_1 u_2 u_{34} + x_4 u_1 u_2 u_{35}) = 2 x_4 u_1 u_2 u_{34},$$

$$\mathcal{R}(x_5 u_1 u_4 u_{23} + x_4 u_1 u_4 u_{23}) = 2 x_4 u_1 u_4 u_{23},$$

$$\mathcal{R}(x_1 u_2 u_{45} + x_1 u_2 u_{45} + x_1 u_2 u_{34} + x_1 u_2 u_{34}) = 2 x_1 u_2 u_{34}.$$

Thus we find 3 summands of $F[4 V_2]^{(1,1,1,2)}$ isomorphic to $V_2$. Restituting the invariants of length 4 we find

$$\mathcal{R}(x_3 x_4 x_5 u_{12}) = x_3 x_4^2 u_{12},$$

$$\mathcal{R}(x_1 x_4 x_5 u_{23}) = x_1 x_4^2 u_{23},$$

$$\mathcal{R}(x_1 x_2 x_3 u_{34} + x_1 x_2 x_4 u_{35}) = 2 x_1 x_2 x_4 u_{34}.$$
Let $L_{ij}$ denote the polarisation corresponding to $\lambda_i = 1$, $\lambda_j = p$, and $\lambda_k = 0$ otherwise. So, for example, $L_{32} = L_{(0,p,1,0,\ldots,0)}$. Define

$$D_m = \left\{ \lambda \in \mathbb{N}^m \mid p \text{ divides } \lambda_i \text{ for all } i \text{ and } \sum_{i=1}^{m} \lambda_i = p(p-1) \right\}.$$

Further define

$$S_m = \{ u_{ij} \mid i < j \leq m \} \cup \{ L_i, L_{ij} \mid i, j \in \{1, \ldots, m\}, i \neq j \} \cup \{ D_\lambda \mid \lambda \in D_m \}.$$

**Theorem 8.1.** The ring of vector invariants, $F[mV_2]^{SL_2(F_p)}$, is generated by $S_m$ and elements from the image of the transfer.

Note that the elements of $S_m$ are clearly $SL_2(F_p)$-invariant and include a system of parameters. Let $A$ denote the algebra generated by $S_m$ and let $a$ denote the ideal in $F[mV_2]^P$ generated by $S_m$. A basis for the finite dimensional vector space $F[mV_2]^P/a$ lifts to a set of $A$-module generators for $F[mV_2]^P$, say $M$. Since the relative transfer homomorphism is a surjective $A$-module morphism, $F[mV_2]^{SL_2(F_p)}$ is generated by $S_m \cup \text{Tr}_{F_p}^{SL_2(F_p)}(M)$. The elements of $M$ may be chosen to be monomials in the generators of $F[mV_2]^P$. Since we are working modulo the image of the transfer, it is sufficient to consider monomials of the form $N(y)^\alpha x^\beta$.

Let $u$ denote the ideal in $F[mV_2]^P$ generated by $\{ u_{ij} \mid i < j \leq m \}$.

**Lemma 8.2.** For $i < j \leq m$, $L_{ij} = x_i N(y_j) + u_{ij} x_j^{p-1}$ and $L_{ji} = x_j N(y_i) - u_{ij} x_i^{p-1}$, giving $L_{ij} \equiv_u x_i N(y_j)$ and $L_{ji} \equiv_u x_j N(y_i)$.

**Proof.** Applying $\nabla_m$ to $L$ gives

$$(x_1 + \cdots + x_m) (y_1^p + \cdots + y_m^p) - (y_1 + \cdots + y_m)(x_1 + \cdots + x_m)^{p-1}.$$

Expanding gives

$$(x_1 + \cdots + x_m) (y_1^p + \cdots + y_m^p) - (y_1 + \cdots + y_m)(x_1 + \cdots + x_m)^p.$$

Collecting the appropriate multi-degrees gives $L_{ij} = x_i y_j^p - y_i x_j^p$ and $L_{ji} = x_j y_i^p - y_j x_i^p$. Using $u_{ij} = x_i y_j - x_j y_i$ and $N(y) = y^p - x^{p-1} y$ gives

$$x_i N(y_j) + u_{ij} x_j^{p-1} = x_i y_j^p - x_j y_j x_j^{p-1} + x_i y_j x_j^{p-1} - x_j y_i^p,$$

and

$$x_j N(y_i) - u_{ij} x_i^{p-1} = x_j y_i^p - x_i y_i x_i^{p-1} - y_j x_i^{p-1} + x_i^{p-1} x_i y_i = L_{ji}.$$

□
Since \( u \subset \mathfrak{a} \), the preceding lemma and the formula \( L_i = x_i N(y_i) \) show that it is sufficient to compute \( \text{Tr}_{P}^{\mathcal{S}L_2(F_p)} \) on monomials of the form \( N(y) \alpha \) or \( x^\beta \).

Let \( B \) denote the Borel subgroup containing \( P \), i.e., the upper triangular elements of \( \mathcal{S}L_2(F_p) \). Define a weight function on \( F[m V_2] \) by \( \text{wt}(x_i) \equiv_{(p-1)} 1 \) and \( \text{wt}(y_i) \equiv_{(p-1)} -1 \). Note that \( N(y_i) \) is isobaric of weight \(-1\). Furthermore, \( F[m V_2]^B \) consists of the span of the the weight zero elements of \( F[m V_2]^P \). The relative transfer \( \text{Tr}_{P}^{B} \) is determined by weight:

\[
\text{Tr}_{P}^{B}(N(y) \alpha x^\beta) = \begin{cases} 
-N(y) \alpha x^\beta, & \text{if } (|\beta| - |\alpha|) \equiv_{(p-1)} 0; \\
0, & \text{otherwise.}
\end{cases}
\]

Since \( \text{Tr}_{P}^{\mathcal{S}L_2(F_p)} = \text{Tr}_{B}^{\mathcal{S}L_2(F_p)} \text{Tr}_{P}^{B} \), it is sufficient to compute \( \text{Tr}_{B}^{\mathcal{S}L_2(F_p)} \) on \( N(y) \alpha \) with \(|\alpha|\) a multiple of \( p-1 \) and \( x^\beta \) with \(|\beta|\) a multiple of \( p-1 \).

However, if \(|\beta| \geq p-1\), then \( x^\beta \in \text{Tr}_{P}^{P}(F[m V_2]) \) and \( \text{Tr}_{P}^{\mathcal{S}L_2(F_p)}(x^\beta) \in \text{Tr}_{B}^{\mathcal{S}L_2(F_p)}(F[m V_2]) \). Thus it is sufficient to compute \( \text{Tr}_{B}^{\mathcal{S}L_2(F_p)}(N(y) \alpha) \) with \(|\alpha|\) a multiple of \( p-1 \).

For \( \lambda \in \mathcal{D}_m \), define

\[
N(y)^{\lambda/p} = \prod_{i=1}^{m} N(y_i)^{\lambda_i/p}.
\]

**Lemma 8.3.** \( \nabla_m(D) \equiv_u (N(y_1) + \cdots + N(y_m))^{p-1} + (x_1 + \cdots + x_m)^{p(p-1)} \), giving \( D_\lambda \equiv_u (\lambda^{1/p}_1)^{N(y)^{\lambda/p}} + x^{\lambda} \) and \( N(y)^{\lambda/p} \equiv_a -x^\lambda \).

**Proof.** The proof is by induction on \( m \). Note that \( \nabla_m = \nabla^{m-1} \). Thus \( \nabla_1(D) = \nabla^0(D) = D = N(y)^{p-1} + x^{p(p-1)} \), as required. Recall that the action of \( \nabla \) on \( F[m V_2] \) is determined by \( \nabla(x_m) = x_m + x_{m+1} \), \( \nabla(y_m) = y_m + y_{m+1} \), \( \nabla(x_i) = x_i \) for \( i < m \), and \( \nabla(y_i) = x_i \) for \( i < m \). Thus \( \nabla(u_{ij}) = u_{ij} \) if \( i < j < m \) and \( \nabla(u_{im}) = y_i(x_m + x_{m+1}) - x_i(y_m + y_{m+1}) = u_{im} + u_{i,m+1} \).

Therefore \( \nabla \) induces an algebra morphism on \( F[m V_2]^P/u \). Furthermore \( \nabla(N(y_i)) = N(y_i) \) if \( i < j < m \) and \( \nabla(N(y_m)) = y_m^p + y_{m+1}^p - (x_m + x_{m+1})^{p-1}(y_m + y_{m+1}) = N(y_m) + N(y_{m+1}) - u_{m,m+1} \sum_{j=0}^{p-2} (-x_m)^j x_{m+1}^{p-2-j} \).

By induction,

\[
\nabla_{m+1}(D) = \nabla(\nabla_m(D)) \in \nabla((N(y_1) + \cdots + N(y_m))^{p-1} + (x_1 + \cdots + x_m)^{p(p-1)} + u).
\]
Theorem 8.4. \(F \leq (A)\) are minimal representatives of their mod-

Proof. We may choose the are minimal representatives of their mod-

As required.

Using the lemma, if \(p - 1\) divides \(|\alpha|\) then \(\text{Tr}_{B}^{S\mathbb{L}_2(F_p)}(N(y)^\alpha)\) is decomposable modulo the image of the transfer, completing the proof of Theorem 8.1.

To complete the calculation of a generating set for \(F[m V_2]^{SL_2(F_p)}\) and compute an upper bound for the Noether number, we need only identify a set of \(A\)-module generators for \(F[m V_2]\). This can be done by applying the Buchberger algorithm to \(S_m\). For example, a Magma [2] calculation for \(m = 3\) and \(p = 3\), produces 522 \(A\)-module generators giving rise to 74 non-zero elements in the image of the transfer. Subducting the transfers against \(S_m\) gives 11 new generators and 29 in total. Magma’s \(\text{MinimalAlgebraGenerators}\) command reduces the number of generators to 28, occurring in degrees 2, 4, 6 and 8. The same calculation for \(p = 5\) and \(m = 3\) gives a Noether number of 24. Thus for \(p \in \{3, 5\}\) and \(m = 3\), the Noether number is \((p + m - 2)(p - 1) = (p + 1)(p - 1)\).

Theorem 8.4. \(F[m V_2]^{SL_2(F_p)}\) is generated as an \(A\)-module in degrees less than or equal to \((p + m - 2)(p - 1)\).

Proof. Define \(\mathfrak{a}'\) to be the ideal in \(F[m V_2]\) generated by \(S_m\), i.e.,

\[\mathfrak{a}' = A^+F[m V_2].\]

A basis for \(F[m V_2]/\mathfrak{a}'\) lifts to a set of \(A\)-module generators for \(F[m V_2]\).

We may choose the \(A\)-module generators to be monomials, \(y^\alpha x^\beta\), which are minimal representatives of their mod-\(\mathfrak{a}'\) congruence class. For convenience, denote \(d = |\alpha| + |\beta|\). For \(i < j\), using \(u_{ij} = x_i y_j - x_j y_i\), if \(x_i\) divides \(y^\alpha x^\beta\), then \(y_j\) does not. For \(j \leq i\), using \(L_i\) and \(L_{ij}\), if \(x_i\) divides \(y^\alpha x^\beta\), then \(y_j\) does not. The remaining representatives fall into two classes: \(y^\alpha\) and \(y_1^{\alpha_1} \cdots y_k^{\alpha_k} x_k^{\beta_k} \cdots x_m^{\beta_m}\) with \(\beta_k \neq 0\) and \(\alpha_i \leq p - 1\).

Case 1: \(y^\alpha\). Using \(D_\lambda\) with \(\lambda \in \mathbb{D}_m\), we see that, for \(|\gamma| \geq p - 1\), \((y^\gamma)^p\) does not divide \(y^\alpha\). Write \(\alpha_i = q_i p + r_i\) with \(r_i < p\). Then \(y^\alpha = (y^q)^p y^r\) with \(|q| \leq p - 2\). Thus \(|\alpha| = p|q| + |r| \leq p(p - 2) + m(p - 1) = (p + m - 1)(p - 1) - 1\). However, \(\text{Tr}(y^\alpha) = 0\) unless \(p - 1\) divides \(|\alpha|\). Therefore, the \(A\)-module generators of the form \(\text{Tr}(y^\alpha)\) satisfy \(d = |\alpha| \leq (p + m - 2)(p - 1)\).

Case 2: \(y_1^{\alpha_1} \cdots y_k^{\alpha_k} x_k^{\beta_k} \cdots x_m^{\beta_m}\) with \(\beta_k \neq 0\) and \(\alpha_i \leq p - 1\). For \(i < j\), let \(x_i^\gamma\) be a monomial in \(x_1, \ldots, x_{j-1}\). If \(|\gamma| = p - 1\), then
\[ x^\gamma L_{ij} = x^\gamma(x_j y_i^p - x_j^p y_i) \equiv_u y^\gamma y_i x_j^p - x^\gamma y_i x_j^p. \]

Therefore, if \( \beta_j \geq p \) for any \( j > k \), then \( |\alpha| < p. \) If \( |\gamma| \leq p - 1 \) then \( x^\gamma L_{ij} = x^\gamma(x_j y_i^p - x_j^p y_i) \equiv_u y^\gamma y_i x_j^p - x^\gamma y_i x_j^p. \) Therefore, if \( \beta_k \geq p \), we also have \( |\alpha| < p. \)

If \( |\beta_j| < p \) for all \( j \leq k \), then \( |\alpha| + |\beta| \leq (m - k + 2)(p - 1) \leq (p + m - 2)(p - 1) \text{ if } k > 1. \) Hence it is sufficient to consider the case \( |\alpha| < p. \)

However the transfer is zero unless \( p - 1 \) divides \( |\alpha| \) so we may assume \( |\alpha| = p - 1. \) If \( |\alpha| = p - 1, \) a straightforward calculation with binomial coefficients gives \( \text{Tr}^P(y^\alpha x^\beta) = -x^{\alpha + \beta}. \) Furthermore, \( \text{Tr}^P_{\gamma}(x^{\alpha + \beta}) = 0 \) unless \( p - 1 \) divides \( |\alpha| + |\beta| \). Write \( \alpha_i + \beta_i = q_i p + r_i \) with \( r_i < p. \)

Then \( x^{\alpha + \beta} = (x^p)^q x^r. \) If \( |q| \geq p - 1 \) and \( |r| > 0, \) we may choose \( i \) so that \( r_i > 0, \) choose \( \lambda_i \in \mathcal{D}_m \) so that \( x^\lambda \) divides \( x^{pq} \) and choose \( j \) so that \( x_j \) divides \( x^\lambda. \) By Lemma 8.2, \( x_j \mathcal{N}(y_j) \in \alpha'. \) Form the S-polynomial between \( D_3 \) and \( x_3 \mathcal{N}(y_j). \) Using Lemma 8.3, this S-polynomial reduces to \( x_3 x^\lambda. \) Thus either \( |q| < p - 1 \) or \( |q| = p - 1 \) and \( |r| = 0. \) If \( |r| = 0 \) and \( |q| = p - 1, \) then \( d = (p - 1)(p - 1) \leq (p + m - 2)(p - 1). \) Suppose \( |q| < p - 1. \) Then \( d \leq m(p - 1) + p(p - 2) = (p + m - 1)(p - 1) - 1. \) Since \( d \) must be a multiple of \( p - 1, \) we have \( d \leq (p + m - 2)(p - 1). \)

**Corollary 8.5.** For \( m > 2, \) the Noether number for \( \mathbf{F}[m V_2]^{SL_2(\mathbf{F}_p)} \) is less than or equal to \( (p + m - 2)(p - 1). \) For \( m = 2 \) and \( p > 2, \) the Noether number is \( p(p - 1) \) and for \( m = 2, p = 2, \) the Noether number is \( p + 1 = 3. \)

**Proof.** The elements of \( \mathcal{S}_m \) lie in degrees \( 2, p + 1 \) and \( p(p - 1). \) Clearly \( L_1 \) and \( D_{(p(p - 1), 0, \ldots, 0)} \) are indecomposable.

For \( p = 2 \) and \( m \in \{3, 4\}, \) Magma calculations give the Noether number \( (p + m - 2)(p - 1) = m. \)

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