ON MONADS OF EXACT REFLECTIVE LOCALIZATIONS OF
ABELIAN CATEGORIES

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ABSTRACT. In this paper we define GABRIEL monads as the idempotent monads associated to exact reflective localizations in ABELian categories and characterize them by a simple set of properties. The coinage of a GABRIEL monad is a SERRE quotient category. The GABRIEL monad induces an equivalence between its coinage and its image, the localizing subcategory of local objects.

1. INTRODUCTION

ABELian categories became, since their introduction in GROTHENDIECK’s Tôhoku paper [Gro57], a central notion in homological algebra. The notion of a computable ABELian category, i.e., an ABELian category in which all existential quantifiers occurring in the defining axioms can be turned into algorithms, was therefore naturally motivated by our attempt to establish an algorithmic context for constructive homological algebra (cf. [BLH11, Chap. 2]). Along these lines we treated in loc. cit. the ABELian categories of finitely presented modules over so-called computable rings and their localization at certain maximal ideals.

Our next goal is to treat the ABELian category $\mathcal{Coh}(X)$ of coherent sheaves on a projective scheme $X$ along the same lines. This category is, by SERRE’s seminal paper [Ser55], equivalent to a quotient category of graded modules.

In fact, the appropriate categorical setup for a SERRE quotient $\mathcal{A}/\mathcal{E}$ of an ABELian category $\mathcal{A}$ modulo a so-called thick subcategory $\mathcal{E} \subset \mathcal{A}$ was introduced later, first by GROTHENDIECK in [Gro57, Chap. 1.11] and then more elaborately in GABRIEL’s thesis [Gab62]. Later GABRIEL and ZISMAN developed in [GZ67] a localization theory of categories, general enough both to recover a SERRE quotient $\mathcal{A}/\mathcal{E}$ as an outcome of a certain localization process $\mathcal{A} \to \Sigma^{-1}\mathcal{A} \simeq \mathcal{A}/\mathcal{E}$ and to enclose VERDIER’s localization process in triangulated categories, which he used in his 1967 thesis (cf. [Ver96]) to define derived categories. Thanks to SIMPSON’s work [Sim06] the GABRIEL-ZISMAN localization is now completely formalized in the proof assistant COQ [Coq04]. In many applications the localization $\Sigma^{-1}\mathcal{A}$ can be realized as a full subcategory of $\mathcal{A}$, the subcategory of all $\Sigma$-local objects of $\mathcal{A}$. This is equivalent to the existence of a so-called idempotent monad (cf. Definition 2.9) associated to the localization. For a further overview on localizations we refer to the arXiv version of [Tho11].

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In our application to $\mathfrak{Coh}(X)$ we are in the setup of SERRE quotient categories $\mathcal{A}/\mathfrak{C}$, which are the outcome of an exact localization having an associated idempotent monad. We call this monad the GABRIEL monad (cf. Definition 3.3). GABRIEL monads satisfy a set of properties which we use as a simple set of axioms to define what we call a $\mathfrak{C}$-saturating monad.

The goal of this paper is to prove the converse by characterizing GABRIEL monads as the $\mathfrak{C}$-saturating monads (Theorem 4.6). This characterization enables us in [BLHa] to give a constructive and simple proof that several known functors in the context of coherent sheaves on a projective scheme produce $\mathfrak{C}$-saturated objects\footnote{In this context the $\Sigma$-local objects are GABRIEL’s $\mathfrak{C}$-saturated objects.} and hence compute the module of twisted global sections. The proof there relies on checking the defining set of axioms of a $\mathfrak{C}$-saturating monad, which turns out to be a relatively easy task. In particular, the proof does not rely on the (full) BGG correspondence [BGG78] of triangulated categories, the SERRE-GROTHENDIECK correspondence [BS98, 20.3.15], or the local duality, as used in [EFS03]. Furthermore, in [BLHb] we use the GABRIEL monads of computable ABELIAN categories to prove that their corresponding SERRE quotients are again computable in the above sense. In particular, the ABELIAN category $\mathfrak{Coh}(X)$ is computable (cf. [BLHa]).

The paper is organized as follows. In Section 2 we recall some basic definitions and facts from category theory and in Section 3 we collect some useful propositions about SERRE quotients, which we need to formulate and prove our main result in Section 4.

2. Preliminaries

In this short section we collect some standard categorical preliminaries, which should make this paper self-contained for the readers interested in our applications to coherent sheaves [BLHa]. For details we refer to [Bor94a, Bor94b] or the active nLab wiki [nLa12].

2.1. Adjoint functors and monads.

**Definition 2.1.** Two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are called a pair of adjoint functors and denoted by $F \dashv G : \mathcal{B} \to \mathcal{A}$ if for each pair of object $(a, b)$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$ there exists an isomorphism

$$\text{Hom}_\mathcal{B}(Fa, b) \cong \text{Hom}_\mathcal{A}(a, Gb).$$

$F$ is called a left adjoint of $G$ and $G$ is right adjoint of $F$.

**Remark 2.2.** Left adjoints and right adjoints are essentially unique.

**Proposition 2.3.** An adjunction $F \dashv G : \mathcal{B} \to \mathcal{A}$ is characterized by the existence of two natural transformations

$$\delta : F \circ G \to \text{Id}_\mathcal{B} \quad \text{and} \quad \eta : \text{Id}_\mathcal{A} \to G \circ F,$$
called the counit\(^2\) and unit of the adjunction\(^3\), respectively, such that the two zig-zag (or unit-counit) identities hold, i.e., the compositions of the natural transformations
\[
\begin{align*}
F &\xrightarrow{F\eta} F \circ G \circ F & \text{and} & G &\xrightarrow{G\eta} G \circ F \circ G
\end{align*}
\]
must be the identity on functors.

**Definition 2.4.** A monad in a category \(\mathcal{A}\) is an endofunctor \(W : \mathcal{A} \to \mathcal{A}\) together with two natural transformations \(\eta : \text{Id}_\mathcal{A} \to W\) and \(\mu : W^2 \to W\), called the **unit** and **multiplication of the monad**, respectively, such that following two coherence conditions hold:
\[
\mu \circ \mu W = \mu \circ \mu W \quad \text{and} \quad \mu \circ \eta W = \mu \circ \eta W = \text{Id}_W.
\]

**Remark 2.5.** Let \(F \dashv G : \mathcal{B} \to \mathcal{A}\) be an adjoint pair with unit \(\eta\) and counit \(\delta\). The composed endofunctor \(W := G \circ F : \mathcal{A} \to \mathcal{A}\) together with the unit of the adjunction \(\eta : \text{Id}_\mathcal{A} \to G \circ F\) as the monad unit \(\eta : \text{Id}_\mathcal{A} \to W\) and
\[
\mu := G\delta F : W^2 \to W
\]
as the monad multiplication is a monad. It is called the **monad associated to the adjoint pair**.

2.2. Idempotent monads and reflective localizations.

**Definition 2.6.** A monad \((W, \eta, \mu)\) is called an **idempotent monad** if the multiplication \(\mu : W^2 \to W\) is a natural isomorphism.

**Definition 2.7.** A full subcategory \(\mathcal{B} \subset \mathcal{A}\) is called **reflective subcategory** if the inclusion functor \(\iota : \mathcal{B} \to \mathcal{A}\) has a left adjoint, called the **reflector**.

**Proposition 2.8** ([Bor94b, §4.2]). Let \((W, \bar{\eta}, \mu)\) be a monad on the category \(\mathcal{A}\). Then the following statements are equivalent:

1. The monad \((W, \bar{\eta}, \mu)\) is idempotent.
2. The two natural transformations \(W\eta, \eta W : W \to W^2\) are equal.
3. The (essential) image of \(W\) in \(\mathcal{A}\) is a reflective subcategory.
4. The corestriction \(\text{cores}_{W(\mathcal{A})} W\) of \(W\) to its (essential) image \(W(\mathcal{A})\) is left adjoint to the inclusion functor of \(W(\mathcal{A})\) in \(\mathcal{A}\).
5. There exists an adjunction \(F \dashv G : \mathcal{B} \to \mathcal{A}\) with unit \(\eta\) and counit \(\delta\) where \(G\) is fully faithful such that its associated monad \((G \circ F, \eta, F\delta G)\) is isomorphic to \((W, \bar{\eta}, \mu)\).

A localization of a category \(\mathcal{A}\) at collection of morphisms \(\Sigma\) is roughly speaking the process of adjoining formal inverses of the morphisms in \(\Sigma\) (cf. [Wei94, Definition 10.3.1]).

**Definition 2.9.** A localization \(\mathcal{A} \to \Sigma^{-1}\mathcal{A}\) (at a subset of morphisms \(\Sigma\) of \(\mathcal{A}\)) is called **reflective localization** if it admits a fully faithful right adjoint \(G : \Sigma^{-1}\mathcal{A} \hookrightarrow \mathcal{A}\).

\(^2\delta_b\) corresponds to \(1_{Gb}\) under the isomorphism \(\text{Hom}_\mathcal{B}((F \circ G)a, b) \cong \text{Hom}_\mathcal{A}(Gb, Gb)\).

\(^3\eta_a\) corresponds to \(1_{Fa}\) under the isomorphism \(\text{Hom}_\mathcal{A}(a, (G \circ F)a) \cong \text{Hom}_\mathcal{B}(Fa, Fa)\).
The following proposition states, among other things, that all functors having fully faithful right adjoints are in fact reflective localizations.

**Proposition 2.10** ([GZ67, Proposition 1.3]). Let \( F \dashv G : \mathcal{B} \rightarrow \mathcal{A} \) be a pair of adjoint functors with unit \( \eta \) and counit \( \delta \). Further let \((W, \eta, \mu) := (G \circ F, \eta, F\delta G)\) be the associated monad. The following statements are equivalent:

1. \( G \) is fully faithful, i.e., \( \mathcal{B} \) is equivalent to its (essential) image under \( G \).
2. The counit \( \delta : F \circ G \rightarrow \text{Id}_A \) is a natural isomorphism\(^4\).
3. The associated monad \((W, \eta, \mu)\) is idempotent.
4. The (essential) image of \( G \) is a reflective subcategory.
5. If \( \Sigma \) is the set of morphisms \( \varphi \in \mathcal{A} \) such that \( F(\varphi) \) is invertible in \( \mathcal{B} \), then \( F : \mathcal{A} \rightarrow \mathcal{B} \) realizes the reflective localization \( \mathcal{A} \rightarrow \Sigma^{-1}\mathcal{A} \).

In particular, \( G \) is conservative, i.e., reflects isomorphism\(^5\).

### 3. Gabriel Localizations

In this section we recall some results about *Serre* quotients arising from reflective localizations.

**Definition 3.1** ([Wei94, Exer. 10.3.2]). A non-empty full subcategory \( \mathcal{C} \) of an Abelian category \( \mathcal{A} \) is called **thick**\(^6\) if it is closed under passing to subobjects, factor objects, and extensions. In other words, for every short exact sequence
\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
\]
in \( \mathcal{A} \) the object \( M \) lies in \( \mathcal{C} \) if and only if \( M' \) and \( M'' \) lie in \( \mathcal{C} \).

*Pierre Gabriel* defines in his thesis [Gab62, §III.1] the **(Serre) quotient category** \( \mathcal{A}/\mathcal{C} \) with the same objects as \( \mathcal{A} \) and with Hom-groups defined by
\[
\text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \lim_{\mathcal{M}', \mathcal{N}'} \text{Hom}_{\mathcal{A}}(M', N/N'),
\]
where the direct limit is taken over all subobjects \( M' \leq M \) and \( N' \leq N \) such that \( M/M' \) and \( N' \) belong to \( \mathcal{C} \). Further he defines the **canonical functor** \( Q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C} \) which is the identity on objects\(^7\) and which maps a morphism \( \varphi \in \text{Hom}_{\mathcal{A}}(M, N) \) to its image in \( \text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \) under the maps
\[
\text{Hom}(M' \hookrightarrow M, N \rightarrow N/N') : \text{Hom}_{\mathcal{A}}(M, N) \rightarrow \text{Hom}_{\mathcal{A}}(M', N/N').
\]

*Gabriel* proves in [Gab62, Proposition §III.1.1] that \( \mathcal{A}/\mathcal{C} \) is an Abelian category and that the canonical functor \( Q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C} \) is exact. The canonical functor \( Q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C} \) fulfills the following universal property.

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\(^4\)In particular, \( F \) is essentially surjective.

\(^5\)\( G(\alpha) \) isomorphism \( \implies \alpha \) isomorphism.

\(^6\)Gabriel uses the notion *épaisse*. Thick subcategories are automatically replete.

\(^7\)So \( Q \) is surjective (on objects) but in general neither faithful nor full.
Proposition 3.2 ([Gab62, Cor. §III.1.2]). Let \( \mathcal{C} \) be a thick subcategory of \( \mathcal{A} \) and \( G : \mathcal{A} \to \mathcal{D} \) an exact functor into the Abelian category \( \mathcal{D} \). If \( G(\mathcal{C}) \) is zero then there exists a unique functor \( H : \mathcal{A}/\mathcal{C} \to \mathcal{D} \) such that \( G = H \circ Q \).

Definition 3.3. The thick subcategory \( \mathcal{C} \) of \( \mathcal{A} \) is called **localizing subcategory** if the canonical functor \( Q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) admits a right adjoint \( S : \mathcal{A}/\mathcal{C} \to \mathcal{A} \), i.e., a functor together with a natural isomorphism

\[
\text{Hom}_{\mathcal{A}/\mathcal{C}}(QM, N) \cong \text{Hom}_{\mathcal{A}}(M, SN).
\]

\( S \) is called the 8 section functor of \( Q \). Denote by

\[
\delta : Q \circ S \to \text{Id}_{\mathcal{A}/\mathcal{C}} \quad \text{and} \quad \eta : \text{Id}_{\mathcal{A}} \to S \circ Q
\]

the counit and unit of the adjunction \( S \dashv Q \), respectively.

We call such a canonical functor \( Q \) a **Gabriel localization** and the associated monad \( (S \circ Q, \eta, \mu = S\delta Q) \) the **Gabriel monad**.

The following definition describes the \( \Sigma \)-local objects in the sense of Gabriel-Zisman, where \( \Sigma \) is the collection of all morphisms in \( \mathcal{A} \) with both kernel and cokernel in \( \mathcal{C} \).

Definition 3.4. Let \( \mathcal{C} \) be a thick subcategory of \( \mathcal{A} \). We call an object \( M \) in \( \mathcal{A}/\mathcal{C} \) **\( \mathcal{C} \)-saturated** (or **\( \mathcal{C} \)-closed**9) if it has no non-trivial subobjects in \( \mathcal{C} \) and every extension of \( M \) by an object in \( \mathcal{C} \) is trivial. Define

\[
\text{Sat}_\mathcal{C}(\mathcal{A}) := \text{full (replete) subcategory of } \mathcal{C}\text{-saturated objects}.
\]

The following proposition together with Proposition 2.10 imply that \( \mathcal{A}/\mathcal{C} \) is a reflective localization; more precisely \( \mathcal{A}/\mathcal{C} \simeq \Sigma^{-1}\mathcal{A} \), where \( \Sigma \) is the collection of all morphisms in \( \mathcal{A} \) with both kernel and cokernel in \( \mathcal{C} \).

Proposition 3.5 ([Gab62, Propositions III.2.2, 2.3 and its corollary]). Let \( \mathcal{C} \subset \mathcal{A} \) be a localizing subcategory and \( S : \mathcal{A}/\mathcal{C} \to \mathcal{A} \) the section functor.

1. \( S : \mathcal{A}/\mathcal{C} \to \mathcal{A} \) is left exact.
2. The counit of the adjunction \( \delta : Q \circ S \sim \text{Id}_{\mathcal{A}/\mathcal{C}} \) is a natural isomorphism.
3. An object \( M \) in \( \mathcal{A} \) is \( \mathcal{C} \)-saturated if and only if \( \eta_M : M \to S(Q(M)) \) is an isomorphism, where \( \eta \) is the unit of the adjunction.

Corollary 3.6. Let \( \mathcal{C} \) be a localizing subcategory of the Abelian category \( \mathcal{A} \). Then \( \mathcal{S}(\mathcal{A}/\mathcal{C}) \simeq \text{Sat}_\mathcal{C}(\mathcal{A}) \) are reflective subcategories of \( \mathcal{A} \), in particular, the following holds:

1. The section functor \( S \) is fully faithful.
2. \( S \) is conservative.
3. \( \eta(S \circ Q) = (S \circ Q) \eta. \)

Proof. Proposition 3.5.(2) establishes the context of Proposition 2.10. \( \square \)

The existence of enough \( \mathcal{C} \)-saturated objects turns \( \text{Sat}_\mathcal{C}(\mathcal{A}) \) into an “orthogonal complement” of \( \mathcal{C} \).

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8Two right adjoints are naturally isomorphic.

9GABRIEL uses fermé.
Definition 3.7. Let $\mathcal{C}$ be a thick subcategory of $\mathcal{A}$. We say that $\mathcal{A}$ has enough $\mathcal{C}$-saturated objects if for each $M \in \mathcal{A}$ there exists a saturated object $N$ and a morphism $\eta_M : M \to N$ such that $H_{\mathcal{C}}(M) := \ker \eta_M \in \mathcal{C}$. It follows that $H_{\mathcal{C}}(M)$ is the maximal subobject of $M$ in $\mathcal{C}$.

Proposition 3.8 ([Gab62, Proposition §III.2.4]). The thick subcategory $\mathcal{C} \subset \mathcal{A}$ is localizing iff $\mathcal{A}$ has enough $\mathcal{C}$-saturated objects.

Remark 3.9. The existence of the maximal subobject in $\mathcal{C}$ simplifies the description of Hom-groups in $\mathcal{A}/\mathcal{C}$, namely

$$\text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \lim_{\longrightarrow} \text{Hom}_{\mathcal{A}}(M', N/H_{\mathcal{C}}(N)).$$

The existence of the section functor $S$ as the right adjoint of $Q$ reduces the computability of Hom-groups in $\mathcal{A}/\mathcal{C} \cong \text{Sat}_\mathcal{C}(\mathcal{A})$ even further to their computability in $\mathcal{A}$

$$\text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \text{Hom}_{\mathcal{A}/\mathcal{C}}(Q(M), Q(N)) \cong \text{Hom}_\mathcal{A}(M, (S \circ Q)(N)),$$

avoiding the direct limit in the definition of $\text{Hom}_{\mathcal{A}/\mathcal{C}}$ completely.

Corollary 3.10. The image $S(\mathcal{A}/\mathcal{C})$ of $S$ is a subcategory of $\text{Sat}_\mathcal{C}(\mathcal{A})$ and the inclusion functor $S(\mathcal{A}/\mathcal{C}) \hookrightarrow \text{Sat}_\mathcal{C}(\mathcal{A})$ is an equivalence of categories with the restricted-corestricted GABRIEL monad $S \circ Q : \text{Sat}_\mathcal{C}(\mathcal{A}) \to S(\mathcal{A}/\mathcal{C})$ as a quasi-inverse. In other words, $\text{Sat}_\mathcal{C}(\mathcal{A})$ is the essential image of $S$ and, hence, of the GABRIEL monad $S \circ Q$.

Proof. The inclusion of $S(Q(\mathcal{A})) = S(\mathcal{A}/\mathcal{C})$ in $\text{Sat}_\mathcal{C}(\mathcal{A})$ will follow from Proposition 3.5.(3) as soon as we can show that $\eta S$, or equivalently $\eta(S \circ Q)$, is a natural isomorphism. Proposition 2.10 applied to Proposition 3.5.(2) states that the multiplication $\mu$ of the GABRIEL monad $S \circ Q$ is a natural isomorphism. Finally, the second coherence condition $\mu \circ \eta(S \circ Q) = \text{Id}_{S \circ Q}$ implies that $\eta(S \circ Q) = \mu^{-1}$ is also a natural isomorphism. 

Corollary 3.11. The restricted canonical functor $Q : \text{Sat}_\mathcal{C}(\mathcal{A}) \to \mathcal{A}/\mathcal{C}$ and the corestricted section functor $S : \mathcal{A}/\mathcal{C} \to \text{Sat}_\mathcal{C}(\mathcal{A})$ are quasi-inverse equivalences of categories. In particular, $\text{Sat}_\mathcal{C}(\mathcal{A}) \simeq S(\mathcal{A}/\mathcal{C}) \simeq \mathcal{A}/\mathcal{C}$ is an ABELIAN category.

The adjunction $\text{core} S_{\text{Sat}_\mathcal{C}(\mathcal{A})}(S \circ Q) \dashv \iota : \text{Sat}_\mathcal{C}(\mathcal{A}) \hookrightarrow \mathcal{A}$ is equivalent to the adjunction $Q \dashv S : \mathcal{A}/\mathcal{C} \to \mathcal{A}$. They both share the same adjunction monad $S \circ Q : \mathcal{A} \to \mathcal{A}$. In particular, $\text{core} S_{\text{Sat}_\mathcal{C}(\mathcal{A})}(S \circ Q)$ is exact and $\iota$ is left exact.

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The notation $H_{\mathcal{C}}$ is motivated by the notation for local cohomology.
Let \( \mathcal{C} \subset \mathcal{A} \) be a localizing subcategory of the ABELian category \( \mathcal{A} \). \( \text{Sat}(\mathcal{A}/\mathcal{C}) \simeq \text{Sat}_{\mathcal{C}}(\mathcal{A}) \) are not in general ABELian subcategories of \( \mathcal{A} \) in the sense of [Wei94, p. 7], as short exact sequences in \( \text{Sat}_{\mathcal{C}}(\mathcal{A}) \) are not necessarily exact in \( \mathcal{A} \). This is due to the fact that the left exact section functor \( S \) (cf. Proposition 3.5(1)) is in general not exact; the cokernel in \( \text{Sat}_{\mathcal{C}}(\mathcal{A}) \subset \mathcal{A} \) differs from the cokernel in \( \mathcal{A} \) (cf. [BLHb]).

The full subcategory \( \mathcal{A}_e \subset \mathcal{A} \) consisting of all objects with no nontrivial subobject in \( \mathcal{C} \) is a pre-ABELian category\(^{11}\). The kernel of a morphisms in \( \mathcal{A}_e \) is its kernel in \( \mathcal{A} \). The cokernel of a morphism in \( \mathcal{A}_e \) is isomorphic to its cokernel as a morphism in \( \mathcal{A} \) modulo the maximal subobject in \( \mathcal{C} \) (cf. Proposition 3.8).

\( \text{Sat}_{\mathcal{C}}(\mathcal{A}) \subset \mathcal{A}_e \) is the completion of \( \mathcal{A}_e \) with respect to the property that every extension by an object in \( \mathcal{C} \) is trivial. This completion is given by the GABRIEL monad \( S \circ Q \) restricted to \( \mathcal{A}_e \).

4. Characterizing reflective GABRIEL localizations and GABRIEL monads

The next proposition states that in fact all exact reflective localizations in ABELian categories are (reflective) GABRIEL localizations.

**Proposition 4.1** ([Gab62, Proposition §III.2.5], [GZ67, Chap. 1.2.5.d]). Let \( \tilde{Q} \vdash \tilde{S} : \mathcal{B} \to \mathcal{A} \) be a pair of adjoint functors of ABELian categories. Assume, that \( \tilde{Q} \) is exact and the counit \( \delta : \tilde{Q} \circ \tilde{S} \to \text{Id}_\mathcal{B} \) of the adjunction is a natural isomorphism. Then \( \mathcal{C} := \ker \tilde{Q} \) is a localizing subcategory of \( \mathcal{A} \) and the adjunction \( \tilde{Q} \vdash \tilde{S} : \mathcal{B} \to \mathcal{A} \) induces an adjoint equivalence from \( \mathcal{B} \) to \( \mathcal{A}/\mathcal{C} \).

Now, we approach the central definition of this paper which collects some properties of GABRIEL monads.

**Definition 4.2.** Let \( \mathcal{C} \subset \mathcal{A} \) be a localizing subcategory of the ABELian category \( \mathcal{A} \) and \( \iota : \text{Sat}_{\mathcal{C}}(\mathcal{A}) \hookrightarrow \mathcal{A} \) the full embedding. We call an endofunctor \( W : \mathcal{A} \to \mathcal{A} \) together with a natural transformation \( \tilde{\eta} : \text{Id}_\mathcal{A} \to W \) \( \mathcal{C} \)-saturating if the following holds:

1. \( \mathcal{C} \subset \ker W \),
2. \( W(\mathcal{A}) \subset \text{Sat}_{\mathcal{C}}(\mathcal{A}) \),
3. \( G := \text{cores}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})} W \) is exact,
4. \( \tilde{\eta}W = W\tilde{\eta} \), and
5. \( \tilde{\eta}_w : \text{Id}_{\mathcal{A}/\text{Sat}_{\mathcal{C}}(\mathcal{A})} \to W|_{\text{Sat}_{\mathcal{C}}(\mathcal{A})} \) is a natural isomorphism\(^{12}\).

Let \( H \) be the unique functor from Proposition 3.2 such that \( G = H \circ Q \). We call the composed functor \( \tilde{H} := \iota \circ H \) the colift of \( W \) along \( Q \), since \( W = \iota \circ G = \tilde{H} \circ Q \).

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\(^{11}\)although not an ABELian category, in general, as monomorphisms need not be kernels of their cokernels. For example, the monomorphism \( 2 : \mathbb{Z} \to \mathbb{Z} \) in \( \mathcal{A}_e = \{\text{f.g. torsion-free ABELian groups}\} \) is not kernel of its cokernel (which is zero), where \( \mathcal{A} := \{\text{f.g. ABELian groups}\} \supset \mathcal{C} := \{\text{f.g. torsion ABELian groups}\} \). This gives an example of a morphism which is monic and epic in \( \mathcal{A}_e \) but not an isomorphism.

\(^{12}\)In particular, \( W(\mathcal{A}) \) is an essentially wide subcategory of \( \text{Sat}_{\mathcal{C}}(\mathcal{A}) \).
**Lemma 4.3.** Let $Q : \mathcal{A} \to \mathcal{A} / \mathcal{C}$ be a Gabriel localization with section functor $S$. Then each $\mathcal{C}$-saturating endofunctor $W$ of $\mathcal{A}$ is naturally isomorphic to $S \circ Q$. Furthermore, the colift $\tilde{H}$ of $W$ along $Q$ is also a section functor naturally isomorphic to $S$.

**Proof.** Consider

$$G = H \circ Q$$

(Proposition 3.2 using 4.2.(1))

$$= H \circ \text{Id}_{\mathcal{A}/\mathcal{C}} \circ Q$$

$$\simeq H \circ Q \circ S \circ Q$$

(Proposition 3.5.(2))

$$= G \circ S \circ Q$$

$$\simeq \text{Id}_{\text{Sat}(\mathcal{A})} \circ S \circ Q$$

(using $S(Q(A)) \subset \text{Sat}(\mathcal{A})$ and 4.2.(5))

$$= \text{core}_{\text{Sat}(\mathcal{A})}(S \circ Q).$$

Then, using the notation of Definition 4.2,

$$W = \tilde{H} \circ Q = \iota \circ G \simeq S \circ Q.$$

This also proves the equivalence $\tilde{H} \simeq S$, as $Q$ is surjective.

**Proposition 4.4.** Let $Q : \mathcal{A} \to \mathcal{A} / \mathcal{C}$ be a Gabriel localization and $(W, \tilde{\eta})$ be a $\mathcal{C}$-saturating endofunctor of $\mathcal{A}$ with colift $\tilde{H}$ along $Q$. Then there exists a natural transformation $\tilde{\delta} : Q \circ \tilde{H} \to \text{Id}_{\mathcal{A} / \mathcal{C}}$ such that $Q$ and $\tilde{H}$ form an adjoint pair $Q \dashv \tilde{H}$ with unit $\tilde{\eta}$ and counit $\tilde{\delta}$.

**Definition 4.5.** Hence, each $\mathcal{C}$-saturating endofunctor $(W, \tilde{\eta})$ is the monad $(W, \tilde{\eta}, \tilde{H}\tilde{\delta}Q)$ associated to the adjunction $Q \dashv \tilde{H}$. We call it a $\mathcal{C}$-saturating monad.

**Proof of Proposition 4.4.** We define a natural transformation $\tilde{\delta} : Q \circ \tilde{H} \to \text{Id}_{\mathcal{A} / \mathcal{C}}$ and check the two zig-zag identities (zz), i.e., that the compositions of natural transformations

$$Q \xrightarrow{Q_{\tilde{\delta}}} Q \circ \tilde{H} \circ Q \xrightarrow{\tilde{\delta}Q} Q$$

and

$$\tilde{H} \xrightarrow{\tilde{\eta}} H \circ Q \circ \tilde{H} \xrightarrow{\tilde{\delta}} \tilde{H}$$

are the identity of functors. By 4.2.(5) we know that $(\tilde{H} \circ Q)\tilde{\eta} = W\tilde{\eta} \overset{4.2.(4)}{=} \tilde{\eta}W = (\tilde{\eta})G$ is an isomorphism. Hence, also $Q\tilde{\eta}$ is a natural isomorphism, because the functor $\tilde{H}$ is equivalent to $S$ by Lemma 4.3 and, thus, reflects isomorphisms by Lemma 3.6.(2). This allows us to define $\tilde{\delta}$ in such a way to satisfy the first zig-zag identity, i.e., set $\tilde{\delta}Q := (Q\tilde{\eta})^{-1}$. This defines $\tilde{\delta}$ as $Q$ is surjective (on objects). The second zig-zag identity is equivalent, again due to the surjectivity of $Q$, to the second zig-zag identity applied to $Q$, i.e., $(\tilde{H}\tilde{\delta}Q) \circ \tilde{\eta}(\tilde{H} \circ Q)$ being the identity transformation of the functor $\tilde{H} \circ Q = W$. Now

$$(\tilde{H}\tilde{\delta}Q) \circ \tilde{\eta}(\tilde{H} \circ Q) = (\tilde{H}(Q\tilde{\eta})^{-1}) \circ \tilde{\eta}(\tilde{H} \circ Q)$$

(by the definition $\tilde{\delta}Q := (Q\tilde{\eta})^{-1}$)

$$= ((\tilde{H} \circ Q)\tilde{\eta})^{-1} \circ \tilde{\eta}(\tilde{H} \circ Q)$$

$$= (\tilde{\eta}(\tilde{H} \circ Q))^{-1} \circ \tilde{\eta}(\tilde{H} \circ Q)$$

(using $\tilde{\eta}(\tilde{H} \circ Q) \overset{4.2.(4)}{=} (\tilde{H} \circ Q)\tilde{\eta}$)

$$= \text{Id}_{\tilde{H} \circ Q}.$$
We now approach our main result.

**Theorem 4.6 (Characterization of Gabriel monads).** Each Gabriel monad is an $E$-saturating monad. Conversely, each $E$-saturating monad is equivalent to a Gabriel monad.

**Proof.** The conditions in Definition 4.2 clearly apply to a Gabriel monad $S \circ Q$ by definition of the canonical functor $Q$, Corollary 3.10, Corollary 3.11, Corollary 3.6.(3), and Proposition 3.5.(3), respectively.

The converse follows directly from Lemma 4.3 and Proposition 4.4 which prove that the two adjunctions $Q \dashv \tilde{H}$ and $Q \dashv S$ are equivalent, and so are their associated monads. □

**References**

[BGG78] I. N. Bernšteı́n, I. M. Gel’fand, and S. I. Gel’fand, *Algebraic vector bundles on $\mathbb{P}^n$ and problems of linear algebra*, Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 66–67. MR MR509387 (80c:14010a) 2

[BLHa] Mohamed Barakat and Markus Lange-Hegermann, *A constructive approach to coherent sheaves*, (in preparation). 2

[BLHb] ————, *Constructive Gabriel localizations*, (in preparation). 2

[BLH11] ————, *An axiomatic setup for algorithmic homological algebra and an alternative approach to localization*, J. Algebra Appl. 10 (2011), no. 2, 269–293, (arXiv:1003.1943). 1

[Bor94a] Francis Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994, Basic category theory. MR 1291599 (96g:18001a) 2

[Bor94b] ————, *Handbook of categorical algebra. 2*, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994, Categories and structures. MR 1313497 (96g:18001b) 2

[BS98] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, vol. 60, Cambridge University Press, Cambridge, 1998. MR 1613627 (99h:13020) 2

[Coq04] Coq development team, *The coq proof assistant reference manual*, LogiCal Project, 2004, Version 8.0. 1

[EFS03] David Eisenbud, Gunnar Fløystad, and Frank-Olaf Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc. 355 (2003), no. 11, 4397–4426 (electronic). MR MR1990756 (2004f:14031) 2

[Gab62] Pierre Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), 323–448. MR 0232821 (38 #1144) 1, 4, 5, 6, 7

[Gro57] Alexander Grothendieck, *Sur quelques points d’algèbre homologique*, Tôhoku Math. J. (2) 9 (1957), 119–221. MR MR0102537 (21 #1328) 1

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967. MR 0210125 (35 #1019) 1, 4, 7

[nLa12] nLab authors, *The nlab*, 2012, (http://ncatlab.org/nlab/) [Online; accessed 10-February-2012]. 2

[Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) 61 (1955), 197–278. MR 0068874 (16,930c) 1
Carlos Simpson, *Explaining Gabriel-Zisman localization to the computer*, J. Automat. Reason. **36** (2006), no. 3, 259–285. MR 2288803 (2007h:68176)

Sebastian Thomas, *On the 3-arrow calculus for homotopy categories*, Homology Homotopy Appl. **13** (2011), no. 1, 89–119, ([arXiv:1001.4536](https://arxiv.org/abs/1001.4536)). MR 2803869

Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. MR 1453167 (98c:18007)

Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR MR1269324 (95f:18001)

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