A LINK BETWEEN THE LOG-SOBOLEV INEQUALITY AND LYAPUNOV CONDITION

YUAN LIU

Abstract. We give an alternative look at the log-Sobolev inequality (LSI in short) for log-concave measures by semigroup tools. The same idea yields a heat flow proof of LSI under the Lyapunov condition for symmetric diffusions on Riemannian manifolds provided the Bakry-Emery’s curvature is bounded from below. This result was first obtained by Cattiaux-Guillin-Wu through a combination of detective $L^2$ transportation-information inequality $W_2I$ and the HWI inequality of Otto-Villani. In comparison, we have a control constant involving a smaller moment of distance function.

Next, we assert a converse implication that the Lyapunov condition can be derived from LSI, which means their equivalence in the above setting. Note that the Lyapunov condition here is slightly weaker than the one in [8].

1. Introduction

In this paper, we will give a direct proof of log-Sobolev inequality (LSI in short) for symmetric diffusions under the Lyapunov condition and curvature condition. The relation between LSI and Lyapunov function will be investigated further.

In the sequel, denote by $E$ a connected complete Riemannian manifold of finite dimension, $d$ the geodesic distance, $dx$ the volume measure, $\mu(dx) = e^{-V(x)}dx$ a probability measure with $V \in C^2(E)$. $L = \Delta - \nabla V \cdot \nabla$ the $\mu$-symmetric diffusion operator, $P_t = e^{tL}$ the semigroup, $\Gamma(f, g) = \nabla f \cdot \nabla g$ the carré du champ operator, and $\mathcal{E}$ the Dirichlet form with domain $\mathcal{D}(\mathcal{E})$ in $L^2(\mu)$. It’s known that the integration by parts formula reads

$$\int \nabla f \cdot \nabla g \ d\mu = -\int f L g \ d\mu, \ \forall f \in \mathcal{D}(\mathcal{E}), g \in \mathcal{D}(L),$$

and $P_t$ is $L^2$-ergodic i.e.

$$\|P_t f - \mu f\|_{L^2(\mu)} \to 0 \text{ as } t \to \infty, \ \forall f \in L^2(\mu).$$

We refer to Bakry-Gentil-Ledoux [5] for a detailed presentation of the fundamentals. For simplicity, write $\mu f = \int f d\mu$ and $\mathcal{E}[f] = \mathcal{E}(f, f)$.

A (tight) LSI means there exists a constant $C > 0$ such that for any $f \in \mathcal{D}(\mathcal{E})$

$$\text{Ent}(f^2) := \mu(f^2 \log f^2) - \mu f^2 \log \mu f^2 \leq C \mathcal{E}(f, f).$$

(1.1)

There have been several classical proofs of LSI for log-concave measures, such as the semigroup argument with gradient estimates in Ledoux [11], or application of Brascamp-Lieb inequality from Bobkov-Ledoux [6], or martingale representation by
Capitaine-Hsu-Ledoux [7]. In particular, Bakry-Emery [4] showed this result purely through derivations. We would like to revisit this case in another simple viewpoint.

**Proposition 1.1.** Suppose \( \mu \) is log-concave in \( \mathbb{R}^n \) such that \( \text{Hess}(V) \geq c \text{Id} \) with \( c > 0 \). Then the LSI \((1.1)\) holds for \( C = \frac{2}{c} \).

Set \( \Phi(t) = 2\mathbb{E}[\sqrt{P_t f}] - c \text{Ent}(P_t f) \) for \( f > 0 \). Heuristically, due to \( P_t f \to \mu f \) in \( L^2 \)-norm, we have \( \Phi(\infty) = 0 \) and then \( \Phi(0) \geq 0 \) if there holds the monotonicity

\[
\frac{d}{dt} \Phi(t) \leq 0.
\]

**Proof.** Write \( \varphi = P_t f \) for positive, bounded and smooth \( f \). Due to the equalities

\[
\frac{\partial}{\partial t} \varphi = L\varphi, \quad \nabla L\varphi = L\nabla \varphi - \text{Hess}(V)\nabla \varphi,
\]

together with the formula for integration by parts, we calculate directly

\[
\frac{d}{dt} \mathbb{E}[\sqrt{\varphi}] = -\sum_{i=1}^n \sum_{j=1}^n \mu \left( \frac{\varphi''_{ij}}{\sqrt{\varphi}} - \frac{\varphi'_{i}'}{\sqrt{\varphi}} \right)^2 - c \mu \langle \nabla \varphi, \text{Hess}(V)\nabla \varphi \rangle
\]

\[
\frac{d}{dt} \text{Ent}(\varphi) = -\mu \frac{|
abla \varphi|^2}{\varphi}.
\]

Using \( \text{Hess}(V) \geq c \text{Id} \) yields \((1.2)\) and thus the desired LSI. 

For general cases, Cattiaux-Guillin-Wu [8] gave a powerful criterion to derive LSI, which is called the Lyapunov condition. With a little relaxation, say \( W \in H^1_{\text{loc}}(\mu) \) (locally \( L^2 \)-integrable Sobolev space, for example see Gilbarg-Trudinger [10]) is a Lyapunov function if \( W > 0 \) everywhere, \( W^{-1} \) is locally bounded and there exist two constants \( c > 0, b \geq 0 \) and some \( x_0 \in \mathbb{R}^n \) such that in the sense of distribution

\[
LW \leq (-cd^2(x, x_0) + b)W.
\]

Note that \( W \geq 1 \) is requested in [8] rather than \( W > 0 \), but the technique of Bakry-Barthe-Cattiaux-Guillin [2, Theorem 1.4] still works for \( W^{-1} \) is locally bounded.

Under the Lyapunov condition and curvature lower bound assumption, the LSI was proved by [8] through a combination of detective \( L^2 \) transportation-information inequality \( W_2I \) and the HWI inequality of Otto-Villani [12]. We give a direct proof of this fact via the idea of monotonicity on heat flow yet.

**Theorem 1.2.** Suppose \( \text{Ric} + \text{Hess}(V) \geq -K \text{Id} \) with \( K \in \mathbb{R}^+ \). Then the LSI \((1.1)\) holds under the Lyapunov condition \((1.4)\).

The control constant \( C \) in \((1.1)\) depends on the curvature bound, spectral gap, parameters in \((1.4)\), and the following quantity associated to \( d \)

\[
- \log ||e^{-d^2(x, x_0)}||_{L^2_K(\mu)},
\]

which is decreasing in \( K \) and always less than \( \mu d^2(x, x_0) \) (by the Jensen’s inequality). It means \( C \) has a smaller moment of distance than the constant in [8] \((3.10)\). One can track details below for more precise comparison.

We further investigate how to derive the Lyapunov condition from LSI.

**Theorem 1.3.** If the LSI \((1.1)\) holds, there exists \( W > 0 \) satisfying \((1.4)\).

As consequence, it follows
Corollary 1.4. Suppose $\text{Ric} + \text{Hess}(V) \geq -K \text{Id}$ with $K \in \mathbb{R}^+$, the LSI (1.1) is equivalent to the Lyapunov condition (1.4).

Remark. It is known that the Poincaré inequality is equivalent to a weak version of Lyapunov condition for some $c > 0$, $b \geq 0$ and closed Petite set $C$ with $W \geq 1$ and $LW \leq -cW + b1_C$, see Bakry-Cattiaux-Guillin [3].

Next two sections will be devoted to the proofs of Theorem 1.2-1.3 respectively.

2. PROOF OF THEOREM 1.2

It is similar to prove Theorem 1.2 as Proposition 1.1, but the new ingredient is how to make a good use of the first quantity on the right hand of (1.3).

Lemma 2.1. ([2, Theorem 1.4]) The Poincaré inequality holds under the Lyapunov condition (1.4). Moreover, for any $h \in D(E)$

$$\int h^2(x) d^2(x, x_0) d\mu(x) \leq \frac{1}{c} \int |\nabla h|^2 d\mu + \frac{b}{c} \int h^2 d\mu. \quad (2.1)$$

Proof. It follows from (1.4) that

$$LW \leq \left[-c + (c + b)1_{B_1(0)}\right] W,$$

which implies the Poincaré inequality by the argument of [2, Page 64]. Moreover,

$$\int h^2(x) d^2(x, x_0) d\mu(x) = \frac{1}{c} \int h^2(cd^2(x, x_0) - b) d\mu + \frac{b}{c} \int h^2 d\mu \leq \frac{1}{c} \int -LW h d\mu + \frac{b}{c} \int h^2 d\mu \leq \int \nabla h \cdot \nabla h d\mu + \frac{b}{c} \int h^2 d\mu \leq \frac{1}{c} \int |\nabla h|^2 d\mu + \frac{b}{c} \int h^2 d\mu.$$

Note that here is no need to assume the integrability of $d^2(x, x_0)$ or $-LW$ for $\mu$, since we can take an approximation sequence in $C^\infty_c(E)$ for given $h$. □

Lemma 2.2. The curvature condition $\text{Ric} + \text{Hess}(V) \geq -K$ with $K \in \mathbb{R}$ implies

1. For any $f \in B_1^+$, $x, y \in E$ and $t > 0$

$$\langle P_t f \rangle^2(x) \leq P_t f^2(y) \exp\left(\frac{Kd^2(x, y)}{1 - e^{-2Kt}}\right). \quad (2.2)$$

2. For any $f \in C_0^1$ and $t > 0$

$$\nabla |P_t f| \leq e^{Kt} |P_t| \nabla f|. \quad (2.3)$$

Proof. Refer to [5, Section 5.5-5.6] or Wang [14, Theorem 2.3.3]. □

Corollary 2.3. If $\text{Ric} + \text{Hess}(V) \geq -K$ with $K \in \mathbb{R}^+$, set $\mu_0 = e^{-Kd^2(x_0, \cdot)}$, then

$$\frac{(P_t f)^2(x)}{\mu f^2} \leq \mu_0 \frac{-1}{1 - e^{-2Kt}} \exp\left(\frac{2K}{1 - e^{-2Kt}} d^2(x_0, x)\right). \quad (2.4)$$
Proof. Denote $\delta(t) = 1 - e^{-2Kt} \leq 1$. Using Lemma \[2.2\] yields
\[
(P_t f^2(y))^{\delta(t)} \geq (P_t f)^{2\delta(t)}(x) \exp \left( -Kd^2(x, y) \right)
\[
\geq (P_t f)^{2\delta(t)}(x) \exp \left( -2Kd^2(x_0, x) - 2Kd^2(x_0, y) \right).
\]
Integrating in $y$ on both sides gives
\[
\mu \left( (P_t f)^{\delta(t)} \right) \geq (P_t f)^{2\delta(t)}(x) \exp \left( -2Kd^2(x_0, x) \right) \mu_0,
\]
which implies \[2.4\] by the Hölder inequality $\mu \left( (P_t f)^{\delta(t)} \right) \leq \mu \left( (P_t f^2)^{\delta(t)} \right)$. \[\square\]

Now we prove Theorem \[1.2\]

Proof. The strategy contains three steps. Assume $f$ is a bounded smooth function.

**Step 1.** Abbreviate $\varphi = P_t f$, we introduce
\[
\text{Ent}^*(\varphi^2) := \int \varphi^2 \log \frac{\varphi^2}{\mu f^2} d\mu,
\]
and with two parameters $A > 0$ and $\eta > 0$
\[
\Psi(t) := \mathcal{E}[\varphi] + A \mu (\varphi - \mu \varphi)^2 - \eta \text{Ent}^*(\varphi^2).
\]
Derivative calculations give respectively
\[
\frac{d}{dt} \mathcal{E}[\varphi] = -2\mu |\nabla \varphi|^2 - 2\mu [\text{Ric} + \text{Hess}(V)](\nabla \varphi, \nabla \varphi),
\]
(2.5)
\[
A \frac{d}{dt} \mu (\varphi - \mu \varphi)^2 = -2A \mu (|\nabla \varphi|^2),
\]
(2.6)
\[
-\eta \frac{d}{dt} \text{Ent}^*(\varphi^2) = 2\eta \mu \left( |\nabla \varphi|^2 \log \frac{\varphi^2}{\mu f^2} \right) + 6\eta \mu |\nabla \varphi|^2.
\]
(2.7)

Here $\nabla \nabla \varphi$ denotes the Hessian of $\varphi$. Due to the curvature condition, (2.5) is less than $-2\mu |\nabla \varphi|^2 + 2K \mu |\nabla \varphi|^2$. To estimate (2.7), using (2.4) gives
\[
\mu \left( |\nabla \varphi|^2 \log \frac{\varphi^2}{\mu f^2} \right) \leq \frac{2K}{1 - e^{-2Kt}} \mu \left( |\nabla \varphi|^2 d^2(x, x_0) \right) - \frac{\log \mu_0}{1 - e^{-2Kt}} \mu |\nabla \varphi|^2.
\]
Set $\eta = e^{\frac{1 - e^{-2Kt}}{2K}}$, applying (2.4) to the above inequality for $h^2 = |\nabla f|^2$ yields
\[
2\eta \mu \left( |\nabla \varphi|^2 \log \frac{\varphi^2}{\mu f^2} \right) \leq 2\mu |\nabla \varphi|^2 + 2b \mu |\nabla \varphi|^2 - \frac{c \log \mu_0}{K} \mu |\nabla \varphi|^2.
\]
(2.8)

Set $A = K + 3\eta + b - \frac{c \log \mu_0}{2K}$, we obtain by combining (2.6) with (2.8)
\[
\frac{d}{dt} \Psi(t) \leq 0, \quad \forall t > 0.
\]

Note that $\eta$ depends on $t$, so we fix $t_0 = 1$ (or any positive number) such that
\[
\Psi(t_0) \geq \Psi(t) \geq \Psi(\infty) \geq 0, \quad \forall t \geq t_0,
\]
namely
\[
\eta \text{Ent}^*(\varphi^2) \leq \mathcal{E}[\varphi] + A \mu (\varphi - \mu \varphi)^2, \quad \forall t \geq t_0.
\]

**Step 2.** We try to compare $\text{Ent}(P_{t_0} f^2)$ with $\text{Ent}^* \left( (P_{t_0} f)^2 \right)$. Define
\[
\Theta_1(t) = \int (P_t f^2 - (P_t f)^2) \log \frac{(P_t f)^2}{\mu f^2} d\mu, \quad \Theta_2(t) = \int P_t f^2 \log \frac{P_t f^2}{(P_t f)^2} d\mu,
\]
\[
\Theta_1(t) = \int (P_t f^2 - (P_t f)^2) \log \frac{(P_t f)^2}{\mu f^2} d\mu, \quad \Theta_2(t) = \int P_t f^2 \log \frac{P_t f^2}{(P_t f)^2} d\mu,
\]
\[
\Theta_1(t) = \int (P_t f^2 - (P_t f)^2) \log \frac{(P_t f)^2}{\mu f^2} d\mu, \quad \Theta_2(t) = \int P_t f^2 \log \frac{P_t f^2}{(P_t f)^2} d\mu,
\]
which satisfy

\[
\text{Ent}(P_{t_0}f^2) - \text{Ent}^* \left( (P_{t_0}f)^2 \right) = \Theta_1(t_0) + \Theta_2(t_0).
\]

Firstly, it follows from (2.4) (note that \( \mu_0 \leq 1 \))

\[
\Theta_1(t_0) \leq \frac{2K}{1 - e^{-2Kt_0}} \mu \left( (P_{t_0}f^2 - (P_{t_0}f)^2) \right)
\]

\[
\leq \frac{2K}{1 - e^{-2Kt_0}} \mu \left( (P_{t_0}f^2 - (P_{t_0}f)^2) \right) - \frac{\log \mu_0}{1 - e^{-2Kt_0}} \mu \left( (P_{t_0}f^2 - (P_{t_0}f)^2) \right).
\]

Using (2.1), (2.3) and the Hölder inequality yields

\[
\mu \left( P_{t_0}f^2 \cdot d^2(x, x_0) \right) \leq c^{-1} \mu \left( \nabla \sqrt{P_{t_0}f} \right)^2 + bc^{-1} \mu f^2
\]

\[
= (4c)^{-1} \mu \left( \nabla P_{t_0}f^2 \right) + bc^{-1} \mu f^2
\]

\[
\leq (4c)^{-1} e^{2Kt_0} \mu \left( \frac{P_{t_0} \left( \nabla f^2 \right)}{P_{t_0}f^2} \right) + bc^{-1} \mu f^2
\]

\[
= c^{-1} e^{2Kt_0} \mu \left( \frac{P_{t_0} \left( \nabla f^2 \right)}{P_{t_0}f^2} \right) + bc^{-1} \mu f^2
\]

\[
\leq c^{-1} e^{2Kt_0} \mu \left( \nabla f^2 \right) + bc^{-1} \mu f^2.
\]

Combining the above estimates gives

\[
(2.9) \quad \Theta_1(t_0) \leq C_1 \mu \left( \nabla f \right)^2 + C_2 \mu f^2 + C_3 \mu (f - \mu f)^2,
\]

where \( C_1 = \frac{2K e^{2Kt_0}}{c(1 - e^{-2Kt_0})} \), \( C_2 = \frac{2K}{c(1 - e^{-2Kt_0})} \) and \( C_3 = \frac{\log \mu_0}{1 - e^{-2Kt_0}} \).

Secondly, due to \( \Theta_2(0) = 0 \), there is the integral representation

\[
\Theta_2(t_0) = \int_0^{t_0} \Theta'_2(t) dt
\]

\[
= \int_0^{t_0} \int L P_t f^2 \log \frac{P_t f^2}{(P_t f)^2} + \frac{P_t f^2}{P_t f} - 2 \frac{P_t f^2}{P_t f} L P_t f \ d\mu dt
\]

\[
= \int_0^{t_0} \int L P_t f^2 \log \frac{P_t f^2}{(P_t f)^2} - 2 \frac{P_t f^2}{P_t f} L P_t f \ d\mu dt
\]

\[
= \int_0^{t_0} \int \frac{\left| \nabla P_t f \right|^2}{P_t f^2} + 4 \frac{\nabla P_t f \cdot \nabla P_t f}{P_t f} - 2 \frac{P_t f^2 \left| \nabla P_t f \right|^2}{(P_t f)^2} \ d\mu dt
\]

\[
\leq \int_0^{t_0} \int \frac{\left| \nabla P_t f \right|^2}{P_t f^2} - 2 \left( \frac{\left| \nabla P_t f \right|^2}{\sqrt{P_t f^2}} \right) \ d\mu dt,
\]

which implies through (2.3) and the Hölder inequality

\[
(2.10) \quad \Theta_2(t_0) \leq \int_0^{t_0} \mu \left| \nabla P_t f \right|^2 dt \leq \int_0^{t_0} 4e^{2Kt} \mu \left( \frac{P_t \left| \nabla f \right|}{P_t f^2} \right)^2 dt
\]

\[
\leq \int_0^{t_0} 4e^{2Kt} \mu \left| \nabla f \right|^2 dt =: C_4 \mathcal{E}[f],
\]
where \( C_4 = \frac{2^{(e^{2K_0} - 1)}}{K} \). Combining (2.10) with (2.11) gives
\[
\text{Ent}(P_t f^2) - \text{Ent}^*(\langle P_0 f \rangle^2) \leq (C_1 + C_4)\mu(\nabla f)^2 + C_2\mu f^2 + C_3\mu(f - \mu f)^2.
\]
Recall the last inequality in Step 1, we obtain from the monotonicity of \( \mathcal{E}[P_t f] \) (see [1, Proposition 3.1.6]) that
\[
\eta \text{Ent}(P_t f^2) \leq [1 + \eta(C_1 + C_4)] \mathcal{E}[f] + \eta C_2\mu f^2 + \eta(A + C_3)\mu(f - \mu f)^2.
\]

**Step 3.** By the same argument as (2.10), we have
\[
\begin{align*}
\text{Ent}(f^2) - \text{Ent}(P_t f^2) &= \int_0^t -\frac{d}{dt}\text{Ent}(P_t f^2) dt \\
&= \int_0^t \int -LP_t f^2 \log \frac{P_t f^2}{\mu f^2} - LP_t f^2 \, d\mu dt \\
&= \int_0^t \int \frac{\nabla P_t f^2}{P_t f^2} \, d\mu dt \leq C_5 \mathcal{E}[f],
\end{align*}
\]
where \( C_5 = \frac{e^{2K_0} - 1}{2K} \). Recall the last inequality in Step 2, it follows
\[
\eta \text{Ent}(f^2) \leq [1 + \eta(C_1 + C_4 + C_5)] \mathcal{E}[f] + \eta C_2\mu f^2 + \eta(A + C_3)\mu(f - \mu f)^2.
\]
Denote by \( \lambda_\mu \) the spectral gap, using the Rothaus’s lemma in [13] yields
\[
\eta \text{Ent}(f^2) \leq C \mathcal{E}[f]
\]
with \( C = 1 + \eta(C_1 + C_4 + C_5) + \eta\lambda_\mu^{-1}(2 + A + C_2 + C_3) \).

\[\square\]

3. PROOF OF THEOREM 1.3

The construction of Lyapunov function comes from solving certain Schrödinger equation. For convenience, suppose a LSI holds as
\[2\rho \text{Ent}(f^2) \leq \mathcal{E}[f].\]
According to the Herbst’s argument in Aida-Masuda-Shigekawa [1], a LSI implies the Gaussian integrability with some \( c > 0 \) and \( x_0 \in E \)
\[
\mu e^{cd^2(x, x_0)} < \infty,
\]
see also Djellout-Guillin-Wu [9] for a characterization of Gaussian integrability via the \( L^1 \) transportation-cost inequality \( W_1 H \), which can be derived from LSI too.
Denote \( \phi(x) = \rho \left( -cd^2(x, x_0) + b \right) \) with \( b = 2\mu e^{cd^2(x, x_0)} \). We introduce
\[
(3.1) \quad Hu := -Lu + \phi u = f, \quad f \in L^2(\mu)
\]
to prove Theorem 1.3.

**Proof.** The strategy contains two steps.

**Step 1.** Definition (3.1) gives \( \mu(u \cdot Hu) \leq \mathcal{E}[u] + \rho b\mu u^2 \) quickly. On the other hand, using the Young’s inequality and LSI yields
\[
\begin{align*}
\mu(u \cdot Hu) &= \mathcal{E}[u] + \rho b\mu u^2 - \rho \mu (cd^2(x, x_0)u^2) \\
&\geq \mathcal{E}[u] + \rho b\mu u^2 - \rho \cdot \mu u^2, \\
&= \mathcal{E}[u] + \rho b\mu u^2 - \frac{b}{2}\rho \mu u^2 + \rho \mu u^2 - \rho \text{Ent}(u^2) \geq \frac{1}{2} (\mathcal{E}[u] + \rho \mu u^2).
\end{align*}
\]
Then $\mu(u \cdot H u)$ determines a coercive Dirichlet form, and $H$ is a positive definite self-adjoint Schrödinger operator with its spectrum contained in $(0, \infty)$. It means $H^{-1}$ exists on $L^2(\mu)$ according to the Lax-Milgram Theorem, i.e. $u = H^{-1}f \in H^1(\mu)$ is a weak solution of Equation (3.1).

Whenever $f \geq 0$, the weak maximum principle yields $u = H^{-1}f \geq 0$ $\mu$-a.e. too. As a routine, we set $u_- = -\min\{u, 0\}$, which is weakly derivative and satisfies

$$\mu(u_- f) = \mu(u_- \cdot H u) = -\mathcal{E}[u_-] - \mu(\phi u_-^2) \leq -\frac{1}{2} \left\{ \mathcal{E}[u_-] + \rho \mu(\phi u_-^2) \right\} \leq 0.$$ 

It follows $u_- = 0$ $\mu$-a.e. and thus $u \geq 0$ $\mu$-a.e.

Now taking $f \equiv 1$ gives $u > 0$ $\mu$-a.e., otherwise there exists $A \subset E$ with $\mu(A) > 0$ and $u \equiv 0$ on $A$, which yields a contradiction as $Hu = 0$ $\mu$-a.e. on $A$ due to (10) Lemma 7.7. Moreover, $u$ is locally Hölder continuous by (10) Theorem 8.22 once we set $f^1 = 0$, $g = -f$ and $L = L - \phi$ such that $Lu = g$ according to the notation therein. Hence $u \in H^1(\mu) \cap C(E)$.

**Step 2.** Assume $u(y) = 0$. With some abuse of notation, denote by $\| \cdot \|_{L^2(B_{2R}(y))}$ the $L^2$-norm on $B_{2R}(y)$ with respect to $dx$ (locally comparable to $d\mu$). Applying (10) Theorem 8.17 to $u$ (as a sub-solution) yields for any small $R$

$$\sup_{B_R(y)} u \leq C \left( R^{-n/2} \| u \|_{L^2(B_{2R}(y))} + R^2 \| f \|_{L^\infty} \right).$$

On the other hand, applying (10) Theorem 8.18 to $u$ (as a super-solution) yields

$$R^{-n/2} \| u \|_{L^2(B_{2R}(y))} \leq C \left( \inf_{B_R(y)} u + R^2 \| f \|_{L^\infty} \right).$$

Combining the above estimates gives

$$\sup_{B_R(y)} u \leq CR^2,$$

which means $u$ is derivative at $y$ with $\nabla u = 0$ and $\partial_t u$ is locally Lipschitz at $y$.

Given $R_0$ and small $\varepsilon$, choose $V_\varepsilon \in \mathcal{C}^2(E)$ such that $V_\varepsilon$ is smooth in $B_{R_0}(y)$, $V_\varepsilon = V$ in $B_{2R_0}(y)$ and $\sup_{x \in E} |\nabla (V_\varepsilon - V)(x)| \leq \varepsilon$. Since for any $v, w \in \mathcal{D}(\mathcal{E})$

$$\mu(v \cdot \partial (V_\varepsilon - V) \cdot \nabla w) \leq \varepsilon \mu(v) |\nabla w| \leq \varepsilon (\mathcal{E}[v] + \mu v^2)^{1/2} (\mathcal{E}[w] + \mu w^2)^{1/2},$$

we can define a second-order elliptic operator $H_\varepsilon := -\Delta + \nabla V_\varepsilon \cdot \nabla + \phi = H + (V_\varepsilon - V) \cdot \nabla$ such that $\mu(v \cdot H_\varepsilon u)$ is a coercive bilinear form, and the spectrum of $H_\varepsilon$ has strictly positive real part. Repeating step 1 yields a nonnegative solution $u_\varepsilon$ on $E$ solving

$$H_\varepsilon u_\varepsilon := -\Delta u_\varepsilon + \nabla V_\varepsilon \cdot \nabla u_\varepsilon + \phi u_\varepsilon = \frac{1}{2}. \quad (3.2)$$

Due to the regularity theory, see (10) Corollary 8.11], $u_\varepsilon$ is smooth in $B_{R_0}(y)$. Hence $u_\varepsilon > 0$ in $B_{R_0}(y)$, otherwise it achieves local minimum at some $z \in B_{R_0}(y)$ such that $\nabla u_\varepsilon(z) = 0$ and $-\Delta u_\varepsilon(z) \leq 0$, which is absurd.

Consider the difference of equation (3.1) with (3.2), we have

$$H_\varepsilon (u - u_\varepsilon) = \frac{1}{2} - \nabla (V - V_\varepsilon) \cdot \nabla u \geq 0,$$

which implies $u \geq u_\varepsilon$ on $E$ and then $u(y) > 0$ makes a contradiction.

As consequence, we obtain $u > 0$ everywhere together with $Lu \leq \phi u$. \qed
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YUAN LIU, INSTITUTE OF APPLIED MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

E-mail address: liuyuan@amss.ac.cn