Symplectic Dirac-Kähler Fields

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Abstract

For the description of space-time fermions, Dirac-Kähler fields (inhomogeneous differential forms) provide an interesting alternative to the Dirac spinor fields. In this paper we develop a similar concept within the symplectic geometry of phase-spaces. Rather than on space-time, symplectic Dirac-Kähler fields can be defined on the classical phase-space of any Hamiltonian system. They are equivalent to an infinite family of metaplectic spinor fields, i.e. spinors of Sp(2N), in the same way an ordinary Dirac-Kähler field is equivalent to a (finite) multiplet of Dirac spinors. The results are interpreted in the framework of the gauge theory formulation of quantum mechanics which was proposed recently. An intriguing analogy is found between the lattice fermion problem (species doubling) and the problem of quantization in general.
1 Introduction

In a classic paper [1] E. Kähler proposed a description of fermions in terms of inhomogeneous differential forms. Rather than by spinor fields, the fermions are represented by a set of antisymmetric tensors in this approach. The role of the Dirac equation is taken over by the so-called Dirac-Kähler equation which involves only tensor manipulations. It imitates the $\gamma$ - matrix algebra with the help of the Clifford product for forms.

At first sight it seems puzzling how a family of tensor fields carrying integer spin can describe a particle of half-integer spin. This paradox is resolved if one notes that (in 4 space-time dimensions) a single Dirac-Kähler field actually corresponds to a multiplet of 4 ordinary Dirac spinors which mix under Lorentz transformations in a nontrivial way ("flavor mixing").

The Dirac-Kähler fermions have attracted a lot of attention both from the physics [2] - [9] and the mathematics [10, 11] point of view. In particular they made their appearance in lattice field theory [12]. It is a well known problem that a straightforward lattice discretization of the ordinary Dirac action does not describe one but rather 16 fermions in the continuum limit. The reason for this replication of fermionic states (usually referred to as the species “doubling” problem) is that the lattice propagator in momentum space has poles at all 16 corners of the Brillouin zone. The Kogut-Susskind [13] or staggered lattice fermions were proposed as an attempt to solve this problem. They are based on a more sophisticated lattice action which reduces the number of fermion species from 16 to 4. Later on it turned out [2, 3] that the Kogut-Susskind fermions are nothing but Dirac-Kähler fields discretized on a hypercubic lattice. As it deals with differential forms only, Dirac-Kähler theory on the lattice can take advantage of all the mathematical tools provided by the algebraic topology of cell complexes. In particular, by a standard procedure, the differential forms of the continuum formulation can be replaced by appropriate cochains.
on the lattice. These cochains are functions defined on the lattice points, links, plaquettes, cubes and hypercubes of the underlying lattice. In this manner it becomes obvious that the extra fermion species implied by the Kogut-Susskind lattice action and the fact that a Dirac-Kähler field contains 4 ordinary Dirac fermions have a common origin.

We only mention that the species doublers on the lattice can be avoided completely by using Wilson fermions or the nonlocal "SLAC derivative" [12], for instance. Alternatively one can regard the 4 Dirac fermions contained in one Kogut-Susskind field as 4 different physical "flavors". As we are interested in Dirac-Kähler fermions here we shall adopt this latter point of view in the following.

Dirac-Kähler (DK) fields can be defined on any Riemannian manifold \((\mathcal{M}_n, g)\), i.e. on any smooth \(n\)-dimensional manifold equipped with a metric \(g\). From the physics point of view this manifold represents \textit{space-time}.

The main purpose of the present paper is to propose an analog of the DK-fields which "live" on symplectic rather than Riemannian manifolds. This means that we are going to study DK-fields not over space-time but rather over a \textit{phase-space}.

A symplectic manifold \((\mathcal{M}_{2N}, \omega)\) is a smooth \(2N\)-dimensional manifold which is endowed with a closed, nondegenerate 2-form \(\omega = \frac{1}{2} \omega_{ab} d\phi^a \wedge d\phi^b\). (The \(\phi^a, a = 1, ..., 2N\), are local coordinates on \(\mathcal{M}_{2N}\).) This manifold should be thought of as the phase-space of a Hamiltonian system with \(N\) degrees of freedom. The corresponding Poisson bracket is given by \(\{\phi^a, \phi^b\} = \omega^{ab}\) where the matrix \((\omega^{ab})\) is the inverse of \((\omega_{ab})\). Using local Darboux coordinates \(\phi^a \equiv (p^i, q^i), i = 1, ..., N\), this matrix is independent of \(\phi^a, \omega^{qp} = -\omega^{pq} = I,\) and the only nonvanishing brackets are \(\{q^i, p^j\} = \delta^{ij}\). If \(\phi^a\) and \(\tilde{\phi}^a\) are local coordinates belonging to two overlapping charts of an atlas covering \(\mathcal{M}_{2N}\) then, by the

\[\text{The pseudo-Riemannian case (Lorentzian space-times) can be dealt with in a completely analogous fashion.}\]
very definition of a symplectic manifold, the coordinate transformation \( \phi \to \tilde{\phi} \) is symplectic, i.e. the Jacobian matrix \( (\partial \tilde{\phi}^a / \partial \phi^b) \) is an element of \( \text{Sp}(2N) \) at every point of the overlap region. \( \text{Sp}(2N) \), the group of linear canonical transformations, plays the same role for phase-space which the Lorentz group plays for space-time. In particular, it is the structure group of the frame bundle over \( \mathcal{M}_{2N} \).

As for introducing DK-fields on symplectic manifolds the first question which we must answer is what kind of spinor field should be used in place of the ordinary Dirac spinors of relativistic field theory. The only natural choice here is to employ the so-called metaplectic spinors \[14\], i.e. the spinors of the metaplectic group \( \text{Mp}(2N) \). Basically \( \text{Mp}(2N) \) is related to \( \text{Sp}(2N) \) in the same way \( \text{Spin}(n) \) is related to \( \text{SO}(n) \). In particular, there exists a two-to-one homomorphism between the two groups, i.e. \( \text{Mp}(2N) \) covers \( \text{Sp}(2N) \) twice. The construction of metaplectic spin bundles and spinor fields over a symplectic manifold proceeds almost literally along the same lines as in the case of space-time spinors, the main difference being that it is \( \text{Mp}(2N) \) now which serves as the structure group. For a detailed exposition we must refer to the literature \[14, 15\].

Metaplectic spinors have been used in many different contexts including geometric quantization \[15\], semi-classical approximations \[16\], Parisi-Sourlas super-symmetry \[17\], string theory \[18, 19\], and anyon super-conductivity \[20\]. Most recently they played an important role in an approach to quantization \[21\] which is based upon a Yang-Mills theory on phase-space with metaplectic “matter” fields. This new formulation of quantum mechanics is one of the main motivations for the present work. We shall come back to it later on.

Let us briefly describe how one can construct representations of \( \text{Mp}(2N) \) \[22\]. One has to associate an operator \( M(S) \) to every matrix \( S \equiv (S^a_b) \in \text{Sp}(2N) \) in such a way that \( M(S_1)M(S_2) = \pm M(S_1S_2) \). These operators can be built up from a kind of “\( \gamma \)-matrices” which constitute a symplectic Clifford
algebra:

\[ \gamma^a \gamma^b - \gamma^b \gamma^a = 2 i \omega^{ab} \]  

(1.1)

We require \( M(S) \) to satisfy the usual compatibility condition between the vector and the spinor representation:

\[ M(S)^{-1} \gamma^a M(S) = S^a_b \gamma^b \]  

(1.2)

Every infinitesimal \( \text{Sp}(2N) \)-transformation is of the form \( S^a_b = \delta^a_b + \omega^{ac} \kappa_{cb} \) with symmetric coefficients \( \kappa_{ab} \). Inserting this together with the ansatz \( M(S) = 1 - \frac{i}{2} \kappa_{ab} \Sigma_{\text{meta}}^{ab} \) into the compatibility condition it is easy to show that the latter is solved by

\[ \Sigma_{\text{meta}}^{ab} = \frac{1}{4} (\gamma^a \gamma^b + \gamma^b \gamma^a) \]  

(1.3)

and that these generators satisfy the \( \text{Sp}(2N) \) commutator relations [22]. Thus every representation of the symplectic Clifford algebra gives rise to a representation of \( \text{Mp}(2N) \).

The most obvious difference between the metaplectic and the space-time spinors is that the symplectic Clifford algebra involves a commutator rather than an anticommutator. As an immediate consequence, this algebra has no finite dimensional matrix representations, and metaplectic spinors are necessarily infinite component objects. What is meant by a “metaplectic representation” is a representation in which \( \gamma^a \) is a hermitian operator on an infinite dimensional Hilbert space \( \mathcal{V} \). Hence the operators \( M \) obtained by exponentiating the generators (1.3) give rise to a unitary representation. (See [22, 23], for further details.)

The symplectic Clifford algebra (1.1) admits a rather intriguing reinterpretation which is also at the heart of the new approach to quantization [21] mentioned above. Assume we are given a quantum mechanical system with a Hilbert space \( \mathcal{V} \) along with \( N \) position and momentum operators \( \hat{x}^i \) and \( \hat{p}^i \)
acting on it. They satisfy the canonical commutator relations \([\hat{x}^i, \hat{\pi}^j] = i\hbar \delta^{ij}\).

By virtue of the identification \(\gamma^i = \kappa \hat{\pi}^i, \quad \gamma^{N+i} = \kappa \hat{x}^i\) for \(i = 1, ..., N\) and with the constant \(\kappa \equiv \sqrt{2/\hbar}\) it is obvious that the “symplectic Clifford algebra” (1.1) is actually nothing but the canonical commutation relations for the \(\hat{x}-\hat{\pi}\)-auxiliary quantum system. We call it an “auxiliary” system because it should not be confused with the actual physical system under consideration, the one whose (curved) phase-space is \(\mathcal{M}_{2N}\). (The classical phase-space pertaining to the auxiliary system is simply \(\mathbb{R}^{2N}\) equipped with the standard symplectic structure.)

The metaplectic spin bundles are bundles over \(\mathcal{M}_{2N}\) with the typical fiber \(V\) and the structure group \(\text{Mp}(2N)\) [14]. At each point \(\phi\) of \(\mathcal{M}_{2N}\) a local copy of \(V\), denoted \(V_\phi\), is attached. Metaplectic spinor fields are sections through these bundles. Locally they are simply functions which assume values in \(V\):

\[
\psi : \mathcal{M}_{2N} \to V, \quad \phi \mapsto |\psi\rangle_\phi \in V_\phi \tag{1.4}
\]

The notation \(|\psi\rangle_\phi\) means that the spinor \(|\psi\rangle \in V\), “lives” in the local Hilbert space at \(\phi\). Upon introducing a basis \(\{|\alpha\rangle\}\) in \(V\) we write \(\psi^\alpha(\phi) \equiv \langle \alpha | \psi \rangle_\phi\) for its components. Here \(\alpha\) is an infinite dimensional generalization of a spinor index. If we take \(\{|\alpha\rangle\}\) to be the \(\hat{x}\)-eigenbasis, for instance, then \(\alpha \equiv (\alpha^1, ..., \alpha^N) \in \mathbb{R}^N\). (See [22, 23] for details.)

In the present paper we shall focus on the local aspects of the bundles involved. We only mention that on certain manifolds there are topological obstructions which prevent them from carrying globally well defined metaplectic spinor fields [14]. In ref. [23] we characterized these obstructions using methods from quantum field theory.

Let us come back to the main question which we are trying to answer in this paper: Do there exist “symplectic Dirac-Kähler fields” which are related to the metaplectic spinors in the same way the ordinary Dirac-Kähler fields are related to Dirac spinors?
Apart from being interesting in its own right, this question is of obvious physical relevance. The fascinating property of metaplectic spinor fields is that, on a purely group theoretical basis, they introduce aspects of quantum mechanics into the geometry of classical phase-spaces. By pure representation theory one is led to the auxiliary quantum system in the local Hilbert spaces $V_\phi$. In ref. [21] we explained in detail how these auxiliary systems relate to the actual physical quantum system with the classical phase-space $M_{2N}$. Using this as our starting point, we showed that it is possible to replace conventional canonical quantization by two new rules with a more transparent physical and geometrical meaning.

Classical mechanics and classical statistical mechanics are geometric theories which are conveniently described in the language of symplectic geometry. Only tensor fields are needed to formulate them. Quantum mechanics, on the other hand, has a natural interpretation in terms of spinor fields on phase-space. Thus, in a sense, the very process of quantization is tantamount to a transition from tensors to spinors. But this is precisely what Dirac-Kähler theory is about: its basic fields are tensors which, however, are equivalent to a multiplet of spinors.

Before embarking on the detailed constructions let us briefly outline the strategy for finding the “symplectic DK-fields” which we shall follow in this paper.

Our main tools are two types of auxiliary quantum systems with Hilbert spaces $V$ and $V^F$, respectively. We mentioned already the (bosonic) $\hat{x}$-$\hat{\pi}$-system on $V$ whose canonical operators realize the metaplectic $\gamma$-matrices $\gamma^a$. We also need a similar fermionic system with a (finite-dimensional) Hilbert space $V^F$ and a set of operators $\hat{\chi}^\mu, \mu = 1, \ldots, n$, satisfying the canonical anticommutator relations $\hat{\chi}^\mu \hat{\chi}^\nu + \hat{\chi}^\nu \hat{\chi}^\mu = i\hbar \delta^{\mu\nu}$. The $SO(n)$-Dirac matrices $\gamma^\mu$ are treated as a special realization of this algebra.

An important technical ingredient is the Weyl symbol calculus [24]-[28]. Let
\( L(V) \) and \( L(V^F) \) be the spaces of linear operators on \( V \) and \( V^F \), respectively. It is possible to uniquely characterize every operator \( \hat{b} \in L(V) \) and \( \hat{f} \in L(V^F) \) in terms of classical phase-functions (symbols) \( b(y) \) and \( f(\theta) \). Here \( y \) and \( \theta \) are coordinates on the (flat) classical phase-spaces which belong to the auxiliary systems. In the bosonic case, \( y \equiv (y^a) \in \mathbb{R}^{2N} \) is a vector with commuting entries, while \( \theta \equiv (\theta^\mu) \) is a set of \( n \) anti-commuting Grassmann numbers. The space of all bosonic (fermionic) symbol functions, equipped with certain algebraic structures, is referred to as the bosonic (fermionic) Weyl algebra \( \mathcal{W}(\mathcal{W}^F) \).

Given a space-time manifold \( \mathcal{M}_n \), we consider fields on this manifold which assume values in \( V^F, L(V^F) \) and \( \mathcal{W}^F \), respectively. In an obvious notation, we denote them \( \psi^\alpha(x), \hat{F}(x), \) and \( F(x,\theta) \).

Similarly, given a phase-space manifold \( \mathcal{M}_{2N} \), we define fields \( \psi^\alpha(\phi), \hat{B}(\phi), \) and \( B(\phi,y) \) which assume values in \( V, L(V) \) and \( \mathcal{W} \), respectively.

In the first part of this paper we shall reformulate standard Dirac-Kähler theory in terms of the fermionic Weyl symbol calculus. We shall see that \( \psi^\alpha(x) \) is an ordinary Dirac spinor and that \( F(x,\theta) \) can be identified with a DK-field. The Grassmann variables \( \theta^\mu \) will play the role of the basis differentials \( dx^\mu \).

This first part of the investigation is quite interesting in its own right. For instance, we shall discover that the Clifford product which is at the heart of DK-theory is basically the same thing as the star product of the fermionic Weyl symbol calculus. As a consequence, \( \mathcal{W}^F \) turns out to be an Atiyah-Kähler algebra [10, 11].

In the second part of this paper we investigate in detail what happens to the standard DK-theory, reformulated in terms of fermionic Weyl symbols, when we replace fermionic symbols by bosonic ones everywhere. This means that we switch from the \( \hat{\chi}^\mu \)- to the \( \hat{x}-\hat{\pi} \)-system. Then \( \psi^\alpha(\phi) \) is a metaplectic spinor field, and by analogy with the fermionic setting we shall argue that \( B(\phi,y) \) is the “symplectic DK field” which we are looking for. Schematically
our approach can be summarized as follows:

\[
\text{DK fields} \quad \iff \quad \text{fermionic symbols} \\
\downarrow \\
\text{symplectic DK fields} \quad \iff \quad \text{bosonic symbols} \quad (1.5)
\]

The rest of this paper is organized as follows: In the second half of this introduction we discuss some aspects of standard DK-theory which will be important later on. Then, in Section 2, we reformulate this theory in terms of fermionic Weyl symbols. Particular attention is paid to the decomposition of DK-fields as a set of Dirac spinors. The construction of the symplectic DK-fields is performed in Section 3. We investigate in detail which properties of \(SO(n)\) DK-fields can be translated to the \(Sp(2N)\)-case and which cannot. Section 4 contains a summary and various remarks on the quantization problem in the light of the present work. Some material needed as a background for Section 2 is relegated to the appendix.

As for its mathematical rigor, the style of this paper is informal. Occasionally the language of fiber bundles is used as a convenient tool but we are mostly interested in the local properties of the bundles involved and no pretense is made as for a rigorous and complete discussion of the global aspects.

—— DK fields on space-time ——

Let us start with an arbitrary (curved) \(n\)-dimensional Riemannian manifold \((\mathcal{M}_n, g)\). Upon introducing local coordinates \(x^\mu\), the tangent space \(T_x\mathcal{M}_n\) and the cotangent space \(T^*_x\mathcal{M}_n\) at the point \(x\) of \(\mathcal{M}_n\) are spanned by the basis vectors \(\partial_\mu \equiv \partial/\partial x^\mu\) and \(dx^\mu, \mu = 1, \ldots, n\), respectively. These spaces constitute the fibers of the (co-)tangent bundle over \(\mathcal{M}_n\). Replacing \(T^*\mathcal{M}_n\) by its \(p\)-fold tensor power we obtain the bundle of (covariant) tensors of rank \(p\). Restricting ourselves to completely antisymmetric tensors we are led to the exterior
algebra \( \bigwedge (T^*_x \mathcal{M}_n) = \bigoplus_{p=0}^n \bigwedge^p (T^*_x \mathcal{M}_n) \). Its elements are the inhomogeneous differential forms

\[
\Phi(x) = \sum_{p=0}^{n} \Phi^{(p)}(x), \quad \Phi^{(p)}(x) \in \bigwedge^p (T^*_x \mathcal{M}_n),
\]

\[
\Phi^{(p)}(x) = \frac{1}{p!} F^{(p)}_{\mu_1 \cdots \mu_p}(x) dx^\mu_1 \wedge \cdots \wedge dx^\mu_p
\]

(1.6)

where \( F^{(p)}_{\mu_1 \cdots \mu_p} \) are completely antisymmetric coefficients. The corresponding algebra multiplication is the wedge product “\( \wedge \)”.

Since we have a metric \( g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu \) at our disposal which gives rise to an analogous bilinear form \( g' = g'^{\mu\nu}(x) \partial_\mu \otimes \partial_\nu \) for the cotangent bundle we can promote the fibers \( \bigwedge (T^*_x \mathcal{M}_n) \) of the exterior algebra bundle to an Atiyah-Kähler algebra \( AK(T^*_x \mathcal{M}_n, g') \) \([1, 10, 11]\).

Quite generally, the Atiyah-Kähler algebra \( AK(V, Q) \) corresponding to an arbitrary vector space \( V \) equipped with a quadratic form \( Q \) consists of the elements of the exterior algebra over \( V \), \( \bigwedge (V) = \bigoplus_p \bigwedge^p (V) \), for which the following three products are defined:

- the exterior product “\( \wedge \)”
- the inner product \( \langle \cdot, \cdot \rangle \) induced by \( Q \)
- the Clifford product “\( \vee \)”

The three products are required to be distributive with respect to the addition and to satisfy the relation

\[
a \vee b = a \wedge b + (a, b)
\]

(1.7)

for all \( a, b \in \bigwedge^1 (V) \). The Clifford product is associative by definition. Hence the basic rule (1.7) is sufficient in order to work out the \( \vee \)-product of two arbitrary elements in \( \bigwedge (V) \). Below we shall give a closed formula for this product.
The Atiyah-Kähler algebra combines the notions of an exterior algebra, a Grassmann algebra and a Clifford algebra in an consistent manner. If we omit the Clifford product it reduces to the Grassmann algebra $\bigwedge(V,Q)$, while omitting both $\vee$ and $(\cdot,\cdot)$ yields the exterior algebra $\bigwedge(V)$. Without the structure of the $\wedge$-product it becomes a Clifford algebra because (1.7) entails $a \vee b + b \vee a = 2(a,b)$.\n
In the case at hand, $V = \bigwedge(T^*_x \mathcal{M}_n)$ and $Q = g'$. This means that for two basis 1-forms the inner product is given by $(dx^\mu, dx^\nu) = g'(dx^\mu, dx^\nu) = g^\mu\nu$ and similarly for higher forms; for instance, $(dx^\mu \wedge dx^\nu, dx^\rho \wedge dx^\sigma) = g^\mu\rho g^{\nu\sigma} - g^\mu\sigma g^{\nu\rho}$.

A bundle over $\mathcal{M}_n$ with typical fiber $\text{AK}(T^*_x \mathcal{M}_n, g')$ is called an Atiyah-Kähler bundle and sections through such bundles are referred to as Dirac-Kähler fields. Locally they are described by a collection of antisymmetric tensor fields $\{ F^{(p)}_{\mu_1 \ldots \mu_p}, p = 0, \ldots n \}$. The three products defined in the fiber give rise to analogous products on the space of sections, for instance $(\Phi_1 \vee \Phi_2)(x) \equiv \Phi_1(x) \vee \Phi_2(x)$. Of course, also all the other operations of the conventional exterior calculus can be applied to Dirac-Kähler fields: the exterior derivative $d$, the coderivative $d^\dagger$, or the contraction with a vector field $v$, $i(v)$, to mention just a few.

In our case the relations defining the Clifford product assume the following form when expressed in terms of the generating elements:

\begin{align*}
1 \vee 1 &= 1, \\
1 \vee dx^\mu &= dx^\mu \vee 1 = dx^\mu \\
dx^\mu \vee dx^\nu &= dx^\mu \wedge dx^\nu + g^{\mu\nu} \quad (1.8)
\end{align*}

By virtue of the postulated associativity of the $\vee$-product, these relations are sufficient in order to determine the Clifford product of two arbitrary differential forms. One finds $1 \vee 1 = 1$, $1 \vee dx^\mu = dx^\mu \vee 1 = dx^\mu$, $dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu}$.

\begin{equation}
\Phi_1 \vee \Phi_2 = \sum_{p=0}^{n} \frac{(-1)^{p(p-1)/2}}{p!}(\mathcal{A}^{\nu} e_{\mu_1} \cdots e_{\mu_p} \Phi_1) \wedge (e^{\mu_1} \cdots e^{\mu_p} \Phi_2) \quad (1.9)
\end{equation}

with $e_{\mu} \equiv i(\partial_\mu)$, $e^{\mu} \equiv g^{\mu\nu} i(\partial_\nu)$ where $i(\partial_\mu)$ denotes the contraction with
the basis vector \( \partial_\mu \). It is an anti-derivation with the properties

\[
\begin{align*}
  i(\partial_\mu)1 &= 0, \\
i(\partial_\mu)dx^\nu &= \delta_\mu^\nu \\
i(\partial_\mu)(\Phi_1 \wedge \Phi_2) &= (i(\partial_\mu)\Phi_1) \wedge \Phi_2 + (\mathcal{A}\Phi_1) \wedge i(\partial_\mu)\Phi_2
\end{align*}
\]

In writing down eqs. (1.9) and (1.10) we used the “main automorphism” \( \mathcal{A} \), a linear map whose action on the DK-field (1.6) is defined as

\[
\mathcal{A}\Phi = \sum_{p=0}^{n} (-1)^p \Phi^{(p)}
\] (1.10)

Later on we shall also need the “main antiautomorphism” \( \mathcal{B} \) which acts according to

\[
\mathcal{B}\Phi = \sum_{p=0}^{n} (-1)^p (p-1)/2 \Phi^{(p)}
\] (1.11)

Obviously, \( \mathcal{A}^2 = \mathcal{B}^2 = 1 \), \( \mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A} \), and also

\[
\begin{align*}
\mathcal{A}(\Phi_1 \wedge \Phi_2) &= (\mathcal{A}\Phi_1) \wedge (\mathcal{A}\Phi_2) \\
\mathcal{B}(\Phi_1 \wedge \Phi_2) &= (\mathcal{B}\Phi_2) \wedge (\mathcal{B}\Phi_1)
\end{align*}
\] (1.12)

for any pair of DK-fields.

As an important special case of (1.9) we note for later use that

\[
dx^\mu \vee \Phi = dx^\mu \wedge \Phi + e^\mu \sigma_\Phi
\] (1.13)

Let us look at the physical interpretation of the DK-fields now. From now on we shall specialize the discussion to a flat space-time \( \mathcal{M}_n = \mathbb{R}^n \) with the metric \( g_{\mu\nu} = \delta_{\mu\nu} \). The generalization of a curved manifold and/or a manifold with Lorentzian signature would be straightforward, but we shall avoid these technical complications here since they are not important for the point we would like to make.

The interpretation of a DK-field as a multiplet of Dirac spinors is based upon the following two logically independent observations.
(i) From (1.8) we obtain for the antisymmetrized Clifford product of two basis differentials

\[ dx^\mu \lor dx^\nu + dx^\nu \lor dx^\mu = 2\delta^{\mu\nu} \] (1.14)

This relation should be compared to the one satisfied by the euclidean Dirac matrices \( \gamma^\mu \):

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu} \] (1.15)

We conclude that the Clifford left multiplication with \( dx^\mu \) defines a representation of the algebra of \( \gamma \)-matrices in the space of (complex) inhomogeneous differential forms: \( \gamma^\mu \equiv dx^\mu \lor \). This representation is reducible though. Assuming \( n \) even from now on, a Dirac spinor has \( 2^{n/2} \) complex components, and an irreducible representation of the algebra (1.15) is in terms of \( 2^{n/2} \times 2^{n/2} \) matrices. On the other hand, the dimension of the exterior algebra is \( 2^n \), i.e. a DK-field \( \Phi \) has \( 2^n \) independent complex component fields. We shall see in a moment that the space \( \mathcal{K} \) of all DK-fields \( \Phi \) can be decomposed into \( 2^{n/2} \times 2^{n/2} \) subspaces \( \mathcal{K}^{(a)} \) which are invariant under Clifford left multiplication, \( \mathcal{K} = \bigoplus_{a=1}^{k} \mathcal{K}^{(a)}, k \equiv 2^{n/2} \). On \( \mathcal{K}^{(a)} \), \( dx^\mu \lor \) gives rise to an irreducible representation of the algebra (1.15).

(ii) From the exterior derivative \( d \) and its adjoint, the coderivative \( d^\dagger \), we can form the so-called Dirac-Kähler operator \( d - d^\dagger \) which has the property that it squares to the Laplacian:

\[ (d - d^\dagger)^2 = -(dd^\dagger + d^\dagger d) = \partial_\mu \partial^\mu \] (1.16)

It shares this property with the Dirac operator \( \gamma^\mu \partial_\mu \) and hence some relationship among the two might be expected. In fact, it turns out that the Dirac-Kähler operator can be expressed in terms of a Clifford multiplication from the left:

\[ (d - d^\dagger)\Phi(x) = dx^\mu \lor \partial_\mu \Phi(x) \] (1.17)
Since we know already that $dx^\mu \vee$ corresponds to a $\gamma$-matrix and leaves the spaces $K^{(\alpha)}$ invariant, we see that the Dirac-Kähler equation

$$(d - d^\dagger + m)\Phi = 0 \quad (1.18)$$

decomposes to a set of equations $(d - d^\dagger + m)\Phi^{(\alpha)} = 0$, $\Phi^{(\alpha)} \in K^{(\alpha)}$, each of which is equivalent to an ordinary Dirac equation $(\gamma^\mu \partial_\mu + m)\psi = 0$.

Following Becher and Joos [3] we can construct the invariant subspaces $K^{(\alpha)}$ as follows. We introduce a new basis $\{Z_{\alpha\beta}\}$ in $K$ whose elements are labeled by a pair of indices $\alpha, \beta = 1, \cdots, 2^{n/2}$ and which are required to satisfy

$$dx^\mu \vee Z_{\alpha\beta} = \sum_{\gamma=1}^{2^{n/2}} (\gamma^\mu T)_{\alpha\gamma} Z_{\gamma\beta} \quad (1.19)$$

where the euclidean Dirac matrices $\gamma^\mu$ are in the irreducible $2^{n/2}$-dimensional representation. They satisfy (1.15) and are assumed to be hermitian, $\gamma_\mu = \gamma_\mu^\dagger$. Frequently we shall regard $Z \equiv (Z_{\alpha\beta})$ as a matrix or, more precisely, as an inhomogeneous differential form which assumes values in the space of spinor matrices. Then (1.19) reads

$$dx^\mu \vee Z = \gamma^\mu T Z \quad (1.20)$$

This equation is satisfied by

$$Z = \sum_{p=0}^{n} \frac{1}{p!} \gamma^T_{\mu_1} \cdots \gamma^T_{\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \quad (1.21)$$

Every DK-field $\Phi$ can be expanded in the basis $\{Z_{\alpha\beta}\}$:

$$\Phi(x) = \sum_{\alpha,\beta} \psi^{(\beta)}(x) Z_{\alpha\beta} \quad (1.22)$$

\[\text{\footnote{We use the notation } \mu, \nu, \cdots = 1, \cdots, n \text{ for Lorentz indices and } \alpha, \beta, \gamma, \cdots = 1, \cdots, 2^{n/2} \text{ for spinor indices.}]}\]
Hence it follows immediately from (1.19) that the invariant subspaces $K^{(\alpha)}$ are spanned by

$$\Phi^{(\beta)} = \sum_{\alpha} \psi^{(\beta)}_\alpha Z_{\alpha\beta} \in K^{(\beta)}, \quad \beta \text{ fixed.} \quad (1.23)$$

In fact, one has

$$dx^\mu \vee \Phi^{(\beta)} = \sum_{\alpha} \left( \sum_{\delta} \gamma^\mu_{\alpha\delta} \psi^{(\beta)}_\delta \right) Z_{\alpha\beta} \quad (1.24)$$

which shows that on $K^{(\beta)}$ Clifford left-multiplication with $dx^\mu$ is equivalent to acting with the Dirac matrix $\gamma^\mu$ on the spinor $\psi^{(\beta)} \equiv \{\psi^{(\beta)}_\alpha; \alpha = 1, \ldots, 2^{n/2}\}$. For every fixed value of $\beta$, $\psi^{(\beta)}$ is an ordinary $2^{n/2}$-component Dirac field. By virtue of the orthogonal decomposition $\Phi = \sum_\beta \Phi^{(\beta)}$, a DK-field describes a multiplet of $2^{n/2}$ Dirac fields.

It is convenient to combine the expansion coefficients $\psi^{(\beta)}_\alpha$ into a spinor matrix $\hat{\psi}$,

$$(\hat{\psi})_{\alpha\beta} \equiv \psi^{(\beta)}_\alpha, \quad (1.25)$$

so that (1.22) reads

$$\Phi(x) = \text{Tr} \left[ \hat{\psi}(x) Z^T \right] \quad (1.26)$$

Writing $\hat{\psi}[\Phi]$ for the matrix related to a given DK-field $\Phi$, eq.(1.24) amounts to

$$\hat{\psi}[dx^\mu \vee \Phi] = \gamma^\mu \hat{\psi}[\Phi] \quad (1.27)$$

Occasionally one finds a slightly different approach in the literature [2]. One assumes that the inhomogeneous form (1.6) is given and one uses its coefficient functions $F^{(p)}_{\mu_1 \cdots \mu_p}$ in order to construct a spinor matrix $\hat{F}$ by simply replacing $dx^\mu \rightarrow \gamma^\mu$ everywhere:

$$\hat{F} \equiv \hat{F}[\Phi] \equiv \sum_{p=0}^n \frac{1}{p!} F^{(p)}_{\mu_1 \cdots \mu_p} \gamma^{\mu_1} \cdots \gamma^{\mu_p} \quad (1.28)$$
Then one verifies that the map $\Phi \mapsto \widehat{F}[\Phi]$ satisfies
\[
\widehat{F}[dx^\mu \lor \Phi] = \gamma^\mu \widehat{F}[\Phi],
\] (1.29)
a property it has in common with $\widehat{\psi}$. Hence we might expect that these two matrix-valued fields are related. Indeed, it turns out that they coincide up to a constant factor. To see this, one inserts the expansions (1.6) and (1.21) into (1.26) and obtains the following formula for the coefficients of $\Phi$, $F_{\mu_1 \cdots \mu_p}$, as a function of $\widehat{\psi}$:
\[
F_{\mu_1 \cdots \mu_p}(x) = (-1)^{p(p-1)/2} \text{Tr} \left[ \widehat{\psi}(x) \gamma_{[\mu_1} \cdots \gamma_{\mu_p]} \right]
\] (1.30)

Because of the orthogonality and completeness relations enjoyed by the Dirac matrices, eq. (1.30) has a unique solution for $\widehat{\psi}$ as a function of the coefficients $F_{\mu_1 \cdots \mu_p}$ which define $\Phi$. One finds that $\widehat{\psi}$ and $\widehat{F}$ are essentially the same thing:
\[
\widehat{\psi}(x) = 2^{-n/2} \sum_{p=0}^{n} \frac{1}{p!} F_{\mu_1 \cdots \mu_p}(x) \gamma^{\mu_1} \cdots \gamma^{\mu_p} \quad (1.31)
\]
\[
= 2^{-n/2} \widehat{F}(x) \quad (1.32)
\]

This formula together with (1.25) gives us a practical tool to compute the projection of $\Phi$ on the invariant subspaces $\mathcal{K}^{(\alpha)}$.

In standard discussions of Dirac-Kähler theory, because of the simple proportionality of $\widehat{\psi}$ and $\widehat{F}$, there is no need for a conceptual distinction between the two matrices. In order to establish their equivalence only familiar identities involving $\gamma$-matrices such as
\[
\text{Tr} \left[ \gamma^{[\mu_1} \cdots \gamma^{\mu_p]} (\gamma_{[\nu_1} \cdots \gamma_{\nu_q]})^\dagger \right] = 2^{n/2} p! \delta^{pq} \delta^{[\mu_1} \cdots \delta^{\mu_p]} \quad (1.33)
\]
are needed. In the symplectic case, the situation will be more complicated and we have to distinguish more carefully $\widehat{\psi}$ which arises from the construction of left-invariant subspaces and $\widehat{F}$ which obtains by replacing $dx^\mu \rightarrow \gamma^\mu$ in $\Phi$. A priori it is not clear that the two objects can easily be related to each
other since the metaplectic $\gamma$-"matrices" are infinite dimensional. Hence the question whether there are trace identities analogous to (1.33) is a nontrivial issue.

2 Dirac-Kähler fields and fermionic Weyl symbols

In this section we describe the relation between the conventional Dirac-Kähler fermions and the Weyl symbol calculus. In subsection 2.1 we summarize various properties of the fermionic Weyl symbol calculus and discuss a number of special aspects and applications which will be relevant. In section 2.2 we show that the fermionic Weyl algebra $W^F$ is an Atiyah-Kähler algebra, and in section 2.3 we introduce $W^F$-valued fields over space-time. In 2.4 we demonstrate that they can be identified with Dirac-Kähler fields. They carry a reducible representation of the Clifford algebra. The decomposition of $W^F$ into invariant subspaces which carry an irreducible representation is performed in subsection 2.5.

2.1 The fermionic Weyl algebra

We consider a set of operators $\hat{\chi}^\mu, \mu = 1, \cdots, n$, which satisfy the canonical anticommutation relations

$$\hat{\chi}^\mu \hat{\chi}^{\nu} + \hat{\chi}^{\nu} \hat{\chi}^\mu = \hbar \delta^{\mu\nu} \quad (2.1)$$

We could think of the $\hat{\chi}$’s as world line fermions which represent the spin of a relativistic particle, for instance [17, 26]. The most general operator we can construct by forming linear combinations of products of $\hat{\chi}$’s has the structure

$$\hat{f} = \sum_{p=0}^{n} \frac{1}{p!} f^{(p)}_{\mu_1 \cdots \mu_p} \hat{\chi}^{\mu_1} \hat{\chi}^{\mu_2} \cdots \hat{\chi}^{\mu_p} \quad (2.2)$$
with arbitrary (complex-valued) constants \( f^{(p)}_{\mu_1 \cdots \mu_p} \).

We would like to establish a linear one-to-one correspondence between the operators (2.2) and functions \( f \) depending on Grassmann numbers \( \theta^1, \theta^2, \cdots, \theta^n \) with \( \theta^\mu \theta^\nu + \theta^\nu \theta^\mu = 0 \). The function \( f(\theta) \) which characterizes the operator \( \hat{f} \) is called the symbol of \( \hat{f} \): \( f = \text{symb}(\hat{f}) \). There are many “symbol maps” which relate operators to classical functions. Here we are interested in the Weyl symbol which is defined as follows. Given the operator (2.2) we define

\[
\begin{align*}
  f(\theta) &= \text{symb}(\hat{f})(\theta) = \sum_{p=0}^{n} \frac{1}{p!} f^{(p)}_{\mu_1 \cdots \mu_p} \theta^{\mu_1} \theta^{\mu_2} \cdots \theta^{\mu_p} \\
  &\quad (2.3)
\end{align*}
\]

which, for a given ordering, is a well defined map from operators to functions. In particular, \( \text{symb}(\hat{\chi}^\mu) = \theta^\mu \) and \( \text{symb}(I) = 1 \) where \( I \) is the unit operator. The inverse mapping is not well defined yet, because in (2.3) we can add to \( f^{(p)}_{\mu_1 \cdots \mu_p} \) arbitrary tensors which are symmetric in at least one index pair. This does not change \( f(\theta) \), but it does change \( \hat{f} \). Specifying a unique operator \( \hat{f} \) for a given \( f(\theta) \) amounts to picking a particular operator ordering prescription. In fact, \( f(\theta) \) can be regarded as a classical phase function of a mechanical system with Grassmann-odd phase-space coordinates \( \theta^\mu \), and the \( \hat{\chi}^\mu \)'s are the corresponding quantum operators. We shall employ the Weyl correspondence rule which means that \( \hat{f} \) follows from \( f(\theta) \) by substituting \( \theta^\mu \to \hat{\chi}^\mu \) in (2.3) and writing all operator products in Weyl ordered, i.e. completely antisymmetrized form. For instance, the product \( \theta^\mu \theta^\nu \) yields the operator \( [\hat{\chi}^\mu \hat{\chi}^\nu]_{\text{Weyl}} = \frac{1}{2}(\hat{\chi}^\mu \hat{\chi}^\nu - \hat{\chi}^\nu \hat{\chi}^\mu) \equiv \hat{\chi}^{[\mu} \hat{\chi}^{\nu]} \). For an arbitrary monomial,

\[
\text{symb}^{-1}(\theta^{\mu_1} \cdots \theta^{\mu_p}) = \hat{\chi}^{[\mu_1} \cdots \hat{\chi}^{\mu_p]} \quad (2.4)
\]

where the square brackets indicate complete antisymmetrization.

In the following we shall require the constants \( f^{(p)}_{\mu_1 \cdots \mu_p} \) appearing in the series expansion of the symbol \( f(\theta) \) to be completely antisymmetric tensors. Then the operator \( \hat{f} \) associated to the series (2.3) is obtained by simply replacing \( \theta^\mu \to \hat{\chi}^\mu \) in this series, and this leads us back to the operator (2.2).
If \( n \) is odd, the inverse symbol map is still not uniquely defined, because in this case the operator \( \hat{\chi}^1 \hat{\chi}^2 \cdots \hat{\chi}^n \) commutes with all operators and is proportional to the identity therefore. By multiplying any operator by \( \hat{\chi}^1 \hat{\chi}^2 \cdots \hat{\chi}^n \) if necessary one can represent all operators by even symbols. This prescription makes the correspondence between operators and symbols bijective. (See [26, 27] for further details.)

If a string of operators is not contracted with an antisymmetric tensor we must reorder it before we can use

\[
\text{symb}(\hat{\chi}^{[\mu_1} \cdots \hat{\chi}^{\mu_p]}) = \theta^{\mu_1} \cdots \theta^{\mu_p}
\]  

in order to read off its symbol. For instance,

\[
\text{symb}(\hat{\chi}^\mu \hat{\chi}^\nu) = \text{symb} \left[ \hat{\chi}^{[\mu} \hat{\chi}^{\nu]} + \frac{\hbar}{2} \delta^{\mu\nu} \right] = \theta^\mu \theta^\nu + \frac{\hbar}{2} \delta^{\mu\nu}
\]

The symbols \( f(\theta) \) are functions of the same type as those considered in Appendix A, to which the reader might turn at this point. Among other things, various linear operations on such functions are discussed there which are particularly useful in the context of the symbol calculus. This includes the “main automorphism” \( A \), the “main antiautomorphism” \( B \), the Hodge operator \( \star \) and the modified Hodge operator \( \diamond \).

While we allow for complex coefficients \( f^{(p)}_{\mu_1 \cdots \mu_p} \), we assume that the operators \( \hat{\chi}^\mu \) are hermitian, \( \hat{\chi}^\mu = (\hat{\chi}^\mu)^\dagger \), and that \( \theta^\mu \) is real, \( \bar{\theta}^\mu = \theta^\mu \). Hence it follows that

\[
\text{symb}(\hat{f}^\dagger) = \overline{\text{symb}(\hat{f})}
\]

where the overbar means complex conjugation.

There is a simple integral formula for the operator \( \hat{f} \) associated to a given Weyl symbol \( f(\theta) \):

\[
\hat{f} = \int \hat{\Omega}(\rho) \, \tilde{f}(\rho) \, d^n \rho
\]
Here $\tilde{f}(\rho)$ is the Fourier transform of $f(\theta)$ as defined in eq.(A.22) of the Appendix, and

$$\hat{\Omega}(\rho) \equiv \exp(-i \chi^\mu \rho_\mu)$$

(2.9)
is the fermionic analogue of the Weyl operators which implement translations on phase-space. Using the identities of appendix A one can verify that eqs. (2.4) and (2.8) are indeed equivalent.

An important concept is the “star product”\textsuperscript{3} or “twisted product” which mimics the multiplication of operators at the level of symbols. It satisfies

$$\text{symb}(\tilde{f} \tilde{g}) = \text{symb}(\tilde{f}) \circ \text{symb}(\tilde{g})$$

(2.10)
for all operators $\tilde{f}$ and $\tilde{g}$. As a consequence, the $\circ$-product is associative, distributive with respect to $+$, but not commutative. It is a deformation of the pointwise product of functions to which it reduces in the limit $\hbar \to 0$.

From

$$\text{symb}(I) = 1, \quad \text{symb}(\hat{\chi}^\mu) = \theta^\mu$$

(2.11)
and eq.(2.4) it follows that

$$1 \circ 1 = 1, \quad 1 \circ \theta^\mu = \theta^\mu \circ 1 = \theta^\mu$$

$$\theta^\mu \circ \theta^\nu = \theta^\mu \theta^\nu + \frac{\hbar}{2} \delta^\mu_\nu$$

(2.12)

By virtue of its postulated distributivity and associativity, the relations (2.12) characterize the $\circ$-product uniquely. They are sufficient to work out the product $f \circ g$ of arbitrary functions $f$ and $g$.

\textsuperscript{3}Both in the fermionic and the bosonic case we keep using the traditional name “star product” even though we write ‘$\circ$’ instead of the usual symbol ‘$*$’. Following refs. \textsuperscript{29} \textsuperscript{30} \textsuperscript{21} this notation indicates that we are dealing with a fiberwise twisted product which should not be confused with the $* \equiv *_{\mathcal{M}}$-product which would refer to the base of the Weyl algebra bundles we are going to construct in section 2.2. It is the $*_{\mathcal{M}}$-product rather than the $\circ$-product which is needed for the deformation quantization \textsuperscript{24} of physical systems on the phase-space $\mathcal{M}$. In the present paper, the $*_{\mathcal{M}}$-product plays no role, however. (Note also that ‘$*$’ stands for the Hodge operator in our case.)
Eq. (2.7) implies that complex conjugation changes the order of the factors in a star product:

$$\bar{f}_1 \circ f_2 = \bar{f}_2 \circ \bar{f}_1$$  \hspace{1cm} (2.13)

The space of functions $f(\theta)$ equipped with the $\circ$-product will be referred to as the fermionic Weyl algebra $\mathcal{W}^F$.

In the literature [26, 27] one finds the following integral representation for the $\circ$-product of two arbitrary functions:

$$\left( f_1 \circ f_2 \right)(\theta) = \epsilon_n \left( \frac{\hbar}{2i} \right)^n \int \exp \left[ -\frac{2}{\hbar}(\theta_1 \theta + \theta \theta_2 + \theta_2 \theta_1) \right] f_1(\theta_1) f_2(\theta_2) \, d^n \theta_1 d^n \theta_2$$  \hspace{1cm} (2.14)

where $\theta_1 \theta \equiv \theta^\mu \theta_\mu$, etc. and with $\epsilon_n$ defined as in eq. (A.23). For our purposes, various alternative representations of the star product are needed. They are derived in appendix B. The first one reads

$$\left( f_1 \circ f_2 \right)(\theta) = \sum_{p=0}^{n} \left( \frac{\hbar}{2} \right)^p \frac{(-1)^{p(p-1)/2}}{p!} \left\{ A^p \frac{\partial}{\partial \theta^{\mu_1}} \frac{\partial}{\partial \theta^{\mu_2}} \cdots \frac{\partial}{\partial \theta^{\mu_p}} f_1(\theta) \right\} \times$$

$$\left\{ \frac{\partial}{\partial \theta^{\mu_1}} \frac{\partial}{\partial \theta^{\mu_2}} \cdots \frac{\partial}{\partial \theta^{\mu_p}} f_2(\theta) \right\}$$  \hspace{1cm} (2.15)

with the automorphism $A : \mathcal{W}^F \to \mathcal{W}^F$ defined in appendix A. An equivalent form involving both left and right derivatives is

$$\left( f_1 \circ f_2 \right)(\theta) = \sum_{p=0}^{n} \left( \frac{\hbar}{2} \right)^p \frac{1}{p!} f_1(\theta) \left\{ \frac{\partial}{\partial \theta^{\mu_1}} \frac{\partial}{\partial \theta^{\mu_p}} \cdots \frac{\partial}{\partial \theta^{\mu_{p-1}}} \frac{\partial}{\partial \theta^{\mu_1}} \frac{\partial}{\partial \theta^{\mu_2}} \cdots \frac{\partial}{\partial \theta^{\mu_p}} f_2(\theta) \right\}$$  \hspace{1cm} (2.16)

The most compact representation reads

$$\left( f_1 \circ f_2 \right)(\theta) = f_1(\theta) \exp \left[ \frac{\hbar}{2} \left\{ \frac{\partial}{\partial \theta^{\mu}} \frac{\partial}{\partial \theta^{\mu}} \right\} f_2(\theta) \right]$$  \hspace{1cm} (2.17)

\(^4\text{Indices are raised and lowered with } g_{\mu\nu} = \delta_{\mu\nu}.\)
where $\frac{\partial}{\partial \theta^\mu}$ is a right derivative acting on $f_1$ and $\frac{\partial}{\partial \theta^\mu}$ a left derivative acting on $f_2$. The result \((2.17)\) looks surprisingly simple and is completely analogous to its bosonic counterpart. All the complicated sign factors which appeared during the calculation, either explicitly or hidden in the $A$-automorphism, conspired to disappear from the final result.

Depending on the problem at hand one or another of the above representations is the most convenient one. Eq.\((2.13)\) we shall relate to Kähler’s formula for the Clifford product shortly. From eq.\((2.16)\) one immediately reads off the important special cases

\[
\theta^\mu \circ f(\theta) = \theta^\mu f(\theta) + \frac{\hbar}{2} \frac{\partial}{\partial \theta^\mu} f(\theta) \tag{2.18}
\]

\[
f(\theta) \circ \theta^\mu = f(\theta) \theta^\mu + \frac{\hbar}{2} f(\theta) \frac{\partial}{\partial \theta^\mu} \tag{2.19}
\]

In order to calculate the product of two $\delta$-functions, which we shall need later on, eq.\((2.14)\) is most suitable:

\[
(\delta \circ \delta)(\theta) = (-1)^{n(n-1)/2} \left( \frac{\hbar}{2} \right)^n \tag{2.20}
\]

Up to now we regarded the $\hat{\chi}^\mu$’s as abstract operators. Let us look at concrete representations on some finite dimensional vector space $V^F$. If $\Gamma^\mu$ is a set of hermitian matrices which satisfy the Clifford algebra relations

\[
\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2 \delta^{\mu\nu} \tag{2.21}
\]

then

\[
\hat{\chi}^\mu = \sqrt{\frac{\hbar}{2}} \Gamma^\mu \tag{2.22}
\]

satisfies the canonical anticommutation relations \((2.1)\). Here $\Gamma^\mu$ denotes the Dirac matrices in an arbitrary, possibly reducible representation. The notation $\gamma^\mu$ is reserved for the (essentially unique) irreducible representation on $V^F = \mathbb{C}^k$, $k \equiv 2^{n/2}$, if $n$ is even. The operators $\hat{f} : \mathcal{W}^F \rightarrow \mathcal{W}^F$ of eq.\((2.2)\) are $k \times k$ matrices then. The space of these operators will be denoted by $\mathcal{L}(\mathcal{W}^F)$.
The identification (2.22) must be interpreted with some care. In setting up the symbol calculus one adopts the rule that the operators \( \hat{\chi}^\mu \) anticommute with numbers of odd Grassmann parity. On the other hand, the entries of the matrices \( \Gamma^\mu \) are ordinary complex numbers, so \( \Gamma^\mu \) commutes with all elements of the Grassmann algebra.

Next we list a few properties of the operators \( \hat{\Omega} \) which we shall need shortly. These operators are reminiscent of the (bosonic) Weyl operators. However, as they stand, they are not unitary but rather hermitian, \( \Omega(\rho) = \Omega(\rho)^\dagger \). This is due to the fact that \( \hat{\chi}^\mu \) anticommutes with the Grassmann-odd \( \rho^\mu \)'s. Actually it is the operators \( \hat{\Omega}(i\rho/\hbar) = \exp(\hat{\chi}^\mu \rho^\mu/\hbar) \) which play the role of the Weyl operators on a fermionic phase-space. They are unitary, \( \hat{\Omega}(i\rho/\hbar)^\dagger = \hat{\Omega}(i\rho/\hbar)^{-1} \), and they shift \( \hat{\chi}^\mu \) by \( \rho^\mu \) times the unit operator:

\[
\hat{\Omega}(i\rho/\hbar)^\dagger \hat{\chi}^\mu \hat{\Omega}(i\rho/\hbar) = \hat{\chi}^\mu + \rho^\mu \quad (2.23)
\]

This leads to a projective representation of the translation group since

\[
\hat{\Omega}(\rho_1) \hat{\Omega}(\rho_2) = \exp(\hbar/2 \rho^\mu_1 \rho^\mu_2) \hat{\Omega}(\rho_1 + \rho_2) \quad (2.24)
\]

The derivative of \( \hat{\Omega}(\rho) \) can be written in either of the two forms

\[
\frac{\partial}{\partial \rho^\mu} \hat{\Omega}(\rho) = \hat{\Omega}(\rho) \left[ i\hat{\chi}^\mu + \frac{\hbar}{2} \rho^\mu \right] \quad (2.25)
\]

\[
= \left[ i\hat{\chi}^\mu - \frac{\hbar}{2} \rho^\mu \right] \hat{\Omega}(\rho) \quad (2.26)
\]

When we replace in \( \hat{\Omega} \) the operators \( \hat{\chi}^\mu \) by the Dirac matrices via (2.22) we are led to

\[
\hat{\Omega}(\rho) = \exp(-i\sqrt{\hbar/2} \Gamma^\mu \rho^\mu) \quad (2.27)
\]

The properties of \( \hat{\Omega} \) are slightly different from those of \( \hat{\Omega} \) because \( \Gamma^\mu \) commutes with \( \rho^\mu \). The \( \hat{\Omega} \)'s are unitary matrices,

\[
\hat{\Omega}(\rho)^\dagger = \hat{\Omega}(-\rho) = \hat{\Omega}(\rho)^{-1} \quad (2.28)
\]
with the composition law

\[
\tilde{\Omega}(\rho_1)\tilde{\Omega}(\rho_2) = \exp\left(-\frac{h}{2}\rho_1^\mu \rho_2^\mu\right) \tilde{\Omega}(\rho_1 + \rho_2)
\]  

(2.29)

The expressions for their derivative are

\[
\frac{\partial}{\partial \rho_\mu} \tilde{\Omega}(\rho) = \tilde{\Omega}(-\rho) \left[ -i \sqrt{\frac{h}{2}} \Gamma^\mu - \frac{h}{2} \rho^\mu \right]
\]  

(2.30)

\[
= \left[ -i \sqrt{\frac{h}{2}} \Gamma^\mu + \frac{h}{2} \rho^\mu \right] \tilde{\Omega}(\rho)
\]  

(2.31)

We shall need these relations when we decompose the reducible Dirac-Kähler representation.

An interesting example where one can see the symbol calculus at work is the generalization of the chirality matrix \(\gamma_5\) in 4 dimensions. We assume that \(n\) is even in the remainder of this subsection and employ the Dirac matrices \(\gamma^\mu\) in the \(2^{n/2}\)-dimensional representation. From (1.15) and \((\gamma^\mu)^\dagger = \gamma^\mu\) it follows that the matrix

\[
\gamma_{n+1} \equiv -i^{n(n-1)/2} \gamma^1 \gamma^2 \ldots \gamma^n
\]  

(2.32)

satisfies \(\gamma^\mu \gamma_{n+1} = -\gamma_{n+1} \gamma^\mu\),

\[
\gamma_{n+1}^2 = 1 \quad \text{and} \quad \gamma_{n+1}^\dagger = \gamma_{n+1}
\]  

(2.33)

in all even dimensions. The sign of (2.32) is chosen such that for \(n = 4\)

\[
\gamma_5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4
\]  

(2.34)

We identify

\[
\gamma^\mu = \kappa \tilde{\chi}^\mu
\]  

(2.35)

where

\[
\kappa \equiv \sqrt{\frac{2}{\hbar}}
\]  

(2.36)
and interpret $\gamma_{n+1}$ as the matrix representation of the abstract operator

$$\tilde{G}_{n+1} = -i^{n(n-1)/2} \kappa^n \hat{\chi}^1 \hat{\chi}^2 \cdots \hat{\chi}^n$$  \hspace{1cm} (2.37)

Its symbol $\text{symb}(\tilde{G}_{n+1}) \equiv G_{n+1}$ follows directly from (2.3) if we note that $\hat{\chi}^1 \cdots \hat{\chi}^n = \epsilon_{\mu_1 \cdots \mu_n} \hat{\chi}^{[\mu_1} \cdots \hat{\chi}^{\mu_n]} / (n!)$:

$$G_{n+1}(\theta) = -i^{n(n-1)/2} \kappa^n \theta^1 \theta^2 \cdots \theta^n \hspace{1cm} (2.38)$$

$$= -(-i)^{n(n-1)/2} \kappa^n \theta^{n-1} \cdots \theta^1$$

Hence, up to a constant, $\gamma_{n+1}$ is represented by the $\delta$-function:

$$G_{n+1}(\theta) = -(-i)^{n(n-1)/2} \kappa^n \delta(\theta) \hspace{1cm} (2.39)$$

As a consequence of eqs. (2.20) and (A.21), this function satisfies

$$G_{n+1} \circ G_{n+1} = 1 \quad \text{and} \quad \tilde{G}_{n+1} = G_{n+1} \hspace{1cm} (2.40)$$

which reflects the properties (2.33) of $\gamma_{n+1}$. By virtue of (A.20) the Fourier transform of $G_{n+1}$ is the constant function

$$\tilde{G}_{n+1}(\rho) = -(-i)^{n(n-1)/2} \kappa^n \hspace{1cm} (2.41)$$

In appendix A we defined the modified Hodge operator $\star$ for a general Grassmann algebra and we showed that it is related to the Fourier transformation via eq. (A.41). Using the latter equation together with the integral representation (2.14) for the star product it is not difficult to see that the application of $\star$ to some $f \in \mathcal{W}^F$ is essentially equivalent to a star-multiplication with $G_{n+1}$ from the right. For a homogeneous function of degree $p$,

$$\star f^{(p)}(\theta) = -(-i)^{n(n-1)/2} \kappa^{2p-n} (f^{(p)} \circ G_{n+1})(\theta) \hspace{1cm} (2.42)$$

If we rescale $\theta$ we can write down a similar equation for inhomogeneous functions even:

$$\star f(\theta / \kappa) = -(-i)^{n(n-1)/2} (f \circ G_{n+1})(\theta / \kappa) \hspace{1cm} (2.43)$$
Finally we remark that the chirality operator \( \hat{G}_{n+1} \) can be expressed as an integral over the Weyl operators:

\[
\hat{G}_{n+1} = -(-i)^{(n-1)/2} \kappa^n \int d^n \rho \hat{\Omega}(\rho)
\]

(2.44)

This is a remarkable relation because contrary to the original definition of \( \gamma_{n+1} \) as the product of all Dirac matrices it carries over to the symplectic case almost literally.

2.2 \( \mathcal{W}^F \) as an Atiyah-Kähler algebra

Let us come back to the abstract Atiyah-Kähler algebra \( \mathbf{AK}(V,Q) \) discussed in the introduction. It is important to observe that the Weyl algebra \( \mathcal{W}^F \) which we reviewed in the previous section contains all the ingredients which make up an Atiyah-Kähler algebra:

(i) The vector space \( V \) is spanned by the basis elements \( \theta^1, \ldots, \theta^n \) and the exterior algebra over this space, \( \Lambda(V) = \bigoplus_{p=0}^n \Lambda^p(V) \), consists of monomials \( \theta^{\mu_1} \cdots \theta^{\mu_p} \in \Lambda^p(V) \). The exterior product \( \Lambda \) on \( \Lambda(V) \) is the pointwise product of (inhomogeneous) functions \( f(\theta) \in \Lambda(V) \).

(ii) By starting from the canonical anticommutation relations (2.1) we have tacitly decided for an inner product \( (\cdot | \cdot) \) on \( \Lambda(V) \). The quadratic form \( Q \) is induced by \( g_{\mu\nu} \equiv \delta_{\mu\nu} \), regarded as an inner product of \( V \). On \( \Lambda^1(V) \) we have

\[
(\theta^\mu | \theta^\nu) = \kappa^{-2} \delta^{\mu\nu}
\]

(2.45)

and similarly for \( p > 1 \) (see below).

(iii) The star product on \( \mathcal{W}^F \) provides a concrete realization of the abstract Clifford product. The Clifford product is associative and distributive over \( + \), and so is the star product. Moreover, \( \vee, \wedge \) and \( (\cdot, \cdot) \) have to

26
satisfy the consistency condition (1.7). From eq.(2.12) it follows that
this relation is indeed satisfied by the star multiplication together with
the pointwise multiplication and the inner product $\langle \cdot | \cdot \rangle$
\[ \theta^\mu \circ \theta^\nu = \theta^\mu \theta^\nu + \langle \theta^\mu | \theta^\nu \rangle \] (2.46)

In particular, upon symmetrization, $\theta^\mu \circ \theta^\nu + \theta^\nu \circ \theta^\mu = 2\langle \theta^\mu | \theta^\nu \rangle$.

Thus we may conclude that \textit{the fermionic Weyl algebra $\mathcal{W}^F$ is a concrete
realization of an Atiyah-Kähler algebra.}

Let us be more explicit about the inner product on $\bigwedge(V)$. Within the
symbol calculus, the standard inner product of DK-theory \cite{11} admits a very
natural representation in terms of the star product:
\[ \langle f_1 | f_2 \rangle = [\bar{f}_1 \circ f_2](\theta = 0) \] (2.47)
Here $f_1$ and $f_2$ are two arbitrary inhomogeneous functions. We allow their
expansion coefficients $f_{\mu_1 \cdots \mu_p}$ to become complex. Note, however, that the
complex conjugation in (2.47) is necessary even if the coefficients are taken to
be real, see eq.(A.10). Using the integral representation
\[ \langle f_1 | f_2 \rangle = (i \kappa^2)^{-n} \epsilon_n \int \bar{f}_1(\theta_1) \exp(\kappa^2 \theta_1^\mu \theta_2^\mu) f_2(\theta_2) d^n \theta_1 d^n \theta_2 \] (2.48)
and expanding $f_1$ and $f_2$ according to
\[ f(\theta) = \sum_{p=0}^{n} \frac{1}{p!} \kappa^p f_{\mu_1 \cdots \mu_p}^{(p)} \theta^{\mu_1} \cdots \theta^{\mu_p} \] (2.49)
with appropriate powers of $\kappa$ separated off from the expansion coefficients, it
is easy to derive that
\[ \langle f_1 | f_2 \rangle = \sum_{p=0}^{n} \frac{1}{p!} \bar{f}_{\mu_1 \cdots \mu_p}^{(p)} f_{\nu_1 \cdots \nu_p}^{(p)} \] (2.50)

The inner products among the basis elements of $p(V)$ (homogeneous func-
tions of degree $p$) can be written down similarly. For $p = 1$ one recovers (2.43),
and for $p = 2$ one has, for instance
\[ \langle \theta^\mu \theta^\nu | \theta^\rho \theta^\sigma \rangle = \kappa^{-4} (\delta^{\mu \rho} \delta^{\nu \sigma} - \delta^{\mu \sigma} \delta^{\nu \rho}) \] (2.51)
We note that $(\cdot | \cdot)$ has the important property of making the star multiplication with $\theta^\mu$ a self-adjoint operator. If we define
\[
C^\mu : \mathcal{W}^F \to \mathcal{W}^F, \quad (C^\mu f)(\theta) = \kappa \theta^\mu \circ f(\theta) \tag{2.52}
\]
then eq.\((2.18)\) tells us that $C^\mu$ is given by the first order differential operator
\[
C^\mu = \kappa \theta^\mu + \frac{1}{\kappa \partial \theta_\mu} \tag{2.53}
\]
If one writes the inner product as in \((2.47)\), the self-adjointness of $C^\mu$ is obvious:
\[
(C^\mu f_1 | f_2) = \kappa [(\theta^\mu \circ f_1) \circ f_2](0) = \kappa [(\theta^\mu \circ f_1 \circ f_2)(0) = \kappa [(\theta^\mu \circ f_1 \circ f_2)](0) = (f_1 | C^\mu f_2) \tag{2.54}
\]
Here we exploited \((2.13)\) and the associativity of the star product.

### 2.3 Symbol-valued fields on space-time

The most familiar application of the above symbol calculus is the deformation theory approach [24, 26] to the quantization of fermionic systems. In this context, the variables $\theta^\mu$ are coordinates on the phase-space of the physical system under consideration. If there are additional bosonic degrees of freedom (such as the position of a spinning particle, say) this fermionic phase-space is embedded in a larger graded phase-space, a supermanifold with both commuting and anticommuting coordinates [17].

In the present paper we are investigating a different setting. Rather than phase-space, the physical arena here is space-time, an ordinary Riemannian manifold $(\mathcal{M}_n, g)$, not a supermanifold. The fermionic Weyl algebra $\mathcal{W}^F$ enters the construction as the fiber of certain bundles over space-time which we shall refer to as “Weyl algebra bundles” [30].
By definition, the base of a Weyl algebra bundle is \((\mathcal{M}_n, g)\) and the typical fiber is \(\mathcal{W}^F\), i.e., at each space-time point \(x\) we attach a local copy \(\mathcal{W}^F_x\) of \(\mathcal{W}^F\). The quadratic form \(Q\) on \(\mathcal{W}^F_x\) is provided by the metric \(g\) evaluated at the point \(x\). Local coordinates on the total space are pairs \((x, f)\) where \(x \equiv (x^\mu)\) are coordinates referring to some chart of \(\mathcal{M}_n\), and \(f\) is a function of the Grassmann variables \(\theta^1, \ldots, \theta^n\). The transition functions are defined in close analogy with the exterior algebra bundle. A coordinate transformation \(x \to \tilde{x}(x)\) on \(\mathcal{M}_n\) is accompanied by \(f \to \tilde{f}\) with \(\tilde{f}(\tilde{\theta}) = f(\theta)\) where \(\tilde{\theta}^\mu \equiv (\partial \tilde{x}^\mu / \partial x^\nu) \theta^\nu\), i.e., \(\theta^\mu\) transforms in the same manner as \(dx^\mu\).

Sections through a Weyl algebra bundle are locally represented by functions

\[ x \mapsto F(x, \cdot) \in \mathcal{W}^F_x \]  

(2.55)

where

\[ F(x, \cdot) : \theta \mapsto F(x, \theta) \]  

(2.56)

is a function of \(n\) commuting and \(n\) anticommuting variables. We define a fiberwise star product of two such sections by

\[ (F_1 \circ F_2)(x, \theta) = \left( F_1(x, \cdot) \circ F_2(x, \cdot) \right)(\theta) \]  

(2.57)

for each point \(x\).

We can apply the inverse symbol map to \(F(x, \cdot)\) and thus obtain a family of operators labelled by the space-time points \(x\):

\[ \hat{F}(x) \equiv \text{symb}^{-1} F(x, \cdot) \]  

(2.58)

If we fix a concrete matrix representation of the fermionic operators on some representation space \(\mathcal{V}^F\), then \(\hat{F}(x)\) acts on a local copy \(\mathcal{V}^F_x\) of \(\mathcal{V}^F\), i.e., \(\hat{F}(x) \in \mathcal{L}(\mathcal{V}^F_x)\). We are particularly interested in the case where \(\mathcal{V}^F\) carries the irreducible \(2^{n/2}\)-dimensional representation of the Clifford algebra (for \(n\) even). Then \(\mathcal{V}^F_x\) is a fiber of the usual spin bundle over \(\mathcal{M}_n\) whose sections are the familiar Dirac spinor fields.
In the present paper we shall not be concerned with the global properties of Weyl algebra bundles. Our main interest is in the metaplectic analog of the Dirac-Kähler construction, and for this purpose it is sufficient to compare to the topologically trivial bundles over the flat space-time $\mathcal{M}_n = \mathbb{R}^n$. An analogous discussion could be given for arbitrary curved space-times as well, but we shall avoid the necessary technical complications here. Thus, in our case, sections can be represented globally by functions $F(x, \theta)$. We remark that there exists a natural inner product on the space of these functions:

$$\langle F_1 | F_2 \rangle = \int d^n x \left( F_1(x, \cdot) | F_2(x, \cdot) \right)$$  \hspace{1cm} (2.59)

### 2.4 Dirac-Kähler fields and symbol calculus

Let us assume we are given an arbitrary DK field $\Phi$ on flat euclidean space-time $\mathbb{R}^n$. It possesses an expansion

$$\Phi(x) = \sum_{p=0}^{n} \frac{1}{p!} F^{(p)}_{\mu_1 \cdots \mu_p}(x) \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$  \hspace{1cm} (2.60)

The (complex) coefficient functions are taken to be completely antisymmetric in all $p$ indices so that there is a bijective correspondence between forms $\Phi$ and sets $\{F^{(p)}_{\mu_1 \cdots \mu_p}\}$ of antisymmetric tensors. Given these tensors, we form the following matrix-valued field:

$$\widehat{F}(x) = \sum_{p=0}^{n} \frac{1}{p!} F^{(p)}_{\mu_1 \cdots \mu_p}(x) \, \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_p}$$  \hspace{1cm} (2.61)

From now on we assume that $n$ is even and that the Dirac matrices are in their irreducible representation. Hence $\widehat{F}(x)$ acts on a local copy $\mathcal{V}^F_x$ of the representation space $\mathcal{V}^F = \mathbb{C}^k$, $k = 2^{n/2}$. By virtue of (2.35) we may regard $\widehat{F}(x)$ as a matrix realization of the abstract operator

$$\widehat{F}(x) = \sum_{p=0}^{n} \frac{K_p}{p!} F^{(p)}_{\mu_1 \cdots \mu_p}(x) \, \chi^{\mu_1} \chi^{\mu_2} \cdots \chi^{\mu_p}$$  \hspace{1cm} (2.62)
For every point $x$, the symbol of this operator is $F(x, \theta) = [\text{symb } \hat{F}(x)](\theta)$, or

$$F(x, \theta) = \sum_{p=0}^{n} \frac{\kappa^p}{p!} F^{(p)}_{\mu_1 \cdots \mu_p}(x) \theta^\mu_1 \theta^\mu_2 \cdots \theta^\mu_p$$

Thus we have set up a linear one-to-one correspondence between differential forms $\Phi(x)$ and symbol functions $F(x, \theta)$. Schematically,

$$\Phi(x) \in \bigwedge (T_x^* M_n) \Rightarrow \hat{F}(x) \in \mathcal{L}(\mathcal{V}_x^F) \Rightarrow F(x, \cdot) \in \mathcal{W}_x^F$$

The first one of the two bijections in (2.64) is the usual “Dirac-Kähler correspondence” $dx^\mu \Rightarrow \gamma^\mu$ which we mentioned already in the introduction, while the second one is the Weyl symbol map. Taken in conjunction, these maps relate DK-fields to symbols. In particular,

$$dx^\mu \Rightarrow \kappa \theta^\mu$$

We shall use the notation $\Phi : F \mapsto [F]$ for the linear map which yields the differential form belonging to a given symbol. For instance,

$$\Phi[\kappa \theta^\mu] = dx^\mu$$

What makes the above construction particularly useful is that under the map $\Phi$ many of the familiar linear and bilinear operations involving differential forms naturally pass over to the symbol functions and vice versa. This is immediately obvious for the automorphism $A$, the antiautomorphism $B$, the Hodge operator $*$ and the modified Hodge operator $\star$. Comparing their definition for symbols in appendix A to their standard definition in terms of differential forms one sees that

$$A\Phi[F] = \Phi[AF], \quad B\Phi[F] = \Phi[BF]$$

$$*\Phi[F] = \Phi[*F], \quad \star\Phi[F] = \Phi[\star F]$$

The exterior derivative $d = dx^\mu \partial_\mu$ translates into $\kappa \theta^\mu \partial_\mu$,

$$d\Phi[F] = \Phi[\kappa \theta^\mu \partial_\mu F]$$
while the contraction $\mathbf{i}(v)$ with a vector field $v = v^\mu \partial_\mu$ becomes a derivative with respect to $\theta$:

$$i(v) \Phi[F] = \Phi[\kappa^{-1} v^\mu \frac{\partial}{\partial \theta^\mu} F]$$

(2.69)

In particular,

$$e_\mu \Phi[F] = \Phi[\kappa^{-1} \frac{\partial}{\partial \theta^\mu} F]$$

(2.70)

The natural inner product on the space of DK-fields is

$$\langle \Phi_1, \Phi_2 \rangle = \int \bar{\Phi}_1 \wedge \ast \Phi_2$$

(2.71)

Its counterpart at the symbol level is (2.59) with (2.47):

$$\langle \Phi[F_1], \Phi[F_2] \rangle = \langle F_1 | F_2 \rangle$$

(2.72)

The coderivative $d^\dagger$ is the formal adjoint of $d$ with respect to $\langle \cdot, \cdot \rangle$. On flat space one has

$$d^\dagger \Phi = -e^\mu \partial_\mu \Phi$$

(2.73)

whence

$$d^\dagger \Phi[F] = \Phi[-\kappa^{-1} \partial_\mu \frac{\partial}{\partial \theta^\mu} F]$$

(2.74)

The wedge product of differential forms is mapped onto the pointwise product of symbol functions:

$$\Phi[F_1] \wedge \Phi[F_2] = \Phi[F_1 F_2]$$

(2.75)

The most important aspect of the form/symbol correspondence is that the image of the Clifford product is precisely the fiberwise star product (2.57):

$$\Phi[F_1] \vee \Phi[F_2] = \Phi[F_1 \circ F_2]$$

(2.76)

\(^5\)All terms which are not of degree $n$ are supposed to be discarded from the integrand in (2.71).
This can be seen for instance by mapping Kähler’s formula (1.9) for the Clifford product on our representation (2.15) of the fermionic Weyl star product:

\[
\Phi[F_1] \vee \Phi[F_2] = \\
= \sum_{p=0}^{n} \frac{(-1)^{p(p-1)/2}}{p!} \left( A^p e_{\mu_1} \cdots e_{\mu_p} \Phi[F_1] \right) \wedge \left( e^{\mu_1} \cdots e^{\mu_p} \Phi[F_2] \right) \\
= \sum_{p=0}^{n} \frac{(-1)^{p(p-1)/2}}{p!} \Phi[k^{-p} A^p \frac{\partial}{\partial \theta_{\mu_1}} \cdots \frac{\partial}{\partial \theta_{\mu_p}} F_1] \wedge \Phi[k^{-p} \frac{\partial}{\partial \theta_{\mu_1}} \cdots \frac{\partial}{\partial \theta_{\mu_p}} F_2] \\
= \Phi \left[ \sum_{p=0}^{n} k^{-2p} \frac{(-1)^{p(p-1)/2}}{p!} \left( A^p \frac{\partial}{\partial \theta_{\mu_1}} \cdots \frac{\partial}{\partial \theta_{\mu_p}} F_1 \right) \left( \frac{\partial}{\partial \theta_{\mu_1}} \cdots \frac{\partial}{\partial \theta_{\mu_p}} F_2 \right) \right] \\
= \Phi[F_1 \circ F_2]
\] (2.77)

Here we used (2.67), (2.70) and (2.75).

One also could prove eq.(2.76) inductively. If we replace \( dx^\mu \) by \( \kappa \theta^\mu \) and “\( \vee \)” by the star product in the relations (1.8) which define the Clifford product we obtain exactly eqs.(2.12) for the star product. Therefore eq.(2.76) is correct for zero-and one-forms. Its generalization for arbitrary \( p \)-forms makes essential use of the associativity of both the Clifford and the star product.

By virtue of our rules for the form/symbol correspondence also the equations (1.13) and (2.18) are now seen to be completely equivalent.

In the DK-equation we need the Clifford product of \( dx^\mu \) with an arbitrary form :

\[
\begin{align*}
\text{\( dx^\mu \vee \Phi[F] \)} & = \Phi [k \theta^\mu] \vee \Phi[F] \\
& = \Phi[k \theta^\mu \circ F] \\
& = \Phi[C^\mu F]
\end{align*}
\] (2.78)

Here \( C^\mu \) is the first order differential operator (2.53). In the introduction we discussed already that \( dx^\mu \vee, \) regarded as an operator on the space of DK-fields, gives rise to the Clifford algebra (1.14). For consistency the same should be true for the star multiplication with \( \kappa \theta^\mu \) and for \( C^\mu \) on the space of symbols.
In fact, it is easy to see that

\[( \kappa \theta^\mu \circ (\kappa \theta^\nu \circ \kappa \theta^\mu \circ) = 2 \delta^\mu{}^\nu \] (2.79)

and

\[C^\mu C^\nu + C^\nu C^\mu = 2 \delta^\mu{}^\nu\] (2.80)

The DK-operator acting on forms reads

\[(d - d^\dagger) \Phi = dx^\mu \wedge \partial_\mu \Phi + e^\mu - \partial_\mu \Phi = dx^\mu \vee \partial_\mu \Phi\] (2.81)

where the second equality follows from (1.13). Therefore \(d - d^\dagger\) becomes \(\kappa \theta^\mu \circ \partial_\mu\) or \(C^\mu \partial_\mu\) at the symbol level:

\[(d - d^\dagger) [F] = \Phi [\kappa \theta^\mu \circ \partial_\mu F] = \Phi [C^\mu \partial_\mu F]\] (2.82)

This converts the DK-equation to

\[\left[ \left( \frac{1}{\kappa \partial \theta_\mu} + \kappa \theta^\mu \right) \partial_\mu + m \right] \Phi(x, \theta) = 0\] (2.83)

In closing we return to the chirality operator \(\gamma_{n+1}\). Under the map \(\Phi\), the image of the delta-function is essentially the volume form \(\text{Vol} \equiv dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n\):

\[\Phi [\kappa^n \delta(\cdot)] = (-1)^{n(n-1)/2} \text{Vol}\] (2.84)

For the chirality operator this means that

\[\Phi [G_{n+1} \circ F] = -i^{n(n-1)/2} \text{Vol} \vee \Phi [F]\] (2.85)

which, at the operator level, corresponds to

\[G_{n+1} \circ F = \gamma_{n+1} \hat{F}\] (2.86)

Thus we see that (up to unimportant constants) the fermionic \(\delta\)-function, the volume form, and the chirality matrix \(\gamma_{n+1}\) are simply different variants of the same object. Furthermore, by eq.(2.43), star multiplication of \(F\) by \(G_{n+1}\) from the right amounts to applying the modified Hodge operator ★.
2.5 Invariant subspaces of $\mathcal{W}^F$

The differential operators $C^\mu$ or the star left-multiplication by $\kappa \theta^\mu$ define a representation of the Clifford algebra (2.21) in the space of symbols $f(\theta)$. As $f(\theta)$ has $2^n$ independent (complex) components, this representation is reducible. It can be decomposed into $2^{n/2}$ representations each of which is isomorphic to the $2^{n/2}$-dimensional irreducible representation provided by the matrices $\gamma^\mu$. (We assume $n$ even in this section.) As a consequence, a symbol-valued field $F(x, \theta)$ describes $2^{n/2}$ ordinary Dirac spinor fields.

In the light of the form/symbol correspondence which we developed in the previous section it is clear that the representation carried by $\mathcal{W}^F$ could be decomposed simply by invoking the standard discussion at the level of differential forms. However, as our main motivation for studying the symbol formulation of DK-fields is to get some understanding of their symplectic analogs we shall reformulate the method of Becher and Joos [3] in symbol language and use this as a guide in the symplectic case. As a by-product we shall find a very elegant derivation of their matrix-valued form $Z$ which puts it in a more general perspective.

We have to decompose the Weyl algebra in orthogonal subspaces,

$$\mathcal{W}^F = \bigoplus_{\alpha=1}^k \mathcal{W}_F^{(\alpha)}, \quad k \equiv 2^{n/2} \quad (2.87)$$

such that $\mathcal{W}_F^{(\alpha)}$ is invariant under star left-multiplication by $\theta^\mu$. Following a strategy similar to the one described in the introduction we look for a $k \times k$-matrix valued function $Z(\theta)$ with the property

$$\kappa \theta^\mu \circ Z_{\alpha\beta}(\theta) = \sum_{\gamma=1}^k (\gamma^\mu)^T_{\alpha\gamma} Z_{\gamma\beta}(\theta) \quad (2.88)$$

The function $Z$ is readily found in our formalism. Since the star product with $\theta^\mu$ involves first derivatives at most, eq. (2.88) is reminiscent of the formulas for the derivative of the Weyl operators $\hat{\Omega}$ and $\hat{\bar{\Omega}}$ which we displayed in section
2.1. In fact, using those formulas together with (2.18) it is easy to show that there exists a rescaling of the arguments of $\hat{\Omega}$ and $\tilde{\Omega}$ in such a way that the star multiplication by $\theta^\mu$ corresponds to an operator multiplication by $\hat{\chi}^\mu$ or $\Gamma^\mu$:

\[
\theta^\mu \circ \hat{\Omega}(\pm i\kappa^2 \theta) = \mp \hat{\Omega}(\pm i\kappa^2 \theta) \hat{\chi}^\mu
\]

(2.89)

\[
\theta^\mu \circ \tilde{\Omega}(\pm \kappa^2 \theta) = \pm i \hat{\chi}^\mu \tilde{\Omega}(\pm \kappa^2 \theta)
\]

(2.90)

\[
\kappa \theta^\mu \circ \tilde{\Omega}(\pm i\kappa^2 \theta) = \pm \Gamma^\mu \tilde{\Omega}(\pm i\kappa^2 \theta)
\]

(2.91)

For the problem at hand, eq.(2.91) is precisely what we need. If $\Gamma^\mu$ constitutes a Clifford algebra, so does $\Gamma^T_\mu$. Hence we may set $Z(\theta) = \tilde{\Omega}(i\kappa^2 \theta)$ with $\Gamma^\mu = \gamma^T_\mu$. Thus

\[
Z(\theta) = \exp \left[ \kappa \theta^\mu \gamma^T_\mu \right]
\]

(2.92)

or in expanded form

\[
Z(\theta) = \sum_{p=0}^n \frac{\kappa^p}{p!} \gamma^T_{\mu_1} \cdots \gamma^T_{\mu_p} \theta^\mu_1 \cdots \theta^\mu_p
\]

(2.93)

Clearly (2.93) is precisely the symbol corresponding to the form (1.21) which was found by Becher and Joos [3] using different techniques. In the context of the present investigation it is important to keep in mind that $Z$ is nothing but a rescaled fermionic Weyl operator since the latter has a well-known bosonic analog.

Because of the completeness properties of the $\gamma$-matrices, $\{Z_{\alpha\beta};\alpha, \beta = 1, \cdots, k\}$ is a basis for $W^F$ and we may expand any symbol as

\[
F(x, \theta) = \sum_{\beta=1}^k \sum_{\alpha=1}^k \psi^{(\beta)}(x) Z_{\alpha\beta}(\theta)
\]

\[
= \sum_{\beta=1}^k F^{(\beta)}(x, \theta)
\]

(2.94)

(Here we use already the notation appropriate for the role of $W^F$ as a fiber at the point $x$.) The rest of the argument parallels our discussion in the
Introduction. We obtain $k \equiv 2^{n/2}$ invariant subspaces $W_{(\alpha)}^F$ (left ideals) which are spanned by

$$F^{(\beta)}(x, \cdot) = \sum_{\alpha=1}^k \psi_{\alpha}^{(\beta)}(x)Z_{\alpha\beta}(\cdot) \in W_{x(\beta)}^F$$

For every fixed value of $\beta$, the expansion coefficients $\psi^{(\beta)} \equiv \{\psi_{\alpha}^{(\beta)}; \alpha = 1, \cdots, k\}$ can be interpreted as an ordinary Dirac spinor. Eq.(2.88) shows that acting with $\kappa\theta^\mu \circ$ on $F^{(\beta)}$ is equivalent to applying $\gamma^\mu$ on $\psi^{(\beta)}$:

$$\kappa\theta^\mu \circ F^{(\beta)} = \sum_\alpha \left( \sum_\delta \gamma^\mu \psi_{\alpha}\bar{\psi}_{\delta} \right) Z_{\alpha\beta} = \sum_\alpha [\gamma^\mu \psi^{(\beta)}]_\alpha Z_{\alpha\beta}$$

Let us arrange the expansion coefficients $\psi_{\alpha}^{(\beta)}$ as a $k \times k$-matrix: $(\hat{\psi})_{\alpha\beta} \equiv \psi_{\alpha}^{(\beta)}$. Then,

$$F(x, \theta) = \text{Tr} \left[ \hat{\psi}(x)Z(\theta)^T \right]$$

Denoting the $\hat{\psi}$-matrix which belongs to a given section $F$ by $\hat{\psi}[F]$ we obtain from (2.96)

$$\hat{\psi}[\kappa\theta^\mu \circ F] = \gamma^\mu \hat{\psi}[F]$$

which mirrors (1.27) at the symbol level.

Given a symbol-valued field $F(x, \theta)$ we can immediately construct the associated spinor matrix-valued field $\hat{F}(x)$ of (2.61) by replacing $\theta^\mu \rightarrow \kappa^{-1}\gamma^\mu$ in its series expansion (2.63). In the process of decomposing the reducible representation carried by $F$ we discovered a second spinor-matrix, $\hat{\psi}$, which is related to $F$ in a canonical way, too. By essentially the same argument as in the introduction it follows that the two matrices are equal up to a constant:

$$\hat{\psi}[F](x) = 2^{-n/2} \hat{F}(x)$$

If we insert the expansions (2.63) and (2.93) for $F(x, \theta)$ and $Z(\theta)$, respectively, into eq.(2.97), we obtain eq.(1.30) for the set $\{F_{\mu_1 \cdots \mu_p}^{(p)}\}$ expressed in terms of
Making an ansatz for \( \hat{\psi} \) in terms of antisymmetrized products of \( \gamma \)-matrices and using the trace identity (1.33), one finds that the expansion coefficients of \( \hat{\psi} \) and \( \hat{F} \) differ by an overall constant only.

While this last step was straightforward for the SO(\( n \))-spinors, it will be much less trivial for metaplectic spinors where the representation space is infinite-dimensional and trace-identities such as (1.33) are not likely to exist. It will be interesting to see how (2.99) is modified then.

**3 Symplectic Dirac-Kähler fields**

In the previous section we reformulated the theory of standard DK-fermions over space-time in terms of fields \( F \) which assume values in the fermionic Weyl algebra \( \mathcal{W}^F \). Now we are going to ask what happens if we replace \( \mathcal{W}^F \) by its (actually much more familiar) bosonic counterpart, the bosonic Weyl algebra \( \mathcal{W} \). Rather than space-time it is now a phase-space \( (\mathcal{M}_{2N}, \omega) \) which plays the role of the base manifold. As we shall argue, replacing the Riemannian structure by a symplectic one, the structure group \( SO(n) \) by \( Sp(2N) \), and, most importantly, fermionic Weyl symbols by bosonic ones, we are led to the notion of a “symplectic DK-field” in a very natural way.

In Subsection 3.1 we begin by working out some special properties of bosonic Weyl symbols which will become important in our construction. In this context, we are basically discussing the conventional quantum mechanics of the auxiliary quantum system with canonical operators \( \hat{x}^i \) and \( \hat{\pi}^i \) which results from quantizing the flat “auxiliary phase-space” \( \mathbb{R}^{2N} \). (Later on the auxiliary phase-space will be identified with the tangent space to the true (physical) phase-space \( \mathcal{M}_{2N} \).) The operators \( \hat{x}^i \) and \( \hat{\pi}^i \) take over the role previously played by \( \hat{\chi}^{\mu} \).

Subsection 3.2 is devoted to the metaplectic \( \gamma \)-matrices. In particular, we propose a symplectic analog of the chirality matrix \( \gamma_5 \) there. The actual
construction of the symplectic DK-fields is performed in Section 3.3, and in
Section 3.4 it is shown how they relate to the metaplectic spinor fields.

3.1 Bosonic Weyl symbols

We consider a hamiltonian system with $N$ degrees of freedom whose classical
phase-space is the symplectic plane $(\mathbb{R}^{2N}, \omega)$. The associated quantum me-
chanical Hilbert space is $\mathcal{V}$ and $\mathcal{L}(\mathcal{V})$ denotes the space of linear operators on
$\mathcal{V}$. The Hilbert space $\mathcal{V}$ carries a representation of the canonical commutation
relations

$$[\hat{\varphi}^a, \hat{\varphi}^b] = i \hbar \omega^{ab}; \quad a, b = 1, \ldots, 2N$$  \hspace{1cm} (3.1)

In a canonical operator basis we split $\hat{\varphi}^a \equiv (\hat{\pi}^i, \hat{x}^i)$, $i = 1, \ldots, N$, so that the
only nonvanishing commutator is between the momenta $\hat{\pi}^i$ and the positions
$\hat{x}^i$: $[\hat{\pi}^i, \hat{x}^j] = -i \hbar \delta^{ij}$. The matrix $(\omega^{ab})$ is the inverse of the constant matrix
$(\omega_{ab})$ formed from the coefficients of the symplectic 2-form $\omega$: $\omega_{ab} \omega^{bc} = \delta^c_a$. On
$(\mathbb{R}^{2N}, \omega)$ we use canonical coordinates $y^a \equiv (y^i_p, y^i_q)$ such that

$$(\omega_{ab}) = \begin{pmatrix} 0 & +I \\ -I & 0 \end{pmatrix}, \quad (\omega^{ab}) = \begin{pmatrix} 0 & -I \\ +I & 0 \end{pmatrix}$$

For the natural skew-symmetric inner product on the symplectic plane we write

$$\omega(y_1, y_2) \equiv y_1^a \omega_{ab} y_2^b$$ \hspace{1cm} (3.2)

The Weyl (or Heisenberg) operators \[\mathbb{I}\]

$$\hat{T}(y) = \exp\left(\frac{i}{\hbar} y^a \omega_{ab} \hat{\varphi}^b\right)$$ \hspace{1cm} (3.3)

implement the translations on phase-space in the Hilbert space $\mathcal{V}$:

$$\hat{T}(y)^\dagger \hat{\varphi}^a \hat{T}(y) = \hat{\varphi}^a + y^a$$ \hspace{1cm} (3.4)

This is a projective representation of the translation group since

$$\hat{T}(y_1) \hat{T}(y_2) = \exp\left[\frac{i}{2\hbar} \omega(y_1, y_2)\right] \hat{T}(y_1 + y_2)$$ \hspace{1cm} (3.5)
The Weyl operators are orthogonal and complete in the sense that

\[
\text{Tr}[\hat{T}(y_1) \hat{T}^\dagger(y_2)] = (2\pi \hbar)^N \delta^{(2N)}(y_1 - y_2) \quad (3.6)
\]

\[
\int d^{2N}y \langle \alpha | \hat{T}(y) \hat{T}^\dagger(y') \rangle = (2\pi \hbar)^N \delta^{(N)}(\alpha - \beta') \delta^{(N)}(\beta - \alpha') \quad (3.7)
\]

Here \{ |\alpha\rangle \} is the basis which diagonalizes the position operators:

\[
\hat{x}^i |\alpha\rangle = \alpha^i |\alpha\rangle, \quad \alpha \equiv (\alpha^1, \cdots, \alpha^N) \quad (3.8)
\]

Sometimes it will be more suggestive to use a tensor notation instead of the bra-ket formalism; for instance, one writes \(\hat{b}_{\alpha}^\beta \equiv \langle \alpha | \hat{b} | \beta \rangle\) for the matrix elements of some arbitrary \(\hat{b} \in \mathcal{L}(V)\) or \(\delta_{\alpha}^\beta \equiv \delta^{(N)}(\alpha - \beta)\) for the identity operator. The eigenvalues \(\alpha \in \mathbb{R}^N\) should be thought of as a continuous analog of a spinor index. In the \(\hat{x}\)-eigenbasis, the Weyl operators are given by

\[
\hat{T}(y)_{\alpha}^\beta = \exp \left[ \frac{i}{\hbar} (y_p\alpha - \frac{1}{2} y_p y_q) \right] \delta^{(N)}(\alpha - \beta - y_q) \quad (3.9)
\]

with \(y_p\alpha \equiv y_p^i \alpha^i\), etc., where the summation over \(i = 1, \cdots, N\) is understood.

From the completeness relation (3.7) it follows that every operator \(\hat{b}\) can be represented as

\[
\hat{b} = (2\pi \hbar)^{-N} \int d^{2N}y \tilde{b}(y) \hat{T}(y) \quad (3.10)
\]

with the complex-valued function \(\tilde{b}\) given by

\[
\tilde{b}(y) = \text{Tr} \left[ \hat{T}(y) \hat{T}^\dagger \right] \quad (3.11)
\]

The function \(\tilde{b}\) (referred to as the alternative Weyl symbol [16]) is closely related to the Weyl symbol of \(\tilde{b}\). In fact, \(b(y) \equiv [\text{symb}(\tilde{b})](y)\) is the Fourier transform of \(\tilde{b}\):

\[
b(y) = (2\pi \hbar)^{-N} \int d^{2N}y_0 \tilde{b}(y_0) \exp \left[ \frac{i}{\hbar} \omega(y_0, y) \right] \quad (3.12)
\]

The inverse transformation reads

\[
\tilde{b}(y) = (2\pi \hbar)^{-N} \int d^{2N}y_0 b(y_0) \exp \left[ \frac{i}{\hbar} \omega(y_0, y) \right] \quad (3.13)
\]
i.e., the symplectic Fourier transformation is an exact involution, \( \tilde{b}(y) = b(y) \)
(and not only an involution up to a reflection of the argument).

Eqs. (3.10)-(3.12) define the (bosonic) Weyl symbol map “symb” from \( \mathcal{L}(\mathcal{V}) \) to the space of (generalized) functions over the symplectic plane, as well as its inverse. The classical phase-function \( b(y) \) uniquely represents an operator \( \hat{b} \) which is Weyl ordered. In particular, the monomial \( y^{a_1} y^{a_2} \cdots y^{a_p} \) stands for the completely symmetrized operator product \( \hat{\varphi}^{(a_1 \cdots a_p)} \). Conversely,

\[
[symb\{\hat{\varphi}^{(a_1 \cdots a_p)}\}](y) = y^{a_1} y^{a_2} \cdots y^{a_p}
\]  
(3.14)

The symmetrization in (3.14) is crucial, otherwise commutator terms would occur. For instance,

\[
[symb\{\hat{\varphi}^a \hat{\varphi}^b\}](y) = y^a y^b + i\frac{\hbar}{2} \omega^{ab}
\]  
(3.15)

An important special class of symbols are those which admit a power series expansion

\[
b(y) = \sum_{p=0}^{\infty} \frac{\kappa^p}{p!} b_{a_1 \cdots a_p}^{(p)} y^{a_1} y^{a_2} \cdots y^{a_p}
\]  
(3.16)

By the symbol map, they are bijectively related to the operators

\[
\hat{b} = \sum_{p=0}^{\infty} \frac{\kappa^p}{p!} b_{a_1 \cdots a_p}^{(p)} \hat{\varphi}^{a_1} \cdots \hat{\varphi}^{a_p}
\]  
(3.17)

provided the tensors \( b_{a_1 \cdots a_p}^{(p)} \) are completely symmetric. If \( b \) is a power series, the “alternative Weyl symbol” \( \tilde{b} \) is a sum of derivatives of \( \delta \)-functions:

\[
\tilde{b}(y) = (2\pi\hbar)^N b \left( i\hbar\omega^{ac} \frac{\partial}{\partial y^c} \right) \delta^{(2N)}(y)
\]  
(3.18)

As in every symbol calculus, the pertinent star product is required to satisfy

\[
symb(\hat{b}_1 \hat{b}_2) = b_1 \circ b_2 \text{ where } b_1 \text{ and } b_2 \text{ are the symbols of } \hat{b}_1 \text{ and } \hat{b}_2, \text{ respectively.}
\]

At least for power series, the bosonic Weyl star product is uniquely determined by its associativity, the distributivity over ‘+’, and the basic relations

\[
1 \circ 1 = 1, \quad 1 \circ y^a = y^a \circ 1 = y^a \quad (3.19)
\]

\[
y^a \circ y^b = y^a y^b + i\frac{\hbar}{2} \omega^{ab}
\]
which follow from (3.14), (3.15) and symb(I) = 1. Explicit formulas for the star product \[25, 28\] of arbitrary symbols include
\[
(b_1 \circ b_2)(y) = b_1(y) \exp \left[ \frac{i}{\hbar} \frac{\partial}{\partial y^a} \omega_{ac} \frac{\partial}{\partial y^c} \right] b_2(y) \tag{3.20}
\]
and
\[
(b_1 \circ b_2)(y) = (\pi \hbar)^{-2N} \int d^{2N} y_1 d^{2N} y_2 \exp \left[ - 2i \{ \omega(y, y_1) + \omega(y_1, y_2) \} / \hbar \right] b_1(y_1) b_2(y_2) \tag{3.21}
\]

The differential operators which effect the star left-multiplication with $\kappa y^a$,
\[
(C^a b)(y) = \kappa y^a \circ b(y), \tag{3.22}
\]
are easily read off from eq.(3.21):
\[
C^a = \kappa y^a + \frac{i}{\kappa} \omega^{ab} \frac{\partial}{\partial y^b} \tag{3.23}
\]

On the space of symbols with an appropriate fall-off behavior we would like to introduce a sesquilinear inner product ($\cdot | \cdot$) with respect to which $C^a$ is selfadjoint,
\[
(C^a b_1 | b_2) = (b_1 | C^a b_2) \tag{3.24}
\]
It is clear from our earlier discussion that the choice
\[
(b_1 | b_2) = [\bar{b}_1 \circ \bar{b}_2](y = 0) \tag{3.25}
\]
meets this requirement. Since $\bar{b}_1 \circ \bar{b}_2 = \bar{b}_2 \circ \bar{b}_1$ also here, the proof is the same as in (2.54). If $b_1$ and $b_2$ are power series of the type (3.16), eq.(3.23) boils down to
\[
(b_1 | b_2) = \sum_{p=0}^{\infty} \frac{i^p}{p!} b_1^{(p)}_{\alpha_1 \ldots \alpha_p} \omega^{\alpha_1 c_1} \ldots \omega^{\alpha_p c_p} b_2^{(p)}_{c_1 \ldots c_p} \tag{3.26}
\]
It is instructive to look at various alternative ways of representing this inner product. There exists the integral representation

$$(b_1 | b_2) = (2\pi)^{-2N} \int d^{2N} y_1 d^{2N} y_2 \bar{b}_1(y_1/\kappa) e^{-i\omega(y_1,y_2)} b_2(y_2/\kappa)$$ \hfill (3.27)

which can be reexpressed in terms of a symplectic Fourier transform:

$$(b_1 | b_2) = (2\pi\hbar)^{-N} \int d^{2N} y \bar{b}_1(\frac{1}{2} y) \tilde{b}_2(y)$$ \hfill (3.28)

Furthermore, if (3.18) can be applied,

$$(b_1 | b_2) = b_2 \left( -i\kappa^{-2} \omega^{ac} \frac{\partial}{\partial y^c} \right) \bar{b}_1(y)|_{y=0}$$ \hfill (3.29)

The above formulae should be compared to their counterparts in the fermionic symbol calculus. Bosonic symbols admitting a power series expansion are characterized by sets \(\{b^{(p)}_{a_1 \cdot \cdot \cdot a_p}, p = 0, 1, 2, \cdot \cdot \cdot\}\) consisting of infinitely many symmetric tensors. Fermionic symbol functions are equivalent to a finite set \(\{f^{(p)}_{\mu_1 \cdot \cdot \cdot \mu_p}, p = 0, 1, \cdot \cdot \cdot, n\}\) of antisymmetric tensors instead.

We saw that the (modified) Hodge operator is essentially the same operation as the Grassmannian Fourier transformation. Omitting all sign factors (which anyhow have no bosonic analog) we have, schematically,

$$\ast f(\theta) \propto \blacklozenge f(\theta) \propto \tilde{f}(\theta) \propto f\left(\frac{\partial}{\partial \theta}\right) \delta(\theta)$$ \hfill (3.30)

Thus one is tempted to define a bosonic version of the Hodge operator simply by setting \((\ast b)(y) = \tilde{b}(y)\) so that \(\ast \ast = 1\) on any \(b\). If \(b\) is a power series, eq.(3.18) is indeed formally analogous to (A.24) for the fermionic Fourier transformation. However, the difference is that the derivative of the fermionic delta-function, \(f(\partial/\partial \theta)\delta(\theta)\), again is a powers series in the \(\theta\)'s, while this is of course not true for the derivatives of the bosonic delta-function, \(\delta^{(2N)}(y)\). In the former case, the monomials \(\theta^{\mu_1} \cdot \cdot \cdot \theta^{\mu_p}\) are mapped onto monomials of the same type. Therefore one set of antisymmetric tensors \(\{f^{(p)}_{a_1 \cdot \cdot \cdot a_p}\}\) is mapped onto another
set of such tensors. In the latter case, the space of symmetric tensors \( b^{(p)}_{a_1\cdots a_p} \) is not mapped onto itself. The image of \( y^{a_1} y^{a_2} \cdots y^{a_p} \) is a singular symbol 
\[ \propto \partial_{y}^{a_1} \cdots \partial_{y}^{a_p} \delta(2N)(y), \quad \partial_{y}^{a} \equiv \omega^{ab} \partial/\partial y^{b}. \]

Nevertheless it will be helpful to think of the symplectic Fourier transformation as the bosonic (symmetric tensor) analog of the Hodge operator. For instance, by (2.48) with (A.22) and (A.40) the fermionic inner product has the same general structure as (3.28):
\[
(f_1|f_2) \propto \int \bar{f}_1(\theta) \bar{f}_2(\theta) d^n\theta 
= \int \bar{f}_1(\theta) (\ast f_2)(\theta) d^n\theta
\]
(3.31)

In the language of differential forms this is nothing but the familiar inner product \( \ast(\bar{\Phi}_1 \wedge \ast \Phi_2) \) in disguise. The product \( (b_1|b_2) \) introduced above is analogous to it, but refers to symmetric rather than antisymmetric tensors.

The space of symbols \( b(y) \) equipped with the pointwise product of functions, the star product, and the inner product constitutes the bosonic Weyl algebra \( \mathcal{W} \). It is the counterpart of the algebra \( \mathcal{W}^F \) which, endowed with analogous structures, had turned out to be an Atiyah-Kähler algebra. Because \( (y^a|y^b) = i\hbar \omega^{ab}/2 \) we see that the three product structures on \( \mathcal{W} \) satisfy the consistency condition
\[
y^a \circ y^b = y^ay^b + (y^a|y^b)
\]
(3.32)

This relation is completely analogous to eq. (2.46) which had been identified with the defining property of an Atiyah-Kähler algebra, eq. (1.7). This supports our point of view that the bosonic Weyl algebra is the natural analog of an Atiyah-Kähler algebra if one works in a symplectic rather than a Riemannian setting.
3.2 \( \gamma \)-Matrices for \( \text{Mp}(2N) \) and the analog of \( \gamma_5 \)

The generators of \( \text{Mp}(2N) \) in the spinor representation are obtained as symmetrized bilinears \( \Sigma_{\text{meta}}^{ab} = (\gamma^a \gamma^b + \gamma^b \gamma^a)/4 \) built from \( 2N \) \( \gamma \)-matrices" satisfying

\[
\gamma^a \gamma^b - \gamma^b \gamma^a = 2i \omega^{ab} \tag{3.33}
\]

Upon identifying

\[
\gamma^a = \kappa \hat{\varphi}^a \tag{3.34}
\]

it is clear that the relations (3.33) coincide precisely with (3.1). Hence, what in the language of group theory is called a “symplectic Clifford algebra” is nothing but the canonical commutation relations of a bosonic quantum system with the canonical operators \( \hat{\varphi}^a = (\hat{\pi}^j, \hat{x}^j) \). For \( N \) finite, all irreducible representations of the canonical commutation relations are unitarily equivalent, so the same is true for the symplectic Clifford algebra. All these representations are infinite dimensional.

We consider representations where \( \gamma^a \) is a hermitian operator on the Hilbert space \( \mathcal{V} \). Frequently \( \mathcal{V} \) is taken to be the Fock space of \( N \) independent harmonic oscillators \([17, 31]\). Then the \( \gamma^a \)'s are linear combinations of the corresponding creation and annihilation operators. Here we shall employ another representation which is particularly natural in the gauge theory approach to quantum mechanics \([21]\). We pick the \( \hat{x} \)-eigenbasis (3.8) with respect to which

\[
\langle \alpha | \hat{x}^j | \beta \rangle = \alpha^j \delta^{(N)}(\alpha - \beta) \quad \text{and} \quad \langle \alpha | \hat{\pi}^j | \beta \rangle = -i \hbar \partial_j \delta^{(N)}(\alpha - \beta). \]

Therefore, in a symbolic matrix notation with \( \langle \alpha | \gamma^a | \beta \rangle \equiv (\gamma^a)^\alpha_\beta \),

\[
(\gamma^j)^\alpha_\beta = -(2i/\kappa) \partial_j \delta^{(N)}(\alpha - \beta) \\
(\gamma^{N+j})^\alpha_\beta = \kappa \alpha^j \delta^{(N)}(\alpha - \beta) \quad ; \quad j = 1, \ldots, N \tag{3.35}
\]

The Hilbert space \( \mathcal{V} \) is the space of square integrable functions \( \psi(\alpha) \equiv \langle \alpha | \psi \rangle \equiv \psi^\alpha \) with its usual inner product. The generators \( \Sigma_{\text{meta}}^{ab} \) act on \( \mathcal{V} \) as second order
differential operators (Schrödinger Hamiltonians with a quadratic potential; see refs. [22, 23] for further details).

Any attempt at putting metaplectic spinors on a similar footing as the \( SO(n) \)-spinors faces the problem that \( V \) is infinite dimensional and that a metaplectic spinor formally is an object \( \psi^\alpha \equiv \psi^{(\alpha)} \) with infinitely many components. As an immediate consequence, trace identities such as (1.33) have no direct counterpart for the metaplectic \( \gamma \)-“matrices”. In the \( \hat{\mathcal{F}} \)-basis, for instance, the trace of an operator \( \hat{b} \in \mathcal{L}(V) \) reads \( \text{Tr}(\hat{b}) = \int d^{N} \alpha \langle \alpha | \hat{b} | \alpha \rangle \), and it is clear that monomials such as \( \gamma^{a_1} \cdots \gamma^{a_p} \) do not possess a trace. Remarkably enough, it turns out that there exist identities similar to (1.33) even in the infinite dimensional case which, however, involve the \( \text{Sp}(2N) \)-analog of \( \gamma_{n+1} \).

We are familiar with the fact that when we are dealing with spinors on an even-dimensional space-time there exists a chirality matrix \( \gamma_{n+1} \), a generalization of \( \gamma_5 \) in 4 dimension, which anticommutes with any \( \gamma^\mu \). Its eigenvalues are \(-1\) and \(+1\), and the corresponding eigenspaces are the left- and right-handed Weyl spinors, respectively. It is quite interesting that we can introduce an analogous concept for metaplectic spinors and that the pertinent “chirality operator” has a very natural interpretation even. Let us try to find an operator \( \gamma_P \in \mathcal{L}(V) \) which anticommutes with all \( \gamma^a \)'s,

\[
\gamma_P \gamma^a + \gamma^a \gamma_P = 0
\]

and satisfies

\[
\gamma_P^\dagger = \gamma^{-1} = \gamma_P
\]

Thus \( \gamma_P \) has the same algebraic properties as \( \gamma_5 \), its eigenvalues are \( \pm 1 \) and, provided it actually exists, we can use it to form the “chiral” projections

\[
\psi_{\pm} = \Pi_{\pm} \psi, \quad \Pi_{\pm} \equiv \frac{1}{2}(1 \pm \gamma_P)
\]

of any \( \psi \in \mathcal{V} \). Since \( \Sigma_{meta}^{ab} \) commutes with \( \gamma_P \), the \( \text{Mp}(2N) \)-transformations
leave the subspaces with $\gamma_P = +1$ and $\gamma_P = -1$ invariant, so that the representation of $\text{Mp}(2N)$ implied by the $\gamma$-matrices (3.33) decomposes accordingly.

Looking at the “metaplectic $\gamma_5$-matrix” from the point of view of the auxiliary quantum mechanics with the $\hat{\varphi}$-degrees of freedom it becomes clear that we may identify $\gamma_P$ with the standard parity operator $P$ in this context. By definition, $P$ changes the sign of both the positions $\hat{x}^i$ and the momenta $\hat{\pi}^i$: $P\hat{x}^i P = -\hat{x}^i$, $P\hat{\pi}^i P = -\hat{\pi}^i$. Hence $P\gamma^a P = -\gamma^a$ for $\gamma^a = \kappa(\hat{\pi}^i, \hat{x}^i)$, which is exactly (3.33) with $\gamma_P \equiv P$. The operator $P$ acts on the wave functions $\psi \in \mathcal{V}$ as $(P\psi)(\alpha) \equiv (\gamma_P\psi)(\alpha) = \psi(-\alpha)$. This means that in the $\hat{x}$-representation

$$\gamma_P |\alpha\rangle = | -\alpha\rangle$$

so that the matrix elements of $\gamma_P$ are given by

$$(\gamma_P)^\alpha_\beta \equiv \langle \alpha |\gamma_P |\beta\rangle = \delta^{(N)}(\alpha + \beta)$$

Thus, “metaplectic chirality” is nothing but “fiberwise parity”, and the projections $\Pi_{\pm} \mathcal{V}$ are simply the subspaces of even and odd wave functions, respectively.

The operator $\gamma_P$ can be written in a manifestly basis independent way:

$$\gamma_P = (4\pi\hbar)^{-N} \int d^{2N}y \hat{T}(y)$$

The general properties of the Weyl operators imply that (3.41) has the desired properties (3.36), (3.37) and using the matrix elements (3.9) one finds that (3.41) coincides with (3.40). Eq.(3.41) is strikingly similar to eq.(2.44) for $\hat{G}_{n+1}$ which confirms our interpretation that the fiberwise parity transformation is the analog of $\gamma_5$.

The operator $\gamma_P$ has a well defined finite trace:

$$\text{Tr}[\gamma_P] = 2^{-N}$$

$^6$Eq.(3.41) shows that $\gamma_P$ belongs to the family of parity-type operators discussed by Grossmann [32] and Royer [33].
This follows from (3.41) with (3.6) or simply by noting that

\[ \text{Tr} [\gamma_P] = \int d^N \alpha (\gamma_P)^\alpha = \int d^N \alpha \delta^{(N)}(2\alpha) = 2^{-N} \]  

(3.43)

While the very existence of this trace is remarkable, we see the first major difference between the bosonic and the fermionic case here. Both \( \gamma_5 \) and \( \gamma_P \) have eigenvalues \( \pm 1 \), but the pairing of positive and negative eigenvalues which leads to \( \text{Tr}(\gamma_5) = 0 \) does not happen for \( \gamma_P \).

Finite products of \( \gamma^a \)-matrices and in particular the unit operator do not possess a well defined trace. On the other hand, traces with a \( \gamma_P \)-insertion,

\[ \text{Tr} [\hat{b} \gamma_P] = \int d^N \alpha \langle \alpha | \hat{b} | -\alpha \rangle \]  

(3.44)

are much better behaved because the reflection \( \alpha \mapsto -\alpha \) removes possible “short distance singularities” (reminiscent of ultraviolet divergences in field theory) which would plague \( \langle \alpha | \hat{b} | \alpha \rangle \).

This situation is quite similar to what one encounters in quantum field theory in the computation of chiral anomalies or, from a mathematical point of view, of the analytical index of the Dirac operator \[34, 35\]. There one considers \( \text{Tr}(I) \) and \( \text{Tr}(\gamma_5) \) where the trace is over the infinite dimensional Hilbert space of Dirac spinor fields. While \( \text{Tr}(I) \) does not exist, \( \text{Tr}(\gamma_5) \) can be interpreted as the index of the Dirac operator.

An important trace of the type (3.44) is

\[ \text{Tr}[\gamma^{(a_1 \cdots a_p)} \gamma_P \gamma^{(b_1 \cdots b_q)}] = i^p 2^{-N} p! \delta^{a_1} \delta^{a_2} \cdots \delta^{a_p} \]  

(3.45)

with the convenient abbreviation \( \gamma_a \equiv \omega_{ab} \gamma^b \), \( \omega^{ab} \gamma_b = \gamma^a \). Eq.(3.45) is similar to (1.33) for the \( \text{SO}(n) \) \( \gamma \)-matrices, but contains an additional factor of \( \gamma_P \) without which the trace would not exist. Eq.(3.43) follows from the properties for the \( \hat{T} \)-operators. First one uses (3.41) with (3.6) to show that

\[ \text{Tr}[\hat{T}(y) \gamma_P] = 2^{-N} \]  

(3.46)
is independent of $y$. Next one writes
\[
\text{Tr}[\hat{T}(y_1)\gamma_P\hat{T}(y_2)] = \text{Tr}[\hat{T}(y_1)\hat{T}(-y_2)\gamma_P] \\
= \exp\left[\frac{i}{2\hbar}\omega(y_1,-y_2)\right] \text{Tr}[\hat{T}(y_1 - y_2)\gamma_P] \\
= 2^{-N}\exp\left[-\frac{i}{2\hbar}y_1^a\omega_{ab}y_2^b\right] 
\]
(3.47)

If one now expands the first and the last expression of (3.47) in powers of $y_1$ and $y_2$ and equates equal powers, the result is precisely eq.(3.45).

Some important special cases of (3.45) include
\[
\text{Tr}[\gamma^a\gamma_P] = 0 \\
\text{Tr}[\gamma^{(a_1\ldots a_p)}\gamma_P] = 0 \\
\text{Tr}[\gamma^a\gamma^b\gamma_P] = 2^{-N}i\omega^{ab} 
\]
(3.48)

The reader is invited to check some of these relations by using the matrix elements of $\gamma^a$ in the $|\alpha\rangle$-basis. It is instructive to see that these calculations involve only well defined manipulations of distributions and that no additional ad hoc regularization is needed. This is different from the derivation of the closely related dimension-counting formulas for the spinors of $OSp(n|2N)$ which appear in certain approaches to the covariant quantization of superstrings [18], for instance.

3.3 The Dirac-Kähler construction on phase-space

Let $(\mathcal{M}_{2N},\omega)$ denote an arbitrary $2N$-dimensional symplectic manifold which serves as the phase-space of some hamiltonian system. Let us consider the Weyl algebra bundle [30, 31] over $\mathcal{M}_{2N}$. Its typical fiber is the bosonic Weyl algebra $\mathcal{W}$, i.e. the space of symbols $b(\cdot)$ equipped with the pointwise product of functions, the star product, and the inner product $(\cdot|\cdot)$. At each point $\phi$ of $\mathcal{M}_{2N}$ we attach a local copy $\mathcal{W}_\phi$ of $\mathcal{W}$. The matrix $(\omega_{ab})$ which enters the definition of the Weyl algebra $\mathcal{W}_\phi$ are the coefficients of the symplectic
2-form $\omega$ evaluated at the point $\phi$. By virtue of Darboux’s theorem, there exist local coordinates $(\phi^a)$ such that those coefficients assume their canonical form on the entire $(\phi^a)$-chart. Local coordinates on the total space are pairs $(\phi, b)$ with $b$ a function $b : \mathbb{R}^{2N} \to \mathbb{C}, y \mapsto b(y)$. The transition functions of the bundle are defined in such a way the variables $(y^1, \ldots, y^{2N})$ on which $b$ depends are the components of a vector $y \in T_\phi M_{2N}$, i.e. $y^a = d\phi^a(y)$. A symplectic change of coordinates $\phi^a \to \tilde{\phi}^a(\phi)$ (canonical transformation) is to be combined with a transformation in the fiber, $b \to \tilde{b}$, such that $\tilde{b}(\tilde{y}) = b(y)$ with $\tilde{y}^a = (\partial\tilde{\phi}^a/\partial\phi^b)y^b$.

Along with the Weyl algebra bundle we also consider the metaplectic spinor bundle over $(M_{2N}, \omega)$ which we described in the Introduction. Its fiber at $\phi$, $V_\phi$, is a copy of the Hilbert space $V$ on which we already constructed a representation of the metaplectic Clifford algebra and, as a consequence, of the structure group $Mp(2N)$.

Let us look at sections through the Weyl algebra bundle. Locally they are specified by functions $\phi \mapsto B(\phi, \cdot) \in \mathcal{W}_\phi$ where $B(\phi, \cdot) : \mathbb{R}^{2N} \to \mathbb{C}, y \mapsto B(\phi, y)$ is a Weyl symbol “living” in the fiber at $\phi$. In this context, the flat “auxiliary phase-space” $\mathbb{R}^{2N}$ is identified with the tangent space $T_\phi M_{2N}$. Hence the function $B(\cdot, \cdot)$ is a map from (a part of ) the total space of the tangent bundle into $\mathbb{C}$.

Many of the concepts which we developed in Section 3.1 for symbols $b \in \mathcal{W}$ naturally pass over to the sections $B$. At every point $\phi$ of $M_{2N}$ we can apply the inverse symbol map to $B(\phi, \cdot) \in \mathcal{W}_\phi$ and obtain a unique operator $\widehat{B}(\phi) = \text{symb}^{-1}B(\phi, \cdot)$ which acts on the local copy $V_\phi$ of the Hilbert space $V$. Thus a section $B$ gives rise to a family of operators $\widehat{B}(\phi) \in \mathcal{L}(V_\phi)$ labeled by the points of phase-space. Its matrix elements with respect to a given basis in $V$ will be denoted $\widehat{B}(\phi)_{\alpha}^\beta \equiv \langle \alpha | \widehat{B}(\phi) | \beta \rangle$. Globally speaking, $\widehat{B}$ is a section through the bundle of $(1,1)$-multispinors $\{22, 21\}$. 
The fiberwise star product of two sections is defined by

\[
(B_1 \circ B_2)(\phi, y) = B_1(\phi, y) \exp \left[ \frac{i\hbar}{2} \frac{\partial}{\partial y^a} \omega^{ab} \frac{\partial}{\partial y^b} \right] B_2(\phi, y)
\]  

(3.49)

This star product has to be carefully distinguished from the \( \ast_{\mathcal{M}} \)-product whose associated Moyal bracket \( \{ f, g \}_M = (f \ast_{\mathcal{M}} g - g \ast_{\mathcal{M}} f) / i\hbar \) replaces the classical Poisson bracket in the deformation quantization approach [24, 36], and which involves derivatives with respect to \( \phi^a \) rather than \( y^a \). In general, the \( \ast_{\mathcal{M}} \)-product is much more complicated than the \( \circ \)-product. It can be constructed iteratively by Fedosov’s method [29, 37, 38, 39, 40], but we shall not need it in the present context.

The fiberwise inner product of two sections is given by \( (B_1 \mid B_2)(\phi) = (\bar{B}_1 \circ B_2)(\phi, 0) \). The natural sesquilinear form on the space of sections is \( \langle B_1 \mid B_2 \rangle = \int d\mu_L(B_1 \mid B_2) \) where \( d\mu_L \) is the Liouville measure.

After these preparations we are now able to construct an analog of the Dirac-Kähler fields on phase-spaces.

Let \( \bigotimes^p_{\text{sym}}(T^*\mathcal{M}_{2N}) \) denote the \( p \)-fold symmetrized tensor power of the cotangent bundle. A section \( \Sigma^{(p)} \) through this bundle is a symmetric tensor field of rank \( p \). We shall also consider the direct sum

\[
\bigotimes_{\text{sym}} (T^*\mathcal{M}_{2N}) = \bigoplus_{p=0}^{\infty} \bigotimes^p_{\text{sym}}(T^*\mathcal{M}_{2N})
\]  

(3.50)

Its sections \( \Sigma = \sum_{p=0}^{\infty} \Sigma^{(p)} \) are analogous to the inhomogeneous differential forms, but with symmetric rather than antisymmetric tensor fields. In local (Darboux) coordinates \( \phi^a \), \( \Sigma \) can be expanded as

\[
\Sigma(\phi) = \sum_{p=0}^{\infty} \frac{1}{p!} B_{a_1 \cdots a_p}^{(p)}(\phi) d\phi^{a_1} \otimes_{\text{sym}} d\phi^{a_2} \otimes_{\text{sym}} \cdots \otimes_{\text{sym}} d\phi^{a_p}
\]  

(3.51)

with \( \otimes_{\text{sym}} \) denoting the symmetric counterpart of the wedge product; for instance, \( d\phi^a \otimes_{\text{sym}} d\phi^b = d\phi^a \otimes d\phi^b + d\phi^b \otimes d\phi^a \). The complex-valued coefficients

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$B^{(p)}_{a_1\cdots a_p}$ are taken to be completely symmetric in all $p$ indices. We shall refer to $\Sigma$ as an “inhomogeneous symmetric tensor" (IST).

Guided by the corresponding construction in the fermionic case we shall now associate an operator $\hat{B}(\phi) \in \mathcal{L}(\mathcal{V}_\phi)$ to $\Sigma(\phi)$ by replacing in eq.(3.51) the differentials $d\phi^a$ with the Gamma-matrices $\gamma^a$:

$$\hat{B}(\phi) = \sum_{p=0}^{\infty} \frac{1}{p!} B^{(p)}_{a_1\cdots a_p}(\phi) \gamma^{a_1} \gamma^{a_2} \cdots \gamma^{a_p}$$

$$= \sum_{p=0}^{\infty} \frac{\kappa^p}{p!} B^{(p)}_{a_1\cdots a_p}(\phi) \hat{\varphi}^{a_1} \hat{\varphi}^{a_2} \cdots \hat{\varphi}^{a_p} \quad (3.52)$$

(As discussed earlier, we interpret the $\gamma$-matrices as $\kappa$ times the canonical operators $\hat{\varphi}^a$ of the auxiliary quantum system in the fiber). Conversely, every Weyl ordered operator on $\mathcal{V}_\phi$ which admits a power series expansion gives rise to a unique IST. The operator can be expanded in the symmetrized monomials $\hat{\varphi}^{(a_1 \cdots a_p)}$ with coefficients which are symmetric tensors and define an IST therefore.

Now we form the symbol of $\hat{B}$: $B(\phi, y) = \text{symb}\hat{B}(\phi)(y)$, i.e.

$$B(\phi, y) = \sum_{p=0}^{\infty} \frac{\kappa^p}{p!} B^{(p)}_{a_1\cdots a_p}(\phi) y^{a_1} y^{a_2} \cdots y^{a_p} \quad (3.53)$$

Taking both steps together we arrive at a one-to-one correspondence between IST’s and symbols with a power series expansion in $y$:

$$\Sigma(\phi) \in \bigotimes_{\text{sym}} (T_\phi^* \mathcal{M}_{2N}) \iff \hat{B}(\phi) \in \widetilde{\mathcal{L}}(\mathcal{V}_\phi) \iff B(\phi, \cdot) \in \widetilde{\mathcal{W}}_\phi \quad (3.54)$$

This chain of bijections is similar to (2.64). However, the difference is that in the fermionic setting every symbol or every Weyl ordered operator gives rise to an inhomogeneous tensor field. This is not true in the bosonic case. We have to explicitly restrict the symbols and operators to those which allow for a power series expansion in $y^a$ or $\hat{\varphi}^a$, respectively. (This is indicated by the notation $\widetilde{\mathcal{L}}(\mathcal{V}_\phi)$ and $\widetilde{\mathcal{W}}_\phi$.) Nevertheless we shall continue to consider also
symbols $B$ which are not analytic in $y$ because they will play a central role in the reduction of symplectic DK-fields.

Now let us look at the rules of the symbol/tensor-correspondence in the bosonic case. Clearly the differentials $d\phi^a$ correspond to $\kappa y^a : d\phi^a \Rightarrow \kappa y^a$. Hence, if we write the (linear, invertible) map from the symbols to the IST’s as $B \mapsto \Sigma[B]$, we have $\Sigma[\kappa y^a] = d\phi^a$ or more generally

$$\Sigma[\kappa^p y^{a_1} \cdots y^{a_p}] = d\phi^{a_1} \otimes_{\text{sym}} \cdots \otimes_{\text{sym}} d\phi^{a_p} \quad (3.55)$$

Up to this point the situation is the same as in Section 2.4 with the commuting $y$’s replacing the anticommuting $\theta$’s. This converts the wedge product to the symmetric tensor product. Differences become manifest when we look at the list of natural operations for IST’s and their realization at the symbol level.

The automorphism $A$ and the antiautomorphism $B$, while important for dealing with the ubiquitous sign factors in exterior algebra computations, are unnecessary for symmetric tensors. As we argued already, the Hodge operator has a natural bosonic translation, the symplectic Fourier transformation. However it does not leave the space $\tilde{W}$ invariant and, as a consequence, does not induce a map of one IST onto another. Furthermore, the exterior derivative is a derivation on the exterior algebra which does not require a connection for its definition. Also this concept has no analog on the bosonic side.

However, every vector field $v = v^a(\phi)\partial_a$ on $\mathcal{M}_{2N}$ gives rise to a contraction operator $i(v)$. By definition, it is a linear operator on the space of IST’s, depending linearly on $v$, and satisfying $i(\partial_a)1 = 0, i(\partial_a)d\phi^b = \delta^b_a$ as well as

$$i(v)[\Sigma_1 \otimes_{\text{sym}} \Sigma_2] = [i(v)\Sigma_1] \otimes_{\text{sym}} \Sigma_2 + \Sigma_1 \otimes_{\text{sym}} [i(v)\Sigma_2] \quad (3.56)$$

Its realization on $\tilde{W}$ reads

$$i(v) \Sigma[B] = \Sigma[\kappa^{-1} v^a \frac{\partial}{\partial y^a} B] \quad (3.57)$$

We also define the operators

$$e_a \equiv i(\partial_a), \quad e^a \equiv \omega^{ab} i(\partial_b) \quad (3.58)$$
with the basis vectors \( \partial_a \equiv \partial/\partial \phi^a \) referring to a system of Darboux local coordinates.

The most important properties of the fermionic Weyl algebra \( \mathcal{W}^F \) were the three different product structures with which it is endowed and which make it an Atiyah-Kähler algebra. The bosonic Weyl algebra \( \mathcal{W} \) is equipped with three analogous products (pointwise multiplication, star product, inner product) which satisfy the basic consistency condition (3.32). At the end of Section 3.1 this led us to the conclusion that \( \mathcal{W} \) is the symplectic counterpart of an Atiyah-Kähler algebra. In the same sense the IST’s \( \Sigma \) are analogous to the Dirac-Kähler fields \( \Phi \).

The product structures on \( \mathcal{W} \) give rise to related products on the space of symmetric tensor fields. One easily verifies that the pointwise product of bosonic symbols is tantamount to the symmetric tensor product:

\[
\Sigma[B_1] \otimes_{\text{sym}} \Sigma[B_2] = \Sigma[B_1 B_2] \tag{3.59}
\]

Furthermore, guided by our experience with the fermionic case, we now define the Clifford product for symmetric tensor field as the image of the bosonic star product under the symbol/tensor correspondence (3.54):

\[
\Sigma[B_1] \lor \Sigma[B_2] = \Sigma[B_1 \circ B_2] \tag{3.60}
\]

By construction, the “symplectic Clifford product”, also denoted ‘\( \lor \)’, is associative and distributive (but not commutative). From eqs. (3.49), (3.57) and (3.58) one obtains the following explicit representation for the product of two IST’s:

\[
\Sigma_1 \lor \Sigma_2 = \sum_{p=0}^{\infty} \frac{i^p}{p!} [e_{a_1} \cdots e_{a_p} - \Sigma_1] \otimes_{\text{sym}} [e^{a_1} \cdots e^{a_p} - \Sigma_2] \tag{3.61}
\]

This equation is strikingly similar to Kähler’s formula (1.9) for the ordinary Clifford product. We emphasize that while eq. (3.61) might look complicated
it is uniquely determined by the fundamental relations

\[ 1 \lor 1 = 1, \quad 1 \lor d\phi^a = d\phi^a \lor 1 = d\phi^a \]

\[ d\phi^a \lor d\phi^b = d\phi^a \otimes_{\text{sym}} d\phi^b + i\omega^{ab} \quad (3.62) \]

if associativity and distributivity are imposed.

Turning to the last product structure on \( W \), there is an obvious choice for a fiberwise inner product \((\cdot, \cdot)\) of symmetric tensor fields: \((\Sigma_1, \Sigma_2) = (B_1|B_2)\) where \(\Sigma_{1,2}\) is related to \(B_{1,2}\) via (3.34). Thus it is clear that the IST's may be regarded as sections through a “symplectic Atiyah-Kähler bundle”.

The left-multiplication by the basis element \( d\phi^a \) reads explicitly

\[ d\phi^a \lor \Sigma = d\phi^a \otimes_{\text{sym}} \Sigma + i e^a \rightarrow \Sigma \quad (3.63) \]

It defines a representation of the symplectic Clifford algebra in the space of inhomogeneous symmetric tensor fields:

\[ d\phi^a \lor d\phi^b - d\phi^b \lor d\phi^a = 2i\omega^{ab} \quad (3.64) \]

Comparing (3.64) to (3.33), \( d\phi^a \lor \) takes the place of the metaplectic Dirac-matrix \( \gamma^a \). Since \( d\phi^a = \Sigma[\kappa y^a]\), \( d\phi^a \lor \) applied to tensors is the same as \( \kappa y^a \circ \) applied to symbols:

\[ d\phi^a \lor \Sigma[B] = \Sigma[\kappa y^a \circ B] = \Sigma[C^a B] \quad (3.65) \]

The differential operators \( C^a \) were introduced in eq.(3.23). They are formally self-adjoint with respect to the inner product \((\cdot|\cdot)\). They constitute a representation of the symplectic Clifford algebra in space of bosonic Weyl symbols:

\[ C^a C^b - C^b C^a = 2i\omega^{ab} \quad (3.66) \]

Since \( \kappa y^a \) is the symbol of \( \kappa \hat{\gamma}^a = \gamma^a \), the operator associated to \( C^a B \) is \( \gamma^a \hat{B} \) with \( \hat{B} = \text{symb}^{-1}(B) \). In summary, we have the chain of correspondences

\[ d\phi^a \lor \Sigma \quad \Rightarrow \quad \gamma^a \hat{B} \quad \Rightarrow \quad C^a B \quad (3.67) \]
Thus we managed to implement the essence of the Dirac-Kähler idea in a symplectic rather than a Riemannian setting. We constructed a representation of the corresponding Clifford algebra on the space of symmetric tensor fields over a phase-space manifold rather than on the exterior algebra over spacetime.

Up to this point our considerations focused on the kinematic aspects of the theory. We have not yet found an analog of the DK-equation. Since $d$ and $d^\dagger$ do not exist for symmetric tensors, the DK-operator $d - d^\dagger$ has no direct counterpart. Still it is possible to write down a “symplectic DK-equation” with the necessary covariance properties:

$$[d \phi^a \vee \nabla_a + m] \Sigma = 0$$ (3.68)

(Here $\nabla$ is a symplectic connection.) This equation could be rewritten as a set of “metaplectic Dirac equations” in the same way as the ordinary DK-equation can be decomposed into a set of ordinary Dirac equations. Metaplectic Dirac operators have been investigated in the mathematical literature recently [41] but no physical application has emerged so far. In Section 4 we shall see that from a kinematical and representation theory point of view the symplectic DK-fields indeed do play an important role in the gauge theory approach to quantization. The interpretation of field equations such as eq.(3.68), if any, will remain an open problem though.

We close this section with a few comments on the “metaplectic $\gamma_5$-matrix” in relation to the DK-fields. In the $SO(n)$-case we saw that $\gamma_{n+1}$, the volume form, and the $\delta$-function are different guises of the same object. Some properties of $\gamma_{n+1}$ are similar in the symplectic case, others are quite different. The symbol of $\gamma_P$, too, is proportional to a $\delta$-function,

$$G_P \equiv \text{symb}(\gamma_P), \quad G_P(y) = (\pi \hbar)^N \delta^{(2N)}(y)$$ (3.69)

This symbol is completely unrelated to the volume form, however. In the
$SO(n)$-case we know that the Clifford right multiplication by $G_{n+1}$ is equivalent to the modified Hodge operator ($\star f \propto f \circ G_{n+1} \propto f \circ \delta^{(n)}$). This property has a partial analog since by virtue of (3.21) the symplectic Fourier transformation which corresponds to $\star$ is essentially the same operation as the star multiplication by $G_P$ from the right:

$$\tilde{b}(y) = 2^{-N} (b \circ G_P)(y/2) \quad (3.70)$$

However, this statement on the space of symbol functions (including distributions) does not imply a corresponding relation for symmetric tensors. The symbol $G_P$ has no IST associated to it.

The matrix $\gamma_P$ makes its appearance also in the natural inner product on $\mathcal{L}(\mathcal{W}_\phi)$. By virtue of the identity

$$\langle B_1 | B_2 \rangle = 2^N \text{Tr} \left[ \hat{B}_1^\dagger \hat{B}_2 \gamma_P \right] \quad (3.71)$$

the inner product on $\mathcal{W}_\phi$ induces a corresponding product for the operators. The latter differs from the familiar Hilbert-Schmidt inner product by the additional $\gamma_P$-matrix which tends to improve the regularity properties of the trace. Eq. (3.71) is most easily proven as follows:

$$\langle B_1 | B_2 \rangle = (B_1 \circ B_2)(y = 0)$$

$$= (\pi \hbar)^{-N} \int d^{2N}y \ (\tilde{B}_1 \circ \tilde{B}_2)(y) G_P(y)$$

$$= (\pi \hbar)^{-N} \int d^{2N}y \ (\tilde{B}_1 \circ \tilde{B}_2 \circ G_P)(y)$$

$$= (\pi \hbar)^{-N} \int d^{2N}y \ [\text{symb} \{ \hat{B}_1^\dagger \hat{B}_2 \gamma_P \}](y)$$

$$= 2^N \text{Tr}[\hat{B}_1^\dagger \hat{B}_2 \gamma_P] \quad (3.72)$$

Here (3.69) was used along with the standard results $\int d^{2N}y b_1(y)b_2(y) = \int d^{2N}y (b_1 \circ b_2)(y)$ and $\text{Tr}(\hat{b}) = (2\pi \hbar)^{-N} \int d^{2N}y b(y)$. 

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3.4 Decomposition of the symplectic DK-representation

We have seen that $d\phi^a \vee$ and $\kappa y^a \circ$ induce a representation of the symplectic Clifford algebra on the space of symmetric tensors and their symbols, respectively. We also saw that the corresponding representations in the $SO(n)$-case are reducible, so it is natural to ask if the same is true in the symplectic setting. We shall demonstrate that at the level of the symbols the representation is indeed reducible. However, in contradistinction to the $SO(n)$-case, the decomposition of $W$ does not induce a concomitant decomposition of the (symmetric) tensor algebra.

We shall see that the representation of the symplectic Clifford algebra carried by the symbol-valued fields $B(\phi, y)$ can be decomposed into infinitely many irreducible representations each of which is equivalent to the one defined by the metaplectic $\gamma$-matrices (3.35). (We recall that this is the representation of the Heisenberg algebra used in conventional canonical quantization.) As a consequence, every field $B(\phi, y)$ amounts to a collection of infinitely many metaplectic spinor fields $\psi^\alpha(\phi)$. Now we discuss the question of the (ir)reducibility for the symbols $B$, the operators $\hat{B}$ and the tensors $\Sigma$ separately.

(a) Symbols

We are going to show that the bosonic Weyl algebra $\mathcal{W}$ admits an orthogonal decomposition

$$\mathcal{W} = \bigoplus_{\alpha \in \mathbb{R}^N} \mathcal{W}_\langle \rangle$$

such that the subspaces $\mathcal{W}_\langle \rangle$ are invariant under star-left multiplication by $y^a$, i.e. $y^a \circ b \in \mathcal{W}_\langle \rangle$ if $b \in \mathcal{W}_\langle \rangle$. To this end we use an infinite dimensional generalization of the Becher-Joos method [3]. We look for a $2N$-parameter
family of operators \( \hat{Z}(y), y \in \mathbb{R}^{2N} \), with the property

\[
y^a \circ \hat{Z}(y) = \hat{Z}(y) \hat{\varphi}^a
\]

One should think of \( \hat{Z}(\cdot) \) as an operator-valued symbol, i.e. the ‘\( y^a \circ \)’ in (3.74) is given by \( \kappa^{-1} C^a \) as if \( \hat{Z} \) was an ordinary symbol. With our experience from the fermionic case we suspect that \( \hat{Z} \) should be closely related to the Weyl operators. It turns out that this is indeed the case. The derivative of the Weyl operators reads

\[
\frac{\partial}{\partial y^a} \hat{T}(y) = i \frac{\hbar}{\omega_{ab}} (\hat{\varphi}^b - \frac{1}{2} y^b) \hat{T}(y) = i \frac{\hbar}{\omega_{ab}} \hat{T}(y) (\hat{\varphi}^b + \frac{1}{2} y^b)
\]

This equation entails that the argument of \( \hat{T} \) can be rescaled in such a way that left multiplication with \( y^a \) is equivalent to the operator multiplication by \( \hat{\varphi}^a \), either from the left or from the right:

\[
y^a \circ \hat{T}(\pm 2iy) = \mp i \hat{\varphi}^a \hat{T}(\pm 2iy)
\]

\[
y^a \circ \hat{T}(\pm 2y) = \mp \hat{T}(\pm 2y) \hat{\varphi}^a
\]

Hence

\[
\hat{Z}(y) = \hat{T}(-2y) = \exp(-i \kappa y^a \omega_{ab} y^b)
\]

is a solution to our problem. In the \( \hat{x} \)-eigenbasis the matrix elements \( \hat{Z}(y)^\alpha_\beta = \langle \alpha | \hat{Z} | \beta \rangle \) are given by

\[
\hat{Z}(y)^\alpha_\beta = \exp[-\frac{i}{\hbar} y_p (\alpha + \beta)] \delta^{(N)} (\alpha - \beta + 2y_q)
\]

They can be used in order to verify that

\[
\langle \alpha | y_q \circ \hat{Z}(y) | \beta \rangle = \beta \langle \alpha | \hat{Z}(y) | \beta \rangle
\]

\[
\langle \alpha | y_p \circ \hat{Z}(y) | \beta \rangle = i\hbar \frac{\partial}{\partial \beta} \langle \alpha | \hat{Z}(y) | \beta \rangle
\]
which is (3.74) in the “position representation”.

We shall need the star product of two different $\hat{Z}$ matrix elements. After some algebra one finds the remarkably simple result

$$\hat{Z}(y)_{\alpha}^{\beta} \circ \hat{Z}(y)_{\bar{\alpha}}^{\bar{\beta}} = 2^{-N} (\gamma_P)_{\alpha}^{\bar{\alpha}} \hat{Z}(y)_{\bar{\beta}}^{\beta}$$

When combined with the identity $\hat{Z}^\dagger = \gamma_P \hat{Z} \gamma_P$ the above equation at $y = 0$ gives rise to the inner product

$$(\hat{Z}_{\beta}^{\alpha} | \hat{Z}_{\bar{\beta}}^{\bar{\alpha}}) = 2^{-N} (\gamma_P)_{\alpha}^{\bar{\alpha}} \delta_{\beta\bar{\beta}}$$

The orthogonality and completeness relations (3.6), (3.7) for $\hat{T}(y)$ imply similar relations for $\hat{Z}(y)$. As a consequence, $\{\hat{Z}(\cdot)_{\alpha}^{\beta} | \alpha, \beta \in \mathbb{R}^N\}$ is a basis in the space of symbol functions $b(\cdot)$. Every $b \in \mathcal{W}$ has an expansion of the form $b(y) = \int d^N \alpha d^N \beta \psi_{(\beta)}^{\alpha} \hat{Z}(y)_{\alpha}^{\beta}$ where the “coefficients” $\psi_{(\beta)}^{\alpha}$ are actually functions $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$.

We continue the discussion directly for the case when $\mathcal{W}$ is the fiber $\mathcal{W}_\phi$ and the symbols $b(\cdot)$ are the $\mathcal{W}$-valued fields $B(\cdot, \phi)$ evaluated at a given point $\phi$. Eqs. (3.6) and (3.7) imply that $B$ can be expanded as

$$B(\phi, y) = \int d^N \alpha \int d^N \beta \psi_{(\beta)}^{\alpha}(\phi) \hat{Z}(y)_{\alpha}^{\beta}$$

and that the expansion coefficients are given by

$$\psi_{(\beta)}^{\alpha}(\phi) = (\pi \hbar/2)^N \int d^N y B(\phi, y) \hat{Z}^\dagger(y)_{\beta}^{\alpha}$$

In a sense which we shall make precise later on, $\psi_{(\beta)}^{\alpha} \equiv \{\psi_{(\beta)}^{\alpha}; \alpha \in \mathbb{R}^N\}$ are the components of infinitely many metaplectic spinors labeled by the “index” $\beta$. If we define

$$B_{(\beta)}(\phi, y) \equiv \int d^N \alpha \psi_{(\beta)}^{\alpha}(\phi) \hat{Z}(y)_{\alpha}^{\beta}$$

so that $B(\phi, y) = \int d^N \beta B_{(\beta)}(\phi, y)$ then eq.(3.74) implies that the invariant subspace $\mathcal{W}_{(\beta)}$ is spanned by precisely the symbols of the type (3.84):

$$y^\alpha \circ B_{(\beta)} = \int d^N \alpha \int d^N \bar{\alpha} \psi_{(\bar{\beta})}^{\alpha} \hat{Z}_{\bar{\alpha}}^{\bar{\beta}} (\mathscr{C})_{\alpha}^{\bar{\beta}}$$
\[ (\bar{\varphi}^a \psi_{(\beta)})^\alpha = \int d^N \alpha (\bar{\varphi}^a)^\alpha \psi_{\alpha} \] (3.85)

Here \((\bar{\varphi}^a \psi_{(\beta)})^\alpha \equiv \int d^N \alpha (\bar{\varphi}^a)^\alpha \psi_{\alpha} \). We see that if the symbol \(B_{(\beta)}\) is related to the spinor \(\psi_{(\beta)}\) by (3.84) then \(y^a \circ B_{(\beta)}\) and \(\bar{\varphi}^a \psi_{(\beta)}\) are related in the same way. Likewise \(\kappa y^a \circ B_{(\beta)}\) corresponds to a multiplication by \(\gamma^a\).

Given an arbitrary symbol in \(\mathcal{W}\) we can project it on any of the subspaces \(\mathcal{W}_{(\beta)}\). We introduce projection operators \(\mathcal{P}_{(\beta)}\) by \(B_{(\beta)} = \mathcal{P}_{(\beta)} B\). If we combine eqs. (3.83) and (3.84) it follows that

\[ B_{(\beta)}(\phi, y) = \int d^2 N y' \mathcal{P}_{(\beta)}(y, y') B(\phi, y') \] (3.86)

where the integral kernel of the projector is given by

\[ \mathcal{P}_{(\beta)}(y, y') = (\pi \hbar / 2)^{-N} \langle \beta | \hat{Z}(y) \hat{Z}^\dagger(y') | \beta \rangle \] (3.87)

Upon using (3.77), (3.5) and (3.9) we obtain explicitly

\[ \mathcal{P}_{(\beta)}(y, y') = (\pi \hbar / 2)^{-N} \exp \left[ -\frac{2i}{\hbar} (\beta + y_q)(y_p - y'_q) \right] \delta^{(N)}(y_q - y'_q) \] (3.88)

The projectors \(\{\mathcal{P}_{(\beta)}; \beta \in \mathbb{R}^N\}\) are orthogonal and complete in the sense that

\[ \int d^2 N y' \mathcal{P}_{(\beta)}(y, y') \mathcal{P}_{(\bar{\beta})}(y', y'') = \delta^{(N)}(\beta - \bar{\beta}) \mathcal{P}_{(\beta)}(y, y'') \]

\[ \int d^N \beta \mathcal{P}_{(\beta)}(y, y') = \delta^{(2N)}(y - y') \] (3.89)

Furthermore, as a consequence of eq.(3.80), the inner product of two different projections reads

\[ (B_{(-\beta_1)} B_{(\beta_2)}) = 2^{-N} \delta^{(N)}(\beta_1 - \beta_2) \int d^N \alpha \bar{\psi}_{(\beta_1)}^\alpha \psi_{(\beta_2)}^\alpha \] (3.90)

Note the sign flip on the LHS of this equation. Obviously \(B_{(-\beta)}\) is the natural dual of \(B_{(\beta)}\) (similar to a spinor adjoint).

To summarize: Every symbol-valued field \(B(\phi, y)\) gives rise to infinitely many projections \(B_{(\beta)}(\phi, y)\) each of which is equivalent to a metaplectic spinor field \(\psi_{(\beta)}(\phi)\) with components \(\psi_{(\beta)}^\alpha(\phi)\) given by (3.83). This is to mean that the
fields $\psi_{(\beta)}$ carry an irreducible representation of the Clifford algebra: $\kappa y^a \circ B_{(\beta)}$ corresponds to the spinor multiplied by a $\gamma$-matrix, $\gamma^a\psi_{(\beta)}$.

Up to this point the situation is similar to the $SO(n)$-case, but differences will show up shortly.

(b) Operators

As in the fermionic case, it proves advantageous to combine the expansion coefficients $\psi^\alpha_{(\beta)}$ as a matrix $\hat{\psi}$:

$$\hat{\psi}^\alpha_\beta \equiv \psi^\alpha_{(\beta)} \equiv \langle \alpha | \hat{\psi} | \beta \rangle$$  \hspace{1cm} (3.91)

(We suppress the argument $\phi$ for the time being.) We shall need some properties of the linear, invertible map $B \mapsto \hat{\psi}[B]$ which relates the symbols to the new operator $\hat{\psi}$.

By definition, $B(y)$ is the ordinary Weyl symbol of the operator $\hat{B}$ introduced earlier. Remarkably enough, this symbol plays a dual role: the same function but with its argument rescaled, $B(\frac{1}{2}y)$, turns out to be the alternative Weyl symbol of the new operator $\hat{\psi}$. This is most easily seen if one uses $\hat{Z}^\dagger(y) = \hat{T}(2y)$ in

$$\hat{\psi} = \left(\frac{\pi \hbar}{2}\right)^{-N} \int d^{2N} y \ B(y) \ \hat{Z}^\dagger(y)$$  \hspace{1cm} (3.92)

$$B(y) = \text{Tr} [\hat{\bar{Z}}(y) \hat{\psi}]$$  \hspace{1cm} (3.93)

which follows from the equations in subsection (a), and then compares (3.92), (3.93) to eqs.(3.10), (3.11). Thus,

$$[\text{symb}\{\hat{B}\}](y) = B(y) \Leftrightarrow [\text{alt-symb}\{\hat{\psi}\}](y) = B(\frac{1}{2}y)$$  \hspace{1cm} (3.94)

This dual role played by $B$ is another hint at the very natural relationship between the Dirac-Kähler idea and the Weyl symbol calculus.

Regarding $\hat{\psi}$ as a functional of $B$ it is not difficult to establish that

$$\hat{\psi}[1] = 2^N \gamma_p$$  \hspace{1cm} (3.95)
\[ \hat{\psi}[\kappa y^a] = 2^N \gamma^a \gamma_P \]  
(3.96)

\[ \hat{\psi}[\kappa y^a \circ B] = \gamma^a \hat{\psi}[B] \]  
(3.97)

\[ \hat{\psi}[B_1 \circ B_2] = 2^{-N} \hat{\psi}[B_1] \gamma_P \hat{\psi}[B_2] \]  
(3.98)

\[ \hat{\psi}[\kappa^p y^{a_1} \circ y^{a_2} \circ \ldots \circ y^{a_p}] = 2^N \gamma^{a_1} \gamma^{a_2} \ldots \gamma^{a_p} \gamma_P \]  
(3.99)

\[ \hat{\psi}[\kappa^p y^a y^b] = 2^N \gamma^a \gamma^b \gamma_P - 2^N i \omega^{ab} \gamma_P \]  
(3.100)

Eq.(3.95) follows directly from the definition of \( \gamma_P \) and eq.(3.97) is our earlier result (3.85), eq.(3.96) being a special case. The most important relation is (3.98). It can be proven by using (3.80) and (3.93) in order to show that \( B_1 \circ B_2 = 2^{-N} \text{Tr}\{\hat{Z} \hat{\psi}[B_1] \gamma_P \hat{\psi}[B_2]\} \). When compared to eq.(3.93), this equation implies (3.98).

Above we had introduced the projectors \( P(\beta) \) which project any symbol on the invariant subspaces \( \mathcal{W}(\beta) \). The map \( B \mapsto \hat{\psi}[B] \) given by (3.92) induces a corresponding projection on the space of operators \( \hat{\psi} \). In the language of our auxiliary quantum mechanical system this projection has a very natural interpretation: it is simply the projection on the position eigenstate \( |\beta\rangle \). From eq.(3.84) we can read off that \( B(\beta) \) has the structure of an expectation value in the state \( |\beta\rangle \)

\[ B(\beta)(y) = \langle \beta| \hat{Z}(y) \hat{\psi}[B]|\beta\rangle \]

\[ = \text{Tr} \left[ \hat{Z}(y) \hat{\psi}[B] \hat{P}(\beta) \right] \]  
(3.101)

Here \( \hat{P}(\beta) \equiv |\beta\rangle\langle \beta| \) is the corresponding projector on the Hilbert space. It follows from (3.101) that symbols \( B \in \mathcal{W}(\beta) \) are associated to operators of the form \( \hat{\psi} P(\beta) \):

\[ \hat{\psi}[P(\beta)B] = \hat{\psi}[B] \hat{P}(\beta) \]  
(3.102)

Finally we have to address the important question of how the operator \( \hat{\psi} \) is related to the operator \( \hat{B} \) which was the central building block in the Dirac-Kähler construction. Imitating the \( SO(n) \)-case, we had obtained \( \hat{B} \) in
eq. (3.52) by replacing $d\phi^a \to \gamma^a$ in the tensor field $\Sigma$. In Section 2 we have seen that for ordinary DK-fields $\hat{\psi}$ and $\hat{F}$ coincide up to a constant factor. It is quite remarkable that, with a minor modification, the same identification is possible in the symplectic situation where $\mathcal{V}$ is infinite dimensional. It turns out that

$$\hat{\psi}[B] = 2^N \hat{B} \gamma_P \quad \text{or} \quad \hat{B} = 2^{-N} \hat{\psi}[B] \gamma_P \quad (3.103)$$

This relationship can be proven in a variety of ways. For instance, we can take advantage of the following very compact representation of operators $\hat{b}$ in terms of their symbols $b$ [32]:

$$\hat{b} = 2^{-N} (2\pi \hbar)^{-N} \int d^{2N} y \ b(\frac{1}{2} y) \hat{T}(y) \gamma_P \quad (3.104)$$

The advantage of (3.104) as compared to the old representation (3.10) is that no Fourier transformation is involved any longer. Eq.(3.104) is easily established by inserting the integral representation for $\gamma_P$ on its RHS and then combining the two Weyl operators with the help of (3.5). From eq.(3.104) we infer that if $B(y)$ is the ordinary symbol of $\hat{B}$ then $B(\frac{1}{2}y)$ is the alternative Weyl symbol of $2^N \hat{B} \gamma_P$. Moreover, we saw already that $B(\frac{1}{2}y)$ is the alternative Weyl symbol of $\hat{\psi}$. As a consequence, $\hat{\psi}$ must coincide with $2^N \hat{B} \gamma_P$.

It is instructive to give a different proof when $B(y)$ is a power series. This is the case for instance when the symbol originates from an IST via the DK-construction. For $\hat{B}$ or $B(y)$ given, the task is to solve $B(y) = \text{Tr}[\hat{Z}(y)\hat{\psi}]$ for the unknown operator $\hat{\psi}$. Using the expansion (3.53) for $\hat{B}$ and the (expanded) exponential (3.77) for $\hat{Z}$, this equation turns into

$$B_{a_1 \ldots a_p}^{(p)} = i^{-p} \text{Tr}[\gamma_{(a_1} \cdots \gamma_{a_p)} \hat{\psi}] \quad (3.105)$$

In the corresponding calculation for the $SO(n)$-case we made an ansatz for $\hat{\psi}$ as a power series in $\gamma^\mu$ and used the $\gamma^\mu$-trace identities in order to project on its coefficients. Because of the additional matrix $\gamma_P$ in the analogous identities
for the metaplectic $\gamma$-matrices the appropriate ansatz for the symplectic $\hat{\psi}$ is a power series in $\gamma^a$ (with coefficients $\psi^{(p)}_{a_1 \cdots a_p}$) times an explicit factor of $\gamma_P$. With this ansatz in (3.105), the trace identities imply $\psi^{(p)}_{a_1 \cdots a_p} = 2^N B^{(p)}_{a_1 \cdots a_p}$ which proves (3.103). In this manner we see that the factor of $\gamma_P$ connecting $\hat{\psi}$ to $\hat{B}$ is simply a reflection of the corresponding factor in the trace identities.

We discussed already that in the infinite dimensional situation the $\gamma_P$ under the traces is crucial in order to make them well defined.

The $\gamma_P$-matrix in (3.103) has the consequence that $\hat{\psi}$ does not admit a power series expansion even if $\hat{B}$ does so. This has important implications for the Dirac-Kähler program. As we are going to discuss next it means that the decomposition of $\mathcal{W}$ into subspaces $\mathcal{W}_{(\alpha)}$ which are invariant under star left multiplication does not translate into a corresponding decomposition of the symmetric tensor algebra into subspaces invariant under (symplectic) Clifford left multiplication. In this respect the $SO(n)$ and the $Sp(2N)$-cases are quite different.

Let us first look at how the space of operators $\hat{B}$ decomposes under $\mathcal{W} = \bigoplus \mathcal{W}_{(\beta)}$. Eqs. (3.102) and (3.103) imply that

$$\hat{P}_{(\beta)} \hat{B} = \hat{B} \gamma_P \hat{P}_{(\beta)} \gamma_P = \hat{B} \hat{P}_{(-\beta)}$$

(3.106)

Hence, at the level of the $\hat{B}$-operators, the projection $P_{(\beta)}$ amounts to a right multiplication by $\hat{P}_{(-\beta)}$.

From eq.(3.106) we can obtain a very useful by-product. If we take the symbol on both sides of this equation and abbreviate $P_{(\alpha)} \equiv \text{symb} \left[ \hat{P}_{(\alpha)} \right]$ then the result is the following compact formula for the projection $B_{(\beta)}$:

$$P_{(\beta)} B \equiv B_{(\beta)} = B \circ P_{(-\beta)}$$

(3.107)

More explicitly, because $P_{(\alpha)}(y) = \delta^{(N)}(y_q - \alpha)$, this means that

$$B_{(\beta)}(y) = B(y) \circ \delta^{(N)}(y_q + \beta)$$

(3.108)
By virtue of (3.21) the latter equation can be brought to the following form which is the most convenient one for practical calculations:

\[
B_{(\beta)}(y_p, y_q) = B \left( y_p - \frac{i\hbar}{2} \frac{\partial}{\partial y_q}, y_q \right) \delta^{(N)}(\bar{y}_q + \beta) \big|_{\bar{y}_q = y_q}
\]

(3.109)

As usual, \( y \equiv (y_p, y_q) \) consists of \( N \)-component momentum- and position-type variables \( y_p \) and \( y_q \).

The structure of \( B_{(\beta)} \) is particularly transparent if \( B(y) \equiv B(y_q) \) does not depend on the momenta. Then its projection on \( \mathcal{W}_{(\beta)} \) reads

\[
B_{(\beta)}(y) = B(y_q) \delta^{(N)}(y_q + \beta),
\]

(3.110)
i.e., it is sharply localized at \( y_q = -\beta \). If \( B \) depends also on \( y_p \) there are additional terms involving derivatives of \( \delta^{(N)}(y_q + \beta) \). Nevertheless, as long as \( B \) depends on \( y \) polynomially, the projected symbol \( B_{(\beta)} \) has support only on the hyperplane \( y_q = -\beta \). This localization of the symbols makes it very easy to visualize the \( \beta \)-subspace of \( \mathcal{W} \). In fact, this intuitive interpretation of \( \mathcal{W}_{(\beta)} \) is the reason why we are using the \( \hat{x} \)-eigenbasis on \( \mathcal{V} \) rather than the harmonic oscillator (Fock space) basis which yields the traditional representation of the \( \gamma^a \)-matrices.

(c) Inhomogeneous symmetric tensors

We know that every symbol-valued field \( B(\phi, y) \) gives rise to infinitely many projections \( B_{(\beta)} \in \mathcal{W}_{(\beta)} \) each of which is equivalent to a spinor \( \psi_{(\beta)} \). On \( \mathcal{W}_{(\beta)} \), \( k y^a \circ B_{(\beta)} \) corresponds to \( \gamma^a \psi_{(\beta)} \) and it represents the Clifford algebra irreducibly. On the other hand, in Section 3.3 we defined the symplectic Clifford product as the image of the star product under the symbol/tensor-correspondence (3.54). It is a natural question therefore whether the representation of the Clifford algebra provided by “\( d\phi^a \vee \)” on the space of symmetric tensors is reducible as well.

At this point we have to recall that the symbol/tensor-correspondence
(3.54) is a bijection between tensors $\Sigma(\phi)$ and symbols $B(\phi, y)$ which are analytic in $y$. Only if $B$ allows for a power series expansion in $y$ the substitution $\kappa y^a \rightarrow d\phi^a$ yields a tensor field. As for the question of the reducibility, the crucial observation is that even if $B(\phi, y)$ is analytic in $y$, the projections $B_{(\beta)}(\phi, y)$ are not in general. This is obvious from eq. (3.109) which shows that $B_{(\beta)}$ is typically a distribution with a sharp localization (in the auxiliary phase-space) on the plane $y_q = -\beta$.

Therefore we must conclude that the decomposition of the bosonic Weyl algebra $\mathcal{W} = \bigoplus \mathcal{W}_{(\beta)}$ does not imply a corresponding decomposition of the space of IST’s. This was different in the fermionic case where the analyticity of $F(x, \theta)$ comes for free and where “symbol-valued fields” and “inhomogeneous differential forms” are two completely equivalent concepts.

From these observations we can learn what the correct notion of a “symplectic Dirac-Kähler field” actually is. Traditionally, in the $SO(N)$-case, a DK-field meant a set of (antisymmetric) tensor fields. This is a historic accident, however, and one could have talked equally well about $\mathcal{W}^F$-valued fields over space-time. When we go from space-time to phase-space and from $SO(n)$ to $Sp(2N)$ we see that the notion which generalizes is not that of a collection (of now symmetric) tensor fields but rather the idea of Weyl symbol-valued fields. On phase-space the fields $B(\phi, y)$, with a not necessarily analytic dependence on $y$, play a role which is completely analogous to that of $F(x, \theta)$ on space-time. The former is equivalent to a set of $Mp(2N)$-spinors in very much the same way as the latter gives rise to a multiplet of $Spin(n)$-spinors.

4 Summary and Discussion

In the first part of this paper we have shown that the theory of space-time DK-fermions allows for a remarkably simple and natural reinterpretation in the framework of the symbol calculus. More precisely, it is the fermionic Weyl
symbol which is to be used here. This symbol was employed in the context of first quantized particle and string theory occasionally, but so far it has not reached the popularity of the Wick symbol which is commonly chosen for fermionic systems.

We have set up a one-to-one correspondence between DK-fields $\Phi(x)$ and symbol-valued fields $F(x, \theta)$ by associating a family of auxiliary quantum systems, with canonical operators $\hat{\chi}_\mu$ and anticommuting phase-space coordinates $\theta^\mu$, to each point $x$ of space-time. The fermionic operators $\hat{\chi}_\mu$ and Grassmann variables $\theta^\mu$ replace the Dirac matrices $\gamma^\mu$ and the differentials $dx^\mu$, respectively. The nontrivial aspect of this correspondence is that it maps all the natural operations which we know for differential forms onto equally natural and well known operations for symbols. For instance, the star product which is at the heart of every symbol calculus turned out to be related to the Clifford product, a pivotal concept in standard DK-theory, in precisely this manner. More generally, we were able to identify all the defining structures of an Atiyah-Kähler algebra on the space of fermionic Weyl symbols.

Our approach provides some new computational tools for calculations involving DK-fields, an integral representation of the Clifford product, for example. More importantly, it sheds new light on the geometrical meaning of various constructions in the standard approach. For instance, the matrix-valued form $Z$ has turned out to be nothing but a fermionic Weyl operator.

In the second part of this paper we developed a symplectic analog of DK-theory. We replaced space-time by phase-space, the “Lorentz group” $SO(n)$ by $Sp(2N)$, Dirac fields by metaplectic spinors, and we then asked if there exists a corresponding notion of a DK-field. The answer turned out to be in the affirmative, but with some qualifications. The crucial step in our construction was switching from the fermionic auxiliary quantum system to a bosonic one whose basic operators $\hat{\varphi}^a$ satisfy canonical commutation relations and thus realize the
symplectic Clifford algebra. Using the Riemannian situation as a guide line we formulated the auxiliary quantum theory in terms of (now bosonic) Weyl symbols. We argued that it is the symbol-valued fields $B(\phi, y)$ which deserve the name of a “symplectic Dirac-Kähler field”. The fields with an analytic dependence on $y$ are equivalent to a set of symmetric tensor fields, the symplectic counterpart of an inhomogeneous differential form. We described in detail which properties of the standard DK-fields pass over to the symplectic case and which don’t. We discovered for example that all the defining structures of an Atiyah-Kähler algebra have analogs in the symplectic setting. In particular, the bosonic Weyl star product gives rise to a “Clifford product”.

It is an interesting feature of this method that both the ordinary and the symplectic Clifford product arise as a quantum deformation (in the sense of [24]) of the corresponding tensor product (wedge product and $\otimes_{\text{sym}}$), the deformation parameter being $\hbar$ or $\kappa^{-2}$.

The most important differences between the Riemannian and the symplectic case occur when it comes to decomposing the representation of the Clifford algebra carried by the symbol-valued fields. While the decomposition of the bosonic Weyl algebra into left invariant subspaces can be carried out along the same lines as for the fermionic algebra, it does not induce a corresponding decomposition of the space of inhomogeneous symmetric tensor fields. The reason is that the projection on the invariant subspaces does not respect the analyticity of $B(\phi, y)$ which is necessary for a tensor interpretation. We take this as a hint that it is actually the concept of a Weyl-algebra-valued field which is at the heart of DK-theory, both on space-time and on phase-space, rather than the idea of inhomogeneous (anti)symmetric tensors. The fields $B(\phi, y)$ are equivalent to a multiplet of metaplectic spinors in the same way an ordinary DK-field is equivalent to a multiplet of Dirac spinors.

\footnote{In order to make this explicit at the tensor level one should refrain from the convenient rescaling of the tensor components by factors of $\kappa$.}
Let us close with a few additional comments.

We begin with a remark on what it precisely means that a DK-field is “equivalent” to a set of spinor fields. This remark applies to $SO(n)$ and $Sp(2N)$ DK-fields alike. Strictly speaking, a metaplectic spinor is defined by its transformation properties under local $Sp(2N)$-transformations, the phase-space analog of the local Lorentz-transformations. Let us fix some point $\phi$ of $\mathcal{M}_{2N}$ and let us change the basis in its tangent space $T_{\phi,\mathcal{M}_{2N}}$ by means of a symplectic matrix $S(\phi) \equiv [S(\phi)]^a_b$. This induces a corresponding unitary transformation $M(S) \in Mp(2N)$ in the local Hilbert space $\mathcal{H}_{\phi}$. The components of a vector and a spinor transform as $y^a \rightarrow (S^{-1})^a_b y^b$ and $\psi^\alpha \rightarrow M(S)^\alpha_\beta \psi^\beta$, respectively. It is important to observe that the spinors contained in a DK-field $B(\phi, y)$ do not individually transform in this manner. In fact, as a direct consequence of (1.2) the $\hat{Z}$-operator transforms according to

$$M(S)^\dagger \hat{Z}(y) M(S) = \hat{Z}(S^{-1}y)$$

Therefore eq.(3.93) reads in the rotated basis

$$B(\phi, S^{-1}y) = \text{Tr} \left[ \hat{Z}(y) M(S) \hat{\psi}(\phi) M(S)^\dagger \right]$$

This means that $\hat{\psi} \equiv (\psi_\alpha^\alpha)_{(\beta)}$ does not transform as a set of independent spinors labeled by the index $\beta$. The index $\beta$, too, is acted upon by a spin matrix:

$$\psi_{(\beta)}^\alpha \rightarrow M(S)^\alpha_\gamma \psi_{(\delta)}^\gamma M^\dagger(S)^\delta_\beta$$

For space-time DK-fields this is a well known phenomenon which is referred to as “flavor mixing” [3]. Among other things it implies that DK-fermions have a nonstandard coupling to gravity [2, 42, 6]. The curved-space Dirac equation for a massless DK-field reads

$$\gamma^\mu \left( \partial_\mu \hat{\psi} - i \omega^{IJ}_\mu [\sigma_{IJ}, \hat{\psi}] \right) = 0$$

See ref. [21] for a detailed discussion of those transformation properties and of the vielbein formalism for phase-spaces.
If \( \hat{\psi} \) was a set of independent spinors the spin connection \( \omega^{IJ}_\mu \sigma_{IJ} \) multiplied \( \hat{\psi} \) from the left only. The flavor mixing caused by the commutator is weak in the Newtonian limit of gravity and most probably cannot be excluded on experimental grounds \[12\].

Finally let us comment on the relation of the symplectic DK-fields to the gauge theory formulation of quantum mechanics \[21\] which was proposed recently. Its basic ingredient is a family of local Hilbert spaces \( \mathcal{V}_\phi \) attached to the points of phase-space. This theory resulted from an attempt at understanding the principles of canonical quantization at a perhaps deeper or at least physically and geometrically more natural level.

The theory is a Yang-Mills-type gauge theory on phase-space. Its “matter fields” are metaplectic spinors \( \psi^\alpha(\phi) \). Canonical quantization is replaced by two new rules. The first one is that in order to go from classical mechanics to semiclassical quantum mechanics we must switch from the vector representation of \( \text{Sp}(2N) \) to its spinor representation. The second rule is a consistency condition which tells us how to sew together local semiclassical expansions so as to recover exact quantum mechanics. It is formulated as symmetry principle: the Yang-Mills theory must be invariant under a new type of background-quantum split symmetry. As it turns out, this implies that the gauge field is a universal, nondynamical abelian connection \( \tilde{\Gamma} \).

The upshot of this construction is the following two-step procedure for the quantization of physical systems on arbitrary curved phase-spaces \( \mathcal{M}_{2N} \):

1. Find an abelian spin connection \( \tilde{\Gamma} \) on \( \mathcal{M}_{2N} \). It is guaranteed to exist on any symplectic manifold and can be constructed iteratively by Fedosov’s

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9 For different formulations of quantum (field) theory using local Hilbert spaces see refs. \[43, 31, 44\]

10 The gauge group is the group of all unitary transformations on \( \mathcal{V} \) and the connection components \( \tilde{\Gamma}_a \) are hermitian operators. Being abelian means that the curvature of \( \tilde{\Gamma} \) is proportional to the unit operator on \( \mathcal{V} \).
(2) Construct (multi) spinor fields which are covariantly constant (possibly up to a phase) with respect to the connection $\tilde{\Gamma}$. They are local generalizations of states and observables.

In particular, states are represented by a covariantly constant spinor field $\psi^\alpha(\phi)$. If the value of this field is known at a fixed reference point $\phi_0$ it is known everywhere in phase-space (up to a physically irrelevant phase). The wave function $\Psi$ of conventional quantum mechanics is identified with $\psi^\alpha(\phi_0) \equiv \Psi(\alpha)$. For further details we refer to [21].

This approach reveals that, in a sense, classical mechanics is related to quantum mechanics in the same way tensor fields (integer spin) relate to spinor fields (half-integer spin) or space-time bosons relate to fermions. What is at the heart of the quantization process is changing the representation of $\text{Sp}(2N)$, the “Lorentz group” of phase-space.

According to the proposal of ref. [21] this change of representation, while very natural from a particle physics point of view, still has to be done “by hand” in the same sense as in the standard approach the canonical commutation relations are imposed “by hand”. One might wonder if there is more natural way of describing this change of representation, and it is here that Dirac-Kähler theory comes into play. DK-theory certainly cannot explain why nature has decided to pick the spinor representation of $\text{Sp}(2N)$ but it can put this question into a novel and perhaps somewhat unexpected perspective.

The symplectic DK-fields give a precise meaning to the idea that classical mechanics “contains” the basic building blocks of quantum mechanics, namely the metaplectic spinor fields. On the one hand, the DK-fields $B(\phi, y)$ belong to the realm of classical mechanics in the sense that they are c-number functions on the classical tangent bundle. On the other hand, $B(\phi, y)$ is equivalent to a family of spinor fields $\psi(\beta) = (\psi^\alpha(\beta))$ whose members are labelled by the
“flavor index” $\beta$. Quantum mechanics is a theory whose basic ingredient is a single metaplectic spinor field. This leads us to conclude that the process of quantization can be understood as the elimination of all but one flavor of metaplectic spinors, i.e. as a projection on a fixed $\beta$-subspace.$^\text{11}$

In the same sense as above, this projection has to be done “by hand”$^\text{12}$. However, with this interpretation, there is an almost perfect analogy between the following two problems which are usually thought of as belonging to rather different branches of physics: the construction of a lattice theory which describes a single species of fermions, and the quantization of physical systems in general. On the Riemannian (or space-time) side, the question is how to avoid the fermion replication which results from the Kogut-Susskind action, and the corresponding symplectic (or phase-space) problem is how to obtain a quantum theory from classical structures. At a heuristic level, the solution to both problems is exactly the same: one must project out a single spinor from a Dirac-Kähler field. Whether this is merely a formal similarity or whether space-time fermions can teach us something about the general structure of quantum mechanics remains to be seen.

$^\text{11}$Note, however, that covariantly constant DK-fields do not amount to covariantly constant projected spinor fields. The reason is the flavor mixing: the condition $\nabla(\bar{\Gamma})B = 0$ involves a commutator of $\bar{\Gamma}$ with $B$, while $\nabla(\bar{\Gamma})\psi = 0$ contains only a left-multiplication by $\bar{\Gamma}$.

$^\text{12}$We mention that also the approach of ref. $^\text{45}$ constructs quantum mechanics from functions on the classical tangent bundle by imposing certain constraints. This approach does not involve metaplectic spinors, however, and the DK-construction seems not to answer the questions raised there.
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Appendix A

In this appendix we collect a number of definitions and identities related to Grassmann algebras which are needed in the main body of the paper. In particular, the main automorphism $A$, the main antiautomorphism $B$, the Hodge operator $\ast$ and the modified Hodge operator $\star$ are discussed and our conventions are specified.

We consider a Grassmann algebra with the real generators $\theta^1, \ldots, \theta^n$, i.e. $\theta^\mu \theta^\nu + \theta^\nu \theta^\mu = 0$ for all $\mu, \nu = 1, \ldots, n$, and introduce functions

$$f(\theta) \equiv f(\theta^1, \ldots, \theta^n) = \sum_{p=0}^{n} f^{(p)}(\theta) \quad (A.1)$$

where $f^{(p)}$ is homogeneous of degree $p$:

$$f^{(p)}(\theta) = \frac{1}{p!} f^{(p)}_{\mu_1 \ldots \mu_p} \theta^{\mu_1} \ldots \theta^{\mu_p} \quad (A.2)$$

The complex-valued constants $f^{(p)}_{\mu_1 \ldots \mu_p}$ are completely antisymmetric in all indices. By definition, the main automorphism $A$ and the main antiautomorphism $B$ act on these functions according to

$$(Af)(\theta) = \sum_{p=0}^{n} (-1)^p f^{(p)}(\theta) \quad (A.3)$$

$$(Bf)(\theta) = \sum_{p=0}^{n} (-1)^{p(p-1)/2} f^{(p)}(\theta) \quad (A.4)$$

Their main properties are

$$A^2 = B^2 = 1, \quad AB = BA$$

$$A(fg) = (Af)(Ag)$$

$$B(fg) = (Bg)(Bf) \quad (A.5)$$

where $(fg)(\theta) \equiv f(\theta)g(\theta)$ is the pointwise product. Some useful identities involving $A$ and $B$ include

$$\theta^\mu f(\theta) = (Af)(\theta) \theta^\mu \quad (A.6)$$
\[ \theta^{\mu_1} \cdots \theta^{\mu_p} f(\theta) = (\mathcal{A}^p f)(\theta) \theta^{\mu_1} \cdots \theta^{\mu_p} \quad (A.7) \]

\[ \theta^{\mu_p} \theta^{\mu_{p-1}} \cdots \theta^{\mu_1} = (-1)^{p(p-1)/2} \theta^{\mu_1} \theta^{\mu_2} \cdots \theta^{\mu_p} = \mathcal{B} \theta^{\mu_1} \theta^{\mu_2} \cdots \theta^{\mu_p} \quad (A.8) \]

Denoting complex conjugation by an overbar we assume \( \bar{\theta}^\mu = \theta^\mu \) and set

\[ \bar{f} g = \bar{g} \bar{f} \quad (A.9) \]

for any two functions \( f \) and \( g \). If one makes the additional assumption that the coefficients \( f^{(p)}_{\mu_1 \cdots \mu_p} \) are real, then eq.(A.8) shows that

\[ \bar{f}(\theta) = (\mathcal{B} f)(\theta) \quad (A.10) \]

Usually we shall allow the coefficients to be complex though. The automorphism \( \mathcal{A} \) can be used in order to convert right-derivatives \( \frac{\partial}{\partial \theta^\mu} \) to left-derivatives \( \frac{\partial}{\partial \bar{\theta}^\mu} \):

\[ f(\theta) \frac{\partial}{\partial \theta^\mu} = \mathcal{A} \frac{\partial}{\partial \bar{\theta}^\mu} f(\theta) \quad (A.11) \]

More generally, one has

\[ f(\theta) \frac{\partial}{\partial \theta^{\mu_p}} \cdots \frac{\partial}{\partial \theta^{\mu_1}} = \mathcal{A}^p \frac{\partial}{\partial \bar{\theta}^{\mu_p}} \cdots \frac{\partial}{\partial \bar{\theta}^{\mu_1}} f(\theta) \quad (A.12) \]

which is easily proven by induction. Since \( \mathcal{A} \) anticommutes with \( \frac{\partial}{\partial \theta^\mu} \), it follows that \( (p, q = 0, 1, 2, \cdots) \)

\[ \mathcal{A}^p \frac{\partial}{\partial \theta^{\mu_1}} \cdots \frac{\partial}{\partial \theta^{\mu_p}} f(\theta) = (-1)^{pq} \frac{\partial}{\partial \bar{\theta}^{\mu_1}} \cdots \frac{\partial}{\partial \bar{\theta}^{\mu_p}} \mathcal{A}^q f(\theta) \quad (A.13) \]

In particular,

\[ \mathcal{A}^p \frac{\partial}{\partial \theta^{\mu_1}} \cdots \frac{\partial}{\partial \theta^{\mu_p}} f(\theta) = (-1)^p \frac{\partial}{\partial \bar{\theta}^{\mu_1}} \cdots \frac{\partial}{\partial \bar{\theta}^{\mu_p}} \mathcal{A}^p f(\theta) \quad (A.14) \]

These identities will be needed in order to establish the equivalence of the Clifford product and the fermionic star product.
Our conventions for the integration are \( \int d\theta^\mu = 0 \) and \( \int \theta^\mu d\theta^\mu = 1 \) (\( \mu \) not summed). We define

\[
d^n\theta \equiv d\theta^1 d\theta^2 \cdots d\theta^n \tag{A.15}
\]

so that

\[
\int \theta^{\mu_n} \theta^{\mu_{n-1}} \cdots \theta^{\mu_1} d^n\theta = \epsilon^{\mu_1 \mu_2 \cdots \mu_n} \tag{A.16}
\]

with \( \epsilon^{12\cdots n} = +1 \). Using (A.11) one can show that

\[
\int f(\theta) \frac{\partial}{\partial \theta^\mu} g(\theta) d^n\theta = \int f(\theta) \frac{\leftarrow}{\partial \theta^\mu} g(\theta) d^n\theta \tag{A.17}
\]

for arbitrary inhomogeneous functions \( f \) and \( g \).

In our conventions, the delta-function is defined to satisfy

\[
\int f(\theta) \delta(\theta - \xi) d^n\theta = f(\xi) \tag{A.18}
\]

(Note the order of the factors.) It is given by

\[
\delta(\theta - \xi) = (\theta^n - \xi^n)(\theta^{n-1} - \xi^{n-1}) \cdots (\theta^1 - \xi^1) \tag{A.19}
\]

or by the Fourier representation

\[
\delta(\theta - \xi) = (-1)^{n(n-1)/2} \int e^{(\theta^\mu - \xi^\mu) \rho_\mu} d^n\rho \tag{A.20}
\]

Here \( \{\xi^1, \cdots, \xi^n\} \) and \( \{\rho_1, \cdots, \rho_n\} \) are two additional sets of real Grassmann variables which anticommute among themselves and with the \( \theta \)'s. (Indices are raised and lowered with the flat metric \( g_{\mu\nu} = \delta_{\mu\nu}. \)) Depending on the value of \( n \), \( \delta \) is either Grassmann-real or purely imaginary:

\[
\overline{\delta(\theta)} = (-1)^{n(n-1)/2} \delta(\theta) \tag{A.21}
\]

The Fourier transform \( \tilde{f} \) is defined according to

\[
\tilde{f}(\rho) = \epsilon_n^{-1} \int e^{i\rho^\mu \theta^\mu} f(\theta) d^n\theta \tag{A.22}
\]
with

\[ \epsilon_n \equiv \begin{cases} 
1 & : \text{for } n \text{ even} \\
-i & : \text{for } n \text{ odd} 
\end{cases} \quad (A.23) \]

The advantage of our conventions is that they give rise to a simple formula for \( \tilde{f} \) in terms of multiple derivatives of the \( \delta \)-function which is free from explicit sign factors and powers of \( i \). One obtains

\[ \tilde{f}(\rho) = f(i \frac{\partial}{\partial \rho}) \delta(\rho) \quad (A.24) \]

because with (A.20)

\[
\tilde{f}(\rho) = \epsilon_n^{-1} \int f(i \frac{\partial}{\partial \rho}) e^{i \theta \rho} d^n \theta \\
= f(i \frac{\partial}{\partial \rho}) \epsilon_n^{-1} (-1)^{n(n-1)/2} \delta(-i \rho) \\
= f(i \frac{\partial}{\partial \rho}) \delta(\rho) 
\]

(A.25)

In particular,

\[ f(\theta) = 1 \implies \tilde{f}(\rho) = \delta(\rho) \quad (A.26) \]

The inverse transformation reads

\[ f(\theta) = \int e^{-i \theta \rho} \tilde{f}(\rho) d^n \rho \quad (A.27) \]

The Grassmann Fourier transformation has the involutive property

\[ \tilde{\tilde{f}}(\theta) = \epsilon_n^{-1} f(\theta) \quad (A.28) \]

i.e. for \( n \) even it is an exact involution. Derivative and multiplication operators are conjugate in the sense that

\[ [\theta^\mu \tilde{f}(\theta)](\rho) = i \frac{\partial}{\partial \rho_\mu} \tilde{f}(\rho) \quad (A.29) \]

\[ [i \frac{\partial}{\partial \theta^\mu} \tilde{f}(\theta)](\rho) = \rho_\mu \tilde{f}(\rho) \quad (A.30) \]

\[ ^{13} \text{All formulas given in this appendix are valid for both } n \text{ even and } n \text{ odd.} \]
Using either (A.22) or (A.24) one can work out the Fourier transform of a product of \( \theta \)'s. The result is

\[
\tilde{\theta}^{\mu_1 \mu_2 \ldots \mu_p}(\rho) = \frac{C_{np}}{(n-p)!} \epsilon^{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_{n-p}} \rho_{\nu_1} \rho_{\nu_2} \cdots \rho_{\nu_{n-p}} 
\]

(A.31)

with the constants

\[
C_{np} \equiv i^p (-1)^{p-1} / (n-1)/2
\]

(A.32)

Identifying \( dx^\mu \equiv \theta^\mu \), the exterior algebra \( \bigwedge (T^*_x \mathbb{R}^n) \) endowed with the inner product coming from \( g_{\mu\nu} = \delta_{\mu\nu} \) provides a special example of a Grassmann algebra. In this context we are familiar with the notion of a Hodge star operator which maps \( p \)-forms onto \( (n-p) \)-forms. In the case at hand we introduce a corresponding linear map \( * : f(\theta) \mapsto (f)(\theta) \) which generalizes this concept.

On the basis elements, the Hodge operator acts according to

\[
* (\theta^{\mu_1} \ldots \theta^{\mu_p}) = \frac{1}{(n-p)!} \epsilon^{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_{n-p}} \theta^{\nu_1} \ldots \theta^{\nu_{n-p}}
\]

(A.33)

and it is extended to arbitrary functions \( f(\theta) \) by linearity. Writing \( (f)(\theta) = \sum_{p=0}^{n} \frac{1}{p!} [f]^{(p)}_{\mu_1 \ldots \mu_p} \theta^{\mu_1} \ldots \theta^{\mu_p} \) one finds for the components\(^{14}\)

\[
[f]^{(n-p)}_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{p!} [f]^{(p)}_{\nu_1 \ldots \nu_p} \epsilon^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_p}
\]

(A.34)

Acting twice with \( * \) on a homogeneous function of degree \( p \) the result is

\[
* * f^{(p)} = (-1)^{p(n-p)} f^{(p)}
\]

(A.35)

Because of the \( p \)-dependent sign factor on the RHS of (A.33) the star operator does not give rise to an involution on the space of all (i.e., inhomogeneous) functions. This motivates us to introduce the operator

\[
\star \equiv *B
\]

(A.36)

\(^{14}\)Note that in parts of the literature a different definition of \( * \) is used which amounts to interchanging the transformation laws of the basis vectors and the components, respectively.
which we shall refer to as the “modified Hodge operator”. For homogeneous functions,

\[ \star f^{(p)} = (-1)^{p(p-1)/2} \star f^{(p)} \]  

(A.37)

which implies

\[ \star \star f^{(p)} = (-1)^{n(n-1)/2} f^{(p)} \]  

(A.38)

with a sign factor independent of \( p \). Hence, for any inhomogeneous function \( f \),

\[ \star \star f = (-1)^{n(n-1)/2} f \]  

(A.39)

For \( n = 4 \), say, \( \star \star = 1 \) so that \( \star \) is an exact involution.

The (modified) Hodge operator is closely related to the Grassmann Fourier transformation. Comparing (A.31) to (A.33) shows that for homogeneous functions

\[ \star f^{(p)} = (-i)^p (-1)^{p(p-1)/2} (-1)^{n(n-1)/2} \tilde{f}(p) \]  

(A.40)

\[ \star f^{(p)} = (-i)^p (-1)^{n(n-1)/2} \tilde{f}(p) \]  

(A.41)

Using (A.24) we may express the Fourier transform by the derivative of a \( \delta \)-function:

\[ \star f^{(p)}(\theta) = (-1)^{p(p-1)/2} (-1)^{n(n-1)/2} f^{(p)} \left( \frac{\partial}{\partial \theta} \right) \delta(\theta) \]  

(A.42)

\[ \star f^{(p)}(\theta) = (-1)^{n(n-1)/2} f^{(p)} \left( \frac{\partial}{\partial \theta} \right) \delta(\theta) \]  

(A.43)

Note that the sign factor on the RHS of (A.43) is independent of \( p \). Hence it follows that for arbitrary inhomogeneous functions

\[ \star f(\theta) = (-1)^{n(n-1)/2} f \left( \frac{\partial}{\partial \theta} \right) \delta(\theta) \]  

(A.44)

This is an interesting representation of the Hodge operator, because in contrast to (A.33), eq.(A.44) continues to be meaningful if we regard \( \theta^\mu \) as a \textit{commuting} variable. This fact will become important in the construction of the metaplectic DK-fields.
Appendix B

In this appendix we derive several important representations of the fermionic star product, eqs. (2.15), (2.16) and (2.17), from the integral representation (2.14).

We start by shifting $\theta_1$ and $\theta_2$ in eq. (2.14):

\[
(f_1 \circ f_2)(\theta) = \epsilon_n \left( \frac{\hbar}{2i} \right)^n \int \exp \left( \frac{2}{\hbar} \theta_1 \theta_2 \right) f_1(\theta_1 + \theta) f_2(\theta_2 + \theta) \, d^n \theta_1 d^n \theta_2 \quad (B.1)
\]

Next we Taylor-expand $f_1$ and $f_2$ with respect to $\theta_1$ and $\theta_2$. Because the exponential produces only terms with equal numbers of $\theta_1$'s and $\theta_2$'s, only those terms in the product of the two Taylor series survive the integration which contain equal numbers as well:

\[
(f_1 \circ f_2)(\theta) = \epsilon_n \left( \frac{\hbar}{2i} \right)^n \sum_{p=0}^{n} \left( \frac{1}{p!} \right)^2 \int \exp \left( \frac{2}{\hbar} \theta_1 \theta_2 \right) \left[ \theta_1^{\mu_1} \cdots \theta_1^{\mu_p} \tilde{\partial}_{\mu_1} \cdots \tilde{\partial}_{\mu_p} f_1(\theta) \right] \\
\times \left[ \theta_2^{\nu_1} \cdots \theta_2^{\nu_p} \tilde{\partial}_{\nu_1} \cdots \tilde{\partial}_{\nu_p} f_2(\theta) \right] d^n \theta_1 d^n \theta_2 \quad (B.2)
\]

Here $\tilde{\partial}_\mu \equiv \frac{\partial}{\partial \theta_\mu}$. Because

\[
(\theta^1 \theta^2 \cdots \theta^p)(\xi^1 \xi^2 \cdots \xi^p) = (-1)^p (\xi^1 \xi^2 \cdots \xi^p)(\theta^1 \theta^2 \cdots \theta^p) \quad (B.3)
\]

for two arbitrary sets of mutually anticommuting Grassmann-odd objects, we may use eqs. (A.7) and (A.14) to write

\[
\theta_1^{\mu_1} \cdots \theta_1^{\mu_p} \tilde{\partial}_{\mu_1} \cdots \tilde{\partial}_{\mu_p} f_1(\theta) = (-1)^p \tilde{\partial}_{\mu_1} \cdots \tilde{\partial}_{\mu_p} (A^p f_1)(\theta) \theta_1^{\mu_1} \cdots \theta_1^{\mu_p} \quad (B.4)
\]

Thus we arrive at

\[
(f_1 \circ f_2)(\theta) = \epsilon_n \left( \frac{\hbar}{2i} \right)^n \sum_{p=0}^{n} \left( \frac{1}{p!} \right)^2 \left[ A^p \tilde{\partial}_{\mu_1} \cdots \tilde{\partial}_{\mu_p} f_1(\theta) \right] I^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_p} \left[ \tilde{\partial}_{\nu_1} \cdots \tilde{\partial}_{\nu_p} f_2(\theta) \right] \quad (B.5)
\]
with
\[ I_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} = \int \exp \left( \frac{2}{\hbar} \theta_1 \theta_2 \right) \theta_1^{\mu_1} \cdots \theta_1^{\mu_p} \theta_2^{\nu_1} \cdots \theta_2^{\nu_p} \, d^n \theta_1 \, d^n \theta_2 \]  
(B.6)

Upon expanding the exponential, only the term of order \( n - p \) can contribute to the integral:
\[ I_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} = \frac{1}{(n - p)!} \left( \frac{2}{\hbar} \right)^{n-p} J_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} \]  
(B.7)
\[ J_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} = \int (\theta_1^a \theta_2^b)^{n-p} \theta_1^{\mu_1} \cdots \theta_1^{\mu_p} \theta_2^{\nu_1} \cdots \theta_2^{\nu_p} \, d^n \theta_1 \, d^n \theta_2 \]  
(B.8)

For symmetry reasons the tensor \( J \) must have the structure
\[ J_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} = \lambda(n, p) \delta_{\nu_1}^{\mu_1} \cdots \delta_{\nu_p}^{\mu_p} \]  
(B.9)

where \( S_p \) is the symmetric group of \( p \) objects. The constants \( \lambda(n, p) \) are most easily determined by choosing the special index combination \( J_{12 \cdots p}^{12 \cdots p} \) for which only the identical permutation contributes in \( \text{(B.9)} \). Furthermore, the summation over \( \alpha \) in \( \text{(B.8)} \) is restricted to \( \alpha > p \) then:
\[ \lambda(n, p) = \]  
(B.10)
\[ = p! \int [\theta_1^{p+1} \theta_2^{p+1} + \cdots + \theta_1^n \theta_2^n]^{n-p} (\theta_1^1 \theta_1^2 \cdots \theta_1^p) (\theta_2^1 \theta_2^2 \cdots \theta_2^p) \, d^n \theta_1 \, d^n \theta_2 \]
\[ = p!(n-p)! \int (\theta_1^1 \theta_1^2 \cdots \theta_1^n) [\theta_1^{p+1} \theta_2^{p+1} \theta_1^{p+2} \theta_2^{p+2} \cdots \theta_1^n \theta_2^n] (\theta_2^1 \theta_2^2 \cdots \theta_2^p) \, d^n \theta_1 \, d^n \theta_2 \]

Commuting the \( \theta \)'s next to the corresponding \( d\theta \)'s produces various sign factors so that finally
\[ \lambda(n, p) = (-1)^n (-1)^{(n-1)/2} (-1)^{p(p-1)/2} n!(n-p)! \]  
(B.11)

If we note that \( \epsilon_n i^n (-1)^{(n-1)/2} = 1 \) both for \( n \) even and \( n \) odd, we see that
\[ \epsilon_n \left( \frac{\hbar}{2i} \right)^n \frac{1}{p!} I_{\mu_1 \cdots \mu_p}^{\nu_1 \cdots \nu_p} = (-1)^{p(p-1)/2} \left( \frac{\hbar}{2i} \right)^p \delta_{\nu_1}^{\mu_1} \cdots \delta_{\nu_p}^{\mu_p} \]  
(B.12)
Inserting (B.12) into (B.5) we obtain precisely the final result given in eq.(2.15) of the main text. The representation (2.16) follows from (2.15) by using (A.12) in order to convert the left derivatives which act on \(f_1\) to right derivatives. One also needs (A.8) to switch from the index sequence \((\mu_1, \mu_2, \cdots, \mu_p)\) to \((\mu_p, \mu_{p-1}, \cdots, \mu_1)\).

The last representation, eq.(2.17), follows from (2.16) by commuting left and right derivatives with the same index next to each other. No sign factor is picked up during this reshuffling.

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