The problem solution on wedge penetration in an initially anisotropic medium within the rigid-plastic scheme

AI Chanyshev\textsuperscript{1,2}\textsuperscript{*} and IM Abdulin\textsuperscript{1}

\textsuperscript{1}Chinakal Institute of Mining, Siberian Branch, Russian Academy of Sciences, Novosibirsk, Russia
\textsuperscript{2}Novosibirsk State University of Economics and Management, Novosibirsk, Russia

E-mail: *a.i.chanyshev@gmail.com

Abstract. Two problems are solved in the paper: on ultimate loads in the initial stage of indentation of an absolutely rigid smooth wedge into a layer of an initially anisotropic plastic medium and in the final stage when the tool penetrates through the layer. The problems are solved with Chanyshev’s constitutive relations of plasticity of the initially anisotropic medium based on use of the eigen elasticity tensors.

The process of wedge tool penetration in rocks is a subject of many researches [1–8]. Sokolovsky [1] studied wedge penetration in a rigid–plastic half-plane earlier investigated by Hill, Lee and Tupper. Some foreign researchers modeled stress state of soil under wedge introduction [2] and solved the problem on penetration of a cylinder in multilayered soil [3]. The model of interaction between a cone and soil with regard to penetration resistance is presented in [4]. Gareeva [5] offers experimental relations for static probing by standard procedures and determines resistances on front and side faces of indenters. Relations for fining optimal parameters of a cone tool penetrating in soil are given in [6], and it is illustrated that the cone is the most rational shape.

This study deals with the penetration of a rigid smooth wedge in a layer of an initially anisotropic material (plain strain deformation). Two limit loads are determined: initial state of the edge penetration and final state when the wedge penetrates the layer and comes out from the other side (an analogous problem for isotropic medium is solved in [9].

Let in the coordinate system $xOy$ the initial medium is deformed elastically by the Hooke law:

\begin{equation}
\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x - a_{12}\sigma_y, \\
\varepsilon_y &= -a_{12}\sigma_x + a_{22}\sigma_y, \\
\varepsilon_{xy} &= a_{33}\tau_{xy},
\end{aligned}
\end{equation}

where $a_{ij}$ — elasticity constants characterizing ductility of the material. The choice of the Hooke law in the form of (1) with four different elasticity constant is only governed by the simplified formula of the solution; the authors will include the rest independent elastic constants $a_{ij}$ (six!) of the plain strain deformation in the further studies.

Aiming the construct plasticity equations for the medium (1), a basis of tensors is introduced [10]:
In the basis (the scalar product is determined as resultant of \( \sigma_{ij} \varepsilon_{ij} \) with summing over repeated indices), the coordinates of the tensors \( T_\sigma, T_\varepsilon \) are, respectively given by:

\[
S_1 = \sigma_x, \quad S_2 = \sigma_y, \quad S_3 = \sqrt{2} \tau_{xy},
\]

\[
\Omega_1 = \varepsilon_x, \quad \Omega_2 = \varepsilon_y, \quad \Omega_3 = \sqrt{2} \varepsilon_{xy}.
\]  

In the coordinates (3), the Hooke law (1) takes on matrix form:

\[
\begin{pmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{pmatrix} = \begin{pmatrix}
a_{11} & -a_{12} & 0 \\
-a_{12} & a_{22} & 0 \\
0 & 0 & a_{33}
\end{pmatrix} \begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}.
\]  

For the symmetrical matrix \( A \) in (4), the eigenvalues [11] are real:

\[
\lambda_1 = \frac{a_{11} + a_{22}}{2} + \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2}, \quad \lambda_2 = \frac{a_{11} + a_{22}}{2} - \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2}, \quad \lambda_3 = a_{33}.
\]

The eigenvectors, pursuant to the symmetry of the matrix \( A \), are mutually orthogonal:

\[
\tilde{b}_1 = \left(\cos \beta, -\sin \beta, 0\right), \quad \tilde{b}_2 = \left(\sin \beta, \cos \beta, 0\right), \quad \tilde{b}_3 = \left(0, 0, 1\right),
\]

where \( \tan 2\beta = 2a_{12}/(a_{11} - a_{22}) \) and the constants \( a_{ij} \) are determined by (1).

Taking into account (5), (6), (2), the eigen tensors are:

\[
\tilde{T}_1 = \begin{pmatrix}
\cos \beta & 0 \\
0 & -\sin \beta
\end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix}
\sin \beta & 0 \\
0 & \cos \beta
\end{pmatrix}, \quad \tilde{T}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]  

Then, the stress tensor \( T_\sigma \) is expanded by the basis (7). As a result, we have the coordinates of the tensor \( T_\sigma \) in the new basis:

\[
\tilde{S}_1 = \sigma_x \cos \beta - \sigma_y \sin \beta, \quad \tilde{S}_2 = \sigma_y \sin \beta + \sigma_x \cos \beta, \quad \tilde{S}_3 = \sqrt{2} \tau_{xy}.
\]

For the coordinates \( \tilde{S}_1, \tilde{\Omega}_i \) in the basis (7), there exists proportional elastic relations:

\[
\tilde{\Omega}_1 = \lambda_1 \tilde{S}_1, \quad \tilde{\Omega}_2 = \lambda_2 \tilde{S}_2, \quad \tilde{\Omega}_3 = \lambda_3 \tilde{S}_3,
\]

where \( \lambda_i \) — the eigenvalues (5). Apparently, Poisson’s ratio is absent, i.e. a force directed along a tensor \( \tilde{T}_i \) causes deformation only along this direction.

The ductilities \( \lambda_1, \lambda_2, \lambda_3 \) for the initially isotropic medium are distributed so that \( \lambda_1 = \lambda_3 > \lambda_2 \) and plasticity develops in a plane intersecting the tensors \( \tilde{T}_1, \tilde{T}_3 \) where there are the highest ductilities. In our case of (1), considering (9) and the hypothesis that \( \lambda_1, \lambda_2, \lambda_3 \) are distributed as \( \lambda_1 > \lambda_3 > \lambda_2 \), we assume that plasticity in the anisotropic medium (1) also develops in the plane \( \tilde{T}_1, \tilde{T}_3 \) where ductilities are higher than in the line of \( \tilde{T}_2 \).

The plasticity criteria in the case of \( \lambda_1 > \lambda_3 > \lambda_2 \) are given by:
where $K_i, K_3$—constants which are the elastic limits in the medium (1) in the lines of the tensors $\vec{T}_i, \vec{T}_3$ (in the plane $\vec{S}_1, \vec{S}_3$ the plasticity conditions is a rectangle with the sides (10)). The idea that the eigen tensors of elasticity are the eigen tensors of plasticity, creep and failure belongs to Anvar Chanyhev [12].

Suing the common scheme [13], we analyze the system of equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

(11)

together with the plasticity conditions (10). Using (11) and the plasticity conditions (10) allows finding the axes in $xOy$ as the straight lines:

$$\frac{dy}{dx} = \pm \sqrt{\text{ctg} \beta}$$

(12)

(the angle $\beta$ is determined from (6)) and the equilibrium equations along them:

$$d\sigma_x + d\tau_{xy} \frac{dy}{dx} = 0.$$

Combined with (10), this yields the stress distribution along the lines (12):

$$\begin{cases}
\frac{dy}{dx} = +\sqrt{\text{ctg} \beta}, \quad \sigma_y + \tau_{xy} \sqrt{\text{ctg} \beta} = \text{const}, \quad \sigma_x = \sigma_y \text{tg} \beta \pm \frac{K_i}{\cos \beta}, \\
\frac{dy}{dx} = -\sqrt{\text{ctg} \beta}, \quad \sigma_y - \tau_{xy} \sqrt{\text{ctg} \beta} = \text{const}, \quad \sigma_x = \sigma_y \text{tg} \beta \pm \frac{K_i}{\cos \beta}.
\end{cases}$$

(13)

Using the second plasticity condition (10) with (11) produces the stress distributions below:

$$\sigma_x = f(y), \quad \sigma_y = \varphi(x), \quad \tau_{xy} = \pm \frac{K_i}{\sqrt{2}}.$$  

(14)

where $f, \varphi$—arbitrary functions of the appropriate coordinates $x, y$.

In layered anisotropic medium, the state (14) conforms with the plastic shearing of the layers relative one another. The formulas (13) describe plastic deformation of the layers.

Finally, we use (13), (14) in solving a problem on penetration of a perfectly rigid smooth wedge in a layer composed of the material (1).

The first illustration is given in Figure 1. A half-plane $xOy$ has a recess $BCD$ where a wedge with a nose angle $2\gamma$ is placed. It is required to find the limit load on the wedge faces $BC$ and $CD$ such that material starts yielding along the slide lines (12) subject to the conditions (13).

Denoting the angle between the characteristics (12) and the axis $x$ as $\alpha$ brings the case of a sharp wedge when $\pi/2 - \gamma > \alpha$ as in Figure 1 or a blunted wedge when $\pi/2 - \gamma < \alpha$. The latter variant will be studied later on.

Let us analyze the straight lines parallel to $CE$ with the equations $dy/dx = \sqrt{\text{ctg} \beta}$. Along them, $\sigma_y + \tau_{xy} \sqrt{\text{ctg} \beta} = 0$ (because of $\sigma_y = \tau_{xy} = 0$ at the boundary $DE$), and, thus, in these lines, $\sigma_y = -\tau_{xy} \sqrt{\text{ctg} \beta}, \sigma_x = \sigma_y \text{tg} \beta \pm \frac{K_i}{\cos \beta}$. 


Figure 1. The wedge $BCD$ penetrating the half-plane with the boundary $y = 0$.

The calculation starts on the face $CD$. The shear stress is zero $CD$ (as the wedge is assumed perfectly smooth). Plot an external normal to $CD$. It follows from Figure 1 that:

$$\vec{n} = (-\cos \gamma, \sin \gamma), \quad \vec{t} = (\sin \gamma, \cos \gamma),$$

where $\vec{t}$ — unit vector of a tangent. Considering (15), we find the stress vector $\vec{p}$ on $CD$:

$$\vec{p} = (-\sigma_x \cos \gamma + \tau_{xy} \sin \gamma) \vec{n} + (-\tau_{xy} \cos \gamma + \sigma_y \sin \gamma) \vec{j},$$

where $\vec{i}, \vec{j}$ — unitary vectors in $xOy$. Using (16), (15), determine the shear and normal stresses on $CD$:

$$p_x = \vec{p} \cdot \vec{i} = -\tau_{xy} \cos 2\gamma - \frac{\sigma_x - \sigma_y}{2} \sin 2\gamma = 0,$$

$$p_y = \vec{p} \cdot \vec{n} = \sigma_x \cos^2 \gamma + \sigma_y \sin^2 \gamma - \tau_{xy} \sin 2\gamma = 0.$$

From the first equation in (17) and the conditions $\sigma_y = -\tau_{xy} \sqrt{\tan \beta}$, $\sigma_x = -\tau_{xy} \sqrt{\tan \beta} - K_1 / \cos \beta$, we determine the stress $\tau_{xy}$ on $CD$ as:

$$\tau_{xy} = \frac{K_1 \sin 2\gamma}{\cos \beta [2 \cos 2\gamma + \sin 2\gamma (\sqrt{\tan \beta} - \sqrt{\tan \beta})]}.$$  

The second equation (17) yields the load $p_y$:

$$p_y = \frac{K_1}{\cos \beta} \left[ \cos^2 \gamma + \frac{\sin 2\gamma (\sqrt{\tan \beta} \cos^2 \gamma + \sin 2\gamma + \sqrt{\tan \beta} \sin^2 \gamma)}{2 \cos 2\gamma + \sin 2\gamma (\sqrt{\tan \beta} - \sqrt{\tan \beta})} \right].$$

With the known load on $CD$, we find the force

$$\vec{F}_n = p_y \frac{h}{\cos \gamma} \vec{n},$$

applied by the wedge to the anisotropic body ($h$ — varied penetration depth of the wedge). Projection of (20) on the axis $y$ and change of the sign in the expression produces the relation to determine the resistance force of the medium to the wedge penetration:

$$2F_y = -2p_y h \tan \gamma = 2|p_y| h \tan \gamma.$$  

(21)
The relation (21) corresponds to the entry of the wedge tip in the layer, i.e. it is valid when \(0 \leq h \leq h_0\) (Figure 2a), where \(h_0\) —height of the conical part of the tool. Considering the entry of the tool in the layer (Figs. 2b and 2c), it is evident that the formulas (19)–(21) hold true upon the assumption that the sides of the tool are free from friction. At the same time, the formulas should assume \(h = h_0\), i.e. along \(h_0 \leq h \leq H\) (\(H\) —layer thickness), the penetration resistance is constant: 

\[2F_y = 2\rho_0 h_0 \tan \gamma.\]

After that, there comes the stage when the wedge tip comes out of the layer (Figure 2d). The calculations show that the resistance is still equal to (21) with the variable \(h\): 

\[0 \leq h \leq h_0.\]

Figure 2. Stages of penetration of the perfectly rigid wedge in the anisotropic body layer:
(a) \(0 \leq h < h_0\); (b) \(h_0 \leq h < H\); (c) \(h = H\); (d) \(h \geq H\): \(h\) —penetration depth; \(h_0\) —height of conical part of the wedge; \(H\) —layer thickness.

The calculations are performed for the maximum possible thick layer breakable by a tool. The tool approaches the body at the velocity \(v_0\). The tool mass is a sum of the mass of the rectangle with a length \(L\) and width \(2h_0 \tan \gamma\) and the mass of the conical head with an area \(h_0^2 \tan \gamma\). The tip penetration velocity is denoted by \(v_1\), the head velocity at the other side of the layer —\(v_2\), and the head velocity outside the layer —\(v_3\). In the limit case, \(v_3 = 0\). Then:

\[
\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = - \frac{ch_0^2}{2},
\]

\[
\frac{mv_2^2}{2} - \frac{mv_1^2}{2} = -ch_0(H - h_0),
\]

\[
\frac{mv_3^2}{2} - \frac{mv_2^2}{2} = - \frac{ch_0^2}{2},
\]

where \(c = 2\rho_0 h_0 \tan \gamma\). Summing up (22)–(24) gives in the limit case \(v_3 = 0\) the relation of the maximum possible penetration depth:

\[
H = \frac{mv_0^2}{2ch_0}.
\]

Inserting the mass 

\[m = \left(2L h_0 \tan \gamma + h_0^2 \tan \gamma\right)\rho g,\]

density \(\rho\) and free fall acceleration \(g\) in equation (25) finally yields:

\[
H = \frac{\rho g}{4\rho_0} \left(\frac{2L}{h_0} + 1\right) v_0^2.
\]
The implications of (26) are quite evident: the penetration depth is higher at the higher initial velocity $v_0$; the penetration depth is lower with the higher rigidity of the medium in the line of the normal drawn to the wedge; the penetration depth is higher with the higher ratio $L / h_0$ (bulkier tail of the tool); the penetration depth is higher with the higher density of the wedge. The most interesting is the dependence of the medium resistance $p_n$ on the medium properties (angle $\beta$) and the wedge nose angle $\gamma$. This study includes calculations of the resistance versus the listed parameters.

Conclusion

In the framework of a rigid–plastic body, the authors have constructed the mathematical model of penetration of a rigid tool in an initially anisotropic medium. The maximum possible penetration depth has been determined. The dependence of the penetration depth on geometry of the wedge and on the anisotropic parameters of the medium has been analyzed.

References

[1] Sokolovsky VV 1969 The Theory of Plasticity Moscow: Vyssh Shkola (in Russian)

[2] Ling-Yu Xu, Fei Cai, Guo-Xin Wang and Keizo Ugai 2013 Nonlinear analysis of laterally loaded single piles in sand using modified strain wedge model Computers and Geotechnics No 51 pp 60–71

[3] Seo H and Prezzi M 2006 Analytical solutions for a vertically loaded pile in multi-layered soil Geomechanics and Geoengineering: An International Journal pp 1–10

[4] Tekeste MZ, Raper RL and Tollner EW 2007 Finite element analysis of cone penetration in soil for prediction of hardpan location American Society of Agricultural and Biological Engineers ISSN 0001-2351 Vol 50 pp 23–31

[5] Gareeva NB 2003 Development of methods for designing foundations of foundations for digital models of soil massifs on the basis of sounding data Author’s Abstract Diss. Doct. Tech. Sci. Ufa (in Russian)

[6] Ponomarenko YuI 2002 Improving the efficiency of the pile foundations in sealed soils Author’s Abstract Diss. Doct. Tech. Sci. Omsk (in Russian)

[7] Kotov VL and Linnik EYu 2014 Numerical calculation of the optimal shape of the body of revolution during motion with a constant speed in a soil medium Vychislit. Mekh. Sploshn. Sred Vol 7 No 2 pp 142–150

[8] Afanasyev AI and Fedoseev AP 2014 Contact stresses in an elliptical working tool of a machine for destruction of oversized material Fundamental. Issled. No 9 pp 497–500

[9] Davydov DV and Myasnikin YuM 2009 On introduction of bodies into a rigid-plastic medium, Vestn. VGU, Ser. Fiz. Mat. Nauk No 1 pp 94–100

[10] Novozhilov VV 1963 On the forms of connection between stresses and strains in initially isotropic inelastic bodies (geometrical standpoint) Prikl. Matem. Mekh. Vol 27 No 5 pp 794–812

[11] Korn GA and Korn TM 1973 Mathematical Handbook for Scientists and Engineers Moscow: Nauka (in Russian)

[12] Chanyshhev Al 1984 Plasticity of anisotropic media J. Appl. Mech. Tech. Phys. Vol 25 No 2 pp 311–314

[13] Kachanov LM 1956 The Fundamentals of the Theory of Plasticity Moscow: Gostekhizdat (in Russian)