Network Synchronization, Diffusion, and the Paradox of Heterogeneity

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Many complex networks display strong heterogeneity in the degree (connectivity) distribution. Heterogeneity in the degree distribution often reduces the average distance between nodes but, paradoxically, may suppress synchronization in networks of oscillators coupled symmetrically with uniform coupling strength. Here we offer a solution to this apparent paradox. Our analysis is partially based on the identification of a diffusive process underlying the communication between oscillators and reveals a striking relation between this process and the condition for the linear stability of the synchronized states. We show that, for a given degree distribution, the maximum synchronizability is achieved when the network of couplings is weighted and directed, and the overall cost involved in the couplings is minimum. This enhanced synchronizability is solely determined by the mean degree and does not depend on the degree distribution and system size. Numerical verification of the main results is provided for representative classes of small-world and scale-free networks.

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I. INTRODUCTION

The interplay between network structure and dynamics has attracted a great deal of attention in connection with a variety of processes, including epidemic spreading, congestion and cascading failures, and synchronization of coupled oscillators. Small-world and scale-free properties are characterized by an algebraic, highly heterogeneous distribution of degrees (number of links per node). Most SFNs also exhibit a small average distance between nodes and high clustering. Scale-free networks are shown to have universal structural features, such as the small-world and scale-free properties. Small-world networks exhibit a small average distance between nodes and high clustering. Scale-free networks have a degree distribution that follows an algebraic law, which is a process involving all possible paths between nodes. Another basic assumption of previous work is that the oscillators are coupled symmetrically with the same coupling strength. Under the assumption of symmetric coupling, the maximum synchronizability may be achieved when the coupling strength is uniform. But to get a better synchronizability, the couplings are not necessarily symmetrical. Many real-world networks are actually directed and weighted, and the communication capacity of a node is likely to saturate when the degree becomes large.

In this paper, we study the effect that asymmetry and saturation of coupling strength have on the synchronizability of complex networks. We identify a physical process of information diffusion that is relevant for the communication between oscillators and we investigate the relation between this process and the stability of synchronized states in directed networks with weighted couplings. We address these fundamental issues using as a paradigm the problem of complete synchronization of identical oscillators.

Previous work has focused mainly on the role played by shortest paths between nodes. By considering only shortest paths it is implicitly assumed that the information spreads only along them. However, the communication between oscillators is more closely related to a process of diffusion on the network, which is a process involving all possible paths between nodes. Another basic assumption of previous work is that the oscillators are coupled symmetrically with the same coupling strength. Under the assumption of symmetric coupling, the maximum synchronizability may be achieved when the coupling strength is uniform. But to get a better synchronizability, the couplings are not necessarily symmetrical. Many real-world networks are actually directed and weighted, and the communication capacity of a node is likely to saturate when the degree becomes large.

We find that the synchronizability is explicitly related to the mixing rate of the underlying diffusive process. For a given degree distribution, the synchronizability is maximum when the diffusion has a uniform stationary state, which in general requires the network of couplings to be weighted and directed. For large sufficiently random networks, the maximum synchronizability is primarily determined by the mean degree of the network and does not depend on the degree distribution and system size, in sharp contrast with the case of unweighted (symmetric) coupling, where the synchronizability is strongly suppressed as the heterogeneity or number of oscillators is increased. Furthermore, we show that the total cost involved in the network coupling is significantly reduced, as compared to the case of unweighted coupling, and is mini-
The synchronizability can be easily enhanced if we modify the topology of the network of couplings. Moreover, the synchronizability of SFNs is strongly enhanced when the network of couplings is suitably weighted. This, in addition to the well-known improved structural robustness of SFNs, may have played a crucial role in the evolution of many SFNs.

The paper is organized as follows. In Sec. III we introduce the synchronization model and the measure of synchronizability. In Sec. III we study the corresponding process of diffusion. In Sec. IV we focus on the case of maximum synchronizability. The problem of cost is considered in Sec. V. In Sec. VI we present direct simulations on networks of maps. Discussion and conclusions are presented in the last section.

II. FORMULATION OF THE PROBLEM

We introduce a generic model of coupled oscillators and we present a condition for the linear stability of the synchronized states in terms of the eigenvalues of the coupling matrix.

A. Synchronization Model

We consider complete synchronization of linearly coupled identical oscillators:

\[
\frac{dx_i}{dt} = f(x_i) - \sigma \sum_{j=1}^{N} G_{ij} h(x_j), \quad i = 1, 2, \ldots, N, \tag{1}
\]

where \( f = f(x) \) describes the dynamics of each individual oscillator, \( h = h(x) \) is the output function, \( G = (G_{ij}) \) is the coupling matrix, and \( \sigma \) is the overall coupling strength. The rows of matrix \( G \) are assumed to have zero sum to ensure that the synchronized state \( \{ x_i(t) = s(t), \forall i \mid \frac{dx_i}{dt} = f(s) \} \) is a solution of Eq. (1).

In the case of symmetrically coupled oscillators with uniform coupling strength, \( G \) is the usual (symmetric) Laplacian matrix \( L = (L_{ij}) \): the diagonal entries are \( L_{ii} = k_i \), where \( k_i \) is the degree of node \( i \), and the off-diagonal entries are \( L_{ij} = -1 \) if nodes \( i \) and \( j \) are connected and \( L_{ij} = 0 \) otherwise. For \( G_{ij} = L_{ij} \), heterogeneity in the degree distribution suppresses synchronization in important classes of networks (see also Ref. [10]). The synchronizability can be easily enhanced if we modify the topology of the network of couplings. Here, however, we address the problem of enhancement of synchronizability for a given network topology.

In order to enhance the synchronizability of heterogeneous networks, we propose to scale the coupling strength by a function of the degree of the nodes. For specificity, we take

\[
G_{ij} = L_{ij} k_i^{\beta}, \tag{2}
\]

where \( \beta \) is a tunable parameter. We say that the network or coupling is weighted when \( \beta \neq 0 \) and unweighted when \( \beta = 0 \). The underlying network associated with the Laplacian matrix \( L \) is undirected and unweighted, but for \( \beta \neq 0 \), the network of couplings becomes not only weighted but also directed because the resulting matrix \( G \) is in general asymmetric. This is a special kind of directed network where the number of in-links is equal to the number of out-links in each node, and the directions are encoded in the strengths of in- and out-links. In spite of the possible asymmetry of matrix \( G \), all the eigenvalues of matrix \( G \) are nonnegative reals and can be ordered as \( 0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N \), as shown below.

B. Basic Spectral Properties

Eq. (3) can be written as

\[
G = D^{-\beta} L, \quad \beta/2
\]

where \( D = \text{diag}\{k_1, k_2, \ldots k_N\} \) is the diagonal matrix of degrees. (We recall that the degree \( k_i \) is the number of oscillators coupled to oscillator \( i \).) From the identity \( \det(D^{-\beta} L - \lambda I) = \det(D^{-\beta/2} L D^{-\beta/2} - \lambda I) \), valid for any \( \lambda \), where \( \det \) denotes the determinant and \( I \) is the \( N \times N \) identity matrix, we have that the spectrum of eigenvalues of matrix \( G \) is equal to the spectrum of a symmetric matrix defined as

\[
H = D^{-\beta/2} L D^{-\beta/2}, \tag{4}
\]

That is, \( \rho(G) = \rho(H) \), where \( \rho \) denotes the set of eigenvalues. From this follows that all eigenvalues of matrix \( G \) are real, as anticipated above. It is worth mentioning that, although the eigenvalues of \( G \) and \( H \) are equal, from the numerical point of view it is much more efficient to compute the eigenvalues from the symmetric matrix \( H \) than from \( G \).

Additionally, all the eigenvalues of matrix \( G \) are nonnegative because \( H \) is positive semidefinite, and the smallest eigenvalue \( \lambda_1 \) is always zero because the rows of \( G \) have zero sum. Moreover, if the network is connected, then \( \lambda_2 > 0 \) for any finite \( \beta \). This follows from the corresponding property for \( L \) and Eq. (4), i.e., the fact that matrices \( H \) and \( L \) are congruent. Naturally, the study of complete synchronization of the whole network only makes sense if the network is connected.

For \( \beta = 1 \), matrix \( H \) is the normalized Laplacian matrix studied in spectral graph theory [20]. In this case, if \( N \geq 2 \) and the network is connected, then \( 0 < \lambda_2 \leq N/(N-1) \) and \( N/(N-1) \leq \lambda_N \leq 2 \). For spectral properties of unweighted SFNs, see Refs. [21][22][23].

C. Synchronizability

The variational equations governing the linear stability of a synchronized state \( \{ x_i(t) = s(t), \forall i \} \) of the system in Eqs. (1) and (2) can be diagonalized into \( N \) blocks of the form

\[
\frac{d\eta}{dt} = [D f(s) - \alpha D h(s)] \eta, \tag{5}
\]

where \( D \) denotes the Jacobian matrix, \( \alpha = \sigma \lambda_i \), and \( \lambda_i \) are the eigenvalues of the coupling matrix \( G \). The largest Lyapunov
exponent $\Gamma(\alpha)$ of this equation can be regarded as a master stability function, which determines the linear stability of the synchronized state for any linear coupling scheme. The synchronized state is stable if $\Gamma(\sigma \lambda_i) < 0$ for $i = 2, \ldots, N$. (The eigenvalue $\lambda_1$ corresponds to a mode parallel to the synchronization manifold.)

For many widely studied oscillatory systems, the master stability function $\Gamma(\alpha)$ is negative in a single, finite interval $(\alpha_1, \alpha_2)$. Therefore, the network is synchronizable for some $\sigma$ when the eigenratio $R = \lambda_N / \lambda_2$ satisfies

$$R < \alpha_2 / \alpha_1.$$  (6)

The right-hand side of this equation depends only on the dynamics ($f$, $h$, and $s$), while the eigenratio $R$ depends only on the coupling matrix $G$. The problem of synchronization is then reduced to the analysis of eigenvalues of the coupling matrix $G$: the smaller the eigenratio $R$, the larger the synchronizability of the network, and vice versa.

### III. DIFFUSION AND BALANCE OF HETEROGENEITY

We study a process of diffusion relevant for the communication between oscillators and we argue that the synchronizability is maximum ($R$ is minimum) for $\beta = 1$.

#### A. Diffusion Process

From the identity

$$\sum_{j=1}^{N} G_{ij} h(x_j) = \sum_{j=1}^{N} k_i^{-\beta} A_{ij} [h(x_i) - h(x_j)]$$  (7)

we observe that the weighted coupling scheme in Eqs. 11 and 24 is naturally related to a diffusive process with absorption and emission described by the transition matrix

$$P = \frac{1}{\Lambda} D^{-\beta} A,$$  (8)

where $A = D - L$ is the adjacency matrix, and $\Lambda$ is the largest eigenvalue of $D^{-\beta} A$. According to this process, if we start with an arbitrary distribution $y = (y^{(1)}, \ldots, y^{(N)})$, where $y^{(i)}$ is associated with the initial state at node $i$, after $n$ time steps the distribution is $P^n y$. This process is different from the usual (conservative) random walk process. In particular, because $\sum_j P_{ij}$ may be different from 1, the diffusion in and out of a node may differ even in the stationary state.

For instance, consider a network of three nodes, $a$-$b$-$c$, where nodes $a$ and $c$ have degree 1 and are both connected to node $b$. For $\beta = 1$, the transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$  (9)

In the stationary state, each of the three nodes has, say, one unity (of the “diffusive quantity”). At each time step, node $b$ receives 1/2 unit from node $a$ and 1/2 unit from node $c$, and each of the nodes $a$ and $c$ receives 1 unit from node $b$. Therefore, node $b$ sends a total of 2 units and receives only 1 unit, while each of the nodes $a$ and $c$ sends 1/2 unit and receives 1 unit. This means that there is an “absorption” of 1/2 unit at each of the nodes $a$ and $c$ and the “emission” of 1 unit at node $b$. It is in this sense that the matrix in Eq. 24 describes a diffusive process with absorption and emission.

On the other hand, the usual random walk process is conservative at each node. Such a process is described by the matrix $D^{-\beta} A G^{-1}$, where $C_{ij} = \delta_{ij} \sum_{k} k_i^{-\beta}$ and the sum is over all the $k_j$ nodes connected to node $j$. For a node $i$ connected to a node $j$ and a uniform distribution, this conservation law implies that the amount of information that node $i$ receives from node $j$ depends on the degree of all the nodes connected to node $j$. But this is not what happens in a network of self-sustained oscillators. In a network of oscillators, the amount of information that oscillator $i$ receives directly from oscillator $j$ can only depend on the strength of the coupling from $j$ to $i$, which is proportional to $1/k_i^\beta$, as in the process described by the transition matrix in Eq. 24.

#### B. Balance of Heterogeneity

Because the master stability function $\Gamma(\alpha)$ is negative in a finite interval $(\alpha_1, \alpha_2)$, increasing (decreasing) the overall coupling strength $\sigma$ beyond a critical value $\sigma_{max}$ ($\sigma_{min}$) destabilizes the synchronized state. Dynamically, the loss of stability is due to a short (long) wavelength bifurcation at $\sigma = \sigma_{max}$ ($\sigma_{min}$) (see also Ref. 11). Physically, this bifurcation excites the shortest (longest) spatial wavelength mode because some oscillators are too strongly (weakly) influenced by the others.

Now, consider the process of diffusion described by matrix $P$ on a network where not all the nodes have the same degree. Starting with an arbitrary distribution $y$, after $n$ steps we have $P^n y$. If we require the (stationary) distribution for $n \to \infty$ to be uniform, we obtain $\beta = 1$ because this is the only case where $y_0 = (1, \ldots, 1)$ is an eigenvector associated with the eigenvalue 1, which is the largest eigenvalue of matrix $P$. For $\beta < 1$, the distribution is more heavily concentrated on nodes with large degree. For $\beta > 1$, the concentration happens on nodes with small degree. Physically, this means that for both $\beta < 1$ and $\beta > 1$ some oscillators are more strongly influenced than others and the ability of the network to synchronize is limited by the least and most influenced oscillators: for small (large) $\sigma$ the system is expected to undergo a long (short) wavelength bifurcation due to the least (most) influenced nodes, as explained above. We then expect the network to achieve maximum synchronizability at $\beta = 1$.

In Fig. 1 we show the numerical verification of this hypothesis for various models of complex networks. The networks are built as follows: (i) Random SFNs

(i) Random SFNs — Each node is assigned to have a number $k_i \geq k_{min}$ of “half-links” according to the probability distribution $P(k) \sim k^{-\gamma}$, where $\gamma$ is a scal-
ing exponent and $k_{\min}$ is a constant integer. The network is generated by randomly connecting these half-links to form links, prohibiting self- and repeated links. In the limit $\gamma = \infty$, all nodes have the same degree $k = k_{\min}$.

(ii) **Networks with expected scale-free sequence** — The network is generated from a sequence $k_1, k_2, \ldots, k_N$, where $k_i \geq k_{\min}$ follows the distribution $P(k) \sim k^{-\gamma}$ and $\max_i k_i^\alpha < \sum_i k_i$. A link is then independently assigned to each pair of nodes $(i, j)$ with probability $p_{ij} = \tilde{k}_i \tilde{k}_j / \sum_i k_i$. In this model, self-links are allowed. We observe, however, that the eigenratio $R$ is insensitive to the removal of self-links.

(iii) **Growing SFNs** — We start with a fully connected network with $m$ nodes and at each time step a new node with $m$ links is added to the network. Each new link is connected to a node $i$ in the network with probability $\Pi_i \sim (1 - p)\tilde{k}_i + p$, where $\tilde{k}_i$ is the updated degree of node $i$ and $0 \leq p \leq 1$ is a tunable parameter. For large degrees, the scaling exponent of the resulting network is $\gamma = 3 + p[m(1-p)]^{-1}$. For $p = 0$, the exponent is $\gamma = 3$ and we recover the Barabási-Albert model.

(iv) **SWNs** — Starting with a ring of $N$ nodes, where each node is connected to $2\kappa$ first neighbors, we add $M \leq N(N - 2\kappa - 1)/2$ new links between randomly chosen pairs of nodes. Self- and repeated links are avoided.

Our extensive numerical computation on the models (i-iv) shows that the eigenratio $R$ has a well defined minimum at $\beta = 1$ in each case [Fig. 1]. The only exception is the class of homogeneous networks, where all the nodes have the same degree $k$. When the network is homogeneous, the weights $k_i^{-\beta}$ can be factored out in Eq. (8) and a uniform stationary distribution is achieved for any $\beta$. In this case, the eigenratio $R$ is independent of $\beta$, as shown in Fig. 1(a) for random homogeneous networks with $k = 10$ (solid line). A random homogeneous network corresponds to a random SFN for $\gamma = \infty$. In all other cases, including the relatively homogeneous SWNs, the eigenratio exhibits a pronounced minimum at $\beta = 1$ (note the logarithmic scale in Fig. 1).

In SWNs, the heterogeneity of the degree distribution increases as the number $M$ of random links is increased. The eigenratio $R$ at $\beta = 0$ reduces as $M$ is increased, but the eigenratio at $\beta = 1$ reduces even more, so that the minimum of the eigenratio becomes more pronounced as the heterogeneity of the degree distribution is increased [Fig. 1(d)]. Similar results are observed in the original Watts-Strogatz model of SWNs, where the mean degree is kept fixed as the number of random links is increased.

In SFNs, the heterogeneity increases as the scaling exponent $\gamma$ is reduced. As shown in Fig. 1(a) for random SFNs, the minimum of the eigenratio $R$ becomes more pronounced as the heterogeneity of the degree distribution is increased.

![Fig. 1: Eigenratio $R$ as a function of $\beta$.](image)

The same tendency is observed across different models of networks. For example, for a given $\gamma$ and $k_{\min} = k_{\min}$, the minimum of the eigenratio is more pronounced in networks with expected scale-free sequence [Fig. 1(b)] than in random SFNs [Fig. 1(a)], because the former may have nodes with degree smaller than $k_{\min}$. For small $\gamma$, the eigenratio in growing SFNs [Fig. 1(c)] behaves similarly to the eigenratio in random SFNs [Fig. 1(a)]. A pronounced minimum for the eigenratio $R$ at $\beta = 1$ is also observed in various other models of complex networks.

### C. Mean Field Approximation

A mean field approximation provides further insight into the effects of degree heterogeneity and the dependence of $R$ on $\beta$.

The dynamical equations (11) can be rewritten as:

$$\frac{dx_i}{dt} = f(x_i) + \sigma k_i^{1-\beta}[\bar{h}_i - h(x_i)],$$  \hspace{1cm} (10)

where

$$\bar{h}_i = \frac{1}{k_i} \sum_j A_{ij} h(x_j)$$  \hspace{1cm} (11)

is the local mean field from all the nearest neighbors of oscillator $i$. If the network is sufficiently random and the system is close to the synchronized state $s$, we may assume that $\bar{h}_i \approx h(s)$ and we may approximate Eq. (10) as

$$\frac{dx_i}{dt} = f(x_i) + \sigma k_i^{1-\beta}[h(s) - h(x_i)],$$  \hspace{1cm} (12)
indicating that the oscillators are decoupled and forced by a common oscillator with output $h(s)$.

From a variational equation analogous to Eq. (5), we have that all oscillators in Eq. (12) will be synchronized by the common forcing when

$$\alpha_1 < \sigma k_i^{1-\beta} < \alpha_2 \quad \forall i.$$  \hspace{1cm} (13)

For $\beta \neq 1$, it is enough to have a single node with degree very different from the others for this condition not to be satisfied for any $\sigma$. In this case, the complete synchronization becomes impossible because the corresponding oscillator cannot be synchronized. Within this approximation, the eigenratio is $R = (k_{\text{max}}/k_{\text{min}})^{1-\beta}$ for $\beta \leq 1$ and $R = (k_{\text{min}}/k_{\text{max}})^{1-\beta}$ for $\beta > 1$, where $k_{\text{min}} = \min_i \{k_i\}$ and $k_{\text{max}} = \max_i \{k_i\}$. The minimum of $R$ is indeed achieved at $\beta = 1$, in agreement with our numerical simulations.

Therefore, this simple mean field approximation not only explains the results of Ref. [3] on the suppression of synchronizability due to heterogeneity in unweighted networks, but also predicts the correct condition for maximum synchronizability in weighted networks.

**IV. MIXING RATE AND SYNCHRONIZABILITY**

We relate the eigenratio $R$ to the mixing rate of the process of diffusion introduced in Sec. III A and we argue that, for $\beta = 1$ and large, sufficiently random networks, the synchronizability depends only on the mean degree of the network.

**A. Mean Degree Approximation**

We now present a general physical theory for the eigenratio $R$. In what follows we focus on the case of maximum synchronizability ($\beta = 1$). For $\beta = 1$ and an arbitrary network, Eq. (8) can be written as

$$P = D^{-1/2}(I - H)D^{1/2},$$  \hspace{1cm} (14)

where $H = D^{-1/2}LD^{-1/2}$ as in Eq. (4). From the identity $\det(P - \lambda I) = \det(I - H - \lambda I)$, valid for any $\lambda$, it follows that the spectra of matrices $P$ and $H$ are related via $\rho(P) = 1 - \rho(H)$, and the uniform stationary state of the process $P^n y$ is associated with the null eigenvalue of the coupling matrix $G$. We then define the mixing rate as $\nu = \ln \mu^{-1}$, where

$$\mu = \lim_{n \to \infty} \|P^n y - y_0\|^{1/n}$$  \hspace{1cm} (15)

is the mixing parameter, $y_0 = (1, 1, \ldots, 1)$ is the stationary distribution discussed in Sec. III B, $y$ is an arbitrary initial distribution normalized as $\sum_i k_i y_i = \sum_i k_i$, and $\|\cdot\|$ is the usual Euclidean norm. In non-bipartite connected networks [20], we have $\lambda_N < 2$ and the initial distribution $y$ always converges to the stationary distribution $y_0$.

The convergence of the limit in Eq. (15) is dominated by the second largest eigenvalue of matrix $P$ in absolute value, namely $\max_{i=2,\ldots,N} |1 - \lambda_i|$. (The largest eigenvalue is associated with the stationary state $y_0$.) As a result, for any network, the mixing parameter is

$$\mu = \max\{1 - \lambda_2, \lambda_N - 1\}.$$  \hspace{1cm} (16)

Therefore, the mixing is faster in networks where the eigenvalues of the coupling matrix are concentrated close to 1.

The condition for the stability of the synchronized states also requires the eigenvalues of the coupling matrix to be close to 1, although through a slightly different relation ($R = \lambda_N/\lambda_2$ to be small). We can combine these two conditions to write an upper bound for the eigenratio $R$ in terms of the mixing parameter:

$$R \leq \frac{1 + \mu}{1 - \mu}.$$  \hspace{1cm} (17)

This relation is relevant because of its general validity and clear physical interpretation. We show that this upper bound is a very good approximation of the actual value of $R$ in many networks of interest.

The mixing parameter $\mu$ can be expressed as an explicit function of the mean degree $k$. Based on results of Ref. [22] for random networks with given expected degrees, we get

$$\max\{1 - \lambda_2, \lambda_N - 1\} = [1 + o(1)]\frac{2}{\sqrt{k}}.$$  \hspace{1cm} (18)

Moreover, the semicircle law holds and the spectrum of matrix $P$ is symmetric around 1 for $k_{\text{min}} \gg \sqrt{k}$ in the thermodynamical limit [22]. These results are rigorous for ensembles of networks with a given expected degree sequence and sufficiently large minimum degree $k_{\text{min}}$, but our extensive numerical computation supports the hypothesis that the approximate relations

$$\lambda_2 \approx 1 - \frac{2}{\sqrt{k}}, \quad \lambda_N \approx 1 + \frac{2}{\sqrt{k}}.$$  \hspace{1cm} (19)

hold under much milder conditions. In particular, relations [19] are expected to hold true for any large, sufficiently random network with $k_{\text{min}} \gg 1$. The rationale for this is that, for $\beta = 1$, the diffusion in each node of one such network is the same as in a random homogeneous network with the same mean degree, where relations [19] are known to be satisfied [20].

Under the assumption that $1 - \lambda_2 \approx \lambda_N - 1$, the eigenratio can be written as

$$R \approx \frac{1 + \mu}{1 - \mu},$$  \hspace{1cm} (20)

where $\mu$ is defined in Eq. (16). Therefore, the larger the mixing rate (smaller $\mu$), the more synchronizable the network (smaller $R$), and vice versa. From Eq. (19), we have that the mixing parameter can be approximated as $\mu \approx 2/\sqrt{k}$ and the eigenratio can be approximated as

$$R \approx \frac{1 + 2/\sqrt{k}}{1 - 2/\sqrt{k}}.$$  \hspace{1cm} (21)
Therefore, for $\beta = 1$, the eigenratio $R$ is primarily determined by the mean degree and does not depend on the number of oscillators and the details of the degree distribution.

This is a remarkable result because, regardless of the degree distribution, the network at $\beta = 1$ is just as synchronizable as a random homogeneous network with the same mean degree, and random homogeneous networks appear to be asymptotically optimal in the sense that $R$ approaches the absolute lower bound in the thermodynamical limit for large enough $k$ \cite{[9]}. Therefore, for $\beta = 1$, the eigenratio $R$ is as a function of the scaling exponent $\gamma$. We present bounds valid for any network weighted at $\beta = 1$ (\bullet) and $\beta = 0$ (\circ). The other curves are the approximations of the eigen ratio in Eqs. (20) (dotted lines) and (21) (solid lines), and the eigenratio for $\beta = 1$ (dashed lines) and $\beta = 0$ (dot-dashed lines) at $\gamma = \infty$. The \circ symbols correspond to random homogeneous networks with the same mean degree of the corresponding SFNs (the degrees are indicated in the figure). The other network parameters are the same as in Fig. 1.

B. Numerical Verification

Now we test our predictions in the models (i-iii) of SFNs, and we show that the synchronizability is significantly enhanced for $\beta = 1$ as compared to the case of unweighted coupling ($\beta = 0$).

As shown in Fig. 2 in unweighted SFNs, the eigenratio $R$ increases with increasing heterogeneity of the degree distribution (see also Ref. \cite{[8]}). But, as shown in the same figure, the eigenratio does not increase with heterogeneity when the coupling is weighted at $\beta = 1$. The difference is especially large for small scaling exponent $\gamma$, where the variance of the degree distribution is large and the network is highly heterogeneous (note that Fig. 2 is plotted in logarithmic scale). The network becomes more homogeneous as $\gamma$ is increased. In the limit $\gamma = \infty$, random SFNs converge to random homogeneous networks with the same degree $k_{min}$ for all the nodes [Fig. 2(a)], while networks with expected scale-free sequence converge to Erdős-Rényi random networks \cite{[31]}, which have links assigned with the same probability between each pair of nodes [Fig. 2(b)], and growing SFNs converge to growing random networks, which are growing networks with uniform random attachment [Fig. 2(c)]. As one can see from Figs. 2(b) and 2(c), the synchronizability is strongly enhanced even in the relatively homogeneous Erdős-Rényi and growing random networks; such an enhancement occurs also in SWNs.

For $\beta = 1$, the eigenratio $R$ is well approximated by the relations in Eq. (20) [Fig. 2 dotted lines] and Eq. (21) [Fig. 2 solid lines] for all three models of SFNs. This confirms our result that the synchronizability is strongly related to the mixing properties of the network \cite{[32]}. For $\beta = 1$, the eigenratio of the SFNs is also very well approximated by the eigenratio of random homogeneous networks with the same number of links [Fig. 2 \circ]. Therefore, for $\beta = 1$, the variation of the eigenratio $R$ with the heterogeneity of the degree distribution in SFNs is mainly due to the variation of the mean degree of the networks, which increases in both random SFNs and networks with expected scale-free sequence as the scaling exponent $\gamma$ is reduced [Figs. 2(a) and 2(b)].

In Fig. 2 we show the eigenratio $R$ as a function of the system size $N$. In unweighted SFNs, the eigenratio increases strongly as the number of oscillators is increased. Therefore, it may be very difficult or even impossible to synchronize large unweighted networks. However, for $\beta = 1$, the eigenratio of large networks appears to be independent of the system size, as shown in Fig. 2 for the models (i-iii) of SFNs. Similar results are observed in many other models of complex networks. All together, these provide strong evidence for our theory.

C. General Bounds

We present bounds valid for any network weighted at $\beta = 1$. In this case, if the network is connected but not globally connected and $N > 2$, we have

$$1 + (N - 1)^{-1} \leq R \leq 2NkD_{max},$$

(22)

where $k$ is the mean degree and $D_{max}$ is the diameter of the network (maximum distance between nodes). This relation follows from the bounds $1/NkD_{max} \leq \lambda_2 \leq 1$ and $1 + (N - 1)^{-1} \leq \lambda_N \leq 2$ \cite{[24]}. For being valid for any network regardless of its structure, the bounds in Eq. (22) are not tight for specific network models, such as random homogeneous networks. Nevertheless, they provide some insight into the problem. In particular, heterogeneity in the degree
distribution is not disadvantageous in this case because it generally reduces the upper bound in Eq. (22), and this is a major difference from the case of unweighted networks considered previously.

V. COUPLING COST

Having shown that weighted networks exhibit improved synchronizability, we now turn to the problem of cost. We show that the total cost involved in the network of couplings is minimum at the point of maximum synchronizability ($\beta = 1$).

The total cost $C$ involved in the network of couplings is defined as the minimum (in the synchronization region) of the total strength of all directed links,

$$ C = \sigma_{\text{min}} \sum_{i=1}^{N} \lambda_i^{1-\beta}, $$

where $\sigma_{\text{min}} = \alpha_1/\lambda_2$ is the minimum coupling strength for the network to synchronize. We recall that $\alpha_1$ is the point where the master stability function first becomes negative. For $\beta = 1$, we have $C = N\alpha_1/\lambda_2$.

In heterogeneous networks, the cost at $\beta = 1$ is significantly reduced as compared to the case of unweighted coupling ($\beta = 0$), as shown in Fig. 4 for random SFNs. The difference becomes more pronounced when the scaling exponent $\gamma$ is reduced and the degree distribution becomes more heterogeneous. The cost for SFNs at $\beta = 1$ is very well approximated by the cost for random homogeneous networks with the same mean degree [Fig. 4, ♦], in agreement with our analysis in Sec. IV A that, at $\beta = 1$, the eigenvalue $\lambda_2$ is fairly independent of the degree distribution.

As a function of $\beta$, the cost has a broad minimum at $\beta = 1$, as shown in Fig. 5 for random SFNs. Similar result is observed in other models of complex networks, including the models (ii-iv) introduced in Sec. III B. This result is important because it shows that maximum synchronizability and minimum cost occur exactly at the same point. Therefore, cost reduction is another important advantage of suitably weighted networks.

VI. DIRECT SIMULATIONS

To confirm our analysis of enhanced synchronizability, we simulate the dynamics on networks of chaotic maps. The example we consider consists of SFNs of logistic maps.
FIG. 6: Synchronization region ($\sigma_{\text{min}}$, $\sigma_{\text{max}}$) as a function of $\gamma > 2.8$ in random SFNs of logistic maps, for $\beta = 1$ (●) and $\beta = 0$ (○). The bifurcation parameter is (a) $a = 3.58$ and (b) $a = 4.0$. Averages are taken over 20 realizations of the networks. The other parameters are the same as in Fig. 4.

$x_{n+1} = f(x_n) = ax_n(1 - x_n)$, where the output function is taken to be $h(x) = f(x)$. In this case, the master stability function is negative for $(1 - e^{-\Gamma_0}) = \alpha_1 < \alpha < \alpha_2 = (1 + e^{-\Gamma_0})$, where $\Gamma_0 > 0$ is the Lyapunov exponent of the isolated chaotic map. In the simulations of the dynamics, the maps are assigned to have random initial conditions close to the synchronization manifold.

We consider two values of the bifurcation parameter $a$ for which the logistic map is chaotic: $a = 3.58$, where $\alpha_2/\alpha_1 \approx 19$, and $a = 4.0$, where $\alpha_2/\alpha_1 = 3$. In both cases, our simulations show that, if $R < \alpha_2/\alpha_1$, then there is a finite interval of the overall coupling strength $\sigma_{\text{min}} < \sigma < \sigma_{\text{max}}$ where the network becomes completely synchronized after a transient time. Moreover, the simulations confirm that $\sigma_{\text{min}} = \alpha_2/\lambda N$ and $\sigma_{\text{max}} = \alpha_2/\lambda N$, as expected. In order to display the synchronization regions for different $\beta$ in the same figure, we introduce $\sigma^* = \sigma \sum_i k_i^{1-\beta}$, $\sigma_{\text{min}} = \sigma_{\text{min}} \sum_i k_i^{1-\beta}$ and $\sigma_{\text{max}} = \sigma_{\text{max}} \sum_i k_i^{1-\beta}$.

In Fig. 6 we show the synchronization region ($\sigma_{\text{min}}$, $\sigma_{\text{max}}$) as a function of the scaling exponent $\gamma$ in random SFNs, for $\beta = 1$ (●) and $\beta = 0$ (○). The factor $\sum_i k_i^{-\beta}$ is $N$ for $\beta = 1$ and $kN$ for $\beta = 0$. For $a = 3.58$, the networks are synchronizable for some $\sigma^*$ in the region $\gamma > 4.0$ if the couplings are unweighted ($\beta = 0$) and in a wider region of $\gamma$ if the couplings are weighted at $\beta = 1$ [Fig. 6(a)]. In the region where both weighted and unweighted networks are synchronizable, the interval ($\sigma_{\text{min}}$, $\sigma_{\text{max}}$), in which synchronization is achieved, is much wider for $\beta = 1$ than for $\beta = 0$. In terms of the original coupling strength $\sigma$, the difference is even larger ($k$ times larger). More strikingly, for $a = 4.0$, unweighted networks do not synchronize for any coupling strength, but the networks weighted at $\beta = 1$ do synchronize for $\gamma \lesssim 4.0$ in a nonzero interval of $\sigma^*$ [Fig. 6(b)].

The cost $C$ defined in Eq. (23) is exactly $\sigma_{\text{min}}^*$. In agreement with the results in Fig. 6, $\sigma_{\text{min}}^*$ is clearly smaller for $\beta = 1$ than for $\beta = 0$ in the interval of $\gamma$ where both weighted and unweighted networks are synchronizable [Fig. 6(a)]. All together, the results in Fig. 6 illustrate the enhancement of synchronizability and the reduction of cost in weighted networks.

VII. CONCLUDING REMARKS

Motivated by the problem of complex-network synchronization, we have introduced a model of directed networks with weighted couplings that incorporates the saturation of connection strength expected in highly connected nodes of realistic networks. In this model, the total strength of all in-links at a node $i$ with degree $k_i$ is proportional to $k_i^{1-\beta}$, where the parameter $\beta$ is a measure of the degree-dependent saturation in the amount of information that a node receives from other nodes. In a network of oscillators, the weights $k_i^{1-\beta}$ can be alternatively interpreted as a property of the (input function of the) oscillators rather than a property of the links. We believe that this model can serve as a paradigm to address many problems of dynamics on complex networks.

Here we have studied complete synchronization of identical oscillators. We have shown that, for a given network topology, the synchronizability is maximum and the total cost involved in the network of couplings is minimum when $\beta = 1$. For large, sufficiently random network with minimum degree $k_{\text{min}} \gg 1$, the synchronizability at $\beta = 1$ is mainly determined by the mean degree and is fairly independent of the number of oscillators and the details of the degree distribution.

This should be contrasted with the case of unweighted coupling ($\beta = 0$), where the synchronizability is strongly suppressed as the number of oscillators or heterogeneity of the degree distribution is increased. In the case $\beta = 1$, the heterogeneity of the degree distribution is completely balanced and the networks are just as synchronizable as random homogeneous networks with the same mean degree.

Our results are naturally interpreted within a framework where the condition for the linear stability of synchronized states is related to the mixing rate of a diffusive process relevant for the communication between oscillators. Under mild conditions, we have shown that, the larger the mixing rate, the more synchronizable the network. In particular, in unweighted networks, the mixing rate decreases with increasing heterogeneity. This, along with the condition $\beta = 1$ for enhanced synchronizability, explains and solves what we call the “paradox of heterogeneity.” This paradox refers to the (apparently) paradoxical relation between the synchronizability and the average distance between oscillators, observed in heterogeneous unweighted networks [8], and is clarified when we observe that synchronizability is ultimately related to the mixing properties of the network.

We expect our results to be relevant for both network design and the understanding of dynamics in natural systems, such as neuronal networks, where the saturation of connection strength is expected to be important. Although we have focused mainly on SFNs, a class of networks that has received most attention, our analysis is general and applies to networks with arbitrary degree distribution.
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