AFFINE MAPS BETWEEN CAT(0) SPACES

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ABSTRACT. We study affine maps between CAT(0) spaces with geometric actions, and show that they essentially split as products of dilations and linear maps (on the Euclidean factor). This extends known results from the Riemannian case. Furthermore, we prove a splitting lemma for the Tits boundary of a CAT(0) space with geometric action, a variant of a splitting lemma for geodesically complete CAT(1) spaces by Lytchak.

1. INTRODUCTION

Let $X$ and $Y$ be geodesic spaces, and $f: X \to Y$ a map. Recall that a geodesic space is a metric space in which every pair of points is joined by a path of shortest length, called a geodesic. We will always parametrize geodesics by arc length. We call $f$ affine if $f$ maps geodesics $\gamma$ in $X$ to geodesics in $Y$, and $f$ rescales $\gamma$ with constant speed $\rho(\gamma)$, which a priori depends on the geodesic $\gamma$. We call $\rho$ the rescaling function. We will not touch on the much more difficult question of the extent to which the set of geodesics determines the metric.

Matveev has obtained strong positive results for Riemannian manifolds with negative curvature [13].

Call an affine map a dilation if there is some constant $c \geq 0$ such that for all $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) = cd_X(x_1, x_2)$. If $X$ and $Y$ are Riemannian, the answer to our problem is classical [10]; any affine map splits as a product of dilations except on the Euclidean factor, where they may be linear maps. Remarkably, Lytchak [12], following work by Ohta [15], extended this result to any arbitrary metric space $Y$ and Riemannian manifold $X$, as long as $X$ is not a product or a higher rank symmetric space. In the latter cases, they produce counterexamples by endowing these spaces with suitable Finsler metrics.

Lytchak and Schröder and later Hitzelberger and Lytchak [8] further investigated the case of real-valued functions on a CAT$(\kappa)$ metric space, and obtained severe restrictions. Understanding affine maps has been important in several applications, particularly to superrigidity problems. We refer the reader to Ohta [15] for a brief discussion.

A group is said to act geometrically on a metric space if the action is properly discontinuous by isometries with compact quotient. This paper considers continuous affine maps on CAT(0) spaces admitting such actions. Caprace and Monod have shown that a cocompact geodesically complete CAT(0) space has an essentially unique factorization of the form $X = X_1 \times \cdots \times X_n \times \mathbb{E}^d$ such that each $X_i \neq \mathbb{R}$ and is irreducible [5] Remarks 1.7, Theorem 1.9].

Recall that a metric space is called proper if all closed balls are compact.

Theorem 1.1 (Main Theorem). Let $X$ be a geodesically complete proper CAT(0) space admitting a geometric group action, and $X = X_1 \times \cdots \times X_n \times \mathbb{E}^d$ be a factorization into irreducible factors such that no $X_i = \mathbb{R}$. Let $Y$ also be a CAT(0) space, and $f: X \to Y$ be a continuous affine map. Then the image of $f$ is convex and hence CAT(0) and splits as $f(X_1) \times \cdots \times f(X_n) \times f(\mathbb{E}^d)$, where $f(\mathbb{E}^d)$ is the Euclidean factor of the image. The restriction of $f$ to every factor $X_i$ is a dilation (possibly with rescaling constant zero) and the restriction to $\mathbb{E}^d$ is a standard affine map between Euclidean spaces.

Combining the Main Theorem with work of Bosché [3], we get two applications to self maps of CAT(0) spaces.

Corollary 1.2 (Self-Affine Maps). Let $X$ be a proper geodesically complete CAT(0) space with geometric action and $f: X \to X$ be a strictly contracting continuous affine map. Then $X$ is flat.

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Corollary 1.3 (Self-Affine Homeomorphisms). Let $X$ be a proper geodesically complete CAT(0) space admitting a geometric group action and $f : X \to X$ be an affine homeomorphism. Assume further that $X$ has no Euclidean factor.

1. If $f$ preserves the factors of $X$ then $f$ is an isometry.
2. If $X$ is irreducible, then $f$ is an isometry.
3. Some power of $f$ is an isometry.
4. The group of isometries has finite index in the group of affine homeomorphisms.

One of the key tools in the proof of the Main Theorem is the following Splitting Lemma, which is an analogue to Lytchak’s [11] Proposition 4.2 for splittings of geodesically complete CAT(1) spaces. We will work with boundaries of CAT(0) spaces admitting geometric actions. Endowed with the Tits metric, such boundaries are always CAT(1) spaces but often not geodesically complete. See, for instance, the Croke-Kleiner examples [6]. A subset $P$ of a CAT(1) space $Y$ is called $\pi$-convex if whenever $x, y \in P$ such that $d(x, y) < \pi$, then the unique geodesic joining them is also in $A$. Given $x, y \in Y$, $x$ and $y$ are called antipodal or antipodes if $d(x, y) \geq \pi$. A nonempty subset $P \subset Y$ is involutive if it contains all of its antipodes.

Theorem 1.4 (Splitting Lemma). Let $X$ be a proper CAT(0) space admitting a geometric group action. Suppose $\partial X$ contains a proper subset $P$ that is $\pi$-convex and involutive and closed in the cone topology. Then $\partial X$ splits as the spherical join $\partial X = P \ast P^\perp$ where $P^\perp$ is the set of points that have Tits distance exactly $\pi/2$ from all points in $P$.

We note that our definition of the perpendicular set $P^\perp$ is different from Lytchak’s Pol($P$), which is the set of points that are at least $\pi/2$ from all points in $P$.

To prove the Splitting Lemma, we critically use the $\pi$-convergence theorem of Papasoglu-Swenson [10] and the theorem of Kleiner [9] that the boundary $\partial X$ of $X$ contains isometrically embedded round spheres of dimension equal to the geometric dimension of the Tits boundary (cf. §2.1).

We then apply the Splitting Lemma to prove the Main Theorem. We first show that asymptotic geodesics are rescaled by the same constant. Thus the rescaling function extends to the boundary $\partial X$. If the rescaling function $\rho$ is not constant on $\partial X$, then let $P$ be the set of points on which $\rho$ attains its maximum. We show that $P$ is closed, $\pi$-convex, and involutive. Therefore the Tits boundary splits off a factor. We can then apply a theorem of Bridson-Haefliger [4] Theorem II.9.24] to get that the underlying CAT(0) space splits as a product.

We do not know if our results extend to affine maps between CAT($\kappa$) spaces, or at least from CAT(0) to CAT($\kappa$) spaces. As we mentioned above, affine maps between irreducible metric spaces are not always dilations. We also do not know if affine maps are always continuous. This is certainly the case in many situations.

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2. CAT(0) Spaces

2.1. Tools From CAT(0) Spaces. In this section we review some definitions and techniques in CAT(0) spaces. For a more thorough treatment the reader should review [4]. Given three distinct points $x, y, z \in X$, choose points $\overline{x}, \overline{y},$ and $\overline{z}$ in the Euclidean plane $\mathbb{E}^2$ such that $d(\overline{x}, \overline{y}) = d(x, y)$, $d(\overline{y}, \overline{z}) = d(x, z)$, and $d(\overline{y}, \overline{z}) = d(y, z)$. We denote the resulting triangle in $\mathbb{E}^2$ by $\overline{xyz}$ and call it a comparison triangle for $\triangle xyz$ in $X$. Choose any $p$ in the geodesic $[x, y]$ and $q$ in $[x, z]$ and get corresponding points $\overline{p} \in [\overline{x}, \overline{y}]$ and $\overline{q} \in [\overline{x}, \overline{z}]$. If for every choice of $p$ and $q$, we have $d(p, q) \leq d(\overline{p}, \overline{q})$, then $\triangle xyz$ is said to be no fatter than $\overline{xyz}$. If every triangle in $X$ is no fatter than its comparison triangle in $\mathbb{E}^2$, then $X$ is called CAT(0).

We will henceforth assume that our CAT(0) spaces are geodesically complete (i.e., that every geodesic segment extends to a geodesic line defined on $(-\infty, \infty)$) and that $X$ is proper (i.e., that closed balls are compact).

Next we recall the definition of Alexandrov angles in CAT(0) spaces. If $x, y, z \in X$, then we denote by $\angle_x(y, z)$ the corresponding angle in the comparison triangle $\overline{xyz}$. If $\alpha$ and $\beta$ are the geodesics $[x, y]$ and
[$x, z$], then the CAT(0) condition implies that
\begin{equation}
    t \mapsto \overline{Z}_x(\alpha(t), \beta(t))
\end{equation}
is a nondecreasing function. Its limit as $t \to 0$ is called the \textit{Alexandrov angle} between $\alpha$ and $\beta$, denoted by $\angle_x(y, z)$, or $\angle_x(\alpha, \beta)$. The condition that $\angle_x(y, z) \leq \overline{Z}_x(y, z)$ for every choice of $x, y, z \in X$ is equivalent to the CAT(0) property for $X$ [4, Proposition II.1.7(4)].

Many geometric properties of nonpositively curved manifolds carry over to the CAT(0) setting. For instance, angle sums of triangles are bounded above by $\pi$. Furthermore we can define the \textit{visual boundary} of $X$, denoted $\partial X$, as the set of equivalence classes of geodesic rays in $X$. Two geodesic rays $\alpha$ and $\beta$ are equivalent, or asymptotic, if one lies in a tubular neighborhood of the other. Equivalently, if a basepoint $x_0 \in X$ is fixed, then $\partial X$ may be defined as the set of geodesic rays emanating from $x_0$ [4, §II.8]. We think of $\partial X$ as “attached to $X$ at infinity” and it captures the notion of infinity of $X$.

We endow $\overline{X} = X \cup \partial X$ with a topology by identifying points in $X$ with geodesics emanating from a common basepoint $x_0$ and points in $\partial X$ with geodesic rays emanating from the same point. Then a sequence of points $(x_n) \subset \overline{X}$ converges to a point $y \in \overline{X}$ if the corresponding geodesics converge uniformly on compact sets. When $X$ is proper, $\overline{X}$ is a compactification for $X$. The subspace topology on $\partial X$ is called the \textit{cone topology}. In this topology geodesic rays are close if they track a long time before diverging.

A second topology on $\partial X$ comes from a metric. Given a pair of points $\zeta, \eta \in \partial X$, the \textit{Tits angle} between them, denoted $\angle_{Tits}(\zeta, \eta)$, is defined as the supremum of Alexandrov angles $\angle_x(\zeta, \eta)$ between the geodesic rays $\alpha$ and $\beta$ emanating from $x$ going out to $\zeta$ and $\eta$ as $x$ ranges over $X$. If $x$ is fixed, we know from [4] Proposition II.9.8(1) that
\begin{equation}
    \lim_{t \to \infty} \overline{Z}_x(\alpha(t), \beta(t)) = \angle_{Tits}(\zeta, \eta)
\end{equation}
where $\overline{Z}_x(\alpha(t), \beta(t))$ is nondecreasing by (1).

The \textit{Tits metric} $d_{Tits}$ is the corresponding length metric (possibly taking the value infinity). This metric induces the \textit{Tits topology}, which is finer than the cone topology. The boundary with the Tits topology is called the \textit{Tits boundary} and denote it by $\partial_{Tits}X$. It is well-known that $d_{Tits}$ is lower semicontinuous in the cone topology [4, Proposition II.9.2(2)].

Amazingly, the Tits boundary has an elegant geometry. Given any real $\kappa$, a geodesic space is called CAT($\kappa$) if triangles are no fatter than comparison triangles in a simply connected Riemannian manifold with constant curvature $\kappa$. The following theorem is due to Gromov in the manifold setting [1], and to Bridson and Haefliger in full generality.

\textbf{Theorem 2.1.} [4, Theorem II.9.13] Let $X$ be a complete CAT(0) space. Then its Tits boundary is a complete CAT(1) space. In particular, any two points of finite Tits distance are connected by a Tits geodesic.

Geodesics in $\partial_{Tits}X$ reflect flatness in $X$. For instance, if $x \in X$ and $\zeta, \eta \in \partial X$ such that $\angle_x(\zeta, \eta) = \angle_{Tits}(\zeta, \eta)$, then the convex hull of the union of the two rays emanating from $x$ going out to $\zeta$ and $\eta$ is isometric to a sector in $\mathbb{E}^2$ [4, Lemma II.9.27].

Another important example of a CAT(1) space is the space of directions based at a point. Given a point $y$ in a CAT($\kappa$) space $Y$, deem two geodesic segments emanating from $y$ to be equivalent if the angle between them is zero. Equivalence classes are called \textit{geodesic germs}. The completion of this space with the metric induced by angles is called the \textit{link} of $y$. Elements in the link are thought of as directions. A direction is \textit{genuine} if it has a geodesic representative. Links in a CAT(1) space are always CAT(1) [43].

Following Kleiner in [9], we define the \textit{geometric dimension} of a CAT($\kappa$) space $Y$ to be the smallest function dim on the class of such spaces (taking on non-negative integer values and infinity) such that

- dim $Y = 0$ if $Y$ is discrete and
- dim $Y$ is strictly greater than the dimension of every link in $Y$

This is equal to the maximal topological dimension of compact subspaces of $Y$ [9, Theorem A].

2.2. DENSITY OF ROUND SPHERES. A \textit{round sphere} in a CAT(1) space $Y$ of geometric dimension $d > 0$ is an isometrically embedded $d$-sphere of curvature 1. In a zero-dimensional CAT(1) space, a round sphere is just a pair of points with distance $\infty$. For boundaries of certain CAT(0) spaces, Kleiner proved the existence of round spheres.
Theorem 2.2. [9] Theorem C Let $X$ be a CAT(0) space of geometric dimension $d$ admitting a geometric group action. Then there is a round sphere in $\partial_{\text{Tits}}X$, which can be chosen to be the boundary of an isometrically embedded copy of $\mathbb{E}^{d+1}$.

We will also need the $\pi$-convergence technique of Papasoglu-Swenson:

Theorem 2.3. [10] Let $G$ be a group acting geometrically on a CAT(0) space $X$ and $\theta \in [0, \pi]$. Then for any sequence of distinct elements $(g_n) \subset G$, there are points $n, p \in \partial X$ and a subsequence $(g_{n_k})$ such that for every compact $K \subset \partial X \setminus B_{\text{Tits}}(n, \theta)$ and neighborhood $U$ of $B_{\text{Tits}}(p, \pi - \theta)$, $g_{n_k}(K) \subset U$ for large enough $k$.

It will be important below to understand how $p$, $n$, and $(g_{n_k})$ arise. Fix $x_0 \in X$. Since $\partial X$ is compact, we can pass to a subsequence so that the sequences $(g_{n_k}x_0)$ and $(g_{n_k}^{-1}x_0)$ converge to points $p$ and $n$ respectively.

The existence of antipodes was proven by Balser-Lytchak.

Lemma 2.4. [2, Lemma 3.1] Let $X$ be a CAT(1) space with geometric dimension $d < \infty$ and $K \subset X$ be a round sphere. Then every point of $X$ has an antipode in $K$.

Corollary 2.5 (Density of Round Spheres). Let $G$ be a group acting geometrically on a CAT(0) space $X$. Then the union of round spheres in $\partial X$ is dense (in the the cone topology).

Proof. Fix $x_0 \in X$. Choose any point $p \in \partial X$. Since the action of $G$ is cocompact, there is a sequence of group elements $(g_n) \subset G$ such that $(g_nx_0)$ converges to $p$. After passing to a subsequence (if necessary), we may assume that $(g_n^{-1}x_0)$ also converges to some point $n \in \partial X$. By [9, Theorem C] we know that there exists a round sphere $K \subset \partial_{\text{Tits}}X$. By Lemma 2.4, we can get a $q \in K$ for which $d_{\text{Tits}}(n, q) \geq \pi$. Apply $\pi$-convergence now to get that $g_nq \to p$.

2.3. A Splitting Lemma. In this section we prove Theorem 1.4 from the introduction. Recall that for a subset $P$ of a metric space $X$,

$$P^\perp = \{ x \in \partial X \mid d_{\text{Tits}}(x, y) = \frac{\pi}{2} \text{ for all } y \in P \}$$

To prove this we first need to establish some lemmata.

Lemma 2.6. Let $P$ be an involutive subset of $\partial X$ and $K$ be a round sphere in $\partial X$. Define

$$P^\perp_K = \{ x \in K \mid d_{\text{Tits}}(x, y) = \frac{\pi}{2} \text{ for all } y \in P \cap K \}.$$ 

Then $P \cap K$ is nonempty and $P^\perp_K \subset P^\perp$.

Proof. By Lemma 2.4, every point of $P$ has an antipode in $K$. Thus $P \cap K$ is nonempty. It is clear that $P^\perp \cap K \subset P^\perp_K$. To prove the converse, let $x \in P^\perp_K$ and $y \in P$. Draw the geodesic $[y, x]$ and extend inside $K$ to get a geodesic $[y, z]$ of length $\pi$ that passes through $x$. Then $d(x, z) = \pi/2$ by definition of $P^\perp_K$ and therefore $d(x, y) = \pi/2$ as well.

Lemma 2.7. If $P \subset \partial X$ is closed in the cone topology, then so is $P^\perp$.

Proof. Let $\{\zeta_n\} \subset P^\perp$ be a sequence of points converging to a point $\zeta \in \partial X$. Then for every $\eta \in P$, $d(\zeta, \eta) \leq \pi/2$ since the Tits metric is lower semicontinuous in the cone topology. But since $X$ is almost geodesically complete, $\zeta$ has an antipode $\zeta'$. Since $d(\zeta', \eta) \leq \pi/2$, we must have $d(\zeta, \eta) = \pi/2$.

Proof of Theorem 1.4. Our goal is to prove that every $x \in \partial X$ lies between points $y \in P$ and $z \in P^\perp$. Let $x \in \partial X$ be given. By Corollary 2.5 we may choose round spheres $K_n$ and $x_n \in K_n$ such that $x_n \to x$. If for large enough $n$, $P \supset K_n$, then $x \in P$ and we are done. On the other hand, every $P_n = P \cap K_n$ is nonempty by the lemma above. So $P_n$ is a proper closed $\pi$-convex subset of $K_n$ and hence a subsphere, hence $K_n = P_n \ast P^\perp_n$ where $P^\perp_n = P^\perp \cap K_n$. Choose $y_n \in P_n$ and $z_n \in P^\perp_n$ such that $x_n \in [y_n, z_n] \subset K_n$. Since $\partial X$ is compact, we may pass to subsequences so that $y_n \to y \in \partial X$ and $z_n \to z \in \partial X$. Since $P$ and $P^\perp$ are both closed, $y \in P$ and $z \in P^\perp$.

It remains to verify that $d(y, x) + d(x, z) = \pi/2$. Since the Tits metric is lower semicontinuous in the cone topology,

$$\pi/2 = d(y, z) \leq d(y, x) + d(x, z) \leq \lim \inf d(y_n, x_n) + d(x_n, z_n) = \pi/2.$$
2.4. Almost-Flat Triangles. Recall that boundaries of flats in $X$ are unions of Tits geodesics in $\partial_{\text{rid}}X$. While the converse is not true in general, it is true approximately as we will see in this section. We will need two lemmata in Euclidean geometry: the well-known Alexandrov Lemma and a controlled version.

**Lemma 2.8.** [4, Lemma I.2.16] Let $\overline{xy}$, $\overline{yz}$, $\overline{zx}$, $\overline{xy}$, $\overline{yz}$, and $\overline{wz}$ be points in $\mathbb{E}^2$ such that

- $\overline{w}$ is between $\overline{y}$ and $\overline{z}$,
- $\overline{z}$ and $\overline{y}$ are on opposite sides of the line passing through $\overline{z}$ and $\overline{w}$,
- $d(\overline{y}, \overline{z}) + d(\overline{z}, \overline{w}) \geq d(\overline{y}, \overline{w})$,
- $d(\overline{w}, \overline{z}) = d(\overline{z}, \overline{w})$,
- $\pi \leq \angle(\overline{w}, \overline{z})$.

Then

- $\angle(\overline{y}, \overline{z}) \geq \angle(\overline{y}, \overline{w}) + \angle(\overline{w}, \overline{z})$,
- $\angle(\overline{w}, \overline{z}) \geq \angle(\overline{w}, \overline{y})$,
- $\angle(\overline{w}, \overline{z}) \geq \angle(\overline{w}, \overline{x})$, and
- $d(\overline{w}, \overline{z}) \geq d(\overline{w}, \overline{y})$.

The next lemma is a modified version of the Alexandrov Lemma, which gives a lower bound on $d(\overline{w}, \overline{z})$ under additional hypotheses.

**Lemma 2.9** (Controlled Alexandrov Lemma). Let $0 < \theta \leq \pi$ be fixed. Given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $\overline{x}$, $\overline{y}$, $\overline{z}$, $\overline{w}$, $\overline{xy}$, $\overline{yz}$, and $\overline{wz}$ are points in $\mathbb{E}^2$ satisfy the conditions of the Alexandrov lemma and in addition:

- $d(\overline{x}, \overline{y}) = d(\overline{y}, \overline{z}) = 1$,
- $\overline{w}$ is the midpoint of the segment $[\overline{x}, \overline{y}]$,
- $\angle(\overline{x}, \overline{y}) = \theta$,
- $\angle(\overline{x}, \overline{y}) \geq \theta - \delta$,

then $d(\overline{x}, \overline{y}) - \epsilon \leq d(\overline{x}, \overline{w})$.

**Proof.** Let $0 < \theta \leq \pi$ be given. Suppose $\overline{x}$, $\overline{y}$, and $\overline{z}$ satisfy the hypotheses. Without loss of generality, choose $\overline{x} = \overline{y}$ and $\overline{y} = \overline{y}$. Let $C$ be the circle of radius 1 centered at $\overline{y}$. Then $\overline{w}$ may be thought of as a continuous function of $\overline{z}$ whose domain is the closed shorter subarc of $C$ joining $\overline{y}$ and $\overline{y}$ (see Figure 1). Since the domain is compact and $\overline{w} \to \overline{w}$ as $\overline{z} \to \overline{z}$, the conclusion follows. \qed

We now return our attention to studying triangles in CAT(0) spaces. The proof of the next lemma echoes that of the Flat Triangle Lemma in [4, Proposition II.2.9].

**Lemma 2.10** (Approximately Flat Triangle Lemma). Let $(X, d)$ be a CAT(0) space and let $\theta \in (0, \pi)$ be fixed. Given $\epsilon > 0$, there is a $\delta > 0$ such that for all $t > 0$, whenever $x, y, z \in X$ with $d(x, y) = d(x, z) = t$ and

$$\angle(y, z) \geq \angle(x, y, z) - \delta$$
and \( \overline{z}(y, z) = \theta \), then for the midpoint \( w \) in the geodesic \( [y, z] \),

\[
d(w, x) \geq d_{\mathbb{E}^2}(\overline{w}, \overline{z}) - ct
\]

where \( \triangle(x, y, z) \) is a comparison triangle in Euclidean space with \( \overline{w} \) the point corresponding to \( w \).

**Proof.** Choose \( \tilde{x}, \tilde{y}, \tilde{z}, \tilde{w} \) in \( \mathbb{E}^2 \) so that \( \triangle(\tilde{x}, \tilde{y}, \tilde{w}) \) and \( \triangle(\tilde{x}, \tilde{z}, \tilde{w}) \) are comparison triangles for \( \triangle(x, y, w) \) and \( \triangle(x, z, w) \) respectively, arranged so that \( \tilde{z} \) and \( \tilde{y} \) are on opposite sides of the the line passing through \( \tilde{x} \) and \( \tilde{w} \). Then

\[
\angle_{\overline{y}}(\overline{\tilde{y}}, \overline{\tilde{z}}) - \delta = \overline{z}(y, z) - \delta \\
\leq \angle_x(y, z) \\
\leq \angle_x(y, w) + \angle_x(w, z) \\
\leq \overline{z}(y, w) + \overline{z}(w, z) \\
= \angle_{\overline{\tilde{y}}}(\overline{\tilde{y}}, \overline{\tilde{w}}) + \angle_{\overline{\tilde{w}}}(\overline{\tilde{w}}, \overline{\tilde{z}}) \\
= \angle_{\overline{\tilde{y}}}(\overline{\tilde{z}})
\]

Thus we may apply Lemma 2.9 after rescaling. \( \square \)

**Corollary 2.11** (Approximately Flat Sectors). Let \( \zeta, \nu \in \partial X \) such that \( \theta = d_{\text{Tits}}(\zeta, \nu) < \pi \) and \( \epsilon > 0 \) be given. Then \( p \in X \) may be chosen so that the following statement holds. If \( \alpha \) and \( \beta \) are the unit speed geodesics emanating from \( p \) going out to \( \zeta \) and \( \nu \) respectively and \( z_t \) is the midpoint of the geodesic \( [\alpha(t), \beta(t)] \), then \( d(p, z_t) \geq t \cos(\theta/2) - ct \).

**Proof.** By definition of \( d_{\text{Tits}}(\zeta, \nu) \), \( p \in X \) may be chosen so that \( \theta - d_{\text{Tits}}(\zeta, \nu) \) is smaller than the \( \delta = \delta(\theta, \epsilon) \) provided in the previous lemma. For any fixed \( t \), the comparison triangle \( \overline{z}(p, \alpha(t), \beta(t)) \) in Euclidean space is an isosceles triangle with two sides of length \( t \) and apex angle of measure \( \angle_{\overline{z}}(\alpha(t), \beta(t)) \). By Equation 2 the latter is nondecreasing in \( t \) with limit \( \theta \) as \( t \to \infty \) and limit \( \angle_{\overline{z}}(\zeta, \nu) \) as \( t \to 0 \) by definition. So when \( t \) is large enough, this apex angle \( \theta_t \) is close to \( \theta \). Applying the Lemma 2.9 gives

\[
d(p, z_t) \geq t \cos(\theta/2) - ct.
\]

3. **Affine Maps**

3.1. **Properties of Affine Maps Between CAT(0) Spaces.** Let \( f : X \to Y \) be a continuous affine map between proper CAT(0) spaces. We first establish Lemma 3.1 below, which allows us to assume that \( f \) is surjective. To this end, recall that CAT(0) metrics are convex, meaning that the distance between a pair of points on geodesics is bounded above by a convex combination of the distances between their endpoints. This implies that geodesic segments are uniquely determined by their endpoints. Moreover, after reparameterizing as constant speed maps over \([0, 1]\), they depend continuously on their endpoints (in the uniform topology on maps).

**Lemma 3.1.** Let \( f : X \to Y \) be a continuous affine map between proper CAT(0) spaces. The image \( Y' \) of \( f \) is a closed, convex subspace of \( Y \).

**Proof.** The fact that \( Y' \) is convex follows from the fact that geodesics are unique. To see that it is also closed, choose any sequence \( \{y_n\} \subset Y' \) converging to a point \( y \in Y \). Choose preimages \( \{x_n\} \subset X \). By passing to a subsequence, we may assume that \( x_n \to x \in X \cup \partial X \). If \( x \in X \), then \( y = f(x) \in Y' \) by continuity and we are done. Otherwise the sequence of geodesics \( [x_0, x_n] \) converges to a ray \( \alpha \). By continuity of geodesics in their endpoints, \( \{y_0, y_n\} \to f(\alpha) \), which is either a ray or a point. If it is a ray, then \( \{y_n\} \) is unbounded, giving us a contradiction. So \( f(\alpha) \) is a point and \( y = y_0 \in Y' \). \( \square \)

Note that the continuous affine image of a CAT(0) space need not be CAT(0). For instance, the identity map \( \mathbb{E}^2 \to (\mathbb{R}^2, l_1) \) is a continuous affine map. Similar examples can be obtained by replacing \( l_1 \) with a norm determined by a suitable centrally symmetric convex body. Indeed, this idea gives rise to the Finsler norms explored in \([12]\).

Recall that for a geodesic \( \alpha \) in \( X \), \( \rho(\alpha) \) denotes the constant by which \( \alpha \) is rescaled by \( f \).
Lemma 3.2. Let \( f : X \to Y \) be a continuous surjective affine map between proper CAT(0) spaces with rescaling function \( \rho \). Then:

1. \( \rho \) determines a function \( \partial X \to [0, \infty) \), which we still call \( \rho \).

2. If \( \rho(\alpha) > 0 \) and \( \beta \) is a geodesic ray asymptotic to \( \alpha \), then \( f(\alpha) \) and \( f(\beta) \) are also asymptotic geodesic rays.

3. \( \rho \) is a continuous function in the cone topology.

4. If there is no \( \zeta \in \partial X \) such that \( \rho(\zeta) = 0 \), then \( f \) extends to a homeomorphism \( X \cup \partial X \to Y \cup \partial Y \).

Proof. We begin by proving (1) and (2). Assume all geodesics in \( X \) are parameterized to have unit speed. Suppose \( \alpha \) and \( \beta \) are a pair asymptotic geodesics in \( X \). First assume \( \rho(\alpha) > 0 \). Then the image of \( \alpha \) is also a geodesic ray (with new speed). Let \( \gamma_n \) be the geodesic joining \( \beta(0) \) to \( \alpha(n) \). Then \( d(\beta(0), \alpha(n)) \to d(\beta(0), \alpha(n)) \) and \( d(f\beta(0), f\alpha(n)) \sim d(f\alpha(0), f\alpha(n)) \) (that is, their ratios go to 1 as \( n \) approaches infinity). This implies that \( \rho(\gamma_n) \to \rho(\alpha) \).

By convexity of the metric, \( \gamma_n \to \beta \) uniformly on compact sets. In \( Y \), \( f(\gamma_n) \) is the geodesic joining \( f(\beta(0)) \) to \( f(\alpha(n)) \) and \( f(\gamma_n) \) converges to the unique geodesic ray \( \widehat{\beta} \) emanating from \( f(\beta(0)) \) which is asymptotic to \( f(\alpha) \). On the other hand, \( f(\gamma_n) \) converges to \( f(\beta) \). This implies that \( f(\beta) = \widehat{\beta} \) and hence \( \rho(\alpha) = \rho(\beta) \). This establishes (1) and (2) when \( \rho(\alpha) \neq 0 \).

To establish (3) and (4) in this setup, observe that whenever \( \gamma_n \) is a sequence of geodesics (either segments or rays) converging to a geodesic ray \( \beta \), then \( f(\gamma_n) \) converges to \( f(\beta) \) and \( \rho(\gamma_n) \) converges to \( \rho(\beta) \).

Finally consider the case where \( \rho(\alpha) = 0 \). Then the image of \( \alpha \) is a single point, and all \( \gamma_n \) have the same image – a finite geodesic segment \( \widehat{\gamma} \) emanating from \( f(\beta(0)) \). Since the lengths of the \( \gamma_n \) go to infinity, \( \rho(\gamma_n) \) converges to zero. Again, \( \gamma_n \) converges to \( \beta \), and so by continuity of \( f \), \( \rho(\beta) = 0 \).

Next we reduce to the case where \( f \) is injective.

Lemma 3.3. Let \( f : X \to Y \) be a continuous affine map between geodesically complete proper CAT(0) spaces. If \( \rho \) is 0 anywhere on \( \partial X \), then \( X \) splits as a product \( X = X_0 \times X_1 \) such that \( \partial X_0 = \rho^{-1}(0) \).

Proof. Our proof resembles the proof of \([4\text{ Proposition II.6.23}]\). Fix \( x \) and \( y \) in \( Y \), and let \( Z_x \) and \( Z_y \) be their preimages under \( f \).

Define \( \phi : Z_x \to [0, \infty) \) by letting \( \phi(p) \) be the distance from \( p \in Z_x \) to the closest point on \( Z_y \). This is a convex function. We will prove that \( \phi \) is constant. Suppose \( \phi(p) > \phi(q) \). Extend \( [q, p] \) to get a geodesic ray \( \alpha \). Observe that \( \alpha \) is contained in \( Z_x \), since \( \rho(\alpha) = 0 \). The fact that \( \rho \) is convex and increasing on \( \phi \) implies that \( \rho \) is unbounded here.

Now, let \( q' \in Z_y \) be the point closest to \( q \) and \( \alpha' \) be the geodesic ray emanating from \( q' \) asymptotic to \( \alpha \). By the previous proposition, \( \rho(\alpha') = \rho(\alpha) = 0 \) and therefore \( f(\alpha') = f(q') \). This shows that \( \alpha' \subset Z_y \). But this means that \( \phi \) is bounded on \( \alpha \), giving us a contradiction.

From this point forward, we will assume that \( \rho \) is bounded away from zero.

We will need to know that \( \rho \) is constant on antipodes. If an antipodal pair \( \zeta, \eta \in \partial X \) is joined by a geodesic line in \( X \), then this statement is obvious. This is guaranteed when \( d_{\pi_{\text{H}}} (\zeta, \eta) > \pi \), for instance, by \([4\text{ Proposition II.9.21(1)}]\). However, one can construct geodesically complete \( \text{CAT}(0) \) 2-complexes in which there is a pair of points \( \zeta, \eta \) in the boundary where \( d_{\pi_{\text{H}}} (\zeta, \eta) = \pi \) but there is no geodesic in the space joining them. Therefore a more robust argument is needed.

Lemma 3.4 (Small Angles Lemma). For every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that whenever \( x, y, z \in X \) are distinct and \( \angle_x(y, z) < \delta \), then \( \angle_f(x, f(y), f(z)) < \epsilon \). Furthermore, for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( x, y, z \in X \) are distinct and \( \angle_x(y, z) < \delta \), then \( \rho([x, y]) - \rho([x, z]) < \epsilon \).

Proof. For \( d \geq 0 \), define

\[
\theta(d) = 2 \sin^{-1} \left( \frac{d}{2} \right).
\]

This is the measure of the apex angle of an isosceles triangle with legs of length 1 and base of length \( d \). The functions \( \theta \) and \( \theta^{-1} \) are increasing continuous functions with \( \theta(0) = 0 \). Let \( M \) and \( m > 0 \) be the maximum and minimum values attained by \( \rho \) on \( \partial X \) and define

\[
\sigma(t) = \theta \left( \frac{M}{m} \theta^{-1}(t) \right).
\]
This is also continuous with \( \sigma(0) = 0 \).

Let \( x, y, z \in X \) be distinct. The first part of the lemma follows from the inequality
\[
\angle_{f(z)}(f(y), f(z)) \leq \sigma\left( \angle_x(y, z) \right).
\]
To prove (3), let \( \alpha \) and \( \beta \) denote the geodesics \([x, y]\) and \([x, z]\) parameterized to have unit speed with \( \alpha(0) = \beta(0) = x \) and choose any \( 0 < t < \min\{d_X(x, y), d_X(x, z)\} \). Recall that \( \angle_x(y, z) \) denotes the angle of a Euclidean comparison triangle at the vertex corresponding to \( x \). Note that an angle in a triangle in Euclidean space grows when the opposite side grows and the adjacent sides shrink. Therefore
\[
\angle_{f(z)}(f(y), f(z)) \leq \angle_{f(z)}(f\alpha(t), f\beta(t))
\]
\[
\leq \theta \left( \frac{M d(\alpha(t), \beta(t))}{mt} \right)
\]
\[
= \sigma \left( \angle_{x}(\alpha(t), \beta(t)) \right)
\]
\[
\to \sigma \left( \angle_{x}(y, z) \right)
\]
as \( t \to 0 \).

Now consider the second part of the lemma. Denote \( A = \rho([x, y]) \) and \( B = \rho([x, z]) \). Applying the triangle inequality to the triple \((f\alpha(t), f\beta(t), f(x))\) for small \( t > 0 \), we get
\[
|At - Bt| \leq \rho(f\alpha(t), f\beta(t))
\]
\[
\leq M d(\alpha(t), \beta(t))
\]
\[
= M t \theta^{-1}(\angle_{x}(\alpha(t), \beta(t))).
\]
Divide both sides by \( t \) and let \( t \to 0 \) to get
\[
|A - B| \leq M \theta^{-1}(\angle_{x}(y, z)).
\]

\( \square \)

**Lemma 3.5.** Assume \( \angle_{T_{\text{its}}}(\zeta, \zeta') = \pi \), \( x_0 \in X \), and \( \alpha \) a geodesic ray from \( x_0 \) to \( \zeta \). Then
\[
\lim_{t \to \infty} \angle_{\alpha(t)}(x_0, \zeta') = 0.
\]

**Proof.** Let \( \beta \) be the geodesic ray emanating from \( x_0 \) going out to \( \zeta' \) and \( \epsilon > 0 \) be fixed. By [4 Proposition II.9.8(3)] we know that when \( s \) and \( t \) are large enough,
\[
\angle_{\alpha(t)}(x_0, \beta(s)) \leq \angle_{\alpha(t)}(x_0, \beta(s)) + \angle_{\beta(s)}(x_0, \alpha(t)) \leq \frac{\epsilon}{2}.
\]
On the other hand, by continuity of Alexandrov angles (with fixed basepoint) we can increase \( s \) even more (if necessary) to guarantee
\[
\angle_{\alpha(t)}(x_0, \beta(s)) \geq \angle_{\alpha(t)}(x_0, \zeta') - \frac{\epsilon}{2}.
\]
Put the two together to get that
\[
\angle_{\alpha(t)}(x_0, \zeta') \leq \epsilon.
\]

\( \square \)

**Lemma 3.6 (Involutive Invariance).** The rescaling function \( \rho \) is constant on pairs of antipodes. Specifically, whenever \( d_{\text{its}}(\zeta, \zeta') \geq \pi \), then \( \rho(\zeta) = \rho(\zeta') \).

**Proof.** Let \( \alpha \) be a geodesic ray going out to \( \zeta \). For each \( t > 0 \), let \( \beta_n \) be the geodesic ray based at \( \alpha(n) \) going out to \( \zeta' \). By [4 Proposition II.9.8(2)],
\[
\lim_{n \to \infty} \angle_{\alpha(n)}(\zeta, \zeta') = \angle_{T_{\text{its}}}(\zeta, \zeta') = \pi.
\]
By Lemma 3.5 \( \angle_{\alpha(n)}(0, \beta_n(1)) \to 0 \), so by Lemma 3.4 \( \rho(\beta_n) \) converges to \( \rho(\alpha) \) as \( n \to \infty \). Thus we establish that \( \rho(\zeta') = \rho(\zeta) \).
3.2. Splitting Affine Maps. Here we prove Theorem 1.1. Recall that a CAT(0) space $X$ is called irreducible if no subspace splits as a product. By [1] Theorem II.9.24, this is equivalent to saying that the boundary does not split as a spherical join in the Tits topology (if $X$ is geodesically complete).

Swenson showed in [17] that cocompact CAT(0) spaces have finite-dimensional boundary. Since our CAT(0) spaces $X$ are cocompact and geodesically complete, we may apply a result of Caprace-Monod to get a decomposition of $X$ into a product of factors which is unique up to permutation [5, Proposition 1.5(iii)] and Theorem 1.9.

Let $f: X \to Y$ be a continuous affine map between proper geodesically complete CAT(0) spaces that is not a dilation. Recall that we may assume that $f$ is bijective. Define $\partial X \subset \partial X$ to be the subset on which $\rho$ attains its maximum. Our goal is to prove that $\partial_{Tits} X$ splits as a spherical join with $\partial X$ as one of the factors. Since $\rho$ is continuous, this set is closed in the cone topology. It is involutive by Lemma 3.6. Therefore the Main Theorem will follow from Lemma 1.4 once we prove

**Lemma 3.7 (Max is \(\pi\)-Convex).** Assume $\zeta, \eta \in \text{Max}$ such that $d(\zeta, \eta) < \pi$, and let $\nu$ be the midpoint of the geodesic $[\zeta, \eta] \subset \partial_{Tits} X$. Then $\nu \in \text{Max}$ as well.

**Proof.** Since $f$ is bijective, $\rho$ cannot attain zero. So by rescaling the metric on $Y$ (if necessary), we may assume that the maximum attained by $\rho$ is 1. Let $\zeta'$ and $\eta'$ be the images of $\zeta$ and $\eta$ and fix $\epsilon > 0$. Set $\theta = d_{Tits}(\zeta, \eta)/2$, and $\theta' = d_{Tits}(\zeta', \eta')/2$. By Corollary 2.11 we may choose $p'$ such that the rays $\alpha'$ from $p'$ to $\zeta'$ and $\beta'$ from $p'$ to $\eta'$ satisfy the following: if $z'_t$ is the midpoint of $[\alpha'(t), \beta'(t)]$, then $d(p', z'_t) \geq t \cos(\theta') - t\epsilon$.

Now let $p$ be the preimage of $\alpha'$ and $\beta'$ be the preimages of $\alpha'$ and $\beta'$. Note that $\alpha$ and $\beta$ determine the points $\zeta$ and $\eta$ at infinity. Denote by $z_t$ the midpoint of $[\alpha(t), \beta(t)]$. By construction, $f(z_t) = z'_t$. By the Law of Cosines and comparison geometry,

$$d(p, z_t) \leq t \cos Z_\rho(\alpha(t), \beta(t)) \leq t \cos(\theta).$$

Thus

$$\frac{d(p', z'_t)}{d(p, z_t)} \geq \frac{\cos(\theta') - \epsilon}{\cos(\theta)}. $$

As shown in the proof of [4, Lemma II.9.14], $z_t$ converges to the midpoint $\nu$ of the Tits geodesic $[\zeta, \eta]$ as $t \to \infty$. Letting $\epsilon \to 0$, we get $\rho(\nu) \geq \cos(\theta')/\cos(\theta)$. Since we assumed $\rho \leq 1$, we have $d(\alpha'(t), \beta'(t)) \leq d(\alpha(t), \beta(t))$. So $\theta' \leq \theta$, and $\cos \theta' \geq \cos \theta$. Thus $\rho(\nu) = 1$. □

**Corollary 3.8.** Let $f: X \to Y$ be an affine homeomorphism between proper CAT(0) spaces, the first of which admits a geometric group action. Then $f$ preserves the prime factorization of $X$ and restricts to a dilation on each factor.

**Proof.** It is clear that $f$ takes closed subsets of $\partial X$ to closed subsets of $\partial Y$. It also preserves convexity of subsets, since it takes midpoints of geodesic segments of length less than $\pi$ to midpoints of geodesic segments. Thus there is a one-to-one correspondence between minimal such subsets of $\partial X$ and minimal such subsets of $\partial Y$. Therefore if $X = X_1 \times X_2$ where $X_1$ is irreducible, then $f(X_1)$ is also an irreducible subspace of $Y$. □

3.3. Self-Affine Maps. Here we consider a self-affine map $f$ of a proper CAT(0) space $X$ admitting a geometric group action. We will prove Corollary 1.2 that $X$ is flat if $f$ is a contraction. We first need to establish some technical lemmata.

**Lemma 3.9.** Let $(X, d)$ be a complete, non-compact metric space. Then there is a $\lambda > 0$ and a sequence $(x_n) \subset X$ such that for every $m \neq n$, $d(x_m, x_n) \geq \lambda$.

**Proof.** Start with a sequence $(x_n)$ that does not have any limit points. Since it cannot be Cauchy, there is an $\epsilon > 0$ such that for every $N \geq 0$, there are $m, n \geq N$ such that $d(x_m, x_n) \geq \epsilon$. Note that if there is an $x_N$ such that $d(x_m, x_N) < \epsilon/2$ for all but finitely many $m > N$, then $d(x_n, x_m) < \epsilon$ for all but finitely many $m$ and $n$, which is a contradiction. This allows us to choose a sequence satisfying the desired property with $\lambda = \epsilon/2$. □

Let $R, \lambda \geq 0$. We say that a subset $\Sigma \subset \partial X$ is $(R, \lambda)$-wide if there is an $x \in X$ such that for every pair $\zeta, \eta \in \Sigma$, there is a $y \in B_\theta(x)$ for which $\angle_{\rho}(\zeta, \eta) \geq \lambda$. We will refer to $x$ as an $(R, \lambda)$-center for $\Sigma$. There is a bound on the cardinality of wide sets:

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Lemma 3.10. Let $X$ be a cocompact proper CAT(0) space and $R, \lambda \geq 0$. Then there is a bound on the cardinality of $(R, \lambda)$-wide subsets of $\partial X$.

Proof. Let $\Sigma \subset \partial X$ be an $(R, \lambda)$-wide subset with $x \in X$ a center. Let $\alpha, \beta$ be a pair of geodesic rays emanating from $x$ going out to a pair of points in $\Sigma$ and set

$$K = \frac{2R + 1}{\sqrt{2 - 2\cos \lambda}}$$

Let $y \in B_R(x)$ and $\alpha'$ and $\beta'$ be geodesic rays starting at $y$ asymptotic to $\alpha$ and $\beta$ such that $\angle y(\alpha', \beta') \geq \lambda$. By [4, Proposition II.1.7(5)], $d(\alpha'(K), \beta'(K))$ is bounded below by the length $B$ of the base of an isosceles triangle with legs of length $K$ and apex angle $\lambda$. Using the Law of Cosines, $B = 2R + 1$. By convexity of metric, $\alpha'(K)$ and $\beta'(K)$ are a distance of at most $R$ away from $\alpha(K)$ and $\beta(K)$. Thus $d(\alpha(K), \beta(K)) \geq 1$.

Therefore there is a subset $\hat{\Sigma} \subset S_{\partial X}$ with the same cardinality as $\Sigma$ such that the distance between every pair of points is at least $1$.

Suppose now that there is a sequence of $(R, \lambda)$-wide subsets $\Sigma_n \subset \partial X$ such that the cardinality of $\Sigma_n$ is at least $n$ with corresponding center $x_n \in X$. Let $\Sigma_n$ be cocompact, and replacing $\Sigma_n$ by translates we may assume that $x_n \to x$. Construct for each $\Sigma_n$ the corresponding set $\hat{\Sigma}_n \subset S_{\partial K}(x_n)$ as in the previous paragraph. Choose $y_{n}^1 \in \hat{\Sigma}_n$, and let $\Sigma_n^1$ be the remaining $n - 1$ points. By properness of $X$, we may pass to a subsequence so that $y_{n}^1 \to y^1 \in S_{\partial K}(x)$. Next choose $y_{n}^2 \in \hat{\Sigma}_n^1$ for $n \geq 2$ and let $\Sigma_n^2$ be the remaining points. Again, pass to a subsequence so that $y_{n}^2 \to y^2 \in S_{\partial K}(x)$. Note that $d(y^1, y^2) \geq 1$. Continuing in this manner, for all $m$ we can find a $y_m \in S_{\partial K}(x)$ such that $d(y^m, y^n) \geq 1$ for every $n \neq m$. Thus we have found an infinite discrete subset of $S_{\partial K}(x)$, contradicting the assumption that $X$ is proper. \hfill \Box

Bosché has classified certain CAT(0) spaces with compact Tits boundaries:

Theorem 3.11. [3] Propositions 3 and 7] Let $X$ be a geodesically complete proper CAT(0) space admitting a geometric group action. If the Tits boundary $\partial_{\text{Tits}}X$ is compact, then $X$ is flat.

Proof of Corollary [1.2] Suppose $\partial_{\text{Tits}}X$ is not compact. By Theorem [1.1], we may pass to a factor of $X$, if necessary, so that $f$ is a dilation. Let $M$ be the maximal size of $(1, \lambda)$-wide sets. Since $\partial_{\text{Tits}}X$ is compact we may apply Lemma [3.9] to get a $\lambda > 0$ and a subset $\Sigma \subset \partial X$ with $M + 1$ elements and such that the Tits distance between every pair of distinct elements is at least $\lambda$.

Form a subset $\Sigma'$ of $X$ by taking for every pair $\zeta, \nu \in \Sigma$ a point $y \in X$ such that $\angle y(\zeta, \nu) \geq \lambda$. Since $f$ is a dilation, it induces an isometry on $\partial_{\text{Tits}}X$. In particular, it preserves the distances between the points in $\Sigma$ while shrinking the diameter of $\Sigma'$. Thus for large enough $k$, $f^k(\Sigma)$ is $(1, \lambda)$-wide, contradicting the assumption that $M$ was the maximal cardinality of such sets. Therefore $\partial_{\text{Tits}}X$ is compact.

Therefore $\partial_{\text{Tits}}X$ is compact and we may apply Bosché’s Theorem to get the desired result. \hfill \Box

Proof of Corollary [1.5] By the Main Theorem, $X$ splits as a product $X = X_1 \times \ldots \times X_n$ of irreducible factors $X_i \neq \mathbb{R}$ such that the restrictions $f|_{X_i}: X_i \to X$ are dilations. Since $f$ is a homeomorphism, no rescaling constant can be zero and hence no $f(X_i)$ is a point. By uniqueness of the splitting and the Main Theorem, $f$ must interchange the factors. If $f$ preserves the factors of $X$, then by Corollary [1.2] it must be an isometry. This proves (1). Statements (2) and (3) follow immediately from (1). For (4), let $j$ be the induced homomorphism from the group of affine homeomorphisms to the symmetric group of rank $n$. Then $\ker j$ is a subgroup of isometries by (1). \hfill \Box

Remark 3.12. Observe that surjectivity is a necessary assumption since a tree may be dilated by constant $\alpha > 1$ to a subtree.

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