Semantic Security and the Second-Largest Eigenvalue of Biregular Graphs

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Abstract

It is investigated how to achieve semantic security for the wiretap channel. It is shown that asymptotically, every rate achievable with strong secrecy is also achievable with semantic security if the strong secrecy information leakage decreases sufficiently fast. If the decrease is slow, this continues to hold with a weaker formulation of semantic security. A new type of functions called biregular irreducible (BRI) functions, similar to universal hash functions, is introduced. BRI functions provide a universal method of establishing secrecy. It is proved that the known secrecy rates of any discrete and Gaussian wiretap channel are achievable with semantic security by modular wiretap codes constructed from a BRI function and an error-correcting code. A concrete universal hash function given by finite-field arithmetic can be converted into a BRI function for certain parameters. A characterization of BRI functions in terms of edge-disjoint biregular graphs on a common vertex set is derived. New BRI functions are constructed from families of Ramanujan graphs. It is shown that BRI functions used in modular schemes which achieve the semantic security capacity of discrete or Gaussian wiretap channels should be nearly Ramanujan. Moreover, BRI functions are universal hash functions on average.

Index Terms

Semantic security, wiretap channel, modular coding scheme, biregular irreducible function, biregular graph, second-largest eigenvalue, Ramanujan graph, Cayley sum graph

I. INTRODUCTION

A. Motivation

In the wiretap channel problem, a sender has a set of messages and would like to transmit one of these messages to a receiver. To this end, the message is encoded and then sent through...
Fig. 1. The modular UHF or BRI scheme. $T$ denotes the physical channel between sender and receiver. $(\phi, \psi)$ is an er-
correcting code and $f$ is a universal hash function or a BRI function (to be defined). $f_s^{-1}$ denotes the randomized inverse of
$f_s$. The seed $s$ has to be known to sender and receiver beforehand.

a noisy channel to the intended receiver, who decodes the channel output. An eavesdropper
observes a different noisy version of the sent codeword. The goal is to find an encoding of the
messages which allows a reliable transmission to the intended message recipient, whereas the
eavesdropper obtains no information about the transmitted message.

What it means that “no information” is obtained by the eavesdropper can be defined in multiple
ways using secrecy measures. In this paper, the secrecy measure of semantic security is focused
on. It is defined asymptotically as the coding blocklength $n$ tends to infinity. For every $n$, a
finite message set $\mathcal{M}_n$ is given. Semantic security holds if $\max_{\mathcal{M}_n} I(\mathcal{M}_n \wedge Z_n)$ tends to zero,
where $\mathcal{M}_n$ ranges over all random variables on $\mathcal{M}_n$, $Z_n$ describes the eavesdropper’s observation
generated by $\mathcal{M}_n$ and $I(X \wedge Y)$ is the mutual information between random variables $X$ and $Y$.

This paper studies modular schemes which enhance ordinary error-correcting codes (ECCs)
in order to provide semantic security against an eavesdropper. In a modular UHF scheme, the
randomized inverse of a universal hash function (UHF) is prefixed to the ECC encoder and the
UHF itself is postfixed to the ECC decoder, see Fig. 1. Recall that a universal hash function
is a function $f : S \times X \rightarrow \mathcal{M}$ such that $\Pr[f(S, x) = f(S, x')] \leq |\mathcal{M}|^{-1}$ for $S$ uniformly
distributed on $S$ and for all $x \neq x'$. For a given seed $s \in S$ and any $m \in \mathcal{M}$, the randomized
inverse $f_s^{-1}(m)$ uniformly at random chooses an element of the set $\{x : f(s, x) = m\}$. Reliable
transmission of messages chosen from $\mathcal{M}$ is possible due to the ECC if the seed is known to
the sender and the receiver.

It is shown by Bellare and Tessaro\textsuperscript{1} [3] and Tal and Vardy [39] that the ordinary secrecy
capacity of any discrete, degraded and symmetric wiretap channel as derived by Wyner [44] and
Csiszár and Körner [17] is achievable by modular UHF schemes such that semantic security
is guaranteed. The proofs make heavy use of the symmetry of the wiretap channel. Since the
seed may be known to the eavesdropper, it can be generated and sent beforehand by the sender,

\textsuperscript{1}[3] and [4] are unpublished extended versions of [5]. We only cite the more detailed unpublished papers.
which however reduces the achievable rate. It is shown in [3] that by reusing the seed not too often, the rate loss due to seed transmission can be made negligible while preserving semantic security.

Bellare and Tessaro also give the example of a finite-field arithmetic UHF $\beta$ which is suitable for being used in modular UHF schemes and is efficiently computable and invertible. In combination with an efficient linear code, the resulting modular UHF scheme is efficient as well and provides a very practical and flexible way of achieving optimal rates and high security for discrete, degraded and symmetric wiretap channels.

Tyagi and Vardy[40] show a leftover hash lemma for the modular UHF scheme. This lemma bounds the mutual information between a uniformly distributed message on $\mathcal{M}$ on the one hand and the pair of eavesdropper’s output and uniformly distributed seed on the other hand. Using this lemma, it is shown that the modular UHF scheme achieves the wiretap capacity with strong secrecy if the wiretap channel is degraded, discrete and symmetric or Gaussian.

By Section II of the present paper, for any strongly secure wiretap code there must exist a large subset of the message set such that the transmission of messages from this subset is semantically secure. In this paper, UHFs are replaced by a new type of functions in the modular coding scheme, called biregular irreducible (BRI) functions. A modular BRI scheme provides semantic security for a message set which is a subset of the image set of the BRI function. On the message set, a BRI function has to have a certain regularity structure.

B. Contributions and overview

a) Section II: In this section, an abstract setting is considered. There is a sender and an eavesdropper, both having access to a seed random variable $S$. The sender uses $S$ to encode a message, which is a random variable independent of $S$. The eavesdropper’s observation is described by the outputs of a channel $K$ whose inputs are the message and $S$. The channel $K$ could be, e.g., an encoder concatenated with the $n$-fold use of a physical channel to the eavesdropper, but no such assumptions are necessary. It is shown that if $I(M \wedge Z, S)$ is small for uniformly distributed message $M$ and $Z$ generated by $M$ and $S$ via $K$, then there is a subset $\mathcal{M}'$ with $|\mathcal{M}'| \geq |\mathcal{M}|/2$ such that $I(\overline{M} \wedge \overline{Z}, S)$ is small for any message random variable $\overline{M}$

[40] is an unpublished, extended version of [41]. Since the latter does not provide all the results and details needed here, we only cite [40].
on $\mathcal{M}'$ and $\mathcal{Z}$ generated by $\mathcal{M}$ and $S$ via $K$. Asymptotically, this means that a message rate which is achievable with strong secrecy is also achievable with semantic security if $I(M \land Z, S)$ decreases sufficiently fast. If the decrease is too slow, then semantic security holds in terms of total variation distance instead of mutual information.

**b) Section III:** Instead of going deeper into the analysis of UHFs in order to find the semantically secure message subset $\mathcal{M}'$, we introduce a new class of functions called BRI functions which replace the UHFs in modular coding schemes, thus giving rise to modular BRI schemes. A BRI function is a function $f : S \times \mathcal{X} \rightarrow \mathcal{N}$ together with a regularity set $\mathcal{M} \subset \mathcal{N}$ which serves as the message set of the modular BRI scheme and on which $f$ has to satisfy certain regularity conditions. In particular, to every $f$ and $m \in \mathcal{M}$, a stochastic $\mathcal{X} \times \mathcal{X}$ matrix $P_{f,m}$ is associated whose $(x, x')$ entry is $\mathbb{P}[f_S(x) = f_S(x') = m]$ divided by a normalizing constant.

Now assume that some channel $K : S \times \mathcal{M} \rightarrow \mathcal{Z}$ describes the eavesdropper’s output given that the message is first passed into the randomized inverse of the BRI function $f$ using seed $S$, and then transmitted through some channel $W$. ($W$ could be the concatenation of error-correcting encoder and physical channel $T$ as in Fig. 1.) One of the central results of this paper, inspired by the proof of the leftover hash lemma of Taygi and Vardy [40], is an upper bound on $\mathbb{E}_S D(K(\cdot|S, m)||Q)$ for every $m \in \mathcal{M}$ and some probability measure $Q$ on $\mathcal{Z}$, where $D(\cdot||\cdot)$ denotes Kullback-Leibler divergence and $\mathbb{E}_S$ is the expectation with respect to $S$. Using this bound, the mutual information $I(M \land Z, S)$ can be upper-bounded for any random variable $\mathcal{M}$ on $\mathcal{M}$ and $\mathcal{Z}$ generated by $\mathcal{M}$ and $S$ via $K$. The main term of this upper bound is the product of the second-largest eigenvalue modulus $\lambda_2(f, m)$ of $P(f, m)$ on the one hand and $\exp(I_{\text{max}}(W))$ on the other hand, where $I_{\text{max}}(W)$ is the $\varepsilon$-smooth max-information of $W$ introduced in [40]. In particular, BRI functions provide a universal method of establishing secrecy for wiretap channels.

A last result of this section which is not needed in the rest of the paper is that BRI functions are UHFs on average.

**c) Section IV:** This section contains the proof of the upper bound stated in Section III. It first upper-bounds Kullback-Leibler divergence by Rényi 2-divergence. An upper bound on the latter yields the result.

**d) Section V:** It is shown that for certain parameters, the finite-field arithmetic UHF $\beta$ mentioned before and used in [3] and [40] is a BRI function with a large and completely known regularity set $\mathcal{M}$ and small $\lambda_2(\beta, m)$ for every $m \in \mathcal{M}$. The analysis rests in large part on the analysis of the eigenvalues of a family of Cayley sum graphs determined by $\beta$. November 20, 2018
Section VI: Here it is shown that in fact, every BRI function $f$ can be described in terms of a family of biregular (bipartite) graphs. Every element $m$ in the image of $f$ induces a graph $G_{f,m}$. The vertex set of $G_{f,m}$ is $S \cup X$ and $(s,x)$ are adjacent if $f(s,x) = m$. The definition of BRI functions can be given an equivalent formulation by requiring $G_{f,m}$ to be biregular and connected for every $m \in \mathcal{M}$. This allows the construction of new examples of BRI functions. An important case is where every $G_{f,m}$ is $(d_1, d_2)$-biregular and Ramanujan, which means that the second-largest eigenvalue modulus of $G_{f,m}$ is at most $\sqrt{d_1 - 1} + \sqrt{d_2 - 1}$. Based on results of Marcus, Spielman and Srivastava [35], a BRI function can be constructed from a biregular connected Ramanujan graph using repeated 2-lifts of graphs. This construction allows for more flexible parameters than the BRI function $\beta$ of the previous section, but so far is not explicit because it relies on a refinement of the probabilistic method.

Section VII: Here some asymptotic consequences of the results of the previous sections are drawn. It is shown that the wiretap capacity of arbitrary discrete and Gaussian wiretap channels is achievable with semantic security by modular BRI schemes. The BRI functions applied in these schemes should be nearly Ramanujan, and the maximum of the associated degree pair has to grow exponentially in the blocklength. Suitable BRI functions constructed from Ramanujan graphs are proven to exist. A consequence of the wiretap coding theorem is an upper bound on the ratio of the size of the message set $\mathcal{M}$ of a BRI function and the maximal $\lambda_2(f,m)$ over $m \in \mathcal{M}$.

Section VIII: This section concludes the paper with a summary and a discussion on some practical aspects of BRI functions and modular BRI schemes, like the constructibility and complexity of BRI functions.

C. Related literature

Semantic security was introduced in information theory by Bellare, Tessaro and Vardy [4], [5]. It is a stronger requirement than strong secrecy as defined by Maurer [37] and Ahlswede and Csiszár [2], where the message is uniformly distributed. It is argued in [4] that semantic security should be adopted as the standard secrecy measure in information-theoretic security, not least because it is the information-theoretic analog to the cryptographic definition of semantic security introduced by Goldwasser and Micali [25] (see also Goldreich’s book [24]).

Semantic security is shown implicitly in resolvability-based proofs of strong secrecy like in Hayashi [26], Devetak [19] and Bloch and Laneman [9]. It is an explicit goal of random coding
in the resolvability-based works of Bunin et al. [12], Frey, Bjelaković and Stańczak [21] and Goldfeld, Cuff and Permuter [22], [23].

The concept of universal hash function is due to Carter and Wegman [13]. UHFs were first used in the context of information theory by Bennett, Brassard and Robert [6]. Modular UHF schemes were proposed as a technique for wiretap coding by Hayashi [27]. The finite-field arithmetic UHF $\beta$ mentioned before which is the main example of a UHF in [3] and [40] seems to go back to the work of Bennett et al. [7].

Liu, Yan and Ling [33] use polar codes to prove that the secrecy capacity of Gaussian wiretap channels is achievable with semantic security. To the authors’ knowledge, no other codes apart from modular UHF schemes, polar codes and random codes have been shown to achieve semantic security for specific scenarios.

The first Ramanujan graphs were constructed independently by Lubotzky, Phillips and Sarnak [34] and Margulis [36]. Ramanujan and nearly Ramanujan graphs are optimal or very good expander graphs, respectively. Expanders are a very active field of research and have many applications in mathematics, computer science and engineering. A good overview is by Hoory, Linial and Wigderson [28].

D. Basic definitions and notation

For a set $A$ and a subset $B \subset A$, by $A \setminus B$ we mean the set difference of $A$ and $B$. If $f : A \to B$ is a function and $b \in B$, then $f^{-1}(b)$ denotes the preimage of $b$ under $f$, i.e., $f^{-1}(b) = \{a \in A : f(a) = b\}$. The randomized inverse of a BRI function (to be defined later) has the same notation, but it should always be clear what is meant. If $E$ is any event, then $1_E$ equals 1 if the event occurs and 0 otherwise. The logarithm $\log$ and the exponential function $\exp$ will always be taken to base 2, the natural logarithm is denoted by $\ln$.

The distribution of a random variable $X$ is denoted by $P_X$. If $X, Y$ are random variables with the joint distribution $P_{XY}$, the conditional distribution of $Y$ given $X$ is written $P_{Y|X}$. The distribution obtained by fixing a realization $x$ of $X$ is denoted by $P_{Y|X=x}$. If $f$ maps realizations of $X$ to the real numbers, then $E_X f(X)$ is the expectation of the random variable $f(X)$.

If $\mathcal{X}$ is any finite set, then $\mathbb{R}^\mathcal{X}$ denotes the set of real-valued functions on $\mathcal{X}$. $\mathbb{R}^\mathcal{X}$ is isomorphic to $\mathbb{R}^{|\mathcal{X}|}$. Similarly, we will work with matrices from $\mathbb{R}^{S \times \mathcal{X}}$. A matrix is called stochastic if it has nonnegative entries and the entries of every row sum to 1. A symmetric matrix $A \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ is diagonalizable with real eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|\mathcal{X}|}$. In this situation, the algebraic
multiplicity of an eigenvalue is the same as its geometric multiplicity, and one can just speak of its multiplicity. If $A$ also has nonnegative entries and constant row sums (e.g., if it is stochastic), then $\mu_1$ is an eigenvalue to the eigenvector $1$, the all-one vector. For such a matrix, its second-largest eigenvalue modulus is $\max(\{|\mu_2|, |\mu_3|\})$, and if this is smaller than $\mu_1$, then $\mu_1$ is a simple eigenvalue of $A$.

II. Semantic Security from Strong Secrecy

In this section we first define the necessary probabilistic concepts like information measures and channels. We then analyze the relation between strong secrecy and semantic security. Finally, we introduce wiretap channels and wiretap codes.

A. Basic probability definitions

Let $Z$ be a measurable space, i.e., $Z$ is equipped with a sigma algebra, which is suppressed in the notation. We will always assume that a probability measure $P$ on $Z$ has a density $p$ with respect to $\mu$, i.e.,

$$P(Z') = \int_{Z'} p(z) \mu(dz)$$

for measurable $Z' \subset Z$.

Example 1. If $Z$ is a discrete set, then we will always assume that $\mu$ is the counting measure defined by $\mu(Z') = |Z'|$. Every probability distribution on $Z$ has a density with respect to $\mu$ and

$$P(Z') = \sum_{z \in Z'} p(z) \mu(z) = \sum_{z \in Z'} p(z).$$

Example 2. The Gaussian distribution on $Z = \mathbb{R}$ with mean $a$ and variance $\sigma^2$ has the usual density

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-a)^2}{2\sigma^2}}$$

with respect to Lebesgue measure.

Example 3. If $P$ has $\mu$-density $p$ and $Q$ has $\nu$-density $q$, then the product $P \otimes Q$ of $P$ and $Q$ has density $r(z_1, z_2) = p(z_1)q(z_2)$ with respect to the product measure $\mu \otimes \nu$ determined by the rule $(\mu \otimes \nu)(Z'_1 \times Z'_2) = \mu(Z'_1)\nu(Z'_2)$. 

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The total variation distance of $P$ and $Q$ is defined by
\[ \|P - Q\| = 2 \sup_{Z'} (P(Z') - Q(Z')), \]
where the supremum is over measurable sets. Since we assume that $P$ and $Q$ both have a density with respect to a measure $\mu$, an alternative expression for $\|P - Q\|$ is
\[ \|P - Q\| = \int |p(z) - q(z)| \mu(dz), \quad (2) \]
i.e., total variation distance is the $L^1$ distance of the densities. The Kullback-Leibler divergence of $P$ and $Q$ is given by
\[ D(P \parallel Q) = \begin{cases} \int p(z) \log \frac{p(z)}{q(z)} \mu(dz) & \text{if } \mu(q = 0, p > 0) = 0, \\ +\infty & \text{else}, \end{cases} \]
the Rényi 2-divergence of $P$ and $Q$ by
\[ D_2(P \parallel Q) = \begin{cases} \int \frac{p(z)^2}{q(z)} \mu(dz) & \text{if } \mu(q = 0, p > 0) = 0, \\ +\infty & \text{else}. \end{cases} \]
If $X$ and $Y$ have joint distribution $P_{XY}$, then the mutual information of $X$ and $Y$ is given by
\[ I(X \land Y) = D(P_{XY} \parallel P_X \otimes P_Y). \]
We also introduce the entropy
\[ H(X) = -\int p_X(x) \log p_X(x) \mu(dx), \]
where $p_X$ is the $\mu$-density of $X$, and conditional entropy
\[ H(X|Y) = \int p_Y(y) H(X|y) \nu(dy), \]
where $p_Y$ is the $\nu$-density of $Y$ and the random variable $X|y$ has distribution $P_{X|Y=y}$. Then
\[ I(X \land Y) = H(X) - H(X|Y). \]
With an additional correlated random variable $S$, such that the joint distribution of $X, Y, S$ is $P_{XYS}$, let $I(X \land Y|s)$ denote the mutual information of the random variables $X$ and $Y$ with joint distribution $P_{XY|S=s}$. Then the conditional mutual information of $X$ and $Y$ given $S$ is
\[ I(X \land Y|S) = \mathbb{E}_S I(X \land Y|S), \]
i.e., the mean of $I(X \land Y|s)$ with respect to $S$. 

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Pinsker’s inequality states that
\[ \|P - Q\|^2 \leq \frac{1}{2 \ln 2} D(P\|Q). \]
If \(X\) is a random variable on the finite set \(\mathcal{X}\) and \(Y\) an arbitrary random variable such that \(P_{XY}\) has a density, then using (2) it was shown in [32] that
\[ I(X \wedge Y) \leq -\|P_{XY} - P_X \otimes P_Y\| \log \|P_{XY} - P_X \otimes P_Y\| / |\mathcal{X}|. \tag{3} \]

B. Channels

A channel \(W\) with input alphabet \(\mathcal{A}\) and output alphabet \(\mathcal{Z}\) assigns to every \(a \in \mathcal{A}\) a probability measure \(W(\cdot|a)\) on \(\mathcal{Z}\). To indicate the input and output alphabets of a channel \(W\), we will often write \(W : \mathcal{A} \to \mathcal{Z}\). This should not lead to confusion with the analogous notation for functions. We will always assume that \(W(\cdot|a)\) has a density \(w(\cdot|a)\) with respect to some reference measure \(\mu\) on \(\mathcal{Z}\), i.e.,
\[ W(\mathcal{Z}'|a) = \int_{\mathcal{Z}'} w(z|a) \mu(dz) \]
for every measurable \(\mathcal{Z}' \subset \mathcal{Z}\).

**Example 4.** If both \(\mathcal{A}\) and \(\mathcal{Z}\) are finite, then \(W\) is a discrete channel. Like for probability measures, the density is always taken with respect to the counting measure \(\mu\) (see Example 1). \(W\) then is determined by the stochastic matrix \((w(z|a))_{a \in \mathcal{A}, z \in \mathcal{Z}}\) satisfying
\[ W(\mathcal{Z}'|a) = \int_{\mathcal{Z}'} w(z|a) \mu(dz) = \sum_{z \in \mathcal{Z}'} w(z|a) \]
for every subset \(\mathcal{Z}'\) of \(\mathcal{Z}\).

**Example 5.** The additive Gaussian noise channel \(W\) with noise variance \(\sigma^2\) has \(\mathcal{A} = \mathcal{Z} = \mathbb{R}\). If \(\mu\) is the Lebesgue measure on \(\mathcal{Z}\) and \(w\) the density of \(W\) with respect to \(\mu\), then \(w(z|a)\) is given by (1).

**Example 6.** If the channel \(W : \mathcal{A} \to \mathcal{Z}\) has density \(w\) with respect to the measure \(\mu\) on \(\mathcal{Z}\), then the blocklength-\(n\) memoryless extension \(W^n\) of \(W\) has density
\[ w^n(z^n|a^n) = \prod_{i=1}^{n} w(z_i|a_i) \]

3This definition of a channel does not encompass all concepts called “channel” in information theory. For example, channels with states (random or arbitrary) are not channels in the sense of this paper.
with respect to the $n$-fold product measure $\mu \otimes \cdots \otimes \mu$, where $a^n = (a_1, \ldots, a_n)$ and $z^n = (z_1, \ldots, z_n)$.

**Example 7.** The conditional probability $P_{X|Y}$ of a random variable $X$ with respect to the random variable $Y$ is a channel.

**Example 8.** Any deterministic function $\phi : \mathcal{X} \to \mathcal{A}$ is a channel.

If $(X,Y)$ is a pair of random variables on $\mathcal{X} \times \mathcal{Y}$ and $P_{Y|X}$ can be described by a channel $W$, then we say that $Y$ is generated by $X$ via $W$. If $P_X = P$, then we often write $I(X \land Y) = I(P, W)$.

If $V : \mathcal{X} \to \mathcal{Y}$ is a discrete channel with density $v$, then the concatenation of $V$ with the arbitrary channel $W : \mathcal{Y} \to \mathcal{Z}$ is the channel $W \circ V : \mathcal{X} \to \mathcal{Z}$ which has the $\mu$-density

$$u(z|x) = \sum_{y \in \mathcal{Y}} w(z|y) v(y|x)$$

if $W$ has the $\mu$-density $w$. The concatenation can be defined analogously if $\mathcal{Y}$ is infinite, but there is a finite set $\mathcal{Y}' \subset \mathcal{Y}$ such that $V(\mathcal{Y}'|x) = 1$ for all $x \in \mathcal{X}$.

**C. Semantic security and strong secrecy**

We will now give meaning to the claim that any strong secrecy rate also is a semantic security rate, perhaps with a weak form of semantic security. We consider an abstract setting not necessarily arising from the wiretap channel problem. Security is an asymptotic property, the index $n$ usually signifies blocklength. For each $n$, there is a finite set $\mathcal{M}_n$ of messages, a set $\mathcal{Z}_n$ to which the eavesdropper has access and a set of seeds which both the sender of the message and the eavesdropper have access to. Every choice of a message and a seed generates an eavesdropper observation in the set $\mathcal{Z}_n$ via the channel $K_n : \mathcal{M}_n \times S_n \to \mathcal{Z}_n$.

**Definition 9.** For every $n$, let $S_n$ be uniformly distributed on $\mathcal{S}_n$.

1) **Strong secrecy** holds if $M_n$ is uniformly distributed on $\mathcal{M}_n$, independent of $S_n$ and $I(M_n \land \mathcal{Z}_n, S_n) \to 0$ as $n \to \infty$, where $\mathcal{Z}_n$ is generated by $M_n$ and $S_n$ via $K_n$.

2) **Semantic security** holds if $\max_{\overline{M}_n} I(\overline{M}_n \land \mathcal{Z}_n, S_n) \to 0$ as $n \to \infty$, where the maximum ranges over all probability distributions on $\mathcal{M}_n$, the message $\overline{M}_n$ is independent of $S_n$ and $\mathcal{Z}_n$ is generated by $\overline{M}_n$ and $S_n$ via $K_n$. 
Clearly, semantic security implies strong secrecy. The next theorem is the nonasymptotic core of the proof that strong secrecy implies semantic security with the same asymptotic message rate. The underlying message set is denoted by $M$ and the seed set by $S$. A channel $K : M \times S \to Z$ determines the eavesdropper’s observations, who also knows the seed. The theorem is slightly more general than needed since $S$ is not required to be uniformly distributed here.

**Theorem 10.** Let $M$ be uniformly distributed on $M$ and independent of $S$, which may have an arbitrary distribution on $S$. Assume that $Z$ is generated by $M$ and $S$ via $K$. If $I(M \land Z, S) \leq \eta$ for some $\eta > 0$, then there exists a subset $M'$ of $M$ with $|M'| \geq |M|/2$ and

$$\max_{P_{M|M'}} I(M \land \overline{Z}, S) \leq -4\sqrt{\eta \ln 2} \log \frac{4\sqrt{\eta \ln 2}}{|M'|},$$

where the maximum is over all probability distributions $P_{M|M'}$ on $M'$, $\overline{M}$ is independent of $S$ and $\overline{Z}$ is generated by $\overline{M}$ and $S$ through $K$.

**Proof.** The theorem immediately follows from Lemmas 12 and 13 below.

The next corollary is an immediate consequence of Theorem 10 and formulates a necessary condition under which the same asymptotic message rate is achievable with semantic security as with strong secrecy.

**Corollary 11.** For every positive integer $n$ let $M_n$ be a message set, $S_n$ a seed set and $K_n : M_n \times S_n \to Z_n$ a channel to some measurable set $Z_n$. Let $(M_n, S_n)$ be uniformly distributed on $M_n \times S_n$ and $Z_n$ generated by $M_n$ and $S_n$ via $K_n$. If strong secrecy holds such that $I(M_n \land Z_n, S_n) \leq \eta_n$ and

$$\tilde{\eta}_n = -\sqrt{\eta_n \log \frac{\sqrt{\eta_n}}{|M_n|}} \to 0$$

as $n$ tends to infinity, then every $M_n$ contains a subset $M'_n$ such that

$$\liminf_{n \to \infty} \frac{\log |M'_n|}{n} = \liminf_{n \to \infty} \frac{\log |M_n|}{n}$$

and semantic security holds for the messages restricted to $M'_n$, more precisely,

$$\max_{P_{M|M_n}} I(M \land \overline{Z}_n, S_n) \leq \tilde{\eta}_n \to 0,$$

where $\overline{M}_n$ ranges over all possible random variables on the smaller message set $M'_n$ and $\overline{Z}_n$ is generated by $\overline{M}_n$ and $S_n$ via $K_n$. 

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Condition (4) yields conditions on the rate of decrease of $I(M_n \land Z_n, S_n)$ which depends on the growth rate of $|M_n|$. For example, if the asymptotic message rate is positive, i.e., $|M_n|$ grows exponentially, then (4) is satisfied if $n^2 I(M_n \land Z_n, S_n)$ goes to zero. Generally, if the rate of decrease of $I(M_n \land Z_n, S_n)$ is not sufficiently large, then semantic security still holds if it is formulated in terms of total variation distance instead of mutual information. More precisely, a slightly weaker definition of semantic security is to require

$$\max_{M_n} \|P_{Z_n S_n M_n} - P_{Z_n S_n} \otimes P_{M_n}\|$$

to tend to zero. The proof of Lemma 13 shows that this can be inferred from strong secrecy no matter what the rate of convergence to zero of $I(M_n \land Z_n, S_n)$ is.

Theorem 10 is proved with the following two lemmas.

Lemma 12. Let $(M, S)$ be independent random variables on $\mathcal{M} \times S$ such that $M$ is uniformly distributed and assume that $Z$ is generated by $M$ and $S$ via $K$. If $I(M \land Z, S) \leq \eta$ for some $\eta > 0$, then there exists a subset $\mathcal{M}'$ of $\mathcal{M}$ such that $|\mathcal{M}'| \geq |\mathcal{M}|/2$ and

$$\mathbb{E}_S D(P_{Z|S,M=m} \| P_{Z|S}) \leq 2\eta$$

for every $m \in \mathcal{M}'$.

Proof. The independence of $M$ and $S$ implies $I(M \land Z, S) = I(M \land Z|S)$. Moreover

$$I(M \land Z|S) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \mathbb{E}_S D(P_{Z|S,M=m} \| P_{Z|S}).$$

By choosing $\mathcal{M}'$ to be the set of those $m \in \mathcal{M}$ which satisfy

$$\mathbb{E}_S D(P_{Z|S,M=m} \| P_{Z|S}) \leq 2\eta,$$

it is straightforward to see that $|\mathcal{M}'| \geq |\mathcal{M}|/2$. \hfill \square

Lemma 13. If

$$\mathbb{E}_S D(K(\cdot|S,m) \| V(\cdot|S))) \leq \eta \tag{5}$$

for some channel $V : S \rightarrow Z$ and every $m \in \mathcal{M}$, then

$$\mathbb{E}_S \|K(\cdot|S,m) - K(\cdot|S,m')\| \leq 2\sqrt{2\eta \ln 2} \tag{6}$$

for all pairs $m, m' \in \mathcal{M}$. In particular, if $\overline{M}$ is independent of $S$ with an arbitrary distribution on $\mathcal{M}$ and $Z$ is generated by $\overline{M}$ and $S$ via $K$, then

$$I(\overline{M} \land Z, S) \leq -2\sqrt{2\eta \ln 2} \log 2\frac{2\sqrt{2\eta \ln 2}}{|\mathcal{M}'|}. \tag{7}$$
Proof. To show that (5) implies (6), observe that

\[
\eta > \mathbb{E}_S D(K(\cdot|S,m)||V(\cdot|S)) \geq \frac{1}{2 \ln 2} \mathbb{E}_S \| K(\cdot|S,m) - V(\cdot|S) \|^2, \tag{8}
\]

where (a) is an application of Pinsker’s inequality. Therefore

\[
\mathbb{E}_S \| K(\cdot|S,m) - K(\cdot|S,m') \| \\
\leq \mathbb{E}_S \| K(\cdot|S,m) - V(\cdot|S) \| + \mathbb{E}_S \| V(\cdot|S) - K(\cdot|S,m') \| \\
\leq \sum_{s \in S} \sqrt{P_S(s)} \sqrt{P_S(s)} \| K(\cdot|s,m) - V(\cdot|s) \| + \sum_{s \in S} \sqrt{P_S(s)} \sqrt{P_S(s)} \| V(\cdot|s) - K(\cdot|s,m') \| \\
\leq \left( \sum_{s \in S} P_S(s) \right)^{\frac{1}{2}} \left( \sum_{s \in S} P_S(s) \| K(\cdot|s,m) - V(\cdot|s) \|^2 \right)^{\frac{1}{2}} \\
+ \left( \sum_{s \in S} P_S(s) \right)^{\frac{1}{2}} \left( \sum_{s \in S} P_S(s) \| V(\cdot|s) - K(\cdot|s,m') \|^2 \right)^{\frac{1}{2}} \\
= \left( \mathbb{E}_S \| K(\cdot|S,m) - V(\cdot|S) \|^2 \right)^{\frac{1}{2}} + \left( \mathbb{E}_S \| V(\cdot|S) - K(\cdot|S,m') \|^2 \right)^{\frac{1}{2}} \\
\leq \sqrt{2 \eta \ln 2} + \sqrt{2 \eta \ln 2},
\]

where (b) is an application of the triangle inequality for total variation distance, (c) is the Cauchy-Schwarz inequality, and (d) follows from (a). This proves (6).

Next we show (7). We have

\[
\| P_{ZS_M} - P_{ZS} \otimes P_M \| \\
\leq \sum_{m \in M} P_M(m) \mathbb{E}_S \| P_{Z|S} - P_{Z|S,M=m} \| \\
\leq \sum_{m,m' \in M} P_M(m) P_M(m') \mathbb{E}_S \| P_{Z|S,M=m} - P_{Z|S,M=m'} \| \\
\leq \sum_{m,m' \in M} P_M(m) P_M(m') 2 \sqrt{2 \eta \ln 2} \\
= 2 \sqrt{2 \eta \ln 2},
\]

where (e) and (f) follow from the convexity of norms and (g) follows from (6). (7) now follows using (3). This completes the proof.

Lemma 13 not only is part of the proof of Theorem 10. It will also be applied to derive the semantic security of modular BRI schemes from a bound of the form (5).
The arguments of this subsection, in particular Lemma 13, suggest that semantic security is a property of messages: The set of messages which satisfy (5) is a message set with (nonasymptotic) semantic security. In contrast, for strong secrecy it is sufficient that the average over (5) be small, hence it is a property of the message set.

D. Wiretap channels and capacities

The security results presented so far were formulated in an abstract scenario. The focus of the rest of this work is on the wiretap channel problem, where the capacity of a wiretap channel has to be found. In this subsection, wiretap channels and wiretap codes are defined.

A wiretap channel is determined by

1) a pair of channels \((T: A \rightarrow \mathcal{Y}, U: A \rightarrow \mathcal{Z})\), where \(A\) is an arbitrary set and \(\mathcal{Y}\) and \(\mathcal{Z}\) are measurable spaces, and

2) a sequence \((\mathcal{A}_n')_{n=1}^{\infty}\) of sets such that \(\mathcal{A}_n' \subset \mathcal{A}_n\), called the input constraint sets. We say a channel has no input constraints if \(\mathcal{A}_n' = \mathcal{A}_n\) for all \(n\).

\(T\) is the physical channel between the sender and the intended receiver and \(U\) is the physical channel between the sender and the eavesdropper. The sequence \((\mathcal{A}_n')\) will usually be omitted in the notation.

Given a wiretap channel \((T,U)\), a (seeded) wiretap code with blocklength \(n\) consists of

1) a discrete channel \(\xi: S \times \mathcal{M} \rightarrow \mathcal{A}_n'\) called the encoder channel, and

2) a measurable mapping \(\zeta: S \times \mathcal{Y}^n \rightarrow \mathcal{M}\), the decoder.

In the special case \(|S| = 1\), the seeded wiretap code \((\xi,\zeta)\) also is called an ordinary wiretap code.

The code rate of a seeded wiretap code is given by \(\log(|\mathcal{M}|)/n\). The (maximal) error incurred by the wiretap code \((\xi,\zeta)\) is defined as

\[ e(\xi,\zeta) = \max_{m \in \mathcal{M}} \max_{s \in S} (T^n \circ \xi)(\mathcal{Y}^n \setminus \zeta^{-1}_s(m)|s,m), \]

where \(\zeta_s(y) = \zeta(s,y)\) for \(y \in \mathcal{Y}^n\), \(\zeta^{-1}_s(m)\) is the preimage of \(m\) under \(\zeta_s\) and \(T^n\) is the blocklength-\(n\) memoryless extension of \(T\). The semantic security information leakage of \((\xi,\zeta)\) is

\[ L_{\text{sem}}(\xi,\zeta) = \max_{\mu_{\mathcal{M}}} I(\mathcal{M} \wedge \mathcal{Z}, S), \]
where the maximum ranges over all probability distributions on $\mathcal{M}$, the seed $S$ is uniformly distributed on $\mathcal{S}$ and independent of $\mathcal{M}$, and $Z$ is generated by $\mathcal{M}$ and $S$ via the channel $U^n \circ \xi$ (recall that $U^n$ denotes the blocklength-$n$ memoryless extension of $U$).

A number $r \geq 0$ is called an achievable semantic security rate if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers and for every $i$ a seeded wiretap code

$$\xi_i : \mathcal{S}_i \times \mathcal{M}_i \to \mathcal{A}'_{n_i}, \quad \zeta_i : \mathcal{S}_i \times \mathcal{Y}_{n_i} \to \mathcal{M}_i$$

of blocklength $n_i$ and code rate $\frac{\log(|\mathcal{M}_i|)}{n_i}$ such that

$$\liminf_{i \to \infty} \frac{\log|\mathcal{M}_i|}{n_i} \geq r, \tag{9}$$

$$\lim_{i \to \infty} e(\xi_i, \zeta_i) = 0 \tag{10}$$

$$\lim_{i \to \infty} L_{\text{sem}}(\xi_i, \zeta_i) = 0. \tag{11}$$

The supremum of all achievable semantic security rates is called the semantic security capacity of the wiretap channel $(T,U)$.

The definitions of achievable strong secrecy rate and strong secrecy capacity are analogous to the above with the exception that in (11), the $L_{\text{sem}}(\xi_i, \zeta_i)$ is replaced by the strong secrecy information leakage $L_{\text{str}}(\xi_i, \zeta_i) = I(M_i \land Z_i, S_i)$, where $M_i$ is uniformly distributed on $\mathcal{M}_i$ and independent of $S_i$, and $Z_i$ is generated by $M_i$ and $S_i$ via $U^{n_i} \circ \xi_i$.

The fact that the codes achieving a semantic security rate only have to be defined along a blocklength subsequence means that this is an optimistic rate definition in Ahlswede’s terminology [1]. A pessimistic formulation would require the codes to exist for every blocklength and properties (9)-(11) to hold along the complete sequence of positive integers (possibly only requiring strong secrecy instead of (11)). A pessimistic formulation is usual in information theory, but the optimistic one is more appropriate when the codes are required to exhibit a certain structure. For example, wiretap codes used in practice might come from a code family equipped with a structure which is available for infinitely many, but not all blocklengths. This will also be the case in our analysis of modular coding schemes.

Another possible variant in the definition of achievable rates and capacities is to allow only ordinary codes instead of seeded ones. Definitions of wiretap channel capacity usually require codes to be ordinary. The definition of achievable rate for seeded wiretap codes assumes that the seed is known both to sender and intended receiver. If the seed is generated by the sender and transmitted to the receiver before the actual message transmission, the achievable rate can
be reduced considerably. We will see in Section VII that this rate reduction can be overcome by reusing the seed.

**Example 14.** If both $T : A \rightarrow Y$ and $U : A \rightarrow Z$ are discrete channels, then the pair $(T, U)$ is called a *discrete wiretap channel*. Its pessimistic strong secrecy capacity without input constraints achieved by ordinary codes is given by

$$\max \left( I(R \wedge Y) - I(R \wedge Z) \right),$$

(12)

where the maximum is over all finite sets $\mathcal{R}$, channels $\rho : \mathcal{R} \rightarrow A$ and random variables $R$ on $\mathcal{R}$ such that $Y$ is generated by $R$ via $T \circ \rho$ and $Z$ is generated by $R$ via $U \circ \rho$ (Csiszár [16]). An analysis of the converse to the coding theorem of the discrete wiretap channel shows that the optimistic strong secrecy capacity with ordinary codes cannot exceed the pessimistic one, so formula (12) remains true also for that case. Moreover, it follows from the results of Wiese, Nötzel and Boche [43] that even if seeded wiretap codes are allowed, no higher rate is achievable.

Since the strong secrecy information leakage can be shown to tend to zero at exponential speed for every achievable strong secrecy rate, condition (4) is satisfied for ordinary codes. Thus by Corollary 11, (12) also equals the pessimistic as well as optimistic semantic security capacity of $(T, U)$ achievable by both seeded and ordinary codes.

**Example 15.** Let $T : \mathbb{R} \rightarrow \mathbb{R}$ and $U : \mathbb{R} \rightarrow \mathbb{R}$ be Gaussian channels with noise variances $\sigma_T^2$ and $\sigma_U^2$, respectively. For any $\Gamma \geq 0$, the pair $(T, U)$ is called a *Gaussian wiretap channel with input power constraint* $\Gamma$ if for every blocklength $n$, the input constraint set is given by the ball

$$B_n(\sqrt{n\Gamma}) = \{ a \in \mathbb{R}^n : \|a\|^2 \leq n\Gamma \},$$

where $\|\cdot\|$ here denotes the Euclidean norm. The Gaussian wiretap channel with input power constraint $\Gamma$ has the pessimistic, ordinary-codes, strong secrecy capacity

$$\begin{cases} \frac{1}{2} \log \left( 1 + \frac{\Gamma}{\sigma_T^2} \right) - \frac{1}{2} \log \left( 1 + \frac{\Gamma}{\sigma_U^2} \right), & \text{if } \sigma_T^2 \geq \sigma_U^2, \\ 0, & \text{else}, \end{cases}$$

(13)

as was shown, e.g., in [40]. Like in the previous example, the optimistic strong secrecy capacity is also given by (13) if ordinary codes are applied. We are not aware of any results upper-bounding the strong secrecy rates for the Gaussian wiretap channel achievable with seeded wiretap codes, but we conjecture them to be no larger than (13) similar to the discrete case.
The strong secrecy information leakage can also be shown to tend to zero at exponential speed [40], and Corollary [11] can be applied to conclude that (13) is the largest achievable pessimistic as well as optimistic semantic security rate when ordinary codes are used.

For the discrete wiretap channel, we will henceforth speak of the secrecy capacity given by (12). For the Gaussian wiretap channel, we will call (13) the ordinary secrecy capacity of the Gaussian wiretap channel.

III. BRI FUNCTIONS

By Corollary [11] every sequence of wiretap codes which ensures strong secrecy ensures semantic security at the same asymptotic rate if condition (4) is satisfied and the message set of every code in the sequence is reduced to a suitable large subset. However, the theorem only is an existence statement, since it does not answer the question how to choose the semantically secure message subsets. This is unsatisfactory from a practical point of view.

In this section, we introduce **BRI functions** which replace UHFs in the modular coding scheme of Fig. 1. We formulate one of the central results of this paper, which is an upper bound on the degree of security which BRI functions can offer in a modular BRI scheme. This bound will be used in the asymptotic analysis to ensure semantic security with wiretap codes constructed from BRI functions and error-correcting codes. The discussion of the efficiency of BRI functions is postponed to the last section. The final result of this section is that BRI functions are UHFs on average. This is noted just for comparison, it will not be used anywhere in this paper.

A. BRI functions and modular BRI schemes

**Definition 16.** A **biregular irreducible (BRI) function** is a function $f : S \times X \to N$, where $S, X, N$ are finite sets, for which there exists a subset $M$ of $N$ such that for every $m \in M$

1) $S$-regularity: $|\{x : f(s, x) = m\}| = d_S$ for some positive integer $d_S$ independent of $s$,

2) $X$-regularity: $|\{s : f(s, x) = m\}| = d_X$ for some positive integer $d_X$ independent of $x$,

3) Irreducibility: 1 is a simple eigenvalue of the stochastic matrix $P_{f,m}$ on $X \times X$ defined by

$$P_{f,m}(x, x') = \frac{|\{s : f(s, x) = f(s, x') = m\}|}{d_S d_X}$$

(see Lemma [17] for a proof that $P_{f,m}$ really is a stochastic matrix).
The second-largest eigenvalue modulus of $P_{f,m}$ is denoted by $\lambda_2(f,m)$. $M$ is called the regularity set of $f$, $\log(|M|)/\log(|\mathcal{X}|)$ is called the rate of $f$. For fixed $s$, we will sometimes write $f_s(x)$ instead of $f(s,x)$.

To prove that BRI functions are well-defined, we note the following lemma.

**Lemma 17.** For any BRI function $f : S \times \mathcal{X} \to \mathcal{N}$ with regularity set $M$, the matrix $P_{f,m}$ as defined in (14) is a stochastic matrix for every $m \in M$.

**Proof.**

$$\sum_{x' \in \mathcal{X}} |\{s : f_s(x) = f_s(x') = m\}| = \sum_{s \in S} 1_{\{f_s(x) = m\}} \sum_{x' \in \mathcal{X}} 1_{\{f_s(x') = m\}} = d_S d_X.$$  

Thus every row sum of $P_{f,m}$ equals 1. \(\square\)

A BRI function can be used together with an ECC to construct a wiretap code. Assume $(T : A \to Y, U : A \to Z)$ is a wiretap channel. $T$ and $U$ can be blocklength-$n$ memoryless extensions of other channels like in the definition of wiretap codes, but the construction is nonasymptotic and further structure of $T$ and $U$ can be ignored. (The case that $T$ and $U$ are memoryless extensions is considered in Section VII.) Let $(\phi, \psi)$ be an ECC with message set $\mathcal{X}$, i.e., $\phi$ is a mapping from $\mathcal{X}$ to $A$ and $\psi$ maps elements of $Y$ back to $\mathcal{X}$. Assume that $(\phi, \psi)$ incurs a transmission error $e(\phi, \psi)$ at most $\epsilon > 0$, meaning that

$$e(\phi, \psi) = \max_{x \in \mathcal{X}} T(\psi^{-1}(x)|\phi(x)) < \epsilon.$$  

Thus, elements of $\mathcal{X}$ can be transmitted from the sender to the intended receiver with a small error probability. Additionally, let $f : S \times \mathcal{X} \to \mathcal{N}$ be a BRI function with regularity set $M$. By some abuse of notation, for each $s \in S$ we introduce a new channel denoted by $f_s^{-1} : M \to \mathcal{X}$ and called the randomized inverse of $f_s$ whose transition probabilities are defined by

$$f_s^{-1}(x|m) = \frac{1}{d_S} 1_{\{f_s(x) = m\}}.$$  

Thus given $s$ and $m$, it chooses an element of the preimage set $\{x : f_s(x) = m\}$ at random.

The ECC $(\phi, \psi)$ and the BRI function $f$ together define a seeded wiretap code $(\xi : S \times M \to A, \zeta : S \times Y \to M)$ by

$$\xi(a|s,m) = (\phi \circ f_s^{-1})(a|m), \quad \zeta(s,y) = f(s,\psi(y)).$$

4We use the term error-correcting code to emphasize the difference to wiretap codes. Our use of the term implies that the mapping of codewords to channel input sequences (channel modulation) is part of the code.
This wiretap code is called a **modular BRI scheme** and denoted by \( \Pi(f, \phi, \psi) \). Modular BRI schemes are a formalization of the modular scheme depicted in Fig. \( \Pi \) with UHFs replaced by BRI functions. Clearly, the **maximal error** \( e(\Pi(f, \phi, \psi)) = e(\xi, \zeta) \) incurred by \( \Pi(f, \phi, \psi) \) satisfies

\[
e(\Pi(f, \phi, \psi)) \leq e(\phi, \psi).
\]

We also define \( L_{sem}(\Pi(f, \phi, \psi)) = L_{sem}(\xi, \zeta) \) and \( L_{str}(\Pi(f, \phi, \psi)) = L_{str}(\xi, \zeta) \).

Clearly, the rate of the modular BRI scheme \( \Pi(f, \phi, \psi) \) is determined by the rate which the pair \((\phi, \psi)\) achieves over \( T \) together with the rate of \( f \). Therefore the regularity set of \( f \) should be large. On the other hand, it will be seen in Theorem 19 that the degree of security which can be achieved by a BRI-prefix scheme depends on \( \lambda_2(f, m) \), more precisely, \( \lambda_2(f, m) \) should be small. Thus we will be interested in making \( \mathcal{M} \) as large as possible, but at the same time to ensure a small \( \lambda_2(f, m) \). Observe that

\[
|\mathcal{M}| \leq \min \left( \frac{|\mathcal{X}|}{d_\mathcal{S}}, \frac{|\mathcal{S}|}{d_\mathcal{X}} \right),
\]

which implies the upper bound

\[
\frac{\log|\mathcal{M}|}{\log|\mathcal{X}|} \leq \min \left( 1 - \frac{\log d_\mathcal{S}}{\log|\mathcal{X}|}, \frac{\log|\mathcal{S}| - \log d_\mathcal{X}}{\log|\mathcal{X}|} \right)
\]

for the rate of \( f \). The inequality (16) also implies

\[
|\mathcal{S}| \geq d_\mathcal{X}|\mathcal{M}|,
\]

in particular, the seed of a BRI function has to be at least as long as the message.

**Example 18.** Let \( \mathcal{X} = \mathcal{S} = \mathbb{F}_{2^\ell}^* \), the multiplicative group of the finite field with \( 2^\ell \) elements. \( \mathbb{F}_{2^\ell} \) is an \( \ell \)-dimensional vector space over \( \mathbb{F}_2 \). Let \( \mathcal{V} \) and \( \mathcal{N} \) be linear subspaces of this vector space with \( \dim \mathcal{V} = b \) and \( \dim \mathcal{N} = k = \ell - b \) such that \( \mathcal{V} + \mathcal{N} = \mathbb{F}_{2^\ell} \). For \( s, x \in \mathbb{F}_{2^\ell}^* \) we define

\[
\beta(s, x) = m \quad \text{if} \quad s \cdot x \in \mathcal{V} + m,
\]

where \( \mathcal{V} + m = \{ v + m : v \in \mathcal{V} \} \). We call \( \beta \) the **seeded coset function determined by \( \mathcal{V} \) and \( \mathcal{N} \)**. We will show in Section \( \mathcal{V} \) that there exist parameters \( b \) and \( k \) for which \( \mathcal{V}, \mathcal{N} \) can be chosen such that \( \beta \) is a BRI function with large regularity set \( \mathcal{M} \) and sufficiently small \( \lambda_2(\beta, m) \).

If one chooses \( \mathcal{M} = \mathcal{N} \), then one obtains the **arithmetic seeded coset function** \( \beta^o \). Choose basis elements \( e_1, \ldots, e_\ell \) of \( \mathbb{F}_{2^\ell} \) over \( \mathbb{F}_2 \) in such a way that \( e_1, \ldots, e_k \) are a basis of \( \mathcal{N} \) and \( e_{k+1}, \ldots, e_\ell \) are a basis of \( \mathcal{V} \). Then the corresponding \( \beta^o \) obtains the form

\[
\beta^o(s, x) = (s \cdot x)|_k,
\]
where every element $x$ of $\mathbb{F}_2^\ell$ is represented by the binary sequence of length $\ell$ given by its coefficients in the basis $e_1, \ldots, e_{2^\ell}$ and $x|_k$ means the restriction of the coefficient sequence to the first $k$ bits, i.e., the coefficients of $e_1, \ldots, e_k$. $\beta^a$ was defined in [7] and proposed as a UHF for modular UHF schemes used in wiretap coding by [3] and [40].

In order to obtain semantic security for a wider class of channels, $\mathcal{N}$ and $\mathcal{V}$ need to be chosen more specifically, and the regularity set $\mathcal{M}$ has to be a nontrivial subset of $\mathcal{N}$. Thus $\mathcal{M}$ is the message set to which $\mathcal{N}$ has to be restricted in order to obtain semantic security. The identification of $\mathcal{M}$ therefore is of utmost importance.

B. Security by BRI functions

In order to analyze the security offered by a BRI-prefix scheme $\Pi(f, \phi, \psi)$ for transmission over a wiretap channel $(T, U)$, the effect of prefixing $f_s^{-1}$ to the channel

$$W = U \circ \phi : \mathcal{X} \rightarrow \mathcal{Z}$$

(19)

has to be investigated. We denote the output distribution induced on $\mathcal{Z}$ by the input $m$ to the channel $W \circ f_s^{-1}$ by $(W \circ f_s^{-1})(m)$. This distribution has the $\mu$-density

$$\frac{1}{d_S} \sum_{x: f(s, x) = m} w(z|x)$$

(20)

if $w$ is the $\mu$-density of $W$. We also denote by $\text{Unif}(\mathcal{X})$ the uniform distribution on $\mathcal{X}$ and define $\text{Unif}(\mathcal{X})W$ as the probability distribution on $\mathcal{Z}$ with $\mu$-density

$$\frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} w(z|x).$$

The central result for the security analysis of $\Pi(f, \phi, \psi)$ is an upper bound on $\mathbb{E}_S D((W \circ f_s^{-1})(m) \| \text{Unif}(\mathcal{X})W)$. The relevance of this expression becomes clear in view of Lemma [13]. The bound and its proof are inspired by the channel leftover hash lemma of Tyagi and Vardy [40]. That $W$ has a structure like in (19) is inessential for this bound.

Like in [40], the upper bound on the security of a modular BRI scheme involves the $\varepsilon$-smooth max-information of subnormalized channels. A subnormalized channel $\tilde{W}$ with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Z}$ (also written $\tilde{W} : \mathcal{X} \rightarrow \mathcal{Z}$ in the following) satisfies the same properties as a channel except that $0 \leq \tilde{W}(\mathcal{Z}|x) < 1$ is allowed, whereas $W(\mathcal{Z}|x) = 1$ for channels. To make the difference clear, we will sometimes call the channels defined in Subsection [1-B]
ordinary channels. Every ordinary channel by definition also is a subnormalized channel. If \( w \) is the density of \( W \) with respect to \( \mu \), then for any measurable subset \( T \) of \( X \times Z \),

\[
w_T(z|x) = w(z|x)1_{\{(x,z) \in T\}}
\]

(21)
defines the \( \mu \)-density of a subnormalized channel \( W_T : X \to Z \).

The max-information of a subnormalized channel \( \tilde{W} : X \to Z \) with finite \( X \) is defined as

\[
I_{\text{max}}(\tilde{W}) = \log \int_{x \in X} \max_{z \in Z} \tilde{w}(z|x) \mu(dz),
\]

where \( \tilde{w} \) is the density of \( \tilde{W} \) with respect to \( \mu \). For \( \varepsilon > 0 \), the \( \varepsilon \)-smooth max-information of an ordinary channel \( W : X \to Z \) with finite \( X \) is defined as

\[
I_{\text{max}}^\varepsilon(W) = \inf_T I_{\text{max}}(W_T),
\]

where \( W_T \) is defined as in (21) and \( T \) ranges over all subsets of \( X \times Z \) satisfying

\[
W(\{z : (x,z) \in T\}|x) \geq 1 - \varepsilon
\]

(22)
for all \( x \in X \). The concepts of max-information and \( \varepsilon \)-smooth max-information go back to [40].

We can now formulate the central result on the degree of security provided by BRI functions.

**Theorem 19.** Let \( f : S \times X \to N \) be a BRI function with regularity set \( M \subset N \) and let \( S \) be uniformly distributed on \( S \). Then for every \( m \in M \) and \( 0 < \varepsilon < 1 - e^{-1} \),

\[
\mathbb{E}_S D((W \circ f_S^{-1})(m)) \leq \frac{1}{\ln 2} \lambda_2(f, m)2^{I_{\text{max}}^\varepsilon(W)} + \log \frac{|X|d_X}{|S|d_S} + \varepsilon \log \frac{|X|}{d_S} - (1 - \varepsilon) \log(1 - \varepsilon).
\]

**Proof.** See Section IV. \( \square \)

Denote the upper bound given in Theorem 19 by \( \eta(f, m, W) \). From Theorem 39, where BRI functions are characterized, it follows that \(|X|d_X = |S|d_S\), which simplifies \( \eta(f, m, W) \). Theorem 19 and Lemma 13 imply the following corollary.

**Corollary 20.** Let \( f : S \times X \to N \) be a BRI function with regularity set \( M \) and \( W : X \to Z \) a channel. Then

\[
\max_{P_{SM}}\|P_{ZSM} - P_{ZS} \otimes P_{SM}\| \leq 2 \max_{m \in M} \sqrt{2\eta(f, m, W) \ln 2},
\]

\[
\max_{P_{SM}} I(M \wedge Z, S) \leq -2 \max_{m \in M} \sqrt{2\eta(f, m, W) \ln 2} \log \frac{2\sqrt{2\eta(f, m, W) \ln 2}}{|M|}.
\]
C. BRI functions and universal hash functions

To conclude this section, we examine the relation between BRI functions and UHFs. A UHF is a function \( f : S \times X \to N \) which satisfies

\[
P[f_s(x) = f_s(x')] \leq \frac{1}{|N|} \tag{23}
\]

if \( S \) is uniformly distributed on the index set \( S \) and \( x \neq x' \), where \( f_s(x) = f(s, x) \). A natural question is whether a BRI function is a universal hash function under the condition that the common value of \( f_s(x) \) and \( f_s(x') \) is an element of \( \mathcal{M} \) and that the right-hand side of (23) is replaced by \( 1/|\mathcal{M}| \). One obtains the following average result, which is not needed in this paper.

**Lemma 21.** If \( f : S \times X \to N \) is a BRI function with regularity set \( \mathcal{M} \), then

\[
\frac{1}{|X| - 1} \sum_{x' \neq x} P[f_s(x) = f_s(x') | f_s(x) \in \mathcal{M}] = \frac{d_S - 1}{|X| - 1} \leq \frac{1}{|\mathcal{M}|}.
\]

**Proof.** Observe that

\[
P[f_s(x) \in \mathcal{M}] = \sum_{m \in \mathcal{M}} \frac{|\{s : f_s(x) = m\}|}{|S|} = \frac{\mathcal{M}|d_X}{|S|}.
\]

Therefore

\[
\frac{1}{|X| - 1} \sum_{x' \neq x} P[f_s(x) = f_s(x') | f_s(x) \in \mathcal{M}]
= \frac{1}{|X| - 1} \sum_{x' \neq x} \frac{P[f_s(x) = f_s(x') \in \mathcal{M}]}{P[f_s(x) \in \mathcal{M}]}
= \frac{1}{|X| - 1} \sum_{x' \neq x} \frac{\sum_{m \in \mathcal{M}} \{|s : f_s(x) = f_s(x') = m\}|}{\mathcal{M}|d_X}
\leq \frac{(d_S - 1)d_X}{(|X| - 1)d_X}
\leq \frac{d_S}{d_S|\mathcal{M}| - 1}
\leq \frac{1}{|\mathcal{M}|},
\]

where (a) is due to (24), (b) holds due to Lemma 17 and (c) is due to (16).

IV. PROOF OF THEOREM 19

The proof of Theorem 19 is divided into four subsections. The first one introduces a few auxiliary concepts and results, the second subsection reduces the statement of the theorem to
A. Auxiliary definitions and results

   a) Kullback-Leibler divergence and Rényi 2-divergence for subnormalized channels: In order to obtain the ε-smooth max-information in the upper bound from Theorem 19, one has to pass from $W$ to subnormalized versions of $W$. Here we introduce a version of Kullback-Leibler divergence and Rényi 2-divergence for subnormalized measures and densities. They appear during the proof of Theorem 19.

   Assume that $f, g : Z \to \mathbb{R}$ are nonnegative functions on a measurable set $Z$ which are integrable with respect to $\mu$, and assume that

   \[ Z_f := \int f \, d\mu, \quad Z_g := \int g \, d\mu \]

   both are positive. Then the Kullback-Leibler divergence of $f$ and $g$ is defined by

   \[ D(f \| g) = Z_f \left( D\left( \frac{f}{Z_f} \bigg\| \frac{g}{Z_g} \right) + \log \frac{Z_f}{Z_g} \right), \]

   where $f/Z_f$ and $g/Z_g$ here denote the corresponding probability measures. The Rényi 2-divergence of $f$ and $g$ is given by

   \[ D_2(f \| g) = D_2\left( \frac{f}{Z_f} \bigg\| \frac{g}{Z_g} \right) + 2 \log Z_f - \log Z_g. \]

   It is straightforward to see that

   \[ D(f \| g) = \int f \log \frac{f}{g} \, d\mu, \quad D_2(f \| g) = \int \frac{f^2}{g} \, d\mu \]

   if $\mu(f > 0, g = 0) = 0$. If $M_1, M_2$ are finite nontrivial measures on $Z$ with respective $\mu$-densities $f, g$, then we define

   \[ D(M_1 \| M_2) = D(f \| g), \quad D_2(M_1 \| M_2) = D_2(f \| g). \]

   It is well-known that Kullback-Leibler divergence is upper-bounded by Rényi 2-divergence for probability measures [42]. This can be extended to the subnormalized case up to some error terms.

   \[ \text{Lemma 22. Let } Z \text{ be a measurable space with measure } \mu \text{ and let } f, g : E \to \mathbb{R} \text{ be nonnegative integrable functions with } Z_f, Z_g > 0 \text{ as in (25). Then} \]

   \[ D(f \| g) \leq Z_f (D_2(f \| g) - \log Z_f). \]
Proof. See Appendix B.

The proof of Theorem 19 actually makes use of the following consequence of Lemma 22.

Lemma 23. Let \( Z \) be a measurable space with measure \( \mu \) and \( S \) a random variable on the finite set \( S \). For every \( s \in S \), let \( f_s, g_s : E \to \mathbb{R} \) be nonnegative integrable functions such that \( Z_{g_s} > 0 \) and \( 1 - \varepsilon \leq Z_{f_s} \leq 1 \) for some \( 0 < \varepsilon < 1 - e^{-1} \). Then

\[
\mathbb{E}_S D(f_S \| g_S) \leq \log \mathbb{E}_S \exp(D_2(f_S \| g_S)) - (1 - \varepsilon) \log(1 - \varepsilon).
\]

Proof.

\[
\mathbb{E}_S D(f_S \| g_S) \overset{(a)}{\leq} \log \mathbb{E}_S \exp(D(f_S \| g_S))
\]

\[
\overset{(b)}{\leq} \log \mathbb{E}_S \exp(Z_{f_S}D_2(f_S \| g_S) - Z_{f_S} \log Z_{f_S})
\]

\[
\overset{(c)}{\leq} \log \mathbb{E}_S \left[ \exp(D_2(f_S \| g_S))(1 - \varepsilon)^{-1} \right]
\]

\[
= \log \mathbb{E}_S \exp(D_2(f_S \| g_S)) - (1 - \varepsilon) \log(1 - \varepsilon),
\]

where (a) is due to the convexity of the exponential function, (b) is a consequence of Lemma 22 and (c) follows from \( 1 - \varepsilon \leq Z_{f_s} \leq 1 \) and the fact that the function \( t \mapsto -t \log t \) decreases between \( e^{-1} \) and 1.

b) Second-largest eigenvalue modulus of stochastic matrices: The next lemma is a well-known result about stochastic matrices.

Lemma 24. Let \( P \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}} \) be a symmetric stochastic matrix. If \( \lambda_2 \) denotes the second-largest eigenvalue modulus of \( P \), then

\[
w^\top Pw \leq \lambda_2 \sum_{x \in \mathcal{X}} w(x)^2 + \frac{1}{|\mathcal{X}|} \left( \sum_{x \in \mathcal{X}} w(x) \right)^2
\]

for every \( w = (w(x))_{x \in \mathcal{X}} \in \mathbb{R}^\mathcal{X} \).

Proof. This result can be found in, e.g., [10]. A proof is included in Appendix B for the sake of self-containedness.

B. Reduction to subnormalized channels

The following reduction to subnormalized channels is proved in exactly the same way in [40] for UHFs instead of BRI functions.
Lemma 25. Let \( T \subset X \times Z \) satisfy (22) for all \( x \in X \). If \( S \) is uniformly distributed on \( S \), then
\[
\mathbb{E}_S D((W \circ f_s^{-1})(m)) \| \text{Unif}(X) W \leq \mathbb{E}_S D((W_T \circ f_s^{-1})(m)) \| \text{Unif}(X) W_T + \varepsilon \log \frac{|X|}{d_S},
\]
where the divergence on the right-hand side was defined in (26).

Proof. The density of \((W \circ f_s^{-1})(m)\) is given in (20). Thus
\[
\mathbb{E}_S D((W \circ f_s^{-1})(m)) \| \text{Unif}(X) W
\]
\[
= \frac{1}{d_S |S|} \sum_{s \in S} \int \sum_{x : f_s(x) = m} w(z|x) \log \frac{\sum_{x' : f_s(x') = m} w(z|x')}{d_S |X|^{-1} \sum_{x'' \in X} w(z|x'')} \mu(dz).
\]
The integrand in (28) equals
\[
\sum_{x \in X} w(z|x) 1\{f_s(x) = m\} \log \frac{\sum_{x' \in X} w(z|x') 1\{f_s(x') = m\}}{d_S |X|^{-1} \sum_{x'' \in X} w(z|x'')}
\]
\[
\overset{(a)}{\leq} \sum_{x : (x, z) \in T} w(z|x) 1\{f_s(x) = m\} \log \frac{\sum_{x' : (x', z) \in T} w(z|x') 1\{f_s(x') = m\}}{d_S |X|^{-1} \sum_{x'' : (x'', z) \in T} w(z|x'')}
\]
\[
+ \sum_{x : (x, z) \notin T} w(z|x) 1\{f_s(x) = m\} \log \frac{\sum_{x' : (x', z) \notin T} w(z|x') 1\{f_s(x') = m\}}{d_S |X|^{-1} \sum_{x'' : (x'', z) \notin T} w(z|x'')}
\]
\[
\leq \sum_{x \in X} w_T(z|x) 1\{f_s(x) = m\} \log \frac{\sum_{x' \in X} w_T(z|x') 1\{f_s(x') = m\}}{d_S |X|^{-1} \sum_{x'' \in X} w_T(z|x'')}
\]
\[
+ \sum_{x : (x, z) \notin T} w(z|x) 1\{f_s(x) = m\} \log \frac{|X|}{d_S},
\]
where \((a)\) is due to the log-sum inequality. If we denote the expression in (29) by \(g(s, z)\), then clearly
\[
\frac{1}{d_S |S|} \sum_{s \in S} \int g(s, z) \mu(dz)
\]
exists and equals the expected divergence in (27). The term (30) is responsible for the error term in (27) due to the assumption on \( T \).

C. Eigenvalue upper bound on Rényi 2-divergence

In view of Lemmas 23 and 25, the main ingredient for the proof of Theorem 19 which is still missing is an upper bound on the expected exponential Rényi 2-divergence between \((W_T \circ f_s^{-1})(m)\) and \(\text{Unif}(X) W_T\).
Lemma 26. Choose any measurable set $\mathcal{T} \subset \mathcal{X} \times \mathcal{Z}$ such that $W_{\mathcal{T}}(\mathcal{Z}|x) > 0$ for all $x \in \mathcal{X}$. For every BRI function $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{N}$ with regularity set $\mathcal{M}$, every $m \in \mathcal{M}$ satisfies

$$
\mathbb{E}_S \exp D_2((W_{\mathcal{T}} \circ f_S^{-1})(m) \| \text{Unif}(\mathcal{X}) W_{\mathcal{T}}) \leq \frac{\lambda_2(f, m) 2^{d_{\max}(W_{\mathcal{T}})}}{|\mathcal{X}| d_S (1 + \lambda_2(f, m) 2^{d_{\max}(W_{\mathcal{T}})})}.
$$

Proof. With the $P_{f,m}$ defined in (14),

\[ 
\begin{align*}
\mathbb{E}_S \exp D_2((W_{\mathcal{T}} \circ f_S^{-1})(m) \| \text{Unif}(\mathcal{X}) W_{\mathcal{T})} & \\
& \overset{(a)}{=} \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \int \left( \frac{d_S^{-1} \sum_{x : f_s(x) = m} w_{\mathcal{T}}(z|x)}{|\mathcal{X}|^{-1} \sum_{x''} w_{\mathcal{T}}(z|x'')} \right)^2 \mu(dz) \\
& = \frac{|\mathcal{X}|}{d_S^2 |\mathcal{S}|} \int \sum_{s \in \mathcal{S}} \sum_{x : f_s(x) = m} \frac{w_{\mathcal{T}}(z|x) \sum_{x' : f_s(x') = m} w_{\mathcal{T}}(z|x')}{\sum_{x'' \in \mathcal{X}} w_{\mathcal{T}}(z|x'')} \mu(dz) \\
& = \frac{|\mathcal{X}|}{d_S^2 |\mathcal{S}|} \int \sum_{x \in \mathcal{X}} w_{\mathcal{T}}(z|x) \frac{\sum_{s \in \mathcal{S}} 1 \{f_s(x) = f_s(x') = m\} \cdot \sum_{x' \in \mathcal{X}} w_{\mathcal{T}}(z|x')}{\sum_{x'' \in \mathcal{X}} w_{\mathcal{T}}(z|x'')} \mu(dz) \\
& = \frac{|\mathcal{X}|}{d_S^2 |\mathcal{S}|} \int \sum_{x' \in \mathcal{X}} \frac{w_{\mathcal{T}}(z|x')}{\sum_{x'' \in \mathcal{X}} w_{\mathcal{T}}(z|x'')} \mu(dz) \\
& \overset{(b)}{=} \frac{|\mathcal{X}| d_{\mathcal{X}}}{d_S^2 |\mathcal{S}|} \int \frac{\sum_{x,x' \in \mathcal{X}} w_{\mathcal{T}}(z|x) P_{f,m}(x,x') w_{\mathcal{T}}(z|x')}{\sum_{x'' \in \mathcal{X}} w_{\mathcal{T}}(z|x'')} \mu(dz) \\
& \overset{(c)}{=} \frac{|\mathcal{X}| d_{\mathcal{X}}}{d_S^2 |\mathcal{S}|} \int \frac{\lambda_2(f, m) (\sum_{x \in \mathcal{X}} w_{\mathcal{T}}(z|x))^2}{\sum_{x'' \in \mathcal{X}} w_{\mathcal{T}}(z|x'')} \mu(dz) \\
& \overset{(d)}{=} \frac{|\mathcal{X}| d_{\mathcal{X}}}{d_S^2 |\mathcal{S}|} \left( \lambda_2(f, m) \int \frac{\sum_{x \in \mathcal{X}} w_{\mathcal{T}}(z|x)^2}{\sum_{x'' \in \mathcal{X}} w_{\mathcal{T}}(z|x'')} \mu(dz) + \frac{1}{|\mathcal{X}|} \int \sum_{x' \in \mathcal{X}} w_{\mathcal{T}}(z|x') \mu(dz) \right) \\
& \leq \frac{|\mathcal{X}| d_{\mathcal{X}}}{d_S^2 |\mathcal{S}|} \left( \lambda_2(f, m) \int \max_x w_{\mathcal{T}}(z|x) \mu(dz) + 1 \right),
\end{align*}
\]

where the form of the density from (20) was inserted in (a) and the definition of $P_{f,m}$ in (b), (c) is due to Lemma 24 and (d) comes from Hölder’s inequality $\sum_x w(x)^2 \leq (\max_x w(x)) \sum_x w(x)$.\qed
D. Completion of the Proof

To complete the proof of Theorem 19, take $0 < \varepsilon \leq 1 - e^{-1}$ and choose any subset $T$ of $X \times Z$ which satisfies (22). It follows that

$$
\mathbb{E}_SD\left((W \circ f_S^{-1})(m)\|\text{Unif}(X)W\right)
$$

\begin{align*}
&\leq \mathbb{E}_SD\left((W_T \circ f_S^{-1})(m)\|\text{Unif}(X)W_T\right) + \varepsilon \log \frac{|X|}{d_S} \\
&\leq \log \mathbb{E}_S \exp\left(D_2((W_T \circ f_S^{-1})(m)\|\text{Unif}(X)W_T)\right) - (1 - \varepsilon) \log(1 - \varepsilon) + \varepsilon \log \frac{|X|}{d_S} \\
&= \log \left(1 + 2^{\lambda_2(f, m)2^{\log(W_T)}}\right) + \log \frac{|X|}{d_S} - (1 - \varepsilon) \log(1 - \varepsilon) + \varepsilon \log \frac{|X|}{d_S} \\
&\leq \frac{1}{\ln 2} \lambda_2(f, m)2^{\lambda_2(f, m)W_T} + \log \frac{|X|}{d_S} - (1 - \varepsilon) \log(1 - \varepsilon) + \varepsilon \log \frac{|X|}{d_S},
\end{align*}

where (a) is due to Lemma 25, (b) is due to Lemma 23, (c) is due to Lemma 26 and (d) is due to the fact that $\log(1 + t) \leq t/\ln 2$ for all positive $t$. By minimizing over $T$, one obtains the desired upper bound. This completes the proof of Theorem 19.

V. Semantic Security with the Seeded Coset BRI Function

A. Conditions for $\beta$ to be a BRI function

This section is devoted to a closer analysis of the seeded coset function $\beta : \mathbb{F}_{q^2}^* \times \mathbb{F}_{q^2}^* \rightarrow \mathcal{N}$ from Example 18. The main results are Theorem 28 and Corollary 29 where some combinations of $\ell, k$ and subspaces $\mathcal{V}, \mathcal{N}$ are found which make $\beta$ a BRI function with large regularity set $\mathcal{M}$ and small $\lambda_2(\beta, m)$ for every $m \in \mathcal{M}$.

To define the seeded coset functions which are good BRI functions, recall the following lemma from finite field theory.

**Lemma 27** (E.g., [31], Theorem 2.6). Let $p$ be a prime number. Every subfield of $\mathbb{F}_{p^n}$ has $p^m$ elements for some positive divisor $m$ of $n$. Conversely, if $m$ is a positive divisor of $n$, then there is exactly one subfield of $\mathbb{F}_{p^n}$ with $p^m$ elements. In particular, the unique subfield of $\mathbb{F}_{p^n}$ with $p^m$ elements can be identified with $\mathbb{F}_{p^m}$. 
We can thus define a seeded coset function by choosing \( V \) to be the unique subspace of \( \mathbb{F}_{2^\ell} \) over \( \mathbb{F}_2 \) which equals \( \mathbb{F}_{2^b} \) for any \( b \) dividing \( \ell \). The properties of the corresponding \( \beta \) are summarized in the following theorem.

**Theorem 28.** Assume that \( b \) divides \( \ell \). Let \( V = \mathbb{F}_{2^b} \) and let \( N \) be any linear subspace of dimension \( k \) satisfying \( \dim(N \cap V) = 0 \). Define

\[
\mathcal{M} := \{ m \in N : \mathbb{F}_{2^b}(m) = \mathbb{F}_{2^\ell} \},
\]

where \( \mathbb{F}_{2^b}(m) \) is the smallest subfield of \( \mathbb{F}_{2^\ell} \) which contains \( \mathbb{F}_{2^b} \) and \( m \). Then the seeded coset function \( \beta : \mathbb{F}_{2^\ell}^* \times \mathbb{F}_{2^\ell}^* \to N \) defined by \( V \) and \( N \) is a BRI function with regularity set \( \mathcal{M} \) satisfying

\[
\lambda_2(\beta, m) \leq \left( \frac{k}{b} \right)^2 2^{-b}.
\]

for every \( m \in \mathcal{M} \). Moreover,

\[
|\mathcal{M}| = \frac{\ell}{b} N_{2^b} \left( \frac{\ell}{b} \right) 2^{-b},
\]

(31)

where

\[
N_q(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d}
\]

is the number of monic irreducible polynomials of degree \( n \) over \( \mathbb{F}_q \) and \( \mu(d) \) is the Möbius function defined by

\[
\mu(d) = \begin{cases} 
1 & n = 1, \\
(-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\
0 & \text{else.}
\end{cases}
\]

**Proof.** See the next subsection.

**Corollary 29.** Let \( Z \) be any measurable space and \( S \) uniformly distributed on \( \mathbb{F}_{2^\ell}^* \). Then for the \( \beta \) from Theorem 28 and any channel \( W : \mathbb{F}_{2^\ell}^* \to Z \) and every \( m \in \mathcal{M} \),

\[
\eta(\beta, m, W) \leq \frac{1}{\ln 2} \left( \frac{k}{b} \right)^2 2^t_{\max}(W)^{-b} + \varepsilon \log k + (1 - \varepsilon) \log(1 - \varepsilon).
\]

**Proof of Corollary 29** \( \beta \) is symmetric, which implies \( |X| = |S| \) and \( d_S = d_X \). It is also clear that \( d_S = |V| = 2^b \). Thus the corollary is an immediate consequence of Theorems 19 and 28.

---

5A polynomial is *monic* if its leading coefficient equals 1.
Note that Corollary 29 gives almost the same bound on the expected divergence between $(W \circ f_S^{-1})(m)$ and $\text{Unif}(\mathcal{X})W$, uniformly in the message, as the leftover hash lemma of [40] does on the mutual information between a uniformly distributed message $M$ on the one hand and the seed $S$ and the eavesdropper’s observation $(W \circ f_S^{-1})(M)$ on the other hand.

A drawback of Theorem 28 is that it only makes a statement for specific relations between $k$ and $b$ ($b$ has to divide $b + k$). In particular, it does not say anything about the case $k < b$. In Section VII a new type of BRI functions is introduced which solves this problem.

The main reason for the inflexible relation of the parameters $k$ and $b$ for seeded coset functions is that, to the authors’ knowledge, no analog to Lemma 35 below exists for arbitrary linear subspaces $\mathcal{V}$ instead of subfields. However, recall that the modular scheme of Bellare and Tessaro with the arithmetic seeded coset function $\beta^o$ has been shown to achieve strong secrecy, e.g., for Gaussian wiretap channels. By Theorem 10 it must be possible to find a large message subset for which $\mathbb{E}_S D((W \circ f_S^{-1})(m)\|Q)$ is uniformly small in $m$ for some probability measure $Q$ on $\mathcal{Z}$. It remains an open problem to identify this message subset.

The following lemma sheds some light on the size of the regularity set $\mathcal{M}$. It will become important in the asymptotic analysis in Section VII.

**Lemma 30.** For all positive integers $a$ and $b$, with $k = (a - 1)b$, 

$$a\mathcal{N}_2^b(a)2^{-b} \geq 2^k \left(1 - \frac{1}{2^{ab/2-1}}\right).$$

**Proof.** One easily shows that

$$\mathcal{N}_q(n) \geq \frac{1}{n}q^n - \frac{q}{n(q - 1)}\left(q^{n/2} - 1\right).$$

for any prime power $q$ and positive integer $n$, see also [31, Exercise 3.27]. Therefore

$$a\mathcal{N}_2^b(a)2^{-b} \geq \left(2^{ab} - \frac{2^b}{2^b - 1}\left(2^{ab/2} - 1\right)\right)2^{-b}$$

$$= 2^k - \frac{2^{ab/2} - 1}{2^b - 1}$$

$$= 2^k \left(1 - \frac{2^{ab/2} - 1}{2^{ab} - 2^k}\right)$$

$$\geq 2^k \left(1 - \frac{1}{2^{ab/2-1}}\right).$$

\[\square\]
B. Proof of Theorem 28

The proof of Theorem 28 has two parts. In the first one, it is shown that $\beta$ is a BRI function with regularity set $M$. The second part lower-bounds the cardinality of $M$.

a) $\beta$ is a BRI function: First observe that $\beta$ is $d_S$-regular and $d_X$-regular with $d_S = d_X = 2^b$ for all $m \in \mathcal{N}$. Therefore it is possible to define a stochastic matrix $P_{\beta,m}$ like in the definition of BRI functions for every $m \in \mathcal{N}$. A central observation is that $P_{\beta,m}$ can be written as the square of another stochastic matrix. This actually holds in a more general setting.

Lemma 31. Let $\mathcal{X}$ and $\mathcal{N}$ be finite sets. Assume that $f : \mathcal{X}^2 \to \mathcal{N}$ is a symmetric function, i.e., $f(s, x) = f(x, s)$ for all $(s, x) \in \mathcal{X}^2$, and that

$$|\{x : f(s, x) = m\}| = d_X$$

for every $s \in \mathcal{X}$ and $m \in \mathcal{N}$. For every $m \in \mathcal{N}$, define the stochastic matrix $Q_{f,m} \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ by

$$Q_{f,m}(x, x') = \frac{1}{d_X} 1_{\{f(x') = m\}}.$$

Then $P_{f,m} = Q_{f,m}^2$.

Proof.

$$Q_{f,m}^2(x, x') = \sum_{s \in \mathcal{X}} Q_{f,m}(x, s)Q_{f,m}(s, x')$$

$$= \frac{1}{d_X^2} \sum_{s \in \mathcal{X}} 1_{\{f_s(s) = f_s(x') = m\}}$$

$$= P_{f,m}(x, x'),$$

where (a) is due to the symmetry of $f$.

Now let $\beta : \mathbb{F}_{2^b}^* \times \mathbb{F}_{2^b}^* \to \mathcal{N}$ again be the Bellaro-Tessaro function defined in Theorem 28. A consequence of Lemma 31 is that the analysis of the eigenvalue structure of $P_{\beta,m}$ is reduced to the analysis of the eigenvalues of $Q_{\beta,m}$ for every $m \in \mathcal{N}$. Since $Q_{\beta,m}$ has entries either $1/d_X = 2^{-b}$ or 0, the matrix $A_{\beta,m} = 2^b Q_{\beta,m}$ is the adjacency matrix of a graph on $\mathbb{F}_{2^b}^*$, which we call $G_{\beta,m}$ (some basic facts and definitions about graphs are collected in Appendix A). Two vertices $x, x'$ of $G_{\beta,m}$ are adjacent if $\beta_x(x') = m$. Since $\beta$ is symmetric, $G_{\beta,m}$ is well-defined, but there may be loops, i.e., edges with the same start and end point (namely, if $\beta_x(x) = m$). $G_{\beta,m}$ is
regular since every \( x \in \mathbb{F}_{2^\ell}^* \) is adjacent to \( |\beta_x^{-1}(m)| = 2^b \) vertices. Thus the largest eigenvalue of \( A_{\beta,m} \) is \( 2^b \). The multiplicity of this eigenvalue and the size of the other eigenvalues is determined in the next lemma.

**Lemma 32.** For the seeded coset function \( \beta : \mathbb{F}_{2^\ell}^* \times \mathbb{F}_{2^\ell}^* \rightarrow N \) with regularity set \( \mathcal{M} \) defined in Theorem 28 and for every \( m \in \mathcal{M} \), the largest eigenvalue \( 2^b \) of \( A_{\beta,m} \) has multiplicity 1, and the absolute value of every eigenvalue not equal to \( 2^b \) is upper-bounded by \( k2^{b/2}/b \).

**Corollary 33.** For every \( m \in \mathcal{M} \), 1 is a simple eigenvalue of \( P_{\beta,m} \) and the second-largest eigenvalue modulus of \( P_{\beta,m} \) satisfies

\[
\lambda_2(\beta, m) \leq \left( \frac{k}{b} \right)^2 2^{-b}.
\]

**Proof of Corollary 33.** Since \( A_{\beta,m} \) has the largest eigenvalue \( 2^b \) with multiplicity 1, the largest eigenvalue of \( Q_{\beta,m} \) is 1 with multiplicity 1 as well. Since the second-largest eigenvalue modulus of \( A_{\beta,m} \) is at most \( k2^{b/2}/b \), the second-largest eigenvalue modulus of \( Q_{\beta,m} \) is at most \( k2^{-b/2}/b \). By Lemma 31 the eigenvalue 1 of \( P_{\beta,m} \) also has multiplicity 1. Also, all other eigenvalues of \( P_{\beta,m} \) are upper-bounded in absolute value by \( k2^{-b}/b^2 \), in particular,

\[
\lambda_2(\beta, m) \leq \left( \frac{k}{b} \right)^2 2^{-b}
\]

for every \( m \in \mathcal{M} \). \( \square \)

The corollary implies that \( \beta \) is a BRI function with regularity set \( \mathcal{M} \). It remains to establish Lemma 32. The proof is based on the fact that \( G_{\beta,m} \) is isomorphic to a special Cayley sum graph. A graph \( G \) on the set \( \{0, \ldots, n-1\} \) is called a Cayley sum graph if there exists a subset \( D \) of \( \{0, \ldots, n-1\} \) such that two numbers \( x, y \in \{0, \ldots, n-1\} \) are adjacent if and only if their sum modulo \( n \) is contained in \( D \).

Two vertices \( s, x \) are adjacent in \( G_{\beta,m} \) if \( s \cdot x \in \mathbb{F}_{2^\ell} + m \). Let \( \alpha \) be a primitive element of \( \mathbb{F}_{2^\ell} \), i.e., \( \alpha \) generates the multiplicative group \( \mathbb{F}_{2^\ell}^* \) of \( \mathbb{F}_{2^\ell} \). Such an \( \alpha \) exists \([31\text{, Theorem 2.8}]\). Thus every nonzero element \( x \) of \( \mathbb{F}_{2^\ell} \) can be written \( x = \alpha^a \) for some unique \( 0 \leq a \leq 2^\ell - 2 \). In particular, there exists a set \( D = \{d_1, \ldots, d_{2^b}\} \) such that \( \mathbb{F}_{2^b} + m = \{\alpha^{d_1}, \ldots, \alpha^{d_{2^b}}\} \) (clearly, \( v + m \neq 0 \) for all \( v \in \mathbb{F}_{2^b} \) since 0 is not contained in \( \mathcal{M} \)). Two elements \( s = \alpha^{a_1} \) and \( x = \alpha^{a_2} \) are adjacent in \( G_{\beta,m} \) if and only if \( a_1 + a_2 \in D \) (mod \( 2^\ell - 1 \)). Therefore \( G_{\beta,m} \) is isomorphic to the Cayley sum graph on \( \{0, \ldots, 2^\ell - 2\} \) determined by \( D \). The eigenvalues of \( G_{\beta,m} \) are determined in the following general result on Cayley sum graphs which is due to Chung.
Lemma 34 ([14], Lemma 2). Let $G$ be a Cayley sum graph on $\{0, \ldots, n-1\}$ determined by the set $D = \{d_1, \ldots, d_k\}$. Then its largest eigenvalue equals $k$. The other eigenvalues have the form

$$\pm \left| \sum_{d \in D} \theta^d \right|,$$

where $\theta$ ranges over the $n$-th unit roots $\theta \neq \pm 1$ with positive real part, and if $n$ is even, an additional eigenvalue is given by

$$\sum_{d \in D} (-1)^d.$$

(32)

Note that (32) is not an eigenvalue in our case because $G_{\beta,m}$ is a graph of $2^{\ell} - 1$ vertices.

Graphs like $G_{\beta,m}$ were already considered by Chung in [14], but explicitly so on only with $\mathbb{F}_{2^b}$ replaced by $\mathbb{F}_p$ with $p$ prime and $\mathbb{F}_{2^\ell}$ by $\mathbb{F}_{p^\ell}$. We give the general argument for completeness. Chung used the following Lemma by Katz [29].

Lemma 35 ([29]). Let $q$ be a prime power and $t$ a nonnegative integer. Let $\psi : \mathbb{F}_{q^t}^* \to \mathbb{C}$ be a nontrivial multiplicative character, i.e., a homomorphism from the multiplicative group $\mathbb{F}_{q^t}^*$ to the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ such that $\psi(x) \neq 1$ for some $x \in \mathbb{F}_{q^t}^*$. Then for any $m$ with $\mathbb{F}_q(m) = \mathbb{F}_{q^t}$,

$$\left| \sum_{x \in \mathbb{F}_q} \psi(x + m) \right| \leq (t-1)\sqrt{q}.$$

To apply Katz’s lemma, let $m \in \mathcal{M}$, i.e., $\mathbb{F}_{2^b}(m) = \mathbb{F}_{2^\ell}$. The mapping $\psi : \mathbb{F}_{2^\ell}^* \to \mathbb{C}$ defined by $\psi(\alpha^a) = \theta^a$ is a nontrivial multiplicative character of $\mathbb{F}_{2^\ell}^*$ for every $(2^\ell - 1)$-th unit root $\theta \neq 1$. It follows that

$$\sum_{d \in D} \theta^d = \sum_{x' \in \mathbb{F}_{2^b}+m} \psi(x') = \sum_{x \in \mathbb{F}_{2^b}} \psi(x - m).$$

Since $\mathbb{F}_{2^b}(-m) = \mathbb{F}_{2^b}(m) = \mathbb{F}_{2^\ell}$, we obtain that $A_{\beta,m}$ apart from the eigenvalue $2^b$ has $2^\ell - 2$ eigenvalues which are upper-bounded by

$$\left(\frac{\ell}{b} - 1\right)2^{b/2} = \frac{k2^{b/2}}{b}.$$

The multiplicity of the eigenvalue $2^b$ is 1. This proves Lemma [32] and completes the first part of the proof of Theorem [28].

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b) Cardinality of $\mathcal{M}$: To complete the proof of Theorem 28, it remains to compute the cardinality of $\mathcal{M}$. An $m \in \mathcal{N}$ does not generate $\mathbb{F}_{2^\ell}$ over $\mathbb{F}_{2^b}$ (i.e., $\mathbb{F}_{2^b}(m) \neq \mathbb{F}_{2^\ell}$) if and only if it is contained in a strict subfield of $\mathbb{F}_{2^\ell}$ containing $\mathbb{F}_{2^b}$. By Lemma 27 every such subfield equals $\mathbb{F}_{2^t}$ for some multiple $t$ of $b$ which divides $\ell$. Therefore our goal is compute

$$\left| \bigcup_{t < \ell, b|t} (\mathcal{N} \cap \mathbb{F}_{2^t}) \right|,$$

where $a|b$ for positive integers $a, b$ means that $a$ divides $b$.

We denote the set of all multiples $t$ of $b$ which are strictly smaller than $\ell$ and divide $\ell$ by $\{t_1, \ldots, t_K\}$. We will need the following simple lemma.

**Lemma 36.** For any subset $\mathcal{I}$ of $\{1, \ldots, K\}$,

$$\dim \left( \mathcal{N} \cap \bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}} \right) = \dim \left( \bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}} \right) - b.$$

**Proof.** Recall the formula

$$\dim(\mathcal{V}_1 + \mathcal{V}_2) = \dim(\mathcal{V}_1) + \dim(\mathcal{V}_2) - \dim(\mathcal{V}_1 \cap \mathcal{V}_2).$$

Then

$$\dim \left( \mathcal{N} \cap \bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}} \right) \overset{(a)}{=} \dim(N) + \dim \left( \bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}} \right) - \dim \left( \mathcal{N} + \bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}} \right)$$

$$\overset{(b)}{=} k + \dim \left( \bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}} \right) - \ell$$

$$= \dim \left( \bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}} \right) - b,$$

where $(a)$ follows from (34) and $(b)$ from the definition of $\mathcal{N}$ and the fact that $\bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}}$ contains $\mathbb{F}_{2^b}$, which implies that the sum of $\mathcal{N}$ and $\bigcap_{i \in \mathcal{I}} \mathbb{F}_{2^{t_i}}$ equals $\mathbb{F}_{2^\ell}$. \qed

The second lemma applied in the proof of (31) is a statement about those elements of $\mathbb{F}_{2^\ell}$ which generate $\mathbb{F}_{2^b}$.

**Lemma 37.** The number of elements $m$ of $\mathbb{F}_{2^\ell}$ which are not contained in a strict subfield of $\mathbb{F}_{2^\ell}$ equals $(\ell/b)N_{2^b}(\ell/b)$.

**Proof.** The elements of $\mathbb{F}_{2^\ell}$ which generate $\mathbb{F}_{2^\ell}$ over $\mathbb{F}_{2^b}$ are exactly the zeros of the monic irreducible polynomials of degree $\ell/b$ with coefficients in $\mathbb{F}_{2^b}$ [31, Section 2.2]. The zero sets of
these polynomials are disjoint, and every monic irreducible polynomial of degree $d$ has exactly $d$ distinct zeros. Therefore the number of generators of $\mathbb{F}_2^\ell$ over $\mathbb{F}_2^b$ equals $\ell/b$ times the number of monic irreducible polynomials of degree $\ell/b$ over $\mathbb{F}_2^b$, which is given by $N_{2^b}(\ell/b)$ [31 Theorem 3.25].

Now the calculation of (33) goes as follows:

$$\sum_{t<\ell/b|\ell} (|\mathcal{N} \cap \mathbb{F}_2^t|) \overset{(a)}{=} \sum_{k=1}^{K} (-1)^{k+1} \sum_{I \subseteq \{1,\ldots,K\} : |I| = k} \mathcal{N} \cap \mathbb{F}_2^{|I|} \overset{(b)}{=} 2^{-b} \sum_{k=1}^{K} (-1)^{k+1} \sum_{I \subseteq \{1,\ldots,K\} : |I| = k} \mathbb{F}_2^{|I|} \overset{(c)}{=} 2^{-b} \left| \bigcup_{t<\ell/b|\ell} \mathbb{F}_2^t \right| \overset{(d)}{=} 2^{-b} \left( 2^\ell - \frac{\ell}{b} N_{2^b}(\ell/b) \right) = 2^k - \frac{\ell}{b} N_{2^b}(\ell/b) 2^{-b},$$

(35)

where (a) follows from the inclusion-exclusion formula, (b) is a consequence of Lemma [36], (c) again is the inclusion-exclusion formula, and (d) follows from Lemma [37]. Thus the cardinality of $\mathcal{M}$ equals $|\mathcal{N}| = 2^k$ minus (35), as claimed. The proof of Theorem [28] is complete.

VI. BRI Functions and Graphs

A. Characterization of BRI functions

The collection of graphs $G_{\beta,m}$ ($m \in \mathcal{N}$) in the previous section determined the function $\beta$ in such a way that $\beta(s,x) = m$ if $s$ is adjacent to $x$ in $G_{\beta,m}$. We will see that in a more general way, BRI functions can be characterized using this relation.

Recall that some basic graph-theoretic terms are defined in Appendix [A]. Additionally, we call a graph $G$ bipartite if its vertex set is the union of two disjoint sets $S$ and $X$ such that every edge in $G$ has one vertex in $S$ and one in $X$. The pair $(S,X)$ is called a bipartition of $G$. A bipartite graph $G$ with bipartition $(S,X)$ is called $(d_S, d_X)$-biregular if every element of $S$ has

$^6$Note that sometimes biregular graphs are defined without having to be bipartite.

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Fig. 2. The complete bipartite graph $K_{\{1,2,3\},\{4,5\}}$ is (2, 3)-biregular.

degree $d_S$ and every element of $\mathcal{X}$ has degree $d_X$. If $d_S = d_X = d$, then the graph is bipartite and $d$-regular.

The complete bipartite graph $K_{S, \mathcal{X}}$ with bipartition $(S, \mathcal{X})$ is the graph on $S \cup \mathcal{X}$ where every element of $S$ is adjacent to every element of $\mathcal{X}$, see Fig. 2. Clearly $K_{S, \mathcal{X}}$ is $(|\mathcal{X}|, |S|)$-biregular. Every function $f : S \times \mathcal{X} \to \mathcal{N}$ is equivalent to a decomposition $(G_m)_{m \in \mathcal{N}}$ of $K_{S, \mathcal{X}}$ into edge-disjoint subgraphs, where two vertices $s \in S$ and $x \in \mathcal{X}$ are adjacent in $G_m$ if $f(s, x) = m$. We say that $f$ is defined by the family $(G_m)_{m \in \mathcal{N}}$.

**Theorem 38.** A function $f : S \times \mathcal{X} \to \mathcal{N}$ is a BRI function with regularity set $\mathcal{M} \subset \mathcal{N}$ if and only if it is defined by a decomposition $(G_m)_{m \in \mathcal{N}}$ of the complete bipartite graph $K_{S, \mathcal{X}}$ into edge-disjoint subgraphs such that $G_m$ is $(d_S, d_X)$-biregular and connected\(^7\) for every $m \in \mathcal{M}$. In this case, if $\lambda_2(G_m)$ is the second-largest eigenvalue modulus of the adjacency matrix of $G_m$, then

$$\lambda_2(f, m) = \frac{\lambda_2(G_m)^2}{d_S d_X} < 1$$

for every $m \in \mathcal{M}$.

**Proof.** It is easy to see that the $S$- and $\mathcal{X}$-regularity of $f$ for every $m \in \mathcal{M}$ is equivalent to the $(d_S, d_X)$-biregularity of $G_m$. We can therefore concentrate on the equivalence of irreducibility of $f$ on $\mathcal{M}$ and the connectedness of $G_m$.

For any $m \in \mathcal{M}$, let $A_m$ be the adjacency matrix of $G_m$. Since $G_m$ is bipartite, it has the form

$$A_m = \begin{bmatrix} 0 & B_m \\ B_m^T & 0 \end{bmatrix}$$

\(^7\)See Appendix A for a definition of connectedness.
for an $X \times S$ matrix $B_m$. The rows of $B_m$ are indexed by the elements of $X$, the columns by $S$, and the $(x, s)$ entry $(B_m)_{x, s}$ of $B_m$ equals 1 if $s$ is adjacent to $x$ in $G_m$ and 0 otherwise. The square of $A_m$ equals

$$A_m^2 = \begin{bmatrix} B_mB_m^T & 0 \\ 0 & B_mB_m^T \end{bmatrix}.$$  

Clearly, every eigenvalue of $A_m^2$ also is an eigenvalue of both $B_mB_m^T$ and $B_mB_m^T$. Since $\text{rank}(A_m^2) = \text{rank}(B_mB_m^T) + \text{rank}(B_mB_m^T)$, $A_m^2$ has the same eigenvalues as both $B_mB_m^T$ and $B_mB_m^T$. It is well-known that the eigenvalue multiplicities of $B_mB_m^T$ and $B_mB_m^T$ coincide. Therefore the multiplicity of an eigenvalue for $A_m^2$ equals twice the multiplicity of this eigenvalue for $B_mB_m^T$.

The $(x, x')$ entry of $B_mB_m^T$ equals

$$(B_mB_m^T)_{x,x'} = \sum_{s \in S} (B_m)_{x,s}(B_m)^{x',s}$$

$$= \sum_{s \in S} 1\{f(s,x)=m\}1\{f(s,x')=m\}$$

$$= |\{s \in S : f_s(x) = f_s(x') = m\}|.$$

Thus $P_{f,m} = d_S^{-1}d_X^{-1}B_mB_m^T$, in particular, $P_{f,m}$ is positive semidefinite. That 1 is a simple eigenvalue of $P_{f,m}$ therefore is equivalent to $d_Sd_X$ being a simple eigenvalue of $B_mB_m^T$, and consequently a double eigenvalue of $A_m^2$. This is the minimal possible multiplicity for this eigenvalue, since the adjacency matrix of a $(d_S, d_X)$-biregular matrix always has eigenvalues $\pm \sqrt{d_Sd_X}$, and these are the maximal eigenvalues by absolute value (this follows from the Perron-Frobenius theorem [11 Theorem 2.2.1] using the fact that $\sqrt{d_Sd_X}$ has the positive eigenvector $w$ with $w(x) = 1$ for $x \in X$ and $w(s) = \sqrt{d_S/d_X}$ for $s \in S$).

It remains to show that $\pm \sqrt{d_Sd_X}$ being simple eigenvalues of $A_m$ is equivalent to $G_m$ being connected. $A_m$ can be written as a block diagonal matrix where every block is the adjacency matrix of one connected component of $G_m$. Clearly, every connected component of $G_m$ is a $(d_S, d_X)$-biregular graph and therefore has the eigenvalues $\pm \sqrt{d_Sd_X}$. It follows that $\pm \sqrt{d_Sd_X}$ are simple eigenvalues of $A_m$ if and only if $G_m$ is connected, and thus also that 1 is a simple eigenvalue of $P_{f,m}$ if and only if $G_m$ is connected.

Now assume that $f$ is a BRI function. Since the second-largest eigenvalue modulus of $B_mB_m^T$ equals the second-largest eigenvalue modulus of $A_m$ by the above considerations, the formula for $\lambda_2(f, m)$ follows. The final claim that $\lambda_2(f, m) < 1$ for every $m \in M$ if $f$ is a BRI function is
immediate from the fact that $P_{f,m}$ is positive semidefinite and the simplicity of 1 as an eigenvalue of $P_{f,m}$. □

From Theorem 38 it is possible to improve Theorem 19. If $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{N}$ is a BRI function with regularity set $\mathcal{M}$ defined by the decomposition $(G_{f,m})_{m \in \mathcal{N}}$, then $G_{f,m}$ is $(d_{S}, d_{X})$-biregular for every $m \in \mathcal{M}$. By a simple and well-known double-counting argument,

\[
d_{X}|\mathcal{X}| = \sum_{x \in \mathcal{X}} \sum_{s \in \mathcal{S}} 1\{s, x \text{ adjacent in } G_{f,m}\} = \sum_{s \in \mathcal{S}} \sum_{x \in \mathcal{X}} 1\{s, x \text{ adjacent in } G_{f,m}\} = d_{S}|\mathcal{S}|. \tag{36}
\]

It follows

**Theorem 39.** Let $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{N}$ be a BRI function with regularity set $\mathcal{M} \subset \mathcal{N}$ and let $S$ be uniformly distributed on $\mathcal{S}$. With the same notation as in Theorem 19, for every $m \in \mathcal{M}$ and $0 < \varepsilon < 1 - e^{-1}$,

\[
\mathbb{E}_{S}D((W \circ f_{S}^{-1})(m) \| \text{Unif}(\mathcal{X})W) \leq \frac{1}{\ln 2} \lambda_2(f, m) 2^{I_{\text{max}}^{m}(W)} + \varepsilon \log \frac{|\mathcal{X}|}{d_{S}} - (1 - \varepsilon) \log(1 - \varepsilon).
\]

**Remark 40.** The use of bipartite graphs in the analysis of the seeded coset function $\beta$ was not necessary due to its symmetry in $s$ and $x$. The adjacency matrices $A_{\beta,m}$ used in Lemma 32 correspond to the $B_{m}$ used here.

**B. Construction of BRI functions**

Theorem 38 allows us to find new BRI functions through their graph-theoretic characterization. In this section, new BRI functions are constructed which have more flexible rates than those derived from a seeded coset function.

In view of the upper bound provided in Theorem 39 whether a BRI function $f$ can provide security depends on the relation between $\log \lambda_2(f, m)$ and $I_{\text{max}}^{m}(W)$. Since $\lambda_2(f, m)$ only depends on $f$ and $I_{\text{max}}^{m}(W)$ only depends on $W$, the goal is to find BRI functions with small $\lambda_2(f, m)$ and large regularity set $\mathcal{M}$. The precise relation between the two terms which is necessary for security will be derived in the context of wiretap coding in the next section. As a hint as to what can be expected from the graph-theoretic side, recall that a $d$-regular graph always has maximal eigenvalue $d$, and $d$ is the largest eigenvalue modulus. If a $d$-regular graph is bipartite, it also has eigenvalue $-d$. Then by the Alon-Boppana bound [38], for every $\varepsilon > 0$ the second-largest eigenvalue modulus of every sufficiently large connected $d$-regular graph is at least $2\sqrt{d - 1} - \varepsilon$.
The analogous statement for \((d_S, d_X)\)-biregular graphs was shown by Feng and Li \cite{20}, i.e., for every \(\varepsilon > 0\), every sufficiently large connected \((d_S, d_X)\)-biregular graph has a second-largest eigenvalue modulus of at least \(\sqrt{d_S - 1} + \sqrt{d_X - 1} - \varepsilon\).

Ramanujan graphs are optimal with respect to the bounds of Al on-Boppana and Feng-Li, respectively. A \(d\)-regular graph \(G\) with adjacency matrix \(A\) is called a Ramanujan graph if every eigenvalue \(\mu\) of \(A\) satisfies \(\mu = \pm d\) or \(|\mu| \leq 2\sqrt{d - 1}\). A \((d_S, d_X)\)-biregular Ramanujan graph \(G\) has the property that every eigenvalue \(\mu\) of \(A\) satisfies \(\mu = \pm \sqrt{d_S d_X}\) or \(|\mu| \leq \sqrt{d_S - 1} + \sqrt{d_X - 1}\). Ramanujan graphs were first constructed by Lubotzky, Phillips and Sarnak \cite{34} and Margulis \cite{36}. Since then, other constructions have followed, see \cite{35} for hints to the literature.

There exist BRI functions \(f : S \times X \to \mathcal{N}\) defined by a graph family \((G_{f,m})_{m \in \mathcal{N}}\) such that \(G_{f,m}\) is a \((d_S, d_X)\)-biregular Ramanujan graph for every \(m \in \mathcal{M}\), and \(\mathcal{M}\) is exponentially large.

**Theorem 41.** For every pair \((d_S, d_X)\) with \(d_S, d_X \geq 3\), every positive integer \(k\) and disjoint sets \(S\) and \(X\) satisfying \(|S| = 2^k d_X\) and \(|X| = 2^k d_S\), there exists a decomposition of \(K_{S,X}\) into \(2^k\) edge-disjoint connected \((d_S, d_X)\)-biregular Ramanujan graphs.

**Proof.** See Subsection VI-C.

**Corollary 42.** For every pair \((d_S, d_X)\) with \(d_S, d_X \geq 3\) and every positive integer \(k\) there exists a BRI function \(f : S \times X \to \mathcal{M}\) with regularity set \(\mathcal{M}\) satisfying

1) \(|S| = 2^k d_X\) and \(|X| = 2^k d_S\) and \(|\mathcal{M}| = 2^k\),

2) \(\lambda_2(f, m) \leq (\sqrt{d_S - 1} + \sqrt{d_X - 1})^2 / (d_S d_X)\) for every \(m \in \mathcal{M}\).

Such a BRI function is called a Ramanujan BRI function.

**Proof.** For every \(m \in \mathcal{M} = \{0, 1\}^k\), let \(G_m\) be one of the \(2^k\) edge-disjoint connected \((d_S, d_X)\)-biregular bipartite Ramanujan graphs whose existence follows from Theorem 41. Define \(f : S \times X \to \mathcal{M}\) by setting \(f(s, x) = m\) if \(s\) is adjacent to \(x\) in \(G_m\). Then \(f\) is well-defined because the number of edges in \(K_{S,X}\) is \(|S||X| = 2^k d_S d_X\), and the number of pairs \((s, x)\) which are adjacent in some \(G_m\) equals \(|\mathcal{M}||S| d_S = 2^k d_S d_X\). The statement about \(\lambda_2(f, m)\) follows immediately from Theorem 38 and the definition of biregular Ramanujan graphs.

The divergence bound of Theorem 39 obtains the following form for a Ramanujan BRI function.
Corollary 43. For a Ramanujan BRI function \( f : S \times X \to \mathcal{M} \) as in Corollary 42,
\[
\mathbb{E}_S D((W \circ f_S^{-1})(m)\| \text{Unif}(\mathcal{X})W) \\
\leq \frac{(\sqrt{d_S - 1} + \sqrt{d_X - 1})^2}{d_Sd_X \ln 2} 2^{f_{\text{max}}(W)} + \varepsilon \log \frac{|X|}{d_S} - (1 - \varepsilon) \log(1 - \varepsilon).
\]

The proof of Theorem 41 is based on a result by Marcus, Spielman and Srivastava [35] about the existence of infinite families of biregular Ramanujan graphs for any given degree pair. This existence result, however, relies on the probabilistic method and is not constructive. This clearly is a disadvantage of Ramanujan BRI functions compared to the seeded coset function \( \beta \).

C. Proof of Theorem 41

Marcus, Spielman and Srivastava in [35] use 2-lifts of graphs to iteratively construct large Ramanujan graphs from smaller ones. Our addition is the observation that one obtains two edge-disjoint Ramanujan graphs on a common vertex set in every step.

a) 2-lifts of graphs: For any graph \( G \) with vertex set \( \mathcal{V}(G) \) and edge set \( \mathcal{E}(G) \), define a signing \( s \) to be a function \( s : \mathcal{E}(G) \to \{-1, 1\} \). We denote edges by their start and end vertices, and when we write \( e = (x, y) \in \mathcal{E} \), then also \( e = (y, x) \) since we only consider undirected graphs. In other words, \( s(x, y) = s(y, x) \) for all vertex pairs \((x, y)\).

Given the signing \( s \) one defines a graph \( \hat{G} \) called the 2-lift of \( G \) associated to \( s \) as follows. The vertex set of \( \hat{G} \) consists of two disjoint copies \( \mathcal{V}_0(G) \) and \( \mathcal{V}_1(G) \) of \( \mathcal{V}(G) \), so that every \( x \) in \( \mathcal{V}(G) \) corresponds to vertices \( x_0, x_1 \) in \( \mathcal{V}(\hat{G}) \). For any edge \( (x, y) \in \mathcal{E}(G) \), the edge set \( \mathcal{E}(\hat{G}) \) contains edges \((x_0, y_0)\) and \((x_1, y_1)\) if \( s(x, y) = 1 \) and \((x_0, y_1)\) and \((x_1, y_0)\) if \( s(x, y) = -1 \). Observe that if \( G \) is bipartite, then so is \( \hat{G} \), and if \( G \) is \((d_1, d_2)\)-biregular, then \( \hat{G} \) is \((d_1, d_2)\)-biregular as well.

The signed adjacency matrix of \( G \) corresponding to the signing \( s \) is the symmetric matrix \( A_s \) with rows and columns indexed by the vertices of \( G \), where the \((x, y)\) entry equals \( s(x, y) \) if \((x, y) \in \mathcal{E}(G)\) and 0 else. Bilu and Linial derived the following result for signed adjacency matrices.

Lemma 44 ([8], Lemma 3.1). Let \( A \) be the adjacency matrix of a graph \( G \) and \( A_s \) the signed adjacency matrix associated with a 2-lift \( \hat{G} \). Then every eigenvalue of \( A \) and every eigenvalue of \( A_s \) are eigenvalues of \( \hat{G} \). Furthermore, the multiplicity of each eigenvalue of \( \hat{G} \) is the sum of its multiplicities in \( A \) and \( A_s \).
This result implies that the negative $-s$ of every signing $s$ gives a 2-lift $\hat{G}_-$ with the same spectrum as the lift $\hat{G}$ obtained from $s$, and which in addition has no edges in common with $\hat{G}$.

**Lemma 45.** If $\hat{G}$ is the 2-lift of a bipartite graph $G$ associated to the signing $s$, then the 2-lift $\hat{G}_-$ of $G$ associated to the signing $-s$ has the same spectrum as $\hat{G}$ and is edge-disjoint from $\hat{G}$.

**Proof.** Denote by $\hat{A}$ the adjacency matrix of $\hat{G}$. Since $\hat{G}$ is bipartite, the spectrum of $\hat{A}$ is symmetric about 0 including multiplicities (see, e.g., [11, Proposition 3.4.1]). Since $G$ is bipartite, the spectrum of $A$ also is symmetric about 0. By Lemma 44, the spectrum of $A_s$ must therefore be symmetric about 0 as well. This implies that $A_{-s} = -A_s$ has the same spectrum as $A_s$, and again by Lemma 44, the adjacency matrix $\hat{A}_-$ of $\hat{G}_-$ has the same spectrum as $\hat{A}$.

That $\hat{G}_-$ is edge-disjoint from $\hat{G}$ is obvious from the definition of 2-lifts.

The other ingredient to our construction is the following result due to Marcus, Spielman and Srivastava.

**Lemma 46 ([35]).** For all $d_1, d_2 \geq 3$ and every connected $(d_1, d_2)$-biregular Ramanujan graph $G$ there exists a signing $s$ such that the 2-lift $\hat{G}$ of $G$ associated to this signing is connected, $(d_1, d_2)$-biregular and Ramanujan as well.

**Proof.** By Theorems 5.3 and 5.6 of [35]. The connectedness follows from the fact that $G$ is connected and that the eigenvalues of $\hat{G}$ which are not eigenvalues of $G$ are bounded by $\sqrt{d_S - 1} + \sqrt{d_X - 1}$, so that the eigenvalue $\sqrt{d_S d_X}$ still has multiplicity 1. This was shown to be equivalent to $\hat{G}$ being connected in the proof of Theorem 38.

**b) Construction of graph family:** Since the vertex set will change in the construction, we notationally decouple the degrees from the vertex set and just call them $d_1, d_2$, where $d_1$ corresponds to $d_S$ and $d_2$ to $d_X$. We start the construction with the complete bipartite graph $G_0 = K_{S_0, X_0}$ on the disjoint union of sets $S_0$ and $X_0$ with $|S_0| = d_2$ and $|X_0| = d_1$. The adjacency matrix of $G_0$ has rank 2 and nonzero eigenvalues $\pm \sqrt{d_1 d_2}$. Therefore $G_0$ is Ramanujan.

Recursively for every $1 \leq t \leq k$ and every sequence $\kappa_1, \ldots, \kappa_t \in \{-1, 1\}^t$ we define a graph $G_{\kappa_1, \ldots, \kappa_t}$ as follows. For any $t \geq 1$, given $\kappa_1, \ldots, \kappa_{t-1}$, we set $G_{\kappa_1, \ldots, \kappa_{t-1}, 1}$ to be any 2-lift of $G_{\kappa_1, \ldots, \kappa_{t-1}}$ which is connected and Ramanujan. Its existence follows from Lemma 46. If $G_{\kappa_1, \ldots, \kappa_{t-1}, 1}$ is the 2-lift associated to the signing $s_t$ of $G_{\kappa_1, \ldots, \kappa_{t-1}}$, then $G_{\kappa_1, \ldots, \kappa_{t-1}, -1}$ is defined to be the 2-lift of $G_{\kappa_1, \ldots, \kappa_{t-1}}$ associated to the signing $-s_t$ of $G_{\kappa_1, \ldots, \kappa_{t-1}}$. By Lemma 45, $G_{\kappa_1, \ldots, \kappa_{t-1}, -1}$
is connected and Ramanujan as well and edge-disjoint from \( G_{\kappa_1, \ldots, \kappa_{t-1}, 1} \). Clearly, the vertex set \( V_k \) of \( G_{\kappa_1, \ldots, \kappa_k} \) has a bipartition into a set of size \( 2^k d_1 \) and one of size \( 2^k d_2 \).

**Lemma 47.** Let \( k \geq 1 \) and let \((\kappa_1, \ldots, \kappa_k) \neq (\kappa'_1, \ldots, \kappa'_k) \in \{-1, 1\}^k \). Then \( G_{\kappa_1, \ldots, \kappa_k} \) and \( G_{\kappa'_1, \ldots, \kappa'_k} \) have disjoint edge sets.

**Proof.** We prove this by induction. The claim follows from Lemma 45 for \( k = 1 \).

Assume \( k > 1 \) and the claim has been proven for every \( 1 \leq t < k \). If \((\kappa_1, \ldots, \kappa_{k-1}) = (\kappa'_1, \ldots, \kappa'_{k-1})\), then the claim follows from Lemma 45. We may therefore assume that \((\kappa_1, \ldots, \kappa_{k-1}) \neq (\kappa'_1, \ldots, \kappa'_{k-1})\).

For any element \( x \) of the common vertex set \( V_k \) of \( G_{\kappa_1, \ldots, \kappa_k} \) and \( G_{\kappa'_1, \ldots, \kappa'_k} \), denote by \( \pi_k(x) \) the element of \( V_{k-1} \) of which \( x \) is a copy. By the definition of 2-lifts, two vertices \( x \) and \( y \) which are adjacent in \( G_{\kappa_1, \ldots, \kappa_k} \) satisfy that \( \pi_k(x) \) and \( \pi_k(y) \) are adjacent in \( G_{\kappa_1, \ldots, \kappa_{k-1}} \). However, the induction hypothesis and \((\kappa_1, \ldots, \kappa_{k-1}) \neq (\kappa'_1, \ldots, \kappa'_{k-1}) \) imply that \( \pi_k(x) \) and \( \pi_k(y) \) are not adjacent in \( G_{\kappa'_1, \ldots, \kappa'_{k-1}} \). Therefore \( x \) and \( y \) cannot be adjacent in \( G_{\kappa'_1, \ldots, \kappa'_k} \). \( \square \)

The proof of Theorem 41 is complete.

**VII. Asymptotic Consequences**

Next we derive some asymptotic consequences of the results obtained in the previous sections. We show in the first subsection that we can achieve the (ordinary) secrecy capacities of discrete and Gaussian wiretap channels with ordinary wiretap codes constructed from ECCs and BRI functions, given that suitable BRI functions exist. BRI functions with the required properties are shown to exist in the second subsection, and their properties are analyzed. Finally, for any BRI function \( f : S \times \mathcal{X} \to \mathcal{N} \) with regularity set \( \mathcal{M} \) and sufficiently large \( \mathcal{X} \), the trade-off between \( |\mathcal{M}| \) and \( \max_{m \in \mathcal{M}} \lambda_2(f, m) \) is characterized using the wiretap coding theorem.

**A. Achievable rates with modular BRI schemes**

Here we show that under the assumption that suitable BRI functions exist, the (ordinary) secrecy capacities of discrete and Gaussian wiretap channels are achievable with modular BRI schemes as well as with ordinary codes constructed from modular BRI schemes. We first recall known upper bounds on the \( \varepsilon \)-smooth max-information of the discrete and the Gaussian wiretap channel. Then we show that every ordinary-codes semantic security rate of the aforementioned wiretap channels is achievable by modular BRI schemes which guarantee semantic security.
In the final step, we construct ordinary wiretap codes from modular BRI schemes incurring a negligible rate loss and degradation of error and semantic security information leakage.

a) Discrete and Gaussian max-informations: To apply Theorem 39 to discrete and Gaussian wiretap channels, upper bounds on the respective $\varepsilon$-smooth max-information are needed. These are provided in the next two lemmas due to Tyagi and Vardy [40]. For the first one, we need to define $\delta$-typical sets. If $\delta > 0$, $A$ is a finite set, $P$ a probability distribution on $A$ and $n$ a positive integer, then the $\delta$-typical set $T_{P,\delta}^n$ of $P$ is defined as the set of $(a_1, \ldots, a_n) \in A^n$ satisfying

$$\frac{|\{|i : a_i = a\}| - P(a)}{n} \leq \delta$$

for every $a \in A$ and where $P(a) = 0$ implies $\{i : a_i = a\} = \emptyset$.

**Lemma 48** ([40], Lemma 5). Let $U : A \rightarrow Z$ be a discrete channel. For every $n$, let $X_n$ be a finite set and $\phi_n : X_n \rightarrow A^n$ any function.

1) Assume there exists a $\delta > 0$ and a probability distribution $P$ on $A$ such that $\phi_n(x) \in T_{P,\delta}^n$ for all $n$ and all $x \in X_n$. Then there exists a positive constant $c = c(|A|, |Z|)$ and a positive $\gamma_d = \gamma_d(\delta, |Z|)$ which tends to 0 as $\delta$ tends to 0 such that

$$\limsup_{n \rightarrow \infty} \frac{I_{\varepsilon_n}^{\max}(U^n \circ \phi_n)}{n} \leq I(P, U) + \gamma_d$$

for $\varepsilon_n = 2^{-n\delta^2}$.

2) If $\delta > 0$ and the $\phi_n$ are arbitrary, then

$$\limsup_{n \rightarrow \infty} \frac{I_{\varepsilon_n}^{\max}(U^n \circ \phi_n)}{n} \leq \max_P I(P, U) + \gamma_d$$

where $P$ varies over the probability distributions on $A$ and for the same $\varepsilon_n$ and $\gamma_d$ as in 1).

**Lemma 49** ([40], Lemma 6). Let $n$ be a positive integer, $\delta > 0$ and set $\varepsilon_n = e^{-n\delta^2/8}$. If $U : \mathbb{R} \rightarrow \mathbb{R}$ is a Gaussian channel with noise variance $\sigma^2$ and $\phi : X \rightarrow \mathbb{R}^n$ is any function satisfying

$$\|\phi(x)\|^2 \leq n \Gamma$$

for all $x \in X$, then there exists a $\gamma_G = \gamma_G(\delta)$ such that

$$\limsup_{n \rightarrow \infty} \frac{I_{\varepsilon_n}^{\max}(U^n \circ \phi)}{n} \leq \frac{1}{2} \log \left(1 + \frac{\Gamma}{\sigma^2}\right) + \gamma_G.$$
b) Basic achievability results for modular BRI schemes: The next two lemmas are proved under the assumption that BRI functions with the required properties exist. That this is indeed the case will be shown in the next subsection. It should also be noted that these lemmas only show the existence of seeded wiretap codes (given by modular BRI schemes).

Lemma 50. Let \((T : \mathcal{A} \to \mathcal{Y}, U : \mathcal{A} \to \mathcal{Z})\) be a discrete wiretap channel without input constraints as defined in Example [14] and \(P\) a probability distribution on \(\mathcal{A}\) satisfying \(I(P, T) > 0\) and \(I(P, U) < I(P, T)\). Choose \(\delta > 0\) and \(0 < \delta' < I(P, T)/(4 - 2\delta)\). Assume that there exists an infinite sequence \(n_1 < n_2 < \cdots\) of positive integers such that there exists a sequence \((f_i : \mathcal{S}_i \times \mathcal{X}_i \to \mathcal{N}_i)_{i=1}^{\infty}\) of BRI functions, with \(\mathcal{M}_i\) the regularity set of \(f_i\), satisfying

\[
I(P, T) - \frac{\delta'}{2} \geq \liminf_{i \to \infty} \frac{\log|\mathcal{M}_i|}{n_i} \geq I(P, T) - \delta',
\]

\[
\liminf_{i \to \infty} \frac{\log|\mathcal{M}_i|}{\log|\mathcal{X}_i|} \geq 1 - \frac{I(P, U)}{I(P, T)} - \delta,
\]

\[
\liminf_{i \to \infty} \frac{\min_{m \in \mathcal{M}_i} (-\log \lambda_2(f_i, m))}{\log|\mathcal{X}_i|} \geq \frac{I(P, U)}{I(P, T)} + \frac{\delta}{2}.
\]

Then there exists a sequence \(\{(\phi_i, \psi_i)\}_{i=1}^{\infty}\) of error-correcting codes such that the sequence of modular BRI schemes \(\{\Pi(f_i, \phi_i, \psi_i)\}_{i=1}^{\infty}\) achieves a semantic security rate of \((1 - \delta)I(P, T) - I(P, U) - \delta'\). Moreover, both \(e(\Pi(f_i, \phi_i, \psi_i))\) and \(L_{\text{sem}}(\Pi(f_i, \phi_i, \psi_i))\) tend to zero at exponential speed.

Proof. Recall the function \(\gamma_d\) defined in Lemma [48]. Choose a \(\delta_1 > 0\) satisfying \(\gamma_d(\delta_1, |\mathcal{Z}|) < \delta I(P, T)/4\). By [18, Theorem 10.2] and the left-hand side of (37) there exists a sequence of error-correcting codes \(\{(\phi_i : \mathcal{X}_i \to \mathcal{A}_i, \psi_i : \mathcal{Y}_i \to \mathcal{X}_i)\}_{i=1}^{\infty}\) which satisfies \(\phi_i(\mathcal{X}_i) \subset T_{P,\delta_1}\) and for which \(e(\phi_i, \psi_i)\) converges to 0 at exponential speed.

We show that the sequence \(\{\Pi(f_i, \phi_i, \psi_i)\}_{i=1}^{\infty}\) of modular BRI schemes has the claimed properties. That \(e(\Pi(f_i, \phi_i, \psi_i))\) tends to 0 exponentially fast is clear from the corresponding property of the ECC sequence and [15]. The rate of \(\Pi(f_i, \phi_i, \psi_i)\) satisfies

\[
\liminf_{i \to \infty} \frac{\log|\mathcal{M}_i|}{n_i} = \liminf_{i \to \infty} \frac{\log|\mathcal{M}_i|}{\log|\mathcal{X}_i|} \cdot \frac{\log|\mathcal{X}_i|}{n_i}
\]

\[
\geq (a) \left( \liminf_{i \to \infty} \frac{\log|\mathcal{M}_i|}{\log|\mathcal{X}_i|} \right) \left( \liminf_{i \to \infty} \frac{\log|\mathcal{X}_i|}{n_i} \right)
\]

\[
\geq (b) \left( 1 - \frac{I(P, U)}{I(P, T)} - \delta \right) \left( I(P, T) - \delta' \right)
\]

\[
\geq (1 - \delta)I(P, T) - I(P, U) - \delta',
\]
where (a) is due to the positivity of the sequences and (b) follows from (37) and (38).

It remains to check whether semantic security is achieved. Write \( W_i = \mathcal{U}^{n_i} \circ \phi_i \). For \( \varepsilon_i = 2^{-n_i c \delta_i^2} \), where \( c = c(|\mathcal{A}|, |\mathcal{Z}|) \) is the constant from Lemma 48 and \( m \in \mathcal{M}_i \)

\[
\limsup_{i \to \infty} \frac{I_{\text{max}}^\varepsilon(W_i) + \log \lambda_2(f_i, m)}{n_i} \leq \limsup_{i \to \infty} \frac{I_{\text{max}}^\varepsilon(W_i)}{n_i} + \limsup_{i \to \infty} \frac{\log \lambda_2(f_i, m)}{n_i}
\]

\[
(c) \leq I(P, U) + \gamma_d - \liminf_{i \to \infty} \frac{-\log \lambda_2(f_i, m) \log |\mathcal{X}_i|}{\log |\mathcal{X}_i|}
\]

\[
(d) \leq I(P, U) - \left( \liminf_{i \to \infty} \frac{-\log \lambda_2(f_i, m)}{\log |\mathcal{X}_i|} \right) \left( \liminf_{i \to \infty} \frac{\log |\mathcal{X}_i|}{n_i} \right) + \gamma_d
\]

\[
(e) \leq I(P, U) - \left( \frac{I(P, U)}{I(P, T)} \right) \delta + \gamma_d
\]

\[
(f) = \frac{-\delta I(P, T)}{4} + \gamma_d
\]

\[
(g) < 0,
\]

where (c) follows from Lemma 50 with the \( \gamma_d = \gamma_d(\delta_1, |\mathcal{Z}|) \) defined there, (d) is possible because the involved sequences are positive, (e) is due to (39) and (37), (f) to the choice of \( \delta' \) and (g) to the choice of \( \delta_1 \). Therefore \( \lambda_2(f_i, m) \exp(I_{\text{max}}^\varepsilon(W_i)) \) tends to zero at exponential speed. This together with the exponential decrease of \( \varepsilon_i \), Theorem 39 and Corollary 20 implies the exponential decrease of \( L_{\text{sem}}(\Pi(f, \phi, \psi)) \).

There exists a result analogous to Lemma 50 for Gaussian wiretap channels.

**Lemma 51.** Let \((T : \mathcal{A} \rightarrow \mathcal{Y}, U : \mathcal{A} \rightarrow \mathcal{Z})\) be a Gaussian wiretap channel as defined in Example 15 where \( \sigma_T^2 \) is the noise variance of \( T \) and \( \sigma_U^2 < \sigma_T^2 \) is the noise variance of \( U \) and \( \Gamma \) is the input power constraint. Let \( \delta > 0 \) and \( 0 < \delta' < \log(1 + \Gamma \sigma_T^2)/(4 - 2\delta) \). Assume there exists an infinite sequence \( n_1 < n_2 < \cdots \) of positive integers such that there exists a sequence \((f_i : \mathcal{S}_i \times \mathcal{A}_i \rightarrow \mathcal{N}_i)\) of BRI functions, with \( \mathcal{M}_i \) the regularity set of \( f_i \), satisfying

\[
\left( 1 + \frac{\Gamma}{\sigma_T^2} \right) - \frac{\delta'}{2} \geq \liminf_{n \to \infty} \frac{\log |\mathcal{X}_i|}{n_i} \geq \frac{1}{2} \log \left( 1 + \frac{\Gamma}{\sigma_T^2} \right) - \delta',
\]

\[
\liminf_{i \to \infty} \frac{\log |\mathcal{M}_i|}{\log |\mathcal{X}_i|} \geq 1 - \frac{\log(1 + \Gamma / \sigma_T^2)}{\log(1 + \Gamma / \sigma_T^2)} - \delta
\]

\[
\liminf_{i \to \infty} \min_{m \in \mathcal{M}_i} \frac{- \log \lambda_2(f_i, m)}{\log |\mathcal{X}_i|} \geq \frac{\log(1 + \Gamma / \sigma_T^2)}{\log(1 + \Gamma / \sigma_T^2)} + \delta/2
\]
Then the sequence of BRI-prefix schemes \((\Pi(f_i, \phi_i, \psi_i))\) achieves a semantic security rate of 
\[(1 - \delta) \log(1 + \Gamma/\sigma^2_U) - \log(1 + \Gamma/\sigma^2_U) - \delta'.\]
The sequences of errors \(e(\Pi(f_i, \phi_i, \psi_i))\) and semantic security information leakages \(L_{\text{sem}}(\Pi(f_i, \phi_i, \psi_i))\) go to zero at exponential speed.

Proof. The proof is analogous to that of Lemma 50.

Corollary 52. If modular BRI schemes as required in Lemmas 50 and 51 exist for every discrete and Gaussian wiretap channel, then the (ordinary) secrecy capacities (12) and (13) are achievable by modular BRI schemes providing exponentially decreasing errors and semantic security information leakages.

Proof. This is clear for the Gaussian wiretap channel. For discrete wiretap channels, it is important that Lemma 50 is valid for all discrete wiretap channels. Thus if \((T, U)\) is the given discrete wiretap channel and \(\rho : \mathcal{R} \rightarrow \mathcal{A}\) is a channel like in (12), then Lemma 50 can be applied to the composite discrete wiretap channel \((T \circ \rho, U \circ \rho)\). This shows that (12) is achievable by modular BRI schemes.

For the Gaussian wiretap channel, the ECCs which can be applied in Lemma 51 as components of a modular BRI scheme achieving (13) have to asymptotically achieve the capacity of the channel between sender and intended receiver (without security requirements). For the discrete memoryless wiretap channel, let the probability distribution \(P\) on \(\mathcal{A}\) be a maximizer in (12). Then any blocklength-\(n\) ECC \((\phi, \psi)\) applied in a modular BRI scheme whose rate is close to (12) has to satisfy two conditions: it should achieve a rate close to \(I(P, T)\), and \(I_{\text{max}}^n(U^n \circ \phi) \approx nI(P, U)\). The latter holds by Lemma 48 if \(\phi\) is a constant-composition encoder in the sense that \(\phi(x) \subset T^n_{P, \delta}\) for all \(x\) in the domain of \(\phi\), for some \(\delta > 0\).

c) From seeded to ordinary codes: Lemmas 50 and 51 are enough to show that the semantic security rates (12) and (13) are achievable using modular BRI schemes. However, without further modification, modular BRI schemes are seeded wiretap codes. Practically, this means a substantial rate loss if the seed is generated anew by the sender and then transmitted to the receiver before every use of the modular BRI scheme (recall that \(|S| \geq |M|\)). By reusing the seed a linear number of times, the rate loss can be made negligible while semantic security is maintained.

For a given wiretap channel \((T : \mathcal{A} \rightarrow \mathcal{Y}, U : \mathcal{A} \rightarrow \mathcal{Z})\), the ordinary wiretap codes considered in the next theorem have the following structure. Let \(f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{N}\) be a BRI function with
regularity set $\mathcal{M}$. For two blocklengths $n$ and $\tilde{n}$, let

$$
\tilde{\phi} : S \rightarrow A'_{\tilde{n}}, \quad \tilde{\psi} : Y_{\tilde{n}} \rightarrow S,
$$

$$
\phi : X \rightarrow A'_n, \quad \psi : Y_n \rightarrow X
$$

be two error-correcting codes. An ordinary wiretap code $(\xi, \zeta)$ is constructed from these components. For some number $N$, the message set of $(\xi, \zeta)$ is $\mathcal{M}^N$ and its blocklength is $\tilde{n} + Nn$. The code is initialized by transmitting the random seed $S$ to the intended receiver using $(\tilde{\phi}, \tilde{\psi})$. Then the modular BRI scheme $\Pi(f, \phi, \psi)$ is applied $N$ times with the same seed $S$ to transmit the $N$ components $m_1, \ldots, m_N$ of a message from $\mathcal{M}^N$, see Fig. 3.

**Theorem 53.** For both discrete and Gaussian wiretap channels, the rates (12) and (13), respectively, are semantic security rates and achievable with ordinary wiretap codes of the structure shown in Fig. 3 assuming BRI schemes exist which satisfy $|S| \leq |X|$ in addition to the requirements of Lemmas 50 and 51.

**Proof.** Let $(\xi, \zeta)$ be the ordinary wiretap code constructed before the statement of the theorem. Using $|S| \leq |X|$, we may without loss of generality assume that $\tilde{n} \leq n$. The rate of $(\xi, \zeta)$ thus equals

$$
\log |\mathcal{M}|^N = \frac{N}{N + 1} \log |\mathcal{M}|^N.
$$

Due to the union bound, its error satisfies

$$
e(\xi, \zeta) \leq e(\tilde{\phi}, \tilde{\psi}) + Ne(\Pi(f, \phi, \psi)).
$$

To bound $L_{sem}(\xi, \zeta)$, fix a seed realization $s$. Denote by $(M_1, \ldots, M_N) = M^N$ any random variable on $\mathcal{M}^N$ and let $Z^N = (Z_1, \ldots, Z_N)$ be generated by $M^N$ via $(U^n \circ \phi \circ f^{-1}_s)^N$. It follows...
from the memorylessness of the discrete and Gaussian wiretap channels that \( P_{Z^N|M^N,S=s} = P_{Z|M,S=s}^N \). Therefore

\[
I(M^N \land Z^N|s) = H(Z^N|s) - H(Z^N|M^N, s)
\]

\[
\leq \sum_{j=1}^{N} \left( H(Z_j|s) - H(Z_j|M_j, s) \right)
\]

\[
= \sum_{j=1}^{N} I(M_j \land Z_j|s),
\]

where the inequality is due to the chain rule of entropy [15, Theorem 8.6.2] and the fact that \( P_{Z^N|M^N,S=s} = P_{Z|M,S=s}^N \). Taking the expectation with respect to \( S \) and maximizing over all possible distributions of \( M^N \) implies \( L_{sem}(\xi, \zeta) \leq NL_{sem}(\Pi(f, \phi, \psi)) \).

Now choose BRI functions \( f_i : S_i \times X_i \rightarrow N_i \) with regularity sets \( M_i \) as required in Lemmas 50 and 51. It follows that there exist blocklength-\( n_i \) ECCs \( (\phi_i, \psi_i) \) such that both \( e(\Pi(f_i, \phi_i, \psi_i)) \) and \( L_{sem}(\Pi(f_i, \phi_i, \psi_i)) \) tend to zero exponentially. Also choose ECCs \( (\bar{\phi}_i, \bar{\psi}_i) \) of blocklengths \( \bar{n}_i \leq n_i \) satisfying that \( e(\bar{\phi}_i, \bar{\psi}_i) \) tends to zero at exponential speed. Let \( N_i = Cn_i \) for some positive \( C \). Denote the wiretap code constructed from these components as in Fig. 3 by \( (\xi_i, \zeta_i) \) (where \( \Pi(f_i, \phi_i, \psi_i) \) is used \( N_i \) times). Then by (43), the asymptotic rate achieved by the sequence \( (\xi_i, \zeta_i) \) equals

\[
\liminf_{i \to \infty} \frac{\log |M_i| N_i}{n + N_i n} \geq \liminf_{i \to \infty} \frac{\log |M_i|}{n}.
\]

The asymptotic error and semantic security information leakages of \( (\xi_i, \zeta_i) \) by (44) and (45) satisfy

\[
\limsup_{i \to \infty} e(\xi_i, \zeta_i) \leq \limsup_{i \to \infty} e(\bar{\phi}_i, \bar{\psi}_i) + N_i \limsup_{i \to \infty} e(\Pi(f_i, \phi_i, \psi_i)),
\]

\[
\limsup_{i \to \infty} L_{sem}(\xi_i, \zeta_i) \leq N_i L_{sem}(\Pi(f_i, \phi_i, \psi_i)).
\]

Thus the ordinary wiretap codes \( (\xi_i, \zeta_i) \) achieve an asymptotically vanishing error and semantic security information leakage while losing nothing in asymptotic rate.

**B. BRI functions which achieve capacity**

Here it is shown that BRI functions as required in Lemmas 50 and 51 indeed exist which additionally satisfy \( |S| \leq |\mathcal{X}| \). Another observation is that the maximum of the \( S \) and the \( \mathcal{X} \) degree have to grow exponentially in the blocklength for such BRI functions, and that these BRI functions are nearly Ramanujan.
**a) Seeded coset BRI functions:** BRI functions constructed from seeded coset functions as in Section [V] have limited rate flexibility. For \( i \geq 1 \), let \( \beta_i : \mathbb{F}_2^* \times \mathbb{F}_2^* \rightarrow \mathbb{F}_2 \) be a seeded coset BRI function with regularity set \( \mathcal{M}_i \) as in Theorem [28]. It holds that \( d_{S_i} = d_{X_i} = 2^{b_i} \) with \( b_i = \ell_i - k_i \), and \( b_i \) divides \( \ell_i \). The symmetry of \( \beta \) implies \( |S| = |X| \). By Lemma [30] the cardinality of the regularity set can be bounded as

\[
k_i + \log \left(1 - \frac{1}{2^{b_i/2}}\right) \leq \log |\mathcal{M}_i| \leq k_i.
\]

Since \( \ell_i \) has to tend to infinity, it holds that

\[
\liminf_{i \to \infty} \frac{\log |\mathcal{M}_i|}{\log |X_i|} = \liminf_{i \to \infty} \frac{k_i}{\ell_i} \tag{46}
\]

\[
\liminf_{i \to \infty} \min_{m \in \mathcal{M}_i} \left(- \log \lambda_2(\beta_i, m) \right) \leq \limsup_{i \to \infty} \frac{b_i}{\ell_i} \tag{47}
\]

Since the right-hand sides of (46) and (47) sum to 1, the achievable rate is directly coupled to the limit superior of \( b_i/\ell_i \). However, since \( b_i \) divides \( \ell_i \), (47) equals the inverse of a positive integer, say \( N \). Therefore asymptotic achievable rate have the form \( 1 - 1/N \). In particular, seeded coset BRI functions cannot provide security against eavesdroppers whose channel noise level is not much worse than that of the channel to the intended message recipient.

**b) Ramanujan BRI functions:** Let \( (T : A \rightarrow \mathcal{Y}, U : A \rightarrow \mathcal{Z}) \) be a discrete or Gaussian wiretap channel. Define

\[
t = I(P, T), \quad r = I(P, U) - I(P, T) \quad \text{in the discrete case,} \tag{48}
\]

\[
t = \frac{1}{2} \log(1 + \Gamma/\sigma_T^2), \quad r = \frac{\log(1 + \Gamma/\sigma_T^2)}{\log(1 + \Gamma/\sigma_T^2)} \quad \text{in the Gaussian case,} \tag{49}
\]

where \( P, \Gamma, \sigma_T^2 \) are as in Lemma [50] and [51] respectively. One may assume \( r \leq 1 \). For any \( \delta, \delta' > 0 \) choose \( d_i, k_i, n_i \) satisfying

\[
t - \delta' \leq \frac{k_i}{n_i} \leq t - \frac{\delta'}{2}, \quad \frac{r + 3\delta/4}{1 - r - 3\delta/4} < \frac{\log d_i}{k_i} < \frac{r + \delta}{1 - r - \delta}.
\]

For every \( i \), construct a Ramanujan BRI function with parameters \( k_i \) and \( d_{S_i} = d_{X_i} = d_i \) as in Theorem [41]. Then \( |S_i| = |X_i| \) and

\[
t - \delta' \geq \frac{\log |X_i|}{n_i} \geq t - \delta'.
\]

Moreover,

\[
\frac{\log |\mathcal{M}_i|}{\log |X_i|} = \frac{k_i}{k_i + \log d_i} \geq 1 - r - \delta
\]
and for $d_i$ and thus $k_i$ sufficiently large,

$$\min_{m \in \mathcal{M}_i} (-\log \lambda_2(f_i, m)) = \frac{2 \log d_i - \log(d_i - 1) - 2}{k_i + \log d_i} \geq r + \frac{\delta}{2}.$$ 

Therefore the conditions of Lemmas 50 and 51 are satisfied. Note that the Ramanujan graphs underlying the above construction of the Ramanujan BRI functions are regular with exponentially increasing degree.

c) Exponential growth of degrees: Here it is shown that for BRI functions satisfying the conditions of Lemmas 50 and 51, the maximum of the $S$-degree and the $X$-degree must grow exponentially. We need the precise form of the Feng-Li bound.

**Lemma 54** ([20]). If $G$ is a $(d_S, d_X)$-biregular graph with diameter $\Delta \geq 8$, then the second-largest eigenvalue modulus $\lambda_2(G)$ of $G$ satisfies

$$\lambda_2(G)^2 \geq d_S + d_X - 2 + 2\sqrt{(d_S - 1)(d_X - 1)} \left(1 - \frac{1}{\Delta - 1}\right).$$

If $G$ is a connected $(d_S, d_X)$-biregular graph with $d_S, d_X \geq 2$ and bipartition $(S, X)$, then it is well-known that its diameter $\Delta$ satisfies

$$\Delta \geq \frac{\log(|X| + |S|)}{\log(d_S) + \log(d_X)}. \quad (50)$$

This can be seen as follows: Starting from any vertex $x$ in $X$, say, every other vertex of $G$ can be reached by a path starting in $x$. Due to the bipartiteness of $G$, an upper bound on the number of vertices which can be reached from $x$ in $n$ steps is

$$\begin{cases} 
\frac{d_{S}^{n/2} d_{X}^{n/2}}{2} & \text{if } n \text{ even}, \\
\frac{d_{S}^{(n-1)/2} d_{X}^{(n+1)/2}}{2} & \text{if } n \text{ odd}.
\end{cases}$$

The expression if one starts in $s \in S$ is analogous. Then a lower bound on $\Delta$ is the smallest $n$ for which one of the above bounds is larger than $|X| + |S|$. A rough lower bound on this minimum is given by (50).

The claim of exponential growth of $\max(d_S, d_X)$ follows from Lemma 55 using the requirements of Lemmas 50 and 51 in particular, that $|X|$ grows exponentially. Lemma 55 itself has slightly weaker assumptions.

---

8The diameter of a graph is defined in Appendix A.
Lemma 55. Let \((f_i : S_i \times X_i \to N_i)_{i=1}^{\infty}\) be a sequence of BRI functions satisfying
\[
\liminf_{i \to \infty} \frac{\log |M_i|}{\log |X_i|} \geq r_1, \quad \liminf_{i \to \infty} \frac{\min_{m \in M_i} (-\log \lambda_2(f_i, m))}{\log |X_i|} \geq r_2,
\]
for numbers \(r_1, r_2 > 0\), where \(M_i\) is the regularity set of \(f_i\). We also assume that \(|X_i|\) increases to infinity. Then there exists a real number \(a > 0\) such that
\[
\liminf_{i \to \infty} \frac{\log \max(d_{S_i}, d_{X_i})}{\log |X_i|} \geq a.
\]

Proof. Let \(f_i\) be defined by the family \((G_{f_i, m})_{m \in M_i}\). We first note that \(d_S\) and \(d_X\) must be at least 2. Otherwise, say if \(d_X = 1\), then \(|S| = 1\) due to the connectedness of \(G_{f_i, m}\). Thus all \(G_{f_i, m}\) coincide for \(m \in M_i\), implying \(|M_i| = 1\), in contradiction to the assumption that \(|M_i|\) tends to infinity.

Let \(\Delta_{f_i, m}\) be the diameter of \(G_{f_i, m}\). If there exist infinitely many \(i\) and for every such \(i\) an \(m \in M_i\) satisfying \(\Delta_{f_i, m} \leq 8\), then for any \(0 < \delta < r_1\)
\[
\log \max(d_{S_i}, d_{X_i}) \geq \frac{\log d_{S_i} + \log d_{X_i}}{2} \geq \frac{\log (|S_i| + |X_i|)}{16} \geq \frac{\log (d_{S_i} + d_{X_i}) + \log |M_i|}{16} \geq \frac{r_1 - \delta}{16} \log |X_i|,
\]
where \((a)\) is due to the assumption \(\Delta_{f_i, m} \leq 8\) and \((50)\), \((b)\) comes from \((16)\) and \((c)\) from \((51)\) if \(i\) is sufficiently large.

If \(\Delta_{f_i, m} \geq 8\) for all \(i\) and \(m \in M_i\), then
\[
\lambda_2(f_i, m) \geq \frac{d_{S_i} + d_{X_i} - 2 + 2\sqrt{(d_{S_i} - 1)(d_{X_i} - 1)(1 - 1/(\Delta_{f_i, m} - 1))}}{d_{S_i} d_{X_i}} \geq \frac{1}{d_{X_i}} \left( 1 - \frac{1}{d_{S_i}} \right) + \frac{1}{d_{S_i}} \left( 1 - \frac{1}{d_{X_i}} \right) \geq \frac{1}{\max(d_{S_i}, d_{X_i})},
\]
where \((d)\) is due to Lemma \((54)\), \((e)\) to \(\Delta_{f_i, m} \geq 8\) and \((f)\) is due to \(d_S, d_X \geq 2\). One thus obtains for any \(0 < \delta < r_2\)
\[
\log \max(d_{S_i}, d_{X_i}) \geq \min_{m \in M_i} (-\log \lambda_2(f_i, m)) \geq (r_2 - \delta) \log |X_i|,
\]
where the first inequality is due to \((52)\) and the second inequality holds for sufficiently large \(i\) by \((51)\). This completes the proof. \(\square\)
d) Good BRI functions are nearly Ramanujan: The sufficient conditions of Lemmas 50 and 51 turn out to be strong restrictions on the BRI functions which can be applied.

**Lemma 56.** Let \( f : S \times \mathcal{X} \to \mathcal{N} \) be a BRI function with regularity set \( \mathcal{M} \). Assume there exists \( 0 < r < 1 \) and \( 0 < \delta < \min(r, 1-r) \) such that

\[
\frac{\log |\mathcal{M}|}{\log |\mathcal{X}|} \geq 1 - r - \delta, \quad \frac{\min_{m \in \mathcal{M}} (-\log \lambda_2(f, m))}{\log |\mathcal{X}|} \geq r + \frac{\delta}{2}.
\]

Then the largest nontrivial eigenvalue modulus of \( G_{f,m} \) is at most \( \sqrt{d_{\mathcal{X}} |\mathcal{X}|^{\delta/2}} \) for every \( m \in \mathcal{M} \).

**Proof.** For any \( m \in \mathcal{M} \)

\[
\log \lambda_2(f, m) \overset{(a)}{=} - \left( r + \frac{\delta}{2} \right) \log |\mathcal{X}|
\]

\[
= (1 - r - \delta) \log |\mathcal{X}| - \left( 1 - \frac{\delta}{2} \right) \log |\mathcal{X}|
\]

\[
\overset{(b)}{\leq} \log |\mathcal{M}| - \left( 1 - \frac{\delta}{2} \right) \log |\mathcal{X}|,
\]

where \((a)\) is due to the right-hand side of (53) and \((b)\) to the left-hand side of (53). In other words

\[
\lambda_2(f, m) \leq \frac{|\mathcal{M}|}{|\mathcal{X}|^{1-\delta/2}} = \frac{d_S |\mathcal{M}|}{d_S |\mathcal{X}|} |\mathcal{X}|^{\delta/2} \leq \frac{|\mathcal{X}|^{\delta/2}}{d_S},
\]

where the second inequality is due to (16). The largest nontrivial eigenvalue modulus of \( G_{f,m} \) thus equals \( \sqrt{d_S d_{\mathcal{X}} \lambda_2(f, m)} \leq \sqrt{d_{\mathcal{X}} |\mathcal{X}|^{\delta/2}}. \)

It is interesting to note that even though the graph degree in the BRI functions used in Lemmas 50 and 51 has to increase exponentially by Lemma 55, in contrast to the usual formulation of the Alon-Boppana and Feng-Li bounds where it is fixed, (nearly) Ramanujan graphs can occur in such a setting as well.

C. Eigenvalues vs. rate of BRI functions

The results of this section so far suggest that the larger \( |\mathcal{M}|/ \max_{m \in \mathcal{M}} \lambda_2(f, m) \) for a BRI function, the better rates are achievable for a wiretap channel. One can thus use the converse for the wiretap channel capacity theorem to derive an asymptotic upper bound on the fraction of \(|\mathcal{M}|\) and \( \max_{m \in \mathcal{M}} \lambda_2(f, m) \).
Lemma 57. Let \( f_i : S_i \times X_i \to N_i \) be a sequence of BRI functions, the regularity set of \( f_i \) denoted by \( M_i \), and with \( |X_i| \) increasing to infinity. Assume that there exists a \( 0 \leq r < 1 \) such that

\[
\lim_{i \to \infty} \frac{\log |M_i|}{\log |X_i|} = 1 - r.
\]

Then

\[
\limsup_{i \to \infty} \min_{m \in M_i} \frac{-\log \lambda_2(f_i, m)}{\log |X_i|} \leq r.
\]

Proof. Choose a sequence of BRI functions \( f_i \) like in the statement and suppose the claim were not true. By passing to a subsequence if necessary, one can without loss of generality assume that there exists a \( 0 < \delta < 1 - r \) such that

\[
\lim_{i \to \infty} \min_{m \in M_i} \frac{-\log \lambda_2(f_i, m)}{\log |X_i|} > r + \delta.
\]

Let \( A = \{0, 1\} \) and define \( T : A \to A \) to be the noiseless binary channel, where \( T(a|a) = 1 \) for \( a \in A \). Further, choose any \( p \) such that

\[
1 - r - \delta < h(p) < 1 - r,
\]

where \( h(p) = -p \log p - (1 - p) \log (1 - p) \) is the binary entropy of \( p \). Define the channel \( U : A \to A \) to be the binary symmetric channel with flipping probability \( p \), i.e.,

\[
U(a|a) = 1 - p, \quad U(1 - a|a) = p
\]

for all \( a \in A \). By [44], the secrecy capacity of \((T, U)\) is given by

\[
\max_P \left( I(P, T) - I(P, U) \right) = h(p),
\]

where the maximum on the left-hand side is over probability distributions on \( A \) and the uniform distribution \( \text{Unif}(A) \) is a maximizer. Thus this rate cannot be exceeded by any modular BRI scheme.

To come to a contradiction, we proceed similarly to the proof of Lemma 50. Choose \( n_i \) such that

\[
2^{n_i - 1} < |X_i| \leq 2^{n_i}.
\]

\( f_i \) determines a blocklength-\( n_i \) wiretap code with encoder \( \xi(\cdot|s, m) = f_i^{-1}(m) \) and decoder \( \zeta(s, y) = f_s(y) \) for \( y \in X_i \) and given seed \( s \). The error \( e(\xi_i, \zeta_i) \) of \((\xi_i, \zeta_i)\) equals 0.
Next we bound \( L_{\text{sem}}(\xi_i, \zeta_i) \). For some \( \delta' > 0 \) to be determined later, let \( \gamma_d = \gamma_d(\delta', |A|) \) and \( \varepsilon_i = 2^{-n_i c \delta'^2} \) as in Lemma 48. Then for sufficiently large \( i \) and any \( m \in \mathcal{M}_i \)

\[
\frac{I_{\max}(U^{n_i}) + \log \lambda_2(f_i, m)}{n_i} \\
\leq 1 - h(p) + \gamma_d - \frac{- \log \lambda_2(f_i, m) \log |X_i|}{\log |X_i|} n_i \\
\begin{aligned}
\leq & 1 - h(p) - (r + \delta) \frac{n_i - 1}{n_i} + \gamma_d \\
= & 1 - r - \delta - h(p) - \frac{r + \delta}{n_i} + \gamma_d,
\end{aligned}
\]

where (a) is due to the second part of Lemma 48 and the fact that \( \max_P I(P, U) = I(\text{Unif}(A), U) = 1 - h(p) \). (b) is due to (54) and (56). Therefore (57) is negative and bounded away from 0 for sufficiently large \( i \) and sufficiently small \( \delta' \) by (55), hence \( L_{\text{sem}}(\xi_i, \zeta_i) \) tends to zero at exponential speed.

The rate of \( (\xi_i, \zeta_i) \) is lower-bounded by

\[
\frac{\log |\mathcal{M}_i|}{n_i} \geq \frac{\log |\mathcal{M}_i|}{\log |X_i|} \frac{n_i - 1}{n_i} > h(p)
\]

for sufficiently large \( i \), where the first inequality is due to (56) and the second to the choice of \( p \) and the assumption on \( \log(|\mathcal{M}_i|)/\log(|X_i|) \).

Thus one obtains a sequence of wiretap codes which asymptotically achieves a semantic security rate strictly larger than \( h(p) \) on \( (T, U) \). This contradicts the fact that \( h(p) \) is the secrecy capacity of \( (T, U) \). Therefore the assumption that a subsequence satisfying (54) for any \( \delta > 0 \) exists must be wrong, and this completes the proof.

In view of the BRI functions constructed in the previous subsection, it is clear that the trade-off derived in the lemma is tight.

VIII. CONCLUSION

A. Summary

It is shown that asymptotically, every rate achievable with strong secrecy is also achievable with semantic security if the strong secrecy information leakage decreases sufficiently fast. If the decrease is slow, this continues to hold if semantic security is given a weaker formulation in terms of total variation distance instead of mutual information.
BRI functions and modular BRI schemes are introduced. A bound quantifying the effect of using the randomized inverse of a BRI function as a prefix to a channel on the secrecy of message transmission is derived. It is shown that the seeded coset function can be converted into a BRI function for suitable parameters. A general characterization of BRI functions in terms of edge-disjoint connected biregular graphs is derived. This characterization is applied to construct Ramanujan BRI functions.

It is shown that the secrecy rates of discrete and Gaussian wiretap channels are achievable by modular BRI schemes and by ordinary wiretap codes constructed from modular BRI schemes. For any sequence of BRI functions which in a modular BRI scheme achieves a positive asymptotic rate for the discrete or Gaussian wiretap channel, the maximum degrees of the corresponding graph families must grow exponentially in the blocklength. The wiretap coding theorem is applied to show a trade-off between the size of the regularity set of a BRI function and the largest associated second-largest eigenvalue modulus. The analysis of BRI functions also shows that they are UHFs on average.

B. Practical aspects

BRI functions provide a universal method of enhancing error-correcting codes in order to provide semantically-secure message transmission through a wiretap channel. By Theorem 39 it only depends on the \( \varepsilon \)-smooth max-information of a channel by how much a given BRI function can decrease the information leakage through this channel. Thus modular BRI schemes are robust towards the channel model. In contrast, the polar wiretap code constructed by Liu, Yan and Ling [33] is tailored to the Gaussian wiretap channel.

The polar wiretap code of [33] can be encoded and decoded efficiently. By efficiency, we mean that the computations can be done in time polynomial in the blocklength. A modular scheme is efficient if its components are, as already observed by Bellare and Tessaro [4] in the case of modular UHF schemes. The conditions which ECCs have to satisfy in the context of Lemmas 50 and 51 were discussed after Corollary 52. Efficient ECCs are known for the binary symmetric and the Gaussian channel. For a BRI function \( f : S \times X \rightarrow N \) with regularity set \( \mathcal{M} \), efficiency does not only mean the efficient computability of \( f(s, x) \) given \( s \) and \( x \), but also efficient invertibility, i.e., the efficient realization of the uniform distribution on the set \( f_s^{-1}(m) \) for any message \( m \) and seed \( s \). If \( f \) is defined by a graph family \( (G_{f,m})_{m \in N^r} \), then the computation of \( f(s, x) \) requires determining in which of the exponentially many graphs \( G_{f,m} \)
the arguments $s$ and $x$ are adjacent. Efficiency of this process would mean that this is possible in time polynomial in $\log |X|$. This is harder than computing $f^{-1}_s(m)$, since in this situation $G_{f,m}$ is determined by the message $m$.

The efficiency of the arithmetic seeded coset function $\beta^o$ was discussed in \cite{40}. A seeded coset BRI function needs four computational steps to be performed efficiently:

1) A mapping and its inverse from $\{0,1\}^{k-\delta}$ to $M = \{m \in \mathcal{N} : \mathbb{F}_{2^b}(m) = \mathbb{F}_{2^t}\}$, for sufficiently large $\delta$, to map binary sequences of data to the regularity set of $\beta$ and back again,

2) the uniform distribution on $\mathbb{F}_{2^b}$ for the calculation of $\beta^{-1}_s(m)$,

3) multiplication on $\mathbb{F}_{2^t}^\ast$ and

4) finding the $m \in \mathcal{N}$ which satisfies $x \in \mathbb{F}_{2t} + m$, for the calculation of $\beta(s,x)$.

We do not go into the discussion of these computational aspects.

Ramanujan BRI functions so far cannot be constructed efficiently. This would be desirable even if the seeded coset BRI functions could be shown to be efficient, since the latter are only available for a restrictive set of parameters. However, explicit constructions of Ramanujan graphs known so far are also less flexible than the method of 2-lifts, and for a BRI function one needs a whole edge-disjoint family of such graphs. One should expect the construction of such families to get easier if one backs off a little bit from the best possible security rates and looks for good edge-disjoint families of non-Ramanujan expanders. Some methods of constructing expanders are presented in \cite{28}.

**Appendix A**

**Graphs: Definitions and Facts**

In this appendix, some definitions and facts from graph theory are collected as a reference.

**Definition 58.** A graph is a pair $G = (\mathcal{X}, \mathcal{E})$, where $\mathcal{X}$ is a finite set and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1$ is a subset of $\{(x,x) : x \in \mathcal{X}\}$ and $\mathcal{E}_2$ is a subset of the set of 2-element subsets of $\mathcal{X}$. The elements of $\mathcal{X}$ are called vertices (singular: vertex) and the elements of $\mathcal{E}$ are called edges. An element of $\mathcal{E}_1$ is also called a loop. Two elements $x, x' \in \mathcal{X}$ are called adjacent if either $x = x'$ and $(x,x) \in \mathcal{E}_1$ or $x \neq x'$ and $\{x,x'\} \in \mathcal{E}_2$. If $\mathcal{E} = \mathcal{E}_2$ (i.e., $G$ has no loops), then $G$ is called simple.
Interpretation: A graph $G$ can be drawn if it is not too big. Vertices are represented by dots and edges by lines connecting these dots. Clearly, an edge which is a loop becomes a loop in the drawing. See Figure 4.

Definition 59. A subgraph of a graph $G = (\mathcal{X}, \mathcal{E})$ is any graph $G' = (\mathcal{X}', \mathcal{E}')$ satisfying $\mathcal{X}' \subset \mathcal{X}$ and $\mathcal{E}' \subset \mathcal{E}$.

Definition 60. For every vertex $x$ of the graph $G = (\mathcal{X}, \mathcal{E})$, its degree $\deg(x)$ is defined as the number of vertices to which $x$ is adjacent. If $\deg(x) = d$ is constant in $x$, then $G$ is called $d$-regular. In this case, $d$ is called the degree of $G$.

Definition 61. The adjacency matrix of the graph $G = (\mathcal{X}, \mathcal{E})$ is an $\mathcal{X} \times \mathcal{X}$ matrix whose $(x, x')$ entry equals 1 if $x$ and $x'$ are adjacent and 0 else.

Every adjacency matrix is symmetric. Therefore it can be diagonalized and has real eigenvalues.

Definition 62. Let $G$ be a graph.

1) A sequence $x_1, \ldots, x_n$ of vertices of $G$ is called a path if $x_\xi$ is adjacent to $x_{\xi+1}$ for $0 \leq \xi \leq n - 1$. In this case, $x_1$ and $x_n$ are called the endvertices of the path.
2) A pair of vertices $x, x'$ is called connected if there exists a path with endvertices $x$ and $x'$.
3) $G$ is called connected if every pair of vertices is connected.
4) The distance of two vertices $x, x'$ is the length of any shortest path connecting $x$ and $x'$.
   If $x, x'$ are not connected, then their distance is set to $+\infty$.
5) The diameter of $G$ is the maximal distance between any two vertices of $G$.

Connectedness is an equivalence relation on the vertex set. The equivalence classes are called...
connected components. Between any two vertices contained in the same connected component, there exists a path connecting the two vertices. If the two vertices are not contained in the same connected component, no such path exists.

**APPENDIX B**

**PROOFS OF LEMMAS**

**Proof of Lemma 22**

\[
D(f \parallel g) = Z_f \left( D \left( \frac{f}{Z_f} \bigg\| \frac{g}{Z_g} \right) + \log \frac{Z_f}{Z_g} \right)
\]

\[
\leq Z_f \left( D_2 \left( \frac{f}{Z_f} \bigg\| \frac{g}{Z_g} \right) + \log \frac{Z_f}{Z_g} \right)
\]

\[
= Z_f \left( D_2(f \parallel g) - 2 \log Z_f + \log Z_g + \log Z_f - \log Z_g \right)
\]

\[
= Z_f \left( D_2(f \parallel g) - \log Z_f \right)
\]

where the inequality is due to the fact that \( D(\cdot \parallel \cdot) \leq D_2(\cdot \parallel \cdot) \) for probability densities [42]. □

**Proof of Lemma 24** The all-one vector \( \mathbf{1} \) is an eigenvector to the eigenvalue 1 of \( P \), in other words,

\[ P \mathbf{1} = \mathbf{1}. \]  (58)

We define the scalar product \( \langle \cdot, \cdot \rangle_X \) on \( \mathbb{R}^X \) by

\[
\langle u, v \rangle_X = \frac{1}{|X|} \sum_x u(x)v(x).
\]

The norm induced by \( \langle \cdot, \cdot \rangle_X \) is denoted by \( \| \cdot \|_X \), in particular, \( \langle w, w \rangle_X = \| w \|_X^2 \). Note that

\[
\| \mathbf{1} \|_X = 1. \]  (59)

For any \( w \in \mathbb{R}^X \) write

\[
\bar{w} = \frac{1}{|X|} \sum_x w(x) = \langle w, \mathbf{1} \rangle_X.
\]
Then
\[
\begin{align*}
w^\top Pw &= \frac{|\mathcal{X}|}{|\mathcal{X}|} \sum_x (Pw)(x)w(x) \\
&= |\mathcal{X}| \langle Pw, w \rangle_{\mathcal{X}} \\
&= |\mathcal{X}| \left[ \langle P(w - \overline{w}1), w - \overline{w}1 \rangle_{\mathcal{X}} + \overline{w} \langle Pw, 1 \rangle_{\mathcal{X}} + \overline{w} \langle P1, w \rangle_{\mathcal{X}} - \overline{w}^2 \langle 1, 1 \rangle_{\mathcal{X}} \right] \\
&\leq (a) |\mathcal{X}| \left[ \lambda_2 \|w - \overline{w}1\|_{\mathcal{X}}^2 + \overline{w} \langle w, P1 \rangle_{\mathcal{X}} + \overline{w} \langle 1, w \rangle_{\mathcal{X}} - \overline{w}^2 \right] \\
&\equiv (b) |\mathcal{X}| \left[ \lambda_2 \|w\|_{\mathcal{X}}^2 - 2\lambda_2 \overline{w} \langle w, 1 \rangle_{\mathcal{X}} + \lambda_2 \overline{w}^2 \langle 1, 1 \rangle_{\mathcal{X}} + \overline{w} \langle w, 1 \rangle_{\mathcal{X}} + \overline{w} - \overline{w}^2 \right] \\
&\equiv (c) |\mathcal{X}| \left[ \lambda_2 \|w\|_{\mathcal{X}}^2 - 2\lambda_2 \overline{w}^2 + \lambda_2 \overline{w}^2 + \overline{w}^2 \right] \\
&= |\mathcal{X}| \left[ \lambda_2 \|w\|_{\mathcal{X}}^2 + (1 - \lambda_2) \overline{w}^2 \right] \\
&= \lambda_2 \sum_{x \in \mathcal{X}} w(x)^2 + (1 - \lambda_2)|\mathcal{X}| \left( \frac{1}{|\mathcal{X}|} \sum_x w(x) \right)^2 \\
&\leq \lambda_2 \sum_{x \in \mathcal{X}} w(x)^2 + \frac{1}{|\mathcal{X}|} \left( \sum_x w(x) \right)^2 
\end{align*}
\]
where (a) is due to the fact that \(w - \overline{w}1\) is orthogonal to the eigenspace of the eigenvector 1 and the variational characterization of eigenvalues, to the symmetry of \(P\) and (58) and (59). In (b), the binomial formula for \(\|\cdot\|^2\) was used, together with (58). (c) is a final application of (59). □

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