W-infinity Algebras from Noncommutative Chern-Simons Theory

A. PINZUL AND A. STERN

Department of Physics, University of Alabama,
Tuscaloosa, Alabama 35487, USA

ABSTRACT

We examine Chern-Simons theory written on a noncommutative plane with a ‘hole’, and show that the algebra of observables is a nonlinear deformation of the $\mathcal{W}_\infty$ algebra. The deformation depends on the level (the coefficient in the Chern-Simons action), and the non-commutativity parameter, which were identified, respectively, with the inverse filling fraction (minus one) and the inverse density in a recent description of the fractional quantum Hall effect. We remark on the quantization of our algebra. The results are sensitive to the choice of ordering in the Gauss law.
An effective hydrodynamic description of the fractional quantum Hall effect (FQHE) in terms of noncommutative Chern-Simons theory was recently proposed in [1]. It connected the area preserving diffeomorphism symmetry of an incompressible Hall fluid and with that present in first order noncommutative Chern-Simons theory. The symmetry is generated by the $w_\infty$ algebra, and it therefore should be present in both contexts. Within the context of Hall fluid, the role of the $w_\infty$ algebra and its quantization to $W_\infty$ (or $W_{1+\infty}$) algebras[2] has been discussed in a number of papers.[3],[4],[5],[6] Here we show how to recover the $w_\infty$ algebra (and deformations thereof) from noncommutative Chern-Simons theory.

It is well known how to write Chern-Simons theory on the noncommutative plane $\times$ time[7], but as with the commutative version, the theory is empty. This can be rectified with the introduction of sources. In the commutative theory, sources can be introduced by punching holes in the plane. The noncommutative analogue of a punctured plane was developed in [8]. A ‘hole’ was introduced by removing low lying states from the Hilbert space. Derivations could be defined, and by utilizing deformed coherent states[9],[10], it was shown how to recover the punctured plane in the commutative limit. Here we write down Chern-Simons theory on such noncommutative spaces. Imposing the necessary boundary conditions on the fields at the ‘hole’, we find the resulting gauge invariant observables on phase space and show that their Poisson bracket algebra is a two parameter nonlinear deformation of the $w_\infty$ algebra. The two parameters are the ‘level’ $k$ (having integer values $\times h$) and the noncommutativity parameter $\Theta_0$. In [1],[11] the integer values were identified with inverse filling fractions $1/\nu$ (minus one), and $\Theta_0$ was identified with the inverse density (in the co-moving frame). We find that the limit $k \to \infty$ gives the contraction to linear deformations $W_\infty$, while $\Theta_0 \to 0$ gives the contraction to $w_\infty$. We thereby recover the symmetry algebra for first order noncommutative Chern-Simons theory. Our results are in contrast to previous quantum mechanical descriptions of the FQHE in terms of the linear deformations $W_\infty$ (or $W_{1+\infty}$) [3],[4],[5],[6]. At the end of this letter, we remark on the quantization of our algebra, where we thus introduce a third deformation parameter $\gamma$, which contains quantum corrections and is sensitive to the choice of ordering in the Gauss law. Initial results suggest that $\gamma$ appears as an overall factor.

We first consider Chern-Simons theory written on the noncommutative plane $\times$ time, which we denote by $\mathcal{M}_F^{(0)} \times \mathbb{R}$. The noncommutative space $\mathcal{M}_F^{(0)}$ is generated by some operator $z$ and its hermitian conjugate $\bar{z}$, satisfying $[z, \bar{z}] = \Theta_0$, $\Theta_0$ being a c-number. $z$ and $\bar{z}$ have infinite dimensional representations. We denote the vector space on which they act by $H^{(0)}$, with basis vectors $|n > \in H^{(0)}$, $n = 0, 1, 2, \ldots$. The space $\mathcal{M}_F^{(0)}$ admits derivations. Derivatives on $\mathcal{M}_F^{(0)}$ will be denoted by $\Delta$ and $\bar{\Delta}$, and assumed to commute

$$[\Delta, \bar{\Delta}] = 0,$$  \hspace{1cm} (1)

Acting on a function $\Phi$ of $z$ and $\bar{z}$,

$$\Delta \Phi = -i[p, \Phi], \hspace{1cm} \bar{\Delta} \Phi = -i[\bar{p}, \Phi].$$ \hspace{1cm} (2)
The operator \( p \) can be taken to be \(-i\Theta_0^{-1}\bar{z}\), with its hermitian conjugate \( \bar{p} = i\Theta_0^{-1}z \). Then
\[
[p, \bar{p}] = -\Theta_0^{-1},
\]
which is consistent with (1).

The degrees of freedom for noncommutative Chern-Simons theory can be taken to be a conjugate pair of potentials \( A \) and \( \bar{A} \). They are functions on \( \mathcal{M}_F^{(0)} \times \mathbb{R} \). Under gauge transformations:
\[
\begin{align*}
A & \rightarrow iU^\dagger \Delta U + U^\dagger AU \\
\bar{A} & \rightarrow iU^\dagger \bar{\Delta} U + U^\dagger \bar{A} U,
\end{align*}
\]
where \( U \) is unitary function. It is convenient to introduce \( X = p + A \) and \( \bar{X} = \bar{p} + \bar{A} \) for they transform covariantly: \( X \rightarrow U^\dagger XU \), \( \bar{X} \rightarrow U^\dagger \bar{X}U \). The field strength is
\[
F = i\Delta \bar{A} - i\bar{\Delta} A + [A, \bar{A}]
= [X, \bar{X}] - [p, \bar{p}],
\]
which then also transforms covariantly. The Chern-Simons Lagrangian can be written
\[
L_{cs} = k\Theta_0 \text{Tr} \left( \frac{i}{2}(\dot{A}\bar{A} - A\dot{\bar{A}}) + A_0F \right)
= k \text{Tr} \left( \frac{i}{2}\Theta_0 \left(D_tX\bar{X} - XD_t\bar{X} \right) + A_0 \right),
\]
where \( D_tX = \dot{X} - i[A_0, X] \), the dot denotes a time derivative and \( \text{Tr} \) is the trace over basis states in \( H^{(0)} \). We have dropped total time derivatives and equated terms related by cyclic permutation in going from the first line to the second in (6). \( A_0 \) plays the role of a Lagrange multiplier. It is assumed to be hermitian and gauge transform as \( A_0 \rightarrow iU^\dagger \dot{U} + U^\dagger A_0 U \), and so \( D_tX \) and its hermitian conjugate \( D_t\bar{X} \) transform covariantly.

For gauge invariance one can assume that \( U \) and \( U^\dagger \) act as the identity on \( |n> \) as \( n \rightarrow \infty \). This corresponds to the requirement in the commutative theory that gauge transformations vanish at spatial infinity. Applying the cyclic property of the trace, \( \text{Tr}D_tX\bar{X} \) and \( \text{Tr}XD_t\bar{X} \) are gauge invariant. Concerning the remaining term in (6), the condition of invariance was shown to lead to level quantization.[12], [13] More precisely, level quantization follows from the demand that \( \exp i \int_\mathbb{R} dt L_{cs} \) is invariant under gauge transformations satisfying
\[
U \rightarrow 1, \quad \text{as} \quad t \rightarrow \pm \infty
\]
The quantization condition is \( k = \text{integer} \times \hbar \), and the integer was identified in [1],[11] with the inverse of the filling fraction \( \nu \) (minus one).

As with Chern-Simons theory on commutative \( \mathbb{R}^3 \), the above theory is empty. This is easily seen in the canonical formalism. The time derivative terms in (6) define the Poisson structure.
The phase space is spanned by matrix elements $\chi_n^m = \langle n|X|m \rangle$ and $\bar{\chi}_n^m = \langle n|\bar{X}|m \rangle$, with Poisson brackets
\[
\{\chi_n^m, \bar{\chi}_r^s\} = -\frac{i}{k\Theta_0} \delta_n^m \delta_r^s.
\] (8)
The remaining terms in the trace in (6) give the Gauss law constraints
\[
G_n^m = \langle n|[X, \bar{X}]|m \rangle > + \Theta_0^{-1} \delta_n^m = \chi_n^r \chi_r^m - \bar{\chi}_n^r \chi_r^m + \Theta_0^{-1} \delta_n^m \approx 0.
\] (9)
They are first class, and from
\[
ike^{\Theta_0} \{\chi_n^m, G_r^s\} = \chi_n^r \delta_n^s - \chi_n^s \delta_r^m
\]
\[
ike^{\Theta_0} \{\bar{\chi}_n^m, G_r^s\} = \bar{\chi}_n^r \delta_n^s - \bar{\chi}_n^s \delta_r^m
\]
These generators act on a Hilbert space $H^{(n_0)}$ which is an infinite dimensional subset of $H^{(0)}$. $H^{(n_0)}$ is spanned by basis vectors $|n \rangle$, $n = n_0, n_0 + 1, n_0 + 2, ..., n_0$ being some positive integer. We have thus put a ‘hole’ in the Hilbert space $H^{(0)}$. $\mathcal{M}_F^{(n_0)}$ was shown to admit derivations. We once again denote derivatives by $\Delta$ and $\bar{\Delta}$, and assume they commute. Now introduce a function $\Phi$ on $\mathcal{M}_F^{(n_0)}$. It can be nonvanishing only on vectors belonging to $H^{(n_0)}$. As before, we assume (2), so we need (3). An explicit expression for $p$ and $\bar{p}$ in terms of $z$ and $\bar{z}$ was given in [8], but it is not necessary here. (3) shows that $p$ and $\bar{p}$ are proportional to the usual raising and lowering operators, respectively. (We take $\Theta_0 > 0$.) But then $\bar{p}$ takes vectors out of $H^{(n_0)}$, while $p$ takes (bra) vectors out of the dual space. For derivations to be well defined we then must impose ‘boundary conditions’ on fields at the ‘hole’. $\bar{\Delta}\Phi$ is well defined on $H^{(n_0)}$ when $\langle n_0|\Phi|n \rangle = 0$, while $\Delta\Phi$ is well defined on the dual space when $\langle n|\Phi|n_0 \rangle = 0$. Stronger boundary conditions are needed for higher derivatives to be defined. On the other hand, for Chern-Simons theory one only needs first order derivatives. More specifically, the derivatives $\Delta\bar{A}$ and $\bar{\Delta}A$ should be well defined as they appear in the field strength (5), and so our boundary conditions are:
\[
\langle n_0|A|n \rangle = \langle n|\bar{A}|n_0 \rangle = 0, \quad \forall \ n \geq n_0
\] (10)
Since $p$ and $\bar{p}$ are proportional to raising and lowering operators, respectively, we can also write
\[
\langle n_0|X|n \rangle = \langle n|\bar{X}|n_0 \rangle = 0, \quad \forall \ n \geq n_0
\] (11)
In order that these boundary conditions are preserved under gauge transformations we need the unitary matrices to satisfy
\[
\langle n_0|U^\dagger|a \rangle = \langle a|U|n_0 \rangle = 0, \quad \forall \ a, b, ... \geq n_0 + 1
\] (12)
We regard $A$ and $\bar{A}$ - and not $X$ and $\bar{X}$ - as the fundamental configuration space variables.
Since gauge transformations are thereby restricted, not all phase space degrees of freedom in Chern-Simons theory can be gauged away, as was the case previously.

For the Chern-Simons Lagrangian we once again assume (6), only now the trace is over a basis in $H^{(n_0)}$. Returning to the Hamiltonian formulation, and now imposing the boundary conditions (11), one is left with the following phase space variables:

$$\chi_a^b = \Theta_0 < a|X|b > \quad \bar{\chi}_a^b = \Theta_0 < a|\bar{X}|b > ,$$
$$\psi_a = < a|X|n_0 > \quad \bar{\psi}^a = < n_0|\bar{X}|a > ,$$

where again $a, b, ... > n_0$, and we have rescaled $\chi$ and $\bar{\chi}$ in order to later obtain the desired commutative limit. The nonzero Poisson brackets are

$$\{\chi_a^b, \chi_c^d\} = -\frac{i}{k \Theta_0} \delta^b_c \delta^d_a \quad \{\psi_a, \bar{\psi}^b\} = -\frac{i}{k \Theta_0} \delta^b_a .$$ (13)

For later convenience we also re-scale the Gauss law constraints:

$$G_a^b = \Theta_0^2 < a|[X, \bar{X}]|b > + \Theta_0 \delta_a^b = \chi_a^c \chi_c^b - \bar{\chi}_a^c \bar{\chi}_c^b + \Theta_0 \delta_a^b + \Theta_0^2 \psi_a \bar{\psi}^b \approx 0$$ (14)

They generate gauge transformations which are consistent with (12):

$$ik\Theta_0^{-1} \{\chi_a^b, G_c^d\} = \chi_c^d \delta_a^b - \chi_a^d \delta_c^b$$
$$ik\Theta_0^{-1} \{\bar{\chi}_a^b, G_c^d\} = \bar{\chi}_c^d \delta_a^b - \bar{\chi}_a^d \delta_c^b$$
$$ik\Theta_0^{-1} \{\psi_a, G_b^c\} = \psi_b \delta_a^c$$
$$ik\Theta_0^{-1} \{\bar{\psi}^a, G_b^c\} = -\bar{\psi}^c \delta_b^a$$ (15)

From a counting argument alone the variables $\chi_a^b$ and $\bar{\chi}_a^b$ can be gauged away, leaving only $\psi_a$ and $\bar{\psi}^a$. But the latter are not gauge invariant. Instead they transform as a vector and conjugate vector, while $\chi_a^b$ and $\bar{\chi}_a^b$ transform as tensors. Then we can construct gauge invariant observables of the form $\bar{\psi} \mathcal{A} \psi$, where $\mathcal{A}$ denotes polynomial functions in the fields $\chi$ and $\bar{\chi}$. It remains to compute their Poisson bracket algebra. For this we can use

$$\{\bar{\psi} \mathcal{A} \psi, \bar{\psi} \mathcal{B} \psi\} = -\frac{i}{k \Theta_0} \bar{\psi} [\mathcal{A}, \mathcal{B}] \psi + \bar{\psi} \bar{\psi} \{ \mathcal{A}, \mathcal{B} \} \psi \psi ,$$ (16)

where $[ , ]$ is the commutator bracket. The labels 1 and 2 indicate two separate vector spaces, where for example $\mathcal{A}$ and $\mathcal{B}$ are the tensor products $\mathcal{A} \otimes \mathbb{1}$ and $\mathbb{1} \otimes \mathcal{B}$, respectively, $\mathbb{1}$ being the unit operator. If we denote the right hand side of (16) by $\bar{\psi} \mathcal{O}_{\mathcal{A}, \mathcal{B}} \psi = -\bar{\psi} \mathcal{O}_{\mathcal{B}, \mathcal{A}} \psi$, then

$$\{\bar{\psi} \mathcal{A} \psi, \bar{\psi} \mathcal{B} \psi\} = \bar{\psi} \mathcal{O}_{\mathcal{A}, \mathcal{B}} \mathcal{C} \psi + \bar{\psi} \mathcal{B} \mathcal{O}_{\mathcal{A}, \mathcal{C}} \psi$$

$^1$For simplicity, we shall assume that there are no further constraints $G_a^{n_0} \approx G_a^{n_0} \approx G_a^{n_0} \approx 0$. For this we may set the $n_0^{th}$ row and column of the Lagrange multiplier $A_0$ equal to zero.
We note that $O_{\mathcal{A}, \mathcal{B}}$ can depend on $\psi$ and $\bar{\psi}$, so that $\bar{\psi}\mathcal{A}\psi$ do not generate a linear algebra. Furthermore, from the Gauss law constraint (14), observables obtained via a reordering of the $\chi$ and $\bar{\chi}$ factors in $\mathcal{A}$ form an equivalence class. We fix a gauge by choosing the following ordering

$$M_{(\alpha, \beta)} = -k \bar{\psi}(\bar{\chi})^{\alpha} \chi^{\beta} \psi$$

From (16), $M_{(0, 0)}$ is a central charge. Examples of nonzero Poisson brackets are:

$$\{M_{(0, 1)}, M_{(1, 0)}\} = -iM_{(0, 0)}$$
$$\{M_{(0, 1)}, M_{(1, 1)}\} = -iM_{(0, 1)}$$
$$\{M_{(1, 1)}, M_{(1, 0)}\} = -iM_{(1, 0)}$$
$$\{M_{(0, 1)}, M_{(2, 0)}\} = -2iM_{(1, 0)}$$
$$\{M_{(0, 2)}, M_{(1, 0)}\} = -2iM_{(0, 1)}$$
$$\{M_{(0, 2)}, M_{(1, 1)}\} = -2iM_{(0, 2)}$$
$$\{M_{(1, 1)}, M_{(2, 0)}\} = -2iM_{(2, 0)}$$
$$\{M_{(0, 2)}, M_{(2, 0)}\} = -4iM_{(1, 1)} + 2i\Theta_0 M_{(0, 0)} - \frac{2i}{k} \Theta_0^2 M_{(0, 0)}^2$$

where we used the Gauss law constraint (14) to do reordering. The last example shows that the algebra is nonlinear.

Although we don’t have a closed form expression for the algebra, there are some familiar contractions. The commutative limit is $\Theta_0 \rightarrow 0$. Both the Poisson bracket and the commutator of any two polynomials $\mathcal{A}$ and $\mathcal{B}$ of $\chi$ and $\bar{\chi}$ are linear in $\Theta_0$ to leading order. For the former the result follows from (13), while for the latter the result follows from the Gauss law constraint (14). Then the second term in (16) can be dropped. Moreover, at the lowest nontrivial order we can represent $\chi$ and $\bar{\chi}$ by commuting numbers $\zeta$ and $\bar{\zeta}$, respectively, and replace the commutator bracket $[ , ]$ by $\{ , \}$, with

$$\{\mathcal{A}, \mathcal{B}\} = \Theta_0 \left( \frac{\partial \mathcal{A}}{\partial \zeta} \frac{\partial \mathcal{B}}{\partial \bar{\zeta}} - \frac{\partial \mathcal{A}}{\partial \bar{\zeta}} \frac{\partial \mathcal{B}}{\partial \zeta} \right)$$

Then in the limit

$$\{M_{(\alpha, \beta)}, M_{(\rho, \sigma)}\} \rightarrow -ik\Theta_0^{-1} \bar{\psi} \{\bar{\zeta}^{\alpha} \zeta^{\beta}, \bar{\zeta}^{\rho} \zeta^{\sigma}\} \psi = -i(\beta \rho - \alpha \sigma) M_{(\alpha + \beta - 1, \rho + \sigma - 1)}$$

This is the ‘classical’ $w_\infty$ algebra which is associated with area preserving diffeomorphisms. (Actually, as in [4] we get only a subalgebra of the $w_\infty$ algebra since no negative values for $\alpha, \beta, ...$ are allowed, restricting to the nonsingular area preserving diffeomorphisms of the plane.) On the other hand, away from the limit of vanishing $\Theta_0$ we get a deformation of the $w_\infty$ algebra. It is in fact a two-parameter nonlinear deformation, the other parameter being the level $k$, which parameterizes the nonlinearity. From (13), any $n$-th order term in the Poisson bracket algebra goes like $k^{1-n}$. Since $k$ is identified with the inverse of the filling fraction $\nu$ in the
FQHE, we get a different algebra for different values of $\nu$. The nonlinear terms tend to zero for large $k$ (small $\nu$) and then we approach the linear ‘quantum’ $W_{\infty}$ algebra.\(^4\)

The role of linear deformations of the $w_{\infty}$ algebra in the quantum mechanical description of the FQHE has been discussed in a number of papers.[3],[4],[5] They make up the edge variables for the system on a finite size domain. In contrast, for arbitrary $k$ and $\Theta_0$, we have obtained a nonlinear deformation of the $w_{\infty}$ algebra in the candidate effective theory for the FQHE. (Nonlinear deformations of the $w_{\infty}$ algebra were obtained previously in different contexts [14], and from [15] such deformations are unique.) A further distinction is that our deformation appears already at the classical level of noncommutative Chern-Simons theory. It is not clear whether $M_{(\alpha,\beta)}$ are ‘edge’ variables. Presumably they live in the vicinity of the ‘puncture’ in the commutative limit. A careful analysis of the continuum limit is required to verify this.

There has been recent work on finite dimensional matrix models with the hope of describing a quantum Hall droplet.[11],[16],[17] In this regard, instead of working with the infinite dimensional Hilbert space $H^{(n_0)}$, as we did above, we can repeat the analysis for its complement $\bar{H}^{(n_0)}$, which is finite dimensional. $\bar{H}^{(n_0)}$ is spanned by basis vectors $|n\rangle, n = 0, 1, 2, ..., n_0 - 1$. We denote the corresponding noncommuting space by $\mathcal{M}_{F}^{(n_0)}$, and its derivatives once again by $\Delta$ and $\bar{\Delta}$, which are assumed to commute. A function $\Phi$ on $\mathcal{M}_{F}^{(n_0)}$ is defined to be nonvanishing only on $\bar{H}^{(n_0)}$. Acting on $\Phi$, $\Delta$ and $\bar{\Delta}$ are written as in (2). This implies (3), so again $p$ and $\bar{p}$ are proportional to the usual raising and lowering operators, respectively. Now $p$ takes vectors out of $H^{(n_0)}$, while $\bar{p}$ takes (bra) vectors out of the dual space. The necessary boundary conditions are $< n_0 - 1|\Phi|n >= 0$ for $\Delta\Phi$ to be well defined, and $< n|\Phi|n_0 - 1 >= 0$ for $\bar{\Delta}\Phi$ to be well defined. Then for Chern-Simons theory, one has: $< n_0 - 1|A|n >=< n|A|n_0 - 1 >= 0 , \forall n \leq n_0 - 1$, with analogous conditions on the matrix elements of $X$ and $\bar{X}$. In order that these boundary conditions are preserved under gauge transformations: $< n_0 - 1|U|a >=< a|U|n_0 - 1 >= 0$, where here $a, b, ... = 0, 1, 2, ..., n_0 - 2$. Gauge transformations are generated by (14), with $a, b, ... = 0, 1, 2, ..., n_0 - 2$. Now all but $2(n_0 - 1)$ phase space variables can be gauged away. The remaining gauge invariant variables can again be written as (17) (where $\psi_a =< a|X|n_0 - 1 >$ and $\bar{\psi}^a =< n_0 - 1|\bar{X}|a >$), although for finite $n_0$ they are not all independent degrees of freedom. For example, the trace of (14) gives $\psi_{a}\bar{\psi}^a = (n_0 - 1)\Theta_0^{-1}$. In the commutative limit we should let $n_0 \rightarrow \infty$ (in addition to $\Theta_0 \rightarrow 0$), so we again recover the $w_{\infty}$ algebra.

The phase space description of the finite dimensional system described above is in agreement with that of Polychronakos[11]. Although we don’t introduce vector degrees of freedom in the Lagrangian as in [11], analogous phase space degrees of freedom $\psi_a$ and $\bar{\psi}^a$ appear at the Hamiltonian level. A potential term is introduced in [11] and the resulting dynamics is claimed to be equivalent the Calogero system.

Additional deformations of our nonlinear $W_{\infty}$ algebra can occur after quantization, with the possible inclusion of central terms. One quantization program is to replace the original

\(^4\)More accurately, due to the absence of negative values for $\alpha, \beta, ...$, we approach a subalgebra of $W_{1+\infty}$, which includes the so-called ‘wedge’ algebra $W_{\Lambda}$ [4].
phase variables $\chi^b_a$, $\bar{\chi}^b_a$, $\psi^a$, $\bar{\psi}^a$ by the quantum operators $\hat{\chi}^b_a$, $\hat{\bar{\chi}}^b_a$, $\hat{\psi}^a$, $\hat{\bar{\psi}}^a$, respectively, and Poisson brackets (13) by the commutation relations:

$$ [\hat{\chi}^b_a, \hat{\bar{\chi}}^d_c] = \frac{\hbar \Theta_0}{k} \delta^a_d \delta^b_c, \quad [\hat{\psi}^a, \hat{\bar{\psi}}^b] = \frac{\hbar}{k \Theta_0} \delta^b_a. \quad (20) $$

thereby introducing the additional deformation parameter $\hbar$. Next we choose the following ordering for the Gauss law operators

$$ \hat{G}^b_a = \hat{\chi}^c_a \hat{\bar{\chi}}^b_c - \hat{\bar{\chi}}^c_a \hat{\chi}^b_c + \Theta_0 \delta^b_a + \Theta_0^2 \hat{\psi}^a \hat{\bar{\psi}}^b. \quad (21) $$

It can be checked that their commutator algebra closes, and so we can consistently impose that they vanish on physical states. The operator analogues $\hat{M}^{(\alpha, \beta)}$ of the gauge invariant quantities (17) can be constructed, and their algebra computed. As in the classical case, $\hat{M}^{(0,0)}$ is central.

Upon computing the quantum analogues of Poisson brackets (18), we get

$$ [\hat{M}^{(0,1)}, \hat{M}^{(1,0)}] = \gamma \hat{M}^{(0,0)} $$

$$ [\hat{M}^{(0,1)}, \hat{M}^{(1,1)}] = \gamma \hat{M}^{(0,1)} $$

$$ [\hat{M}^{(1,1)}, \hat{M}^{(1,0)}] = \gamma \hat{M}^{(1,0)} $$

$$ [\hat{M}^{(0,1)}, \hat{M}^{(2,0)}] = 2\gamma \hat{M}^{(1,0)} $$

$$ [\hat{M}^{(0,2)}, \hat{M}^{(1,0)}] = 2\gamma \hat{M}^{(0,1)} $$

$$ [\hat{M}^{(0,2)}, \hat{M}^{(1,1)}] = 2\gamma \hat{M}^{(0,2)} $$

$$ [\hat{M}^{(1,1)}, \hat{M}^{(2,0)}] = 2\gamma \hat{M}^{(2,0)} $$

$$ [\hat{M}^{(0,2)}, \hat{M}^{(2,0)}] = \gamma \left( 4\hat{M}^{(1,1)} - 2\Theta_0 \hat{M}^{(0,0)} + \frac{2}{k} \Theta_0^2 \hat{M}^{(2,0)} \right) \quad (22) $$

where the $\gamma$ factor contains $\hbar$ corrections. From (22) it appears that $\gamma$ may be an overall factor in the quantum commutators. For the choice of ordering in (21), one gets $\gamma = \hbar(1 + \hbar/k)$. On the other hand, if $\hat{\psi}^a$ and $\hat{\bar{\psi}}^b$ are switched in the last term of (21), then $\gamma = \hbar$. $\gamma$ can be re-expressed in terms of the filling fraction $\nu$. According to [11], $\nu^{-1} = 1 + k/\hbar$.

The task of writing down a closed form expression for the full quantum algebra appears to be nontrivial. After this hurdle, one is next faced with the task of finding unitary representations. Although representation theory for linear deformations of the $w_\infty$ algebra is known[18], the same cannot be said for the nonlinear deformations. If the quantization program can be successfully carried out it should offer a nice test for the noncommutative Chern-Simons description of the FQHE. Lastly, we remark that the exhibition of the noncommutative $W_\infty$ algebra could be helpful in recovering the commutative limit. In that limit, we should somehow recover Chern-Simons theory on a domain with a boundary. (An attempt along these lines was made in [16].) The latter is known to have all its degrees of freedom at the spatial boundary. These are the so called ‘edge states’, which are associated with a conformal algebra, or more generally a $w_\infty$ algebra. Thus our gauge invariant observables should get mapped to the edge states in the limit.

REFERENCES
[1] L. Susskind, hep-th/0101029.
[2] For a review, see X. Shen, Int. J. Mod. Phys. A 7, 6953 (1992).
[3] S. Iso, D. Karabali and B. Sakita, Phys. Lett. B 296, 143 (1992).
[4] A. Cappelli, C. A. Trugenberger and G. R. Zemba, Nucl. Phys. B 396, 465 (1993); Phys. Rev. Lett. 72, 1902 (1994); Annals Phys. 246, 86 (1996); Annals Phys. 246, 86 (1996); Nucl. Phys. B 448, 470 (1995); Nucl. Phys. Proc. Suppl. 45A, 112 (1996); Int. J. Mod. Phys. A 12, 1101 (1997).
[5] J. Martinez and M. Stone, Int. J. Mod. Phys. B 7, 4389 (1993).
[6] M. Huerta, Int. J. Mod. Phys. A 15, 915 (2000); G. Cristofano, G. Maiella and V. Marotta, Mod. Phys. Lett. A 15, 547 (2000).
[7] C-S. Chu, Nucl.Phys. B580 352 (2000); A.A. Bichl, J.M. Grimstrup, V. Putz, M. Schweda, JHEP 0007 046 (2000); J. Kluson, Phys.Lett. B505 243 (2001); M.M. Sheikh-Jabbari, Phys.Lett. B510 247 (2001); N. Grandi, G.A. Silva, Phys.Lett. B507 345 (2001).
[8] A. Pinzul and A. Stern, JHEP 0203, 039 (2002) [hep-th/0112220].
[9] V.I. Man’ko, G. Marmo, E.C.G. Sudarshan, F. Zaccaria, Physica Scripta 55, 528 (1997).
[10] G. Alexanian, A. Pinzul and A. Stern, Nucl. Phys. B600, 531 (2001), hep-th/0010187.
[11] A. P. Polychronakos, JHEP 0011 008 (2000); 0104 011 (2001); hep-th/0106011.
[12] V. P. Nair and A. P. Polychronakos, Phys. Rev. Lett. 87, 030403 (2001).
[13] D. Bak, K. M. Lee and J. H. Park, Phys. Rev. Lett. 87, 030402 (2001).
[14] I. Bakas and E. Kiritsis, Int. J. Mod. Phys. A 7, 55 (1992); F. Yu and Y.S. Wu, Phys. Rev. Lett. 68, 2996 (1992).
[15] F. Yu and Y.S. Wu, Nucl.Phys. B373 713 (1992).
[16] A. Pinzul and A. Stern, JHEP 0111, 023 (2001) [hep-th/0107179].
[17] A. R. Lugo, Mod. Phys. Lett. A 17, 141 (2002).
[18] V. Kac and A. Radul, Commun. Math. Phys. 157, 429 (1993); hep-th/9512150; E. Frenkel, V. Kac, A. Radul and W. Q. Wang, Commun. Math. Phys. 170, 337 (1995).