A NOTE ON THE GAGLIARDO-NIRENBERG INEQUALITY IN A BOUNDED DOMAIN

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Abstract. The classical Gagliardo-Nirenberg inequality was established in \( \mathbb{R}^n \). An extension to a bounded domain was given by Gagliardo in 1959. In this note, we present a simple proof of this result and prove a new Gagliardo-Nirenberg inequality in a bounded Lipschitz domain.

1. Introduction

In this short note, we prove two kinds of the Gagliardo-Nirenberg inequalities in a bounded domain based on the Gagliardo-Nirenberg inequality in \( \mathbb{R}^n \). First, we introduce some notations. Denote by \( |E| \) the Lebesgue measure of \( E \subset \mathbb{R}^n \). Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( u : \Omega \to \mathbb{R} \). For \( 0 < \alpha \leq 1 \), define

\[
[u]_{C^\alpha(\overline{\Omega})} = \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.
\]

For \( 0 < p \leq \infty \), denote

\[
\|u\|_{p,\Omega} = \left( \int_{\Omega} |u|^p \right)^{1/p}.
\]

For \( -\infty < p \leq +\infty \), set

\[
|u|_{p,\Omega} = \begin{cases} 
\|u\|_{p,\Omega} & p > 0; \\
\|\nabla^k u\|_{\infty,\Omega} & p < 0, -n/p = k \in \mathbb{N}; \\
[\nabla^k u]_{C^\alpha(\overline{\Omega})} & p < 0, -n/p = k + \alpha, k \in \mathbb{N}, 0 < \alpha < 1.
\end{cases}
\]

If \( |u|_{p,\Omega} < \infty \), we say that \( u \in L^p(\Omega) \). For simplicity, we also write \( \|u\|_p \) and \( |u|_p \) instead of \( \|u\|_{p,\Omega} \) and \( |u|_{p,\Omega} \) if \( \Omega \) is clearly understood.

The famous Gagliardo-Nirenberg inequality was first proved by Gagliardo [7] and Nirenberg [13] independently restricted in Sobolev spaces \( W^{k,p}(\mathbb{R}^n) \) where \( k \in \mathbb{N} \) (i.e. nonnegative integers) and \( 1 \leq p \leq \infty \). More precisely (see [10, Theorem 12.87]),

Theorem 1.1. Let \( 1 \leq q, r \leq +\infty, k, j \in \mathbb{N} \) with \( j < k \), \( j/k \leq \theta \leq 1 \) and \( p \in \mathbb{R} \) such that

\[
\frac{n}{p} - j = \theta \left( \frac{n}{r} - k \right) + (1 - \theta) \frac{n}{q}.
\]

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Then there exists a constant $C$ depending only on $n, k, q, r$ and $\theta$ such that for any $u \in W^{k,r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,
\[
|\nabla^j u|_p \leq C\|\nabla^k u\|_r^\theta \|u\|_q^{1-\theta}
\]
with the exception that if $1 < r < +\infty$ and $k - j - n/r \in \mathbb{N}$, we must take $j/k \leq \theta < 1$.

Remark 1.2. The relation (1.1) is a necessary requirement of the scaling consideration in (1.2).

The Gagliardo-Nirenberg inequality has been extended greatly in different directions, such as inequalities in other function spaces [8, 9, 12, 17], involving fractional derivatives [3, 4], aim at best constants [5, 11], in bounded domains[2, 3, 4] and manifolds [1, 6] ect.

Our main results are the following:

**Theorem 1.3.** Let $\Omega$ be a bounded Lipschitz domain. Assume that $1 \leq q, r \leq +\infty$, $k, j \in \mathbb{N}$ with $j < k$, $j/k \leq \theta \leq 1$ and $p \in \mathbb{R}$ such that
\[
\frac{n}{p} - j = \theta \left(\frac{n}{r} - k\right) + (1 - \theta)\frac{n}{q}.
\]
Then there exists a constant $C$ depending only on $n, k, q, r, \theta$ and $\Omega$ such that for any $u \in W^{k,r}(\Omega) \cap L^q(\Omega)$,
\[
|\nabla^j u|_p \leq C\|\nabla^k u\|_r^\theta \|u\|_q^{1-\theta} + C\|u\|_q
\]
with the exception that if $1 < r < +\infty$ and $k - j - n/r \in \mathbb{N}$, we must take $j/k \leq \theta < 1$.

Remark 1.4. Theorem 1.3 is proved in [7] and stated in [14]. Here, we give a simple proof.

**Theorem 1.5.** Let $\Omega$ be a bounded Lipschitz domain. Assume that $1 \leq q, r \leq +\infty$, $k, j \in \mathbb{N}$ with $j < k$, $j/k \leq \theta \leq 1$ and $p \in \mathbb{R}$ such that
\[
\frac{n}{q} > \frac{n}{r} - k
\]
and
\[
\frac{n}{p} - j = \theta \left(\frac{n}{r} - k\right) + (1 - \theta)\frac{n}{q}.
\]
Then for any $E \subset \Omega$ with $|E| > 0$, there exists a constant $C$ depending only on $n, k, q, r, \theta, E$ and $\Omega$ such that for any $u \in W^{k,r}(\Omega)$,
\[
|\nabla^j u - \nabla^j P_E|_p \leq C\|\nabla^k u\|_r^\theta \|u - P_E\|_q^{1-\theta}
\]
with the exception that if $1 < r < +\infty$ and $k - j - n/r \in \mathbb{N}$, we must take $j/k \leq \theta < 1$. The $P_E$ is the unique polynomial of degree $k - 1$ such that for any multi-index $0 \leq |\gamma| \leq k - 1$,
\[
\int_E \nabla^\gamma u = \int_E \nabla^\gamma P_E.
\]

Remark 1.6. The (1.5) is a technical assumption and we don’t know whether it can be removed or not.
2. Proof of the main results

To prove Theorem 1.3, we need the following two lemmas. The first lemma (see [15] and [16, Theorem 5, Chapter IV]) deals with extension from a bounded domain to the whole space. This allows us to use the Gagliardo-Nirenberg inequality in the whole space.

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. Then there exists a bounded linear extension operator \( T : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n) \) \( (k \geq 0, 1 \leq p \leq \infty) \) such that

\[
\|T f\|_{W^{k,p}(\mathbb{R}^n)} \leq C \|f\|_{W^{k,p}(\Omega)}, \quad \forall f \in W^{k,p}(\Omega),
\]

where \( C \) depends only on \( n, k, p \) and \( \Omega \).

The next is an interpolation in a bounded domain.

**Lemma 2.2.** Let \( \Omega \) be a bounded Lipschitz domain and \( u \in W^{k,r}(\Omega) \) \( (k \geq 0, 1 \leq r \leq \infty) \). Then for any \( \varepsilon, q > 0, 0 \leq j < k \) and \( p \in \mathbb{R} \) with

\[
\frac{n}{p} - j > \frac{n}{r} - k,
\]

there exists a constant \( C \) depending only on \( n, k, q, \varepsilon \) and \( \Omega \) such that

\[
\|\nabla^j u\|_p \leq \varepsilon \|\nabla^k u\|_r + C \|u\|_q.
\]

**Proof.** We prove the lemma by contradiction. Suppose that the conclusion is false. Then there exist \( \varepsilon_0 > 0 \) and a sequence of \( \{u_m\} \subset W^{k,r}(\Omega) \) such that

\[
\|\nabla^j u_m\|_p \geq \varepsilon_0 \|\nabla^k u_m\|_r + m \|u_m\|_q.
\]

Without loss of generality, we assume that \( \|u_m\|_{W^{k-1,r}(\Omega)} + |\nabla^j u_m|_p = 1 \). Then

\[
1 \geq \varepsilon_0 \|\nabla^k u_m\|_r + m \|u_m\|_q.
\]

Hence, \( \|u_m\|_{W^{k,r}(\Omega)} \leq 1 + 1/\varepsilon_0 \). By (2.2), \( W^{k,r} \) is compactly embedded into \( L^p \).

Thus, there exist a subsequence (denoted by \( \{u_m\} \) again) and \( \bar{u} \in W^{k,r}(\Omega) \) such that as \( m \to \infty \),

\[
\|u_m - \bar{u}\|_{W^{k-1,r}(\Omega)} + |\nabla^j u_m - \nabla^j \bar{u}|_p \to 0.
\]

Therefore, \( \|\bar{u}\|_{W^{k-1,r}(\Omega)} + |\nabla^j \bar{u}|_p = 1 \). However, from (2.3),

\[
\|\bar{u}\|_q = \lim_{m \to \infty} \|u_m\|_q = 0.
\]

That is, \( \bar{u} \equiv 0 \) and we arrive at a contradiction. \( \square \)

Now, we can give the

**Proof of Theorem 1.3.** Extend \( u \) to \( Tu \in W^{k,r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \). By the Gagliardo-Nirenberg inequality in \( \mathbb{R}^n \) (see Theorem 1.1) and Lemma 2.2 with \( \varepsilon = 1 \),

\[
|\nabla^j u|_{p,\Omega} \leq |\nabla^j Tu|_{p,\mathbb{R}^n}
\]

\[
\leq C \|\nabla^k Tu\|_{r,\mathbb{R}^n} \|Tu\|_{q,\mathbb{R}^n}^{1-\theta}
\]

\[
\leq C \left( \sum_{i=1}^{k} \|\nabla^i u\|_{r,\Omega} \right)^{\theta} \|u\|_{q,\Omega}^{1-\theta}
\]

\[
\leq C (\|u\|_{q,\Omega} + \|\nabla^k u\|_{r,\Omega})^{\theta} \|u\|_{q,\Omega}^{1-\theta}
\]

\[
\leq C \|\nabla^k u\|_{r,\Omega}^{\theta} \|u\|_{q,\Omega}^{1-\theta} + C \|u\|_{q,\Omega}.
\]
To prove Theorem 1.5, we need the following lemma.

**Lemma 2.3.** Let $\Omega$ be a bounded Lipschitz domain, $E \subset \Omega$ with $|E| > 0$ and $u \in W^{k,r}(\Omega)$ ($k \geq 0, 1 \leq r \leq \infty$). Then there exists a unique polynomial $P_E$ of degree $k - 1$ with

\begin{equation}
\int_E \nabla^\gamma u = \int_E \nabla^\gamma P_E, \quad \forall \ 0 \leq |\gamma| \leq k - 1
\end{equation}

such that for any $0 \leq j < k$ and $p \in \mathbb{R}$ with

$$\frac{n}{p} - j > \frac{n}{r} - k,$$

we have

$$|\nabla^j u - \nabla^j P_E|_p \leq C\|\nabla^k u\|_r,$$

where $C$ depends only on $n, k, r, \Omega$ and $E$.

**Remark 2.4.** Lemma 2.3 can be regarded as an extension of the Poincaré inequality.

**Proof.** The proof is similar to that of Lemma 2.2. Suppose that the conclusion is false. Then there exist a sequence of $\{u_m\} \subset W^{k,r}(\Omega)$ and $\{P_m\}$ such that (2.4) holds and

$$|\nabla^j u_m - \nabla^j P_m|_p \geq m\|\nabla^k u_m\|_r.$$

Let $v_m = u_m - P_m$. Without loss of generality, we assume that $\|v_m\|_{W^{k-1,r}(\Omega)} + |\nabla^j v_m|_p = 1$. Then

\begin{equation}
1 \geq m\|\nabla^k v_m\|_r.
\end{equation}

Hence, $\|v_m\|_{W^{k,r}(\Omega)} \leq 2$. By the compact embedding, there exist a subsequence (denoted by $\{v_m\}$ again) and $\bar{v} \in W^{k,r}(\Omega)$ such that as $m \to \infty$,

$$\|v_m - \bar{v}\|_{W^{k-1,r}(\Omega)} + |\nabla^j v_m - \nabla^j \bar{v}|_p \to 0.$$

Thus,

\begin{equation}
\|\bar{v}\|_{W^{k-1,r}(\Omega)} + |\nabla^j \bar{v}|_p = 1.
\end{equation}

However, from (2.5),

$$\|\nabla^k \bar{v}\|_r = 0.$$

That is, $\bar{v}$ is a polynomial of degree $k - 1$. From (2.4), $\int_E \nabla^\gamma v_m = 0$ for any $m \geq 1$ and $0 \leq |\gamma| \leq k - 1$. Let $m \to \infty$, we have the same equalities for $\bar{v}$. Hence $\bar{v} \equiv 0$, which contradicts with (2.6). □

Now, we can give the **Proof of Theorem 1.5.** Since $n/q > n/r - k$, by Lemma 2.3,

$$\|u - P_E\|_q \leq C\|\nabla^k u\|_r.$$

Hence, from Theorem 1.3,

$$|\nabla^j u - \nabla^j P_E|_p \leq C\|\nabla^k u\|_r^\theta\|u - P_E\|_q^{1-\theta} + \|u - P_E\|_q \leq C\|\nabla^k u\|_r^\theta\|u - P_E\|_q^{1-\theta}.$$

□
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