There was movement that was stationary, for the four-velocity had passed around

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ABSTRACT
Is the Doppler interpretation of galaxy redshifts in a Friedmann-Lemaître-Robertson-Walker (FLRW) model valid in the context of the approach to comoving spatial sections pioneered by de Sitter, Friedmann, Lemaître and Robertson, i.e. according to which the 3-manifold of comoving space is characterised by both its curvature and topology? Holonomy transformations for flat, spherical and hyperbolic FLRW spatial sections are proposed. By quotienting a simply-connected FLRW spatial section by an appropriate group of holonomy transformations, the Doppler interpretation in a non-expanding Minkowski space-time, obtained via four-velocity parallel transport along a photon path, is found to imply that an inertial observer is receding from herself at a speed greater than zero, implying contradictory world-lines. The contradiction in the multiply-connected case occurs for arbitrary redshifts in the flat and spherical cases, and for certain large redshifts in the hyperbolic case. The link between the Doppler interpretation of redshifts and cosmic topology can be understood physically as the link between parallel transport along a photon path and the fact that the comoving spatial geodesic corresponding to a photon’s path can be a closed loop in an FLRW model of any curvature. Closed comoving spatial loops are fundamental to cosmic topology.

Key words: cosmology: theory – relativity – reference systems – time

1 INTRODUCTION
Much debate has recently taken place regarding the interpretation of the redshifts of comoving galaxies in cosmological models as a special-relativistic Doppler effect in the absence of the concept of expanding space (Chodorowski 2007a; Barnes et al. 2006; Chodorowski 2007b; Abramowicz et al. 2007; Francis et al. 2007; Lewis et al. 2008; Bunn & Hogg 2009; Peacock 2008; Abramowicz et al. 2009; Chodorowski 2009; Faraoni 2009). Here, it is shown that in the flat and spherical Friedmann-Lemaître-Robertson-Walker (FLRW) models, where galaxies are comoving massless test objects in the standard comoving-coordinate system, the Doppler interpretation in non-expanding Minkowski space-time leads to a contradiction for galaxies at distances from the observer that are arbitrarily small, provided that these galaxies are considered to be comoving. The corresponding (weaker) argument in hyperbolic FLRW models is also presented.

In Sect. 2 some mathematical properties of the comoving spatial sections in FLRW models are recalled. The methods of using these properties in flat, positively curved, and negatively curved FLRW models are presented in Sect. 2.1 Sect. 2.2 and Sect. 2.3 respectively. The consequences are described in Sect. 3. The way that density perturbations modify these consequences is discussed in Sect. 4.1. The failure of the Doppler interpretation to arbitrarily small separations for non-negatively curved exact-FLRW models may seem to be inconsistent with the Minkowski nature of FLRW models in the limit towards a point in space-time. This paradox is explained in Sect. 4.2. A definition of “expansion of space” and related statements from the literature are briefly discussed in Sect. 4.3 and the notion of an expanding Minkowski space-time is mentioned in Sect. 4.4. Conclusions are presented in Sect. 5.

For clarity, the terms “hyperbolic” and “spherical” are used for negatively and positively curved FLRW models, respectively, rather than the ambiguous terms “open” and “closed”. The term “galaxy” is used for an external massless galaxy located at some non-zero distance from the observer in the covering space. The observer should be considered to be located in our (massless) Galaxy. The “galaxy” and the “Galaxy” are massless in order not to violate homogeneity. A “comoving spatial geodesic” is distinct from a space-time geodesic. The former refers to a geodesic (a curve that min-
imises the metric distance between any close pair of points on that curve) of a comoving spatial section (where the metric only has a spatial component). It can also be thought of as a space-time geodesic (e.g. the path of a photon) projected to a comoving spatial section by ignoring the cosmological time coordinate. For example, in a positively curved FLRW model, a comoving spatial geodesic is an arc that is part of a great circle of the hypersphere $S^2$, since great circles are straight lines.

For a short introduction to the topology of FLRW models and observational approaches to measuring it, see Roukema (2000). For reviews, see Lachièze-Rey & Luminet (1995); Luminet (1998); Starkman (1998); Luminet & Roukema (1999); Blankent & Roukema (2000); Reboçâs & Gomerê (2004). Analyses of WMAP data presently include analyses suggesting that sub-latitude 3-manifold whose curvature may be zero, positive or negative constant curvature 3-manifold whose curvature may be zero, positive or negative.

The comoving spatial section of an FLRW model is a "fundamental domain", with identified faces. It can also be thought of as being tiled by a set of mirror copies of the fundamental domain of $M$, except that instead of a reflection (generated by a mirror), a holonomy transformation (an isometry that allows $M$ to be completely and exactly tiled by copies of the fundamental domain of $M$) must be used. The galaxies are chosen to be massless in order to avoid violating the homogeneity condition of the FLRW models.

Consider a galaxy at redshift $z > 0$ in a simply-connected FLRW model of comoving spatial section $\tilde{M} = R^3$, $\tilde{M} = S^3$, or $\tilde{M} = H^3$, i.e. a galaxy that emitted photons towards the observer “o” when the scale factor was $a < 1$ and the present scale factor (at the observation epoch) is $a_0 = a_0 = 1$. Let both the observer and the galaxy at $z$ be at rest in comoving coordinates. Let us also assume that the observer has been at rest in comoving coordinates since the epoch $a$.

Now let us attempt to make a Doppler interpretation of the galaxy’s redshift. A global concept of velocity is not a standard part of the FLRW model. However, it is possible to parallel-transport the four-velocity of the galaxy to the observer, typically along the path taken by a photon (e.g., Synge 1960; Narlikar 1994; Peacock 2003; Dunn & Hogg 2004; Fairon 2004), so that it can be locally compared with the four-velocity of the observer. The resulting velocity difference $\beta$ in units of the speed of light $c$ satisfies

$$1 + z = \sqrt{\frac{1 + \beta}{1 - \beta}}. \tag{1}$$

Does this imply that the galaxy can be thought of as receding with velocity $\beta$ from the observer in non-expanding Minkowski space-time, and be redshifted in the way implied by the Lorentz transformation?

2 METHOD

The comoving spatial section of an FLRW model is a constant curvature 3-manifold whose curvature may be zero, positive or negative and whose topology ($\pi_1$ homotopy group) may be trivial or non-trivial (de Sitter 1917; Friedmann 1923; 1924; Lemaître 1931; Robertson 1932). The curvature is related to the matter-energy density through the Einstein field equations, since the FLRW model constitutes a family of solutions to these equations. In contrast, so far there only exist some initial hints as to what might eventually contribute to a theory of the topology of FLRW spatial sections (e.g., Hawking 1984; Masañumi 1996; Dowker & Surva 1998; Anderson et al. 2004; Roukema et al. 2007; Roukema & Różański 2009).

Nevertheless, it is clear that in an exact-FLRW model, i.e. a model without any density perturbations of any sort, the quotienting of the spatial comoving covering 3-manifold $\tilde{M}$ by a group of holonomy transformations $\Gamma$ has no effect on the metric. The local cosmological parameters (“local” in the sense of representing a limit towards a space-time point in an exact-FLRW model), such as the Hubble constant $H_0$ and the total density parameter $\Omega_0$, are unchanged by the transformation from $\tilde{M}$ to $M := \tilde{M}/\Gamma$, so $M$ is also an FLRW model.

Another way of thinking about the relation between the covering space $\tilde{M}$ and the 3-manifold $M$ itself is that if we populate $M$ by a set of massless galaxies, then $\tilde{M}$ can be thought of as the observer’s apparent space containing multiple topological images of each galaxy. Loosely speaking, $\tilde{M}$ can be thought of as being tiled by a set of mirror copies of the fundamental domain of $M$, except that instead of a reflection (generated by a mirror), a holonomy transformation (an isometry that allows $M$ to be completely and exactly tiled by copies of the fundamental domain of $M$) must be used. The galaxies are chosen to be massless in order to avoid violating the homogeneity condition of the FLRW models.

Consider a galaxy at redshift $z > 0$ in a simply-connected FLRW model of comoving spatial section $\tilde{M} = R^3$, $\tilde{M} = S^3$, or $\tilde{M} = H^3$, i.e. a galaxy that emitted photons towards the observer “o” when the scale factor was $a < 1$ and the present scale factor (at the observation epoch) is $a_0 = a_0 = 1$. Let both the observer and the galaxy at $z$ be at rest in comoving coordinates. Let us also assume that the observer has been at rest in comoving coordinates since the epoch $a$.

2.1 Zero spatial curvature

In the case of zero spatial curvature of the underlying FLRW model, let us introduce the holonomy transformation $f_0((x, y, z')) = (x + L, y, z'), \forall (x, y, z') \in \tilde{M}$ where $(x, y, z')$ is an arbitrary comoving position in the spatial section $\tilde{M}$, $L > 0$ is the comoving distance from the observer to the galaxy, and the orientation of the fundamental

1 The order $\{0, +, -\}$ is chosen for consistency of the presentation. This relates to the physical characteristics of the three types of FLRW spatial sections: negatively curved spaces are the least convenient for the problem dealt with in this paper (see Sect. 2).

2 Since the position variable $z'$ is of peripheral importance, the unprimed variable $z$ is retained for the redshift.
domain is chosen so that the $x$ direction joins the comoving positions of the observer and the galaxy. Now quotient $M = \mathbb{R}^3$ by the group generated by $f_0$, obtaining

$$M = \mathbb{R}^3 / \{ i f_0, i \in \mathbb{Z} \} = S^1 \times \mathbb{R}^2.$$  

(3)

This can be loosely referred to as a 1-torus, $T^1$ (where the 3-dimensionality is implicit).

The physical difference between $\mathbb{R}^3$ and $T^1$ models is enormous. However, changing from the former to the latter does not change the metric. The most popular observational cosmology tests on various classes of extragalactic objects cannot distinguish an $\mathbb{R}^3$ spatial section from a $T^1$ spatial section. Moreover, if the Doppler interpretation of galaxy redshifts only depends on local, metric properties, then it should not be affected by the change between these two spatial sections. The only change relevant to the present discussion is that a copy of the observer is now located at the same comoving spatial position as the (comoving) galaxy. Moreover, the “copy” of the observer and the observer herself now constitute a single physical observer. Consequences of the change from one space to another are considered in Sect. 3.2.

### 2.2 Positive spatial curvature

Now consider the case where the FLRW model has positive curvature, with radius of curvature

$$R_C = \frac{c}{H_0 \sqrt{\Omega_{\text{tot}} - 1}}.$$  

(4)

Let us represent the comoving spatial section $\hat{M} = S^3$ as a subset of Euclidean 4-space $\{ (x, y, z', w) \in \mathbb{R}^4 : x^2 + y^2 + z'^2 + w^2 = R_C^2 \}$, and place the observer at $(0, 0, 0, R_C)$ and the galaxy at $[R_C \sin(L/R_C), 0, 0, R_C \cos(L/R_C)]$, with $L > 0$ as the comoving distance from the observer to the galaxy, as above.

Choose $m, n \in \mathbb{Z}$ such that

$$0 < m < n, \quad \left| \frac{2\pi m}{n} - \frac{L}{R_C} \right| < \delta$$

(5)

for some arbitrarily small comoving distance $\delta > 0$, since the rationals $\mathbb{Q}$ are dense in $\mathbb{R}$. Let us introduce the holonomy transformation

$$f_+ \left[ \begin{array}{c} x \\ y \\ z' \\ w \end{array} \right] = \left[ \begin{array}{cccc} \cos \theta & 0 & 0 & -\sin \theta \\ 0 & \cos \theta & -\sin \theta & 0 \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & \cos \theta \end{array} \right] \left[ \begin{array}{c} x \\ y \\ z' \\ w \end{array} \right]$$

\forall (x, y, z', w) \in S^3$$

(6)

where $(x, y, z', w)$ are arbitrary comoving positions in the spatial section $S^3 \subset \mathbb{R}^4$, $\theta := 2\pi / n$, and an arc of a great circle in the $x$-$w$ plane joins the comoving positions of the observer and the galaxy. Now quotient $S^3$ by the group generated by $f_+$, obtaining

$$M = S^3 / \{ i f_+, i \in \mathbb{Z} \},$$

(7)

which is known as the lens space $L(n, 1)$, since the fundamental domain is a dihedron\(^4\) with two hexagonal (flat) faces (e.g., [Gaussmann et al. 2001]). This holonomy transformation, a double rotation in two orthogonal 2-planes, has the elegant property that every point is transported the same distance under the action of the holonomy, i.e. it is a Clifford translation.

As in the zero curvature case, this topology change does not change the metric. On the other hand, what is relevant to the present discussion is that a copy of the observer now exists at a comoving spatial location closer than a comoving distance of $\delta > 0$ from the galaxy at the redshift $z$, for an arbitrarily small $\delta$. This can be seen in the covering space by applying $f_+$ [the holonomy transformation (6)] to the observer $m$ times, keeping in mind the choice of $m, n$ [cf. Eq. (5)]. Consequences of this are considered in Sect. 3.2.

### 2.3 Negative spatial curvature

For negatively curved FLRW models, let us consider a less general case than for non-negatively curved models. Consider a covering space $M = \mathbb{H}^3$ of absolute curvature radius $|R_C|$. Choose a galaxy at redshift $z > 0$ and comoving distance $L > 0$ from the observer that satisfies $L = 2r_{\text{inj}}(M)$ for an hyperbolic 3-manifold $M$ whose shortest closed geodesic is of length $2r_{\text{inj}}$ (inj is called the injectivity radius). Values of $2r_{\text{inj}}$ for small hyperbolic 3-manifolds can typically be as low as $\gtrsim 0.31|R_C|$. For example, see table 1 of [Luminet & Roukema 1999] for a few small hyperbolic 3-manifolds that could be used to find some galaxies satisfying this condition\(^5\).

Now orient the group of holonomy transformations $\Gamma$ that gives $M = \mathbb{H}^3 / \Gamma$ in the appropriate sense in the covering space $\mathbb{H}^3$ so that the holonomy transformation $f_- \in \Gamma$ associated with twice the injectivity radius, i.e. $2r_{\text{inj}}$, matches a copy of the observer to the position of the galaxy, i.e. so that

$$f_-(0) = x,$$

(8)

where $0$ and $x$ are the observer’s and galaxy’s comoving positions in $\mathbb{H}^3$, respectively. In general, it will not be possible to generate a 3-manifold from the group $\{ i f_-, i \in \mathbb{Z} \}$ alone, in contrast with the non-negatively curved cases. This is because of the mathematically more complicated nature of hyperbolic 3-manifolds. The situation is now similar to the zero curvature case (Sect. 2.2), i.e. the copy of the observer is located at the same comoving position as the galaxy, with the distinction that the choice of galaxy had to be strongly constrained to a countably infinite number\(^6\) of possibilities. The consequences of changing from $\mathbb{H}^3$ to $\mathbb{H}^3 / \Gamma$ are considered in Sect. 3.5.

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3 A polyhedron with two (flat) faces is called a dihedron.

4 A polygon with one (straight) edge is called a hexagon.

5 A closed open model.

6 These spaces could be called “closed open” models using the confusing terminology frequently used in modern cosmology.

7 Taking into account the matter horizon may reduce this to a finite number.
the copy of the observer has also been at rest in comoving coordinates since the epoch $a$. Hence, the copy of the observer is also comoving, the copy of the observer is also co-

**3 RESULTS**

### 3.1 Zero spatial curvature

As explained in Sect. 2.1 changing the comoving spatial section from $\mathbb{R}^3$ to $T^3$ does not change the metric, and should not change the Doppler effect interpretation of the redshift $z$ of a galaxy if that interpretation is valid.

Figure 1 shows the observer, the galaxy, the photon emitted by the latter and absorbed by the former, the copy of the observer, and the world-lines of the observer, galaxy and copy of the observer, according to the FLRW model of spatial section $M$, in the covering space $\hat{M}$ in comoving coordinates. A copy of the observer is located at $x_{o'} = L$, i.e. at the same comoving position as the galaxy. Since the observer is comoving, the copy of the observer is also comoving, so the copy of the observer and the galaxy are both comoving at the same spatial position. Hence, the copy of the observer and the galaxy are at rest with respect to one another.

It was assumed above (Sect. 2) that the observer has been at rest in comoving coordinates since the epoch $a$, so the copy of the observer has also been at rest in comov-

**Figure 1.** An observer “o” and a galaxy “g” both at rest in comoving coordinates in an FLRW model, showing one spatial dimension of the comoving covering space $\hat{M}$ and the comoving time coordinate. The galaxy emits a photon at cosmological time $t_e$ that is observed by the observer at cosmological time $t_o$ to have redshift $z > 0$. A holonomy transformation identifies the galaxy and the observer so that the comoving space is $\hat{M}/\Gamma$ for a group of holonomy transformations $\Gamma$, that varies according to curvature, as specified in Sect. 2.1, Sect. 2.2 and Sect. 2.3 respectively. The identification is exact in the zero and negatively curved cases (for arbitrarily selected and specially selected galaxies, respectively) and approximate to within comoving separation $\delta$ in the positively curved case [see Eq. (5)]. The world-lines of the observer, the galaxy, and the copy of the observer “o’” are shown as thick vertical line segments. The world-line of the copy of the observer coincides exactly with the world-line of the galaxy in the non-positively curved cases.

**Figure 2.** As in Fig. 1 shown in non-expanding Minkowski space-time in order to attempt to interpret the galaxy’s redshift as a Doppler effect. The location of the world-line of the copy of the observer in this diagram is ambiguous, as indicated by the question marks and discussed in the text (Sect. 3): how is it possible that the (inertial) observer is moving at constant speed away from herself in (non-expanding) Minkowski space-time?

This includes the space-time event $(L, t_e)$ of the galaxy emitting the photons that are later on observed at $(0, t_o)$. Hence, we can consider the copy of the observer to have emitted photons at the same space-time event $(L, t_e)$ in the same direction as the photons emitted by the galaxy. It follows that if a Doppler interpretation of the galaxy’s redshift $z > 0$ is attempted based on parallel transport of four-velocities along the photon path, leading to an inferred velocity $\beta > 0$ [Eq. (4)], then the same Doppler interpretation implies that the copy of the observer at $x_{o'} = L$ must also be inferred to be receding at velocity $\beta > 0$ with respect to the “original” observer. However, the observer and the copy of the observer are physically identical. Hence, the observer is inferred to be receding at velocity $\beta > 0$ with respect to herself.

This is impossible unless the physical concept of expanding space is added to the Doppler interpretation. In a non-expanding Minkowski space-time diagram for a given inertial observer, the slope of any inertially moving object’s world-line is constant and the slope defines the object’s velocity relative to that of the observer. The slope of the observer’s world-line is exactly vertical. A change of global topology does not change this [Brans & Stewart 1973, Peters 1983, Low 1990, Uzan et al. 2002, Barrow & Levin 2001, Roukema & Bajitlik 2008]. An inertial observer in a non-expanding Minkowski space-time is moving at zero velocity with respect to herself. She cannot recede at velocity $\beta > 0$ from herself. Since $x_{o'} = x_o$ and $\dot{x}_{o'} = \dot{x}_o$, there is no peculiar velocity difference between the galaxy and the copy of the observer that could avoid this contradiction.

Figure 2 shows the equivalent information with an at-
tempt to support the Doppler interpretation of the galaxy’s redshift. The galaxy is receding at a Doppler-inferred velocity of $\beta > 0$, so in a non-expanding Minkowski space-time diagram labelled in units with $c = 1$, the galaxy must have a world-line at an angle of $\tan \beta$ from the vertical. Since $\dot{x}_\| = x_\| \dot{\gamma}$ and $\dot{x}_\perp = x_\perp \dot{\delta}$ where $x$ is a comoving coordinate, equality must also hold in the Minkowski diagram. That is, in Fig. 2 the copy of the observer must also be receding at $\beta > 0$ from herself, i.e. must also have a world-line at an angle of $\tan \beta$ from the vertical, that coincides exactly with the galaxy’s world-line. However, the world-line of the galaxy and copy of the observer in Fig. 2 are labelled with a question mark, since the observer must be stationary with respect to herself in a non-expanding Minkowski space-time, i.e. the copy of the observer must have a vertical world-line. The latter is also indicated by a question mark in Fig. 2. The copy of the observer cannot have two physically distinct world-lines. The Doppler interpretation is clearly self-contradictory.

3.2 Positive spatial curvature

The situation for spherical FLRW models (Sect. 2.2) is nearly the same as for flat FLRW models, with the difference that the $m$-th copy of the observer is located arbitrarily close (less than $\delta$) from the galaxy [see Eq. 2], rather than exactly at the galaxy. The assumption that parallel transport of a four-velocity enables inference of a Doppler recession velocity can be used to show that the velocity between the copy of the observer (within $\delta$ of the galaxy) and the galaxy itself can be made arbitrarily small. This is sufficient to show that the copy of the observer is receding at a speed $\beta' \lesssim \beta$ arbitrarily close to $\beta > 0$, i.e. the observer is receding from herself at non-zero velocity $\beta'$. Again this is impossible in a non-expanding Minkowski space-time.

3.3 Negative spatial curvature

The hyperbolic FLRW models (Sect. 2.3) have a much more restrained domain where the Doppler interpretation in non-expanding Minkowski space-time fails. For simplicity, only galaxies at distances $L = 2 r_{\text{inj}}(M)$ for an hyperbolic 3-manifold $M$ are considered in this paper. For example, with $\Omega_{\text{tot}} = 1.015$, $R_C = 24 h^{-1}$ Gpc, and $r_{\text{inj}} \gtrsim 0.31$, this requires $z \gtrsim 21$. Given this restriction, the Doppler interpretation of the redshift of the would-be massless extremely high redshift galaxy again leads to a contradiction: the observer must be receding with respect to herself, which is absurd unless an expanding space concept is introduced.

4 DISCUSSION

4.1 Almost-FLRW models

In the flat case, the self-contradiction of the Doppler interpretation is limited below in $z$ only by the requirement that $z$ be large enough that the (massless) galaxy be at rest in comoving coordinates. In an exact-FLRW model, i.e. in the absence of density perturbations, there is no gravitational collapse of the cosmic web of filaments of large-scale structure, galaxies and clusters of galaxies, and no turnaround radius, so comoving “galaxies” exist arbitrarily close to the observer, at any distance $L > 0$. In this case, even with a “galaxy” at $0 < z \ll 1$, the application of the holonomy transformation (2) will still lead to a contradiction: the copy of the observer at the location of the comoving galaxy Doppler-interpreted still recedes at velocity $\beta > 0$ from herself.

However, in reality, density perturbations certainly do exist, so the FLRW model is certainly wrong. The standard (formally inconsistent, but practical) approach of using an FLRW model together with perturbations implies a turnaround radius, so that the comoving coordinate system of the FLRW model only makes physical sense for galaxies at comoving distances $\gg 10 h^{-1}$ Mpc. Hence, in a perturbed FLRW model, the failure of the Doppler interpretation of galaxy redshifts occurs only for $L \gg 10 h^{-1}$ Mpc, where $L$ is the comoving distance to the galaxy, i.e. for $z \gg 0.003$.

The situation is similar for a perturbed spherical FLRW model. Again, there is some domain where the Doppler interpretation of cosmological redshifts is valid, and it should no longer be possible to obtain $\beta > 0$ arbitrarily small. An evaluation of the domain of parameter space where there would be no self-contradiction in the Doppler interpretation of redshifts in non-expanding Minkowski space-time in the perturbed spherical FLRW case would require calculations that depend on $R_C$, the matter density parameter $\Omega_m$, the dark energy parameter/cosmological constant $\Omega_\Lambda$, and $z$. For some parameter combinations sufficiently far from realistic estimates, even some redshifts $z \gg 0.003$ might be interpreted as a parallel-transported relativistic Doppler effect without leading to an observer receding from herself in a non-expanding Minkowski space-time.

On the other hand, in contrast with the flat and spherical cases, allowing perturbations in an hyperbolic FLRW model should not avoid the self-contradiction of the Doppler interpretation, since the possible “galaxies” chosen in this case are at very high redshifts.

The requirement that an FLRW model be perturbed in order for the Doppler interpretation to be valid at small redshifts ($z \lesssim 0.003$) seems to be an implicit assumption in presentations such as Sect. II of Bunn & Hogg (2009). In a perfectly homogeneous FLRW model with a massless police officer aiming a radar at a nearly massless, comoving, “speeding” car, the road itself is stretching and the local environment of the car is comoving with that car. The police officer can aim the radar at a building adjacent to the car and will infer that the building is also receding at a recklessly high speed, e.g., 150 km/h. Moreover, the driver can open his door and touch the road with his fingers, without any risk of injury, since both the car and the patch of road in the car’s immediate neighbourhood are receding at 150 km/h. Let us now quotient the space by a group of holonomy transformations as described above, without modifying the FLRW nature of the model, and the police officer suddenly notices that it is he himself who, along with the patch of road, is receding at a speed of 150 km/h . . . from himself. He is reluctant to give himself a speeding ticket, since he is convinced of his innocence—he is stationary with respect to his local surroundings. He decides that there is something
unphysical about interpreting the redshift as a velocity, and concludes that space itself is small and expanding.

This analogy seems strange because roads are normally of fixed length in “physical” units. For this to be possible, density perturbations and gravitational collapse to structures that stabilise in physical spatial coordinates must be added to the model. If perturbations are added, then small, local regions that are not comoving are created. However, in an exact-FLRW model of zero or positive curvature, the contradiction as derived above occurs, no matter how small the length scale, since “galaxies” at arbitrarily small distances are comoving in this idealised model.

4.2 Is the failure of the Doppler interpretation consistent with the Minkowski limit of an FLRW space-time?

An FLRW model must necessarily approach Minkowski space-time in the limit towards any point in space-time. Is this consistent with the derivation above of a Doppler interpretation failure for arbitrarily small \( z \), or equivalently, arbitrarily small \( L \)? Clearly yes, since

\[
\forall L > 0, \exists \epsilon : 0 < \epsilon \ll L.
\]

For any small \( L \), the limit of space-time properties towards a point must be evaluated through even smaller neighbourhoods (e.g. 4-cylinders of size \( |\delta x| < \epsilon, |\delta t| < \epsilon \)) around the observer. Hence, in an exactly homogeneous FLRW model, there is no problem. In an almost-FLRW model, as mentioned above, some restrictions to the results apply.

4.3 Expanding space and finiteness

Francis et al. (2007) clarify the notion of expanding space by defining it as the physical situation where “the distance between observers at rest with respect to the fluid increases with time”. Using standard FLRW notation, an equivalent, but more compact definition of “expanding space” is:

**Definition 1.** “expanding space” means \( \dot{a}(t) > 0 \).

Chodorowski (2007b) introduces the terminology “expansion of the cosmic substratum” for this FLRW notion of expanding space.

The expansion of space becomes even clearer when Peacock (2008)’s statement that “undoubtedly space is expanding” in the spherical case and Chodorowski (2007b)’s equivalent statement that “the proper volume of a spherical FL universe increases as \( a(\tau)^3 \); more and more space thus appears” are extended to early twentieth-century results (de Sitter 1917; Friedmann 1922, 1924; Lemaître 1931; Robertson 1935) regarding the role of the topology of the 3-spatial section of the Universe as a property to be determined by observations. The latter approach leads to the present-day understanding of 3-manifolds, according to which a flat or hyperbolic FLRW universe can have a finite spatial section, i.e. there is no physical reason why a realisation of a constant curvature 3-manifold should be simply connected. Hence, Peacock’s and Chodorowski’s statements can be updated in the spirit of de Sitter, Friedmann, Lemaître and Robertson: space is expanding in a compact FLRW model of zero, positive, or negative curvature during epochs when \( \dot{a} > 0 \).

In other words, Peacock and Chodorowski agree that it is natural to see finite FLRW models as (globally) expanding, and it is hard to imagine any physical reason why this should apply only to finite spherical FLRW models and not to finite flat and hyperbolic FLRW models.

The notion of expanding space in infinite models, e.g. the simply-connected non-positively curved models, is more difficult to conceive of globally, since \( \infty \not\in \mathbb{R} \) and even in typical extensions of \( \mathbb{R} \) that include \( \infty \), the expression \( \infty - \infty \) is usually undefined. On the other hand, a local concept of expanding space, where “local” means any finite (greater than zero) region of the comoving spatial section, i.e. a region attached to a framework of galaxies, gives a notion of expanding space. Such a region expands as \( a^3 \) when \( \dot{a} > 0 \). The condition that every finite (greater than zero) region should satisfy this in a non-compact exact-FLRW model provides an intuitive way of understanding Defn [1] in the case of an infinite comoving space. A minor caveat is that in order to avoid ambiguity, in an almost-FLRW model, this use of “local” would have to be distinguished from the even more local concept of regions smaller than the turnaround radius.

Nevertheless, it is the finite models (or, at least, models that are compact in at least one direction) that are the most interesting not only for extending the concept of expanding space as presented by Peacock and Chodorowski, but also for providing the holonomy transformations used above to lead to the contradiction of an inertial observer receding from herself.

4.4 Would an expanding Minkowski space-time make physical sense?

Although quotienting a Minkowski space-time by a group of spatial holonomy transformations at constant time for a preferred inertial observer retains the common sense physical notion that an inertial observer is not moving with respect to herself (Brans & Stewart 1973; Peters 1983; Lestr 1990; Uzan et al. 2002; Barrow & Levin 2001; Roukema & Bajtlik 2008), it is possible to add the notion of expanding space to a Minkowski space-time in order to make it an “expanding-space-time” (Peters 1984). In this case, an observer can have a non-zero velocity with respect to herself. Hence, ironically in the context of the recent debate, parallel-transport of four-velocities along photon paths can allow cosmological redshifts to be interpreted as a relativistic Doppler effect without the contradiction presented here, provided that the concept of expanding space is added to the Minkowski space-time used for this interpretation, and provided that the velocity is thought of as being tied to a path and not as a global concept.

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9 The author is not aware of discussion of multiply-connected spatial sections by A. G. Walker.
4.5 Can assuming trivial topology avoid the contradiction?

In order to avoid the contradiction in the Doppler interpretation of galaxy redshifts in non-expanding Minkowski spacetime as presented here, would it be sufficient to consider a limited spatial region within an FLRW model, or to consider FLRW models whose spatial sections are simply-connected (where every closed loop is continuously contractible to a point)?

The former would be sufficient to avoid the contradiction, if defined appropriately. If the galaxy of interest in a specific FLRW model is closer to the observer than twice the injectivity radius, i.e. $2r_{inj}$, then by the definition of the injectivity radius, a closed spatial loop cannot occur. Nevertheless, given the present empirical estimates of the metric parameters of the Concordance Model, even the most restrictive of the three curvatures, i.e. the hyperbolic case, would still imply a contradiction that could be applicable to the earliest galaxy building blocks. For example, with $\Omega_m = 0.30, \Omega_\Lambda = 0.72$, and the hyperbolic FLRW spatial section $v_2959(3,4)$ [Weeks, 1985] which has $2r_{inj} \approx 0.300$, the comoving length of the shortest closed spatial geodesic would be about 6304 $h^{-1}$ Mpc, so that it would be possible for a closed loop from the observer to herself at $z \approx 8.30$ to occur, if she is appropriately positioned.

The latter possibility would partially avoid the contradiction, by making the link between global spatial topology and the Doppler interpretation explicit as a case to exclude. The simply-connected spherical case would also require a domain restriction, depending on the values of $\Omega_m$ and $\Omega_\Lambda$ and the epoch of the observer, since as the universe recollapses, a cosmological time can occur when the observer sees a distant (blueshifted) copy of herself approaching.

However, in either case, the apparent simplicity of the Doppler interpretation in non-expanding Minkowski space-time would be complicated by adding the global-topology related exclusion.

5 CONCLUSIONS

Cosmic topology is not just a luxury referred to by de Sitter, Friedmann, Lemaître and Robertson (and Schwarzschild in the pre-relativistic era) [1]. Apart from providing a competitor to the infinite flat model for understanding WMAP observational data, cosmic topology can also help to improve physical insight into fundamentals of FLRW models that initially appear to be unrelated to global topology. If the parallel-transported four-velocity Doppler interpretation of galaxy redshifts depends only on the metric (including $a(t)$), then it leads to the physical contradiction of an inertial observer in a non-expanding Minkowski space-time receding from herself. This contradiction can be resolved, but only at the cost of introducing the notion of expanding space, in which case the motivation for a Doppler interpretation is weakened, or by explicitly stating that the comoving distance to the galaxy must be less than twice the injectivity radius of comoving space.

The link between the Doppler interpretation of redshifts and cosmic topology can be summarised physically as follows.

(i) The relativistic Doppler interpretation of a cosmological redshift is obtained by parallel transporting the emitting galaxy’s four-velocity along a photon path to the observer.

(ii) For an FLRW model of any of the three curvatures (with the restrictions as detailed above), the comoving spatial geodesic of the photon’s path can be considered to be a closed path by changing from a simply-connected FLRW model to a multiply-connected FLRW model, without any modification of the metric. This should not affect the Doppler interpretation if that interpretation depends only on the metric.

(iii) Closed comoving spatial paths constitute the mathematical foundation of cosmic topology (e.g., Sect. 3.4, Lachièze-Rey & Luminet, 1993).

Hence, it is unsurprising that insistence on the possibility of interpreting galaxy redshifts as a relativistic Doppler effect leads to the properties of cosmic topology, and in turn requires the physical concept of expanding space, i.e. epochs when $\dot{a} > 0$ (Defn 1).

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