On some kinds of factorizable topological groups

Meng Bao\textsuperscript{a}, Xiaoquan Xu\textsuperscript{b,*}

\textsuperscript{a}College of Mathematics, Sichuan University, Chengdu 610064, China
\textsuperscript{b}Fujian Key Laboratory of Granular Computing and Applications, Minnan Normal University, Zhangzhou 363000, China

Abstract

Based on the concepts of $R$-factorizable topological groups and $M$-factorizable topological groups, we introduce four classes of factorizabilities on topological groups, named $PM$-factorizabilities, $Pm$-factorizabilities, $SM$-factorizabilities and $PSM$-factorizabilities, respectively. Some properties of the four classes of spaces are investigated.

Keywords: Topological groups, feathered, $PM$-factorizable, $Pm$-factorizable.

2000 MSC: 22A05; 54A25; 54H11.

1. Introduction

In the field of Topological Algebra, topological groups are standard researching objects and have been studied for many years, see [1]. A topological group is a group equipped with a topology such that the multiplication on the group is jointly continuous and the inverse mapping is also continuous. It is well-known that for every continuous real-valued function $f$ on a compact topological group $G$, there exists a continuous homomorphism $p : G \to L$ onto a second-countable topological group $L$ and a continuous real-valued function $h$ on $L$ such that $f = h \circ p$. Then, Tkachenko posed the concept of $R$-factorizable topological groups, see [4]. A topological group $G$ is called $R$-factorizable if for every continuous real-valued function $f$ on $G$, we can find a continuous homomorphism $\pi : G \to H$ onto a second-countable topological group $H$ such that $f = g \circ \pi$, for some continuous real-valued function $g$ on $H$. We know that $R$-factorizable topological groups are generalizations of compact groups and separable metrizable groups. For more interesting properties about $R$-factorizable topological groups, see [2, 6, 7]. However, since a metrizable topological group need not to be $R$-factorizable, it follows that H. Zhang, D. Peng and W. He in [10] posed the notion of $M$-factorizable topological group. A topological group $G$ is called $M$-factorizable if for every continuous real-valued function $f$ on $G$, there is a continuous homomorphism $\varphi : G \to H$ onto a metrizable topological group $H$ such that $f = g \circ \varphi$, for some continuous real-valued function $g$ on $H$. Since all first-countable topological groups are metrizable, it is trivial to see that all first-countable topological groups are $M$-factorizable. Moreover, it was proved in [10, Theorem 3.2] that a topological group is $R$-factorizable if and only if it is $M$-factorizable and $\omega$-narrow.

By further researches of $R$-factorizable topological groups, L. Peng and Y. Liu introduced the concept of $PR$-factorizable topological groups, that is, a topological group $G$ is called $PR$-factorizable if for every continuous real-valued function $f$ on $G$, there exists a perfect homomorphism $\pi : G \to H$ onto a second-countable topological group $H$ such that $f = g \circ \pi$, for some continuous real-valued function $g$ on $H$. They gave the characterizations of $PR$-factorizable topological groups in [3, Theorem 2.6]. In particular,
a topological group is \( P \mathbb{R} \)-factorizable if and only if it is Lindel"of feathered. Moreover, since every \( \omega \)-narrow feathered topological group is Lindel"of (see \([1, 4.3.A]\)), it is easy to see that a topological group is \( P \mathbb{R} \)-factorizable if it is \( \omega \)-narrow and feathered. Then, we introduce the following notion.

**Definition 1.1.** A topological group \( G \) is called \( PM \)-factorizable if for every continuous real-valued function \( f \) on \( G \), there exists a perfect homomorphism \( \pi : G \to H \) onto a metrizable topological group \( H \) such that \( f = g \circ \pi \), for some continuous real-valued function \( g \) on \( H \).

Clearly, each \( P \mathbb{R} \)-factorizable topological group is \( PM \)-factorizable. L. Peng and Y. Liu introduced an example \([3, Example 3.13]\) which is an \( \mathbb{R} \)-factorizable topological group, but not \( P \mathbb{R} \)-factorizable. Indeed, the topological group \( G \) in \([3, Example 3.13]\) is not feathered. Since all \( \mathbb{R} \)-factorizable topological groups are \( M \)-factorizable, it is a \( M \)-factorizable topological group. However, by the definition of \( M \)-factorizability, it is easy to that every \( PM \)-factorizable topological group is feathered, hence the topological group \( G \) of \([3, Example 3.13]\) is not \( PM \)-factorizable.

In this paper, we give some characterizations of \( PM \)-factorizable topological groups, such as a topological group \( G \) is \( PM \)-factorizable if and only if \( G \) is feathered \( M \)-factorizable. We also shown that a topological group \( G \) is \( P \mathbb{R} \)-factorizable if and only if \( G \) is \( PM \)-factorizable and \( \omega \)-narrow. Then it is natural to deduce that a topological group \( G \) is \( P \mathbb{R} \)-factorizable if and only if \( G \) is feathered \( \mathbb{R} \)-factorizable. W. He et al. in \([3, Proposition 2.1]\) proved the following proposition.

**Proposition 1.2.** Let \( G = \prod_{i \in I} G_i \) be the product of an uncountable family of non-compact separable metrizable topological groups. Then the group \( G \) is \( \mathbb{R} \)-factorizable, but it fails to be feathered.

Therefore, the result also can present that the product group \( G \) is not \( P \mathbb{R} \)-factorizable. Of course, \( G \) is an \( M \)-factorizable topological group, but not \( PM \)-factorizable. Moreover, some interesting properties of \( M \)-factorizable topological groups in \([3, 10]\) are strengthened to \( PM \)-factorizable topological groups.

For example, the product \( G = \prod_{n \in \mathbb{N}} G_n \) of countably many \( PM \)-factorizable topological groups is \( PM \)-factorizable if and only if every factor \( G_n \) is metrizable or every \( G_n \) is \( P \mathbb{R} \)-factorizable, the product of a \( PM \)-factorizable topological group with a compact metrizable topological group is \( PM \)-factorizable.

Then, according to the concept of \( m \)-factorizable topological groups, that is, a topological group \( G \) is called \( m \)-factorizable if for every continuous mapping \( f : G \to M \) to a metrizable space \( M \), there exists a continuous homomorphism \( \pi : G \to K \) onto a second-countable group \( K \) such that \( f = g \circ \pi \), for some continuous homomorphism \( g \) from \( K \) onto \( M \), we introduce \( Pm \)-factorizable topological groups by strengthening the continuous homomorphism \( \pi \) to a perfect homomorphism. Then we show that a topological group \( G \) is \( Pm \)-factorizable if and only if \( G \) is \( P \mathbb{R} \)-factorizable and pseudo-\( K_1 \)-compact, which deduces that a topological group \( G \) is \( Pm \)-factorizable if and only if \( G \) is feathered \( m \)-factorizable. Furthermore, it is claimed that the product of a \( Pm \)-factorizable topological group with an arbitrary compact group is \( Pm \)-factorizable.

Finally, we pose the concepts of \( SM \)-factorizable topological groups and \( PSM \)-factorizable topological groups. A topological group \( G \) is called \( strongly \ M \)-factorizable (\( SM \)-factorizable for short) if for every continuous mapping \( f : G \to M \) to a metrizable space \( M \), there exists a continuous homomorphism \( \pi : G \to H \) onto a metrizable group \( H \) and a continuous mapping \( g : H \to M \) such that \( f = g \circ \pi \). In particular, if the continuous homomorphism \( \pi : G \to H \) is perfect, we call \( G \) \( PSM \)-factorizable. Since a topological group \( G \) is \( PM \)-factorizable if and only if \( G \) is feathered \( M \)-factorizable, and a topological group \( G \) is \( Pm \)-factorizable if and only if \( G \) is feathered and \( m \)-factorizable, see Theorem \([3,5]\) and Corollary \([5,3]\), it is natural to consider whether it is equivalent between \( PSM \)-factorizability and feathered \( SM \)-factorizability. Then we show that it also holds in Theorem \([6,5]\). Furthermore, the four classes of factorizable properties are preserved by open continuous homomorphisms on topological groups, see Theorem \([3,11]\) Corollary \([5,6]\) Theorem \([6,7]\) and Corollary \([6,8]\).

2. Preliminary

Throughout this paper, all topological spaces are assumed to be Hausdorff, unless otherwise is explicitly stated. Let \( \mathbb{N} \) be the set of all positive integers and \( \omega \) the first infinite ordinal. The readers may consult
3. Some properties of \( P_M \)-factorizable topological groups

In this section, we give some characterizations of \( P_M \)-factorizable topological groups, such as a topological group \( G \) is \( P_M \)-factorizable if and only if \( G \) is feathered \( M \)-factorizable. We also showed that a topological group \( G \) is \( PR \)-factorizable if and only if \( G \) is \( P_M \)-factorizable and \( \omega \)-narrow. Then it is natural to deduce that a topological group \( G \) is \( PR \)-factorizable if and only if \( G \) is feathered \( R \)-factorizable.

Recall that paracompact \( p \)-spaces are the preimages of metrizable spaces under perfect mappings. By the definitions of \( P_M \)-factorizable topological groups, the following result is clear.

**Proposition 3.1.** Every \( P_M \)-factorizable topological group is a paracompact \( p \)-space.

It was proved in \([1\), Theorem 4.3.35] that a topological group is feathered if and only if it is \( p \)-space, and if it is a paracompact \( p \)-space.

**Proposition 3.2.** Every \( P_M \)-factorizable topological group is feathered.

Then according to the concept of \( PR \)-factorizable topological groups, every \( PR \)-factorizable topological group is \( P_M \)-factorizable. Therefore, every compact topological group is \( P_M \)-factorizable. Moreover, it is not difficult to see that a feathered \( M \)-factorizable topological group is \( P_M \)-factorizable.

**Theorem 3.3.** A topological group \( G \) is \( P_M \)-factorizable if and only if \( G \) is feathered \( M \)-factorizable.

**Proof.** The necessity is trivial.

The sufficiency of Theorem 3.3 can be shown as follows. If \( G \) is a feathered \( M \)-factorizable topological group, it follows from \([3\), Theorem 3.3] that either \( G \) is metrizable or \( G \) is \( R \)-factorizable. If \( G \) is metrizable, it is trivial. On the other hand, if \( G \) is an \( R \)-factorizable topological group, \( G \) is \( \omega \)-narrow by \([1\), Proposition 8.1.3]. Since a topological group is \( PR \)-factorizable if and only if it is \( \omega \)-narrow and feathered by \([3\), Theorem 2.6]. Therefore, \( G \) is a \( PR \)-factorizable topological group, and is also \( PM \)-factorizable, naturally.

Moreover, we show that each \( \omega \)-narrow \( PM \)-factorizable topological group is \( PR \)-factorizable.
**Theorem 3.4.** A topological group $G$ is PR-factorizable if and only if $G$ is PM-factorizable and $\omega$-narrow.

**Proof.** Since every PR-factorizable topological group is $\omega$-narrow, the necessity is trivial.

Let’s prove the sufficiency. Suppose that $G$ is a PM-factorizable and $\omega$-narrow topological group, $f : G \to \mathbb{R}$ is a continuous real-valued function. Then there exists a perfect homomorphism $\varphi : G \to K$ onto a metrizable topological group $K$ such that $f = g \circ \varphi$, where $g : K \to \mathbb{R}$ is continuous. Since $G$ is $\omega$-narrow, so is $K$. Since every first-countable $\omega$-narrow topological group is second-countable, we obtain that $G$ is PR-factorizable.

By Theorems 3.3 and 3.4, we obtain the following result.

**Corollary 3.5.** A topological group $G$ is PR-factorizable if and only if $G$ is feathered $\mathbb{R}$-factorizable.

Indeed, it was proved in [8, Theorem 3.4] that for a feathered topological group $G$, $G$ is $\omega$-narrow if and only if it is $\mathbb{R}$-factorizable. Moreover, [3, Theorem 2.6] presented that a topological group $G$ is PR-factorizable if and only if $G$ is $\omega$-narrow and feathered. As a topological group $G$ is $\mathbb{R}$-factorizable if and only if it is $\mathcal{M}$-factorizable and $\omega$-narrow by [10, Theorem 3.2], the Corollary 3.5 also can be obtained.

**Proposition 3.6.** A topological group $G$ is PR-factorizable if and only if $G$ is a feathered Lindelöf $\Sigma$-group.

**Proof.** The sufficiency is clear. Indeed, a feathered Lindelöf $\Sigma$-group $G$ is Lindelöf feathered, so $G$ is PR-factorizable, as a topological group is Lindelöf feathered if and only if it is PR-factorizable by [8, Theorem 2.5].

Then we show the necessity. Let $G$ be a PR-factorizable topological group. It follows from [8, Theorem 2.6] that $G$ is $\omega$-narrow and feathered. Then, by [3, Theorem 3.4], for a feathered topological group $G$, $G$ is $\omega$-narrow if and only if $G$ is a Lindelöf $\Sigma$-group.

**Theorem 3.7.** A topological group $G$ is PM-factorizable if and only if one of the following holds:

1. $G$ is metrizable;
2. $G$ is PR-factorizable.

**Proof.** Since every PR-factorizable topological group is PM-factorizable and every metrizable topological group is also PR-factorizable, the sufficiency is clear.

Then suppose that a PM-factorizable topological group $G$ is not metrizable. By Proposition 3.2, $G$ is feathered. Since a non-metrizable PM-factorizable topological group is $\mathbb{R}$-factorizable by [8, Theorem 3.3], $G$ is $\mathbb{R}$-factorizable and so $G$ is a PR-factorizable topological group.

Then from [11, Theorem 4.8], a locally compact group $G$ is $\mathcal{M}$-factorizable if and only if $G$ is metrizable or $G$ is $\sigma$-compact. Then, it is well-known that every locally compact topological group is feathered, so the characterization also holds for PM-factorizable topological groups by Theorem 3.3.

**Proposition 3.8.** A locally compact group $G$ is PM-factorizable if and only if one of the following conditions holds:

1. $G$ is metrizable;
2. $G$ is $\sigma$-compact.

It is well-known that a subgroup of an $\mathbb{R}$-factorizable topological group may not be $\mathbb{R}$-factorizable. Indeed, by [1, Example 8.2.1], there is an Abelian $P$-group $G$ and a dense subgroup $H$ of $G$ such that $G$ is Lindelöf, hence $R$-factorizable, but $H$ is not $R$-factorizable. In particular, $H$ is $\omega$-narrow. Therefore, $H$ is not $\mathcal{M}$-factorizable. Since every $\omega$-narrow topological group can be embedded into an $\mathbb{R}$-factorizable group as a closed invariant subgroup, see [1, Theorem 8.2.2], hence $\mathbb{R}$-factorizability is not closed-heredity for topological groups. However, W. He et al. showed that every subgroup of an $\mathcal{M}$-factorizable feathered group is $\mathcal{M}$-factorizable, it also means that every subgroup of a PM-factorizable is $\mathcal{M}$-factorizable. So it is easy to see that PM-factorizability is closed-heredity for topological groups.
Proposition 3.9. Every closed subgroup of a PM-factorizable topological group is PM-factorizable.

Proof. Let \( G \) be a PM-factorizable topological group and \( H \) a closed subgroup of \( G \). By Theorem 3.3, \( G \) is \( M \)-factorizable and feathered. According to [8, Lemma 4.1], every subgroup of an \( M \)-factorizable feathered group is \( M \)-factorizable, so \( H \) is a \( M \)-factorizable topological group. Moreover, it is well-known that a closed subspace of a feathered space is feathered. Hence, \( H \) is \( M \)-factorizable and feathered. We have that \( H \) is PM-factorizable by Theorem 3.3.

From [1, Theorem 3.4.4], every subgroup of an \( \omega \)-narrow topological group is \( \omega \)-narrow, so the following is clear.

Corollary 3.10. Every closed subgroup of a P\( \mathbb{R} \)-factorizable topological group is P\( \mathbb{R} \)-factorizable.

Theorem 3.11. If \( f : G \to H \) is an open continuous homomorphism of a PM-factorizable topological group onto a topological group \( H \), then \( H \) is PM-factorizable.

Proof. Indeed, it was proved in [10, Corollary 3.8] that a quotient group of a \( M \)-factorizable topological group is also \( M \)-factorizable. Moreover, by [1, Corollary 4.3.24], if \( f : G \to H \) is an open continuous homomorphism of a feathered topological group onto a topological group \( H \), then \( H \) is also feathered. Therefore, if \( G \) is PM-factorizable, that is, feathered and \( M \)-factorizable by Theorem 3.3, then the topological group \( H \) is also PM-factorizable as an open continuous homomorphic image.

From [1, Proposition 3.4.2], if a topological group \( H \) is a continuous homomorphic image of an \( \omega \)-narrow topological group \( G \), then \( H \) is also \( \omega \)-narrow. The following corollary is follows from Theorem 3.3.

Corollary 3.12. If \( f : G \to H \) is an open continuous homomorphism of a P\( \mathbb{R} \)-factorizable topological group onto a topological group \( H \), then \( H \) is P\( \mathbb{R} \)-factorizable.

4. Products of PM-factorizable topological groups

In this section, we investigate some properties about products of PM-factorizable topological groups. In particular, some interesting properties of \( M \)-factorizable topological groups in [5, 11] are strengthened to PM-factorizable topological groups. For example, the product \( G = \prod_{n \in \mathbb{N}} G_n \) of countably many PM-factorizable topological groups is PM-factorizable if and only if every factor \( G_n \) is metrizable or every \( G_n \) is P\( \mathbb{R} \)-factorizable, the product of a PM-factorizable topological group with a compact metrizable topological group is PM-factorizable.

First, according to the result that a locally compact group \( G \) is \( M \)-factorizable if and only if \( G \) is metrizable or \( G \) is \( \sigma \)-compact, H. Zhang et al. gave an example to show that a product of two \( M \)-factorizable topological groups may fail to be \( M \)-factorizable. By further observation about the example, we find that a product of two PM-factorizable topological groups may not be \( M \)-factorizable, so naturally not be PM-factorizable.

Example 4.1. Assume that \( G \) is a metrizable locally compact group which is not \( \sigma \)-compact and \( H \) is a compact and non-metrizable group. Obviously, both \( G \) and \( H \) are \( M \)-factorizable. Moreover, each locally compact topological group is feathered, then \( G \) and \( H \) both are PM-factorizable. However, the product group \( G \times H \) is neither metrizable nor \( \sigma \)-compact. Therefore, the product group \( G \times H \) is not PM-factorizable by Proposition 3.8. Indeed, since \( G \times H \) is feathered but not \( M \)-factorizable, it can also be yielded that it is not PM-factorizable by Theorem 3.3.

Theorem 4.2. The product \( G = \prod_{n \in \mathbb{N}} G_n \) of countably many PM-factorizable topological groups is PM-factorizable if and only if every factor \( G_n \) is metrizable or every \( G_n \) is P\( \mathbb{R} \)-factorizable.
Proof. It follows from [1, Proposition 4.3.13] that the product space $G$ is feathered. If $G$ is $PM$-factorizable, by Theorem [3.7] $G$ is either metrizable or $PR$-factorizable. If $G$ is metrizable, each $G_n$ is also metrizable. If $G$ is $PR$-factorizable, so is every $G_n$ by Corollary [3.12].

On the contrary, if every $G_n$ is metrizable, it is clear that $G$ is also metrizable. If every $G_n$ is a $PR$-factorizable topological group, it is easy to see that $G$ is $R$-factorizable. Moreover, $G$ is feathered, we conclude that $G$ is $PR$-factorizable by Corollary [3.9].

The product of countably many $PR$-factorizable topological groups is also $PR$-factorizable, see [3, Proposition 2.7], then it is clear to achieve the following by Theorems 3.7 and 4.2.

Proposition 4.3. If $G$ is a $PM$-factorizable topological group, then so is $G^\omega$.

Remark 4.4. Let $G$ be a compact group with $w(G) > \omega$ and $D$ an uncountable discrete group. Since $G$ and $D$ both are feathered, it is clear that $G \times D$ is feathered. However, $G$ is not metrizable and $D$ is not $PR$-factorizable, then $G \times D$ is not $PM$-factorizable, hence is also not $M$-factorizable by Theorem 5.3.

By Lemma 3.1 and Theorem 4.7 of [3], an $M$-factorizable topological group which contains a non-metrizable pseudocompact subspace is $\omega$-narrow.

Theorem 4.5. Let $G$ and $H$ be topological groups, where $G$ contains a non-metrizable pseudocompact subspace. If $G \times H$ is $PM$-factorizable, then $G \times H$ is $PR$-factorizable.

Proof. Since every $PM$-factorizable topological group is $M$-factorizable, it is clear that $H$ is $\omega$-narrow by [3, Theorem 5.7]. Since projection is an open continuous homomorphism, $G$ and $H$ both are $PM$-factorizable by Theorem 3.11. Then $G$ is $\omega$-narrow since $G$ contains a non-metrizable pseudocompact subspace. We have that the $PM$-factorizable topological group $G \times H$ is also $\omega$-narrow, hence is $PR$-factorizable by Theorem 3.4.

Theorem 4.6. Let $G$ be a feathered group and $K$ a pseudocompact feathered group. Then $G \times K$ is $PM$-factorizable if and only if either both $G$ and $K$ are metrizable or $G$ is $PR$-factorizable.

Proof. Let the product group $G \times K$ be $PM$-factorizable. Then the factors $G$ and $K$ are $PM$-factorizable as the open continuous images by Theorem 3.11. If $G \times K$ is not metrizable, then either $G$ or $K$ is not metrizable. If $G$ is not metrizable, it follows from Theorem 3.7 that $G$ is $PR$-factorizable. On the other case, if $K$ is not metrizable, $K$ is a non-metrizable pseudocompact topological group, then $G$ is $\omega$-narrow. By [3, Theorem 2.6], a topological group is feathered and $\omega$-narrow if and only if it is $PR$-factorizable, hence $G$ is a $PR$-factorizable topological group.

On the contrary, if $G$ and $K$ are metrizable topological groups, it is clear that $G \times K$ is $PM$-factorizable. If $G$ is $PR$-factorizable, then $G \times K$ is $R$-factorizable as $K$ is pseudocompact. Moreover, both $G$ and $K$ are feathered, then $G \times K$ is also feathered, which deduces that $G \times K$ is a $PR$-factorizable topological group by Corollary 5.5 hence is $PM$-factorizable.

Theorem 4.7. Let $G$ and $H$ be topological groups, where the Raikov completion $\hat{G}$ of $G$ contains a non-metrizable compact subspace. If $G \times H$ is $PM$-factorizable, then $H$ is $PR$-factorizable.

Proof. Since every $PM$-factorizable topological group is $M$-factorizable, it follows from [3, Theorem 3.11] that $H$ is pseudo-\(\omega\)-compact, hence $H$ is $\omega$-narrow by [1, Proposition 3.4.31]. Since the product $G \times H$ is $PM$-factorizable, so is $H$ by Theorem 3.11. Therefore, we have that $H$ is a $PR$-factorizable topological group by Theorem 3.4.

Proposition 4.8. If the product $G \times H$ of topological groups is $PM$-factorizable and the group $G$ is precompact, then either $G$ is second countable or $H$ is $PR$-factorizable.

Proof. The first part that $G$ is second countable follows just from [3, Proposition 3.12] and the second part is deduced by Theorem 4.7.
Theorem 4.9. Let $G$ be a feathered group and $H$ a precompact feathered group. Then $G \times H$ is $PM$-factorizable if and only if either both $G$ and $H$ are metrizable or $G$ is Lindelöf $\Sigma$-group.

Proof. The necessity is claimed in [3, Theorem 3.13], where $H$ just need to be precompact.

It suffices to prove the sufficiency. On the first case, if both $G$ and $H$ are metrizable, it is trivial that $G \times H$ is $PM$-factorizable. On the other case, let $G$ be a feathered Lindelöf $\Sigma$-group and $H$ a precompact feathered group. Then the Raikov completion $gH$ of $H$ is compact. $G \times H$ is $R$-factorizable as a subgroup of the Lindelöf $\Sigma$-group $G \times gH$. Moreover, since both $G$ and $H$ are feathered, $G \times H$ is also feathered, hence is $PR$-factorizable by Corollary 4.14. We obtain that $G \times H$ is a $PM$-factorizable topological group.

W. He et al. proved that the product of an $M$-factorizable topological group with a locally compact separable metrizable topological space is $M$-factorizable, see [3, Theorem 3.14].

Theorem 4.10. Let $G$ be a $PM$-factorizable topological group and $H$ a locally compact separable metrizable topological group. Then $G \times H$ is $PM$-factorizable.

Proof. Since the product of an $M$-factorizable topological group with a locally compact separable metrizable topological group is $M$-factorizable, $G \times H$ is $M$-factorizable. Moreover, both $G$ and $H$ are feathered as every locally compact group is feathered, so $G \times H$ is also feathered. Therefore, $G \times H$ is $PM$-factorizable by Theorem 4.10.

Remark 4.11. Indeed, by revising the proof to [3, Theorem 3.14], we can give a direct proof of Theorem 4.10.

Proof. Since $H$ is a locally compact separable metrizable group, $H$ is $\sigma$-compact. Then there is an increasing sequence $\{H_n : n \in \mathbb{N}\}$ of compact subsets of $H$ such that $H = \bigcup_{n \in \mathbb{N}} H_n$ and $H_n$ is contained in the interior of $H_{n+1}$ for each $n \in \mathbb{N}$. Let $f$ be a continuous real-valued function on $G \times H$. Denote by $C(H_n)$ the space of continuous real-valued functions on $H_n$ with sup-norm, for each $n \in \mathbb{N}$. Then define a mapping $\Psi_n : G \to C(H_n)$ by $\Psi_n(x)(y) = f(x, y)$ for all $x \in G$ and $y \in H_n$. Since $H_n$ is compact and second countable, $\Psi_n$ is continuous and $C(H_n)$ is second countable. Put $\Psi$ the diagonal product of $\{\Psi_n : n \in \mathbb{N}\}$. Since $\prod_{n \in \mathbb{N}} C(H_n)$ is second countable, it is clear that $\Psi(G)$ is also second countable.

By the hypothesis, $G$ is $PM$-factorizable, we can find a perfect homomorphism $\pi$ of $G$ onto a metrizable group $K$ and a continuous mapping $\psi$ of $K$ to $\Psi(G)$ such that $\Psi = \psi \circ \pi$. Take $y \in H$ and choose $n \in \mathbb{N}$ with $y \in H_n$. Let $x, x' \in G$. If $\pi(x) = \pi(x')$, then $\Psi(x) = \Psi(x')$. Then $\Psi_n(x) = \Psi_n(x')$, that is, $f(x, y) = f(x', y)$. Therefore, we can define a mapping $h : K \times H \to \mathbb{R}$ such that $f = h \circ (\pi \times id_H)$. It is not difficult to verify that $h$ is continuous. Since both $K$ and $H$ are metrizable topological groups, $G \times H$ is also metrizable. Moreover, $\pi$ is a perfect mapping, so is the mapping $\pi \times id_H$. Thus, we conclude that the product $G \times H$ is $PM$-factorizable.

Theorem 4.12. Let $G$ be a $PR$-factorizable topological group and $H$ a locally compact separable metrizable topological group. Then $G \times H$ is $PR$-factorizable.

Proof. First, $G \times H$ is $PM$-factorizable by Theorem 4.10. Moreover, since both $G$ and $H$ are $\omega$-narrow, then $G \times H$ is $\omega$-narrow and it is achieved that it is $PR$-factorizable by Theorem 3.4.

Corollary 4.13. Let $G$ be a $PM$-factorizable topological group and $H$ a compact metrizable topological group. Then $G \times H$ is $PM$-factorizable.

Corollary 4.14. Let $G$ be a $PR$-factorizable topological group and $H$ a compact metrizable topological group. Then $G \times H$ is $PR$-factorizable.

A topological space $X$ is called *pseudo-$\aleph_1$-compact* if every discrete family of open subsets of $X$ is countable. As we all know, every separable space is pseudo-$\aleph_1$-compact and every pseudo-$\aleph_1$-compact topological group is $\omega$-narrow. Therefore, the following result follows from Theorem 3.4.

Corollary 4.15. A pseudo-$\aleph_1$-compact $PM$-factorizable topological group is $PR$-factorizable.
Corollary 4.16. A product of a pseudo-$\aleph_1$-compact $PM$-factorizable topological group and a compact group is $P\mathbb{R}$-factorizable.

Proof. Indeed, let $G$ be a pseudo-$\aleph_1$-compact $PM$-factorizable topological group and $H$ a compact group. It was proved in [10, Corollary 5.2] that a product of a pseudo-$\aleph_1$-compact $M$-factorizable topological group and a compact group is $\mathbb{R}$-factorizable. Then the product $G \times H$ is $\mathbb{R}$-factorizable. Moreover, $G$ is feathered by Proposition 4.19 and $H$ is also feathered, so $G \times H$ is a feathered group, which deduces that $G \times H$ is $P\mathbb{R}$-factorizable by Corollary 4.3.

It was proved in [11, Theorem 5.4] that if $G \times K$ is $M$-factorizable, where $G$ is an $M$-factorizable group and $K$ is compact group, then $G$ is pseudo-$\aleph_1$-compact or $K$ is metrizable. Then we have the following by Corollaries 4.13 and 4.16.

Theorem 4.17. Let $G$ be a $PM$-factorizable topological group and $K$ a compact group. Then $G \times K$ is $PM$-factorizable if and only if one of the following conditions holds:

1. $K$ is metrizable;
2. $G$ is pseudo-$\aleph_1$-compact.

Recall that a mapping $f : X \to Y$ is $d$-open if for every open set $U$ in $X$, the image $f(U)$ is contained in the interior of the closure of $f(U)$. The following results was proved in [8], see Proposition 6.3 and Theorem 6.5.

Proposition 4.18. An image of a feathered topological group under a continuous $d$-open homomorphism is feathered.

Proposition 4.19. Let $p$ be a continuous $d$-open homomorphism from a topological group $G$ onto a topological group $H$. If $G$ is $M$-factorizable, so is $H$.

Since it is proved in Theorem 5.3 that a topological group $G$ is $PM$-factorizable if and only if $G$ is feathered $M$-factorizable, the following is deduced by two propositions above.

Corollary 4.20. If a topological group $H$ is a continuous $d$-open homomorphic image of a $PM$-factorizable topological group $G$, then $H$ is also $PM$-factorizable.

5. On $Pm$-factorizable topological groups

In this section, according to the concept of $m$-factorizable topological groups, that is, a topological group $G$ is called $m$-factorizable if for every continuous mapping $f : G \to M$ to a metrizable space $M$, there exists a continuous homomorphism $\pi : G \to K$ onto a second-countable group $K$ such that $f = g \circ \pi$, for some continuous homomorphism $g$ from $K$ onto $M$, we introduce $Pm$-factorizable topological groups as the following by strengthening the continuous homomorphism $\pi$ to a perfect homomorphism.

Definition 5.1. [1] A topological group $G$ is called $m$-factorizable if for every continuous mapping $f : G \to M$ to a metrizable space $M$, there exists a continuous homomorphism $\pi : G \to K$ onto a second-countable group $K$ such that $f = g \circ \pi$, for some continuous homomorphism $g$ from $K$ onto $M$.

In particular, if the continuous homomorphism $\pi$ is perfect, we call the topological group $G$ $Pm$-factorizable. Clearly, every $Pm$-factorizable topological group is $m$-factorizable and every $Pm$-factorizable topological group is $P\mathbb{R}$-factorizable. Then we show that a topological group $G$ is $Pm$-factorizable if and only if $G$ is $P\mathbb{R}$-factorizable and pseudo-$\aleph_1$-compact and the product of a $Pm$-factorizable topological group with an arbitrary compact group is $Pm$-factorizable.

First, by [1, Proposition 8.5.1], we just need to revise the continuous homomorphism $\pi$ to a perfect homomorphism $\pi$, we can obtain the following.
Proposition 5.2. A topological group $G$ is $Pm$-factorizable if and only if for every continuous pseudometric $d$ on $G$, there exist a perfect homomorphism $\pi : G \to K$ onto a second-countable topological group $K$ and a continuous pseudometric $g$ on $K$ such that $d(x,y) = g(\pi(x),\pi(y))$, for all $x,y \in G$.

Lemma 5.3. \[x] Lemma 3.2] Suppose that $f : G \to X$ is a continuous mapping of a $PR$-factorizable topological group $G$ to a Tychonoff space $X$ with $w(X) \leq \tau$. Then one can find a perfect homomorphism $\pi : G \to K$ onto a topological group $K$ with $w(K) \leq \tau$ such that $f = h \circ \pi$ for some continuous mapping $h : g(G) \to K$.

Theorem 5.4. A topological group $G$ is $Pm$-factorizable if and only if $G$ is $PR$-factorizable and pseudo-$\aleph_1$-compact.

Proof. Since every $Pm$-factorizable topological group is $m$-factorizable and every $Pm$-factorizable topological group is $PR$-factorizable, the necessity is clear since all $m$-factorizable topological groups are pseudo-$\aleph_1$-compact. It suffices to claim the sufficiency.

Assume that a topological group $G$ is $PR$-factorizable and pseudo-$\aleph_1$-compact and $f : G \to M$ is a continuous mapping of $G$ onto a metrizable space $M$. Since continuous mapping preserve pseudo-$\aleph_1$-compactness, $M$ is also pseudo-$\aleph_1$-compact, then $w(M) \leq \omega$. Since $G$ is $PR$-factorizable, it follows from Lemma 5.3 that we can find a perfect homomorphism $\pi : G \to K$ onto a second-countable topological group $K$ such that $f = g \circ \pi$ for some continuous mapping $g : K \to M$ onto the metrizable space $M$. Therefore, we have that $G$ is a $Pm$-factorizable topological group.

Since a topological group $G$ is $m$-factorizable if and only if $G$ is $\mathbb{R}$-factorizable and pseudo-$\aleph_1$-compact by \[x] Theorem 8.5.2, it is clear to deduce the following by Corollary 3.5.

Corollary 5.5. A topological group $G$ is $Pm$-factorizable if and only if $G$ is feathered and $m$-factorizable.

The following corollary is from Theorem 5.4 and Corollary 3.12.

Corollary 5.6. Let $\pi : G \to H$ be an open continuous homomorphism of a topological group $G$ onto $H$. If $G$ is $Pm$-factorizable, so is $H$.

By \[x] Theorem 8.5.8, if an $\mathbb{R}$-factorizable group $G$ satisfies $w(G) \leq \tau \geq \aleph_0$, then $|C(G)| \leq \tau^\omega$, where $C(X)$ denotes the set of continuous real-valued functions on a space $X$. Indeed, if $G$ is an $\mathbb{R}$-factorizable group with $w(G)^\omega < 2^{\aleph_1}$, then $G$ is pseudo-$\aleph_1$-compact. Then the following result follows from Theorem 5.4.

Corollary 5.7. Every $PR$-factorizable group $G$ with $w(G)^\omega < 2^{\aleph_1}$ is $Pm$-factorizable.

Corollary 5.8. Let $G$ be a $PR$-factorizable group with $w(G)^\omega < 2^{\aleph_1}$. Then the product $G \times K$ is $Pm$-factorizable, for every compact group $K$.

Lemma 5.9. Let $G$ be a topological group with the property that for every continuous function $f : G \to \mathbb{R}$, there exists a perfect homomorphism $\pi : G \to H$ onto a $PR$-factorizable group $H$ such that $f = g \circ \pi$ for some continuous function $g : H \to \mathbb{R}$. Then the group $G$ is $PR$-factorizable.

Proof. Let $f : G \to \mathbb{R}$ be a continuous function. By the hypothesis, there exists a perfect homomorphism $\pi : G \to H$ onto a $PR$-factorizable group $H$ and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ \pi$. Since $H$ is $PR$-factorizable, we can find a perfect homomorphism $p : H \to K$ onto a second-countable topological group $K$ and a continuous real-valued function $h$ on $K$ such that $g = h \circ p$.

$$
\begin{array}{ccc}
G & \xrightarrow{f} & \mathbb{R} \\
\downarrow{\pi} & & \downarrow{g} \\
H & \xrightarrow{h} & K \\
\downarrow{p} & & \\
K & & \\
\end{array}
$$

Let $\varphi = p \circ \pi$. Since both $p$ and $\pi$ are perfect, $\varphi$ is also perfect. Therefore, there is a perfect homomorphism $\varphi$ of $G$ onto $K$ satisfying $f = h \circ \varphi$. We conclude that $G$ is a $PR$-factorizable topological group.
Theorem 5.10. Let \( G \) be a \( Pm\)-factorizable topological group and \( K \) an arbitrary compact group, then the topological group \( G \times K \) is \( Pm\)-factorizable.

Proof. By Corollary 5.5, \( G \) is feathered and \( m\)-factorizable. Since the product group of every \( m\)-factorizable group with an arbitrary compact group is also \( m\)-factorizable, it is clear that \( G \times K \) is \( m\)-factorizable. Moreover, every \( Pm\)-factorizable topological group is feathered and every locally compact group is also feathered, so \( G \times K \) is feathered, which deduces that \( G \times K \) is \( Pm\)-factorizable. By Corollary 5.5.

Remark 5.11. The following is a direct proof of Theorem 5.10.

Proof. Let \( f : G \times K \to \mathbb{R} \) be a continuous function. \( C(K) \) denotes the space of all continuous real-valued functions on \( K \) with the sup-norm topology and consider the mapping \( \Psi : G \to C(K) \) defined by \( \Psi(x)(y) = f(x, y) \), for all \( x \in G \) and \( y \in K \). Since \( K \) is compact, \( \Psi \) is continuous. Since \( G \) is a \( Pm\)-factorizable topological group, it follows from Theorem 5.4 that \( G \) is \( P\mathbb{R}\)-factorizable and pseudo-\( K\)-compact, so the subspace \( \Psi(G) \) of the metric space \( C(K) \) is pseudo-\( K\)-compact and hence is second-countable. Since \( G \) is \( P\mathbb{R}\)-factorizable, we can find a perfect homomorphism \( \pi : G \to H \) onto a second-countable topological group \( H \) and a continuous mapping \( \psi : H \to C(K) \) such that \( \Psi = \psi \circ \pi \) by Lemma 5.3.

Then if \( x_1, x_2 \in G \) and \( \pi(x_1) = \pi(x_2) \), then \( f(x_1, y) = f(x_2, y) \) for each \( y \in K \). Suppose on the contrary, if \( f(x_1, y) \neq f(x_2, y) \) for some \( x_1, x_2 \in G \) and \( y \in K \), then \( \Psi(x_1)(y) = \Psi(x_2)(y) \). It is easy to see that \( \pi(x_1) \neq \pi(x_2) \). Therefore, we can define a mapping \( h : H \times K \to \mathbb{R} \) such that \( h \circ (\pi \times id_K) = f \), where \( id_K \) is the identity mapping of \( K \) onto itself.

\[
\begin{tikzcd}
G \times K \ar{r}{f} \ar{d}[swap]{\pi \times id_K} & \mathbb{R} \\
H \times K \ar{u}{h}
\end{tikzcd}
\]

Moreover, \( H \) is continuous. For an arbitrary point \((s, y) \in H \times K\) and a number \( \varepsilon > 0 \). Let \( x^* \in G \) be such that \( \pi(x^*) = s \). Since \( \psi \) is continuous, there exists an open neighborhood \( U \) of \( s \) in \( H \) such that \( ||\Psi(t) - \Psi(s)|| < \varepsilon/2 \) for each \( t \in U \). We can find a neighborhood \( V \) of \( y \) in \( K \) such that \( |f(x^*, z) - f(x^*, y)| < \varepsilon/2 \) for each \( z \in V \). Let \((t, z) \in U \times V \) and \( x \in G \) such that \( \pi(x) = t \). Then

\[
|h(t, z) - h(s, y)| \leq |f(x, z) - f(x^*, z)| + |f(x^*, z) - f(x^*, y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Therefore, \( h \) is continuous.

Since \( H \) is a second-countable topological group and \( K \) is a compact group, we know that \( H \times K \) is Lindelöf. Moreover, it is well-known that every metrizable topological group is feathered and every locally compact topological group is feathered, it follows that \( H \times K \) is feathered. As every Lindelöf topological group is \( \mathbb{R}\)-factorizable, we obtain that \( H \times K \) is \( P\mathbb{R}\)-factorizable by Corollary 5.5. Since \( f = h \circ (\pi \times id_K) \), it follows from Lemma 5.9 that \( G \times K \) is \( P\mathbb{R}\)-factorizable. Finally, as \( G \) is pseudo-\( K\)-compact and \( K \) is compact, the product \( G \times K \) is pseudo-\( K\)-compact and we conclude that \( G \times K \) is \( Pm\)-factorizable by Theorem 5.4.

In [1. Theorem 8.5.11], if the product \( G \times K \) of an \( \mathbb{R}\)-factorizable group \( G \) and a compact group \( K \) is \( \mathbb{R}\)-factorizable, then either \( G \) is pseudo-\( K\)-compact or \( K \) is metrizable. Therefore, by Theorem 5.10 and Corollary 4.14 we have the following result which is similar to Theorem 4.17.

Theorem 5.12. Let \( G \) be a \( P\mathbb{R}\)-factorizable topological group and \( K \) a compact group. Then \( G \times K \) is \( P\mathbb{R}\)-factorizable if and only if one of the following conditions holds:

1. \( K \) is metrizable;
2. \( G \) is pseudo-\( K\)-compact.

Proposition 5.13. A \( C\)-embedded closed subgroup of a \( P\mathbb{R}\)-factorizable topological group is \( P\mathbb{R}\)-factorizable.
Proof. Let $H$ be a $C$-embedded closed subgroup of a $PR$-factorizable topological group $G$. Then every continuous function $f : H \to \mathbb{R}$ admits an extension to a continuous function $g : G \to \mathbb{R}$. Since $G$ is $PR$-factorizable, there exists a perfect homomorphism $\pi : G \to K$ onto a second-countable group $K$ such that $g = h \circ \pi$ for some continuous function $h$ on $K$. Since $H$ is a closed subgroup of $G$, the homomorphism $\pi|_H$ of $H$ onto the subgroup $\pi(H)$ of $K$ is also perfect and factorizes $f$.

The following follows from the fact that every retract of a space $X$ is $C$-embedded in $X$.

**Corollary 5.14.** Let $G$ be a $PR$-factorizable topological group and $H$ a closed subgroup of $G$. If $H$ is a retract of $G$, then $H$ is also $PR$-factorizable.

**Corollary 5.15.** Let $G$ be a topological group. If $G \times \mathbb{Z}(2)^{\omega_1}$ is $PR$-factorizable, then $G$ is $Pm$-factorizable.

Proof. Let $K = \mathbb{Z}(2)^{\omega_1}$, where $\mathbb{Z}(2) = \{0, 1\}$ is the discrete group. Indeed, it was proved in [1] Theorem 8.5.5] that if $G \times \mathbb{Z}(2)^{\omega_1}$ is $R$-factorizable, $G$ is pseudo-$\aleph_1$-compact. Then it suffices to show that $G$ is $PR$-factorizable by Theorem 6.4.

Let $e_K$ be the identity of the group $K$. It is clear that $G \cong G \times \{e_K\}$ is closed in $G \times K$ and is a retract of $G \times K$. By Corollary 5.14, $G$ is $PR$-factorizable, which deduces that $G$ is a $Pm$-factorizable topological group.

6. On $SM$-factorizable topological groups

In this section, we pose the concepts of $SM$-factorizable topological groups and $PSM$-factorizable topological groups. We show that a topological group $G$ is $PSM$-factorizable if and only if $G$ is feathered $SM$-factorizable, and the properties of $SM$-factorizabilities and $PSM$-factorizabilities are preserved by open continuous homomorphisms on topological groups. Indeed, by the definition of $m$-factorizable topological groups, it is natural to extend $\mathbb{R}$ to a metrizable space, so we pose the following concept.

**Definition 6.1.** A topological group $G$ is called strongly $M$-factorizable ($SM$-factorizable for short) if for every continuous mapping $f : G \to M$ to a metrizable space $M$, there exists a continuous homomorphism $\pi : G \to H$ onto a metrizable group $H$ and a continuous mapping $g : H \to M$ such that $f = g \circ \pi$. In particular, if the continuous homomorphism $\pi : G \to H$ is perfect, we call $G$ $PSM$-factorizable.

\[
\begin{array}{ccc}
G & \xrightarrow{f} & M \\
\pi & \downarrow & \\
H & \xleftarrow{g} & \\
\end{array}
\]

Clearly, every $SM$-factorizable topological group is $M$-factorizable and every $m$-factorizable topological group is $SM$-factorizable.

**Theorem 6.2.** A topological group $G$ is $m$-factorizable if and only if $G$ is $SM$-factorizable and $\omega$-narrow.

Proof. Since every $m$-factorizable topological group is $R$-factorizable, hence is $\omega$-narrow, the necessity is trivial.

Let’s prove the sufficiency. Suppose that $G$ is an $SM$-factorizable and $\omega$-narrow topological group, $f : G \to M$ is a continuous mapping to a metrizable space $M$. Then there exists a continuous homomorphism $\varphi : G \to K$ onto a metrizable topological group $K$ such that $f = g \circ \varphi$, where $g : K \to M$ is a continuous mapping. Since $G$ is $\omega$-narrow, so is $K$. Since every first-countable $\omega$-narrow topological group is second-countable, we obtain that $G$ is $m$-factorizable.

Similarly, the following result also holds.
Proposition 6.3. A topological group $G$ is $P_m$-factorizable if and only if $G$ is $PSM$-factorizable and $\omega$-narrow.

Proposition 6.4. A topological group $G$ is $SM$-factorizable if and only if for every continuous pseudometric $d$ on $G$, one can find a continuous homomorphism $\pi : G \to K$ onto a metrizable topological group $K$ and a continuous pseudometric $g$ on $K$ such that $d(x, y) = g(\pi(x), \pi(y))$ for all $x, y \in G$.

Proof. Let $G$ be an $SM$-factorizable topological group and $d$ a continuous pseudometric on $G$. Consider the metric space $M = G/d$ with the associated metric $d^*$, obtained from $G$ by identifying points at zero distance with respect to $d$. Let $p: G \to G/d$ be the projection assigning to a point $x \in G$ the equivalence class $[x]$ consisting of all $z \in G$ with $d(x, z) = 0$. Then $d(x, y) = d^*(p(x), p(y))$, for all $x, y \in G$. By the hypothesis, $G$ is $SM$-factorizable, there exists a continuous homomorphism $\pi : G \to K$ onto a metrizable topological group $K$ and a continuous pseudometric $g$ on $K$ such that $p = h \circ \pi$. Define a continuous pseudometric $\rho$ on $G$ by $\rho(s, t) = d^*(h(s), h(t))$ for all $s, t \in K$. Then for all $x, y \in G$,

$$d(x, y) = d^*(p(x), p(y)) = d^*(h(\pi(x)), h(\pi(y))) = \rho(\pi(x), \pi(y)).$$

On the other hand, let $f : G \to M$ be a continuous mapping to a metric space $M$ with a metric $\kappa$. Define a continuous pseudometric $d$ on $G$ by $d(x, y) = \kappa(f(x), f(y))$ for all $x, y \in G$. By the hypothesis, there exists a continuous homomorphism $\pi : G \to K$ onto a metrizable topological group $K$ and a continuous pseudometric $\rho$ on $K$ such that $\rho(\pi(x), \pi(y)) = \kappa(f(x), f(y))$, for all $x, y \in G$. Therefore, $\rho(\pi(x), \pi(y)) = \kappa(f(x), f(y))$, for all $x, y \in G$. Then $\pi(x) = \pi(y)$ always implies $f(x) = f(y)$. We can find a continuous mapping $h : K \to M$ such that $f = h \circ \pi$. Thus, we conclude that $G$ is $SM$-factorizable.

By Proposition 6.3, it is not difficult to see that if we change the continuous homomorphism $\pi$ onto a perfect homomorphism, it also holds for $PSM$-factorizable topological groups. Then we show that every feathered $SM$-factorizable is $PSM$-factorizable.

Theorem 6.5. A topological group $G$ is $PSM$-factorizable if and only if $G$ is feathered $SM$-factorizable.

Proof. The necessity is trivial, it suffices to claim the sufficiency.

Let $G$ be a feathered $SM$-factorizable topological group and $f : G \to H$ a continuous mapping to a metrizable space $E$. Then we can find a continuous homomorphism $\pi : G \to H$ onto a metrizable topological group $H$ and a continuous mapping $g : H \to E$ such that $f = g \circ \pi$. Since every $M$-factorizable group is $\omega$-balanced by [10, Theorem 3.1] and $G$ is feathered, there exists a perfect homomorphism $p : G \to K$ onto a metrizable topological group $K$. Let $\varphi$ be the diagonal product of the homomorphisms $\pi$ and $p$ and $M = \varphi(G) \subseteq H \times K$. Since $p$ is perfect, the homomorphism $\varphi$ is also perfect by [2, Theorem 3.7.11]. By the definition of $\varphi$, we can find continuous homomorphisms $q_H : M \to H$ and $q_K : M \to K$ satisfying $\pi = q_H \circ \varphi$ and $p = q_K \circ \varphi$. Then from [1, Proposition 3.7.5], it follows that $q_K$ is perfect.

Then the subgroup $M$ of $H \times K$ is metrizable. Indeed, since $H$ and $K$ are both first-countable and the property of first-countability is hereditary, it is clear that the subgroup $M$ of $H \times K$ is first-countable, hence $M$ is metrizable.

```
                  E
                   ↗
                    f
                     ↗
                     g
                    ↗
                   G
                  ↗   ↗
                 z   ϕ
                 ↗   ↗
                H   q_H
                  ↗         ↗
                 p   ↘   ↘
                ↗   ↗           ↗
               K   ϕ   q_K
                ↗   ↗
               q_K
```

We define a continuous mapping $h : M \to E$ by $h = q \circ q_H$. Then, for each continuous mapping $f : G \to E$ to a metrizable space $E$, we can find a perfect homomorphism $\varphi : G \to M$ onto a metrizable topological group $M$ and a continuous mapping $h : M \to E$ such that $f = h \circ \varphi$. Therefore, we conclude that $G$ is $PSM$-factorizable.
Lemma 6.6. Let \( \{U_n : n \in \mathbb{N}\} \) be a family of neighborhoods of the identity \( e \) in an \( \omega \)-balanced group \( G \). Then there exists a continuous homomorphism \( p : G \to H \) onto a metrizable topological group \( H \) and a family \( \{V_n : n \in \mathbb{N}\} \) of open neighborhoods of the identity \( e_H \) in \( H \) such that \( p^{-1}(V_n) \subseteq U_n \), for each \( n \in \mathbb{N} \).

**Proof.** It is well-known that if \( G \) is an \( \omega \)-balanced topological group, for each open neighborhood \( U \) of the identity element \( e \) in \( G \), there exists a continuous homomorphism \( \pi \) of \( G \) onto a metrizable group \( H \) such that \( \pi^{-1}(V) \subseteq U \), for some open neighborhood \( V \) of the identity element \( e_H \) of \( H \), see [2, Theorem 4.3.18]. Then for every \( n \in \mathbb{N} \), we can find a continuous homomorphism \( \pi_n \) of \( G \) onto a metrizable topological group \( H_n \) and an open neighborhood \( W_n \) of the identity element in \( H_n \) such that \( \pi_n^{-1}(W_n) \subseteq U_n \). Let \( \pi \) be the diagonal product of the family \( \{\pi_n : n \in \mathbb{N}\} \). Then \( \pi = \pi(G) \) is a subgroup of the product \( P = \prod_{n \in \mathbb{N}} H_n \), so the group \( H \) is also metrizable. Denote by \( p_n \) the projection of \( P \) onto the factor \( H_n \). Then \( \pi_n = p_n \circ \pi \) for each \( n \in \mathbb{N} \). The open neighborhoods \( V_n = H \cap \pi_n^{-1}(W_n) \) of the identity in \( H \) is such that \( \pi^{-1}(V_n) = \pi_n^{-1}(W_n) \subseteq U_n \).

**Theorem 6.7.** Let \( \pi : G \to H \) be an open continuous homomorphism of a topological group \( G \) onto \( H \). If \( G \) is SM-factorizable, so is \( H \).

**Proof.** Let \( f : H \to M \) be a continuous mapping to a metrizable space \( M \). Then \( f \circ \pi : G \to M \) is a continuous mapping. Since \( G \) is SM-factorizable, there exists a continuous homomorphism \( \varphi : G \to K \) onto a metrizable topological group \( K \) and a continuous mapping \( g : K \to M \) such that \( f \circ \pi = g \circ \varphi \). Since \( K \) is a first-countable topological group, we can find a countable local base \( \{U_n : n \in \mathbb{N}\} \) at the identity \( e_K \) of \( K \) and put \( V_n = \pi(\varphi^{-1}(U_n)) \), for each \( n \in \mathbb{N} \), \( V_n \) is an open neighborhood of the identity \( e_H \) in \( H \), for each \( n \in \mathbb{N} \).

In the metrizable space \( M \), the metric which generates the original topology is denoted by \( d \). For each \( m \in M \) and arbitrary \( \varepsilon > 0 \), let \( O_\varepsilon(m) = \{t \in M : d(m, t) < \varepsilon\} \). For each \( h \in H \) and \( \varepsilon > 0 \), choose \( g \in G \) with \( \pi(g) = h \) and put \( x = \varphi(g) \). Then \( f(h) = f(\pi(g)) = g(\varphi(g)) = g(x) \). Since \( g \) is continuous, there exists \( n \in \mathbb{N} \) such that \( g(xU_n) \subseteq O_\varepsilon(f(h)) \). Since \( V_n = \pi(\varphi^{-1}(U_n)) \), for each \( n \in \mathbb{N} \), it is easy to see that \( f(hV_n) \subseteq O_\varepsilon(f(h)) \) for some \( n \in \mathbb{N} \).

Since \( \pi : G \to H \) is an open continuous homomorphism of an \( \omega \)-balanced topological group \( G \) onto topological group \( H \), \( H \) is also \( \omega \)-balanced. Then there exists a continuous homomorphism \( p : H \to L \) onto a metrizable topological group \( L \) with a local base \( \{W_n : n \in \mathbb{N}\} \) at the identity such that \( p^{-1}(W_n) \subseteq V_n \) for each \( n \in \mathbb{N} \) by Lemma 6.6. Put \( N \) the kernel of \( p \). Then \( N \subseteq \bigcap_{n \in \mathbb{N}} V_n \). Therefore, \( f \) is constant on every coset \( hN \) in \( H \). Then we can define a mapping \( h : L \to M \) such that \( h \circ p = f \). It is not difficult to see that for every \( g \in L \) and arbitrary \( \varepsilon \), there exists \( n \in \mathbb{N} \) such that \( h(gW_n) \subseteq O_\varepsilon(h(y)) \), which means that \( h \) is continuous. We conclude that \( H \) is also SM-factorizable.

If \( f : G \to H \) is an open continuous homomorphism of a feathered group \( G \) onto a group \( H \), \( H \) is also feathered, see [1, Corollary 4.3.24], the following corollary is clear by Theorems 6.5 and 6.7.

**Corollary 6.8.** Suppose that \( \pi : G \to H \) is an open continuous homomorphism of a PSM-factorizable topological group \( G \) onto \( H \), then \( H \) is also PSM-factorizable.

It was proved in [10, Proposition 5.1] that a pseudo-\( N_1 \)-compact and \( M \)-factorizable topological group is \( R \)-factorizable. Moreover, it is well-known that a topological group is \( m \)-factorizable if and only if it is \( R \)-factorizable and pseudo-\( N_1 \)-compact.

**Proposition 6.9.** Every pseudo-\( N_1 \)-compact \( M \)-factorizable topological group is SM-factorizable.
During the following figure, for convenience, $\mathbb{R}$ denotes $\mathbb{R}$-factorizable topological groups, $\mathcal{M}$ denotes $\mathcal{M}$-factorizable topological groups, and so on. Moreover, the two arrows represent stronger properties, such as every $P\mathbb{R}$-factorizable topological group is $\mathbb{R}$-factorizable, etc. The notions $+nar$, $+fea$ and $+pse$ represent adding the properties of $\omega$-narrow, feathered and pseudo-$\aleph_1$-compact, respectively, and then the single arrow will hold, that is, two spaces at both ends of the arrow will be equivalent.

However, the property of pseudo-$\aleph_1$-compact is too strong in Proposition 6.9, so it is natural to consider what properties can be added to an $\mathcal{M}$-factorizable topological group to imply it is $\mathcal{SM}$-factorizable and every $\mathcal{SM}$-factorizable topological group obtains the properties at the same time.

Finally, the following question is posed naturally.

**Question 6.10.** Is there an $\mathcal{SM}$-factorizable topological group but not $m$-factorizable?

**References**

[1] A.V. Arhangel’skii, M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Press and World Sci., 2008.
[2] R. Engelking, *General Topology (revised and completed edition)*, Heldermann Verlag, Berlin, 1989.
[3] L. Peng, Y. Liu, *On Lindelöf feathered topological groups*, Topol. Appl., 285 (2020) 107405.
[4] M. G. Tkachenko, *Factorization theorems for topological groups and its applications*, Topol. Appl., 38 (1991) 21–37.
[5] M. G. Tkachenko, *Homomorphic images of $\mathbb{R}$-factorizable groups*, Comment. Math. Univ. Carol., 47 (3) (2006) 525–537.
[6] M. G. Tkachenko, *Hereditarily $\mathbb{R}$-factorizable paratopological groups*, Topol. Appl., 157 (2010) 1548–1557.
[7] M. G. Tkachenko, *Products of $\mathbb{R}$-factorizable groups*, Topol. Proc., 39 (2012) 167–184.
[8] W. He, D. Peng, M. Tkachenko, H. Zhang, *$\mathcal{M}$-factorizable feathered topological groups*, Topol. Appl., 289 (2021) 107481.
[9] W. He, D. Peng, M. Tkachenko, H. Zhang, *$\mathcal{M}$-factorizability of products and $\tau$-fine topological groups*, Topol. Appl., 296 (2021) 107674.
[10] H. Zhang, D. Peng, W. He, *On $\mathcal{M}$-factorizable topological groups*, Topol. Appl., 274 (2020) 107126.