Superadditivity of convex roof coherence measures

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Abstract
In this paper, we examine the superadditivity of convex roof coherence measures. We put forward a theorem on the superadditivity of convex roof coherence measures, which provides a sufficient condition to identify the convex roof coherence measures fulfilling the superadditivity. By applying the theorem to each of the known convex roof coherence measures, we prove that the coherence of formation and the coherence concurrence are superadditive, while the geometric measure of coherence, the convex roof coherence measure based on linear entropy, the convex roof coherence measure based on fidelity, and convex roof coherence measure based on $\frac{1}{2}$-entropy are non-superadditive.

Keywords: convex roof, coherence measures, superadditivity

1. Introduction
Quantum coherence is an essential feature of quantum mechanics which is responsible for the departure between the classical and quantum world. It is an important component in quantum information processing [1], and plays a central role in emergent fields, such as quantum metrology [2, 3], nanoscale thermodynamics [4–6], and quantum biology [7–10]. Recently, the quantification of coherence has attracted a growing interest due to the development of quantum information science [11–52].

By adopting the viewpoint of coherence as a physical resource, Baumgratz et al proposed a seminal framework for quantifying coherence [14]. In that framework, a functional of states can be taken as a coherence measure if it fulfills four conditions, namely, the coherence being zero (positive) for incoherent states (all other states), the monotonicity of coherence under incoherent operations, the monotonicity of coherence under selective measurements on average, and the nonincreasing of coherence under mixing of quantum states. By following the framework, a number of coherence measures have been found. Some of them are defined based on the distance between the state under consideration to the set of incoherent states.
such as the $l_1$ norm of coherence \cite{14}, the relative entropy of coherence \cite{14} and the robustness of coherence \cite{16}, while others are defined based on the convex roof construction \cite{11, 22, 44–47, 49}, such as the coherence of formation \cite{11, 33, 45}, the geometric measure of coherence \cite{22}, and the coherence concurrence \cite{46}, where the coherence of a mixed state is quantified by the weighted sum of the coherence of the pure states in a decomposition of the mixed state, minimized over all possible decompositions. With these coherence measures, various topics of quantum coherence, such as the dynamics of coherence \cite{28, 35}, the distillation of coherence \cite{13, 33, 45}, and the relations between quantum coherence and quantum correlations \cite{17, 20–25, 41} have been investigated.

Another interesting topic of quantum coherence is the superadditivity of a coherence measure. A coherence measure $C$ is said to be superadditive if the relation,

$$C(\rho_{A|B}) \geq C(\rho_A) + C(\rho_B), \quad (1.1)$$

is valid for all density matrices $\rho_{A|B}$ of a finite-dimensional system with respect to a particular reference basis $\{|i\rangle_A \otimes |j\rangle_B\}$, where $\rho_A = \text{tr}_B \rho_{A|B}$ and $\rho_B = \text{tr}_A \rho_{A|B}$ are with respect to the basis $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$, respectively. The superadditivity of a coherence measure describes the trade-off relations between the coherence of a bipartite system and that of its subsystems and it is a precondition of defining a discordlike correlation based on the coherence measure \cite{24, 25}. Investigations on this topic have been started recently \cite{21, 24–27}. The superadditivity of the relative entropy of coherence was first proved in \cite{21}, and based on the superadditivity of the relative entropy of coherence, the discordlike correlations were established \cite{25, 26}. The superadditivity of the $l_1$ norm of coherence was then proved in \cite{24}, and based on it a correlated coherence describing the relationship between bipartite coherence and quantum correlations is defined. It was recently proved that the robustness of coherence is non-superadditive, i.e. not satisfying the superadditivity \cite{27}. Therefore, the superadditivity or non-superadditivity of all the known three coherence measures defined based on distance have been resolved. However, the superadditivity of convex roof coherence measures remains unresolved. Since convex roof coherence measures involve an optimization process, they usually do not admit a closed form expression for mixed states although they typically admit a closed form expression for pure states. Thus, it is more difficult in general to prove whether the superadditivity is valid for a convex roof coherence measure than that for a distance-based coherence measure.

In this paper, we address the issue: which of the known convex roof coherence measures are superadditive and which are non-superadditive? To examine the superadditivity of a convex roof coherence measure, we will put forward a theorem, which provides a sufficient condition to identify the convex roof coherence measures fulfilling the superadditivity. By applying the theorem to each of the known convex roof coherence measures, we find that the coherence of formation and the coherence concurrence are superadditive, while the geometric measure of coherence, the convex roof coherence measure based on linear entropy, the convex roof coherence measure based on fidelity, and convex roof coherence measure based on $\frac{1}{2}$-entropy are non-superadditive.

2. Convex roof coherence measures

To present our findings clearly, we first recapitulate some notions related to our topic. Coherence of a state is measured with respect to a particular reference basis, whose choice is dictated by the physical scenario under consideration. If the particular basis is denoted as $\{|i\rangle, \; i = 1, 2, \cdots, d\}$, an incoherent state is then defined as $\delta = \sum_i p_i |i\rangle \langle i|$, where $p_i$ are probabilities with $\sum_i p_i = 1$. The set of all incoherent states is denoted by $\mathcal{I}$. All other states
which cannot be written as diagonal matrices in this basis are called coherent states. We use ρ to represent a general state, and δ specially to denote an incoherent state. An incoherent operation is defined as a completely positive trace-preserving map, Λ(ρ) = ∑nKnρK† n, where the Kraus operators Kn satisfy not only ∑nKnK† n = I but also KnIK† n < I for each Kn, i.e. each Kn maps an incoherent state to an incoherent state. With these notions, Baumgratz et al proposed a rigorous framework for quantifying coherence, which can be stated as follows [14].

A functional C can be taken as a coherence measure if it satisfies the four conditions:

(C1) C(ρ) ≥ 0, and C(ρ) = 0 if and only if ρ ∈ I;
(C2) Monotonicity under incoherent operations, C(ρ) ≥ C(Λ(ρ)) if Λ is an incoherent operation;
(C3) Monotonicity under selective incoherent operations, C(ρ) ≥ ∑n pnC(ρn), where

\[ p_n = \text{Tr}(K_nρK_n^†), \quad ρ_n = K_nρK_n^†/p_n, \quad \text{and} \quad Λ(ρ) = ∑n K_nρK_n^† \]

is an incoherent operation;
(C4) Non-increasing under mixing of quantum states, i.e. convexity, C(∑n pnρn) ≥ C(∑npnρn) for any set of states {ρn} and any probability distribution {pn}.

Based on the rigorous framework, various coherence measures can be constructed. A main family of them are so called convex roof coherence measures, which are defined by extending the J. Phys. A: Math. Theor. which can steer clear of the difficulty. It can be stated as a theorem. Put forward an approach to examine the superadditivity of a convex roof coherence measure, and mixed states. The difficulty appears in calculating the coherence of mixed states. We here measure being superadditive means that equation (1.1) is valid for all states, including all pure states. A coherence measure can be said to prove a convex roof coherence measure non-superadditive. Indeed, a coherence measure can be easy to prove a convex roof coherence measure non-superadditive if a counterexample of violating equation (1.1) is found, but a coherence measure being superadditive means that equation (1.1) is valid for all states, including all pure and mixed states. The difficulty appears in calculating the coherence of mixed states. We here put forward an approach to examine the superadditivity of a convex roof coherence measure, which can steer clear of the difficulty. It can be stated as a theorem.

Theorem. A convex roof coherence measure C_f is superadditive if the inequality,

\[ C_f(|ϕ⟩⟨ϕ|_A) ≥ C_f \left( \sum_i q_i |ϕ⟩⟨ϕ|_A^i \right) + \sum_i q_i C_f (|ϕ_i⟩⟨ϕ_i|_B), \]

is satisfied for all pure states |ϕ⟩⟨ϕ|_A = ∑q_i |ϕ_i⟩⟨ϕ_i|_A with \( \sum_i |q_i|^2 = 1 \), where \( q_i = \sum_j |c_{ij}|^2 \) and \( |ϕ_i⟩_B = \frac{1}{\sqrt{q_i}} \sum_j c_{ij} |j⟩_B \).
We prove the theorem as follows. First, we prove that if a coherence measure $C_f$ satisfies equation (3.1), then the superadditivity relation (1.1) is fulfilled for all pure state $|\varphi\rangle_{AB}$. To this end, we only need to prove

$$C_f\left(\sum_i \sqrt{q_i} |i\rangle_A\right) \geq C_f(\rho_A),$$

(3.2)

and

$$\sum_i q_i C_f(|\varphi_i\rangle_B) \geq C_f(\rho_B),$$

(3.3)

where $\rho_A = \text{tr}_B \rho_{AB} = \sum_{ij} c_{ij} c_{ij}^* |i\rangle_A \langle j|$ and $\rho_B = \text{tr}_A \rho_{AB} = \sum_i q_i |\varphi_i\rangle_B \langle \varphi_i|.$

To prove equation (3.2), we demonstrate that there exists an incoherent operation that can map $\sum_i \sqrt{q_i} |i\rangle_A$ to $\rho_A$. In fact, such an operation can be simply taken as $\Lambda(\cdot) = \sum_{i=1}^{d^a} K_i \cdot I_i$ with $K_i = \sum_{i=1}^{d^a} \frac{1}{\sqrt{q_i}} |i\rangle \langle i|$. Obviously, the operation defined by $\Lambda$ is incoherent, and it is straightforward to verify that $\Lambda(\sum_i \sqrt{q_i} |i\rangle_A) = \rho_A$. Noting that an incoherent operation can never increase the coherence of a state, we then obtain $C_f\left(\sum_i \sqrt{q_i} |i\rangle_A\right) \geq C_f(\rho_A)$, i.e. equation (3.2).

To prove equation (3.3), we use $\rho_B = \sum_i p_i |\psi_i\rangle_B \langle \psi_i|$ to represent the optimal decomposition of $\rho_B$ that achieves the infimum in equation (2.1). Since $\rho_B = \sum_i q_i |\varphi_i\rangle_B \langle \varphi_i|$ is also an ensemble decomposition of $\rho_B$, there must be $\sum_i q_i C_f(|\varphi_i\rangle_B) \geq \sum_i p_i C_f(|\psi_i\rangle_B) = C_f(\rho_B)$, i.e. equation (3.2). We then obtain

$$C(|\varphi\rangle_{AB}) \geq C(\rho_A) + C(\rho_B).$$

(3.4)

Second, we prove that $C_f$ is superadditive for all states if it is superadditive for pure states $\rho_{AB} = |\varphi\rangle_{AB} \langle \varphi|$. To this end, we use $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$ to represent one of the optimal decompositions that give $C_f(\rho_{AB})$. By using equation (3.4), we have

$$C_f(\rho_{AB}) = \sum_i p_i C_f(|\psi_i\rangle_{AB}) \geq \sum_i p_i \left( C_f(\rho^A_i) + C_f(\rho^B_i) \right),$$

(3.5)

where $\rho^A_i = \text{tr}_B |\psi_i\rangle_{AB} \langle \psi_i|$ and $\rho^B_i = \text{tr}_A |\psi_i\rangle_{AB} \langle \psi_i|$. According to the convexity of a coherence measure, i.e. condition (C4), there are $\sum_i p_i C_f(\rho^A_i) \geq C_f(\sum_i p_i \rho^A_i)$ and $\sum_i p_i C_f(\rho^B_i) \geq C_f(\sum_i p_i \rho^B_i)$, which lead to

$$C_f(\rho_{AB}) \geq C_f(\sum_i p_i \rho^A_i) + C_f(\sum_i p_i \rho^B_i).$$

(3.6)

Noting that $\rho_A = \sum_i p_i \rho^A_i$ and $\rho_B = \sum_i p_i \rho^B_i$, we finally obtain

$$C_f(\rho_{AB}) \geq C_f(\rho_A) + C_f(\rho_B).$$

(3.7)

This completes the proof of the theorem.

4. Applications of the theorem

The above theorem only involves pure states but has nothing to do with mixed states. By verifying the validity of the inequality (3.1) for pure states $|\varphi\rangle_{AB}$, one can conclude that equation (1.1) is valid for all states $\rho_{AB}$, i.e. $C_f$ is of superadditivity. This greatly simplifies the calculations and makes it possible to prove whether a convex roof coherence measure is superadditive. In the following, we will apply our theorem to each of the known convex roof coherence measures to find which of them are superadditive.
4.1. The coherence of formation

We show that the coherence of formation is superadditive.

The coherence of formation is defined as

\[
C_{\text{for}}(\rho) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i C_r(|\phi_i\rangle),
\]

(4.1)

where \(C_r(|\phi\rangle) = S(\Delta(|\phi\rangle\langle\phi|))\) with \(S(\rho) = -\text{Tr} \rho \log_2 \rho\) being the von Neumann entropy. Hereafter, we use \(\Delta(\rho)\) to denote the diagonal part of \(\rho\), i.e. \(\Delta(\rho) = \sum_i \rho_{ii} |i\rangle \langle i|\). The coherence of formation was first put forward in [4], and it was proved to be a coherence measure, i.e. satisfying the conditions (C1)–(C4), later in [45].

To prove the superadditivity of the coherence of formation, we only need to examine the inequality (3.1) for pure states \(|\phi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A |j\rangle_B\). Substituting \(|\phi\rangle_{AB}\) into \(C_r(|\phi\rangle_{AB}) = S(\Delta(|\phi\rangle_{AB}\langle\phi|))\), we have

\[
C_{\text{for}}(|\phi\rangle_{AB}) = -\sum_j |c_{ij}|^2 \log_2 |c_{ij}|^2.
\]

(4.2)

On the other hand, there are

\[
C_{\text{for}}(\sum_i \sqrt{q_i} |i\rangle_A) = - \sum_i q_i \log_2 q_i,
\]

(4.3)

\[
\sum_j q_j C_{\text{for}}(|\phi_j\rangle_B) = \sum_j q_j \left( \sum_i \frac{|c_{ij}|^2}{q_i} \log_2 \frac{|c_{ij}|^2}{q_i} \right) = -\sum_j |c_{ij}|^2 \log_2 |c_{ij}|^2 q_i,
\]

(4.4)

and therefore

\[
C_{\text{for}}(\sum_i \sqrt{q_i} |i\rangle_A) + \sum_j q_j C_{\text{for}}(|\phi_j\rangle_B) = \sum_i q_i \log_2 q_i - \sum_{ij} |c_{ij}|^2 \log_2 \frac{|c_{ij}|^2}{q_i} = -\sum_j |c_{ij}|^2 \log_2 |c_{ij}|^2 q_i,
\]

(4.5)

Comparing equation (4.2) with equation (4.5), we immediately obtain

\[
C_{\text{for}}(|\phi\rangle_{AB}) = C_{\text{for}}(\sum_i \sqrt{q_i} |i\rangle_A) + \sum_j q_j C_{\text{for}}(|\phi_j\rangle_B),
\]

which means that equation (3.1) is fulfilled and therefore the coherence of formation is superadditive.

4.2. The coherence concurrence

We show that the coherence concurrence is superadditive.

The coherence concurrence is defined as

\[
C_C(\rho) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i C_l(|\phi_i\rangle),
\]

(4.6)

where \(C_l(\rho) = \sum_{i \neq j} |\rho_{ij}|\) is the \(l_1\) norm of coherence [14]. The coherence concurrence was first put forward in [44], and rigourously proved in [46].

To prove the superadditivity of the coherence concurrence, we calculate \(C_C(|\phi\rangle_{AB})\) with \(|\phi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A |j\rangle_B\), and have
\[ C_C(|\varphi\rangle_{AB}) = \left( \sum_{i,k} |c_{ik}|^2 \right)^2 - 1. \]  

(4.7)

On the other hand, there are

\[ C_C\left( \sum_i \sqrt{q_i} |i\rangle_A \right) = \sum_{i\neq j} \sqrt{\sum_{k,l} |c_{ik} c_{jl}|}, \]

(4.8)

\[ \sum_i q_i C_C(|\varphi_i\rangle_B) = \sum_{i,k,l} |c_{ik} c_{jl}| - 1, \]

(4.9)

and therefore

\[ C_C\left( \sum_i \sqrt{q_i} |i\rangle_A \right) + \sum_i q_i C_C(|\varphi_i\rangle_B) = \sum_{i\neq j} \sqrt{\sum_{k,l} |c_{ik} c_{jl}|^2} + \sum_{i,k,l} |c_{ik} c_{jl}| - 1 \]

\[ \leq \sum_{i\neq j} \sum_{k,l} |c_{ik} c_{jl}| + \sum_{i,k,l} |c_{ik} c_{jl}| - 1 \]

\[ = \sum_{i,k,l} |c_{ik} c_{jl}| - 1. \]  

(4.10)

Comparing equation (4.7) with equation (4.10), we immediately obtain \( C_C(|\varphi\rangle_{AB}) \geq C_C\left( \sum_i \sqrt{q_i} |i\rangle_A \right) + \sum_i q_i C_C(|\varphi_i\rangle_B) \), which means that equation (3.1) is fulfilled and therefore the coherence concurrence is superadditive.

### 4.3. The geometric measure of coherence

We show that the geometric measure of coherence is non-superadditive.

The geometric measure of coherence is defined as

\[ C_{g}\langle \rho \rangle = \inf_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i (1 - F(|\varphi_i\rangle, \delta)), \]

(4.11)

where \( F(\rho, \delta) = (\text{Tr}(\sqrt[\delta]{\rho} \sqrt[\delta]{\rho}))^2 \) is the Uhlmann fidelity [53]. This measure was put forward in [22]. There is \( C_{g}\langle |\varphi\rangle \rangle = 1 - |c|^2_{\text{max}} \) for pure states \( |\varphi\rangle = \sum_i c_i |i\rangle \) [52].

To prove the geometric measure of coherence non-superadditive, we give a counterexample to inequality (3.1). The counterexample can be taken as \( |\varphi\rangle_{AB} = \frac{1}{2}(|11\rangle + |12\rangle + |21\rangle + |22\rangle) \). For this state, we have \( C_{g}\langle |\varphi\rangle_{AB} \rangle = \frac{1}{2} \), \( C_{g}\left( \sum_i \sqrt{q_i} |i\rangle_A \right) = C_{g}\left( \frac{1}{\sqrt{2}} |1\rangle_A + \frac{1}{\sqrt{2}} |2\rangle_A \right) = \frac{1}{2} \), and \( \sum_i q_i C_{g}\langle |\varphi_i\rangle_B \rangle = C_{g}\left( \frac{1}{\sqrt{2}} |1\rangle_B + \frac{1}{\sqrt{2}} |2\rangle_B \right) = \frac{1}{2} \). Then, there is \( C_{g}\langle |\varphi\rangle_{AB} \rangle \leq \frac{1}{2} < C_{g}\left( \sum_i \sqrt{q_i} |i\rangle_A \right) + \sum_i q_i C_{g}\langle |\varphi_i\rangle_B \rangle = 1 \), which violates the condition in the theorem. In this case, it is suspected that the geometric measure of coherence is non-superadditive. However, its non-superadditivity cannot be decided only by the violation of the inequality (3.1), since the inequality in our theorem is only a sufficient condition of superadditivity. To confirm the non-superadditivity of \( C_g \), we use the definition relation of superadditivity, i.e. equation (1.1). In fact, since \( |\varphi\rangle_{AB} = \frac{1}{2}(|11\rangle + |12\rangle + |21\rangle + |22\rangle) \) is a separable state, there are always \( C_{g}\left( \sum_i \sqrt{q_i} |i\rangle_A \right) = C_{g}\langle \rho_A \rangle \) and \( \sum_i q_i C_{g}\langle |\varphi_i\rangle_B \rangle = C_{g}\langle \rho_B \rangle \), and therefore equation (1.1) is not valid, too.
4.4. Convex roof coherence measure based on fidelity and that based on linear entropy

We show that both convex roof coherence measure based on fidelity and convex roof coherence measure based on linear entropy are non-superadditive, too.

Convex roof coherence measure based on fidelity is defined as

$$C_F(\rho) = \inf_{\{\rho_i, \phi_i\}} \sum_i p_i \sqrt{1 - F(\phi_i, \rho)},$$

(4.12)

where $F(\rho, \phi) = (\text{Tr}(\sqrt{\sqrt{\rho} \phi \sqrt{\rho}}))^2$ is the Uhlmann fidelity. It was put forward in [47].

Convex roof coherence measure based on linear entropy is defined as

$$C_L(\rho) = \inf_{\{\rho_i, \phi_i\}} \sum_i p_i C_L(|\phi_i\rangle),$$

(4.13)

where $C_L(|\phi\rangle) = \sum_i |c_i|^4$ for $|\phi\rangle = \sum_i c_i |i\rangle$. It was put forward in [48].

To prove the convex roof coherence measure based on fidelity non-superadditive, we take the same state $|\phi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |12\rangle + |21\rangle + |22\rangle)$, as done in section 4.3. There are $C_F(|\phi\rangle) = \frac{50}{11}$, $C_F(\sum_i \sqrt{q_i} |i\rangle) = \frac{1}{2}$, and $\sum_i q_i C_L(|\phi\rangle) = \frac{1}{2}$, which does not fulfill equation (3.1), as well as equation (1.1). Similarly, to prove the convex roof coherence measure based on linear entropy non-superadditive, we again take $|\phi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |12\rangle + |21\rangle + |22\rangle)$. There are $C_L(|\phi\rangle) = \frac{1}{2}$, $C_L(\sum_i \sqrt{q_i} |i\rangle) = \frac{1}{2}$, and $\sum_i q_i C_L(|\phi\rangle) = \frac{1}{2}$, which does not fulfill equation (3.1), as well as equation (1.1), too. Hence, both the convex roof coherence measure based on fidelity and that based on linear entropy are non-superadditive.

4.5. The convex roof coherence measure based on $\frac{1}{\sqrt{2}}$-entropy

We show that the convex roof coherence measure based on $\frac{1}{\sqrt{2}}$-entropy is non-superadditive.

Convex roof coherence measure based on $\frac{1}{\sqrt{2}}$-entropy is defined as

$$C_\frac{1}{\sqrt{2}}(\rho) = \inf_{\{\rho_i, \phi_i\}} \sum_i p_i C_\frac{1}{\sqrt{2}}(|\phi_i\rangle),$$

(4.14)

where $C_\frac{1}{\sqrt{2}}(|\phi\rangle) = 2 \log_2(\sum^d_i |c_i|^2)$ for $|\phi\rangle = \sum_i c_i |i\rangle$. It was proposed in [44]. To show that this measure does not fulfill the inequality (3.1), a counterexample can be taken as $|\phi\rangle = \frac{\sqrt{5}}{\sqrt{2}} |11\rangle + \frac{\sqrt{10}}{\sqrt{2}} |12\rangle + \frac{\sqrt{10}}{\sqrt{2}} |21\rangle + \frac{\sqrt{3}}{\sqrt{2}} |22\rangle$. For this state, there are $C_\frac{1}{\sqrt{2}}(|\phi\rangle) = 2 \log_2 \frac{\sqrt{5} + \sqrt{10} + \sqrt{10} + \sqrt{3}}{\sqrt{2}}$, $C_\frac{1}{\sqrt{2}}(\sum_i \sqrt{q_i} |i\rangle) = 2 \log_2 \frac{1 + \sqrt{2}}{\sqrt{2}}$, and $\sum_i q_i C_\frac{1}{\sqrt{2}}(|\phi\rangle) = \frac{1}{2} \log_2 \frac{5 + \sqrt{2}}{\sqrt{2}} + \frac{1}{2} \log_2 \frac{5 + \sqrt{2}}{\sqrt{2}}$. We then have $C_\frac{1}{\sqrt{2}}(|\phi\rangle) < C_\frac{1}{\sqrt{2}}(\sum_i \sqrt{q_i} |i\rangle) + \sum_i q_i C_\frac{1}{\sqrt{2}}(|\phi\rangle)$, which means that $C_\frac{1}{\sqrt{2}}$ does not fulfill equation (3.1).

To confirm that the convex roof coherence measure based on $\frac{1}{\sqrt{2}}$-entropy is non-superadditive.

We need to examine equation (1.1) with $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|$. By following the same method used for obtaining $C_F(\rho)$ in [47], we can obtain the expression of $C_\frac{1}{\sqrt{2}}(\rho)$ for single-qubit states $\rho$,

$$C_\frac{1}{\sqrt{2}}(\rho) = 2 \log_2 \left( \frac{1 + \sqrt{1 - C_l(\rho)^2}}{2} + \frac{\sqrt{1 - 1 - C_l(\rho)^2}}{2} \right),$$

(4.15)

where $C_l(\rho)$ is the $l_1$ norm of coherence. With the aid of equation (4.15), it is easy to work out $C_\frac{1}{\sqrt{2}}(|\phi\rangle_{AB}) - C_\frac{1}{\sqrt{2}}(\rho_A) - C_\frac{1}{\sqrt{2}}(\rho_B) = -0.0096$ with $\rho_A = \text{Tr}_{AB}|\phi\rangle_{AB}\langle\phi|$ and $\rho_B = \text{Tr}_A|\phi\rangle_{AB}\langle\phi|$. This indicates that $C_\frac{1}{\sqrt{2}}$ is non-superadditive.
5. Remarks and conclusions

Quantifying coherence has received increasing attention, and considerable work has been directed towards finding relations between coherence measures and quantum correlations. Superadditivity of a coherence measure describes the trade-off relations between the coherence of a bipartite system and that of its subsystems, and it is a precondition of defining a discordlike correlation based on the coherence measure. In this paper, we have put forward a theorem on the superadditivity of convex roof coherence measures, which provides a sufficient condition to identify the convex roof coherence measures fulfilling the superadditivity.

We should stress that equation (3.1) in our theorem only involves pure states but has nothing to do with mixed states. Therefore, when the theorem is used to examine the superadditivity of a convex roof coherence measure, one only needs to check equation (3.1) for pure states but does not need to consider any mixed states. This avoids the difficulty of calculating a convex roof coherence measure of mixed states and greatly simplifies the calculations needed. A convex roof coherence measure $C(\rho)$ satisfies equation (1.1) for all states, i.e. being superadditive, as long as it satisfies equation (3.1) for pure states.

By applying our theorem to each of the known convex roof coherence measures, we prove that the coherence of formation and the coherence concurrence are superadditive, while the geometric measure of coherence, the convex roof coherence measure based on linear entropy, the convex roof coherence measure based on fidelity, and convex roof coherence measure based on $\frac{1}{2}$-entropy are non-superadditive. Noting that some distance-based coherence measures have been used to define a discordlike correlation [24–26], our results indicate that a discordlike correlation of the form $I_C(\rho_{AB}) = C(\rho_{AB}) - C(\rho_A) - C(\rho_B)$ can be defined based on the convex roof coherence measures with the superadditivity, such as the coherence of formation and the coherence concurrence.

In passing, we would like to point that the expression of the sufficient condition in our theorem is not unique. In stead of equation (3.1), an alternative expression of the sufficient condition can be taken as

$$C_f(\ket{\varphi}_{AB}) \geq \sum_j p_j C_f(\ket{\varphi_j}_A) + \sum_i q_i C_f(\ket{\varphi_i}_B),$$

(5.1)

where $p_j = \sum_i |c_{ij}|^2, \ket{\varphi_j}_A = \frac{1}{\sqrt{p_j}} \sum_i c_{ij} \ket{i}_A$, and all the others are the same as in the theorem. Compared with equations (3.1) and (5.1) is more accuracy in the sense that the right hand side of equation (5.1) is smaller than that of equation (3.1), but equation (3.1) is more convenient to use.

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