COUNTING INTEGRAL POINTS ON SOME HOMOGENEOUS VARIETIES WITH LARGE REDUCTIVE STABILIZERS

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ABSTRACT. Let $G$ be a semisimple group over rational numbers and $H$ is a subgroup over rational numbers. Given a representation of $G$ and an integral vector $x$ whose stabilizer is equal to $H$. In this paper we investigate the asymptotic of integral points on $Gx$ with bounded height. We find its asymptotic up to an implicit constant when $H$ is “large” in $G$ but we allow the presence of intermediate subgroups. This is achieved by a novel combination of two equidistribution results in two different settings: one is that of Esikin–Mozes–Shah on a Lie group modulo a lattice and the other one is a result of Chamber-Loir–Tschinkel on a smooth projective variety with a normal crossing divisor.

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1. Introduction

Let $V_0$ be a finite dimensional $\mathbb{Q}$-vector space. Identify $V_0(\mathbb{R}) \cong \mathbb{R}^N$ and hence $V_0(\mathbb{R})$ is equipped with an Euclidean norm given by
\[||(x_1, ..., x_N)|| := (x_1^2 + ... + x_N^2)^{1/2}.\]
Let $B_R := \{v \in V_0(\mathbb{R}) | \|v\| \leq R\}$. We also fix an $\mathbb{Z}$-structure $V_0(\mathbb{Z})$ on $V_0$.

Let $X \hookrightarrow V_0$ be an affine variety over $\mathbb{Q}$, a natural question is whether $X(\mathbb{Z}) \cap B_R$ is asymptotic to $\text{Vol}(X(\mathbb{R}) \cap B_R)$ for certain natural measure $\text{Vol}$ on $X(\mathbb{R})$. When $X = V_0$, this is true but it fails in general. However, one might hope this remains true, at least in some weak sense, in the homogeneous setting.

So let us assume that $X$ is stable and homogeneous (i.e. the action is transitive on $\mathbb{C}$-points) under the action of $G$, a closed connected $\mathbb{Q}$-subgroup of $\text{GL}(V_0)$. We assume that $G$ is semisimple. Also assume $X(\mathbb{Z})$ is non-empty. Then there exists an arithmetic lattice $\Gamma \leq G := G(\mathbb{R})$ such that $\Gamma$ preserves the set $X(\mathbb{Z})$. A theorem of Borel–Harish-Chandra [BHC62] asserts that $X(\mathbb{Z})$ decomposes into finitely many $\Gamma$-orbits. Thus it is natural to investigate $|\Gamma \cdot v_0 \cap B_R|$ for $v_0 \in X(\mathbb{Z})$. The following asymptotic has been established in [EMS96, Theorem 1.11] (compare [EM93] and [DRS93]) using Ratner’s theorem [Ra91] on the classification of unipotent-invariant ergodic measures and the linearization technique of Dani–Margulis [DM93].

**Theorem 1.1.** Assume that $H \leq G$ is maximal among proper connected $\mathbb{Q}$-subgroups and $H := H(\mathbb{R})$ has finite volume modulo $H \cap \Gamma$, then
\[
\lim_{R \to +\infty} \frac{|\Gamma \cdot v_0 \cap B_R|}{\mu_{G/H}(G \cdot v_0 \cap B_R)} = 1.
\]

Going beyond this situation, it has been noted in [EMS96, Section 7] that “focusing” might happen. As a consequence, they found examples where the above limit exists but may not be equal to 1. However, they are not able to make a general statement beyond the case of maximal subgroups, though in some special cases results of this kind exist (see [GTBT15] and [Yan18]). On the positive side, they are able to establish this asymptotic 1 for certain special $(\rho_0, v_0)$ in the presence of intermediate subgroups. For instance they proved

**Theorem 1.2.** Assume $G = \text{SL}_n$ and $H \leq G$ is a maximal $\mathbb{Q}$-torus that is $\mathbb{Q}$-anisotropic. Let $\rho_0$ be the Adjoint representation $G \to \text{GL}(\mathfrak{g})$ and $v_0$ is a rational matrix in $\mathfrak{g}$ whose centralizer in $G$ is equal to $H$, then asymptotic 1 holds.

The purpose of the paper is to establish the weaker asymptotic

\[
\lim_{R \to +\infty} \frac{|\Gamma \cdot v_0 \cap B_R|}{\mu_{G/H}(G \cdot v_0 \cap B_R)} = c
\]

for certain special $H$ that may not be maximal and for general $(\rho_0, v_0)$.

We shall require $H$ to have finite volume modulo $H \cap \Gamma$ (it is expected the weaker asymptotic is false without this assumption and we do not go into that direction in this paper, see [OS14, KK18, SZ19, Zha19, Zha20]) and we denote the probability measure supported on $H\Gamma/\Gamma$ by $\mu_H$. Additionally we require the following conditions.

**Condition 1.1.** $H$ is reductive and $(Z_GH)^\circ$, the connected component of the centralizer of $H$ in $G$, is contained in $H$. 

Condition 1.2. For all connected \( \mathbb{Q} \)-subgroup \( L \) with \( H \subset L \subset G \), we have \( H \) intersect every connected component of \( L \) non-trivially.

The first condition implies that (see Lemma 3.9)

- there are only finitely many intermediate subgroups between \( H \) and \( G \).

The second condition is fulfilled if (see [Mat64] or [BT65, Theorem 14.4])

- \( H \) contains a maximal \( \mathbb{R} \)-split torus.

Now we are ready to state our main theorem.

Theorem 1.3. let \( G, H, \Gamma, \rho_0, v_0, |\cdot| \) be same as above. In addition we assume that \( H \) satisfies condition 1.1 and 1.2. Then the weaker asymptotic 2 holds.

Via a theorem of Borel–Harish-Chandra [BHC62] and Chamber-Loir–Tschinkel [CLT10], one can deduce the following.

Corollary 1.4. Same assumption as above. Fix an integral structure on \( V_0 \). It induces an integral model \( X \) of \( G \cdot v_0 \). Assume \( X(\mathbb{Z}) \neq \emptyset \). Then there exists a constant \( c' > 0 \) and \( a, b \geq 0 \) such that

\[
\lim_{R \to +\infty} \frac{|X(\mathbb{Z}) \cap B_R|}{c' R^a \ln R^b} = 1.
\]

When the height function is geometric, then \( a = \sigma_U \) and \( b = b'_U \), where \( U \) is defined to be union of orbits of \( G(\mathbb{R}) \) on \( X(\mathbb{R}) \) that contain an integral point.

Being geometric refers to those height functions constructed in Section 4.1. This may be viewed as an integral version of Manin conjecture with an implicit constant. Note that \( U \) is not very explicit in general, but when \( G \)-acts transitively on \( G/H(\mathbb{R}) \) (for instance, when the stabilizer group is an \( \mathbb{R} \)-split torus), one can take \( U := G/H(\mathbb{R}) \) and the constants \( a_U, b'_U \) agree with the normal one. We do not know whether \( a \) being equal to 0 could actually happen.

One may also put weights on orbits and ask whether the weighted version of asymptotic 2 holds. For this we provide the following equidistribution result.

Theorem 1.5. Same assumption as above. Then

\[
\lim_{R \to \infty} \frac{1}{c\mu_{G/H}(B_R)} \sum_{x \in \Gamma \cdot v_0 \cap B_R} \delta_x
\]

converges to a probability measure on the closure of \( G/H \) in \( \mathbb{P}(V_0 \oplus \mathbb{Q}) \).

Similar statements hold replacing \( \mathbb{P}(V_0 \oplus \mathbb{Q}) \) by \( \mathbb{P}(V_0) \). When \( H \) is a symmetric subgroup a more precise statement has been established in [GOS09]. In the current situation we do not know if the limit coincides with the following

\[
\lim_{R \to \infty} \frac{1}{c\mu_{G/H}(B_R)} \mu_{G/H}|_{B_R}.
\]

When focusing does not arise, this is expected to be true.

Let us list some remarks.

- Our results do not provide an explicit expression on \( c \). A priori, it depends on all the data. In particular methods from [GR95] or [WX16] does not apply to give a local-to-global type expression for the asymptotic.
- Our results do not cover the example in [EMS96, Section 7].
• The height function that can be dealt with must be geometric. We do not know how to handle a general algebraic proper function defined on $G/H$.
• We suspect the second condition 1.2 is not needed. To remove that condition one might have to generalize [CLT10] to a setting more general than varieties.
• It is tempting to apply the same methods to the rational counting problem, as the corresponding results from ergodic theory are available thanks to the work of Oh–Gorodnik [GO11], assuming $H$ is semisimple. But an issue similar to condition 1.2 prevents us from doing so.
• We do not know whether it is possible for focusing to happen in our set-up.

Now let us briefly discuss our approach.

We are going to attack this counting problem via equidistribution results as in [DRS93]. For the approach using height zeta function, the reader is referred to [CLT10, CLT12, TBT13]. However rather than showing certain families of measure converges to the full Haar measure on $G/\Gamma$, we shall be content with showing

$$\lim_{R \to \infty} \frac{1}{\mu_{G/H}(B_R)} \int_{B_R} g \ast \mu_H \mu_{G/H}([g]) \text{ exists.}$$

Note that this is really a double integral. In previous work one usually only take advantage of only one of the integral.

When $H$ is very large (say, maximal or in the setting of [GTBT15] and [Yan18]), one choose to show that for generic sequences, the inner integral always converges to the $G$-invariant measure $\mu_G$.

When $H$ is trivial, one choose to study the outer integral using spectral methods or mixing. For example see [Man07, GN12]. However, their approach only works for certain norms that are stable under perturbation on both sides, a property which does not hold in our set-up.

Once this is done, one has the number of integer points is, up to an implicit constant, asymptotic to the volume, which one can compute via the method of [CLT10].

The novelty of the present paper is to, in addition to [EMS96], put into use of the result of [CLT10] to analyze the double integral above.

Outline. In Section 2 we translate the counting problem to an equidistribution problem mentioned above.

In Section 3 we recall the equidistribution result of [EMS96], using which we define a compactification of $G/H$ inside $\text{Prob}(G/\Gamma)$. When $H$ satisfies condition 1.1, the boundary decomposes into finitely many $G$-orbits. Though this is only a topological object, we show that there is an algebraic one that covers it assuming additionally that 1.2.

In Section 4 we recall the equidistribution result of [CLT10]. To use their theorem we apply equivariant resolution of singularity first and then verify that the Haar measure on $G/H$ and the height function from Euclidean norm do arise from the construction of their paper.

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2. Equidistribution and counting

2.1. Convention. Let $G$ be a linear algebraic group over $\mathbb{Q}$ and $H$ be a connected subgroup over $\mathbb{Q}$. The Roman letter $H$ is reserved for $H(\mathbb{R})$. Let $\Gamma$ be an arithmetic lattice. Let $\mu_H$ be the Haar measure supported on $H\Gamma/\Gamma$. We always assume $\mu_H$ is finite and normalize it to be a probability measure. Similarly, $\mu_H^\circ$ denotes the probability measure supported on $H^\circ\Gamma/\Gamma$. We also let $\tilde{\mu}_H$ denote the Haar measure of $H$ whose restriction to the fundamental domain of $H/H\cap\Gamma$ coincide with $\mu_H$. Then let $\mu_{G/H}$ be the unique $G$-invariant locally finite measure on $G/H$ such that for all continuous compactly supported function $f$ on $G$, one has
\[
\int f(x)\tilde{\mu}_G(x) = \int \int f(gh)\tilde{\mu}_H(h)\mu_{G/H}([g]).
\]

2.2. Counting follows from equidistribution. Given a continuous transitive $G$-action on a topological space $X$ and $x_0 \in X$ such that $g \mapsto g \cdot x_0$ gives a homeomorphism from $G/H$ to $X$. Given a proper function $l : X \to [0, \infty)$ we let
\[ B_R := \{ x \in X | l(x) \leq R \} \]
As $R$ tends to $\infty$, we ask whether
\[ \lim_{R \to \infty} \frac{|B_R \cap \Gamma \cdot x_0|}{\mu_{G/H}(B_R)} \]
exists and what it is equal to.

We shall be concerned with situations where $\{B_R\}$ enjoys the following properties:

- for any $0 < \varepsilon < 1$ there exists $U_\varepsilon$, a neighborhood of identity in $G$, such that $B_{(1-\varepsilon)R} \subset gB_R \subset B_{(1+\varepsilon)R}$ for all $R \geq 1$ and $g \in U_\varepsilon$;
- there exists constants $a, b, c_1, c_2$ with $c_1, c_2 > 0$ such that
\[ c_1 \leq \lim_{R \to \infty} \frac{\mu_{G/H}(B_R)}{R^a \ln R^b} \leq c_2. \]
In this case we say that the family $\{B_R\}$ is nice. The first condition implies that
\[ B_{(1-\varepsilon)R} \subset \bigcap_{g \in U_\varepsilon} gB_R \subset \bigcup_{g \in U_\varepsilon} gB_R \subset B_{(1+\varepsilon)R}. \]
Thus our condition is strictly stronger than the well-roundedness condition in [EM93]. For instance, the distance function induced from the natural Riemannian metric on the associated symmetric space is excluded.

**Proposition 2.1.** Assume $\{B_R\}$ is nice and
\[
\lim_{R \to \infty} \frac{1}{\mu_{G/H}(B_R)} \int_{B_R} g_*\mu_H\mu_{G/H}([g]) \quad \text{exists.}
\]
If we denote the limit measure by $\mu_\infty$, then
- $\mu_\infty$ is absolutely continuous with respect to $\mu_G$;
- the Radon–Nikodym derivative can be represented by a strictly positive continuous function $f_\infty$;
- and this function satisfies
\[
\lim_{R \to \infty} \frac{|B_R \cap \Gamma \cdot x_0|}{\mu_{G/H}(B_R)} = f_\infty([g]).
\]
Proof. Step 1, limit exists and is continuous.
We first prove the limit in the third part exists. So take \([g_0] \in G/H\).
For \(0 < \epsilon < 1\), choose \(U_\epsilon\) satisfying properties as in the definition of being nice.
By shrinking to a smaller one we assume \(U_\epsilon = U_\epsilon^{-1}\).
Then we choose \(V'_\epsilon \subset V_\epsilon\) two families of open neighborhood of identity contained in \(U_\epsilon\) such that
- the closure of \(V'_\epsilon\) is contained in \(V_\epsilon\);
- \[\lim_{\epsilon \to 0} \frac{\mu_\infty(V'_\epsilon[g_0])}{\mu_\infty(V_\epsilon[g_0])} = \lim_{\epsilon \to 0} \frac{\mu_G(V'_\epsilon[g_0])}{\mu_G(V_\epsilon[g_0])} = 1.\]
Then we choose a continuous function \(f_\epsilon\) with \(1_{V'_\epsilon[g_0]} \leq f_\epsilon \leq 1_{V_\epsilon[g_0]}\).
We write \(\Phi_R([g]) := \frac{|B_R \cap g\Gamma \cdot x_0|}{\mu_{G/H}(B_R)}\sum_{\gamma \in \Gamma \cap H} 1_{B_R}(g \gamma \cdot x_0)\)
Then we have (5)
\[\limsup_R \langle \Phi_R, 1_{V'_\epsilon[g_0]} \rangle_{\mu_G} \leq \limsup_R \langle \Phi_R, f_\epsilon \rangle_{\mu_G} = \int f_\epsilon(x) \mu_\infty(x) \leq \liminf_R \langle \Phi_R, 1_{V_\epsilon[g_0]} \rangle_{\mu_G} \mu_\infty(V'_\epsilon[g_0]) \leq \int f_\epsilon(x) \mu_\infty(x) \leq \mu_\infty(V_\epsilon[g_0]).\]
Now we can start to estimate. Assume \(\epsilon\) is small enough so that \(U_\epsilon 	o U_\epsilon[g_0]\) is a homeomorphism. Then \(\bar{\mu}_G |_{U_\epsilon} \) is identified with \(\mu_G |_{U_\epsilon[g_0]}\) under this homeomorphism.
\[\langle \Phi_R, 1_{V'_\epsilon[g_0]} \rangle_{\mu_G} = \frac{1}{\mu_{G/H}(B_R)} \int_{[g] \in V'_\epsilon[g_0]} |B_R \cap g\Gamma \cdot x_0| \mu_G([g])\]
\[= \frac{1}{\mu_{G/H}(B_R)} \int_{g \in V'_\epsilon} |g^{-1}B_R \cap g_0\Gamma \cdot x_0| \bar{\mu}_G(g)\]
\[\geq \frac{\mu_G(V'_\epsilon[g_0])}{\mu_{G/H}(B_R)} \left| \bigcap_{g \in U_\epsilon} gB_R \cap g_0\Gamma \cdot x_0 \right|\]
\[\geq \frac{\mu_G(V'_\epsilon[g_0])}{\mu_{G/H}(B_R)} |B_{(1-\epsilon)R} \cap g_0\Gamma \cdot x_0|\]
By taking the \(\limsup\) as \(R\) tends to \(\infty\) and using Equation (5) above we get
\[\mu_G(V'_\epsilon[g_0]) \limsup \frac{|B_{(1-\epsilon)R} \cap g_0\Gamma \cdot x_0|}{\mu_{G/H}(B_R)} \leq \limsup \langle \Phi_R, 1_{V'_\epsilon[g_0]} \rangle_{\mu_G} \leq \mu_\infty(V_\epsilon[g_0]).\]
Similarly we have
\[\mu_G(V'_\epsilon[g_0]) \liminf \frac{|B_{(1+\epsilon)R} \cap g_0\Gamma \cdot x_0|}{\mu_{G/H}(B_R)} \geq \mu_\infty(V'_\epsilon[g_0]).\]
Combining these two while replacing \((1 - \epsilon)R\) by \(R\) in the first inequality and \((1 + \epsilon)R\) by \(R\) in the second inequality, we get
\[\frac{\mu_\infty(V'_\epsilon[g_0])}{\mu_G(V'_\epsilon[g_0])} \liminf \frac{\mu_{G/H}(B_R/(1+\epsilon))}{\mu_{G/H}(B_R)} \leq \liminf \frac{|B_R \cap g_0\Gamma \cdot x_0|}{\mu_{G/H}(B_R)}\]
\[\leq \limsup \frac{|B_R \cap g_0\Gamma \cdot x_0|}{\mu_{G/H}(B_R)} \leq \frac{\mu_\infty(V'_\epsilon[g_0])}{\mu_G(V'_\epsilon[g_0])} \limsup \frac{\mu_{G/H}(B_R/(1-\epsilon))}{\mu_{G/H}(B_R)}\]
Taking \( \limsup_{\varepsilon \to 0} \) on the left end and recall the definition of being nice, we have

\[
\lim_{\varepsilon \to 0} \sup_{R \to \infty} \frac{\mu_{G/H}(B_R/(1+\varepsilon))}{\mu_{G/H}(B_R)} = 1
\]

\[
\Rightarrow \lim_{\varepsilon \to 0} \sup_{R \to \infty} \frac{\mu_{\infty}(V'[g_0])}{\mu_{\infty}(V[g_0])} = \lim_{\varepsilon \to 0} \sup_{R \to \infty} \frac{\mu_{\infty}(V'[g_0])}{\mu_{G/H}(B_R)}
\]

which is equal to the limit sup \( \varepsilon \to 0 \) of the right end. Thus

\[
\liminf_{\varepsilon \to 0} \left\{ \frac{|B_R \cap g_0 \Gamma \cdot x_0|}{\mu_{G/H}(B_R)} \right\} = \limsup_{\varepsilon \to 0} \left\{ \frac{|B_R \cap g_0 \Gamma \cdot x_0|}{\mu_{G/H}(B_R)} \right\} = \lim \left\{ \frac{|B_R \cap g_0 \Gamma \cdot x_0|}{\mu_{G/H}(B_R)} \right\} \text{ exists.}
\]

Let us note that the above limit is equal to

\[
\lim_{\varepsilon \to 0} \frac{\mu_{\infty}(V[g_0])}{\mu_{G/H}(B_R)} = \lim_{\varepsilon \to 0} \mu_{\infty}(V'[g_0])
\]

Call this limit \( f([g_0]) \), which is continuous because for \( u \in U \),

\[
f(u[g_0]) = \lim_{\varepsilon \to 0} \frac{|B_R \cap u g_0 \Gamma \cdot x_0|}{\mu_{G/H}(B_R)} = \lim_{\varepsilon \to 0} \frac{|u^{-1} B_R \cap g_0 \Gamma \cdot x_0|}{\mu_{G/H}(B_R)}
\]

\[
\geq \limsup_{\varepsilon \to 0} \frac{|B_R/(1-\varepsilon) \cap g_0 \Gamma \cdot x_0|}{\mu_{G/H}(B_R)} = \limsup_{\varepsilon \to 0} \frac{|B_R \cap g_0 \Gamma \cdot x_0|}{\mu_{G/H}(B_R/(1-\varepsilon))}
\]

converges to 0 as \( \varepsilon \to 0 \).

By the same argument it is not hard to see that for any two \( g_1, g_0 \) in \( G \), there exists a positive constant \( c_{g_1, g_0} \) such that \( f([g_1 g_0]) \geq c_{g_1, g_0} f([g_0]) \). Therefore \( f \) is either constantly equal to 0 or strictly positive.

**Step 2, absolute continuity**

Now we switch our attention to \( \mu_{\infty} \) and show that it is absolutely continuous with respect to \( \mu_{G} \). That is to say, we need to show \( \mu_{\infty}(E) = 0 \) whenever \( \mu_{G}(E) = 0 \).

Recall that \( \tilde{\mu}_{\Gamma} \) is the Haar measure on \( G \) whose restriction to a fundamental set gives \( \mu_{G} \). We first observe that if a set \( E \) has measure 0 with respect to \( \mu_{G} \), then for any probability measure \( \lambda \) and any bounded open set \( U \) in \( G \), we have \( \int_{u \in U} u \lambda \tilde{\mu}_{\Gamma}(u)(E) = 0 \). Indeed by decomposing \( E \) into smaller pieces and shrinking \( U \) we assume that \( U = U^{-1} \) and \( E \) is the image of some \( \tilde{E} \) under the natural projection \( G \to G/\Gamma \) and the projection is injective when restricted to \( U^{-1} \tilde{E} \). Using this bijection (onto \( U^{-1} E \)) we lift \( \lambda|_{U^{-1} E} \) to \( \tilde{\lambda} \) supported on \( U^{-1} \tilde{E} \). Moreover, we require that for each \( \varepsilon \in E \), the map from \( U \) to \( G/\Gamma \) via \( u \mapsto u \varepsilon \) is injective.

Now by interchanging the order of integration,

\[
\int_{u \in U} u \lambda(E) \tilde{\mu}_{\Gamma}(u) = \int_{u \in U} \int_{G/\Gamma} 1_{u^{-1}E}(x) \lambda(x) \tilde{\mu}_{\Gamma}(u)
\]

\[
= \int_{u \in U} \int_{G} 1_{u^{-1}E}(x) \tilde{\lambda}(x) \tilde{\mu}_{\Gamma}(u)
\]

\[
= \int \int 1_{\{u, x\} \in U \times \tilde{E}}(u, x) \tilde{\mu}_{\Gamma}(u) \tilde{\lambda}(x)
\]

\[
= \int \tilde{\mu}_{\Gamma}(\tilde{E}) \tilde{\lambda}(x) = 0,
\]

which confirms our observation.
On the other hand, for an $E$ with $\mu_G(E) = 0$, we can find a family of shrinking measurable sets $E_i$ such that $E = \cap E_i$ and $\mu_\infty(\partial E_i) = 0$. For $u \in G$,

$$
\mu_\infty(E_i) = \lim_{R \to \infty} \frac{1}{\mu_G(B_R)} \int_{[g] \in B_R} g_* \mu_H(E_i) \mu_G([g])
$$

$$
u_* \mu_\infty(E_i) = \lim_{R \to \infty} \frac{1}{\mu_G(B_R)} \int_{[g] \in \nu_* B_R} g_* \mu_H(E_i) \mu_G([g]).
$$

The following difference can be estimated by

$$
\left| \mu_\infty(E_i) - \frac{1}{\mu_G(U_\e)} \int_{u \in U_\e} u_* \mu_\infty(E_i) \tilde{\mu}_G(u) \right|
$$

$$\leq \lim_{\epsilon \to 0} \frac{1}{\mu_G(U_\e)} \int_{u \in U_\e} \mu_G(H_\e) \int_{B_{(1+\epsilon)R} \setminus B_{(1-\epsilon)R}} g_* \mu_H(E_i) \mu_G([g]) \tilde{\mu}_G(u)
$$

$$\leq \lim_{\epsilon \to 0} \frac{\mu_G(H_\e) \int_{B_{(1+\epsilon)R} \setminus B_{(1-\epsilon)R}} g_* \mu_H(E_i) \mu_G([g]) \tilde{\mu}_G(u)}{\mu_G(H_\e) \int_{B_{(1+\epsilon)R} \setminus B_{(1-\epsilon)R}}} \leq \epsilon',
$$

for some $\epsilon'$, which is independent of $i$ and converges to 0 as $\epsilon$ does so. So if we let $i$ tends to infinity, we get

$$
\left| \mu_\infty(E) - \frac{1}{\mu_G(U_\e)} \int_{u \in U_\e} u_* \mu_\infty(E) \tilde{\mu}_G(u) \right| \leq \epsilon'.
$$

However, by our observation, the measure of $E$ against “Haar average” of any probability measure is 0. So we are left with

$$
\mu_\infty(E) \leq \epsilon'
$$

for $\epsilon'$ arbitrarily small, which forces $\mu_\infty(E) = 0$. This ends the proof of absolute continuity.

**Step 3, finishes the proof**

Now we write $\mu_\infty = \psi \mu_G$ for some non-negative function $\psi$ in $L^1(\mu_G)$.

By what has been shown in Step 1, for any $[g_0] \in G/\Gamma$,

$$f([g_0]) = \lim_{\epsilon \to 0} \frac{\mu_G(V_\epsilon [g_0])}{\mu_G(V_\epsilon [g_0])} = \lim_{\epsilon \to 0} \frac{1}{\mu_G(V_\epsilon)} \int_{u \in V_\epsilon} \psi(u[g_0]) \tilde{\mu}_G(u)
$$

Therefore $f = \psi$ almost surely. As $\psi \mu_G$ is a probability measure, we see that $f$ can not be the 0 function. As said in step 1, this implies that $f$ is strictly positive. \(\square\)

2.3. **A reformulation.** We reformulate the required equidistribution assumption (4) from Proposition 2.1 in a form that will be actually proved later.

Let Ave be the continuous map $\text{Prob}(\text{Prob}(G/\Gamma)) \to \text{Prob}(G/\Gamma)$ defined by

$$
\int f(x) \text{Ave}(\lambda)(x) := \int \left( \int f(x) \mu(x) \right) \lambda(\mu) \quad \forall f \in C_c(G/\Gamma)
$$

where both spaces are equipped with the weak-* topology. Note that this makes sense as the map $\mu \mapsto \int f(x) \mu(x)$ is a bounded continuous map on $\text{Prob}(G/\Gamma)$ if $f$ is continuous and bounded. For a point $x \in \text{Prob}(G/\Gamma)$ we let $\delta_x$ be the probability measure supported on $\{x\}$.
Proposition 2.2. In the set-up of Theorem 1.3, the limit of the following
\[
\frac{1}{\mu_{G/H}(B_R)} \int_{[g] \in B_R} g \delta_{\mu_H} \mu_{G/H}([g])
\]
exists in \( \text{Prob}(\text{Prob}(G/\Gamma)) \).

Proof of Theorem 1.3 assuming Proposition 2.2. By Proposition 2.1, it suffices to show that the Equation 4 holds. But by Proposition 2.2 and the fact that Ave is continuous we have that
\[
\lim_R \text{Ave} \left( \frac{1}{\mu_{G/H}(B_R)} \int_{[g] \in B_R} g \delta_{\mu_H} \mu_{G/H}([g]) \right)
\]
exists which is nothing but Equation 4.

\( \square \)

3. Compactifications

3.1. Convention. Throughout this section we make the following assumption unless otherwise noted. Take \( H \leq G \) to be connected reductive groups defined over \( \mathbb{Q} \) and assume that \( H \) satisfies both condition 1.1 and 1.2. Let \( \Gamma \) be an arithmetic lattice in \( G \) and assume \( H \cap \Gamma \) is a lattice in \( H \).

3.2. Translates of homogeneous measures. We need the following three inputs from the work of [EMS96, EMS97].

Theorem 3.1. For any sequence \( (g_n) \) in \( G \), there exists a subsequence \( (g_{n_k}) \), a connected \( \mathbb{Q} \)-subgroup \( L \) of \( G \) with non-trivial \( \mathbb{Q} \)-characters, a bounded sequence \( (\delta_n) \) in \( G \), \( (\gamma_n) \) in \( \Gamma \), \( (h_n) \) in \( H^\circ \) such that
1. \( g_n = \delta_n \gamma_n h_n \);
2. \( \lim \delta_{n_k} \) exists, which we denote by \( \delta_\infty \);
3. \( \gamma_{n_k} H^{\gamma_{n_k}^{-1}} \subset L \);
4. \( \lim_k (g_{n_k}) \mu_{L^\circ} = \delta_\infty \mu_{L^\circ} \).

Theorem 3.2. If \( (\lambda_n) \) is a sequence in \( \Gamma \) and \( L \) is a connected \( \mathbb{Q} \)-subgroup of \( G \) satisfying that
- \( \lambda_n H \lambda_n^{-1} \) is contained in \( L \) for all \( n \) and every proper connected \( \mathbb{Q} \)-subgroup \( L' \) of \( L \) only contains \( \lambda_n H \lambda_n^{-1} \) for at most finitely many \( n \),
then \( \lim_n \lambda_n \mu_{H^\circ} = \mu_{L^\circ} \).

The following, [EMS96, Lemma 5.1, Lemma 5.2], is the algebra behind the focusing criterion in [EMS96, Corollary 1.15].

Theorem 3.3. Let \( M \) and \( L \) be two reductive \( \mathbb{Q} \)-subgroup of \( G \). Let \( X(M, L) \) denote the set \( \{g \in G \mid g M g^{-1} \subset L \} \). Then there is a finite set \( D \subset X(M, L) \cap \Gamma \) such that \( X(M, L) \cap \Gamma = L \cap \Gamma \cdot D \cdot (Z_G M)^\circ \cap \Gamma \). Also, there exists a finite set \( D' \subset X(M, L) \) such that \( X(M, L) = L \cdot D' \cdot (Z_G M)^\circ \).

Theorem 3.4. Let \( L \leq G \) be a reductive subgroup over \( \mathbb{Q} \) and assume that \( Z_G L / Z(L) \) is \( \mathbb{Q} \)-anisotropic. Then there exists a bounded set \( B \subset G \) such that \( G = B \cdot \Gamma \cdot L \).

When \( Z_G L \) is \( \mathbb{Q} \)-anisotropic, this is [EMS97, Theorem 1.3]. In general this follows from [Zha20, Theorem 1.7].

As we are going to concern with translates of \( \mu_H \) rather than \( \mu_{H^\circ} \), let us prove the following corollary.
Corollary 3.5. Same notation as in Theorem 3.1, 3.2. Moreover we have that

- \( \lim_k g_n \mu_H = \delta_{\infty} \mu_L; \)
- \( \lim_n \lambda_n \mu_H = \mu_L. \)

Proof. It suffices to show the second one.

By assumption, \( x \mapsto \lambda_n x \lambda_n^{-1} \) defines a homomorphism from \( H \) to \( L \), which maps \( H^o \) to \( L^o \). Thus it descends to a homomorphism \( c_{\lambda_n} : H/H^o \to L/L^o \). By our assumption 1.2, this is surjective for all \( n \). Also for \( y \in L/L^o \), the notation \( yL^\Gamma / \Gamma \) or \( y\mu_{L^o} \) makes sense.

Now we claim that fix \( y_0 \in L/L^o \), for any \( x_n \in c_{\lambda_n}^{-1}(y_0) \),

3.2.1. Claim. \( \lim \lambda_n x_n \mu_{H^o} = y_0 \mu_{L^o}. \)

Proof of the claim. It suffices to show that for any infinite subsequence, there exists a further subsequence where the limit exists and is the expected one. Therefore we may feel free to pass to a further subsequence whenever needed.

We may assume that, by passing to a subsequence, \( \lambda_n x_n = \delta_n \gamma_n' h_n' \) with \( \delta_n' \) converging to \( \delta' \), \( \gamma_n' \in \Gamma \) and \( h_n' \in H^o \). Moreover, there exists a connected \( \mathbb{Q} \)-subgroup \( M \) that contains \( \gamma_n' H \gamma_n'^{-1} \) for all \( n \) and every proper connected \( \mathbb{Q} \)-subgroup of \( M \) contains at most finitely many \( \gamma_n' H \gamma_n'^{-1} \). Hence \( \lambda_n x_n \mu_{H^o} \) converges to \( \delta' \mu_{M^o} \) by Theorem 3.2.

On the other hand, as \( \lambda_n x_n H^o \lambda_n^{-1} \) is contained in \( y_0 L^o \), we have \( \lambda_n x_n H^o \Gamma \subset y_0 L^o \Gamma \) for all \( n \). Thus \( \delta' M^o \Gamma \) is contained in \( y_0 L^o \Gamma \). By the lemma below, there exists \( l_M \in L^o \) and \( \gamma_M \in \Gamma \) such that \( \delta' = y_0 l_M \gamma_M \) and \( \gamma_M M^o \gamma_M^{-1} \subset L^o \). Hence \( \lambda_n x_n = (\delta_n' \delta_n'^{-1} y_0 l_M) \cdot (\gamma_M \gamma_n'') \cdot (h_n) \).

Therefore, by replacing \( \delta_n' \) with \( \delta_n' \delta_n'^{-1} y_0 l_M \), \( \gamma_n' \) with \( \gamma_M \gamma_n'' \) and \( M \) with \( \gamma_M M^o \gamma_M^{-1} \) we may assume that

- \( \delta_n' \) converges to \( y_0 l_M \) for some \( l_M \in L^o \);
- \( \gamma_n' H \gamma_n'^{-1} \subset M \subset L \) for all \( n \).

By switching the role played by \( \gamma_n' \) and \( \lambda_n \), we see that \( \dim M = \dim L \). And the proof completes.

Now we continue to prove the corollary.

\[
\lambda_n \mu_H = \frac{1}{|H/H^o|} \sum_{x \in H/H^o} \lambda_n x \mu_{H^o} = \frac{1}{|H/H^o|} \sum_{y \in L/L^o} \sum_{x \in c_{\lambda_n}^{-1}(y)} \lambda_n x \mu_{H^o}.
\]

By our claim for any \( x_n \in c_{\lambda_n}^{-1}(y) \), \( \lambda_n x_n \mu_{H^o} \) converges to \( y \mu_{L^o} \). Therefore

\[
\lim_{n \to \infty} \lambda_n \mu_H = \frac{1}{|H/H^o|} \sum_{y \in L/L^o} y \mu_{L^o} = \mu_L.
\]
Proof. By assumption there are $b \in B^2$, $\gamma \in \Gamma$ such that $g = hb\gamma$. So we have

$$hb\gamma A^k \cap hB^2 \Gamma \text{ which is equivalent to } \gamma A^k \gamma^{-1} \cap \bigcup_{k \in \Gamma} B^2 k.$$ 

By Baire Category theorem there exists $k \in \Gamma$ such that $B^2 k$ contains a nonempty open set in $A^k$, which is Zariski dense in $A^k$. Hence we must have $\gamma A^k \gamma^{-1} \subset B^2 k$ for this $k \in \Gamma$. By observing that the left hand side contains the identity, we have $B^2 k = B^2$. It then follows that $\gamma A^k \gamma^{-1} \subset B$. □

3.3. Compactification in the space of measures.

**Definition 3.7.** Let $\text{Prob}(G/\Gamma)$ be the space of probability measures on $G/\Gamma$ equipped with the weak-$\ast$ topology. We define

$$t_0 : G/H \rightarrow \text{Prob}(G/\Gamma)$$

$$[g] \mapsto g_\ast \mu_H$$

and let $X_0 := \overline{t_0(G/H)}$ and $D_0 := X_0 \setminus t_0(G/H)$.

Note that $X_0$ is compact by Theorem 3.1. This is always true when the centralizer of $H$ in $G$ is $\mathbb{Q}$-anisotropic and both $H$, $G$ are reductive.

**Definition 3.8.**

$$\mathcal{A} := \{ L \leq G | L \text{ connected and } \exists \text{ a sequence } (\gamma_n) \text{ in } \Gamma \text{ such that } \gamma_n \mu_H \rightarrow \mu_L \}$$

and we define an equivalence relation on $\mathcal{A}$ by

$$L \sim L' \iff \exists \gamma \in \Gamma, \gamma L \gamma^{-1} = L'.$$

The set $\mathcal{A}/\sim$ is finite due to the next lemma.

**Lemma 3.9.** Assume that $L$ is a connected subgroup of some connected reductive $\mathbb{Q}$-group $M$ and satisfies condition 1.1, then there are only finitely many intermediate reductive groups between $L$ and $M$. Everything is over $\mathbb{C}$ here.

**Proof.** First we claim that up to $G(\mathbb{C})$-conjugacy, there are only finitely many subgroup (over $\mathbb{C}$) $A$ satisfying condition 1.1 that is isomorphic to some fixed linear algebraic group $A_0$. Indeed for each $A$, we write $A = Z \cdot S$ where $Z$ is the connected center and $S$ is a semisimple group. By [EMV09, Lemma A.1], there are only finitely many possibilities for $S$ up to $G(\mathbb{C})$-conjugacy. So suppose $A_i = Z_i \cdot S_i$ for $i = 1, 2$ are two groups isomorphic to $A$ and $g \in G(\mathbb{C})$ is such that $gS_1g^{-1} = S_2$. Hence $gZ_1 g^{-1} \subset Z G S_2$. By condition 1.1, we have that $gZ_1 g^{-1}$ and $Z_2$ are both maximal tori in $Z G S_2$. So $h g Z_1 g^{-1} h^{-1} = Z_2$ for some $h \in Z G S_2(\mathbb{C})$. Hence $h g A_1 g^{-1} h^{-1} = A_2$. This proves the claim.

Now fix $A$ that is connected, reductive and contains $L$. It suffices to show that the collection (denoted by $\mathcal{B}$) of $B$ that is conjugate to $A$ and contains $L$ is finite. By Theorem 3.3, there exists a finite set $D \subset X(L, A)$ such that $X(L, A) = A \cdot D \cdot (Z_M L)^\circ$. For each $B \in \mathcal{B}$, find $g$ such that $gB g^{-1} = A$. Then $g \in X(L, A)$. So $g = adz$ for some $a \in A$, $d \in D$ and $z \in (Z_M L)^\circ$, which is contains in $L$ by assumption. So $z$ is also contained in $d^{-1} Ad$. Hence $B = g^{-1} Ag = z^{-1} d^{-1} Adz = d^{-1} Ad$. This finishes the proof.

□

Corollary 3.5 and Lemma 3.6 imply that

**Corollary 3.10.** As a set $X_0 = \bigcup_{[L] \in \mathcal{A}/\sim} G \cdot \mu_L$. 
3.4. An algebraic compactification. For each \( L \in \mathcal{A} \) that contains \( H \), fix a representation \((\rho_L, V_L)\) and a \( \mathbb{Q} \)-vector \( v_L \in V_L \) whose stabilizer in \( G \) is exactly equal to \( L \). So we have fixed finitely many vectors by Lemma 3.9. For a finite dimensional linear space \( V \), we let \( \mathbb{P}(V) \) be the associated projective variety.

Definition 3.11.

\[
\iota_1 : G/H \rightarrow \bigoplus V_L \rightarrow \bigoplus \mathbb{P}(V_L \oplus \mathbb{Q})
\]

\[
g \mapsto \oplus g \cdot v_L, \oplus v \mapsto \oplus [v : 1]
\]

We define \( X_1 := \iota_2(G/H) \) and \( X_1 := \iota_1(G/H) \), a projective variety over \( \mathbb{Q} \).

Here and elsewhere we use something to denote the closure in Hausdorff topology while something is reserved for Zariski topology.

By extending the action of \( G \) to \((\rho_L \oplus 1, V_L \oplus \mathbb{Q})\). We see that \( \iota_1 \) is \( G \)-equivariant.

Definition 3.12. Given a sequence \((g_n)\) in \( G \) such that \( \iota_1([g_n]) \) converges. We let \( \mathcal{A}_0(g_n) \) be the collection of \( L \) such that \( (g_n v_L) \) converges in \( V_L \).

Lemma 3.13. If \( L_1, L_2 \in \mathcal{A}_0(g_n) \), then \( (L_1 \cap L_2)^\circ \in \mathcal{A}_0(g_n) \). Therefore there exists a minimal element in \( \mathcal{A}_0(g_n) \).

Proof. By assumption \( \lim g_n v_{L_i} \) (i=1,2) exists in \( V_{L_i} \) (i=1,2), therefore \( \lim g_n (v_{L_1} \oplus v_{L_2}) \) exists in \( V_{L_1} \oplus V_{L_2} \). Hence \( \lim [g_n] \) exists in \( G/L_1 \cap L_2 \) and so \([g_n] \) is bounded in \( G/(L_1 \cap L_2)^\circ \). So \( g_n v_{(L_1 \cap L_2)^\circ} \) is bounded in \( V_{(L_1 \cap L_2)^\circ} \). But we already knew \( \lim g_n v_{(L_1 \cap L_2)^\circ} \) exists in \( \mathbb{P}(V_{(L_1 \cap L_2)^\circ} \oplus \mathbb{Q}) \), hence the limit has to be in \( V_{(L_1 \cap L_2)^\circ} \).

3.5. Relation between these two compactifications. Given a sequence \((g_n)\) in \( G \) such that \( \iota_1([g_n]) \) converges, by Lemma 3.13 we find a minimal element \( L_0 \) in \( \mathcal{A}(g_n) \) and find \( b \in G \) such that \( g_n v_{L_0} = b v_{L_0} \).

Proposition 3.14. Under the assumption above, \( \lim(g_n)_* \mu_H = b_\mu_{L_0} \).

Proof. It suffices to show that for any subsequence \( n_k \), there exists a further subsequence \( n_k \) such that \( \lim g_{n_k} \mu_H = b_\mu_{L_0} \). Therefore we shall feel free to pass to a subsequence whenever necessary.

So we may assume that \( g_n = \delta_n \gamma_n h_n \) with \( \delta_n \rightarrow \delta_\infty \), that there exists a connected \( \mathbb{Q} \)-subgroup \( L \leq G \) containing \( \gamma_n H \gamma_n^{-1} \) for all \( n \) and that no proper subgroup of \( L \) contains \( \gamma_n H \gamma_n^{-1} \) for infinitely many \( n \). Hence by Corollary 3.5, \( g_n \mu_H \) converges to \( \delta_\infty \mu_L \).

By Theorem 3.3 and passing to a subsequence, there exists \( d \in X(H, L) \cap \Gamma \), \( z_n \in (\mathbb{Z}_G H)^\circ \cap \Gamma \), and \( l_n \in L \cap \Gamma \) such that \( \gamma_n = l_n d z_n \).

Let \( L' := d^{-1} L d = (l_n^{-1} \gamma_n z_n^{-1})^{-1} L (l_n^{-1} \gamma_n z_n^{-1}) \), which contains \( H \).

Then

\[
\lim g_n v_{L'} = \delta_n l_n d z_n h_n v_{L'} \Rightarrow \delta_n d l_n d z_n v_{L'}
\]

By our assumption \((\mathbb{Z}_G H)^\circ \subset H\), so

\[
g_n v_{L'} = \delta_n d v_{L'} \text{ is bounded} \Rightarrow L' \in \mathcal{A}(g_n) \Rightarrow L_0 \subset L'.
\]

Also we know that \( g_n \mu_H \) converges to \( \delta_\infty \mu_L = \delta_\infty d \mu_{L'} \).

On the other hand, as \( \lim g_n v_{L_0} = b v_{L_0} \), we can find \( l_0^n \in L_0 \) and \( b_n \in G \) such that \( g_n = b_n l_0^n \) and \( b_n \rightarrow b_\infty \). Hence \( g_n H \Gamma \subset b_n l_0^n L_0 \Gamma = b_n L_0 \Gamma \). Since \( \delta_\infty d L' \Gamma / \Gamma \subset L' \in \mathcal{A}(g_n) \Rightarrow L_0 \subset L'.

\[
\]
\[ bl_0 \Gamma / \Gamma. \]
\[ \delta_\infty dL' \subset bL_0 \Gamma \implies \exists \gamma_0, \delta_\infty dL' \subset bL_0 \gamma_0 \implies \exists l_0, \delta_\infty d = bl_0 \gamma_0 \]
\[ \implies bl_0 \gamma_0 L' \subset bL_0 \gamma_0 \iff \gamma_0 L' \gamma_0^{-1} \subset L_0 \]
But we already knew \( L_0 \subset L' \), so \( \gamma_0 L' \gamma_0 = L_0 = L' \) and
\[ \delta_\infty d \mu_{L'} = bl_0 \gamma_0 \mu_{L'} = b \mu_{L_0}. \]
And the proof is complete. \( \square \)

Now we are ready to define a map \( \pi_1 \) from \( X_1 \) to \( X_0 \). For any \( x \in X_1 \), choose a sequence \( (g_n) \) such that \( \lim g_n \oplus \nu_L = x \), then \( \pi_1(x) := \lim g_n \mu_H \). By Proposition 3.14 above, this is well-defined and does not depend on the choice of \( (g_n) \).

**Lemma 3.15.** \( \pi_1 \) is continuous.

**Proof.** It suffices to show that if \( x_n \to x_\infty \) in \( X_1 \), then \( \pi_1(x_n) \to \pi_1(x_\infty) \) in \( X_0 \).

For each \( x_n \), we choose a sequence \( (g^n_i) \) such that \( \lim g^n_i \oplus \nu_L = x_n \). Then we can find \( i_n \) such that for all \( i_n' \geq i_n \), \( x_\infty = \lim g^n_i \oplus \nu_L \).

On the other hand, by passing to a subsequence we may assume that \( \lim \pi_1(x_n) \) exists. By definition, \( \pi_1(x_n) = \lim g^n_i \mu_H \). Note that the space of probability measures equipped with the weak-* topology has a countable basis. Hence we can find \( j_n \) such that for all \( j_n' \geq j_n \), \( \lim \pi_1(x_n) = \lim g^n_i \mu_H \). Taking \( k_n := \max\{i_n, j_n\} \) finishes the proof. \( \square \)

4. Equidistribution on varieties

4.1. Equidistribution on smooth varieties with nice boundaries. The setup is the following (see [CLT10, Section 4.2]). Note that Cartier divisors and Weil divisors are equivalent in our setting, so we will just call them divisors for simplicity.

- \( X \) is a smooth projective variety of equal dimension defined over \( \mathbb{R} \);
- \( D \) is an effective divisor on \( X \) whose base change to \( \mathbb{C} \) has strict normal crossing;
- \( \iota_D \) is the canonical section of the line bundle \( O_X D \);
- \( U \) is the complement of \( \text{supp}(D) \) in \( X \);
- \( \omega_X \) is the canonical line bundle on \( X \). Its sections consist of top-degree differential forms;
- For \( V \) an open set in \( X \), each local section \( \omega \in \Gamma(V, \omega_X) \) gives rise to a measure on \( V(\mathbb{R}) \) which we denote by \( |\omega| \);
- Let \( \omega_X D \) be the line bundle whose local sections consist of those meromorphic top-degree differential forms whose poles can be cancelled by the zeros of \( D \), thought of as a Cartier divisor;
- Assume \( \omega_X D \) is equipped with a smooth metric, by which we mean the following. Forget the algebraic structure and move to the category of smooth manifolds by taking the real points. A metric is a collection of functions
  \[ || \cdot ||_x : \omega_X D(x) \to \mathbb{R}_{\geq 0} \text{ for every } x \in X(\mathbb{R}) \]
where \( \omega_X D(x) \) denotes the fibre at \( x \), such that \( ||l||_x = 0 \) iff \( l = 0 \) for any \( l \in \omega_X D(x) ; ||a||_x = ||a||_x \) for every \( a \in \mathbb{R} \) and \( l \in \omega_X D(x) ; x \mapsto ||l(x)||_x \) is a smooth function for every \( V \subset X(\mathbb{R}) \) open and \( l \in \Gamma(V, \omega_X D) \); we shall drop the dependence on \( x \) from the notation if no confusion might arise;
Depending on this metric we define a locally finite measure \( \tau_{X,D} \). Let \( \omega \) be a local section of \( \omega_X \) then locally \( \tau_{X,D} \) is represented by \( \frac{[\omega]}{[\omega \otimes f_D]} \);

- \( L \) is another effective divisor whose support contains \( \text{supp}(D) \) and \( O_X(L) \) is endowed with a smooth metric;
- Define \( \text{ht} : U(\mathbb{R}) \rightarrow \mathbb{R}_{>0} \) by \( \text{ht}(x) := ||f_L(x)||^{-1} \) and

\[
B_R := \left\{ x \in U(\mathbb{R}) \mid ||f_L(x)|| \geq \frac{1}{R} \right\} = \left\{ x \in U(\mathbb{R}) \mid \text{ht}(x) \leq R \right\},
\]

a compact set contained in \( U(\mathbb{R}) \).

It is proved in [CLT10, Corollary 4.8] that

**Theorem 4.1.** The family \( \{B_R\} \) is nice (see section 2.2) and the limit of the measures

\[
\frac{1}{\tau_{X,D}(B_R)} \tau_{X,D}\big|_{B_R}
\]

exists as \( R \) tends to \(+\infty\). Moreover the limit measure is a sum of continuous measures on the manifolds defined by some connected components of the \( \mathbb{R}\)-points of the intersection of certain irreducible components of \( D_\mathbb{C} \).

Their proof also yields the following, which will be actually applied. For each connected component \( U \) of \( U(\mathbb{R}) \), let \( B_{R,U} := B_R \cap U \).

**Theorem 4.2.** The family \( \{B_R \cap U\} \) is nice and the limit of the measures

\[
\frac{1}{\tau_{X,D}(B_R \cap U)} \tau_{X,D}\big|_{B_R \cap U}
\]

exists as \( R \) tends to \(+\infty\). Moreover the limit measure is a sum of continuous measures on the manifolds defined by some connected components of the \( \mathbb{R}\)-points of the intersection of certain irreducible components of \( D_\mathbb{C} \).

Actually this limit measure has a more precise description in [CLT10, Corollary 4.8] but we are content with this weaker statement. Let us give a more precise description of the asymptotic of \( \tau_{X,D}(B_R) \) though.

Let \( \mathcal{B} \) be an index set of irreducible components of \( D_\mathbb{C} \) and \( \mathcal{P}(\mathcal{B}) \) be the collection of subsets of \( \mathcal{B} \). For \( A \in \mathcal{P}(\mathcal{B}) \), let \( D_A := \cap_{\alpha \in A} D_\alpha \). Let Gal be the Galois group of \( \mathbb{C} \) over \( \mathbb{R} \). It acts on \( \mathcal{P}(\mathcal{B}) \) and let \( \mathcal{P}(\mathcal{B})^{\text{Gal}} \) be the subset of \( \mathcal{B} \) that is Gal-stable.

Define the analytic Clemens complex by

\[
C^n_\mathbb{R}(D) := \left\{ (A, Z) \mid \begin{array}{l} A \in \mathcal{P}(\mathcal{B})^{\text{Gal}}, \text{Z is a } \mathbb{C}\text{-irreducible component of } D_A \\
\text{defined over } \mathbb{R}, \text{Z}(\mathbb{R}) \text{ is non-empty} \end{array} \right\}
\]

equipped with the partial order \( \leq \) defined by

\[
(A, Z) \leq (A', Z') \iff A \subset A', \text{Z } \supset Z'.
\]

For \( (A, Z) \in C^n_\mathbb{R}(D) \), define

\[
\dim(A, Z) := \max \{ n \mid \exists (A_0, Z_0) \leq \cdots \leq (A_n, Z_n) = (A, Z) \}.
\]

The dimension of a poset (poset = partially ordered set) would then be the maximal of dimensions of elements contained in it.
Now we take $U$ to be a union of some connected components (in the Hausdorff topology) of $U(\mathbb{R})$. When $U = U(\mathbb{R})$, definitions below would coincide with the corresponding one in \cite{CLT10}.

Define $\mathcal{C}^\alpha_{R,U}(D)$ to be a sub-poset of $\mathcal{C}^\alpha_R(D)$ by

$$
\mathcal{C}^\alpha_{R,U}(D) := \{(A,Z) \in \mathcal{C}^\alpha_R(D) \mid Z(\mathbb{R}) \cap U \neq \emptyset\}.
$$

Define

$$
\sigma_U := \left\{ \frac{d_\alpha - 1}{\lambda_\alpha} \biggm| \alpha \in \mathcal{R}_a(\mathbb{R}) \cap U \neq \emptyset \right\}
$$

where $D_a(\mathbb{R})$, by definition, is $D_a(\mathbb{C}) \cap X(\mathbb{R})$ in the case when $D_a$ may not be defined over $\mathbb{R}$. We collect the indices where the maximal is attained to form

$$
\mathcal{R}_{\max,U} \subset \mathcal{R}.
$$

Define

$$
\mathcal{C}^\alpha_{R,(L,D),U}(D) := \{(A,Z) \in \mathcal{C}^\alpha_{R,U}(D) \mid A \subset \mathcal{R}_{\max,U}\}
$$

and $b_U := \dim \mathcal{C}^\alpha_{R,(L,D),U}$. The following is a (again we are content with a weaker statement) variant of \cite[Theorem 4.7]{BM97} which follows from the same proof. For the sake of notation let $b'_U := b_U$ if $\sigma_U > 0$ and $b'_U := b_U + 1$ if $\sigma_U = 0$.

**Theorem 4.3.** Assumptions same as above. Then there exists a constant $C > 0$ such that

$$
\lim_{R \to \infty} \frac{\tau_{X,D}(B_R \cap U)}{CR^{\sigma_U} \log(R)^{b'_U-1}} = 1.
$$

4.2. Resolution of singularities.

**Theorem 4.4.** There exists a smooth projective $G$-variety $X_2$ over $\mathbb{R}$ with a $G$-equivariant proper morphism $\pi_2 : X_2 \to X_1$ such that:

1. Let $U_2 := \pi_2^{-1}(G/H)$, then $\pi_1|_{U_2}$ is an isomorphism;
2. Let $D_2 := \pi_2^{-1}(D_1)$, then $(D_2)_C$ is a strict normal crossing divisor.

See \cite{BM97} and \cite{EV98} for instance. And \cite[Proposition 3.9.1]{Kol07} explains how to lift the algebraic action of an algebraic group.

4.3. Divisor of the $G$-invariant top-degree differential form. The result of this subsection is valid more generally for $G$ semisimple and $H$ with $Z_GH/Z(H)$ being $\mathbb{Q}(i)$-anisotropic. $Z(H)$ is defined as the center of $H$.

We verify that the Haar measure $\mu_{G/H}$ can be constructed from $D_2$ via the method of Section 4.1. The whole subsection is devoted to proving the following.

**Proposition 4.5.** There exists an effective divisor $D'_2$ whose support is equal to supp$(D_2)$ and a smooth metric on $\mathcal{O}_X(D'_2)$ such that $\tau_{X,D'_2}|_{G/H} = \mu_{G/H}$.

To prove this it suffices to show that

**Claim.** The $G$-invariant top-degree differential form $\omega_0$ on $G/H$, when viewed as a meromorphic section of the canonical line bundle, has poles at every $D_\alpha$ for every irreducible component $D_\alpha$.

Once this is proved, $D'_2$, defined as $-\text{div}(\omega_0)$, is an effective divisor whose support is equal to supp$(D_2)$ and $\omega_0$ becomes a global nowhere vanishing section of $\omega_{X_2}(D'_2)$. And we simply define a metric on $\omega_{X_2}(D'_2)$ by imposing $||\omega_0|| \equiv 1$. This metric is smooth and $\tau_{X_2,D'_2}$ is equal to the Haar measure $|\omega_0|$ on $G/H$. 


To prove the claim it suffices to show that for each irreducible component $D_n$ of $D_C$, there exists $x \in D_n(\mathbb{C})$ and a section $D$ in the dual of $\omega_X$ such that $D_x = 0$ but $\langle D, \omega_0 \rangle_x$ is bounded away from 0 from below for a sequence $x_i$ converging to $x$. As $\omega_0$ is $G$-invariant, it is allowed to replace $x$ by $gx$ for some $g \in G$. Let us note this approach has been applied to the setting of bi-equivariant compactification of $G$ in [CLT02, Section 2] and the proof below is inspired by theirs.

As $U_2(\mathbb{C})$ is dense in $X_2(\mathbb{C})$ in the Hausdorff topology, we may find a sequence $(g_n)$ from $G(\mathbb{C})$ such that $\lim g_n \oplus v_L = x$. Now apply Theorem 3.4 to

- $G' := R_{Q(i)}/qG$;
- $H' := R_{Q(i)}/qH$;
- $\Gamma'$ to be a lattice commensurable with $G'(\mathbb{Z})$;
- $(g_n) \subset G'(\mathbb{C}) \cong G'(\mathbb{R})$.

Note that under our condition 1.1, we still have $(Z_{G'}H')^0 \subset H'$. We find a bounded sequence $(\delta_n)$ in $G(\mathbb{C})$, a sequence $(\gamma_n)$ in $\Gamma'$, which is commensurable with $G(\mathbb{Z})$ when identifying $G'(\mathbb{R}) \cong G(\mathbb{C})$, and $(h_n)$ in $H(\mathbb{C})$ such that $g_n = \delta_n \gamma_n h_n$. As $g_n \oplus v_L = \delta_n \gamma_n v_L$, we may and do assume that $h_n = \text{id}$. By passing to a subsequence also assume that $\delta_n$ converges to $\delta_\infty$. Now $x = \lim \delta_\infty \gamma_n \oplus v_L$ so we assume $\delta_n = \delta_\infty$. Replacing $x$ by $\delta_\infty^{-1} x$, we assume $g_n = \gamma_n$.

Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) denote the Lie algebra of $G$ (resp. $H$). By passing to a further subsequence we find a $Q(i)$-subspace $W$ in $\mathfrak{g}_{Q(i)}$ such that $W \cap \text{Ad} \gamma_n \mathfrak{h} = \{0\}$ for all $n$.

Our strategy of proving the claim is to construct a section $D$ of $\Lambda^\text{top} \mathcal{F}_{X_2}$, where $\mathcal{F}_{X_2}$ denotes the tangent bundle on $X_2$, such that $\langle D, \omega_0 \rangle \neq 0$ but $D$ vanishes at $x$. It suffices to show that the quantity $|\langle D, \omega_0 \rangle_{\gamma_n \oplus v_L}| > \delta_0$ for some $\delta_0 > 0$ independent of $n$.

Indeed for a vector $v \in \mathfrak{g}$, define $D_v$ by

$$\langle D_v, f \rangle_x := \left. \frac{d}{dt} \right|_{t=0} f(\exp(tv))$$

for every function $f$ defined locally at $x$. For $v = v_1 \wedge v_2 \wedge \ldots \wedge v_m$, we let $D_v := D_{v_1} \wedge \ldots \wedge D_{v_m}$. Then we choose $D$ to be $D_w$ for some set $\{w_1, \ldots, w_m\}$ of $Q(i)$-basis of $W$.

We let $\pi_{\mathfrak{h}}$ denote the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Let $m := \dim G/H$. We fix a set of basis $\{x_1, \ldots, x_m\}$ of $(\mathfrak{g}/\mathfrak{h})^*$. Choose a $Q(i)$-subspace $\mathfrak{h}'$ of $\mathfrak{g}$ complementary to $\mathfrak{h}$ and we pull back $x_1$ to be linear functionals $x'_i$ on $\mathfrak{h}'$. Identify a small neighborhood of 0 in $\mathfrak{h}'(\mathbb{C})$ with a small neighborhood of $[e]$ in $G/H$ via $v \mapsto [\exp(v)]$. Under this identification, each $x'_i$ becomes a function $\bar{x}_i$ on a small neighborhood of $[e]$ in $G/H$. Then $\omega_0$ is a $G$-invariant differential form whose localization at identity coset is equal to some non-zero constant multiple of $d \bar{x}_1 \wedge \ldots \wedge d \bar{x}_m$. Without loss of generality we just assume they are equal. In light of the following lemma, this definition does not depend on the choice of $\mathfrak{h}'$.

**Lemma 4.6.** $\langle D_v, d \bar{x}_i \rangle_{[e]} = \langle \pi_{\mathfrak{h}}(v), x_i \rangle$. 

Proof. Write \( v = v_1 + v_2, \) then \( D_v = D_{v_1} + D_{v_2} \) and \( \langle D_{v_2} \rangle_\varepsilon = 0. \)

\[
\langle D_v, d\tilde{x}_i \rangle_\varepsilon = \langle D_{v_1}, d\tilde{x}_i \rangle_\varepsilon = \frac{d}{dt}_{t=0} \tilde{x}_i([\exp(v_1t)]) = \frac{dx'_i(v_1t)}{dt}_{t=0} = \langle \pi_\varepsilon(v_1), x_i \rangle = \langle \pi_\varepsilon(v), x_i \rangle.
\]

\[\square\]

Lemma 4.7. \( \langle D_{w}, \omega_0 \rangle_{[\varepsilon]} = \det \langle (\pi_\varepsilon(\text{Ad} \gamma^{-1} g \cdot w_i), x_j) \rangle_{i,j}. \)

Proof. Note that \( (\omega_0)_{[\varepsilon]} = L_{g^{-1}}^*(\omega_0)_{[\varepsilon]} \) where \( L_g \) denotes the left action of \( G \) on \( G/H \). So we just need to show that \( \langle D_{w_1}, L_{g^{-1}}^*(dx_j) \rangle_{[\varepsilon]} = \langle \pi_\varepsilon(\text{Ad} \gamma^{-1} w_i), x_j \rangle. \) Indeed

\[
\text{LHS} = \frac{d}{dt}_{t=0} L_{g^{-1}}(\tilde{x}_j)(\exp(w_i t))_{[\varepsilon]} = \frac{d}{dt}_{t=0} (\tilde{x}_j)(g^{-1} \exp(w_it)g[e]) = \langle D_{(\text{Ad} g^{-1} w_i)} d\tilde{x}_j \rangle_{[\varepsilon]},
\]

which is equal to the right hand side by the lemma above. \[\square\]

Now we can prove our claim. As \( w_i \)'s are \( \mathbb{Q}(i) \)-vectors and \( \gamma_n \)'s are contained in \( \Gamma' \), which is commensurable with \( G(\mathbb{Z}[i]) \), we have that \( \{ \text{Ad} \gamma_n^{-1} \cdot w_i \}_{i,n} \) are vectors with bounded denominators. As \( x_i, \pi_\varepsilon \) are all defined over \( \mathbb{Q}(i) \), we have that the entries of the matrices \( \langle (\pi_\varepsilon \text{Ad} \gamma_n^{-1} w_i, x_j) \rangle_{i,j} \) as \( n \) varies are \( \mathbb{Q}(i) \)-vectors with bounded denominators. Therefore there exists a constant \( \varepsilon_0 > 0 \) such that

\[
\langle D_{w}, \omega_0 \rangle_{[\varepsilon_0]} = \det \langle (\pi_\varepsilon \text{Ad} \gamma_n^{-1} w_i, x_j) \rangle_{i,j}
\]

is either 0 or at least \( \varepsilon_0 \). By our choice of \( w_i \), it is not equal to 0, so we are done.

4.4. One source of examples of heights. By partition of unity one can construct smooth metrics for any line bundle, though it is not clear every proper (in the sense of topology) real algebraic function \( l : G/H \rightarrow \mathbb{R}_{\geq 0} \) can be obtained this way. In this subsection we explain that those arising from a representation and an Euclidean norm does arise this way.

Let \( \rho_0 : G \rightarrow \text{GL}(V_0) \) be a representation of \( G \) over \( \mathbb{Q} \). Let \( v_0 \in V_0(\mathbb{Q}) \) be a vector whose stabilizer in \( G \) is exactly equal to \( H \). Fix an isomorphism \( V_0(\mathbb{R}) \cong \mathbb{R}^N \) for some \( N \) and write

\[
\|x\| := \left(x_1^2 + \ldots + x_N^2\right)^{1/2}.
\]

Consider the function \( l : G/H(\mathbb{R}) \rightarrow \mathbb{R} \) defined by

\[
l([g]) := \|g \cdot v_0\|.
\]

Let \( X_{v_0} \) be the compactification of \( G \cdot v_0 \) in \( \mathbb{P}(V_0 \oplus \mathbb{Q}) \). By considering the diagonal embedding \( G/H \rightarrow X_1 \times X_{v_0} \) and replacing \( X_1 \) by the compactification of this one, we may assume that there exists \( \pi_{v_0} : X_1 \rightarrow X_{v_0} \) that is \( G \)-equivariant.

Lemma 4.8. Under the assumption in the last paragraph, there exists a line bundle \( L \) on \( X_2 \) whose support contains \( D_2 \) and a smooth metric such that \( l \) coincides with the corresponding \( h^* \) defined as in 4.1.

Proof. Indeed the function \( l \) may be viewed as obtained from certain metric on the pull-back line bundle \( \mathcal{O}_{\mathbb{P}(V_0 \oplus \mathbb{Q})}(1) \) and the canonical section associated with the divisor \( \mathbb{P}(V_0 \oplus \{0\}) \). As there is a morphism from \( X_2 \) to \( X_{v_0} \) we can further pull back the metrized line bundle and the effective divisor to \( X_2 \). It only remains
to check the metric is indeed smooth. Indeed the metric on $X_{v_0}$, which by itself may not be smooth, is smooth in the sense that it can be extended to a smooth function in the ambient space. Therefore its pull-back to $X_2$ also satisfies the same property. But $X_2$ is smooth by itself, hence this metric is also smooth in the usual sense. And the proof completes. □

4.5. Wrap-up. Now we prove Proposition 2.2, which implies Theorem 1.3, Corollary 1.4, and Theorem 1.5 in this section.

Let $X_1 := \pi_{-1}^{-1}(X_0)$, $X_2 := \pi_{-1}^{-1}(X_1)$ and $X_{v_0} := G \cdot v_0$. Thus we have the following $G$-equivariant picture.

$$
\begin{array}{c}
X_2 \\
\downarrow \pi_2 \\
X_1 \\
\downarrow \pi_1 \\
X_0 \\
\downarrow \pi_{v_0} \\
X_{v_0}
\end{array}
$$

Let $U_2 := (\pi_1 \circ \pi_2)^{-1}U_0$ and $\lambda_2 := \lim_{R \to \infty} \frac{\mu_{G/H}([BR \cap U_2])}{\mu_{G/H}(BR)}$ in $\Prob(X_2)$ exists by Theorem 4.2 with the help of Theorem 4.4, Proposition 4.5 and Lemma 4.8. And $\lambda_0 := (\pi_1 \circ \pi_2)_* \lambda_2$. Recall Ave : $\Prob(\Prob(G/\Gamma)) \to \Prob(G/\Gamma)$ from section 2.3.

Then we have

$$
\lim_{R \to \infty} \frac{1}{\mu_{G/H}(BR)} \int_{BR} g_* \mu_{G/H}([g]) = \text{Ave}(\lambda_0) = \int \mu \lambda_0(\mu).
$$

This proves Proposition 2.2 and hence Theorem 1.3. Together with Theorem 4.3, Corollary 1.4 follows. To prove Theorem 1.5, a few words are needed.

By Theorem 4.2, $\lambda_2$ is absolutely continuous with respect to the smooth measure on intersection of some divisors with continuous density. As each such component is preserved by $G$, $\lambda_0$ also has the same property. Note that there is no notion of smooth measure on $\Prob(G/\Gamma)$ but $X_0$, according to Corollary 3.10, is a finite disjoint union $\bigsqcup_{L \in \mathcal{A}/\sim} G \cdot \mu_L$. Each strata $G \cdot \mu_L$ has a natural smooth structure. So we have

$$
\lambda_0 = \sum \lambda_0|_{G \cdot \mu_L} = \sum f_L \mu_{G/L}
$$

for some continuous, non-negative $f_L : G/L \to \mathbb{R}$. Here, by abuse of notation, we write $f_L \mu_{G/L}$ for the push-forward of it on $G \cdot \mu_L$.

Now we fix a smooth function $P : X_2 \to \mathbb{R}$. Let $P' := (\pi_1 \circ \pi_2)_* P$. Namely $P'$ is a continuous function on $X_0$ satisfying

$$
\int P(x) \lambda_2(x) = \int P'(x) \lambda_0(x).
$$

Let $P_L$ be the restriction of $P'$ to $G \cdot \mu_L$. Again by abuse of notation we think of $P_L$ as a function on $G/L$. So we have

$$
(\pi_1 \circ \pi_2)_*(P \cdot \lambda_2) = \sum_{L \in \mathcal{A}/\sim} P_L f_L \mu_{G/L}.
$$

As usual we have chosen $\mu_{G/L}$ that is compatible with $\tilde{\mu}_G$ and $\tilde{\mu}_L$. 

Now we apply Ave. Using the diagram

\[ \begin{array}{c}
G/L \\
\downarrow p_L \\
\downarrow q_L
\end{array} \]

\[ \begin{array}{c}
G/\Gamma \\
\downarrow G/L
\end{array} \]

we have

\[
\text{Ave}(\lambda_0) = \text{Ave}(\sum f_L \mu_{G/L}) = \sum ((p_L)_*(q_L)^* f_L) \mu_{G/\Gamma}
\]

\[
\text{Ave}(P'\lambda_0) = \text{Ave}(\sum P_L f_L \mu_{G/L}) = \sum ((p_L)_*(q_L)^* (P_L f_L)) \mu_{G/\Gamma}
\]

and we define

\[ P^\vee([g]) := \sum ((p_L)_*(q_L)^* (P_L f_L)). \]

Therefore we have the following (note that \( P' = P \) when restricted to \( G/H \))

\[
\lim_{R \to \infty} \frac{1}{\mu_{G/H}(B_R)} \int_{B_R} P([g]) g_* \mu_H \mu_{G/H}([g]) = P^\vee \cdot \mu_G.
\]

By the same argument as in the proof of Prop 2.?, we may conclude that

\[
\lim_{R \to \infty} \frac{1}{\mu_{G/H}(B_R)} \sum_{\gamma \in \Gamma/\Gamma \cap H} P(g\gamma \cdot v_0) 1_{B_R}(g\gamma \cdot v_0) = P^\vee([g]) \quad \forall [g] \in G/\Gamma.
\]

Note that for each \([g]\) the map

\[ P \mapsto \frac{P^\vee([g])}{1^\vee([g])} = \frac{\sum_{L \in A/\sim} \sum_{\gamma \in \Gamma/\Gamma \cap L} P_L(g\gamma) f_L(g\gamma)}{\sum_{L \in A/\sim} \sum_{\gamma \in \Gamma/\Gamma \cap L} P_L(g\gamma)} \]

defines a positive bounded linear functional on the class of continuous functions on \( X_2 \). Thus it is represented by a Radon measure probability measure \( \nu_{[g]} \). In particular this proves Theorem 1.5.

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