A doubly stochastic block Gauss-Seidel algorithm for solving linear equations

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Abstract

We propose a simple doubly stochastic block Gauss-Seidel algorithm for solving linear systems of equations. By varying the row partition parameter and the column partition parameter of the coefficient matrix, we recover the Landweber algorithm, the randomized Kaczmarz algorithm, the randomized Gauss-Seidel algorithm, and the doubly stochastic Gauss-Seidel algorithm. For general linear systems (consistent or inconsistent), we show the exponential convergence of the norms of the expected iterates via exact formulas. For consistent linear systems, we prove the exponential convergence of the expected norms of the error or the residual.

Keywords: Randomized Kaczmarz, Randomized Gauss-Seidel, Doubly stochastic Gauss-Seidel, Doubly stochastic block Gauss-Seidel, Exponential convergence

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1. Introduction

Randomized iterative algorithms for solving a linear system of equations

\[ Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m \]

have attracted much attention recently; see, for example, [1–4, 6–13]. At each step, to generate the next iterate from the current iterate, the randomized Kaczmarz algorithm [12] uses a randomly picked row, the randomized Gauss-Seidel (i.e., randomized coordinate descent) algorithm [6] uses a randomly picked column, and the doubly stochastic Gauss-Seidel algorithm [11] uses a randomly picked entry of the coefficient matrix \( A \). It is natural to ask whether one can design a randomized algorithm which uses a randomly picked submatrix of \( A \).

In this paper, we propose a doubly stochastic block Gauss-Seidel algorithm which uses a submatrix of \( A \) at each step (see Algorithm 1 in §2). The Landweber iterative algorithm [3], the randomized Kaczmarz algorithm, the randomized Gauss-Seidel algorithm, and the doubly stochastic Gauss-Seidel algorithm are special cases of our algorithm. Our algorithm does not need to use projections and Moore-Penrose pseudoinverses of submatrices, so it is different from the block algorithms in [4, 8, 10]. Moreover, our algorithm can utilize fast matrix multiplies and efficient implementation, yielding remarkable improvements in computational time.

Main contributions. We propose a simple doubly stochastic block Gauss-Seidel algorithm for solving linear equations and prove its convergence theory. More specifically, we show the exponential convergence of the norms of the expected iterates via exact formulas (see Theorems 2.2 and 2.5) for general linear systems (consistent or inconsistent), and prove the exponential convergence of the expected norms of the error or the residual (see Theorems 2.8 and 2.11) for consistent linear systems.
2.1. The exponential convergence of the norms of the expected iterates

In this subsection we show the exponential convergence of the norms of the expected iterates for general (consistent or inconsistent) linear systems. Note that the conditioned expectation on $x^{k-1}$

$$
\mathbb{E}[x^k | x^{k-1}] = x^{k-1} - \alpha \mathbb{E} \left[ \frac{I_{I,J}(A_{I,J})^T(I_{I,J})^T}{\|A_{I,J}\|^2_F} \right] (Ax^{k-1} - b)
$$

$$
= x^{k-1} - \alpha \sum_{(I,J) \sim \mathcal{P}} \frac{I_{I,J}(A_{I,J})^T(I_{I,J})^T}{\|A_{I,J}\|^2_F} \frac{\|A_{I,J}\|^2_F}{\|A\|^2_F} (Ax^{k-1} - b)
$$

$$
= x^{k-1} - \alpha \frac{A^T}{\|A\|^2_F} (Ax^{k-1} - b),
$$

which is one-step Landweber’s update from $x^{k-1}$ with step size $\alpha/\|A\|^2_F$. The following lemma will be used to prove the exponential convergence of the norms of the expected iterates. Its proof (via singular value decomposition) is straightforward and we omit the details.
**Lemma 2.1.** Let $\alpha > 0$ and $A$ be any nonzero real matrix. For every $u \in \text{range}(A)$, it holds

$$\left\| \left( \mathbf{I} - \frac{\alpha A A^T}{\|A\|_F^2} \right)^k u \right\|_2 \leq \left( \max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(A)}{\|A\|_F^2} \right| \right)^k \|u\|_2.$$  

In the following theorem, we show the exponential convergence of the norm of the expected error for the consistent linear system $Ax = b$.

**Theorem 2.2.** Let $x^k$ denote the $k$th iterate of DSBGS applied to the consistent linear system $Ax = b$ with arbitrary $x^0 \in \mathbb{R}^n$. In exact arithmetic, it holds

$$\|\mathbb{E}[x^k - x^0]\|_2 \leq \left( \max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(A)}{\|A\|_F^2} \right| \right)^k \|x^0 - x^0\|_2,$$

where $x^0 = (I - A^\dagger A)x^0 + A^\dagger b$ i.e., the projection of $x^0$ onto the set $\{x \in \mathbb{R}^n \mid Ax = b\}$.

**Proof.** Note that the conditioned expectation on $x^{k-1}$

$$\mathbb{E}[x^k - x^0 | x^{k-1}] = \mathbb{E}[x^k | x^{k-1}] - x^0 = x^{k-1} - \alpha \frac{A^T}{\|A\|_F^2} (Ax^{k-1} - b) - x^0$$

$$= x^{k-1} - \alpha \frac{A^T}{\|A\|_F^2} (Ax^{k-1} - x^0) - x^0$$

$$= (I - \frac{\alpha A^T A}{\|A\|_F^2}) (x^{k-1} - x^0).$$

Taking expectation gives

$$\mathbb{E}[x^k - x^0] = \mathbb{E}[\mathbb{E}[x^k - x^0 | x^{k-1}]] = \left( I - \frac{\alpha A^T A}{\|A\|_F^2} \right)^k \mathbb{E}[x^{k-1} - x^0]$$

$$= \left( I - \frac{\alpha A^T A}{\|A\|_F^2} \right)^k (x^0 - x^0).$$

Applying the norms to both sides we obtain

$$\|\mathbb{E}[x^k - x^0]\|_2 \leq \left( \max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(A)}{\|A\|_F^2} \right| \right)^k \|x^0 - x^0\|_2.$$  

Here the inequality follows from the fact that $x^0 - x^0 = A^\dagger Ax^0 - A^\dagger b \in \text{range}(A^T)$ and Lemma 2.1.

**Remark 2.3.** If $x^0 \in \text{range}(A^T)$, then $x^0 = A^\dagger b$.

**Remark 2.4.** To ensure convergence of the expected iterate, it suffices to have

$$\max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(A)}{\|A\|_F^2} \right| < 1 \quad \text{i.e.,} \quad 0 < \alpha < \frac{2\|A\|_F^2}{\sigma_1^2(A)}.$$  

In the following theorem, we show the exponential convergence of the norm of $\mathbb{E}[Ax^k - Ax^*]$ for the consistent or inconsistent linear system $Ax = b$, where $x^*$ is any solution of $A^T Ax = A^T b$.

**Theorem 2.5.** Let $x^k$ denote the $k$th iterate of DSBGS applied to the consistent or inconsistent linear system $Ax = b$ with arbitrary $x^0 \in \mathbb{R}^n$. In exact arithmetic, it holds

$$\|\mathbb{E}[Ax^k - Ax^*]\|_2 \leq \left( \max_{1 \leq i \leq r} \left| 1 - \frac{\alpha \sigma_i^2(A)}{\|A\|_F^2} \right| \right)^k \|Ax^0 - Ax^*\|_2,$$

where $x^*$ is any solution of $A^T Ax = A^T b$.  

\[3\]
Proof. Note that the conditioned expectation on $x^{k-1}$

$$
\mathbb{E}[Ax^k - Ax_\star | x^{k-1}] = A(\mathbb{E}[x^k | x^{k-1}] - x_\star) = A \left( x^{k-1} - \alpha \frac{A^T}{\|A\|_F^2} (Ax^{k-1} - b) - x_\star \right)
$$

$$
= A \left( x^{k-1} - \alpha \frac{A^T}{\|A\|_F^2} (Ax^{k-1} - Ax_\star) - x_\star \right) \quad \text{(by } A^T b = A^T Ax_\star) \nonumber
$$

$$
= Ax^{k-1} - Ax_\star - \alpha \frac{A A^T}{\|A\|_F^2} (Ax^{k-1} - Ax_\star)
$$

Taking expectation gives

$$
\mathbb{E}[Ax^k - Ax_\star] = \mathbb{E}[\mathbb{E}[Ax^k - Ax_\star | x^{k-1}]] = \left( I - \alpha \frac{A A^T}{\|A\|_F^2} \right) (Ax^0 - Ax_\star).
$$

Applying the norms to both sides we obtain

$$
\|\mathbb{E}[Ax^k - Ax_\star]\|_2 \leq \left( \max_{1 \leq i \leq r} \left| 1 - \alpha \sigma_i^2(A) \frac{\|A\|_F^2}{\|A\|_F^2} \right| \right)^k \|Ax^0 - Ax_\star\|_2.
$$

Here the inequality follows from the fact that $Ax^0 - Ax_\star \in \text{range}(A)$ and Lemma 2.1.

\[\square\]

2.2. The exponential convergence of the expected norms of the error or the residual

In this subsection we prove the exponential convergence of the expected norms of the error or the residual for consistent linear systems. The following two lemmas will be used. Their proofs are straightforward and we omit the details.

Lemma 2.6. For any vector $u \in \mathbb{R}^m$ and any matrix $A \in \mathbb{R}^{m \times n}$, it holds $u^TAA^T u \leq \|A\|_F^2 u^T u$. 

Lemma 2.7. For any matrix $A \in \mathbb{R}^{m \times n}$ with rank $r$ and any vector $u \in \text{range}(A)$, it holds

$$
u^TAA^T u \geq \sigma^2_r(A) \|u\|_2^2.
$$

For full column rank consistent systems, we prove the exponential convergence of the expected norm of the error in the following theorem. We recall that in this case $A^\dagger b$ is the unique solution of $Ax = b$.

Theorem 2.8. Let $x^k$ denote the $k$th iterate of DSBGS applied to the full column rank consistent linear system $Ax = b$ with arbitrary $x^0 \in \mathbb{R}^n$. Assume $0 < \alpha < 2/t$. In exact arithmetic, it holds

$$
\mathbb{E}[\|x^k - A^\dagger b\|_2^2] \leq \left( 1 - \frac{2\alpha - \alpha^2 \sigma^2_r(A)}{\|A\|_F^2} \right)^k \|x^0 - A^\dagger b\|_2^2.
$$
Remark 2.10. If \( \text{Theorem 2.8 are given in [12, Theorem 2]} \) and \([11, \text{Theorem 1}]\), respectively.

Remark 2.9. Proof. Note that

\[
\mathbb{E}[\|x^k - A^\dagger b\|_2^2 | x^{k-1}] \leq \|x^{k-1} - A^\dagger b\|_2^2 - 2\alpha(x^{k-1} - A^\dagger b)^T \left( \frac{\text{tr}A^T A}{\|A\|_F^2} \right) (x^{k-1} - A^\dagger b)
\]

Taking expectation again gives

\[
\mathbb{E}[\|x^k - A^\dagger b\|_2^2] = \mathbb{E}[\mathbb{E}[\|x^k - A^\dagger b\|_2^2 | x^{k-1}]]
\]

The last equality follows from \((I_{s,J})^T I_{s,J} = I\) and Lemma 2.6. Taking expectation gives

\[
\mathbb{E}[\|x^k - A^\dagger b\|_2^2 | x^{k-1}] \leq \|x^{k-1} - A^\dagger b\|_2^2 - 2\alpha(x^{k-1} - A^\dagger b)^T \left( \frac{\text{tr}A^T A}{\|A\|_F^2} \right) (x^{k-1} - A^\dagger b)
\]

Taking expectation again gives

\[
\mathbb{E}[\|x^k - A^\dagger b\|_2^2] = \mathbb{E}[\mathbb{E}[\|x^k - A^\dagger b\|_2^2 | x^{k-1}]]
\]

Taking expectation again gives

\[
\mathbb{E}[\|x^k - A^\dagger b\|_2^2] = \mathbb{E}[\mathbb{E}[\|x^k - A^\dagger b\|_2^2 | x^{k-1}]]
\]

Remark 2.9. For the case \( s = m \), \( t = 1 \), \( \alpha = 1 \) (i.e., the randomized Kaczmarz algorithm) and the case \( s = m \), \( t = n \), \( \alpha = 1/n \) (i.e., the doubly stochastic Gauss-Seidel algorithm), the results of Theorem 2.8 are given in [13, Theorem 2] and [14, Theorem 1], respectively.

Remark 2.10. If \( t = 1 \) and \( x^0 \in \text{range}(A^T) \), we can show \( x^k - x^0 \in \text{range}(A^T) \) by induction, where \( x^0 = (I - A^\dagger A)x^0 + A^\dagger b \) i.e., the projection of \( x^0 \) onto the set \( \{x \in \mathbb{R}^n \mid Ax = b\} \). Then for rank deficient consistent linear systems, by the same approach, we can prove the convergence bound

\[
\mathbb{E}[\|x^k - x^0\|_2^2] \leq \left( 1 - \frac{2\alpha - \alpha^2 \sigma_2^2(A)}{\|A\|_F^2} \right) \|x^0 - x^0\|_2^2.
\]

The result for the special case \( s = m \) and \( t = 1 \) (i.e., RK) was already given in [13, Theorem 3.4].

Next, we prove the exponential convergence of the expected norm of the residual for consistent linear systems.

**Theorem 2.11.** Let \( x^k \) denote the \( k \)th iterate of DSBGS applied to the consistent linear system (full column rank or rank-deficient) \( Ax = b \) with arbitrary \( x^0 \in \mathbb{R}^n \). If \( t = n \) and \( 0 < \alpha < 2\sigma_2^2(A)/\|A\|_F^2 \), then

\[
\mathbb{E}[\|Ax^k - b\|_2^2] \leq \left( 1 + \alpha^2 \frac{2\alpha \sigma_2^2(A)}{\|A\|_F^2} \right)^k \|Ax^0 - b\|_2^2.
\]
If \( t < n \) and \( 0 < \alpha < 2\sigma_T^2(A)/(t\rho) \), then

\[
\mathbb{E}[\|Ax^k - b\|_2^2] \leq \left(1 - \frac{2\alpha\sigma_T^2(A) - t\rho\alpha^2}{\|A\|_F^2}\right)^k \|Ax^0 - b\|_2^2,
\]

where

\[
\rho = \max_{1 \leq j \leq t} \sigma_T^2(A_{:,j}).
\]

**Proof.** Note that

\[
\|Ax^k - b\|_2^2 = \|Ax^{k-1} - \alpha \left(\frac{AI_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b) - b\|_2^2
\]

\[
= \|Ax^{k-1} - b\|_2^2 - 2\alpha(Ax^{k-1} - b)^T \left(\frac{AI_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b)
\]

\[
+ \alpha^2(Ax^{k-1} - b)^T \left(\frac{I_{:,J}A_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b).
\]

If \( t = n \), then it follows from \((I_{:,J})^T A^T A_{:,J} = \|A_{:,J}\|_F^2 \) (since \( A_{:,J} = A_{:,J} \) is a column vector) that

\[
\|Ax^k - b\|_2^2 = \|Ax^{k-1} - b\|_2^2 - 2\alpha(Ax^{k-1} - b)^T \left(\frac{AI_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b)
\]

\[
+ \alpha^2(Ax^{k-1} - b)^T \left(\frac{I_{:,J}A_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b).
\]

Taking expectation gives

\[
\mathbb{E}[\|Ax^k - b\|_2^2 | x^{k-1}] \leq (1 + \alpha^2)\|Ax^{k-1} - b\|_2^2 - 2\alpha(Ax^{k-1} - b)^T \left(\frac{A\mu^T}{\|A\|_F^2}\right)(Ax^{k-1} - b)
\]

\[
\leq \left(1 + \alpha^2 - \frac{2\alpha\sigma_T^2(A)}{\|A\|_F^2}\right)\|Ax^{k-1} - b\|_2^2.
\]

The last inequality follows from \( Ax^{k-1} - b \in \text{range}(A) \) and Lemma 27. Taking expectation again gives

\[
\mathbb{E}[\|Ax^k - b\|_2^2] = \mathbb{E}[\mathbb{E}[\|Ax^k - b\|_2^2 | x^{k-1}]]
\]

\[
\leq \left(1 + \alpha^2 - \frac{2\alpha\sigma_T^2(A)}{\|A\|_F^2}\right)^k \|Ax^0 - b\|_2^2.
\]

If \( t < n \), then it follows from \((I_{:,J})^T A^T A_{:,J} = A_{:,J}^T A_{:,J} \leq \rho I \) (since \( \rho = \max_{1 \leq j \leq t} \sigma_T^2(A_{:,J}) \)) that

\[
\|Ax^k - b\|_2^2 \leq \|Ax^{k-1} - b\|_2^2 - 2\alpha(Ax^{k-1} - b)^T \left(\frac{AI_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b)
\]

\[
+ \alpha^2(Ax^{k-1} - b)^T \left(\frac{\rho I_{:,J}A_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b)
\]

\[
\leq \|Ax^{k-1} - b\|_2^2 - 2\alpha(Ax^{k-1} - b)^T \left(\frac{AI_{:,J}(A_{:,J})^T(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b)
\]

\[
+ \alpha^2(Ax^{k-1} - b)^T \left(\frac{I_{:,J}(I_{:,J})^T}{\|A_{:,J}\|_F^2}\right)(Ax^{k-1} - b). \quad \text{(by Lemma 27)}
\]
Taking expectation gives

\[
\mathbb{E}[\|Ax^k - b\|_2^2 | x^{k-1}] \leq \left(1 + \frac{t \rho^2}{\|A\|_F^2}\right) \|Ax^{k-1} - b\|_2^2 - 2\alpha(Ax^{k-1} - b)^T \left(\frac{AA^T}{\|A\|_F^2}\right)(Ax^{k-1} - b)
\]

\[
\leq \left(1 - \frac{2\alpha\sigma^2(A) - t \rho^2}{\|A\|_F^2}\right)\|Ax^{k-1} - b\|_2^2.
\]

The last inequality follows from \(Ax^{k-1} - b \in \text{range}(A)\) and Lemma 2.7. Taking expectation again gives

\[
\mathbb{E}[\|Ax^k - b\|_2^2] = \mathbb{E}[\mathbb{E}[\|Ax^k - b\|_2^2 | x^{k-1}]]
\]

\[
\leq \left(1 - \frac{2\alpha\sigma^2(A) - t \rho^2}{\|A\|_F^2}\right)\mathbb{E}[\|Ax^{k-1} - b\|_2^2]
\]

\[
\leq \left(1 - \frac{2\alpha\sigma^2(A) - t \rho^2}{\|A\|_F^2}\right)^k\|Ax^0 - b\|_2^2. \quad \square
\]

Remark 2.12. For the case \(s = 1, t = n, \alpha = \sigma^2_f(A)/\|A\|_F^2\) (i.e., the randomized Gauss-Seidel algorithm) and the case \(s = m, t = n, \alpha = \sigma^2_t(A)/\|A\|_F^2\) (i.e., the doubly stochastic Gauss-Seidel algorithm), the results of Theorem 2.11 are given in [6, Theorem 3.2] and [7, Theorem 2], respectively.

Remark 2.13. Note that \((A_{-s,t})^T b = (A_{-s,t})^T Ax_s\), where \(x_s\) is any solution of \(A^T Ax = A^T b\). Then if \(s = 1\), we can prove the convergence bounds in Theorem 2.11 (replacing \(b\) by \(Ax_s\)) still hold for inconsistent linear systems. The result for the special case \(s = 1\) and \(t = n\) (i.e. RGS) was already given in the literature, for example, [6, Theorem 3.2], [7, Lemma 4.2] and [8, Theorem 3].

3. Numerical results

In this section, we compare the performance of the doubly stochastic block Gauss-Seidel (DSBGS) algorithm proposed in this paper against the randomized Kaczmarz (RK) algorithm for solving consistent linear systems. All experiments are performed using MATLAB on a laptop with 2.7-GHz Intel Core i7 processor, 16 GB memory, and Mac operating system.

![Figure 1: Left: The error \(\|x^k - x_*\|_2\) for RK with \(\alpha = 1\), DSBGS(m/2,1) with \(\alpha = 1\), and DSBGS(m/10,2) with \(\alpha = 1/2\). Right: The error \(\|x^k - x_*\|_2\) for DSBGS(m/10,2) with \(\alpha = 2, 3, 4, 5, 6, 7\).](image)

The coefficient matrix \(A \in \mathbb{R}^{m \times n}\) and the solution \(x_* \in \mathbb{R}^n\) are randomly generated by using the MATLAB function \texttt{randn}. Then the right-hand side \(b\) is taken to be \(Ax_*\). We use DSBGS(s, t) to denote the doubly stochastic block Gauss-Seidel algorithm employing the row partition \(\{I_1, I_2, \ldots, I_s\}\) with

\[I_i = \{(i - 1)m/s + 1, (i - 1)m/s + 2, \ldots, im/s\}, \quad i = 1, 2, \ldots, s,\]

and the column partition \(\{J_1, J_2, \ldots, J_t\}\) with

\[J_j = \{(j - 1)n/t + 1, (j - 1)n/t + 2, \ldots, jn/t\}, \quad j = 1, 2, \ldots, t.\]
All algorithms are started from the initial guess \( x^0 = 0 \), terminated if \( \|x^k - x^*_k\|_2 \leq 10^{-8} \). We set \( m = 1000 \) and \( n = 100 \). In Figure 1 we plot the convergence history of the error \( \|x^k - x^*_k\|_2 \) for RK (i.e., DSBGS\((m,1)\)), DSBGS\((m/2,1)\), and DSBGS\((m/10,2)\). The average elapsed CPU times (in second) with respect to 20 repeated runs of RK with \( \alpha = 1 \), DSBGS\((m/2,1)\) with \( \alpha = 1 \), and DSBGS\((m/10,2)\) with \( \alpha = 1/2 \) are 0.1475, 0.1390, and 0.2430, respectively. DSBGS\((m/10,2)\) with \( \alpha = 5 \) has the best performance among the step size \( \alpha = 2, 3, 4, 5, 6, 7 \).

4. Concluding remarks

We have proposed a doubly stochastic block Gauss-Seidel algorithm and prove its convergence theory. The randomized Kaczmarz algorithm, the randomized Gauss-Seidel algorithm, and the doubly stochastic Gauss-Seidel algorithm are special cases of the doubly stochastic block Gauss-Seidel algorithm. Numerical experiments show that appropriate step size and partition parameters significantly improve the performance. Finding appropriate variable step size, proposing more effective sampling strategies for submatrices, and designing other block variants via the idea in [8] should be valuable topics in the future study.

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