Existence and nonlinear stability of steady states of the Schrödinger-Poisson system

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Abstract

We consider the Schrödinger-Poisson system in the attractive (plasma physics) Coulomb case. Given a steady state from a certain class we prove its nonlinear stability, using an appropriately defined energy-Casimir functional as Lyapunov function. To obtain such steady states we start with a given Casimir functional and construct a new functional which is in some sense dual to the corresponding energy-Casimir functional. This dual functional has a unique maximizer which is a steady state of the Schrödinger-Poisson system and lies in the stability class. The steady states are parametrized by the equation of state, giving the occupation probabilities of the quantum states as a strictly decreasing function of their energy levels.

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1 Introduction

A large ensemble of charged quantum particles interacting only by the electrostatic field which they create collectively can be described by the Schrödinger-Poisson system:

\[
\begin{align*}
&i \frac{\partial \psi_k}{\partial t} = -\Delta \psi_k + V \psi_k, \\
&\Delta V = -n, \\
&n = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2.
\end{align*}
\]

Here \(\psi_k = \psi_k(t, x)\) is the wave function of the \(k\)th state, \(k \in \mathbb{N}\), \(\lambda_k \geq 0\) denote the corresponding occupation probabilities normalized such that \(\sum \lambda_k = 1\), \(n = n(t, x)\) is the number density, and \(V = V(t, x)\) the self-consistent potential of the ensemble. In order to avoid continuous spectra we shall analyze this system on a bounded domain \(\Omega \subset \mathbb{R}^3\) with sufficiently smooth boundary, and we supplement it with Dirichlet boundary conditions:

\[
\psi_k(t, x) = 0, \ V(t, x) = 0, \ t \geq 0, \ x \in \partial \Omega, \ k \in \mathbb{N}.
\]

We could also consider the system on the whole space \(\mathbb{R}^3\) and add to \(V\) an appropriate exterior potential \(V_e\). Initial data are given by a complete orthonormal system \((\psi_k(\cdot, 0))\) in \(L^2(\Omega)\). We refer to [1, 4, 9, 16] for background information on the Schrödinger-Poisson system (1.1), (1.2), (1.3).

In terms of the density operator \(R(t)\) of the system, a time dependent, hermitian, positive trace-class operator acting on the Hilbert space \(L^2(\Omega)\), the time evolution is given by the von-Neumann-Heisenberg equation

\[
i \frac{\partial R}{\partial t} = [H_V, R].
\]

Here the Hamiltonian is defined as \(H_V := -\Delta + V(t, x)\) with potential \(V\) given as the solution of the Poisson equation (1.2) with Dirichlet boundary condition, and \(n(t, x) := \rho(t, x, x)\) where \(\rho(t, x, y)\) is the kernel of the operator \(R(t)\). The Schrödinger-Poisson picture and the Heisenberg picture are equivalent: Let \((\phi_k)\) be a complete orthonormal sequence of eigenvectors of \(R(0)\) with eigenvalues \(\lambda_k\) and let \((\psi_k(t, \cdot))\) be the solution of the Schrödinger-Poisson system (1.1), (1.4) with initial data \(\psi_k(0) = \phi_k\). Then
\( \rho(t, x, y) = \sum_k \lambda_k \psi_k(t, x) \bar{\psi}_k(t, y) \) defines the kernel of an operator \( R(t) \) which solves the von-Neumann-Heisenberg equation with the corresponding initial datum, and vice versa.

The Schrödinger-Poisson picture is more suitable for our present purposes, which are as follows: We investigate the nonlinear stability of certain steady states of the Schrödinger-Poisson system, i.e., of solutions of the form 
\( \psi_k(t, x) = e^{i\mu_k t} \phi_k(x) \) with energy levels \( \mu_k \in \mathbb{R} \), and we prove the existence of such steady states. To our knowledge, the stability problem has not yet been investigated. The existence of steady states has been considered by different methods before, cf. [16, 17, 18, 19].

Our approach is motivated by analogous results for the Vlasov-Poisson system which arises as the classical limit of the Schrödinger-Poisson system. Both systems share the following property: The total energy of the system is conserved along solutions—indeed, the dynamics can be interpreted as the “Hamiltonian flow” induced by the energy functional—, but the steady states are not critical points of the energy. On the other hand, there exist additional conserved quantities, the so-called Casimir functionals \( \mathcal{H}_C \), such that a given steady state is a critical point for the appropriately chosen energy-plus-Casimir functional \( \mathcal{H}_C \). The energy-Casimir method was first used to prove genuine, nonlinear stability for fluid-flow problems by Arnol’d in the 1960’s, cf. [2, 3]. Some of the background of this method can be found in [15]. More recently, the energy-Casimir method was adapted to problems in kinetic theory, in particular the Vlasov-Poisson system, cf. [10, 11, 12, 13, 14, 20, 21, 22, 23]. When applying this method there is a sharp contrast between the plasma physics situation and the stellar dynamics one, where the sign in the Poisson equation is reversed: The quadratic part in the expansion of the energy-Casimir functional at the steady state is positive definite in the plasma physics case while it is indefinite in the stellar dynamics case. Therefore, in the former case the method applies in a straightforward manner, cf. [20], while in the latter case a careful investigation of the behavior of the energy-Casimir functional along minimizing sequences is needed and leads to nonlinear stability only for such steady states which are obtained as minimizers of this functional. The present paper addresses the plasma-physics case for the Schrödinger-Poisson system, and thus the approach should be more like the former case for the Vlasov-Poisson system.

This is indeed so: In Section 3 we show that steady states \((\psi_0, \lambda_0)\) from a specified class are nonlinearly stable, and we do so by estimating
$\mathcal{H}_C(\psi, \lambda) - \mathcal{H}_C(\psi_0, \lambda_0)$ from below by an expression which is quadratic in $(\psi, \lambda) - (\psi_0, \lambda_0)$, where $(\psi, \lambda)$ is some other, ‘close-by’ state. In Section 4 we construct a functional which is in some sense dual to a given energy-Casimir functional. As shown in Section 5 this dual functional has a unique maximizer, which is a steady state, and nonlinearly stable by Section 3. We emphasize that—as opposed to the stellar-dynamics situation for the Vlasov-Poisson system—the stability analysis and the existence analysis are independent from each other; the connecting Section 4 is included to put both parts into a common perspective. Before going into all this we introduce the class of steady states respectively Casimir functionals under consideration, derive some preliminary estimates, and fix some notation.

2 Preliminaries

As state space for the Schrödinger-Poisson system we use the set

$$\mathcal{S} := \left\{ (\psi, \lambda) \mid \psi = (\psi_k)_{k \in \mathbb{N}} \subset H^1_0(\Omega) \cap H^2(\Omega) \right\}$$

is a complete orthonormal system in $L^2(\Omega)$,

$\lambda = (\lambda_k)_{k \in \mathbb{N}} \in l^1$ with $\lambda_k \geq 0$, $k \in \mathbb{N}$,

$$\sum_k \lambda_k \int |\Delta \psi_k|^2 < \infty;$$

$\sum_k$ always means $\sum_{k=1}^{\infty}$. Our notation for the Sobolev spaces $H^2$ and $H^1_0$ is the standard one; by $\| \cdot \|_p$ we will denote the norm in the usual $L^p$ space. For $(\psi, \lambda) \in \mathcal{S}$ we have

$$n_{\psi, \lambda} := \sum_k \lambda_k |\psi_k|^2 \in L^2(\Omega),$$

and $V_{\psi, \lambda}$ denotes the Coulomb potential induced by $n_{\psi, \lambda}$, i. e.,

$$\nabla V_{\psi, \lambda} = -n_{\psi, \lambda} \text{ on } \Omega, \quad V_{\psi, \lambda} = 0 \text{ on } \partial \Omega;$$

note that $V_{\psi, \lambda} \in H^1_0(\Omega) \cap H^2(\Omega)$ by the energy bound and Sobolev inequalities. For every initial state $(\psi(0), \lambda) \in \mathcal{S}$ there is a unique strong solution $[0, \infty[ \ni t \mapsto \psi(t)$ of (1.1)–(1.4) with $(\psi(t), \lambda) \in \mathcal{S}$, cf. [1]. Throughout the paper, potentials $V$ are real-valued while quantum states $\psi_k$ are complex-valued. The energy of a state $(\psi, \lambda) \in \mathcal{S}$ is defined as

$$\mathcal{H}(\psi, \lambda) := \sum_k \lambda_k \int |\nabla \psi_k|^2 + \frac{1}{2} \int n_{\psi, \lambda} V_{\psi, \lambda}$$
\[ = \sum_k \lambda_k \int |\nabla \psi_k|^2 + \frac{1}{2} \int |\nabla V \psi, \lambda|^2; \]

integrals always extend over the set \( \Omega \). The energy is conserved along solutions of the Schrödinger-Poisson system, indeed, the system (1.1)–(1.4) can be written in the form

\[
\begin{align*}
 i \frac{\partial \psi_k}{\partial t} &= -\frac{1}{2\lambda_k} \delta \bar{\psi}_k \mathcal{H}, \\
 i \frac{\partial \bar{\psi}_k}{\partial t} &= -\frac{1}{2\lambda_k} \delta \psi_k \mathcal{H}, \\
 \frac{d\lambda_k}{dt} &= 0,
\end{align*}
\]

where the bar denotes complex conjugation.

To assess the stability of a given steady state we employ an energy-Casimir functional

\[ \mathcal{H}_C(\psi, \lambda) := \sum_k C(\lambda_k) + \mathcal{H}(\psi, \lambda) \]

with the real-valued function \( C \) defined appropriately. Clearly, \( \mathcal{H}_C \) is a conserved quantity for the Schrödinger-Poisson system.

The class of functions which generate the Casimir functionals will now be specified: We say that a function \( f : \mathbb{R} \to \mathbb{R} \) is of Casimir class \( \mathcal{C} \) iff it has the following properties:

(i) \( f \) is continuous with \( f(s) > 0, s \leq s_0 \) and \( f(s) = 0, s \geq s_0 \) for some \( s_0 \in [0, \infty] \),

(ii) \( f \) is strictly decreasing on \( (-\infty, s_0] \) with \( \lim_{s \to -\infty} f(s) = \infty \),

(iii) there exist constants \( \epsilon > 0 \) and \( C > 0 \) such that

\[ f(s) \leq C(1 + s)^{-7/2 - \epsilon}, s \geq 0. \]

For \( f \in \mathcal{C} \),

\[ F(s) := \int_s^\infty f(\sigma) \, d\sigma, \quad s \in \mathbb{R}, \quad (2.1) \]

defines a decreasing, continuously differentiable, and non-negative function which is strictly convex on its support, and

\[ F(s) \leq C(1 + s)^{-5/2 - \epsilon}, s \geq 0. \]
In passing we note that by adjusting various exponents our results easily extend to general space dimensions.

**Remark 1**

(a) A typical example for \( f \in C \) is the Boltzmann distribution \( f(s) = e^{-\beta s} \) with \( \beta > 0 \), where the cut-off level \( s_0 = \infty \). Another example, which also decays exponentially for \( s \to \infty \), is given by the Fermi-Dirac statistics:

\[
f(s) := C \int_{\mathbb{R}^3} \frac{dv}{\epsilon + e^{\frac{|v|^2}{2} + s}}, \quad s \in \mathbb{R},
\]

where \( C > 0 \) and \( \epsilon > 0 \) are positive parameters.

A function \( f \) with \( f(s) = 0 \) for \( s > s_0 \) with \( s_0 \in \mathbb{R} \) will yield a steady state consisting of a finite number of quantum oscillators.

(b) We could generalize the assumption (iii) to requiring that both \( f(-\Delta + V) \) and \( F(-\Delta + V) \) are of trace class for (smooth) potentials \( V \geq 0 \), cf. Lemma 1 (b) below. However, we prefer to keep our assumptions on \( f \) explicit.

**Lemma 1** Let \( f \in C \).

(a) For every \( \beta > 1 \) there exists \( C = C(\beta) \in \mathbb{R} \) such that

\[
F(s) \geq -\beta s + C, \quad s \leq 0.
\]

(b) Let \( V \in H^1_0(\Omega) \) be non-negative on \( \Omega \). Then both \( f(-\Delta + V) \) and \( F(-\Delta + V) \) are trace class.

**Proof.** Part (a) is straightforward from assumption (ii) and the definition of \( F \). As to (b), let \( (\mu_k) \) denote the sequence of eigenvalues of \(-\Delta + V\). Then, since \( V \) is non-negative and \( F \) decreasing,

\[
\sum_k F(\mu_k) \leq \sum_k F(\mu_k^0)
\]

where \( \mu_k^0 \) denote the eigenvalues of \(-\Delta\). For the latter we have the well-known estimate that the number of such eigenvalues less than some \( \mu \in \mathbb{R} \) grows like \( \mu^{3/2} \) for \( \mu \to \infty \), which implies that the right hand sum is finite, and \( F(-\Delta + V) \) is trace class. Since \( f \) decays faster than \( F \) the same holds true for \( f(-\Delta + V) \). \( \square \)

At several points the following technical observation will be useful:
Lemma 2 For \( \psi \in H^1_0(\Omega) \cap H^2(\Omega) \) with \( \| \psi \|_2 = 1 \) and \( V \in H^1_0(\Omega) \), \( V \geq 0 \), we have
\[
F(\langle \psi, (-\Delta + V)\psi \rangle) \leq \langle \psi, F(-\Delta + V)\psi \rangle
\]
with equality if \( \psi \) is an eigenstate of \( -\Delta + V \).

Proof. Denoting the spectral measure associated with \( -\Delta + V \) and \( \psi \) by \( d\sigma \) the claim translates into the inequality
\[
F\left( \int \sigma d\sigma \right) \leq \int F(\sigma) d\sigma
\]
which holds due to the convexity of \( F \) and Jensen’s inequality. \( \square \)

To conclude this section we make precise the notion of a steady state of the Schrödinger-Poisson system: A quadruple \( (\psi_0, \lambda_0, \mu_0, V_0) \) with \( (\psi_0, \lambda_0) \in S \), \( \mu_0 = (\mu_{0,k}) \in \mathbb{R}^\mathbb{N} \), and \( V_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) is a steady state of the Schrödinger-Poisson system \( (1.1) - (1.4) \) iff
\[
(-\Delta + V_0)\psi_{0,k} = \mu_{0,k}\psi_{0,k}, \ k \in \mathbb{N}, \tag{2.2}
\]
and
\[
\Delta V_0 = -n_0 = -\sum_k \lambda_{0,k} |\psi_{0,k}|^2, \tag{2.3}
\]
where the energy levels \( \mu_{0,k} \) and occupation probabilities \( \lambda_{0,k} \) are related through an equation of state of the form
\[
\lambda_{0,k} = f(\mu_{0,k}), \ k \in \mathbb{N}, \tag{2.4}
\]
with some \( f \in C \).

Remark 2 If \( (\psi_0, \lambda_0, \mu_0, V_0) \) satisfies the equations \( (2.2), (2.3), (2.4) \) with \( f \in C \) then the estimate
\[
\sum_k \lambda_{0,k} \|\psi_{0,k}\|_{H^2}^2 < \infty.
\]
follows and thus in particular \( (\psi, \lambda) \in S \). To see this we use \( (2.2) \) and estimate
\[
\sum_k \lambda_{0,k} \| \nabla \psi_{0,k} \|_2^2 + \int |\nabla V_0|^2 = \sum_k \mu_{0,k} f(\mu_{0,k}) \leq C \sum_k (1 + \mu_{0,k})^{-5/2 + \epsilon} < \infty
\]
by assumption (iii) on \( f \) and the asymptotic behaviour of \( \mu_{0,k} \). Thus, by the Sobolev inequality,
\[
\| n_0 \|_3 \leq \sum_k \lambda_{0,k} \|\psi_{0,k}\|_6^2 < \infty,
\]

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and \( V_0 \in W^{2,3}(\Omega) \subset L^\infty(\Omega) \) follows. Again from (2.2) we conclude that
\[
\sum_k \lambda_{0,k} \| \Delta \psi_{0,k} \|_2^2 \leq C \left( \sum_k \lambda_{0,k} \mu_{0,k}^2 + \sum_k \lambda_{0,k} \right)
\leq C \left( 1 + \sum_k (1 + \mu_{0,k})^{-(3/2+\epsilon)} \right) < \infty.
\]

Given \( f \in C \) we still need to specify the corresponding Casimir functional: With \( F \) given by (2.1), its Legendre or Fenchel transform is defined by
\[
F^*(s) := \sup_{\lambda \in \mathbb{R}} (\lambda s - F(\lambda)), \ s \in \mathbb{R}, \tag{2.5}
\]
and the energy-Casimir functional corresponding to \( f \) is
\[
\mathcal{H}_C(\psi, \lambda) := \sum_k F^*(-\lambda_k) + \mathcal{H}(\psi, \lambda), \ (\psi, \lambda) \in S. \tag{2.6}
\]
Note that since \( F' = -f \) has an inverse on \([-\infty, 0]\),
\[
F^*(s) = \int_{-s}^0 f^{-1}(\sigma) \, d\sigma \tag{2.7}
\]
for \(-\infty = -f(-\infty) < s \leq 0\), and all \(-\lambda_k\) lie in this interval.

Obviously, only the values of \( f \in C \) on the interval \( 0, \infty \) are significant for the following theory. However, for technical reasons we consider the functions \( f \) defining the equations of state as defined on all of \( \mathbb{R} \).

3 Stability

In the present section we shall establish the following result on nonlinear stability:

**Theorem 1** Let \((\psi_0, \lambda_0, \mu_0, V_0)\) be a steady state of the Schrödinger-Poisson system with
\[
\lambda_{0,k} = f(\mu_{0,k}), \ k \in \mathbb{N},
\]
for some \( f \in C \), and \((\psi_0, \lambda_0) \in S\). Then this steady state is nonlinearly stable in the following sense: If \( t \mapsto (\psi(t), \lambda) \) is a solution of the Schrödinger-Poisson system with initial datum \((\psi(0), \lambda) \in S\) then
\[
\frac{1}{2} \left\| \nabla V_{\psi(t), \lambda} - \nabla V_0 \right\|_2^2 \leq \mathcal{H}_C(\psi(0), \lambda) - \mathcal{H}_C(\psi_0, \lambda_0), \ t \geq 0.
\]
We recall that $H_C$ is defined by (2.6) for the given function $f$ and note that, clearly, the right hand side in the estimate above becomes arbitrarily small if $(\psi(0), \lambda)$ is close to $(\psi_0, \lambda_0)$ in the appropriate topology. The main step in the proof of Theorem 1 is to show the following estimate:

**Lemma 3** Let $V \in H^1_0(\Omega)$, $V \geq 0$. Then

$$\sum_k \left[ F^*(\lambda_k) + \lambda_k \int \left[ |\nabla \psi_k|^2 + V |\psi_k|^2 \right] \right] \geq -\text{Tr}[F(-\Delta + V)], (\psi, \lambda) \in \mathcal{S},$$

with equality for $(\psi, \lambda) = (\psi_V, \lambda_V)$, where $\psi_V = (\psi_{V,k}) \in H^1_0(\Omega)^N$ is an orthonormal sequence of eigenfunctions of $-\Delta + V$ with eigenvalues $\mu_V = (\mu_{V,k})$, and $\lambda_V = (\lambda_{V,k}) = (f(\mu_{V,k}))$.

**Proof.** The fact that $F$ and $F^*$ are related by conjugacy implies that

$$F^*(-\lambda) + \lambda \mu \geq \inf_{s \in \mathbb{R}} [F^*(-s) + s \mu]$$

$$= -\sup_{s \in \mathbb{R}} [-F^*(s) + s \mu] = -F^{**}(\mu)$$

$$= -F(\mu), \lambda, \mu \in \mathbb{R}. \quad (3.1)$$

We substitute $\lambda_k$ for $\lambda$ and

$$\mu_k := \int \left[ |\nabla \psi_k|^2 + V |\psi_k|^2 \right] = \langle \psi_k, (-\Delta + V) \psi_k \rangle$$

for $\mu$ and sum over $k$ to find

$$\sum_k \left[ F^*(-\lambda_k) + \lambda_k \int \left[ |\nabla \psi_k|^2 + V |\psi_k|^2 \right] \right] \geq -\sum_k F(\langle \psi_k, (-\Delta + V) \psi_k \rangle)$$

$$\geq -\sum_k \langle \psi_k, F(-\Delta + V) \psi_k \rangle$$

$$= -\text{Tr}[F(-\Delta + V)]$$

by Lemma 2 and the definition of trace.

Now suppose that $(\psi, \lambda) = (\psi_V, \lambda_V)$. Since by definition each $\psi_{V,k}$ is an eigenfunction of $-\Delta + V$ the $\mu_k$ defined above are the corresponding eigenvalues $\mu_{V,k}$, and

$$\text{Tr}[F(-\Delta + V)] = \sum_k F(\mu_{V,k}).$$

On the other hand we have $\lambda_{V,k} = f(\mu_{V,k}) = -F'(\mu_{V,k})$ which by conjugacy is equivalent to $\mu_{V,k} = F^{**}(-\lambda_{V,k}), k \in \mathbb{N}$. This implies that

$$\sum_k F(\mu_{V,k}) = -\sum_k [F^*(-\lambda_{V,k}) + \lambda_{V,k} \mu_{V,k}],$$

and the proof is complete. □
Remark 3 In Lemma 3 equality holds if and only if \((\psi, \lambda) = (\psi_V, \lambda_V)\). This follows from the strict convexity of \(F\), but we make no use of this observation in the rest of the paper.

Proof of Theorem 1. Let \(V = V_{\psi, \lambda}\) be the potential induced by \((\psi, \lambda) \in \mathcal{S}\). Then

\[
\frac{1}{2} \|\nabla V - \nabla V_0\|^2 = \frac{1}{2} \int |\nabla V|^2 + \int \Delta V V_0 + \frac{1}{2} \int |\nabla V_0|^2
\]

\[
= H_C(\psi, \lambda) - \left[ \sum_k \left( F^*(-\lambda_k) + \lambda_k \int |\nabla \psi_k|^2 \right) - \frac{1}{2} \int |\nabla V_0|^2 - \int \Delta V V_0 \right]
\]

\[
= H_C(\psi, \lambda) - \left[ \sum_k \left( F^*(-\lambda_k) + \lambda_k \int \left[ |\nabla \psi_k|^2 + V_0 |\psi_k|^2 \right] \right) - \frac{1}{2} \int |\nabla V_0|^2 \right]
\]

\[
\leq H_C(\psi, \lambda) - \left[ -\text{Tr}[F(-\Delta + V_0)] - \frac{1}{2} \int |\nabla V_0|^2 \right]
\]

\[
= H_C(\psi, \lambda) - \left[ \sum_k \left( F^*(-\lambda_{0,k}) + \lambda_{0,k} \int \left[ |\nabla \psi_{0,k}|^2 + V_0 |\psi_{0,k}|^2 \right] \right) - \frac{1}{2} \int |\nabla V_0|^2 \right]
\]

where we have used Lemma 3 twice. Given a solution with \((\psi(0), \lambda) \in \mathcal{S}\) we may substitute \((\psi(t), \lambda) \in \mathcal{S}\) into this estimate, and since \(H_C\) is constant along solutions the assertion follows. \(\square\)

4 Dual functionals

Our aim for the rest of this paper is to prove the existence of steady states which satisfy the assumption of our stability result. For each \(f \in \mathcal{C}\) a corresponding steady state will be obtained as the unique maximizer of an appropriately defined functional. In the present section we derive this dual functional from the energy-Casimir functional used in the stability analysis. The relation between these functionals is of interest in itself, but it is not used in the proofs of our results. Throughout this section we fix an element \(f \in \mathcal{C}\). We move to the dual functional in two steps. First we apply the saddle point principle and define, for \(\Lambda > 0\) fixed,

\[
G(\psi, \lambda, V, \sigma) := \sum_k \left[ F^*(-\lambda_k) + \lambda_k \int \left[ |\nabla \psi_k|^2 + V |\psi_k|^2 \right] \right] - \frac{1}{2} \int |\nabla V|^2
\]
\[ + \sigma \left[ \sum_k \lambda_k - \Lambda \right] \]

where \( \psi = (\psi_k) \) is again an orthonormal system in \( L^2(\Omega) \), \( \lambda \in l^1_+ = \{(\sigma_k) \in l^1 | \sigma_k \geq 0, k \in \mathbb{N}\} \), and \( V \in H^1_0(\Omega) \) may now vary independently of \( \psi \) and \( \lambda \). The role of the parameter \( \sigma \in \mathbb{R} \) (Lagrange multiplier) will become clear shortly; the relation between \( \mathcal{H}_C \) and this new functional is as follows:

**Remark 4** For any \( \psi, \lambda, \sigma \),

\[
\sup_V G(\psi, \lambda, V, \sigma) = \mathcal{H}_C(\psi, \lambda) + \sigma \left[ \sum_k \lambda_k - \Lambda \right],
\]

and the supremum is attained at \( V = V_{\psi, \lambda} \). In fact, integration by parts and some computations show that

\[
G(\psi, \lambda, V, \sigma) = \mathcal{H}_C(\psi, \lambda) + \sigma \left[ \sum_k \lambda_k - \Lambda \right] - \frac{1}{2} \left\| \nabla V_{\psi, \lambda} - \nabla V \right\|_2^2.
\]

As second step on our way to a dual variational formulation we reduce the functional \( G \) to a functional of \( V \) and \( \sigma \) as follows:

\[
\Phi(V, \sigma) := \inf_{\psi, \lambda} G(\psi, \lambda, V, \sigma)
\]

where the infimum is taken over all \( \lambda \in l^1_+ \) and all orthonormal sequences \( \psi \) in \( L^2(\Omega) \). It is this functional which will have a unique maximizer in the next section, which is then a steady state. First however, we need to bring it into a different form:

**Remark 5** The infimum in the definition of \( \Phi \) is attained at \( \psi = (\psi_{V,k}) \), an orthonormal sequence of eigenstates of \( -\Delta + V \) with corresponding eigenvalues \( \mu_{V,k} \), and \( \lambda = \lambda_V \) where \( \lambda_{V,k} = f(\mu_{V,k} + \sigma) \), \( k \in \mathbb{N} \). Moreover,

\[
\Phi(V, \sigma) = -\frac{1}{2} \int |\nabla V|^2 - \text{Tr} [F(-\Delta + V + \sigma)] - \sigma \Lambda.
\]

To see this, recall Lemma 3 and Remark 4 and observe that \( f(\cdot + \sigma) \in \mathcal{C} \) for any \( \sigma \in \mathbb{R} \), provided \( f \in \mathcal{C} \).
5 Existence of steady states

In the present section we shall for each state relation \( f \in \mathcal{C} \) and each total charge \( \Lambda > 0 \) construct a unique maximizer of the functional \( \Phi \), which is then a steady state of the Schrödinger-Poisson system. We consider only non-negative potentials and use the notation

\[
H_{0,+}^1(\Omega) := \{ V \in H_0^1(\Omega) | V \geq 0 \}.
\]

**Theorem 2** Let \( f \in \mathcal{C} \) and \( \Lambda > 0 \) be given. The functional

\[
\Phi : H_{0,+}^1(\Omega) \times \mathbb{R} \ni (V, \sigma) \mapsto -\frac{1}{2} \int |\nabla V|^2 - \text{Tr} [F(-\Delta + V + \sigma)] - \sigma \Lambda
\]

is continuous, strictly concave, bounded from above, and coercive. In particular, there exists a unique maximizer \((V_0, \sigma_0)\) of \( \Phi \). If we define \( \psi_0 = (\psi_{0,k}) \) as the orthonormal sequence of eigenstates of the operator \(-\Delta + V_0\) with corresponding eigenvalues \( \mu_{0,k} \) and \( \lambda_{0,k} := f(\mu_{0,k} + \sigma_0) \), then \((\psi_0, \lambda_0, \mu_0, V_0)\) is a steady state of the Schrödinger-Poisson system with \( \sum_k \lambda_{0,k} = \Lambda \) and \((\psi_0, \lambda_0) \in S\).

Note that \( \sigma_0 \) plays the role of a (constant) Fermi level here.

**Proof.** \( \Phi \) is strictly concave: The first term of \( \Phi \) is evidently concave. To show the strict concavity of the second term i.e., the strict convexity of \( \text{Tr} [F(-\Delta + V + \sigma)] \), let \((V_j, \sigma_j) \in H_{0,+}^1 \times \mathbb{R}, j=1,2, \alpha \in ]0,1[\), and \( \phi \in H^2 \cap H_0^1 \). By convexity of \( F \) and Lemma 3,

\[
F(\langle \phi, \alpha(-\Delta + V_1 + \sigma_1)\phi + (1-\alpha)(-\Delta + V_2 + \sigma_2)\phi \rangle) \\
\leq \alpha \langle \phi, F(-\Delta + V_1 + \sigma_1)\phi \rangle + (1-\alpha) \langle \phi, F(-\Delta + V_2 + \sigma_2)\phi \rangle.
\]

Now we substitute \( \psi_k \) for \( \phi \), \( (\psi_k) \) an orthonormal sequence of eigenstates of \( \alpha(-\Delta + V_1 + \sigma_1) + (1-\alpha)(-\Delta + V_2 + \sigma_2) \), and sum over \( k \) to obtain the convexity estimate for \( \text{Tr} [F(-\Delta + V + \sigma)] \). If we have equality in this estimate then

\[
\langle \psi_k, F(-\Delta + V_1 + \sigma_1)\psi_k \rangle = \langle \psi_k, F(-\Delta + V_2 + \sigma_2)\psi_k \rangle, \ k \in \mathbb{N}
\]

and thus \( V_1 = V_2 \) and \( \sigma_1 = \sigma_2 \).
Φ is bounded from above and coercive: Since F is non-negative, the critical case in the coercivity estimate is \( \sigma < 0 \). Let \( \mu_V \) denote the ground state energy of \(-\Delta + V\) with corresponding ground state \( \psi_V \). Since \( F \) is non-negative and satisfies estimate (a) in Lemma 1 we have for \( \sigma \leq -\mu_V \),

\[
\Phi(V, \sigma) \leq -\frac{1}{2} \int |\nabla V|^2 - \langle \psi_V, F(-\Delta + V + \sigma)\psi_V \rangle - \sigma \Lambda
\]

\[
= -\frac{1}{2} \int |\nabla V|^2 - F(-\mu_V + \sigma) - \sigma \Lambda
\]

\[
\leq -\frac{1}{2} \int |\nabla V|^2 + (\beta - \Lambda) \sigma + \beta \mu_V - C,
\]

where we choose \( \beta > \Lambda \). Also

\[
\mu_V = \inf_{\phi \in H_0^1, \|\phi\|_2 = 1} \int \left[ -|\nabla \phi|^2 + V|\phi|^2 \right] \leq \frac{1}{\text{vol} \Omega} \int V \leq C_1\|V\|_{H_0^1},
\]

choosing \( \phi := 1/\sqrt{\text{vol} \Omega} \). Together with the estimate above and Poincaré’s inequality this implies that for \( \sigma \leq -C_1\|V\|_{H_0^1} \) we have

\[
\Phi(V, \sigma) \leq -C_2\|V\|_{H_0^1}^2 + C_3\|V\|_{H_0^1} + (\beta - \Lambda) \sigma + C_4
\]

(5.1)

where the constants \( C_1, C_2, C_3, C_4 \) are positive and \( \beta > \Lambda \), cf. Lemma 1 (a). On the other hand, by the non-negativity of \( F \) and Poincaré’s inequality,

\[
\Phi(V, \sigma) \leq -C_2\|V\|_{H_0^1}^2 - \sigma \Lambda,
\]

(5.2)

and (5.1) and (5.2) together imply that \( \Phi \) is bounded from above and coercive.

Existence of a unique maximizer: The existence of a unique maximizer of \( \Phi \) is standard, cf. for example [8, Ch. II, Prop. 1.2], provided \( \Phi \) is upper semi-continuous. This in turn follows from the fact that \( \Phi \) is concave and bounded from below, at least locally, cf. [8, Ch. I, Lemma 2.1]: The only term for which this may not be immediately obvious is the trace term, but

\[
\text{Tr}[F(-\Delta + V + \sigma)] \leq \sum_k F(\mu_k + \sigma_0) < \infty
\]

where \( \mu_k \) are the eigenvalues of \(-\Delta + V\) and \( \sigma \geq \sigma_0 \) for arbitrary \( \sigma_0 \in \mathbb{R} \).
(ψ₀, λ₀, μ₀, V₀) is a steady state: Since \(F' = - f\), the stationarity of \(\Phi(V₀, σ)\) with respect to \(σ\) implies

\[
0 = \frac{d\Phi(V₀, σ)}{dσ} \bigg|_{σ₀} = \operatorname{Tr} [f (-Δ + V₀ + σ₀)] - Λ
\]

\[
= \sum_k f(μ₀, k + σ₀) - Λ = \sum_k λ₀, k - Λ
\]

so that \(\sum_k λ₀, k = Λ\) as claimed. In order that \((ψ₀, λ₀, μ₀, V₀)\) is a steady state we need to show that

\[
ΔV₀ + \sum_k λ₀, k |ψ₀, k|^2 = 0. \tag{5.3}
\]

To verify this we observe that \(V₀\), being a maximizer of \(\Phi(⋅, σ₀)\), satisfies the Euler-Lagrange equation

\[
ΔV₀(x) + K_f(Δ + V₀ + σ₀)(x, x) = 0, \quad x ∈ Ω, \tag{5.4}
\]

where \(K_L\) is the kernel associated with a trace-class operator \(L\). In our case

\[
K_f(Δ + V₀ + σ₀)(x, x) = \sum_k f(μ₀, k + σ₀)|ψ₀, k|^2(x) \tag{5.5}
\]

and \((5.3)\) follows from \((5.4), (5.5)\), and the fact that by definition, \(λ₀, k = f(μ₀, k + σ₀)\). As to the proof for \((ψ₀, λ₀) ∈ S\) we refer to Remark 2.

In view of the relations between our various functionals derived in the previous section it is of interest to note:

**Remark 6** If \((V₀, σ₀)\) is the maximizer obtained in Theorem 2 and \((ψ₀, λ₀, μ₀, V₀)\) is the corresponding steady state, then

\[
Φ(V₀, σ₀) = H_C(ψ₀, λ₀).
\]

To see this, note that by \((4.2)\) we have

\[
Φ(V₀, σ₀) = G(ψ₀, λ₀, V₀, σ₀) ≤ H_C(ψ₀, λ₀),
\]

where equality holds iff \(V₀\) is the maximizer of \(G(ψ₀, λ₀, V, σ₀)\) on \(H₁\); note that here \(G\) is independent of \(σ\) since \(\sum_k λ₀, k = 1\). This, on the other hand, is equivalent to the fact that \(V₀\) is the solution of the Poisson equation \((5.3)\).
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