Computing the Pseudospectral Abscissa of Time-Delay Systems

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Abstract: The pseudospectra of a linear time-invariant system are the sets in the complex plane consisting of all the roots of the characteristic equation when the system matrices are subjected to all possible perturbations with a given upper bound. The pseudospectral abscissa are defined as the maximum real part of the characteristic roots in the pseudospectra and, therefore, they are for instance important from a robust stability point of view. In this paper we present a numerical method for the computation of the pseudospectral abscissa of retarded delay differential equations with discrete pointwise delays. Our approach is based on the connections between the pseudospectra and the level sets of an appropriately defined complex function. These connections lead us to a bisection algorithm for the computation of the pseudospectral abscissa, where each step relies on checking the presence of imaginary axis eigenvalues of an appropriately defined operator. Because this operator is infinite-dimensional a predictor-corrector approach is taken. In the predictor step the bisection algorithm is applied where the operator is discretized into a matrix, yielding approximations for the pseudospectral abscissa. The effect of the discretization is fully characterized in the paper. In the corrector step, the approximate pseudospectral abscissa are corrected to any given accuracy, by solving a set of nonlinear equations that characterize extreme points in the pseudospectra contours.

Keywords: pseudospectra, pseudospectral abscissa, computational methods, time-delay, delay equations, robustness, stability.

1. INTRODUCTION

The pseudospectra provide information about the characteristic roots of the system when the system matrices in the characteristic equation are subject to perturbations. They are closely related with the robust stability of the system and distance to instability, Trefethen (1997). We consider the characteristic equation of the time-delay systems:

$$\det F(\lambda) = 0$$  \(1\)

where

$$F(\lambda) := \lambda I_{n \times n} - \left( \sum_{i=0}^{m} A_i e^{-\lambda \tau_i} \right),$$ \(2\)

where

$$A_i \in \mathbb{C}^{n \times n}, \tau_i \in \mathbb{R}_0^+, \text{ for } i = 0, \ldots, m \text{ and } \tau_0 = 0.$$

The maximum real part of the characteristic roots is the spectral abscissa,

$$\alpha_0 = \sup_{\lambda \in \mathbb{C}} \{\Re(\lambda) : \det F(\lambda) = 0\}.$$ \(3\)

When the system matrices in (2) are subject to the perturbations, the pseudospectra of the characteristic equation (1) is defined as

$$A_{\varepsilon} = \{\lambda \in \mathbb{C} : \det (F(\lambda) + \Delta F(\lambda)) = 0\}$$ \(4\)

where perturbations on the systems matrices are represented as

$$\Delta F(\lambda) := -\left( \sum_{i=0}^{m} \delta A_i e^{-\lambda \tau_i} \right),$$ \(5\)

$$\delta A_i \in \mathbb{C}^{n \times n} \text{ and satisfying } \|\delta A_i\|_2 \leq \frac{\epsilon}{w_i} \text{ for } i = 0, \ldots, m.$$

Here the $$w_i \in \mathbb{R}_0^+, i = 0, \ldots, m$$ are some weights on the perturbations which can be chosen apriori. The maximum real part in the pseudospectra is the pseudospectral abscissa which is defined as

$$\alpha_\varepsilon = \sup_{\lambda \in \mathbb{C}} \{\Re(\lambda) : \lambda \in A_{\varepsilon}(F)\},$$ \(6\)

The computation of the pseudospectral abscissa for finite dimensional systems corresponds to the special case of (1):

$$F_0(\lambda) = \lambda I_n - A_0.$$

In this particular case, the pseudospectra can be equivalently expressed as Boyd et al. (1989)

$$A_\varepsilon^0 = \left\{ \lambda \in \mathbb{C} : \sigma_{\max} \left( F_0^{-1}(\lambda) \right) > \frac{1}{\varepsilon} \right\}$$ \(7\)

where $$\sigma_{\max}(A)$$ is the largest singular value of the matrix A. Note that this definition reduces the pseudospectra boundary to the level set of the resolvent norm. This connection is used to compute the distance to instability and pseudospectral abscissa via a bisection algorithm in Byers (1988) and Burke et al. (2003a) respectively. A quadratically convergent algorithm for the pseudospectral abscissa computation is given in Burke et al. (2003b) based on a ‘criss-cross’ procedure. In Michiels et al. (2006), these results are extended to matrix functions of the form (2) where the perturbations take the form of (5). In particular, it is shown in Theorem 1 that the pseudospectral abscissa as defined in (4) can be expressed in the following way:
where
\[
f(\lambda) = w(\lambda)\sigma_{\max}(F^{-1}(\lambda)), \quad w(\lambda) = \sum_{i=0}^{m} \frac{|e^{-\lambda\tau_i}|}{w_i}.
\]

Using the pseudospectra definition in (8), the pseudospectral abscissa in (6) can be rewritten as
\[
\alpha_\epsilon = \sup_{\lambda \in \mathbb{C}} \left\{ \Re(\lambda) : f(\lambda) = \frac{1}{\epsilon} \right\}.
\] (10)

The supremum in the definition (10) is well-defined since \(F^{-1}(\lambda)\) is a strictly proper function and \(w(\lambda)\) is uniformly bounded on any right half complex plane in (9).

In Section 2, a bisection algorithm is given for pseudospectral abscissa computation of time-delay systems based on the connection between the pseudospectra and the level sets of the function \(f(\lambda)\).

This algorithm is implemented in two steps: first the approximate pseudospectral abscissa is computed by the prediction step in Section 3 and then the approximate results are corrected in Section 4.

The overall algorithm for the pseudospectral abscissa computation is outlined in Section 5. A numerical example and concluding remarks can be found in Sections 6 and 7.

**Notation:**
The notation in the paper is standard and given below.

- \(\mathbb{C}, \mathbb{R}\): the field of the complex and real numbers,
- \(\mathbb{R}_0^+\): the positive real numbers including zero,
- \(\Im(u)\): imaginary part of the complex number \(u\),
- \(|u|\): magnitude of the complex number \(u\),
- \(\bar{u}\): conjugate of the complex number \(u\),
- \(A^*\): complex conjugate transpose of the matrix \(A\),
- \(I_{n \times n}\): identity matrix with dimension \(n\),
- \(\sigma_{\max}(A)\): the largest singular value of the matrix \(A\),
- \(\|F(j\omega)\|_\infty\): \(L_\infty\) norm of the transfer function \(F\).

**2. THE BISECTION ALGORITHM FOR THE PSEUDOSPECTRAL ABSCissa COMPUTATION**

Given the function \(f\) in (9), define the function \(\alpha_f(\sigma)\) as:
\[
\alpha_f(\sigma) := \sup_{\omega \in \mathbb{R}} f(\sigma + j\omega)
\] (11)
over \(\sigma \in (\alpha_0, \infty)\).

**Proposition 1.** The function \(\alpha_f(\sigma)\) is strictly monotonically decreasing over the interval \(\sigma \in (\alpha_0, \infty)\).

**Proof.** The proof is by contradiction. Assume that the function \(\alpha_f(\sigma)\) is strictly increasing for some \(\sigma\). Then there exists a level set \(\frac{1}{\epsilon}\) such that there are at least two disjoint pseudospectra regions in the complex plane due to (8) and the strictly properness of \(F^{-1}(\lambda)\) (see the blue lines in Figure 1). Since \(\alpha_f(\sigma)\) is a continuous function, one of the disjoint sets in the pseudospectra disappears without merging to other pseudospectra sets for a higher level set \(\frac{1}{\epsilon}\) (red lines in the Figure). This is a contradiction with the continuity of the individual eigenvalues and the fact that \(F(\sigma)^{-1}\) is strictly proper (preventing eigenvalues to move off to infinity).

Similarly, it can be shown that the case \(\alpha_f(\sigma_1) = \alpha_f(\sigma_2)\) with \(\sigma_0 < \sigma_1 < \sigma_2\) is a contradiction with the continuity of the individual eigenvalues. \(\square\)

**Proposition 2.** The function \(\alpha_f(\sigma)\) satisfies
\[
\lim_{\sigma \to \alpha_0(\inf)} \alpha_f(\sigma) = \infty,
\]
\[
\lim_{\sigma \to -\infty} \alpha_f(\sigma) = 0.
\]

**Proof.** The first assertion in the Proposition follows from the fact that there are characteristic roots on the boundary \(\Re(s) = \alpha_0\). Therefore, the \(H_\infty\) norm of \(F^{-1}(\sigma + j\omega)\) diverges to infinity since the denominator becomes singular. The second assertion is the result of that \(F^{-1}\) is a strictly proper function. \(\square\)

Using the definition in (10), the pseudospectral abscissa is the \(\sigma\) value of the function where \(\alpha_f(\sigma) = \frac{1}{\epsilon}\). Since the function \(\alpha_f(\sigma)\) is strictly decreasing by Proposition 1 and it attains all the values from 0 to \(\infty\) by Proposition 2, the pseudospectral abscissa can be calculated by the following bisection algorithm.

**Algorithm 1.**

1) \(\sigma_L = \alpha_0, \sigma_R = \infty, \Delta \sigma = \text{tol},\)
2) while \((\sigma_R - \sigma_L) > 2 \times \text{tol}\)
   2.1) \(\Delta \sigma = 2 \times \Delta \sigma,\)
   2.2) if \((\sigma_R = \infty)\)
       then \(\sigma_M = \sigma_L + \Delta \sigma,\)
       else \(\sigma_M = \frac{\sigma_R + \sigma_L}{2},\)
   2.3) determine if \(\alpha_f(\sigma_M) > \frac{1}{\epsilon}\)
       then \(\sigma_L = \sigma_M,\)
       else \(\sigma_R = \sigma_M,\)
{result: the approximate pseudospectral abscissa \(\tilde{\sigma} = \sigma_L\)}

The main computation in the bisection algorithm is to check whether the inequality in step 2.3 is satisfied. By algebraic computation, the function \(\alpha_f(\sigma)\) is equivalent to
\[
\alpha_f(\sigma) = \sup_{\omega \in \mathbb{R}} f(\sigma + j\omega),
\]
\[
= \sup_{\omega \in \mathbb{R}} \left\{ w(\sigma + j\omega)\sigma_{\max}(F^{-1}(\sigma + j\omega)) \right\},
\]
\[
= w(\sigma) \sup_{\omega \in \mathbb{R}} \left\{ \sigma_{\max}(F^{-1}(\sigma + j\omega)) \right\},
\]
\[
= w(\sigma)\|F^{-1}_\sigma\|_\infty
\] (12)
where

$$F^{-1}_σ(j\omega) = \left( j\omega I_{n \times n} - \left( \sum_{i=0}^{m} A_{\sigma,i} e^{-j\omega\tau_i} \right) \right)^{-1}$$

(13)

and

$$A_{\sigma,0} = A_0 - \sigma I_{n \times n},$$
$$A_{\sigma,i} = A_i e^{-\tau_i}, \text{ for } i = 1, \ldots, m.$$ (14)

The inequality \( \alpha_f(\sigma) > \frac{1}{\epsilon} \) in step 2.3 is satisfied if and only if \( F^{-1}_σ(j\omega) \) has a singular value equal to \( \frac{1}{\epsilon \alpha_f(\sigma)} \) for some value of \( \omega \). This condition can be reduced into the verification of the imaginary axis eigenvalues of an infinite dimensional operator as shown in the following Theorem.

**Theorem 3.** The inequality

$$\alpha_f(\sigma) > \frac{1}{\epsilon}$$

(15)

is satisfied if and only if the linear infinite dimensional operator \( L_σ \) has eigenvalues on the imaginary axis where \( L_σ \) is defined on \( X := C([-\tau_{\text{max}}, \tau_{\text{max}}], \mathbb{C}^{2n}) \) by

$$\mathcal{D}(L_σ) = \{ \phi \in X : \phi' \in X, \phi(0) = M_0 \phi(0) + \sum_{i=1}^{m} (M_i \phi(-\tau_i) + M_{-i} \phi(\tau_i)) \},$$

$$L_σ \phi = \phi', \phi \in \mathcal{D}(L_σ)$$

(17)

with

$$M_0 = \begin{bmatrix} A_{\sigma,0} - (\epsilon w(\sigma))^2 I_{n \times n} \\ -I_{n \times n} \end{bmatrix},$$
$$M_i = \begin{bmatrix} A_{\sigma,i} \\ 0 \\ 0 \\ -A_{\sigma,i} \end{bmatrix}, \quad M_{-i} = \begin{bmatrix} 0 \\ 0 \\ \epsilon w(\sigma) \end{bmatrix}, \quad 1 \leq i \leq N.$$

**Proof.** A similar proof is given in Gumussoy and Michiels (2009). This Theorem generalizes Proposition 28 of Genin et al. (2002).

Using Theorem 3, we can refine the conceptual algorithm for the pseudospectral abscissa computation,

**Algorithm 2.**

1. \( \sigma_L = \sigma_0, \sigma_R = \infty, \Delta \sigma = \text{tol}, \)
2. while \( (\sigma_R - \sigma_L) > 2 \times \text{tol} \)
   2.1) \( \Delta \sigma = 2 \times \Delta \sigma, \)
   2.2) if \( (\sigma_R = \infty) \)
      then \( \sigma_M = \sigma_L + \Delta \sigma, \)
      else \( \sigma_M = \sigma_L + \frac{\sigma_R - \sigma_L}{2}, \)
   2.3) if \( L_{\sigma_M} \) has imaginary axis eigenvalues
      then \( \sigma_L = \sigma_M, \)
      else \( \sigma_R = \sigma_M. \)
{result: the approximate pseudospectral abscissa \( \hat{\sigma} = \sigma_L \)}

Note that step 2.3 in the Algorithm 3 requires solving a linear infinite dimensional eigenvalue problem for \( L_σ \) which needs to be discretized in a practical implementation. We will do this using a spectral method (see, e.g. Trefethen (2000); Breda et al. (2005, 2006)) and calculate the approximate solution by solving the standard linear eigenvalue problem. This approach is described in the next section.

### 3. Predicting the Pseudospectral Abscissa

Given a positive integer \( N \), we consider a mesh \( \Omega_N \) of \( 2N + 1 \) distinct points in the interval \([-\tau_{\text{max}}, \tau_{\text{max}}]\):

$$\Omega_N = \{ \theta_{N,i}, \ i = -N, \ldots, N \},$$

(18)

where

$$-\tau_{\text{max}} = \theta_{N,-N} < \cdots < \theta_{N,-1} < \theta_{N,0} = 0 < \theta_{N,1} < \cdots < \theta_{N,N} \leq \tau_{\text{max}}.$$

This allows to replace the continuous space \( X \) with the space \( X_N \) of discrete functions defined over the mesh \( \Omega_N \), i.e. any function \( \phi \in X \) is discretized into a block vector \( x = [x_{-N}^T \cdots x_N^T]^T \in X_N \) with components

$$x_i = \phi(\theta_{N,i}) \in \mathbb{C}^{2n}, \ i = -N, \ldots, N.$$

Let \( \mathcal{P}_N x, x \in X_N \) be the unique \( \mathbb{C}^{2n} \) valued interpolating polynomial of degree \( \leq 2N \) satisfying

$$\mathcal{P}_N x(\theta_{N,i}) = x_i, \ i = -N, \ldots, N.$$

In this way, the operator \( L_σ \) over \( X \) can be approximated with the matrix \( L_σ^N : X_N \rightarrow X_N \), defined as

$$(L_σ^N x)_i = (\mathcal{P}_N x)'(\theta_{N,i}), \ i = -N, \ldots, -1,$$

$$(L_σ^N x)_0 = M_0 \mathcal{P}_N x(0) + \sum_{i=1}^{m} (M_i \mathcal{P}_N x(-\tau_i) + M_{-i} \mathcal{P}_N x(\tau_i))$$

(19)

Using the Lagrange representation of \( \mathcal{P}_N x \),

$$\mathcal{P}_N x = \sum_{k=-N}^{N} l_{N,k} x_k, \, ,$$

where the Lagrange polynomials \( l_{N,k} \) are real valued polynomials of degree \( 2N \) satisfying

$$l_{N,k}(\theta_{N,i}) = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases}$$

we obtain the explicit form

$$\mathcal{L}_σ^N = \begin{bmatrix} d_{-N, -N} & \ldots & d_{-N, -N} \\ \vdots & \ddots & \vdots \\ d_{d, -N} & \ldots & d_{d, -N} \end{bmatrix} \in \mathbb{R}^{(2N+1)(2n) \times (2N+1)2n},$$

(20)

where

$$d_{i,k} = l_{N,k}(\theta_{N,i})I, \ i, k \in \{ -N, \ldots, N \}, i \neq 0,$$
$$a_0 = M_0 x_0 + \sum_{i=1}^{m} (M_i l_{N,0}(-\tau_i) + M_{-i} l_{N,0}(\tau_i))$$
$$a_k = \sum_{i=1}^{m} (M_i l_{N,k}(-\tau_i) + M_{-i} l_{N,k}(\tau_i))$$

for \( k \in \{ -N, \ldots, N \}, \ k \neq 0. $$

Note that all the problem specific information and the parameter \( \sigma \) are concentrated in the middle row of \( \mathcal{L}_σ^N \), i.e. the elements \( (a_{-N}, \ldots, a_N) \), while all other elements of \( \mathcal{L}_σ^N \) can be computed beforehand.

Since step 2.3 of Algorithm 2 is based on checking the presence of eigenvalues of \( L_σ \) on the imaginary axis and
thus strongly rely on the symmetry of the eigenvalues with respect to the imaginary axis, it is important that this property is preserved in the discretization. The following Proposition gives the condition on the mesh such that this symmetry holds.

**Proposition 4.** If the mesh $\Omega_N$ satisfies
\[
\theta_{N,-i} = -\theta_{N,i}, \ i = 1, \ldots, N, \tag{20}
\]
then the following result hold: for all $\lambda \in \mathbb{C}$, we have
\[
\det(\lambda I - \mathcal{L}_N^N) = 0 \iff \det(-\lambda - \mathcal{L}_N^N) = 0. \tag{21}
\]

**Proof.** Consider the differentiation matrix with elements
\[
\Delta_{k,l} = l_{k,k-N-1}(\theta_{N,k-N-1}), \ k, l \in \{1, \ldots, 2N+1\},
\]
and let $U$ be such that $U^{-1}\Delta U = \Delta T$. Define the matrix $S \in \mathbb{R}^{2N+1 \times 2N+1}$ with terms equal to 1 on the main skew diagonal and 0 elsewhere, the symmetry property of the mesh (20) assures that
\[
(U^{-1}S) \otimes \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \mathcal{L}_N^N (US) \otimes \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} = -(\mathcal{L}_N^N)^*,
\]
that is, the matrices $\mathcal{L}_N^N$ and $-(\mathcal{L}_N^N)^T$ are similar. The proposition directly follows. \hfill \Box

Based on the discretization of $\mathcal{L}_\sigma$ into $\mathcal{L}_N^N$, we propose the following algorithm to approximate (predict) the pseudospectral abscissa. It corresponds to Algorithm 2 where step 2.3 is replaced with the matrix $\mathcal{L}_N^N$ for a fixed $N$.

**Algorithm 3.**

1) $\sigma_L = \sigma_0$, $\sigma_R = \infty$, $\Delta \sigma = \text{tol}$,
2) while $(\sigma_R - \sigma_L) > 2 \times \text{tol}$
   2.1) $\Delta \sigma = 2 \times \Delta \sigma$,
   2.2) if $(\sigma_R = \infty)$
      then $\sigma_M = \sigma_L + \Delta \sigma$,
      else $\sigma_M = \frac{\sigma_R + \sigma_L}{2}$,
   2.3) if $\mathcal{L}_{\sigma_M}^N$ has imaginary axis eigenvalues
      then $\sigma_L = \sigma_M$,
      else $\sigma_R = \sigma_M$.

{result: the approximate pseudospectral abscissa, $\hat{\sigma} = \sigma_L$}

In what follows we clarify the effect of using the discretized operator in the algorithm. The next Theorem establishes the link between the imaginary axis eigenvalues of $\mathcal{L}_N^N$ and the corresponding inequality check condition similar to the connection between $\mathcal{L}_\sigma$ and the inequality (15) in Theorem 3.

**Theorem 5.** Assume that the mesh $\Omega_N$ is symmetric around the zero as given in (20). Let $p_N$ be the polynomial of the degree 2N satisfying the conditions,
\[
p_N(0; \lambda) = 1, \tag{22}
\]
\[
p_N'(\theta; \lambda) = \lambda p_N(\theta; \lambda), \ i = -N, \ldots, -1, 1, \ldots, N. \tag{23}
\]

The matrix $\mathcal{L}_N^N$ has an imaginary axis eigenvalue $\sigma = j\omega$ if and only if the inequality
\[
\alpha_N^N(\sigma) > \frac{1}{\epsilon} \tag{23}
\]
holds where
\[
\alpha_N^N(\sigma) := \sup_{\omega \in \mathbb{R}} f_N(\sigma + j\omega) \tag{23}
\]
\[
f_N(\sigma + j\omega) = w(\sigma)(j\omega - A_{\sigma,0} - \sum_{i=1}^m A_{\sigma,i} p_N(-\tau_i; j\omega)).
\]

Therefore, the effect of using $\mathcal{L}_N^N$ instead of $\mathcal{L}_\sigma$ corresponds to computing the approximate pseudospectral abscissa
\[
\alpha_N^N = \sup_{\lambda \in \mathbb{C}} \{ \Re(\lambda) : f_N(\lambda) = \frac{1}{\epsilon} \}. \tag{23}
\]

The accuracy of the approximation depends on the discretization parameter $N$. Therefore, the accuracy can be chosen arbitrarily close to $\alpha_\epsilon$ by increasing $N$. Note that at each iteration of step 2.3 in Algorithm 3, an eigenvalue problem of size $(2N)(2N+1)$ needs to be solved which may be computationally very demanding for large $N$.

However, because the eigenvalues of $\mathcal{L}_N^N$ exhibit the spectral convergence to the corresponding eigenvalues of $\mathcal{L}_\sigma$ (following the lines of Breda et al. (2005)) and because the approximation error can be corrected in a very cheap way as we shall see in the next section, it is sufficient to have a small value of $N$ for most practical applications.

4. CORRECTING THE PSEUDOSPECTRAL ABSICSSA

The Bisect Algorithm 3 finds the complex points $\hat{\lambda}_i = \hat{\sigma} + j\hat{\omega}_i$ for $i = 1, \ldots, \tilde{n}$ achieving the approximate pseudospectral abscissa $\hat{\sigma}$ which is close to the $\alpha_\epsilon$ given tolerance and discretization points $N$. These approximate results are corrected by using the property that the eigenvalues of the pseudospectral abscissa appear as solutions of a finite dimensional nonlinear eigenvalue problem. The following theorem establishes the link between this nonlinear eigenvalue problem and the linear eigenvalue problem of $\mathcal{L}_\sigma$.

**Theorem 6.** $\lambda$ is an eigenvalue of linear operator $\mathcal{L}_\sigma$ if and only if
\[
det H_\sigma(\lambda) = 0, \tag{24}
\]
where
\[
H_\sigma(\lambda) := \lambda I - M_0 - \sum_{i=1}^m (M_i e^{-\lambda \tau_i} + M_{-i} e^{\lambda \tau_i}) \tag{25}
\]
and the matrices $M_0$, $M_i$, $M_{-i}$ are defined in Theorem 3.

The solutions of (24) can be found by solving $H_\sigma(\lambda) v = 0$, $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^{2n}$, $v \neq 0$, (26) which in general has an infinite number of solution.

The correction method is based on the property that if $\omega(\sigma) || F_{\sigma}^{-1}(j\omega)|| = \frac{1}{\epsilon}$, then the operator $\mathcal{L}_\sigma$, or equivalently, (26), has a multiple non-semisimple eigenvalue as shown in Figure 2:

If $\lambda_\sigma = \alpha_\epsilon + j\omega_\epsilon$ are such that
\[
w(\sigma) || F_{\sigma}^{-1}(j\omega)|| = \frac{1}{\epsilon} = w(\alpha_\epsilon) \sigma_{\text{max}}(F_{\alpha_\epsilon}^{-1}(j\omega_\epsilon)), \tag{27}
\]
then setting
\[
h_\sigma(\lambda) = \det H_\sigma(\lambda), \tag{28}
\]
the pair $\omega(\sigma), \alpha(\sigma)$ satisfies
\[
h_\sigma(\omega_\epsilon) = 0, \ h_\sigma(\epsilon \omega_\epsilon) = 0. \tag{28}
\]
These complex-valued equations seem over-determined but this is not the case due to the spectral properties of $-H_\sigma(\lambda)$. Using the symmetry of the eigenvalues of the nonlinear eigenvalue problem (26) with respect to imaginary axis, we can write the following:
For Corollary 7.

and Substituting

Taking into account the above results, we end up with (30).

From the symmetry property of the eigenvalues and the assertions follow. According to Hryniv and Lancaster (1999), we get

where \( n \) is a normalizing condition. Given the structure of \( H_n \) it can be verified that a corresponding left eigenvector is given by \([-v^* u^*]^\top\). According to Hryniv and Lancaster (1999), we get

A simple computation yields:

\[
[-v^* u^*] H_n^\top(j\omega) \begin{bmatrix} u \\ v \end{bmatrix} = 2 \Re \left\{ v^* \left( I + \sum_{i=1}^{p} A_{\sigma,i} \tau_i e^{-j\omega \tau_i} \right) u \right\},
\]

which is always real. This is a consequence of the property (30).

Taking into account the above results, we end up with \( 4n + 3 \) real equations

\[
\begin{aligned}
\{ H_n(j\omega, \sigma) \begin{bmatrix} u \\ v \end{bmatrix} = 0, \quad n(u, v) = 0 \\
\Re (j\omega) \begin{bmatrix} u \\ v \end{bmatrix} = 0, \quad n(u, v) = 0 \\
3 \left\{ v^* \left( I + \sum_{i=1}^{p} A_{\sigma,i} \tau_i e^{-j\omega \tau_i} \right) u \right\} = 0
\end{aligned}
\]

\tag{34}

in the \( 4n + 2 \) unknowns \( \Re(v), \Re(u), \Im(u), \Im(v), \omega \) and \( \sigma \). These equations are still overdetermined because the property (29) is not explicitly exploited in the formulation, unlike the property (30). However, it makes the equations (34) exactly solvable, and the \((\omega, \sigma)\) components have a one-to-one-correspondence with the solutions of (31).

For \( \omega \geq 0 \), we have

\[
\exists h_\sigma(j\omega) = 0
\]

and

\[
\Re h_\sigma(j\omega) = 0.
\]

\tag{30}

Proof. From the symmetry property of the eigenvalues with respect to the imaginary axis,

\[
h_\sigma(\lambda) = h_\sigma(-\lambda), \quad h_\sigma^\top(\lambda) = -h_\sigma^\top(-\lambda).
\]

Substituting \( \lambda = j\omega \) yields

\[
h_\sigma(j\omega) = h_\sigma(-j\omega) = (h_\sigma(j\omega))^*,
\]

\[
h_\sigma^\top(j\omega) = -h_\sigma^\top(-j\omega) = -(h_\sigma^\top(j\omega))^*,
\]

and the assertions follow. \( \Box \)

Using Corollary 7 we can simplify the conditions (28) to:

\[
\begin{aligned}
\Re h_\sigma(j\omega) = 0 \\
\Im h_\sigma(j\omega) = 0.
\end{aligned}
\]

\tag{31}

Hence, the pair \((\omega, \alpha)\) satisfying (27) can be directly computed from the two equations (31), e.g. using Newton’s method, provided that good starting values are available.

The drawback of working directly with (31) is that an explicit expression for the determinant of \( H_n \) is required. To avoid this, let \( u, v \in \mathbb{C}^n \) be such that

\[
H_n(j\omega) \begin{bmatrix} u \\ v \end{bmatrix} = 0, \quad n(u, v) = 0,
\]

\tag{32}

in conclusion, as a result of the bisection algorithm in the prediction step, the approximate pseudospectral abscissa \( \tilde{\sigma} \) and the corresponding critical frequencies \( \tilde{\omega}_i \), \( i = 1, \ldots, \tilde{n} \) are calculated. Note that these computations are based on the approximation of \( L_n^\tilde{n} \) into a matrix \( L_n^\tilde{n} \). Using these approximate results as estimates of \((\omega_i, \alpha_i)\) (27), we can compute the approximate eigenvectors \( u \) and \( v \). These approximate values improved in the correction step by solving (34). At the end of the correction step, the pseudospectral abscissa \( \sigma = \alpha \), and the achieved frequency \( \omega = \omega_\tilde{n} \) are obtained within predefined tolerance.

5. ALGORITHM

The overall algorithm for computing the pseudospectral abscissa consists of two steps: the prediction step and the correction step. The first step requires a repeated computation of the eigenvalues of a \((2N + 1)2n \times (2N + 1)2n\) matrix \( L_n^N \). The second step solves (34) with \( 4n + 3 \) equations and \( 4n + 2 \) unknowns using Gauss-Newton algorithm. Our method chooses \( N \) sufficiently large such that the results of the prediction step are good starting values for the correction step. Note that by increasing \( N \) and using only the prediction step, the approximate pseudospectral abscissa can be computed arbitrarily close to \( \alpha \). However, this approach has more numerical cost than the combined approach when \( N \) is large.

Algorithm 4.

Input: system data, tolerance tol for prediction step, discretization points \( N \)

Output: pseudospectral abscissa \( \alpha \)

Prediction Step:

1) Calculate the spectral abscissa \( \alpha_0 \) of \( F \) (2),
2) \( \sigma_L = \alpha_0, \sigma_R = \infty, \Delta \sigma = \text{tol}, \)
3) while \((\sigma_R - \sigma_L) > 2 \times \text{tol} \)
3.1) \( \Delta \sigma = 2 \times \Delta \sigma, \)
3.1) if \( (\sigma_R = \infty) \) then \( \sigma_M = \sigma_L + \Delta \sigma, \) else \( \sigma_M = \frac{\sigma_R + \sigma_L}{2}, \)
3.2) if \( L_n^N \) has imaginary axis eigenvectors then \( \sigma_L = \sigma_M, \)
else \( \sigma_R = \sigma_M. \)

\{ result: the approximate pseudospectral abscissa \( \tilde{\sigma} = \sigma_L \) and the corresponding frequencies \( \tilde{\omega}_i \), \( i = 1, \ldots, \tilde{n} \) of \( L_n^N \) \}

Correction Step:

1) calculate the approximate null vectors \{ \( x_1, \ldots, x_{\tilde{n}} \) \} of \( H_n(j\tilde{\omega}_i) \), \( i = 1, \ldots, \tilde{n} \),
problems. We generated the following difficult example to test the numerical method on several benchmark LAB function.

The prediction step computes the pseudospectral abscissa by solving equations based on the nonlinear eigenvalue problem. The method is successfully applied to the moderate size example and its effectiveness is shown.

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REFERENCES

Boyd, S., Balakrishnan, V., and Kabamba, P. (1989). A bisection method for computing the $H_{\infty}$-norm of a transfer matrix and related problems. *Mathematics of Control, Signals, and Systems*, 2, 207–219.

Breda, D., Maset, S., and Vermiglio, R. (2005). Pseudospectral differencing methods for characteristic roots of delay differential equations. *SIAM Journal on Scientific Computing*, 27, 482–495.

Breda, D., Maset, S., and Vermiglio, R. (2006). Pseudospectral approximation of eigenvalues of derivative operators with non-local boundary conditions. *Applied Numerical Mathematics*, 56, 318–331.

Burke, J., Lewis, A., and Overton, M. (2003a). Optimization and pseudospectra, with applications to robust stability. *SIAM Journal on Matrix Analysis and Applications*, 25, 80–104.

Burke, J., Lewis, A., and Overton, M. (2003b). Robust stability and a criss-cross algorithm for pseudospectra. *IMA Journal of Numerical Analysis*, 23, 359–375.

Byers, R. (1988). A bisection method for measuring the distance of a stable matrix to the unstable matrices. *SIAM Journal on Scientific and Statistical Computing*, 9, 875–881.

Engelborghs, K., Luzyanina, T., and Roose, D. (2002). Numerical bifurcation analysis of delay differential equations. *SIAM Journal on Scientific Computing*, 25, 105–136.

Genin, Y., Stefan, R., and Dooren, P.V. (2002). Real and complex stability radii of polynomial matrices. *Linear Algebra and its Applications*, 351-352, 381–410.

Gumussoy, S. and Michiels, W. (2009). Computing $H_\infty$ norms of time-delay systems. *submitted to IEEE Conference on Decision and Control*.

Hryniv, R. and Lancaster, P. (1999). On the perturbation of analytic matrix functions. *Integral Equations and Operator Theory*, 34, 325–338.

Michiels, W., Green, K., Wagenknecht, T., and Niculescu, S.I. (2006). Pseudospectra and stability radii for analytic matrix functions with application to time-delay systems. *Linear Algebra and its Applications*, 418, 315–335.

Trefethen, L. (1997). Pseudospectra of linear operators. *SIAM Review*, 39, 383–406.

Trefethen, L. (2000). *Spectral methods in MATLAB*. volume 10 of Software, Environments, and Tools, SIAM.