On the propagation of electromagnetic radiation in the field of a plane gravitational wave

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The propagation of free electromagnetic radiation in the field of a plane gravitational wave is investigated. A solution is found one order of approximation beyond the limit of geometrical optics in both transverse–traceless (TT) gauge and Fermi Normal Coordinate (FNC) system. The results are applied to the study of polarization perturbations. Two experimental schemes are investigated in order to verify the possibility to observe these perturbations, but it is found that the effects are exceedingly small.

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I. INTRODUCTION

The propagation of electromagnetic fields in curved space–time has always been an important problem since the formulation of general relativity. Gravitational Doppler shift, deflection of light rays, and gravitational lensing have been useful tests for general relativity and have represented important tools in astronomical observations (see for instance [13]).

In the first approach to this problem, the gravitational background was considered as a sort of anisotropic medium (a good introduction to this kind of approach is found in [3], while a recent exhaustive treatise is [4]; for the passage from wave optics to geometric optics in curved space–time see [3]). In the literature, on the level of geometrical optics, many authors pointed their efforts at gravitational fields of isolated physical systems (such as a star at rest or a rotating body) which were not radiative ones (see [6–11]). It was also considered the problem of light fluctuations in a gravitational wave background (see [12]); however in this work the author did not address his attention to possible polarization effects.

In the following years, in the framework of the linearized theory of general relativity, much work has been done to describe the propagation of electromagnetic fields in a gravitational wave background, using a different approach with respect to the usual one. In fact many authors have split the electromagnetic tensor (or the 4–vector potential) in a sum of two terms: the first one is the flat space–time solution, while the second term describes the perturbation due to the weak gravitational field (see [13–20]). Nevertheless, as it will be shown in Sec. II, in this way the original physical problem is changed. It is for this reason that we think that the former approach is the most suitable for the problem we are concerned with. An attempt in this direction has been recently done (see [21]) but the Lorentz condition is not properly handled, being fulfilled only for particular directions of propagation and amplitudes of the 4–vector potential.

It is therefore seen that, as far as the problem of the propagation of electromagnetic radiation in the field of a gravitational wave is concerned, a satisfactory solution is lacking which takes into account the tensor nature of Maxwell equations, even in the geometrical optics limit. Purpose of this paper is to investigate this problem. Therefore, besides the usual phase perturbation, the consequences on the polarization of the electromagnetic field are evaluated. It is a result of the present paper that, when Lorentz condition is met, nontrivial results in the polarization perturbations are found (see Sec. VII and the conclusions).

As it is well known, Maxwell equations are a set of redundant equations. If the problem is considered from the electromagnetic 4–potential point of view this redundance could be partially eliminated. The free electromagnetic equations in curved space time for the vector potential (de Rahm equations) read

\[ A^{\mu\nu}_{\nu} - A^{\nu\mu}_{\nu} = 0 \]  

(1.1)

(all through the paper, notations and conventions as in [3]). In order to get a simple set of equations, adopting the standard approach, we impose the Lorentz gauge condition on vector potential, bringing Eq. (1.1) into the system

\[
\begin{align*}
A^{\mu\nu} &= 0 \\
A^{\nu}_{\mu} &= 0 
\end{align*}
\]  

(1.2)

Once a solution of the previous system is found, the antisymmetric tensor of the electromagnetic field, which is the physical measurable quantity, is obtained by the usual relation
\[ F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (1.3) \]

From the above equations it is clear that the direct interaction of a gravitational wave with an electromagnetic field is due only to the metric tensor. This means that in curved space–time the differential equations of electromagnetism have variable coefficients. However the explicit form of the dependence from the metric tensor depends critically on the reference frame used to perform calculation. Our strategy is to consider system (1.2) in a reference frame in which calculation is easier. This system is the so called transverse–traceless (TT) gauge (see [1]). However the reference frame in which measures are performed is the Fermi Normal Coordinate (FNC) system (see for instance [2]). Therefore once the antisymmetric tensor of the electromagnetic field is known in the TT gauge we find its expression in FNC by means of the usual transformation rules (the electromagnetic field tensor is indicated by \( F_{\alpha\beta} \) in TT gauge; \( x^\mu \) are the FNC while \( y^\mu \) are the TT gauge ones):

\[ F_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} F_{\alpha\beta} \quad (1.4) \]

The connection between the two frames is known for every point of the space–time in the linear approximation of weak gravitational fields. This result is found in [22,23].

The paper is organized as follows. In Sec. II we find a solution to the de Rahm equations (1.2) in TT gauge. In Sec. III and Sec. IV we write the electromagnetic tensor components in TT gauge and FNC respectively. Finally in Sec. VI we apply our results to the calculation of the polarization effects.

II. SOLUTION OF THE DE RAHM EQUATIONS IN TT GAUGE

For weak gravitational field, to which we shall limit ourselves in this paper, the metric tensor could be written as

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << 1 \quad (2.1) \]

where \( \eta_{\mu\nu} \) is the Minkowski metric tensor with positive signature. In this approximation the particular coordinate frame called TT gauge (see [1]) is characterized by \( h = \eta^{\mu\nu} h_{\mu\nu} = 0, h_{0\mu} = 0 \) and \( h_{ij} = 0 \). Neglecting second order terms in \( h_{\mu\nu} \) the linearized homogeneous de Rahm equations with the Lorentz condition are

\[
\begin{aligned}
\mathcal{A}_{\mu,\nu} - h^{\alpha\beta} A_{\alpha,\beta} + \left( h^{\alpha\beta} + h^{\mu \beta} - h^{\beta \mu} \right) A_{\alpha,\beta} &= 0 \\
A_{\mu,\nu} &= 0
\end{aligned}
\quad (2.2)
\]

This system is a set of differential equations with variable coefficients. In the approach of [13–19] the vector potential \( A^\mu \) is split into two parts:

\[ A^\mu = 0 A^\mu + 1 A^\mu \quad (2.3) \]

where the first term is a solution of the flat space–time wave equation while the second one represents the perturbation due to the weak gravitational field. In this approach all the quadratic terms in the perturbation vector are neglected. With this assumptions system (2.2) becomes a set of differential equations for the perturbed vector potential with constant coefficients (the d’Alembert operator) and a source term which involves the unperturbed potential and \( h_{\mu\nu} \) (e.g. [17]). It is our opinion that this procedure could in general lead to some misunderstanding on the nature of the interaction when the “unperturbed” electromagnetic field is oscillatory (this is not the case of [14] in which the “unperturbed” electromagnetic field is constant). To show this point let us consider the case of an ordinary differential equation, the Mathieu equation (see for instance [24,25]), which may be written as

\[ \ddot{q} + \omega^2 [1 + h(t)] q = 0 \quad (2.4) \]

and which describes, for instance, the charge on the plates of a condenser with a variable capacitance in an oscillating LC circuit [24]. Supposing \( h(t) << 1 \), a splitting \( q = 0 q + 1 q \) in which \( 0 q \) is the “unperturbed” solution of Eq. (2.4) when \( h(t) = 0 \) and \( 1 q \) is the perturbation due to the presence of \( h(t) \) brings to the equation of a harmonic oscillator for \( 1 q \) with a driven force proportional to the product of \( h(t) \) and \( 0 q \). It is a matter of fact that, in general, the new set of equations is not equivalent to Eq. (2.4) [24]. The physical reason lies in the fact that splitting of the charge changes the original problem of a free circuit with variable capacitance in a different problem of a circuit with constant capacitance and an external electromotive force. In the same manner, splitting of the vector potential changes the
problem of the propagation of a free electromagnetic field in a gravitational background, acting as a sort of anisotropic medium, in the problem of the generation of a field by a given current. For these reasons our approach will be that of performing calculations without splitting of the vector potential. This strategy was yet carried out for the problem of the interaction of a gravitational wave with electromagnetic circuits in [26].

Let us now state precisely the physical problem which we are intended to solve. Consider a TT gauge reference frame in which a plane gravitational wave propagates in the $z$ direction. In this system the only non–vanishing components of $h_{\mu\nu}$ are $[\Phi_g = \chi (y^3 - y^0)]:$

$$h_{11}(\Phi_g) = -h_{22}(\Phi_g) = h_{+}(\Phi_g); \quad h_{12}(\Phi_g) = h_{21}(\Phi_g) = h_{\times}(\Phi_g) \quad (2.5)$$

Let us consider now a free electromagnetic field which in the absence of gravitational waves propagates in a given direction described by the spatial component of the four wave vector $k^\mu$. The vector potential for such a field is given by $A^\mu = C^\mu g(k^\nu y^\nu)$, where $C^\mu$ are four constants that satisfy Lorentz condition $C^\mu k_\mu = 0$. A gravitational wave, modifying the geometry of the space–time, acts as an anisotropic medium whose dielectric and magnetic properties change in space and in time, $1/\chi$ and $1/(c\chi)$ being the order of magnitude of space distances and time intervals for which variations take place appreciably. If $|k_0| = k >> \chi, \quad (2.6)$

then the amplitude, polarization and direction of the electromagnetic field remain practically constant over distances of the order of $1/k$ and times of the order of $1/(ck)$. Therefore we can assume that the form of a solution of Eqs. (2.2) would be

$$A^\mu = a^\mu f(I) \quad (2.7)$$

where $a^\mu$ is a 4-vector function of space and time and $I$ is a scalar function. In the framework of assumption (2.6) we can suppose that first derivatives of $a^\mu$ and second derivatives of $I$ are small quantities with respect to $a^\mu$ and $k_\mu$ respectively, so we can neglect superior order derivatives. This approach is one order of approximation beyond the geometrical optics limit allowing therefore to find not only the phase shift (that is to say the scalar function $I$, the eikonal), but also the change in the polarization of the electromagnetic wave (for all these considerations see for instance [27]).

In order to solve the problem it is important to find, within the Lorentz condition, an electromagnetic gauge for which the equations are easier to solve. One can see immediately that, because of the space-time dependence of the gravitational wave [see Eq. (2.3)], the first equation of system (2.2) is the same for $\mu = 3, 0$. Therefore we fix a gauge for which

$$A^3 = A^0 \quad (2.8)$$

With this choice the equations for $A^\nu$ (from now on indices $r, s, t$ run from 1 to 2) are independent on $A^3 = A^0$. Once $A^\nu$ are found, by means of the Lorentz condition one can find immediately $A^3 = A^0$ which fulfill also the first equation of (2.3). To carrying on this program it is convenient to perform a coordinate change. We put

$$\begin{cases} T = (y^3 - y^0)/\sqrt{2} \\ X = y^1 \\ Y = y^2 \\ Z = (y^3 + y^0)/\sqrt{2} \end{cases} \quad (2.9)$$

The new metric tensor $G_{\mu\nu}$ can be written as [see Eq. (1.4)]

$$G_{\mu\nu} = \frac{\partial y^\alpha}{\partial X^\mu} \frac{\partial y^\beta}{\partial X^\nu} g_{\alpha\beta} = N_{\mu\nu} + h_{\mu\nu}(\sqrt{2}\chi T) \quad (2.10)$$

where

$$N_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (2.11)$$

If we call $A^\mu$ and $K^\mu$ the vector potential and the four wave vector in this new frame we get
\[ \mathcal{A}^\mu = \left( \frac{(A^3 - A^0)/\sqrt{2}}{A^1}, A^2, \frac{(A^3 + A^0)/\sqrt{2}}{A^1} \right) \]

where \( A_\mu = \begin{pmatrix} k_3 - k_0, k_1, k_3 + k_0 \end{pmatrix} \)

In this frame the fixed gauge is written as the usual Coulomb gauge in flat space–time, that is to say:

\[ \mathcal{A}^0 = 0; \quad \frac{\partial \mathcal{A}^i}{\partial X^i} = 0. \]  

(2.13)

The first equation of (2.2) for the \( t \) components is written as

\[ N^{\alpha\beta} A_t,_{\alpha,\beta} - h^{rs} A_t,_{r,s} + h^t_{s,0} \mathcal{A}_{3} = 0 \]  

(2.14)

while the equation for \( A^3 \) is given by

\[ N^{\alpha\beta} A^3,_{\alpha,\beta} - h^{rs} A^3,_{r,s} - h^s_{r,0} \mathcal{A}_{r} = 0 \]  

(2.15)

Once \( \mathcal{A}^t \) are found, from Eqs. (2.13) we can find a general expression for \( \mathcal{A}^3 \); we can write

\[ \mathcal{A}^3 = -\int_0^Z \frac{\partial \mathcal{A}^r}{\partial X^r} dZ' + g(X,Y,T) \]  

(2.16)

where, in order to fulfill Eq. (2.17), \( g(X,Y,T) \) must be a solution of

\[ \left[ \delta^{rs} - h^{rs}(\sqrt{2}\chi T) \right] \frac{\partial^2 g(X,Y,T)}{\partial X^r \partial X^s} = 2 \left. \frac{\partial^2 \mathcal{A}^r}{\partial X^r \partial T} \right|_{Z=0} + h_{r,0}(\sqrt{2}\chi T) \frac{\partial \mathcal{A}^r}{\partial X^r} |_{Z=0} \]  

(2.17)

As we shall see later this function could be chosen in such a manner to have a solution which describes “plane waves” travelling in some direction.

In order to find a solution of Eq. (2.14) we exploit consequences of assumptions (2.6) and (2.7). For what was said we neglect second order derivatives of \( a^r \), and terms like \( h_{\mu\nu,\alpha} a^r,_{\alpha} \). Therefore Eq. (2.14) is divided into the following equations for \( a^\alpha \) and \( \mathcal{I} \):

\[ G^{\alpha\beta} \mathcal{I}_{\alpha} \mathcal{I}_{\beta} = 0 \]  

(2.18)

\[ G^{\alpha\beta} \left( 2a^t,_{\alpha} \mathcal{I}_{,\beta} + a^t \mathcal{I},_{\alpha,\beta} \right) + \sqrt{2}\chi h^{(1)}_{s,0} a^s \mathcal{I}_{,3} = 0 \]  

(2.19)

where \( h^{(n)}_{rs} \) means the n–order derivative of \( h_{rs} \) with respect to the argument \( \sqrt{2}\chi T \).

Now we notice that the “unperturbed” polarization amplitude and eikonal of the electromagnetic field are not oscillating quantity: indeed the polarization amplitude is a constant and the eikonal is a big quantity, changing by \( 2\pi \) when we move through one wavelength, while the gravitational perturbation occurs on the scale of \( 1/\chi \gg 1/k \). Because of the smallness of the metric perturbation [see Eq. (2.1)] and of its coordinate dependence we can therefore safely put

\[ \mathcal{I} = S + u(T) \]  

\[ a^t = B^t + b^t(T) \]  

(2.20)

where \( S = k_\alpha x^\alpha + \phi \) is the usual phase of the flat space–time, \( B^t \) are constant and finally \( u(T) \) and \( b^t(T) \) are the small perturbations due to the gravitational wave background. With these assumptions Eq. (2.19) become

\[ \dot{u} = \frac{1}{2} \frac{K_r K_s}{K_3} h^{rs} \]  

(2.22)

\[ \dot{b}^r = -\frac{1}{2} h^{rs} B^s \]  

(2.23)

where \( \dot{\cdot} \) means derivation with respect to \( T \). The solutions to these equations are simply given by

\[ u = \frac{1}{2} \frac{K_r K_s}{\sqrt{2}\chi K_3} \left[ H^{rs}(\sqrt{2}\chi T) - H^{rs}(\Phi_0) \right] + u_0 \]  

(2.24)

\[ b^r = -\frac{1}{2} h^{rs}(\sqrt{2}\chi T) B^s + b^r_0 \]  

(2.25)
where $H^{rs}(x)$ is a primitive of $h^{rs}(x)$. As far as $A^3$ is concerned, from Eqs. (2.14), (2.20), and (2.21) we can write

$$A^3 = - \frac{K_r}{K_3} [B^r + b^r(T)] \{ f[S + u(T)] - f[S_0 + u(T)] \} + g(X,Y,T)$$

(2.26)

where $S_0 = S|_{Z=0}$. Let us now focus our attention on the function $g(X,Y,T)$. Taking into account Eqs. (2.22), (2.23), and the relation $N^{\alpha\beta}K_{\alpha}K_{\beta} = 0$, then Eq. (2.17) could be written as

$$\left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} - h^{rs} \frac{\partial}{\partial X^r} \frac{\partial}{\partial X^s} \right) g = - \frac{K_r}{K_3} (B^r + b^r) (\delta^{rs} - h^{rs}) K_tK_s f^{(2)}(S_0 + u)$$

(2.27)

A solution of this equation is

$$g(T) = - \frac{K_r}{K_3} [B^r + b^r(T)] f(S_0 + u)$$

(2.28)

Putting this solution in Eq. (2.26) we find

$$A^3 = - \frac{K_r}{K_3} [B^r + b^r(T)] f(S + u)$$

(2.29)

The solution described by Eqs. (2.13), (2.20), (2.21), and (2.29) represents a “plane wave like” solution for the electromagnetic field.

Getting back to the TT gauge system in which we wrote Eqs. (2.2) we finally find

$$A^r = \left[ B^r - \frac{1}{2} h^{rs}(\Phi_g)B^s + b_r^0 \right] f(S + u)$$

(2.30)

$$A^0 = A^3 = - \frac{k_r}{k_3 + k_0} \left[ B^r - \frac{1}{2} h^{rs}(\Phi_g)B^s + b_r^0 \right] f(S + u)$$

where

$$u = \frac{1}{2} \frac{k_r k_s}{\chi(k_3 + k_0)} [H^{rs}(\Phi_g) - H^{rs}(\Phi_0)] + u_0$$

(2.31)

is the phase shift function. We have therefore written a plane wave like solutions of Eqs. (2.2) in the particular Lorentz gauge for which $A^3 = A^0$. As one can see from Eqs. (2.30) the amplitude of $A^3 = A^0$ tends to infinity when $k_0 + k_3$ goes to zero (that is to say for parallel interaction). However this is only a result due to the particular chosen gauge, without any physical meaning. In fact in the next section we shall see that the electromagnetic field does not present such infinity problems.

### III. ELECTROMAGNETIC FIELD IN TT GAUGE

In this section we find the components of the antisymmetric tensor of the electromagnetic field in interaction with a gravitational wave in the TT gauge. To this aim we shall use Eq. (1.3). In order to do that we must before write the covariant component of $A^\mu$. We have

$$A_\nu = g_{\nu\mu} A^\mu \Rightarrow \begin{cases} A_3 = A^3 = A^0 = - A_0 \\ A_r = A^r + h_{rs} A^s \end{cases}$$

(3.1)

We therefore have:

$$A_3 = - A_0 = - \frac{k_r}{k_3 + k_0} (B^r - \frac{1}{2} h^{rs}B^s + b_r^0) f(S + u)$$

(3.2)

$$A_r = (B^r + \frac{1}{2} h^{rs}B^s + b_r^0) f(S + u)$$
In order to be consistent with the approximations made [see Eq. (2.6) and its consequences] we neglect first order derivatives of $h_{\mu\nu}$ in Eq. (3.3). The components of the electromagnetic field tensor in the TT gauge are therefore written as

$$F_{\mu\nu} = -k_r \left( B^r - \frac{1}{2} h^r_s B^s + b_0^r \right) f'(S + u)$$

and

$$F_{v0} = -\left[ -k_0 \left( B^r + \frac{1}{2} h^r_s B^s + b_0^r \right) + u_3 B^r + \frac{k_r k_s}{k_0 + k_3} \left( B^s - \frac{1}{2} h^s_t B^t + b_0^s \right) \right] f'(S + u)$$

where we have used the fact that $u_{,0} = -u_{,3}$ and

$$u_{,3} = \frac{1}{2} \frac{k_r k_s}{k_0 + k_3} h^{rs} (\Phi_2)$$

### IV. ELECTROMAGNETIC FIELD IN FNC SYSTEM

The aim of this section is to write the expression for the electromagnetic field tensor in FNC. In the framework of linear approximation, TT-coordinates are given by FNC by means of

$$y^\mu(x^\alpha) = x^\mu + \epsilon^\mu(x^\beta)$$

which involves the infinitesimal vector field $\epsilon^\mu$; this vector field is known in every point of the space (see [23]) therefore solving the problem of the connection between FNC and TT-gauge. If the TT-gauge perturbation of the Minkowskian metric is a plane wave propagating in the $z$ direction, with angular frequency $\chi$ then the infinitesimal vector field $\epsilon^\mu$ may be written as (see [23])

$$\epsilon^v = \frac{1}{2x^3} \left\{ \frac{H_{rs}[\chi(x^3 - x^0)] - H_{rs}(-\chi x^0)}{\chi x^3} - \frac{h_{rs}(-\chi x^0)}{\chi x^3} \right\} x^r x^s$$

$$\epsilon^r = -\left\{ \frac{H_{rs}[\chi(x^3 - x^0)] - H_{rs}(-\chi x^0)}{\chi x^3} - \frac{1}{2} h_{rs}(-\chi x^0) \right\} x^s$$

where $v = 0, 3$, and we remind that $H_{rs}(\phi)$ is a primitive of $h_{rs}(\phi)$ and that $r, s = 1, 2$. Obviously a plane wave is only an approximation of the gravitational field in a relatively small region around a point far away from the source. This implies that (4.2) represents the infinitesimal vector field $\epsilon^\mu$ only in the region in which the plane wave propagation is valid. In which follows we consider source distances and gravitational wavelengths for which Eqs. (4.2) hold in the region of interest.

The electromagnetic tensor field components in FNC are given in terms of the TT gauge ones by means of Eq. (4.4). If the transformation rule between the two system is given by Eq. (1.1) then we have

$$F_{\mu\nu} = F_{\mu\nu} + F_{\mu\alpha} \epsilon^\alpha + F_{\alpha\nu} \epsilon^\alpha_{\mu}$$

In the range of validity of Eqs. (1.2) for plane monochromatic waves from the $z$ direction we can write [$a = \chi(x^3 - x^0)$, $a_0 = -\chi x^0$]
\begin{align*}
F_{0t} &= F_{0t} - \left[ \frac{H_t^s(a) - H_t^s(a_0)}{\chi^s} - \frac{1}{2} h_t^s(a_0) \right] F_{0s} + \\
&+ \frac{x^s}{x^3} \left[ \frac{H_{st}(a) - H_{st}(a_0)}{\chi^s} - h_{st}(a_0) \right] F_{03} + \\
&- \frac{\chi x^r x^s}{2x^3} \left[ \frac{h_{rs}(a) - h_{rs}(a_0)}{\chi^s} - h_{rs}(a_0) \right] (F_{0t} - F_{t3}) + \\
&+ \chi x^r \left[ \frac{h_{r}^s(a) - h_{r}^s(a_0)}{\chi^s} - \frac{1}{2} h_{r}^{(1)}(a_0) \right] F_{st} \\
\end{align*}

\begin{align*}
F_{03} &= F_{03} + \frac{x^r x^s}{z} \left[ \frac{H_{r}^s(a) - H_{r}^s(a_0)}{\chi^s} - h_{r}^s(a) \right] F_{0s} + \\
&- \frac{x^r x^s}{x^3 x^3} \left[ \frac{H_{rs}(a) - H_{rs}(a_0)}{\chi^s} - h_{rs}(a_0) - \frac{\chi x^3}{2} h_{rs}^{(1)}(a_0) \right] F_{03} + \\
&+ \chi x^r \left[ \frac{h_{r}^s(a) - h_{r}^s(a_0)}{\chi^s} - \frac{1}{2} h_{r}^{(1)}(a_0) \right] F_{s3} \\
\end{align*}

\begin{align*}
F_{12} &= F_{12} + \frac{x^r}{x^3} \epsilon_{ik3} \left[ \frac{H_{r}^s(a) - H_{r}^s(a_0)}{\chi^s} - h_{r}^s(a_0) \right] \delta^{ij} (F_{j3} - F_{0j}) \\
\end{align*}

\begin{align*}
F_{t3} &= F_{t3} - \left[ \frac{H_r^s(a) - H_r^s(a_0)}{\chi^s} - \frac{1}{2} h_r^s(a_0) \right] F_{c3} + \\
&+ \frac{x^r}{x^3} \left[ \frac{H_{rt}(a) - H_{rt}(a_0)}{\chi^s} - h_{rt}(a_0) \right] F_{03} + \\
&+ \frac{x^r}{x^3} \left[ \frac{H_r^s(a) - H_r^s(a_0)}{\chi^s} - h_r^s(a) \right] F_{t3} + \\
&+ \frac{x^r x^s}{x^3 x^3} \left[ \frac{H_{rs}(a) - H_{rs}(a_0)}{\chi^s} - \frac{1}{2} h_{rs}(a) - \frac{1}{2} h_{rs}(a_0) \right] (F_{0t} - F_{t3}) \\
\end{align*}

where \( \epsilon_{ijk} \) is the completely antisymmetric unit pseudotensor whose components are zero unless \( i \neq j \neq k \) and in this case they are equal to 1 or \(-1\) according to the fact that \( ijk \) is an even or odd permutation of 123.

If the electromagnetic field which is under investigation is localized in a space region with linear dimensions small compared with \( 1/\chi \) (for instance the electromagnetic field of a resonant cavity), then it is appropriate to expand the relations (4.4)-(4.7) in power of \( \chi x^s \). The first three terms of the expansion are given by:

\begin{align*}
F_{0t} &= F_{0t} - \frac{1}{2} h_t^r F_{0r} + \frac{\chi x^r}{2} \left( h_{(1)}^r F_{0s} + h_{(1)}^r F_{03} \right) - \frac{\chi x^3 x^3}{2} h_{(1)}^r F_{0r} + \\
&- \frac{\chi x^r x^s}{4} h_{(2)}^{rs} (F_{0t} - F_{t3}) + \\
&+ \frac{\chi^2 x^r x^s}{2} \left( \frac{1}{3} h_{(2)}^{rs} F_{0s} + h_{(2)}^{rs} F_{s3} \right) - \frac{\chi^2 x^3 x^3}{6} h_{(2)}^{rs} F_{0s} \\
\end{align*}

\begin{align*}
F_{03} &= F_{03} - \frac{\chi x^r}{2} h_{(1)}^r (F_{0s} - F_{s3}) + \\
&- \frac{\chi^2 x^r x^s}{6} h_{(2)}^{rs} F_{03} - \chi^2 x^r x^3 h_{(2)}^{rs} \left( \frac{1}{3} F_{0s} - \frac{1}{2} F_{s3} \right) \\
\end{align*}

\begin{align*}
F_{12} &= F_{12} - \frac{\chi x^r}{2} h_{(1)}^r \epsilon_{ij3} \delta^{ij} (F_{j3} - F_{0j}) + \\
&+ \frac{\chi^2 x^r x^3}{6} \epsilon_{ij3} h_{(2)}^{rs} \delta^{ij} (F_{j3} - F_{0j}) \\
\end{align*}
then for consistency reasons we must also expand those equations in powers of polarization of a photon propagating in a gravitational wave field in a region small enough to use Eqs. (4.8)–(4.11), along the two arms:

In which explicit $h_{rs}$ and its derivatives are functions of $(-\chi x^0)$ and the tensor part of $\mathcal{F}_{\mu\nu}$ in the first term of (4.8)–(4.11) is to be expanded in power of $\chi x^3$ until second order. We see immediately that the $z$ components of both electric and magnetic fields have the first term of the expansion vanishing. Therefore in the first approximation the difference with the TT-gauge arises in the electromagnetic field component normal to the direction of propagation of the gravitational wave.

If the problem is to “follow” a photon in its motion, that is to say to consider the phase shift and the change in the polarization of a photon propagating in a gravitational wave field in a region small enough to use Eqs. (4.8)–(4.11), then for consistency reasons we must also expand those equations in powers of $\chi x^0$. In this case we have:

\[
F_{0t} = \mathcal{F}_{0t} - \frac{1}{2} h_t^r \mathcal{F}_{r0} + \frac{\chi x^r}{2} \left( (h^{(1)}_t)^r_{,s} \mathcal{F}_{st} + (h^{(1)}_t)^r_{,r} \mathcal{F}_{03} - \frac{\chi (x^3 - x^0)}{2} h^{(1)}_t^r \mathcal{F}_{0r} \right) - \frac{\chi^2 x^r x^s}{4} h^{(2)}_{rs} (\mathcal{F}_{0t} - \mathcal{F}_{t3}) + \frac{\chi^2 x^r (x^3 - x^0)}{2} h^{(2)}_r s \mathcal{F}_{st} + \frac{\chi^2 x^r (x^3 - 3x^0)}{6} h^{(2)}_{rt} \mathcal{F}_{03} - \frac{\chi^2 (2x^3 x^3 - 6x^3 x^0 + 3x^0 x^0)}{12} h^{(2)}_r \mathcal{F}_{0r} + \frac{\chi^2 x^r (x^3 - x^0)}{6} h^{(2)}_r s \mathcal{F}_{s3} \quad (4.12)
\]

\[
F_{03} = \mathcal{F}_{03} - \frac{\chi x^r}{2} h^{(1)}_r s (\mathcal{F}_{0s} - \mathcal{F}_{s3}) - \frac{\chi x^r x^s}{6} h^{(2)}_{rs} \mathcal{F}_{03} + \frac{\chi^2 x^r (x^3 - 3x^0)}{6} h^{(2)}_r s \mathcal{F}_{0s} + \frac{\chi^2 x^r (x^3 - x^0)}{2} h^{(2)}_r s \mathcal{F}_{s3} \quad (4.13)
\]

\[
F_{12} = \mathcal{F}_{12} - \frac{\chi x^r}{2} h^{(1)}_r t \epsilon_{tst3} \delta^{ij} (\mathcal{F}_{0j} - \mathcal{F}_{j3}) + \frac{\chi^2 x^r x^3}{6} \epsilon_{tst3} h^{(2)}_r s \delta^{ij} (\mathcal{F}_{j3} - \mathcal{F}_{0j}) + \frac{\chi^2 x^r (x^3 - 3x^0)}{6} h^{(2)}_r t \epsilon_{tst3} \delta^{ij} (\mathcal{F}_{0j} - \mathcal{F}_{j3}) \quad (4.14)
\]

\[
F_{13} = \mathcal{F}_{13} - \frac{1}{2} h_t^r \mathcal{F}_{r3} + \frac{\chi x^r}{2} \left( (h^{(1)}_t)^r_{,s} \mathcal{F}_{st} - (h^{(1)}_t)^r_{,r} \mathcal{F}_{ts} \right) - \frac{\chi (x^3 - x^0)}{2} h^{(1)}_t \mathcal{F}_{r3} + \frac{\chi^2 x^r x^s}{12} h^{(2)}_{rs} (\mathcal{F}_{0t} - \mathcal{F}_{t3}) - \frac{\chi^2 x^r (2x^3 - 3x^0)}{6} h^{(2)}_r s \mathcal{F}_{ts} + \frac{\chi^2 x^r (x^3 - 3x^0)}{12} h^{(2)}_{rt} \mathcal{F}_{03} - \frac{\chi^2 (2x^3 x^3 - 6x^3 x^0 + 3x^0 x^0)}{12} h^{(2)}_r \mathcal{F}_{r3} \quad (4.15)
\]

in which now explicit $h_{rs}$ and its derivatives are intended calculated in zero and also the tensor part of $\mathcal{F}_{\mu\nu}$ in the first term of (4.12)–(4.13) is to be expanded in power of $\chi (x^3 - x^0)$ until second order.

**V. PHASE-SHIFT**

As a first application of the obtained results we calculate the total phase–shift between the light beams in an interferometer with arms in the directions defined by the azimuthal and polar angles $(\theta_r, \phi_r)$ $(r = 1, 2)$. The gravitational wave propagates in the positive $z$ direction. The total phase–shift is obtained by the difference of the phase change along the two arms:

\[
\delta \Phi = \delta u_{1f} - \delta u_{2f} + \delta u_{1b} - \delta u_{2b} \quad (5.1)
\]

where lower indices 1, 2 refers to the two arms, and $f, b$ to forward and backward path.
The phase is a scalar quantity, therefore we can perform the calculation in TT gauge; the result holds true also in FNC, the laboratory frame. In TT gauge, free falling bodies initially at rest, stay at rest at any subsequent time (as it is easily seen by inspection of motion equations). Therefore, if the beam splitter is in the origin 0 of the coordinates and the two mirrors are placed at a distance \( L \) along the directions \( \mathbf{n}_r = (\sin \theta_r \cos \phi_r, \sin \theta_r \sin \phi_r, \cos \theta_r) \) \((r = 1, 2)\), the forward phase change is given by (see eq. (2.31)): 

\[
\delta u_{rf} = \frac{1}{2} \frac{k}{\chi} (1 + \cos \theta_r) \left[ H_{\phi_r} (\phi_0 - \chi y^0 - \chi L (1 - \cos \theta_r)) - H_{\phi_r} (\phi_0 - \chi y^0) \right] \tag{5.2}
\]

where \( H_{\phi} = \cos 2\phi H_+ + \sin 2\phi H_\times \). When light is travelling backward the reverse direction of propagation is given by the angles \((\pi - \theta_r, \pi + \phi_r)\). Therefore the backward phase change is given by

\[
\delta u_{rb} = \frac{1}{2} \frac{k}{\chi} (1 - \cos \theta_r) \left[ H_{\phi_r} (\phi_0 - \chi y^0 - 2\chi L) - H_{\phi_r} (\phi_0 - \chi y^0 - \chi L (1 - \cos \theta_r)) \right] \tag{5.3}
\]

This holds true for any waveform of the gravitational radiation. Now we consider a plane monochromatic gravitational wave. We set

\[
h_{\phi} (\Phi_g) = A_{\phi} \sin (\Phi_g + \alpha) ; \quad A_{\phi} = \cos 2\phi \ A_+ + \sin 2\phi \ A_\times \tag{5.4}
\]

where the constants \( A_+ \), \( A_\times \) are the polarization amplitudes \((h_+ (\Phi_g) = A_+ \sin (\Phi_g + \alpha) \), \( h_\times (\Phi_g) = A_\times \sin (\Phi_g + \alpha) \)).

In this case we have

\[
\delta u_{rf} = A_{\phi_r} \frac{k}{\chi} (1 + \cos \theta_r) \sin \left[ \frac{\chi L}{2} (1 - \cos \theta_r) \right] \sin \left[ \frac{\chi L}{2} (1 - \cos \theta_r) + \chi y^0 - \alpha \right] \\
\delta u_{rb} = A_{\phi_r} \frac{k}{\chi} (1 - \cos \theta_r) \sin \chi L \sin (\chi L + \chi y^0 - \alpha) + \\
- A_{\phi_r} \frac{k}{\chi} (1 - \cos \theta_r) \sin \left[ \frac{\chi L}{2} (1 - \cos \theta_r) \right] \sin \left[ \frac{\chi L}{2} (1 - \cos \theta_r) + \chi y^0 - \alpha \right]
\]

Therefore the phase change in each arm is given by

\[
\delta u_{rf} + \delta u_{rb} = A_{\phi_r} \frac{k}{\chi} \left\{ 2 \cos \theta_r \sin \left[ \frac{\chi L}{2} (1 - \cos \theta_r) \right] \sin \left[ \frac{\chi L}{2} (1 - \cos \theta_r) + \chi y^0 - \alpha \right] + \\
+ (1 - \cos \theta_r) \sin \chi L \sin (\chi L + \chi y^0 - \alpha) \right\} \tag{5.5}
\]

For small \( \theta_r \) angle, when one arm direction is very near to that of \( z \) axis one obtains

\[
\delta u_{rf} + \delta u_{rb} \xrightarrow{\theta_r \rightarrow 0} A_{\phi_r} \frac{k \theta_r^2}{\chi} \left[ \sin \chi L \sin (\chi L + \chi y^0 - \alpha) + \chi L \sin (\chi y^0 - \alpha) \right] \tag{5.6}
\]

while for almost antiparallel direction one has

\[
\delta u_{rf} + \delta u_{rb} \xrightarrow{\theta_r \rightarrow \pi} A_{\phi_r} \frac{k (\pi - \theta_r)^2}{\chi} \left[ \sin \chi L \sin (\chi L + \chi y^0 - \alpha) + \chi L \sin (2\chi L + \chi y^0 - \alpha) \right] \tag{5.7}
\]

We see therefore that for parallel and antiparallel directions there is no phase change.

In the limit of small distances \( \chi L << 1 \) one has

\[
\delta u_{rf} + \delta u_{rb} \xrightarrow{\chi L \ll 1} A_{\phi_r} k L \sin^2 \theta \sin (\chi y^0 - \alpha) \tag{5.8}
\]

Now let us consider an interferometer with arms in the positive \( x \) and \( y \) directions respectively. In this case \( A_{\phi_1} = A_+ \) and \( A_{\phi_2} = -A_+ \). Therefore the total phase–shift is given by

\[
\delta \Phi = 2 (\delta u_{1f} + \delta u_{1b}) = A_+ k (2L) \frac{\sin \chi L}{\chi L} \sin (\chi L + \chi y^0 - \alpha) \tag{5.9}
\]

which is in agreement with known results (see for instance [28]).
VI. POLARIZATION

There are two possible state of polarization. If the electromagnetic wave is propagating in the direction we are naturally led to describe the electromagnetic field in a frame whose three axis directions are:

\[ \mathbf{e}_{(1)} = (-\cos \theta \cos \phi, -\cos \theta \sin \phi, \sin \theta) \]
\[ \mathbf{e}_{(2)} = (\sin \phi, -\cos \phi, 0) \]
\[ \mathbf{e}_{(3)} = \frac{k}{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]

In flat space–time, the possible polarization of the electric field are along the \( \mathbf{e}_{(1)} \) and \( \mathbf{e}_{(2)} \) directions. They are obtained when

\[ B_{(1)}^1 = -\cos \phi \frac{E_0}{k}; \quad B_{(1)}^2 = -\sin \phi \frac{E_0}{k} \]  
\[ B_{(2)}^1 = -\sin \phi \frac{E_0}{k}; \quad B_{(2)}^2 = \cos \phi \frac{E_0}{k} \]  

When a polarized wave interacts with a gravitational wave, the electric field rotates and changes in magnitude. We can write

\[ (F_{k0})_{(r)} = \left[ (E_1)_{(r)} \mathbf{e}_{(1)}^k + (E_2)_{(r)} \mathbf{e}_{(2)}^k + (E_3)_{(r)} \mathbf{e}_{(3)}^k \right] f'(S + u) \]  

in which \( r = 1, 2 \)

\[ (E_1)_{(1)} = E_0 \left[ 1 + \frac{1}{2} (\cos^2 \theta - \cos \theta - 1) h_\phi - \frac{k}{E_0} (\cos \phi b_0^1 + \sin \phi b_0^2) \right] \]
\[ (E_2)_{(1)} = -E_0 \left[ \frac{1}{2} h_\phi' + \frac{k}{E_0} (\sin \phi b_0^1 - \cos \phi b_0^2) \right] \]  
\[ (E_3)_{(1)} = \frac{E_0}{2} \sin \theta (1 - \cos \theta) h_\phi \]  
\[ (E_1)_{(2)} = E_0 \left[ \frac{1}{2} (1 + 2 \cos \theta) h_\phi' - \frac{k}{E_0} (\cos \phi b_0^1 + \sin \phi b_0^2) \right] \]
\[ (E_2)_{(2)} = E_0 \left[ 1 - \frac{1}{2} (2 + \cos \theta) h_\phi - \frac{k}{E_0} (\sin \phi b_0^1 - \cos \phi b_0^2) \right] \]  
\[ (E_3)_{(2)} = -E_0 \sin \theta h_\phi' \]

where

\[ h_\phi = \cos 2\phi h_+ + \sin 2\phi h_\times \]  
\[ h_\phi' = -\sin 2\phi h_+ + \cos 2\phi h_\times \]  

As an example of application, let us consider the following "gedanken Experiment". A system is made by a polarizer in the origin of a TT reference frame and a particle with mass \( m \) and charge \( q \) at a distance \( L \) along the direction \( (\theta, \phi) \); both of them are freely falling and point–like with respect to the gravitational wavelength. An electromagnetic wave pulse is propagating in the same direction. The duration \( \tau_p \) of the pulse fulfills the condition \( 1/(ck) << \tau_p << 1/(c\chi) \), namely it is very small compared with the gravitational wave period but it is very long with respect to the electromagnetic period. In this way the electromagnetic pulse is short enough that the metric tensor could be considered constant in the region in which the electromagnetic field exists but also long enough that this could be considered practically a plane monochromatic wave. A charged particle in the presence of an electromagnetic field in curved space–time undergoes a force whose expression is

\[ mc \left( \frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \right) = \frac{q}{c} F_{\nu\lambda} u^\nu \]  

(6.9)
where \( ds \) is the line element, \( u^\mu \) the four–velocity of the particle and \( \Gamma^\mu_{\alpha\beta} \) are the Christoffel symbols. If the particle is at rest in TT gauge, then \( u^\mu = (1, 0, 0, 0) \). In this way one obtains
\[
\frac{d^2 y^0}{ds^2} = 0 \quad \implies \quad dy^0 = ds
\]
\[
\frac{d^2 y^3}{ds^2} = \frac{g}{mc^2} F^3_0
\]
\[
\frac{d^2 y^i}{ds^2} = \frac{g}{mc^2} F^i_0
\]

Let us suppose that the polarization is in the \( e_{(2)} \) direction and that the electromagnetic beam reach the polarizer at the time \( t_i \). Therefore at the particle position
\[
F^3_0 = -\frac{1}{2} E_0 \sin \theta \left[ h'_\phi(a_f) - h'_\phi(a_i) \right] f'[k_\mu y^\mu + u(a_f)] \tag{6.11}
\]
where \( a_f = \chi(L \cos \theta - t_i - L) \), \( a_i = \chi(L \cos \theta - t_i) \) and \( y^0 \in [t_i + L, t_i + L + \tau_p] \), \( y^j \in \left[ L e^j_{(3)} \right] \). The particle has therefore a TT acceleration in the direction \( z \) given by
\[
\frac{d^2 y^3}{dt^2} = -\frac{q E_0}{2m} \sin \theta \left[ h'_\phi(a_f) - h'_\phi(a_i) \right] f'[k_\mu y^\mu + u(a_f)] \tag{6.12}
\]

This holds true for any waveform of the gravitational radiation. Now we consider a plane monochromatic gravitational wave [see eqs. (6.3)]. Therefore [see also Eq. (6.3)] \( h'_\phi(\Phi_\gamma) = \tilde{A}_\phi' \sin (\Phi_\gamma + \alpha) \), \( A'_\phi = -\sin 2\phi \ A_+ + \cos 2\phi \ A_\times \), and
\[
\frac{d^2 y^3}{dt^2} = -\frac{q E_0 A'_\phi}{m} \sin \theta \sin \left[ \frac{\chi L}{2} \right] \cos \left[ \alpha - \chi t_i - \frac{\chi L}{2} \right] f'[k_\mu y^\mu + u(a_f)] \tag{6.13}
\]

It is immediately seen that for parallel and antiparallel interaction the acceleration vanishes. For perpendicular interaction one has
\[
\frac{d^2 y^3}{dt^2} = -\frac{q E_0 A'_\phi}{m} \sin \left( \frac{\chi L}{2} \right) \cos \left( \chi t_i + \frac{\chi L}{2} - \alpha \right) f'[k_\mu y^\mu + u(a_f)] \tag{6.14}
\]

For \( \chi L << 1 \)
\[
\frac{d^2 y^3}{dt^2} = -\frac{q E_0 A'_\phi \chi L}{2m} \sin \theta (1 - \cos \theta) \cos (\chi t_i - \alpha) f'[k_\mu y^\mu + u(a_f)] \tag{6.15}
\]

from which one could easily see that the greater acceleration is caused when \( \theta = 2 \pi/3 \). In order to get the observable acceleration, one must get its expression in FNC, being this the laboratory reference frame. However the acceleration is a first order quantity in \( h_{\mu\nu} \); therefore its expression in FNC is the same as in TT gauge.

It has been shown that, in the laboratory reference frame, the particle is accelerated in the \( z \) direction only because of the direct interaction between the incoming electromagnetic and gravitational waves. In the absence of one of the two wave the acceleration would vanish, thus being a peculiar feature of the electromagnetic propagation in a gravitational background. This result contains an apparent paradox: the appearance of a component of the electric field along the gravitational wave propagation direction with the consequent acceleration of the particle along the wave vector of the gravitational wave, even though this last one is a transverse wave. This is due to the fact that the components of the electromagnetic tensors are not independent (because of Maxwell equations).

In order to get the order of magnitude of the effect described by Eqs. (6.12)–(6.13) let us suppose that the charged particle is a proton. Taking \( \theta = \pi/2 \), a laser beam with intensity \( I = 50 \text{ W/mm}^2 \) and wavelength \( \lambda = 2\pi/k = 1 \mu\text{m} \), and putting the particle at a distance \( L = \pi/\chi \) from the polarizer (in such a way to have the greatest acceleration) we find that, after the passage of the electromagnetic wave pulse, the drift velocity in the \( z \) direction is [the factor \( \gamma \) takes into account the effect of the phases of the last two factors in Eq. (6.13)]
\[
v_d = \gamma \ 0.7 \ A'_\phi \text{ cm/sec}, \quad -1 \leq \gamma \leq 1; \tag{6.16}
\]
thus, even in this optimistic situation, the effect is exceedingly small and, most likely, out of the range of an experimental verification.
As a second example of application let us consider the possibility to perform a polarization rotation measure. PVLAS is an experiment under construction which is designed to measure the vacuum magnetic birefringence [29]. The achievable sensitivity of about $5 \times 10^{-9} \text{ rad/}\sqrt{\text{Hz}}$, with an integration time of $10^7 \text{ sec}$ [30], would allow the measurement of an ellipticity of $10^{-12} \text{ rad}$. With a simple modification of the experimental setup (the collocation of a quarter-wave plate) the same sensitivity could be reached in the rotation angle of a linearly polarized electromagnetic radiation [30]. This is the best result to date. The description of the apparatus is found in [29]; however, in the last years there have been many improvements in the performances. For instance the finesse of the 4.5 m Fabry–Perot cavity is expected to be $10^5 \div 10^6$, while the laser have an intensity of $\sim 100 \text{ mW/mm}^2$ [30]. In the same assumptions as in the previous example for the electromagnetic pulse, and for gravitational waves of $\sim 100 \text{ Hz}$, the approximation $\chi L \ll 1$ could be used, and then Eq. (6.11) gives on the analyzer prism

$$F_{03} = F_{03} = F \frac{\chi L}{2} \sin \theta \cos (\chi t_i - \alpha) f' [k_{\mu} y^\mu + u(a_f)]$$

(6.17)

where $F$ is the finesse of the cavity and the first equality holds for first order quantities in $h_{\mu\nu}$. The rotation angle is therefore given by

$$\sin \theta F \frac{\chi L}{2} A'_\phi \sim \sin \theta A'_\phi.$$  

(6.18)

It is therefore seen once again that also taking the most optimistic value of $10^{-24}$ for $A'_\phi$ (gravitational wave from a pulsar) the effect is too small to be detected.

By the way we note that in Ref. [20] authors claimed that no rotation of the plane of the polarization of the electromagnetic wave occurs to first order in $h$ calculations. Indeed this result was obtained for a particular linear polarization of the gravitational wave. In fact authors set $h \times = 0$. In this particular case also our solution is consistent with the previous claim. However in the general case there is a rotation of the plane of the polarization of the electromagnetic wave. A comparison showing more in detail the differences between the approach of this paper and the splitting procedure in the physical situation described in [20] is found in [31].

VII. CONCLUSION

We have found a solution in TT gauge to the free de Rahm equations with Lorentz condition for a gravitational background described by a plane wave, within the framework of two approximations. The first one is the linear gravitational approximation. The second one is that the gravitational wavelength is much smaller than the electromagnetic one; this allowed us to find a solution one order of approximation beyond the limit of geometrical optics. The result was used to write the components of the electromagnetic tensor in TT gauge and FNC which automatically fulfill free Maxwell equations in curved space–time up to terms of order $(\chi/k)^0$.

We have applied these solutions to the problem of polarization of an electromagnetic field, exploiting an interesting feature of the propagation of electromagnetic waves in a gravitational wave background. In fact we have found that, when an electromagnetic field linearly polarized interacts with a gravitational wave whose direction of propagation is perpendicular to the electric field, then the electromagnetic field changes in magnitude and rotates in such a way that the electric field gets a component parallel to the gravitational wave vector. This result could seem in disagreement with the transversal nature of the gravitational waves: however this fact arise from the mutual dependence of the electromagnetic tensor components through Maxwell equations.

Two possible applications of this result have been investigated. However the effects are so small that are out of the range of current experimental techniques, and unlikely to be ever observed in the future. We think that a study is recommended in order to check if possible measurable effects could be found on light beam coming from cosmological distances (analogously to what was done in [12] for amplitude fluctuations).

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