Existentially Restricted Quantified Constraint Satisfaction

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Abstract

The quantified constraint satisfaction problem (QCSP) is a powerful framework for modelling computational problems. The general intractability of the QCSP has motivated the pursuit of restricted cases that avoid its maximal complexity. In this paper, we introduce and study a new model for investigating QCSP complexity in which the types of constraints given by the existentially quantified variables, is restricted. Our primary technical contribution is the development and application of a general technology for proving positive results on parameterizations of the model, of inclusion in the complexity class coNP.

1 Introduction

1.1 Background

The constraint satisfaction problem (CSP) is a general framework in which many combinatorial search problems can be conveniently formulated. Intuitively, the CSP involves deciding if a collection of constraints on a set of variables can be simultaneously satisfied. The CSP can be formalized as the problem of deciding the truth of a first-order sentence consisting of a conjunction of constraints, in front of which all variables are existentially quantified.

A natural and useful generalization of the CSP is the quantified constraint satisfaction problem (QCSP). The definition of the QCSP is similar to that of the CSP, but variables may be both universally and existentially quantified. While the CSP lies in the complexity class NP and hence can only be used to model other problems in NP, the higher expressivity of the QCSP permits the modelling of problems in the (presumably) larger complexity class PSPACE. Such problems arise naturally in a wide variety of domains, for example, logic, artificial intelligence, verification, combinatorics, and game theory.

In their general formulation, the CSP and QCSP are intractable, being NP-complete and PSPACE-complete, respectively; this intractability motivates the pursuit of restricted cases of these problems that avoid “maximal” complexity, and fall into complexity classes strictly below NP and PSPACE, respectively. It is possible to parameterize these problems by restricting the constraint language, or the types of constraints that are permitted in problem instances. This form of restriction captures and places into a unified framework many particular cases of the CSP and QCSP that have been independently investigated, including the HORN SATISFIABILITY and 2-SATISFIABILITY problems, and their quantified versions. The notion of constraint language has its roots in the classic dichotomy theorem of Schaefer [24], which shows that every constraint language over a two-element domain gives rise to a case of the CSP that is either in P, or is NP-complete. The research program of classifying the CSP complexity of all constraint languages over domains of arbitrary finite size has attracted significant attention; see for instance [23, 19, 18, 8, 5, 4, 6, 7, 17].

On the QCSP front, previous work on constraint language restrictions is as follows. An analog of Schaefer’s theorem is known [16, 15], which shows that all constraint languages over a two-element domain give rise to a case of the QCSP that is either in P, or is PSPACE-complete. A “finer” version studying the
alternation-bounded QCSP over a two-element domain has also been obtained \[20\]. Recently, the study of constraint languages in domains of size larger than two has been initiated \[3, 11, 13\]. Results include the development of an algebraic theory for studying the QCSP \[3\], general technology for proving positive complexity results \[11\], and some broad classification results \[13\].

1.2 A New Model: Existentially Restricted Quantified Constraint Satisfaction

In all previous work on QCSP complexity, constraint language restrictions are applied equally to both universally and existentially quantified variables; that is, in constraints, for any position where an existentially quantified variable can occur, a universally quantified variable can also occur in that position, and vice-versa. This paper introduces and studies a new model for investigating QCSP complexity, where constraint language restrictions are applied only to existentially quantified variables. We call our new model existentially restricted quantified constraint satisfaction, and refer to the previously studied QCSP model simply as the standard model.

Both our new model and the standard model are generalizations of the usual CSP model: when these two models are restricted to instances having only existential quantification, they coincide and yield the CSP model. However, there is a principal difference between these two QCSP models which rears its head as soon as universal quantification is permitted: although it is possible to obtain polynomial-time tractability results in the standard model for interesting constraint languages, under extremely mild assumptions on the constraint language, our new model is at least coNP-hard. This, of course, reflects the definition of our model, in which the universal variables do not need to observe any form of restriction. In consequence, the best type of positive complexity result that one can reasonably hope for in our new model is a demonstration of containment inside coNP. Accordingly, the main technical contribution of this paper is the development and application of a general technology for proving coNP-inclusion results in our new model.

Our new model, as with the standard model and the CSP model, constitutes a simple, syntactic means of restricting a generally intractable computational problem. Although this paper is the first to systematically investigate this model, we view it as being at least as natural as the standard model. We believe that this model gives rise to beautiful theory at the interface of logic, algebra, and computational complexity. We now turn to articulate some concrete reasons for interest in our new model.

First, our model allows us to obtain positive complexity results—of inclusion in coNP—for classes of QCSP instances for which the only complexity result that can be derived in the standard model is the trivial PSPACE upper bound. Roughly speaking, this is because there are QCSP instances where the constraint language of the existential variables has tractable structure, but the overall constraint language lacks tractable structure. One example is the class of extended quantified Horn formulas, defined by Kleine Büning et al. \[10\] as the boolean (two-element) QCSP where all constraints must be clauses that, when restricted to existential variables, are Horn clauses. A second example is the boolean QCSP where each constraint must be a clause in which there are at most two existential variables; this class forms a natural generalization of the classical 2-SATISFIABILITY problem. We in fact obtain the first non-trivial complexity upper bounds for both of these classes in this paper.

In addition, there are situations in which our new model can be used to derive exact complexity analyses for constraint languages under the standard model. Previous work has revealed that there are constraint languages that are coNP-hard under the standard model \[13\]. We obtain coNP-inclusion results, and hence coNP-completeness results, for some of these constraint languages in the standard model, via our new model. In particular, we obtain—in the standard model—the first non-trivial upper bounds for constraint languages having a set function polymorphism, a robust class of constraint languages that, in a precise sense, capture the arc consistency algorithm that has been studied heavily in constraint satisfaction \[13\]. These standard model coNP-inclusion results are obtained by first performing a reduction to our new model, and then by establishing a coNP inclusion in our new model.
1.3 Overview of results

As we have mentioned, the primary technical contribution of this paper is the development and application of a general technology for proving coNP-inclusion results in our new model. This technology is centered around a new notion which we call fingerprint (presented in Section 4). Intuitively, a fingerprint is a succinct representation of a conjunction of constraints. We require fingerprints to have a number of properties, and highlight two of these now. First, we require that there is a polynomial-time algorithm that, given a conjunction of constraints, computes a fingerprint that represents the constraints. Second, fingerprints must “encode sufficient information”: from a fingerprint representing a conjunction of constraints, it must be possible to construct a satisfying assignment for the conjunction (assuming that the conjunction was satisfiable in the first place). We will say that a constraint language for which there is a set of fingerprints obeying these conditions as well as some further conditions has a fingerprint scheme.

Our goal of proving coNP-inclusion results leads naturally to the idea of a proof system in which proofs certify the falsity of QCSP instances. After introducing the concept of a fingerprint scheme, we indeed present a proof system applicable to any QCSP instance over a constraint language having a fingerprint scheme (Section 5). The key feature of this proof system is that it supports polynomially succinct proofs. Using this proof system, we show that for any constraint language having a fingerprint scheme, any class of QCSP instances over the constraint language having bounded alternation is contained in coNP. (By bounded alternation, we mean that there is a constant that upper bounds the number of quantifier alternations.)

We then apply the developed technology by showing that a number of classes of constraint languages have fingerprint schemes, and hence that the described coNP-inclusion result applies to them (Section 6). Recent work on the complexity of constraint satisfaction has heavily exploited the fact that each constraint language gives rise to a set of operations called polymorphisms which are strongly tied to and can be used to study complexity by means of algebraic methods [23, 21, 8]. Correspondingly, these classes are described using polymorphisms, and are constituted of the constraint languages having the following types of polymorphisms: set functions, near-unanimity operations, and Mal’tsev operations. Moreover, for our new model, we observe a dichotomy theorem for two-element constraint languages: they are either in coNP under bounded alternation, or of the highest complexity possible for their quantifier prefix.

Lastly, we study the set function polymorphisms giving rise to constraint languages that are coNP-hard in the standard model (Section 7). We use the developed theory to observe that such constraint languages are in coNP, in the standard model, under bounded alternation. We then investigate the case of unbounded alternation, and show that in this case, such constraint languages are \( \Pi^p_2 \)-hard, and thus not in coNP, unless the polynomial hierarchy collapses. We accomplish this hardness result by showing the \( \Pi^p_2 \)-hardness of extended quantified Horn formulas, and then reducing from these formulas to the described constraint languages. Our \( \Pi^p_2 \)-hardness result on extended quantified Horn formulas gives the first non-trivial complexity lower bound proved on such formulas. Finally, we observe that—since extended quantified Horn formulas can be captured by our model—the \( \Pi^p_2 \)-hardness of these formulas implies that the bounded-alternation coNP-inclusion results we have obtained cannot, in general, be extended to the case of unbounded alternation.

One might summarize the technical contributions of this paper as follows. We make significant advances in understanding our new model in the case of bounded alternation, for which we prove a number of coNP-inclusion results; and, we establish that the case of unbounded alternation behaves in a provably different manner.

2 Preliminaries

2.1 Definitions

Throughout this paper, we use \( D \) to denote a domain, which is a nonempty set of finite size.
Definition 2.1 A relation (over $D$) is a subset of $D^k$ for some $k \geq 1$, and is said to have arity $k$. A constant relation is an arity one relation of size one. A constraint is an expression of the form $R(w_1, \ldots, w_k)$, where $R$ is an arity $k$ relation and the $w_i$ are variables. A constraint language is a set of relations, all of which are over the same domain.

An arity $k$ constraint $R(w_1, \ldots, w_k)$ is true or satisfied under an interpretation $f$ defined on the variables \{w_1, \ldots, w_k\} if $(f(w_1), \ldots, f(w_k)) \in R$.

Definition 2.2 A quantified formula is an expression of the form $\exists X_1 \forall Y_1 \exists X_2 \forall Y_2 \ldots \exists X_tC$ such that $t \geq 1$, the sets $X_1, Y_1, X_2, \ldots$ are pairwise disjoint sets of variables called quantifier blocks, and none of the sets $X_1, Y_1, X_2, \ldots$ are empty except possibly $X_1$. Each $X_i$ is called an existential block, and each $Y_i$ is called a universal block. The expression $C$ is a quantifier-free first-order formula with free variables $X_1 \cup Y_1 \cup X_2 \cup \ldots$.

We will denote the first existential block of a quantified formula $\phi$ by $X_1^\phi$, the first universal block of $\phi$ by $Y_1^\phi$, and so forth. Note that, in this paper, we will primarily consider quantified formulas that do not have any free variables. Truth of a quantified formula is defined as in first-order logic: a quantified formula is true if there exists an assignment to $X_1$ such that for all assignments to $Y_1$, there exists an assignment to $X_2$, $\ldots$ such that $C$ is true. A strategy for a quantified formula $\phi$ is a sequence of mappings $\{\sigma_x\}$ where there is a mapping $\sigma_x$ for each existentially quantified variable $x$ of $\phi$, whose range is $D$, and whose domain is the set of functions mapping from the universally quantified variables $Y_x$ preceding $x$, to $D$. We say that a strategy $\{\sigma_x\}$ for a quantified formula $\phi$ is a winning strategy if for all assignments $\tau$ to the universally quantified variables of $\phi$ to $D$, when each universally quantified variable is set according to $\tau$ and each existentially quantified variable $x$ is set according to $\sigma_x(\tau|_{Y_x})$, the $C$ part of $\phi$ is satisfied. It is well-known that a quantified formula has a winning strategy if and only if it is true.

We now define the “standard model” of quantified constraint satisfaction. In this paper, the symbol $\Gamma$ will always denote a constraint language.

Definition 2.3 The QCSP($\Gamma$) problem is to decide the truth of a quantified formula $\exists X_1 \forall Y_1 \exists X_2 \forall Y_2 \ldots \exists X_tC$ where $C$ is a conjunction of constraints, each of which has relation from $\Gamma$ and variables from $X_1 \cup Y_1 \cup X_2 \cup \ldots$.

Definition 2.4 For $t \geq 1$, the QCSP$_t$($\Gamma$) problem is the restriction of the QCSP($\Gamma$) problem to instances having $t$ or fewer non-empty quantifier blocks.

Observation 2.5 For all constraint languages $\Gamma$ and $t \geq 1$, the problem QCSP$_t$($\Gamma$) is in the complexity class $\Sigma^p_t$ if $t$ is odd, and in the complexity class $\Pi^p_t$ if $t$ is even.

The usual CSP model—the class of problems CSP($\Gamma$)—can now be easily defined.

Definition 2.6 The CSP($\Gamma$) problem is defined to be QCSP$_1$($\Gamma$).

We now formalize our new model of existentially restricted quantified constraint satisfaction. In the standard model, quantified formulas contain a conjunction of constraints; in this new model, quantified formulas contain a conjunction of extended constraints.

Definition 2.7 An extended constraint (over $D$) is an expression of the form $(y_1 = d_1) \land \ldots \land (y_m = d_m) \Rightarrow R(x_1, \ldots, x_k)$ where $m \geq 0$, each $y_i$ is a universally quantified variable, each $x_i$ is an existentially quantified variable, each $d_i$ is an element of $D$, and and $R \subseteq D^k$ is a relation.
We apply the usual semantics to extended constraints, that is, an extended constraint is true if \( R(x_1, \ldots, x_k) \) is true or there exists an \( i \) such that \( y_i \neq d_i \). Note that we permit \( m = 0 \), in which case an extended constraint is just a normal constraint having existentially quantified variables.

Our new model concerns quantified formulas with extended constraints, that is, formulas of the form

\[
\exists X_1 \forall Y_1 \exists X_2 \forall Y_2 \ldots \exists X_t C
\]

where \( C \) is the conjunction of extended constraints. However, the QCSP is typically defined as the problem of deciding such a formula where \( C \) is the conjunction of constraints. We would like to point out that any instance of the QCSP can be converted to a quantified formula with extended constraints. Let \( R(w_1, \ldots, w_k) \) be a constraint within a quantified formula, and assume for the sake of notation that \( w_1, \ldots, w_j \) are universally quantified variables and that \( w_{j+1}, \ldots, w_k \) are existentially quantified variables. The constraint \( R(w_1, \ldots, w_k) \) is semantically equivalent to the conjunction of the extended constraints

\[
(w_1 = d_1) \land \ldots \land (w_j = d_j) \Rightarrow R_{(d_1, \ldots, d_j)}(w_{j+1}, \ldots, w_k)
\]

to all tuples \((d_1, \ldots, d_j) \in D^j\), where \( R_{(d_1, \ldots, d_j)} \) denotes the relation \( \{(d_{j+1}, \ldots, d_k) : (d_1, \ldots, d_k) \in R\} \). That is, we create an extended constraint for every possible instantiation to the universally quantified variables of the constraint. With this observation, we can see that all constraints can be converted to extended constraints in an instance of the QCSP, in polynomial time. We can thus conclude that, as with the QCSP, quantified formulas with extended constraints are PSPACE-complete in general.

We now give the official definitions for our new model.

**Definition 2.8** The \( \text{QCSP}^3(\Gamma) \) problem is to decide the truth of a quantified formula \( \exists X_1 \forall Y_1 \exists X_2 \forall Y_2 \ldots \exists X_t C \) where \( C \) is a conjunction of extended constraints, each of which has relation from \( \Gamma \).

**Definition 2.9** For \( t \geq 1 \), the \( \text{QCSP}^3(\Gamma) \) problem is the restriction of the \( \text{QCSP}^3(\Gamma) \) problem to instances having \( t \) or fewer non-empty quantifier blocks.

**Observation 2.10** For all constraint languages \( \Gamma \) and \( t \geq 1 \), the problem \( \text{QCSP}^3(\Gamma) \) is in the complexity class \( \Sigma^p_t \) if \( t \) is odd, and in the complexity class \( \Pi^p_t \) if \( t \) is even.

**Observation 2.11** For all constraint languages \( \Gamma \), the problem \( \text{QCSP}^3(\Gamma) \) is equivalent to the problem CSP(\( \Gamma \)).

The reader may ask why we did not define \( \text{QCSP}^3(\Gamma) \) in terms of quantified formulas with constraints: we could have defined \( \text{QCSP}^3(\Gamma) \) as the problem of deciding those quantified formulas \( \phi \) with constraints such that after the above conversion process is applied to \( \phi \) to obtain a formula \( \phi' \) with extended constraints, each of the resulting extended constraints in \( \phi' \) has relation from \( \Gamma \). The main reason that we chose the given definition is that we are most interested in positive results, and any positive result concerning the model as we have defined it implies a positive result on the alternative definition; this is because converting constraints to extended constraints in a quantified formula can be carried out in polynomial time, as noted above. We also believe that the definition of our model is very robust, and cite the connections with the standard model developed in Section 7 as evidence for this.

In this paper, we will use reductions to study the complexity of the problems we have defined. We say that a problem reduces to another problem if there is a many-one polynomial-time reduction from the first problem to the second. We say that a class of quantified formulas uniformly reduces to another class of quantified formulas if the first class reduces to the second via a reduction that does not increase the number of non-empty quantifier blocks. Hence, if for instance \( \text{QCSP}^3(\Gamma_1) \) uniformly reduces to \( \text{QCSP}^3(\Gamma_2) \), then for all \( t \geq 1 \), \( \text{QCSP}^3_t(\Gamma_1) \) many-one polynomial-time reduces to \( \text{QCSP}^3_t(\Gamma_2) \).
2.2 Polymorphisms

We now indicate how the algebraic, polymorphism-based approach that has been used to study CSP(Γ) and QCSP(Γ) complexity can be used to study QCSP^3(Γ) complexity. We adapt this algebraic approach in a straightforward way, and refer the reader to [23][21] for more information on this approach.

The first point we wish to highlight is that, up to some mild assumptions, the set of relations expressible by a constraint language Γ characterizes the complexity of QCSP^3(Γ).

**Definition 2.12** (see [27] for details) When Γ is a constraint language over D, define ⟨Γ⟩, the set of relations expressible by Γ, to be the smallest set of relations containing Γ ∪ {=} and closed under permutation, extension, truncation, and intersection. (Here, =_D denotes the equality relation on D.)

**Proposition 2.13** Let Γ_1, Γ_2 be constraint languages (over D) where Γ_1 is finite and Γ_2 contains =_D. If ⟨Γ_1⟩ ⊆ ⟨Γ_2⟩, then QCSP^3(Γ_1) uniformly reduces to QCSP^3(Γ_2).\(^1\) (Intuitively, the more relations that a constraint language Γ can express, the higher in complexity it is.)

**Proof.** If ⟨Γ_1⟩ ⊆ ⟨Γ_2⟩ and Γ_2 contains =_D, then every constraint over Γ_1 is equivalent to a formula consisting of existentially quantified variables and a conjunction of constraints over Γ_2 (see for instance the discussion in [9]). Let φ be an instance of QCSP^3(Γ_1). We create an instance of QCSP^3(Γ_2) from φ as follows. For each extended constraint

\[(y_1 = d_1) \land \ldots \land (y_m = d_m) \Rightarrow R(x_1, \ldots, x_k)\]

in φ, let

\[\exists w_1 \ldots \exists w_n(T_1(v_1, \ldots, v_{k_1}) \land \ldots \land T_p(v_1, \ldots, v_{k_p}))\]

be a formula that is equivalent to R(x_1, \ldots, x_k) and where the T_i are contained in Γ_2. Replace the extended constraint

\[(y_1 = d_1) \land \ldots \land (y_m = d_m) \Rightarrow R(x_1, \ldots, x_k)\]

by the p extended constraints of the form

\[(y_1 = d_1) \land \ldots \land (y_m = d_m) \Rightarrow T_i(v_1, \ldots, v_{k_i})\]

and add the variables \{w_1, \ldots, w_n\} to the innermost block of existentially quantified variables. □

From Proposition 2.13 we can see that when investigating finite constraint languages containing the equality relation, any two such constraint languages expressing exactly the same relations are uniformly reducible to one another, and hence of the same complexity in both the QCSP^3(Γ) and QCSP^3(Γ) frameworks.

**Definition 2.14** An operation µ : D^k → D is a polymorphism of a relation R ⊆ D^n if for any choice of k tuples

\[(t^1_1, \ldots, t^1_m), \ldots, (t^k_1, \ldots, t^k_m) \in R,\]

the tuple \((µ(t^1_1, \ldots, t^k_1)), \ldots, (µ(t^1_m, \ldots, t^k_m))\) is in R. When an operation µ is a polymorphism of a relation R, we also say that R is invariant under µ. An operation µ is a polymorphism of a constraint language Γ if µ is a polymorphism of all relations R ∈ Γ.

\(^1\)We remark that this proposition is not true if one removes the assumption that Γ_2 contains =_D, assuming that P does not equal NP. Let |D| > 1, let Γ_2 be the set of all constant relations, and let Γ_1 be equal to Γ_2 ∪ {=} then, we have that ⟨Γ_1⟩ ⊆ ⟨Γ_2⟩, QCSP^3(Γ_1) is coNP-hard (see Example 3.3), and QCSP^3(Γ_2) is in P.
We will be interested in the set of all polymorphisms of a constraint language $\Gamma$, as well as the set of all relations invariant under all operations in a given set.

**Definition 2.15** Let $O_D$ denote the set of all finitary operations on $D$, and let $R_D$ denote the set of all finite arity relations on $D$. When $\Gamma \subseteq R_D$ is a constraint language, we define

$$\text{Pol}(\Gamma) = \{\mu \in O_D \mid \mu \text{ is a polymorphism of } \Gamma\}.$$  

When $F$ is a set of operations over $D$, we define

$$\text{Inv}(F) = \{R \in R_D \mid R \text{ is invariant under all } \mu \in F\}.$$  

It is known that the expressive power of a constraint language is determined by its polymorphisms, that is, $(\Gamma) = \text{Inv}(\text{Pol}(\Gamma))$. Consequently, the complexity of a constraint language is determined by its polymorphisms, and we have the following analog of Proposition 2.13.

**Proposition 2.16** Let $\Gamma_1, \Gamma_2$ be constraint languages (over $D$) where $\Gamma_1$ is finite and $\Gamma_2$ contains $=_D$. If $\text{Pol}(\Gamma_2) \subseteq \text{Pol}(\Gamma_1)$, then $\text{QCSP}^3(\Gamma_1)$ uniformly reduces to $\text{QCSP}^3(\Gamma_2)$.

In light of the above discussion, it makes sense to directly define $\text{QCSP}^3(F)$ for a set of operations $F$: we define $\text{QCSP}^3(F)$ as the problem $\text{QCSP}^3(\text{Inv}(F))$, and we define $\text{QCSP}_{\ell}^3(F)$ analogously.

From Proposition 2.16 it can be seen that constraint languages having “many” polymorphisms are easier than those having “fewer”. Correspondingly, many of the results on CSP($\Gamma$) complexity show that the presence of a certain type of polymorphism implies polynomial-time decidability. The positive complexity results in this paper will also have this form, that is, we will prove results showing that if a constraint language $\Gamma$ has a polymorphism of a certain type, then $\text{QCSP}^3_{\ell}(\Gamma)$ is in $\text{coNP}$.

### 2.3 Relational structures

It has been observed [19] that the CSP can be formulated as the homomorphism problem of deciding, given a pair $(A, B)$ of relational structures, whether or not there is a homomorphism from $A$ to $B$. We will make use of relational structures in this paper, and introduce them here. A **vocabulary** $\sigma$ is a collection of **relation symbols**, each of which has an associated arity. A **relational structure** $A$ (over vocabulary $\sigma$) consists of a **universe** $A$, which is a set of size greater than or equal to one, and a relation $R^A \subseteq A^k$ for each relation symbol $R$ of $\sigma$, where $k$ is the arity associated to $R$. In this paper, we only consider relational structures having finite-size universes. When $A$ and $B$ are relational structures over the same vocabulary $\sigma$, a **homomorphism** from $A$ to $B$ is a mapping $h$ from the universe of $A$ to the universe of $B$ such that for every relation symbol $R$ of $\sigma$ and every tuple $(a_1, \ldots, a_k) \in R^A$, it holds that $(h(a_1), \ldots, h(a_k)) \in R^B$. Two relational structures $A$ and $B$ are **homomorphically equivalent** if there is a homomorphism from $A$ to $B$ and a homomorphism from $B$ to $A$.

We say that a constraint language $\Gamma$ corresponds to a relational structure $B$ (over $\sigma$) if each relation $S$ of $\Gamma$ can be put in a one-to-one correspondence with a relation symbol $R$ of $\sigma$ so that $S = R^B$. We define $\text{QCSP}^3(B)$ as the problem $\text{QCSP}^3(\Gamma)$ for the constraint language $\Gamma$ corresponding to $B$; and, we use $B^\Gamma$ to denote a relational structure corresponding to a constraint language $\Gamma$. We can translate a conjunction of constraints $C$ over $\Gamma$ to the instance $(A, B^\Gamma)$ of the homomorphism problem where the universe of $A$ contains all variables occurring in $C$ and, for each relation symbol $R$, the relation $R^A$ contains all tuples $(a_1, \ldots, a_k)$ such that the constraint $R^{B^\Gamma}(a_1, \ldots, a_k)$ appears in $C$.

For a relational structure $B$ with universe $B$, the relational structure $\varphi(B)$ is defined as follows. The universe of $\varphi(B)$ is $\varphi(B)$, where $\varphi(B)$ denotes the power set of $B$ excluding the empty set. For every relation symbol $R$ of arity $k$ and non-empty subset $S \subseteq R^B$, the relation $R^{\varphi(B)}$ contains the tuple $(S_1, \ldots, S_k)$ where $S_i = \{b_i : (b_1, \ldots, b_k) \in S\}$.
3 Complexity

This section demonstrates some basic complexity properties of our new model QCSP$^3(\Gamma)$. First, we show that under the very mild assumption of criticality, a constraint language is coNP-hard in this model.

**Definition 3.1** A constraint language $\Gamma$ is critical if there is an algorithm that, given a positive integer $n \geq 2$, outputs in time polynomial in $n$, sets of constraints $C_1, \ldots, C_n$ over $\Gamma$ such that $\bigcup_{i \in \{1, \ldots, n\}} C_i$ is unsatisfiable, but for any $j \in \{1, \ldots, n\}$, $\bigcup_{i \in \{1, \ldots, n\}\setminus\{j\}} C_n$ is satisfiable.

**Proposition 3.2** If $\Gamma$ is a critical constraint language, then for all $t \geq 2$, the problem QCSP$^3_t(\Gamma)$ is coNP-hard. In particular, the $\forall \exists$ formulas of QCSP$^3(\Gamma)$ are coNP-hard.

**Proof.** We reduce from the complement of the 3-SAT problem. Take an instance $\phi$ of the 3-SAT problem with variables $Y$ and clauses $C_1, \ldots, C_n$. Compute sets of constraints $C_1, \ldots, C_n$ with the property given in the definition of critical constraint language, and let $X$ denote the variables occurring in the constraints $C_i$.

We create an instance $\phi'$ of QCSP$^3(\Gamma)$ with quantifier prefix $\forall Y \exists X$. The extended constraints in $\phi'$ are created as follows. Fix two distinct elements $a_0, a_1$ of the domain of $\Gamma$. For every clause $C_i$, for every constraint $R(x_1, \ldots, x_k)$ in $C_i$, and for every variable $v$ occurring in $C_i$, create an extended constraint

$$(v = d) \Rightarrow R(x_1, \ldots, x_k)$$

where $d$ is equal to $a_0$ if $v$ is negated in $C_i$, and equal to $a_1$ otherwise.

Observe that if $f : Y \to \{0, 1\}$ is an assignment satisfying the instance $\phi$ of 3-SAT, then the created extended constraints are false under the assignment $g : Y \to \{a_0, a_1\}$ defined by $g(y) = a_{f(y)}$, and so the created instance $\phi'$ is false. Conversely, if the instance $\phi'$ is false, then the created extended constraints must be false under some assignment $g : Y \to \{a_0, a_1\}$; in this case, it can be verified that the assignment $f : Y \to \{0, 1\}$ defined by $g(y) = a_{f(y)}$ satisfies all clauses of $\phi$.

**Example 3.3** Suppose $\Gamma$ is a constraint language containing the equality relation $=D$ and (at least) two different constant relations $R_a, R_b$. The constraint language $\Gamma$ is critical: for any $n \geq 2$, the sets of constraints $\{=D(v_i, v_{i+1})\}$, $\{R_a(v_n)\}$, and $\{=D(v_i, v_{i+1})\}$ for $i \in \{2, \ldots, n-1\}$ has the desired property, and can easily be generated in polynomial time.

Although our QCSP$^3(\Gamma)$ model is in general coNP-hard, we can use it to obtain positive complexity results for QCSP instances for which no complexity result can be derived in the standard model, other than the trivial PSPACE upper bound. We discuss this phenomenon in the following examples.

**Example 3.4** Let us consider extended quantified 2-SAT formulas, which we define to be instances of the boolean QCSP where each constraint must be a clause in which there are at most two occurrences of existential variables. We call such a constraint an extended 2-clause. Recall that a clause is a disjunction of literals, where a literal is a variable or its negation. The following are examples of extended 2-clauses:

$$\overline{y_1} \lor y_4 \lor \overline{x_1} \lor \overline{x_2}$$

$$x_1 \lor y_2 \lor \overline{x_3} \lor \overline{y_5} \lor y_8$$

Here, the $y_i$ denote universal variables, and the $x_i$ denote existential variables. For any tuple $(a_1, \ldots, a_k) \in \{0, 1\}^k$, let $R_{(a_1, \ldots, a_k)}$ denote the relation $\{0, 1\}^k \setminus \{(a_1, \ldots, a_k)\}$. Notice that each extended 2-clause is equivalent to a constraint of the form $R_{(a_1, \ldots, a_k)}(v_1, \ldots, v_k)$. For example, the two given clauses are equivalent to the constraints:

$$R_{(1,0,1,1)}(y_1, y_4, x_1, x_2)$$
If we are to model extended quantified 2-SAT formulas using the standard model QCSP(Γ), we take as our constraint language the set of all relations that can appear in constraints, that is, the set of all relations \( R(a_1, \ldots, a_k) \). This constraint language is easily seen to give rise to a PSPACE-complete case of the QCSP, under the standard model: it can directly encode QUANTIFIED 3-SAT.

On the other hand, we can model extended quantified 2-SAT formulas under our new model as the problem QCSP\(^3\)(Γ)\(t\), where Γ\(t\) is the constraint language \{ \( R(0) \), \( R(1) \), \( R(0,0) \), \( R(0,1) \), \( R(1,1) \) \}. For example, the two given clauses are equivalent to the extended constraints:

\[
(y_1 = 1) \land (y_4 = 0) \Rightarrow R(1,1)(x_1, x_2) \\
(y_2 = 0) \land (y_5 = 1) \land (y_8 = 0) \Rightarrow R(0,1)(x_1, x_3)
\]

Let \( m : \{0,1\}^3 \rightarrow \{0,1\} \) be the majority operation on \{0,1\}, that is, the symmetric operation that returns 0 if two or three of its arguments are equal to 0, and 1 if two or three of its arguments are equal to 1. It can be verified that the constraint language Γ\(t\) has \( m \) as a polymorphism. The operation \( m \) is an example of a near-unanimity operation; later in this paper (Theorem 6.3), we show that for any constraint language Γ having a near-unanimity polymorphism, the problem QCSP\(^3\)(Γ) is in \( \text{coNP} \) (for all \( t \geq 2 \)). Hence, we have in particular that QCSP\(^3\)(Γ\(t\)) is in \( \text{coNP} \) (for all \( t \geq 2 \)), that is, extended quantified 2-SAT formulas are in \( \text{coNP} \), under bounded alternation.

**Example 3.5** Extended quantified Horn formulas were introduced by Kleine Büning et al. [10]. An extended quantified Horn formula is an instance of the boolean QCSP where every constraint is an extended Horn clause, that is, a clause in which there is at most one positive literal of an existential variable. In other words, an extended Horn clause is a clause where removing all literals of universal variables results in a Horn clause.

Let \( H \subseteq \{0,1\}^3 \) be the relation \( \{0,1\}^3 \setminus \{(1,1,0)\} \), and let Γ\(H\) be the constraint language \{ \( H \), \( R_0 \), \( R_1 \) \}, where \( R_0 \) and \( R_1 \) are defined as the constant relations \{ \{0\} \} and \{ \{1\} \}, respectively. A given extended quantified Horn formula can easily be translated (in polynomial time) into an instance of the problem QCSP\(^3\)(Γ\(H\)). In particular, any extended Horn clause can be translated into an existentially quantified conjunction of extended constraints over Γ\(H\), and so the idea of the proof of Theorem 2.4 can be applied. For example, consider the following two extended Horn clauses:

\[
x_1 \lor \overline{x_2} \lor y_1 \lor \overline{y_2} \lor \overline{x_3} \lor x_4 \\
\overline{y_1} \lor x_1 \lor x_2
\]

They are equivalent to the following existentially quantified formulas over Γ\(H\):

\[
\exists v[(y_1 = 0) \land (y_2 = 1) \Rightarrow H(x_2, x_3, v)] \\
[(y_1 = 0) \land (y_2 = 1) \Rightarrow H(v, x_4, x_1)] \\
\exists v[(R_1(v)) \land ((y_1 = 1) \Rightarrow H(v, x_1, x_2))]
\]

The boolean AND operation is a polymorphism of the constraint language Γ\(H\), and is an example of a semilattice operation; later in the paper (Corollary 6.2), we show that for any constraint language Γ having a semilattice polymorphism, the problem QCSP\(^3\)(Γ) is in \( \text{coNP} \) (for all \( t \geq 2 \)). This implies that extended quantified Horn formulas are in \( \text{coNP} \), under bounded alternation; this is the first non-trivial complexity upper bound on this class of formulas.

We observe that our model is at least as hard as the standard model, with respect to constraint languages containing all constants.
**Proposition 3.6** If $\Gamma$ be a constraint language containing all constant relations, then QCSP($\Gamma$) uniformly reduces to QCSP$^3(\Gamma)$. 

**Proof.** Given an instance $\phi$ of QCSP($\Gamma$), we create an instance $\phi'$ of QCSP$^3(\Gamma)$ as follows. The quantifier prefix of $\phi'$ is equal to that of $\phi$, except there are $|D|$ extra existentially quantified variables (which may be existentially quantified anywhere), denoted by $\{v_d : d \in D\}$. The instance $\phi'$ contains the extended constraints $\{R_d(v_d) : d \in D\}$, where $R_d$ denotes the constant relation $\{(d)\}$; these extended constraints “force” each variable $v_d$ to take on the value $d$. For each constraint $C$ of $\phi$, we also create extended constraints in $\phi'$, as follows. For the sake of notation, let us denote $C$ by $R(y_1, \ldots, y_m, x_1, \ldots, x_n)$ where the $y_i$ are universally quantified and the $x_i$ are existentially quantified. For each tuple $(d_1, \ldots, d_m) \in D^m$, create an extended constraint

$$(y_1 = d_1) \land \ldots \land (y_m = d_m) \Rightarrow R(v_{d_1}, \ldots, v_{d_m}, x_1, \ldots, x_n).$$

We now turn to look at some algebraic properties of our new model.

**Theorem 3.7** If $B$, $B'$ are homomorphically equivalent relational structures (over a finite vocabulary), then QCSP$^3(B)$ and QCSP$^3(B')$ uniformly reduce to each other.

**Proof.** First, suppose that $C$ and $C'$ are relational structures such that $C$ has universe $\{c\}$ of size one, and there is a homomorphism $h$ from $C$ to $C'$. Then, every relation of $C'$ is either empty or contains the tuple $(h(c), \ldots, h(c))$ with all coordinates equal to $h(c)$. It follows that QCSP($C'$) can be easily decided in polynomial time: an instance is true as long as there is no extended constraint $(y_1 = d_1) \land \ldots \land (y_m = d_m) \Rightarrow R(x_1, \ldots, x_k)$ such that $R$ is empty and there is an assignment mapping $y_i$ to $d_i$ for all $i$. (Likewise, every relation of $C$ is either empty or contains the tuple $(c, \ldots, c)$ with all coordinates equal to $c$; and QCSP($C$) can be easily decided in polynomial time.) We thus assume that both $B$ and $B'$ has universe of size strictly greater than one.

We show how to reduce from QCSP$^3(B)$ to QCSP$^3(B')$. Let $h$ be a homomorphism from $B$ to $B'$, let $h'$ be a homomorphism from $B'$ to $B$, and let $\phi$ be an arbitrary instance of QCSP$^3(B)$. We create an instance of QCSP$^3(B')$ as follows. First, let $\phi'$ be the quantified formula obtained from $\phi$ by replacing each relation $R^B$ with $R^{B'}$. Letting it be understood that the universally quantified variables of $\phi'$ are still quantified over the universe $B$ of $B$, it can be verified that if $\{\sigma_x\}$ is a winning strategy for $\phi$, then $\{h\sigma_x\}$ is a winning strategy for $\phi'$; and likewise, that if $\{\sigma'_x\}$ is a winning strategy for $\phi'$, then $\{h'\sigma'_x\}$ is a winning strategy for $\phi$. Now, it remains to modify $\phi'$ so that the universally quantified variables are quantified over the universe $B'$ of $B'$. If $|B'| \geq |B|$, it suffices to take any injective mapping $i : B \rightarrow B'$ and simply replace all instances of $b$ in $\phi'$ by $i(b)$. If $|B'| < |B|$, then each universally quantified variable over $B$ can be simulated by $s$ universally quantified variables over $B'$, where $s$ is a sufficiently large constant so that $|B'|^s \geq |B|$. Notice that such a constant $s$ exists since $|B'| \geq 2$. □

When $A$ is an algebra with operations $F$, we define QCSP$^3(A)$ as QCSP$^3(F)$.

**Theorem 3.8** Let $A$ be a finite algebra.

- If $B$ is a subalgebra of $A$, then QCSP$^3(B)$ uniformly reduces to QCSP$^3(A)$.
- If $B$ is a homomorphic image of $A$, then QCSP$^3(B)$ uniformly reduces to QCSP$^3(A)$.

**Proof.** The proof of this theorem follows proofs of similar results in [8]. If $B$ is a subalgebra of $A$, then QCSP$^3(B)$ reduces to QCSP$^3(A)$ by the identity mapping. Now let $B$ be a homomorphic image of $A$,
and let \( f \) be a surjective homomorphism from \( A \) to \( B \). Let \( \phi \) be an instance of QCSP(\( B \)), and let \( B_\phi \) the relational structure whose relations are exactly the relations occurring in \( \phi \). Define a relational structure \( B_A \) over the same vocabulary as \( B_\phi \) where for each relation symbol \( R \), the relation \( R^{B_A} \) is defined as \( \{(a_1, \ldots, a_k) : (f(a_1), \ldots, f(a_k)) \in R^{B_\phi}\} \). All relations \( R^{B_A} \) are invariant under the operations of \( A \). Let \( f' \) be any mapping from the universe \( B \) of \( B \) to the universe \( A \) of \( A \) such that \( f'(b) \in f^{-1}(b) \). Then, \( f \) is a homomorphism from \( B_A \) to \( B_\phi \), \( f' \) is a homomorphism from \( B_\phi \) to \( B_A \), and the reduction of Theorem 3.7 can be employed. □

4 Fingerprints

This section introduces the notion of a fingerprint along with various associated notions. In this section and the next, we are concerned primarily with our new existentially restricted model of quantified constraint satisfaction, and so we assume that all quantified formulas under discussion contain extended constraints.

We define a projection operator \( pr_k \) which projects “onto the first \( k \) coordinates”: formally, when \( R \) is a relation of arity \( n \) and \( k \in \{0, \ldots, n\} \), we define \( pr_k R = \{(d_1, \ldots, d_k) : (d_1, \ldots, d_n) \in R\} \). At this point, we adopt the convention that there is a unique tuple of arity \( 0 \), and hence a unique non-empty relation of arity \( 0 \). When a non-empty (empty) relation is projected down to an arity \( 0 \) relation, we consider the result to be the unique non-empty (respectively, empty) relation of arity \( 0 \).

Definition 4.1 A fingerprint collection is a set \( F \) with an associated domain \( D \) whose elements are called fingerprints. Each fingerprint \( F \in F \) has an associated arity, denoted by \( \text{arity}(F) \), and specifies a relation \( R(F) \subseteq D^{\text{arity}(F)} \). We require that there is a fingerprint \( \top \in F \) such that \( R(F) \) is the non-empty relation of arity \( 0 \). We require that there is a projection function \( \pi \) such that

- given any fingerprint \( F \in F \) and \( k \in \{0, \ldots, \text{arity}(F)\} \), outputs a fingerprint \( \pi_k F \) such that \( R(\pi_k F) = \text{pr}_k R(F) \), and
- when \( 0 \leq k \leq l \leq n \) and \( F \in F \) is of arity \( n \), it holds that \( \pi_k F = \pi_k(\pi_l F) \).

Also, we require that there is a preorder \( \sqsubseteq \) such that

- \( F \sqsubseteq F' \) implies \( \text{arity}(F) = \text{arity}(F') \) and \( R(F) \subseteq R(F') \),
- if \( F \sqsubseteq F' \) and \( k \in \{0, \ldots, \text{arity}(F)\} \), then \( \pi_k F \sqsubseteq \pi_k F' \), and
- with respect to \( \sqsubseteq \), chains are of polynomial length; that is, there exists a polynomial \( p \) such that if \( F_1, \ldots, F_m \in F \) are distinct fingerprints of arity \( n \) and \( F_1 \sqsubseteq \cdots \sqsubseteq F_m \), then \( m \leq p(n) \).

Finally, we require that each fingerprint \( F \in F \) has a representation with size polynomial in its arity, that is, there is a function \( r : F \to \{0,1\}^* \) and a polynomial \( q \) such that \( |r(F)| \leq q(\text{arity}(F)) \) for all \( F \in F \).

For ease of notation we will always denote the representation \( r(F) \) of a fingerprint \( F \) simply by \( F \), although technically all of the algorithms that we will discuss manipulate representations of fingerprints.

Example 4.2 A simple example of a fingerprint collection is as follows. Let \( D \) be any element of \( D \). Define \( F_d = \{\top_0, \bot_1, \top_1, \bot_1, \ldots\} \) where each \( \bot_i \) has arity \( i \) and has \( R(\bot_i) = \emptyset \), and each \( \top_i \) has arity \( i \) and is non-empty with \( R(\top_i) \) containing the unique tuple of arity \( i \) equal to \( d \) at all coordinates, that is, \((d, d, \ldots, d)\). We can define the projection function \( \pi \) by \( \pi_k \bot_n = \bot_k \) and \( \pi_k \top_n = \top_k \), for \( n \geq k \). We define the preorder \( \sqsubseteq \) by \( \bot_i \sqsubseteq \top_i \). Chains are clearly of length at most two. The fingerprint collection \( F_d \) clearly admits a polynomial size representation, as each fingerprint \( F \in F_d \) can simply be encoded by its arity along with a bit denoting whether it is of type \( \top \) or \( \bot \).
Example 4.3  A perhaps more interesting example of a fingerprint collection is as follows. Fix a domain $D$ and let $\mathcal{F}_{\psi(D)}$ contain all tuples $(D_1, \ldots, D_n)$ where each $D_i$ is a non-empty subset of $D$. We define $\mathcal{R}((D_1, \ldots, D_n))$ to be the set of all $n$-tuples $(d_1, \ldots, d_n)$ such that $d_i \in D_i$ for all $i \in \{1, \ldots, n\}$. Let $\mathcal{F}_{\psi(D)}$ also contain elements $\bot_0, \bot_1, \ldots$ where $\bot_i$ is of arity $i$ with $\mathcal{R}(\bot_i) = \emptyset$. We can define the projection function $\pi$ by $\pi_k(D_1, \ldots, D_n) = (D_1, \ldots, D_k)$ and $\pi_\bot = \bot_k$ for $n \geq k$.

We define the preorder $\subseteq$ by the following rule: $(D_1, \ldots, D_n) \subseteq (D'_1, \ldots, D'_n)$ if and only if $D_i \subseteq D'_i$ for all $i \in \{1, \ldots, n\}$. Notice that if $F_1, \ldots, F_m \in \mathcal{F}_{\psi(D)}$ are distinct fingerprints of arity $n$ and $F_1 \subseteq \cdots \subseteq F_m$, then $m \leq n|D|$, and so chains are of linear length. (Recall that all domains $D$ in this paper are finite).

It is straightforward to give a polynomial (in fact, linear) size representation for the fingerprints of $\mathcal{F}_{\psi(D)}$, as there are a constant number of subsets of $D$.

Definition 4.4  A fingerprint collection is a fingerprint $F$ paired with a tuple of variables $\langle x_1, \ldots, x_k \rangle$ of length $k = \text{arity}(F)$, and is denoted by $F(x_1, \ldots, x_k)$. It is considered to be true under an assignment $f$ defined on $\{x_1, \ldots, x_k\}$ if $(f(x_1), \ldots, f(x_k)) \in \mathcal{R}(F)$.

While a fingerprint application is similar to a constraint, we use the “angle brackets” notation for fingerprint applications to differentiate between the two.

Definition 4.5  A fingerprint scheme for a constraint language $\Gamma$ consists of:

- A fingerprint collection $\mathcal{F}$ over the domain $D$ of $\Gamma$.
- A projection algorithm running in polynomial time that, given a fingerprint $F \in \mathcal{F}$ of arity $n$ and an integer $k \in \{0, \ldots, n\}$, computes the fingerprint $\pi_k F$.
- An inference algorithm $\text{Inf}$ running in polynomial time that, given
  - a fingerprint application $F(x_1, \ldots, x_k)$, and
  - a conjunction of constraints $C$ over $\Gamma$ and variables $\{x_1, \ldots, x_n\}$, where $n \geq k$,
computes a fingerprint $F'$ of arity $n$ where

1. (soundness) for all assignments $f : \{x_1, \ldots, x_n\} \rightarrow D$, if both $F(x_1, \ldots, x_k)$ and $C$ are true under $f$, then $F'(x_1, \ldots, x_n)$ is also true under $f$, and
2. (progress) it holds that $\pi_k F' \subseteq F$.

Under the two given assumptions, we say that $F'(x_1, \ldots, x_n)$ is a fingerprint application suitable for $C$ if there exists a fingerprint application $F(x_1, \ldots, x_k)$ such that $\text{Inf}(F(x_1, \ldots, x_k), C) = F'$.

- A construction mapping $\text{Cons} : \mathcal{F} \rightarrow D$ such that when $F(x_1, \ldots, x_n)$ is a fingerprint application suitable for a conjunction of constraints $C$ (over $\Gamma$ and variables $\{x_1, \ldots, x_n\}$ with $n \geq 1$) and $\mathcal{R}(F) \neq \emptyset$, the mapping taking $x_i$ to $\text{Cons}(\pi_i F)$, for all $i \in \{1, \ldots, n\}$, satisfies $C$.

Example 4.6  Let $D$ be any domain and $d \in D$ be an element of $D$. Suppose that $\Gamma$ is a constraint language over $D$ that is $d$-valid in that every non-empty relation $R \in \Gamma$ contains the all-$d$ tuple $(d, \ldots, d)$ having the arity of $R$. We demonstrate that there is a fingerprint scheme for $\Gamma$. The fingerprint collection is $\mathcal{F}_d$, defined in Example 4.2. It is clear that projections can be computed in polynomial time. The inference algorithm, given $F(x_1, \ldots, x_k)$ and a conjunction of constraints $C$ over $\Gamma$ and variables $\{x_1, \ldots, x_n\}$, outputs $\bot_n$ if $F = \bot_k$ or $C$ contains a constraint with empty relation, and outputs $\top_n$ otherwise. It is straightforward to verify that
this inference algorithm obeys the soundness and progress conditions. The construction mapping for our fingerprint scheme simply always outputs $d$. This mapping satisfies the requirement for a construction mapping: if $F$ is a fingerprint with $\mathcal{R}(F) \neq \emptyset$, then $F = \bigcap_n (\text{where } n \text{ is the arity of } F)$; when $\bigcap_n \langle x_1, \ldots, x_n \rangle$ is a fingerprint application suitable for constraints $C$ over $\Gamma$ and $\{x_1, \ldots, x_n\}$, no constraint in $C$ may have empty relation, so every constraint in $C$ is $d$-valid, and thus the assignment mapping every $x_i$ to $d$ satisfies $C$.

Example 4.7 We continue Example 4.3 by giving a fingerprint scheme using the fingerprint collection defined there. This example makes use of ideas concerning set functions in the context of constraint satisfaction as well as the notion of arc consistency; for more information, we refer the readers to the papers [13,14]. We consider a set function to be a mapping $f : \varphi(D) \rightarrow D$, where $\varphi(D)$ denotes the power set of $D$ excluding the empty set. We say that $f$ is a polymorphism of a constraint language $\Gamma$ if all of the functions $f_i : D^i \rightarrow D$ defined by $f_i(x_1, \ldots, x_i) = f(\{x_1, \ldots, x_i\})$, for $i \geq 1$, are polymorphisms of $\Gamma$. Equivalently, $f$ is a polymorphism of $\Gamma$ if $f$ is a homomorphism from $\varphi(B^\Gamma)$ to $B^\Gamma$.

Let $\Gamma$ be a constraint language over domain $D$ having a set function $f : \varphi(D) \rightarrow D$ as polymorphism. We demonstrate a fingerprint scheme for $\Gamma$. The fingerprint collection is $F_{\varphi(D)}$, from Example 4.3. Projections of fingerprints can clearly be computed in polynomial time. The inference algorithm is arc consistency. More specifically, the inference algorithm takes as input a fingerprint application $(D_1, \ldots, D_k)\langle x_1, \ldots, x_k \rangle$ and a conjunction of constraints $(A, B^\Gamma)$ over $\Gamma$ and variables $\{x_1, \ldots, x_n\}$. The algorithm tries to establish arc consistency on $(A, B^\Gamma)$ with additional constraints stating that $x_i \in D_i$ for all $i \in \{1, \ldots, k\}$. The result is either a homomorphism $g : A \rightarrow \varphi(B^\Gamma)$ with $g(x_i) \subseteq D_i$ for all $i \in \{1, \ldots, n\}$ such that the fingerprint $(g(x_1), \ldots, g(x_n))$ satisfies the soundness and progress requirements; or, certification that arc consistency cannot be established, in which case the fingerprint $(\emptyset, \ldots, \emptyset)$ satisfies the soundness and progress requirements.

The construction mapping is defined by $\text{Cons}((D_1, \ldots, D_n)) = f(D_n)$. To see that this algorithm satisfies the given criterion, assume that $(D_1, \ldots, D_n)\langle x_1, \ldots, x_n \rangle$ is a fingerprint application suitable for a conjunction of constraints $(A, B^\Gamma)$. The mapping $h : \{x_1, \ldots, x_n\} \rightarrow D$ given by the construction mapping obeys $h(x_i) = f(D_i)$. By assumption, the mapping taking $x_i$ to $D_i$ is a homomorphism $g : A \rightarrow \varphi(B^\Gamma)$. Observe that $h$ is the composition of $g : A \rightarrow \varphi(B^\Gamma)$ and $f : \varphi(B^\Gamma) \rightarrow B^\Gamma$, and so $h$ is a homomorphism from $A$ to $B^\Gamma$, as desired.

5 Proof System

This section presents the proof system that will permit us to derive coNP-inclusion results for constraint languages having fingerprint schemes. The proof system gives rules for deriving, from a quantified formula $\phi$ and a fingerprint application for $\phi$, further fingerprint applications for $\phi$.

Before giving the proof system, we require some notation. We assume that, for every quantified formula $\phi = \exists X_1 \forall Y_1 \exists X_2 \forall Y_2 \cdots \exists X_k \mathcal{C}$, there is an associated total order $\leq^\phi$ on the existential variables $\bigcup_{i=1}^k X_i$ that respects the quantifier prefix in that $x \leq^\phi x'$ if $x \in X_i, x' \in X_j$, and $i < j$. We say that a tuple of variables $\langle x_1, \ldots, x_k \rangle$ is a prefix of $\phi$ if it is an “initial segment” of the existential variables of $\phi$ under the $\leq^\phi$ ordering, that is, all $x_i$ are existentially quantified in $\phi$, $x_i \leq^\phi x_{i+1}$ for all $i \in \{1, \ldots, k - 1\}$, and $x \leq^\phi x_j$ implies that $x = x_i$ for some $i \leq j$. When $\langle x_1, \ldots, x_k \rangle$ is a prefix of a quantified formula $\phi$ that is understood from the context and $X = \{x_1, \ldots, x_k\}$, we use the “set notation” $\langle X \rangle$ to denote $\langle x_1, \ldots, x_k \rangle$.

We now give the proof system, which consists of rules for deriving expressions of the form $\phi, F(\langle X \rangle) \vdash F'(\langle X' \rangle)$. In such expressions, $\phi$ is always a quantified formula, $F(\langle X \rangle)$ and $F'(\langle X' \rangle)$ are fingerprint applications, and it is always assumed that $X \subseteq X_1^\phi$. Moreover, note that if $\phi, F(\langle X \rangle) \vdash F'(\langle X' \rangle)$, it will always hold that $X' = X_1^\phi$. 

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Definition 5.1 The proof system for a fingerprint scheme with fingerprint collection $F$ and inference algorithm $Inf$ consists of the following three rules.

\[
\begin{align*}
&\exists X' C, F(X) \vdash \text{Inf}(F(X), C)(X') \\
&\phi, F(X) \vdash F'(X') \quad \phi, F'(X') \vdash F''(X'') \\
&\phi[g], F(X) \vdash F''(X') \quad X' \supseteq X_1^g \\
&\phi_1, F(X) \vdash (\pi_{X_1^g}F')(X_1^0)
\end{align*}
\]

The last rule is applicable when $\phi$ has more than one existential block, and $g : Y_1^0 \rightarrow D$ is an assignment to the first universal block. The formula $\phi[g]$ is defined to be the formula derived from $\phi$ by removing $\forall Y_1^0$ from the quantifier prefix and instantiating each variable occurrence $y \in Y_1^0$ in $C$ with the constant $g(y)$.

Proposition 5.2 The proof system for any fingerprint scheme is sound in the following sense: for any quantified formula $\phi = \exists X_1 \ldots \exists X_t C$ and any fingerprint application $F(X)$ for $\phi$, if it holds that $\phi, F(X) \vdash F'(X')$, then the formulas $\exists X_1 \ldots \exists X_t C \wedge F'(X')$ and $\exists X_1 \ldots \exists X_t C \wedge F(X) \wedge F'(X')$ have the same winning strategies (and hence the same truth value).

Proof. Straightforward. Note that the soundness of the first proof rule relies on the soundness of the inference algorithm $Inf$. □

Let us say that $\phi$ has a proof of falsity if $\phi, \top \vdash \bot$ for a fingerprint application $\bot = F'(X')$ with $R(F') = \emptyset$. The previous proposition implies that if a formula $\phi$ has a proof of falsity, then $\phi$ is indeed false. Our next theorem implies that, when a fingerprint scheme for a constraint language $\Gamma$ exists, the proof system for this scheme is complete for the class of formulas QCSP$^3(\Gamma)$ in that all false formulas have proofs of falsity. It in fact demonstrates that for false alternation-bounded instances, there are polynomial-size proofs of falsity.

Theorem 5.3 Suppose that $\Gamma$ is a constraint language having a fingerprint scheme. In the above proof system, the false formulas of QCSP$^3(\Gamma)$ have polynomial-size proofs of falsity (for each $t \geq 1$).

Theorem 5.3 is proved in the appendix. We derive the following consequence, the principal result of this section, from Theorem 5.3.

Theorem 5.4 If $\Gamma$ is a constraint language having a fingerprint scheme, then the problem QCSP$^3(\Gamma)$ is in coNP (for each $t \geq 2$).

Proof. Observe that proofs in the above proof system can be verified in polynomial time; in particular, instances of the third proof rule can be verified in polynomial time as a fingerprint scheme is required to have a polynomial-time projection algorithm. The theorem is immediate from Proposition 5.2, Theorem 5.3 and this observation. □

6 Applications

The previous section gave a proof system for constraint languages having fingerprint schemes, and moreover demonstrated that the given proof system implies coNP upper bounds on the complexity of such constraint languages, in our new model. In this section, we derive a number of coNP upper bounds by demonstrating that various classes of constraint languages have fingerprint schemes. All of these classes have been previously studied in the CSP($\Gamma$) model, see [18, 23, 22, 17]. If for all $t \geq 2$ it holds that the problem QCSP$^3(\Gamma)$ is in coNP, we will simply say that the problem QCSP$^3(\Gamma)$ is in coNP.
Theorem 6.1  If $\Gamma$ is a constraint language having a set function polymorphism, then the problem $\text{QCSP}_t^\exists(\Gamma)$ is in $\text{coNP}$.

Proof.  If $\Gamma$ is a constraint language having a set function polymorphism, then it has a fingerprint scheme by the discussion in Examples 4.3 and 4.7. The theorem thus follows from Theorem 5.4. □

From Theorem 6.1, we can readily derive a $\text{coNP}$ bound on constraint languages having a semilattice polymorphism. A semilattice operation is a binary operation that is associative, commutative, and idempotent.

Corollary 6.2  If $\Gamma$ is a constraint language having a semilattice polymorphism, then the problem $\text{QCSP}_t^\exists(\Gamma)$ is in $\text{coNP}$.

Proof.  Suppose that $\Gamma$ is a constraint language over domain $D$ having a semilattice polymorphism $\oplus : D^2 \to D$. Let $\Gamma'$ be defined as the set containing all relations in $\Gamma$ as well as all constant relations of $D$. Since $\Gamma \subseteq \Gamma'$, it suffices to show the result for $\Gamma'$. Observe that $\Gamma'$ is also invariant under $\oplus$: all constant relations are preserved by $\oplus$ as $\oplus$ is idempotent. It is known that any constraint language having a semilattice polymorphism also has a set function polymorphism [18], and hence $\text{QCSP}_t^\exists(\Gamma')$ is in $\text{coNP}$ by Theorem 6.1. □

A near-unanimity operation is an idempotent operation $f : D^k \to D$ with $k \geq 3$ and such that when all but one of its arguments are equal to an element $d \in D$, then $f$ returns $d$; that is, it holds that $f(a, b, \ldots, b) = \cdots = f(b, \ldots, b, a) = b$ for all $a, b \in D$.

Theorem 6.3  If $\Gamma$ is a constraint language having a near-unanimity polymorphism, then the problem $\text{QCSP}_t^\exists(\Gamma)$ is in $\text{coNP}$.

A Mal’tsev operation is an operation $f : D^3 \to D$ such that $f(a, b, b) = f(b, b, a) = a$ for all $a, b \in D$.

Theorem 6.4  If $\Gamma$ is a constraint language having a Mal’tsev polymorphism, then the problem $\text{QCSP}_t^\exists(\Gamma)$ is in $\text{coNP}$.

Theorems 6.3 and 6.4 are proved in the appendix.

Using the results in this section thus far, we can readily obtain a complexity classification theorem for constraint languages over a two-element domain, in our new model. Let us say that the problem $\text{QCSP}_t^\exists(\Gamma)$ has maximal complexity if $\text{QCSP}_t^\exists(\Gamma)$ is PSPACE-complete and for all $t \geq 1$, the problem $\text{QCSP}_t^\exists(\Gamma)$ is $\Sigma_t^p$-complete for odd $t$, and $\Pi_t^p$-complete for even $t$. (We will also apply this terminology to problems of the form $\text{QCSP}(\Gamma)$.) The following theorem shows that in our model of QCSP complexity, there is a dichotomy in the behavior of constraint languages over a two-element domain: either they are in $\text{coNP}$ under bounded alternation, or of maximal complexity.

Theorem 6.5  Let $\Gamma$ be a constraint language over a two-element domain $D$, containing $=_{D}$. The problem $\text{QCSP}_t^\exists(\Gamma)$ is in $\text{coNP}$ if $\Gamma$ has

- a constant polymorphism,
- a semilattice polymorphism,
- a near-unanimity polymorphism, or
- a Mal’tsev polymorphism;

otherwise, $\text{QCSP}_t^\exists(\Gamma)$ has maximal complexity.
A conjunction of QCSP has maximal complexity by [16, 15, 20], by Proposition 3.6, it suffices to give a uniform reduction from with 

\[ \text{quantified formula, there is a winning strategy where} \]

\[ \text{variables} \]

\[ \text{known that} \]

\[ \text{has a set function} \]

\[ \text{Theorem 6.1. Otherwise, denote} \] 

\[ D = \{0, 1\}; \text{by Post’s lattice} \]

\[ \text{it is known that} \]

\[ \text{is contained in the clone of operations} F_u \text{ generated by the unary operation mapping} 0 \text{ to} 1, \text{and} 1 \text{ to} 0. \text{Let} \text{NAE denote the “not-all-equal” relation} \]

\[ \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}. \text{It is known that} \]

\[ F_u \text{ is exactly the set of polymorphisms of NAE} \]

\[ \text{is in} \]

\[ \text{denote the “not-all-equal” relation} \]

\[ \text{is set to} \]

\[ \text{variables} \]

\[ \text{coNP is} \]

\[ \text{Let} \]

\[ \text{is easy, then} \]

\[ \text{is in} \]

\[ \text{is hard} \]

\[ \text{if it is non-empty and for all non-empty} \]

\[ \text{for all} \]

\[ \text{is hard} \]

\[ \text{if it has two disjoint coherent sets; otherwise, we say that} \]

\[ \text{is easy.} \]

\[ \text{Let} \]

\[ \text{be an idempotent set function. Say that} \]

\[ \text{is} \]

\[ \text{coNP-hard, or QCSP}(f) \text{ is in P. In particular, the following is known.} \]

\[ \text{Definition 7.1} \]

\[ \text{Let} f : \varphi(D) \rightarrow D \text{ be an idempotent set function. Say that} \]

\[ C \subseteq D \text{ is coherent (with respect to} f) \text{ if it is non-empty and} \]

\[ \text{for all non-empty} \]

\[ A \subseteq D, \text{ it holds that} f(A) \subseteq C \text{ implies} A \subseteq C. \text{ We say that} \]

\[ f \text{ is hard if it has two disjoint coherent sets; otherwise, we say that} \]

\[ f \text{ is easy.} \]

\[ \text{Theorem 7.2} \]

\[ \text{Let} f : \varphi(D) \rightarrow D \text{ be an idempotent set function. If} f \text{ is hard, then} \]

\[ \text{QCSP}(f) \text{ is coNP-hard for all} t \geq 2. \text{ If} f \text{ is easy, then} \]

\[ \text{QCSP}(f) \text{ is in P.} \]

\[ \text{As the complexity of easy set functions is known, our focus here is on the hard set functions. Our first observation is that, under bounded alternation, hard set functions are coNP-complete.} \]
Theorem 7.3 If \( f \) is a hard set function, then QCSP\(_t(f)\) is coNP-complete (for all \( t \geq 2 \)).

Proof. Hardness for coNP is immediate from Theorem 7.2. Inclusion in coNP follows from Theorem 6.1 and Proposition 3.6. □

We have that for a hard set function \( f \), the problem QCSP\(_t(f)\) is in coNP, that is, QCSP\(_t(f)\) is in coNP under bounded alternation. This result naturally prompts the question of whether or not QCSP\(_t(f)\) is in coNP under unbounded alternation. We are able to answer this question in the negative: we demonstrate that such a QCSP\(_t(f)\) is \( \Pi_2^P \)-hard, implying that if QCSP\(_t(f)\) were in coNP, we would have \( \Pi_2^P = \text{coNP} \) and that the polynomial hierarchy collapses. We in fact show \( \Pi_2^P \)-hardness of such QCSP\(_t(f)\) by first showing that extended quantified Horn formulas reduce to QCSP\(_t(f)\), and then that extended quantified Horn formulas are \( \Pi_2^P \)-hard.

Theorem 7.4 If \( f \) is a hard set function, then QCSP\(_t(f)\) is \( \Pi_2^P \)-hard.

Proof. Immediate from Theorems 7.3 and 7.6 proved below. □

Theorem 7.5 Let \( f \) be a hard set function. The problem of deciding the truth of extended quantified Horn formulas uniformly reduces to QCSP\(_t(f)\).

Proof. Let \( C_0 \) and \( C_1 \) be disjoint coherent sets with respect to \( f \), let \( C \) be a coherent set with respect to \( f \) that is not equal to \( D \), and let \( c_i \in C \) be a fixed element of \( C \).

Given an extended quantified Horn formula \( \phi \), we create an instance \( \phi' \) of QCSP\(_t(f)\) as follows. First, note that by the introduction of extra existentially quantified variables, we can transform \( \phi \) in polynomial time into another extended quantified Horn formula in which each clause has a constant number of literals. We thus assume that each clause of \( \phi \) has a constant number of literals.

We create a constraint invariant under \( f \) for each extended Horn clause of \( \phi \). In particular, for each extended Horn clause \( l_1 \lor \ldots \lor l_k \) of \( \phi \), we create in \( \phi' \) the constraint \( m_1 \lor \ldots \lor m_k \), where

- \( m_i = (y \notin C_0) \) if \( l_i = y \) and \( y \) is a universally quantified variable,
- \( m_i = (y \notin C_1) \) if \( l_i = \overline{y} \) and \( y \) is a universally quantified variable,
- \( m_i = (x = c_i) \) if \( l_i = x \) and \( x \) is an existentially quantified variable, and
- \( m_i = (x \notin C) \) if \( l_i = \overline{x} \) and \( x \) is an existentially quantified variable.

Let us verify that any such constraint \( M = m_1 \lor \ldots \lor m_k \) is invariant under \( f \). Let \( a_1, \ldots, a_n \) be assignments to the variables \( V \) of \( M \) satisfying \( M \). We want to show that the assignment \( a \) defined by \( a(v) = f(\{a_1(v), \ldots, a_n(v)\}) \) for all variables \( v \in V \), also satisfies \( M \). If any \( a_i \) satisfies \( M \) by satisfying an \( m_j \) of the form \( (y \notin C_0) \), \( (y \notin C_1) \), or \( (x \notin C) \), then \( a \) also satisfies the \( m_j \) by the coherence of \( C_0 \), \( C_1 \), and \( C \). (Suppose for instance \( a_i(y) \notin C_0 \). Then we have \( a(y) = a(S) \) for a set \( S \) including \( a_i(y) \), which is not in \( C_0 \), and so by the coherence of \( C_0 \), we have \( a(y) \notin C_0 \).) Otherwise, there exists an \( m_j \) with \( m_j = (x = c_i) \). But since the original clause \( l_1 \lor \ldots \lor l_k \) was an extended Horn clause, it contained at most one existentially quantified variable appearing positively, and there is a unique such \( m_j = (x = c_i) \). We have \( a_i(x) = c_i \) for all \( i = 1, \ldots, n \) and thus, by the idempotence of \( f \), \( a(x) = f(\{c_i\}) = c_i \).

We have shown that \( \phi' \) is indeed an instance of QCSP\(_t(f)\). It remains to show that \( \phi' \) is true if and only if \( \phi \) is true. Notice that in \( \phi' \), a strategy is winning as long as it is winning with respect to all assignments \( \tau : Y \to D \) to the universally quantified variables \( Y \) where \( \tau(y) \in C_0 \cup C_1 \) for all \( y \in Y \); the sets \( C_0 \) and \( C_1 \) in \( \phi' \) encode the values 0 and 1 for the universally quantified variables in \( \phi \). Let \( \{\sigma_y\} \) be a winning strategy for \( \phi \). Define \( \sigma_y' \) to be equal to \( c_i \) whenever \( \sigma_y \) is equal to 1, and to be an element of \( D \setminus C \) whenever
σ_x is equal to 0. Associating the sets C_0 and C_1 with 0 and 1 as discussed, it is straightforward to verify that \{σ'_x\} is a winning strategy for φ'. Likewise, if \{σ'_x\} is a winning strategy for φ', define σ_x to be 1 whenever σ'_x is equal to c, and 0 otherwise; using the same association, it is straightforward to verify that \{σ'_x\} is a winning strategy for φ. □

**Theorem 7.6** The problem of deciding the truth of extended quantified Horn formulas (without any bound on the number of alternations) is Π^2_p-hard.

The proof of this theorem is inspired by the proof of [10, Theorem 3.2].

**Proof.** We first prove that the problem is NP-hard, then indicate how the proof can be generalized to yield Π^2_p-hardness. We reduce from CNF satisfiability. Let φ be a CNF formula over variable set \{y_1, \ldots, y_n\}. We create an extended quantified Horn formula φ' based on φ as follows. The quantifier prefix of φ' is \((∃x_0 y_1 ∃x_1 y_2) \ldots (∃x_0 y_{n−1} ∃x_1 y_n) ∃d.

The clauses of φ' are as follows. We call the following the core clauses of φ'; they do not depend on φ.

\[
x^0_1 \land x^1_1 \Rightarrow \text{False}
\]

\[
y_i \land x^0_{i+1} \land x^1_{i+1} \Rightarrow x^1_i \text{ for all } i \in \{1, \ldots, n−1\}
\]

\[
\overline{y}_i \land x^0_{i+1} \land x^1_{i+1} \Rightarrow x^0_i \text{ for all } i \in \{1, \ldots, n−1\}
\]

\[
y_n \land d \Rightarrow x^1_n
\]

\[
\overline{y}_n \land d \Rightarrow x^0_n
\]

In addition, for each clause \(l_1 \lor l_2 \lor l_3\) of φ, there is a clause in φ' of the form

\[
\overline{l_1} \land \overline{l_2} \land l_3 \Rightarrow d
\]

We claim that φ is satisfiable if and only if φ' is true.

**φ is unsatisfiable implies φ' is false.** Suppose that φ is unsatisfiable. It is straightforward to verify that the following clauses can be derived from the clauses for φ', by which we mean that any winning strategy for φ' must satisfy the following clauses.

We can derive

\[
z_1 \land \ldots \land z_n \Rightarrow d
\]

for any choice of \(z_1, \ldots, z_n\) with \(z_i \in \{y_i, \overline{y}_i\}\) because any assignment to the variables \(\{y_1, \ldots, y_n\}\) falsifies a clause, and hence makes the left-hand side of a clause

\[
\overline{l_1} \land \overline{l_2} \land l_3 \Rightarrow d
\]

true.

Now, by induction (starting from \(n−1\)), it can be shown that for all \(k = n−1, \ldots, 0\), the clauses

\[
z_1 \land \ldots \land z_k \Rightarrow x^0_{k+1}
\]

\[
z_1 \land \ldots \land z_k \Rightarrow x^1_{k+1}
\]

can be derived, for any choice of \(z_1, \ldots, z_k\) with \(z_i \in \{y_i, \overline{y}_i\}\). This is done by using the core clauses (other than the first core clause).

By this induction, we obtain that \(x^0_1\) and \(x^1_1\) can be derived (by setting \(k = 0\)); using the clause

\[
x^0_1 \land x^1_1 \Rightarrow \text{False}
\]

we can then derive False.
The clauses of \( QCSP \) are in fact \( \Pi^P_1 \)-hard. Let \( \phi = \forall w_1 \ldots \forall w_m \exists y_1 \ldots \exists y_n \mathcal{C} \) be a quantified boolean formula where \( \mathcal{C} \) is a 3-CNF. The extended quantified Horn formula \( \phi' \) that we create has quantifier prefix 

\[
(\forall w_1 \ldots \forall w_m)(\exists x_0^0 \exists x_1^1 \forall y_1) \ldots (\exists x_m^0 x_n^1 \forall y_n) \exists d.
\]

The clauses of \( \phi' \) are defined as above. The key point is that, under any assignment to the variables \( \{w_1, \ldots, w_m\} \), the formula \( \phi' \) is true if and only if the formula \( \phi' \) is true, by using the reasoning in the NP-hardness proof.

Theorem 7.6 has an interesting implication for our model of existentially restricted quantified constraint satisfaction. This paper has focused mainly on proving that for certain constraint languages \( \Gamma \), it holds that \( QCSP^3_1(\Gamma) \) is in coNP, that is, the bounded alternation formulas for \( QCSP^3(\Gamma) \) are in coNP. This theorem implies that such results, in general, can not be extended to the case of unbounded alternation. In particular, we have such a constraint language \( \Gamma \) whose \( QCSP^3(\Gamma) \) complexity—in the case of unbounded alternation—is not in coNP, unless coNP = \( \Pi^P_2 \) and the polynomial hierarchy collapses.

**Theorem 7.7** Let \( \Gamma_H \) be the constraint language defined in Example 8.8. It holds that \( QCSP^3_1(\Gamma_H) \) is in coNP for all \( t \geq 2 \), but \( QCSP^3(\Gamma_H) \) is \( \Pi^P_2 \)-hard.

**Proof.** Inclusion of \( QCSP^3_1(\Gamma_H) \) in coNP is discussed in Example 8.5 there, it is also pointed out that \( QCSP^3(\Gamma_H) \) is equivalent to the problem of deciding the truth of extended quantified Horn formulas, so the \( \Pi^P_2 \)-hardness of \( QCSP^3(\Gamma_H) \) follows from Theorem 7.6. \( \square \)
8 Discussion

8.1 Comparison with the standard model

As we have discussed in the introduction, our new model and the standard model are both natural generalizations of the CSP model. However, we would like to argue here that our new model is, in certain respects, a more faithful generalization of the CSP model.

In the CSP model, it is possible to use algebraic notions such as subalgebra and homomorphic image to study the complexity of constraint languages [8]. We have shown that these notions can also be used to study complexity in our new model (Theorem 5.8), indicating that our model is algebraically robust. In contrast, the first property of Theorem 5.8 does not hold in the standard model: using the results of [11, Chapter 5], it is easy to construct a semilattice $A$ having a subalgebra $B$ such that QCSP($A$) is polynomial-time decidable, but QCSP($B$) is coNP-hard.

Along these lines, it is known that there are constraint languages with trivial CSP complexity—where all CSP instances are satisfiable by a mapping taking all variables to the same value—but maximal QCSP complexity, under the standard model. This “wide complexity gap” phenomenon of the standard model does not appear to occur in our new model. To understand why, it is didactic to consider the case of QCSP instances with quantifier type $\forall \exists$. Fix any constraint language giving rise to a polynomial-time tractable case of the CSP. In our new model, $\forall \exists$ instances over the constraint language are immediately seen to be in coNP. The argument is simple: once the universal variables are instantiated, the result is an instance of the CSP over the constraint language, which can be decided in polynomial time. In contrast, in the standard model, $\forall \exists$ instances over the constraint language may be $\Pi^P_2$-complete, that is, maximally hard given the quantifier prefix. The argument given for our new model does not apply: even if the constraints of a $\forall \exists$ instance are originally over the constraint language, after the universal variables have been instantiated with values, new constraints may be created that are not over the original constraint language. The key point is that while instantiation of universally quantified variables may “disrupt” the constraint language in the standard model, it does not do so in our new model. All in all, the faithfulness of our model to the original CSP model affords a fresh opportunity to enlarge the repertoire of positive QCSP complexity results, by way of extending existing CSP tractability results.

8.2 Conclusions

We have introduced and studied a new model for restricting the QCSP, a generally intractable problem. We presented powerful technology for proving coNP-inclusion results in this new model, under bounded alternation, and have applied this technology to a variety of constraint languages. We also derived new results on the standard model using results on our new model, in particular, new results on the complexity of constraint languages having a set function polymorphism. In addition, we demonstrated that, in general, coNP-inclusion results for our new model in the case of bounded alternation, can not be extended to the case of unbounded alternation.

One interesting direction for future research is to classify the complexity of all constraint languages in our new model under bounded alternation. At this moment, a plausible conjecture is that all constraint languages that are tractable in the CSP model are in coNP in our new model, under bounded alternation. A second direction is to investigate further the case of unbounded alternation. In particular, one could investigate the unbounded-alternation complexity of constraint languages that are known to be in coNP under bounded alternation; Theorem 7.7 represents one step in this direction.

---

2 An example of a constraint language having the mentioned properties is the constraint language over the domain $\{0, 1\}$ consisting of all arity four relations that contain the all-0 tuple $(0, 0, 0, 0)$. This is PSPACE-complete in the alternation-unbounded standard model by the results [16, 15], and complete for the various levels of the polynomial hierarchy in the alternation-bounded standard model by the result [20].
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A Proof of Theorem 5.3

Our first step is to show completeness of the proof system, that is, if $\phi$ is false, then $\phi, \top \vdash \bot$. We accomplish this in a sequence of lemmas. Throughout, we assume that the quantified formulas under discussion are all instances of QCSP$(\exists, \forall, \Gamma)$, and we fix a fingerprint scheme for $\Gamma$.

Lemma A.1 For any quantified formula $\phi$ with $t$ existential blocks and any fingerprint application $F(X)$ for $\phi$ with $X \subseteq X_1^\phi$, there exists a proof of size $|\pi| F$ for some fingerprint application $F$ with $\pi_{|X|} F \subseteq F$.

Proof. The proof is by induction on $t$. If $t = 1$, the proof consists of one instance of the first proof rule; the $\pi_{|X|} R(F') \subseteq F$ criterion (of the lemma) holds because of the “progress” requirement on inference algorithms. If $t > 1$, then let $g : Y_1^\phi \to D$ be any mapping. By induction, there is a proof of size $t - 1$ that $\phi[g], F(X) \vdash F'(X_1^\phi \cup X_2^\phi)$ for some fingerprint $F'$ with $\pi_{|X|} F' \subseteq F$. Applying the third proof rule, we obtain a proof of size $t$ that $\phi, F(X) \vdash (\pi_{|X|}' F'')(X_1^\phi)$. Set $F'' = (\pi_{|X|}' F'')$. By Definition 4.1 we have $\pi_{|X|} F'' = \pi_{|X|} F'$. Since $\pi_{|X|} F'' \subseteq F$, we have the lemma. □

We say that a fingerprint application $F(X)$ is stable for a quantified formula $\phi$ if $X = X_1^\phi$, $F \neq \bot$, it holds that $\phi, \top \vdash F(X)$, and $F' \subseteq F$ implies $\phi, \top \not\vdash F'(X)$. That is, a fingerprint application is stable for $\phi$ if it can be derived, but no fingerprint application “lower” than it can be derived.

Lemma A.2 If $\phi$ is a quantified formula such that $\phi, \top \not\vdash \bot$, then $\phi$ has a stable fingerprint application.

Proof. Immediate from the definition of stable fingerprint application and Lemma A.1 which implies that for some fingerprint application $F(X)$ with $X = X_1^\phi$, it holds that $\phi, \top \vdash F(X)$. □

Let us say that a fingerprint $F$ extends another fingerprint $F'$ if $\text{arity}(F) \geq \text{arity}(F')$ and $\pi_{\text{arity}(F')} F = F'$.

Lemma A.3 If $\phi$ is a quantified formula with a stable fingerprint application $F(X)$ and having two or more existential blocks, then for any mapping $g : Y_1^\phi \to D$, the quantified formula $\phi[g]$ has a stable fingerprint application $F'(X')$ where $F'$ extends $F$.

Proof. By assumption, we have $\phi, \top \vdash F(X)$, with $X = X_1^\phi$. Let $g : Y_1^\phi \to D$ be any mapping. By Lemma A.1 there exists a fingerprint $F_2$ of arity $|X_1^\phi \cup X_2^\phi|$ having the properties that $\phi[g], F(X) \vdash F_2(X')$ and $\pi_{|X|} F_2 \subseteq F$, where $X' = [X_1^\phi \cup X_2^\phi]$. Now let $F'$ be a “minimal” fingerprint with the above properties, that is, a fingerprint such that for no other $F_2$ having the above properties does it hold that $F_2 \subseteq F'$.

We claim that $F'(X')$ is a stable fingerprint application for $\phi[g]$ where $F'$ extends $F$. It suffices to show that $F'$ extends $F$. Suppose not; then $\pi_{|X|} F' \subseteq F$ and $\pi_{|X|} F' \neq F$. By the third rule of the proof system, $\phi, F(X) \vdash (\pi_{|X|} F')(X)$ and by the second rule, $\phi, \top \vdash (\pi_{|X|} F')(X)$, contradicting that $F(X)$ is stable for $\phi$. □

Lemma A.4 If $\phi$ is a quantified formula with a stable fingerprint application, then $\phi$ is true.
Proof. We prove the following claim: if $F(X^\phi_1)$ is a stable fingerprint application for $\phi$, then $\phi$ is true via the assignment $f : X^\phi_1 \to D$ defined by the construction mapping Cons and the fingerprint application $F(X^\phi_1)$, as in Definition 4.5.

We prove this claim by induction on the number $t$ of existential blocks in $\phi$. Let $C$ denote the constraints of $\phi$.

When $t = 1$, it is straightforward to verify that any fingerprint application derivable from $\phi, \top$, and hence any stable fingerprint application, is suitable for $C$. Note that only the first two proof rules can be used to perform derivations from $\phi, \top$.

When $t > 1$, by Lemma A.3 we have that for all $g : Y^\phi_1 \to D$, there is a stable fingerprint application $F_g(X_1^\phi[g])$ for $\phi[g]$ where $F_g$ extends $F$. By induction, for all $g$, the quantified formula $\phi[g]$ is true via the assignment $f_g : X^\phi_1 \to D$ defined by Cons and $F_g(X^\phi_1)$, as in Definition 4.5. The claim follows from the observation that for all $g : Y^\phi_1 \to D$, the restriction of $f_g$ to $X^\phi_1$ is equal to $f : X^\phi_1 \to D$. □

We are now able to observe that the proof system is complete.

Lemma A.5 If the quantified formula $\phi$ is false, then $\phi$ has a proof of falsity (that is, $\phi, \top \vdash \bot$).

Proof. If $\phi, \top \not\vdash \bot$, then by Lemma A.2, the formula $\phi$ has a stable fingerprint application; it follows from Lemma A.4 that the formula $\phi$ is true. □

However, we want to show something stronger than Lemma A.5: that every false quantified formula has a succinct proof of falsity. The following lemma is key; roughly speaking, it shows that for alternation-bounded formulas, if there is a proof of falsity at all, then there is a succinct proof of falsity.

Lemma A.6 For each $t \geq 1$, there exists a polynomial $p_t$ such that if $\phi$ has $t$ existential blocks, $n$ existential variables, and $\phi, F(X) \vdash F'(X^\phi_1)$, then there is a proof of $\phi, F(X) \vdash F'(X^\phi_1)$ of size bounded above by $p_t(n)$.

Proof. We first observe that the proof system has the following monotonicity property: if $\phi, F(X) \vdash F'(X')$, then $\Pr_{|X|}(F') \subseteq R(F)$. This is straightforward to verify.

We prove the lemma by induction on $t$. When $t = 1$, inspecting a proof of $\phi, F(X) \vdash F'(X^\phi_1)$, it can be seen that only the first two proof rules are applicable, and by monotonicity there must be distinct fingerprints $F_1, \ldots, F_m$ with $F_m = F'$ and $F_1 \sqsubseteq \cdots \sqsubseteq F_m$ such that

$$\phi, F(X) \vdash F_1(X^\phi_1)$$

and

$$\phi, F_k(X^\phi_1) \vdash F_{k+1}(X^\phi_1)$$

for all $k = 1, \ldots, m - 1$. The above proofs can be combined into a proof of $\phi, F(X) \vdash F'(X^\phi_1)$ using no more than $(m - 1)$ applications of the second rule. We thus obtain a proof of size $m + (m - 1)$, which is polynomial in $n = |X^\phi_1|$ because of the requirement that in a fingerprint collection, chains are of polynomial length.

When $t > 1$, inspecting a proof of $\phi, F(X) \vdash F'(X^\phi_1)$ and using monotonicity, by induction on the structure of the proof it can be seen that there must be mappings $g_1, \ldots, g_m : Y^\phi_1 \to D$, distinct fingerprints $F_1, \ldots, F_m$ of arity $|X^\phi_1|$ with $F_m = F'$, and fingerprints $F'_1, \ldots, F'_m$, where

$$F_1 \sqsubseteq \cdots \sqsubseteq F_m$$

and

$$\phi, F_k(X^\phi_1) \vdash F'_k(X^\phi_1)$$

for all $k = 1, \ldots, m - 1$. The above proofs can be combined into a proof of $\phi, F(X) \vdash F'(X^\phi_1)$ using no more than $(m - 1)$ applications of the second rule. We thus obtain a proof of size $m + (m - 1)$, which is polynomial in $n = |X^\phi_1|$ because of the requirement that in a fingerprint collection, chains are of polynomial length.
and such that

\[
\phi[g_1], F(X) \vdash F'_1(\langle X_1^\phi[g_1] \rangle) \quad X_1^\phi[g_1] \supseteq X_1^\phi
\]

and

\[
\phi[g_k], F_k(\langle X_1^\phi \rangle) \vdash F'_{k+1}(\langle X_1^\phi[g_k] \rangle) \quad X_1^\phi[g_k] \supseteq X_1^\phi
\]

for all \( k = 1, \ldots, m - 1 \). As before, the above proofs can be combined into a proof of \( \phi, F(X) \vdash F'(X_1^\phi) \) using no more than \((m - 1)\) applications of the second rule. By induction, the hypothesis of each rule instance given above has a proof of size \( p_{l-1}(n) \). We thus obtain a proof of size bounded above by \( m(p_{l-1}(n) + 1) + (m - 1) \). This expression is polynomial in \( n \): \( m \) is polynomial in \( |X_1^\phi| \leq n \) because of the requirement that in a fingerprint collection, chains are of polynomial length. \( \square \)

**Proof.** (Theorem 5.3) By Lemma A.5, for all false formulas \( \phi \) of QCSP\(\bar{\tau}(\Gamma) \), it holds that \( \phi, \top \vdash \bot \). By Lemma A.6, there is a proof of \( \phi, \top \vdash \bot \) of size bounded above by \( p_l(n) \) where \( n \) is the number of variables of \( \phi \), and \( p_l \) is a polynomial. \( \square \)

### B  Proof of Theorem 6.3

**Proof.** Let \( \Gamma \) be a constraint language over domain \( D \) having \( f : D^k \to D \) as near-unanimity polymorphism. By Theorem 5.4, it suffices to show that \( \Gamma \) has a fingerprint scheme.

We first describe the fingerprint collection.

The fingerprints of arity \( n \) are sets of constraints on \( \{v_1, \ldots, v_n\} \). In particular, a fingerprint of arity \( n \) contains exactly one constraint for each possible variable set of size less than or equal to \( k \); formally, there is a constraint over each variable tuple of the form \( \langle v_{i_1}, \ldots, v_{i_m} \rangle \) with \( 1 \leq i_1 < \cdots < i_m \leq n \) and \( 1 \leq m \leq k \).

The relation specified by a fingerprint \( F \) is the set of all tuples \( (d_1, \ldots, d_n) \) such that the mapping \( v_i \to d_i \) satisfies all constraints in \( F \).

The projection function \( \pi \), given a fingerprint \( F \) and \( k \), simply projects all constraints onto the variables \( \{v_1, \ldots, v_k\} \).

For two fingerprints \( F, F' \) of arity \( n \), we define \( F \subseteq F' \) if and only if for each constraint \( R(v_{i_1}, \ldots, v_{i_m}) \) in \( F \), the corresponding constraint \( R'(v_{i_1}, \ldots, v_{i_m}) \) in \( F' \) satisfies \( R \subseteq R' \).

Chains are of polynomial length, since the length of a chain is bounded above by the total number of tuples that can appear in a fingerprint. This total number is equal to the number of constraints, \( \binom{n}{k} + \binom{n}{k-1} + \cdots + \binom{n}{1} \), times \( |D|^k \), which upper bounds the number of tuples in each constraint; this is clearly polynomial in \( n \), for fixed \( D \) and \( k \).

We now describe the fingerprint scheme.

The inference algorithm, given a fingerprint application \( F\langle v_1, \ldots, v_k \rangle \) and a conjunction of constraints \( C \) over \( \Gamma \) and \( \{v_1, \ldots, v_n\} \), establishes strong \( k \)-consistency on the constraints \( C \cup F \) to obtain \( C' \); the result is the fingerprint \( F' \) of arity \( n \) where each constraint \( R(v_{i_1}, \ldots, v_{i_m}) \) contains the solutions to \( C' \) restricted to \( \{v_{i_1}, \ldots, v_{i_m}\} \). Please see [22] for the definition of strong \( k \)-consistency and solution.

We define the construction mapping as follows. Let \( F \) be a fingerprint of arity \( n \), with \( \mathcal{R}(F) \neq \emptyset \), that is suitable for a conjunction of constraints \( C \). Due to the definition of our inference algorithm, we know that \( F \) is strongly \( k \)-consistent, and that any assignment satisfying the constraints in \( F \) also satisfies \( C \). The construction mapping is defined inductively: it simply computes the mapping \( a : \{v_1, \ldots, v_{n-1}\} \to D \) defined by \( a(v_i) = \text{Cons}(\pi_i F) \), and then outputs a value \( d \) such that the extension of \( a \) mapping \( v_n \) to \( d \)
satisfies the constraints in \( F \); such a value is guaranteed to exist by the fact that \( F \) is strongly \( k \)-consistent and \cite{22} Theorem 3.5. Notice that if \( F(v_1, \ldots, v_n) \) is a fingerprint application suitable for \( C \), then the mapping defined in Definition 4.5 satisfies the constraints in \( F \), which in turn (as we pointed out) implies that this mapping satisfies the constraints of \( C \), as desired. \( \square \)

C Proof of Theorem 6.4

Proof. Let \( \Gamma \) be a constraint language over domain \( D \) having \( \phi : D^3 \rightarrow D \) as Mal’tsev polymorphism. By Theorem \ref{thm:mal'tsev}, it suffices to show that \( \Gamma \) has a fingerprint scheme.

We assume basic familiarity with the paper \cite{17}, and use the terminology of that paper.

We first describe the fingerprint collection.

The fingerprints are the relations that are compact representations, that is, the relations \( F \subseteq D^k \) having the property that \( F \) is a compact representation of some relation.

The relation specified by a fingerprint \( F \) is \( \mathcal{R}(F) = \langle F \rangle_\phi \) (the notation \( \langle F \rangle_\phi \) denotes the smallest relation containing \( F \) and closed under \( \phi \)). Observe that any such fingerprint \( F \) can be represented\(^3\) in size polynomial in its arity \( n \), since for any compact representation \( F \), it holds that \( |F| \leq 2^\star n \star |D|^2 \), since \( F \) has at most two tuples for each element of some signature, and \( n \star |D|^2 \) is an upper bound on the size of a signature.

The projection function \( \pi \), given a fingerprint \( F \) and \( k \), simply yields \( \text{pr}_k F \).

For two fingerprints \( F, F' \) of arity \( n \), we define \( F \subseteq F' \) if and only if \( \mathcal{R}(F) \subseteq \mathcal{R}(F') \).

Chains are of polynomial length: suppose \( F, F' \) are fingerprints with \( \mathcal{R}(F) \subseteq \mathcal{R}(F') \). Then \( \text{Sig}_{\mathcal{R}(F)} \subseteq \text{Sig}_{\mathcal{R}(F')} \); this follows immediately from the definition of signature. But in fact it holds that \( \text{Sig}_{\mathcal{R}(F)} = \text{Sig}_{\mathcal{R}(F')} \), for if \( \text{Sig}_{\mathcal{R}(F)} = \text{Sig}_{\mathcal{R}(F')} \), then \( \mathcal{R}(F) \) is a representation of \( \mathcal{R}(F') \), and it would follow from \cite{17} Lemma 1\(^{4}\) that \( \langle \mathcal{R}(F) \rangle_\phi = \mathcal{R}(F) \) was equal to \( \mathcal{R}(F') \). Since signatures of arity \( n \) are subsets of a set with \( n \star |D|^2 \) elements, chains are of polynomial length.

We now describe the fingerprint scheme.

The inference algorithm, given a fingerprint application \( F(x_1, \ldots, x_k) \) and a conjunction of constraints \( C \) over variables \( \{x_1, \ldots, x_n\} \), first computes from \( F \) a fingerprint \( F_e \) such that \( \mathcal{R}(F_e) = \mathcal{R}(F) \times D^{n-k} \). Then, starting from \( F_e \), the constraints of \( C \) are processed by \text{Next} one by one (as in the algorithm \text{Solve} in order to obtain a fingerprint \( F' \) such that the tuples of \( \langle F' \rangle_\phi \) are exactly the satisfying assignments of \( C \) that also satisfy \( F_e(x_1, \ldots, x_k) \). Notice that \( F' \subseteq F_e \), and so \( \text{pr}_k F' \subseteq \text{pr}_k F_e = F \).

The construction mapping is defined inductively. Given a fingerprint \( F \) of arity \( n \) with \( \mathcal{R}(F) \neq \emptyset \), it simply computes the tuple \( \tilde{t} = (\text{Cons}(\pi_1 F), \ldots, \text{Cons}(\pi_{n-1} F)) \); by induction, we may assume that this tuple is in \( \text{pr}_{n-1} \mathcal{R}(F) \). There thus exists an element \( d \) such that \( (\tilde{t}, d) \) is in \( \mathcal{R}(F) \), which is the output of the mapping. \( \square \)

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\(^{3}\)Other than in this instance, all uses of the term “representation” within this proof will refer to the notion of representation defined in \cite{17}.
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