MULTIVARIATE GROUP ENTROPIES,
SUPER-EXPONENTIALLY GROWING SYSTEMS
AND FUNCTIONAL EQUATIONS

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Abstract. We define the class of multivariate group entropies, first inves-
tigated in [22], as a natural generalization of the Z-entropies introduced in
[32]. We propose new examples related to the “super-exponential” univer-
sality class of complex systems; in particular, we introduce a general entropy,
suitable for this class. We also show that the group-theoretical structure asso-
ciated with our multivariate entropies can be used to define a large family of
exactly solvable discrete dynamical models. The natural mathematical frame-
work allowing us to formulate this correspondence is offered by the theory of
formal groups and rings.

Keywords: Group Entropies, Information Theory, Dynamical Systems

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1. Introduction

The study of generalized entropies in information theory, and in particular in
information geometry, has been actively pursued in the last decades (see, e.g. [2],
[3] [17], [24], [25]).

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The combination of group theory with the concept of generalized entropy has led to the notion of \textit{group entropy}, first introduced in [29] and further investigated in [26], [31], [32], [11], [16], [22], [33], [8], [13]. Basically, a group entropy is a functional $S: P_W \rightarrow \mathbb{R}^+ \cup \{0\}$ on the probability space $P_W$ associated with a complex system, possessing several interesting mathematical properties: it satisfies the first three Shannon-Khinchin (SK) axioms (continuity, maximum entropy principle, expansibility) and the \textit{composability axiom}. This axiom requires that given two independent systems $A$, $B$, the entropy of the compound system $A \cup B$ must be described by the relation $S(A \cup B) = \Phi(S(A), S(B))$, where $\Phi(x, y)$ is a commutative (formal) group law (see [16] for a recent review).

In [16], [33], universality classes of complex systems, defined in terms of a specific state space growth rate $W = W(N)$ have been studied in the context of information theory. Here $N$ is the number of particles, or constituents of the system. For each class there exists a specific group entropy, which is extensive over the class, and constructed in a purely axiomatic way from the Shannon-Khinchin axioms. In many respects, group entropies represent a versatile tool for generating a wide set of information measures, potentially useful in the theory of complex systems, biology, social sciences, etc.

The main purpose of this note is to further extend this ideas by formalizing and analyzing a new class of group entropies that we shall call \textit{multivariate group entropies}, since they are multivariate functions constructed from a probability distribution. This new class has been introduced in [22], where some concrete examples have also been proposed. We shall clarify the properties of the multivariate family; in particular, we define a realization in terms of a simple class of functionals: the \textit{multivariate $Z$-entropies}. They represent a natural generalization of the notion of $Z$-entropy introduced in [32], which refers to the standard, \textit{univariate} case of group entropies.

The theory of group entropies is primarily a mathematical one: we are essentially constructing a family of functionals on a probability space, which share with Rényi’s entropy many useful properties. This allows us to give a natural interpretation of our entropies in the context of information geometry (as in [22]).

Besides, in this paper we shall construct explicitly new concrete examples of both univariate and multivariate group entropies. The new functionals introduced here are designed in such a way that they are extensive over the universality class of systems whose state space possesses a \textit{super-exponential growth} in the number $N$ of its constituents.

The \textit{super-exponential class} has been first explored in [15] within the framework of group entropies. In particular, a new entropy was proposed, extensive over the class characterized by a state space growth rate $W(N) \sim N^N$. We remind that the case of a \textit{sub-exponential} growth rate of the kind $W(N) \sim N^a$ has been largely explored in the context of non-extensive statistical mechanics [36], [37]. A different approach for super-exponential systems, based on scaling-law analysis has been developed in [18].

We shall introduce a new family of group entropies parametrized by an “interpolation function” that can be easily fine-tuned to cover different universality classes, ranging from the exponential class to the super-exponential one (including the cases of both sub-factorial and super-factorial growth of the state space).
Finally, we establish a connection between the theory of group entropies and that of functional equations. A group entropy inherits in an obvious manner a functional equation, expressed by its own group law. We further investigate the set of functional equations related to group entropies by means of the algebraic structure of formal rings, recently introduced in [9]. A formal ring is essentially a formal group endowed with a second composition law, which can be naturally realized as a deformed product, compatible with the deformed sum expressed by the group law.

Thus, starting from the formal ring structure associated with a group entropy, a set of functional equations is defined, whose solutions can be explicitly determined.

In this framework, we propose an application of the present approach to discrete dynamical systems. Indeed, by discretizing over a regular lattice the functional equations related with the formal ring structure of a group entropy, we obtain classes of discrete dynamical systems that, once again, possess exact solutions (here we do not focus on the initial value problem), and in some cases related sequences of integer numbers.

Although the purpose of this work is essentially of a mathematical nature, we mention that from an applicative point of view we aim to construct new information measures, potentially useful in the detection of complexity in different scenarios, from an information-geometrical perspective. Indeed, new applications can be envisaged, ranging from statistical inference theory to the study of biological models (from this respect, see e.g. [7], [35], [34], [33]). Work along this lines is in progress.

The paper is organized as follows. In section 2, the new class of multivariate Z-entropies is discussed. In section 3, the theory of formal groups and rings is briefly reviewed. In section 4, some new examples of multivariate entropies are defined. In particular, in section 5 a new, general family of super-exponential entropies is introduced. A study of functional equations and discrete dynamical systems naturally associated with group entropies is proposed in the final section 6.

### 2. Multivariate Group Entropies

In order to introduce the notion of group entropy, we first remind the content of the Shannon-Khinchin (SK) axioms [24], [25], [17]. Let \( S(p) \) be a function on a set of probability distributions. The first three SK axioms essentially amount to the following properties:

- **SK1** \( S(p) \) is continuous with respect to all variables \( p_1, \ldots, p_W \).
- **SK2** \( S(p) \) takes its maximum value over the uniform distribution.
- **SK3** \( S(p) \) is expansible: adding an event of zero probability does not affect the value of \( S(p) \).

These axioms represent a minimal set of “non-negotiable” requirements that a function \( S(p) \) over a set of probability distributions should satisfy necessarily to be meaningful, both from a physical and information-theoretical point of view. The fourth axiom, requiring specifically additivity on conditional distributions, leads to Boltzmann’s entropy [17].

Instead, we replace the additivity axiom by a more general statement.

#### 2.1. Composability axiom

An entropy is said to be composable if there exists a smooth function \( \Phi(x, y) \) such that, given two statistically independent subsystems...
A and B of a complex system,

\[(1) \quad S(A \cup B) = \Phi(S(A), S(B)),\]

when the two subsystems are defined over any arbitrary probability distribution of \(P_W\). Hereafter, all quantities will be assumed to be dimensionless.

The relation (1) has been introduced in [37]. However, we prefer to further specialize this definition (as in [29], [26], [31], [32], [11], [16], [22], [33]). Precisely, in addition to eq. (1), we shall also require the following properties:

- **(C1) Symmetry:** \(\Phi(x, y) = \Phi(y, x)\).
- **(C2) Associativity:** \(\Phi(x, \Phi(y, z)) = \Phi(\Phi(x, y), z)\).
- **(C3) Null-Composability:** \(\Phi(x, 0) = x\).

Observe that, indeed, the requirements \((C1)-(C3)\) are crucial ones: they impose the independence of the composition process with respect to the order of \(A\) and \(B\), the possibility of composing three independent subsystems in an arbitrary way, and the requirement that, when composing a system with another one having zero entropy, the total entropy remains unchanged. In our opinion, these properties are also fundamental: indeed, no thermodynamical or information-theoretical applications would be easily conceivable without these properties. From an algebraic point of view, the requirements \((C1)-(C3)\) define a formal group law.

Consequently, infinitely many choices allowed by eq. (1), like \(\Phi(x, y) = \sin(x + y)\) or \(\Phi(x, y) = x + y + x^2 y\), or even the simple one \(\Phi(x, y) = xy\) are discarded. In this respect, the theory of formal groups [4], [12], [23] offers a natural language in order to formulate the theory of generalized entropies in a consistent way.

We remind that another independent and interesting approach, which defines the pseudo-additivity class of entropies, has been formulated in [14].

Let \(p : \{p_i\}_{i=1}^W\), with \(W > 1\), \(\sum_{i=1}^W p_i = 1\) be a discrete probability distribution; we denote by \(P_W\) the set of all discrete probability distributions with \(W\) entries \(^1\).

**Definition 1.** A group entropy is a function \(S : P_W \to \mathbb{R}^+ \cup \{0\}\) which satisfies the Shannon-Khinchin axioms (SK1)-(SK3) and the composability axiom.

We shall distinguish two classes: the univariate group entropies, represented by \(Z\)-entropies, and the multivariate ones.

In [22], the following family of entropy functions has been introduced:

\[(2) \quad S(p) := F(S_1(p), \ldots, S_n(p)),\]

where \(\{S_1(p), \ldots, S_n(p)\}\) are all group entropies and \(F : \mathbb{R}^n \to \mathbb{R}\) is a suitable function. In particular, the conditions ensuring that the entropy (2) is still a group entropy have been clarified.

Besides, a general procedure has been proposed, allowing to realize a new group entropy as the result of a multivariate composition process of other group entropies, all of them sharing the same formal group law \(\Phi(x, y)\). This reminds us the typical procedure of Lie group theory, when applied to partial differential equations: one can generate new solutions of an equation from known ones [20]. This, in turn, shows the power and the versatility of the group-theoretical approach for the study of generalized entropies. The previous construction naturally leads us to define

\(^1\)We shall identify \(W\) with the integer part of \(W(N)\).
Multivariate group entropies, namely group entropies expressed in terms of suitable multivariate functions.

There is a simple realization of the multivariate class that will be defined in this article: the multivariate \( Z \)-entropies.

In order to make a self-contained presentation, I will first review some basic aspects of the theory of formal groups and rings, that provide us with an elegant algebraic language for the formulation of the theory of generalized entropies.

3. Formal groups and rings

The mathematical form of the composability axiom leads us naturally to the theory of formal groups \([4]\), which has found a wide range of applications, from algebraic topology to the theory of elliptic curves, arithmetic number theory and combinatorics (see also \([27]\), \([28]\), \([30]\) for recent applications). Classical reviews are \([23]\), \([6]\), \([19]\) and the monograph \([12]\), that we shall follow closely for notation.

Let \( R \) be a commutative ring with identity, and \( R[[x_1, x_2, \ldots]] \) be the ring of formal power series in the variables \( x_1, x_2, \ldots \) with coefficients in \( R \).

**Definition 2 (Formal group).** A commutative one–dimensional formal group law over \( R \) is a formal power series \( \Phi(x, y) \in R[[x, y]] \) such that \([4]\)

1) \( \Phi(x, 0) = \Phi(0, x) = x \),
2) \( \Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z)) \).

When \( \Phi(x, y) = \Phi(y, x) \), the formal group law is said to be commutative.

The existence of an inverse formal series \( \varphi(x) \in R[[x]] \) such that \( \Phi(x, \varphi(x)) = 0 \) follows from the previous definition. Let \( G(t) \) be the series:

\[
G(t) = \sum_{k=0}^{\infty} a_k \frac{t^{k+1}}{k+1}.
\]

Each coefficient of the inverse series \( G^{-1}(s) \) can be explicitly computed. We have \( a_0 = 1, a_1 = -b_1, a_2 = \frac{1}{2}b_1^2 - b_2, \ldots \).

**Definition 3.** The Lazard formal group law \([12]\) is defined by the formal power series

\[
\Phi(s_1, s_2) = G(G^{-1}(s_1) + G^{-1}(s_2)).
\]

For any commutative one-dimensional formal group law \( \Psi(x, y) \) over any ring \( R \), there exists a series \( \psi(x) \in R[[x]] \otimes \mathbb{Q} \) such that

\[
\psi(x) = x + O(x^2), \quad \text{and} \quad \Psi(x, y) = \psi^{-1}(\psi(x) + \psi(y)) \in R[[x, y]] \otimes \mathbb{Q}.
\]

Standard examples are the additive group law

\[
\Phi(x, y) = x + y,
\]

and the so called multiplicative group law

\[
\Phi(x, y) = x + y + \sigma xy, \quad \sigma \in \mathbb{R}.
\]

Let us also define the notion of formal ring, recently introduced in \([9]\).

**Definition 4 (Formal Ring).** Let \((R, +, \cdot)\) be a unital ring. A formal ring is a triple \( \mathcal{A} = (R, \Phi, \Psi) \) where \( \Phi, \Psi \in R[[x, y]] \) are formal power series such that

1) \( \Phi \) is a commutative formal group law.
(2) \( \Psi \) satisfies the relations
\[
\Psi(\Psi(x, y), z) = \Psi(x, \Psi(y, z)) , \\
\Psi(x, \Phi(y, z)) = \Phi(\Psi(x, y), \Psi(x, z)) , \\
\Psi(\Phi(x, y), z) = \Phi(\Psi(x, z), \Psi(y, z)) .
\]

The formal ring will be said to be commutative if \( \Psi(x, y) = \Psi(y, x) \).

As has been proved in [9], a one-dimensional realization of the previous definition is given by
\[
\Phi(s_1, s_2) = G(G^{-1}(s_1) + G^{-1}(s_2)) \\
\Psi(s_1, s_2) = G(G^{-1}(s_1) \cdot G^{-1}(s_2))
\]
for a suitable \( G(t) \in R[t] \). Besides, an \( n \)-dimensional generalization of both Definition 4 and equations (6)-(7) has been proposed in [9].

A simple example of one-dimensional formal ring structure is provided by the couple \( \{ \Phi(x, y), \Psi(x, y) \} \) where \( \Phi(x, y) \) is given by eq. (5) jointly with the new product
\[
\Psi(x, y) = \exp \left( \frac{(1/\sigma) \log(1 + \sigma x) \log(1 + \sigma y)}{\sigma} - 1 \right).
\]

Remark 1. In the subsequent applications to group entropies we will realize the equations (6), (7) in terms of standard real-valued functions (this, a priori, may require suitable constraints: for instance, \( x, y \geq 0 \) and \( \sigma > 0 \) in eq. (8)).

3.1. The multivariate Z-entropies. The notion of group logarithm [32] provides us with a group-theoretical deformation of the standard logarithm. It can be defined with different regularity properties. For the purposes of this paper, it is sufficient to think of a group logarithm as a function \( \ln_\chi(x) := \chi(\ln x) \), where \( \chi \in C^1(\mathbb{R}_0) \) is a strictly increasing function, with \( \chi(t) = t + O(t^2) \), taking positive values over \( \mathbb{R}^+ \).

Definition 5. A multivariate Z-entropy (MZE) is a function \( Z : P_W \rightarrow \mathbb{R}^+ \cup \{0\} \) of the form
\[
Z_{\chi, \alpha_1, \ldots, \alpha_n}(p) := \ln_\chi \left( \left( \sum_{i_1=1}^W p_{i_1}^{\alpha_1} \right)^{\frac{1}{\alpha_1}} \left( \sum_{i_2=1}^W p_{i_2}^{\alpha_2} \right)^{\frac{1}{\alpha_2}} \cdots \left( \sum_{i_n=1}^W p_{i_n}^{\alpha_n} \right)^{\frac{1}{\alpha_n}} \right),
\]
where \( \alpha_k > 0, k = 1, \ldots, n \).

The univariate Z-entropies [32] (for short, Z-entropies) have the form
\[
Z_{G, \alpha}(p) = \frac{\ln_\chi \left( \sum_{i=1}^W p_i^{\alpha} \right)}{1 - \alpha}.
\]
For \( n = 1 \) the multivariate class coincides with the Z-class: each entropy of the first class has an entropy of the second one with a suitable \( G(t) \) given by means of the identification \( \chi(t) = \frac{G(1 - \alpha t)}{1 - \alpha} \). Observe that a natural possibility is to choose \( \chi \) to be a concave function, and to consider \( 0 < \alpha < 1 \). More generally, if we wish to deal with Schur-concave functionals, these restrictions can be dropped.

It is straightforward to prove the following

Proposition 1. The MZEs (5) satisfy the first three Shannon-Khinchin axioms.
The main result of this section concerns the composability property of the MZE-class.

**Theorem 1.** The multivariate Z-entropies (5) satisfy the composability axiom, with composition law given by \( \Phi(x, y) = \chi(x^{-1}(x) + \chi^{-1}(y)) \).

**Proof.** Let \( P_{ij}^{A \times B} = P_i^A P_j^B \) be the joint probability distribution associated with the independent systems \( A \) and \( B \). Then we have that the entropies (5) satisfy the relation

\[
Z_{\chi,\alpha_1,\ldots,\alpha_n}(A \times B) = \ln \left( \prod_{k=1}^{n} \sum_{i_k,j_k} p_{ik,jk}^{\alpha_k} \right) = \ln \left( \prod_{k=1}^{n} \left( \sum_{i_k} p_{ik}^{\alpha_k} \right) \prod_{j_k} \left( \sum_{j_k} p_{jk}^{\alpha_k} \right) \right)
\]

\[
= \chi \left( \ln \left( \prod_{k=1}^{n} \sum_{i_k} p_{ik}^{\alpha_k} \right) + \ln \prod_{j_k} \left( \sum_{j_k} p_{jk}^{\alpha_k} \right) \right) = \chi \left( \chi^{-1} \left( \ln \left( \prod_{k=1}^{n} \sum_{i_k} p_{ik}^{\alpha_k} \right) \right) + \chi^{-1} \left( \ln \prod_{j_k} \left( \sum_{j_k} p_{jk}^{\alpha_k} \right) \right) \right) = \chi \left( \chi^{-1}(Z_{\chi,\alpha_1,\ldots,\alpha_n}(A)) + \chi^{-1}(Z_{\chi,\alpha_1,\ldots,\alpha_n}(B)) \right) = \Phi(Z_{\chi,\alpha_1,\ldots,\alpha_n}(A), Z_{\chi,\alpha_1,\ldots,\alpha_n}(B)).
\]

Therefore we deduce the following

**Corollary 1.** The multivariate Z-entropies are group entropies.

The physical and information-theoretical meaning of the function \( \chi \) can be grasped by close analogy with the approach introduced in [16], [33].

Let us assume that the state space growth function \( W \in C^1(\mathbb{R}^+) \) is strictly increasing, and takes positive values over \( \mathbb{R}^+ \). Also, we shall assume

\[
(W^{-1})'(1) \neq 0.
\]

Consequently, we get the useful representation

\[
Z_{\chi,\alpha_1,\ldots,\alpha_n}(\lambda) = \lambda \left( W^{-1} \left( \left( \sum_{i_1=1}^{W} p_{i_1}^{\alpha_1} \right)^{-1/\alpha_1} \left( \sum_{i_2=1}^{W} p_{i_2}^{\alpha_2} \right)^{-1/\alpha_2} \ldots \left( \sum_{i_n=1}^{W} p_{i_n}^{\alpha_n} \right)^{-1/\alpha_n} \right) - W^{-1}(1) \right),
\]

where \( \lambda \in \mathbb{R}^+ \), \( \alpha_1, \ldots, \alpha_n > 0 \). A natural choice is \( \lambda = \frac{1}{(W^{-1})'(1)} \). The form (12) will be taken into account in the forthcoming considerations.

The composition law associated is given by

\[
\phi(x, y) = \lambda \left\{ W^{-1} \left[ W \left( \frac{x}{\lambda} + W^{-1}(1) \right) \right] W \left( \frac{y}{\lambda} + W^{-1}(1) \right) - W^{-1}(1) \right\}.
\]

This formula, for the univariate case only, has been derived in [16] and rigorously proved in [33]. However, one can show that it is still valid in the multivariate case.

4. **NEW EXAMPLES OF MULTIVARIATE Z-ENTROPIES**

To our knowledge, the first examples of multivariate entropies are proposed in [22]. In this section, we shall present new examples of multivariate entropy functionals.
4.1. A linear combination of Rényi’s entropies. Let \( R(\alpha) = \frac{1}{1-\alpha} \ln(\sum_{i=1}^{W} p_i^\alpha) \) denote the Rényi entropy \([1]\). The simplest example is of course provided by the linear combination
\[
R_{\alpha_1,\ldots,\alpha_n}(p) = \lambda_1 R_{\alpha_1}(p) + \ldots \lambda_n R_{\alpha_n}(p),
\]
where \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}^+ \cup \{0\} \). The composition law is obviously the additive one, with \( \chi(t) = t \).

4.2. A very rapidly growing state space. Let
\[
W(N) \sim \exp \left[ k_2 \exp(k_3 N) - k_1 \right],
\]
with \( k_1, k_2, k_3 \in \mathbb{R}^+ \). The new group entropy extensive over the class of systems characterized by the growth rate \((15)\) is
\[
Z_{\alpha,k_1}(p) := k_1 \ln \left( \frac{\ln \left( \sum_i p_i^{\alpha} \right)}{k_1(1-\alpha)} + 1 \right).
\]
Its multivariate extension is straightforward. The composition law associated \((13)\) reads
\[
\Phi(x,y) = k_1 \ln \left( e^{\frac{x}{k_1}} + e^{\frac{y}{k_1}} - 1 \right).
\]

5. A new, general entropy for the super-exponential class

In this section, we shall introduce an entropy designed to be extensive in different super-exponential regimes. In this context, we are naturally led to the classical functional equation
\[
y e^y = x.
\]
For \( x \geq 0 \), it admits in \( \mathbb{R} \) the solution \( y = L(x) \); we denote here by \( L(x) \) the branch of the real \( W \)-Lambert function commonly denoted by \( W_0(x) \).

Generally speaking, let us consider the state space growth rate
\[
W(N) = e^{g(N \ln N)},
\]
where \( g \) is a strictly increasing \( C^1(\mathbb{R}_{\geq 0}) \) invertible function, with
\[
g(x) \to \infty \quad \text{for } x \to \infty.
\]
In the spirit of formal group theory, we also suppose
\[
g(x) = x + O(x^2).
\]
For notational simplicity, let us denote by \( \gamma(x) \) the compositional inverse of \( g \):
\[
\gamma(g(x)) = x. \quad \text{We observe that } \gamma'(1) \neq 0, \text{coherently with the assumption (11).}
\]
The function \( g(x) \) plays the role of “interpolating function”, allowing us to deform the growth rate of the considered state space and to explore different regimes.

The natural choice \( g(x) = c \cdot x \), with \( c \in \mathbb{R}^+ \), into eq. \((18)\) corresponds to the interesting case
\[
W(N) \equiv N^{CN},
\]
which was studied in \([15]\). Indeed, a Hamiltonian model, called the pairing model, was introduced as an example of a complex system in the class whose state space growth is expressed by the function \((21)\).

The growth rate \((18)\) depends on the asymptotic behaviour of \( g(x) \) when \( x \to \infty \).

\footnote{This to avoid confusion with the growth function \( W(x) \).}
We propose the new (univariate) entropy

\[ Z_{\gamma, \alpha}(p) := \exp \left[ L\left( \gamma \left( \frac{\ln(\sum_{i=1}^{W} p_{i}^{\alpha})}{1 - \alpha} \right) \right) \right] - 1 , \]

which can be extensive in different regimes. Precisely, let us consider the limit

\[ l = \lim_{x \to \infty} \frac{g(x)}{x} . \]

We can distinguish several cases.

a) If \( g(x) \) for \( x \to \infty \) grows faster than any linear function (i.e. \( l = \infty \)), the corresponding entropy (22) is extensive in super-factorial regimes. Clearly, these regimes can be further discriminated by specific choices of \( g(x) \) and consequently of \( \gamma(x) \).

b) If \( g(x) \) is a linear function, with \( l = c \in \mathbb{R}^+ \), the entropy (22) reproduces the group entropy introduced in [15]. In particular, for \( 0 < c < 1 \) the entropy (22) is extensive in the sub-factorial regime.

c) If

\[ g(x) = \exp(L(x)) - 1, \quad W(N) \sim e^N , \]

then \( l = 0 \), and the entropy (22) is extensive in the standard exponential regime. Thus, according to the previous discussion, the functional form (22) allows us to “sweep out” several universality classes in a simple and direct way. Obviously, for \( \gamma(x) = L^{-1}(\ln(x + 1)) \), we recover Rényi’s entropy.

5.1. The group-theoretical structure. In full generality, the composition law satisfied by entropies (22) reads

\[ \Phi(x, y) = \exp \left[ L(\phi((x + 1) \ln(x + 1), (y + 1) \ln(y + 1))) \right] - 1 , \]

where

\[ \phi(x, y) = g^{-1}(g(x) + g(y)) \]

is the composition law induced by the function \( g(x) \) (20). Thus, eq. (25) represents a family of group-theoretical structures, depending on the choice of \( g(x) \). Consequently, the functional (22) defines a family of new group entropies that cover different universality classes of systems. Therefore, we expect that the entropy (22), as well as its multivariate version

\[ Z_{\gamma(x), \alpha_1, \ldots, \alpha_n}(p) := \exp \left[ L\left( \gamma \left( \frac{\ln(\sum_{i=1}^{W} p_{i1}^{\alpha_1})}{1 - \alpha_1}, \ldots, \frac{\ln(\sum_{i=1}^{W} p_{in}^{\alpha_n})}{1 - \alpha_n} \right) \right) \right] - 1 \]

could be useful in applications to complex systems, biology and social sciences. These aspects will be studied elsewhere.

6. Formal Rings, Functional Equations and Discrete Systems

Group entropies by definition are intimately related with the theory of functional equations [1]: indeed, each group entropy, due to the composability axiom, satisfies a specific functional equation, expressing its composition law. In this section, we shall propose a general Lemma that describes, in the one-dimensional case, a more general connection among entropies and functional equations by means of the notion of formal rings. Also, we shall introduce families of difference equations, which
admit exact solutions constructed once again from the algebraic structure associated with group entropies.

The following simple Lemma is at the basis of this correspondence.

**Lemma 1.** Let \( \{ \Phi(x, y), \Psi(x, y) \} \) be two compatible composition laws of the form

\[
\Phi(x, y) = G(G^{-1}(x) + G^{-1}(y)) , \quad \Psi(x, y) = G(G^{-1}(x) \cdot G^{-1}(y)) ,
\]

where \( G(t) = t + O(t^2) \) is a real, analytic and invertible function on \( \mathbb{R} \).

i) The functional equation

\[
(27) \quad f(x + y) = \Phi(f(x), f(y))
\]

admits the solution

\[
f(x) = G(x) .
\]

ii) The functional equation

\[
(28) \quad f(x + y) = \Psi(f(x), f(y))
\]

admits the solution given by

\[
f(x) = G(\exp(x)) .
\]

iii) The functional equation

\[
(29) \quad f(xy) = \Phi(f(x), f(y))
\]

admits the solution expressed by

\[
f(x) = G(\ln(x)) .
\]

iv) The functional equation

\[
(30) \quad f(xy) = \Psi(f(x), f(y))
\]

admits the solution

\[
f(x) = G(x) .
\]

**Proof.** We shall treat explicitly the cases ii) and iii) only. The proof of the other cases is immediate.

ii) Equation (27) can be written in the form

\[
(31) \quad f(x + y) = \Psi(f(x), f(y)) = G(G^{-1}(f(x)) \cdot G^{-1}(f(y))) .
\]

Therefore, for \( f(x) = G(\exp(x)) \), we have the identity

\[
f(x + y) = G(G^{-1}(G(\exp(x))) \cdot G^{-1}(G(\exp(y)))) = G(\exp(x + y)) .
\]

iii) We have

\[
\Phi(f(x), f(y)) = G(G^{-1}(f(x)) + G^{-1}(f(y))) = G(G^{-1}(G(\ln x)) + G^{-1}(G(\ln y)))
\]

\[
= G(\ln x + \ln y) = f(xy) .
\]

\( \Box \)
Hereafter, we shall propose several examples of application of the previous results to the solution of suitable families of functional equations. The trivial case is represented by the standard Cauchy functional equation
\begin{equation}
 f(x + y) = f(x) + f(y),
\end{equation}
which corresponds to the additive group law (4) and to the choice \( G(t) = t \). We have immediately the general linear solution \( f(x) = ax, a \in \mathbb{R} \).

### 6.1. The formal ring structure associated with Tsallis entropy

The formal ring structure given by eqs. (5), (8) allows us to define the functional equations
\begin{equation}
 f(x + y) = f(x) + f(y) + \sigma f(x)f(y),
\end{equation}
\begin{equation}
 f(xy) = \exp\left(\frac{1}{\sigma}\log(1 + \sigma f(x)) \log(1 + \sigma f(y))\right) - 1,
\end{equation}
with \( \sigma > 0 \). The structure is generated by the group exponential \( G(t) = \frac{\exp(\sigma t) - 1}{\sigma} \), which provides an exact solution for both eqs. (33), (34) according to Lemma 1.

### 6.2. A rational group law

Consider the formal group law
\begin{equation}
 \Phi(x,y) = \frac{x + y + axy}{1 + bxy}.
\end{equation}

Notice that when \( a = b = 0 \), we recover the standard additive law (4); for \( b = 0 \), we recover the case (5). We mention that a specific one-parametric realization of the formal group (35), i.e.
\begin{equation}
 \Phi(x,y) = \frac{x + y + (\alpha - 1)xy}{1 + \alpha xy}
\end{equation}
plays an important role in algebraic topology. Precisely, for \( \alpha = -1, 0, 1 \), we obtain group laws associated with the Euler characteristic, with the Todd genus and the Hirzebruch L-genus [6], respectively. The functional equation
\begin{equation}
 f(x + y) = \frac{f(x) + f(y) + af(x)f(y)}{1 + bf(x)f(y)}
\end{equation}
with \( a, b > 0 \) admits the solution
\begin{equation}
 f(x) = \frac{2(e^{rx} - 1)}{-a(e^{rx} - 1) + \sqrt{a^2 + 4b(e^{rx} + 1)}}.
\end{equation}

This solution coincides with the group logarithm introduced in [10], where a generalized, bi-parametric Tsallis entropy has been proposed. In turn, the multiplicative equation
\begin{equation}
 f(xy) = \frac{f(x) + f(y) + af(x)f(y)}{1 + bf(x)f(y)}
\end{equation}
admits the solution
\begin{equation}
 f(x) = \frac{2(x^r - 1)}{-a(x^r - 1) + \sqrt{a^2 + 4b(x^r + 1)}} \quad x, r > 0.
\end{equation}

The formal product \( \Psi(x,y) \) associated with the group law (36) has been computed in [9]. Thus, a similar analysis can be extended to the other functional equations related with the given ring structure.

Interesting examples of functional equations arising from generalized cohomology theories have been discussed in [5] in the context of formal group theory.
6.3. **Discrete Systems and Sequences of Integer Numbers.** We wish to establish a connection between group entropies and discrete dynamical systems. To this aim, we shall consider a regular lattice of points parametrized by \( n \in \mathbb{Z} \) and the correspondence \( f(n) \rightarrow z_n, \quad n \in \mathbb{Z} \). In this way, we induce an application \( z : \mathbb{N} \rightarrow \mathbb{R} \) associated with \( f \). Consequently, given a formal ring structure \( \mathcal{A} = \{ \Phi(x, y), \Psi(x, y) \} \), we can introduce the independent discrete equations

\[
\begin{align*}
(39) \quad z_{n+m} &= \Phi(z_n, z_m) \quad \text{(DE1)} \\
(40) \quad z_{n+m} &= \Psi(z_n, z_m) \quad \text{(DE2)} \\
(41) \quad z_{n-m} &= \Phi(z_n, z_m) \quad \text{(DE3)} \\
(42) \quad z_{n-m} &= \Psi(z_n, z_m) \quad \text{(DE4)}
\end{align*}
\]

obtained from the functional equations \((27)-(30)\) by means of the previous correspondence. Exact solutions of eqs. \((DE1)-(DE4)\) can be constructed applying Lemma 1. As a specific example of realization of \((DE1)\), let us consider the equation

\[ z_{n+m} = z_n + z_m + p z_n z_m, \quad p \in \mathbb{R} . \]

Its solution is, group-theoretically,

\[ z_n = \frac{\exp(pn) - 1}{p} . \]

The related form \((41)\)

\[ w_{n,m} = w_n + w_m + p w_n w_m, \quad p \in \mathbb{R} , \]

admits the solution \( w_n = G(\ln(z_n)) = \frac{n^p-1}{p} \).

We mention that for \( p \in \mathbb{N} \setminus \{0\} \), we can easily generate sequences of integers with interesting properties. Indeed, by means of a rescaling, we obtain that \( q_n := n^p - 1 \) satisfies the recurrence

\[ q_{n,m} = q_n + q_m + q_n q_m, \quad n, m \in \mathbb{Z} . \]

For instance, for \( p = 3 \), we generate the sequence of integers

\[ \ldots, -513, -344, -217, -126, -65, -28, -9, -2, -1, 0, 7, 26, 63, 124, 215, 342, 511, \ldots \]

or, for \( p = 5 \), the sequence of integers

\[ \ldots, -32769, -16808, -7777, -3126, -1025, -244, -33, -2, -1, 0, 31, 242, 1023, 3124, 7775, 16806, 32767, \ldots \]

all of them satisfying the recurrence \((44)\).

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