Varieties defined by natural transformations

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Abstract

We define varieties of algebras for an arbitrary endofunctor on a cocomplete category using pairs of natural transformations. This approach is proved to be equivalent to the one of equational classes defined by equation arrows. Free algebras in the varieties are investigated and their existence is proved under the assumptions of accessibility.

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In universal algebra we deal with varieties - classes of algebras satisfying a certain collection of identities (pairs of terms of corresponding language). This concept was generalized by Adámek and Porst in [3]. They worked with algebras for an endofunctor on a cocomplete category and used the free-algebra construction developed by Adámek in [1] (a chain of term-functors) to define the equation arrows as certain regular epimorphisms. Using them they defined equational categories as analogy to varieties. These categories are later studied in [5].

We focus on another approach to varieties of algebras for a functor. We also use the free-algebra construction and we define natural term as a natural transformation with codomain in a term-functor. A pair of natural terms with the common domain will be called natural identity and will be satisfied on an algebra if its both natural transformations have the same term-evaluation on this algebra. Natural identities induce classes of algebras, which are proved to be precisely the classes defined by means of equation arrows. We present several examples of such classes and show that in some cases this approach essentially simplifies the presentation.

In the second chapter we investigate free algebras in a variety. The induction by natural identities allows us to make the restriction on the identities with the domain preserving the colimits of some small chains. Such identities
will be called accessible. These cases still cover most of the usual examples and we prove that such varieties have free algebras. The proof uses the conversion of variety to a category of algebras for a diagram of monads used by Kelly in [7] to define algebraic colimit of monads. His theorem proving the existence of free objects of this category yields the existence of free algebras in the variety induced by accessible identities.

**Notational convention**  The constant functor mapping the objects to object $X$ will be denoted by $C_X$. The initial object in a cocomplete category will be denoted by $0$. For functors, we omit the brackets and the composition mark $\circ$, when possible. The class of objects and morphisms of a category will be denoted by Ob and Mor, respectively. The concrete isomorphism between two concrete categories over $\mathcal{C}$ (i.e. the isomorphism preserving the forgetful functor) will be denoted by $\cong_C$.

# 1 Classes of algebras

## 1.1 Algebras and equational classes

**Definition 1.1** Let $F$ be an endofunctor on a category $\mathcal{C}$. By $\text{Alg } F$ we denote the category of $F$-algebras - its objects are pairs $(A, \alpha)$, where $\alpha : FA \to A$ is a morphism in $\mathcal{C}$. The morphism $\phi_F : (A, \alpha) \to (B, \beta)$ of $F$-algebras is a morphism $\phi : A \to B$ such that $\phi \circ \alpha = \beta \circ F\phi$. The subscript $F$ in the notation of morphism is usually omitted.

**Remark 1.1** There is a forgetful functor $Z_F : \text{Alg } F \to \mathcal{C}$ assigning to an algebra $(A, \alpha)$ its underlying object $A$.

From now on, let $\mathcal{C}$ be a cocomplete category. Let us recall the free-algebra construction (introduced in [1], generalized in [3]). We show the definition in the functorial form.

**Definition 1.2** By transfinite induction we define term functors $F_n : \mathcal{C} \to \mathcal{C}$, for $n \in \text{Ord}$ and natural transformations $w_{m,n} : F_m \to F_n$, for $m \leq n$:

- **Initial step:** $F_0 = \text{Id}_\mathcal{C}$, $w_{0,0} = \text{id}$
- **Isolated step:** Let $F_{n+1} = FF_n + \text{Id}_\mathcal{C}$, the transformations $w_{0,n+1} = t_{n+1}$ and $q_n : FF_n \to F_{n+1}$ are the canonical injections of $\text{Id}_\mathcal{C}$ and
$FF_n$, respectively, into the coproduct and $w_{m+1,n+1} = [Fw_{m,n}, \text{id}_{\mathcal{C}}]$ for $m \leq n$ is defined by

$$
\begin{array}{ccc}
FF_m & \xrightarrow{Fw_{m,n}} & FF_n \\
q_m & & q_n \\
F_{m+1} & \xrightarrow{w_{m+1,n+1}} & FF_n + \text{Id}_{\mathcal{C}} = F_{n+1} \\
\epsilon_{m+1} & & \epsilon_{n+1} \\
\text{Id}_{\mathcal{C}} & & \\
\end{array}
$$

If $m$ is a limit ordinal, then we define $w_{m,m+1}$ as the unique factorization of $\{w_{k,m+1} | k < m\}$ over the colimit cocone $\{w_{k,m} | k < m\}$.

**Limit step:** $F_n = \text{colim}_{m<n} F_m$ and $w_{m,n}$ is the corresponding component of colimit cocone.

The construction gives rise to the transformation $y_n : F \rightarrow F_n$ defined by

$$y_n = w_{1,n} \circ q_0$$

for every ordinal $n > 0$.

To distinguish the transformations for different functors we denote the name of the functor in the superscript: $w^F_{m,n}$, $q^F_n$, $\epsilon^F_n$, $y^F_n$.

The construction yields for every $m \leq n$ the property:

$$w_{m,n} \circ q_m = q_n \circ Fw_{m,n}. \quad (1)$$

**Remark 1.2** If we substitute $\text{Id}_{\mathcal{C}}$ by $C_0$ in initial step of construction, then we get equivalent concept.

As a consequence of the definition we get the following properties (see [3]).

**Remark 1.3** Given an $F$-algebra $(A, \alpha)$, for every $n \in \text{Ord}$ there is a morphism (a term-evaluation on $(A, \alpha)$)

$$\epsilon_{n,(A,\alpha)} : F_n A \rightarrow A$$

defined recursively by: $\epsilon_{0,(A,\alpha)} = \text{id}_A$, $\epsilon_{n+1,(A,\alpha)} = [\alpha \circ F\epsilon_{n,(A,\alpha)}, \text{id}_A]$.

$$
\begin{array}{ccc}
FF_n(A) & \xrightarrow{F\epsilon_{n,(A,\alpha)}} & F(A) \\
q_{n,A} & & \alpha \\
F_{n+1}(A) & \xrightarrow{\epsilon_{n+1,(A,\alpha)}} & A \\
\epsilon_{n+1,A} & & \text{id}_A \\
A & & \\
\end{array}
$$
and by \( \epsilon_{t,(A,\alpha)} = \colim_{m<l} \epsilon_{m,(A,\alpha)} \) for a limit ordinal \( l \). Then for every \( n, m \leq n \) we have:

\[
\begin{align*}
\epsilon_{n+1,(A,\alpha)} \circ q_{n,A} &= \alpha \circ F \epsilon_{n,(A,\alpha)} \quad (2) \\
\epsilon_{n,(A,\alpha)} \circ \iota_{n,A} &= \text{id}_A \quad (3) \\
\epsilon_{m,(A,\alpha)} &= \epsilon_{n,(A,\alpha)} \circ w_{m,n,A} \quad (4) \\
\epsilon_{n,(A,\alpha)} \circ y_{n,A} &= \alpha, \quad (5)
\end{align*}
\]

where the last property requires \( n > 0 \). We write the name of the functor in the superscript \( \epsilon_{k,(A,\alpha)} = \epsilon_{k,(A,\alpha)}^{F} \), if necessary.

We recall here the notion of equational category of \( F \)-algebras introduced in [3].

**Definition 1.3** Let \( X \) be an object of \( \mathcal{C} \), \( n \in \text{Ord} \). An equation arrow of arity \( n \) over \( X \) is defined as a regular epimorphism \( e : F_{n}X \to E \). The object \( X \) is called a variable-object of \( e \).

We say, that an \( F \)-algebra \((A,\alpha)\) satisfies an equation arrow \( e : F_{n}X \to E \) if for every \( f : X \to A \) there is a morphism \( h : E \to A \) such that \( \epsilon_{n,(A,\alpha)} \circ F_{n}f = h \circ e \).

For a class \( \mathcal{E} \) of equation arrows we define an equational class of \( F \)-algebras induced by \( \mathcal{E} \) as the class of all algebras satisfying all equations \( e \in \mathcal{E} \). Considered as a full subcategory of \( \text{Alg} F \) it is called an equational category and denoted by \( \text{Alg} (F,\mathcal{E}) \). If \( \mathcal{E} \) is a singleton, we say \( \text{Alg} (F,\mathcal{E}) \) is single-based.

As shown in [3], this approach generalizes classical universal algebra on sets, since every identity uniquely determines the regular epimorphism on the set of all terms, which is given by unification of terms included in identity.

### 1.2 Naturally Induced Classes

Now we introduce the concept of algebras induced by natural transformations.

**Definition 1.4** Let \( n \) be an ordinal and \( G \) be a \( \mathcal{C} \)-endofunctor. A natural transformation \( \phi : G \to F_{n} \) is called a natural term, more precisely a \( G \)-term.

By \( G \)-identity we mean a pair of \( G \)-terms. Such pairs are called natural identities. Let \( \phi \) and \( \psi \) be \( m \)-ary and \( n \)-ary \( G \)-terms, respectively. The functor \( G \) is called a domain and \((m,n)\) is an arity-couple of identity \((\phi, \psi)\). If \( m = n \), we say \((\phi, \psi)\) has the arity \( n \).
We say, that an \( F \)-algebra \((A, \alpha)\) satisfies the \( G \)-identity \((\phi, \psi)\), if

\[
\epsilon_{m,(A, \alpha)} \circ \phi_A = \epsilon_{n,(A, \alpha)} \circ \psi_A.
\]

Then we write

\[
(A, \alpha) \models (\phi, \psi).
\]

For a class \( \mathcal{I} \) of natural identities we define a naturally induced class of \( F \)-algebras as the class of all algebras satisfying all identities \((\phi, \psi) \in \mathcal{I}\). The corresponding full subcategory of \( \text{Alg} \, F \) is denoted by \( \text{Alg} \, (F, \mathcal{I}) \). If \( \mathcal{I} \) is a singleton, we say \( \text{Alg} \, (F, \mathcal{I}) \) is single-induced.

Two natural identities are said to be algebraically equivalent iff they induce the same classes of \( F \)-algebras. Analogously we define the algebraic equivalence of classes of natural identities. The algebraic equivalence relation will be denoted by \( \approx \).

\begin{remark}
1. Arities of components of a natural identity can be arbitrarily raised. Clearly for an identity \((\phi, \psi)\) of arity-couple \((m_1, m_2)\) we have \((\phi, \psi) \approx (w_{m_1,n} \circ \phi, w_{m_2,n} \circ \psi)\) for every \( n \geq \max\{m_1, m_2\}\). Hence every natural identity is algebraically equivalent to the one consisting of natural terms of the same arity.

2. Every set \( \mathcal{N} = \{(\phi_i, \psi_i)|i \in I\} \) of \( n \)-ary natural identities is algebraically equivalent to the singleton. Clearly \( \mathcal{N} \approx \{\phi, \psi\} \), where \( \phi, \psi \) are the unique factorizations of the cocones \( \phi_i, \psi_i \), respectively, over the coproduct of domains of single identities.

3. As a consequence, every class naturally induced by a set of identities is single-induced.
\end{remark}

### 1.3 Examples of naturally induced classes

In section \[\text{(1.4)}\] we show, that every equational class is naturally induced and vice versa. And since the concept of equational classes generalizes the varieties in universal algebra, every variety of algebras in the classical sense is naturally induced class. Explicit correspondence is shown in the following example.

\begin{example}
Let \( \mathcal{C} = \text{Set} \), \( \Sigma \) be a signature consisting of operation symbols \( \sigma \) of (possibly infinite) arities \( \text{ar}(\sigma) \). Let \( F = \bigsqcup_{\sigma \in \Sigma} \text{hom}(\text{ar}(\sigma), -) \) and \( u_{\sigma} : \text{hom}(\text{ar}(\sigma), -) \to F \) be the canonical inclusion for every \( \sigma \in \Sigma \). Then the category of \( \Sigma \)-algebras is isomorphic to \( \text{Alg} \, F \). For each \( \Sigma \)-term \( \tau \) let \( X_\tau \) be the set of variables occurring in \( \tau \) and let \( d(\tau) \) be the depth of \( \tau \) (supremum
of ordinals corresponding to chains of the proper subterms of $\tau$ ordered by subterm-relation).

For a given term $\tau$ we assign a $d(\tau)$-ary $G_{\tau}$-term $\phi_{\tau}$, where $G_{\tau} = \mathrm{hom}(X_\tau, -)$. The transformation $\phi_{\tau}$ is defined inductively: if $\tau$ is a variable $x$, then $\phi_{\tau} : \mathrm{hom}([x], -) \to F_0$ is obvious isomorphism. If $\tau = \sigma(p_i; i \in \mathrm{ar}(\sigma))$ and we have $\phi_{\rho_i} : \mathrm{hom}((\rho(i)), -) \to F_{d(\rho_i)}$ for each $i \in \mathrm{ar}(\sigma)$, then we can extend all transformations $\phi_{\rho_i}$ to $\phi'_i : \mathrm{hom}((\rho(i)), -) \to F_n$, where $n = \sup\{d(\rho_i)|i \in \mathrm{ar}(\sigma)\}$. We define $\phi_{\tau}$ the following way. Since $X_{\rho_i} \subseteq X_\tau$ for every $i$, we have $p_i : \mathrm{hom}((X_\tau), -) \to \mathrm{hom}((X_{\rho_i}), -)$, hence the factorization over the limit cone yields the unique $r : \mathrm{hom}((X_\tau), -) \to \prod_{i \in \mathrm{ar}(\sigma)} \mathrm{hom}((X_{\rho_i}), -)$. We define $\phi_{\tau}$ as the following composition:

$$
\begin{align*}
\hom(X_\tau, -) &\xrightarrow{r} \prod_{i \in \mathrm{ar}(\sigma)} \hom(X_{\rho_i}, -) \\
F_{n+1} &\xrightarrow{q_n} \coprod_{i \in \mathrm{ar}(\sigma)} \hom(X_{\rho_i}, -) \xrightarrow{\prod \phi'_i} \prod_{i \in \mathrm{ar}(\sigma)} F_n
\end{align*}
$$

Observe, that $n + 1 = d(\tau)$.

For each $\Sigma$-term we have assigned a natural term. Now for identity $(\tau_1, \tau_2)$ consisting of two $\Sigma$-terms with variables in $X$ we assign a pair of corresponding natural $\mathrm{hom}(X, -)$-terms. It is easy to see, that we get the identity, which induces exactly the variety given by $(\tau_1, \tau_2)$. For example, monoids are objects of $\mathbf{Alg}((\mathrm{hom}(2, -) + \mathrm{hom}(0, -), \{i, j, k\})$, where $i$ is binary identity with domain $\mathrm{hom}(3, -)$ and stands for associativity while $j, k$ are unary with domain Id and correspond to left and right neutrality of 1.

The concept can be used to define naturally induced classes of algebras even on some illegitimate categories.

**Example 1.2** Let $\mathcal{C} = \mathbf{End}A$ be the illegitimate category of endofunctors on some cocomplete category $\mathcal{A}$. The composability of objects of $\mathcal{C}$ yields for every $k \in \omega$ the existence of the ”composition power functor” $S_k : \mathcal{C} \to \mathcal{C}$ such that

$$
S_k(P) = \underbrace{P \circ P \circ \ldots \circ P}_{k \text{ times}}.
$$

We can define analogies for universal algebras - all we need to do, is to substitute products of sets by composition of functors and each $\mathrm{hom}(k, -)$ by $S_k$ in the description above. As analogy of monoids we get the category $\mathbf{Monad} \mathcal{A}$ of monads on $\mathcal{A}$. Namely, $\mathbf{Monad} \mathcal{A} = \mathbf{Alg}((S_2 + S_0), \{i, j, k\})$, where domains of identities $i, j, k$ are $S_3, S_1, S_1$, respectively. Each operation
\[ \pi : (S_2 + S_0)(P) \to P \text{ decomposes into } \mu : S_2(P) = PP \to P \text{ and } \eta : S_0 = \text{Id} \to P \text{ and the satisfaction of identities fully corresponds to usual condition claimed on } \mu \text{ and } \eta. \]

Theorem 3.6 in [3] describes the equational presentation of category of algebras for a monad. The following example shows their the presentation by natural identities.

**Example 1.3** Given a monad \( M = (M, \eta, \mu) \) on \( C \), then its Eilenberg-Moore category \( M\text{-alg} \) is a class of \( M \)-algebras induced by two natural identities:

\[
\begin{align*}
M^2 & \xrightarrow{\mu} M \xrightarrow{q_0} M_1 \\
M_{q_0} & \xrightarrow{MM_{q_1}} M_1 \\
\text{Id} & \xrightarrow{id} M_0
\end{align*}
\]

Therefore

\[ M\text{-alg} = \text{Alg} (M, \{(q_0 \circ \eta, \text{id}_{\text{Id}}), (q_1 \circ Mq_0, q_0 \circ \mu)\}). \]

**Example 1.4** Consider the power-set monad on \( \text{Set} \) defined by power-set functor \( P \) and transformation \( \eta : \text{Id}_{\text{Set}} \to P, \mu : P^2 \to P \) given by assignments \( \eta_X(x) = \{x\}, \mu_X(\{X_i | i \in I\}) = \bigcup_{i \in I} X_i. \) As a concrete instance of the previous case for power-set monad \( (P, \eta, \mu) \) we get the category of join-complete semilattices \( \text{JCSlat} \). Hence this class is presentable by a pair of naturally induced identities - compare with presentation by a proper class of equation arrows (see [3], Example 3.3 - we need equation arrows \( e_X : F_3X \to E_X \) for every set \( X \)).

### 1.4 Conversion theorem

Our aim is to prove that naturally induced classes and equational classes coincide. At first we show that every single-based equational class is naturally induced. Then, conversely, we prove that every class induced by a single natural identity is equational. The crucial point of the proof is the local smallness of category \( C \).

**Remark 1.5** Within the proof we use the copower functor:

*Given an object \( Q \in C \), there is a functor \(- \bullet Q : \text{Set} \to C\), which is left adjoint to \( \text{hom}(Q, -) : C \to \text{Set}\). It assigns to a set \( M \) the coproduct of \( M \) copies of \( Q \) (the "\( M \)-th" copower of \( Q \)) and for a mapping \( h : M \to N \) we*
define $h \cdot Q$ as the unique factorization of cocone $u_{h(m)} : Q \to \coprod_{j \in M} Q, m \in M$, over a colimit cocone $u_m : Q \to \coprod_{j \in M} Q$.

Then we get the adjunction $(\eta, \varepsilon) : (- \cdot Q) \dashv \hom(Q, -) : C \to \text{Set}$, where the unit morphism $\eta_X : X \to \hom(Q, X \cdot Q)$ for a set $X$ and $x \in X$ is defined by $\eta_X(x) = u_x : Q \to X \cdot Q$, i.e. the $x$-labeled canonical injection into the coproduct. Moreover, for an object $A$ of $C$, the counit $\varepsilon : \hom(Q, A) \cdot Q \to A$ is defined as the unique factorization of a cocone $\{ \phi : Q \to A \}$ over the colimit.

**Lemma 1.6** Every single-based equational class is naturally single-induced class.

**Proof:** Let $\mathcal{S}$ be a single-based equational class of $F$-algebras defined by an equation arrow $e$, where $e$ is a regular epimorphism $F_n X \to E$ such that $(E, e)$ is a coequalizer of $\phi_0, \psi_0 : Q \rightrightarrows F_n X$. We define a mapping $\theta_{\phi,A} : \hom(X, A) \cdot Q \to F_n A$. For every $f : X \to A$ let $\theta_{\phi,A}(f) = F_n f \circ \chi_0 : Q \to F_n A$. Now let

$$G = (- \cdot Q) \circ \hom(X, -),$$

$$\phi_A = \widetilde{\theta_{\phi,A}} : \hom(X, A) \cdot Q \to F_n A.$$  

Clearly $\phi_A$ is the component of a natural transformation $\phi : G \to F_n$. Observe, that for every $f : X \to A$, holds

$$\phi_A \circ u_f = \theta_{\phi,A}(f) = F_n f \circ \phi_0.$$  

Analogously we define the natural transformations $\theta_{\psi,-} : G \to F_n$ and $\psi : G \to F_n$ satisfying $\psi_A \circ u_f = F_n f \circ \psi_0$. Now we have the functor $G$ and $G$-identity $(\phi, \psi)$. It remains to show, that it induces exactly the equational class $\mathcal{S}$.

Let $(A, \alpha)$ satisfy the equation arrow $e$. Then for every $f : X \to A$ there is $h : E \to A$, such that $\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e$. Then we have

$$\epsilon_{n,(A,\alpha)} \circ \phi_A \circ u_f = \epsilon_{n,(A,\alpha)} \circ F_n f \circ \phi_0 = h \circ e \circ \phi_0 = h \circ e \circ \psi_0,$$

and by symmetry we get $\epsilon_{n,(A,\alpha)} \circ \phi_A \circ u_f = \epsilon_{n,(A,\alpha)} \circ \psi_A \circ u_f$. Since $f$ was chosen arbitrarily and injections $u_f$ form a colimit cocone, we have $\epsilon_{n,(A,\alpha)} \circ \phi_A = \epsilon_{n,(A,\alpha)} \circ \psi_A$, i.e. $(A, \alpha)$ satisfies the $G$-identity $(\phi, \psi)$.  

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Now let \((B, \beta)\) be an \(F\)-algebra in a class induced by the \(G\)-identity \((\phi, \psi)\). Let \(g : X \to B\) be a morphism in \(C\). Then we have

\[
\epsilon_{n,(B,\beta)} \circ F_n g \circ \phi_0 = \epsilon_{n,(B,\beta)} \circ \phi_B \circ u_g \\
= \epsilon_{n,(B,\beta)} \circ \psi_B \circ u_g
\]

and again by symmetry we get \(\epsilon_{n,(B,\beta)} \circ F_n g \circ \phi_0 = \epsilon_{n,(B,\beta)} \circ F_n g \circ \psi_0\), hence \(\epsilon_{n,(B,\beta)} \circ F_n g\) coequalizes the pair \((\phi_0, \psi_0)\) and there is a unique \(h : E \to B\) such that \(\epsilon_{n,(B,\beta)} \circ F_n g = h \circ e\). Thus \((B, \beta)\) satisfies the equation arrow \(e\).

\[\square\]

**Lemma 1.7** Every naturally single-induced class is equational.

**Proof:** Let \(G\) be a \(C\)-endofunctor. Let \(N\) be a class induced by a \(G\)-identity \((\phi, \psi)\). Due to Remark 1.4 we may assume that \(\phi\) and \(\psi\) have the same arity, say \(n\), therefore both are the natural transformations \(G \to F_n\). Let \((E, e)\) be the coequalizer of \(\phi\) and \(\psi\). Then for every object \(X\) of \(C\) we have a morphism \(e_X : F_n X \to EX\). Let \(E = \{e_X | X \in \text{Ob} C\}\). We will prove \(N = \text{Alg}(F, E)\).

Let \((A, \alpha)\) satisfy \((\phi, \psi)\). Then for every \(X \in \text{Ob} C\) and \(f : X \to A\) we have

\[
\epsilon_{n,(A,\alpha)} \circ F_n f \circ \phi_X = \epsilon_{n,(A,\alpha)} \circ \phi_A \circ Gf \\
= \epsilon_{n,(A,\alpha)} \circ \psi_A \circ Gf \\
= \epsilon_{n,(A,\alpha)} \circ F_n f \circ \psi_X,
\]

therefore we have coequalizing morphism \(\epsilon_{n,(A,\alpha)} \circ F_n f\) for \((\phi_X, \psi_X)\). Since the colimits of functors are calculated componentwise, \(e_X\) is a coequalizer of \((\phi_X, \psi_X)\), therefore there is a unique \(h : EX \to A\), such that \(\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e_X\).

Given an \(F\)-algebra \((B, \beta)\) satisfying all equation arrows from \(E\), then it satisfies the arrow \(e_B : F_n B \to EB\) and there is \(h : EB \to B\) (chosen for \(\text{id}_B : B \to B\)) such that \(\epsilon_{n,(B,\beta)} = h \circ e_B\), therefore the property is satisfied, since \(e_B\) coequalizes the pair \((\phi_B, \psi_B)\).

\[\square\]

**Theorem 1.8** Let \(F\) be an endofunctor on a cocomplete category \(C\). Then the equational classes of \(F\)-algebras coincide with naturally induced classes of \(F\)-algebras.

**Proof:** Every equational class \(S\) is a (possibly large) intersection of single-based ones and those are by the Lemma 1.6 naturally induced, hence we get \(S\) to be naturally induced by the class of all natural identities present in some of the collection. Conversely, the naturally induced class \(N\) is a (possibly large) intersection of the ones induced by a single natural identity, which are due to Lemma 1.7 equational classes induced by a class of equation arrows.

Union of these classes defines the class of all equation arrows defining the class \(N\) as an equational class.

\[\square\]
Definition 1.5 Class of algebras induced by equations or natural identities is called a variety.

2 Free algebras in the variety

Our aim is to answer the question of existence of free algebras in varieties. At first we recall well-known results involving free algebras, which will be used to solve this question.

2.1 Free algebras and monads

We will work with $\mathcal{C}$-endofunctors preserving the colimits of $\lambda$-labeled chains, where $\lambda$ is infinite limit ordinal - let the class containing these functors be denoted by $\text{End}_\lambda \mathcal{C}$. Since colimits commute with colimits we get the following property (see also [7], 2.4.).

Proposition 2.1 The class $\text{End}_\lambda \mathcal{C}$ is closed under colimits and compositions.

Definition 2.1 Let $G$ be an endofunctor on $\mathcal{C}$. The natural $G$-identity is called accessible if $G$ preserves the colimits of $\lambda$-labeled chains for some infinite limit ordinal $\lambda$.

Definition 2.2 The functor $F$ admitting free $F$-algebras is called a varietor.

Let $F$ preserve the colimits of $\lambda$-indexed chains. Then, as shown in [1], $F$ is a varietor. Since $F$ preserves the $\lambda$-labeled chains, $FF_\lambda$ is a colimit of chain $\{FF_n|n<\lambda\}$. Hence one can see that $w_{\lambda,\lambda+1}$ is isomorphism. In such a case we say that the free $F$-algebra construction stops after $\lambda$ steps. If we set $v = \text{colim}_{n<\lambda} q_n$, on every object $A$ we get the free $F$-algebra

$$V_F(A) = (F_\lambda A, v_A).$$

If necessary, we write the name of functor $F$ in the superscript: $v = v^F$.

This construction gives rise to the functor $V_F : \mathcal{C} \rightarrow \text{Alg} F$, $V_F = (F_\lambda, v)$ together with transformation $\epsilon_\lambda : V_FZ_F \rightarrow \text{Id}_{\text{Alg} F}$, $\epsilon_{\lambda,(A,\alpha)} : (F_\lambda A, v_A) \rightarrow (A, \alpha)$. Hence we have got the free functor $V_F$ and adjunction $V_F \dashv Z_F$, where the unit and counit are $\iota_\lambda$ and $\epsilon_\lambda$, respectively.

It is well known fact, that this adjunction yields the free monad over a functor $F$ - see [2], Theorem 20.56. Hence the free monad over $F$ is

$$\mathcal{M}(F) = (F_\lambda, \eta^F, \mu^F),$$
where $\eta^F = \iota^F_\lambda$ and $\mu^F = Z_F \epsilon_F \nu_F$ and universal morphism for $F$ is $y^F_\lambda : F \to F_\lambda$. More detailed approach to the theory of monads can be found in [2], [6] and [7].

Now we use another functor $G$ in $\text{End}_\lambda C$ and we work with its algebras. We point out the important consequences of the discussion above:

**Proposition 2.2** Let there be a transformation $\rho : G \to F_\lambda$. Then there is a transformation $\sigma : G_\lambda \to F_\lambda$, subject to the conditions:

1. $\sigma = \overline{\rho}$ is given by the freeness of $\mathcal{M}(F)$ as the unique monad transformation $\mathcal{M}(G) \to \mathcal{M}(F)$ corresponding to $\rho : G \to F_\lambda$; thus
   
   $\sigma \circ y^G_\lambda = \rho$.

2. $\sigma_A = \overline{\eta^F_\lambda}$ is given by the adjunction $\mathcal{V}_G \dashv Z_G$ as the unique $G$-algebra morphism $\mathcal{V}_G(A) \to P(A)$ corresponding to $\eta^F_\lambda : A \to F_\lambda A = Z_G P(A)$, where $P : C \to \text{Alg} G$ is the functor assigning to an object $A$ an algebra $(F_\lambda A, \beta_A)$ and $\beta_A = (\mu^F \circ \rho F_\lambda)_A$. Hence
   
   $\sigma \circ \iota^G_\lambda = \eta^F.$

3. For $k \leq \lambda$ let $\epsilon^G_{k,P} : G_k F_\lambda \to F_\lambda$ be the obvious transformation with the components $\epsilon^G_{\lambda,P(A)}$. Then the following property holds:
   
   $\sigma = \epsilon^G_{\lambda,P} \circ G \lambda \eta^F$.

We will show that $\sigma$ defined above from the transformation $\rho : G \to F_\lambda$ can be gained via the colimit construction, which will be later useful.

**Definition 2.3** For all $k \in \mathbb{N}$ we define the transformations $\rho_k : G_k \to F_\lambda$ inductively: $\rho_1 = [\rho, \eta^F], \rho_{k+1} = [\mu^F \circ \rho F_\lambda \circ G \rho_k, \eta^F]$

**Lemma 2.3** For every $j < k \in \mathbb{N}$ holds $\rho_k \circ w^G_{j,k} = \rho_j$.

**Proof:** For every natural $j < k$ we have

\[
\rho_k \circ w^G_{j,k} = [\mu^F \circ \rho F_\lambda \circ G \rho_k, \eta^F] \circ w^G_{j,k} = [\mu^F \circ \rho F_\lambda \circ G \rho_{k-1} \circ G w^G_{j-1,k-1}, \eta^F] = [\mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w^G_{j-1,k-1}, \eta^F)]
\]

If $j = 1$ then holds

\[
\mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w^G_{j-1,k-1}) = \mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w^G_{0,k-1}) = \mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ \iota^G_{k-1}) = \mu^F \circ \rho F_\lambda \circ G \eta^F = \mu^F \circ \rho \lambda \eta^F = \rho
\]

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hence the property holds for $j = 1$ and every $k > 1$.

Now let $1 < j < k$ and assume the validity of $\rho_{k-1} \circ w_{j-1,k-1}^G = \rho_{j-1}$. Then we have: $\mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w_{j-1,k-1}^G) = \mu^F \circ \rho F_\lambda \circ G\rho_{j-1}$ hence $\rho_k \circ w_{j,k}^G = \rho_j$ and the proof is complete. \hfill \Box

Now the transformations $\rho_k$ form a compatible cocone for $w_{j,k}^G$, hence we can extend it to the infinite limit case:

**Definition 2.4** For a limit ordinal $l$ let $\rho_l = \text{colim}_{k<l} \rho_k$.

Since the $w$-compatibility clearly holds, our construction extends to the ordinal chain with analogous definition for the isolated step as for finite indices. The definition yields the property for every $k \leq \lambda$:

$$\rho_k \circ \iota_k^G = \iota_\lambda^F \quad (6)$$

To prove that this chain of transformations converges to $\sigma$, we will show that its colimit is $G$-algebra morphism.

**Lemma 2.4** The transformation $\rho_\lambda : G_\lambda \to F_\lambda$ underlies the natural transformation $V_G(A) \to P(A)$ of functor $\mathcal{C} \to \text{Alg}\ G$, where $P$ is the functor used in Proposition 2.2.

**Proof:** What we need to prove is that for an object $A$ in $\mathcal{C}$ the morphism $\rho_{\lambda,A} : G_\lambda A \to F_\lambda A$ is a $G$-algebra morphism $(G_\lambda A, \nu_A^G) \to (F_\lambda A, \beta_A)$. It suffices to prove the equality of natural transformations: $\beta \circ G \rho_\lambda = \rho_\lambda \circ \nu^G$. Let $k < \lambda$, then we have

$$\rho_\lambda \circ \nu^G \circ G w_{k\lambda}^G = \rho_\lambda \circ w_{k+1\lambda}^G \circ q_k^G = \rho_{k+1} \circ q_k^G = \mu^F \circ \rho F_\lambda \circ G \rho_k = \beta \circ G \rho_k = \beta \circ G \rho_\lambda \circ G w_{k\lambda}^G$$

and since $G$ preserves the colimits of $\lambda$-indexed chains, $\{G w_{k\lambda}^G | k \leq l < \lambda\}$ is the colimit cocone. From the uniqueness of factorization over the colimit we get the required equality. \hfill \Box

**Lemma 2.5** The transformations $\rho_\lambda, \sigma : G_\lambda \to F_\lambda$ coincide.

**Proof:** Given a $\mathcal{C}$-object $A$, then due to previous lemma, $\rho_{\lambda,A}$ is a $G$-algebra morphism $\rho_{\lambda,A} : (G_\lambda A, \nu_A^G) \to (F_\lambda A, \beta_A)$ and by (6) we have $\rho_\lambda \circ \iota_\lambda^G = \iota_\lambda^F = \eta^F$, hence by uniqueness of factorization of $\eta_\lambda^F : A \to Z_G(F_\lambda A, \beta_A)$ over $\eta_A^G = \iota_\lambda^G$ we get $\rho_{\lambda,A} = \widetilde{\eta}_A^\lambda$ which is due to Proposition 2.2 equal to $\sigma_A$. \hfill \Box
Lemma 2.6 Let $\phi, \psi : G \to F_\lambda$ be natural transformations. Then for every $k \leq \lambda$ we have the algebraic equivalence:

$$(\phi, \psi) \approx (\phi_k, \psi_k),$$

where $\phi_k, \psi_k$ are derived from $\phi, \psi$, respectively, as in Definition 2.3 [2.4]

Proof: Let $(A, \alpha)$ be an $F$-algebra. Then for every $k \leq \lambda$

$$(*) \quad \epsilon_{\lambda, (A, \alpha)} \circ \phi_{k,A} \circ \iota_k^G = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{k,A} \circ \iota_k^G,$$

since $\epsilon_{\lambda, (A, \alpha)} \circ \phi_{k,A} \circ \iota_k^G = \epsilon_{\lambda, (A, \alpha)} \circ \eta^F = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{1,A} \circ \iota_1^G$ which together with $(*)$ yields $\epsilon_{\lambda, (A, \alpha)} \circ \phi_{1,A} = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{1,A}$.

1. $k = 1$: Since $\epsilon_{\lambda, (A, \alpha)} \circ \phi_{1,A} \circ \iota_0^G = \epsilon_{\lambda, (A, \alpha)} \circ \phi_A$, then from $(h)$ and symmetry we get $\epsilon_{\lambda, (A, \alpha)} \circ \psi_{1,A} \circ \iota_0^G = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{1,A}$ which together with $(*)$ yields $\epsilon_{\lambda, (A, \alpha)} \circ \phi_{1,A} = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{1,A}$.

Assume the hypothesis

$$(h_k) \quad \epsilon_{\lambda, (A, \alpha)} \circ \phi_{k,A} = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{k,A}$$

Recall, that $\epsilon_{\lambda, (A, \alpha)} : F_\lambda A \to A$ is a morphism $(F_\lambda A, \mu^F) \to (A, \epsilon_{\lambda, (A, \alpha)})$, i.e. holds $\epsilon_{\lambda, (A, \alpha)} \circ \mu^F = \epsilon_{\lambda, (A, \alpha)} \circ F_\lambda \epsilon_{\lambda, (A, \alpha)}$. Then we have:

$$\begin{align*}
\epsilon_{\lambda, (A, \alpha)} \circ \phi_{k+1,A} \circ \iota_k^G &= \epsilon_{\lambda, (A, \alpha)} \circ \mu_k^F \circ \phi_{k,A} \circ G \phi_{k,A} \\
&= \epsilon_{\lambda, (A, \alpha)} \circ F_\lambda \epsilon_{\lambda, (A, \alpha)} \circ F_\lambda \phi_{k,A} \circ \phi_{Gk,A} \\
(h_k) \quad &= \epsilon_{\lambda, (A, \alpha)} \circ F_\lambda \epsilon_{\lambda, (A, \alpha)} \circ F_\lambda \psi_{k,A} \circ \phi_{Gk,A} \\
&= \epsilon_{\lambda, (A, \alpha)} \circ F_\lambda \epsilon_{\lambda, (A, \alpha)} \circ F_\lambda \psi_{k,A} \circ G \phi_{k,A} \\
&= \epsilon_{\lambda, (A, \alpha)} \circ \phi_{A} \circ \epsilon_{\lambda, (A, \alpha)} \circ \psi_{A} \circ \phi_{Gk,A} \\
(h) \quad &= \epsilon_{\lambda, (A, \alpha)} \circ \psi_{k+1,A} \circ \iota_k^G
\end{align*}$$

and together with $(*)$ we get $\epsilon_{\lambda, (A, \alpha)} \circ \phi_{k+1,A} = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{k+1,A}$.

2. $l \leq \lambda$, $l$ limit: Assume $(h_k)$ for every $k < l$. Then from the uniqueness of factorization of cocone $\epsilon_{\lambda, (A, \alpha)} \circ \phi_{l,A} : G_l \to A$ over the colimit cocone $w_{k,l,A}^G$ we get $\epsilon_{\lambda, (A, \alpha)} \circ \phi_{l,A} = \epsilon_{\lambda, (A, \alpha)} \circ \psi_{l,A}$.
We have proved for every $k$:

$$(A, \alpha) \models (\phi, \psi) \Rightarrow (A, \alpha) \models (\phi_k, \psi_k).$$

However $(A, \alpha) \models (\phi_k, \psi_k)$ easily implies $(A, \alpha) \models (\phi, \psi)$ since $\phi = \phi_1 \circ q_G^0 = \phi_k \circ w_{1,k}^G \circ q_G^0 = \phi_k \circ y_k^G$.

\[\square\]

2.2 Algebras for a diagram of monads

This section refers to the paper [7] of G. M. Kelly, chapter VIII., which deals with colimits of monads. It is well known (see e.g. [2], Corollary 20.57), that for every varietor $F$ the categories of its algebras and algebras for a monad $M(F)$ are concretely isomorphic via the comparison functor.

Let $D : \mathcal{D} \to \text{Monad} \mathcal{C}$ be a diagram and $D(x) = (M_x, \eta^x, \mu^x)$ for every object $x \in \mathcal{D}$. Consider the category $D-\text{alg}$ of algebras for a diagram $D$ of monads, whose objects are collections of $\mathcal{C}$-morphisms $\{\alpha_x : M_x A \to A \mid x \in \mathcal{D}\}$, where $\alpha_x$ is in $D(x)-\text{alg}$ for every object $x \in \mathcal{D}$ and for each $f : x \to y$ in $\mathcal{D}$ the $D$-compatibility condition $\alpha_y \circ D(f)_A = \alpha_x$ is satisfied. The morphisms in $D-\text{alg}$ are the morphisms of algebras for each $x$, i.e. $\phi : (A, \alpha) \to (B, \beta)$ is a morphism if $\phi \circ \alpha_x = \beta_x \circ M_x(\phi)$ for every $x$. If there is a monad $K$ such that $K-\text{alg} \cong \mathcal{C} D-\text{alg}$, then this monad is called algebraic colimit of $D$.

Kelly asked about existence of this algebraic colimit. It came out to be equivalent to existence of the free objects in $D-\text{alg}$. He proved the existence in his Theorem 27.1 in [7] under the general assumptions of existence of suitable factorization systems and some smallness requirements for the monads. Using the trivial factorization system (Iso, Mor) and preservation of $\lambda$-labeled chains, we get this theorem in the following form:

**Theorem 2.7** Let underlying functor of each $D(x)$ preserve the colimits of $\lambda$-labeled chains. Then $D-\text{alg}$ has the free objects.

This theorem will be used to prove the existence of free object in a variety induced by accessible identities. Let $F$ be a functor in $\text{End}_\kappa \mathcal{C}$ for some infinite limit ordinal $\kappa$ and consider the variety of $F$-algebras induced by a set of accessible natural identities. Since the free $F$-algebra construction stops after $\kappa$ steps, we may consider arity of each natural term to be less or equal to $\kappa$. Then, due to the Remark 1.4 the set of natural identities can be substituted by a single identity $(\phi, \psi)$. Its domain, denoted by $G$, is the coproduct of domains of single identities, hence, due to 2.1 it preserves colimits of $\nu$-indexed chains for some large enough limit ordinal $\nu$. Let $\lambda = \max\{\kappa, \nu\}$, then $F, G \in \text{End}_\lambda \mathcal{C}$. Hence the arity of $(\phi, \psi)$ can be set to $\lambda$.

Let $\mathcal{D}$ be a category consisting of two objects 0, 1, their identities and two more morphisms $f, g : 0 \to 1$. Let $D : \mathcal{D} \to \text{Monad} \mathcal{C}$ be a diagram.
such that \( D(0) = \mathcal{M}(G) \), \( D(1) = \mathcal{M}(F) \), \( D(f) = \overline{\phi} \), \( D(g) = \overline{\psi} \), where \( \overline{\phi}, \overline{\psi} \) are the monad transformations given by Proposition 2.2. We will prove the concrete equivalence of \( \text{Alg} (F, (\phi, \psi)) \) and \( D - \text{alg} \).

**Lemma 2.8** For the \( \lambda \)-ary \( G \)-identity \( (\phi, \psi) \) and diagram \( D \) defined above holds:

\[
\text{Alg} (F, (\phi, \psi)) \cong C \ D - \text{alg}.
\]

**Proof:** Consider the comparison functor \( I : \text{Alg} F \rightarrow \mathcal{M}(F) - \text{alg} \) assigning to an \( F \)-algebra \( (A, \alpha) \) the \( F_\lambda \)-algebra \( (A, \epsilon_{\lambda,(A,\alpha)}) \). Then due to Lemma 2.6

\[
(A, \alpha) \models (\phi, \psi) \iff (A, \alpha) \models (\phi_\lambda, \psi_\lambda) \iff \epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda = \epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda.
\]

For every \( F_\lambda \)-algebra \( (A, \beta) \) holds \( \epsilon_{F_\lambda,1(A,\beta)} \circ q_{F_\lambda}^0 = \beta \), hence we get

\[
\epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda = \epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda \iff I(A, \alpha) \models (\phi^*, \psi^*),
\]

where \( \phi^* = q_{F_\lambda}^0 \circ \phi_\lambda \). Since \( I \) is the isomorphism with an inverse given by \( (A, \beta) \mapsto (A, \beta \circ y_{\lambda,A}) \), we get

\[
\text{Alg} (F, (\phi, \psi)) \cong C \mathcal{M}(F) - \text{alg} \cap \text{Alg} (F_\lambda, (\phi^*, \psi^*)).
\]

Due to Proposition 2.2 we have \( \phi_\lambda = \overline{\phi} \), which is a monad transformation (and analogously for \( \psi \)), hence we get \( \mathcal{M}(F) - \text{alg} \cap \text{Alg} (F_\lambda, (\phi^*, \psi^*)) \) to be concretely isomorphic to \( D - \text{alg} \).

Now we can use Kelly’s theorem to conclude our investigation:

**Theorem 2.9** Let \( F \) preserve the colimits of \( \lambda \)-indexed chains for some limit ordinal \( \lambda \). Then the free algebra exists in every variety induced by a set of accessible identities.

To express the consequence for the varieties presented by equation arrows, recall the notion of presentability of an object (see [4]):

**Definition 2.5** Let \( \lambda \) be a regular cardinal. An object \( A \) of a category is called \( \lambda \)-presentable provided that its hom-functor \( \text{hom}(A, -) \) preserves \( \lambda \)-directed colimits. An object is called presentable if it is \( \lambda \)-presentable for some \( \lambda \).

**Corollary 2.10** Let \( F \) preserve the colimits of \( \lambda \)-indexed chains for some limit ordinal \( \lambda \). Then the free algebra exists in every variety induced by a set of equation arrows with presentable variable-objects.
**Proof:** As shown in the proof of Lemma 1.6 an equation arrow $e : F_n X \to E$ converts to a natural identity with the domain $G = (\cdot Q) \circ \text{hom}(X, -)$ for some $Q \in \text{ObC}$. If the variable-object $X$ is presentable, $\text{hom}(X, -)$ preserves $\kappa$-directed colimits for some $\kappa$ and since $(\cdot Q)$ is left adjoint, $G$ preserves $\kappa$-directed colimits too. Therefore the colimits of $\kappa$-indexed chains are preserved and due to Theorem 2.9 the corresponding variety has free objects. The rest is obvious. \qed

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