Stress Intensity Factor of Mode III Cracks in Thin Sheets.

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The stress field at the tip of a crack of a thin plate of elastic material that is broken due to a mode III shear tearing has a universal form with a non-universal amplitude, known as the stress intensity factor, which depends on the crack length and the boundary conditions. We present in this paper exact analytic results for this stress intensity factor, thus enriching the small number of exact results that can be obtained within Linear Elastic Fracture Mechanics (LEFM).

I. INTRODUCTION

Linear Elastic Fracture Mechanics (LEFM) \[1\] deals with the dynamics of cracks in elastic materials subject to the assumption that linear elasticity is applicable everywhere in the material except for a very small region around the crack tip known as the process zone. While recent experiments and theories indicate that there exist examples where nonlinear elasticity may be important for a full characterization of the stress field before a dynamical crack, some exact results of LEFM continue to have a very important role in describing and understanding the dynamics of cracks in a variety of fracture contexts. Indeed, one of the most celebrated results of LEFM is the analytic description of the universal stress distribution in front of a crack. This analytic result is particularly simple when the crack is developing under one of the three fundamental shearing modes that a material can suffer. The simplest form results from mode III cracking which is obtained under an out-of-plane shear tearing. In a cylindrical coordinate system \(r, \theta, z\), where \(r = 0\) coincide with the crack tip and \(\theta = \pm \pi\) with the crack faces, the stress in front of a crack takes on the form

\[
\sigma_{\theta z} = \frac{K_{III}}{\sqrt{\pi r}} \cos \left( \frac{\theta}{2} \right),
\]

where \(K_{III}\) is the so-called “Stress Intensity Factor”. The results \([1]\) underlines the universality of the angular dependence and of the square-root singularity of the stress at the crack tip \([2, 3]\). All the non-universal aspects of the stress distribution are collected in the Stress Intensity Factor which depends on everything, including the crack length, the boundary conditions and the history of the loads that drive the crack evolution. The value of the stress intensity factor determines whether the crack propagates or not; Irwin \([4]\) established a relationship between the energy release rate and the stress intensity factor for a crack growth. A crack propagates as long as the energy release rate into the crack tip balance the dissipation involved in the crack propagation, i.e. the Griffith criterion \([1, 3]\).

Solving for mode III is relatively easy since the fundamental equation to be dealt with is the Laplace equation \([6]\) rather than the bi-Laplace equation which is the relevant one for modes I and II. On the other hand in 3-dimensional materials mode III is unstable and attempts to break in this mode quickly turn to cracks developing under effective modes I or II. This is not the case for thin plates where mode III can be sustained in experiments as had been demonstrated recently in Ref. \([7]\). The theory of thin plates is however more cumbersome, using again the bi-Laplace operator even for mode III cracks. Nevertheless the problem can be treated analytically \([8–10]\) with the result that the leading singularity at the crack tip is higher, going like \(r^{-3/2}\). In this case, the shear force close to the crack tip can be written as

\[
Q_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta z} dz = \frac{K_{III}}{(\pi r)^{3/2}} \cos \left( \frac{\theta}{2} \right),
\]

we assign tilde on \(K_{III}\) to distinguish it from the known stress intensity factor for this mode. \(h\) is the thickness of the plate.

We may mention that the Kirchhoff plate theory predicts the stresses very accurately for small deflection and at some distance from the crack tip \((r/h > 1)\) (cf. \([11]\) and references therein). However, for large deflection or near the crack tip \((r/h < 0.1)\), the singularity of the stress field weakens \([10, 11]\), and therefore it may diverge from the actual results.

The aim of the present paper is to present further analytic progress that culminates in the calculation of the stress intensity factor for cracks developing in thin sheets. The structure of the paper is as follows: in Sect. \(\text{III}\) we set up the problem for mathematical analysis. In Sect. \(\text{IV}\) we study the properties of the analytic functions that need to be solved for and the boundary conditions. In Sect. \(\text{V}\) we introduce the conformal map from the strip to the upper half plane which is used to advantage in simplifying the solution of the problem. Sect. \(\text{V}\) wraps up the problem and presents the analytic result for the Stress Intensity Factor.

II. SETTING UP THE PROBLEM

Consider a semi-infinite strip of a thin sheet with a cut in the in the middle of the left-most end of the strip, along the symmetry axis. The left boundary is subjected to anti-symmetric shear load \(f_s\), i.e. mode III configuration, cf. Fig. \([1]\) For small deflection, we can neglect the in-plane deformation of the middle surface. According to
the Kirchhoff plate theory [12], the deflection \( w \) in the z-axis for pure bending becomes
\[
\Delta^2 w = 0 ,
\]
where the operator ‘delta’ stands for the Laplace operator \( \nabla \cdot \nabla \); the bending moments and the twisting moment are
\[
M_x = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{xx} z dz = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) ,
\]
\[
M_y = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{yy} z dz = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) ,
\]
\[
M_{xy} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{xy} z dz = D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} ,
\]
and the anti-plane shear stress
\[
Q_x = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{xz} dz = -D \frac{\partial}{\partial x}(\Delta w) ,
\]
\[
Q_y = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{yz} dz = -D \frac{\partial}{\partial y}(\Delta w) .
\]
where the sheet is on the \( xy \) plane, and the bending occurs along the z direction. \( \sigma_{ij} \) is a component of the stress tensor, \( D = Eh^3/12(1-\nu^2) \) is the so-called the flexural rigidity of a plate, \( h \) is the thickness of the plate, and \( E, \nu \) are the Young’s modulus and Poisson’s ratio, respectively.

Introduce now the complex notation \( z = x + iy, \bar{z} = x - iy \), such that
\[
\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y) \equiv \partial_z ,
\]
\[
\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \equiv \partial_{\bar{z}} .
\]
Since \( w \) is a solution of the bi-Laplace equation, \( w \) can be expressed as [13]
\[
w(z) = \Re[\bar{z}\varphi(z) + \chi(z)] ,
\]
where \( \Re \) denotes “the real part”, and \( \varphi(z), \chi(z) \) are both analytic functions, accept maybe on the boundary.

The complex first derivative becomes
\[
\partial_z w + i\partial_{\bar{z}} w = 2\bar{z}w_w(\bar{z}) + \chi(z) ,
\]
\[
\partial_{\bar{z}} w + i\partial_z w = 2zw_{\bar{z}}(\bar{z}) + \varphi(z) .
\]
The bending moments and the twisting moment are
\[
M_x = -D \left( (1-\nu)\Re(\bar{z}\varphi'' + \chi'') + 2(1+\nu)\Re(\varphi') \right) ,
\]
\[
M_y = -D \left( -(1-\nu)\Re(\bar{z}\varphi'' + \chi'') + 2(1+\nu)\Re(\varphi') \right) ,
\]
\[
M_{xy} = D(1-\nu)\Im(\bar{z}\varphi'' + \chi''). \tag{12}
\]
Summing over the two bending moments gives
\[
M_x + M_y = -D(1+\nu)\Delta w = -4D(1+\nu)\Re(\varphi') . \tag{13}
\]
and the complex representation of the shear force becomes
\[
Q_x + iQ_y = -4D\varphi''(z) . \tag{14}
\]

Next, the shear force exerted normal to the boundary is
\[
Q_n = Q_x \frac{dy}{ds} - Q_y \frac{dx}{ds} \tag{15}
\]
where \( n, s \) are the normal and the tangent direction oriented with respect to each other as the axes \( x, y \). Multiply the last equation by \( ds \) and using the complex representation in Eq. (14) gives
\[
Q_n ds = Q_x dy - Q_y dx = 2Di(\varphi'' dz - \overline{\varphi''} d\bar{z}) . \tag{16}
\]

### III. Properties of \( \varphi \) and Boundary Conditions

In the case of mode III symmetry, the load and the deflection are anti-symmetric with respect to the \( x \) axis, (cf. Fig. 1). We consider only one crack lying on part of the \( x \) axis (C-D). The upper (B-C) and the lower left edges are displaced with a constant distance from each other, in a direction perpendicular to the strip plane. Then, \( w \) is anti-symmetric, but also \( \Delta w = 4\Re[\varphi'(z)] \) is anti-symmetric. This implies that
\[
\Re[\varphi'(z)] = -\Re[\varphi'(\bar{z})] \tag{17}
\]
the left and the right members give
\[
\varphi'(z) + \overline{\varphi'(z)} = -\varphi'(\bar{z}) - \overline{\varphi'(\bar{z})} \tag{18}
\]
which may be arranged to
\[
\varphi'(z) + \overline{\varphi'(z)} = -\overline{\varphi'(z)} + \overline{\varphi'(\bar{z})} . \tag{19}
\]
Thus, if \( \varphi(z) \) is analytic on both the upper and the lower half plane, and not necessary on the boundary, the function \( \bar{\varphi}(z) \) is also analytic on the same domain. Thus the left side of Eq. (19) is analytic in the whole domain (excluding the boundary). Since it equals to minus its complex conjugate, it must be a pure imaginary function. However, according to the Cauchy-Riemann equations, an imaginary analytic function is a constant. Thus,

\[
\varphi'(z) = -\varphi'(z) + iC, \quad (20)
\]

Where \( C \) is a real number. Taking the complex conjugate of this expression, followed by the complex conjugation of \( z \), gives \( C = 0 \). Thus, the anti-symmetry with respect to the \( x \) axis prevails

\[
\varphi'(z) = -\varphi'(z). \quad (21)
\]

Since \( \varphi'(z) \) is not necessary analytic on the \( x \) axis and on the boundary, it is advantageous to separate notations, \( \varphi'_+(z) \) and \( \varphi'_-(z) \), for its parts on the upper and lower half-planes, respectively.

A. Ahead of the crack

In the case of a symmetric body with an anti-symmetric loading, it can be assumed that \( w(x,0) = 0 \) on the symmetry axis ahead of the crack (D-E). Moreover, the anti-symmetry of \( w \) dictates that \( \Delta w(x > 0,0) = 0 \). From Eqs. (13) and (24)

\[
\frac{1}{2} \Delta w_+(x,0) = 2\Re[\varphi'(z)] = \varphi'(x + i0) + \varphi'(x - i0) = \varphi'(x + i0) - \varphi'(x - i0) = \varphi'_+(x) - \varphi'_-(x) = 0 \quad (22)
\]

We used the subscripts + and – for the the limiting values of the functions reaching from the upper and the lower half of the plane, respectively.

B. Crack face

The crack face (C-D) is assumed to be traction free. Thus, the stress components acting normal to the crack’s face become zero. This condition gives

\[
M_y = M_{xy} = Q_y = 0, \text{ for } C < x < D, y = 0 \quad (23)
\]

Integrating Eq. (15) along the boundary from a point \( x \) on the crack face to end of the strip, i.e. point \( A \to \infty \), gives

\[
\int_{x+0}^{A} Q_n ds = 2Di \int_{x+0}^{A} \left[ \varphi'' dz - \bar{\varphi''} dz \right], \quad (24)
\]

or

\[
\int_{x+0}^{A} Q_n ds = -2Di \{ \varphi'(x+i0) - \bar{\varphi'}(x+i0) - [\varphi'(\infty) - \bar{\varphi'}(\infty)] \}. \quad (25)
\]

Due to a finite slope at infinity, we can assume that \( \lim_{A \to \infty} \varphi'(z = A) = 0 \). Also, on the crack face and on the upper free edge, \( Q_y = 0 \). Thus, Eq. (25) becomes

\[
\varphi'(x + i0) - \bar{\varphi'}(x + i0) = \frac{i}{2D} \int_{C}^{B} Q_n ds. \quad (26)
\]

and by (21), we find

\[
\varphi'_+(x) + \varphi'_-(x) = i\beta, \quad (27)
\]

and

\[
\beta = \frac{f_b}{2D} = \frac{1}{2D} \int_{C}^{B} Q_n^0 ds.
\]

where \( f_b \) is the total applied force on the upper left boundary to create a constant deflection.

C. The loaded edge

This edge (B-C) is considered as a simply supported edge i.e. \( \partial w(-b, y)/\partial y = 0 \) and \( M_x(-b, y) = 0 \), for \( x = -b, y > 0 \). These boundary conditions annul also \( M_y(-b, y) = 0 \), and by the same consideration of Sec. IIIA the complex function becomes

\[
\varphi'_+(-b + iy) - \varphi'_-(-b - iy) = 0 \quad (28)
\]

D. Free upper boundary

As earlier, a free edge (B-A) feels a zero normal stress. Thus,

\[
M_y = M_{xy} = Q_y = 0
\]

with the last condition giving

\[
\varphi''(x + ia) - \bar{\varphi''}(x + ia) = 0, \text{ for } -b < x, \ y = a. \quad (29)
\]

Applying the same integration as in Eq. (25) for the upper edge gives,

\[
\varphi'(x + ia) - \bar{\varphi'}(x + ia) = 0 \quad (30)
\]

and by Eq. (21), we obtain

\[
\varphi'_+(x + ia) + \varphi'_-(x - ia) = 0. \quad (31)
\]
By introducing an auxiliary function \( G \), with a choice of a branch where \( G \) is imaginary, \( i \), the boundary conditions become

\[
ζ = \text{cosh}(\frac{πb}{a}), \quad \text{and the crack tip is at } ξ_D = \text{cosh}(πb/a).
\]

**IV. CONFORMAL MAPPING**

The upper half of the strip is mapped onto the upper half of \( ζ \)-plane, where \( ζ = ξ + iη \). The mapping function is (cf. Fig. 2)

\[
z = Ω(ζ) = \frac{a}{π} \text{cosh}^{-1}(ζ) - b. \tag{32}
\]

The analytic function can be written as

\[
φ'(z) = φ'(Ω(ζ)) = F(ζ). \tag{33}
\]

and the boundary conditions become

\[
\begin{align*}
F_+(ξ) - F_-(ξ) &= 0, \quad ξ_D < ξ < η = 0, \\
F_+(ξ) + F_-(ξ) &= iβ, \quad 1 < ξ < ξ_D, \quad η = 0, \\
F_+(ξ) - F_-(ξ) &= 0, \quad 1 < ξ < 1, \quad ζ = 0, \\
F_+(ξ) + F_-(ξ) &= 0, \quad ξ < 1, \quad η = 0.
\end{align*} \tag{34}
\]

These four equations constitute a Hilbert problem [13]. By introducing an auxiliary function \( G(ζ) \), we can write the boundary equations as a single equation. In our case, the function

\[
G(ζ) = (ζ + 1)^{1/2}(ζ - 1)^{1/2}(ζ - ξ_D)^{1/2} \tag{35}
\]

with branch cut along \( η = 0 \) and \( ξ < -1, 1 < ξ < ξ_D \) with a choice of a branch where \( G(ζ) → ζ^{1/2} \) as \( ζ → ∞ \). Then,

\[
\frac{G_-(ζ)}{G_+(ζ)} = \begin{cases} 
+1 & \text{for } -1 < ξ < 1, \quad ξ_D < ξ < ξ_D \\
-1 & \text{for } ξ < -1, 1 < ξ < ξ_D
\end{cases} \tag{36}
\]

and equations (34) can be written as a single equation

\[
F_+(ξ) - \frac{G_- (ξ)}{G_+ (ξ)} F_-(ξ) = iβ[H(ξ - 1) - H(ξ - ξ_D)] \tag{37}
\]

where \( H(ξ) \) is Heaviside step function.

Multiplication of the both sides with \( G_+(ξ) \) gives

\[
G_+(ξ)F_+(ξ) - G_-(ξ)F_-(ξ) = iG_+(ξ)β[H(ξ - 1) - H(ξ - ξ_D)] \tag{38}
\]

Using the Plemelj’s formulae [3], we can define an analytic function in both upper and the lower half plane of \( ζ \),

\[
G(ζ)F(ζ) = \frac{1}{2πi} \int_1^{ξ_D} \frac{iG_+(ξ)β}{ξ - ζ}dξ. \tag{39}
\]

However, this solution of the Hilbert problem is not complete and we must add an analytic function in the whole plane, since it vanished in the left hand side of Eq. (37). Thus,

\[
F(ζ) = \frac{β}{2πG(ζ)} \int_1^{ξ_D} \frac{G_+(ζ)β}{ζ - ξ}dξ + P_0(ζ) \tag{40}
\]

where \( P_0(ζ) \) is analytic in the whole plane and therefore, by Liouville’s theorem a polynomial of finite degree.

Since the moments are bounded as \( |ζ| → ∞ \), the degree of \( P(ζ) \) can not be greater than 1. In addition, upon the condition in Eq. (21), the shear force at the point \( t > ξ_D \) along the ξ-axis becomes

\[
Q_y(t) = \frac{2πD}{Ω'(t)} \left[ F'_+(t) + F'_-(t) \right]. \tag{42}
\]

here, we use the fact that \( Ω'(t) = Ω'(ζ) \) in this section.

Differentiate the solution of \( F(ζ) \) from the preceding section, Eq. (40), gives

\[
F'(ζ) = -iβG'(ζ)I(ζ) + \frac{iβ}{G(ζ)}I'(ζ) - \frac{i(π^2 + p_0)G'(ζ)}{G(ζ)^2} + ip_1 \tag{43}
\]

where

\[
I(ζ) = \frac{1}{2πi} \int_1^{ξ_D} \frac{G_+(ζ)}{ζ - ζ}dζ.
\]

notice that \( G_+(ζ) \) is imaginary.

Now, the shear force away from the crack at the symmetry axis is

\[
Q_y(t) = -4D\frac{Ω'(t)G(t)}{Ω(t)G(t)} \left[ \frac{G'(t)}{G(t)}(βI(t) + p_1 t + p_0) - (βI'(t) + p_1) \right] \tag{44}
\]
where
\[ G'(t) = \left( \frac{t}{\sqrt{t^2 - 1}} + \frac{1}{2(t - \xi_D)} \right). \]

\( I(t) \) is the Cauchy’s principal value,
\[ I(t) = \frac{1}{2\pi i} \int^\xi_D_{\xi-t} G_+(\xi) \, d\xi, \]
and \( I'(t) \) is its derivative.

The constants \( p_1 \) and \( p_0 \) can be found from the boundary conditions. At the edge \( B - C \), \( Q_y(t) \) has the same expression as in Eq. \([45]\). Since we assume a finite applied load at this edge, the singularity of the shear force at \( t = \pm 1 \) disappears if
\[ \beta I(t) + p_1 t + p_0 = 0. \]  
which gives
\[ p_1 = \frac{1}{2} \beta [I(-1) - I(1)], \]
\[ p_0 = -\frac{1}{2} \beta [I(-1) - I(1)]. \]  

Finally, the stress intensity factor close to the crack tip in thin plates is obtained from
\[ \tilde{K}_{III} = \lim_{x \to 0} (\pi x)^{\frac{2}{3}} Q_y(t), \]
where \( x = \Omega(t) \). For \( x \to 0 \), we can write
\[ x = \frac{a}{\pi \sinh \frac{\pi b}{a}} (t - \xi_D) \]
thus, Eq. \([45]\) becomes
\[ \tilde{K}_{III} = \left( \frac{a}{\sinh \frac{\pi b}{a}} \right)^{\frac{2}{3}} \lim_{t \to \xi_D} (t - \xi_D)^{\frac{2}{3}} Q_y(t) \]

Applying the expression found in Eq. \([45]\) gives
\[ \tilde{K}_{III} = -\frac{2D\pi \sqrt{a}}{\sinh^2 \left( \frac{\pi b}{a} \right)} (\beta I(\xi_D) + p_1 \xi_D + p_0). \]  

Using \([47]\)
\[ \tilde{K}_{III} = -\frac{D\pi \sqrt{a}}{i \sin^2 \left( \frac{\pi b}{a} \right)} \int^\xi_D_0 \sqrt{\xi^2 - 1} \sqrt{\frac{1}{\xi^2 - 1}} \, d\xi \]
notice that \( \xi_D^2 - 1 = \sinh^2 \left( \frac{\pi b}{a} \right) \). Changing the variable of integration to \( \lambda = \frac{1}{\sqrt{\xi^2 - 1}} \) with the parameter \( k^2 = \frac{\xi_D^2 - 1}{\xi_D^2 + 1} \), one obtains
\[ \tilde{K}_{III} = 2D\beta \sqrt{a} \left\{ \frac{\sinh \left( \frac{\pi b}{a} \right)}{\cosh \left( \frac{\pi b}{a} \right) + 1} \right\} K(k), \]
and finally, the stress intensity factor for mode III in thin sheets becomes
\[ \tilde{K}_{III} = f_b \sqrt{a} \left\{ \frac{\sinh \left( \frac{\pi b}{a} \right)}{\cosh \left( \frac{\pi b}{a} \right) + 1} \right\} K(k), \]
where \( 2a \) is the strip width, \( b \) is the crack length, \( f_b \) is the total load and \( K(k) \) is the complete elliptic integral of the first kind.

For a crack larger than the strip’s width \( b \gg a \), we can estimate the stress intensity factor in term of the deflection \( w_0 \) \( \left( \frac{\pi b}{a} \ll 1 \right) \) instead of the load. In this case, \( K(k) \sim b/a \) and the stress intensity factor becomes \( \tilde{K}_{III} \sim f_b b/\sqrt{a} \). An estimate of the load from the bending theory gives \( f_b \sim \frac{Dw_0}{b^2} \) \([14]\), and we obtain \( \tilde{K}_{III} \sim \frac{D\pi w_0}{b^2} \). Hence, in particular, we see that the stress intensity factor decreases as the square of the crack length.

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