The Profile of A Thin Piece of Paper with Both Ends Fixed

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Abstract. In this manuscript, utilizing the controlled variable method, the three different profiles of thin paper with fixed ends are investigated. The results indicate that the specific profile of the paper depends on the length of the paper and the fixed distance between the sheets. Meanwhile, the mass of the paper has little effect on its profiles and so it can be ignored. Also, the symmetry-broken phenomenon about the thin paper is observed. Finally, according to the Principle of Least Action, it proposes a simple model to simulate the thin paper profiles.

1. Introduction
On the students' desks, there are always a lot of books and draft papers. If we put a piece of paper between two books. An interesting phenomenon was noticed as shown in Fig 1: the profile of the paper is varied obviously when the distance between the two books is tuned. It can be symmetrical, asymmetric, and even is close to the table. Symmetry and asymmetry, which are often appeared in our middle school physics textbooks. It is realized that this is a very interesting phenomenon. What factors determine the symmetrical and asymmetrical contours of paper, and how they changed. This paper focuses on exploring the causes of the paper outline.

Figure 1. The demonstration experiment on the desk. (a) (b) (c) correspond to three different paper outlines respectively.

2. Experiment Design
To facilitate analysis and understanding while keep simplifying the experiment, it only considers the paper boundary curve.

(1) Place two pieces of wood on the horizontal table, and suppose the distance between the two pieces of wood on the basis to make sure that the two pieces of wood are parallel. Then, hold up the paper between two pieces of wood. At the same time, the paper will form a curved surface due to extrusion, as shown in Fig 2 (a).
(2) Due to the measurement of the paper curve relative to the desktop, a lot of repetitive measurement will be very time-consuming and labor-consuming. A scheme about the ink bucket and ink line is proposed and used by carpenters. Thus, when the paper stops vibrating, we start to smear wet pigment on the edge of the paper and use coordinate paper to fit the edge of the paper. At last, print the paper arc curve on the coordinate paper, as shown in Fig 2 (b).

(3) And then, the coordinate system is established on the coordinate paper, and the paper contour data is recorded, as shown in Fig. 2 (c). Finally, complete the data recording, and draw the experimental data diagram, as shown in Fig2 (d).

It found that the factors influencing the paper outline may include the following four aspects: The length of paper L; the distance between the two blocks; the mass of the paper; the material of paper. Using the above experimental device, the influence of these four factors on the paper contour through the controlled variable method. Finally, based on the Principle of Least Action, a physical model to simulate the symmetrical paper contour was put forward.

3. Results and Discussion

Based on the above experimental device, the experiments were carried out through the controlled variable method. To observe the three phenomena mentioned in the introduction, three groups of experiments that the distances between the corresponding blocks (17.0cm, 20.0cm, and 22.0cm) were selected. In each large group of experiments, through changing the length of the paper, the contour of the paper was observed. Also, it needs to keep the distance between the two pieces of wood constant and adjust the length of the paper.

3.1 Changing paper length

As shown in Fig. 3 (a), shows the profile of the paper measured by our experiment, which is symmetrical as a whole. The red scatter is the highest point of the paper contour, approximately forming a straight line perpendicular to the horizontal axis. It means that when the distance between the blocks is 17.00cm, the abscissa of the highest point of the paper contour is approximately unchanged, just at the center of the block distance of 8.50cm. Fig. 3 (b) displays the relationship between the vertical coordinates of the highest point of the paper contour and the paper length. It can be seen that the vertical height of the highest point has a fitted linear relationship with the paper length.

In this experiment, only the symmetrical shape of the paper contour was observed. To explore the other two cases, it adjusted the distance between the two blocks to 20.00cm and 22.00cm. The results are as follows:
As can be seen from Fig. 4 (a), the abscissa of the highest point of the paper is no longer fixed and has a certain offset. It means that the paper profile is no longer symmetrical and begins to tilt. We observe the second phenomenon in the introduction part. At the same time, we also record the relationship between the vertical coordinate of the corresponding highest point and the paper length, as shown in Fig. 4 (b). Clear, it can see that they are still approximately linear.

It can be seen from Fig. 5 (a) that the paper contour is symmetrical when the paper length is not more than 34.00 cm, similar to Fig. 3 (a). When the length of the paper is 36.00 cm, the third phenomenon in the introduction appears: a part of the paper begins to stick to the desktop. In Fig. 5 (b), when the paper length changes from 34.00 cm to 36.00 cm, the highest point of paper contour also has the obvious mutation. Therefore, when the paper length changes from 34.00 cm to 36.00 cm, the contour of the paper changes abruptly.

3.2 Changing the distance between blocks
In this experiment, the paper length was fixed to 29.70 cm, and the woodblock was moved. As shown in Fig. 6, when the distance between the blocks \( d = 18.00 \) cm, the paper presents a symmetrical contour. When the paper length is more than 20.00 cm, the highest point of the paper contour deviates from the midpoint and appears randomly on both sides of the midpoint. The appearance of left-right deviation reminds us whether symmetry is allowed. For example, in Fig. 7, \( d = 25.00 \) cm, and the highest point is to the left. And then taking the midpoint between the boards as the center of symmetry.
3.3 The exploration of symmetry
In the introduction section, it discusses the influence of asymmetry on the two kinds of contours. Fig. 7 (a) and (c) show two cases of off-center. Fig. 7 (b) and (d) are the highest points of the contours in figures (a) and (c). It can be seen that if the relative center point deviates to the left (right), there must be one corresponding to the right (left). The asymmetric outline also contains the idea of symmetry.

3.4 Changing paperweight
Based on the fixed distance between two pieces of wood and the length of the paper, the weight of paper can be changed by changing the number of paper sheets. In the experiment, increasing the number of sheets of paper each time to explore the change of paper contour. The experimental results are shown in Fig. 8. The paper contour does not change significantly with the change of paper mass in a reasonable area. Therefore, the influence of paper mass on the contour can be ignored.
4. Theoretical analysis

Suppose that the coordinate system of the worktable is the x-o-z plane and the bending direction of the paper is along the x-axis. Meanwhile, for simplicity, it is assumed that the paper does not change along the z-axis. The outline of the paper on the x-o-y plane can be represented by the single-valued function y = f(x) and the schematic diagram is shown in Fig. 9. Therefore, the curvature of the paper [1] is:

\[ \rho = \frac{|y''|}{(1 + y')^{3/2}} \]

The elastic potential energy of the whole paper [2,3,4,5] is as follows:

\[ E_p = A \int_0^d \rho^2 \sqrt{1 + y'^2} \, dx = A \int_0^d \frac{y''^2}{(1 + y')^{5/2}} \, dx \]

Due to the quality of the paper, its gravitational potential energy is as follows:

\[ E_g = \int_0^d ugy\sqrt{1 + y'^2} \, dx \]

The restrictions are as follows:

\[ L = \int_0^d \sqrt{1 + y'^2} \, dx \]

At the same time, as a second-order differential equation, the following restrictions need to be added:

\[ y(0) = y(d) = 0 \]

When the curvature radius of the highest point is large enough, the paper quality can be ignored. So the paper gravitational potential energy can be ignored. As shown in Fig. 10, when \( \theta \) is small enough, the arc length is equal to the paper length, so it can figure out:

\[ \sin \theta = \frac{d/2}{L_c} = \sqrt{\frac{(L/2)^2 - (d/2)^2}{L/2}} \]

\[ L_c = \frac{Ld}{2\sqrt{L^2 - d^2}} \]

![Figure 9. Schematic diagram of paper simulation](image1)

![Figure 10. Schematic diagram of paper length simulation](image2)

Set the function with minimum elastic potential and limited by paper length be \( y = f(x) \). When neglecting the quality of the paper and assuming that \( \theta \) is small enough, the Lagrange multiplier [6] method is used to solve the problem:

\[ \delta \left( E_p + \lambda \int_0^d \sqrt{1 + y'^2} \, dx \right) = 0 \]
By replacing $E_p$ to the formula, the differential and variation of the above formula [7,8,9,10] are obtained as follows:

$$\delta [A \int_0^d \frac{y''^2}{(1+y'^2)^{3/2}} \, dx + \lambda \int_0^d \sqrt{1+y'^2} \, dx] \approx \delta [A \int_0^d y''^2 \, dx + \lambda \int_0^d \sqrt{1+y'^2} \, dx] = A \int_0^d 2y'' \delta y'' \, dx + \lambda \int_0^d y' \delta y' \, dx = \int_0^d (2Ay'' - 2Ay'''\beta y') \delta y \, dx + (\lambda y' - 2Ay'') \delta y |^d_0 = 0$$

The solution is as follows: $2Ay'''' - \lambda y''' = 0$ Set $y'' = g$.

$$2Ag'' - \lambda g = 0$$

After combining the results with the original formula, the following results are as follow:

$$y = C \sin(\beta x), \quad \beta = \frac{\pi}{d}, \quad \lambda = -2A \frac{\pi}{d^2}$$

From the formula, it can figure out:

$$L = \int_0^d \sqrt{1+y'^2} \, dx = \frac{1}{\sqrt{1+c^2\beta^2}} \int_0^d \sqrt{1 - \left[ \frac{c^2\beta^2}{1+c^2\beta^2} \sin(\beta x) \right]^2} \, dx$$

The Lagrange equation can be obtained by analyzing the deviation:

$$\frac{\partial F}{\partial y''} = \frac{2Ay''}{(1+y'^2)^{3/2}}$$

5. **Expectation and Conclusion**

For the above calculation, it needs further experiments to verify on this basis. A large number of experiments have confirmed that the two key factors for paper contour are paper length $L$ and fixed distance $d$. The specific expression of the contour may be related to the power law of $L$ and $d$. Therefore, it designs to change $L$ and $d$ at the same time, and keep their power-law. In the simplest case, $L$ and $d$ should be magnified at the same scale. Further, change the ratio of $L$ and $d$, and then change the paper sample for the test. This has arrived at the Scale theory. These conjectures, along with the improvement of the theoretical model and the further completion of the experiment, will be verified by us one by one.

Through a large number of experiments, it systematically explores the causes of the contour of the paper fixed at both ends. The paper length $L$ and the fixed distance $d$ have a decisive effect on the paper contour. Finally, combining the Principle of Least Action to give a simple physical model to explain the symmetrical profile of paper. This study will help to understand the profile of the suspension bridge and offer new insight into the model. Given physics, it gives a more profound basis.

**Reference:**

[1] Calculus, Vol. 1, Wiley, 1991.
[2] Classical Mechanics, Pearson Education, 2011
[3] Introduction to Classical Mechanics, Cambridge University Press, 2008.
[4] Introduction to Continuum Mechanics, Elsevier, 2009.
[5] Mechanics of Materials, Butterworth-Heinemann, 2013.
[6] The Feynman Lectures on Physics (1 Volume), Pearson, 2012.
[7] Ordinary Differential Equations, Dover Publications, 1985.
[8] Ordinary Differential Equations, XLIBRIS, 2014.
[9] Student Solution Manual for Mathematical Methods for Physics and Engineering, Cambridge University Press, 2006.
[10] Mathematics for Physicists, Cambridge University Press, 2019