On Information-Theoretic Classical Verification of Quantum Computers

Ayal Green

School of Computer Science and Engineering, The Hebrew University, Jerusalem 91904, Israel

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Abstract

Quantum inspired protocols e.g. [AAV13,AG17] attempt to achieve a single-prover interactive protocol where a classical machine can verify quantum computations in an information-theoretically secure manner. We define a family of protocols which seem natural for verifying quantum computations and generalizes such known protocols, namely those of [AAV13,AG17]. We show that any protocol from this family is bound to require an extremely powerful prover, much like the classical protocols of [LFKN92] and [Sha92]. Using our analysis, we also hint at possible ways one might try to realize a protocol where the prover can be weaker, namely a quantum computer (i.e. a BQP machine).

1 Introduction

One of the main open questions in Quantum Complexity Theory has to do with delegation of quantum computations. More specifically, whether a BPP verifier can delegate polynomial time quantum computations (namely computations in BQP) to a BQP prover which can not be trusted. The initial protocols for the problem [ABOE10, BFK09] (with follow up work [MF16, ABOEM17, FK17]) required the BPP verifier to process constantly many quantum bits. Alternatively, [RUV13] considered a setting where a strictly classical verifier is given access to multiple, entangled, non-interacting provers. However, despite remarkable advancement in this line of work over the past decade (see e.g. [BH13, FH13, FV15, Ji16, Ji17, NV17, CGJV19] and notably [NV18, JNV+20]), the basic requirements on the provers were not relaxed. Revisiting the original setting where a strictly classical verifier interacts with a single BQP prover, another notable result by [Mah18] showed a protocol which achieves the goal, but where soundness relies on computational assumptions. It is still open whether information-theoretically sound verification (where security is required against dishonest provers with unbounded computational power) can be done with a single prover, when the verifier is entirely classical, namely a BPP machine.

On the other hand, we do know that strictly classical interactive protocols are strong enough for information-theoretic verification of BQP computations, if the prover is sufficiently strong. The celebrated P#P ⊆ IP and IP = PSPACE results [LFKN92, Sha92] show this, as BQP ⊆ PP [BV97]. Alas, attempting to adapt the protocols of [LFKN92, Sha92], namely the sum-check protocol, to be used with a weaker BQP prover seems to be problematic. In order to prove a BQP problem using the sum-check protocol, one must first reduce it to a general #P problem - which is no longer solvable by a BQP machine. To address this issue, [AG17] suggest an alternative protocol, which is more natural to the quantum setting. In [AG17] the problem is formulated in terms of local quantum operations and measurements, and they show that a PostBQP [Aar05] prover, which may evaluate quantum values to within inverse-exponential precision, suffices for it. Unfortunately, [AG17] also show that if their protocol is relaxed by allowing inverse-polynomial errors, such that an honest BQP prover may pass it, the modified protocol is no longer secure and there exists a cheating strategy.
Our contribution. We present an in depth discussion of the problems that arise when trying to devise protocols for verifying quantum computations using a single BQP prover. To do so, we define a family of protocols, called *Inexact Linear Scalar Consistency Checking (ILSCC)* protocols - and prove strong limitations on them. In ILSCC protocols, for each round \(i\), there is a scalar \(u_i\) which is evaluated at the beginning of the round, and a scalar \(v_i\) which is evaluated at the end of the round. A key characteristic of ILSCC protocols is that the verification process only consists of comparing \(u_i\) and \(v_{i-1}\) at each round. Another key characteristic of these protocols is that these values are computed by linear functionals. Moreover, we consider such protocols where the comparisons are performed up to an inverse polynomial accuracy (see Section 3 for exact definition). These are natural protocols for verifying BQP computations, as according to the circuit model all that a BQP machine does are linear manipulations of vector states, and as BQP machines can only provide an inverse-polynomially accurate estimation of their inner state, due to statistical errors. In particular, the ILSCC family of protocols generalizes the protocols of [AAV13, AG17]. For an in depth discussion of the generality of ILSCC protocols, see Section 5. We analyze ILSCC protocols based on a parameter which we call the *expected next value*. This value has to do with the maximal expected decrease of the consistency scalar \(v_i\) between one round to the following\(^1\). Based on this expected decrease we show the following (see Section 4 for the exact statements and proofs):

**Theorem 1** (Informal): There is a cheating strategy against all ILSCC protocols where the scalar consistency value is expected to decrease by a multiplicative constant factor.

**Proof overview:** Theorem 1 follows from showing that if the scalar consistency value is expected to decrease by a constant factor in each round - then a cheating prover can send erroneous values in each round \(i\) such that the error \(\delta_i\), which is the difference between the actual evaluated \(v_i\) and the value \(v_i'\) which would have been evaluated in this round had the prover been honest, will also reduce by a constant factor. This error reduction in every round adds up to an exponential reduction at the end of the protocol. But such errors are not caught by the verifier, which allows errors of up to an inverse polynomial.

**Theorem 2** (Informal): There is no interactive proof for BQP computations of the form of ILSCC, where the scalar consistency value doesn’t decrease (i.e. stays the same\(^2\)), unless BQP = BPP.

**Proof overview:** Theorem 2 follows from the following reasoning: if the evaluated values do not change between rounds, then these rounds are essentially redundant. This means that the intermediate rounds can be removed, resulting with a protocol where there are only 2 rounds of communication between the verifier and the prover. However, as the protocol is only accurate up to an inverse-polynomial, these rounds only include logarithmically many bits of information sent from the prover to the verifier. Therefore, the prover can iterate over all possible values he expects from the prover, and by that simulate the protocol on his own. This means that, had there existed such a protocol to verify BQP computations, this protocol could also be used to solve BQP in BPP.

Our analysis highlights inherent problems in devising protocols for verifying BQP computations. We hope this steers future research towards approaches which may be viable for verifying quantum computers. In particular, our analysis brings forth open questions regarding cases where it does not apply. Notably, it still leaves some window of values for the *expected next value* in which a secure ILSCC protocol might exists. This is discussed at length in Section 5.

Related work. As we had already mentioned, our ILSCC family of protocols generalizes the protocols of [AAV13, AG17]. It is also natural to consider the relation between ILSCC and known classical protocols, namely [LFKN92, Sha92]. At first glance, it might seem like the protocol of [LFKN92] is indeed an ILSCC protocol. The protocol consists of the prover sending a polynomial at each round, and the verifier comparing an evaluation of the polynomial from round \(i\) with a linear combination of evaluations of the polynomial from round \(i+1\). These are comparisons of scalar values which result from linear operations. However, a closer examination of the ILSCC definition in Section 3 also constrains the operator norm of an associated linear operator, which

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\(^1\)For a complete explanation of what the maximum and expectation are taken over, see Section 4.

\(^2\)The exact definitions are such that an *increase* is not possible.
is undefined when working in a finite field. The protocol of [Sha92] shares this dissimilarity. In addition, the consistency checks of [Sha92] also include multiplying different evaluations of a polynomial - which is not a linear operation.

**Paper organization.** General preliminaries and notations are provided in Section 2. We define a family of generalized protocols, namely *ILSCC*, and associated notions in Section 3. We analyse the inherent difficulty of using *ILSCC* protocols with a BQP prover in Section 4. Finally, we discuss the generality of *ILSCC* protocols as well as open questions regarding possible approaches to devise verification protocols with a BQP prover in Section 5.

## 2 Background

### 2.1 Notations

We use the following notations throughout the paper:

- For a local gate $g_i$, we define $G_i = g_i \otimes I$ to be its $n$ qubits expansion (where $n$ is the number of qubits in the entire system).
- Similarly, for local unitaries $u_i$ or $h_i$ we use $U_i$ and $H_i$ (respectively) to denote their $n$ qubits expansions.
- Let $\rho$ be an operator on hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and let $z$ be an operator on $\mathcal{H}_A$. Given an orthonormal basis $\{|i\rangle\}_i$ for $\mathcal{H}_A$ and $\{|l\rangle\}_l$ for $\mathcal{H}_B$, such that we can write $\rho = \sum_{i,k,j,l} \rho_{i,j,k,l} |i\rangle \langle j| \otimes |k\rangle \langle l|$, we define the reduction of $\rho$ to the qubits on which $z$ operates as

$$\rho|_z = tr_{\mathcal{H}_B}(\rho) = tr_{\mathcal{H}_B} \left( \sum_{i,j,k,l} \rho_{i,j,k,l} |i\rangle \langle j| \otimes |k\rangle \langle l| \right) = \sum_{i,j} \sum_k \rho_{i,j,k,k} |i\rangle \langle j|$$

- We use the notation $\approx_\mu$ to denote equality to within $\mu$ precision. That is, for scalars $a, b$:

$$a \approx_\mu b \iff |a - b| \leq \mu$$

- We use $\mathbb{C}$ to denote the set of complex numbers.

### 2.2 BQP

**Definition 2.1** (BQP, due to [BV97]). The complexity class BQP is the set of languages which are accepted with probability 2/3 by some polynomial time Quantum Turing Machine.

### 2.3 Top Row Matrix [AG17]

A *top row matrix* is defined as follows:

**Definition 2.2** (top row matrix). Given a $2^n \times 2^n$ matrix $B$, we define it’s top row matrix to be: $|0^n\rangle \langle 0^n| B$, where $|0^n\rangle$ is the computational basis state on $n$ qubits $|00\cdots0\rangle$. By referring to the top row matrix of a quantum circuit on $n$ qubits which is given as a sequence of local gates $g_T \cdots g_1$, we mean the top row matrix of $B = G_T \cdot G_{T-1} \cdots G_1$.

[AG17] also show the simple fact that calculating the trace of the top row matrix of a matrix $B$ with precision $\pm \frac{1}{5}$, where $B$ is the operator which results from compositing an arbitrary sequence of local gates $g_T, \ldots, g_1$, such that each gate $g_i$ acts non-trivially on 3 out of $n$ input qubits and such that $T = \text{poly}(n)$, is BQP-hard.
2.4 The AG protocol

We feel our discussion of ILSCC protocols is easier to understand in the context of a specific protocol. To this end, we will present the protocol of [AG17] in which a verifier \( V \) verifies the trace \( C \) of a top row matrix \( A \) for a quantum computation given by a sequence of local quantum gates \( g_1, \ldots, g_T \), by interacting with a prover \( P \). More formally, the protocol verifies \( \text{tr} (A) = C \) for \( A = |0^n\rangle \langle 0^n| G_T \cdots G_1 \) and value \( C \).

The protocol uses the following measure to generate random unitaries on a single qubit:

Definition 2.3 (The measure \( U(2) \)). The measure \( U(2) \) on unitary operators on a single qubit is defined as choosing angles \( \theta, \varphi_1, \varphi_2 \) uniformly at random from \([0, 2\pi]\), and pick the resulting unitary \( u \) to be:

\[
\begin{bmatrix}
\cos \theta \cdot e^{i\varphi_1} & \sin \theta \cdot e^{i\varphi_2} \\
-\sin \theta \cdot e^{-i\varphi_2} & \cos \theta \cdot e^{-i\varphi_1}
\end{bmatrix}
\]

With this measure defined, we can present the AG protocol:

The AG protocol: Given a top row matrix \( A_{2^n \times 2^n} = |0^n\rangle \langle 0^n| G_T \cdots G_1 \), and a claim \( \text{tr} (A) = C \), the AG protocol between the verifier \( V \) and prover \( P \) goes as follows:

1. In the 0'th round – \( V \) asks for \( M_0 = A|g_1\rangle \), receives back a matrix \( m_0 \) from \( P \), and verifies that \( C = \text{tr} (m_0) \) (rejects otherwise).

2. In the \( i \)'th round – \( V \) chooses \( u_1^i, u_2^i, u_3^i \sim U(2) \), sets \( u_i = u_1^i \otimes u_2^i \otimes u_3^i \) on the qubits on which \( g_i \) operates, and asks for:

\[
M_i = (U_i \cdot U_{i-1} \cdots U_1 \cdot |0^n\rangle \langle 0^n| G_T \cdots G_{i+1})|g_{i+1}\rangle
\]

\( V \) receives back a matrix \( m_i \) from \( P \), and verifies that \( \text{tr} (m_i) = \text{tr} (m_{i-1} \cdot g_i^{-1} \cdot u_i) \) (rejects otherwise).

3. After the \( T \)'th round – \( V \) computes \( \text{tr} (M_T) = \text{tr} (U_T \cdot U_{T-1} \cdots U_1 \cdot |0^n\rangle \langle 0^n|) \) on its own (the various \( U_i \) all act on single qubits in tensor product, so this can be done efficiently) and accepts if

\[
\text{tr} (M_T) = \text{tr} (M_T)
\]

Completeness and soundness [AG17] show that due to the algebraic properties of the trace operator:

\[
\text{tr} (A) = \text{tr} (M_0), \quad \text{tr} (M_i) = \text{tr} (M_{i-1} \cdot g_i^{-1} \cdot u_i)
\]

(1)

And so their protocol has completeness 1. They also show that the protocol’s soundness is 0 (see Section 5 in [AG17]). In order to prove the soundness parameter, [AG17] present the error matrix \( \Delta_i \) for each round \( i \), which is defined to be:

\[
\Delta_i = M_i - m_i
\]

(2)

And by Equations 1, 2, along with the linearity of the trace operator, they get:

\[
\text{tr} (m_i) = \text{tr} (m_{i-1} \cdot g_i^{-1} \cdot u_i) \iff \text{tr} (\Delta_i) = \text{tr} (\Delta_{i-1} \cdot g_i^{-1} \cdot u_i)
\]

(3)

So, in order to pass round \( i \)'s verification it has to be the case that

\[
\text{tr} (\Delta_i) = \text{tr} (\Delta_{i-1} \cdot g_i^{-1} \cdot u_i)
\]

(4)

They also show that:

\[
\text{tr} (\Delta_{i-1}) \neq 0 \Rightarrow \Pr_u [\text{tr} (\Delta_{i-1} \cdot g_i^{-1} \cdot u_i) = 0] = 0
\]

(5)

In other words, if \( m_i \) is different than \( M_i \) then the prover has to send \( m_{i+1} \) which is different than \( M_{i+1} \) - otherwise he will be caught in the round’s verification. This will continue until the final round, in which the verifier computes the final value on its own (as it is now a local operation) and can catch the cheating prover.
**Precision errors**  The AG protocol above assumed infinite precision in computations, values and probability measure used. In order to modify the protocol such that it can actually be realized, [AG17] also present a variant in which all the values are given using polynomially many bits - resulting in inverse-exponential precision. In this variant, the completeness can no longer be maintained using exact equation checks. Instead, the protocol is adapted to use approximated comparisons $\approx_\mu$ where $\mu$ is inverse-exponential. As a by product, in this variant the verification is for the claim $\text{tr} (A) \approx_\mu C$. [AG17] show that the adapted protocol maintains a perfect completeness parameter, and has soundness $\frac{1}{2}$. To prove the soundness parameter for this imprecise protocol, [AG17] consider the probability with which $P$ may decrease the error value $\text{tr} (\Delta_i)$ throughout the protocol. They show that the probability with which a malicious prover may pass the protocol while decreasing a large $\text{tr} (\Delta_0)$ throughout the protocol until making it small enough such that $\text{tr} (\Delta_T) \approx_\mu 0$ is at most $\frac{1}{2}$, and otherwise $V$ will reject in the final round.

2.5 Cheating strategy against the AG protocol for BQP verification

In light of the imprecise AG protocol, a natural question arises: what if the protocol is further adapted, to use approximated comparisons $\approx_\mu$ where $\mu$ is inverse-polynomial, rather than inverse-exponential? If such an adapted protocol is still sound and complete, this protocol could be used to verify BQP computations using a BQP prover! [AG17] address this issue, and show that their protocol is no longer sound if such a relaxation is made. To show this, they present a strategy by which a cheating prover can convince the verifier to accept the false claim $\text{tr} (A) = C$ for $A = |0^n\rangle\langle0^n|G_T \cdot G_{T-1} \cdots G_1$ and value $C = \text{tr} (A) + \frac{\delta}{2}$. In this strategy, the prover adaptively picks the error matrix $\Delta_i$, and chooses an appropriate matrix $m_i = M_i - \Delta_i$ (according to Equation 2) to send to the verifier. The strategy goes as follows:

1. In the 0’th round – Denoting $\delta_0 = \text{tr} (A) - C$, $P$ picks $\Delta_0 = \frac{\delta_0}{\text{tr} (I)} I$, and sends $m_0 = M_0 - \Delta_0$ to $V$.

2. In the $i$’th round – Denoting $\delta_i = \text{tr} (\Delta_{i-1} \cdot g_i^{-1} \cdot u_i)$, $P$ picks $\Delta_i = \frac{\delta_i}{\text{tr} (I)} I$, and sends $m_i = M_i - \Delta_i$ to $V$.

By the choice of $\delta_i$, $\Delta_i$, it is easy to see that

$$\text{tr} (\Delta_i) = \frac{\delta_i}{\text{tr} (I)} \text{tr} (I) = \delta_i = \text{tr} (\Delta_{i-1} \cdot g_i^{-1} \cdot u_i) \tag{6}$$

$$\delta_i = \text{tr} (\Delta_{i-1} \cdot g_i^{-1} \cdot u_i) = \delta_{i-1} \frac{\text{tr} (g_i^{-1} \cdot u_i)}{\text{tr} (I)} \tag{7}$$

By Equations 3, 6, $P$ will manage to pass the first $T - 1$ rounds. In addition, as $I$ has the maximal trace over all unitaries, Equation 7 means that $\delta_i \leq \delta_{i-1}$, and that there is some constant $\varepsilon < 1$ such that $\delta_i \leq \varepsilon \cdot \delta_{i-1}$ for at least $\frac{T}{\varepsilon}$ rounds\(^3\). This means, that with extremely high probability, $\delta_T$ is inverse-exponentially small, and so $V$ accepts after the final round as $\delta_T = \text{tr} (\Delta_T) \approx_\mu 0$.

3 Linear Scalar Consistency Checking protocols

We would now like to define a family of protocols of a particular fairly natural structure, generalizing the AG protocol (as well as the protocol of [AAV13]). Recall that in The AG protocol the consistency checks are of the following form: multiply a given matrix $m_{i-1}$ passed from the prover to the verifier in the $(i-1)$’th round by some other matrix $g_i^{-1} \cdot u_i$, calculate the trace, and compare it to the trace of a matrix $m_i$ which is sent by the prover on the i’th round. This is a special case of what we call “linear scalar consistency checks”, where in order to compare two matrices we compute their values under some linear functional (the trace operator, in the AG protocol), up to some linear transformation on one of the matrices. Consistency means that the two values are equal. With this in mind, we use $\mathcal{M}_{k \times k}$ to denote the space of $k \times k$ matrices and define the following family of protocols, where at each round a consistency check is made using just a single, linearly calculated scalar:

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\(^3\)disregarding negligible imprecisions in order to only use polynomially many bits for each value.

\(^4\)With probability (over the choices of the unitaries $u_1, \ldots, u_T$) that approaches 1.
Definition 3.1. (Linear-Scalar Consistency Checking) We say that a $T(n)$-round interactive protocol between a verifier $V$ and a prover $P$ is a Linear-Scalar Consistency Checking (LSCC) protocol for a scalar function $C : \{0,1\}^n \rightarrow \mathbb{C}$ if there exists a linear scalar function $F : M_{k \times k} \rightarrow \mathbb{C}$ such that given an input $x \in \{0,1\}^n$ and value $C \in \mathbb{C}$:

1. At round 0: $V$ asks $P$ for some matrix $M_0 \in M_{k \times k}$ s.t. $C(x) = F(M_0)$, receives a matrix $M_0$, and verifies that $C = F(m_0)$ (rejects otherwise).
2. At each round $0 < i \leq T$: $V$ picks $T_i \sim D_i(x)$ where $D_i$ is a distribution on the set of linear transformations on $M_{k \times k}$, with operator norm at most 1, such that $D_i$ is computable by $V$. $V$ then sends $T_i$ to $P$, and expects $P$ to send back a matrix $M_i \in M_{k \times k}$ s.t. $F(T_i(M_{i-1})) = F(M_i)$. $V$ receives a matrix $m_i$, and checks for consistency by verifying $F(T_i(M_{i-1})) = F(m_i)$ (rejects otherwise).
3. After round $T$: $V$ accepts iff $F(m_T) = f(x, T_1, \ldots, T_T)$, where $f$ is some function computable by $V$ (rejects otherwise).

To fully understand Definition 3.1, consider the AG protocol. There, the input $x$ is a description of a quantum gate sequence $g_1, \ldots, g_T$, the function $C$ is the trace of the top row matrix of the gate sequence (see Definition 2.2), $F$ is the trace operator, $T_i = g_i^{-1} \cdot u_i$, and $f = tr(U_T \cdot U_{T-1} \cdots U_1 \cdot |0^n\rangle\langle 0^n|)$.

So we have a family of protocols which generalizes the AG protocol\(^5\). However, in LSCC protocols the consistency checks are exact; if we want to study protocols which are meant to be carried out with BQP provers it is necessary to allow approximations in consistency checks, since a BQP prover necessarily introduces inverse polynomial inaccuracies to its estimations of the matrix elements due to statistical errors. This suggests that we should consider protocols where such inverse polynomial inaccuracies in the verifier’s consistency checks are allowed, unlike in the LSCC family. We note that [AG17] already allowed inaccuracies in their adapted (imprecise) protocol. There, the inaccuracies were inverse-exponential. To allow inaccuracies in general LSCC protocols, we define the following family called ILSCC.

Definition 3.2. (Inexact Linear-Scalar Consistency Checking) (Informal) An Inexact Linear Scalar Consistency Checking (ILSCC) protocol with precision $\mu(n)$ for a scalar function $C : \{0,1\}^n \rightarrow \mathbb{C}$ is defined similarly to an LSCC protocol, except that in the ILSCC protocol all equality comparisons are done to within precision $\hat{\mu}$, where $\hat{\mu} = \mu(n) \cdot \max_{y \in \{0,1\}^n} |C(y)|$. For a formal definition, see Appendix C.

Having defined ILSCC protocols, we would like to understand whether they can be used in interactive proofs for functional problems. That is, given an input, we would like to verify the value of some function on this input. In order to speak in a formal, general way about verification of such functional problems, we define a language $L$ based on the function. Given a function $C(\cdot)$, the language $L$ contains all pairs of the form $(x,C(x))$. However, similarly to previous notions we have presented - we would also like to allow a tolerance in our functional verification, so $L$ also contains pairs $(x,C)$ where $C$ is an approximation of $C(x)$. We pick an approximation that allows BQP verification (see Section 2.3). Formally, we define the following:

Definition 3.3. (Scalar Function Language) For a given function $C : \{0,1\}^* \rightarrow \mathbb{C}$, we define the set $SF_n(C)$ as $SF_n(C) = \{(x,C) | x \in \{0,1\}^n, |C(x) - C| \leq \frac{1}{2} \cdot \max_{y \in \{0,1\}^n} |C(y)|\}$. We define the Scalar Function Language associated with the function $C$ as

$$SF(C) \equiv \cup_{n=0}^\infty SF_n(C).$$

It will also be useful to define the class of all Scalar Functional Languages:

Definition 3.4. (Scalar Function Class) We define the class SFL (for Scalar Function Languages) to be the set of all scalar function languages, over all functions.

Furthermore, for any language $L = SF(C)$ we define a Polynomial Linear-Scalar Consistency Proof System:

\(^5\)as well as that of [AAV13], as all the consistency checks there consist of comparing the trace of a new matrix to the trace of a previous matrix, under some linear projection.
Definition 3.5. (Polynomial Linear-Scalar Consistency Proof System) Let $\mathcal{L} = SF(C)$ be a Scalar Functional Language associated with the function $C : \{0,1\}^* \rightarrow \mathbb{C}$. We say that $\mathcal{L}$ has a Polynomial Linear-Scalar Consistency Proof System if there exists a protocol $\mathcal{W}$ between a verifier $\mathcal{V} \in \mathbb{BPP}$ and a prover $\mathcal{P}$ such that:

1. $\mathcal{W}$ is an $ILSCC$ protocol for $C$ with precision $\mu(n) = \frac{1}{\text{poly}(n)}$, and $T(n)$ rounds such that $\Omega(n) \leq T(n) \leq \text{poly}(n)$
2. (Exact completeness) $\forall x \in \{0,1\}^n$, $\text{Pr}[\mathcal{W}(x,C(x)) = \text{Accept}] = 1$. Moreover, this property holds even if we set the precision $\mu$ of $\mathcal{W}$ to be arbitrarily small.
3. (Soundness) For any protocol $\tilde{\mathcal{W}}$ between a verifier $\mathcal{V}$ and a prover $\mathcal{P}$, where $\mathcal{V}$ behaves the same as in $\mathcal{W}$ (but the prover may act adversarially): $\forall (x,c) \notin \mathcal{L}$, $\text{Pr}[\tilde{\mathcal{W}}(x,c) = \text{Accept}] \leq \frac{1}{3}$.

And we define the class of Polynomial Linear-Scalar Checkable languages:

Definition 3.6. (Polynomial Linear-Scalar Checkable) The class Polynomial Linear-Scalar Checkable ($PLSC$) is defined as the class of all Scalar Functional Languages which have a Polynomial Linear-Scalar Consistency Proof System. That is:

$$PLSC = \{\mathcal{L} \in SFL \mid \mathcal{L} \text{ has a Polynomial Linear-Scalar Consistency Proof System}\}$$

Finally, we define the following:

Definition 3.7. (Viable protocols) Let $\mathcal{L} \in PLSC$. We call any protocol $\mathcal{W}$ for which the three conditions of Definition 3.5 are met, a Viable protocol for $\mathcal{L}$. We denote the set of all Viable protocols for $\mathcal{L}$ by $\text{Viable}(\mathcal{L})$.

Definition 3.8. (Dishonest protocols) Let $\mathcal{L} \in PLSC$, and let $\mathcal{W} \in \text{Viable}(\mathcal{L})$. We call any protocol $\tilde{\mathcal{W}}$ where $\mathcal{V}$ behaves the same as in $\mathcal{W}$ (but the prover may act adversarially) a Dishonest protocol for $\mathcal{L}$ with respect to $\mathcal{W}$. We denote the set of all dishonest protocols for $\mathcal{L}$ with respect to $\mathcal{W}$ by $\text{Dishonest}(\mathcal{L})_\mathcal{W}$.

4 Limitations on $ILSCC$ protocols for verifying $BQP$

To gain insight on the class $PLSC$ of languages that are verifiable by $ILSCC$ protocols, we will prove two theorems concerning their possible viable protocols. To this end, we associate each $ILSCC$ protocol with a parameter which we call the expected next value. Intuitively, this parameter has to do with the maximal reduction of the scalar value from round to round: At round $i$ the verifier receives the matrix $m_i$, and evaluates its associated value $F(m_i)$. So for the matrix $m_i$, the next value is $F(T_{i+1}(m_i))$. Now, let

$$Q = \arg\max_{\|Q\|_F = 1} (|F(Q)|), \quad q^* = \max_{\|Q\|_F = 1} (|F(Q)|)$$

Where we use $\|\cdot\|_F$ to denote the Frobenius norm. We note that for any $Q \in Q$, the next value which corresponds to $Q$ can not be greater than $Q$’s value $q^*$, due to the fact that $T_{i+1}$ has operator norm at most 1. We define the expected next value with respect to such $Q$, where the expectation is taken over the rounds and choices of random operators:

Definition 4.1. (Expected next value) Let $\mathcal{L} \in PLSC$, and let $\mathcal{W} \in \text{Viable}(\mathcal{L})$. We define $\mathcal{W}$’s expected next value on inputs in $\{(x,C) \mid x \in \{0,1\}^n, C \in \mathbb{C}\}$, denoted $E_{\mathcal{W}}^n$, to be the minimal expectation value

$$E_{\mathcal{W}}^n = \min_{x \in \{0,1\}^n} \left( \min_{Q \in \mathcal{Q}} \left( \frac{\mathbb{E}_{T_i \sim D_i(x)} \left[ \|F(T_{i+1}(Q))\| \right]}{0 \leq i \leq T(n)} \right) \right)$$

We note that $E_{\mathcal{W}}^n \leq q^*$, and state our two theorems:
Theorem 1. ∀L ∈ PLSC, ∀W ∈ Viable (L), ∀ε > 0, ∃N | ∀n > N : E^n_W > (1−ε)q^*

Theorem 2. Let L ∈ PLSC. If L has a viable protocol W ∈ Viable (L) such that ∀n ∈ N : E^n_W = q^*, then L ∈ BPP

In other, less formal words, Theorem 1 means that for any constant ε > 0 there is no viable protocol W with expected next value E_W ≤ (1−ε)q^*. On the other hand, Theorem 2 says that any language L ∈ PLSC which has a viable protocol W with E_W = q^* is in BPP, so in particular L is not complete for BQP unless BQP ⊆ BPP.

The perceptive reader may have noticed that the previous Theorem 1 and Theorem 2 did not cover all possible values for the expected next value E_W. Indeed, there remains a band of values for E_W which is somewhat mysterious. We discuss this "band of mystery" in Section 5.

4.1 Theorem 1 proof overview

Given a language L ∈ PLSC with an associated function C : {0,1}^* → ℂ and W ∈ Viable (L), we would like to analyze the execution of any dishonest protocol W for L with respect to W on input (x,C) for some x ∈ {0,1}^n, C ∈ ℂ. For this purpose, we define an error term Δ_i for each round of a specific execution of W on (x,C). To do so, we first define the honest matrix M_i to be the matrix that is received from P in the i'th round when running the (honest) viable protocol W on input (x,C(x)) with the same selection of T_1, . . . , T_n by the verifier as in the run of W. Keeping in mind that in our execution of W the prover sends a matrix m_i in the i'th round, we define:

Δ_i = M_i − m_i

By exact completeness, we have

F(T_i(M_{i−1})) = F(M_i)

(9)

So by the linearity of F, we get that the consistency condition

F(T_i(m_{i−1})) ≈_µ F(m_i)

(10)

is equivalent to the condition

F(T_i(Δ_{i−1})) ≈_µ F(Δ_i)

(11)

Now, to prove Theorem 1, we show that if there exists ε > 0 such that E_W ≤ (1−ε)q^*, then there exists a cheating strategy for the prover where |F(Δ_T)| is exponentially smaller than |F(Δ_0)|. This strategy is realized by choosing values such that Δ_i = k_i · Q^*, where k_i is a scalar which is iteratively picked as to pass the i'th round. Due to Q^*'s maximality under F, the error terms |F(Δ_i)| = |k_i · q^*| are guaranteed to be monotonically decreasing. Moreover, since E_W ≤ (1−ε)q^*, there exists an input for which the error terms are likely to decrease by some constant factor in a constant fraction of the rounds - resulting with an exponential overall decrease. This is a generalization of the cheating strategy against the AG protocol (see Section 2.5), where Q^* is just the (normalized) identity matrix I. For a detailed proof, see Appendix A.

4.2 Theorem 2 proof

Proof. Let L ∈ PLSC, W ∈ Viable (L) with respective operator F and distributions {D_i}_{i=1}^{T(n)} such that ∀n ∈ N : E^n_W = E_W = q^*. In order to show that L ∈ BPP we first show that for any A ∈ M_{k×k}, its value under the operation of F isn’t changed by first operating on it with T_i ∼ D_i (up to a phase change which doesn’t depend on A). More formally, we claim the following:

Claim 1. Let L ∈ PLSC, W ∈ Viable (L) with respective operator F and distributions {D_i}_{i=1}^{T}, and let A ∈ M_{k×k}. If E_W = q^* then for all 0 < i ≤ T:

Pr_{T_i ∼ D_i}[F(A) = λ(T_i) · F(T_i (A))] = 1

Where λ(T_i) is a scalar that only depends on T_i, such that |λ| = 1.


Claim 1 is proven in Appendix B.

Now we explain why Claim 1 entails \( \mathcal{L} \in \text{BPP} \). Let us consider any accepting run of the protocol \( \mathcal{W} \) (with a choice of \( T_1, \ldots, T_T \) and \( m_0, \ldots, m_T \) by the verifier \( V \) and prover \( P \) respectively). By recursively applying Claim 1 we get that, up to phase changes due to \( \lambda(T_1), \ldots, \lambda(T_T) \):

\[
\mathcal{F}(m_0) = \cdots = \mathcal{F}(m_T)
\]

So \( P \) could pass the first \( T-1 \) rounds by simply sending \( m_0 \) over and over again\(^6\). Then, to pass the \( T \)'th round, \( P \) can simply send \( m_T \). This makes it clear that the intermediate rounds of interaction are in fact completely redundant. The protocol is exactly equivalent to a 2 round protocol \( \mathcal{W}_2 \) where

- in the first round: \( P \) sends \( m_0 \) to \( V \)
- in the second round: \( V \) sends \( T_1, \ldots, T_T \) to \( P \), and \( P \) sends back \( m_T \) (\( V \) then performs its verification on \( m_T \) using \( T_1, \ldots, T_T \))

This equivalence means that \( \mathcal{W}_2 \) has the same soundness and completeness properties as \( \mathcal{W} \), which puts \( \mathcal{L} \) is in the complexity class \( \text{MA}[3] \) (Merlin Arthur Merlin) [Bab85]. Now, if we take into consideration that the equality checking of \( V \) is only done with inverse polynomial precision, it follows that \( P \) can also pass the protocol using the inverse-polynomial approximations \( m_0, m_T \) which are obtained from \( m_0, m_T \) by only keeping logarithmically many bits to describe each. This means that there are only polynomially many possible choices of \( m_0 \), and for each such choice - given also a choice of \( T_1, \ldots, T_T \) - there are only polynomially many possible options for \( m_T \). This means that \( V \) can simulate \( P \) on her own, which results in \( \mathcal{L} \in \text{BPP} \) as required.

We note that the final argument in the proof of Theorem 2 relied on an implicit assumption that the matrices are in \( \mathcal{M}_{k \times k} \) where \( k \) is constant. But one could also consider protocols where matrix sizes are polynomial - for instance, if they represent a reduced density matrix on a logarithmic number of qubits. This change will imply that the verifier could no longer simulate the protocol on her own, resulting in \( \mathcal{L} \in \text{MA}[3] = \text{AM} \) [BM88] instead of \( \mathcal{L} \in \text{BPP} \). This change would mean that in particular \( \mathcal{L} \) is not complete for \( \text{BQP} \) unless \( \text{BQP} \subseteq \text{AM} \), which is weaker than the original corollary but still seems rather unlikely.

5 Discussion and open questions

We hope our definition and analysis of \( \text{ILSCC} \)s help shed light on the innate difficulties of using natural approaches for \( \text{BQP} \) verification. To this end, we remark that our discussion of \( \text{ILSCC} \) protocols might be more general than it appears in first glance.

1. We had already mentioned in Section 1 that protocols where consistency checks are linear are natural to consider with respect to \( \text{BQP} \) problems, as quantum computers are considered to be linear beings. However, one could consider using vectors for consistency checking, rather than scalars. We note that this only has the potential to circumvent Theorem 1 if the consistency checking involved non-linear operations on the vectors (e.g. a consistency check which involves norms). Otherwise, there is an equivalent protocol where the consistency test only involves scalars.

2. We would also like to address the fact that \( \text{ILSCC} \) protocols use a fixed functional \( \mathcal{F} \) for consistency checking. Indeed, the cheating strategy (Section 4.1; Appendix A) uses the prover’s knowledge of \( \mathcal{F} \) to pick the error matrices \( \Delta_i \) according to \( Q^* \). To circumvent this, one could consider a protocol where the consistency tests are different for each round. So in the \( i \)’th round the consistency test will be of the form \( \mathcal{F}_i(m_{i-1}) = \mathcal{F}_i(m_i) \) for \( \hat{F}_i \), \( \tilde{F}_i \) which are distinct random linear functionals which are chosen separately for each round\(^7\). The caveat is that the prover still needs to be told which matrix \( m_i \) he should send in order to pass the \( i \)’th round, and the prover could use this knowledge to infer the functional \( \tilde{F}_i \) - resulting with the same problem.

\(^6\)W.l.o.g., \( V \) can be assumed to make the phase changes on its own, as \( \lambda(T_i) \) only depends on \( T_i \).

\(^7\)\( \text{ILSCC} \) protocols can be generalized to include such protocols by introduction another random linear transformation \( \tilde{T}_i \) for the \( i \)’th round and defining \( \hat{F}_i = \mathcal{F} \circ \tilde{T}_i \); \( \tilde{F}_i \) = \( \hat{F}_{i-1} \circ \tilde{T}_i \), and using consistency checks of the form \( \mathcal{F}_i(m_{i-1}) = \hat{F}_i(m_i) \).
That being said, our analysis does bring forth open questions regarding the possibility of BQP verification, even without finding non-linear structure in a BQP complete problem. We address a couple of them in particular:

1. **Number of rounds vs. precision** The cheating strategy presented in Section 4.1 (and Appendix A) is based on repeatedly decreasing an initial error term such that by the last round it is smaller than an inaccuracy parameter $\mu$ which is needed for the protocol’s completeness. This suggests another approach to mitigate the cheating strategy: one could look for protocols where the number of rounds is such that even if at each round the error term is decreased by e.g. a constant factor - it is still not smaller than the inaccuracy value. To this end one could try two venues:
   - **Decrease the number of rounds** - One could try to find a protocol which verifies a BQP-complete problem and has a sufficiently sub-linear number of rounds that a final error term is inverse-polynomial even if an initial error term decreases by a constant factor at each round with high probability. We note that such a result will put BQP in a subclass of IP which will be an interesting result in its own merits.
   - **Decrease the allowed inaccuracy** - On the other hand, one could look for a protocol where the prover is not allowed inverse-polynomial inaccuracy to begin with. The motivation behind allowing the prover such inaccuracies was that a BQP machine experiences such inaccuracies in its measurements. But what if the prover could be expected to compute matrix entries with higher precision? For example, if there were only a polynomial number of discrete possible values\(^8\), which are all inverse-polynomially separated from each-other, then the prover can be expected to infer the exact values with high probability by inverse-polynomially approximating them.

2. **ILSCC band of mystery** As previously mentioned, the analysis of Section 4 did not cover all possible \textit{expected next values}. One could consider a protocols $W$ where $E_W = (1 - \varepsilon)q^*$ where $\varepsilon$ is not bounded away from 0 by a constant. We first mention that indeed, our analysis was not tight. Instead of Claim 1, it is easy to prove the following inexact variant:

   \textbf{Claim 2.} For any $A \in \mathcal{M}_{k \times k}$, if $E_W \geq \left(1 - \frac{1}{T(n)}\right) q^*$ for a super-polynomial $f(n)$ then there exist super-polynomial functions $f(n)$, $f(n)$ such that if we denote $\alpha = \frac{1}{f(n)}$, $\beta = \frac{1}{f(n)}$

   \[ \Pr\left[ \exists_{i \sim D_i} \left[ F(A) \approx_{\alpha} F(T_i(A)) \right] \approx_{\beta} 1 \right] \approx_{\beta} 1 \]  

(12)

Using the union bound, Equation (12) can then be used to extend Theorem 2 to exclude the region $E_W \geq \left(1 - \frac{1}{T(n)}\right) q^*$. In the other direction, Theorem 1 can be extended to show $E_W > \left(1 - \frac{1}{g(T)}\right) q^*$ for some sub-linear function $g(T)$, which excludes the region where $E_W \leq \left(1 - \frac{1}{g(T)}\right) q^*$. But still, there remains a band which is not covered by the two cases. This is not a coincidence. To demonstrate the band where neither of the cases apply, let us consider a hypothetical example where $T(n) = n$, with a choice of $D_i$’s such that

\[ \Pr\left[ \exists_{i \sim D_i} \left[ F(T_i(Q^*)) = \left(1 - \frac{1}{n}\right) F(Q^*) \right] = 1 \right] \]  

(13)

In this scenario, the cheating strategy and arguments of Theorem 1 only imply that $F(\Delta_T)$ is smaller than $F(\Delta_0)$ by a constant factor. So the prover can not decrease his error enough as to pass the verifier’s final verification, and his proposed cheating strategy is not valid. On the other hand, in this example the verifier’s consistency tests can be accurate enough to pick up on the change of the values $F(m_i)$ of the intermediate $m_i$ matrices. This means the intermediate rounds can no longer be skipped - and the arguments of Theorem 2 are no longer valid.

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\(^8\)Assuming there are still enough possible values, so the protocol’s soundness isn’t hurt due to insufficient randomness.
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A Theorem 1 proof

Theorem 1. \( \forall L \in PLSC, \forall W \in Viable(L), \forall \varepsilon > 0, \exists N \mid \forall n > N : E^W_L > (1 - \varepsilon) q^* \)

Proof. Let \( L \in PLSC \) be a language with viable protocol \( W \in Viable(L) \) such that \( E^W_L = (1 - \varepsilon) q^* \) for some \( \varepsilon > 0, n \in \mathbb{N} \), and let \( x \in \{0,1\}^n \), \( C \in \mathbb{C} \). We define the following cheating strategy\(^9\) for the prover \( P \) on input \((x,C)\) by recursively defining for each round an error value \( \delta_i \) along with a choice of error term \( \Delta_i \) to satisfy this error value, which in turn commands the choice of matrices \( m_i \) that the prover sends in each round (according to the definition of \( \Delta_i \) in Section 4.1):

1. At round 0 we define \( \delta_0 = C(x) - C, \Delta_0 = \frac{C(x)Q^*}{1}\)
2. At each round \( 0 < i \leq T \): we define \( \delta_i = F(T_i(\Delta_{i-1})), \Delta_i = \frac{\delta_i}{q^*} |C^*(T_i(x))| \)

We now use the cheating strategy to show that if \( n \) is sufficiently large, \( W \) is not sound. This means it can’t be a viable protocol, thus proving the theorem. To do so, let \( C = C(x) + \frac{1}{2} \max y \in \{0,1\}^n |C(y)| \). We look at the probability that the cheating strategy will make \( V \) accept the input \((x,C) \notin L\):

1. At round 0: \( V \) doesn’t reject, as by linearity of \( F \) we have
   \[ F(\Delta_0) = \frac{\delta_0}{q^*} F(Q^*) = \delta_0 \]
   and so
   \[ C = C(I) - \delta_0 = F(M_0) = F(M_0 - M_0 + m_0) = F(m_0) \]

2. At each round \( V \) doesn’t reject, as the condition of Equation 11 is met:
   \[ F(T_i(\Delta_{i-1})) = \delta_i = \frac{\delta_i}{q^*} F(Q^*) = F(\Delta_i) \]

3. Now we just need to show that \( V \) accepts after round \( T \) w.p. greater than \( \frac{1}{3} \). To do so, we note that by perfect completeness:
   \[ f(x, T_1, \ldots, T_T) = F(M_T) = F(M_T - m_T + m_T) = F(\Delta_T + m_T) + \delta_T \]

So, keeping in mind that \( |\delta_0| = \frac{1}{2} \max y \in \{0,1\}^n |C(y)| \), it suffices to show that
   \[ \Pr[|\delta_T| < \frac{|\delta_0|}{poly(n)}] > \frac{1}{3} \]

Which would mean that \( \Pr[|V \text{ Accepts}|] > \frac{1}{3} \) in the dishonest protocol where \( V \) acts according to \( W \) and the prover acts according to the cheating strategy, making \( W \) not sound. For each round \( 0 < i \leq T \), we define the shrinkage \( S_i = \frac{\delta_i}{\delta_{i-1}} \), and note that by linearity
   \[ \delta_i = \frac{\delta_{i-1}}{q^*} F(T_i(Q^*)) \Rightarrow S_i = \frac{|F(T_i(Q^*))|}{q^*} \]

We recall that \( E^W_L = (1 - \varepsilon) q^* \), so there are at least \( \frac{\varepsilon T}{2} \) rounds \( i \) for which \( E_{T_i}[S_i] \leq 1 - \frac{\varepsilon}{2} \). Let us denote the set of such rounds by \( \mathcal{S} \). Similarly, \( \forall i \in \mathcal{S} : \Pr[S_i \leq 1 - \frac{\varepsilon}{2}] > \frac{\varepsilon}{2} \). Now we can denote \( \xi \) the indicator variable for the event that \( S_i \leq 1 - \frac{\varepsilon}{2} \), and denote \( \xi = \{ i \in \mathcal{S} \text{ s.t } \xi = 1 \} \). A simple chernoff bound now gets us that \( \Pr[|\xi| \leq \frac{1}{2} \cdot \frac{\varepsilon T}{2} - \frac{\varepsilon}{2}] \) is inverse-exponentially small.

It remains to note that by the maximality of \( q^* \) in equation 19, \( S_i \leq 1 \) for every round \( 0 < i \leq T \). So we have \( |\delta_T| \leq \frac{|\delta_0|}{(1 - \frac{\varepsilon}{2})^T} \). This means that inequality 18 holds, as we have shown that the probability of \( |\xi| \) to be linear in \( T \) is inverse-exponentially close to 1.

\( \square \)

\(^9\)Throughout this analysis, we disregard inverse-exponential inaccuracies that occur as all communication in the protocol must be polynomial, as these inaccuracies are negligible with respect to the inverse-polynomial inaccuracies allowed in the comparisons.
B Claim 1 proof

Claim 1. Let \( \mathcal{L} \in \text{PLSC}, \ W \in \text{Viable}(\mathcal{L}) \) with respective operator \( \mathcal{F} \) and distributions \( \{D_i\}_{i=1}^T \), and let \( A \in \mathcal{M}_{k \times k} \). If \( E_W = q^* \) then for all \( 0 < i < T \):

\[
\Pr_{T_i \sim D_i} \left[ \mathcal{F}(A) = \lambda(T_i) \cdot \mathcal{F}(T_i(A)) \right] = 1
\]

(20)

Where \( \lambda(T_i) \) is a scalar that only depends on \( T_i \), such that \( |\lambda| = 1 \).

Proof. To prove Claim 1, we first note that by the condition \( E_W = q^* \) we get that:

\[
\Pr_{T_i \sim D_i} [\mathcal{F}(Q^*)] = |\mathcal{F}(T_i(Q^*))| = 1
\]

(21)

But why does (21) imply (20)? Why can’t it be the case that \( T_i \) doesn’t change the value of \( Q^* \), but decreases the values of other matrices? To understand this question better and answer it, we use the linear nature of \( \mathcal{F} \)’s linearity means that it is simply an inner product with a vector \( F \) of the same dimension\(^{10} \) (e.g. the vector which corresponds to the trace operator is just the vector representation of the Identity matrix), whereas \( T_i(A) = T_i \cdot A \). Let us denote \( F \)’s normalized vector by \( e := \frac{F}{\|F\|} \), so we have

\[
\|e\| = 1, \ F = \|F\| \cdot e
\]

(22)

It follows that, up to a phase change which we can omit w.l.o.g.:

\[
Q^* = e, \ q^* = \|F\|, \ \mathcal{F}(x) = q^* \langle e | x \rangle
\]

(23)

Now, let us consider a basis \( \{\hat{e}\} \) for \( \mathcal{M}_{k \times k} \) where \( e \) is the first basis vector \( e_1 \), and denote \( B \) the base change matrix, so that \( B \cdot e = e_1 \). In this basis we map \( T_i \) to the operator \( \mathcal{F} \) by conjugating it by \( B \):

\[
\mathcal{F} = B \circ T_i \circ B^\dagger
\]

(24)

From Equations (21), (23) and (24) we get:

\[
\Pr_{\mathcal{F}}[\|e_1 | \mathcal{F} e_1 \| = 1] = 1
\]

(25)

Given the fact that \( \mathcal{F} \) has operator norm at most 1, Equation 25 means that \( e_1 \) is an eigen vector of \( \mathcal{F} \) with eigen value \( \lambda_1 \) such that \( |\lambda_1| = 1 \). So we can rewrite (25) as

\[
\Pr_{\mathcal{F}}[\lambda^1_1 (e_1 | \mathcal{F} e_1 ) = 1] = 1
\]

(26)

Now we are ready to rewrite (20) as

\[
\Pr_{\mathcal{F}}[\langle e_1 | A \rangle = \lambda^1_1 (e_1 | \mathcal{F} \cdot A )] = 1
\]

(27)

And ask the (restated) question of whether (26) implies (27) in a more concrete way: why can’t it be the case that \( \mathcal{F} \) is chosen such that it multiplies \( e_1 \) by \( \lambda_1 \), but acts on the rest of the space in a very random way? We use the fact that \( \mathcal{F} \) is also a linear transformation on \( \mathcal{M}_{k \times k} \), and hence can be represented by a \( k^2 \times k^2 \) matrix. The answer is quite simple when considering the properties of this matrix. By (26) we get that \( \mathcal{F} \)’s top left entry has to be \( \lambda_1 \), as it sends \( e_1 \) to \( \lambda_1 \cdot e_1 \). This means that the rest of the top row entries are 0; otherwise, if there is a column \( i \neq 1 \) where the top row has an entry \( \varepsilon \neq 0 \), we get for \( v = \frac{1}{\sqrt{1+\varepsilon^2}} (e_1 + \varepsilon \cdot e_i) \) that \( \|v\| = 1 \) but \( \|v \cdot \mathcal{F} \cdot v\| = \sqrt{1+\varepsilon^2} > 1 \), in contradictions to \( \mathcal{F} \) having a maximal eigen value of 1. So if we denote the subspace which is orthogonal to \( e_1 \) by \( e_1^\perp \), we get that while \( \mathcal{F} \) might indeed act in a non-trivial way on any vector \( v_\perp \in e_1^\perp \), it will not change its projection on \( e_1 \). It is now easy to see that for a general vector \( v \in \mathcal{M}_{k \times k} \), if we decompose it as \( v = c \cdot e_1 + v_\perp \) (for some \( v_\perp \in e_1^\perp \) and scalar \( c \)) then:

\[
\lambda^1_1 (e_1 | \mathcal{F} v) = \lambda^1_1 (e_1 | c \cdot \mathcal{F} e_1 + \mathcal{F} v_\perp) = c = \langle e_1 | c \cdot e_1 + v_\perp \rangle = \langle e_1 | v \rangle
\]

(28)

as required, and we have proven Claim 1.

\footnote{\( \mathcal{F} \) is a linear function on a \( k^2 \) dimension space, so it is a matrix with \( k^2 \) columns. and it is scalar, so it has a single row.}
C  **ILSCC protocols formal definition**

**Definition 3.2 (Linear-Scalar Consistency Checking)  (formal)**

We say that a $T(n)$-round interactive protocol between a verifier $V$ and a prover $P$ is an *Inexact Linear Scalar Consistency Checking (ILSCC)* protocol with precision $\mu(n)$ for a scalar function $C : \{0,1\}^* \rightarrow \mathbb{C}$ if there exists a linear scalar function $F : \mathcal{M}_{k \times k} \rightarrow \mathbb{C}$ such that given an input $x \in \{0,1\}^n$ and $C \in \mathbb{C}$:

1. The maximal possible value of $|C(\cdot)|$ over $\{0,1\}^n$ is $C_{\text{max}}$, and we denote $\hat{\mu} = \mu(n) \cdot C_{\text{max}}$
2. At round 0: $V$ asks $P$ for some matrix $M_0 \in \mathcal{M}_{k \times k}$ s.t. $C = F(M_0)$, receives a matrix $m_0$, and verifies that $C \approx \hat{\mu} F(m_0)$ (rejects otherwise).
3. At each round $0 < i \leq T$: $V$ picks $T_i \sim D_i(x)$ where $D_i$ is a distribution on the set of linear transformations on $\mathcal{M}_{k \times k}$ with operator norm at most 1, such that $D_i$ is computable by $V$. $V$ then asks $P$ for a matrix $M_i \in \mathcal{M}_{k \times k}$ s.t. $F(T_i(M_{i-1})) = F(M_i)$, receives a matrix $m_i$, and checks for consistency by verifying $F(T_i(m_{i-1})) \approx \hat{\mu} F(m_i)$ (rejects otherwise).
4. After round $T$: $V$ accepts iff $F(m_T) \approx \hat{\mu} f(x, T_1, \ldots, T_T)$, where $f$ is some function computable by $V$ (rejects otherwise).