An *hp*-version interior penalty discontinuous Galerkin method for the quad-curl eigenvalue problem

Jiayu Han¹ · Zhimin Zhang²

Received: 28 November 2022 / Accepted: 10 October 2023 © The Author(s), under exclusive licence to Springer Nature B.V. 2023

Abstract

An *hp*-version interior penalty discontinuous Galerkin method under nonconforming meshes is proposed to solve the quad-curl eigenvalue problem. We prove well-posedness of the numerical scheme for the quad-curl equation and then derive an error estimate in a mesh-dependent norm, which is optimal with respect to *h* but has different *p*-version error bounds under conforming and nonconforming tetrahedron meshes. The *hp*-version discrete compactness of the DG space is established for the convergence proof. The performance of the method is demonstrated by numerical experiments using conforming/nonconforming meshes and *h*-version/*p*-version refinement. The optimal *h*-version convergence rate and the exponential *p*-version convergence rate are observed.

Keywords *hp* discontinuous Galerkin method · Quad-curl eigenvalue problem · Error estimate · Discrete compactness

Mathematics Subject Classification 65N25 · 65N30 · 35Q60 · 35B45

Communicated by Martin Kronbichler.

Project supported by the National Natural Science Foundation of China (Grant Nos. 12001130, 12131005 and 12361084), and China Postdoctoral Science Foundation No. 2020M680316.

✉ Jiayu Han
hanjiayu@gznu.edu.cn

Zhimin Zhang
ag7761@wayne.edu

¹ School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, China

² Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

Published online: 10 November 2023
1 Introduction

The quad-curl eigenvalue problem holds significant importance in the field of inverse electromagnetic scattering theory of inhomogeneous media [16, 40, 51] and magnetohydrodynamics equations [56]. As a classic electromagnetic model, the Maxwell eigenvalue problem has become a prominent topic in the realm of numerical mathematics and computational electromagnetism (see, e.g., [2, 9, 11–14, 22, 33, 38, 39, 41, 48, 49, 58]). Over the past few years, there has been a growing interest in the scientific community towards the numerical treatment of the quad-curl equation and its associated eigenvalue problem. Some early works on the quad-curl equation include a nonconforming element method by Zheng et al. [56] and an $h$-version IPDG method by Hong et al. [34]. Chen et al. [19] and Sun et al. [52] further established new a-priori error estimates and a multigrid method, respectively, for this equation. Sun et al. [51] also introduced an $h$-version weak Galerkin method for the quad-curl equation. Mixed element methods for this equation were studied by Sun [50] and Zhang [57], respectively. Lagrange finite element methods on planar domains for the quad-curl equation were proposed by Brenner et al. [15]. More recently, Zhang and Hu et al. [31, 32, 55] proposed several families of $H(\text{curl}^2)$-conforming finite elements in both two and three dimensions. A quadrilateral spectral element method for the quad-curl eigenvalue problem was further developed in [54]. The $H(\text{curl}^2)$-conforming virtual element methods [59] and the decoupled finite element method [18] were also proposed.

The $hp$ finite element methods are popular in scientific computing due to their flexibility and high accuracy. DG methods provide general framework for $hp$-adaptivity by employing the discontinuous finite element spaces, allowing for great flexibility in mesh design and polynomial bases. For a historical overview of the development of $hp$-version DG methods, we refer to, e.g., articles [1, 6], monographs [23, 25] and the references therein. For the second-order elliptic problem and thebiharmonic problem, a significant amount of research has been done on $hp$-version IPDG methods, as seen in [27–29, 35, 37, 42, 43, 45, 53]. It is important to note that the conventional approaches used to enable $hp$ adaptive methods on meshes with hanging nodes, but conforming methods via constraints, which are popular for elliptic PDEs (see, e.g., [7] and the references therein), can also be applied to the quad-curl eigenvalue problem.

However, to the best of our knowledge, there is a lack of existing literature on $hp$-version DG methods for the quad-curl equation and its eigenvalue problem. This paper aims to bridge this gap by proposing an $hp$-version IPDG method with two interior penalty parameters to solve the quad-curl equation and its associated eigenvalue problem. The paper also establishes $hp$-version error estimates for the DG solution of the quad-curl equation (without the div-free constraint) under the assumption that the exact solution $w \in H^3(\Omega)$ and $\text{curl} w \in H^{1/2+\sigma}(\Omega)$. To bound the error of the IPDG method for the eigenvalue problem, we analyze the $hp$-version DG discretization of the quad-curl equation with a div-free constraint. We establish a discrete Poincaré inequality in the discrete div-free space to ensure the well-posedness of the DG discretization scheme. Then, we establish $hp$-version discrete compactness on the discrete div-free space to prove the uniform convergence of discrete solution operators. Finally, we use the well-known Babuška-Osborn theory [4] to prove the convergence
of the IPDG method for the quad-curl eigenvalue problem. The a priori error bound of the DG eigenvalues is optimal in $h$ on both conforming and nonconforming meshes, and suboptimal in $p$ by 3 order on nonconforming simplex mesh and by 2 order on conforming simplex mesh.

This paper is structured as follows. In Sect. 2, an $hp$-version IPDG scheme is given for the quad-curl equation without div-free condition. In Sect. 3, we will discuss the stability of the IPDG scheme and its a-priori error estimates in DG norm under conforming/nonconforming mesh. An IPDG scheme for the quad-curl eigenvalue problem is proposed in Sect. 4. The discrete Poincaré inequality and discrete compactness of discrete div-free space are established. The error bound for IPDG eigenvalues and the error estimates of eigenfunctions in low norms will follow. In the end of this paper, we present several numerical examples to validate the efficiency of our methods under both $h$-refinement and $p$-refinement modes.

Throughout this paper, we use the symbol $a \lesssim b$ and $a \gtrsim b$ to mean that $a \leq Cb$ and $a \geq Cb$ respectively, where $C$ denotes a positive constant independent of mesh parameters and polynomial degrees and may not be the same in different places.

## 2 An $hp$-version IPDG method for the quad-curl problem

Consider the quad-curl problem

\begin{align*}
curl^4 w + w &= f \quad \text{in } \Omega, \\
curl w \times n &= 0 \quad \text{on } \partial \Omega, \\
\curl w &\times n = 0 \quad \text{on } \partial \Omega,
\end{align*}

where $\Omega$ is a bounded simply-connected Lipschitz polyhedron domain in $\mathbb{R}^d (d = 2, 3)$, and $n$ is the unit outward normal to $\partial \Omega$.

We adopt the following function space

$$
H(\curl^6, \Omega) := \{ v \in L^2(\Omega) : \curl^j v \in L^2(\Omega), 1 \leq j \leq 6 \}
$$

equipped with the norm $\| \cdot \|_{6, \curl}$ and

$$
H_0(\curl^2, \Omega) := \{ v \in L^2(\Omega) : \curl^j v \in L^2(\Omega), \curl^{j-1} v \times n|_{\partial \Omega} = 0, 1 \leq j \leq 2 \}
$$

The weak form of (2.1)–(2.3) is to find $w \in H_0(\curl^2, \Omega)$ such that

$$
a(w, v) = (f, v), \quad \forall v \in H_0(\curl^2, \Omega),
$$

where

$$
a(w, v) = (\curl^2 w, \curl^2 v) + (w, v).
$$
We consider the shape regular and affine meshes $T_h = \{ K \}$ that partition the domain $\Omega$ into tetrahedra in $\mathbb{R}^3$ (or triangles in $\mathbb{R}^2$), and introduce the finite element space

$$V_h = \{ v_h | K \in P(K), \forall K \in T_h, \text{ with } p_K \geq 2 \}.$$  \hspace{1cm} (2.5)

where $P(K) = (P_{p_K})^d$ with the polynomial space of total degree $\leq p_K$. Let $h = \max_{K \in T_h} h_K$ and $p = \min_{K \in T_h} p_K$. Let $E^0_h$ and $E^3_h$ be respectively the set of internal faces and the set of boundary faces of partition $T_h$, $E_h := E^0_h \cup E^3_h$, $f \in E^0_h$ be the interface of two adjacent elements $K^\pm$, and $n^\pm$ be the unit outward normal vector of the face $f$ associated with $K^\pm$. We use $h_f := \max(h_{K^+}, h_{K^-})$ and $p_f := \min(p_{K^+}, p_{K^-})$ to represent the maximum diameter of the circumcircle and the minimum polynomial degree of the elements sharing the face $f$, respectively. We denote by $v^\pm = (v|_{K^\pm})_f$ and introduce three notations as follows:

$$[v] = v^+ \times n^+ + v^- \times n^-, \quad [\text{curl}v] = \text{curl}v^+ \times n^+ + \text{curl}v^- \times n^-,$$

$$\{\text{curl}^2v\} = (\text{curl}^2v^+ + \text{curl}^2v^-)/2.$$  

If $f \in E^0_h$, we define $[v]$, $[\text{curl}v]$ and $\{\text{curl}^2v\}$ on $f$, respectively, as follows:

$$[v] = v \times n, \quad [\text{curl}v] = \text{curl}v \times n, \quad \{\text{curl}^2v\} = \text{curl}^2v.$$  

Pick up any $v_h$ in $V_h$. For any $K \in T_h$, by using Green’s formula we have

$$\int_K f \cdot v_h = \int_{\partial K} \text{curl}^3w \cdot v_h \times n ds + \int_K \text{curl}^2w \cdot \text{curl}^2v_h dx + \int_K w \cdot v_h dx$$

$$+ \int_{\partial K} \text{curl}^2w \cdot (\text{curl}v_h \times n) ds. \hspace{1cm} (2.6)$$

Denoting $\|\text{curl}_h(\cdot)\| := \left( \sum_{K \in T_h} \|\text{curl}(\cdot)\|_{0,K}^2 \right)^{1/2}$, this infers that

$$\int_\Omega f \cdot v_h dx = \int_\Omega \text{curl}_h^3w \cdot \text{curl}_h^2v_h dx + \int_{f \in E_h} \text{curl}_h^3w \cdot [v_h] ds + \int_\Omega w \cdot v_h dx$$

$$+ \int_{f \in E_h} \text{curl}_h^2w \cdot [\text{curl}v_h] ds \hspace{1cm} (2.7)$$

whose right-hand side can be rewritten as the bilinear form

$$a_h(w, v_h) = \int_\Omega \text{curl}_h^2w \cdot \text{curl}_h^2v_h dx + \int_\Omega w \cdot v_h dx$$

$$+ \int_{f \in E_h} \{\text{curl}_h^3w\} \cdot [v_h] ds + \int_{f \in E_h} \{\text{curl}_h^2w\} \cdot [\text{curl}v_h] ds$$

$$+ \int_{f \in E_h} \{\text{curl}_h^3v_h\} \cdot [w] ds + \int_{f \in E_h} \{\text{curl}_h^2v_h\} \cdot [\text{curl}w] ds.$$
+ \int_{f \in \mathcal{E}} \eta_1 p_f^2 \left[ \text{curl} \mathbf{v}_h \right] \cdot \left[ \text{curl} \mathbf{w} \right] ds + \int_{f \in \mathcal{E}} \eta_2 p_f^6 \left[ \mathbf{v}_h \right] \cdot \left[ \mathbf{w} \right] ds \quad (2.8)

where we have used the jump $\left[ \text{curl} \mathbf{w} \right]$ and $\left[ \mathbf{w} \right]$ vanishes across faces $f$ and $\eta_1$ and $\eta_2$ are two positive constant to be determined. Finally we reach at the following relation

$$a_h(\mathbf{w}, \mathbf{v}_h) = (f, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$ (2.9)

Hence the IPDG discretization of (2.4) is to find $\mathbf{w}_h \in \mathbf{V}_h$ such that

$$a_h(\mathbf{w}_h, \mathbf{v}_h) = (f, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (2.10)$$

The above two equalities give the following Galerkin orthogonality

$$a_h(\mathbf{w}_h - \mathbf{w}, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (2.11)$$

**Lemma 2.1** (Lemma 4.5 in [5]) There are linear continuous operators $\Pi^K_p : \mathbf{H}^0(K) \to P(K)$ such that for $\mathbf{v} \in \mathbf{H}^0(K)$

$$\| \mathbf{v} - \Pi^K_p \mathbf{v} \|_{q,K} \lesssim h_K^{\min(p_K+1,s)-q} p_K^{-s} \| \mathbf{v} \|_{s,K}, \quad 0 \leq q \leq s. \quad (2.12)$$

Let $\mathbf{v} \in \mathbf{V}_h$ be equipped with the following norms and semi-norms

$$\| \mathbf{v} \|^2_h := \sum_{K \in \mathcal{T}_h} \left( \| \text{curl}^2 \mathbf{v} \|^2_{0,K} + \| \mathbf{v} \|^2_{0,K} \right) + \sum_{f \in \mathcal{E}_h} \left( h_f^{-3} p_f^6 \| \left[ \mathbf{v} \right] \|^2_{0,f} \right)$$

$$+ \sum_{f \in \mathcal{E}_h} \left( h_f^3 p_f^{-6} \| \{ \text{curl}^2 \mathbf{v} \} \|^2_{0,f} + h_f p_f^{-2} \| \{ \text{curl}^2 \mathbf{v} \} \|^2_{0,f} + h_f^{-1} p_f^2 \| \{ \text{curl} \mathbf{v} \} \|^2_{0,f} \right), \quad (2.13)$$

$$| \mathbf{v} |^2_h := \| \mathbf{v} \|^2_h - \| \mathbf{v} \|^2. \quad (2.14)$$

Next we shall discuss the well-posedness of the discrete variational problem. It is obvious that the bilinear form $a_h(\cdot, \cdot)$ is bounded with respect to the norm $\| \cdot \|_h$ in $\mathbf{V}_h + \mathbf{H}_0(\text{curl}^2, \Omega) \cap \{ \mathbf{v} : \sum_{j=0}^3 \| \text{curl}^j \mathbf{v} \|_{1/2+\sigma, \Omega} < \infty \}$ with $\sigma > 0$.

### 3 Error estimate in DG-norm

**Lemma 3.1** (Lemma 6.1 in [30]) For any polynomial $\mathbf{v} \in P(K)$, there holds

$$\| \mathbf{v} \|_{0,\partial K} \lesssim h_K^{-1/2} p_K \| \mathbf{v} \|_{0,K} \quad \text{and} \quad | \mathbf{v} |_{1,K} \lesssim h_K^{-1} p_K^2 \| \mathbf{v} \|_{0,K}. \quad (3.1)$$

The following theorem shows the coerciveness of $a_h(\cdot, \cdot)$ in $\mathbf{V}_h$, which guarantees the well-posedness of the discrete problem (2.10).
Theorem 3.1  For sufficiently large $\eta_1$ and $\eta_2$, we have

$$a_h(\mathbf{v}, \mathbf{v}) \gtrsim \| \mathbf{v} \|^2_h, \forall \mathbf{v} \in \mathbf{V}_h.$$  (3.2)

Proof  Let us pick up any $\mathbf{v} \in \mathbf{V}_h$ we deduce

$$a_h(\mathbf{v}, \mathbf{v}) \gtrsim \sum_{K \in T_h} \left( \| \text{curl}^2 \mathbf{v} \|^2_{0,K} + \| \mathbf{v} \|^2_{0,K} \right)$$

$$+ \sum_{f \in E_h} \left( \eta_1 h_f p_f^2 \| \text{curl} \mathbf{v} \|^2_{0,f} + \eta_2 h_f^{-3} p_f^6 \| \mathbf{v} \|^2_{0,f} \right)$$

$$- \rho_1^{-1} \sum_{f \in E_h} h_f^2 p_f^{-2} \| \text{curl}^2 \mathbf{v} \|^2_{0,f} - \rho_1 \sum_{f \in E_h} h_f^{-1} p_f^2 \| \text{curl} \mathbf{v} \|^2_{0,f}$$

$$- \rho_2^{-1} \sum_{f \in E_h} h_f^3 p_f^{-6} \| \text{curl}^3 \mathbf{v} \|^2_{0,f} - \rho_2 \sum_{f \in E_h} h_f^{-3} p_f^6 \| \mathbf{v} \|^2_{0,f}$$  (3.3)

with $\rho_1$ and $\rho_2$ to be determined later. Using trace theorem and inverse estimate in Lemma 3.1, we have for $f = K_1 \cap K_2$

$$h_f^2 p_f^{-2} \| \text{curl}^2 \mathbf{v} \|^2_{0,f} \leq 0.5 C_1 \| \text{curl}^2 \mathbf{v} \|^2_{0,K_1 \cup K_2},$$

$$h_f^3 p_f^{-6} \| \text{curl}^3 \mathbf{v} \|^2_{0,f} \leq C_2 h_f^2 p_f^{-2} \| \text{curl}^2 \mathbf{v} \|^2_{0,f} \leq 0.5 C_2 C_1 \| \text{curl}^2 \mathbf{v} \|^2_{0,K_1 \cup K_2}.$$  (3.5)

Then we deduce

$$a_h(\mathbf{v}, \mathbf{v}) \gtrsim \sum_{K \in T_h} \left( \| \text{curl}^2 \mathbf{v} \|^2_{0,K} + \| \mathbf{v} \|^2_{0,K} \right)$$

$$+ \sum_{f \in E_h} \left( \eta_1 h_f^{-1} p_f^2 \| \text{curl} \mathbf{v} \|^2_{0,f} + \eta_2 h_f^{-3} p_f^6 \| \mathbf{v} \|^2_{0,f} \right)$$

$$- 2 \rho_1^{-1} C_1 \sum_{K \in T_h} \| \text{curl}^2 \mathbf{v} \|^2_{0,K} - \rho_1 \sum_{f \in E_h} h_f^{-1} p_f^2 \| \text{curl} \mathbf{v} \|^2_{0,f}$$

$$- 2 \rho_2^{-1} C_1 C_2 \sum_{K \in T_h} \| \text{curl}^2 \mathbf{v} \|^2_{0,K} - \rho_2 \sum_{f \in E_h} h_f^{-3} p_f^6 \| \mathbf{v} \|^2_{0,f}.$$  (3.6)

Hence

$$a_h(\mathbf{v}, \mathbf{v}) \gtrsim \sum_{K \in T_h} \left( (1 - 2 \rho_1^{-1} C_1 - 2 \rho_2^{-1} C_1 C_2) \| \text{curl}^2 \mathbf{v} \|^2_{0,K} + \| \mathbf{v} \|^2_{0,K} \right)$$

$$+ \sum_{f \in E_h} \left( h_f^{-1} (\eta_1 - \rho_1) p_f^2 \| \text{curl} \mathbf{v} \|^2_{0,f} + (\eta_2 - \rho_2) h_f^{-3} p_f^6 \| \mathbf{v} \|^2_{0,f} \right)$$  (3.7)
Choose the large $\rho_1, \rho_2, \eta_1$ and $\eta_2$ such that $2\rho_1^{-1}C_1 + 2\rho_2^{-1}C_1C_2 < 1$, $\eta_1 > \rho_1$ and $\eta_2 > \rho_2$. Finally the inequality (3.5) together with (3.4) yields the assertion. \hfill $\square$

In the following theorem, we prove the error estimate of the discrete problem (2.10) by establishing the interpolation error estimates in DG-norm.

**Theorem 3.2** Assume that \( \{T_h\} \) is a family of nonconforming meshes. Let \( w \in H_0(\text{curl}^2, \Omega) \) and \( A_K(w) < \infty \) for any \( K \in T_h \) then

\[
\| w - w_h \|_h \lesssim \left( \sum_{K \in T_h} \left( h_K^{\min(p_K+1,s_K) - 2} p_K^{3.5-s_K} A_K(w) \right)^2 \right)^{1/2}, \quad p := \min_{K \in T_h} p_K \geq 2
\]  

(3.8)

where \( A_K(w) := \| w \|_{s_K,K} \) if \( \| w \|_{s_K,K} < \infty \) for the local regularity index \( s_K \geq 4 \), otherwise \( A_K(w) := \| w \|_{7/2+\delta,K} (\delta > 0) \) and \( s_K = 3 \).

**Proof** Let \( \Pi_p \) be the global projection operator which equals \( \Pi^K_p \) on \( K \). First of all we shall prove

\[
\| w - \Pi_p w \|_h \lesssim \left( \sum_{K \in T_h} \left( h_K^{\min(p_K+1,s_K) - 2} p_K^{3.5-s_K} A_K(w) \right)^2 \right)^{1/2}, \quad p_K, s_K \geq 2
\]  

(3.9)

Let us denote \( E_h(w) = w - \Pi_p w \) then

\[
\| E_h(w) \|_h^2 = \sum_{K \in T_h} \left( \| \text{curl}^2 E_h(w) \|_{0,K}^2 + \| E_h(w) \|_{0,K}^2 \right) + \sum_{f \in E_h} h_f^3 p_f^{-6} \| \text{curl}^3 E_h(w) \|_{0,f}^2 + \sum_{f \in E_h} \left( p_f^6 h_f^{-3} \| E_h(w) \|_{0,f}^2 + p_f^2 h_f^{-1} \| \text{curl} E_h(w) \|_{0,f}^2 \right) + p_f^2 h_f \| \text{curl}^2 E_h(w) \|_{0,f}^2
\]

:= I_1 + I_2 + I_3.

We first estimate \( I_1 \). Using the error estimate of \( \Pi^K_p \) in Lemma 2.1, we have

\[
\| E_h(w) \|_{0,K} + \| \text{curl}^2 E_h(w) \|_{0,K} \lesssim h_K^{\min(p_K+1,s_K) - 2} p_K^{2-s_K} \| w \|_{s_K,K}.
\]

Let \( f = K_1 \cap K_2 \) and \( s_f := \max(s_{K_1}, s_{K_2}) \). For \( p_f \geq 3 \) we have from Lemma 2.1...
We can estimate $I_p f$ while for $56$ Page 8 of 29 BIT Numerical Mathematics (2023) 63:56

Remark 3.2

The convergence rate is optimal with respect to $\in [26]$. This together with (3.9) yields the assertion.

Remark 3.1

The error estimate (3.9) on the general polygonal element, like the way in Lemma 23 in [17] or Lemma 4.12 in [26].

Then we estimate $I_2$: For $p_f = 2$ we have for $i = 1, 2$

$$\| \text{curl}^3 E_h(w) \|_{\Omega, K_i} \leq h_f^{-1} \| \text{curl}^3 w \|_{\Omega, K_i}^2 + h_f^{25} \| \text{curl}^3 w \|_{1/2+\delta, K_i}$$

while for $p_f \geq 3$ from Lemma A.3 in [46] we have for $i = 1, 2$

$$\| \text{curl}^3 E_h(w) \|_{\Omega, K_i} \leq h_f^{-1/2} \| \text{curl}^3 E_h(w) \|_{\Omega, K_i} + (\| \text{curl}^3 E_h(w) \|_{\Omega, K_i} \| \text{curl}^3 E_h(w) \|_{1/2+\delta, K_i})^{1/2}$$

$$\leq (h_f^{-\min(p_f+1,s_f)-3.5} p_f^{-3+3.5-s_f} + h_f^{-\min(p_f+1,s_f)-3.5} p_f^{-3.5-s_f}) \| w \|_{s_f, K_i}$$

$$\leq h_f^{-\min(p_f+1,s_f)-3.5} p_f^{-3.5-s_f} \| w \|_{s_f, K_i}.$$ 

We can estimate $I_3$ similarly:

$$\| E_h(w) \|_{\Omega, K_i} \leq h_f^{-1/2} \| E_h(w) \|_{\Omega, K_i} + (\| E_h(w) \|_{\Omega, K_i} \| E_h(w) \|_{1/2+\delta, K_i})^{1/2}$$

$$\leq (h_f^{-\min(p_f+1,s_f)-0.5} p_f^{-0.5-s_f} + h_f^{-\min(p_f+1,s_f)-0.5} p_f^{-0.5-s_f}) \| w \|_{\Omega, K_i} for i = 1, 2.$$ (3.10)

Hence we have proved (3.9). Using (2.11) and Theorem 3.1 we have

$$\| w_h - v_h \|_h \leq a_h(w_h - v_h, w - v_h) \leq \| w_h - v_h \|_h \| w - v_h \|_h, \quad \forall v_h \in V_h.$$ 

Then

$$\| w - w_h \|_h \leq \inf_{v_h \in V_h} \| w - v_h \|_h. \quad (3.11)$$

This together with (3.9) yields the assertion. □

Remark 3.1

The error estimate (3.9) on the general polygonal $(d = 2)$ or polyhedral $(d = 3)$ meshes can be proved by using a triangle or a tetrahedron to cover the polygonal or polyhedral element, like the way in Lemma 23 in [17] or Lemma 4.12 in [26].

Remark 3.2

The convergence rate is optimal with respect to $h$ but the convergence rate in polynomial degree $p$ is not optimal under nonconforming mesh. The convergence rate can be improved under conforming meshes. The utilization of the $H^1$-conforming element interpolation can yield the following theorem.

Theorem 3.3

Assume that $\{T_h\}$ is a family of conforming meshes. Let $w \in H_0(curl^2, \Omega)$ and $A_\Omega(w) := (\sum_{K \in T_h} A_K(w)^2)^{1/2} < \infty$ with $\min_{K \in T_h} s_K := s$ then

$$\| w - w_h \|_h \leq h^{\min(p+1,s)-2} p^{3-s} A_\Omega(w), \quad p \geq 2. \quad (3.12)$$

Springer
Proof Let $Q_h$ be the $C^0$-conforming finite element space defined in [5] with the polynomial degree $p$ and the mesh size $h$. There is a projection $P_h : H^1(\Omega) \rightarrow Q_h \times Q_h$ (see Theorem 4.8 in [5] or Theorem 2 in [3]) such that

$$\| P_h v - v \|_{1, \Omega} + h^{-1} p \| P_h v - v \|_{0, \Omega} \lesssim h^{\min(p+1,l)-1} \| v \|_{L^l, \Omega}, \quad p, l \geq 1.$$  

(3.13)

Since

$$|\operatorname{curl}(P_h w - w)|_{1, K} \leq |\operatorname{curl}P_h w - P_h \operatorname{curl}w|_{1, K} + |P_h \operatorname{curl}w - \operatorname{curl}w|_{1, K} \lesssim p^2 h^{-1} |\operatorname{curl}P_h w - \operatorname{curl}w|_{1, K} + \| P_h \operatorname{curl}w - \operatorname{curl}w \|_{1, K},$$

we have

$$\left( \sum_{K \in T_h} |\operatorname{curl}(P_h w - w)|_{1, K}^2 \right)^{1/2} \lesssim h^{\min(p+1,s)-2} p^{3-s} \| w \|_{s, \Omega}. \quad (3.14)$$

Similarly as above, we have

$$\left( \sum_{K \in T_h} \| \operatorname{curl}^2(P_h w - w) \|_{1, K}^2 + h^2 P^{-4} \| \operatorname{curl}^3(P_h w - w) \|_{1, K}^2 \right)^{1/2} \lesssim h^{\min(p+1,s)-3} p^{5-s} \| w \|_{s, \Omega}. \quad (3.15)$$

Denoted by $\widetilde{E}_h(w) := P_h w - w$, since for $i = 1, 2$

$$\| \operatorname{curl}^2 \widetilde{E}_h(w) \|_{K_i, 0, f} \lesssim h^{-1/2} \| \operatorname{curl}^2 \widetilde{E}_h(w) \|_{0, K_1 \cup K_2} \quad + \quad (\| \operatorname{curl}^2 \widetilde{E}_h(w) \|_{0, K_1 \cup K_2} \| \operatorname{curl}^2 \widetilde{E}_h(w) \|_{1, K_1 \cup K_2})^{1/2}$$

we have

$$\left( \sum_{f \in E_h} \| \{ \operatorname{curl}^2 \widetilde{E}_h(w) \} \|_{0, f}^2 \right)^{1/2} \lesssim (h^{\min(p+1,s)-2.5} p^{3-s} + h^{\min(p+1,s)-2.5} p^{4-s}) \| w \|_{s, \Omega} \quad \lesssim h^{\min(p+1,s)-2.5} p^{4-s} \| w \|_{s, \Omega} \quad (3.16)$$

and similarly for $i = 1, 2$

$$\left( \sum_{f \in E_h} \| \{ \operatorname{curl}^3 \widetilde{E}_h(w) \} \|_{0, f}^2 \right)^{1/2} \lesssim h^{\min(p+1,s)-3.5} p^{6-s} \| w \|_{s, \Omega}. \quad (3.17)$$

Springer
\[
\left( \sum_{f \in E} \left\| \text{curl} \tilde{E}_h(f) \right\|_{0,f}^2 \right)^{1/2} \lesssim h^{\min(p+1,s)-1.5} p^{2-s} \left\| w \right\|_{s,\Omega}. \tag{3.18}
\]

The estimates (3.14)–(3.18) give
\[
\left\| \tilde{E}_h(w) \right\|_h \lesssim h^{\min(p+1,s)-2} p^{3-s} \left\| w \right\|_{s,\Omega}. \tag{3.19}
\]

The assertion can be done by the inequality (3.11).

\section{IPDG method for the quad-curl eigenvalue problem}

In this section we restrict our analysis on the case \( P(K) = (P_p(K))^d \) on a simplex \( K \). The quad-curl eigenvalue problem reads: Find \( \lambda \in \mathbb{R}, u \in H_0(\text{curl}^2, \Omega) \) and \( \tilde{p} \in H^1_0(\Omega) \) such that

\[
(\text{curl}^2 u, \text{curl}^2 v) + (\nabla \tilde{p}, v) = \lambda (u, v), \quad \forall v \in H_0(\text{curl}^2, \Omega),
\]
\[
(u, \nabla q) = 0, \quad \forall q \in H^1_0(\Omega). \tag{4.1}
\]

One readily verifies that

\[
\tilde{a}_h(u, v) + (\nabla \tilde{p}, v) = \lambda (u, v), \quad \forall v \in V_h,
\]

where

\[
\tilde{a}_h(u, v) = a_h(u, v) - (u, v).
\]

We introduce the following function spaces:

\[
X = \{ v \in H_0(\text{curl}, \Omega) | (v, \nabla q) = 0, \forall q \in H^1_0(\Omega) \},
\]
\[
U_h = \{ q_h \in H^1_0(\Omega) | q_h|_K \in P_{p+1}(K), \forall K \in T_h \},
\]
\[
X_h = \{ v_h \in V_h | (v_h, \nabla q) = 0, \forall q \in U_h \}. \tag{4.3}
\]

The discrete form of the eigenvalue problem (4.1) is given by: Find \( (\lambda_h, u_h, \tilde{p}_h) \in \mathbb{R} \times V_h \times U_h \) with \( u_h \neq 0 \) such that

\[
\tilde{a}_h(u_h, v_h) + (\nabla \tilde{p}_h, v_h) = \lambda_h (u_h, v_h), \quad \forall v_h \in V_h,
\]
\[
(u_h, \nabla q) = 0, \quad \forall q \in U_h. \tag{4.6}
\]

To analyze the convergence of the discretization (4.6), we consider the source problem with div-free constraint: Find \( T f \in H_0(\text{curl}^2, \Omega) \) and \( S f \in H^1_0(\Omega) \) such that

\[
(\text{curl}^2 T f, \text{curl}^2 v) + (\nabla S f, v) = (f, v), \quad \forall v \in H_0(\text{curl}^2, \Omega),
\]
\[
(\nabla q, T f) = 0, \quad \forall q \in H^1_0(\Omega). \tag{4.7}
\]
Its DG discretization is to seek \( T_h f \in V_h \) and \( S_h f \in U_h \):

\[
\tilde{a}_h(T_h f, v_h) + (\nabla S_h f, v_h) = (f, v_h), \quad \forall v_h \in V_h, \\
(\nabla q, T_h f) = 0, \quad \forall q \in U_h.
\]  

(4.8)

It is easy to verify that \( S f = S_h f = 0 \) for \( f \in X \).

**Assumption 4.1** (Propositions 4.5 in [36]) Let \( v_h \in V_h \). Then there is \( v_h^c \in H_0(\text{curl}, \Omega) \cap V_h \) such that

\[
h_f^{-2} \| v_h - v_h^c \|^2 + \| \text{curl}_h(v_h - v_h^c) \|_0^2 \lesssim \sum_{f \in E_h} h_f^{-1} \| [v_h] \|_{0, f}^2.
\]  

(4.9)

where \( \| \text{curl}_h(\cdot) \| := \| (\sum_{K \in \mathcal{T}_h} \| \text{curl}(\cdot) \|_{0, K}) \|^{1/2} \) and the hidden constant is independent of the mesh size \( h \) and the polynomial degree \( p \).

**Remark 4.1** The above conclusion is valid on nonconforming meshes since they can be conformed by adding some edges \((d = 2)\) or faces \((d = 3)\). The fact that the hidden constant in (4.9) is independent of the mesh size \( h \) is verified in [36]. However, its independence on the polynomial degree \( p \) is not verified yet. Here we give an argument for the case of \( H(\text{curl}) \)-conforming rectangular element method as follows.

Let \( \{ \varphi^i_{K,e} \}_{4p}^{i=1} \) and \( \{ \varphi^i_{K,b} \}_{2p^2-2p}^{i=1} \) be the edge-based basis functions and the cell-based basis functions on \( Q_{p-1}^p(K) \times Q_{p-1}^p(K) \) (see Appendix A), respectively. Any \( v \in Q_{p-1}^p(K) \times Q_{p-1}^p(K) \) can be written as

\[
v = \sum_{e \in E(K)} \sum_{i=1}^{4p} \varphi^i_{K,e} \varphi^i_{K,e} + \sum_{i=1}^{2p^2-2p} \varphi^i_{K,b} \varphi^i_{K,b}.
\]  

(4.10)

Let \( v^c \in H_0(\text{curl}, \Omega) \cap V_h \) be the function whose edge moments are

\[
\bar{v}^i_{K,e} = \begin{cases} 
\frac{1}{2} \sum_{e \subset K} \varphi^i_{K,e} & \text{if } e \not\subset \partial \Omega \\
0 & \text{if } e \subset \partial \Omega
\end{cases}
\]  

for \( i = 1, \ldots, 4p \) and whose cell moments are

\[
\bar{v}^i_{K,b} = \varphi^i_{K,b}
\]  

for \( i = 1, \ldots, 2p^2-2p \). According to the transformation \( v \cdot F_K = B^{-T}_K \tilde{v} \),

\[
\| v \|_{0,K} \lesssim \| \tilde{v} \|_{0,\tilde{K}}, \quad \| \text{curl} v \|_{0,K} \lesssim h^{-1}_K \| \text{curl} \tilde{v} \|_{0,\tilde{K}}.
\]  

(4.11)
Note that $\hat{v} - \hat{v}^c \in Q^1_p([-1, 1]^2) + Q^p([-1, 1]^2)$. By the orthogonality of basis functions we have
\[
\|\text{curl}(\hat{v} - \hat{v}^c)\|_{0,K}^2 \lesssim \sum_{e \subset \partial K} \sum_{i=1}^{N_e} (v^i_{K,e} - \hat{v}^i_{K,e})^2 \lesssim \sum_{e \subset \partial K} \sum_{i=1}^{N_e} (v^i_{K,e} - \hat{v}^i_{K',e})^2 \\
\lesssim \sum_{e \subset \partial K} h_e \int_e |(v|_K - v|_{K'}) \cdot \tau|^2 ds \tag{4.12}
\]
and similarly by (6.2) and (6.3) in Appendix A
\[
\|\hat{v} - \hat{v}^c - \Pi(\hat{v} - \hat{v}^c)\|_{0,K}^2 + \|\Pi(\hat{v} - \hat{v}^c)\|_{0,K}^2 \lesssim \sum_{e \subset \partial K} \sum_{i=1}^{N_e} (v^i_{K,e} - \hat{v}^i_{K,e})^2 \\
\lesssim \sum_{e \subset \partial K} h_e \int_e |(v|_K - v|_{K'}) \cdot \tau|^2 ds. \tag{4.13}
\]
Finally we have from the above two estimates and (4.11)
\[
h^{-2}_K \|v - v^c\|_{0,K}^2 + \|\text{curl}(v - v^c)\|_{0,K}^2 \lesssim \sum_{e \subset \partial K} h^{-1}_K \|v\|_{0,e}^2. \tag{4.14}
\]
Then (4.9) follows.

The Hodge operator is a useful tool in our error analysis. It is defined as $Hg \in H^1_0(curl, \Omega)$ and $\rho \in H^1_0(\Omega)$ for $g \in H^1_0(curl, \Omega)$ such that
\[
(curl Hg, curl v) + (\nabla \rho, v) = (curl g, curl v), \quad \forall v \in H^1_0(curl, \Omega), \quad (\nabla q, Hg) = 0, \quad \forall q \in H^1_0(\Omega). \tag{4.15}
\]
We introduce the following curl-conforming finite element spaces [44]
\[
\tilde{V}_h = \{v_h \in H^1_0(curl, \Omega) | v_h|_K \in (P_p(K))^d, K \in T_h\}, \tag{4.16}
\]
\[
\tilde{X}_h = \{v_h \in \tilde{V}_h \text{ and } (v_h, \nabla q) = 0, \forall q \in U_h\}, \tag{4.17}
\]
and give the corresponding interpolation error estimates in virtue of Lemma 3.1 in [10].

**Lemma 4.1** Let $R_h$ be the edge element interpolation associated with $\tilde{V}_h$. If $v \in X \subset H^0(\Omega)$ for some $r_0 \in (1/2, 1]$ and $\text{curl} v \in \text{curl}\tilde{V}_h$ then
\[
\|R_h v - v\| \lesssim h^{-r_0} p^{-1/2}(\|v\|_{r_0} + \|\text{curl} v\|). \tag{4.18}
\]
Proof Denote $\tilde{v} = (\nabla F)^{-T}v \circ F$ where $F$ is the affine mapping between $K$ and its reference element $\tilde{K}$. Lemma 3.1 (Inequality (3.22)) in [10] together with Theorem 5.3 in [24] and Theorem 4.1 in [8] gives

$$\|\tilde{R}_K \tilde{v} - \tilde{v}\|_{0,\tilde{K}} \lesssim p^{-1/2}(\|\tilde{v}\|_{0,\tilde{K}} + \|\nabla \tilde{v}\|_{0,\tilde{K}})$$  \hspace{1cm} (4.19)

where $\tilde{R}_K \tilde{v} = (\nabla F)^{-T}R_h v \circ F$. The assertion can be deduced via the scaling argument.

Lemma 4.2 (Lemma 7.6 in [39]) Let $v_h \in \tilde{X}_h$ then $H v_h \in X$ such that $\text{curl} v_h = \text{curl} H v_h$ and

$$\|H v_h\|_{0} \lesssim \|\text{curl} v_h\|, \hspace{1cm} \|H v_h - v_h\| \leq \|R_h H v_h - H v_h\|.$$  \hspace{1cm} (4.20) \hspace{1cm} (4.21)

Lemma 4.3 For any $v_h \in V_h$ it is valid that

$$\sum_{K \in T_h} \|\text{curl} v_h\|^2_{0,K} \lesssim \sum_{K \in T_h} \|\text{curl}^2 v_h\|^2_{0,K} + \sum_{f \in E_h} \left( h_f^{-1} \|\text{curl} v_h\|^2_{0,f} + h_f^{-2} p_f^4 \|v_h\|^2_{0,f} \right).$$  \hspace{1cm} (4.22)

Proof This proof follows the argument of Theorem 3.1 in [47]. We introduce the auxiliary problem: Find $\sigma_h \in U_h$ such that

$$(\nabla \sigma_h, \nabla s) = ((\text{curl} v_h)^c, \nabla s), \hspace{1cm} \forall s \in U_h.$$ 

Then using Assumption 4.1 and Lemma 3.1 we have

$$(\nabla \sigma_h, \nabla \sigma_h) = ((\text{curl} v_h)^c - \text{curl} v_h + \text{curl} v_h, \nabla \sigma_h) \lesssim \|\sigma_h\|_1 \left( \sum_{f \in E_h} h_f^{-1} \|\text{curl} v_h\|_{0,f}^2 \right)^{1/2} + \left| \sum_{f \in E_h} \int_f \text{div}(\|v_h\|) \sigma_h ds \right|$$

$$\lesssim \|\sigma_h\|_1 \left( \sum_{f \in E_h} h_f^{-1} \|\text{curl} v_h\|_{0,f}^2 + \sum_{f \in E_h} h_f^{-2} p_f^4 \|v_h\|_{0,f}^2 \right)^{1/2}.$$ 

Note that $\nabla \sigma_h - (\text{curl} v_h)^c \in \tilde{X}_h$. We have from Assumption 4.1

$$\|(\text{curl} v_h)^c - \nabla \sigma_h\| \lesssim \|\text{curl}((\text{curl} v_h)^c - \nabla \sigma_h)\| \lesssim \|\text{curl}((\text{curl} v_h)^c - \text{curl} v_h + \text{curl} v_h)\|$$

$$\lesssim \left( \sum_{f \in E_h} h_f^{-1} \|\text{curl} v_h\|_{0,f}^2 \right)^{1/2} + \|\text{curl}^2 v_h\|.$$
The combination of the above two inequalities and Assumption 4.1 yields the assertion.

The following result guarantees the well-posedness of the problem (4.8).

**Lemma 4.4** There holds the following discrete Poincaré inequality

\[ \|v_h\|_h \lesssim |v_h|_h, \quad \forall v_h \in X_h. \] (4.23)

**Proof** Let \( v_h^c \) be defined as in Assumption 4.1 for any \( v_h \in X_h \). We have the Helmholtz decomposition \( v_h^c = v_h^0 \oplus \nabla e_0 \) with \( v_h^0 \in \tilde{X}_h \) and \( e_0 \in U_h \). Due to the fact \( (Hv_h^0 - v_h, \nabla e_0) = 0 \) we have

\[ (Hv_h^0 - v_h, Hv_h^0 - v_h) = (Hv_h^0 - v_h, Hv_h^0 - v_h + v_h^c - v_h) \]

which together with (4.9), Lemma 4.2 and (4.19) leads to

\[ \|Hv_h^0 - v_h\| \lesssim \|Hv_h^0 - v_h^0\| + \|v_h^c - v_h\| \]

\[ \lesssim h^\alpha p^{-1/2}\|\text{curl} v_h^c\| + \left( \sum_{f \in F_h} h_f^{-1}\|\|v_h\|_{0,f}\|^2 \right)^{1/2} \]

\[ \lesssim h^\alpha p^{-1/2} \left( \sum_{K \in T_h} \|\text{curl} v_h\|^2_{0,K} \right)^{1/2} + hp^{-3}|v_h|_h. \] (4.24)

This combines with (4.9) and (4.22) to get

\[ \|v_h\| \lesssim \|Hv_h^0\| + h^\alpha p^{-1/2} \left( \sum_{K \in T_h} \|\text{curl} v_h\|^2_{0,K} \right)^{1/2} + hp^{-3}|v_h|_h \]

\[ \lesssim \|\text{curl} v_h^c\| + h^\alpha p^{-1/2}|v_h|_h \]

\[ \lesssim \left( \sum_{K \in T_h} \|\text{curl} v_h\|^2_{0,K} \right)^{1/2} + hp^{-3}|v_h|_h + h^\alpha p^{-1/2}|v_h|_h \]

\[ \lesssim |v_h|_h + h^\alpha p^{-1/2}|v_h|_h. \] (4.25)

This leads to the assertion directly.

The estimate (3.7) shows that \( \tilde{a}_h(v, v) \gtrsim |v|^2_h \) for any \( v \in V_h \). According to Lemma 4.4 we have the following.

**Corollary 4.1** It holds the a-priori estimate

\[ \|(T - T_h)f\|_h \lesssim \inf_{v_h \in V_h} \|Tf - v_h\|_h + \inf_{v_h \in U_h} |Sf - v_h|_{1,\Omega}. \]
Proof The combination of (4.7) and (4.8) leads to
\[ \tilde{a}_h((T - T_h)f, v_h) + (\nabla (S - S_h)f, v_h) = 0, \quad \forall v_h \in V_h. \]
It follows that for any \( v_h \in X_h \)
\[ \tilde{a}_h(v_h - T_h f, v_h - T_h f) = \tilde{a}_h(v_h - f, v_h - T_h f) - (\nabla (S - S_h)f, v_h - T_h f). \]
Hence by the discrete Poincaré inequality (4.25) we have
\[ \|v_h - T_h f\|_h \leq |T f - v_h|_h + |(S - S_h)f|_{1, \Omega}, \quad \forall v_h \in X_h. \]
This together with the triangle inequality infers that
\[ \|T f - T_h f\|_h \leq \|T f - w_h\|_h + |(S - S_h)f|_{1, \Omega} \]
where \( w_h \) is as in (3.11) with \( w := T f \). Then the proof is finished. \( \square \)

We can prove the following \( hp \)-version error estimates for of the IPDG solution.

**Theorem 4.1** Assume one of the following regularities for the equations (4.7) is valid:

\[ \|T f\|_{4, \Omega} \lesssim \|f\|, \quad (4.26) \]
\[ \|T f\|_{r_0, \Omega} + \|\text{curl} T f\|_{1+r_1, \Omega} + \|\text{curl}^3 T f\|_{1/2+\delta, \Omega} \lesssim \|f\|, \quad (4.27) \]

where \( r_0 \in (1/2, 1] \) is as in Lemma 4.1, \( r_1 \in (1/2, 1] \), and \( \delta > 0 \). Let \( \{T_h\} \) be a family of conforming meshes. Let \( g \in L^2(\Omega) \) with \( \text{div} g = 0 \), \( T g \in H_0(\text{curl}^2, \Omega) \) and \( A_K(T g) < \infty \) for any \( K \in T_h \) then
\[ \|T g - T_h g\| \lesssim \varepsilon(h, p)h^{\min(p+1,s)-2}p^{3-s}A_\Omega(T g), \quad \text{with} \quad s := \min_{K \in T_h} s_K, \quad (4.28) \]
where \( \varepsilon(h, p) = h^{r_0}p^{-1/2} \) under Assumption (4.26) and \( \varepsilon(h, p) = h^{\min(r_0,r_1)} \) under Assumption (4.27).

**Proof** For \( g \in L^2(\Omega) \) with \( \text{div} g = 0 \) we have \( S_h g = S g = 0 \). We derive
\[ \|T g - T_h g\|^2 = (T g - T_h g, T g - P_h T g + P_h T g - (T_h g) + (T_h g) - (T_h g)^c - T_h g) \]
Using the Helmholtz decomposition \( w_0 := P_h T g - (T_h g)^c = w_0^c \oplus \nabla e \) with \( w_0^c \in X_h \) and \( e \in U_h \), we have from (4.9) in Assumption 4.1
\[ \|T g - T_h g\|^2 \lesssim \left( \|T g - P_h T g\| + \|(T_h g)^c - T_h g\| \right) \|T g - T_h g\| + |(T g - T_h g, w_0^c)|. \quad (4.29) \]
Note that $S_h g = S g = S_h H w_0^c = S H w_0^c = 0$. Let $\Pi_h^c$ be the orthogonal projection onto $V_h \cap H(\text{curl}, \Omega)$ such that for any $v \in H(\text{curl})$,

$$a_h(v - \Pi_h^c v, q) = 0, \forall q \in V_h \cap H(\text{curl}, \Omega).$$ \hfill (4.30)

If Assumption (4.26) holds then by (3.19)

$$|T H w_0^c - \Pi_h^c T H w_0^c| \leq \|T H w_0^c - P_h T H w_0^c\|_h \lesssim h^{-1} \|T H w_0^c\|_{4, \Omega} \lesssim \varepsilon(h, p) \|H w_0^c\|.$$

If Assumption (4.27) holds then the similar argument as those in Theorem 5.6 of [19] shows

$$|T H w_0^c - \Pi_h^c T H w_0^c| \lesssim h^{\min(r_0, r_1)}(\|T H w_0^c\|_{r_0, \Omega} + \|\text{curl}T H w_0^c\|_{r_1, \Omega} + \|\text{curl}^3 T H w_0^c\|_{1/2 + \delta, \Omega}) \lesssim \varepsilon(h, p) \|H w_0^c\|.$$

Using Lemmas 4.2 and 4.1, the third term at the right-side hand of (4.29) is estimated as follows:

$$|(T g - T_h g, w_0^c)| \leq |(T g - T_h g, w_0^c - H w_0^c)| + |(T g - T_h g, H w_0^c)|$$

$$\lesssim h^{r_0} p^{-1/2} \|\text{curl} w_0^c\| \|T g - T_h g\| + |\tilde{a}_h(T g - T_h g, T H w_0 - \Pi_h^c T H w_0)|$$

$$\lesssim h^{r_0} p^{-1/2} \|P_h T g - (T_h g)^c\| \|T g - T_h g\| + |T g - T_h g| \varepsilon(h, p) \|H w_0^c\|$$ \hfill (4.31)

where $\|H w_0^c\|$ can be estimated by Lemmas 4.2 and 4.1:

$$\|H w_0^c\| \lesssim \|w_0\| + h^{r_0} p^{-1/2} \|\text{curl}(P_h T g - (T_h g)^c)\|$$

$$\lesssim \|(P_h - I) Tg + T g - T_h g + T_h g - (T_h g)^c\|$$

$$+ h^{r_0} p^{-1/2} \|P_h T g - (T_h g)^c\|_{\text{curl}}.$$

The substitution of (4.31) into the estimate (4.29) gives

$$\|T g - T_h g\| \lesssim h^{r_0} p^{-1/2} \|P_h T g - (T_h g)^c\|_{\text{curl}} + |T g - T_h g| \varepsilon(h, p)$$

$$+ \|P_h T g - T g\| + \|T_h g - (T_h g)^c\|$$

$$\lesssim \|T g - T_h g\| \varepsilon(h, p) + \|P_h T g - T g\| + \|T_h g - (T_h g)^c\|$$

$$+ h^{r_0} p^{-1/2} (\|P_h T g - T g\|_h + \|\text{curl}(T_h g - (T_h g)^c)\|) \text{by (4.22)}$$

$$\lesssim \varepsilon(h, p) h^{\min(p + 1, s) - 2} p^{3 - s} A_{\Omega}(T g) + h^{r_0} \left( h^{-1} \|T g\|_{1, f} \right)^2$$ \hfill (4.9),

Cor. 4.1 & Thm. 3.3
Let \( h \in \mathbb{R} \) and \( p \in \mathbb{R}^{+} \) be such that \( h \) is uniformly bounded w.r.t. \( \| \cdot \|_{h} \).

\[
\lesssim \epsilon(h, p) h^{p+1} \leq 3 - s A_{\Omega}(Tg) + h^{p+1} p^{-3} \| T_{h}g - Tg \|_{h}.
\] (4.32)

Then the estimate (4.28) is obtained by using Corollary 4.1 and Theorem 3.3. \( \square \)

**Remark 4.2** Theorem 4.1 shows that \( \| (T_{h} - T)g \| \) is of higher order than \( \| (T_{h} - T)g \|_{h} \) by an infinitesimal \( \epsilon(h, p) \). To avoid technical difficulty, the strong regularity (4.26) is used to derive the \( h/p \)-version estimate \( \epsilon(h, p) = h^{p-1/2} \), while the weaker regularity (4.27) is used to derive the \( h \)-version estimate \( \epsilon(h, p) = h^{p} \).

Let \( \mathcal{H} \) be a sequence of \((h, p)\) with \( h/p \) converging to 0.

**Lemma 4.5** (Discrete compactness property) Any sequence \( \{v_{h}\}_{(h, p) \in \mathcal{H}} \) with \( v_{h} \in X_{h} \) that is uniformly bounded w.r.t \( \| \cdot \|_{h} \) contains a subsequence that converges strongly in \( L^{2}(\Omega) \).

**Proof** Let \( \{v_{h}\}_{(h, p) \in \mathcal{H}} \) with \( \| v_{h} \|_{h} < M \) for a positive constant \( M \). It is trivial to assume that the sequence \((h_{i}, p_{i}) \in \mathcal{H} \) satisfies \( h_{i}/p_{i} \to 0 \) as \( i \to \infty \). According to (4.24), \( \| Hv_{h_{i}}^{0} - v_{h_{i}} \| \to 0 \) as \( i \to \infty \). Note that by (4.9) we deduce

\[
\| \text{curl}Hv_{h_{i}}^{0} \| = \| \text{curl}v_{h_{i}}^{0} \| \lesssim \| v_{h_{i}} \|_{h_{i}}.
\]

This means that \( \{Hv_{h_{i}}^{0}\} \) is bounded in \( H(\text{curl}, \Omega) \). Since \( X \) is compactly imbedded into \( L^{2}(\Omega) \), there is a subsequence of \( \{Hv_{h_{i}}^{0}\} \) converging to some \( v_{0} \) in \( L^{2}(\Omega) \). Hence a subsequence of \( \{v_{h_{i}}\} \) will converge to \( v_{0} \) in \( L^{2}(\Omega) \) as well. \( \square \)

The following uniform convergence can be derived from the discrete compactness property of \( X_{h} \).

**Theorem 4.2** There holds the uniform convergence

\[
\| T_{h} - T \|_{L^{2}(\Omega) \to L^{2}(\Omega)} \to 0, \ h \to 0, \ p \to \infty.
\]

**Proof** Since \( \cup_{(h, p) \in \mathcal{H}} U_{h} \) is dense in \( H^{1}_{0}(\Omega) \), we deduce from Corollary 4.1: for any \( f \in L^{2}(\Omega) \)

\[
\| (T - T_{h})f \| \leq \|(T - T_{h})f \|_{h} \to 0.
\] (4.33)

That is, \( T_{h} \) converges to \( T \) pointwisely in \( L^{2}(\Omega) \). Thanks to the discrete compactness of \( X_{h} \), \( \cup_{(h, p) \in \mathcal{H}} T_{h}B \) is a relatively compact set in \( L^{2}(\Omega) \) where \( B \) is the unit ball in \( L^{2}(\Omega) \). In fact, Let us choose any sequence \( \{v_{h}\}_{h \in \mathcal{H}} \subset B \). Note that \( T \) is compact from \( X \) to \( L^{2}(\Omega) \) then \( \{Tv_{h}\}_{h \in \mathcal{H}} \) is a relatively compact set in \( L^{2}(\Omega) \). Hence it holds the collectively compact convergence \( T_{h} \to T \) in \( L^{2}(\Omega) \) as \( h \to 0, p \to \infty \). Noting \( T, T_{h} : L^{2}(\Omega) \to L^{2}(\Omega) \) are self-adjoint, due to Proposition 3.7 or Table 3.1 in [21] the assertion is valid. \( \square \)
Remark 4.3  The uniform convergence in Theorem 4.2 is valid on the mild polygonal 
\((d = 2)\) or polyhedral \((d = 3)\) meshes, provided that \(\inf_{v \in U_h} |Sf - v|_1 \to 0 \quad (h \to 0)\) in Corollary 4.1. The fact \(\inf_{v \in U_h} |Sf - v|_1 \to 0\) cannot be guaranteed 
when the edge(or face) number of polygonal (or polyhedral) elements is so large that 
their nodes (or faces) is shared by few elements.

Using the spectral approximation theory in [4] we are in a position to give the 
estimate for IPDG eigenvalues.

Theorem 4.3  Let \(\lambda_h\) be an eigenvalue of (4.2) converging to the eigenvalue \(\lambda\) of (4.6) 
and \(M(\lambda) \subset \{ v : A_K(v) < \infty, \forall K \in T_h \}\). When \(\{ T_h \}\) is a family of nonconforming 
meshes

\[
|\lambda - \lambda_h| \lesssim h^{2 \min(s,p+1)-4} p^{7-2s},
\]

\[
\|u - u_h\|_h \lesssim h^{\min(s,p+1)-2} p^{3.5-s};
\]

when \(\{ T_h \}\) is a family of conforming meshes

\[
|\lambda - \lambda_h| \lesssim h^{2 \min(s,p+1)-4} p^{6-2s},
\]

\[
\|u - u_h\| \lesssim \varepsilon(h, p) h^{\min(s,p+1)-2} p^{3-s},
\]

\[
\|u - u_h\|_h \lesssim h^{\min(s,p+1)-2} p^{3-s};
\]

where \(s = \min_{K \in T_h} s_K\) and \(M(\lambda)\) denotes the space spanned by all eigenfunctions 
corresponding to the eigenvalue \(\lambda\).

Proof  Let \(\lambda_h\) and \(\lambda\) be the \(mth\) eigenvalues of (4.2) and (4.6), respectively, and 
dim \(M(\lambda) = q\). From Theorem 7.2 (inequality (7.12)), Theorem 7.3 and Theorem 7.4 in [4] we get

\[
|\lambda - \lambda_h| \lesssim \sum_{i,j = m}^{m+q-1} |((T - T_h)\varphi_i, \varphi_j)| + \|(T - T_h)|M(\lambda)||^2_{L^2(\Omega) \to L^2(\Omega)},
\]

\[
\|u - u_h\| \lesssim \|(T - T_h) |M(\lambda)||^2_{L^2(\Omega) \to L^2(\Omega)},
\]

where \(\varphi_m, \ldots, \varphi_{m+q-1}\) are a a set of basis functions for \(M(\lambda)\). Note that \(Sf = S_h f = 0\) in (4.7) and (4.8) for \(f \in X\). According to Theorem 4.1, the estimate \(4.37\) 
is obtained from (4.40). Hence the following Garlerkin orthogonality holds:

\[
((T - T_h)\varphi_i, \varphi_j) = \tilde{a}_h((T - T_h)\varphi_i, T\varphi_j) = \tilde{a}_h((T - T_h)\varphi_i, (T - T_h)\varphi_j).
\]

Substituting it into (4.39), we deduce (4.34) and (4.36) from Theorems 3.2 and 3.3. 
Note that \(u_h = \lambda_h T_h u_h\) and \(u = \lambda T u\). By the boundedness of \(T_h\) and Corollary 4.1 we derive

\[
\|u_h - u\|_h = \|\lambda_h T_h u_h - \lambda T u\|_h
\]
The quadrilateral meshes on the square (left top, 1428 DOFs) and on the L-shape domain (right top, 3024 DOFs); the triangular meshes with hanging nodes on the square (left bottom, 7176 DOFs, $\lambda_1, h \sim \lambda_5, h$: 7.0E+02, 7.07E+02, 2.3E+03, 4.2E+03, 5.0E+03) and on the L-shape domain (right bottom, 3816 DOFs, $\lambda_1, h \sim \lambda_5, h$: 33, 98, 3.8E+02, 4.0E+02, 6.8E+02)

\[
\leq \| \lambda_h T_h u_h - \lambda T_h u_h \|_h + \| \lambda T_h u - \lambda T u \|_h \\
\lesssim \| \lambda_h u_h - \lambda u \| + \lambda \| T_h u - T u \|_h \\
\lesssim |\lambda_h - \lambda| + \| u - u_h \| + \inf_{v \in V_h} \| T u - v \|_h
\]

which together with (4.37) and Theorem 3.3(or Theorem 3.2) yields the estimate (4.38) (or (4.35)).

5 Numerical experiment

In this section, we will adopt h-refinement on the mesh size and p-refinement on the polynomial degree in the IPDG discretization of the quad-curl eigenvalue problem. We utilize the FE mesh data structure from the iFEM package [20] in the MATLAB environment.

We consider the eigenvalue problem on three domains: the square $\Omega_S := (-1, 1)^2$, the L-shape domain $\Omega_L := (-1, 1)^2 \setminus \{(-1, 0) \times [0, 1]\}$, and the cube $(-1, 1)^3$. 
First we solve the eigenvalue problem using $P_2$ polynomial space on triangular meshes. For comparative purposes, we also compute the eigenvalue problem on quadrilateral meshes. The choice of $\eta_1$ and $\eta_2$ is not highly sensitive to the computational accuracy on the triangular meshes. We set $\eta_1 = 2.5, \eta_2 = 1.6$ a quadrilateral mesh, and then compute the quad-curl eigenvalues on $\Omega_L$ and $\Omega_S$ (see the top of Fig. 1 for the coarsest mesh). The same $\eta_1$ and $\eta_2$ are used for the computation on the uniform triangular meshes. The computational results are listed in Tables 1, 2, 3 and 4. We calculate the convergence rates for numerical eigenvalues using the approximate formula $\log\left(\frac{|\lambda_{i,h} - \lambda_1|}{|\lambda_{j,h} - \lambda_1|}\right) / \log\left(\frac{h_i}{h_j}\right)$. Since the exact eigenvalues are unknown, we use the numerical eigenvalues computed by the curl-curl-conforming elements in [55] on the finest mesh as the reference values.

### 5.1 Numerical results via $P_2$ and $P_3$ polynomial spaces

From Tables 1, 2, 3 and 4, we observe that the asymptotic convergence rates of $\lambda_{1,h}, \lambda_{2,h}$ on $\Omega_S$ and $\lambda_{2,h}$ on $\Omega_L$ are approximately 2. However, the convergence rate of $\lambda_{1,h}$ on $\Omega_L$ using triangular mesh is less than 2, while using quadrilateral meshes yields a convergence rate of approximately 2. Additionally, we notice that the numerical eigenvalues obtained from triangular meshes exhibit higher accuracy compared to those obtained from quadrilateral meshes. This is because a DG space with $P_2$ elements has twice the number of unknowns on a mesh consisting of triangles than on one consisting of quadrilaterals.

Furthermore, we conduct numerical experiments on non-uniform triangular meshes with hanging nodes (shown at the bottom of Fig. 1). By utilizing 7176 degrees of free-
Table 2  Numerical eigenvalues on the L-shape using quadrilateral meshes

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| $h$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
| $\lambda_{1,h}$ | 35.3209 | 34.0426 | 33.6188 | 33.4995 | 33.4685 |
| $\lambda_{2,h}$ | 58.0695 | 100.9415 | 99.1287 | 98.6054 | 98.4652 |
| $\lambda_{3,h}$ | 72.6584 | 392.0341 | 384.0611 | 381.7741 | 381.1653 |
| $\lambda_{4,h}$ | 106.2517 | 412.9257 | 402.516 | 399.245 | 398.2771 |
| $\lambda_{5,h}$ | 440.7255 | 501.4575 | 688.5538 | 683.8663 | 682.5761 |
| $|\lambda_{1,h} - \lambda_{1}|/\lambda_{1}$ | 5.57e-02 | 1.75e-02 | 4.82e-03 | 1.25e-03 | 3.27e-04 |
| Rate | 1.9386 | 1.9329 | | | |
| $|\lambda_{2,h} - \lambda_{2}|/\lambda_{2}$ | 4.10e-01 | 2.56e-02 | 7.23e-03 | 1.92e-03 | 4.93e-04 |
| Rate | 1.9135 | 1.9475 | | | |
| $|\lambda_{3,h} - \lambda_{3}|/\lambda_{3}$ | 8.09e-01 | 2.91e-02 | 8.22e-03 | 2.15e-03 | 5.51e-04 |
| Rate | 1.9231 | 1.9635 | | | |
| $|\lambda_{4,h} - \lambda_{4}|/\lambda_{4}$ | 7.32e-01 | 3.77e-02 | 1.16e-02 | 3.34e-03 | 9.13e-04 |
| Rate | 1.7890 | 1.8716 | | | |
| $|\lambda_{5,h} - \lambda_{5}|/\lambda_{5}$ | 3.53e-01 | 2.65e-01 | 9.43e-03 | 2.56e-03 | 6.68e-04 |
| Rate | 1.8817 | 1.9369 | | | |

Table 3  Numerical eigenvalues on the square using uniform triangular meshes

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| $h$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
| $\lambda_{1,h}$ | 697.66 | 707.89 | 708.32 | 708.11 | 708.01 |
| $\lambda_{2,h}$ | 703.97 | 709.47 | 708.67 | 708.19 | 708.03 |
| $\lambda_{3,h}$ | 2294.94 | 2354.17 | 2353.56 | 2351.21 | 2350.34 |
| $\lambda_{4,h}$ | 4112.82 | 4251.31 | 4259.09 | 4257.22 | 4256.24 |
| $\lambda_{5,h}$ | 4912.41 | 5027.41 | 5028.64 | 5025.61 | 5024.45 |
| $|\lambda_{1,h} - \lambda_{1}|/\lambda_{1}$ | 1.46e-02 | 1.12e-04 | 4.92e-04 | 1.95e-04 | 5.79e-05 |
| Rate | 1.3344 | 1.7510 | | | |
| $|\lambda_{2,h} - \lambda_{2}|/\lambda_{2}$ | 5.66e-03 | 2.12e-03 | 9.84e-04 | 3.04e-04 | 8.33e-05 |
| Rate | 1.6968 | 1.8655 | | | |
| $|\lambda_{3,h} - \lambda_{3}|/\lambda_{3}$ | 2.34e-02 | 1.78e-03 | 1.52e-03 | 5.22e-04 | 1.49e-04 |
| Rate | 1.5412 | 1.8097 | | | |
| $|\lambda_{4,h} - \lambda_{4}|/\lambda_{4}$ | 3.36e-02 | 1.06e-03 | 7.70e-04 | 3.29e-04 | 1.00e-04 |
| Rate | 1.2244 | 1.7186 | | | |
| $|\lambda_{5,h} - \lambda_{5}|/\lambda_{5}$ | 2.22e-02 | 6.80e-04 | 9.25e-04 | 3.21e-04 | 9.14e-05 |
| Rate | 1.5257 | 1.8141 | | | |
Table 4 Numerical eigenvalues on the L-shape using uniform triangular meshes

| $h$  | 1/8   | 1/16  | 1/32  | 1/64  | 1/128 |
|------|-------|-------|-------|-------|-------|
| $\lambda_{1,h}$ | 33.4608 | 33.4824 | 33.4664 | 33.4603 | 33.4589 |
| $\lambda_{2,h}$ | 98.5124 | 98.5541 | 98.4667 | 98.4312 | 98.4204 |
| $\lambda_{3,h}$ | 381.4047 | 381.6117 | 381.19 | 381.0231 | 380.9735 |
| $\lambda_{4,h}$ | 396.4531 | 398.4319 | 398.177 | 397.9822 | 397.9045 |
| $\lambda_{5,h}$ | 677.3528 | 682.6124 | 682.4738 | 682.2339 | 682.1436 |
| $|\lambda_{1,h} - \lambda_{1}|/\lambda_{1}$ | 9.86e−05 | 7.44e−04 | 2.66e−04 | 8.37e−05 | 4.18e−05 |
| Rate | 1.6684 | 1.0000 |
| $|\lambda_{2,h} - \lambda_{2}|/\lambda_{2}$ | 9.72e−04 | 1.40e−03 | 5.08e−04 | 1.47e−04 | 3.76e−05 |
| Rate | 1.7859 | 1.9705 |
| $|\lambda_{3,h} - \lambda_{3}|/\lambda_{3}$ | 1.18e−03 | 1.72e−03 | 6.16e−04 | 1.78e−04 | 4.78e−05 |
| Rate | 1.7915 | 1.8973 |
| $|\lambda_{4,h} - \lambda_{4}|/\lambda_{4}$ | 3.67e−03 | 1.30e−03 | 6.64e−04 | 1.73e−04 | 2.26e−05 |
| Rate | 1.9443 | 2.9323 |
| $|\lambda_{5,h} - \lambda_{5}|/\lambda_{5}$ | 6.99e−03 | 7.22e−04 | 5.19e−04 | 1.67e−04 | 3.48e−05 |
| Rate | 1.6343 | 2.2661 |

Table 5 Numerical eigenvalues on the square by different polynomial degrees $p$: uniform triangular mesh $h = 1/8$

| $p$  | 2   | 3   | 4   | 5   | 6   | 7   |
|------|-----|-----|-----|-----|-----|-----|
| $\lambda_{1,h}$ | 697.664507 | 708.181644 | 707.993329 | 707.971329 | 707.971765 | 707.971509 |
| $\lambda_{2,h}$ | 703.966943 | 708.349258 | 707.989007 | 707.971929 | 707.971702 | 707.971551 |
| $\lambda_{3,h}$ | 2294.939257 | 2353.247385 | 2350.206945 | 2349.986846 | 2349.987798 | 2349.985790 |
| $\lambda_{4,h}$ | 4112.822158 | 4260.588288 | 4256.264643 | 4255.816486 | 4255.817946 | 4255.814058 |
| $\lambda_{5,h}$ | 4912.406669 | 5029.614780 | 5024.259053 | 5023.992937 | 5023.992162 | 5023.992341 |

dom (DOF), we obtain approximate eigenvalues 7176 degrees of freedom (DOF) we obtain approximate eigenvalues $\lambda_{1,h} \sim \lambda_{5,h}$ on the square up to 2–3 digits: 7.0E+02, 7.07E+02, 2.3E+03, 4.2E+03, 5.0E+03. Similarly, with 3816 DOFs, we obtain approximate eigenvalues $\lambda_{1,h} \sim \lambda_{5,h}$ on the L-shape domain up to 2 digits: 33, 98, 3.8E+02, 4.0E+02, 6.8E+02. This demonstrates the robustness of the IPDG method in solving quad-curl eigenvalues on nonconforming triangular meshes.

Next, we solve for the lowest 5 eigenvalues using a $P_3$ polynomial space. The corresponding numerical eigenvalues and their error curves are presented in Table 5 and the left side of Fig. 2, respectively. We observe that the convergence rate in this case is approximately $O(h^4)$ for all computed eigenvalues.

In the end of this subsection, we discuss the impact of the setting of $\eta_1$ and $\eta_2$ on the numerical eigenvalues. By setting $\eta_1 = 0.02 \sim 4.22$ and $\eta_2 = 0.06 \sim 4.26$, we solve for the first numerical eigenvalue on the square $(-1, 1)^2$ with $h = 1/8$. Figure 3 illustrates that the value of $\lambda_{1,h}$ fluctuates when $\eta_1$ or $\eta_2$ is small. When $\eta_1$ and $\eta_2$ are
large enough, for example, $\eta_1, \eta_2 > 1$, $\lambda_{1,h}$ increases as $\eta_1$ and $\eta_2$ increase. Generally, it is advisable to choose $\eta_1$ and $\eta_2$ greater than 1 during computation.

5.2 Numerical results via $p$-version refinement

As the last numerical example in 2D, we apply the IPDG method to solve the eigenvalue problem on the square using different polynomial degrees while keeping the mesh size fixed. The right hand side of Fig. 2 shows the error curves in a semi-log chart with fixed $h = 1/8$ and polynomial degrees $p$ ranging from 2 to 7. It can be observed that the errors of the computed DG eigenvalues exhibit a linear trend with respect to the local polynomial degrees on the semi-log scale, indicating an exponential rate of convergence. Numerically, the convergence is approximately $exp(-rp)$ with $r = 2.5 \sim 3$. 
Table 6 Numerical eigenvalues on the cube by different polynomial degrees $p$: 6 uniform tetrahedra

| $p$ | 5   | 6   | 7   | 8   | 9   |
|-----|-----|-----|-----|-----|-----|
| $\lambda_{1,h}$ | 168.4810 | 112.2701 | 109.7608 | 106.8450 | 106.6736 |
| $\lambda_{2,h}$ | 191.1329 | 117.3074 | 112.3088 | 106.8450 | 106.6834 |
| $\lambda_{3,h}$ | 191.1329 | 117.3074 | 112.3088 | 106.9811 | 106.6737 |
| $\lambda_{4,h}$ | 466.1273 | 302.4162 | 264.9300 | 253.0533 | 247.4130 |
| $\lambda_{5,h}$ | 466.1273 | 302.4162 | 264.9300 | 253.0533 | 247.4129 |

Fig. 4 Error curves on the cube using different polynomial degrees with fixed mesh

Table 7 Convergence rates for the solution (5.1) using $P_2$ polynomial space

| $h$     | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
|---------|-----|------|------|------|-------|
| $\frac{\|T_h-T\|}{\|T\|}$ | 3.26E−01 | 2.08E−02 | 4.73E−03 | 1.18E−03 | 2.98E−04 |
| Rate    | 2.0000 | 1.9852 |
| $\frac{\|\text{curl}_h(T_h-T)\|}{\|\text{curl}T\|}$ | 1.34E−01 | 3.92E−02 | 1.03E−02 | 2.61E−03 | 6.57E−04 |
| Rate    | 1.9817 | 1.9891 |
| $\frac{\|T_h-T\|}{\|T\|}$ | 5.43E−01 | 2.78E−01 | 1.40E−01 | 7.00E−02 | 3.50E−02 |
| Rate    | 0.9990 | 1.0000 |

In addition, we present a numerical example on a three dimensional cube $[-1, 1]^3$ with polynomial degrees $p$ ranging from 5 to 9. The cube is divided into six uniform tetrahedra, and we set $\eta_1 = 15.6$ and $\eta_2 = 1.35$. Table 6 lists the computed lowest five DG eigenvalues, and Fig. 4 displays the corresponding error curves. Numerically, the convergence rates of the lowest five DG eigenvalues are approximately $exp(-rp)$ with $r = 0.10 \sim 0.15$. 

Springer
5.3 Numerical verification for Theorem 4.1

In this subsection, in order to verify the convergence order in the estimate (4.28) of Theorem 4.1, we shall compute the convergence rate of the DG discretization (4.8) using $P_2$ polynomial space. Here we set

$$T_g \left( \begin{array}{c} 3\pi \sin(\pi x_2)^2 \cos(\pi x_2) \sin(\pi x_1)^3 \\ -3\pi \sin(\pi x_1)^2 \cos(\pi x_1) \sin(\pi x_2)^3 \end{array} \right), (x_1, x_2) \in (0, 1)^2$$

(5.1)

where $g$ can be determined from direct calculation. We list both the relative error and the convergence rates of $T_h g$ in Table 7. It can be seen that the convergence rates of $T_h g$ in $\| \cdot \|$, $\| \text{curl}_h (\cdot) \|$ and $\| \cdot \|_h$ are $O(h^2)$, $O(h^2)$ and $O(h)$, respectively. This is consistent with the theoretical result.

6 Discussion versus $H(\text{curl}^2)$-conforming methods

The quadrilateral spectral element method [54] achieves super accuracy but is only developed for two dimensions. The curl-curl element method [31, 32, 55], on the other hand, has fewer degrees of freedom per element and results in a well-conditioned algebraic system. Both methods have been developed for both two and three dimensions. However, neither of these methods can be applied to nonconforming meshes or realize the hp-adaptive version. The IPDG method, on the other hand, allows for nonconforming meshes and supports hp-adaptivity. However, it shares common issues with DG methods, such as ill-conditioned algebraic systems and a large number of degrees of freedom.

Acknowledgements We cordially thank the referee for helpful suggestions which improve the quality of this manuscript significantly. This work was supported in part by the National Natural Science Foundation of China Grants: 12001130, 12131005 and 12361084, and China Postdoctoral Science Foundation No. 2020M680316. We thank Baijun Zhang of CSRC for his extensive discussions on this topic.

Declarations

Conflict of interest The authors declare that there are no conflicts of interest.

Appendix A

We introduce the following function (with the Legendre polynomial $L_i(\xi)$ of degree $i$)

$$\hat{\phi}_i(\xi) = \begin{cases} 1 - \frac{\xi}{2}, & i = 0, \\ \frac{1+\xi}{2}, & i = 1, \\ \frac{L_i(\xi) - L_{i-2}(\xi)}{\sqrt{2(2i-1)}}, & i \geq 2. \end{cases}$$

(6.1)
The basis functions of $\mathbf{H}(\text{curl})$-conforming rectangular element in $Q^{p-1,p}([-1,1]^2) \times Q^{p,p-1}([-1,1]^2)$ is as follows

A. Cell-based basis functions:

I. $\Phi_{i,j}^- := \nabla \hat{\phi}_i(x_1) \hat{\phi}_j(x_2) - \hat{\phi}_i(x_1) \nabla \hat{\phi}_j(x_2)$ for $2 \leq i, j \leq p$. ((p - 1)$^2$ in total)

II. $\Phi_{i,j}^+ := \nabla \hat{\phi}_i(x_1) \hat{\phi}_j(x_2) + \hat{\phi}_i(x_1) \nabla \hat{\phi}_j(x_2)$ for $2 \leq i, j \leq p$. ((p - 1)$^2$ in total)

III. $\Phi_{1,j} := \hat{\phi}_j(x_2)e_1$ and $\Phi_{i,1} := \hat{\phi}_i(x_1)e_2$ for $2 \leq i, j \leq p$. (2(p - 1) in total)

B. Edge-based basis functions

I. $\Phi_{i,j}^+ := \hat{\phi}_i(x_1) \hat{\phi}_j(x_2) + \hat{\phi}_i(x_1) \hat{\phi}_j(x_2)$ for $i = 0, 1$ and $2 \leq j \leq p$. (2(p - 1) in total)

II. $\Phi_{i,j}^- := \nabla \hat{\phi}_i(x_1) \hat{\phi}_j(x_2) + \hat{\phi}_i(x_1) \nabla \hat{\phi}_j(x_2)$ for $j = 0, 1$ and $2 \leq i \leq p$. (2(p - 1) in total)

III. $\Phi_{1,j} := \hat{\phi}_j(x_2)e_1$ and $\Phi_{i,1} := \hat{\phi}_i(x_1)e_2$ for $i, j = 0, 1$. (4 in total)

The above basis functions are orthogonal in the sense of $(\text{curl}, \text{curl})$ and $<\gamma \cdot, \gamma \cdot>$ where $\gamma v := \nabla \cdot \tau$ is the tangential trace of $v$. Let $\tilde{v} \in \nabla (Q^{1,p} + Q^{p,1})$. Denote $\Pi(\tilde{v}) := (\Pi_1(\tilde{v}), \Pi_2(\tilde{v}))^T$ with

$$\Pi_1(\tilde{v}) = \frac{1}{2} \int_{-1}^{1} \tilde{v}_1(-1, 1)d\tilde{x}_1 \phi_0(x_2) + \frac{1}{2} \int_{-1}^{1} \tilde{v}_1(1, 1)d\tilde{x}_1 \phi_1(x_2),$$

$$\Pi_2(\tilde{v}) = \frac{1}{2} \int_{-1}^{1} \tilde{v}_2(-1, 1)d\tilde{x}_2 \phi_0(x_1) + \frac{1}{2} \int_{-1}^{1} \tilde{v}_2(1, 1)d\tilde{x}_2 \phi_1(x_1).$$

(6.2)

Then $\tilde{v} = r + t + \Pi(\tilde{v})$ with $r = \Sigma_{i=0}^{1} \Sigma_{j=2}^{p} C_{i,j}^+ \Phi_{i,j}^+$ and $t = \Sigma_{j=0}^{p} \Sigma_{i=2}^{1} C_{i,j}^+ \Phi_{i,j}^-$. One can verify that

$$\int_{[-1,1]^2} (r_1^2(1 - \tilde{x}_2^2)^{-1} + r_2^2) d\mathbf{x} \lesssim \Sigma_{i=0}^{1} \Sigma_{j=2}^{p} (C_{i,j}^+)^2((j^2 - j)^{-1} + 1),$$

$$\int_{[-1,1]^2} (t_1^2(1 - \tilde{x}_1^2)^{-1} + t_2^2) d\mathbf{x} \lesssim \Sigma_{j=0}^{p} \Sigma_{i=2}^{1} (C_{i,j}^-)^2((i^2 - i)^{-1} + 1).$$

(6.3)

References

1. Arnold, D.N., Brezzi, F., Cockburn, B., Marini, L.D.: Unified analysis of discontinuous Galerkin methods for elliptic problems. SIAM J. Numer. Anal. 39, 1749–1779 (2001)

2. Ainsworth, M., Coyle, J.: Computation of Maxwell eigenvalues on curvilinear domains using hp-version Nédélec elements. In: Numerical Mathematics and Advanced Applications: Proceedings of ENUMATH 2001 the 4th European Conference on Numerical Mathematics and Advanced Applications Ischia, July 2001, pp. 219–231. (2003)

3. Apel, T., Melenk, J.M.: Interpolation and quasi-interpolation in h- and hp-version finite element spaces. In: Encyclopedia of Computational Mechanics Second Edition, pp. 1–33. (2017). https://doi.org/10.1002/9781119176817.ecm2002m
4. Babuška, I., Osborn, J.: Eigenvalue problems. In: Ciarlet, P.G., Lions, J.L. (eds.) Finite Element Methods (Part 1), Handbook of Numerical Analysis, vol. 2, pp. 640–787. Elsevier Science Publishers, North-Holland (1991)
5. Babuška, I., Suri, M.: The hp version of the finite element method with quasiuniform meshes. M2AN 21, 199–238 (1987)
6. Baker, G.A.: Finite element methods for elliptic equations using nonconforming elements. Math. Comput. 31, 45–59 (1977)
7. Bangerth, W., Kayser-Herold, O.: Data structures and requirements for hp finite element software. ACM Trans. Math. Softw. 36, 1–31 (2009)
8. Bespalov, A., Heuer, N.: Optimal error estimation for H(curl)-conforming p-interpolation in two dimensions. SIAM J. Numer. Anal. 47, 3977–3989 (2009)
9. Boffi, D.: Fortin operator and discrete compactness for edge elements. Numer. Math. 87, 229–246 (2000)
10. Boffi, D., Costabel, M., Dauge, M., Demkowicz, L., Hiptmair, R.: Discrete compactness for the p-version of discrete differential forms. SIAM J. Numer. Anal. 49, 135–158 (2011)
11. Boffi, D., Guzman, P., Neilan, M.: Convergence of Lagrange finite elements for the Maxwell eigenvalue problem in 2d. IMA J. Numer. Anal. 43(2), 663–691 (2022)
12. Boffi, D., Ciarlet, P., Jamelot, E.: Solving electromagnetic eigenvalue problems in polyhedral domains with nodal finite elements. Numer. Math. 113, 497–518 (2009)
13. Cangiani, A., Dong, Z., Georgoulis, E.H., Houston, P.: hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes. Springer, Berlin (2017)
14. Cao, S., Chen, L., Huang, X.: Error analysis of a decoupled finite element method for quad-curl problems. J. Sci. Comput. 90, 1–25 (2022). https://doi.org/10.1007/s10915-021-01705-7
15. Chen, G., Qiu, W., Xu, L.: Analysis of an interior penalty DG method for the quad-curl problem. IMA J. Numer. Anal. 00, 1–34 (2020)
16. Ciarlet, P., Jr., Hechme, G.: Computing electromagnetic eigenmodes with continuous Galerkin approximations. Comput. Methods Appl. Mech. Eng. 198, 358–365 (2008)
17. Cockburn, B., Karniadakis, G.E., Shu, C.-W.: 2000 The development of discontinuous Galerkin methods. In: B. Cockburn et al. (eds.) Discontinuous Galerkin Methods (Newport, RI, 1999), Vol. 11 of Lecture Notes in Computational Science and Engineering, pp. 3–50. Springer, Berlin (2000)
18. Demkowicz, L.: Polynomial exact sequences and projection-based interpolation with applications to Maxwell equations. In: Boffi, D., Gastaldi, L. (eds.) Mixed Finite Elements, Compatibility Conditions, and Applications, Vol. 1939 of Lecture Notes in Mathematics, pp. 101–158. Springer, Berlin (2008)
19. Di Pietro, D., Ern, A.: Mathematical aspects of discontinuous Galerkin methods. In: Rémi Abgrall et al. (eds.) Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 69. Springer, Heidelberg (2012)
20. Dong, Z.: Discontinuous Galerkin methods for the biharmonic problem on polygonal and polyhedral meshes. Int. J. Numer. Anal. Mod. 16, 825–846 (2018)
21. Feng, X., Karakashian, O.A.: Two-level non-overlapping Schwarz preconditioners for a discontinuous Galerkin approximation of the biharmonic equation. J. Sci. Comput. 22(23), 289–314 (2005)
22. Georgoulis, E.H., Houston, P.: Discontinuous Galerkin methods for the biharmonic problem. IMA J. Numer. Anal. 29, 573–594 (2009)
23. Georgoulis, E.H., Houston, P., Virtanen, J.: An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems. IMA J. Numer. Anal. 31, 281–298 (2011)
24. Herbert, E., Christian, W.: hp analysis of a hybrid DG method for Stokes flow. IMA J. Numer. Anal. 2, 687–721 (2013)
31. Hu, K., Zhang, Q., Zhang, Z.: Simple curl-curl-conforming finite elements in two dimensions. SIAM J. Sci. Comput. 42, A3859–A3877 (2020)
32. Hu, K., Zhang, Q., Zhang, Z.: A family of finite element Stokes complexes in three dimensions. SIAM J. Sci. Comput. (2022). https://doi.org/10.1137/20M1358700
33. Hiptmair, R.: Finite elements in computational electromagnetism. Acta Numer. 11, 237–339 (2002)
34. Hong, Q., Hun, J., Shu, S., Xu, J.: A discontinuous Galerkin method for the fourth-order curl problem. J. Comput. Math. 30, 565–578 (2012)
35. Houston, P., Schwab, C., Süli, E.: Discontinuous hp-finite element methods for advection-diffusion-reaction problems. SIAM J. Numer. Anal. 39, 2133–2163 (2002)
36. Houston, P., Perugia, I., Schneebeli, A., Schötzau, D.: Interior penalty method for the indefinite time-harmonic Maxwell equations. Numer. Math. 100, 485–518 (2005)
37. Karakashian, O., Collins, C.: Two-level additive Schwarz methods for discontinuous Galerkin approximations of the biharmonic equation. J. Sci. Comput. 74, 573–604 (2018)
38. Kikuchi, F.: Weak formulations for finite element analysis of an electromagnetic eigenvalue problem. Sci. Pap. Coll. Arts Sci. Univ. Tokyo 38, 43–67 (1988)
39. Monk, P.: Finite Element Methods for Maxwell’s Equations. Oxford University Press, Oxford (2003)
40. Monk, P., Sun, J.: Finite element methods for Maxwell’s transmission eigenvalues. SIAM. J. Sci. Comput. 34, B247–B264 (2012)
41. Monk, P.: On the p- and hp-extension of Nedelec’s curl-conforming elements. J. Comput. Appl. Math. 53, 117–137 (1994)
42. Mozolevski, I., Süli, E.: A priori error analysis for the hp-version of the discontinuous Galerkin finite element method for the biharmonic equation. Comput. Methods Appl. Math. 3, 596–607 (2003)
43. Mozolevski, I., Süli, E., Böning, P.R.: hp-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation. J. Sci. Comput. 30, 465–491 (2007)
44. Nédélec, J.C.: Mixed finite elements in $R^3$. Numer. Math. 35, 315–341 (1980)
45. Pan, J., Li, H.: A penalized weak Galerkin spectral element method for second order elliptic equations. J. Comput. Appl. Math. 113228 (2021)
46. Prudhomme, S., Pascal, F., Oden, J.T., Romkes A.: Review of a priori error estimation for discontinuous Galerkin methods. Tech. Report 2000-27, TICAM, University of Texas at Austin (2000)
47. Qiu, W., Shi, K.: A mixed DG method and an HDG method for incompressible magnetohydrodynamics. IMA J. Numer. Anal. 40(2), 1356–1389 (2019)
48. Reddy, C. J., Deshpande, M. D., Cockrell, C. R., Beck, F. B.: Finite Element Method for Eigenvalue Problems in Electromagnetics. Nasa Sti/recon Technical Report N, 95(1995)
49. Russo, A.D., Alonso, A.: Finite element approximation of Maxwell eigenproblems on curved Lipschitz polyhedral domains. Appl. Numer. Math. 59, 1796–1822 (2009)
50. Sun, J.: A mixed FEM for the quad-curl eigenvalue problem. Numer. Math. 132, 185–200 (2016)
51. Sun, J., Zhang, Q., Zhang, Z.: A curl-conforming weak Galerkin method for the quad-curl problem. BIT Numer. Math. 59, 1093–1114 (2019)
52. Sun, Z., Cui, J., Gao, F., Wang, C.: Multigrid methods for a quad-curl problem based on $C^0$ interior penalty method. Comput. Math. Appl. 76(9), 2192–2211 (2018)
53. Süli, E., Mozolevski, I.: hp-version interior penalty DGFEms for the biharmonic equation. Comput. Methods Appl. Mech. Eng. 196, 1851–1863 (2007)
54. Wang, L., Shan, W., Li, H., Zhang, Z.: $H$(curl$^2$)-conforming quadrilateral spectral element method for quad-curl problems. Math. Mod. Methods Appl. Sci. 31, 1951–1986 (2021)
55. Zhang, Q., Wang, L., Zhang, Z.: An $H$(curl$^2$)-conforming finite element in 2 dimensions and applications to the quad-curl problem. SIAM J. Sci. Comput. 41, A1527–A1547 (2019)
56. Zheng, B., Hu, B., Xu, Q.: A nonconforming element method for fourth order curl equations in $R^3$. Math. Comput. 276, 1871–1886 (2011)
57. Zhang, S.: Mixed schemes for quad-curl equations. ESAIM: M2AN 52, 147–161 (2018)
58. Zhou, J., Hu, X., Zhong, L., Shu, S., Chen, L.: Two-grid methods for Maxwell eigenvalue problem. SIAM J. Numer. Anal. 52(4), 2027–2047 (2014)
59. Zhao, J., Zhang, B.: The curl-curl conforming virtual element method for the quad-curl problem. Math. Mod. Methods Appl. Sci. 31, 1659–1690 (2021)
