1. Introduction

Conley index theory, initiated by Conley [Con78], is an important tool for the study of topological dynamical systems. In this series of papers, we propose a new framework for this theory. We do not assume any prior knowledge of Conley index theory.
In this introduction, we explain only the discrete time case for simplicity, although we treat both the discrete time case and the continuous time case in this paper.

First, let us briefly summarize the traditional construction of the Conley index. Suppose that one is given a topological dynamical system, say, a self-homeomorphism \( f: X \to X \) on a locally compact metrizable space \( X \) (in this paper, we work in a more general setting, namely, a continuous partial self-map on a locally compact Hausdorff space). Let \( S \) be an isolated \( f \)-invariant subset of \( X \). In the traditional Conley index theory, one first defines the notion of index pairs for \( S \), whose precise definition varies among researchers (cf. Conley [Con78, Ch. III, Defn. 4.1], Salamon [Sal85, Defns. 4.1 and 5.1], Robbin–Salamon [RS88, Defn. 5.1], Mrozek [Mro90, Defn. 2.1], Szymczak [Szy95, Defn. 3.4], Franks–Richeson [FR00, Defn. 3.1]). Among them, the most general one is Robbin–Salamon’s [RS88, Defn. 5.1]. According to their definition, an index pair for \( S \) is a pair \((N, L)\) satisfying the following three conditions:

- \( N \) and \( L \) are compact subsets of \( X \) satisfying \( L \subset N \).
- \( N \setminus L \) is an isolating neighbourhood of \( S \).
- The self-map \( f_{(N,L)}: N/L \to N/L \) defined by
  \[
  f_{(N,L)}(x) = \begin{cases} 
  f(x) & \text{(if } x \in (N \setminus L) \cap f^{-1}(N \setminus L)), \\
  * & \text{(otherwise)}
  \end{cases}
  \]

is continuous.

One proves, then, the following results:

- Every isolated \( f \)-invariant subset \( S \) of \( X \) admits an index pair.
- For any two index pairs \((N, L)\) and \((N', L')\) for \( S \), one can canonically construct a family of continuous maps from \( N/L \) to \( N'/L' \) in a coherent way.

The Conley index of \( S \) relative to \( f \) is the image of the above based continuous self-map \( f_{(N,L)}: N/L \to N/L \), seen as an object of the endomorphism category (cf. Definition 2.1 (1)), under an appropriate functor \( Q \). Here, the word ‘appropriate’ means that, roughly speaking, the following two conditions are satisfied:

- The Conley index is independent of the choice of the index pair. More precisely, one can canonically construct, from the above family of continuous maps, a single isomorphism from \( Q(f_{(N,L)}) \) to \( Q(f_{(N',L')}) \).
- The Conley index is invariant under continuation of \( f \).

Various functors \( Q \) satisfying these conditions are used (cf. Robbin–Salamon [RS88, Defn. 9.1], Mrozek [Mro90, Defn. 2.8], [Mro94, Defn. 1.3], Szymczak [Szy95, Defn. 4.1], Franks–Richeson [FR00, Defn. 4.8]). Among them, the most general one is Szymczak’s [Szy95, Defn. 4.1].

Next, let us explain our new approach. Our key observation is that, in the above definition of index pairs by Robbin–Salamon, \( N \) and \( L \) themselves are not important; we only need the information on the difference set \( N \setminus L \). To see this, let us introduce the following two notions:
We say that a locally compact Hausdorff subset $E$ of $X$ is $f$-compactifiable if the self-map $f^+_E : E^+ \to E^+$ defined by

$$f^+_E(x) = \begin{cases} f(x) & \text{(if } x \in E \cap f^{-1}(E)), \\ * & \text{(otherwise)} \end{cases}$$

is continuous, where $E^+$ is the one-point compactification of $E$ (cf. Definition 4.6).

We say that a neighbourhood $E$ of $S$ is an index neighbourhood if it is isolating and $f$-compactifiable (cf. Definition 5.9).

Then, a pair $(N, L)$ of compact subsets of $X$ satisfying $L \subset N$ is an index pair for $S$ if and only if $N \setminus L$ is an index neighbourhood of $S$. Also, if $E$ is an index neighbourhood of $S$, then $(\overline{E}, \overline{E} \setminus E)$ is an index pair for $S$, where $\overline{E}$ is the closure of $E$. This observation suggests the possibility that the notion of index pairs is a red herring, and one can develop Conley index theory in a more efficient and more transparent way by using the notion of index neighbourhoods instead. The purpose of this series of papers is to convince the readers that it is indeed the case.

In fact, we can go further. One sees that crucially used in the construction of the family of continuous maps mentioned above is the following property of index neighbourhoods (cf. Salamon [Sal85, Lem. 4.7] and Theorem 5.8):

Let $E$ and $E'$ be two index neighbourhoods of $S$. Then, there exist $a, b, a', b' \in \mathbb{N}$ such that

$$\bigcap_{i=0}^{a+b} f^{-i}(E) \subset f^{-a}(E'), \quad \bigcap_{i'=0}^{a'+b'} f^{-i'}(E') \subset f^{-a'}(E).$$

Let us write $E \sim_f E'$ when this condition is satisfied (cf. Definition 3.5). Then, one notices that the relation $\sim_f$ makes sense not only for isolating neighbourhoods of $S$ but more generally for all subsets of $X$. It is possible to construct a canonical family of based continuous maps from $E^+$ to $E'^+$ for any two $f$-compactifiable subsets $E$ and $E'$ satisfying $E \sim_f E'$ (cf. Definition 3.8 and Theorem 4.5). Thus, one might be able to say that the notions of isolated $f$-invariant subsets and index neighbourhoods are still somewhat superficial, and the truly important notions in Conley index theory are $f$-compactifiable subsets and the relation $\sim_f$ (this position is perhaps too extreme).

Rewriting the proofs in Conley index theory using $f$-compactifiable subsets instead of index pairs is not a trivial task (at least to the author). One problem is that, in the paper of Robbin–Salamon, they rephrased the continuity of the self-map $N/L \to N/L$ into a certain condition which refers $N$ and $L$, not the difference set $N \setminus L$ ([RS88, Thms. 4.2 and 4.3]), and develop the theory using this rephrased condition. Also, their construction of index pairs uses the Lyapunov function ([RS88, Thms. 5.3 and 5.4]), which is not available in our more general setting. Mrozek [Mro90, §5], [Mro94, §4] gave another construction of index pairs that does not use the Lyapunov function, but his argument seems to be difficult to translate into the language of $f$-compactifiable subsets. Many proofs that we give, including the
proof of the existence of index neighbourhoods, are based on the following elementary observation:

- Let \( f : X \to Y \) be a continuous partial map of locally compact Hausdorff spaces. Then, the map \( f^+ : X^+ \to Y^+ \) between the one-point compactifications defined by

\[
f^+(x) = \begin{cases} 
  f(x) & \text{(if } x \in \text{Dom } f) \\
  \ast & \text{(otherwise)}
\end{cases}
\]

is continuous if and only if \( f \) is proper and openly defined (cf. Definitions 4.2 and 4.3).

This observation enables us to separate the continuity of \( f^+ \) into two conditions, the properness and the openly definedness of \( f \). The author hopes that the proofs that we give are more transparent than the existing proofs.

In this first paper of the series, we define the Conley index as an object at the point-set topology level, not at the homotopy category level, and study its point-set level properties. Homotopical properties of the Conley index, including the homotopy invariance under continuation, will be studied in the forthcoming paper [Mor].

**Remark 1.1.** There is another approach to developing Conley index theory without using index pairs by Sánchez-Gabites [San11].

Now, let us speculate some possibilities to apply our new framework to Floer theory of three-manifolds, without much supporting evidence.

Using Conley index theory, Manolescu [Man03] constructed the Seiberg–Witten Floer stable homotopy type, which has more information than the usual monopole Floer homology. It is expected that there is an extension of the monopole Floer homology which assigns an \( A_\infty \)-category with a distinguished object to each compact three-manifold with boundary, and it can be used for formulating a gluing formula for the monopole Floer homology. One can envision that there is also an extension of the Seiberg–Witten Floer stable homotopy type which assigns a stable \((\infty, 1)\)-category with a distinguished object to each compact three-manifold with boundary (very roughly speaking, a stable \((\infty, 1)\)-category is something like a stable homotopy category equipped with all the information of homotopy coherence, and can be seen as a nonlinear version of the \( A_\infty \)-category). However, the Conley index is usually defined at the level of the plain homotopy category, and because of that, Manolescu’s Seiberg–Witten Floer stable homotopy type is defined as an object in a variant of the plain stable homotopy category. In this paper, we construct the Conley index purely at the point-set level so that, in the forthcoming paper [Mor], the homotopy invariance under continuation can be established in a homotopy coherent manner. This could be a first step toward defining the above-mentioned extension of the Seiberg–Witten Floer stable homotopy type for compact three-manifolds with boundary.

Unlike the Seiberg–Witten case, it is still not known if there is a homotopy refinement of the instanton Floer homology for closed three-manifolds. In the Seiberg–Witten case, the finite-dimensional approximation of the formal gradient flow satisfies a certain compactness condition ([Man03 Prop. 3]) that ensures the existence of an isolating neighbourhood. This is crucial for
defining the Floer stable homotopy type via Conley index theory, because
the Conley index is traditionally defined for isolated invariant subsets. An
analogous compactness condition fails in the instanton case, and it seems
very difficult to define the instanton Floer stable homotopy type by using
the traditional Conley index theory. However, our new formulation does
not necessarily require the existence of isolating neighbourhood; we only
need a compactifiable subset. Thus, we can imagine the possibility that our
framework works also for the instanton case, even though, at this moment,
there is no circumstantial evidence for this speculation.

Outline of the paper. In the first half of this paper, we treat the discrete
time case. In Section 2, we give the definition of the Szymczak category,
which is the universal receiver of the Conley index. In Section 3, we give
basic definitions and results on partial maps on sets, and define a functor
Conl_\text{f} for partial self-maps on sets. In Section 4, we study proper and
openly defined continuous partial maps, and refine Conl_\text{f} for continuous
partial self-maps on locally compact Hausdorff spaces using the notion of f-
compactifiable subsets. In Section 5, we define the Conley index of isolated
f-invariant sets using the notion of index neighbourhoods.

In the second half of this paper, we treat the continuous time case. Many
definitions, theorems, lemmas, and proofs are parallel to the discrete time
case, while some definitions and proofs are more involved. We omit proofs
when they are completely the same as the discrete time case. In Section 6, we
define an analogue of the Szymczak category in the continuous time setting.
In Section 7, we give basic definitions and results on partial semiflows on
sets, and define a functor Conl_\text{F} for them. In Section 8, we study finite-time
proper and openly defined continuous partial semiflows, and refine Conl_\text{F} for
continuous partial semiflows on locally compact Hausdorff spaces using the
notion of F-compactifiable subsets. In Section 9, we define the Conley index of
isolated F-invariant sets using the notion of index neighbourhoods.

In Appendix A, we prove some results on proper maps that are needed
in this paper. In Appendix B, we explain a categorical meaning of the
Szymczak category using shift equivalences.

Convention. In this paper, every topological space is assumed to be comp-
actly generated weak Hausdorff. Products, pullbacks, and subspaces are
taken in the category of compactly generated weak Hausdorff spaces.

Part 1. The discrete time case

2. The Szymczak category

The Conley index is defined as an object of a certain category, which we
call the Szymczak category:

Definition 2.1. Let C be a category.

1. We define the category of endomorphisms in C (or the category of
N-equivariant objects in C), for which we write End(C), as follows:
   • An object of End(C) is an endomorphism f: X \to X in C.
• For each two endomorphisms \( f: X \to X \) and \( g: Y \to Y \) in \( C \), the hom-set \( \text{End}(C)(f, g) \) is given by

\[
\text{End}(C)(f, g) = \varphi \in C(X, Y) \mid \varphi f = g \varphi.
\]

(2) We define the Szymczak category of \( C \), for which we write \( \text{Sz}(C) \), as follows:
• An object of \( \text{Sz}(C) \) is an endomorphism \( f: X \to X \) in \( C \).
• For each two endomorphisms \( f: X \to X \) and \( g: Y \to Y \) in \( C \), the hom-set \( \text{Sz}(C)(f, g) \) is given by

\[
\text{Sz}(C)(f, g) = (\text{End}(C)(f, g) \times \mathbb{N})/\sim,
\]
where \((\varphi, k) \sim (\varphi', k')\) if and only if there exists \( n \in \mathbb{N} \) such that \( \varphi f^{k+n} = \varphi' f^{k+n} \) (or equivalently, \( g f^{k+n} \varphi = g f^{k+n} \varphi' \)).
• The composition of two morphisms \((\varphi, k) \in \text{Sz}(C)(f, g)\) and \((\psi, \ell) \in \text{Sz}(C)(g, h)\) is given by \((\psi, \ell) \circ (\varphi, k) = (\varphi \psi, k + \ell)\), which does not depend on the choice of the representatives.

Remark 2.2. The Szymczak category is introduced by Szymczak [Szy95, §2.1] under the name ‘the category of objects equipped with a morphism’.

We can easily see that an equivalence of two categories \( \Phi: C \to D \) induces equivalences \( \Phi: \text{End}(C) \to \text{End}(D) \) and \( \Phi: \text{Sz}(C) \to \text{Sz}(D) \).

3. Partial maps of sets

Definition 3.1. Let \( X \) and \( Y \) be two sets. A partial map from \( X \) to \( Y \) is a pair \((D, f)\), where \( D \) is a subset of \( X \), which we call the domain of the partial map, and \( f: D \to Y \) is a map.

We use the symbol \( \twoheadrightarrow \) to signify a partial map. We write \( \text{Dom} f \) rather than \( D \) to avoid specifying the domain of a partial map. We define the composite \( g f: X \to Z \) of two partial maps \( f: X \to Y \) and \( g: Y \to Z \) by

\[
\text{Dom}(g f) = f^{-1}(\text{Dom} g), \quad (g f)(x) = g(f(x)).
\]

Let \( \text{Set}_0 \) be the category of sets and partial maps. Let \( \text{Set}_\ast \) be the category of based sets and based maps. Then, we can define the following functor:

Definition 3.2. We define a functor \((-)\#: \text{Set}_0 \to \text{Set}_\ast \) as follows:
• Given a set \( X \), we put \( X^+ = X \sqcup \{\ast\} \), with base point \( \ast \).
• Given a partial map \( f: X \to Y \) of sets, we define a based map \( f^+: X^+ \to Y^+ \) by

\[
f^+(x) = \begin{cases} f(x) & \text{if } x \in \text{Dom} f, \\ \ast & \text{otherwise.} \end{cases}
\]

Proposition 3.3. \((-)\#: \text{Set}_0 \to \text{Set}_\ast \) is an equivalence of categories.

Proof. The following functor \( U: \text{Set}_\ast \to \text{Set}_0 \) is an inverse of \((-)\#:\)
• Given a based set \((X, x_0)\), put \( U(X, x_0) = X \setminus \{x_0\} \).
• Given a based set \( f: (X, x_0) \to (Y, y_0) \), define a partial map \( U f: X \setminus \{x_0\} \to Y \setminus \{y_0\} \) by

\[
\text{Dom}(U f) = f^{-1}(Y \setminus \{y_0\}), \quad (U f)(x) = f(x).
\]
\( \Box \)
For $n \in \mathbb{N}$ and a partial self-map $f : X \twoheadrightarrow X$, we put

$$f^n = f \circ \cdots \circ f : X \twoheadrightarrow X,$$

and for a subset $E$ of $X$, we use the notation $f^{-n}(E)$ to mean the inverse image $(f^n)^{-1}(E)$.

**Definition 3.4.** Let $f : X \twoheadrightarrow X$ be a partial self-map on a set $X$. For each subset $E$ of $X$, we define a partial self-map $f_E : E \to E$ by

$$\text{Dom } f_E = E \cap f^{-1}(E), \quad f_E(x) = f(x)$$

and call it the **induced partial self-map on $E$**.

In this paper, by $f^n_E$ ($n \in \mathbb{N}$), we always mean $(f_E)^n$, not $(f^n)_E$. Thus,

$$\text{Dom } f^n_E = \bigcap_{i=0}^{n} f^{-i}(E), \quad f^n_E(x) = f^n(x).$$

**Definition 3.5.** Let $f : X \twoheadrightarrow X$ be a partial self-map on a set $X$. Let $E$ and $E'$ be two subsets of $X$.

1. We say that a triple $(a, b, c) \in \mathbb{N}^3$ is $(E, E')$-admissible relative to $f$ if

$$\bigcap_{i=0}^{a+b} f^{-i}(E) \subset f^{-a}(E'), \quad \bigcap_{i'=0}^{b+c} f^{-i'}(E') \subset f^{-b}(E).$$

We write $A_f(E, E') \subset \mathbb{N}^3$ for the set of all $(E, E')$-admissible triples relative to $f$.

2. We use the notation $E \sim_f E'$ to mean $A_f(E, E') \neq \emptyset$.

**Remark 3.6.** The relation $E \sim_f E'$ holds if and only if there exist $a, a', b, b' \in \mathbb{N}$ such that

$$\bigcap_{i=0}^{a+b} f^{-i}(E) \subset f^{-a}(E'), \quad \bigcap_{i'=0}^{a'+b'} f^{-i'}(E') \subset f^{-a'}(E).$$

**Lemma 3.7.** Let $f : X \twoheadrightarrow X$ be a partial self-map on a set $X$.

1. For any subset $E$ of $X$, we have $A_f(E, E) = \mathbb{N}^3$. In particular, \((0, 0, 0) \in A_f(E, E)\).

2. Let $E, E'$, and $E''$ be three subsets of $X$. If $(a, b, c) \in A_f(E, E')$ and $(a', b', c') \in A_f(E', E'')$, then $(a + a', b + b', c + c') \in A_f(E, E'')$.

**Proof.** (1) Trivial.

(2) We see that

$$\bigcap_{j=0}^{a+a'+b+b'} f^{-j}(E) = \bigcap_{i'=0}^{a'+b'} f^{-i'} \left( \bigcap_{i=0}^{a+b} f^{-i}(E) \right)$$

$$\subset \bigcap_{i'=0}^{a'+b'} f^{-i'}(f^{-a}(E')) = f^{-a} \left( \bigcap_{i=0}^{a'+b'} f^{-i'}(E') \right)$$

$$\subset f^{-a}(f^{-a'}(E'')) = f^{-(a+a')}(E'').$$
Similarly, we have
\[ \bigcap_{j''=0}^{b+y+c'} f^{-j''}(E'') \subset f^{-(b+y')}(E). \]

It follows from Remark 3.6 and Lemma 3.7 that \( \sim_f \) is an equivalence relation on the set of all subsets of \( X \).

**Definition 3.8.** Let \( f: X \to X \) be a partial self-map on a set \( X \). Let \( E \) and \( E' \) be two subsets of \( X \). For each \( (a, b, c) \in A_f(E, E') \), define a partial map \( f^{(a,b,c)}_{E'E} : E \to E' \) by
\[ \text{Dom } f^{(a,b,c)}_{E'E} = \bigcap_{i=0}^{a+b+c} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E'), \quad f^{(a,b,c)}_{E'E}(x) = f^{a+b+c}(x). \]

**Theorem 3.9.** Let \( f: X \to X \) be a partial self-map on a set \( X \).

1. Let \( E \) be a subset of \( X \). Then, for any \( (a, b, c) \in \mathbb{N}^3 \), we have \( f^{(a,b,c)}_{EE} = f^{a+b+c} \).

In particular, \( f^{(0,0,0)}_{EE} = \text{id}_E \).

2. Let \( E, E', \) and \( E'' \) be three subsets of \( X \). Then, for any \( (a, b, c) \in A_f(E, E') \) and \( (a', b', c') \in A_f(E', E'') \), we have \( f^{(a,a',b,b',c,c')}_{E'E'} \circ f^{(a,b,c)}_{E'E} = f^{(a+a',b+b',c+c')}_{E'E} \).

3. Let \( E \) and \( E' \) be two subsets of \( X \). Then, for any \( (a, b, c) \in A_f(E, E') \), we have \( f^{(a,b,c)}_{E'E} \circ f^{(a,b,c)}_{EE} = f^{(a,b,c)}_{EE} \).

4. Let \( E \) and \( E' \) be two subsets of \( X \). Then, for any \( (a, b, c) \in A_f(E, E') \), we have \( f^{(a,b,c)}_{E'E} \circ f^{(a,b,c)}_{E'E} = f^{(a,b,c)}_{E'E} \).

**Proof.**

1. Trivial.

2. It suffices to verify that \( \text{Dom } (f^{(a',b',c')}_{E'E'} \circ f^{(a,b,c)}_{E'E}) = \text{Dom } f^{(a+a',b+b',c+c')}_{E'E} \). By definition,
\[ \text{Dom } (f^{(a',b',c')}_{E'E'} \circ f^{(a,b,c)}_{E'E}) = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E') \]
\[ \cap f^{-a+b+c}(E') \cap f^{(a'+b')}(E') \cap f^{(a'+b'+c')}(E'') \]
\[ = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E') \cap \bigcap_{i''=a+a'+b+c}^{a+a'+b+b'+c+c'} f^{-i''}(E''). \]
Since
\[
\bigcap_{i' = a} f^{-i'}(E') = \bigcap_{j = a} f^{-j} \left( \bigcap_{j' = 0} f^{-j}(E') \right)
\]
and
\[
\bigcap_{j'' = a} f^{-j''} (f^{-a'}(E'')) = \bigcap_{i'' = a + a'} f^{-i''}(E'').
\]
we have
\[
\bigcap_{i = a + b} f^{-i}(E) \cap \bigcap_{i' = a} f^{-i'}(E') \cap \bigcap_{i'' = a + a'} f^{-i''}(E'').
\]

Therefore,
\[
\bigcap_{i = 0} f^{-i}(E) \cap \bigcap_{i'' = a + a'} f^{-i''}(E'') = \bigcap_{i'' = a + a'} f^{-i''}(E'').
\]

On the other hand, by definition,
\[
\text{Dom } f^{(a_a', b', c + c')}_{E' \cap E} = \bigcap_{i = 0} f^{-i}(E) \cap \bigcap_{i'' = a + a'} f^{-i''}(E'').
\]

Since
\[
\bigcap_{i = 0} f^{-i}(E) = \bigcap_{j' = 0} f^{-j'}(a + b) \left( \bigcap_{j = 0} f^{-j}(E) \right)
\]
and
\[
\bigcap_{i' = a} f^{-i'}(E') = \bigcap_{i'' = a + a'} f^{-i''}(E'').
\]
we have
\[
\bigcap_{i = 0} f^{-i}(E) = \bigcap_{i = 0} f^{-i}(E) \cap \bigcap_{i' = a} f^{-i'}(E') \cap \bigcap_{i'' = a + a'} f^{-i''}(E'').
\]
Since
\[ a + a' + b + b' + c + c' \]
\[ \bigcap_{i'' = a + a'} f^{-i''}(E'') = \bigcap_{j'' = a + a'} f^{-j''} \left( \bigcap_{i'' = 0} f^{-j''}(E'') \right) \]
\[ \subset \bigcap_{j' = a + a'} f^{-j'}(f^{-b'}(E')) = \bigcap_{i' = a + a'} f^{-i'}(E'), \]
we have
\[ \bigcap_{i'' = a + a'} f^{-i''}(E'') \subset \bigcap_{i'' = a + a'} f^{-i''}(E''). \]

Therefore,
\[ \bigcap_{i'' = a + a'} f^{-i''}(E'') = \bigcap_{i'' = a + a'} f^{-i''}(E'') \]
\[ = \bigcap_{i'' = a + a'} f^{-i''}(E'') \]
This completes the proof of (2).

(3) By (1) and (2),
\[ f_{(a,b,c)} \circ f_E = f_{(a,b,c)} \circ f_{(0,0,1)} = f_{(a,b,c+1)} = f_E \circ f_{(a,b,c)}. \]

(4) By (1) and (2),
\[ f_{(a,b,c)} \circ f'_{E'} = f_{(a,b,c)} \circ f'_{(a',b',c')} = f_{(a',b',c')} \circ f_E. \]

Definition 3.10. Let \( f : X \to X \) be a partial self-map on a set \( X \).

(1) We define a small category \( \text{Sub}_f \) as follows:
- An object of \( \text{Sub}_f \) is a subset of \( X \).
- For each two subsets \( E \) and \( E' \) of \( X \), the hom-set \( \text{Sub}_f(E, E') \)
  is given by \( \text{Sub}_f(E, E') = A_f(E, E') \).
- The composition of two morphisms is given by addition.

(2) We define a functor \( \text{Conl}_f : \text{Sub}_f \to \text{End}(\text{Set}_*) \) as follows:
- For each subset \( E \) of \( X \), we put \( \text{Conl}_f(E) = (f_E^+ : E^+ \to E^+) \).
- For each two subsets \( E \) and \( E' \) of \( X \), we assign to \( (a, b, c) \in A_f(E, E') \)
  the morphism \( (f_{(a,b,c)}^+ : f_{E'}^+ \to f_E^+) \).

By Lemma 3.17 \( \text{Sub}_f \) is indeed a category. Let us verify that \( \text{Conl}_f \) is a
functor. First, by Proposition 3.3 and Theorem 3.9 (3), \( f_{(a,b,c)}^+ \) is a morphism
from \( f_{E'}^+ \) to \( f_E^+ \) in \( \text{End}(\text{Set}_*) \). Second, we see from Theorem 3.9 (1) and (2)
that \( \text{Conl}_f \) is compatible with the identity morphisms and composition. Thus,
\( \text{Conl}_f \) is a functor.

For many purposes, the functor \( \text{Conl}_f \) has too much information. One can
argue that the important thing is the relation \( \sim_f \), and it is not appropriate
to ask if \((a, b, c) \in A_f(E, E')\) or not for a particular \((a, b, c) \in \mathbb{N}^3\). For this reason, we introduce the following truncations of \(\widetilde{\text{Sub}}_f\) and \(\widetilde{\text{Conl}}_f\):

**Definition 3.11.** Let \(f : X \to X\) be a partial self-map on a set \(X\).

1. We define a small category \(\text{Sub}_f\) as follows:
   - An object of \(\text{Sub}_f\) is a subset of \(X\).
   - For each two subsets \(E\) and \(E'\) of \(X\), the hom-set \(\text{Sub}_f(E, E')\) is given by
     \[
     \text{Sub}_f(E, E') = \begin{cases} * & \text{if } E \sim_f E' \text{,} \\ \emptyset & \text{otherwise.} \end{cases}
     \]
2. We define a functor \(\text{Conl}_f : \text{Sub}_f \to \mathbb{Sz}(\text{Set}_*)\) as follows:
   - For each subset \(E\) of \(X\), we put \(\text{Conl}_f(E) = (f_E^+: E^+ \to E^+)\).
   - To each two subsets \(E\) and \(E'\) of \(X\) with \(E \sim_f E'\), we assign the morphism \((f_{E', E}^{(a, b, c)})^+, a + b + c\) defined by
     \[
     (f_{E', E}^{(a, b, c)})^+, a + b + c = (f_{E', E}^{(a', b', c')})^+, a' + b' + c'.
     \]

If \((a, b, c), (a', b', c') \in A_f(E, E')\), it follows from Theorem 3.9 (4) that

\[
(f_{E', E}^{(a, b, c)})^+, a + b + c = (f_{E', E}^{(a', b', c')})^+, a' + b' + c'.
\]

Thus, the definition of \(f_{E', E}^{+}\) does not depend on the choice of \((a, b, c) \in A_f(E, E')\), and \(\text{Conl}_f\) is a well-defined functor.

**Remark 3.12.** Since \(\sim_f\) is a symmetric relation, every morphism in \(\text{Sub}_f\) is an isomorphism.

4. **Continuous partial maps of topological spaces**

**Definition 4.1.** Let \(X\) and \(Y\) be two topological spaces. We say that a partial map \(f : X \rightharpoonup Y\) is **continuous** if \(f\) is continuous as a map from \(\text{Dom} f\) to \(Y\).

The composite of two continuous partial maps is continuous.

**Definition 4.2.** Let \(f : X \rightharpoonup Y\) be a continuous partial map of topological spaces.

1. We say that \(f\) is **proper** if \(f\) is proper as a map from \(\text{Dom} f\) to \(Y\).
2. We say that \(f\) is **openly defined** if \(\text{Dom} f\) is open in \(X\).

The composite of two proper (resp. openly defined) partial maps is proper (resp. openly defined).

We write \(\mathcal{LCHaus}^\partial\) for the category whose objects are locally compact Hausdorff spaces and whose morphisms are proper and openly defined partial maps. Let \(\mathcal{CHaus}_*\) be the category of based compact Hausdorff spaces.
and based continuous maps. The significance of proper and openly defined partial maps is that we can define the following functor:

**Definition 4.3.** We define a functor \((-)^+ : \text{LCHaus}^0 \to \text{CHaus}_*\) as follows:

- Given a locally compact Hausdorff space \(X\), we put \(X^+\) to be the one-point compactification of \(X\), with base point \(*\).
- Given a proper and openly defined partial map \(f : X \to Y\) of locally compact Hausdorff spaces, we define a based continuous map \(f^+ : X^+ \to Y^+\) by
  \[
  f^+(x) = \begin{cases} 
  f(x) & \text{(if } x \in \text{Dom } f) \\
  * & \text{(otherwise)}
  \end{cases}
  \]

Let us verify that \(f^+\) is indeed continuous. Suppose that \(L\) is a closed subset of \(Y^+\) that does not contain \(*\). We have \((f^+)^{-1}(L) = f^{-1}(L)\).

Since \(L\) is a compact subset of \(Y\) and \(f : \text{Dom } f \to Y\) is a proper map, \((f^+)^{-1}(L)\) is a compact subset of \(X^+\). Suppose that \(L\) is a closed subset of \(Y^+\) that contains the base point \(*\). We have \((f^+)^{-1}(L) = f^{-1}(L \setminus \{\ast\}) \cup (X \setminus \text{Dom } f) \cup \{\ast\}\).

Since \(L \setminus \{\ast\}\) is a closed subset of \(Y\) and \(\text{Dom } f\) is an open subset of \(X\), \(f^{-1}(L \setminus \{\ast\}) \cup (X \setminus \text{Dom } f)\) is closed in \(X\). Thus, \((f^+)^{-1}(L)\) is closed in \(X^+\).

**Proposition 4.4.** \((-)^+ : \text{LCHaus}^0 \to \text{CHaus}_*\) is an equivalence of categories.

**Proof.** The following functor \(U : \text{CHaus}_* \to \text{LCHaus}^0\) is an inverse of \((-)^+:

- Given a based compact Hausdorff space \((X,x_0)\), put \(U(X,x_0) = X \setminus \{x_0\}\).
- Given a based continuous map \(f : (X,x_0) \to (Y,y_0)\), define a partial map \(Uf : X \setminus \{x_0\} \to Y \setminus \{y_0\}\) by
  \[
  \text{Dom}(Uf) = f^{-1}(Y \setminus \{y_0\}), \quad (Uf)(x) = f(x).
  \]

For the sake of completeness, let us verify that \(Uf\) is indeed proper and openly defined. Consider the following pullback diagram:

\[
\begin{array}{ccc}
\text{Dom}(Uf) & \xrightarrow{Uf} & Y \setminus \{y_0\} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}
\]

The bottom map is proper since \(X\) is compact Hausdorff (Lemma \([A.7]\)). Thus, the top map is also proper (Lemma \([A.3]\)). Since \(Y \setminus \{y_0\}\) is open in \(X\), the inverse image \(f^{-1}(Y \setminus \{y_0\}) = \text{Dom}(Uf)\) is open in \(X\). Since \(\text{Dom}(Uf)\) is contained in \(X \setminus \{x_0\}\), it is also open in \(X \setminus \{x_0\}\). \(\Box\)

**Theorem 4.5.** Let \(f : X \to X\) be a continuous partial self-map on a topological space \(X\). Let \(E\) and \(E'\) be two subsets of \(X\). Let \((a,b,c) \in A_f(E, E')\).

1. If \(f_E\) and \(f_{E'}\) are proper, \(f_{E\cap E'}^{a,b,c}\) is also proper.
(2) If \( f_E \) and \( f_{E'} \) are openly defined, \( f_{E'}^{a,b,c} \) is also openly defined.

Proof. (1) The map \( f_{E}^{a,b,c} : \text{Dom} \ f_{E}^{a,b,c} \to E' \) is factorized as follows:

\[
\text{Dom} \ f_{E}^{a,b,c} = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E')
\]

\[
\xymatrix{ \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E') \ar[r]^-{f_{E}^{a,b,c}} \ar[d] & \bigcap_{i=0}^{b+c} f^{-i}(E) \cap \bigcap_{i'=0}^{b+c} f^{-i'}(E') \ar[d] \ar[r]^-{f_{E}^{b+c}} & E' }.
\]

Let us prove that the three maps in this factorization are all proper.

Since \( f_{E'} : E' \to E' \) is proper, the third map \( f_{E}^{b+c} \) is proper.

The first map \( f_{E}^{a} \) fits into the following pullback diagram:

\[
\xymatrix{ \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E') \ar[r]^-{f_{E}^{a}} \ar[d] & \bigcap_{i=0}^{b+c} f^{-i}(E) \cap \bigcap_{i'=0}^{b+c} f^{-i'}(E') \ar[d] \ar[r]^-{f_{E}^{b+c}} & E' }.
\]

Since \( f_{E} : E \to E \) is proper, the bottom map is proper. Hence the top map is also proper (Lemma A.3).

The second map fits into the following pullback diagram:

\[
\xymatrix{ \bigcap_{i=0}^{b} f^{-i}(E) \cap \bigcap_{i'=0}^{b+c} f^{-i'}(E') \ar[r]^-{f_{E}^{b}} \ar[d] & \bigcap_{i=0}^{b+c} f^{-i}(E) \cap \bigcap_{i'=0}^{b+c} f^{-i'}(E') \ar[d] \ar[r]^-{f_{E}^{b+c}} & E' }.
\]

To see that the top map is proper, it suffices to see that the bottom map is proper (Lemma A.3). Consider the following commutative diagram:

\[
\xymatrix{ \bigcap_{i=0}^{b} f^{-i}(E) \ar[r]^-{f_{E}^{b}} \ar[d] & f^{-b}(E) \ar[d] & f_{E}^{b} \ar[d] \ar[r]^-{f_{E}^{b}} & E. }
\]

Since \( f_{E} : E \to E \) is proper, the diagonal map is proper. By Lemma A.2(2), the top map is also proper.

(2) The inclusion map \( \text{Dom} \ f_{E}^{a,b,c} \to E \) is factorized as follows:

\[
\text{Dom} \ f_{E}^{a,b,c} = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E') \cap \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=0}^{a+b} f^{-i'}(E).
\]

Let us prove that the two maps in this factorization are open inclusions.
Since $f_E: E \twoheadrightarrow E$ is openly defined, the second map is an open inclusion. Let us prove that the first map is an open inclusion. Consider the following pullback diagram:

\[
\begin{array}{ccc}
\bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E') & \hookrightarrow & \bigcap_{i=0}^{a+b} f^{-i}(E) \\
\downarrow & & \downarrow \\
\bigcap_{i'=a}^{a+b+c} f^{-i'}(E') & \hookrightarrow & f^{-a}(E') \\
\bigcap_{i'=0}^{b+c} f^{-i'}(E') & \hookrightarrow & E'.
\end{array}
\]

Since $f_{E'}: E' \twoheadrightarrow E'$ is openly defined, the bottom map is an open inclusion. Thus, the top map is also an open inclusion. □

**Definition 4.6.** Let $f: X \twoheadrightarrow X$ be a continuous partial self-map on a locally compact Hausdorff space $X$. We say that a locally closed subset $E$ of $X$ is **$f$-compactifiable** if the induced partial self-map $f_E: E \twoheadrightarrow E$ is proper and openly defined.

**Remark 4.7.** In the above definition, $f$-compactifiable subsets are assumed to be locally closed in $X$ so that they are locally compact Hausdorff (cf. [Bou71, Ch. I, §9, Props. 12 and 13]).

Let $\widetilde{\text{CptifSub}}_f$ (resp. $\text{CptifSub}_f$) be the full category of $\widetilde{\text{Sub}}_f$ (resp. $\text{Sub}_f$) consisting of $f$-compactifiable subsets of $X$. By Proposition 4.4 and Theorem 4.5, the functor $\widetilde{\text{Conl}}_f: \widetilde{\text{Sub}}_f \to \text{End}(*\text{Set}_*)$ (resp. $\text{Conl}_f: \text{Sub}_f \to \text{Sz}(*\text{Set}_*)$) induces a functor from $\widetilde{\text{CptifSub}}_f$ to $\text{End}(\text{CHaus}_*)$ (resp. from $\text{CptifSub}_f$ to $\text{Sz}(\text{CHaus}_*)$).

**Definition 4.8.** Let $f: X \twoheadrightarrow X$ be a continuous partial self-map on a locally compact Hausdorff space $X$. We use the same notations $\widetilde{\text{Conl}}_f: \widetilde{\text{CptifSub}}_f \to \text{End}(\text{CHaus}_*)$ and $\text{Conl}_f: \text{CptifSub}_f \to \text{Sz}(\text{CHaus}_*)$ for the functors defined above.

5. The Conley index of isolated $f$-invariant subsets

**Definition 5.1.** Let $f: X \twoheadrightarrow X$ be a partial self-map on a set $X$. We say that a subset $S$ of $X$ is **$f$-invariant** if $S \subseteq \text{Dom } f$ and $f(S) = S$.

**Definition 5.2.** Let $f: X \twoheadrightarrow X$ be a partial self-map on a set $X$. Let $E$ be a subset of $X$. We define the **$f$-invariant part** $I_f(E)$ of $E$ by

\[
I_f(E) = \bigcap_{a,b \in \mathbb{N}} f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(E) \right).
\]
Remark 5.3. The family
\[
\left( f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(E) \right) \right)_{a,b \in \mathbb{N}}
\]
is decreasing in the following sense: for any \(a, b, a', b' \in \mathbb{N}\) satisfying \(a \leq a'\) and \(b \leq b'\), we have
\[
f^{a'} \left( \bigcap_{i=0}^{a'+b'} f^{-i}(E) \right) \subset f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(E) \right).
\]

The above definition of the \(f\)-invariant part agrees with the usual one, namely:

**Lemma 5.4.** Let \(f: X \rightarrow X\) be a partial self-map on a set \(X\). Let \(E\) be a subset of \(X\). Then, \(I_f(E)\) is the largest \(f\)-invariant subset contained in \(E\).

**Proof.** We remark that the largest \(f\)-invariant subset of \(E\) exists. Indeed, if \((S_i)_{i \in I}\) is a family of \(f\)-invariant subsets of \(E\), the union \(\bigcup_{i \in I} S_i\) is also \(f\)-invariant.

Let us prove that \(I_f(E)\) is \(f\)-invariant. We have \(I_f(E) \subset \text{Dom } f\) and \(I_f(E) \subset E\). By Remark 5.3 we have
\[
f(I_f(E)) = f \left( \bigcap_{a,b \in \mathbb{N}} f^a \left( \bigcap_{i=0}^{a+b+1} f^{-i}(E) \right) \right) = \bigcap_{a,b \in \mathbb{N}} f^{a+1} \left( \bigcap_{i=0}^{a+b+1} f^{-i}(E) \right) = I_f(E).
\]

Let us prove that every \(f\)-invariant subset of \(E\) is contained in \(I_f(E)\). It suffices to see that, if \(S\) is an \(f\)-invariant subset of \(E\), then
\[
S \subset f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(E) \right)
\]
holds for any \(a, b \in \mathbb{N}\). Take any \(x \in S\). Since \(S \subset f^a(S)\), there exists \(x' \in S\) such that \(f^a(x') = x\). For each \(i \in \mathbb{N}\) with \(0 \leq i \leq a + b\), we have \(x' \in f^{-i}(E)\) since \(f^i(S) \subset S \subset E\). Thus,
\[
x = f^a(x') \in f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(E) \right).
\]

**Definition 5.5.** Let \(f: X \rightarrow X\) be a continuous partial self-map on a locally compact Hausdorff space \(X\). Let \(S\) be an \(f\)-invariant subset of \(X\).

1. A neighbourhood \(E\) of \(S\) is called **isolating** if the following three conditions are satisfied:
   - \(E\) is relatively compact.
   - \(\overline{E} \subset \text{Dom } f\).
   - \(S = I_f(\overline{E})\).

   Here, \(\overline{E}\) denotes the closure of \(E\).

\(^1\)This Lemma is wrong. See Correction.
(2) $S$ is called isolated if it admits an isolating neighbourhood.

**Remark 5.6.** If $E$ is an isolating neighbourhood of $S$, the closure $\overline{E}$ is also isolating. If $E$ is an isolating neighbourhood of $S$ and $E'$ is a neighbourhood of $S$ contained in $E$, then $E'$ is also isolating.

**Lemma 5.7.** Let $f: X \rightarrow X$ be a continuous partial self-map on a locally compact Hausdorff space $X$. Let $S$ be an isolated $f$-invariant subset of $X$. Then, $S$ is compact.

**Proof.** Take an isolating neighbourhood $K$ of $S$. By Remark 5.6, we may assume that $K$ is compact. We have

$$S = \bigcap_{a,b \in \mathbb{N}} f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(K) \right).$$

Observe that, for each $a, b \in \mathbb{N}$,

$$f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(K) \right)$$

is compact and closed in $K$. Thus, $S$ is compact. □

**Theorem 5.8.** Let $f: X \rightarrow X$ be a continuous partial self-map on a locally compact Hausdorff space $X$. Let $S$ be an isolated $f$-invariant subset of $X$. Then, for any two isolating neighbourhoods $E$ and $E'$ of $S$, we have $E \sim_f E'$.

**Proof.** Let $K = \overline{E}$ be the closure of $E$ and $U = (E')^\circ$ be the interior of $E'$. We have

$$(K \setminus U) \cap \bigcap_{a,b \in \mathbb{N}} f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(K) \right) = (K \setminus U) \cap S = \emptyset.$$ 

By the compactness of $K$ and Remark 5.3, there exist $a_0, b_0 \in \mathbb{N}$ such that

$$(K \setminus U) \cap f^{a_0} \left( \bigcap_{i=0}^{a_0+b_0} f^{-i}(K) \right) = \emptyset.$$ 

This can be rephrased as

$$\bigcap_{i=0}^{a_0+b_0} f^{-i}(K) \subset f^{-a_0}(U).$$

In particular,

$$\bigcap_{i=0}^{a_0+b_0} f^{-i}(E) \subset f^{-a_0}(E').$$

We can similarly prove that there exist $a'_0, b'_0 \in \mathbb{N}$ with $a'_0 \leq b'_0$ such that

$$\bigcap_{i'=0}^{a'_0+b'_0} f^{-i'}(E') \subset f^{-a'_0}(E).$$

□

**Definition 5.9.** Let $f: X \rightarrow X$ be a continuous partial self-map on a locally compact Hausdorff space $X$. Let $S$ be an $f$-invariant subset of $X$. We say that a neighbourhood $E$ of $S$ is an index neighbourhood if it is isolating and $f$-compactifiable.
Theorem 5.10. Let $f : X \rightarrow X$ be a continuous partial self-map on a locally compact Hausdorff space $X$. Let $S$ be an isolated $f$-invariant subset of $X$. Then, the set of index neighbourhoods of $S$ forms a neighbourhood base for $S$ (in particular, there exists an index neighbourhood of $S$).

Proof. Take any neighbourhood $N$ of $S$. Since $S$ is compact by Lemma 5.7 and $X$ is locally compact Hausdorff, we can take a compact neighbourhood $K$ of $S$ contained in $N$. Let $U = K^o$ the interior of $K$. By Remark 5.6, we may assume that $K$ and $U$ are isolating neighbourhoods of $S$. By Theorem 5.8, we have $K \sim f U$. Take $(a, b, c) \in A_f(K, U)$ arbitrarily, and put

$$E = \bigcap_{i=0}^{a+b} f^{-i}(K) \cap \bigcup_{i'=a}^{a+b+c} f^{-i'}(U).$$

By Remark 5.6, $E$ is an isolating neighbourhood of $S$. We have $E \subset N$. Since $K \subset \text{Dom } f$, we see that $K \cap f^{-1}(K)$ is closed in $K$, hence compact. Thus, the induced partial self-map $f_K : K \rightarrow K$ is proper (Lemma A.7). Since $U \subset \text{Dom } f$, we see that $U \cap f^{-1}(U)$ is open in $U$, i.e. the induced partial self-map $f_U : U \rightarrow U$ is openly defined. Also, $E$ is locally closed in $X$. Now, it is enough to prove the following lemma:

Lemma 5.11. Let $f : X \rightarrow X$ be a continuous partial self-map on a topological space $X$. Let $E$ and $E'$ be two subsets of $X$ and $(a, b, c) \in A_f(E, E')$. Assume that $f_E : E \rightarrow E$ is proper and $f_{E'} : E' \rightarrow E'$ is openly defined. Put

$$E'' = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcup_{i'=a}^{a+b+c} f^{-i'}(E').$$

Then, the induced partial self-map $f_{E''} : E'' \rightarrow E''$ is proper and openly defined.

Remark 5.12. In the setting of Lemma 5.11, $E''$ satisfies $E'' \sim f E$ (and hence, $E'' \sim f E'$). Indeed, we have $E'' \subset E$ and

$$\bigcap_{i=0}^{a+2b+c} f^{-i}(E) \subset E''.$$

Proof of Lemma 5.11. Let us prove that $f_{E''}$ is proper. Consider the following commutative diagram:

$$
\begin{array}{ccc}
E'' \cap f^{-1}(E'') & \xrightarrow{f_{E''}} & E'' \\
\downarrow & & \downarrow \\
E \cap f^{-1}(E) & \xrightarrow{f_E} & E.
\end{array}
$$

Since the bottom map is proper, in order to see that the top map is proper, it is enough to verify that the square is a pullback, i.e. $f_{E}^{-1}(E'') = E'' \cap f^{-1}(E'')$ (Lemma A.3). We see that

$$f_{E}^{-1}(E'') = \bigcap_{i=0}^{a+b+1} f^{-i}(E) \cap \bigcup_{i'=a+1}^{a+b+c+1} f^{-i'}(E').$$
and
\[ E'' \cap f^{-1}(E') = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E'). \]

Since
\[ \bigcap_{i=0}^{a+b+1} f^{-i}(E) \subseteq \bigcap_{i=0}^{a+b} f^{-i}(E) \subseteq f^{-a}(E'), \]
we obtain
\[ \bigcap_{i=0}^{a+b+1} f^{-i}(E) = \bigcap_{i=0}^{a+b+1} f^{-i}(E) \cap f^{-a}(E'), \]
and
\[ \bigcap_{i=0}^{a+b+1} f^{-i}(E) \cap \bigcap_{i' = a+1}^{a+b+c+1} f^{-i'}(E') = \bigcap_{i=0}^{a+b+1} f^{-i}(E) \cap \bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E'). \]
We have thus proved \( f_{E''}^{-1}(E') = E'' \cap f^{-1}(E'). \)

Let us prove that \( f_{E''} \) is openly defined. Consider the following diagram:

\[
\begin{array}{ccc}
E'' \cap f^{-1}(E'') & \xrightarrow{f_{a+b+c}} & E'' \\
\downarrow & & \downarrow \quad f_{a+b+c} \\
E' \cap f^{-1}(E') & \xrightarrow{f_{a+b+c}} & E'.
\end{array}
\]

Since \( E' \cap f^{-1}(E') \) is open in \( E' \), in order to see \( E'' \cap f^{-1}(E'') \) is open in \( E'' \), it is enough to verify that the square is a pullback, i.e. \( E'' \cap f^{-(a+b+c)}(E' \cap f^{-1}(E')) = E'' \cap f^{-1}(E'') \). We see that
\[
E'' \cap f^{-(a+b+c)}(E' \cap f^{-1}(E')) = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E')
\]
and
\[
E'' \cap f^{-1}(E'') = \bigcap_{i=0}^{a+b+1} f^{-i}(E) \cap \bigcap_{i' = a+1}^{a+b+c+1} f^{-i'}(E').
\]
Since
\[
\bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E') \subseteq f^{-(a+1)} \left( \bigcap_{i' = a}^{b+c} f^{-i'}(E') \right) \subseteq f^{-(a+1)}(f^{-b}(E)) = f^{-(a+b+1)}(E),
\]
we obtain
\[
\bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E') = f^{-(a+b+1)}(E) \cap \bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E')
\]
and
\[
\bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E') = \bigcap_{i=0}^{a+b+1} f^{-i}(E) \cap \bigcap_{i' = a}^{a+b+c+1} f^{-i'}(E').
\]
We have thus proved \( E'' \cap f^{-(a+b+c)}(E' \cap f^{-1}(E')) = E'' \cap f^{-1}(E''). \) \( \square \)
The proof of Theorem 5.10 is completed. □

**Definition 5.13.** Let \( f : X \rightarrow X \) be a continuous partial self-map on a locally compact Hausdorff space \( X \). Let \( S \) be an isolated \( f \)-invariant subset of \( X \). We define the *Conley index of \( S \) relative to \( f \)* to be \( \text{Conl}_f(E) \), where \( E \) is an index neighbourhood of \( S \).

By Theorems 5.8 and 5.10, the full subcategory of \( \text{CptifSub}_f \) consisting of all index neighbourhoods of \( S \) is equivalent to the category with one object and one morphism. Thus, the Conley index \( \text{Conl}_f(E) \) is well-defined as an object of \( \text{Sz}(\text{CHaus}^{\ast}) \) up to unique isomorphism.

**Example 5.14.** There are \( \sim_f \)-equivalence classes of \( f \)-compactifiable subsets that cannot be obtained from Theorems 5.8 and 5.10:

1. Fix \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \). Define a continuous map \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by
   \[
   f(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta).
   \]
   For each \( r \in \mathbb{R}_{\geq 0} \),
   \[
   U_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}
   \]
   and
   \[
   K_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}
   \]
   are both \( f \)-compactifiable subsets of \( \mathbb{R}^2 \).
2. Define a continuous map \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by \( f(x, y) = (x, y + 1) \). For each continuous function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \),
   \[
   E_{\varphi} = \{(x, y) \in \mathbb{R}^2 \mid y < \varphi(x)\}
   \]
   is an \( f \)-compactifiable subset of \( \mathbb{R}^2 \). Note that a pair of two continuous functions \( \varphi, \psi : \mathbb{R} \rightarrow \mathbb{R} \) satisfies \( \sim_f \) if and only if \( \varphi - \psi \) is a bounded function.

**Part 2. The continuous time case**

6. A continuous time analogue of the Szymczak category

**Definition 6.1.**

1. Let \( (X, x_0) \) be a based set and
   \[
   F : (\mathbb{R}_{\geq 0} \times X, \mathbb{R}_{\geq 0} \times \{x_0\}) \rightarrow (X, x_0)
   \]
   a map. For each \( t \in \mathbb{R}_{\geq 0} \), we write \( f^t : (X, x_0) \rightarrow (X, x_0) \) for the based map defined by \( f^t(x) = F(t, x) \). We say that \( F \) is a *based semiflow* on \( X \) if the following two conditions are satisfied:
   - \( f^t f^u = f^{t+u} \) (\( t, u \in \mathbb{R}_{\geq 0} \)).
   - \( f^0 = \text{id}_X \).
2. Let \( (X, x_0) \) be a based compact Hausdorff space. We say that a based semiflow \( F \) on \( X \) is *continuous* if it is continuous as a map from \( \mathbb{R}_{\geq 0} \times X \) to \( X \).

We write \( \text{SFlow}(\text{Set}_\ast) \) for the category of based semiflows on sets, i.e.

- An object of \( \text{SFlow}(\text{Set}_\ast) \) is a based semiflow on a set.
Definition 6.2. 

A morphism from a based semiflow $F$ on $(X, x_0)$ to a based semiflow $G$ on $(Y, y_0)$ in $\text{SFlow}(\text{Set}_*)$ is a based map $\varphi: (X, x_0) \rightarrow (Y, y_0)$ such that $\varphi f^t = g^t \varphi$ for any $t \in \mathbb{R}_{\geq 0}$.

Similarly, we write $\text{SFlow}(\text{CHaus}_*)$ for the category of based continuous semiflows on compact Hausdorff spaces, i.e.

- An object of $\text{SFlow}(\text{Set}_*)$ is a based continuous semiflow on a compact Hausdorff space.
- A morphism from a based continuous semiflow $F$ on $(X, x_0)$ to a based continuous semiflow $G$ on $(Y, y_0)$ in $\text{SFlow}(\text{CHaus}_*)$ is a based continuous map $\varphi: (X, x_0) \rightarrow (Y, y_0)$ such that $\varphi f^t = g^t \varphi$ for any $t \in \mathbb{R}_{\geq 0}$.

To define the point-set level Conley index in the continuous time setting, we need an analogue of the Szymczak category, which uses the category of based semiflows instead of the category of endomorphisms.

Definition 7.1.

(1) We define a category $\text{Sz}_{\text{cont}}(\text{Set}_*)$ as follows:

- An object of $\text{Sz}_{\text{cont}}(\text{Set}_*)$ is a based semiflow on a set.
- For each two based semiflows $F$ and $G$ on sets, the hom-set $\text{Sz}_{\text{cont}}(\text{Set}_*)(F, G)$ is given by
  $$\text{Sz}_{\text{cont}}(\text{Set}_*)(F, G) = (\text{SFlow}(\text{Set}_*)(F, G) \times \mathbb{R}_{\geq 0}) / \sim,$$
  where $(\varphi, t) \sim (\varphi', t')$ if and only if there exists $s \in \mathbb{R}_{\geq 0}$ such that $\varphi f^{t+s} = \varphi' f^{t+s}$ (or equivalently, $g^{t+s} \varphi = g^{t+s} \varphi'$).
- The composition of two morphisms $(\varphi, t) \in \text{Sz}_{\text{cont}}(\text{Set}_*)(F, G)$ and $(\psi, u) \in \text{Sz}_{\text{cont}}(\text{Set}_*)(G, H)$ is given by $(\psi \varphi, t + u)$, which does not depend on the choice of the representatives.

(2) We define a category $\text{Sz}_{\text{cont}}(\text{CHaus}_*)$ as follows:

- An object of $\text{Sz}_{\text{cont}}(\text{CHaus}_*)$ is a based continuous semiflow on a compact Hausdorff space.
- For each two based continuous semiflows $F$ and $G$ on compact Hausdorff spaces, the hom-set $\text{Sz}_{\text{cont}}(\text{CHaus}_*)(F, G)$ is given by
  $$\text{Sz}_{\text{cont}}(\text{CHaus}_*)(F, G) = (\text{SFlow}(\text{CHaus}_*)(F, G) \times \mathbb{R}_{\geq 0}) / \sim,$$
  where $(\varphi, t) \sim (\varphi', t')$ if and only if there exists $s \in \mathbb{R}_{\geq 0}$ such that $\varphi f^{t+s} = \varphi' f^{t+s}$ (or equivalently, $g^{t+s} \varphi = g^{t+s} \varphi'$).
- The composition of two morphisms $(\varphi, t) \in \text{Sz}_{\text{cont}}(\text{CHaus}_*)(F, G)$ and $(\psi, u) \in \text{Sz}_{\text{cont}}(\text{CHaus}_*)(G, H)$ is given by $(\psi \varphi, t + u)$, which does not depend on the choice of the representatives.

7. Partial semiflows on sets

Definition 7.1. Let $X$ be a set and $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ a partial map. For each $t \in \mathbb{R}_{\geq 0}$, we write $f^t: X \rightarrow X$ for the partial map defined by

$$\text{Dom } f^t = \{ x \in X \mid (t, x) \in \text{Dom } F \}, \quad f^t(x) = F(t, x).$$
We say that $F$ is a partial semiflow on $X$ if the following two conditions are satisfied:

- $f^t f^u = f^{t+u}$ ($t, u \in \mathbb{R}_{\geq 0}$).
- $f^0 = \text{id}_X$.

**Definition 7.2.** We define a category $\text{SFlow}(\text{Set})$ as follows:

- An object of $\text{SFlow}(\text{Set})$ is a partial semiflow on a set.
- A morphism from a partial semiflow $F$ on $X$ to a partial semiflow $G$ on $Y$ in $\text{SFlow}(\text{Set})$ is a partial map $\varphi: X \rightarrow Y$ such that $\varphi f^t = g^t \varphi$ for any $t \in \mathbb{R}_{\geq 0}$.

**Definition 7.3.** We define a category $\text{SFlow}(\text{Set}_*)$ by assigning to each partial semiflow $F$ on $X$ the based semiflow $F^+$ on $X^+$ defined by

$$ F^+(t, x) = \begin{cases} F(t, x) & \text{if } (t, x) \in \text{Dom } F, \\ \ast & \text{(otherwise)}. \end{cases} $$

**Proposition 7.4.** $(-)^+: \text{SFlow}(\text{Set}) \rightarrow \text{SFlow}(\text{Set}_*)$ is an equivalence of categories.

*Proof.* Define a functor $U: \text{SFlow}(\text{Set}_*) \rightarrow \text{SFlow}(\text{Set})$ by assigning to each based semiflow $F$ on $(X, x_0)$ the partial semiflow $UF$ on $X \setminus \{x_0\}$ defined by

$$ \text{Dom}(UF) = F^{-1}(X \setminus \{x_0\}), \quad (UF)(t, x) = F(t, x). $$

Then, $U$ is an inverse of $(-)^+$. \qed

**Definition 7.5.** Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a partial semiflow on a set $X$. For each subset $E$ of $X$, we define a partial semiflow $F_E: \mathbb{R}_{\geq 0} \times E \rightarrow E$ on $E$ by

$$ \text{Dom } F_E = \left\{ (t, x) \in \mathbb{R}_{\geq 0} \times X \mid x \in \bigcap_{s \in [0, t]} f^{-s}(E) \right\}, \quad F_E(t, x) = F(t, x), $$

and call it the induced partial semiflow on $E$.

In this paper, by $f^t_E$ ($t \in \mathbb{R}_{\geq 0}$), we always mean $(f_E)^t$ (i.e. the time-$t$ partial map for the partial semiflow $F_E$), not $(f^t)_E$. Thus,

$$ \text{Dom } f^t_E = \{ x \in X \mid (t, x) \in \text{Dom } F_E \} = \bigcap_{s \in [0, t]} f^{-s}(E), \quad f^t_E(x) = f^t(x). $$

**Definition 7.6.** Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a partial semiflow on a set $X$. Let $E$ and $E'$ be two subsets of $X$.

1. We say that a triple $(a, b, c) \in \mathbb{R}_{\geq 0}^3$ is $(E, E')$-admissible relative to $F$ if

$$ \bigcap_{t \in [0, a+b]} f^{-t}(E) \subset f^{-a}(E'), \quad \bigcap_{t \in [0, b+c]} f^{-t}(E') \subset f^{-b}(E). $$

We write $A_F(E, E') \subset \mathbb{R}_{\geq 0}^3$ for the set of all $(E, E')$-admissible triples relative to $F$.

2. We use the notation $E \sim_F E'$ to mean $A_F(E, E') \neq \emptyset$. 

Remark 7.7. The relation $E \sim_F E'$ holds if and only if there exist $a, a', b, b' \in \mathbb{R}_{\geq 0}$ such that
\[
\bigcap_{t \in [0, a+b]} f^{-t}(E) \subset f^{-a}(E'), \quad \bigcap_{t' \in [0, a'+b']} f^{-t'}(E') \subset f^{-a'}(E).
\]

Lemma 7.8. Let $F: \mathbb{R}_{\geq 0} \times X \to X$ be a partial semiflow on a set $X$.
(1) For any subset $E$ of $X$, we have $A_F(E, E) = \mathbb{R}_{\geq 0}^3$. In particular, $(0, 0, 0) \in A_F(E, E)$.
(2) Let $E, E'$, and $E''$ be three subsets of $X$. If $(a, b, c) \in A_F(E, E')$ and $(a', b', c') \in A_F(E', E'')$, then $(a + a', b + b', c + c') \in A_F(E, E'')$.

Proof. The same as the proof of Lemma 7.7. □

It follows from Remark 7.7 and Lemma 7.8 that $\sim_F$ is an equivalence relation on the set of all subsets of $X$.

Definition 7.9. Let $F: \mathbb{R}_{\geq 0} \times X \to X$ be a partial semiflow on a set $X$. Let $E$ and $E'$ be two subsets of $X$. For each $(a, b, c) \in A_F(E, E')$, define a partial map $f^{(a,b,c)}_{E' E}: E \to E'$ by
\[
\text{Dom} f^{(a,b,c)}_{E' E} = \bigcap_{t \in [0, a+b]} f^{-t}(E) \cap \bigcap_{t' \in [a, a+b+c]} f^{-t'}(E'),
\]
\[
f^{(a,b,c)}_{E' E}(x) = f^{a+b+c}(x).
\]

Theorem 7.10. Let $F: \mathbb{R}_{\geq 0} \times X \to X$ be a partial semiflow on a set $X$.
(1) Let $E$ be a subset of $X$. Then, for any $(a, b, c) \in \mathbb{R}_{\geq 0}^3$, we have
\[
f^{(a,b,c)}_{E E} = f^{a+b+c}_E.
\]
In particular, $f^{(0,0,0)}_{E E} = \text{id}_E$.
(2) Let $E, E'$, and $E''$ be three subsets of $X$. Then, for any $(a, b, c) \in A_F(E, E')$ and $(a', b', c') \in A_F(E', E'')$, we have
\[
f^{(a', b', c')}_{E'' E'} \circ f^{(a,b,c)}_{E' E} = f^{(a+a', b+b', c+c')}_{E'' E}.
\]
(3) Let $E$ and $E'$ be two subsets of $X$. Then, for any $(a, b, c) \in A_F(E, E')$ and any $t \in \mathbb{R}_{\geq 0}$, we have
\[
f^{(a,b,c)}_{E E} \circ f^t_E = f^t_{E'} \circ f^{(a,b,c)}_{E' E}.
\]
(4) Let $E$ and $E'$ be two subsets of $X$. Then, for any $(a, b, c), (a', b', c') \in A_F(E, E')$, we have
\[
f^{(a,b,c)}_{E E} \circ f^{a'+b'+c'}_E = f^{(a', b', c')}_E \circ f^{a+b+c}_E.
\]

Proof. The same as the proof of Theorem 3.9. □

Definition 7.11. Let $F: \mathbb{R}_{\geq 0} \times X \to X$ be a partial semiflow on a set $X$.
(1) We define a small category $\tilde{\text{Sub}}_F$ as follows:
- An object of $\tilde{\text{Sub}}_F$ is a subset of $X$.
- For each two subsets $E$ and $E'$ of $X$, the hom-set $\tilde{\text{Sub}}_F(E, E')$ is given by $\tilde{\text{Sub}}_F(E, E') = A_F(E, E')$.
- The composition of two morphisms is given by addition.
(2) We define a functor $\widetilde{\text{Conl}}_F : \widetilde{\text{Sub}}_F \to \text{SFlow} \text{(Set)}^*$ as follows:

- For each subset $E$ of $X$, we put $\widetilde{\text{Conl}}_F(E) = F^+_E$.
- For each two subsets $E$ and $E'$ of $X$, we assign to $(a, b, c) \in A_{F}(E, E')$ the morphism $(f^{(a,b,c)}_{E', E})^+ : F^+_E \to F^+_E$.

As in Definition 3.10, we can see that $\widetilde{\text{Sub}}_F$ is a category and $\widetilde{\text{Conl}}_F$ is a functor.

Definition 7.12. Let $F : R^+ \times X \rightharpoonup X$ be a partial semiflow on a set $X$.

1. We define a small category $\text{Sub}_F$ as follows:
   - An object of $\text{Sub}_F$ is a subset of $X$.
   - For each two subsets $E$ and $E'$ of $X$, the hom-set $\text{Sub}_F(E, E')$ is given by
     \[
     \text{Sub}_F(E, E') = \begin{cases} 
     \ast & \text{(if } E \sim_F E'), \\
     \emptyset & \text{(otherwise)}
     \end{cases}
     \]

2. We define a functor $\text{Conl}_F : \text{Sub}_F \to \text{SCont} \text{(Set)}^*$ as follows:
   - For each subset $E$ of $X$, we put $\text{Conl}_F(E) = F^+_E$.
   - To each two subsets $E$ and $E'$ of $X$ with $E \sim_F E'$, we assign the morphism $f^+_{E', E} : F^+_E \to F^+_E$ defined by
     \[
     f^+_{E', E} = (f^{(a,b,c)}_{E', E})^+, a + b + c,
     \]
     where $(a, b, c) \in A_{F}(E, E')$.

As in Definition 3.11, we can see that $\text{Conl}_F$ is a well-defined functor.

Remark 7.13. Since $\sim_F$ is a symmetric relation, every morphism in $\text{Sub}_F$ is an isomorphism.

8. Continuous partial semiflows on topological spaces

Definition 8.1. Let $X$ be a topological space. We say that a partial semiflow $F$ on $X$ is continuous if $F$ is continuous as a partial map from $R^+ \times X$ to $X$.

Definition 8.2. Let $F : R^+ \times X \rightharpoonup X$ be a continuous partial semiflow on a topological space $X$.

1. We say that $F$ is finite-time proper if, for any $t \in R^+$, the restriction $F|_{[0, t] \times X} : [0, t] \times X \rightharpoonup X$ is proper.
2. We say that $F$ is openly defined if it is openly defined as a continuous partial map from $R^+ \times X$ to $X$.

Definition 8.3. We define a category $\text{SFlow}(LCHaus^0)$ as follows:

- An object of $\text{SFlow}(LCHaus^0)$ is a finite-time proper and openly defined partial semiflow on a locally compact Hausdorff space.
- A morphism from $F : R^+ \times X \rightharpoonup X$ to $G : R^+ \times Y \rightharpoonup Y$ in $\text{SFlow}(LCHaus^0)$ is a proper and openly defined partial map $\varphi : X \rightharpoonup Y$ such that $\varphi f^t = g^t \varphi$ for any $t \in R^+$.

The significance of finite-time proper and openly defined partial semiflows is that we can define the following functor:
Definition 8.4. We define a functor \((-)^+\): SFlow(LCHaus$^0$) $\to$ SFlow(CHaus$_*$) by assigning to each finite-time proper and openly defined partial semiflow $F$ on $X$ the continuous based semiflow $F^+$ on $X^+$ defined by

$$F^+(t, x) = \begin{cases} F(t, x) & \text{if } (t, x) \in \text{Dom } F, \\ \ast & \text{(otherwise)} \end{cases}$$

Let us verify that $F^+$ is indeed continuous. Suppose that $K$ is a closed subset of $X^+$ that does not contain $. For each $t \in \mathbb{R}_{\geq 0}$, we have

$$(F^+)^{-1}(K) \cap ([0, t] \times X) = (F|_{[0, t] \times X})^{-1}(K).$$

Since $K$ is a compact subset of $X$ and $F|_{[0, t] \times X} : \text{Dom}(F|_{[0, t] \times X}) \to X$ is a proper map, $(F^+)^{-1}(K) \cap ([0, t] \times X)$ is compact, hence closed in $[0, t] \times X^+$. Since $t$ is arbitrary, we conclude that $(F^+)^{-1}(K)$ is closed in $\mathbb{R}_{\geq 0} \times X^+$. Suppose that $K$ is a closed subset of $X^+$ that contains $. We have

$$(F^+)^{-1}(K) = F^{-1}(K \setminus \{\ast\}) \cup (\mathbb{R}_{\geq 0} \times X \setminus \text{Dom } F) \cup (\mathbb{R}_{\geq 0} \times \{\ast\}).$$

Since $K \setminus \{\ast\}$ is a closed subset of $X$ and Dom $F$ is an open subset of $\mathbb{R}_{\geq 0} \times X$, we see that $F^{-1}(K \setminus \{\ast\}) \cup (\mathbb{R}_{\geq 0} \times X \setminus \text{Dom } F)$ is closed in $\mathbb{R}_{\geq 0} \times X$. Thus, $(F^+)^{-1}(K)$ is closed in $\mathbb{R}_{\geq 0} \times X^+$.

Proposition 8.5. \((-)^+\): SFlow(LCHaus$^0$) $\to$ SFlow(CHaus$_*$) is an equivalence of categories.

Proof. Define a functor $U : \text{SFlow(CHaus$_*$)} \to \text{SFlow(LCHaus$^0$)}$ by assigning to each based continuous semiflow $F$ on $(X, x_0)$ the partial semiflow $UF$ on $X \setminus \{x_0\}$ defined by

$$\text{Dom}((UF)) = F^{-1}(X \setminus \{x_0\}), \quad (UF)(t, x) = F(t, x),$$

which is finite-time proper and openly defined. Then, $U$ is an inverse of \((-)^+\).

For the sake of completeness, let us verify that $UF$ is indeed finite-time proper and openly defined. Take any $t \in \mathbb{R}_{\geq 0}$. We have

$$\text{Dom}((UF)|_{[0, t] \times (X \setminus \{x_0\})}) = F^{-1}(X \setminus \{x_0\}) \cap ([0, t] \times X).$$

Consider the following pullback diagram:

$$\begin{array}{ccc}
\text{Dom}((UF)|_{[0, t] \times X}) & \xrightarrow{(UF)|_{[0, t] \times (X \setminus \{x_0\})}} & X \setminus \{x_0\} \\
\downarrow & & \downarrow \\
[0, t] \times X & \xrightarrow{F} & X.
\end{array}$$

The bottom map is proper since it is a continuous map from a compact Hausdorff space (Lemma [A.7]). Thus, the top map is also proper (Lemma [A.3]). Since $X \setminus \{x_0\}$ is open in $X$, the inverse image $F^{-1}(X \setminus \{x_0\}) (= \text{Dom}(UF))$ is open in $\mathbb{R}_{\geq 0} \times X$. Since Dom$(UF)$ is contained in $\mathbb{R}_{\geq 0} \times (X \setminus \{x_0\})$, it is also open in $\mathbb{R}_{\geq 0} \times (X \setminus \{x_0\})$. \(\square\)

Let us prove some basic properties of finite-time proper partial semiflows and openly defined partial semiflows.
**Lemma 8.6.** Let \( F : \mathbb{R}_{\geq 0} \times X \to X \) be a continuous partial semiflow on a topological space \( X \).

1. If \( F \) is finite-time proper, \( f^t \) is proper for any \( t \in \mathbb{R}_{\geq 0} \).
2. If \( F \) is openly defined, \( f^t \) is openly defined for any \( t \in \mathbb{R}_{\geq 0} \).

**Proof.** (1) Let \( t : X \to [0,t] \times X \) be the closed inclusion defined by \( t(x) = (t,x) \). We can factorize \( f^t : X \to X \) as follows:

\[
X \ni [0,t] \times X \overset{F}{\longrightarrow} X.
\]
Since \( t \) and \( F \mid_{[0,t] \times X} \) are both proper, their composite \( f^t \) is also proper.

(2) Let \( t : X \to \mathbb{R}_{\geq 0} \times X \) be the closed inclusion defined by \( t(x) = (t,x) \). Since \( \text{Dom} f^t = i^{-1}(\text{Dom} F) \), we see that \( \text{Dom} f^t \) is open in \( X \). □

**Lemma 8.7.** Let \( F : \mathbb{R}_{\geq 0} \times X \to X \) be a continuous partial semiflow on a topological space \( X \).

1. \( F \) is finite-time proper if and only if \( F \mid_{[0,\varepsilon] \times X} \) is proper for some \( \varepsilon \in \mathbb{R}_{\geq 0} \).
2. \( F \) is openly defined if and only if \( F \mid_{[0,\varepsilon] \times X} \) is openly defined for some \( \varepsilon \in \mathbb{R}_{\geq 0} \).

**Proof.** (1) The ‘only if’ part is trivial. Let us prove the ‘if’ part. For each \( n \in \mathbb{N} \), put

\[
F_n = F \mid_{[n\varepsilon,(n+1)\varepsilon] \times X} : [n\varepsilon,(n+1)\varepsilon] \times X \to X.
\]
We first prove that \( F_n \) is a proper partial map for any \( n \in \mathbb{N} \). By assumption, \( F_0 \) is proper. Suppose that \( F_n \) is proper. Let \( \varepsilon \in \mathbb{R} \). Then, we can factorize \( F_{n+1} \) as follows:

\[
[(n+1)\varepsilon,(n+2)\varepsilon] \times X \xrightarrow{g_{n+1} \times \text{id}_X} [n\varepsilon,(n+1)\varepsilon] \times X \xrightarrow{F_n} X \xrightarrow{F_0} X.
\]
This shows that \( F_{n+1} \) is proper. The induction is completed.

Take any \( s \in \mathbb{R}_{\geq 0} \). Choose \( N \in \mathbb{N} \) so that \( N\varepsilon \geq s \). Patching \( F_{n+1} \) for \( 0 \leq n \leq N - 1 \), we see that

\[
F \mid_{[0,N\varepsilon] \times X} : [0,N\varepsilon] \times X \to X
\]
is proper (Lemma A.8). Now, \( F \mid_{[0,s] \times X} : [0,s] \times X \to X \) is factorized as

\[
[0,s] \times X \hookrightarrow [0,N\varepsilon] \times X \overset{F \mid_{[0,N\varepsilon] \times X}}{\longrightarrow} X,
\]
and thus proper.

(2) The ‘only if’ part is trivial. Let us prove the ‘if’ part. As in the proof of (1), we can prove that

\[
F_n = F \mid_{[n\varepsilon,(n+1)\varepsilon] \times X} : [n\varepsilon,(n+1)\varepsilon] \times X \to X.
\]
is openly defined for any \( n \in \mathbb{N} \). Patching \( F_n \) for all \( n \in \mathbb{N} \), we see that \( F \) is openly defined.
Theorem 8.8. Let $F : \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a topological space $X$. Let $E$ and $E'$ be two subsets of $X$. Let $(a, b, c) \in A_F(E, E')$.

1. If $F_E$ and $F_{E'}$ are finite-time proper, $f_{(a,b,c)}^{E,E'}$ is proper.
2. If $F_E$ and $F_{E'}$ are openly defined, $f_{(a,b,c)}^{E,E'}$ is openly defined.

Proof. The same as the proof of Theorem 4.5. We use Lemma 8.6. □

Definition 8.9. Let $F : \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a topological space $X$. Let $E$ and $E'$ be two subsets of $X$. Let $(a, b, c) \in A_F(E, E')$.

1. If $F_E$ and $F_{E'}$ are finite-time proper, $f_{(a,b,c)}^{E,E'}$ is proper.
2. If $F_E$ and $F_{E'}$ are openly defined, $f_{(a,b,c)}^{E,E'}$ is openly defined.

Proof. The same as the proof of Theorem 4.5. We use Lemma 8.6. □

Definition 8.10. Let $F : \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a topological space $X$. Let $E$ and $E'$ be two subsets of $X$. Let $(a, b, c) \in A_F(E, E')$.

1. If $F_E$ and $F_{E'}$ are finite-time proper, $f_{(a,b,c)}^{E,E'}$ is proper.
2. If $F_E$ and $F_{E'}$ are openly defined, $f_{(a,b,c)}^{E,E'}$ is openly defined.

Proof. The same as the proof of Theorem 4.5. We use Lemma 8.6. □

9. Some Lemmas

In this section, we prepare some lemmas needed in the next section.

Lemma 9.1. Let $F : \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a topological space $X$.

1. Let $K$ be a compact Hausdorff subset of $X$ such that $[0, \varepsilon] \times K \subset \text{Dom } F$ for some $\varepsilon \in \mathbb{R}_{> 0}$. Then, the induced partial semiflow $F_K : \mathbb{R}_{\geq 0} \times K \rightarrow K$ is finite-time proper.
2. Let $U$ be an open subset of $X$ such that $[0, \varepsilon] \times U \subset \text{Dom } F$ for some $\varepsilon \in \mathbb{R}_{> 0}$. Then, the induced partial semiflow $F_U : \mathbb{R}_{\geq 0} \times U \rightarrow U$ is openly defined.

Proof. (1) By Lemma 8.7 (1), it suffices to verify that $F_K|[0,\varepsilon] \times K : [0,\varepsilon] \times K \rightarrow K$ is proper. By Lemma 8.7 (1), it suffices to see that this partial map is proper, it is enough to verify that $\text{Dom}(F_K|[0,\varepsilon] \times K)$ is compact Hausdorff. Since $[0,\varepsilon] \times K$ is compact Hausdorff, it suffices to see that $\text{Dom}(F_K|[0,\varepsilon] \times K)$ is closed in $[0,\varepsilon] \times K$. Define a closed subset $\Delta_\varepsilon$ of $[0,\varepsilon] \times [0,\varepsilon]$ and two continuous maps $\pi_1, \pi_2 : \Delta_\varepsilon \rightarrow [0,\varepsilon]$ by $\Delta_\varepsilon = \{(s,t) \in [0,\varepsilon] \times [0,\varepsilon] \mid s \leq t\}$, $\pi_1(s,t) = s$, $\pi_2(s,t) = t$. 


Consider the following diagram:

\[
\begin{array}{ccc}
\Delta_{\varepsilon} \times K & \xrightarrow{\pi_1 \times \text{id}_K} & [0, \varepsilon] \times K \\
\downarrow \quad \pi_2 \times \text{id}_K & & \downarrow \quad F|_{[0,\varepsilon] \times K} \\
[0, \varepsilon] \times K & & X.
\end{array}
\]

Then, we have

\[
\text{Dom}(F_K|_{[0,\varepsilon] \times K}) = ([0, \varepsilon] \times K) \setminus ((\pi_2 \times \text{id}_K)((\pi_1 \times \text{id}_K)^{-1}(([0, \varepsilon] \times K) \setminus F^{-1}(K))).
\]

Since \(\pi_2: \Delta_{\varepsilon} \to [0, \varepsilon]\) is an open map, the base change \(\pi_2 \times \text{id}_X\) is also an open map (Lewis [Lew85, Lem. 1.6]). Thus, \(\text{Dom}(F_K|_{[0,\varepsilon] \times K})\) is closed in \([0, \varepsilon] \times K\).

(2) By Lemma 8.7 (2), it suffices to verify that

\[
\text{Dom}(F_U|_{[0,\varepsilon] \times U}) = \left\{ (t, x) \in [0, \varepsilon] \times U \mid x \in \bigcap_{s \in [0, t]} f^{-s}(E) \right\}
\]

is open in \([0, \varepsilon] \times U\). Let \(\Delta_{\varepsilon}\) and \(\pi_1, \pi_2: \Delta_{\varepsilon} \ni [0, \varepsilon]\) be as in the proof of (1). Consider the following diagram:

\[
\begin{array}{ccc}
\Delta_{\varepsilon} \times U & \xrightarrow{\pi_1 \times \text{id}_U} & [0, \varepsilon] \times U \\
\downarrow \quad \pi_2 \times \text{id}_U & & \downarrow \quad F|_{[0,\varepsilon] \times U} \\
[0, \varepsilon] \times U & & X.
\end{array}
\]

Then, we have

\[
\text{Dom}(F_U|_{[0,\varepsilon] \times U}) = ([0, \varepsilon] \times U) \setminus ((\pi_2 \times \text{id}_U)((\pi_1 \times \text{id}_U)^{-1}(([0, \varepsilon] \times U) \setminus F^{-1}(U))).
\]

Since \(\pi_2: \Delta_{\varepsilon} \to [0, \varepsilon]\) is a proper map, the base change \(\pi_2 \times \text{id}_X\) is a closed map (Lemma A.5). Thus, \(\text{Dom}(F_U|_{[0,\varepsilon] \times U})\) is open in \([0, \varepsilon] \times U\). \qed

Lemma 9.2. Let \(F: \mathbb{R}_{\geq 0} \times X \to X\) be a continuous partial semiflow on a topological space \(X\). Let \(K\) be a compact Hausdorff subset of \(X\) such that \([0, \varepsilon] \times K \subset \text{Dom} F\) for some \(\varepsilon \in \mathbb{R}_{> 0}\). Then, for any \(a, b \in \mathbb{R}_{\geq 0}\),

\[
f^a\left( \bigcap_{t \in [0, a+b]} f^{-t}(K) \right)
\]

is a compact Hausdorff subset of \(X\).

Proof. By Lemma 9.1 (1), the induced partial semiflow \(F_K: \mathbb{R}_{\geq 0} \times K \to K\) is finite-time proper. By Lemma 8.5 (1), the induced partial self-map \(f^{a+b}_K: K \to K\) is proper. Since \(K\) is compact Hausdorff, the inverse image

\[
f^{-(a+b)}_K(K) = \text{Dom} f^{a+b}_K = \bigcap_{t \in [0, a+b]} f^{-t}(K)
\]

is a compact Hausdorff subset of \(X\).
must be compact Hausdorff. Thus,
\[ f^a \left( \bigcap_{t \in [0, a+b]} f^{-t}(K) \right) \]
is also compact Hausdorff. \qed

Remark 9.3. The assumption that \([0, \varepsilon] \times K \subset \text{Dom } F\) (resp. \([0, \varepsilon] \times U \subset \text{Dom } F\)) in Lemmas 9.1 (1) and 9.2 (resp. Lemma 9.1 (2)) is very mild. For example, this trivially holds when \(F\) is an everywhere defined map. We also have the following result:

**Lemma 9.4.** Let \(F: \mathbb{R}_{\geq 0} \times X \rightarrow X\) be a continuous partial semiflow on a topological space \(X\). Suppose that \(F\) is openly defined. Then, for any relatively compact subset \(E\) of \(X\), there exists \(\varepsilon \in \mathbb{R}_{> 0}\) such that \([0, \varepsilon] \times E \subset \text{Dom } F\).

**Proof.** Let \(K = \overline{E}\) be the closure of \(E\). It suffices to prove that \([0, \varepsilon] \times K \subset \text{Dom } F\) for some \(\varepsilon \in \mathbb{R}_{> 0}\). Since \(\text{Dom } F\) is open in \(\mathbb{R}_{\geq 0} \times X\) and contains \(\{0\} \times X\), we have
\[
\bigcup_{t \in \mathbb{R}_{> 0}} \text{Dom } f^t = X \cup K.
\]
It follows from Lemma 8.6 (2) that \(\text{Dom } f^t\) is open in \(X\) for each \(t \in \mathbb{R}_{> 0}\). Since \(K\) is compact and the family \(\text{Dom } f^t\) is decreasing with respect to \(t\), there exists \(\varepsilon \in \mathbb{R}_{> 0}\) such that \(K \subset \text{Dom } f^\varepsilon\). This is equivalent to saying that \([0, \varepsilon] \times K \subset \text{Dom } F\). \qed

10. The Conley Index of Isolated \(F\)-Invariant Subsets

**Definition 10.1.** Let \(F: \mathbb{R}_{\geq 0} \times X \rightarrow X\) be a partial semiflow on a set \(X\). We say that a subset \(S\) of \(X\) is \(F\)-invariant if \(S \subset \text{Dom } f^t\) and \(f^t(S) = S\) hold for any \(t \in \mathbb{R}_{\geq 0}\).

**Definition 10.2.** Let \(F: \mathbb{R}_{\geq 0} \times X \rightarrow X\) be a partial semiflow on a set \(X\). Let \(E\) be a subset of \(X\). We define the \(F\)-invariant part \(I_E(F)\) of \(E\) by
\[
I_E(F) = \bigcap_{a, b \in \mathbb{R}_{\geq 0}} f^a \left( \bigcup_{t \in [0, a+b]} f^{-t}(E) \right).
\]

**Remark 10.3.** The family
\[
\left( f^a \left( \bigcap_{t \in [0, a+b]} f^{-t}(E) \right) \right)_{a, b \in \mathbb{R}_{\geq 0}}
\]
is decreasing in the following sense: for any \(a, b, a', b' \in \mathbb{R}_{\geq 0}\) satisfying \(a \leq a'\) and \(b \leq b'\), we have
\[
f^{a'} \left( \bigcup_{t \in [0, a'+b']} f^{-t}(E) \right) \subset f^a \left( \bigcup_{t \in [0, a+b]} f^{-t}(E) \right).
\]
The above definition of the \(F\)-invariant part agrees with the usual one, namely:
Lemma 10.4. Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a partial semiflow on a set $X$. Let $E$ be a subset of $X$. Then, $I_F(E)$ is the largest $F$-invariant subset contained in $E$.

Proof. The same as the proof of Lemma 5.4. □

Definition 10.5. Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a locally compact Hausdorff space $X$. Let $S$ be an $F$-invariant subset of $X$.

(1) A neighbourhood $E$ of $S$ is called isolating if the following three conditions are satisfied:
   - $E$ is relatively compact.
   - There exists $\varepsilon \in \mathbb{R}_{> 0}$ such that $[0, \varepsilon] \times \overline{E} \subset \text{Dom } F$.
   - $S = I_F(\overline{E})$.
   Here, $\overline{E}$ denotes the closure of $E$.

(2) $S$ is called isolated if it admits an isolating neighbourhood.

Remark 10.6. By Lemma 9.4, if $F$ is openly defined, the second condition in Definition 10.5 (1) is automatically satisfied.

Remark 10.7. If $E$ is an isolating neighbourhood of $S$, the closure $\overline{E}$ is also isolating. If $E$ is an isolating neighbourhood of $S$ and $E'$ is a neighbourhood of $S$ contained in $E$, then $E'$ is also isolating.

Lemma 10.8. Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a locally compact Hausdorff space $X$. Let $S$ be an isolated $F$-invariant subset of $X$. Then, $S$ is compact.

Proof. The same as the proof of Lemma 5.7. We use Lemma 9.2. □

Theorem 10.9. Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a locally compact Hausdorff space $X$. Let $S$ be an isolated $F$-invariant subset of $X$. Then, for any two isolating neighbourhoods $E$ and $E'$ of $S$, we have $E \sim_F E'$.

Proof. The same as the proof of Theorem 5.8. We use Lemma 9.2. □

Definition 10.10. Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a locally compact Hausdorff space X. Let $S$ be an $F$-invariant subset of $X$. We say that a neighbourhood $E$ of $S$ is an index neighbourhood if it is isolating and $F$-compactifiable.

Theorem 10.11. Let $F: \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a locally compact Hausdorff space $X$. Let $S$ be an isolated $F$-invariant subset of $X$. Then, the set of index neighbourhoods of $S$ forms a neighbourhood base for $S$ (in particular, there exists an index neighbourhood of $S$).

Proof. The proof is similar to that of Theorem 5.10 although slightly more involved.

Take any neighbourhood $N$ of $S$. Since $S$ is compact by Lemma 10.8 and $X$ is locally compact Hausdorff, we can take a compact neighbourhood $K$ of $S$ contained in $N$. Let $U = K^\circ$ the interior of $K$. By Remark 10.7, we may

\[2\text{This Lemma is wrong. See Correction.}\]
assume that $K$ and $U$ are isolating neighbourhoods of $S$. By Theorem 10.9 we have $K \sim_F U$. Take $(a, b, c) \in A_F(K, U)$ arbitrarily, and put
\[ E = \bigcap_{t \in [0, a+b+c]} f^{-t}(K) \cap \bigcap_{t' \in [a, a+b+c]} f^{-t'}(U). \]
Then, $E$ is an isolating neighbourhood of $S$. Indeed,
\[ U' = \bigcap_{t \in [0, a+b+c]} f^{-t}(U) \]
is open in $X$ by Lemmas 8.6 (2) and 9.1 (2), and $S \subset U' \subset E$ holds. We have $E \subset N$. By Lemma 10.11 (1), the induced partial semiflow $F_K : \mathbb{R}_{\geq 0} \times K \rightarrow K$ is finite-time proper. By Lemma 9.1 (2), the induced partial semiflow $F_U : \mathbb{R}_{\geq 0} \times U \rightarrow U$ is openly defined. Also, $E$ is locally closed in $X$. Now, it is enough to prove the following lemma:

**Lemma 10.12.** Let $F : \mathbb{R}_{\geq 0} \times X \rightarrow X$ be a continuous partial semiflow on a topological space $X$. Let $E$ and $E'$ be two subsets of $X$ and $(a, b, c) \in A_F(E, E')$. Assume that $F_E : \mathbb{R}_{\geq 0} \times E \rightarrow E$ is finite-time proper and $F_E' : \mathbb{R}_{\geq 0} \times E' \rightarrow E'$ is openly defined. Put
\[ E'' = \bigcap_{t \in [0, a+b]} f^{-t}(E) \cap \bigcap_{t' \in [a, a+b+c]} f^{-t'}(E'). \]
Then, the induced partial semiflow $F_{E''} : \mathbb{R}_{\geq 0} \times E'' \rightarrow E''$ is finite-time proper and openly defined.

**Remark 10.13.** In the setting of Lemma 10.12 $E''$ satisfies $E'' \sim_F E$ (and hence, $E'' \sim_F E'$). Indeed, we have $E'' \subset E$ and
\[ \bigcap_{t \in [0, a+2b+c]} f^{-t}(E) \subset E''. \]

**Proof of Lemma 10.12.** (1) Let us prove that $F_{E''}$ is finite-time proper. Take any $s \in \mathbb{R}_{\geq 0}$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Dom}(F_{E''}|_{[0, s] \times E''}) & \xrightarrow{F_{E''}|_{[0, s] \times E''}} & E'' \\
\downarrow & & \downarrow \\
\text{Dom}(F_E|_{[0, s] \times E}) & \xrightarrow{F_E|_{[0, s] \times E}} & E.
\end{array}
\]

Since the bottom map is proper, in order to see that the top map is proper, it is enough to verify that the square is a pullback, i.e. $(F_E|_{[0, s] \times E})^{-1}(E'') = \text{Dom}(F_{E''}|_{[0, s] \times E''})$ (Lemma A.3). We see that
\[
(F_E|_{[0, s] \times E})^{-1}(E'') = \left\{ (t, x) \in [0, s] \times X \mid x \in \bigcap_{u \in [0, t]} f^{-u}(E) \cap f^{-t}(E'') \right\}
\]
and

\[
\text{Dom}(F_{E''}|_{[0,s] \times E''}) = \left\{(t, x) \in [0, s] \times X \mid x \in \bigcap_{u' \in [0,t]} f^{-u''}(E'') \right\}
\]

\[
= \left\{(t, x) \in [0, s] \times X \mid x \in \bigcap_{u \in [0,t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a,t+a+b+c]} f^{-u'}(E') \right\}.
\]

Since

\[
\bigcap_{u \in [0,t+a+b]} f^{-u}(E) \subseteq \bigcap_{u' \in [0,t]} f^{-u'}(f^{-u}(E')) = \bigcap_{u' \in [a,t+a]} f^{-u'}(E'),
\]

we obtain

\[
\bigcap_{u \in [0,t+a+b]} f^{-u}(E) = \bigcap_{u \in [0,t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a,t+a]} f^{-u'}(E').
\]

and

\[
\bigcap_{u \in [0,t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a,t+a+b+c]} f^{-u'}(E') = \bigcap_{u \in [0,t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a,t+a+b+c]} f^{-u'}(E').
\]

We have thus proved \((F_E|_{[0,s] \times E})^{-1}(E'') = \text{Dom}(F_{E''}|_{[0,s] \times E''}).\)

Let us prove that \(F_{E''}\) is openly defined. Since \(E'' \subset f^{-(a+b+c)}(E')\), we can define a continuous map \(g: \mathbb{R}_{\geq 0} \times E'' \to \mathbb{R}_{\geq 0} \times E'\) by \(g(t, x) = (t, f^{a+b+c}(x))\). Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Dom} F_{E''} & \longrightarrow & \mathbb{R}_{\geq 0} \times E'' \\
g \downarrow & & \downarrow g \\
\text{Dom} F_{E'} & \longrightarrow & \mathbb{R}_{\geq 0} \times E'.
\end{array}
\]

Since \(\text{Dom} F_{E'}\) is open in \(E'\), in order to see that \(\text{Dom} F_{E''}\) is open in \(E''\), it is enough to verify that the square is a pullback, i.e. \(g^{-1}(\text{Dom} F_{E'}) = \text{Dom} F_{E''}\).

We see that

\[
g^{-1}(\text{Dom} F_{E'}) = g^{-1} \left\{(t, x) \in \mathbb{R}_{\geq 0} \times X \mid x \in \bigcap_{u' \in [0,t]} f^{-u'}(E') \right\}
\]

\[
= \left\{(t, x) \in \mathbb{R}_{\geq 0} \times E'' \mid x \in f^{-(a+b+c)} \left( \bigcap_{u' \in [0,t]} f^{-u'}(E') \right) \right\}
\]

\[
= \left\{(t, x) \in \mathbb{R}_{\geq 0} \times X \mid x \in \bigcap_{u \in [0,a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a,t+a+b+c]} f^{-u'}(E') \right\}
\]
and
\[
\text{Dom } F_{E''} = \left\{ (t, x) \in \mathbb{R}_{\geq 0} \times X \mid x \in \bigcap_{u' \in [0, t]} f^{-u''}(E'') \right\} = \left\{ (t, x) \in \mathbb{R}_{\geq 0} \times X \mid x \in \bigcap_{u \in [0, t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a, t+a+b+c]} f^{-u'}(E') \right\}.
\]
Since
\[
f_{u'}(E') = f^{-u}(E) \cap \bigcap_{u' \in [a, t+a+b+c]} f^{-u'}(E')
\]
we obtain
\[
\bigcap_{u' \in [a, t+a+b+c]} f^{-u'}(E') = \bigcap_{u \in [0, t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a, t+a+b+c]} f^{-u'}(E')
\]
and
\[
\bigcap_{u \in [0, t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a, t+a+b+c]} f^{-u'}(E') = \bigcap_{u \in [0, t+a+b]} f^{-u}(E) \cap \bigcap_{u' \in [a, t+a+b+c]} f^{-u'}(E').
\]
We have thus proved \( \text{Dom } F_{E''} = g^{-1}(\text{Dom } F_{E'}) \).

The proof of Theorem \[10.11\] is completed.

**Definition 10.14.** Let \( F : \mathbb{R}_{\geq 0} \times X \to X \) be a continuous partial semiflow on a locally compact Hausdorff space \( X \). Let \( S \) be an isolated \( F \)-invariant subset of \( X \). We define the *Conley index of \( S \) relative to \( F \) to be \( \text{Conl}_F(S) \), where \( E \) is an index neighbourhood of \( S \).

By Theorems \[10.9\] and \[10.11\] the full subcategory of \( \text{CptifSub}_F \) consisting of all index neighbourhoods of \( S \) is equivalent to the category with one object and one morphism. Thus, the Conley index \( \text{Conl}_F(S) \) is well-defined as an object of \( \text{Sz}_{\text{cont}}(\text{Chaus}_S) \) up to unique isomorphism.

**Remark 10.15.** We can see in the same way as Example \[5.14\] that there are \( \sim_F \)-equivalence classes of \( F \)-compactifiable subsets that cannot be obtained from Theorems \[10.9\] and \[10.11\].

**Part 3. Appendix**

**Appendix A. Proper maps of compactly generated weak Hausdorff spaces**

In this appendix, we prove some results on proper maps that are needed in this paper. As in the other parts of the paper, we assume every topological space to be compactly generated weak Hausdorff.

We define proper maps as follows:
**Definition A.1.** Let \( f : X \to Y \) be a continuous map of topological spaces. We say that \( f \) is **proper** if, for any compact Hausdorff subset \( L \) of \( Y \), the inverse image \( f^{-1}(L) \) is compact Hausdorff.

**Lemma A.2.** Let \( f : X \to Y \) and \( g : Y \to Z \) be two continuous maps.

(1) If \( f \) and \( g \) are proper, \( gf \) is also proper.
(2) If \( gf \) is proper, \( f \) is also proper.

**Proof.** (1) Obvious.
(2) Let \( L \) be a compact Hausdorff subset of \( Y \). We have
\[
(f^{-1}(L) \subset (gf)^{-1}(g(L))).
\]
Since \( Z \) is weak Hausdorff, \( g(L) \) is a compact Hausdorff subset of \( Z \). Since \( gf \) is proper, the inverse image \( (gf)^{-1}(g(L)) \) is a compact Hausdorff subset of \( X \). On the other hand, since \( Y \) is weak Hausdorff, \( L \) is closed in \( Y \). Hence, \( f^{-1}(L) \) is closed in \( X \) (and therefore, in \( (gf)^{-1}(g(L)) \)). Thus, \( f^{-1}(L) \) is compact Hausdorff.

**Lemma A.3.** Proper maps are stable under base change.

**Proof.** Let \( f : X \to Y \) be a proper map and \( g : Y' \to Y \) be a continuous map. Consider the following pullback diagram:
\[
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y.
\end{array}
\]
Take any compact Hausdorff subset \( L' \) of \( Y' \). Since \( Y \) is weak Hausdorff, \( g(L') \) is a compact Hausdorff subset of \( Y \). Since \( f \) is proper, the inverse image \( f^{-1}(g(L')) \) (\( = g'(f'^{-1}(L')) \)) is a compact Hausdorff subset of \( X \). The following is a pullback diagram:
\[
\begin{array}{ccc}
f'^{-1}(L) & \xrightarrow{g'} & f^{-1}(g(L')) \\
\downarrow{f'} & & \downarrow{f} \\
L' & \xrightarrow{g} & g(L').
\end{array}
\]
We see that \( f'^{-1}(L) \) is compact Hausdorff because it is a fibre product of two compact Hausdorff spaces. Thus, \( f' \) is proper.

**Lemma A.4.** Let \( f : X \to Y \) be a continuous map of topological spaces. Then, the following two conditions are equivalent:

(i) \( f \) is proper.
(ii) For any continuous map \( g : L \to Y \) with \( L \) compact Hausdorff, the fibre product \( L \times_Y X \) is compact Hausdorff.

**Proof.** (ii) \( \Rightarrow \) (i): Consider the case when \( g \) is an inclusion.
(i) \( \Rightarrow \) (ii): By Lemma A.3, the base change \( f' : L \times_Y X \to L \) is proper. Since \( L \) is compact Hausdorff, it follows that \( L \times_Y X \) (\( = f'^{-1}(L) \)) is also compact Hausdorff.

**Lemma A.5.** Proper maps are universally closed.
Proof. By Lemma A.3, it suffices to verify that proper maps are closed. Let \( f : X \to Y \) be a proper map. Let \( A \) be a closed subset of \( X \). Since \( Y \) is compactly generated, it suffices to verify that, for any continuous map \( g : L \to Y \) with \( L \) compact Hausdorff, \( g^{-1}(f(A)) \) is closed in \( L \). Consider the following pullback diagram:

\[
\begin{array}{ccc}
L \times_Y X & \xrightarrow{f'} & L \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\]

By Lemma A.4, \( L \times_Y X \) is compact Hausdorff. We see that \( f' : L \times_Y X \to L \) is a continuous map between compact Hausdorff spaces, hence closed. Thus, \( g^{-1}(f(A)) = f'(g^{-1}(A)) \) is closed in \( L \).

\[\square\]

Lemma A.6. An inclusion is proper if and only if it is closed.

Proof. The ‘if’ part holds because closed subsets of compact Hausdorff spaces are compact Hausdorff. The ‘only if’ part follows from Lemma A.5. \[\square\]

Lemma A.7. Let \( f : K \to X \) be a continuous map with \( K \) compact Hausdorff. Then, \( f \) is proper.

Proof. Take any compact Hausdorff subset \( L \) of \( X \). Since \( X \) is weak Hausdorff, \( L \) is closed in \( X \), and the inverse image \( f^{-1}(L) \) is closed in \( K \). Since \( K \) is compact Hausdorff, \( f^{-1}(L) \) is also compact Hausdorff. \[\square\]

Lemma A.8. Let \( f : X \to Y \) be a continuous map of topological spaces. Suppose that there exists a finite closed cover \((A_i)_{i \in I}\) of \( X \) such that \( f|_{A_i} : A_i \to Y \) is proper for each \( i \in I \). Then, \( f \) is proper.

Proof. Take any compact Hausdorff subset \( L \) of \( Y \). For each \( i \in I \), we see that \( f^{-1}(L) \cap A_i \) is compact Hausdorff. We have a continuous map

\[
g : \bigsqcup_{i \in I} (f^{-1}(L) \cap A_i) \to X
\]

induced from the inclusions. Since \( X \) is weak Hausdorff, \( f^{-1}(L) \), which is the image of \( g \), is compact Hausdorff. \[\square\]

Appendix B. Shift equivalences and the Szymczak category

In this appendix, we introduce a class of morphisms, called shift equivalences, and explain its relation to the Szymczak category, following the idea of Franks–Richeson [FR00]. The results in this appendix are not used in this paper. We include them, however, because they clarify a categorical meaning of the Szymczak category.

We use the language of localization of categories. For basics on this subject, see e.g. [Stacks §04VB].

Definition B.1. Let \( C \) be a category.

1. Let \( f : X \to X \) be an endomorphism in \( C \). Then, we write \( \hat{f} \) for \( f \) seen as a morphism from \( f \) to itself in \( \text{End}(C) \).
(2) Let \( f: X \to X \) and \( g: Y \to Y \) be two endomorphisms in \( C \). We say that a morphism \( \varphi: f \to g \) in \( \text{End}(C) \) is a shift equivalence if there exist a morphism \( \psi: g \to f \) in \( \text{End}(C) \) and \( n \in \mathbb{N} \) such that \( \psi f = \hat{f}^n \) and \( \varphi \psi = \hat{g}^n \).

We write \( \text{ShiftEq}(C) \) for the class of all shift equivalences in the category \( \text{End}(C) \).

**Lemma B.2.** Let \( C \) be a category. Then, \( \text{ShiftEq}(C) \) is a saturated multiplicative system in \( \text{End}(C) \) (in the sense of [Stacks Defn. 05Q8](#)).

**Proof.** Obviously, \( \text{ShiftEq}(C) \) is closed under composition and contains the isomorphisms. Suppose that we are given a diagram

\[
\begin{array}{ccc}
  f & \xrightarrow{\chi} & h \\
  \varphi \downarrow & & \downarrow \hat{h}^n \\
  g & \xrightarrow{\chi \psi} & h.
\end{array}
\]

in \( \text{End}(C) \) such that \( \varphi \in \text{ShiftEq}(C) \). Let us take \( \psi: g \to f \) and \( n \in \mathbb{N} \) such that \( \psi \varphi = \hat{f}^n \) and \( \varphi \psi = \hat{g}^n \). Then, the following diagram is commutative, and the right map \( \hat{h}^n \) is a morphism in \( \text{ShiftEq}(C) \):

\[
\begin{array}{ccc}
  f & \xrightarrow{\chi} & h \\
  \varphi \downarrow & & \downarrow \hat{h}^n \\
  g & \xrightarrow{\chi \psi} & h.
\end{array}
\]

Suppose that \( \varphi, \psi: f \to g \) are two morphisms in \( \text{End}(C) \) and \( \chi: h \to f \) is a morphism in \( \text{ShiftEq}(C) \) such that \( \varphi \chi = \psi \chi \). Let us take \( \omega: f \to h \) and \( n \in \mathbb{N} \) such that \( \omega \chi = \hat{h}^n \) and \( \chi \omega = \hat{f}^n \). Then, \( \hat{g}^n \varphi = \hat{g}^n \psi \), and \( \hat{g}^n \) is a morphism in \( \text{ShiftEq}(C) \). Thus, \( \text{ShiftEq}(C) \) is a left multiplicative system. One can see that \( \text{ShiftEq}(C) \) is a right multiplicative system in the same way.

Let us prove that \( \text{ShiftEq}(C) \) is saturated. Suppose that we are given a sequence

\[
\begin{array}{ccc}
  f & \xrightarrow{\varphi} & g \\
  \downarrow \psi & & \downarrow \tau \\
  h & \xrightarrow{\chi} & i
\end{array}
\]

in \( \text{End}(C) \) such that \( \psi \varphi \) and \( \chi \psi \) are morphisms in \( \text{ShiftEq}(C) \). We want to prove that \( \psi \) is in \( \text{ShiftEq}(C) \). Take \( \omega: h \to f \), \( \tau: i \to g \), and \( m, n \in \mathbb{N} \) such that

\[
\omega \varphi = \hat{f}^m, \quad \psi \varphi \omega = \hat{h}^m, \quad \tau \chi \psi = \hat{g}^n, \quad \chi \psi \tau = \hat{h}^n.
\]

Put \( \sigma = \varphi \omega \hat{h}^n \). Then, we have \( \sigma \psi = \hat{g}^{m+n} \) and \( \psi \sigma = \hat{h}^{m+n} \).

**Definition B.3.** Let \( C \) be a category. We define a functor \( Q: \text{End}(C) \to \text{Sz}(C) \) as follows:

- \( Q \) is identity on the objects.
- \( Q \varphi = (\varphi, 0) \) for each morphism \( \varphi \) in \( \text{End}(C) \).

**Proposition B.4.** Let \( C \) be a category.

1. \((\text{Sz}(C), Q)\) is a localization of the category \( \text{End}(C) \) at the class of morphisms \( \text{ShiftEq}(C) \).
2. A morphism \( \varphi \) in \( \text{End}(C) \) is a shift equivalence if and only if \( Q \varphi \) is an isomorphism.
Proof. (1) Let $f$ and $g$ be two endomorphisms in $\mathcal{C}$. Let $\varphi: f \to g$ be a shift equivalence. Take $\psi: g \to f$ and $n \in \mathbb{N}$ such that $\psi\varphi = \hat{f}^n$ and $\varphi\psi = \hat{g}^n$. We see that $Q\varphi = \psi$ is an isomorphism in $\mathcal{Sz}(\mathcal{C})$ with inverse $(\psi, n)$.

Let $\Phi: \text{End}(\mathcal{C}) \to \mathcal{D}$ be a functor that sends shift equivalences to isomorphisms in $\mathcal{D}$. We want to show that there exists a unique functor $\Phi: \mathcal{Sz}(\mathcal{C}) \to \mathcal{D}$ such that $\Phi Q = \Phi$. Obviously, such $\Phi$ must be equal to $\Phi$ on the objects. Notice that, for any morphism $\varphi: f \to g$ in $\text{End}(\mathcal{C})$ and any $n \in \mathbb{N}$, we have $(\varphi, n) = Q\varphi \circ (Q\hat{f})^{-n}$. This means that $\Phi$ must satisfy $\Phi(\varphi, n) = \varphi \circ (\Phi \hat{f})^{-n}$. To the contrary, if we define $\Phi$ in this way, it is indeed a well-defined functor satisfying $\Phi Q = \Phi$.

(2) This follows from (1), Lemma B.2, and [Stacks, Lem. 05Q9]. □

Next, let us consider the continuous time case.

**Definition B.5.** Let $\mathcal{C}$ be either $\text{Set}_*$ or $\text{CHaus}_*$. 

(1) Let $F$ be an object of $\text{SFlow}(\mathcal{C})$. Then, for each $t \in \mathbb{R}_{\geq 0}$, we write $\hat{f}^t$ for $f^t$ seen as a morphism from $F$ to itself in $\text{SFlow}(\mathcal{C})$.

(2) Let $F$ and $G$ be two objects in $\text{SFlow}(\mathcal{C})$. We say that a morphism $\varphi: F \to G$ in $\text{SFlow}(\mathcal{C})$ is a shift equivalence if there exist a morphism $\psi: G \to F$ in $\text{SFlow}(\mathcal{C})$ and $s \in \mathbb{R}_{\geq 0}$ such that $\psi\varphi = \hat{f}^s$ and $\varphi\psi = \hat{g}^s$.

We write $\text{ShiftEq}_{\text{cont}}(\mathcal{C})$ for the class of all shift equivalences in the category $\text{SFlow}(\mathcal{C})$.

**Lemma B.6.** Let $\mathcal{C}$ be either $\text{Set}_*$ or $\text{CHaus}_*$. Then, $\text{ShiftEq}_{\text{cont}}(\mathcal{C})$ is a saturated multiplicative system in $\text{SFlow}(\mathcal{C})$.

**Proof.** The same as the proof of Lemma B.2. □

**Definition B.7.** Let $\mathcal{C}$ be either $\text{Set}_*$ or $\text{CHaus}_*$. We define a functor $Q: \text{SFlow}(\mathcal{C}) \to \mathcal{Sz}_{\text{cont}}(\mathcal{C})$ as follows:

- $Q$ is identity on the objects.
- $Q\varphi = (\varphi, 0)$ for each morphism $\varphi$ in $\text{SFlow}(\mathcal{C})$.

**Proposition B.8.** Let $\mathcal{C}$ be either $\text{Set}_*$ or $\text{CHaus}_*$. 

(1) $(\mathcal{Sz}_{\text{cont}}(\mathcal{C}), Q)$ is a localization of the category $\text{SFlow}(\mathcal{C})$ at the class of morphisms $\text{ShiftEq}_{\text{cont}}(\mathcal{C})$.

(2) A morphism $\varphi$ in $\text{SFlow}(\mathcal{C})$ is a shift equivalence if and only if $Q\varphi$ is an isomorphism.

**Proof.** The same as the proof of Proposition B.4. □

**Remark B.9.** The relation between the Szymczak category and shift equivalences are explained in Franks–Richeson [FR00]. However, in [FR00], localization is not used; the Conley index is thus defined as a shift equivalence class in the endomorphism category. In other words, it is defined as an isomorphism class of the objects in the Szymczak category. The Conley index defined in Szymczak [Szy95] carries more information: it is a connected simple system in the Szymczak category, i.e. an object of the Szymczak category defined up to unique isomorphism, which retains the information of automorphisms.
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REFERENCES

[Bou71] N. Bourbaki, Éléments de mathématique. Topologie générale. Chapitres 1 à 4. Hermann, Paris, 1971.

[Con78] C. Conley, Isolated invariant sets and the Morse index. CBMS Regional Conference Series in Mathematics, 38. American Mathematical Society, Providence, R.I., 1978.

[FR00] J. Franks and D. Richeson, Shift equivalence and the Conley index. Trans. Amer. Math. Soc. 352 (2000), 3305–3322.

[Lew85] L. G. Lewis Jr., Open maps, colimits, and a convenient category of fibre spaces. Topology Appl. 19 (1985), 75–89.

[Man03] C. Manolescu, Seiberg–Witten–Floer stable homotopy type of three-manifolds with $b_1 = 0$. Geom. Topol. 7 (2003), 889–932; Errata: arXiv:math/0104024v5.

[Mor] Y. Morita, Conley index theory without index pairs. II. In preparation.

[Mro94] M. Mrozek, Leray functor and cohomological Conley index for discrete dynamical systems. Trans. Amer. Math. Soc. 318 (1990), 149–178.

[RS88] J. W. Robbin and D. Salamon, Dynamical systems, shape theory and the Conley index. Ergodic Theory Dynam. Systems 8 * (1988), Charles Conley Memorial Issue, 375–393.

[Sal85] D. Salamon, Connected simple systems and the Conley index of isolated invariant sets. Trans. Amer. Math. Soc. 291 (1985), 1–41.

[Sán11] J. J. Sánchez-Gabites, An approach to the shape Conley index without index pairs. Rev. Mat. Complut. 24 (2011), 95–114.

[Stacks] The Stacks project authors, The Stacks project. https://stacks.math.columbia.edu 2021.

[Szy95] A. Szymczak, The Conley index for discrete semidynamical systems. Topology Appl. 66 (1995), 215–240.

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Corrections to: Conley Index Theory without Index Pairs. I: The Point-set Level Theory

Yosuke Morita

Lemmas 5.4 and 10.4 in the original article are wrong. The first and the third paragraphs of the proof of Lemma 5.4 are correct, hence the largest $f$-invariant subset of $E$ exists and is contained in $I_f(E)$. However, the second paragraph is incorrect. In fact, the following counterexample exhibits that the subset $I_f(E)$ is not necessarily $f$-invariant:

Example 1. Let $X = (\mathbb{N} \times \mathbb{N})/\sim$, where the equivalence relation $\sim$ is defined by

$$(a, b) \sim (a', b') \iff ((a = a' \text{ and } b = b') \text{ or } (a \geq b \text{ and } a - a' = b - b')).$$

Define a self-map $f: X \to X$ by $f(a, b) = (a + 1, b)$. Thus, $f$ can be described by the following diagram:

\[
\begin{array}{cccccc}
(0, 1) & \mapsto & (0, 0) & \mapsto & (1, 0) & \mapsto & (2, 0) & \mapsto & (3, 0) & \mapsto & \cdots \\
(0, 2) & \mapsto & (1, 2) \\
(0, 3) & \mapsto & (1, 3) & \mapsto & (2, 3) \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

Put $E = X$. We see that

$$I_f(E) = \bigcap_{a,b \in \mathbb{N}} f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(E) \right) = \bigcap_{a \in \mathbb{N}} f^a(X) = \left\{ \overline{(n, 0)} \mid n \in \mathbb{N} \right\},$$

and hence

$$f(I_f(E)) = \left\{ \overline{(n, 0)} \mid n \in \mathbb{N}_{\geq 1} \right\}.$$ 

Thus, we have $I_f(E) \neq f(I_f(E))$, i.e. $I_f(E)$ is not $f$-invariant.

Remark 2. It is obvious that $I_f(E)$ is a subset of $E$ satisfying $I_f(E) \subset \text{Dom } f$ and $f(I_f(E)) \subset I_f(E)$. The issue is that $I_f(E) \subset f(I_f(E))$ may not hold.

There is an analogous counterexample to Lemma 10.4:

Example 3. Put $X = (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})/\sim$, where $\sim$ is the equivalence relation defined by

$$(a, b) \sim (a', b') \iff ((a = a' \text{ and } b = b') \text{ or } (a \geq b \text{ and } a - a' = b - b')).$$

Define a semiflow $F: \mathbb{R}_{\geq 0} \times X \to X$ by $F(t, (a, b)) = (a + t, b)$. Put $E = X$. Then, $I_f(E)$ is not $F$-invariant.
Fortunately, the failures of Lemmas 5.4 and 10.4 do not affect the other parts of the original article. This is because the following two weaker lemmas, which are enough for our purpose, hold true:

**Lemma 4.** Let \( f : X \to X \) be a continuous partial self-map on a locally compact Hausdorff space \( X \). Let \( K \) be a compact subset of \( \text{Dom} \ f \). Then, \( I_f(K) \) is the largest \( f \)-invariant subset of \( K \).

**Proof.** It suffices to verify that \( I_f(K) \subset f(I_f(K)) \). In other words, it is enough to see that the intersection \( f^{-1}(x) \cap I_f(K) \) is nonempty for any \( x \in I_f(K) \). Observe that, for each \( a, b \in \mathbb{N} \),

\[
f^{-1}(x) \cap f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(K) \right)
\]

is a closed subset of \( K \). Furthermore, it is nonempty since

\[
x \in I_f(K) \subset f^{a+1} \left( \bigcap_{i=0}^{a+b} f^{-i}(K) \right) = f \left( f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(K) \right) \right).
\]

From Remark 5.3 in the original article and the compactness of \( K \), we conclude that

\[
f^{-1}(x) \cap I_f(K) = \bigcap_{a,b \in \mathbb{N}} f^{-1}(x) \cap f^a \left( \bigcap_{i=0}^{a+b} f^{-i}(K) \right) \neq \emptyset. \tag*{\Box}
\]

**Lemma 5.** Let \( F : \mathbb{R}_{\geq 0} \times X \to X \) be a continuous partial semiflow on a locally compact Hausdorff space \( X \). Let \( K \) be a compact subset of \( X \) such that \([0, \varepsilon] \times K \subset \text{Dom} \ F \) for some \( \varepsilon \in \mathbb{R}_{>0} \). Then, \( I_F(K) \) is the largest \( F \)-invariant subset of \( K \).

**Proof.** Using Lemma 9.2 in the original article and arguing as in the proof of Lemma 4, we see that \( I_f(K) \subset f^t(I_f(K)) \) (and hence, \( I_f(K) = f^t(I_f(K)) \)) holds for \( t \in [0, \varepsilon] \). This implies \( I_f(K) = f^t(I_f(K)) \) for any \( t \in \mathbb{R}_{\geq 0} \). \( \Box \)

We finally remark that there is another important situation in which Lemmas 5.4 and 10.4 are true:

**Definition 6.** We say that a partial map \( f : X \to Y \) is **injective** if it is injective as a map from \( \text{Dom} \ X \) to \( Y \).

**Lemma 7.** Let \( f : X \to X \) be an injective partial self-map on a set \( X \). Let \( E \) be a subset of \( X \). Then, \( I_f(E) \) is the largest \( f \)-invariant subset of \( E \).

**Proof.** If \( f \) is injective, the second paragraph of the proof of Lemma 5.4 in the original article is valid as written; direct images under injective maps commute with intersections. \( \Box \)

**Lemma 8.** Let \( F : \mathbb{R}_{\geq 0} \times X \to X \) be a partial semiflow on a set \( X \) such that \( f^t : X \to X \) is injective for any \( t \in \mathbb{R}_{\geq 0} \). Let \( E \) be a subset of \( X \). Then, \( I_F(E) \) is the largest \( F \)-invariant subset of \( E \).

**Proof.** The same as the proof of Lemma 7. \( \Box \)
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