ON THE TEMPERED $L$-FUNCTION CONJECTURE

VOLKER HEIERMANN AND ERIC OPDAM

ABSTRACT. We give a general proof of Shahidi’s tempered $L$-function conjecture, which has previously been known in all but one case. One of the consequences is the standard module conjecture for $p$-adic groups, which means that the Langlands quotient of a standard module is generic if and only if the standard module is irreducible and the inducing data generic. We have also included the result that every generic tempered representation of a $p$-adic group is a sub-representation of a representation parabolically induced from a generic supercuspidal representation with a non-negative real central character.

1. INTRODUCTION

Let $F$ be a non archimedean local field of characteristic 0. Let $G$ be the group of points of a quasi-split connected reductive $F$-group.

By a parabolic subgroup (Borel subgroup, Levi subgroup, torus, split torus) of $G$ we will mean the group of points of an $F$-parabolic subgroup ($F$-Borel subgroup, $F$-Levi subgroup, $F$-torus, $F$-split torus) of the algebraic group underlying $G$.

Fix a Borel subgroup $B = TU$ of $G$, and let $T_0 \subset T$ be the maximal split torus in $T$. If $M$ is any semi-standard Levi subgroup of $G$ (i.e. a Levi subgroup which contains $T_0$), a standard parabolic subgroup of $M$ will be a parabolic subgroup of $M$ which contains $B \cap M$.

Denote by $W$ the Weyl group of $G$ defined with respect to $T_0$ and by $w_0^G$ the longest element in $W$. By (Sh3, section 3) we can fix a non degenerate character $\psi$ of $U$ which, for every Levi subgroup $M$, is compatible with $w_0^Gw_0^M$. We will still denote $\psi$ the restriction of $\psi$ to $M \cap U$. Every generic representation $\pi$ of $M$ becomes generic with respect to $\psi$ after changing the splitting in $U$.

Let $P = MU$ be a standard parabolic subgroup of $G$ and $T_M$ the maximal split torus in the center of $M$. We will write $a_M^*$ for the dual of the real Lie-algebra $a_M$ of $T_M$, $a_{M,\mathbb{C}}^*$ for its complexification and $a_{M,+}$ for the positive Weyl chamber in $a_{M,\mathbb{C}}^*$ defined with respect to $P$. Following [W], we define a map $H_M : M \to a_M^*$, such that $|\chi(m)|_F = q^{-\langle \chi, H_M(m) \rangle}$ for every $F$-rational character $\chi \in a_M^*$. If $\pi$ is a smooth representation of $M$ and $\nu \in a_{M,\mathbb{C}}^*$, we denote by $\pi_\nu$ the smooth representation of $M$ defined by $\pi_\nu(m) = q^{-\langle \nu, H_M(m) \rangle} \pi(m)$. (Remark that, although the sign in the definition of $H_M$ has been changed compared to the one due to Harish-Chandra, the meaning of $\pi_\nu$ is unchanged.) The symbol $i_P^G$ will denote the
functor of parabolic induction normalized such that it sends unitary representations to unitary representations, $G$ acting on its space by right translations.

The parabolic subgroup of $G$ which is opposite to $P$ will be denoted by $\mathcal{P} = M \mathcal{U}$.

Let $(\tau, E)$ be an irreducible tempered $\psi$-generic representation of $M$.

Put $\tilde{w} = w^G_P w^M$ and fix a representative $w$ of $\tilde{w}$ as in [Sh3]. Then $w\mathcal{P}w^{-1}$ is a standard parabolic subgroup of $G$. For any $\nu \in a^*_M$ there is a Whittaker functional $\lambda_p(\nu, \tau, \psi)$ on $i^G_P V$. It is a linear functional on $i^G_P V$, which is holomorphic in $\nu$, such that for all $\nu \in i^G_P V$ and all $u \in U$ one has $\lambda_p(\nu, \tau, \psi)((i^G_P \tau_\nu)(u)v) = \psi(u)\lambda_p(\nu, \tau, \psi)(v)$. More precisely, assuming that the space of $\tau$ is formed by Whittaker functions, one can define $\lambda_p(\nu, \tau, \psi)$ by (cf. [Sh1], proposition 3.1)

$$\lambda_p(\nu, \tau, \psi)(v) = \int_U (v(wu))(1)\psi(w)du,$$

where $(v(wu))(1)$ denotes the value in 1 of the Whittaker function $v(wu)$ in the space of $\tau_\nu$. Remark that by Rodier’s theorem [Ro], $i^G_P \tau_\nu$ has a unique $\psi$-generic irreducible sub-quotient.

For all $\nu$ in an open subset of $a^*_M$ we have an intertwining operator $J_{\mathcal{P}|P}(\tau_\nu) : i^G_P \tau_\nu \to i^G_P \tau_\nu$. For $\nu$ in $(a^*_M)^+$ far away from the walls, it is defined by a convergent integral

$$\int_U J_{\mathcal{P}|P}(\tau_\nu)v(g) = \int_U v(ug)du.$$ 

It is meromorphic in $\nu$ and the map $J_{\mathcal{P}|P}J_{\mathcal{P}|P}$ is scalar. Its inverse equals Harish-Chandra’s $\mu$-function up to a constant and will be denoted $\mu(\tau, \nu)$.

Let $t(w)$ be the map $i^G_P \mathcal{V} \to i^G_P w \mathcal{V}$, which sends $v$ to $v(w^{-1})$. There is a complex number $C_\psi(\nu, \tau, w)$ [Sh1] such that $\lambda_p(\nu, \tau, \psi) = C_\psi(\nu, \tau, w)\lambda_{\mathcal{P}}(wv, w\tau, \psi)t(\mathcal{V})$. The function $a^*_M \to C, \nu \mapsto C_\psi(\nu, \tau, w)$ is meromorphic.

The local coefficient $C_\psi$ satisfies the equality $C_\psi(\cdot, \tau, w)C_\psi(w(\cdot), w\tau, w^{-1}) = \mu(\tau, \nu)$ [Sh1].

In [Sh3], F. Shahidi attached to each irreducible component $v_i$ of the adjoint action of the $L$-group $L M$ of $M$ on $\text{Lie}(L U)$, an $L$-function $L(s, \tau, v_i)$, an $\epsilon$-factor $\epsilon(s, \tau, r_i, \psi)$, and a $\gamma$-factor $\gamma(s, \tau, r_i, \psi)$, such that

$$\gamma(s, \tau, r_i, \psi) = \epsilon(s, \tau, r_i, \psi)L(1 - s, \tau, r_i^\vee)/L(s, \tau, r_i).$$

In fact, $L(s, \tau, v_i)$ equals the reciprocal of the numerator of $\gamma(s, \tau, r_i, \psi)$.

He showed that the local coefficient $C_\psi$ is equal to the product of the factors $\gamma(is, \tau, r_i, \psi)$ with a holomorphic and non-vanishing function (cf. [Sh3], identity 3.11).

The aim of this paper is to prove the following result:

**Theorem 1.1.** The local coefficient $\nu \mapsto C_\psi(\nu, \tau, w)$ is holomorphic in the negative Weyl chamber, i.e. for $\nu \in -(a^*_M)^+$, and the $L$-functions $L(s, \tau, r_i)$ are holomorphic for $s > 0$.

Remark that the holomorphicity of the local coefficient $C_\psi$ is by the product formula for the local coefficient a consequence of the holomorphicity of the $L$-functions, although we will prove both parallel. The holomorphicity of the $L$-function is known as Shahidi’s tempered $L$-function conjecture. It was originally stated in [Sh3], conjecture 7.1. It was later proved in all, but one case by different authors ([CSh],...
The remaining case concerned a group of type $E_8$ and its maximal Levi of type $E_6 \times A_1$. If $\tau$ is supercuspidal, the holomorphicity had already been shown in the original paper of F. Shahidi [Sh3, proposition 7.3].

As a corollary, one gets by [HM] the following result, which is called the standard modules conjecture:

**Corollary 1.2.** Let $\nu \in a^\ast_M$. Denote by $J(\tau, \nu)$ the Langlands quotient of the induced representation $i^G_\nu \tau_\nu$. Then, the representation $J(\tau, \nu)$ is generic if and only if $i^G_\nu \tau_\nu$ is irreducible.

The paper is organized as follows: in section 2, we prove a result which is not needed in the rest of the paper, but which seems to us interesting in the context. It tells that any generic irreducible tempered representations of $G$ is a subrepresentation parabolically induced by a supercuspidal representation of a standard Levi subgroup with non negative central character.

In section 3 the holomorphicity conjectures are reduced to properties of functions, which can be defined in an affine Hecke algebra context. The main ingredient here is the description of the supercuspidal support of discrete series representations of $p$-adic groups given in [H2]. In section 4, we show that the holomorphicity property for these functions holds under some condition on the parameters which appear. We deduce this from the unramified principal series case for split groups which is proved in [MSH]. In section 5, we finally prove that the parameters coming from generic tempered representations of standard Levi subgroups of $G$ satisfy this condition.

We thank F. Shahidi for some useful conversations and providing the proof of lemma 6.1.

### 2. An Embedding Property for Generic Discrete Series

The aim of this section is the proof of the proposition 2.5. The proof has been inspired by the paper [Re].

**Lemma 2.1.** Let $P = M U$ and $P_\nu = M_\nu U_\nu$ be two standard parabolic subgroups of $G$, $P \subseteq P_\nu$. Let $\sigma$ be a unitary $\psi$-generic supercuspidal representation of $M$ and $\nu \in a^\ast_M$. Write $\tilde{P}_1 = w(P \cap M_\nu)U_\nu w^{-1}$ and $\tilde{P} = w\tilde{P}w^{-1}$.

The intertwining operator $A_w = t(w^{-1}) J_{\tilde{P}_1 \tilde{P}}(w\sigma_\nu)$ is well defined and $\lambda_P(\nu, \sigma, \psi) A_w = c\lambda_{\tilde{P}_1}(\tilde{w}\nu, w\sigma, \psi)$, where $c$ is a non zero constant.

**Proof.** The intertwining operator $A_w$ is well defined, because any root $\alpha$ which is positive for $\tilde{P}$ and negative for $\tilde{P}_1$ verifies $(\tilde{w}\nu, \alpha^\vee) > 0$. One shows as in the case of opposite parabolic subgroups that there is a meromorphic function $C(\nu', w\sigma)$ depending on $\nu' \in a^\ast_{wM_\nu w^{-1}}$ such that $\lambda_{\tilde{P}_1}(\tilde{w}\nu', w\sigma, \psi) = C(\tilde{w}\nu', w\sigma) \lambda_{\tilde{P}}(\nu', \sigma, \psi) t(w^{-1}) J_{\tilde{P}_1 \tilde{P}}(w\sigma_\nu)$. As the intertwining operator depends effectively on a representation induced from $M_\nu$ and $\tilde{w}\nu$ is in the negative Weyl chamber of $a^\ast_{wM_\nu w^{-1}}$ with respect to $w^{-1}M_\nu w\tilde{P}_1 = \tilde{P}_1$, it follows from the product formular for the C-function and the fact that theorem 1.1 is known in the supercuspidal case, that $C(\cdot, w\sigma, \psi)$ is holomorphic in $\tilde{w}\nu$. As in the supercuspidal case the zeroes of the local coefficient $C_\psi$ lie on the unitary axis, this proves the lemma.

The following result is due to W. Casselman [Ca], proposition 4.1.4 and 4.1.6:
Proposition 2.2. Let \((\pi, V)\) be an admissible representation of \(G\), \(P_1 = M_1 U_1\) a semi-standard parabolic subgroup and \(H\) an open compact subgroup of Iwahori type with respect to \((P_1, M_1)\), which means that \(H = (H \cap U_1)(H \cap M)(H \cap U_1)\).

Then there is an open compact subgroup \(U'_1\) of \(U_1\) such that \(V^H \cap V(U_1) \subseteq V(U'_1)\). The spaces \((V^H)^_a := \pi(1_{H a})V\) with \(a \in T_{M_1}\) positive for \(P_1\) and such that \(a U'_1 a^{-1} \subseteq H \cap U\) are all equal to the same space, denoted \(S^H_{P_1}(V)\). The Jacquet function \(j^G_{P_1}\) induces an isomorphism \(S^H_{P_1}(V) \rightarrow (V)^H\).

Lemma 2.3. (with the assumptions and notations of proposition 2.2) If \((\pi', V')\) is a sub-representation of \((\pi, V)\), then one has \(S^H_{P_1}(V) \cap V' = S^H_{P_1}(V')\).

Proof. By definition, it is clear that \(S^H_{P_1}(V) \subseteq S^H_{P_1}(V) \cap V'\). On the other hand, if \(v\) is an element of \(S^H_{P_1}(V) \cap V'\), then there is by proposition 2.2 an element \(v'\) in \(S^H_{P_1}(V')\) such that \(j^G_{P_1} v = j^G_{P_1} v'\). As \(S^H_{P_1}(V') \subseteq S^H_{P_1}(V)\), it follows from proposition 2.2 that \(v = v'\). \(\square\)

Lemma 2.4. Let \(P_1 = M U_1\) be a semi-standard parabolic subgroup with Levi factor \(M\) and denote by \(\tilde{P}_1\) the semi-standard parabolic subgroup which is conjugated by \(w\) to \(P_1\). Let \((\sigma, E)\) be an admissible representation of \(M\), let \(H\) be an open compact subgroup of \(G\) of Iwahori type with respect to \(P_1\), such that there is a nonzero element \(e \in E^H\). Then there is a well defined element \(v\) in \((i_G^{\tilde{P}_1} w E)^H\) with support in \(\tilde{P}_1 w H\) such that \(v(w) = e\). It lies in \(S^H_{\tilde{P}_1}(i_G^{\tilde{P}_1} w E)\).

Proof. Choose an element \(a \in T_M\) which satisfies the assumptions of the proposition relative to \(P_1\) and \(i_G^{\tilde{P}_1} w E\). One observes that \(\sigma(a^{-1}) e\) lies in \(E^{a^{-1}(H \cap M)a}\). There is a well defined element \(\tilde{v}\) in \((i_G^{\tilde{P}_1} w E)^H\) with support contained in \(\tilde{P}_1 w(a^{-1} H a)\) verifying \(\tilde{v}(w) = \sigma(a^{-1}) e\): this follows easily from the fact that \(a^{-1} H a\) is also of Iwahori type relative to \(P_1\) and consequently \(\tilde{P}_1 w(a^{-1} H a) = \tilde{P}_1 w(a^{-1} H \cap U_1)a\). A computation analog to the one in the proof of lemma 5.1 in [4] gives then that \((i_G^{\tilde{P}_1} w \sigma)(1_{H a}) \tilde{v}\), multiplied by a convenient nonzero constant, has the desired properties. \(\square\)

Proposition 2.5. Let \(\pi\) be a \(\psi\)-generic irreducible discrete series representation of \(G\). There exists a standard parabolic subgroup \(P = MU\) of \(G\), a unitary \(\psi\)-generic supercuspidal representation \((\sigma, E)\) of \(M\) and \(\nu \in a^\circ_M\), such that \(\pi\) is a sub-representation of \(i_G^P \sigma_\nu\).

Proof. It follows from results of [4] that there exist \(P = MU\), \(\sigma\) and \(\nu\) as in the statement such that \(\pi\) is a sub-quotient of \(i_G^P \sigma_\nu\). In addition, \(\pi\) is the only irreducible \(\psi\)-generic sub-quotient of \(i_G^P \sigma_\nu\). From this one sees, that it is enough to show that there is an irreducible sub-space of \(i_G^P \sigma_\nu\), on which the Whittaker functional \(\lambda_\nu(\nu, \sigma, \psi)\) does not vanish.

Denote by \(\Sigma(P)\) the set of reduced roots of \(T_M\) in \(\text{Lie}(U)\), by \(\Sigma_\nu\) the subset of roots \(\alpha\) such that \(\langle \nu, \alpha' \rangle = 0\) and by \(M_\nu\) the semi-standard Levi subgroup of \(G\) containing \(M\) obtained by adjoining the roots in \(\Sigma_\nu\) to \(M\).

One has \(\nu \in a^\circ_{M_\nu}\) and there is a parabolic subgroup \(P_\nu = M_\nu U_\nu\) such that \(\nu\) lies in the positive Weyl chamber of \(a^\circ_{M_\nu}\) with respect to this parabolic subgroup. The parabolic \(P_\nu\) may not be standard, but \(P_\nu\) is conjugated in \(G\) to a standard parabolic subgroup. By conjugation \(\sigma\) and \(\nu\) in the same manner and conjugating then \(\sigma\) and
M inside $M_\nu$, so that $M$ becomes the Levi factor of a standard parabolic subgroup $P$, one can finally assume $P_\nu$ standard and $\nu \in a_{M_\nu}^+$.

One can then write $i^G_P \sigma_\nu = i^G_P (i_{M_\nu}^{M_P} \sigma_\nu)$. The representation $\tau = i^M_{P \cap M_\lambda} \sigma$ is a direct sum of irreducible tempered representations $(\tau_i, E_i)$. (Some of them may be isomorphic).

Write $\tilde{P}$ for the standard parabolic subgroup which is conjugated to $\tilde{P}$ by $w$. Put $P_1 = \tilde{P} \cap \tilde{M}_\nu U_\nu$ and denote by $\tilde{P}_1$ the parabolic subgroup of $G$ which is conjugated to $\tilde{P}_1$ by $w$.

Denote by $\mathcal{F}_{\tilde{P}_1 w P_1}$ the subspace of $i^G_{\tilde{P}_1} w E$ formed by the functions with support in the open set $\tilde{P}_1 w P_1$. It follows from the geometric lemma that the Jacquet functor $j^G_{\tilde{P}_1}$ sends $\mathcal{F}_{\tilde{P}_1 w P_1}$ to a subspace of $j^G_{\tilde{P}_1} i^G_{\tilde{P}_1} w E$ on which $M$ acts by the representation $\sigma_\nu$.

Choose a Whittaker function $e$ in the space of $\sigma$ with nonzero value in 1 and an open compact subgroup $H$ of $G$ of Iwahori type with respect to $(P_1, M)$, such that $e$ is $H \cap M$-invariant. By the lemma 2.4, there is an element $v_0$ in $S^H_{\tilde{P}_1}(i^G_{\tilde{P}_1} w E)$ with support in $\tilde{P}_1 w H$ such that $v_0(w) = e$. Recall that $\tilde{P}_1 w H = \tilde{P}_1 w (H \cap U) \subseteq \tilde{P}_1 w P_1$.

It follows directly from the definition that $\lambda_{\tilde{P}_1}(\tilde{w} \nu, w \sigma, \psi)$ does not vanish in $v_0$.

By the lemma 2.1, the intertwining operator $A_{w} = t(w^{-1}) j^G_{\tilde{P}_1}(w \sigma_\nu)$ is well defined and $\lambda_P(\nu, \sigma, \psi)A_{w} = c \lambda_{\tilde{P}_1}(\tilde{w} \nu, w \sigma, \psi)$, where $c$ is a non zero constant. In particular, $\lambda_P(\nu, \sigma, \psi)$ is non zero in $A_{w} v_0$. It remains to show that $A_{w} v_0$ lies in the subspace $(i^G_P E_\nu)_{0}$ of $i^G_P E_\nu$ spanned by the irreducible sub-representations. For this we will show on the one hand that the Jacquet functor $j^G_{\tilde{P}_1}$ sends $A_{w} v_0$ to a nonzero element of the subspace $(j^G_{\tilde{P}_1} i^G_P E_\nu)_0$ of $j^G_{\tilde{P}_1} i^G_P E_\nu$ generated by the sub-representations which admit a generalized central character with real part $\nu$. On the other hand we will show that the Jacquet functor $j^G_{\tilde{P}_1}$ sends the subspace $(i^G_P E_\nu)_0$ onto $(j^G_{\tilde{P}_1} i^G_P E_\nu)_0$. As $A_{w} v_0$ is by [HI] proposition 4.1.1 an element of $S^H_{\tilde{P}_1}(i^G_P E_\nu)$, it follows then from lemma 2.3 that $A_{w} v_0$ lies in $S^H_{\tilde{P}_1}(i^G_P E_\nu)_0$ and consequently in $(i^G_P E_\nu)_0$. This finishes the proof.

Let us show first that $j^G_{\tilde{P}_1}$ sends $A_{w} v_0$ to a nonzero element of $(j^G_{\tilde{P}_1} i^G_P E_\nu)_0$. As $A_{w} v_0$ is a nonzero element in $S^H_{\tilde{P}_1}(i^G_P E_\nu)$, $j^G_{\tilde{P}_1} A_{w} v_0$ is nonzero by the proposition 2.2. It is then enough to show that $T_\nu M$ acts on $j^G_{\tilde{P}_1} A_{w} v_0$ by a character equal to the central character $\chi_\nu$ of $\sigma_\nu$. For every $a \in T_\nu M$, $(i^G_P E_\nu)(a) v_0 - \chi_\nu(a) v_0$ has trivial image in $j^G_{\tilde{P}_1} i^G_P w E_\nu$, because $j^G_{\tilde{P}_1} v_0$ lies in a subspace isomorphic to $\sigma_\nu$. This means that there are $u_1, \ldots, u_t \in U_1$ and $v_1, \ldots, v_t$ in $i^G_{\tilde{P}_1} w E_\nu$, such that

$$(i^G_{\tilde{P}_1} E_\nu)(a) v_0 - \chi_\nu(a) v_0 = \sum_i [(i^G_{\tilde{P}_1} w \sigma_\nu)(u_i) v_i - v_i].$$

Applying on both sides $A_{w}$, one gets

$$i^G_P \sigma_\nu(a) A_{w} v_0 - \chi_\nu(a) A_{w} v_0 = \sum_i [(i^G_{\tilde{P}_1} \sigma_\nu)(u_i) A_{w} v_i - A_{w} v_i].$$

It follows that $(j^G_{\tilde{P}_1} i^G_P \sigma_\nu)(a)$ acts on $j^G_{\tilde{P}_1} (A_{w} v_0)$ by the character $\chi_\nu(a)$.

It remains to show that the irreducible subspaces $\pi_i$ of $i^G_P \sigma_\nu$ are the only subquotients such that $j^G_{\tilde{P}_1} \pi_i$ admits as exponent a generalized character with real part
As $\nu$ is a regular element of $a_{M_1}^*$, the length of $(j_{P_1}^G i_{\nu}^G E)^{\infty}$ is by the geometric lemma equal to the cardinality $l$ of the subset of $W_M \mathcal{W}_{M_{\nu}} / W_M$ formed by the elements which stabilize $M$. It equals the length of $(j_{P_1 \cap M_{\nu}}^G \tau_\nu)^{\infty}$. Denote by $l_i$ the length of $(j_{P_1 \cap M_{\nu}}(\tau_\nu))^{\infty}$. An irreducible sub-representation $\pi_i$ of $i_{\nu}^G \sigma_\nu$ is a sub-representation of some $i_{\nu}^G (\tau_\nu)$. It is enough to show that the length of $(j_{P_1}^G \pi_1)^{\infty}$ is at least $l_i$, because the $l_i$ sum up to $l$.

By the Frobenius reciprocity, one has,
\[
\text{Hom}_G(\pi_i, i_{\nu}^G (\tau_\nu)) = \text{Hom}_M(j_{P_1}^G \pi_i, (\tau_\nu))
\]
which means that $(\tau_\nu)$ is a quotient of $j_{P_1}^G \pi_i$. From the transitivity of the Jacquet functor, if follows that $j_{P_1 \cap M_{\nu}}^G (\tau_\nu)$ is a sub-quotient of $j_{P_1}^G \pi_i$. As $(j_{P_1 \cap M_{\nu}}(\tau_\nu))^{\infty}$ has length $l_i$, it follows that the length of $(j_{P_1}^G \pi_1)^{\infty}$ is at least $l_i$. $$ \square $$

3. Reduction to an affine Hecke algebra setting

Let $P = MU$ be a maximal standard parabolic subgroup of $G$. Denote by $\alpha$ the unique simple $F$-root for $G$ which is not a root for $M$ and by $\rho$ half the sum of the $F$-roots whose root spaces span Lie$(U)$. Remark that $\rho$ lies in $a_M^*$. For an $F$-root $\beta$, denote by $\beta^\vee$ a root in the absolute root system that restricts to $\beta$ and by $\beta^\vee$ the coroot corresponding to $\beta$. Write $\langle \cdot, \cdot \rangle$ for the duality between $a_M^*$ and $a_M$. For $\lambda \in a_M^*$ and an $F$-root $\beta$, we will sometimes write $\lambda \langle \beta \rangle$. Here, $\beta^\vee$ will be identified with its orthogonal projection on $a_M$. Put $\tilde{\alpha} = \langle \rho, \alpha^\vee \rangle^{-1} \rho$. Let $(\tau, V)$ be an irreducible discrete series representation of $M$. By proposition 2.5, there is a standard parabolic subgroup $P_1 = M_1 U_1$ of $G$ contained in $P$, a unitary $\psi$-generic irreducible supercuspidal representation $\sigma$ of $M_1$ and $\nu_\tau \in a_M^{* \mathfrak{S}}$, $\nu_\tau \geq M \cap P_1$, such that $\tau$ is a sub-representation of $i_{\nu}^G (\sigma \otimes \chi_{\nu_\tau})$. (Remark that we do not need for the sequel such a strong result, but only the well known existence of a generic supercuspidal support.)

Denote by $\Sigma_{\text{red}}(P_1)$ the set of reduced roots for the action of the split center of $M_1$ on Lie$(U_1)$. Remark that to any $\tilde{\beta} \in \Sigma_{\text{red}}(P_1)$ one can associate a parabolic subgroup $P_{1, \tilde{\beta}} = M_{1, \tilde{\beta}} U_{1, \tilde{\beta}}$, such that $P_1 \cap M_{1, \tilde{\beta}}$ is a maximal standard parabolic subgroup of $M_{1, \tilde{\beta}}$. For $\tilde{\beta} \in \Sigma_{\text{red}}(P_1)$, we will denote $\beta$ the unique simple root for $M_{1, \tilde{\beta}}$ which projects to $\tilde{\beta}$ and write then also $M_{1, \beta}$, $U_{1, \beta}$ and $P_{1, \beta}$.

Harish-Chandra’s $\mu$-function $\mu(\sigma \otimes \chi_{\nu})$ is a product
\[
\prod_{\tilde{\beta} \in \Sigma_{\text{red}}(P_1)} \mu_{M_{1, \beta}}(\sigma \otimes \chi_{\nu}).
\]
The set of roots $\tilde{\beta}$ such that $\mu_{M_{1, \beta}}(\sigma \otimes \chi_{\nu})$ is not holomorphic on $a_{M_1}^*$ as a function in $\nu$ is the set of positive roots of a root system in $a_{M_1}^*$ (cf. [Si2], proposition 3.5).

We will denote this root system by $\Sigma_{\sigma}$. Denote by $\tilde{\beta}^\vee$ the coroot of a root $\tilde{\beta}$ in $\Sigma_{\sigma}$. Remark that by the main result of [H2], $\nu_\tau$ is a residue point in $a_{M_1}^*$ for Harish-Chandra’s $\mu$-function $\nu \mapsto \mu_{M}(\sigma \otimes \chi_{\nu})$, defined relative to $M$. (The precise definition of a residue point, which is given in [O], does not matter here.)

Fix $\tilde{\beta} \in \Sigma_{\sigma}$. In [Sh3], F. Shahidi has associated to each irreducible component $\rho_{1, \beta}$ of the adjoint action of $L M_1$ on Lie$(L U_{1, \beta})$ a meromorphic function $\gamma_{M_{1, \beta}}(s, \sigma, r_{1, \beta}, \psi)$. He showed that there is at most one index $\beta$ such that $\gamma_{M_{1, \beta}}(s, \sigma, r_{1, \beta}, \psi)$
has a zero on the real axis and that this index equals in fact either 1 or 2. We will denote it in the sequel by $\epsilon_\beta$, put $\epsilon_\overline{\beta} = \frac{\langle \overline{\alpha}, \beta' \rangle}{2} \epsilon_\beta$ and $i_{\overline{\beta}} = \langle \overline{\alpha}, \beta' \rangle$.

**Proposition 3.1.** There are meromorphic functions $f$ and $f_i$ which are holomorphic and non-vanishing on the real axis, such that

$$C_\psi(s\overline{\alpha}, \tau) = f(s) \prod_{\overline{\beta} \in \Sigma^+_{\delta} - \Sigma_{\delta}^+} \frac{1 - q^{-\langle \nu_\tau + s\overline{\alpha}, \overline{\beta}' \rangle}}{1 - q^{\frac{1}{2} + \langle \nu_\tau + s\overline{\alpha}, \overline{\beta}' \rangle}}$$

and

$$\gamma(is, \tau, r_i, \psi) = f_i(s) \prod_{\overline{\beta}, \epsilon_{\overline{\beta}} = i} \frac{1 - q^{-\langle \nu_\tau + s\overline{\alpha}, \overline{\beta}' \rangle}}{1 - q^{\frac{1}{2} + \langle \nu_\tau + s\overline{\alpha}, \overline{\beta}' \rangle}}.$$

**Proof.** Denote by $r$ the adjoint action of $^L M$ on $V = Lie(^L U)$. This action decomposes in irreducible sub-representations $r_i$ corresponding to the weights of $^L _i M$. The space $V_i$ of $r_i$ is generated by the root spaces $\eta_\beta'$ corresponding to the roots $\beta'$ which have the same restriction to $^L _i M$ as $i\alpha'$. The local coefficient $C_\psi$ can be expressed by the $\gamma$-function defined in [Sh3]: up to a product by a holomorphic function, $C_\psi(s\overline{\alpha}, \tau)$ equals $\prod_i \gamma(is, \tau, r_i, \psi)$ (cf. identity (3.11) in [Sh3]). Write $\gamma(is, \tau, r_i, \psi) = \gamma(\tau \otimes \chi_{s\overline{\alpha}}, r_i, \psi)$. For $\beta \in \Sigma_{\text{red}}(P_1)$ denote by $r_{1,i,\beta}$ the restriction of $r_i$ to $^L M_1 \to Lie(^L U_{1,\beta})$. Then, by the product formula for the $\gamma$-function (cf. identity (3.13) in [Sh3]), one has

$$\gamma(\tau \otimes \chi_{s\overline{\alpha}}, r_i, \psi) = \prod_{\beta} \gamma_{M_{1,\beta}}(\sigma \otimes \chi_{\nu_\tau + s\overline{\alpha}}, r_{1,i,\beta}, \psi),$$

the roots $\overline{\beta}$ being taken in $\Sigma_{\text{red}}(P_1) - \Sigma_{\text{red}}(P_1 \cap M)$.

Define $i_\beta = \langle \overline{\alpha}, \beta' \rangle$. The representation $r_{1,i,\beta}$ can only be nonzero if $i_\beta i$. Then, $\gamma_{M_{1,\beta}}(\sigma \otimes \chi_{\nu_\tau + s\overline{\alpha}}, r_{1,i,\beta}, \psi)$ is equal to $\gamma_{M_{1,\beta}}(\sigma \otimes \chi_{\nu_\tau + s\overline{\alpha}}, r_{1,i,\beta}, \psi)$. This function is holomorphic and nonzero for $s \in \mathbb{R}$, except perhaps if $i = \epsilon_\beta i_\beta$ with $\epsilon_\beta \in \{1, 2\}$. This can then only happen at one of these two values for $\epsilon_\beta$ (cf. Corollary 7.6 of [Sh3]). Then $\gamma_{M_{1,\beta}}(\sigma \otimes \chi_{\nu_\tau + s\overline{\alpha}}, r_{1,i,\beta}, \psi)$ is equal to the product of a function $\gamma(\tau \otimes \chi_{s\overline{\alpha}}, r_i, \psi)$ is equal to the product of a function on the real axis by $L(1 - \epsilon_\beta \langle \nu_\tau + s\overline{\alpha}, \beta' \rangle, \sigma^\vee, r_i) / L(\epsilon_\beta \langle \nu_\tau + s\overline{\alpha}, \beta' \rangle, \sigma, r_i)$ (cf. identity (7.4) of [Sh3]). Up to a product by a holomorphic non-vanishing function on the real axis, this quotient equals $(1 - q^{-\epsilon_\beta \langle \nu_\tau + s\overline{\alpha}, \beta' \rangle})/(1 - q^{-1 + \epsilon_\beta \langle \nu_\tau + s\overline{\alpha}, \beta' \rangle})$.

Denote by $\Sigma'$ the subset of the roots $\overline{\beta} \in \Sigma_{\text{red}}(P_1) \setminus \Sigma_{\text{red}}(P_1 \cap M)$ such that $\gamma_{M_{1,\beta}}(\sigma \otimes \chi_{\nu_\tau + s\overline{\alpha}}, r_{1,i,\beta}, \psi)$ has a pole or a zero in some $s \in \mathbb{R}$. We have just proved that $C_\psi(s\overline{\alpha}, \tau)$ is, up to the product by a meromorphic function without poles and zeroes on the real axis, equal to

$$\prod_{\overline{\beta} \in \Sigma'} \frac{1 - q^{-\epsilon_\beta \langle \nu_\tau + s\overline{\alpha}, \overline{\beta}' \rangle}}{1 - q^{-1 + \epsilon_\beta \langle \nu_\tau + s\overline{\alpha}, \overline{\beta}' \rangle}}.$$
This expression is the product of a regular function on \( \mathbb{R} \) depending on \( s \) without zeroes on the real axis by
\[
\prod_{\beta \in \Sigma'} \frac{1 - q^{(\nu_\tau + s\alpha, \beta')}}{1 - q^{\frac{1}{\epsilon_\beta} + (\nu_\tau + s\alpha, \beta')}}.
\]

Recall that Harish-Chandra’s \( \mu \)-function \( \mu(\sigma \otimes \chi_\nu) \) is a product
\[
\prod_{\beta \in \Sigma_{\text{red}}(P_1)} \mu_{M_1, \beta}(\sigma \otimes \chi_\nu).
\]
The factor \( \mu_{M_1, \beta}(\sigma \otimes \chi_\nu) \) has a zero or a pole in \( \nu \in a_{M_1}^* \), if and only if \( \beta \in \Sigma_{\sigma}^+ \).
Then, there is a positive real number \( \epsilon_\Sigma \), such that \( \mu_{M_1, \beta}(\sigma \otimes \chi_\nu) \) is the product of a function without zeros and poles on \( a_{M_1}^* \) by \( \prod_{\beta' \in (\pm \beta)} \frac{1 - q^{(\nu_\tau, \beta')}}{1 - q^{\frac{1}{\epsilon_\beta} + (\nu_\tau, \beta')}} \) (cf. [Sh3], identity 1.4), applied to \( \mu_{M_1, \beta}(\sigma \otimes \chi_\nu) \) for each \( \beta \), and the above relation between local coefficient \( C_\psi \) and the \( \gamma \)-function, applied to \( \gamma_{M_1, \beta} \) for each \( \beta \), it follows that \( \Sigma' = \Sigma_{\sigma}^+ - \Sigma_{\text{red}}(P_1 \cap M) \). One deduces from this also that, for \( \beta \in \Sigma' \), the functions \( s \mapsto -\frac{1}{\epsilon_\beta} + (\nu_\tau + s\alpha, \beta) \) and \( s \mapsto -\frac{1}{\epsilon_\beta} + (\nu_\tau + s\alpha, \beta') \) must have the same zeroes on the real axis. As \( \beta' \) is a scalar multiple of the projection of \( \beta \) to \( a_{M_1} \), it follows that \( \epsilon_\beta \) is equal to the product of \( \epsilon_\beta' \) by \( \frac{\langle \beta', \beta' \rangle}{\langle \beta, \beta \rangle} \).

Going back to the expressions for the \( \gamma \)-factors and remarking that \( \epsilon_{\beta i, \beta} = \epsilon_{\beta i, \beta}^{\prime} \epsilon_\beta \epsilon \), one gets the statement for the different \( \gamma \)-factors.

\[\square\]

4. The conjectures for affine Hecke algebras

Let \( \Sigma \) be a reduced root system in a vector space \( a_{M_1}^* \). Let \( a_{M_1}^{M_*} \) be a subspace of codimension one, generated by a subset \( \Sigma^{M_*} \) of positive roots in a standard sub-root system \( \Sigma^{M} \) of \( \Sigma \). For each positive root \( \beta \in \Sigma \), let \( \epsilon_\beta \) be a number \( > 0 \) such that \( \epsilon_\beta = \epsilon_\alpha \) if \( \beta \) and \( \alpha \) are conjugated.

Let \( \mu \) be the meromorphic function on \( a_{M_1}^* \) in \( \nu \) defined by
\[
\prod_{\beta \in \Sigma} \frac{1 - q^{(\nu, \beta')}}{1 - q^{\frac{1}{\epsilon_\beta} + (\nu, \beta')}},
\]
and let \( \mu^{M} \) be the factor of \( \mu \) given by
\[
\prod_{\beta \in \Sigma^{M}} \frac{1 - q^{(\nu, \beta')}}{1 - q^{\frac{1}{\epsilon_\beta} + (\nu, \beta')}}.
\]

Let \( \nu_\tau \) be a residue point \( [O] \) for \( \mu^{M} \) in \( a_{M_1}^{M_*} \). Denote by \( \omega_\alpha \) the fundamental weight in \( a_{M_1}^* \), which corresponds to the simple root \( \alpha \) of \( \Sigma \) which does not lie in \( a_{M_1}^{M_*} \). Consider the functions
\[
C(s) = \prod_{\beta \in \Sigma_{\sigma}^+ \Sigma^{M,*}} \frac{1 - q^{(\nu_\tau + s\omega_\alpha, \beta')}}{1 - q^{\frac{1}{\epsilon_\beta} + (\nu_\tau + s\omega_\alpha, \beta')}}
\]
and 
\[ \gamma_i(s\omega, \tau, \psi) = f_i(s) \prod_{\beta, \epsilon, \beta'(\omega_i, \beta') = i} \frac{1 - q^{-(\nu_r + s\omega, \beta')}}{1 - q^{-(\nu_r + s\omega, \beta')}}. \]

**Theorem 4.1.** For each irreducible component of $\Sigma$, suppose either that all the labels $\epsilon_\beta$ are equal, or that $\epsilon_\beta'/\epsilon_\beta$ equals the ratio of the square of the lengths of $\beta'$ and $\beta$.

Then the function $C(s)$ is holomorphic for $s < 0$ and the functions $\gamma_i(s)$ are non-vanishing for $s > 0$.

**Proof.** Suppose first all $\epsilon_\beta = 1$. Denote by $G_\Sigma$ the group of $F$-points of a split connected reductive group defined over $F$ with root system $\Sigma$ and by $B_\Sigma = T_\Sigma U_\Sigma$ a Borel subgroup which is standard with respect to the choice of the ordering of $\Sigma$. Then $\Sigma^M$ corresponds to a standard maximal parabolic subgroup $P = MU$ of $G_\Sigma$. As $\nu_r$ is a residue point, the representation $i_{B\cap M}^M \chi_{\nu_r}$ has a sub-quotient which is a discrete series representation. By [MSh], proposition 3.1, it has also a generic discrete series sub-quotient. There is an element $w$ in the Weyl group for $M$, such that $\tau$ is a sub-representation of $i_{B\cap M}^M \chi_{\nu_r}$. By [MSh], $C_\psi(s\omega_\alpha)$ is holomorphic for $s < 0$. By proposition 3.1, $C_\psi(s\omega_\alpha)$ is, up to a factor which is holomorphic and non-vanishing on the real line, equal to

\[ C_\psi(s\omega_\alpha, \tau) = f(s) \prod_{\beta \in \Sigma^+ - \Sigma^M} \frac{1 - q^{-(\nu_r + s\omega, \beta')}}{1 - q^{-(\nu_r + s\omega, \beta')}}. \]

As $w$ leaves the set $\Sigma^+ - \Sigma^M$, the element $\omega_\alpha$ and the product $\langle \cdot, \cdot \rangle$ invariant, the statement follows. The set over which factors the function $\gamma_i$ is also invariant by the Weyl group of $M$. As the numerator of $\gamma_i$ is just the reciprocal of the $i$th $L$-function of $\tau$, its non-vanishing property follows from the holomorphicity of the corresponding $L$-function proved in [MSh].

Denote by $z_n(s)$ (resp. $z_p(s)$) the number of roots $\beta \in \Sigma^+ - \Sigma^M$, such that $\langle \nu_r + s\omega, \beta' \rangle = 0$ (resp. $\langle \nu_r + s\omega, \beta' \rangle = 1$) and $z_n, i$ (resp. $z_p, i$) the subsets corresponding to the roots $\beta$ such that $\langle \omega_i, \beta' \rangle = i$. The holomorphicity of $C(-s)$ in $s$ is equivalent to $z_n(-s) \geq z_p(-s)$ and the non-vanishing of $\gamma_i(s)$ to $z_n, i(s) \leq z_p, i(s)$. By what we just remarked this is true for $s > 0$, when all the $\epsilon_\beta$ are equal to 1.

Suppose now all $\epsilon_\beta$ equal to 0. Multiplying the equations above by $\epsilon$, $z_n(s)$ is the number of roots $\beta \in \Sigma^+ - \Sigma^M$, such that $\langle \nu_r + \epsilon s\omega, \beta' \rangle = 0$, and $z_p(s)$ the number of roots $\beta \in \Sigma^+ - \Sigma^M$, such that $\langle \nu_r + \epsilon s\omega, \beta' \rangle = 1$. Observe that, if $\nu_r$ is a residue point for all $\epsilon_\beta = \epsilon$, then $\nu_r$ is a residue point for all $\epsilon_\beta = 1$. Consequently, we are in the situation of equal parameters 1, where the holomorphicity and non-vanishing results have just been proved.

Suppose now $\Sigma$ of type $B_n$, $C_n$, $F_4$ or $G_2$. Denote by $\kappa$ the ratio of the square of the length of a long root by the one of a short root. Suppose $\epsilon_\beta'/\epsilon_\beta = \kappa$, if $\beta'$ is a long root and $\beta$ a short root. Write $\tilde{\beta} = \beta/\kappa$, if $\beta$ is a long root, $\tilde{\beta} = \beta$, if $\beta$ is a short root, and denote by $\tilde{\Sigma}$ the set of the $\tilde{\beta}$. Then $\tilde{\Sigma}$ is a root system of type $C_n$, if $\Sigma$ was of type $B_n$, of type $B_n$, if $\Sigma$ was of type $C_n$, and of type $F_4$ (resp. $G_2$), if $\Sigma$ was of type $F_4$ (resp. $G_2$). Let $\epsilon$ be the common value of the $\epsilon_\beta$ with $\beta$ a short root. Then, $z_n(s)$ is the number of roots $\tilde{\beta} \in \tilde{\Sigma}^+ - \tilde{\Sigma}^M$, such that $\langle \nu_r + s\omega, \beta' \rangle = 0$, and $z_p(s)$ the number of roots $\beta \in \Sigma^+ - \Sigma^M$, such that $\langle \nu_r + s\omega, \beta' \rangle = 1/\epsilon$. Remark that $\nu_r$ is a residue point for the set of roots $\tilde{\Sigma}^M$ with all labels equal $\epsilon$. So, we
are back in the equal parameter case, where the holomorphicity and non-vanishing result have already been considered above, adding that $\epsilon_\beta(\omega_\alpha, \beta^\vee) = \epsilon_\beta(\omega_\alpha, \beta^\vee)$.  \(\square\)

5. The Conjectures in the p-adic case

Recall that $P = MU$ denotes a maximal standard parabolic subgroup of $G$, $\alpha$ the unique simple $F$-root for $G$ which is not a root for $M$, $\rho$ half the sum of the $F$-roots whose root spaces span $\text{Lie}(U)$ and that $\widetilde{\alpha} = (\rho, \omega^\vee)^{-1}\rho$.

**Theorem 5.1.** Let $(\tau, V)$ be an irreducible tempered representation of $M$. The function $C_\psi(-s\widetilde{\alpha}, \tau)$ and the functions $L(\tau, s, r_i)$ are regular for $s > 0$.

**Proof.** By the product formula for the local coefficient $C_\psi$ and the $\gamma$–functions, one is reduced to consider the case, where $\tau$ is a discrete series representation. Here the theorems 3.1 and 4.1 apply. So, it remains to show that the labels $\epsilon_{\overline{\tau}}$ satisfy the assumption in the statement of the theorem 4.1. Denote by $\Sigma$ the reduced $F$-root system for $G$, by $P_1 = M_1U_1$ and $\sigma$ respectively the standard parabolic subgroup and the generic supercuspidal representation of $M_1$ from which $\tau$ is induced and by $\Sigma^{M_1}$ the reduced $F$-root system of the Levi subgroup $M_1$.

One has to show that for two roots $\beta'$ and $\beta$ in $\Sigma_\sigma$ of the quotient $\epsilon_{\overline{\tau}}/\epsilon_{\overline{\sigma}}$ satisfies the assumptions in the statement of theorem 4.1. We will prove first that one can reduce to the case where $\Sigma$ is irreducible and $\Sigma^{M_1}$ of corank 2.

Remark that the labels $\epsilon_{\overline{\tau}}$ and $\epsilon_{\overline{\sigma}}$ do not change if one conjugates $\beta'$ and $\beta$ by an element of the Weyl group of $\Sigma_\sigma$. So, we may suppose that $\beta + \beta'$ is a root in $\Sigma_\sigma$. Suppose that the corank of $\Sigma^{M_1}$ in $\Sigma$ is $> 2$ and denote by $\Sigma^{M'}$ the sub-root system of $\Sigma$ of the minimal Levi sub-group $M'$ of $G$ containing $\Sigma^{M_1}$, $\beta$ and $\beta'$. Then, possibly after conjugation, $\Sigma^{M_1}$ is a standard corank 2 sub-root system in $\Sigma^{M'}$ and the values of the numbers defined in the proposition 3.1 are the same with respect to $\Sigma^{M'}$ or $\Sigma$. If $\Sigma^{M'}$ is not irreducible, then $\beta$ and $\beta'$ must be projections of roots in a same irreducible component $\Sigma_1$ of $\Sigma$, because $\beta + \beta'$ is a root in $\Sigma_\sigma$. The system $\Sigma^{M_1} \cap \Sigma_1$ is a sub-root system of corank 2 in $\Sigma_1$, and one is reduced to study the subgroup of $G$ generated by $\Sigma^{M_1} \cap \Sigma_1$ relative to the one generated by $\Sigma_1$ with the restriction of $\sigma$ to this subgroup. So, one is finally reduced to the case, where $\Sigma^{M_1}$ is a sub-root system of corank 2 of $\Sigma$. This situation is considered case by case in the next section, using the following lemma.  \(\square\)

**Lemma 5.2.** Denote by $(.,.)$ the Weyl group-invariant scalar product in the space spanned by the absolute roots of $G$ and, for a root $\overline{\beta}$ in $\Sigma_\sigma$, by $\omega^{M_1, \beta}$ the fundamental weight corresponding to $\beta$ relative to the root system $\Sigma^{M_1, \beta}$ and by $\overline{\beta}$ the scalar multiple of $\omega^{M_1, \beta}$ that verifies $(\overline{\beta}, \beta^\vee) = 1$.

The labels $\epsilon_{\overline{\tau}}, \epsilon_{\overline{\sigma}}$ and $\epsilon_{\beta'}, \epsilon_{\beta}$ defined in section 3 verify the formula

$$\frac{\epsilon_{\overline{\tau}}}{\epsilon_{\overline{\sigma}}} = \frac{\epsilon_{\beta'}(\beta', \beta)}{\epsilon_{\beta}(\beta, \beta)} = \frac{\epsilon_{\beta'}(\beta', \beta)}{\epsilon_{\beta}(\beta, \beta)}.$$  

**Proof.** Recall that $\epsilon_{\overline{\beta}} = \frac{(\overline{\beta}, \beta^\vee)}{2(\beta, \beta)} \epsilon_{\beta}$. So, it is enough to show that

$$\overline{\beta} = \frac{(\beta, \beta)}{2(\beta, \beta)} \beta.$$
Remark first that for every \( \lambda \) in \( a_1^T \) and every root \( \gamma \), one has \( \langle \lambda, \gamma^\vee \rangle = (\frac{2}{2\pi})(\lambda, \gamma) \).

It is clear that \( \beta = \kappa \beta \) for some constant \( \kappa \), because both lie in the one-dimensional vector space \( d_{M_1}^{M, \beta^*} \). Then, one computes

\[
\kappa = \frac{\langle \beta, \beta \rangle}{\langle \beta, \beta \rangle} = \frac{\langle \beta, \beta \rangle}{2\langle \beta, \beta \rangle} = \frac{\langle \beta, \beta \rangle}{2\langle \beta, \beta \rangle}.
\]

\[\square\]

6. Labels of supercuspidal \( \mu \)-functions in the generic case

Remark first that our situation is invariant for restriction of scalars: if \( H \) is a quasi-split connected reductive \( F \)-group, \( F'/F \) a finite Galois extension and \( G = \text{Res}_{F'/F} H \), then the absolute root system for \( G \) is a union of copies of the absolute root system of \( H \) with an action of the Galois group permuting these copies. In particular, the absolute roots (resp. duals of the absolute roots) for \( G \) restrict to \( F \)-roots as do the absolute roots (resp. duals of the absolute roots) for \( H \). So, as every \( F \)-quasi-simple group \( G \) is the restriction of scalars of an absolute quasi-simple group, it is enough to consider the latter ones. (Of course, in the split case, this does not make any difference.)

In this section, we give for every absolute root system of an absolute quasi-simple quasi-split group over \( F \) its Dynkin-diagram, its \( F \)-root system \( \Sigma \), the list of the standard sub-root systems \( \Sigma^M \) of corank 2 of \( \Sigma \) and the set of quotient roots \( \Sigma(T_M) \). We consider then the subset \( \Sigma_\mu \) formed by the roots \( \beta \) in \( \Sigma(T_M) \) such that \( \Sigma^M_\beta \) is self-conjugated as a corank one sub-root system of \( \Sigma^{M_\beta} \). It turns out that \( \Sigma_\mu \) is always a root system, and it is clear that any root system \( \Sigma_\sigma \) which may appear from the above context must be a sub-root system of \( \Sigma_\mu \).

One does not have to study further the cases where \( \Sigma_\mu \) is a product of irreducible root systems of type \( A \), because in this case all roots which lie in a same irreducible component are conjugated. So, only the cases where \( \Sigma_\mu \) is of type \( B_2 \) or \( G_2 \) will require further attention. We call these cases the relevant cases. With help of lemma 5.2, we compute in these cases the possible values of the labels \( \epsilon_\gamma \) corresponding to the long and short root, using the list in [L] completed in [Sh2]. In some cases, we will need in addition the following lemma to prove that unwanted ratios for the labels do not appear.

**Lemma 6.1.** Let \( \sigma \) be a generic supercuspidal representation of a maximal Levi subgroup \( M' \) of a quasi-split connected reductive group \( G' \) defined over \( F \). The second \( L \)-function \( L(s, \sigma, r_2) \) attached to \( \sigma \) is constant in the following cases:

(i) \( G' \) is split of type \( D_5 \) and \( M' \) is of type \( A_2 \times A_1 \times A_1 \),

(ii) \( G' \) is split of type \( D_7 \) and \( M' \) is of type \( A_2 \times D_4 \),

(iii) \( G' \) is split of type \( C_3 \) and \( M' \) is of type \( A_2 \).

(iv) \( G' \) is quasi-split of type \( 2A_5 \) and \( M' \) is the restriction of scalars of a group of type \( A_2 \) relative to a cyclic extension of \( F \) of degree 2.

(v) \( G' \) is quasi-split of type \( 2D_4 \) and \( M' \) a split group of type \( A_2 \).

**Proof.** The second \( L \)-function is here in fact the one attached to the exterior square \( L \)-function of the \( A_2 \) part which can be reinterpreted as the first and only \( L \)-function in a non associated setting. So it follows from [Sh3, lemma 7.4] that the \( L \)-function is 1. \[\square\]
6.1. The split cases: Here $\tilde{\beta}$ is always equal to the fundamental weight $\omega^M_{\beta}$ in $\Sigma^M_{\beta}$.

$A_n$:

$\Delta - \Delta^M = \{\alpha_i, \alpha_j\}$, $1 \leq i < j \leq n$, $M$ is of type $A_{i-1} \times A_{j-i-1} \times A_{n-j}$, $\Sigma_{\text{red}}(T_M) = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j\}$, so that $\Sigma_{\mu}$ is always a product of root systems of type $A$. Consequently, there are no relevant cases.

$B_n$:

(1) $\Delta - \Delta^M = \{\alpha_i, \alpha_j\}$, $1 \leq i < j \leq n-1$, $M$ is of type $A_{i-1} \times A_{j-i-1} \times B_{n-j}$, $\Sigma_{\text{red}}(T_M) = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j, \alpha_i + 2\alpha_j\}$ is of type $B_2$, $\alpha_i$ is the long root, $M_{\alpha_i}$ is of type $A_{j-1} \times B_{n-j}$ and $M_{\alpha_j}$ is of type $A_{i-1} \times B_{n-i}$. In order of $\Sigma_{\mu}$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means that $j = 2i$. Then $(\omega_{\alpha_j}^{M_{\alpha_j}}, \omega_{\alpha_i}^{M_{\alpha_i}}) = j/2$, $(\omega_{\alpha_j}^{M_{\alpha_j}}, \omega_{\alpha_i}^{M_{\alpha_i}}) = j$, $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j) = 2$, $\epsilon_{\alpha_i}$ is necessarily 1 and $\epsilon_{\alpha_j}$ may be 1 or 2. One deduces that the assumptions are satisfied.

(2) $\Delta - \Delta^M = \{\alpha_i, \alpha_n\}$, $1 \leq i < n$, $M$ is of type $A_{i-1} \times A_{n-i-1}$, $\Sigma_{\text{red}}(T_M) = \{\alpha_i, \alpha_n, \alpha_i + \alpha_n, \alpha_i + 2\alpha_n\}$ is of type $B_2$, $\alpha_i$ is the long root. In order of $\Sigma_{\mu}$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means that $n = 2i$. Then $(\omega_{\alpha_i}^{M_{\alpha_i}}, \omega_{\alpha_n}^{M_{\alpha_n}}) = n/2$, $(\omega_{\alpha_i}^{M_{\alpha_i}}, \omega_{\alpha_n}^{M_{\alpha_n}}) = n/4$, $(\alpha_i, \alpha_i) = 2$, $(\alpha_n, \alpha_n) = 1$, $\epsilon_{\alpha_i}$ and $\epsilon_{\alpha_n}$ are necessarily 1. One deduces that the assumptions are satisfied.

$C_n$:

(1) $\Delta - \Delta^M = \{\alpha_i, \alpha_j\}$, $1 \leq i < j \leq n-1$, $M$ is of type $A_{i-1} \times A_{j-i-1} \times C_{n-j}$, $\Sigma_{\text{red}}(T_M) = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j, \alpha_i + 2\alpha_j\}$ is of type $B_2$, $\alpha_i$ is the long root, $M_{\alpha_i}$ is of type $A_{j-1} \times C_{n-j}$ and $M_{\alpha_j}$ is of type $A_{i-1} \times C_{n-i}$. In order of $\Sigma_{\mu}$ to
Here the relevant cases are:

- $\Delta^-\Delta^M = \{(\alpha_i, \alpha_n)\}, 1 \leq i < n, M$ is of type $A_{i-1} \times A_{n-i-1}$, $\Sigma_{red}(T_M) = \{\alpha_i, \alpha_n, \alpha_i + \alpha_n, 2\alpha_i + \alpha_n\}$ is of type $B_2$, $\alpha_n$ is the long root. In order of $\Sigma_\mu$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means that $n = 2i$. Consequently, there are no relevant cases.

$D_n$:

1. $\Delta - \Delta^M = \{(\alpha_i, \alpha_j)\}, 1 \leq i < j \leq n - 2$, $M$ is of type $A_{i-1} \times A_{j-i-1} \times D_{n-j}$, $\Sigma_{red}(T_M) = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j, \alpha_i + 2\alpha_j\}$ is of type $B_2$, $\alpha_i$ is the long root, $M_{\alpha_i}$ is of type $A_{j-1} \times D_{n-j}$, $M_{\alpha_j}$ is of type $A_{i-1} \times D_{n-i}$. In order of $\Sigma_\mu$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means that $j = 2i$. Then $\omega_{\alpha_i}^{M_{\alpha_i}}, \omega_{\alpha_i}^{M_{\alpha_j}} = j/2$, $\omega_{\alpha_i}^{M_{\alpha_j}}, \omega_{\alpha_j}^{M_{\alpha_i}} = j$, $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j) = 2, \epsilon_{\alpha_i}$ is necessarily 1 and $\epsilon_{\alpha_j}$ may be 1 or 2. One deduces that the assumptions are satisfied.

2. $\Delta - \Delta^M = \{(\alpha_i, \alpha_j)\}, 1 \leq i < n, j = n-1$ or $j = n$, $M$ is of type $A_{i-1} \times A_{n-i-1}$, $\Sigma_{red}(T_M) = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j\}$ is of type $A_2$. Consequently, there are no relevant cases.

$E_6$:

Here the only relevant case is

1. $\Delta - \Delta^M = \{(\alpha_2, \alpha_4)\}, M$ is of type $A_2 \times A_2$, $\Sigma_\mu = \{\alpha_2, \alpha_4, \alpha_2 + \alpha_4, \alpha_2 + 2\alpha_4, \alpha_2 + 3\alpha_4\}$ is of type $G_2$, $\alpha_2$ is the long root. As $M_{\alpha_2}$ and $M_{\alpha_4}$ are both of $A$-type, $(\omega_{\alpha_2}^{M_{\alpha_2}}, \omega_{\alpha_2}^{M_{\alpha_2}}) = 1/2$ and $(\omega_{\alpha_4}^{M_{\alpha_4}}, \omega_{\alpha_4}^{M_{\alpha_4}}) = 3/2$, $\epsilon_{\alpha_2}$ and $\epsilon_{\alpha_4}$ are necessarily 1. One deduces that the assumptions are satisfied.

$E_7$:

Here the relevant cases are:
The relevant cases are:

1. $\Delta - \Delta^M = \{\alpha_1, \alpha_3\}$, $M$ is of type $A_5$, $\Sigma_\mu = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_3, \alpha_1 + 2\alpha_3, \alpha_1 + 3\alpha_3, 2\alpha_1 + 3\alpha_3\}$ is of type $G_2$, $\alpha_1$ is the long root, $M_{\alpha_1}$ is of type $A_1 \times A_5$, $M_{\alpha_3}$ is of type $D_6$, $(\omega^{M_{\alpha_1}}, \omega^{M_{\alpha_3}}) = 1/2$ and $(\omega^{M_{\alpha_3}}, \omega^{M_{\alpha_3}}) = 3/2$, $\epsilon_{\alpha_1}$ and $\epsilon_{\alpha_3}$ are always 1 and it follows from lemma 6.1 that $\epsilon$ is always 1 and it deduces that the assumptions are satisfied.

2. $\Delta - \Delta^M = \{\alpha_1, \alpha_6\}$, $M$ is of type $D_4 \times A_1$, $\Sigma_\mu = \{\alpha_1, \alpha_6, \alpha_1 + \alpha_6, \alpha_1 + 2\alpha_6\}$ is of type $B_2$, $\alpha_1$ is the long root, $M_{\alpha_1}$ and $M_{\alpha_6}$ are both of type $D_5$, $(\omega^{M_{\alpha_1}}, \omega^{M_{\alpha_1}}) = 1$ and $(\omega^{M_{\alpha_6}}, \omega^{M_{\alpha_6}}) = 2$, $\epsilon_{\alpha_1}$ is always 1 and $\epsilon_{\alpha_6}$ can be 1 or 2. One deduces that the assumptions are satisfied.

3. $\Delta - \Delta^M = \{\alpha_4, \alpha_6\}$, $M$ is of type $A_2 \times A_1 \times A_1 \times A_1$, $\Sigma_\mu = \{\alpha_4, \alpha_6, \alpha_4 + \alpha_6, 2\alpha_4 + \alpha_6, 3\alpha_4 + \alpha_6, 3\alpha_4 + 2\alpha_6\}$ is of type $G_2$, $\alpha_6$ is the long root, $M_{\alpha_4}$ is of type $D_5$, $M_{\alpha_6}$ is of type $A_2 \times A_1 \times A_3$, $(\omega^{M_{\alpha_4}}, \omega^{M_{\alpha_4}}) = 3$ and $(\omega^{M_{\alpha_6}}, \omega^{M_{\alpha_6}}) = 1$, $\epsilon_{\alpha_6}$ is always 1 and it follows from lemma 6.1 that $\epsilon_{\alpha_4}$ is always 1, too. One deduces that the assumptions are satisfied.

$E_8$:
The relevant cases are:

1. $\Delta - \Delta^M = \{\alpha_1, \alpha_5\}$, $M$ is of type $A_3 \times A_3$, $\Sigma_\mu = \{\alpha_1, \alpha_5, \alpha_1 + \alpha_5, \alpha_1 + 2\alpha_5, \alpha_1 + 3\alpha_5\}$ is of type $B_2$, $\alpha_1 + \alpha_5$ is the long root, $M_{\alpha_5}$ is of type $D_7$, $M_{\alpha_1 + \alpha_5}$ is of type $D_4$, $(\omega^{M_{\alpha_5}}, \omega^{M_{\alpha_5}}) = 4$ and $(\omega^{M_{\alpha_1 + \alpha_5}}, \omega^{M_{\alpha_1 + \alpha_5}}) = 2$, and $\epsilon_{\alpha_1 + \alpha_5}$ is always 1 and $\epsilon_{\alpha_1}$ can be 1 or 2. One deduces that the assumptions are satisfied.

2. $\Delta - \Delta^M = \{\alpha_1, \alpha_6\}$, $M$ is of type $D_4 \times A_2$, $\Sigma_\mu = \{\alpha_1, \alpha_6, \alpha_1 + \alpha_6, \alpha_1 + 2\alpha_6, \alpha_1 + 3\alpha_6, 2\alpha_1 + 3\alpha_6\}$ is of type $G_2$, $\alpha_1$ is the long root, $M_{\alpha_1}$ is of type $D_5 \times A_2$, $M_{\alpha_6}$ is of type $D_7$, $(\omega^{M_{\alpha_1}}, \omega^{M_{\alpha_1}}) = 1$ and $(\omega^{M_{\alpha_6}}, \omega^{M_{\alpha_6}}) = 3$, $\epsilon_{\alpha_1}$ is always 1 and it follows from lemma 6.1 that $\epsilon_{\alpha_6}$ is always 1, too. One deduces that the assumptions are satisfied.

3. $\Delta - \Delta^M = \{\alpha_1, \alpha_8\}$, $M$ is of type $D_6$, $\Sigma_\mu = \{\alpha_1, \alpha_8, \alpha_1 + \alpha_8, 2\alpha_1 + \alpha_8\}$ is of type $B_2$, $\alpha_8$ is the long root, $M_{\alpha_1}$ is of type $E_7$, $M_{\alpha_8}$ is of type $D_7$, $(\omega^{M_{\alpha_8}}, \omega^{M_{\alpha_8}}) = 2$ and $(\omega^{M_{\alpha_1}}, \omega^{M_{\alpha_1}}) = 1$, and $\epsilon_{\alpha_8}$ is always 1 and $\epsilon_{\alpha_1}$ can be 1 or 2. One deduces that the assumptions are satisfied.

4. $\Delta - \Delta^M = \{\alpha_2, \alpha_5\}$, $M$ is of type $A_3 \times A_3$, $\Sigma_\mu = \{\alpha_2, \alpha_2 + \alpha_5, \alpha_2 + 2\alpha_5, 2\alpha_2 + 3\alpha_5\}$ is of type $B_2$, $\alpha_5$ is the long root, $M_{\alpha_5}$ is of type $A_7$, $M_{\alpha_2 + \alpha_5}$ is of type $D_7$. 

Figure 6. Dynkin diagram for $E_7$

Figure 7. Dynkin diagram for $E_8$
The relevant cases are satisfied.

\( \Delta - \Delta^M = \{ \alpha_4, \alpha_6 \} \), \( M \) is of type \( A_2 \times A_1 \times A_1 \times A_2 \), \( \Sigma_{\mu} = \{ \alpha_4, \alpha_4 + \alpha_6, 2\alpha_4 + \alpha_6, 3\alpha_4 + 2\alpha_6 \} \) is of type \( B_2 \), \( \alpha_4 + \alpha_6 \) is of type \( D_5 \), \( M_{\alpha_4 + \alpha_6} \) is of type \( E_6 \), \( (\omega_{\alpha_4}, \omega_{\alpha_4}) = 3 \) and \( (\omega_{\alpha_4 + \alpha_6}, \omega_{\alpha_4 + \alpha_6}) = 6 \), \( \epsilon_{\alpha_4 + \alpha_6} \) can be 1 or 2, and it follows from lemma 6.1 that \( \epsilon_{\alpha_4} \) is always 1. One deduces that the assumptions are satisfied.

\( \Delta - \Delta^M = \{ \alpha_4, \alpha_7 \} \), \( M \) is of type \( A_2 \times A_1 \times A_2 \times A_1 \), \( \Sigma_{\mu} = \{ \alpha_4, \alpha_4 + \alpha_7, 2\alpha_4 + \alpha_7, 3\alpha_4 + \alpha_7 \} \) is of type \( B_2 \), \( \alpha_4 + \alpha_7 \) is of type \( D_5 \), \( M_{\alpha_4 + \alpha_7} \) is of type \( E_6 \), \( M_{\alpha_4 + \alpha_7} \) is of type \( E_7 \), \( \alpha_8 \) is of type \( E_6 \times A_1 \), \( (\omega_{\alpha_8}, \omega_{\alpha_8}) = 1/2 \), \( \epsilon_{\alpha_8} \) is always 1 and \( \epsilon_{\alpha_7} \) may be 1 or 2. One deduces that the assumptions are satisfied.

\( \Delta - \Delta^M = \{ \alpha_7, \alpha_8 \} \), \( M \) is of type \( E_6 \), \( \Sigma_{\mu} = \{ \alpha_7, \alpha_8, \alpha_7 + \alpha_8, 2\alpha_7 + \alpha_8, 3\alpha_7 + \alpha_8, 3\alpha_7 + 2\alpha_8 \} \) is of type \( G_2 \), \( \alpha_8 \) is of type \( E_7 \), \( M_{\alpha_8} \) is of type \( E_6 \times A_1 \), \( (\omega_{\alpha_8}, \omega_{\alpha_8}) = 1/2 \), \( \epsilon_{\alpha_8} \) is always 1 and \( \epsilon_{\alpha_7} \) may be 1 or 2. One deduces that the assumptions are satisfied.

6.2. The non-split quasi-split cases: Here the absolute root system differs from the \( F \)-root system. The question of self-conjugacy can be dealt with the \( F \)-root system. For the formula which relates \( \epsilon_\beta \) and \( \epsilon_{\overline{\beta}} \), one has now to use \( \tilde{\beta} \), which is a multiple of \( \omega_{\beta}^{M_\beta} \) by a nonzero scalar. This scalar is determined by the relation between the restrictions of \( \beta^\vee \) and \( \beta^\vee \). Remark that all the absolute root systems
below are simply laced, so that the absolute roots have in each case the same length. We will also use the fact that the $\epsilon_\alpha$ are invariant by restriction of scalars.

$^2A_{2n-1}$

This absolute root system corresponds to quasi-split groups which split over a quadratic extension $F'$ of $F$. The $F$-root system is of type $C_n$. Hence we have the same relevant cases as discussed in the split $C_n$ case. We will denote by $\tilde{A}_i$ the type of a quasi-split group which is the restriction of scalars with respect to $F'/F$ of a split group of type $A_i$.

(1) $\Delta - \Delta^M = \{\alpha_i, \alpha_j\}, \ 1 \leq i < j \leq n - 1$, $M$ is of type $\tilde{A}_{i-1} \times \tilde{A}_{j-i-1} \times ^2A_{2(n-j)-1}$, $\Sigma_{\text{red}}(T_M) = \{\alpha_i, \alpha_n, \alpha_i + \alpha_n, \alpha_i + 2\alpha_j\}$ is of type $B_2$, $\alpha_i$ is the long root. If $\Sigma_\mu$ is properly contained in $\Sigma(T_M)$, $M_{\alpha_i}$ is of type $\tilde{A}_{j-1} \times ^2A_{2(n-j)-1}$ and $M_{\alpha_j}$ of type $\tilde{A}_{i-1} \times ^2A_{2(n-i)-1}$. In order of $\Sigma_\mu$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means $j = 2i$. Then $(\tilde{\alpha}_i, \tilde{\alpha}_i) = j/2$ and $(\tilde{\alpha}_j, \tilde{\alpha}_j) = j$. As the $\epsilon_\beta$ are invariant for restriction of scalars, we have always $\epsilon_{\alpha_i} = 1$, $\epsilon_{\alpha_j}$ may be 1 or 2. One deduces that our assumptions are satisfied.

(2) $\Delta - \Delta^M = \{\alpha_i, \alpha_n\}, \ 1 \leq i < n$, $M$ is of type $\tilde{A}_{i-1} \times \tilde{A}_{n-i-1}$, $\Sigma_{\text{red}}(T_M) = \{\alpha_i, \alpha_n, \alpha_i + \alpha_n, 2\alpha_i + \alpha_n\}$ is of type $B_2$, $\alpha_n$ is the long root. In order of $\Sigma_\mu$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means $n = 2i$. Then $(\tilde{\alpha}_i, \tilde{\alpha}_i) = n/2$, $(\tilde{\alpha}_n, \tilde{\alpha}_n) = n/4$, $\epsilon_\alpha$ is always 1 (as in the previous case) and $\epsilon_{\alpha_n} = 1$ by [Sh2, diagram $^2A_{2k-1}-2$]. One deduces that the assumptions are satisfied.

$^2A_{2n}$
This absolute root system corresponds to $F$-groups which split over a quadratic extension $F'/F$. The reduced $F$-roots system is of type $B_n$. Hence we have the same relevant cases as discussed in the split $B_n$ case.

(1) $\Delta - \Delta^F = \{\alpha_i, \alpha_j\}, 1 \leq i < j \leq n-1$, $M$ is of type $\tilde{A}_{i-1} \times \tilde{A}_{j-i-1} \times 2A_2(n-j)$, $\Sigma_{red}(T_M) = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j, \alpha_i + 2\alpha_j\}$ is of type $B_2$, $\alpha_i$ is the long root, the relevant first factor of $M_{\alpha_i}$ is of type $\tilde{A}_{j-1}$, the relevant second factor of $M_{\alpha_j}$ is of type $2A_{2(n-i)}$. In order of $\Sigma_{\mu}$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means $j = 2i$. Then $(\tilde{\alpha}_i, \tilde{\alpha}_i) = j/2, (\tilde{\alpha}_j, \tilde{\alpha}_j) = j, \epsilon_{\alpha_i} = 1$ as above and, by [Sh2] diagram $2A_{2k-1} - 1, 4$, $\epsilon_{\alpha_j}$ may be 1 or 2. One deduces that our assumptions are satisfied.

(2) $\Delta - \Delta^F = \{\alpha_i, \alpha_n\}, 1 \leq i < n$, $M$ is of type $\tilde{A}_{i-1} \times \tilde{A}_{n-i-1}$, $\Sigma_{red}(T_M) = \{\alpha_i, \alpha_n, \alpha_i + \alpha_n, \alpha_i + 2\alpha_n\}$ is of type $B_2$, $\alpha_i$ is the long root. In order of $\Sigma_{\mu}$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means $n = 2i$. Then $(\tilde{\alpha}_i, \tilde{\alpha}_i) = n/2, (\tilde{\alpha}_n, \tilde{\alpha}_n) = n, \epsilon_{\alpha_i} = 1$ (as in the previous case) and $\epsilon_{\alpha_n}$ can be 1 or 2 by [Sh2] diagram $2A_{2k-1} - 3$. One deduces that our assumptions are satisfied.

$2D_{n+1}$:

This absolute root system corresponds to $F$-groups which split over a quadratic extension $F'/F$. The reduced $F$-roots system is of type $B_n$. Hence we have the same relevant cases as discussed in the split $B_n$ case.

(1) $\Delta - \Delta^F = \{\alpha_i, \alpha_j\}, 1 \leq i < j \leq -1$, $M$ is of type $A_{i-1} \times A_{j-i-1} \times 2D_{n-j+1}$, $\Sigma_{red}(T_M) = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j, \alpha_i + 2\alpha_j\}$ is of type $B_2$, $\alpha_i$ is the long root, the relevant first factor of $M_{\alpha_i}$ is of type $A_{j-1}$, the relevant second factor of $M_{\alpha_j}$ is of type $2D_{n-i+1}$. In order of $\Sigma_{\mu}$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means $j = 2i$. Then $(\tilde{\alpha}_i, \tilde{\alpha}_i) = j/2, (\tilde{\alpha}_j, \tilde{\alpha}_j) = j$, clearly $\epsilon_{\alpha_i} = 1$ and, by [Sh2] diagram $2D_n - 1, 2$, $\epsilon_{\alpha_j}$ may be 1 or 2. One deduces that our assumptions are satisfied.

(2) $\Delta - \Delta^F = \{\alpha_i, \alpha_n\}, 1 \leq i < n$, $M$ is of type $A_{i-1} \times A_{n-i-1}$, $\Sigma_{red}(T_M) = \{\alpha_i, \alpha_n, \alpha_i + \alpha_n, \alpha_i + 2\alpha_n\}$ is of type $B_2$, $\alpha_i$ is the long root. In order of $\Sigma_{\mu}$ to be of type $B_2$, $M$ must be self-conjugate in $M_{\alpha_i}$, which means $n = 2i$. Then $(\tilde{\alpha}_i, \tilde{\alpha}_i) = n/2, (\tilde{\alpha}_n, \tilde{\alpha}_n) = n, \epsilon_{\alpha_i} = 1$ (as in the previous case) and $\epsilon_{\alpha_n}$ may be 1 or 2 by [Sh2] diagram $2D_n - 3$. One deduces that our assumptions are satisfied.

Figure 11. Index and relative Dynkin diagram for $2D_{n+1}$.
$3D_4$ and $6D_4$:  
These are the two quasi-split triality $D_4$ groups. The group $3D_4$ splits over a (cyclic) extension of degree 3 and the group $6D_4$ over a Galois extension of degree 6 with Galois group $S_3$. So, in both cases the absolute root system is the same, only the action of the Galois group differs. The $F$-root system is in both cases of type $G_2$, which is already of rank 2. So the only relevant case is, when $\Sigma_\mu$ equals the $F$-root system. Denote by $\alpha_1$ the short root and by $\alpha_2$ the long root. As $M_{\alpha_1}$ is of type $A_1$, one has always $\epsilon_{\alpha_1} = 1$. The group $M_{\alpha_2}$ is of type $\tilde{A}_1$, which means that the root system of its $L$-group is the union of three root systems of type $A_1$ with a transitive action of the Galois group. One deduces that $\epsilon_{\alpha_2}$ is always 1, too. As $(\tilde{\alpha}_1, \tilde{\alpha}_1) = 2$ and $(\tilde{\alpha}_2, \tilde{\alpha}_2) = 2/3$, our assumptions are satisfied.

$2E_6$:  
The two quasi-split cases of $2E_6$ type (one has an unramified quadratic extension as “splitting field”, the other a ramified extension of degree 2) give rise to a relative Dynkin diagram of type $F_4$ (which dictates the analysis of the relevant cases). In these cases the analysis is exactly the same. We denote by $F'$ the splitting field (a quadratic extension of $F$).

(1) $\Delta^M = \Delta - \{\alpha_1, \alpha_2\}$, $M$ is of type $\tilde{A}_2$, $\Sigma_\mu = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$ is of type $G_2$, $\alpha_1$ is the long root, $M_{\alpha_1}$ is of type $A_1 \times \tilde{A}_2$, $M_{\alpha_2}$ is of type $2A_5$, $(\tilde{\alpha}_1, \tilde{\alpha}_1) = 1/2$ and $(\tilde{\alpha}_2, \tilde{\alpha}_2) = 3/2$, $\epsilon_{\alpha_1}$ is always 1 and $\epsilon_{\alpha_2}$ may be 1 or 2 (by [Sh2, diagram $2E_6 - 1$]), and it follows from lemma 6.1 that $\epsilon_{\alpha_2}$ is always 1.
\[ \Delta^M = \Delta - \{\alpha_1, \alpha_2\}, \] 
\[ M \text{ is of type } \text{ of type } 2A_3, \Sigma_{\mu} = \{\alpha_1, \alpha_4, \alpha_1 + \alpha_4, \alpha_3 + 2\alpha_4\} \]
is of type \(B_2\), \(\alpha_1\) is the long root, \(M_{\alpha_1}\) is of type \(2D_4\), \(M_{\alpha_4}\) is of type \(2A_5\), 
\((\tilde{\alpha}_1, \alpha_1) = 1\) and 
\((\tilde{\alpha}_4, \alpha_4) = 2\), \(\epsilon_{\alpha_1}\) is always 1, and \(\epsilon_{\alpha_4}\) may be 1 or 2 by [Sh2] Diagram \(2D_4 - 1\). One deduces that the assumptions are satisfied.

\[ \Delta^M = \Delta - \{\alpha_3, \alpha_4\}, \]
\[ M \text{ is of type } A_2, \Sigma_{\mu} = \{\alpha_3, \alpha_4, \alpha_3 + \alpha_4, 2\alpha_3 + \alpha_4, 3\alpha_3 + \alpha_4, 3\alpha_3 + 2\alpha_4\} \]
is of type \(G_2\), \(\alpha_4\) is the long root, \(M_{\alpha_3}\) is of type \(2D_4\), \(M_{\alpha_4}\) is of type \(A_2 \times A_1\), 
\((\tilde{\alpha}_3, \tilde{\alpha}_3) = 3\) and 
\((\tilde{\alpha}_4, \tilde{\alpha}_4) = 1\), \(\epsilon_{\alpha_4}\) is always 1 and \(\epsilon_{\alpha_3}\) may 
be 1 or 2 (for the first, use [Sh2] Diagram \(2D_4\), and it follows from lemma 6.1 that \(\epsilon_{\alpha_4}\) is always 1.

References

[B] Bourbaki, N. “Groupes et Algèbres de Lie”, Chap. 6, Masson, Paris, 1981.

[Ca] Casselman, W., “Introduction to the theory of admissible representations of p-adic reductive groups”, non publié.

[CSh] Casselman, W. and Shahidi, F., “On irreducibility of standard modules for generic representations”, Ann. Sc. e. Norm. Sup. 31 (1998), 561–589.

[H1] Heiermann, V., “Une formule de Plancherel pour l’algèbre de Hecke d’un groupe réductif p-adique”, Comm. Math. 76 (2001), 388–415.

[H2] Heiermann, V., “Décomposition spectrale et représentations spéciales d’un groupe réductif p-adique”, Journ. Inst. Math. Jussieu 3 (2004), 327–395.

[HM] Heiermann, V. and Muic, G., “The standard modules conjecture”, Math. Zeitschr. 255 (2007), no. 1, 19–37.

[KH] Kim, H., “On local L-functions and normalized intertwining operators”, Can. J. Math. 57 (2005), no. 3, 535–597.

[KK] Kim, H., Kim, W. “On local L-functions and normalized intertwining operators II; quasi-split groups”, to appear in Shahidi’s birthday conference volume 2008

[KW1] Kim, W., “Square Integrable Representations and the Standard Module Conjecture for General Spin Groups”, Can. J. Math. 61 (2009), no. 3, 617–640.

[KW2] Kim, W., “Holomorphy of L-functions and intertwining operators”, preprint, 2009.

[L] Langlands, R.P. “Euler products”, Yale University, New Haven, 1971.

[MSh] Muic, G., Shahidi, F. “Irreducibility of standard representations for Iwahori-Spherical Representations”, Math. Ann. 312 (1998), 151–165.

[O] Opdam, E., “On the spectral decomposition of Affine Hecke Algebras”, Journ. Inst. Math. Jussieu 3 (2004), no. 1, 531–648.

[Re] Reeder, M., “p-adic Whittaker functions and vector bundles on flag manifolds”, Comp. Mathem. 5 (1993), 9–36.

[Ro] Rodier, F., “Whittaker models for admissible representations”, Proc. Sympos. Pure Math. AMS 26 (1973), 425–430.

[Sh1] Shahidi, F., “On certain L-functions”, Amer. J. Math. 103 (1981), 297–356.

[Sh2] Shahidi, F., “On the Ramanujan conjecture and finiteness of poles for certain L-functions”, Ann. Math. 127 (1981), 547–584.

[Sh3] Shahidi, F., “A proof of Langlands’ conjecture on Plancherel measures; complementary series for p-adic groups”, Ann. Math. 132 (1990), 273–330.

[Si1] Silberger, A., “Special representations of reductive p-adic groups are not integrable”, Ann. Math. 103 (1980), 571–587.

[Si2] Silberger, A., “Discrete series and classification of p-adic groups I”, Am. J. Math. 103 (1981), 1241–1321.

[W] Waldspurger, J.-L., “La formule de Plancherel pour les groupes p-adiques (d’après Harish-Chandra)”, J. Inst. Math. Jussieu 2 (2003), 235–333.