Deconfinement phase transition in $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$ from supersymmetric matrix quantum mechanics

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We test the recent claim that supersymmetric matrix quantum mechanics with mass deformation preserving maximal supersymmetry can be used to study $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$ in the planar limit. When the mass parameter is large, we can integrate out all the massive fluctuations around a particular classical solution, which corresponds to $R \times S^3$. The resulting effective theory for the gauge field moduli at finite temperature is studied both analytically and numerically, and shows that the deconfinement phase transition in $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$ at weak coupling. This transition was speculated to be a continuation of the conjectured phase transition at strong coupling, which corresponds to the Hawking-Page transition based on the gauge-gravity duality. By choosing a different classical solution of the same model, one can also reproduce results for gauge theories on other space-time such as $R \times S^3/Z_n$ and $R \times S^3$. All these theories can be studied at strong coupling by the new simulation method, which was used successfully for supersymmetric matrix quantum mechanics without mass deformation.

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Introduction. — Supersymmetric large-$N$ gauge theories have attracted much attention in the past decade due to its conjectured duality [1] to classical gravity theories. In order to investigate the duality further and to make use of it, one has to study the gauge theories in the strongly coupled regime. Monte Carlo simulation can be a powerful approach as in QCD, but the problem is that the naive lattice regularization breaks supersymmetry. Recently there have been considerable developments in constructing lattice theories preserving part of supersymmetry [2]. However, in some cases including theories of interest in the context of gauge-gravity duality, there might be a regularization scheme alternative to the lattice, which respects supersymmetry maximally. In the case of 1d gauge theory or matrix quantum mechanics (MQM), for instance, one can fix the gauge nonperturbatively and introduce a Fourier mode cutoff [3]. Using this method, the case with 16 supercharges has been simulated at various coupling constant [4], and the results interpolated nicely the weak-coupling behavior obtained by high temperature expansion [5] and the strong-coupling behavior predicted from the gauge-gravity duality [6]. (Consistent results were obtained [7] also by using the lattice approach [8].) The aim of this Letter is to make a first step towards an extension of this work to higher dimensions.

In Ref. [9] the use of supersymmetric MQM with mass deformation preserving supersymmetry has been proposed as a regularization of $\mathcal{N} = 4$ super Yang-Mills theory (SYM) on $R \times S^3$ in the planar limit. In fact the supersymmetry of the target theory is enhanced to the superconformal symmetry, which includes 32 supercharges. The regularized theory has only 16 supercharges, but we consider it optimal since the conformal symmetry is necessarily broken by any regularizations. We test this proposal when the mass parameter is large, in which case we can integrate out all the massive fluctuations around a particular classical solution, which corresponds to $R \times S^3$. The resulting effective theory for the gauge field moduli at finite temperature reproduces the deconfinement phase transition in $\mathcal{N} = 4$ SYM on $R \times S^3$ at weak coupling. This transition [10, 11] was speculated to be a continuation of the conjectured phase transition at strong coupling [12], which corresponds to the Hawking-Page transition [13] based on the gauge-gravity duality.

The model and the classical vacuum. — The model we study is defined by [14]

$$S = \frac{1}{g^2} \int dt \text{tr} \left[ \frac{1}{2} (D_t X_M)^2 - \frac{1}{4} [X_M, X_N]^2 + \frac{1}{2} \Psi^\dagger D_t \Psi - \frac{1}{2} \Psi^\dagger \gamma_M [X_M, \Psi] + \frac{\mu^2}{2} (X_i)^2 + \frac{\mu^2}{8} (X_i)^2 + i \mu \epsilon_{ijk} X_i X_j X_k + \frac{3 \mu}{8} \Psi^\dagger \gamma_{123} \Psi \right], \quad (1)$$

where the range of indices is given by $1 \leq M, N \leq 9$, $1 \leq i, j, k \leq 3$ and $4 \leq a \leq 9$. The covariant derivative is defined by $D_t = \partial_t - i [A, \cdot]$, where $A$ as well as $X_M$ and $\Psi$ in (1) are $N \times N$ matrices depending on $t$. The model has SU(2|4) symmetry, which includes 16 supercharges, and for $\mu = 0$ it reduces to the supersymmetric MQM studied recently by Monte Carlo simulation [14].
The model possesses many vacua

\[ X_\nu = \mu \bigoplus_{I=1}^{\nu} \left( L_i^{(n_I)} \otimes 1_{k_I} \right) \]  

(2)

representing multi fuzzy spheres, where \( L_i^{(n_I)} \) are the \( n_I \)-dimensional irreducible representation of the SU(2) generators obeying \( [L_i^{(n_I)} , L_j^{(n_J)}] = i \varepsilon_{ijk} L_k^{(n_I)} \). The parameters \( n_I \) and \( k_I \) in (2) satisfy the relation \( \sum_{I=1}^\nu n_I k_I = N \). These vacua preserve the SU(2)|4 symmetry and they are all degenerate. Let us consider the theory (1) around the background (2) in the case

\[ k_I = k, \quad n_I = n + I - \frac{\nu + 1}{2} \]  

(3)

for \( I = 1, \cdots, \nu \), assuming \( \nu \) to be odd, and take the limit

\[ n \to \infty, \quad \nu \to \infty, \quad k \to \infty, \quad n - \frac{\nu + 1}{2} \to \infty \]

with \( g^2 k / n = \lambda / \nu = \text{fixed} \),

(4)

where \( \nu = 2 \pi^2 (2/\mu)^3 \). It is claimed \( (9) \) (See also Refs. \( [12, 16] \) for earlier proposals.) that the resulting theory is actually equivalent to the planar limit of \( N = 4 \) SYM on \( R \times S^3 \), where \( S^3 \) is considered as an \( S^1 \) bundle over \( R^2 \). By making the Kaluza-Klein (KK) reduction along the \( S^1 \) fiber direction, one obtains the KK modes on \( S^2 \). Reflecting the non-trivial fiber structure, however, the KK modes should be expanded in terms of monopole harmonics on \( S^2 \) —which are not single-valued in general unlike ordinary spherical harmonics—is in the presence of the monopole charges corresponding to the KK momenta. On the other hand, the “fuzzy regularization” of the monopole harmonics is given by \( (n + 2q) \times n \) rectangular matrices, where \( q \) represents the monopole charge. The fluctuations in the \( (I,J) \) block around the background (2) for the case (3) can therefore be regarded as the KK mode with the KK momentum \( (I - J)/2 \). Note that we obtain precisely the right KK momentum spectrum albeit with the cutoff \( (\nu - 1)/2 \). The parameter \( n \) plays the role of the cutoff for the angular momentum on \( S^2 \). Let us emphasize that these two momentum cutoffs preserve the gauge invariance and the SU(2)|4 symmetry, which is a subgroup of the SU(2, 2)|4 superconformal symmetry.

Except for the existence of the cutoffs, planar diagrams obtained by expanding the supersymmetric MQM around the background (3) agree with planar diagrams in the \( N = 4 \) SYM on \( R \times S^3 \). Nonplanar diagrams do not agree for two reasons. One is that the \( S^2 \) is constructed as a fuzzy sphere, and the fuzziness affects nonplanar diagrams. The other is that the reduction in the \( S^3 \) direction is expected to occur due to the mechanism analogous to the quenched Eguchi-Kawai model \( [17, 18] \), which works only for planar diagrams. The role of the quenched momentum variables are played by the monopole charges. Unlike in the usual quenched Eguchi-Kawai model \( [18] \), however, the momentum has a discrete spectrum corresponding to \( S^1 \). Due to this difference, sending \( \nu \) to infinity alone does not remove nonplanar diagrams, and therefore one needs to take the large-\( k \) limit.

Let us also emphasize that the model (1) is a massive theory, which has no flat direction. Furthermore, the background (3) is stable against quantum fluctuations thanks to the SU(2)|4 symmetry at least at zero temperature. Tunneling to the other vacua through the instanton effects is suppressed in the large-\( k \) limit. Hence there is no need to do something like momentum quenching, which is necessary in the quenched Eguchi-Kawai model.

Effective theory for the gauge field moduli.— Let us introduce finite temperature \( T \) by compactifying the Euclidean time \( t \) in (1) to a circle with the circumference \( \beta = T^{-1} \). Unlike the \( T = 0 \) case, one cannot set the gauge field to zero due to the nontrivial holonomy along the \( t \) direction. In fact the gauge field contains some moduli given by \( A(t) = \bigoplus_{I=1}^\nu \{ 1_{n_I} \otimes \tilde{A}(t) \} \) around the general background (2), where \( \tilde{A}(t) \) are \( k_I \times k_I \) hermitian matrices. One can choose a gauge in which \( \tilde{A}(t) \) takes the form \( \tilde{A}(t) = \frac{1}{\beta} \text{diag}(\alpha_1^{(I)}, \cdots, \alpha_{k_I^{(I)}}) \), where \( \alpha_a^{(I)} \in (-\pi, \pi) \) \( (a = 1, \cdots, k_I) \). At small \( g^2 \) or at large \( \mu \), one can integrate over all the massive fluctuations around the general background (2), and the effective action for the gauge field moduli is given by eq. (3.10) in Ref. \( [19] \).

Here we restrict ourselves to the particular case (3), and study the effective action analytically in the large-\( k \) limit to test the equivalence to the planar limit of \( N = 4 \) SYM on \( R \times S^3 \) at small \( \lambda \) (See eq. (4.3)). For that purpose we rewrite the effective action in terms of the distribution functions of \( \alpha_a^{(I)} \) defined by

\[ \rho^{(I)}(\theta) = \frac{1}{k} \sum_{a=1}^k \delta(\theta - \alpha_a^{(I)}) \]  

(5)

One can easily arrive at the form

\[ S_{\text{MQM}} = k^2 \sum_{I,J=1}^\nu \int d\theta d\theta' \rho^{(I)}(\theta) V^{(I,J)}(\theta - \theta') \rho^{(J)}(\theta') \]  

(6)

\[ V^{(I,J)}(\theta) = \sum_{p=1}^\infty \tilde{V}_p^{(I,J)} \cos(p\theta) \]  

(7)

\[ \tilde{V}_p^{(I,J)} = \frac{1}{p} \left\{ \delta_{I,J} - 6z_{x^{(I,J)}}(x^p) - z_{v^{(I,J)}}(x^p) \right\} - 4(-1)^{p+1} z_{j^{(I,J)}}(x^p) \]  

(8)

where we have introduced a dimensionless parameter

\[ x = e^{-\beta \mu/2} \]  

(9)
and the functions
\[
\begin{align*}
\hat{z}^{(I,J)}_s(x) &= x \frac{\partial}{\partial x} \left( \frac{x^{n_I-n_J}+1}{1-x^2} \right), \\
\hat{z}^{(I,J)}_v(x) &= x^2 \frac{\partial}{\partial x} \left( \frac{x^{n_I-n_J}-1+2\delta_{IJ}}{1-x^2} \right) \\
\hat{z}^{(I,J)}_f(x) &= x^2 \frac{\partial}{\partial x} \left( \frac{x^{n_I-n_J}+2\delta_{IJ}}{1-x^2} \right)
\end{align*}
\] (10)

with \( n_{IJ} = n_I + n_J - |n_I - n_J| \), which can be interpreted as the single-particle partition functions for the scalars, the vector and the fermions, respectively.

Since \( k \) appears only as an overall coefficient in the effective action \([10, 11]\), one can solve the theory exactly in the large-\( k \) limit by the saddle-point equation
\[
\sum_{J=1}^{\nu} \int d\theta' V^{(I,J)}(\theta - \theta') \rho^{(I)}(\theta') = 0 . \tag{13}
\]
The free energy is obtained by \( F_{\text{MQM}} = T S_{\text{MQM}} \), where we use the solution which minimizes the action when we evaluate \( S_{\text{MQM}} \) on the right hand side.

**Results for \( N = 4 \) SYM on \( R \times S^3 \).** To proceed further, let us recall some known results for \( N = 4 \) U(\( k \)) SYM on \( R \times S^3 \) in the planar large-\( k \) limit at weak coupling. Integrating out all the massive fields, one obtains the effective action \([10, 11]\)
\[
S_{\text{SYM}} = k^2 \int d\theta d\theta' \rho(\theta) V(\theta - \theta') \rho(\theta') \tag{14}
\]
for the distribution \( \rho(\theta) \) of the gauge field moduli, where the kernel \( V(\theta) \) is expanded as
\[
V(\theta) = \sum_{p=1}^{\infty} \hat{V}_p \cos(p\theta) , \tag{15}
\]
\[
\hat{V}_p = \frac{1}{p} \left\{ 1 - 6z_s(x^p) - z_v(x^p) - 4(-1)^{p+1}z_f(x^p) \right\} .
\]
The single partition functions are written as
\[
z_s(x) = \frac{x + x^2}{(1-x)^3}, \quad z_v(x) = \frac{6x^2 - 2x^3}{(1-x)^3}, \quad z_f(x) = \frac{4x^2}{(1-x)^3} .
\]

Obviously the uniform distribution is always a solution to the saddle-point equation. At low temperature, it gives the absolute minimum of the effective action. One can show that there is a first order phase transition at a critical point determined by \( \frac{\partial}{\partial x} \hat{V}_1 = 0 \) as \( x_\omega = 7 - 4\sqrt{3} \) \([10, 11]\) in terms of the dimensionless parameter \( \frac{1}{\sqrt{\nu}} \).

Above the critical temperature, the dominant solution has a compact support \([-\theta_0, \theta_0]\) with \( \theta_0 < \pi \), and near the critical temperature, in particular, its explicit form is given by the Gross-Witten form \([10, 11]\)
\[
\rho(\theta) = \frac{1}{\pi\omega} \left( \cos \frac{\theta}{2} \right)^{1/2} \sqrt{\omega - \sin^2 \frac{\theta}{2}} \tag{16}
\]
for \( |\theta| \leq \theta_0 \), where \( \theta_0 = 2\sin^{-1} \sqrt{\omega} \) and \( \omega = 1 - \sqrt{1-\nu^3} \).

Results for MQM. — First let us note that the kernel \( V^{(I,J)}(\theta) \) in \( [8] \) decreases exponentially as \( |I - J| \) becomes large. Moreover, one finds that
\[
\sum_{J=1}^{\nu} V^{(I,J)}(\theta) = \sqrt{\nu}
\]
with \( V(\theta) \) is the kernel \([10]\) for the \( N = 4 \) SYM, and the remaining \( I \)-dependent part \( \Delta V^{(I)}(\theta) \) decreases exponentially as one moves away from the edges \( I = 1 \) and \( I = \nu \). (See below for an explicit confirmation by simulation.) Substituting \([13]\) into \([6]\), we obtain
\[
\frac{1}{k^2} F_{\text{MQM}} = \int d\theta d\theta' \rho(\theta) V(\theta - \theta') \rho(\theta') + \cdots ,
\]
where the abbreviated terms vanish as \( 1/\nu \) in the limit \([4]\). Thus in that limit we obtain the relationship
\[
\frac{1}{k^2} F_{\text{MQM}} = \frac{1}{k^2} F_{\text{SYM}} . \tag{19}
\]

In fact the condition \( n - \nu/2 \rightarrow \infty \) in \([4]\) is not needed in deriving \([19]\). Therefore it may not be necessary for the equivalence.

We can also show \([20]\), that the critical temperatures of the two theories agree without assuming the property \([18]\). The critical temperature of our effective theory \([8]\) can be determined by \( \frac{\partial}{\partial p} \hat{V}_p^{(I,J)} = 0 \) for some \( p \). By noticing that \( \hat{V}_p^{(I,J)} \) becomes a Toeplitz matrix in the \( n \rightarrow \infty \) limit, and using a property of its minimum eigenvalue in the \( \nu \rightarrow \infty \) limit \([21]\), we obtain precisely the same condition \( \hat{V}_1 = 0 \) for the critical point of the effective theory \([13]\) for the \( N = 4 \) SYM.

Monte Carlo Simulation. — As is done in Ref. \([13]\), for simpler cases, we can perform Monte Carlo simulation of the effective theory for the gauge field moduli in our case \([8]\). For instance, we have confirmed the statement \([19]\) explicitly \([20]\). Also we have checked that the distribution \( \rho^{(I)}(\theta) \) of the gauge field moduli has the expected property \([18]\). Figure \([1]\) shows that as one goes towards the midpoint \( I = (\nu + 1)/2 \), the distribution converges rapidly to the result \([19]\) for the \( N = 4 \) SYM on \( R \times S^3 \) represented by the solid line.
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Summary and outlook. — In this Letter we presented explicit computations at weak coupling and at finite temperature that confirm the equivalence between the supersymmetric MQM around a particular background and the $\mathcal{N} = 4$ SYM on $R \times S^3$ in the planar limit [22]. We may therefore hope to study this 4d theory at various \textit{t}

Hooft coupling constant by simulating supersymmetric MQM as has been done already for $\mu = 0$ [4, 7].

By expanding the same model around a different classical solution, we also reproduce [20] analogous phase transitions in supersymmetric gauge theories on other spacetime such as $R \times S^3/Z_q$ [24] and $R \times S^2$ [25]. We can study these theories also at strong coupling and compare the results with their gravity duals [26].

The idea that dimensionally reduced large-$N$ gauge theories can retain information of the theory before reduction dates back to early eighties [17]. Recently it is commonly considered in the opposite way that actually the reduced models are more fundamental, and that the space [27] and possibly also time [28] are merely an emergent notion. Our results show that the reduced model reproduces gauge theory results in various space-time depending on which vacuum one expands the theory around. We consider that this sheds light on the aspects of reduced models as a background independent formulation of quantum gravity.

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