THE LOOP COHOMOLOGY OF A SPACE WITH THE POLYNOMIAL COHOMOLOGY ALGEBRA

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Abstract. We prove that for a simply connected space $X$ with the polynomial cohomology algebra $H^*(X; k) = S(U)$, the loop space cohomology $H^*(\Omega X; k) = \Lambda(s^{-1}U)$ is the exterior algebra if $k$ is a commutative unital ring without 2-torsion, or if and only if $k = \mathbb{Z}_2$ and the Steenrod operation $Sq_1$ is multiplicatively decomposable on $H^*(X; \mathbb{Z}_2)$. The last statement in fact contains a converse of a theorem of A. Borel.

1. Introduction

Let $X$ denote a simply connected topological space. In [3] A. Borel gave a condition for the cohomology $H^*(X; \mathbb{Z}_2)$ to be polynomial in terms of a simple system of generators of the loop space cohomology $H^*(\Omega X; \mathbb{Z}_2)$ that are transgressive (see also [10], [9]). This was one of the first nice applications of spectral sequences that has been introduced in [15], and led in particular to calculations of the cohomology of the Eilenberg-MacLane spaces (see [9]). However, for the converse direction, that is to determine $H^*(\Omega X; \mathbb{Z}_2)$ as an algebra for a given $X$ with $H^*(X; \mathbb{Z}_2)$ polynomial, a spectral sequence argument no longer works. On the other hand, the above calculation is closely related to the problem whether the shuffle product on the bar construction $BH^*(X; \mathbb{Z}_2)$ is geometric; in the case of the affirmative answer we would get $H^*(\Omega X; \mathbb{Z}_2)$ as an exterior algebra. Indeed, in many cases of spaces with the polynomial cohomology algebra, including classifying spaces of topological groups, the loop space cohomology is identified as an exterior algebra ([11], [1]). However, in [12] was given a counterexample to geometricity of the shuffle product for $X = K(\mathbb{Z}_2, n)$, $n \geq 2$, in which case $H^*(X; \mathbb{Z}_2)$ was known to be polynomial.

Note that our calculation of the multiplicative structure of $H^*(\Omega X; k)$ heavily depends on the coefficient (commutative, unital) ring $k$ is 2-torsion free or not: In the first case there is no additional requirement on the polynomial algebra $H^*(X; k)$ the loop space cohomology $H^*(\Omega X; k)$ to be exterior, while the second case involves the Steenrod cohomology operation $Sq_1$. More precisely, we have the following

**Theorem 1.** Let $X$ be a simply connected space with the cohomology algebra $H = H^*(X; k) = S(U)$ to be polynomial.

(i) If $k$ has no 2-torsion, then the loop space cohomology $H^*(\Omega X; k) = \Lambda(s^{-1}U)$ is the exterior algebra.

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(ii) If $k = \mathbb{Z}_2$, then $H^*(\Omega X; \mathbb{Z}_2) = \Lambda(s^{-1}U)$ is the exterior algebra if and only if $S_{q_1}(H) \subset H^+ \cdot H^+$.

Our method of proving the theorem consists of using the so-called Hirsch algebra model $(RH, d + h) \rightarrow C^*(X; k)$ of $X$ ([14], see also below). Note that the underlying differential (bi)graded algebra $(RH, d)$ is a non-commutative version of Tate-Jozefiak resolution of the commutative algebra $H$ ([16], [7]), while $h$ is a perturbation of $d$ similar to [6], and the tensor algebra $RH = T(V)$ is endowed with higher order operations $E_{p, q}$ that extend $\cup$-product measuring the non-commutativity of the product on $RH$; moreover, there also is a binary operation \( \cup_2 \) on $RH$ measuring the non-commutativity of the $\cup$-product. In the case of polynomial $H$ we describe the multiplicative generators $V$ of $RH$ in terms of the above operations that allows an explicit calculation of the cohomology algebra $H^*(BH) = \text{Tor}_H^*(k, k)$, and, consequently, of the loop space cohomology $H^*(\Omega X; k)$.

Note that if $U$ is finite dimensional, then item (i) of Theorem 1 agrees with the main result of [1]; furthermore, for $k$ a field, it can be in fact deduced from the Eilenberg-Moore spectral sequence (see, for example, [9], also for references of using $\cup_1$-product when dealing with polynomial cohomology rings).

2. Hirsch resolutions of polynomial algebras

Graded modules $A^*$ over $k$ are assumed to be connected and 1-reduced, i.e., $A^0 = k$ and $A^1 = 0$. For example, the cochain complex $C^*(X; k)$ of a space $X$ is 1-reduced, since by definition $C^*(X; k) = C^*(\text{Sing}^1 X; k)/C^\geq 0(\text{Sing} x; k)$, in which $\text{Sing}^1 X \subset \text{Sing} X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard $n$-simplex $\Delta^n$ to the base point $x$ of $X$.

Here we construct an explicit multiplicative resolution for a polynomial algebra $H$. Without loss of generality for the application (see section 3) we assume below that when the signs are actually needed $H$ is evenly graded.

First recall the definition of a Hirsch algebra.

Definition 1. A Hirsch algebra is a 1-reduced associative dga $A$ equipped with multilinear maps

$$E_{p, q} : A^p \otimes A^q \rightarrow A,$$  

satisfying the following conditions:

(i) $E_{p, q}$ is of degree $1 - p - q$;

(ii) $E_{1, 0} = Id = E_{0, 1}$ and $E_{p > 1, 0} = 0 = E_{0, q > 1}$;

(iii) The homomorphism $E : BA \otimes BA \rightarrow A$ defined by

$$E([\bar{a}_1] \cdots [\bar{a}_p] \otimes [\bar{b}_1] \cdots [\bar{b}_q]) = E_{p, q}(a_1, ..., a_p; b_1, ..., b_q)$$

and extended as a coderivation induces a chain map $\mu_E : BA \otimes BA \rightarrow BA$.

A morphism $f : A \rightarrow B$ between two Hirsch algebras is a dga map $f$ commuting with all $E_{p, q}$. Condition (iii) implies that $BA$ is a dg Hopf algebra with the product $\mu_E$; though we do not require necessarily $\mu_E$ to be associative (compare [5]), in the sequel we assume the associativity relation given by (2.1) that minimizes the resolution of $H$ involved. Note also that, the component $\mu_{E_{10} + E_{01}}$ of $\mu_E$, denoted by $\mu_{sh}$, coincides...
with the shuffle product on $BA$. In particular, the operation $E_{1,1}$ satisfies conditions similar to Steenrod’s cochain $\cap$-product:

$$dE_{1,1}(a; b) - E_{1,1}(da; b) + (-1)^{|a|}E_{1,1}(a; db) = (-1)^{|a|}ab - (-1)^{|a|(|b|+1)}ba,$$

so it measures the non-commutativity of the $\cdot$ product on $A$. Thus, a Hirsch algebra with $E_{p,q} = 0$ for $p, q \geq 1$ is just a commutative differential graded algebra (cdga). Below we interchangeably denote $E_{1,1}$ by $\cap$.

For a general definition of a bigraded multiplicative resolution $\rho : (RH, d) \to H$ for a cga $H$ see [13, 14]. Here for $H$ polynomial we construct such a resolution as follows. Let $R^*H^* = T(V^{*,*})$ with

$$V^{*,*} = V^{0,*} \oplus E^{<0,*} \oplus T^{-2r,*} = U^*, \quad r \geq 1,$$

where $E^{<0,*} = \{ E^{<0,*}_{p,q} \}_{p, q \geq 1}$ with $E^{<0,*}_{p,q}$ spanned on the set of (formal) expressions $E_{p,q}(a_1, ..., a_p; b_1, ..., b_q)$, $a_i \in R^{*-i}H^*$, $b_m \in R^{-j}H^*$, $n = \sum_{r=1}^p i_r + \sum_{m=1}^q j_m + p + q - 1$, while $T^{-2r,*}$ is spanned on the set of expressions $a_1 \cup a_2 \cup \cdots \cup a_{r+1}$ with $a_i \in V^{0,*}$ and $a_1 \cup a_2 = a_1 \cup a_1$; given a triple $(a; b; c) = (a_1, ..., a_k; b_1, ..., b_l; c_1, ..., c_r)$, $a_i, b_j, c_l \in RH$, the elements of $E$ are subjected to the relations

$$R_{k, \ell, r}(a; b; c) = R_{k, \ell, r}(a; b; c)$$

with

$$R_{k, \ell, r}(a; b; c) = \sum_{k_1 + \cdots + k_p = k \atop \ell_1 + \cdots + \ell_p = \ell \atop 1 \leq p \leq k+\ell} \sgn(a, b) E_{p, r}(E_{k_1, \ell_1}(a_1, ..., a_{k_1}; b_1, ..., b_{\ell_1}), \ldots, E_{k_p, \ell_p}(a_{k_{p-1}}, ..., \tilde{a}_k; b_{\ell_{p-1}}, ..., b_p); c_1, ..., c_r)$$

and

$$R_{k, \ell, r}(a; b; c) = \sum_{k_1 + \cdots + k_p = k \atop \ell_1 + \cdots + \ell_q = \ell \atop 1 \leq q \leq k+\ell} \sgn(a, c) E_{q, r}(a_1, ..., a_k; E_{\ell_1, r_1}(b_1, ..., b_{\ell_1}; c_1, ..., c_{r_1}), \ldots, E_{\ell_q, r_q}(b_{\ell_{q-1}}, ..., b_q; c_{r_q+1}, ..., c_q)).$$

The differential $d$ is defined: On $V^{0,*}$ by $dV^{0,*} = 0$; on $E$ by

$$dE_{p,q}(a_1, ..., a_p; b_1, ..., b_q) = \sum_{1 \leq i \leq p} (-1)^{i+1}E_{p,q}(a_1, ..., da_i, ..., a_p; b_1, ..., b_q)$$

$$+ \sum_{1 \leq j \leq q} (-1)^{p+j+1}E_{p,q}(a_1, ..., a_p; b_1, ..., db_j, ..., b_q)$$

$$+ \sum_{1 \leq i < p} (-1)^iE_{p-1,q}(a_1, ..., a_i, a_{i+1}, ..., a_p; b_1, ..., b_q)$$

$$+ \sum_{1 \leq j < q} (-1)^{p+j}E_{p,q-1}(a_1, ..., a_p; b_1, ..., b_j, b_{j+1}, ..., b_q)$$

$$- \sum_{0 \leq i \leq p \atop 0 \leq j \leq q} (-1)^{(p+i+1)+j}E_{i,j}(a_1, ..., a_i; b_1, ..., b_j) \cdot E_{p-i,q-j}(a_{i+1}, ..., a_p; b_{j+1}, ..., b_q),$$
and on $T$ by
\begin{align}
\label{2.3}
d(a_1 \cup a_2 \cdots \cup a_n) &= \sum_{(i,j)} (a_i \cup a_j \cdots \cup a_n) \sim_1 (a_j \cup a_i \cdots \cup a_n)
\end{align}
where the summation is over unshuffles $(i:j) = (i_1 < \cdots < i_k; j_1 < \cdots < j_k)$ of $n$ with $(a_{i_1}, \ldots, a_{i_k}) = (a_{j_1}, \ldots, a_{j_k})$ if and only if $i = j$.

In particular, $R^0 H^* = T(V^0,*)$, $V^{-1,*} = E_{11}(R^0 H \otimes R^0 H)$, $R^{-1} H^* = V^{-1,*} \oplus (R^{-1} H)^+$ with $(R^{-1} H)^+ = R^{0H+} \cdot V^{-1,*} \oplus V^{-1,*} \cdot R^{0H+}$,

\begin{align*}
V^{-2,*} &= (E_{11}(E_{11} \otimes 1 + 1 \otimes E_{11}) + E_{12} + E_{21})(R^0 H \otimes R^0 H \otimes R^0 H)/\sim \\
&\bigoplus E_{11}((R^{-1} H)^+ \otimes R^0 H \otimes (R^{-1} H)^+) \bigoplus (V^{0,*} \cup_2 V^{0,*}),
\end{align*}
where $\sim$ implies the relation \eqref{2.1} for $(a; b; c) = (a; b; c), a, b, c \in R^0 H^*$.

\begin{align}
E_{1,2}(a; b, c) - E_{1,2}(a; c, b) + a \sim_1 (b \sim_1 c) \\
= (a \sim_1 b) \sim_1 c - E_{2,1}(a, b; c) + E_{2,1}(b, a; c);
\end{align}
and for $a, b \in V^{0,*}, d(a \sim_1 b) = ab - ba, d(a \cup_2 b) = a \sim_1 b + b \sim_1 a, d(a \cup_2 a) = a \sim_1 a, d(a \cup_2 a \cup_2 a) = a \sim_1 (a \cup_2 a) + (a \cup_2 a) \sim_1 a$.

Clearly, $(RH, d)$ is endowed with the operations $E_{p,q}$ and together with the $\cdot$ product on $RH$ is converted into a Hirsch algebra $(RH, d, \cdot, E_{p,q})$. The following proposition proved in section 5 finishes the verification of the projection $\rho : RH \to H$ to be a Hirsch resolution map.

**Proposition 1.** The chain complex $(R^* H^*, d)$ is acyclic in the negative resolution degrees, i.e., $H^i(R^* H^*, d) = 0, i < 0$.

![Figure 1. Geometrical interpretation of some canonical syzygies in the Hirsch resolution RH.](image)

Note that in the above figure the symbol $"\sim"$ assumes equality \eqref{2.4}.

We also need the following multiplicative resolution of $H$ derived from $RH$.

Represent $T$ as
\[ T = T_0 \oplus T', \quad T_0 = \langle a \cup_2 a \mid a \in V^{0,*} \rangle. \]
Let $R_\nu H = RH/\nu'$ where $\nu'$ is a Hirsch ideal in $RH$ spanned on $T' \oplus dT'$. Thus, $R_\nu H = T(V_\nu)$ with $V_\nu = V^{0,*} \oplus E_\nu \oplus T_\nu$, where $E_\nu = E/\langle E \cap \nu' \rangle$, and the inclusion
of modules $V_\nu \subset V$ induces the inclusion of dg algebras
\begin{equation}
R_\nu H \subset RH.
\end{equation}
The argument of the proof that $\rho$ is a resolution map also implies that $I'$ is acyclic and $\rho_\nu = \rho_{R_\nu H} : (R_\nu H, d) \to H$ is a multiplicative resolution of $H$. Clearly, $R_\nu H$ inherits a (non-free) Hirsch algebra operations, $E_{p,q}'$, too.

3. Hirsch models of $X$ with $H^*(X; k)$ polynomial

Recall (2, 8) that given a space $X$, there are operations $E_{p,q}$ on the cochain complex $C^*(X; k)$ making it into a Hirsch algebra. Note that in the simplicial case one can choose $E_{p,q} = 0$ for $q \geq 2$. In particular, the product $\mu_E$ on $BC^*(X; k)$ is geometric, i.e., there is an algebra isomorphism
\begin{equation}
H^*(\Omega X; k) \approx H(BC^*(X; k), d_{AC}, \mu_E).
\end{equation}
In [14] a Hirsch model
\begin{equation}
f : (RH, d + h) \to C^*(X; k)
\end{equation}
of $X$ is constructed. Below such a model can be specialized for $H^*(X; k)$ polynomial as follows.

(i) Let $H^*(X; k)$ be polynomial with $k$ having no 2-torsion. The equality $d(a \wedge_1 a) = -2a^2$ in $C^*(X; k)$, some odd dimensional cocycle $a$, implies that $H = H^*(X; k)$ is evenly graded. Then in (3.2) the perturbation $h$ can be taken to be zero to obtain
\begin{equation}
f : (RH, d) \to C^*(X; k).
\end{equation}
Indeed, first define $f$ on $V$ and then extend multiplicatively. On $V^0, \ast$ by choosing cocycles $f^0 : U^0 \to C^0(X; k)$; on $E$: by
\[ fE_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q) = E_{p,q}(fa_1, \ldots, fa_p; fb_1, \ldots, fb_q), \]
and on $T$ the chain map $f$ is defined inductively: the construction is non-obstructive, since both the cochain complex in the domain and the cohomology of the target are evenly graded.

(ii) Let $H^*(X; k)$ be polynomial with $k = \mathbb{Z}_2$. Then in (3.2) $h|_{V^0, \ast \oplus E} = 0$ and $f$ is defined on $V^0, \ast \oplus E$ as in item (i), while on $T_\nu$ the pair $(h, f)$ is defined by $h^2(a \cup_2 a) = \mathcal{P}_1(a)$ and $df(a \cup_2 a) = f^0\mathcal{P}_1(a) + f^0a \sim f^0a$ for cocycles $a, \mathcal{P}_1(a) \in R^0H$ with $\rho_{\mathcal{P}_1(a)} = Sq_1(\rho a)$.

Note that (2.5) extends to the inclusion of dga’s (not of Hirsch algebras!)
\begin{equation}
i_\nu : (R_\nu H, d + h^2) \subset (RH, d + h),
\end{equation}
where $h^2$ is obtained by restricting of $h$ to $T_\nu$, and $(R_\nu H, d + h^2, \cdot, E_{p,q}^\nu)$ is again a Hirsch algebra.

3.1. A canonical Hirsch algebra structure on $H^*(X; \mathbb{Z}_2)$. Observe that the resolution map $\rho : RH \to H$ no longer remains to be chain with respect to the perturbed differential $d + h$ on $RH$ in item (ii) above. Instead we introduce a Hirsch algebra structure on $H$, this time denoted by
\[ Sq_{p,q} : H^{op} \otimes H^{op} \to H, \]
in a canonical way, and then change $\rho$ to construct a map of Hirsch algebras
\[ f_\nu : (R_\nu H, d + h^2, \cdot, E_{p,q}^\nu) \to (H, 0, \cdot, Sq_{p,q}) \]
as follows. Let \( \mathcal{U} \) be a set of polynomial generators of \( H \). Define first a map \( \tilde{S}_q_{1,1} : \mathcal{U} \times \mathcal{U} \to H \) by

\[
\tilde{S}_q_{1,1}(a; b) = \begin{cases} 
S_q(a), & a = b, \\
0, & \text{otherwise}
\end{cases}
\]

and extend it to the binary operation \( S_{q_{1,1}} : H \otimes H \to H \) as a (both side) derivation with respect to the \( \cdot \) product; in particular, \( S_{q_{1,1}}(u; u) = S_q(u) \) for all \( u \in H \), and \( S_{q_{1,1}}(ab; ab) \) is decomposable via the Cartan formula

\[
S_{q_{1,1}}(ab; ab) = S_q(ab) = S_q(a) \cdot S_q(b) + S_{q_0}(a) \cdot S_q(b).
\]

Furthermore, extend \( S_{q_{1,1}} \) to higher order operations \( S_{q_{p,q}} : H^{\otimes p} \otimes H^{\otimes q} \to H \) with \( S_{p,q}(a_1, ..., a_p; b_1, ..., b_q) = S_{q_{p,q}}(b_1, ..., b_q; a_1, ..., a_p) \) to obtain the Hirsch algebra \((H, 0, \cdot, S_{q_{p,q}})\) as follows. Represent the both sides of (2.1) as

\[
R_{k,l,r}(\langle a; b; c \rangle) = S_{q_{k+l,r}}(sh(\langle a; b; c \rangle)) + \tilde{R}_{k,l,r}(\langle a; b; c \rangle),
\]

where \( sh \) denotes the shuffle multiplication, and for pairs \( u_1 \otimes v_1 \in H^{\otimes p_1} \otimes H^{\otimes q_1} \), \( (p_1, q_1) = (k + l, r) \), and \( u_2 \otimes v_2 \in H^{\otimes p_2} \otimes H^{\otimes q_2} \), \( (p_2, q_2) = (k, l + r) \), of the form \( u_1 \otimes v_1 = sh(\langle a; b; c \rangle) \), \( u_2 \otimes v_2 = a \cdot sh(\langle a; b; c \rangle) \), define

\[
S_{q_{p_1,q_1}}(u_1; v_1) + S_{q_{p_2,q_2}}(u_2; v_2) = \tilde{R}_{k,l,r}(\langle a; b; c \rangle) + \tilde{R}_{k,l,r}(\langle a; b; c \rangle).
\]

Next, for pairs \( u'_1 \otimes v'_1 \in H^{\otimes p_1-1} \otimes H^{\otimes q} \) and \( u'_2 \otimes v'_2 \in H^{\otimes p} \otimes H^{\otimes q-1} \) of the form

\[
\left( \sum_{1 \leq i < j} u_1 \otimes \cdots \otimes u_i u_{i+1} \otimes \cdots \otimes u_p \right) \otimes v \quad \text{and} \quad u'_2 \otimes v'_2 = u \otimes \left( \sum_{1 \leq j < q} v_1 \otimes \cdots \otimes v_j v_{j+1} \otimes \cdots v_q \right)
\]

with some \( u \otimes v \in H^{\otimes p} \otimes H^{\otimes q} \), define

\[
S_{q_{p-1,q}}(u'_1; v'_1) + S_{q_{p,q-1}}(u'_2; v'_2) = \sum_{0 \leq i \leq p \atop 0 \leq j \leq q} S_q_{i,j}(u_1, ..., u_i; u_{i+1}, ..., u_{j-1}; v_1, ..., v_{j-1}) \cdot S_q_{p-i,q-j}(u_{i+1}, ..., u_p; v_{j+1}, ..., v_q).
\]

Note that the left hand sides of (3.5) and (3.6) coincide when \( \langle a; b; c \rangle \) in (3.5) and \( u \otimes v \) in (3.6) are respectively specialized as:

1. \( \langle a; b; c \rangle = (u_3, ..., u_p; u_1 v_2; u_1 u_2) \) and \( u_1 = u_3 = \cdots = u_{2i-1} = \cdots , u_2 = u_4 = \cdots = u_{2i} = \cdots , v_1 = u_1 u_2, q = 1; \)
2. \( \langle a; b; c \rangle = (u_{i+1}, ..., u_p; v_1 v_2; v_3, ..., v_q) \) and \( u_1 = v_1 v_2, p = 1, v_1 = v_3 = \cdots = v_{2j-1} = \cdots , v_2 = v_4 = \cdots = v_{2j} = \cdots ; \)
3. \( \langle a; b; c \rangle = (a; b; a) \) and \( u = (a, b, a), v = (a); \)
4. \( \langle a; b; c \rangle = (a; a; b) \) and \( u = (a, b, a), v = (a, b, a). \)

It is straightforward to check that the right hand sides of (3.5) and (3.6) also coincide for the above cases. Thus formulas (3.5) and (3.6) imply (2.1) and (2.2) for \( S_{q_{p,q}} \) respectively. By this we can obviously achieve that each \( S_{q_{p,q}}, p, q \geq 1 \), is expressed in terms of the \( \cdot \) product and the cohomology operation \( S_q \) on \( H \).

Now define \( f_\nu \) first on the generators and then extend multiplicatively on \( R_\nu H \).

For \( x \in V_\nu \), let

\[
f_\nu(x) = \begin{cases} 
\rho(x), & x \in V^{0,*}, \\
S_{q_{p,q}}(\rho a_1, ..., \rho a_p; \rho b_1, ..., \rho b_q), & x = E_{p,q}(a_1, ..., a_p; b_1, ..., b_q), \\
0, & x \in T_\nu.
\end{cases}
\]
Remark 1. The weak equivalences $\rho$ and $f$ in item (i) above show that $X$ is Hirsch $\mathbb{k}$-formal, while in item (ii), $X$ can be thought of as weak Hirsch $\mathbb{Z}_2$-formal, since $i_\nu$ is not a map of Hirsch algebras.

4. Proof of Theorem 1

To prove the theorem we need the following propositions.

Proposition 2. A morphism $f : A \to A'$ of Hirsch algebras induces a Hopf dga map of the bar constructions

$$Bf : BA \to BA'$$

being a homology isomorphism, if $f$ does and the modules $A, A'$ are $\mathbb{k}$-free.

Proof. The proof is standard by using a spectral sequence comparison argument. □

Proposition 3. By hypotheses of Theorem 1:

(i) If $\mathbb{k}$ has no 2-torsion, then the cohomology algebra $H^*(\Omega X; \mathbb{k})$ is naturally isomorphic to $H(BH, d_{BH}, \mu_{sh})$.

(ii) If $\mathbb{k} = \mathbb{Z}_2$, then the cohomology algebra $H^*(\Omega X; \mathbb{Z}_2)$ is naturally isomorphic to $H(BH, d_{BH}, \mu_{sq})$ with the product $\mu_{sq}$ on $BH$ corresponding to the Hirsch algebra $(H, 0, \cdots, S_{q,p,q})$.

Proof. (i) View the resolution map $\rho : (RH, d) \to H$ as a map of Hirsch algebras with the trivial Hirsch algebra structure $E = \{E_{1,0}, E_{0,1}\}$ on $H$. In particular, $\mu_E = \mu_{sh}$ on $BH$. Then apply Hirsch model (3.3) and Proposition 2 to obtain the following sequence of algebra isomorphisms

$$H(BC^*(X; \mathbb{k}), d_{BC}, \mu_E) \xrightarrow{Bi_\nu^*} H(B(RH), d_{B(RH)}, \mu_E) \xrightarrow{Bi_\nu^*} H(BH, d_{BH}, \mu_{sh}),$$

and the proof is finished by using (3.3).

(ii) This time consider the following sequence of isomorphisms

$$H(BC^*(X; \mathbb{Z}_2), d_{BC}, \mu_E) \xrightarrow{Bf^*} H(B(RH), d_{B(RH)}, \mu_E) \xrightarrow{Bf^*} H(B(R_\nu H), d_{R_\nu H}, \mu_{E_\nu}) \xrightarrow{Bf^*} H(BH, d_{BH}, \mu_{sq}),$$

where $f, f_\nu$ and $i_\nu$ are constructed in section 3 and the first two maps are Hirsch algebra ones. Since $i_\nu$ is not a Hirsch algebra map it remains to show that $Bi_\nu^*$ is multiplicative. Indeed, as an element $y \in H^*(BH)$ has a canonical representative cocycle $b \in BH$ with $b = \sum_{\sigma \in S_n} [b_{\sigma(1)}| \cdots | b_{\sigma(n)}]$, $b_i \neq b_j$, $i \neq j$, and $n \geq 1$, an element $x \in H^*(B(R_\nu H))$ with $x = Bf_\nu^*(y)$ has a canonical representative cocycle $a \in B(R_\nu H)$, $Bf_\nu(a) = b$, $\rho(a_i) = b_i$, $a_i \in R^0 H$, of the form

$$a = \sum_{(i_1; \cdots; i_k)} [\bar{a}_{i_1}| \cdots | \bar{a}_{i_k}] + \sum_{\sigma \in S_n} [\bar{a}_{\sigma(1)}| \cdots | \bar{a}_{\sigma(n)}]$$

where the summation is over unshuffles

$$(i_1; \ldots; i_k) = (i_1 < \cdots < i_1; \ldots; i_{k-1} < \cdots < i_k)$$ of $\underline{n}$.
with $t_\ell > 1$ for some $1 \leq \ell \leq k$, and $a_{i_{t+1}} \in R^{1-t_\ell+1}_p H$,

$$a_{i_{t+1}} = \sum_{0 < r, q < i_{t+1}} a_{i_{t+r-1}} \cdots \vdash 1 \cdots \vdash 1 \vdash 1$$

\((j_1; \ldots; j_q)\) is unshuffle of \((i_{t+1}, \ldots, i_{t+r})\), and \(\vdash 1\)-products are taken in the right most association; in particular, \(a_2 = a_1 \vdash 1 \vdash 2\) for \(n = 2, k = 1, a_3 = a_1 \vdash 1 (a_2 \vdash 1 a_3) + E_{1,2}(a_1; a_2, a_3) + E_{1,2}(a_1; a_3, a_2)\) for \(n = 3, k = 1\). It is easy to see that \(B_i\) is multiplicative restricted to such cocycles. Again the proof is finished by using \((4.1)\).

**Proof of Theorem**

(i) The proof follows from Proposition \((3)\) and the isomorphism \(H(BH, d_{BH}, \mu_{s_q}) \approx \Lambda(s^{-1}U)\).

(ii) By Proposition \((3)\) it suffices to calculate \(H_\Omega := H(BH, d_{BH}, \mu_{s_q})\). If \(Sq_1(H) \subseteq H^+ \cdot H^+\) then by definition of \(Sq_{p,q}\) we also have \(Sq_{p,q}(H^p \otimes H^q) \subseteq H^+ \cdot H^+\) for \(p, q \geq 1\); this means that the induced multiplication \(\mu^*_{s_q}\) is the same as \(\mu^*_{s_{q_{30}}+s_{q_{31}}} = \mu^*_{s_1}\) on \(H_\Omega\), and we get the isomorphism \(H_\Omega \approx H(BH, d_{BH}, \mu_s) \approx \Lambda(s^{-1}U)\) as in item (i).

Conversely, let \(H_\Omega \approx \Lambda(s^{-1}U)\) be an exterior algebra. Suppose that for some \(a, b \in U\) we have \(Sq_1(a) = b\). Then for the elements \(\bar{a}, \bar{b} \in s^{-1}U \subset H_\Omega\), we would have \(\mu^*_{s_q}(\bar{a}; \bar{a}) = \bar{b}\) that contradicts \(H_\Omega\) to be exterior. Theorem \((4)\) is proved.

**Corollary 1.** Every loop space cohomology \(H^*(\Omega X; \mathbb{k})\) with \(H^*(X; \mathbb{k})\) evenly generated polynomial is exterior.

## 5. Proof of Proposition \((1)\)

It suffices to construct a linear map \(s : RH \rightarrow RH\) of total degree -1 with the following property: For each element \(a \in \text{Ker } \rho\) there is an integer \(n(a) \geq 1\) such that \(n(a)^{th}\)-iteration of the operator \(sd + ds - Id : RH \rightarrow RH\) evaluated on \(a\) is zero, i.e.,

\[(sd + ds - Id)^{(n(a))}(a) = 0.\]

The idea of constructing such a map is borrowed from \((4)\). Assume that the set of polynomial generators \(U^* \subset H^*\) is linearly ordered. Let \(V^* = (V^*; \ast)\) with \(V^0, \ast \approx U^*\) and \(\ast \ast = (\Upsilon^*; \ast)\). Given \((v, t) \in V^0, \ast \times \Upsilon (V^0, \ast \times \Upsilon)\), \(t = a_1 \cdots \cdots \vdash a_n (t = a_1 \cup \ldots \cup a_n), a_i \in V^0, \ast\), we write \(v \leq t\) or \(v \geq t\), if \(v \leq a_i\) or \(v \geq a_i\), for all \(i\) respectively. Below we need the subsets \(E_\ast, E_{op} \subset E_1\) and \(E_1, \hat{E} \subset \Upsilon\) defined as follows: Let \(E_1\) be the set of all iterations \(a_1 \cdots \vdash a_n, n \geq 2\), with \(a_i \in V^0, \ast \cup \Upsilon, 1 \leq i \leq n; E_\ast \subset E_1\) be the subset of the right most iterations with all \(a_i \in V^0, \ast\) and \(a_1 < \cdots < a_n; E_{op} \subset E_1\) be the subset of those iterations that contain \(a_k \vdash a_{k+1}\)-product such that \(a_1 < \cdots < a_k\) and \(a_k \geq a_{k+1}\) for \(a_j \in V^0, \ast, 1 \leq j \leq k;\) and, finally, \(E_1, \hat{E} \subset \Upsilon\) be the subset of any iterations of \(E_{p,q}\) evaluated on strings of variables \(a_1, \ldots, a_n, n \geq 2, a_i = x_{i_1} \cdots x_{i_{k_i}}, x_{i,j} \in W\) with \(W = V^0, \ast \cup E_1\) such that at least one \(k_i > 1\) and if \(\epsilon\) is
the smallest number with $k_r > 1$ and $E_{p,q}(u_1, ..., u_p; u_{p+1}, ..., v_{p+q})$ is the operation with $u_r = a_r$, then $r = 1$ for $p \geq 1$ or $r = 2$ for $p = 1$ and $u_1 \in W$. 

Given an element $x \in E^{op}$, let $\tilde{x} \in \tilde{E} \cup \tilde{Y}$ denote the generator obtained from $x$ by replacing $a_k \sim a_{k+1}$ by $a_k \cup a_{k+1}$.

Given an element $x \in \tilde{E}$, choose $E_{p,q}(u; v) = E_{p,q}(u_1, ..., u_p; v_1, ..., v_q)$ with $u_1$ or $v_1$ to be $a$, and let $x' \in E$ denote the generator obtained from $x$ by replacing $E_{p,q}(u; v)$ by

$$E_{p', q'}(u'; v') = \begin{cases} 
E_{p+1,q}(x, y, u_2, ..., u_p; v_1, ..., v_q), & p \geq 1, \quad u_1 = x \cdot y, \quad x \in W, \\
E_{1,q+1}(u_1; x, y, v_2, ..., v_q), & p = 1, \quad v_1 = x \cdot y, \quad u_1, x \in W.
\end{cases}$$

Then define $s$ for a monomial $x_1 \cdot \cdot \cdot x_n \in RH$ and $1 \leq i < k < j \leq n$ by

$$s(x_1 \cdot \cdot \cdot x_n) = \begin{cases} 
x_1 \cdot \cdot \cdot x_{k-1} \sim x_k \cdot \cdot \cdot x_n, & x_i \in \mathcal{Y}_i^0, x_k \in \mathcal{Y}_i^{0} \cup \mathcal{E}_o, x_j \in W \setminus E^{op}, \\
x_1 \cdot \cdot \cdot x_{k-1} \sim x_k \cdot \cdot \cdot x_n, & x_1 \geq \cdot \cdot \cdot \geq x_{k-1} < x_k, \\
x_1 \cdot \cdot \cdot x_{k-1} \sim x_k \cdot \cdot \cdot x_n, & x_i \in W \setminus E^{op}, x_k \in E^{op}, x_j \in W, \\
x_1 \cdot \cdot \cdot x_{k-1} \sim x_k \cdot \cdot \cdot x_n, & x_1 \in W; x_k \in \tilde{E}, x_j \in V, \\
0, & \text{otherwise.}
\end{cases}$$

The map $s$ has a property that if for a generator $v \in V$, $s(v) = 0$, then there is a unique summand component $x$ of $dv$ such that $s(x) = v$ (see Fig. 1). This in fact makes the verification of equality \[5.1\] straightforward.

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