HIGHER DIMENSIONAL GAP THEOREMS
FOR THE MAXIMUM METRIC

ALAN HAYNES, JUAN J. RAMIREZ

Abstract. Recently, the first author together with Jens Marklof studied generalizations
of the classical three distance theorem to higher dimensional toral rotations, giving upper
bounds in all dimensions for the corresponding numbers of distances with respect to any
flat Riemannian metric. In dimension two they proved a five distance theorem, which
is best possible. In this paper we establish analogous bounds, in all dimensions, for the
maximum metric. We also show that in dimensions two and three our bounds are best
possible.

1. Introduction

Suppose that $d \in \mathbb{N}$, let $\mathcal{L}$ be a unimodular lattice in $\mathbb{R}^d$, and define $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$. For
each $\vec{\alpha} \in \mathbb{R}^d$ and $N \in \mathbb{N}$ let $S_N = S_N(\vec{\alpha}, \mathcal{L})$ denote the $d$-dimensional Kronecker sequence
defined by

$$S_N = \{ n\vec{\alpha} + \mathcal{L} : 1 \leq n \leq N \} \subseteq \mathbb{T}^d.$$  

Given a metric $d$ on $\mathbb{R}^d$ we define, for each $1 \leq n \leq N$,

$$\delta_{n,N}^d = \min\{ d(n\vec{\alpha}, m\vec{\alpha} + \vec{\ell}) > 0 : 1 \leq m \leq N, \vec{\ell} \in \mathcal{L} \}.  \tag{1.1}$$

The quantity $\delta_{n,N}^d$ is the smallest positive distance in $\mathbb{R}^d$ from $n\vec{\alpha}$ to an element of the set $S_N + \mathcal{L}$. As a natural generalization of the well known three distance theorem [5, 6, 7, 8], we are interested in understanding, for each $\vec{\alpha}$ and $N$, the number

$$g_{n,N}^d = g_{n,N}^d(\vec{\alpha}, \mathcal{L}) = |\{ \delta_{n,N}^d : 1 \leq n \leq N \}|$$
of distinct values taken by $\delta_{n,N}^d$, for $1 \leq n \leq N$. We will focus our discussion on two
metrics: the Euclidean metric (for which we will write $\delta_{n,N}^d = \delta_{n,N}$ and $g_{n,N}^d = g_N$), and the
maximum metric (for which we will write $\delta_{n,N}^d = \delta_{n,N}^*$ and $g_{n,N}^d = g_N^*$). To be clear, by the
maximum metric on $\mathbb{R}^d$ we mean the metric defined by

$$d(\vec{x}, \vec{y}) = \max_{1 \leq i \leq d} |x_i - y_i|.$$  

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For the case of the Euclidean metric it is known that, for any \( L, \vec{\alpha}, \) and \( N, \)

\[
g_N(\vec{\alpha}, L) \leq \begin{cases} 
3 & \text{if } d = 1, \\
5 & \text{if } d = 2, \\
41 & \text{if } d = 3, \\
\sqrt{\pi} \nu_d \frac{\Gamma(d/2)}{\Gamma(d-1)} (d-1) \left[ \ln(d-1) + 4 \ln \ln(d-1) + 10 \right] + 1 & \text{if } d \geq 4,
\end{cases}
\]

with the constant

\[
\nu_d = \left( \int_0^{\pi/6} \sin^{d-2}(\phi) \, d\phi \right)^{-1}.
\]

The bound for \( d = 1 \) (in which case the Euclidean and maximum metrics coincide) follows from the three distance theorem. The bounds for \( d \geq 2 \) were recently established in [3]. For \( d = 1 \) and \( 2 \) there are examples of \( L, \vec{\alpha}, \) and \( N \) for which the upper bounds above are actually obtained (see the introduction of [3]), therefore those bounds are best possible. For \( d \geq 3 \) the upper bounds above are probably far from best possible, but improving them substantially may require new ideas.

In this paper, motivated both by historical precedent and by questions which were asked of us after the publication of [3], we will show how the machinery from that paper can be used to easily bound the corresponding quantity \( g_N^* \) for the maximum metric, and even to obtain the best possible bounds in dimensions \( d = 2 \) and \( 3 \). To our knowledge, the only known result about this problem is due to Chevallier [1, Corollaire 1.2], who showed that \( g_N^* \leq 5 \) when \( d = 2 \) and \( L = \mathbb{Z}^2 \). Chevallier also gave an example in this case (see remark at end of [1, Section 1]) for which \( g_N^* = 4 \). We will prove the following theorem.

**Theorem 1.** For any \( d, L, \vec{\alpha}, \) and \( N, \) we have that

\[
g_N^*(\vec{\alpha}, L) \leq 2^d + 1.
\]

Furthermore, when \( d = 2 \) or \( 3 \) this bound is, in general, best possible.

To prove Theorem 1 we will first realize the quantity \( g_N^* \) as the value of a function \( G \) defined on the space \( \text{SL}(d+1, \mathbb{Z}) \setminus \text{SL}(d+1, \mathbb{R}) \) of unimodular lattices in \( \mathbb{R}^{d+1} \). This part of the proof, carried out in Section 2, is exactly analogous to the development in [2] and [3], which in turn is an extension of ideas originally presented by Marklof and Strömbergsson in [4]. In Section 3 we will use a simple geometric argument to bound \( G(M) \), when \( M \) is an arbitrary unimodular lattice in \( \mathbb{R}^d \), and for \( d = 2 \) and \( 3 \) we will give examples of \( L, \vec{\alpha}, \) and \( N \) for which our upper bounds are attained. Such examples, especially when \( d = 3 \), appear to be quite difficult to find.

Finally we remark that for \( d = 2 \) the conclusions of Theorem 1 also hold for the Manhattan metric. To see this, observe that the unit ball for this metric is a rotated and homothetically scaled copy of the unit ball for the maximum metric. It follows that, if \( d = 2 \) and if \( d \) is the Manhattan metric on \( \mathbb{R}^2 \), then there is a matrix \( R \in \text{SO}(2, \mathbb{R}) \) (rotation by \( \pi/4 \)) with the property that, for every \( L, \vec{\alpha}, \) and \( N, \)

\[
g_N^d(\vec{\alpha}, L) = g_N^*(R\vec{\alpha}, RL).
\]
Therefore \( g^d_N \leq 5 \) for this metric also, and this bound is best possible.

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2. Lattice formulation of the problem

As mentioned above, the observations in this section are very similar to those in [3, Section 2]. Therefore we will omit some of the details, which are explained in full in that paper.

Let \( |\cdot|_\infty \) denote the maximum norm on \( \mathbb{R}^d \). By a linear change of variables in the definition (1.1), we have that

\[
\delta^*_{n,N} = \min \left\{ |k\tilde{\alpha} + \ell|_\infty > 0 : -n < k < N_+ - n, \ell \in \mathcal{L} \right\},
\]

where \( N_+ := N + \frac{1}{2} \). Choose \( M_0 \in \text{SL}(d, \mathbb{R}) \) so that \( \mathcal{L} = \mathbb{Z}^d M_0 \), and let

\[
A_N(\tilde{\alpha}) = A_N(\tilde{\alpha}, \mathcal{L}) = \begin{pmatrix} 1 & 0 & \alpha \cr 0 & M_0 & 1 \cr \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \cr 0 & N^{1/d} \cr \end{pmatrix}.
\]

Then, for all \( 1 \leq n \leq N \), we have that

\[
\delta^*_{n,N} = N_+^{-1/d} \min \left\{ |\tilde{v}|_\infty > 0 : (u, \tilde{v}) \in \mathbb{Z}^{d+1} A_{N_+}(\tilde{\alpha}), -\frac{n}{N_+} < u < 1 - \frac{n}{N_+} \right\}.
\]

Now write \( G = \text{SL}(d+1, \mathbb{R}) \) and \( \Gamma = \text{SL}(d+1, \mathbb{Z}) \) and, for \( M \in G \) and \( t \in (0, 1) \), define

\[
F(M, t) = \min \left\{ |\tilde{v}|_\infty > 0 : (u, \tilde{v}) \in \mathbb{Z}^{d+1} M, -t < u < 1 - t \right\}.
\]

It follows from the proof of [3, Proposition 1] that \( F \) is well-defined as a function from \( \Gamma \setminus G \times (0, 1) \) to \( \mathbb{R}_{>0} \). It is also clear that

\[
\delta^*_{n,N} = N_+^{-1/d} F \left( A_{N_+}(\tilde{\alpha}), \frac{n}{N_+} \right).
\]

Furthermore, if we set

\[
\mathcal{G}(M) = |\{ F(M, t) \mid 0 < t < 1 \}|
\]

and, for \( N \in \mathbb{N} \),

\[
\mathcal{G}_N(M) = |\{ F(M, \frac{n}{N_+}) \mid 1 \leq n \leq N \}|
\]

then it follows from the definitions that

\[
g^*_N = \mathcal{G}_N(A_{N_+}(\tilde{\alpha})) \leq \mathcal{G}(A_{N_+}(\tilde{\alpha})�)
\]

This is the key connection which we will use in our proof of Theorem 1.
3. Proof of Theorem 1

To prove the first part of Theorem 1, in light of (2.1) it is sufficient to show that, for any \( M \in \Gamma \backslash G \),

\[
G(M) \leq 2^d + 1.
\]

(3.1) Suppose that \( M \in \Gamma \backslash G \) and choose vectors \((u_1, \vec{v}_1), \ldots, (u_K, \vec{v}_K) \in \mathbb{Z}^{d+1}M\), with \( K = G(M) \), so that the following conditions hold:

- \( 0 < |\vec{v}_1|_{\infty} < |\vec{v}_2|_{\infty} < \cdots < |\vec{v}_K|_{\infty} \).
- For each \( t \in (0,1) \), there exists an \( 1 \leq i \leq K \) such that \( |\vec{v}_i|_{\infty} = F(M,t) \).
- For each \( 1 \leq i \leq K \), there exists a \( t \in (0,1) \) such that \(-t < u_i < 1 - t\) and \( |\vec{v}_i|_{\infty} = F(M,t) \).

Note that each \( u_i \) lies in the interval \((-1,1)\). We make the following basic observation.

**Proposition 2.** If for some integer \( 1 \leq i \leq K \), we have that \( |u_i| < 1/2 \), then we must have that \( i = K \).

**Proof.** If \( u_i \in [0,1/2) \) then for any \( 0 < t < 1 - u_i \), we have that \( F(M,t) \leq |\vec{v}_i|_{\infty} \). Noting that \((-u_i, -\vec{v}_i) \in \mathbb{Z}^{d+1}M\), we see that this inequality also holds for any \( t \) satisfying \( u_i < t < 1 \). Since \( u_i < 1/2 \), we conclude that \( F(M,t) \leq |\vec{v}_i|_{\infty} \) for all \( 0 < t < 1 \), which proves that \( i = K \). The case when \( u_i \in (-1/2, 0] \) follows from the same argument.

Next we use geometric information to place restrictions on the vectors \((u_i, \vec{v}_i)\). This is where we will use the fact that we are working with the maximum norm.

**Proposition 3.** For \( 1 \leq i < j \leq K \), if \( \text{sgn}(u_i)\vec{v}_i \) and \( \text{sgn}(u_j)\vec{v}_j \) lie in the same orthant of \( \mathbb{R}^d \), then \( j = K \).

**Proof.** If \( |u_j| < 1/2 \) then what we are trying to prove follows from the previous proposition. Therefore suppose that \( |u_j| \geq 1/2 \). By Proposition 2, this also forces \( |u_j| \geq 1/2 \).

If \( \text{sgn}(u_i) = \text{sgn}(u_j) \) and if \( \vec{v}_i \) and \( \vec{v}_j \) lie in the same orthant, then \( |u_i - u_j| < 1/2 \), and

\[
0 < |\vec{v}_i - \vec{v}_j|_{\infty} \leq \max \{|\vec{v}_i|_{\infty}, |\vec{v}_j|_{\infty}\} = |\vec{v}_j|_{\infty}.
\]

Since \((u_i - u_j, \vec{v}_i - \vec{v}_j) \in \mathbb{Z}^{d+1}M\), it follows from the proof of Proposition 2 that

\[
F(M,t) \leq |\vec{v}_i - \vec{v}_j|_{\infty} \leq |\vec{v}_j|_{\infty}
\]

(3.2) for all \( 0 < t < 1 \). Therefore we conclude that \( j = K \).

Similarly, if \( \text{sgn}(u_i) = -\text{sgn}(u_j) \) and if \( \vec{v}_i \) and \( -\vec{v}_j \) lie in the same orthant, then \( |u_i + u_j| < 1/2 \), and

\[
0 < |\vec{v}_i + \vec{v}_j|_{\infty} \leq \max \{|\vec{v}_i|_{\infty}, |\vec{v}_j|_{\infty}\} = |\vec{v}_j|_{\infty}.
\]

Since \((u_i + u_j, \vec{v}_i + \vec{v}_j) \in \mathbb{Z}^{d+1}M\), this again implies that (3.2) holds, and we conclude that \( j = K \).
By Proposition 3, each of the vectors \( \text{sgn}(u_i)v_i \), for \( 1 \leq i \leq K - 1 \), must lie in a different orthant of \( \mathbb{R}^d \). This immediately gives the bound in (3.1), and therefore completes the proof of the first part of Theorem 1.

Finally, we give examples with \( d = 2 \) and \( 3 \) for which the bound in Theorem 1 is attained. In what follows, for \( \vec{x} \in \mathbb{R}^d \) we write \( \| \vec{x} \| = \min \{ |\vec{x} - \vec{\ell}|_\infty : \vec{\ell} \in \mathcal{L} \} \).

For \( d = 2 \) take \( \mathcal{L} = \mathbb{Z}^2 \), \( \vec{\alpha} = \left( \frac{157}{500}, -\frac{23}{200} \right) \), and \( N = 11 \). Then we have that
\[
\begin{align*}
\delta_{1,N}^* = \|10\vec{\alpha}\| &= \frac{3}{20}, & \delta_{2,N}^* = \|9\vec{\alpha}\| &= \frac{87}{500}, & \delta_{4,N}^* = \|7\vec{\alpha}\| &= \frac{99}{500}, \\
\delta_{5,N}^* = \|6\vec{\alpha}\| &= \frac{31}{100}, & \quad \text{and} \quad \delta_{6,N}^* = \|\vec{\alpha}\| &= \frac{157}{500},
\end{align*}
\]
therefore \( g_N^*(\alpha, \mathcal{L}) = 5 \).

For \( d = 3 \) take \( \mathcal{L} = \mathbb{Z}^3 \), \( \vec{\alpha} = \left( -\frac{157}{10000}, -\frac{742}{3125}, -\frac{23}{400} \right) \), and \( N = 73 \). Then we have that
\[
\begin{align*}
\delta_{1,N}^* = \|72\vec{\alpha}\| &= \frac{7}{50}, & \delta_{2,N}^* = \|71\vec{\alpha}\| &= \frac{443}{3125}, & \delta_{5,N}^* = \|68\vec{\alpha}\| &= \frac{456}{3125}, \\
\delta_{6,N}^* = \|67\vec{\alpha}\| &= \frac{59}{400}, & \delta_{18,N}^* = \|55\vec{\alpha}\| &= \frac{13}{80}, & \delta_{19,N}^* = \|54\vec{\alpha}\| &= \frac{557}{3125}, \\
\delta_{22,N}^* = \|51\vec{\alpha}\| &= \frac{1993}{10000}, & \delta_{23,N}^* = \|50\vec{\alpha}\| &= \frac{43}{20}, & \quad \text{and} \quad \delta_{24,N}^* = \|4\vec{\alpha}\| &= \frac{23}{100},
\end{align*}
\]
therefore \( g_N^*(\alpha, \mathcal{L}) = 9 \).

This completes the proof of our main result. We conclude by remarking that, for \( d \geq 4 \), it seems likely that the upper bound of Theorem 1 is too large. In fact, for \( d = 4 \), we have not found any examples so far with \( g_N^* > 9 \). Establishing optimal upper bounds in these cases is an interesting open problem.

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AH, JR: Department of Mathematics, University of Houston, Houston, TX, United States.
haynes@math.uh.edu, juan.ramirez789456@gmail.com