Neutron stars and strong-field effects of general relativity

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Abstract. The basic observed properties of neutron stars are reviewed. I suggest that neutron stars in low-mass X-ray binaries are the best of all known sites for testing strong-field effects of general relativity.

1. Validity of General Relativity

General Relativity (GR) is the correct description of gravity and space-time. The phenomena verified with three classic tests of GR are so well established that they are now used as tools in every-day astronomical practice and even in technological applications.

The gravitational bending of light, famously detected in Eddington’s solar eclipse expedition is today used to determine the stellar content of our Galaxy and the Magellanic Clouds (from stellar micro-lensing events detected by the OGLE, MACHO and EROS experiments). Lensing of distant galaxies by intervening galaxy clusters is used to determine the (dark) matter distribution in the latter.

Gravitational redshift, first observed in spectra of the white dwarf Sirius B in 1925, has since been detected in the laboratory (Pound-Rebka experiment) and is now of necessity taken into account in surveying practice (the GPS system). The effect is also essential in timing radio pulsars—when compared to some millisecond pulsars, terrestrial clocks clearly run slower at full moon than at new moon.

The magnitude of precession of the perihelion of Mercury is dwarfed by the same effect in the Hulse-Taylor pulsar, where the periastron shifts by 4.2° per year. A similar system, Wolszczan’s binary pulsar, allows a confirmation of the Shapiro delay.

Of course, GR also provides the framework for understanding the evolution of our expanding Universe. All these successes allow us to confidently use general relativity, even in domains where its validity has not yet been strictly proven.

Observations of certain X-ray binaries (e.g., Cygnus X-1 and the so called X-ray novae), as well as of stellar motions in our Galaxy, and of velocities in the inner cores
of other galaxies, strongly suggest the existence of black holes. However, the laws of GR have not yet been truly tested in the strong field regime.

1.1. Why neutron stars

The strength of gravity is conveniently parametrized by the mass to size ratio, $(M/R)(G/c^2)$. For black holes, of course, $GM/(Rc^2) \sim 1$, as for the Schwarzschild radius $R_{\text{Sch}} = 2MG/c^2$. For the Sun, $GM\odot/c^2 \approx 1.5 \text{ km}$, while the solar radius $R\odot \approx 300000 \text{ km}$, which yields $M\odot/R\odot \sim 10^{-5}$ (in units of $c^2/G$). A similar value is obtained for mass/distance in the binary Hulse-Taylor pulsar, where relativistic effects in the orbital motion are so clearly detected (because the pulsar period is so short $\approx 0.06 \text{ s}$, and known to 10 significant figures). For white dwarfs, $M/R \sim 10^{-3}$. But for neutron stars, $M/R \sim 10^{-1}$, and GR effects just outside their surface are about as important as near the black hole surface.

As a testbed for GR, neutron stars have one great advantage over black holes—they have a tangible surface which can support magnetic fields and can emit X-rays and other radiation. A great deal can be learned about neutron stars without assuming the validity of GR. Hence, a great deal can be learned about GR by observing neutron stars. Today, about 1000 radio pulsars are known and about 100 X-ray binaries containing neutron stars, so also in sheer numbers neutron stars have an advantage over black holes.

1.2. Basic references

The narrative presented in Sections 1 and 2, to a large extent relies on well established observations and theories, which have made their way into excellent textbooks, where detailed references can be found to the literature. Among those, particularly useful in the context of these lectures are the ones by Shapiro and Teukolsky (1983), Lipunov (1992), Mészáros (1992), Glendenning (1997), and Frank, King and Raine (1985).

2. A brief history of neutron stars

Before discussing in detail the properties of rapidly rotating, (at most) weakly magnetized, compact stars—which are ideal astrophysical objects for testing strong-field predictions of General Relativity—let us recount how they were identified.
2.1. Key dates

The basic chronology of the discovery of neutron stars can be found, together with the references, e.g., in the text by Shapiro and Teukolsky. The following selection reflects my bias of what seems particularly important with the hindsight of today.

1914: Adams discovered that the rather dim, $L \approx 3 \times 10^{-3} L_\odot$, star Sirius B (orbiting Sirius), whose mass had been determined to be $M \approx 0.85 \pm 0.10 M_\odot$, has the spectrum of a “white” star—hence the name white dwarf. The unusual combination of low luminosity and high temperature implied a small radius, $R \approx 2 \times 10^4 \text{km}$. This conclusion was based on an application of the black-body formula

$$L = 4\pi\sigma_B R^2 T^4.$$  

1925: Adams measures the redshift, $z$, of certain lines in Sirius B. Applying general relativity, one can infer the value of $M/R$ from $z$, and from the known mass a value of the stellar radius, $R \sim 10^4 \text{km}$. The agreement with the spectroscopically determined value was a great triumph of GR.

1926: The Fermi-Dirac statistic is discovered.

1926 (December): Fowler identifies the agent holding up white dwarfs against gravity—it is the degeneracy pressure of electrons.

1930: Chandrasekhar discovers theoretical models of white dwarfs, from which the maximum value for white dwarf mass follows, the famous $1.4 M_\odot$. Incidentally, $M \sim 1 M_\odot$ and $R \sim \text{few} \times 10^3 \text{km}$ imply a density $\rho \sim 10^6 \text{g/cm}^3$, which in turn implies a minimum period of possible rotation or vibration of a few seconds: $(G\rho)^{-1/2} \sim 3 \text{s}$.

1932: Chadwick discovers the neutron.

1932: Landau discusses cold, degenerate stars composed of neutrons.

1934: Baade and Zwicky write: “With all reserve we advance the view that supernovae represent the transition from ordinary star to neutron stars.” This remains a remarkable contribution—two years after the discovery of neutrons, Baade and Zwicky correctly explain the mechanism of Supernovae (type II) explosions, find the correct value for the gravitational binding energy released in the creation of a neutron star, $\sim 10^{53} \text{erg}$, and even identify a site where a neutron star is present (and was discovered 35 years later!): the Crab nebula.

1938: Landau discusses the energy released inside ordinary stars with neutron-star cores (a theoretical precursor of what is now known as a Thorne-Żytkow object). At the time, the energy source of the Sun was not known. The great contribution here is the pointing out of the enormous energy released in accretion onto neutron stars.

1939: Oppenheimer and Volkoff solve the relativistic equations of stellar structure for a fermi gas of neutrons, and thus construct the first detailed model of a neutron star.
They find a maximum mass ($\approx 0.7 M_\odot$, lower than the one for modern equations of state), above which the star is unstable to collapse. Thus the road to the theoretical discovery of black holes is paved.

1940’s are lost to the Second World War.

1950’s: The basic physics of the interior of neutron stars is worked out by the Soviet school, including a detailed understanding of the superfluid phase.

1962: Giacconi et al. discover the first extrasolar source of X-rays, Sco X-1.

1967: Shklovsky derives a model for Sco X-1, in which the X-ray source is an accreting neutron star in a binary system.

1967: Pacini points out that neutron stars should rotate with periods $P << 1$ s, and may have magnetic fields of surface value $B \sim 10^{12}$ G. The ensuing dipole radiation is not directly observable, as its frequency $2\pi/P$ is below the plasma frequency of interstellar space.

1967: Radio pulsars with $P \leq 3$ s discovered by Hewish, Bell et al.

1968: Gold gives the “lighthouse” model of radio pulsars.

1968: Spin-down of radio pulsars is measured, $\dot{P} > 0$. From this moment, it is clear that pulsars are rotating, compact objects, ultimately powered by the kinetic energy of their rotation.

1971: Giacconi et al. discover the first of accreting counterparts of radio pulsars, the X-ray pulsar Cen X-3, of period 4.84 s. Today, many are known, in the period range $0.7 \text{s} \leq P \leq 10000 \text{s}$.

1978: Trümper et al. discover the $\sim 40$ keV cyclotron line in the spectrum of the accreting X-ray pulsar Her X-1. From the formula $h\nu = 1 \text{keV} \times (B/10^8 \text{G})$, the inferred value of the magnetic field at the stellar surface is $B_p = \text{few} \times 10^{12} \text{G}$, in agreement with the estimates of the dipole strength of ordinary radio pulsars.

1982: The discovery of millisecond pulsars by Backer, Kulkarni et al.

1996: The discovery of kHz quasi-periodic oscillations (QPOs) in the X-ray flux of low-mass X-ray binaries (LMXBs).

1998: The discovery of 2.5 ms pulsar in the transient LMXB SAX J 1808.4-3658 by Wijnands and van der Klis.

2.2. The physics of identifying neutron stars

It should be apparent from the above review, that the basic physics behind identifying neutron stars is fairly simple. Of course, the discovery was possible only after decades of sustained technological development, particularly in the field of radio and X-ray detectors, as well as much observational effort. Also, the existence of neutron stars would not have been so readily accepted without the solid theoretical foundations laid down over a period
of many years. But the basic, incontrovertible, observational arguments are really based on two or three simple formulae.

Let us accept the theoretical result, that a neutron star is a body of mass $M \sim 1M_\odot$ and radius $R \sim 10\text{ km}$, hence of mean density $\bar{\rho} > 10^{14}\text{ g/cm}^3$. How can we be certain that such bodies have been discovered?

a) The mass can be determined directly in some binary systems by methods of classical astronomy (as developed for spectroscopic binaries), essentially by an application of Kepler's laws. For the binary X-ray pulsars, the errors are rather large, but it is clear that one or two solar masses is the right value. For the binary radio pulsars (the Hulse-Taylor and Wolszczan pulsars), where the pulse phase can be determined very precisely and relativistic effects give much redundancy, the mass has been measured very accurately (to $0.01M_\odot$) and is close to $1.4M_\odot$. For binary (millisecond) radio pulsars with white dwarf companions, the mass function is always consistent with these values.

b) In bright, steady, X-ray sources, and especially in X-ray bursters (where the X-ray flux briefly saturates at a certain peak value), one can assume that the radiative flux is limited, at the so called Eddington value, by a balance between radiation pressure on electrons and gravitational pull on protons. Since both forces are proportional to $(\text{distance})^{-2}$, there is a direct relation between flux and mass. Again, $M \sim 1M_\odot$ is obtained, for $L_X \approx 10^{38}\text{ erg/s}$.

c) The radius can be determined whenever a thermal spectrum is detected, by a combination of the black-body formula, eq. (1), and of Wien's law giving the characteristic temperature of a body emitting the thermal spectrum. Thus, for X-ray pulsars, such as Her X-1, the spectrum gives a characteristic temperature of $T \sim 10\text{ keV} \sim 10^8\text{ K}$, which in combination with the luminosity $L_X \sim 10^{37}\text{ erg/s}$ gives an area of $\sim 10^{10}\text{ cm}^2$, consistent with the area of a "polar cap." This is the area through which open magnetic field lines pass for a $R \sim 10\text{ km}$ star, rotating at $P = 1.24\text{ s}$, with a $B \sim 10^{12}\text{ G}$ field.

For the non-pulsating bright X-ray source Sco X-1, $T \sim 1\text{ keV}$, $L \sim 10^{38}\text{ erg/s}$, i.e., $R \sim 10\text{ km}$ directly, as expected if the accreting material is spread over the whole surface.

d) For pulsars, an upper limit to the stellar radius follows from causality, $\omega R < c$, hence $R < cP/(2\pi)$. For millisecond pulsars, this gives $R < 100\text{ km}$.

e) The moment of inertia of certain pulsars (if they are powered by rotation) can be measured directly in "cosmic calorimeters." If the luminosity of the Crab nebula ($\approx 5 \times 10^{38}\text{ erg/s}$) is equated to $I\omega\dot{\omega}$, for the known period ($P = 33\text{ ms}$) and its derivative of the Crab pulsar (or the known age of the nebula), the value $I \approx 10^{45}\text{ g\cdot cm}^2$ is obtained.

A similar, but less secure, argument can be given for the famous eclipsing pulsar PSR 1957+20 ($P = 1.6\text{ ms}$, $\dot{P} \approx 10^{-19}$). It is thought that the power needed to ablate the
0.02M⊙ companion is ∼10^{38} \text{ erg/s} (assuming isotropic emission from the pulsar). Again, I ∼10^{40} \text{ g·cm}^2 is obtained.

f) Finally, a lower limit to the density can at once be derived for rotating objects from Newton’s formula for keplerian orbital motion: ω_K = \sqrt{GM/R^3} = \sqrt{4\pi\rho'}/3. Since 2\pi/P = ω ≤ ω_K, for any star rotating at a period P, the mean density satisfies \bar{ρ} ≥ 3\pi G^{-1}P^{-2}. With the known value of Newton’s constant, this gives directly \bar{ρ} > 2 × 10^{14} \text{ g/cm}^3, for SAX J 1808.4-3658 (P = 2.5 ms) or the millisecond pulsars, such as PSR 1957+20 (P = 1.6 ms).

These basic results are subject to many consistency checks, which in all cases support the basic result that objects with a solid or fluid surface (i.e., they are not black holes!) have been identified of dimensions M ∼ 1M⊙ and R ∼ 10 km:

i) The gravitational energy released in accretion L ∼ GM\dot{M}/R is consistent (for the discussed values M ∼ M⊙ and R ∼ 10 km) with the mass accretion rate inferred from theoretical studies of binary evolution.

ii) In some X-ray bursters, the photosphere clearly expands. Again, spectral fits for the temperature and for the radius of the photosphere (eq. [1]), assuming Eddington luminosity, constrain the M – R relationship, in a manner consistent with the values discussed above.

iii) The surface magnetic field measured from the cyclotron line in X-ray pulsars agrees, to an order of magnitude (B_p ∼ 10^{12±1} G), with the one inferred for radio pulsars, by applying the notion that the spin down in the latter sources is obtained through balancing the energy loss in the simple dipole formula \dot{E} = -2|m|^2/(3c^2), where |m| = B_p R^3/2, with the kinetic energy loss of a body of moment of inertia I = 10^{45} \text{ g·cm}^2.

Incidentally, for millisecond pulsars, the value inferred from spin-down, B_p ∼ 10^{9±1} G, is consistent with the absence of polar cap accretion (and of associated pulsations) in X-ray bursters and other LMXBs. Thus, as far as the magnetic field is concerned, two or three classes of neutron stars are known—ordinary radio pulsars and accreting X-ray pulsars (B ∼ 10^{12±1} G), millisecond radio pulsars (B ∼ 10^{9±1} G), and low-mass X-ray binaries, where there is no evidence for such strong magnetic fields (i.e., B < 10^9 G).

iv) The observed long-term spin-up and spin-down of accreting X-ray pulsars is also consistent with a moment of inertia I ∼ 10^{45} \text{ g·cm}^2 , for torques which are expected at the mass-accretion rates derived from the observed X-ray flux, assumed to be \dot{L}_X ∼ GM\dot{M}/R ∼ 0.1\dot{M}c^2, and the assumption that the lever arm corresponds to an Alfvénic radius, obtained by balancing the ram pressure with the dipole magnetic pressure, i.e., B^2/(8\pi) ∼ \rho v_A^2 at r = r_A, \dot{M} = \epsilon 4\pi r^2 \rho v_r, B = B_p R^3/r^3, where \epsilon ∼ 1 is a geometric factor.
3. The maximum mass of compact stars

3.1. Neutron stars or quark stars?

It is clear that radio pulsars and some accreting X-ray sources contain compact objects of properties closely resembling those known from theoretical models of neutron stars. Specifically, there can be no doubt that rotating stars of \( M \sim M_\odot \) and \( R \sim 10 \) km exist. However, their internal constitution is not yet known. The expected mass and radius of “strange” (quark) stars is similar, the main difference being that quark stars of small masses would have small radii—unlike neutron stars whose radius generally grows with decreasing mass—(Alcock et al. 1986). The observed “neutron stars” could be made up mostly of neutrons, but some of them could also be composed partly, or even mostly, of quark matter.

From the point of view of testing GR, the internal constitution of static (non-rotating) stars would matter little, as their external metric, directly accessible to observations, would be independent of their nature—the only parameter in the unique static, spherically symmetric, asymptotically flat solution (the Schwarzschild metric) is the gravitational mass, \( M \), of the central body. However, for rapidly rotating stars, the metric does vary with properties of the body other than its mass, and it would be good to know the precise form of the equation of state (e.o.s.) of matter at supranuclear density.

As we have seen, at least some low-mass X-ray binaries (LMXBs) contain stellar remnants of extremely high density, exceeding \( 10^{14} \) g cm\(^{-3} \), and many of them are not black holes because they exhibit X-ray bursts of the type thought to result from a thermonuclear flash on the surface of an ultra-compact star. Further, in these long-lived accreting systems the mass of the compact star is thought to have increased over time by several tenths of a solar mass above its initial value, and in the process the stars should have been spun up to short rotational periods. The compact objects in the persistent LMXBs are expected to be the most massive stellar remnants other than black holes, hence the most stringent limits on the e.o.s. of dense matter is expected to be derived from the mass of the X-ray sources in low-mass X-ray binaries. Before we discuss how this can be done, let us turn to the maximum mass.

3.2. The maximum mass of neutron stars

One quantity that depends sensitively on the e.o.s. is the maximum mass of a fluid configuration in hydrostatic equilibrium. For neutron stars this maximum mass, and in general the mass–radius relationship, is known from integrating the TOV equations for a wide variety of e.o.s. (Arnett and Bowers 1977). The mass of rotating configurations is also known (Cook et al. 1994). Here, I will only briefly review the basic physics behind
the existence of the maximum mass and then give an example for strange stars, where the e.o.s. is so simple that the variation of mass with the parameter describing the interactions can be determined analytically.

As we know from the work of Chandrasekhar and others, the maximum mass is reached when the adiabatic index reaches a sufficiently low value that the star becomes unstable to collapse. In the Newtonian case, this critical index is $4/3$, corresponding to the extreme relativistic limit for fermions supplying the degeneracy pressure, when the formula for kinetic energy of a particle $E = \sqrt{p^2c^2 + m_f^2c^4}$ reduces to $E = pc$.

The very simple argument explaining the instability, due to Landau, goes like this. There is a balance between the increasingly negative gravitational binding energy when a massive sphere of fermions is compressed, and the increasing kinetic energy of each fermion as it is squeezed into an increasingly confined volume—each fermion likes to live in phase space of volume $\sim \hbar$. Of course, as the star is compressed when its mass is increased, the fermion momenta increase and the extreme relativistic regime is approached, with a corresponding softening of the adiabatic index. The total energy of $N$ particles in a star of mass $M$ bound by gravity, is up to factors of order unity, $E_{\text{tot}} = -GM^2/R + N\bar{E}$, where $\bar{E}$ is the mean kinetic energy of the particles. If the particles are fermions, of mass $m_f$, their momentum following from the uncertainty principle is $p = \hbar(N/V)^{1/3}$, and we can take $V = R^3$ for the volume of the star. In the non-relativistic case, $E = p^2/(2m_f)$ so $N\bar{E} = \hbar^2N^{5/3}/(2m_fR^2)$, and a stable configuration can be found by minimizing $E_{\text{tot}}$ with respect to $R$. But in the extreme relativistic case, $E = pc$, $N\bar{E} = hcn^{4/3}/R$, and both terms in $E_{\text{tot}}$ are now proportional to $1/R$, so no minimum energy configuration is found.

In reality, to find the maximum mass configuration, one has to solve the TOV equations using a plausible e.o.s. The TOV equations have essentially the same scaling properties as the familiar equations of Newtonian hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2},$$
$$\frac{dm}{dr} = 4\pi r^2 \rho,$$

i.e., if the pressure and density scale with some fiducial density, $P \propto \rho \propto \rho_0$, then $m \propto r \propto \rho_0^{-1/2}$. Such scalings allow some general statements to be made about the maximum mass, such as the Rhoads-Ruffini limit: $M < 3M_\odot$ if $\rho \geq \rho_0 > 2 \times 10^{14}\text{ g/cm}^3$.

### 3.3. Quark stars

Conversion of some up and down quarks into strange quarks is energetically favorable in bulk quark matter (because the Fermi energy is so high) and it has been suggested that at large atomic number, matter in its ground state is in the form of “collapsed nuclei”
with strangeness about equal to the baryon number (Bodmer 1971). On this assumption, Witten (1984) discussed the possible transformation of neutron stars to stars made up of matter composed of up, down, and strange quarks in equal proportions, and found the maximum mass of such quark stars as a function of the density of (self-bound) quark matter at zero pressure is \( \rho_0 \geq 4 \times 10^{14} \text{ g/cm}^3 \). Detailed models of these “strange” stars have been constructed (Alcock et al. 1986, Haensel et al. 1986). Here, I discuss only the maximum mass of such stars.

Following Alcock (1991), take a gas of any relativistic particles—the e.o.s. is \( P_g = \rho_g c^2 / 3 \). If these are moving in a background of vacuum with uniform energy density \( \rho_v c^2 = B \), i.e., negative pressure \( p_v = -B \), then the e.o.s. connecting the total pressure \( p = p_g + p_v \), with the total density \( \rho = \rho_g + \rho_v \), is

\[
p = (\rho - \rho_0) c^2 / 3,
\]

with \( \rho_0 c^2 = 4B \). Witten (1984) showed that for this simple e.o.s. the maximum mass from the TOV equation is \( M = 2M_\odot \sqrt{\rho_1 / \rho_0} \), with \( \rho_1 \equiv 4.2 \times 10^{14} \text{ g/cm}^3 \). The scaling \( \rho_0^{-1/2} \) is discussed in the previous subsection.

The physical interpretation of the result is that the relativistic particles are in fact quarks, and the “bag constant” \( B \), is a device invented at MIT to simulate their confinement. The e.o.s. \( p = (\rho - \rho_0) c^2 / 3 \), then, describes interacting quarks in an approximation to quantum chromodynamics (QCD) known as the MIT bag model (Farhi and Jaffe 1984). Thus, the maximum mass found is the maximum mass of static strange (quark) stars. However, it still depends on the free parameter \( \rho_0 \).

3.4. The maximum mass of strange stars

To illustrate the utility of the scaling law, I will now discuss the maximum mass of a strange star. First, as already noted by Oppenheimer and Volkoff (1939), the stellar mass decreases with the fermion mass, so to find the maximum mass of a quark star it is enough to consider massless quarks. In view of the scaling of TOV equations, the question reduces to that of finding \( \rho_0 \), the density of strange matter at zero pressure. In short, the maximum mass of a strange star in the model considered is \( M_{\text{max}} = 1.98M_\odot \times (59.8 \text{ MeV}/B)^{1/2} \), and the least upper bound to the mass of the strange star is given by the same formula, with \( B = B_{\text{min}} \), the lowest possible value of the bag constant. Realistically, the actual maximum mass of a (non-rotating) strange star will be smaller by about 10% because, because in fact, \( m_s > 0 \).

Currently, the actual value of \( B \) cannot be reliably derived from fits to hadronic masses of the quark-model of nucleons. Its lowest possible value can be found by requiring that neutrons do not combine to form plasma of deconfined up and down quarks, or
equivalently, that quark matter composed of up and down quarks in 1:2 ratio is unstable to emission of neutrons through the reaction \( u + 2d \rightarrow n \). This implies that the baryonic chemical potential at zero pressure of such quark matter satisfies (Haensel 1996)

\[
\mu_{u,d}(0) > 939.57 \text{ MeV.} \tag{3}
\]

As we neglect the masses of up and down quarks in our considerations, the baryonic chemical potential at pressure \( P \) is given by the expression (Chapline and Nauenberg 1976)

\[
\mu(P) = (P + \rho c^2)/n = 4(A/3)^{3/4}(P + B)^{1/4}, \tag{4}
\]

where \( n \) is the baryon number density, and \( \rho c^2 = An^{4/3} + B \) is the energy density. For matter (not in beta equilibrium) composed of deconfined up and down quarks in 1:2 ratio, \( n = n_u = n_d/2 \) and hence \( A = (1 + 2^{4/3})(3\hbar c/4)\pi^{2/3}C^{-1/3} \), i.e., \( \mu(0) \propto (B/C)^{1/4} \), where \( C \equiv 1-2\alpha_c/\pi \) and \( \alpha_c \) is the QCD coupling constant. Inequality (1) then becomes

\[
\frac{B}{C} > 58.9 \text{ MeV fm}^{-3} \equiv B_1. \tag{5}
\]

Thus, \( B_{\text{min}} = (1 - 2\alpha_c/\pi)B_1 \), through lowest order in quark-gluon coupling. So, for massless interacting quarks, the energy density at zero pressure is \( \rho_0 c^2 = 4B \geq (1 - 2\alpha_c/\pi)\rho_1 c^2 \). For massive quarks the expression for minimum density becomes more complicated, but we will not need it to determine the upper bound to the mass of a static strange star in the MIT bag model—it is enough to consider the e.o.s. of an ultra relativistic Fermi gas in a volume with vacuum energy density \( B > 0 \).

For strange matter in beta equilibrium the number densities of the (massless for now) up, down, and strange quarks are equal, \( n_u = n_d = n_s \), and the energy density is \( \rho c^2 = A_s n^{4/3} + B_1 \), with \( A_s = 9\hbar c\pi^{2/3}C^{-1/3}/4 \), as is appropriate for three colors per flavour. This gives an equation of state identical to that of non-interacting quarks, eq. (2), the only difference being in that the lower bound on the density at zero pressure, following from conditions of neutron stability (eqs. [3], [5]), is decreased by the factor \( C \) with respect to the value for an ideal Fermi gas in a bag:

\[
\rho_0(\alpha_c) = \left( 1 - \frac{2\alpha_c}{\pi} \right) \rho_0(0).
\]

Thus, through lowest order in the QCD interaction, the fiducial density is changed, but not the e.o.s. Since the stellar mass scales as \( \rho_0^{-1/2} \), this implies that the least upper bound on the mass of the star as a function of the QCD coupling constant is given for non-rotating strange stars by

\[
M_{\text{max}}(\alpha_c) = \left( 1 - \frac{2\alpha_c}{\pi} \right)^{-1/2} M_{\text{max}}(0) \tag{6}
\]

through first order in \( \alpha_c \). For \( \alpha_c = 0.6 \) this gives a maximum strange star mass of \( 2.54M_\odot \), higher by 27% than the maximum mass which is obtained for \( \alpha = 0 \).
4. Measuring the mass of accreting neutron (or strange) stars

Finally, we have to confront the question how the mass of the compact objects in LMXBs may be determined. Hopefully, a mass will be measured which will eliminate a class of equations of state of dense matter. Unfortunately, application to X-ray bursters of standard methods for determining the mass function of the binary—and hence constraining the mass of the compact X-ray source—is exceedingly difficult, as the optical emission is usually dominated by that of the accretion disk (e.g. van Paradijs et al., 1996). However, reliable mass values obtained by this method may soon become available, particularly for transient sources, such as the accreting millisecond pulsar SAX J1808.4–3658.

The mass of the compact object in an X-ray binary may also be determined by studying the time variability of the radiation flux formed in the accretion flow. Specifically, for sufficiently weakly magnetized stars, a maximum frequency is expected corresponding to the presence of the innermost (marginally) stable circular orbit allowed in general relativity (Kluźniak, Michelson and Wagoner, 1990). It has been reported that such a maximum frequency may have been observed, at least in one system where quasi periodic oscillations (QPOs) in the X-ray flux saturate at a particular value (Zhang et al. 1998). In this manner, several e.o.s. were excluded (Kluźniak 1998) on the understanding that the maximum observed kHz QPO frequency implies a mass in excess of $2M_\odot$; see also Kaaret et al. (1997). Similar considerations (Bulik et al. 1999) exclude static (or slowly rotating) quark stars if the minimum density of quark matter is $\rho_0 > 4.2 \times 10^{14}$ g/cm$^3$.

The overall conclusion (Kluźniak 1998) is that neutron-star matter may be composed simply of neutrons with some protons, electrons and muons, as models of more exotic neutron-star matter (including hyperons or pion and kaon condensates) do not agree with the simplest interpretation of the kHz QPO data, namely that the maximum frequency observed in the low-mass X-ray binary 4U 1820-30, i.e., 1066 Hz (Zhang et al. 1998), is attained in the marginally stable orbit around a neutron star. If the compact stellar remnants in these systems are slowly rotating, the same conclusion would apply to ultradense matter in general, at densities greater than $4.2 \times 10^{14}$ g/cm$^3$, as matter composed of massless quarks would also be excluded for such densities (Bulik et al. 1999). However, as we have seen, minimum densities smaller than $4.2 \times 10^{14}$ g/cm$^3$ seem possible for more realistic models of self-bound quark matter, and this would change the conclusion.

For rapidly rotating strange stars the conclusion may be drastically different, as the metric is greatly modified by a pronounced flattening of the star (this effect is less important for neutron stars). In general, the marginally stable orbit is pushed out by this effect, and a fairly low orbital frequency can be obtained for a low mass star. This is illustrated in Fig. 1 (taken from Stergioulas et al. 1999) which exhibits the frequency in the
innermost (marginally) stable circular orbit of general relativity (ISCO) as a function of stellar mass, $M$, for the Schwarzschild metric [the hyperbola $f_+ = 2.2 \, \text{kHz}(M_{\odot}/M)$], as well as the ISCO frequency for strange stars rotating at Keplerian frequencies (i.e., maximally rotating, at the equatorial mass-shedding limit), for various values of the density at zero pressure, $\rho_0$ of eq. (2). It turns out that for these maximally rotating models, the ISCO is always at 1.7 to 1.8 km above the stellar surface, the increase of the ISCO orbital frequency for these models can then be understood in terms of Kepler’s law: $2\pi f \sim \sqrt{G\bar{\rho}}$, where $\bar{\rho}$ is the mean density of matter inside the orbit.

5. Testing strong-field general relativity with accreting neutron stars

There are really two types of objects where strong-field effects of general relativity are crucial: black holes and accreting neutron (or quark) stars. Black holes are both attractive and difficult in this context—on the one hand, their very existence would be impossible in many other theories of gravity, on the other, their existence is a hypothesis which must experimentally verified. Possibly this will eventually be achieved by careful observations of motions in the inner accretion disk in AGNs and/or black hole binaries.

The existence of neutron stars (or quark stars) would be perfectly possible in Newtonian gravity (although their detailed properties would be different from those expected in a general-relativistic world). But from the point of view of determining the metric, they have the great advantage, that not only their mass can be measured (as for binary black holes), but also, at least in some cases, other basic parameters such as the rotational period and the radius can be determined directly. Hopefully, this would allow relativistic effects in the accretion flow to be unambiguously resolved.

One class of phenomena which may be helpful in pinning down the external metric of accreting sources is the relativistic trapping of vibrational modes in the inner accretion disk. Indeed, it has been suggested that the 67 Hz oscillation seen in the source GRS 1915+105 has this origin, and is a signature of the Kerr metric (Nowak et al. 1997). This is perhaps the most convincing relativistic effect discovered to date in accreting sources. Unfortunately, the mass of GRS 1915+105 is not known, and there is no independent knowledge of its angular momentum (the source is a black hole candidate).

Another promising avenue is the search for the marginally stable orbit (ISCO), expected to exist in accreting neutron stars (Kluźniak and Wagoner 1995) and to show up as a maximum frequency in the X-ray spectra of LMXBs (Kluźniak, Michelson and Wagoner 1990). Indeed, the recently discovered kHz QPOs in X-ray bursters and other probable neutron star systems do show some features which are consistent with their observed frequency being the Keplerian frequency in an accretion disk terminating close
Fig. 1. The frequency of the co-rotating innermost stable circular orbit as a function of mass for static models (thin, continuous line) and for strange stars rotating at the equatorial mass-shedding limit (thick lines, in the style of Fig. 1). For the static models, this frequency is given by the keplerian value at $r = 6GM/c^2$, i.e., by $f_+ = 2198 \text{Hz}(M_\odot/M)$, and the minimum ISCO frequency corresponds to the maximum mass, denoted by a filled circle, an empty circle, and a star, respectively for $\rho_0/(10^{14} \text{g cm}^{-3}) = 4.2, 5.3, \text{and } 6.5$. Note that the ISCO frequencies for rapidly rotating strange stars can have much lower values, and $f_+ < 1 \text{kHz}$ can be achieved for strange stars of fairly modest mass, e.g. $1.4M_\odot$, if the star rotates close to the equatorial mass-shedding limit. This figure is from Stergioulas et al. 1999.

to the marginally stable orbit (Kaaret 1997, Zhang 1998, Kluźniak 1998). But with the data gathered to date, it seems easier to constrain the e.o.s. of dense matter, on the assumption that the QPO frequency saturates in the ISCO, than to show that this assumption is indeed correct. One difficulty is that the physics of accretion disks is still very poorly understood.
New data is being gathered daily and new experiments are planned which may lead to a break-through in this field.

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