Dense subgraphs in the $H$-free process

Lutz Warnke
University of Oxford

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Random graph processes

Random graph process
(a) Start with empty graph on $n$ vertices
(b) Add edges, one at a time, chosen uniformly at random from all remaining pairs.

Random $H$-free graph process
(a) Start with empty graph on $n$ vertices
(b) Add edges, one at a time, chosen uniformly at random from all remaining pairs that do not complete a copy of $H$.

Basic questions:
(1) Final number of edges?
   (Erdős-Suen-Winkler, 1995)
(2) Subgraph counts?
Properties of the $H$-free process

In this talk $H$ satisfies some ‘density condition’

**Final number of edges**

- known for the $K_3$-free process up to constants (Bohman, 2009)
- known for the $H$-free process only up to log-factors (Osthus-Taraz, 2001)

**Subgraph counts**

- Comparable to normal random graph during the first $m$ steps, where

\[ m \approx \delta n^{2-1/d_2(H)} (\log n)^{c(H)} \]

(Bohman-Keevash, 2009+)
- Behaviour in later steps remains open
Recap: Small subgraphs in $G(n, i)$

Maximum Density

For a graph $F$ it is determined by ‘densest subgraph’:

$$m(F) := \max_{J \subseteq F, e_J \geq 1} \left\{ \frac{e_J}{v_J} \right\}.$$ 

Small subgraphs theorem (Bollobás)

Suppose we have a fixed graph $F$ and $i = n^{2-1/\alpha}$. Then we have a ‘threshold phenomenon’ in $G(n, i)$:

$$\text{whp} \begin{cases} 
\text{no copy of } F & \text{if } m(F) > \alpha \\
\text{‘many’ copies of } F & \text{if } m(F) < \alpha
\end{cases}$$
Results of Bohman-Keevash

Fixed subgraphs in the $H$-free process

Bohman-Keevash showed that for fixed $F$ with $H \not\subseteq F$, in the graph produced by the $H$-free process after the first $m = \delta n^{2-1/d_2(H)}(\log n)^{c(H)}$ steps:

\[
\text{whp } \begin{cases} 
\text{no copy of } F & \text{if } m(F) > d_2(H) \\
\text{‘many’ copies of } F & \text{if } m(F) < d_2(H)
\end{cases}
\]

$$\Rightarrow$$ $H$-free process ‘looks’ almost like a normal random graph, but it has no copies of $H$!

What happens in later steps?

• can ‘very dense’ subgraphs appear?
Previous results

\textbf{$K_3$-free process (Gerke-Makai, 2010+)}

There exists $c > 0$ such that whp any fixed $F$ with

\[ m(F) \geq c \]

does not appear in the $K_3$-free process.

\[ \implies \text{No ‘very dense’ subgraphs in later steps!} \]

\textbf{What happens in the $H$-free process?}

- what about graphs with $v_F = \omega(1)$ vertices?
Our result

$H$-free process (W., 2010+)

There exists $c(H), d(H) > 0$, such that the $H$-free process has whp no subgraph $J$ with density

$$m(J) \geq c(H)$$

on $v_J \leq n^{d(H)}$ vertices.

$\implies$ No ‘very dense’ subgraphs in later steps, even if they are ‘large’ (‘many’ vertices)!

Remarks

- extends/generalizes results for $K_3$-free process
- tight up to the constant:
  - whp fixed $F$ with $m(F) < d_2(H)$ appear
Proof idea

Goal:
- whp no copy of $J$ appears in the $H$-free process

Main idea
- We prove that whp already after the first $m$ steps:
  - for every possible placement of $J$, at least one of its pairs is ‘closed’ (i.e. can not be added in later steps)

Proof Strategy
- Show that whp in each step there are ‘many’ pairs that would close at least one pair of $J$
- Avoiding those pairs in all $m$ steps is ‘very unlikely’
Summary

For $H$ that satisfies some ‘density condition’:

**The $H$-free process contains no ‘very dense’ subgraphs**

- Whp the $H$-free process has no subgraphs $J$ on $v_J \leq n^{d(H)}$ vertices with density $m(J) \geq c(H)$

**Conjecture (W., 2010+)**

- The $H$-free process contains whp no copy of a fixed graph $F$ with $m(F) > d_2(H)$. 