A UNIVERSAL GENERALIZATION OF THE AMBROSE-SINGER THEOREM

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Abstract. Let $Q \to M$ be a principal $G$-bundle, and $B_0$ a connection on $Q$. We introduce an infinitesimal homogeneity condition for sections in an associated vector bundle $P \times_G V$ with respect to $B_0$, and, inspired by the well known Ambrose-Singer theorem, we prove the existence of a connection which satisfies a system of parallelism conditions. We explain how this general theorem can be used to prove all known Ambrose-Singer type theorems by an appropriate choice of the initial system of data. We also obtain new applications, which cannot be obtained using the known formalisms, e.g., structure and classification theorems for locally homogeneous spinors, locally homogeneous and locally symmetric triples.

Key Words: Geometric structures, Infinitesimally homogeneous, Ambrose-Singer theorem, Principal bundles, Connections

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1. Introduction

A Riemannian manifold $(M,g)$ is called symmetric (locally symmetric) if, for each point $x \in M$, there exists an isometry $s_x : M \to M$ (respectively a local isometry $s_x : U \to U$ defined on an open neighborhood $U$ of $x$) such that $s_x(x) = x$ and $d_x s_x = -\text{id}_{T_x M}$.

An explicit characterization of these spaces is given by the following theorem due to E. Cartan:
Theorem 1.1. [KN2, Theorem 6.2, Theorem 6.3] A Riemannian manifold \((M, g)\) is locally symmetric if and only if \(\nabla^g R^g = 0\), where \(\nabla^g\) is the Levi-Civita connection of \(g\) and \(R^g\) is the Riemann curvature tensor. Moreover, any connected, simply connected, complete locally symmetric Riemannian manifold is (globally) symmetric.

Symmetric (locally symmetric) Riemannian manifolds are homogeneous (resp. locally homogeneous). Recall that a Riemannian manifold \((M, g)\) is called homogeneous (resp. locally homogeneous) if for any two points \(x_1, x_2 \in M\) there is an isometry \(\phi : M \to M\) (resp. \(\phi : U_1 \to U_2\) between open neighborhoods \(U_i \ni x_i\)) with \(\phi(x_1) = x_2\).

Cartan’s result (Theorem [1]) has been extended by Ambrose and Singer to the locally homogeneous framework:

Theorem 1.2. [AS, Si] A Riemannian manifold \((M, g)\) is locally homogeneous if and only if there exists a metric connection \(\nabla\) such that \(\nabla R^\nabla = \nabla T^\nabla = 0\), where \(T^\nabla \in \mathcal{A}^0(L^2_{\text{alt}}(T_M, T_M))\) and \(R^\nabla \in \mathcal{A}^2(\text{gl}(T_M))\) denote the torsion and the curvature of \(\nabla\). Moreover, any connected, simply connected, complete locally homogeneous Riemannian manifold is (globally) homogeneous.

A metric connection satisfying the above conditions is called an Ambrose-Singer connection by some authors [TV, Tr, NT]. Obviously the Levi-Civita connection of a locally symmetric space is an Ambrose-Singer connection.

In this definition \((M, g)\) is locally symmetric if and only if there exists a connection \(\nabla\) on a suitable bundle \(P \to M\) such that \(\nabla^g R^g = 0\), where \(\nabla^g\) is the Levi-Civita connection of \((M, g)\). Theorem 1.1 can be reformulated as follows: \((M, g)\) is locally homogeneous if and only if it admits an Ambrose-Singer connection.

This statement is a special case of a very general principle in modern differential geometry: a local homogeneity condition for a class of geometric structures on a given manifold is equivalent to the existence of a connection (on a suitable bundle) which satisfies an Ambrose-Singer type condition. The goal of this article is a general theorem which implies all known results of this type, i.e. all Ambrose-Singer type theorems.

Our approach and our formalism are motivated by the following bundle versions of Cartan’s and Ambrose-Singer’s theorem. In order to state these theorems, which will be proved in section 4, we define the bundle analogues of the notions “(locally) symmetric space”, “(locally) homogeneous space”:

Definition 1.3. Let \(M\) be a differentiable manifold, and \(K\) a Lie group. A triple \((g, P \xrightarrow{\phi} M, A)\) consisting of a Riemannian metric \(g\) on \(M\), a principal \(K\)-bundle \(P \xrightarrow{\phi} M\) on \(M\), and a connection \(A\) on \(P\) is called:

1. Symmetric (locally symmetric) if, for each point \(x \in M\), there exists:
   (i) An isometry \(\phi : M \xrightarrow{s_x} M\) (resp. a local isometry \(U \xrightarrow{s_x} U\) such that \(s_x(x) = x\) and \(d_x s_x = -\text{id}_{T_x M}\)),
   (ii) An \(s_x\)-covering bundle isomorphism \(P \xrightarrow{\phi x} P\) (resp. \(P_U \xrightarrow{\phi x} P_U\)) which leaves any point \(y \in P_x\) and the connection \(A\) (resp. \(A_U\)) invariant.

2. Homogeneous (locally homogeneous) if for any two points \(x_1, x_2 \in M\) there exists:
   (a) An isometry \(\phi : M \xrightarrow{\phi} M\) (resp. a local isometry \(x_1 \in U_1 \xrightarrow{\phi} U_2 \ni x_2\)) with \(\phi(x_1) = x_2\).
   (b) A \(\phi\)-covering bundle isomorphism \(P \xrightarrow{\phi} P\) (resp. \(P_U \xrightarrow{\phi} P_U\)) which leaves \(A\)-invariant (resp. such that \(\phi^*(A_U) = A_U\)).

In this definition \(P_x\) stands for the fiber \(p^{-1}(x)\), and for an open set \(U \subset M\), we denoted by \(P_U\) (\(A_U\)) the restriction of the bundle \(P\) (the connection \(A\)) to
U (respectively $P_U$). With these definitions we will prove the following bundle analogs of Cartan’s, respectively Ambrose-Singer’s theorem:

**Theorem 1.4.** Let $M$ be a differentiable manifold, and $K$ a compact Lie group.

1. A triple $(g, P \xrightarrow{\lambda} M, A_0)$ as above is locally symmetric if and only if 
   \[ \nabla^g R^g = 0, \ (\nabla^g \otimes \nabla^A)F^A = 0. \]

2. A triple $(g, P \xrightarrow{\lambda} M, A_0)$ as above is locally homogeneous if and only if there exists a pair $(\nabla, A)$ consisting of a metric connection on $M$ and a connection $A$ on $P$ such that 
   \[ \nabla R^\nabla = 0, \ \nabla T^\nabla = 0, \ (\nabla \otimes \nabla^A)F^A = 0, \ (\nabla \otimes \nabla^A)(A - A_0) = 0. \]

In this statement we used the affine space structure of the space $\mathcal{A}(P)$ of all connections on $P$, so $A - A_0$ is an element of $\mathcal{A}(M, \text{ad}(P))$.

Why are we interested in these bundle analogues of the classical notions and results we have recalled? Note first that a result of Itoh, which we explain below, on connections on $P$.

**Example 1.1.** Let $(G, H)$ be a reductive pair. In other words, $G$ is a connected Lie group, $H$ a closed subgroup of $G$ such that $\mathfrak{h}$ admits an $\text{ad}_H$-invariant complement $\mathfrak{s}$ in $\mathfrak{g}$. Such a complement defines a $G$-invariant connection $\nabla^\mathfrak{s}$ on the principal bundle $G \rightarrow G/H$. For a Lie group $K$, the data of a $G$-homogeneous $K$-bundle on $M := G/H$ is equivalent to the data of a morphism $\lambda : H \rightarrow K$. The bundle corresponding to $\mathfrak{h}$ is $P_\lambda := G \times_\lambda K$. This bundle comes with an obvious $G$-invariant connection, namely $\nabla^\mathfrak{s}_\lambda := (\rho_\lambda)_*(\nabla^\mathfrak{s})$, where $\rho_\lambda : G \rightarrow P_\lambda$ is the obvious bundle morphism. For a symmetric space $(G, H, \sigma)$ (see [KN2 Section XI.2]) one has a canonical choice of $\mathfrak{s}$ (see [KN2 Proposition 2.2]), so any $G$-homogeneous $K$-bundle $P$ on $M$ comes with a canonical connection, which will be denoted by $A_\lambda$. With these preparations we can state the following result of Itoh [1].

**Theorem 1.5.** Let $(G, H, \sigma)$ be a symmetric space with $H$ compact, and $M = G/H$ compact and oriented. Let $P$ be a $G$-homogeneous principal $K$-bundle on $M$ with $K$ compact. Let $g$ be the Riemannian metric on $M$ associated with an $\text{ad}_H$-invariant inner product on the canonical complement $\mathfrak{s}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Then, the curvature of the canonical invariant connection $A_\lambda$ on $P$ is parallel with respect to $\nabla^g \otimes \nabla^A_\lambda$.

Therefore, the triple $(g, P_\lambda \rightarrow M, A_\lambda)$ is locally symmetric; it will be symmetric when $M$ is simply connected.

Note also that

**Example 1.2.** For a locally symmetric (locally homogeneous) Riemannian manifold $(M, g)$, let $C_0 \in \mathcal{A}(O(M))$ denote the Levi-Civita connection on the orthonormal frame bundle $O(M)$ of $(M, g)$. The associated triple $(g, O(M) \rightarrow M, C_0)$ is locally symmetric (respectively locally homogeneous) in the sense of Definition 1.3.

Our bundle versions of Cartan’s and Ambrose-Singer’s theorem (Theorem 1.4) have important consequences: we will prove (see Theorem 1.24 Corollary 1.25) that any locally symmetric (homogeneous) triple on a simply connected, complete base $(M, g)$ is globally symmetric (homogeneous). This will allow us to prove a classification theorem for locally symmetric (homogeneous) triples: any locally symmetric (homogeneous) triple on a compact base $M$ is a quotient of a globally symmetric (homogeneous) triple on the universal cover $\tilde{M}$ (see Theorem 1.26 Corollary 1.28). These results play an important role and can be used effectively for the classification of geometric manifolds (in the sense of Thurston) which are principal bundles.
over a given geometric base (see [B52, BT]).

In this article we prove a general theorem, explained in the next subsection, which yields not only Theorem 1.4, but all Ambrose-Singer type theorems which we found in the literature. Moreover, as special cases of our theorem, we will obtain new such results, for instance a characterization of locally homogeneous pairs \((g, s)\) consisting of a Riemannian metric and a spinor.

1.1. Infinitesimally homogeneous sections. Let \(M\) be a differentiable manifold of dimension \(n\), and let \(\theta\) be a finite dimensional vector space, \(\rho : G \to GL(V)\) be a morphism of Lie groups, and \(\pi : Q \to L(M)\) be an \(r\)-morphism of principal bundles over \(M\). Let also \(V\) be a finite dimensional vector space, \(\rho : G \to GL(V)\) be a morphism of Lie groups, and \(E := Q \times_{\rho} V\) be the associated vector bundle.

For \(B \in A(Q)\), the linear connection on the tangent bundle \(T_M\) associated with \(f_*(B)\) is denoted by \(\nabla^B_M\); it coincides with the linear connection associated with \(B\) on vector bundle \(Q \times_{\rho} \mathbb{R}^n \simeq T_M\). The linear connections on vector bundles \(E = Q \times_{\rho} V\) (\(\text{ad}(Q) = Q \times_{\text{ad}} \mathfrak{g}\)) associated with \(B\) will be denoted by \(\nabla^E_k\) (respectively \(\nabla^B_k\)).

Let \(B_0\) be a fixed connection on \(Q\), and \(\sigma \in \Gamma(E)\). Put

\[
\sigma^{(i)}_{B_0} := ((\nabla^{B_0}_M) \otimes (\nabla^{B_0}_E)) \otimes \cdots (\nabla^{B_0}_M \otimes \nabla^{B_0}_E) \sigma \in \Gamma((\Lambda^1_M) \otimes \otimes E),
\]

(with the usual convention \(\sigma^{(0)}_{B_0} = \sigma\)). For any \(k \in \mathbb{N}\) and \(x \in M\) put

\[
b^E_{\sigma}(k) := \{b \in \text{ad}(Q_x) | b \cdot \{\sigma^{(i)}_{B_0}\}_x = 0 \text{ for } 0 \leq i \leq k\},
\]

and note that \(b^E_{\sigma}(k)\) is a Lie subalgebra of the finite dimensional Lie algebra \(\text{ad}(Q_x)\). One has \(b^E_{\sigma}(k+1) \subset b^E_{\sigma}(k)\) for any \(k\). Put

\[
k^E_{\sigma} := \min\{k \in \mathbb{N} | b^E_{\sigma}(k+1) = b^E_{\sigma}(k)\}.
\]

If \((x_1, x_2) \in M \times M\) then for any \(G\)-equivariant isomorphism \(\theta : Q_{x_1} \to Q_{x_2}\), we can define the next linear isomorphisms

\[
\theta_V : E_{x_1} \to E_{x_2}, \quad \theta_{\text{ad}} : \text{ad}(Q_{x_1}) \to \text{ad}(Q_{x_2}).
\]

We also obtain a linear isomorphism \(\theta_M : T_{x_1}M \to T_{x_2}M\) via the bundle morphism \(f\). Now, denoting by \(\theta^k\) the induced isomorphism \(\{\Lambda^1_{x_1}\} \otimes^k E_{x_1} \to \{\Lambda^1_{x_2}\} \otimes^k E_{x_2}\), we define

**Definition 1.6.** Let \(B_0 \in A(Q)\) be a connection on \(Q\). A section \(\sigma \in \Gamma(E)\) is called infinitesimally homogeneous with respect to \(B_0\), if for any pair \((x_1, x_2) \in M \times M\) there exists a \(G\)-equivariant isomorphism \(\theta : Q_{x_1} \to Q_{x_2}\) such that

\[
\theta^k((\sigma^{(i)}_{B_0})_{x_1}) = (\sigma^{(i)}_{B_0})_{x_2} \text{ for } 0 \leq i \leq k^E_{\sigma} + 1.
\]

Let \(\theta : Q_{x_1} \to Q_{x_2}\) be a \(G\)-equivariant isomorphism such that (2) holds. Then \(\theta_{\text{ad}}\) applies isomorphically \(b^E_{\sigma}(k)\) on \(b^E_{\sigma}(k)\) for \(0 \leq k \leq k^E_{\sigma} + 1\). This implies

**Remark 1.7.** If \(\sigma \in \Gamma(E)\) is an infinitesimally homogeneous section with respect to \(B_0\), then \(k^E_{\sigma}\) is independent of \(x\). The obtained constant will be denoted by \(k^\sigma\).

The main result is Theorem 1.8, which states:

**Theorem 1.8.** Suppose that \(\sigma\) is infinitesimally homogeneous with respect to \(B_0\). Fix \(\theta_{\text{ad}} \in Q\), and suppose that the pair \((G, H^\sigma_{\text{ad}})\) is reductive. There exists a connection \(B \in A(Q)\) such that:

\[
\begin{align*}
(1) & \quad (\nabla^B_k \otimes \nabla^B_\sigma)(\sigma^{(k)}_{B_0}) = 0 \quad \text{for } 0 \leq k \leq k^\sigma + 1. \\
(2) & \quad (\nabla^B_m \otimes \nabla^B_\sigma)(B - B_0) = 0.
\end{align*}
\]
The reductivity condition needed in the theorem (see section 2.2) is automatically satisfied when $G$ is compact. We will see that this theorem can be used to prove Theorem 1.4 stated above. Moreover, it provides a general approach for the study of locally homogeneous objects by investigating their corresponding infinitesimally homogeneous analogues. In each specific case we will choose an appropriate system of data

$$(M, Q, p, f : Q \to L(M), B_0, \sigma)$$

and, using Theorem 2.13, we will prove in each case a result of the type “locally homogeneous is equivalent to infinitesimally homogeneous, and is equivalent to globally homogeneous in the complete, simply connected case”.

Section 2 of this article is devoted to the proof of Theorem 1.8. In section 3, this result will be first used to prove known Ambrose-Singer type theorems:

1. The classical Ambrose-Singer theorem (Theorem 1.2).
2. Kirichenko’s theorem for locally homogeneous systems $(g, P_1, \ldots, P_k)$, where $g$ is a Riemannian metric, and $P_i$ are tensor fields (Theorem 3.5). In particular we will consider the case of locally homogeneous almost Hermitian manifolds (Theorem 3.6).
3. Opozda’s theorem on locally homogeneous $G$-structures (Theorem 3.9).

Note that, using a lemma based on elliptic regularity (see Lemma 3.12), we show that the analyticity condition in Opozda’s theorem is not necessary.

At the end of the section we prove a new Ambrose-Singer type theorem which concerns locally homogeneous pairs $(g, s)$ consisting of a Riemannian metric and a spinor on a spin Riemannian manifold $M$. This result is not a special case of Kiritchenko’s theorem, because a spinor on $M$ cannot be considered as a tensor field. Indeed, $s$ is a section in the spinor bundle, which is a vector bundle associated with the principal bundle of the fixed spin structure, not with the frame bundle $L(M)$. This shows already the advantage of our formalism (which uses a general principal $G$-bundle $Q$ instead of the classical $L(M)$).

Section 4 is dedicated to locally symmetric (homogeneous) triples in the sense of Definition 1.3. We will prove Theorem 1.4 and we will explain how this result can be used for the classification of the locally symmetric (homogeneous) triples on a given compact Riemannian manifold in terms of globally symmetric (homogeneous) triples on its universal cover.

2. THE MAIN RESULT

In this section we will give the proof of our main theorem (Theorem 2.13).

2.1. A Leibniz formula. Let $M$ be a differentiable manifold of dimension $n$ and $L(M) \to M$ be its frame bundle. Let $G$ be a Lie group and let $r : G \to GL(n)$ be a morphism of Lie groups, $\pi : Q \to M$ be a principal $G$-bundle over $M$. Moreover, let $V$ be a finite dimensional vector space, $\rho : G \to GL(V)$ be a morphism of Lie groups, and $E := Q \times_\rho V$ be the associated vector bundle. Put

$$W_{ijpq} := (\mathbb{R}^n)^{\otimes i} \otimes (\mathbb{R}^n)^{\otimes j} \otimes V^{\otimes p} \otimes V^{\otimes q},$$

and let $R : G \to GL(W_{ijpq})$ be the linear representation induced by $r$ and $\rho$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and consider the Lie algebra morphism $\mathfrak{g} \to gl(W_{ijpq})$. This infinitesimal action defines a $G$-invariant pairing

$$\mathfrak{g} \times W_{ijpq} \to W_{ijpq},$$

which induces a pairing of associated vector bundles

$$ad(Q) \times (T_{M}^{ij} \otimes (\Lambda_{M}^{i})^{\otimes j} \otimes E^{\otimes p} \otimes E^{\otimes q}) \to T_{M}^{ij} \otimes (\Lambda_{M}^{i})^{\otimes j} \otimes E^{\otimes p} \otimes E^{\otimes q}.$$
The pairings (3), (4) will be denoted by \((h, \eta) \mapsto b \cdot \eta\) to save on notations. For instance, if \(b \in \text{ad}(Q)\) and \(\eta \in \{\Lambda^1_M\}^{\otimes j} \otimes E\) then the pairing
\[
\text{ad}(Q) \times_M \left( \{\Lambda^1_M\}^{\otimes j} \otimes E \right) \to \{\Lambda^1_M\}^{\otimes (j+k)} \otimes E
\]
is given by the formula
\[
(b \cdot \eta)(w_1, \ldots, w_j) := b \cdot (\eta(w_1, \ldots, w_j)) = \sum_{i=1}^j \eta(w_1, \ldots, b \cdot w_i, \ldots, w_j).
\]
The fact that the pairing (3) is \(G\)-invariant, has an important consequence:

**Remark 2.1.** The pairing of associated vector bundles given in (4) is parallel with respect to any connection on \(Q\).

We shall also need the pairing
\[
\left( \{\Lambda^1_M\}^{\otimes k} \otimes \text{ad}(Q) \right) \times_M \left( \{\Lambda^1_M\}^{\otimes j} \otimes E \right) \to \{\Lambda^1_M\}^{\otimes (j+k)} \otimes E
\]
given by
\[
(u \otimes b) \cdot \eta := u \otimes (b \cdot \eta).
\]
Explicitly, for \(\beta \in \{\Lambda^1_M\}^{\otimes k} \otimes \text{ad}(Q_x)\) and \(\eta \in \{\Lambda^1_M\}^{\otimes j} \otimes E_x\), one has
\[
(\beta \cdot \eta)(v_1, \ldots, v_k, w_1, \ldots, w_j) = \left( \beta(v_1, \ldots, v_k) \cdot \eta \right)(w_1, \ldots, w_j) =
\]
\[
= \beta(v_1, \ldots, v_k) \cdot \eta(w_1, \ldots, w_j) - \sum_{i=1}^j \eta(w_1, \ldots, \beta(v_1, \ldots, v_k) \cdot w_i, \ldots, w_j).
\]
Note also that, if \(\beta = (\omega_1 \otimes \cdots \otimes \omega_k) \otimes b\) is a tensor monomial with \(\omega_i \in \Lambda^1_M\) and \(b \in \text{ad}(Q_x)\), then
\[
\left( (\omega_1 \otimes \cdots \otimes \omega_k) \otimes b \right) \cdot \eta = (\omega_1 \otimes \cdots \otimes \omega_k) \otimes (b \cdot \eta).
\]

Let \(B\) be a connection on the principal \(G\)-bundle \(Q\) as above and \(f : Q \to L(M)\) be an \(r\)-morphism of principal bundles over \(M\). Let \(\nabla^B_M\) denote the linear connection on the tangent bundle \(T_M\) associated with \(f_*(B)\). This connection corresponds to the linear connection associated with \(B\) on the vector bundle \(Q \times_r \mathbb{R}^n \simeq T_M\). Let \(\nabla^B_E, \nabla^B_M\) denote the linear connections on the vector bundles \(E = Q \times_r V\), respectively \(\text{ad}(Q) = Q \times_r \mathfrak{g}\) which are associated with \(B\). We also need the linear connection \(\nabla^B_{ijpq}\) on the vector bundle \(T_M^{\otimes i} \otimes (\Lambda^1_M)^{\otimes j} \otimes E^{\otimes p} \otimes E^{\otimes q}\) associated with \(B\). This connection can be written as
\[
\nabla^B_{ijpq} = (\nabla^B_M)^{\otimes i} \otimes (\nabla^B_E)^{\otimes j} \otimes (\nabla^B_M)^{\otimes p} \otimes (\nabla^B_E)^{\otimes q}.
\]
Taking into account Remark 2.1 it follows

**Remark 2.2.** For any \(x \in M\) and tangent vector \(\xi \in T_x M\) the following Leibniz rule holds:
\[
\nabla^B_{ijpq, \xi}(b \cdot \eta) = (\nabla^B_{ijpq})_\xi(b) \cdot \eta + b \cdot \nabla^B_{ijpq, \xi}(\eta).
\]
In particular, for any pairs \((b, v) \in \Gamma(\text{ad}(Q)) \times \mathfrak{X}(M)\) one has
\[
\nabla^B_{ijpq, \xi}(b \cdot v) = (\nabla^B_{ijpq})_\xi(b) \cdot v + b \cdot (\nabla^B_{ijpq, \xi}v),
\]
and for any \((b, y) \in \Gamma(\text{ad}(Q)) \times \Gamma(E)\)
\[
\nabla^B_{ijpq, \xi}(b \cdot y) = (\nabla^B_{ijpq})_\xi(b) \cdot y + b \cdot (\nabla^B_{ijpq, \xi}y).
\]

The tensor product of connections induces the connection \((\nabla^B_M)^{\otimes j} \otimes \nabla^B_E\) on the vector bundle \(\{\Lambda^1_M\}^{\otimes j} \otimes E\) and the connection \((\nabla^B_M)^{\otimes j} \otimes \nabla^B_M\) on the vector bundle \(\{\Lambda^1_M\}^{\otimes j} \otimes \text{ad}(Q)\). The pairing (6) will be used to obtain the following variation formula for the connection \((\nabla^B_M)^{\otimes j} \otimes \nabla^B_E\) with respect to \(B\):
Lemma 2.4. For any $\beta \in \Gamma(\{A^1_M\}^{\otimes k} \otimes E)$, $\eta \in \Gamma(\{A^1_M\}^{\otimes j} \otimes E)$, and any tangent vector $\xi \in T_xM$ one has

\[
((\nabla^{B'}_x)^{\otimes j} \otimes \nabla^{B'}_x) \eta = ((\nabla^B_x)^{\otimes j} \otimes \nabla^B_x) \eta + \beta \cdot \eta. \tag{8}
\]

Explicitly, for any $x \in M$ and $\xi \in T_xM$

\[
((\nabla^{B'}_x)^{\otimes j} \otimes \nabla^{B'}_x) \eta = ((\nabla^B_x)^{\otimes j} \otimes \nabla^B_x) \xi \eta + \beta(\xi) \cdot \eta.
\]

With the notations introduced above one can prove the following Leibniz formula.

\[\text{Remark 2.6.}\] Let $B \in \mathcal{A}(Q)$ and $\beta \in A^1(M, \text{ad}(Q))$. Put $B' := B + \beta$. For any $\eta \in \Gamma(\{A^1_M\}^{\otimes j} \otimes E)$ one has

\[
((\nabla^{B'}_x)^{\otimes j} \otimes \nabla^{B'}_x) \eta = ((\nabla^B_x)^{\otimes j} \otimes \nabla^B_x) \eta + \beta \cdot \eta. \tag{8}
\]

Proof. We give the proof in the case $\beta$ is a tensor monomial, so it has the form $\beta = \omega \otimes b$, where $\omega \in \Gamma(\{A^1_M\})$, and $b \in \Gamma(\text{ad}(Q))$. Using Remark 2.2 we obtain

\[
((\nabla^{B'}_x)^{\otimes (1+j)} \otimes \nabla^{B'}_x) \xi \beta \cdot \eta = ((\nabla^{B'}_x)^{\otimes (1+j)} \otimes \nabla^{B'}_x) \xi (\omega \otimes b) \cdot \eta
\]

\[\omega \otimes \omega \otimes (b \cdot \eta) + \omega \otimes ((\nabla^{B'}_x)^{\otimes (1+j)} \otimes \nabla^{B'}_x) \xi (\omega \otimes b) \cdot \eta + ((\nabla^{B'}_x)^{\otimes (1+j)} \otimes \nabla^{B'}_x) \xi (\omega \otimes b) \cdot \eta = ((\nabla^{B'}_x)^{\otimes (1+j)} \otimes \nabla^{B'}_x) \xi \beta \cdot \eta + \beta(\xi) \cdot \eta.
\]

- \[\blacksquare\]

2.2. The existence of an adapted connection. We will need the following standard general results.

\[\text{Lemma 2.5.}\] Let $\pi: Q \to M$ be a principal $G$-bundle over a manifold $M$, and let $H$ be a closed subgroup of $G$. There is a bijection between the set of $H$-reductions of $Q$ and the set of pairs

\[
(\varphi, u) \in \Gamma(M, Q \times_G (G/H)) \times (G/H).
\]

The $H$-reduction associated with a pair $(\varphi, u)$ is the pre-image $\Phi^{-1}(u)$, where

\[\Phi: Q \to G/H\]

is the equivariant map associated with $\varphi$.

- Let $P \subset Q$ be the $H$-reduction of $Q$. A connection $B \in \mathcal{A}(Q)$ will be called compatible with $P$ if the restriction of $B$ to $P$ is tangent to $P$ (and hence defines a connection on $P$).

- \[\text{Remark 2.6.}\] Let $P \subset Q$ be the $H$-reduction associated with the pair $(\varphi, u)$. Then the direct image map $\mathcal{A}(P) \hookrightarrow \mathcal{A}(Q)$ whose image is the space of connections on $Q$ which are compatible with $P$, and whose associated linear map is the inclusion $A^1(\text{ad}(P)) \subset A^1(\text{ad}(Q))$.

- Taking into account Remark 2.6 in the presence of a fixed $H$-reduction $P \subset Q$, we will identify a connection $B \in \mathcal{A}(P)$ with its image in $\mathcal{A}(Q)$, so any connection $B \in \mathcal{A}(P)$ will also be regarded as connection on $Q$.

- Let $\sigma$ be an infinitesimal section with respect to $B_0 \in \mathcal{A}(Q)$. We will define a closed subgroup $H^\sigma \subset G$ and a $H^\sigma$-reduction $P^\sigma \subset Q$ such that for any connection $B'$ on $P^\sigma$ one has

\[
((\nabla^{B'}_x)^{\otimes k} \otimes \nabla^{B'}_x)\sigma^{(k)}_{B_0} = 0 \quad \text{for } 0 \leq k \leq k^\sigma + 1.
\]
For $x \in M$ we will identify the fibre $L(M)_x$ of the frame bundle $L(M)$ with the space of linear isomorphisms $\mathbb{R}^n \to T_x M$. Therefore, with the notations introduced at the beginning of this section, a point $q \in Q$ defines a linear isomorphism

$$f(q) : \mathbb{R}^n \to T_x M.$$  

Using the $k$-order covariant derivative $\sigma_{B_0}^{(k)}$ of $\sigma$ we obtain a $G$-equivariant map

$$\varphi_k : Q \to L^k(\mathbb{R}^n, V)$$

defined by the formula

$$\sigma_{B_0}^{(k)}(f(q)(\xi_1), \ldots, f(q)(\xi_k)) = [q, \varphi_k(q)(\xi_1, \ldots, \xi_k)] \forall(\xi_1, \ldots, \xi_k) \in (\mathbb{R}^n)^k.$$  

In other words, $\varphi_k$ is the $G$-equivariant map $Q \to L^k(\mathbb{R}^n, V)$ associated with $\sigma_{B_0}^{(k)}$ regarded as a section in the associated bundle

$$(\Lambda^k T)^{\otimes k} \otimes E = Q \times_G L^k(\mathbb{R}^n, V).$$

Put $W := \bigoplus_{k=0}^{k^*+1} L^k(\mathbb{R}^n, V)$ and define a $G$-equivariant map $\Phi : Q \to W$ by

$$\Phi(q) := (\varphi_k(q))_{0 \leq k \leq k^*+1}.$$  

(10)

Since the section $\sigma \in \Gamma(E)$ is infinitesimally homogeneous, it follows that $\Phi(Q)$ is a single $G$-orbit of $W$. Indeed, let $q_0 \in Q$, and put $x_0 := \pi(q_0)$. For a point $q \in Q$, let $x = \pi(q)$ and $\theta : Q_{x_0} \to Q_x$ be a $G$-equivariant isomorphism such that

$$\theta^k(\{\sigma_{B_0}^{(k)}\}_{x_0}) = \{\sigma_{B_0}^{(k)}\}_{x} \text{ for } 0 \leq k \leq k^*+1.$$  

(11)

(see Definition 1.6). This implies the equality

$$[\theta(q_0), \varphi_k(q_0)] = [q, \varphi_k(q)]$$

in $Q_x \times_G L^k(\mathbb{R}^n, V)$. Choosing $a \in G$ such that $\theta(q_0) = qa$, we obtain

$$\varphi_k(q) = a \varphi_k(q_0) \text{ for } 0 \leq k \leq k^*+1,$$

which shows that $\Phi(q) \in G\Phi(q_0)$. Therefore $\Phi(Q) \subset G\Phi(q_0)$. Using the $G$-equivariance property of $\Phi$ we get $\Phi(Q) = G\Phi(q_0)$, as claimed. Put

$$H^\sigma_{q_0} := G\Phi(q_0), \quad P^\sigma_{q_0} := \Phi^{-1}(\Phi(q_0)).$$

Using Lemma 2.5, it follows that $P^\sigma_{q_0}$ is a $H^\sigma_{q_0}$-reduction of $Q$. Since $\Phi$ is obviously constant on $P^\sigma_{q_0}$, it follows that the restrictions $\varphi_k |_{P^\sigma_{q_0}}$ are all constant on $P^\sigma_{q_0}$, so the corresponding sections will be parallel with respect to any connection on $P^\sigma_{q_0}$. Therefore

Remark 2.7. The sections $\sigma_{B_0}^{(k)}$ ($0 \leq k \leq k^*+1$) are parallel with respect to any connection on the bundle $P^\sigma_{q_0}$, so with respect to any connection on $Q$ which is compatible with $P^\sigma_{q_0}$. In other words, for any $B \in \mathcal{A}(P^\sigma_{q_0})$ we have

$$(\nabla^B_x \otimes \nabla^B_x)(\sigma_{B_0}^{(k)}) = 0 \text{ for } 0 \leq k \leq k^*+1.$$  

(12)

Remark 2.8. For any $x \in M$ one has $h^\sigma_x(k^*+1) = \text{ad}(P^\sigma_{q_0})_x$.

Proof. By definition $h^\sigma_x(k^*+1)$ is the infinitesimal stabilizer in $\text{ad}(Q_x)$ of the system $\{\sigma_{B_0}^{(k)}\}_{x} |_{0 \leq k \leq k^*+1}$, regarded as an element of the fiber $Q_x \times_G W$ of the associated vector bundle $Q \times_G W$. An element $q \in Q_x$ identifies $\text{ad}(Q_x)$ with $g$, the system $\{\sigma_{B_0}^{(k)}\}_{x} |_{0 \leq k \leq k^*+1}$ with $\Phi(q)$ and the fiber $Q_x \times_G W$ with $W$. For $q \in P^\sigma_{q_0} := \Phi^{-1}(\Phi(q_0))$, we have $\Phi(q) = \Phi(q_0)$, so

$$h^\sigma_x(k^*+1) = \{[q, u] \mid u \in g_{\Phi(q_0)}\} = \{[q, u] \mid u \in h^\sigma_{q_0}\} = \text{ad}(P^\sigma_{q_0}).$$
Remark 2.8 shows in particular that the union
\[ h^\sigma := \bigcup_{x \in M} h^\sigma_x(k^\sigma_x + 1) \]
is a Lie algebra sub-bundle of \( \text{ad}(Q) \).

**Proposition 2.9.** Suppose that \( \sigma \) is infinitesimally homogeneous with respect to \( B_0 \). Let \( B' \in \mathcal{A}(Q) \) be a connection compatible with \( P^\sigma_{q_0} \). Then

1. The sub-bundle \( h^\sigma \subset \text{ad}(Q) \) is \( \nabla^B_{\text{ad}} \)-parallel.
2. One has \( (\nabla^B_{\text{ad}} \otimes \nabla^B_{\text{ad}})(B' - B_0) \in \Gamma(\Lambda_M^1 \otimes \Lambda_M^1 \otimes h^\sigma) \).

**Proof.** (1) Let \( \nu : [0,1] \to M \) be a smooth path in \( M \). By Remarks 2.2 and 2.7, the parallel transport with respect to the connection \( \nabla^B_{\text{ad}} \) maps isomorphically \( h^\nu_{\nu(0)} \) onto \( h^\nu_{\nu(1)} \).

(2) Put \( \beta := B' - B_0 \in \mathfrak{A}^1(\text{ad}(Q)) \). Using Remark 2.3, we obtain, for \( 0 \leq k \leq k^\sigma + 1 \)
\[ 0 = ((\nabla^B_{\text{ad}} \otimes \nabla^B_{\text{ad}})_{\xi}(\beta) \cdot e^{(k)}_{B_0}) = 0 \quad \text{for} \quad 0 \leq k \leq k^\sigma. \]

Taking into account formula (7), it follows that, for any \( v \in T_x M \) one has
\[ ((\nabla^B_{\text{ad}} \otimes \nabla^B_{\text{ad}})_{\xi}(\beta) \cdot e^{(k)}_{B_0}) = 0. \]

Therefore for any \( (\xi, v) \in T_x M \times T_x M \) one has
\[ ((\nabla^B_{\text{ad}} \otimes \nabla^B_{\text{ad}})_{\beta}(\xi, v) \in h^\sigma_x(k^\sigma_x) = h^\sigma_x(k^\sigma_x + 1), \]
which shows that
\[ (\nabla^B_{\text{ad}} \otimes \nabla^B_{\text{ad}})_{\beta} \in \Gamma(\Lambda^1_M \otimes \Lambda^1_M \otimes h^\sigma(k^\sigma + 1)) = \Gamma(\Lambda^1_M \otimes \Lambda^1_M \otimes h^\sigma). \]

We recall that a pair \((G,H)\), where \( G \) is a Lie group, and \( H \subset G \) a closed subgroup, is called reductive if \( h \) admits an \( \text{ad}_H \)-invariant complement in \( g \) [SW, Example 4, p. 165]. Note that

**Remark 2.10.** Any pair \((G,H)\) with \( H \) compact is reductive. In particular, when \( G \) is compact, any pair \((G,H)\) with \( H \subset G \) a closed subgroup, is reductive.

Let \( \sigma \) be a infinitesimally homogeneous section with respect to \( B_0 \). The following result shows that, assuming that \((G,H^\sigma_{q_0})\) is reductive, any connection \( B' \in \mathcal{A}(P^\sigma_{q_0}) \) can be modified, such that the obtained connection \( B \) satisfies
\[ (\nabla^B_{\text{ad}} \otimes \nabla^B_{\text{ad}})(B' - B_0) = 0, \]
which is a much stronger property than Proposition 2.9 (2). In order to prove this note first that

**Remark 2.11.** Suppose that \((G,H^\sigma_{q_0})\) is reductive, and let \( \mathfrak{k} \) be an \( \text{ad}_H^\mathfrak{k} \)-invariant complement of \( h^\mathfrak{k}_{\mathfrak{k}} \) in \( g \). The direct sum decomposition
\[ g = h^\mathfrak{k}_{\mathfrak{k}} \oplus \mathfrak{k}, \]
is \( \text{ad}_H^\mathfrak{k} \)-invariant. Putting \( \mathfrak{b}^\sigma := P^\sigma_{q_0} \times H^\mathfrak{k}_{\mathfrak{k}} \mathfrak{k} \) we obtain a vector bundle decomposition
\[ \text{ad}(Q) = Q \times_G g = P^\sigma_{q_0} \times H^\mathfrak{k}_{\mathfrak{k}} \mathfrak{k} \mathfrak{g} = h^\sigma \oplus \mathfrak{b}^\sigma, \]
which is parallel with respect to any connection on $P_{q_0}$ (so to any connection on $Q$ which is compatible with $P_{q_0}^\sigma$).

Note that the Lie algebra isomorphism $\theta_B : \text{ad}(Q)_{x_1} \to \text{ad}(Q)_{x_2}$ associated with a $G$-isomorphism $\theta : Q_{x_1} \to Q_{x_2}$ satisfying \(^{(2)}\) maps isomorphically $\mathfrak{t}^\sigma_{x_1}$ onto $\mathfrak{t}^\sigma_{x_2}$.

**Proposition 2.12.** Suppose that $\sigma$ is infinitesimally homogeneous with respect to $B_0$, and the pair $(G, H^\sigma_{q_0})$ is reductive. Then there exists a connection $B \in \mathcal{A}(P_{q_0}^\sigma)$ such that

\[
(\nabla^B_M \otimes \nabla^B_M)(B - B_0) = 0.
\]

**Proof.** Let $B' \in \mathcal{A}(P_{q_0}^\sigma)$, so that Proposition \(2.9\) applies. The problem is to find $\beta \in \Gamma(A^1_{\mathcal{M}} \otimes \mathfrak{h}^\sigma)$ such that the equation \(14\) holds for $B = B' + \beta$.

We have the following direct sum decompositions

\[
\text{ad}(Q) = \mathfrak{b}^\sigma \oplus \mathfrak{t}^\sigma, \quad (\Lambda^1_M \otimes \text{ad}(Q)) = (A^1_M \otimes \mathfrak{b}^\sigma) \oplus (A^1_M \otimes \mathfrak{t}^\sigma).
\]

Using the splitting \(16\) we can decompose $B' - B_0 \in A^1_M \otimes \text{ad}(Q)$ in a unique way as follows

\[B' - B_0 = b_h + b_t,\]

where $b_h \in A^1_M \otimes \mathfrak{b}^\sigma$ and $b_t \in A^1_M \otimes \mathfrak{t}^\sigma$.

Put $B := B' - b_h = B_0 + b_t, \beta := -b_h \in A^1_M \otimes \mathfrak{b}^\sigma$. By Remark \(2.11\) the decomposition \(15\) is parallel with respect to $B$. Similarly, the decomposition \(16\) will be $(\nabla^B_M \otimes \nabla^B_M)$-parallel, so $A^1_M \otimes \mathfrak{t}^\sigma$ is a $(\nabla^B_M \otimes \nabla^B_M)$-parallel sub-bundle of $A^1_M \otimes \text{ad}(Q)$. Since $b_t$ is a section of $A^1_M \otimes \mathfrak{t}^\sigma$ we obtain

\[
(\nabla^B_M \otimes \nabla^B_M)\xi b_t \in \Gamma(A^1_M \otimes A^1_M \otimes \mathfrak{t}^\sigma) \quad \forall \xi \in T_M.
\]

On the other hand, using Proposition \(2.9\) we obtain

\[
(\nabla^B_M \otimes \nabla^B_M)\xi b_t \in \Gamma(A^1_M \otimes A^1_M \otimes \mathfrak{b}^\sigma) \quad \forall \xi \in T_M.
\]

Therefore $(\nabla^B_M \otimes \nabla^B_M)\xi b_t = 0$, which proves \(14\) because $b_t = B - B_0$.

\[\square\]

**Theorem 2.13.** Suppose that $\sigma$ is infinitesimally homogeneous with respect to $B_0$. Fix $q_0 \in Q$, and suppose that the pair $(G, H^\sigma_{q_0})$ is reductive. There exists a connection $B \in \mathcal{A}(Q)$ such that:

\begin{enumerate}
\item [(1)] $(\nabla^B_M \otimes \sigma_{\mathfrak{b}^\sigma})(\sigma_{\mathfrak{b}^\sigma}) = 0$ for $0 \leq k \leq k^\sigma + 1$.
\item [(2)] $(\nabla^B_M \otimes \nabla^B_M)(B - B_0) = 0$.
\end{enumerate}

A connection $B \in \mathcal{A}(Q)$ satisfying the conclusion of Theorem \(2.13\) will be called an adapted, or Ambrose-Singer type connection for the infinitesimally homogeneous section $\sigma$. In particular if, $k = 0$ then $\sigma^{(0)}_{B_0} = \sigma$, and we obtain the following result:

**Corollary 2.14.** Suppose that $\sigma$ is infinitesimally homogeneous with respect to $B_0$. Fix $q_0 \in Q$, and suppose that the pair $(G, H^\sigma_{q_0})$ is reductive. Then there exists a connection $B \in \mathcal{A}(Q)$ with the properties:

\[
\nabla^B_M \sigma = 0, \quad (\nabla^B_M \otimes \nabla^B_M)(B - B_0) = 0.
\]

It is important to note that if $G$ is compact, then the reductivity condition in Theorem \(2.13\) Corollary \(2.14\) will be automatically satisfied.
3. Applications

In this section we will see that our results provide a general framework to study locally homogeneous objects by investigating the corresponding infinitesimally homogeneous analogues. For each class of locally homogeneous objects one chooses a system of data $(M, Q, \rho, f : Q \to L(M), B_0, \sigma)$ of the type considered in the previous section, and considers the corresponding infinitesimally homogeneity condition, which a priori is weaker than local homogeneity. In the presence of an infinitesimally homogeneous object our general Theorem 2.13 will yield an adapted connection $\mathcal{B}$, which can be used to prove that any infinitesimally homogeneous object is locally homogeneous, and is even globally homogeneous if certain topological and completeness conditions are satisfied.

3.1. LH Riemannian manifolds, Ambrose-Singer Theorem. In this section we explain briefly how the Ambrose-Singer theorem (Theorem 1.2) can be obtained using our Theorem 2.13. As explained above we need first to chose an appropriate system of data $(M, Q, \rho, f : Q \to L(M), B_0, \sigma)$.

Let $Q := O(M)$ be the orthonormal frame bundle of $(M, g)$ and $C_0$ be the Levi-Civita connection on it. Let $f : Q \to L(M)$ be the inclusion bundle map and $\sigma := R^g$ be the Riemann curvature tensor of $g$, regarded as a section of the vector bundle $E := (A_1)^{\otimes 4} \cong Q \times_\rho (\mathbb{R}^{n^4})^{\otimes 4}$. It is easy to see that Singer’s infinitesimally homogeneous condition for $g$ is equivalent to our infinitesimally homogeneous condition for the section $\mathcal{R}^g \in \Gamma(E)$ with respect to $C_0$ (see Definition 1.6).

Suppose that the Riemannian curvature tensor $R^g$ is infinitesimally homogeneous with respect to $C_0$. By Corollary 2.14 there exists a connection $C \in \mathcal{A}(O(M))$ such that:

$$\nabla^C R^g = 0, \quad (\nabla^C \otimes \nabla^C)(C - C_0) = 0.$$  

Let $\nabla$ be the metric connection on $M$ induced by $C$, then $\nabla R^g = 0$, $\nabla(C - C_0) = 0$, and Corollary 2.14 below shows that $\nabla$ is indeed an Ambrose-Singer connection.

**Proposition 3.1.** Let $\nabla_0$ (resp. $\nabla$) be the linear connections on smooth manifold $M$ associated with $C_0 \in \mathcal{A}(L(M))$ (resp. $C \in \mathcal{A}(L(M))$). Let

$$S := \nabla - \nabla_0 = C - C_0 \in A^1(\text{End}(T_M)) = A^1(ad(L(M))).$$

Then the following conditions are equivalent:

1. $\nabla R^\nabla = \nabla T^\nabla = \nabla S = 0$,
2. $\nabla R^\nabla = \nabla T^\nabla = \nabla S = 0$.

**Proof.** The difference $T^\nabla - T^\nabla_0$ is the image of $C - C_0$ under the bundle morphism $A^1(\text{End}(T_M)) \to L^2_{\text{alt}}(T_M, T_M)$ given by $S \mapsto T_S$, where

$$T_S(X, Y) := S(X)(Y) - S(Y)(X).$$

This morphism is induced by an $\text{GL}(n)$-equivariant isomorphism

$$\mathbb{R}^{n^4} \otimes gl(n) \to L^2_{\text{alt}}(\mathbb{R}^{n^4}, \mathbb{R}^{n^4}),$$

so it is parallel with respect to any linear connection on $M$. Therefore $\nabla S = 0$ implies $\nabla T_S = 0$, so, under the assumption $\nabla S = 0$, the conditions $\nabla T^\nabla = 0$, $\nabla T^\nabla_0 = 0$ are equivalent.

Since $\nabla_0 = \nabla - S$, we have $R^\nabla_0 = R^\nabla - d^\nabla S + \frac{1}{2}[S \wedge S]$. A direct computation shows that the assumption $\nabla S = 0$ implies

1. $\nabla[S \wedge S] = 0$,
2. $(d^\nabla S)(X, Y) = S(T^\nabla(X, Y))$ for any vector fields $X, Y \in \mathfrak{X}(M)$. 

Therefore, under the assumptions $\nabla S = 0$, $\nabla T^\nabla = 0$, the conditions $\nabla R^\nabla = 0$ and $\nabla R^\nabla = 0$ are equivalent.

Suppose now that $C_0$ is the Levi-Civita connection of a Riemannian manifold $(M, g)$, and let $C \in \mathcal{A}(\mathcal{O}(M))$. The morphism $\Lambda^1 \rightarrow \mathcal{L}_{2n} \rightarrow \mathcal{L}_{2n}(M, TM)$ given by $S \mapsto T_S$ is a bundle isomorphism, which is $O(n)$-equivariant, hence parallel with respect to any metric connection on $M$. Therefore, in this case the conditions $\nabla T^\nabla = 0$ and $\nabla (C - C_0) = 0$ are equivalent, so Proposition 3.1 gives

**Corollary 3.2.** Let $C_0 \in \mathcal{A}(\mathcal{O}(M))$ be the Levi-Civita connection of $(M, g)$, and $R^g$ be the Riemann curvature tensor. Let $\nabla$ be a linear metric connection on $M$ associated with $C \in \mathcal{A}(\mathcal{O}(M))$. The following conditions are equivalent.

1. $\nabla R^g = \nabla (C - C_0) = 0$,
2. $\nabla T^\nabla = \nabla T^\nabla = 0$.

Note that the existence of an Ambrose-Singer connection has a fundamental consequence, namely Singer’s theorem which states that any infinitesimally homogeneous (in particular any locally homogeneous) complete and simply connected Riemannian manifold is globally homogeneous [Si]. This result holds in the differentiable framework (does not need the real analyticity). More precisely one can prove the next theorem by applying the [KN1, Corollary 7.9] to an Ambrose-Singer connection.

**Theorem 3.3.** A Riemannian manifold $(M, g)$ is infinitesimally homogeneous if and only if it is locally homogeneous. Any connected, simply connected, complete infinitesimally homogeneous Riemannian manifold is homogeneous.

The Ambrose-Singer theorem has been extended to the general framework of locally homogeneous pseudo-Riemannian manifolds. Since the structure group of the frame bundle is not necessary compact, one needs an additional reductivity condition. These generalizations can also be obtained using our general theorem (Theorem 2.3), see [GO, Lm, CL] for details.

### 3.2. Kirichenko’s theorem for LH systems $(g, P_1, \ldots, P_k)$

A system $(g, P_1, \ldots, P_k)$ consisting of a Riemannian metric $g$ and tensor fields $P_1, \ldots, P_k$ on $M$ is called locally homogeneous if for any two points $x_1, x_2 \in M$ there is an isometry $\varphi: U_1 \rightarrow U_2$ between open neighborhoods $U_i \ni x_i$ such that $\varphi(x_1) = x_2$ and $\varphi_\ast (P_j)_{v_1} = P_j_{v_2}$.

A locally homogeneous system $(g, P_1, \ldots, P_k)$ defines a pseudogroup of local isometries of $(M, g)$ which acts transitively on $M$. So, in the terminology used by Kirichenko [Kl], one obtains a “geometric structure” on $M$ which is associated with the metric $g$ and the system $(P_1, \ldots, P_k)$.

Each tensor field $P_j$ is a section in the vector bundle $T^j \otimes (\Lambda^1 \otimes s_j)$. The following result due to Kirichenko [Kl], extends the Ambrose-Singer theorem to manifolds with a geometric structure defined by a locally homogeneous system $(g, P_1, \ldots, P_k)$.

**Theorem 3.5.** A system $(g, P_1, \ldots, P_k)$ consisting of a Riemannian metric $g$ and tensor fields $P_1, \ldots, P_k$ on $M$ is locally homogeneous if and only if there exists a metric connection $\nabla$ such that

$$\nabla R^\nabla = \nabla T^\nabla = \nabla P_1 = \cdots = \nabla P_k = 0.$$  

(17)

If $M$ is simply connected, then any locally homogeneous system $(g, P_1, \ldots, P_k)$ with $(M, g)$ complete is globally homogeneous in the following sense: the exists a transitive group of isometries of $(M, g)$ leaving the tensor fields $P_j$ invariant.
In the presence of locally homogeneous system \((g, P_1, \ldots, P_k)\), a metric connection satisfying (17) is called an Ambrose-Singer-Kiričenko connection for the system \((P_1, \ldots, P_k)\) (or for the geometric structure defined by this system) [14].

Let \(Q := O(M)\) be the orthonormal frame bundle of \((M, g)\), \(C_0\) be its Levi-Civita connection, and \(f : Q \to L(M)\) be the obvious inclusion bundle map. The system \((R^g, P_1, \cdots, P_k)\) defines a section \(\sigma\) of the vector bundle

\[
E := (A_{M}^{1})^{\otimes 4} \otimes T_{s1}^{*}(M) \otimes T_{s2}^{*}(M) \otimes \cdots \otimes T_{sk}^{*}(M).
\]

Since the system \((g, P_1, \ldots, P_k)\) is locally homogeneous, the section \(\sigma \in \Gamma(E)\) is infinitesimally homogeneous with respect to \(C_0\) (see Definition [10]). Using Corollary 2.14 there exists a connection \(C \in A(O(M))\) such that:

\[
\nabla^C \sigma = 0, \quad (\nabla^C \otimes \nabla^C)(C - C_0) = 0.
\]

Let \(\nabla\) be the metric connection on \(M\) induced by \(C\). Then, one obtains

\[
\nabla R^g = 0, \quad \nabla(C - C_0) = 0, \quad \nabla P_1 = \nabla P_2 = \cdots = \nabla P_k = 0.
\]

By Corollary 3.2 we see that \(\nabla\) is indeed an Ambrose-Singer-Kiričenko connection. The converse implication is proved by applying the [KN1] Corollary 7.5 (and its proof method) to a connection satisfying (17). More precisely, for top points \(x_1, x_2 \in M\) we choose a smooth path \(\gamma : [0, 1] \to M\) with \(\gamma(0) = x_1, \gamma(1) = x_2\). The parallel transport with respect to \(\nabla\) defines an isometric isomorphism \(F : T_{x_1}M \to T_{x_2}M\) mapping \(R_{x_1}^{\nabla}, T_{x_1}^{\nabla}, P_{x_1}\) to \(R_{x_2}^{\nabla}, T_{x_2}^{\nabla}, P_{x_2}\), respectively. As in [KN1] Corollary 7.5 \(F\) gives a local \(\nabla\)-affine isomorphism \(\varphi\), which will be isometric and leave the tensor fields \(P_j\) invariant.

In the complete, simply connected case the \(\nabla\)-affine isomorphisms \(\varphi\) obtained in this way extend to global isometries of \((M, g)\).

Note that a generalization of Theorem 3.5 to pseudo-Riemannian manifolds can be found in [GO, La].

Example 3.1. The case of locally homogeneous almost Hermitian manifolds is considered in [Sk4]. An almost Hermitian manifold \((M, g, J)\) is locally homogeneous if, for every two points \(x_1, x_2 \in M\) there is a Hermitian isometry \(\varphi : U_1 \to U_2\) between open neighborhoods \(U_1 \ni x_1, U_2 \ni x_2\) sending \(x_1\) to \(x_2\). By a Hermitian isometry we mean a diffeomorphism which is compatible with both the almost complex structure and the Hermitian metric.

A characterization theorem similar to Theorem 1.2 was proved by Sekigawa [Sk4] for almost Hermitian manifolds. The following local version can be obtained using Kiričenko’s theorem by choosing \(k = 1\) and \(P_1 = J\).

**Theorem 3.6.** [Sk4, CN Theorem 1] An almost Hermitian manifold \((M, g, J)\) is locally homogeneous if and only if there exists a metric connection \(\nabla\) such that

\[
\nabla R^\nabla = \nabla T^\nabla = \nabla J = 0.
\]

If \(M\) is simply connected, then any complete locally homogeneous almost Hermitian structure on \(M\) is (globally) homogeneous.

3.3. Opozda’s theorem on locally homogeneous \(G\)-structures. In [O2] [O3] affine connections and their affine transformations has been investigated by Opozda and in [O1] the notion of infinitesimal homogeneity is extended to arbitrary connections on \(G\)-structures. In this section we show that Opozda’s theorem can be proved using Theorem 2.13. Our Lemma 3.12 will allow us to show that the analyticity condition required in Opozda’s statement is not necessary.
Let $M$ be an $n$-dimensional manifold, and $G$ be a Lie subgroup of the linear group $GL(n, \mathbb{R})$. We recall that a $G$-structure on $M$, is a sub-bundle $P \subset L(M)$ where $P$ is a principal $G$-bundle, and the inclusion map is $G$-equivariant.

**Definition 3.7.** Let $P \subset L(M)$ be a $G$-structure on $M$. A connection $C_0$ on $P$ is called locally homogeneous, if for any two points $x_1, x_2 \in M$ there exists a diffeomorphism $x_1 \in U_1 \rightarrow U_2 \ni x_2$ between open neighborhoods of $x_i$, such that $df(P_{x_1}) \subset P_{x_2}$, and $f$ is a $\nabla_0$-affine isomorphism, where $\nabla_0$ is the linear connection associated with $C_0$.

Opozda also introduces a natural infinitesimal homogeneity condition for connections on the principal bundle of a $G$-structure (see [O1, Definitions 1.3, 1.5]). Using our formalism (see Definition 1.6), this condition is equivalent to

**Definition 3.8.** Let $P \subset L(M)$ be a $G$-structure on $M$. A connection $C_0$ on $P$ is called infinitesimally homogeneous, if the pair $(T\nabla_0, R\nabla_0)$, regarded as a section in the associated vector bundle

$$E := P \times_G (L^2_{alt}(\mathbb{R}^n, \mathbb{R}^n) \oplus L^2_{alt}(\mathbb{R}^n, \text{End}(\mathbb{R}^n))),$$

is infinitesimally homogeneous with respect to $C_0$.

Suppose that $C_0$ is infinitesimally homogeneous, i.e. the section $\sigma_0 = (T\nabla_0, R\nabla_0)$ is infinitesimally homogeneous with respect to $C_0$. By Theorem 2.11 we obtain a connection $C \in \mathcal{A}(P)$ such that for $0 \leq k \leq k_{\sigma_0} + 1$:

$$((\nabla^C_0)^{\otimes k} \otimes \nabla^C_0)(\sigma^{(k)}_{C_0}) = 0 \, , \, (\nabla^C_0 \otimes \nabla^C_0)(C - C_0) = 0. \quad (18)$$

Let $\nabla$ denote the linear connection on $M$ induced by $C$. Taking $k = 0$, one obtains

$$\nabla^T\nabla_0 = 0 \, , \, \nabla^R\nabla_0 = 0 \, , \, \nabla(C - C_0) = 0.$$

Proposition 3.11 with $S := \nabla - \nabla_0$ gives

$$\nabla^S = \nabla^T = \nabla^R = \nabla S = 0.$$

Lemma 3.12 proved below shows that the atlas consisting of $\nabla$-normal coordinate systems is analytic, and with respect to the corresponding analytic structure, the connection $\nabla_0 = \nabla - S$ is analytic. The equation (18) implies that for $0 \leq k \leq k_{\sigma_0} + 1$:

$$\nabla((\nabla^0)^{(k)}_0 \nabla_0) = 0 \, , \, \nabla((\nabla^0)^{(k)}_0 R\nabla_0) = 0.$$

Using [O1] Lemma 2.1 the above equations are verified for all $k \geq 0$.

For two points $x_1, x_2 \in M$ we choose a smooth path $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = x_1$, $\gamma(1) = x_2$. The parallel transport with respect to $\nabla$ defines a linear isomorphism $F : T_{x_1}M \rightarrow T_{x_2}M$ mapping $(\nabla^0_0)^{(k)}_0 \nabla_0)_{x_1}$, $(\nabla^0_0 R\nabla_0)_{x_1}$ to $(\nabla^0_0)^{(k)}_0 \nabla_0)_{x_2}$, $(\nabla^0_0 R\nabla_0)_{x_2}$ respectively. Using [KN1] Theorem 7.2 $F$ defines a local $\nabla_0$-affine isomorphism $\varphi$ sending $x_1$ to $x_2$.

Therefore, we obtain Opozda’s theorem ([O1] Theorem 2.2 (i)):

**Theorem 3.9.** Let $P \subset L(M)$ be a $G$-structure on a manifold $M$ and $C_0$ be an infinitesimally P-homogeneous connection on $P$. Fix $q_0 \in P$, and suppose that the pair $(G, H_{q_0}^G)$ is reductive. Then $\nabla_0$ is locally homogeneous.

**Lemma 3.10.** Let $M$ be an analytic n-manifold, $G$ be a Lie group, $p : P \rightarrow M$ be an analytic principal $G$-bundle on $M$, and $A \in \mathcal{A}(P)$ be an analytic connection on $P$. Let $\rho : G \rightarrow GL(F)$ be a representation of $G$ on an r-dimensional vector space $F$, $E := P \times_{\rho} F$ be the associated vector bundle, and $\nabla^0_E$ be the linear connection on $E$ associated with $A$. Then

1. Let $\sigma \in \mathcal{A}^0(M, E)$ such that $\nabla^0_E \sigma$ is an analytic $E$-valued 1-form. Then $\sigma$ is analytic.
(2) Let $\nabla$ be an analytic connection on $M$, and $\alpha \in A^1(M, \text{ad}(P))$ such that the derivative $(\nabla \otimes \nabla^A_{ad})(\alpha)$ vanishes in $A^0(\Lambda^1_M \otimes \Lambda^1_M \otimes \text{ad}(P))$. Then $\alpha$ is analytic, so $B := A + \alpha$ is an analytic connection on $P$.

**Proof.** (1) The problem is local, so it suffices to prove that $\alpha$ is analytic around any point $x \in M$. With respect to

- an analytic chart $b : U \rightarrow V \subset \mathbb{R}^n$,
- an analytic trivialization $\tau : P_U \rightarrow U \times G$ of $P$,
- a basis $e$ of $F$,

the restriction of the operator $\nabla^a$ to a sufficiently small open neighborhood $U$ of $x$ can be identified with a differential operator of the form

$$d + a_h,\tau,b : A^0(V, \mathbb{R}^r) \rightarrow A^1(V, \mathbb{R}^r),$$

where $a_h,\tau,b \in A^1(V, \mathfrak{g}(r))$ is analytic. Let $\sigma_{h,\tau,b} \in A^0(V, \mathbb{R}^r)$ be the $\mathbb{R}^r$-valued map associated with $\sigma$. The hypothesis implies that $(d + a_h,\tau,b)\sigma_{h,\tau,B}$ is analytic, hence

$$d^*(d + a_h,\tau,b)\sigma_{h,\tau,b}$$

is also analytic. But $d^*(d + a_h,\tau,b)$ is an elliptic operator with analytic coefficients, so, by analytic elliptic regularity [Be, Theorem 40, p. 467], it follows that $\sigma_{h,\tau,b}$ is analytic. Therefore $\sigma_{h,\tau,b}$ is analytic.

(2) Let $C \in \mathcal{A}(L(M))$ be the connection on the frame bundle $L(M)$ which correspond to $\nabla$. The statement follows by applying (1) to the connection defined by the pair $(C, A)$ on the product bundle $L(M) \times_M P$ and the representation $\text{GL}(\mathbb{R}) \times G \rightarrow \text{GL}(\mathbb{R}^n \otimes \mathfrak{g})$ given by $(u, l) \mapsto (u)^{-1} \otimes \text{ad}$.

**Lemma 3.11.** Let $M$ be an analytic manifold, and $\nabla$ be an analytic connection on $M$. Let $S \in A^1(\text{End}(T_M))$ be such that $\nabla S$ is analytic. Then $S$ is analytic, so the connection $\nabla + S$ is also analytic.

**Lemma 3.12.** Let $M$ be a differentiable manifold, let $\nabla$ be a linear connection on $M$ such that $\nabla T^V = 0$, $\nabla R^V = 0$, and let $S \in A^1(\text{End}(T_M))$ be such that $\nabla S = 0$. The atlas consisting of $\nabla$-normal coordinate systems is analytic, and with respect to the corresponding analytic structure, the connections $\nabla$ and $\nabla + S$ are analytic.

**Proof.** By [KN1, Theorem 7.7] the atlas consisting of $\nabla$-normal coordinate systems is analytic and with respect to the corresponding analytic structure the connection $\nabla$ is analytic. By Lemma 3.11, the connection $\nabla + S$ is analytic.

### 3.4 Locally homogeneous spinors.

We have seen that the case of LH almost Hermitian structures can be obtained as a special case of Kirichenko’s formalism dedicated to LH tensors. We give now an example which cannot be obtained as a special case of this formalism. In this example we have to consider a section in a vector bundle which is not associated with the frame bundle of the base manifold.

Let $(M, g)$ be an oriented Riemannian $n$-manifold, and $\Lambda : Q \rightarrow \text{SO}(M)$ be Spin structure on $M$, where $Q$ is a $\text{Spin}(n)$-bundle. Let $r : \text{Spin}(n) \rightarrow \text{SO}(n)$ be the canonical epimorphism, and $\kappa : \text{Spin}(n) \rightarrow \text{GL}(\Delta_n)$ be the spin representation and $S := P \times_\kappa \Delta_n$ be the associated spinor bundle, where $\Delta_n$ is the vector space of Dirac spinors defined by

$$\Delta_n = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \quad \text{for} \quad n = 2k, 2k + 1.$$  

Recall that, if $n$ is even, then $S$ comes with a Levi-Civita parallel orthogonal decomposition $S = S^+ \oplus S^-$ (see [ES]). Since the Lie group morphism $\text{Spin}(n) \rightarrow \text{SO}(n)$
is a local isomorphism it follows that $\Lambda$ induces a bijection $\mathcal{A}(Q) \to \mathcal{A}(\text{SO}(M))$ between the spaces of connections on the two bundles.

**Definition 3.13.** Let $(M,g)$ be a LH Riemannian manifold. A spinor $s \in \Gamma(M,S)$ will be called LH if for any pair $(x_1,x_2) \in M \times M$ there exists an isometry $U_1 \xrightarrow{\phi} U_2$ between open neighborhoods $U_1 \ni x_i$, and a $\varphi$-covering bundle isomorphism $P_{U_1} \xrightarrow{\Phi} P_{U_2}$ such that $\Phi_*(s|_{U_1}) = s|_{U_2}$.

We also have a natural infinitesimal homogeneity condition: this is just the infinitesimal homogeneity condition $s$ regards as a section of $S$ with respect to the lift $\tilde{C}_0$ of the Levi-Civita connection $C_0$ to $P$. This condition is apparently weaker than the LH condition.

**Remark 3.14.** Let $(M,g)$ be a spin LH Riemannian manifold, and $s$ a LH spinor, then the section $(R^g,s)$ is infinitesimally homogeneous with respect to $\tilde{C}_0 \in \mathcal{A}(Q)$.

**Theorem 3.15.** Let $s \in \Gamma(M,S)$ be a locally homogeneous spinor. There exists an Ambrose-Singer connection $\nabla$ on $(M,g)$ which leaves $s$ invariant.

**Proof.** Put $G := \text{Spin}(n)$. Consider the associated vector bundle $E := (\Lambda_M^1)^{\otimes 4} \oplus S$, associated with representation $\rho : G \to \text{GL}((\mathbb{R}^{n*})^{\otimes 4} \oplus \Delta_n)$ induced by $(r,\kappa)$.

Let $R^g$ be the Riemann curvature tensor and $s \in \Gamma(M,S)$ be a spinor. The pair $(R^g,s)$ defines a section $\sigma$ of vector bundle $E$, which is obviously infinitesimally homogeneous with respect to $\tilde{C}_0$. Corollary 2.14 gives a connection $\tilde{C} \in \mathcal{A}(Q)$ such that

$$\nabla^C \sigma = 0, \quad (\nabla^C_M \otimes \nabla^C_{\text{ad}})(\tilde{C} - \tilde{C}_0) = 0.$$ Recalling that we have an obvious identification $\mathcal{A}(\text{SO}(M)) = \mathcal{A}(Q)$, we can denote by the same symbol $\nabla$ the connections on the vector bundles $T_M, S$ induced by $\tilde{C}$. The above equations are equivalent to

$$\nabla R^g = 0, \quad \nabla(\tilde{C} - \tilde{C}_0) = 0, \quad \nabla s = 0.$$ By Corollary 3.12 we see that $\nabla$ is an Ambrose-Singer connection which leaves the spinor $s$ invariant.

**Corollary 3.16.** Suppose that $(M,g)$ is simply connected and complete, and $s \in \Gamma(M,S)$ is a locally homogeneous spinor. There exists a Lie group $G$ and an action $\beta : G \times Q \to Q$ with the following properties:

1. $\beta$ lifts a transitive action by isometries $\alpha : G \times M \to M$.
2. $\beta$ leaves $s$ invariant.

### 4. Locally homogeneous triples

#### 4.1. Infinitesimally homogeneous triples. Let $(M,g)$ be a connected, compact, locally homogeneous Riemannian manifold, $K$ be a connected, compact Lie group, and $p : P \to M$ be a principal $K$-bundle on $M$. Let $A$ be a connection on $P$ and fix an ad-invariant inner product on the Lie algebra $\mathfrak{k}$ of $K$; these data define a $K$-invariant Riemannian metric $g_A$ on $P$ which makes $p$ a Riemannian submersion. Such metrics on principal bundles are called connection metrics and have been studied in the literature [FZ, CR, WZ].

The connection metric $g_A$ associated with any LH triple $(g, P \xrightarrow{p} M, A)$ is locally homogeneous, hence it defines a geometric structure in the sense of Thurston [Th] on the total space $P$. Therefore the classification of LH triples on a given connected, compact LH Riemannian manifold $(M,g)$ is related to the classification of compact geometric manifolds which are principal bundles over a geometric base. We refer to
for LH triples on Riemann surfaces. In [BT] we proved explicit classifications theorems for LH triples on Riemann surfaces.

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \(K\) be a compact Lie group, \(P\) be a principal \(K\)-bundle on \(M\), and \(A_0\) be a connection on \(P\). Put

\[
G := O(n) \times K, \quad Q := O(M) \times_M P, \quad V := (\mathbb{R}^{n*})^{\otimes 4} \oplus ((\mathbb{R}^{n*})^{\otimes 2} \otimes \mathfrak{t}).
\]

Note that \(G\) comes with obvious representations \(r : G \to GL(n)\), \(\rho : G \to GL(V)\), and \(Q\) comes with an obvious bundle morphism \(f : Q \to L(M)\) of type \(r\) given by the projection \(O(M) \times_M P \to O(M) \subset L(M)\).

The Riemann curvature tensor \(R^g\) of \(g\) will be regarded as a section of \((\Lambda^1_M)^{\otimes 4}\), and the curvature \(F^{A_0}\) of connection \(A_0\) will be regarded as a section of the vector bundle \(\Lambda^2_M \otimes \text{ad}(P) \subset (\Lambda^1_M)^{\otimes 2} \otimes \text{ad}(P)\). The pair \(\sigma := (R^g, F^{A_0})\) is a section of the associated vector bundle \(E := Q \times_G V\). Let \(C_0\) denote the Levi-Civita connection on \(O(M)\) and let \(B_0\) be the connection on \(Q\) defined by the pair of connections

\[
B_0 := (C_0, A_0) \in \mathcal{A}(O(M)) \times \mathcal{A}(P) = \mathcal{A}(Q).
\]

**Definition 4.1.** The triple \((g, P, \mathcal{A}_0, M, A_0)\) will be called infinitesimally homogeneous if the section \(\sigma := (R^g, F^{A_0}) \in \Gamma(E)\) is infinitesimally homogeneous with respect to \(B_0\) in the sense of Definition [7,0].

This condition can be reformulated explicitly as follows:

Let \(\nabla^g\) denote the Levi-Civita connection of \(g\) on \(M\) and, let \(\nabla^{A_0}\) denote the associated connection on adjoint bundle \(\text{ad}(P)\). We denote by \(\nabla^g \otimes \nabla^{A_0}\) the tensor product connection on \(\Lambda^2_M \otimes \text{ad}(P)\). More generally, we obtain a tensor product connection \((\nabla^g \otimes \nabla^{A_0})^i = (\nabla^g)^{\otimes (i-1)} \otimes (\nabla^g \otimes \nabla^{A_0})\) on the vector bundle \((\Lambda^1_M)^{\otimes (i-1)} \otimes \Lambda^2_M \otimes \text{ad}(P)\).

For \(x \in M\), the Lie algebra of skew-symmetric endomorphism of the Euclidean space \(T_xM, g_x\) will be denoted by \(\text{so}(T_xM)\). For any \(k \in \mathbb{N}\) and \(x \in M\) we define \(h^g_{\eta^{A_0}}(k)\) to be the set of all pairs \((u, v) \in \text{so}(T_xM) \oplus \text{ad}(P_x)\) such that

\[
u \cdot ((\nabla^g)^{\otimes k})^{R^g}_{x} = 0, \quad (u, v) \cdot ((\nabla^g \otimes \nabla^{A_0})^i F^{A_0})_{x} = 0 \quad \text{for} \quad 0 \leq i \leq k.
\]

Note that \(h^g_{\eta^{A_0}}(k)\) is a Lie subalgebra of \(\text{so}(T_xM) \oplus \text{ad}(P_x)\), and that for any \(k \in \mathbb{N}\)

\[
h^g_{\eta^{A_0}}(k+1) \subset h^g_{\eta^{A_0}}(k).
\]

Put

\[
k^g_{\eta^{A_0}} := \min\{k \in \mathbb{N} \mid h^g_{\eta^{A_0}}(k+1) = h^g_{\eta^{A_0}}(k)\}.
\]

Using these definitions and notations we see that

**Remark 4.2.** The triple \((g, P, \mathcal{A}_0, M, A_0)\) is infinitesimally homogeneous if and only if for any \((x_1, x_2) \in M \times M\), there exists a pair \((f, \phi)\) where \(f : T_{x_1}M \to T_{x_2}M\) is a linear isometry, and \(\phi : P_{x_1} \to P_{x_2}\) is \(K\)-equivariant isomorphism, such that for any \(0 \leq k \leq k^g_{\eta^{A_0}} + 1\) one has:

\begin{enumerate}
  \item \(f((\nabla^g)^{\otimes k})^{R^g}_{x_1} = ((\nabla^g)^{\otimes k})^{R^g}_{x_2}\).
  \item \((f, \phi)((\nabla^g \otimes \nabla^{A_0})^{k} F^{A_0})_{x_1} = ((\nabla^g \otimes \nabla^{A_0})^{k} F^{A_0})_{x_2}\).
\end{enumerate}

The second condition can be formulated explicitly as follows: \((\nabla^g \otimes \nabla^{A_0})^{k} F^{A_0}\) is a section of the vector bundle \((\Lambda^1_M)^{\otimes k} \otimes \Lambda^2_M \otimes \text{ad}(P)\). The fiber \(\text{ad}(P)_x\) at \(x \in M\) for the adjoint bundle \(\text{ad}(P)\) is given by \(\text{ad}(P)_x = (P_x \times \mathfrak{t})/K\), so a \(K\)-equivariant isomorphism \(\phi : P_{x_1} \to P_{x_2}\) induces a linear isomorphism

\[
\text{ad}(\phi) : \text{ad}(P)_{x_1} \to \text{ad}(P)_{x_2}, \quad [y, \alpha] \mapsto [\phi(y), \alpha].
\]
The second condition in Remark 4.2 can be written in \( \text{ad}(P)_{x_2} \) as
\[
\{(\nabla^{\sigma} \otimes \nabla^{A_0})^k F^{A_0}\}_{x_2} (f(v_1), \ldots, f(v_k), f(w_1), f(w_2)) = \text{ad}(\phi) \left( \{(\nabla^{\sigma} \otimes \nabla^{A_0})^k F^{A_0}\}_{x_1} (v_1, \ldots, v_k, w_1, w_2) \right)
\]
for any tangent vectors \( v_i \in T_{x_2} M, w_1, w_2 \in T_{x_2} M \).

The isomorphism pair \((f, \phi)\) defines a Lie algebra isomorphism
\[
\text{so}(T_{x_1} M) \oplus \text{ad}(P_{x_1}) \rightarrow \text{so}(T_{x_2} M) \oplus \text{ad}(P_{x_2}),
\]
which isomorphically maps \( h_{x_1}^{g,A_0}(k) \) onto \( h_{x_2}^{g,A_0}(k) \) for \( 0 \leq k \leq k_{x_1}^{g,A_0} + 1 \). This implies

\textbf{Remark 4.3.} Let \((g, P \xrightarrow{p} M, A_0)\) be an infinitesimally homogeneous triples. Then \( k_{x}^{g,A_0} \) is independent of \( x \). We will denote by \( k_{x}^{g,A_0} \) the obtained constant.

Applying Theorem 2.13 to our situation we obtain:

\textbf{Theorem 4.4.} Suppose \((g, P \xrightarrow{p} M, A_0)\) is an infinitesimally homogeneous triple on \((M, g)\). Then, there exists a pair \((\nabla, A)\) consisting of a metric connection on \(M\) and \(A \in \mathcal{A}(P)\) with the following properties:
\[
\nabla R^g = 0, \quad \nabla T^g = 0, \quad (\nabla \otimes \nabla A) F^A = 0, \quad (\nabla \otimes \nabla A)(A - A_0) = 0.
\]

\textbf{Proof.} Let \( Q := O(M) \times_M P \) and \( E := Q \times_G V \) be as above. Let \( \sigma \in \Gamma(E) \) be the section defined by the pair \((R^g, F^{A_0})\). By Corollary 2.14 there exists a connection \( B = (C, A) \in \mathcal{A}(O(M)) \times \mathcal{A}(P) = \mathcal{A}(Q) \) such that:
\[
\nabla^B \sigma = 0, \quad (\nabla^B \otimes \nabla^B)(B - B_0) = 0.
\]

Let \( \nabla \) denote the induced metric connection on \( M \) by \( C \) and \( \nabla A \) denote the induced connection on the adjoint bundle \( \text{ad}(P) \) by \( A \). From \( \nabla^B \sigma = 0 \) we obtain
\[
\nabla R^g = 0, \quad (\nabla \otimes \nabla A) F^A_0 = 0.
\]

The difference \( B - B_0 \) can be identified with the pair \((C - C_0, A - A_0)\). Hence, equation \((\nabla^B \otimes \nabla^B)(B - B_0) = 0\) is equivalent to
\[
\nabla(C - C_0) = 0, \quad (\nabla \otimes \nabla A)(A - A_0) = 0.
\]

The conditions \( \nabla R^g = 0, \quad \nabla(C - C_0) = 0 \) and Corollary 3.2 imply \( \nabla T^g = 0 \).

The conditions \( \nabla T^g = 0, \quad (\nabla \otimes \nabla A)(A - A_0) = 0 \) and Lemma 4.4 below give \((\nabla \otimes \nabla A) F^A = 0\). \( \Box \)

\textbf{Lemma 4.5.} Retain the notation used above. If \( \nabla T^g = 0, \quad (\nabla \otimes \nabla A)(A - A_0) = 0 \), then \((\nabla \otimes \nabla A) F^A = 0 \) if and only if \((\nabla \otimes \nabla A) F^A = 0 \).

\textbf{Proof.} Put \( \alpha := A - A_0 \in A^1(M, \text{ad}(P)) \). One has the following variation formula for curvature 2-form
\[
F^A_0 = F^{A - \alpha} = F^A - d^A \alpha + \frac{1}{2}[\alpha \wedge \alpha].
\]

Let \( X, Y, Z \in \mathfrak{X}(M) \) be arbitrary vector fields on \( M \). In one hand,
\[
(d^A \alpha)(X,Y) = \nabla^A_X (\alpha(Y)) - \nabla^A_Y (\alpha(X)) - \alpha([X,Y]).
\]

On the other hand, since \( \alpha \in A^1(M, \text{ad}(P)) \) is \((\nabla \otimes \nabla A)\)-parallel, one has
\[
0 = ((\nabla \otimes \nabla A) \alpha)(X,Y) = \nabla^A_X (\alpha(Y)) - \alpha(\nabla_X Y).
\]

Using (22) and (24), one obtains
\[
(d^A \alpha)(X,Y) = \alpha(\nabla_X Y) - \alpha(\nabla_Y X) - \alpha([X,Y]) = \alpha(T^g(X,Y)).
\]
and from (24) one gets $\nabla^A_X(\alpha(T^\nabla(Y, Z))) = \alpha(\nabla_X T^\nabla(Y, Z))$. Using (24) and (25) one obtains

\begin{align*}
((\nabla \otimes \nabla)d^A\alpha)(X, Y, Z) &= \nabla^A_X((d^A\alpha)(Y, Z)) - (d^A\alpha)(\nabla_X Y, Z) - (d^A\alpha)(Y, \nabla_X Z) \\
&= \nabla^A_X(\alpha(T^\nabla(Y, Z))) - \alpha(T^\nabla(\nabla_X Y, Z)) - \alpha(T^\nabla(Y, \nabla_X Z)) \\
&= \alpha(\nabla_X T^\nabla(Y, Z)) - \alpha(T^\nabla(\nabla_X Y, Z)) - \alpha(T^\nabla(Y, \nabla_X Z)) \\
&= \alpha((\nabla_X T^\nabla)(Y, Z)) = 0.
\end{align*}

Put $\eta = \frac{1}{2}[\alpha \wedge \alpha]$. For arbitrary vector fields $Y, Z \in \mathfrak{X}(M)$

\[2\eta(Y, Z) = [\alpha \wedge \alpha]|(Y, Z) = [\alpha(Y), \alpha(Z)] - [\alpha(Z), \alpha(Y)] = 2[\alpha(Y), \alpha(Z)].\]  

(26)

Therefore,

\begin{align*}
\nabla^A_X(\eta(Y, Z)) &= \nabla^A_X([\alpha(Y), \alpha(Z)]) = \nabla^A_X(\alpha(Y)), \alpha(Z)) + [\alpha(Y), \nabla^A_X(\alpha(Z))]. \tag{27}
\end{align*}

Using (24) and (27) we obtain

\begin{align*}
((\nabla \otimes \nabla)\eta)(Y, Z) &= \nabla^A_X(\eta(Y, Z)) - \eta(\nabla_X Y, Z) - \eta(Y, \nabla_X Z) \\
&= \nabla^A_X(\alpha(Y)), \alpha(Z)) + [\alpha(Y), \nabla_X(\alpha(Z))] - [\alpha(Z), \alpha(Y)], \alpha(\nabla_X Z) \\
&= \nabla^A_X(\alpha(Y)), \alpha(Z)) + [\alpha(Y), (\nabla \otimes \nabla)_X(\alpha(Z))] = 0.
\end{align*}

So, we proved that the ad-valued 2-forms $d^A\alpha$ and $[\alpha \wedge \alpha]$ are $(\nabla \otimes \nabla)$-parallel. Using (22), we conclude: $(\nabla \otimes \nabla)^F = 0$ if and only if $(\nabla \otimes \nabla)^F = 0$. \[\square\]

Using Ambrose-Singer’s theorem (see Theorem 1.2) we obtain:

**Remark 4.6.** Let $(g, P \xrightarrow{\varphi} M, A_0)$ be an infinitesimally homogeneous triple over $(M, g)$. Then, the Riemannian metric $g$ is locally homogeneous.

A pair $(\nabla, A)$ of a metric connection on $M$ and $A \in A(P)$ defines in a canonical way a linear connection $\tilde{\nabla}$ on the tangent bundle $T_P$. This construction will play a major role in what follows, so we explain this construction in detail:

Recall that, in general, a connection $A$ on the principal $K$-bundle $p : P \rightarrow M$ can be identified with the corresponding horizontal distribution. Using the bundle isomorphism $J : p^*T_M \rightarrow A$ (defined by the inverse of $p_1A$), we obtain a linear connection $\nabla^{h,A} := J(p^*\nabla)$ on the horizontal subbundle $A \subset T_P$.

On the other hand we can define a linear connection $\nabla^{v,A}$ on the vertical bundle $V_P$ via the canonical bundle isomorphism $V_P \simeq P \times \mathfrak{t}$. More precisely, if $a^\#$, $b^\#$ denote the fundamental fields corresponding to $a, b \in \mathfrak{t}$ and if $\tilde{X}$ denotes the $A$-horizontal lift of $X \in \mathfrak{X}(M)$, then we define

\[\nabla^{v,A}_a b^\# = [a, b]^\#, \quad \nabla^{v,A}_X = 0.\]

Using the direct sum decomposition $T_P = A \oplus V_P$, the linear connection $\nabla^{h,A}$ on the horizontal subbundle $A$, and the linear connection $\nabla^{v,A}$ on vertical subbundle $V_P$, we obtain a linear connection on $P$ defined by

\[\tilde{\nabla} := \nabla^{h,A} \oplus \nabla^{v,A}.\]

The straightforward calculations show that [Ba2, Proposition 2.6], the connection $\tilde{\nabla}$ has the following properties:

**Proposition 4.7.** Let $\tilde{\nabla}$ be the linear connection on the tangent bundle $T_P$ defined as above. Let $\tilde{X}, \tilde{Y}$ be the $A$-horizontal lift of vector fields $X, Y \in \mathfrak{X}(M)$ and $Z$ be a vertical vector field on $P$. Then

\begin{enumerate}
\item \(\tilde{\nabla}_X \tilde{Y} = (\nabla_X Y)^\#, \quad \tilde{\nabla}_X Z = [\tilde{X}, Z].\)
\item \(\nabla_a b^\# = [a, b]^\#, \quad \nabla_a Z = [a^\#, Z].\)
\item \(\nabla_{\tilde{X}} a^\# = \nabla_a \tilde{X} = [a^\#, \tilde{X}] = 0.\)
\end{enumerate}
The canonical bundle isomorphism $\xi : P \times \mathfrak{t} \to V_P$ can be used to identify the space of smooth maps $C^\infty(P, \mathfrak{t})$ with the Lie algebra of vertical vector fields on $P$. This identification is given by $f \mapsto \xi(f)$, where, for $f \in C^\infty(P, \mathfrak{t})$, $\xi(f) : P \to V_P$ is the vertical vector field given by

$$\xi(f)_y := f(y)\bigg|_0 = \left. \frac{d}{dt} \right|_{t=0} (y \exp tf(y)).$$

The following propositions can be proved easily [Ba, Lemma 2.7, Prop. 2.8.]

**Proposition 4.8.** Let $\nu : P \to M$ be a principal $K$-bundle and $A \in \mathcal{A}(P)$. Let $\nu \in \mathcal{A}^0(\text{ad}(P))$ and $a^\#$ denote the fundamental vector field corresponding to $a \in \mathfrak{t}$. Let $\tilde{X}$ be the $A$-horizontal lift of vector field $X \in \mathfrak{X}(M)$. Then, one has

$$\tilde{\nabla}_a \xi(\nu) = [a^\#, \xi(\nu)] = 0, \quad \tilde{\nabla}_X \xi(\nu) = [X, \xi(\nu)] = \xi(\nabla_X^A \nu).$$

**Theorem 4.9.** Let $(\nabla, A)$ be a pair consisting of a metric connection on $M$ and a connection $A \in \mathcal{A}(P)$. If $\nabla R^\nabla = 0$, $\nabla T^\nabla = 0$, $(\nabla \otimes \nabla^A) F^A = 0$. Then, the associated connection $\tilde{\nabla} := \nabla^{b,A} \oplus \nabla^A$ satisfies the following conditions:

$$\nabla R^\nabla = 0, \quad \nabla T^\nabla = 0.$$

**Proof.** Denote the torsion tensor and the curvature tensor of connection $\nabla$ by $\tilde{T}$, $\tilde{R}$ to save on notation. Let $a^\#, b^\#$ be the fundamental vector fields corresponding to $a,b \in \mathfrak{t}$, and let $X, Y$ denote the $A$-horizontal lifts of vector fields $X, Y$ on $M$. The properties of $\nabla$ in Proposition 4.7 imply

$$\tilde{T}(\tilde{X}, a^\#) = 0, \quad \tilde{T}(a^\#, b^\#) = [a^\#, b^\#]. \quad (28)$$

Using the equation $\xi(F^A(X, Y)) = [X, Y] - [X, Y]_v$ for the vertical component of the Lie bracket of two horizontal lifts [GH, P. 257], one obtains

$$\tilde{T}(\tilde{X}, \tilde{Y}) = T^\nabla(X, Y)_v + \xi(F^A(X, Y)). \quad (29)$$

To prove $\nabla \tilde{T} = 0$, it suffices to show that $(\nabla \tilde{T})(U, V, W) = 0$ for vector fields $U, V, W \in \mathfrak{X}(P)$ in the special cases when each one of these three vector fields is either a horizontal lift, or a fundamental vector field. One has

$$(\nabla \tilde{T})(V, W) = \nabla_U \tilde{T}(V, W) - \tilde{T}(\nabla_U V, W) - \tilde{T}(V, \nabla_U W). \quad (30)$$

If $U = \tilde{X}, V = \tilde{Y}, W = \tilde{Z}$ then

$$\nabla_U \tilde{T}(V, W) = \nabla_X ((T^\nabla(Y, Z)) - \xi(F^A(Y, Z))) = \nabla_X T^\nabla(Y, Z) - \xi(\nabla_X^A F^A(Y, Z)). \quad (31)$$

In the same way one obtains

$$\tilde{T}(\nabla_U V, W) = (T^\nabla(\nabla_X Y, Z)) - \xi(F^A(\nabla_X Y, Z)), \quad (32)$$

$$\tilde{T}(V, \nabla_U W) = (T^\nabla(Y, \nabla_X Z)) - \xi(F^A(Y, \nabla_X Z)). \quad (33)$$

Using the equations (31), (32), (33) and Proposition 4.8:

$$(\nabla_U \tilde{T})(V, W) = \nabla_U \tilde{T}(V, W) - \tilde{T}(\nabla_U V, W) - \tilde{T}(V, \nabla_U W) = \nabla X T^\nabla(Y, Z) - (T^\nabla(\nabla_X Y, Z)) - \xi(\nabla_X^A F^A(Y, Z)) = \xi(\nabla(\nabla^A F^A)(X, Y, Z)) = 0.$$

As $\nabla T^\nabla = 0$ and $(\nabla \otimes \nabla^A) F^A = 0$, the right hand side of above equation vanishes. If $U = a^2, V = b^2, W = c^2$, where $a,b,c \in \mathfrak{t}$ then by the Jacobi identity

$$(\nabla_{ab} \tilde{T})(b^2, c^2) = \left( [a, [b, c]] + [b, [c, a]] + [c, [a, b]] \right)^\# = 0.$$
If $U = a$ and $V = \tilde{Y}$, $W = \tilde{Z}$ are the $A$-horizontal lifts of vector fields $Y, Z \in \mathfrak{X}(M)$ then using Propositions 4.8, 4.7 we get

$$\nabla_{a^\sharp} ((T^a_{\#}(Y, Z))_{\#}) + (T^a_{\#}(Y, Z))_{\#}) = \tilde{T}(\nabla_{a^\sharp} \tilde{Y}, \tilde{Z}) = T(\tilde{Y}, \nabla_{a^\sharp} \tilde{Z}) = 0.$$

in any other possible cases the claim follows directly from the definition of $\nabla$. For the curvature tensor the only non vanishing cases are:

$$R(\tilde{X}, \tilde{Y})\tilde{Z} = (R^\#_{\#}(X, Y)Z)_{\#}, \quad \tilde{R}(\tilde{X}, \tilde{Y})c_{\#}^\# = \nabla_{\xi(F^A(X,Y))}c_{\#}^\#.$$

(34)

and with the same kind of arguments as above, one obtains $\tilde{\nabla}\tilde{R} = 0$. \hfill \blacksquare

Recall that the space of connections $\mathcal{A}(P)$ for a principal $K$-bundle $P$ over $M$ is an affine space with model space $A^1(\text{ad}(P))$ [DK]. For a connection $A_0 \in \mathcal{A}(P)$, and a $1$-form $\alpha \in A^1(\text{ad}(P))$ the connection form $\omega_A$ of $A := A_0 + \alpha$ is given by $\omega_A = \omega_{A_0} + \alpha$, where the second term on the right has been identified with the associated torsorial $1$-form of type $\text{ad}$ on $P$ (see [KN1, Example 5.2 p.76]). In other words, for a tangent vector $v \in T_y P$ the element $\alpha_y(v) \in \mathfrak{k}$ is defined by the equality $\alpha(p_*v) = [y, \alpha_y(v)] \in \text{ad}(P_y)$. So, regarding $\alpha(p_*v)$ as a $K$-equivariant map $P_y \to \mathfrak{k}$, one has $\alpha_y(v) = \alpha(p_*v)(y)$.

**Lemma 4.10.** Let $A = A_0 + \alpha$, where $A, A_0 \in \mathcal{A}(P)$ and $\alpha \in A^1(\text{ad}(P))$. Let $Z$ be a vector field on $M$, $\tilde{Z}^{A_0}$ be its $A_0$-horizontal lift, and $\tilde{Z}^A$ be its $A$-horizontal lift (which coincides with the $A$-horizontal projection of $\tilde{Z}^{A_0}$). Then

$$\tilde{Z}^A = \tilde{Z}^{A_0} - \xi(\alpha(Z)),$$

**Proof.** For any vector field $Y \in \mathfrak{X}(P)$ the $A$-horizontal projection of $Y$ is given by $Y^A = Y - \xi(\omega_A(Y))$. If now $Y = \tilde{Z}^{A_0}$, we have $\omega_{A_0}(Y) = 0$, so for any $y \in P$ we have $\omega_{A_0}(Y) = \omega_y(Y) = \alpha(y(y))$. \hfill \blacksquare

**Lemma 4.11.** Let $\nabla$ be a linear connection on $M$ and $A \in \mathcal{A}(P)$ such that

$$\nabla R^\# = 0, \quad \nabla T^\# = 0, \quad (\nabla \otimes \nabla^A)F^A = 0, (\nabla \otimes \nabla^A)(A - A_0) = 0.$$

Then the distributions $A$, $A_0$ are $\nabla$-parallel, where $\nabla = \nabla^h.A \oplus \nabla^v.A$.

**Proof.** To prove that $A$ is $\nabla$-parallel, let $\tilde{X}$ be the $A$-horizontal lift of vector field $X$ on $M$. We have to show that for any vector field $Y$ on $P$ the vector field $\nabla_Y \tilde{X}$ is $A$-horizontal. If $Y := \tilde{Z}$ is $A$-horizontal lift of a vector fields $Z$ on $M$ then $\nabla_Y X$ is $A$-horizontal lift of $\nabla_Z X$. If $Y \in \mathfrak{X}(P)$ is a vertical vector field, then $\nabla_Y \tilde{X} = 0$ and therefore $\nabla_Y \tilde{X} \in \Gamma(A)$. For the second property, we will denote by $\tilde{Z}^{A_0}$ the $A_0$-horizontal lift of a vector field $Z \in \mathfrak{X}(M)$. Since the vector fields of the form $\tilde{Z}^{A_0}$ generate $\Gamma(A_0)$ as a $C^\infty(P, \mathbb{R})$-module, it suffices to prove that, for arbitrary vector fields $Y \in \mathfrak{X}(P)$ and $Z \in \mathfrak{X}(M)$ one has $\nabla_Y (\tilde{Z}^{A_0}) \in \Gamma(A_0)$. Putting $A = A - A_0$ and using Lemma 4.10 we obtain

$$\tilde{Z}^A = \tilde{Z}^{A_0} - \xi(\alpha(Z)).$$

If $Y = \tilde{U}$ is the $A$-horizontal lift of a vector field $U$ on $M$, then, using the parallelism assumption $(\nabla \otimes \nabla^A)\alpha = 0$, the definition of $\nabla$ and Proposition 4.8 we obtain

$$\nabla_Y \tilde{Z}^{A_0} = \nabla_Y (\tilde{Z}^A + \xi(\alpha(Z))) = (\nabla_U Z)^{-A} + \nabla_U \xi(\alpha(Z)) = (\nabla_U Z)^{-A} + \xi(\alpha(Z)).$$

But, by Lemma 4.10 the right hand side of the above equation is the $A_0$-horizontal lift of $\nabla_U Z$, which proves the claim in this case. If now $Y = a^\#$ is a fundamental field, then

$$\nabla_Y \tilde{Z}^{A_0} = \nabla_{a^\#} (\tilde{Z}^A + \xi(\alpha(Z))) = \nabla_{a^\#} \tilde{Z}^A + \nabla_{a^\#} \xi(\alpha(Z)).$$
Conversely, [KN1, Theorem 7.4] can be reformulated as follows:

\[ \bar{\nabla} \circ \xi(A) = 0. \]

Therefore, \( \bar{\nabla} \circ \xi(A) \) is \( A_0 \)-horizontal.

In conclusion, using Theorems 4.14, Theorem 4.9, Lemma 4.11 and Proposition 4.14, we obtain

**Theorem 4.12.** Let \( (g, P \overset{p}{\to} M, A_0) \) be infinitesimally homogeneous. There exists a pair \((\bar{\nabla}, A)\) consisting of a metric connection on \((M, g)\) and \(A \in A(P)\) such that

\[ \nabla R^\nabla = 0, \quad \nabla T^\nabla = 0, \quad (\nabla \otimes \nabla^A)F^A = 0, \quad (\nabla \otimes \nabla^A)(A - A_0) = 0. \]

The linear connection \( \bar{\nabla} = \nabla^h \oplus \nabla^v \) on the tangent bundle \( T_P \) associated with this pair has the following properties:

1. \( \nabla R^\nabla = \nabla T^\nabla = 0 \)
2. The vector fields \( a^\# \) are \( \bar{\nabla} \)-parallel along the \( A \)-horizontal curves,
3. The distributions \( A, A_0 \) are \( \nabla \)-parallel.

### 4.2. Structure and classification theorems

We begin by recalling the results proved in [KN1] Ch.VI Section 7 on the existence and extension properties of local affine isomorphisms with respect to a linear connection satisfying the conditions \( \nabla R^\nabla = \nabla T^\nabla = 0 \). Compared to [KN1], our presentation uses a new formalism: the space of germs of \( \nabla \)-affine isomorphisms.

Let \( M \) be a differentiable \( n \)-manifold, and let \( \nabla \) be a linear connection on \( M \) satisfying the conditions \( \nabla R^\nabla = \nabla T^\nabla = 0 \). Let \( \mathcal{S}^\nabla \) be the space of germs of \( \nabla \)-affine isomorphisms defined between open sets of \( M \), and let \( s : \mathcal{S}^\nabla \to M, t : \mathcal{S}^\nabla \to M \) be the source, respectively the target map on this space. \( \mathcal{S}^\nabla \) has a canonical structure of a differentiable manifold and, with respect to this structure, \( s \) and \( t \) are local diffeomorphisms. A germ \( \varphi \in \mathcal{S}^\nabla \) defines an isomorphism \( \varphi_* : T_{s(\varphi)}M \to T_{t(\varphi)}M \) with the property

\[ \varphi_*(R^\nabla_s(\varphi)) = R^\nabla_t(\varphi), \quad \varphi_*(T^\nabla_s(\varphi)) = T^\nabla_{t(\varphi)}. \]

Conversely, [KN1] Theorem 7.4] can be reformulated as follows

**Remark 4.13.** Let \((u, v) \in M \times M\). For any linear isomorphism \( f : T_uM \to T_vM \) satisfying

\[ f(R^\nabla_u) = R^\nabla_v, \quad f(T^\nabla_u) = T^\nabla_v \]

there exists a unique germ \( \varphi_\infty \in (s, t)^{-1}(u, v) \) (of a \( \nabla \)-affine isomorphism) such that \( (\varphi_\infty)_* = f \).

Let now \( \sigma : M \times M \to M, \tau : M \times M \to M \) be the projections on the two factors, and let \( \text{Iso}(\sigma^*(T_M), \tau^*(T_M)) \subset \text{Hom}(\sigma^*(T_M), \tau^*(T_M)) \) be the (locally trivial) fibre bundle of isomorphisms between the two pull-backs of \( T_M \). Conditions [43] define a closed, locally trivial subbundle \( S^\nabla \subset \text{Iso}(\sigma^*(T_M), \tau^*(T_M)) \). Remark 4.14 shows that

**Remark 4.14.** The natural map \( \delta : \mathcal{S}^\nabla \to S^\nabla \) given by \( \varphi \mapsto \varphi_* \) is bijective.

This map is an injective immersion, and it is bijective, but it is not a diffeomorphism. \( \mathcal{S}^\nabla \) can be identified with the union of leaves of a foliation of \( S^\nabla \) with \( n \)-dimensional leaves. The topology of \( \mathcal{S}^\nabla \) is finer than the topology of \( S^\nabla \). The leaves of this foliation are the integrable submanifolds of the involutive distribution \( \mathcal{D}^\nabla \subset T_{S^\nabla} \) defined in the following way: Let \( f \in S^\nabla \). Put \( u := \sigma(f), v := \tau(f), \varphi := \delta^{-1}(f) \). For a tangent vector \( \xi \in T_uM \), let \( \gamma : (-\varepsilon, \varepsilon) \to M \) be a smooth curve such that \( \gamma(0) = u, \gamma(0) = \xi \). Using parallel transport with respect to \( \nabla \) along
the curves $\gamma$, $\varphi \circ \gamma$ we obtain, for any sufficiently small $t \in (-\varepsilon, \varepsilon)$, isomorphisms $a_t : T_u M \to T_{\gamma(t)} M$, $b_t : T_v M \to T_{\varphi(\gamma(t))} M$. Define

$$\lambda_f(\xi) := \frac{d}{dt}_{t=0} (b_t \circ f \circ a_t^{-1}) \in T_{(u,v)}(S^\nabla).$$

The distribution $D^\nabla$ is defined by

$$D^\nabla_f := \{ \lambda_f(\xi) \mid \xi \in T_{\sigma(f)} M \}.$$

The curve $t \mapsto b_t \circ f \circ a_t^{-1}$ will be an integral curve of this distribution. Note that we may take $\gamma$ to be the $\nabla$-geodesic with initial condition $(u, \xi)$, and then $\varphi \circ \gamma$ will be the $\nabla$-geodesic with initial condition $(v, f(\xi))$. Using this remark, we obtain

**Remark 4.15.** Let $\varphi \in \mathcal{S}^\nabla$, and $\xi \in T_u(\varphi) M$. Suppose that $\nabla$-geodesics $\gamma$, $\eta$ with initial conditions $(s(\varphi), \xi)$, $(t(\varphi), \varphi_*(\xi))$ respectively can be both extended on the interval $(\alpha, \beta) \ni 0$. Then $\gamma$ has a smooth lift in $\mathcal{S}^\nabla$ with initial condition $\varphi$ via the source map $s : \mathcal{S}^\nabla \to M$.

Using this remark one can prove:

**Proposition 4.16.** Let $\nabla$ be a connection on a connected manifold $M$ such that $\nabla R^\nabla = \nabla T^\nabla = 0$. Suppose that $\nabla$ is complete. Then the source map $s : \mathcal{S}^\nabla \to M$ is a covering map. In particular, when $\nabla$ is complete and $M$ is simply connected, any element $\varphi \in \mathcal{S}^\nabla$ extends to a unique global $\nabla$-affine isomorphism $M \to M$. In particular, for any pair $(x, x') \in M \times M$ there exists a unique global $\nabla$-affine isomorphism mapping $x$ to $x'$.

**Proof.** It suffices to note, that for a $\nabla$-convex open set $U \subset M$ the following holds: any germ $\varphi \in \mathcal{S}^\nabla$ with $s(\varphi) \in U$ has an extension on $U$. The point is that, since $\nabla$ is complete, for any geodesic $\gamma : (\alpha, \beta) \to U$ passing through $s(\varphi)$, the composition $\varphi \circ \gamma$ can be extended on the whole $(\alpha, \beta)$. Therefore all $\nabla$-geodesics in $U$ passing through $s(\varphi)$ admit lifts with initial condition $\varphi$. Using these lifts it follows that the connected components of $s^{-1}(U)$ are identified with $U$ via $s$. \hfill $\blacksquare$

An important special case (which intervenes in the proof of Singer’s theorem) concerns a linear connection $\nabla$ satisfying the conditions $\nabla R^\nabla = \nabla T^\nabla = 0$ which is a metric connection, i.e. there exists a Riemannian metric $g$ on $M$ such that $\nabla g = 0$. In this case one defines submanifolds

$$\mathcal{S}^\nabla_g := \{ \varphi \in \mathcal{S}^\nabla \mid \varphi_* \text{ is an isometry} \}, \quad S^\nabla_g := \{ f \in S^\nabla \mid f \text{ is an isometry} \}$$

of $\mathcal{S}^\nabla$, $S^\nabla$ respectively, $\mathcal{S}^\nabla_g$ is open in $\mathcal{S}^\nabla$. In other words $S^\nabla_g$ is a union of integral submanifolds (of maximal dimension) of the involutive distribution $D^\nabla$. On the other hand, an important result in Riemannian geometry states [4V, Proposition 1.5]:

**Proposition 4.17.** Let $(M, g)$ be a complete Riemannian manifold. Then any metric connection $\nabla$ on $M$ is complete.

Using these facts Proposition 4.16 gives

**Corollary 4.18.** Let $(M, g)$ be a connected Riemannian manifold endowed with a metric connection $\nabla$ such that $\nabla R^\nabla = \nabla T^\nabla = 0$. Suppose that $(M, g)$ is complete. Then the source map $s : \mathcal{S}^\nabla_g \to M$ is a covering map. In particular, when $(M, g)$ is complete and simply connected, any element $\varphi \in \mathcal{S}^\nabla_g$ extends to a unique global $\nabla$-affine isometry $M \to M$. In particular, for any pair $(x, x') \in M \times M$ there exists a unique global $\nabla$-affine isometry mapping $x$ to $x'$. 

Now we come back to the connection $\nabla = \nabla^{h,A} \oplus \nabla^{v,A}$ on $T_P$ associated with a pair $(\nabla, A)$ of a metric connection on $M$ and $A \in \mathcal{A}(P)$ which satisfy the conditions

$$\nabla R^\nabla = 0, \ \nabla T^\nabla = 0, \ (\nabla \otimes \nabla^A)^F = 0, \ (\nabla \otimes \nabla^A)(A - A_0) = 0.$$ 

We know that $\nabla R^\nabla = \nabla T^\nabla = 0$, so all constructions and results above apply to $\nabla$. Using the additional structure we have on $P$ (the $K$-action, the two connections $A$, $A_0$, and the metric $g$ on $P/K$) we will define an open submanifold $S_{g,K}^\nabla$ of $\mathfrak{S}^\nabla$ consisting of germs of affine transformations which are compatible with this structure:

Since $\nabla$ is $K$-invariant, the manifolds $\mathfrak{S}^\nabla$, $S^\nabla$ come with natural right $K$-actions given by $(\varphi, k) \mapsto R_k \circ \varphi \circ R_k^{-1}$, $(f, k) \mapsto (R_k)_* \circ f \circ (R_k)_*$, and the maps $s$, $t : \mathfrak{S}^\nabla \to P$, $\sigma, \tau : S^\nabla \to P$ are $K$-equivariant.

We define a submanifold $S_{g,K}^\nabla \subset S^\nabla$ by

$$S_{g,K}^\nabla := \left\{ f \in S^\nabla \mid f(A_{\sigma(f)}) = A_{\tau(f)} \circ f(A_{0,\sigma(f)}) = A_{0,\tau(f)} \right\},$$ 

$$f(a^\#_{\sigma(f)}) = a^\#_{\tau(f)} \forall a \in \mathfrak{t}, f \text{ induces an isometry } T_{p(\sigma(f))}M \to T_{p(\tau(f))}M. \right\} \ .$$

**Lemma 4.19.** $S_{g,K}^\nabla$ is $K$-invariant, and is a union of integral submanifolds of the distribution $\mathcal{D}^\nabla$. In particular $\mathfrak{S}_{g,K}^\nabla := \delta^{-1}(S_{g,K}^\nabla)$ is a $K$-invariant open submanifold of $\mathfrak{S}^\nabla$.

**Proof.** We have to prove that for any $f \in S_{g,K}^\nabla$ one has $\mathcal{D}_f \subset T_f S_{g,K}^\nabla$, i.e. that for any $\xi \in T_{\sigma(f)}P$ we have $\lambda_f(\xi) \in T_f S_{g,K}^\nabla$. It suffices to prove that for any $a \in \mathfrak{t}$ and any $\zeta \in T_{p(\sigma(f))}M$ one has (denoting by $\tilde{\zeta}_{\sigma(f)}$ the $A$-horizontal lift of $\zeta$ at $\sigma(f)$):

$$\lambda_f(a^\#_{\sigma(f)}) \in T_f S_{g,K}^\nabla, \ \lambda_f(\tilde{\zeta}_{\sigma(f)}) \in T_f S_{g,K}^\nabla .$$

The first formula is obtained using the curve $t \mapsto \sigma(f) \exp(ta)$, and the second is obtained using the curve $t \mapsto \tilde{\eta}_{\sigma(f)}(t)$, where $\eta : (-\varepsilon, \varepsilon) \to M$ is a $\nabla$-geodesic such that $\eta(0) = p(\sigma(f))$, $\tilde{\eta}(0) = \zeta$. One uses the fact that the vector fields $a^\#$ are $\nabla$-parallel along $A$-horizontal curves, and that the distributions $A$, $A_0$ are $\nabla$-parallel.

**Lemma 4.20.** The restrictions,

$$\left\{ (p \circ \sigma, p \circ \tau) \middle| S_{g,K}^\nabla : S_{g,K}^\nabla \to M \times M, \ (p \circ a, p \circ t) \middle| S_{g,K}^\nabla : S_{g,K}^\nabla \to M \times M \right\}$$

are surjective.

**Proof.** Let $x_0, x_1 \in M$, and let $\eta : [0, 1] \to M$ be a smooth path in $M$ such that $\eta(0) = x_0$, $\eta(1) = x_1$. Choose a point $y_0 \in P_{x_0}$, let $\tilde{\eta}$ be the $A$-horizontal lift of $\eta$ with the initial condition $\tilde{\eta}(0) = y_0$, and let $y_1 := \tilde{\eta}(1)$. Using $\nabla$-parallel transport along $\tilde{\eta}$, we obtain an element $f \in S^\nabla$ with $\sigma(f) = y_0$, $\tau(f) = y_1$. Using Theorem 4.12, we see that $f \in S_{g,K}^\nabla$.

**Theorem 4.21.** Let $(M, g)$ be a connected Riemannian manifold, $(g, P \xrightarrow{\rho} M, A_0)$ be a triple consisting of a principal $K$-bundle $P$ on $M$, and a connection $A_0$ on $P$. The following conditions are equivalent:

1. $(g, P \xrightarrow{\rho} M, A_0)$ is locally homogeneous.
2. $(g, P \xrightarrow{\rho} M, A_0)$ is infinitesimally homogeneous.
3. There exists a pair $(\nabla, A)$ consisting of a metric connection on $(M, g)$ and a connection $A \in \mathcal{A}(P)$ such that

$$\nabla R^\nabla = 0, \ \nabla T^\nabla = 0, \ (\nabla \otimes \nabla^A)^F = 0, \ (\nabla \otimes \nabla^A)(A - A_0) = 0. \ \ (37)$$
Proof. The implication (1)⇒(2) is obvious, and the implication (2)⇒(3) stated by Theorem 4.13. For the implication (3)⇒(1), let \( \nabla = \nabla^{h,A} \oplus \nabla^{v,A} \) be the connection on \( T_xM \) with a pair \( (\nabla, A) \). Let \( (x_0, x_1) \in M \times M \). By Lemma 4.20 there exists \( \varphi \in \mathfrak{g}_xK \) such that \( y_0 := \varphi(x_0) \in P_{x_0} \) and \( y_1 := \varphi(x_1) \). The orbit \( \varphi K \) is a submanifold of \( \mathfrak{g}_xK \), which is mapped diffeomorphically onto \( y_0K \) via \( \varphi \), and onto \( y_1K \) via \( \varphi \). Regarding the maps \( s, t : \mathfrak{g}_xK \to P \) as sheaves of sets over \( P \), and using [God] Théorème 3.3.1, p. 150 we obtain an open neighbourhood \( U \subset \mathfrak{g}_xK \) of \( \varphi K \) in \( \mathfrak{g}_xK \), which is mapped injectively onto an open neighborhood \( U \) of \( y_0K \) via \( s \), and is mapped injectively onto an open neighborhood \( V \) of \( y_1K \) via \( t \). Since \( K \) is compact, we may suppose that \( U \) is \( K \)-invariant. Therefore \( U \) and \( V \) will be also \( K \)-invariant, so \( U = p^{-1}(U), V = p^{-1}(V) \) for open neighborhoods \( U, V \) of \( x_0, x_1 \) respectively. \( U \) defines a \( K \)-equivariant, \( \nabla \)-affine isomorphism \( U \to V \) which maps \( A_0U \) onto \( A_0V \) and induces an isometry \( U \to V \). This shows that \( (g, P \overset{\nabla}{\to} M, A_0) \) is locally homogeneous.

For the symmetric case we have:

**Theorem 4.22.** Let \( (M, g) \) be a connected Riemannian manifold, \( (g, P \overset{\nabla}{\to} M, A_0) \) be a triple consisting of a principal \( K \)-bundle \( P \) on \( M \), and a connection \( A_0 \) on \( P \). The following conditions are equivalent:

1. \( (g, P \overset{\nabla}{\to} M, A_0) \) is locally symmetric.
2. One has
   \[
   \nabla^gR^g = 0, (\nabla^g \otimes \nabla^{A_0})F^{A_0} = 0.
   \]

**Proof.** Suppose that (1) holds, and, for a point \( x \in M \), let \( U := U_x, s := s_x, \Phi := \Phi_x \) be as in Definition 4.13. Taking into account that \( \nabla^g \) is \( s \)-invariant, and \( A_0 \) is \( \Phi \)-invariant, it follows that \( \nabla^gR^g, (\nabla^g \otimes \nabla^{A_0})F^{A_0} \) are invariant sections of \( (\Lambda^2_U)^{\otimes g}, (\Lambda^2_U)^{\otimes g} \oplus \text{ad}(P_L) \) which are invariant with respect to the involution defined by \( s \), respectively by the pair \( (s, \Phi) \). Since the degree of these sections with respect to \( M \) are odd, the argument of [KN2] Section XI.1, Theorem 1.1] applies.

Conversely, suppose (2) holds. Note that the pair \( (\nabla^g, A_0) \) satisfies the conditions in Lemma 4.11. Using the same construction as in the proof of Theorem 4.21, we obtain a linear connection \( \nabla \) on \( P \) such that \( \nabla R^\nabla = \nabla T^\nabla = 0 \) and leaves the horizontal distribution of \( A_0 \) invariant (see Theorem 4.22).

Let \( x \in M \). For any point \( y \in P_x \) put
\[
f_y := \begin{pmatrix}
-\text{id}(A_0)_y & 0 \\
0 & \text{id}_{T_y(P)}
\end{pmatrix} \in \text{GL}(T_yP),
\]
where \( (A_0)_y \subset T_yP \) stands for the \( A_0 \)-horizontal space at \( y \). Using formulae (28), (29), (31) it is easy to check that for any \( y \in P_x \) one has \( f_y \in \mathfrak{g}_xK \). Moreover, it is easy to see that \( F_x := \{ f_y \mid y \in P_x \} \) is a \( K \)-orbit with respect to the \( K \)-action on \( \mathfrak{g}_xK \). Using the bijection \( \mathfrak{g}_xK \to S^\mathfrak{g}_xK \) we obtain a \( K \)-orbit \( \{ \varphi y \mid y \in P_x \} \) for the \( K \)-action on \( \mathfrak{g}_xK \), which is mapped diffeomorphically onto \( P_x \) via the maps \( s \) and \( t \). The same arguments as in the proof of Theorem 4.21 apply, and give a bundle isomorphism \( \Phi : p^{-1}(U) \to p^{-1}(V) \), where \( U, V \) are open neighborhoods of \( x \), and the induced map \( s : U \to V \) is an isometry. Moreover, one has \( \Phi \circ s = f_y \) for any \( y \in P_x \), in particular \( s_{xx} = -\text{id}_{T_xM} \). Therefore \( s \) induces an isometric involution on sufficiently small normal neighborhood \( W \subset M \) of \( x \). It suffices to note that the restriction of \( \Phi \) to \( p^{-1}(W) \) is an involution. This is obvious taking into account the next remark which gives an explicit geometric construction of \( \Phi \):
Remark 4.23. Let \( \gamma : [0,1] \to W \) be a smooth path with \( \gamma(0) = x, \gamma' := s \circ \gamma, \) let \( y \in P, \) and let \( \tilde{\gamma}, \tilde{\gamma}' \) be the \( A_0 \)-horizontal lifts with initial conditions \( \tilde{\gamma}(0) = \gamma(0) = y. \) Then \( \Phi(\gamma(1)) = \tilde{\gamma}'(1), \) and \( \Phi(\tilde{\gamma}'(1)) = \tilde{\gamma}(1). \)

The remark is proved using that \( \Phi \) lifts \( s \) and leaves the connection \( A_0 \) invariant.

For the case when \( (M, g) \) is complete, we have

**Theorem 4.24.** Let \( (g, P \xrightarrow{\bar{P}} M, A_0) \) be a locally homogeneous triple with \( M \) connected. If \( (M, g) \) is complete, then the map \( \mathcal{S}^s_{g,K}/K \to M \) induced by \( s \) is a covering map. If \( (M, g) \) is complete and \( M \) is simply connected, then any germ \( \varphi \in \mathcal{S}^s_{g,K} \) can be extended to a unique bundle isomorphism \( \Phi : P \to P \) which covers an isometry \( M \to M, \) and has the property \( \Phi(A_0) = A_0. \) In particular, for any \( (x, x') \in M \times M \) there exists such a bundle isomorphism with \( \Phi(P_x) = P_{x'} \).

**Proof.** We use the same method as in the proof of Proposition 4.16 Corollary 4.18. The fact that \( \mathcal{S}^s_{g,K}/K \to M \) is a covering map is obtained using parallel transport with respect to \( \nabla \) along \( A \)-horizontal lifts of \( \nabla \)-geodesics in \( M. \)

**Corollary 4.25.** Let \( (g, P \xrightarrow{\bar{P}} M, A_0) \) be a locally symmetric triple with \( M \) simply connected, and \( (M, g) \) complete. Then \( (g, P \xrightarrow{\bar{P}} M, A_0) \) is a symmetric triple.

**Proof.** By Theorem 4.24 any locally symmetric triple is locally homogeneous, so Theorem 4.24 applies, so the locally defined involutive bundle isomorphisms \( \Phi_x \) given by the locally symmetric condition (see Definition 1.3) extend to the whole total space \( P. \)

Using theorem 4.24 we can also prove the following classification theorem for LH triples:

**Theorem 4.26.** Let \( M \) be a compact manifold, and \( K \) be a compact Lie group. Let \( \pi : \tilde{M} \to M \) be the universal cover of \( M, \) \( \Gamma \) be the corresponding covering transformation group. Then, for any locally homogeneous triple \( (g, P \xrightarrow{\bar{P}} M, A) \) with structure group \( K \) on \( M \) there exists

1. A connection \( B \) on the pull-back bundle \( \bar{Q} := \pi^*(P). \)
2. A closed subgroup \( G \subseteq \text{Iso}(\tilde{M}, \pi^*(g)) \) acting transitively on \( \tilde{M} \) which contains \( \Gamma \) and leaves invariant the gauge class \( [B] \in B(Q). \)
3. A lift \( \iota : \Gamma \to G \bar{Q}(Q) \) of the inclusion monomorphism \( \iota : \Gamma \to G, \) where \( G \bar{Q}(Q) \) stands for the group of automorphisms of \( (Q, B) \) which lift transformations in \( G. \)
4. An isomorphism between the \( \Gamma \)-quotient of \( (\pi^*g, Q \to \tilde{M}, B) \) and the initial triple \( (g, P \xrightarrow{\bar{P}} M, A). \)

**Proof.** Let \( G \subseteq \text{Iso}(\tilde{M}, \bar{g}) \) be the subgroup defined by

\[ G := \{ \psi \in \text{Iso}(\tilde{M}, \bar{g}) \mid \exists \Psi : Q \to Q \text{ \( \psi \)-covering bundle isom., } \Psi^*(B) = B \}. \]

Using the fact that \( K \) is compact, it follows by Lemma 4.27 below, that \( G \) is a closed subgroup of the Lie group \( \text{Iso}(\tilde{M}, \bar{g}). \) Note that Lemma 4.27 applies because the action of the Lie group \( \text{Iso}(\tilde{M}, \bar{g}) \) on \( \tilde{M} \) is smooth.

Applying Theorem 4.24 to \((\pi^*g, Q, B),\) it follows that \( G \) acts transitively on \( \tilde{M}, \) and leaves invariant the gauge class \([B].\) Moreover, the definition of \( G \) shows that it contains \( \Gamma. \) The lift \( \iota \) is obtained as follows: for \( \varphi \in \Gamma \) we define \( \iota(\varphi) : Q \to Q \) by

\[ \iota(\varphi)(\bar{x}, y) := (\varphi(\bar{x}), y) \forall (\bar{x}, y) \in Q := \tilde{M} \times_{\pi} P. \]

Note that the map \( \Gamma \ni \varphi \mapsto \iota(\varphi) \in G \bar{Q}(Q) \) is group morphism (as required).
Lemma 4.27. [Ba1] Let $M$ be a differentiable manifold, $K$ be a compact Lie group, $p : P \to M$ be principal $K$-bundles over $M$, and $A \in \mathcal{A}(P)$ be a connection on $P$. Let $\alpha : L \times M \to M$ be a a smooth action of a Lie group $L$ on $M$. For $l \in L$ denote by $\varphi_l : M \to M$ the corresponding diffeomorphism. The subspace
\[ L_A := \{ l \in L \mid \exists \Phi \in \text{Hom}(\varphi_l(P), P) \text{ such that } \Phi^*(A) = A \} \]
is a closed Lie subgroup of $L$.

Theorem 4.26 shows that:

Corollary 4.28. Let $M$ be a connected compact manifold, and $K$ be a compact Lie group. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$, $\Gamma$ be the corresponding covering transformation group. Then any locally homogeneous triple $(g,P \xrightarrow{p} M,A)$ with structure group $K$ on $M$ can be identified with a $\Gamma$-quotient of a homogeneous triple $(\pi^*g,Q := \pi^*(P) \to \tilde{M},B)$ on the universal cover $\tilde{M}$.

References

[AS] Ambrose, W., Singer, I.M.: On homogeneous Riemannian manifolds, Duke Math. J., 25, 647-669 (1958).
[Ba1] Bazdar, A.: Locally homogeneous triples: Extension theorems for parallel sections and parallel bundle isomorphisms, Mediterr. J. Math. (2017).
[Ba2] Bazdar, A.: Geometric principal bundles over geometric Riemannian manifolds, Ph.D. thesis, Aix-Marseille University (2017). http://www.theses.fr/2017AIXM0210
[BT] Bazdar, A., Teleman, A.: Locally homogeneous connections on principal bundles over hyperbolic Riemann surfaces, (2018). https://arxiv.org/abs/1811.07995
[Be] Besse, A.: Einstein manifolds, Classics in Mathematics, Springer (1987).
[CL] Calvaruso, G., Lopez, M.C.: Pseudo-Riemannian Homogeneous Structures, Developments in Mathematics, Vol. 56, Springer (2019).
[CN] Console, S., Nicolodi, L.: Infinitesimal characterization of almost Hermitian homogeneous spaces, Commentationes Mathematicae Universitatis Carolinae, Vol. 40, 713-721 (1999).
[DK] Donaldson, S., Kronheimer, P.: The Geometry of Four-Manifolds, Oxford University Press (1990).
[FZ] Florit, L., Ziller, W.: Topological obstructions to fatness revisited, Geometry & Topology 15, 891-925 (2011).
[Fr] Friederich, Th.: Dirac Operators in Riemannian Geometry, Graduate Studies in Mathematics vol. 25, AMS (2000).
[GO] Gadea, P.M., Oubiña, J. A.: Reductive Homogeneous Pseudo-Riemannian manifolds, Monatsh. Math. 124, 17-34 (1997).
[God] Godement, R.: Topologie Algébrique et Théorie des Faisceaux, Hermann, Paris (1964),
[GH] Greuel, W., Halperin, S., Vanstone, R.: Connections, Curvature and Cohomology II, Pure Appl. Math., 47-IL, Academic Press, London, (1973).
[Ito] Itoh, M.: Invariant connections and Yang-Mills solutions. Trans. Amer. Math. Soc. 267(1), 229-236 (1981).
[JG] Jensen, G.: Einstein metrics on principal fiber bundles, J. Diff. Geom. 8, 599-614 (1973).
[KN1] Kobayashi S., Nomizu, K.: Foundations of Differential Geometry I, Interscience Publ., New York, Vol. I (1963).
[KN2] Kobayashi S., Nomizu, K.: Foundations of Differential Geometry I, Interscience Publ., New York, Vol. II (1969).
[Ki] Kiričenko, V. F.: On homogeneous Riemannian spaces with invariant tensor structure, Soviet Math. Dokl., Vol. 21 No. 3, 734-737 (1980).
[Lu] Lujan, I.: Reductive locally homogeneous pseudo-Riemannian manifolds and Ambrose-Singer connections, Differ. Geom. Appl. 41, 65-90 (2015).
[NT] Nicolodi, L., Tricerri, F.: On two theorems of I. M. Singer about homogeneous spaces, Ann. Global Anal. Geom. 8, 193-200 (1990).
[O1] Opozda, B.: On locally homogeneous $G$-structures, Geom. Dedicata, 73, 215-223 (1998).
[O2] Opozda, B.: Curvature homogeneous and locally homogeneous affine connections, Proc. Amer. Math. Soc. 124, 1889-1893 (1996).
[O3] Opozda, B.: Affine versions of Singer’s theorem on curvature homogeneous spaces, Ann. Global Anal. Geom., 187-199 (1997).
[SW] A. Sagle, R. Walde: Introduction to Lie groups and Lie algebras, Academic Press (1973).
[Se] Sekigawa, K.: Notes on homogeneous almost Hermitian manifolds, Hokkaido Math. J. 7, 206-213 (1978).

[Si] Singer, I. M.: Infinitesimally homogeneous spaces, Comm. Pure Appl. Math., 13, 685-697 (1960).

[Th] Thurston, W. P.: Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, AMS, Bulletin. New Series, vol. 6 (3) 357-381 (1982).

[Tr] Tricerri, F.: Locally homogeneous Riemannian manifolds, Rend. Sem. Mat. Univ. Politec. Torino 50:4, 411-426 (1993).

[TV] Tricerri, F., Vanhecke, L.: Homogeneous Structures on Riemannian Manifolds, London Math. Soc. Lect. Notes, vol. 83. Cambridge Univ. Press, Cambridge (1983).

[WZ] Wang, M., Ziller, W.: Einstein metrics on principal torus bundles, J. Diff. Geom. 31, 215-248 (1990).

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