Two-loop Feynman Diagrams in Yang-Mills Theory from Bosonic String Amplitudes

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Abstract

We present intermediate results of an ongoing investigation which attempts a generalization of the well known one-loop Bern Kosower rules of Yang-Mills theory to higher loop orders. We set up a general procedure to extract the field theoretical limit of bosonic open string diagrams, based on the sewing construction of higher loop world sheets. It is tested with one- and two-loop scalar field theory, as well as one-loop and two-loop vacuum Yang-Mills diagrams, reproducing earlier results. It is then applied to two-loop two-point Yang-Mills diagrams in order to extract universal renormalization coefficients that can be compared to field theory. While developing numerous technical tools to compute the relevant contributions, we hit upon important conceptual questions: Do string diagrams reproduce Yang-Mills Feynman diagrams in a certain preferred gauge? Do they employ a certain preferred renormalization scheme? Are four gluon vertices related to three gluon vertices? Unfortunately, our investigations remained inconclusive up to now.

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1 Introduction

Achievements developed by Bern and Kosower [1, 2, 3] allow to deduce the contributions of a large number of one-loop Feynman diagrams in Yang-Mills gauge theory from a single string scattering diagram [4]. This was first noticed by analyzing amplitudes of some heterotic string model, but later on it was realized that only the bosonic degrees of freedom were relevant in the appropriate field theoretical limit. In this sense one can say that the bosonic string amplitude adds up implicitly all particle diagrams of a given loop order. To extract these contributions one has to get rid of the massive modes of the string as well as its tachyonic excitation. If one were able to compute the corresponding zero Regge slope limit of the entire amplitude also in higher loop orders, a tremendous simplification of the computation of loop corrections in gauge theories might follow, which could be of great impact in perturbative techniques. It was further noticed that the string theoretical input to this method can be reduced to the quantum mechanics of particles moving on some world line which is reminiscent of the string world sheet at infinite string tension. Thus the World Line Formalism [5, 6, 7] was established and its relation to string theory explored [8, 9]. During this investigation it was also realized that the tachyonic mode of the string could be employed to calculate scattering amplitudes of scalar field theory despite its unphysical mass. As well a conclusive first quantized treatment of QED could be defined [10, 11] without regarding the string theoretical origin of the formalism any more. On the other hand, the foremost challenge of the entire enterprise still remains unsolved, which is the generalization of the Bern Kosower rules of Yang-Mills theory to higher loop orders. In fact the World Line formalism did allow higher order computations in QED and scalar field theory [7, 12, 13]. It was also possible to extract such information directly from string diagrams [14, 15, 16, 17]. Any attempt to master two-loop Yang-Mills theory starting from string diagrams remained inconclusive up to now. In [18, 19] a worldline approach to two-loop Yang-Mills theory was proposed, whose
connection to string theory has not yet been explored.

In this article which is an extension of relevant parts of [20, 21] we try to define such a generalization in the spirit of the mentioned earlier works. Thus we first review parts of these and deduce a general procedure how to project the string amplitude onto the massless vector boson mode. Our method attempts to unify the formerly employed ones, reproducing all their results and making the two-loop extension straightforward at the first glance. We are able to present an exact computation of all contributions of the two-loop two gluon string amplitude, that are relevant to obtain the two-loop coefficient of the Yang-Mills $\beta$ function. But in the end we shall uncover two severe difficulties which we are unable to deal with, preventing us from identifying two-loop Bern-Kosower rules. One of these is the unanswered question of how the gauge choice of field theory enters into the string amplitude. We demonstrate that a comparison of Feynman diagrams of a particular “topology” to their string counterpart does not allow any of the so-called covariant background gauges. At the one-loop level the diagrams were reproduced in the particular Feynman background gauge and this was therefore supposed to be the preferred gauge in which the string amplitudes naturally appear in the field theory limit. The second open question is, how the string diagrams deal with renormalization. While string theory contains a scale at which the massive string modes cut off the “divergencies” of local field theories, these require renormalization. This, for instance, calls for the presence of counter term insertions contributing at the two-loop level of perturbation theory, a procedure which in general will depend on the renormalization scheme chosen. In which manner these contributions are included in string diagrams is rather obscure. Using background techniques, like we do in this paper, one also has to inspect IR regularization, before one can compare with the conventional $\beta$ function [19].

A further technical obstruction is our present inability to give a consistent treatment of four-gluon vertices. These do not generically occur in string diagrams which are built up by sewing together three-point string vertices. In the limit when the geometry of the string world sheet tends to a diagram involving four-point vertices some of the moduli are frozen and their integrations have to be removed, which leaves one with divergencies that cannot be explained or regularized in an obvious manner. Thus our work unfortunately remains inconclusive in these respects but still allows some insight into the problems that prevent a further progress so far.

The work on these topics is still in progress and we would like to express our appreciation for the discussion on this paper with P. Di Vecchia, A. Lerda, R. Marotta and R. Russo in fall 1998 in Copenhagen.

2 The field theoretical limit of string amplitudes

The observation that field theoretical amplitudes can be recovered from string theory traces back to Scherk [22], who found that expressions obtained from the dual operator formalism coincide precisely with field theoretical result for tree-level and one-loop diagrams of $\Phi^3$ theory in the limit of vanishing Regge slope. One only has to introduce a relation between the string and field theoretical couplings, the whole kinematics of Feynman diagrams follows automatically.

From the string theoretical point of view one expects that in this limit the massless modes of the string particle spectrum form an effective low energy field theory. In the simple bosonic model these particle-like states are the scalar tachyonic excitation and the massless vector boson of the spectrum of the bosonic string. When the string tension goes to infinity, i.e. $\alpha' \to 0$, the former will be identified as a divergent $1/\alpha'$ contribution of the amplitude while the latter is given by its constant term.

We shall identify these divergent, respectively constant contributions by introducing proper time variables, playing the same role as Schwinger proper times (SPT) in field theory. These are
introduced when rewriting the momentum integrals over the internal momenta as
\[ \int \frac{d^dp}{(2\pi)^d} p_\mu p_\nu \ldots = \int_0^\infty dt \int \frac{d^dp}{(2\pi)^d} P_\mu P_\nu \ldots e^{-t(p^2 + m^2)}. \] (1)

One can now perform the Gaussian momentum integrations to obtain a SPT integral. This gives the type of integral which is naturally derived from string diagrams, by the conservation of difficulties it is technically not easier to solve than the Feynman momentum integral itself. The integration region is singled out by reducing the moduli space which has to be integrated over in the amplitude to that small region where the SPT variables stay finite [23]. The SPT variables are defined in string theory by identifying
\[ \delta \tau (L_0 - a) = \delta \tau \alpha' \left( p^2 + m^2 \right) \equiv \delta t \left( p^2 + m^2 \right), \] (2)
or
\[ t = \alpha' \tau + t_0 = \alpha' \ln |z| + t_0. \] (3)

Logarithms of moduli correspond to proper time variables in the field theory. This reveals a truly geometrical interpretation of Feynman diagrams, which in this picture represent particles whose propagation is parametrized by proper time variables. The length of a particular propagator is defined by the fact that \( x^{-L_0} \), \( x \) ranging from 0 to 1, is the operator that propagates the external states in the string diagrams along the boundary of the open world sheet. It leads to
\[ t = -\alpha' \ln(x) \quad \text{and} \quad x = e^{-t/\alpha'}. \] (4)

We thus have found means to extract the tachyonic and vector contribution from the string amplitude in terms of different powers in the moduli of the world sheet. The tachyon part of the amplitude is exponentially divergent in \( \alpha' \), or proportional to \( 1/x \), the gluon part is constant when \( x \to 0 \) and all massive states are exponentially suppressed. The full integration region in the moduli space is defined by proper times \( t \in [0, \infty] \), which translates to \( x \in [0, 1] \). The insertion points of the external states are being integrated over all connected components of the world sheet boundary. On the other hand, from the point of view of field theory we could already be satisfied with \( x \in [0, \epsilon] \).

We shall now set up a systematic three step procedure to extract the field theoretical contribution from string amplitudes following the sewing procedure in the form of [14, 16]. We first replace moduli by proper time variables according to (4) and then, secondly, eliminate all terms proportional to higher powers of moduli, keeping only those which are constant for gluon diagrams or the divergent parts, proportional to inverse powers of moduli, for scalar field theory. This method does to the present stage not enable to extract any mixed diagram, which involves couplings of two different types of particles. Finally we determine the integration region of the moduli. This program will be demonstrated to work on scalar theory and one-loop Yang-Mills diagrams, before we come to our main topic, the computation of two-loop Yang-Mills diagrams and the extraction of renormalization constants.

### 3 Scalar theory

The limit leading to the tachyonic mode of the string spectrum can be used to reproduce results for single Feynman diagrams as well as formulas for complete \( n \)-point functions, which are known from the World Line approach to field theory [8, 11]. All field theoretical results quoted will refer to the \( \Phi^3 \) theory defined by the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{m^2}{2} \Phi^2 + \frac{\lambda}{3!} \Phi^3. \] (5)
We first briefly discuss the World Line Formalism, show how the Green’s function of the bosonic string reduces to the World Line Green’s function [8], and afterwards proceed to Feynman diagrams [14, 16, 17].

3.1 The bosonic Green’s function

The Green’s function of the bosonic string can be written

\[ G(h)(z_i, z_j) = \ln \left( \frac{E(h)(z_i, z_j)}{\sqrt{V_i'(0)V_j'(0)}} \right) - \frac{1}{2} \int_{z_i}^{z_j} \omega^\mu (2\pi \Im(t_{\mu\nu}))^{-1} \int_{z_i}^{z_j} \omega^\nu. \]  

(6)

It is discussed in greater detail in the appendix which we refer to for the geometrical description of world sheets. While the full string amplitude does not depend on any change of coordinate, the tiny part which is extracted from it in the chosen limit does, and one has to take a choice which set of local coordinates \( V_i(z) \) one wants to use. The tree-level Green’s function reduces to the inverse of the Laplacian on the flat plane:

\[ G(0)(z_1, z_2) = \ln \left| \frac{z_1 - z_2}{\sqrt{V_1'(0)V_2'(0)}} \right|. \]  

(7)

The local coordinates in general are specified in a way that the insertion points of the external states are being integrated over only one component of the boundary, which corresponds in the field theoretical picture to selecting a particular loop for any external particle. If one then neglects the radii of the isometrical circles, the boundary component of a single loop becomes the interval between two fixed points on the real axis, both, circles and fixed points, referring to the Schottky map of this loop. We then use the explicit form

\[ V_i'(0) = \left| \frac{(z_i - \eta_j)(z_i - \xi_j)}{(\xi_j - \eta_j)} \right| \]  

(8)

for the local coordinates. They specify the loop \( j \) that is generated by the Schottky maps whose fixed points are \( \xi_j \) and \( \eta_j \) and to which the external state \( i \) is attached. In fact, they are just the inverse of the first abelian differentials on the world sheet

\[ V_i'(0) = \left( \frac{\omega^j(z_i)}{dz_i} \right)^{-1}. \]  

(9)

On the tree-level world sheet one then defines SPT variables by \( \ln(z_{1,2}) = -t_{1,2}/\alpha' \) and find

\[ G(0)(z_1, z_2) \sim \ln \left| \frac{z_1}{z_2} \right| = \pm \frac{(t_1 - t_2)}{2\alpha'}, \]  

(10)

depending on which \( z_i \) is the larger one. This matches precisely to the tree-level World Line Green’s function.

Proceeding to loop-level needs an interpretation of the further moduli of the world sheet, which are the fixed points and multipliers of the generators of the Schottky group. The latter are connected to the length of that loop by

\[ T = -\alpha' \ln |k|, \]  

(11)

while the fixed points shall be treated later. We now have to expand the prime form and the period matrix of the world sheet into a power series in the multipliers of the Schottky group and...
neglect all terms which are suppressed when $\alpha' \to 0$. The expansion is discussed thoroughly in appendix A.2. Using the results one finds:

$$E(z_1, z_2) = (z_1 - z_2) + o(k),$$  \hspace{1cm} (12)

$$\left(2\pi \Im (\tau_{\mu\nu})\right) = -\delta_{\mu\nu} \ln(k_\mu) - (1 - \delta_{\mu\nu}) \ln \left| \frac{(\eta_\mu - \eta_\nu)(\xi_\mu - \xi_\nu)}{\eta_\mu - \xi_\nu} \right| + o(k),$$

$$\int_{z_1}^{z_2} \omega^\mu = \ln \left| \frac{(z_1 - \eta_\mu)(z_2 - \xi_\mu)}{(z_1 - \xi_\mu)(z_2 - \eta_\mu)} \right| + o(k).$$

This simplifies for the one-loop result by choosing $\eta = 0$ and $\xi = \infty$, so that we obtain:

$$E^{(1)}(z_1, z_2) \sim (z_1 - z_2),$$  \hspace{1cm} (13)

$$\left(2\pi \Im (\tau_{11}^{(1)})\right)^{-1} \sim \frac{1}{\alpha'T},$$

$$\left(\int_{z_1}^{z_2} \omega^1 \right)^{(1)} \sim \frac{|t_1 - t_2|}{\alpha'}. $$

This has to be substituted into (9) and we recover the World Line one-loop Green’s function. A more complete discussion of the whole procedure is given in [8] and also the case of higher loop orders is verified. This establishes the relation

$$G^{(g)}(z_1, z_2) \to \frac{1}{2\alpha'} G_B^{(g)}(t_1, t_2)$$

between the Green’s function of the bosonic string and that of the scalar World Line formalism of field theory to any order in perturbation theory.

We now proceed to discuss the two-loop case. We always divide the integration measure of the integrals over moduli by the volume of the modular group of the Riemann surface by fixing three of the four fixed points to the values

$$\eta_2 = 0, \quad \xi_2 = \infty, \quad \xi_1 = 1$$  \hspace{1cm} (15)

on the complex sphere, calling $\eta_1 = \eta$. This standard choice is illustrated in figure 1.

![Figure 1: The two-loop world sheet of the open string](image-url)
If we for instance intend to compute the diagram of figure 2 we can cut the world sheet as in figure 3. The conformal invariance allows to cut in a way that \( z_1, z_2, \eta < 1 \).

We then read off

\[
\begin{align*}
\eta_2 &= 0 \\
\eta_1 &= \eta \\
\xi_1 &= 1 \\
\xi_2 &= \infty
\end{align*}
\]

and satisfy this relation by defining new moduli \( A_i \in [0, 1] \) for \( i = 1, 2, 3 \), which we substitute for the former ones:

\[
z_1 = A_1, \quad \eta = A_1 A_2, \quad z_2 = A_1 A_2 A_3.
\]

The \( A_i \) are interpreted as sewing parameter which parametrize the length of the according propagator. Their logarithms are translated into the proper time variables \( t_i \) of the field theoretical diagrams. The loops support two more such SPT variables by their Schottky multipliers which are replaced by \( t_4 \) and \( t_5 \) so that the prescription altogether reads

\[
\begin{align*}
t_i &= -\alpha' \ln(A_i), \quad \text{for } i = 1, 2, 3, \\
t_5 &= -\alpha' \ln(k_1) - t_1 - t_2, \\
t_4 &= -\alpha' \ln(k_2) - t_1 - t_2 - t_3.
\end{align*}
\]

This parametrization is depicted in figure 4. Any other logarithm occurring in the formula for the Green’s function will be translated analogously:

\[
\begin{align*}
\alpha' \ln(z_1 - z_2) &= -t_1 + \alpha' \ln(1 - A_2 A_3), \\
\alpha' \ln(z_1 z_2) &= -2t_1 - t_2 - t_3, \\
\alpha' \ln \left( \frac{z_1}{z_2} \right) &= t_2 + t_3,
\end{align*}
\]

\[
\alpha' \ln \left( \frac{(z_2 - \eta)(z_1 - 1)}{(z_2 - 1)(z_1 - \eta)} \right) = -t_2 + \alpha' \ln \left( \frac{(1 - A_3)(1 - A_1)}{(1 - A_1 A_2 A_3)(1 - A_2)} \right)
\]

and we finally find

\[
G^{(2)}(z_1, z_2) = \ln(z_1 - z_2) - \frac{1}{2} \ln(z_1 z_2) - \frac{1}{\ln(k_1) \ln(k_2) - \ln^2(\eta)} \tag{20}
\]
Furthermore a treatment of the integration measure and the normalization constants must be 
from (5) has to be chosen 
The nature of the normalization factor 
will also be discussed later on more thoroughly. The integration measure is given by:

$$\int d^d p d^d k \frac{1}{(2\pi)^{2d} p^2 k^2 (q-k)^2 (p-q)^2 (k-p)^2} = \int_0^\infty \left( \prod_{i=1}^5 dt_i \right) \left( t_1 t_2 (t_3 + t_4 + t_5) + (t_3 + t_4) t_5 \right)^{-d/2} \times \exp \left( -d^2 t_1 t_2 (t_3 + t_4 + t_5) + t_1 t_3 (t_4 + t_5) + t_2 t_4 (t_3 + t_5) + t_3 t_4 t_5 \right).$$

The nature of the normalization factor

$$C_2 \equiv \left( (2\pi)^2 \sqrt{2\alpha'} \right)^{-d} g_S^2$$

will also be discussed later on more thoroughly. The integration measure is given by:

$$[dm]^2 = \frac{dk_1 dk_2 d\eta}{k_1^2 k_2^2 (\eta - 1)^2 V_1'(0) V_2'(0)} \frac{d\zeta_2 d\zeta_4}{(1 - k_1)^2 (1 - k_2)^2} \times \det^{-d/2} (-i\tau_{\mu\nu}) \prod_{\beta} \left( \prod_{n=1}^\infty (1 - k_\beta^n)^{2-d} (1 - k_\beta)^{-2} \right).$$

The volume of the projective group has already been divided out by introducing the standard 
choice (15) of coordinates. For the local coordinates we use the usual $V_i'(0) = z_i$. Next we
and following (17) and (18) we get:

\[ \eta_{k_1 k_2} = \exp \left( -m^2 \sum_{i=1}^{5} t_i \right). \]  

As we intend to project exclusively onto the tachyonic excitations we can employ their mass-shell condition

\[ \alpha' m^2 = -1 \]  

to define a particle mass for the scalar field. We then discard all terms proportional to powers of any moduli and retain a field theory diagram of a massive scalar field, while all other contributions are suppressed exponentially or, in the case of the gluon, stay finite when \( \alpha' \to 0 \). In regard of the prefactor we can omit terms proportional to \( k_1, k_2 \) in the integrand

\[ \prod_{\beta} \left( \prod_{i=1}^{\infty} (1 - k_\beta)^{2-d}(1 - k_\beta)^{-2} \right) (1 - k_1)^2(1 - k_2)^2 = 1 + o(k_1, k_2) \]  

and following (17) and (18) we get:

The final result for the two-loop two-tachyon amplitude reads:

\[ \mathcal{A}^{(2)}_2(p^2) = \frac{\lambda^4}{2^9(4\pi)^d} \int_0^\infty \prod_{i=1}^{5} \left( dt_i \ e^{-m^2 t_i} \right) \left( (t_1 + t_2)(t_3 + t_3 + t_5) + (t_3 + t_4)t_5 \right)^{-d/2} \times \exp \left( -p^2 t_1 t_2(t_3 + t_4 + t_5) + t_1 t_3(t_4 + t_5) + t_2 t_4(t_3 + t_5) + t_3 t_4 t_5 \right) . \]

It reproduces the field theoretical result for the Feynman diagrams from figure 3 including any combinatorial factors. In [14] this method is also applied to reducible diagrams and the same positive result was confirmed.

This procedure not only works for \( \Phi^3 \) theory but can also be extended to scalar \( \Phi^4 \) theory [16, 17] by a different matching of the coupling constants. A problem arises with the moduli or SPT variables of those propagators which shrink to zero length. Their integrations need to be regularized by hand, a method which is conceptually not very satisfactory. Still it can be performed to get contributions consistent with results for scalar field theories, while we will not be able to extrapolate the method to gauge theories. Another open question which one should be able to tackle already in two-loop scalar diagrams concerns the presence, or absence, of counterterm diagrams. It has not been investigated, how their contributions to the two-loop scattering are somehow implicitly contained in the string diagram. If not, the missing part could depend on the renormalization scheme.

### 4 Pure Yang-Mills gauge theory

After having analyzed the scalar field theory in terms of Green’s functions and Feynman diagrams we shall now treat the most interesting case of Yang-Mills gauge theory. The challenging aim of our investigation is the generalization of the Bern-Kosower rules which allow a greatly simplified
computation of one-loop $n$-point functions in pure gauge theories \[2\]. We shall present the means to calculate exactly all contributions of the two-point two-loop string amplitude that survive in the field theory limit. In this context we shall particularly have to point out the difficulties arising from the unknown relation between different choices for the local coordinate on the world sheet and different gauge choices in the field theory. Furthermore the problem of constructing four-gluon vertices in string theory will be left a riddle.

Starting point is the $n$-gluon $h$-loop amplitude of the bosonic string which is given by the expectation value of $n$ gluon vertex operators computed in the background of a bosonic string theory on a Genus $h$ Riemann surface. The contractions of the world sheet coordinates are again performed using the Green’s function from (3). We briefly collect the constituents of this amplitude

$$A_n^{(h)}(p_1, ..., p_n) = N^h \text{tr} (\lambda^{a_1} \cdots \lambda^{a_n}) C_h A^{(h)} \int [dm]_h^n \prod_{i<j} \exp \left( 2\alpha' p_ip_j G^{(h)}(z_i, z_j) \right)$$

$$\times \exp \left( \sum_{i \neq j} \sqrt{2\alpha'} p_i \epsilon_i \partial_{z_i} G^{(h)}(z_i, z_j) + \frac{1}{2} \sum_{i \neq j} \epsilon_i \epsilon_j \partial_{z_i} \partial_{z_j} G^{(h)}(z_i, z_j) \right).$$

The prefactor $N^h \text{tr} (\lambda^{a_1} \cdots \lambda^{a_n})$ is the Chan Paton factor of the diagram with external gauge charges $\lambda^{a_i}$ of some $SU(N)$ gauge group, whose adjoint representation is normalized as follows:

$$\text{tr} (\lambda^{a_i} \lambda^{a_j}) = \frac{1}{2} \delta_{a_i a_j}. \quad (31)$$

The $\epsilon_i$ are the polarization vectors of the external gluons and from the expansion of the exponential one only has to keep the terms that are multilinear in the $\epsilon_i$. The general integration measure can be written in terms of the Schottky parameters that span the moduli space $\Gamma$:

$$[dm]_h^n = \prod_{m=1}^h \left( \frac{dk_m d\xi_m d\eta_m}{k_m^2 (\xi_m - \eta_m)^2} \right) \prod_{m=1}^n (dz_m) \frac{1}{dV_{abc}}$$

$$\times (\det(-i\tau_{\mu\nu}))^{-d/2} \prod_{\beta} \left( \int_{m=1}^\infty (1 - k_m) \prod_{m=2}^\infty (1 - k_m^\beta)^2 \right).$$

It contains the normalization constant involving the product over primary classes and the volume of the modular group $dV_{abc}$ in the denominator. The bosonic Green’s function is again be given by

$$G^{(h)}(z_i, z_j) = \ln \left( \frac{E^{(h)}(z_i, z_j)}{\sqrt{V_i'(0)V_j'(0)}} \right) - \frac{1}{2} \int_{z_i}^{z_j} \omega^\mu (2\pi \Im (\tau_{\mu\nu}))^{-1} \int_{z_i}^{z_j} \omega^\nu$$

$$\text{Note the difference in the depence on the local coordinates compared to (13), which has to be obeyed to get correct results in higher than one-loop order (3). We are discussing off-shell amplitudes and therefore do not demand the mass-shell and transversality relations}$$

$$p^2 = 0, \quad pe = 0 \quad (34)$$

to hold, although for the sake of brevity we often restrict ourselves to the term proportional to $(p_1 \epsilon_i)(\epsilon_j p_i)$. The derivation of the normalization constant is found in (13). It uses tree-level three- and four-point amplitudes and factorization arguments to fix the dependence of the prefactor on the scale $\alpha'$ and the dimensionless string coupling constant

$$gS = \frac{1}{2} (2\alpha')^{1-d/4}. \quad (35)$$
It is then found:

\[ N_0 = 2g_S(2\alpha')^{(d-6)/4}, \]

\[ C_h = \left((2\pi)^h \sqrt{2\alpha'}\right)^{-d} g_S^{2h-2}. \]

The first factor reproduces the normalization of the Fourier transformations which come with loop calculations and also assigns the correct physical dimension to the amplitude, while the second carries the necessary power in the string coupling that is dictated by the Euler characteristic \( \chi(M) = 2 - 2h \) of the Riemann surface.

### 4.1 One-loop diagrams

Following the rules we have established when addressing scalar field theory we shall now proceed to gauge fields [15]. We expand the integrand of the amplitude in the moduli, keep only the finite and non-vanishing part of the expansion when \( \alpha' \rightarrow 0 \) and translate everything into the language of SPT variables. The results allow to confirm the Ward identities of the Feynman background gauge and to read off the coefficient of the Yang-Mills \( \beta \) function from the wave function renormalization constant of the gauge field, which at one-loop order receives contributions only from a single Feynman diagram.

We first specify the formulas for the measure (32), the Green’s function (33) and the constants (36):

\[ [dm]^n_1 = \prod_{m=1}^n (dz_m) \frac{1}{dV_{abc}} \frac{dkd\eta d\xi}{k^2(\eta - \xi)^2} \left( -\frac{\ln(k)}{2\pi} \right)^{-d/2} \prod_{m=1}^\infty (1 - k^m)^{-2-d} \]

\[ G^{(1)}(z_i, z_j) = \ln(z_i - z_j) - \frac{1}{2} \ln(z_i z_j) + \frac{1}{2 \ln(k)} \ln^2 \left( \frac{z_i}{z_j} \right) \]

\[ + \ln \left( \prod_{m=1}^\infty \frac{1 - k^m z_i}{z_i} \frac{1 - k^m z_j}{z_j} \right), \]

\[ N_0^n C_h = \left(2\pi \sqrt{2\alpha'}\right)^{-d} (2\alpha' g^2)^n/2. \]

We have used the usual fixed point choice \( \eta = 0 \) and \( \xi = \infty \) as well as the Lovelace type local coordinates \( V'_i(0) = z_i \). Further \( z_2 = 1 \) has been fixed by a projective transformation.

\[ \eta = 0 \hspace{2cm} z_1 \hspace{1cm} \xi = \infty \hspace{1cm} z_2 = 1 \]

Figure 5: Parametrization of the one-loop two-point world sheet

The only diagram can be parametrized as in figure 3 which leads to the integration region \( \Gamma = \{ z_1 | 0 < z_1 < 1 \} \). Substituting the sewing parameter \( z_1 = A \) into (37) and defining the proper times

\[ T = -\alpha' \ln(k), \quad t = -\alpha' \ln(A), \quad [0, 1] \rightarrow [\infty, 0], \]

(38)
we find

\[ [dm]_1^2 = \frac{dkdz_1}{k^2} \left( -\ln(k) \right)^{-d/2} \prod_{m=1}^{\infty} (1 - k^m)^{2-d} \]

\[ = \left( \frac{1}{\alpha'} \right)^2 (2\pi)^{d/2} \frac{AT^{-d/2}dTdt}{k} \left( 1 + (d-2)k + o(k^2) \right), \]

\[ G(1)(z_1, z_2) = \ln(1 - A) - \frac{1}{2} \ln(A) + \frac{1}{2 \ln(k)} \ln^2(A) + \ln \left( \prod_{m=1}^{\infty} \frac{(1-k^m/A)(1-Ak^m)}{(1-k^m)^2} \right) \]

\[ = \ln(1 - A) + \frac{t}{2\alpha'} - \frac{t^2}{2\alpha'T} + \sum_{n=1}^{\infty} \left( \ln(1 - Ak^n) + \ln(1 - k^n/A) - 2 \ln(1 - k^n) \right). \]

In the same manner one can take the derivatives of the Green’s function. We now only have to keep the term in the integrand that is proportional to \( A_0, k_0, \alpha_0' \), and thus finally find:

\[ A_2^{(2)}(p^2) = N\text{tr} (\lambda^1 \lambda^2) \frac{4g^2}{(4\pi)^{d/2}} \int_0^\infty dT \left( T^{-d/2} \right) dt \int_0^\infty dt \left( T^{-d/2} \right) \]

\[ \times \left( \frac{1}{2} - \frac{t}{T} \right)^2 (d-2) - 2 + o(k, A) \right) \exp \left( -p^2 \left( t - \frac{t^2}{T} + o(\alpha', k, A) \right) \right). \]

Rescaling the integration over \( t \) by a factor \( T \) leads to the result of \[15\]. All integrations can easily be performed using some formulas given in appendix \[D\] and the precise value that comes out of the computation of the sum of the two Feynman diagrams of Yang-Mills theory in the Feynman background gauge \[24\], drawn in \[4\] is recovered.

Figure 6: The one-loop Feynman diagrams that contribute to the Yang-Mills \( \beta \) function

In particular from the wave function one-loop renormalization constant

\[ Z_A = 1 + \frac{g^2 N \frac{11}{12} \frac{1}{3}}{\epsilon} \]  

the correct coefficient of the Yang-Mills \( \beta \) function \[24\] is deduced. It is interesting to note that these notions obviously depend on the choice of local coordinates \( V_i(z) \).

We now only briefly address the one-loop three-gluon diagrams. This is the first and most simple example how to construct four-gluon vertices in string theory, which will be defined by extracting those regions in the moduli space where a propagator between two three-gluon vertices is short in terms of field theoretical proper time. We use the parametrization of figure \[7\].

The integration region for the sewing parameters from \(17\) will be as usually \([0, 1]^3\). They are defined by

\[ z_1 = A_1 A_2, \quad z_2 = A_2 \]
and lead to proper time variables

\[
\begin{align*}
t_i &= -\alpha' \ln(A_i), \quad \text{for } i = 1, 2, \\
t_3 &= -\alpha' \ln(k) - t_1 - t_2.
\end{align*}
\]  

(43)

We then get the sum of the two Feynman diagrams without four-gluon vertices by expanding the integrand and taking the appropriate limit. On the other hand, if we let \(\alpha' \to 0\) such that e.g. \(A_1\) stays finite, we have \(t_1 \to 0\), although the insertion points \(z_1\) and \(z_2\) remain widely separated on the world sheet. The problem of this naive definition of four-gluon vertices in string amplitudes, the so-called pinching, is that it remains completely unclear what should happen to the free modulus \(A_1\). In [15] the term in the integrand, that stays finite when \(A_1 \to 1\), is chosen to be relevant and the rest is being ignored. In fact, this method did allow to verify the relation between the three and four gluon diagrams deriving from the Ward identities of the Feynman background gauge for Yang-Mills theories and might thus be called heuristically successful.

### 4.2 Two-loop vacuum diagrams

The most simple two-loop Feynman diagrams one can think of are the vacuum diagrams drawn in figure 8.

![Figure 8: The three two-loop vacuum diagrams for Yang-Mills gauge theory](image)

They were discussed in [25]. The generalization will not be straightforward as we shall have to deal with the announced difficulties concerning the diagram (b) with a four-gluon vertex. Another more fundamental question arises in the context of gauge invariance. The diagrams (a) and (b) clearly have to correspond to different regions in the moduli space of the string world sheet, simply because the number of field theoretical propagators is not the same, and therefore a different number of proper time variables has to be introduced. On the other hand their respective contributions in the field theoretical calculation will depend on the gauge one has chosen. One might expect that the statement which was true at one-loop level, that string contribution just come out in the Feynman background gauge, is still valid. The least one would like to require would be that added up they give a gauge invariant result that can be compared to field theory. Actually the question cannot be answered in the case of the vacuum diagrams, as their contributions vanish identically in dimensional regularization. This follows the principle that vacuum
of \( \alpha \) of the derivation of these expressions is summarized in appendix A. We have always added powers
and regarding all relevant powers of the string tension, we are left with an overall factor 1
chapter.

We now specialize all the relevant expressions in the amplitude, dropping the colour factor
which is empty. To then work out the amplitude
\[
\mathcal{A}_0^{(2)} = C_2 \int_{\Gamma} [dm]_2^0 = \frac{g^2}{4} \left( 2\pi \sqrt{2\alpha'} \right)^{-2d} (2\alpha')^2 \int_{\Gamma} [dm]_2^0
\]
we only need the measure of the integration over the moduli of which some are fixed as in (45).
It involves the period matrix
\[
2\pi \Im (\tau_{11}) = -\frac{\alpha' \ln(k_1)}{\alpha'} - \frac{2k_2(\eta - 1)^2}{\eta} + o \left( k_1^2, k_2^2 \right),
\]
and its determinant
\[
\det^{-d/2}(-i\tau_{\mu\nu}) = (2\pi)^d \left( \frac{\alpha' \ln(k_1)\alpha' \ln(k_2) - \alpha'^2 \ln^2(\eta)}{\alpha'^2} \right)^{-d/2}
\]
\[
\times \left( 1 - \alpha' \left( \frac{(\eta - 1)^2(\alpha' \ln(k_1)k_1 + \alpha' \ln(k_2)k_2)}{\eta(\alpha' \ln(k_1)\alpha' \ln(k_2) - \alpha'^2 \ln^2(\eta))} \right) \right.
\]
\[
\left. - \frac{2k_1k_2\alpha' \ln(\eta)(\eta + 1)(\eta - 1)^3}{\eta^2(\alpha' \ln(k_1)\alpha' \ln(k_2) - \alpha'^2 \ln^2(\eta))} \right) + o(\alpha'^2, k_1^2, k_2^2). \]
Together we find:
\[
[dm]_2^0 = \frac{dk_1dk_2d\eta}{k_1^2k_2^2(1 - \eta)^2} \det^{-d/2}(-i\tau_{\mu\nu}) \]
\[
\times \prod_{m=1}^{\infty} \prod_{\beta} (1 - k_{\beta}^m)^{-d} \prod_{m=2}^{\infty} (1 - k_{\beta}^m)^2 (1 - k_1)^2(1 - k_2)^2
\]
\[
= \frac{dk_1dk_2d\eta}{k_1^2k_2^2} \det^{-d/2}(-i\tau_{\mu\nu}) \left( 1 + (d - 2)(k_1 + k_2) \right.
\]
\[
\left. + \left( (d - 2)^2 + \frac{d(1 - \eta)^2(1 + \eta^2)}{\eta^2} \right) k_1k_2 + o \left( k_1^2, k_2^2 \right) \right). \]
The derivation of these expressions is summarized in appendix A. We have always added powers
of \( \alpha' \) to get finite SPT variables in the end. After substituting for the integration over moduli
\[
\int_{\Gamma} \frac{dk_1dk_2d\eta}{k_1k_2\eta} = \left( \frac{1}{\alpha'} \right)^3 \int_0^{\infty} \prod_{i=1}^3 (dt_i)
\]
and regarding all relevant powers of the string tension, we are left with an over all factor \( 1/\alpha' \).
The parametrization of figure 5 using \( \eta = A \) translates into
\[
t_1 = -\alpha' \ln(A), \quad t_2 = -\alpha' \ln(k_1) - t_1, \quad t_3 = -\alpha' \ln(k_2) - t_1. \]
Now we have to extract the term proportional to $A^0, k^0, \alpha'$ from the remaining integrand, which should belong to that part of the amplitude, i.e. that region in the moduli space, where all proper times stay finite. This corresponds to the sum of the two diagrams in (a) of figure 8. We get

$$A^{(2)}_0\big|_{(a)} = \frac{g^2}{(4\pi)^d} d(d-2) \int_0^{\infty} \prod_{i=1}^{3} (dt_i) \frac{2t_1 + t_2 + t_3}{(t_1 t_2 + t_1 t_3 + t_2 t_3)^{1+d/2}},$$

which differs from the result in [25] by the symmetrized integration region. The integral vanishes anyway in dimensional regularization.

Investigating the possible pinching contributions to diagram (b) of figure 8 we find the curious situation that three different limits of the string amplitude contribute to a single Feynman diagram. The regions where $t_2$ or $t_3$ vanish, are defined by finite values for the moduli $k_1/A$ and $k_2/A$. We set $k_i \to A k_i$, expand in $A$ and the other multiplier, keeping the finite part. The expansion in $\alpha'$ has to be performed without any prefactor in $\alpha'$ as the missing proper time integration leads to the missing of a factor $1/\alpha'$. The integrand we get is finite when $k_i \to 1$ in both cases and the $k$-integration can be split off from the rest, giving the unique result

$$A^{(2)}_0\big|_{t_1, t_2, t_3} \big|_{(b)} = \frac{g^2}{(4\pi)^d} d(d-2) \int \frac{dk}{k^2} \int_0^{\infty} \prod_{i=1}^{2} (dt_i) (t_1 t_2)^{-d/2}.$$  

The question what should happen to the integration over $k$ was answered in [25] in the sense that one has to take an infinitesimal region around $k = 1$ in the integrand and drop the integration. Even more problematic is the treatment of the integral over the free modulus in the case of the third possible pinching. This we get by having $t_1 \to 0$, i.e. $A$ finite. We do the same expansion as before:

$$A^{(2)}_0\big|_{t_1} \big|_{(b)} = \frac{g^2}{(4\pi)^d} \int \frac{dA}{A^2(1-A)^2} (d(1 + A^4) - 2d(1 + A^2)A + (d^2 - 2d + 4)A^2)$$

$$\times \int_0^{\infty} \prod_{i=1}^{2} (dt_i) (t_1 t_2)^{-d/2}.$$  

The integral over $A$ is divergent at 0 and 1. In [25] this was cured by a rather arbitrary zeta function regularization after expanding the integrand around $A = 1$ and omitting the divergency at $A = 0$ completely:

$$\int_0^1 \frac{dA}{(1-A)^2} \sim \zeta(0) = -\frac{1}{2}.$$  

Similar divergent integrals have also been encountered in scalar $\Phi^4$ theory and similar methods, involving “world sheet cut-offs”, have been used for regularization [16, 17]. The result is

$$A^{(2)}_0\big|_{t_1} \big|_{(b)} = -\frac{g^2}{2(4\pi)^d} (d-2)^2 \int_0^{\infty} \prod_{i=1}^{2} (dt_i) (t_1 t_2)^{-d/2}.$$  

Figure 9: Parametrization of the two-loop vacuum world sheet
As we are unable to propose any reasonable alternatives, we do not intend to criticise these methods in detail. As mentioned a comparison of the contributions found from the string amplitude to field theory is impossible anyway, both are vanishing by definition. But we have to conclude, that certain divergent integrals over free moduli are appearing, if one tries to follow the strategies of the naive pinching procedures. These divergencies appear exactly in two different types which cover all the cases we investigated. If the proper time variable which is associated with the multiplier $k$ of a loop becomes small, we find an integrand finite when $k \to 1$ and the integration divergent of the kind $dk/k^2$, while any other sewing parameter $\eta$ leads to integrands that diverge like $d\eta/(\eta^2(1 - \eta)^2)$. This shall be verified in the next chapter to be a generic feature of the pinching. The nature of the diverging integrals is very similar to those types of integrals that have to be performed when doing the trace over the Hilbert space of the intermediate string states when sewing together the world sheets of two strings, thus building up world sheets of higher Genus. In this sense we believe them to be related to the tachyon exchange in the shrinking propagator.

4.3 Problems concerning the overlapping of isometric circles

In this subsection we will explain a problem which arises when performing the $k \to 0$ limit of the string worldsheet. Up to now, and as well in the following, we completely ignore the requirement, that the circles cut out off the complex plane around the fixed points in the Schottky construction of the Riemann surface are not overlapping. Strictly speaking, this induces further restrictions on the moduli space, the allowed values of Schottky parameters, which in principle can be relevant also in the low energy limit. For all contributions to scalar field theory this apparently has not been the case, because the naive sewing has been successful. But as we are unable in general to resolve the same task for Yang-Mills diagrams, this cannot be ruled out. In fact, in [25] the prescription derived in [15] was used, which implied a further restriction on the sewing parameters, as compared to our results above. But, luckily or not, the difference can be tracked down to an overall factor of 2 from symmetrizing the integrands in [25], which does not appear to be very significant in deciding whether the modification is necessary or not. Still it is worthwhile to be investigated in more detail.

In [15] the Schottky parametrization was employed with circles different from the isometric ones. The latter are located at infinity, which makes the same treatment more difficult. Starting from the parametrization of diagram 9 one adds the isometric circle around $\eta_1 = \eta$ and $\xi_1 = 1$ and a circle of radius $\sqrt{k_2}$ around $\eta_2 = 0$ and its image with infinite radius but finite position at $1/\sqrt{k_2}$ around $\xi_2 = \infty$. Now $\eta$ can no longer vary freely between 0 and 1, but the two circles are further required not to intersect. This translates into the inequalities

$$\sqrt{k_2} \leq \frac{\eta - \sqrt{k_1}}{1 - \sqrt{k_1}}, \quad 0 \leq \frac{1 - \sqrt{k_1}}{1 + \sqrt{k_1}} (1 - \eta).	ag{55}$$

The second inequality reduces to $\eta \leq 1$ for small $k_1$, whereas the first one is still a complicated relation between fixed points and multipliers, and several regions of their values will contribute. In [15] the single inequality

$$0 \leq \sqrt{k_2} \leq \sqrt{k_1} \leq \eta \leq 1 \tag{56}$$

was used. It is not sufficient to solve the above requirements, but was enough to produce results consistent with scalar field theory expectations. The same inequality, translated to a different region of integration, was also used in [24] for the vacuum diagrams of the Yang-Mills theory, but again we cannot judge their results to be decisive. Anyway, in more complicated diagrams, a systematic evaluation of all regions in the moduli space which respect the generalized version of the above inequalities may be necessary. One can easily convince oneself that for diagrams with external legs, the inequalities get much more complicated and a variety of distinctions arises. For some scalar $\Phi^4$ diagrams this task has been performed in [17]. Also the methods we shall be employing to solve the unrestricted integrals do not allow a straightforward extension to this
case, which leaves us with very little hope to be able to evaluate the analogous integrals.

4.4 Two-loop diagrams involving external gluons

We now come to our main topic, two-loop string diagrams with two external gluon vertex operators inserted. We shall again proceed along the lines we have established in the chapter on scalar theory and also we shall have to deal with the problems we already encountered in the previous two sections about Yang-Mills diagrams. This method should in principle allow to compute the two-loop coefficient of the Yang-Mills $\beta$ function

$$\beta = -g \frac{\partial Z_g}{\partial \mu} = -g \left( \beta_0 \left( \frac{g}{4\pi} \right)^2 + \beta_1 \left( \frac{g}{4\pi} \right)^4 + o(g^6) \right).$$

It can be extracted from the two point function, if the field theoretical gauge is any background type gauge. We shall admit an arbitrary covariant gauge of the gauge field fluctuations, which is still allowed after the background gauge is chosen. In this kind of gauge the dependence of the charge renormalization constant on the subtraction scale can be deduced from the gauge field wave function renormalization constant alone. This in turn is given by the two point function:

$$Z_A = 1 + \beta_0 \frac{\epsilon}{\epsilon} \left( \frac{g}{4\pi} \right)^2 + \frac{\beta_1}{2\epsilon} \left( \frac{g}{4\pi} \right)^4 + o(g^6).$$

The coefficients take the values $\beta_0 = \frac{11}{3}$, $\beta_1 = \frac{34}{3}$. They are gauge invariant and therefore pose a good criterion to test the validity of the methods employed to higher loop order. All purely gluonic diagrams that contribute in field theory are drawn in figure 10.

![Figure 10: The gluonic two-loop contributions to the Yang-Mills two-point function](image)

The further diagrams involving ghost fields are obtained by substituting gluon loops by ghost loops. All such are listed in reference [24], where also their contributions in the Feynman background gauge are displayed. By the help of M. Peter from the Heidelberg University we are also
able to discuss the contributions of these diagrams in an arbitrary covariant background gauge. In table 4 they are summarized for \( d = 4 - 2\epsilon \), using the same letters for the diagrams as in reference [24], which are not to be confused with our notation of figure 10. We see that to the leading order in the \( 1/\epsilon \) expansion all diagrams are transverse, while this order necessarily adds up to zero. The next to leading order is not transverse diagram by diagram but the sum of all contributions of course is. It remains unclear what we should expect to happen to the counter term diagrams, a problem that could not be investigated at one-loop order and has not been in scalar field theory. We should be able to notice their absence by a deviation of our result from the correct \( \beta \) function coefficient by just their contribution. A very natural assumption seems, that we are dealing with a “naked” theory with string derived diagrams, where no counter terms are present. It is unclear, how the string diagrams, which originally have a natural cut-off, implement renormalization. The explicit form of counter terms conventionally depends on the renormalization scheme employed, which would pose another puzzle. In other words: If the effects of the one-loop renormalization are included automatically, the result is universal, while if not, it will depend on the scheme. As we are unable to resolve the problems concerning four-gluon vertices, we shall not be in the position to unravel this question anyway.

We now take (30) for \( h = 2 \) and \( n = 2 \) and use results from appendix B employing the usual coordinate fixing (13) to get

\[
A_2^{(2)}(p, -p) = N^2 \text{tr}(\lambda^{a_1} \lambda^{a_2}) C_2 N_0^2 \int \frac{dk_1 dk_2 dq_1 dq_2}{k_1 k_2 (1 - \eta)^2} (1 - k_1)^2 (1 - k_2)^2 \\
\times \delta^{-d/2}(-i\tau_{\mu\nu}) \prod_\beta \left( \prod_{m=1}^{\infty} (1 - k_\beta^m)^{-d} \prod_{m=2}^{\infty} (1 - k_\beta^m)^2 \right) \exp \left( -2\alpha' p^2 G^{(2)}(z_1, z_2) \right) \\
\times \left( (2\alpha')(\epsilon_1 p_2)(\epsilon_2 p_1) \partial_{z_1} G^{(2)}(z_1, z_2) \partial_{z_2} G^{(2)}(z_1, z_2) + (\epsilon_1 \epsilon_2) \partial_{z_1} \partial_{z_2} G^{(2)}(z_1, z_2) \right),
\]

where the expansion in the multipliers and \( \alpha' \) is partly done already. The determinant of the period matrix can be read off from (30). The prime form is

\[
E^{(2)}(z_1, z_2) = \frac{\alpha' \ln(z_1 - z_2) - (z_1 - z_2)^2 \left( \frac{k_1 (\eta - 1)^2}{(z_1 - \eta)(z_2 - \eta)(z_1 - \eta - z_2)} + \frac{k_2}{z_1 z_2} \right)}{\eta^2} + o(k_1^2, k_2^2)
\]

and the abelian integrals follow

\[
\left( \int_{z_1}^{z_2} \omega^1 \right)^{(2)} = \frac{\alpha' \ln \left( \frac{(z_1 - \eta)(z_2 - \eta)}{(z_1 - 1)(z_2 - 1)} \right) - k_2 (\eta - 1)(z_2 - z_1)(z_1 z_2 + \eta)}{\eta z_1 z_2} + k_1 k_2 (\eta - 1)^3 (z_1 - z_2) \frac{\eta + 1}{\eta^2} \left( \frac{\eta + 1}{(z_1 - 1)(z_2 - 1)} + \frac{\eta (\eta + 1)}{(z_1 - \eta)(z_2 - \eta)} \right) + o(k_1^2, k_2^2),
\]

\[
\left( \int_{z_1}^{z_2} \omega^2 \right)^{(2)} = \frac{\alpha' \ln \left( \frac{z_1}{z_2} \right) + k_1 (\eta - 1)^2 (z_1 - z_2) \frac{1}{(z_1 - 1)(z_2 - 1)} + \frac{\eta}{(z_1 - \eta)(z_2 - \eta)}}{\eta^2} + k_1 k_2 (\eta - 1)^2 (z_1 - z_2) \frac{\eta + 1}{\eta} \left( \eta + 1 + \frac{\eta (\eta + 1)}{z_1 z_2} \right) + o(k_1^2, k_2^2),
\]

finally the inverse of the period matrix:

\[
(2\pi \Im(\tau_{\mu\nu}))^{-1} = \det^{-1} (2\pi \Im(\tau_{\mu\nu})) \begin{pmatrix} \tau_{22} & -\tau_{12} \\ -\tau_{21} & \tau_{11} \end{pmatrix},
\]

\[
\det^{-1} (2\pi \Im(\tau_{\mu\nu})) = \left( \frac{\alpha' \ln(k_1) \alpha' \ln(k_2) - \alpha'^2 \ln^2(\eta)}{\alpha'^2} \right)^{-1},
\]

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Table 1: Contributions of two-gluon two-loop diagrams in Yang-Mills theory in a general covariant background gauge

| Terms proportional to $\epsilon^2 p^2$ | Terms proportional to $(\epsilon p)^2$ |
|----------------------------------------|----------------------------------------|
| (a) $\left( \frac{1}{9} + \frac{5}{33} \right) \epsilon^{-2} + \left( \frac{1}{12} + \frac{35}{9} \right) \epsilon^{-1}$ | $\left( \frac{1}{9} + \frac{5}{33} \right) \epsilon^{-2} + \left( 1 + \frac{5}{3} \right) \epsilon^{-1}$ |
| (b) $\left( \frac{25}{6} + \frac{55}{3} + \frac{2}{8} \right) \epsilon^{-2}$ | $\left( \frac{25}{6} + \frac{55}{3} + \frac{2}{8} \right) \epsilon^{-2}$ |
| + $\left( \frac{215}{12} + \frac{115}{6} + \frac{192}{18} + \frac{3^2}{16} \right) \epsilon^{-1}$ | + $\left( \frac{55}{3} + \frac{67}{72} + \frac{192}{18} + \frac{3^2}{16} \right) \epsilon^{-1}$ |
| (c) $\left( \frac{1}{9} - \frac{5}{33} \right) \epsilon^{-1}$ | $-\frac{\sqrt{2}}{21} \epsilon^{-1}$ |
| (d) $\left( \frac{25}{6} + \frac{55}{3} + \frac{2}{8} \right) \epsilon^{-2}$ | $\left( \frac{5}{6} - \frac{2}{3} + \frac{3^2}{16} \right) \epsilon^{-1}$ |
| (e) $\left( \frac{9}{3} - \frac{5}{24} \right) \epsilon^{-2}$ | $\left( -6 + 3\xi - \frac{3^2}{8} \right) \epsilon^{-2}$ |
| + $\left( -24 + 18\xi - \frac{96}{21} + \frac{3^2}{8} \right) \epsilon^{-1}$ | + $\left( -24 + 18\xi - \frac{96}{21} + \frac{3^2}{8} \right) \epsilon^{-1}$ |
| (f) $\left( \frac{1}{9} - \frac{5}{33} \right) \epsilon^{-2} + \left( \frac{13}{12} - \frac{35}{9} \right) \epsilon^{-1}$ | $\left( \frac{1}{9} - \frac{5}{33} \right) \epsilon^{-2} + \left( -3 - \frac{55}{24} \right) \epsilon^{-1}$ |
| (g) $\left( -\frac{5}{24} + \frac{136}{64} - \frac{2^2}{16} \right) \epsilon^{-2}$ | $\left( -\frac{5}{24} + \frac{136}{64} - \frac{2^2}{16} \right) \epsilon^{-2}$ |
| + $\left( -\frac{41}{24} + \frac{106}{36} - \frac{96}{21} \right) \epsilon^{-1}$ | + $\left( -\frac{41}{24} + \frac{106}{36} - \frac{96}{21} \right) \epsilon^{-1}$ |
| (h) $\left( -\frac{9}{3} - \frac{5}{24} + \frac{3^2}{32} \right) \epsilon^{-2}$ | $\left( -\frac{9}{3} - \frac{5}{24} + \frac{3^2}{32} \right) \epsilon^{-2}$ |
| + $\left( \frac{57}{16} - \frac{196}{64} + \frac{125^2}{64} - \frac{96^2}{32} \right) \epsilon^{-1}$ | + $\left( \frac{57}{16} - \frac{196}{64} + \frac{125^2}{64} - \frac{96^2}{32} \right) \epsilon^{-1}$ |
| (i) $\left( \frac{1}{24} - \frac{5}{24} \right) \epsilon^{-2} + \left( \frac{19}{48} - \frac{116}{48} \right) \epsilon^{-1}$ | $\left( \frac{1}{24} - \frac{5}{24} \right) \epsilon^{-2} + \left( \frac{19}{48} - \frac{116}{48} \right) \epsilon^{-1}$ |
| (j) $\left( \frac{1}{9} - \frac{5}{33} \right) \epsilon^{-2} + \left( \frac{9}{3} - \frac{75}{48} + \frac{3^2}{18} \right) \epsilon^{-1}$ | $\left( \frac{1}{9} - \frac{5}{33} \right) \epsilon^{-2} + \left( -\frac{9}{3} - \frac{75}{48} + \frac{3^2}{18} \right) \epsilon^{-1}$ |
| (k) $\left( \frac{27}{16} - \frac{416}{12} + \frac{3^2}{32} \right) \epsilon^{-2}$ | $\left( \frac{27}{16} - \frac{416}{12} + \frac{3^2}{32} \right) \epsilon^{-2}$ |
| + $\left( \frac{213}{10} - \frac{1216}{64} + \frac{3^2}{2} \right) \epsilon^{-1}$ | + $\left( \frac{213}{10} - \frac{1216}{64} + \frac{3^2}{2} \right) \epsilon^{-1}$ |

\[
\times \left( 1 - \alpha' \left( \frac{2(\eta - 1)^2(\alpha' \ln(k_1)k_1 + \alpha' \ln(k_2)k_2)}{\eta(\alpha' \ln(k_1)\alpha' \ln(k_2) - \alpha'^2 \ln^2(\eta))} - \frac{4k_1k_2\alpha' \ln(\eta)(\eta + 1)(\eta - 1)^3}{\eta^2(\alpha' \ln(k_1)\alpha' \ln(k_2) - \alpha'^2 \ln^2(\eta))} \right) + o(\alpha'^2, k_1^2, k_2^2) \right). 
\]

For the product over the primary classes we use the expansion

\[
\prod_{\beta} \left( \prod_{n=1}^{\infty} (1 - k_{\beta}^n)^{-d} \prod_{n=2}^{\infty} (1 - k_{\beta}^n)^{2} \right)^{2} (1 - k_1)^2 (1 - k_2)^2
\]

\[= 1 + (d - 2)(k_1 + k_2) + \left( (d - 2)^2 + \frac{d(1 - \eta)^2(1 + \eta^2)}{\eta^2} \right) k_1 k_2 + o(k_1^2, k_2^2)
\]

(64)

and the prefactor is

\[C_2 N_0^2 = \frac{g^4}{4} \left( 2\pi \sqrt{2\alpha'} \right)^{-2d} (2\alpha')^3.
\]

(65)
The integrals \( I(P) \) further (19) and substitute everything into (60) we find states, which does not give any different result, as has been checked explicitly in all cases. Using loops is then given by the parametrization of figure 3, except up to interchanging the external dimensions by rescaling the integration variables. In the second line we have extracted the momentum dependence as in (17) and finally define proper time variables by (18). The integration is transformed as

$$
\int \frac{dk_1 dk_2 d\eta d_{z_1} d_{z_2}}{k_1 k_2 \eta z_1 z_2} = \left( \frac{1}{\alpha'} \right)^5 \int_0^\infty \prod_{i=1}^5 (dt_i). \tag{66}
$$

This corresponds to a sewing procedure that sewed the diagram together as depicted in figure 8. We next choose the insertion points of the external states to lie at the “large” loop, i.e. \( \eta_2 = 0 \) and \( \xi_2 = \infty \). The possibly astonishing fact, that the fixing of coordinates on the string world sheet leads to a distinction of different loops when looking for contributions to particular field theoretical Feynman diagrams has already been discussed in [8, 9]. It has been found how the local coordinates have to be chosen in order to get correct Green’s functions, which in our case requires us to take \( V'_t(0) = z_i \), as expected. The only way to have the external states on different loops is then given by the parametrization of figure 3 except up to interchanging the external states, which does not give any different result, as has been checked explicitly in all cases. Using further (19) and substitute everything into (60) we find

$$
A_2^{(2)}(p^2) \left|_{(a)} \right. = N^2 \delta_{a_1 a_2} \frac{g^4}{(4\pi)^d} \int_0^\infty \prod_{i=1}^5 (dt_i) \times \left( (\epsilon_1 p_2)(\epsilon_2 p_1)P_{5}^{(a)}(t_1, \ldots, t_5, d) + e^2 P_{4'}^{(a)}(t_1, \ldots, t_5, d) \times \exp \left( -p^2 t_1 t_2 (t_3 + t_4 + t_5) + t_1 t_3 (t_4 + t_5) + t_2 t_4 (t_3 + t_5) + t_3 t_4 t_5 \right) \right)
$$

$$
= N^2 \delta_{a_1 a_2} \left( \frac{g}{4\pi} \right)^4 \left( \frac{4\pi}{\sqrt{p^d}} \right)^{-\epsilon} \left( (\epsilon_1 p_2)(\epsilon_2 p_1)P_{5}^{(a)}(\epsilon) + (e^2 p^2) P_{4'}^{(a)}(\epsilon) \right).
$$

The integrals \( I^{(a)} \) must be computed in dimensional regularization. Each of the polynomials \( P^{(a)} \) contains a couple of hundred terms and is therefore not written explicitly, instead we only display its degree as an index. In the second line we have extracted the momentum dependence in \( d = 4 + \epsilon \) dimensions by rescaling the integration variables.

Power counting reveals that we have to deal at least with some logarithmic UV divergencies proportional to \( 1/\epsilon \) for small values of the SPT variables, whereas for \( t_i \to \infty \) all integrals are IR finite. In a first step one can substitute

$$
t_1 = t x_1 x_2 x_3 x_4,
$$

$$
t_2 = t(1 - x_1) x_2 x_3 x_4,
$$

$$
t_3 = t(1 - x_2) x_3 x_4,
$$

$$
t_4 = t(1 - x_3) x_4,
$$

$$
t_5 = t(1 - x_4),
$$

$$
\int_0^\infty \prod_{i=1}^5 (dt_i) = \int_0^1 \prod_{i=1}^4 (dx_i) \int_0^\infty dt_4
$$

$$
to do the trivial integration over the sum of all \( t_i \), which gives

$$
I_5^{(a)}(\epsilon) = \int_0^\infty dt \ t^{-\epsilon/2} e^{-t} \int_0^1 \prod_{i=1}^4 (dx_i) \frac{x_2 x_3^2 x_4^2 P_{5}^{(a)}(x_1, x_2, x_3, x_4, \epsilon)}{(x_4(1 - x_4(1 - x_2 x_3(1 - x_2 x_3))))^{5+\epsilon/2}} \tag{69}
$$
We have to perform the limit in a way that the integration over \( t \) can be done using (D.4) and yields
\[
\Gamma(-\epsilon) = -\frac{1}{\epsilon} - \gamma + o(\epsilon).
\] (71)

We shall later find that the first term of the integrand in (68) after substituting according to (70) has only 1/\( \epsilon \) divergencies. This allows to extract the 1/\( \epsilon^2 \) divergencies of the integrals, which originate from the poles of this first term, without regarding corrections proportional to \( \epsilon \) from the second factor. This calculation is therefore able to reveal the leading order divergencies. It is completed in appendix \[\text{3}]. There we also present an algorithm which enables an exact computation of all the integrals considered. Both methods lead to consistent results that are independently obtained. Hence we are very confident about the correctness of our calculations.

Regarding figure 4 we notice that diagram (a) can have pinching limits contributing to diagram (c) if \( t_i \to 0 \) for \( i \in \{1, ..., 4\} \), while it reduces to diagram (e) when \( t_5 \to 0 \). Further we have to perform the limit in a way that \( t_1 \) and \( t_3 \to 0 \) or \( t_2 \) and \( t_4 \to 0 \) to end up with diagram (d). The choices \( t_1, t_2, t_3 \to 0 \) are obtained by assigning finite values to the sewing parameters \( A_i \) and after permuting the integration variables they are all of the form
\[
I^{(c)}_4(\epsilon) = \int \frac{dA}{A^2(1-A)^2} \int_0^\infty \prod_{i=1}^4 (dt_i) \frac{P_4^{(c)}(t_1, ..., t_4; A, \epsilon)}{((t_1 + t_2)(t_3 + t_4) + t_3t_4)^{4+\epsilon/2}} \times \exp \left( -\frac{P_3^{(c)}(t_1, ..., t_4)}{(t_1 + t_2)(t_3 + t_4) + t_3t_4} \right). \tag{72}
\]

The exponent and the denominator are simply got by setting the appropriate proper time variables in (65) to zero, whereas the numerator has to be computed separately by adding a power of \( \alpha' \) into the expansion of the integrand. The integration of those sewing variables that are left finite displays precisely the divergencies already encountered in the previous chapter when we discussed two-loop vacuum diagrams. We do not decide what to do with them at this stage but expand the integrand around \( A = 1 \) and keep the finite as well as the divergent term. We then compute the proper time integration and mark the respective terms of the expansion in \( A \) by their possibly divergent prefactors
\[
O_2 \equiv \int \frac{dA}{(1-A)^2}, \quad O_0 \equiv \int dA. \tag{73}
\]

We have for some cases even computed the integrations over proper times without doing any expansion of the integrand in the free moduli finding a more complicated structure of five instead of two terms but no deeper insight into the uncertainties of the pinching procedure. Therefore we restrict ourselves at this point to state the two types of terms we mentioned. The expression resulting from letting \( t_4 \to 0 \), which contributes to diagram (c), differs only in the respect, that keeping \( k_2/(A_1A_2A_3) \) finite we end up with the undefined integral \( dk_2/k_2^2 \), which we, too, already found in the two-loop vacuum case. Then, of course, also \( P_4^{(c)} \) depends on \( k_2 \) but is finite in the limit \( k_2 \to 1 \). We expand around \( k_2 = 1 \), mark the finite term by the factor
\[
O_1 \equiv \int dk_2 \quad \text{and} \quad O_0 \equiv \int dA. \tag{74}
\]
and omit terms of the order $o(k_2 - 1)$.

We are now only left with contributions to (d) and (e). The case $t_5 \to 0$ is completely analogous to the one mentioned previously with the only exception that the integration looks somewhat easier:

$$I^{(e)}_4(\epsilon) = \int \frac{dk_1}{k_1^2} \int_0^\infty \prod_{i=1}^4 (dt_i) \frac{P^{(e)}_4(t_1, \ldots, t_4, k_1, \epsilon)}{(t_1 + t_2)(t_3 + t_4)^{1+\epsilon/2}} \exp \left( -\frac{P^{(e)}_3(t_1, \ldots, t_4)}{(t_1 + t_2)(t_3 + t_4)} \right). \quad (75)$$

We shall explore this by factorizing the integrals. In the last case of diagram (d) we do not find any contributions at all, simply because there are two proper time variables that are absent. This leads to two factors of $\alpha'$ in front of the integral, which then vanishes proportional to the inverse string tension.

Having completed the study of contributions to diagram (a) and any diagrams including four-gluon vertices that arise from it, we now proceed to diagram (b) and its possible pinchings. We have to distinguish, which internal propagator the external states are sitting at, by regarding both of the two parametrizations drawn in figure 11 and 13.

\begin{align*}
\eta_2 = 0 & \quad \eta_1 = \eta \quad \xi_1 = 1 \quad \xi_2 = \infty \\
z_2 & \quad z_1 \quad \xi_1 \\
\eta_1 = \eta & \quad \xi_1 = 1 \quad \xi_2 = \infty \\
z_1 & \quad z_2 \quad \xi_2 \\
\eta_2 = 0 & \quad \eta_1 = \eta \quad \xi_1 = 1 \quad \xi_2 = \infty \\
z_2 & \quad z_1 \quad \xi_1 \\
\eta_1 = \eta & \quad \xi_1 = 1 \quad \xi_2 = \infty \\
z_1 & \quad z_2 \quad \xi_2
\end{align*}

Figure 11: Parametrization (i) of world sheet (b)

Figure 12: Sewing (i) of diagram (b)

Figure 13: Parametrization (ii) of the world sheet (b)

Figures 12 and 14 demonstrate how the left and right loop are to be distinguished. The sewing parameters of the configuration (i) are given by

$$\eta = A_1, \quad z_1 = A_1 A_2, \quad z_2 = A_1 A_2 A_3, \quad (76)$$

those of (ii) by

$$z_1 = A_1, \quad z_2 = A_1 A_2, \quad \eta = A_1 A_2 A_3. \quad (77)$$
For both cases all but one proper time variables can be defined in a unique manner

\[ t_i = -\alpha' \ln(A_i) \quad \text{for } i = 1, 2, 3, \]
\[ t_4 = -\alpha' \ln(k) - t_1 - t_2 - t_3, \]

but the one of the smaller loop is different:

\[ t_5 = -\alpha' \ln(k_1) - t_4 \quad \text{for (i)}, \]
\[ t_5 = -\alpha' \ln(k_1) - t_1 - t_2 - t_3 \quad \text{for (ii)}. \]

This makes a distinction in the translation of further logarithms of moduli appearing in the integrand of the amplitude necessary. Configuration (i) has to be used with

\[ \alpha' \ln(z_1 - z_2) = -t_1 - t_2 + \alpha' \ln(1 - A_3), \]
\[ \alpha' \ln(z_1 z_2) = -2t_1 - 2t_2 - t_3, \]
\[ \alpha' \ln \left( \frac{z_1}{z_2} \right) = t_3, \]
\[ \alpha' \ln \left( \frac{(z_1 - \eta)(z_2 - 1)}{(z_1 - 1)(z_2 - \eta)} \right) = \alpha' \ln \left( \frac{(1 - A)(1 - A_1 A_2 A_3)}{(1 - A_1)(1 - A_2 A_3)} \right), \]

whereas (ii) with:

\[ \alpha' \ln(z_1 - z_2) = -t_1 + \alpha' \ln(1 - A_2), \]
\[ \alpha' \ln(z_1 z_2) = -2t_1 - t_2, \]
\[ \alpha' \ln \left( \frac{z_1}{z_2} \right) = t_2; \]
\[ \alpha' \ln \left( \frac{(z_1 - \eta)(z_2 - 1)}{(z_1 - 1)(z_2 - \eta)} \right) = t_2 + \alpha' \ln \left( \frac{(1 - A_2 A_3)(1 - A_1 A_2)}{(1 - A_1)(1 - A_3)} \right). \]

Despite the variables \( t_2 \) and \( t_4 \) featuring independently in the case (i), the integrand will only depend on their sum \( t_2 + t_4 \), which is also true for \( t_1 \) and \( t_3 \) in the case of (ii). This reflects that the two propagators parametrized by these two proper times carry the same field theoretical momentum. By doing the usual expansion of the integrand we then obtain the amplitude

\[
A^{(2)}(\rho^2) \big|_{(b)} = N^2 \delta_{\alpha_1, \alpha_2} \frac{g^4}{(4\pi)^d} \int_0^\infty \prod_{i=1}^5 (dt_i) \times \left( (\epsilon_1 p_2)(\epsilon_2 p_1)P^{(b)}_1(t_1, ..., t_5, d) + \epsilon^2 P^{(b)}_1(t_1, ..., t_5, d) \right) \times \exp \left( -p^2 \frac{t_3(t_1 + t_5)(t_2 + t_3 + t_4) + t_1 t_5}{(t_1 + t_5)(t_2 + t_3 + t_4) + t_1 t_5} \right) \times \left( \epsilon_1 p_2)(\epsilon_2 p_1)I^{(b)}_1(c) + (\epsilon^2 p^2) I^{(b)}_2(c) \right). \]
The sewing configurations (i) und (ii) lead to different explicit polynomials in the numerator but all are of identical structure, so that we do not have to treat them separately in this general discussion. We now follow the same strategies as earlier and cut our arguments short accordingly. From the figures 12 and 14 one immediately notices all relevant pinching contributions. Having \( t_2 \to 0 \) or \( t_4 \to 0 \) in region (i) we get diagram (c), if both vanish (d). The same is true in region (ii) for \( t_1 \) and \( t_3 \). The moduli divergencies we find are identical to those found in the pinching contributions derived from diagram (a). We use again for vanishing \( t_1 \), \( t_2 \) or \( t_3 \) the prefactors \( O_0 \) and \( O_2 \) for the divergency and if \( t_4 \) is small we have \( O_1 \). A contribution to diagram (e) does not exist and contributions to (d) go to zero proportional to \( \alpha' \) again.

We have finally got all possible contributions to field theoretical Feynman diagrams that can be extracted from the appropriate string amplitude given a particular choice of local coordinates and fixed points. The results of the calculation of the integrals are summarized in table 2 and 3, while the details of the integration are postponed to appendix B.

### Table 2: Contributions to Feynman diagrams derived from (a)

| Diagram | Pinching | Order \( \epsilon^{-2} \) | Order \( \epsilon^{-1} \) |
|---------|----------|-----------------|-----------------|
| (a)     |          | \( \frac{35}{12} \)  | \( \frac{335}{24} \)  |
| (c)     | \( t_1 \to 0 \) | \( O_2 \frac{1}{6} + O_0 \frac{5}{16} \) | \( O_2 \frac{5}{6} + O_0 \frac{37}{32} \) |
| (c)     | \( t_2 \to 0 \) | \( O_2 \frac{1}{6} + O_0 \frac{5}{16} \) | \( O_2 \frac{5}{6} + O_0 \frac{37}{32} \) |
| (c)     | \( t_3 \to 0 \) | \( O_2 \frac{5}{24} + O_0 \frac{7}{16} \) | \( O_2 \frac{7}{24} + O_0 \frac{25}{32} \) |
| (c)     | \( t_4 \to 0 \) | \( O_1 \frac{25}{48} \) | \( O_1 \frac{205}{96} \) |
| (e)     | \( t_5 \to 0 \) | 0 | 0 |
| (d)     | \( t_1, t_3 \to 0 \) | 0 | 0 |
| (d)     | \( t_2, t_4 \to 0 \) | 0 | 0 |

### Table 3: Contributions to Feynman diagrams derived from (b)

| Diagram | Parametrization | Pinching | Order \( \epsilon^{-2} \) | Order \( \epsilon^{-1} \) |
|---------|-----------------|----------|-----------------|-----------------|
| (b)     | (i)             |          | \( \frac{11}{6} \)  | \( \frac{127}{18} \)  |
| (c)     | (i)             | \( t_2 \to 0 \) | \( O_2 \frac{1}{6} + O_0 \frac{5}{16} \) | \( O_2 \frac{5}{6} + O_0 \frac{37}{32} \) |
| (c)     | (i)             | \( t_4 \to 0 \) | \( O_1 \frac{7}{24} \) | \( O_1 \frac{55}{48} \) |
| (d)     | (i)             | \( t_2, t_4 \to 0 \) | 0 | 0 |
| (b)     | (ii)            |          | 0 | \( -\frac{14}{9} \) |
| (c)     | (ii)            | \( t_1 \to 0 \) | \( O_2 \frac{5}{24} + O_0 \frac{7}{16} \) | \( O_2 \frac{7}{24} + O_0 \frac{47}{48} \) |
| (c)     | (ii)            | \( t_3 \to 0 \) | \( O_2 \frac{5}{24} + O_0 \frac{7}{16} \) | \( O_2 \frac{7}{24} + O_0 \frac{55}{48} \) |
| (d)     | (ii)            | \( t_1, t_3 \to 0 \) | 0 | 0 |

All terms proportional to the Euler constant or \( \ln (4\pi/p^2) \) are dropped as they come out
correctly automatically and can be restored easily. We have also changed our conventions in the favour of using \(d = 4 - 2\epsilon\) in order to compare results to the field theory. The given numbers still leave a lot of space to possible interpretations. Even only regarding the contributions to diagram (a) from figure 10 reveal that there is no choice of the field theoretical gauge parameter, which lets the results coincide with the sum of the diagrams (i) + (j) + (k) of the same topology from table 3, if one also demands the sum of the two parametrizations (i) and (ii) of diagram (b) to coincide with (a) and (b) from table 3. An identification diagram by diagram seems to be ruled out therefore. The next would be to try to compare the sum of all diagrams. Unambiguously are

\[
\left(\frac{35}{12} + \frac{11}{6}\right)\epsilon^{-2} + \left(\frac{335}{24} + \frac{127}{18} - \frac{14}{9}\right)\epsilon^{-1} = \frac{19}{4}\epsilon^{-2} + \frac{467}{24}\epsilon^{-1}
\]  

(83)

from diagram (a) and (b) without any four-gluon vertices, while the diagrams including such give

\[
\left(3\left(O_2\frac{9}{24} + O_0\frac{12}{16}\right) + O_1\frac{9}{48}\right)\epsilon^{-2} + \left(3\left(O_2\frac{87}{48} + O_0\frac{92}{32}\right) + O_1\frac{315}{96}\right)\epsilon^{-1}.
\]  

(84)

They depend on the undefined diverging integrals over the free moduli, which should be replaced by some finite number in the manner of a regularization prescription. The most simple criterion for consistency could be seen to be the vanishing of the leading order of the expansion in \(1/\epsilon\). As we deal with two unknown parameters this cannot give a unique answer and the general simplicity of the occurring numbers, which on the other hand might seem encouraging, allows more speculation about such a regularization prescription than we dare to present in this spot. For instance, using only the quadratic divergency \(O_2\) and dropping \(O_0\) completely, then demanding the \(1/\epsilon^2\) term to vanish, one finds \((7/2)\epsilon^{-1}\). A surprisingly simple number, but it is simply wrong, neither is such a kind of discussion sufficiently rigorous in any manner.

There are a couple of possibly sensible modifications of the general expansion method we used, that may have passed unnoticed in scalar theory and the one-loop Yang-Mills computation. One can for instance think of the ambiguity concerning which loop an external particle sits at and demand to add up all possibilities to attach a given number of external states to a diagram, by distinguishing all the different loops, even if the topology of the diagram created is identical. Following [8] this would mean that one had to choose different local coordinates in the vicinity if the external states

\[
V'_i(0) = \left|\frac{(z_i - \eta_1)(z_i - \xi_1)}{\eta_1 - \xi_1}\right| = \left|\frac{(z_i - \eta)(z_i - 1)}{\eta - 1}\right|.
\]  

(85)

If we omit reducible diagrams, whose identification has been demonstrated in 14, the existing possibilities are listed diagrammatically in appendix C, where we also give the appropriate sewing translation and the results that are computed by solving the integrals which are found after expanding in all the relevant parameters. We have not displayed the pinching contributions which can be derived from these diagrams. The whole situation seems far too involved to allow any conclusion concerning the related field theoretical gauge choice and its relation to the coordinate fixing on the world sheet of the string, as the divergencies that are appearing when performing the pinching of these kind of diagrams are of the form

\[
\int \frac{1}{A^3(1 - A)} \quad \text{and} \quad \int \frac{1}{A(1 - A)^3}.
\]  

(86)

which is different from that already encountered. We can neither draw any new conclusion with respect to the values of the ambiguous integrals.

At the moment we are therefore unable to give any definite result.
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A The Schottky representation of Riemann surfaces

The most important issues about the Schottky group and the representation of Riemann surfaces by its means are given in [28, 29], while the method used to expand geometrical quantities in the Schottky multipliers is not published in the literature. We follow [30] in most respects. Some more general useful facts about abelian integrals and the prime form are detailed in [31].

A.1 The Schottky group

The convenient parametrization of automorphisms \( t(z) \) of the compactified complex plane is written

\[
t(z) = \frac{az + b}{cz + d} \quad \text{with} \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc = 1.
\]  

(A.1)

The matrices

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

with \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \) (A.2)

are in \( SL(2, \mathbb{C}) \), which is even isomorphic to the automorphism group. The Schottky parameters are then defined by

\[
\frac{t(z) - \xi}{t(z) - \eta} = k \frac{z - \xi}{z - \eta},
\]

(A.3)

where \( \xi \) and \( \eta \) are the fixed points of the mapping \( t(z) \), \( k \) its multiplier. The invariance under \( \eta \leftrightarrow \xi \) and \( k \leftrightarrow k^{-1} \) allows to choose \( |k| \leq 1 \). The relation to the former parametrization is found to be

\[
t(z) = \frac{\eta(z - \xi) - k\xi(z - \eta)}{(z - \xi) - k(z - \eta)}.
\]

(A.4)

And we read off:

\[
\begin{align*}
a &= \frac{\eta - k\xi}{\sqrt{|k||\eta - \xi|}}, & b &= \frac{-\eta(1 - k)}{\sqrt{|k||\eta - \xi|}}, \\
c &= \frac{1 - k}{\sqrt{|k||\eta - \xi|}}, & d &= \frac{k\eta - \xi}{\sqrt{|k||\eta - \xi|}}.
\end{align*}
\]

The Schottky group \( G_S^g \) is defined by composition and inversion of a given set of generators \( \{t_1, \ldots, t_g\} \). It therefore consists of all mappings of the form \( t_1^{i_1}t_2^{i_2} \cdots t_m^{i_m} \), having indices in \( \{1, \ldots, g\} \) and integer exponents.

Projective mappings transform circles into circles and for each there is a particular pair of isometric circles which have identical radii. The interior of the original circle is mapped onto the exterior of its image and the exterior of the origin onto the interior of its image. This induces a one to one identification of points and a fundamental region can be chosen to be the exterior of both circles which are identified. This construction of equivalence classes is a manifold that has the topology of a Riemann surface \( \Sigma_g \) of Genus \( g \), if this is the number of identified, disjoint circles. For technical reasons one excludes the set of points \( \mathcal{D} \), where the orbits of the Schottky group become dense:

\[
\{C \cup \{\infty\}\}/G_S - \mathcal{D} \simeq \Sigma_g
\]

(A.5)
Figure 15: The non contractable cycles on a Torus

The attractive and repulsive fixed points then lie inside the original, respectively the image isometric circle. Their radii \( r \) and \( \bar{r} \) and the coordinates of their centres are:

\[
\begin{align*}
  r &= \bar{r} = \sqrt{|k| |\eta - \xi|/|1 - k|}, \\
  c &= \frac{\xi - k\eta}{1 - k}, \quad \bar{c} = \frac{\eta - k\xi}{1 - k}.
\end{align*}
\]

The dual of the homology basis of non contractable cycles is the cohomology basis in the sense of de Rham cohomology. It consists of the first abelian differentials

\[
\omega_\mu \equiv \sum_{G(2,k)} \left( \frac{1}{z - t(\eta_\mu)} - \frac{1}{z - t(\xi_\mu)} \right) dz.
\]

The sum is performed over all Schottky mappings that do not carry a power of \( t_\mu \) on their right. The \( \omega_\mu \) are analytic on all of \( \Sigma_g \). One further has to demand the normalization of the integrals

\[
\oint_{a_\nu} \omega_\mu = 2\pi i \delta_{\mu\nu} \tag{A.8}
\]

along the paths \( a_\nu \), that constitute one half of the homology basis to render the \( \omega_\mu \) unique. The other half of the integrals

\[
\oint_{b_\nu} \omega_\mu \equiv 2\pi i \tau_{\mu\nu} \tag{A.9}
\]

are the entries of the period matrix \( \tau_{\mu\nu} \) of the Riemann surface. Another geometrical function we shall need is the prime form which is characterized by its local behaviour

\[
E(z_i, z_j) \to (z_i - z_j), \quad \text{for } z_i \to z_j \tag{A.10}
\]

and its transformation property under conformal changes of coordinates

\[
E(V_i(z_i), V_j(z_j)) = \frac{E(z_i, z_j)}{\sqrt{V_i^{1/2}(z_i)V_j^{1/2}(z_j)}}. \tag{A.11}
\]

The definition is unique, if also the invariance under transport around \( a_\nu \) cycles and a phase factor for \( b_\nu \) cycles is demanded. One can always choose coordinates so that all the fixed points and the centre coordinates of isometric circles are lying on the real axis, which greatly simplifies all figures we display. The coordinate invariance of string theory further allows to fix three of the
moduli of the world sheet by global diffeomorphisms, which is conveniently employed to fix one at 0, another at \( \infty \) and finally one at 1.

The two generalizations of the representation of Riemann surfaces one eventually needs in string theory are those to surfaces with boundaries and to surfaces that are not orientable. The former type of world sheets for open string theories is obtained by cutting holes into the complex plane, the closed string worldsheet. If one exclusively has to deal with open strings it is convenient to take only half of the complex plane and choose all the fixed points of the Schottky group that generates the desired loops as well as the insertion points of the external states to lie on its boundary, taken to be the real axis. Passing from points to equivalence classes then includes identifying \( 2g \) semi-circles whose centre coordinates are real.

Non-orientable manifolds are constructed by including inversions into the generators of the Schottky group, which results in inserting projective planes \( \mathbb{RP}^2 \) into the world sheet.

### A.2 Expanding in Schottky multipliers

We now demonstrate the techniques which allow to expand the geometrical quantities involved in the two-loop string amplitude into power series in the two Schottky multipliers \( k_1 \) and \( k_2 \), which then enables us to extract the relevant part of the amplitude. It is necessary to identify all terms up to the order \( k_1^2 k_2^2 \), to cancel the prefactors coming from the integration measure of the amplitude.

We exploit the isomorphism from the projective group onto \( SL(2, \mathbb{C}) \) and use the basis invariance of trace and determinant. One can read off the matrix \( T(z) \) corresponding to a Schottky generator \( t(z) \) from (A.4)

\[
T(z) = \frac{1}{\sqrt{|k||\eta - \xi|}} \begin{pmatrix} \xi - k\eta & -\eta \xi (1 - k) \\ 1 - k & -\eta + k\xi \end{pmatrix}
\]

and gets:

\[
\frac{\det(T(z))}{(\text{tr}(T(z)))^2} = \frac{k(\eta - \xi)^2}{-(1 + k)(\eta - \xi))^2} = \frac{k}{(1 + k)^2} = k + o(k^2).
\]

This gives a first order expression for the multiplier \( k(t_1(z)t_2(z)) \) of \( t(z) = t_1(z)t_2(z) \):

\[
k(t_1(z)t_2(z)) = \frac{\det(T_1(z)T_2(z))}{(\text{tr}(T_1(z)T_2(z)))^2} + o(k^2(t_1(z)), k^2(t_2(z)))
\]

\[
= \frac{\det(T_1(z))\det(T_2(z))}{(\text{tr}(T_1(z)T_2(z)))^2} + o(k^2(t_1(z)), k^2(t_2(z)))
\]
A useful asymptotic expression for the mapping itself is

\[ k(t_1(z))k(t_2(z)) \frac{(\eta_1 - \xi_1)^2(\eta_2 - \xi_2)^2}{(\xi_1 - \eta_1)^2(\xi_2 - \eta_2)^2} + o(k^2(t_1(z)), k^2(t_2(z))), \]

where we had to use

\[ \text{tr}(T_1(z)T_2(z)) = \frac{1}{|k_1||k_2||\eta_1 - \xi_1||\eta_2 - \xi_2|} ((\xi_1 - \eta_2)(\xi_2 - \eta_1) + o(k(t_1(z)), k(t_2(z)))) \]  \hspace{0.5cm} (A.15)

We simplify our notation and write \( k_t \equiv k(t(z)) \), dropping the argument \( z \). Multipliers of mappings of the kind \( t_1t_2^{-1} \) are obtained by replacing \( \eta_2 \leftrightarrow \xi_2 \) in (A.14):

\[ k(t_1t_2^{-1}) = k_1k_2 \frac{(\eta_1 - \xi_1)^2(\eta_2 - \xi_2)^2}{(\xi_1 - \eta_1)^2(\eta_2 - \xi_2)^2} + o(k_1^2, k_2^2). \]  \hspace{0.5cm} (A.16)

In fact, the two cases we have computed are already covering all types of multipliers needed. The symmetries of (A.14) and (A.15) show that

\[ k(t_1t_2) = k(t_2t_1) = k(t_1^{-1}t_2^{-1}) = k(t_2^{-1}t_1^{-1}), \]

\[ k(t_1^{-1}t_2) = k(t_1t_2^{-1}) = k(t_2^{-1}t_1) = k(t_2t_1^{-1}) \]  \hspace{0.5cm} (A.17)

holds. One easily deduces the complete set of Schottky mappings that might have multipliers which are only of first order in both the multipliers of the two generators to be the subset

\[ G_S^2 \supset G_S^{(2)} \equiv \{ \text{id}, t_1, t_1^{-1}, t_2, t_2^{-1}, t_1t_2, t_2t_1, t_2^{-1}t_1, t_1^{-1}t_2, t_2t_1^{-1}, t_1t_2^{-1}, t_2^{-1}t_1^{-1}, t_2^{-1}t_1t_2^{-1}, t_2^2t_1^{-1}, t_2^{-1}t_1^2, t_2^{-1}t_1^2 \} \]

A useful asymptotic expression for the mapping itself is

\[ t(z) = \xi + k(\eta - \xi) \frac{z - \xi}{z - \eta} + o(k^2), \]  \hspace{0.5cm} (A.18)

which is obtained from (A.4). Because of \(|k| \leq 1\) this explains the asymptotic properties of a Schottky map:

\[ \lim_{n \to \infty} t^n(z) = \xi, \quad \lim_{n \to \infty} t^{-n}(z) = \eta. \]  \hspace{0.5cm} (A.19)

We shall also need

\[ \frac{(z_1 - t(w_1))(z_2 - t(w_2))}{(z_1 - t(w_2))(z_2 - t(w_1))} = 1 + k \frac{(\xi - \eta)^2(z_1 - z_2)(w_1 - w_2)}{(z_1 - \xi)(z_2 - \xi)(w_1 - \eta)(w_2 - \eta)} + o(k^2). \]

Now we can expand the period matrix, the abelian integrals, the normalization constant \( N \) of the partition function and the prime form to first order. An explicit expression for the prime form can easily be found from the definition given in the previous section:

\[ E(z_1, z_2) \equiv (z_1 - z_2) \prod_{t \in G_S \setminus \{\text{id}\}} \sqrt{\frac{(z_1 - t(z_2))(z_2 - t(z_1))}{(z_1 - t(z_1))(z_2 - t(z_2))}} \]  \hspace{0.5cm} (A.20)

which has the expansion

\[ E^{(2)}(z_1, z_2) = (z_1 - z_2) \prod_{t \in G_S^{(2)} \setminus \{\text{id}\}} \sqrt{\frac{(z_1 - t(z_2))(z_2 - t(z_1))}{(z_1 - t(z_1))(z_2 - t(z_2))}} + o(k_1^2, k_2^2) \]  \hspace{0.5cm} (A.21)

\[ = (z_1 - z_2) \left( 1 - k_1 \frac{(\eta_1 - \xi_1)^2(z_1 - z_2)^2}{(z_1 - \eta_1)(z_2 - \eta_1)(z_1 - \xi_1)(z_2 - \xi_1)} + o(k_1^2, k_2^2) \right) \]

\[ \times \left( 1 - k_2 \frac{(\eta_2 - \xi_2)^2(z_1 - z_2)^2}{(z_1 - \eta_2)(z_2 - \eta_2)(z_1 - \xi_2)(z_2 - \xi_2)} + o(k_1^2, k_2^2) \right) \]
The integrals over the abelian differentials can be explicitly computed elementarily. They are

\[ \times \left( 1 - k(t, t^{-1}) t^2 \right) \frac{(\eta_2 - \xi_1)^2 (z_1 - z_2)^2}{(z_1 - \eta_2)(z_2 - \eta_2)(z_1 - \xi_1)(z_2 - \xi_1)} \]

\[ \times \left( 1 - k(t, t^{-1}) t^2 \right) \frac{(\eta_2 - \xi_2)^2 (z_1 - z_2)^2}{(z_1 - \eta_2)(z_2 - \eta_2)(z_1 - \xi_2)(z_2 - \xi_2)} \]

\[ \times \left( 1 - k(t, t^{-1}) t^2 \right) \frac{(\xi_2 - \xi_1)^2 (z_1 - z_2)^2}{(z_1 - \eta_2)(z_2 - \eta_2)(z_1 - \eta_2)(z_2 - \eta_1)} + o(k_1^2, k_2^2). \]

The normalization constant of the partition function is being combined with two factors from the integration measure

\[ \mathcal{N}(1 - k_1)^2 (1 - k_2)^2 \equiv \prod_{\beta} \left( \prod_{n=1}^{\infty} (1 - k_\beta^n)^{-d} \prod_{n=2}^{\infty} (1 - k_\beta^n)^2 \right) (1 - k_1)^2 (1 - k_2)^2, \quad (A.22) \]

where the product over \( \beta \) extends over the primary classes of the Schottky group, which are equivalence classes under conjugation. Those which contribute to the required order may be represented by \( G^\text{pr}_S \equiv \{ t_1, t_2, t_1 t_2, t_1^{-1} t_2, t_1 t_2^{-1} \} \). The expansion reads:

\[ \mathcal{N}^{(2)}(1 - k_1)^2 (1 - k_2)^2 = \prod_{\beta \in G^\text{pr}_S} \left( \prod_{n=1}^{\infty} (1 - k_\beta^n)^{-d} \prod_{n=2}^{\infty} (1 - k_\beta^n)^2 \right) (1 - k_1)^2 (1 - k_2)^2 + o(k_1^2, k_2^2) \]

\[ = 1 + (d - 2)(k_1 + k_2) + \left( d - 2 \right) + d \left( \frac{(\eta_1 - \xi_1)^2 (\eta_2 - \xi_2)^2}{(\eta_1 - \eta_2)^2 (\xi_2 - \eta_2)^2} \right) k_1 k_2 + o(k_1^2, k_2^2). \quad (A.23) \]

The integrals over the abelian differentials can be explicitly computed elementarily. They are then defined by sums over logarithms

\[ \int_{z_1}^{z_2} \omega^\mu = \sum_{t \in G_{S}(0, \mu)} \ln \left( \frac{(z_2 - t(\xi_\mu))(z_1 - t(\eta_\mu))}{(z_1 - t(\xi_\mu))(z_2 - t(\eta_\mu))} \right), \quad (A.24) \]

where the sum is performed over elements of \( G_{S}^{(2)} \) which do not have a power of \( t_\mu \) to the right, e.g. for \( \mu = 1 \) just over \( \{ id, t_2, t_2^{-1}, t_1 t_2, t_1^{-1} t_2, t_1 t_2^{-1}, t_1^{-1} t_2^2 \} \). The expansion is:

\[ \exp \left( \int_{z_1}^{z_2} \omega^\mu \right)^{(2)} = \exp \left( \sum_{t \in G_{S}(0, 1)} \ln \left( \frac{(z_2 - t(\xi))(z_1 - t(\eta))}{(z_1 - t(\xi))(z_2 - t(\eta))} \right) \right) + o(k_1^2, k_2^2) \quad (A.25) \]

\[ = \frac{(z_2 - \xi_1)(z_1 - \eta_1)}{(z_2 - \eta_1)(z_1 - \xi_1)} \left( 1 - k_2 \frac{(\eta_2 - \xi_2)^2 (\xi_1 - \eta_1)(z_1 - z_2)}{(\eta_2 - \xi_1)(\eta_2 - \eta_1)(z_1 - \xi_2)(z_2 - \xi_2)} \right) \]

\[ \times \left( 1 - k(t, t^{-1}) t^2 \right) \frac{(\eta_2 - \xi_1)^2 (\xi_1 - \eta_1)(z_1 - z_2)}{(\xi_2 - \xi_1)(\xi_2 - \eta_1)(z_1 - \xi_2)(z_2 - \xi_2)} \]

\[ \times \left( 1 - k(t, t^{-1}) t^2 \right) \frac{(\xi_2 - \xi_1)^2 (\xi_1 - \eta_1)(z_1 - z_2)}{(\xi_2 - \xi_1)(\xi_2 - \eta_1)(z_1 - \xi_2)(z_2 - \xi_2)} \]

\[ \times \left( 1 - k(t, t^{-1}) t^2 \right) \frac{(\eta_2 - \eta_1)^2 (\xi_1 - \eta_1)(z_1 - z_2)}{(\eta_2 - \xi_1)(\eta_2 - \eta_1)(z_1 - \eta_2)(z_2 - \eta_1)} + o(k_1^2, k_2^2), \]
We have found the term proportional to \( \epsilon \) in (67) up to powers of \( \epsilon \). This procedure is nothing but the usual one.

\[
\exp \left( \int_{z_1}^{z_2} \omega^2 \right)^{(2)} = \exp \left( \sum_{t \in G_S^{(2)} \setminus \{0,2\}} \ln \left( \frac{(z_2 - t(\xi_2))(z_1 - t(\eta_2))}{(z_1 - t(\xi_2))(z_2 - t(\eta_2))} \right) \right) + o(k_1^2, k_2^2) \quad \text{(A.26)}
\]

The period matrix looks explicitly

\[
2\pi i \tau_{\mu\nu} \equiv \delta_{\mu\nu} \ln(k_\mu) + \sum_{t \in G_S(\mu, \nu)} \ln \left( \frac{(\eta_\mu - t(\eta_\nu))(\xi_\mu - t(\xi_\nu))}{(\eta_\mu - t(\xi_\nu))(\xi_\mu - t(\eta_\nu))} \right) \quad \text{(A.27)}
\]

with summation over elements of \( G_S^{(2)} \) that neither have a power of \( t_\mu \) to the left nor one of \( t_\nu \) to the right, for diagonal elements of \( \tau_{\mu\nu} \) also id is excluded. For instance for \( \mu = 1, \nu = 1 \) we only have to sum over \( \{t_2, t_2^{-1}\} \) while \( \mu = 1, \nu = 2 \) just allows \( \{id, t_2t_1, t_2^{-1}t_1, t_2^{-1}t_1^{-1}\} \). The expansion is obtained to be

\[
2\pi i \tau_{11}^{(2)} = \ln(k_1) + \sum_{t \in G_S^{(2)} \setminus \{1, 1\}} \ln \left( \frac{(\eta_1 - t(\eta_1))(\xi_1 - t(\xi_1))}{(\eta_1 - t(\xi_1))(\xi_1 - t(\eta_1))} \right) + o(k_1^2, k_2^2) \]

\[
= \ln(k_1) + 2 \frac{(\xi_1 - \eta_1)^2}{(\eta_1 - \xi_2)(\eta_1 - \eta_2)(\eta_1 - \xi_1)} + o(k_1^2, k_2^2),
\]

\[
2\pi i \tau_{22}^{(2)} = (1 \leftrightarrow 2),
\]

\[
2\pi i \tau_{12}^{(2)} = 2\pi i \tau_{21}^{(2)} = \sum_{t \in G_S^{(2)} \setminus \{1, 2\}} \ln \left( \frac{(\eta_1 - t(\eta_2))(\xi_1 - t(\xi_2))}{(\eta_1 - t(\xi_2))(\xi_1 - t(\eta_2))} \right) + o(k_1^2, k_2^2)
\]

\[
= \ln \left( \frac{(\eta_1 - \eta_2)(\xi_1 - \xi_2)}{(\eta_1 - \xi_2)(\xi_1 - \eta_2)} \right) + 2k(t_1 t_2) \frac{(\eta_1 - \xi_1)(\xi_2 - \eta_2)}{(\eta_1 - \eta_2)(\xi_1 - \xi_2)} + 2k(t_1 t_2^{-1}) \frac{(\eta_1 - \xi_1)(\xi_2 - \eta_2)}{(\xi_1 - \eta_2)(\eta_1 - \xi_2)} + o(k_1^2, k_2^2). \quad \text{(A.28)}
\]

In all the given expressions we still have got the freedom to fix three of the four fixed points to arbitrary values and will conveniently choose these to be 0, 1, \( \infty \).

B Computation of SPT integrals

In this appendix we explicitly compute the integrals over proper time variables we have obtained in chapter 4.4 by performing the low energy limit of the two-loop string diagram extracting the gluon contribution. We use the conventional dimensional regularization scheme in \( d = 4 + \epsilon \) dimensions. It is usually not necessary to distinguish between IR and UV by using different \( \epsilon \) parameter as long as one can be sure that the overall amplitude is IR finite, which is obvious from \( \text(M) \). For computing the integrals we therefore freely use the formulas \( \text(D.1), \text(D.2) \) and \( \text(D.3) \) by analytic extension outside the region, where the integrals on the left hand sides of these equations converge. We then finally expand the result in powers of \( \epsilon \) and extract the minimal subtraction terms. This procedure is nothing but the usual one.

B.1 Computation of the leading divergence

We have found the term proportional to \( (e^p)(ep) \) in \( \text[(M)]{\text{(M)}} \) up to prefactors of the kind

\[
\int_0^\infty \prod_{i=1}^{5-v} (dt_i) \left( \frac{\text{Polynomial of (5-v)th degree in } t_i}{\text{Polynomial of second degree in } t_i} \right)^{5-v+\epsilon/2} \quad \text{(B.1)}
\]

\[\times \exp \left( \frac{\text{Polynomial of third degree in } t_i}{\text{Polynomial of second degree } t_i} \right),\]
if \( v = 0, 1 \) is the number of four gluon vertices we consider. The term proportional to \( \epsilon^2 p^2 \) can be computed in completely the same manner and for the sake of brevity we concentrate on the one given above. If we are only interested in the leading \( 1/\epsilon^2 \) divergence, we can meanwhile omit the exponential from the integrand of (B.3) and only after performing all but the last integration use it as an IR regulator. We have already done the substitution that lead to (69) and justified to replace the second term deriving from the exponent as in (70) by \( 1 + o(\epsilon) \). Then reverting the first substitution we get (67) back up to the simplified exponent:

\[
\int_0^\infty \prod_{i=1}^{5-v} (dt_i) \frac{\text{Polynomial of } (5-v)\text{th degree in } t_i}{\text{Polynomial of second degree in } t_i} \exp \left( -\sum_{i=1}^{5} t_i \right).
\]

We actually need this integral only in an infinitesimal vicinity of the origin, where we are allowed to expand the integrand because of continuity. If we take care of not getting any divergencies from large proper times we can finally extend the integration of the modified integrand to the full interval from zero to infinity again. We had to check that all the conditions mentioned are satisfied holding in all the cases we shall be regarding in the following. As the integrals obtained by these means can then be computed, we get a result for the \( 1/\epsilon^2 \) divergent term that can be compared to results we shall get later on by more sophisticated methods that are less transparent on the other hand. We write the integrals thus obtained in the following manner

\[
J_3(\epsilon) = \int_0^\infty \prod_{i=1}^{5} (dt_i) \frac{P_5(t_1, ..., t_5, \epsilon)}{(P_2(t_1, t_2, t_3, t_4, t_5))^{5+\epsilon/2}}.
\]

In this subsection we use the notation \( J_{5-v}^{(x)}|_{i...}(\epsilon) \) for these integrals without the exponential factor in the integrand, using the lower index for the number \( 5-v \) of integrations and the additional ones \( i... \) for the possibly missing proper time variables, as well as an additional letter \( (x) \) for the according diagram of figure 10, sticking strictly to the notation introduced in (18) and (78). In the end we shall reintroduce the exponential as cut-off without change of notation for the integral. The polynomials are carrying indices signaling their degree.

The diagram (a) has the contribution given in (63) which has a polynomial

\[
P_2^{(a)}(t_1, ..., t_5) = \alpha^2 \left( \ln(k_1) \ln(k_2) - \ln^2(\eta) \right)
= (t_1 + t_2)(t_3 + t_4 + t_5) + (t_3 + t_4)t_5
= a(t_1 + t_2) + b
\]

in the denominator. For the polynomial \( P_5^{(a)} \) in the numerator we use

\[
P_5^{(a)}(t_1, ..., t_5, \epsilon) = \sum_{i=0}^{3} c_i t_1^i = \sum_{i=0}^{3} \sum_{j=0}^{3-i} c_{ij} t_1^i t_2^j = \sum_{i=0}^{3} \sum_{j=0}^{3-i} \sum_{k=0}^{3-j} c_{ijk} t_1^i t_2^j t_3^k
\]

The summation limits are to be understood in the sense that \( 0, 2 - i - j - k \) stands for the larger value of 0 or \( 2 - i - j - k \), alternatively, if upper limits are double valued, they stand for the lower one. We now use for the integrations over \( t_1 \) and \( t_2 \) the formula (D.1), for that over \( t_3 \) we have (D.2) and finally for \( t_4 \) it is (D.3):

\[
J_5^{(a)}(\epsilon) = \int_0^\infty \prod_{i=1}^{5} (dt_i) \frac{\sum_{i,j} c_{ij} t_1^i}{(a(t_1 + t_2) + b)^{5+\epsilon/2}}
\]

\[
= \int_0^\infty \prod_{i=2}^{5} (dt_i) \sum_{ij} c_{ij} t_2^j (a(t_2 + b))^{i-4-\epsilon/2} a^{-i-1} B(i + 1, 4 - i + \epsilon/2)
\]
For doing the $t_4$ integration we had to substitute

$$t_4 = \frac{(1-x)t_5}{x}, \quad dt_4 = -\frac{t_5dx}{x^2}, \quad [0, \infty] \to [1, 0]. \quad (B.7)$$

The cut-off integral is

$$\int_0^\infty dt_5 \, t_5^{-1-\epsilon} e^{-t_5} = \Gamma(-\epsilon) = -\frac{1}{\epsilon} - \gamma + o(\epsilon). \quad (B.8)$$

To compute the integrals derived from (a) by pinching is even easier because of their polynomials being of lower degree. For $t_1 \to 0$ we get a contribution to diagram (c), having the polynomial

$$P_2^{(c)}(t_2, t_3, t_4, t_5) = t_2(t_3 + t_4 + t_5) + (t_3 + t_4)t_5 \quad (B.9)$$

in the denominator and

$$P_4^{(c)}(t_2, t_3, t_4, t_5, \epsilon) = \sum_{i=0}^2 c_i t_2^i = \sum_{i=0}^1 \sum_{j=0}^1 c_{ij} t_2^i t_3^j = \sum_{i=0}^2 \sum_{j=0}^1 \sum_{k=0}^{2-i-j} c_{ijk} t_2^i t_3^j t_4^k \quad (B.10)$$

in the numerator, where all coefficients whose indices do not satisfy $j + k \leq 2$ vanish. The integrals are solved by (D.1), (D.2) and (D.3) using again the substitution (B.7).

$$J_4^{(c)} = \int_0^\infty \prod_{i=2}^5 dt_i \, \frac{\sum c_i t_2^i}{(at_2 + b)^{4+\epsilon/2}} \quad (B.11)$$

$$= \int_0^\infty dt_3 dt_4 dt_5 \sum_{ijk} c_{ijk} \, t_3^i \, b^{i-j} \, a^{i-1} \, B(i + 1, 3 - i + \epsilon/2)$$

$$= \int_0^\infty dt_4 dt_5 \sum_{ijk} c_{ijk} \, t_4^{i+j+k-2-\epsilon/2} \, t_5^{i-j} \, (t_4 + t_5)^{-i-1}$$

$$\times B(i + 1, 3 - i + \epsilon/2) \, B(j + 1, 3 - j + \epsilon/2)$$

$$\times 2F_1(i + 1, j + 1; 4 + \epsilon/2; t_5/(t_4 + t_5))$$

$$= \int_0^\infty dt_5 \sum_{ijk} c_{ijk} \, t_5^{i+j+k-5-\epsilon} \, B(i + 1, 3 - i + \epsilon/2) \, B(j + 1, 3 - j + \epsilon/2)$$

$$\times B(i + j + k - 1 - \epsilon/2, 2 - j - k + \epsilon/2)$$

$$\times 3F_2(i + 1, j + 1, 2 - j - k + \epsilon/2; 4 + \epsilon/2, i + 1; 1).$$
The $t_5$ integration again has to be treated by the cut-off prescription as in (B.8). The integral deduced from $t_2 \to 0$ has identical structure as this one with $t_1 \to 0$, while $t_3 \to 0$ can be handled even more simply. This then is equivalent to $t_4 \to 0$ again. We get in both cases

$$P_{2}^{(c)}(t_1, t_2, t_4, t_5) = (t_1 + t_2)(t_4 + t_5) + t_4t_5$$

\[= a(t_1 + t_2) + b \] (B.12)

in the denominator and

$$P_{4}^{(c)}(t_1, t_2, t_4, t_5, \epsilon) = \sum_{i=0}^{1} c_i t_i^1 = \sum_{i=0}^{1} \sum_{j=0}^{2-i} c_{ij} t_i^1 t_j^2 = \sum_{i=0}^{1} \sum_{j=0}^{2-i-j} c_{ijk} t_i^1 t_j^2 t_k^3$$ (B.13)

in the numerator. The first integrations can all be done applying (D.3). Afterwards the cut-off is employed and the result reads

$$J_{4}^{(c)}|_3(\epsilon) = \int_{0}^{\infty} dt_1 dt_2 dt_4 dt_5 \frac{\sum_{i} c_i t_i^1}{(a(t_1 + t_2) + b)^{i+\epsilon/2}}$$ (B.14)

\[= \int_{0}^{\infty} dt_2 dt_4 dt_5 \sum_{ij} c_{ij} t_i^2 a^{-i-1}(at_2 + b)^{i-3-\epsilon/2}B(i + 1, 3 - i + \epsilon/2) \]

\[= \int_{0}^{\infty} dt_4 dt_5 \sum_{ijk} c_{ijk} t_i^{j+k-2-\epsilon/2} t_i^{j+k-2-\epsilon/2} (t_4 + t_5)^{i-j-2} \]

\[\times B(i + 1, 3 - i + \epsilon/2)B(j + 1, 2 - i - j + \epsilon/2) \]

\[= \int_{0}^{\infty} dt_5 \sum_{ijk} c_{ijk} t_i^{j+k-5-\epsilon} B(i + 1, 3 - i + \epsilon/2) \]

\[= \int_{0}^{\infty} dt_5 \sum_{ijk} c_{ijk} t_i^{j+k-5-\epsilon} B(i + 1, 3 - i + \epsilon/2) \]

\[\times B(j + 1, 2 - i - j + \epsilon/2)B(i + j + k - 1 - \epsilon/2, 3 - k + \epsilon/2). \]

We shall not present the computation of the contributions to diagram (e) as they are vanishing, which makes it a little boring. This is in fact a consequence of numerous cancellations, whereas the vanishing of contributions to diagram (d) is due to the overall prefactor $\alpha'$ as already mentioned.

We now proceed to diagram (b) of figure [10], where we get

$$P_{2}^{(a)}(t_1, ..., t_5) = \alpha^2 \left(\ln(k_1) \ln(k_2) - \ln^2(\eta)\right)$$ (B.15)

\[= (t_1 + t_2)(t_2 + t_3 + t_4) + t_1 t_5 \equiv a(t_2 + t_3 + t_4) + b \]

in the denominator which suggests to integrate by the order $t_2 \to t_3 \to t_4 \to t_1$. The polynomial in the numerator then is

$$P_{5}^{(b)}(t_1, ..., t_5, \epsilon) = \sum_{i=0}^{3} c_i t_i^1 = \sum_{i=0}^{3} \sum_{j=0}^{3-i} c_{ij} t_i^1 t_j^2$$ (B.16)

\[= \sum_{i=0}^{3} \sum_{j=0}^{3-i-j} \sum_{k=0}^{3-i-j} c_{ijk} t_i^1 t_j^2 t_k^3 + \sum_{i=0}^{3} \sum_{j=0}^{3-i-j} \sum_{k=0}^{3-i-j-k} c_{ijk} t_i^1 t_j^2 t_k^3 \]

The integrations can be performed be using exclusively (D.3) and reveal

$$J_{5}^{(b)}(\epsilon) = \int_{0}^{\infty} dt_1 dt_2 dt_3 dt_4 dt_5 \frac{\sum_{i} c_i t_i^1}{(a(t_2 + t_3 + t_4) + b)^{i+\epsilon/2}}$$ (B.17)

\[= \int_{0}^{\infty} dt_1 dt_3 dt_4 dt_5 \sum_{ij} c_{ij} t_i^1 a^{-i-1}(a(t_3 + t_4) + b)^{i-4-\epsilon/2} \]
\[ xB(i + 1, 4 - i + \epsilon/2) \]
\[ = \int_0^\infty dt_1 dt_4 dt_5 \sum_{ijkl} C_{ijkl} t_4^k t_5^l a^{i-j-2} t_1 t_4 + b^{i-j-k-3} \]
\[ xB(i + 1, 4 - i + \epsilon/2) B(j + 1, 3 - i - j + \epsilon/2) \]
\[ = \int_0^\infty dt_1 dt_5 \sum_{ijkl} C_{ijkl} t_1^{i+j+k+l-2} t_5^{i+j+k-2} \]
\[ \times B(i + 1, 4 - i + \epsilon/2) B(j + 1, 3 - i - j + \epsilon/2) \]
\[ \times B(k + 1, 2 - i - j - k + \epsilon/2) \]
\[ \times B(4 - l + \epsilon/2, i + j + k + l - 1 - \epsilon/2), \]

where again the final \( t_5 \) integration has to be done by (B.8). All contributions to diagrams including four gluon vertices can be reduced to cases we have already covered when discussing those derived from (a). The necessary replacements are quite obvious.

### B.2 Exact computation of SPT integrals

For the integrals from (B.7) we can define an algorithm that even allows an exact computation. This will reveal further information to us, which hints, how the rest of the integrals corresponding to all the other diagrams can be solved by explicit calculation, using only the formulas displayed in appendix D. We have to thank M. Peter from Heidelberg for his advice and active help on this subject. Our notation is adopted from the previous chapter, only replacing \( J \) by \( I \).

The most important observation is that the proper time integrals can formally be written as ordinary momentum integrals of a scalar field theory, which are known from solving the integrals emerging from Feynman rules for such a theory. To do this we have to treat each term of the polynomial in the numerators separately and exploit the identity

\[
I(q^2, n_1, ..., n_5) = \int_0^\infty \frac{d^Dp d^Dk}{(2\pi)^{2D}} \frac{1}{(p^2)^{n_1} (p - q)^2} \frac{(k^2)^{n_2} ((k - q)^2)^{n_3} ((p - k)^2)^{n_4} ((k - p)^2)^{n_5}}{\Gamma(n_1)} \]
\[
= \int_0^\infty \prod_{i=1}^5 \left( \frac{dt_i}{\Gamma(n_i)} \right) \left( (t_1 + t_2)(t_3 + t_4 + t_5) + (t_3 + t_4 + t_5) \right)^{-D/2} \]
\[
\times \exp \left( -q^2 \frac{t_1 t_2 (t_3 + t_4 + t_5) + t_1 t_3 (t_4 + t_5) + t_2 t_4 (t_3 + t_5) + t_3 t_4 t_5}{(t_1 + t_2)(t_3 + t_4 + t_5) + (t_3 + t_4 + t_5)} \right). \tag{B.18}
\]

The SPT integral of the second line with \( D = 10 + \epsilon \) exactly resembles the integrals we are trying to solve when dealing with Yang-Mills theory. We use the variable \( t_i \) for the propagator with exponent \( n_i \) as in (B.7) and then do the Gaussian integrations by (B.8). Such momentum integrals one knows how to handle. The recursion relation

\[
0 = \int_0^\infty \frac{d^Dp d^Dk}{(2\pi)^{2D}} \frac{\partial}{\partial p_\mu} \left( \frac{p_\mu - k_\mu}{(p^2)^{n_1} (p - q)^2} \frac{(k^2)^{n_2} ((k - q)^2)^{n_3} ((p - k)^2)^{n_4} ((k - p)^2)^{n_5}}{\Gamma(n_1)} \right) \]
\[
= (D - n_1 - n_2 - 2n_5) I(q^2, n_1, ..., n_5) \]
\[
- n_1 \left( I(q^2, n_1 + 1, n_2, n_3, n_4, n_5 - 1) - I(q^2, n_1 + 1, n_2, n_3, n_4, n_5) \right) \]
\[
- n_2 \left( I(q^2, n_1, n_2 + 1, n_3, n_4, n_5 - 1) - I(q^2, n_1, n_2 + 1, n_3, n_4, n_5) \right) \]
\[
\right) \]

relates any given integral successively to a number of integrals that has \( n_5 = 0, n_3 = 0 \) or \( n_4 = 0 \). In the first case both integrals factorize as one propagator vanishes, in the latter case we get two
one-loop integrals that can be easily computed, which will be done explicitly in the following. This recursion thus allows to reduce all terms to a small number of simple types, but it creates a couple of thousand terms of such. Therefore we have used the algebraic computer programs MAPLE and FORM to do the task of performing the recursion and the expansion of its results in $1/\epsilon$. The results are summarized in several tables chapter [4].

We next demonstrate how the pinching contributions of (a) can be calculated and shall find that we have to explicitly compute only the two integrals just cited. To find the most appropriate order of substitutions for the proper time variables it is essential to understand the translation into momentum integrals. Take the case $t_1 \to 0$ whose contribution is

$$I^{(c)}_4 |_{1}(\epsilon) = \int_0^\infty dt_3 \int_0^\infty \frac{P^{(c)}_4(t_2, \ldots, t_5, \epsilon)}{(t_2(t_3 + t_4 + t_5) + (t_3 + t_4)t_5)^{4+\epsilon/2}} \exp \left( -\frac{t_2 t_4(t_3 + t_5) + t_3 t_4^2}{t_2(t_3 + t_4 + t_5) + (t_3 + t_4)t_5} \right) \quad (B.20)$$

and use

$$t_2 = x t, \quad t_5 = (1-x)t, \quad t_3 \to t t_3, \quad t_4 \to t t_4, \quad (B.21)$$

rescaling $t_2$ and $t_5$ to a unit square. On the contrary, a simultaneous scaling of all the integration variables would have not been successful. The integration over $t$ is trivial and gives

$$I^{(c)}_4 |_{1}(\epsilon) = \Gamma(-\epsilon) \int_0^1 dx \int_0^\infty dt_3 dt_4 \frac{x(1-x)t_4}{(t_3 + t_4 + x(1-x))^4} \frac{\hat{P}^{(c)}_4(t_3, t_4, x, \epsilon)}{(t_3 + t_4 + x(1-x))^{4+\epsilon/2}}. \quad (B.22)$$

We next define

$$\hat{P}^{(c)}_4(t_3, t_4, x, \epsilon) = \sum_i c_i(t_3, x, \epsilon) t_i^4 \equiv \sum_{ij} c_{ij}(x, \epsilon) t_4 t_3^j x^i$$

and integrate over $t_3$ and $t_4$ by using (D.1):

$$I^{(c)}_4 |_{1}(\epsilon) = \Gamma(-\epsilon) \int_0^1 dx \sum_{ijk} c_{ijk}(\epsilon) x^{i+j+k-2+\epsilon/2} (1-x)^{i+j-2+\epsilon/2} \quad (B.24)$$

$$\times B(i + 1 + \epsilon, 3 - i - \epsilon/2) B(j + 1, 2 - i - j - \epsilon/2)$$

$$= \Gamma(-\epsilon) \sum_{ijk} c_{ijk}(\epsilon) B(i + 1 + \epsilon, 3 - i - \epsilon/2) B(j + 1, 2 - i - j - \epsilon/2)$$

$$\times B(i + j - 1 + \epsilon/2, i + j + k - 1 + \epsilon/2).$$

The cases $t_2, t_3, t_4 \to 0$ can be treated in precisely the same manner by a simple permutation of indices, which can be read off from (B.22) by inspecting (B.20). We only remain with computing the contribution coming from $t_5 \to 0$ to the diagram (e):

$$I^{(c)}_4 |_{5}(\epsilon) = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 \int_0^\infty dt_4 \frac{P^{(c)}_4(t_1, \ldots, t_4, \epsilon)}{(t_1 + t_2)(t_3 + t_4)^{4+\epsilon/2}}$$

$$\times \exp \left( -\frac{t_1(t_2 t_3 + t_2 t_4 + t_3 t_4) + t_3 t_4^2}{(t_1 + t_2)(t_3 + t_4)} \right)$$

$$= \int_0^\infty dt_3 \int_0^\infty dt_4 \frac{P^{(c)}_4(t_1, t_2, t_4, \epsilon)}{(t_1 + t_2)(t_3 + t_4)^{4+\epsilon/2}} \exp \left( -\frac{t_1 t_2}{t_1 + t_2} + \frac{t_3 t_4}{t_3 + t_4} \right). \quad (B.25)$$

We rescale $t_1$ and $t_2$

$$t_1 = x t, \quad t_2 = (1-x)t, \quad (B.26)$$

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and as the polynomial $P_4^{(e)}$ in the numerator is always quadratic in these two variables, the integration over $t$ can be split off from the rest:

$$I_4^{(e)}|_5(\epsilon) = \Gamma(-\epsilon/2) \int_0^1 dx \int_0^\infty dt_3 dt_4 \ (x(1-x))^{\epsilon/2} \ 
\times \frac{\tilde{P}_4^{(e)}(t_3, t_4, x, \epsilon)}{(t_3 + t_4)^{1+\epsilon/2}} \exp\left(-\frac{t_3 t_4}{t_3 + t_4}\right).$$

After another substitution

$$t_3 = y \rho, \ t_4 = (1-y) \rho,$$

and defining

$$\tilde{P}_4^{(e)}(t_3, t_4, x, \epsilon) \equiv \sum_i c_i(y, \epsilon) x^i \equiv \sum_{ij} c_{ij}(\epsilon) x^i y^j$$

we finally obtain

$$I_4^{(e)}|_5(\epsilon) = \Gamma(-\epsilon/2)^2 \int_0^1 dy dy \sum_{ij} c_{ij}(\epsilon) (x(1-x))^{\epsilon/2} (y(1-y))^{\epsilon/2} x^i y^j$$

$$= \Gamma(-\epsilon/2)^2 \sum_{ij} c_{ij}(\epsilon) B(i + 1 + \epsilon/2, 1 + 1 + \epsilon/2) B(j + 1 + \epsilon/2, 1 + \epsilon/2).$$

We have such got all the pinching contributions derived from (a).

The only case we have not covered yet is the diagram (b) itself. All its pinching limits can again be reduced to the former $t_1 \to 0$ integral type. From (B.23) we get, following the notation of figure 12.

$$I_5^{(b)}(\epsilon) = \int_0^\infty \prod_{i=1}^5 (dt_i) \frac{P_5^{(b)}(t_1, ..., t_5, \epsilon)}{((t_1 + t_5)(t_2 + t_3 + t_4) + t_1 t_5)^{5+\epsilon/2}} \ 
\times \exp\left(-\frac{t_3((t_1 + t_5)(t_2 + t_3 + t_4) + t_1 t_5)}{(t_1 + t_5)(t_2 + t_3 + t_4) + t_1 t_5}\right).$$

In fact, by explicit inspection the polynomial $P_5^{(b)}$ turns out to depend only on the sum of $t_2$ and $t_4$. This corresponds to the fact that from the point of view of the field theory, one of the two variables is superfluous, as both propagators carry the same momentum, are thus identical. We then use their sum as a new variable

$$t_2 = xt, \ t_4 = (1-x)t, \ t_1 \to tt_1, \ t_3 \to tt_3, \ t_5 \to tt_5.$$

The integrand does not depend on $x$ and the integration over $t$ is trivial, so that we get

$$I_5^{(b)}(\epsilon) = \Gamma(-\epsilon) \int_0^\infty dt_1 dt_3 dt_5 \frac{\tilde{P}_5^{(b)}(t_1, t_3, t_5, \epsilon)}{((t_1 + t_5)(t_3 + 1) + t_1 t_5)^{5+\epsilon/2}} \ 
\times \left(\frac{t_3(t_1 + t_5 + t_1 t_5)}{(t_1 + t_5)(t_3 + 1) + t_1 t_5}\right)^\epsilon.$$
integrate over $t_3$ by (D.1), next over $t_1$ using (D.2) and finally over $t_5$ substituting
\[ t_5 = \frac{y}{1-y}, \quad dt_5 = \frac{dy}{(1-y)^2}, \quad [0, \infty] \rightarrow [0, 1] \] (B.35)
and by (D.3). The result of all this is
\[
I_5^{(b)}(\epsilon) = \Gamma(-\epsilon) \sum_{ijk} c_{ijk}(\epsilon) B(i + 1 + \epsilon, 4 - i + \epsilon/2) B(j + 1, 4 - j + \epsilon/2)
\times B(j + k - 3 - \epsilon/2, 4 - k + \epsilon/2)
\times 3 F_2(i + 1 + \epsilon, j + 1, j + k - 3 - \epsilon/2; 5 + \epsilon/2, j + 1; 1). \] (B.36)

We have been able to exactly solve all the SPT integrals.

C Two-loop diagrams with external states attached to the small loop

This is an extension to chapter 4.4, where we treat diagrams with one or both external states sitting at the interior, “small” loop. The procedure is very much equivalent to the usual and differs mainly in the choice of local coordinates one uses in the vicinity of the external legs of the diagrams, as well as in the parametrization of the sewing variables that is derived from this. We shall therefore be very brief and present all possible combinations of contributions in terms of the world sheet diagrams we have introduced earlier, then display in table 4 the appropriate sewing parameters and finally simply cite the results one obtains from solving the integrals in table 5. For the local coordinates of those external states that are supposed to be attached to the interior loop we now take
\[
V'_{i}(0) = \left| \frac{(z_i - 1)(z_i - \eta)}{1 - \eta} \right|, \] (C.1)
while for those which are sitting at the large loop we have $V'_{i}(0) = z_i$ as before. The parametrization we define in table 4 is given by the methods of 8, which necessarily leads to the correct Green's function.

\[ \eta_2 = 0 \quad \eta_1 = \eta \quad z_2 \quad \xi_1 = 1 \quad \xi_2 = \infty \]

Figure 17: Parametrization (I) of the world sheet of diagram (a)
\[ \eta_2 = 0 \quad \eta_1 = \eta \quad \zeta_2 = \infty \]

\[ \eta_2 = 0 \quad \eta_1 = \eta \quad \zeta_1 = 1 \quad \zeta_2 = \infty \]

Figure 18: Parametrization (II) of the world sheet of diagram (a)

\[ \eta_2 = 0 \quad \eta_1 = \eta \quad \zeta_1 = 1 \quad \zeta_2 = \infty \]

Figure 19: Parametrization (III) of the world sheet of diagram (a)

\[ \eta_2 = 0 \quad \eta_1 = \eta \quad \zeta_1 = 1 \quad \zeta_2 = \infty \]

Figure 20: Parametrization (IV) of the world sheet of diagram (a)
Figure 21: Parametrization (I) of the world sheet of diagram (b)

Figure 22: Parametrization (II) of the world sheet of diagram (b)

Figure 23: Parametrization (III) of the world sheet of diagram (b)
Table 4: Translation of the moduli into sewing parameters

| Diagram | Parametrization | $z_1$    | $z_2$    | $\eta$ |
|---------|-----------------|----------|----------|--------|
| (a)     | (I)             | $A_1$    | $1 - A_3$| $A_1 A_2$|
| (a)     | (II)            | $A_1 A_2$| $1 - A_3$| $A_1$  |
| (a)     | (III)           | $A_1$    | $A_1 A_2 (1 - A_3)$| $A_1 A_2$|
| (a)     | (IV)            | $A_1 A_2$| $A_1 (1 - A_3)$| $A_1$  |
| (b)     | (I)             | $1 - A_2$| $1 - A_2 A_3$| $A_1$  |
| (b)     | (II)            | $A_1 (1 - A_2)$| $A_1 (1 - A_2 A_3)$| $A_1$  |
| (b)     | (III)           | $A_1 (1 - A_2)$| $1 - A_3$| $A_1$  |

Table 5: Contributions to Feynman diagrams

| Diagram | Parametrization | Coefficient |
|---------|-----------------|-------------|
| (a)     | (I)             | $\frac{25}{9} \epsilon^{-2} + \frac{181}{9} \epsilon^{-1}$|
| (a)     | (II)            | $\frac{121}{36} \epsilon^{-2} + \frac{3721}{216} \epsilon^{-1}$|
| (a)     | (III)           | $\frac{41}{6} \epsilon^{-2} + \frac{2427}{216} \epsilon^{-1}$|
| (a)     | (IV)            | $\frac{91}{36} \epsilon^{-2} + \frac{3199}{216} \epsilon^{-1}$|
| (b)     | (I)             | $\frac{293}{144} \epsilon^{-2} + \frac{26999}{4320} \epsilon^{-1}$|
| (b)     | (II)            | $\frac{521}{144} \epsilon^{-2} + \frac{54799}{4320} \epsilon^{-1}$|
| (b)     | (III)           | $-\frac{13}{12} \epsilon^{-2} - \frac{369}{80} \epsilon^{-1}$|
D Useful formulas and functions

Formulas used to compute SPT integrals [32]:

\[ \int_0^\infty \frac{x^{\mu - 1}}{(\beta x + 1)^\nu} \, dx = \beta^{-\mu} B(\mu, \nu - \mu), \quad (D.1) \]

\[ \left[ \Re(\nu) > \Re(\mu) > 0, \ |\arg(\beta)| < \pi \right], \]

\[ \int_0^\infty x^{\nu - 1}(\beta + x)^{-\mu} (\gamma + x)^{-\rho} \, dx = \beta^{-\mu} - \mu B(\nu, \mu - \nu + \rho) \]

\[ \times {}_2F_1(\mu, \nu; \mu + \rho; 1 - \gamma/\beta), \]

\[ \left[ \Re(\nu) > 0, \ Re(\mu) > Re(\nu - \rho), \ |\arg(\beta)| < \pi, \ |\arg(\gamma)| < \pi \right], \]

\[ \int_0^1 x^{\rho - 1}(1-x)^{\sigma - 1} {}_2F_1(\alpha, \beta; \gamma; x) \, dx = B(\rho, \sigma) \]

\[ {}_3F_2(\alpha, \beta, \gamma; \rho, \rho + \sigma; 1), \quad (D.2) \]

\[ \left[ \Re(\rho) > 0, \ Re(\sigma) > 0, \ Re(\gamma + \sigma - \alpha - \beta) > 0 \right]. \]

The Gamma function:

\[ \Gamma(z) = \int_0^\infty dt \, e^{-t} t^{z-1}, \quad [\Re(z) > 0]. \quad (D.4) \]

The Euler Beta function:

\[ B(\mu, \nu) = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)} \]

\[ = \int_0^1 x^{\nu - 1}(1-x)^{\mu - 1} \, dx, \quad [\Re(\nu), \Re(\mu) > 0]. \quad (D.5) \]

The generalized hypergeometric series:

\[ {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!}, \quad (D.6) \]

where \((\alpha)_k \equiv \alpha(\alpha+1) \cdots (\alpha+k-1)\) and \((\alpha)_0 \equiv 1\). For \(_2F_1\) there exists the integral representation

\[ {}_2F_1(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta - 1}(1-t)^{\gamma - 1 - \beta}(1-tz)^{-\alpha} \, dt, \quad (D.7) \]

\[ \left[ \Re(\gamma) > \Re(\beta) > 0 \right]. \]

For the integration of Gaussian integrals after applying the Schwinger trick to Feynman propagators

\[ \int_{-\infty}^{\infty} \frac{d^d p d^d k}{(2\pi)^{2d}} \exp \left( -\alpha p^2 - \beta k^2 - \delta q^2 + 2\gamma pk + 2xpq + 2ykq \right) = \]

\[ (\alpha \beta - \gamma^2)^{-d/2} \exp \left( -q^2 \frac{\alpha \delta \beta - \delta \gamma^2 - 2xy \gamma - x^2 \beta - y^2 \alpha}{\alpha \beta - \gamma^2} \right). \quad (D.8) \]

The Riemann Zeta function:

\[ \zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \quad [\Re(z) > 1]. \quad (D.9) \]
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