Point symmetries of the Robinson–Trautman equation

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Abstract. We present our results on point symmetries and invariant solutions of the Robinson–Trautman equation.

1. Introduction
Robinson–Trautman metrics were introduced [1, 2] as a simple model of gravitational radiation outgoing from a spatially bounded source. Although asymptotically flat Robinson–Trautman metrics were shown to exist [3, 4, 5], the only known explicit examples are the Minkowski and the Schwarzschild metrics.

In this short communication we discuss point symmetries of the Robinson–Trautman equation (obtained by reduction of vacuum Einstein equations) and its symmetric solutions. We hope that our research can help finding solutions representing realistic physical situations. For details and further results we refer the reader to our preprint [6].

In terms of standard coordinates $u, r, \xi, \bar{\xi}$, the Robinson–Trautman metric is given by [7]

$$g = 2du(Hdu + dr) - 2r^2P^{-2}d\xi d\bar{\xi}, \quad P_r = 0.$$  

(1)

Vacuum Einstein equations imply

$$H = -r\partial_u \ln P - m(u)/r + P^2\partial_\xi \partial_{\bar{\xi}} \ln P,$$

and a fourth order PDE for the function $P$

$$P^2\partial_\xi \partial_{\bar{\xi}}(P^2\partial_\xi \partial_{\bar{\xi}} \ln P) + 3m\partial_u (\ln P) - \partial_u m = 0,$$

referred to as the Robinson–Trautman equation. Coordinate freedom allows one to choose $u$ such that $m$ becomes constant [7].

2. Point symmetries for $m \neq 0$
Consider equation (3) rewritten in the gauge $m = 1$:

$$P^2\partial_\xi \partial_{\bar{\xi}}(P^2\partial_\xi \partial_{\bar{\xi}} \ln P) + 3\partial_u \ln P = 0.$$  

(4)
It is easy to prove that (4) is preserved by the following point transformations:

\[
\begin{align*}
u &\mapsto u' = a^4 u + b, \\
\xi &\mapsto \xi' = f(\xi), \\
P &\mapsto P' = a^{-1}|f_\xi|P,
\end{align*}
\]

where \( a \neq 0 \) and \( b \) are real constants and \( f \) is a holomorphic function of \( \xi \) and \( f_\xi \neq 0 \).

Infinitesimal transformations (5) are generated by the vector field

\[
k = (4Au + B)\partial_u + F(\xi)\partial_\xi + \tilde{F}\partial_{\bar{\xi}} + (\text{Re} F_\xi - A)P\partial_P,
\]

where \( A \) and \( B \) are real constants and \( F \) is a holomorphic function of \( \xi \). Fields (8) form the symmetry algebra \( g \).

Suppose that the function \( P \) is a solution of the Robinson–Trautman equation. Invariance with respect to (8) implies the following additional condition:

\[
(4Au + B)P,u + FP_\xi + \tilde{F}P_{\bar{\xi}} + (A - \text{Re} F_\xi)P = 0.
\]

In what follows we will perform the symmetry reduction of (4) assuming that its solution is preserved by vector fields (8) forming a one- or two-dimensional subalgebra of \( g \). The case of \( \text{dim} \ g \geq 3 \) does not lead to any new interesting solutions [6].

3. Solutions with one or two symmetries

Consider a one-dimensional algebra generated by vector field (8). The coefficients \( A \) and \( B \) and the function \( F \) can be simplified by an appropriate choice of coordinates. For instance, if \( A = 0 \), \( B \neq 0 \) and \( F \neq 0 \) we can scale \( u \) and \( \xi \) so that \( B = 1 \) and \( F = \xi \). This leads to

\[
k = \partial_u + \xi \partial_\xi + \bar{\xi} \partial_{\bar{\xi}} + P\partial_P.
\]

We can distinguish five types of invariant solutions listed in Table 1, together with corresponding vector \( k \), form of the invariant solution \( P \) and the reduced Robinson–Trautman equation. Throughout the paper, the real and imaginary part of \( \xi \) are denoted by \( x \) and \( y \), respectively.

Take now into account the case of two-dimensional subalgebra \( g_2 \) of \( g \). We denote the basis vectors of \( g_2 \) by \( k_1 \) and \( k_2 \). There are two nonisomorphic two-dimensional Lie algebras such that either

\[
[k_1, k_2] = 0
\]

or

\[
[k_1, k_2] = k_2.
\]

The commutator of two fields (8) reads

\[
[k_1, k_2] = 4(B_1A_2 - B_2A_1)\partial_u + (F_1F_2,\xi - F_2F_1,\xi)\partial_\xi + \\
+ (\tilde{F}_1\tilde{F}_2,\xi - \tilde{F}_2\tilde{F}_1,\xi)\partial_{\bar{\xi}} + P\text{Re}(F_1F_2,\xi,\xi - F_2F_1,\xi,\xi)\partial_P
\]

(indices 1, 2 refer to vectors \( k_1, k_2 \), respectively.)

In the Abelian case, equation (11) implies

\[
A_1B_2 = A_2B_1,
\]

\[
F_1F_2,\xi = F_2F_1,\xi,
\]

\[
\text{Re}(F_1F_2,\xi,\xi - F_2F_1,\xi,\xi) = 0.
\]
Equation (17) follows from (16) and \( F_1 \) is proportional to \( F_2 \). Due to (15) and (11) we can assume without loss of generality that

\[
k_1 = (4Au + B)\partial_u + C\partial_x - AP\partial_P, \quad k_2 = \partial_y, \quad C \in R.
\]

(18)

Using symmetry transformations we can distinguish five subclasses of invariant solutions depending on values of \( A, B \) and \( C \). We present these results in Table 2.

In the non-Abelian case, equation (12) gives

\[
4(A_2B_1 - A_1B_2) = 4A_2u + B_2,
\]

(19)

\[
F_1F_2,\xi - F_2F_1,\xi = F_2,
\]

(20)

\[
PRe (F_1F_2,\xi - F_2F_1,\xi) = P(ReF_2,\xi - A_2).
\]

(21)

It follows from (19) that

\[
A_2 = 0, \quad B_2(1 + 4A_1) = 0
\]

(22)

and, similarly as in the Abelian case, (20) implies (21) Proceeding as in the Abelian case, we obtain four types of invariant solutions presented in Table 3.

Discussion

We have examined point symmetries of the Robinson–Trautman equation with nonzero \( m \) (3). Forms of invariant solutions in the case of one or two symmetries were given, as well as the corresponding reductions of equation (3). Results [6] obtained for the case of vanishing \( m \) and \( \text{dim } g \geq 3 \) show that all known [7] exact solutions of (3) have two or three symmetries.

In the case A.3 (Table 2) the Robinson-Trautman equation can be solved analytically. This way one obtains the so-called C-metrics [8]. Possible physical interpretation of these metrics was considered recently [9, 10]. It is still an open question whether other classes of invariant solutions contain examples corresponding to nontrivial asymptotically flat metrics.

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### Table 1. Invariant solutions with single symmetry.

| CC | $k$ | $P$ | RT equation |
|----|----|----|-------------|
| 1  | $\partial_u$ | $p(\xi, \xi')$ | $p^2 \partial_\xi \partial_{\xi'} \ln p = \text{Re} \xi$. |
| 2  | $4u \partial_u - P \partial_P$ | $u^{-1/4}(\xi, \xi')$ | $p^2 \partial_\xi \partial_{\xi'} (p^2 \partial_\xi \partial_{\xi'} \ln p) = \frac{3}{4}$. |
| 3  | $\partial_u - \xi \partial_\xi - \xi \partial_\xi' - P \partial_P$ | $e^\phi p(z, \xi)$ | $p^2 \partial_\xi \partial_{\xi'} (p^2 \partial_\xi \partial_{\xi'} \ln p) - 3(z \partial_z + \xi \partial_\xi) \ln p + 3 = 0, \ z = e^{-u} \xi$. |
| 4  | $\partial_u$ | $p(u, x)$ | $p^2 \partial_u^2 (p^2 \partial_u^2 \ln p) + 3 \partial_u \ln p = 0$. |
| 5  | $4Au \partial_u + \xi \partial_\xi + \xi \partial_\xi' + P(1 - A) \partial_P$ | $u^{1/4} \delta(\xi, \xi')$ | $4Ap^2 \partial_\xi \partial_{\xi'} (p^2 \partial_\xi \partial_{\xi'} \ln p) + 3(A - 1) + 3(z \partial_z + \xi \partial_\xi) \ln p = 0, \ z = u^{-(4A)^{-1}} \xi$. |

### Table 2. Invariant solutions with two Abelian symmetries.

| CC | $k_1$ | $k_2$ | $P$ | RT equation |
|----|------|------|----|-------------|
| A.1 | $\partial_u$ | $\partial_u$ | $p(x)$ | $p^2 (\ln p)'' = x$. |
| A.2 | $\partial_x$ | $\partial_u$ | $p(u)$ | $p' = 0$. |
| A.3 | $\partial_u + \partial_x$ | $\partial_u$ | $p(u - x)$ | $p^2 (p^2 (\ln p)'')'' + 3 (\ln p)' = 0$. |
| A.4 | $4u \partial_u - P \partial_P$ | $\partial_u$ | $u^{-1/4} p(x)$ | $p^2 (p^2 (\ln p)'')'' = \frac{3}{4}$. |
| A.5 | $4Au \partial_u + \partial_x - A P \partial_P$ | $\partial_u$ | $u^{-1/4} p(4A)^{-1} \ln u - x)$ | $p^2 (p^2 (\ln p)'')'' + \frac{3}{4} \ln (\ln p)' = \frac{3}{4}$. |

### Table 3. Invariant solutions with two non-Abelian symmetries.

| CC | $k_1$ | $k_2$ | $P$ | RT equation |
|----|------|------|----|-------------|
| NA.1 | $\epsilon \partial_u - x \partial_x - y \partial_y - P \partial_P$, $\epsilon = 0$ or $\epsilon = 1$ | $\partial_y$ | $x p(u + e \ln x)$ | $\epsilon [p^2 (p^2 (\ln p)'')'' - 2 p^2 (\ln p)''' - p^2 (\ln p)'' + 2 p^2] + 3 (\ln p)' = 0$. |
| NA.2 | $4Au \partial_u - x \partial_x - y \partial_y - (1 + A) P \partial_P$ | $\partial_y$ | $u \frac{4 + 4}{\partial_x} p(u + e \ln x) \frac{1}{x}$ | $p^2 (p^2 (\ln p)'')'' + 3 X (\ln p)' = 3(1 + A)$. |
| NA.3 | $-u \partial_u + \partial_x + \frac{1}{4} P \partial_P$ | $\partial_u$ | $p(x)$ | $p^2 (\ln p)''' = x$. |
| NA.4 | $-u \partial_u + \partial_x + \frac{1}{4} P \partial_P$ | $\partial_u + e \partial_\xi + e \partial_\xi' + P \text{Re} e \partial_P$ | $u^{-\frac{1}{4}} [1 + u^{-\frac{1}{4}} e^{-\xi} \frac{1}{2} \tilde{p}(z), z = \frac{1 + u^{-\frac{1}{4}} - e^{-\xi}}{1 + u^{-\frac{1}{4}} - e^{-\xi}}$ | $p^2 (p^2 (\ln p)'')'' + \frac{1}{4} p^2 (\ln p)''' - 36 \cos z + 48 (\ln p)' \sin z = 0$. |