Explicit real-part estimates for high order derivatives of analytic functions

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Abstract. The representation for the sharp constant $K_{n,p}$ in an estimate of the modulus of the $n$-th derivative of an analytic function in the upper half-plane $C_+$ is considered. It is assumed that the boundary value of the real part of the function on $\partial C_+$ belongs to $L^p$. The representation for $K_{n,p}$ comprises an optimization problem by parameter inside of the integral. This problem is solved for $p = 2(m+1)/(2m+1-n)$, $n \leq 2m+1$, and for some first derivatives of even order in the case $p = \infty$. The formula for $K_{n,2(m+1)/(2m+1-n)}$ contains, for instance, the known expressions for $K_{2m+1,\infty}$ and $K_{m,2}$ as particular cases. Also, a two-sided estimate for $K_{2m,\infty}$ is derived, which leads to the asymptotic formula $K_{2m,\infty} = 2((2m-1)!!)^2/\pi + O((2m-1)!!^2/(2m-1))$ as $m \to \infty$. The lower and upper bounds of $K_{2m,\infty}$ are compared with its value for the cases $m = 1, 2, 3, 4$. As applications, some real-part theorems with explicit constants for high order derivatives of analytic functions in subdomains of complex plane are described.

2010 MSC. Primary: 30A10; Secondary: 30H10

Keywords: analytic functions, asymptotic formula, explicit real-part estimates, high order derivatives

0 Introduction

In this paper we deal substantially with the coefficient $K_{n,p}(\alpha)$ in the inequality

$$|\Re\{e^{i\alpha}f^{(n)}(z)\}| \leq \frac{K_{n,p}(\alpha)}{(3z)^{n+\frac{1}{p}}} ||\Re f||_p,$$  \hspace{1cm} (0.1)

where $z$ is a point in the half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$. Here $f$ is an analytic function in $\mathbb{C}_+$ represented by the Schwarz formula

$$f(z) = \frac{1}{\pi i} \int_\infty^\infty \frac{\Re f(\zeta)}{\zeta - z} \, d\zeta$$ \hspace{1cm} (0.2)
and such that the boundary values on $\partial C_+$ of the real part of $f$ belong to the space $L^p(-\infty, \infty), 1 \leq p < \infty$.

Here and in what follows we adopt the notation $||\Re f||_p$ for $||\Re f|_{\partial C_+}||_p$, where $|| \cdot ||_p$ stands for the norm in $L^p(-\infty, \infty)$. Note that the value $K_{n,\infty}(\alpha)$ is obtained by passage to the limit of $K_{n,p}(\alpha)$ as $p \to \infty$.

Inequality (0.1) with the best possible coefficient in front of $||\Re f||_p$ was obtained by Kresin and Maz’ya [8]. In [8] it was shown that

$$K_{n,p}(\alpha) = \frac{n!}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} \cos \left( \alpha - (n+1)\varphi + \frac{n\pi}{2} \right) \cos^{(n+1)} \varphi d\varphi \right\}^{1/q}, \quad (0.3)$$

where $1/p + 1/q = 1$. So, the sharp constant $K_{n,p}$ in the inequality

$$|f^{(n)}(z)| \leq \frac{K_{n,p}}{(3z)^{n+1}} ||\Re f||_p$$

is given by

$$K_{n,p} = \max_{\alpha} K_{n,p}(\alpha). \quad (0.5)$$

Note that inequalities (0.1) and (0.4) for analytic functions belong to the class of sharp real-part theorems (see Kresin and Maz’ya [6] and references there) which go back to Hadamard’s real-part theorem [2].

The present article extends the topic of papers by Kresin and Maz’ya [7, 8]. In [7] the explicit formulas for $K_{0,p}$ for $p \in [1, \infty)$ and for $K_{1,p}$ for $p \in [1, \infty]$ were found. In [8] the case of $n \geq 2$ was considered and the explicit formulas for $K_{n,p}$ were derived for $n = 2m+1, 2, 4$ and $p = \infty$ as well as for arbitrary $n$ and $p = 1, 2$. Namely, in [8] it was shown that

$$K_{2m+1,\infty} = \frac{2}{\pi} \left( \frac{(2m+1)!}{2m+1} \right)^2, \quad m = 0, 1, 2, \ldots, \quad (0.6)$$

$$K_{2,\infty} = \frac{3\sqrt{3}}{2\pi}, \quad K_{4,\infty} = \frac{3}{4\pi} \left( 16 + 5\sqrt{5} \right), \quad (0.7)$$

and

$$K_{n,1} = \frac{n!}{\pi}, \quad K_{n,2} = \sqrt{\frac{(2n)!}{2^{2n+1}\pi}}. \quad (0.8)$$

In this paper the optimization problem (0.5) is solved in a series of cases described below. In these cases we obtain the explicit formulas for the sharp constant $K_{n,p}$. In a complicated case $n = 2m, p = \infty$ we prove a two-sided estimate for $K_{2m,\infty}$. In conclusion, some applications of obtained results to estimates of high order derivatives of analytic functions in subdomains of $\mathbb{C}$ are described.

Now we describe the results of the present paper in more detail. Introduction is followed by four sections. The first of them is auxiliary. It concerns the integral

$$Q_{p,n,\gamma}(\beta) = \int_{-\pi/2}^{\pi/2} \cos \left( \beta - (n+1)\varphi \right)^\gamma \cos^n \varphi d\varphi, \quad (0.9)$$
depending on the parameter $\beta$. We consider the problem on maximum of $Q_{\mu,n,\gamma}(\beta)$ in $\beta$. In what follows, by $\mathbb{N}$ we mean the set of the natural numbers and by $[a]$ we denote the integer part of the number $a$. Assuming that $m, n \in \{0\} \cup \mathbb{N}$, $m \geq n + 1$ and $\gamma > 2 \left\lfloor \frac{m}{n+1} \right\rfloor - 2$, we prove the equality

$$\max_{\beta} Q_{2m,n,\gamma}(\beta) = Q_{2m,n,\gamma}(0) = \int_{-\pi/2}^{\pi/2} \left| \cos(n+1)\varphi \right|^\gamma \cos^{2m} \varphi d\varphi$$

and find the last integral. If $m \leq n$ and $\gamma > -1$, it is shown that the function $Q_{2m,n,\gamma}(\beta)$ is independent of $\beta$ and its value is given.

Concretizing result of Section 1 for (0.3) with $q = 2(m+1)/(n+1)$ and $n \leq 2m+1$, in Section 2 we obtain the explicit formula for $K_{n,2(m+1)/2m+1-n}(\alpha)$ is independent of $\alpha$ for the case $m \leq n$ and

$$K_{n,2m+1} = n! \left\{ \frac{(2m-1)!!}{(2m)!!} B \left( \frac{m+1}{n+1}, \frac{1}{2}, \frac{1}{2} \right) \right\} \frac{n+1}{2m+1},$$

where by $B$ is denoted the Beta-function. We note, that the above-mentioned formulas for $K_{2m,\infty}$ and $K_{n,2}$ are particular cases of (0.10) for $n = 2m+1$ and $n = m$, correspondingly.

Section 3 is devoted to derivatives of even order in the case $p = \infty$. First, we solve the optimization problem (0.5) with $n = 6, 8$ and $p = \infty$, and find the values of the sharp constants

$$K_{6,\infty} = \frac{105\sqrt{2}}{4\pi} \left( 9\cos \frac{\pi}{28} + 3\cos \frac{3\pi}{28} + \cos \frac{5\pi}{28} \right),$$

$$K_{8,\infty} = \frac{315}{8\pi} \left\{ 175 + 9\sqrt{2} \left( 17\cos \frac{\pi}{36} + 9\cos \frac{5\pi}{36} + 11\cos \frac{7\pi}{36} \right) \right\}.$$  

Further, using the result of Section 1, we obtain the two-sided estimate

$$\frac{2}{\pi} ((2m-1)!!)^2 < K_{2m,\infty} < \frac{2m}{2m-1} \frac{2}{\pi} ((2m-1)!!)^2,$$

which leads to the asymptotic formula

$$K_{2m,\infty} = \frac{2}{\pi} ((2m-1)!!)^2 + O \left( \frac{((2m-1)!!)^2}{2m-1} \right)$$

as $m \to \infty$.

Let us denote by

$$L_{2m} = \frac{2}{\pi} ((2m-1)!!)^2, \quad U_{2m} = \frac{2m}{2m-1} \frac{2}{\pi} ((2m-1)!!)^2$$

the values of the lower and upper bounds in two-sided estimate (0.13), correspondingly. We can compare these bounds with the sharp constant in inequality
for $n = 2, 4, 6, 8$ and $p = \infty$. Using (0.7), (0.11) and (0.12), we get

$$\frac{L_2}{K_{2,\infty}} \approx 0.7698, \quad \frac{L_4}{K_{4,\infty}} \approx 0.8830, \quad \frac{L_6}{K_{6,\infty}} \approx 0.9204, \quad \frac{L_8}{K_{8,\infty}} \approx 0.9396,$$

$$\frac{U_2}{K_{2,\infty}} \approx 1.5396, \quad \frac{U_4}{K_{4,\infty}} \approx 1.2141, \quad \frac{U_6}{K_{6,\infty}} \approx 1.1045, \quad \frac{U_8}{K_{8,\infty}} \approx 1.0738.$$  

In concluding Section 4 we collect some real-part estimates with explicit constants in the majorizing part of inequality for the modulus of derivatives of analytic functions in subdomains of $\mathbb{C}$.

## 1 The main lemma

First we prove the following auxiliary assertion.

**Lemma 1.** Let $m, n \in \{0\} \cup \mathbb{N}$. If $m \geq n + 1$ and $\gamma > 2 \left[\frac{m}{n+1}\right] - 2$, then

$$\max_{\beta} Q_{2m,n,\gamma}(\beta) = Q_{2m,n,\gamma}(0) = \int_{-\pi/2}^{\pi/2} |\cos(n+1)\varphi|^\gamma \cos^{2m} \varphi d\varphi \quad (1.1)$$

$$= \frac{(2m-1)!!}{(2m)!!} B\left(\frac{\gamma+1}{2}, \frac{1}{2}\right) + \pi \frac{m!}{2^{2m+\gamma-1}(\gamma+1)} \sum_{j=1}^{\frac{m}{\gamma+1}} \frac{2^m}{B\left(\frac{\gamma+1}{2}, \frac{1}{2} - j\right)} \quad (1.2)$$

where by $B$ is denoted the Beta-function.

If $m \leq n$ and $\gamma > -1$, then the function $Q_{2m,n,\gamma}(\beta)$ is independent of $\beta$, and it is given by

$$Q_{2m,n,\gamma}(\beta) = \frac{(2m-1)!!}{(2m)!!} B\left(\frac{\gamma+1}{2}, \frac{1}{2}\right) \quad (1.3)$$

**Proof.** Making the change of variable $\psi = \beta - (n+1)\varphi$ in (1.3) with $\mu = 2m$, we obtain

$$Q_{2m,n,\gamma}(\beta) = \frac{1}{n+1} \int_{\beta-(n+1)\pi}^{\beta+(n+1)\pi} |\cos \psi|^\gamma \cos^{2m} \frac{\psi - \beta}{n+1} d\psi.$$  

Since the integrand is $(n+1)\pi$-periodic, it follows that

$$Q_{2m,n,\gamma}(\beta) = \frac{1}{n+1} \int_{0}^{(n+1)\pi} |\cos \psi|^\gamma \cos^{2m} \frac{\psi - \beta}{n+1} d\psi$$

$$= \frac{1}{n+1} \sum_{j=0}^{n} \int_{j\pi}^{(j+1)\pi} |\cos \psi|^\gamma \cos^{2m} \frac{\psi - \beta}{n+1} d\psi.$$  

The change of variable $\psi - j\pi = \vartheta$ implies

$$Q_{2m,n,\gamma}(\beta) = \frac{1}{n+1} \int_{0}^{\pi} |\cos \vartheta|^\gamma g_{m,n}(\vartheta - \beta) d\vartheta, \quad (1.4)$$
where
\[ g_{m,n}(\theta) = \sum_{j=0}^{n} \cos^{2m} \left( \frac{\theta + j\pi}{n+1} \right). \] (1.5)

Since
\[ \cos^{2m} x = \frac{1}{2^{2m}} \left\{ \binom{2m}{m} + 2 \sum_{k=0}^{m-1} \binom{2m}{k} \cos(2(m-k)x) \right\}, \]
we can write (1.5) in the form
\[ g_{m,n}(\theta) = \frac{n+1}{2^{2m}} \binom{2m}{m} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \sum_{j=0}^{n} \cos \left( \frac{2(m-k)(\theta + j\pi)}{n+1} \right). \]

Putting here \( k = m - l \) \( (l = 1, 2, \ldots, m) \) and taking into account that
\[ \frac{1}{2^{2m}} \binom{2m}{m} = \frac{(2m)!!}{(2m)!} \]
we obtain
\[ g_{m,n}(\theta) = \frac{(2m-1)!!(n+1)}{(2m)!!} + \frac{1}{2^{2m-1}} \sum_{l=1}^{m} \binom{2m}{m-l} \sum_{j=0}^{n} \cos \left( \frac{2l(\theta + j\pi)}{n+1} \right). \] (1.6)

Consider the interior sum in (1.6). We have
\[ \sum_{j=0}^{n} \cos \left( \frac{2l(\theta + j\pi)}{n+1} \right) = \Re \left\{ e^{\frac{2\pi li}{n+1}} \sum_{j=0}^{n} e^{\frac{2\pi j}{n+1}} \right\}. \]
So, for \( l \in \{1, \ldots, m\} \),
\[ \sum_{j=0}^{n} \cos \left( \frac{2l(\theta + j\pi)}{n+1} \right) = \Re \left\{ e^{\frac{2\pi li}{n+1}} \frac{1 - e^{2\pi i}}{1 - e^{\frac{2\pi i}{n+1}}} \right\} = 0, \] (1.7)
if \( s = \frac{l}{n+1} \notin \mathbb{N} \), and
\[ \sum_{j=0}^{n} \cos \left( \frac{2l(\theta + j\pi)}{n+1} \right) = (n+1) \cos 2s\theta, \] (1.8)
if \( s = \frac{l}{n+1} \in \mathbb{N} \).

Taking into account (1.7) and (1.8), we can rewrite (1.6) in the form
\[ g_{m,n}(\theta) = \frac{(2m-1)!!(n+1)}{(2m)!!} + \frac{n+1}{2^{2m-1}} \sum_{s=1}^{\left\lfloor \frac{m}{n+1} \right\rfloor} \binom{2m}{m-s(n+1)} \cos 2s\theta. \]
Combining this with (1.4), we obtain

\[ Q_{2m,n,\gamma}(\beta) = \frac{(2m - 1)!!}{(2m)!!} \int_0^\pi |\cos \theta|^\gamma \, d\theta \]

\[ + \frac{1}{2^{2m-1}} \sum_{s=1}^{\left\lfloor \frac{m}{n+1} \right\rfloor} \left( \frac{2m}{m-s(n+1)} \right) \int_0^\pi |\cos \theta|^\gamma \cos 2s(\theta - \beta) \, d\theta. \]

Since the integrands in the last equality are \(\pi\)-periodic, it follows that

\[ Q_{2m,n,\gamma}(\beta) = \frac{(2m - 1)!!}{(2m)!!} \int_{-\pi/2}^{\pi/2} \cos^\gamma \theta \, d\theta \]

\[ + \frac{1}{2^{2m-1}} \sum_{s=1}^{\left\lfloor \frac{m}{n+1} \right\rfloor} \left( \frac{2m}{m-s(n+1)} \right) \int_{-\pi/2}^{\pi/2} \cos^\gamma \theta \cos 2s(\theta - \beta) \, d\theta, \]

that is

\[ Q_{2m,n,\gamma}(\beta) = \frac{2(2m - 1)!!}{(2m)!!} \int_0^{\pi/2} \cos^\gamma \theta \, d\theta \]

\[ + \frac{1}{2^{2(m-1)}} \sum_{s=1}^{\left\lfloor \frac{m}{n+1} \right\rfloor} \left( \frac{2m}{m-s(n+1)} \right) \cos 2s\beta \int_0^{\pi/2} \cos^\gamma \theta \cos 2s\theta \, d\theta. \quad (1.9) \]

Let \( m \geq n + 1 \). Taking into account the formula (see, e.g., Gradshtein and Ryzhik [1], 3.631(9))

\[ \int_0^{\pi/2} \cos^{\nu-1} x \cos ax \, dx = \frac{\pi}{2^{\nu-1} \nu B\left(\frac{\nu+a}{2}, \frac{\nu-a+1}{2}\right)}, \quad (1.10) \]

where \( \Re \nu > 0 \), and the condition \( \gamma > 2 \left\lfloor \frac{m}{n+1} \right\rfloor - 2 \) of the present lemma, we conclude that

\[ \int_0^{\pi/2} \cos^\gamma \theta \cos 2s\theta \, d\theta > 0 \]

for any \( s \in \{1, 2, \ldots, \left\lfloor \frac{m}{n+1} \right\rfloor\} \). This and (1.9) imply that the maximum of \( Q_{2m,n,\gamma}(\beta) \) in \( \beta \) is attained at \( \beta = 0 \). Hence, by (1.9) we obtain (1.1). Calculating by (1.10) the integrals in (1.9), we arrive at (1.2).

The sum in (1.9) vanishes in the case \( m \leq n \). Hence, the function \( Q_{2m,n,\gamma}(\beta) \) is independent of \( \beta \) under condition \( m \leq n \), which proves (1.3).

\section{Sharp estimates for derivatives of analytic functions with \( \Re f \in h^p(\mathbb{R}_+^2), p = 2(m+1)/(2m+1-n) \)}

In what follows, by \( h^p(\mathbb{R}_+^2) \), \( 1 \leq p \leq \infty \), we mean the Hardy space of harmonic functions in the upper half-plane \( \mathbb{R}_+^2 \) which are represented by the Poisson inte-
gral with a density in $L^p(-\infty, \infty)$. It is well known (see, e.g. Levin [11], Sect. 19.3) that $f$ belongs to the Hardy space $H^p(\mathbb{C}_+)$ of analytic functions in $\mathbb{C}_+$ if $\Re f \in H^p(\mathbb{R}^2_+), 1 < p < \infty$. Besides, any function $f \in H^p(\mathbb{C}_+), 1 < p < \infty$, admits the representation (0.2) since $\Re f \in h^p(\mathbb{R}^2_+)$.

Now we consider the case $p = 2(m+1)/(2m+1-n)$ in inequality (0.4), that is $q = 2(m+1)/(n+1)$. We suppose that $n \leq 2m+1, n \geq 1$. In the case $n = 2m+1$ we put $p = \infty$.

**Theorem 1.** Let $\Re f \in H^p(\mathbb{R}^2_+)$ with $p = 2(m+1)/(2m+1-n)$, and let $z$ be an arbitrary point in $\mathbb{C}_+$. The sharp constant $K_{n,2(m+1)/(2m+1-n)}$ in the inequality

$$|f^{(n)}(z)| \leq \frac{K_{n,2(m+1)/(2m+1-n)} \||\Re f||_p}{(3z)^{n+\frac{1}{p}}},$$

is given by

$$K_{n,2(m+1)/(2m+1-n)} = \frac{n!}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} \left| \cos(n+1)\phi \right|^{2(m+1)/n+1} \cos^m \phi d\phi \right\}^{\frac{n+1}{n+\frac{1}{p}}}$$

$$= \frac{n!}{\pi} \left\{ \frac{(2m-1)!!}{(2m)!!} B\left(\frac{m+1}{n+1} + \frac{1}{2}, \frac{1}{2}\right) \right\}$$

$$\quad + \frac{\pi(n+1)}{2^{2m-1+2(m+1)/n+1}} \sum_{j=1}^{\frac{m}{n+1}} B\left(\frac{m+1}{n+1} + j + 1, \frac{m+1}{n+1} - j + 1\right) \right\}^{\frac{n+1}{n+\frac{1}{p}}}.$$ (2.2)

In particular,

$$K_{n,2(m+1)/(2m+1-n)} = \frac{n!}{\pi} \left\{ \frac{(2m-1)!!}{(2m)!!} B\left(\frac{m+1}{n+1} + \frac{1}{2}, \frac{1}{2}\right) \right\}^{\frac{n+1}{n+\frac{1}{p}}}$$

for $m \leq n$.

**Proof.** Putting $q = 2(m+1)/(n+1)$ in (0.3)-(0.5) and $\gamma = 2(m+1)/(n+1)$, $\mu = 2m$ in (0.9), we can write the sharp constant $K_{n,p}$ in inequality (2.1) as follows

$$K_{n,2(m+1)/(2m+1-n)} = \frac{n!}{\pi} \max_{\alpha} \left\{ Q_{2m,n,2(m+1)/(n+1)} \left(\alpha + \frac{n\pi}{2}\right) \right\}^{\frac{n+1}{n+\frac{1}{p}}}$$

$$= \frac{n!}{\pi} \max_{\beta} \left\{ Q_{2m,n,2(m+1)/(n+1)} \left(\beta\right) \right\}^{\frac{n+1}{n+\frac{1}{p}}}.$$ (2.5)

For $\gamma = 2(m+1)/(n+1)$ we have

$$\gamma = \frac{m+1}{n+1} + 1 > \left\lfloor \frac{m}{n+1} \right\rfloor + 1$$
that is, the condition
\[ \gamma > 2 \left[ \frac{m}{n+1} \right] - 2 \]
of Lemma 1 is satisfied. Applying Lemma 1 to \((2.5)\), we complete the proof.

Two following consequences of Theorem 1 contain explicit formulas for \(K_{n,p}\) in particular cases.

**Corollary 1.** If \(n = 2m, m \in \mathbb{N}\), then
\[ K_{2m,2m+2} = \frac{(2m)!}{\pi} \left\{ \frac{\sqrt{\pi}(2m-1)!! \Gamma \left( \frac{m+1}{2m+1} + \frac{1}{2} \right)}{(2m)!! \Gamma \left( \frac{m+1}{2m+1} + 1 \right)} \right\}^{\frac{2m+1}{2(m+1)}}. \] (2.6)

**Corollary 2.** If \(m = k(n+1) - 1, k \in \mathbb{N}\), then
\[ K_{n,2k-1} = \frac{n!}{\pi} \left\{ \frac{\sqrt{\pi}(2k(n+1)-3)!! \Gamma \left( k + \frac{1}{2} \right)}{(2k(n+1)-2)!!k!} \right\}^{\frac{1}{2k}} + \frac{\pi}{2^{2k(n+2)-3}} \sum_{j=1}^{k-1} \left( \frac{2k(n+1)-2}{(k-j)(n+1)-1} \right) \left( \frac{2k}{k-j} \right)^{\frac{1}{2k}}. \]

3 Estimates for even order derivatives of analytic functions with \(\Re f \in h^\infty(\mathbb{R}_+^2)\)

By (0.3),
\[ K_{2m,\infty}(\alpha) = \frac{(2m)!}{\pi} \int_{-\pi/2}^{\pi/2} |\cos(\alpha - (2m+1)\varphi)| \cos^{2m-1} \varphi d\varphi. \] (3.1)

The starting point of this section is the following assertion from the paper by Kresin and Maz’ya [8].

**Lemma 2.** The equality
\[ \frac{dK_{2m,\infty}}{d\alpha} = \frac{(2m)!}{\pi(2m+1)^22^{2m-1}} \int_0^{\pi/2} (|\cos(\alpha - \varphi)| - |\cos(\alpha + \varphi)|) \Lambda_m(\varphi) d\varphi \] (3.2)
holds with
\[ \Lambda_m(\varphi) = \sum_{\ell=1}^m (-1)^\ell(2\ell - 1) \left( \frac{2m-1}{m-\ell} \right) \sin \left( \frac{(2\ell-1)\varphi}{2m+1} \right) \sin \left( \frac{(2\ell-1)\pi}{2(2m+1)} \right). \] (3.3)
**Remark 1.** Before passing to applications of Lemma 2 we make two remarks. The first one concerns the range of $\beta$ in the evaluation of the maximum

$$
\max_{\beta} Q_{\mu,n,\gamma}(\beta),
$$

where the function $Q_{\mu,n,\gamma}(\beta)$ is defined by (0.9). It is clear, that $Q_{\mu,n,\gamma}(\beta)$ is $\pi$-periodic and even function in $\beta$. Therefore, we can limit our consideration of $Q_{\mu,n,\gamma}(\beta)$ to the interval $[0, \pi/2]$.

The second remark relates the sign of the function $|\cos(\beta - \varphi)| - |\cos(\beta + \varphi)|$, which appear inside of integral (3.2). We show that

$$
|\cos(\beta - \varphi)| \geq |\cos(\beta + \varphi)|
$$

(3.4)

for $\beta, \varphi \in [0, \pi/2]$. In fact, since

$$
|\cos(\beta - \varphi)| - |\cos(\beta + \varphi)| = \begin{cases} 
\cos(\beta - \varphi) - \cos(\beta + \varphi) & \text{for } \varphi \in \left[0, \frac{\pi}{2} - \beta\right], \\
\cos(\beta - \varphi) + \cos(\beta + \varphi) & \text{for } \varphi \in \left(\frac{\pi}{2} - \beta, \frac{\pi}{2}\right),
\end{cases}
$$

it follows that

$$
|\cos(\beta - \varphi)| - |\cos(\beta + \varphi)| = \begin{cases} 
2 \sin \varphi \sin \beta & \text{for } \varphi \in \left[0, \frac{\pi}{2} - \beta\right], \\
2 \cos \varphi \cos \beta & \text{for } \varphi \in \left(\frac{\pi}{2} - \beta, \frac{\pi}{2}\right),
\end{cases}
$$

and hence (3.4) holds for $\beta, \varphi \in [0, \pi/2]$. Besides, the equality sign in (3.4) holds only for $\beta = 0$ or for $\beta = \pi/2$ provided that $\varphi \in (0, \pi/2)$.

In the next two assertions we deal with the values of constants $K_{6,\infty}$ and $K_{8,\infty}$.

**Corollary 3.** Let $\Re f \in \mathcal{H}^{\infty}(\mathbb{R}^2_+)$, and let $z$ be an arbitrary point in $\mathcal{C}_+$. The sharp constant $K_{6,\infty}$ in the inequality

$$
|f^{(6)}(z)| \leq K_{6,\infty} \|\Re f\|_{\infty}
$$

is given by

$$
K_{6,\infty} = \frac{105 \sqrt{2}}{4\pi} \left(9 \cos \frac{\pi}{28} + 3 \cos \frac{3\pi}{28} + \cos \frac{5\pi}{28}\right).
$$

(3.6)

**Proof.** By Lemma 2

$$
\frac{dK_{6,\infty}}{d\alpha} = \frac{45}{49\pi} \int_{\alpha}^{\pi/2} (|\cos(\alpha - \varphi)| - |\cos(\alpha + \varphi)|) \Lambda_{3}(\varphi) d\varphi,
$$

where

$$
\Lambda_{3}(\varphi) = 5 \left(-2 \frac{\sin \frac{\varphi}{4}}{\sin \frac{\pi}{14}} + 3 \frac{\sin \frac{3\varphi}{7}}{\sin \frac{\pi}{14}} - \frac{\sin \frac{5\varphi}{7}}{\sin \frac{\pi}{14}}\right).
$$

(3.8)
Using the identities $\sin 3x = 3 \sin x - 4 \sin^3 x, \sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$ in (3.8), we find

$$
\Lambda_3(\varphi) = 80 \frac{\sin \frac{\pi}{14} \sin \frac{\varphi}{7}}{\sin \frac{\pi}{14} \sin \frac{5\pi}{14}} \left( \sin^2 \frac{\pi}{14} - \sin^2 \frac{\varphi}{7} \right) F_3(\varphi),
$$

where

$$
F_3(\varphi) = \left( 8 \sin^2 \frac{\pi}{14} - 7 \right) \sin^2 \frac{\pi}{14} + \left( 3 - 4 \sin^2 \frac{\pi}{14} \right) \sin^2 \frac{\varphi}{7}.
$$

Since $3 - 4 \sin^2(\pi/14) > 3 - 4 \sin^2(\pi/6) > 0$, we have

$$
F_3(\varphi) < \left( 8 \sin^2 \frac{\pi}{14} - 7 \right) \sin^2 \frac{\pi}{14} + \left( 3 - 4 \sin^2 \frac{\pi}{14} \right) \sin^2 \frac{\pi}{14} = -4 \cos^2 \frac{\pi}{14} \sin^2 \frac{\pi}{14},
$$

which together with (3.9) proves the inequality $\Lambda_3(\varphi) < 0$ for $\varphi \in (0, \pi/2)$.

Now, by (3.7) and (3.4) we conclude that

$$
\frac{dK_{6,\infty}}{d\alpha} < 0
$$

for $\alpha \in (0, \pi/2)$. Thus, by (3.11),

$$
K_{6,\infty} = K_{6,\infty}(0) = \frac{6!}{\pi} \int_{-\pi/2}^{\pi/2} \left| \cos (\alpha - \varphi) \right| \left| \cos (\alpha + \varphi) \right| d\varphi = \frac{2 \cdot 6!}{\pi} \left\{ \int_0^{\pi/14} \cos 7\varphi \cos 5\varphi d\varphi - \int_{3\pi/14}^{\pi/14} \cos 7\varphi \cos 5\varphi d\varphi + \int_{3\pi/14}^{5\pi/14} \cos 7\varphi \cos 5\varphi d\varphi + \int_{5\pi/14}^{\pi/2} \cos 7\varphi \cos 5\varphi d\varphi \right\}.
$$

Evaluating the integrals on the right-hand side of the last equality, we arrive at (3.6).

\textbf{Corollary 4.} Let $\Re f \in h^\infty(\mathbb{R}_+^2)$, and let $z$ be an arbitrary point in $C_+$. The sharp constant $K_{8,\infty}$ in the inequality

$$
|f^{(8)}(z)| \leq \frac{K_{8,\infty}}{(3z)^8} ||\Re f||_\infty
$$

is given by

$$
K_{8,\infty} = \frac{315}{8\pi} \left\{ 175 + 9\sqrt{2} \left( 17 \cos \frac{\pi}{36} + 9 \cos \frac{5\pi}{36} + 11 \cos \frac{7\pi}{36} \right) \right\}.
$$

\textbf{Proof.} By Lemma 2,

$$
\frac{dK_{8,\infty}}{d\alpha} = \frac{70}{9\pi} \int_0^{\pi/2} \left| \cos (\alpha - \varphi) \left| - \cos (\alpha + \varphi) \right| \Lambda_4(\varphi) \right| d\varphi,
$$

where

$$
\Lambda_4(\varphi) = 7 \left( -5 \frac{\sin \frac{\varphi}{9}}{\sin \frac{\pi}{18}} + 18 \sin \frac{3\varphi}{9} - 5 \frac{\sin \frac{5\varphi}{9}}{\sin \frac{\pi}{18}} + \frac{\sin \frac{7\varphi}{9}}{\sin \frac{\pi}{18}} \right).
$$
Using the identities
\[\sin 3x = 3\sin x - 4\sin^3 x, \sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x\]
and
\[\sin 7x = 7\sin x - 56\sin^3 x + 112\sin^5 x - 64\sin^7 x\]
in (3.8), we find
\[\Lambda_4(\phi) = 896 \frac{\sin^2 \frac{\pi}{18} \sin \frac{\phi}{9}}{\sin \frac{\pi}{18} \sin \frac{\phi}{9}} \left( \sin^2 \frac{\pi}{18} - \sin^2 \frac{\phi}{9} \right) F_4(\phi),
\]
where
\[F_4(\phi) = \left( 155 - 604\sin^2 \frac{\pi}{18} + 768\sin^4 \frac{\pi}{18} - 320\sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18}
+ \left( -90 + 396\sin^2 \frac{\pi}{18} - 560\sin^4 \frac{\pi}{18} + 256\sin^6 \frac{\pi}{18} \right) \sin^2 \frac{\pi}{18} \sin^2 \frac{\phi}{9}
+ \left( 15 - 80\sin^2 \frac{\pi}{18} + 128\sin^4 \frac{\pi}{18} - 64\sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\phi}{9}.
\]
It follows
\[F_4(\phi) > \left( 155 - 604\sin^2 \frac{\pi}{18} + 768\sin^4 \frac{\pi}{18} - 320\sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18}
- 10 \left( 9 + 56\sin^4 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18} - 16 \left( 5 + 4\sin^4 \frac{\pi}{18} \right) \sin^6 \frac{\pi}{18}
= \left( 65 - 684\sin^2 \frac{\pi}{18} + 208\sin^4 \frac{\pi}{18} - 384\sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18},
\]
Using this inequality and \(\sin(\pi/12) = (\sqrt{6} - \sqrt{2})/4\), we obtain
\[F_4(\phi) > \left( 65 - 684\sin^2 \frac{\pi}{18} - 384\sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18}
> \left( 65 - 684\sin^2 \frac{\pi}{12} - 384\sin^6 \frac{\pi}{12} \right) \sin^4 \frac{\pi}{18} = \left( 261\sqrt{3} - 433 \right) \sin^4 \frac{\pi}{18} > 0.
\]
The last estimate together with (3.14) proves the inequality \(\Lambda_4(\phi) > 0\) for \(\phi \in (0, \pi/2)\). Now, by (3.12) and (3.3) we conclude that
\[\frac{dK_{8, \infty}}{d\alpha} > 0\]
for \(\alpha \in (0, \pi/2)\). Thus, by (3.1),
\[K_{8, \infty} = K_{8, \infty}(\pi/2) = \frac{8!}{\pi} \int_{-\pi/2}^{\pi/2} |\sin 9\phi| \cos^7 \phi \, d\phi = 2\frac{8!}{\pi} \left\{ \int_{0}^{\pi/9} \sin 9\phi \cos^7 \phi \, d\phi
- \int_{\pi/9}^{2\pi/9} \sin 9\phi \cos^7 \phi \, d\phi + \int_{2\pi/9}^{\pi/3} \sin 9\phi \cos^7 \phi \, d\phi
- \int_{\pi/3}^{4\pi/9} \sin 9\phi \cos^7 \phi \, d\phi + \int_{4\pi/9}^{\pi/2} \sin 9\phi \cos^7 \phi \, d\phi \right\}.
\]
After evaluating the integrals on the right-hand side of the last equality, we arrive at (3.11). \(\square\)
Now we apply Lemma 1 in the proof of the following assertion.

**Theorem 2.** The following two-side inequality

\[
\frac{2}{\pi} ((2m-1)!!)^2 < K_{2m,\infty} < \frac{2m}{2m-1} \frac{2}{\pi} ((2m-1)!!)^2 
\]  

(3.15)

holds.

**Proof.** By (3.1),

\[
K_{2m,\infty}(\alpha) < \frac{(2m)!}{\pi} \int_{-\pi/2}^{\pi/2} |\cos (\alpha - (2m+1)\varphi)| \cos^{2m-1} \varphi d\varphi .
\]

From this and equality (1.3) with \(m - 1\) instead of \(m\) and \(n = 2m, \gamma = 1\), we obtain

\[
K_{2m,\infty}(\alpha) < \frac{2(2m)!}{\pi} \frac{(2m-3)!!}{(2m-2)!!} = \frac{2m}{\pi} \frac{2m}{2m-1} \frac{(2m-1)!!}{(2m-2)!!} .
\]  

(3.16)

which together with (0.5) proves the upper estimate in (3.15).

Now we turn to the inverse estimate of \(K_{2m,\infty}\) in (3.15). It follows from (0.5) and (3.1),

\[
K_{2m,\infty} \geq K_{2m,\infty}(\alpha) > \frac{(2m)!}{\pi} \int_{-\pi/2}^{\pi/2} |\cos (\alpha - (2m+1)\varphi)| \cos^{2m} \varphi d\varphi .
\]

Using the last estimate and (1.3) with \(n = 2m\) and \(\gamma = 1\), we find that

\[
K_{2m,\infty} > \frac{2(2m)!}{\pi} \frac{(2m-1)!!}{(2m)!!} ,
\]

which is equivalent to the lower estimate in (3.15).

In final section we describe some real-part estimates which take the explicit form by combination with formulas for \(K_{n,p}\) from (0.6)-(0.8), Theorem 1, Corollaries 1-4 and the estimate of Theorem 2.

4 Explicit estimates for derivatives of analytic functions in domains

The next two assertions were proved in paper by Kresin and Maz’ya [10].

**Proposition 1.** Let \(\Omega = \mathbb{C} \setminus G\), where \(G\) is a convex domain in \(\mathbb{C}\), and let \(f\) be a holomorphic function in \(\Omega\) with bounded real part. Then for any point \(z \in \Omega\) the inequality

\[
|f^{(n)}(z)| \leq \frac{K_{n,\infty}}{d_z^n} \sup_{\Omega} |\Re f| , \quad n = 1, 2, \ldots ,
\]

holds.
holds with $d_z = \text{dist} \left( z, \partial \Omega \right)$, where

$$
K_{n,\infty} = \frac{n!}{\pi} \max_{\beta} \int_{-\pi/2}^{\pi/2} \left| \cos \left( \beta - (n + 1)\varphi \right) \right| \cos^{n-1} \varphi \, d\varphi
$$

is the best constant in the inequality

$$
|f^{(n)}(z)| \leq \frac{K_{n,\infty}}{(3z)^n} \|\Re f\|_{\infty}
$$

for holomorphic functions $f$ in the half-plane $\mathbb{C}_+$ with the bounded real part.

**Proposition 2.** Let $\Omega$ be a domain in $\mathbb{C}$, and let $\mathcal{R}(\Omega)$ be the set of holomorphic functions $f$ in $\Omega$ with $\sup_{\Omega} |\Re f| \leq 1$. Assume that a point $\zeta \in \partial \Omega$ can be touched by an interior disk $D$. Then

$$
\limsup_{z \to \zeta} \sup_{f \in \mathcal{R}(\Omega)} |z - \zeta|^n |f^{(n)}(z)| \leq K_{n,\infty}, \quad n = 1, 2, \ldots,
$$

where $z$ is a point of the radius of $D$ directed from the center to $\zeta$. Here the constant $K_{n,\infty}$ is the same as in Proposition 1 and cannot be diminished.

By (0.6) and (3.15), the constant $K_{n,\infty}$ in Propositions 1 and 2 obeys the relations

$$
K_{2m-1,\infty} = \frac{2}{\pi} \frac{(2m-1)!!}{2m-1}, \quad K_{2m,\infty} < \frac{2m}{2m-1} \frac{2}{\pi} \frac{(2m-1)!!}{2m-1}
$$

for any $m \in \mathbb{N}$. The values of the constants $K_{2,\infty}$, $K_{4,\infty}$, $K_{6,\infty}$ and $K_{8,\infty}$ in these statements are given by (0.7), (3.6) and (3.11), correspondingly.

Now we turn to a real-part estimate for derivatives of analytic functions in the disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$. By $h^p(D)$, $1 \leq p \leq \infty$, we mean the Hardy space of harmonic functions in the real unit disk $D$ which are represented by the Poisson integral with a density in $L^p(\partial D)$. Below by $|| \cdot ||_p$ we denote the norm in the space $L^p(\partial D)$.

The inequality, obtained by Khavinson [5]

$$
|f'(z)| \leq \frac{4}{\pi(1-r^2)}||\Re f||_{\infty},
$$

contains the best possible coefficient in front of $||\Re f||_{\infty}$, where $r = |z| < 1$.

The next estimate for derivatives of analytic functions with $\Re f \in h^p(D)$

$$
|f^{(n)}(z)| \leq \frac{C_{n,p}}{(1-r^2)^{n+\frac{1}{p}}}||\Re f||_p \tag{4.1}
$$

was proved in the paper by Kalaj and Elkes [3]. The representation of the constant $2^{-\left(n+\frac{1}{2}\right)}C_{n,p}$ in [3] is equivalent to the representation (0.5) with (0.3) for the sharp constant $K_{n,p}$ in inequality (0.4). The case $n = 1$ in (4.1) was considered by Kalaj and Marković [4].

The assertion below was established in paper by Kresin [9].
**Proposition 3.** Let $f$ be an analytic function in $D$ with $\Re f \in h^p(D)$. The inequality

$$\sup_{|z| < 1} \sup_{||\Re f||_p \leq 1} (1 - |z|^2)^{n + \frac{1}{p}} |f^{(n)}(z)| \geq 2^{n + \frac{1}{p}} K_{n,p}$$

holds, where $K_{n,p}$ is the sharp constant in inequality (0.4).

Proposition 3 together with (4.1) leads to relation

$$\sup_{|z| < 1} \sup_{||\Re f||_p \leq 1} (1 - |z|^2)^{n + \frac{1}{p}} |f^{(n)}(z)| = 2^{n + \frac{1}{p}} K_{n,p},$$

which shows that the constant $C_{n,p} = 2^{n + \frac{1}{p}} K_{n,p}$ in estimate (4.1) cannot be diminished.

The explicit expression for $C_{2m-1,\infty}$ was established by Kalaj and Elkies [3]. The formulas for $C_{2m,\infty} = 2^{2m} K_{2m,\infty}$ with $m = 1, 2, 3, 4$ can be obtained by (0.7), (3.6) and (3.11). Other examples of the explicit formulas for the constant $C_{n,p}$ in (4.1) can be derived by relation $C_{n,p} = 2^{n + \frac{1}{p}} K_{n,p}$ and Theorem 1 as well as Corollaries 1, 2.

The next two-sided inequality

$$\frac{2^{2m+1}}{\pi} \left((2m - 1)!!\right)^2 < C_{2m,\infty} < \frac{2m}{2m - 1} \frac{2^{2m+1}}{\pi} \left((2m - 1)!!\right)^2$$

follows from equality $C_{2m,\infty} = 2^{2m} K_{2m,\infty}$ and estimate (3.15). This implies

$$C_{2m,\infty} \sim \frac{2^{2m+1}}{\pi} \left((2m - 1)!!\right)^2$$

as $m \to \infty$.

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