Josephson junctions of multiple superconducting wires

Oindrila Deb\textsuperscript{1}, K. Sengupta\textsuperscript{2}, and Diptiman Sen\textsuperscript{1}

\textsuperscript{1}Centre for High Energy Physics, Indian Institute of Science, Bengaluru 560012, India
\textsuperscript{2}Theoretical Physics Department, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700032, India

(Dated: May 24, 2018)

We study the spectrum of Andreev bound states and Josephson currents across a junction of \(N\) superconducting wires which may have \(s\)- or \(p\)-wave pairing symmetries and develop a scattering matrix based formalism which allows us to address transport across such junctions. For \(N \geq 3\), it is well known that Berry curvature terms contribute to the Josephson currents; we chart out situations where such terms can have relatively large effects. For a system of three \(s\)- or three \(p\)-wave superconductors, we provide analytic expressions for the Andreev bound state energies and study the Josephson currents in response to a constant voltage applied across one of the wires; we find that the integrated transconductance at zero temperature is quantized to integer multiples of \(4e^2/h\), where \(e\) is the electron charge and \(h = 2\pi\hbar\) is Planck’s constant. For a sinusoidal current with frequency \(\omega\) applied across one of the wires in the junction, we find that Shapiro plateaus appear in the time-averaged voltage \(\langle V_1 \rangle\) across that wire for any rational fractional multiple (in contrast to only integer multiples in junctions of two wires) of \(2e\langle V_1 \rangle/(h\omega)\). We also use our formalism to study junctions of two \(p\)- and one \(s\)-wave wires. We find that the corresponding Andreev bound state energies depend on the spin of the Bogoliubov quasiparticles; this produces a net magnetic moment in such junctions. The time variation of these magnetic moments may be controlled by an external voltage applied across the junction. We discuss experiments which may test our theory.

I. INTRODUCTION

Josephson junctions have constituted a fascinating aspect of superconducting systems from the very beginning; see, for example, Ref. 1. Such junctions exhibit a variety of phenomena, such as the dc Josephson effect (a constant current flows in the absence of any voltage biases between the different superconductors), the ac Josephson effect (an alternating current flows in the presence of constant voltage biases), and Shapiro steps (these appear as plateaus in plots of the average voltage versus average current when the currents are made to vary periodically). The physics of such junctions is known to rely crucially on the pairing symmetry of its constituent superconductors. For example, a junction of two \(p\)-wave superconductors exhibits a fractional Josephson effect\textsuperscript{2,3} which manifests itself in a fractional Josephson frequency \(\omega_f = eV/h\) or the absence of odd-integer Shapiro steps. The latter property of such junctions has been used experimentally for the detection of Majorana modes\textsuperscript{4}. Furthermore, a junction of two superconducting wires of \(s\)- and \(p\)-wave symmetries is known to generate a magnetic moment at the interface whose time variation can be controlled by an external applied voltage\textsuperscript{5,6}. Multiple junctions with \(s\)-wave superconductors have been studied using a scattering matrix formalism\textsuperscript{6,7}, and voltage-induced Shapiro steps have been studied in a junction of three \(s\)-wave superconductors\textsuperscript{10}. However, no such studies have been carried out for multi-terminal junctions involving unconventional superconductors.

In recent years, topological phases of matter have also been studied extensively \textsuperscript{11,12}. These are usually characterized by bulk band structures which have non-zero values of some topological invariant, such as the Chern number in two-dimensional systems. The value of the topological invariant determines several properties of the system such as the number of boundary modes and their contribution to electron transport. Recently it has been shown that Josephson junctions of three or more superconductors can exhibit interesting topological properties\textsuperscript{6,7,13-18} as follows. First, there is a Berry curvature associated with the Andreev bound-state wave functions; this curvature contributes to the Josephson currents. Second, the current-voltage relation can, in certain situations, involve a Chern number which is given by the integral of the Berry curvature over a two-dimensional space of the superconducting phases. A three-terminal Josephson interferometer has been realized experimentally and some topological transitions have been observed recently\textsuperscript{19}. However, such studies have not been extended to multi-wire junctions involving both \(s\)- and \(p\)-wave superconductors.

In this work we develop a scattering matrix based approach which allows us to address transport properties of multi-wire Josephson junctions involving both \(s\)- and \(p\)-wave superconducting wires; our work therefore constitutes a generalization of similar multi-wire junctions involving only \(s\)-wave superconductors. The plan of our paper and the key results obtained are as follows. In Sec. 11 we introduce the model of \(N\) superconducting wires with either \(s\)- or \(p\)-wave symmetries meeting at a junction which is characterized by a scattering matrix \(S\);
we obtain an expression for the Andreev bound-state energies presented in terms of $S$ and the pairing phase $\delta$. It is well known that for $N \geq 3$, Berry curvature terms appear and can contribute to the Josephson currents in the different wires; we show that for junctions involving both $s$- and $p$-wave wires, such terms, along with the Andreev bound-state energies, have a non-trivial dependence on the spin of the quasiparticles. Second, we provide a detailed study of the dc and ac Josephson effects in junctions with all $s$- or all $p$-wave wires. We show that a constant voltage $V_j$ applied across one of the wires (say the $j$th wire) in such a junction leads to a finite constant current in a different wire (say the $i$th wire) which receives contributions from the Berry curvature terms and leads to a quantized zero temperature integrated transconductance $G_{ij} = \int d\phi_i/(2\pi)(d\langle I_i \rangle/dV_j) = 4e^2p/h$, where $\phi_i$ is the superconducting phase of the $i$th wire, $p$ is an integer, and $\langle \ldots \rangle$ denotes time average over a time period $T = h/(2eV_j)$. We also consider an RC circuit in which the Josephson current in each wire flows in parallel with a resistance and a capacitance. We show that if the applied current in one of the wires has a sinusoidally varying term characterized by a frequency $\omega$, Shapiro plateaus can appear in the plot of the time-averaged voltage $\langle V_j \rangle$ versus the average current in that wire. Our results show that Shapiro plateaus for such junctions occur when $\langle V_j \rangle$ and $\omega$ satisfy $e\langle V_j \rangle/(\hbar \omega) = m/\ell$, for integers $m$ and $\ell$. This indicates that plateaus can appear when $2e\langle V_j \rangle/(\hbar \omega)$ is any rational fraction (in contrast to only integer values for standard Shapiro plateaus in two-terminal junctions), and this may lead, in principle, to a devil’s staircase structure of such plateaus. In Sec. III we discuss the case of three $s$-wave superconducting wires in detail; the Andreev bound states can be analytically found in this case. We will take a simple example of an $S$-matrix which is time-reversal symmetric and a randomly generated example of an $S$-matrix which is not time-reversal symmetric; both give rise to a Berry curvature but the Chern number (given by a two-dimensional integral of the Berry curvature) is zero in the first case and non-zero in the second case. We also discuss the cases of three $p$-wave wires, and two $p$-wave and one $s$-wave wire; in the latter case, the energies of spin-up and -down Andreev bound states are not identical. In Sec. IV we present numerical results for different three-wire systems. For the case of three $s$-wave superconducting wires, we first discuss the ac Josephson effect and find how this can clearly show the effects of the Berry curvature. We then show that Shapiro plateaus can appear in the plot of the voltage in a particular wire versus the dc part of the current in the same wire when the current has an ac part which varies sinusoidally with a frequency $\omega$. We find that Shapiro plateaus can appear at both integer and fractional multiples of $\hbar \omega/(2e)$. We provide an understanding of the widths of the Shapiro plateaus by relating them to the Fourier transforms of the energies of the Andreev bound states. Similar Shapiro plateaus appear in the case of three $p$-wave wires. For a system of two $p$-wave and one $s$-wave wires, the asymmetry between the energies of spin-up and -down states implies that there can be interesting spin-dependent effects; in particular, we show that the junction region can have a net magnetic moment whose time variation can be controlled by an external applied voltage. We conclude in Sec. V by summarizing our results and suggesting some experimental tests of our results.

II. JUNCTION OF $N$ SUPERCONDUCTING WIRES

A. Model

We consider a system consisting of $N$ wires which meet at a junction; a schematic picture for $N = 3$ is shown in Fig. 1. Each wire, labeled $i$, consists of a normal part (shown in orange color) where the coordinate denoted by $x_i$ increases from zero (which is exactly at the junction of the $N$ wires) to a small value $\delta$. Beyond $x_i = \delta$, the wire is superconducting (shown in dark blue color). Beyond a large length, the superconducting part of the wire is connected to a normal metal lead which is at a potential $V_i$, and there is an incoming current on that lead given by $I_i$; these leads are not shown in Fig. 1. We will assume

FIG. 1: Schematic picture of a junction of three superconducting wires. The superconducting wires (marked SC1, SC2 and SC3) meet at a junction which is a normal region which is characterized by an $S$-matrix. The coordinate $x_1$ and the size of the junction $\delta$ are indicated for the wire SC1.
that $\delta$ (the length of the normal part of each wire) is small enough that we can approximately set $e^{i k_F \delta} = 1$, where $k_F$ is the Fermi momentum (taken to be the same in all the wires).

We will be interested in both $s$- and $p$-wave superconductors. We therefore recall the following facts about such SCs; see, for instance, Ref. [23]. In terms of the spin Pauli matrices $\sigma^x, \sigma^y, \sigma^z$, the pairing part of the Hamiltonian of an $s$-wave SC has the second-quantized form

$$H_{\text{pair}} = \int dx \left[ \Delta e^{i \phi} \left( \Psi^\dagger \downarrow \Psi^\dagger \uparrow \right) i \sigma^y \left( \Psi^\dagger \uparrow \Psi^\dagger \downarrow \right) + \text{H.c.} \right]$$

$$= \int dx \left[ \Delta e^{i \phi} (\Psi^\dagger \downarrow \Psi^\dagger \uparrow - \Psi^\dagger \uparrow \Psi^\dagger \downarrow) + \text{H.c.} \right].$$

Equation (1) describes Cooper pairs which are in the spin-singlet state $| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle$. In a $p$-wave SC, the pairing part of the Hamiltonian is given by

$$H_{\text{pair}} = \int dx \left[ -i \frac{\Delta e^{i \phi}}{k_F} \left( \Psi^\dagger \uparrow \right) \vec{d} \cdot \vec{\sigma} i \sigma^y \frac{\partial}{\partial x} \left( \Psi^\dagger \downarrow \right) + \text{H.c.} \right],$$

where $\vec{d}$ is a unit vector; for $\vec{d} = \hat{x}$, $\hat{y}$ and $\hat{z}$, the Cooper pairs are in the spin triplet states $| \uparrow \uparrow \rangle - | \downarrow \downarrow \rangle, | \uparrow \uparrow \rangle + | \downarrow \downarrow \rangle$ and $| \uparrow \downarrow \rangle + | \downarrow \uparrow \rangle$ respectively. The factor of $-i/k_F \partial / \partial x$ in Eq. (2) gives $+1$ if we consider electrons near the Fermi momenta $\pm k_F$. We will restrict ourselves to the case where all $p$-wave superconducting wires have the same orientation of $\vec{d}$. In the absence of an external magnetic field, we can then choose $\vec{d} \parallel \hat{z}$ without loss of generality. In this case, Eq. (2) takes the form

$$H_{\text{pair}} = \int dx \left[ -i \frac{\Delta e^{i \phi}}{k_F} \left( \Psi^\dagger \uparrow \right) \Psi^\dagger \downarrow + \Psi^\dagger \uparrow \frac{\partial}{\partial x} \left( \Psi^\dagger \downarrow \right) + \text{H.c.} \right].$$

Thus the Cooper pairs are in the spin triplet state with $S^z = 0$; hence Cooper pairs in $s$- and $p$-wave SC wires will have the same value of $S^z$. This will make it possible to study a system with both $s$-wave and $p$-wave SC wires since all the wires will be compatible with each other at the junction where an electron or hole from one of the wires can scatter into another wire. We note that the choice of pairing in the $S^z = 0$ channel for all the $p$-wave SC wires in the junction does not lead to a loss of generality for the following reasons. First, for junctions with a single $p$-wave wire, the choice of pairing in the $S^z = 0$ sector is merely a choice of the spin quantization axis to be along $\vec{d} \parallel \hat{z}$; it does not alter the physical properties of the junction. Second, the choice of the same direction $\vec{d}$ for the different $p$-wave wires is not artificial. It is well known that the direction of $\vec{d}$ depends on the material properties and geometry of the wires; hence multiple wires constructed out of the same material are expected to have $\vec{d}$ in the same direction. Finally, a junction between two $p$-wave superconductors with orthogonal $\vec{d}$-vectors is known not to support a Josephson current at zero temperature. Thus, taking these issues into consideration we choose all the $p$-wave SC wires in our junction to have pairings in the $S^z = 0$ channel.

We first consider a system of $N$ wires which are all $s$-wave SCs. For spin-up quasiparticles, the annihilation operators are given by superpositions of $\Psi^\dagger$ and $\Psi^\dagger$, and we will denote the corresponding wave functions by $\psi_{c \uparrow}$ and $\psi_{h \uparrow}$. For spin-down quasiparticles, the annihilation operators are superpositions of $\Psi_\downarrow$ and $\Psi_\downarrow^\dagger$, and the corresponding wave functions are $\psi_{c \downarrow}$ and $\psi_{h \downarrow}$. We introduce a symbol $\sigma = +1 \ (-1)$ to denote spin up (spin down) respectively, and $\bar{\sigma} = -1 \ (+1)$ if $\sigma = +1 \ (-1)$. Hence, the wave function of a spin-$\sigma$ quasiparticle is given by a combination of $\psi_{c \sigma}$ and $\psi_{h \bar{\sigma}}$. For a spin-up electron, let $\phi_i$ be the phase of the SC pairing amplitude $\Delta_i$ on wire $i$, namely, $\Delta_i = \Delta e^{i \phi_i}$. We will take the magnitude of the pairing, $| \Delta_i | = \Delta$, to be the same in all the wires. Due to the spin-singlet nature of the Cooper pairs in an $s$-wave SC, spin-down electrons will have the pairing amplitude $\Delta_i = - \Delta e^{i \phi_i}$; this is clear from Eq. (1). We can therefore write the pairing amplitude as $\sigma \Delta e^{i \phi_i}$ for spin-$\sigma$ electrons on wire $i$.

We now recall the derivation of the energy $E_\sigma$ of an Andreev bound state (ABS) which lies in the SC gap, i.e., $| E_\sigma | < \Delta$. To this end, we assume that the normal region lying at the center of the system (i.e., the orange region in Fig. 1) is characterized by a unitary $N \times N$ scattering matrix $S$. (If we impose time-reversal symmetry, $S$ will also be a symmetric matrix). We will assume that $S$ does not depend on the energy; this is because the magnitude of the pairing, $\Delta$, is typically much smaller than the Fermi energy $E_F = h^2 k_F^2 / (2m)$. We therefore take $S$ to be constant in the range of energies from $-\Delta$ to $+\Delta$ (note that the energies are defined taking $E_F$ to be the zero level). The fact that the central region is normal means that the form of $S$ should be the same for spin-up and spin-down electrons even though electrons may have different pairing amplitudes in the SC regions. Furthermore, the region being normal implies that the incoming and outgoing electron wave functions in wire $i$, $\psi_{\text{in}}^{c \sigma i}(E_\sigma)$ and $\psi_{\text{out}}^{c \sigma i}(E_\sigma)$, are related as

$$\psi_{\text{out}}^{c \sigma i}(E_\sigma) = \sum_{j=1}^{N} S_{ij} \psi_{\text{in}}^{c \sigma j}(E_\sigma),$$

the incoming and outgoing hole wave functions, $\psi_{\text{h}}^{c \sigma i}(E_\sigma)$ and $\psi_{\text{h}}^{c \sigma j}(E_\sigma)$.
and \( \psi_{h\sigma}^{\text{out}}(E_\sigma) \), are related as
\[
\psi_{h\sigma}^{\text{out}}(E_\sigma) = \sum_{j=1}^{N} S_{ij}^{*} \psi_{h\sigma j}^{\text{in}}(E_\sigma),
\] (5)
and electron and hole wave functions are not coupled to each other through \( S \). Next, when an outgoing electron (hole) in the normal region in wire \( i \) strikes the junction with the SC at \( x_i = \delta \), it is Andreev reflected back to the normal region as an incoming hole (electron). Namely,\( \text{a}_{\sigma}(E_\sigma) = e^{i\phi_i} \psi_{h\sigma i}^{\sigma}(E_\sigma), \)
\[
\psi_{h\sigma i}^{\text{in}}(E_\sigma) = a_{\sigma}(E_\sigma) \sigma e^{-i\phi_i} \psi_{h\sigma i}^{\sigma}(E_\sigma),
\] (6)
where
\[
a_{\sigma}(E_\sigma) = \frac{E_\sigma - i \sqrt{\Delta^2 - E_\sigma^2}}{\Delta}.
\] (7)
(Note that we are ignoring any phases picked up by the electrons or holes while propagating between the junction at \( x_i = 0 \) and the SC at \( x_i = \delta \) due to the approximation \( e^{ik_\delta} = 1 \). In Eq. (7) we note that the real part of \( a_{\sigma}(E_\sigma) \) can be positive, negative or zero, while the imaginary part can only be negative or zero; this is important since the eigenvalue equations given below will only fix the value of \( a_{\sigma}^2(E_\sigma) \), and we then have to take the appropriate square root of that to obtain \( a_{\sigma}(E_\sigma) \). Combining Eqs. (6) and (7), we find that
\[
\psi_{e\sigma i}^{\sigma}(E_\sigma) = a_{\sigma}^2(E_\sigma) \sum_{j,k=1}^{N} S_{ij} e^{-i\phi_i} S_{jk} e^{-i\phi_k} \psi_{e\sigma k}^{\sigma}(E_\sigma).
\] (8)
Introducing an \( N \)-dimensional column \( \psi_{e\sigma i}(E_\sigma) \) whose entries are given by \( \psi_{e\sigma i}^{\sigma}(E_\sigma) \), a diagonal matrix \( e^{i\phi} \) whose diagonal entries are given by \( e^{i\phi_i} \), and its inverse matrix \( e^{-i\phi} \), Eq. (8) takes the form of an eigenvalue equation
\[
S e^{i\phi} S^* e^{-i\phi} \psi_{e\sigma i}(E_\sigma) = \frac{1}{a_{\sigma}^2(E_\sigma)} \psi_{e\sigma i}(E_\sigma).
\] (9)
It is clear from Eq. (9) that the ABS energies and the corresponding wave functions do not change if any of the phases \( \phi_i \) are shifted by \( 2\pi \). In addition, the ABS energies and wave functions remain unchanged if all the phases \( \phi_i \) are shifted by the same constant, since that constant will cancel out between \( e^{i\phi} \) and \( e^{-i\phi} \). As a result, we have the identities
\[
\sum_{i=1}^{N} \frac{\partial E_{\sigma}}{\partial \phi_i} = 0,
\] (10)
\[
\sum_{i=1}^{N} \frac{\partial \psi_{e\sigma i}^{\sigma}}{\partial \phi_i} = 0, \quad \text{and} \quad \sum_{i=1}^{N} \frac{\partial \psi_{h\sigma i}^{\text{in}}}{\partial \phi_i} = 0.
\] (11)
We will use this symmetry to set one of the phases (for example, \( \phi_1 \) for a three-wire system) equal to zero in many of the calculations.

Equation (9) implies that
\[
S e^{i\phi} S^* e^{-i\phi} S e^{i\phi} \psi_{e\sigma i}^{\sigma}(E_\sigma) = \frac{1}{a_{\sigma}^2(E_\sigma)} S e^{i\phi} \psi_{e\sigma i}^{\sigma}(E_\sigma).
\] (12)
Since \( a_{\sigma}^2(E_\sigma) = |\left( -E_\sigma - i \sqrt{\Delta^2 - E_\sigma^2} \right)/\Delta |^2 \), Eqs. (9) and (12) imply that if there is an ABS at energy \( E_\sigma \) with wave function \( \psi_{e\sigma i}(E_\sigma) \), there must be an ABS at energy \( -E_\sigma \) with wave function \( \psi_{e\sigma i}(-E_\sigma) = S e^{i\phi} \psi_{e\sigma i}^{\sigma}(E_\sigma) \). An exception to this statement can occur if \( a_{\sigma}^2(E_\sigma) = 1 \) is real in which case there may be only one ABS with no degeneracy. In particular, if \( a_{\sigma}^2(E_\sigma) = -1 \), there may be only one state lying at \( E_\sigma = 0 \), and if \( a_{\sigma}^2(E_\sigma) = 1 \), there may be only one state lying at \( E_\sigma^2 = \Delta^2 \) (however, this should not really be considered to be a bound state since its energy lies at the edge between the SC gap and the bulk states, and its localization length is therefore large).

We note that Eq. (9) has the same form for \( \sigma = \pm 1 \). Namely, the ABS energies \( E_{\sigma} \) and wave functions \( \psi_{e\sigma i} \) are identical for spin-up and -down quasiparticles in a system in which all the wires are s-wave SCs. Hence each energy will have a two-fold degeneracy.

Next, we discuss a system in which some of the SC wires are s-wave and the others are p-wave. Comparing Eqs. (1) and (4), we see that the pairing amplitudes for spin-up and -down electrons have opposite signs for an s-wave SC but the same sign for a p-wave SC. Hence if wire \( i \) is an s-wave SC, Eq. (10) holds. But if wire \( i \) is a p-wave SC, we find that the factors of \( \sigma \) do not appear; however one of the equations in Eq. (10) picks up a minus sign. Namely, we find that
\[
\psi_{h\sigma i}^{\text{in}}(E_\sigma) = a_{\sigma}(E_\sigma) e^{-i\phi_i} \psi_{e\sigma i}^{\sigma}(E_\sigma),
\] (13)
This leads us to define a diagonal matrix \( \eta \) whose \( i \)-th diagonal entry \( \eta_i \) is equal to \( +1 \) (\(-1 \)) if the \( i \)-th SC wire is s-wave (p-wave). To go from spin-up quasiparticles to spin-down quasiparticles, we have to change \( e^{i\phi_i} \rightarrow -\eta_i e^{i\phi_i} \) and \( e^{-i\phi_i} \rightarrow -\eta_i e^{-i\phi_i} \) on wire \( i \). We thus see that to account for spin, it is useful to consider a diagonal matrix which is equal to \( -\eta \). We then find that the ABS energy is given by the eigenvalue equation
\[
S \eta e^{i\phi} S^* e^{-i\phi} \psi_{e\sigma i}(E_\sigma) = \frac{1}{a_{\sigma}^2(E_\sigma)} \psi_{e\sigma i}(E_\sigma),
\] (14)
for spin-up quasiparticles (\( \sigma = +1 \)), and
\[
S e^{i\phi} S^* \eta e^{-i\phi} \psi_{e\sigma i}(E_\sigma) = \frac{1}{a_{\sigma}^2(E_\sigma)} \psi_{e\sigma i}(E_\sigma),
\] (15)
for spin-down quasiparticles (\( \sigma = -1 \)). We then see that for each value of \( \sigma \), the ABS energies have an \( E \rightarrow -E \)
symmetry only if all the SC wires are s-wave or all are p-wave. If some of them are s-wave and some are p-wave, there is no $E \rightarrow -E$ symmetry in general.

We now note that Eq. (14) implies

$$Se^{i\phi}S^*\eta e^{-i\phi}Se^{i\phi}\psi^*_\sigma(E_\sigma) = \frac{1}{a_{\sigma}^2(E_\sigma)}Se^{i\phi}\psi^*_\sigma(E_\sigma). \tag{16}$$

Hence, if there is an ABS at energy $E$ with wave function $\psi_\sigma(E)$ for spin-up quasiparticles separately do not show that our formalism reproduces the known result for (one $|E|$ two ABS energies. For two $s$ energies are given by

$$E_\sigma = \pm \Delta \sqrt{1 - |S_{12}|^2 \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right)} \tag{17}$$

for both spin-up and -down quasiparticles. For two $p$-wave wires, the ABS energies are

$$E_\sigma = \pm \Delta |S_{12}| \sin\left(\frac{\phi_1 - \phi_2}{2}\right) \tag{18}$$

for both spin-up and spin-down quasiparticles. If one wire is an s-wave SC and the other is p-wave, the ABS energies are

$$E_\sigma = \sigma \text{ sgn} (\sin(\phi_1 - \phi_2)) \times \Delta \sqrt{1 \pm \frac{1}{2} - \frac{|S_{12}|^2 \sin^2(\phi_1 - \phi_2)}{2}}, \tag{19}$$

where sgn denotes the signum function.

To compare the expressions in Eqs. (17-19) with those given in Ref. 3, we have to note the following. At a point $x$ in a SC wire, an s-wave SC has a pairing of the form given in Eq. (1) while a p-wave SC has a pairing of the form given in Eq. (2). On comparing the two expressions, we see that an s-wave pairing phase $\phi$ is insensitive to the sign of the coordinate $x$ while a p-wave pairing phase depends on the sign of $x$ because of the $\partial/\partial x$. As mentioned at the beginning of this section, we are taking the coordinates $x_i$ in every wire to increase from zero at the junction; this is a convenient choice for three or more wires. However, for a two-wire system, it is conventional to take $x$ to go from $-\infty$ to $\infty$ with the junction being at $x = 0$; hence the coordinate increases from zero to $\infty$ in one of the wires and decreases from zero to $-\infty$ in the other wire. We can change our notation to agree with this convention by changing the coordinate in, say, wire 1 from $x_1$ to $-x_1$. If wire 1 has p-wave pairing, this must be compensated by changing the phase $\phi_1 \rightarrow \phi_1 + \pi$. No such change in the phase is required if wire 1 has s-wave pairing. Hence, for two s-wave SC wires the ABS energies are still given by Eq. (17). But for two $p$-wave wires, we have to shift either $\phi_1$ or $\phi_2$ by $\pi$. This changes the expression for the ABS energies from Eq. (18) to

$$E_\sigma = \pm \Delta |S_{12}| \cos\left(\frac{\phi_1 - \phi_2}{2}\right). \tag{20}$$

If wire 1 is s-wave and wire 2 is p-wave, we do not have to change $\phi_1$ and $\phi_2$, and the ABS energies are again given by Eq. (19). But if wire 1 is p-wave and wire 2 is s-wave, we have to shift $\phi_1$ by $\pi$ and the ABS energies are then given by Eq. (19) multiplied by $-1$. The expressions in Eqs. (17), (20), and (19) (up to a sign) agree with those given in Ref. 3.

### C. Berry curvature

Returning to the case of $N$ wires, we now look at the wave function $\psi_{n,\sigma}(E_{n,\sigma}) = \psi_{n,\sigma}^\text{out}(E_{n,\sigma})$ given by Eq. (9) for an ABS with spin $\sigma$ and energy $E_{n,\sigma}$ in the band labeled as $n$ (where $n = 1, 2, \cdots, N$). Following Ref. 6, we define the Berry curvature matrix

$$B_{n,\sigma,ij}(\phi_1, \phi_2, \cdots, \phi_N) = -2 \text{ Im} \left[ \frac{\partial \psi_{n,\sigma}^\text{out}}{\partial \phi_i} \frac{\partial \psi_{n,\sigma}}{\partial \phi_j} \right]. \tag{21}$$

This is a real antisymmetric matrix with the following properties. Equation (11) implies that each row and each column of $B_{n,\sigma,ij}$ adds up to zero, i.e.,

$$\sum_{i=1}^N B_{n,\sigma,ij} = \sum_{j=1}^N B_{n,\sigma,ij} = 0. \tag{22}$$

These identities along with the antisymmetry imply that, for each value of $n$ and $\sigma$, the matrix $B_{n,\sigma,ij}$ contains only $(N - 1)(N - 2)/2$ independent real parameters which we can take to be the values of $B_{n,\sigma,ij}$ for $1 \leq i < j \leq N - 1$. 

---

**Note:** The above text contains mathematical expressions and equations that are typical of a physics or mathematics document. The symbols and notation used are common in the field of quantum mechanics and condensed matter physics. The content discusses the behavior of quasi-particles in superconducting wires and the implications of different pairing mechanisms (s-wave, p-wave) on the energy spectra and Berry curvatures. The text also references previous works for further details and derivations.
Thus the Berry curvature can be non-zero only if $N \geq 3$. We note here, that in contrast to junctions of all $s$-wave wires or all $p$-wave wires, the Berry curvature for a mixed $s$-$p$ junction depends on $\sigma$. In this paper, we will use the technique discussed in Ref. \[20\] to calculate the Berry curvature. [Note that Eq. \[21\] would have given the same values of $B_{n,\sigma,ij}$ if we had used $\psi_{e,n,\sigma}^\dagger$ instead of $\psi_{e,n,\sigma}(E_{n,\sigma}) = \psi_{e,n,\sigma}^{\text{out}},$ since these are related by the matrix $S$ as in Eq. \[13\] and $S$ is independent of the $\phi_i$’s].

Holding the phases $\phi_3, \phi_4, \cdots, \phi_N$ fixed, we can define a Chern number $Ch_{n,\sigma,12}$ by integrating $B_{n,\sigma,12}$ over $\phi_1$ and $\phi_2$,

$$Ch_{n,\sigma,12} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 B_{n,\sigma,12}. \tag{23}$$

This is always quantized to have integer values.

Following Eq. \[16\], we can derive a relation between the Berry curvature for a positive energy, spin-up band (denoted as $n$, $\sigma = +1$), and a negative energy, spin-down band (denoted as $n'$, $\sigma = -1$). The wave functions in the two bands are related as $\psi_{n',-1}(E) = Se^{i\phi}\psi_{n,+1}(E)$, where $E > 0$. Let us denote the $j$-th component of the wave function $\psi_{n,+1}$ by $\psi_{n,j}$, where $j = 1, 2, \cdots, N$. We can then show that

$$B_{n',-1,ij}(-E) = -B_{n,+1,ij}(E) - \frac{\partial|\psi_{n,j}|^2}{\partial \phi_i} + \frac{\partial|\psi_{n,i}|^2}{\partial \phi_j}. \tag{24}$$

[We will see later that the last two terms in Eq. \[24\] are sometimes much smaller than the first term; we then have $B_{n',-1,ij}(-E) \simeq -B_{n,+1,ij}(E)$]. Equation \[24\] implies the exact relation

$$Ch_{n',-1,ij} = -Ch_{n,+1,ij}. \tag{25}$$

For any value of the pairing phases $\phi_i$, we can derive a sum rule for the ABS energies and Berry curvatures of all the bands and the two possible spins. According to Eq. \[14\], $\psi_{n,j}$ is the $j$-th component of the $n$-th eigenstate of the unitary matrix $U = S\beta e^{i\phi}S^*e^{-i\phi}$. The orthonormality of the eigenstates of a unitary matrix implies that $\sum_n |\psi_{n,j}|^2 = 1$ for each value of $j$; hence $\sum_n \partial|\psi_{n,j}|^2/\partial \phi_i = 0$ for each value of $i$ and $j$. Equation \[24\] then leads to the identity

$$\sum_{\sigma=\pm 1} \sum_{n=1}^N B_{n,\sigma,ij} = 0. \tag{26}$$

Furthermore, since the energies of the $\sigma = +1$ bands have opposite signs to the energies of the $\sigma = -1$ bands (Eq. \[10\]), we have

$$\sum_{\sigma=\pm 1} \sum_{n=1}^N E_{n,\sigma} = 0. \tag{27}$$

We now consider a situation in which a voltage $V_i$ is applied to wire $i$; the phase $\phi_i$ then varies in time according to

$$\dot{\phi}_i = \frac{2e}{\hbar} V_i, \tag{28}$$

where $\dot{\phi}_i = d\phi/dt$. Next, the contribution of a particular ABS with energy $E_{n,\sigma}$ to the Josephson current in the $i$-th wire is given by

$$I_i = \frac{1}{2} \sum_{\sigma=\pm 1} \sum_{n=1}^N \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \times \left[ \frac{2e}{\hbar} \frac{\partial E_{n,\sigma}}{\partial \phi_i} - \frac{2e}{\hbar} \sum_{j=1}^N B_{n,\sigma,ij} \dot{\phi}_j \right]. \tag{29}$$

(The prefactor of $1/2$ has been included to avoid double counting of spin; see Ref. \[3\]. Note that while summing over $n$ and $\sigma$, we have included the effect of a temperature $T$ through the Fermi function

$$f(E_{n,\sigma}) = \frac{1}{e^{\beta E_{n,\sigma}} + 1}. \tag{30}$$

where $\beta = 1/(k_B T)$, $k_B$ is the Boltzmann constant, and we have taken the chemical potential to lie at the center of the SC gap; this will be discussed in detail in Sec. \[III\]. We can see that Eq. \[29\] satisfies current conservation, $\sum_{i=1}^N I_i = 0$ due to Eqs. \[10\] and \[22\]. In Eq. \[29\] we have introduced the temperature dependence as $f(E_{n,\sigma}) - 1/2$ following Ref. \[3\]. However, Eqs. \[29\] imply that the value of $I_i$ would not change if we dropped the factor of $1/2$.}

### D. Josephson effects

In this section, we study the ac Josephson effect in these junctions. To this end, we first note that if $V_i = 0$, the $\phi_i$’s are constant in time and we get constant currents given by

$$I_i = \frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \left[ \frac{2e}{\hbar} \frac{\partial E_{n,\sigma}}{\partial \phi_i} \right]. \tag{31}$$

This is called the dc Josephson effect. The Berry curvature does not contribute in this case.

To study the ac Josephson effect, we first take the $V_i$ to be time-independent non-zero constants leading to $\phi_i = (2e/\hbar)V_i t$

$$I_i = \frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \times \left[ \frac{2e}{\hbar} \frac{\partial E_{n,\sigma}}{\partial \phi_i} - \frac{4e^2}{\hbar} \sum_{j=1}^N B_{n,\sigma,ij} V_j \right]. \tag{32}$$
These currents vary with time since $\phi_i$ and therefore $B_{n,\sigma,ij}$ vary with time. Equation (32) can be written in units of energy (eV) as

$$\frac{\hbar I_i}{2e} = \frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \times \left[ \frac{\partial E_{n,\sigma}}{\partial \phi_i} - \sum_{j=1}^{N} B_{n,\sigma,ij} 2eV_j \right].$$  \hspace{1cm} (33)

Let us consider the case where only $V_1 \neq 0$ and $V_2 = V_3 = \cdots = 0$. Then $\phi_1 = (2e/h)V_1 t$ and $\phi_2, \phi_3, \cdots$ are some constants. Since the system remains invariant when $\phi_1 \to \phi_1 + 2\pi$ keeping the other $\phi_i$’s fixed, we see that $E_{n,\sigma}$ and $B_{n,\sigma,ij}$ vary in time with a period $T = 2\pi \hbar/(2eV_1)$. Equation (33) implies that the total charge flowing in wire $i$ in one time period $T$ is given by

$$Q_i = \frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \times \left[ 2e \int_0^T dt \frac{\partial E_{n,\sigma}}{\partial \phi_i} - \frac{4e^2}{h} V_1 \int_0^T dt B_{n,\sigma,i1} \right].$$  \hspace{1cm} (34)

The average current flowing over time $T$ is then given by

$$\langle I_i \rangle = \frac{Q_i}{T}.$$  \hspace{1cm} (35)

(We are interested in $\langle I_i \rangle$ since it gives the dc part of the current). Since $E_{n,\sigma}$ is a periodic function of $\phi_i$, and $\phi_1$ varies linearly in time with a period $T$, we see that

$$\int_0^T dt \frac{\partial E_{n,\sigma}}{\partial \phi_1} = 0.$$  \hspace{1cm} (36)

This equation, along with $B_{n,\sigma,11} = 0$, implies that $Q_1 = 0$; hence $\langle I_1 \rangle = 0$.

For the case of two wires ($N = 2$), the facts that $E_{n,\sigma}$ is a function of $\phi_1 - \phi_2$ and $B_{n,\sigma,1j} = 0$ can be used to analytically show that $\langle I_1 \rangle = \langle I_2 \rangle = 0$. But for $N \geq 3$, we find numerically that $\langle I_i \rangle$ is generally not equal to zero for $i = 2, 3, \cdots, N$. Thus the application of a constant voltage bias on one wire produces a constant current in all the other wires (in addition to a time-dependent current which gives zero when integrated over time $T$). This phenomenon of transconductance (defined as $G_{ij} = \langle I_i \rangle/|V_j|$ where $i \neq j$) has been studied earlier. However, those papers discussed this phenomenon when incommensurate voltage biases are applied to two of the wires (the time-averaged transconductance is then quantized), whereas we have shown here that the transconductance is non-zero even if a voltage bias is applied to only one wire. The transconductance for a particular set of values of $\phi_i$ is not quantized in our case; however, the integral of the transconductance over one of the phases $\phi_i$ is quantized at zero temperature as we will now show.

If we hold $\phi_3, \phi_4, \cdots$ fixed, the invariance of $E_{n,\sigma}$ under $\phi_2 \to \phi_2 + 2\pi$ implies that $f_{0}^{2\pi} d\phi_2 \partial E_{n,\sigma}/\partial \phi_2 = 0$, while the linear variation of $\phi_1$ with $t$ implies that $(1/T) \int_0^T dt B_{n,\sigma,12} = (1/2\pi) \int_0^{2\pi} d\phi_1 B_{n,\sigma,12}$. Using Eqs. (23), (24) and (32), we obtain, at zero temperature where $f(E_{n,\sigma}) - 1/2 = -\text{sgn}(E_{n,\sigma})/2$,

$$\int_0^{2\pi} \frac{d\phi_2}{2\pi} \langle I_2 \rangle = \frac{2e^2}{h} \sum_{n,\sigma} \left( - \frac{\text{sgn}(E_{n,\sigma})}{2} \right) Ch_{n,\sigma,12}$$

$$= \frac{4e^2}{h} \sum_{n} \left( - \frac{\text{sgn}(E_{n+1})}{2} \right) Ch_{n+1,12};$$  \hspace{1cm} (37)

where $h = 2\pi \hbar$, and we have used Eq. (25) and the fact that $E_{n,-1} = -E_{n,1}$ to derive the second line in Eq. (37) from the first line (i.e., we have removed the sum over $\sigma$ and changed the prefactor from 2 to 4). Since $Ch_{n+1,12}$ is an integer for all values of $n$, and Chern numbers of positive and negative energy bands have opposite signs, we see that the integral of the transconductance over $\phi_2$ is quantized in units of $4e^2/h$.

### E. Shapiro plateaus

In this section, we consider a sinusoidal applied voltage, which, for two-wire junctions, is well known to lead
to Shapiro plateaus. To understand this, it is common to consider an RC circuit in which a resistance $R_{ij}$ and a capacitance $C_{ij}$ are placed between every pair of leads $i$ and $j$ so that the currents flowing through them are in parallel to the Josephson currents; a schematic picture of this for a three-wire system is shown in Fig. 2.

The equations for the current in the $i$-th wire are then modified from Eq. (28) to

$$I_i = \frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \left( 2e \frac{\partial E_{n,\sigma}}{\partial \phi_1} - 2e \sum_{j \neq i} B_{n,\sigma,ij} \dot{\phi}_j \right)$$

$$+ \sum_{j \neq i} \frac{\hbar}{2e} \left[ C \ddot{\phi}_i - \ddot{\phi}_j \right] + \frac{V_i - V_j}{R_{ij}}, \quad (38)$$

where we have used the fact that $B_{n,\sigma,ij} = 0$ if $i = j$. For convenience, let us assume that $R_{ij} = R$ and $C_{ij} = C$ for all pairs of leads. Using Eq. (28), we then obtain

$$I_i = \frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \left( 2e \frac{\partial E_{n,\sigma}}{\partial \phi_1} - 2e \sum_{j \neq i} B_{n,\sigma,ij} \dot{\phi}_j \right)$$

$$+ \frac{\hbar}{2e} \sum_{j \neq i} \left[ C \ddot{\phi}_i - \ddot{\phi}_j \right]$$

$$\quad + \frac{V_i - V_j}{R_{ij}}, \quad (39)$$

We now consider the case where

$$I_1 = I + A \sin(\omega t), \quad (40)$$

and $V_2 = V_3 = \cdots = 0$. Then $\phi_2, \phi_3, \cdots$ are all constant in time, while $\phi_1$ will vary with time. Equation (39) then gives

$$\frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \left( 2e \frac{\partial E_{n,\sigma}}{\partial \phi_1} \right)$$

$$C \ddot{\phi}_1 + \frac{\ddot{\phi}_1}{R} = I + A \sin(\omega t), \quad (41)$$

where $N$ is the number of wires, and

$$\frac{1}{2} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \left( 2e \frac{\partial E_{n,\sigma}}{\partial \phi_1} \right)$$

$$C \ddot{\phi}_1 + \frac{\ddot{\phi}_1}{R} = I_j, \quad (42)$$

for $j \neq 1$. We observe that Eq. (41) does not contain any Berry curvature terms; this is therefore the simplest equation to solve. Given some initial values of $\phi_1$ and $\dot{\phi}_1$ at time $t = 0$, we can solve this numerically. We can write Eq. (41) in the form

$$\dot{\phi}_1 + \gamma \ddot{\phi}_1 + \frac{2e^2}{(N-1)\hbar^2 C} \sum_{n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right] \left( \frac{\partial E_{n,\sigma}}{\partial \phi_1} \right)$$

$$= \frac{2e}{(N-1)\hbar C} \left[ I + A \sin(\omega t) \right], \quad (43)$$

where $\gamma = 1/(RC)$ is a positive quantity.

We will now present a perturbative argument to understand how Shapiro plateaus can arise from Eq. (13). (A similar procedure has been presented in Ref. [24] and applied in Ref. [21].) The perturbation parameter will be taken to be the SC pairing amplitude $\Delta$; note that the ABS energies $E_{n,\sigma} = 0$ if $\Delta = 0$, as we can see from Eq. (7). To zeroth order in $\Delta$, therefore, the third term on the left hand side of Eq. (13) (namely, the Josephson current) is equal to zero. Note that this term is a nonlinear function of $\phi_1$; omitting this term therefore gives a simple linear equation. After a long time, when a transient term decaying as $e^{-\gamma t}$ has gone to zero, the general solution of this equation is given by

$$\phi_1 = at + a \sin(\omega t + \chi) + \phi_0,$$

where

$$a = \frac{2e I}{(N-1)\hbar C},$$

$$\alpha = \frac{2e A}{(N-1)\hbar C \omega \sqrt{\gamma^2 + \omega^2}},$$

$$\chi = \tan^{-1} \left( \frac{\omega}{\gamma} \right), \quad (44)$$

and $\phi_0$ is an arbitrary constant.

We can now use the above result to put back the third term on the left hand side of Eq. (13). We do this as follows. For a fixed set values of $\phi_2, \phi_3, \cdots$, we know that $E_{n,\sigma}$ is a periodic function of $\phi_1$ with a period $2\pi$. We can therefore write

$$\frac{\partial E_{n,\sigma}}{\partial \phi_1} = \sum_{l=\pm\infty} c_l e^{il\phi_1},$$

where

$$c_l = \int_{0}^{2\pi} d\phi_1 \frac{\partial E_{n,\sigma}}{\partial \phi_1} e^{-il\phi_1}. \quad (45)$$

The Fourier coefficients $c_l$ are functions of $\phi_2, \phi_3, \cdots$. (The coefficient $c_0$ must be equal to zero since we know that $\int_{0}^{2\pi} d\phi_1 \partial E_{n,\sigma}/\partial \phi_1 = 1 - A \sin(\omega t)$, $\gamma C$, $\phi_0$ is an arbitrary constant, and $\phi_0$ is determined by the initial conditions. Substituting Eq. (44) in Eq. (45) and using the identity

$$e^{iz \sin \theta} = \sum_{m=-\infty}^{\infty} J_m(z) e^{im\theta}, \quad (46)$$

where $J_m(z)$ denotes the Bessel function of order $m$, we obtain

$$e^{il\phi_1} = e^{il(\alpha t + \phi_0)} \sum_{m=-\infty}^{\infty} J_m(i\alpha) e^{im(\omega t + \chi)}. \quad (47)$$

For a function $f(t)$, we define the long-time average value as

$$\langle f \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt f(t). \quad (48)$$
Using the result
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, e^{i\epsilon t} = 0 \quad \text{if} \quad \epsilon \neq 0, \]
\[ = 1 \quad \text{if} \quad \epsilon = 0, \] (49)
we find from Eqs. (46) and (47) that
\[ \frac{\partial E_{n,\sigma}}{\partial \phi_1} = \sum_{l,m} c_l J_m(la) e^{imx+i\phi_0}, \] (50)
whenever \( \alpha = -(m/l)\omega \) for a pair of integers \( m \) and \( l \), and is equal to zero otherwise. On the other hand, Eqs. (25) and (44) give \((2e/h)(V_i) = \alpha\). We therefore conclude that Eq. (50) is non-zero only if \( \langle V_i \rangle \) is a rational multiple of \( h\omega/(2e) \).

Next, if Eq. (50) is non-zero, we see from Eqs. (40) and (48) that this effectively changes the dc part of \( I_1 \) from \( I \) to \( I + (e/h) \sum_{n,\sigma} [f(E_{n,\sigma}) - 1/2] \langle \partial E_{n,\sigma}/\partial \phi_1 \rangle \); however, \( \phi_1 \) continues to be given by Eq. (44) and \( \langle V_i \rangle \) therefore remains equal to \( h\omega/(2e) \). Since Eq. (50) can have a range of values depending on \( \phi_0 \), we see that \( I + (e/h) \sum_{n,\sigma} [f(E_{n,\sigma}) - 1/2] \langle \partial E_{n,\sigma}/\partial \phi_1 \rangle \) can have a range of values while \( \langle V_i \rangle \) has a fixed value. This corresponds to a Shapiro plateau in a plot of \( \langle V_i \rangle \) versus the dc part of \( I_1 \).

The discussion presented above and the expression in Eq. (50) imply that the width of the plateau at \((2e/h)\langle V_i \rangle = -(m/l)\omega \) will be proportional to the Fourier coefficient \( c_l \) multiplied by \( J_m(la) \). We thus expect that \( c_l \) can give an idea of the plateau widths at integer multiples of \( \omega/|l| \). The numerical results presented in Sec. IV confirm this expectation.

The above arguments imply that in contrast to the standard Josephson junctions made out of two wires, junctions of multiple wires exhibit Shapiro plateaus for drive frequencies for which \( 2e\langle V_i \rangle/(h\omega) \) is a rational number \( m/l \) (where \( m \) and \( l \) are integers). Since there is a rational number lying between any two rational numbers, there should be a plateau lying between any two plateaus leading to a devil’s staircase structure.\(^{21,22}\) However the plateau width quickly goes to zero and therefore become difficult to see as either \( m \) or \( l \) becomes large; this is because \( J_m(la) \to 0 \) very rapidly as \( |m| \to \infty \) for a fixed value of \( la \), and the Fourier coefficient \( c_l \) to \( 0 \) as \( |l| \to \infty \) for any smooth periodic function \( \partial E_{n,\sigma}/\partial \phi_1 \).

III. JUNCTION OF THREE SUPERCONDUCTING WIRES

A. Three \( s\)-wave superconducting wires

We first consider a system of three \( s\)-wave SC wires. (We will only consider spin-up quasiparticles here; the ABS energies, wave functions and Berry curvature are identical for spin-down quasiparticles in this case). Equation \(^{20\text{i}}\) implies that \( 1/\alpha^2_s(E_\sigma) \) is an eigenvalue of \( S e^{i\phi} S^* e^{-i\phi} \), and there must be three such eigenvalues. The \( E \to -E \) symmetry implies that two of the ABS must have energies \( \pm E_\sigma \) (which are generally not equal to either zero or \( \pm \Delta \)), while the third ABS must have \( E_\sigma \) equal to either zero or \( \pm \Delta \). To see which of these two possibilities occur for the third ABS, we first consider the trivial case \( S = I \). It is then clear that all the ABS energies lie at \( \pm \Delta \). If we now smoothly move \( S \) away from \( I \), the \( E \to -E \) symmetry implies that two of the ABS can move away from \( \pm \Delta \) as a pair, but the third state must remain fixed at \( E_\sigma^2/\Delta^2 = 1 \) (which is not a bound state). We thus conclude in general that one of the eigenvalues of \( S e^{i\phi} S^* e^{-i\phi} \) must be equal to 1, while the other two eigenvalues form a complex conjugate pair \( a_{\sigma}^2(E_\sigma) \) and \( a_{\sigma'}^2(E_\sigma) \). Using the fact that the sum of the eigenvalues of a matrix is equal to its trace, we find that the energies of two of the ABS are given by:

\[ E_\sigma = \pm \frac{1}{2} \sqrt{1 + \text{Tr}(S e^{i\phi} S^* e^{-i\phi})}. \] (51)

If the matrix \( S \) is symmetric, Eq. (51) implies that two of the ABS have energies

\[ E_\Delta = \pm |1 - |S_{12}|^2 \sin^2(\phi_1 - \phi_2)| - |S_{13}|^2 \sin^2(\phi_1 - \phi_3) \]
\[ - |S_{23}|^2 \sin^2(\phi_2 - \phi_3)|^{1/2}. \] (52)

A general parametrization of \( 3 \times 3 \) unitary matrices \( S \) has been given in Ref. \(^{17\text{a}}\). Instead of looking at the most general case, however, we will first consider a special family of matrices which is completely symmetric under all possible permutations of the three wires. Apart from an overall phase (which is unimportant since it cancels out between \( S \) and \( S^* \) in Eq. (52)), it turns out that such completely symmetric matrices are labeled by a single real parameter \( \lambda \) and have the form:

\[ S = \begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix}, \]

where \( r = \frac{1 + i\lambda}{3 + i\lambda} \),
\[ t = \frac{2}{3 + i\lambda}. \] (53)

The physical significance of \( \lambda \) is that it is the strength of a barrier between the three wires.\(^{20\text{a}}\). For \( \lambda = 0 \), there is no barrier and the transmission probability \(|t|^2 = 4/9\) has the maximum possible value allowed by unitarity for three wires which are completely symmetric with respect
to each other. For \( \lambda = \infty \), the barrier is infinitely large and \( |t|^2 = 0 \). With \( S_{12} = S_{13} = S_{23} = t \), Eq. (52) gives

\[
\frac{E_\sigma}{\Delta} = \pm \left[ \sum_{i=1}^{3} e^{i\phi_i^2} + \lambda^2 \right]^{1/2}.
\] (54)

For \( \lambda = 0 \), we get \( E_\sigma/\Delta = \pm |\sum_{i=1}^{3} e^{i\phi_i}|/3 \), while \( \lambda \to \pm \infty \) gives \( E_\sigma/\Delta \to \pm 1 \). To understand the form of \( E_\sigma/\Delta \) better, it is useful to study the function

\[
F = \sum_{i=1}^{3} e^{i\phi_i} = e^{i\phi_1} + e^{i\phi_2} + e^{i\phi_3}.
\] (55)

We can show that \( F = 0 \) if the three phases \( \phi_1, \phi_2, \phi_3 \) are \( 2\pi/3 \) apart. For instance, if we set \( \phi_3 = 0 \), we get \( F = 0 \) if \( \phi_1, \phi_2 \) is equal to either \( (2\pi/3, 4\pi/3) \) or \( (4\pi/3, 2\pi/3) \). If we expand around one of these points, say, \( \phi_1 = 2\pi/3 + \delta\phi_1 \) and \( \phi_2 = 4\pi/3 + \delta\phi_2 \), we obtain

\[
F = -\frac{\sqrt{3}}{2} (\delta\phi_1 - \delta\phi_2) + \frac{i}{2} (\delta\phi_1 + \delta\phi_2)
\] (56)
to first order in \( \delta\phi_1, \delta\phi_2 \). Equation (54) then takes the form

\[
\frac{E_\sigma}{\Delta} = \pm \left[ (3/4)(\delta\phi_1 - \delta\phi_2)^2 + (1/4)(\delta\phi_1 + \delta\phi_2)^2 + \lambda^2 \right]^{1/2}.
\] (57)

We can think of Eq. (57) as the energy-momentum dispersion of a particle moving in two dimensions with momentum \( k_x = \sqrt{3}(\delta\phi_1 - \delta\phi_2) \) and \( k_y = \delta\phi_1 + \delta\phi_2 \); the dispersion has the relativistic form

\[
E_\sigma^2 = v^2 (k_x^2 + k_y^2) + m^2,
\]

where

\[
v = \frac{\Delta}{2 \sqrt{9 + \lambda^2}}
\]

and

\[
m = \frac{\Delta \lambda}{\sqrt{9 + \lambda^2}}
\] (58)

are the ‘velocity’ and ‘mass’ respectively. The situation described above is similar to what happens in graphene close to the two Dirac points, a mass term can be induced there by applying a sublattice potential or a spin-orbit interaction.

Turning to the Berry curvature matrix \( B_{n,\sigma} \), we find that Eqs. (22) and the antisymmetry imply that the matrix is described by a single real parameter \( b_{n,\sigma} \) as

\[
B_{n,\sigma} = \begin{pmatrix} 0 & b_{n,\sigma} & -b_{n,\sigma} \\ -b_{n,\sigma} & 0 & b_{n,\sigma} \\ b_{n,\sigma} & -b_{n,\sigma} & 0 \end{pmatrix}.
\] (59)

The value of \( b_{n,\sigma} \) depends on the \( \phi_i \)'s, \( S \) and the particular ABS band \( n \) and spin \( \sigma \) which is being considered.

FIG. 3: (a) Surface plot of ABS energies (in units of \( \Delta \)) vs \( (\phi_1, \phi_2) \) for the \( S \)-matrix given in Eq. (58) with \( \lambda = 0.1 \). For all values of \( (\phi_1, \phi_2) \), one of the energies lies at \( E_\sigma/\Delta = 1 \) and the other two ABS energies appear as a \( \pm E_\sigma \) pair; the gap between these two bands is minimum at \( (2\pi/3, 4\pi/3) \) and \( (4\pi/3, 2\pi/3) \). (b) Surface plot of Berry curvature \( B_{n,12} \) vs \( (\phi_1, \phi_2) \) for the ABS band with \( E_\sigma/\Delta < 0 \). There are peaks at \( (2\pi/3, 4\pi/3) \) and \( (4\pi/3, 2\pi/3) \) with negative and positive signs respectively, and \( C_{n,12} = 0 \).

Typically, we find that the Berry curvature is large near those values of the \( \phi_i \)'s where two of the ABS are almost degenerate in energy.

In the rest of this section, we will set \( \phi_3 = 0 \) for convenience. For the \( S \)-matrix given in Eq. (58) and \( |t| \ll 1 \), we numerically find the following results in the \( (\phi_1, \phi_2) \) plane. The parameter \( B_{n,12} = b_{n,\sigma} \) is peaked near the two points \( (\phi_1, \phi_2) = (2\pi/3, 4\pi/3) \) and \( (4\pi/3, 2\pi/3) \) and
is close to zero everywhere else; the peaks occur where two of the ABS have energies close to zero and are therefore almost degenerate. Furthermore, for the ABS with one of the energies given in Eq. (57), the sign of the peak is given by \( \text{sgn}(\lambda E) \) at the first point and \(-\text{sgn}(\lambda E)\) at the second point. In all cases, we find that the Chern number \( C_{n,\sigma,12} \) given by (65) is equal to zero as the contributions from the two peaks cancel each other. Figure 3 shows surface plots versus \((\phi_1, \phi_2)\) of (a) the three bands of ABS energies and (b) the Berry curvature \( B_{n,\sigma,12} \) for the ABS band with \( E_{n,\sigma}/\Delta < 0 \), for the \( S\)-matrix given in Eq. (65) with \( \lambda = 0.1 \). (We have set \( \phi_3 = 0 \). We see that \( B_{n,\sigma,12} \) has peaks at \((2\pi/3,4\pi/3)\) and \((4\pi/3,2\pi/3)\) with negative and positive signs respectively, and the Chern number \( C_{n,\sigma,12} \) is equal to zero. The fact that \( C_{n,\sigma,12} = 0 \) is generally true if \( S \) is symmetric, i.e., time-reversal symmetric.

If \( S \) is not a symmetric matrix the Chern number \( C_{n,\sigma,12} \) can be non-zero. In that case we typically find that there is a single peak in \( B_{n,\sigma,12} \) and \( C_{n,\sigma,12} = \pm 1 \). Furthermore, the peak in \( B_{n,\sigma,12} \) usually occurs at a point in the \((\phi_1, \phi_2)\) plane where one of the ABS is almost degenerate with the ABS which always lies at \( E_{\sigma}/\Delta^2 = 1 \). Figure 4 shows surface plots versus \((\phi_1, \phi_2)\) of (a) the three bands of ABS energies and (b) the Berry curvature versus \((\phi_1, \phi_2)\) for the ABS band with \( E_{\sigma}/\Delta < 0 \), for a randomly generated \( S\)-matrix given by

\[
S = \begin{pmatrix}
0.8389 - 0.0346i & -0.2399 + 0.3146i & -0.3542 + 0.1138i \\
-0.0163 + 0.3446i & 0.7254 - 0.0341i & -0.4820 + 0.3483i \\
0.2591 + 0.3299i & 0.3589 + 0.4328i & 0.7119 - 0.0341i \\
\end{pmatrix}
\]

(60)

(We again set \( \phi_3 = 0 \)). We see that the Berry curvature has a peak with a negative sign at \((\phi_1, \phi_2) = (2.04, 1.57)\), and \( C_{n,\sigma,12} \) is equal to \(-1\).

**Large \( \lambda \) limit of the symmetric \( S\)-matrix:** It is interesting to consider what happens if we take the parameter \( \lambda \) to be large in Eq. (53): as mentioned there, this corresponds to having a large barrier at the junction between the three wires. Keeping terms only up to order \( 1/\lambda \), we find that \( r \simeq -1 - 2i/\lambda \) and \( t \simeq -2i/\lambda \). The operator in Eq. (44) then takes the form

\[
S e^{i\phi} S^* e^{-i\phi} \simeq I_3 + iM,
\]

\[
M = \frac{2}{\lambda} \begin{pmatrix}
0 & 1 - e^{i(\phi_1 - \phi_2)} & 1 - e^{i(\phi_1 - \phi_3)} \\
1 - e^{i(\phi_2 - \phi_1)} & 0 & 1 - e^{i(\phi_2 - \phi_3)} \\
1 - e^{i(\phi_3 - \phi_1)} & 1 - e^{i(\phi_3 - \phi_2)} & 0
\end{pmatrix}
\]

(61)

where \( I_3 \) is the \( 3 \times 3 \) identity matrix and \( M \) is a Hermitian matrix. The eigenvalues of \( M \) turn out to be zero and

\[
\pm 2 \sqrt{\sin^2(\frac{\phi_1 - \phi_2}{2}) + \sin^2(\frac{\phi_1 - \phi_3}{2}) + \sin^2(\frac{\phi_2 - \phi_3}{2})}
\]

(62)

It follows from Eqs. (44), (61) and (62) that one of the ABS energies lies at \( E_{\sigma}/\Delta^2 = 1 \), while the other two are given by

\[
\frac{E_{\sigma}}{\Delta} \simeq \pm \left[ 1 - \frac{2}{\lambda^2} \left\{ \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right) + \sin^2\left(\frac{\phi_1 - \phi_3}{2}\right) + \sin^2\left(\frac{\phi_2 - \phi_3}{2}\right) \right\} \right]
\]

(63)

up to terms of order \( 1/\lambda^2 \). For the ABS with negative energy, Eq. (63) implies that

\[
\frac{\partial E_{\sigma}}{\partial \phi_1} \simeq \frac{\Delta}{\lambda^2} [\sin(\phi_1 - \phi_2) + \sin(\phi_1 - \phi_3)]
\]

(64)

We thus see that the contribution of this term to the current (in Eqs. (42) and (43) for instance) scales as \( \Delta/\lambda^2 \) for large \( \lambda \). This observation will be useful later.

It is clear from Eqs. (1) and (61) that in the large \( \lambda \) limit, the ABS wave functions and hence the Berry curvature depend only on the phases \( \phi_1 \) and not on \( \lambda \). The Berry curvature is large near the point \( \phi_1 = \phi_2 = \phi_3 \) since the three eigenvalues of \( M \) are degenerate there. However we find that the peak value of the Berry curvature is much smaller here compared to its value for small \( \lambda \) as shown in Fig. 4.

**B. Systems with some \( p\)-wave superconducting wires**

We first consider a system in which all the three SC wires have \( p\)-wave pairing. The diagonal matrix \( \eta \) defined after Eq. (13) is then equal to \(-I\), and the ABS energies will be given by the eigenvalue equation

\[
S e^{i\phi} S^* e^{-i\phi} \eta(E_{\sigma}) = \frac{1}{\alpha_{\sigma}^2(E_{\sigma})} \psi_{\sigma}(E_{\sigma})
\]

(65)

for both spin-up and -down quasiparticles; hence each of the energies will have a two-fold degeneracy. Then an
while the other two energies must be of the form \( \pm \frac{E}{\Delta} \). This implies that the energies of two of the \( \Delta = 1 \) is equal to its trace implies that the energies of two of the two ABS energies appear as a \( \pm \) pair. The gap between the band with \( E_\sigma = 0 \) corresponding to \( \alpha^2_\sigma(E_\sigma) = -1 \). Thus, if \( \phi_3 \) is fixed, there is a line in the \( (\phi_1, \phi_2) \) plane on which one of the ABS energies is equal to zero.

**IV. NUMERICAL RESULTS FOR THREE SUPERCONDUCTING WIRES**

We will now present our numerical results. When calculating the currents \( I_i \), we have to sum over spin-up and -down quasiparticles and also over the three bands of ABS energies given by functions of \( \phi_1 \) and \( \phi_2 \) (we will generally set \( \phi_1 = 0 \)). As usual we will denote the ABS energies by \( E_{n,\sigma} \), where \( n = 1, 2, 3 \) labels the bands and \( \sigma = \pm 1 \). If \( I_{i,n,\sigma} \) is the contribution to the current in wire \( i \) from band \( n \) and spin \( \sigma \), the total current in wire \( i \) is given by

\[
I_i = \frac{1}{2} \sum_{n,\sigma} I_{i,n,\sigma} \left[ f(E_{n,\sigma}) - \frac{1}{2} \right].
\]  

In general, the SC phases \( \phi_i \) and the ABS energies \( E_{n,\sigma} \) can vary with time. We will assume that Eq. (67) is valid at all times with \( E_{n,\sigma} \) being the instantaneous energy.

In the numerical calculations presented below, we have chosen the SC gap to be \( \Delta = 10^{-6} \) eV in order to obtain experimentally reasonable values of the currents \( I_i \) (of the order of nA) and the frequency \( \omega \) (of the order of GHz) used to study Shapiro plateaus. Another reason for choosing \( \Delta = 10^{-6} \) eV, along with appropriate values of the elements of the \( S \) matrix, is to show large variations and striking peaks in the ABS energies and Berry curvature in Fig. 3. We note, however, that the value of \( \Delta = 10^{-6} \) eV is much smaller than typical experimental values of the order of \( 10^{-3} \) eV. As stated after Eq. (64), if we scale \( \Delta \) up from \( 10^{-6} \) to \( 10^{-3} \) eV and \( \lambda^2 \) up by the same factor of \( 10^3 \), i.e., scale \( \lambda \) up by a factor of...
about 30, the values of the currents will not change drastically (the currents will change to some extent because the Berry curvature changes as we vary $\lambda$ from small to large values). To conclude, we will present numerical results for $\Delta = 10^{-6} \text{ eV}$ so as to illustrate various ideas (such as transconductance, Shapiro plateaus and junction magnetic moment) most clearly, with the understanding that the results will not change qualitatively if we use more realistic values of $\Delta$ along with suitable values of $S$. We also note that in what follows, we will only consider the contribution to the currents from the Andreev bound states.

### A. Three s-wave superconducting wires

To begin, we will examine the case of three $s$-wave SC wires. In this case, the energies, wave functions and Berry curvature are identical for spin-up and -down quasiparticles. We will therefore consider only spin-up quasiparticles in the following. Next, we know that one of the ABS energy bands always lies at the edge between the gap and the bulk ($E_{\gamma}/\Delta^2 = 1$); hence it is not really a bound state. We will therefore not consider this band. The other two bands have $\pm E_\sigma$. We will only consider cases where these energies are gapped away from zero. At temperatures much lower than the gap, the band with positive (negative) energy will be unoccupied (occupied) and will therefore have $f(E_{\pm\sigma}) - 1/2$ equal to $-1/2$ ($1/2$) respectively. Since the energies come in $\pm E$ pairs, we have $\sum_{n=\pm} E_{n,\sigma}[f(E_{n,\sigma}) - 1/2] = E_{-\sigma}$, where $E_{\pm\sigma}$ denotes the positive (negative) energy values for spin $\sigma$. For the Berry curvature, we have $\sum_{n=\pm} B_{n,\sigma} f(E_{n,\sigma}) - 1/2] \simeq (B_{+,1,12} - B_{-,1,12})/2$, where $B_{\pm\sigma,12}$ denotes the Berry curvatures in the positive (negative) energy bands. (We find numerically that for each value of the $\phi_i$'s, $B_{+,\sigma,12}$ and $B_{-\sigma,12}$ have almost the same magnitude but opposite signs, as mentioned after Eq. (24).) We also note that $E_{\pm\sigma}$ and $B_{\pm\sigma,12}$ have the same values for $\sigma = \pm 1$. Hence, in many of the equations below, we will write only $\sigma = +1$ and include a factor of 2 for spin.

We first look at the ac Josephson effect, namely, the case of constant voltages $V_i$; this is discussed in Eq. (52), but we have to multiply the expression in that equation by 2 to account for spin. In Fig. 5, we show the average currents $\langle I_1 \rangle$, $\langle I_2 \rangle$ and $\langle I_3 \rangle$ as functions of $\phi_2$ for a system with $V_1 = 5 \times 10^{-5}$ V (so that $2eV_1 = 10^{-4}$ eV), $V_2 = V_3 = 0$, $\phi_3 = 0$, $\Delta = 10^{-6}$ eV, and an $S$-matrix of the form given in Eq. (53) with $\lambda = 0.1$. (As described in Eq. (33), we will use the units of current to be $e/h$ times eV which is equal to $(2\pi V)/[(e/h)^2] \approx 2.43 \times 10^{-4}$ A). We can understand the main features of Fig. 5 as follows. First, as noted after Eq. (30), $\langle I_1 \rangle = 0$; hence current conservation implies that $\langle I_2 \rangle = -\langle I_3 \rangle$. Second, we find numerically that the entire contribution to $\langle I_2 \rangle$ and $\langle I_3 \rangle$ comes from the Berry curvature term, i.e., the second term on the right hand side of Eq. (34). Since $V_1$ is constant, $\phi_1 = (2e/h)V_1 t$ varies linearly with time and covers the full range of $2\pi$ in a time $T = 2\pi h/(2eV_1)$. Equations (34-35) then imply that $\langle I_1 \rangle = 0$, and

\[
\langle I_2 \rangle = -\langle I_3 \rangle = \frac{2e^2V_1}{h} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \times \left[ B_{-,1,12}(\phi_1, \phi_2) - B_{+,1,12}(\phi_1, \phi_2) \right]
\]

(68)

Since the Berry curvature $B_{-,1,12}$ has large values around $(\phi_1, \phi_2) = (2\pi/3, 4\pi/3)$ with a negative sign and around $(4\pi/3, 2\pi/3)$ with a positive sign (as shown in Fig. 3), we see from Eq. (68) that for $2eV_1 = 10^{-4}$ eV, $\langle I_2 \rangle$ will be large and positive around $\phi_2 = 2\pi/3$ (with $I_2(t)$ getting its maximum contribution at the time when $\phi_1$ passes through $4\pi/3$) and negative around $\phi_2 = 4\pi/3$ (with the maximum contribution to $I_2(t)$ coming from the time when $\phi_1$ passes through $2\pi/3$). This explains the locations and signs of the peaks in $\langle I_2 \rangle$ in Fig. 5. In addition, we have verified numerically that Eq. (67) is satisfied with $Ch_{-,1,12} = Ch_{+,1,12} = 0$.

For $2eV_1 = 10^{-4}$ eV, Eq. (68) and the fact that $B_{+,1,12}(\phi_1, \phi_2) \simeq -B_{-,1,12}(\phi_1, \phi_2)$ imply that as a function of $\phi_2$, $\langle I_2 \rangle = -\langle I_3 \rangle$ is equal, in units of $10^{-4} \times (e/h)(eV) \approx 24$ nA, to twice the average Berry
curvature which is defined as

$$B_{12}(\phi_2) = \int_0^{2\pi} \frac{d\phi_1}{2\pi} B_{-,+1,12}(\phi_1, \phi_2),$$

(69)

For instance, we see in Fig. 6 that $2B_{12} \approx 2.68$ for $\phi_2 = 2\pi/3$ and $\phi_3 = 0$; this value will be used later.

Similarly, in Fig. 6 we show $\langle I_i \rangle$ as functions of $\phi_2$ for a system with $V_1 = 5 \times 10^{-5}$ V (so that $2eV_1 = 10^{-4}$ eV), $V_2 = V_3 = 0$, $\Delta = 10^{-6}$ eV, and an $S$-matrix of the form given in Eq. (60). Once again we find that $\langle I_1 \rangle = 0$, $\langle I_2 \rangle = -\langle I_3 \rangle$, and the entire contribution to $\langle I_2 \rangle$ and $\langle I_3 \rangle$ comes from the Berry curvature term. Furthermore, we see only a single peak in $\langle I_2 \rangle$ with a negative sign; the peak is located at $\phi_2 = 1.57$ which is consistent with the sign and location of the peak in the Berry curvature shown in Fig. 4. Once again, we have confirmed numerically that Eq. (37) holds with $Ch_{-,+1,12} = -Ch_{+,+1,12} = -1$.

![Figure 6](image)

**FIG. 6:** Plot of average currents $\langle I_i \rangle$ (red solid line which lies at zero), $\langle I_2 \rangle$ (blue solid line), and $\langle I_3 \rangle$ (black dashed line) vs $\phi_2$ lying in the range $[0, 2\pi]$. We have chosen $V_1 = 5 \times 10^{-5}$ V, $V_2 = V_3 = 0$, $\Delta = 10^{-6}$ eV, and the $S$-matrix has the form given in Eq. (60). The currents are in units of $10^{-4} \times (e/\hbar)(eV) \approx 24$ nA.

We next look for Shapiro plateaus in an $RC$ circuit involving three wires; we will assume that the all resistances (capacitances) are equal to $R$ ($C$). As discussed in Eq. (40), we consider a case where $I_1 = I + A\sin(\omega t)$, and $V_2 = V_3 = 0$ so that $\phi_2$ and $\phi_3$ are constant in time. Using Eqs. (63) and (43) we then obtain the following equations

$$\ddot{\phi}_1 + \gamma \dot{\phi}_1 + \frac{2e^2}{\hbar C} \frac{\partial E_{-,+1}}{\partial \phi_1} = \frac{e}{\hbar C} [I + A\sin(\omega t)],$$

(70)

$$I_2 - I_3 = \frac{2e}{\hbar} \left( \frac{\partial E_{-,+1}}{\partial \phi_2} - \frac{\partial E_{-,+1}}{\partial \phi_3} \right)$$

$$+ 2e(\bar{B}_{-,+1,12} - B_{+,+1,12}) \dot{\phi}_1,$$

(71)

$$I_2 + I_3 = -I_1 = -I - A\sin(\omega t),$$

(72)

where we have included a factor of 2 for spin. We can solve these equations numerically. Equation (70) does not involve the Berry curvature and can be solved without using the next two equations. Equation (71) involves the Berry curvature and $\dot{\phi}_1$ which can be found using Eq. (70). Equation (72) has a trivial form. After solving Eqs. (70,71) over a long time $T$, we can calculate the average values

$$\langle V_1 \rangle = \frac{1}{T} \int_0^T dt V_1(t) = \frac{\hbar}{2e} \left( \frac{\phi(T) - \phi(0)}{T} \right),$$

$$\langle I_2 - I_3 \rangle = \frac{1}{T} \int_0^T dt \left( I_2(t) - I_3(t) \right).$$

(73)

We can then plot these average values versus $I$ for a particular set of values of $A$, $\omega$, $\phi_2$ and $\phi_3$.

To fix the values of the various parameters, it is convenient to re-define time in dimensionless units of $\omega t$; $\phi_1$ and $\dot{\phi}_1$ are then dimensionless. Next, we define the dimensionless quantities

$$\alpha_1 = \frac{\hbar \omega C}{e^2} \quad \text{and} \quad \alpha_2 = \frac{\hbar}{e^2 R}$$

(74)

Equations (70,71) can then be rewritten as

$$\frac{\hbar \omega}{2} [\alpha_1 \dot{\phi}_1 + \alpha_2 \phi_1] + \frac{\partial E_{-,+1}}{\partial \phi_1} = \frac{e}{2e} [I + \alpha_1 \sin t],$$

(75)

$$\frac{\hbar}{2e} (I_2 - I_3) = \left( \frac{\partial E_{-,+1}}{\partial \phi_2} - \frac{\partial E_{-,+1}}{\partial \phi_3} \right) + \hbar \omega (B_{-,+1,12} - B_{+,+1,12}) \dot{\phi}_1.$$  

(76)

All the terms in Eqs. (75,76) have the dimensions of energy. In addition, we have $V_1 = (\hbar \omega / 2e) \dot{\phi}_1$ in units of energy; hence

$$\langle V_1 \rangle = \frac{\hbar \omega}{2e} \left( \phi_1 \right).$$

(77)

In Fig. 7 we show a plot of $\langle V_1 \rangle$ vs $I$ for $\hbar \omega = 10^{-6}$ eV, $A = 4$, $\alpha_1 = \alpha_2 = 0.5$, $\phi_2 = 2\pi/3$, $\phi_3 = 0$, and $\Delta = 10^{-6}$ eV. Note that $\hbar \omega = 10^{-6}$ eV corresponds to $\omega \approx 1.52$ GHz. Furthermore, Eq. (74) and $\alpha_1 = \alpha_2 = 0.5$ imply that $C \approx 0.080$ pF and $R \approx 8.22$ kΩ. We have taken the $S$-matrix to be of the form given in Eq. (53) with $\lambda = 0.1$. 
There are prominent plateaus at $\langle V_1 \rangle = 0.5, 1, 1.5, 2, 2.5, 3$ and 3.5 times $10^{-6}$ V (corresponding to integer multiples of $h\omega/(2e)$) and narrower plateaus at subharmonic values given by $\langle V_1 \rangle = 0.25, 0.75, 1.75$ and 2.25 times $10^{-6}$ V (namely, odd integer multiples of $h\omega/(4e) = 0.25$ V). Figure 8 shows a plot of $\langle I_2 - I_3 \rangle$ vs $I$ for the same system parameters; we see plateau-like features around the same values of $I$ as in Fig. 7. In Fig. 9 we show a plot of $\langle V_1 \rangle$ vs $I$ for the same system parameters as in Fig. 7 except that we have chosen $\phi_2 = \phi_3 = 0$. The plots in Figs. 7 and 9 look similar, except that the subharmonic plateaus at odd-integer multiples of $h\omega/4e$ are somewhat less prominent in Fig. 9. We note that the choices of $\phi_2$ and $\phi_3$ are such that the system passes through a region of large Berry curvature in Fig. 7 but not in Fig. 9 (see Fig. 3); however this makes very little difference to the behavior of $\langle V_1 \rangle$ since this quantity is obtained from Eq. (75) which does not contain the Berry curvature. On the other hand, we find numerically that the quantity $\langle I_2 - I_3 \rangle$ shown in Fig. 8 mainly gets a contribution from the Berry curvature term in Eq. (77); the first term in that equation (of the form $\partial E/\partial \phi_1$) makes only a small contribution. For the parameters chosen in Fig. 3 we have $\phi_2 = \phi_3$; hence there is perfect symmetry between wires 2 and 3. In this case, therefore, we have $\langle I_2 - I_3 \rangle = 0$ for all values of $I$.

We can relate the plateaus in $\langle V_1 \rangle$ in Fig. 7 and the plateau-like features in $\langle I_2 - I_3 \rangle$ in Fig. 8 using the following qualitative argument. Since $\langle I_2 - I_3 \rangle$ mainly gets a contribution from the Berry curvature term in Eq. (76), and $B_{+,+1,12} \approx -B_{-,+1,12}$, we can write that equation approximately as

$$I_2 - I_3 \approx \frac{e}{\hbar} 4\hbar\omega B_{-,+1,12}\dot{\phi}_1. \quad (78)$$

Let us now replace all the quantities in the above equation by their average values; this gives

$$\langle I_2 - I_3 \rangle \approx \frac{e}{\hbar} 4\hbar\omega \langle B_{12}\rangle \langle \dot{\phi}_1 \rangle. \quad (79)$$

Next, $\langle \dot{\phi}_1 \rangle$ is related to $\langle V_1 \rangle$ through Eq. (77). On the prominent plateaus in Fig. 7, $\langle V_1 \rangle$ is equal to integer
multiples of $\hbar \omega/(2e)$. Putting all this together, Eq. (79) can be written as

$$\langle I_2 - I_3 \rangle \simeq \frac{e}{\hbar} 4n\hbar B_{12},$$

(80)

where $n$ is an integer. Since we have taken $\hbar \omega = 10^{-6}$ eV, Eq. (80) implies that $\langle I_2 - I_3 \rangle$ should, in units of $10^{-6} \times (e/\hbar)(eV) \simeq 0.24$ nA, have plateau-like features at integer multiples of $4B_{12} \simeq 5.36$, where we have used the value of $B_{12}$ quoted after Eq. (48). This agrees fairly well with what we observe in Fig. 8.

Interestingly, the presence of two phases, $\phi_2$ and $\phi_3$, allows us to vary the widths of the Shapiro plateaus in this three-wire junction, which would not be possible to do in a two-wire junction. We have seen in Figs. 7 and 9 that the plateaus are quite wide when $\phi_2 = 2\pi/3$ or zero (recall that we have fixed $\phi_3 = 0$). However, if we set $\phi_2 = \pi$, we find from Eq. (74) that $E_3$ is independent of $\phi_1$ for any value of $\lambda$. The third term on the left hand side of Eq. (75) is then equal to zero, and we get a linear equation in $\phi_1$: hence the Shapiro plateaus disappear completely. Thus the value of $\phi_2$ can be used to tune the widths of the Shapiro plateaus.

We have studied plots analogous to Figs. 7 and 9 when the S-matrix is of the form given in Eq. (10), $\hbar \omega = 10^{-6}$ eV, $A = 4$ (in units of $10^{-6} \times (e/\hbar)(eV) \simeq 0.24$ nA), $\alpha_1 = \alpha_2 = 0.5$, $\phi_2 = 1.57$, $\phi_3 = 0$, and $\Delta = 10^{-6}$ eV. We find that the Shapiro plateaus are very narrow at $\langle V_1 \rangle$ equal to integer multiples of $\hbar \omega/(2e)$ and no plateaus are visible at odd integer multiples of $\hbar \omega/(4e)$. Similarly, no plateau-like features are visible in a plot of $\langle I_2 - I_3 \rangle$ versus $l$ unlike the plot shown in Fig. 8.

As discussed at the end of Sec. II E, the relative widths of the Shapiro plateaus can be understood to some extent by looking at the absolute values of the Fourier coefficients $|c_l|$ of $\partial E_{-l+1}/\partial \phi_1$. This is shown in Fig. 11 for three cases: (a) S-matrix given in Eq. (53) with $\omega = 0.1$, $\Delta = 10^{-6}$ eV, $\phi_2 = 2\pi/3$, and $\phi_3 = 0$, where the Berry curvature has a peak around $\phi_1 = 4\pi/3$ (see Fig. 7), (b) S-matrix given in Eq. (53) with $\omega = 0.1$, $\Delta = 10^{-6}$ eV, and $\phi_2 = \phi_3 = 0$, where the Berry curvature has no peak at any value of $\phi_1$ (Fig. 3), and (c) S-matrix given in Eq. (60), $\Delta = 10^{-6}$ eV, $\phi_2 = 1.57$, and $\phi_3 = 0$, where the Berry curvature has a peak at $\phi_1 = 2.04$ (Fig. 4). (In each case, we see that $c_0 = 0$ and $|c_l| = |c_1|$ for all values of $l$ as noted after Eq. (15)). In the system shown in Fig. 11(a), we see that $|c_1|$ is quite large at $l$ equal to both $+1$ and $+2$, although $|c_2|/|c_1|$ is small; this explains why there are wide Shapiro plateaus at $\langle V_1 \rangle$ equal to integer multiples of $\hbar \omega/(2e)$ and narrower plateaus at odd integer multiples of $\hbar \omega/(4e)$ (see Fig. 7). In Fig. 11(b), we see that $|c_1|$ is quite large at $l = \pm 1$, but the value of $|c_2|/|c_1|$ is smaller than in Fig. 11(a); this explains why the Shapiro plateaus are wide at integer multiples $\hbar \omega/(2e)$ but quite narrow at odd integer multiples of $\hbar \omega/(4e)$ (Fig. 7). In Fig. 11(c), we see that $|c_1|$ is quite small at all values of $l$; this explains why the Shapiro plateaus are so narrow in this case.

**B. Three p-wave superconducting wires**

We have found that Shapiro plateaus also appear in a system with three p-wave superconducting wires. Considering Eqs. 9 and (53) for the ABS energies and wave functions for three s-wave and three p-wave wires, we
find that the band at \( E_\sigma = \pm \Delta \) in the first case maps to \( E_\sigma = 0 \) in the second case; the other two energies, given in Eqs. (21) and (30), map from \( E_\sigma/\Delta > 0 \) (< 0) in the first case to \( E_\sigma/\Delta < 0 \) (> 0) in the second case; with these mappings, the corresponding wave functions and Berry curvatures are identical in the two cases. Figures 11 and 12 show \( \langle V_1 \rangle \) and \( \langle I_2 - I_3 \rangle \) versus \( I \) when the S-matrix has the form given in Eq. (53) with \( \lambda = 0.1 \), \( \hbar \omega = 10^{-6} \) eV, \( A = 4 \) (in units of \( 10^{-6} \times (e/\hbar)(eV) \approx 0.24 \text{nA} \), and \( V_1 \) is in units of \( 10^{-6} \) eV). The S-matrix has the form given in Eq. (53) with \( \lambda = 0.1 \).

![Figure 11](image1)

**FIG. 11:** Plot of \( \langle V_1 \rangle \) vs \( I \) for three \( p \)-wave SC wires. We have chosen \( \hbar \omega = 10^{-6} \) eV, \( A = 4 \), \( \alpha_1 = \alpha_2 = 0.5 \), \( \phi_2 = 2\pi/3 \), \( \phi_3 = 0 \), and \( \Delta = 10^{-6} \) eV. For the ranges of currents in Fig. 11 where \( V_1 \) shows plateaus, we see bumps (rather than plateau-like features) in \( \langle I_2 - I_3 \rangle \) in Fig. 12. We find numerically that \( \langle I_2 - I_3 \rangle \) gets a contribution mainly from the first term (of the form \( \partial E/\partial \phi_1 \)) in Eq. (70) which dominates over the Berry curvature term. Thus, the effects of the Berry curvature are not easily visible in this system, unlike the case of three \( s \)-wave wires.

### C. Two \( p \)-wave and one \( s \)-wave superconducting wire

We now look at a system in which wires 1 and 2 are \( p \)-wave SCs and wire 3 is a \( s \)-wave SC. In this case, we find that the ABS energies (denoted by \( E_{n,\sigma} \), where \( n \) labels the bands and \( \sigma \) labels the spin) are generally not equal to zero or \( \pm \Delta \). (An exception to this occurs when \( \phi_1 = \phi_2 \); in this case, one of the ABS energies is exactly equal to zero). Furthermore, the energies of the spin-up and -down quasiparticles are generally not equal to each other. As discussed after Eq. (16), we will have \( E_{n,\sigma} = -E_{n,-\sigma} \) for all values of \( (\phi_1, \phi_2) \).

An interesting feature of this system is that one of the ABS energies can suddenly change from \( +\Delta \) to \( -\Delta \) as we vary the phases \( \phi_i \). This happens whenever the value of one of the \( a_\sigma^2 (E_\sigma) \)'s given by Eq. (14) goes through 1. If we write \( a_\sigma^2 (E_\sigma) = e^{2\beta E_\sigma} \), we find, using the fact that the imaginary part of \( a_\sigma (E_\sigma) \) in Eq. (2) must be negative or zero, that \( E_\sigma/\Delta \) is equal to \( -\cos \theta \) if \( \theta \) is small and positive and is equal to \( \cos \theta \) if \( \theta \) is small and negative. Thus, \( E_\sigma \) changes abruptly when \( \theta \) goes through zero.

The fact that the ABS energies are not identical for spin-up and spin-down quasiparticles implies that there can be some spin-dependent effects in this system. One such effect is that the region of the junction and the SC wires can have a net magnetic moment. The appearance of a non-zero magnetic moment requires breaking of time-reversal symmetry. In our system, this will happen if any of the SC phases \( \phi_i \) is not equal to 0 or \( \pi \). This is because \( \phi_i \to -\phi_i \) under time-reversal, as we can see from the complex conjugation of Eq. (1) or (3). Hence \( \phi_i \neq 0 \) or \( \pi \) breaks time-reversal symmetry. In the numerical results presented below, we will ensure time-reversal breaking by setting one of the phases equal to \( 2\pi/3 \).

We define the magnetic moment as

\[
m_z = -\frac{\mu_B}{2} \sum_{n,\sigma} \sigma \left[ f(E_{n,\sigma}) - \frac{1}{2} \right],
\]

\[
= \frac{\mu_B}{2} \sum_n \tanh(\beta E_{n+1}/2),
\]  

(81)
where $\mu_B$ is the Bohr magneton, the prefactor of 1/2 has been put in to avoid double counting of spin, and we have used the symmetry $E_{n,\sigma} = -E_{n,-\sigma}$ to write the second line of Eq. (51). In Fig. 14 we show a surface plot of $m_z$ (in units of $\mu_B/2$) versus $(\phi_1, \phi_2)$ for a system with the $S$-matrix of the form given in Eq. (53) with $\lambda = 0.1$, and a low temperature given by $T = 0.1 \Delta/k_B$. We have set $\phi_3 = 0$. We see that in large regions of Fig. 13 $m_z$ is close to $\pm 1$. This happens because typically two of the ABS energies $E_{n,\lambda=1}$ have the same sign and the other one has the opposite sign, and $\tanh(z) \to \pm 1$ as $z \to \pm \infty$. However, near the line $\phi_1 = \phi_2$, one of the energies gets close to zero and does not contribute much to $m_z$. The other two energies turn out to have the same sign if $\phi_1 = \phi_2$. Hence the two contribute with the same sign, either $+1$ or $-1$, producing values of $m_z$ close to $\pm 2$ which is what we observe in Fig. 14. In the figure, the vertical faces with long crisscross lines appear because one of the ABS energies abruptly changes from $+\Delta$ to $-\Delta$ as we move across the $(\phi_1, \phi_2)$ plane, and this leads to an abrupt change in $m_z$. This abrupt change occurs when either $\phi_1 + \phi_2 = 2\pi$ or $\phi_1 = \phi_2 \pm \pi$.

Next, we consider what happens when a constant voltage bias $V_1$ is applied (i.e., to one of the wires with $p$-wave SC) keeping $V_2 = V_3 = 0$. Then $\phi_1$ varies linearly with time according to $\phi_1 = (2e/h)V_1$ whereas $\phi_2$ and $\phi_3$ remain fixed. As a result, the ABS energies $E_{n,\sigma}$ vary with time and hence so does $m_z$. This is shown in Fig. 14 for a system in which the $S$-matrix is given by Eq. (53) with $\lambda = 0.1$, $\phi_2 = 2\pi/3$, $\phi_3 = 0$, and $\phi_1$ varies with time as $\phi_1 = 0.1 t$ (this corresponds to $(2e/h)V_1 = 0.1 \, s^{-1}$ and the initial value $\phi_1(t = 0) = 0$). Figure 14 shows the results over two time periods of $\phi_1$, given by $0 \leq t \leq 4\pi/0.1$. Figure 14 (a) shows the ABS energies of spin-up (thick blue lines) and spin-down (thin red lines) quasiparticles as a function of time; we see that one of the energies for each spin changes suddenly.

Next, we consider what happens when a constant voltage bias $V_1$ is applied (i.e., to one of the wires with $p$-wave SC) keeping $V_2 = V_3 = 0$. Then $\phi_1$ varies linearly with time according to $\phi_1 = (2e/h)V_1$ whereas $\phi_2$ and $\phi_3$ remain fixed. As a result, the ABS energies $E_{n,\sigma}$ vary with time and hence so does $m_z$. This is shown in Fig. 14 for a system in which the $S$-matrix is given by Eq. (53) with $\lambda = 0.1$, $\phi_2 = 2\pi/3$, $\phi_3 = 0$, and $\phi_1$ varies with time as $\phi_1 = 0.1 t$ (this corresponds to $(2e/h)V_1 = 0.1 \, s^{-1}$ and the initial value $\phi_1(t = 0) = 0$). Figure 14 shows the results over two time periods of $\phi_1$, given by $0 \leq t \leq 4\pi/0.1$. Figure 14 (a) shows the ABS energies of spin-up (thick blue lines) and spin-down (thin red lines) quasiparticles as a function of time; we see that one of the energies for each spin changes suddenly.
between $+\Delta$ and $-\Delta$ at certain times. Figure 14 (b) shows $m_z$ calculated at the same low temperature equal to 0.1 $\Delta/k_B$ as in Fig. 13 we see that $m_z$ changes suddenly between +1 and −1 at the same times as one of the ABS energies. When $\phi_1 = \phi_2 = 2\pi/3$, one of the ABS energies for each spin becomes equal to zero; this happens at $t = (2\pi/3)/0.1 \approx 20.9$ and $(8\pi/3)/0.1 \approx 83.8$. At those times $m_z$ reaches its minimum value of −2 as we see in Fig. 14 (b). (Note that Fig. 14 (b) is essentially a projection of Fig. 13 on to the line $\phi_2 = 2\pi/3$).

We obtain somewhat different results if a constant voltage bias $V_3$ is applied (i.e., to the wire with $s$-wave SC) keeping $V_1 = V_2 = 0$. Then $\phi_3$ varies linearly with time according to $\phi_3 = (2e/h)V_3$ whereas $\phi_1$ and $\phi_2$ remain fixed. The variations of the ABS energies $E_{n,\sigma}$ and $m_z$ with time are shown in Fig. 15 over two time periods for a system in which the $S$-matrix is given by Eq. (53) with $\lambda = 0.1$, $\phi_1 = 2\pi/3$, $\phi_2 = 0$, and $\phi_3$ varies with time as $\phi_3 = 0.1 t$. Figure 15 (a) shows the ABS energies of spin-up (thick blue lines) and spin-down (thin red lines) quasiparticles as a function of time; we again see that one of the energies for each spin changes suddenly between $+\Delta$ and $-\Delta$ at certain times. Figure 15 (b) shows $m_z$ calculated at a temperature equal to 0.1 $\Delta/k_B$: we see that $m_z$ changes suddenly between +1 and −1 at the same times as one of the ABS energies. This occurs when $\phi_1 + \phi_2 = 2\phi_3 \bmod 2\pi$. Note that none of the ABS energies ever become equal to zero since $\phi_1 \neq \phi_2$; hence $m_z$ does not become equal to ±2 at any time.

V. DISCUSSION

In this paper, we have studied a system of several SC wires which meet at a junction. The junction is parametrized by a scattering matrix $S$ which is of a normal form that does not mix electrons and holes. The SC wires can have $s$-wave or $p$-wave pairings, with the pairing phases denoted by $\phi_i$. We have discussed how the ABS energies and wave functions can be determined for any combination of $s$-wave and $p$-wave wires and any spin of the quasiparticles. Our results provide a scattering matrix based formalism for studying junctions involving superconducting wires with both $s$- and $p$-wave pairings; these results generalize the earlier existing results for multi-terminal $s$-wave wires [6, 7]. Various symmetries of the energies and wave functions have been pointed out. In particular, spin-up and -down ABS have the same energies if the three wires are of the same type (all $s$-wave or all $p$-wave), but they do not have the same energies if some of the wires are $s$-wave and others are $p$-wave. We have then studied the Berry curvature and Chern numbers which appear if the number of SC wires is three or more.

Next, we have discussed the ac Josephson effect in which a constant voltage bias is applied to one of the SC wires. We find that an time-averaged current flows only in the other two wires; this current is sensitive to the Berry curvature and the corresponding integrated transconductance is quantized in units of $4e^2/h$. We then discuss what happens if resistances and capacitance are placed between every pair of wires and if the current flowing in one of the wires has both a constant piece $I$ and a
piece which is sinusoidally varying with a frequency $\omega$. In such an $RC$ circuit Shapiro plateaus can appear in a plot of the time-averaged voltage bias $\langle V_1 \rangle$ in the same wire versus $I$. The plateaus can occur at values of $\langle V_1 \rangle$ equal to any rational multiple of $\hbar \omega/(2e)$ leading, in principle, to a devil’s staircase structure. However, in practice, the plateau width goes to zero rapidly as the denominator of the rational number becomes large. We have shown that the plateau widths are related to Fourier transforms of the ABS energies as functions of the $\phi_i$’s.

Next, we have studied several systems of three SC wires in detail. As two examples of the scattering matrix $S$, we have considered a highly symmetric and time-reversal invariant form and a randomly generated asymmetric form which is not time-reversal invariant. Throughout our analysis, we have assumed that the matrix $S$ is spin independent even when time-reversal symmetry is broken. This would be true if, for example, the junction consists of a loop which is threaded by a magnetic flux (thus breaking time-reversal symmetry); such a flux would affect the transport around the loop through an Aharonov-Bohm phase which does not depend on the spin. [We present explicit expressions for the ABS energies for the cases of three $s$-wave and three $p$-wave wires. The forms of the Berry curvature and the values of the Chern numbers depend crucially on the form of $S$.]

Finally, we have numerically studied a number of three-wire systems to test the various ideas presented in the earlier sections. For three $s$-wave wires, we have shown that the dependence of the ac Josephson current in one wire on the phase $\phi_i$ in one of the other wires can directly provide information about the Berry curvature. Next, we have shown that when the time-averaged $V$ in one wire is plotted versus $I$ when the current also contains an oscillating piece in the same wire, Shapiro plateaus with discernible widths appear at both integer and half-odd-integer multiples of $\hbar \omega/(2e)$. Furthermore, the currents in the other two wires show plateau-like features in the ranges of the $I$ where $V$ shows plateaus; these features are mainly due to the Berry curvature terms in the currents. For three $p$-wave wires, we find Shapiro plateaus in the plot of $V$ versus $I$; the currents in the other wires show some bumps in the same ranges of $I$ but these are not primarily due to the Berry curvature. For a system with two $p$-wave wires and one $s$-wave wire, the fact that the ABS energies are not identical for spin-up and -down ABS leads to spin-dependent effects. For instance, we find that the junction region can have a non-zero magnetic moment which depends on the phases $\phi_i$. We note that this allows for a direct control of the time variation of such a magnetic moment by externally applying a voltage across the wires. In the ac Josephson effect, where one of these phases varies linearly with time, we find that the magnetic moment varies periodically in time, showing large jumps when one of the ABS energies either touches zero or changes abruptly between the top and the bottom of the SC gap.

There are several experiments which could verify our theoretical results. First, we predict that standard $I-V$ characteristics measurements with three-wire $s$-wave or $p$-wave junctions in the presence of an external microwave radiation of frequency $\omega$ would detect Shapiro plateaus at rational fractional values of $2e\langle V_1 \rangle/(\hbar \omega)$. Such experiments are routine for two-wire junctions and could, in principle, be carried out for multi-wire systems. Second, for three-wire junctions involving both $p$- and $s$-wave wires, one may detect the effect of an oscillating magnetic moment through the resulting radiated electric field near the junction; such experiments have been discussed in a different context. Finally, the measurement of the integrated transconductance in junctions involving three $s$-wave wires by using an appropriate four-terminal setup would constitute an experimental way of ascertaining the quantization predicted by Eq. (37).

In conclusion, we have studied multi-wire junctions of $s$- and $p$-wave superconductors and have developed a scattering matrix based approach which can be used to describe such junctions. We have studied the ac Josephson effect in such junctions pointing out the presence of Shapiro plateaus. We have also shown that such systems involving both $s$- and $p$-wave superconducting wires lead to presence of magnetic moments in the junction whose time variation can be controlled via an external applied voltage. We have suggested several experiments which can test our theory.

**Acknowledgments**

D.S. thanks Department of Science and Technology, India for Project No. SR/S2/JCB-44/2010 for financial support.

---

1. J. B. Ketterson and S. N. Song, *Superconductivity* (Cambridge University Press, Cambridge, 1999).
2. A. Kitaev, Physics-Uspekhi 44, 131 (2001).
H.-J. Kwon, K. Sengupta, and V. M. Yakovenko, Eur. Phys. J. B 37, 349 (2004).
L. P. Rokhinson, X. Liu, and J. K. Furdyna, Nature Physics 8, 795 (2012).
K. Sengupta and V. M. Yakovenko, Phys. Rev. Lett. 101, 187003 (2008).
R.-P. Riwar, M. Houzet, J. S. Meyer, and Y. V. Nazarov, Nature Communications 7, 11167 (2016).
T. Yokoyama and Y. V. Nazarov, Phys. Rev. B 92, 155437 (2015).
D. A. Savinov, Physica C 509, 22 (2015).
D. A. Savinov, Physica C: Superconductivity and its applications 527, 80 (2016).
J. C. Cuevas and H. Pothier, Phys. Rev. B 75, 174513 (2007).
M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
Y. Peng, F. Pientka, E. Berg, Y. Oreg, and F. von Oppen, Phys. Rev. B 94, 085409 (2016).
E. Eriksson, R.-P. Riwar, M. Houzet, J. S. Meyer, and Y. V. Nazarov, Phys. Rev. B 95, 075417 (2017).
J. S. Meyer and M. Houzet, Phys. Rev. Lett. 119, 136807 (2017).
A. Zazunov, R. Egger, M. Alvarado, and A. Levy Yeyati, Phys. Rev. B 96, 024516 (2017).
H.-Y. Xie, M. G. Vavilov, and A. Levchenko, Phys. Rev. B 96, 161406(R) (2017).
H.-Y. Xie, M. G. Vavilov, and A. Levchenko, Phys. Rev. B 97, 035443 (2018).
E. Strambini, S. D’Ambrosio, F. Vischi, F. S. Bergeret, Y. V. Nazarov, and F. Giazotto, Nature Nanotechnology 11, 1055 (2016).
C. W. J. Beenakker, Phys. Rev. Lett. 67, 3836 (1991).
M. Maiti, K. M. Kulikov, K. Sengupta, and Y. M. Shukrinov, Phys. Rev. B 92, 224501 (2015).
Y. M. Shukrinov, S. Y. Medvedeva, A. E. Botha, M. R. Kolahchi, and A. Irie, Phys. Rev. B 88, 214515 (2013).
A. Soori, O. Deb, K. Sengupta, and D. Sen, Phys. Rev. B 87, 245435 (2013).
A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).
G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B 25, 4515 (1982).
T. Fukui, Y. Hatsugai, and H. Suzuki, J. Phys. Soc. Jpn. 74, 1674 (2005).
K. K. Likharev, Dynamics of Josephson Junctions and Circuits (Taylor and Francis, London, 1986).
M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1972).
S. Lal, S. Rao, and D. Sen, Phys. Rev. B 66, 165327 (2002).
A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. 81, 109 (2009).
J. Clarke, Phys. Rev. B 4, 2963 (1971); H. Dayem and J. J. Wiegand, Phys. Rev. 155, 419 (1967).
K. Y. Constantiniyan, G. A. Ovsyannikov, Y. V. Kislinskii, A. V. Shadrin, I. V. Borisenko, P. V. Komissinskii, A. V. Zaitsev, J. Mygind, and D. Winkler, J. Phys. Conf. Ser. 234, 042004 (2010).
N. Kanda, T. Higuchi, H. Shimizu, K. Konishi, K. Yoshikawa, and M. Kuwata-Gonokami, Nature Communications 2, 362 (2011).