The abelianization of the level $L$ mapping class group

Andrew Putman

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Abstract

We calculate the abelianizations of the level $L$ subgroup of the genus $g$ mapping class group and the level $L$ congruence subgroup of the $2g \times 2g$ symplectic group for $L$ odd and $g \geq 3$.

1 Introduction

Let $\Sigma_{g,n}$ be an orientable genus $g$ surface with $n$ boundary components and let $\text{Mod}_{g,n}$ be its mapping class group, that is, the group $\pi_0(\text{Diff}^+(\Sigma_{g,n}, \partial \Sigma_{g,n}))$. This is the (orbifold) fundamental group of the moduli space of Riemann surfaces and has been intensely studied by many authors. For $n \in \{0, 1\}$, the action of $\text{Mod}_{g,n}$ on $H_1(\Sigma_{g,n}; \mathbb{Z})$ induces a surjective representation of $\text{Mod}_{g,n}$ into the symplectic group whose kernel $\mathcal{I}_{g,n}$ is known as the Torelli group. This is summarized by the exact sequence

$$1 \rightarrow \mathcal{I}_{g,n} \rightarrow \text{Mod}_{g,n} \rightarrow \text{Sp}_{2g}(\mathbb{Z}) \rightarrow 1.$$ 

For $L \geq 2$, let $\text{Sp}_{2g}(\mathbb{Z}, L)$ denote the level $L$ congruence subgroup of $\text{Sp}_{2g}(\mathbb{Z})$, that is, the subgroup of matrices that are equal to the identity modulo $L$. The pull-back of $\text{Sp}_{2g}(\mathbb{Z}, L)$ to $\text{Mod}_{g,n}$ is known as the level $L$ subgroup of $\text{Mod}_{g,n}$ and is denoted by $\text{Mod}_{g,n}(L)$. The group $\text{Mod}_{g,n}(L)$ can also be described as the group of mapping classes that act trivially on $H_1(\Sigma_{g,n}; \mathbb{Z}/L\mathbb{Z})$. It fits into an exact sequence

$$1 \rightarrow \mathcal{I}_{g,n} \rightarrow \text{Mod}_{g,n}(L) \rightarrow \text{Sp}_{2g}(\mathbb{Z}, L) \rightarrow 1.$$ 

In [6], Hain proved that the abelianization of $\text{Mod}_{g,n}(L)$ consists entirely of torsion for $g \geq 3$ (an alternate proof was given by McCarthy in [12]). In this note, we compute this torsion for $L$ odd.

To state our theorem, we need some notation. Denoting the $n \times n$ zero matrix by $0_n$ and the $n \times n$ identity matrix by $1_n$, let $\Omega_g$ be the matrix

$$\left( \begin{array}{cc} 0_g & 1_g \\ -\bar{g}_g & 0_g \end{array} \right)$$

(we will abuse notation and let the entries of $\Omega_g$ lie in whatever ring we are considering at the moment). By definition, the group $\text{Sp}_{2g}(\mathbb{Z})$ consists of $2g \times 2g$ integral matrices $X$ that satisfy $X^t \Omega_g X = \Omega_g$. We
will denote by $\mathfrak{sp}_{2g}(L)$ the additive group of all $2g \times 2g$ matrices $A$ with entries in $\mathbb{Z}/L\mathbb{Z}$ that satisfy $A^t\Omega_g + \Omega_g A = 0$. To put it another way, $\mathfrak{sp}_{2g}(L)$ is the additive group obtained from the symplectic Lie algebra over $\mathbb{Z}/L\mathbb{Z}$ by forgetting the bracket.

Our main theorem is as follows, and is proven in §4.

**Theorem 1.1** (Integral $H_1$ of level $L$ subgroups). For $g \geq 3$, $n \in \{0,1\}$, and $L$ odd, set $H(L) = H_1(\Sigma_{g,n}; \mathbb{Z}/L\mathbb{Z})$. We then have an exact sequence

$$0 \to K \to H_1(\text{Mod}_{g,n}(L); \mathbb{Z}) \to \mathfrak{sp}_{2g}(L) \to 0,$$

where $K = \wedge^3 H(L)$ if $n = 1$ and $K = (\wedge^3 H(L))/H(L)$ if $n = 0$.

**Remark.** The condition $g \geq 3$ is necessary, since in [12] McCarthy proves that if 2 or 3 divides $L$, then $\text{Mod}_2(L)$ surjects onto $\mathbb{Z}$. A computation of $H_1(\text{Mod}_2, n(L); \mathbb{Z})$ (or even $H_1(\text{Mod}_2, n(L); \mathbb{Q})$) would be very interesting.

We now describe the sources for the terms in the exact sequence of Theorem 1.1. The kernel $K$ comes from the **relative Johnson homomorphisms** of Broaddus-Farb-Putman [4]. For $\text{Mod}_{g,n}(L)$, these are surjective homomorphisms

$$\tau_{g,1}(L) : \text{Mod}_{g,1}(L) \to \wedge^3 H(L)$$

and

$$\tau_g(L) : \text{Mod}_g(L) \to (\wedge^3 H(L))/H(L)$$

which are related to the celebrated Johnson homomorphisms on the Torelli group (see §3 and [4]).

The cokernel $\mathfrak{sp}_{2g}(L)$ is the abelianization of $\text{Sp}_{2g}(\mathbb{Z}, L)$. Now, the isomorphism

$$H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sp}_{2g}(L)$$

can be deduced from general theorems of Borel on arithmetic groups (see [3, §2.5]); however, Borel’s results are much more general than we need and it takes some work to derive the desired result from them. We instead imitate a beautiful argument of Lee-Szczarba [11], who prove that

$$H_1(\text{SL}_n(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sl}_n(L)$$

for $n \geq 3$. Here $\text{SL}_n(\mathbb{Z}, L)$ is the level $L$ congruence subgroup of $\text{SL}_n(\mathbb{Z})$ and $\mathfrak{sl}_n(L)$ is the additive group of $n \times n$ matrices with coefficients in $\mathbb{Z}/L\mathbb{Z}$ and trace 0. The proof of the following theorem is contained in [42].

**Theorem 1.2** (Integral $H_1$ of $\text{Sp}_{2g}(\mathbb{Z}, L)$). For $g \geq 3$ and $L$ odd, we have

$$H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \mathfrak{sp}_{2g}(L).$$

Moreover, $[\text{Sp}_{2g}(\mathbb{Z}, L), \text{Sp}_{2g}(\mathbb{Z}, L)] = \text{Sp}_{2g}(\mathbb{Z}, L^2)$. 

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Remark. It is unclear whether the hypothesis that $L$ is odd is necessary for Theorems 1.1 or 1.2 but it is definitely used in both proofs.

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2 The abelianization of $\text{Sp}_{2g}(\mathbb{Z}, L)$

We will need the following notation.

**Definition 2.1.** For $1 \leq i, j \leq n$, let $E_{i,j}^n(r)$ be the $n \times n$ matrix with an $r$ at position $(i, j)$ and 0’s elsewhere. Similarly, let $S_{i,j}^n(r)$ be the $n \times n$ matrix with an $r$ at positions $(i, j)$ and $(j, i)$ and 0’s elsewhere.

**Definition 2.2.** For $1 \leq i, j \leq g$, denote by $X_{i,j}^g(r)$ the matrix \[
\begin{pmatrix}
I_g & 0_g \\
0_g & I_g
\end{pmatrix}
\] by $Y_{i,j}^g(r)$ the matrix \[
\begin{pmatrix}
I_g & S_{i,j}^g(r) \\
0_g & I_g
\end{pmatrix}
\] and by $Z_{i,j}^g(r)$ the matrix \[
\begin{pmatrix}
I_g & \varepsilon_{i,j}^g(r) \\
0_g & I_g
\end{pmatrix}.
\]

Observe that $X_{i,j}^g(L), Y_{i,j}^g(L) \in \text{Sp}_{2g}(\mathbb{Z}, L)$ for all $1 \leq i, j \leq g$ and that $Z_{i,j}^g(L) \in \text{Sp}_{2g}(\mathbb{Z}, L)$ for $1 \leq i, j \leq g$ with $i \neq j$. The following theorem forms part of Bass-Milnor-Serre’s solution to the congruence subgroup problem for the symplectic group.

**Theorem 2.3** (Bass-Milnor-Serre [11 Theorem 12.4, Corollary 12.5]). For $g \geq 2$ and $L \geq 1$, the group $\text{Sp}_{2g}(\mathbb{Z}, L)$ is generated by \{ $X_{i,j}^g(L) \mid 1 \leq i, j \leq g$ \} \{ $Y_{i,j}^g(L) \mid 1 \leq i, j \leq g$ \}.

Remark. We emphasize that the matrices $Z_{i,j}^g(L)$ are not needed – the proof of [11 Lemma 13.1] contains an explicit formula for them in terms of the $X_{i,j}^g$ and the $Y_{i,j}^g$.

Using this, we can prove the following.

**Lemma 2.4.** For $g \geq 3$ and $L$ odd, we have $\text{Sp}_{2g}(\mathbb{Z}, L^2) \leq [\text{Sp}_{2g}(\mathbb{Z}, L), \text{Sp}_{2g}(\mathbb{Z}, L)]$.

**Proof.** We must show that each generator of $\text{Sp}_{2g}(\mathbb{Z}, L^2)$ given by Theorem 2.3 is contained in $[\text{Sp}_{2g}(\mathbb{Z}, L), \text{Sp}_{2g}(\mathbb{Z}, L)]$. We will do the case of $X_{i,j}^g(L^2)$; the other case is similar. Assume first that $i \neq j$. Since $g \geq 3$, there is some $1 \leq k \leq g$ so that $k \neq i, j$. The following matrix identity then proves the desired claim:

$$X_{i,j}^g(L^2) = [X_{i,k}^g(L), Z_{k,j}^g(L)].$$

Now assume that $i = j$. Again, there exists some $1 \leq k_1 < k_2 \leq g$ so that $k_1, k_2 \neq i$. Also, since $L$ is odd there exists some integer $N$ so that $2N + L = 1$. We thus have

$$X_{i,i}^g(L^2) = X_{i,i}^g((2N + L)L^2) = X_{i,i}^g(2NL^2) \cdot X_{i,i}^g(L^3),$$

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so the following matrix identities complete the proof:

\[
\begin{align*}
X^g_{i,i}(2NL^2) &= [X^g_{i,k_1}(NL), Z^g_{k_1,i}(L)], \\
X^g_{i,i}(L^3) &= [X^g_{k_1,k_1}(L), Z^g_{k_1,i}(L)] \cdot [Z^g_{k_2,i}(L), X^g_{k_1,k_2}(L)].
\end{align*}
\]

**Proof of Theorem 1.2.** We begin by defining a function \(\phi : Sp_{2g}(Z, L) \to sp_{2g}(L)\). Consider any matrix \(X \in Sp_{2g}(Z, L)\). Write \(X = I_{2g} + LA\), and define

\[
\phi(X) = A \pmod{L}.
\]

We claim that \(\phi(X) \in sp_{2g}(L)\). Indeed, by the definition of the symplectic group we have \(X^t \Omega g X = \Omega g\). Writing \(X = I_{2g} + LA\) and expanding out, we have

\[
\Omega g + L(A^t \Omega g + \Omega g A) + L^2(A^t \Omega g A) = \Omega g.
\]

We conclude that modulo \(L\) we have \(A^t \Omega g + \Omega g A = 0\), as desired.

Next, we prove that \(\phi\) is a homomorphism. Consider \(X, Y \in Sp_{2g}(Z, L)\) with \(X = I_{2g} + LA\) and \(Y = I_{2g} + LB\). Thus \(XY = I_{2g} + L(A + B) + L^2AB\), so modulo \(L\) we have \(\phi(XY) = A + B\), as desired.

Observe now that \(\ker(\phi) = Sp_{2g}(Z, L^2)\). Since \(sp_{2g}(L)\) is abelian, this implies that \([Sp_{2g}(Z, L), Sp_{2g}(Z, L)] < Sp_{2g}(Z, L^2)\). Lemma 2.4 then allows us to conclude that \(\ker(\phi) = Sp_{2g}(Z, L^2) = [Sp_{2g}(Z, L), Sp_{2g}(Z, L)]\), and the theorem follows.

3 The Torelli group

We now review some facts about \(I_{g,n}\).

**Definition 3.1.** Let \(n \in \{0, 1\}\). A *bounding pair* on \(\Sigma_{g,n}\) is a pair \(\{x_1, x_2\}\) of disjoint nonhomotopic nonseparating curves on \(\Sigma_{g,n}\) so that \(x_1 \cup x_2\) separates \(\Sigma_{g,n}\). Letting \(T_\gamma\) denote the Dehn twist about a simple closed curve \(\gamma\), the *bounding pair map* associated to a bounding pair \(\{x_1, x_2\}\) is \(T_{x_1} T_{x_2}^{-1}\).

Observe that if \(\{x_1, x_2\}\) is a bounding pair, then \(T_{x_1} T_{x_2}^{-1} \in I_{g,n}\). Building on work of Birman [2] and Powell [14], Johnson proved the following.

**Theorem 3.2** (Johnson, [7]). For \(g \geq 3\) and \(n \in \{0, 1\}\), the group \(I_{g,n}\) is generated by bounding pair maps.

**Remark.** In fact, under the hypotheses of this theorem Johnson later proved that finitely many bounding pair maps suffice [9]. This should be contrasted with work of McCullough-Miller [13] that says that for \(n \in \{0, 1\}\), the group \(I_{2,n}\) is *not* finitely generated.

We will also need Johnson’s computation of the abelianization of \(I_{g,n}\).
Theorem 3.3 (Johnson, [10]). Let $g \geq 3$, and set $H = H_1(\Sigma_g; \mathbb{Z}) \cong H_1(\Sigma_{g,1}; \mathbb{Z})$. Then
\[ H_1(\mathcal{I}_{g,1}; \mathbb{Z}) \cong \wedge^3 H \oplus (2\text{-torsion}) \]
and
\[ H_1(\mathcal{I}_g; \mathbb{Z}) \cong (\wedge^3 H/ H) \oplus (2\text{-torsion}). \]
The maps
\[ \tau_{g,1} : \mathcal{I}_{g,1} \longrightarrow H_1(\mathcal{I}_{g,1}; \mathbb{Z})/(2\text{-torsion}) \cong \wedge^3 H \]
and
\[ \tau_g : \mathcal{I}_g \longrightarrow H_1(\mathcal{I}_g; \mathbb{Z})/(2\text{-torsion}) \cong (\wedge^3 H)/ H \]
are known as the Johnson homomorphisms and have many remarkable properties. For a survey, see [8].

4 The abelianization of $\text{Mod}_{g,n}(L)$

Partly to establish notation, we begin by recalling the statement of the 5-term exact sequence in group homology.

Theorem 4.1 (see, e.g., [5, Corollary VII.6.4]). Let
\[ 1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1 \]
be a short exact sequence of groups and let $R$ be a ring. There is then an exact sequence
\[ H_2(G; R) \longrightarrow H_2(Q; R) \longrightarrow H_1(K; R) \longrightarrow H_1(G; R) \longrightarrow H_1(Q; R) \longrightarrow 0, \]
where $H_1(K; R) \longrightarrow Q$ is the ring of co-invariants of $H_1(K; R)$ under the natural action of $Q$, that is, the quotient of $H_1(K; R)$ by the ideal generated by \( \{ q(k) - k \mid q \in Q \text{ and } k \in K \} \).

We will need a special case of a theorem of Broaddus-Farb-Putman that gives “relative” versions of the Johnson homomorphisms on certain “homologically defined” subgroups of $\text{Mod}_{g,b}$. In our situation, the result can be stated as follows.

Theorem 4.2 (Broaddus-Farb-Putman, [4, Example 5.3 and Theorem 5.8]). Fix $L \geq 2$, $g \geq 3$, and $n \in \{0, 1\}$. Set $H = H_1(\Sigma_{g,n}; \mathbb{Z})$ and $H(L) = H_1(\Sigma_{g,n}; \mathbb{Z}/L \mathbb{Z})$, and define $X$ and $X(L)$ to equal $H$ and $H(L)$ if $n = 0$ and to equal $0$ if $n = 1$. Hence $(\wedge^3 H)/X$ is the target for the Johnson homomorphism on $\mathcal{I}_{g,n}$. Then there exist homomorphisms $\tau_{g,n}(L) : \text{Mod}_{g,1}(L) \rightarrow (\wedge^3 H(L))/ X(L)$ that fit into the commutative diagram
\[ \begin{array}{ccc}
\mathcal{I}_{g,n} & \xrightarrow{\tau_{g,n}} & (\wedge^3 H)/X \\
\downarrow & & \downarrow \\
\text{Mod}_{g,n}(L) & \xrightarrow{\tau_{g,n}(L)} & (\wedge^3 H(L))/ X(L)
\end{array} \]
Here the right hand vertical arrow is reduction mod $L$. 

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We preface the proof of Theorem 1.1 with two lemmas. Our first lemma was originally proven by McCarthy [12, proof of Theorem 1.1]. We give an alternate proof. If \( G \) is a group and \( g \in G \), then denote by \([g]\) the corresponding element of \( H_1(G; \mathbb{Z})\).

**Lemma 4.3.** For \( n \in \{0, 1\} \), let \( \{x_1, x_2\} \) be a bounding pair on \( \Sigma_{g,n} \). Then 
\[
L[T_x x_1 T_x^{-1} x_2] = 0
\]
in \( H_1(\text{Mod}_{g,n}(L); \mathbb{Z})\).

**Proof.** Embed \( \{x_1, x_2\} \) in a 2-holed torus as in Figure 1. We will make use of the crossed lantern relation from [16]. Letting \( \{y_1, y_2\} \) and \( \{z_1, z_2\} \) be the other bounding pair maps depicted in Figure 1 this relation says that
\[
(T_y y_1 T_y^{-1} y_2)(T_x x_1 T_x^{-1} x_2) = (T_z z_1 T_z^{-1} z_2).
\]
Observe that for \( i = 1, 2 \) we have \( z_i = T_{x_2}(y_i) \). The key observation is that for all \( n \geq 0 \) we have another crossed lantern relation
\[
(T_{T_{x_2}^n(y_1)} T_{T_{x_2}^{n+1}(y_2)}^{-1})(T_x x_1 T_x^{-1} x_2) = (T_{T_{x_2}^{n+1}(y_1)} T_{T_{x_2}^{n+1}(y_2)}^{-1}).
\]
Since \( T_{x_2}^L \in \text{Mod}_{g,n}(L) \), we conclude that in \( H_1(\text{Mod}_{g,n}(L); \mathbb{Z}) \) we have
\[
[T_{y_1} y_2^{-1}] = [T_{x_2}^L] + [T_{y_1} y_2^{-1}] - [T_{x_2}^L] = [T_{x_2}^L (T_{y_1} y_2^{-1}) T_{x_2}^{-L}] = [(T_{T_{x_2}^n(y_1)} T_{T_{x_2}^{n+1}(y_2)}^{-1})]
\]
\[
= [T_{x_1} x_2^{-1}] + [(T_{T_{x_2}^{n}(y_1)} T_{x_2}^{-L+1}(y_2))]
\]
\[
= 2[T_{x_1} x_2^{-1}] + [(T_{T_{x_2}^{n-2}(y_1)} T_{x_2}^{-L+2}(y_2))]
\]
\[
= \cdots
\]
\[
= L[T_{x_1} x_2^{-1}] + [T_{y_1} y_2^{-1}],
\]
so \( L[T_{x_1} x_2^{-1}] = 0 \), as desired. \( \square \)

For the statement of the following lemma, recall that if a group \( G \) acts on a ring \( R \), then the coinvariants of that action are denoted \( R_G \).
Lemma 4.4. For $L \geq 2$, define $H = H_1(\Sigma_g; \mathbb{Z})$ and $H(L) = H_1(\Sigma_g; \mathbb{Z}/L\mathbb{Z})$. Then
\[
(\wedge^3 H)_{\text{Sp}_{2g}(\mathbb{Z}, L)} \cong \wedge^3 H(L)
\]
and
\[
((\wedge^3 H)/H)_{\text{Sp}_{2g}(\mathbb{Z}, L)} \cong (\wedge^3 H(L))/H(L).
\]

Proof. Letting $S = \{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H$, the groups $\wedge^3 H$ and $(\wedge^3 H)/H$ are generated by $T := \{x \wedge y \wedge z \mid x, y, z \in S \text{ distinct}\}$. Consider $x \wedge y \wedge z \in T$. It is enough to show that in the indicated rings of coinvariants we have $L(x \wedge y \wedge z) = 0$. Now, one of $x, y, z$ must have algebraic intersection number 0 with the other two terms. Assume that $x = a_1$ and $y, z \in \{a_2, b_2, \ldots, a_g, b_g\}$ (the other cases are similar). There is then some $\phi \in \text{Sp}_{2g}(\mathbb{Z}, L)$ so that $\phi(b_1) = b_1 + L a_1 = b_1 + L x$ and so that $\phi(y) = y$ and $\phi(z) = z$. We conclude that in the indicated ring of coinvariants we have $b_1 \wedge y \wedge z = (b_1 + L x) \wedge y \wedge z$, so $L(x \wedge y \wedge z) = 0$, as desired. \qed

Remark. Lemma 4.4 would not be true if $\wedge^3 H$ were replaced by $\wedge^2 H$, as $\wedge^2 H$ contains a copy of the trivial representation of $\text{Sp}_{2g}(\mathbb{Z})$.

Proof of Theorem 1.1. We will do the proof for $\text{Mod}_{g, 1}(L)$; the other case is similar. Let $H$ and $H(L)$ be as in Theorem 4.2. Associated to the short exact sequence
\[1 \to I_{g, 1} \to \text{Mod}_{g, 1} \to \text{Sp}_{2g}(\mathbb{Z}, L) \to 1\]
is the 5-term exact sequence in homology given by Theorem 4.1. Theorem 3.3 says that $H_1(I_{g, 1}; \mathbb{Z}) \cong \wedge^3 H \oplus (2\text{-torsion})$ and Theorem 1.2 says that $H_1(\text{Sp}_{2g}(\mathbb{Z}, L); \mathbb{Z}) \cong \text{sp}_{2g}(\mathbb{Z}/L\mathbb{Z})$. The last 3 terms of our 5-term exact sequence are thus
\[
(\wedge^3 H \oplus (2\text{-torsion}))_{\text{Sp}_{2g}(\mathbb{Z}, L)} \xrightarrow{i} H_1(\text{Mod}_{g, 1}(L); \mathbb{Z}) \to \text{sp}_{2g}(\mathbb{Z}/L\mathbb{Z}) \to 0.
\]
Since $L$ is odd, Lemma 4.3 together with Theorem 3.2 say that if $x \in (\wedge^3 H \oplus (2\text{-torsion}))_{\text{Sp}_{2g}(\mathbb{Z}, L)}$ is 2-torsion then $i(x) = 0$. Moreover, Lemma 4.4 says that
\[
(\wedge^3 H)_{\text{Sp}_{2g}(\mathbb{Z}, L)} \cong \wedge^3 H(L).
\]
We thus obtain an exact sequence
\[
\wedge^3 H(L) \xrightarrow{j} H_1(\text{Mod}_{g, 1}(L); \mathbb{Z}) \to \text{sp}_{2g}(\mathbb{Z}/L\mathbb{Z}) \to 0.
\]
Theorem 4.2 then implies that $j$ is an injection, and the proof is complete. \qed
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Department of Mathematics; MIT, 2-306
77 Massachusetts Avenue
Cambridge, MA 02139-4307
E-mail: andyp@math.mit.edu