Gapless state of interacting Majorana fermions in a strain-induced Landau level

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Mechanical strain can generate a pseudo-magnetic field, and hence Landau levels (LLs), for low energy excitations of quantum matter in two dimensions. We study the collective state of the fractionalised Majorana fermions arising from residual generic spin interactions in the central LL, where the projected Hamiltonian reflects the spin symmetries in intricate ways: emergent U(1) and particle-hole symmetries forbid any bilinear couplings, leading to an intrinsically strongly interacting system; also, they allow the definition of a filling fraction, which is fixed at 1/2. We argue that the resulting many-body state is gapless within our numerical accuracy, implying ultra-short-ranged spin correlations, while chirality correlators decay algebraically. This amounts to a Kitaev ‘non-Fermi’ spin liquid, and shows that interacting Majorana Fermions can exhibit intricate behaviour akin to fractional quantum Hall physics in an insulating magnet.

Introduction: Majorana Fermions, elusive as elementary particles, have been the subject of intense interest as emergent quasiparticles in condensed matter physics [1–12]. Their practical relevance derives from the appearance of symmetry protected Majorana zero modes in topological quantum computing [1, 13]. In addition, as fractionalised degrees of freedom they arise as novel collective excitations in long range entangled quantum phases of matter [14–17], to the study of which this work is devoted.

Platforms proposed for collective Majorana phases include superconductor-topological insulator heterostructures [13, 18], vortex matter in chiral-superconductors [19] and the ν = 5/2 fractional quantum Hall (FQH) liquid [11, 20]. An intriguing alternative is given by Kitaev’s honeycomb spin liquid [4] (QSL). The starting point of our work is the exact solution of the eponymous honeycomb model which identifies Majorana fermions as effective low-energy degrees of freedom arising from fractionalisation of the microscopic spin degrees of freedom [4]. However, their Dirac dispersions imply a vanishing low energy density of states (DOS), so that residual spin interactions that lead to short-range four-Majorana interactions are apriori irrelevant for the pure model at the free Majorana fixed point.

Application of mechanical strain, by contrast, modifies this situation drastically given it acts as a synthetic magnetic field to low energy excitations resulting in non-dispersing Landau levels (LLs) of non-interacting Majorana excitations [15] like in graphene [21–24] with characteristic signatures in experimental probes [25]. These LLs provide a non-vanishing DOS for Majorana fermions, allowing for the residual spin interactions, inevitably present in any real material, to become extremely interesting. We explore the resulting collective behaviour. These extensively degenerate Landau levels lead to an intrinsically strongly interacting problem with the potential for the fractionalised Majoranas of the Kitaev Z2 QSL to exhibit manifold non-Fermi liquid instabilities, as is famously the case in FQH at ν = 1/2 [19, 26–28].

We thus pose the general question how the many-body state of the degenerate Majorana fermions changes upon addition of generic perturbations allowed by symmetry; and for our concrete example of the strained Kitaev model, how the collective state is reflected in the correlations of the spins?

Our analysis points to a gapless QSL which is reminiscent of the composite Fermi liquid originally proposed for the FQH problem at filling ν = 1/2 [19, 26–28]. This exhibits spin correlators even more short-range than the unstrained Kitaev QSL, while the ‘chiral’ three-spin correlators decay algebraically with distance, ∼ r−4. Constitutive to our analysis is the non-trivial (projective) implementation of the microscopic symmetries on the fractionalised Majoranas not unlike the low energy effective molecular orbitals of the recently studied twisted bi-layer graphene [29, 30]. This moves the study of the interplay of symmetry and long-range entanglement from the soluble Kitaev QSL physics into the realm of a gapless strongly interacting setting, towards quantum ‘non-Fermi’ spin liquids, so to speak.

The rest of this paper is organised as follows. Starting with the strained Kitaev model, we present the fate of spin interactions upon projection onto the central Landau level (cLL), deriving the terms present in, and the symmetries of, a generic effective Hamiltonian. This is followed by a numerical analysis using exact diagonalisation and density matrix renormalization group (DMRG), and a study of more tractable related models. We conclude with an outlook.
Effective Hamiltonian: (a) Each hexagon has three chirality terms $F_x$, $F_y$, $F_z$ defined in terms of three spin operators which couple two Majoranas on A sites in the zero flux sector. Tri-axial strain C acts in the directions shown. (b) Each hexagon has three rhombic plaquettes (yellow, blue, red) that signify a 4-Majorana interaction which can arise due to product of $F_a$ operators on neighboring hexagons.

Model: We consider the Kitaev honeycomb model, $H = \sum_{ij} J_{ij,\alpha} \sigma_i^\alpha \sigma_j^\alpha$ [4], with its bond-dependent nearest-neighbour Ising exchanges, $\alpha \in x, y, z$ for the three different bond directions (see Fig. 1). Representing each spin $S = 1/2$ in terms of four Majorana fermions $b_i^x, b_i^y, b_i^z$ and $c_j$ such that $\sigma_i^\alpha = ib_i^\alpha c_j$ yields the ground state, a $Z_2$ QSL with dynamic gapless Majorana fermions, $c_i$, minimally coupled to non-dynamical $Z_2$ fluxes, with flux gap $\equiv \Delta_f$, formed by the product of $b_i^z$s around the hexagonal plaquettes. The ground state lies in the zero flux sector where the Majorana fermions have a linearly dispersing spectrum at the two Dirac points $\pm K$.

Tri-axial strain (C) is known to generate a uniform pseudo-magnetic field [15] for a flake in a region of radius, $l_r$, with an effective magnetic length $l_B \sim \frac{1}{\sqrt{\delta}}$ [23]. This strain breaks lattice inversion—but not time-reversal symmetry, with the resulting pseudo-magnetic field having opposite direction at the two Dirac points $\pm K$, as in graphene [31, 32]. We work in the physically relevant hierarchy of length scales $\frac{1}{\sqrt{\delta}} < l_B < l_r$ and hence restrict our analysis to the zero $Z_2$ flux sector and uniform pseudo-magnetic field regime. The low energy physics thus naturally maps to the problem of non-dispersive matter Majoranas in the cLL.

The wavefunctions of the cLL reside on only one sublattice [23, 33] (say A, see Fig. 1) leading to the following soft mode expansion for the lattice matter Majoranas on A sites ($\equiv c_{iA}$) (see SM [34])

$$c_{iA} = \sum_m [\Phi_0(m, r_i)e^{iK_{\alpha}r_i}f_m + \Phi^*_0(m, r_i)e^{-iK_{\alpha}r_i}f^*_m]$$

where $\Phi_0(m, r_i) \propto z^m e^{-\frac{x}{2}} (z = x - iy)$ is the cLL form factor in the symmetric gauge and $m$ the angular momentum [35]. $r$ is measured in units of $l_B$ here and in the rest of the paper.

The canonical $f$-fermions, $\{f^*_m, f_m\} = \delta_{m, -m}$, are obtained from combining the Majoranas in the two different valleys. Crucial is how they transform under different microscopic symmetries. Recall that in the QSL the Majorana fermions $c_i$ transform under a projective representation of various symmetries [36–38]. Since the flux gap remains intact, the projective symmetry group (PSG) are the same as those of the unstrained system except for spatial symmetries explicitly broken by the application of strain. The transformation of $(f_m, f^*_m)$ under threefold rotations, $C_3$, time-reversal, $T$ (TRS) and rotoreflection, $h_x$ ([34]) is

$$
\begin{array}{|c|c|c|c|}
\hline
f^*_m & f_m & T & h_x \\
\hline
f^*_m & -f^*_m e^{-i\frac{\pi}{3}(m+1)} & f_m & -f_m e^{-i\frac{\pi}{3}(m-2)} \\
\hline
\end{array}
$$

Crucially, these forbid any quadratic term, in the hopping $f^*_m f_m$ or pairing $f_m f^*_m$ channels, as seen from the TRS operation $(a_m f^*_m f_m \rightarrow a_m^* f^*_m f_m)$.

This impossibility of a quadratic term seems to indicate that at the level of free fermions, the flatness of the cLL is symmetry protected. In addition,TRS corresponds to a particle-hole transformation within the cLL, taking the occupation of the $m$-th orbital

$$n_m = f^*_m f_m \rightarrow 1 - n_m.$$  

Thus, as long as TRS is not broken spontaneously, this directly implies a half-filled cLL.

Further, for an appropriate gauge choice for the $Z_2$ gauge field [4], the matter Majoranas are manifestly invariant under honeycomb lattice translations in the zero flux sector. This is enhanced to a continuous translation symmetry for the soft modes where translations by a vector $\mathbf{a}$ changes $f^*_m \rightarrow f^*_m e^{iK_{\alpha}a}$. For the interaction terms this leads to an emergent number conservation for the $f_m$, taking the form of a global $U(1)$ symmetry. Thus, quartic Majorana interactions lead to number-conserving quartic terms for the $f$’s. We neglect higher-order terms such as an eight Majorana term reducing $U(1)$ to $Z_6$ ([34]).

Generic spin-interactions and effective Hamiltonian: The generic symmetry allowed form of the leading order effective Hamiltonian in the cLL thus reads

$$H = \frac{1}{2} \sum_{m_1 \cdots m_4} [J_{m_1 m_2 m_3 m_4} f^*_m f^*_n f_{m_2} f_{m_3} f_{m_4} + h.c.]$$

where $m_1 \cdots 4$ are angular momentum indices. The coupling constants, determined from the non-Kitaev interactions, satisfy $J_{m_1 m_2 m_3 m_4} = -J_{m_2 m_1 m_3 m_4} = -J_{m_3 m_1 m_2 m_4} = J_{m_4 m_3 m_2 m_1}$ from fermion antisymmetry.

Generic spin interactions beyond the soluble Kitaev ones are both symmetry allowed and important for the material candidates. These include short range Heisenberg and pseudo-dipolar spin-spin interactions [39]. Characteristic to degenerate perturbation theory of strongly correlated systems [40], both these interactions have a zero projection in the low energy sector.
but lead to virtual tunneling between the cLL states at higher order—specifically through the generation of six spin terms (34).

Interestingly, the leading six-spin term so generated is a product of two spin-chirality terms \( F_x(I) \) and \( F_x(L) \) of two neighbouring hexagons (labelled \( I \) and \( L \)), centered at positions \( r_i \) and \( r_j \) (see Fig. 1(a) and (b)). After projection, this gives rise to

\[
-\mathcal{V}_f(|r_i - r_j|) \mathcal{g}_{\Gamma + \varepsilon} \mathcal{g}_{\Gamma + \varepsilon} \mathcal{g}_{\Gamma + \varepsilon} \mathcal{g}_{\Gamma + \varepsilon}
\]

where \( \mathcal{V}_f(|r|) \) is the strength of the interaction, with \( r \) measured from the centre of the flake. In particular, for a flake under tri-axial strain, a nearest neighbour Heisenberg spin exchanges with amplitude \( J_3 \), connecting sites of the same sub-lattice, gives \( \mathcal{V}(r) = \mathcal{V}_0(|r|) \) with \( \mathcal{V}_0 \sim J_3^4 \). We find it useful to consider a family generalisation \( \mathcal{V}(r) = \mathcal{V}_0(|r|)^\beta \) where \( \beta \geq 0 \) is an integer. The low energy couplings in eqn. (4) then reads

\[
J_{m_1,m_2,m_3,m_4} = \frac{i\mathcal{V}(m_1 + m_2 - m_3 - m_4)}{(2\pi)^2 \sqrt{2m_1 + m_2 + m_3 + m_4}} \frac{m_1m_2m_3m_4}{m_1m_2m_3m_4} \times (m_1 - m_2)(m_3 - m_4)(-m_1m_2 + m_3m_4) \times \Gamma \left( m_1 + m_2 + m_3 + m_4 - 2 + \beta \right)
\]

where \( \mathcal{V} = \mathcal{V}_0a^4 \) (a is lattice constant) is set to unity.

Despite the striking similarity with that of half filled LL in the FQH problem, note that the present one is time reversal invariant. Also, the Hamiltonian given in eqn. (4) corresponds to correlated pair-hopping processes rather than projected density-density interactions for the \( \nu = 1/2 \) lowest Landau level(LLL) case [19, 26–28, 41–43].

**Ground state:** We have performed exact diagonalisation (ED) studies on finite flakes, where restricting the angular momentum indices \( \{m_1,\ldots,m_4\} < m_{\text{max}} \) sets the size of the flake, which can be systematically increased. First, note that the total angular momentum, up to commensurability effects [34], closely follows the time-reversal symmetric value of \( m_o = m_{\text{max}}(m_{\text{max}} - 1)/4 \) (see Fig. 2(a)), as in a uniform droplet state in FQH physics, \( \langle f_{m}^\dagger f_{m'} \rangle = \frac{1}{2} \delta_{m,m'} \) [44].

With an interaction potential scaling as \( V_0|m|^\beta \), and the flake radius \( \sqrt{m_{\text{max}}} \), we normalise the ground state energy by \( m_{\text{max}}^{3/2+1} \); this normalized ground-state energy, \( \epsilon_{gs} \), slowly saturates with increasing \( m_{\text{max}} \). The real space density profile \( \rho(r) = \sum_m |\Phi_m(m,r)|^2 \langle n_m \rangle \) is shown in Fig. 2(b). The state appears gapless, as the energy gap, \( \Delta \), to the first two excitations vs. \( 1/m_{\text{max}} \) (see Fig. 2(c)) falls linearly. These excitations in the half-filled sector correspond to density fluctuations over the ground state (see Fig. 2(d)). These ED results taken together show that the system hosts a uniform droplet ground state which is gapless, time-reversal symmetric, and hosts density fluctuations as low energy excitations.

A self-consistent mean field theory for a state \( |\psi_{MF}\rangle \) can be obtained by decoupling the Hamiltonian (eqn. (4)) as

\[
f_{m_1}^\dagger f_{m_2} f_{m_3} f_{m_4} \rightarrow -\chi_{m_1,m_2} f_{m_3}^\dagger f_{m_4} + \cdots \text{ where } \chi_{m_1,m_2} = \langle f_{m_1}^\dagger f_{m_2} \rangle. \]

Such a state breaks time-reversal symmetry since \( \chi_{m_1,m_2} \neq 0 \) for \( m_1 \neq m_2 \) at odds with the ED results and hence fails to capture the essential features of the above gapless state. Also its energy \( \epsilon_{var} = \langle \psi_{MF}|H|\psi_{MF}\rangle/(m_{\text{max}}^{3/2}) \) (\( \beta = 1 \)), Fig. 2(a), is unsurprisingly higher than the ED ground state.

**Simplified models:** Our numerical results are limited by the finite size accessible in the ED calculations. In the following, we construct illustrative limits sharing some essential features of the exact system (eqn. (6)), namely angular momentum conservation and absence of quadratic terms. We show that these also stabilize time-reversal symmetric gapless states. These models we call long- (LR) and short-range (SR), with the exact eqn. (6) reduced to \( J_{m_1,m_2,m_3,m_4}^{LR} = i\text{sign}(m_1 - m_2)\text{sign}(m_3 - m_4)\text{sign}(m_3m_4 - m_1m_2)\delta(m_1 + m_2 + m_3 - m_4) \) and \( J_{m_1,m_2,m_3,m_4}^{SR} = i\text{sign}(m_1 - m_2)\text{sign}(m_3 - m_4)\text{sign}(m_3m_4 - m_1m_2)\delta(|m_2 - m_1| - 3)|\delta(|m_3 - m_4| - 1). \)

Both capture the fundamental microscopic process of pair hopping of fermions which lies at the heart of eqn. (4). The LR model also is reminiscent of SYK [45, 46] physics but with angular-momentum conservation and non-random couplings.

Our analysis of LR is still restricted to the small sys-
FIG. 3. **SR and LR model**: (a) Gap to first excited state for both short-ranged (SR) and long-ranged (LR) goes to zero with increasing $m_{\text{max}}$. (For LR, the value in y axis should be multiplied by a factor 5) (b) Density oscillations in the first excited state (DMRG for SR, ED for LR).

FIG. 4. **Density correlator**: Behavior of the density-density correlator in $m$ space for the ground state for SR, LR and exact system ($\beta = 1$, average density $n = 1/2$). The behavior in intermediate $m$ ($2 \lesssim m \lesssim 12$) goes as $1/m^2$. The exact and LR saturate ($m_{\text{max}} = 24$, ED results) while the SR system shows a clear $1/m^2$ behavior ($m_{\text{max}} = 10^2$, DMRG results).

The unperturbed Kitaev QSL [47], and is due to the sublattice selectivity of the cLL. The simplest nontrivial correlations are those of chirality operators, such as $\langle F_y(r_i) \rangle = S^z_i \delta^{x}_i S^{x}_i - \delta^y_i S^y_i$. While $\langle F_x \rangle = 0$, as expected for time reversal symmetry, the 2-point correlator is $\langle F_x(0)F_x(r) \rangle = \sum_m \frac{(-1)^m/(2\pi)^22^nnm!}{m^{2}}(\langle n_o n_m \rangle - \langle n_o \rangle \langle n_m \rangle)$. (8)

For LR and the exact system (where ED studies are done) $\langle n_o n_m \rangle - \langle n_o \rangle \langle n_m \rangle$, away from the boundaries ($2 \lesssim m \lesssim 12$), seems to go as $\sim 1/m^2$ (see Fig. 4) and appear to saturate as the system size is approached. The SR model, in DMRG studies on much bigger systems, shows a persistent $1/m^2$ behavior even at large $m$. This translates to a $\langle F_x(0)F_x(r) \rangle \sim 1/r^4$ for the radial direction of the droplet in real space.

**Outlook**: We have engineered and analyzed a system of strongly interacting Majorana fermions. Its genesis in a microscopic spin model has allowed us to derive nature and symmetries of a generic Hamiltonian in the central LL. It differs from the conventional half filled LL [19, 26–28] in the presence of time reversal symmetry and consequently an exact particle-hole symmetry. This provides an entirely novel and concrete setting to explore the interplay of symmetries and interactions in a flat band.

From the QSL perspective, our results imply strain can singularly enhance residual interactions in a Kitaev magnet, generating qualitatively new interacting gapless QSLs whose properties presently seem to defy a free particle-type understanding. This is a QSL analog of a strongly correlated gapless phases– commonly dubbed as non-Fermi liquids– beyond the enigmatic spinon-Fermi surface [48–50]. The generality of our considerations means that studying strain engineering among the slew of candidate Kitaev QSL materials [51] may be an auspicious experimental proposition.

The low energy effective field theoretic understanding of the present phase and its robustness to disorder – somewhat natural for Kitaev candidate materials [52] remain natural questions for future work. More generally, our work at the crossroads of flat-band systems and symmetry protected phases provides a microscopic route to non-Fermi liquid physics [45, 46] traditionally studied in the context of the quantum Hall effect and more recently in twisted bi-layer graphene [29, 53, 54].

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**LOW ENERGY EFFECTIVE HAMILTONIAN**

**Low energy non-interacting problem:** To derive the low energy theory we use notation detailed in Fig. S1 [37]. The itinerant Majorana modes can be soft mode decomposed as

\[ c_n(i) = c_{1A}(i)e^{i\mathbf{K}\cdot \mathbf{r}_i} + c_{2A}(i)e^{i\mathbf{K}'\cdot \mathbf{r}_i} \]  

(S.9)

where \( \alpha = A, B \) denotes the sublattice index and \( \mathbf{K} \) and \( \mathbf{K}' = -\mathbf{K} \) are Dirac cones.

The continuum description of matter Majoranas under triaxial strain in the zero flux sector is given by [15, 25] \( H = \frac{3\hbar}{4} \int dr C(r) \Gamma H C(r) \) where

\[
H = \begin{pmatrix}
0 & \Pi_x + i\Pi_y & 0 & 0 \\
-\left(\Pi_x - i\Pi_y\right) & 0 & 0 & 0 \\
0 & 0 & \left(\Pi_x' + i\Pi_y'\right) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(S.10)

and \( C(r) \) = Transpose\{\( c_{1A}, c_{1B}, c_{2A}, c_{2B} \)\} and

\[
\Pi_\alpha = p_\alpha + A_\alpha, \quad \text{and} \quad \Pi'_x = p_\alpha - A_\alpha \]

(S.11)

where \( p_\alpha = -i\partial_\alpha \). Similar to the treatment of quantum Hall, one can diagonalize this in the symmetric gauge-as, near Dirac cone \( \mathbf{K} \) and \( \mathbf{K}' \) with eigenvalues \( E_n^{(1)} = \)

\[
\begin{pmatrix}
0 & \Psi_{n=0} = (|0\rangle_{\mathbf{K} A}, 0)^T \\
\frac{3\sqrt{2}\hbar}{4\ell_B} & \Psi_{n>0} = (|n\rangle_{\mathbf{K} A}, -i|n-1\rangle_{\mathbf{K} B})^T
\end{pmatrix}
\]  

(S.12)

and \( E_n^{(2)} = \)

\[
\begin{pmatrix}
0 & \Psi_{n=0} = (|0\rangle_{\mathbf{K} A}, 0)^T \\
\frac{3\sqrt{2}\hbar}{4\ell_B} & \Psi_{n>0} = (|n\rangle_{\mathbf{K} A}, i|n-1\rangle_{\mathbf{K} B})^T
\end{pmatrix}
\]  

for \( n = 0 \)

\[
\forall n > 0
\]

(S.13)

where \(|n\rangle\) label the single particle LL wavefunctions in symmetric gauge [35]. Note that the zero energy states on both the cones have weights only on the \( A \) (same) sublattice. Moreover the states near \(-\mathbf{K}\) are time-reversal partners of those at \( \mathbf{K} \). Defining the cLL projected Majorana operators,

\[
f_m^\dagger \equiv c_{n=0,m,\mathbf{K}} \approx \sum_i e^{-i\mathbf{K}\cdot \Phi_0(m, \mathbf{r}_i)} \hat{P} \hat{c}_A \hat{P} \]  

(S.14)

\[
f_m \equiv c_{n=0,m,\mathbf{K}} \approx \sum_i e^{i\mathbf{K}\cdot \Phi_0(m, \mathbf{r}_i)} \hat{P} \hat{c}_A \hat{P} \]  

(S.15)

where \( \hat{P} \) is the projector, we find that they satisfy

\[
\{f_n, f_m^\dagger\} = \delta_{nm}, \quad \{f_n, f_m\} = 0, \quad \{f_n^\dagger, f_m^\dagger\} = (S.16)
\]

reflecting the canonical fermionic algebra of these operators. Here \( \Phi_0(m, \mathbf{r}_i) = \langle n = 0, m \rangle \mathbf{r}_i \rangle \). The zeroth Landau level projection therefore implies

\[
c_{iA} = \sum_m \Phi_0(m, \mathbf{r}_i) e^{i\mathbf{K}\cdot \mathbf{r}_i} f_m^\dagger + \Phi_0^*(m, \mathbf{r}_i) e^{i\mathbf{K}'\cdot \mathbf{r}_i} f_m \]  

(S.17)

which is the eqn. (1) in the main text.

**Symmetry analysis:** The Kitaev spin model has the following underlying microscopic symmetries [36–38] (i) Two lattice translations corresponding to the triangular Bravais lattice, \( T_1 \) and \( T_2 \) (ii) A six fold \( C_6 \) spin rotation about \([111]\) (this is combined with a reflection over the plane). (iii) Reflection about the \( z \) bonds: \( \sigma \) and (iv) Time reversal, \( T \). [37] defines \( h_x = \sigma \lambda_6 \) to discuss it as a useful symmetry operator. Lattice matter Majorana fermions transform according to the following PSG [36].

\[
\begin{array}{cccc}
T_{1,2} & C_6 & \sigma & T \\
c_A & c_A & c_B & C_A \\
c_B & -c_A & -c_B & -c_B \\
\end{array}
\]

(S.18)

Under tri-axial strain the two lattice are no longer equivalent and hence the surviving symmetries are given by (i) Translational symmetry (the continuum state) (ii) \( C_3 \) symmetry and (iii) time reversal symmetry \( T \). Given the flux gap remains intact [15], it is justified to assume that the PSG of the Majoranas for the surviving symmetries of the strained system does not change. Note that while \( \sigma \) and \( C_6 \) are separately not the symmetries of the system under distortion \( h_x \) is. The PSG transformation of Majorana fermions under these residual symmetries is

\[
\begin{array}{cccc}
T_{1,2} & C_3 & \sigma & T \\
c_A & c_A & -c_A & -c_A \\
\end{array}
\]

(S.19)
Starting with the PSG on the lattice matter Majoranas [37], it is straightforward to work out the symmetries of the soft modes, \(a_{1\alpha}, a_{2\alpha}\) (\(\alpha = A, B\)) and hence the cLL modes \(f_m f_m^\dagger\). This is then given in eqn. (2) in the main text.

Given these symmetries, given time-reversal and hermiticity, no quadratic terms are allowed (either number conserving or number conservation breaking term) \((a_{mm} f_m f_m^\dagger \to \mathcal{T} a_{mm} f_m^\dagger f_m)\).

*Umklapp like terms:* Up to conditions of hermiticity and time-reversal we now check about when Umklapp like (slow varying) terms could be important. This corresponds to the case when the momentum factors are not fast oscillating. For \(m f\)'s, positioned at \(r + \delta_i; i = 1, \ldots, m\) and \(n f\)s with lattice labels \(r + \delta_j; j = 1, \ldots, n\) provides a term of the kind \(\sim e^{i\mathbf{a}_i \cdot \mathbf{r}} f_1 f_2 \ldots f_m f_n f_2 \ldots f_n\) has a phase \(a = \mathbf{K} \cdot \left((m - n)\mathbf{r}\right) + \mathbf{K} \cdot \left(\sum_{i=1}^m \delta_i - \sum_{j=1}^n \delta_j\right)\).

Given \(r = p\delta_1 + q\delta_2\) we have \(a = \left((m - n)(p - q) \frac{2\pi r}{3}\right) + \mathbf{K} \cdot \left(\sum_{i=1}^m \delta_i - \sum_{j=1}^n \delta_j\right)\). For this to not oscillate we have \(m - n = 3s\) where \(s\) is an integer. The minimal term which can be non-number conserving and even, corresponds to a spin term \((s = 2, n = 1, m = 7)\). This breaks the fermionic \(U(1)\) to \(Z_6\).

**Angular momentum conservation:** A term of the kind \(f_m f_n f_m f_n\) puts an constraint \((m_1 + m_2 - m_3 - m_4 = 3n)\) under \(C_3\) and \((m_1 + m_2 - m_3 - m_4 = 6n)\) under \(h_x\). For \(n = 0\) we have angular momentum conservation, which is the microscopic term we have focused on as the leading contribution motivated from the microscopics. Other microscopic terms, in particular warping effects, can lead to breaking of these angular momentum conservation where \((m_1 + m_2 - m_3 - m_4 = 6)\).

**Derivation of interaction vertex:**

**General projection:** The spin-spin terms are projected to the zero flux sector, and then to the central Landau level. Given cLL has weight on only one of the sublattice – microscopics \(\mathcal{T}\) allows for couplings only between \(4n\) number of Majoranas operators on the same sublattice, say at positions \(i, j, k, l \sim gc_{iA}c_jc_kc_l\). Projecting this to the cLL (using eqn. (S.17)), keeping slowly varying terms and ignoring Umklapp processes provides an emergent number conservation \(U(1)\) symmetry for the \(f\) operators leading to a form of Hamiltonian given by

\[
\begin{align*}
H &= \sum_{m_1, m_2, m_3, m_4} \left(g_{1,2,3,4} f_{m_1}^\dagger f_{m_2}^\dagger f_{m_3} f_{m_4} + g_{1,2,3,4} f_{m_1}^\dagger f_{m_2} f_{m_3}^\dagger f_{m_4} + g_{1,2,3,4} f_{m_1}^\dagger f_{m_2} f_{m_3} f_{m_4}^\dagger + g_{1,2,3,4} f_{m_1} f_{m_2}^\dagger f_{m_3}^\dagger f_{m_4} + g_{1,2,3,4} f_{m_1} f_{m_2} f_{m_3}^\dagger f_{m_4}^\dagger + g_{1,2,3,4} f_{m_1} f_{m_2} f_{m_3} f_{m_4}^\dagger + g_{1,2,3,4} f_{m_1} f_{m_2}^\dagger f_{m_3}^\dagger f_{m_4}^\dagger \right) \\
&\quad + \left(g_{1,2,3,4} f_{m_1}^\dagger f_{m_2} f_{m_3} f_{m_4} + g_{1,2,3,4} f_{m_1} f_{m_2}^\dagger f_{m_3}^\dagger f_{m_4} + g_{1,2,3,4} f_{m_1} f_{m_2} f_{m_3}^\dagger f_{m_4}^\dagger + g_{1,2,3,4} f_{m_1} f_{m_2}^\dagger f_{m_3} f_{m_4}^\dagger \right)
\end{align*}
\]

(S.20)

Note that the sum is unrestricted over all \(m\). This can be reorganized where the first two and last two indices can be anti-symmetrized such that pair of indices \((m_1, m_2)\) can be restricted to \(m_2 > m_1\) and the antisymmetrized \(g_{1,2,3,4} = g_{1,2,3,4} - g_{2,1,3,4} - g_{1,2,4,3} + g_{2,1,4,3}\) can be used to define \(J_{m_1, m_2, m_3, m_4} = \frac{1}{2} \left(g_{1,2,3,4} - g_{1,3,2,4} + g_{1,3,4,2} + g_{1,4,2,3} + g_{4,1,2,3} + g_{4,1,3,2}\right)\). The effective Hamiltonian is

\[
H = \sum_{(m_1 < m_2), (m_3 < m_4)} J_{m_1, m_2, m_3, m_4} f_{m_1}^\dagger f_{m_2}^\dagger f_{m_3} f_{m_4} + h.c. \\
+ J_{m_1, m_2, m_3, m_4} f_{m_1} f_{m_2}^\dagger f_{m_3}^\dagger f_{m_4} + h.c.
\]

(S.21)

One can rewrite this into an unrestricted sum with

\[
H = \frac{1}{2} \left(\sum_{m_1, m_2, m_3, m_4} J_{m_1, m_2, m_3, m_4} f_{m_1}^\dagger f_{m_2} f_{m_3}^\dagger f_{m_4} + h.c.\right)
\]

(S.22)

It is useful to project the hermitian partner of the microscopic terms together to keep track of the quadratic terms which eventually cancel.

**Microscopic spin terms:** We motivate the nature of the interaction vertex we choose below from the microscopic spin-spin interactions. Consider a hexagon labelled \(I\) centered at position \(r_1 = i\). Three \(A\) type Majoranas are located at three vectors \(\vec{x}, \vec{y}\) and \(\vec{z}\) surrounding the centre (see Fig. 1 in main text). Three kinds of chiral three spin terms can exist which, under projection, can couple two \(A\) site Majoranas.

\[
F_x(I) = S_{i+\hat{x}}^x S_{i+\hat{y}}^y S_{i+\hat{z}}^z \rightarrow -ic_{i+\hat{x}}c_{i+\hat{y}}
\]

(S.23)

\[
F_y(I) = S_{i+\hat{x}}^y S_{i+\hat{y}}^x S_{i+\hat{z}}^z \rightarrow -ic_{i+\hat{x}}c_{i+\hat{z}}
\]

(S.24)

\[
F_z(I) = S_{i+\hat{x}}^z S_{i+\hat{y}}^x S_{i+\hat{z}}^y \rightarrow -ic_{i+\hat{x}}c_{i+\hat{z}}
\]

(S.25)

Although every \(F\) operator couples two \(A\) sites, they are odd under time-reversal symmetry and are therefore not individually allowed. However pair of such terms can engineer an interaction term between four \(A\) sites which forms a rhombic plaquette. Focusing on a hexagon there are three kinds of rhombic plaquettes which can be engineered; these are \(C_3\) related to each other (see Fig. 1 in main text).

For instance a six spin term projects to the following quartic Majorana term

\[
V_f(|d_{I,L}|) F_x(I) F_x(L) \rightarrow -V_f(|d_{I,L}|) c_{i+\hat{x}}c_{i+\hat{y}}c_{i+\hat{z}}c_{i+\hat{y}}
\]

(S.26)
For each such rhombic plaquette, however, there are two kinds of 6 spin terms which can couple the same four Majorana operators on the A sites (see Fig. 1 and Fig. S2) for e.g.

\[
V_f(|d_{1,L}|)F_x(I)F_z(L) + V_f(|d_{1,J}|)F_x(I)F_z(J) \\
= - \left( V_f(|d_{1,L}|) - V_f(|d_{1,J}|) \right) c_{i+\hat{z}}c_{i+\hat{y}}c_{i+\hat{z}}c_{i+\hat{y}} \tag{S.27}
\]

These two terms cancel each other when \(|d_{1,J}| = |d_{1,L}|\) i.e., in absence of strain. However in presence of strain these two distances are not the same and for a short ranged interaction of the functional form, \(V_f(d) \sim e^{-d}\) this, therefore generates a term of the kind \(v_1(\hat{t})c_{i+\hat{z}}c_{i+\hat{y}}c_{i+\hat{z}}c_{i+\hat{y}}\) where \(v_1\) is dependent on the value of strain \(C\) and on the position of the hexagon \(i, v_1\) therefore has a varying strength over all in the flake. Including the other two interactions \((v_2\) and \(v_3)\), these interactions centered at hexagon \(I\) couple the following Majorana terms:

\[
\begin{align*}
|v_1(\hat{t})| & = c_{i+\hat{z}}c_{i+\hat{y}}c_{i+\hat{z}}c_{i+\hat{y}} \\
|v_2(\hat{t})| & = c_{i+\hat{z}}c_{i+\hat{y}}c_{i+\hat{z}}c_{i+\hat{y}} \\
|v_3(\hat{t})| & = c_{i+\hat{z}}c_{i+\hat{y}}c_{i+\hat{z}}c_{i+\hat{y}} \tag{S.28}
\end{align*}
\]

To track their behavior we construct a vector \(\hat{V}\) using \((v_1, v_2, v_3)\) as \(\hat{V} = v_1\hat{x} + v_2\hat{y} + v_3\hat{z}\) which reflects a plaquette directed in the direction of \(\hat{V}\), with the strength \(|\hat{V}|\). One finds that \(|\hat{V}|\) has a rotational symmetry around the flake with a strength which linearly increases as one goes away from the center. The value of \(|\hat{V}|\) increases linearly with \(C\) as shown in Fig. S2.

**Form factor:** To capture the essential microscopic phenomenology as discussed above we consider a hexagon centered at \(r = re^{i\theta}\). The Majorana operators which couple at position \(r\) are, in the coarse grained picture, centered at \((r-a)e^{i\theta+\alpha}r e^{i\theta-\alpha}\) and \((r+a)e^{i\theta}\) where \(\alpha = \arctan(a/r)\) and \(a\) is order lattice constant. Under cLL projection (using \(\Phi_{\alpha}(m, r) = \frac{1}{\sqrt{2\pi^2 m!}}e^{-r^2/4} = \frac{1}{\sqrt{2\pi^2 m!}}e^{-r^2/4} = B_4 r^m e^{-r^2/4} = B_4 r^m e^{-r^2/4} = \frac{1}{\sqrt{2\pi^2 m!}}e^{-r^2/4}\)

\[
g_{1,2,3,4} = \int dr dr \Phi_0(m_1, r-a)e^{i\theta}\Phi_0(m_2, r+a)e^{i\theta}\Phi_0(m_3, r+a)e^{i\theta}\Phi_0(m_4, r-a)e^{i\theta} \tag{S.29}
\]

\[
= B_{m_1} B_{m_2} B_{m_3} B_{m_4} \delta(m_1 + m_2 - m_3 - m_4) \int dr V(r)(r-a)^{m_1}r^{m_2+a}r^{m_3+a}r^{m_4}e^{i(m_2-m_3)\alpha(r)}e^{-\frac{2r^2(r-a)^2+(r-a)^2}{4}} \tag{S.30}
\]

\[
\tilde{g}_{1,2,3,4} = B_{m_1} B_{m_2} B_{m_3} B_{m_4} \delta(m_1 + m_2 - m_3 - m_4) \times \int dr V(r) \times \left( \frac{r-a}{r} \right)^{m_1}r^{m_2+a}r^{m_3+a} \tag{S.31}
\]

and using the discussion near eqn. (S.22)

\[
J_{m_1, m_2, m_3, m_4} = \frac{\Pi_{i=1,4} B_{m_i}}{2} \int dr V(r) \left( 4i r^{m_1}r^{m_2+a}r^{m_3+a}r^{m_4}g_{i, m_1, m_2, m_3, m_4}a^3 \right) e^{-r^2} \tag{S.32}
\]

**ADDITIONAL NUMERICAL RESULTS**

Fig. S3 shows the energy eigenvalues (displaced from the ground state energy) with respect to the expectation of \(M\) operator (displaced with the TR symmetric value \(M_0\)). Since TR symmetric value has \((n_m) = 1/2\), the expectation of \(M\) operator is \(M_0 = \frac{m_{max}(m_{max}-1)}{4}\). For finite sized systems when \(M_0\) is not an integer, it can lead to degenerate pair of TR partners ground states leading
to commensuration effects in the gap scaling. For $\beta = 1$ the ground state density and its excitations is shown in Fig. S4. The angular momentum and variation of gap for $\beta = 0$ is shown in Fig. S5. Even for $\beta = 0$ the system is remains in time-reversal symmetric state and the gap seems to fall linearly with $1/m_{\max}$. For SR system, the entanglement of a sub-region as a function of $m/m_{\max}$ and central charge behavior showing $c = 1$ (see Fig. S6).