STOCHASTIC PDES: DOMAIN FORMATION IN DYNAMIC TRANSITIONS

GRANT LYTHE∗

Abstract. Spatiotemporal evolution in the real Ginzburg-Landau equation is studied with space-time noise and a slowly increasing critical parameter. Analytical estimates for the characteristic size of the domains formed in a slow sweep through the critical point agree with the results of finite difference solution of the stochastic PDEs.

Noise is an indispensable part of the description of symmetry-breaking transitions, although it is normally included only implicitly in mathematical models. This work examines a dynamic transition ("quench") in a spatially extended system. The model contains a bifurcation parameter having a critical value at which the state with zero order parameter becomes unstable to two equivalent states with non-zero order parameter. Above this critical value, the whole system does not choose the same state. Rather, a pattern of domains and defects ("kinks") is formed [19, 3, 16]. Here it is shown that the characteristic size of the domains formed when the bifurcation parameter is made explicitly time-dependent depends logarithmically on the magnitude of additive noise: additive noise is thus a determining influence even when extremely small. The effect of low-intensity multiplicative noise is much less dramatic. Noise is added in such a way that it has no correlation length of its own (white in space and time). Numerical results are reported, obtained using a finite difference algorithm whose continuum limit is an SPDE. The simplest model with the essential features, the real Ginzburg-Landau equation, is considered here. The dynamics take the system from a potential with a single minimum to one with two minima (Figure 1). Similar results have been found for the Swift-Hohenberg equation, which is more explicitly designed to model Rayleigh-Benard convection [19].

The Ginzburg-Landau SPDE. The order parameter $Y_t(x)$ is a function of space and time and the bifurcation parameter $g$. For $g < 0$ fixed, the state $\{Y_t(x) = 0 \forall x\}$ is stable, and only noise prevents the system from approaching it arbitrarily closely. For $g > 0$ fixed, one sees a pattern of regions in which $Y_t(x)$ is positive and regions in which $Y_t(x)$ is negative (domains) separated by narrow transition layers (kinks). The subject of this paper is not the slow merging of domains or the nucleation of new kinks - these happen on timescales much longer than those considered here [11, 3]. This paper derives the number of domains formed when the parameter $g$ is explicitly time-dependent, starting from $g < 0$ and ending with $g > 0$.

The SPDE describing the dynamics is written in the following dimensionless form:

\[
\text{(0.1)} \quad dY_t(x) = (g(t)Y_t(x) - Y_t(x)^3 + \mathcal{L}Y_t(x))dt + \epsilon dW_t(x).
\]

Here $Y_t(x) : [0, L]^m \times [-\frac{1}{\mu}, \frac{1}{\mu}] \times \Omega \to \mathbb{R}$, $\Omega$ is a probability space and $W_t(x)$ is the Brownian sheet. The equations were solved as initial value problems, with

\[
\text{(0.2)} \quad g(t) = \mu t
\]

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∗Centre for Nonlinear Studies, Los Alamos National Laboratory, New Mexico NM87545, USA
slowly increased from −1 to 1. Periodic boundaries in $x$ are used so that any spatial structure is not a boundary effect. The constants $\mu$, $\epsilon$ and $\frac{1}{L}$ are all $\ll 1$. The spatial operator $L = \Delta$ where $\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}$, the Laplacian in $\mathbb{R}^m$.

An alternative scaling of (0.1) is sometimes illuminating: if $x$ is rescaled so that $[0, L] \to [0, 1]$ then (0.1) becomes

\[
(0.3) \quad dY_t(x) = (g(t)Y_t(x) - Y_t(x)^3 + D\Delta Y_t(x)) dt + D^2 \epsilon dW_t(x),
\]

where $D = L^2$.

The Brownian Sheet. To add noise to an ordinary differential equation, one adds increments of the Wiener process at each time step; the idea is of a particle continuously subject to small impulses \[3\]. To add noise to a partial differential equation (PDE) one adds increments of the Brownian sheet, that are random in space and time \[24\]. One can imagine, for example, the motion of a flexible sheet in a sand-storm. Typically the motivation for adding noise to PDEs is to make allowances for small-scale, rapidly-varying effects with mean zero that have been neglected in the derivation of an equation.

The Wiener process can be thought of as assigning to each successive interval of the time axis a Gaussian random variable with variance proportional to the length of the interval. The Brownian sheet assigns to each volume element in $\mathbb{R}_+ \times \mathbb{R}^m$ a Gaussian random variable whose variance is proportional to the volume of the element \[24\]. More precisely, it is possible to define a map $A$ from $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^m)$ to a probability space such that for each $h \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^m)$, $A(h)$ is a Gaussian random variable with mean zero and $\langle A(h_1) A(h_2) \rangle = l(h_1 \cap h_2)$ where $l$ is the Lebesgue measure \[24\]. The Brownian sheet, so called because its realisations in $m = 1$ look like a ruffled bed-sheet tucked in on two adjacent sides, is defined as $W_t(x) = A([0, t] \times [0, x])$, where $[0, x]$ is the element (interval, square, cube, . . .) with opposite corners at the origin and at $x \in \mathbb{R}^m$. Thus

\[
(0.4) \quad W_t(x) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}.
\]

The set $\Omega$ is the set of labels for realisations; averages over realisations are denoted by angled brackets. Each $W_t(x)$ is a real-valued Gaussian random variable with mean zero and variance $\langle W_t^2(x) \rangle = tx^m$. Stochastic processes will be denoted in bold with the subscript $t$. For example, the Wiener process is denoted $W_t$ and is constructed by defining $W_t = A([0, t])$. We denote processes satisfying SPDEs with upper case bold Roman letters with subscript $t$ and $x$ in brackets.

The action of $W_t(x)$ as an integrator is easily described in the case where the integrand is a deterministic function. If $f_1(x, t)$ and $f_2(x, t)$ are continuous functions on $D \times [0, t]$, where $D \in \mathbb{R}^m$, then

\[
I_t = \int_0^t \int_D f_1(x, s) \, dx \, dW_s(x) \quad \text{and} \quad J_t = \int_0^t \int_D f_2(x, s) \, dx \, dW_s(x)
\]

are Gaussian random variables with $< I_t > = 0$, $< J_t > = 0$ and

\[
< I_t J_t > = \int_0^t \int_D f_1(x, s) f_2(x, s) \, dx \, ds.
\]
**Numerical Solution.** An approximate numerical solution of an SPDE is produced by solving a large set of SDEs. The finite difference method for a parabolic SPDE consists of replacing the infinite dimensional system (0.1) by \(N^m\) ordinary SDEs on a grid of equally-spaced points in \([0,L]^m\) separated by \(\Delta x\). The SDE at site \(i\) is
\[
(0.5) \ dY_i(t) = (g(t)Y_i(t) + \gamma^2 Y_i(t)) \ dt + (\Delta x)^{-2} \delta Y_i(t) \ dt + (\Delta x)^{-m/2} \epsilon dW_i(t),
\]
where the discrete Laplacian \(\delta\) is defined by
\[
(0.6) \ \delta Y(i) = \sum_{i'} Y(i') - 2m Y(i)
\]
and the sum is over the \(2m\) nearest neighbours of \(x\). The random variables added at neighbouring grid points are independent no matter how small \(\Delta x\) is. Stochastic PDEs like (0.1) use space-time white noise along with a deterministic operator that includes some spatial coupling (here the Laplacian) to generate fields with non-delta-function correlations in space and time. Linear SPDEs can be very efficiently solved in Fourier space \([18, 21, 22, 23]\). However, the finite difference method is more easily adapted to nonlinear SPDEs. One finite-difference numerical realisation of an SPDE generates an approximate solution for one \(\omega \in \Omega\), at a discrete set of times \(\{t_i\}\) and positions \(\{x_i\}\). The time-stepping used to generate the figures in this work is the stochastic analogue of the second-order Runge-Kutta method \([10, 14, 13]\).

**Fourier decomposition.** For \(k \in Z^m\), let the Fourier coefficient \(y_t(k)\) be defined by
\[
(0.7) \ y_t(k) = L^{-\frac{m}{2}} \int_{[0,L]^m} e^{ixk} Y_t(x) \ d^m x \quad \text{where} \quad \kappa = \frac{2\pi}{L} k.
\]
Each \(y_t(k)\) is a complex-valued stochastic process satisfying the following SDE when \(k \geq 1\):
\[
(0.8) \ \ dy_t(k) = (g(t) - \kappa^2) y_t(k) \ dt + \frac{\epsilon}{\sqrt{2}} dW_t(k) + \text{ nonlinear terms}.
\]
Since \(Y_t(x)\) is real-valued, for \(k \leq -1\) we obtain \(y_t(k)\) from the relation \(y_t(-k) = (y_t(k))^*\). The imaginary part of \(y(0)\) is always 0; the real part satisfies
\[
(0.9) \ \ dY_t(0,r) = g(t)Y_t(0,r) \ dt + \epsilon dW_t(0,r).
\]
Each \(W_t(k)\) is an independent complex-valued Wiener process. When \(|y_t(k)| \ll 1\) for all \(k\), the nonlinear terms that involve products of \(y_t(k)\) are unimportant and the SDEs (0.8) can be solved separately. It then becomes necessary to evaluate the value of \(g\) at which these nonlinear terms become important. This is called an exit value problem in a dynamic bifurcation \([19, 21, 22, 23]\).

**Dynamic bifurcation.** We consider a stochastic ordinary differential equation slightly more general than (0.3), including also multiplicative noise:
\[
(0.10) \ \ dy_t = g(t)y_t \ dt + \epsilon dW_t^{(1)} + \gamma y_t dW_t^{(2)},
\]
where \(W_t^{(1)}\) and \(W_t^{(2)}\) are independent real-valued Wiener processes, \(g = \mu t\) and the initial condition is \(y_t = y_0\) at \(g = \mu t_0 = -1\). \(0 < \epsilon, \gamma < \sqrt{\mu} < 1\). The solution of (0.10) is
\[
(0.11) \ \ y_t = y_0 \exp \left( \frac{1}{2} \mu (t^2 - t_0^2) - \gamma^2 t + \gamma w_t^{(2)} \right)
\]
\[
(0.11') \ \ \exp \left( \frac{1}{2} \mu t^2 - \frac{1}{2} \gamma^2 t + \gamma w_t^{(2)} \right) \int_{t_0}^t \exp \left( -\frac{1}{2} \mu s^2 + \frac{1}{2} \gamma^2 s - \gamma w_s^{(2)} \right) \ dw_s^{(1)}.
\]
The second term in (0.11) has zero mean and variance given by

\[
\langle y_t^2 \rangle - \langle y_t \rangle^2 = \epsilon^2 \exp(\mu t^2 + \gamma^2 t) \int_0^t \exp(-\mu s^2 - \gamma^2 t) \, ds. \tag{0.12}
\]

For \( g > \sqrt{\mu} \), this variance is well approximated by

\[
\langle y_t^2 \rangle - \langle y_t \rangle^2 \approx \epsilon^2 \sqrt{\frac{\pi}{\mu}} e^{(g + \frac{1}{2} \gamma^2)^2}. \tag{0.13}
\]

Comparing the mean and standard deviation of \( y_t \) for \( g > \sqrt{\mu} \), we find that, for \( \mu, \gamma \ll 1 \) and \( |y_0| \leq \mathcal{O}(1) \), the standard deviation is much larger than the mean if \( \mu |\log \epsilon| < \frac{1}{2} \). Then the dynamic bifurcation is noise-dominated. If \( y_t \) remains small until well after \( g \) passes through 0, and finally attains an \( \mathcal{O}(1) \) magnitude at a value of \( g \) with mean value \( g = g_c + \mathcal{O}(\mu) \) and standard deviation proportional to \( \mu \) where

\[
g_c = \sqrt{2\mu |\log \epsilon|}. \tag{0.14}
\]

Stochastic equations like (0.1) and (0.10) with \( g \) constant and \( \gamma \neq 0 \) have attracted much attention because multiplicative noise acts in a way that can be interpreted as a shift of the bifurcation point from \( g = 0 \). However, in all cases the shift is proportional to \( \gamma^2 \) compared and is dwarfed by the large dynamic delay of the bifurcation of interest here. In (0.13) we see that, for the variance of \( y_t \), \( g \) is replaced by \( g + \frac{1}{2} \gamma^2 \). Note, however, that for \( \gamma \neq 0 \) the distribution of \( y_t(x) \) is not Gaussian. A slight non-Gaussianity is the main effect of multiplicative noise in the SDE and SPDE systems of interest here and will not be further discussed.

Space-time evolution. Now return to the spatially extended system. We first make the hypothesis that, as in the dynamic bifurcation, \( Y_t(x) \) remains everywhere small for \( g < g_c = \sqrt{2\mu |\log \epsilon|} \). Then the emerging pattern of domains can be studied.
from the linearised version of (0.1) (i.e. without the cubic term). The solution of the linearised version of (0.1) is:

\[
(0.15) Y_t(x) = \int_{[0,L]^m} G(t, t_0, x, v) f(v) dv + \epsilon \int_{t_0}^t \int_{[0,L]^m} G(t, s, x, v) dv dW_s(v),
\]

where \( G(t, s, x, v) = \frac{2}{\sqrt{\pi}} \frac{e^{-|x-v|^2/2}}{|t-s|^{3/2}} \).

The first term in (0.15), dependent on the initial data \( f(x) \), relaxes quickly to very small values. The correlation function is therefore obtained from the second, stochastic, integral in (0.15). Performing the integration over space, assuming that \( L > \frac{2\sqrt{\pi}}{\mu} \), gives

\[
(0.17) c(x) = \langle Y_t(x') Y_t(x' + x) \rangle = \epsilon^2 \int_{t_0}^t \frac{\epsilon \mu (t^2 - s^2) e^{-|x-v|^2/4(t-s)}}{(8\pi(t-s))^{3/2}} ds.
\]

Since \( Y_t(x) \) satisfies a non-autonomous SPDE, the correlation function is explicitly a function of time. In the early part of the evolution, however, the deviation from that obtained from the corresponding static \((g=\text{constant}) \) equation is small. We consider this quasi-static period by making the change of variables \( u = t - s \) in (0.17). Then

\[
(0.18) \exp(\mu(t^2 - s^2)) = \exp(2\mu tu - \mu u^2) = \exp(\mu tu)(1 - \mu u^2 + \ldots).
\]

and

\[
(0.19) c(x) = \frac{\epsilon^2}{(8\pi)^{3/2}} \left( \int_0^\infty \frac{e^{-2|g| u}}{u^{3/2}} e^{-\frac{|x|^2}{4u}} du - \mu \int_0^\infty \frac{e^{-2|g| u}}{u^{3/2} - 2} e^{-\frac{|x|^2}{4u}} du + \ldots \right).
\]

Thus

\[
(0.20) c(x) = \frac{\epsilon^2 |g|}{2(2\pi)^{3/2}} x^{1-\frac{m}{2}} \left( K_{\frac{m}{2}-1}(x\sqrt{|g|}) + \frac{\mu}{g^2} x^2 16 |g| K_{\frac{m}{2}-3}(x\sqrt{|g|}) + \ldots \right).
\]

where \( K_m \) is the modified Bessel function of order \( m \). For example, when \( m = 1 \),

\[
(0.21) c(x-x') = \frac{\epsilon^2}{4\sqrt{|g|}} e^{-|x-x'|\sqrt{|g|}} \left( 1 - \frac{3}{16} \frac{\mu}{g^2} (1 + x\sqrt{|g|}) + \frac{1}{3} x^2 |g| + \ldots \right).
\]

The first term in (0.20) is the (long-time) correlation function for the SPDE obtained by fixing \( g < 0 \) \[13\]. Clearly the expansion in \( \frac{\mu}{g^2} \) is no longer useful for \( g > -\sqrt{\mu} \). The correlation function itself, however, remains well-behaved as \( g \) passes through 0; the divergences associated with critical slowing down are not present.

In one space dimension, the solution of the SPDE (0.1) is a stochastic process with values in a space of continuous functions \([24, 3]\). That is, for fixed \( \omega \in \Omega \) and \( t \in [-\frac{1}{\mu}, \frac{1}{\mu}] \), one obtains a configuration, \( Y_t(x) \), that is a continuous function of \( x \). This can be pictured as the shape of a string at time \( t \) that is constantly subject to small random impulses all along its length. In more than one space dimension, however, the \( Y_t(x) \) are not continuous functions but only distributions \([24, 3]\) and...
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Fig. 0.2. Mean squared value of $Y_t(x)$. For $g > \sqrt{\mu}$, the variance of $Y_t(x)$ is proportional to $\exp(\mu^2)$. This exponential rise eventually takes $Y_t(x)$ to $O(1)$ values. The solid line is (0.24) and the dots are from a numerical realisation in two space dimensions with $\mu = 0.002$, $\epsilon = 10^{-10}$, and $L = 12\pi$, carried out on a $256 \times 256$ grid. The vertical dotted line is at $g = g_c = \sqrt{2\mu|\log \epsilon|}$.

the correlation function $c(x)$ diverges at $x = 0$. In the non-autonomous equations studied here, however, the divergent part does not grow exponentially for $g > 0$, and by $g = \sqrt{2\mu|\log \epsilon|}$ it is only apparent on extremely small scales, beyond the resolution of any feasible finite difference algorithm.

We now examine the evolution for $g > \sqrt{\mu}$, where we can approximate (0.17) using Laplace’s formula:

$$c(x) \approx \frac{\epsilon^2}{\sqrt{\mu}} \frac{e^{\mu t^2}}{(8t)^{3/2}} e^{-\frac{x^2}{8t}}. \tag{0.22}$$

Thus typical values of $Y_t(x)$ increase exponentially fast and the correlation length at time $t$ is $\sqrt{8t}$. Once $c(0) > O(\epsilon)$, the noise no longer greatly influences the evolution; its effect can be thought of as wiping out the memory of the initial condition at $g < 0$ and replacing it with an effective random initial condition. From (0.22), we see that, at $g = \sqrt{2\mu|\log \epsilon|}$, $c(0) = O(1)$ and the cubic nonlinearity can no longer be ignored. (Figure 2.)

**Density of kinks.** The second hypothesis that enables the characteristic domain size to be estimated is that the effect of the cubic nonlinearity, when it finally makes itself felt, is to freeze in the spatial structure. This is indeed the case in numerical simulations: no perceptible changes occur between $g = g_c$ and $g = 1$ [17, 18]. Thus the correlation length at $g = g_c$,

$$\lambda = \frac{\sqrt{8g_c/\mu}}{\mu} = 2^{7/4} \left(\frac{|\log \epsilon|}{\mu}\right)^{1/4}, \tag{0.23}$$

becomes the characteristic length for spatial structure after $g = g_c$. In one space dimension it is possible to put the scenario just described to quantitative test by
producing numerous realisations of the non-autonomous SPDE \((0.1)\) and recording \(r\), the number of times that \(Y_t(x)\) crosses upwards through 0 in the domain \([0, L]\) at \(g = 1\). The average can then be compared with the mean number of upcrossings for Gaussian random field with correlation function \((0.17)\) at \(g = g_c\).

Consider a homogeneous Gaussian random field \(Y_t(x)\) with correlation function \(c(x)\). Then \(Y_t(x + \Delta x) - Y_t(x)\) is a Gaussian random variable with mean zero and variance \(⟨(Y_t(x + \Delta x) - Y_t(x))^2⟩ = b(\Delta x)\), where

\[
(0.24) \quad b(\Delta x) = 2 (c(0) - c(\Delta x)) = 2c'(0)\Delta x + c''(0)\Delta x^2 + \ldots .
\]

The probability that \(Y_t\) has an upcrossing of 0 between \(x\) and \(x + \Delta x\) is given by

\[
P\{\text{upcrossing} \in (x, x + \Delta x)\} = \int_{-\infty}^{0} P\{Y_t(x) = u\}P\{Y_t(x + \Delta x) - Y_t(x) > u\} du
\]

\[
= (2\pi)^{-1} (b(\Delta x)c(0))^{-\frac{1}{2}} \int_{0}^{\infty} e^{-u^2/2c(0)} \int_{u}^{\infty} e^{-v^2/2b(\Delta x)} dv du
\]

\[
= \frac{1}{2} \left( \frac{b(\Delta x)}{\pi c(0)} \right)^{\frac{1}{2}} \int_{0}^{\infty} \exp\left(-u^2 \frac{b(\Delta x)}{c(0)}\right)(1 - \text{erf}(u)) du
\]

\[
(0.25) \quad = \frac{1}{2\pi} \arctan\left( \left( \frac{b(\Delta x)}{c(0)} \right)^{\frac{1}{2}} \right).
\]

Now consider a grid of total length \(L\) made up of \(N\) sites separated by \(\Delta x\). Let \(\Delta x \to 0\) with \(L\) fixed, i.e. let \(N \to \infty\). When \(c'(0) \neq 0\), the number of zero crossings of \(Y_t\) defined on this grid is proportional to \(\Delta x^{-\frac{1}{2}}\) for \(\Delta x \to 0\). When, as in \((0.17)\), \(c'(0) = 0\), the mean number of zero crossings per unit length approaches a finite number as \(\Delta x \to 0\) given by \([1, 12]\):

\[
r/L = \frac{1}{2\pi} \sqrt{-c''(0)/c(0)}.
\]

(0.26)

In Fig.3 this average is displayed as a function of the sweep rate for the case \((0.22)\), evaluated at \(g = g_c\):

\[
r = \frac{L}{4\pi} \left( \frac{\mu}{2|\log \epsilon|} \right)^{\frac{1}{2}}.
\]

(0.27)

**Summary.** When the critical parameter in the Ginzburg-Landau equation is slowly increased through 0 a characteristic length is produced as follows. The field remains everywhere small until well after the critical value for the loss of stability of the uniform symmetric state. Meanwhile, the correlation length grows proportional to \(\sqrt{t}\). At \(g \approx g_c = \sqrt{2\mu|\log \epsilon|}\), where \(\mu\) is the rate of increase of the parameter and \(\epsilon\) is the amplitude of the noise, the field at last becomes \(O(1)\) and the spatial pattern present is frozen in by the nonlinearity. Thereafter one observes spatial structure with characteristic size proportional to \(\left(\frac{1}{\sqrt{\mu}}\right)^{\frac{1}{2}}\). The increase of \(g\) need not be uniform, but the analytical estimates presented here rely on the rate of increase of \(g\) as it passes through 0 being small.
Fig. 0.3. Mean number of zero crossings at $g = 1$. The solid lines are $0.27$ with $L = 2048$ and $\mu = 0.01$ (upper line) and $\mu = 0.001$ (lower line). The dots are obtained from finite difference simulations of (0.1).

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