Topological matchings and amenability

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Abstract. We introduce a novel quantity for general dynamical systems, which we shall call the mean topological matching number. We prove that a dynamical system is amenable if its mean topological matching number equals one. Furthermore, we show that the converse is true provided that the considered dynamical system does not contain any isolated points. We conclude that a Hausdorff topological group \( G \) is amenable if and only if the mean topological matching number of the associated action of \( G \) on itself equals one.

1. Introduction

Let \( G \) be a group of automorphisms of a uniform space \( X \). We call \((X, G)\) a dynamical system and say that it is amenable if there exists a \( G \)-invariant mean on the algebra of uniformly continuous, bounded, and real-valued functions on \( X \). An important case is the case of a discrete countable group acting on itself by translations. It is classical \([ \text{Føl55} ]\) that in this case amenability can be characterized by the existence of so-called Følner sets – finite subsets of the group that are almost invariant with respect to a finite set of translations. Our motivation to write this note was to provide a Følner-type characterization of amenability of general dynamical systems. We will show that \((X, G)\) is amenable if and only if for every finite uniform covering of \( X \), there exists a finite subset \( F \subseteq X \), such that a finite set of \( G \)-translations of \( F \) can be almost matched with respect to the uniform covering with \( F \). Here, a matching of subsets with respect to a covering is a bijection which respects the covering. Even in the classical case of discrete groups discussed above, this provides a new characterization of amenability in terms of existence of a weak form of Følner sets, see Corollary \( 6.14 \). For alternative weak forms of Følner sets see the work of J. Moore, \([ \text{Moo13} ]\).

In this article we introduce an invariant for general dynamical systems on uniform spaces. This quantity is defined in terms of matching numbers of a family of bipartite graphs associated with a given dynamical system and therefore called the mean topological matching number. It will be shown that a dynamical system is amenable if its mean topological matching number equals one (Theorem \( 6.3 \)). Furthermore, we shall prove that the converse is true provided that the considered dynamical system is Hausdorff and perfect, i.e., it does not contain any isolated points (Theorem \( 6.4 \)). Utilizing these results, we will show that a Hausdorff
topological group $G$ is amenable if and only if the mean topological matching number of the associated action of $G$ on itself equals one (Theorem 6.10).

The paper is organized as follows. In Section 2 we will recollect the very basics concerning means on function spaces. In Section 3 we recall the necessary background knowledge regarding uniform spaces. In Section 4 we briefly discuss the concept of an invariant mean for dynamical systems in general and for topological groups in particular. In Section 5 we recall Hall’s theorem concerning matchings in bipartite graphs. In Section 6 we prove the aforementioned main results. In Section 7 we discuss the connection between our characterization of amenable topological groups (Theorem 6.10) and a characterization of extremely amenable topological groups due to Pestov [Pes05a].

2. Means on function spaces

In this section we want to recall some general basics concerning means on function spaces. For this purpose, we follow the presentation in [BJM89].

Let $X$ be a set. The set of all finite subsets of $X$ shall be denoted by $\mathcal{F}(X)$. Additionally, let $\mathcal{F}_+(X) := \mathcal{F}(X) \setminus \{\emptyset\}$. Furthermore, we denote by $B(X)$ the set of all bounded real-valued functions on $X$. For $f \in B(X)$, we define $\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$. Let $H$ be a linear subspace of $B(X)$. A mean on $H$ is a linear map $m : H \to \mathbb{R}$ such that

$$\inf\{f(x) \mid x \in X\} \leq m(f) \leq \sup\{f(x) \mid x \in X\}$$

for all $f \in H$. The set of all means on $H$ is denoted by $M(H)$. For each $x \in X$, we obtain a mean on $H$ by $\eta(x) : H \to \mathbb{R}$, $f \mapsto f(x)$.

Suppose $X$ to be a topological space. We denote by $C(X)$ the set of all continuous real-valued functions on $X$. Additionally, let $C_b(X) := C(X) \cap B(X)$. Finally, if $f \in C(X)$, then we define $\text{spt}(f) := \{x \in X \mid f(x) \neq 0\}$.

**Theorem 2.1** ([BJM89]). Let $X$ be a set and let $H$ be a linear subspace of $B(X)$ containing the constant functions. Then the following statements hold.

1. $M(H)$ is convex and weak-* compact.
2. The convex hull of $\eta(X)$ is weak-* dense in $M(H)$.
3. If $X$ is a topological space and $H \subseteq C(X)$, then $\eta : X \to M(H)$ is weak-* continuous.

**Lemma 2.2.** Suppose $X$ to be a topological space such that every open non-empty subset of $X$ is infinite. Let $H$ be a linear subspace of $C_b(X)$ containing the constant functions. Then $\{|f|^{-1} \sum_{x \in F} \eta(x) \mid F \in \mathcal{F}_+(X)\}$ is weak-* dense in $M(H)$.

**Proof.** Let $m \in M(H)$, $H_0 \in \mathcal{F}(H)$ and $\varepsilon \in (0, \infty)$. We define $s := \sup_{f \in H_0} \|f\|_\infty + 1$ and $\theta := \frac{\varepsilon}{2s}$. According to Theorem 2.1 (2), there exist $F \in \mathcal{F}_+(X)$ and $\alpha : F \to (0, 1]$ such that $\sum_{x \in F} \alpha(x) = 1$ and $|m(f) - \sum_{x \in F} \alpha(x)f(x)| \leq \frac{\varepsilon}{2}$ for all $f \in H_0$. It is well-known that $\{\beta \in (0, 1] \cap \mathbb{Q}^F \mid \sum_{x \in F} \beta(x) = 1\}$ is dense in $\{\beta \in (0, 1]^F \mid \sum_{x \in F} \beta(x) = 1\}$. Hence, there
exists \( \beta : F \to (0, 1] \cap \mathbb{Q} \) such that \( \sum_{x \in F} \beta(x) = 1 \) and \( \sum_{x \in F} |\alpha(x) - \beta(x)| \leq \theta \). The latter assertion readily implies that

\[
\left| \sum_{x \in F} \alpha(x)f(x) - \sum_{x \in F} \beta(x)f(x) \right| \leq \sum_{x \in F} |\alpha(x) - \beta(x)||f|_\infty \leq \frac{\varepsilon}{3}
\]

for each \( f \in H_0 \). There exist \( n \in \mathbb{N} \setminus \{0\} \) and \( \gamma : F \to \{1, \ldots, n\} \) such that \( \beta(x) = \frac{\gamma(x)}{n} \) for all \( x \in F \). Now, if \( x \in F \), then \( V(x) := \bigcap \{f^{-1}((f(x) - \frac{\varepsilon}{3}, f(x) + \frac{\varepsilon}{3})) | f \in H_0\} \) is an open non-empty subset of \( X \). Since every open non-empty subset of \( X \) is infinite, there is \( \Phi : F \to \mathcal{F}_+(X) \) such that

1. \( \Phi(x) \subseteq V(x) \) for every \( x \in F \),
2. \( |\Phi(x)| = \gamma(x) \) for every \( x \in F \),
3. \( \Phi(x) \cap \Phi(y) = \emptyset \) for any two distinct \( x, y \in F \).

Let \( E := \bigcup \{\Phi(x) \mid x \in F\} \). We observe that \( |E| = n \). For every \( f \in H_0 \), it follows that

\[
\left| \sum_{x \in F} \beta(x)f(x) - \frac{1}{|E|} \sum_{y \in E} f(y) \right| \leq \frac{1}{n} \left| \sum_{x \in F} \gamma(x)f(x) - \sum_{x \in F} \sum_{y \in \Phi(x)} f(y) \right|
\]

\[
\leq \frac{1}{n} \left| \sum_{x \in F} \gamma(x)f(x) - \sum_{y \in \Phi(x)} f(y) \right|
\]

\[
\leq \frac{1}{n} \left| \sum_{x \in F} \sum_{y \in \Phi(x)} |f(x) - f(y)| \right|
\]

\[
\leq \frac{1}{n} \sum_{x \in F} \frac{\varepsilon \gamma(x)}{3} = \frac{\varepsilon}{3}
\]

and therefore

\[
\left| m(f) - \frac{1}{|E|} \sum_{y \in E} f(y) \right| \leq m(f) - \sum_{x \in F} \alpha(x)f(x) + \sum_{x \in F} |\alpha(x)f(x) - \sum_{x \in F} \beta(x)f(x)|
\]

\[
+ \left| \sum_{x \in F} \beta(x)f(x) - \frac{1}{|E|} \sum_{y \in E} f(y) \right|
\]

\[
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

This finishes the proof.

Recall that a topological space is \textit{perfect} if it does not contain any isolated points. Furthermore, a topological space \( X \) is called \textit{homogeneous} if, for any two points \( x, y \in X \), there exists a homeomorphism \( g : X \to X \) such that \( g(x) = y \). It is easy to see the following:

\textbf{Remark 2.3.} Let \( X \) be a topological space. The following statements hold.
(1) If $X$ is homogeneous and not discrete, then $X$ is perfect.
(2) If $X$ is $T_1$ and perfect, then every open non-empty subset of $X$ is infinite.

3. Uniform spaces

In this section we shall recall the very basics concerning uniform spaces. For this purpose, we will follow the approach of [Isb64].

In order to introduce the concept of a uniform space, we shall need some set-theoretic basics. Let $X$ be a set. We denote by $\mathcal{P}(X)$ the set of all subsets of $X$. Let $U, V \subseteq \mathcal{P}(X)$. We say that $V$ refines $U$ and write $U \preceq V$ if

$$\forall V \in V \exists U \in U: V \subseteq U.$$ 

Furthermore, let $U \wedge V := U \cup V$ and $U \vee V := \{ U \cap V \mid U \in U, V \in V \}$. More generally, if $(U_i)_{i \in I}$ is a family of subsets of $\mathcal{P}(X)$, then we define $\bigwedge_{i \in I} U_i := \bigcup_{i \in I} U_i$ and

$$\bigvee_{i \in I} U_i := \left\{ \bigcap_{i \in I} U_i \mid (U_i)_{i \in I} \in \prod_{i \in I} U_i \right\}.$$

For a subset $S \subseteq X$, we call $\text{St}(S, U) := \bigcup\{ U \in U \mid U \cap S \neq \emptyset \}$ the star of $S$ with respect to $U$. Likewise, given any $x \in X$, we call $\text{St}(x, U) := \text{St}\{\{x\}, U\}$ the star of $x$ with respect to $U$. Moreover, the star of $U$ is defined to be $U^* := \{ \text{St}(U, U) \mid U \in U \}$. Besides, let $U^{*,0} = U$ and $U^{*,n+1} := (U^{*,n})^*$ for every $n \in \mathbb{N}$. We say that $U$ is a star-refinement of $V$ and write $U \preceq^* V$ if $U \preceq V^*$. We shall call $U$ a covering of $X$ if $X = \bigcup U$. We denote by $\mathcal{C}(X)$ the set of all coverings of $X$. A uniformity on $X$ is a non-empty subset $D \subseteq \mathcal{C}(X)$ such that

(1) $\forall U \in D \forall V \in \mathcal{C}(X): V \preceq U \Rightarrow V \in D,$
(2) $\forall U, V \in D \exists W \in D: U \preceq^* W, V \preceq^* W.$

Now we come to uniform spaces. A uniform space is a non-empty set $X$ equipped with a uniformity on $X$, whose elements are called the uniform coverings of the uniform space $X$. Let $X$ be a uniform space. The set of all finite uniform coverings of $X$ shall be denoted by $\mathcal{N}(X)$. The topology of $X$ is defined as follows: a subset $S \subseteq X$ is open in $X$ if, for every $x \in S$, there exists a uniform covering $U$ of $X$ such that $\text{St}(x, U) \subseteq S$. Let $Y$ be another uniform space. A map $f: X \to Y$ is said to be uniformly continuous if $f^{-1}(U) := \{ f^{-1}(U) \mid U \in U \}$ is a uniform covering of $X$ whenever $U$ is a uniform covering of $Y$. We denote by $\text{UC}(X, Y)$ the set of all uniformly continuous functions from $X$ to $Y$. A bijection $f: X \to Y$ is called an isomorphism if both $f$ and $f^{-1}$ are uniformly continuous maps. By an automorphism of $X$, we mean an isomorphism from $X$ to itself. The group of all automorphisms of $X$ shall be denoted by $\text{Aut}(X)$. Note that any uniformly continuous map between uniform spaces is continuous with regard to the respective topologies.

Furthermore, we shall be concerned with uniform convergence. Let $X, Y$ be uniform spaces. Concerning a function $f \in \text{UC}(X, Y)$ and a uniform covering $U$ of $Y$, we define $[f, U] := \{ g \in \text{UC}(X, Y) \mid \forall x \in X: g(x) \in \text{St}(f(x), U) \}$. It is straightforward to check that
{\mathcal{V} \in \mathcal{P}(\text{UC}(X,Y)) \mid \exists \mathcal{U} \text{ uniform covering of } Y: \mathcal{V} \preceq \{[f,\mathcal{U}] \mid f \in \text{UC}(X,Y)\}\} \text{ constitutes uniformity on } \text{UC}(X,Y), \text{ which we refer to as the uniformity of uniform convergence. The induced topology on } \text{UC}(X,Y) \text{ is called the topology of uniform convergence.}

As one might expect, any metric space constitutes a uniform space: if \( X \) is a metric space, then we may consider \( X \) as a uniform space by equipping it with the induced uniformity, that is, \( \{\mathcal{U} \in \mathcal{P}(\mathcal{P}(X)) \mid \exists r \in (0,\infty): \mathcal{U} \preceq \{B(x,r) \mid x \in X\}\} \). This particularly applies to the space of real numbers. Concerning a uniform space \( X \), we denote by \( \text{UC}(X) \) the set of all uniformly continuous functions from \( X \) to \( \mathbb{R} \), and we put \( \text{UC}_b(X) := \text{UC}(X) \cap B(X) \).

For our considerations in Section 6, we will need some further observations concerning finite uniform coverings.

**Lemma 3.1 (\cite{Isb64}).** Let \( X \) be a uniform space. If \( \mathcal{U} \) is a finite uniform covering of \( X \), then there exists a finite uniform covering \( \mathcal{V} \) of \( X \) such that \( \mathcal{U} \preceq \mathcal{V} \).

Among all finite coverings of a uniform space, the uniform ones are precisely those which admit a subordinate uniform partition of the unity.

**Lemma 3.2 (\cite{Isb64}).** Let \( X \) be a uniform space. A finite covering \( \mathcal{U} \) of \( X \) is uniform if and only if there exists a family of uniformly continuous functions \( f_\mathcal{U}: X \to [0,1] \) \((U \in \mathcal{U})\) such that

1. \( \text{spt}(f_\mathcal{U}) \subseteq U \) for every \( U \in \mathcal{U} \),
2. \( \sum_{U \in \mathcal{U}} f_\mathcal{U}(x) = 1 \) for all \( x \in X \).

For later use, let us also note the following observation.

**Lemma 3.3.** Let \( X \) be a uniform space, let \( H \subseteq \text{UC}_b(X) \) be finite and \( \varepsilon \in (0,\infty) \). Then there exists \( \mathcal{U} \in \mathcal{N}(X) \) such that \( \text{diam } f(U) \leq \varepsilon \) for all \( U \in \mathcal{U} \) and \( f \in H \).

**Proof.** Let \( f \in H \). Since \( f(X) \) is relatively compact in \( \mathbb{R} \), there is a finite subset \( F \subseteq \mathbb{R} \) such that \( f(X) \subseteq \bigcup\{B(y,\varepsilon/8) \mid y \in F\} \). Now, \( \mathcal{V}(f) := \{B(y,\varepsilon/2) \mid y \in F\} \cup \{\mathbb{R} \setminus f(X)\} \) is a finite covering of \( \mathbb{R} \). We argue that \( \mathcal{V}(f) \) is a uniform covering of \( \mathbb{R} \). To this end, let \( x \in \mathbb{R} \). If \( B(x,\varepsilon/8) \cap f(X) \neq \emptyset \), then there exists some \( y \in F \) such that \( x \in B(y,\varepsilon/4) \), which implies that \( B(x,\varepsilon/4) \subseteq B(y,\varepsilon/2) \). Otherwise, \( B(x,\varepsilon/8) \subseteq \mathbb{R} \setminus f(X) \). This proves our claim. Since \( f \) is uniformly continuous, it follows that \( \mathcal{U}(f) := f^{-1}(\mathcal{V}(f)) \) constitutes a finite uniform covering of \( X \). Consequently, \( \mathcal{U} := \bigvee_{f \in H} \mathcal{U}(f) \) is a member of \( \mathcal{N}(X) \). Finally, we observe that \( \text{diam } f(U) \leq \varepsilon \) for all \( U \in \mathcal{U} \) and \( f \in H \). \( \square \)

As it is well known and as we shall briefly recall here, any topological group may be considered as a uniform space. In order to explain this, let \( G \) be an arbitrary topological group. We denote by \( \mathcal{U}(G) \) the filter of all neighborhoods of the neutral element in \( G \). We define \( G_r \) to be the uniform space obtained by endowing \( G \) with the right uniformity, i.e.,

\[ \{\mathcal{U} \in \mathcal{P}(\mathcal{P}(G)) \mid \exists U \in \mathcal{U}(G): \mathcal{U} \preceq \{U x \mid x \in G\}\} \]
It is easy to see that the topology generated by the right uniformity is precisely the original topology of $G$. Besides, let us note that an injective homomorphism is given by the map $\lambda_G: G \to \text{Aut}(G_r)$ where $\lambda_G(g)(x) := gx$ for all $g, x \in G$. It is also straightforward to prove that $\lambda_G: G \to \text{Aut}(G_r)$ is continuous with respect to the topology of uniform convergence. For more details concerning uniform structures on topological groups, we refer to [RD81].

We conclude this section with a short remark about topological automorphism groups of uniform spaces: if $X$ is a uniform space, then $\text{Aut}(X)$ endowed with the topology of uniform convergence constitutes a topological group, and the corresponding right uniformity is just the uniformity of uniform convergence restricted to $\text{Aut}(X)$.

4. Amenable dynamical systems and topological groups

Let us briefly discuss the concept of amenability for dynamical systems in general and for topological groups in particular. For a more elaborate study, we refer to [BO08, Run02, Pat88].

**Definition 4.1.** Let $(X, G)$ be a dynamical system, i.e., a pair consisting of a uniform space $X$ and a subgroup $G$ of $\text{Aut}(X)$. An invariant mean on $(X, G)$ is a mean $m: \text{UC}_b(X) \to \mathbb{R}$ such that $m(f) = m(f \circ g)$ for all $f \in \text{UC}_b(X)$ and $g \in G$. The set of all invariant means on $(X, G)$ shall be denoted by $M(X, G)$. We say that $(X, G)$ is amenable if $M(X, G) \neq \emptyset$.

As pointed out at the end of Section 3, any topological group may be considered as a uniform space, wherefore the previous definition particularly applies to topological groups.

**Definition 4.2.** Let $G$ be a topological group. Then $G$ is said to be amenable if the dynamical system $(G_r, \lambda_G(G))$ is amenable, i.e., there is a mean $m: \text{UC}_b(G_r) \to \mathbb{R}$ such that $m(f) = m(f \circ \lambda_G(g))$ for all $f \in \text{UC}_b(G_r)$ and $g \in G$.

It is well known that if $G$ is an amenable topological group, then the dynamical system $(X, h(G))$ is amenable for any uniform space $X$ and any homomorphism $h: G \to \text{Aut}(X)$ being continuous with respect to the topology of uniform convergence. For a more elaborate study of the concept of amenability for topological groups, the reader is referred to [BO08, Run02, Pat88]. Finally in this section, let us recollect a well-known characterization of discrete amenable groups, which we shall need for the proof of Theorem 6.10.

**Theorem 4.3 ([Føl55]).** A discrete group $G$ is amenable if and only if, for all $\theta \in [0, 1)$ and $E \in \mathcal{F}(G)$, there is some $F \in \mathcal{F}_+(G)$ such that $|F \cap gF| \geq \theta |F|$ for all $g \in E$.

5. Matchings in bipartite graphs

In this section we briefly recall some very few combinatorial concepts. More precisely, we will be concerned with matchings in bipartite graphs.
**Definition 5.1.** By a bipartite graph, we mean a triple $\mathcal{B} = (X, Y, R)$ consisting of two finite sets $X$ and $Y$ and a relation $R \subseteq X \times Y$. Let $\mathcal{B} = (X, Y, R)$ be a bipartite graph. If $S \subseteq X$, then we define $N_B(S) := \{ y \in Y \mid \exists x \in S : (x, y) \in R \}$. A matching in $\mathcal{B}$ is an injective map $\varphi: D \to Y$ such that $D \subseteq X$ and $(x, \varphi(x)) \in R$ for all $x \in D$. A matching $\varphi$ in $\mathcal{B}$ is said to be perfect if $\text{dom}(\varphi) = X$. Furthermore, we call $\nu(\mathcal{B}) := \sup\{|\text{dom } \varphi| \mid \varphi \text{ matching in } \mathcal{B}\}$ the matching number of $\mathcal{B}$.

We will need Hall’s well-known matching theorem, which we restate for convenience.

**Theorem 5.2 ([Hal35], [Ore55]).** If $\mathcal{B} = (X, Y, R)$ is a bipartite graph, then

$$\nu(\mathcal{B}) = |X| - \sup\{|S| - |N_B(S)| \mid S \subseteq X\}.$$ 

**Corollary 5.3.** Let $\mathcal{B} = (X, Y, R)$ be a bipartite graph. The following are equivalent:

1. There exists an injective map $\varphi: X \to Y$ such that $(x, \varphi(x)) \in R$ for all $x \in X$.
2. $|S| \leq |N_B(S)|$ for every subset $S \subseteq X$.

In what follows, we shall have a closer look at bipartite graphs arising from uniform coverings of uniform spaces. For this purpose, let us introduce some additional notation.

**Definition 5.4.** Let $X$ be a set, $\mathcal{U} \in \mathcal{C}(X)$ and $E, F \in \mathcal{F}(X)$. We consider the bipartite graph $\mathcal{B}(E, F, \mathcal{U}) := (E, F, R(E, F, \mathcal{U}))$ where

$$R(E, F, \mathcal{U}) := \{(x, y) \in E \times F \mid y \in \text{St}(x, \mathcal{U})\} = \{(x, y) \in E \times F \mid \exists U \in \mathcal{U}: \{x, y\} \subseteq U\}.$$ 

Furthermore, we define $\mu(E, F, \mathcal{U}) := \nu(\mathcal{B}(E, F, \mathcal{U}))$.

**Remark 5.5.** Let $\mathcal{U}$ be a covering of a set $X$. If $E, F \in \mathcal{F}(X)$, then $E \cap F \leq \mu(E, F, \mathcal{U})$.

The subsequent observations will prove useful in Section 6.

**Lemma 5.6.** Let $\mathcal{U}$ be a covering of a set $X$ and let $F_0, F_1, F_2 \in \mathcal{F}(X)$. Then

$$\mu(F_0, F_2, \mathcal{U}^*) \geq \mu(F_0, F_1, \mathcal{U}) + \mu(F_1, F_2, \mathcal{U}) - |F_1|.$$ 

**Proof.** Suppose $\varphi_0$ and $\varphi_1$ to be matchings in $\mathcal{B}(F_0, F_1, \mathcal{U})$ and $\mathcal{B}(F_1, F_2, \mathcal{U})$ such that $|\text{dom}(\varphi_0)| = \mu(F_0, F_1, \mathcal{U})$ and $|\text{dom}(\varphi_1)| = \mu(F_1, F_2, \mathcal{U})$, respectively. Put $D_i := \text{dom}(\varphi_i)$ for each $i \in \{0, 1\}$. Let $D := \varphi_0^{-1}(D_1)$ and define $\psi: D \to F_2$, $x \mapsto \varphi_1(\varphi_0(x))$. Evidently, $\psi$ is injective. Besides, $\psi(x) = \varphi_1(\varphi_0(x)) \in \text{St}(\varphi_0(x), \mathcal{U}) \subseteq \text{St}(x, \mathcal{U}^*)$ for every $x \in D$. Hence, $\psi$ is a matching in $\mathcal{B}(F_0, F_2, \mathcal{U}^*)$. Furthermore,

$$|F_1 - |D| = |F_1| - |\varphi_0(D)| = |F_1 \setminus \varphi_0(D)| = |F_1 \setminus (\varphi_0(D) \cap D_1)|$$

$$= |(F_1 \setminus \varphi_0(D)) \cup (F_1 \setminus D_1)| \leq |F_1 \setminus \varphi_0(D)| + |F_1 \setminus D_1|$$

$$= 2|F_1| - \mu(F_0, F_1, \mathcal{U}) - \mu(F_1, F_2, \mathcal{U})$$

and thus $\mu(F_0, F_2, \mathcal{U}^*) \geq |D| \geq \mu(F_0, F_1, \mathcal{U}) + \mu(F_1, F_2, \mathcal{U}) - |F_1|$. 

\qed
Corollary 5.7. Let $\mathcal{U}$ be a covering of a set $X$ and let $F_0, \ldots, F_n \in \mathcal{F}(X)$. Then

$$
\mu(F_0, F_n, U^{*n-1}) \geq \sum_{i=0}^{n-1} \mu(F_i, F_{i+1}, \mathcal{U}) - \sum_{i=1}^{n-1} |F_i|.
$$

6. Mean topological matching number

In this section we establish a novel invariant for general dynamical systems and explore its connection to the concept of amenability.

Definition 6.1. If $(X, G)$ is a dynamical system, then we define the mean topological matching number of $(X, G)$ to be the quantity

$$
\mu(X, G) := \inf_{E \in \mathcal{F}(G)} \inf_{U \in \mathcal{N}(X)} \sup_{F \in \mathcal{F}_+(X)} \inf_{g \in E} \frac{\mu(F, g(F), U)}{|F|} \in [0, 1].
$$

Remark 6.2. Let $X$ and $Y$ be uniform spaces and let $f: X \to Y$ be injective and uniformly continuous. Let $G$ be a group and let $h_0: G \to \text{Aut}(X)$ and $h_1: G \to \text{Aut}(Y)$ be homomorphisms. If $f$ is $G$-equivariant, i.e., $f(h_0(g)(x)) = h_1(g)(f(x))$ for all $x \in X$ and $g \in G$, then $\mu(X, h_0(G)) \leq \mu(Y, h_1(G))$.

Theorem 6.3. Let $(X, G)$ be a dynamical system. If $\mu(X, G) = 1$, then $(X, G)$ is amenable.

Proof. Let $\varepsilon \in (0, \infty)$, $H \in \mathcal{F}(\text{UC}_b(X))$ and $E \in \mathcal{F}_+(G)$. We observe that

$$
A(H, E, \varepsilon) := \{m \in M(\text{UC}_b(X)) \mid \forall f \in H \forall g \in E: |m(f) - m(f \circ g)| \leq \varepsilon\}
$$

is closed in the compact Hausdorff space $M(\text{UC}_b(X))$. We shall prove that $A(H, E, \varepsilon) \neq \emptyset$. To this end, we put $\theta := \varepsilon/(1 + 2 \sup_{f \in H} \|f\|_\infty)$. By Lemma 3.3, there exists $\mathcal{U} \in \mathcal{N}(X)$ such that $\text{diam } f(U) \leq \theta$ for all $U \in \mathcal{U}$ and $f \in H$. Since $\mu(X, G) = 1$, there exists $F \in \mathcal{F}_+(X)$ such that $|F| - \mu(F, g(F), \mathcal{U}) \leq \theta |F|$ for all $g \in E$. Note that

$$
m: \text{UC}_b(X) \to \mathbb{R}, \ f \mapsto \frac{1}{|F|} \sum_{x \in F} f(x)
$$
is an element of $M(\mathcal{U}C_b(X))$. Let $g \in E$. Let $\varphi : D \to g(F)$ be an injective map such that
$D \subseteq F$, $|D| = \mu(F, g(F), \mathcal{U})$, and $\varphi(x) \in \text{St}(x, \mathcal{U})$ for all $x \in D$. If $f \in H$, then

$$|m(f) - m(f \circ g)| = \frac{1}{|F|} \left| \sum_{x \in F} f(x) - \sum_{x \in F} f(g(x)) \right|$$

$$= \frac{1}{|F|} \left| \sum_{x \in D} (f(x) - f(\varphi(x))) + \sum_{x \in F \setminus D} f(x) - \sum_{x \in g(F) \setminus \varphi(D)} f(x) \right|$$

$$\leq \frac{1}{|F|} \left( \sum_{x \in D} |f(x) - f(\varphi(x))| + \sum_{x \in F \setminus D} |f(x)| + \sum_{x \in g(F) \setminus \varphi(D)} |f(x)| \right)$$

$$\leq \theta \frac{\mu(F, g(F), \mathcal{U})}{|F|} + 2\frac{|F| - \mu(F, g(F), \mathcal{U})}{|F|} \|f\|_{\infty}$$

$$\leq \theta + 2\|f\|_{\infty} \theta = (1 + 2\|f\|_{\infty}) \theta \leq \varepsilon.$$

Thus, $m$ is a member of $A(H, E, \varepsilon)$. Therefore, $A(H, E, \varepsilon) \neq \emptyset$. Since

$$A(H_0 \cup H_1, E_0 \cup E_1, \varepsilon_0 \wedge \varepsilon_1) \subseteq A(H_0, E_0, \varepsilon_0) \cap A(H_1, E_1, \varepsilon_1)$$

for all $H_0, H_1 \in \mathcal{F}(\mathcal{U}C_b(X))$, $E_0, E_1 \in \mathcal{F}(G)$ and $\varepsilon_0, \varepsilon_1 \in (0, \infty)$, we conclude that

$$\mathcal{A} := \{A(H, E, \varepsilon) \mid H \in \mathcal{F}(\mathcal{U}C_b(X)), E \in \mathcal{F}(G), \varepsilon \in (0, \infty)\}$$

has the finite intersection property. By Theorem 2.1, $M(\mathcal{U}C_b(X))$ is compact. Consequently,
$\bigcap \mathcal{A} \neq \emptyset$. Finally, we observe that $M(X, G) = \bigcap \mathcal{A}$, wherefore $(X, G)$ is amenable. \hfill $\square$

**Theorem 6.4.** Let $X$ be a uniform space such that every open non-empty subset of $X$ is infinite, and let $G$ be a subgroup of $\text{Aut}(X)$. If $(X, G)$ is amenable, then $\mu(X, G) = 1$.

**Proof.** Let $\theta \in [0, 1], \mathcal{U} \in \mathcal{N}(X)$ and $E \in \mathcal{F}_+(G)$. By Lemma 3.2, there exists a family of uniformly continuous functions $f_U : X \to [0, 1]$ $(U \in \mathcal{U})$ such that

1. $\text{spt}(f_U) \subseteq U$ for every $U \in \mathcal{U}$,
2. $\sum_{U \in \mathcal{U}} f_U(x) = 1$ for all $x \in X$.

Since $(X, G)$ is amenable and every open non-empty subset of $X$ is infinite, Lemma 2.2 asserts that there exists $F \in \mathcal{F}_+(X)$ such that

$$\left| \frac{1}{|F|} \sum_{x \in F} f_U(x) - \sum_{x \in F} f_U(g(x)) \right| \leq \frac{1 - \theta}{|E| + 1}$$

for all $U \in \mathcal{U}$ and $g \in E$.

We show that $\mu(F, g(F), \mathcal{U}) \geq \theta |F|$ for all $g \in E$. To this end, let $g \in E$. We consider the bipartite graph $\mathcal{B} := \mathcal{B}(F, g(F), \mathcal{U})$. If $S \subseteq F$, then we put $\mathcal{V} := \{U \in \mathcal{U} \mid U \cap S \neq \emptyset\}$ and
$T := N_B(S)$, and we observe that

$$|S| = \sum_{x \in S} 1 = \sum_{S \in \mathcal{S}} \sum_{U \in \mathcal{U}} f_U(x) \leq \sum_{U \in \mathcal{V}} \sum_{x \in S} f_U(x) \leq \sum_{U \in \mathcal{V}} \sum_{y \in g(F)} f_U(y)$$

\[ \leq (1 - \theta)|F| + \sum_{y \in T} \sum_{U \in \mathcal{V}} f_U(y) \]

\[ \leq (1 - \theta)|F| + \sum_{y \in T} \sum_{U \in \mathcal{V}} f_U(y) \]

that is, $|S| - |N_B(S)| \leq (1 - \theta)|F|$. According to Theorem 5.2, it follows that

$$\mu(F, g(F), \mathcal{U}) = \frac{|F| - \sup_{S \subseteq F} (|S| - |N_B(S)|)}{|F|} \geq \frac{|F| - (1 - \theta)|F|}{|F|} = \theta.$$ 

This substantiates that $\mu(X, G) = 1$. □

**Corollary 6.5.** Let $X$ be a perfect Hausdorff uniform space and let $G$ be a subgroup of $\text{Aut}(X)$. Then $(X, G)$ is amenable if and only if $\mu(X, G) = 1$.

**Proof.** This follows from Theorem 6.3, Theorem 6.4 and Remark 2.3. □

The results above motivate to study the mean topological matching number of a dynamical system. Of course, we are particularly interested in the case where the mean topological matching number equals one. This situation can be characterized with regard to generating subsets of the acting group.

**Proposition 6.6.** Let $(X, G)$ be a dynamical system and let $S$ be a symmetric generating subset of $G$ containing the identity map. Then $\mu(X, G) = 1$ if and only if

$$\inf_{E \in \mathcal{F}(S)} \inf_{U \in \mathcal{N}(X)} \sup_{F \in \mathcal{F}(X)} \inf_{g \in E} \frac{\mu(F, g(F), \mathcal{U})}{|F|} = 1.$$

**Proof.** ($\Rightarrow$) This is obvious.

($\Leftarrow$) Let $\theta_0 \in [0, 1]$, $E_0 \in \mathcal{F}(G)$ and $\mathcal{U} \in \mathcal{N}(X)$. Since $E_0$ is finite, there exist a finite subset $E \subseteq S$ as well as $n \in \mathbb{N} \setminus \{0\}$ such that $E_0 \subseteq E^n$. By Lemma 3.1, there exists $\mathcal{V} \in \mathcal{N}(X)$ such that $U \leq \mathcal{V}_{\ast,n-1}$. Consider $W := \bigvee_{g \in E^n} g^{-1}(\mathcal{V})$ and $\theta := \frac{n + \theta_0 - 1}{n} \in [0, 1]$. By assumption, there exists $F \in \mathcal{F}_+(X)$ such that $\inf_{s \in E} \mu(F, s(F), W) \geq \theta |F|$. To this end, let $g \in E_0$. Then there exist $g_1, \ldots, g_\ell \in E$ such that $g = g_\ell \cdots g_1$. For each $i \in \{1, \ldots, n\}$, let $s_i := g_\ell \cdots g_i$. Note that

$$\mu(s_{i+1}(F), s_i(F), \mathcal{V}) = \mu(s_{i+1}(F), s_{i+1}(g_i(F)), \mathcal{V}) = \mu(F, g_i(F), s_{i+1}^{-1}(\mathcal{V}))$$
for each \( i \in \{1, \ldots, n-1\} \). Hence,
\[
\mu(F,g(F),\mathcal{U}) \geq \mu(F,g_n(F),\mathcal{V}) + \sum_{i=1}^{n-1} \mu(s_{i+1}(F),s_i(F),\mathcal{V}) - \sum_{i=1}^{n-1} |s_i(F)|
\]
\[
= \mu(F,g_n(F),\mathcal{V}) + \sum_{i=1}^{n-1} \mu(F,g_i(F),s_{i+1}^{-1}(\mathcal{V})) - (n-1)|F|
\]
\[
\geq \mu(F,g_n(F),\mathcal{W}) + \sum_{i=0}^{n-1} \mu(F,g_i(F),\mathcal{W}) - (n-1)|F|
\]
\[
\geq n\theta|F| - (n-1)|F| = \theta_0|F|.
\]
Consequently, \( \inf_{g \in E_0} \mu(F,g(F),\mathcal{U}) \geq \theta_0|F| \). This substantiates that \( \mu(X,G) = 1 \).

**Corollary 6.7.** Let \((X,G)\) be a dynamical system and let \(S\) be a finite symmetric generating subset of \(G\) containing the identity map. Then \(\mu(X,G) = 1\) if and only if
\[
\inf_{\mathcal{U} \in \mathcal{N}(X)} \sup_{F \in \mathcal{F}_{+}(X)} \inf_{g \in S} \frac{\mu(F,g(F),\mathcal{U})}{|F|} = 1.
\]

Furthermore, the mean topological matching number of a dynamical system is invariant under passing to dense subgroups of the acting group.

**Proposition 6.8.** Let \((X,G)\) be a dynamical system and let \(H\) be a subgroup of \(G\). If \(H\) is dense in \(G\) with respect to the topology of uniform convergence, then \(\mu(X,G) = \mu(X,H)\).

**Proof.** Evidently, \(\mu(X,G) \leq \mu(X,H)\). In order to prove the converse inequality, let \(\varepsilon \in (0,\infty)\), \(E \in \mathcal{F}(G)\) and \(\mathcal{U} \in \mathcal{N}(X)\). By Lemma 3.1, there exists \(\mathcal{V} \in \mathcal{N}(X)\) such that \(\mathcal{U} \preceq^* \mathcal{V}\). Since \(H\) is dense in \(G\), for each \(g \in E\) there exists some \(\hat{g} \in H \cap [g,\mathcal{V}]\). Let \(F \in \mathcal{F}_{+}(X)\) such that \(\inf_{g \in E} \mu(F,\hat{g}(F),\mathcal{V}) \geq (\mu(X,H) - \varepsilon)|F|\). Since \(\hat{g}(x) \in \text{St}(g(x),\mathcal{V})\) for all \(x \in F\) and \(g \in E\), we conclude that \(\mu(F,g(F),\mathcal{U}) \geq \mu(F,\hat{g}(F),\mathcal{V})\) for every \(g \in G\). Hence,
\[
\inf_{g \in E} \frac{\mu(F,g(F),\mathcal{U})}{|F|} \geq \inf_{g \in E} \frac{\mu(F,\hat{g}(F),\mathcal{V})}{|F|} \geq \mu(X,H) - \varepsilon.
\]
This shows that \(\mu(X,G) \geq \mu(X,H)\).

Subsequently, we shall have a closer look at the mean topological matching number of the dynamical system associated to an arbitrary topological group in Definition 4.2. This will lead to a novel characterization of amenability for Hausdorff topological groups.

**Definition 6.9.** If \(G\) is a topological group, then we define the **mean topological matching number** of \(G\) to be the quantity
\[
\mu(G) := \mu(G_r,\lambda_G(G)) = \inf_{E \in \mathcal{F}(G)} \inf_{U \in \mathcal{N}(G_r)} \sup_{F \in \mathcal{F}_{+}(G)} \inf_{g \in E} \frac{\mu(F,gF,\mathcal{U})}{|F|}.
\]

**Theorem 6.10.** A Hausdorff topological group \(G\) is amenable if and only if \(\mu(G) = 1\).
Proof. ($\iff$) This is due to Theorem 6.3.

($\implies$) Suppose $G$ to be amenable. We argue that $\mu(G) = 1$. The proof proceeds by case analysis depending on whether $G$ is discrete. Let us first assume $G$ to be discrete. On account of Theorem 4.3 and Remark 5.5,

$$\mu(G) = \inf_{E \in F(G)} \inf_{\mu \in N(G_r)} \sup_{g \in E} \inf_{F \in F_+(G)} \frac{\mu(F, gF, U)}{|F|} \geq \inf_{E \in F(G)} \inf_{g \in E} \frac{|F| - |F \setminus gF|}{|F|} = 1.$$ 

Thus, $\mu(G) = 1$. To complete the case analysis, finally suppose that $G$ is not discrete. Since $G$ is a homogeneous Hausdorff space, Remark 2.3 implies that every open non-empty subset of $G$ is infinite. Hence, $\mu(G) = \mu(G_r, \lambda_G(G)) = 1$ by Theorem 6.4. This finishes the proof. 

Corollary 6.11. Let $G$ be a Hausdorff topological group and let $S \subseteq G$ be a symmetric subset containing the neutral element and generating a dense subgroup of $G$. Then $G$ is amenable if and only if

$$\inf_{E \in F(S)} \inf_{\mu \in N(G_r)} \sup_{g \in E} \inf_{F \in F_+(G)} \frac{\mu(F, gF, U)}{|F|} = 1.$$ 

Proof. Denote by $H$ the subgroup of $G$ generated by $S$. Since the homomorphism $\lambda_G: G \to \text{Aut}(G_r)$ is continuous, the subgroup $\lambda_G(H)$ is dense in $\lambda_G(G)$ with respect to the topology of uniform convergence. Accordingly,

$$G \text{ amenable } \iff \mu(G_r, \lambda_G(G)) = 1 \iff \mu(G_r, \lambda_G(H)) = 1 \iff \inf_{E \in F(S)} \inf_{\mu \in N(G_r)} \sup_{g \in E} \inf_{F \in F_+(G)} \frac{\mu(F, gF, U)}{|F|} = 1. \quad \square$$

Note that compact topological groups satisfy the following strong matching condition.

Proposition 6.12. Let $G$ be a compact topological group. If $U$ is a uniform covering of $G_r$, then there exists $F \in F_+(G)$ such that $\mu(F, gF, U) = |F|$ for all $g \in G$.

Proof. Let $U$ be a uniform covering of $G_r$. Then there exists an open neighborhood $U$ of the neutral element in $G$ such that $U \preceq \{ U^{-1}Ux \mid x \in G \}$. Since $G$ is compact, $V := \bigcap_{g \in G} g^{-1}Ug$ is an open neighborhood of the neutral element in $G$. Besides, $gV = Vg$ for all $g \in G$. Let $F \in F_+(G)$ such that $G = VF$ and $\inf\{|E| \mid E \in F_+(G), G = VE\} = |F|$. Let $g \in G$. Note that $VgF = gVF = G$. We consider the bipartite graph $B := (F, gF, R)$ where $R := \{(x, y) \in F \times gF \mid Vx \capVy \neq \emptyset\}$. Let $S \subseteq F$ and $T := N_B(S)$. We show that $|S| \leq |T|$. To this end, let $E := (F \setminus S) \cup T$. We argue that $G = VE$. Clearly, if $z \in V(F \setminus S)$, then $z \in VE$. Otherwise, there exist $x \in S$ and $y \in gF$ such that $z \in Vx \cap Vy$, which readily implies that $y \in N_B(S)$ and thus $z \in VT \subseteq VE$. Therefore, $G = VE$. Accordingly, $|F| \leq |E|$ and hence $|S| \leq |T|$. Consequently, Corollary 5.3 asserts that $B$ admits a perfect matching. That is, $\mu(B) = |F|$ and thus $\mu(F, gF, U) = |F|$. \hfill \square
Recall that a topological group $G$ has small open subgroups if every neighborhood of the neutral element in $G$ contains an open subgroup of $G$. It is well known that any topological subgroup of $S_\alpha$ as well as any totally disconnected locally compact Hausdorff topological group (in particular, any discrete group) has small open subgroups (see [AT08]). With regard to the associated uniform structure, we observe the following: if $G$ is a topological group having small open subgroups, then the right uniformity of $G$ is given by

$$\{U \in \mathcal{P}(\mathcal{P}(G)) \mid \exists H \text{ open subgroup of } G : U \leq \{Hx \mid x \in G\}\}.$$

Therefore, we immediately obtain the subsequent consequence of Theorem 6.10 and Corollary 6.11 for Hausdorff topological groups having small open subgroups.

**Corollary 6.13.** Let $G$ be a Hausdorff topological group having small open subgroups and let $S \subseteq G$ be a symmetric subset containing the neutral element and generating a dense subgroup of $G$. The following conditions are equivalent.

1. $G$ is amenable.
2. For all $\varepsilon \in (0, \infty)$ and $n \in \mathbb{N}$, every open subgroup $H$ of $G$, every right coset coloring $\varphi : \{Hx \mid x \in G\} \to \{1, \ldots, n\}$, and every finite subset $E \subseteq G$, there exists a finite subset $F \subseteq G$ such that

$$\forall i \in \{1, \ldots, n\} \forall g \in E : \| (\varphi \circ \pi_H)^{-1}(i) \cap F \| - \| (\varphi \circ \pi_H)^{-1}(i) \cap gF \| < \varepsilon |F|.$$ 

3. For all $\varepsilon \in (0, \infty)$ and $n \in \mathbb{N}$, every open subgroup $H$ of $G$, every right coset coloring $\varphi : \{Hx \mid x \in G\} \to \{1, \ldots, n\}$, and every finite subset $E \subseteq S$, there exists a finite subset $F \subseteq G$ such that

$$\forall i \in \{1, \ldots, n\} \forall g \in E : \| (\varphi \circ \pi_H)^{-1}(i) \cap F \| - \| (\varphi \circ \pi_H)^{-1}(i) \cap gF \| < \varepsilon |F|.$$ 

Even in the case of discrete groups, this gives an interesting characterization of amenability that we record in the following corollary.

**Corollary 6.14.** Let $G$ be a discrete group and let $S$ be a symmetric generating subset of $G$ containing the neutral element. The following conditions are equivalent.

1. $G$ is amenable.
2. For all $\varepsilon \in (0, \infty)$ and $n \in \mathbb{N}$, every coloring $\varphi : G \to \{1, \ldots, n\}$, and every finite subset $E \subseteq G$, there exists a finite subset $F \subseteq G$ such that

$$\forall i \in \{1, \ldots, n\} \forall g \in E : \| \varphi^{-1}(i) \cap F \| - \| \varphi^{-1}(i) \cap gF \| < \varepsilon |F|.$$ 

3. For all $\varepsilon \in (0, \infty)$ and $n \in \mathbb{N}$, every coloring $\varphi : G \to \{1, \ldots, n\}$, and every finite subset $E \subseteq S$, there exists a finite subset $F \subseteq G$ such that

$$\forall i \in \{1, \ldots, n\} \forall g \in E : \| \varphi^{-1}(i) \cap F \| - \| \varphi^{-1}(i) \cap gF \| < \varepsilon |F|.$$ 

It is clear that Condition (3) in the previous corollary is a priori a weaker requirement than asking for the existence of Følner sets. It is a consequence of Moore’s work [Moo13] that
in order characterize discrete amenable groups one can weaken Condition (3) even further and consider only colorings with two colors. It will be subject of further study to find a suitable analogue of this result for Hausdorff topological groups.

7. Extremely amenable topological groups

In this section we want to draw a connection between Theorem 6.10 and a characterization of extremely amenable topological groups due to Pestov [Pes05a] (see also [Pes02, Pes05b]). Recall that topological group $G$ is said to be extremely amenable if every continuous action of $G$ on a non-empty compact Hausdorff space admits a fixed point. In order to state and discuss Pestov’s result, let us recall some additional terminology from [Pes05b].

**Definition 7.1.** We say that a topological group $G$ has the Ramsey-Dvoretzky-Milman property if, for all $\varepsilon \in (0, \infty)$ and $f \in \text{UC}_b(G)$ and every finite subset $E \subseteq G$, there exists some $g \in G$ such that $\text{diam } f(Eg) \leq \varepsilon$.

For the sake of convenience, let us furthermore mention the subsequent slight, but useful reformulation of the Ramsey-Dvoretzky-Milman property.

**Proposition 7.2 ([Pes05b]).** A topological group $G$ has the Ramsey-Dvoretzky-Milman property if and only if, for every $\varepsilon \in (0, \infty)$, every finite subset $H \subseteq \text{UC}_b(G)$ and every finite subset $E \subseteq G$, there exists $g \in G$ such that $\text{diam } f(Eg) \leq \varepsilon$ for each $f \in H$.

The following result reveals the link between the Ramsey-Dvoretzky-Milman property and extreme amenability.

**Theorem 7.3 ([Pes05b]).** A topological group is extremely amenable if and only if it has the Ramsey-Dvoretzky-Milman property.

Now let us restate Pestov’s result in terms of finite uniform coverings.

**Corollary 7.4.** A topological group $G$ is extremely amenable if and only if, for every finite subset $E \subseteq G$ and each $U \in \mathcal{N}(G)$, there exist $g \in G$ and $U \in \mathcal{U}$ such that $Eg \subseteq U$.

**Proof.** $(\implies)$ Let $E \subseteq G$ be finite and let $U \in \mathcal{N}(G)$. Without loss of generality, assume $E$ to be non-empty. By Lemma 3.2, there exists a family of uniformly continuous functions $f_U: G \to [0, 1]$ ($U \in \mathcal{U}$) such that

1. $\text{spt}(f_U) \subseteq U$ for every $U \in \mathcal{U}$,
2. $\sum_{U \in \mathcal{U}} f_U = 1$.

Due to Theorem 7.3, $G$ has the Ramsey-Dvoretzky-Milman property. Hence, by Proposition 7.2, there exists $g \in G$ such that $\text{diam } f(Eg) \leq \frac{1}{|\mathcal{U}|+1}$ for each $f \in H$. Let $h_0 \in E$. By (2), there exists $U \in \mathcal{U}$ such that $f_U(h_0g) > \frac{1}{|\mathcal{U}|+1}$. We conclude that $f_U(hg) > 0$ for each $h \in E$. Consequently, $Eg \subseteq U$ due to (1). This proves the claim.
Let $\varepsilon \in (0, \infty)$, $f \in \text{UC}_b(G_r)$ and $E \in \mathcal{F}(G)$. Due to Lemma 3.3, there exists $\mathcal{U} \in \mathcal{N}(G_r)$ such that $\text{diam} f(U) \leq \varepsilon$ for all $U \in \mathcal{U}$. By assumption, there exist $g \in G$ and $U \in \mathcal{U}$ such that $Eg \subseteq U$. Hence, $\text{diam} f(Eg) \leq \varepsilon$ and this completes the proof. □

Finally, let us briefly discuss the connection between Theorem 6.10 and Corollary 7.4. To this end, suppose $G$ to be an extremely amenable topological group. Let $\mathcal{U} \in \mathcal{N}(G_r)$ and let $E$ be a finite subset of $G$. By Corollary 7.4, there exist $g \in G$ and $U \in \mathcal{U}$ such that $(E \cup EE)g \subseteq U$. Let $F := Eg \subseteq G$. Then $F \cup hF \subseteq (E \cup EE)g \subseteq U$ and thus $\mu(F, hF, U) = |F|$ for each $h \in E$. This shows that $\mu(G) = 1$. Hence, $G$ is amenable due to Theorem 6.10.

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References

[AT08] Alexander Arhangel’skii and Mikhail Tkachenko, Topological groups and related structures, Atlantis Studies in Mathematics, vol. 1, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[BJM89] John F. Berglund, Hugo D. Junghenn, and Paul Milnes, Analysis on semigroups, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1989, Function spaces, compactifications, representations, A Wiley-Interscience Publication.

[BO08] Nathanial P. Brown and Narutaka Ozawa, $C^*$-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[Føl55] Erling Følner, On groups with full Banach mean value, Math. Scand. 3 (1955), 243–254.

[Hal35] Philip Hall, On representatives of subsets, Journal of the London Mathematical Society 10 (1935), 26–30.

[Isb64] John R. Isbell, Uniform spaces, Mathematical Surveys, No. 12, American Mathematical Society, Providence, R.I., 1964.

[MO03] Justin T. Moore, Amenability and Ramsey theory, Fund. Math. 220 (2013), no. 3, 263–280.

[Ore55] Oystein Ore, Graphs and matching theorems, Duke Math. J. 22 (1955), 625–639.

[Pat88] Alan L. T. Paterson, Amenability, Mathematical Surveys and Monographs, vol. 29, American Mathematical Society, Providence, RI, 1988.

[Pes02] Vladimir Pestov, Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups, Israel J. Math. 127 (2002), 317–357.

[Pes05a] ________, A corrigendum to: “Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups”, Israel J. Math. 145 (2005), 375–379.

[Pes05b] ________, Dynamics of infinite-dimensional groups and Ramsey-type phenomena, Publicações Matemáticas do IMPA. [IMPA Mathematical Publications], Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2005, 25o Colóquio Brasileiro de Matemática. [25th Brazilian Mathematics Colloquium].

[RD81] Walter Roelcke and Susanne Dierolf, Uniform structures on topological groups and their quotients, McGraw-Hill International Book Co., New York, 1981, Advanced Book Program.

[Runde02] Volker Runde, Lectures on amenability, Lecture Notes in Mathematics, vol. 1774, Springer-Verlag, Berlin, 2002.
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