MATRIX FACTORIZATIONS FOR NONAFFINE LG–MODELS

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ABSTRACT. We propose a natural definition of a category of matrix factorizations for nonaffine Landau-Ginzburg models. For any LG-model we construct a fully faithful functor from the category of matrix factorizations defined in this way to the triangulated category of singularities of the corresponding fiber. We also show that this functor is an equivalence if the total space of the LG-model is smooth.

INTRODUCTION

In the paper [8] we established a connection between categories of D-branes of type B (B-branes) in affine Landau-Ginzburg models and triangulated categories of singularities of the singular fibers. A mathematical definition for the category of B-branes in affine Landau-Ginzburg models was proposed by M. Kontsevich. According to his proposal the superpotential $W$ will deform complexes of coherent sheaves to “$W$-twisted” complexes. These are 2-periodic chains of maps of vector bundles in which the composition of two consecutive maps is no longer required to be zero, but instead is equal to multiplication by $W$. Such chains are called matrix factorizations of $W$ and so Kontsevich predicted that the category of B-branes should be the category of matrix factorizations. Kapustin and Li verified [4] the equivalence of this definition with the physics notion of B-branes in LG-models in the case of the usual quadratic superpotential and gave physics arguments supporting Kontsevich’s proposal in the case of general superpotentials.

The triangulated category of singularities is defined for any noetherian scheme as a quotient of the bounded derived category of coherent sheaves by the subcategory of perfect complexes. On the other hand, the definition of the category of matrix factorization was given only for affine LG-models. Under the conditions that the total space $X$ of an LG-model is affine and smooth we showed in the paper [8] that the categories of matrix factorizations are equivalent to the triangulated categories of singularities of the corresponding fibers. In particular, this implies that, in spite of the fact that the category of B-branes is defined using the total space $X$, it depends only on the singular fibers of the superpotential.

In this paper we propose a natural generalization of the definition of the category of matrix factorizations which makes sense for arbitrary, not necessarily affine LG-models. We show that with our general definition the same result as in affine case holds true as long as the total space of an LG-model is smooth. In the case of a singular total space we have only a full embedding of the category of matrix factorizations in the triangulated category of the corresponding fiber. Studying the difference between these categories in the case of singular total spaces is certainly very interesting and we plan to return to this question in the future.

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When this paper was finished I was informed that a category of matrix factorizations in the nonaffine case is also discussed in a forthcoming paper of Kevin Lin and Daniel Pomerleano [6].

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1. Triangulated categories of singularities

Let $X$ be a noetherian scheme over a field $k$. Denote by $\mathbf{D}(\text{Qcoh } X)$ the unbounded derived category of quasi-coherent sheaves on $X$. The derived category $\mathbf{D}(\text{Qcoh } X)$ has arbitrary coproducts. We can consider the full triangulated subcategory of compact objects, i.e. such objects $C \in \mathbf{D}(\text{Qcoh } X)$ for which the functor $\text{Hom}(C, -)$ commutes with all coproducts. It is known (see [7, 3]) that the subcategory of compact objects coincides with the subcategory of perfect complexes $\text{Perf}(X)$. Recall that a complex on a scheme is called perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite type.

For a noetherian scheme it is also natural to consider the abelian subcategory $\text{coh}(X) \subset \text{Qcoh}(X)$ of coherent sheaves and the bounded derived categories $\mathbf{D}^b(\text{coh } X)$ of coherent sheaves on $X$. The natural functor from $\mathbf{D}^b(\text{coh } X)$ to the unbounded derived category of quasi-coherent sheaves $\mathbf{D}(\text{Qcoh } X)$ is fully faithful and identifies $\mathbf{D}^b(\text{coh } X)$ with the full subcategory $\mathbf{D}^{0,b}_{\text{coh}}(\text{Qcoh } X)$ consisting of all cohomologically bounded complexes with coherent cohomology ([2], Ex.II, 2.2.2). In particular $\text{Perf}(X)$ can be viewed as a full triangulated subcategory of $\mathbf{D}^b(\text{coh } X)$. For a regular scheme the inclusion $\text{Perf}(X) \subseteq \mathbf{D}^b(\text{coh } X)$ is an equivalence while for a singular scheme this inclusion is strict. The triangulated category of singularities of $X$ is defined as the Verdier quotient of these two natural categories that we can attach to a noetherian scheme $X$:

**Definition 1.1.** The triangulated category of singularities of $X$, denoted by $\mathbf{D}_{Sg}(X)$, is the Verdier quotient of the bounded derived category $\mathbf{D}^b(\text{coh } X)$ by the full triangulated subcategory of perfect complexes $\text{Perf}(X)$.

We say that $X$ satisfies the condition (ELF) if

(ELF) $X$ is separated, noetherian, of finite Krull dimension, and has enough locally free sheaves.

The last condition means that for any coherent sheaf $\mathcal{F}$ there is an epimorphism $\mathcal{E} \to \mathcal{F}$ with a locally free sheaf $\mathcal{E}$. It also implies that any perfect complex is globally (not only locally) quasi-isomorphic to a bounded complex of locally free sheaves of finite type. For example, any quasi-projective scheme satisfies these conditions. Note that if $X$ satisfies (ELF), then any closed and any open subscheme of $X$ is also noetherian, finite dimensional and has enough locally free sheaves. This is automatic for a closed subscheme, while for an open subscheme $U \subset X$ it follows from the fact that any coherent sheaf on $U$ can be obtained as the restriction of a coherent sheaf on $X$. From now on we will assume that all schemes we work with satisfy the condition (ELF).

The definition of triangulated category of singularities extends verbatim to the case of stacks and noncommutative spaces. One can also define a graded version of this category, which is a central object in the so called Landau-Ginzburg/Calabi-Yau correspondence discussed in the paper [10].
It is possible to reformulate Definition 1.1 so that it will make sense intrinsically in any triangulated category \( D \). We say that an object \( A \in D \) is \textit{homologically finite} if for any object \( B \in D \) all \( \text{Hom}(A, B[i]) \) are trivial except for finite number of \( i \in \mathbb{Z} \). Such objects form a full triangulated subcategory \( D_{hf} \subset D \). We define a triangulated category \( D_{Sg} \) as the quotient \( D/ D_{hf} \).

It is proved in [9] that if a scheme \( X \) satisfies (ELF). Then the subcategory \( D_{hf} \) of homologically finite objects in \( D = \text{D}^b(\text{coh}(X)) \) coincides with the subcategory of perfect complexes \( \text{Perf}(X) \) and, hence, \( D_{Sg} \cong D_{Sg}(X) \). Thus the triangulated category of singularities can be defined internally starting only with the bounded derived category of coherent sheaves.

A simple but fundamental property of triangulated categories of singularities is the fact that they are local in the Zariski topology. Explicitly this means that for any Zariski open embedding \( j : U \hookrightarrow X \), for which \( \text{Sing}(X) \subset U \), the functor \( j^* : D_{Sg}(X) \to D_{Sg}(U) \) is an equivalence of triangulated categories [8].

On the other hand, two analytically isomorphic singularities can have non-equivalent (but closely related) triangulated categories of singularities. The main reason why such categories can fail to be equivalent is that a triangulated category of singularities is not necessarily idempotent complete.

However, for any triangulated category \( D \) we can consider its so called idempotent completion (or Karoubian envelope) \( \overline{D} \). This is a category that consists of all kernels of all projectors. It has a natural structure of a triangulated category and the canonical functor \( D \to \overline{D} \) is an exact full embedding [1]. In [11] we show that for any two schemes \( X \) and \( X' \) satisfying (ELF), whose formal completions \( \overline{X} \) and \( \overline{X}' \) along the singularities are isomorphic, we have that the idempotent completions of the triangulated categories of singularities \( D_{Sg}(\overline{X}) \) and \( D_{Sg}(\overline{X}') \) are equivalent.

Triangulated categories of singularities have additional good properties in the case of Gorenstein scheme. If \( X \) is Gorenstein and has finite dimension, then \( \mathcal{O}_X \) is a dualizing complex for \( X \), in the sense that it has a finite injective dimension as a quasi-coherent sheaf and the natural map

\[
\mathcal{F} \to R\mathcal{H}\text{om}(R\mathcal{H}\text{om}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)
\]

is an isomorphism for any coherent sheaf \( \mathcal{F} \). In particular, there is an integer \( n_0 \) such that \( \mathcal{E}\text{xt}^i(\mathcal{F}, \mathcal{O}_X) = 0 \) for each quasi-coherent sheaf \( \mathcal{F} \) and all \( i > n_0 \).

The following statements and their proofs can be found in [8].

\textbf{Lemma-Definition 1.2.} ([8], Lemma 1.19) Let \( X \) be a Gorenstein scheme satisfying (ELF). We say that a coherent sheaf \( \mathcal{F} \) is \textit{Cohen-Macaulay} if the following equivalent conditions hold.

1) The sheaves \( \mathcal{E}\text{xt}^i(\mathcal{F}, \mathcal{O}_X) \) are trivial for all \( i > 0 \).

2) There is a right locally free resolution \( 0 \to \mathcal{F} \to \{Q^0 \to Q^1 \to Q^2 \to \cdots\} \).

\textbf{Proposition 1.3.} ([8], Prop. 1.23) Let \( X \) be a Gorenstein scheme satisfying (ELF). Then any object \( A \in D_{Sg}(X) \) is isomorphic to the image of a Cohen-Macaulay sheaf.

For a Gorenstein scheme \( X \) that satisfy condition (ELF) we can calculate morphisms between objects in the triangulated category of singularities in terms of morphisms between coherent sheaves on \( X \) (see [8], Prop 1.21). In particular, it follows that if the closed subset \( \text{Sing}(X) \) is complete then the spaces of morphisms between any two objects in the triangulated category of singularities \( D_{Sg}(X) \) are finite dimensional.
2. Matrix factorizations for non-affine Landau-Ginzburg models

By a Landau-Ginzburg model we will mean the following data: a scheme $X$ over a field $k$ and a regular function $W$ on $X$ such that the morphism $W : X \to \mathbb{A}^1_k$ is flat. (It is equivalent to say that the map of algebras $k[x] \to \Gamma(O_X)$ is an injection.) In general, the data of a LG model should also include a Kähler form on $X$ but we will ignore this extra piece of information since it is not needed for the definition of B-branes. As it was mentioned before in this paper we assume that the scheme $X$ satisfies the condition (ELF).

**Remark 2.1.** Usually in the definition of a LG-model we ask that $X$ be regular and our final equivalence result holds precisely under this condition. However for all other considerations regularity of $X$ is not necessary. Moreover, it seems that it is very interesting to consider the case of a singular $X$ as well.

With any $k$-point $w_0 \in \mathbb{A}^1$ we can associate a differential $\mathbb{Z}/2\mathbb{Z}$-graded category $DG_{w_0}(X,W)$, an exact category $Pair_{w_0}(X,W)$, and a triangulated category $H^0DG_{w_0}(X,W)$ that is the homotopy category for DG category $DG_{w_0}(X,W)$.

Objects of all these categories are ordered pairs

$$\mathbf{E} := \left( \begin{array}{c} E_1 \\ \begin{array}{c} e_1 \\ e_0 \\ E_0 \end{array} \end{array} \right),$$

where $E_0,E_1$ are locally free sheaves of finite type on $X$ and the compositions $e_0 e_1$ and $e_1 e_0$ are the multiplications by the element $(W - w_0 \cdot 1) \in \Gamma(O_X)$.

Morphisms from $\mathbf{E}$ to $\mathbf{F}$ in the category $DG_{w_0}(W)$ form $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$\mathbb{H} \text{om}(\mathbf{E}, \mathbf{F}) = \bigoplus_{0 \leq i,j \leq 1} \text{Hom}(E_i, F_j)$$

with a natural grading $(i-j) \mod 2$, and with a differential $D$ acting on a homogeneous element $p$ of degree $k$ as

$$Dp = f \cdot p - (-1)^k p \cdot e.$$

The space of morphisms $\text{Hom}(\mathbf{E}, \mathbf{F})$ in the category $Pair_{w_0}(X,W)$ is the space of morphisms in $DG_{w_0}(X,W)$ which are homogeneous of degree 0 and commute with the differential.

The space of morphisms in the category $H^0DG_{w_0}(W)$ is the space of morphisms in $Pair_{w_0}(X,W)$ modulo null-homotopic morphisms, i.e.

$$\text{Hom}_{Pair_{w_0}(X,W)}(\mathbf{E}, \mathbf{F}) = Z^0(\mathbb{H} \text{om}(\mathbf{E}, \mathbf{F})), \quad \text{Hom}_{H^0DG_{w_0}(X,W)}(\mathbf{E}, \mathbf{F}) = H^0(\mathbb{H} \text{om}(\mathbf{E}, \mathbf{F})).$$

Thus, a morphism $p : \mathbf{E} \to \mathbf{F}$ in the category $Pair_{w_0}(X,W)$ is a pair of morphisms $p_1 : E_1 \to F_1$ and $p_0 : E_0 \to F_0$ such that $p_1 e_0 = f_0 p_0$ and $f_1 p_1 = p_0 e_1$. The morphism $p$ is null-homotopic if there are two morphisms $s_0 : E_0 \to F_1$ and $s_1 : E_1 \to F_0$ such that $p_1 = f_0 s_1 + s_0 e_1$ and $p_0 = s_1 e_0 + f_1 s_0$.

It is clear that the category $Pair_{w_0}(X,W)$ is an exact category with respect to componentwise monomorphisms and epimorphisms.

The category $H^0DG_{w_0}(X,W)$ can be endowed with a natural structure of a triangulated category. To specify it we have to define a translation functor [1] and a class of exact triangles.
The translation functor can be defined as a functor that takes an object $E$ to the object $E[1] = \left( E_0 \xrightarrow{-e_0} E_1 \xleftarrow{-e_1} \right)$, i.e. it changes the order of the modules and the signs of the maps, and takes a morphism $p = (p_0, p_1)$ to the morphism $p[1] = (p_1, p_0)$. We see that the functor [2] is the identity functor.

For any morphism $p : E \to F$ from the category Pair$_{w_0}(X, W)$ we define a mapping cone $\text{Cone}(p)$ as an object

$$\text{Cone}(p) = \left( F_1 \oplus E_0 \xrightarrow{c_1} F_0 \oplus E_1 \right)$$

such that

$$c_0 = \begin{pmatrix} f_0 & p_1 \\ 0 & -e_1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} f_1 & p_0 \\ 0 & -e_1 \end{pmatrix}.$$

There are maps $q : F \to \text{Cone}(p)$, $g = (\text{id}, 0)$ and $r : \text{Cone}(p) \to E[1]$, $r = (0, -\text{id})$.

The standard triangles in the category $H^0 DG_{w_0}(X, W)$ are defined to be the triangles of the form $E \xrightarrow{p} F \xrightarrow{q} \text{Cone}(p) \xrightarrow{r} E[1]$ for some $p \in \text{Pair}_{w_0}(X, W)$.

**Definition 2.2.** A triangle $E \xrightarrow{p} F \xrightarrow{q} \text{Cone}(p) \xrightarrow{r} E[1]$ in $H^0 DG_{w_0}(X, W)$ is called an exact triangle if it is isomorphic to a standard triangle.

**Proposition 2.3.** The category $H^0 DG_{w_0}(X, W)$ endowed with the translation functor [1] and the above class of exact triangles becomes a triangulated category.

**Proof.** The proof is straightforward. □

We define a triangulated category $\text{MF}_{w_0}(X, W)$ of matrix factorizations on $(X, W)$ as a Verdier quotient of $H^0 DG(X, W)$ by a triangulated subcategory of “acyclic” objects. This quotient will also be called a triangulated category of D-branes of type B in the LG-model $(X, W)$ over $w_0$.

More precisely, for any complex of objects of the category $\text{Pair}_{w_0}(X, W)$

$$E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \ldots \xrightarrow{d^{i-1}} E^j$$

we can consider a totalization $\mathbb{T}$ of this complex. It is a pair with

$$\mathbb{T}_1 = \bigoplus_{k+m \equiv 0 \mod 2} E^m_k, \quad \mathbb{T}_0 = \bigoplus_{k+m \equiv 1 \mod 2} E^m_k,$$

and with $t_l = d^m_k + (-1)^m l e_k$ on the component $E^m_k$, where $l = (k + m) \mod 2$.

Denote by $Ac_{w_0}(X, W)$ the minimal full triangulated subcategory that contains totalizations of all acyclic complexes in the exact category $\text{Pair}_{w_0}(X, W)$. It is easy to see that $Ac_{w_0}(X, W)$ coincides with the minimal full triangulated subcategory containing totalizations of all short exact sequences in $\text{Pair}_{w_0}(X, W)$.

**Definition 2.4.** We define the triangulated category of matrix factorizations $\text{MF}_{w_0}(X, W)$ on $X$ with a superpotential $W$ as the Verdier quotient $H^0 DG_{w_0}(X, W)/Ac_{w_0}(X, W)$.
In particular, this definition implies that any short exact sequence in Pair\(_{w_0}(X, W)\) becomes an exact triangle in MF\(_{w_0}(X, W)\).

**Remark 2.5.** Triangulated categories of this type appeared in [12] under name “derived categories of the second kind”. This construction of matrix factorizations is also discussed in [5].

**Remark 2.6.** Note that the category of matrix factorization MF\(_{w_0}(X, W)\) defined above has a natural differential graded enhancement obtained as a DG quotient of the DG category DG\(_{w_0}(X, W)\) by the corresponding DG subcategory of all totalizations of acyclic objects. The existence of this DG quotient follows from general results of Bernhard Keller and Vladimir Drinfeld.

### 3. Matrix factorizations and categories of singularities

In this section we discuss a connection between the category of matrix factorizations MF\(_{w_0}(X, W)\) introduced above and the triangulated category of singularities D\(_{Sg}(X_{w_0})\) of the fiber \(X_{w_0}\) of the map \(W\) over a k-point \(w_0\). Without loss of generality we can take \(w_0 = 0\).

Denote by \(X_0\) the fiber of \(W\): \(X \to \mathbb{A}^1\) over the point 0 and by \(i : X_0 \hookrightarrow X\) the closed embedding.

With any pair \(E\) we can associate a short exact sequence

\[
0 \longrightarrow E_1 \xrightarrow{e_1} E_0 \longrightarrow \text{Coker } e_1 \longrightarrow 0
\]

of coherent sheaves on \(X\).

We can attach to an object \(E\) the sheaf \(\text{Coker } e_1\). This is a sheaf on \(X\). But the multiplication with \(W\) annihilates it. Hence, we can consider \(\text{Coker } e_1\) as a sheaf on \(X_0\), i.e. there is a sheaf \(E\) on \(X_0\) such that \(\text{Coker } e_1 \cong i_*E\). Any morphism \(p : E \to F\) in Pair\(_0(X, W)\) gives a morphism between cokernels. In this way we get a functor \(\text{Cok} : \text{Pair}_0(X, W) \to \text{coh}(X_0)\).

**Lemma 3.1.** Let \(E\) and \(F\) be two pairs on \(X\). Then we have

a) If \(p : E \to F\) is a morphism which induces an isomorphism between the sheaves \(\text{Coker } f_1\) and \(\text{Coker } e_1\), then \(p\) becomes an isomorphism in MF\(_0(X, W)\).

b) For any morphism \(a : \text{Coker } f_1 \to \text{Coker } e_1\) there is a pair \(E'\) together with morphisms \(p : E' \to E\) and \(s : E' \to F\) such that \(\text{Cok}(s)\) is an isomorphism and \(a = \text{Cok}(p)\text{Cok}(s)^{-1}\). Moreover, if \(a\) is a surjection, then \(E'\) can be chosen so that \(p\) is a surjection as well.

c) If the sheaves \(\text{Coker } f_1\) and \(\text{Coker } e_1\) are isomorphic, then \(E\) and \(F\) are isomorphic in MF\(_0(X, W)\).

**Proof.** a) Let us take a locally free covering \(G \to E_1\) and consider a trivial pair \(G\) with \(G_1 = G_0 = G\) and \(g_1 = \text{id}, g_0 = W\). We have a map of pairs \(G \to E\) that induces a natural map \(p' : E' \oplus G \to E\). If \(p : E \to F\) gives an isomorphism between sheaves \(\text{Coker } f_1\) and \(\text{Coker } e_1\) then \(p'\) is a surjection and the kernel of \(p'\) is a 0-homotopic pair. Hence, \(p'\) and \(p\) are isomorphisms in MF\(_0(X, W)\).

b) A morphism \(a : \text{Coker } f_1 \to \text{Coker } e_1\) induces a morphism \(a' : F_0 \to \text{Coker } e_1\). Consider a pull back of \(a'\) along the surjection \(E_0 \to \text{Coker } e_1\) and denote by \(U\) a Cartesian product

...
Since \( X \) has enough locally free sheaves we can find a locally free sheaf \( F'_0 \) with a surjection on \( U \). It induces a map \( p_0 : F'_0 \to E_0 \) and a surjection \( s_0 : F'_0 \to F_0 \).

The surjection \( s_0 \) gives a map \( F'_0 \to \text{Coker } f_1 \). Denote by \( f'_1 : F'_1 \to F'_0 \) the kernel of the latter map. The surjection \( s_0 \) defines a surjection \( s_1 : F'_1 \to F_1 \) and the sheaf \( F'_1 \) is locally free as extension of \( F_1 \) with the kernel of \( s_0 \). Since the multiplication by \( W \) acts trivially on \( \text{Coker } f_1 \) there is a map \( f'_0 : F'_0 \to F'_1 \) such that the composition \( f'_1 f'_0 \) coincides with the multiplication by \( W \) on \( F'_0 \). It is easy to see that the composition \( f'_0 f'_1 \) is equal to multiplication with \( W \) on \( F'_1 \) as well. Thus, we obtain a pair \( F' \) and a surjection \( s : F' \to F \) for which \( \text{Cok}(s) \) is an isomorphism.

On the other hand, we have a map \( p_0 : F'_0 \to E_0 \) that can be included in the following commutative square

\[
\begin{array}{c}
F'_0 & \longrightarrow & \text{Coker } f_1 \\
p_0 & \downarrow & a \\
E_0 & \longrightarrow & \text{Coker } e_1
\end{array}
\]

This commutative diagram can be extended (uniquely) to a map of the pairs \( p : F' \to E \). If now \( a \) was a surjection, then \( p_0 : F'_0 \to E_0 \) is also surjection. As in a) above take a locally free covering \( G \to E_1 \) and consider a trivial pair \( G \) with \( G_1 = G_0 = G \) and \( g_1 = \text{id}, g_0 = W \). We have a map of pairs \( G \to E \). Changing \( F' \) to the direct sum \( F' \oplus G \) we obtain a surjection on \( E \). Thus, b) is proved. The statements b) and a) imply c) directly.

One of the main benefits of this lemma is that it allows us to replace a given pair \( E \) by pairs \( E' \) which are isomorphic to \( E \) in the category \( \text{MF}_0(X,W) \) but are better suited to our purposes. Indeed, consider a pair \( E \) and the coherent sheaf \( E \in \text{coh}(X_0) \) that is the cokernel of the map \( e_1 \). Since \( X \) has enough locally free sheaves we can find a locally free sheaf \( L \) on \( X \) with a surjection \( i^*L \to E \). By Lemma 3.1 b) there is a pair \( F \) such that \( \text{Coker } f_1 \cong i_*i^*L \) and a surjection \( p : L \to F \) that induces the surjection \( i^*L \to E \). Taking the kernel of \( p \) we obtain a pair \( F \) that is isomorphic to \( F[1] \) in \( \text{MF}_0(X,W) \). In particular it follows that \( L \) is the 0-object in \( \text{MF}_0(X,W) \) and that the sequence \( F \to L \to E \) gives an exact triangle in \( \text{MF}_0(X,W) \). Repeating this procedure for \( E \) we get \( E' \cong F'[1] \cong E[2] \cong E \).

With these preliminaries we are ready to show that the functor \( \text{Cok} \) induces exact functors from the triangulated categories \( H^0D\text{G}_0(X,W) \) and \( \text{MF}_0(X,W) \) to the triangulated category of singularities \( D_{Sg}(X_0) \) of the fiber \( X_0 \).

**Proposition 3.2.** The functor \( \text{Cok} : \text{Pair}_0(X,W) \to \text{coh}(X_0) \) induces exact functors

\( \Pi : H^0D\text{G}_0(X,W) \to D_{Sg}(X_0) \) and \( \Sigma : \text{MF}_0(X,W) \to D_{Sg}(X_0) \) between triangulated categories.

**Proof.** We have a functor \( \text{Pair}_0(X,W) \to D_{Sg}(X_0) \) which is the composition of \( \text{Cok} \) and the natural functor from \( \text{coh}(X_0) \) to \( D_{Sg}(X_0) \). To prove the existence of the functor \( \Pi \) we must
show that any morphism $p = (p_1, p_0) : E \to F$ that is homotopic to 0 goes to the 0-morphism in $D_{Sg}(X_0)$. Fix a homotopy $(s_1, s_0)$, where $s_1 : E_1 \to F_0$ and $s_0 : E_0 \to F_1$. Consider the following decomposition of $p$

This gives the decomposition of $\text{Cok}(p)$ through a locally free object $i^*F_0$. Hence, $\text{Cok}(p) = 0$ in the category $D_{Sg}(X_0)$ and we obtain a functor $\Pi$ from $H^0DG_0(X, W)$ to $D_{Sg}(X_0)$. It is easy to check that the functor $\Pi$ is exact.

To see that the functor $\Pi$ induces a functor from the quotient category $MF_0(X, W)$ to $D_{Sg}(X_0)$, we should to check that for any object $T \in Ac_0(X, W)$ the cokernel of $t_1$ is locally free on $X_0$. The subcategory $Ac_0(X, W)$ is generated by totalizations of short exact sequences in $\text{Pair}_0(X, W)$.

Let us consider a short exact sequence

\[(3) \quad 0 \to G \xrightarrow{g} E \xrightarrow{p} F \to 0\]

and denote by $T$ its totalization as in (1). We also consider a complex

\[
\begin{array}{c}
0 \to G_0 \xrightarrow{(g_0, 0)} E_0 \oplus G_1 \xrightarrow{(p_0 - e_0, q_1)} F_0 \oplus E_1 \xrightarrow{(f_0, p_1)} F_1 \to 0
\end{array}
\]

that is a totalization of the complex (3) in the category of complexes of coherent sheaves on $X$. This complex is acyclic. We denote by $U$ the image of the middle map and by $\psi = (\psi_0, \psi_1) : E_0 \oplus G_1 \to U$ and $\phi : U \to F_0 \oplus E_1$ the canonical surjection and injection. We have the following commutative diagram

in which all columns and all rows are short exact sequences on $X$. Hence, the object $T$ is an extension of $i^*F_0$ and $\mathcal{H}$. On the other hand, we also have the commutative diagram
which shows that $\mathcal{H} \cong i^*G_0$ is locally free on $X_0$. Thus, the object $T$, which is the cokernel of the map $t_1$, is an extension of two locally free sheaves $i^*F_1$ and $i^*G_0$ on $X_0$. Therefore, it is locally free as well. \qed

**Proposition 3.3.** If $\Sigma(E) \cong 0$, then $E \cong 0$ in the category $\text{MF}_0(X,W)$.

**Proof.** Let $E = \Sigma(E)$. If $E \cong 0$ in $D_{Sg}(X_0)$ then $E$ is a perfect complex. On the other hand, we have 2-periodic resolution of the form

$\cdots \to i^*E_0 \to i^*E_1 \to \cdots \to i^*G_0 \to 0$

Since $E$ is perfect then the kernel of the first map of the brutal truncation $\sigma \geq 2n + 1$ of this resolution for sufficient negative $n$ is a locally free sheaf. On the other hand it is isomorphic to $E$. Hence, $E$ is locally free if it is perfect.

Note that if $E \cong i^*F$, then by Lemma 3.1 the object $E$ is isomorphic in the category $\text{MF}_0(X,W)$ to the object $F$ with $F_1 \cong F_0 \cong F$ and $f_1 = W, f_0 = \text{id}$. This implies that $E \cong 0$ in the category $\text{MF}_0(X,W)$, because $F$ is 0-homotopic.

If now a locally free sheaf $E$ on $X_0$ has a right resolution of the form

$(4) \quad 0 \to E \to i^*G_1^{-k} \to i^*G_1^{-k+1} \to \cdots \to i^*G_0 \to 0$

then by Lemma 3.1 b) we can construct an exact sequence of pairs of the form

$(5) \quad 0 \to E' \to L_k^{-k} \to L_k^{-k+1} \to \cdots \to L_0 \to 0$

such that the functor Cok sends the sequence $(5)$ to the sequence $(4)$. As we noted above, all pairs $L^{-i}$ are isomorphic to 0 in $\text{MF}_0(X,W)$. Hence, the object $E' \cong 0$ in $\text{MF}_0(X,W)$ too. And by Lemma 3.1 c) the object $E$ is isomorphic to 0 in the category $\text{MF}_0(X,W)$ as well.

Finally, consider a general case. Let $E = \Sigma(E)$ be a locally free sheaf on $X_0$. Let us consider the following complex

$(6) \quad \begin{array}{c} i^*E_1 \to \cdots \to i^*E_0 \to i^*E_0 \end{array}$

which is concentrated in degrees $(-2m + 1)$ to 0 for $2m > \dim X_0$. It has two nontrivial cohomology $H^{-2m+1}$ and $H^0$, both of which are isomorphic to $E$. Since $E$ is locally free and $2m > \dim X_0$ this complex in $D^b(\text{coh } X_0)$ is isomorphic to the direct sum of its cohomology. Hence, there is a map from $E$ to the complex $(6)$ in $D^b(\text{coh } X_0)$. Therefore, we can find a locally free resolution $P$ of $E$ of the form

$(7) \quad \begin{array}{c} \cdots \to i^*P^{-k} \to \cdots \to i^*P^0 \end{array} \to E$
and a map from this resolution to the complex \([5]\) that acts identically on the 0-th cohomology. Take the brutal truncation \(\sigma^{\geq -2m+2}(P)\) of the resolution \([7]\) and consider the corresponding composition map from the complex \(\sigma^{\geq -2m+2}(P)\) to \([3]\)

\[
\begin{array}{ccccccc}
0 & \rightarrow & i^*P^{-2m+2} & \rightarrow & \cdots & \rightarrow & i^*P^0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & i^*E_1 & \rightarrow & i^*E_0 & \rightarrow & \cdots & \rightarrow & i_*E_0 & \rightarrow & 0
\end{array}
\]

Since this map induces an isomorphism on the 0-th cohomology, the cone of this map

\[
(8) \quad i^*E_1 \oplus i^*P^{-2m+2} \rightarrow \cdots \rightarrow i^*E_1 \oplus i^*P^0 \rightarrow i^*E_0
\]

has only one nontrivial cohomology \(H^{-2m+1}\). Denote this cohomology by \(T\). It is easy to see that \(T\) is isomorphic to the direct sum of \(E\) and \(H^{-2m+2}(\sigma^{\geq -2m+2}P)\). This follows from the fact that in the derived category \(D^b(\text{coh}X_0)\) the complex \([8]\) is a cone of the map \(\sigma^{\geq -2m+2}(P) \rightarrow E \oplus E[2m-1]\) which factors through \(E\).

Thus, the locally free sheaf \(E\) is a direct summand of the locally free sheaf \(T\) that has a right resolution of the form \([8]\). It was shown above that any pair \(\underline{\tau}\) for which \(\text{Cok}(\underline{\tau}) \cong \tau\), is isomorphic to 0 in the category \(\text{MF}_0(X,W)\). Now consider the maps \(E \xrightarrow{i} T \xrightarrow{\pi} E\), whose composition is the identity. By Lemma \([3.1] b\) there is a pair \(\underline{\tau}\) and a surjection \(p : \underline{\tau} \rightarrow \underline{E}\) such that \(\text{Cok}(p) = \pi\). Applying again Lemma \([3.1] b\) we find a pair \(\underline{E}'\) and a map \(q : \underline{E}' \rightarrow \underline{\tau}\) such that \(\text{Cok}(q) = i\). This implies that the pairs \(\underline{E}'\) and \(\underline{E}\) are isomorphic in \(\text{MF}_0(X,W)\) and are isomorphic to a direct summand of the object \(\underline{\tau}\). Since \(\underline{\tau}\) is isomorphic to the 0-object we have that \(\underline{E} \cong 0\) in the category \(\text{MF}_0(X,W)\) as well.

**Theorem 3.4.** Let \(X\) be a scheme that satisfies condition (ELF). Then the natural functor \(\Sigma : \text{MF}_0(X,W) \rightarrow D_{Sg}(X_0)\) is fully faithful.

**Proof.** First we will show that \(\Sigma\) is full. Let \(\mathcal{F}\) and \(\mathcal{E}\) be sheaves on \(X_0\) that are cokernels of \(f_1\) and \(e_1\) for two pairs \(\underline{F}\) and \(\underline{E}\), respectively. By the definition of a localization any morphism \(u\) from \(\mathcal{F}\) to \(\mathcal{E}\) in \(D_{Sg}(X_0)\) can be represented by a pair of morphisms in \(D^b(\text{coh}(X_0))\) of the form

\[
(9) \quad \mathcal{F} \xleftarrow{s} A \xrightarrow{a} \mathcal{E}
\]

such that the cone \(C(s)\) is a perfect complex. The two cohomology sheaves of the complex \(i^*E_1 \rightarrow i^*E_0\) are isomorphic to \(E\) and we obtain a canonical map \(\phi : \mathcal{E} \rightarrow \mathcal{E}[2]\) in \(D^b(\text{coh}(X))\) that becomes an isomorphism in the quotient category \(D_{Sg}(X_0)\). The morphism \(u\) induces a morphism \(u_k\) from \(\mathcal{F}\) to \(\mathcal{E}[2k]\) in \(D_{Sg}(X_0)\) that is represented by the roof

\[
\mathcal{F} \xleftarrow{s} A \xrightarrow{a_k} \mathcal{E}[2k]
\]

with \(a_k = \phi^k a\). Since the cone of \(s\) is perfect, it is isomorphic to a bounded complex of locally free sheaves. This means that for a sufficiently large \(k\) there is no nontrivial map from \(C(s)\) and \(C(s)[-1]\) to \(\mathcal{E}[2k]\). Hence, the map \(u_k\) in \(D_{Sg}(X_0)\) is represented by a map \(\bar{u}\) from \(\mathcal{F}\) to \(\mathcal{E}[2k]\) in \(D^b(\text{coh}(X_0))\).
Now we take a surjective morphism \( \pi : i^*G^0 \to F \) that erases the map \( \tilde{u} : F \to E[2k] \). Denote by \( \mathcal{H} \) the kernel of \( \pi \). The map \( \tilde{u} \) induces a map \( \tilde{u} : \mathcal{H} \to \mathcal{E}[2k-1] \). By the same argument we find a surjective morphism \( i^*G^{-1} \to \mathcal{H} \) that erases the map \( \tilde{u} : \mathcal{H} \to \mathcal{E}[2k-1] \).

Repeating the above procedure we construct an acyclic complex of the form

\[
0 \to F' \to i^*G^{-2k+1} \to \cdots \to i^*G^0 \to F \to 0
\]

and a map \( u' : F' \to \mathcal{E} \) such that \( \tilde{u} \) is the composition of the canonical map \( F \to F'[2k] \) and \( u'[2k] : F'[2k] \to \mathcal{E}[2k] \).

By Lemma 3.1 b) we can construct an exact sequence of pairs of the form

\[
0 \to \mathcal{E}' \to \mathbb{L}^{-2k+1} \to \cdots \to \mathbb{L}^0 \to \mathbb{E} \to 0
\]

such that the application of functor \( \text{Cok} \) to the sequence (11) gives the sequence (10). All pairs \( \mathbb{L}^{-i} \) are isomorphic to 0 in \( \text{MF}_0(X,W) \). Hence, the object \( \mathcal{E}' \cong \mathbb{E} \) in \( \text{MF}_0(X,W) \). Moreover, there is an equality \( u' \cdot \Sigma(\alpha) = u \) of morphisms in \( \text{D}_{Sg}(X_0) \), where \( \alpha : \mathcal{E} \xrightarrow{\sim} \mathcal{E}' \) is the isomorphism in the category \( \text{MF}_0(X,W) \).

Thus, it is enough to show that the morphism \( u' : F' \to \mathcal{E} \) is equal to \( \Sigma(q) \) for some map \( \beta : \mathcal{E}' \to \mathcal{E} \) in the category \( \text{MF}_0(X,W) \). By Lemma 3.1 b) there is a pair \( \mathcal{E}'' \) and morphisms of pairs \( p : \mathcal{E}'' \to \mathcal{E} \) and \( s : \mathcal{E}'' \to \mathcal{E}' \) such that \( \text{Cok}(s) \) is an isomorphism and \( u' = \text{Cok}(p) \cdot \text{Cok}(s)^{-1} \).

Again by Lemma 3.1 a) the morphism \( s \) becomes an isomorphism in the category \( \text{MF}_0(X,W) \). Hence, for \( \beta = ps^{-1} \) in \( \text{MF}_0(X,W) \) we obtain that \( \Sigma(\beta) = u' \). Therefore, the functor \( \Sigma \) is full.

Now we prove that \( \Sigma \) is faithful. It is a standard statement asserting that a full exact functor between triangulated categories with trivial kernel is also faithful. Indeed, let \( p : \mathcal{E} \to \mathcal{E} \) be a morphism for which \( \Sigma(p) = 0 \). Complete \( p \) to an exact triangle

\[
\mathcal{E} \xrightarrow{p} \mathcal{E} \xrightarrow{q} G \xrightarrow{id} \mathcal{E}[1].
\]

Then the identity map of \( \Sigma(\mathcal{E}) \) factors through the map \( \Sigma(\mathcal{E}) \xrightarrow{\Sigma(q)} \Sigma(G) \). Since \( \Sigma \) is full, there is a map \( s : \mathcal{E} \to \mathcal{E} \) factoring through \( q : \mathcal{E} \to \mathcal{G} \) such that \( \Sigma(s) = \text{id} \). Hence, the cone \( C(s) \) of the map \( s \) goes to zero under the functor \( \Sigma \). By Proposition 3.3 the object \( C(s) \) is the zero object too, so \( s \) is an isomorphism. Thus, \( q : \mathcal{E} \to \mathcal{G} \) is a split monomorphism and \( p = 0 \). \( \square \)

In general the functor \( \Sigma \) is not necessarily an equivalence and it is very interesting to understand a difference between \( \text{MF}_0(X,W) \) and \( \text{D}_{Sg}(X_0) \) for singular LG-models. However, for regular schemes we have an equivalence.

**Theorem 3.5.** Let \( X \) be a scheme that satisfies condition (ELF). If \( X \) is regular then the functor \( \Sigma : \text{MF}_0(X,W) \to \text{D}_{Sg}(X_0) \) is an equivalence of triangulated categories.

**Proof.** As we showed above the functor \( \Sigma \) is fully faithful. To complete the proof that \( \Sigma \) is an equivalence we need to check that every object \( A \in \text{D}_{Sg}(X_0) \) is isomorphic to \( \Sigma(\mathcal{E}) \) for some pair \( \mathcal{E} \). By Proposition 1.3 since \( X_0 \) is Gorenstein any object \( A \in \text{D}_{Sg}(X_0) \) is isomorphic to the image of a coherent sheaf \( \mathcal{E} \) such that \( \mathcal{E} \text{ext}^i_{X_0}(\mathcal{E},\mathcal{O}_{X_0}) = 0 \) for all \( i > 0 \). Consider an epimorphism \( \mathcal{E}_0 \to i_*\mathcal{E} \) of sheaves on \( X \) with locally free \( \mathcal{E}_0 \). Denote by \( e_1 : \mathcal{E}_1 \to \mathcal{E}_0 \) the kernel of this map. Since the multiplication with \( W \) gives the zero map on \( \mathcal{E} \), there is a map \( e_0 : \mathcal{E}_0 \to \mathcal{E}_1 \) such
that $e_0e_1 = W$ and $e_1e_0 = W$. We get a pair

$\mathbb{E} := \begin{pmatrix} e_1 \\ e_0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_0 \end{pmatrix}$

and we need only to check that $\mathbb{E}_1$ is locally free. It follows from the fact that for any closed point $t : x \hookrightarrow X$ we have

$\text{Ext}^i_X(\mathbb{E}_1, t_*\mathcal{O}_x) = 0$

for all $i > 0$. To show it we note that by Lemma 1.2 the sheaf $\mathcal{E}$ has a right locally free resolution on $X_0$. For any local free sheaf $\mathcal{P}$ on $X_0$ we have $\text{Ext}^i_X(i_*\mathcal{P}, t_*\mathcal{O}_x) = 0$ for $i > 1$. Since $X$ is regular, the abelian category of coherent sheaves on $X$ has finite cohomological dimension. Therefore, we obtain $\text{Ext}^i_X(i_*\mathcal{E}, t_*\mathcal{O}_x) = 0$ for $i > 1$. This implies (12) and the theorem. □

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