Special precovering classes in comma categories

Jiangsheng Hu¹ & Haiyan Zhu²,*

¹School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, China; ²College of Science, Zhejiang University of Technology, Hangzhou 310023, China

Email: jiangshenghu@jsut.edu.cn, hyzhu@zjut.edu.cn

Received May 1, 2020; accepted September 28, 2020; published online May 7, 2021

Abstract Let T be a right exact functor from an abelian category B into another abelian category A. Then there exists a functor p from the product category A × B to the comma category (T ↓ A). In this paper, we study the property of the extension closure of some classes of objects in (T ↓ A), the exactness of the functor p and the detailed description of orthogonal classes of a given class p(X, Y) in (T ↓ A). Moreover, we characterize when special precovering classes in abelian categories A and B can induce special precovering classes in (T ↓ A). As an application, we prove that under suitable conditions, the class of Gorenstein projective left Λ-modules over a triangular matrix ring Λ = (RM0S) is special precovering if and only if both the classes of Gorenstein projective left R-modules and left S-modules are special precovering. Consequently, we produce a large variety of examples of rings such that the class of Gorenstein projective modules is special precovering over them.

Keywords abelian category, comma category, special precovering class, cotorsion pair, Gorenstein projective object

MSC(2020) 18A25, 16E30, 18G25

1 Introduction

Recall that for any abelian categories A and B and any right exact functor T : B → A, there exists an abelian category, denoted by (T ↓ A), consisting of all the triplets (B, ϕ), where ϕ : T(B) → A is a morphism in A. We note that this new abelian category is called a comma category in [11, 21]. The detailed definitions can be found in Definition 2.1 below. The examples of comma categories include but are not limited to: the category of modules or complexes over a triangular matrix ring, the morphism category of an abelian category and so on (see Example 2.2). It should be noted that comma categories not only give rise to adjoint functors for constructing recollements in abelian categories and triangulated categories (see [4, 24]) and establishing various derived equivalences between triangular matrix algebras (see [17]), but also are used in the study of Auslander-Reiten quivers and tilting modules (see [21]). We refer to a lecture due to Fossum et al. [11] for more details.

Originating from the concept of injective envelopes, the approximation theory has attracted increasing interest and hence obtained considerable development in the context of module categories or abelian

*Corresponding author
categories. Independent research by Auslander et al. [1] in the finite-dimensional case, and by Enochs and Jenda [6] and Xu [31] for arbitrary modules, created a general theory of left and right approximations—or preenvelopes and precovers—of modules. The notions of a preenvelope and a precover, tied up by a homological notion of a complete cotorsion pair observed by Salce [28] in the 1970s, are dual in the category theoretic sense. The point is that, though there is no duality between the categories of all the modules, complete cotorsion pairs make it possible to produce special preenvelopes once we know special precovers exist and vice versa. Considerable energy has been used for proving that concrete classes are special precovering or covering under suitable conditions. The examples include the classes of modules which are flat, Gorenstein projective and Gorenstein flat. A number of results can be found in [2,3,7–9,16,23,29].

The main objective of this paper is to study special precovering classes in the comma category \((T \downarrow \mathcal{A})\). More precisely, we characterize when special precovering classes in abelian categories \(\mathcal{A}\) and \(\mathcal{B}\) can induce special precovering classes in \((T \downarrow \mathcal{A})\).

When one deals with the above problem, two technical obstacles occur. The first one is that one has to choose an appropriate special precovering class in \((T \downarrow \mathcal{A})\) from those of \(\mathcal{A}\) and \(\mathcal{B}\). To overcome this obstacle, we introduce a functor \(p\) from the product category \(\mathcal{A} \times \mathcal{B}\) into the comma category \((T \downarrow \mathcal{A})\) and give a detailed description of orthogonal classes of a given class induced by \(p\), and then establish certain relations connecting orthogonal classes of a given subclass of \((T \downarrow \mathcal{A})\) induced by \(p\) with some corresponding classes in \(\mathcal{A}\) and \(\mathcal{B}\). The next technical problem we encounter is that special precovering classes are not closed under direct summands in general. For example, the class of free \(R\)-modules is special precovering over any ring \(R\) but it is not closed under direct summands. So a crucial ingredient for constructing complete cotorsion pairs from special precovering classes used in [12, Lemma 2.2.6] is missing. To circumvent this problem here, we first replace the class \(\mathcal{X}\) with the class of direct summands of objects in \(\mathcal{X}\), and then demonstrate that this kind of replacement can preserve the property of special precovering under certain conditions.

To state our main result more precisely, let us first introduce some definitions.

Throughout this paper, \(\mathcal{A}\) and \(\mathcal{B}\) are abelian categories. For any ring \(R\), \(\text{Mod}R (\text{mod}R)\) is the class of (finitely generated) left \(R\)-modules and \(\text{Ch}(R)\) is the class of complexes of left \(R\)-modules. For unexplained ones, we refer the reader to [6,11,12].

Let \(\mathcal{X}\) be a subclass of \(\mathcal{A}\). For convenience, we set

\[
\mathcal{X}^+ := \{M : \text{Ext}^1_{\mathcal{A}}(X, M) = 0 \text{ for every } X \in \mathcal{X}\}, \quad \mathcal{X} := \{M : \text{Ext}^1_{\mathcal{A}}(M, X) = 0 \text{ for every } X \in \mathcal{X}\}.
\]

Recall that a morphism \(f : G \rightarrow M\) is called a special \(\mathcal{X}\)-precover of an object \(M\) if \(f\) is surjective, \(G \in \mathcal{X}\) and ker\(f \in \mathcal{X}^+\). Dually, a morphism \(g : N \rightarrow H\) is called a special \(\mathcal{X}\)-preenvelope of an object \(N\) if \(g\) is injective, \(H \in \mathcal{X}\) and coker\(g \in \mathcal{X}^+\). The class \(\mathcal{X}\) is called special precovering (resp. special preenveloping) in \(\mathcal{A}\) if every object has a special \(\mathcal{X}\)-precover (resp. special \(\mathcal{X}\)-preenvelope).

Let \(\mathcal{Y}\) be a subclass of \(\mathcal{B}\). The functor \(T : \mathcal{B} \rightarrow \mathcal{A}\) is called \(\mathcal{Y}\)-exact if \(T\) preserves the exactness of the exact sequence \(0 \rightarrow B \rightarrow B' \rightarrow Y \rightarrow 0\) in \(\mathcal{B}\) with \(Y \in \mathcal{Y}\).

It is well known that the product category \(\mathcal{A} \times \mathcal{B}\) is also an abelian category whenever both \(\mathcal{A}\) and \(\mathcal{B}\) are abelian categories. Thus there exists a functor \(p\) from \(\mathcal{A} \times \mathcal{B}\) into \((T \downarrow \mathcal{A})\). The detailed definition can be seen in Definition 2.3 below. It should be noted that the functor \(p\) was used by Enochs et al. [5] in the study of Gorenstein conditions over triangular matrix rings. Let \(\mathcal{X}\) be a subclass of \(\mathcal{A}\) with \(0 \in \mathcal{X}\) and \(\mathcal{Y}\) be a subclass of \(\mathcal{B}\) with \(0 \in \mathcal{Y}\). We set

\[
\langle p(\mathcal{X}, \mathcal{Y}) \rangle := \left\{ \begin{array}{c} A \quad 0 \rightarrow \begin{pmatrix} X' \\ Y' \end{pmatrix} \\ B \end{array} : \begin{pmatrix} X' \\ Y' \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow 0 \text{ is exact with } \begin{pmatrix} X' \\ Y' \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} \in p(\mathcal{X}, \mathcal{Y}) \right\}.
\]

If \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under extensions in \(\mathcal{A}\) and \(\mathcal{B}\), respectively, then \(\langle p(\mathcal{X}, \mathcal{Y}) \rangle\) is the smallest subclass of \((T \downarrow \mathcal{A})\) containing \(p(\mathcal{X}, \mathcal{Y})\) and closed under extensions (see Proposition 2.5). In particular, if we choose \(\mathcal{X} = \text{mod}R\) and \(\mathcal{Y} = \text{mod}S\), then \(\langle p(\mathcal{X}, \mathcal{Y}) \rangle\) defined here is just the monomorphism category \(\mathcal{M}(R, M, S)\) introduced by Xiong et al. [30] (see Corollary 2.8).
Let $\mathcal{A}$ be an abelian category with enough projective objects. Recall that an object $M$ in $\mathcal{A}$ is called *Gorenstein projective* if $M = T^i(P^*)$ for some exact complex $P^*$ of projective objects which remains exact after $\text{Hom}_{\mathcal{A}}(-, P)$ is applied for any projective object $P$. The complex $P^*$ is called a *complete $\mathcal{A}$-projective resolution*. In what follows, we denote by $\mathcal{GP}_\mathcal{A}$ the subcategory of $\mathcal{A}$ consisting of Gorenstein projective objects and by $\mathcal{GP}(R)$ the class of Gorenstein projective left $R$-modules for any ring $R$.

Now, our main result can be stated as follows.

**Theorem 1.1.** Let $\mathcal{A}$ and $\mathcal{B}$ both have enough projective objects and enough injective objects.

1. Assume that $\mathcal{X}$ is a subclass of $\mathcal{A}$ with $0 \in \mathcal{X}$, $\mathcal{Y}$ is a subclass of $\mathcal{B}$ with $0 \in \mathcal{Y}$ and $T : \mathcal{B} \to \mathcal{A}$ is a $\mathcal{Y}$-exact functor. If $\mathcal{X}$ and $\mathcal{Y}$ are special precovering, then $\langle \mathcal{p}(\mathcal{X}, \mathcal{Y}) \rangle$ is also special precovering in $(T \downarrow \mathcal{A})$. Moreover, the converse holds when $T(\mathcal{Y} \cap \mathcal{Y}^\perp) \subseteq \mathcal{X}^\perp$ and $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions.

2. If $T : \mathcal{B} \to \mathcal{A}$ is a compatible functor, then $\mathcal{GP}_\mathcal{A}$ and $\mathcal{GP}_\mathcal{B}$ are special precovering in $\mathcal{A}$ and $\mathcal{B}$, respectively if and only if $\mathcal{GP}_{(T \downarrow \mathcal{A})}$ is special precovering in $(T \downarrow \mathcal{A})$.

We note that the condition $T$ is $\mathcal{Y}$-exact in Theorem 1.1(1) cannot be omitted in general (see Remark 3.10). As a consequence, we refine a main result obtained by Mao [20, Theorem 5.6(1)] by deleting two superfluous assumptions (see Remark 3.8).

The study of the existence of special Gorenstein projective precovers has been the subject of much research in recent years. So far the existence of special Gorenstein projective precovers (of right modules) is known over a right coherent ring for which the projective dimension of any flat left module is finite (see [8]). The examples of such rings include but are not limited to: Gorenstein rings (see [6]), commutative noetherian rings of finite Krull dimension (see [29]), as well as two-sided noetherian rings $R$ such that the injective dimension of $R$ (as a left $R$-module) is finite (see [8]). But for arbitrary rings this is still an open question. Work on this problem can be seen in [3, 8, 9, 16, 23] for example.

As a direct consequence of Theorem 1.1(2) and Proposition 4.4 below, we have the following result which gives more examples of rings (not necessary coherent) such that the class of Gorenstein projective modules is special precovering over them.

**Corollary 1.2.** Let $\Lambda = (R \quad M \quad S)$ be a triangular matrix ring. Assume that $RM$ and $MS$ have finite projective dimension and finite flat dimension, respectively. Then $\mathcal{GP}(\Lambda)$ is special precovering in $\text{Mod}\Lambda$ if and only if $\mathcal{GP}(R)$ and $\mathcal{GP}(S)$ are special precovering in $\text{Mod}R$ and $\text{Mod}S$, respectively.

The rest of this paper is arranged as follows. In Section 2, we study the homological behavior of the functor $\mathcal{p}$ from product categories into comma categories. In Section 3, we first characterize when complete hereditary cotorsion pairs in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce complete hereditary cotorsion pairs in $(T \downarrow \mathcal{A})$ (see Proposition 3.4). This is based on the homological behavior of the functor $\mathcal{p}$ established in Section 2. As a result, we give the proof of Theorem 1.1(1). In Section 4, we first give an explicit description for an arbitrary object in the comma category $(T \downarrow \mathcal{A})$ to be Gorenstein projective (see Proposition 4.7), and then give the proof of Theorem 1.1(2).

In the following sections, we always assume that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, $\mathcal{X}$ is a subclass of $\mathcal{A}$ with $0 \in \mathcal{X}$ and $\mathcal{Y}$ is a subclass of $\mathcal{B}$ with $0 \in \mathcal{Y}$.

## 2 The functor $\mathcal{p}$ and its homological behavior

This section is devoted to preparations for the proofs of our main results in this paper. First, we introduce a functor $\mathcal{p}$ from the product category $\mathcal{A} \times \mathcal{B}$ into the comma category $(T \downarrow \mathcal{A})$, and then discuss the homological behavior of the functor $\mathcal{p}$, including the property of the extension closure of some classes of objects in $(T \downarrow \mathcal{A})$, the exactness of the functor $\mathcal{p}$ and the detailed description of orthogonal classes of a given class $\mathcal{p}(\mathcal{X}, \mathcal{Y})$ in $(T \downarrow \mathcal{A})$.

**Definition 2.1.** (See [11, 21]). Let $T : \mathcal{B} \to \mathcal{A}$ be a right exact functor. Then the comma category $(T \downarrow \mathcal{A})$ is defined as follows:

1. The objects are triplets $(\begin{smallmatrix} A \\ B \\ \varphi \end{smallmatrix})$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\varphi : T(B) \to A$ being a morphism in $\mathcal{A}$. 

Hu J S et al. Sci China Math May 2022 Vol. 65 No. 5 935
Consider the following conditions

Proposition 2.5.

(1) Let $\mathcal{A}$ and $\mathcal{B}$ be two categories, $\mathcal{M}$ be an $\mathcal{R}$-$\mathcal{S}$-bimodule, and $\Lambda = (\Lambda^R \Lambda^S)$ be the triangular matrix ring. If we define $T \cong \mathcal{M} \otimes \mathcal{S} \rightarrow \text{Mod}\mathcal{S} \rightarrow \text{Mod}\mathcal{R}$, then we get that $\text{Mod}\mathcal{A}$ is equivalent to the comma category $(T \downarrow \text{Mod}\mathcal{R})$.

(2) Let $\Lambda = (\Lambda^R \Lambda^S)$ be a triangular matrix ring. If we define $T \cong \mathcal{M} \otimes \mathcal{S} : \text{Ch}(\mathcal{S}) \rightarrow \text{Ch}(\mathcal{R})$, then $\text{Ch}(\mathcal{A})$ is equivalent to the comma category $(T \downarrow \text{Ch}(\mathcal{R}))$.

(3) If $\mathcal{A} = \mathcal{B}$ and $T$ is the identity functor, then the comma category $(T \downarrow \mathcal{A})$ coincides with the morphism category $\text{mor}(\mathcal{A})$ of $\mathcal{A}$.

(4) Let $\mathcal{A} = \text{Mod}\mathcal{R}$ and $\mathcal{B} = \text{Ch}(\mathcal{R})$. If we define $e : \mathcal{B} \rightarrow \mathcal{A}$ via $C^* \mapsto C^0$ for any $C^* \in \mathcal{B}$, then $e$ is an exact functor and we have a comma category $(e \downarrow \mathcal{A})$.

The following functor $p$ was introduced by Mitchell [22, p.29], which is a particular case of the additive left Kan extension functor. Recently, it was used by Enochs et al. [5] in the study of Gorenstein conditions over triangular matrix rings.

Definition 2.3. Let $T : \mathcal{B} \rightarrow \mathcal{A}$ be a right exact functor. Then we have the following functor:

- $p : \mathcal{A} \times \mathcal{B} \rightarrow (T \downarrow \mathcal{A})$ via $p(A, B) = (A \otimes T(B))_0$ and $p(a, b) = (a \otimes T(b))_0$, where $(A, B)$ is an object in $\mathcal{A} \times \mathcal{B}$ and $(a, b)$ is a morphism in $\mathcal{A} \times \mathcal{B}$.

Remark 2.4. Let $A$ be an object in $\mathcal{A}$ and $B$ be an object in $\mathcal{B}$. It is trivial to obtain that $p(A, B) = p(A, 0) \oplus p(0, B)$. Moreover, $p$ preserves projective objects if $\mathcal{A}$ and $\mathcal{B}$ have enough projective objects.

- If we define $q : (T \downarrow \mathcal{A}) \rightarrow \mathcal{A} \times \mathcal{B}$ via $q(A, B) = (A, B)$ and $q(a, b) = (a, b)$ for any object $(A, B)$ in $(T \downarrow \mathcal{A})$ and any morphism $(a, b)$ in $(T \downarrow \mathcal{A})$, then $p$ is a left adjoint of $q$. Hence $p$ is a right exact functor.

Recall that a class $\mathcal{L}$ of objects in an abelian category $\mathcal{B}$ is said to be closed under extensions if whenever $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is exact in $\mathcal{B}$ with $X, Z \in \mathcal{L}$, then $Y \in \mathcal{L}$. For convenience, we set

$$\mathcal{B}_\mathcal{Y} := \left\{ \left( \begin{array}{c} X \\ Y \end{array} \right)_{\mathcal{P}} \mid X \in (T \downarrow \mathcal{A}) : Y \in \mathcal{Y}, \mathcal{P} \text{ is monic and coker} \mathcal{P} \in \mathcal{X} \right\}.$$ 

Recall from the introduction that

$$\langle p(\mathcal{X}, \mathcal{Y}) \rangle := \left\{ \left( \begin{array}{c} A \\ B \end{array} \right) : 0 \rightarrow \left( \begin{array}{c} X \\ Y' \end{array} \right) \rightarrow \left( \begin{array}{c} A \\ B \end{array} \right) \rightarrow \left( \begin{array}{c} X \\ Y \end{array} \right) \rightarrow 0 \text{ is exact with } \left( \begin{array}{c} X' \\ Y'' \end{array} \right), \left( \begin{array}{c} X \\ Y \end{array} \right) \in p(\mathcal{X}, \mathcal{Y}) \right\},$$

and the functor $T : \mathcal{B} \rightarrow \mathcal{A}$ is called $\mathcal{Y}$-exact if $T$ preserves the exactness of the exact sequence $0 \rightarrow B \rightarrow B' \rightarrow Y \rightarrow 0$ in $\mathcal{B}$ with $Y \in \mathcal{Y}$.

Proposition 2.5. Consider the following conditions:

(1) $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions in $\mathcal{A}$ and $\mathcal{B}$, respectively.

(2) $\langle p(\mathcal{X}, \mathcal{Y}) \rangle = \mathcal{B}_\mathcal{Y}$ is the smallest subclass of $(T \downarrow \mathcal{A})$ containing $p(\mathcal{X}, \mathcal{Y})$ and closed under extensions.

(3) $\mathcal{B}_\mathcal{Y}$ is closed under extensions.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The converses hold if $T : \mathcal{B} \rightarrow \mathcal{A}$ is $\mathcal{Y}$-exact.

Proof. (1) $\Rightarrow$ (2). Let $(\alpha_B)_{\mathcal{P}} \in \langle p(\mathcal{X}, \mathcal{Y}) \rangle$. Then there is an exact sequence

$$0 \rightarrow \left( \begin{array}{c} X' \oplus T(Y') \\ Y'' \end{array} \right) \rightarrow \left( \begin{array}{c} A \\ B \end{array} \right) \rightarrow \left( \begin{array}{c} X \oplus T(Y) \\ Y \end{array} \right) \rightarrow 0$$
in \((T \downarrow \mathcal{A})\) with \(X, X' \in \mathcal{X}\) and \(Y, Y' \in \mathcal{Y}\). Thus the sequence \(0 \to Y' \to B \to Y \to 0\) is exact in \(\mathcal{B}\), and hence we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
T(Y') & \longrightarrow & T(B) & \longrightarrow & T(Y) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \uparrow & & \\
0 & \longrightarrow & X' \oplus T(Y') & \longrightarrow & A & \longrightarrow & X \oplus T(Y) \longrightarrow 0.
\end{array}
\]

Clearly, \(\varphi\) is monic and \(B \in \mathcal{Y}\) since \(\mathcal{Y}\) is closed under extensions. Moreover, by the snake lemma, we have an exact sequence \(0 \to X' \to \text{coker}\varphi \to X \to 0\) which implies \(\text{coker}\varphi \in \mathcal{X}\), as desired.

Conversely, assume that \((\frac{X}{Y})\varphi\) is an object in \((T \downarrow \mathcal{A})\) such that \(Y \in \mathcal{Y}\), \(\varphi\) is monic and \(\text{coker}\varphi \in \mathcal{X}\). So \((\frac{X}{Y})\varphi \in \langle p(\mathcal{X}, \mathcal{Y})\rangle\) follows from the exact sequence

\[0 \to \left(\frac{T(Y)}{Y}\right) \to \left(\frac{X}{Y}\right) \to \left(\text{coker}\varphi\right) \to 0\]

in \((T \downarrow \mathcal{A})\).

By the proof above, one can get that \(\langle p(\mathcal{X}, \mathcal{Y})\rangle\) is closed under extensions. So \(\langle p(\mathcal{X}, \mathcal{Y})\rangle\) is the smallest subclass of \((T \downarrow \mathcal{A})\) containing \(p(\mathcal{X}, \mathcal{Y})\) and closed under extensions.

(2) \(\Rightarrow\) (3) \(\Rightarrow\) (1).

(3) \(\Rightarrow\) (1). Let \(0 \to A' \to A \to A'' \to 0\) be an exact sequence in \(\mathcal{A}\) with \(A', A'' \in \mathcal{X}\). Then \(0 \to \left(\frac{A'}{0}\right) \to \left(\frac{A}{0}\right) \to \left(\frac{A''}{0}\right) \to 0\) is an exact sequence in \((T \downarrow \mathcal{A})\) with \(\left(\frac{A'}{0}\right), \left(\frac{A''}{0}\right) \in \mathcal{B}_{\mathcal{X}}\). Thus \(\left(\frac{A}{0}\right) \in \mathcal{B}_{\mathcal{X}}\), and hence we get that \(A \in \mathcal{X}\) by hypothesis. So \(\mathcal{X}\) is closed under extensions, as desired. On the other hand, let \(0 \to B' \to B \to B'' \to 0\) be an exact sequence in \(\mathcal{B}\) with \(B', B'' \in \mathcal{Y}\). Note that \(T\) is \(\mathcal{Y}\)-exact by hypothesis. Then \(0 \to \left(\frac{T(B')}{B'}\right) \to \left(\frac{T(B)}{B}\right) \to \left(\frac{T(B'')}{B''}\right) \to 0\) is an exact sequence in \((T \downarrow \mathcal{A})\). Thus we can get that \(\left(\frac{T(B')}{B'}\right), \left(\frac{T(B'')}{B''}\right) \in \mathcal{B}_{\mathcal{Y}}\) by hypothesis. It follows that \(\left(\frac{T(B)}{B}\right) \in \mathcal{B}_{\mathcal{X}}\), which implies that \(B \in \mathcal{Y}\).

This completes the proof. \(\square\)

**Lemma 2.6.** Let \(\Lambda = (\frac{R}{0}M_S)\) be a triangular matrix ring.

(1) (See [13, Theorem 3.1]) A left \(\Lambda\)-module \((\frac{X}{Y})\varphi\) is a projective left \(\Lambda\)-module if and only if \(Y\) is a projective left \(S\)-module and \(\varphi : M \otimes_S Y \to X\) is an injective \(R\)-morphism with a projective cokernel.

(2) (See [11, Proposition 1.14]) A left \(\Lambda\)-module \((\frac{X}{Y})\varphi\) is a flat left \(\Lambda\)-module if and only if \(Y\) is a flat left \(S\)-module and \(\varphi : M \otimes_S Y \to X\) is an injective \(R\)-morphism with a flat cokernel.

For any ring \(R\), the classes of projective and flat left \(R\)-modules will be denoted by \(\mathcal{P}(R)\) and \(\mathcal{F}(R)\), respectively. By Proposition 2.5 and Lemma 2.6, we have the following corollary.

**Corollary 2.7.** Let \(\Lambda = (\frac{R}{0}M_S)\) be a triangular matrix ring. Then

1. \(\mathcal{F}(\Lambda) \cong \langle p(\mathcal{F}(R), \mathcal{F}(S))\rangle\);
2. \(\mathcal{P}(\Lambda) \cong \langle p(\mathcal{P}(R), \mathcal{P}(S))\rangle\).

Let \(_RM_S\) be a finitely generated \(R\)-\(S\)-bimodule over a pair of Artin algebras \(R\) and \(S\), and \(\Lambda = (\frac{R}{0}M_S)\) be the triangular matrix algebra. Recall from [30] that the monomorphism category \(\mathcal{M}(R, M, S)\) induced by a bimodule \(_RM_S\) is the subcategory of finitely generated left \(\Lambda\)-modules consisting of \((\frac{X}{Y})\varphi\) such that \(\varphi : M \otimes_S Y \to X\) is an injective \(R\)-morphism. When \(_RM_S =_RR_S\), it is the classical submodule category \(\mathcal{F}(R)\) in [25–27].

**Corollary 2.8.** Let \(R\) and \(S\) be Artin algebras, and \(\Lambda = (\frac{R}{0}M_S)\) be a triangular matrix algebra. Then \(\mathcal{M}(R, M, S) \cong \langle p(\text{mod} R, \text{mod} S)\rangle\).

The following proposition characterizes when the functor \(p\) is exact.

**Proposition 2.9.** The following are true for any right exact functor \(T : \mathcal{B} \to \mathcal{A}\):

1. \(T\) is \(\mathcal{Y}\)-exact if and only if two exact sequences \(0 \to A' \to A \to A'' \to 0\) in \(\mathcal{A}\) and \(0 \to B' \to B \to Y \to 0\) in \(\mathcal{B}\) with \(Y \in \mathcal{Y}\) induce an exact sequence \(0 \to p(A', B') \to p(A, B) \to p(A'', Y) \to 0\) in \((T \downarrow \mathcal{A})\).

2. \(T\) is exact if and only if \(p : \mathcal{A} \times \mathcal{B} \to (T \downarrow \mathcal{A})\) is exact.
Proof. We only need to prove (1), because (2) is a direct consequence of (1). Assume that \( T : \mathcal{B} \to \mathcal{A} \) is \( \mathcal{Y} \)-exact. Let \( 0 \to A' \to A \to A'' \to 0 \) be an exact sequence in \( \mathcal{A} \) and \( 0 \to B' \to B \to Y \to 0 \) be an exact sequence in \( \mathcal{B} \) with \( Y \in \mathcal{Y} \). Then \( 0 \to T(B') \to T(B) \to T(Y) \to 0 \) is an exact sequence in \( \mathcal{A} \). So the sequence \( 0 \to p(A', B') \to p(A, B) \to p(A'', Y) \to 0 \) is exact in \( (T \downarrow \mathcal{A}) \).

Conversely, let \( 0 \to B' \to B \to Y \to 0 \) be an exact sequence in \( \mathcal{B} \) with \( Y \in \mathcal{Y} \). Note that \( 0 \to p(0, B') \to p(0, B) \to p(0, Y) \to 0 \) is an exact sequence in \( (T \downarrow \mathcal{A}) \) by hypothesis. So \( 0 \to T(B') \to T(B) \to T(Y) \to 0 \) is an exact sequence in \( \mathcal{A} \). This completes the proof.

\( \square \)

**Proposition 2.10.** If \( T : \mathcal{B} \to \mathcal{A} \) is \( \mathcal{Y} \)-exact, then \( (p(X, Y))^\perp = (X^\perp_Y) \) holds in the category \( (T \downarrow \mathcal{A}) \).

**Proof.** In the sequel, we need the following identities:
\[
(p(X, Y))^\perp = p(X, Y)^\perp = p(X, 0)^\perp \cap p(0, Y)^\perp,
\]
which hold by Remark 2.4(1).

At first, we claim that \( (X^\perp_Y) \subseteq p(X, Y)^\perp \). Let \( (A \ B) \in (X^\perp_Y) \). It is sufficient to show the following exact sequences:
\[
\zeta : 0 \to \left( \begin{array}{c} A \\ B \end{array} \right) \to \left( \begin{array}{c} M \\ N \end{array} \right) \to \left( \begin{array}{c} T(Y) \\ Y \end{array} \right) \to 0, \quad \xi : 0 \to \left( \begin{array}{c} A \\ B \end{array} \right) \to \left( \begin{array}{c} D \\ B \end{array} \right) \to \left( \begin{array}{c} X \\ 0 \end{array} \right) \to 0
\]
are split for any \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \). In fact, for \( \zeta \), we have \( g : Y \to N \) such that \( ng = 1_Y \) by hypothesis. It follows that \( m(g(T(g))) = T(nT(g)) = T(n) = 1_{T(Y)} \), i.e., \( (m(g(T(g))))^{\perp} \cap (X^\perp_Y) = 1 \) which implies that the sequence \( \zeta \) is split, as desired. In addition, \( \zeta = 0 \) since the exact sequence \( 0 \to A \to D \to X \to 0 \) is split.

For the reverse containment \( (p(X, Y))^\perp \subseteq (X^\perp_Y) \), we only need to show that \( p(X, 0)^\perp \subseteq (X^\perp_Y) \) and \( p(0, Y)^\perp \subseteq (X^\perp_Y) \). Let \( (A \ B) \) be an object in \( p(X, 0)^\perp \). Assume that \( \varepsilon : 0 \to A \to D \to X \to 0 \) is an exact sequence in \( \mathcal{A} \) with \( X \in \mathcal{X} \). Then the following exact sequence:
\[
0 \to \left( \begin{array}{c} A \\ B \end{array} \right) \to \left( \begin{array}{c} D \\ B \end{array} \right) \to \left( \begin{array}{c} X \\ 0 \end{array} \right) \to 0
\]
is split in \( (T \downarrow \mathcal{A}) \) by hypothesis. It follows that the sequence \( \varepsilon \) is split, and hence \( A \in X^\perp \). So \( p(X, 0)^\perp \subseteq (X^\perp_Y) \), as desired.

On the other hand, let \( (A \ B) \) be an object in \( p(0, Y)^\perp \) and \( \zeta : 0 \to B \to N \to Y \to 0 \) be an exact sequence in \( \mathcal{B} \) with \( Y \in \mathcal{Y} \). Note that \( T \) is \( \mathcal{Y} \)-exact. It follows that \( 0 \to T(B) \to T(N) \to T(Y) \to 0 \) is an exact sequence in \( \mathcal{A} \). Consider the following pushout diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & T(B) \\
\Bigg\downarrow & & \Bigg\downarrow \\
0 & \longrightarrow & A \\
\end{array}
\quad
\begin{array}{ccc}
& & T(N) \\
\Bigg\downarrow & & \Bigg\downarrow \\
& & T(Y) \\
\end{array}
\quad
\begin{array}{ccc}
& & 0 \\
\Bigg\downarrow & & \Bigg\downarrow \\
& & 0 \\
\end{array}
\]
Thus we have an exact sequence \( g : 0 \to (A \ B \to (L \to (T(Y) \to 0 \) in \( (T \downarrow \mathcal{A}) \). Hence the sequence \( g \) is split by hypothesis, so the sequence \( \zeta \) is split, i.e., \( B \in Y^\perp \). Consequently, \( p(0, Y)^\perp \subseteq (X^\perp_Y) \). This completes the proof.

\( \square \)

**Lemma 2.11.** If \( \mathcal{A} \) has enough injective objects, then \( (p(\mathcal{A}, \mathcal{B})) = (\mathcal{I}) \), where \( \mathcal{I} \) is the class of injective objects in \( \mathcal{A} \).

**Proof.** Assume that \( \mathcal{A} \) has enough injective objects and \( \mathcal{I} \) is the class of injective objects in \( \mathcal{A} \). Let \( (A \ B) \in (\mathcal{I}) \). Then we have a monomorphism \( \sigma : T(B) \to I \) with \( I \) injective. Thus we have an exact sequence \( \zeta : 0 \to (A \to (T(B) \to (A \to 0 \) in \( (T \downarrow \mathcal{A}) \) which is induced from the exact commutative diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\Bigg\downarrow & & \Bigg\downarrow \\
I & \longrightarrow & I \oplus A \\
\Bigg\downarrow & & \Bigg\downarrow \\
0 & \longrightarrow & 0.
\end{array}
\]
By hypothesis, $\xi$ is split. It follows that there is a morphism \((f, g) : I \oplus A \to I\) such that \((f, g)(\epsilon) = 1_I\) and \((f, g)(\sigma) = 0\) which implies \(g\varphi = -\sigma\). So \(\varphi\) is monic, i.e., \((\beta')_{\varphi} \in (p(\mathcal{A}, \mathcal{B}))\) by Proposition 2.5.

Conversely, assume that \((\beta')_{\varphi} \in (p(\mathcal{A}, \mathcal{B}))\). Consider the exact sequence

\[
\xi : 0 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} A \\ \varphi \end{pmatrix} \to 0
\]

with \(I\) injective. Then there is a map \(p : A \to M\) such that \(\pi p = 1\) since \(I\) is injective. It follows that \(\pi(\phi - p\varphi) = 0\). By the universal property of the kernel, we get a map \(\alpha : T(B) \to I\) such that \(\phi - p\varphi = \alpha \varphi\). Since \(\varphi\) is monic (by Proposition 2.5) and \(I\) is injective, there is a map \(\beta : A \to I\) such that \(\alpha = \beta \varphi\). Setting \(q = p + i\beta\), we have \(q\varphi = (p + i\beta)\varphi = p\varphi + i\beta \varphi = p\varphi + i\alpha = \phi\) and \(\pi q = \pi p + \pi i\alpha = \pi p = 1\).

Thus \((\xi')(\xi') = 1\), and hence \(\xi\) is split.

**Proposition 2.12.** If \(\mathcal{A}\) has enough injective objects, then \((p(\mathcal{X}, \mathcal{Y})) = \mathcal{X}(\mathcal{Y}) \cap \mathcal{Y}(\mathcal{X})\), where \(I\) is the class of injective objects in \(\mathcal{A}\).

**Proof.** Assume that \(\mathcal{A}\) has enough injective objects and \(I\) is the class of injective objects in \(\mathcal{A}\). To prove that \(\mathcal{X}(\mathcal{Y}) \cap \mathcal{Y}(\mathcal{X}) \subseteq (p(\mathcal{X}, \mathcal{Y}))\), we suppose that \((\beta')_{\varphi} \in (\mathcal{X}(\mathcal{Y}) \cap \mathcal{Y}(\mathcal{X}))\). It follows from Proposition 2.5 and Lemma 2.11 that \(\varphi\) is monic. Note that both \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under extensions, so it is sufficient to show \(B \in \mathcal{Y}\) and \(\text{coker} \varphi \in \mathcal{X}\) by Proposition 2.5.

Let \(\xi : 0 \to Y \to N \to B \to 0\) be an exact sequence in \(\mathcal{B}\) with \(Y \in \mathcal{Y}\). Then there is an exact sequence \(\xi : 0 \to (\mathcal{Y}) \to (\mathcal{X}) \to (\mathcal{X}) \to 0\) in \((T \downarrow \mathcal{A})\). By hypothesis, the sequence \(\xi\) is split since \((\mathcal{Y}) \in (\mathcal{X})\). So the sequence \(\xi\) is split and \(B \in \mathcal{Y}\), as desired.

Let \(0 \to X \xrightarrow{f} M \xrightarrow{\text{coker} \varphi} 0\) be an exact sequence in \(\mathcal{A}\) with \(X \in \mathcal{X}\). Then we have a pullback diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & X \\
\downarrow & & \downarrow \\
0 & \to & M \\
\downarrow & & \downarrow \\
0 & \to & \text{coker} \varphi \\
\end{array}
\]

which induces an exact sequence

\[
\xi' : 0 \to \begin{pmatrix} X \\ 0 \end{pmatrix} \to \begin{pmatrix} L \\ B \end{pmatrix} \to \begin{pmatrix} A \\ \varphi \end{pmatrix} \to 0
\]

in \((T \downarrow \mathcal{A})\). Since \(\xi'\) is split by hypothesis, there is a morphism \(\tilde{g} : L \to X\) such that \(\tilde{g}f = 1\) and \(\tilde{g}\varphi = 0\). By the universal property of the cokernel, there exists a morphism \(\tilde{g} : L \to X\) such that \(\tilde{g} = gp\). Thus we have \(gf = gp\tilde{f} = \tilde{g}f = 1\), which means that the third row in the above diagram is split. So \(\text{coker} \varphi \in \mathcal{X}\), as required.

For the reverse containment \((p(\mathcal{X}, \mathcal{Y})) \subseteq \mathcal{X}(\mathcal{Y}) \cap \mathcal{Y}(\mathcal{X})\), by Remark 2.4(1) and Lemma 2.11, we only need to show that \(p(A, 0) \in (\mathcal{X}(\mathcal{Y}))\) for any \(A \in \mathcal{X}\) and \(B \in \mathcal{Y}\).

Let \(\varepsilon : 0 \to (\mathcal{X}) \to (\mathcal{Y}) \to (\mathcal{X}) \to 0\) be an exact sequence in \((T \downarrow \mathcal{A})\) with \(A \in \mathcal{X}\), \(X \in \mathcal{X}\) and \(Y \in \mathcal{Y}\). Note that the exact sequence \(0 \to X \to M \to A \to 0\) in \(\mathcal{A}\) is split. It follows that the sequence \(\varepsilon\) is split. So \(p(A, 0) \in (\mathcal{X}(\mathcal{Y}))\). Similarly, one can prove that \(p(0, B) \in (\mathcal{X}(\mathcal{Y}))\). This completes the proof. \(\Box\)
Recall that an exact category $\mathcal{D}$ is said to be Frobenius provided that it has enough projective objects and enough injective objects, and the class of projective objects coincides with the class of injective objects. We end this section with the following result which is a generalization of [30, Corollary 2.3].

**Corollary 2.13.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough projective objects and enough injective objects. If $T : \mathcal{B} \to \mathcal{A}$ is an exact functor, then $\langle p(\mathcal{A}, \mathcal{B}) \rangle$ is a Frobenius category if and only if $\mathcal{A}$ and $\mathcal{B}$ are Frobenius and $T$ preserves projective objects.

**Proof.** Clearly, $\langle p(\mathcal{A}, \mathcal{B}) \rangle$ is an exact category, where the exact structure inherits from $(T \downarrow \mathcal{A})$. Let $\mathcal{P}$ (resp. $\mathcal{T}$) be the class of projective (resp. injective) objects in $\mathcal{A}$ and $\mathcal{Q}$ (resp. $\mathcal{J}$) the class of projective (resp. injective) objects in $\mathcal{B}$.

Then we have

$$\downarrow p(\mathcal{A}, 0) \cap \downarrow \left( \begin{array}{c} \mathcal{I} \\ 0 \end{array} \right) = \downarrow \left( \begin{array}{c} \mathcal{A} \\ 0 \end{array} \right) \cap \downarrow \left( \begin{array}{c} \mathcal{I} \\ 0 \end{array} \right) \quad \text{since } p(\mathcal{A}, 0) = \langle \mathcal{A} \rangle$$

$$= \langle p(\downarrow \mathcal{A}, \downarrow 0) \rangle \quad \text{by Proposition 2.12}$$

$$= \langle p(\mathcal{P}, \mathcal{B}) \rangle$$

and

$$\downarrow p(0, \mathcal{B}) \cap \downarrow \left( \begin{array}{c} \mathcal{I} \\ 0 \end{array} \right) = \downarrow \left( \begin{array}{c} T(\mathcal{B}) \\ \mathcal{B} \end{array} \right) \cap \downarrow \left( \begin{array}{c} \mathcal{I} \\ 0 \end{array} \right) \quad \text{since } p(0, \mathcal{B}) = \langle T(\mathcal{B}) \rangle$$

$$= \langle p(\downarrow T(\mathcal{B}), \downarrow \mathcal{B}) \rangle \quad \text{by Proposition 2.12}$$

$$= \langle p(\downarrow T(\mathcal{B}), \mathcal{Q}) \rangle.$$  

Thus, we get

$$\downarrow \langle p(\mathcal{A}, \mathcal{B}) \rangle = \downarrow p(\mathcal{A}, \mathcal{B}) = \downarrow p(\mathcal{A}, 0) \cap \downarrow p(0, \mathcal{B}) \quad \text{by Remark 2.4(1)}$$

$$= \downarrow p(\mathcal{A}, 0) \cap \downarrow \left( \begin{array}{c} \mathcal{I} \\ 0 \end{array} \right) \cap \downarrow p(0, \mathcal{B}) \quad \text{since } \mathcal{I} \subset \mathcal{A}$$

$$= \left( \downarrow p(\mathcal{A}, 0) \cap \downarrow \left( \begin{array}{c} \mathcal{I} \\ 0 \end{array} \right) \right) \cap \left( \downarrow p(0, \mathcal{B}) \cap \downarrow \left( \begin{array}{c} \mathcal{I} \\ 0 \end{array} \right) \right)$$

$$= \langle p(\mathcal{P}, \mathcal{B}) \rangle \cap \langle p(\downarrow T(\mathcal{B}), \mathcal{Q}) \rangle \quad \text{by the above identities}$$

$$= \langle p(\mathcal{P}, \mathcal{Q}) \rangle.$$  

Moreover, the class of injective objects in the exact category $\langle p(\mathcal{A}, \mathcal{B}) \rangle = (\frac{\mathcal{J}}{\mathcal{I}}) \cap (p(\mathcal{A}, \mathcal{B}))$. Therefore, $\langle p(\mathcal{A}, \mathcal{B}) \rangle$ is Frobenius if and only if $T(\mathcal{Q}) \subset \mathcal{I} = \mathcal{P}$ and $\mathcal{J} = \mathcal{Q}$, as desired.\hfill $\Box$

### 3 Complete cotorsion pairs and the proof of Theorem 1.1(1)

In this section, we shall use our results in Section 2 to show the first statement of the main result, Theorem 1.1. More precisely, we first recall the definition of complete hereditary cotorsion pairs in abelian categories, and then characterize when complete hereditary cotorsion pairs in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce complete hereditary cotorsion pairs in $(T \downarrow \mathcal{A})$. Especially, we shall establish a crucial result, Proposition 3.4, which will play a role in the proof of Theorem 1.1(1).

Let $\mathcal{L}$ be a class of objects in an abelian category $\mathcal{D}$. Recall that $\mathcal{L}$ is said to be resolving if whenever $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{D}$ with $Z \in \mathcal{L}$, then $X \in \mathcal{L}$ if and only if $Y \in \mathcal{L}$. Dually, $\mathcal{L}$ is said to be coresolving if whenever $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{D}$ with $X \in \mathcal{L}$, then $Y \in \mathcal{L}$ if and only if $Z \in \mathcal{L}$.

**Definition 3.1.** Let $\mathcal{D}$ be an abelian category.

1. A cotorsion pair (see [28]) is a pair of classes $(\mathcal{H}, \mathcal{G})$ of objects in $\mathcal{D}$ such that $\mathcal{H}^\perp = \mathcal{G}$ and $^\perp \mathcal{G} = \mathcal{H}$. 

(2) A cotorsion pair \((\mathcal{H}, \mathcal{G})\) is said to be hereditary (see [15]) if Ext\(_i^{\mathcal{H}}(X, Y) = 0\) for every \(X \in \mathcal{H}\) and \(Y \in \mathcal{G}\) and \(i > 0\). Moreover, if \(\mathcal{D}\) has enough projectives and injectives, the condition that \((\mathcal{H}, \mathcal{G})\) is hereditary is equivalent to that \(\mathcal{D}\) is resolving and \(\mathcal{G}\) is coresolving (see [10, Proposition 2.1]).

(3) A cotorsion pair \((\mathcal{H}, \mathcal{G})\) is said to be complete (see [12]) if \(\mathcal{H}\) is special precovering and \(\mathcal{G}\) is special preenveloping. Moreover, if \(\mathcal{D}\) has enough projective objects, the condition that \((\mathcal{H}, \mathcal{G})\) is complete is equivalent to that \(\mathcal{G}\) is special preenveloping. Similarly, if \(\mathcal{D}\) has enough injective objects, the condition that \((\mathcal{H}, \mathcal{G})\) is complete is equivalent to that \(\mathcal{H}\) is special precovering (see [14, Subsection 6.3, p. 595]).

**Lemma 3.2.** The following are true for any comma category \((\mathcal{T} \downarrow \mathcal{A})\):

1. \((\mathcal{X}, \mathcal{Y})\) is coresolving in \((\mathcal{T} \downarrow \mathcal{A})\) if and only if \(\mathcal{X}\) and \(\mathcal{Y}\) are coresolving in \(\mathcal{A}\) and \(\mathcal{B}\), respectively.
2. If \(T : \mathcal{B} \rightarrow \mathcal{A}\) is \(\mathcal{A}\)-exact and \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under extensions in \(\mathcal{A}\) and \(\mathcal{B}\), respectively, then \((\mathcal{p}(\mathcal{X}, \mathcal{Y}))\) is resolving in \((\mathcal{T} \downarrow \mathcal{A})\) if and only if \(\mathcal{X}\) and \(\mathcal{Y}\) are resolving in \(\mathcal{A}\) and \(\mathcal{B}\), respectively.

**Proof.** We only prove (2); the proof of (1) is straightforward. Assume that \(T\) is \(\mathcal{A}\)-exact. For the “only if” part, we assume that \((\mathcal{p}(\mathcal{X}, \mathcal{Y}))\) is resolving in \((\mathcal{T} \downarrow \mathcal{A})\). Let \(0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0\) be an exact sequence in \(\mathcal{A}\) with \(X'' \in \mathcal{X}\). It follows that the sequence \(0 \rightarrow \mathcal{p}(X', 0) \rightarrow \mathcal{p}(X, 0) \rightarrow \mathcal{p}(X'', 0) \rightarrow 0\) is exact in \((\mathcal{T} \downarrow \mathcal{A})\). Thus we get that \(\mathcal{p}(X, 0) \in (\mathcal{p}(\mathcal{X}, \mathcal{Y}))\) if and only if \(\mathcal{p}(X', 0) \in (\mathcal{p}(\mathcal{X}, \mathcal{Y}))\) by hypothesis, and hence \(X \in \mathcal{X}\) if and only if \(X' \in \mathcal{X}\). So \(\mathcal{X}\) is resolving in \(\mathcal{A}\). Similarly, one can prove that \(\mathcal{Y}\) is resolving in \(\mathcal{B}\).

For the “if” part, we assume that \(\mathcal{X}\) and \(\mathcal{Y}\) are resolving in \(\mathcal{A}\) and \(\mathcal{B}\), respectively. Then \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under extensions in \(\mathcal{A}\) and \(\mathcal{B}\), respectively. Let \(0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0\) be an exact sequence in \((\mathcal{T} \downarrow \mathcal{A})\) with \((\mathcal{X}, \mathcal{Y})\) \(T\) is \(\mathcal{A}\)-exact. Then we have the following commutative diagram of exact sequences:

\[
\begin{array}{cccc}
0 & \rightarrow & T(Y') & \rightarrow & T(Y) & \rightarrow & T(Y'') & \rightarrow & 0 \\
& \downarrow \varphi' & & \varphi & & \downarrow \varphi'' & & \\
0 & \rightarrow & X' & \rightarrow & X & \rightarrow & X'' & \rightarrow & 0.
\end{array}
\]

Note that \(\varphi''\) is monic and \(\text{coker}\varphi'' \in \mathcal{X}\) by Proposition 2.5. Then we have that \(\text{ker}\varphi' \cong \text{ker}\varphi\) and \(0 \rightarrow \text{coker}\varphi' \rightarrow \text{coker}\varphi \rightarrow \text{coker}\varphi'' \rightarrow 0\) is exact in \(\mathcal{A}\). Therefore, we get that \(\varphi\) is monic if and only if \(\varphi'\) is monic. Since \(\mathcal{X}\) is resolving in \(\mathcal{A}\) by hypothesis, we have that \(\text{coker}\varphi \in \mathcal{X}\) if and only if \(\text{coker}\varphi' \in \mathcal{X}\).

Note that \(0 \rightarrow X' \rightarrow Y \rightarrow Y'' \rightarrow 0\) is an exact sequence in \(\mathcal{B}\) and \(\mathcal{Y}\) is resolving in \(\mathcal{B}\). It follows that \(Y' \in \mathcal{Y}\) if and only if \(Y \in \mathcal{Y}\). So \((\mathcal{X}, \mathcal{Y})\) \(\mathcal{B}\) follows that \(\mathcal{X} \subseteq \mathcal{X}\) and \(\mathcal{Y} \subseteq \mathcal{Y}\). Let \(A \in \mathcal{X}\) and \(B \in \mathcal{Y}\). Then \(x \in \mathcal{X}\) and \(y \in \mathcal{Y}\). Therefore, \((x, y) \in \mathcal{X} \times \mathcal{Y}\).

**Lemma 3.3.** Let \(\mathcal{A}\) and \(\mathcal{B}\) both have enough projective objects and enough injective objects. If \(T : \mathcal{B} \rightarrow \mathcal{A}\) is \(\mathcal{A}\)-exact, then \((\mathcal{X}, \mathcal{X})\) and \((\mathcal{Y}, \mathcal{Y})\) are (hereditary) cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{B}\), respectively if and only if \((\mathcal{p}(\mathcal{X}, \mathcal{Y})), (\mathcal{X^+}, \mathcal{Y^+})\) is a (hereditary) cotorsion pair in \((\mathcal{T} \downarrow \mathcal{A})\) and \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under extensions.

**Proof.** By [10, Proposition 2.1] and Lemma 3.2, it suffices to show that \((\mathcal{p}(\mathcal{X}, \mathcal{Y})), (\mathcal{X^+}, \mathcal{Y^+})\) is a cotorsion pair in \((\mathcal{T} \downarrow \mathcal{A})\) if and only if \((\mathcal{X}, \mathcal{X^+})\) and \((\mathcal{Y}, \mathcal{Y^+})\) are cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{B}\), respectively.

“\(\Rightarrow\)”. Let \((\mathcal{X}, \mathcal{X^+})\) be a cotorsion pair in \(\mathcal{A}\) and \((\mathcal{Y}, \mathcal{Y^+})\) a cotorsion pair in \(\mathcal{B}\). Then all the injective objects belong to \(\mathcal{X^+}\). By Proposition 2.12, we have the following:

\[\mathcal{p}(\mathcal{X}, \mathcal{Y}) = \mathcal{p}(\mathcal{X}^+, \mathcal{Y}^+) = \mathcal{p}(\mathcal{X}^+, \mathcal{Y}^+) = \mathcal{p}(\mathcal{X}^+, \mathcal{Y}^+).
\]

Note that \(\mathcal{p}(\mathcal{X}, \mathcal{Y}) = \mathcal{p}(\mathcal{X}^+, \mathcal{Y}^+)\) by Proposition 2.10. It follows that \((\mathcal{p}(\mathcal{X}, \mathcal{Y})), (\mathcal{X^+}, \mathcal{Y^+})\) is a cotorsion pair in \((\mathcal{T} \downarrow \mathcal{A})\).

“\(\Leftarrow\)”. Now, we assume that \((\mathcal{p}(\mathcal{X}, \mathcal{Y})), (\mathcal{X^+}, \mathcal{Y^+})\) is a cotorsion pair in \((\mathcal{T} \downarrow \mathcal{A})\) and \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under extensions. It is sufficient to show that \(\mathcal{X^+} \subseteq \mathcal{X}\) and \(\mathcal{Y^+} \subseteq \mathcal{Y}\). Let \(A \in \mathcal{X^+}\) and
$B \in \perp(Y^\perp)$. Then for any $M \in X^\perp$ and $N \in Y^\perp$, it is clear that the following exact sequences:

$$0 \rightarrow \begin{pmatrix} M \\ N \end{pmatrix} \rightarrow \begin{pmatrix} C \\ N \end{pmatrix} \rightarrow \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow 0$$

and

$$0 \rightarrow \begin{pmatrix} M \\ N \end{pmatrix} \rightarrow \begin{pmatrix} D \\ L \end{pmatrix} \rightarrow \begin{pmatrix} T(B) \\ B \end{pmatrix} \rightarrow 0$$

are split. This implies that $(\begin{pmatrix} A \\ 0 \end{pmatrix}, (T(B)) \in \perp(X^\perp, Y^\perp) = \langle p(X, Y) \rangle$, and so $A \in X, B \in Y$ by Proposition 2.5. This completes the proof.

The following proposition is crucial to the proof of Theorem 1.1(1).

**Proposition 3.4.** Let $\mathcal{A}$ and $\mathcal{B}$ both have enough projective objects and enough injective objects. Assume that $T : \mathcal{B} \rightarrow \mathcal{A}$ is $\mathcal{Y}$-exact. If $(X, X^\perp)$ and $(Y, Y^\perp)$ are complete cotorsion pairs in $\mathcal{A}$ and $\mathcal{B}$, respectively, then so is $(\langle p(X, Y) \rangle, (X^\perp, Y^\perp))$. Moreover, the converse holds when $T(Y \cap Y^\perp) \subseteq X^\perp$ and $X$ and $Y$ are closed under extensions.

**Proof.** Assume that $(X, X^\perp)$ is a complete cotorsion pair in $\mathcal{A}$ and $(Y, Y^\perp)$ is a complete cotorsion pair in $\mathcal{B}$. For any $(\begin{pmatrix} A \\ 0 \end{pmatrix}) \in (T \downarrow \mathcal{A})$, there is an exact sequence $0 \rightarrow B \rightarrow V \rightarrow Y \rightarrow 0$ in $\mathcal{B}$ with $V \in Y^\perp$ and $Y \in \mathcal{Y}$. Noting that $T$ is $\mathcal{Y}$-exact, we get the following pushout diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & T(B) & \rightarrow & T(V) & \rightarrow & T(Y) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A & \rightarrow & C & \rightarrow & T(Y) & \rightarrow & 0.
\end{array}
$$

Furthermore, we have an exact sequence $0 \rightarrow C \rightarrow U \rightarrow X \rightarrow 0$ in $\mathcal{A}$ with $U \in X^\perp$ and $X \in X$. Consider the following pushout diagram:

$$
\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & C & \rightarrow & T(Y) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A & \rightarrow & U & \rightarrow & D & \rightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X & \rightarrow & X & \rightarrow & X & \rightarrow & X & \rightarrow & 0.
\end{array}
$$

Thus we get an exact sequence $0 \rightarrow (\begin{pmatrix} A \\ 0 \end{pmatrix}) \rightarrow (\begin{pmatrix} V \\ B \end{pmatrix}) \rightarrow 0$ in $(T \downarrow \mathcal{A})$ with $(\begin{pmatrix} V \\ B \end{pmatrix}) \in (X^\perp, Y^\perp)$ and $(\begin{pmatrix} V \\ B \end{pmatrix}) \in \langle p(X, Y) \rangle$. Note that $\mathcal{A}$ and $\mathcal{B}$ have enough projective objects. Then $(T \downarrow \mathcal{A})$ has enough projective objects. So $(\langle p(X, Y) \rangle, (X^\perp, Y^\perp))$ is a complete cotorsion pair in $(T \downarrow \mathcal{A})$ by [14, Subsection 6.3, p. 595], as desired.

Conversely, we assume that $(\langle p(X, Y) \rangle, (X^\perp, Y^\perp))$ is a complete cotorsion pair in $(T \downarrow \mathcal{A})$, $T(Y \cap Y^\perp) \subseteq X^\perp$ and $X$ and $Y$ are closed under extensions. Then for any $B \in \mathcal{B}$, we have an exact sequence $0 \rightarrow (\begin{pmatrix} U \\ B \end{pmatrix}) \rightarrow 0$ in $(T \downarrow \mathcal{A})$ with $(\begin{pmatrix} U \\ B \end{pmatrix}) \in \langle p(X, Y) \rangle$ and $(\begin{pmatrix} U \\ B \end{pmatrix}) \in (X^\perp, Y^\perp)$. Thus we have an exact sequence $0 \rightarrow V \rightarrow Y \rightarrow B \rightarrow 0$ in $\mathcal{B}$ with $Y \in \mathcal{Y}$ and $V \in Y^\perp$, which means that $(\mathcal{Y}, Y^\perp)$ is a complete cotorsion pair in $\mathcal{B}$.

Furthermore, for any $A \in \mathcal{A}$, we have an exact sequence $0 \rightarrow (\begin{pmatrix} A \\ 0 \end{pmatrix}) \rightarrow (\begin{pmatrix} M \\ B \end{pmatrix}) \rightarrow 0$ in $(T \downarrow \mathcal{A})$ with $(\begin{pmatrix} M \\ B \end{pmatrix}) \in \langle p(X, Y) \rangle$ and $(\begin{pmatrix} M \\ B \end{pmatrix}) \in (X^\perp, Y^\perp)$. Thus we obtain that $Y \in \mathcal{Y} \cap Y^\perp, \operatorname{coker} \varphi \in X, K \in X^\perp$.
and $\varphi$ is monic. Consequently, we have the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & & 0 & & T(Y) & & T(Y) & & 0 \\
& & & & \downarrow & & \downarrow & & \\
0 & \to & K & \xrightarrow{\phi} & M & \to & A & \to & 0 \\
& & \downarrow & \uparrow & \phi & & \downarrow & \uparrow & 0 \\
0 & \to & \text{coker}\phi & \xrightarrow{i} & \text{coker}\varphi & \to & A & \to & 0.
\end{array}
\]

By hypothesis, $T(Y) \in X^\perp$. It follows that the middle column splits which implies that the left column is split, and hence $\text{coker}\phi$ is a direct summand of $K$. Thus $\text{coker}\varphi \in X^\perp$, i.e., the exact sequence $0 \to \text{coker}\phi \to \text{coker}\varphi \to A \to 0$ implies $(X, X^\perp)$ is a complete cotorsion pair in $\mathcal{A}$.

**Remark 3.5.** If we replace the condition “$T(Y \cap Y^\perp) \subseteq X^\perp$” with “$T(Y \cap Y^\perp) \subseteq X$” in Proposition 3.4, the result still holds. In the case where $T(Y \cap Y^\perp) \subseteq X$, for any $A \in \mathcal{A}$, it is easy to check that the exact sequence $0 \to K \to M \to A \to 0$ in the third commutative diagram in Proposition 3.4 satisfies that $K \in X^\perp$ and $M \in X$. This implies that $(X, X^\perp)$ is a complete cotorsion pair in $\mathcal{A}$.

As a corollary of Proposition 3.4 and Corollary 2.8, we re-obtain [30, Theorem 1.1(2)].

**Corollary 3.6.** (See [30].) Let $R$ and $S$ be Artin algebras, and $\Lambda = (R_S M_S)$ be a triangular matrix algebra. If $M$ is a finitely generated projective left $S$-module, then $\mathcal{M}(R, M, S)$ is a functorially finite subcategory of $\text{mod}\Lambda$, and has Auslander-Reiten sequences.

**Proof.** By the proof of Theorem 1.1(2) in [30, p. 34], it suffices to show that $\mathcal{M}(R, M, S)$ is precovering in $\text{mod}\Lambda$. This is true by Proposition 3.4 and Corollary 2.8.

**Corollary 3.7.** Let $\Lambda = (R_S M_S)$ be a triangular matrix ring, and let $X_1$ and $X_2$ be two classes of left $R$-modules, and $Y_1$ and $Y_2$ be two classes of left $S$-modules. Assume that $\text{Tor}_i^R(M, Y) = 0$ for any $Y \in Y_1$. If $(X_1, Y_2)$ and $(Y_1, X_2)$ are hereditary complete cotorsion pairs in $\text{Mod}R$ and $\text{Mod}S$, respectively, then so is $(\mathcal{B}_{X_1}^{Y_1}, \mathcal{B}_{Y_2}^{X_2})$. Moreover, the converse holds when $M \otimes_S N \subseteq X_2$ or $M \otimes_S N \subseteq X_1$ for any $N \in Y_1 \cap Y_2$.

**Proof.** If we define $T \equiv M \otimes_S - : \text{Mod}S \to \text{Mod}R$, then $\text{mod}\Lambda$ is equivalent to the comma category $(T \downarrow \text{Mod}R)$ by Example 2.2(1). Assume that $(\mathcal{B}_{X_1}^{Y_1}, \mathcal{B}_{Y_2}^{X_2})$ is a cotorsion pair. Then $0 \in X_1 \cap X_2$, $0 \in Y_1 \cap Y_2$, and $\mathcal{B}_{X_1}^{Y_1}$ is closed under extensions. It follows from Proposition 2.5 that $(p(X_1, Y_1)) = \mathcal{B}_{X_1}^{Y_1}$. Let $M$ be a left $R$-module in $X_2$. Then $(M_0) \in (X_1 \cap Y_2) = (p(X_1, Y_1)) = (\mathcal{B}_{X_1}^{Y_1})$ by Proposition 2.10. This implies $M \in X_1^\perp$. On the other hand, let $N$ be a left $R$-module in $X_2^\perp$. It follows that $(N_0) \in (X_1 \cap Y_2) = (p(X_1, Y_1)) = (\mathcal{B}_{X_1}^{Y_1})$ by Proposition 2.10. So we have $N \in X_2$ and $X_1^\perp = X_2$. Similarly, one can show $Y_1^\perp = Y_2$. So the result holds by Lemma 3.3, Proposition 3.4 and Remark 3.5.

**Remark 3.8.** We note that Corollary 3.7 refines a result obtained by Mao [20]. More precisely, the conditions that “$\text{Tor}_i^R(U, C_2) = 0$ for any $i \geq 1$” and “$\text{Tor}_i^R(U, C_1) = 0$ for any $i \geq 2$” in [20, Theorem 5.6(1)] are superfluous. Also, our proof here is different from that in [20].

Let $\mathcal{L}$ be a class of objects in an abelian category $\mathcal{D}$. We denote by $\text{Smd}(\mathcal{L})$ the class of direct summands of objects in $\mathcal{L}$.

**Lemma 3.9.** Let $\mathcal{D}$ be an abelian category with enough injective objects. If $\mathcal{L}$ is special precovering in $\mathcal{D}$, then $(\text{Smd}(\mathcal{L}), \text{Smd}(\mathcal{L})^\perp)$ is a complete cotorsion pair.
Proof. Note that $\text{Smd}(\mathcal{L})^\perp = \mathcal{L}^\perp$. Then it is clear that $\text{Smd}(\mathcal{L})$ is a special precovering class. In the sequel, we claim that $\perp(\text{Smd}(\mathcal{L})^\perp) \subseteq \text{Smd}(\mathcal{L})$. For any $L \in \perp(\text{Smd}(\mathcal{L})^\perp)$, there is an exact sequence $0 \to K \to \mathcal{L} \to L \to 0$ with $\mathcal{L} \in \mathcal{L}$ and $K \in \mathcal{L}^\perp$ since $\mathcal{L}$ is special precovering. Thus the above sequence is split as $\text{Smd}(\mathcal{L})^\perp = \mathcal{L}^\perp$. It follows that $L \in \text{Smd}(\mathcal{L})$, so $\perp(\text{Smd}(\mathcal{L})^\perp) = \text{Smd}(\mathcal{L})$. We complete the proof.

We are now in a position to prove Theorem 1.1(1).

Proof of Theorem 1.1(1). At first, we claim that $T$ is $\text{Smd}(\mathcal{Y})$-exact. For any exact sequence $0 \to B_1 \to B_0 \to Y \to 0$ in $\mathcal{B}$ with $Y \in \text{Smd}(\mathcal{Y})$, there is an induced exact sequence $0 \to B_1 \to B_0 \oplus M \to Y \oplus M \to 0$, where $Y \oplus M \in \mathcal{Y}$. Then we obtain the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & T(B_1) & \longrightarrow & T(B_0) & \longrightarrow & T(Y) & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | \\
0 & \longrightarrow & T(B_1) & \longrightarrow & T(B_0) \oplus T(M) & \longrightarrow & T(Y) \oplus T(M) & \longrightarrow & 0,
\end{array}
$$

where the bottom row is exact since $T$ is $\mathcal{Y}$-exact. It follows that the first row is exact.

Since $(\text{Smd}(\mathcal{X}), \text{Smd}(\mathcal{Y}))$ and $(\text{Smd}(\mathcal{X}), \text{Smd}(\mathcal{Y}))$ are complete cotorsion pairs by Lemma 3.9, the pair $((p(\text{Smd}(\mathcal{X}), \text{Smd}(\mathcal{Y}))), (\text{Smd}(\mathcal{X})^\perp))$ is a complete cotorsion pair in $((T \downarrow \mathcal{A}))$ by Proposition 3.4. Thus for any object $\left( \begin{array}{c} A \\ B \end{array} \right) \in (T \downarrow \mathcal{A})$, there is an exact sequence

$$
0 \rightarrow \left( \begin{array}{c} C \\ D \end{array} \right) \psi \rightarrow \left( \begin{array}{c} M \\ Y \end{array} \right) \varphi \rightarrow \left( \begin{array}{c} A \\ B \end{array} \right) \rightarrow 0
$$

with $\left( \begin{array}{c} C \\ D \varphi \end{array} \right) \in \langle p(\text{Smd}(\mathcal{X}), \text{Smd}(\mathcal{Y})) \rangle$ and $\left( \begin{array}{c} C \\ D \psi \end{array} \right) \in \text{Smd}(\mathcal{X})^\perp$.

Since $\mathcal{Y}$ is special precovering, we have an exact sequence $0 \to K \to Y \to \overline{Y} \to 0$ in $\mathcal{B}$ with $Y \in \mathcal{Y}$ and $K \in \mathcal{Y}^\perp$. It follows from $\mathcal{Y}^\perp = \text{Smd}(\mathcal{Y})^\perp$ that $Y \cong \overline{Y} \oplus K$. Similarly, one can show that there exists an exact sequence $0 \to U \to X' \to \text{coker } \overline{\varphi} \to 0$ in $\mathcal{A}$ with $U \oplus \text{coker } \overline{\varphi} \cong X' \in \mathcal{X}$ and $U \in \mathcal{X}^\perp$ by the facts that $\text{coker } \overline{\varphi} \in \text{Smd}(\mathcal{X})$ and $X'$ is special precovering. For $T(K) \in \mathcal{A}$, there is an exact sequence $0 \to T(K) \to N \to X \to 0$ in $\mathcal{A}$ with $N \in \text{Smd}(\mathcal{X})^\perp = \mathcal{X}^\perp$ and $X \in \text{Smd}(\mathcal{X})$. Moreover, there is an exact sequence $0 \to L \to X \to \overline{X} \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $L \in \mathcal{X}^\perp$, so $X \cong \overline{X} \oplus L$. Therefore, we have an exact sequence in $((T \downarrow \mathcal{A}))$:

$$
0 \rightarrow \left( \begin{array}{c} C \\ D \end{array} \right) \psi \rightarrow \left( \begin{array}{c} M \\ Y \end{array} \right) \varphi \rightarrow \left( \begin{array}{c} A \\ B \end{array} \right) \rightarrow 0
$$

with

$$
D \cong \overline{D} \oplus K, \quad Y \cong \overline{Y} \oplus K, \quad C \cong \overline{C} \oplus U \oplus N \oplus L, \quad M \cong \overline{M} \oplus U \oplus N \oplus L, \quad \psi = \left( \begin{array}{c} \overline{\psi} \\ 0 \\ 0 \\ 0 \\ 0 \\ i \\ 0 \end{array} \right), \quad \varphi = \left( \begin{array}{c} \overline{\varphi} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right).
$$

Clearly, $\left( \begin{array}{c} C \\ D \end{array} \right) \psi \in \langle \mathcal{X}^\perp \rangle$. Note that we have an exact sequence

$$
0 \rightarrow \left( \begin{array}{c} T(Y) \\ Y \end{array} \right) \psi \rightarrow \left( \begin{array}{c} M \\ Y \end{array} \right) \varphi \rightarrow \left( \begin{array}{c} X' \oplus X \\ 0 \end{array} \right) \rightarrow 0 \quad \text{in } (T \downarrow \mathcal{A}).
$$

Then $\left( \begin{array}{c} M \\ Y \varphi \end{array} \right) \in \langle p(\mathcal{X}, \mathcal{Y}) \rangle$. So we obtain that $\langle p(\mathcal{X}, \mathcal{Y}) \rangle$ is special precovering.

Conversely, assume that $\langle p(\mathcal{X}, \mathcal{Y}) \rangle$ is special precovering for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then there is an exact sequence $\xi : 0 \rightarrow \left( \begin{array}{c} C \\ D \end{array} \right) \psi \rightarrow \left( \begin{array}{c} M \\ Y \varphi \end{array} \right) \rightarrow \left( \begin{array}{c} A \\ B \end{array} \right) \rightarrow 0$ in $((T \downarrow \mathcal{A}))$ with $\left( \begin{array}{c} M \\ Y \end{array} \right) \varphi \in \langle p(\mathcal{X}, \mathcal{Y}) \rangle$ and $\left( \begin{array}{c} C \\ D \end{array} \right) \psi \in \langle \mathcal{X}^\perp \rangle$.
By Proposition 2.10, \((\mathcal{P}(\mathcal{X}, \mathcal{Y}))^\perp\) is the subcategory of \(\mathcal{Y}\) consisting of objects with special precovering properties. Thus, the exact sequence \(0 \to D \to Y \to B \to 0\) implies that \(Y\) is special precovering in general.

### Remark 3.10

We cannot omit the condition that \(T\) is \(\mathcal{Y}\)-exact in Theorem 1.1(1). For example, let \(\Lambda\) be the \(k\)-algebra given by the quiver \(\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet\) with a relation \(\beta\alpha \) and \(P(i)\) (resp. \(I(i)\) and \(S(i)\)) denote the indecomposable projective (resp. injective and simple) module corresponding to the vertex \(i\).

In fact, \(\Lambda = (k, k, k, k, k_2)\) with \(A_2 := \bullet \xrightarrow{\alpha} \bullet\). Let 
\[
(\mathcal{X}, \mathcal{X}^\perp) = (\text{Mod}_k, \text{Mod}_k) \quad \text{and} \quad (\mathcal{Y}, \mathcal{Y}^\perp) = (\text{Mod}_kA_2, I(kA_2)).
\]

Applying \(T = M \otimes_{kA_2} \) to the exact sequence \(0 \to P(2) \xrightarrow{i} P(3) \to S(3) \to 0\) in \(\text{Mod}_kA_2\), we obtain that \(T\) is not \(\mathcal{Y}\)-exact. Note that \(\langle \mathcal{P}(\text{Mod}_k, \text{Mod}_kA_2) \rangle^\perp = \{I(1), I(2), I(3)\}\). This implies that \(S(2)\) does not have special \(\langle \mathcal{P}(\text{Mod}_k, \text{Mod}_kA_2) \rangle\)-precovers. So \(\langle \mathcal{P}(\text{Mod}_k, \text{Mod}_kA_2) \rangle\) is not special precovering in general.

## 4 Gorenstein projective objects and the proof of Theorem 1.1(2)

In this section, to get the proof of Theorem 1.1(2), we first characterize when the functor \(\mathcal{P} : \mathcal{A} \times \mathcal{B} \to (T \downarrow \mathcal{A})\) preserves Gorenstein projective objects. It should be noted that the functor \(\mathcal{P}\) does not preserve Gorenstein projective objects by [33, Example 1].

Throughout this section, \(\mathcal{A}\) and \(\mathcal{B}\) always have enough projective objects.

### Definition 4.1

The right exact functor \(T : \mathcal{B} \to \mathcal{A}\) is compatible, if the following two conditions hold:

1. \((C1)\) \(T(Q^\bullet)\) is exact for any projective object \(Q^\bullet\) in \(\mathcal{B}\).
2. \((C2)\) \(\text{Hom}_{\mathcal{A}}(P^\bullet, T(Q))\) is exact for any complete \(\mathcal{A}\)-projective resolution \(P^\bullet\) and any projective object \(Q\) in \(\mathcal{B}\).

Moreover, \(\mathcal{A} \to \mathcal{A}\) is called weakly compatible, if it satisfies the conditions (W1) and (C2), where \(W1\) \(T(Q^\bullet)\) is exact for any complete \(\mathcal{B}\)-projective resolution \(Q^\bullet\).

### Remark 4.2

1. We note that the exact functor \(c : \text{Ch}(\text{Mod}) \to \text{Mod}\) defined in Example 2.2(4) is compatible.
2. Let \(R\) and \(S\) be Artin algebras, and \(\Lambda = (R, R, S)\) be a triangular matrix algebra. If we define \(T \cong M \otimes_S \) be a triangular matrix algebra. If we define \(T \cong M \otimes_S \text{mod}\) by \(T \cong M \otimes_S \text{mod}\), it is easy to check that \(T\) is compatible if and only if \(M\) is a compatible \(R\)-module as defined by Zhang [33, Definition 1.1].
(3) It should be noted that a weakly compatible functor $T$ is not compatible in general as the following example shows.

**Example 4.3.** Let $\Lambda = kQ/I$ with the quiver 

![Quiver Diagram]

and $I = \langle x^2, ax, a\beta, x\beta \rangle$, i.e., $\Lambda = (\begin{array}{ccc} R & \text{by} & S \\
0 & \text{by} & 0 \end{array})$ with $R = k$ and $S = (k[x]/(x^2))$. Note that the algebra $S$ is Cohen-Macaulay (CM)-free, i.e., every finitely generated Gorenstein projective left $S$-module is projective, so each compatible $\mathcal{B}$-projective resolution is always split, where $\mathcal{B}$ is the category of finitely generated left $S$-modules. It follows that $T = M \otimes_S \cdot$ must be weakly compatible and the $\mathcal{B}$-projective resolution

$$Q^* = \cdots \to \left( \frac{k[x]}{(x^2)} \right) \to \left( \frac{k[y]}{(x^2)} \right) \to \left( \frac{k[z]}{(x^2)} \right) \to \cdots$$

is not complete. Note that $T(Q^*) = \cdots \to k \to k \to \cdots$ is not exact, so $T$ is not compatible.

For any ring $R$, the projective (resp. injective) dimension of a left $R$-module $M$ will be denoted by $\text{pd}_R M$ (resp. $\text{id}_R M$) and the flat dimension of a right $R$-module $N$ will be denoted by $\text{fd}_R N$. The following proposition gives more examples of compatible functors.

**Proposition 4.4.** Let $M$ be an $R$-$S$-bimodule and $T = M \otimes_S \cdot$. Then the following hold:

1. If $\text{fd}_M S$ is finite, then $T$ satisfies (C1).
2. If $\text{pd}_R M$ is finite, then $T$ satisfies (C2).
3. If $R$ is a left noetherian ring and $\text{id}_R M$ is finite, then $T$ satisfies (C2).

**Proof.**

1. Let $Q^* = \cdots \to Q^{-1} \to Q^0 \to Q^1 \to \cdots$ be an exact sequence of projective left $S$-modules and $\text{fd}_M S = n$. By dimension shifting, we have $\text{Tor}^1_S(M, \ker d^i) = \text{Tor}^{n+1}_S(M, \ker d^{n+i}) = 0$ for any integer $i$. It follows that $T(Q^*)$ is still exact.

2. (2) and (3). For any projective left $S$-module $Q$, there is an index $I$ such that $Q$ is a direct summand of $S^I$. Then $M \otimes_S Q$ is a direct summand of $M \otimes_S S^I \cong M^I$, and hence $\text{pd}_R M \otimes_S Q$ is finite if $\text{pd}_R M$ is finite and $\text{id}_R M \otimes_S Q$ is finite if $\text{id}_R M$ is finite and $R$ is left noetherian. Let $P^* = \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ be a complete $\mathcal{A}$-projective resolution. Then (2) and (3) hold by the isomorphism $\text{Ext}^{n+1}_R(\ker d^i, M \otimes_S Q) \cong \text{Ext}^{n+1}_R(\ker d^{n+i}, M \otimes_S Q)$ for any integer $i$.

**Lemma 4.5.** Let $G$ be an object in $\mathcal{B}$ and $L$ an object in $\mathcal{A}$.

1. If $p(0, G)$ is a Gorenstein projective object, then $G$ is Gorenstein projective.
2. If $p(L, 0)$ is a Gorenstein projective object, then $L$ is Gorenstein projective.

**Proof.**

1. Let $p(0, G)$ be a Gorenstein projective object. Then there is a complete $(T \downarrow \mathcal{A})$-projective resolution $\mathcal{P}(p(G, Q^*)) = \cdots \to \mathcal{P}(p^{-1}, Q^1) \to \mathcal{P}(p^0, Q^0) \to \mathcal{P}(p^1, Q^1) \to \cdots$ with $Z^0(p(G, Q^*)) = p(0, G)$. Then $p(G, Q^*)$ is $\text{Hom}_{T/\mathcal{A}}(-, p(0, Q))$ exact for any projective object $Q$ in $\mathcal{B}$, which implies that $Q^*_G = \cdots \to Q^{-1} \to Q^0 \to Q^1 \to \cdots$ is $\text{Hom}_{\mathcal{A}}(-, Q)$ exact since $\text{Hom}_{T/\mathcal{A}}(p(P^i, Q^i), p(0, Q)) \cong \text{Hom}_{\mathcal{A}}((P^i, Q^i), q(p(0, Q))) \cong \text{Hom}_{\mathcal{A}}(P^i, T(Q)) \times \text{Hom}_{\mathcal{A}}(Q, Q)$ by Remark 2.4. Clearly, $G = Z^0_G(Q^*)$. Then $G$ is Gorenstein projective.

2. Let $p(L, 0)$ be a Gorenstein projective object. Then there is a complete $(T \downarrow \mathcal{A})$-projective resolution $\mathcal{P}(p(G, Q^*)) = \cdots \to \mathcal{P}(p^{-1}, Q^1) \to \mathcal{P}(p^0, Q^0) \to \mathcal{P}(p^1, Q^1) \to \cdots$ with $Z^0(p(G, Q^*)) = p(L, 0)$. Then $p(G, Q^*)$ is $\text{Hom}_{T/\mathcal{A}}(-, p(L, 0))$ exact for any projective object $P$ in $\mathcal{A}$, which implies that $P^* = \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ is $\text{Hom}_{\mathcal{A}}(-, P)$ exact since $\text{Hom}_{T/\mathcal{A}}(p(P^i, Q^i), p(P, 0)) \cong \text{Hom}_{\mathcal{A}}((P^i, Q^i), q(p(P, 0))) \cong \text{Hom}_{\mathcal{A}}(P^i, P)$ by Remark 2.4. Clearly, $L = Z^0_G(P^*)$. Then $L$ is Gorenstein projective.
Lemma 4.6. For the comma category \((T \downarrow \mathcal{A})\), we have

1. \(T\) satisfies (W1) if and only if \(p(0,G)\) is a Gorenstein projective object for any Gorenstein projective object \(G\) in \(\mathcal{B}\).
2. \(T\) satisfies (C2) if and only if \(p(L,0)\) is a Gorenstein projective object for any Gorenstein projective object \(L\) in \(\mathcal{A}\).

Proof. (1) “\(\Rightarrow\)”. Let \(Q^* = \cdots \rightarrow Q^{-1} \rightarrow Q^0 \xrightarrow{d^0} Q^1 \rightarrow \cdots\) be a complete \(\mathcal{B}\)-projective resolution. Then it is sufficient to show that \(T\) is exact corresponding to the exact sequence \(0 \rightarrow G^0 \xrightarrow{q} Q^0 \rightarrow G^1 \rightarrow 0\), where \(G^0 = \ker d^0\) and \(G^1 = \im d^0\). Since \(G^0\) is Gorenstein projective, \(p(0,G^0)\) is Gorenstein projective by hypothesis. Thus there is an exact sequence \(0 \rightarrow p(0,G^0) \xrightarrow{Q^0} (\frac{G}{Q^0}) \rightarrow 0\) in \((T \downarrow \mathcal{A})\) with \((\frac{G}{Q^0})\) projective. Since \(Q_B\) is projective, we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & G^0 & \rightarrow & Q^0 & \rightarrow & G^1 & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & G^0 & \rightarrow & Q_B & \rightarrow & G_B & \rightarrow & 0.
\end{array}
\]

Therefore, we obtain the following exact commutative diagram:

\[
\begin{array}{cccccc}
T(G^0) & \xrightarrow{T(i)} & T(Q^0) & \xrightarrow{T(i)} & T(G^1) & \xrightarrow{T(i)} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
T(G^0) & \xrightarrow{T(i)} & T(Q_B) & \xrightarrow{T(i)} & T(G_B) & \xrightarrow{T(i)} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
T(G^0) & \xrightarrow{T(i)} & Q_A & \xrightarrow{T(i)} & G_A & \xrightarrow{T(i)} & 0,
\end{array}
\]

which implies that \(T(i)\) is monic, as required.

“\(\Rightarrow\)”. Let \(G\) be a Gorenstein projective object in \(\mathcal{B}\). Then there is a complete \(\mathcal{B}\)-projective resolution \(Q^* = \cdots \rightarrow Q^{-1} \rightarrow Q^0 \xrightarrow{d^0} Q^1 \rightarrow \cdots\) such that \(G = \ker d^0\). By hypothesis, \(p(0,Q^*) = \cdots \rightarrow p(0,Q^{-1}) \rightarrow p(0,Q^0)\) is a projective resolution. It follows that

\[\text{Hom}_{(T \downarrow \mathcal{A})}(p(0,Q^*),p(P,Q)) \simeq \text{Hom}_{\mathcal{A} \times \mathcal{B}}((0,Q^*),q(p(P,Q))) = \text{Hom}_{\mathcal{A}}(Q^*,Q)\]

for any projective object \(p(P,Q)\) in \((T \downarrow \mathcal{A})\). Thus, \(p(0,Q^*)\) is complete. Then \(p(0,G)\) is a Gorenstein projective object by the fact that \(p(0,G) = \ker(p(0,d^0))\).

(2) Let \(P^* = \cdots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots\) be a complete \(\mathcal{A}\)-projective resolution. Then there exists an exact sequence \(p(P^*,0) = \cdots \rightarrow p(P^{-1},0) \xrightarrow{d^{-1}} p(P^0,0) \xrightarrow{d^0} p(P^1,0) \xrightarrow{d^1} \cdots\) in \((T \downarrow \mathcal{A})\) with each term projective. For any projective object \(p(P,Q)\) in \((T \downarrow \mathcal{A})\), one has

\[\text{Hom}_{(T \downarrow \mathcal{A})}(p(P^*,0),p(P,Q)) \simeq \text{Hom}_{\mathcal{A} \times \mathcal{B}}((P^*,0),q(p(P,Q))) = \text{Hom}_{\mathcal{A}}(P^*,P \oplus T(Q)).\]

Consequently, we get that \(\text{Hom}_{\mathcal{A}}(P^*,T(Q))\) is exact for any projective object \(Q\) in \(\mathcal{B}\) if and only if \(p(P^*,0)\) is a complete \((T \downarrow \mathcal{A})\)-resolution for any complete \(\mathcal{A}\)-resolution \(P^*\), and so (2) holds.

The following proposition is crucial to the proof of Theorem 1.1(2) which characterizes when the functor \(p\) preserves Gorenstein projective objects.

Proposition 4.7. Let \((T \downarrow \mathcal{A})\) be a comma category. Then \(\langle p(\mathcal{G}_\mathcal{A},\mathcal{G}_\mathcal{B})\rangle \subseteq \mathcal{G}_T(\downarrow \mathcal{A})\) if and only if \(T\) is weakly compatible. Moreover, if \(T\) is compatible, then \(\mathcal{G}_T(\downarrow \mathcal{A}) = \langle p(\mathcal{G}_\mathcal{A},\mathcal{G}_\mathcal{B})\rangle\).

Proof. By Lemmas 4.5 and 4.6, it is sufficient to show \(\mathcal{G}_T(\downarrow \mathcal{A}) \subseteq \langle p(\mathcal{G}_\mathcal{A},\mathcal{G}_\mathcal{B})\rangle\) provided that \(T\) is compatible. For any \(\langle \frac{\mathcal{G}}{\mathcal{H}} \rangle \in \mathcal{G}_T(\downarrow \mathcal{A})\), there is a complete \((T \downarrow \mathcal{A})\)-projective resolution

\[p(P^*,Q^*) = \cdots \rightarrow p(P^{-1},Q^{-1}) \xrightarrow{d^{-1}} p(P^0,0) \xrightarrow{d^0} p(P^1,0) \xrightarrow{d^1} \cdots\]
with $(\frac{H}{G})_{\varphi} \in \ker(d^i, \delta^i)$. Then $Q^\bullet$ is a $\mathcal{B}$-projective resolution, so $T(Q^\bullet)$ is exact since $T$ is compatible. Thus we have the following exact commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & T(G^i) & \longrightarrow & T(Q^i) & \longrightarrow & T(G^{i+1}) & \longrightarrow & 0 \\
\downarrow{}^{\varphi^i} & & \downarrow{}^{(0)} & & \downarrow{}^{\varphi^{i+1}} & & \downarrow{} & & \\
0 & \longrightarrow & H^i & \longrightarrow & P^i \oplus T(Q^i) & \longrightarrow & H^{i+1} & \longrightarrow & 0,
\end{array}
$$

where $G^i = \ker(\delta^i)$, $(\frac{H^i}{G^i})_{\varphi^i} \in \ker(d^i, \delta^i)$ and $\varphi^i$ is canonically induced. In particular, $G^0 = G$, $H^0 = H$ and $\varphi^0 = \varphi$. It follows that each $\varphi^i$ is monic and there is an exact sequence $0 \rightarrow \ker(\varphi^i) \rightarrow P^i \rightarrow \ker(\varphi^{i+1}) \rightarrow 0$. Therefore, we have an $\mathcal{A}$-projective resolution $P^\bullet = \cdots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots$ with $\ker(0) = \ker(\varphi)$. For any projective object $P$ in $\mathcal{A}$, one has $\hom_{\mathcal{T}[\mathcal{A}]}(p(P^0, Q^i), \mathbf{p}(P, 0)) \simeq \hom_{\mathcal{A}[\mathcal{B}]}((P^0, Q^i), \mathbf{p}(P, 0)) = \hom_{\mathcal{A}}(P^0, P)$. Note that $\hom_{\mathcal{T}[\mathcal{A}]}(p(P^0, Q^i), \mathbf{p}(P, 0))$ is exact. It follows that $P^\bullet$ is complete. This implies that $\ker(\varphi)$ is a Gorenstein projective object in $\mathcal{A}$.

Let $Q$ be a projective object in $\mathcal{B}$. Applying $\hom_{\mathcal{T}[\mathcal{A}]}(\mathbf{p}(0, Q))$ to the complete $(T \downarrow \mathcal{A})$-projective resolution $\mathbf{p}(P^0, Q^i)$ above, we obtain that $Q^\bullet = \cdots \rightarrow Q^{-1} \rightarrow Q^0 \rightarrow Q^1 \rightarrow \cdots$ is $\hom_{\mathcal{A}}(\mathbf{p}(0, Q))$ exact and the proof is similar to that of Lemma 4.5(1). So $G = Z^0(Q^\bullet)$ is Gorenstein projective. This completes the proof.

**Remark 4.8.** We note that Proposition 4.7 generalizes [33, Theorem 1.4] and [19, Theorem 1.1]. More precisely, assume that $(T \downarrow \text{Mod}(R)) = \text{Mod}(\Lambda)$, where $\Lambda = (\begin{smallmatrix} 0 & S \\ S & 0 \end{smallmatrix})$ is a triangular matrix ring. If $R$ and $S$ are Artin algebras and $M$ is a compatible $R$-$S$-bimodule, then Proposition 4.7 here is just [33, Theorem 1.4]. On the other hand, if $R$ and $S$ are arbitrary rings and $\text{pd}_R M < \infty$ and $\text{fd}_S M < \infty$, then Proposition 4.7 here is just [19, Theorem 1.1].

Recall from [32, Theorem 2.2] that a complex $C^\bullet$ in $\text{Ch}(R)$ is Gorenstein projective if and only if $G^n$ is a Gorenstein projective left $R$-module for all $n \in \mathbb{Z}$. As a consequence of Remark 4.2(1) and Proposition 4.7, we have the following corollary.

**Corollary 4.9.** Let $(e \downarrow \mathcal{A})$ be a comma category in Example 2.2(4). Then $(\frac{Y}{X})_{\varphi}$ is a Gorenstein projective object in $(e \downarrow \mathcal{A})$ if and only if $Y^\bullet$ is a Gorenstein projective object in $\text{Ch}(R)$ and $\varphi : Y^0 \rightarrow X$ is an injective $R$-morphism with a Gorenstein projective cokernel.

We are now in a position to prove the second statement of the main result, Theorem 1.1.

**Proof of Theorem 1.1(2).** Assume that $T : \mathcal{B} \rightarrow \mathcal{A}$ is a compatible functor. Note that $\mathcal{GP}_{\mathcal{B}} \cap \mathcal{GP}_{\mathcal{A}}^{\perp}$ is the class of projective objects in $\mathcal{B}$. It follows from (C2) in Definition 4.1 that $T(\mathcal{GP}_{\mathcal{B}} \cap \mathcal{GP}_{\mathcal{A}}^{\perp}) \subseteq \mathcal{GP}_{\mathcal{A}}^{\perp}$. By Theorem 1.1(1) and Proposition 4.7, it suffices to show that $T$ is $\mathcal{GP}_{\mathcal{A}}$-exact.

In fact, for any exact sequence $0 \rightarrow A \rightarrow B \rightarrow Y \rightarrow 0$ in $\mathcal{B}$ with $Y \in \mathcal{GP}_{\mathcal{B}}$, there is an exact sequence $0 \rightarrow K_1 \rightarrow P_0 \rightarrow Y \rightarrow 0$ in $\mathcal{B}$ such that $K_1$ is a Gorenstein object and $P_0$ is a projective object. We choose an exact sequence $0 \rightarrow L_1 \rightarrow Q_0 \rightarrow A \rightarrow 0$ in $\mathcal{B}$ with $Q_0$ a projective object. Thus we have the following exact commutative diagram:
where the second row is split. Note that $0 \to T(K_1) \to T(P_0) \to T(Y) \to 0$ is an exact sequence in $\mathcal{A}$ by (C1) in Definition 4.1. Applying the functor $T$ to the above diagram, we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccc}
 & & & & & \\
 & & & & & \\
 & & & & & \\
T(L_1) & \to & T(Z_1) & \to & T(K_1) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & T(Q_0) & \to & T(W_0) & \to T(P_0) \\
\downarrow & & \downarrow & & \downarrow & \\
T(A) & \to & T(B) & \to & T(Y) & \to 0.
\end{array}
\]

By the snake lemma, we have that $0 \to T(A) \to T(B) \to T(Y) \to 0$ is an exact sequence in $\mathcal{A}$. This completes the proof.

The following example shows that one can get many rings which are not necessarily coherent such that the class of Gorenstein projective modules is special precovering over them.

**Example 4.10.** Let $S$ be a Gorenstein ring and $R = (\begin{smallmatrix} S & S^{(r)} \\ 0 & S \end{smallmatrix})$. By Corollary 1.2, $\mathcal{GP}(R)$ is a special precovering class. Note that $M = (\begin{smallmatrix} 0 \\ S \\ \bar{S} \end{smallmatrix})$ is a projective right $S$-module and an injective left $R$-module. Applying Corollary 1.2 again, we get that $\mathcal{GP}(\Lambda)$ is a special precovering class in Mod$\Lambda$ for $\Lambda = (\begin{smallmatrix} R \\ M \\ S \end{smallmatrix})$.

We set $a = (\begin{smallmatrix} 0 \\ 0 \\ 1 \bar{s} \end{smallmatrix})$ and $I = \{x \in \Lambda : xa = 0\}$. It follows that $I = (\begin{smallmatrix} 0 \\ S^{(r)} \\ 0 \\ \bar{S} \end{smallmatrix}) \subseteq I$, and hence $I$ is not finitely generated. So $\Lambda$ is not left coherent by [18, Corollary 4.60].

**Remark 4.11.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, then there is also a comma category, denoted by $(\mathcal{B} \downarrow F)$ (see [11,21]). Thus we have the functor $h : \mathcal{A} \times \mathcal{B} \to (\mathcal{B} \downarrow F)$ via $h(A, B) = (F(1)A)_{\mathcal{B}}(1, 0)$ and $h(a, b) = (F(a)_{\mathcal{B}})_{\mathcal{B}}$, where $(A, B)$ is an object in $\mathcal{A} \times \mathcal{B}$ and $(a, b)$ is a morphism in $\mathcal{A} \times \mathcal{B}$. Dually, one can also study when complete hereditary cotorsion pairs in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce complete hereditary cotorsion pairs in $(\mathcal{B} \downarrow F)$, and characterize when special preenveloping classes in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce special preenveloping classes in $(\mathcal{B} \downarrow F)$. All the results concerning the comma category $(T \downarrow \mathcal{A})$ have their counterparts with the help of the comma category induced by left exact functors.

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant Nos. 11671069 and 11771212), Zhejiang Provincial Natural Science Foundation of China (Grant No. LY18A010032), Qing Lan Project of Jiangsu Province and Jiangsu Government Scholarship for Overseas Studies (Grant No. JS-2019-328). Part of the work was done during a visit of the first author to Charles University in Prague with the support by Jiangsu Government Scholarship. The first author thanks Professors Jan Trlifaj and Jan Stůvříček, and the Department of Algebra for their hospitality. The authors thank Professor Changchang Xi for helpful discussions on parts of this article. The authors are grateful to the referees for reading the paper carefully and for many suggestions on mathematics and English expressions.

**References**

1. Auslander M, Reiten I, Smalø S O. Representation Theory of Artin Algebras. Cambridge Studies in Advanced Mathematics, vol. 36. Cambridge: Cambridge University Press, 1995
2. Bican L, El Bashir R, Enochs E E. All modules have flat covers. Bull Lond Math Soc, 2001, 33: 385–390
3. Bravo D, Hovey M, Gillespie J. The stable module category of a general ring. arXiv:1405.5768, 2014
4 Chen X W, Le J. Recollements, comma categories and morphic enhancements. Proc Roy Soc Edinburgh Sect A, 2022, in press
5 Enochs E E, Cortés-Izurdiaga M, Torrecillas B. Gorenstein conditions over triangular matrix rings. J Pure Appl Algebra, 2014, 218: 1544–1554
6 Enochs E E, Jenda O M G. Relative Homological Algebra. De Gruyter Expositions in Mathematics, vol. 30. Berlin-New York: Walter de Gruyter, 2000
7 Enochs E E, Torrecillas B. Flat covers over formal triangular matrix rings and minimal Quillen factorizations. Forum Math, 2011, 23: 611–624
8 Estrada S, Iacob A, Odabasi S. Gorenstein flat and projective (pre)covers. Publ Math Debrecen, 2017, 91: 111–121
9 Estrada S, Iacob A, Yeomans K. Gorenstein projective precovers. Mediterr J Math, 2017, 14: 33
10 Estrada S, Pérez M A, Zhu H Y. Balanced pairs, cotorsion triplets and quiver representations. Proc Edinb Math Soc (2), 2020, 63: 67–90
11 Fossum R M, Griffith P A, Reiten I. Trivial Extension of Abelian Categories. Lecture Notes in Mathematics, vol. 456. Berlin: Springer-Verlag, 1975
12 Göbel R, Trlifaj J. Approximations and Endomorphism Algebras of Modules. Berlin: Walter de Gruyter, 2006
13 Haghany A, Varadarajan K. Study of modules over formal triangular matrix rings. J Pure Appl Algebra, 2000, 147: 41–58
14 Holm H, Jørgensen P. Cotorsion pairs in categories of quiver representations. Kyoto J Math, 2019, 59: 575–606
15 Hovey M. Cotorsion pairs and model categories. In: Interactions between Homotopy Theory and Algebra. Contemporary Mathematics, vol. 436. Providence: Amer Math Soc, 2007, 277–296
16 Jørgensen P. Existence of Gorenstein projective resolutions and Tate cohomology. J Eur Math Soc (JEMS), 2007, 9: 59–76
17 Ladkani S. Derived equivalences of triangular matrix rings arising from extensions of tilting modules. Algebr Represent Theory, 2011, 14: 57–74
18 Lam T Y. Lectures on Modules and Rings. Berlin: Springer, 1999
19 Li H H, Zheng Y F, Hu J S, et al. Gorenstein projective modules and recollements over triangular matrix rings. Comm Algebra, 2020, 48: 4932–4947
20 Mao L X. Cotorsion pairs and approximation classes over formal triangular matrix rings. J Pure Appl Algebra, 2020, 224: 106271
21 Marmaridis N. Comma categories in representation theory. Comm Algebra, 1983, 11: 1919–1943
22 Mitchell B. Rings with several objects. Adv Math, 1972, 8: 1–161
23 Murfet M, Salarian S. Totally acyclic complexes over noetherian schemes. Adv Math, 2011, 226: 1096–1133
24 Psaroudakis C. Homological theory of recollements of abelian categories. J Algebra, 2014, 398: 63–110
25 Ringel C M, Schmidmeier M. Submodule categories of wild representation type. J Pure Appl Algebra, 2006, 205: 412–422
26 Ringel C M, Schmidmeier M. The Auslander-Reiten translation in submodule categories. Trans Amer Math Soc, 2008, 360: 691–716
27 Ringel C M, Schmidmeier M. Invariant subspaces of nilpotent linear operators, I. J Reine Angew Math, 2008, 614: 1–52
28 Salce L. Cotorsion theories for abelian groups. Symp Math, 1979, 23: 11–32
29 Šaroch J, Štěpánek J. Singular compactness and definability for $\Sigma$-cotorsion and Gorenstein modules. Selecta Math (NS), 2020, 26: 23
30 Xiong B L, Zhang P, Zhang Y H. Bimodule monomorphism categories and RSS equivalences via cotilting modules. J Algebra, 2018, 503: 21–55
31 Xu J. Flat Covers of Modules. Lecture Notes in Mathematics, vol. 1634. Berlin-Heidelberg-New York: Springer-Verlag, 1996
32 Yang X Y, Liu Z K. Gorenstein projective, injective, and flat complexes. Comm Algebra, 2011, 39: 1705–1721
33 Zhang P. Gorenstein-projective modules and symmetric recollements. J Algebra, 2013, 388: 65–80