SUBSHIFTS WITH LEADING SEQUENCES, UNIFORMITY OF COCYCLES AND SPECTRA OF SCHREIER GRAPHS

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Abstract. We introduce a class of subshifts governed by finitely many two-sided infinite words. We call these words leading sequences. We show that any locally constant cocycle over such a subshift is uniform. From this we obtain Cantor spectrum of Lebesgue measure zero for associated Jacobi operators if the subshift is aperiodic. Our class covers all simple Toeplitz subshifts as well as all Sturmian subshifts. We apply our results to the spectral theory of Schreier graphs for uncountable families of groups acting on rooted trees.

Introduction

This article is concerned with matrix valued cocycles over dynamical systems, with applications to spectral theory of Jacobi operators. More specifically, we consider a uniquely ergodic dynamical system $(\Omega, T)$, where $\Omega$ is a compact metric space, $T : \Omega \to \Omega$ is a homeomorphism and there is only one $T$-invariant probability measure on $\Omega$. It is well-known that such systems admit a uniform ergodic theorem for continuous functions, see e.g. [41]. Hence, for any continuous $f : \Omega \to \mathbb{C}$, there exists $\lambda \in \mathbb{C}$ with

$$
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(T^j \omega),
$$

uniformly in $\omega \in \Omega$. A natural question in this context is whether a similar result holds for matrix-valued functions if summation is replaced by multiplication and averaging is done by a logarithmic mean. To be more precise, to any continuous function $A : \Omega \to SL(2, \mathbb{R})$ let us associate the cocycle

$$A(\cdot, \cdot) : \mathbb{Z} \times \Omega \to SL(2, \mathbb{R})$$

defined by

$$A(n, \omega) := \begin{cases} 
A(T^{n-1} \omega) \cdots A(\omega) : & n > 0 \\
Id : & n = 0 \\
A^{-1}(T^n \omega) \cdots A^{-1}(T^{-1} \omega) : & n < 0.
\end{cases}$$

Following [16] (cf. [41] as well), we call the continuous function $A : \Omega \to SL(2, \mathbb{R})$ uniform if the limit

$$\Lambda = \lim_{n \to \infty} \frac{1}{n} \log \| A(n, \omega) \|$$

exists for all $\omega \in \Omega$ and the convergence is uniform on $\Omega$. If $A$ is uniform, we also call the associated cocycle $A(\cdot, \cdot)$ uniform. With these definitions, the question about existence of ergodic averages uniformly in $\omega$ becomes the question about uniformity of all continuous
SL(2, \mathbb{R})\)-valued functions, or equivalently, of their associated cocycles. If \((\Omega, T)\) is minimal (i.e. \(\{T^n\omega : n \in \mathbb{Z}\}\) is dense in \(\Omega\) for each \(\omega \in \Omega\)) the definition can be simplified. Then, uniform convergence already follows if the limit exists for all \(\omega \in \Omega\). This observation has been shared with us by Benjamin Weiss in a private communication, and we present its proof at the end of Section 1.

Existence or non-existence of uniform cocycles has been studied by various people, e.g. in [41, 24, 16, 30]. In fact, Walters raises in [41] the question whether every uniquely ergodic dynamical system with non-atomic invariant measure \(\mu\) admits a non-uniform cocycle. Walters presents a class of examples admitting non-uniform cocycles based on results of Veech [40]. He also discusses a further class of examples, namely suitable irrational rotations, for which non-uniformity was shown by Herman [24]. Furman carries out in [16] a careful study of uniformity of cocycles. For strictly ergodic dynamical systems, he characterizes uniform cocycles with positive \(\Lambda\) in terms of uniform diagonalizability, which in turn is equivalent to uniform hyperbolicity of the cocycle. In this way the question of existence of non-uniform cocycles becomes the question of existence of hyperbolic but not uniformly hyperbolic cocycles.

A particular class of dynamical system of interest are subshifts. For them, there exists a finite set \(A\) such that \(\Omega\) is a closed subset of \(A^\mathbb{Z}\) (equipped with product topology), which is invariant under the shift
\[
T : A^\mathbb{Z} \rightarrow A^\mathbb{Z}, (Ts)(n) = s(n + 1).
\]
For subshifts, one may first restrict attention to those \(A : \Omega \rightarrow SL(2, \mathbb{R})\) which are locally constant (i.e. admit an \(N > 0\) such that \(A(\omega)\) only depends on \(\omega(-N)\ldots \omega(N)\)). We will then also call the associated cocycle locally constant.

Uniformity of locally constant cocycles is relevant for applications to the spectral theory of self-adjoint operators arising in the study of aperiodic order. Such operators have attracted considerable attention in recent decades, see e.g. the surveys [39, 11, 11] for more information on such operators, and [11, 2, 20] for background on aperiodic order. In particular, it was shown in [28], that uniformity of locally constant cocycles implies Cantor spectrum of Lebesgue measure zero for discrete Schrödinger operators associated with aperiodic subshifts. As a consequence, all aperiodic linearly repetitive subshifts yield Cantor spectrum of Lebesgue measure zero. In [10] this was generalized to the substantially larger class of aperiodic subshifts satisfying the so-called Boshernitzan condition. These results do not only hold for discrete Schrödinger operators but also for the larger class of Jacobi operators [5]. By now it seems fair to say that establishing validity of Boshernitzan condition has become the standard method of proving Cantor spectrum in this context.

Quite remarkably, [32] gave results showing Cantor spectrum for Schrödinger operators associated to certain subshifts, which do not satisfy Boshernitzan condition. These results do not only hold for discrete Schrödinger operators but also for the larger class of Jacobi operators [5]. By now it seems fair to say that establishing validity of Boshernitzan condition has become the standard method of proving Cantor spectrum in this context. Quite remarkably, [32] gave results showing Cantor spectrum for Schrödinger operators associated to certain subshifts, which do not satisfy Boshernitzan condition. These results do not only hold for discrete Schrödinger operators but also for the larger class of Jacobi operators [5]. By now it seems fair to say that establishing validity of Boshernitzan condition has become the standard method of proving Cantor spectrum in this context. Quite remarkably, [32] gave results showing Cantor spectrum for Schrödinger operators associated to certain subshifts, which do not satisfy Boshernitzan condition. These results do not only hold for discrete Schrödinger operators but also for the larger class of Jacobi operators [5]. By now it seems fair to say that establishing validity of Boshernitzan condition has become the standard method of proving Cantor spectrum in this context. Quite remarkably, [32] gave results showing Cantor spectrum for Schrödinger operators associated to certain subshifts, which do not satisfy Boshernitzan condition. These results do not only hold for discrete Schrödinger operators but also for the larger class of Jacobi operators [5]. By now it seems fair to say that establishing validity of Boshernitzan condition has become the standard method of proving Cantor spectrum in this context. Quite remarkably, [32] gave results showing Cantor spectrum for Schrödinger operators associated to certain subshifts, which do not satisfy Boshernitzan condition. These results do not only hold for discrete Schrödinger operators but also for the larger class of Jacobi operators [5]. By now it seems fair to say that establishing validity of Boshernitzan condition has become the standard method of proving Cantor spectrum in this context.
A main result of the present article, Theorem \[5.5\], shows uniformity of all locally constant cocycles over simple Toeplitz subshifts. In particular, we provide the first examples of subshifts without Boshernitzan condition which satisfy uniformity of all locally constant cocycles. As a consequence we obtain Cantor spectrum of Lebesgue measure zero for aperiodic Jacobi operators associated to simple Toeplitz subshifts.

The application to Jacobi operators is particularly relevant as it allows us to apply our results to a problem from spectral graph theory. Indeed, as has recently been observed in \[21\], Laplacians on certain infinite graphs arising naturally in the theory of groups of intermediate growth can be realized as Jacobi operators on subshifts. In \[21\], the important example of the so-called “first Grigorchuk’s group”\(^1\), introduced in \[18\], was studied from this viewpoint. The corresponding subshift turned out to be defined by a primitive substitution and thus linearly repetitive. Hence the Cantor spectrum of Lebesgue measure zero holds for the anisotropic Laplacians on the infinite Schreier graphs of this group.

In fact, this group belongs to an uncountable family of groups \[19\] that are non-isomorphic and even non-quasi-isometric, but nevertheless share many properties, in particular, they all act by automorphisms on the infinite binary tree and are of intermediate growth, i.e., have word growth strictly between polynomial and exponential. It is then natural to ask (see \[20, 21\]) whether the spectral result from \[21\] holds for other groups in the family, and even to members of other related uncountable families of groups, so called spinal groups. Our second main result (see Corollary 7.2 and Section 7 in general) shows that this is indeed the case. The part of the proof that realizes the Laplacians on Schreier graphs as Jacobi operators on subshifts carries over from \[21\]. However, the subshifts that arise will no longer be linear repetitive, and in general will not even satisfy the Boshernitzan condition. However, we show that they are all simple Toeplitz subshifts, and therefore our main result applied and yields Cantor spectrum of Lebesgue measure zero in the aperiodic (anisotropic) case.

A few words on methods may be in order. The basic task in proving uniformity of cocycles is to provide lower bounds on the growth of products of matrices. For locally constant \(\mathbf{A}\) this amounts to providing lower bounds on growth along all finite words. Inspired by the considerations for Sturmian subshifts in \[31\], we introduce here a new method to achieve this. This method may be of interest in itself (beyond the applications given in the main two results). It relies on proving growth along finitely many infinite words, which control the whole subshift in a meaningful way. We call these infinite words leading sequences and call the subshifts admitting them subshifts satisfying the leading sequence condition (LSC). On the conceptual level, putting forward this class of subshifts is a key insight of the article. The corresponding technical result claims uniformity of locally constant cocycles over (LSC) subshifts, see Theorem \[2.5\]. We show that this class contains all simple Toeplitz subshifts as well as all Sturmian subshifts.

The article is organized as follows. In Section 1 we recall some basic notation and results concerning subshifts and cocycles. In Section 2 we introduce the leading sequence condition

\(^1\)This is how the group is generally known and how we refer to it, in spite of the first author’s reluctance.
(LSC) for subshifts. We discuss general properties of (LSC) subshifts and prove the main technical result: uniformity of locally constant cocycles on (LSC) subshifts (Theorem 2.5). In Section 3, we derive Cantor spectrum for Jacobi operators associated to aperiodic (LSC) subshifts. In Section 4, we provide combinatorial conditions ensuring (LSC). Equipped with these combinatorial conditions, we then show that simple Toeplitz subshifts and Sturmian subshifts satisfy condition (LSC) in Section 5 and Section 6 respectively. Finally, Section 7 contains the application of our results to the study of spectra of Laplacians on infinite Schreier graphs of Grigorchuk groups and spinal groups acting on the infinite binary tree. The material of the Sections 2 to 6 constitutes a part of the PhD thesis of one of the authors [38].

Acknowledgements. R. G. and D. L. gratefully acknowledge hospitality of the Department of Mathematics of the University of Geneva on various occasions during the last five years. The work of R. G. is supported by the Simons Foundation through Collaboration Grant 527814. R. G. and T. N. gratefully acknowledge support of the Swiss National Science Foundation and of the grant of the Government of the Russian Federation No 14.W03.31.0030. The work of D. S. is supported by a PhD scholarship from Landesgraduiertenstipendium - Thüringen. D. S. would also like to express his gratitude for an invitation to the Department of Mathematics of the University of Geneva in 2017.

1. Background on subshifts and uniform cocycles

In this section we recall some background on our main actors. These are subshifts over a finite alphabet and uniform cocycles.

Let \( A \) be a finite set called the alphabet. The elements of the free monoid \( A^* = \cup_{n=0}^{\infty} A^n \) are denoted as finite words over \( A \), where \( A^0 = \{ \epsilon \} \) with the empty word \( \epsilon \). We will freely use standard notation concerning words (see e.g. [33]). In particular, we define the length of a word \( v \) by \( |v| = n \) if \( v \in A^n \) for some \( n \in \mathbb{N} \). Moreover, we let the concatenation \( xy \) of \( x = x(1)\ldots x(N) \) and \( y = y(1)\ldots y(M) \) be given by \( xy = x(1)\ldots x(N)y(1)\ldots y(M) \) and call \( x \) then a prefix of \( xy \).

A pair \((\Omega, T)\) is called a subshift over \( A \), if \( \Omega \) is a closed subset of \( A^\mathbb{Z} \) (with product topology) and invariant under the shift \( T : A^\mathbb{Z} \rightarrow A^\mathbb{Z} \), \((Ts)(n) := s(n+1)\). Whenever \((\Omega, T)\) is a subshift over \( A \) and \( v \) is a word of length \( n \) over \( A \), we say that \( v \) occurs in \( \omega \in \Omega \) at \( k \in \mathbb{Z} \) if

\[
\text{occurs in } \omega \in \Omega \text{ at } k \in \mathbb{Z} \text{ if } v = \omega(k)\ldots \omega(k+n-1).
\]

We denote the set of all words of length \( n \) occurring in \( \omega \in \Omega \) by \( \mathcal{W}(\omega)_n \) and define the set of all finite words associated to \((\Omega, T)\) by

\[
\mathcal{W}(\Omega) := \bigcup_{n \in \mathbb{N}, \omega \in \Omega} \mathcal{W}(\omega)_n.
\]

As is well known, both minimality and unique ergodicity can be characterized via \( \mathcal{W}(\Omega) \) (see for example Section 5.1 in [15]). More specifically, the subshift is minimal if \( \mathcal{W}(\Omega)_n = \)
\(W(\omega)_n\) for any \(\omega \in \Omega\) for any \(n \in \mathbb{N}\). A subshift is uniquely ergodic if the limit
\[
\lim_{n \to \infty} \frac{\# \omega(1) \ldots \omega(n)}{n} =: \nu(v)
\]
exists uniformly in \(\omega \in \Omega\) for any \(v \in W(\Omega)\). Here, \(\# \omega(1) \ldots \omega(n)\) denotes the number of occurrences of \(v\) in \(\omega(1) \ldots \omega(n)\) i.e. the cardinality of the set
\[
\{ k \in \{1, \ldots, n - |v| + 1\} : \omega(k) \ldots \omega(k + |v| - 1) = v \}.
\]
The term \(\nu(v)\) is then called the frequency of \(v\) and uniformity of the limit above is referred to as uniform existence of frequencies.

For a finite word \(w = w(1) \ldots w(n)\) we define the reflected word \(w^R\) via \(w^R = w(n) \ldots w(1)\). Similarly, for a one-sided infinite word \(\eta : \mathbb{N} \to A\) we denote by \(\eta^R\) the one-sided infinite word arising by reflecting \(\eta\) (i.e. \(\eta^R : -\mathbb{N} \to A, \eta^R(n) = \eta(-n)\)). We write \(|\cdot|\) to denote the position of the origin. More specifically for an \(\omega : \mathbb{Z} \to A\) we write
\[
\omega = \varrho|\eta
\]
whenever \(\eta : \mathbb{N} \cup \{0\} \to A\) agrees with the restriction of \(\omega\) to \(\mathbb{N} \cup \{0\}\) and \(\varrho : -\mathbb{N} \to A\) agrees with the restriction of \(\omega\) to \(-\mathbb{N}\). A function \(f\) from a subshift is called locally constant if there exists an \(N \in \mathbb{N}\) such that
\[
f(\omega) = f(\varrho), \text{ whenever } \omega(-N) \ldots \omega(N) = \varrho(-N) \ldots \varrho(N).
\]
Clearly, any locally constant function into a topological space is continuous. As mentioned already in the introduction our main concern are locally constant functions \(A : \Omega \to SL(2, \mathbb{R})\). Any such \(A\) gives rise to a cocycle and we will also call this cocycle locally constant. For our subsequent dealing with locally constant cocycles we include two simple observations. One observation shows that we can 'chop off' finite pieces of cocycles without changing the exponential behaviour, the other is that we can take inverses without changing norms.

**Proposition 1.1.** Let \((\Omega, T)\) be a subshift and \(A : \Omega \to SL(2, \mathbb{R})\) locally constant. Then the following holds:

(a) \(\|A(-n, \omega)\| = \|A(n, T^{-n}\omega)\|\) for all \(n \in \mathbb{N}\) and \(\omega \in \Omega\).

(b) There exists for any \(N \in \mathbb{N}\) a \(c(N) > 0\) with
\[
\log \|A(n - N, T^N\omega)\| - c(N) \leq \log \|A(n, \omega)\| \leq \log \|A(n - N, T^N\omega)\| + c(N)
\]
for all \(\omega \in \Omega\) and all \(n \in \mathbb{Z}\).

**Proof.** (a) This is a direct consequence of the definition of \(A(\cdot, \cdot)\) and \(\|B\| = \|B^{-1}\|\) for any \(B \in SL(2, \mathbb{R})\).

(b) By continuity of \(A\) we easily infer
\[
c(N) := \sup_{\omega \in \Omega} \{ \log \|A(N, \omega)\| \} < \infty.
\]
Since \( A(n, \omega) = A(n - N, T^N\omega)A(N, \omega) \), the desired statement follows now from the inequality
\[
\frac{1}{\|D\|} \|C\| \leq \|CD\| \leq \|D\| \|C\|
\]
for matrices \( C, D \in SL(2, \mathbb{R}). \) \( \square \)

We finish this section by discussing how the definition of uniformity of cocycles can be simplified in the case of minimal dynamical systems. More specifically, in that case uniformity of convergence follows once convergence holds for all elements of the dynamical system. We learned this result from Benjamin Weiss. The result and its proof do not seem to have appeared in print and with his kind permission we include it here.

**Theorem 1.2.** Let \((\Omega, T)\) be a minimal dynamical system and \( f_n : \Omega \to \mathbb{R} \) be continuous functions for \( n \in \mathbb{N} \) with
\[
f_{n+m}(\omega) \leq f_n(\omega) + f_m(T^n\omega)
\]
for all \( \omega \in \Omega \) and \( n, m \in \mathbb{N} \). If \( f_n(\omega)/n \to \phi(\omega) \) for all \( \omega \in \Omega \) then \( \phi \) is constant and the convergence is uniform.

The proof is split in two parts:
(I) Let \( a_n := \min\{f_n(\omega)/n : \omega \in \Omega\} \) and \( A := \liminf a_n \). Then, \( \phi(\omega) \leq A \) for a dense \( G_\delta \) set of \( \omega \)’s.
(II) Let \( b_n := \max\{f_n(\omega)/n : \omega \in \Omega\} \) and \( B := \limsup b_n \). Then, \( \phi(\omega) \geq B \) for a dense \( G_\delta \) set of \( \omega \)’s.

It is clear that the theorem follows from (I) and (II). It remains to show (I) and (II).

**Proof of (I):** Fix \( \alpha > A \) and consider
\[
E^m_\alpha := \{\omega : \inf_{k \geq n} f_k(\omega)/k < \alpha\}.
\]
Then, \( E^m_\alpha \) is open for any \( n \in \mathbb{N} \). It remains to show that each \( E^m_\alpha \) is dense. (Then, the statement will follow from the Baire category theorem by taking intersections over \( n \) and suitable sequences \( \alpha \) converging to \( A \).) To show denseness of \( E^m_\alpha \) we let \( W \) be an open set in \( X \). By minimality there exists an \( N \) such that \( \cup_{j=0}^N T^j W = X \). We now choose \( m \) sufficiently large with \( a_m < \alpha \). Then,
\[
V := \{\omega : f_m(\omega)/m < \frac{a_m + \alpha}{2}\}
\]
is open and non-empty (by definition of \( a_m \)). Thus, we can find \( x \in W \) and \( j \in \{0, \ldots, N\} \) with \( T^j x \in V \). By subadditivity we then have
\[
\frac{f_{m+j}(x)}{m+j} \leq \frac{f_j(x)}{j} + \frac{f_m(T^j x) \cdot m}{m+j} < \alpha.
\]
Here, the last inequality follows from the definition of \( V \) and the largeness of \( m \). This, shows \( x \in E^m_\alpha \). As \( W \) was arbitrary the desired denseness statement holds.

**Proof of (II):** The condition on the \( f_n \) easily yields
\[
f_{m-n}(\omega) \geq -f_n(T^{-n}\omega) + f_m(T^{-n}\omega)
\]
for all $m, n$ with $n \leq m$. We can now consider $\beta < B$,

$$F^n_\beta := \{ \omega : \sup_{k \geq n} f_k(\omega)/k > \beta \}$$

and use $\bigcup_{j=0}^N T^{-j} W = X$ and

$$U := \{ \omega : f_m(\omega)/m > \frac{b_m + \beta}{2} \}$$

to mimick the proof of (I).

This finishes the proof of the theorem. $\square$

As a consequence of the previous theorem a cocycle $A$ over a minimal dynamical system $(\Omega, T)$ is uniform if and only if

$$\lim_{n \to \infty} \frac{1}{n} \log \| A(n, \omega) \|$$

exists for all $\omega \in \Omega$.

2. Subshifts satisfying (LSC) and the main (technical) result

In this section we introduce the class of subshifts that is the main concern in this article, discuss some of their basic properties and state our main technical result for these subshifts.

Consider a subshift $(\Omega, T)$. The subshift is said to satisfy the combinatorial leading sequence condition if there exists a natural number $r \in \mathbb{N}$ and finitely many $\omega(j) \in \Omega$, $j = 1, \ldots, r$, such that the following holds:

$(\alpha)$ There exists $N \in \mathbb{N}$ with

$$W(\Omega)_n = \bigcup_{j=1}^r \{ \omega(j)(-k+1)\ldots \omega(j)(-k+n) : k = 0, \ldots, n \}$$

for all $n \in \mathbb{N}$ with $n \geq N$.

In this case the words $\omega(j)$, $j = 1, \ldots, r$, are called the leading words or leading sequences of the subshift. Clearly, $(\alpha)$ is a condition concerning combinatorics on words. A subshift satisfying the combinatorial leading sequence condition with leading words $\omega(j)$, $j = 1, \ldots, r$, is said to satisfy the cocycle leading sequence condition if, for every locally constant function $A : \Omega \to SL(2, \mathbb{R})$, the following two statements holds:

$(\beta)$ For every $j \in \{1, \ldots, r\}$ the limits

$$\lim_{n \to \pm \infty} \frac{1}{|n|} \log \| A(n, \omega(j)) \|$$

exist. Moreover, all limits have the same value.

$(\gamma)$ For every $j \in \{1, \ldots, r\}$ and every $v \in \mathbb{R}^2 \setminus \{0\}$, at most one of the limits

$$\lim_{n \to \pm \infty} \frac{1}{|n|} \log \| A(n, \omega(j))v \|$$

is negative.
The conditions \((\beta), (\gamma)\) involve cocycles and are - apriori - hard to check. In Section 4 we will provide sufficient combinatorial conditions for validity of \((\beta)\) and \((\gamma)\). These sufficient conditions will be shown to hold in the case of simple Toeplitz subshifts and Sturmian subshifts in Section 5 and Section 6.

**Definition 2.1** (Leading sequence condition (LSC)). A subshift satisfying \((\alpha), (\beta)\) and \((\gamma)\) is said to satisfy the leading sequence condition (LSC).

We next gather some simple properties of subshifts satisfying (LSC).

**Proposition 2.2.** Let \((\Omega, T)\) be a subshift satisfying (LSC) with leading words \(\omega(j), j = 1, \ldots, r\). Then, the following holds:

(a) The subshift is uniquely ergodic.

(b) The inequality \(\|W(\Omega)_n \leq r(n + 1)\) holds for all \(n \in \mathbb{N}\) larger than a suitable \(N \in \mathbb{N}\).

**Proof.** (a) It suffices to show uniform existence of frequencies of words. Let \(v \in W(\Omega)\) be arbitrary. Define \(A : \Omega \to SL(2, \mathbb{R})\) by \(A(\omega) = I\) if \(\omega(0) \ldots \omega(|v| - 1) \neq v\) and

\[
A(\omega) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}
\]

otherwise. Clearly, \(A\) is locally constant. Now, it it not hard to see that

\[
\log \|A(n, \omega(j))\|_n = \log 2^{\sharp v \omega(j)(0) \ldots \omega(j)(|v| + n - 2)}_n
\]

for \(n > 0\) and, similarly,

\[
\log \|A(n, \omega(j))\|_{|n|} = \log 2^{\sharp v \omega(j)(n) \ldots \omega(j)(|v| - 2)}_{|n|}
\]

for \(n < 0\). So, from \((\beta)\) we infer that the frequencies of words exist along all half-sided sequences of the form \(\omega(j)(1)\omega(j)(2) \ldots \omega(j)(|v| - 1)\omega(j)(0), j = 1, \ldots, r\). From \((\alpha)\) we then easily obtain uniform existence of frequencies of words.

(b) This is immediate from \((\alpha)\). \(\square\)

**Remark 2.3.** While we do not need it, we note that (LSC) is stable under morphism. More specifically, consider a subshift \((\Omega, T)\) over \(A\) satisfying (LSC). Let \(B\) be a finite set and \(\varphi : A \to B^n\) be arbitrary. Define \(\Omega_\varphi \subset B^\mathbb{Z}\) as the set of all translates of sequences of the form

\[
\ldots \varphi(\omega(-1))\varphi(\omega(0))\varphi(\omega(1))
\]

for \(\omega \in \Omega\). Then, \((\Omega_\varphi, T)\) is a subshift satisfying (LSC) as well. We leave the details to the reader.

We now turn towards our main result. We start by recalling a lemma essentially due to Ruelle [36] (see [30] [27] as well).
Lemma 2.4. Let \((A_n)\) be a sequence of matrices in \(SL(2, \mathbb{R})\) with \(\sup_{n \in \mathbb{N}} \|A_{n+1}^{-1}A_n\| < \infty\) and assume that \(\Lambda = \lim_{n \to \infty} \frac{\log \|A_n \cdots A_1\|}{n}\) exists and is positive. Then, there exists a unique one-dimensional subspace \(V \subset \mathbb{R}^2\) with
\[
\lim_{n} \frac{\log \|A_n \cdots A_1v\|}{n} = -\Lambda \quad \text{and} \quad \lim_{n} \frac{\log \|A_n \cdots A_1u\|}{n} = \Lambda
\]
for all \(v \in V\) with \(v \neq 0\) and all \(u \notin V\).

After this preparation we can come to the main technical result of the article.

Theorem 2.5 (Main technical result). Assume that the minimal uniquely ergodic subshift \((\Omega, T)\) satisfies (LSC). Then, every locally constant function \(A : \Omega \to SL(2, \mathbb{R})\) is uniform.

Proof. By \((\beta)\) there exist a \(\Lambda \geq 0\) with \(\Lambda = \lim_{n \to \pm \infty} \frac{\log \|A(n, \omega(j))\|}{n}\) for every \(j = 1, \ldots, r\). If \(\Lambda = 0\) the desired statement follows rather easily from \((\alpha)\). So, we consider now the case \(\Lambda \neq 0\). By Theorem 3 of [30] it suffices to show that there exists a \(\delta > 0\) with
\[
\frac{\log \|A(n, \omega)\|}{n} \geq \delta
\]
for all \(\omega \in \Omega\) and all sufficiently large \(n\). This follows from \((\beta)\) and \((\gamma)\). Here are the details: Assume without loss of generality that \(A(\omega)\) only depends on \(\omega(0)\). (The general case can be treated by Proposition 1.1.) By Lemma 2.4 there exists for each \(j \in \{1, \ldots, r\}\) a one dimensional subspace \(V_+^{(j)} \subset \mathbb{R}^2\) with
\[
\frac{\log \|A(n, \omega(j))v\|}{n} \to -\Lambda, n \to \infty,
\]
whenever \(v \in V_+^{(j)} \setminus \{0\}\) and a one dimensional subspace \(V_-^{(j)} \subset \mathbb{R}^2\) with
\[
\frac{\log \|A(-n, \omega(j))v\|}{n} \to -\Lambda, n \to \infty,
\]
whenever \(v \in V_-^{(j)} \setminus \{0\}\). Now, by \((\gamma)\) we infer that \(V_+^{(j)} \neq V_-^{(j)}\) for each \(j \in \{1, \ldots, r\}\). So, again, by Lemma 2.4 we have
\[
\frac{\log \|A(n, \omega(j))v\|}{n} \to \Lambda, n \to \infty
\]
whenever \(v \in V_-^{(j)} \setminus \{0\}\). Now fix \(\hat{v} \in V_-^{(j)} \setminus \{0\}\). Then there exists \(n_1 \in \mathbb{N}\) with
\[
\frac{\log \|A(-n, \omega(j))\hat{v}\|}{n} \leq -\frac{\Lambda}{2} \quad \text{and} \quad \frac{\log \|A(n, \omega(j))\hat{v}\|}{n} \geq \frac{\Lambda}{2}
\]
for all \(n \geq n_1\). Moreover, there exists a number \(n_2\) with
\[
\frac{1}{n} \cdot \max_{l \mid |l| < n_1} (\log \|A(l, \omega(j))\hat{v}\|) \leq \frac{\Lambda}{8}
\]
for all $n \geq n_2$. Let now $n_-, n_+ \geq 0$ with $N := n_- + n_+ \geq \max\{2n_1, n_2\}$. It is easy to see that

$$A(N, T^{-n_-} \omega^{(j)}) = A(n_+, \omega^{(j)}) \cdot A(-n_-, \omega^{(j)})^{-1}.$$ 

We denote $\hat{u} := A(-n_-, \omega^{(j)})\hat{v}$ and obtain

$$\log\|A(N, T^{-n_-} \omega^{(j)})\| \geq \log \left( \frac{\|A(N, T^{-n_-} \omega^{(j)})\hat{u}\|}{\|\hat{u}\|} \right) = \log\|A(n_+, \omega^{(j)})\hat{v}\| - \log\|A(-n_-, \omega^{(j)})\hat{v}\|.$$ 

Clearly, $n_-$ and $n_+$ cannot both be smaller than $n_1$, since we assumed $n_- + n_+ \geq 2n_1$. If $n_- + n_+ \geq n_1$ holds, then the definitions of $n_1$ and $\hat{v}$ imply

$$\frac{\log\|A(N, T^{-n_-} \omega^{(j)})\|}{N} \geq \frac{\log\|A(n_+, \omega^{(j)})\hat{v}\|}{n_+} + \frac{\log\|A(-n_-, \omega^{(j)})\hat{v}\|}{n_-} \geq \frac{\Lambda}{2}.$$ 

If $n_- < n_1$ and $n_+ \geq n_1$ hold, we can use $\frac{n_-}{N} \geq \frac{1}{2}$ as well as $n_- < n_1$ and $N \geq n_2$ to obtain

$$\frac{\log\|A(N, T^{-n_-} \omega^{(j)})\|}{N} \geq \frac{\log\|A(n_+, \omega^{(j)})\hat{v}\|}{n_+} + \frac{\log\|A(-n_-, \omega^{(j)})\hat{v}\|}{n_-} \geq \frac{\Lambda}{8}. $$

The remaining case ($n_- \geq n_1$ and $n_+ < n_1$) can be treated similarly. By Proposition 1.1 we infer that there exists a $\delta > 0$ with

$$\frac{\log\|A(N, T^{-n_-+1} \omega^{(j)})\|}{N} \geq \delta.$$ 

Now, by (α), for every $\omega$ and every sufficiently large $n$, there exists $j \in \{1, \ldots, r\}$ and $n_-, n_+ \geq 0$ such that

$$\omega(0) \ldots \omega(n-1) = \omega^{(j)}(-n_- + 1) \ldots \omega^{(j)}(0) \omega^{(j)}(1) \ldots \omega^{(j)}(n_+).$$

Here, for $n_- = 0$, the part $\omega^{(j)}(-n_-) \ldots \omega^{(j)}(-1)$ denotes the empty word and similarly for $n_+ = 0$. We obtain $A(n, \omega) = A(n_- + n_+, T^{-n_-+1} \omega^{(j)})$ and the desired statement follows. \hfill \Box

**Remark 2.6.** The definition of (LSC) may be weakened and still allow for the above result to hold. Details are discussed in this remark:

(a) Invoking the avalanche principle of Bourgain / Jitomirskaya [8] as in in [10] it is not hard to see that the previous result remains valid if condition (α) is weakened to (α): There exists an $r \in \mathbb{N}$, finitely many $\omega^{(j)} \in \Omega$, $j = 1, \ldots, r$, as well as a sequence $(l_n)$ of natural numbers with $l_n \to \infty$, such that the following holds:

(α) There exists $N \in \mathbb{N}$ with

$$\mathcal{W}(\Omega)_{l_n} = \bigcup_{j=1}^{r} \{\omega^{(j)}(-k + 1) \ldots \omega^{(j)}(-k + l_n) : k = 0, \ldots, l_n\}$$

for all $n \in \mathbb{N}$ with $n \geq N$. 

We refrain from giving details as our main examples satisfy condition \((\alpha)\).

(b) Note that one could also allow for infinitely many leading sequences provided the convergence in \((\beta)\) is uniform over the family of leading sequences.

3. Spectral theory of Jacobi operators associated to (LSC) subshifts

In this section we introduce the Jacobi operators associated to a subshift and then present our first main result, which is a spectral consequence of Theorem 2.5.

Consider a dynamical system \((\Omega, T)\). To continuous functions \(f : \Omega \to \mathbb{R} \setminus \{0\}, g : \Omega \to \mathbb{R}\) we associate a family of discrete operators \((H_\omega)_{\omega \in \Omega}\). Specifically, for each \(\omega \in \Omega\), \(H_\omega\) is a bounded selfadjoint operator from \(\ell^2(\mathbb{Z})\) to \(\ell^2(\mathbb{Z})\) acting via

\[
(H_\omega u)(n) = f(T^n \omega)u(n-1) + f(T^{n+1} \omega)u(n+1) + g(T^n \omega)u(n)
\]

for \(u \in \ell^2(\mathbb{Z})\) and \(n \in \mathbb{Z}\). In the case \(f \equiv 1\) these operators are known as discrete Schrödinger operators. For general \(f\) the name Jacobi operators is often used in the literature. The spectrum of \(H_\omega\), i.e. the set of \(E \in \mathbb{R}\) such that \((H_\omega - E)\) is not invertible, is denoted by \(\sigma(H_\omega)\). If \((\Omega, T)\) is minimal, then the spectrum of \(H_\omega\) does not depend on \(\omega \in \Omega\). We denote it by \(\Sigma(f, g)\). Thus, we have

\[
\Sigma(f, g) = \sigma(H_\omega)
\]

for all \(\omega \in \Omega\) in the minimal case. The spectral theory of the \(H_\omega\) is intimately linked with behaviour of solutions \(u : \mathbb{Z} \to \mathbb{C}\) to the equation

\[
f(T^n \omega)u(n-1) + f(T^{n+1} \omega)u(n+1) + g(T^n \omega)u(n) - Eu(n) = 0
\]

for \(E \in \mathbb{R}\). The behaviour of such solutions in turn can be captured via the function

\[
M^E : \Omega \to SL(2, \mathbb{R}), M^E(\omega) := \begin{pmatrix}
E - g(T^n \omega) & -1 \\
E - f(T^{n+1} \omega) & f(T^n \omega)
\end{pmatrix} \begin{pmatrix}
0 \\
1
\end{pmatrix}.
\]

More specifically, as is well known and not hard to see, \(u\) is a solution of the preceding equation if and only if

\[
\tilde{u}(n) = M^E(n, \omega)\tilde{u}(0)
\]

for all \(n \in \mathbb{Z}\), where

\[
\tilde{u}(n) = \begin{pmatrix}
u(n+1) \\
f(T^{n+1} \omega)u(n)
\end{pmatrix}.
\]

We call \(M^E\) the Jacobi cocycle associated to the energy \(E\). Note that it belongs indeed to \(SL(2, \mathbb{R})\) (and it is exactly to ensure this that we introduced the factor \(f(T^{n+1} \omega)\) in various places above.)

Now, uniformity of the cocycles \(M^E\) implies Cantor spectrum of Lebesgue measure zero. In the Schrödinger case this was shown in [28]. This result was then extended to Jacobi operators in [3]. To state the result in a precise manner, we need one more piece of notation. We say that \((f, g) : \Omega \to \mathbb{R}^2\) is periodic if there exists a natural number \(P \geq 1\) with \((f(T^P \omega), g(T^P \omega)) = (f(\omega), g(\omega))\) for all \(\omega \in \Omega\).
Lemma 3.1 (Theorem 3 in [5]). Let \((\Omega, T)\) be a uniquely ergodic minimal dynamical system and \(f, g : \Omega \to \mathbb{R}\) continuous with \(f(\omega) \neq 0\) for all \(\omega \in \Omega\). Assume that \((f, g)\) is not periodic. Then, \(\Sigma(f, g)\) is a Cantor set of Lebesgue measure zero if \(M^E\) is uniform for every \(E \in \mathbb{R}\).

Given this result it is not hard to prove the main application of our article.

Theorem 3.2. Let \((\Omega, T)\) be a uniquely ergodic minimal subshift satisfying (LSC). Let \(f, g : \Omega \to \mathbb{R}\) be continuous functions taking only finitely many values such that \(f(\omega) \neq 0\) for all \(\omega \in \Omega\) and \((f, g) : \Omega \to \mathbb{R}^2\) is not periodic. Then, the spectrum \(\Sigma(f, g)\) of the associated Jacobi operators is a Cantor set of Lebesgue measure zero.

Proof. As the continuous functions \(f, g\) take only finitely many values, they are locally constant. Thus, the associated cocycle \(M^E\) is locally constant for every \(E \in \mathbb{R}\). Hence, by (LSC) and Theorem 2.5 all \(M^E\) are uniform. Now, the desired statement follows directly from Lemma 3.1. \(\square\)

As we will see in the subsequent sections, there are ample classes of examples to which this theorem applies.

4. Combinatorial criteria for (LSC)

In this section we discuss combinatorial conditions on the subshift ensuring (LSC). Clearly, \((\alpha)\) is already a combinatorial condition. So, we mainly work on providing combinatorial criteria for \((\beta)\) and \((\gamma)\). In the end, we provide a sufficient condition for a subshift to satisfy (LSC). While this condition may not seem particularly pleasing at first sight, it turns out that it can rather easily be checked in examples. In fact, this is how we will treat the examples discussed in the subsequent sections of the article.

Let \(A\) be a finite alphabet and \((\Omega, T)\) a subshift over \(A\). A function \(F : W(\Omega) \to \mathbb{R}\) is called subadditive if \(F(xy) \leq F(x) + F(y)\) holds for all \(x, y \in W(\Omega)\) with \(xy \in W(\Omega)\). For a subadditive function, we define \(F^{(n)} := \max\{\frac{F(x)}{n} : |x| = n\}\) and \(\phi : \mathbb{N} \to \mathbb{R}, \phi(n) := nF^{(n)}\). Then, \(\phi(n + m) \leq \phi(n) + \phi(m)\) for all \(n, m \in \mathbb{N}\) by subadditivity of \(F\). Thus, we infer

\[
\lim_{n \to \infty} \frac{\phi(n)}{n} = \inf \frac{\phi(n)}{n} =: \overline{F}.
\]

The sequence \(p : \mathbb{N} \to A\) is said to satisfy the condition (PQ) if there exists a \(c > 0\) such that for any prefix \(v\) of \(p\) the inequality

\[
\liminf_{n \to \infty} \frac{\tilde{\#}_wp(1) \ldots p(n)}{n} |v| \geq c
\]

holds. Here, \(\tilde{\#}_w\) denotes the maximal number of mutually disjoint copies of \(v\) in \(w\).

After these preparations we can now provide a characterization of those \(p\) for which \(\lim_n \frac{F(p(1) \ldots p(n))}{n} = \overline{F}\) holds. The statement can be seen as a variant of a main result in [29] and the proof is inspired by methods from [29].
Lemma 4.1 (Combinatorial condition for \((\beta)\)). Let \((\Omega, T)\) be a subshift, \(\omega \in \Omega\) and \(p = \omega(1)\omega(2) \ldots\).

(a) The limit \(\lim_{n \to \infty} \frac{F(p(1) \ldots p(n))}{n}\) exists for a subadditive function \(F : \mathcal{W}(\Omega) \to \mathbb{R}\) if the following two assumptions hold:

- \(p\) satisfies \((PQ)\);
- there exists a sequence \(p^{(n)}\) of prefixes of \(p\) with \(\lim_{n \to \infty} \frac{F(p^{(n)})}{|p^{(n)}|} = F\).

(b) If \(\lim_{n \to \infty} \frac{F(p(1) \ldots p(n))}{n}\) exists for every subadditive \(F\), then \(p\) satisfies \((PQ)\).

Remark 4.2. It is possible to replace \((PQ)\) in the previous lemma by the seemingly weaker condition \((PW)\) given as follows: there exists a \(c > 0\) such that for any prefix \(v\) of \(p\) the inequality

\[
\liminf_{n \to \infty} \frac{\#_vp(1) \ldots p(n)}{n} |v| \geq c
\]

holds. This can be shown by the same argument as in [29].

Proof. (a) Let \(v_k, k \in \mathbb{N}\), be an arbitrary sequence of prefixes of \(p\) with \(|v_k| \to \infty\). Then,

\[
\frac{F(v_k)}{|v_k|} \leq \frac{\phi(|v_k|)}{|v_k|}
\]

by definition of \(\phi\). This implies

\[
\limsup_k \frac{F(v_k)}{|v_k|} \leq \limsup_k \frac{\phi(v_k)}{|v_k|} = F.
\]

Thus, it remains to show

\[
\liminf_k \frac{F(v_k)}{|v_k|} \geq F.
\]

Assume the contrary. Then, going to a subsequence if necessary we can assume without loss of generality that there exists \(\delta > 0\) with

\[
(*) \quad \frac{F(v_k)}{|v_k|} \leq F - \delta
\]

for all \(k \in \mathbb{N}\). Let \(\varepsilon > 0\) be arbitrary. By definition of \(F\) and as \(|v_k| \to \infty\), there exists then \(k_0 \in \mathbb{N}\) with

\[
(\diamond) \quad \frac{F(w)}{|w|} \leq F + \varepsilon
\]

for all \(|w| \geq |v_{k_0}|\). Consider now an arbitrary \(k \geq k_0\) and \(p(1) \ldots p(N)\) for a large \(N \in \mathbb{N}\). Then, by \((PQ)\) (and as \(N\) is large) we can write

\[
p(1) \ldots p(N) = x_0v_kv_1v_k \ldots v_kx_m
\]

with suitable (possibly empty) words \(x_k\) and at least \(\frac{Nc}{2|v_k|}\) copies of \(v_k\). After removing every other copy of \(v_k\) we arrive at

\[
p(1) \ldots p(N) = y_0v_ky_2v_k \ldots v_ky_r,
\]
where now

$$|y_j| \geq |v_k|$$

for all $j \in \{1, \ldots, r\}$ and the number $r$ of copies of $v_k$ is still at least $\frac{Nc}{4|v_k|}$. Then, by subadditivity of $F$ we can calculate

\[
\frac{F(p(1)\ldots p(N))}{N} \leq \frac{r|v_k|}{N} \cdot \frac{F(v_k)}{|v_k|} + \sum_{j=1}^{r} \frac{|y_j|}{N} \cdot \frac{F(y_j)}{|y_j|} 
\]

(b) For every prefix $v$ of $p$, we define

$$l_v : \mathcal{W}(\Omega) \to \mathbb{R}, l_v(x) := \tilde{\nu}_v(x) \cdot |v|.$$ 

Then, $-l_v$ is subadditive and, by assumption, the limit

$$\nu(v) := \lim_{N \to \infty} \frac{l_v(p(1)\ldots p(N))}{N}$$

exists. Assume now that $p$ does not satisfy (PQ). Then there exists a sequence $(v_n)$ of prefixes of $p$ with

$$|v_n| \to \infty, \text{ for } n \to \infty, \text{ and } \sum_{n=1}^{\infty} \nu(v_n) < \frac{1}{2}.$$ 

Set $l_n := l_{v_n}$ for $n \in \mathbb{N}$. By the preceding considerations we can choose inductively for each $k \in \mathbb{N}$ a number $n(k)$, with

$$\sum_{j=1}^{k} \frac{l_n(j)(w)}{|w|} < \frac{1}{2}$$

for every prefix $w$ of $p$ with $|w| \geq \frac{|v_n(k+1)|}{2}$. Note that the preceding inequality implies

$$|v_{n(k)}| < \frac{|v_n(k+1)|}{2}.$$
as $\frac{l_n(v_n)}{|v_n|} = 1$. Define the function $l : \mathcal{W}(\Omega) \rightarrow \mathbb{R}$ by

$$l(w) := \sum_{j=1}^{\infty} l_{n(j)}(w).$$

Note that the sum is actually finite for each $w \in \mathcal{W}(\Omega)$ (as all but finitely many of its terms vanish). Obviously, $(-l)$ is subadditive. Thus, by assumption, the limit $\lim_{|w| \rightarrow \infty} \frac{l(w)}{|w|}$ exists. On the other hand, we clearly have

$$\frac{l(v_n)}{|v_n|} \geq \frac{l_{n(k)}(v_{n(k)})}{|v_{n(k)}|} \geq 1$$

as well as by the induction construction

$$\frac{l(w)}{|w|} = \sum_{j=1}^{k} \frac{l_{n(j)}(w)}{|w|} < \frac{1}{2}$$

for any prefix $w$ of $p$ with $\frac{|v_{n(k+1)}|}{2} \leq |w| < |v_{n(k+1)}|$. This gives a contradiction proving (b).

**Lemma 4.3** (Combinatorial condition for $(\gamma)$). Let $(\Omega, T)$ be a subshift and $\omega \in \Omega$. Assume that there exists a sequences $(w_k)_{k \in \mathbb{N}}$, of finite words of increasing length such that $\omega$ looks around the origin as

$$\omega = \ldots w_k w_k w_k \quad \text{or as} \quad \omega = \ldots w_k w_k w_k$$

for each $k \in \mathbb{N}$. Then, for every locally constant $A : \Omega \rightarrow SL(2, \mathbb{R})$ and every $v \in \mathbb{R}^2 \setminus \{0\}$, at most one of the limits $\lim_{n \rightarrow \pm \infty} \frac{1}{|v_n|} \log \|A(n, \omega^{(j)})v\|$ is negative.

**Proof.** This follows by a variant of the so-called Gordon argument (see e.g. [9] for background on this argument in the context of aperiodic order). We provide the details in the case that $A(\omega)$ only depends on $\omega(0)$. The general case follows after some slight modifications invoking Proposition [11]. Assume that $w_k$ exists such that $\omega = \ldots w_k w_k w_k$ (the proof for $\omega = \ldots w_k w_k w_k$ is similar). Define $v_n := A(n, \omega)v$ for $n \in \mathbb{Z}$. As $A$ takes values in $SL(2, \mathbb{R})$ we obtain from Cayley Hamilton theorem that

$$A(|w_k|, \omega)^2 - tr(A(|w_k|, \omega))A(|w_k|, \omega) + Id = 0$$

for all $k \in \mathbb{N}$. Here $Id$ is the $2 \times 2$ identity matrix. Now, by definition of $v_n$ and our assumption we have

$$A(|w_k|, \omega)^2 v = v_{2|w_k|} \text{ and } A(|w_k|, \omega)v = v_{|w_k|}.$$ 

So, we then obtain

$$\|v_{2|w_k|} - tr(A(|w_k|, \omega))v_{|w_k|} + v\| = 0$$

and after multiplication by $A(|w_k|, \omega)^{-1}$ then also

$$\|v_{|w_k|} - tr(A(|w_k|, \omega))v + v_{-|w_k|}\| = 0.$$
Given these equalities a short computation shows
\[
\max \{ \|v_{-|w_k|}\|, \|v_{|w_k|}\|, \|v_{2|w_k|}\| \} \geq \frac{1}{2} \|v\|.
\]
Hence, we infer that \(\|v_n\|\) cannot tend to zero for \(n \to \pm \infty\) if \(\|v\| \neq 0\). So, in particular, we cannot have exponential decay of \(\|v_n\|\) for \(n \to \pm \infty\).

For the general case, where \(A(\omega)\) depends on a finite word around the origin, the values of \(A(|w_k|, T^{-|w_k|} \omega)\), \(A(|w_k|, \omega)\) and \(A(|w_k|, T^{|w_k|} \omega)\) are not necessarily equal. More precisely, there is a finite number \(N\) such that the leftmost and rightmost \(N\) matrices in the products may differ. We can now consider \(T^N \omega\), which satisfies
\[
A(|w_k|, T^{-|w_k|} \cdot g) = A(|w_k|, g) = B \cdot A(|w_k| - 2N, T^{w_k} \cdot g),
\]
where \(B\) is a product of \(4N\) cocycle values and hence bounded. Similar to the computations above, the Cayley-Hamilton theorem yields
\[
\max \{ \|v_{-|w_k|}\|, \|v_{|w_k|}\|, \|v_{2|w_k|}\| \} \geq \frac{\|v\|}{2 \|B\|}.
\]
This shows the claim for \(T^N \omega\). By Proposition 4.1 \(\omega\) and \(T^N \omega\) have the same exponential behaviour, which finishes the proof.

\[\square\]

**Remark 4.4.** Note that the proof actually shows that there does not exist a \(v \in \mathbb{R}^2 \setminus \{0\}\) such that \(\|v_n\|\) tends to zero for both \(n \to \infty\) and \(n \to -\infty\). As discussed in Section 3 the Jacobi cocycle \(M^E, E \in \mathbb{R}\), describes solutions \(u : \mathbb{Z} \to \mathbb{C}\) of the equation
\[
f(T^n \omega)u(n - 1) + f(T^{n+1} \omega)u(n + 1) + g(T^m \omega)u(n) - Eu(n) = 0.
\]
Thus, the Jacobi operator \(H_\omega\) does not have eigenvalues if \(\omega\) satisfies the condition of the Lemma.

**Proposition 4.5 (Sufficient condition for (LSC)).** Let \((\Omega, T)\) be a uniquely ergodic subshift. Then, \((\Omega, T)\) satisfies (LSC) if there exists a one-sided infinite word \(p\) and \(r \in \mathbb{N}\) and finite words \(v^{(1)}, \ldots, v^{(r)}\) and the following conditions hold:

(a′) The words \(\omega^{(j)} := p^{R^{(j)}}|p\) all belong to \(\Omega\) and condition (a) holds.

(b′) The sequence \(p\) satisfies the condition (PQ) and there is a sequence \(p^{(n)}\) of prefixes of \(p\) with \(\lim_{n} F(p^{(n)}/|p^{(n)}|) \to F\) for all subadditive \(F\).

(γ′) For any \(j = 1, \ldots, r\), there exists a sequence \(w_k, k \in \mathbb{N}\), of finite words of increasing length such that \(\omega^{(j)}\) looks around the origin as
\[
\omega^{(j)} = \ldots w_k w_k w_k \text{ or as } \omega^{(j)} = \ldots w_k w_k w_k \ldots.
\]

**Proof.** Clearly \((a′)\) implies validity of \((a)\). Thus, it remains to show that the subshift satisfies \((β)\) and \((γ)\):

The subshift satisfies \((β)\): We first gather a few simple consequences of \((β′)\) and \((γ′)\). Clearly, the subshift is palindromic (i.e. \(w \in \mathcal{W}(\Omega)\) implies \(w^R \in \mathcal{W}(\Omega)\)). Indeed, by minimality any \(w \in \mathcal{W}(\Omega)\) must appear in \(p\) infinitely often and then \(w^R\) appears in \(p^R\) infinitely often. Moreover, with \(F : \mathcal{W}(\Omega) \to \mathbb{R}\) also the function
\[
F^R : \mathcal{W}(\Omega) \to \mathbb{R}, F^R(w) := F(w^R),
\]
is subadditive. Hence, by (PQ) and Lemma 4.1 (a) for any subadditive $F$ also the limit

$$\lim_{n \to \infty} \frac{F(p(n) \ldots p(1))}{n}$$

exists. If $F$ is induced by a locally constant cocycle $A$ via

$$F(w) = \max\{\log \|A(|w|, \omega)\| : \omega \in \Omega \text{ with } \omega(1) \ldots \omega(|w|) = w\}$$

it is easy to see from Proposition 1.1 that

$$\Lambda_+ := \lim_{n \to \infty} \log \|A(n, \omega^{(j)})\| = \lim_{n \to \infty} \frac{F(p(1) \ldots p(n))}{n}$$

and

$$\Lambda_- := \lim_{n \to \infty} \log \|A(-n, \omega^{(j)})\| = \lim_{n \to \infty} \frac{F(p(n) \ldots p(1))}{n}$$

both exist. Moreover, since $\omega^{(j)} = p^{Rv^{(j)}}p$ and $(\gamma')$ yield

$$p(|w_k| - |v^{(j)}|) \ldots p(1) = p(1) \ldots p(|w_k| - |v^{(j)}|)$$

and the words $w_k$ get arbitrary large, we must have $\Lambda_+ = \Lambda_-$. This gives $(\beta)$.

The subshift satisfies $(\gamma)$: This follows from the previous lemma. \qed

5. Simple Toeplitz subshifts satisfy (LSC)

In this section we consider simple Toeplitz subshifts. It is well-known that these are aperiodic. We show that they satisfy (LSC). Then, by Theorem 2.5 every locally constant cocycle over such a subshift is uniform and by Theorem 3.2 the associated Jacobi operators have Cantor spectrum. As discussed in the introduction, this generalizes the results of [32] and can be seen as the main results of this article. We only have to show that simple Toeplitz subshifts satisfy the conditions of Proposition 4.5.

First, we recall the definition of a (simple) Toeplitz subshift: Let $\omega \in \mathcal{A}^\mathbb{Z}$ be a two-sided infinite word such that for all $j \in \mathbb{Z}$ there exists $p \in \mathbb{N}$ with $\omega(j + kp) = \omega(j)$ for all $k \in \mathbb{Z}$. The orbit closure of such an $\omega$ under the shift action is called a Toeplitz subshift. For more details, we refer the reader to the survey [14] and the references therein. Here, we will only consider the subclass of so called simple Toeplitz subshifts, which exhibit additional structure and are defined as follows: Let $(a_k)_k \in \mathcal{A}^{\mathbb{N} \setminus \{0\}}$ be a sequence of letters and $(n_k)_k \in (\mathbb{N} \setminus \{1\})^{\mathbb{N} \setminus \{0\}}$ a sequence of period lengths that are greater or equal two. Those sequences are called coding sequences of a simple Toeplitz subshift. Let $\mathcal{A}$ denote the eventual alphabet, that is, the set of letters that appear infinitely often in $(a_k)$. In the following, we will always assume $\#\mathcal{A} \geq 2$ in order to exclude periodic words. Moreover, we assume $a_{k+1} \neq a_k$, since consecutive occurrences of the same letter can be expressed as a single occurrence if $n_k$ is increased accordingly. We use $K$ to denote a number such that $a_k \in \mathcal{A}$ holds for all $k \geq K$. We define the subshift from palindromic blocks: Let

$$p^{-1} := \epsilon \quad \text{and} \quad p^{(k+1)} := p^{(k)} a_{k+1} p^{(k)} \ldots p^{(k)} a_{k+1} p^{(k)}$$
with \( n_{k+1} \)-many \( p^{(k)} \)-blocks and \( (n_{k+1} - 1) \)-many \( a_{k+1} \)’s. Clearly, \( p^{(k)} \) is a palindrome for every \( k \in \mathbb{N} \cup \{0\} \). Note that \( |p^{(k)}(k+1)| + 1 = n_{k+1}(|p^{(k)}| + 1) \) holds for all \( k \geq -1 \), which implies \( |p^{(k)}(k+1)| + 1 = \prod_{j=0}^{k+1} n_j \). Moreover, \( p^{(k)} \) converges to a one-sided infinite word \( p^{(\infty)} := \lim_{k \to \infty} p^{(k)} \). Define \( \Omega := \{ \omega \in A^\mathbb{Z} : \mathcal{W}(\omega) \subseteq \mathcal{W}(p^{(\infty)}) \} \).

Alternatively, simple Toeplitz subshifts can also be defined via a “hole filling procedure”. As this is instructive, we include some details next (see Section 7 and [32] as well): Let \( p \) with \( k \) for every \( k \in \mathbb{N} \cup \{0\} \) be an additional letter which represents the “hole”. In addition to the sequences \((a_k)\) and \((n_k)\) above, let \((r_k)_{k \in \mathbb{N} \cup \{0\}}\) be a sequence of integers with \( 0 \leq r_k < n_k \). We define the two-sided infinite, periodic words

\[
(a_{n_k-1}^k) : = \ldots a_k \ldots a_k \; ? \; a_k \ldots a_k \; ? \; a_k \ldots a_k \; ? \; \ldots
\]

with period length \( n_k \) and holes at \( n_k \mathbb{Z} + r_k \). We now insert \((a_1^{n_1-1})\) into the holes of \((a_0^{n_0-1})\), that is, we define a new word \((a_0^{n_0-1}) \prec (a_1^{n_1-1})\) by

\[
((a_0^{n_0-1}) \prec (a_1^{n_1-1}))((j)) := \begin{cases} (a_0^{n_0-1})(j) & \text{for } j \notin n_0 \mathbb{Z} + r_0 \\ (a_1^{n_1-1})(\frac{j-r_0}{n_0}) & \text{for } j \in n_0 \mathbb{Z} + r_0 . \end{cases}
\]

By inserting \((a_2^{n_2-1})\) into the holes of the obtained word, then inserting \((a_3^{n_3-1})\), and so on, we obtain a sequence of two-sided infinite words

\[
\omega_k := \ldots \omega_k \prec (a_1^{n_1-1}) \prec (a_2^{n_2-1}) \prec \ldots \prec (a_k^{n_k-1}) \prec \ldots
\]

When we take the limit \( \omega_{\infty} := \lim_{k \to \infty} \omega_k \) in \((A \cup \{\?\})^\mathbb{Z}\), there is at most one position where \( \omega_{\infty} \) has a hole. If such a position exists, then we fill the hole by an arbitrary letter from \( \mathcal{A} \). If now \( \omega \in \mathcal{A}^\mathbb{Z} \) denotes the word that was obtained this way, we define the simple Toeplitz subshift as \( \Omega := \{T^k \omega : k \in \mathbb{N}\} \). It is equal to the subshift that was defined above in terms of \( p^{(\infty)} \) (see for example [37], Proposition 2.6).

It was shown in [32], Corollary 2.1 that every simple Toeplitz subshift \((\Omega, T)\) is minimal and uniquely ergodic. In addition, \#\( \bar{\mathcal{A}} \geq 2 \) implies that every simple Toeplitz word defined by \((a_k)\) is non-periodic (see for example [37], Proposition 2.2). Conversely, \#\( \bar{\mathcal{A}} = 1 \) clearly gives periodicity of the subshift. In Proposition 2.4 in [32] it was shown that every element in the subshift can be obtained by the hole filling procedure with the same coding sequences \((a_k)\) and \((n_k)\). From this, it easily follows that for every \( k \in \mathbb{N} \cup \{0\} \) and every \( \omega \in \Omega \), there is a unique decomposition of \( \omega \) in the form

\[
\omega = \ldots p^{(k)} \star p^{(k)} \star p^{(k)} \star p^{(k)} \ldots
\]

where \( \star \) denotes elements from \( \{a_j : j \geq k + 1\} \).

To prove that simple Toeplitz subshifts satisfy (LSC), we will show that they satisfy the sufficient conditions of Proposition 1.5. This will be done in two steps. In the first one, we discuss the words \( \omega^{(j)} \) and study their combinatorial properties. In the second one, we treat asymptotic averages of subadditive functions.

We start with the discussion of the words \( \omega^{(j)} \). Consider the one-sided infinite word \( p = p^{(\infty)} \). With \( r = \#\mathcal{A} \), we write \( \mathcal{A} = \{a^{(1)}, \ldots, a^{(r)}\} \) and define the words \( v^{(j)} \) of length
one by $v^{(j)} = a^{(j)}$. Then for every $j$, there are arbitrary large $k$ with $a_{k+1} = a^{(j)}$. Thus the words $p^{(k)} a^{(j)} p^{(k)}$ occur in the subshift and hence all

$$\omega^{(j)} := p^R v^{(j)}|p = (p^{(\infty)})^R a^{(j)}|p^{(\infty)}$$

belong to $\Omega$. Next we check that all sufficiently long finite words occur in some $\omega^{(j)}$ close to the origin. Recall that $\tilde{K}$ denotes a number such that $a_k \in \tilde{A}$ holds for all $k \geq \tilde{K}$.

**Proposition 5.1** (Occurrence of words around the origin). For all $L \geq |p^{(\tilde{K})}|$ and all $u \in \mathcal{W}(\Omega)_L$, there are $j \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, L\}$ such that $u = \omega^{(j)}(-k) \ldots \omega^{(j)}(-k + L - 1)$ holds.

**Proof.** For every $u \in \mathcal{W}(\Omega)_L$ there exists an $\omega \in \Omega$ such that $u$ occurs in $\omega$. Let $k$ denote the unique number such that $|p^{(k)}| < L \leq |p^{(k+1)}|$ holds and decompose $\omega$ as $\omega = \ldots p^{(k+1)} \ast p^{(k+1)} \ast p^{(k+1)} \ldots$ with single letters $\ast \in \{a_j : j \geq k + 2\}$. We distinguish two cases: Firstly, if $u$ is completely contained in $p^{(k+1)} = p^{(k)} a_{k+1} p^{(k)} \ldots p^{(k)}$, then $u$ contains at least once the single letter $a_{k+1}$. Choose $j$ such that $a^{(j)} = a_{k+1}$ holds. Around the origin, $\omega^{(j)}$ has the form

$$\ldots p^{(k+1)} a^{(j)}|p^{(k+1)} \ldots = \ldots p^{(k)} a_{k+1} p^{(k)} \ldots p^{(k)} a_{k+1} p^{(k)} \ldots p^{(k)} \ldots \ldots \ldots$$

Now the claim follows by aligning an occurrence of $a_{k+1}$ in $u$ with $\omega^{(j)}(-1)$. Secondly, if $u$ is not contained in a single $p^{(k+1)}$-block, then there is a letter $a \in \{a_j : j \geq k + 2\} \subseteq \tilde{A}$ such that $u$ is contained in $p^{(k+1)} a p^{(k+1)}$ and $a$ is contained in $u$. Choose $j$ such that $a^{(j)} = a$. Around the origin, $\omega^{(j)}$ is of the form $\ldots p^{(k+1)} a^{(j)}|p^{(k+1)} \ldots$ and thus $u$ occurs in $\omega^{(j)}$ as claimed. \hfill \Box

Finally, we check property $(\gamma')$ of the sufficient conditions in Proposition 4.5.

**Proposition 5.2** (Occurrence of three blocks). For any $j = 1, \ldots, r$, there exists a sequence $w_i$ of words of increasing length such that $\omega^{(j)} = \ldots w_i w_i|w_i \ldots$ holds for all $i \in \mathbb{N}$.

**Proof.** For every $a^{(j)} \in \tilde{A}$, there is an increasing sequence $k$, with $a_k = a^{(j)}$. Consider the finite words $w_i = p^{(k_i-1)} a_k$. Since every $p^{(k)}$ is a prefix as well as a suffix of $p^{(k+1)}$, the element $\omega^{(j)}$ looks around the origin like

$$\ldots p^{(k)} a^{(j)}|p^{(k)} \ldots = \ldots p^{(k_i-1)} a_k p^{(k_i-1)} a_k |p^{(k_i-1)} a_k|p^{(k_i-1)} a_k \ldots = \ldots w_i w_i|w_i \ldots$$

\hfill \Box

After having discussed the words $\omega^{(j)}$, we now discuss averages of subadditive functions. First we show that the blocks $p^{(k)}$ are prefixes with the limit property that is required in condition $(\beta')$ in Proposition 5.3.

**Proposition 5.3.** Let $F : \mathcal{W}(\Omega) \rightarrow \mathbb{R}$ be a subadditive function and let

$$\overline{F} := \lim_{L \to \infty} \max_{\|x\| = L} \frac{F(x)}{L}.$$

Then $\lim_{k \to \infty} \frac{F(p^{(k)})}{|p^{(k)}|} = \overline{F}$ holds.
Proof. By definition of $\bar{F}$, we have $\limsup_{k \to \infty} \frac{F(p^{(k)})}{|p^{(k)}|} \leq \bar{F}$. Therefore, it only remains to show that $\liminf_{k \to \infty} \frac{F(p^{(k)})}{|p^{(k)}|} \geq \bar{F}$ holds: Let $D := \max\{F(a) : a \in A\}$ and fix an arbitrary $k \in \mathbb{N}$. Let $L \geq |p^{(k)}|$ and let $x$ be an arbitrary word of length $L$. Then $x$ is contained in some element $\omega \in \Omega$ and every $\omega$ can be decomposed as $\omega = \ldots p^{(k)} \ast p^{(k)} \ast p^{(k)} \ldots$ with single letters $\ast \in A$. We obtain $x = u \ast p^{(k)} \ast p^{(k)} \ldots v$, where $u$ is a suffix and $v$ is a prefix of $p^{(k)}$. Note that we have at most $\frac{L}{|p^{(k)}|}$ blocks $p^{(k)}$ and $\frac{L}{|p^{(k)}|} + 1 < \frac{2L}{m|k|}$ single letters $\ast$ in $x$. Hence we obtain

$$\frac{F(x)}{L} \leq \frac{F(u)}{L} + \frac{2D}{|p^{(k)}|} \cdot D + \frac{L}{|p^{(k)}|} : F(p^{(k)}) + \frac{F(v)}{L} \leq \frac{|p^{(k)}|D}{L} + \frac{2D}{|p^{(k)}|} + \frac{F(p^{(k)})}{|p^{(k)}|} + \frac{|p^{(k)}|D}{L}.$$ 

Since $x$ was arbitrary, the above yields for every $k$ and every $L \geq |p^{(k)}|$ the inequality

$$\max_{x:|x|=L} \frac{F(x)}{L} \leq \frac{|p^{(k)}|D}{L} + \frac{2D}{|p^{(k)}|} + \frac{F(p^{(k)})}{|p^{(k)}|} + \frac{|p^{(k)}|D}{L}.$$ 

In particular we can take the limit $L \to \infty$ and obtain for every $k$

$$\bar{F} \leq \frac{2D}{|p^{(k)}|} + \frac{F(p^{(k)})}{|p^{(k)}|} \quad \text{and thus} \quad \bar{F} \leq \liminf_{k \to \infty} \frac{F(p^{(k)})}{|p^{(k)}|}. \quad \Box$$

To prove (LSC) for simple Toeplitz subshifts, it only remains to show that condition (PQ) is satisfied.

**Proposition 5.4** ( Validity of (PQ)). For every prefix $v$ of $p^{(\infty)}$, the inequality

$$\liminf_{k \to \infty} \frac{\#_{\ast}p^{(\infty)}(1) \ldots p^{(\infty)}(L)}{L} |v| \geq \frac{1}{8}\quad$$

holds, that is, the sequence $p^{(\infty)}$ satisfies (PQ).

Proof. Let $v$ be a prefix of $p^{(\infty)}$. Let $K$ be such that $|p^{(K)}| < |v| \leq |p^{(K+1)}|$ holds and let $m$ be such that $m(|p^{(K)}| + 1) \leq |v| < (m + 1)(|p^{(K)}| + 1)$ holds. Note that this implies $1 \leq m < n_{K+1}$. First we compute an auxiliary result. Clearly

$$\frac{\#_{\ast}p^{(K+1)} |v|}{|p^{(K+1)}| + 1} \geq \frac{m(|p^{(K)}| + 1)}{n_{K+1}} \frac{m}{n_{K+1}(|p^{(K)}| + 1)} = \frac{n_{K+1}}{m + 1} \frac{m}{n_{K+1}}$$

holds. To see that this term is bounded away from zero, we distinguish two cases:

- If $\frac{n_{K+1}}{m + 1} < 2$ holds, then we obtain $\frac{m}{n_{K+1}} \cdot \frac{m}{n_{K+1}} = 1 \cdot \frac{m}{n_{K+1}} > \frac{1}{4}$.
- If $\frac{n_{K+1}}{m + 1} \geq 2$ holds, then we obtain $\frac{n_{K+1}}{m + 1} \cdot \frac{m}{n_{K+1}} + 1 \cdot \frac{m}{n_{K+1}} = \frac{(n_{K+1} + 1)}{m + 1} \cdot \frac{m}{n_{K+1}} \geq \frac{1}{4}.$
It is now easy to provide the necessary bound on \( \tilde{\gamma}_v p^{(\infty)}(1) \ldots p^{(\infty)}(L) \) from below:

\[
\tilde{\gamma}_v p^{(\infty)}(1) \ldots p^{(\infty)}(L) \cdot \frac{|v|}{L} \geq \frac{|v|}{p^{(K+1)}(1) \ldots p^{(K+1)}(L)} \cdot \frac{|v|}{p^{(K+1)}(1) \ldots p^{(K+1)}(L)} = \frac{|v|}{L} \geq \frac{|v|}{p^{(K+1)}(1) \ldots p^{(K+1)}(L)} \cdot \frac{1}{L} 
\]

For all sufficiently large \( L \) we have \( \frac{|p^{(K+1)}(1) + 1}{L} \leq \frac{1}{4} \), which yields the claim. \( \square \)

We summarize the content of the preceding propositions in the next theorem.

**Theorem 5.5.** Any simple Toeplitz subshift satisfies (LSC). In particular, all locally constant cocycles over simple Toeplitz subshifts are uniform.

**Proof.** The preceding propositions show that the assumptions of Proposition 4.5 are satisfied. This proves the first statement. The last statement is an immediate consequence of the first statement and Theorem 3.2. \( \square \)

**Remark 5.6 (Purely singular continuous spectrum).** When combined with Theorem 3.2, the previous theorem implies that the spectrum of an aperiodic Jacobi operator associated to a simple Toeplitz subshift is a Cantor set of Lebesgue measure zero. Thus, the spectrum is singular. Moreover, it can be shown that, for almost all \( \omega \in \Omega \) with respect to the unique ergodic probability measure, \( H_\omega \) does not have eigenvalues. If \( n_k \geq 4 \) holds for all \( k \geq 0 \), then the spectrum of \( H_\omega \) is actually purely singular continuous for all \( \omega \in \Omega \), that is, no \( H_\omega \) has eigenvalues (see also Theorem 1.3 in [32]).

As mentioned above, simple Toeplitz subshifts with eventual alphabet containing at least two letters are aperiodic. This aperiodicity is stable under taking (suitable) morphisms. This will be relevant in the application to Jacobi operators. Specifically, we will need the following proposition.

**Proposition 5.7.** Let \((\Omega, T)\) be a simple Toeplitz subshift with eventual alphabet \( \tilde{A} \) containing at least two letters. Let \( \mathcal{B} \) be an arbitrary finite set and \( \Phi : \tilde{A} \rightarrow \mathcal{B} \) not constant on \( \tilde{A} \). Define for \( \omega \in \Omega \) the word \( \Phi(\omega) \) in \( \mathcal{B}^{\mathbb{Z}} \) via

\[
\Phi(\omega)(n) := \Phi(\omega(n)).
\]

Then, \((\Phi(\Omega), T)\) is an aperiodic simple Toeplitz subshift.

**Proof.** Clearly, \( \Phi(\Omega) \) is simple Toeplitz with eventual alphabet \( \Phi(\tilde{A}) \). By assumption on \( \Phi \) this alphabet has at least two elements and the statement follows. \( \square \)

**Remark 5.8 (Boshernitzan condition and simple Toeplitz subshifts).** As mentioned in the introduction of this article not all simple Toeplitz satisfy Boshernitzan condition. Indeed, an explicit characterization of the Toeplitz subshifts satisfying this condition is given in [32]. In Section 7 it will be seen that the class of simple Toeplitz subshifts where \( n_k \) is a power
of two for all \( k \in \mathbb{N} \cup \{0\} \), will be of particular interest to us. When restricted to this case one obtains that validity of Boshernitzan condition is equivalent to existence of a natural number \( C \) and a sequence \((t_r)\) of natural numbers with \( t_r \to \infty \) and \( \{a_{t_r}, \ldots, a_{t_r+C}\} = \{a_s : s \geq t_r\} \) for all \( r \) (Corollary 6.5 in [37]).

6. **Sturmian Subshifts satisfy (LSC)**

In this section we show that Sturmian subshifts satisfy (LSC). By our main results the associated Jacobi operators then have Cantor spectrum of Lebesgue measure zero. Of course, this is well known, [7], but we include the discussion for completeness.

We will show that Sturmian subshifts satisfy the conditions of Proposition 4.5. Let us first recall how Sturmian subshifts are defined. Here we freely follow [6] (see [31] as well). Let \( \alpha \) be an irrational number with continued fraction expansion
\[
\alpha = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}
\]
Define recursively the words \( s_n, n \in \mathbb{N} \cup \{0\} \) over the alphabet \{0, 1\} via
\[
s_{-1} := 1, \quad s_0 := 0, \quad s_1 := s_0^{a_1-1}s_{-1}, \quad s_n := s_{n-1}s_{n-2}, \quad n \geq 2.
\]
Define
\[
\mathcal{W}(\alpha) := \bigcup_n \mathcal{W}(s_n)
\]
and set
\[
\Omega(\alpha) := \{\omega \in \{0, 1\}^\mathbb{Z} : \mathcal{W}(\omega) \subset \mathcal{W}(\alpha)\}.
\]
Then, \( \Omega(\alpha) \) is invariant under \( T \) and closed, and \((\Omega(\alpha), T)\) is called the Sturmian subshift with rotation number \( \alpha \).

There exist palindromes \( \pi_n, n \geq 2 \) with
\[
s_{2n} = \pi_{2n}10,
\]
\[
s_{2n+1} = \pi_{2n+1}01.
\]
Moreover, clearly \( s_{n-1} \) is a prefix of \( s_n \) for \( n \geq 2 \) and we can therefore define the 'right limit'
\[
c_\alpha := \lim_{n \to \infty} s_n.
\]
Analogously, for \( n \geq 2 \) also \( s_{n-1} \) is a suffix of \( s_{n+1} \). This gives existence of the 'left-limits'
\[
d_\alpha := \lim_{n \to \infty} s_{2n}, \quad e_\alpha := \lim_{n \to \infty} s_{2n+1}.
\]
Define \( u_\alpha \) to be the two-sided infinite word which agrees with \( c_\alpha \) on \( \mathbb{N} \) and with \( d_\alpha \) on \( -\mathbb{N} \cup \{0\} \) and define \( v_\alpha \) to be the two-sided infinite word which agrees with \( c_\alpha \) on \( \mathbb{N} \) and with \( e_\alpha \) on \( -\mathbb{N} \cup \{0\} \). Then, both \( u_\alpha \) and \( v_\alpha \) belong to \( \Omega(\alpha) \).

**Lemma 6.1.** Every Sturmian subshift satisfies (LSC) with \( r = 2 \) and \( \omega^{(1)} = u_\alpha \) and \( \omega^{(2)} = v_\alpha \).
Proof. We show that the assumptions of Proposition 4.5 are satisfied with $p = c \alpha$, $v^{(1)} = 01$ and $v^{(2)} = 10$:

Condition ($\alpha'$) is satisfied: As discussed above $s_n = \pi_n ab$ with $a, b \in \{0, 1\}$ for $n \geq 2$. From the definition of $u_\alpha$ and $v_\alpha$ we then directly infer that $\omega^{(1)} = pRv^{(1)}p$ and $\omega^{(2)} = pRv^{(2)}p$. So, it remains to show the statement on occurrences of words around the origin in $\omega^{(1)}$ and $\omega^{(2)}$. This is not hard to see and is given e.g. in Lemma 6.6 of [31]. This finishes the proof of ($\alpha'$).

Condition ($\beta'$) is satisfied: This follows from Theorem 11 of [31] (but it is not hard to give a direct proof).

Condition ($\gamma'$) is satisfied: A short computation shows that $(s_{n}s_{n+1})^* = (s_{n+1}s_{n})^*$ for $n \geq 3$, where the * indicates that the last two letters are removed. From this we easily see that

$$\omega^{(1)} = \ldots s_{2n+1} | s_{2n+1} s_{2n+1} \ldots$$

and

$$\omega^{(2)} = \ldots s_{2n} | s_{2n} s_{2n} \ldots$$

for each $n \geq 1$. □

7. Spectra of Schreier graphs of spinal groups

In this section we will apply our results to an interesting class of examples that served as an initial motivation to this work. Our examples are from a seemingly different context, that of finitely generated groups of automorphisms of rooted trees. These groups came into light chiefly after the discovery in this class of first examples of groups on intermediate growth [19]. The construction provides an uncountable family $\{G_\xi\}, \xi \in \{0, 1, 2\} \N$ of groups of automorphisms of the infinite binary tree with growth strictly between polynomial and exponential, if the sequence $\xi$ is not eventually constant. Each group $G_\xi$ is generated by 4 involutions. One example that has been particularly well studied and enjoys additional nice properties, like being generated by a finite automaton, is the so-called “first Grigorchuk’s group” $G = \langle a, b, c, d \rangle$ (see e.g. Chap. VIII in [13]). In the family $\{G_\xi\}$ it corresponds to $\xi = (012)^\infty$.

An action of a group $G$ by automorphisms on a rooted spherically homogeneous tree $T$ (that we will always assume transitive on the levels of the tree) extends to an action by homeomorphisms on the boundary of the tree $\partial T$. If a finite generating set $S$ is chosen in $G$, then the orbital partition corresponding to the action of $G$ on $\partial T$ gives rise to a map $\Phi$ from $\partial T$ to the space of (isomorphism classes of) rooted regular graphs with edges labelled by elements of $S$. The closure of its image with isolated points removed is called the space of (orbital) Schreier graphs of $G$ (with respect to $S$) and is denoted $Sch(G)$. The group acts on it by changing the root. Then, the dynamical system $(Sch(G), G)$ is minimal and uniquely ergodic (with unique invariant probability measure given by the pushforward of the uniform measure on the boundary of the tree under $\Phi$). Schreier graphs are interesting objects in their own right and serve as a useful tool in the study of the group. More generally, given a finitely generated group $G$ and a subset $S \subset G$, a Schreier graph can be defined for any transitive action of $G$ on a set $X$: the vertex set of the graph is the set $X$ and the set of (labelled oriented) edges is $\{(x, s \cdot x) | x \in X, s \in S\}$. The graph is connected
if and only if \( S \) generates \( G \). It is regular of degree \(|S|\). As a particular case, if \( S = S^{-1} \) and the action of \( G \) on \( X \) is free, we get the Cayley graph of \( G \) with respect to \( S \).

For a finitely generated group \( G \) and a chosen finite symmetric generating set \( S \), the corresponding Cayley graph and Schreier graphs for natural group actions present in particular an interesting class of examples in spectral graph theory that investigates the spectra of Laplacians acting on the \( l^2 \)-space on the vertices of the graph. Here, we prefer to consider Markov operators

\[
M = \sum_{s \in S} p_s s
\]

with \( p_s > 0 \), \( p_s = p_{s^{-1}} \), for all \( s \in S \), and \( \sum_{s \in S} p_s = 1 \) (or, more generally, \( \sum_{s \in S} p_s \leq 1 \)).

The corresponding Laplacian is then just \( 1 - M \). Clearly, the spectral type does not change if we add constants to the operator. Thus, we can deal with Laplacians as well.

The question about spectral type of Schreier graphs and Cayley graphs of finitely generated groups is in general widely open.

The paper [3] was one of the first to address the spectral theory of Schreier graphs of groups acting on rooted trees. It presents an example of a group whose orbital Schreier graphs for the action on the boundary of the tree have Cantor spectrum of Lebesgue measure 0 and a countable set of points, and another example, the group \( G \) mentioned above, where this spectrum is a union of two disjoint intervals. They only considered the isotropic Markov operator, i.e., with \( p_s = 1/|S| \) for all \( s \in S \). Note that, as the groups in question are amenable, the spectrum as a set is an invariant of the space of Schreier graphs and does not depend on a particular orbit.

The construction from [19] has been generalized in a number of ways. One way, initially suggested in [4], leads to the so-called spinal groups, of which we describe here one particular construction. For each \( d \geq 2, m \geq 1 \), we consider an uncountable family of groups \( \{G_{\xi}\} \) acting by automorphisms on the infinite \( d \)-regular rooted tree \( T_d \), with \( \xi \in \Xi_{d,m} \subset \text{Epi}(B, A)^\mathbb{N} \), where \( A = \mathbb{Z}/d\mathbb{Z} \), \( B = (\mathbb{Z}/d\mathbb{Z})^m \) and \( \Xi_{d,m} \) consists of all infinite sequences of epimorphisms that have trivial intersection of kernels over any tail. For \( \xi \in \Xi_{d,m} \), the group \( G_{\xi} \) is generated by the automorphism \( a \) that cyclically permutes the branches at the root of the tree and a copy \( B_{\xi} \) of \( B \) in \( \text{Aut}(T_d) \). The action of the elements from \( B_{\xi} \) on the tree can be described as follows: Any \( b_{\xi} \in B_{\xi} \) acts trivially everywhere but on the subtrees rooted at the vertices of the rightmost infinite ray in the tree. In the subtree rooted in the vertex at the \( r \)-th level of \( T_d \), it acts by permuting the branches at the root of the subtree as \( \xi_r(b) \) (see [35, 22] for a more detailed description of this specific class of spinal groups). For \( d = 2, m = 2 \) we recover the family from [19]. More generally, all of these examples with \( d = 2 \) are of intermediate growth.

For \( d = 2 \), the Schreier graphs of a spinal group \( G_{\xi} \) with respect to the generating set \( S_{\xi} = \{a\} \cup B_{\xi} \setminus \{\text{id}\} \) have the similar structure: they are lines with loops and multiple edges.

The linear structure of Schreier graphs allows to associate a subshift to the dynamical system \( (\text{Sch}(G_{\xi}), G_{\xi}) \). It was shown in [21] that the Markov operators on the Schreier graphs become then unitary equivalent to the Schrödinger operators on the associated subshift. It is also shown there that the subshift associated with the first example \( G \) is
linearly repetitive, and hence the Cantor spectrum of Lebesgue measure 0 theorem for Schrödinger operators on linearly repetitive subshifts applies and yields new information about the Laplacian spectrum on the orbital Schreier graphs for the action of $G$ on the boundary of $T_d$. Namely, while the periodic potential corresponds to the isotropic Markov operator whose spectrum was already known from [3], the case of aperiodic potential shows that the spectrum of the anisotropic Markov operator $(p_b, p_c, p_d$ not all equal) is a Cantor set of Lebesgue measure $0$. This has interesting consequences, as it implies in particular Cantor spectrum of Lebesgue measure $0$ for the isotropic Markov operator on the Schreier graph of $G$ with the minimal generating set $\{a, b, c\}$, see [22].

The question arises as to which extent our results hold for other spinal groups acting on the binary tree. It turns out that very few groups in the families $\{G_\xi\}, \xi \in \Xi_{2,m}, m \geq 2$ give rise to linearly repetitive subshifts. It follows from [35] that the set of $\xi \in \Xi_{d,m}$ with linearly repetitive Schreier graphs is of measure $0$ with respect to the Bernoulli measure on the set of parameters. The more general Boshernitzan condition which also would be enough to imply the Cantor spectrum in the aperiodic case is verified on Schreier graphs of groups forming a set of measure $1$ in the space of parameters [35], but not all $G_\xi$ satisfy it either (compare Remark 5.8 as well). Here we prove that for all $m \geq 2$ and all $\xi \in \Xi_{2,m}$, the subshift defined by Schreier graphs of $G_\xi$ is simple Toeplitz. The results from the previous sections then allow us to extend the result from [21] and to deduce the Cantor spectrum of Lebesgue measure $0$ for anisotropic Markov operators on orbital Schreier graphs of an arbitrary spinal group $G_\xi, \xi \in \Xi_{2,m}, m \geq 2$.

Indeed, consider a new alphabet $A = \{a\} \cup \{\alpha_\phi \in Epi(B, A)\}$, so that a letter in the alphabet $A$ is either $a$ or represents $B \setminus Ker(\phi), \phi \in Epi(B, A)$, a possible set of labels on a multi-edge between two vertices in the Schreier graph. Consider the one-sided infinite sequence $\eta$ in this alphabet that we read on the Schreier graph rooted at the boundary point $1^n$, and associate to $G_\xi$ the corresponding two-sided subshift. We now observe that, as the infinite, rooted at $1^{\infty}$ Schreier graph is the limit, as $n \to \infty$, of the finite Schreier graphs on the vertex set of the $n$-th level of $T_d$ rooted at $1^n$, the sequence $\eta$ is the limit of the words in the alphabet $A$ read on these finite Schreier graphs starting from the root $1^n$. The structure of the finite Schreier graphs for the action of spinal groups on the levels of $T_d$ is well understood and can be described recursively, see [35] and also [34] for the special case of Grigorchuk’s family. Translated in words in $A$, this recursion means that $\eta$ is the limit of words of the shape

$$p^{(n+1)} = p^{(n)}\alpha_{\xi_j(n+1)}p^{(n)},$$

with $n > 0$ and $p^{(0)} = a$. Hence, by the definition of a simple Toeplitz subshift as given in Section 5, we conclude that our subshift is simple Toeplitz with eventual alphabet

$$\tilde{A}_\xi = \{\alpha_\phi \in A : \xi_j = \phi \text{ for infinitely many } j \in \mathbb{N}\}.$$ 

By definition of $\Xi_{2,m}$, for every $\xi \in \Xi_{2,m}$, the intersection of kernels of the epimorphisms $\xi_j$ is trivial along any tail of $\xi$, hence the eventual alphabet consists of at least two letters for every $\xi \in \Xi_{2,m}, m \geq 2$. The subshifts that we associate to the groups $\{G_\xi \mid \xi \in \Xi_{2,m}\}, m \geq 2$, are therefore all aperiodic. We arrive at the following result.
Theorem 7.1. The subshift over the alphabet $\mathcal{A}$ associated with a spinal group $G_\xi$, $\xi \in \Xi_{2,m}$, is an aperiodic simple Toeplitz subshift.

This allows us to apply the Theorems 3.2 and 5.5 to conclude Cantor spectrum result for anisotropic Markov operators. The anisotropy of the Markov operator translates into a condition on the weights attached to the letters of the alphabet $\mathcal{A}$ that describe connections in the Schreier graph. Namely, to the letter $\alpha_\phi \in \mathcal{A}$ is attached the weight $q_\phi = \sum_{b \in B \setminus \text{Ker}(\phi)} p_b$, which is positive as we have assumed $p_s > 0$ for all $s \in S_\xi$. Here is the precise statement.

Corollary 7.2. Let $G_\xi$ be a spinal group with $\xi \in \Xi_{2,m}$, $m \geq 2$. If $M$ is a Markov operator on a graph $X \in \text{Sch}(G_\xi)$ such that the numbers $q_\phi$ are not all equal over the essential alphabet $\tilde{\mathcal{A}}_\xi$, then the spectrum of $M$ is a Cantor set of Lebesgue measure 0.

Proof. By the previous theorem and the construction of the subshift, the Markov operators in question can be considered as Jacobi operators on a simple Toeplitz subshift. By Theorem 5.5 simple Toeplitz subshifts satisfy (LSC). In fact, the Markov operator on the Schreier graph becomes exactly the Jacobi operator on the subsshift with the function $f$ taking values $p_a$ and $q_\phi$ over the essential alphabet $\tilde{\mathcal{A}}_\xi$. Proposition 5.7 then ensures, in notations of Theorem 3.2, that $(f, g)$ is aperiodic and that $f(\omega) \neq 0$ for all $\omega$, as required in Theorem 3.2. $\square$

Remark 7.3 (Condition on the weights). From the proof above we see that the conditions on the weights that we really need are: $p_s = p_{s-1}$ for all $s \in S_\xi$, $p_a > 0$ as well as $q_\phi > 0$ and not all equal over the essential alphabet $\tilde{\mathcal{A}}_\xi$. The positivity conditions ensure that the weighted graph is connected and the last condition, as mentioned above, is the anisotropicity of $M$ and is necessary for the aperiodicity of the Jacobi operator on the corresponding subshift.

Remark 7.4 (Spectrum in the isotropic case). The previous corollary deals with the spectrum of the anisotropic Markov operator. Note that the spectrum of the isotropic Markov operator can be computed explicitly. Namely, it is proven in [22] that if $G_\xi$ is a spinal group with $\xi \in \Xi_{2,m}$, $m \geq 2$, then the spectrum of the isotropic Markov operator on the orbital Schreier graph for the action on the boundary of the tree is

$$\left[ -\frac{1}{2^{m-1}}, 0 \right] \cup \left[ 1 - \frac{1}{2^{m-1}}, 1 \right].$$

Moreover it coincides with the spectrum of the isotropic Markov operator on the Cayley graph of this group with respect to the generating set $S_\xi$.

Remark 7.5 (Spectrum of the Schreier graph vs spectrum of the Cayley graph). As noted in the previous remark, for spinal groups $G_\xi$, $\xi \in \Xi_{2,m}$, the spectrum of the Markov operator on the orbital Schreier graph for the action on the boundary of the infinite binary

\[^2\]Should on the other hand $q_\phi = 0$ hold for one of the $\phi$ appearing in $\xi$, the corresponding graphs will be an infinite union of finitely many finite graphs each. Hence, their spectrum consists of finitely many eigenvalues (each with infinite multiplicity).
tree coincides with that of the Markov operator on the Cayley graph, with respect to the spinal generators $S_\xi$. This result is certainly not true in general, and even for these groups it is now known if the spectra are the same in the anisotropic case. Note however that under certain natural conditions the spectrum of a Schreier graph embeds in the spectrum of the Cayley graph (see [12]). More specifically, whenever the group has intermedium growth part (b) of that theorem directly gives inclusion of spectra. This inclusion holds also for amenable groups acting on trees. Indeed, then the infinite Schreier graph $\Gamma$ is a limit of finite Schreier graphs $\Gamma_n$ (corresponding to levels of the tree), and in the case this infinite graph is amenable then its spectrum is the closure of the union of spectra of graphs $\Gamma_n$ (see [3]).

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