COMPARISON OF INVARIANT METRICS ON THE SYMMETRIZED BIDISC

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ABSTRACT. We show that the Bergman metric, the Kobayashi-Royden metric and the complete Kähler-Einstein metric with Ricci curvature equal to \(-1\) are uniformly equivalent on the symmetrized bidisc \(G_2\). Furthermore, we prove that the complete Kähler-Einstein metric on \(G_2\) can be written as the summation of the Ricci-\((1, 1)\) form of the Bergman metric and the complex hessian of some smooth function on \(G_2\) by showing that the Bergman metric admits the quasi-bounded geometry.

1. Introduction and results

In this paper, we study the comparisons between classical invariant metrics on the symmetrized bidisc \(G_2 := \{(z + w, zw) : z, w \in \mathbb{D}\}\), where \(\mathbb{D}\) is the unit disk in \(\mathbb{C}\). A metric \(h\) on a complex manifold \(M\) is called invariant if every biholomorphic map \(f\) is an isometry \(f^*h = h\). Invariant metrics on negatively curved complex manifolds are interesting in complex geometry because geometric structures of invariant metrics depend only on the complex structure. The classical invariant metrics include the Bergman metric, the Carathéodory-Reiffen metric, the Kobayashi-Royden metric and the complete Kähler-Einstein metric of Ricci curvature equal to \(-1\).

The original motivation of the study of \(G_2\) is the robust control theory (for example, see [2]). The symmetrized bidisc \(G_2\) is a bounded inhomogeneous pseudoconvex domain without \(C^1\) boundary which is neither strictly pseudoconvex domain, geometric convex domain nor a pseudoconvex domain of finite type in \(\mathbb{C}^2\). The study of invariant metrics on symmetrized bidisc \(G_2\) is interesting because \(G_2\) serves as the first non-trivial example which is not biholomorphic to any geometric convex domains but still, the Carathéodory-Reiffen metric and the Kobayashi-Royden metric are the same (see [1],[2],[10] and [15]). However, the complete Kähler-Einstein metric of negative scalar curvature on \(G_2\) is much less understood compared to other invariant metrics (for example, see [21]).

The complete Kähler-Einstein metric of negative scalar curvature has been studied extensively on bounded strictly pseudoconvex domains (for example, see [6]). For a bounded weakly pseudoconvex domain, the uniqueness and existence of the complete Kähler-Einstein metric with negative scalar curvature were proved in [19]. Moreover, for a complete noncompact Kähler manifold \(M\), the existence of a complete Kähler-Einstein metric of negative scalar curvature also characterizes the positivity of the canonical line bundle. It was recently shown by Wu-Yau that if a complete Kähler
manifold \((M, \omega)\) has holomorphic sectional curvature negatively pinched, then \(M\) admits a unique complete Kähler-Einstein metric \(\omega_{KE}\) with Ricci curvature equal to \(-1\) and \(\omega_{KE}\) is uniformly equivalent to the Kobayashi metric and the base metric \(\omega\) [22].

Note that the equivalence of those invariant metrics have been established for strictly pseudoconvex domains and geometric convex domains and pseudoconvex domains of finite type (for example, see [4], [7], [12] and [17]). Also, invariant metrics on Kähler manifolds with the uniform squeezing property are equivalent (cf. [23]). Moreover, for a bounded strictly pseudoconvex domain with \(C^4\) boundary in \(\mathbb{C}^n\), there exists a complete Kähler metric whose Riemannian sectional curvature is bounded between two negative numbers (see [16], or Proposition 3.6.1 in the book [13]). It then follows from [22] that such a metric must be uniformly equivalent to the complete Kähler-Einstein metric, Kobayashi-Royden metric, and also with the Bergman metric if the domain is additionally assumed to be simply-connected (see theorem 2 and 3, 7 in [22]). On the contrary, on weakly pseudoconvex domains, the relation of those metrics is less clear than that of strictly pseudoconvex domains (for example, see [8], [7], [17]). In particular, without assuming certain regularity on the boundary, it is hard to describe the complete Kähler-Einstein metric in a more specific manner (for example, see [3] and [20], [9] for some possible cases).

Based on these motivations, we study the complete Kähler-Einstein metric as well as other invariant metrics on \(G^2\) and we prove following results by applying the fundamental results proved in [22]:

**Theorem 1.** The Bergman metric, the Kobayashi-Royden metric and the complete Kähler-Einstein metric with Ricci curvature equal to \(-1\) on the symmetrized bidisc \(G^2\) are uniformly equivalent.

**Theorem 2.** There exists a smooth function \(u\) on \(G^2\) such that the complete Kähler-Einstein metric \(\omega_{KE}\) with Ricci curvature equal to \(-1\) is given by

\[
\omega_{KE} = \sqrt{-1} \partial \bar{\partial} \log \det(g^B_{G^2}) + \sqrt{-1} \partial \bar{\partial} u,
\]

where \(g^B_{G^2}\) is the Bergman metric on \(G^2\).

By Theorem 2, with the global coordinates \((w_1, w_2) \in \mathbb{C}^2\), the metric components of the complete Kähler-Einstein metric \(\omega_{KE}\) on \(G^2\) are given by

\[
\omega_{KE,i\bar{j}} = u_{i\bar{j}} - \text{Ric}_{B,i\bar{j}}
\]

for some real-valued smooth function \(u\), where \(u_{i\bar{j}} = \frac{\partial^2 u}{\partial w_i \partial \bar{w}_j}\) on \(G^2\) and \(\text{Ric}_{B,i\bar{j}}\) is the Ricci curvature of the Bergman metric \(g^B_{G^2}\) on \(G^2\) for \(i, j = 1, 2\).

As a byproduct of Theorem 1, one can immediately construct the example of weakly pseudoconvex domain of any dimension \(n \geq 3\):

**Corollary 3.** For any \(n \geq 3\), a bounded weakly pseudoconvex domain in \(\mathbb{C}^n\):

\[
\Omega = G^2 \times \mathbb{D}^{n-2},
\]
is neither strictly pseudoconvex domain, geometric convex domain nor pseudoconvex domain of finite type, but the Bergman metric, the Kobayashi-Royden metric and the complete Kähler-Einstein metric $\omega_{KE}$ with Ricci curvature equal to $-1$ are uniformly equivalent on $\Omega$; and Kobayashi-Royden metric and the Carathéodory-Reiffen metric are the same on $\Omega$.

2. Curvature tensors of the Bergman metric

$G_2 = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D}\}$ is defined as the image of the bidisc $\mathbb{D}^2$ under $\Phi$, where

$$\Phi : \mathbb{D} \times \mathbb{D} \rightarrow G_2, (z_1, z_2) \mapsto (z_1 + z_2, z_1 z_2) =: (w_1, w_2).$$

The Bergman kernel $B_{G_2}(w, w)$ of $G_2$ was explicit (cf. [11], [18]) and here we describe it by using $B = \Phi^* B_{G_2}$, the pull-back of the Bergman kernel on $\mathbb{D}^2$, given by

$$B(z, z) = \frac{1}{2\pi^2} \frac{1}{(z_1 - z_2)(\overline{z_1} - \overline{z_2})} \left\{ \frac{1}{(1 - z_1 \overline{z_1})^2(1 - z_2 \overline{z_2})^2} - \frac{1}{(1 - z_1 \overline{z_2})^2(1 - z_2 \overline{z_1})^2} \right\}$$

(cf. page 12 in [5]).

Now we recall the characterization of the automorphism group of $G_2$ (cf. [14]).

**Theorem 4.** Any automorphism $H$ of $G_2$ is in the form of $H(\Phi(z_1, z_2)) = \Phi(h(z_1), h(z_2))$ for $h \in \text{Aut}(\mathbb{D})$, where $z_1, z_2 \in \mathbb{D}$.

**Corollary 5.** For any $(w_1, w_2) \in G_2$, there exists $H \in \text{Aut}(G_2)$ such that $H(w_1, w_2) = (x, 0)$ for $x \in [0, 1)$.

**Proof.** For any $z_1 \in \mathbb{D}$, there exists $h \in \text{Aut}(\mathbb{D})$ such that $h(z_1) = 0$. For any $z_2 \in \mathbb{D}$, there exists $\theta \in [0, 2\pi)$ such that $e^{i\theta} h(z_2) = x \in [0, 1)$. Therefore, $\Phi(e^{i\theta} h(z_1), e^{i\theta} h(z_2)) = (x, 0)$. This finishes the proof. $\square$

Since Bergman metric is invariant under automorphism, in order to estimate Bergman metric and its covariant derivatives, it suffices to evaluate at $(x, 0) \in G_2$ or equivalently $(x, 0) \in \mathbb{D} \times \mathbb{D}$ for $x \in [0, 1)$. We will use the coordinate $w_1 = z_1 + z_2, w_2 = z_1 z_2$ on $G_2$ for vector fields $\frac{\partial}{\partial w_i}, i = 1, 2$. Then the metric component of the pullback Bergman metric is given by

$$g_{ij} = \frac{\partial^2 \log B_{G_2}(w, w_i)}{\partial w_i \partial w_j} = B^{-2}_{G_2} \frac{\partial^2}{\partial w_i \partial w_j} B_{G_2} - \partial_i B_{G_2} \partial_j B_{G_2}, i = 1, 2. \tag{2.2}$$

We use the notation $\frac{\partial}{\partial w_1} = \partial_1, \frac{\partial}{\partial w_2} = \partial_2, \frac{\partial}{\partial w_1} = \partial_{\overline{w}_1}, \frac{\partial}{\partial w_2} = \partial_{\overline{w}_2}$. To use the map $\Phi$ in computations, we convert from $\frac{\partial}{\partial w_i}$ to $\frac{\partial}{\partial z_i}$ by the inverse function theorem, and
expressions of \( \frac{\partial z_i}{\partial w_j} \) are given by
\[
\frac{\partial z_1}{\partial w_1} = \frac{z_1}{z_1 - z_2}, \quad \frac{\partial z_1}{\partial w_2} = -1, \quad \frac{\partial z_2}{\partial w_1} = -\frac{z_2}{z_1 - z_2}, \quad \frac{\partial z_2}{\partial w_2} = \frac{1}{z_1 - z_2}.
\] (2.3)
where \( z_1, z_2 \) satisfy \( w_1 = z_1 + z_2, w_2 = z_1 z_2 \). Since we will use \( d\Phi^{-1} = \left( \frac{\partial z_i}{\partial w_j} \right)_{i,j=1,2} \) for computations, we shall use the notation \( \Phi^{-1} \) which makes sense only in the relation \( B_{G^2} = B \circ \Phi^{-1} \) on that given point.

The following proposition follows from direct computations.

**Proposition 6.** The derivatives of \( B \) in (2.1) at \( (x, 0) \in \mathbb{D} \times \mathbb{D}, 0 \leq x < 1 \) are given by
\[
\begin{align*}
\partial_1 B &= \partial_2 B = \frac{x (x^2 - 3)}{2\pi^2 (x^2 - 1)^3}, \\
\partial_1^2 B &= \partial_2^2 B = \frac{x^2 - 3}{2\pi^2 (x^2 - 1)^3}, \\
\partial_1^3 B &= \partial_2^3 B = \frac{-x^4 + 4x^2 + 3}{2\pi^2 (x^2 - 1)^4}, \\
\partial_1^4 B &= \partial_2^4 B = \frac{-x^2 + 3}{2\pi^2 (x^2 - 1)^4}, \\
\partial_1^5 B &= \partial_2^5 B = \frac{-x^2 - 3}{2\pi^2 (x^2 - 1)^4}, \\
\partial_1^6 B &= \partial_2^6 B = \frac{-x^2 + 3}{2\pi^2 (x^2 - 1)^4}, \\
\partial_1^7 B &= \partial_2^7 B = \frac{-x^2 - 3}{2\pi^2 (x^2 - 1)^4}, \\
\partial_1^8 B &= \partial_2^8 B = \frac{-x^2 + 3}{2\pi^2 (x^2 - 1)^4}.
\end{align*}
\]

**Remark 7.** One can verify from computations that all formulas in Proposition 6 at \( (x, 0), 0 \leq x < 1 \in \mathbb{D} \times \mathbb{D} \) coincide at the value \( (0, x), 0 \leq x < 1 \). Hence we can use either \( (x, 0) \) or \( (0, x) \) on \( \mathbb{D} \times \mathbb{D} \) as the elements of the inverse image of \( \Phi \) at \( (x, 0) \in G_2 \).
Proposition 8. The components of the Bergman metric $g_{ij}$ at $(x,0), 0 \leq x < 1 \in G_2$ are given as follows:

\[
\begin{align*}
g_{1\overline{1}} &= \frac{6 - 4x^2}{(x^4 - 3x^2 + 2)^2}, \\
g_{1\overline{i}} = g_{i\overline{1}} &= \frac{2x(x^2 - 2)}{(x^2 - 1)^2}, \\
g_{2\overline{2}} &= -2 \frac{(2x^4 - 6x^2 + 5)}{(x^2 - 2)(x^2 - 1)^2}.
\end{align*}
\]

Proof. The first derivatives of $B \circ \Phi^{-1}$ are

\[
\partial_i B_{G_2} = \frac{\partial}{\partial w_i}(B \circ \Phi^{-1}) = \partial_1 B \frac{\partial z_1}{\partial w_i} + \partial_2 B \frac{\partial z_2}{\partial w_i}, \quad i = 1, 2,
\]

and similar formulas hold for complex conjugate case. So with Proposition 6, computations give that at $(x,0), 0 \leq x < 1$,

\[
\begin{align*}
\partial_1 B_{G_2} &= \partial_1^1 B_{G_2} = \frac{x(x^2 - 3)}{2\pi^2(x^2 - 1)^3}, \\
\partial_2 B_{G_2} &= \partial_2^2 B_{G_2} = -\frac{x^2(x^2 - 2)}{\pi^2(x^2 - 1)^3}.
\end{align*}
\]

For second derivatives of $B \circ \Phi^{-1}$, since

\[
\frac{\partial}{\partial \overline{w}_j} ((\partial_i B) \circ \Phi^{-1}) = \frac{\partial}{\partial \overline{z}_1}(\partial_i B) \frac{\partial \overline{z}_1}{\partial \overline{w}_j} + \frac{\partial}{\partial \overline{z}_2}(\partial_i B) \frac{\partial \overline{z}_2}{\partial \overline{w}_j},
\]

we have

\[
\begin{align*}
\partial^2_{\overline{w}_j} B_{G_2} &= \frac{\partial^2}{\partial w_i \partial \overline{w}_j}(B \circ \Phi^{-1}) = \frac{\partial}{\partial \overline{w}_j} \left( \partial_1 B \frac{\partial z_1}{\partial w_i} \right) + \frac{\partial}{\partial \overline{w}_j} \left( \partial_2 B \frac{\partial z_2}{\partial w_i} \right) \\
&= \frac{\partial}{\partial \overline{w}_j} ((\partial_1 B) \circ \Phi^{-1}) \frac{\partial z_1}{\partial w_i} + \frac{\partial}{\partial \overline{w}_j} ((\partial_2 B) \circ \Phi^{-1}) \frac{\partial z_2}{\partial w_i} + \partial_1 B \frac{\partial^2 z_1}{\partial w_i \partial \overline{w}_j} + \partial_2 B \frac{\partial^2 z_2}{\partial w_i \partial \overline{w}_j} \\
&= \partial^2_{1\overline{1}} B \frac{\partial \overline{z}_1}{\partial \overline{w}_j} \frac{\partial z_1}{\partial w_i} + \partial^2_{1\overline{2}} B \frac{\partial \overline{z}_2}{\partial \overline{w}_j} \frac{\partial z_1}{\partial w_i} + \partial^2_{2\overline{1}} B \frac{\partial \overline{z}_1}{\partial \overline{w}_j} \frac{\partial z_2}{\partial w_i} + \partial^2_{2\overline{2}} B \frac{\partial \overline{z}_2}{\partial \overline{w}_j} \frac{\partial z_2}{\partial w_i},
\end{align*}
\]

because $\frac{\partial^2 z_1}{\partial w_i \partial \overline{w}_j} = \frac{\partial^2 z_2}{\partial w_i \partial \overline{w}_j} = 0$ where $i, j = 1, 2$. Hence from computation with Proposition 6, at $(x,0), 0 \leq x < 1$,

\[
\begin{align*}
\partial^2_{1\overline{1}} B_{G_2} &= \frac{-x^4 + 4x^2 + 3}{2\pi^2(x^2 - 1)^4}, \\
\partial^2_{1\overline{2}} B_{G_2} = \partial^2_{2\overline{1}} B_{G_2} &= \frac{x(x^2 - 4)}{\pi^2(x^2 - 1)^4}, \\
\partial^2_{2\overline{2}} B_{G_2} &= \frac{-2x^6 + 6x^4 - 6x^2 + 5}{\pi^2(x^2 - 1)^4}.
\end{align*}
\]

Now proposition follows from computations with (2.2). \qed
Proposition 9. The component of inverse metric of the Bergman metric \( g \) at \((x,0) \in G_2, 0 \leq x < 1\) are given as follows:

\[
\begin{align*}
g_{11}^\uparrow &= \frac{(x^2 - 2)^2 (2x^4 - 6x^2 + 5)}{2 (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}, \\
g_{12}^\uparrow &= g_{21}^\uparrow = \frac{x (x^2 - 2)^4}{2 (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}, \\
g_{22}^\uparrow &= \frac{2x^4 - 7x^2 + 6}{2x^8 - 16x^6 + 46x^4 - 60x^2 + 30}.
\end{align*}
\]

Proof. All formulas of \( g \) at \((x,0), 0 \leq x < 1\) follows from direct computations with Proposition 8. For the record, the determinant of \( g \) is precisely given by

\[
deg(g) = -\frac{4 (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}{(x^2 - 2)^3 (x^2 - 1)^2}.
\]

Recall that the Christoffel symbols \( \Gamma_{ij}^k \) of a Kähler metric \( g = (g_{ij}) \) is written in local coordinates by

\[
\Gamma_{ij}^k = g^{kj} \partial_i g_{j\ell}.
\]

(2.4)

On \( G_2 \), we have the following formulas of all \( \Gamma_{ij}^k \):

Proposition 10. The Christoffel symbols \( \Gamma_{ij}^k \) of the Bergman metric \( g \) at \((x,0) \in G_2, 0 \leq x < 1\) are given as follows:

\[
\begin{align*}
\Gamma_{11}^1 &= \frac{2x (x^6 - 2x^4 - x^2 + 3)}{(x^2 - 2) (x^2 - 1) (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}, \\
\Gamma_{12}^1 &= \frac{6 (x^2 - 2)}{x^8 - 8x^6 + 23x^4 - 30x^2 + 15}, \\
\Gamma_{21}^1 &= \Gamma_{12}^1 = \frac{2x^2 (x^2 - 2)^2}{(x^2 - 1) (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}, \\
\Gamma_{11}^2 &= \frac{2x^3 (x^2 - 2)^3}{(x^2 - 1) (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}, \\
\Gamma_{21}^2 &= \Gamma_{12}^2 = -\frac{x (x^8 - 10x^6 + 37x^4 - 62x^2 + 39)}{(x^2 - 2) (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}, \\
\Gamma_{22}^1 &= \frac{2x^2 (x^2 - 3) (x^2 - 2)^2}{x^8 - 8x^6 + 23x^4 - 30x^2 + 15}.
\end{align*}
\]
\[ \partial_i g_{jj} = \partial_i g_j = -2B_{Gz}^{-2} \partial_i B_{Gz}(B_{Gz} \partial_j^2 B_{Gz} - \partial_j B_{Gz} \partial_i B_{Gz}) \]
\[ + B_{Gz}^{-2} \left( \partial_i B_{Gz} \partial_j^2 B_{Gz} + B_{Gz} \partial_j^2 B_{Gz} - \partial_j^2 B_{Gz} \partial_i B_{Gz} - \partial_j B_{Gz} \partial_i^2 B_{Gz} \right). \] (2.5)

Since the formulas of \( \partial_i B_{Gz} \) are given in the proof of Proposition 8, we should compute \( \partial_j^2 B_{Gz} \) and \( \partial_j^2 B_{Gz} \) to get all formulas of Christoffel symbols. Elementary calculus computations with a chain-rule give for any indices \( i, j, k \),

\[ \partial_{ij}^2 B_{Gz} = \frac{\partial^2}{\partial w_i \partial w_j} (B \circ \Phi)^{-1} \]
\[ = \partial_1^2 B \frac{\partial z_1}{\partial w_j} \frac{\partial z_1}{\partial w_i} + \partial_2^2 B \frac{\partial z_2}{\partial w_j} \frac{\partial z_1}{\partial w_i} + \partial_2^2 B \frac{\partial z_1}{\partial w_j} \frac{\partial z_2}{\partial w_i} + \partial_2^2 B \frac{\partial z_2}{\partial w_j} \frac{\partial z_2}{\partial w_i} + \partial_1 B \frac{\partial z_1}{\partial w_i} \frac{\partial z_2}{\partial w_j} + \partial_2 B \frac{\partial z_2}{\partial w_i} \frac{\partial z_2}{\partial w_j}, \]
\[ \partial_{ijk}^2 B_{Gz} = \frac{\partial}{\partial w_k} \frac{\partial^2}{\partial w_j \partial w_i} (B \circ \Phi)^{-1} = \]
\[ \left( \left( \partial_1^2 B \right) \frac{\partial z_1}{\partial w_k} + \left( \partial_2^2 B \right) \frac{\partial z_2}{\partial w_k} \right) \frac{\partial \pi_1}{\partial w_j} \frac{\partial \pi_1}{\partial w_i} \]
\[ + \left( \left( \partial_1^2 B \right) \frac{\partial z_1}{\partial w_k} + \left( \partial_2^2 B \right) \frac{\partial z_2}{\partial w_k} \right) \frac{\partial \pi_2}{\partial w_j} \frac{\partial \pi_2}{\partial w_i} \]
\[ + \partial_1^2 B \frac{\partial \pi_1}{\partial w_j} \frac{\partial \pi_1}{\partial w_i} \frac{\partial^2 z_1}{\partial w_i \partial w_k} + \partial_2^2 B \frac{\partial \pi_2}{\partial w_j} \frac{\partial \pi_2}{\partial w_i} \frac{\partial^2 z_1}{\partial w_i \partial w_k} \]
\[ + \partial_1^2 B \frac{\partial \pi_1}{\partial w_j} \frac{\partial \pi_1}{\partial w_i} \frac{\partial^2 z_2}{\partial w_i \partial w_k} + \partial_2^2 B \frac{\partial \pi_2}{\partial w_j} \frac{\partial \pi_2}{\partial w_i} \frac{\partial^2 z_2}{\partial w_i \partial w_k}. \]

From above, it suffices to determine all formulas of \( \frac{\partial^2 z_1}{\partial w_i \partial w_j} \). With (2.3) at \((x, 0)\),

\[ \frac{\partial^2 z_1}{\partial w_1 \partial w_1} = 0, \quad \frac{\partial^2 z_1}{\partial w_1 \partial w_2} = 0, \quad \frac{\partial^2 z_1}{\partial w_2 \partial w_2} = -2 \frac{x^2}{x^3} \]
\[ \frac{\partial^2 z_2}{\partial w_1 \partial w_1} = 0, \quad \frac{\partial^2 z_2}{\partial w_1 \partial w_2} = 0, \quad \frac{\partial^2 z_2}{\partial w_2 \partial w_2} = 2 \frac{x^2}{x^3}. \]

Now each formula \( \Gamma_{jk}^i \) follows from computations with putting all necessary terms in (2.4). \( \square \)
Proposition 11. The curvature components of the Bergman metric at \((x, 0) \in G_2, 0 \leq x < 1\) are given by

\[
R_1 \text{TT} = \frac{4 \left(9x^{16} - 108x^{14} + 551x^{12} - 1552x^{10} + 2605x^8 - 2598x^6 + 1410x^4 - 300x^2 - 18\right)}{(x^4 - 3x^2 + 2)^4(x^8 - 8x^6 + 23x^4 - 30x^2 + 15)},
\]

\[
R_2 \text{T} \bar{T} = R_1 \bar{T} \text{T} = R_1 \text{T} \bar{T} = \frac{4 \left(x^{16} - 12x^{14} + 68x^{12} - 248x^{10} + 627x^8 - 1074x^6 + 1170x^4 - 726x^2 + 195\right)}{(x^2 - 2)^3(x^8 - 8x^6 + 23x^4 - 30x^2 + 15)},
\]

\[
R_1 \text{T} \bar{T} = R_1 \bar{T} \text{TT} = \frac{4 \left(2x^{10} - 10x^8 + 76x^6 - 147x^4 + 138x^2 - 51\right)}{(x^2 - 2)^4(x^8 - 8x^6 + 23x^4 - 30x^2 + 15)},
\]

\[
R_1 \bar{T} \text{TT} = R_2 \text{T} \bar{T} = R_2 \bar{T} \text{T} = \frac{4 \left(x^{12} - 10x^{10} + 47x^8 - 130x^6 + 207x^4 - 174x^2 + 60\right)}{(x^2 - 1)^4(x^8 - 8x^6 + 23x^4 - 30x^2 + 15)},
\]

\[
R_2 \text{TT} = \frac{4 \left(7x^{16} - 84x^{14} + 423x^{12} - 1156x^{10} + 1829x^8 - 1614x^6 + 624x^4 + 60x^2 - 90\right)}{(x^2 - 2)^4(x^8 - 8x^6 + 23x^4 - 30x^2 + 15)}.
\]

Proof. We will compute the components of curvature tensor \(R = R_{\bar{a}\bar{b}c\bar{d}} dz^c \otimes dz^d\) associated with given Hermitian metric \(g\) by well-known formula:

\[
R_{\bar{a}\bar{b}c\bar{d}} = -\frac{\partial^2 g_{\bar{a}\bar{b}}}{\partial z_c \partial \bar{z}_d} + \sum_{p,q=1}^l g^{\bar{p}\bar{q}} \frac{\partial g_{\bar{p}\bar{c}}}{\partial z_d} \frac{\partial g_{\bar{q}\bar{d}}}{\partial z_c}.
\]

(2.6)

For the Bergman metric \(g_{ij}\) on \(G_2\), we already obtained \(\frac{\partial}{\partial w_i} g_{\bar{j} \bar{p}} = \partial_i g_{\bar{j} \bar{p}}\) in (2.5). Also, the inverse metric was obtained in Proposition 9. From (2.5), \(\frac{\partial^2 g_{\bar{a}\bar{b}}}{\partial z_c \partial \bar{z}_d}\) is written in terms of the Bergman kernel \(B_{G_2}\) as follows:

\[
\partial^2_{ij} g_{kl} = 6B_{G_2}^4 \partial_i B_{G_2} \partial_j B_{G_2} B_{G_2} B_{G_2} \partial^2_{kl} B_{G_2} - 2B_{G_2}^3 \partial_i^2 B_{G_2} B_{G_2} \partial^2_{kl} B_{G_2} - 4B_{G_2} \partial_i B_{G_2} \partial_j B_{G_2} \partial^2_{kl} B_{G_2} - 2B_{G_2}^3 \partial_i^2 B_{G_2} \partial_j^2 B_{G_2} \partial^2_{kl} B_{G_2}
\]

\[
- 2B_{G_2}^3 \partial_i B_{G_2} \partial_j B_{G_2} \partial^2_{kl} B_{G_2} - 6B_{G_2}^3 \partial_i B_{G_2} \partial_j B_{G_2} \partial_k B_{G_2} \partial_l B_{G_2} + 2B_{G_2}^3 \partial^2_{ij} B_{G_2} \partial_k B_{G_2} \partial_l B_{G_2}
\]

\[
+ 2B_{G_2}^3 \partial_i B_{G_2} \partial^2_{kl} B_{G_2} + 2B_{G_2}^3 \partial_j B_{G_2} \partial^2_{kl} B_{G_2} + B_{G_2}^2 \partial^2_{ij} B_{G_2} \partial^2_{kl} B_{G_2}
\]

\[
+ B_{G_2}^2 \partial_i B_{G_2} \partial^2_{kl} B_{G_2} + B_{G_2}^2 \partial_j B_{G_2} \partial^2_{kl} B_{G_2} + B_{G_2} \partial^2_{ij} B_{G_2} \partial^2_{kl} B_{G_2}
\]

\[
+ 2B_{G_2}^2 \partial_i B_{G_2} \partial^2_{kl} B_{G_2} B_{G_2} - B_{G_2}^2 \partial_i B_{G_2} \partial^2_{kl} B_{G_2} - B_{G_2}^2 \partial^2_{ij} B_{G_2} \partial^2_{kl} B_{G_2}
\]

\[
+ 2B_{G_2}^2 \partial_i B_{G_2} \partial^2_{kl} B_{G_2} - B_{G_2}^2 \partial_i B_{G_2} \partial^2_{kl} B_{G_2} - B_{G_2}^2 \partial^2_{ij} B_{G_2} \partial^2_{kl} B_{G_2}
\]

\[
+ 2B_{G_2} \partial_i B_{G_2} \partial^2_{kl} B_{G_2} - B_{G_2} \partial_i B_{G_2} \partial^2_{kl} B_{G_2} - B_{G_2} \partial^2_{ij} B_{G_2} \partial^2_{kl} B_{G_2},
\]
With all formulas in the proof of Proposition 10, the only missing term is $\partial^4_{klij}B_{G_2}$, which is written as

$$
\partial^4_{iklj}B_{G_2} = \frac{\partial}{\partial w_k} \frac{\partial^3}{\partial w_i \partial w_j \partial w_l} (B \circ \Phi^{-1}) =
$$

$$
\left( (\partial^4_{1111}B) \frac{\partial z_1}{\partial w_i} \frac{\partial z_1}{\partial w_j} \frac{\partial z_2}{\partial w_k} \frac{\partial z_2}{\partial w_l} (B \circ \Phi^{-1}) \right) =
$$

Then each formula of $R_{abcd}$ can be obtained from elementary but lengthy computations. 

To compute the holomorphic sectional curvature of the Bergman metric on $G_2$, we proceed the Gram-Schmidt process to determine the orthonormal basis $X, Y$. Take the first unit vector field

$$
X = \frac{\partial_1}{\sqrt{g_{11}}}.
$$

Then another vector field $\tilde{Y}$ which is orthogonal to $X$ is given by

$$
\tilde{Y} = \frac{\partial_2}{\sqrt{g_{22}}} - g(\frac{\partial_2}{\sqrt{g_{22}}}, X)X = a_1 \partial_1 + a_2 \partial_2,
$$
where \( a_1 = \frac{g_{11}}{\sqrt{g_{11}} \sqrt{g_{22}}} \), \( a_2 = \frac{1}{\sqrt{g_{22}}} \). Since \( g(\tilde{Y}, \tilde{Y}) = a_1 \bar{a}_1 g_{11} + a_1 \bar{a}_2 g_{12} + a_2 \bar{a}_1 g_{21} + a_2 \bar{a}_2 g_{22} \), we will use

\[
Y = \frac{\tilde{Y}}{\sqrt{g(\tilde{Y}, \tilde{Y})}} = \frac{a_1 \delta_1 + a_2 \delta_2}{\sqrt{a_1 \bar{a}_1 g_{11} + a_1 \bar{a}_2 g_{12} + a_2 \bar{a}_1 g_{21} + a_2 \bar{a}_2 g_{22}}} =: t_1 \delta_1 + t_2 \delta_2, \tag{2.8}
\]

where

\[
t_i = \frac{a_i}{\sqrt{a_1 \bar{a}_1 g_{11} + a_1 \bar{a}_2 g_{12} + a_2 \bar{a}_1 g_{21} + a_2 \bar{a}_2 g_{22}}}, \quad i = 1, 2. \tag{2.9}
\]

**Proposition 12.** Let \( H(Z) = R(Z, \tilde{Z}, Z, \tilde{Z}) \) for \( Z \in \{X, Y\} \). The holomorphic sectional curvatures \( H(X), H(Y) \) of the Bergman metric at \((x, 0) \in G_2, 0 \leq x < 1\) are given as below:

\[
H(X) = \frac{9x^{16} - 108x^{14} + 551x^{12} - 1552x^{10} + 2605x^8 - 2598x^6 + 1410x^4 - 300x^2 - 18}{(3 - 2x^2)^2 (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)},
\]

\[
H(Y) = \frac{(3 - 2x^2)^2(x^4 - 5x^2 + 5)^3 (x^4 - 3x^2 + 3)^2}{9x^{28} - 225x^{26} + 2575x^{24} - 17844x^{22} + 83941x^{20} - 278485x^{18} + 681267x^{16} - 1237584x^{14} + 1668725x^{12} - 1646775x^{10} + 1150505x^8 - 531240x^6 + 137820x^4 - 9810x^2 - 2430}.
\]

In particular, all values of \( H(X) \) and \( H(Y) \) are negative at \((x, 0) \in G_2, 0 \leq x < 1\) and

\[
\lim_{x \to 1} H(X) = \lim_{x \to 1} H(Y) = -1.
\]

**Proof.** From the definition of the holomorphic sectional curvature, compute \( H(X), H(Y) \) which become

\[
H(X) = \frac{R_{11TT}}{g_{11} g_{11}}
\]

and

\[
H(Y) = \sum_{i,j,k,l=1}^2 t_i t_j t_k t_l R_{ijkl}.
\]

Then formulas of \( H(X), H(Y) \) follow from the direct elementary computations and one can check that all values of \( H(X), H(Y) \) are negative. \( \square \)

However, we can also compute the bisectional curvature of the Bergman metric on \( G_2 \) based on Proposition 11.

**Proposition 13.** Let \( B(X, Y) := R(X, \bar{X}, Y, \bar{Y}) \). Then at \((x, 0) \in G_2, 0 \leq x < 1\),

\[
B(X, Y) = \frac{(x^2 - 1)^2 f_1(x)}{(3 - 2x^2)^2 (x^8 - 8x^6 + 23x^4 - 30x^2 + 15)^2},
\]

where

\[
\frac{a_1}{\sqrt{a_1 \bar{a}_1 g_{11} + a_1 \bar{a}_2 g_{12} + a_2 \bar{a}_1 g_{21} + a_2 \bar{a}_2 g_{22}}}, \quad i = 1, 2.
\]
where

\[ f_1(x) = 9x^{20} - 162x^{18} + 1297x^{16} - 6074x^{14} + 18412x^{12} - 37738x^{10} + 52968x^8 - 50274x^6 + 30876x^4 - 11070x^2 + 1755. \]

In particular,

\[
\lim_{x \to 1} B(X,Y) = 0, \\
B(X,Y)(0.9,0.9,0,0) = 0.00679073.
\]

Consequently, the bisectional curvature of the Bergman metric on \( G_2 \) is not negatively pinched.

**Proof.** By (2.7) and (2.8),

\[
B(X,Y) = \frac{t_1 t_1}{g_{1T}} R_{1T1T} + \frac{t_1 t_2}{g_{1T}} R_{1T12} + \frac{t_2 t_2}{g_{1T}} R_{1T22} + \frac{t_2 t_1}{g_{1T}} R_{1T21}.
\]

Now proposition follows from direct computations with Proposition 11 and (2.9). \( \square \)

It follows by the similar argument that

**Lemma 14.** At \((x,0) \in G_2, 0 \leq x < 1,\)

\[
R(X,X,Y,X) = R(X,X,Y,Y) = -\frac{3x(2-x^2)\frac{1}{2} \left(1 - x^2\right)^3 \left(3x^8 - 24x^6 + 71x^4 - 92x^2 + 45\right)}{(3 - 2x^2)^2 \sqrt{(2x^4 - 6x^2 + 5)(3 - 2x^2)(4x^6 - 18x^4 + 28x^2 - 15)} \left(f_2(x)\right)^{\frac{1}{2}}},
\]

\[
R(Y,Y,X,X) = R(X,Y,Y,Y) = \frac{x(2-x^2)^{\frac{3}{2}} \left(x^2 - 1\right)^2 (9x^{14} - 126x^{12} + 739x^{10} - 2335x^8 + 4276x^6 - 4545x^4 + 2610x^2 - 630)}{(3 - 2x^2)^2 \sqrt{(2x^4 - 6x^2 + 5)(3 - 2x^2)(4x^6 - 5x^2 + 3)^2 \left(x^4 - 3x^2 + 3\right) \sqrt{f_2(x)}}},
\]

\[
R(X,Y,X,Y) = R(Y,X,Y,X) = \frac{-3x^2 \left(x^2 - 2\right)^3 \left(x^2 - 1\right)^2 \left(3x^8 - 27x^6 + 89x^4 - 124x^2 + 62\right)}{(3 - 2x^2)^2 \left(x^4 - 5x^2 + 5\right) \left(x^4 - 3x^2 + 3\right)},
\]

where

\[ f_2(x) = \frac{x^8 - 8x^6 + 23x^4 - 30x^2 + 15}{4x^6 - 18x^4 + 28x^2 - 15}. \]

**Proposition 15.** The holomorphic sectional curvature of the Bergman metric on \( G_2 \) is negatively pinched.
Proof. Take any unit vector field \( V = aX + bY \) with respect to the Bergman metric with \( |a|^2 + |b|^2 = 1 \). Then at \((x, 0) \in G_2, 0 \leq x < 1\),
\[
R(V, \bar{V}, V, \bar{V}) = |a|^4R(X, \bar{X}, X, \bar{X}) + |a|^2\bar{a}bR(Y, \bar{X}, X, \bar{X}) + |a|^2ab\bar{R}(X, \bar{Y}, X, \bar{X})
\]
\[
+ |a|^2|b|^2R(Y, \bar{Y}, X, \bar{X}) + |a|^2\bar{a}b\bar{R}(X, \bar{X}, X, \bar{Y}) + |a|^2|b|^2R(Y, \bar{X}, X, \bar{Y}) + a^2\bar{b}^2R(X, \bar{Y}, X, \bar{Y})
\]
\[
+ ab|b|^2R(Y, \bar{Y}, X, \bar{Y}) + |a|^2|b|^2R(X, \bar{X}, \bar{Y}, \bar{Y}) + \bar{a}b|b|^2R(X, \bar{Y}, \bar{Y}, \bar{Y})
\]
\[
+ |b|^4R(Y, \bar{Y}, Y, \bar{Y}) = |a|^4H(X) + |b|^4H(Y) + 4|a|^2|b|^2B(X, Y) + 4\text{Re}(\bar{a}b)\left( |a|^2R(X, \bar{X}, X, \bar{Y}) + |b|^2R(Y, \bar{Y}, Y, \bar{X}) \right)
\]
\[
+ 2\text{Re}(\bar{a}b)^2R(Y, \bar{X}, \bar{Y}, \bar{X}).
\]

With Proposition 12, Proposition 13 and Lemma 14, one can show that \( R(V, \bar{V}, V, \bar{V}) \) is negatively pinched for \( x \in [0, 1) \). In fact, letting \( L(V) = (3 - 2x^2)^2 R(V, \bar{V}, V, \bar{V}) \), one can show that \(-10 \leq L(V) \leq -1/2\). By Corollary 5, the holomorphic sectional curvature of the Bergman metric on \( G_2 \) is negatively pinched between \(-10\) and \(-1/18\).

\[\square\]

Proof of Theorem 1. With Proposition 15, Theorem 1 follows from Theorem 2 and Theorem 3 in [22]. \[\square\]

Also, Corollary 3 easily follows from well known product properties of invariant metrics (for example, see page 669 in [15]).

3. Quasi-bounded geometry of Bergman metric and consequences

The notions of quasi-bounded geometry is introduced by S.T Yau and S.Y Cheng in [6]) and we will show that the Bergman metric on \( G_2 \) admits the quasi-bounded geometry. By Theorem 9 in [22]), it suffices to prove any \( k \)-th covariant derivative of the curvature tensor of the Bergman metric on \( G_2 \) is bounded. By Proposition 11, while any curvature component \( R_{ijkl} \) blows up as \( x \to 1 \),
\[
\lim_{x \to 1} g^1_B = \lim_{x \to 1} g^1_{\bar{B}} = \lim_{x \to 1} g^2_B = \lim_{x \to 1} g^2_{\bar{B}} = 1/2
\]
by Propositions 9. On the other hand, from Proposition 15, we have \(-a < h(g_B) < -b < 0\) for some \( a, b > 0 \), which implies
\[
\sup_{x \in G_2} |R_{ijkl} R^{ijkl}| \leq \frac{136}{3} (b - a)
\]
(3.1)

(for example, see the proof of Lemma 13 in [22]). Here instead of using the global coordinate vector fields \( \frac{\partial}{\partial w_i}, i = 1, 2 \), we will use the orthonormal frames \( X, Y \) in (2.7) and (2.8) for computing any \( k \)-th covariant derivative of the curvature tensor.
Proposition 16. For any \( m \in \mathbb{N} \), the \( m \)-th covariant derivative of the curvature tensor of the Bergman metric on \( G_2 \) is bounded.

Proof. Since we are going to evaluate at \((x,0)\) for \( x \in [0,1)\), the singularities will only occur in the term \( \frac{1}{1-z\bar{z}} \) as in (2.1). We are going to trace the order of \( \frac{1}{1-z\bar{z}} \) in the curvature tensors. By straightforward calculation, it is easy to see that the lowest order of \( \frac{1}{1-z\bar{z}} \) in \( R_{ijkl} \) and \( \Gamma^k_{ij} \) are -4 and -1 respectively. As a consequence, the lowest order of \( \frac{1}{1-z\bar{z}} \) in \( \nabla^m R_{ijkl} \) is \(-4 - m\). Here we use \( i,j,k,l \) to denote either holomorphic or anti-holomorphic directions. On the other hand, by the expressions of \( X,Y \) in (2.7) and (2.8), they both contain \( 1 - z\bar{z} \) of order 1. Therefore, if we evaluate \( \nabla^m R_{ijkl} \) to unit vectors at \((x,0)\), it is obviously bounded near \( x = 1 \). The proposition is thus proved. \( \square \)

Now Theorem 2 follows from Proposition 16, with Lemma 31 in [22].

Acknowledgement: The second author is supported by National Science Foundation grant DMS-1412384, Simons Foundation grant #429722 and CUSE grant program at Syracuse University. Both authors thank Damin Wu for the very helpful discussions and thank Lixin Shen for his help on the numerical analysis.

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