TWISTED CONFORMAL KILLING TENSORS ARE GENERICALLY TRIVIAL

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Abstract. We show that twisted Conformal Killing Tensors (CKTs) are generically trivial on closed Riemannian manifolds. In negative curvature, it is known that the existence of twisted CKTs is the only obstruction to solving exactly the twisted cohomological equations which may appear in various geometric problems such as the study of transparent connections. The main result of this paper says that these equations can be generically solved. The proof relies on the introduction of a new microlocal property for (pseudo)differential operators called operators of uniform divergence type.

1. Introduction

Consider \((\mathcal{E}, \nabla^\mathcal{E})\) a Hermitian vector bundle equipped with a unitary connection over a smooth Riemannian \(n\)-manifold \((M, g)\) with \(n \geq 2\). Let \(SM\) be the unit sphere bundle and \(\pi : SM \to M\) be the projection. We consider the pullback bundle \((\pi^*\mathcal{E}, \pi^*\nabla^\mathcal{E})\) over \(SM\) and we will forget the \(\pi^*\) in the following in order to simplify the notations. Denote by \(X\) the geodesic vector field and define the operator \(X := \nabla^\mathcal{E}_X\), acting on sections of \(C^\infty(SM, \mathcal{E})\). By standard Fourier analysis, we can write \(f \in C^\infty(SM, \mathcal{E})\) as
\[
\sum_{m \geq 0} f_m,
\]
where \(f_m \in C^\infty(M, \Omega_m \otimes \mathcal{E})\) and pointwise in \(x \in M\):
\[
\Omega_m \otimes \mathcal{E} := \ker(\Delta^\mathcal{E}_v + m(m + n - 2)),
\]
is the kernel of the vertical Laplacian (note that this Laplacian is independent of the connection \(\nabla^\mathcal{E}\), it only depends on \(\mathcal{E}\) and on \(g\)). Elements in this kernel are called the twisted spherical harmonics of degree \(m\). We will say that \(f \in C^\infty(SM, \mathcal{E})\) has finite Fourier content if its expansion in spherical harmonics only contains a finite number of terms. The operator \(X\) maps
\[
X : C^\infty(M, \Omega_m \otimes \mathcal{E}) \to C^\infty(M, \Omega_{m-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})
\]
and can be decomposed as \(X = X_+ + X_-\), where, if \(u \in C^\infty(M, \Omega_m \otimes \mathcal{E})\), \(X_+ u \in C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})\) denotes the orthogonal projection on the twisted spherical harmonics of degree \(m + 1\). The operator \(X_+\) is elliptic and thus has finite-dimensional kernel whereas \(X_-\) is of divergence type (see §3 for definitions). Moreover, \(X_+^* = -X_-\), where the adjoint is computed with respect to the canonical \(L^2\) scalar product on \(SM\) induced by

Date: August 24, 2020.
the Sasaki metric. We refer to the original articles of Guillemin-Kazhdan [GK80a, GK80b] for a description of these facts and to [GPSU16] for a more modern exposition.

**Definition 1.1.** We call twisted Conformal Killing Tensors (CKTs) elements in the kernel of $X_+|_{C^\infty(M,\Omega_m \otimes \mathcal{E})}$.

We say that the twisted CKTs are trivial when the kernel is reduced to $\{0\}$. The goal of this paper is to investigate the generic properties of non trivial twisted CKTs. It is known that the kernel of $X_+$ is invariant by a conformal change of the metric $g$ (see [GPSU16, Section 3.6]). Moreover, in negative curvature, it is known that there exists $m_0 \geq 0$ such that for all $m \geq m_0$, $\ker X_+|_{C^\infty(M,\Omega_m \otimes \mathcal{E})}$ is trivial (see [GPSU16, Theorem 4.5]). The number $m_0$ can be estimated explicitly in terms of an upper bound of the sectional curvature of $(M,g)$ and the curvature of the vector bundle $\mathcal{E}$. This should still be true for Anosov manifolds (namely, manifolds whose geodesic flow is uniformly hyperbolic on the unit tangent bundle) although it is still an open question for the moment. In particular, this is true for Anosov surfaces since any Anosov surface $(M,g)$ is conformally equivalent to a metric of negative curvature and twisted CKTs are invariant by conformal changes. We insist on the fact that it is not even known if there are examples of negatively-curved manifolds (or even Anosov manifolds) of dimension $n \geq 3$ which exhibit non trivial twisted CKTs. Nevertheless, it was conjectured in [GPSU16, Section 1] that the absence of twisted CKTs should be a generic property. In this paper, we answer positively to this question:

**Theorem 1.2.** Let $\mathcal{E}$ be a Hermitian vector bundle over a smooth Riemannian manifold $(M,g)$ with $\dim M \geq 3$\(^1\). Then, for any $k \geq 2$, there is a residual set of unitary connections with regularity $C^k$ such that $(\mathcal{E},\nabla^{\mathcal{E}})$ has no CKTs.

For fixed $m \in \mathbb{N}$, absence of twisted CKTs is an open property: this is a mere consequence of standard elliptic theory for (pseudo)differential operators. As a consequence, in order to prove that connections without CKTs form a residual set (i.e. a countable intersection of open and dense sets), it is sufficient to show that for fixed $m \in \mathbb{N}$, unitary connections without twisted CKTs of degree $m$ form a dense set of unitary connections with regularity $C^k$: we prove this property in Theorem 4.3. In order to do so, we use a perturbative argument for the eigenvalues of a certain natural Laplacian operator $\Delta_+ = (X_+)^*X_+ \geq 0$ acting on $C^\infty(M,\Omega_m \otimes \mathcal{E})$ (see §4.1) whose kernel is given by the twisted CKTs of degree $m$. Due to the affine structure of the set of connections, it is possible to compute explicitly the first and second variation (with respect to the connection) of the 0 eigenvalue of the operator $\Delta_+$. The first variation vanishes but the second variation\(^2\) is given by some explicit

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\(^1\)In the case $\dim M = 2$, the operators $X_{\pm}$ are elliptic which changes the problem. For our results in this case, see Proposition 4.5.

\(^2\)We also point out that we initially tried to apply the perturbation theory of Uhlenbeck [Uhl76, Theorem 1] but we could not make it work. The main reason, it seems, is that we need to look at a second order
non-negative quantity (see Lemma 4.2). It is then sufficient to produce a perturbation of the connection such that this second variation is strictly positive.

In order to do so, we introduce a new notion of microlocal analysis which we call operators of uniform divergence type (see §3). These (pseudo)differential operators are of the form $Q : C^\infty(M, F) \to C^\infty(M, E)$ where $\text{rank}(F) > \text{rank}(E)$. They are of divergence type in the sense that the principal symbol of the adjoint is injective for all $(x, \xi) \in T^*M \setminus \{0\}$, i.e. $\sigma_Q^*(x, \xi) \in \text{Hom}(E_x, F_x)$ is injective (equivalently the principal symbol $\sigma_Q(x, \xi) \in \text{Hom}(F_x, E_x)$ is surjective). The uniform divergence type property asserts that

$$\bigoplus_{|\xi|=1} \ker \sigma_Q(x, \xi) = F_x,$$

for all $x \in M$ and allows to describe the values that elements in $\ker Q|_{C^\infty(M, F)}$ can take at a given point $x$. In particular, under this property, the map

$$\text{ev}_x : \ker Q|_{C^\infty(M, F)} \to F_x, \text{ ev}_x(f) := f(x)$$

is surjective for all $x \in M$.

We also point out that the perturbation used is a priori global on $M$ but it could be interesting to see whether our result can be made local in the following sense: given an open subset $\Omega \subset M$, the equation $X_\mu u = 0$ on $\Omega$ has generically (with respect to the connection) only trivial solutions. Such a local perturbative argument is developed in [KM16] who show that generically a metric has no Killing tensors (see §2.3 for a definition). This would require extra work on operators of uniform divergence type and we plan to investigate this in the future. More generally, the method developed in the present article seems fairly robust in order to deal with general linear perturbations of gradient-type or Laplacian-type operators.

Finally, let us briefly mention that in negative curvature, twisted CKTs are an obstruction to solving exactly some transport equations called twisted cohomological equations which appear in some geometric settings such as the study of transparent pairs of connections. If $(\mathcal{E}, \nabla^\mathcal{E})$ is a smooth vector bundle of rank $r$ with unitary connection over the negatively-curved Riemannian manifold $(M, g)$, one can look at the pullback $(\pi^*\mathcal{E}, \pi^*\nabla^\mathcal{E})$ over the unit tangent bundle $SM$, where $\pi : SM \to M$ is the projection on the base. A closed geodesic on $M$ can be identified with a periodic orbit $\gamma$ for the geodesic flow on $SM$, and one can look at the holonomy induced by the pullback connection along $\gamma$, i.e. the parallel transport of sections of $\pi^*\mathcal{E}$ along the geodesic lines. We say that a connection is transparent if the holonomy is trivial along all periodic orbits of the geodesic flow (see [Pat09, Pat11, Pat12, Pat13, GPSU16, CL] for the study of this question). In this case, it is known that $\pi^*\mathcal{E}$ is trivial over $SM$ (see [CL] for instance) and one can variation of the eigenvalues at 0 whereas the transversality theory in [Uhl76] is a “first-order perturbation” theory.
prove that there exists a smooth family \((e_1, \ldots, e_r) \in C^\infty(SM, \pi^*E)\) which is independent at every point \((x, v) \in SM\) (it trivializes the bundle \(\pi^*E\) over \(SM\)) and such that \(\pi^*\nabla_X e_i = 0\) for \(i = 1, \ldots, r\). We call such an equality/equation a twisted cohomological equation: it is a transport equation on the unit tangent bundle involving some vector bundle. In negative curvature, it is known that such a twisted cohomological equation imply that the sections \(e_i \in C^\infty(SM, \pi^*E)\) have finite Fourier content (see [GPSU16, Theorem 4.1]). If one can prove that the \(e_i\) are actually independent of the velocity variable i.e. \(e_i \in C^\infty(M, \Omega_0 \otimes E) \cong C^\infty(M, E)\), then this implies that they are actually sections living on the base manifold \(M\) and the equation \(\pi^*\nabla_X e_i = 0\) is equivalent to \(\nabla^E e_i = 0\), i.e. these sections are parallel. In other words the vector bundle \((E, \nabla^E)\) over \(M\) is isomorphic to the trivial bundle \((C^r, d)\) equipped with the trivial flat connection. In order to prove that the \(e_i\) are indeed independent of the velocity variable, it is sufficient to know that the connection \(\nabla^E\) has no non trivial twisted CKTs (see [GPSU16, Theorem 5.1]), hence the importance of their study. Note that one can also study a more general question than that of uniqueness for transparent connections and ask the following inverse problem: does the holonomy of the connection along closed geodesics stably determine the connection? We refer to [CL] for an extensive discussion of this question which is intimately related to the existence of non-trivial CKTs.

The existence/non-existence of CKTs can also be investigated on manifolds with boundary: it is proved in [DS10, GPSU16] that there are no (twisted) CKTs which identically vanish on the boundary or on a hypersurface.

Acknowledgement: M.C. and T.L. have received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 725967).

2. Symmetric tensor analysis

We recall some elementary properties of symmetric tensors on Riemannian manifolds. The reader is referred to [DS10] for an extensive discussion.

2.1. Definitions and first properties.

2.1.1. Symmetric tensors in Euclidean space. We consider a \(n\)-dimensional Euclidean vector space \((E, g_E)\) with orthonormal frame \((e_1, \ldots, e_n)\). We denote by \(\otimes^m E^*\) the \(m\)-th tensor power of \(E^*\) and by \(\otimes^m_s E^*\) the symmetric tensors of order \(m\), namely the tensors \(u \in \otimes^m E^*\) satisfying:

\[ u(v_1, \ldots, v_m) = u(v_{\sigma(1)}, \ldots, v_{\sigma(m)}), \]
for all \( v_1, \ldots, v_m \in E \) and \( \sigma \in \mathfrak{S}_m \), the permutation group of \( \{1, \ldots, m\} \). If \( K = (k_1, \ldots, k_m) \in \{1, \ldots, n\}^m \), we define \( e^*_{K} = e^*_{k_1} \otimes \cdots \otimes e^*_{k_m} \), where \( e^*_j(v) := \delta_{ij} \). We introduce the symmetrization operator \( S : \otimes^m E^* \to \otimes^m_S E^* \) defined by:

\[
S(\eta_1 \otimes \cdots \otimes \eta_m) := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(m)},
\]

where \( \eta_1, \ldots, \eta_m \in E^* \). Given \( v \in E \), we define \( v^* \in E^* \) by \( v^*(w) := g_E(v, w) \) and call \( \varepsilon : E \to E^* \) the musical isomorphism, following the usual terminology. Its inverse is denoted by \( \sharp : E^* \to E \). The scalar product \( g_E \) naturally extends to \( \otimes^m E^* \) (and thus to \( \otimes^m_S E^* \)) using the following formula:

\[
g_{\otimes^m E^*}(v_1 \otimes \cdots \otimes v_m, w_1 \otimes \cdots \otimes w_m) := \prod_{j=1}^m g_E(v_j, w_j),
\]

where \( v_i, w_i \in E \). In particular, if \( u = \sum_{i_1, \ldots, i_m=1}^n u_{i_1 \cdots i_m} e^*_1 \otimes \cdots \otimes e^*_m \), then \( \|u\|_{\otimes^m E^*}^2 = \sum_{i_1, \ldots, i_m=1}^n |u_{i_1 \cdots i_m}|^2 \). For the sake of simplicity, we will still write \( g_E \) instead of \( g_{\otimes^m E^*} \). The operator \( S \) is an orthogonal projection with respect to this scalar product.

There is a natural trace operator \( T : \otimes^m E^* \to \otimes^{m-2} E^* \) (it is formally defined to be 0 for \( m = 0, 1 \)) given by:

\[
Tu := \sum_{i=1}^n u(e_i, e_i, \ldots, \cdot),
\]

and it also maps \( T : \otimes^m E^* \to \otimes^{m-2} E^* \). Its adjoint (with respect to the metric \( g_{\otimes^m E^*} \)) on symmetric tensors is the map \( J : \otimes^m_S E^* \to \otimes^{m+2} E^* \) given by \( Ju := S(g_E \otimes u) \). It is easy to check that the map \( J \) is injective. This implies by standard linear algebra that one has the decomposition, where \( \otimes^m_S E^*|_{0-Tr} = \ker T \cap \otimes^m_S E^* \) denotes the trace-free symmetric \( m \)-tensors:

\[
\otimes^m S E^* = \otimes^m S E^*|_{0-Tr} \oplus \frac{1}{k} J \otimes^{m-2} E^* = \oplus_{k \geq 0} J^k \otimes^{m-2k} E^*|_{0-Tr}. \quad (2.1)
\]

Given \( K = (k_1, \ldots, k_m) \in \{1, \ldots, n\}^m \), we introduce \( \Theta(K) := (\theta_1(K), \ldots, \theta_n(K)) \), where \( \theta_i(K) = \sharp \{ k_j = i \mid j = 1, \ldots, m \} \). Observe that \( Se^*_K = Se^*_{K'} \) if and only if \( \Theta(K) = \Theta(K') \) and \( g_E(Se^*_K, Se^*_{K'}) = 0 \), if \( \Theta(K) \neq \Theta(K') \). In other words, there exists a subset \( A \subset \{1, \ldots, n\}^m \) such that \( \{Se^*_K \mid K \in A\} \) forms an orthogonal family (not orthonormal though since the elements are not unitary) for the scalar product \( g_E \): this subset is chosen of maximal size and so that if \( K, K' \in A \) with \( K \neq K' \), then \( \Theta(K) \neq \Theta(K') \).

2.1.2. Homogeneous polynomials. There is a natural identification of \( \otimes^m_S E^* \) with the vector space \( P_m(E) \) of homogeneous polynomials on \( E \), namely polynomials of the form

\[
f(v_1, \ldots, v_m) := \sum_{|\alpha|=m} c_\alpha v_1^{\alpha_1} \cdots v_m^{\alpha_m},
\]
by considering the isomorphism, for \( u \in \otimes_S^m E^* \)

\[
P_m(E) \ni \lambda_m u : v \mapsto u(v, ..., v).
\]

Note that \( \lambda_m = \lambda_m S \), i.e. \( \lambda_m \) vanishes on the orthogonal of symmetric tensors (with respect to the metric \( g_{\otimes^m E^*} \)). In particular, given \( K = (k_1, ..., k_m) \in \{1, ..., n\}^m \), we have

\[
\lambda_m \left( S e^K_\alpha \right) = \lambda_m e^K_\alpha = \prod_{j=1}^m v_{k_j}.
\]

The complex/real dimension of \( P_m(E) \) is \( p(n, m) := \binom{n+m-1}{m} \). We denote by \( H_m(E) \) the subspace of harmonic homogeneous polynomials, namely the polynomials \( u \in P_m(E) \) which satisfy the extra condition that \( \Delta_E u = 0 \), where the Laplacian is computed with respect to the metric \( g_E \). On \( P_m(E) \), we introduce the operator

\[
\partial \left( \sum_{|\alpha| = m} c_\alpha v_1^{\alpha_1} ... v_n^{\alpha_n} \right) := \sum_{|\alpha| = m} c_\alpha \partial_{v_1}^{\alpha_1} ... \partial_{v_n}^{\alpha_n},
\]

and we define the scalar product

\[
\langle P, Q \rangle := \partial(P)\overline{Q} \in \mathbb{C}.
\]

Note that if \( P = \sum_{|\alpha| = m} a_\alpha v_\alpha, \; Q = \sum_{|\alpha| = m} b_\alpha v_\alpha \), then

\[
\langle P, Q \rangle = \sum_{|\alpha| = m} \alpha! a_\alpha \overline{b}_\alpha.
\]

The operator \( \partial \) also satisfies the relation \( \partial(PQ) = \partial(P)\partial(Q) = \partial(Q)\partial(P) \). This implies the following:

**Lemma 2.1.** Let \( R \in P_k(E) \) and let \( \Phi : P_m(E) \to P_{m+k}(E) \) be the map defined by \( \Phi(Q) = RQ \). Then, \( \Phi^* : P_{m+k}(E) \to P_m(E) \) is given by \( \Phi^*(P) = \partial(R)P \).

**Proof.** This is straightforward:

\[
\langle \Phi(Q), P \rangle = \langle RQ, P \rangle = \partial(RQ)\overline{P} = \partial(Q)\overline{\partial(R)P} = \langle Q, \partial(R)P \rangle.
\]

We have \( \partial(|v|^2) = \partial^2_{v_1} + ... + \partial^2_{v_n} = \Delta_E \) and thus:

\[
\langle |v|^2 P, Q \rangle = \langle P, \partial(|v|^2)Q \rangle = \langle P, \Delta_E Q \rangle,
\]

that is the adjoint of \( \Delta_E \) with respect to \( \langle \cdot, \cdot \rangle \) is \( |v|^2 \). The map \( |v|^2 : P_m(E) \to P_{m+2}(E) \) is clearly injective (and thus \( \Delta_E : P_{m+2}(E) \to P_m(E) \) is surjective) and we thus have the decomposition:

\[
P_m(E) = H_m(E) \oplus \perp |v|^2 P_{m-2}(E) = \oplus_{k \geq 0} |v|^{2k} H_{m-2k}(E),
\]
where $H_m(E) = \{0\}$ for $m < 0$. This implies that:

$$\dim(H_m(E)) := h(n,m) = \frac{(n + m - 1)}{m} - \frac{(n + m - 3)}{m - 2}. \quad (2.2)$$

Moreover, we have:

**Lemma 2.2.** We have $m(m - 1)\lambda_{m-2} T = \Delta_E \lambda_m$ and $\lambda_m J = |v|^2 \lambda_{m-2}$. As a consequence $\lambda_m : \otimes^n_S E^*|_{0-\text{Tr}} \to H_m(E)$ and $\lambda_m : J \otimes^{m-2} S_E^*|_{0-\text{Tr}} \to |v|^2 H_{m-2}(E)$ are both isomorphisms. Moreover, there exists a constant $c_m > 0$ such that if $f \in \otimes^n_S E^*|_{0-\text{Tr}}$, then $\|\lambda_m f\|_{H_m(E)} = c_m \|f\|_{\otimes^n_S E^*|_{0-\text{Tr}}}$ i.e. $\lambda_m : \otimes^m S_E^*|_{0-\text{Tr}} \to H_m(E)$ is (up to a constant factor) an isometry.

**Proof.** We have for $v \in E$ and $u \in \otimes^m S_E^*$

$$\lambda_m J u(v) = \lambda_m S(g_E \otimes u)(v) = \lambda_m (g_E \otimes u)(v) = g_E(v,v)u(v,\ldots,v) = |v|^2 \lambda_{m-2} u(v).$$

Then, we compute for any $m$-tuple $(i_1,\ldots,i_m)$

$$m(m - 1)\lambda_{m-2} T \mathbf{e}_{i_1}^\ast \otimes \cdots \otimes \mathbf{e}_{i_m}^\ast = \frac{1}{(m-2)!} \sum_{\sigma \in S_m} \sum_{j=1}^n \delta_{j_{i(1)}} \delta_{j_{i(2)}} v_{i(3)} \cdots v_{i(m)}$$

$$= \sum_{1 \leq k \neq l \leq m} \delta_{i_k,i_l} \cdot \hat{v}_{i_k} \cdots \hat{v}_{i_l} \cdots v_{i_m}$$

$$= \Delta_E \lambda_m \mathbf{e}_{i_1}^\ast \otimes \cdots \otimes \mathbf{e}_{i_m}^\ast.$$

This proves the claims made in the first two sentences. The remaining claim follows from Schur’s Lemma. Indeed, we have two natural unitary representations of $O(n)$ on $\otimes^m S_E^*$ and $\mathbf{P}_m(E)$ given by the action by pullback (the second action is obvious; for the first one, see the proof of Lemma 3.7) and the operator $\lambda_m$ is an intertwining operator. Since $H_m(E)$ is well-known to be an irreducible representation of $O(n)$, so is $\otimes^m S_E^*|_{0-\text{Tr}}$ and by Schur’s Lemma, $\lambda_m \lambda_m : \otimes^m S_E^*|_{0-\text{Tr}} \to \otimes^m S_E^*|_{0-\text{Tr}}$ is a (positive) multiple of the identity. $\square$

**2.1.3. Spherical harmonics.** We define the operator of restriction $r_m : \mathbf{P}_m(E) \to C^\infty(\mathcal{S}_E)$, where $\mathcal{S}_E$ denotes the unit sphere in $E$ by $r_m(u) := u|_{\mathcal{S}_E}$. It is well-known that the operator maps isomorphically $r_m : H_m(E) \to \Omega_m$, where $\Omega_m := \ker(\Delta S_E + m(m + n - 2))$ and $\Delta_S$ denotes the Laplacian on the unit sphere of $E$, is an isomorphism. Indeed, this follows from the following formula (see [GHL04, Proposition 4.48] for instance):

$$\Delta_E(u)|_{\mathcal{S}_E} = \Delta_{S_E}(u|_{\mathcal{S}_E}) + \frac{\partial^2 u}{\partial r^2}|_{\mathcal{S}_E} + (n - 1) \frac{\partial u}{\partial r}|_{\mathcal{S}_E},$$

where $r$ is the radial coordinate, and using the homogeneity of $u$. We endow $L^2(\mathcal{S}_E)$ with the canonical $L^2$ scalar product of functions induced by the round metric, namely

$$\langle u_1, u_2 \rangle_{L^2(\mathcal{S}_E)} := \int_{\mathcal{S}_E} u_1(v) \overline{u_2(v)} d\text{vol}_{\mathcal{S}_E}(v).$$
Lemma 2.3. The following lemma is standard but we still provide a proof for the sake of completeness:

\[ \otimes^m_S E^* = \oplus_{k \geq 0} J^k \otimes_S^m E^* \mid_{\text{Tr} - \text{Tr}} \rightarrow_{\lambda_m} P_k(E) = \oplus_{k \geq 0} |v|^{2k} \mathcal{H}_{m-2k}(E) \rightarrow_{r_m} \oplus_{k \geq 0} \Omega_{m-2k}(E) \]

are isomorphisms which act diagonally on these decompositions. Moreover, they act on each diagonal term (up to a constant factor) as isometries. For the sake of simplicity, we also introduce the notation:

\[ S_m(E) := \oplus_{k \geq 0} \Omega_{m-2k}(E). \]

2.2. Multiplication by a connection 1-form. We now twist with a complex inner product space \( \mathcal{E} \) of dimension \( r \) and consider the tensor product \( \otimes^m_S E^* \otimes \mathcal{E} \) which consists of elements

\[ f = \sum_{k=1}^r u_k \otimes e_k, \]

where \( u_k \in \otimes^m_S E^* \) and \((e_1, ..., e_r)\) forms an orthonormal basis of \( \mathcal{E} \). Using the map \( \lambda_m \) (resp. \( \pi_m^\ast \)), we will also identify \( \otimes^m_S E^* \otimes \mathcal{E} \) with \( P_m(E) \otimes \mathcal{E} \) (resp. \( S_m(E) \otimes \mathcal{E} \)). If \( \Gamma \in E^* \otimes \text{End}(\mathcal{E}) \), and \( f = \sum_{k=1}^r u_k \otimes e_k \in P_m(E) \otimes \mathcal{E} \), we can define \( \Gamma f \in P_{m+1}(E) \otimes \mathcal{E} \) by:

\[ \Gamma f(v) = \sum_{k=1}^r u_k(v) \otimes \Gamma(v)e_k. \]

The following lemma is standard but we still provide a proof for the sake of completeness:

**Lemma 2.3.** Let \( \Gamma \in E^* \otimes \text{End}(\mathcal{E}) \). Then we have the mapping

\[ \mathcal{H}_m(E) \otimes \mathcal{E} \ni f \mapsto \Gamma f \in (\mathcal{H}_{m+1}(E) \otimes \mathcal{E}) \oplus (|v|^2 \mathcal{H}_{m-1}(E) \otimes \mathcal{E}). \]

**Proof.** We have

\[ \Gamma f(v) = \sum_{k=1}^r u_k(v) \otimes \Gamma(v)e_k = \sum_{k,j=1}^r \Gamma_{jk}(v)u_k(v) \otimes e_j, \]

where \( E \ni v \mapsto \Gamma_{jk}(v) \in \mathbb{C} \) is a linear form i.e. a homogeneous polynomial of degree 1 (which is in particular harmonic). Thus, the lemma boil's down to proving that if \( u \in \mathcal{H}_m(E) \) and \( \eta \in E^* \), then \( v \mapsto \eta(v)u(v) \) is an element of \( \mathcal{H}_{m+1}(E) \oplus |v|^2 \mathcal{H}_{m-1}(E) \). To see this, define \( b := (n + m - 1)^{-1} \nabla \eta \cdot \nabla u \in \mathcal{H}_{m-1}(E) \), as \( \nabla \eta \) is a constant vector so it commutes with \( \Delta_E \). Next, we claim that \( a := \eta \cdot u - |v|^2 b \in \mathcal{H}_{m+1}(E) \), so we compute

\[
\Delta_E a = \Delta_E(\eta)u + 2\nabla \eta \cdot \nabla u + \eta \Delta_E u - \Delta_E(|v|^2)b - 2\nabla(|v|^2) \cdot \nabla b - |v|^2 \Delta_E b = 0
\]

using Euler’s formula since \( b \) is \((m-1)\)-homogeneous and the definition of \( b \). This completes the proof.
Following the previous lemma, we define $\Gamma_- : H_m(E) \otimes \mathcal{E} \rightarrow H_{m-1}(E) \otimes \mathcal{E}$ as the orthogonal projection onto the lower-order harmonic polynomials. First of all, we prove the following result, forgetting about the twist by $\mathcal{E}$ (equivalently $\mathcal{E} = \mathbb{C}$ in the next lemma).

**Lemma 2.4.** Let $\Gamma \in E^* \setminus \{0\}$. Then $\Gamma_- : H_m(E) \rightarrow H_{m-1}(E)$ is surjective.

**Proof.** Up to a preliminary change of coordinates (a rotation), we can always assume that $\Gamma = \mu e_i^*$ with $\mu \neq 0$. By the previous Lemma, we then have

$$\Gamma_- u(v) = \mu (n + m - 1)^{-1} \partial_{v_1} u(v).$$

Let us compute the dimension of the kernel of $\Gamma_-$. We have $\Gamma_- u = 0$ if and only if $\partial_{v_1} u = 0$ i.e. $u$ is independent of $v_1$. Since $u$ is a homogeneous polynomial, this means that $u = \sum_{|\alpha| = m, \alpha_1 = 0} c_{\alpha} v^\alpha$. Moreover, since $u$ is harmonic, this also means that $\Delta u = 0 = \Delta' u$, where $\Delta' = \partial_{v_2}^2 + \ldots + \partial_{v_n}^2$ and thus $u$ is harmonic polynomial of degree $m$ in $H_m(\mathbb{R}^{n-1})$. The other inclusion being obvious, we thus have $\ker \Gamma_- \simeq H_m(\mathbb{R}^{n-1})$. Thus:

$$\dim(\ker \Gamma_-) = h(n-1,m) = \binom{n-2+m}{m} - \binom{n-4+m}{m-2}.$$ 

As a consequence, using the Pascal’s rule for binomial coefficients

$$\dim(\text{ran} \Gamma_-) = \dim(H_m(\mathbb{R}^n)) - \dim(\ker \Gamma_-) = \binom{n-1+m}{m} - \binom{n-3+m}{m-2} - \left( \binom{n-2+m}{m} - \binom{n-4+m}{m-2} \right)$$

$$= \binom{n-2+m}{m-1} - \binom{n-4+m}{m-3} = \dim(H_{m-1}(\mathbb{R}^n)),$$

thus $\Gamma_-$ is surjective. □

Note that, using the restriction map $r_m : H_m(E) \rightarrow \Omega_m(E)$, the last lemma is equivalent to saying that $\Gamma_- : \Omega_m \rightarrow \Omega_{m-1}$ is surjective. Eventually, we will need this last Lemma, where $\text{End}_{sk}(\mathcal{E})$ denotes the skew-Hermitian endomorphisms

**Lemma 2.5.** Let $u \in \Omega_m \otimes \mathcal{E}$. Then, there exists $\Gamma \in \text{End}_{sk}(\mathcal{E})$ and $w \in \Omega_{m+1} \otimes \mathcal{E}$ such that $u = \Gamma_- w$.

**Proof.** We write $u = \sum_{k=1}^{r} u_k \otimes e_k$, where $u_k \in \Omega_m(E)$ are spherical harmonics (possibly complex). For each $k = 1, \ldots, r$, we choose an arbitrary real-valued $\Gamma_k \in E^* \setminus \{0\}$ and we define $\Gamma \in E^* \otimes \text{End}_{sk}(\mathcal{E})$ by, in the $(e_1, \ldots, e_r)$ basis:

$$\Gamma(v) := \begin{pmatrix} i\Gamma_1(v) & 0 & \cdots & 0 \\ 0 & i\Gamma_2(v) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & i\Gamma_r(v) \end{pmatrix}.$$
where \( v \in E \). By the previous Lemma, for all \( k = 1, \ldots, r \) we can always find \( w_k \in \Omega_{m+1} \) such that \( i\Gamma_k w_k = u_k \). We then set \( w := \sum_{k=1}^r w_k \otimes e_k \), so \( \Gamma w = u \).

\[ \square \]

2.3. Twisted tensor analysis on the manifold. Given a section \( u \in C^\infty(M, \otimes^m_{S^*} T^* M \otimes \mathcal{E}) \), the connection \( \nabla^\mathcal{E} \) produces an element \( \nabla^\mathcal{E} u \in C^\infty(M, T^* M \otimes (\otimes^m_{S^*} T^* M) \otimes \mathcal{E}) \). In coordinates, if \( (e_1, \ldots, e_r) \) is a local orthonormal frame for \( \mathcal{E} \) and \( \nabla^\mathcal{E} = d + \Gamma \), for some one-form with values in skew-hermitian matrices \( \Gamma \), we have:

\[
\nabla^\mathcal{E} \left( \sum_{k=1}^r u_k \otimes e_k \right) = \sum_{k=1}^r \nabla u_k \otimes e_k + u_k \otimes \nabla^\mathcal{E} e_k = \sum_{k=1}^r \left( \nabla u_k + \sum_{l=1}^r \sum_{i=1}^n \Gamma_{il}^k u_l \otimes dx_i \right) \otimes e_k, \tag{2.3}
\]

where \( u_k \in C^\infty(M, \otimes^m_{S^*} T^* M) \) and \( \nabla \) is the Levi-Civita connection. The symmetrization operator \( S_\mathcal{E} : C^\infty(M, \otimes^m T^* M \otimes \mathcal{E}) \rightarrow C^\infty(M, \otimes^1_{S^*} T^* M \otimes \mathcal{E}) \) is defined by:

\[
S_\mathcal{E} \left( \sum_{k=1}^r u_k \otimes e_k \right) = \sum_{k=1}^r S(u_k) \otimes e_k,
\]

where \( u_k \in C^\infty(M, \otimes^m_{S^*} T^* M) \) and \( S \) is the symmetrization operators of tensors previously introduced. We can symmetrize (2.3) to produce an element \( D_\mathcal{E} := S_\mathcal{E} \nabla^\mathcal{E} u \in C^\infty(M, \otimes^{m+1}_{S^*} T^* M \otimes \mathcal{E}) \) given in coordinates by:

\[
D_\mathcal{E} \left( \sum_{k=1}^r u_k \otimes e_k \right) = \sum_{k=1}^r \left( Du_k + \sum_{l=1}^r \sum_{i=1}^n \Gamma_{il}^k S(u_l \otimes dx_i) \right) \otimes e_k, \tag{2.4}
\]

where \( D := S \nabla \) (\( \nabla \) being the Levi-Civita connection) is the usual symmetric derivative of symmetric tensors. Elements of the form \( Du \in C^\infty(M, \otimes^{m+1}_{S^*} T^* M) \) are called potential tensors. By comparison, we will call elements of the form \( D_\mathcal{E} f \in C^\infty(M, \otimes^{m+1}_{S^*} T^* M \otimes \mathcal{E}) \) twisted potential tensors. The operator \( D_\mathcal{E} \) is a first order differential operator and the expression of its principal symbol

\[
\sigma_{\text{princ}}(D_\mathcal{E}) \in C^\infty(T^* M, \text{Hom}(\otimes^m_{S^*} T^* M \otimes \mathcal{E}, \otimes^{m+1}_{S^*} T^* M \otimes \mathcal{E}))
\]

can be read off from (2.4), namely \( \sigma_{\text{princ}}(D_\mathcal{E}) = \sigma_{\text{princ}}(D) \otimes \text{id}_\mathcal{E} \):

\[
\sigma_{\text{princ}}(D_\mathcal{E})(x, \xi) \cdot \left( \sum_{k=1}^r u_k(x) \otimes e_k(x) \right) = \sum_{k=1}^r (\sigma_{\text{princ}}(D)(x, \xi) \cdot u_k(x)) \otimes e_k(x) = i \sum_{k=1}^r S(\xi \otimes u_k(x)) \otimes e_k(x),
\]
where \( e_k(x) \in \mathcal{E}_x \), \( u_k(x) \in \otimes_S^{m+1} T^* M \) and the basis \((e_1(x), ..., e_r(x))\) is assumed to be orthonormal. One can check that this is an injective map, which means that \( D_\mathcal{E} \) acting on twisted symmetric tensors of order \( m \) is a left-elliptic operator and can be inverted on the left modulo a smoothing remainder; its kernel is finite-dimensional and consists of elements called \textit{twisted Killing Tensors}.

In the particular case where \( \mathcal{E} = \mathbb{C} \) (i.e. there is no twist), the elements in the kernel of \( D \) are called \textit{Killing Tensors} (for \( m = 1 \), they generate infinitesimal isometries). It is known that if the flow is ergodic, the kernel of \( D \) is trivial in the sense that it is reduced to \( \{0\} \) when \( m \) is odd and \( \mathbb{C} \cdot \mathcal{S}(g^\otimes m/2) \) when \( m \) is even. This simply follows from the well-known conjugation relation \( \pi^\ast_{m+1} D = \pi^\ast_m X \). Moreover, it is known that the kernel of \( D \) is generically trivial \( \text{(KM16)} \) (with respect to the metric \( g \)). In the presence of a twist by a vector bundle \( \mathcal{E} \), one can also analyse the kernel of \( D_\mathcal{E} \); it is proved in \( \text{(GPSU16)} \) that on a negatively-curved manifold, if \((\mathcal{E}, \nabla^\mathcal{E})\) has no CKTs, then the kernel of \( D_\mathcal{E} \) is trivial (in the same sense as before). This also relies on the conjugation relation \( \pi^\ast_{m+1} D_\mathcal{E} = \pi^\ast_m X_\mathcal{E} \).

The adjoint \( D_\mathcal{E}^\ast : C^\infty(M, \otimes_S^{m+1} T^* M \otimes \mathcal{E}) \to C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E}) \) has a surjective principal symbol given by

\[
\sigma_{D_\mathcal{E}^\ast}(x, \xi) \cdot \left( \sum_{k=1}^r u_k(x) \otimes e_k(x) \right) = -i \sum_{k=1}^r \xi^k u_k(x) \otimes e_k(x).
\] (2.5)

As we will see in the next section (see Definition 3.1), such an operator is called of divergence type. Using the correspondence between trace-free twisted symmetric tensors of degree \( m \) and twisted spherical harmonics of degree \( m \), there is an explicit explicit the link between \( X_\mathcal{E}/D_\mathcal{E} \) and \( X_\mathcal{E}/D_\mathcal{E} \). More precisely, we introduce \( \mathcal{P} : C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E}) \to C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E}) \) the pointwise orthogonal projection on trace-free twisted symmetric tensors. We then have the following equalities (see \( \text{(GPSU16, p. 22)} \)) on \( C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E}) \):

\[
X_\mathcal{E} \pi^\ast_m = \pi^\ast_{m+1} \mathcal{P} D_\mathcal{E}, \quad X_\mathcal{E} \pi^\ast_m = -\frac{m}{m-2+2m} \pi^\ast_{m+1} D_\mathcal{E}^\ast.
\] (2.6)

The kernel of \( X_\mathcal{E} \) is therefore in one-to-one correspondence with the kernel of \( \mathcal{P} D_\mathcal{E} \). In particular, we have the mapping

\[
D_\mathcal{E}^\ast : C^\infty(M, \otimes_S^{m+1} T^* M \otimes \mathcal{E}) \to C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E}).
\]

3. Microlocal preliminaries

3.1. An abstract result.
3.1.1. **Statement of the result.** Let $P : C^\infty(M, E) \to C^\infty(M, F)$ be a differential operator of order $m \geq 0$ between two vector bundles such that $\text{rank}(F) > \text{rank}(E)$ and let $\sigma_P \in C^\infty(T^*M, \text{Hom}(E, F))$ be its principal symbol.

**Definition 3.1.** We say that $P$ is of **gradient type** (or equivalently that $P^*$ is of **divergence type**) if $\sigma_P(x, \xi)$ is injective for all $(x, \xi) \in T^*M \setminus \{0\}$ (equivalently, $\sigma_{P^*}(x, \xi)$ is surjective for all $(x, \xi) \in T^*M \setminus \{0\}$).

**Lemma 3.2.** Assume that $P$ is of gradient type. Then for all $s \in \mathbb{R}$, $\ker(P^*|_{H^s(M, F)})$ is infinite dimensional.

**Proof.** Injectivity of the principal symbol implies the existence of a pseudodifferential operator $Q \in \Psi^{-m}(M; E, F)$ such that $QP = 1_E + R$, where $R$ is a smoothing operator. By classical arguments, this implies that for any $s \in \mathbb{R}$, the image $P(H^{s+m}(M, E)) \subset H^s(M, F)$ is closed. This implies the decomposition

$$
H^s(M, F) = \ker(P^*|_{H^s(M, F)}) \oplus P(H^{s+m}(M, E)),
$$

which is orthogonal for the $L^2$ scalar product. By ellipticity, the kernel of $P$ is finite dimensional. We introduce the formally self-adjoint operator $\Delta := P^*P$ and denote by $\Pi_0$ the $L^2$-orthogonal projection on $\ker \Delta = \ker P$. Thus, any section $f \in H^s(M, F)$ can be uniquely decomposed as $f = Pu + v$, where $v \in \ker(P^*|_{H^s(M, F)})$ and $u \in H^{s+m}(M, E) \cap \ker \Pi_0$. The $L^2$-orthogonal projection on the image of $P$ is a self-adjoint pseudodifferential operator of order 0, defined by

$$
\pi_{\text{ran}(P)} := P\Delta^{-1}P^*,
$$

where $\Delta^{-1}$ is the operator defined by the 0 in restriction to $\text{ran}(\Pi_0)$ and by the inverse of $\Delta$ on $\ker(\Pi_0)$. The $L^2$-orthogonal projection on the kernel of $P^*$ is then given by

$$
\pi_{\text{ker}P^*} := 1_F - \pi_{\text{ran}(P)}. \quad \text{Note that } f \in \ker(P^*|_{H^s(M, F)}) \text{ if and only if } \pi_{\text{ran}(P)}f = 0.
$$

We first show that $\ker(P^*|_{H^s(M, F)}) \neq \{0\}$. Assume it is not the case, that is any $f \in H^s(M, F)$ is of the form $f = Pu$, where $u \in H^{s+m}(M, E)$. We can then consider for $h > 0$ and $(x_0, \xi_0) \in T^*M$ a section $f \in C^\infty(M, F)$ such that $0 \neq f(x_0) \in \ker \sigma_{\pi_{\text{ran}(P)}}(x_0, \xi_0)$ (note that $\sigma_{\pi_{\text{ran}(P)}}(x_0, \xi_0)$ is a symbol of order 0; it is the orthogonal projection on the image $\sigma_P(x_0, \xi_0)(E_{x_0}) \subset F_{x_0}$). This is always possible since $\text{rank}(F) > \text{rank}(E)$. We further assume that $\|f(x)\|_F = 1$ for all $x$ in a neighborhood of $x_0$. We consider a Lagrangian state $e^{\hat{H}S}$ such that $S(x_0) = 0, dS(x_0) = \xi_0$. Then, we have $\pi_{\text{ran}(P)}(e^{\hat{H}S})f(x_0) = \sigma_{\pi_{\text{ran}(P)}}(x_0, \xi_0)f(x_0) + \mathcal{O}(h) = \mathcal{O}(h)$. But we have:

$$
1 = \|e^{\hat{H}S}f(x_0)\|_{F_{x_0}} = \|(\pi_{\text{ran}(P)}(e^{\hat{H}S})f)(x_0)\|_{F_{x_0}} = \mathcal{O}(h),
$$

which is a contradiction.
We now assume that \( \ker(P^*|_{H^s(M,F)}) \) is finite dimensional. Writing \( \mathbb{I}_F = \pi_{\text{ran}}(P) + \pi_{\ker(P^*)} \), we can construct a Gaussian state\(^3\) \( \varphi_{x_0,\xi_0} f \), where \( \varphi \) is a local cut-off function with \( \varphi = 1 \) near \( x_0 \). We assume \( f(x_0) \in \ker \sigma_{\pi_{\text{ran}}(P)}(x_0, \xi_0) \) and \( \|f\|_F \leq 1 \) close to \( x_0 \). It can be checked that \( \|\varphi_{x_0,\xi_0}(h)f\|_{L^2} = c + o(1) \), for some \( c > 0 \). Moreover, a computation in local coordinates gives (see also [DG75, Equation 1.5] and [Zwo12, p. 102])

\[
\|\pi_{\text{ran}}(P)\varphi_{x_0,\xi_0}(h)f\|_{L^2}^2 = \langle \pi_{\text{ran}}(P)\varphi_{x_0,\xi_0}(h)f, \varphi_{x_0,\xi_0}(h)f \rangle_{L^2} = \langle \sigma_{\pi_{\text{ran}}(P)}(x_0, \xi_0)f(x_0), f(x_0) \rangle_{F_{x_0}} + o(1) = o(1),
\]

and so we obtain

\[
\varphi_{x_0,\xi_0}(h)f = \underbrace{\pi_{\text{ran}}(P)\varphi_{x_0,\xi_0}(h)f + \pi_{\ker(P^*)}\varphi_{x_0,\xi_0}(h)f}_{o_{L^2}(1)}.
\]

Since \( \ker(P^*) \) is finite dimensional, we can always assume that \( \pi_{\ker(P^*)}\varphi_{x_0,\xi_0}(h)f \to_{h \to 0} v \in \ker(P^*|_{H^s(M,F)}) \), that is \( \varphi_{x_0,\xi_0}(h)f \to v \in \ker(P^*|_{H^s(M,F)}) \) (the convergence takes place in \( L^2 \) but the limit \( v \) is in \( H^s \)). But this can always be achieved by taking an arbitrary large number of such \( \varphi_i e_{x_i,\xi_i}(h)f_i \), \( i = 1, \ldots, N \) with disjoint supports on the manifold (\( \varphi_i \) is supported near \( x_i \)). This produces non-zero elements \( v_i \in \ker(P^*|_{H^s(M,F)}) \) which are all pairwise orthogonal, contradicting the finite-dimensionality of \( \ker(P^*) \). \( \square \)

We introduce the following property.

**Definition 3.3.** We say that \( P^* \) is uniformly of divergence type if it is of divergence type and for all \( x_0 \in M \):

\[
\bigoplus_{|\xi|=1} \ker \sigma_{P^*}(x_0, \xi) = F_{x_0}.
\]

Note that \( \ker \sigma_{P^*}(x_0, \xi) = \ker \sigma_{\pi_{\text{ran}}(P)}(x_0, \xi) \). The restriction \( \xi \neq 0 \) is due to the fact that the principal symbol is 0-homogeneous in \( \xi \) and thus only makes sense for large \( \xi \). We then have the following result:

**Lemma 3.4.** Assume \( P \) is of gradient type. Let \( s > n/2 \), \( x \in M \) and define the map \( \text{ev}_x : \ker P^*|_{H^s(M,F)} \to F_x \) by \( \text{ev}_x(f) := f(x) \). Then:

\[
\text{ev}_x : \ker P^*|_{H^s(M,F)} \to \bigoplus_{|\xi|=1} \ker \sigma_{P^*}(x_0, \xi)
\]

is surjective. In particular, if \( P^* \) is of uniform divergence type, then \( \text{ev}_x : \ker P^*|_{H^s(M,F)} \to F_x \) is surjective.

\(^3\)Here, in local coordinates, \( e_{x_0,\xi_0} \) has the form:

\[
e_{x_0,\xi_0}(x) = (\pi h)^{-n/4} e^{|\xi||x-x_0| - \frac{1}{4h}|x-x_0|^2}.
\]
The choice of $s > n/2$ is simply there to ensure that $H^s$ embeds continuously into $C^0$ and thus $ev_x$ is well-defined. The previous Lemma gives a lower bound on the possible values that elements in $\ker P^*$ can take at a given point.

**Proof.** Fix $x_0 \in M$. It is sufficient to prove that for $\xi \in T^*M \setminus \{0\}$, one has $\ker \sigma_{P^*}(x_0, \xi) \subset \text{ran}(ev)$. We consider a Lagrangian state $f(h) := e^{P^*S} f$, for some smooth section $f$ (independent of $h > 0$) where $dS(x_0) = \xi$, $S(x_0) = 0$, and $f(x_0) \in \ker \sigma_{P^*}(x_0, \xi)$. Then

$$f(x_0) = (f(h))(x_0) = (\pi_{\text{ran}(P)}f(h))(x_0) + (\pi_{\ker(P^*)}f(h))(x_0)$$

$$= \sigma_{\pi_{\text{ran}(P)}}(x_0, \xi)f(x_0) + \mathcal{O}(h) + (\pi_{\ker(P^*)}f(h))(x_0).$$

Composing with the orthogonal projection $\pi_{\text{ran}(ev)}^\perp$ on $\text{ran}(ev)^\perp$, we then obtain that pointwise at $x_0$:

$$\pi_{\text{ran}(ev)}^\perp f(x_0) = \mathcal{O}(h),$$

and thus, since $f(x_0)$ is independent of $h$, $\pi_{\text{ran}(ev)}^\perp f(x_0) = 0$ i.e. $f(x_0) \in \text{ran}(ev)$. 

3.1.2. **Example: the divergence of a vector field.** Let us illustrate the preceding property by a simple example i.e. the divergence of a vector field. Let $(M, g)$ be a smooth Riemannian manifold. Given $X \in C^\infty(M, TM)$, the divergence $\delta X \in C^\infty(M)$ of $X$ is defined as minus the $L^2$ formal adjoint of the gradient operator, namely:

$$\langle f, \delta X \rangle := -\langle \nabla f, X \rangle.$$

**Lemma 3.5.** The divergence operator $\delta$ is of uniform divergence type.

**Proof.** We first prove that $\delta$ is of divergence type. For that, it is sufficient to compute its principal symbol. It is an elementary computation to show that $\sigma_{\nabla}(x, \xi) = i \times \xi^2$ and thus for $v \in T_xM$, one has $\sigma_{\delta}(x, \xi)v = i\langle \xi, v \rangle$. Observe that $\sigma_{\nabla}(x, \xi)$ is injective for all $\xi \neq 0$ with constant rank equal to 1, i.e. $\delta$ is of divergence type. We now show that $\delta$ satisfies Definition 3.3. Pick $x_0 \in M, v \in T_xM$ and consider $\xi \in T^*_{x_0}M \setminus \{0\}$ such that $\xi^2 \perp v$. Then:

$$\sigma_{\delta}(x_0, \xi)v = i\langle \xi, v \rangle = ig(\xi^2, v) = 0.$$

\[\Box\]

3.1.3. **Example: differential forms.** More generally, consider the bundle of differential $k$-forms, $\Omega^k = \Lambda^k(T^*M)$, the exterior derivative $d$ and its formal adjoint $d^*$ acting on sections of $\Omega^k$. It can be checked that for $\alpha \in \Omega^k(x)$

$$\sigma_d(x, \xi)\alpha = i\xi \wedge \alpha, \quad \sigma_{d^*}(x, \xi)\alpha = -i\xi^2 \alpha.$$

In fact one may show $\ker \sigma_d(x, \xi)|_{\Omega^k(x)} = \xi \wedge \Omega^{k-1}(x)$, so $d$ is of gradient type if and only if $k = 0$. Equivalently $d^*$ is of divergence type if and only if $k = 1$, which by metric duality is the content of Lemma 3.5. Again by duality, we obtain $\ker \sigma_{d^*}(x, \xi)|_{\Omega^k(x)} = \iota_{\xi^2} \Omega^{k+1}(x)$ and
since pointwise in local coordinates every differential \( k \)-form \( dx_{i_1} \wedge \ldots \wedge dx_{i_k} \) is obtained by contracting a suitable \((k + 1)\)-form, we obtain
\[
\oplus_{|\xi|=1} \ker d^* (x, \xi) = \Omega^k(x). \tag{3.1}
\]

Observe that Lemma 3.2 does not apply directly to \( d \) but by the Hodge decomposition \( \ker d^*|_{\Omega^k} = d^* C^\infty(M; \Omega^{k+1}) \oplus \mathcal{H}^k \) is infinite dimensional, where \( \mathcal{H}^k \) are harmonic \( k \)-forms. However, setting \( \Delta = dd^* + d^*d \) in the proof of the same lemma would produce the analogous result with minor corrections. Finally, Lemma 3.4 also does not apply directly, but by using the Hodge decomposition and (3.1) we obtain the analogous result: for \( x \in M \), the map \( \text{ev}_x : \ker d^*|_{H^s(M, \Omega^k)} \to \Omega^k(x) \) is surjective for \( s > n/2 \).

3.1.4. **Counterexample: a divergence type operator that is not uniform.** This is a very elementary example constructed by hand so that it does not work, but it is very likely that one can find more elaborate examples. Consider for \((M, g)\) a smooth Riemannian manifold and a vector bundle \( E \to M \) over \( M \). Consider an elliptic selfadjoint differential operator \( P : C^\infty(M, E) \to C^\infty(M, E) \) and assume \( P \) is invertible. Let \( Q : C^\infty(M, E) \to C^\infty(M, E) \) defined by \( Qf := (Pf, - Pf) \), then \( \sigma_Q(x, \xi) u = (\sigma_P(x, \xi) u, - \sigma_P(x, \xi) u) \) and \( \sigma_Q(x, \xi)(u_1, u_2) = \sigma_P(x, \xi)(u_1 - u_2) \). Thus \( Q \) is of gradient type or equivalently \( Q^* \) is of divergence type. But \( Q^* \) is not of uniform divergence type since
\[
\oplus_{|\xi|=1} \ker \sigma_{Q^*}(x, \xi) = \{(u, u) \mid u \in E_x \} \approx E_x \neq E_x \oplus E_x.
\]
In particular, it is easy to describe the kernel of \( Q^* \) since \( P \) is invertible, namely
\[
\ker Q^*|_{C^\infty(M, E \oplus E)} = \{(f, f) \mid f \in C^\infty(M, E) \},
\]
and thus for every \( x \in M \), the map \( \text{ev}_x : \ker Q^*|_{C^\infty(M, E \oplus E)} \to E_x \oplus E_x \) defined by \( \text{ev}_x(f_1, f_2) = (f_1(x), f_2(x)) \) is not surjective. Note that this example shows that the lower bound given by Lemma 3.4 is sharp. Also observe that, taking \( Qf = (\Delta f, - \Delta f) \in C^\infty(M, \mathbb{C}^2) \), where \( \Delta : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C}) \) is the Laplacian induced by \( g \) and acting on functions, one obtains an operator which is divergence type but not uniform. However this time, the map \( \text{ev}_x \) is surjective for every \( x \in M \). This comes from the fact that the kernel of \( \Delta \) is not trivial (and given by the constants).

3.2. **Application to trace-free divergence-free tensors.** We now study the operators \( X_+ \) and \( X_- \) (see (1.1)) in the light of the preceding paragraph. We first have the

**Lemma 3.6.** The operator \( X_- \) is of divergence type.

**Proof.** By definition, it is sufficient to prove that \( X_- \) is of gradient type i.e. that its principal symbol is injective. By (2.6), the principal symbol of \( X_+ \) is given (up to conjugating by
the map \( \pi_m^* \) by
\[
\sigma_{x_\xi}(x, \xi) \left( \sum_{k=1}^r u_k(x) \otimes e_k(x) \right) = \sum_{k=1}^r i\mathcal{PS}(\xi \otimes u_k(x)) \otimes e_k(x),
\]
where \( u_k(x) \in \otimes_S^m T_x^* M_{|_{0-\text{Tr}}} \). Thus, it is sufficient to prove that
\[
\otimes_S^m T_x^* M_{|_{0-\text{Tr}}} \ni u \mapsto \mathcal{PS}(\xi \otimes u)
\]
is injective. This is the content of [DS10, Theorem 5.1]. It can also be found in [GK80b]. \( \square \)

As a direct application of the preceding paragraph, we obtain that \( \ker X_-|_{H^s(M, \Omega_m)} \) is infinite-dimensional for all \( s \in \mathbb{R} \). We also have:

**Lemma 3.7.** The operator \( X_- \) is of uniform divergence type if and only if \( n \geq 3 \).

As a consequence of the previous paragraph, for all \( s > n/2 \) and \( x_0 \in M \), the map \( \text{ev} : \ker X_-|_{H^s(M, \Omega_m \otimes \mathcal{E})} \to \Omega_m \otimes \mathcal{E}(x_0) \) defined by \( \text{ev}(w) := w(x_0) \) is surjective.

**Proof.** We use that according to (2.6), the operator \( X_- \) on \( C^\infty(M, \Omega_m \otimes \mathcal{E}) \) is equivalent to the operator \( D^*_s \) acting on \( C^\infty(M, \otimes_S^m T_x^* M_{|_{0-\text{Tr}}} \otimes \mathcal{E}) \). Since the principal symbol of the operator acts diagonally on \( \mathcal{E} \), it is sufficient to prove it for \( \mathcal{E} = \mathbb{C} \), i.e. there is no twist.

It is classical that \( D^* \sim \eta_- \) is elliptic if \( n = 2 \) (see e.g. [GK80a]) so its symbol is injective and so \( D^* \) is not of uniform divergence type. Now assume \( n \geq 3 \) and recall that \( \sigma(D^*)(x, \xi) = -i\xi_\xi : \Omega_m(x) \to \Omega_m(x) \). Note that \( \dim \ker t_{\xi_\xi} > 0 \) by dimension counting (2.2). Consider the subspace
\[
W := \otimes_{|I| = 1} \ker t_{\xi_\xi}|_{\Omega_m(x)} \subset \Omega_m(x).
\]
We claim first that \( W \) is invariant under the action of \( O(n) \). To see this, let \( A \in O(n) \); it suffices to show that \( A \ker t_{\xi_\xi} \subset \ker t_{\xi_\xi} \). Let \( s \in \ker t_{\xi_\xi}|_{\Omega_m(x)} \) and denote by \( e_1, \ldots, e_n \) an orthonormal basis at \( T_x M \)
\[
s = \sum_{I} s_I e_{i_1}^* \otimes \cdots \otimes e_{i_m}^*, \quad I = (i_1, \ldots, i_m) \subset \{1, \ldots, n\}^m.
\]
Note simply that \( A\xi_\xi = (A\xi)_\xi \), where \( A\xi = \xi \circ A^T \) is the left group action so
\[
A t_{\xi_\xi} s = \sum_I s_I t_{\xi_\xi}(Ae_{i_1}^*) \otimes \cdots \otimes (Ae_{i_m})^* = \sum_I s_I (Ae_{i_1})^* \otimes \cdots \otimes (Ae_{i_m})^*
\]
\[
= A \sum_I s_I (\xi_\xi, e_{i_1}) (e_{i_2}^*) \otimes \cdots \otimes (e_{i_m}^*) = A t_{\xi_\xi} s.
\]
Here, we used that by definition \( A \) preserves the inner product. This proves the observation
and so \( W \neq \{0\} \) is a sub-representation of \( \Omega_m \). But it is well-known that the representation of \( O(n) \) on \( \Omega_m(x) \) is irreducible, thus \( W = \Omega_m(x) \) completing the proof. \( \square \)
It is straightforward to extend this claim to all symmetric tensors of some order.

**Proposition 3.8.** The operator $D^*_E$ acting on all symmetric tensors $C^\infty(M; \otimes^n T^* M \otimes \mathcal{E})$ is of uniform divergence type.

**Proof.** It suffices to consider $\mathcal{E} = \mathbb{C}$. Next, it is sufficient to recall the decomposition in (2.1): for any $k$, consider

$$\{0\} \neq W := \oplus_{|\xi|=1} \ker \iota_\xi \bigcup \mathcal{J}_k \otimes \Omega^m_{-2k} T^*_x M |_{0_\mathcal{V}} \subset \mathcal{J}_k \otimes \Omega^m_{-2k} T^*_x M |_{0_\mathcal{V}} \equiv \Omega_{m-2k}(x).$$

One checks that $O(n)$ acts on the left on $\mathcal{J}_k \otimes \Omega^m_{-2k} T^*_x M |_{0_\mathcal{V}}$ and the computation in (3.2) remains valid to show $O(n)$ acts on $W$. As the representation of $O(n)$ on $\Omega_{m-2k}(x)$ is irreducible we get $W = \Omega_{m-2k}(x)$, proving the claim. $\square$

4. Perturbation of the Laplacian

**4.1. Perturbing the sum of the eigenvalues.** Consider a connection $\nabla^\mathcal{E}$ with CKTs. We denote by $X^\Gamma := X + \Gamma(v)$ the operators induced by the unitary connections $\nabla^\mathcal{E} + \Gamma$, where $\Gamma \in C^\infty(M, T^* M \otimes \text{End}_{sk}(\mathcal{E}))$ is small enough. We introduce $\Delta_+^\Gamma := -X^\Gamma X^{\Gamma*} \geq 0$. Each $\Delta_+^\Gamma$ is a Laplacian type operator in the sense that it is non-negative, formally selfadjoint (and with principal symbol given by $\sigma_{\Delta_+}^\Gamma(x, \xi) = |\xi|^2 \mathbb{1}_{\Omega_m}$). In particular, it is selfadjoint with domain $H^2$, its $L^2$-spectrum is discrete, contained in the positive real line and accumulates near $+\infty$. The eigenstates are smooth. By assumption, we have assumed that there are CKTs for the connection obtained with $\Gamma = 0$. We denote by $\Pi$ the $L^2$-orthogonal projection on the eigenstates at $0$ (the CKTs): it can be written as

$$\Pi = \sum_{i=1}^d \langle \cdot, u_i \rangle_{L^2(M, \Omega_m)} u_i,$$

where $(u_1, ..., u_d)$ forms an orthonormal family for the $L^2$ scalar product and $d$ is the dimension of the CKTs.

We choose a small (counter clockwise oriented) circle $\gamma$ around $0$ so that inside $\gamma$, $0$ is the only eigenvalue for the operator $\Delta_+^{\Gamma=0}$. Of course, this is an open property in the sense that it is still true for any small perturbation $\Delta_+^{\Gamma}$ with $\Gamma \neq 0$ of the operator. We introduce

$$\Pi^\Gamma := \frac{1}{2\pi i} \int_\gamma (z - \Delta_+^{\Gamma})^{-1} \, dz.$$

For $\Gamma = 0$, we have $\Pi^\Gamma = \Pi$ is the $L^2$-orthogonal projection on the CKTs. For $\Gamma \neq 0$, some eigenvalues may leave $0$ (but they still have to be contained in the positive real line) and $\Pi^\Gamma$ is the $L^2$-orthogonal projection on all the eigenvalues contained inside the circle $\gamma$. We then define

$$\lambda^\Gamma := \text{Tr} \left( \Delta_+^{\Gamma} \Pi^\Gamma \right),$$
which is the sum of the eigenvalues contained inside $\gamma$. Of course, for $\Gamma = 0$, $\lambda_{\Gamma=0} = 0$. The map

$$C^\infty(M, T^*M \otimes \text{End}_{sk}(E)) \ni \Gamma \mapsto (\lambda_{\Gamma}, \Pi_{\Gamma}) \in \mathbb{R} \times \mathcal{L}(L^2)$$

is smooth and we are going to compute its first and second derivatives at $\Gamma = 0$. We start with the first derivative.

**Lemma 4.1.** For all $A \in C^\infty(M, T^*M \otimes \text{End}_{sk}(E))$, $d\lambda_{\Gamma=0}(A) = 0$.

**Proof.** We consider for small $s \in \mathbb{R}$ the family of operators $\Delta^s_+ = -X_-^sX_+^s$. We have:

$$\frac{d}{ds} \Pi_{sA} \bigg|_{s=0} = \frac{d}{ds} \frac{1}{2\pi i} \int_\gamma (z - \Delta^s_+)^{-1} dz \bigg|_{s=0} = \frac{1}{2\pi i} \int_\gamma (z - \Delta_+)^{-1} \dot{\Delta}_+(z - \Delta_+)^{-1} dz,$$

and:

$$\frac{d}{ds} \lambda_{sA} \bigg|_{s=0} = \text{Tr} \left( \frac{d}{ds} \Delta^s_+ \bigg|_{s=0} \Pi \right) + \text{Tr} \left( \Delta_+ \frac{d}{ds} \Pi_{sA} \bigg|_{s=0} \right).$$

We claim that both terms vanish (this is always the case for the second term, whatever the perturbation actually). Indeed, for the first term, we observe that

$$\frac{d}{ds} \Delta^s_+ \bigg|_{s=0} = -X_-A_+ - A_-X_+$$

and thus, using that $\ker \Delta_+ = \ker X_+$, we obtain:

$$\text{Tr} \left( \frac{d}{ds} \Delta^s_+ \bigg|_{s=0} \Pi \right) = -\text{Tr}((X_-A_+ + A_-X_+)\Pi) = -\text{Tr} \left( \sum_i \langle \cdot, u_i \rangle_{L^2} X_-A_+ u_i \right) = -\sum_i \langle X_-A_+ u_i, u_i \rangle_{L^2} = \sum_i \langle A_+ u_i, X_+ u_i \rangle_{L^2} = 0.$$

As far as the second term is concerned, we have using that $(z - \Delta_+)^{-1} = \Pi/z + R(z)$, where $R$ is holomorphic:

$$\Delta_+ \frac{d}{ds} \Pi_{sA} \bigg|_{s=0} = \frac{1}{2\pi i} \int_\gamma \Delta_+(z - \Delta_+)^{-1} \dot{\Delta}_+(z - \Delta_+)^{-1} dz$$

$$= \frac{1}{2\pi i} \int_\gamma (\Delta_+ - z)(z - \Delta_+)^{-1} \dot{\Delta}_+(z - \Delta_+)^{-1} dz$$

$$+ \frac{1}{2\pi i} \int_\gamma (z - \Delta_+)^{-1} \dot{\Delta}_+(z - \Delta_+)^{-1} dz = -\dot{\Delta}_+ \Pi + \Pi \dot{\Delta}_+ \Pi,$$
and taking the trace, we obtain:

\[
\text{Tr} \left( \Delta_+ \frac{d}{ds} \Pi^s A \right)_{s=0} = \text{Tr} \left( -\Delta_+ \Pi + \Pi \Delta_+ \Pi \right) = \text{Tr} \left( -\Delta_+ \Pi + \Delta_+ \Pi^2 \right) = \text{Tr} \left( -\Delta_+ \Pi + \Delta_+ \Pi \right) = 0.
\]

This concludes the proof. \(\square\)

We now compute the second variation of \(\lambda\).

**Lemma 4.2.** For all \(A \in C^\infty(M, T^*M \otimes \text{End}_{sk}(\mathcal{E}))\):

\[
d^2 \lambda_{\Gamma=0}(A, A) = \sum_{i=1}^d \| \pi_{\ker X^- A^+ u_i} \|^2_{L^2}.
\]

**Proof.** This is a rather tedious computation. We have:

\[
d^2 \lambda_{\Gamma=0}(A, A) = \text{Tr} \left( \dot{\Delta}_+ \Pi \right) + 2 \text{Tr} \left( \dot{\Delta}_+ \dot{\Pi} \right) + \text{Tr} \left( \Delta_+ \ddot{\Pi} \right)
\]

Since \(\dot{\Delta}_+ = 2(\dot{\mathbf{X}}_+)^* \dot{\mathbf{X}}_+\), the first term gives:

\[
(\text{I}) = \text{Tr} \left( \dot{\Delta}_+ \Pi \right) = 2 \text{Tr}((\dot{\mathbf{X}}_+)^* \dot{\mathbf{X}}_+ \Pi) = 2 \sum_i \| \dot{\mathbf{X}}_+ u_i \|^2_{L^2} = 2 \sum_i \| A^+ u_i \|^2_{L^2}.
\]

For the second term, we use that

\[
\dot{\Pi} = \Pi \dot{\Delta}_+ R(0) + R(0) \Delta_+ \Pi,
\]

where we recall that \(R\) is defined by \((z - \Delta_+)^{-1} = \Pi/z + R(z)\). Thus:

\[
(\text{II}) = 2 \text{Tr}(\dot{\Delta}_+ \Pi \dot{\Delta}_+ R(0)) + 2 \text{Tr}(\dot{\Delta}_+ R(0) \Delta_+ \Pi).
\]

Note that the first term vanishes as \(\mathbf{X}_+ \Pi = 0\) and \(\mathbf{X}^*_+ = -\mathbf{X}_+\). And last but not least, we compute the third term. First of all, we have:

\[
\ddot{\Pi} = 2 \times \frac{1}{2\pi i} \int_\gamma \frac{1}{(z - \Delta_+)^{-1}} \dot{\Delta}_+ (z - \Delta_+)^{-1} \ddot{\Delta}_+(z - \Delta_+)^{-1} dz
\]

\[
+ \frac{1}{2\pi i} \int_\gamma (z - \Delta_+)^{-1} \dot{\Delta}_+ (z - \Delta_+)^{-1} dz.
\]
This implies after some simplification that:
\[
\Delta_+ \bar{\Pi} = -2\hat{\Delta}_+ \frac{1}{2\pi i} \int_\gamma (z - \Delta_+)^{-1} \hat{\Delta}_+ (z - \Delta_+)^{-1} dz \\
+ 2 \times \frac{1}{2\pi i} \int_\gamma z(z - \Delta_+)^{-1} \hat{\Delta}_+ (z - \Delta_+)^{-1} \hat{\Delta}_+ (z - \Delta_+)^{-1} dz \\
- \hat{\Delta}_+ \Pi \\
+ \frac{1}{2\pi i} \int_\gamma z(z - \Delta_+)^{-1} \hat{\Delta}_+ (z - \Delta_+)^{-1} dz \\
= -2(\hat{\Delta}_+ \Pi \hat{\Delta}_+ R(0) + \hat{\Delta}_+ R(0) \hat{\Delta}_+ \Pi) \\
+ 2(\Pi \hat{\Delta}_+ \Pi \hat{\Delta}_+ R(0) + \Pi \hat{\Delta}_+ R(0) \hat{\Delta}_+ \Pi + R(0) \hat{\Delta}_+ \Pi \hat{\Delta}_+ \Pi) \\
- \hat{\Delta}_+ \Pi \\
+ \Pi \hat{\Delta}_+ \Pi.
\]

It is an elementary computation, using that $X_+ \Pi = 0$ and $\Pi X_- = 0$, to show that $\Pi \hat{\Delta}_+ \Pi = 0$. Thus:
\[
\Delta_+ \bar{\Pi} = -2(\hat{\Delta}_+ \Pi \hat{\Delta}_+ R(0) + \hat{\Delta}_+ R(0) \hat{\Delta}_+ \Pi) \\
+ 2(\Pi \hat{\Delta}_+ \Pi \hat{\Delta}_+ R(0) + \Pi \hat{\Delta}_+ R(0) \hat{\Delta}_+ \Pi + R(0) \hat{\Delta}_+ \Pi \hat{\Delta}_+ \Pi).
\]

Taking the trace, using that $\Pi^2 = \Pi$, we get:
\[
\text{Tr}(\Delta_+ \bar{\Pi}) = -2 \text{Tr}(\hat{\Delta}_+ \Pi \hat{\Delta}_+ R(0)).
\]

Summing the contributions (I, II, III), we obtain:
\[
\frac{d^2}{ds^2} \lambda_{sA} \bigg|_{s=0} = 2 \sum_i ||\dot{X}_+ u_i||^2_{L^2} + 2 \text{Tr}(\hat{\Delta}_+ R(0) \hat{\Delta}_+ \Pi).
\]

It remains to study this last term. After some computations, one can show that:
\[
\text{Tr}(\hat{\Delta}_+ R(0) \hat{\Delta}_+ \Pi) = \sum_{i=1}^d \langle \dot{X}_+ u_i, X_+ R(0) X_+^* \dot{X}_+ u_i \rangle.
\]

We now study $Q := -X_+ R(0) X_+^*$. Recall that any $f \in C^\infty(M, \Omega_{m+1})$ can be uniquely decomposed as $f = X_+ u + h$, where $h \in \ker X_-$ and $u \in \ker \Pi$. Applying $(X_+)^*$, we get $(X_+)^* f = -X_- f = (X_+)^* X_+ u + 0 = \Delta_+ u$. Using the equality $1 - \Pi = -R(0) \Delta_+$, and the fact that $X_+ \Pi = 0$, we then obtain that:
\[
X_+ u = -X_+ R(0) (X_+)^* f = Q f.
\]
In other words, $Q = \pi_{\text{ran}(X_+)}$ is the $L^2$-orthogonal projection on $\text{ran}(X_+)$. Thus:
\[
\frac{\text{d}^2}{\text{d}s^2} \lambda_{sA} \bigg|_{s=0} = 2 \sum_i \|\dot{X}_+ u_i\|_{L^2}^2 - \langle \dot{X}_+ u_i, \pi_{\text{ran}(X_+)}\dot{X}_+ u_i \rangle_{L^2} = 2 \sum_i \|\pi_{\text{ker} X_-} X_+ u_i\|_{L^2}^2 = 2 \sum_i \|\pi_{\text{ker} X_-} A_+ u_i\|_{L^2}^2.
\]

4.2. Proof of the main Theorem. As mentioned in the introduction, Theorem 1.2 follows from the following result.

**Theorem 4.3.** Let $(\mathcal{E}, \nabla^\mathcal{E})$ be Hermitian vector bundle over the Riemannian manifold $(M, g)$, equipped with a smooth unitary connection. Assume that $\ker \left( X_+ \big|_{C^\infty(M, \Omega_m)} \right)$ is not trivial. Then, for all $k \geq 2$, for all $\varepsilon > 0$, there exists a unitary $\nabla^\mathcal{E}$ such that $\|\nabla^\mathcal{E} - \nabla^\mathcal{E}\|_{C^k(M, T^*M \otimes \text{End}^g(\mathcal{E}))} < \varepsilon$ and $(\mathcal{E}, \nabla^\mathcal{E})$ has no twisted CKTs of degree $m$.

Note that the definition of the $C^k$ norms (which may depend on some choice of coordinates) is irrelevant.

**Proof.** Assume that $\ker \left( X_+ \big|_{C^\infty(M, \Omega_m)} \right)$ is $d$-dimensional. Consider a small circle around 0 in $\mathbb{C}$ in which 0 is the only eigenvalue (with multiplicity $d$). It is sufficient to prove that we can produce an arbitrary small perturbation $\nabla^\mathcal{E} + \Gamma$ such that the sum of the eigenvalues inside the circle is strictly positive. Since the Laplacians $\Delta^\mathcal{E}_i \geq 0$ are self-adjoint and non-negative, this means that at least one of the eigenvalues at 0 was ejected, namely $\ker \left( X_+ \big|_{C^\infty(M, \Omega_m)} \right)$ is at most $d - 1$ dimensional. Then, repeating the process finitely many times, one can eject all the resonances out of 0, i.e. one obtains a connection $\nabla^\mathcal{E} + \Gamma$, where $\Gamma$ is arbitrarily small (in $C^k$) such that $X^\Gamma_+$ has no eigenvalues at 0.

Now, using Lemma 4.2, in order to produce a perturbation $\nabla^\mathcal{E} + \Gamma$ of the connection $\nabla^\mathcal{E}$ such that the sum of the eigenvalues of $\Delta^\mathcal{E}_i$ inside the circle is strictly positive, it is sufficient to take $\nabla^\mathcal{E} + s A$ (where $A$ is $C^k$), for $s$ small enough and where $\pi_{\text{ker} X_-} A_+ u_i \neq 0$ (for some $i \in \{1, ..., d\}$). Therefore, this boils down to the following result:

**Lemma 4.4.** Assume $u_0 \in \ker \left( X_+ \big|_{C^\infty(M, \Omega_m \otimes \mathcal{E})} \right)$. Then, there exists $\Gamma \in C^\infty(M, T^*M \otimes \text{End}^g(\mathcal{E}))$ such that $\pi_{\text{ker} X_-} \Gamma_+ u_0 \neq 0$.

**Proof.** Assume this is not the case. Then, using the splitting
\[
C^\infty(M, \Omega_{m+1} \otimes \mathcal{E}) = \ker \left( X_- \big|_{C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})} \right) \oplus^{\perp} \left( C^\infty(M, \Omega_m \otimes \mathcal{E}) \right),
\]
we obtain that for all $\Gamma \in C^\infty(M, T^*M \otimes \text{End}^g(\mathcal{E}))$, there exists $f_\Gamma \in C^\infty(M, \Omega_m \otimes \mathcal{E})$ such that $\Gamma_+ u_0 = X_+ f_\Gamma$. Thus, for all $w \in \ker \left( X_- \big|_{C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})} \right)$:
\[
\langle \Gamma_+ u_0, w \rangle_{L^2} = \langle X_+ f_\Gamma, w \rangle_{L^2} = -\langle f_\Gamma, X_- w \rangle_{L^2} = 0 = \langle u_0, \Gamma_- w \rangle_{L^2}.
\]
We claim that this implies that pointwise in $x \in M$, one has $0 = \langle u_0(x), \Gamma_-(x)w(x) \rangle_{\Omega_m(x) \otimes \mathcal{E}_x}$, for all $\Gamma \in C^\infty(M, T^*M \otimes \text{End}_{sk}(\mathcal{E}))$ and for all $w \in \ker X_-|_{C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})}$. Assuming the claim, we then use that $X_-$ is of uniform divergence type (as proved in Lemma 3.7): by Lemma 3.4, it implies that the map

$$\ker X_-|_{C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})} \ni w \mapsto w(x) \in \Omega_{m+1}(x) \otimes \mathcal{E}_x$$

is surjective for all $x \in M$. We then apply Lemma 2.5 which allows to find $\Gamma$ and $w$ such that $\Gamma_-(x)w(x) = u_0(x)$. Thus $\langle u_0(x), u_0(x) \rangle = 0$ for all $x \in M$ that is $u_0 = 0$. This is a contradiction.

It now remains to prove that pointwise in $x \in M$, we have $0 = \langle u_0(x), \Gamma_-(x)w(x) \rangle_{\Omega_m(x) \otimes \mathcal{E}_x}$. We fix $x_0 \in M$ and consider an arbitrary $w \in \ker X_-|_{C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})}$, $\Gamma \in C^\infty(M, T^*M \otimes \text{End}_{sk}(\mathcal{E}))$. We consider a sequence of (real-valued) functions $\varphi_h \in C^\infty(M)$ such that $\varphi_h \to_{h \to 0} (x_0) \in \mathcal{D}'(M)$, where the convergence takes place in the sense of distributions, i.e. $\langle \varphi_h, f \rangle_{L^2(M)} \to_{h \to 0} f(x_0)$, for all $f \in C^\infty(M)$. We then have:

$$0 = \langle u_0, (\varphi_h \Gamma)_- w \rangle_{L^2} = \int_M \varphi_h(x) \langle u_0(x), \Gamma_-(x)w(x) \rangle_{\Omega_m(x) \otimes \mathcal{E}_x} \, d\text{vol}_g(x)$$

$$\to_{h \to 0} \langle u_0(x_0), \Gamma_-(x_0)w(x_0) \rangle_{\Omega_m(x_0) \otimes \mathcal{E}_{x_0}}.$$ 

This concludes the proof of the main theorem.

In the case of surfaces the operator $X_-$ is not of uniformly divergent type, but we may still perturb the CKTs in some cases. Let $(\mathcal{E}, \nabla^\mathcal{E})$ and $(M, g)$ be as in Theorem 1.2 and assume $\dim M = 2$ with $M$ orientable of genus $g$. As $M$ admits a complex structure, we may consider $(T^*M)^{0,1} := \mathcal{K}$ the canonical bundle spanned locally by the forms $dz$ and analogously $\mathcal{K}^{-1} := (T^*M)^{1,0}$ locally spanned by $d\bar{z}$. One checks that $\otimes^m_s T^*M|_{0-\mathcal{T}_0} = \mathcal{K}^\otimes_m \oplus \mathcal{K}^\otimes(-m)$ and using the map $\pi_m^*$ there is a splitting for each $m \not= 0$: $\Omega_m = H_m \oplus H_{-m}$. We write $\Omega_0 = H_0$. The operators $X_\pm$ for $m > 0$ decompose as $X_\pm|_{\Omega_m \otimes \mathcal{E}} = \mu_+ \oplus \mu_-$, where $\mu_+ : H_m \otimes \mathcal{E} \to H_{m+1} \otimes \mathcal{E}$ for any $m$ (see e.g. [Pat09] for details). Generalising our earlier approach, we obtain for surfaces:

**Proposition 4.5.** Let $m > 0$. If $\ker X_-|_{\Omega_{m+1} \otimes \mathcal{E}} \neq \{0\}$, there is a perturbation that reduces the dimension of $\ker X_+|_{\Omega_m \otimes \mathcal{E}}$ by at least one. Consequently, if the index $\text{ind} \mu_+|_{H_m \otimes \mathcal{E}} \leq 0$, then it is possible to perturb the connection to eject all the twisted CKTs; if $\text{ind} \mu_+|_{H_m \otimes \mathcal{E}} > 0$, then if necessary we may perturb the connection to obtain

$$\dim \ker \mu_+|_{H_m \otimes \mathcal{E}} = \text{ind} \mu_+|_{H_m \otimes \mathcal{E}}.$$ 

**Proof.** We first consider an equivalent of Lemma 2.5 for the case of surfaces. In local isothermal coordinates $g = e^{2\lambda}|dz|^2$, we may write the connection form as $\Gamma = \Gamma(\partial_z)dz +$
\( \Gamma(\partial_{\bar{z}})dz; \) set \( \Gamma_+ = \Gamma(\partial_{\bar{z}})dz. \) Then \( \Gamma_- : H_m \otimes \mathcal{E} \to H_{m-1} \otimes \mathcal{E} \) is given by

\[
\Gamma_-(e^{m\lambda}(dz^m \otimes s) = e^{(m-1)\lambda}(dz)^{m-1} \otimes (e^{-\lambda}\partial_{\bar{z}})s.
\]

Therefore as soon as \( \Gamma(\partial_{\bar{z}})(x) : \mathcal{E}_x \to \mathcal{E}_x \) is invertible, we have \( \Gamma_-(x) : H_m(x) \otimes \mathcal{E}_x \to H_{m-1}(x) \otimes \mathcal{E}_x \) an isomorphism.

Coming back to the main proof, we may without loss of generality assume that \( u_0 \in C^\infty(M; H_m \otimes \mathcal{E}). \) Arguing by contradiction as in the proofs of Theorem 4.3 and Lemma 4.4 shows \( \langle u_0(x), \Gamma_-(x)w(x)\rangle_{\Omega_m(x)\otimes \mathcal{E}_x} = 0 \) for all \( x \in M, \Gamma \in C^\infty(M; T^*M \otimes \text{End}_{\mathcal{E}} \mathcal{E}) \) and \( w \in \ker \mu_-|_{H_{m+1}\otimes \mathcal{E}}. \) Assume \( 0 \neq w \in \ker \mu_-|_{H_{m+1}\otimes \mathcal{E}}. \) Since \( \mu_- \) is elliptic, \( \{w = 0\} \subset M \) is nowhere dense by the unique continuation principle. Picking a suitable \( \Gamma \) according to the previous paragraph, we obtain \( u_0 = 0 \) on \( \text{supp}(w) \), thus \( u_0 \equiv 0 \), contradiction. Therefore to reduce the dimension of \( \ker X_+|_{\Omega_{m+1}\otimes \mathcal{E}} \) by at least one, it suffices to produce a single non-trivial element in \( \ker \mu_-|_{H_{m+1}\otimes \mathcal{E}} \), proving the first part of the claim.

If \( \eta_{\pm} \) denote the raising/lowering operators for \( \mathcal{E} = \mathbb{C} \), then by (2.5) and (2.6) \( \sigma_{\mu_{\pm}}(x, \xi) = \sigma_{\eta_{\pm}}(x, \xi) \otimes \text{id}_\mathcal{E}, \) so the value of \( \text{ind} \mu_-|_{H_{m+1}\otimes \mathcal{E}} \) is topological.\(^4\) Note that \( \mu_{+}^* = -\mu_- \) implies \( \text{ind} \mu_+|_{H_m \otimes \mathcal{E}} = -\text{ind} \mu_-|_{H_{m+1}\otimes \mathcal{E}}. \) Thus by the previous paragraph, as long as \( \ker \mu_-|_{H_{m+1}\otimes \mathcal{E}} \neq \{0\}, \) an inductive argument ejecting the eigenvalues one by one shows that we may reduce the dimension of \( \ker \mu_+|_{H_m \otimes \mathcal{E}} \) to \( \max\{0, \text{ind} \mu_+|_{H_m \otimes \mathcal{E}}\} \), completing the proof.

\[\square\]

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\(^4\)By the Atiyah-Singer index theorem, it may be computed explicitly as a function of the first Chern class \( c_1(\mathcal{E}) \) and \( \text{ind} \eta_-|_{H_{m+1}\otimes \mathcal{E}} = (2m + 1)(g - 1) \) (for the latter see [PSU14, Lemma 2.1]).
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