A characterization of 3+1 spacetimes via the Simon-Mars tensor

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We present the 3+1 decomposition of the Simon-Mars tensor, which has the property of being identically zero for a vacuum and asymptotically flat spacetime if and only if the latter is locally isometric to the Kerr spacetime. Using this decomposition we form two dimensionless scalar fields. Computing these scalars provides a simple way of comparing locally a generic (even non vacuum and non analytic) stationary spacetime to Kerr. As an illustration, we evaluate the Simon-Mars scalars for numerical solutions of the Einstein equations generated by boson stars and neutron stars, for analytic solutions of the Einstein equations such as Curzon-Chazy spacetime and δ = 2 Tomimatsu-Sato spacetime, and for an approximate solution of the Einstein equations: the modified Kerr metric, which is an example of a parametric deviation from Kerr spacetime.

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I. INTRODUCTION

The Kerr metric [1] is an exact solution of the Einstein equations describing rotating black holes. It is generally accepted that the compact object at our Galactic Center SgrA* is described by this geometry [2]. But, alternative asymptotically flat compact objects are also studied in the literature as possible models for SgrA* (a recent example is rotating boson stars [3, 4]). It could be interesting to have a mathematical tool measuring the deviation from these spacetimes with respect to the Kerr one. To achieve this goal, the Simon-Mars tensor has been chosen.

The Simon-Mars tensor has been introduced by Mars in [5], under the name of Spacetime Simon tensor. This tensor and its ancestors, the Cotton tensor [6] and the Simon tensor [7], were defined to understand what singles out Kerr among the family of stationary and axisymmetric metrics. In [5], Mars proved a theorem stating that if a spacetime satisfies the Einstein vacuum field equations, is asymptotically flat compact objects are also studied in the literature as possible models for SgrA* (a recent example is rotating boson stars [3, 4]). It could be interesting to have a mathematical tool measuring the deviation from these spacetimes with respect to the Kerr one. To achieve this goal, the Simon-Mars tensor has been chosen.

In the following section, the definition and properties of the Simon-Mars tensor are briefly reviewed. In Section III, the 3+1 decomposition of this tensor in 8 components is performed step by step. Then the definition of the Simon-Mars scalar fields is given. In the next section (Section IV), the specific case of axisymmetric spacetimes is considered. The last part, Sections V is devoted to the applications: first we present numerical computation of the 3+1 components of the Simon-Mars for a Kerr space-

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time to verify that all of them are identically zero. Then we compute those 8 components in adapted coordinates for two numerical solutions of Einstein equations: rotating boson stars and neutron stars, and we do the same for the 2 scalars. To do the link between our characterization and the one of [11], we tackle two examples given in [11]: Curzon-Chazy and $\delta = 2$ Tomimatsu-Sato spacetimes, which are exact analytic solutions of the Einstein equations. Finally, we compute the Simon-Mars scalars for a family of spacetimes deviating from Kerr by a continuous parameter: the modified Kerr metric [13].

II. SIMON-MARS TENSOR

A. Definition

Let $M$ be a 4-dimensional manifold endowed with a smooth Lorentzian metric $g_{\alpha\beta}$ of signature $(-, +, +, +)$. Greek indices vary from 0 to 4 while Latin indices are only 1 to 3. The Einstein summation convention is used unless specified otherwise. We work in geometric units for which $G = c = 1$. The Levi-Civita covariant derivative associated to $g_{\alpha\beta}$ is the operator $\nabla_\alpha$ while $R^\mu_{\nu\sigma\tau}$ and $R_{\alpha\beta\gamma\delta}$ denote respectively the Riemann and Ricci tensors. From these two tensors one builds the Weyl tensor $C_{\alpha\beta\gamma\delta}$ which corresponds to the traceless part of the Riemann tensor$^1$:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{R}{6} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) - \frac{1}{2} (g_{\alpha\gamma} R_{\beta\delta} - g_{\alpha\delta} R_{\beta\gamma} - g_{\beta\gamma} R_{\alpha\delta} + g_{\beta\delta} R_{\alpha\gamma}).$$

Furthermore we use the right self-dual Weyl tensor, which is defined as

$$C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + i C^*_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{1}{2} \eta_{\gamma\delta\rho\sigma} C_{\alpha\beta}^{\rho\sigma},$$

where $\eta_{\gamma\delta\rho\sigma}$ is the volume 4-form associated with $g$ and the star denotes Hodge duality: the Hodge dual of a 2-form is given by

$$F^*_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu\lambda\rho} F^{\lambda\rho}.$$ (3)

To construct the Simon-Mars tensor one assumes the existence of a Killing vector field in spacetime. Let us recall that a vector field $\xi^\mu$ on $M$ is called a Killing vector field if it verifies the following condition:

$$\mathcal{L}_\xi g_{\alpha\beta} = 0,$$ (4)

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to $\xi$. As we are interested in this work by stationary spacetimes, they all possess the Killing vector field $\xi = \partial_t$. Besides, a Killing vector field satisfies the identity

$$\nabla_\alpha (\xi_\beta) = 0.$$ (5)

To this property follow the definition of the Papapetrou field given by

$$F_{\alpha\beta} = \nabla_\alpha \xi_\beta.$$ (6)

$F_{\alpha\beta}$ is antisymmetric (i.e. is a 2-form) because of (5). We will also use the self-dual form of the Papapetrou field defined as we have seen in (2) and (5) by

$$F^*_{\alpha\beta} = F_{\alpha\beta} + i F^*_{\alpha\beta} = \nabla_\alpha \xi_\beta + \frac{i}{2} \eta_{\alpha\beta\lambda\mu} \nabla^\lambda \xi^\mu.$$ (7)

From $F^*_{\alpha\beta}$ we define the twist 1-form

$$\omega_\alpha = F^*_{\alpha\beta} \xi^\beta,$$ (8)

which is closed for a vacuum spacetime. In such a case we can define a local potential which is called the twist potential $\omega$: $\omega_\alpha = \nabla_\alpha \omega$. The norm of the Killing vector field is $\lambda = \xi_\alpha \xi^\alpha$ and no restriction is imposed on its sign. The last ingredient needed to construct the Simon-Mars tensor is the Ernst 1-form:

$$\sigma_\mu = 2 \xi^\nu F_{\alpha\mu}.$$ (9)

In vacuum spacetimes, the Ernst 1-form is closed so one can define a local potential $\sigma$ called the Ernst potential which can be written in terms of the norm and twist of the Killing vector field

$$\sigma = \lambda + 2i \omega,$$ (10)

but this is not the case for non-vacuum spacetimes such as boson star and neutron stars spacetimes. At last we can give the definition of the Simon-Mars tensor, given by Mars [5], and based on the work of Simon [7] :

$$S_{\alpha\beta\nu} = 4 \xi^\mu \xi^\nu C_{\mu\alpha\beta\nu} \sigma_{\nu} + \gamma_{(\alpha} \beta) C_{\nu\rho\sigma} F^{\rho\sigma} \xi_{\nu},$$ (11)

where we used the following abbreviation

$$\gamma_{\alpha\beta} = -\lambda g_{\alpha\beta} + \xi_\alpha \xi_\beta.$$ (12)

As stated in the introduction, Bini et al. [8] are calling Simon-Mars tensor the 2-tensor $\tilde{S}$ linked to $S$ by

$$\tilde{S}_{\alpha\beta} = \eta^{\nu} \eta_{\nu\sigma\lambda\mu} S_{\beta}^{\lambda\mu}.$$ (13)

B. Properties

The Simon-Mars tensor (11) has the algebraic properties of a Lanczos potential, namely:

$$\tilde{S}_{\beta\nu} = 0,$$

$$S_{[\alpha\beta\nu]} = S_{\alpha\beta\nu},$$

$$S_{\alpha\beta\nu} + S_{\beta\nu\alpha} + S_{\nu\alpha\beta} = 0.$$ (14)
But the fundamental property of this tensor used in this work and derived in [5] is the following: the Simon-Mars tensor $S_{αβρ}$ vanishes identically for an asymptotically flat spacetime which verifies the Einstein vacuum field equations if and only if this spacetime is locally isometric to the Kerr spacetime. A geometric interpretation of this statement is discussed in [14].

In the sense of the preceding quoted theorem, the Simon-Mars tensor characterizes the Kerr spacetime. It is then interesting to calculate its value for other asymptotically flat and stationary spacetimes.

III. ORTHOGONAL SPLITTING OF THE SIMON-MARS TENSOR

A. Basis of 3+1 formalism and useful formulas

In this article, we only consider globally hyperbolic spacetimes. Such spacetimes admit a foliation by a one-parameter family of spacelike hypersurfaces denoted by $Σ$. This is the 3+1 decomposition, references on this formalism can be found in the literature, see for instance [15, 16]. Here we review only the formulas which are useful for this work.

The unit vector which determines the unique direction normal to $Σ$, denoted by $n^α$, is also the 4-velocity of an observer called the Eulerian observer. Because $Σ$ is spacelike, the following property is verified by the normal vector

$$n^α n_α = -1.$$  
(15)

On each hypersurface the metric induced by $g$ is

$$h_{αβ} = g_{αβ} + n_α n_β,$$  
(16)

and the extrinsic curvature is given by

$$K_{αβ} = -\frac{1}{2} L_n h_{αβ}.$$  
(17)

We will also use the following abbreviation

$$l_{αβ} = h_{αβ} + n_α n_β = g_{αβ} + 2 n_α n_β.$$  
(18)

The orthogonal splitting of the volume element is given by

$$η_{αβγδ} = -n_α ε_{βγδ} + n_β ε_{αγδ} + n_γ ε_{αβδ} + n_δ ε_{αβγ},$$  
(19)

where $ε_{αβγ} = n^λ η_{λαβγ}$ is the spatial volume element which is a fully antisymmetric spatial tensor and which verifies

$$ε_{ijk} ε^{ijl} = 2 h_k^l,$$  
(20)

$$ε_{ijk} ε^{lmn} = h_i^l h_j^m h_k^n + h_i^m h_j^l h_k^n + h_i^m h_j^n h_k^l - h_i^l h_j^m h_k^n - h_i^n h_j^m h_k^l.$$  
(21)

In all the calculation, we suppose the existence of the Killing vector field $ξ^μ = ∂^μ t$ (because all the spacetimes considered are stationary). Its orthogonal splitting is given by:

$$ξ^μ = ∂^μ t = N n^μ + β^μ,$$  
(22)

where $N$ is called the lapse because it is related to the time lapse between two slices of the foliation and $β^μ$ the shift because it tells how the coordinates are shifted from one slice to another, cf [15] for details. The line element of a spacetime expressed with the 3+1 formalism is the following:

$$g_{αβ} dx^α dx^β = -(N^2 - h_{ij} β^i β^j) dt^2 + h_{ij} dx^i dx^j + 2 h_{ij} β^j dt.$$  
(23)

We end this section by writing, for the specific case of stationarity (all the partial derivatives with respect to the time are zero), every useful formula for the next section (each of them can be easily derived):

- The 3+1 decomposition of the derivative of the unit normal vector:

$$∇α n_β = -K_α β - n_α D_β ln N,$$  
(24)

where $D_α$ is the covariant derivative associated with the 3-dimensional metric $h$.

- The derivative of the lapse:

$$∇α N = D_α N + n_α β^γ D_γ ln N.$$  
(25)

- The Lie derivative of the shift:

$$L_\alpha β_α = -2 β^γ K_α γ.$$  
(26)

- The derivative of the shift:

$$∇α β_γ = D_α β_γ - n_α β_δ K_α δ - n_α (n_γ β^δ D_δ ln N - β^δ K_δ γ).$$  
(27)

B. Orthogonal splitting of the self-dual Weyl tensor

First we introduce the electric and magnetic parts of the Weyl tensor given in [16] (and first defined in [17])

$$E_αβ = C_αγβδ n^γ n^δ,$$  
(28)

$$B_αβ = \frac{1}{2} η_δσρ C_αγ δρσ n^γ n^δ.$$  
(29)

The decomposition of the Weyl tensor (1) using (28) and (29) is also given in [16] using (18) (for a demonstration see [18])

$$C_αβγδ = 2 (l_α [γ E_β] δ + l_β [δ E_γ] α) - 2 (ε_δρ η_α β B_ρ δ + ε_αβρ n_γ B_ρ δ),$$  
(30)
so (28) and (29) are given by (again see [16])
\[
E_{ij} = R_{ij} + KK_{ij} - K_{il}K^{l}_{j} \\
-4\pi \left[ S_{ij} + \frac{h_{ij}}{3} (4\rho - S) \right], \quad (31)
\]
\[
B_{ij} = \varepsilon^{mn}_i (D_mK_{nj} - 4\pi h_{jm}p_n), \quad (32)
\]
with $S_{\alpha\beta}$, $p_{\alpha}$ and $\rho$ the components of the orthogonal splitting of the stress energy tensor $T_{\mu\nu}$:
\[
\rho = T_{\mu\nu}n^\mu n^\nu, \quad (33)
\]
\[
p_{\alpha} = -T_{\mu\nu}h^\mu_{\alpha} n^\nu, \quad (34)
\]
\[
S_{\alpha\beta} = T_{\mu\nu}h^\mu_{\alpha} h^\nu_{\beta}. \quad (35)
\]

Thanks to (2), (30) is also the 3+1 decomposition of the real part of the self-dual Weyl tensor. As the magnetic part of the Weyl tensor is the Hodge dual of the electric part, the imaginary part of the self-dual Weyl tensor reads
\[
\text{Im} (C_{\alpha\beta\gamma\delta}) = 2 \left( l_{[\gamma} B_{\delta][\alpha} + l_{[\alpha} B_{\gamma][\delta} \right) \\
+2i \left( \epsilon_{\gamma\rho}n_{[\alpha} E_{\beta]}^\rho + \epsilon_{\alpha\beta\rho} n_{[\beta} E^\rho_{\alpha]} \right). \quad (36)
\]

### C. Orthogonal splitting of $F_{\alpha\beta}$, $\sigma_\mu$ and $\gamma_{\alpha\beta}$

The self-dual Papapetrou field is defined by (7). As $F_{\alpha\beta}$ is complex, we perform the 3+1 decomposition first for the real part and then for the imaginary one. The real part is (6), so we take the 3+1 decomposition of the Killing vector field $\xi_\beta$ given by (22), we develop, then we use (24), (25) and (27). The antisymmetric part reads³
\[
\text{Re} (F_{\alpha\beta}) = D_{[\alpha \beta]} - 2n_{[\alpha} D_{\beta]} N + 2n_{[\alpha} \beta^\delta K_{\beta] \delta}. \quad (37)
\]

We checked also that we recover the Killing equation (5) with the symmetric part. Let us do the same for the imaginary part, after development and using the expression of the volume form (19) and its antisymmetry we obtain²
\[
\text{Im} (F_{\alpha\beta}) = -n_{[\alpha} \epsilon_{\beta\lambda\mu} D^\lambda \beta^\mu \\
-\epsilon_{\alpha\beta\mu} \left( D^\mu N - \beta^\delta K_{\mu \delta} \right). \quad (38)
\]

Let us tackle the decomposition of the Ernst potential given by (9)
\[
\sigma_\mu = 2\xi^\alpha \nabla_\alpha \xi_\mu + i\xi^\alpha \eta_{\alpha\mu\lambda\nu} \nabla^\lambda \xi^\nu. \quad (39)
\]

Using (22) and (37) (resp. (38)) the 3+1 decomposition of the real part (resp. imaginary part) is given by⁴
\[
\text{Re} (\sigma_\mu) = 2ND_{\mu} N - 2N K_{\mu \beta} \beta^\beta + 2\beta^\lambda D_{[\beta} \beta^\mu], \\
+2n_\mu \left( \beta^\beta D_{\mu} N - \beta_\beta K_{\mu \beta} \right), \quad (40)
\]
\[
\text{Im} (\sigma_\mu) = \epsilon_{\mu\nu} \left( 2\beta^\beta D^\beta N - 2K^{\beta \mu} \beta^\beta + ND^\beta \beta^\beta \right) \\
+n_\mu \epsilon_{ij\nu} \beta^\beta D^\beta D^\nu \beta^k. \quad (41)
\]

Finally we can split $\gamma_{\alpha\beta}$ given by (12) simply using (16) and (22)
\[
\gamma_{\alpha\beta} = \left( N^2 - \beta_i \beta^i \right) h_{\alpha\beta} + \beta_{\alpha\beta} \\
+N (n_{\alpha\beta} + \beta_{\alpha} n_{\beta}) + \beta_i \beta^i n_{\alpha\beta}. \quad (42)
\]

### D. Orthogonal splitting of the Simon-Mars tensor

Let us recall the definition of the Simon-Mars tensor (11)
\[
S_{\alpha\beta \nu} = 4 \epsilon^{\mu\nu} C_{\mu\alpha\rho [\beta} \sigma_{\nu]} + \gamma_{[\beta} C_{\nu] \mu \rho \delta} F^{\rho \mu} \xi_{\alpha}. \quad (43)
\]

As this is a complex tensor, we decompose independently the real and the imaginary parts, and we also consider one of the two terms of (11) at a time.

#### 1. First term

Let us then first consider the fourth of the real part of the first term and develop it
\[
\text{Re} (F T) = \underbrace{\xi^{\mu} \xi^{\nu} \text{Re} (C_{\mu \alpha \rho [\beta} \Re (\sigma_{\nu]}))}_{I_{\alpha \beta \nu}} \\
-\xi^{\mu} \xi^{\nu} \text{Im} (C_{\mu \alpha \rho [\beta} \text{Im} (\sigma_{\nu]})). \quad (43)
\]

Using (22) and (30), we obtain the 3+1 decomposition of the first part of $I_{\alpha \beta \nu}$,
\[
\xi^{\mu} \xi^{\nu} \text{Re} (C_{\mu \alpha \rho [\beta} \Re (\sigma_{\nu]})) = V_{\alpha \beta} + n_{\alpha} W_{\beta} + n_{\beta} W_{\alpha} + n_{\alpha} n_{\beta} Y, \quad (44)
\]
with
\[
V_{ij} = \left( N^2 + \beta_k \beta^k \right) E_{ij} - \beta_{ij} \beta^k E_{ik} - \beta^k \beta_i E_{kj} \\
+h_{ij} \beta^k \beta^\rho E_{k \rho} + N (\epsilon_{kjs} \beta^k B_j^* + \epsilon_{kis} \beta^k B_j^*) \quad (45)
\]
\[
W_i = N \beta^k E_{ik} + \epsilon_{kis} \beta^k B_j^* \quad (46)
\]
\[
Y = \beta^i \beta^j E_{ij}. \quad (47)
\]

² same as equation (4.35) in [10]
³ except for the sign misprint in the second term, same as equation (4.36) in [10]
⁴ same as real and imaginary part of equation (6.5) in [10], except for one term forgotten in the imaginary part
Rewriting (40) we have directly the decomposition of the second part
\[ \text{Re}(\sigma_\nu) = Z_\nu + n_\nu T, \] (48)
with
\[ Z_i = 2ND_i N - 2NK_k \beta^k + 2\beta^k D_{[k} \beta_{i]}, \] (49)
\[ T = 2\beta^i D_i N - 2K_{ij} \beta^j, \] (50)
so
\[ I_{\alpha\beta\nu} = 2V_{\alpha[\beta} Z_{\nu]} + 2TV_{\alpha[\beta} n_{\nu]}, \]
\[ + 2n_\alpha W_{[\beta} Z_{\nu]} + 2W_{\alpha} n_{[\beta] Z_{\nu]} \]
\[ + 2T W_{[\beta} n_{\nu]} n_\alpha + 2Y n_\alpha n_{[\beta] Z_{\nu]}. \] (51)

Using (36), we do the same for the first part of \( I_{\alpha\beta\nu} \) of (43), we obtain
\[ \xi^\nu \xi^\nu \text{Im}(C_{\mu\alpha\beta}) = \bar{V}_{\alpha} + n_\alpha \bar{W}_{\beta} + n_\beta \bar{W}_{\alpha} + n_\alpha \bar{n}_\beta \bar{Y}, \] (52)
with
\[ \bar{V}_{ij} = \left(N^2 + \beta^k \beta^k\right) B_{ij} - \beta_j \beta^k B_{ik} - \beta^k \beta^k B_{ij} \]
\[ + h_{ij} \beta^k \beta^l B_{kl} - N \left(\epsilon_{ij} \epsilon^k \epsilon^l E_i^* + \epsilon_{ij} \epsilon^k \epsilon^l E_k^*\right) \] (53)
\[ \bar{W}_i = N \beta^i B_{ik} - i \epsilon_{ijs} \beta^j \beta^k E_k^*, \] (54)
\[ \bar{Y} = \beta^i \beta^j B_{ij}, \] (55)

Rewriting (41) we have also
\[ \text{Im}(\sigma_\nu) = \bar{Z}_\nu + n_\nu \bar{T}, \] (56)
with
\[ \bar{Z}_i = -\epsilon_{ijk} \left(2K_i \beta^k \beta^j - 2\beta^j D^k N - ND^j \beta^k\right), \] (57)
\[ \bar{T} = \epsilon_{ijk} \beta^j D^k, \] (58)
so
\[ I_{\alpha\beta\nu} = 2V_{\alpha[\beta} \bar{Z}_{\nu]} + 2\bar{T}V_{\alpha[\beta} n_{\nu]}, \]
\[ + 2n_\alpha \bar{W}_{[\beta} \bar{Z}_{\nu]} + 2W_{\alpha} n_{[\beta] \bar{Z}_{\nu]} \]
\[ + 2T \bar{W}_{[\beta} n_{\nu]} n_\alpha + 2Y n_\alpha n_{[\beta] \bar{Z}_{\nu]}. \] (59)

Finally, using (51) and (59), and the antisymmetry in \( \beta \) and \( \nu \), (43) can be written
\[ \text{Re}(FT) = 2 \left[ V_{\alpha[\beta} Z_{\nu]} - \bar{V}_{\alpha[\beta} \bar{Z}_{\nu]} \right] \\
+ \left( TV_{\alpha[\beta} - \bar{T} V_{\alpha[\beta} - W_\alpha Z_{[\beta} + \bar{W}_\alpha \bar{Z}_{[\beta} \right) n_{\nu]} \]
\[ + \left( TW_{[\beta} - \bar{T} W_{[\beta} - Y Z_{[\beta} + \bar{Y} \bar{Z}_{[\beta} \right) n_{\nu]} n_\alpha \]
\[ + n_\alpha \left( W_{[\beta} Z_{\nu]} - W_{[\beta} \bar{Z}_{\nu]} \right) \], \] (60)

For the imaginary part we do the same :
\[ \text{Im}(FT) = \xi^\nu \xi^\nu \text{Re}(C_{\mu\alpha\beta}) \text{Im}(\sigma_\nu) \]
\[ + \xi^\nu \xi^\nu \text{Im}(C_{\mu\alpha\beta}) \text{Re}(\sigma_\nu), \] (61)
and we only have to use (44) with (56), and (52) with (48) to obtain
\[ \text{Im}(FT) = 2 \left[ V_{\alpha[\beta} Z_{\nu]} + \bar{V}_{\alpha[\beta} \bar{Z}_{\nu]} \right] \\
+ \left( TV_{\alpha[\beta} + \bar{T} V_{\alpha[\beta} - W_\alpha \bar{Z}_{[\beta} - \bar{W}_\alpha Z_{[\beta} \right) n_{\nu]} \]
\[ + \left( TW_{[\beta} + \bar{T} W_{[\beta} - Y \bar{Z}_{[\beta} - Y Z_{[\beta} \right) n_{\nu]} n_\alpha \]
\[ + n_\alpha \left( W_{[\beta} \bar{Z}_{\nu]} + \bar{W}_{[\beta} Z_{\nu]} \right) \], \] (62)
where \( V_{\alpha\beta}, \bar{V}_{\alpha\beta}, Z_\nu, \bar{Z}_\nu, T, \bar{T}, W_\alpha, \bar{W}_\alpha, Y \) and \( \bar{Y} \) are given respectively by (45), (53), (49), (57), (50), (58), (46), (54), (47) and (55).

2. Second term

We obtain the same kind of decomposition for the real part and the imaginary part of the second term
\[ \text{Re}(ST) = \gamma_{\alpha\beta} \xi^\alpha \text{Re}(C_{\nu\mu\rho}) \text{Re}(F^\rho) \]
\[ \tfrac{IV_{\alpha\beta\nu}}{I'_{\alpha\beta\nu}}, \] (63)
but the contractions are a little more difficult to do for this term. Using (22), (30) and (37) to calculate \( I_{\alpha\beta\nu} \) and using (22), (36), (38) and the properties of the spatial volume form : (20) and (21) for \( I'_{\alpha\beta\nu} \), we obtain
\[ \text{Re}(\gamma_{\alpha\beta} \xi^\alpha \text{Im}(C_{\nu\mu\rho}) \text{Im}(F^\rho)) \]
\[ \tfrac{IV_{\alpha\beta\nu}}{I'_{\alpha\beta\nu}}, \] (64)
with
\[ L_t = 2 \left(D_j \beta^k - D^k \beta^j\right) \beta_k E_{ij} + 2\beta_k E_{jk} \left(D_i \beta^j - D^j \beta_i\right) \]
\[ - 2N \epsilon_{ijkl} B^j B^k - 4A_\beta \left(\beta_m K^m - D^j N\right), \] (65)
\[ M = -2 \epsilon_{ijkl} \beta^j B^k - 4\beta^j E_{ij} \left(K^j \beta_k - D^j N\right), \] (66)
where
\[ A_{il} = \epsilon_{ijkl} \beta^k B^l - \epsilon_{ijkl} \beta_k B^l \epsilon_{ijkl} B^l - 2NE_{il}. \] (67)

We can also rewrite (42) in the following form
\[ \gamma_{\alpha\beta} = G_{\alpha\beta} + N \left(\beta_\beta n_\alpha + \beta_\alpha n_\beta\right) + n_\alpha n_\beta \beta_i \beta^i, \] (68)
with
\[ G_{ij} = \left(N^2 - \beta^k \beta^k\right) h_{ij} + \beta_i \beta_j, \] (69)
so we have
\[ \text{Re}(ST) = 2G_{\alpha[\beta} L_{\nu]} + 2N n_\alpha \beta_{[\beta} L_{\nu]} \]
\[ + 2N \left(MG_{\alpha[\beta} - N\beta_\alpha L_{[\beta} \right) n_{\nu]} \]
\[ + 2n_\alpha \left(M N \beta_{[\beta} - \beta_\beta \beta_{[\beta} L_{[\beta} \right) n_{\nu]}. \] (70)

We do the same for the imaginary part
\[ \text{Im}(ST) = \gamma_{\alpha\beta} \xi^\alpha \text{Im}(C_{\nu\mu\rho}) \text{Im}(F^\rho) \]
\[ \tfrac{IV_{\alpha\beta\nu}}{I'_{\alpha\beta\nu}} \]
\[ \left(t \frac{IV_{\alpha\beta\nu}}{I'_{\alpha\beta\nu}} \right), \] (71)
To calculate \( IV_{\alpha\beta\nu} \) we just have to take the formula of \( I'_{\alpha\beta\nu} \) and make the changes \( E \to B \) and \( B \to -E, \) and
to calculate $III'_{\alpha\beta\nu}$ we take $II'_{\alpha\beta\nu}$ and make the changes $B \to E$ and $E \to -B$, so we obtain

$$\Im \left( ST \right) = 2G_{\alpha[\beta} \bar{L}_{\nu]} + 2Nn_{\alpha}[\beta} \bar{L}_{\nu]}$$
$$+ 2\left( MG_{\alpha[\beta} - N\beta_{\alpha} \bar{L}_{[\beta]} \right) n_{\nu]},
$$

(72)

with

$$\bar{L}_{i} = 2 \left( D^{i} \beta^{k} - D^{k} \beta^{i} \right) \beta_{k} B_{j} + 2 \left( D_{i} \beta^{j} - D^{i} \beta_{j} \right) \beta^{k} B_{jk}$$
$$+ 2N_{\epsilon jkl} E^{l} \left( D^{i} \beta^{j} - D^{k} \beta^{l} \right) \beta^{j} B_{ij},$$

(73)

$$\bar{M} = 2\epsilon_{ijk} \beta^{j} E^{l}_{k} \left( D^{i} \beta^{j} - 4\beta^{j} B_{ij} \left( K^{j} \beta^{k} - D^{k} N \right) \right),$$

(74)

where

$$\bar{A}_{il} = \epsilon_{ijkl} \beta^{k} E^{j} \beta^{j} i - \epsilon_{ijkl} \beta^{j} E^{k}_{l} - 2N B_{il}.$$  

(75)

3. Final decomposition

Gathering all those calculations, we obtain the real part of the decomposition of the Simon-Mars tensor coming from (60) (with a factor 4) and (70):

$$\Re \left( S_{\alpha[\beta} \bar{\nu]} \right) = 2 \left( S_{\alpha[\beta}^{1} n_{\nu} - S_{\alpha\nu} n_{\beta} \right)$$
$$+ S_{\alpha[\beta}^{3} n_{\nu} + S_{\beta\nu} n_{\alpha} - S_{\alpha\nu} n_{\beta},$$  

(76)

with

$$S_{\alpha[\beta}^{1} = 4 \left( V_{ij} Z_{k} - \bar{V}_{ij} \bar{Z}_{k} \right) + G_{ij} L_{k},$$

(77)

$$S_{\alpha[\beta}^{2} = 4 \left( TV_{ij} - \bar{T}V_{ij} + \bar{W}_{i} \bar{Z}_{j} - W_{i} Z_{j} \right) + MG_{ij} - N\beta_{i} L_{j},$$

(78)

$$S_{\alpha[\beta}^{3} = 4 \left( W_{i} \bar{Z}_{j} - \bar{W}_{i} \bar{Z}_{j} + N\beta_{i} L_{j} \right),$$

(79)

$$S_{\alpha[\beta}^{4} = 4 \left( TW_{i} - \bar{T}W_{i} + \bar{Y} \bar{Z}_{i} - Y Z_{i} \right) + N\beta_{i} M - \beta_{i} \beta^{l} L_{i}.$$  

(80)

We see that $S_{\alpha[\beta}^{1}$ is antisymmetric in its 2 last indices and that $S_{\alpha[\beta}^{3}$ is antisymmetric.

We do the same for the imaginary part coming from (62) (with also a factor 4) and (72):

$$\Im \left( S_{\alpha[\beta} \bar{\nu]} \right) = 2 \left( S_{\alpha[\beta}^{1} n_{\nu} - S_{\alpha\nu} n_{\beta} \right)$$
$$+ S_{\alpha[\beta}^{3} n_{\nu} + S_{\beta\nu} n_{\alpha} - S_{\alpha\nu} n_{\beta},$$  

(81)

with

$$S_{\alpha[\beta}^{1} = 4 \left( V_{ij} Z_{k} + \bar{V}_{ij} \bar{Z}_{k} \right) + G_{ij} L_{k},$$

(82)

$$S_{\alpha[\beta}^{2} = 4 \left( TV_{ij} + \bar{T}V_{ij} - W_{i} \bar{Z}_{j} - \bar{W}_{i} Z_{j} \right) + MG_{ij} - N\beta_{i} L_{j},$$

(83)

$$S_{\alpha[\beta}^{3} = 4 \left( W_{i} \bar{Z}_{j} - \bar{W}_{i} \bar{Z}_{j} + N\beta_{i} L_{j} \right),$$

(84)

$$S_{\alpha[\beta}^{4} = 4 \left( TW_{i} + \bar{T}W_{i} - Y \bar{Z}_{i} - \bar{Y} Z_{i} \right) + N\beta_{i} M - \beta_{i} \beta^{l} L_{i}.$$  

(85)

All these terms must be zero for Kerr spacetime (this is checked in V B 1), the goal is to compute them for other spacetimes. But, before doing so, we build two scalar fields to be able to compare coordinate independent quantities, according to the spirit of general relativity.

E. Simon-Mars scalars

The simplest scalar we can form with the Simon-Mars tensor is its “square”:

$$S_{\alpha\beta\nu} S^{\alpha\beta\nu}.$$  

(86)

We decompose it in two scalars, the absolute value of its real part and of its imaginary part, within the 3+1 formalism using the decomposition of the Simon-Mars tensor given in the preceding section.

First let us consider the real part using (76) and (15) and the symmetries. All the spatial indices give zero contracted with $\bar{n}$, so we obtain

$$\bar{b} = \left| \Re \left( S_{\alpha\beta\nu} S^{\alpha\beta\nu} \right) \right|$$
$$= 4 \left| S_{ijk}^{1} S_{ijk}^{1} - S_{ijk}^{1} S_{ijk}^{1} - 2 \left( S_{ijk}^{2} S_{ij}^{2} - S_{ijk}^{2} S_{ij}^{2} \right) \right| + 2 \left( S_{ijk}^{1} S_{ijk}^{1} - S_{ijk}^{1} S_{ijk}^{1} \right),$$

(87)

and doing the same for the imaginary part, we obtain

$$\bar{b} = \left| \Im \left( S_{\alpha\beta\nu} S^{\alpha\beta\nu} \right) \right|$$
$$= 4 \left| S_{ijk}^{1} S_{ijk}^{1} + S_{ijk}^{1} S_{ijk}^{1} - 2 \left( S_{ijk}^{2} S_{ij}^{2} + S_{ijk}^{2} S_{ij}^{2} \right) \right| + 2 \left( S_{ijk}^{1} S_{ijk}^{1} + S_{ijk}^{1} S_{ijk}^{1} \right).$$

(88)

We will also calculate these scalars for different spacetimes.

IV. AXISYMMETRIC SPACETIMES

Generally, alternatives to the Kerr Black Hole spacetime are axisymmetric, so in this way we consider the specific case of stationary and axisymmetric spacetimes. We use the quasi-isotropic coordinate system which is adapted to these symmetries.

A. Quasi-isotropic coordinates

This work deals with stationary, axisymmetric and circular spacetimes. For spacetimes with those symmetries, we can use quasi-isotropic coordinates (see [19, 20]). In these coordinates, the line element can be written

$$ds^{2} = -N^{2} dt^{2} + A \left( dr^{2} + r^{2} d\theta^{2} \right)$$
$$+ B^{2} r^{2} \sin^{2} \theta \left( d\varphi + \beta r dt \right)^{2}.$$  

(89)

Comparing with (16), we identify the 3+1 quantities of this metric : $N$ is the lapse, the shift is given by $\beta^{i} = \left( 0, 0, \beta^{\varphi} \right)$ and the spatial metric reads

$$h_{ij} = \begin{pmatrix}
A^{2} & 0 & 0 \\
0 & A^{2} r^{2} & 0 \\
0 & 0 & B^{2} r^{2} \sin^{2} \theta
\end{pmatrix},$$  

(90)

where $\beta^{\varphi}, A$ and $B$ depend only on $r$ and $\theta$. 
In these coordinates, we can easily check that we have always
\[ T = \bar{T} = 0. \]

Furthermore, we have \( R_{r\phi} = R_{\theta \phi} = 0 \) which implies
\[ E_{r\phi} = E_{\theta \phi} = B_{r\phi} = B_{\theta \phi} = 0, \]
for Kerr (for which the stress energy tensor \( T_{\mu \nu} = 0 \)) and for boson stars (for which \( T_{r\phi} = T_{\theta \phi} = 0 \)). Thus in this case we have also always
\[ M = \bar{M} = 0, \]
and only the following projections are non zero (the 2-tensors are all symmetric) : \( V_{rr}, V_{\theta \theta}, V_{r\phi}, V_{\phi \phi}, V_{rr}, V_{\theta \theta}, V_{r\phi}, V_{\phi \phi}, V_{\phi \phi}, W_{\phi \phi}, W_{\phi \phi}, W_{\phi \phi}, W_{\phi \phi}, W_{\phi \phi}, W_{\phi \phi} \), \( Z_{rr}, Z_{\theta \theta}, Z_{r \phi}, Z_{\theta \phi}, Z_{r \phi}, Z_{\theta \phi}, \) \( G_{rr}, G_{\theta \theta}, G_{r \phi}, G_{\theta \phi}, L_{r}, L_{\theta}, L_{r \theta}, L_{\theta \theta} \). This implies the following equalities
\[ S_{ij}^2 = -2S_{ij}^2, \]
\[ S_{ij}^2 = -2S_{ij}^2. \]

Thus, we need only to compute the tensors \( S_{ijk}, S_{ij}^2, \) and \( S_{ij}^4 \) given by (77), (78) and (80) for the real part and \( S_{ijk}, S_{ij}^2, \) and \( S_{ij}^4 \) given by (82), (83) and (85) for the imaginary part. Furthermore, the non zero components of these tensors are : \( S_{1rr\theta}^1, S_{1rr\phi}^1, S_{1rr\phi}^2, S_{rr \phi \phi}^2, S_{rr \phi \phi}^2, S_{\phi \phi \theta}^4, S_{\phi \phi \theta}^4 \) and exactly the same for the imaginary part.

Finally, because of (94) and (95), we can write \( \beta \) and \( \bar{\beta} \) as
\[ \beta = 4 \left[ S_{ijk}^1 S_{ijk}^1 - S_{ijk}^1 S_{ijk}^1 + 2 \left( S_{ij}^1 S_{ij}^4 - S_{ij}^4 S_{ij}^1 \right) \right] - 9 \left( S_{ij}^1 S_{ij}^2 - S_{ij}^2 S_{ij}^2 \right), \]
\[ \bar{\beta} = 4 \left[ S_{ijk}^1 S_{ijk}^1 + S_{ijk}^1 S_{ijk}^1 + 2 \left( S_{ij}^1 S_{ij}^4 - S_{ij}^4 S_{ij}^1 \right) \right] - 9 \left( S_{ij}^1 S_{ij}^2 - S_{ij}^2 S_{ij}^2 \right). \]

Let us do the calculation for the simplest case : the Schwarzschild spacetime.

B. Schwarzschild spacetime

For the spherically symmetric Schwarzschild spacetime, we have \( K_{ij} = 0 \) and \( \beta_k = 0 \). Thanks to this symmetry, the quasi-isotropic coordinates become isotropic because \( A = B \) in (90) such that
\[ h_{ij} = A^2 f_{ij}, \]
where \( f_{ij} \) is a flat metric (98) means that the Schwarzschild metric is conformally flat. In these coordinates, the electric (28) and magnetic (29) parts of the self-dual Weyl tensor are given by
\[ E_{ij} = R_{ij}, \]
\[ B_{ij} = 0. \]

The only non null terms are (45), (49), (69) and (65)
\[ V_{ij} = N^2 R_{ij}, \]
\[ Z_i = 2 N D_i N, \]
\[ G_{ij} = N^2 h_{ij}, \]
\[ L_i = 4 N R_{ij} D_j N, \]
so in this case \( S_{ij}^1 \) (77) is the only component non trivially zero :
\[ S_{ij}^{1 \text{Sch}} = 4 V_{ij} Z_k | + G_{ij} L_k | = 4 N^3 \left( 2 R_{ij} D_k N + h_{ij} R_{k \ell} D_j^\ell N \right). \]

But, in isotropic coordinates, with (98), we can calculate (repeated indices are not summed here)
\[ R_{r \phi} = R_{\theta \phi} = R_{r \theta} = 0, \]
\[ R_{\theta \theta} = - \frac{h_{\theta \theta} h_{rr}}{2}, \]
\[ R_{r \phi} = - \frac{h_{r \phi} h_{rr}}{2}, \]
\[ R_{r \phi} = - \frac{r^2 \sin^2 \theta}{2} R_{r r}. \]

This is logical because the Ricci scalar for Schwarzschild is zero so we have
\[ 0 = R = 1 \frac{1}{A^2} \left( R_{rr} + \frac{R_{\theta \theta}}{r^2} + \frac{R_{r \phi}}{r^2 \sin^2 \theta} \right). \]

Writing (105) in components and using (106)-(108), we conclude that
\[ S_{ij}^{1 \text{Sch}} = 0, \]
which is the result we expect.

V. APPLICATIONS

Once we have all the components (77) to (85) and the scalars (96) and (97), we can evaluate them for different spacetimes and develop a quantification of the “non-Kerness” of those spacetimes. Thanks to the 3+1 decomposition, we can compute the Simon-Mars tensor components and scalars in every kind of stationary spacetimes. To illustrate this, we considered two examples of purely numerical spacetimes, two analytical spacetimes and a modified Kerr metric which is a parametric deviation from Kerr. Each example required an adapted tool : for numerical solutions we used codes based on the Kadath library [22] available on line [23], and for analytic solutions we used the SageManifolds extension of the open-source computer algebra system Sage [24, 25].

A. Units

In this section we use geometrical units for which \( c = G = 1 \). In those units, the dimension of the Simon-Mars tensor is \( 1/L^3 \), so the dimension of the two scalar
fields $\beta$ and $\bar{\beta}$ is $1/L^6$. To manipulate dimensionless quantities, we chose to scale by $M$, the ADM (Arnowitt, Deser and Misner) mass of each spacetime considered. In geometrical units, a length has the same dimension as a mass, so in this part every length is given in units of the ADM mass, and $\beta$ means $\beta \times M^6$, same thing for $\bar{\beta}$.

B. Numerical solutions of the Einstein equations

First we present the case of Kerr spacetime in quasi-isotropic coordinates [21], which is analytic, to test the validity of the calculation. Then, we apply it on rotating boson star spacetimes and rotating neutron stars which are solutions of the Einstein equation with a matter content solved by making use of the Kadath library (see [3] for boson stars and [26] for neutron stars).

1. Kerr black hole

First we had to encode Kerr spacetime in the Kadath library. The 3D spacetime $\Sigma$ is decomposed into several numerical domains. The first one starts outside the event horizon, which is located at $r = \frac{1}{2}\sqrt{M^2 - a^2}$ in quasi-isotropic coordinates, and the last one is compactified. In each domain the fields are described by their development on chosen basis functions (typically Chebyshev polynomials). For each domain we define the lapse, shift and 3D metric. For Kerr spacetime we choose to write these 3+1 quantities in quasi-isotropic coordinates, and the last one is compactified. In each domain the fields are described by their development on chosen basis functions (typically Chebyshev polynomials). For each domain we define the lapse, shift and 3D metric. For Kerr spacetime we choose to write these 3+1 quantities in quasi-isotropic coordinates, and the last one is compactified. In each domain the fields are described by their development on chosen basis functions (typically Chebyshev polynomials).

Then we want to verify that the Simon-Mars tensor is identically null for Kerr spacetime, so we compute the 8 components (by virtue of (94) and (95) only 6 of them are meaningful) of the Simon-Mars tensor for Kerr spacetime with $a = 0.8M$. In the Fig. 1 is shown the maximal value of each component of $S_{i j k}^1$, $S_{i j}^2$, $S_4^3$, $S_{i j k}^3$, $S^4_{i j}$, and $S_i^4$ and of $S_{i j k}^1$, $S_{i j}^2$, and $S_4^3$ in the first panel and of $\beta$ and $\bar{\beta}$ in the second panel for the Kerr spacetime characterized by $M = 1$ and $a = 0.8M$ as a function of the number of spectral coefficients in both the radial and angular dimension (we chose the same number for these two dimensions).

2. Rotating boson stars

Boson stars are localized configurations of a complex self-gravitating field $\Phi$. Their study is motivated by the fact that they can play the role of black hole mimickers [27]. For instance, this model presents a viable alternative to the Kerr black hole for the description of the Galactic Center. Mathematically, this is a solution to the coupled system Einstein-Klein-Gordon equations

$$R_{\alpha \beta} - \frac{1}{2} R g_{\alpha \beta} = 8\pi T_{\alpha \beta},$$

$$\nabla_\alpha \nabla^\alpha \Phi = \frac{dV}{d|\Phi|^2} \Phi,$$

where

$$T_{\alpha \beta} = \nabla_{(\alpha} \Phi \nabla_{\beta)} \Phi - \frac{1}{2} g_{\alpha \beta} \left[ \nabla^\mu \Phi \nabla_\mu \Phi + V \left( |\Phi|^2 \right) \right].$$

The potential can take different forms depending on the model we choose for the boson star, here we consider only "mini" boson star formed with a free field potential:

$$V \left( |\Phi|^2 \right) = \frac{m^2}{\hbar^2} |\Phi|^2,$$

where $m$ is the boson mass.
To solve the coupled equations (111) and (112), the following ansatz is used:

$$\Phi = \phi (r, \theta) \exp \left[ i (\omega t - k \varphi) \right].$$

(115)

A specific boson star is characterized then by the values of $\omega$, which is a real parameter, and $k$, which is an integer (non-rotating boson stars have $k = 0$). Given this, the 3+1 decomposition of the matter part (33)-(35) of a boson star is

$$\rho = \left[ \frac{(\omega + k \beta \varphi)^2}{N^2} + k^2 h^\varphi \varphi \right] \phi^2 + \frac{h_{ij}}{2} \partial_i \phi \partial_j \phi + \frac{V}{2},$$

(116)

$$p_\varphi = \frac{k}{N} (\omega + k \beta \varphi) \phi^2,$$

(117)

$$S_{ij} = \partial_k (\bar{\phi} \partial_k \phi + V).$$

(118)

We write then the Einstein-Klein-Gordon equations (111) and (112) in 3+1 form using quasi-isotropic coordinates (see [19]), and those equations are numerically solved by Kadath (see [3] for details). More on boson stars can be found in [4, 28] (and in references of [3]).

Let us plot in Fig. 2 the same quantities we did for Kerr in Fig. 1, that is to say the maximal values of $\beta$ and $\bar{\beta}$ for several rotating neutron stars as functions of the number of spectral coefficients in both the radial and angular dimension for boson stars with the free field potential (114) for $\omega = 0.8 \ m/h = 1.05 \ M^{-1}$ and $k = 1, 2$.

Contrary to the case of Kerr spacetime, the maximal values of $\beta$ and $\bar{\beta}$ do not depend on the resolution up to a certain precision. The conclusion is that it is not zero. The rest of this paper, exploring the same quantities for other spacetimes will permit us to give a meaning to the maximal value found for $\beta$ and $\bar{\beta}$. Furthermore, thanks to Fig. 2, we can choose a fine resolution for the following plots, which will be 17 points in $r$ and $\theta$.

We can also explore the global behavior of the maximal values of $\beta$ and $\bar{\beta}$ for different boson stars for a fixed resolution. For instance, we plot in Fig. 3 those values as functions of $\omega$ for various boson stars.

![Fig. 3. Maximal values $\beta$ and $\bar{\beta}$ as functions of $\omega$ for boson stars with $k = 1, 2, 3$. As we recover Minkowski spacetime when $\omega \rightarrow 1$ (in units of $[m/h]$, see [3]), we expect that these values tend to zero which is the case.]

To see how the Simon-Mars scalars behave locally, we present in Fig. 4 and 5 contour plots of $\log (\beta)$ and $\log (\bar{\beta})$ in the equatorial plane as functions of the quasi-isotropic version of the Weyl-Papapetrou coordinates : $r$ and $z$. As the scalar field $\phi$ decreases exponentially fast after reaching its maximum, it makes sense to plot $\log (\beta)$ and $\log (\bar{\beta})$.

3. Rotating neutron stars

Another example of stationary, axisymmetric and asymptotically flat spacetime with a matter content, for which the metric can also be expressed in quasi-isotropic coordinates is the model of rotating neutron stars. Contrary to boson stars, the matter content is a perfect fluid. The stress tensor has the following form

$$T_{\alpha \beta} = (\varepsilon + p) \ u_\alpha \ u_\beta + pg_{\alpha \beta},$$

(119)

where $u^\alpha$ is the unit timelike vector field representing the fluid 4-velocity, $\varepsilon$ and $p$ are the two scalar fields representing respectively the energy density and the pressure of the fluid. The 3+1 decomposition of the 4-velocity of the fluid with respect to the Eulerian observer 4-velocity $\bar{\eta}$ is

$$u^\alpha = \Gamma (n^\alpha + U^\alpha),$$

(120)
where $\Gamma = -n^\mu u_\mu$ is the Lorentz factor of the fluid with respect to the Eulerian observer and the 3-velocity of the fluid with respect also to this Eulerian observer is \cite{10}

$$U^\alpha = \frac{B}{N} (\Omega - N^\nu) r \sin \theta,$$

(121)

where $\Omega$ is the orbital angular velocity with respect to a distant inertial observer. The normalization condition $u^\mu u_\mu = -1$ gives

$$\Gamma = \frac{1}{\sqrt{1 - U^2}}.$$

(122)

So the 3+1 matter content (33)-(35) can be written

$$\rho = \Gamma^2 (\varepsilon + p) - p$$

(123)

$$p_\alpha = \Gamma^2 (\varepsilon + p) U_\alpha$$

(124)

$$S_{\alpha\beta} = \Gamma^2 (\varepsilon + p) U_\alpha U_\beta + p h_{\alpha\beta}.$$ 

(125)

In order to close the system, we consider a polytropic equation of state with $\gamma = 2$. To read more about neutron stars and how these objects are computed numerically see \cite{26} and references therein.

We plot in Fig. 6 the maximal values of $\beta$ and $\bar{\beta}$ for a static and several rotating neutron stars as functions of the resolution.

As in the boson star case, the maximal values of $\beta$ and $\bar{\beta}$ do not depend on the resolution up to a certain precision. We choose the resolution of 17 points in $r$ and
\[ \theta \] for the following plots.

We can explore the global behavior of the maximal values of \( \beta \) and \( \bar{\beta} \) for different neutron stars for a fixed resolution. For instance, we plot in Fig. 7 those maximal values as functions of \( \Omega \) for different rotating neutron stars.

Figures 8 and 9 show contours of \( \log(\beta) \) and \( \log(\bar{\beta}) \) for two different neutron stars.

C. Exact analytic solutions of the Einstein equations

It is interesting also to compare the values of the Simon-Mars scalars we obtain in the two preceding ex-
amples to other spacetimes. We chose to do it with two specific spacetimes for which the difference to Kerr has been already quantified in [11], one is the Curzon-Chazy spacetime first studied by Curzon in [29] and Chazy in [30], the other is the $\delta = 2$ Tomimatsu-Sato spacetime defined in [31].

Calculating the Simon-Mars scalars for these spacetimes permits us to put a link between our characterization of the “non-Kerness” and the characterization with invariants given in [11]. This classification is more fine than ours because is uses more than the Simon-Mars tensor. But it is based on vacuum spacetime, or very small deviation from vacuum spacetimes while ours can be applied to all spacetimes. Thus, our scalar calculation can advantageously complete their classification providing invariants that can be calculated even for non vacuum spacetimes such as the two examples treated in the preceding section.

Even if there are more examples in [11], in most of them (which are not vacuum solutions of the Einstein equations, so Mars theorem does not hold), the Simon-Mars is zero, so we focus in the solutions that are in vacuum and not locally isomorphic to Kerr (their Simon-Mars tensor cannot be zero). For this part, we used the SageManifolds free package [24, 25], which is an extension towards differential geometry and tensor calculus of the open-source mathematics software Sage [32].

1. Exact vacuum axisymmetric solutions

As we saw in section IV A, the metric of a generic stationary, axisymmetric and circular spacetime can be written in quasi-isotropic coordinates (89) : $(t, r, \theta, \varphi)$. The related cylindrical coordinates $(t, \rho, z, \varphi)$ with $\rho = r \sin \theta$ and $z = r \cos \theta$ are the Weyl-Lewis-Papapetrou coordinates :

$$\begin{align*}
\text{ds}^2 &= -e^{2U} \left( dt + \beta^r \rho d\varphi \right)^2 \\
&\quad + e^{-2U} \left[ e^{2\gamma} \left( d\rho^2 + dz^2 \right) + \rho^2 d\varphi^2 \right], \quad (126)
\end{align*}$$

where $U$ and $\gamma$ are functions of $\rho$ and $z$. In these coordinates, $U$ and $\gamma$ play the same role of $A$, $B$ and $N$ in the quasi-isotropic coordinates : indeed, in vacuum, $B = 1/N$ (see eq. (3.16) of [19]). Writing the vacuum Einstein field equations satisfied by this metric, we obtain a single differential equation, called the Ernst equation, for a complex potential, called the Ernst potential [34]. This equation is integrable so it can be solved by various solutions generating techniques : in the static case ($\beta^r = 0$), a family of asymptotically flat solutions can be found expressing $U$ as a sum of Legendre polynomials. This family is called the Weyl class [35] and Curzon-Chazy spacetime is the simplest member of this class.

For stationary (but not static) axisymmetric solutions, it is convenient to write the Ernst equation in prolate spherical coordinates $(x, y)$ related to the Weyl-Lewis-Papapetrou coordinates $(\rho, z)$ by

$$\begin{align*}
x &= \frac{1}{2} \left( \sqrt{\rho^2 + (z + 1)^2} + \sqrt{\rho^2 + (z - 1)^2} \right) \quad (127) \\
y &= \frac{1}{2} \left( \sqrt{\rho^2 + (z + 1)^2} - \sqrt{\rho^2 + (z - 1)^2} \right), \quad (128)
\end{align*}$$

where $x \geq 1$ and $y \in [-1, 1]$. Tomimatsu and Sato [31, 33] have labeled the solutions by a deformation parameter called $\delta$. The solution corresponding to $\delta = 1$, the simplest one, is the Kerr one. The solution which we consider in this paper is the $\delta = 2$ Tomimatsu-Sato solution.

2. Kerr spacetime

As a test of the SageManifolds code, we evaluated the Simon-Mars tensor (11) for the Kerr metric in Boyer-Lindquist coordinates and we evaluated also each of the eight 3+1 components of the Simon-Mars tensor : (77)- (80) and (82)-(85). We found identically zero. This computations confirms that the same results are found with the original definition and with the 3+1 decomposition. Furthermore, it validates the use of SageManifolds worksheets. The latter are freely downloadable from [36].

3. Curzon-Chazy spacetime

The Curzon-Chazy spacetime is a static, axisymmetric and asymptotically flat solution of the Einstein equations [29, 30]. Its line element is given by the Weyl-Lewis-Papapetrou one with $\beta^r = 0$ and

$$U = -\frac{M}{r}, \quad \gamma = -\frac{M^2 \sin^2 \theta}{2r^2}. \quad (129)$$

It has a spherically symmetric Newtonian potential, corresponding to the potential of a point particle of mass $M$ located at $r = 0$, where lies a singularity of complex nature, but this spacetime is not spherically symmetric. Because the shift is zero for this spacetime, $\beta$ is identically null, so we can only show the contour plot of $\log(\beta)$ in Fig. 10.

The computation of the Simon-Mars scalars has been performed with SageManifolds (the corresponding worksheet is available at [36]). As this solution is analytic, we identified those scalars both with their 3+1 definitions (87) and (88), but also as the real and imaginary part of the “square” of the 4-dimensional Simon-Mars scalar (86).

First we see that the Simon-Mars scalars diverge at the singularity. For vacuum spacetimes, the Weyl tensor is equal to the Riemann tensor, so the Simon-Mars are
close to the Kretschmann with diverge at each real singularity (not coordinate singularities) of the metric, so we can expect this behavior. Furthermore, comparing with Fig. 1 of [11], we can confirm that the adapted quantity is log (ß) and not ß by comparing the shape of the contours, we can also see that a spherically symmetric spacetime differs little to the Kerr spacetime when log (ß) < −8. In this sense, at the singularity the Curzon-Chazy spacetime is infinitely far from the Kerr spacetime, which is not false.

4. δ = 2 Tomimatsu-Sato spacetime

This spacetime, which is defined in [31, 33], is a stationary, axisymmetric vacuum solution of the Einstein equations which is also asymptotically flat (see [37]). This metric is one of the rare stationary and axisymmetric exact solutions of Einstein equations. The non zero elements of the metric are

$$g_{tt} = - \frac{A(x, y)}{B(x, y)},$$

$$g_{t\varphi} = - \frac{2Mq(x, y)C(x, y)(1 - y^2)}{B(x, y)},$$

$$g_{xx} = \frac{M^2B(x, y)}{\rho^2\delta^2(x^2 - 1)(x^2 - y^2)^3},$$

$$g_{yy} = \frac{M^2B(x, y)}{\rho^2\delta^2(y^2 - 1)(y^2 - x^2)^3},$$

$$g_{\varphi\varphi} = \frac{M^2(y^2 - 1)p^2B^2(x, y)(x^2 - 1)}{A(x, y)B(x, y)\delta^2} + 4q^2\delta^2C^2(x, y)(y^2 - 1),$$

with

$$A(x, y) = \left[p^2(x^2 - 1)^2 + q^2(1 - y^2)^2\right]^2 - 4p^2q^2(x^2 - 1)(1 - y^2)(x^2 - y^2)^2,$$

$$B(x, y) = (p^2x^4 + q^2y^4 - 1 + 2px^3 - 2py^3)^2 + 4p^2y^2(px^3 - pxy^2 + 1 - y^2)^2,$$

$$C(x, y) = p^2(x^2 - 1)[(x^2 - 1)(1 - y^2) - 4x^2(x^2 - y^2)] - p^3x(x^2 - 1)[2(x^4 - 1) + (x^2 + 3)(1 - y^2)] + q^2(1 + px)(1 - y^2)^3,$$

where $p$ and $q$ are real constant satisfying the constraint $p^2 + q^2 = 1$, and $M$ is the ADM mass, here we choose $p = 1/5$\(^5\), and $M = 1$.

This spacetime contains two degenerated Killing horizons at $x = 1$, $y = \pm 1$, and a naked ring singularity (more on the $\delta = 2$ Tomimatsu-Sato spacetime in [38]). The computation of the two Simon-Mars scalars starting from the metric (130)-(137) is rather formidable. We performed it by means of the SageManifolds code mentioned above. The corresponding worksheet is available at [36].

In Fig. 11, we show the contour plots of log (ß) and log ($\tilde{B}$) as functions of $X = -1/x$ and $y$ (given by (127) and (128)) to compare our result to the Fig. 2 of [11]. We see that the order of magnitude of the two scalars is the same. But is is harder to make a comparison with Fig. 2 of [11] in this case because the scalars grow exponentially around the singularity, which is not the case in [11]. Nevertheless we can compare the value of the scalars along the horizontal axis ($y = 0$), and we see that in this case an axisymmetric spacetime seems to differ little to the Kerr spacetime when log (ß) < −6. Outside the ergosphere the values of log (ß) and log ($\tilde{B}$)

\(^5\) same as in [11]
diminish frankly.

We do the same plot in the Weyl-Lewis-Papapetrou coordinates which are more usual coordinates (see [38]), in Fig. 12.

D. Approximate solution : the Modified Kerr metric

The modified Kerr metric proposed by Johannsen and Psaltis [13] is a family of approximate solutions of the Einstein field equations, which are non linear parametric deviations from the Kerr metric. The line element is given in Boyer-Lindquist coordinates by

$$ds^2 = - [1 + h(r, \theta)] \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4aMr \sin^2 \theta}{\Sigma} \, dr^2$$

$$+ [1 + h(r, \theta)] \, d\varphi^2 + \frac{\Sigma [1 + h(r, \theta)]}{\Delta + a^2 \sin^2 \theta} \, dr^2$$

$$+ \Sigma \, d\theta^2 + \left[ \sin^2 \theta \left( r^2 + a^2 \right) + \frac{2a^2Mr \sin^2 \theta}{\Sigma} \right] \, d\varphi^2,$$

(138)

where $a$ is the specific angular momentum of the black hole, $M$ its ADM mass and

$$\Sigma = r^2 + a^2 \cos^2 \theta,$$

$$\Delta = r^2 - 2Mr + a^2.$$  

(139)  

(140)

The simplest choice for $h(r, \theta)$ in accordance with the observational constraints on weak-field deviations from general relativity is given by

$$h(r, \theta) = \epsilon_3 \frac{M^3r}{\Sigma^2}.$$  

(141)

This spacetime is a black hole only for certain values of $\epsilon_3$ and $a$ (see Fig. 3 of [39]) : if $0 \leq a \leq 0.8M$ and $-5 \leq \epsilon_3 \leq 0$, it is always the case. In this paper, we focus on this particular zone, so we use the positive parameter $\epsilon = -\epsilon_3$.

In Fig. 13, we plot the maximal values of $\bar{\beta}$ and $\bar{\bar{\beta}}$ as functions of $\epsilon$ for different values of the spin. We see that the bigger $a$ is, the bigger is the lowest maximal value of the scalars, meaning that the more this modified Kerr black hole is rapidly rotating, the more it differs for the Kerr original solution. The idea was to quantify the deviation to the Kerr spacetime of the preceding examples by seeing how fast the two Simon-Mars scalars evaluated in the modified Kerr metric deviate from zero.

For spacetimes containing singularities as the Curzon-Chazy solution and the $\delta = 2$ Tomimatsu-Sato...
spacetime, the characterization of their “non-Kerrness” can be only local because the two scalars diverge at the singularities. But for smooth spacetimes such as boson stars or neutron stars, we can report the maximal values of the scalars on the Fig. 13, as we did for one specific example of each. We can say that the spacetime generated by the neutron star chosen is more close to the Kerr spacetime than the spacetime generated by the boson star. This corresponds to our intuition because the boson star shape is a torus, and it seems a more exotic object than the neutron star.

![Graph](image)

**FIG. 13.** Maximal values $\beta$ and $\bar{\beta}$ as functions of $\epsilon$ for modified Kerr metrics with $a = 0, 0.5, 0.8 \, M$. As the deviation from the Kerr metric is non linear, we use a log-log scale. The violet dotted line corresponds to the maximal value of $\beta$ ($\beta = 3.48826.10^{-6}$) for the boson star with $\omega = 1.05 \, M^{-1}$ and $k = 1$, and the cyan dotted line corresponds to the maximal value of $\bar{\beta}$ ($\bar{\beta} = 3.34572.10^{-8}$) for the neutron star with $\Omega = 0.039 \, M^{-1}$.

To see local configurations of the scalar fields for examples of modified Kerr spacetimes, we present contour plots of $\log(\beta)$ and $\log(\bar{\beta})$ for chosen values of $\epsilon$ and $a$ in Fig. 14 and 15. As the topology of the event horizon is not simple in this geometry (see [39]) and as it is located in the zone where $r < 2M$, we choose to plot only the outer domain for $r \geq 2M$. The scalar values are very small compared to the other solutions, which is reassuring because this example is supposed to be very close to the Kerr spacetime.

![Graph](image)

**FIG. 14.** Contour plot of $\log(\beta)$ and $\log(\bar{\beta})$ as functions of $r \geq 2M$ and $z$ for a modified Kerr metric with $a = 0.5 \, M$ and $\epsilon = 10^{-8}$.

**VI. CONCLUSION**

We performed the 3+1 decomposition of the Simon-Mars tensor, and defined two scalar fields from it. It permitted us to quantify the deviation of different spacetimes to the Kerr one by evaluating these scalars. This classification works only for stationary spacetimes, but can be applied to non vacuum spacetimes, and especially in numerical spacetimes with a matter content, what could not be done before, as far as we know. Nevertheless, in some of these non-vacuum spacetimes, the scalar fields can be identically zero even if the spacetime is not locally isometric to the Kerr spacetime, because the Mars theorem does not hold in non-vacuum spacetimes. Thus one has to be careful using this characterization of Kerr spacetime, which provides an efficient way to compare different spacetimes to one another.
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