On the higher order corrections to the Fokker-Planck equation

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The Rayleigh model of nonlinear Brownian motion is revisited in which the heavy particle of mass \( M \) interacts with ideal gas molecules of mass \( m \ll M \) via instantaneous collisions. Using the van Kampen method of expansion of the master equation, non-linear corrections to the Fokker-Planck equation are obtained up to sixth order in the small parameter \( \lambda = \sqrt{m/M} \), improving earlier results. The role and origin of non-Gaussian statistics of the random force in the corresponding Langevin equation are also discussed.

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I. INTRODUCTION

There has been renewed discussion recently of the Rayleigh model of nonlinear Brownian motion, stimulated primarily by findings of new qualitative effects governed by nonlinear stochastic processes such as rectification of fluctuations [1-4], stochastic resonance [5], etc. In the Rayleigh model the heavy Brownian particle of mass \( M \) is immersed in an ideal gas of molecules of mass \( m \ll M \) and interacts with them through instantaneous elastic collisions. The gas is assumed to be so rarefied that collisions of its molecules with the particle do not affect the distribution of incident molecules, and also re-collisions can be neglected. In lowest order in the small parameter \( \lambda = \sqrt{m/M} \), the velocity distribution function \( f(V, t) \) of the particle satisfies the second order Fokker-Planck equation which can be obtained by the conventional method, i.e. truncating the Kramers-Moyal expansion of the master equation to the first two terms and evaluating coefficients integrating the corresponding Langevin equation [6, 7]. This procedure is consistent if the random force in the Langevin equation can be treated as white (with negligible correlation time) and Gaussian. However these two assumptions may be justified strictly speaking only in the ultimate limit \( \lambda \rightarrow 0 \). For finite \( \lambda \), the Kramers-Moyal expansion has generally an infinite number of non-zero terms, and more careful analysis is needed to study effects of higher order in \( \lambda \) [8]. In this case, as was first shown by van Kampen, the differential equations for the distribution function do not have the form of the second order Fokker-Planck equation but rather involve derivatives of order higher than two [8]. Although these higher order equations are linear in the distribution function, they are sometimes referred to as nonlinear Fokker-Planck equations because the corresponding Langevin equations involve nonlinear corrections of higher orders in \( \lambda \) to the linear damping force. These corrections may lead to interesting physical consequences, such as additional terms in the fluctuation spectrum and modification of decay rate coefficients [9].

Also, going beyond the lowest order is often necessary to account for delicate fluctuation-induced phenomena such as directional drift in the absence of systematic forces.

If the temperatures on the left and right sides of the Rayleigh particle are different, the particle undergoes directional movement even when the pressure on both sides is the same [2-4]. The systematic average velocity of the particle can be calculated as a perturbative solution of the Fokker-Planck equation with non-linear corrections to the damping force [8]. Alternatively, one can use the corresponding nonlinear Fokker-Planck equation or an equivalent set of equations for the moments [10].

To derive a differential equation for the distribution function \( f(V, t) \) to desirable order in \( \lambda \), one has to extract an explicit \( \lambda \)-dependence in the Kramers-Moyal expansion, transforming it into the expansion in powers of \( \lambda \), often referred to as the van Kampen expansion (VKE). The method essentially relies on an assumption of certain scaling properties of transition rates in the master equation. For the Rayleigh model an explicit expression for the transition rates is available and all coefficients in the VKE can be found analytically. Note that another popular exactly solvable model of Brownian motion, namely that of a particle coupled to a bath of harmonic oscillators, is degenerate in the sense that all nonlinear corrections vanish identically.

The Rayleigh model may be generalized in many ways to study effects of asymmetry of the surrounding bath [2-4], possibility of non-canonical distributions [10], finite-range interaction [11], strong non-equilibrium fluctuations [12], etc. To analyze these generalizations it would be of help to compare their predictions with the results of the original model. In doing that we have found that the VKE for the Rayleigh model often appeared in the literature in forms which are either not quite accurate or incomplete, and lead to nonlinear Fokker-Planck equations with serious defects. For instance, these equations may have a non-Gaussian stationary solution even when the Maxwellian distribution is an exact solution of the original master equation. Violation of the equilibrium condition can lead to large errors, such as an exponentially overestimated activation flux in the barrier crossing problem [13]. As was noted by van Kampen in [8], the Maxwellian distribution must satisfy each term of the VKE for the Rayleigh particle separately. While the equation of order \( \lambda^2 \) derived in [8] has the Maxwellian
stationary solution, the equations obtained later by other authors do not always possess this virtue. The extraction of explicit dependence on \( \lambda \) for the Rayleigh model is slightly more complicated comparing with other applications of the van Kampen method and requires some care. One purpose of this paper is to give an accurate derivation of the equations for the velocity distribution of the Rayleigh particle up to order \( \lambda^5 \). These equations have the Maxwellian stationary solution and preserve positivity of the distribution function when deviations from equilibrium are small. Another purpose is to discuss some subtle points relevant to the derivation, such as the origin and consequences of non-Gaussian statistics of the random force in the corresponding Langevin equation, which has not been clearly articulated in the literature so far. Although the assumption of Gaussian random force was criticized in the literature, it is still widely applied to derive the second order Fokker-Planck equation, often without proper justification. In the last section we discuss why the random force is approximately Gaussian in lowest order in \( \lambda \), while corrections of higher orders may be essentially non-Gaussian. This fact has important implications which go far beyond the specific model considered here.

\[ \text{II. KRAMERS-MOYAL AND VAN KAMPEN EXPANSIONS} \]

The conventional starting point in the derivation of the Fokker-Planck equation, as well as equations of higher order, is the master equation for the velocity distribution function \( f(V, t) \)

\[
\frac{\partial f(V, t)}{\partial t} = \int dV' \left\{ f(V', t)W(V' \rightarrow V) - f(V, t)W(V \rightarrow V') \right\}, \tag{1}
\]

It is convenient to express the transition rates \( W(V_1 \rightarrow V_2) \) as a function of the initial state \( V_1 \) and the transition length \( \Delta V = V_2 - V_1 \), that is \( W(V_1 \rightarrow V_2) = W(V_1 | \Delta V) \). Then the master equation takes a more suggestive form

\[
\frac{\partial f(V, t)}{\partial t} = \int d\Delta V \left\{ f(V - \Delta V, t)W(V - \Delta V | \Delta V) - f(V, t)W(V | \Delta V) \right\}
\]

\[
= \int d(\Delta V) \left\{ \Psi(V - \Delta V, \Delta V) - \Psi(V, \Delta V) \right\}, \tag{2}
\]

where \( \Psi(V, \Delta V) \equiv f(V, t)W(V | \Delta V) \). Making in (2) the expansion

\[
\Psi(V - \Delta V, \Delta V) = \Psi(V, \Delta V) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\Delta V \frac{\partial}{\partial V} \right)^n \Psi(V, \Delta V) \tag{3}
\]

leads immediately to the Kramers-Moyal expansion,

\[
\frac{\partial f(V, t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial V} \right)^n \left\{ a_n(V)f(V, t) \right\} \tag{4}
\]

with coefficients \( a_n \) given by

\[
a_n(V) = \int d(\Delta V)(\Delta V)^nW(V | \Delta V). \tag{5}
\]

If the random force exerted on the particle by the bath molecules is white and Gaussian, one can show that only two first terms survive in the Kramers-Moyal expansion, which therefore turns into the second order Fokker-Planck equation. This can be proved using another equivalent representation for \( a_n \),

\[
a_n(V) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} \langle (V(t + \tau) - V(t))^n \rangle, \tag{6}
\]

which can be obtained from writing the transition rates \( W \) in terms of the transition probability \( P(V, t|V', t + \tau) \) as \( W(V \rightarrow V') = \lim_{\tau \rightarrow 0} \frac{1}{\tau} P(V, t|V', t + \tau) \). Then \( a_n \) takes the form

\[
a_n(V) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int dV'(V' - V)^nP(V, t|V', t + \tau),
\]

which is equivalent to (4).
The expression (6) for $a_n$ is often more useful than (5) since it does not involves transition rates $W$, which are usually unknown. On the other hand the average $\left<(V(t + \tau) - V(t))^n\right>$, appearing in (6), can be readily found integrating the Langevin equation

$$\dot{V} = A(V) + F(t),$$

(7)

where $A(V)$ is the damping force and $F(t)$ is the random force with negligible correlation time,

$$\langle F(t)F(0) \rangle = \Gamma \delta(t).$$

(8)

Integrating (7) one gets

$$V(t + \tau) - V(t) = \int_t^{t+\tau} dt' \{A(V(t')) + F(t')\},$$

(9)

which for small $\tau$ can be consistently approximated as

$$V(t + \tau) - V(t) \approx A(V(t))\tau + \int_t^{t+\tau} dt' F(t').$$

(10)

This approximation corresponds to a coarse-grained description with a time resolution $\tau_0$ much shorter than the characteristic time for the relaxation of the particle’s velocity $\tau_V$ and much longer than the correlation time for the random force $\tau_F$. $\tau_F \ll \tau_0 \ll \tau_V$. The limit $\tau \to 0$ in (9) means actually $\tau \to \tau_0$. Therefore, in (9) one can neglect time dependence of $A(V(t))$, but not of the random force which evolves significantly within the integration range $\tau \sim \tau_0 \gg \tau_F$.

Using the approximation (10) one can readily find $\langle(V(t + \tau) - V(t))^n\rangle$ and then, from Eq. (9), the coefficients $a_n$. For a white and Gaussian random force only two first terms survive in the Kramers-Moyal expansion,

$$a_1 = A(V), \quad a_2 = \Gamma, \quad a_n = 0, n > 2,$$

(11)

which therefore turns into the second order Fokker-Planck equation

$$\frac{\partial f(V,t)}{\partial t} = -\frac{\partial}{\partial V} A(V)f(V,t) + \frac{\Gamma}{2} \frac{\partial^2}{\partial V^2} f(V,t).$$

(12)

The result that in the Kramers-Moyal expansion only two first terms do not vanish relies entirely on the assumptions that the random force in the Langevin equation is Gaussian white noise. The above approach does not involve any parameters controlling applicability of this assumption, and the range of validity of the Fokker-Planck equation is difficult to analyze. This difficulty does not arise in situations when transition rates in the master equation are known explicitly. In this case one can find $a_n$ directly from Eq. (4) without appealing to the Langevin equation and therefore without making any assumption about statistical properties of the random force. Using this method one finds that the condition $a_n = 0, \quad n > 2$ does not hold invariably, and the Kramers-Moyal expansion contains in general infinite number of terms. Nevertheless, it is still possible to get for $f(V,t)$ a differential equation of finite order analyzing dependence of coefficients $a_n$ on the small parameter $\lambda$ and neglecting terms of higher order in $\lambda$. The Kramers-Moyal expansion in the form (4) is not appropriate for such a perturbation analysis because the dependence on $\lambda$ is implicit in (4). Assuming that transition rates have certain scaling properties with respect to $\lambda$, van Kampen modified the Kramers-Moyal expansion transforming it into the form of the expansion in powers of $\lambda$.

### III. VAN KAMPEN EXPANSION FOR THE RAYLEIGH PARTICLE

For the one-dimensional Rayleigh model the transition rate has the form

$$W(V|\Delta V) = \frac{\nu}{4}e^{-|\Delta V|}f_M\left(V + \frac{1}{2}e^{-2|\Delta V|}\right),$$

(13)

where $f_M(v)$ is the Maxwell distribution for the gas molecules, $\nu$ is the number of particle per unit length, and

$$\epsilon = \sqrt{\frac{m}{M + m}} = \lambda \sqrt{\frac{1}{1 + \lambda^2}}.$$  

(14)
The transition rate has the scaling property
\[ e^2 W(V|\Delta V) = \Phi(V|\epsilon^2 \Delta V), \tag{15} \]
where
\[ \Phi(V|\xi) = \frac{\nu}{4} |\xi| f_M \left( V + \frac{1}{2} \xi \right). \tag{16} \]
To extract an explicit dependence of \( a_n \) on \( \lambda \) it is convenient to re-write Eq. (5) as follows
\[ a_n(V) = \int d(\Delta V)(\Delta V)^n W(V|\Delta V) \]
\[ = \epsilon^{2n} \int d(\epsilon^{-2} \Delta V)(\epsilon^{-2} \Delta V)^n \left( \epsilon^2 W(V|\Delta V) \right), \tag{17} \]
or using (15),
\[ a_n(V) = \epsilon^{2n} \int d\xi \xi^n \Phi(V|\xi). \tag{18} \]
Then the Kramers-Moyal expansion takes the form
\[ \frac{\partial f(V,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\epsilon^2 \frac{\partial}{\partial V} \right)^n \{ \alpha_n(V)f(V,t) \}, \tag{19} \]
where
\[ \alpha_n(V) = \int d\xi \xi^n \Phi(V|\xi). \tag{20} \]
In (19) the dependence on \( \lambda \) enters in two ways. First, \( \epsilon^{2n} \) as a function of \( \lambda \) has the form
\[ \epsilon^{2n} = \lambda^{2n} \varphi_n(\lambda), \tag{21} \]
where
\[ \varphi_n(\lambda) = \left( \frac{1}{1 + \lambda^2} \right)^n = 1 - n\lambda^2 + \frac{n(n+1)}{2} \lambda^4 + \cdots \tag{22} \]
Second, the expansion (19) involves dependence on the velocity the particle \( V \) which is obviously a function of \( \lambda \). If we restrict ourselves to small-range fluctuations about the equilibrium state, it may be reasonably expected from the equipartition theorem that the ratio of the particle’s typical velocity to that of surrounding molecules is of order \( \sqrt{m/M} \). This leads to the second scaling assumption
\[ V = \lambda x \tag{23} \]
where the scaled velocity \( x \sim \lambda^0 \). Then the next step in the extraction of the explicit dependence on \( \lambda \) is the expansion of \( \alpha_n(V) = \alpha_n(\lambda x) \) near \( V = 0 \),
\[ \alpha_n(\lambda x) = \sum_{p=0}^{\infty} \alpha_n^{(p)}(\lambda) \frac{(\lambda x)^p}{p!} \tag{24} \]
where \( \alpha_n^{(p)} \) is \( p \)-th derivative of \( \alpha_n(V) \) at \( V = 0 \). Physically this expansion reflects the fact that the heavy particle is much slower than surrounding light molecules. The first term with \( p = 0 \) corresponds to the particle which does not move at all (approximation of the infinitely heavy particle, \( \lambda \rightarrow 0 \)), while the next terms successively take into account the finite inertia of the particle.
Substitution of the above expansions for \( \epsilon^{2n} \) and \( \alpha_n \) in (19) gives finally the desirable van Kampen expansion in powers of \( \lambda \)
\[ \frac{\partial f(x,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n \varphi_n(\lambda) \sum_{p=0}^{\infty} \frac{\lambda^p}{p!} \alpha_n^{(p)}(\lambda) \frac{\partial^n}{\partial x^n} (x^p f(x,t)). \tag{25} \]
This form of the VKE is slightly different from that one usually finds in the literature and has the advantage that the
dependence on \( \lambda \) is entirely contained in the product \( \lambda^{n+p} \varphi_n(\lambda) \), whereas the coefficients \( \alpha_n^{(p)} \) are \( \lambda \)-independent.
The VKE \( \text{25a} \) can be written in the form
\[
\frac{\partial f(x,t)}{\partial t} = \sum_{k=1}^{\infty} \frac{\partial^k}{\partial x^k} S_k(x,\lambda) f(x,t),
\tag{26}
\]
where the first six coefficients are
\[
S_1(x,\lambda) = -(\lambda - \lambda^3 + \lambda^5 + \cdots) \times \left\{ \alpha_1^{(0)} + \alpha_2^{(1)} x + \frac{1}{2!} \lambda^2 \alpha_3^{(2)} x^2 + \frac{1}{3!} \lambda^3 \alpha_4^{(3)} x^3 + \frac{1}{4!} \lambda^4 \alpha_5^{(4)} x^4 + \frac{1}{5!} \lambda^5 \alpha_6^{(5)} x^5 + \cdots \right\},
\tag{27}
\]
\[
S_2(x,\lambda) = \frac{1}{2!}(\lambda^2 - 2\lambda^4 + 3\lambda^6 + \cdots) \times \left\{ \alpha_2^{(0)} + \alpha_3^{(1)} x + \frac{1}{2!} \lambda^2 \alpha_4^{(2)} x^2 + \frac{1}{3!} \lambda^3 \alpha_5^{(3)} x^3 + \frac{1}{4!} \lambda^4 \alpha_6^{(4)} x^4 + \cdots \right\},
\]
\[
S_3(x,\lambda) = \frac{1}{3!}(\lambda^3 - 3\lambda^5 + \cdots) \left\{ \alpha_3^{(0)} + \alpha_4^{(1)} x + \frac{1}{2!} \lambda^2 \alpha_5^{(2)} x^2 + \frac{1}{3!} \lambda^3 \alpha_6^{(3)} x^3 + \cdots \right\},
\]
\[
S_4(x,\lambda) = \frac{1}{4!}(\lambda^4 - 4\lambda^6 + \cdots) \left\{ \alpha_4^{(0)} + \alpha_5^{(1)} x + \frac{1}{2!} \lambda^2 \alpha_6^{(2)} x^2 + \cdots \right\},
\]
\[
S_5(x,\lambda) = \frac{1}{5!}(\lambda^5 + \cdots) \left\{ \alpha_5^{(0)} + \alpha_6^{(1)} x + \cdots \right\},
\]
\[
S_6(x,\lambda) = \frac{1}{6!}(\lambda^6 + \cdots) \left\{ \alpha_6^{(0)} + \cdots \right\}.
\]
It is usually more convenient to write the result in the form of the expansion in powers of \( \lambda \), collecting in (27) terms
of the same order,
\[
\frac{\partial f(x,t)}{\partial t} = \sum_{k=1}^{\infty} \lambda^k D_k f(x,t).
\tag{28}
\]
The first six differential operators \( D_k \) are
\[
D_1 = -\alpha_1^{(0)} \frac{\partial}{\partial x},
\tag{29}
\]
\[
D_2 = -\alpha_1^{(1)} \frac{\partial}{\partial x} x + \frac{1}{2} \alpha_2^{(0)} \frac{\partial^2}{\partial x^2},
\]
\[
D_3 = \alpha_1^{(0)} \frac{\partial}{\partial x} - \frac{1}{2} \alpha_1^{(2)} \frac{\partial}{\partial x} x^2 + \frac{1}{2} \alpha_2^{(1)} \frac{\partial^2}{\partial x^2} x - \frac{1}{6} \alpha_3^{(0)} \frac{\partial^3}{\partial x^3},
\]
\[
D_4 = \alpha_1^{(1)} \frac{\partial}{\partial x} x - \frac{1}{6} \alpha_1^{(3)} \frac{\partial}{\partial x} x^3 - \alpha_2^{(0)} \frac{\partial^2}{\partial x^2} + \frac{1}{4} \alpha_2^{(2)} \frac{\partial^2}{\partial x^2} x^2 - \frac{1}{6} \alpha_3^{(1)} \frac{\partial^3}{\partial x^3} x + \frac{1}{24} \alpha_3^{(0)} \frac{\partial^4}{\partial x^4},
\]
\[
D_5 = -\alpha_1^{(0)} \frac{\partial}{\partial x} x + \frac{1}{2} \alpha_1^{(2)} \frac{\partial}{\partial x} x^2 - \frac{1}{24} \alpha_1^{(4)} \frac{\partial^2}{\partial x^2} x^4 - \alpha_2^{(1)} \frac{\partial^2}{\partial x^2} x^2 + \frac{1}{12} \alpha_2^{(3)} \frac{\partial^2}{\partial x^2} x^3 + \frac{1}{2} \alpha_3^{(0)} \frac{\partial^3}{\partial x^3} x^3 - \frac{1}{12} \alpha_3^{(2)} \frac{\partial^3}{\partial x^3} x^4 - \frac{1}{24} \alpha_4^{(0)} \frac{\partial^4}{\partial x^4} x^3 + \frac{1}{120} \alpha_5^{(0)} \frac{\partial^5}{\partial x^5},
\]
\[
D_6 = -\alpha_1^{(1)} \frac{\partial}{\partial x} x + \frac{1}{6} \alpha_1^{(3)} \frac{\partial}{\partial x} x^3 + \frac{1}{12} \alpha_1^{(5)} \frac{\partial}{\partial x} x^5 + \frac{3}{2} \alpha_2^{(0)} \frac{\partial^2}{\partial x^2} x^2 - \frac{1}{2} \alpha_2^{(2)} \frac{\partial^2}{\partial x^2} x^2 + \frac{1}{48} \alpha_2^{(4)} \frac{\partial^2}{\partial x^2} x^4 + \frac{1}{720} \alpha_3^{(0)} \frac{\partial^3}{\partial x^3} x^6 - \frac{1}{120} \alpha_5^{(1)} \frac{\partial^5}{\partial x^5} x^4 + \frac{1}{120} \alpha_5^{(1)} \frac{\partial^5}{\partial x^5} x^4 + \frac{1}{720} \alpha_6^{(0)} \frac{\partial^6}{\partial x^6}.
\]
These formulas are valid not only for the original Rayleigh model, but also for asymmetric models [2-4] when properties
of the bath on the left and on the right of the particle are different. Of course, the explicit form of the coefficients \( \alpha_n^{(p)} \)
are different for different models. In what follows we restrict ourselves to the symmetric problem when the transition rate
is given by Eq.\( \text{13} \). In this case, according to \( \text{20} \) and \( \text{16} \), one gets
\[
\alpha_n(V) = \frac{\mu}{4} \int d\xi \xi^n |\xi| f_M(V + \xi/2)
\]
and therefore
\[ \alpha_n^{(p)} = \frac{\nu}{4} \int d\xi \xi^n \xi f_M^{(p)}(\xi/2). \] (30)

Here the Maxwellian distribution for the bath molecules is
\[ f_M(v) = \left( \frac{\sigma}{2\pi} \right)^{1/2} \exp \left( -\frac{1}{2} \sigma v^2 \right), \quad \sigma = \frac{m}{k BT}, \] (31)

and \( f_M^{(p)}(\xi/2) = \frac{d^p}{d\xi^p} f_M(v) \big|_{v=\xi/2}. \)

One can observe that \( \alpha_n^{(p)} = 0 \) if \( n + p \) is odd. Then the equation (28) contains only operators of even indices \( D_{2k} \), which involve \( \alpha_n^{(p)} \) only with even \( n + p \). These coefficients \( \alpha_n^{(p)} \) can be expressed as linear combinations of the integrals \( I_k = \int_{-\infty}^{\infty} d\xi \xi^{2k} \xi | f_M(\xi/2) \). Using the first three of them
\[ I_1 = \frac{64}{\sqrt{2\pi}} \sigma^{-3/2}, \quad I_2 = \frac{1024}{\sqrt{2\pi}} \sigma^{-5/2}, \quad I_3 = \frac{24576}{\sqrt{2\pi}} \sigma^{-7/2} \] (32)

one can calculate all coefficients \( \alpha_n^{(p)} \) appearing in the equation of order \( \lambda^6 \),
\[
\begin{align*}
\alpha_1^{(1)} & = -\frac{\nu \sigma}{8} I_1 = -a \sigma^{-1/2}, \\
\alpha_1^{(3)} & = \frac{3\nu \sigma^2}{8} I_1 - \frac{\nu \sigma^3}{32} I_2 = -a \sigma^{1/2}, \\
\alpha_1^{(5)} & = -\frac{15\nu \sigma^3}{8} I_1 + \frac{5\nu \sigma^4}{16} I_2 - \frac{\nu \sigma^5}{128} I_3 = a \sigma^{3/2}, \\
\alpha_2^{(0)} & = \frac{\nu}{4} I_1 = 2a \sigma^{-3/2}, \\
\alpha_2^{(2)} & = -\frac{\nu \sigma}{4} I_1 + \frac{\nu \sigma^2}{16} I_2 = 6a \sigma^{-1/2}, \\
\alpha_2^{(4)} & = \frac{3\nu \sigma^2}{4} I_1 - \frac{3\nu \sigma^3}{8} I_2 + \frac{\nu \sigma^4}{64} I_3 = 6a \sigma^{1/2}, \\
\alpha_3^{(1)} & = \frac{\nu \sigma}{8} I_2 = -16a \sigma^{-3/2}, \\
\alpha_3^{(3)} & = \frac{3\nu \sigma^2}{8} I_2 - \frac{\nu \sigma^3}{32} I_3 = -48a \sigma^{-1/2}, \\
\alpha_4^{(0)} & = \frac{\nu}{4} I_2 = 32a \sigma^{-5/2}, \\
\alpha_4^{(2)} & = -\frac{\nu \sigma}{4} I_2 + \frac{\nu \sigma^2}{16} I_3 = 160a \sigma^{-3/2}, \\
\alpha_5^{(1)} & = -\frac{\nu \sigma}{8} I_3 = -384a \sigma^{-5/2}, \\
\alpha_6^{(0)} & = \frac{\nu}{4} I_3 = 768a \sigma^{-7/2}. 
\end{align*}
\] (33)

In these formulas \( a = \frac{8\nu}{\sqrt{2\pi}} \).

**IV. FOKKER-PLANCK AND HIGHER ORDER EQUATIONS**

The results of the previous section allow to write the van Kampen expansion for the Rayleigh model in the explicit form up to order \( \lambda^6 \). Truncation of (28) to terms of order \( \lambda^2 \) leads to the second order Fokker-Planck equation
\[
\frac{\partial f(x,t)}{\partial t} = \lambda^2 D_2 f(x,t),
\] (34)

where
\[
D_2 = \frac{8\nu}{\sqrt{2\pi}} \left\{ \sigma^{-1/2} \frac{\partial}{\partial x} x + \sigma^{-3/2} \frac{\partial^2}{\partial x^2} \right\}.
\] (35)
and $\sigma = m/k_BT$. It was shown recently that the equation (34), first derived by Rayleigh, can be recovered within a more general approach expressing coefficients in the $\lambda$-expansion in terms of correlation functions for the random force and then taking the Markovian limit (11).

The next approximation is the equation of order $\lambda^4$,

$$\frac{\partial f(x, t)}{\partial t} = \left\{ \lambda^2 D_2 + \lambda^4 D_4 \right\} f(x, t), \quad (36)$$

with

$$D_4 = \frac{8\nu}{\sqrt{2\pi}} \left\{ -\sigma^{-1/2} \frac{\partial}{\partial x} x + \frac{1}{6} \sigma^{1/2} \frac{\partial}{\partial x} x^3 - 2\sigma^{-3/2} \frac{\partial^2}{\partial x^2} + \frac{3}{2} \sigma^{-1/2} \frac{\partial^2}{\partial x^2} x^2 \
+ \frac{8}{3} \sigma^{-3/2} \frac{\partial^3}{\partial x^3} x + \frac{4}{3} \sigma^{-5/2} \frac{\partial^4}{\partial x^4} \right\}. \quad (37)$$

This equation is equivalent to that obtained by van Kampen in \[5\].

The equation of order $\lambda^6$ reads as

$$\frac{\partial f(x, t)}{\partial t} = \left\{ \lambda^2 D_2 + \lambda^4 D_4 + \lambda^6 D_6 \right\} f(x, t), \quad (38)$$

where

$$D_6 = \frac{8\nu}{\sqrt{2\pi}} \left\{ \sigma^{-1/2} \frac{\partial}{\partial x} x - \frac{1}{6} \sigma^{1/2} \frac{\partial}{\partial x} x^3 - \frac{1}{120} \sigma^{3/2} \frac{\partial}{\partial x} x^5 + 3\sigma^{-3/2} \frac{\partial^2}{\partial x^2} - 3\sigma^{-1/2} \frac{\partial^2}{\partial x^2} x^2 + \frac{8}{8} \sigma^{1/2} \frac{\partial^2}{\partial x^2} x^4 - 8\sigma^{-3/2} \frac{\partial^3}{\partial x^3} x + \frac{16}{3} \sigma^{-1/2} \frac{\partial^3}{\partial x^3} x^3 \right.\n- \frac{16}{3} \sigma^{-5/2} \frac{\partial^4}{\partial x^4} + \frac{16}{3} \sigma^{-3/2} \frac{\partial^4}{\partial x^4} x^2 + \frac{16}{5} \sigma^{-5/2} \frac{\partial^5}{\partial x^5} x + \frac{16}{15} \sigma^{-7/2} \frac{\partial^6}{\partial x^6} \right\}. \quad (39)$$

This equation appeared previously in an incomplete form in \[14\] missing the term $\alpha^{(1)}_5 \partial^5 \sigma^2 x$ in $D_6$.

One can show \[17\] that the stationary solution of the master equation with transition rates in the form (39) is the Maxwellian distribution which for the scaled velocity $x = \lambda^{-1} V$ has the form $f_\lambda(x) = C \exp \left( -\frac{1}{2} \sigma x^2 \right)$. This distribution does not depend on $\lambda$ and therefore must satisfy each term in the expansion (28) separately,

$$D_n f_\lambda(x) = 0. \quad (40)$$

One can immediately check that the above expressions for $D_2$, $D_4$, and $D_6$ do satisfy this condition.

Eqs. (36) or (38) can be solved using an appropriate perturbation technique, however it is often easier to handle the equations for the moments $\langle x^n \rangle$ \[2\] \[14\].

It is known that approximations of the master equation by a differential equation involving derivatives of order higher than two may lead to solutions which are not positive definite \[6\] \[10\]. The reason why the Fokker-Planck equation (34) preserves positivity of $f$ is because the right hand side of the equation, and therefore the time derivative $f_t$, are always positive at the points where $f(x, t)$ as a function of $x$ has minima. As a result, the minima become less deep with time, the initially positive solution remains positive for all times. This is not true in general for Eqs. (36) and (38) involving $x$-derivatives of order higher than two. These derivatives can be of any sign at extreme points, which in principle may result in negative $f_t$ at minimum points. However, since the terms with higher order derivatives are of higher order in $\lambda$, one may expect that for sufficiently smooth initial distributions the sign of the right sides of Eqs. (36) and (38) is determined by the term with $f_{xx}$. For example, the only terms in the right side of Eq. (36) which can be negative at minimum points are those involving $\lambda^4 \sigma^{-3/2} f_{xxx}$ and $\lambda^6 \sigma^{-5/2} f_{xxxx}$. They are smaller than the term $\lambda^2 \sigma^{-3/2} f_{xx}$, which gives the positive contribution, by the factors of order $\lambda^2(x/x_c)$ and $\lambda^2(x_{th}/x_c)^2$, respectively. Here $x_c$ is the characteristic length of the distribution $f(x, t)$ for a given $t$, and $x_{th}$ is the scaled thermal velocity of the particle, $x_{th} = \lambda^{-1} \sqrt{k_BT/M}$. Recall that the expansion method we used implies that the system is close to equilibrium and that $x \sim x_{th} \sim \lambda^0$. Under this condition the term with $f_{xx}$ dominates and determines the sign of the right hand side of Eq. (36) at minimum points, which guarantees preservation of the positivity of the solution.

The expansion method may be implemented also for non-equilibrium fluctuations \[12\] but in that case the possibility to introduce a stochastic variable $x$, which would be of order one for all relevant times, is less obvious and has to be justified a posteriori.
V. DISCUSSION

If the white random force $F(t)$ in the Langevin equation $\dot{V}(t) = A(V) + F(t)$ is also assumed to be a Gaussian process, then the conventional procedure outlined in Section 2 leads invariably to the second order Fokker-Planck equation (42), no matter whether the damping force $A(V)$ is linear or not. On the other hand, the van Kampen method, which does not require any assumptions about statistics of the random force, leads to the second order equation only to the lowest order in the expansion parameter $\lambda$, when the damping force is linear $A(V) = -\gamma V$. In higher orders in $\lambda$, when the damping force $A(V)$ involves nonlinear corrections $A(V) = -\gamma_1 V - \gamma_3 V^3 - \cdots$ (where $\gamma_3/\gamma_1 \sim \lambda^2$), the VKE leads to equations with $V^2$-derivatives of order higher than two. This means that the approximation of a Gaussian random force is legitimate to lowest order in $\lambda$ but not to higher orders. It is therefore inconsistent to take into account nonlinear corrections to the linear damping force $A(V) = -\gamma V$, and at the same time to assume that the random force is Gaussian. Failure to appreciate this point may lead to wrong conclusions. For example, for the Rayleigh model the Maxwell distribution is a correct stationary solution to any order in $\lambda$. On the other hand the assumption of a Gaussian random force would lead to the Fokker-Planck equation (12) which for nonlinear $A(V)$ has a non-Maxwellian stationary solution.

The Gaussian property of the random force is usually expected to hold interpreting the force as a result of many uncorrelated collisions, and appealing to the central limit theorem. However, one has to keep in mind that a decomposition of the total force exerted on the particle into a regular damping and “random” parts is a purely mathematical procedure, which in general can be performed with an appropriate projection operator technique. The random force $F(t)$, obtained in this way generally can not be interpreted as a superposition of many “physical” forces. Moreover, it is not even a dynamical variable: its evolution in time is not governed by the Newtonian propagator $\exp(tL)$ with the Liouville operator $L$, but by a more complicated “projected” propagator. One can get some insight in properties of the random force expanding it in powers of $\lambda$ which has the form

$$F(t) = F_0(t) + \lambda \int_0^t dt_1 S(t-t_1)F_0(t_1) + \lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2 S(t-t_1)S(t_1-t_2)F_0(t_2) + \cdots$$

where $S(t)$ is a non-Newtonian propagator the explicit form of which is not important for our purpose here. The first term $F_0(t)$ in the expansion (12) is a dynamical variable and has a well defined physical meaning: it is a force exerted on the particle fixed in space (the limit of infinitely heavy particle). Conventional qualitative reasoning to justify the Gaussian property is quite applicable for this term. Moreover, for a large Brownian particle, interacting simultaneously with many bath molecules, the Gaussian property of $F_0(t)$ can be proved analytically (11). To lowest order in $\lambda$ the random force is just $F_0(t)$ and therefore is Gaussian. On the other hand, there is no reason to expect that the same argument should work for the “unphysical forces” represented in (12) by terms of higher orders in $\lambda$ involving the non-Newtonian propagator. These terms in general are not Gaussian. In a future publication we shall explicitly evaluate correlation functions $(F(t_1)F(t_2)\cdots F(t_k))$ for an exactly solvable model with parabolic interaction suggested in (11). This would allow to construct the expansion similar to the VKE but expressing coefficients in terms of correlation functions for the random force rather than transition rates $W$. In lowest order in $\lambda$ it was done in (11), recovering in the Markovian limit the Fokker-Planck equation (43).

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