On maximum intersecting sets in direct and wreath product of groups

Ademir Hujdurović, Klavdija Kutnar, Dragan Marušić, Štefko Miklavič

Abstract

For a permutation group $G$ acting on a set $V$, a subset $I$ of $G$ is said to be an intersecting set if for every pair of elements $g, h \in I$ there exists $v \in V$ such that $g(v) = h(v)$. The intersection density $\rho(G)$ of a transitive permutation group $G$ is the maximum value of the quotient $|I|/|G_v|$ where $G_v$ is a stabilizer of a point $v \in V$ and $I$ runs over all intersecting sets in $G$. If $G_v$ is the largest intersecting set in $G$ then $G$ is said to have the Erdős-Ko-Rado (EKR)-property, and moreover, $G$ has the strict-EKR-property if every intersecting set of maximum size in $G$ is a coset of a point stabilizer. Intersecting sets in $G$ coincide with independent sets in the so-called derangement graph $\Gamma_G$, defined as the Cayley graph on $G$ with connection set consisting of all derangements, that is, fixed-point free elements of $G$.

In this paper a conjecture regarding the existence of transitive permutation groups whose derangement graphs are complete multipartite graphs, posed by Meagher, Razafimahatratra and Spiga in [J. Combin. Theory Ser. A 180 (2021), 105390], is proved. The proof uses direct product of groups. Questions regarding maximum intersecting sets in direct and wreath products of groups and the (strict)-EKR-property of these group products are also investigated. In addition, some errors appearing in the literature on this topic are corrected.

Keywords: intersection density, intersecting set, derangement graph, transitive permutation group.

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1 Introduction

The Erdős-Ko-Rado theorem [7] is one of the central results in extremal combinatorics. It gives a bound on the size of a family of intersecting $k$-subsets of a set and classifies the families satisfying the bound. Besides many interesting proofs of this theorem, it has also been extended in various directions. For example, Hsieh [13] and Frankl and Wilson [8] give a version for intersecting subspaces of a vector space over a finite field, Rands [20] extends it to intersecting blocks in a design and Meagher and Moura [17] prove a version for partitions.

In this paper, we focus on the extension of the Erdős-Ko-Rado theorem to permutation groups. For a finite set $V$ let $\text{Sym}(V)$ denote the corresponding symmetric group, and if $|V| = n$, we will...
use the notation $S_n$. Given a permutation group $G \leq \text{Sym}(V)$, a subset $I$ of $G$ is called intersecting if, for any two permutations $g$ and $h$ in $I$, there exists $v \in V$ such that $g(v) = h(v)$.

The problem of determining the intersecting sets of a permutation group $G$ can be formulated in a graph-theoretic terminology. We denote by $\Gamma_G$ the derangement graph of $G$, a Cayley graph whose vertex set coincides with $G$ and whose edges are the unordered pairs $\{g, h\}$ such that $gh^{-1}$ is a derangement, that is, $gh^{-1}$ is fixed-point-free. Now, an intersecting set of $G$ is simply an independent set or a coclique of $\Gamma_G$, and similarly the classical Erdős-Ko-Rado theorem translates into a classification of independent sets of maximal cardinality of the Kneser graphs.

We say the group $G$ has the EKR property if the size of any intersecting subset of $G$ is bounded above by the size of the largest point-stabilizer in $G$.

A permutation group $G$ is said to have the Erdős-Ko-Rado property (in short EKR-property), if the size of a maximum intersecting set is equal to the order of the largest point stabilizer, and is said to have the strict Erdős-Ko-Rado property (in short strict-EKR-property) if every maximum intersecting set of $G$ is a coset of a point stabilizer. It is clear that strict-EKR-property implies EKR-property, but the converse does not hold. For example, the alternating group $A_4$ in its natural action on $\{1, 2, 3, 4\}$ has the EKR-property, but not the strict-EKR-property. Namely, $\{id, (132), (142)\}$ is a maximum intersecting set not coinciding with a coset of a point stabilizer in $A_4$. In fact, for the natural action of the alternating groups, $A_4$ is the only alternating group without the strict-EKR-property (see [11]). Furthermore, the symmetric group $S_n$, $n \geq 2$, in its natural action has the strict-EKR-property (see [1, 6, 10, 15]), and $p$-groups, $p$ a prime, have the EKR-property (see [12, 16]). As for doubly transitive permutation groups it was proved in [18] that they have the EKR-property, but there are infinitely many examples of such groups without the strict-EKR-property.

The other end of the spectrum is more interesting: there exist transitive permutation groups having the size of a maximum intersecting set bigger than the order of a point stabilizer. For example, the alternating group $A_5$ acting on 2-subsets of $\{1, 2, 3, 4, 5\}$ admits intersecting sets twice as big as is the order of a point stabilizer in this action. In [16] a measure of how far a permutation group is from having the EKR property is introduced via the concept of the so-called intersection density. The intersection density of a transitive permutation group $G$, denoted by $\rho(G)$, is defined by $\rho(G) = \frac{|I|}{|G_v|}$, where $I$ is an intersecting set in $G$ of a maximum size, and $G_v$ is a point stabilizer. Clearly, $\rho(G) \geq 1$ with the equality holding if and only if $G$ has the EKR-property. As proved in [10] intersection densities of transitive permutation groups can be arbitrarily large, that is, for every $M > 0$ there exists a transitive permutation group $G$, with $\rho(G) > M$. Intersection densities of transitive permutation groups of degree $2p$, $p$ a prime, have been determined to be either 1 or 2 (see [12, 21] for details).

In [19] Meagher, Razafimahatratra, and Spiga proved that the derangement graph of an arbitrary transitive permutation group of degree at least 3 contains a triangle and among other posed the following conjecture which is the main motivation for this paper.

**Conjecture 1.1** [19 Conjecture 6.6(1)] *If $n$ is even but not a power of 2, then there is a transitive permutation group $G$ of degree $n$ such that $\Gamma_G$ is a complete multipartite graph with $n/2$ parts.*

The paper is organized as follows. In Section 2 we gather some useful results needed later on in the paper. In Section 3 we prove Conjecture 1.1 (see Theorem 3.4). As a byproduct a characterization of the derangement graphs of direct products of groups is obtained (see Theorem 3.11). In particular, we prove that for transitive permutation groups $G$ and $H$, the direct product $G \times H$ (in the canonical action) has the (strict)-EKR-property if and only if both $G$
and $H$ have the (strict)-EKR-property, and that $\rho(G \times H) = \rho(G)\rho(H)$. This implies that by taking direct products of sufficiently many transitive permutation groups that do not have the EKR property a construction of transitive permutation groups with arbitrarily large intersection densities can be obtained.

In Section 4 derangement graphs of wreath products of transitive permutation groups are characterized (see Lemma 4.2 and Proposition 4.3). Next, given transitive permutation groups $G$ and $H$ with $G \wr H$ denoting the corresponding wreath product, we show that $\rho(G) \leq \rho(G \wr H) \leq \rho(G)\rho(H)$ (see Proposition 4.4). Also, we give sufficient conditions for $G \wr H$ to have the strict-EKR-property (see Proposition 4.10) and necessary and sufficient conditions for $G \wr H$ with $H$ being regular to have the strict-EKR-property (see Proposition 4.8).

Along the way certain inconsistencies in the material on this topic (see [2] and Remarks 3.6 and 4.11) are taken care of.

2 Preliminaries

Let $\mathcal{D}$ be the set of all derangements of a permutation group $G$. Following [19] we define the derangement graph of $G$ to be the graph $\Gamma_G = \text{Cay}(G, \mathcal{D})$ with vertex set $G$ and edge set consisting of all edges $\{g, h\}$ such that $gh^{-1} \in \mathcal{D}$. Therefore $\Gamma_G$ is the Cayley graph of $G$ with connection set $\mathcal{D}$, which is a loop-less simple graph since $\mathcal{D}$ does not contain the identity element of $G$ and $\mathcal{D}$ is inverse-closed. In the terminology of derangement graphs an intersecting set of $G$ is an independent set or a coclique in $\Gamma_G$.

The largest size of a clique, the so-called clique number, in a graph $X$ is denoted by $\omega(X)$, and the largest size of an independent set, that is, the independence number, in $X$ is denoted by $\alpha(X)$. The following classical upper bound on the size of the largest coclique in vertex-transitive graphs turns out to be quite useful when considering intersection densities of permutation groups. Namely, the derangement graph $\Gamma_G$ of a permutation group $G$ is always vertex-transitive.

Lemma 2.1 [4, Corollary 4] Let $\Gamma$ be a vertex-transitive graph. Then the largest coclique in $\Gamma$ is of size $\alpha(\Gamma)$ bounded by

$$\alpha(\Gamma) \leq \frac{|V(\Gamma)|}{\omega(\Gamma)},$$

where $\omega(\Gamma)$ is the size of a maximum clique in $\Gamma$.

The next result is usually referred to as the “No-Homomorphism Lemma”.

Lemma 2.2 [3, Theorem 2] Let $X$ be a graph and $Y$ be a vertex-transitive graph admitting a homomorphism from $X$ to $Y$. Then

$$\frac{|V(X)|}{\alpha(X)} \leq \frac{|V(Y)|}{\alpha(Y)}.$$

For graphs $X$ and $Y$ the strong product $X \boxtimes Y$ is the graph with vertex set $V(X) \times V(Y)$ such that $(x_1, y_1)$ is adjacent to $(x_2, y_2)$ if and only if one of the following three conditions holds:

(i) $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$;
(ii) $y_1 = y_2$ and $\{x_1, x_2\} \in E(X)$;
(iii) $\{x_1, x_2\} \in E(X)$ and $\{y_1, y_2\} \in E(Y)$.
The following result regarding maximum cliques in the strong product of graphs will be used for determining the intersection densities of direct product of groups. We use the following notation: for graphs $X$, $Y$ and for a set $C$ of vertices of $X \boxtimes Y$ we denote by $p_X(C) = \{x \in V(X) \mid \exists y \in V(Y) \text{ such that } \{x, y\} \in C\}$ and $p_Y(C) = \{y \in V(Y) \mid \exists x \in V(X) \text{ such that } \{x, y\} \in C\}$ the projection of $C$ onto $V(X)$ and $V(Y)$, respectively.

**Lemma 2.3** [11] Lemma 7.3] Let $X$ and $Y$ be graphs and $C$ a maximum clique in $X \boxtimes Y$. Then $C = p_X(C) \times p_Y(C)$, where $p_X(C)$ and $p_Y(C)$ is, respectively, a maximum clique in $X$ and $Y$.

For graphs $X$ and $Y$, the *direct product* (also called the *tensor product*) $X \times Y$ is the graph with vertex set $V(X) \times V(Y)$ such that $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $\{x_1, x_2\} \in E(X)$ and $\{y_1, y_2\} \in E(Y)$.

The following result on the independence number of vertex-transitive graphs together with its direct consequence in Lemma 2.5 will be needed in Section 3.

**Lemma 2.4** [22] Corollary 1.5] Let $X$ and $Y$ be vertex-transitive graphs. Then

$$\alpha(X \times Y) = \max\{\alpha(X)|V(Y)|, \alpha(Y)|V(X)|\}.$$  

**Lemma 2.5** Let $X$ be a vertex-transitive graph. Then $\alpha(X \times \cdots \times X) = \alpha(X)|V(X)|^{n-1}$.

For graphs $X$ and $Y$, the *wreath product* $X[Y]$ of $X$ by $Y$ (also called the *lexicographic product*) is the graph with vertex set $V(X) \times V(Y)$ such that $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $\{x_1, x_2\} \in E(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$. The following result will be useful later in the paper.

**Lemma 2.6** [9] Theorem 1] Let $X$ and $Y$ be graphs. Then

$$\alpha(X[Y]) = \alpha(X)\alpha(Y).$$

Finally, the next result will be used in our analysis of intersection densities of group products.

**Lemma 2.7** [10] Lemma 6.5] If $H \leq G$ are transitive permutation groups then $\rho(G) \leq \rho(H)$.

## 3 Derangement graphs of direct product of groups

Given finite sets $V$ and $W$ let $G \leq \text{Sym}(V)$ and $H \leq \text{Sym}(W)$. We will consider two actions of $G \times H$. Namely, $G \times H$ acts naturally on $V \times W$ by the rule

$$(g, h) : (v, w) \mapsto (g(v), h(w)) \text{ for } g \in G \text{ and } h \in H.$$  

Following [2] this action will be called the *external direct product*. The other action of $G \times H$ is the action on $V \cup W$ (where $V$ and $W$ are considered as disjoint sets regardless of their structure), defined by the rule

$$(g, h) : x \mapsto g(x) \text{ if } x \in V \text{ and } (g, h) : x \mapsto h(x) \text{ if } x \in W.$$  

Again, following [2] this action will be called the *internal direct product*. Observe that the external direct product is transitive if and only if two factor groups are transitive, while the internal direct product is never transitive.
3.1 External direct products

The derangement graphs of the external direct products of groups were studied in [2]. In [2, Lemma 4.1] it is claimed that $\Gamma_{G \times H} \cong \overline{\Gamma_G \times \Gamma_H}$. This is correct only if the complement $\overline{X}$ of a graph $X$ is defined in such a way that a loop is added to each vertex without a loop in $X$. Since this is not a standard definition of a graph complement, which is not explicitly stated in [2], we give below a different characterization of derangement graphs of external direct products that uses the standard definition of a graph complement.

**Theorem 3.1** For finite sets $V$ and $W$ let $G \leq \text{Sym}(V)$ and $H \leq \text{Sym}(W)$. Then the derangement graph $\Gamma_{G \times H}$ of the external direct product $G \times H$ acting on $V \times W$ satisfies

$$\Gamma_{G \times H} \cong \overline{\Gamma_G \boxtimes \Gamma_H}.$$ 

**Proof.** Let $(g_1, h_1), (g_2, h_2)$ be two different elements in $G \times H$. Then

$$\{(g_1, h_1), (g_2, h_2)\} \in E(\Gamma_{G \times H}) \iff$$

$$(\exists (v, w) \in V \times W) \text{ such that } (v, w) \text{ is fixed by } (g_1g_2^{-1}, h_1h_2^{-1}) \iff$$

$$(\exists (v, w) \in V \times W) \text{ such that } (g_1g_2^{-1} \in G_v \land h_1h_2^{-1} \in H_w) \iff$$

$$(g_1 = g_2 \land g_1 \sim_{\Gamma_G} g_2) \land (h_1 = h_2 \lor h_1 \sim_{\Gamma_H} h_2) \iff$$

$$(g_1 = g_2 \land h_1 \sim_{\Gamma_H} h_2) \lor (h_1 = h_2 \land g_1 \sim_{\Gamma_G} g_2) \lor (g_1 \sim_{\Gamma_G} g_2 \land h_1 \sim_{\Gamma_H} h_2) \iff$$

$$\{(g_1, h_1), (g_2, h_2)\} \in E(\overline{\Gamma_G \boxtimes \Gamma_H}).$$

Observe that if $H$ is a regular group, then $\Gamma_H$ is a complete graph, and so $\overline{\Gamma_H}$ is the empty graph, giving us two immediate corollaries of Theorem 3.1.

**Corollary 3.2** For finite sets $V$ and $W$ let $G \leq \text{Sym}(V)$ and $H \leq \text{Sym}(W)$. If $H$ is regular, then $\overline{\Gamma_{G \times H}}$ is isomorphic to a disjoint union of $|H|$ copies of $\overline{\Gamma_G}$.

**Corollary 3.3** For finite sets $V$ and $W$ let $G \leq \text{Sym}(V)$ and $H \leq \text{Sym}(W)$. If $H$ is regular and $\Gamma_G$ is a complete multipartite graph with $k$ parts, then $\overline{\Gamma_{G \times H}}$ is a complete multipartite graph with $k|H|$ parts.

**Proof.** If $\Gamma_G$ is a complete multipartite graph with $k$ parts, then $\overline{\Gamma_G}$ is a disjoint union of $k$ cliques. By Corollary 3.2 it follows that $\overline{\Gamma_{G \times H}}$ is isomorphic a disjoint union of $|H|$ copies of $\overline{\Gamma_G}$. Hence $\overline{\Gamma_{G \times H}}$ is a disjoint union of $k|H|$ cliques, and therefore $\overline{\Gamma_{G \times H}}$ is a complete multipartite graph with $k|H|$ parts.

We are now ready to prove Conjecture 1.1.

**Theorem 3.4** If $n$ is even but not a power of 2, then there is a transitive permutation group $G$ of degree $n$ such that $\Gamma_G$ is a complete multipartite graph with $n/2$ parts.

**Proof.** Let $n = 2^a k$ with $k \geq 3$ odd and $a \geq 1$. By [19, Lemma 5.3] there exists a transitive permutation group $K$ of degree $2k$ such that $\Gamma_K$ is a complete multipartite graph with $k$ parts.
Let $H$ be a regular group of degree $2^a - 1$. Then by Corollary 3.3 it follows that $\Gamma_{K \times H}$ is a complete multipartite graph with $2^a - 1k = n/2$ parts.

Recall that maximum cocliques in $\Gamma_G$ (equivalently, maximum cliques of $\overline{\Gamma_G}$) correspond to maximum intersecting sets of $G$. Consequently, if $G$ is transitive, $\rho(G) = \omega(\overline{\Gamma_G})/|G_v|$.

**Theorem 3.5** Let $G \leq \text{Sym}(V)$ and $H \leq \text{Sym}(W)$ be transitive permutation groups and let $G \times H$ be the external direct product acting on $V \times W$. Then

(i) $\rho(G \times H) = \rho(G)\rho(H)$, and

(ii) $G \times H$ has the strict-EKR-property if and only if both $G$ and $H$ have the strict-EKR-property.

**Proof.** Recall that $\rho(G \times H) = \omega(\overline{\Gamma_{G \times H}})/|(G \times H)_{(v,w)}|$ where $(v,w) \in V \times W$. Since $\overline{\Gamma_{G \times H}} \cong \overline{\Gamma_G} \boxtimes \overline{\Gamma_H}$, by Lemma 2.3 it follows that $\omega(\overline{\Gamma_{G \times H}}) = \omega(\overline{\Gamma_G})\omega(\overline{\Gamma_H})$. Observe that $(G \times H)_{(v,w)} = G_v \times H_w$. It follows that $\rho(G \times H) = \omega(\overline{\Gamma_{G \times H}})/|(G \times H)_{(v,w)}| = \omega(\overline{\Gamma_G})\rho(H)/|G_v||H_w| = \rho(G)\rho(H)$, proving (i).

As for (ii), let $\mathcal{F}$ be a maximum intersecting set of $G \times H$ containing $(1_G, 1_H)$ and let $Q$ be the corresponding maximum independent set in $\Gamma_{G \times H}$, or equivalently a maximum clique in $\overline{\Gamma_{G \times H}}$. By Theorem 3.1 we have $\Gamma_{G \times H} \cong \overline{\Gamma_G} \boxtimes \overline{\Gamma_H}$. By Lemma 2.3 it follows that $Q = p_{\overline{\Gamma_G}}(Q) \times p_{\overline{\Gamma_H}}(Q)$ where $p_{\overline{\Gamma_G}}(Q)$ and $p_{\overline{\Gamma_H}}(Q)$ are maximum cliques of $\overline{\Gamma_G}$ and $\overline{\Gamma_H}$ containing $1_G$ and $1_H$, respectively. Now, if $G$ and $H$ have the strict-EKR-property, then $p_{\overline{\Gamma_G}}(Q)$ and $p_{\overline{\Gamma_H}}(Q)$ (viewed as subsets of $G$ and $H$ respectively) are vertex stabilizers in $G$ and $H$, and consequently $Q$ (and equivalently $\mathcal{F}$) is a vertex-stabilizer in $G \times H$. Conversely, if $Q$ is a vertex-stabilizer in $G \times H$, it follows that $p_{\overline{\Gamma_G}}(Q)$ and $p_{\overline{\Gamma_H}}(Q)$ are vertex-stabilizers in $G$ and $H$, respectively.

**Remark 3.6** In [2] Theorem 4.3 it is claimed that the external direct product of groups has the (strict)-EKR-property if and only if each of the factors has the (strict)-EKR-property, a claim made in Theorem 3.5 too. We would like to note, however, that the proof of [2] Theorem 4.3 is incomplete, as it relies on [2] Lemma 4.1 (which is incorrect unless the complement of a graph is defined to include loops) and [2] Lemma 4.2., the proof of which seems to be missing some details.

**Remark 3.7** Given a group $G$ with $\rho(G) > 1$, one can construct a group with arbitrarily large intersection density by taking sufficiently many copies of $G$ in the direct product $G \times G \times \cdots \times G$.

### 3.2 Internal direct products

The study of intersecting sets in internal direct products was initiated by Ku and Wong in [14]. They gave necessary and sufficient conditions for the direct product of symmetric groups to have the strict-EKR-property. In [2] the derangement graphs of internal direct products were characterized.

**Lemma 3.8** [2] Let $V_1, \ldots, V_n$ be disjoint sets and $G_i \leq \text{Sym}(V_i)$ $i \in \{1, \ldots, n\}$. Let $G = G_1 \times \cdots \times G_n$ be the internal direct product acting on $V_1 \cup \cdots \cup V_n$. Then $\Gamma_G \cong \Gamma_{G_1} \times \cdots \times \Gamma_{G_n}$. Moreover, if all groups $G_i$ have the EKR-property, then $G$ also has the EKR-property.
No analogue to Lemma 3.8 holds for the strict-EKR-property. Namely the internal direct product of groups each having the strict-EKR-property does not need to have the strict EKR-property. For example, $S_3 \times S_3$ does not have the strict EKR-property.

We are interested in determining conditions under which does the internal direct product of the form $G \times G \times \cdots \times G$ have the strict-EKR-property. As we will see in Section 4, this is important for understanding when does the wreath product of groups have the strict-EKR-property. With the exception of the internal direct products of full symmetric groups, not much has been done on this problem. This is not surprising, as on the one hand, these groups are not transitive (and restriction to investigate transitive permutation groups is common), and on the other, the problem is quite hard. Namely, one would need to characterize maximum independent sets in the direct product of derangement graphs of the factor groups, which is not an easy task in view of the fact that the structure of maximum independent sets in direct products is usually not all that simple. The situation changes somewhat with the so-called MIS-normal direct products introduced below.

The direct product $X_1 \times \cdots \times X_n$ of graphs $X_1, \ldots, X_n$ is said to be MIS-normal (maximum-independent-set-normal) if every maximum independent set of it is the preimage of an independent set of one factor under projections. Following Zhang, [22] we say that a graph $\Gamma$ is IS-primitive if there is no non-maximum independent set $A$ such that $|A|/|N[A]| = \alpha(\Gamma)/|V(\Gamma)|$, where $N[A]$ denotes the closed neighbourhood of $A$. Note that a disconnected vertex-transitive graph is not IS-primitive. The following result follows from [22] Theorem 1.4] and [23] Corollary 3.2].

**Lemma 3.9** Let $X$ be a non-bipartite vertex-transitive graph. Then $X \times \cdots \times X$ is MIS-normal if and only if $X$ is IS-primitive.

**Example 3.10** Let $G$ be a regular group. Then $\Gamma_G$ is IS-primitive. Namely, the graph $\Gamma_G$ is a complete graph, and hence maximum independent sets are of size 1.

**Lemma 3.11** Let $G \leq \text{Sym}(V)$ be a transitive permutation group with $|V| \geq 3$. Then the internal direct product $G \times \cdots \times G$ has the strict-EKR-property if and only if $G$ has the strict-EKR-property and $\Gamma_G$ is IS-primitive.

**Proof.** Let $G^n = G \times \cdots \times G$. Suppose first that $G^n$ has the strict-EKR-property. If $G$ does not have the strict-EKR-property, then there exists a maximum intersecting set $S$ in $G$ containing the identity which is not a point stabilizer. Then $I = S \times G \times \cdots \times G$ is a maximum intersecting set in $G^n$ which is not a point stabilizer, contradicting the assumption that $G^n$ has the strict-EKR-property. Since $G$ is of degree at least 3, it follows by [19] Theorem 1.5] that $\Gamma_G$ is non-bipartite. Therefore, if $\Gamma_G$ is not IS-primitive, then Lemma 3.8 implies that $\Gamma_{G^n}$ is not MIS-normal, that is, there exists a maximum independent set in $\Gamma_{G^n}$ which is not a preimage of a maximum independent set in $\Gamma_G$. As vertex stabilizers in $G^n$ have the form $G \times \cdots \times G \times G_0 \times G \times \cdots \times G$, we conclude that $G^n$ does not have the strict-EKR-property, a contradiction showing that $\Gamma_G$ is IS-primitive.

Conversely, suppose that $G$ has the strict-EKR-property and that $\Gamma_G$ is IS-primitive. Lemma 3.8 implies that $\Gamma_{G^n} \cong \Gamma_G \times \cdots \times \Gamma_G$. Since $G$ is of degree at least 3, it follows by [19] Theorem 1.5] that $\Gamma_G$ is non-bipartite. Since $\Gamma_G$ is IS-primitive, it follows by Lemma 3.9 that $\Gamma_{G^n}$ is MIS-normal. Let $I$ be a maximum intersecting set in $G^n$ containing the identity. Then $I$ is a maximum independent set in $\Gamma_{G^n}$, and since the graph is MIS-normal, we may conclude, without loss of generality, that $I = I_0 \times G \times \cdots \times G$, where $I_0$ is a maximum independent set in $\Gamma_G$. Since $G$ has the strict-EKR-property, it follows that $I_0$ is a stabilizer of a point in $G$. Hence $I$ is a stabilizer of a point in $G^n$, showing that $G^n$ has the strict-EKR-property.

We close the section with an open problem.
Problem 3.12 Characterize transitive permutation groups \( G \) such that \( \Gamma_G \) is IS-primitive.

4 Derangement graphs of wreath products

Let \( G \) be a permutation group acting on a set \( V \) and \( H \leq S_n \) a permutation group acting on the set \( N = \{1, \ldots, n\} \). The wreath product of \( G \) by \( H \) denoted by \( G \wr H \) is the set of all permutations \( ((g_1, \ldots, g_n), h) \) of \( V \times N \) (where \( g_1, \ldots, g_n \in G \) and \( h \in H \)) such that \( ((g_1, \ldots, g_n), h) : (a, i) \mapsto (g_i(a), h(i)) \).

Observe that for \( ((g_1, \ldots, g_n), h), ((g'_1, \ldots, g'_n), h') \in G \wr H \) we have
\[
((g_1, \ldots, g_n), h) \cdot ((g'_1, \ldots, g'_n), h') = (gh'_1g'_1, \ldots, gh'_n(n)g'_n(n), hh')
\]
and
\[
((g_1, \ldots, g_n), h)^{-1} = ((g_{h^{-1}(1)}, \ldots, g_{h^{-1}(n)}), h^{-1}).
\]

Observe that the wreath product \( G \wr H \) is isomorphic to the semidirect product \( G^m \times H \), where \( G^m \) is the internal direct product of \( G \) with \( n \) factors. The main goal of this section is to characterize derangement graphs of wreath products, and to study the intersection density, EKR-property and strict-EKR-property of wreath products. We start with the following lemma.

Lemma 4.1 Let \( G \leq \text{Sym}(V) \) and \( H \leq S_n \) be permutation groups. Vertices \( ((g_1, \ldots, g_n), h) \) and \( ((g'_1, \ldots, g'_n), h') \) are adjacent in \( \Gamma_{G \wr H} \) if and only if \( g_i \) is adjacent with \( g'_i \) in \( G \) for every \( i \in \{1, \ldots, n\} \) such that \( h(i) = h'(i) \).

Proof.

Clearly, two vertices \( g = ((g_1, \ldots, g_n), h) \) and \( g' = ((g'_1, \ldots, g'_n), h') \) are adjacent if and only if \( g'g^{-1} = ((g'_h^{-1}(1)g_{h^{-1}(1)}), \ldots, g'_h^{-1}(n)g_{h^{-1}(n)}), h'h^{-1}) \) is a derangement. If \( j \notin \text{fix}(h'h^{-1}) \) then \( g'g^{-1} \) does not fix a point in \( V \times \{j\} \) regardless of what are the values of \( g_k \) and \( g'_k \) for \( k \in \{1, \ldots, n\} \).

Suppose that \( j \in \text{fix}(h'h^{-1}) \). Observe that \( g'g^{-1}(v, j) = (g'_h^{-1}(j)g_{h^{-1}(j)}(v), j) \). Hence \( g'g^{-1} \) does not fix a point in \( V \times \{j\} \) if and only if \( g'_h^{-1}(j)g_{h^{-1}(j)} \) is a derangement. We conclude that \( g'g^{-1} \) is a derangement if and only if \( g'_h^{-1}(j)g_{h^{-1}(j)} \) is a derangement for each \( j \in \text{fix}(h'h^{-1}) \). This is equivalent with \( g'_hi^{-1}(j)g_{h^{-1}(j)} \) being a derangement for each \( i \in \{1, \ldots, n\} \) such that \( h(i) \in \text{fix}(h'h^{-1}) \).

The last condition is equivalent with \( h(i) = (h'h^{-1})(h(i)) = h'(i) \). To summarize, \( ((g_1, \ldots, g_n), h) \) and \( ((g'_1, \ldots, g'_n), h') \) are adjacent in \( \Gamma_{G \wr H} \) if and only if \( g_i \) is adjacent with \( g'_i \) in \( G \) for every \( i \in \{1, \ldots, n\} \) such that \( h(i) = h'(i) \).

Let \( G \leq \text{Sym}(V) \) and \( H \leq S_n \) be permutation groups, and let \( h, h' \in H \). Denote with \( \Gamma_{G \wr H}[h, h'] \) the subgraph of \( \Gamma_{G \wr H} \) with vertex set \( G^m \times \{h, h'\} \) containing all the edges with one endvertex in \( G^m \times \{h\} \) and the other in \( G^m \times \{h'\} \). Observe that for \( h = h' \), the graph \( \Gamma_{G \wr H}[h, h] \) is the subgraph of \( \Gamma_{G \wr H} \) induced by the set \( G^m \times \{h\} \). We will denote it by \( \Gamma_{G \wr H}[h] \). As for \( h \neq h' \), the subgraph \( \Gamma_{G \wr H}[h, h] \) is the bipartite subgraph of \( \Gamma_{G \wr H} \) containing all the edges between sets \( G^m \times \{h\} \) and \( G^m \times \{h'\} \). We now give a characterization of the derangement graph of a wreath product of groups. To simplify the notation, we denote by \( K_n^* \) the complete graph of order \( n \) with a loop at each vertex.

Lemma 4.2 Let \( G \leq \text{Sym}(V) \) and \( H \leq S_n \) be permutation groups and let \( h, h' \in H \) be distinct. Then:
Proof. Applying Lemma 4.1 for vertices \((g_1, \ldots, g_n), h\) and \((g_1', \ldots, g_n'), h\) it follows that \((g_1, \ldots, g_n), h\) and \((g_1', \ldots, g_n'), h\) are adjacent in \(\Gamma_{G[H]}\) if and only if \(g_i\) is adjacent to \(g_i'\) in \(G\) for each \(i \in \{1, \ldots, n\}\). Now the claim (i) holds directly by the definition of direct product of graphs. Similarly, (ii) follows from Lemma 4.1 and the definition of direct product of graphs. 

In the case when \(H \leq S_n\) is regular, the derangement graph of \(G \wr H\) has a particularly nice structure, given as follows.

**Proposition 4.3** Let \(G \leq \text{Sym}(V)\) and \(H \leq S_n\). If \(H\) is regular, then \(\Gamma_{G[H]} \cong K_n[\Gamma_G \times \Gamma_G \times \cdots \times \Gamma_G]\).

**Proof.** Since \(H\) is regular, we have \(h(i) \neq h'(i)\) for any two distinct elements \(h, h' \in H\) and for every \(i \in \{1, \ldots, n\}\). Then Lemma 4.2 implies that \(\Gamma_{G[H]}[h, h'] \cong K^*_G \times \cdots \times K^*_G \times K_2\). Observe that the last graph is isomorphic to a complete bipartite graph of order \(2|G|^n\), that is, there are all possible edges between sets \(G^n \times \{h\}\) and \(G^n \times \{h'\}\) in \(\Gamma_{G[H]}\). Since by Lemma 4.2(i) the subgraph induced by \(G^n \times \{h\}\) is isomorphic to \(\Gamma_G \times \cdots \times \Gamma_G\), by the definition of wreath product of graphs, it follows that \(\Gamma_{G[H]} \cong K_n[\Gamma_G \times \Gamma_G \times \cdots \times \Gamma_G]\).

The next proposition gives bounds on the intersection density of the wreath product of groups.

**Proposition 4.4** Let \(G \leq \text{Sym}(V)\) and \(H \leq S_n\) be transitive permutation groups. Then \(\rho(G) \leq \rho(G \wr H) \leq \rho(G)\rho(H)\).

**Proof.** Let \(I_G\) be a maximum intersecting set in \(G\) and let \(I = I_G \times G \times \cdots \times G \times H \subseteq G \wr H\). We claim that \(I\) is an intersecting set in \(G \wr H\). Let \(g = (g_1, \ldots, g_n, h)\) and \(g' = (g_1', \ldots, g_n', h')\) be arbitrary elements of \(I\). Since \(I_G\) is an intersecting set in \(G\) there exists \(v \in V\) such that \(g_1(v) = g'_1(v)\). It follows that \(g(v, 1) = (g_1(v), 1) = (g'_1(v), 1) = g'(v, 1)\). We conclude that \(I\) is indeed an intersecting set in \(G \wr H\).

Observe that \(|I| = |I_G||H_1||G|^{n-1}\) and \(|(G \wr H)_{(v,i)}| = \frac{|G|^n|H|}{|V||n|}\). Using the orbit stabilizer theorem for \(G\) and \(H\) it follows that \(\frac{|I|}{|(G \wr H)_{(v,i)}|} = \frac{|I_G|}{|G_1|} = \rho(G)\). Since \(I\) is an intersecting set in \(G \wr H\) we have \(\rho(G \wr H) \geq \frac{|I|}{|(G \wr H)_{(v,i)}|} = \rho(G)\).

Lemma 4.2 implies that \(\Gamma_H[\Gamma_G \times \cdots \times \Gamma_G]\) is a subgraph of \(\Gamma_{G[H]}\), and so there exists a homomorphism from \(\Gamma_H[\Gamma_G \times \cdots \Gamma_G]\) to \(\Gamma_{G[H]}\). Using Lemmas 2.2, 2.5 and 2.6 it follows that

\[
\frac{|G|^n \cdot |H|}{\alpha(\Gamma_H) \cdot \alpha(\Gamma_G) \cdot |G|^{n-1}} \leq \frac{|G|^n \cdot |H|}{\alpha(\Gamma_G[H])}.
\]

Equation 11 implies that \(\alpha(\Gamma_{G[H]}) \leq \alpha(\Gamma_G)\alpha(\Gamma_H)|G|^{n-1}\). It follows that

\[
\rho(G \wr H) = \frac{\alpha(\Gamma_{G[H]})}{|G_v||G|^{n-1}|H_1|} \leq \frac{\alpha(\Gamma_H)\alpha(\Gamma_G)}{|G_v||H_1|} = \rho(G) \cdot \rho(H).
\]

\]
Corollary 4.5 Let $G \leq \text{Sym}(V)$ be a transitive permutation group, and let $H \leq S_n$ be a transitive permutation group having the EKR-property. Then $\rho(G \wr H) = \rho(G)$.

Determining the exact value of $\rho(G \wr H)$ for all transitive permutation groups $G$ and $H$ is not an easy task. We conjecture the following is true.

Conjecture 4.6 Let $G \leq \text{Sym}(V)$ and $H \leq S_n$ be transitive permutation groups. Then $\rho(G \wr H) = \rho(G)$.

Now we turn our attention to study the following problem.

Problem 4.7 Let $G \leq \text{Sym}(V)$ and $H \leq S_n$. Give necessary and sufficient conditions for $G \wr H$ to have the strict-EKR-property.

If $H$ is regular, then $\Gamma_{G\wr H}$ is isomorphic to $K_n[\Gamma_G \times \cdots \times \Gamma_G]$, where $K_n$ denotes the complete graph on $n$ vertices. Observe that every independent set of $K_n[\Gamma_G \times \cdots \times \Gamma_G]$ is of the form $\{v\} \times I$, where $v$ is a vertex of $K_n$ and $I$ is an independent set of $\Gamma_G \times \cdots \times \Gamma_G$. This shows that in the case when $H$ is regular, Problem 4.7 reduces to the question when does the internal direct product $G \times \cdots \times G$ have the strict-EKR-property, which was answered in Lemma 3.11.

Proposition 4.8 Let $G \leq \text{Sym}(V)$ be transitive with $|V| \geq 3$ and let $H \leq S_n$ be regular. Then $G \wr H$ has the strict-EKR-property if and only if $G$ has the strict-EKR-property and $\Gamma_G$ is IS-primitive.

Corollary 4.9 The group $S_3 \wr S_2$ does not have the strict-EKR-property.

Proof. Note that $S_2$ is regular. Observe that $\Gamma_{S_3}$ is disconnected (a disjoint union of two triangles), hence $\Gamma_{S_3}$ is not IS-primitive. The result now follows from Proposition 4.8.  

The next proposition generalizes [2 Proposition 6.5].

Proposition 4.10 Let $G \leq \text{Sym}(V)$, $|V| \geq 3$, and $H \leq S_n$ be transitive permutation groups. If the internal direct product $G^n$ of $n$ factors of the group $G$ has the strict-EKR-property and $H$ has the EKR-property, then $G \wr H$ has the strict-EKR-property.

Proof. Since $P = G^n$ has the strict-EKR-property, it follows by Lemma 3.11 that $G$ has the strict-EKR-property. Let $S$ be an intersecting set of $G \wr H$ of maximum size, that is $|S| = |G|^{n-1} |G_v||H_1|$. Without loss of generality we may assume that $S$ contains the identity of $G \wr H$. Since $\Gamma_{G\wr H}$ contains a subgraph isomorphic to $\Gamma_H[\Gamma_P]$ it follows that $S$ is also an independent set in $\Gamma_H[\Gamma_P]$. Observe that $\alpha(\Gamma_H[\Gamma_P]) = \alpha(\Gamma_H)\alpha(\Gamma_P) = |S|$. Hence $S$ is an intersecting set of $\Gamma_H[\Gamma_P]$ of maximum size.

For $h \in H$ let $S_h = S \cap (G^n \times \{h\})$. Then for every $h \in H$, we have $S_h = \emptyset$ or $S_h$ is a maximum intersecting set in $P$. Since $P$ has the strict-EKR-property, it follows that $S_h \neq \emptyset$ implies that $S_h$ is a coset of a point stabilizer in $P$. Since $S$ contains the identity of $G \wr H$, it follows that $S_h = P(v(h),i(h))$ for some $v(h) \in V$ and some $i(h) \in \{1,\ldots,n\}$. Let $h$ and $h'$ be two distinct elements of $H$ such that $S_h$ and $S_{h'}$ are non-empty. It follows that $h'h^{-1}$ is not a derangement.

Suppose that $i(h) \neq i(h')$. Let $g \in G$ be a derangement. Consider elements $((g_1,\ldots,g_n),h)$ and $((g'_1,\ldots,g'_n),h')$ in $G \wr H$ where $g_j = g$ for $j \neq i(h)$ and $g_{i(h)} = 1$, while $g'_j = 1$ for $j \neq i(h)$.
and \(g_j i(h) = g\). Observe that \(g_j g_j^{-1}\) is a derangement for each \(j \in \{1, \ldots, n\}\) hence \((g_1, \ldots, g_n), h)\) and \((g_1', \ldots, g_n'), h')\) are adjacent in \(\Gamma_{G|H}\) which contradicts the fact that they are both contained in \(S\). Consequently, \(i(h) = i(h')\).

To simplify the notation, let \(i(h) = i(h') = k\). Suppose that \(P_{v(h), k} \neq P_{v(h'), k}\), that is, \(G_{v(h)} \neq G_{v(h')}\). Choose \(f \in G_{v(h)}\) and \(f' \in G_{v(h')}\) in such a way that \(f' f^{-1}\) is a derangement. Such elements always exist for otherwise \(G_{v(h)} \cup G_{v(h')}\) would be an independent set in \(\Gamma_G\) of size greater than the size of a point stabilizer. Consider elements \((g_1, \ldots, g_n), h\) and \((g_1', \ldots, g_n'), h')\) in \(G \wr H\) where \(g_j = g\) for \(j \neq k\) and \(g_k = f\), while \(g_j' = 1\) for \(i \neq k\) and \(g_k' = f'\). As in the previous paragraph it follows that \((g_1, \ldots, g_n), h)\) and \((g_1', \ldots, g_n'), h')\) are adjacent in \(\Gamma_{G|H}\), contradicting the fact that they are both contained in \(S\), an independent set in \(\Gamma_{G|H}\). This shows that \(P_{v(h), k} = P_{v(h'), k}\). Let \(w = v(h)\). It follows that \(S = P_{(w, k)} \times S_H\) where \(S_H\) is a maximum intersecting set in \(H\). Since \(H\) has the EKR-property, it follows that \(|S_H| = |H_k|\). Since \(S\) contains the identity of \(G \wr H\), it follows that \(1_H \in S_H\).

Suppose that \(S_H \neq H_k\). Then there exist \(h, h' \in S_H\) such that \(h(k) \neq h'(k)\). Namely, if that was not the case we would have that \(h^{-1} h'(k) = k\) for every \(h, h' \in S_H\). Then by taking \(h' = 1_H\) we would have \(h(k) = k\) for every \(h \in H_j\) implying that \(S_H = H_k\), contrary to the assumption that \(S_H \neq H_k\). Let us now consider elements \((g_1, \ldots, g_n), h)\) and \((g_1', \ldots, g_n'), h')\) in \(G \wr H\) where \(g_j = g\) for \(j \neq k\) and \(g_k = 1\), while \(g_j' = 1\) for all \(i \in \{1, \ldots, n\}\). As in the previous paragraph it follows that \((g_1, \ldots, g_n), h)\) and \((g_1', \ldots, g_n'), h')\) are adjacent in \(\Gamma_{G|H}\), contradicting the fact that they both belong to \(S\). This shows that \(S_H = H_k\), and so \(S\) is the stabilizer of the point \((w, k)\), completing the proof of Proposition 4.10.

**Remark 4.11** [2] Proposition 6.5], which claims that groups \(S_m \wr S_n\) \((m, n \geq 1)\) have the strict-EKR-property, is wrong, as Corollary 4.9 shows that \(S_3 \wr S_2\) does not have the strict-EKR-property. The proof of [2] Proposition 6.5] uses the result that internal direct product \(S_m \times S_m \times \cdots \times S_m\) has the strict-EKR-property. However, this is not true for \(m \in \{2, 3\}\) hence the proof of [2] Proposition 6.5] is correct only for \(m \geq 4\).

We now investigate when is it that the group \(S_m \wr S_n\) has the strict-EKR-property. As explained in Remark 4.11, one only needs to consider the cases \(m \in \{2, 3\}\). We will need the following two lemmas. Observe that \(S_n\) has the strict-EKR-property and admits an element fixing exactly one point.

**Lemma 4.12** Let \(H \leq S_n\) be a transitive permutation group having the strict-EKR-property admitting an element fixing exactly one point. Then the group \(S_2 \wr H\) has the strict-EKR-property.

**Proof.** Since \(H\) has the strict-EKR-property, it follows from Corollary 4.5 that \(S_2 \wr H\) has the EKR-property. Let \(I\) be a maximum intersecting set in \(S_2 \wr H\) containing the identity. As \(S_2 \wr H\) has the EKR-property, we get that \(|I|\) is equal to the order of the stabilizer of \((1, 1)\) in \(S_2 \wr H\), which is further equal to \(2^{n-1}|H|/n\) by the Orbit-stabilizer lemma. For \(h \in H\) let \(I_h = I \cap (S_2^n \times \{h\})\) and observe that \(|I_h| \leq 2^{n-1}\). Let \(I_H = \{h \in H \mid I_h \neq \emptyset\}\) and observe that \(I_H\) is an intersecting set in \(H\). As \(H\) has the strict-EKR-property, we have that \(|I_H| \leq |H_1| = |H|/n\) by the Orbit-stabilizer lemma. Therefore, we have that
\[
\frac{2^{n-1}|H|}{n} = |I| = \sum_{h \in I_H} |I_h| \leq |I_H|2^{n-1} \leq \frac{2^{n-1}|H|}{n}.
\]
This shows that $|I_h| = 2^{n-1}$ for each $h \in I_H$ and that $|I_H| = |H|/n$, and so $I_H$ is an intersecting set in $H$ of maximum size. Hence $I_H$ is a stabilizer of a point. Without loss of generality, we assume that $I_H$ is the stabilizer of 1 in $H$.

Let $d \in I_H$ be such that $fix(d) = \{1\}$ (such an element exists by the assumption on $H$). Since the identity is contained in the intersecting set $I$, and $|I_d| = 2^{n-1}$, it follows that $I_d = (\{1\} \times S_2 \times \cdots \times S_2) \times \{d\}$.

Suppose now that there exists $g = ((g_1, \ldots, g_n), h) \in I$ with $g_1 \neq 1$. Let $g' = ((g_2', \ldots, g_n'), d)$ where $g_i' = g_i t$ with $t = (12) \in S_2$. Observe that $g$ and $g'$ both belong to $I$, which contradicts the fact that they are adjacent in $\Gamma_{S_3 \wr H}$. This shows that for each $h \in I_H$ we have that $I_h = (\{1\} \times S_2 \times \cdots \times S_2) \times \{h\}$ and we conclude that $I = (\{1\} \times S_2 \times \cdots \times S_2) \times I_H$. Hence $I$ is the stabilizer of the point $(1, 1)$.

\textbf{Lemma 4.13} Let $H \leq S_n$ be a transitive permutation group having the strict-EKR-property admitting an element fixing exactly one point, and let $n \geq 3$. Then the group $S_3 \wr H$ has the strict-EKR-property.

\textbf{Proof.} Since $H$ has the strict-EKR-property, it follows from Corollary 4.9 that $S_3 \wr H$ has the EKR-property. Let $I$ be a maximum intersecting set in $S_3 \wr H$ containing the identity. For $h \in H$ let $I_h = I \cap (S_3^n \times \{h\})$ and let $I_H = \{h \in H \mid I_h \neq \emptyset\}$. Similarly as in the proof of Lemma 4.12 we get that $|I_h| = 2 \cdot 6^{n-1}$ for every $h \in I_H$ and that $I_H$ is a stabilizer of a point. Without loss of generality, we may assume that $I_H$ is the stabilizer of 1 in $H$.

By the assumption on $H$ there exists $d \in I_H$ such that $fix(d) = \{1\}$. It follows that for every $(g_1, \ldots, g_n, d) \in I$ we have that $g_1$ fixes a point. Moreover, if $g_1 \neq 1$, then every element in $I_d$ has the first coordinate equal to 1 or $g_1$, as otherwise we would obtain two adjacent elements of $I_d$. Observe that not every element in $I_d$ has the first coordinate equal to 1, as then we would have $|I_d| \leq 6^{n-1}$. Let $f \in S_3 \setminus \{1\}$ be a non-derangement that appears as the first coordinate of some element from $I_d$. Then $I_d = (\{1, f\} \times S_3 \times \cdots \times S_3) \times \{d\}$.

Suppose that there exists $g = ((g_1, \ldots, g_n), h) \in I$ with $g_1 \not\in \{1, f\}$. Further, let $g' = ((g_2', \ldots, g_n'), d) \in I_d$ where $g_i' = 1$ if $g_i \in \{(123), (132)\}$ and $g_i' = f$ otherwise, while $g_1' = g_1(123)$ for $i \neq 1$. Observe that $g$ and $g'$ both belong to $I$, contradicting the fact that they are adjacent in $\Gamma_{S_3 \wr H}$. This shows that for each $h \in I_H$ we have $I_h = (\{1, f\} \times S_3 \times \cdots \times S_3) \times \{h\}$. We conclude that $I = (\{1, f\} \times S_3 \times \cdots \times S_3) \times I_H$. Hence $I$ is a stabilizer of the point $(x, 1)$, where $fix(f) = \{x\}$.\[ 

\textbf{Proposition 4.14} Let $m$ and $n$ be positive integers. The group $S_m \wr S_n$ has the strict-EKR-property if and only if $(m, n) \neq (3, 2)$.

\textbf{Proof.} The proof follows combining Proposition 4.10, Corollary 4.9 and Lemmas 4.12 and 4.13.

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