Total order compatible with addition on commutative semigroups

Askold Khovanskii *

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Abstract

In the paper we present a detailed exposition of mainly known results (for example, see [1]). We describe all total orders $\succ$ compatible with addition on additive subsemigroup $S$ of finite dimensional spaces over rational numbers. We provide a necessary and sufficient condition under which a finitely generated semigroups $S$ equipped with an order $\succ$ is a well-ordered set. We also present some auxiliary results on orders compatible with addition on additive subsemigroups of finite dimensional spaces over real numbers.

All arguments in this paper are based on two simple theorems in the geometry of convex (not necessarily closed) sets. Proofs of these theorems are presented for readers’s convenience.

A first version of this paper was written as a handout for my graduate course on the theory of Newton–Okounkov bodies.

1 Introduction

A total order $\succ$ on a commutative semigroup $S$ is compatible with addition if, for any triple $x, y, a \in S$ such that $x \succ y$, the inequality $x + a \succ y + a$ holds.

We are interested in all such orders on subsemigroups of the $n$-dimensional lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. We will completely describe such orders on additive subsemigroups of the $n$-dimensional space $\mathbb{Q}^n$ over the field of rational numbers $\mathbb{Q}$.

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We are also interested in all such orders on the semigroup $\mathbb{Z}_{\geq 0}^n \subset \mathbb{Z}^n$ (consisting of all integral points in $\mathbb{R}^n$ with nonnegative coordinates), which make $\mathbb{Z}_{\geq 0}^n$ a well-ordered set. We will completely describe all such orders on any finitely generated subsemigroup of the additive group of the space $\mathbb{Q}^n$.

Our arguments use geometry of convex subsets (not necessarily closed or bounded) in real affine spaces. We use the classical Caratheodory Theorem, which describes the convex hull $\Delta(A)$ (the smallest convex set containing $A$) of a set $A \subset \mathbb{R}^n$. We also use a version of the Separation Theorem which holds for any convex set $\Delta \subset \mathbb{R}^n$ (not necessarily closed or bounded) and for any point $a$ in its complement $a \in \mathbb{R}^n \setminus \Delta$. For readers’s convenience, we present proofs of both these theorems in convex geometry.

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2 Order compatible with addition on general commutative semigroups

We are mainly interested in the class of additive subsemigroups of real vector spaces.

The following (obvious) Lemma on general commutative semigroups automatically holds for semigroups belonging to this class.

**Lemma 1.** If commutative semigroup $S$ has a total order compatible with addition, then $S$ has a cancelation property; and an identity $nx = ny$ implies $x = y$, where $x, y \in S$ and $n$ is a natural number.

**Proof.** If $x \prec y$ or $y \succ x$, then, for any $a \in S$, we correspondingly have that $x + a \succ y + a$ or $y + a \succ x + a$. So, if $x + a = y + a$, then $x = y$. Thus, the semigroup $S$ has the cancelation property.

If $x \succ y$ or $y \succ x$, then we correspondingly have that $nx \succ ny$ or $ny \succ nx$. So, if $nx = ny$, then $x = y$. \qed

**Corollary 2.** If $S$ satisfies assumption of Lemma 1, then $S$ can be naturally embedded to its Grothendieck group $G$; and the group $G$ is a free commutative group (i.e. $G$ has no torsion).

**Corollary 3.** Any total order compatible with addition on a commutative semigroup $S$ can be uniquely extended to the total order compatible with addition on the Grothendieck group $G$ of the semigroup $S$. 

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Proof. Any elements $a_1, a_2 \in G$ can be represented in the form $x_1 - y_1 = a_1, x_2 - y_2 = a_2$. We say that $a_1$ is bigger (or correspondingly, smaller) than $a_2$ if $x_1 + y_2$ is bigger (or correspondingly, smaller) than $x_2 + y_1$. The above order on $G$ is well defined (i.e. is independent of representations of $a_1, a_2$ as the difference of elements from $S$) and is the only possible extension of the order on $S$ to an order on $G$.

Thus, a description of all total orders compatible with addition on a commutative semigroup $S$ is reduced to a description of all total orders compatible with addition on its Grothendieck group $G$.

On any free commutative group $G$ there is a total order compatible with addition. We will construct such order later (see Lemma 9), when we will discuss lexicographic orders on real vector spaces. So, the conditions on semigroup $S$ from Lemma 1 are not only necessary but also sufficient for existence of a total order on $S$ compatible with addition.

One can easily check the following two lemmas.

**Lemma 4.** For any total order $\succ$ compatible with addition on a commutative group $G$ the set $G_+ \subset G$ defined by condition $x \in G_+ \iff x \succ 0$ has the following properties:

1. the set $G_+$ is a semigroup with respect to addition;
2. zero is not in $G_+$;
3. for any $x \neq 0$ exactly one element from the couple $(x, -x)$ belongs to $G_+$.

**Lemma 5.** If a subset $G_+ \subset G$ satisfies the conditions 1)–3) from the previous lemma then the relation $x \succ y \iff x - y \in G_+$ defines a total order on the group $G$ compatible with addition.

Let us prove a simple general lemma on well-ordered commutative semigroups, assuming that the ordering is compatible with addition.

**Lemma 6.** If a commutative semigroup $S$ equipped with a total order $\succ$ compatible with addition is a well-ordered set, then, for any nonzero element $a \in S$, the condition $2a \succ a$ holds. (If $S$ contains the origin, then this condition means that the origin is the smallest element in $S$.)

Proof. If for some $a \in S$ the condition $a \succ 2a$ holds, then the sequence $a, 2a, \ldots, na, \ldots$ is strictly decreasing and does not contain a smallest element.
3 Lexicographic orders

Lexicographic orders on finite dimensional real vector spaces are very important for us. Let us start with a formal definition, which works even for infinite dimensional real vector spaces.

Consider a real vector space $L$. Let $\{e_\lambda\}, \lambda \in \Lambda$ be any basis in $L$, where $\Lambda$ is an index set. Choose any well-order on the set $\Lambda$.

Remark 1. Note that there are well-orders on any set $\Lambda$. If the set $\Lambda$ is infinite such order could be very exotic; but, on set containing $n < \infty$ elements, all orders are in one-to-one correspondence with all enumerations of elements in the set $\Lambda$ by indices $1 \leq i \leq n$.

Using the chosen well-order on $\Lambda$, one can define a total order on $L$ compatible with addition. Each vector $v \in L$ has a unique representation of the form $v = \sum x_\lambda(v)e_\lambda$, where only finitely many coefficients $x_\lambda(v)$ are not equal to zero.

Definition 1. Let $a = \sum x_\lambda(a)e_\lambda$ and $b = \sum x_\lambda(b)e_\lambda$ be two vectors in the space $L$. Let $\Lambda_{a,b} \subset \Lambda$ be the set of indices such that $x_\lambda(a) \neq x_\lambda(b)$. Denote by $\lambda_0$ the smallest element in $\Lambda_{a,b}$. We say that $a$ is bigger than $b$ in the lexicographic order associated with the well-ordered basis $\{e_\lambda\}$ of $L$ if $x_{\lambda_0}(a) > x_{\lambda_0}(b)$.

Definition 2. An order $> \!$ on a real vector space $L$ is compatible with multiplication on positive numbers if, for any $x, y \in L$ and any $\mu > 0$ such that $x > y$, the relation $\mu x > \mu y$ holds.

The following Lemma is obvious:

Lemma 7. Any lexicographic order on a real vector space $L$ is compatible with addition and with multiplication by positive numbers.

Lemma 8. For any total order $> \!$ compatible with addition and multiplication by positive numbers on a real vector space $L$ the set $L_+$, defined by the condition $x \in L_+ \iff x > 0$, is convex. In particular this condition holds for any lexicographic order on $L$. 

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Proof. Since the order $\succ$ is compatible with addition and with multiplication by positive numbers, for any two points $x, y \in L_+$ and real number $0 \leq \lambda \leq 1$, the set $L_+$ contains the point $\lambda x + (1 - \lambda y)$.

Lemma 9. On any free commutative group $G$, there is a total order compatible with addition.

Proof. Any free commutative group $G$ can be naturally embedded in the real vector space $L = G \otimes \mathbb{Z} \mathbb{R}$. The lexicographic order on $L$ induces a total order compatible with addition on any subgroup of $L$.

Thus we see that a total order compatible with addition on a commutative semigroup $S$ exists if and only if $S$ satisfies the assumptions of Lemma 1.

From now on, we will deal only with the lexicographic order on real finite-dimensional vector spaces.

Definition 3. We will call the lexicographic order associated with the ordered basis of an $n$-dimensional real vector space $L$ the lexicographic order related to the coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$, defined by the basis $e_1, \ldots, e_n$. We will denote this order by the symbol $\succ_{\mathbf{x}}$.

Let us discuss the geometrical meaning of the order $\succ_{\mathbf{x}}$ on $L$ related to a coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 4. With the coordinate system $\mathbf{x}$ one associates a flag of subspaces $L = L_0 \supset L_1 \supset \cdots \supset L_n = 0$, where $L_1$ is defined by the equation $x_1 = 0$; $L_2$ by the equations $x_1 = x_2 = 0$; so on, up to $L_n$, defined by the equations $x_1 = \cdots = x_n = 0$ (i.e. $L_n = 0$).

With the coordinate system $\mathbf{x}$, one associates the collection of open half spaces $L_i^+ \subset L_i$ of the space $L_i$; with the boundary $L_{i+1}$ specified by the condition that $x_{i+1} > 0$ on $L_i^+$.

The flag $L = L_0 \supset L_1 \supset \cdots \supset L_n = 0$, together with the collection of chosen half spaces $L_0^+, \ldots, L_{n-1}^+$, totally determines the lexicographic order on $L$.

Definition 5. The set $X_+ = \cup_{0 \leq i < n} L_i^+$ we will call the $\mathbf{x}$-half space related to the coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$.
The following two Lemmas are obvious.

**Lemma 10.** The $x$-half space $X_+$ of $L$ totally determines the lexicographic order on $L$ related to the coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$. Moreover the identity $X_+ = L_+(\mathbf{x})$ holds, where $L_+(\mathbf{x})$ is the set of points $a \in L$ such that $a \succ_x 0$.

**Lemma 11.** The orders $\succ_x$ and $\succ_y$ related to coordinate systems $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ on $L$ coincide if and only if the linear map $A : L \to L$ which transforms the coordinate system $\mathbf{x}$ to the coordinate system $\mathbf{y}$ is given by an upper triangular matrix having positive entries on its main diagonal.

For any coordinate system $\mathbf{x}$ on $L$ let $L_-(\mathbf{x})$ be the set defined by the following condition: $x \in L_-(\mathbf{x}) \iff -x \in L_+(\mathbf{x})$ (where $L_+(\mathbf{x})$ is the $x$-half space of $L$).

**Lemma 12.** For any coordinate system $\mathbf{x}$ on $L$ the sets $L_+(\mathbf{x})$ and $L_-(\mathbf{x})$ are convex. These sets satisfy the following conditions:

\[
L_+(\mathbf{x}) \cup L_-(\mathbf{x}) = L \setminus \{0\},
\]

\[
L_+(\mathbf{x}) \cap L_-(\mathbf{x}) = \emptyset.
\]

Let us formulate two Theorems which geometrically characterize lexicographic orders on $L$ without using coordinate systems. We will prove these theorems in the section.

**Theorem 13.** Let $\succ$ be a total order on a real finite-dimensional vector space $L$ that is compatible with addition and with multiplication by positive numbers. Then $\succ$ is the lexicographic order $\succ_x$ related to some coordinate system $\mathbf{x}$ on $L$.

**Remark 2.** On a real vector space $L$ of any dimension $n > 0$ there are a many total orders compatible with addition which are not lexicographic orders related to some coordinate system (over real numbers). Indeed, one can consider $L$ as a free commutative group with respect to addition and use an (exotic) order on it such as that described in Lemma.
Theorem 14. Let $X_+ \subset L$ be a convex set and let $X_-$ be the set defined by condition $x \in X_- \iff -x \in L_+$. Assume that the following conditions hold:

\[ X_+ \cup X_- = L \setminus \{0\}, \]
\[ X_+ \cap X_- = \emptyset. \]

Then there is a (unique) coordinate system $x$ on $L$, such that $X_+ = L_+(x)$, where $L_+(x)$ is the $x$-half space of $L$.

4 Orders on subgroups and subsemigroups of real numbers

First, we will consider results for $n = 1$. Let $G \subset \mathbb{R}$ be an additive subgroup of $\mathbb{R}$ equipped with some total order $\succ$ compatible with addition. As above, we denote by $G_+$ a semigroup $G_+ \subset G$ containing all points $a \in G$ such that $a \succ 0$.

Lemma 15. If the convex hull $\Delta(G_+)$ of the set $G_+$ does not contain the origin, then the order $\succ$ is either induced by the natural order on the line of real numbers, or by the opposite order, i.e. $a \succ b$ if and only if $a < b$.

Proof. The semigroup $G_+$ can contain only positive points, or only negative points. Indeed, if $a > 0$ and $b < 0$ are contained in $G_+$ then $\Delta(G_+)$ contains the origin, which is not possible. If $G_+$ contains only positive points, then the intersection of the open ray $x > 0$ with $G$ is equal to $G_+$. Indeed, assume that a point $a \in G$ does not belong to $G_+$, but belongs to the ray. Then, the negative point $-a$ must belong to $G_+$, which contradicts our assumption. Thus, $a \succ b$ if and only if $a - b > 0$.

If $G_+$ belongs to the negative ray $x < 0$, then similar arguments show that $a \succ b$ if and only if $a - b < 0$. \hfill \Box

Lemma 16. If a semigroup $S \subset \mathbb{Q} \subset \mathbb{R}$ contains only rational points, then the convex hull $\Delta(S)$ contains the origin if and only if the origin belongs to $S$.

Proof. If $0 \in \Delta(S)$, then $S$ contains some positive point $\lambda > 0$ and some negative point $\mu < 0$. The ratio $\frac{\lambda}{\mu}$ is a negative rational number, so $\frac{\lambda}{\mu} = -\frac{p}{q}$ where $p$ and $q$ are natural numbers. We have that $q\lambda + \mu p = 0$, which means that the semigroup $S$ with the points $\lambda, \mu$ contains the origin. \hfill \Box
Theorem 17. Let $G \subset \mathbb{Q} \subset \mathbb{R}$ be an additive group which contains only points with rational coordinates. Then there are exactly two total orders on $G$ compatible with addition: the natural order ($a \succ b$ if $a > b$) and the reverse order ($a \succ b$ if $a < b$).

Proof. The semigroup $G_+ \subset G$ related to the order $\succ$ cannot contain the origin. Thus, by Lemma 16 $\Delta(G)$ does not contain the origin either. Thus, the required statement follows from Lemma 15.

In the section 6 we generalize Theorem to the multidimensional case. The multidimensional statement analogous to Lemma 15 is based on a version of the Separation Theorem for convex sets (see Theorem 21). The multidimensional statement analogous to Lemma 16 is based on Caratheodory’s theorem (see Theorem 24).

Lemma 18. Let $S \subset \mathbb{R}$ be a finitely generated semigroup which contains only nonnegative numbers. Then, $S$ with the natural order induced by $\mathbb{R}$ is a well-ordered set. Moreover, for any $l \in \mathbb{R}$, there are only finitely many elements in $S$ which are smaller than $l$.

Proof. Indeed, assume that $C$ is the smallest nonzero number among generators of $S$. Then, on any segment $0 \leq x \leq l$, there are at most $\frac{l}{C} + 1$ elements of the semigroup $S$. So, any subset of $S$ contains a smallest element.

Theorem 19. A finitely generated semigroup $\subset \mathbb{R}$ which contains only rational points, i.e. $S \subset \mathbb{Q}$; and, which as a total order $\succ$ compatible with addition; is a well-ordered set if and only, for every nonzero element $a \in S$, the inequality $2a \succ a$ holds.

Proof. In one direction, the Theorem follows from Lemma 6.

Let us prove it in the opposite direction.

Since $S$ contains rational only points and the order $\succ$ is compatible with addition, the order $\succ$ is induced either by the natural order on $\mathbb{R}$, or by the opposite order on $\mathbb{R}$. By the condition in the Theorem, either $S$ belongs to the ray of nonnegative numbers (if $\succ$ is induced from the natural order); or $S$ belongs to the ray of nonpositive numbers (if $\succ$ is induced from the opposite order). In both cases, Lemma 18 implies that $S$ is a well-ordered set.
5 Two theorems in the geometry of convex sets

In this section, we discuss two theorems from convex geometry which we will use later.

5.1 Version of the Separation Theorem for non-necessarily closed convex sets

Let us recall the classical Separation Theorem for closed convex sets in a real finite dimensional space $L$ (for example, see [2]).

**Theorem 20 (Separation Theorem).** For any closed convex set $\Delta \subset L$ and for any point $a \in L$ not belonging to $\Delta$, there is a linear function $x : L \to \mathbb{R}$ such that for any point $b \in \Delta$ the inequality $x(a) < x(b)$ holds.

**Proof.** Choose any Euclidean metric on $L$. Denote by $\rho(v_1, v_2)$ the distance between points $v_1, v_2 \in L$. Denote by $f : L \to \mathbb{R}$ the function whose value at a point $y \in L$ is equal to $\rho(a, y)$. The function $F$ is smooth on $L \setminus \{a\}$ and it tends to infinity as $y$ tends to infinity.

Since $\Delta$ is closed, the function $f$ attains its minimum on $\Delta$ at some point $b \in \Delta$.

Let $x$ be the linear function on $L$ defined by relation $x(y) = \langle y, b - a \rangle$, where $\langle v_1, v_2 \rangle$ is the inner product of the vectors $v_1, v_2 \in L$.

The gradient $\nabla F_b$ of the function $F$ at the point $b$ is equal to $\frac{b - a}{|b - a|}$. For any point $y \in \Delta$, the segment joining $b$ and $y$ belongs to $\Delta$. Since $f$ attains its minimum on $\Delta$ at the point $b$, the inner product $\langle \nabla F_b, c - b \rangle$ is a nonnegative number.

The inequality $\langle \nabla F_b, c - b \rangle \geq 0$ means that the set $\Delta$ belongs to the closed half space where the function $x$ is bigger than or equal to $x(b)$; while the point $a$ is located in the open half space where $x$ is smaller than $x(b)$. \(\square\)

With any system of coordinates $x = (x_1, \ldots, x_n)$ on $L$ one associates the lexicographic order $\succ_x$ on $L$. 
Definition 6. For any point \( a \in L \), denote by \( L_+(a, x) \) the set of points \( y \in L \) satisfying the inequality \( y \succ_x a \). We will call the set \( L_a(x) \) the \( x \)-half space with the vertex \( a \).

The set \( L_+(a, x) \) is equal to the \( x \)-half space \( L_+(x) \) shifted by vector \( a \).

Theorem 21 (Version of the Separation Theorem, see see [1]). Let \( \Delta \subset L \) be a (not-necessarily closed) convex set, and let \( a \in L \setminus \Delta \) be a point not belonging to \( \Delta \). Then, there is a coordinate system \( x = (x_1, \ldots, x_n) \) in \( L \) such that \( \Delta \) belongs to the \( x \)-half space with vertex \( a \), i.e. \( \Delta \subset L_+(a, x) \).

Proof. Let \( \overline{\Delta} \) be the closure of \( \Delta \). If \( a \) does not belong to \( \overline{\Delta} \), then by the Separation Theorem for closed convex sets, there is a linear function \( x : L \to \mathbb{R} \) such that, for any \( b \in \overline{\Delta} \), the inequality \( x(a) < x(b) \) holds. Let us choose an arbitrary system of coordinates \( x_1 = (x_1, x_2, \ldots, x_n) \), with \( x_1 = x \). Then,

\[
L_+(a, x) \supset \Delta.
\]

So, for the case under consideration the Theorem is proven.

Now, assume that \( a \in \overline{\Delta} \). Since \( a \) is not in \( \Delta \), there is a support hyperplane \( LH \) for \( \overline{\Delta} \) at the point \( a \). In relation to \( H \), one can consider a linear function \( x : L \to \mathbb{R} \) such that, for any \( b \in \overline{\Delta} \), the inequality \( x(a) < x(b) \) holds. Let us choose an arbitrary system of coordinates \( x = (x_1, x_2, \ldots, x_n) \), with \( x_1 = x \). Then,

\[
L_+(a, x) \supset \Delta.
\]

Consider a convex set \( \Delta_1 = H \cap \Delta \) in the affine space \( H \). If \( 0 \in H \), then \( H \) is a linear space of dimension \( (n - 1) \).

If \( H \) does not contain the origin, choose any point \( O_1 \in H \), and consider \( H \) a linear space with origin \( O_1 \).

Note that any linear function \( \tilde{x} \) which is defined on \( H \) can be extended to a linear function \( x \) on \( L \) (if \( 0 \in H \) there is a one parameter family of such extensions; if the origin is not in \( H \) such an extension is unique).

By induction, we can assume that the Theorem is proven for all \( (n - 1) \)-dimensional spaces. So, there is a coordinate system \( \tilde{x} = (\tilde{x}_2, \ldots, \tilde{x}_n) \) on \( H \) such that the set \( H_+(a, \tilde{x}) \subset H \) contains the set \( \Delta_1 \).

To complete the proof, one can extend the functions \( \tilde{x}_2, \ldots, \tilde{x}_n \) (which are defined on \( H \)) to linear functions \( x_2, \ldots, x_n \) on \( L \). Consider the coordinate system \( x = x_1, x_2, \ldots, x_n \), with \( x_1 = x \), where the function \( x \) is defined in the first step of our inductive proof.
It is easy to check that for the coordinate system \( x = x_1, x_2, \ldots, x_n \) the \( x \)-half space \( L_+(a, x) \) with vertex \( a \) contains the set \( \Delta \).

Now, we are ready to prove Theorem 14.

**Proof of Theorem 14.** By assumption the origin does not belong to the convex set set \( L_+ \). So, by our version of Separation Theorem, there is a coordinate system \( x \) on \( L_+ \) such that the \( x \)-half space \( L_+(a, x) \) contains the set \( L_+ \). These assumptions imply that the sets \( L_+(a, x) \) and \( L_+ \) are equal. Indeed, if there is a point \( x \in L_+(a, x) \) which is not in \( L_+ \), then \( x \) must belong to the set \( -L_- \). This means that \( -x \in L_+ \). We obtain a contradiction since the point \( -x \) is not in set \( L_+(a, x) \). This contradiction proves the Theorem.

The version of Separation Theorem implies the following corollary:

**Corollary 22.** A subset \( \Delta \) in a real \( n \)-dimensional space \( L \) is convex, if and only if it is equal to the intersection (over all choices of coordinate systems \( x \) and points \( a \in L \)) of all sets \( L_+(a, x) \) containing \( \Delta \).

**Proof.** All sets \( L_+(a, x) \) are convex. For any point \( a \) not in \( \Delta \) there is a set \( L_+(a, x) \) which contains \( \delta \).

5.2 Convex geometry related to Caratheodory’s theorem

Let \( L \) be any real vector space (perhaps of infinite dimension). For a set \( A \subset L \) let us denote by \( \Delta(A) \) the convex hull of \( A \) (which is not necessarily is closed).

**Lemma 23.** A point \( x \in L \) belongs to the convex hull \( \Delta(A) \) of a set \( A \subset L \) if and only if \( x \) belongs to the convex hull of some finite subset \( B \) of the set \( A \).

**Proof.** On the one hand the convex set \( \Delta(A) \) must contain the convex hull of each finite set \( B \subset A \). On the other hand if \( x_1, x_2 \) belong to the convex hulls of finite sets \( B_1, B_2 \), then the segment joining \( x_1 \) and \( x_2 \) belongs to the convex hull of the finite set \( B_1 \cup B_2 \).

**Definition 7.** A set \( B \subset L \) containing \( k + 1 \)-points is affinely independent if it does not belong to any affine subspace \( L_B \subset L \), with \( \dim L_B < k \).
Caratheodory’s Theorem (for example, see [2]) improves Lemma 23.

**Theorem 24 (Caratheodory Theorem).** A point \( x \in L \) belongs to the convex hull \( \Delta(A) \) of a set \( A \subset L \) if and only if \( x \) belongs to the convex hull of some finite subset \( B \subset A \) which is affinely independent. Any point \( x \) in the smallest affine space containing the set \( B \) has a unique representation of the form \( x = \sum \lambda_i b_i \), where \( b_i \) are points of the set \( B \) and \( \lambda \) are real numbers such that \( \sum \lambda_i = 1 \).

Caratheodory’s Theorem follows from the geometric Lemma 25 (see below).

Let \( L_1 \) and \( L_2 \) be affine spaces of dimensions \( n \) and \( k \), respectively. Let \( \Delta \subset L_1 \) be a convex \( n \)-dimensional polyhedron.

**Lemma 25.** Let \( A : L_1 \to L_2 \) be an affine map. Then the image \( A(\Delta) \subset L_2 \) is equal to the union \( \bigcup A(\Gamma_i) \) of the images \( A(\Gamma_i) \) of all faces \( \Gamma_i \) of \( \Delta \) such that \( \dim \Gamma_i \leq k \).

**Proof.** Let \( a \in A(\Delta) \) be any point in the image of \( \Delta \). Its preimage \( L_3 = A^{-1}(a) \subset L_1 \) is an \((n-k)\)-dimensional affine subspace of \( L_1 \) which intersects the polyhedron \( \Delta \). Let \( \Gamma \) the lowest dimensional face of \( \Delta \) which has non empty intersection with \( L_3 \). Thus, \( L_3 \) cannot intersect the boundary of \( \Gamma \). This condition implies that \( L_3 \cap \Gamma \) is an interior point of \( \Gamma \). Thus \( \dim \Gamma \leq k \).

We will need one more observation from linear algebra.

Let \( \mathbb{R}^{k+1} \) be the standard linear space with the standard basis \( e_1, \ldots, e_{k+1} \) and standard the coordinates \( \lambda_1, \ldots, \lambda_{k+1} \).

Let \( \mathcal{L}_k \) be the \( k \)-dimensional hyperplane in \( \mathbb{R}^{k+1} \) defined by the equation

\[
\lambda_1 + \cdots + \lambda_{k+1} = 1.
\]

Each point \( x \) in \( \mathcal{L}_k \) has a unique representation of the form \( \sum \lambda_i e_i \) where \( \sum \lambda_1 = 1 \).

Let \( \Delta_k \) be the standard simplex in \( \mathcal{L}_k \subset \mathbb{R}^{k+1} \), defined by the inequalities \( \lambda_1 \geq 0, \ldots, \lambda_{k+1} \geq 0 \). The vertices of the polyhedron \( \Delta_k \) are the endpoints \( P_1, \ldots, P_{k+1} \) of the vectors \( e_1, \ldots, e_{k+1} \).
Proof of Caratheodory’s Theorem. For a point \( x \in \Delta(A) \), choose a set \( B \subset A \) having the smallest number \( k + 1 \) of elements \( b_1, \ldots, b_{k+1} \), such that \( x \) belongs to the convex hull \( \Delta(b) \) of \( B \). Let \( L_B \) be the the smallest affine space containing \( \Delta(B) \); so \( \dim \Delta(B) = \dim L_B \leq k \).

Let us show that \( \dim L_B = k \). Consider an affine map \( A : \mathcal{L}_k \to L_B \) which maps the vertices \( p_1, \ldots, p_{k+1} \) to the points \( b_1, \ldots, b_{k+1} \) (thus \( A(p_i) = b_i \)).

The image \( A(\Delta_k) \) is the union of images \( A(\Gamma_i) \) of the faces \( \Gamma_i \) of \( k \)-dimensional simplex \( \Delta_k \), such that \( \dim \Gamma_i = \dim L_B \). Since \( x \in A(\Delta_k) \) and \( x \) does not belong to an image of a proper face \( \Gamma_i \subset \Delta_k \) one concludes that \( \dim L_B = k \) and the map \( A : \mathcal{L}_k \to L_B \) is one-to-one affine map. So the set \( B \) is affinely independent, and each point \( y \in L_B \) has a unique representation of the form \( y = \sum \lambda_i a_i, \sum \lambda_i = 1 \).

6 Total orders compatible with addition on an additive subgroup on \( \mathbb{R}^n \) and \( \mathbb{Q}^n \)

In this section we will generalize to the multidimensional case the one-dimensional results which we presented earlier. We will use as our main tools the two theorems from convex geometry presented in the previous section.

6.1 Lexicographic orders and orders compatible with addition on subgroups of \( \mathbb{R}^n \)

Any total order compatible with addition on a commutative group \( G \) is determined by a semigroup \( G_+ \subset G \) (see Lemma 5).

In this subsection we will consider subgroups \( G \) of the additive group of a real finite dimensional real vector space \( L \) equipped with the order defined by a subsemigroup \( G_+ \subset G \subset L \). We are interested in the following question:

Under what conditions on the semigroup \( G_+ \) is the total order on \( G \) induced by a lexicographic order on \( L \) which is related to some coordinates system \( \mathbf{x} \) on \( L \)?

The following Theorem provides the answer.
Theorem 26. The total order on a group $G \subset L$ which is defined by semigroup $G_+ \subset G$ is induced by the order $\succ_x$ on $L$ related to some coordinate system $x$ if and only if the convex hull $\Delta(G_+)$ of the set $G_+$ does not contain the origin.

**Proof.** Assume that the order on $G$ is induced by the lexicographic order $\succ_x$ on $L$. Then, the set $G_+$ is contained in the convex set $L_+(x)$, which does not contain the origin. So, the convex hull $\Delta(G_+)$ does not contain the origin either. This proves the Theorem in one direction.

If the origin does not belong to $\Delta(G_+)$, then, by our version of the Separation Theorem, there is a system of coordinates $x$ such that $G_+ \subset L_+(x)$.

Let us show that $G \cap L_+(x) = G_x$. Indeed, if there is a point $a \in L_+(x) \cap G$ not belonging to $G_+$, then $-a \in G_+$. But $-a$ does not belong to $L_+(x)$. We obtain a contradiction which proves the needed statement.

This identity $G \cap L_+(x) = G_+$ shows that the order on $G$ is induced by the lexicographic order $\succ_x$ on $L$. \hfill \Box

**Proof of Theorem 13** Theorem 13 follows from Theorem 26. Indeed, if a total order $\succ$ on $L$ is compatible with addition and with multiplication on positive number, then by Lemma 8 the set $L_+$ responsible of the order $\succ$ is convex. By Theorem 26 there is a coordinate system $x$ on $L$ such that $L_+ \subset L_+(x)$. Moreover from the proof of Theorem 26 one can see that $L \cap L_+(x) = L_+$. \hfill \Box

**Proof of Theorem 14** One can be prove Theorem 14 in the same way as Theorem 26. Indeed, since the set $X_+$ is convex and does not contain the origin. Thus by version of Separation Theorem there is a system of coordinates $x$ such that $X_+ \subset L_+(x)$.

Let us show that $L_+(x) \subset X_+$. Indeed, if there is a point $a \in L_+(x)$ not belonging to $X_+$, then $-a \in X_+$. But $-a$ do not belong to $L_+$. We obtain a contradiction which proves the needed statement. \hfill \Box

6.2 Orders compatible with addition on subgroups of $\mathbb{Q}^n$

The following Theorem holds.
Theorem 27. Let \( A \) be any subset of the \( n \)-dimensional vector space \( \mathbb{Q}^n \) over the field \( \mathbb{Q} \) of rational numbers. Then, the semigroup \( S_A \) generated by the set \( A \) contains the origin if and only if the convex hull \( \Delta(A) \) of \( A \) contains the origin.

**Proof.** Assume that \( 0 \in \Delta(A) \). By Caratheodory’s theorem there is a set \( B \subset A \) of affinely independent points \( \{a_1, \ldots, a_{k+1}\} \) and a \((k+1)\)-tuple of nonnegative numbers \( \{\lambda_i\} \) such that \( 0 = \lambda_i a_i \) and \( \sum \lambda_i = 1 \).

Since the points \( a_i \) and the origin belong to \( \mathbb{Q}^n \), and \( a_1, \ldots, a_k \) are affinely independent, all numbers \( \lambda_1, \ldots, \lambda_k \) are rational.

Multiplying the identity \( \sum \lambda_i a_i \) by the product of the denominators of rational numbers \( \lambda_i \), we obtain the relation

\[
\sum q_i a_i = 0,
\]

where \( q_i \) are natural numbers.

This identity means that the semigroup \( S_A \), together with the set \( A \), contains the origin.

On the other hand, if \( 0 \in S_A \), then there are points \( a_i \in A \) and natural numbers \( q_i \), such that \( \sum q_i a_i = 0 \). Dividing this identity by \( Q = \sum q_i \), and putting \( \lambda_i = \frac{q_i}{Q} \), we obtain the representation of the origin in the form \( 0 = \sum \lambda_i a_i \) where \( \lambda_i > 0 \) and \( \sum \lambda_i = 1 \). This means that the origin belongs to the convex hull of the set \( A \).

Theorem 28. A total order \( \succ \) on a subgroup \( G \) of the additive group of the \( n \)-dimensional vector space \( \mathbb{Q}^n \subset \mathbb{R}^n \) over rational numbers is compatible with addition if and only if the order \( \succ \) is induced from the some lexicographic order \( \succ_x \) on \( \mathbb{R}^n \).

**Proof.** Assume that the order \( \succ \) on \( G \) is compatible with addition. The semigroup \( G_+ \subset G \subset \mathbb{Q}^n \) which is responsible for the order \( \succ \) cannot contain the origin. Thus, by Theorem 27, the convex hull \( \Delta(G_+) \) of this semigroup also does not contain the origin. So, by Theorem 26, the order \( \succ \) is induced by some lexicographic order \( \succ_x \) on \( \mathbb{R}^n \).

On the other hand, the order \( \succ \) induced by any lexicographic order \( \succ_x \) on \( \mathbb{R}^n \) is compatible with addition. \( \square \)
7 Well-ordered semigroups

In this section we discuss well-ordered finitely generated subsemigroups of the additive group of finite dimensional vector spaces over the real numbers and over the rational numbers.

7.1 Well-ordered semigroups of \( \mathbb{R}^n \)

The following theorem holds:

**Theorem 29.** Assume that an order \( \succ \) of a finitely generated semigroup \( S \subset L \) of the additive group of a real \( n \)-dimensional space \( L \) is induced by the lexicographic order \( \succ_x \) related to some coordinate system \( x \) on \( L \). Then, \( S \) is a well-ordered set with respect to the order \( \succ \) if and only if the ordered semigroup \( S \) satisfies the condition from Lemma 6.

**Proof.** If \( S \) is a well-ordered set, then \( S \) satisfies the conditions of Lemma 6. Let us prove the Theorem in the opposite direction.

We will use induction on the dimension \( n \) of the ambient space \( L \). Note that for \( n = 1 \), the Theorem is already proven above (see Lemma 18).

Assume that the order on \( S \subset L \), where \( \dim L = n \), is induced by the lexicographic order \( \succ_x \) on \( L \) related to the coordinates system \( x = (x_1, \ldots, x_n) \).

Let \( A \) be a set of generators of the semigroup \( S \) and let \( B \) be a subset of \( A \) on which the function \( x_1 \) is positive. Since \( S \) satisfies the condition in Lemma 6, the function \( x \) is nonnegative on \( S \). So, the function \( x \) is equal to zero on the set \( C = A \setminus B \).

Denote by \( S_{x_1} \) the semigroup \( S \cap \{x_1 = 0\} \) lying in the space \( x_1 = 0 \) of dimension \( n - 1 \). The semigroup \( S_{x_1} \) is generated by elements from the set \( C \), and equipped with the lexicographic order related to the coordinate system \( (x_2, \ldots, x_n) \) on the hyperplane \( x_1 = 0 \). By induction, the semigroup \( S_{x_1} \) is well-ordered.

Let us consider the image \( x_1(S) \) of the semigroup \( S \) under the map \( x_1 : L \to \mathbb{R} \). The set \( x_1(S) \) is an additive subsemigroup of \( \mathbb{R} \) generated by elements from the set \( x_1(B) \) (and by zero, if the set \( S_{x_1} \) is not empty).

The finitely generated semigroup \( x_1(S) \subset \mathbb{R} \) is a well-ordered set by Lemma 18.
Let us show that any set \( D \subset S \) contains a smallest element. First, there is a smallest value \( d = \min x_1(D) \) of the function \( x_1 \) on the set \( D \) since the set \( x_1(s) \) is well-ordered.

Consider a semigroup \( S(B) \) generated by elements of the set \( B \). Let \( F \subset S(B) \) be a subset on which the function \( x_1 \) is equal to \( d \). The set \( f \) is finite since the function \( x_1 \) is positive on the finite set \( B \).

The set \( \{ x_1 = d \} \cap S \subset S \) on which \( x_1 \) is equal to \( d \) is covered by a finite collection of shifted copies \( S x_1 + f \) of the semigroup \( S x_1 \), where \( f \) is any element of \( F \).

Each such copy \( S x_1 + f \) is a well-ordered set, since \( S x_1 \) is well-ordered and its order is compatible with addition. So, the set \( \{ x_1 = d \} \cap S \) is well-ordered.

We see that the set \( D \subset S \) contains a smallest element: the smallest element of the set \( \{ x_1 = d \} \cap D \), where \( d \) is the smallest value of \( x_1 \) on \( d \).

### 7.2 Well-ordered semigroups of \( \mathbb{Q}^n \)

**Theorem 30.** Assume that an order \( \succ \) on a finitely generated semigroup \( S \subset \mathbb{Q}^n \) of the additive group of an \( n \)-dimensional vector space \( \mathbb{Q}^n \) over rational numbers is compatible with addition. Then, \( S \) is a well-ordered set with respect to the order \( \succ \) if and only if the ordered semigroup \( S \) satisfies the condition in Lemma 7.

**Proof.** An order \( \succ \) compatible with addition can be uniquely extended to an order compatible with addition to the group \( G \), generated by the semigroup \( S \).

By Theorem 28, any order compatible with addition on the group \( S \subset \mathbb{Q}^n \) is induced by a lexicographic order. To complete the proof it is enough to use Theorem 29.

### 8 Subgroups and Subsemigroups of the lattice \( \mathbb{Z}^n \)

Let us apply the results discussed above to additive subgroups and subsemigroups of the standard lattice \( \mathbb{Z}^n \).

The lattice \( \mathbb{Z}^n \) is naturally embedded in the spaces \( \mathbb{Q}^n \subset \mathbb{R}^n \).
**Theorem 31.** Each total order on the group $\mathbb{Z}^n$ and on any of its subssemi-group $S \subset \mathbb{Z}^n$ that is compatible with addition is induced by a lexicographic order $\succ_x$ on $\mathbb{R}^n$ that is related to some coordinate system $x = (x_1, \ldots, x_n)$.

Such an order $\succ_x$ on a finitely generated semigroup $S = \mathbb{Z}^n$ is a well-order on $S$ if and only if any nonzero element $a \in S$ satisfies the inequality $2a \succ_x a$.

In particular the semigroup $\mathbb{Z}^n_{\geq 0}$ consisting of integral points with nonnegative coordinates is a well-ordered set with respect to the lexicographic order $\succ_x$ if and only if the origin is the smallest element of the semigroup $\mathbb{Z}^n_{\geq 0}$.

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