Solution of the constant radial acceleration problem

using Weierstrass elliptic and related functions

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We consider the constant radial acceleration problem. While the problem is known to be integrable and has received some recent attention in an orbital mechanics context, a closed form explicit solution, relating the state variables to an anomaly, has eluded all researchers so far. We show how such a solution exists and is elegantly expressed in terms of the Weierstrass elliptic and related functions \( \wp, \zeta \) and \( \sigma \). We show how previously known facts can be derived from the new explicit solution and discuss new insights such an approach reveal.

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Nomenclature

\( \wp \) = Weierstrass \( \wp \) function

\( \zeta \) = Weierstrass \( \zeta \) function

\( \sigma \) = Weierstrass \( \sigma \) function

\( K(.) \) = Complete elliptic integral of the first type

\( \omega, \omega' \) = Periods of the Weierstrass elliptic functions

\( \omega_1, \omega_2, \omega_3 \) = Periods of the Weierstrass elliptic functions associated to the roots of \( g \)

\( g_2, g_3 \) = Invariants of the Weierstrass elliptic functions

\( f(.) \) = Third order polynomial associated to the spacecraft dynamics

\( g(.) \) = Third order polynomial associated to the Weierstrass functions

\( e_1, e_2, e_3 \) = Roots of the polynomial \( f \)

\( \rho_1, \rho_2, \rho_3 \) = Roots of the polynomial \( f \) in the special case \( \gamma_0 = 0 \)

\( \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \) = Roots of the polynomial \( g \)

\( w_1, w_2, w_3 \) = Roots of the polynomial \( g \) in the special case \( \gamma_0 = 0 \)

\( \alpha \) = Constant radial acceleration

\( \mu \) = Gravitational constant

\( t, \tau \) = Time and pseudo-time

\( h \) = Orbital angular momentum

\( \mathcal{E} \) = Energy

\( v \) = Velocity vector magnitude

\( r \) = Position vector magnitude

\( \mathbf{v} \) = Velocity vector

\( \mathbf{r} \) = Position vector

\( \gamma \) = Orbital flight path angle

\( \theta \) = Anomaly between \( \mathbf{r}_0 \) and \( \mathbf{r} \)

\( T_r, T_t \) = Pseudo-period and period for \( r \)
I. Introduction

The motion of a point mass particle subject to a central gravity field and to an additional radial force is described by one of the few known integrable dynamical systems. Its practical interest is related, among the other things, to spacecraft low-thrust propulsion [1–4], to controversial models in modern physics such as that of the Rindler acceleration [5] or anomalies of the gravitational field in the solar system such as that of the Pioneer anomaly [6].

While its solution can be found in terms of Jacobi elliptic integrals, such a solution is hardly ever discussed nor used as it results in equations expressing the time as a function of the state variables and not vice-versa. Without a solution, basic manipulations of the energy equation still allow to analyse the problem [3, 4] deriving classical results [2] and to discriminate, in special cases, between bounded and unbounded motion. In applications related to spacecraft trajectory design [1, 7] it is of great importance, on the other hand, to have access to an explicit solution to the problem. In the recent work of Quarta et al. [1] such a solution is proposed in an approximated form making use of circular functions and limited to the bounded case. In that work the authors prefer the use of approximating circular function expressions to the implicit exact solution in terms of Jacobi elliptic functions lamenting the lack of physical insight connected to these mathematical functions. In [8], instead, the solution is computed, again for a special case, in terms of the Jacobi elliptic functions, confirming how such a solution is implicit and requires the numerical inversion of complex relations.

In this paper we find and discuss the general explicit solution to the problem. To our knowledge it is the first time such a solution is given. Our solution is allowed by the careful use of Weierstrass elliptic functions \( \wp, \zeta \) and \( \sigma \) (see [9] for a good introduction to these functions). These functions are no longer part of the typical background of engineers and physicists though they appear in the solution to many problems in classical mechanics and they are a superior tool with respect to the more popular Jacobi expressions [10], whenever the main polynomial expression is parametric.

The full solution to the constant radial acceleration problem, is here elegantly expressed by simple explicit equations describing the complex physical nature of the motion. The newly found expressions are valid in general for bounded and unbounded motion, they have no restrictive hypothesis and can be thus used directly in the design of interplanetary trajectories. Interestingly,
the solution to the constant radial thrust problem involves all the steps needed to solve the more famous Kepler problem: a) the introduction of an ad-hoc anomaly, b) finding an explicit solution in terms of this anomaly and c) the definition of a Kepler’s equation to recover the solution in the time domain. It was recently pointed out by [11], how elliptic functions in general and Weierstrass formalism in particular, while part of common knowledge at the beginning of this century, are no longer part of the curricula of engineers or physicists. We hope that with our results we will also contribute to convince of the use and importance of these beautiful mathematical tools facilitating their use in modern science.

II. Problem formulation

Consider a point mass subject to a central gravity field and to a constant acceleration directed radially and having magnitude $\alpha$. Negative $\alpha$ values will account for inward accelerations. Without loss of generality, consider the central field gravitational constant to be $\mu = 1$. The conservation of the angular momentum $h$ and the conservation of the energy $\mathcal{E}$ can thus be written as:

$$h = r^2 \dot{\theta}$$

(1)

$$\mathcal{E} = \frac{v^2}{2} - \frac{1}{r} - \alpha r$$

(2)

where we have introduced the particle distance from the attracting body $r$, the particle velocity $v$ and the anomaly $\theta$ determining the particle position. Expressing $v$ in terms of $r$ and $\theta$, we have
\[ v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 \]  

substituting Eq.(3) back into Eq.(2) and expressing \( \dot{\theta} \) in terms of \( r \) using Eq.(1) we get:

\[ 2\mathcal{E} = \dot{r}^2 + h^2/r^2 - 2/r - 2\alpha r \]

and solving for \( \dot{r} \):

\[ r\dot{r} = \pm \sqrt{2\alpha r^3 + 2\mathcal{E} r^2 + 2r - h^2} = \pm \sqrt{f(r)} \]  

In the expression above, the sign indicates whether \( r \) is increasing or decreasing and changes at \( f(r) = 0 \) resulting in the radicand being always greater than zero. The solution by quadratures of the constant radial thrust problem (assuming \( r \) monotonously increasing in \([t_0, t]\)) is:

\[
\begin{align*}
\int_{t_0}^{t} du &= \int_{r_0}^{r} \frac{udu}{\sqrt{2\alpha u^3 + 2\mathcal{E} u^2 + 2u - h^2}} \\
\int_{\theta_0}^{\theta} du &= h \int_{r_0}^{r} \frac{du}{u\sqrt{2\alpha u^3 + 2\mathcal{E} u^2 + 2u - h^2}}
\end{align*}
\]

In the general case in which \( r \) is not monotonous in \([t_0, t]\) the integrals above need to be subdivided accordingly accounting for the mentioned sign change. Note that the above integrals define the time as a function of the state variables, while it is the inverse of such a relation, i.e. expressing the state variables as a function of time, that is of much greater interest and will here be derived.

### A. The polynomial \( f(r) \)

A number of interesting considerations may be done, preliminarily, studying the polynomial \( f(r) \) which defines entirely the spacecraft dynamics in the \( r, \dot{r} \) phase-space via Eq.(4). We indicate with \( e_1, e_2 \) and \( e_3 \) the three roots of the third order polynomial \( f(r) \) sorted in descending order (first of the imaginary part, then the real part) and with \( \Delta \) the polynomial discriminant, so that the convention reported in Table 1, and adopted in [12] is followed.

The three roots \( e_i \) of \( f(r) \) define entirely the problem taxonomy as they define the sign of the polynomial \( f(r) \): only regions where \( f(r) > 0 \) are allowed. In Figure 1, we show the three cases that can be encountered: one, two or three positive real roots for \( f(r) \). Only the area \( r > 0, f(r) > 0 \) delimits allowed motion and is shown. For \( \Delta < 0 \) only one real positive root exists (apply the
\[ \Delta > 0 \quad \Delta < 0 \]

\[ e_1 \geq e_2 > e_3 \quad e_1 = a + ib, \quad e_3 = a - ib \]

Table 1 Convention on the polynomial roots \( e_i \) ordering.

Descartes rule and remember that the other two roots must be complex and conjugate) and thus the motion is allowed only for \( r \geq e_2 \). For \( \Delta > 0 \) we need to distinguish two cases. The first case is when \( \alpha < 0 \) (inward acceleration). In this case, applying again the Decartes rule to the polynomial \( f(r) \) defined in Eq.(4), one can conclude that \( f \) has always two positive real roots and the motion is thus bounded as \( r \in [e_2, e_1] \). The second case is when \( \alpha \geq 0 \) (outward acceleration). In this case \( f(r) \) has three real roots of which either one or three will be positive. The motion will then be confined in the area defined by the starting condition \( r_0 \). It is helpful to visualize the phase-state trajectories plotting Eq. (4) in a suitable parametrization. We write the polynomial \( f(r) \) as a function of the initial conditions \( r_0, v_0 \) and the initial flight-path angle \( \gamma_0 \):

\[ f(r) = 2\alpha r^3 + 2(v_0^2/2 - 1/r_0 - \alpha r_0)r^2 + 2r - v_0^2 r_0^2 \cos^2 \gamma_0 \quad (6) \]

We may now plot the resulting trajectory in the phase-space using the radial acceleration \( \alpha \) as a parameter and considering the initial conditions as fixed. As an example, we show such a plot in Figure 2 for the particular case treated also by [2, 3] where circular initial conditions are assumed: \( r_0 v_0^2 = 1 \) and \( \cos \gamma = 0 \). For this particular case, all trajectories to the left of the line defined by \( r = r_0 \) correspond to negative values of \( \alpha \) resulting in inner orbits, while for \( r > r_0 \) we have the trajectories resulting from a positive \( \alpha \) and which eventually open up and become unbounded. In this case, the three roots of the polynomial \( f(r) \) admit a simple expression:

\[ \rho_1 = r_0, \quad \rho_2 = \frac{1 - \sqrt{1 - 8\alpha r_0^2}}{4\alpha r_0}, \quad \rho_3 = \frac{1 + \sqrt{1 - 8\alpha r_0^2}}{4\alpha r_0} \quad (7) \]

It is then trivial to conclude that, in order for the resulting motion to be bounded, we need to have:

\[ \alpha r_0^2 < \frac{1}{8} \]
Fig. 2 Phase-space trajectories for different $\alpha$ values for the case $r_0 = v_0 = 1$, $\gamma_0 = 0$. The case $\alpha = 1/8$ is also visualized resulting in the homoclinic connection at $r = 2$. 

in accordance to the classic result reported for example in [2, 3].

III. Time as a function of the state (implicit solution)

Consider the first of Eq. (5). Following the general integration method described in [10], we apply the Tschirnaus transformation (see [13] for a general introduction) to reduce the third degree polynomial to a depressed cubic:

$$ u = \sqrt{\frac{2}{\alpha}} w - \frac{\varepsilon}{3\alpha} $$

and define $\tilde{r}_0 = \sqrt{\alpha/2}(r_0 + \frac{\varepsilon}{3\alpha})$ and $\tilde{r} = \sqrt{\alpha/2}(r + \frac{\varepsilon}{3\alpha})$. We have:

$$ \Delta t = 2\sqrt{\frac{4}{\alpha^2}} \int_{r_0}^{\tilde{r}} \frac{wdw}{\sqrt{4w^3 - g_2w - g_3}} - \frac{\varepsilon}{3\alpha} \sqrt{\frac{2}{\alpha}} \int_{\tilde{r}_0}^{\tilde{r}} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}} $$

(8)
We recognize in Eq.(8) the Weierstrass elliptic integrals of the first and second kind [10]. We thus introduce the Weierstrass elliptic function \( \wp(z, g_2, g_3) \) with invariants \( g_2 \) and \( g_3 \) defined as follow:

\[
\begin{align*}
g_2 &= \frac{2}{\alpha} \sqrt{2} \left( \frac{E^2}{T} - \alpha \right) \\
g_3 &= h^2 + \frac{2E}{3\alpha} - \frac{4E^3}{27\alpha^2}
\end{align*}
\]

For a complete treatment of Weierstrass elliptic functions one can refer to [9], it is here sufficient to remember that \( \wp \) is a the solution to the following differential equation:

\[
\wp' = \frac{4}{\wp^3} - \frac{g_2}{\wp} - \frac{g_3}{\wp}
\]

and that it is defined in the complex plane where it is holomorphic and doubly periodic with half-periods which we indicate with \( \omega \) and \( \omega' \). In this paper we will make use of the notation used by [12] when dealing with the elliptic functions. We also introduce the Weierstrass \( \zeta \) function simply defined as

\[
\zeta'(z, g_2, g_3) = -\wp(z, g_2, g_3),
\]

where the derivative with respect to the complex variable \( z \) is indicated with a prime. For notation sake, the invariants \( g_2 \) and \( g_3 \) will be dropped and we will thus write \( \wp(z), \zeta(z) \) instead of \( \wp(z, g_2, g_3), \zeta(z, g_2, g_3) \). To solve the integral in Eq.(8), the simple substitution \( w = \varphi(v) \) leads to:

\[
\Delta t = \sqrt[3]{\frac{4}{\alpha^2}} \int_{\rho_0}^{\rho} \zeta'(v) dv - \frac{E}{3\alpha} \sqrt[3]{\frac{2}{\alpha}} \int_{\rho_0}^{\rho} dv
\]

where \( \rho = \varphi^{-1}(r) \) and \( \rho_0 = \varphi^{-1}(r_0) \). The inverse of the Weierstrass \( \varphi \) function appears in the above expression indicated with the symbol \( \varphi^{-1} \). We have thus found the problem quadrature which can now be formally expressed as:

\[
\Delta t = \sqrt[3]{\frac{4}{\alpha^2}} (\zeta(\rho_0) - \zeta(\rho)) + \frac{E}{3\alpha} \sqrt[3]{\frac{2}{\alpha}} (\rho_0 - \rho)
\]

The above expression relates the time as a function of the initial conditions and the current state. It is the Weierstrassian counterpart to the equivalent expression in terms of the Jacobi elliptic functions (see [14] §3.132 for the general case, or [8] for a particular case) and can be regarded as an "implicit" solution to the problem. While more compact than previously known results (and valid in general for all initial conditions), to get \( r \) as a function of the time \( t \) one still needs to invert Eq.(10) which requires a numerical procedure. This problem, shared with the known expressions in terms of the Jacobi elliptic integrals, is the reason the use of analytical solutions for the radial
thrust problem is unpopular. Many commented how it hinders the physical insight into the problem while not even being computationally efficient, thus suggesting the use of approximate approaches. Contrary to this common knowledge, in the next sections we will show that it is possible to derive the explicit and closed form analytical solution in terms of the Weierstrass elliptic functions. Such expressions are valid for all values of $\alpha$ and $\Delta$, for bounded and unbounded motion and provide, straight-forwardly, a great physical insight into the problem.

IV. The state as a function of a time (explicit solution)

Consider now Eq.(4) and introduce the Sundmann transformation to regularize the problem:

$$dt = rd\tau$$

indicating now with a prime $r'$ the derivative with respect to the new time variable $\tau$ we have:

$$r' = \pm \sqrt{2\alpha r^3 + 2E r^2 + 2mu - h^2}$$

and the quadrature becomes:

$$\int_{r_0}^{r} du = \int_{r_0}^{r} \frac{du}{\sqrt{2\alpha u^3 + 2Eu^2 + 2mu - h^2}} = \int_{r_0}^{r} \frac{du}{\sqrt{f(u)}}$$

This integral can be solved and inverted by the direct application of a result which, according to Whittaker [9] (pag.454, example 2), is due to Weierstrass and which, in our case, may be written:

$$r = r_0 + \frac{1}{2} \left( \frac{\varphi(\Delta \tau)}{f''(r_0)} \right)^2 - \frac{1}{24} f''(r_0) + \frac{1}{2} f'(r_0) \left[ \varphi(\Delta \tau) - \frac{1}{24} f''(r_0) \right]$$

where $\varphi(\tau)$ is the Weierstrass $\varphi(\tau,g_2,g_3)$ function with invariants:

$$g_2 = \frac{e^2}{3} - \alpha$$

$$g_3 = \frac{e^2}{4} (h^2 + \frac{2e}{9\alpha} - \frac{4e^3}{27\alpha^2})$$

We define here the polynomial $g(s) = s^3 - g_2 s - g_3$ associated to these invariants and that will be important later in this paper. We may then introduce $r_m$ as the pericenter radius and start
counting \( t \) and \( \tau \) from there. Then, by definition, \( \dot{r}(0) = 0 \) and from Eq.(4) we get \( f(r_m) = 0 \) and the equation above may be written in the simple and elegant form:

\[
    r = r_m + \frac{1}{24} f''(r_m) \frac{r_m}{\wp(\tau) - \frac{1}{24} f''(r_m)}
\]

which expresses one of the state variable (the radius) directly as a function of the Sundmann pseudo-time \( \tau \).

To search for an equivalent expression for the other state variable \( \theta \) lets start from the momentum conservation:

\[
    \frac{d\theta}{dt} = \frac{h}{r^2} \rightarrow \frac{d\theta}{d\tau} = \frac{h}{r}
\]

Using Eq.(14) it is easy to see that:

\[
    \int \frac{1}{r} d\tau = \int \frac{\wp(\tau) + \beta}{\gamma \wp(\tau) + \delta} d\tau
\]

which is a known integral (see [14] §5.141), and hence obtain the analytical expression:

\[
    \theta = \frac{h}{\gamma} \tau + h \frac{\beta \gamma - \delta}{\gamma^2 \wp(v)} \left[ \ln \frac{\sigma(v - \tau)}{\sigma(\tau + v)} + 2 \tau \zeta(v) \right]
\]

where:

\[
    \beta = -\frac{1}{24} f''(r_m), \quad \gamma = r_m, \quad (16)
\]

\[
    \delta = \frac{f'(r_m)}{4} + \beta r_m, \quad \wp(v) = -\frac{\delta}{\gamma}, \quad (17)
\]

and \( \sigma \) is the Weierstrass \( \sigma \) function defined as \( \sigma'/\sigma = \zeta \). Note that we have also assumed \( \theta_m = 0 \), as we start counting \( \theta \) from the pericenter. Eq.(15) above could already be considered the solution we are seeking as it relates simply and with one short expression the state variable \( \theta \) to the pseudo time \( \tau \). The presence of the logarithm of a complex variable makes the expression not “usable” as the phase ambiguity cannot be resolved. The following few steps address this issue. Using the definition of the Weierstrass elliptic function: \( \wp^2 = 4\wp^3 - g_2\wp - g_3 \), we have

\[
    \wp'(v) = \frac{v_m (\alpha v_m^2 + r_m v_m^2 - 1)}{2r_m}
\]

where \( v_m \) is the pericenter velocity. Using this expression together with Eq.(17) eventually we find the remarkable simplification:

\[
    \frac{h \beta \gamma - \delta}{\gamma^2 \wp'(v)} = \pm i
\]
Fig. 3 Example of an exact periodic solution obtained in terms of the Weierstrass elliptic functions $\wp$, $\sigma$ and $\zeta$. Initial conditions are $r_p = 1.0$, $v_p = 1.26014$ $\alpha = -0.05$.

We select among the two possible values of $v$ such that $\wp(v) = -\delta/\gamma$ in the fundamental rectangle, by forcing in the above expression the plus sign. This allows us to rewrite Eq.(15) in the form:

$$\exp i(v_m \tau - \theta) = \frac{\sigma(v - \tau)}{\sigma(\tau + v)} \exp 2\tau \zeta(v)$$

(18)

which is not affected by any phase ambiguity any more. Thus, the solution to the constant acceleration radial problem, in the new pseudo-time, is described in its most general case by the following
compact expressions:

\[
\begin{cases}
  r = r_m + \frac{1}{4} \frac{f'(r_m)}{\psi(r_m)} \\
  \exp i(v_m \tau - \theta) = \frac{\sigma(v - \tau)}{\sigma(\tau + v)} \exp 2\tau \zeta(v)
\end{cases}
\]

One may explicit further the second of the above relations by the use of the Euler formula for the exponential and obtain:

\[
\begin{align*}
  \sin \theta &= x(\tau) \sin(v_m \tau) - y(\tau) \cos(v_m \tau) \\
  \cos \theta &= y(\tau) \sin(v_m \tau) + x(\tau) \cos(v_m \tau)
\end{align*}
\]

having introduced \(x\) and \(y\) as, respectively, the real and the imaginary part of \(z = \frac{\sigma(v - \tau)}{\sigma(\tau + v)} \exp 2\tau \zeta(v)\).

We give, in Figure 3, an example of a trajectory plotted using the new expressions found. It is worth to mention here again that Eq.(19) is “universal” in the sense that it is valid for bounded and unbounded motion.

V. The radial Kepler’s equation

As shown in the previous sections the constant radial acceleration problem admits explicit solutions relating the state variables \(r\) and \(\theta\) directly to the pseudo-time \(\tau\). One may look at \(\tau\) as the eccentric/hyperbolic anomaly of the keplerian problem: also in that more famous case it is a Sundmann transformation that relates these anomalies to the time \(t\). In particular, for the eccentric anomaly \(dt = \frac{n}{a} dE\), where \(n\) is the mean motion and \(a\) the semi-major axis of the conic. It is well known how, to go back to the time \(t\) one has to solve the Kepler’s equation. In the case of the constant radial thrust problem things are very similar: the equivalent to Kepler’s equation relates the pseudo-time \(\tau\) to the time \(t\). We refer to this equation as the “radial Kepler equation”. From the definition we know that \(dt = rd\tau\). Using for \(r\) the newly found expression in Eq.(14), we have:

\[
t(\tau) = \int_0^\tau \left[ r_m + \frac{1}{4} \frac{f'(r_m)}{\psi(u)} - \frac{1}{24} f''(r_m) \right] du
\]

It is possible to prove, by direct substitution, that \(\frac{1}{24} f''(r_m)\) is always a root of \(g(s)\), say \(\tilde{e}_k\). More specifically, following to the convention in Table 1, it will be \(\tilde{e}_3\) if the motion is unbounded, \(\tilde{e}_2\) if the motion is bounded. Then one can write the above integral in the form:

\[
t(\tau) = \int_0^\tau \left[ r_m + \frac{1}{4} \frac{f'(r_m)}{\psi(u)} - \tilde{e}_k \right] du
\]
This integral is a known integral (see [10] §1037.07-09). Exploiting the identity \( \hat{e}_i \hat{e}_j = \frac{m}{4n_x} \) we can further simplify the expressions reported in [10] and write:

\[
t(\tau) = r_m \tau - \frac{\hat{e}_k f'(r_m)}{g_3 + 8\hat{e}_k^3} \left[ \hat{e}_k \tau + \zeta(\tau) + \frac{1}{2} \phi'(\tau) \right]
\]

where we remind again that \( \hat{e}_k = \frac{1}{24} f''(r_m) \). The above equation is undetermined in \( \tau = 0 \) (and thus numerically unstable) where both \( \phi \) and its derivative \( \phi' \) are infinite. To remove such an indetermination at \( \tau = 0 \) we use the identity:

\[
\frac{\phi'(a)}{\phi(b) - \phi(a)} = \zeta(b - a) - \zeta(b + a) + 2\zeta(a) = \frac{d}{d\tau} \ln \left( \frac{\sigma(b - a)}{\sigma(b + a)} \right)
\]

and \( \hat{e}_k = \phi(\omega_k) \) (see [12] §18.3.1) to conclude:

\[
t(\tau) = r_m \tau - \frac{\hat{e}_k f'(r_m)}{2g_3 + 16\hat{e}_k^3} \left[ 2\hat{e}_k \tau + \zeta(\tau - \omega_k) + \zeta(\tau + \omega_k) \right]
\]

which now holds the correct value \( t(0) = 0 \). For the sake of the reader’s convenience we report the definitions of \( \omega_k \) (from [12] Figure 18.1) as a function of the complex half-periods \( \omega \) and \( \omega' \) of the elliptic function \( \phi \):

\[
\omega_1 = \omega \\
\omega_2 = \omega + \omega' \\
\omega_3 = \omega'
\]

Remarkably, Eq.(22) is “universal” being formally valid in this form for bounded and unbounded motion. It is Eq.(22) that we call “the radial Kepler equation”. The role it plays in the solution of the constant radial acceleration problem is the same as that of the Kepler equation in the Kepler problem.

VI. Use of the new solution

A. Periodicity of \( r \)

In case of bounded motion it is of interest to compute the period of \( r \). In the \( \tau \) domain \( r \) is periodical and its period is the period of \( \phi(\tau, g_2, g_3) \) as can be derived trivially from Eq.(14). As we only are interested in the periodicity of \( \phi \) for the case of bounded motion, we may restrict our analysis to the case of a positive discriminant for \( g(s) \). We compute the two half-periods \( \omega \) and \( \omega' \)

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of the doubly periodic complex function $\wp(z, g_2, g_3)$ using the known relations with the complete elliptic integral of the first type $K$ valid for $g_3 > 0$:

$$\omega = K(m)/\sqrt{\tilde{e}_1 - \tilde{e}_3}$$

$$\omega' = iK(1 - m)/\sqrt{\tilde{e}_1 - \tilde{e}_3}$$

where $m = (\tilde{e}_2 - \tilde{e}_3)/(\tilde{e}_1 - \tilde{e}_3)$. The period on the real axis will then be $T = \omega$. In the case $g_3 < 0$, we may use the homogeneity condition $\wp(z, g_2, g_3) = -\wp(iz, g_2, -g_3)$. Eventually one can show that in all cases (i.e. $\forall g_3$), the following holds:

$$T_\tau = 2K(m)/\sqrt{\tilde{e}_1 - \tilde{e}_3}$$

(23)

In Figure 5 we plot the pseudo-period $T_\tau$ against the value of the radial acceleration $\alpha$ for different initial conditions. For any chosen value of $\tau$ we can always find a value of $\alpha$ resulting in an orbit with that period. Computing then the radial Kepler equation for $\tau = T_\tau$, we find an expression for the true period $T_t$ of the radius $r$:

$$T_t = r_m T_\tau - \frac{\tilde{e}_k f'(r_m)}{2g_3 + 16\tilde{e}_3^2} \left[ 2\tilde{e}_k T_\tau + 4\zeta(T_\tau/2) \right]$$

where we have exploited the quasi-periodicity of the $\zeta$ functions (see [12] §18.2.19).
Fig. 5 Pseudo-period of $r$ at different thrust levels (Eq.(VI A) for different $\alpha$). Different curves corresponds to different starting $v_p$ sampled in $[0.5, 1.5]$.

but numerically equivalent, expression for $T_t$ can be also found computing from Eq.(10) the time to travel from pericenter to apocenter (i.e. half-period):

$$T_t/2 = \frac{3}{4\alpha^2} (\zeta(r_m) - \zeta(r_M)) + \frac{\mathcal{E}}{3\alpha} \frac{3^2}{\alpha} (r_m - r_M)$$

B. Computing $r_m$, $v_m$ and $\tau_0$

In Eq.(14)-Eq(17) the pericenter radius $r_m$ appear. The pericenter radius can be determined by looking at the roots $e_i$ of the polynomial $f(r)$ and setting $r_m = e_i$ where $e_i$ is the closest real root to $r_0$ such that $e_i \leq r_0$. The initial conditions will, in general, be not given at the pericenter, in which case the initial pseudo-time can be computed directly from Eq.(14) as:

$$\varphi(\tau_0) = \frac{1}{24} f''(r_m) + \frac{1}{4} \frac{f'(r_m)}{r_0 - r_m}$$

we must then select the appropriate value for the inversion of $\varphi^{-1}$ by looking only within the first $T_\tau$ and choosing the solution with the correct $\dot{r}$. 

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Fig. 6 Plot of Eq.(14) for the case $r_0 = r_m = 1$, $v_0 = 1.2$. The unbounded case (left) corresponds to $\alpha = 0.1$ and the bounded case (right) corresponds to $\alpha = 0.02$

C. The condition for bounded motion

Once $r_m$ is computed the pseudo-time dependency of $r$ can be computed. In Figure 6 we plot Eq.(14) assuming as a starting position the pericenter radius $r_0 = r_m$, and as a starting velocity $v_0 = 1.2$. Two cases are shown: one unbounded, obtained for $\alpha = 0.1$ and one bounded, obtained for $\alpha = 0.02$. In order for the motion to be bounded it is clear from Eq.(13) that the denominator cannot vanish. Introducing $\wp_{\text{min}}$ as the minimum value assumed by $\wp$ on the real axis, we may then write the condition to have bounded motion as $24\wp_{\text{min}} > f''(r_0) = 12\alpha r_0 + 4\mathcal{E}$. The minimum value assumed on the real axis by the Weiestrass elliptic function is computed introducing the three roots $\tilde{e}_i$ of the polynomial $g(s) = 4s^3 - g_2s - g_3$. The greatest real root is $\wp_{\text{min}}$. If we indicate it with $\tilde{e}$ we have the simple final relation:

$$\tilde{e} > \frac{1}{2} \alpha r_0 + \frac{1}{6} \mathcal{E} = \frac{1}{24} f''(r_m)$$

which is the generic condition to obtain bounded motion in the constant radial thrust problem.

Note how an equivalent to this relation was previously known only for the special case of a starting circular orbit. That result is now extended to the most general case thanks to the use of Weierstrass elliptic functions. Take as an example $\alpha = 0.02$, $r_0 = 1.1$, $r_p = 1$, $v_p = 1.2$. Evaluating the three roots of $g(s)$ we get:

$$\tilde{e}_1 = -0.0402894, \quad \tilde{e}_2 = -0.0170428, \quad \tilde{e}_3 = 0.0573322$$
and thus $\dot{c} = 0.0573322$. If we now compute $\frac{1}{2} \alpha r_0 + \frac{1}{6} E = -0.039333$ we may immediately conclude that the motion will be bounded by direct application of Eq. (25). The search for a particular value of $\alpha$ or of the initial velocity $v_0$ which results in an escape trajectory can then be made efficiently, e.g. using a simple bisection algorithm.

Consider now the more restrictive case in which $r_0 = r_m$ and $v_0 = m$. We may express the three roots of $g(s)$ in a simple form, by exploiting the relation $h = r_0 v_0$:

$$w_1 = \frac{1}{2} \alpha r_0 + \frac{1}{6} E = \frac{1}{24} f''(r_m)$$

$$w_{2,3} = -\frac{1}{2} \left(\frac{1}{2} \alpha r_0 + \frac{1}{6} E\right) \pm \frac{1}{8r_0} \sqrt{(2 - r_0 v_0^2)^2 - 8\alpha r_0^3 v_0^2}$$

Note how $\frac{1}{24} f''(r_m)$ is always a root of $g(s)$, a fact that will have a great importance later. Applying again Eq. (25) we see that the motion will be unbounded if and only if $\dot{c} = w_1$. This last condition, after some manipulations, can be shown to be equivalent to the set of conditions:

$$r_0 v_0^2 < \frac{2}{3} \quad \text{and} \quad \alpha < \min\left(\frac{1-r_0 v_0^2}{r_0^2}, \frac{(2-r_0 v_0^2)^2}{8r_0^3 v_0^2}\right)$$

$$\frac{2}{3} \leq r_0 v_0^2 \leq 2 \quad \text{and} \quad \alpha < \frac{(2-r_0 v_0^2)^2}{8r_0^3 v_0^2}$$

$$r_0 v_0^2 > 2 \quad \text{and} \quad \alpha < 0$$

(27)

In case of a starting circular orbit, we have $r_0 v_0^2 = 1$ and the above conditions all collapse into the classical result $\alpha r_0^2 < \frac{1}{8}$. We have thus generalized this classical result derived in [2, 3].

D. The condition for periodic motion

While, in a bounded motion case, $r$ is always a periodic function of both the time and the pseudo-time, the whole trajectory will only be periodic if and only if there exist two numbers $M, N \in \mathbb{N}$ such that $\theta(N T_r) = 2M \pi$. Let us compute the value $\Delta \theta$ reached by the variable $\theta$ after $N$ full periods $T_r$. Starting from Eq. (18) we write:

$$e^{i(v_m N T_r - \Delta \theta)} = \frac{\sigma(v - N T_r)}{\sigma(N T_r + v)} e^{2N T_r \zeta(v)}$$

Consider:

$$\phi(\tau) = \frac{\sigma(v - \tau)}{\sigma(v + \tau)}$$
first we compute $\phi(\tau + T\tau)$ using the quasi-periodicity of the $\sigma$ function (see [12] §18.2.20) and the fact that, for $g_3 > 0$ we have $NT\tau = 2N\omega$:

$$\phi(\tau + NT\tau) = \phi(\tau + 2N\omega) = -\frac{\sigma(\tau - v + 2N\omega)}{\sigma(\tau + v + 2N\omega)} = -\frac{\sigma(\tau - v)}{\sigma(\tau + v)} (-1)^{N} e^{(\tau - v + N\omega)(2N\zeta(\omega))}$$

which becomes:

$$\phi(\tau + NT\tau) = \phi(\tau)e^{-4Nv\zeta(T\tau/2)}$$  \hspace{1cm} (28)

valid also in the case of $g_3 < 0$ as can be shown repeating the above computation for $T\tau = -i\omega$ and using the identity $\varphi(z, g_2, g_3) = -\varphi(i\omega, g_2, -g_3)$. We may now write:

$$e^{i(v_m NT\tau - \Delta \theta)} = e^{4N((T\tau/2)\zeta(v) - v\zeta(T\tau/2))}$$

and, for $N = 1$:

$$\Delta \theta = v_m T\tau - 4\text{Im} [T\tau/2\zeta(v) - v\zeta(T\tau/2)]$$

hence the condition for periodic motion:

$$v_m T\tau - 4\text{Im} [T\tau/2\zeta(v) - v\zeta(T\tau/2)] = 2q\pi$$

where $q = M/N \in \mathbb{Q}$ is rational. The trajectory plotted in Figure 3 was found iteratively by finding $v_p$ so that in the above equation $q = 1/10$.

\section*{VII. Conclusions}

An exact, explicit, closed form, solution of the constant radial acceleration problem is found. The solution is elegantly expressed, in all cases, by an expression involving Weierstrass elliptic functions and as a function of a pseudo-time. Just like in the keplerian mechanics, a radial Kepler equation must then be solved to recover the time dependance. Our solution adds to the list of interesting problem of classical mechanics that can be solved by the use of Weierstrass elliptic functions and provides a new useful tool for aerospace engineers and physicists who deal with the application of this dynamics.
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