Vertex operators of ghost number three in Type IIB supergravity

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Abstract

We study the cohomology of the massless BRST complex of the Type IIB pure spinor superstring in flat space. In particular, we find that the cohomology at the ghost number three is nontrivial and transforms in the same representation of the supersymmetry algebra as the solutions of the linearized classical supergravity equations. Modulo some finite dimensional spaces, the ghost number three cohomology is the same as the ghost number two cohomology. We also comment on the difference between the naive and semi-relative cohomology, and the role of b-ghost.
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1 Introduction

Vertex operators are one of the central objects in string theory. They represent cohomology classes of the BRST operator. The BRST cohomology depends on the chosen background, and in fact describes the tangent space to the moduli space of backgrounds at the chosen point.

In particular, let us look at the pure spinor superstring theory in expansion around flat space. The structure of massless BRST cohomology in flat space is more or less clear, but it appears that it has never been explicitly spelled out in the literature. The present paper is aimed at filling this gap.

For the closed bosonic string the cohomology was computed in [1]. We will here do a similar computation for the pure spinor superstring, but with the following difference. It is well known that the physically relevant cohomology problem is the so-called semirelative cohomology [2], which is $Q_{BRST}$ acting on the vertex operators $V$ satisfying the following condition:

$$(b_0 - \bar{b}_0)V = 0 \quad (1)$$

This condition was built-in into the computations of [1]. In the pure spinor superstring, the construction of the $b$-ghost is very subtle. In our paper we will compute the "naive" cohomology of $Q_{BRST}$, without taking into account (1). Failure to take into account (1) leads to some strange results:
1. Nonphysical vertex operators, \textit{i.e.} elements of the BRST cohomology which do not correspond to any linearized SUGRA solutions

2. Absence of the dilaton zero mode

3. Nontrivial cohomology at the ghost number three

Problems 1 and 2 are removed if we require the existence of the dilaton superfield $\Phi$ (see \cite{3} and the discussion in Section 5.3). To defeat the ghost number three cohomology is more difficult. It is dangerous as a potential obstacle for continuing an infinitesimal solution to a finite solution (\textit{i.e.} obstructed deformations of the flat spacetime). Such obstructions would render the theory physically inconsistent. In bosonic string, all linearized deformations are unobstructed. One explanation is that the \textit{semi-relative} cohomology at the ghost number three is zero, and therefore there is no obstacle. More precisely, the higher order correction to $V$ are controlled by the string field equation \cite{4, 5}:

\begin{equation}
QV = (b_0 - \bar{b}_0)(V^2) + \ldots
\end{equation}

Since the ghost number four cohomology is zero, $V^2$ is in the image of $Q$. In fact, the pre-image could be chosen to be annihilated by $L_0 - \bar{L}_0$, and this shows that Eq. (2) can be resolved order by order in the deformation parameter.

Unfortunately, we do not have such a proof in the pure spinor formalism. It follows from the consistency of \cite{6} that there is actually no obstacle in extending the infinitesimal deformation to higher orders. Even though the ghost number three cohomology is nonzero, the actual obstruction vanishes for physical states. It would be good to have a transparent proof of this fact using the language of BRST cohomology and vertex operators. This would probably require the use of the composite $b$-ghost.

\subsection{Plan of the paper}

In the rest of this introductory section we will review general facts about the BRST cohomology and its relation to the deformations of the worldsheet sigma-model. Then in Section 2 we will review the cohomology of the classical electrodynamics, and explain how to reduce the cohomology of the Type IIB BRST operator in flat space to the cohomology of electrodynamics.
The relation will involve the computation of the cohomology of the algebra of translations with coefficients in the space of solutions of SUSY Maxwell equations (Section 3) and the tensor produce of two copies of such spaces (Section 4). The results on BRST cohomology are summarized in Sections 5 and 6.

1.2 Classical sigma-model and its deformations

It was shown in [6] that classical solutions of the Type IIB supergravity are in one-to-one correspondence with two-dimensional sigma-models satisfying certain axioms. Most importantly, there should be two nilpotent odd symmetries $Q_L$ and $Q_R$:

$$Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0 \tag{3}$$

Also, there should be conserved charge known as the “ghost number”, with both $Q_L$ and $Q_R$ having ghost number +1.

Suppose that we are given such a sigma-model. A natural question is, how can it be deformed? Deformations of the sigma-model are the deformations of the action:

$$S \rightarrow S + \varepsilon \int U \tag{4}$$

where $U$ is some operator. If $U$ vanishes on-shell, then such deformation is trivial, as it can be undone by a field redefinition. Suppose that the deformation is non-trivial.

1.3 From integrated vertex to unintegrated vertex

The condition that the deformed action still has a pair of nilpotent symmetries is equivalent to requiring the existence of $X_L$ and $X_R$ such that on-shell:

$$Q_L U \simeq dX_L \quad \text{and} \quad Q_R U \simeq dX_R \tag{5}$$

Here $\simeq$ means “equivalent on-shell”, i.e. “equivalent modulo the equations of motion”. Explicitly, (5) implies the existence of infinitesimal transformations $q_L$ and $q_R$ (vector fields on the field space) such that:

$$Q_L U + \varepsilon q_L \mathcal{L} = d\tilde{X}_L \quad \text{and} \quad Q_R U + \varepsilon q_R \mathcal{L} = d\tilde{X}_R \tag{6}$$

were $\mathcal{L}$ is the sigma-model Lagrangian. (The $\tilde{X}_{L/R}$ of (6) may be different from the $X_{L/R}$ of (5) because the variation of the Lagrangian is proportional
to the equations of motion only modulo a total derivative). Then $Q_L + \varepsilon q_L$ and $Q_R + \varepsilon q_R$ are both symmetries of the deformed action [4]. Actually they are nilpotent:

$$(Q_L + \varepsilon q_L)^2 = (Q_R + \varepsilon q_R)^2 = \{Q_L + \varepsilon q_L, Q_R + \varepsilon q_R\} = O(\varepsilon^2) \quad (7)$$

This is automatically true because all those anticommutators would be conserved charges of the ghost number two. In this paper we study vertices which are homogeneous polynomials of $x$ and $\theta$. The conserved charges of the ghost number two are polynomials of low degree. Therefore if $U$ is of large enough degree in $x$ and $\theta$, then the nilpotence condition (7) is satisfied.

It is enough to verify (5) for $Q = Q_L + Q_R$:

$$\exists X \text{ such that } QU = dX \quad (8)$$

where $Q = Q_L + Q_R \quad (9)$

Conditions (5) and (8) are equivalent, because $Q_L U$ and $Q_R U$ are independent, as both left and right ghost number are conserved. In fact, any linear combination $\alpha Q_L + \beta Q_R$ with nonzero constant $\alpha$ and $\beta$ can be chosen as a BRST operator; all such complexes are quasi-isomorphic to each other.

Operators $U$ satisfying the condition (8) are called “integrated vertices”. Notice that $X$ is a one-form of the ghost number one, and $d(QX) = 0$; this typically implies $QX = dV$, because there are no conserved charges of the ghost number two. This $V$ is called the unintegrated vertex corresponding to the integrated vertex $U$:

$$(Q_L + Q_R)X = dV \quad (10)$$

It is also possible to revert this procedure and go from $V$ back to $U$. This involves the assumption about the vanishing of the cohomology at the nonzero conformal weight. Although (to the best of our knowledge) the proof of this vanishing theorem has never been given, we feel that the statement is true. Notice that the construction of [7] establishes the correspondence between

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1In this paper we will study vertices which are homogeneous polynomial of $x$ and $\theta$; some of our results are only valid under the assumption that the degree of the polynomial is large enough; exceptions may happen for vertices which do not depend on $x$.

2Going from the deformation of the action to the cohomology of $Q_L + Q_R$ requires the absence of local conserved charges with nonzero ghost number; going back (from $V$ to $U$) requires the vanishing of the cohomology in the sector with positive conformal dimension.
integrated and unintegrated vertices independently of this assumption. Although (in its current form) it only works in flat space and in $AdS_5 \times S^5$, it also teaches us something about the generic curved background. For example, it tells us that the map $U \mapsto V$ is injective. Indeed, suppose that existed an integrated vertex $U$ such that $QU = dX$ and $QX = 0$ (i.e. nonzero $U$ gives $V$). Let us expand such $U$ in Taylor series around a fixed point in the curved space-time, and take the leading term. This should give us the flat space vertex. Since the map $U \mapsto V$ is injective in flat space, the leading term in $V$ should also be nonzero. This means that, if $V$ gets killed, then $U$ cannot survive either.

In any case, our working hypothesis is:

- at the linearized level the deformations of the action are in one-to-one correspondence with the BRST cohomology of $Q = Q_L + Q_R$ at the ghost number two

### 1.4 Ghost number three vertices as obstacles to deformations

If $U$ is an integrated vertex operator, then (4) defines a deformation of the sigma-model action to the first order in $\varepsilon$. It is natural to ask, if the deformation can be continued to higher orders of $\varepsilon$. An obstacle can, in principle, arise already at the order $\varepsilon^2$. Once we deform the action as in (4), the BRST operator gets deformed:

$$Q \to Q + \varepsilon q$$

(11)

Here $q$ is such that:

$$QU + qL = dX$$

(12)

where $L$ is the sigma-model Lagrangian (the existence of such $q$ follows from the fact that $QU$ is a total derivative on-shell, this is in the definition of an integrated vertex operator). Let us consider the following expression: $Q(qX - I_{q^2})$ where $I_{q^2}$ is the Hamiltonian generating $q^2$:

$$q^2L = dI_{q^2} \text{ on-shell}$$

(13)

It was proven in [8] that exists a ghost-number-three operator $W$ such that:

$$Q(qX - I_{q^2}) = dW$$

(14)

with $QW = 0$ (15)
Moreover, the cohomology class of $W$ is the obstacle for extending the deformation to the order $\varepsilon^2$. The same analysis can be extended to higher orders in $\varepsilon$.

**Conclusion:** If the BRST cohomology at the ghost number three is zero then any infinitesimal deformation can be continued to a finite deformation, at least as a power series in $\varepsilon$. However, if the BRST cohomology at the ghost number three is nonzero, then there is a potential obstacle.

**Comment on the derivation in [8]** In [8] we concentrated on the perturbation theory around $AdS_5 \times S^5$, while in the present paper we work in flat space. Some of the assumptions leading to Eq. (14) do not work literally in flat space. For example, conserved charges with nonzero ghost number (besides the BRST charge) do exist in flat space [3]. However, these charges do not depend on $\mathbf{x}$. If we restrict ourselves to the polynomial expressions with large enough degree, then the arguments of [8] do apply.

**Another way of looking at the obstacle** Suppose that we have an unintegrated vertex operator $V$ of the ghost number two. Suppose that we deform the action as in (4) by some integrated operator $\tilde{U}$ (which is related by the descent procedure to some other integrated vertex $\tilde{V}$). The BRST operator gets deformed: $Q \mapsto Q + \varepsilon \tilde{q}$. The question is, does $V$ survive such a deformation? In other words it is possible to correct $V \mapsto V + \varepsilon v$ in such a way that $(Q + \varepsilon \tilde{q})(V + \varepsilon v) = o(\varepsilon^2)$? If the cohomology at the ghost number three is trivial, then this is always possible. Otherwise, further analysis is needed: one has to prove that the ghost number three vertex $\tilde{q}V$ is $Q$-exact.

**A simpler related phenomenon** Similar thing happens at the ghost number one. In flat space, there is a nontrivial cohomology at the ghost number one, corresponding to the global symmetries. However, a generic perturbation of the flat space will kill all this ghost number one cohomology. This is obvious, as generic linearized SUGRA solution does not have any global symmetries. What we want to stress, is the cohomological interpretation of why the ghost number one cohomology gets killed: the existence of the ghost number two cohomology.
1.5 Ghost number three cohomology is nonzero

In this paper we will show that the ghost number three cohomology is nonzero.

The more or less general example of a cohomologically nontrivial ghost number three vertex can be obtained as follows. Let us consider a ghost number two vertex for an exponential linearized solution, for example a Ramond-Ramond excitation:

\[ V_2 = e^{(k \cdot x)} \left( (\theta_L \Gamma^m \lambda_L)(\theta_L \Gamma_m) + [\lambda_L \theta^\geq_4] \right)_\alpha P^{\alpha \beta} \left( (\theta_R \Gamma^m \lambda_R)(\theta_R \Gamma_m) + [\lambda_R \theta^\geq_4] \right)_\beta \]

where \( P^{\alpha \beta} \) is a constant polarization tensor, \( \hat{k}P = \hat{P}k = 0 \). Suppose that \( a_m \) is a constant vector such that \( (a \cdot k) \neq 0 \). Let us consider:

\[ V_3 = (a_m(\lambda_L \Gamma^m \theta_L) - a_m(\lambda_R \Gamma^m \theta_R)) V_2 \tag{17} \]

Notice that \( V_3 \) is BRST closed. We will prove in Section 4.1 that it is not BRST exact. \(^3\) Also notice that \( (\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R) \) is the ghost number one unintegrated vertex corresponding to the global conserved charge of translations (the momentum of the string). Vertices of the ghost number three transform in the same representation of the super-Poincare algebra as the linearized SUGRA solutions. (In particular, the obstacle for \( V_3 \) to be BRST-exact is in fact the scalar \( (k \cdot a) \), so all the polarization is in \( P^{\alpha \beta} \).)

The integrated vertex corresponding to (17) can be constructed as follows. Let \( U_2 \) be the integrated vertex corresponding to \( V_2 \). Let \( j \) be the conserved current corresponding to \( (\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R) \):

\[ Qj = d(\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R) \tag{18} \]

Since \( U_2 \) is an integrated vertex, exists a 1-form \( X \) such that \( QU_2 + qL = dX \). Let us denote:

\[ U_3 = ((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)) U - j \wedge X \tag{19} \]

We have:

\[ QU_3 = -((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)) dX - j \wedge dV_2 - d((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)) \wedge X \equiv d\left(jV_2 - ((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)) X \right) \tag{20} \]

\(^3\)Notice that \( a_m(\lambda_L \Gamma^m \theta_L) + a_m(\lambda_R \Gamma^m \theta_R) = Q(a \cdot x) \), but the relative sign in (17) is minus. With the plus sign it would be \( Q((a \cdot x)V_2) \)

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The next step is:

\[ Q \left( jV_2 - ((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)) X \right) = d\left( ((\lambda_L \Gamma^m \theta_L) - (\lambda_R \Gamma^m \theta_R)) V_2 \right) \]  

(21)

We conclude that \( U_3 \) is the integrated vertex operator corresponding to \( V_3 \). It is a two-form of the ghost number one.

In this paper we will study polynomial vertices, i.e. vertices depending on \( x \) polynomially. The exponential vertices (16) and (17) are sums of infinitely many polynomial vertices. Indeed, \( e^{(k \cdot x)} \) can be decomposed in the Taylor series, and the BRST operator preserves the degree of a polynomial (we assign degree 1 to \( x \) and degree 1/2 to \( \theta \) and \( \lambda \)). Polynomial vertices are, essentially, harmonic polynomials of \( x \) dressed with some appropriate \( \theta \)-dependence.

A low degree example of a polynomial vertex of the ghost number three has been previously constructed in the revised version of [9]. It is equivalent to the linear term in the expansion of \( V_3 \) in powers of \( x \).

1.6 Cohomology at ghost number four and higher is zero

We will prove in Section 5.5 that the pure spinor cohomology is zero at the ghost number four. We have proven in [9] that the pure spinor cohomology is zero at the ghost number greater than four.

This implies that the ghost number three cohomology survives the deformation from flat space-time to generic curved space-time. (However, in the case of a generic curved space-time, there are no ghost number one vertices; therefore the construction of Section 1.5 does not work.)

1.7 Argument for vanishing of the obstruction based on symmetry

Consider an unintegrated vertex operator \( V \) and the corresponding deformation of the sigma-model. Can we extend it to the second order in the deformation parameter? The potential obstacle is the ghost number 3 cohomology class \( W \) defined in Eq. (14). It is bilinear in \( V \):

\[ W = \{ V, V \} \]  

(22)
We will show that $W$ transforms in the linearized supergravity multiplet \textit{(i.e.} in the same representation as $V$, modulo some discrete states). The map $V \otimes V \to W$ given by (22) defined by (22) should commute with the action of the supersymmetry, in particular with the translations. Moreover, one can see that:

\[ \text{deg}(W) = 2 \text{deg}(V) - 2 \]  

(23)

(For example, for the linear dilaton background analyzed in \cite{3}, $V \simeq [\lambda^2 \theta^4]$ and $q \simeq [\lambda \theta^2 \theta]$. ) This implies that $[ V_1, V_2 ]$ can only be nonzero if either $V_1$ or $V_2$ is a low degree polynomial.

It should be possible to complete this argument, which would provide a proof of the vanishing of the obstructions to most of the deformations of the flat space at the second order (but this proof will not work at higher orders).

\section{Type IIB BRST complex vs Maxwell complex}

We will compute the cohomology of the Type IIB BRST complex by relating it to the super-Maxwell BRST complex.

\subsection{Super-Maxwell BRST complex}

The cohomology of the super-Maxwell BRST complex:

\[ Q_{\text{SMaxw}} = \lambda^\alpha \left( \frac{\partial}{\partial \theta^\alpha} + \Gamma^m_{\alpha \beta} \theta^\beta \frac{\partial}{\partial x^m} \right) \]  

(24)

is only nontrivial at the ghost numbers 0 and 1. At the ghost number 0 the cohomology is constants: $V(\theta_L, \theta_R, x) = \text{const}$. At the ghost number 1, the cohomology is in one-to-one correspondence with the solutions of the free Maxwell equation and the free Dirac equation. The vanishing of the cohomology at the ghost numbers two and three is equivalent to the following statements:

1. For any current $j_m$ such that $\partial_m j_m = 0$ always exists the gauge field $F_{mn}$ satisfying $\partial_k F_{lm} = 0$ and $\partial_m F_{mn} = j_n$

2. For any antichiral spinor $\psi$ exists a chiral spinor $\phi$ such that $\Gamma^m \partial_m \phi = \psi$

3. For any $\rho$ exists $j_m$ such that $\partial_m j_m = \rho$
Example: Let us look at the ghost number two cohomology. The leading term in the $\theta$-expansion is either $(\theta \Gamma^m \lambda)(\theta \Gamma^n \lambda)(\theta \Gamma_{mn} \psi(x))$ or $(\theta \Gamma^m \lambda)(\theta \Gamma^n \lambda)(\theta \Gamma_{mnl} \theta) A^l(x)$. Let us for example investigate the first possibility. The following expression is in the cohomology of $\lambda^\alpha \frac{\partial}{\partial \theta^\alpha}$:

\[
(\theta \Gamma^m \lambda)(\theta \Gamma^n \lambda)(\theta \Gamma_{mn} \psi(x))
\]  

(25)

Now let us study the effect of the $\frac{\partial}{\partial x}$-term in (24). For (25) to survive the action of $(\lambda \Gamma^m \theta) \frac{\partial}{\partial x^m}$ we need:

\[
(\lambda \Gamma^l \theta) \frac{\partial}{\partial x^l} ((\theta \Gamma^m \lambda)(\theta \Gamma^n \lambda)(\theta \Gamma_{mn} \psi(x))) = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} \text{(something)}
\]  

(26)

The “something” on the right hand side always exists, because any expression of the form $[\lambda^3 \theta^4]$ annihilated by $\lambda^\alpha \frac{\partial}{\partial \theta^\alpha}$ is automatically in the image of $\lambda^\alpha \frac{\partial}{\partial \theta^\alpha}$. It remains to investigate the possibility of (25) being $Q$-exact:

\[
(\theta \Gamma^m \lambda)(\theta \Gamma^n \lambda)(\theta \Gamma_{mn} \psi(x)) =
\]

\[
= (\lambda \Gamma^l \theta) \frac{\partial}{\partial x^l} ((\theta \Gamma^k \lambda)(\theta \Gamma_k \phi(x)) + \text{(terms of higher orders in } \theta) +
\]

\[
+ \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} \text{(something)}
\]  

(27)

This is possible iff $\psi(x) = \Gamma^m \frac{\partial}{\partial x^m} \phi(x)$. But for any $\psi(x)$ we can find $\phi(x)$ such that $\psi(x) = \Gamma^m \frac{\partial}{\partial x^m} \phi(x)$. This implies that any expression of the type (25) is always BRST-trivial. The class with the leading term $(\theta \Gamma^m \lambda)(\theta \Gamma^n \lambda)(\theta \Gamma_{mnl} \theta) A^l(x)$ is analyzed similarly.

Conclusion:

\[
H^0(Q_{\text{SMaxw}}) = \mathbb{C}
\]  

(28)

\[
H^1(Q_{\text{SMaxw}}) = \text{Maxwell} \bigoplus \text{Dirac}
\]  

(29)

\[
H^{>1}(Q_{\text{SMaxw}}) = 0
\]  

(30)

Here “Maxwell $\bigoplus$ Dirac” stands for the direct sum of the space of solutions of the Maxwell equations and the space of solutions of the Dirac equation.

We now want to relate the super-Maxwell complex to the Type IIB SUGRA complex.
2.2 Definition of the doubled complex

Let us consider the tensor product of two SMaxwell complexes:

\[ Q_{\text{SMaxw} \otimes \text{SMaxw}} = \lambda_L^\alpha \left( \frac{\partial}{\partial \theta_L^\alpha} + \Gamma_m^{\alpha \beta} \frac{\partial}{\partial x_L^m} \right) + \lambda_R^\hat{\alpha} \left( \frac{\partial}{\partial \theta_R^{\hat{\alpha}}} + \Gamma_m^{\hat{\alpha} \hat{\beta}} \frac{\partial}{\partial x_R^m} \right) \]  \( (31) \)

The operator \( Q_{\text{SMaxw} \otimes \text{SMaxw}} \) acts on the space of functions \( F(\lambda_L, \lambda_R, \theta_L, \theta_R, x_L, x_R) \). We will denote \( Q_L \) and \( Q_R \) the two terms on the right hand side of \( (31) \). This is the “doubled” BRST complex. The difference with the Type IIB SUGRA BRST complex is the splitting \( x = x_L + x_R \). In the Type IIB BRST complex there is no separation of \( x \) into \( x_L \) and \( x_R \):

\[ Q_{\text{SUGRA}} = \lambda_L^\alpha \left( \frac{\partial}{\partial \theta_L^\alpha} + \Gamma_m^{\alpha \beta} \frac{\partial}{\partial x_L^m} \right) + \lambda_R^\hat{\alpha} \left( \frac{\partial}{\partial \theta_R^{\hat{\alpha}}} + \Gamma_m^{\hat{\alpha} \hat{\beta}} \frac{\partial}{\partial x_R^m} \right) \]  \( (32) \)

The difference with \( (31) \) is that the left and the right parts have a common \( x \) instead of separate \( x_L \) and \( x_R \); the operator \( Q_{\text{SUGRA}} \) acts on the space of functions \( F(\lambda_L, \lambda_R, \theta_L, \theta_R, x) \).

The computation of the cohomology of \( (31) \) is straightforward, because it is just the tensor product of two Maxwell complexes \( (24) \); therefore the cohomology is:

\[ H^n(Q_{\text{SMaxw} \otimes \text{SMaxw}}) = \bigoplus_{p+q=n} H^p(Q_{\text{SMaxw}}) \otimes H^q(Q_{\text{SMaxw}}) \]  \( (33) \)

where the spaces \( H^p(Q_{\text{SMaxw}}) \) are given by Eqs. \( (28) \), \( (29) \) and \( (30) \).

2.3 Spectral sequence \( \mathcal{E}^{p,q}_r \)

To compute the cohomology of \( (32) \), we relate it to the cohomology of \( (31) \) by the following trick. Let us introduce a formal fermionic variable \( c^m \) and the operator:

\[ Q_{\text{Lie}} = c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) \]  \( (34) \)

(We call it \( Q_{\text{Lie}} \) because it can be thought of as the cohomology of the abelian Lie algebra of translations.) Let us consider the bicomplex:

\[ Q_{\text{tot}} = Q_L + Q_R + Q_{\text{Lie}} \]  \( (35) \)
Consider two ways of computing the cohomology of $Q_{\text{tot}}$. We can either compute first the cohomology of $Q_{\text{Lie}}$, and then consider $Q_L + Q_R$ as a perturbation. Or, first compute $H(Q_L + Q_R)$ and then act on it by $Q_{\text{Lie}}$. This means that there are two different spectral sequences, both converging to $H(Q_{\text{tot}})$.

First $Q_{\text{Lie}}$, then $Q_L + Q_R$: Because of the Poincare lemma, the cohomology of $Q_{\text{Lie}}$ is only nontrivial in the ghost number 0, and is represented by the functions $f(\lambda_L, \lambda_R, \theta_L, \theta_R, x_L + x_R)$. Therefore the Type IIB BRST complex is equivalent to the cohomology of $Q_L + Q_R$ acting on the cohomology of $Q_{\text{Lie}}$:

$$H(Q_{\text{SUGRA}}) = H(Q_L + Q_R, H(Q_{\text{Lie}})) = H(Q_{\text{tot}})$$ (36)

First $Q_L + Q_R$, then $Q_{\text{Lie}}$: now let us first compute the cohomology of $Q_L + Q_R$, and then consider $Q_{\text{Lie}}$ as a perturbation. The resulting spectral sequence will be denoted $E^{p,q}_r$. It computes the cohomology of the SUGRA BRST complex:

$$E^{p,q}_1 = H^p(Q_{\text{Lie}}, \bigoplus_{q_L + q_R = q} H^{q_L}(Q_L) \otimes H^{q_R}(Q_R))$$ (37)

$$E^{p,q}_r \Rightarrow E^{p,q}_\infty$$ (38)

$$\bigoplus_{p+q=n} E^{p,q}_\infty = H^n(Q_{\text{SUGRA}})$$ (39)

Therefore, the only nontrivial components are:

$$E^{p,0}_1 = \Lambda^p C^{10}$$ (40)

$$E^{p,1}_1 = H^p(Q_{\text{Lie}}, \text{SMaxw}_L \bigoplus \text{SMaxw}_R)$$ (41)

$$E^{p,2}_1 = H^p(Q_{\text{Lie}}, \text{SMaxw}_L \otimes \text{SMaxw}_R)$$ (42)

All other components are zero. The only potentially nonzero differentials are:

$$E^{p,0}_1 \rightarrow E^{p+1,0}_1, E^{p,1}_1 \rightarrow E^{p+1,1}_1, E^{p,2}_1 \rightarrow E^{p+1,2}_1$$ (43)

$$E^{p,2}_2 \rightarrow E^{p+2,1}_2, E^{p,1}_2 \rightarrow E^{p+2,0}_2$$ (44)

$$E^{p,2}_3 \rightarrow E^{p+3,0}_3$$ (45)
Therefore, in order to compute the BRST cohomology of SUGRA, we have to compute the cohomology of \( Q_{\text{Lie}} \) with coefficients in various spaces of solutions of the classical electrodynamics, and then compute the differentials \( d_r \).

### 3 Cohomology of classical electrodynamics

In the previous section we related the cohomology of the SUGRA complex to the Lie algebra cohomology of the algebra of translations \( \mathbb{R}^{10} \) with coefficients in the tensor product of solutions of Maxwell and Dirac equations. In order to compute it, we will first compute the cohomology with coefficients in the single space of solutions of Maxwell and Dirac equations. Then, in the next section, we will proceed to compute the cohomology with coefficients in the tensor product of two such spaces.

#### 3.1 Cohomology of \( \mathbb{R}^{10} \) with values in solutions of Maxwell equations

Consider the space of solutions of the vacuum Maxwell equations:

\[
\frac{\partial}{\partial x^m} \frac{\partial}{\partial x^m} A_n = 0
\]  

depending on a parameter \( c^m \), a free Grassmann variable. We need to calculate the cohomology of the operator \( c^m \frac{\partial}{\partial x^m} \) acting on this space.
We will start by computing the cohomology of divergenceless currents. Consider the space \( J \) of one-forms \( j_m(x, c)dx^m \) satisfying \( \frac{\partial}{\partial x^m} j_m(x, c) = 0 \). This is a subspace of the space of all 1-forms \( \Omega^1 \):

\[
0 \to J \xrightarrow{\delta} \Omega^1 \to \Omega^0 \to 0
\]

This gives the long exact sequence of cohomology:

\[
0 \to C^{10} \to C^{10} \to C \to H^1(J) \to 0 \to 0 \to H^2(J) \to 0 \to \ldots
\]

We conclude:

\[
H^0(J) = C
\]

\[
H^1(J) = C
\]

\[
H^{>1}(J) = 0
\]

Now we proceed to the cohomology of the Maxwell complex. A solution of the Maxwell equation is completely characterized by its curvature. The space of solutions is therefore the same as the space of closed 2-forms \( F_{mn}dx^m \wedge dx^n \) satisfying \( \partial^m F_{mn} = 0 \). It is included in the following short exact sequence:

\[
0 \to F \to Z^2 \to J \to 0
\]

where \( Z^2 \) is the space of all closed 2-forms. The corresponding long exact sequence reads:

\[
\Lambda^2 C^d \to \Lambda^2 C^d \to C^d \to H^1(\text{Maxwell}) \to H^1(Z^2) \to C \to H^2(\text{Maxwell}) \to H^2(Z^2) \to 0 \to \ldots
\]

To calculate the cohomology of \( Z^2 \) we use:

\[
0 \to Z^1 \to \Omega^1 \to Z^2 \to 0
\]

and

\[
0 \to C \to \Omega^0 \to Z^1 \to 0
\]

This implies that for \( k > 0 \):

\[
H^k(Z^2) = H^{k+1}(Z^1) = H^{k+2}(C) = \Lambda^{k+2} C^d.
\]

Therefore, we obtain from (53):

\[
H^0(\text{Maxwell}) = \Lambda^2 C^d : f_{[mn]}dx^m \wedge dx^n
\]

\[
H^1(\text{Maxwell}) = C^d \oplus \Lambda^3 C^d : c_k f_{li}dx^k \wedge dx^l \text{ and } f_{[klm]}c^k dx^l \wedge dx^m
\]

\[
H^2(\text{Maxwell}) = C \oplus \Lambda^4 C^d : c_k q_{ij}dx^k \wedge dx^j \text{ and } f_{[ijkl]}c^{ij} dx^k \wedge dx^l
\]

\[
H^{n>2}(\text{Maxwell}) = \Lambda^{n+2} C^d : f_{[ij_1 \ldots j_{n+2}]^j} c^{i_1} \ldots c^{i_n} dx^{j_1} \wedge dx^{j_{n+2}}
\]
Notice that all these cohomology classes are represented by the constant field strength. In other words, the dilatation symmetry $x^m \frac{\partial}{\partial x^m}$ acts as zero in cohomology.

### 3.2 Cohomology of $R^{10}$ with values in solutions of Dirac equations

Let $\mathcal{D}$ denote the space of solutions of the Dirac equations, and $\mathcal{S}$ the space of chiral-spinor-valued functions, and $\mathcal{S}^*$ the antichiral-spinor-valued functions. There is a short exact sequence:

$$0 \to \mathcal{D} \to \mathcal{S} \xrightarrow{\Gamma^m \frac{\partial}{\partial x^m}} \mathcal{S}^* \to 0 \quad (60)$$

This leads to the long exact sequence of the cohomologies:

$$0 \to C^{16} \to C^{16} \to H^1(\mathcal{D}) \to 0 \to H^2(\mathcal{D}) \to 0 \to \cdots \quad (61)$$

Therefore:

$$H^0(\text{Dirac}) = C^{16} : \text{constant spinors} \quad (62)$$

$$H^1(\text{Dirac}) = C^{16} : \hat{c}\Psi \text{ where } \Psi \text{ is constant} \quad (63)$$

$$H^{n>1}(\text{Dirac}) = 0 \quad (64)$$

### 4 Cohomology of the tensor product of two classical electrodynamics

Having computed the cohomology of $Q_{\text{Lie}}$ with values in Maxwell and Dirac solutions, we will now use it to compute the cohomology with values in the tensor product $\text{SMaxw}_L \otimes \text{SMaxw}_R$. Again, we will use some spectral sequence. In order to distinguish it from the spectral sequence of Section 2, we will use the notation $E^{p,q}_r$ instead of $E^{p,q}_r$.

\footnote{Unfortunately, because of certain limitations of LaTeX, we can not afford similar notations for the differentials $d_r$.}
4.1 Dirac-Dirac sector

4.1.1 Spectral sequence $E_p^{p,q}$

The following group is part of the ghost number 3 cohomology:

$$H^1(Q_{\text{Lie}}, \text{Dirac} \otimes \text{Dirac})$$  \hspace{1cm} (65)

In this section we will calculate this cohomology group.

The differential $Q_{\text{Lie}}$ is realized on the space of bispinors $P^{\alpha\dot{\beta}}(x_L, x_R, c)$ satisfying:

$$\frac{\partial}{\partial x^m_L} \Gamma^m_{\alpha\alpha'} P^{\alpha'\dot{\beta}}(x_L, x_R, c) = 0 \hspace{1cm} (66)$$

$$\frac{\partial}{\partial x^m_R} P^{\alpha\dot{\beta}'}(x_L, x_R, c) \Gamma^m_{\dot{\beta}'\dot{\beta}} = 0 \hspace{1cm} (67)$$

The differential $Q_{\text{Lie}}$ acts as follows:

$$Q_{\text{Lie}} P^{\alpha\dot{\beta}} = c^m \left( \frac{\partial}{\partial x^m_L} - \frac{\partial}{\partial x^m_R} \right) P^{\alpha\dot{\beta}}$$ \hspace{1cm} (68)

Let us introduce the filtration by the degree $N$:

$$N = \frac{1}{2} \left( c \frac{\partial}{\partial c} + x_L \frac{\partial}{\partial x_L} - x_R \frac{\partial}{\partial x_R} \right)$$ \hspace{1cm} (69)

Then $c^m \frac{\partial}{\partial x^m_L}$ is the leading (of degree zero) term in $Q_{\text{Lie}}$ and $-c^m \frac{\partial}{\partial x^m_R}$ is subleading (of degree one). Let us calculate the cohomology of $Q_{\text{Lie}}$ using the spectral sequence of this filtration. The first page $E_1^{p,q}$ is:

$$E_1^{p,q} = H^{p+q} \left( c^m \frac{\partial}{\partial x^m_L}, \frac{F^p}{F^{p+1}}(\text{Dirac} \otimes \text{Dirac}) \right)$$ \hspace{1cm} (70)

$$d_1 = -c^m \frac{\partial}{\partial x^m_R} : E_1^{p,q} \rightarrow E_1^{p+1,q}$$ \hspace{1cm} (71)

where $F^p$ consists of polynomials with $N \geq p$. Schematically, $E_1^{p,q}$ consists of expressions of the form

$$P^{\alpha\dot{\beta}} = [c^{p+q} x_L^{n+p} x_R^{n+q}]$$ \hspace{1cm} (72)
satisfying both left and right Dirac equations, representing the cohomology of \( c^m \frac{\partial}{\partial x^m_R} \). Because of Section 3.1, the cohomology of \( c^m \frac{\partial}{\partial x^m_L} \) is localized on \( n + p = 0 \), and either \( p + q = 0 \) or \( p + q = 1 \). This means that the only nontrivial components of \( E_1^{p,q} \) are the ones represented by the following expressions:

\[
E_1^{-m,m} : P(x_R^{\otimes 2m}) \\
E_1^{-m+1,m} : \hat{c}R(x_R^{\otimes(2m-1)})
\]

(73)

(74)

Here, as usual, we denote \( \hat{c} = c^m \Gamma_m \).

The only nontrivial differential is \( d_1 : E_1^{-m,m} \rightarrow E_1^{-m+1,m} \). The cohomology of this differential is \( E_2^{p,q} \). Notice that \( d_2 = 0 \). Indeed, the construction of \( d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1} \) involves the inversion of \( c^m \frac{\partial}{\partial x^m_L} \) and therefore any expression in the image of \( d_2 \) is necessarily in the image of \( x^m_L \frac{\partial}{\partial x^m_L} \). But \( x^m_L \frac{\partial}{\partial x^m_L} \) acts as zero on \( E_1 \) and therefore also on \( E_2 \).

Therefore our spectral sequence converges at the second page: \( E_2 = E_\infty \).

4.1.2 The image of \( d_1(E_1^{-m,m}) \)

The condition that the cohomology class of an expression of the form (74) is cancelled by the \( d_1 \) of an expression of the form (73) is:

\[
R = -\frac{1}{10} \Gamma^m \frac{\partial}{\partial x^m_R} P \quad (75)
\]

with

\[
\frac{\partial}{\partial x^m_R} P \Gamma^m = 0 \quad (76)
\]

Indeed, for any \( P(x_R) \) solving the right Dirac equation (76) we can tautologically write:

\[
\hat{c}R = -c^m \frac{\partial}{\partial x^m_R} P + c^m \frac{\partial}{\partial x^m_L} \left( \hat{x}_L R + x^m_L \frac{\partial}{\partial x^m_R} P \right) 
\]

(77)

Then (75) is the necessary and sufficient condition that \( \Psi := \hat{x}_L R + x^m_L \frac{\partial}{\partial x^m_R} P \) satisfies both \( \frac{\partial}{\partial x^m_R} \Psi \Gamma^m = 0 \) and \( \Gamma^m \frac{\partial}{\partial x^m_R} \Psi = 0 \). (And, moreover, any presentation of \( \hat{c}R \) as \( -c^m \frac{\partial}{\partial x^m_R} P \) plus \( c^m \frac{\partial}{\partial x^m_L} \) (smth) will necessarily be of the form (77).)
Comment Those $\mathcal{P}$ which satisfy $\Gamma^m \frac{\partial}{\partial x^m_R} \mathcal{P} = 0$ are in the kernel of $d_1$, and therefore they form $E_2^{-m,m}$. They are in the ghost number two cohomology (the Ramond-Ramond fields). We have previously explained that $d_2$ is zero; if it were not zero, it would have killed the ghost number two cohomology.

Notice that any $\mathcal{P}$ satisfying (75) and (76) is automatically harmonic: $\Delta \mathcal{P} = 0$, therefore (75) and (76) imply that $\mathcal{R}$ satisfies the left Dirac equation:

$$\Gamma^m \frac{\partial}{\partial x^m_R} \mathcal{R} = 0 \quad (78)$$

This means that:

$$\Gamma^m \frac{\partial}{\partial x^m_R} \mathcal{R} \text{ is an obstacle for the triviality of } \mathcal{R} \quad (79)$$

In the rest of this section we will prove that this is the only obstacle, i.e. any $\mathcal{R}$ satisfying (78) can be represented as (75), (76).

4.1.3 Proof that (79) is the only obstacle to the triviality of $\mathcal{R}$

In this section we will prove that if $\mathcal{R}$ is a polynomial of nonzero degree (i.e. not a constant), than (79) is the only obstacle to the triviality of $\mathcal{R}$.

Notice that it is always possible to solve for $\mathcal{P}$ to satisfy (75), but $\mathcal{P}$ will not necessarily satisfy (76). But if the Dirac equation (78) is satisfied, then we have:

$$\Gamma^m \partial_m \partial_n \mathcal{P} \Gamma_n = 0 \quad (80)$$

$$\Delta \mathcal{P} = 0 \quad (81)$$

We will now prove that (80) and (81) imply that exist $\mathcal{P}_L$ and $\mathcal{P}_R$ such that:

$$\mathcal{P} = \mathcal{P}_L + \mathcal{P}_R \quad (82)$$

where

$$\Gamma^m \frac{\partial}{\partial x^m_R} \mathcal{P}_L = 0 \quad \text{and} \quad \frac{\partial}{\partial x^m_R} \mathcal{P}_R \Gamma^m = 0 \quad (83)$$

This implies that $\mathcal{P}$ can be chosen to satisfy the right Dirac equation, and therefore $\mathcal{R}$ is in the image of $d_1$. 
**Proof.** Let us switch from the bispinor notations to the forms notations. The left Dirac operator corresponds to $D_L = d + \delta$ while the right Dirac operator is $D_R = (-1)^{F+1}(d - \delta)$. Eq. (81) implies that $(\delta d + d\delta)P = 0$ while Eq. (80) implies that $(\delta d - d\delta)P = 0$. Therefore we have:

$$d\delta P = \delta dP = 0 \quad (84)$$

We will now prove that under the condition (84) exist $P_L$ and $P_R$ such that:

$$P = P_L + P_R$$
$$D_L P_L = D_R P_R = 0 \quad (85)$$

It is useful to keep in mind the cohomology of the de Rham $d$ on harmonic forms is:

$$H^0(d, \ker\Delta) = H^1(d, \ker\Delta) = \mathbb{C}, \quad H^{\geq 1}(d, \ker\Delta) = 0 \quad (86)$$

(the $H^1(d, \ker\Delta)$ is generated by $x^m dx^m$).

**Case when $P$ is a 5-form** In this case we will write $P^{(5)}$ instead of $P$. Since $d\delta P^{(5)} = 0$, exists a harmonic 3-form $P^{(3)}$ such that:

$$\delta P^{(5)} = dP^{(3)} \quad (87)$$

Similarly, as $\delta dP^{(5)} = 0$, exists a harmonic 7-form $P^{(7)}$ such that:

$$dP^{(5)} = \delta P^{(7)} \quad (88)$$

Furthermore, there exist harmonic $P^{(1)}$ and $P^{(9)}$ such that:

$$\delta P^{(3)} = dP^{(1)} \quad \text{and} \quad dP^{(7)} = \delta P^{(9)} \quad (89)$$

This implies that $\delta P^{(1)} = 0$ and therefore exists a harmonic form $S^{(2)}$ such that $P^{(1)} = \delta S^{(2)}$. Similarly, $P^{(9)} = dS^{(8)}$. Therefore the following $P_L$ and $P_R$ satisfy (85):

$$P_L = \frac{1}{2} \left( (P^{(5)} - (P^{(3)} + dS^{(2)}) - (P^{(7)} + \delta S^{(8)})) \right) \quad (90)$$
$$P_R = \frac{1}{2} \left( (P^{(5)} + (P^{(3)} + dS^{(2)}) + (P^{(7)} + \delta S^{(8)})) \right) \quad (91)$$
Case when \( P \) is a 3-form plus 7-form  The 7-form part of \( P \) is related to the 3-form part by the condition that \( P \) is self-dual. In this case we will write \( P^{(3)} + P^{(7)} \) instead of \( P \). (This \( P^{(3)} \) has nothing to do with the \( P^{(3)} \) of the previous paragraph.) Since \( d\delta P^{(3)} = 0 \) and \( \delta \delta P^{(3)} = 0 \), exists harmonic \( P^{(1)} \) such that:

\[
\delta P^{(3)} = dP^{(1)} \tag{92}
\]

This implies that \( \delta P^{(1)} = 0 \). Similarly, exists a harmonic \( P^{(5)} \) such that:

\[
dP^{(3)} = \delta P^{(5)} \tag{93}
\]

This automatically implies:

\[
dP^{(5)} = \delta P^{(7)} \tag{94}
\]

Also exists a harmonic \( P^{(9)} \) such that:

\[
\delta P^{(9)} = dP^{(7)} \text{ and } dP^{(9)} = 0 \tag{95}
\]

We take:

\[
P_L = \frac{1}{2} \left( -P^{(1)} + P^{(3)} - P^{(5)} + P^{(7)} - P^{(9)} \right) \tag{96}
\]

\[
P_R = \frac{1}{2} \left( P^{(1)} + P^{(3)} + P^{(5)} + P^{(7)} + P^{(9)} \right) \tag{97}
\]

Case when \( P \) is a 1-form plus a 9-form  Now suppose that \( P = P^{(1)} + P^{(9)} \). Let us first assume that the degree of \( P \) is more than 1. We have:

\[
d\delta P^{(1)} = 0 \Rightarrow \delta P^{(1)} = 0 \Rightarrow P^{(1)} = \delta S^{(2)} \tag{98}
\]

Similarly \( P^{(9)} = dS^{(8)} \). Now we have:

\[
P_L = \frac{1}{2} \left( (\delta + d)S^{(2)} + \frac{1}{2}(d + \delta)S^{(8)} \right) \tag{99}
\]

\[
P_R = \frac{1}{2} \left( (\delta - d)S^{(2)} + \frac{1}{2}(d - \delta)S^{(8)} \right) \tag{100}
\]

Now consider the case when the degree of \( P \) is one, i.e. \( P \) is linear in \( x \). In this case we can have \( \delta P^{(1)} = \text{const} \). This corresponds to the \( R \) of \( \{75\} \) a constant proportional to unit matrix. The corresponding element of \( H^1(Q_{\text{Lie}} \otimes \text{Dirac} \otimes \text{Dirac}) \) is:

\[
(\theta_L \Gamma^m \lambda_L) (\theta_L \Gamma_m \theta_R) (\lambda_R \Gamma^n \theta_R) \tag{101}
\]

It corresponds to the following ghost number three vertex:

\[
(\theta_L \Gamma^m \lambda_L)(\theta_L \Gamma^p \lambda_L) (\theta_L \Gamma_m \Gamma_p \theta_R) (\lambda_R \Gamma^n \theta_R) \tag{102}
\]
Conclusion  We conclude that the main obstacle for (74) to be trivial is 
$\Gamma^m \partial_m R \neq 0$. (And besides that, there is also a case when $R$ is a constant 
times a unit matrix, which results in a nontrivial vertex (102).) If $\Gamma^m \partial_m R \neq 0$, then there is a nontrivial cohomology class of the form:
\[ c^n \Gamma^n R + r_1 [cx_L x_R^{(2m-2)}] + r_2 [cx_L x_R^{(2m-3)}] + \ldots + r_{2m-1} [cx_L^{(2m-1)}] \]  (103)

Indeed, acting on the leading term $c^n \Gamma^n R$ with $-c^m \frac{\partial}{\partial x_R}$ we get an expres-
sion of the form $[c^2 x_R^{2m-2}]$, which does not depend on $x_L$ and therefore is 
annihilated by $c^m \frac{\partial}{\partial x_L}$. But since $H^2 (c \frac{\partial}{\partial x}, \text{Dirac}) = 0$, this expression is 
 automatically of the form $c^m \frac{\partial}{\partial x_L} [cx_L x_R^{(2m-2)}]$. Continuing this process we get

4.1.4  Proof that $V_3$ of Eq. (17) is BRST nontrivial

Let us consider the ghost number three vertex $V_3$ given by Eq. (17), and 
expand it in the Taylor series in $x$ and $\theta$. We assign to $x$ degree 1 and to $\lambda$ 
and $\theta$ degree 1/2. The BRST operator preserves this degree. In particular, 
every term in the expansion is a BRST-closed polynomial of $x, \lambda, \theta$. It is 
enough to prove the nontriviality term by term. Let us consider the extended 
 space $(x_L, x_R, \lambda_L, \lambda_R, \theta_L, \theta_R)$. In this extended space, we get:
\[ V_3 = (Q_L + Q_R) ((a \cdot (x_L - x_R))V_2) \]  (104)

The corresponding element of $H^1 \left( c \left( \frac{\partial}{\partial x_L} - \frac{\partial}{\partial x_R} \right), \text{Dirac} \otimes \text{Dirac} \right)$ is given by:
\[ (a \cdot c)Pe^{k(x_L + x_R)} \]  (105)

Consider the expansion in powers of $x_L$. The leading term is $(a \cdot c)Pe^{kz_R}$. 
We observe:
\[ (10(c \cdot a) - \partial \dot{a}) = c \frac{\partial}{\partial x_L} (4\dot{x}_L \dot{a} + 5\dot{a} \dot{x}_L) \]  (106)

and $(4\dot{x}_L \dot{a} + 5\dot{a} \dot{x}_L)Pe^{kz_R}$ satisfies the left Dirac equation. Therefore (105) is 
equivalent to $\frac{1}{10} \partial \dot{a}Pe^{kz_R}$. Comparing this with (103), we get:
\[ R = \frac{1}{10}\hat{a}Pe^{kx_R} \]  

(H7)

\[ \Gamma^m \frac{\partial}{\partial x^m_R} R = \frac{1}{5}(a \cdot k)Pe^{kx_R} \neq 0 \]  

(108)

Then (79) implies that \( V_3 \) represents a nontrivial cohomology class.

**Ghost number three vertex of [9]** can be obtained as the first order of expansion of (105) in powers of \( x \). Indeed, at the first order of the \( x \)-expansion

\[ R = \frac{1}{10}\hat{a}(k \cdot x_R)P. \]

Notice that the expression:

\[ \left( \hat{a}(k \cdot x_R) - \frac{1}{5}\hat{x}_R(k \cdot a) \right)P \]  

(109)

satisfies the left Dirac equation (we use \( \hat{k}P = 0 \)). Therefore

\[ R = \frac{1}{10}\hat{a}(k \cdot x_R)P \]

is equivalent to

\[ R = \frac{1}{50}\hat{x}_R(k \cdot a)P. \]

Therefore the leading term of the \( x \)-linear part of (105) is equivalent to \( \frac{1}{50}\hat{c}\hat{x}_R(k \cdot a)P \) which is the leading term of the vertex constructed in [9].

**4.2 Maxwell-Maxwell sector**

In this section we will compute the cohomology of \( c^m \left( \frac{\partial}{\partial x^m_L} - \frac{\partial}{\partial x^m_R} \right) \) on the solutions of bi-Maxwell equations.

**4.2.1 Bi-Maxwell equations**

Solutions of bi-Maxwell equations are defined as expressions of the form:

\[ dx^p_L \wedge dx^q_L \left( \frac{\partial}{\partial x^p_L} A(x_L, x_R)_q ; [m] \frac{\partial}{\partial x^m_R} \right) dx^m_R \wedge dx^n_R \]  

(110)

satisfying the left and right Maxwell equations:

\[ \frac{\partial}{\partial x^p_L} \frac{\partial}{\partial x^p_L} A(x_L, x_R)_q ; [m] \frac{\partial}{\partial x^m_R} = 0 \]  

(111)

\[ \frac{\partial}{\partial x^p_L} A(x_L, x_R)_q ; [m] \frac{\partial}{\partial x^m_R} \frac{\partial}{\partial x^n_R} = 0 \]  

(112)
Notice that we have left and right indices, separated with the semicolon. We use the notations \( \frac{\partial}{\partial x} \). The expression \( \frac{\partial}{\partial x} \phi \) means the same as \( \partial \phi / \partial x \). The sole purpose of such notations is to improve the readability of the formulas, as they allow us to naturally separate left and right indices.

### 4.2.2 Spectral sequence \( E_{r}^{p,q} \)

**Definition** As in Section 4.1, we will use the filtration by the powers of \( x_{L} \), i.e. treat \( x_{L} \) as being small. The elements of \( E_{r}^{p,q} \) are of the type:

\[
E_{r}^{p,q} : dx_{L} \wedge dx_{L} \left[ c^{p+q} x_{L}^{p+q} x_{R}^{n+q} \right] dx_{R} \wedge dx_{R} + \ldots
\] (113)

where \( \ldots \) stands for terms of the type \( dx_{L} \wedge dx_{L} \left[ c^{p+q} x_{L}^{p+s} x_{R}^{n+q-s} \right] dx_{R} \wedge dx_{R} \) with \( s > 0 \), which are factored out when we consider \( F^{p} (\text{Maxwell} \otimes \text{Maxwell}) \) modulo \( F^{p+1} (\text{Maxwell} \otimes \text{Maxwell}) \). For a polynomial element \( A_{q;m} \), of the total order \( M \) in \( x_{L} \) and \( x_{R} \), there is an expansion in powers of \( x_{L} \):

\[
A(x_{L}, x_{R}) = A_{q;m}(x_{R}) + A_{q;m}^{(1)}(x_{L}, x_{R}) + \ldots + A_{q;m}^{(N)}(x_{L})
\] (114)

where \( A_{q;m}^{(0)} \) does not depend on \( x_{L} \), \( A_{q;m}^{(1)} \) is linear in \( x_{L} \), etc.

**The structure of \( E_{2}^{p,q} \)** The following is the most general (up to the \( c \frac{\partial}{\partial x_{L}} \)-exact terms) ansatz for the leading term \( A_{q;m}^{(0)} \):

\[
A_{q;m}^{(0)} = c_{p} d_{L}^{p} \wedge d_{L}^{q} A(x_{R})_{q;m} \left[ m \left[ \frac{\partial}{\partial n} \right] \right] dx_{R}^{m} \wedge dx_{R}^{n} + \ldots
\] (115)

with

\[
A(x_{R})_{q;m} \left[ m \left[ \frac{\partial}{\partial n} \right] \right] \frac{\partial}{\partial n} = 0
\] (116)

\[
B(x_{R})_{pqr;m} \left[ m \left[ \frac{\partial}{\partial n} \right] \right] \frac{\partial}{\partial n} = 0
\] (117)

\[
\partial^{q} A(x_{R})_{q;m} \left[ m \left[ \frac{\partial}{\partial n} \right] \right] = 0
\] (118)

\[
\partial^{p} B(x_{R})_{qrs;m} \left[ m \left[ \frac{\partial}{\partial n} \right] \right] = 0
\] (119)

\( A_{q;m}^{(0)} \) represents an element of \( E_{2}^{M-1, M+1} \), see (113) (120)

Here \( A(x_{R})_{q;m} \) and \( B(x_{R})_{pqr;m} = B(x_{R})_{pqr;m} \) are polynomials in \( x_{R} \) of the order \( M \). They correspond to the two terms in (57). Eqs. (116) and (117) enforce the right Maxwell equation. (The left Maxwell equation is
automatically satisfied because \( A \) does not depend on \( x_L \). Eqs. (118) and (119) are the conditions for being in the kernel of \( d_1 \). In other words, those are the conditions for the existence of \( A^{(1)}(c, x_L, x_R)_p; m \) linear in \( x_L \) and \( c \) such that:

\[
e_{\tau(L)} \frac{\partial}{\partial x_R} \left( e^p dx^p_L \land dx^q_L A(x_R)_q; m \overset{\leftarrow}{\partial n} + e^p dx^q_L \land dx^r_L B(x_R)_{pqr}; m \overset{\leftarrow}{\partial n} \right) dx^m_R \land dx^n_R = 0
\]

and \( A^{(1)}(x_L, x_R)_q; m \overset{\leftarrow}{\partial n} = 0 \) (122)

Eq. (118) is the vanishing of the obstacle proportional to the first term in (58), and Eq. (119) is to avoid hitting the second term in (58).

Remember that we are working in the polynomial sector, i.e. \( B_{pqr} : [m(x_R) \overset{\leftarrow}{\partial n}] \) is a homogeneous polynomial in \( x_R \). Moreover, we assume that the degree of the polynomial is nonzero:

\[
B_{pqr} : [m(x_R) \overset{\leftarrow}{\partial n}] \neq \text{const}
\]

(123)

Then (119) implies that we can remove the term \( e^p dx^q_L \land dx^r_L B(x_R)_{pqr}; m \overset{\leftarrow}{\partial n} dx^m_R \land dx^n_R \), by adding to \( A^{(0)} \) an element in \( d_1(E_1^{M_L 1, M_R 1}) \). Indeed, this is equivalent to the existence of the following two objects:

- \( C(x_R)_{pq}; m \) satisfying \( C(x_R)_{pq}; m \overset{\leftarrow}{\partial n} \overset{\leftarrow}{\partial m} = 0 \) and

- \( G(x_L, x_R)_{pq}; m \) linear in \( x_L \) satisfying left and right Maxwell equations:

\[
\frac{\partial}{\partial x_L^p} G(x_L, x_R)_{qr}; m \overset{\leftarrow}{\partial n} = 0
\]

\[
\frac{\partial}{\partial x_R^p} G(x_L, x_R)_{pq}; m \overset{\leftarrow}{\partial n} = 0
\]

\[
G(x_L, x_R)_{pq}; m \overset{\leftarrow}{\partial n} \overset{\leftarrow}{\partial m} = 0
\]
such that:

\[ c^p dx^q_L \wedge dx^r_L B(x_R)_{pq} ; [m \n \partial_n] \ dx^m_R \wedge dx^n_R + O(x_L) = \]

\[ c^k \left( \frac{\partial}{\partial x^k_L} - \frac{\partial}{\partial x^k_R} \right) \left( dx^p_L \wedge dx^q_L C(x_R)_{pq} ; [m \n \partial_n] \ dx^m_R \wedge dx^n_R + \right. \]

\[ \left. + dx^p_L \wedge dx^q_L G(x_L, x_R)_{pq} ; [m \n \partial_n] \ dx^m_R \wedge dx^n_R \right) + \]

\[ + c_p dx^p_L \wedge dx^q_L \tilde{A}(x_R)_{q ; [m \n \partial_n] \ dx^m_R \wedge dx^n_R + O(x_L) \] (124)

Here \( \tilde{A}(x_R)_{q ; [m \n \partial_n] \) is some correction to the \( A(x_R)_{q ; [m \n \partial_n] \) of (115). (In other words, when we gauge away the \( B \)-term, this leads to some change in the \( A \) term: \( A \to A + \tilde{A}. \)) The existence of such \( C(x_R)_{pq ; m} \) and \( G(x_L, x_R)_{pq ; m} \) follows from (119) and the fact that \( H^3(Q_{\text{Lie}}, \text{Maxw}) \) is zero in polynomials of the degree \( > 0 \), in the following way. Eq. (119) implies that exists \( C(x_R)_{pq ; m} \) satisfying the right Maxwell equation, such that:

\[ B_{pqr} ; [m \n \partial_n] = -\partial_p C_{qr} ; [m \n \partial_n] \] (125)

Therefore, in computing the first line on the RHS of (124), the \( c^k \frac{\partial}{\partial x^k_L} \) gives zero as \( C_{pq ; m} \) does not depend on \( x_L \), and when acting with \( -c^k \frac{\partial}{\partial x^k_R} \), we get:

\[ -c^k \frac{\partial}{\partial x^k_R} dx^p_L \wedge dx^q_L C(x_R)_{pq} ; [m \n \partial_n] \ dx^m_R \wedge dx^n_R \] (126)

This has to be understood as an element of \( E^{-M-1/2, M+1/2}_1 \), i.e. modulo the image of \( c^k \frac{\partial}{\partial x^k_L} \). This ambiguity is described by the second line on the RHS of (124), the term containing \( G(x_L, x_R)_{pq ; m} \). This term can be used to remove the components other than those listed in Eq. (57); the component \( C^d \) corresponds to \( \tilde{A}_{q ; m} \), and the component \( \Lambda^3 C^d \) kills the \( B \)-term.

We conclude that we can get rid of the \( B \)-term in (115) by adding to \( A \) an element in \( d_1(E^{M+1/2, M+1/2}_1) \).

\(^5\)Notice that we are using the results about \( H(Q_{\text{Lie}}, \text{Maxw}) \) in two different ways. First, we use \( H^1(Q_{\text{Lie}}, \text{Maxw}_L) \) to argue that the leading term can be reduced to the form (115). Then we use the vanishing of \( H^3(Q_{\text{Lie}}, \text{Maxw}_R) \) in polynomials of high enough degree to remove the \( B \)-term by adding \( d_1(\text{smth}) \).
Now let us consider the case when $B_{pq};[m(x_R) \partial n]$ is constant:
\[
B_{pq};[m(x_R) \partial n]= \text{const} \quad (127)
\]
Consider the total antisymmetrization:
\[
B_{[pqrmn]} = B_{[pq];m(x_R) \partial n]}. \quad (128)
\]
In this case the $B$-term in (115) cannot be gauged away, as $B_{[pqrmn]}$ represents a nontrivial cohomology class of $H^3(\text{Maxw}) = \Lambda^5 \mathbf{C}^{10}$. However, we will show in Section 5.5 that this is cancelled by the $d_2 : \mathcal{E}_2^{1,2} \to \mathcal{E}_2^{3,1}$. In other words, for our ansatz to survive on $\mathcal{E}_3^{1,2}$ we need to put $B_{[pqrmn]}$ to zero:
\[
B_{[pqrmn]} = 0 \quad (129)
\]

4.2.3 Double field strength

Let us therefore assume that $B(x_R)_{pq};m = 0$. Can the remaining $A$-term also be in the image of $d_1$? Let us define the double field strength $F_{[pq];[mn]}$ as follows:
\[
F_{[pq];[mn]} = \partial_{[p}A_{q];[m} \partial_n] \quad (130)
\]
This double field strength has the following properties:
\[
F_{[pq];[mn]} = 0 \quad \text{(total antisymmetrization)} \quad (131)
\]
\[
\partial_{[p}F_{q];mn] = 0 \quad (132)
\]
\[
F_{qr};[mn] \partial_k = 0 \quad (133)
\]
\[
\partial^p F_{[pq];[mn]} = 0 \quad (134)
\]
\[
F_{[pq];[mn]} \partial^m = 0 \quad (135)
\]
\[
\Delta F_{[pq];[mn]} = 0 \quad (136)
\]

4.2.4 Double field strength is the obstacle to triviality

We will now show that $A$ is trivial iff $F_{pq};mn = 0$.

We have to understand under which conditions the class with the leading term (115) is trivial, i.e. can be obtained by acting with $c^m \left( \frac{\partial}{\partial x^p} - \frac{\partial}{\partial x^R} \right)$ on
something:

\[ c_p dx_L^p \wedge dx_L^q A(x_R)^q;[m \stackrel{\leftarrow}{\partial}_n] \ dx_R^m \wedge dx_R^n + \ldots = \]

\[ = c^j \left( \frac{\partial}{\partial x_L^j} - \frac{\partial}{\partial x_R^j} \right) \left( dx_L^p \wedge dx_L^q W(x_R)^{pq};[m \stackrel{\leftarrow}{\partial}_n]\ dx_R^m \wedge dx_R^n + \ldots \right) \]

This property is equivalent to the existence of \( W(x_R)^{pq};[m \stackrel{\leftarrow}{\partial}_n] \) satisfying:

\[ A_q;[m \stackrel{\leftarrow}{\partial}_n] = \partial^p W_{pq};[m \stackrel{\leftarrow}{\partial}_n] \]

\[(138)\]

with \( \partial^p W_{qr};[m \stackrel{\leftarrow}{\partial}_n] = 0 \)

\[(139)\]

and \( W_{pq};[m \stackrel{\leftarrow}{\partial}_n] \partial_n = 0 \)

\[(140)\]

Notice that the ghost number two vertices correspond to \( W_{pq};[m \stackrel{\leftarrow}{\partial}_n] \) satisfying \[(139), (140)\] and \( \partial^p W_{pq};[m \stackrel{\leftarrow}{\partial}_n] = 0 \) instead of \[ (138) \].

If \( A \) can be expressed through \( W \) as in \[ (138) \], then we have:

\[ F_{pq;}mn = \partial^p A_q;[m \stackrel{\leftarrow}{\partial}_n] = - \partial^p \partial^r W_{qr};[m \stackrel{\leftarrow}{\partial}_n] = \frac{1}{2} \partial^r \partial^r W_{pq};[m \stackrel{\leftarrow}{\partial}_n] = 0 \]

\[(141)\]

This means that:

\[ F_{pq;}mn \neq 0 \text{ is an obstacle to the triviality of } A \]

\[(142)\]

We will now prove that this is the only obstacle. In other words, if \( F_{pq;}mn = 0 \), then \[ (114) \] is cohomologically trivial.

Let \( M \) be the space of polynomial expressions of the form:

\[ \Phi(dx_L, x);[mn] \text{ satisfying } \Phi[mn \stackrel{\leftarrow}{\partial}_k] = 0 \text{ and } \Phi_{mn} \stackrel{\leftarrow}{\partial}_n = 0 \]

\[(143)\]

Let \( M^N \) be the subspace of \( M \) consisting of polynomials of the order \( N \) in \( x \), i.e. \( x^p \partial_{dx^p} \Phi = N \Phi \). Notice that such \( \Phi_{mn} \) are automatically harmonic. The operator \( d_L + \delta_L \) acts on such expressions, and is nilpotent:

\[ \ldots \xrightarrow{d_L + \delta_L} M^N \xrightarrow{d_L + \delta_L} M^{N-1} \xrightarrow{d_L + \delta_L} \ldots \]

\[(144)\]
Lemma

$$H^N(d_L + \delta_L, \mathcal{M}) = H^N(d_L, \mathcal{M}) = H^N(\delta_L, \mathcal{M}) = 0 \text{ for } N > 0 \quad (145)$$

Indeed, $d_L$ is acyclic, as $H^N(d_L)$ is the same as already computed in Section 3.1 cohomology of the translations algebra on the solutions of the Maxwell equations, and it is zero for $N > 0$. This means that it is always possible to gauge away the term with the highest number of $dx_L$, and therefore the cohomology of $d_L + \delta_L$ is zero. The proof of $H^N(\delta_L) = 0$ is identical to the proof of $H^N(d_L) = 0$ after applying the Hodge dual operation on the c ghosts.

Eq. (116) implies that the expression $dx_L p A_p; [m \partial_n] \in \mathcal{M}$. Eq. (118) implies that it is annihilated by $\delta_L$. Since $H^N(\delta_L) = 0$, exists $\Phi^{(2)} \in \mathcal{M}$ such that:

$$\begin{align*}
dx_L A_p; [m \partial_n] &= \delta_L (dx_L p \wedge dx_L \Phi^{(2)}_{pq}; mn) \\
&= \delta_L (\Phi^{(2)}_{mn} + \Phi^{(4)}_{mn} + \Phi^{(6)}_{mn} + \Phi^{(8)}_{mn} + \Phi^{(10)}_{mn})
\end{align*} \quad (146)$$

(Here each $\Phi^{(2j)}_{mn}$ is a polynomial of the degree $2j$ in $dx_L$.) Indeed, as elements of $\mathcal{M}$ are harmonic functions, $d_L \delta_L \Phi^{(2)}_{mn} = 0$ implies $\delta_L d_L \Phi^{(2)}_{mn} = 0$ and therefore the existence of $\Phi^{(4)}$ such that $d_L \Phi^{(2)}_{mn} + \delta_L \Phi^{(4)}_{mn} = 0$. And so on until $\Phi^{(10)}_{mn}$.

Since $\Phi^{(10)}_{mn}$ is a top form, exists $\Psi^{(9)} \in \mathcal{M}$ such that $\Phi^{(10)} = d_L \Psi^{(9)}$. Furthermore, $d_L (\Phi^{(8)} - \delta_L \Psi^{(9)}) = 0$, therefore exists $\Psi^{(7)} \in \mathcal{M}$ such that $\Phi^{(8)} - \delta_L \Psi^{(9)} = d_L \Psi^{(7)}$. Continuing, we get $\Phi^{(6)} - \delta_L \Psi^{(7)} = d_L \Psi^{(5)}$, $\Phi^{(4)} - \delta_L \Psi^{(5)} = d_L \Psi^{(3)}$ and finally $d_L (\Phi^{(2)} - \delta_L \Psi^{(3)}) = 0$. Let us denote:

$$\begin{align*}
\Phi &= \Phi^{(2)} + \Phi^{(4)} + \Phi^{(6)} + \Phi^{(8)} + \Phi^{(10)} \\
\Psi &= \Psi^{(3)} + \Psi^{(5)} + \Psi^{(7)} + \Psi^{(9)}
\end{align*} \quad (148)$$

Then we get:

$$\begin{align*}
dx_L A_p; [m \partial_n] &= \delta_L + d_L \left( \Phi - (\delta_L + d_L) \Psi \right) \\
&= \delta_L + d_L \left( \Phi - (\delta_L + d_L) \Psi \right)
\end{align*} \quad (150)$$

Notice that $\Phi - (\delta_L + d_L) \Psi$ is of the form:

$$\Phi - (\delta_L + d_L) \Psi = dx_L p \wedge dx_L \tilde{\Psi}_{pq}; mn \quad (151)$$

This concludes the proof that the ansatz (115) is trivial iff $F_{[pq]}; [mn] = 0$. 

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Case $N = 0$ The vanishing lemma \(\text{[145]}\) does not work in the case $N = 0$, in this case the cohomology of $d_L$ is given by the formulas of Section 3.1 with the replacement $c^m \mapsto dx_L^m$, $dx^m \mapsto dx_R^m$. Similarly, the cohomology of $\delta_L$ is obtained via the Hodge duality. Therefore, it is necessary to repeat the analysis taking into account this nontrivial cohomology. There is no obstacle to satisfy \(\text{[146]}\), even if $A_p; [m \partial_n] = \text{const}$, because there are no 11-forms and therefore the cohomology of $\delta_L$ vanishes on expressions which are monomials of the first order in $dx_L$. There are potential obstacles in completing the chain \(\text{[147]}\). We will not do the analysis here, but just point out that by rotational symmetry, the potential obstacles are proportional to the following constant tensors: the total antisymmetrization and the contraction:

\[
C_{pmn} = A_{[p;m \partial_n]} \\
C_n = g^{pm} A_{[p;m \partial_n]} \tag{152, 153}
\]

4.3 Dirac-Maxwell sector

Consider the following ansatz for the leading term of the expansion in powers of $x_L$:

\[
\hat{c} \Psi_{[m}(x_R) d_m x_R^n \wedge dx_R \tag{154}
\]

where $\Psi$ satisfies the Maxwell equation $\Psi \hat{M} = 0$. This is in the image of $d_1$ when exists $\Phi$ such that:

\[
\Psi_{[m \partial_n]} = \Gamma^k \partial_k \Phi_{[m \partial_n]} \tag{155}
\]

and $\Phi \hat{M} = 0$

Then it follows that $\Delta \Phi_{[\bullet \partial_{\bullet}]} = 0$ and therefore:

\[
\Gamma^m \partial_m \Psi_{[\bullet \partial_{\bullet}]} = 0 \tag{156}
\]

If $\Psi$ does not satisfy this equation, then the trivialization \(\text{[155]}\) is impossible. Notice that $\Psi \hat{M} = 0$, therefore $\Psi_{[\bullet \partial_{\bullet}}$ is automatically annihilated by $\Delta$. But it is not necessarily annihilated by the left Dirac operator.

We conclude that $\Gamma^m \partial_m \Psi_{[\bullet \partial_{\bullet}]}$ is an obstacle for \(\text{[154]}\) to be trivial.
4.4 Maxwell-Dirac sector

Consider the following ansatz for the leading term:

\[ \Psi_m(x_R) \ c_n dx^L_n \wedge dx^L_m \]  

(157)

where \( \Psi \) satisfies the right Dirac equation \( \Psi_m \overset{\partial_n}{\rightarrow} \Gamma^n = 0 \) and also \( \partial^m \Psi_m = 0 \). This is trivial if exists \( A_m \) such that:

\[ \Psi_m = \partial_n (\partial_n A_m - \partial_m A_n) \]  

(158)

with \( A_k \overset{\partial_k}{\rightarrow} \Gamma^k = 0 \) (159)

This implies that \( \Delta A = 0 \) and therefore \( \partial_m \Psi_m = 0 \). Therefore \( \partial_m \Psi_m \) is an obstacle for (157) to be trivial. For the polynomials of nonzero degree this is the only obstacle. Indeed, suppose that \( \partial_{[m} \Psi_{n]} = 0 \). As the cohomology of Dirac solutions at the nonzero degree is zero, this implies that:

\[ \Psi_m = \partial_m \Xi \]  

(160)

where \( \Xi = \Xi(x_R) \) satisfies the right Dirac equation. The cohomology of \( H^{|g(\delta)} \) on the solutions of the Dirac equation is zero, therefore exists \( A_n \) such that \( \Xi = -\partial^n A_n \) where \( \Phi_n \) satisfies the Dirac equation. This implies (158).

5 BRST cohomology

We are now ready to compute the cohomology of \( Q_{\text{SUGRA}} \).

5.1 Ghost number one

The corresponding part of \( \mathcal{E}_2 \) consists of two parts:

\[ \mathcal{E}^{1,0}_2 = C^{10} \]  

(161)

\[ \mathcal{E}^{0,1}_2 = H^0(\text{SMaxw}_L) \bigoplus H^0(\text{SMaxw}_R) = \]  

\[ = \Lambda^2 C^{10} \bigoplus \Lambda^2 C^{10} \bigoplus C^{16} \bigoplus C^{16} \]  

(162)
However, there is a nontrivial $d_2 : \mathcal{E}^{0,1}_2 \to \mathcal{E}^{2,0}_2 = \Lambda^2 C^{10}$, which cancels the $L \leftrightarrow R$ antisymmetric part of $\Lambda^2 C^{10} \oplus \Lambda^2 C^{10} \subset \mathcal{E}^{0,1}_2$ with $\mathcal{E}^{2,0}_2$. We are left with:

\[
\begin{align*}
\mathcal{E}^{1,0}_\infty &= C^{10} \\
\mathcal{E}^{0,1}_\infty &= \Lambda^2 C^{10} \bigoplus C^{16} \bigoplus C^{16} \\
\mathcal{E}^{2,0}_\infty &= 0
\end{align*}
\]

These vertices are in one-to-one correspondence with the generators of the super-Poincare algebra.

### 5.2 Ghost number two

The corresponding part of $\mathcal{E}_2$ consists of three parts:

\[
\begin{align*}
\mathcal{E}^{2,0}_2 &= \Lambda^2 C^{10} \\
\mathcal{E}^{1,1}_2 &= H^1(S\text{Maxw}_L) \bigoplus H^1(S\text{Maxw}_R) \\
\mathcal{E}^{0,2}_2 &= H^0(S\text{Maxw}_L \otimes S\text{Maxw}_R)
\end{align*}
\]

#### 5.2.1 $\mathcal{E}^{2,0}_2$

We have already seen that $\mathcal{E}^{2,0}_2$ gets killed by the $d_2$:

\[
\mathcal{E}^{2,0}_\infty = 0
\]

#### 5.2.2 $\mathcal{E}^{1,1}_2$

Let us look at $\mathcal{E}^{1,1}_2$. We have:

\[
\mathcal{E}^{1,1}_2 = (C^{10} \oplus \Lambda^3 C^{10} \oplus C^{16}) \bigoplus (C^{10} \oplus \Lambda^3 C^{10} \oplus C^{16})
\]

The interpretation is as follows:

- $C^{10} \oplus C^{10}$ corresponds to the linear dilaton and the “asymmetric linear dilaton” (the nonphysical vertex of $R$ with constant $A_m$)
- One copy of $\Lambda^3 C^{10}$ cancels under $d_2$ with $\mathcal{E}^{3,0}_2$
- Another copy of $\Lambda^3 C^{10}$ is the NSNS $B$-field strength $H = dB$
- Two copies of $C^{16}$ are both unphysical
5.2.3 $\mathcal{E}^{0,2}_2$

This is the direct sum:

\[
H^0(\text{Maxw}_L \otimes \text{Maxw}_R) \oplus H^0(\text{Dirac}_L \otimes \text{Dirac}_R) \oplus \\
\oplus H^0(\text{Maxw}_L \otimes \text{Dirac}_R) \oplus H^0(\text{Dirac}_L \otimes \text{Maxw}_R)
\]  

(171)

The space $H^0(\text{SMaxw}_L \otimes \text{SMaxw}_R)$ can be thought of as the space of functions:

\[
F_{[mn\,\,;\,\,pq]}(x)
\]

(172)

satisfying:

\[
\partial_{[k} F_{mn\,\,;\,\,pq]} = 0
\]  

(173)

\[
F_{[mn\,\,;\,\,pq]} \tilde{\partial}_r = 0
\]  

(174)

\[
\partial_m F_{[mn\,\,;\,\,pq]} = 0
\]  

(175)

\[
F_{[mn\,\,;\,\,pq]} \tilde{\partial}^q = 0
\]  

(176)

Eqs. (173) and (174) together imply that:

\[
F_{[mn\,\,;\,\,pq]} = \text{const}
\]  

(177)

We can write:

\[
F_{[mn\,\,;\,\,pq]} = \partial_{[m} A^L_{n]\,\,;\,\,pq]} = A^R_{[mn\,\,;\,\,p]} \tilde{\partial}_{q]}
\]

(178)

A consequence of Eqs. (173), (174), (175), (176) is the existence of $\phi^R_q$ and $\phi^L_m$ such that:

\[
\partial^m A^R_{[mn\,\,;\,\,p]} = \partial_p \phi^R_m
\]  

(179)

\[
A^L_{m\,\,;\,\,pq]} \tilde{\partial} = \partial_m \phi^L_p
\]  

(180)

This implies:

\[
g^{np} F_{[mn\,\,;\,\,pq]}(x) = \frac{1}{2} \partial_q \left(g^{np} A^R_{[mn\,\,;\,\,p]} - \phi^R_m\right) = \frac{1}{2} \partial_m \left(g^{np} A^L_{n\,\,;\,\,pq]} - \phi^L_q\right)
\]  

(181)

Let us denote:

\[
B^R_m = g^{np} A^R_{[mn\,\,;\,\,p]} - \phi^R_m,
\]

(182)

\[
B^L_q = g^{np} A^L_{n\,\,;\,\,pq]} - \phi^L_q
\]  

(183)
Notice that $\partial_m B^R_n = \text{const}$ and $\partial_q B^L_q = \text{const}$. Let us denote:

$$B^L_m \pm B^R_m = A^\pm_m$$

Then:

$$\partial_q A^+_m = 0$$
$$\partial_q A^-_m = 0$$

Therefore $A^+_m = \partial_m \Phi$ (the gradient of the dilaton) and $A^-_m$ is the unphysical state of [9]. Notice that Eq. (186) implies that $\partial_p \partial_q A^-_m = 0$, i.e. $A^-_m$ is a linear function of $x$. Notice that Eqs. (180) and (179) define $\phi^{L|R}$ only up to a constant, and therefore $A^\pm_m$ is defined only up to a constant. This is because linear dilaton and linear asymmetric dilaton have already been counted in $E^{1,1}_\infty$.

**Conclusion** As expected, $F_{pq;mn}$ has the quantum numbers of the NSNS sector of the linearized Type IIB SUGRA, modulo some zero mode subtleties. The symmetric part $F_{pq;mn} + F_{mn;pq}$ corresponds to the Riemann curvature tensor $R_{[pq][mn]}$, and the antisymmetric part $F_{pq;mn} - F_{mn;pq}$ to $\partial_p B^\text{NSNS}_{q[m} \partial_{n]}$.

### 5.3 Comment on nonphysical states

There are the following nonphysical states:

| State | Description |
|-------|-------------|
| $C^{10}$ | from $E^{1,1}_\infty$: constant $A^-_m$ |
| $C^{16} \oplus C^{16}$ | from $E^{1,1}_\infty$ |
| $\Lambda^2 C^{10}$ | from $E^{0,2}_\infty$: $\partial_q A^-_m$ |

They have the quantum numbers of the adjoint representation of the super-Poincaré algebra.

In the bosonic string theory, the nonphysical states were removed by imposing the constraint $(b_0 - \bar{b}_0) V = 0$ [2]. This is probably possible also in the pure spinor approach, as the pure spinor $b$-ghost was constructed in the nonminimal formalism [10]. But there is also another way of removing the nonphysical states, which we will now describe.
As we discussed in the Introduction, the BRST closedness of the vertex operator is a necessary and sufficient condition for the corresponding deformation of the classical worldsheet action to have the classical BRST invariance. However, at the one-loop level there is an anomaly which is cancelled by the Fradkin-Tseytlin term [6]:

$$\alpha' \int d^2 \tau \Phi R$$

Here $\Phi$ is the dilaton superfield. The only place where $\Phi$ enters is the Fradkin-Tseytlin term (187), which does not matter at the classical level. It is, in this sense, “invisible” in the classical theory. The condition of the one-loop BRST invariance implies that $\Phi$ is related to the “visible” superfields (those which enter in the main part of the worldsheet action) by some equations:

$$D_{\alpha} \Phi = \Omega_{\alpha}$$

$$\hat{D}_{\hat{\alpha}} \Phi = \hat{\Omega}_{\hat{\alpha}}$$

where $\Omega_{\alpha}$ and $\hat{\Omega}_{\hat{\alpha}}$ on the right hand side are some function of the “visible” superfields. In this sense, $\Phi$ is determined, unambiguously up to a constant, from the “visible” superfields.

However, it turns out that for some classical backgrounds the equations (188) and (189) are incompatible [3]. Such backgrounds, in our terminology, are nonphysical. Being perfectly consistent from the point of view of the classical worldsheet sigma-model, they however fail at the one-loop level.

This is somewhat unusual, as the typical situation is that differential equations are “generally speaking incompatible, but sometimes become compatible”. Here we have the opposite situation. Equations (188) and (189) for $\Phi$ are compatible for the vast majority of backgrounds, but become incompatible on a finite-dimensional nonphysical component. In other words, physical and non-physical deformations are “mutually obstructed”.

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We observe that the nonphysical operators seem to be in correspondence with the global symmetries. This should have a natural interpretation in terms of the action of the $b$-ghost:

\[ \text{nonphysical, ghost number 2} \xrightarrow{b_0 - \tilde{b}_0} \text{ghost number 1} \quad (\text{global symmetries}) \quad (190) \]

But, as we explained:

- instead of imposing the condition $(b_0 - \tilde{b}_0)V = 0$, one can request the existence of the dilaton superfield $\Phi$

Notice that including $\Phi$ also solves the following problem. Our analysis, based on the naive BRST cohomology, failed to identify the dilaton zero mode. But once we include $\Phi$, the dilaton is identified as the lowest component of $\Phi$, and in particular the zero mode of the dilaton is recovered.

### 5.4 Ghost number three

Most of the ghost number three vertex operators transform in the same representation as ghost number two vertex operators. This is in line with the picture:

\[ \text{ghost number 3} \xrightarrow{b_0 - \tilde{b}_0} \text{ghost number 2} \quad (191) \]

Notice that the map (191) lowers the polynomial degree of the vertex by 2, as the $b$-ghost should. For example, in the Dirac-Dirac sector, the ghost number 3 vertex is of the form $\hat{c}R$; to produce the bispinor field we remove $\hat{c}$ and then act with the left Dirac operator:

\[ \hat{c}R \mapsto \Gamma^m \frac{\partial}{\partial x^m_R} R \quad (192) \]

Removing $\hat{c}$ lowers the degree by one, and then $\frac{\partial}{\partial x^m_R}$ again lowers the degree by one.

Let us look more carefully at the subtleties which arise when we consider polynomial vertices of low degree.

#### 5.4.1 $\mathcal{E}_2^{3,0}$

This is $\Lambda^3 C^{10}$. It cancels with part of $\mathcal{E}_2^{1,1}$ — see Section 5.2.2.
This is $\Lambda^4C^{10} \oplus \Lambda^4C^{10}$. It cancels in the following way. First of all, we have restrict to the kernel of $d_2 : \mathcal{E}_2^{2,1} \rightarrow \mathcal{E}_2^{4,0}$. This kills one copy of $\Lambda^4C^{10}$. But also, we have to take a factorspace over the image of $d_2 : \mathcal{E}_2^{0,2} \rightarrow \mathcal{E}_2^{2,1}$. This should cancel against the (177).

**5.4.3 $\mathcal{E}_2^{1,2}$**

**Dirac-Dirac sector** There are the following obstacles to triviality:

1. The bispinor $\Gamma^m \frac{\partial}{\partial x^m} R$, which satisfies both left and right Dirac equations

2. There is also a discrete state (102) which corresponds to $R$ being a constant times the unit matrix

**Maxwell-Maxwell sector** There are the following obstacles to triviality:

1. The double field strength $F_{[mn]} ; [pq]$ of Section 4.2.3

2. If the double field strength is zero, then there are constant tensors $C_{kln}$ and $C_k$ defined in (152) and (153)

First let us look at the double field strength. Notice that the formulas of Section 4.2.3 are almost identical to Section 5.2.3. The only difference is that the $F_{[mn]} ; [pq]$ of Section 5.2.3 is not required to be of the form (130). As the $F_{[mn]} ; [pq]$ of Section 4.2.3 is required to be of the form (130), it automatically satisfies:

$$g^{np} F_{[mn]} ; [pq] (x) = \frac{1}{2} \partial_{[p} g^{np} A_{n]} ; [p \partial_q]$$

Eq. (118) implies the existence of $\Phi$ such that $\partial^n A_n ; p = \partial_p \Phi$. Taking also into account Eq. (116), we get:

$$g^{np} F_{[mn]} ; [pq] (x) = \partial_m \partial_q \left[ \frac{1}{8} (g^{pm} A_p ; n - \Phi) \right]$$

This is the same equation as we got in Section 5.2.3, except there is no unphysical $A_m^-$. 

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On the other hand, there are $C_{klm}$ and $C_k$ defined in (152) and (153), which should be mapped by $b_0 - \bar{b}_0$ to $H_{klm}^{NSNS}$ and the dilaton gradient. Also, there is the discrete state (102), which is probably mapped by $b_0 - \bar{b}_0$ to the dilaton zero mode. Notice that in our computation we missed the dilaton zero mode, as the corresponding vertex is probably a BRST variation of something that is not annihilated by $b_0 - \bar{b}_0$ [2].

All this seems to confirm Eq. (191).

### 5.5 Ghost number four

The cohomology at the ghost number four is zero. The space $H^2(Q_{\text{Lie}}, \text{SMaxw}_L \otimes \text{SMaxw}_R) = 0$ because the leading term would be:

$$c^p c^q dx^r_L \wedge dx^s_L G(x^r_R)_{pqrs} : mn dx^m_R \wedge dx^n_R$$

and this precisely cancels with the $B$-term in (115).

The space $H^4(Q_{\text{Lie}}, C) = \Lambda^4 C_{10}^{10}$ is nonzero, but it cancels with the $d_2$ of $H^2(Q_{\text{Lie}}, \text{SMaxw}_L \oplus \text{SMaxw}_R)$. Indeed, let us consider the following element of $H^1(Q_{\text{Lie}}, \text{SMaxw}_L \otimes \text{SMaxw}_R)$ with constant $B_{pq} : \partial_{nm}$:

$$c^p dx^q_L \wedge dx^r_L B_{pq} : \partial_{nm} dx^m_R \wedge dx^n_R$$

(196)

(This is a particular case of (115) with zero $A$ and constant $B$.) Being an element of $H^1(Q_{\text{Lie}}, \text{SMaxw}_L \otimes \text{SMaxw}_R)$, this is a $c$-dependent element of the cohomology of $Q_L + Q_R$, parametrized by a left times right field strength. We need to act on this by the $d_2 : E_2^{1,2} \to E_2^{3,1}$. For that, we need to know the actual (c-dependent) vertex, which is built using the left and right vector potentials, i.e. $c^p x^q_L (\theta^r_L \Gamma^m \lambda_L) B_{pq} : \partial_{nm} x^m_R (\theta^r_R \Gamma^n \lambda_R)$. The $Q_{\text{Lie}}$ on the vertex is not zero:

$$-c^p c^q (\theta^r_L \Gamma^r \lambda_L) B_{pq} : \partial_{nm} x^m_R (\theta^r_R \Gamma^n \lambda_R) - c^p x^q_L (\theta^r_L \Gamma^r \lambda_L) B_{pq} : \partial_{nm} c^m (\theta^r_R \Gamma^n \lambda_R)$$

but is a pure gauge, namely it is $Q_L + Q_R$ of:

$$-c^p x^q_L B_{pq} : \partial_{nm} x^m_R (\theta^r_R \Gamma^n \lambda_R) + c^p x^q_L (\theta^r_L \Gamma^r \lambda_L) B_{pq} : \partial_{nm} c^m x^m_R$$

(197)
And the $Q_{\text{Lie}}$ of this gives:

\begin{align*}
- c^\mu c^\nu B_{\mu\nu} : [m \overset{\partial}{\to} n] x^m_R (\theta_R \Gamma^m \lambda_R) + c^\mu x^m_R (\theta_L \Gamma^m \lambda_L) B_{\mu\nu} : [m \overset{\partial}{\to} n] c^m c^n & \quad (199) \\
+ c^\mu c^\nu x^m_L B_{\mu\nu} : [m \overset{\partial}{\to} n] c^m (\theta_R \Gamma^m \lambda_R) - c^\mu c^\nu (\theta_L \Gamma^m \lambda_L) B_{\mu\nu} : [m \overset{\partial}{\to} n] c^m x^n_R & \quad (200)
\end{align*}

The second row is $-(Q_L + Q_R) c^\mu c^\nu x^m_L B_{\mu\nu} : mn c^m x^n_R$. And the first row is equivalent, in the Maxwell cohomology, to the expression:

\begin{align*}
- c^\mu c^\nu c^\rho B_{[\mu\nu\rho]} (dx^m_R \land dx^n_R - dx^m_L \land dx^n_L) & \quad (201)
\end{align*}

where $B_{[\mu\nu\rho]}$ is defined in (128). This can be used to kill any class of the form:

\begin{align*}
 c^\mu c^\nu c^\rho G_{[\mu\nu\rho]} (dx^s_R \land dx^t_R - dx^s_L \land dx^t_L) & \quad (202)
\end{align*}

in $H^3(Q_{\text{Lie}}, \text{SMaxw}_L \oplus \text{SMaxw}_R)$. The classes of the form:

\begin{align*}
 c^\mu c^\nu c^\rho H_{[\mu\nu\rho]} (dx^s_R \land dx^t_R + dx^s_L \land dx^t_L) & \quad (203)
\end{align*}

are not in the image of $d_2$. However, the $d_2$ of them is nonzero, giving an element of $\mathcal{E}^{5,0} = H^5(C^{10})$ of the form $c^\mu c^\nu c^\rho c^\sigma H_{[\mu\nu\rho\sigma]}$.

We conclude that $\mathcal{E}_3^{3,1} = 0$.

6 Action of the supersymmetry on the ghost number three vertices

In this section we will study the action of the supersymmetry on the ghost number three vertices. We will first act by the left supersymmetry on the element of the Maxwell-Dirac sector, and see that the result is some element of the Dirac-Dirac sector. Then we will act by the left supersymmetry on the Dirac-Dirac sector, which will bring us back to the Maxwell-Dirac sector. We will verify that the anticommutator of two supersymmetries is a translation.

6.1 Left supersymmetry on the Maxwell-Dirac sector

Let us consider an element of the Maxwell-Dirac sector, of the following form:

\begin{align*}
\Psi_m (x_R) c_n dx^n_R \land dx^m_L + \ldots & \quad (204)
\end{align*}
where... stand for elements of the lower degree in $x_R$ (which have dependence on $x_L$). Let us act on it by the left supersymmetry with the parameter $\epsilon^\alpha$, which we will call $S_\epsilon$. To evaluate the action of this supersymmetry, we will use the formulas from Section 6.1.3 of [11] (where $S_\epsilon$ was denoted $Q^R_{\text{Lie}}$, and $\epsilon^\alpha$ was called $\xi^\alpha$). We get the following element of the Dirac-Dirac sector:

$$-\frac{2}{3} \times \frac{1}{2} [\hat{c}, \Gamma^m] \epsilon \Psi_m + \ldots$$

(205)

We observe:

$$\Gamma^j \frac{\partial}{\partial x^j_L} \left( \frac{1}{2} [\hat{x}_L, \Gamma^m] \epsilon \Psi_m \right) = \Gamma^j \frac{\partial}{\partial x^j_L} \left( \frac{9}{10} \hat{x}_L \Gamma^m \epsilon \Psi_m \right)$$

(206)

This implies that (205) gives the same cohomology class as:

$$-\frac{2}{3} \times \frac{9}{10} \hat{c} \Gamma^m \epsilon \Psi_n(x_R) + \ldots$$

(207)

In notations of Section 4.1.2 we have:

$$R = -\frac{2}{3} \times \frac{9}{10} \Gamma^m \epsilon \Psi_n(x_R)$$

(208)

The obstacle to the triviality is:

$$\Gamma^m \partial_m R = -\frac{2}{3} \times \frac{9}{10} \Gamma^m \Gamma^n \epsilon \partial_m \Psi_n(x_R) = -\frac{2}{3} \times \frac{9}{10} \Gamma^{mn} \epsilon \partial_{[m} \Psi_{n]}(x_R)$$

(209)

(This is a bispinor: $(\Gamma^{mn} \epsilon)^\alpha (\partial_{[m} \Psi_{n]}(x_R))^\beta$)

### 6.2 Left supersymmetry on the Dirac-Dirac sector

We want to calculate the action of the supersymmetry with the parameter $\epsilon$ on the class:

$$\hat{c}R(x_R) + \ldots$$

(210)

This is a bit tricky, because the leading term $\hat{c}R(x_R)$ does not contribute, and we have to analyze the subleading term proportional to $x_L$:

$$\hat{c}R(x_R) +$$

$$+ \frac{5}{6} \hat{c}x_k \partial_k R(x_R) - \frac{1}{6} \hat{x}_L c^k \partial_k R(x_R) + \frac{5}{6} (c \cdot x_L) \Gamma^k \partial_k R + \ldots$$

(211)
where \( \ldots \) stand for terms of the higher order in \( x_L \). Again, we use the formulas from [11]. When acting by the supersymmetry with the parameter \( \epsilon \), we are getting the following element of the Maxwell-Dirac sector:

\[
- \frac{3}{2} \epsilon \Gamma_{[n} \left( \frac{5}{6} \hat{c} \partial_{m]} R(x_R) - \frac{1}{6} \Gamma_{m]} c^k \partial_k R(x_R) + \frac{5}{6} c_m \Gamma^k \partial_k R(x_R) \right) dx_L^n \wedge dx_L^m
\]  

(212)

This can be written as follows:

\[
- \frac{3}{2} \epsilon c^k Y_{kmn} \partial_l R \, dx_L^n \wedge dx_L^m
\]

where

\[
Y_{kmn}^l = - \frac{5}{6} \Gamma_{[n} \delta_{m]l} - \frac{5}{6} \Gamma_{k[n} \delta_{m]l} - \frac{1}{6} \Gamma_{nm} \delta_{lk}
\]  

(213)

We can add \( Q_{\text{Lie}} (\epsilon \Gamma_{nm} R dx_L^n \wedge dx_L^m) \) then we get:

\[
- \frac{3}{2} \epsilon c^k \tilde{Y}_{kmn} \partial_l R \, dx_L^n \wedge dx_L^m
\]

where

\[
\tilde{Y}_{kmn}^l = - \frac{5}{6} \Gamma_{[n} \delta_{m]l} - \frac{5}{6} \Gamma_{k[n} \delta_{m]l} + \frac{5}{6} \Gamma_{nm} \delta_{lk}
\]  

(214)

Now \( \tilde{Y}_{[kmn]} = 0 \) and \( \tilde{Y}_{mmn} = -5 \Gamma_{[n} \). Consider the following tensor:

\[
Z_{kmn}^l = \frac{5}{18} \Gamma_{[n} \delta_{m]l} - \frac{5}{6} \Gamma_{k[n} \delta_{m]l} + \frac{5}{6} \Gamma_{nm} \delta_{lk}
\]  

(215)

It satisfies \( Z_{[kmn]}^l = 0 \) and \( Z_{mmn}^l = 0 \). Therefore the cohomology class does not change if we replace \( \tilde{Y}_{kmn}^l \) as follows:

\[
\tilde{Y}_{kmn}^l \mapsto \tilde{Y}_{kmn}^l - Z_{kmn}^l = - \left( \frac{10}{9} \Gamma_{[n} \delta_{m]} \right)
\]  

(216)

Indeed:

\[
\epsilon c^k Z_{kmn}^l \partial_l R \, dx_L^n \wedge dx_L^m + \ldots =
\]

\[
= c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) (x_L^k Z_{kmn}^l \partial_l R \, dx_L^n \wedge dx_L^m + \ldots)
\]  

(217)

We conclude that the supersymmetry with the parameter \( \epsilon \) brings \( \epsilon R + \ldots \) to \(-\frac{10}{9} \times (-\frac{3}{2}) \epsilon \Gamma_{m} \epsilon^k \partial_k dx_L^n \wedge dx_L^n \). When \( R \) is given by (208), we get:

\[
- \epsilon \Gamma_{m} \epsilon^k \psi_j c_k dx_L^k \wedge dx_L^n = -(\epsilon \Gamma_{m} \epsilon) \frac{\partial}{\partial x_R^j} (\psi_j (x_R) c_k dx_L^k \wedge dx_L^j)
\]  

(218)

This is in agreement with the fact that the anticommutator of two SUSY transformations is a translation.
6.3 Conclusion

Ghost number three vertices transform in the linearized Type IIB supergravity supermultiplet.

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