Discrete nonlinear hyperbolic equations. Classification of integrable cases

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February 8, 2022

Abstract

We consider discrete nonlinear hyperbolic equations on quad-graphs, in particular on \( \mathbb{Z}^2 \). The fields are associated to the vertices and an equation \( Q(x_1, x_2, x_3, x_4) = 0 \) relates four fields at one quad. Integrability of equations is understood as 3D-consistency. The latter is a possibility to consistently impose equations of the same type on all the faces of a three-dimensional cube. This allows to set these equations also on multidimensional lattices \( \mathbb{Z}^N \). We classify integrable equations with complex fields \( x \), and \( Q \) affine-linear with respect to all arguments. The method is based on analysis of singular solutions.

1 Introduction

The idea of consistency (or compatibility) is at the core of the theory of integrable systems. It appears already in the very definition of complete integrability of a Hamiltonian flow in the Liouville–Arnold sense, which says that the flow may be included into a complete family of commuting (compatible) Hamiltonian flows [1]. Similarly, it is a characteristic feature of soliton (integrable) partial differential equations that they appear not separately but are always organized in hierarchies of commuting (compatible) systems. The condition of the existence of a number of commuting systems may be taken as the basis of the symmetry approach which is used to single out integrable systems in some general classes and to classify them [18]. Another way of relating continuous and discrete systems, connected with the idea of compatibility, is based on the notion of Bäcklund transformations and the Bianchi permutability theorem [9]. The latter developed into one of the fundamental principles of discrete differential geometry [12].

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So, the consistency of discrete equations takes center stage in the integrability theater. We say that a \(d\)-dimensional discrete equation possesses the consistency property, if it may be imposed in a consistent way on all \(d\)-dimensional sublattices of a \((d + 1)\)-dimensional lattice (a more precise definition will be formulated below). As the above remarks show, the idea that this notion is closely related to integrability, is not new. In the case \(d = 1\) it was used as a possible definition of integrability of maps in [24]. For \(d = 2\) a decisive step was made in [10] and independently in [19]; it was shown that the integrability in the usual sense of soliton theory (as existence of a zero curvature representation) follows for two-dimensional systems from the three-dimensional consistency. So the latter property may be considered as a definition of integrability. It is a criterion which may be checked in a completely algorithmic manner starting with no more information than the equation itself. Moreover, in if this criterion gives a positive result, it delivers also the discrete zero curvature representation.

Basic building blocks of systems on quad-graphs are quad-equations, which are equations on quadrilaterals

\[
Q(x_1, x_2, x_3, x_4) = 0, \tag{1}
\]

where the field variables \(x_i \in \mathbb{CP}^1\) assigned to the four vertices of the quadrilateral as shown in Figure 1. On \(\mathbb{Z}^2\) equations of this type can be treated as discrete analogues of nonlinear hyperbolic equations. Boundary value problems of Goursat type for such systems were studied in [6].

**Assumption.** In this paper we assume that \(Q\) is affine-linear, i.e. a polynomial of degree one in each argument. This implies that equation (1) can be solved for any variable and the solution is a rational function of other three variables.

![Figure 1: A quad-equation \(Q(x_1, x_2, x_3, x_4) = 0\); the variables \(x_i\) are assigned to vertices](image)

The general idea of integrability as consistency in this case is shown in Figure 2. We put six quad-equations on the faces of a coordinate cube. The subscript \(j\) corresponds to the shift in the \(j\)-th coordinate direction. If one starts with arbitrary values \(x, x_1, x_2, x_3\) then the values \(x_{12}, x_{13}, x_{23}\) are found from three equations on the left, front and bottom faces and the equations on the right, back and top faces yield, in general, three different values of \(x_{123}\). The system is called 3D-consistent, if these three values are identical for arbitrary initial data \(x, x_1, x_2, x_3\).
In [3] we classified 3D-consistent systems of a particular type. The equations on all faces coincided up to the values of parameters associated with three types of the edges. Moreover, cubic symmetry was imposed as well as a certain additional condition called the tetrahedron property. A 3D-consistent system without the tetrahedron property was found in [13]. Later, this system was shown to be linearizable in [22]. In [26] it was shown that the 3D-consistent equations classified in [3] satisfy the integrability test based on the notion of algebraic entropy.

The consistency approach was generalized in various directions. Systems with the fields on edges lead to Yang-Baxter maps [25, 23, 20]. Quadrirational Yang-Baxter maps were classified in [4]. The 4D-consistency of discrete 3D-systems is related to the functional tetrahedron equation studied in [17, 16, 15, 8].

In the present paper we classify 3D-consistent affine-linear quad-equations in a more general setup. The faces of the consistency cube can carry a priori different quad-equations. Neither symmetry nor the tetrahedron property are assumed. This leads to a general classification of integrable quad-equations.

The outline of our approach is the following.

a) By applying discriminant-like operators to successively eliminate variables one can descend from an affine-linear polynomial of four variables, associated to a quadrilateral, to quadratic polynomials of two variables, associated to its edges, and finally to quartic polynomials of one variable, associated to its vertices (Section 2).

b) By analysis of singular solutions, we prove that the biquadratic polynomials which come to an edge of the cube from two adjacent faces coincide up to a constant factor (see Section 3). At this point an additional non-degeneracy assumption is needed. We assume that all the biquadratic polynomials do not have factors of the form $x - c$ with constant $c$. (Examples of equations without this property are presented in Section 7).

c) This allows us to associate to each vertex of the cube a quartic polynomial in the respective variable; the admissible sets of polynomials are classified modulo Möbius transformations, each variable is transformed independently (Section 4).

d) Finally we reverse the procedure and reconstruct the biquadratic polynomials on the edges of the cube and the affine-linear equations themselves (Section 6).

Figure 2: A 3D consistent system of quad-equations. The equations are associated to faces of the cube.
2 Affine-linear and biquadratic polynomials

Our approach is based on the descent from the faces to the edges and from the edges to the vertices of the cube. In this Section we consider a single face and describe this descent irrespective of 3D-consistency. Let \( P_n^m \) denote the set of polynomials in \( n \) variables which are of degree \( m \) in each variable. We consider the following action of Möbius transformations on polynomials \( f \in P_n^m \):

\[
M[f](x_1, \ldots, x_n) = (c_1x_1 + d_1)^m \cdots (c_nx_n + d_n)^m f \left( \frac{a_1x_1 + b_1}{c_1x_1 + d_1}, \ldots, \frac{a_nx_n + b_n}{c_nx_n + d_n} \right),
\]

where \( a_i d_i - b_i c_i = \Delta_i \neq 0 \). The operations

\[
\mathcal{P}_1^4 \xrightarrow{\delta_x, \delta_y} \mathcal{P}_2^4 \xrightarrow{\delta_x} \mathcal{P}_1^4, \quad \delta_{x,y}(Q) = Q x Q_y - Q Q_{xy}, \quad \delta_x(h) = h_x^2 - 2h h_{xx}
\]

are covariant with respect to Möbius transformations. (The subscripts denote partial differentiation). More precisely: if \( Q \in \mathcal{P}_1^4, \ h \in \mathcal{P}_2^4 \), then

\[
\delta_{x,i,j}(M[Q]) = \Delta_i \Delta_j M[\delta_{x,i,j}(Q)], \quad \delta_{x,i}(M[h]) = \Delta_i^2 M[\delta_{x,i}(h)]. \tag{2}
\]

Further on we will make an extensive use of relative invariants of polynomials under Möbius transformations. For quartic polynomials \( r \in \mathcal{P}_1^4 \) these relative invariants are well known and can be defined as the coefficients of the Weierstrass normal form \( r = 4x^3 - 2gx - g \). For a given polynomial \( r(x) = r_4x^4 + r_3x^3 + r_2x^2 + r_1x + r_0 \) they are given by (see e.g. [28])

\[
g_2(r, x) = \frac{1}{48} (2r_4 r_{IV} - 2r'_4 r'' + (r'')^2) = \frac{1}{12} (12r_0 r_4 - 3r_1 r_3 + r_2^2),
\]

\[
g_3(r, x) = \frac{1}{3456} (12r_4 r'' r_{IV} - 9(r')^2 r_{IV} - 6r(r''')^2 + 6r' r'' r'' + 2(r''')^3)
\]

\[
= \frac{1}{432} (72r_0 r_2 r_4 - 27r_1 r_3 + 9r_1 r_2 r_3 - 27r_0 r_2 r_3 - 2r_2^3).
\]

Under the Möbius change of \( x = x_1 \) these quantities are just multiplied by the constant factors:

\[
g_k(M[r], x) = \Delta_1^{2k} g_k(r, x), \quad k = 2, 3.
\]

For biquadratic polynomials \( h \in \mathcal{P}_2^4 \),

\[
h(x, y) = h_{22} x^2 y^2 + h_{21} x^2 y + h_{20} x^2 + h_{12} x y^2 + h_{11} x y + h_{10} x + h_{02} y^2 + h_{01} y + h_{00}, \tag{3}
\]

the relative invariants are

\[
i_2(h, x, y) = 2h h_{xyy} - 2h_x h_{xy} + 2h_y h_{yy} + h_{xy}^2 = 8h_{00} h_{22} - 4h_{01} h_{21} - 4h_{10} h_{12} + 8h_{02} h_{20} + h_{11},
\]

\[
i_3(h, x, y) = \frac{1}{4} \det \begin{pmatrix} h & h_x & h_{xx} \\ h_y & h_{xy} & h_{xxy} \\ h_{yy} & h_{xyy} & h_{xxyy} \end{pmatrix} = \det \begin{pmatrix} h_{22} & h_{21} & h_{20} \\ h_{12} & h_{11} & h_{10} \\ h_{02} & h_{01} & h_{00} \end{pmatrix}.
\]
Notice that \( i_3 \) can be defined as well by the formula

\[
-4i_3(h, x, y) = \delta_{x,y}(\delta_{x,y}(h))/h.
\]

Under the M"obius change of \( x = x_1 \) and \( y = x_2 \),

\[
i_k(M[h], x, y) = \Delta^k_1 \Delta^k_2 i_k(h, x, y), \quad k = 2, 3.
\]

The following properties of the operations \( \delta_{x,y}, \delta_x \) are proved straightforwardly.

**Lemma 1.** For any affine-linear polynomial \( Q(x_1, x_2, x_3, x_4) \in \mathcal{P}_4^1 \) there holds:

\[
\delta_{x_3}(\delta_{x_1,x_2}(Q)) = \delta_{x_2}(\delta_{x_1,x_3}(Q)), \quad (4)
\]

\[
i_k(\delta_{x_1,x_2}(Q), x_3, x_4) = i_k(\delta_{x_3,x_4}(Q), x_1, x_2), \quad k = 2, 3. \quad (5)
\]

For any biquadratic polynomial \( h(x_1, x_2) \in \mathcal{P}_2^2 \) there holds:

\[
g_k(\delta_{x_1}(h), x_2) = g_k(\delta_{x_2}(h), x_1), \quad k = 2, 3. \quad (6)
\]

Denote \( h^{ij} = h^{ji} = \delta_{x_i,x_j}(Q) \) where \( \{i, j, k, l\} = \{1, 2, 3, 4\} \). Then Lemma 1 implies the commutativity of the diagram

\[
\begin{array}{ccc}
r_4(x_1) & \xrightarrow{\delta_{x_1}} & h_{34}(x_3, x_4) & \xrightarrow{\delta_{x_3}} & r_3(x_3) \\
\delta_{x_3} & & \delta_{x_1,x_2} & & \delta_{x_2} \\
\delta_{x_1} & & \delta_{x_2,x_3} & & \delta_{x_2} \\
& & Q(x_1, x_2, x_3, x_4) & \xrightarrow{\delta_{x_1,x_4}} & h_{23}(x_2, x_3) \\
r_1(x_3) & \xleftarrow{\delta_{x_2}} & h_{12}(x_1, x_2) & \xrightarrow{\delta_{x_1}} & r_2(x_2) \\
& & \delta_{x_3} & & \delta_{x_3} \\
& & \delta_{x_3} & & \delta_{x_3}
\end{array}
\]

Moreover, biquadratic polynomials on the opposite edges have the same invariants \( i_2, i_3 \), and all four quartic polynomials \( r_i \) have the same invariants \( g_2, g_3 \). This diagram can be completed by the polynomials \( h_{13}, h_{24} \) corresponding to the diagonals (so that the graph of the tetrahedron appears), but we will not need them. The introduced polynomials satisfy also a number of other interesting relations.

**Lemma 2.** For any affine-linear polynomial \( Q(x_1, x_2, x_3, x_4) \in \mathcal{P}_4^1 \) and with the notations \( h^{ij}(x_i, x_j) = \delta_{x_i,x_j}(Q) \in \mathcal{P}_2^2 \), the following identities hold:

\[
4i_3(h^{12}, x_1, x_2)h^{14} = \det \begin{pmatrix} h^{12} & h^{12}_x & \ell \\ h^{12}_x & h^{12}_{xx} & \ell_x \\ h^{12}_{xx} & h^{12}_{xx} & \ell_{xx} \end{pmatrix}, \quad (8)
\]

where \( \ell = h^{23}_{x_4} h^{34}_{x_4} - h^{23}_{x_3} h^{34}_{x_3} + h^{23}_{x_3} h^{34}_{x_3} \).
known, then their derivatives. Therefore, if the biquadratic polynomials on the three edges (out of four) is set of three biquadratic polynomials comes as a polynomial in its equivalence class with respect to Möbius transformations is divisible by a factor $x - c$ or $y - c$ (with $c =$ const).

\[
h^{12}h^{34} - h^{14}h^{23} = PQ, \quad P = \text{det}\begin{pmatrix} Q & Q_{x_1} & Q_{x_3} \\ Q_{x_2} & Q_{x_1x_2} & Q_{x_2x_3} \\ Q_{x_4} & Q_{x_1x_4} & Q_{x_3x_4} \end{pmatrix} \in \mathcal{P}_4^1; \tag{9}
\]

\[
\frac{2Q_{x_1}}{Q} = \frac{h^{12}h^{34} - h^{14}h^{23} + h^{23}h^{34} - h^{23}h^{34}}{h^{12}h^{34} - h^{14}h^{23}}. \tag{10}
\]

Identity (8) shows that $h^{14}$ can be expressed through three other biquadratic polynomials (provided $i_3(h^{12}) \neq 0$). Identity (3) defines $Q$ as one of the factors in a simple expression built from $h^{ij}$. Finally, differentiating (10) with respect to $x_2$ or $x_4$ leads to a relation of the form $Q^2 = F[h^{12}, h^{23}, h^{34}, h^{14}]$, where $F$ is a rational expression on $h^{ij}$ and their derivatives. Therefore, if the biquadratic polynomials on three edges (out of four) is known, then $Q$ can be found explicitly. Of course it is seen from Lemma 2 that not any set of three biquadratic polynomials comes as $h^{ij}$ from some $Q \in \mathcal{P}_4^1$.

Biquadratic polynomials $h^{ij}$ for a given $Q \in \mathcal{P}_4^1$ are closely related to singular solutions of the affine-linear equation

\[
Q(x_1, x_2, x_3, x_4) = 0. \tag{11}
\]

The polynomial $Q \in \mathcal{P}_4^1$ is assumed irreducible (in particular, $Q_{x_i} \neq 0$): otherwise the polynomial $Q$ should be considered as reducible, since under the change of the variable $x_i \mapsto 1/x_i$ it turns into $x_i Q$. Obviously, equation (11) can be solved with respect to any variable: let $Q = p(x_j, x_k, x_l)x_i + q(x_j, x_k, x_l)$ then $x_i = -q/p$ for the generic values of $x_j, x_k, x_l$. However, $x_i$ is not determined if the point $(x_j, x_k, x_l)$ lies on the curve in $(\mathbb{C} \mathbb{P}^1)^3$

\[
S_i : \quad p(x_j, x_k, x_l) = q(x_j, x_k, x_l) = 0, \quad Q \equiv px_i + q. \tag{12}
\]

The projection of this curve onto the coordinate plane $(j, k)$ is exactly the biquadratic $h^{jk} = px_i + q = 0$.

**Definition 1.** A solution $(x_1, x_2, x_3, x_4)$ of equation (11) is called singular with respect to $x_i$, if it satisfies also the equation $Q_{x_i}(x_1, x_2, x_3, x_4) = 0$. The curve $S_i$ is called the singular curve for $x_i$.

**Lemma 3.** If a solution $(x_1, x_2, x_3, x_4)$ of equation (11) is singular with respect to $x_i$, then $h^{jk} = h^{jl} = h^{kl} = 0$. Conversely, if $h^{jk} = 0$ for some solution, then this solution is singular with respect to either $x_i$ or $x_i$.

**Proof.** Since $h^{jk} = Q_{x_j}Q_{x_i} - QQ_{x_i}$, the equations $h^{jk} = 0$ and $Q_{x_i}Q_{x_j} = 0$ are equivalent for the solutions of equation $Q = 0$.

We will use the following notion of non-degeneracy for biquadratic polynomials.

**Definition 2.** A biquadratic polynomial $h(x, y) \in \mathcal{P}_2^2$ is called non-degenerate, if no polynomial in its equivalence class with respect to Möbius transformations is divisible by a factor $x - c$ or $y - c$ (with $c =$ const).
According to this definition, a non-degenerate polynomial $h(x, y) \in P_2^2$ is either irreducible, or of the form $(\alpha_1 xy + \beta_1 x + \gamma_1 y + \delta_1)(\alpha_2 xy + \beta_2 x + \gamma_2 y + \delta_2)$ with $\alpha_i \delta_i \neq \beta_i \gamma_i$. In both cases the equation $h = 0$ defines $y$ as a two-valued function on $x$ and vice versa. On the other hand, for example the polynomial $h(x, y) = x - y^2$ (considered as element of $P_2^2$) is, according to Definition 2, a degenerate biquadratic, since under the inversion $x \mapsto 1/x$ it turns into $x(1-xy^2)$.

A fundamental role in our studies will be played by the following notion.

**Definition 3.** An affine-linear function $Q \in P_1^4$ is said to be of type $Q$, if all four its accompanying biquadratics $h^{jk} \in P_2^2$ are non-degenerate, and is said to be of type $H$ otherwise.

## 3 3D-consistency and biquadratic curves

Consider the system of equations

$$
\begin{align*}
A(x, x_1, x_2, x_{12}) &= 0, & \bar{A}(x_3, x_{13}, x_{23}, x_{123}) &= 0, \\
B(x, x_1, x_3, x_{13}) &= 0, & \bar{B}(x_2, x_{12}, x_{23}, x_{123}) &= 0, \\
C(x, x_2, x_3, x_{23}) &= 0, & \bar{C}(x_1, x_{12}, x_{13}, x_{123}) &= 0
\end{align*}
$$

(13)

on a cube, see Figure 3. The functions $A, \ldots, \bar{C}$ are affine linear (from $P_1^4$) and are not a priori supposed to be related to each other in any way. We will use the notation $A^{ij} = \delta_{x_i x_j} A$ for the accompanying biquadratic polynomials.

**Theorem 1.** Let all six functions $A, \ldots, \bar{C}$ be of the type $Q$, and let equations (13) be 3D-consistent. Then

Figure 3: A 3D consistent system of quad-equations. The equations are associated to faces of the cube: $A$ and $\bar{A}$ on the bottom and on the top ones, $B$ and $\bar{B}$ on the front and on the back ones, $C$ and $\bar{C}$ on the left and on the right ones, respectively.
1) for any edge of the cube, the two biquadratic polynomials corresponding to this edge (coming from the two faces sharing this edge) coincide up to a constant factor;
2) the product of these factors around any vertex is equal to $-1$, for example
\[ A^{0.1}B^{0.3}C^{-0.2} + A^{0.2}B^{0.1}C^{0.3} = 0; \] (14)
3) the system \([13]\) possesses the tetrahedron property $\frac{\partial x_{123}}{\partial x} = 0$.

Proof. Elimination of $x_{12}$, $x_{13}$ and $x_{23}$ leads to equations
\[
F(\bar{x}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_{123}) = \bar{A}_{x_{13}x_{23}} BC - \bar{A}_{x_{23}} B_{x_{13}} C - \bar{A}_{x_{13}} BC_{x_{23}} + \bar{A}_{x_13} C_{x_{23}} = 0,
\]
\[
G(\bar{x}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_{123}) = \bar{B}_{x_{12}x_{23}} AC - \bar{B}_{x_{23}} A_{x_{12}} C - \bar{B}_{x_{12}} AC_{x_{23}} + \bar{B}_{x_{12}} A_{x_{13}} C_{x_{23}} = 0,
\]
\[
H(\bar{x}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_{123}) = \bar{C}_{x_{12}x_{13}} AB - \bar{C}_{x_{13}} A_{x_{12}} B - \bar{C}_{x_{12}} AB_{x_{13}} + \bar{C}_{x_{12}} A_{x_{13}} B_{x_{13}} = 0.
\]
Here the numbers over the arguments of $F, G, H$ indicate the degrees of the right hand side in the variables (the degree is understood in the projective sense, as in the example at the end of the previous Section). Due to 3D-consistency, the expressions for $x_{123}$ as functions of $\bar{x}, \bar{x}_1, \bar{x}_2, \bar{x}_3$ found from these equations, coincide. Therefore these factorizations hold:
\[
F = f(\bar{x}, \bar{x}_3)K, \quad G = g(\bar{x}, \bar{x}_2)K, \quad H = h(\bar{x}, \bar{x}_1)K, \quad K = K(\bar{x}, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_{123}), \quad (15)
\]
where $f, g, h$ are some polynomials of degree $2$ in the second argument. The degree in $\bar{x}$ remains to be determined.

Let the initial data $\bar{x}, \bar{x}_1, \bar{x}_2$ be free variables, and $\bar{x}_3$ chosen to satisfy the equation $f(\bar{x}, \bar{x}_3) = 0$. Then $F \equiv 0$, and thus the system $B = C = \bar{A} = 0$ does not determine the value of $x_{123}$. Moreover, the equation $B = 0$ can be solved with respect to $x_{13}$ since otherwise the initial data must be constrained by equation $B^{0.1}(\bar{x}, \bar{x}_1) = 0$. Analogously, the equation $C = 0$ is solvable with respect to $x_{23}$. Therefore, the uncertainty appears from the singularity of equation $\bar{A} = 0$ with respect to $x_{123}$. Hence, the relation $\bar{A}^{0.13}(\bar{x}, \bar{x}_{13}) = 0$ is valid. In virtue of the assumption of the theorem, $x_{13}$ is a (two-valued) function of $\bar{x}_3$ and does not depend on $\bar{x}_1$. This means that the equation $B = 0$ is singular with respect to $\bar{x}_1$ and therefore $B^{0.3}(\bar{x}, \bar{x}_3) = 0$. Analogously, $C^{0.3}(\bar{x}, \bar{x}_3) = 0$.

Thus, we have proven that if $x_3 = \varphi(\bar{x})$ is a zero of the polynomial $f$ then it is also a zero of the polynomials $B^{0.3}, C^{0.3}$. If one of these three polynomials is irreducible, then this already implies that they coincide up to a constant factor. If the polynomials are reducible this may be wrong since it is possible that $f = a^2, B^{0.3} = ab, C^{0.3} = ac$, where $a, b, c$ are affine-linear in $\bar{x}, \bar{x}_3$. In any case we have $\text{deg}_x f = 2$ and this is sufficient to complete the proof.

Indeed, this implies $\text{deg}_x K = 0$, that is the tetrahedron property is valid. In turn, this implies the relation \([13]\), as was shown in \([3]\). Recall this calculation: let us rewrite the system \([13]\) in the form
\[
\begin{align*}
x_{12} &= a(x, x_1, x_2), \quad x_{13} = b(x, x_1, x_3), \quad x_{23} = c(x, x_2, x_3), \\
x_{123} &= d(x_1, x_2, x_3) = \bar{a}(x_3, x_{13}, x_{23}) = \bar{b}(x_2, x_{12}, x_{23}) = \bar{c}(x_1, x_{12}, x_{13})
\end{align*}
\]
and find by differentiation
\[
\begin{align*}
    d_{x_1} &= \bar{a}_{x_1} b_{x_1}, \\
    d_{x_2} &= \bar{a}_{x_2} c_{x_2}, \\
    d_{x_3} &= \bar{b}_{x_2} c_{x_3}, \\
    d_{x_3} &= \bar{c}_{x_1} b_{x_3},
\end{align*}
\]
These equations readily imply the relation
\[
a_{x_2} b_{x_1} c_{x_2} + a_{x_1} b_{x_3} c_{x_2} = 0.
\]
The latter is equivalent to (14) by virtue of the identity \(a_{x_2}/a_{x_1} = A^{0.1}/A^{0.2}\). The variables in equation (14) separate: \(B^{0.3}/C^{0.3} = -A^{0.2}/C^{0.2} \cdot B^{0.1}/A^{0.1}\), so that \(B^{0.3}/C^{0.3}\) may only depend on \(x\). In view of the assumption of the theorem this ratio is constant.

There exist 3D-consistent systems whose equations are not of type \(Q\). The statements of Theorem 1 may or may not be valid for such a system, as the following examples demonstrate.

**Example 1.** The simplest 3D-consistent equation is the linear one
\[
x + x_i + x_j + x_{ij} = 0.
\]
In this case, all the biquadratic polynomials are equal to 1, so that the statement 1) is fulfilled and statement 2) is not. Since 2) is a consequence of the tetrahedron property 3), the latter cannot hold either. Indeed,
\[
x_{123} = 2x + x_1 + x_2 + x_3.
\]
The factor \(f\) in this example is also equal to 1, but this is a coincidence, destroyed by Möbius changes of variables. Indeed, in this case \(\deg_x K = 1\), and after the inversion of all variables \(x_i \to 1/x_i\) we come to \(f = xx_3^2\), while \(B^{0.3}\) turns into \(x^2x_3^3\).

**Example 2.** The Hietarinta equation (13)
\[
(x - e^{(j)})(x_{ij} - o^{(j)})(x_i - o^{(i)})(x_j - e^{(i)})(x_{ij} - o^{(i)})(x_i - e^{(j)})(x_j - o^{(j)}) = 0 \ (16)
\]
is 3D-consistent, but the statement 1) does not hold:
\[
\begin{align*}
    B^{0.3} &= (e^{(3)} - o^{(1)})(o^{(1)} - o^{(3)})(x - e^{(3)})(x - e^{(1)})(x_3 - e^{(1)})(x_3 - o^{(3)}), \\
    C^{0.3} &= (e^{(3)} - o^{(2)})(o^{(2)} - o^{(3)})(x - e^{(3)})(x - e^{(2)})(x_3 - e^{(2)})(x_3 - o^{(3)}).
\end{align*}
\]
The factor \(f\) is proportional to \((x - e^{(3)})(x_3 - e^{(1)})(x_3 - e^{(2)})\). Consequently, \(\deg_x K = 1\) and the tetrahedron property does not hold.

**Example 3.** Probably the best known example of a 3D-consistent system is given by the discrete potential KdV equation
\[
(x - x_{ij})(x_i - x_j) + \alpha^{(i)} - \alpha^{(j)} = 0. \quad \ (17)
\]
In this case all statements of the theorem are valid, in spite of the degeneracy of biquadratics:
\[
B^{0.3} = \alpha^{(1)} - \alpha^{(3)}, \quad C^{0.3} = \alpha^{(2)} - \alpha^{(3)}, \quad f = 1.
\]
(Recall that the degree is understood in the projective sense. Under the inversion these polynomials turn into \(x^2x_3^2\).)
Example 4. Equation \([Q_1]\)

\[
Q(x, x_1, x_2, x_{12}; \alpha^{(1)}, \alpha^{(2)}; \delta)
= \alpha^{(1)}(x - x_2)(x_1 - x_{12}) - \alpha^{(2)}(x - x_1)(x_2 - x_{12}) + \delta\alpha^{(1)}\alpha^{(2)}(\alpha^{(1)} - \alpha^{(2)}) = 0
\]

is consistent not only with its own copies (see [3] and Theorem [4] below), but also with the linear equations. Namely, the system formed of the equations

\[
Q(x, x_1, x_{12}, x_2; \alpha^{(1)}, \alpha^{(2)}; \delta) = 0, \quad x_{13} - x_3 = x_1 - x, \quad x_{23} - x_3 = x_2 - x
\]

and their copies on the opposite faces, is 3D-consistent. In this case the edge \((x, x_3)\) carries the polynomials

\[
B^{0,3} = C^{0,3} = -1, \quad f = 1.
\]

However, in contrast to the previous example, the tetrahedron property is not valid and \(\text{deg}_x K = 2\). This means that the polynomial \(f\) is not biquadratic and its image under inversion is \(x_3^2\). Moreover, the biquadratic polynomials corresponding to the edge \((x, x_1)\) do not coincide:

\[
A^{0,1} = \alpha^{(2)}(\alpha^{(1)} - \alpha^{(2)})((x_1 - x)^2 - \delta(\alpha^{(1)})^2), \quad B^{0,1} = -1, \quad h = 1.
\]

We see in this example that it is possible that some of the biquadratic polynomials satisfy the assumption of the theorem and the others do not.

### 4 Classification of biquadratic polynomials

Diagram (7) suggests an algorithm for the classification of affine-linear equations \(Q = 0\) modulo Möbius transformations. The first step is to use Möbius transformations to bring the polynomials \(r_i(x_i)\) associated to the vertices of the quadrilateral into canonical form. According to formulas (2),

\[
\delta_{x_i}(\delta_{x_j,x_k}(M[Q])) = \Delta_i^2\Delta_j^2\Delta_k^2 M[\delta_{x_i}(\delta_{x_j,x_k}(Q))] = \frac{C}{\Delta_i^2} M[r_i],
\]

where \(C = \Delta_1^2\Delta_2^2\Delta_3^2\Delta_4^2\). Since the polynomial \(Q\) is defined up to an arbitrary factor, we may assume that Möbius changes of variables in the equation \(Q = 0\) induce transformations

\[
r_i \mapsto \frac{1}{\Delta_i^2} M[r_i]
\]

of the polynomials \(r_i\). This allows us to bring each \(r_i\) into one of the following six forms:

\[
r = (x^2 - 1)(k^2x^2 - 1), \quad r = x^2 - 1, \quad r = x^2, \quad r = x, \quad r = 1, \quad r = 0,
\]

according to the six possibilities for the root distribution of \(r\): four simple roots, two simple roots and one double, two pairs of double roots, one simple root and one triple, one quadruple root, or, finally, \(r\) vanishes identically. Note that in the first canonical form it is always assumed that \(k \neq 0, \pm 1\), so that the second and third forms are not considered as particular cases of the first one.
These polynomials \( h \) and their relative invariants \( i_2, i_3 \) are:

\[
\begin{align*}
(r(x), r(y)), \quad r(x) &= (x^2 - 1)(k^2x^2 - 1) : \\
&= \frac{1}{2}\left(\frac{k^2\alpha^2 - \alpha^2}{x^2 - 1} - 2Axy - x^2 - y^2\right), \quad A^2 = r(\alpha), \quad (20) \\
i_2 &= 3\left(\frac{k^2\alpha^2 + \alpha^2}{x^2 - 1}\right) - k^2 - 1, \quad 4i_3 = A(k^2\alpha - \alpha^{-3}); \quad (21) \\
(x, y) : \quad h &= \frac{1}{\alpha}\left(\frac{1}{x} - \frac{1}{y}\right) - \frac{1}{\alpha^2}\left(\frac{x}{y} + 1\right) + \frac{3}{4}\alpha^3, \quad i_2 = \frac{3}{\alpha^2}, \quad i_3 = \frac{1}{32\alpha^3}; \quad (22) \\
(x^2, y^2) : \quad h &= \frac{\lambda^2 x^2 + \mu xy + \nu y^2}{\mu^2 - 4\lambda\nu}, \quad i_2 = 1 + 12\lambda\nu, \quad i_3 = -\lambda\mu\nu; \quad (23) \\
(1, 1) : \quad h &= \lambda(x + y)^2 + \mu(x - y) + \nu, \quad \mu^2 - 4\lambda\nu = 1, \quad i_2 = 12\lambda^2, \quad i_3 = \mp 2\lambda^3; \quad (24) \\
(0, 0) : \quad h &= (\lambda(xy + \alpha x + \mu y + \nu)^2, \quad i_2 = 12(\lambda\nu - \lambda\mu)^2, \quad i_3 = 2(\lambda\nu - \lambda\mu)^2; \quad (25) \\
(x^2 - 1, y^2) : \quad h &= \alpha y^2 \pm xy + \frac{1}{4\alpha}, \quad i_2 = 1, \quad i_3 = 0; \quad (26) \\
(x, 1) : \quad h &= \pm \frac{1}{4}(y - \alpha)^2 \mp x, \quad i_2 = 0, \quad i_3 = 0; \quad (27) \\
(1, 0) : \quad h &= \lambda y^2 + \mu y + \nu, \quad \mu^2 - 4\lambda\nu = 1, \quad i_2 = 0, \quad i_3 = 0. \quad (28)
\end{align*}
\]

\textbf{Proof.} The list is obtained by solving the system \((18)\) for different canonical pairs \((r_1, r_2)\).

The exhaustion of cases is shortened if we notice that \(g_2^2 \neq 27g_3^2\) in one case only, and that the relative invariants for the polynomial \(r_1 = ax^2 + bx + c\) are \(12g_2 = a^2, 216g_3 = -a^3\), so that the second polynomial must be of the form \(r_2 = ay^2 + by + \hat{c}\). The solution for the pair \((x, 0)\) turns out to be empty. \(\square\)
5 Classification of affine-linear equations of type $Q$

It is important to note that after bringing the polynomials $r_i(x_i)$ into canonical forms, one still has some freedom. Namely, one can use Möbius transformations which do not change the form of $r$ to further simplify the biquadratics $h$ and the affine-linear equation $Q$. In particular, the list of Theorem 2 is slightly more detailed than the list of biquadratics modulo Möbius transformations.

Indeed, the polynomial (22) turns into (23) under the inversion of $x$; the change $x \mapsto -x$ allows to fix the signs in the polynomials (24), (26); in the case (27), the sign is fixed by the change $x \mapsto -x, y \mapsto iy$; the polynomials (25), (28) admit a further simplification.

However, a transformation of any one of the four variables for a quadrilateral influences biquadratic polynomials on two edges adjacent to the correspondent vertex, and therefore it cannot a priori be guaranteed that all four biquadratics can be brought to some definite form simultaneously. For example, if each vertex corresponds to the polynomial (22) or of the form (23), the edges may correspond to the polynomials either of the form (24) or of the form (27). We do not know a priori that these polynomials can be always brought into the same form (even with different coefficients). Actually, this is possible, as the proof of the following theorem shows.

The next step is the reconstruction of the affine-linear polynomials from the biquadratic ones. Since our goal is only the classification of systems of type $Q$ equations, we will not solve this problem in its full generality. We leave aside the cases (26), (27) and (28), since the corresponding biquadratics are degenerate. For the same reason, the additional restrictions on the values of parameters are imposed: $\lambda \nu \neq 0$ in the cases (22), (23), $\lambda \neq 0$ in the case (24), and $\nu \alpha - \lambda \mu \neq 0$ in the case (25).

**Theorem 3.** Any affine-linear equation of type $Q$ is equivalent, up to Möbius transformations, to one of the equations in the following list:

\[
\begin{align*}
\text{sn}(\alpha) \text{sn}(\beta) \text{sn}(\alpha + \beta)(k^2 x_1 x_2 x_3 x_4 + 1) - \text{sn}(\alpha)(x_1 x_2 + x_3 x_4) \\
- \text{sn}(\beta)(x_1 x_4 + x_2 x_3) + \text{sn}(\alpha + \beta)(x_1 x_3 + x_2 x_4) &= 0, \\
(\alpha - \alpha^{-1})(x_1 x_2 + x_3 x_4) + (\beta - \beta^{-1})(x_1 x_4 + x_2 x_3) - (\alpha \beta - \alpha^{-1} \beta^{-1})(x_1 x_3 + x_2 x_4) \\
+ \delta(\alpha - \alpha^{-1})(\beta - \beta^{-1})(\alpha \beta - \alpha^{-1} \beta^{-1}) &= 0, \\
\alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) \\
- \alpha \beta(\alpha + \beta)(x_1 + x_2 + x_3 + x_4) + \alpha \beta(\alpha + \beta)(\alpha^2 + \alpha \beta + \beta^2) &= 0, \\
\alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \delta \alpha \beta(\alpha + \beta) &= 0.
\end{align*}
\]

**Proof.** Let the polynomials $h^{12}, h^{23}, h^{34}$ and $h^{14}$ be of the form (29), with the parameters $(\alpha, A), (\beta, B), (\tilde{\alpha}, \tilde{A})$ and $(\tilde{\beta}, \tilde{B})$, respectively, lying on the elliptic curve $A^3 = r(\alpha)$. The relative invariants $i_2, i_3$ of $h^{12}$ and $h^{34}$ must coincide as a corollary of (5), and it is easy to check that this condition allows only the following possible values for $(\tilde{\alpha}, \tilde{A})$:

\[
(\alpha, A), \quad (\alpha, -A), \quad \frac{1}{k \alpha^2}(\alpha, -A), \quad \frac{1}{k \alpha^2}(-\alpha, A)
\]

and analogously for $(\tilde{\beta}, \tilde{B})$. At first glance, it seems like we had to examine 16 quadruples of $h^{ij}$, but actually the situation is much more favorable. Indeed, according to (2), a
Möbius change of variables in the equation \( Q = 0 \) yields
\[
\delta_{x_k,x_l}(M[Q]) = \Delta_k \Delta_l M[\delta_{x_k,x_l}(Q)] = \frac{C}{\Delta_i \Delta_j} M[h^{ij}],
\]
where \( C = \Delta_1 \Delta_2 \Delta_3 \Delta_4 \). Since \( Q \) is only defined up to a multiplicative constant, we may assume that a Möbius change of variables induces transformations
\[
h^{ij} \mapsto \frac{1}{\Delta_i \Delta_j} M[h^{ij}]
\]
on the biquadratic polynomials \( h^{ij} \). In particular, if
\[
h^{34} = h(x_3, x_4, -\alpha, -A) \quad \text{or} \quad h^{34} = h\left(x_3, x_4, \frac{1}{k \alpha} - \frac{A}{k \alpha^2}\right),
\]
then the corresponding one of the Möbius transformations \( x_3 \mapsto -x_3 \) or \( x_3 \mapsto 1/(k x_3) \) will change \( h^{34} \) to
\[
-h(-x_3, x_4; -\alpha, -A), \quad \text{resp.} \quad -k x_3^2 h\left(\frac{1}{k x_3}, x_4; \frac{1}{k \alpha}, -\frac{A}{k \alpha^2}\right),
\]
both of which coincide with \( h(x_3, x_4, \alpha, A) \) due to symmetries of the polynomial \( Q \). Thus, performing a suitable Möbius transformation of the variable \( x_3 \) (which does not affect the polynomial \( r(x_3) \)), we may assume without loss of generality that \( (\tilde{\alpha}, \tilde{A}) = (\alpha, A) \). After that, the polynomial \( h^{14} \) is uniquely found according to formula \( 3, \) and it turns out that the equality \((\tilde{\beta}, \tilde{B}) = (\beta, B) \) is fulfilled automatically. Thus, the change of one variable allows to achieve the equality of the parameters corresponding to the opposite edges of the square. A direct computation using formula \( 10 \) yields the equation
\[
\alpha \beta \gamma (k^2 x_1 x_2 x_3 x_4 + 1) + \alpha (x_1 x_2 + x_3 x_4) + \beta (x_1 x_4 + x_2 x_3) + \gamma (x_1 x_3 + x_2 x_4) = 0,
\]
where \( \gamma = (\alpha B + \beta A)/(k^2 \alpha^2 \beta^2 - 1) \), and the final change \( \alpha \to \text{sn}(\alpha), A \to \text{sn}'(\alpha) \) and analogously for \( \beta \) brings it to the form \( 29 \).

Also in the other cases, suitable Möbius changes of the variables \( x_2, x_3, x_4 \) allow us to bring the polynomials into the form \( h^{12} = h(x_1, x_2, \alpha), h^{23} = h(x_2, x_3, \beta), h^{34} = h(x_3, x_4, \alpha) \). Moreover, a direct computation using formula \( 3 \) proves that also \( h^{14} = h(x_1, x_3, \beta) \). Then the answer is found by use of \( 10 \).

To give a few more details, polynomials \( 20 \) give rise to equation \( 30 \). In this case equations \( 6 \) imply that the parameters \( \alpha \) of the polynomials \( h^{12} \) and \( h^{34} \) differ at most by sign. This is compensated by the change \( x_3 \to -x_3 \) which is possible due to the symmetry \( h(x, y, \alpha) = -h(-x, y, -\alpha) \).

In the cases \( 22, 23 \), the appropriate scalings and, if necessary, inversions of the variables \( x_2, x_3, x_4 \) allow us to bring \( h^{12}, h^{23}, h^{34} \) into the form \( 20 \) without the constant term; therefore we simply obtain this case at \( \delta = 0 \).

Polynomial \( 21 \) corresponds to equation \( 31 \). This is the most simple case since the parameters are fixed already by condition \( 5 \).

In the case \( 24 \), appropriate shifts and, if necessary, changes of sign of the variables \( x_2, x_3, x_4 \) allow us to bring \( h^{12}, h^{23}, h^{34} \) into the form \( 2h(x, y, \alpha) = \alpha^{-1}(x - y)^2 - \delta \alpha \) with \( \delta = 1 \). Analogously, in the case \( 25 \) appropriate Möbius transforms of the general form bring \( h^{12}, h^{23}, h^{34} \) into the same form with \( \delta = 0 \). In both cases, the invariants are \( i_2 = 3 \alpha^{-2}, 4i_3 = \alpha^{-3} \), therefore the parameters of \( h^{12} \) and \( h^{34} \) coincide and no further changes are necessary. The resulting equation is \( 32 \).
6 Classification of 3D-consistent systems of type $Q$

Theorem 1 provides very strong necessary conditions for 3D-consistency in the case when all equations are of type $Q$. This will allow us to classify such systems in this section. In this final step we have to arrange the obtained equation around the cube and to choose the parameters in such a way that the condition (14) is fulfilled. The effect of this condition may be a change of sign or an inversion of one of the parameters.

In the following Theorem we return to the notation of the variables and parameters according to the shifts on the lattice. The ordering of the equations corresponds to the previous Theorem, and we name these equations as in $\mathbf{3}$.

**Theorem 4.** Any 3D-consistent system (13) of type $Q$ is, up to Möbius transformations, one of the following list:

\[
\begin{align*}
\text{sn}(\alpha(i)) & \text{sn}(\alpha(j)) \text{sn}(\alpha(i) - \alpha(j)) \left( k^2 x_i x_j x_{ij} + 1 \right) + \text{sn}(\alpha(i))(x_i + x_j x_{ij}) \\
& - \text{sn}(\alpha(j))(x_j + x_i x_{ij}) - \text{sn}(\alpha(i) - \alpha(j))(x_{ij} + x_i x_j) = 0, \\
\left(\alpha(i) - \frac{1}{\alpha(j)}\right)(x_i + x_j x_{ij}) & - \left(\alpha(j) - \frac{1}{\alpha(i)}\right)(x_j + x_i x_{ij}) - \left(\frac{\alpha(i)}{\alpha(j)} - \frac{\alpha(j)}{\alpha(i)}\right)(x_{ij} + x_i x_j) \\
& - \frac{\delta}{4} \left(\alpha(i) - \frac{1}{\alpha(j)}\right) \left(\alpha(j) - \frac{1}{\alpha(i)}\right) \left(\frac{\alpha(i)}{\alpha(j)} - \frac{\alpha(j)}{\alpha(i)}\right) = 0, \\
\alpha(i)(x - x_j)(x_i - x_{ij}) - \alpha(j)(x - x_i)(x_j - x_{ij}) + \alpha(i)\alpha(j) & (\alpha(i) - \alpha(j))(x + x_i + x_j + x_{ij}) \\
& - \alpha(i)\alpha(j) (\alpha(i) - \alpha(j)) (\alpha(i) + \alpha(j))^2 - \alpha(i)\alpha(j) + (\alpha(j))^2 = 0, \\
\alpha(i)(x - x_j)(x_i - x_{ij}) - \alpha(j)(x - x_i)(x_j - x_{ij}) & + \delta\alpha(i)\alpha(j)(\alpha(i) - \alpha(j)) = 0.
\end{align*}
\]

**Proof.** First of all, note that equations of the different types (29)–(32) cannot be consistent with each other since the corresponding singular curves are different. In particular, the parameters $k^2$ in the case (29) and $\delta$ in the cases (30), (32) must be the same on each face of the cube. Moreover, each equation of the list possesses the square symmetry, that is, it is invariant with respect to the changes $(x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4)$ and $(x_1 \leftrightarrow x_3, \alpha \leftrightarrow \beta)$.

Therefore, the equations on all faces may differ only by the values of $\alpha$ and $\beta$. Consider the equations corresponding to three faces meeting in one vertex, say $x$:

\[
Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = 0, \quad Q(x, x_2, x_3, x_{23}, \beta, \tilde{\gamma}) = 0, \quad Q(x, x_3, x_1, x_{13}, \gamma, \tilde{\alpha}) = 0.
\]

Let

\[
\delta_{x_2, x_{12}} Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = \kappa(\alpha, \tilde{\beta}) h(x, x_1, \alpha).
\]

Then, due to the symmetry,

\[
\delta_{x_1, x_{12}} Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = \kappa(\tilde{\beta}, \alpha) h(x, x_2, \tilde{\beta})
\]

and, according to the Theorem 1, the parameters must be related as follows:

\[
\begin{align*}
h(x, x_1, \alpha) & = m(\alpha, \tilde{\alpha}), \quad h(x, x_2, \beta) = m(\beta, \tilde{\beta}), \quad h(x, x_3, \gamma) = m(\gamma, \tilde{\gamma}), \\
\frac{\kappa(\alpha, \tilde{\beta}) \kappa(\beta, \tilde{\gamma}) \kappa(\gamma, \tilde{\alpha})}{\kappa(\tilde{\beta}, \alpha) \kappa(\tilde{\gamma}, \beta) \kappa(\tilde{\alpha}, \gamma)} m(\alpha, \tilde{\alpha}) m(\beta, \tilde{\beta}) m(\gamma, \tilde{\gamma}) & = -1.
\end{align*}
\]

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In the case (29), a direct computation proves that $\kappa(\alpha, \beta) = 2 \text{sn}(\alpha) \text{sn}(\beta) \text{sn}(\alpha + \beta)$ and 

$$h(x, y, \alpha) = \frac{1}{2 \text{sn}(\alpha)} (k^2 \text{sn}(\alpha) x^2 y^2 + 2 \text{sn}'(\alpha) xy - x^2 - y^2 + \text{sn}^2(\alpha)),$$

therefore $\tilde{\alpha}$ may take the values $\pm \alpha$ and analogously for $\beta, \gamma$. Obviously, up to a change of the numeration, two cases are possible:

$$\tilde{\alpha} = -\alpha, \quad \tilde{\beta} = -\beta, \quad \tilde{\gamma} = -\gamma \quad \text{or} \quad \tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta, \quad \tilde{\gamma} = -\gamma.$$

Moreover, this is actually only one case since we can make the change $(\alpha, \tilde{\beta}) \to (-\alpha, -\tilde{\beta})$ which keeps the equation $Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = 0$ invariant, as one can easily see from (29). It is not difficult to check that we can always adjust the signs on the whole cube as in the system $(Q_4)$.

Consider now the case (30). Here

$$\kappa(\alpha, \beta) = -\frac{(1 - \alpha^2 \beta^2)(1 - \alpha^2)(1 - \beta^2)}{\alpha^2 \beta^2},$$

$$h(x, y, \alpha) = \frac{\alpha}{1 - \alpha^2} (x^2 + y^2) - \frac{1 + \alpha^2}{1 - \alpha^2} xy + \frac{(1 - \alpha^2)\delta}{4\alpha},$$

and $\tilde{\alpha} = \alpha$ or $\tilde{\alpha} = 1/\alpha$. Taking into account the invariance of equation (30) with respect to the simultaneous inversion of $\alpha, \beta$, we can set, without loss of generality,

$$\tilde{\alpha} = 1/\alpha, \quad \tilde{\beta} = 1/\beta, \quad \tilde{\gamma} = 1/\gamma$$

resulting in the system $(Q_3)$. In the cases (31), (32) we have respectively

$$\kappa(\alpha, \beta) = -4\alpha \beta (\alpha + \beta), \quad h(x, y, \alpha) = \frac{1}{4\alpha} (x - y)^2 - \frac{\alpha}{2}(x + y) + \frac{\alpha^3}{4},$$

$$\kappa(\alpha, \beta) = -2\alpha \beta (\alpha + \beta), \quad h(x, y, \alpha) = \frac{1}{2\alpha} (x - y)^2 - \frac{\alpha \delta}{2},$$

and we may set $\tilde{\alpha} = -\alpha, \quad \tilde{\beta} = -\beta, \quad \tilde{\gamma} = -\gamma$ exactly as before. This leads to the systems $(Q_2), (Q_1)$.

The master equation $(Q_4)$ of the list was first derived in [2] and further investigated in [5]. A Lax representation for $(Q_4)$ was found in [19] with the help of the method based on the three-dimensional consistency. Equations $(Q_1)$ and $(Q_3)_{\delta=0}$ go back to [21]. Equations $(Q_2)$ and $(Q_3)_{\delta=1}$ first appeared explicitly in [3].

7 Examples of type $H$ systems

In contrast to the type $Q$ systems, the systems of type $H$ can be considered as “degenerate”. Their classification seems to be a rather tedious task. Presently we cannot suggest any effective procedure to solve this problem. On the other hand, the examples given in the Section 3 demonstrate that this class should not be just neglected as “pathological”. Indeed, the discrete KdV example (17) suggests that in some cases the degeneracy of the
biquadratics is just an unessential coincidence which does not spoil the integrability properties of an equation. Here we consider several more examples of this kind, corresponding to the cases (22), (23) at $\lambda \mu = 0$, (24) at $\lambda = 0$, and (25) at $\kappa \nu - \lambda \mu = 0$ which were excluded in the previous Section. It turns out that if we apply the same algorithm in these cases (in spite of the fact that there is no justification for this) then the list $H$ from our previous paper [3] will be reproduced:

\[
\begin{align*}
\alpha^{(i)}(xx_i + x_jx_{ij}) - \alpha^{(j)}(xx_j + x_ix_{ij}) + \delta((\alpha^{(i)})^2 - (\alpha^{(j)})^2) &= 0, \\
(x - x_{ij})(x_i - x_j) + (\alpha^{(i)} - \alpha^{(j)})(x + x_i + x_j + x_{ij}) + (\alpha^{(j)})^2 - (\alpha^{(i)})^2 &= 0, \\
(x - x_{ij})(x_i - x_j) + \alpha^{(j)} - \alpha^{(i)} &= 0.
\end{align*}
\]

(H3) (H2) (H1)

One may check directly that all statements of the Theorem remain valid for these equations, in spite of the degeneracy of the biquadratics.

Considering the asymmetric cases (26), (27), (28) with the different polynomials associated to the different vertices, one finds that the following variants are possible, up to the permutations. (There is clearly no distinction between edges and diagonals when we are dealing with a single equation.)

\[
(x_1^2 - 1, x_2^2, x_3^2, x_4^2), \quad (x_1^2 - 1, x_2^2 - 1, x_3^2, x_4^2), \quad (x_1^2 - 1, x_2^2 - 1, x_3^2 - 1, x_4^2), \\
(x_1, 1, 1, 1), \quad (x_1, x_2, 1, 1), \quad (x_1, x_2, x_3, 1), \\
(1, 0, 0, 0), \quad (1, 1, 0, 0), \quad (1, 1, 1, 0).
\]

A direct check shows that the variants of type \(\begin{pmatrix} r_2(x_1) & r_1(x_3) \\ r_1(x_1) & r_2(x_2) \end{pmatrix}\) are realizable and lead to the following list of 3D-consistent equations:

\[
\begin{align*}
\alpha(x_1x_2 + x_3x_4) - \beta(x_1x_4 + x_2x_3) + (\alpha^2 - \beta^2)\left(\delta + \frac{\varepsilon x_2x_4}{\alpha\beta}\right) &= 0, \\
(x_1 - x_3)(x_2 - x_4) + (\beta - \alpha)(x_1 + x_2 + x_3 + x_4) + \beta^2 - \alpha^2 \\
+ \varepsilon(\beta - \alpha)(2x_2 + \alpha + \beta)(2x_4 + \alpha + \beta) + \varepsilon(\beta - \alpha)^3 &= 0, \\
(x_1 - x_3)(x_2 - x_4) + (\beta - \alpha)(1 + \varepsilon x_2x_4) &= 0.
\end{align*}
\]

(H3) (H2) (H1)

This list can be considered as a deformation of the list $H$ which corresponds to the case $\varepsilon = 0$. However, we use the notation with cyclic indices rather than shifts since due to the lack of symmetry the arrangement of the equations on the faces of a cube requires a more explicit description (see below). Note that in (H1) the polynomial $1 + \varepsilon x_2x_4$ can be replaced by the polynomial $\kappa x_2x_4 + \mu(x_2 + x_4) + \nu$ with arbitrary coefficients. The corresponding biquadratic polynomials and their discriminants are given in the following table (up to multiplication by a suitable constant, $Q \rightarrow \mu(\alpha, \beta)Q$):

|       | $h(x_1, x_2)$ | $r_1(x_1)$ | $r_2(x_2)$ |
|-------|---------------|-------------|-------------|
| (H3) | $x_1x_2 + \varepsilon \alpha^{-1}x_2^2 + \delta \alpha$ | $x_1^2 - 4\delta \varepsilon$ | $x_2^2$ |
| (H2) | $x_1 + x_2 + \alpha + 2\varepsilon(x_2 + \alpha)^2$ | $1 - 8\varepsilon x_1$ | $1$ |
| (H1) | $1 + \varepsilon x_2^2$ | $-4\varepsilon$ | $0$ |
Each of these equations possesses the rhombic symmetry

\[ Q(x_1, x_2, x_3, x_4, \alpha, \beta) = -Q(x_3, x_2, x_1, x_4, \beta, \alpha) = -Q(x_1, x_4, x_3, x_2, \beta, \alpha), \]

but not the square symmetry since the vertices \(x_1, x_2\) correspond to polynomials with zeroes of different multiplicities. The equation is 3D-consistent on the black-white lattice \(i + j + k \pmod{2}\). That is, each face must carry a copy of the equation in such way that the parameters on opposite edges coincide and the vertices \(x, x_{12}, x_{13}, x_{23}\) are of the same type (here we switch again to the notation where indices denote shifts, as in Figure 3):

\[ Q(x, x_i, x_{ij}, x_j, \alpha^{(i)}, \alpha^{(j)}) = 0, \quad Q(x_{ik}, x_{jk}, x_{123}, \alpha^{(i)}, \alpha^{(j)}) = 0, \quad \{i, j, k\} = \{1, 2, 3\}. \]

Obviously, the equations on opposite faces of the cube do not coincide, but the equation may nevertheless be extended to the whole lattice \(\mathbb{Z}^3\). The tetrahedron property is fulfilled.

Finally, we notice that it is also possible to combine equations with the square and trapezoidal symmetry. Consider equation (\(Q_1\)) again. Let one pair of opposite faces carry the equations

\[ Q_1(x, x_1, x_{12}, x_2; \alpha^{(1)}, \alpha^{(2)})_{\delta=1} = 0, \quad Q_1(x_3, x_{13}, x_{123}, \alpha^{(1)}, \alpha^{(2)})_{\delta=0} = 0, \]

and let two other pairs carry the equations

\[ Q(x, x_i, x_{i3}, x_3, \alpha^{(i)}, \varepsilon) = 0, \quad Q(x_j, x_{ij}, x_{123}, \alpha^{(i)}, \varepsilon) = 0, \quad \{i, j\} = \{1, 2\}, \]

where the polynomial

\[ Q(x_1, x_2, x_3, x_4, \gamma, \varepsilon) = (x_1 - x_2)(x_3 - x_4) + \gamma(\varepsilon^{-1} - \varepsilon x_3 x_4) \]

actually coincides with (\(H\)) up to the permutation of \(x_2, x_3\). This awkward structure is 3D-consistent and, surprisingly, satisfies the tetrahedron property. It can be also extended to the lattice \(\mathbb{Z}^3\).

8 Concluding remarks

Non-commutative analogues of some of the equations in the Q-list are known. In particular, a quantum version of (\(Q_1\)) appeared in [27]. In [11] the consistency approach was extended to the non-commutative context, when the fields take values in an arbitrary associative algebra. The definition of three-dimensional consistency remains the same in this case, however, the assumption on the affine-linearity is replaced by the requirement that the equation can be brought to the linear form \(px = q\) with respect to any variable \(x\). These two properties are not equivalent in the noncommutative case, as is seen from the following examples. The first one was found in [11] and the other two by V.V. Sokolov and V. Adler (unpublished):

\[ \alpha^{(1)}(x - x_2)(x_2 - x_{12})^{-1} = \alpha^{(2)}(x - x_1)(x_1 - x_{12})^{-1}, \quad (\hat{Q}_1)_{\delta=0} \]

\[ \alpha^{(1)}(x_1 - x_{12} + \alpha^{(2)})(x - x_1 - \alpha^{(1)})^{-1} = \alpha^{(2)}(x_2 - x_{12} + \alpha^{(2)})(x - x_2 - \alpha^{(2)})^{-1}, \quad (\hat{Q}_1)_{\delta=1} \]

\[ (1 - (\alpha^{(1)})^2)(x_1 - \alpha^{(2)}x_{12})(\alpha^{(1)}x - x_1)^{-1} = (1 - (\alpha^{(2)})^2)(x_2 - \alpha^{(1)}x_{12})(\alpha^{(2)}x - x_2)^{-1}. \quad (\hat{Q}_3)_{\delta=0} \]
The existence of non-commutative analogs of \( Q_2 \), \( Q_3 \), \( Q_4 \) remains an open question. Although the analysis of the singular solutions may still be useful as a general principle, our technique is based on the algebraic properties of affine-linear and biquadratic polynomials and is therefore not applicable to this problem.

More general quantum systems with consistency property were found recently in [8, 7].

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