Hopf algebras of primitive Lie pseudogroups and Hopf cyclic cohomology

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Abstract

We associate to each infinite primitive Lie pseudogroup a Hopf algebra of ‘transverse symmetries’, by refining a procedure due to Connes and the first author in the case of the general pseudogroup. The affiliated Hopf algebra can be viewed as a ‘quantum group’ counterpart of the infinite-dimensional primitive Lie algebra of the pseudogroup. It is first constructed via its action on the étale groupoid associated to the pseudogroup, and then realized as a bicrossed product of a universal enveloping algebra by a Hopf algebra of regular functions on a formal group. The bicrossed product structure allows to express its Hopf cyclic cohomology in terms of a bicocyclic bicomplex analogous to the Chevalley-Eilenberg complex. As an application, we compute the relative Hopf cyclic cohomology modulo the linear isotropy for the Hopf algebra of the general pseudogroup, and find explicit cocycle representatives for the universal Chern classes in Hopf cyclic cohomology. As another application, we determine all Hopf cyclic cohomology groups for the Hopf algebra associated to the pseudogroup of local diffeomorphisms of the line.

Introduction

The transverse characteristic classes of foliations with holonomy in a transitive Lie pseudogroup Γ of local diffeomorphisms of \( \mathbb{R}^n \) are most effectively described in the framework of the Gelfand-Fuks [11] cohomology of the Lie

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algebra of formal vector fields associated to \( \Gamma \) (cf. e.g. Bott-Haefliger [2]).

In the dual, \( K \)-homological context, the transverse characteristic classes of general foliations have been expressed by Connes and the first author (cf. [6, 7, 8]) in terms of the Hopf cyclic cohomology of a Hopf algebra \( \mathcal{H}_n \) canonically associated to the group \( \text{Diff} \mathbb{R}^n \).

In this paper we construct similar Hopf algebras for all classical groups of diffeomorphisms, or equivalently for the infinite primitive Lie-Cartan pseudogroups [3] of local \( C^\infty \)-diffeomorphisms. The Hopf algebra \( \mathcal{H}_\Pi \) associated to such a pseudogroup \( \Pi \) can be regarded as a ‘quantum group’ analog of the infinite dimensional primitive Lie algebra of \( \Pi \) (cf. Singer-Sternberg [26], Guillemin [13]). It is initially constructed via its tautological action on the \'{e}tale groupoid associated to \( \Pi \), and is then reconstructed, in a manner reminiscent of a ‘quantum double’, as the bicrossed product of a universal enveloping algebra by a Hopf algebra of regular functions on a formal group.

In turn, the bicrossed product structure is employed to reduce the computation of the Hopf cyclic cohomology of \( \mathcal{H}_\Pi \) to that of a bicocyclic bicomplex analogous to the Chevalley-Eilenberg complex. This apparatus is then applied to compute the relative Hopf cyclic cohomology of \( \mathcal{H}_\Pi \) modulo \( \mathfrak{gl}_n \). We actually find explicit cocycles representing the Hopf cyclic analogues of the universal Chern classes. In the case of \( \mathcal{H}_1 \), the improved technique allows us to refine our previous computations [24] and completely determine the non-periodized Hopf cyclic cohomology of the Hopf algebra \( \mathcal{H}_1 \) affiliated with the pseudogroup of local diffeomorphisms of the line.

We now give a brief outline of the main results. Our construction of the Hopf algebra associated to a primitive Lie pseudogroup \( \Pi \) is modeled on that of \( \mathcal{H}_n \) in [6], which in turn was inspired by a procedure due to G. I. Kac [18] for producing non-commutative and non-cocommutative quantum groups out of ‘matched pairs’ of finite groups. It relies on splitting the group \( \text{Diff}_\Pi \) of globally defined diffeomorphisms of type \( \Pi \) as a set-theoretical product of two subgroups,

\[
\text{Diff}_\Pi = G_\Pi \cdot N_\Pi, \quad G_\Pi \cap N_\Pi = \{e\}.
\]

For a flat (i.e. containing all the translations) primitive pseudogroup \( \Pi \), \( G_\Pi \) is the subgroup consisting of the affine transformations of \( \mathbb{R}^n \) that are in \( \text{Diff}_\Pi \), while \( N_\Pi \) is the subgroup consisting of those diffeomorphisms in \( \text{Diff}_\Pi \) that preserve the origin to order 1.

The pseudogroup of contact transformations is the only infinite primitive pseudogroup which is not flat. In that case, we identify \( \mathbb{R}^{2n+1} \) with the Heisenberg group \( H_n \). Instead of the vector translations, we let \( H_n \) act on
itself by group left translations, and define the group of ‘affine Heisenberg transformations’ \( G_\Pi \) as the semidirect product of \( H_n \) by the linear isotropy group \( G_\Pi^0 \) consisting of the linear contact transformations. As factor \( N_\Pi \) we take the subgroup of all contact diffeomorphisms preserving the origin to order 1 in the sense of Heisenberg calculus, \textit{i.e.} whose differential at 0 is the identity map of the Heisenberg tangent bundle (\textit{cf.} e.g. \cite{25}).

The factorization \( \text{Diff}_\Pi = G_\Pi \cdot N_\Pi \) allows to represent uniquely any \( \phi \in \text{Diff}_\Pi \) as a product \( \phi = \varphi \cdot \psi \), with \( \varphi \in G_\Pi \) and \( \psi \in N_\Pi \). Factorizing the product of any two elements \( \varphi \in G_\Pi \) and \( \psi \in N_\Pi \) in the reverse order, 

\[ \psi \cdot \varphi = (\psi \triangleright \varphi) \cdot (\psi \triangleleft \varphi), \]

one obtains a left action \( \psi \mapsto \tilde{\psi}(\varphi) := \psi \triangleright \varphi \) of \( N_\Pi \) on \( G_\Pi \), along with a right action \( \triangleleft \) of \( G_\Pi \) on \( N_\Pi \). Equivalently, these actions are restrictions of the natural actions of \( \text{Diff}_\Pi \) on the coset spaces \( \text{Diff}_\Pi / N_\Pi \cong G_\Pi \) and \( G_\Pi \setminus \text{Diff}_\Pi \cong N_\Pi \).

The ‘dynamical’ definition of the Hopf algebra \( H_\Pi \) associated to the pseudogroup \( \Pi \) is obtained by means of its action on the (discrete) crossed product algebra \( A_\Pi = C^\infty(G_\Pi) \rtimes \text{Diff}_\Pi \), which arises as follows. One starts with a fixed basis \( \{ X_i \}_{1 \leq i \leq m} \) for the Lie algebra \( g_\Pi \) of \( G_\Pi \). Each \( X \in g_\Pi \) gives rise to a left-invariant vector field on \( G_\Pi \), which is then extended to a linear operator on \( A_\Pi \), in the most obvious fashion:

\[ X(f U_{\phi^{-1}}) = X(f) U_{\phi^{-1}}, \]

where \( f \in C^\infty(G_\Pi) \) and \( \phi \in \text{Diff}_\Pi \). One has

\[ U_{\phi^{-1}} X_i U_{\phi} = \sum_{j=1}^{m} \Gamma^j_i(\phi) X_j, \quad i = 1, \ldots, m, \]

with \( \Gamma^j_i(\phi) \in C^\infty(G_{cn}) \), and we define corresponding multiplication operators on \( A_\Pi \) by taking

\[ \Delta^j_i(f U_{\phi^{-1}}) = (\Gamma(\phi)^{-1})^j_i f U_{\phi}, \quad \text{where} \quad \Gamma(\phi) = (\Gamma^j_i(\phi))_{1 \leq i,j \leq m}. \]

As an algebra, \( H_\Pi \) is generated by the operators \( X_k \)'s and \( \Delta^j_i \)'s. In particular, \( H_\Pi \) contains all iterated commutators

\[ \Delta^j_{i_1, \ldots, i_r} := [X_{k_r}, \ldots, [X_{k_1}, \Delta^j_i] \ldots], \]

which are multiplication operators by the functions

\[ \Gamma^j_{i_1, \ldots, i_r}(\phi) := X_{k_r} \ldots X_{k_1}(\Gamma^j_i(\phi)), \quad \phi \in \text{Diff}_\Pi. \]
For any $a, b \in A$, one has
\[ X_k(ab) = X_k(a) b + \sum_j \Delta^j_k(a) X_j(b), \]
\[ \Delta^j_i(ab) = \sum_k \Delta^j_k(a) \Delta^i_k(b), \]
and by multiplicativity every $h \in H = U(A)$ satisfies a ‘Leibniz rule’ of the form
\[ h(ab) = \sum h(1)(a) h(2)b, \quad \forall a, b \in A. \]
The operators $\Delta^i_{j,k}$ satisfy the following Bianchi-type identities:
\[ \Delta^k{i,j} - \Delta^k{j,i} = \sum_{r,s} c^k_{r,s} \Delta^r{i} \Delta^s{j} - \sum_\ell c^k_{i,j} \Delta^k{\ell}, \]
where $c^k_{i,j}$ are the structure constants of the Lie algebra $g$.

**Theorem 0.1.** Let $H = U(A)$ be the abstract Lie algebra generated by the operators $\{X_k, \Delta^j_{i,k1...kr}\}$ and their commutation relations.

1. The algebra $H$ is isomorphic to the quotient of the universal enveloping algebra $U(g)$ by the ideal $B$ generated by the Bianchi identities.
2. The Leibniz rule determines uniquely a coproduct, with respect to which $H$ a Hopf algebra and $A$ an $H$-module algebra.

We next describe the bicrossed product realization of $H$. Let $F$ denote the algebra of functions on $N$ generated by the jet ‘coordinates’
\[ n^j_{i,k1...kr}(\psi) := \Gamma^j_{i,k1...kr}(\psi)(e), \quad \psi \in N; \]
the definition is obviously independent of the choice of basis for $g$. Furthermore, $F$ is a Hopf algebra with coproduct uniquely and well-defined by the rule
\[ \Delta f(\psi_1, \psi_2) := f(\psi_1 \circ \psi_2), \quad \forall \psi_1, \psi_2 \in N, \]
and with antipode
\[ Sf(\psi) := f(\psi^{-1}), \quad \psi \in N. \]
Now the universal enveloping algebra $U = U(g)$ can be equipped with a right $F$-comodule coalgebra structure $\nabla : U \to U \otimes F$ as follows. Let
\{X_I = X_{i_1}^{i_1} \cdots X_{i_m}^{i_m} ; i_1, \ldots, i_m \in \mathbb{Z}^+ \} \text{ be the PBW basis of } U_\Pi \text{ induced by the chosen basis of } \mathfrak{g}_\Pi. \text{ Then }

\[ U_{\psi^{-1}}X_I U_\psi = \sum_J \beta^J_I(\psi) X_J, \]

with \( \beta^J_I(\psi) \) in the algebra of functions on \( G_\Pi \) generated by \( \Gamma^j_{i,K}(\psi) \). One obtains a coaction \( \nabla : U_\Pi \rightarrow U_\Pi \otimes F_\Pi \) by defining

\[ \nabla(X_I) = \sum_J X_J \otimes \beta^J_I(\cdot)(e); \]

again, the definition is independent of the choice of basis.

The right action \( \triangleleft \) of \( G_\Pi \) on \( N_\Pi \) induces an action of \( G_\Pi \) on \( F_\Pi \) and hence a left action of \( U_\Pi \) on \( F_\Pi \), that makes \( F_\Pi \) a left \( U_\Pi \)-module algebra.

**Theorem 0.2.** With the above operations, \( U_\Pi \) and \( F_\Pi \) form a matched pair of Hopf algebras, and their bicrossed product \( F_\Pi \bowtie U_\Pi \) is canonically isomorphic to the Hopf algebra \( H_{\Pi}^{\text{cop}} \).

The Hopf algebra \( H_\Pi \) also comes equipped with a modular character \( \delta = \delta_\Pi \), extending the infinitesimal modular character \( \delta(X) = \text{Tr}(\text{ad}\, X), \ X \in \mathfrak{g}_\Pi \). The corresponding module \( C_\delta \), viewed also as a trivial comodule, defines a ‘modular pair in involution’, cf. [7], or a particular case of an ‘SA YD module-comodule’, cf. [16]. Such a datum allows to specialize Connes’ \( Ext_\Lambda \)-definition [4] of cyclic cohomology to the context of Hopf algebras (cf. [6]), [7]). The resulting Hopf cyclic cohomology, introduced in [6] and extended to SA YD coefficients in [17], incorporates both Lie algebra and group cohomology and provides the appropriate cohomological tool for the treatment of symmetry in noncommutative geometry. However, its computation for general, \( i.e. \) non-commutative and non-cocommutative, Hopf algebras poses quite a challenge. In the case of \( H_\Pi \), it has been shown in [6] that \( HP^*(H_\Pi; \mathbb{C}_\delta) \) is canonically isomorphic to the Gelfand-Fuks cohomology of the Lie algebra \( \mathfrak{a}_n \) of formal vector fields on \( \mathbb{R}^n \), result which allowed to transfer the transverse characteristic classes of foliations from \( K \)-theory classes into \( K \)-homology characteristic classes. There are very few instances of direct calculations so far (see e.g. [24], where the periodic Hopf cyclic cohomology of several variants of \( H_1 \) has been directly computed), and a general machinery for performing such computations is only beginning to emerge.

We rely on the bicrossed product structure of the Hopf algebra \( H_\Pi \) to reduce the computation of its Hopf cyclic cohomology to that of a simpler, bicyclic
bicomplex. The latter combines the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g} = \mathfrak{g}_\Pi$ with coefficients in $\mathcal{C}_\delta \otimes \mathcal{F}^{\otimes \bullet}$ and the coalgebra cohomology complex of $\mathcal{F} = \mathcal{F}_\Pi$ with coefficients in $\wedge^\bullet \mathfrak{g}$:

![Diagram of the bicomplex]

**Theorem 0.3.** 1. The above bicomplex computes the periodic Hopf cyclic cohomology $HP^*(\mathcal{H}_\Pi; \mathcal{C}_\delta)$.

2. There is a relative version of the above bicomplex that computes the relative periodic Hopf cyclic cohomology $HP^*(\mathcal{H}_\Pi, \mathcal{U}(\mathfrak{h}); \mathcal{C}_\delta)$, for any reductive subalgebra $\mathfrak{h}$ of the linear isotropy Lie algebra $\mathfrak{g}_\Pi$.

As the main application in this paper, we compute the periodic Hopf cyclic cohomology of $\mathcal{H}_n$ relative to $\mathfrak{gl}_n$ and find explicit cocycle representatives for its basis, as described below.

For each partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k)$ of the set $\{1, \ldots, p\}$, where $1 \leq p \leq n$, we let $\lambda \in S_p$ also denote a permutation whose cycles have lengths $\lambda_1 \geq \ldots \geq \lambda_k$, i.e. representing the corresponding conjugacy class $[\lambda] \in [S_p]$.

We then define

$$C_{p,\lambda} := \sum (-1)^\mu 1 \otimes \eta_{\mu(1),j_{\lambda(1)}}^{j_1} \wedge \cdots \wedge \eta_{\mu(p),j_{\lambda(p)}}^{j_p} \otimes X_{\mu(p+1)} \wedge \cdots \wedge X_{\mu(n)},$$

where the summation is over all $\mu \in S_n$ and all $1 \leq j_1, j_2, \ldots, j_p \leq n$.

**Theorem 0.4.** The cochains $\{C_{p,\lambda}; 1 \leq p \leq n, [\lambda] \in [S_p]\}$ are cocycles and their classes form a basis of the group $HP^\epsilon(\mathcal{H}_n, \mathcal{U}(\mathfrak{gl}_n); \mathcal{C}_\delta)$, where $\epsilon \equiv n \mod 2$, while $HP^{1-\epsilon}(\mathcal{H}_n, \mathcal{U}(\mathfrak{gl}_n); \mathcal{C}_\delta) = 0$.

The correspondence with the universal Chern classes is obvious. Let $\mathcal{P}_n[c_1, \ldots, c_n] = \mathbb{C}[c_1, \ldots, c_n]/\mathcal{I}_n$. 

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denote the truncated polynomial ring, where \( \deg(c_j) = 2j \), and \( \mathcal{I}_n \) is the ideal generated by the monomials of degree \( > 2n \). To each partition \( \lambda \) as above, one associates the degree 2p monomial

\[
c_{p,\lambda} := c_{\lambda_1} \cdots c_{\lambda_k}, \quad \lambda_1 + \ldots + \lambda_k = p;
\]

the corresponding classes \( \{c_{p,\lambda}; 1 \leq p \leq n, \ \lambda \in [S_p]\} \) form a basis of the vector space \( P_n[c_1, \ldots, c_n] \).

A second application is the complete determination of the Hochschild cohomology of \( \mathcal{H}_1 \) and of the non-periodized Hopf cyclic cohomology groups \( HC^q(\mathcal{H}_1; \mathbb{C}_\delta) \), where \( \delta \) is the modular character of the Hopf algebra \( \mathcal{H}_1 \).

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1 Construction via Hopf actions

This section is devoted to the ‘dynamical’ construction of the Hopf algebras associated to primitive Lie pseudogroups of infinite type. For the sake of clarity, we start with the case of the general pseudogroup (cf. also [9]), where the technical details can be handled in the most transparent fashion. The
other cases of flat pseudogroups can be treated in a similar manner. In order to illustrate the slight adjustments needed to cover them, we describe in some detail the Hopf algebras affiliated to the volume preserving and the symplectic pseudogroups. On the other hand, the case of the contact pseudogroup, which is the only non-flat one, requires a certain change of geometric viewpoint, namely replacing the natural motions of $\mathbb{R}^{2n+1}$ with the natural motions of the Heisenberg group $H_n$.  

1.1 Hopf algebra of the general pseudogroup  
Let $F^{\mathbb{R}^n} \to \mathbb{R}^n$ be the frame bundle on $\mathbb{R}^n$, which we identify to $\mathbb{R}^n \times \text{GL}(n, \mathbb{R})$ in the obvious way: the 1-jet at $0 \in \mathbb{R}^n$ of a germ of a local diffeomorphism $\phi$ on $\mathbb{R}^n$ is viewed as the pair  

\[(x := \phi(0), y := \phi'_0(0)) \in \mathbb{R}^n \times \text{GL}(n, \mathbb{R}), \quad \phi_0(x) := \phi(x) - \phi(0). \tag{1.1}\]

The flat connection on $F^{\mathbb{R}^n} \to \mathbb{R}^n$ is given by the matrix-valued 1-form $\omega = (\omega^i_j)$ where, with the usual summation convention,  

\[\omega^i_j := (y^{-1})^i_\mu dy^\mu_j = (y^{-1} dy)_j^i, \quad i, j = 1, \ldots, n, \tag{1.2}\]

and the canonical form is the vector-valued 1-form $\theta = (\theta^k)$,  

\[\theta^k := (y^{-1})^k_\mu dx^\mu = (y^{-1} dx)^k, \quad k = 1, \ldots, n \tag{1.3}\]

The basic horizontal vector fields for the above connection are  

\[X_k = y^\mu_k \partial_\mu, \quad k = 1, \ldots, n, \quad \text{where} \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{1.4}\]

and the fundamental vertical vector fields associated to the standard basis of $\text{gl}(n, \mathbb{R})$, formed by the elementary matrices $\{E^j_i; 1 \leq i, j \leq n\}$, have the expression  

\[Y^j_i = y^\mu_i \partial^j_\mu, \quad i, j = 1, \ldots, n, \quad \text{where} \quad \partial^j_\mu := \frac{\partial}{\partial y^j_\mu}. \tag{1.5}\]

Let $G := \mathbb{R}^n \times \text{GL}(n, \mathbb{R})$ denote the group of affine motions of $\mathbb{R}^n$.  

**Proposition 1.1.** The vector fields $\{X_k, Y^j_i; i, j, k = 1, \ldots, n\}$ form a basis of left-invariant vector fields on the group $G$.  

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Proof. Represent $G$ as the subgroup of $\text{GL}(n + 1, \mathbb{R})$ consisting of the matrices $a = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, with $y \in \text{GL}(n, \mathbb{R})$, and $x \in \mathbb{R}^n$. Let $\{e_k, E^j_i; i, j, k = 1, \ldots, n\}$ be the standard basis of the Lie algebra $g := \mathbb{R}^n \rtimes \text{gl}(n, \mathbb{R})$, and denote by $\{\tilde{e}_k, \tilde{E}^j_i; i, j, k = 1, \ldots, n\}$ the corresponding left-invariant vector fields. By definition, at the point $a$, $\tilde{e}_k$ is tangent to the curve $t \mapsto \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & te_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y & tye_k + x \\ 0 & 1 \end{pmatrix}$ and therefore coincides with $X_k = \sum_\mu y^\mu_k \partial_\mu$, while $\tilde{E}^j_i$ is tangent to $t \mapsto \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{E^j_i} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y e^{E^j_i} & 0 \\ 0 & 1 \end{pmatrix}$ which is precisely $Y^j_i = \sum_\mu y^\mu_i \partial^j_\mu$. \hfill \Box

The group of diffeomorphisms $G := \text{Diff} \mathbb{R}^n$ acts on $F \mathbb{R}^n$, by the natural lift of the tautological action to the frame level:

$$\varphi(x, y) := (\varphi(x), \varphi'(x) \cdot y), \quad \text{where} \quad \varphi'(x)^j_i = \partial_j \varphi^i(x).$$

(1.6)

Viewing here $G$ as a discrete group, we form the crossed product algebra

$$\mathcal{A} := C^\infty_c(F \mathbb{R}^n) \rtimes G.$$

As a vector space, it is spanned by monomials of the form $f U^*_\varphi$, where $f \in C^\infty_c(F \mathbb{R}^n)$ and $U^*_\varphi$ stands for $\varphi^{-1}$, while the product is given by the multiplication rule

$$f_1 U^*_\varphi_1 \cdot f_2 U^*_\varphi_2 = f_1(f_2 \circ \varphi_1) U^*_\varphi_2 \varphi_1.$$  

(1.7)

Alternatively, $\mathcal{A}$ can be regarded as the subalgebra of the endomorphism algebra $\mathcal{L}(C^\infty_c(F \mathbb{R}^n)) = \text{End}_C (C^\infty_c(F \mathbb{R}^n))$, generated by the multiplication and the translation operators

$$M_f(\xi) = f(\xi), \quad f \in C^\infty_c(F \mathbb{R}^n), \xi \in C^\infty_c(F \mathbb{R}^n)$$

(1.8)

$$U^*_\varphi(\xi) = \xi \circ \varphi, \quad \varphi \in G, \xi \in C^\infty_c(F \mathbb{R}^n).$$

(1.9)

Since the right action of $\text{GL}(n, \mathbb{R})$ on $F \mathbb{R}^n$ commutes with the action of $G$, at the Lie algebra level one has

$$U_\varphi Y^j_i U^*_\varphi = Y^j_i, \quad \varphi \in G.$$  

(1.10)
This allows to promote the vertical vector fields to derivations of $\mathcal{A}$. Indeed, setting
\[ Y^j_i(f U_\varphi^*) = Y^j_i(f) U_\varphi^*, \quad f U_\varphi^* \in \mathcal{A}, \] (1.11)
the extended operators satisfy the derivation rule
\[ Y^j_i(a b) = Y^j_i(a) b + a Y^j_i(b), \quad a, b \in \mathcal{A}, \] (1.12)
We also prolong the horizontal vector fields to linear transformations $X_k \in \mathcal{L}(\mathcal{A})$, in a similar fashion:
\[ X_k(f U_\varphi^*) = X_k(f) U_\varphi^*, \quad f U_\varphi^* \in \mathcal{A}. \] (1.13)
The resulting operators are no longer $G$-invariant. Instead of (1.10), they satisfy
\[ U_\varphi^* X_k U_\varphi = X_k - \gamma^j_{ik}(\varphi) Y^j_i, \] (1.14)
where
\[ \gamma^j_{ik}(\varphi)(x, y) = (y^{-1} \cdot \varphi'(x)^{-1} \cdot \partial_\mu \varphi'(x) \cdot y)^i_j y^\mu_k. \] (1.15)
Using the left-invariance of the vector fields $X_k$ and (1.14), or just the explicit formula (1.15), one sees that $\varphi \mapsto \gamma^j_{ik}(\varphi)$ is a group 1-cocycle on $G$ with values in $C^\infty(F \mathbb{R}^n)$; specifically,
\[ \gamma^i_{jk}(\varphi \circ \psi) = \gamma^i_{jk}(\varphi) \circ \tilde{\psi} + \gamma^i_{jk}(\psi), \quad \forall \varphi, \psi \in G. \] (1.16)
As a consequence of (1.14), the operators $X_k \in \mathcal{L}(\mathcal{A})$ are no longer derivations of $\mathcal{A}$, but satisfy instead a non-symmetric Leibniz rule:
\[ X_k(a b) = X_k(a) b + a X_k(b) + \delta^j_{ik}(a) Y^j_i(b), \quad a, b \in \mathcal{A}, \] (1.17)
where the linear operators $\delta^j_{ik} \in \mathcal{L}(\mathcal{A})$ are defined by
\[ \delta^j_{ik}(f U_\varphi^*) = \gamma^j_{ik}(\varphi) f U_\varphi^*. \] (1.18)
Indeed, on taking $a = f_1 U_{\varphi_1}^*$, $b = f_2 U_{\varphi_2}^*$, one has
\[ X_k(a \cdot b) = X_k(f_1 U_{\varphi_1}^* \cdot f_2 U_{\varphi_2}^*) = X_k(f_1 \cdot U_{\varphi_1}^* f_2 U_{\varphi_1}^*) U_{\varphi_2}^* \]
\[ = X_k(f_1) U_{\varphi_1}^* \cdot f_2 U_{\varphi_2}^* + f_1 U_{\varphi_1}^* \cdot X_k(f_2 U_{\varphi_2}^*) \]
\[ + f_1 U_{\varphi_1}^* \cdot (U_{\varphi_1} X_k U_{\varphi_1}^* - X_k)(f_2 U_{\varphi_2}^*), \]
which together with (1.14) and the cocycle property (1.16) imply (1.17).
The same cocycle property shows that the operators $\delta_{jk}^i$ are derivations:

$$\delta_{jk}^i(ab) = \delta_{jk}^i(a) b + a \delta_{jk}^i(b), \quad a, b \in \mathcal{A},$$  \hspace{1cm} (1.19)

The operators $\{X_k, Y_j^i\}$ satisfy the commutation relations of the group of affine transformations of $\mathbb{R}^n$:

$$[Y_j^i, Y_k^\ell] = \delta_k^j Y_i^\ell - \delta_i^j Y_k^\ell,$$  \hspace{1cm} (1.20)

$$[Y_j^i, X_k] = \delta_k^i X_j, \quad [X_k, X_\ell] = 0.$$

The operators $\{X_k, Y_j^i, \delta_{jk}^i, \delta_{jk}^i_{\ell_1...\ell_r}; i, j, k, \ell_1 ... \ell_r \in \mathbb{N}\}$ form a Lie algebra. We let $\mathcal{H}_n$ denote the unital subalgebra of $\mathcal{L}(\mathcal{A})$ generated by $\mathfrak{h}_n$.

Evidently, they commute among themselves:

$$[\delta_{jk}^i, \delta_{jk'}^{i'}_{\ell_1...\ell_r}] = 0.$$  \hspace{1cm} (1.23)

The commutators between the $Y_j^i$'s and $\delta_{jk}^i$'s, which can be easily obtained from the explicit expression (1.15) of the cocycle $\gamma$, are as follows:

$$[Y_j^i, \delta_{jk}^i] = \delta_{jk}^i Y_j^i - \delta_j^i Y_k^i.$$  \hspace{1cm} (1.24)

More generally, one checks by induction the relations

$$[Y_j^i, \delta_{jk}^i_{\ell_1...\ell_r}] = \sum_{s=1}^r \delta_j^s \delta_{jk}^i_{\ell_1...\ell_s \ell_{s+1}...\ell_r} - \delta_j^i \delta_{jk}^i_{\ell_1...\ell_r}.$$  \hspace{1cm} (1.25)

The commutator relations (1.20), (1.21), (1.23), (1.24) show that the subspace $\mathfrak{h}_n$ of $\mathcal{L}(\mathcal{A})$ generated by the operators

$$\{X_k, Y_j^i, \delta_{jk}^i_{\ell_1...\ell_r}; i, j, k, \ell_1 \ldots \ell_r = 1, \ldots, n, r \in \mathbb{N}\}$$  \hspace{1cm} (1.25)

forms a Lie algebra.
$\delta_{jk \ell_1 \ldots \ell_r}^i$ are not all distinct; the order of the first two lower indices or of the last $r$ indices is immaterial. Indeed, the expression of the cocycle $\gamma$,

$$\gamma^i_{jk}(\varphi)(x,y) = (y^{-1})^i_{\lambda} (\varphi'(x)^{-1})^\lambda_\rho \partial_\rho \varphi^\rho(x) y^\mu_j y^\mu_k,$$  

(1.26)
is clearly symmetric in the indices $j$ and $k$. The symmetry in the last $r$ indices follows from the definition of $\gamma^i_{jk}$ and the fact that, the connection being flat, the horizontal vector fields commute. It can also be directly seen from the explicit formula

$$\gamma^i_{jk \ell_1 \ldots \ell_r}(\varphi)(x,y) = (y^{-1})^i_{\lambda} \partial_{\lambda_1} \ldots \partial_{\lambda_r} ((\varphi'(x)^{-1})^\lambda_\rho \partial_\rho \varphi^\rho(x)) y^\mu_j y^\mu_k y_{\ell_1} \ldots y_{\ell_r}.$$ 

(1.27)

**Proposition 1.2.** The operators $\delta_{\bullet \ldots \bullet}^i$ satisfy the identities

$$\delta_{j\ell k}^i - \delta_{j k \ell}^i = \delta_{j k}^s \delta_{s \ell}^i - \delta_{j \ell}^s \delta_{s k}^i.$$  

(1.28)

**Proof.** These Bianchi-type identities are an expression of the fact that the underlying connection is flat. Indeed, if we let $a, b \in A$ and apply (1.17) we obtain

$$X_\ell X_k(ab) = X_\ell X_k(a) b + X_k(a) X_\ell(b) + \delta_{j\ell}^i(X_k(a)) Y^j_i(b)$$

$$+ X_\ell(a) X_k(b) + a X_\ell X_k(b) + \delta_{j k}^i(a) Y^j_i(X_k(b))$$

$$+ X_\ell(\delta_{j k}^i(a)) Y^j_i(b) + \delta_{j k}^i(a) X_\ell(Y^j_i(b)) + \delta_{s \ell}^r(\delta_{j k}^i(a)) Y^r_s(Y^j_i(b)).$$

Since $[X_k, X_\ell] = 0$, by antisymmetrizing in $k, \ell$ it follows that

$$\delta_{j \ell}^i(X_k(a)) Y^j_i(b) + \delta_{j k}^i(a) Y^j_i(X_k(b)) + X_\ell(\delta_{j k}^i(a)) Y^j_i(b)$$

$$+ \delta_{j \ell}^i(a) X_\ell(Y^j_i(b)) + \delta_{s \ell}^r(\delta_{j k}^i(a)) Y^r_s(Y^j_i(b))$$

$$= \delta_{j k}^i(X_\ell(a)) Y^j_i(b) + \delta_{j k}^i(a) Y^j_i(X_\ell(b)) + X_k(\delta_{j \ell}^i(a)) Y^j_i(b)$$

$$+ \delta_{j \ell}^i(a) X_k(Y^j_i(b)) + \delta_{s k}^r(\delta_{j \ell}^i(a)) Y^r_s(Y^j_i(b)).$$

Using the ‘affine’ relations (1.20) and the symmetry of $\delta^i_{jk}$ in the lower indices one readily obtains the equation (1.28).

In view of this result, the algebra $\mathcal{H}_n$ admits a basis similar to the Poincaré-Birkhoff-Witt basis of a universal enveloping algebra. The notation needed
to specify such a basis involves two kinds of multi-indices. The first kind
are of the form
\[ I = \left\{ i_1 \leq \ldots \leq i_p \mid \left( j_{k_1} \right) \leq \ldots \leq \left( j_{k_q} \right) \right\}, \tag{1.29} \]
while the second kind are of the form
\[ K = \left\{ \kappa_1 \leq \ldots \leq \kappa_r \right\}, \]
where
\[ \kappa_s = \left( j_{s \kappa_1} \leq \kappa_s \leq \ell_{s \kappa_1} \leq \ldots \leq \ell_{s p_s} \right), \quad s = 1, \ldots, r; \tag{1.30} \]
in both cases the inner multi-indices are ordered lexicographically. We then
denote
\[ Z_I = X_{i_1} \ldots X_{i_p} Y_{k_1}^{j_1} \ldots Y_{k_q}^{j_q} \quad \text{and} \quad \delta_K = \delta_{j_1 k_1}^{i_1} \ell_{1 \kappa_1} \ldots \delta_{j_r k_r}^{i_r} \ell_{r \kappa_r} \ldots \ell_{r p_r}. \tag{1.31} \]

**Proposition 1.3.** The monomials \( \delta_K Z_I \), ordered lexicographically, form a linear basis of \( \mathcal{H}_n \).

**Proof.** We need to prove that if \( c_{I,\kappa} \in \mathbb{C} \) are such that
\[ \sum_{I,\kappa} c_{I,\kappa} \delta_K Z_I (a) = 0, \quad \forall a \in \mathcal{A}, \tag{1.32} \]
then \( c_{I,\kappa} = 0 \), for any \((I, K)\).

To this end, we evaluate (1.32) on all elements of the form \( a = f U_{\zeta}^* \) at the point
\[ e = (x = 0, y = \mathbf{I}) \in F^{\mathbb{R}^n} = \mathbb{R}^n \times \text{GL}(n, \mathbb{R}). \]
In particular, for any fixed but arbitrary \( \varphi \in \mathbf{G} \), one obtains
\[ \sum_{I} \left( \sum_{K} c_{I,\kappa} \gamma_{\kappa}(\varphi)(e) \right) (Z_I f) (e) = 0, \quad \forall f \in C_c^\infty(F\mathbb{R}^n). \tag{1.33} \]
Since the \( Z_I \)'s form a PBW basis of \( \mathfrak{A}(\mathbb{R}^n \times \mathfrak{gl}(n, \mathbb{R})) \), which can be viewed as the algebra of left-invariant differential operators on \( F\mathbb{R}^n \), the validity of (1.33) for any \( f \in C_c^\infty(F\mathbb{R}^n) \) implies the vanishing for each \( I \) of the corresponding coefficient. One therefore obtains, for any fixed \( I \),
\[ \sum_{K} c_{I,\kappa} \gamma_{\kappa}(\varphi)(e) = 0, \quad \forall \varphi \in \mathbf{G}. \tag{1.34} \]
To prove the vanishing of all the coefficients, we shall use induction on the \textit{height} of $K = \{\kappa_1 \leq \ldots \leq \kappa_r\}$; the latter is defined by counting the total number of horizontal derivatives of its largest components:

$$|K| = \ell_1 + \cdots + \ell_r.$$

We start with the case of height 0, when the identity (1.34) reads

$$\sum_K c_{I,K} \gamma^{i_1}_{j_1 k_1}(\varphi)(e) \cdots \gamma^{i_r}_{j_r k_r}(\varphi)(e) = 0, \quad \forall \varphi \in \mathcal{G}.$$

Let $\mathcal{G}_0$ be the subgroup of all $\varphi \in \mathcal{G}$ such that $\varphi(0) = 0$. Choosing $\varphi$ in the subgroup $\mathcal{G}^{(2)}(0) \subset \mathcal{G}_0$ consisting of the diffeomorphisms whose 2-jet at 0 is of the form

$$J^2_0(\varphi)^i(x) = x^i + \frac{1}{2} \sum_{j,k=1}^n \xi^{i}_{jk} x^j x^k, \quad \xi \in \mathbb{R}^{n^3}, \quad \xi^i_{jk} = \xi^i_{kj},$$

and using (1.26), one obtains:

$$\sum_K c_{I,K} \xi^{i_1}_{j_1 k_1} \cdots \xi^{i_r}_{j_r k_r} = 0, \quad \xi^i_{jk} \in \mathbb{R}^{n^3}, \quad \xi^i_{jk} = \xi^i_{kj}.$$

It follows that all coefficients $c_{I,K} = 0$.

Let now $N \in \mathbb{N}$ be the largest height of occurring in (1.34). By varying $\varphi$ in the subgroup $\mathcal{G}^{(N+2)}(0) \subset \mathcal{G}_0$ of all diffeomorphisms whose $(N + 2)$-jet at 0 has the form

$$J^{N+2}_0(\varphi)^i(x) = x^i + \frac{1}{(N+2)!} \sum_{j,k,\alpha_1,\ldots,\alpha_{N+2}} \xi^{i}_{j k \alpha_1 \cdots \alpha_N} x^j x^k x^{\alpha_1} \cdots x^{\alpha_N},$$

$$\xi^{i}_{j k \alpha_1 \cdots \alpha_N} \in \mathbb{C}^{n^{N+3}}, \quad \xi^{i}_{j k \alpha_1 \cdots \alpha_N} = \xi^{i}_{k j \alpha_{(1)} \cdots \alpha_{(N)}}, \quad \forall \text{ permutation } \sigma,$$

and using (1.27) instead of (1.26), one derives as above the vanishing of all coefficients $c_{I,\kappa}$ with $|\kappa| = N$. This lowers the height in (1.34) and thus completes the induction.

\[\square\]

Let $\mathfrak{B}$ denote the ideal of $\mathfrak{A}(h_n)$ generated by the combinations of the form

$$\delta^i_{j \ell k} - \delta^i_{j k \ell} - \delta^s_{j k} \delta^i_{\ell s} + \delta^s_{j \ell} \delta^i_{s k}, \quad (1.35)$$

which according to (1.28) vanish when viewed in $\mathcal{H}_n$. 

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Corollary 1.4. The algebra $\mathcal{H}_n$ is isomorphic to the quotient of the universal enveloping algebra $\mathfrak{A}(\mathfrak{h}_n)$ by the ideal $\mathfrak{B}$.

Proof. Extending the notation for indices introduced above, we form multi-
indices of the third kind, $K' = \{\kappa'_1 \leq \ldots \leq \kappa'_r\}$, which are similar to those of the second kind ($1.30$) except that we drop the requirement $k_s \leq \ell^s_1$, i.e.

$$\kappa'_s = \left(\begin{array}{c} i_s \\ j_s \leq k_s, \ell^s_1 \leq \ldots \leq \ell^s_{p_s}\end{array}\right), \quad s = 1, \ldots, r. \quad (1.36)$$

We then form a Poincaré-Birkhoff-Witt basis of $\mathfrak{A}(\mathfrak{h}_n)$ out of the monomials $\delta_{K'} Z_I$, ordered lexicographically. Let $\pi : \mathfrak{A}(\mathfrak{h}_n) \to \mathcal{H}_n$ be the tautological algebra homomorphism, which sends the generators of $\mathfrak{h}_n$ to the same symbols in $\mathcal{H}_n$. In particular, it sends the basis elements of $\mathfrak{A}(\mathfrak{h}_n)$ to the corresponding elements in $\mathcal{H}_n$

$$\pi(\delta_{K'} Z_I) = \delta_{K'} Z_I,$$

but the monomial in the right hand side belongs to the basis of $\mathcal{H}_n$ only when the components of all $\kappa'_s \in K'$ are in the increasing order. Evidently, $\mathfrak{B} \subset \text{Ker} \pi$. To prove the converse, assume that

$$u = \sum_{I,K'} c_{I,K'} \delta_{K'} Z_I \in \mathfrak{A}(\mathfrak{h}_n)$$

satisfies

$$\pi(u) \equiv \sum_{I,K'} c_{I,K'} \delta_{K'} Z_I = 0. \quad (1.37)$$

In order to show that $u$ belongs to the ideal $\mathfrak{B}$, we shall again use induction, on the height of $u$. The height 0 case is obvious, because the 0-height monomials remain linearly independent in $\mathcal{H}_n$, so $\pi(u) = 0$ implies $c_I = 0$ for each $I$, and therefore $u = 0$.

Let now $N \geq 1$ be the largest height of occurring in $u$. For each $K'$ of height $N$, denote by $K$ the multi-index with the corresponding components $\kappa'_s \in K'$ rearranged in the increasing order. In view of ($1.28$), one can replace each $\delta_{K'}$ in the equation ($1.37$) by $\delta_K + \text{lower height}$, because the difference belongs to $\mathfrak{B}$. Thus, the top height part of $u$ becomes

$$\sum_{I,|K|=N} c_{I,K'} \delta_K Z_I,$$

and so

$$u = v + \sum_{I,|K|=N} c_{I,K'} \delta_K Z_I \quad (\text{mod } \mathfrak{B}),$$
where \( v \) has height at most \( N - 1 \). Then (1.37) takes the form

\[
\pi(v) + \sum_{I,|K| = N} c_{I,K'} \delta_K Z_I = 0,
\]

and from Proposition 1.3 it follows that the coefficient of each \( \delta_K Z_I \) vanishes. One concludes that

\[
u = v \pmod{\mathcal{B}}.
\]

On the other hand, by the induction hypothesis, \( \pi(v) = 0 \) implies \( v \in \mathcal{B} \). \( \square \)

In order to state the next result, we associate to any element \( h^1 \otimes \ldots \otimes h^p \in \mathcal{H}_n^{\otimes p} \) a multi-differential operator, acting on \( \mathcal{A} \), by the following formula

\[
T(h^1 \otimes \ldots \otimes h^p)(a^1 \otimes \ldots \otimes a^p) = h^1(a^1) \cdots h^p(a^p),
\]

where \( h^1, \ldots, h^p \in \mathcal{H}_n \) and \( a^1, \ldots, a^p \in \mathcal{A} \);

the linear extension of this assignment will be denoted by the same letter.

**Proposition 1.5.** For each \( p \in \mathbb{N} \), the linear transformation \( T : \mathcal{H}_n^{\otimes p} \to \mathcal{L}(\mathcal{A}^{\otimes p}, \mathcal{A}) \) is injective.

**Proof.** For \( p = 1 \), \( T \) gives the standard action of \( \mathcal{H}_n \) on \( \mathcal{A} \), which was just shown to be faithful. To prove that \( \text{Ker} T = 0 \) for an arbitrary \( p \in \mathbb{N} \), assume that

\[
H = \sum_p h^1_p \otimes \cdots \otimes h^p_p \in \text{Ker} T.
\]

After fixing a Poincaré-Birkhoff-Witt basis as above, we may uniquely express each \( h^j_p \) in the form

\[
h^j_p = \sum_{I_j,K_j} C_{\rho,I_j,K_j} \delta_{K_j} Z_{I_j}, \quad \text{with} \quad C_{\rho,I_j,K_j} \in \mathbb{C}.
\]

Evaluating \( T(H) \) on elementary tensors of the form \( f_1 U_{\varphi_1}^* \otimes \cdots \otimes f_p U_{\varphi_p}^* \), one obtains

\[
\sum_{\rho,I,K} C_{\rho,I_1,K_1} \cdots C_{\rho,I_p,K_p} \delta_{K_1} (Z_{I_1}(f_1)U_{\varphi_1}^*) \cdots \delta_{K_p} (Z_{I_p}(f_p)U_{\varphi_p}^*) = 0.
\]

Evaluating further at a point \( u_1 = (x_1, y_1) \in F^\mathbb{R} \), and denoting

\[
u_2 = \tilde{\varphi}_1(u_1), \ldots, u_p = \tilde{\varphi}_{p-1}(u_{p-1}),
\]

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the above identity gives

\[
\sum_{\rho, I, K} C_{\rho, I_1, K_1} \cdots C_{\rho, I_p, K_p} \cdot \gamma_{K_1}(\varphi_1)(u_1) \cdots \gamma_{K_p}(\varphi_p)(u_p) \\
\cdot Z_{I_1}(f_1)(u_1) \cdots Z_{I_p}(f_p)(u_p) = 0.
\]

Let us fix points \( u_1, \ldots, u_p \in F\mathbb{R}^n \) and then diffeomorphisms \( \psi_0, \psi_1, \ldots, \psi_p \), such that

\[
u_2 = \tilde{\psi}_1(u_1), \ldots, u_p = \tilde{\psi}_{p-1}(u_{p-1}).
\]

Following a line of reasoning similar to that of the preceding proof, and iterated with respect to the points \( u_1, \ldots, u_p \), we can infer that for each \( p \)-tuple of indices of the first kind \( (I_1, \ldots, I_p) \) one has

\[
\sum_{\rho, K} C_{\rho, I_1, K_1} \cdots C_{\rho, I_p, K_p} \cdot \gamma_{K_1}(\varphi_1)(u_1) \cdots \gamma_{K_p}(\varphi_p)(u_p) = 0.
\]

Similarly, making repeated use of diffeomorphisms of the form

\[
\psi_k \circ \varphi \quad \text{with} \quad \varphi \in G^{(N)}(u_k), \quad k = 1, \ldots, p,
\]

for sufficiently many values of \( N \), we can eventually conclude that for any \( (K_1, \ldots, K_p) \)

\[
\sum_{\rho} C_{\rho, I_1, K_1} \cdots C_{\rho, I_p, K_p} = 0.
\]

This proves that \( H = 0 \).

The crossed product algebra \( \mathcal{A} = C_c^\infty(F\mathbb{R}^n) \rtimes G \) carries a canonical trace, uniquely determined up to a scaling factor. It is defined as the linear functional \( \tau : \mathcal{A} \to \mathbb{C} \),

\[
\tau(f U_{\varphi}^*) = \begin{cases} 
\int_{F\mathbb{R}^n} f \varpi, & \text{if } \varphi = \text{Id}, \\
0, & \text{otherwise.}
\end{cases}
\]  

(1.39)

Here \( \varpi \) is the volume form attached to the canonical framing given by the flat connection

\[
\varpi = \bigwedge_{k=1}^n \theta^k \wedge \bigwedge_{(i,j)} \omega^i_j \quad \text{(ordered lexicographically)}.
\]
The tracial property
\[ \tau(ab) = \tau(ba), \quad \forall a, b \in A, \]
is a consequence of the \( G \)-invariance of the volume form \( \varpi \). In turn, the latter follows from the fact that
\[ \tilde{\varphi}^*(\theta) = \theta \quad \text{and} \quad \tilde{\varphi}^*(\omega) = \omega + \gamma \cdot \theta; \]
indeed,
\[ \tilde{\varphi}^*(\varpi) = \bigwedge_{k=1}^n \theta^k \land \bigwedge_{(i,j)} \left( \omega^i_j + \gamma^i_{j\ell}(\varphi)\theta^\ell \right) = \bigwedge_{k=1}^n \theta^k \land \bigwedge_{(i,j)} \omega^i_j. \]
This trace satisfies an invariance property relative to the modular character of \( \mathcal{H}_n \). The latter, \( \delta : \mathcal{H}_n \to \mathbb{C} \), extends the trace character of \( \mathfrak{gl}(n, \mathbb{R}) \), and is defined on the algebra generators as follows:
\[ \delta(Y^i_j) = \delta^i_j, \quad \delta(X_k) = 0, \quad \delta(\delta^i_{jk}) = 0, \quad i, j, k = 1, \ldots, n. \quad (1.40) \]
Clearly, this definition is compatible with the relations (1.28) and therefore extends to a character of the algebra \( \mathcal{H}_n \).

**Proposition 1.6.** For any \( a, b \in A \) and \( h \in \mathcal{H}_n \) one has
\[ \tau(h(a)) = \delta(h) \tau(a). \quad (1.41) \]

**Proof.** It suffices to verify the stated identity on the algebra generators of \( \mathcal{H}_n \). Evidently, both sides vanish if \( h = \delta^i_{jk} \). On the other hand, its restriction to the Lie algebra \( \mathfrak{g} = \mathbb{R}^n \ltimes \mathfrak{gl}(n, \mathbb{R}) \) is just the restatement, at the level of the Lie algebra, of the invariance property of the left Haar measure on \( G = \mathbb{R}^n \ltimes \text{GL}(n, \mathbb{R}) \) with respect to right translations. \( \square \)

**Proposition 1.7.** There exists a unique anti-automorphism \( \tilde{S} : \mathcal{H}_n \to \mathcal{H}_n \) such that
\[ \tau(h(a)b) = \tau(a \tilde{S}(h)(b)), \quad (1.42) \]
for any \( h \in \mathcal{H}_n \) and \( a, b \in A \). Moreover, \( \tilde{S} \) is involutive:
\[ \tilde{S}^2 = \text{Id}. \quad (1.43) \]
Proof. Using the ‘Leibnitz rule’ (1.12) for vertical vector fields, and the invariance property (1.41) applied to the product \(ab, a, b \in A\), one obtains
\[
\tau(Y^i_j(a)b) = -\tau(aY^i_j(b)) + \delta^i_j \tau(ab), \quad \forall a, b \in A. \quad (1.44)
\]
On the other hand, for the basic horizontal vector fields, (1.17) and (1.41) give
\[
\tau(X_k(a)b) = -\tau(aX_k(b)) - \tau(\delta^i_jk(a)Y^i_j(b)) = -\tau(aX_k(b)) + \tau(a\delta^i_jk(Y^i_j(b))), \quad (1.45)
\]
the second equality uses the 1-cocycle nature of \(\gamma^i_{jk}\). The same property implies
\[
\tau(\delta^i_jk(a)b) = -\tau(a\delta^i_jk(b)), \quad \forall a, b \in A. \quad (1.46)
\]
Thus, the generators of \(\mathcal{H}_n\) satisfy integration by parts identities of the form (1.42), with
\[
\tilde{S}(Y^i_j) = -Y^i_j + \delta^i_j \quad (1.47)
\]
\[
\tilde{S}(X_k) = -X_k + \delta^i_jkY^i_j \quad (1.48)
\]
\[
\tilde{S}(\delta^i_jk) = -\delta^i_jk \quad (1.49)
\]
Since the pairing \((a, b) \mapsto \tau(ab)\) is non-degenerate, the above operators are uniquely determined.

Being obviously multiplicative, the ‘integration by parts’ rule extends from generators to all elements \(h \in \mathcal{H}_n\), and uniquely defines a map \(\tilde{S} : \mathcal{H}_n \to \mathcal{H}_n\) satisfying (1.42). In turn, this very identity implies that \(\tilde{S}\) is a homomorphism from \(\mathcal{H}_n\) to \(\mathcal{H}_n^{op}\), as well as the fact that \(\tilde{S}\) is involutive.

Relying on the above results, we are now in a position to equip \(\mathcal{H}_n\) with a canonical Hopf structure.

**Theorem 1.8.** There exists a unique Hopf algebra structure on \(\mathcal{H}_n\) with respect to which \(A\) is a left \(\mathcal{H}_n\)-module algebra.

Proof. The formulae (1.12), (1.17) and (1.19) extend by multiplicatively to a general ‘Leibnitz rule’ satisfied by any element \(h \in \mathcal{H}_n\), of the form
\[
h(ab) = \sum_{(h)} h_{(1)}(a)h_{(2)}(b), \quad h_{(1)}, h_{(2)} \in \mathcal{H}_n, \quad a, b \in A \quad (1.50)
\]
By Proposition 1.5 this property uniquely determines the coproduct map 
\( \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \),
\[
\Delta(h) = \sum_{(h)} h^{(1)} \otimes h^{(2)},
\]
that satisfies
\[
T(\Delta h)(a \otimes b) = h(ab).
\]
Furthermore, the coassociativity of \( \Delta \) becomes a consequence of the associativity of \( \mathcal{A} \), because after applying \( T \) it amounts to the identity
\[
h((ab)c) = h(a(bc)), \quad \forall h \in \mathcal{H}, a, b \in \mathcal{A}.
\]
Similarly, the property that \( \Delta \) is an algebra homomorphism follows from the fact that \( \mathcal{A} \) is a left \( \mathcal{H} \)-module. By the very definition of the coproduct, \( \mathcal{A} \) is actually a left \( \mathcal{H} \)-module algebra.
The counit is defined by
\[
\varepsilon(h) = h(1);
\]
when transported via \( T \), its required properties amount to the obvious identities
\[
h(a1) = h(1a) = h(a), \quad \forall h \in \mathcal{H}, a \in \mathcal{A}.
\]
It remains to show the existence of antipode. We first check that the anti-
automorphism \( \tilde{S} \) is a twisted antipode, i.e. satisfies for any \( h \in \mathcal{H} \),
\[
(\text{Id} \ast \tilde{S})(h) := \sum_{(h)} h^{(1)} \tilde{S}(h^{(2)}) = \delta(h) 1,
\]
\[
(\tilde{S} \ast \text{Id})(h) := \sum_{(h)} \tilde{S}(h^{(1)}) h^{(2)} = \delta(h) 1.
\]
Indeed, with \( a, b \in \mathcal{A} \) arbitrary, one has
\[
\tau(a \delta(h)b) = \tau(h(ab)) = \sum_{(h)} \tau(h^{(1)}(a) h^{(2)}(b)) = \sum_{(h)} \tau(a (\tilde{S}(h^{(1)}) h^{(2)})(b)),
\]
which proves (1.54). Similarly, but also using the tracial property,
\[
\tau(a \delta(h)b) = \tau(h(ba)) = \sum_{(h)} \tau(h^{(2)}(a) h^{(1)}(b)) = \sum_{(h)} \tau(a (\tilde{S}(h^{(2)}) h^{(1)})(b)),
\]
or equivalently
\[
\sum_{(h)} \tilde{S}(h^{(2)}) h^{(1)} = \delta(h) 1;
\]
applying $\tilde{S}$ to both sides yields (1.55).

Now let $\tilde{\delta} \in \mathcal{H}_n^*$ denote the convolution inverse of the character $\delta \in \mathcal{H}_n^*$, which on generators is given by

$$
\tilde{\delta}(Y_i^j) = -\delta_i^j, \quad \tilde{\delta}(X_k) = 0, \quad \tilde{\delta}(\delta_{jk}) = 0, \quad i, j, k = 1, \ldots, n.
$$

Then $S := \tilde{\delta} * \tilde{S}$ is an algebra anti-homorphism which satisfies the antipode requirement

$$
\sum_{(h)} S(h(1))h(2) = \varepsilon(h) 1 = \sum_{(h)} h(1)S(h(2))
$$
on the generators, and hence for any $h \in \mathcal{H}_n$.

## 1.2 The general case of a flat primitive pseudogroup

Let $\Pi$ be a flat primitive Lie pseudogroup of local $C^\infty$-diffeomorphisms of $\mathbb{R}^m$. Denote by $F_{\Pi}\mathbb{R}^m$ the sub-bundle of $F\mathbb{R}^m$ consisting of the $\Pi$-frames on $\mathbb{R}^m$. It consists of the 1-jets at $0 \in \mathbb{R}^n$ of the germs of local diffeomorphisms $\phi \in \Pi$. Since $\Pi$ contains the translations, $F_{\Pi}\mathbb{R}^m$ can be identified, by the restriction of the map (1.1), to $\mathbb{R}^m \times G_0(\Pi)$, where $G_0(\Pi) \subset \text{GL}(m, \mathbb{R})$ is the linear isotropy group, formed of the Jacobians at 0 of the local diffeomorphisms $\phi \in \Pi$ preserving the origin.

The flat connection on $F\mathbb{R}^m$ restricts to a connection form on $F_{\Pi}\mathbb{R}^m$ with values in the Lie algebra $g_0(\Pi) = g_0(\Pi)$,

$$
\omega_\Pi := y^{-1} dy \in g_0(\Pi), \quad y \in G_0(\Pi).
$$

The basic horizontal vector fields on $F_{\Pi}\mathbb{R}^m$ are restrictions of those on $F\mathbb{R}^m$,

$$
X_k = y_\mu^k \frac{\partial}{\partial x_\mu}, \quad k = 1, \ldots, 2n, \quad y = (y_i^j) \in G_0(\Pi), \quad (1.57)
$$

and the fundamental vertical vector fields are

$$
Y_i^j = y_\mu^i \frac{\partial}{\partial y_j^\mu}, \quad i, j = 1, \ldots, 2n, \quad y = (y_i^j) \in G_0(\Pi); \quad (1.58)
$$

when assembled into a matrix-valued vector field, $Y = (Y_i^j)$ takes values in the Lie subalgebra $g_0(\Pi) \subset \mathfrak{gl}(m, \mathbb{R})$.

By virtue of Proposition 1.1 (1.57), (1.58) are also left-invariant vector fields, that give a framing of the group of affine $\Pi$-motions $G(\Pi) := \mathbb{R}^m \rtimes G_0(\Pi)$.
The group \( G(\Pi) := \text{Diff}(\mathbb{R}^m) \cap \Pi \) of global \( \Pi \)-diffeomorphisms acts on \( F_{\Pi}^i \mathbb{R}^m \) by prolongation, and the corresponding crossed product algebra

\[
\mathcal{A}(\Pi) := C^\infty_c(F_{\Pi}^i \mathbb{R}^m) \rtimes G(\Pi)
\]

is a subalgebra of \( \mathcal{A} \). After promoting the above vector fields \( X_k \) and \( Y^j \) to linear transformations in \( \mathcal{L}(\mathcal{A}(\Pi)) \), one automatically obtains, as in \( \S 1.1 \), the affiliated multiplication operators \( \delta^i_{jk} \in \mathcal{L}(\mathcal{A}(\Pi)) \),

\[
\delta^i_{jk}(f U^* \varphi) = \gamma^i_{jk}(\varphi) f U^* \varphi, \quad \varphi \in G(\Pi),
\]

and then their higher ‘derivatives’ \( \delta^i_{jk \ell_1 ... \ell_r \varphi} \in \mathcal{L}(\mathcal{A}(\Pi)) \),

\[
\delta^i_{jk \ell_1 ... \ell_r}(f U^* \varphi) = \gamma^i_{jk \ell_1 ... \ell_r}(\varphi) f U^* \varphi, \quad \forall \varphi \in G(\Pi).
\]

These are precisely the restrictions of the corresponding operators in \( \mathcal{L}(\mathcal{A}) \), characterized by the property that the \( m \times m \)-matrix defined by their ‘isotropy part’ is \( g_0(\Pi) \)-valued:

\[
(\gamma^i_{j...i}(\varphi))_{1 \leq i,j \leq m} \in g_0(\Pi) \subset \mathfrak{gl}(m, \mathbb{R}), \quad \varphi \in G(\Pi).
\]

They form a Lie subalgebra \( \mathfrak{h}(\Pi) \) of \( \mathfrak{h}_m \) and satisfy the Bianchi-type identities of Proposition 1.2.

We let \( \mathcal{H}(\Pi) \) denote the generated subalgebra of \( \mathcal{L}(\mathcal{A}(\Pi)) \) generated by the above operators, while \( \mathcal{B}(\Pi) \) stands for the ideal generated by the identities \( (1.28) \).

**Theorem 1.9.** The algebra \( \mathcal{H}(\Pi) \) is isomorphic to the quotient of the universal enveloping algebra \( \mathfrak{A}(\mathfrak{h}(\Pi)) \) by the ideal \( \mathcal{B}(\Pi) \), and can be equipped with a unique Hopf algebra structure with respect to which \( \mathcal{A}(\Pi) \) is a left \( \mathcal{H}(\Pi) \)-module algebra.

**Proof.** The proof amounts to a mere repetition of the steps followed in the previous subsection to establish Theorem 1.8. It suffices to notice that all the arguments remain valid when the isotropy-type of the generators is restricted to the linear isotropy Lie algebra \( g_0(\Pi) \).

To illustrate the construction of the Hopf algebra \( \mathcal{H}(\Pi) \) in a concrete fashion, we close this section with a more detailed discussion of the two main subclasses of flat primitive Lie pseudogroups: volume preserving and symplectic.
1.2.1 Hopf algebra of the volume preserving pseudogroup

The sub-bundle $F_0\mathbb{R}^n$ of $F\mathbb{R}^n$ consists in this case of all special (unimodular) frames on $\mathbb{R}^n$, defined by taking the 1-jet at $0 \in \mathbb{R}^n$ of germs of local diffeomorphisms $\phi$ on $\mathbb{R}^n$ that preserve the volume form, i.e.

$$\phi^*(dx^1 \wedge \cdots \wedge dx^n) = dx^1 \wedge \cdots \wedge dx^n.$$  

By means of the identification (1.1), $F_0\mathbb{R}^n \cong \mathbb{R}^n \times \text{SL}(n, \mathbb{R})$.

The flat connection is given by the $\mathfrak{sl}(n, \mathbb{R})$-valued 1-form $s\omega = (s\omega^i_j)$

$$s\omega^i_j := \omega^i_j = (y^{-1})^i_j dy^\mu_j, \quad i \neq j = 1, \ldots, n;$$

$$s\omega^i_i := \omega^i_i - \omega^a_i, \quad i = 1, \ldots, n - 1. \quad (1.60)$$

The basic horizontal vector fields on $F_0\mathbb{R}^n$ are restrictions of those on $F\mathbb{R}^n$,

$$sX_k = y^\mu_k \partial_\mu, \quad k = 1, \ldots, n, \quad y \in \text{SL}(n, \mathbb{R})$$

while the fundamental vertical vector fields are

$$sY^i_j := Y^i_j = y^\mu_i \partial_\mu_j, \quad i \neq j = 1, \ldots, n,$$

$$sY^i_i := \frac{1}{2}(Y^i_i - Y^a_i), \quad i = 1, \ldots, n - 1.$$

By Proposition 1.1, these are also the left-invariant vector fields on the group $G_s := \mathbb{R}^n \rtimes \text{SL}(n, \mathbb{R})$ associated to the standard basis of $\mathfrak{g}_s := \mathbb{R}^n \rtimes \mathfrak{sl}(n, \mathbb{R})$.

The group $G_s := \text{Diff}(\mathbb{R}^n, \text{vol})$ of volume preserving diffeomorphisms acts on $F_0\mathbb{R}^n$ as in (1.6), and the corresponding crossed product algebra

$$\mathcal{A}_s := C^\infty_c(F_0\mathbb{R}^n) \rtimes G_s$$

is a subalgebra of $\mathcal{A}$.

The analogue of (1.14) is

$$U^*_\phi sX_k U^*_\phi = sX_k - s\gamma^i_{jk}(\phi)sY^i_j, \quad (1.61)$$

where

$$s\gamma^i_{jk} = \gamma^i_{jk}, \quad i \neq j = 1, \ldots, n, \quad \text{and}$$

$$s\gamma^i_{ik} = \gamma^i_{ik} - \gamma^a_{ik}, \quad i = 1, \ldots, n - 1.$$

One thus obtains the multiplication operators $s\delta^i_{jk \ell_1 \cdots \ell_r} \in \mathcal{L}(\mathcal{A}_s)$,

$$s\delta^i_{jk \ell_1 \cdots \ell_r}(f U^*_\phi) = s\gamma^i_{jk \ell_1 \cdots \ell_r}(\phi) f U^*_\phi,$$
which continue to satisfy the Bianchi-type identities of Proposition 1.2.
Denoting by $\mathcal{SH}_n$ the subalgebra of $\mathcal{L}(\mathcal{A}_s)$ generated by the above operators, one equips it with the canonical Hopf structure with respect to which $\mathcal{A}_s$ is a left $\mathcal{SH}_n$-module algebra. We remark that the Hopf algebra $\mathcal{SH}_n$ is unimodular, in the sense that its antipode is involutive.
The conformal volume preserving case is similar, except that the linear isotropy subgroup is $\text{CSL}(n, \mathbb{R}) = \mathbb{R}^+ \times \text{SL}(n, \mathbb{R})$.

1.2.2 Hopf algebra of the symplectic pseudogroup
Let $G_{sp} \subset G$ be the subgroup of diffeomorphisms of $\mathbb{R}^{2n}$ preserving the symplecting form,

$$\Omega = dx^1 \wedge dx^{n+1} + \cdots + dx^n \wedge dx^{2n}. \tag{1.62}$$

Denote by $F_{sp}\mathbb{R}^{2n}$ the sub-bundle of $F\mathbb{R}^{2n}$ formed of symplectic frames on $\mathbb{R}^{2n}$, i.e. those defined by taking the 1-jet at $0 \in \mathbb{R}^{2n}$ of germs of local diffeomorphisms $\phi$ on $\mathbb{R}^{2n}$ preserving the form $\Omega$. Via (1.1) it can be identified to $\mathbb{R}^{2n} \times \text{Sp}(n, \mathbb{R})$. In turn, $\text{Sp}(n, \mathbb{R})$ is identified to the subgroup of matrices $A \in \text{GL}(2n, \mathbb{R})$ satisfying

$$^tAJA = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix}, \tag{1.63}$$

while its Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ is formed of matrices $a \in \mathfrak{gl}(2n, \mathbb{R})$ such that

$$^tJa + Ja = 0, \tag{1.64}$$

Thus the flat connection is given by the $\mathfrak{sp}(n, \mathbb{R})$-valued 1-form

$$\mathfrak{sp}\omega := y^{-1} dy \in \mathfrak{sp}(n, \mathbb{R}), \quad y \in \text{Sp}(n, \mathbb{R}),$$

the basic horizontal vector fields on $F_{sp}\mathbb{R}^{2n}$ are restrictions of those on $F\mathbb{R}^{2n}$,

$$\mathfrak{sp}X_k = y^\mu_k \partial_\mu, \quad k = 1, \ldots, 2n, \quad y \in \text{Sp}(n, \mathbb{R}),$$

and the fundamental vertical vector fields are given by the $\mathfrak{sp}(n, \mathbb{R})$-valued vector field $(\mathfrak{sp}Y^j_i) \in \mathfrak{sp}(n, \mathbb{R})$

$$\mathfrak{sp}Y^j_i = y^\mu_i \partial^j_\mu, \quad i, j = 1, \ldots, 2n, \quad y \in \text{Sp}(n, \mathbb{R}).$$

They also form a basis of left-invariant vector fields on the group $G_{sp} := \mathbb{R}^{2n} \times \text{Sp}(n, \mathbb{R})$ associated to the standard basis of $\mathfrak{g}_{sp} := \mathbb{R}^{2n} \times \mathfrak{sp}(n, \mathbb{R})$. 

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The group $G_{sp} := \text{Diff}(\mathbb{R}^n, \Omega)$ of symplectic diffeomorphisms acts on $F_{sp}\mathbb{R}^{2n}$ by prolongation, and the corresponding crossed product algebra

$$A_{sp} := C^\infty_c(F_{sp}\mathbb{R}^{2n}) \rtimes G_{sp}$$

is a subalgebra of $A$. The vector fields $sp X_k$ and $sp Y^j_i$ extend to linear transformations in $\mathcal{L}(A_{sp})$, and their action brings in multiplication operators $sp \delta^i_{jk \ell_1...\ell_r} \in \mathcal{L}(A_{sp})$.

$$sp \delta^i_{jk \ell_1...\ell_r} (f U^s_{\varphi}) = sp \gamma^i_{jk \ell_1...\ell_r} (\varphi) f U^s_{\varphi}.$$  

We let $Sp\mathcal{H}_n$ denote the generated subalgebra of $\mathcal{L}(A_{sp})$ generated by the above operators. It acquires a unique Hopf algebra structure such that $A_{sp}$ is a left $Sp\mathcal{H}_n$-module algebra. Like $S\mathcal{H}_n$, the Hopf algebra $Sp\mathcal{H}_n$ is unimodular.

The conformal symplectic case is again similar, except that the linear isotropy subgroup is $CSp(n, \mathbb{R}) = \mathbb{R}^+ \times Sp(n, \mathbb{R})$.

### 1.3 Hopf algebra of the contact pseudogroup

We denote by $G_{cn} \subset G$ the subgroup of orientation preserving diffeomorphisms of $\mathbb{R}^{2n+1}$ which leave invariant the contact form

$$\alpha := -dx^0 + \sum_{i=1}^{n} (x^i dx^{n+i} - x^{n+i} dx^i).$$  

(1.65)

The vector field $E_0 := \partial_0 \equiv \frac{\partial}{\partial x^0}$ satisfies $\iota_{E_0}(\alpha) = 1$ and $\iota_{E_0}(d\alpha) = 0$, i.e. represents the Reeb vector field of the contact structure. The contact distribution $\mathcal{D} := \text{Ker} \alpha$ is spanned by the vector fields $\{E_1, \ldots, E_{2n}\}$,

$$E_i = \partial_i - \frac{1}{2} x^{n+i} E_0, \quad E_{n+i} = \partial_{n+i} + \frac{1}{2} x^i E_0, \quad 1 \leq i \leq n.$$  

(1.66)

The Lie brackets between these vector fields are precisely those of the Heisenberg Lie algebra $\mathfrak{h}_n$:

$$[E_i, E_{j+n}] = \delta^i_j E_0, \quad [E_0, E_i] = 0, \quad [E_0, E_{j+n}] = 0, \quad i, j = 1, \ldots, n.$$

This gives an identification of $\mathbb{R}^{2n+1}$, whose standard basis we denote by $\{e_0, e_1, \ldots, e_{2n}\}$, with the Heisenberg group $H_n$, whose group law is given
therefore they form a group of strict contact transformations which is isomorphic to $\mathfrak{h}$.

Proof. Straightforward, given that in the above realization the exponential map $\exp: \mathfrak{h} \to H_n$ coincides with the identity map $\text{Id}: \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$. □

We recall that a contact diffeomorphism is a diffeomorphism $\phi: \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ such that $\phi_* (\text{Ker } \alpha) = \text{Ker } \alpha$, or equivalently $\phi^* (\alpha) = f \alpha$ for a nowhere vanishing function $f: \mathbb{R}^{2n+1} \to \mathbb{R}$; $\phi$ is orientation preserving iff $f > 0$ and is called a strict contact diffeomorphism if $f \equiv 1$.

In particular, the group left translations $L_\mathbf{a}: H_n \to H_n$, $\mathbf{a} \in H_n$,

$$L_\mathbf{a}(x) = \mathbf{a} \ast x = (a^0 + x^0 + \beta(\mathbf{a}', x'), a' + x') \quad \forall x \in H_n,$$

are easily seen to be strict contact diffeomorphism. They replace the usual translations in $\mathbb{R}^{2n+1}$. Together with the linear transformations preserving the symplectic form $\beta: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of (1.68),

$$S_A(x) = (x^0, Ax'), \quad A \in \text{Sp}(n, \mathbb{R}), \quad x' = (x^1, \ldots, x^{2n}),$$

they form a group of strict contact transformations which is isomorphic to $H_n \rtimes \text{Sp}(n, \mathbb{R})$. Indeed, one has

$$S_A \circ L_\mathbf{a}(x) = S_A(x^0 + a^0 + \beta(\mathbf{a}', x'), x' + \mathbf{a}')$$

$$= (x^0 + a^0 + \beta(\mathbf{a}', x'), A(x' + \mathbf{a}')) = (x^0 + a^0 + \beta(A\mathbf{a}', Ax'), A(x' + \mathbf{a}'))$$

$$= L_{S_A(a)}(S_A(x)) = (L_{S_A(a)} \circ S_A)(x),$$

therefore $S_A \circ L_\mathbf{a} \circ S_A^{-1} = L_{S_A(a)}$; also

$$(L_\mathbf{a} \circ S_A) \circ (L_B \circ S_B) = L_\mathbf{a} \circ (S_A \circ L_B) \circ S_B = L_\mathbf{a} \circ (L_{S_A(b)} \circ S_A) \circ S_B$$

$$= L_{\mathbf{a} \ast S_A(b)} \circ S_{AB}.$$
We shall enlarge this group the 1-parameter group of \textit{contact homotheties} 
\( \{ \mu_t; t \in \mathbb{R}_+^* \} \), defined by 
\[
\mu_t(x) = (t^2x^0, tx'), \quad x \in H_n;
\]
one has 
\[
\mu_t \circ S_A = S_A \circ \mu_t \quad \text{and} \quad \mu_t \circ L_a = L_{\mu_t(a)} \circ \mu_t.
\]
We denote by \( G_{\text{cnt}} \) the group generated by the contact transformations 
\( \{ L_a, S_A, \mu_t; a \in H_n, A \in \text{Sp}(n, \mathbb{R}), t \in \mathbb{R}_+^* \} \). It can be identified with the semidirect product 
\[
G_{\text{cnt}} \cong H_n \rtimes \text{CSp}(n, \mathbb{R}),
\]
where \( \text{CSp}(n, \mathbb{R}) = \text{Sp}(n, \mathbb{R}) \times \mathbb{R}_+^* \) is the \textit{conformal symplectic group}. Via the identification 
\[
L_a \circ S_A \circ \mu_t \cong (a, A, t), \quad a \in H_n, A \in \text{Sp}(n, \mathbb{R}), t \in \mathbb{R}_+^*,
\]
the multiplication law is 
\[
(a, A, t) \cdot (b, B, s) = (a * S_A(\mu_t(b)), AB, ts).
\]

Besides the usual tangent bundle \( T \mathbb{R}^{2n+1}_2 \), it will be convenient to introduce a version of it that arises naturally when the contact structure is treated as special case of a Heisenberg manifold (cf. e.g. \cite{25}). Denoting by \( \mathcal{R} := T \mathbb{R}^{2n+1}_2 / \mathcal{D} \) the line bundle determined by the class of the Reeb vector field modulo the contact distribution, the \textit{H-tangent bundle} is the direct sum 
\( T^H \mathbb{R}^{2n+1}_2 := \mathcal{R} \oplus \mathcal{D} \). If \( \phi \in G_{\text{cnt}} \) is a contact diffeomorphism, then its tangent map at \( x \in \mathbb{R}^{2n+1}_2 \), \( \phi \in \text{H-tangent map at } x \), \( \phi : T_x \mathbb{R}^{2n+1}_2 \rightarrow T_{\phi(x)} \mathbb{R}^{2n+1}_2 \), leaves the contact distribution \( \mathcal{D} \) invariant, and hence induces a corresponding \textit{H-tangent map}, which will be denoted \( \phi^H : T^H \mathbb{R}^{2n+1}_2 \rightarrow T^H_{\phi(x)} \mathbb{R}^{2n+1}_2 \).

\textbf{Lemma 1.11.} \textit{Given any contact diffeomorphism } \phi \in G_{\text{cnt}}, \text{ there is a unique } \varphi \in G_{\text{cnt}}, \text{ such that } \psi = \varphi^{-1} \circ \phi \text{ has the properties}
\[
\psi(0) = 0 \quad \text{and} \quad \psi^H_{x_0} = \text{Id} : T^H_{0} \mathbb{R}^{2n+1}_2 \rightarrow T^H_{0} \mathbb{R}^{2n+1}_2.
\]

\textit{Proof.} Consider the H-tangent map \( \phi^H : T^H_{0} \mathbb{R}^{2n+1}_2 \rightarrow T^H_{a} \mathbb{R}^{2n+1}_2 \), where \( a = \phi(0) \). Then relative to the moving frame \( \{ E_1, \ldots, E_{2n} \} \) for \( \mathcal{D}, \phi^H_{x_0} | \mathcal{D} :
\( \mathcal{D}_0 \to \mathcal{D}_a \) is given by a conformal symplectic matrix \( A_\phi(0) \in \text{CSp}(n, \mathbb{R}) \) with conformal factor \( t_\phi(0)^2 > 0 \). Furthermore, one can easily see that

\[
\phi_{*0}(e_0) = t_\phi(0)^2 E_0 \mid_a + w_\phi, \quad \text{with} \quad w_\phi \in \mathcal{D}_a,
\]

and so the restriction \( \phi^H_{*0} \mid \mathcal{R} : \mathcal{R}_0 \to \mathcal{R}_a \) can be identified with the scalar \( t_\phi(0)^2 \) relative to the moving frame \( E_0 \) of \( \mathcal{R} \).

Since \( A = t_\phi^{-1}(0) A_\phi(0) \in \text{Sp}(n, \mathbb{R}) \), we can form

\[
\varphi := L_a \circ S_A \circ \mu_t \in \text{G}_cn, \quad a = \phi(0), \quad t = t_\phi(0),
\]

and we claim that this \( \varphi \in \text{G}_cn \) fulfills the required property. Indeed, \( L_a^{-1} \circ \varphi \) has the expression

\[
(L_a^{-1} \circ \varphi)(x) = (t_\phi(0)^2 x^0, t_\phi(0) A x')
\]

and therefore the matrix of \( \varphi_{*0} : T_{*0} \mathbb{R}^{2n+1} \to T_{*0} \mathbb{R}^{2n+1} \) is the same as that of \( \phi^H_{*0} : T_{*0} \mathbb{R}^{2n+1} \to T_{*0} \mathbb{R}^{2n+1} \), i.e.

\[
(L_a^{-1} \circ \varphi)_{*0} = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & t_\phi(0) A
\end{pmatrix}
= \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & A_\phi(0)
\end{pmatrix}.
\]  

Thus, \( \psi = \varphi^{-1} \circ \phi \) satisfies (1.71).

To prove uniqueness, observe that if \( \varphi = L_a \circ S_A \circ \mu_t \in \text{G}_cn \) fulfills (1.71), then \( a = 0 \) and, since \( \varphi_{*0} : T_{*0} \mathbb{R}^{2n+1} \to T_{*0} \mathbb{R}^{2n+1} \) is of the form (1.73), \( A = \text{Id} \) and \( t = 1 \).

The subgroup of \( \text{G}_cn \) defined by the two conditions in (1.71) will be denoted \( N_{cn} \). Lemma 1.11 gives a Kac decomposition for the group of orientation preserving contact diffeomorphisms:

\[
\text{G}_cn = \text{G}_cn \cdot N_{cn}.
\]  

Note though that the group \( N_{cn} \) is no longer pro-unipotent. However, it has a pro-unipotent normal subgroup of finite codimension, namely

\[
U_{cn} := \{ \psi \in N_{cn} \mid \psi(0) = 0, \quad \psi_{*0} = \text{Id} : T_{*0} \mathbb{R}^{2n+1} \to T_{*0} \mathbb{R}^{2n+1} \};
\]

this gives rise to a group extension

\[
\text{Id} \to U_{cn} \to N_{cn} \to \mathbb{R}^{2n} \to 0,
\]  

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with the last arrow given by the tangent map at 0.

The decomposition [1,74] gives rise to a pair of actions of $G_{cn}$: a left action $\triangleright$ on $G_{cn} \cong G_{cn}/N_{cn}$, and a right action $\triangleleft$ on $N_{cn} \cong G_{cn} \backslash G_{cn}$. To understand the left action, let $\phi \in G_{cn}$ and $\varphi = L_x \circ S_A \circ \mu_t \in G_{cn}$. By Lemma [1,11]

$$\phi \circ \varphi = (\phi \triangleright \varphi) \circ \psi, \quad \text{with} \quad \phi \triangleright \varphi \in G_{cn} \quad \text{and} \quad \psi \in N_{cn}.$$ 

Write $\phi \triangleright \varphi = L_b \circ S_B \circ \mu_s$. If in (1.72) one replaces $\phi$ by $\phi \circ \varphi$ then,

$$A_{\phi \circ \varphi}(0) = A_\phi(x)A_{\varphi}(0) = tA_\phi(x)A, \quad t_{\phi \circ \varphi}(0) = t_\phi(x)t,$$

hence

$$\phi \triangleright \varphi = L_{\phi(x)} \circ S_{t_\phi(x)}^{-1}A_\phi(x)A \circ \mu_t t.$$

Using the parametrization (1.69), the explicit description of the action can be recorded as follows.

**Lemma 1.12.** Let $\phi \in G_{cn}$ and $(a, A, t) \in G_{cn}$. Then

$$\tilde{\phi}(x,A,t) := \phi \triangleright (x,A,t) = (\phi(x), t_\phi^{-1}(x)A_\phi(x)A, t_\phi(x)t). \quad (1.76)$$

With the above notational convention, let $R_{(b,B,s)}$ denote the right translation by an element $(b,B,s) \in \text{Sp}(n,\mathbb{R})$, and let $\phi \in G_{cn}$. We want to understand the commutation relationship between these two transformations. By (1.70) and (1.76), one has

$$(\tilde{\phi} \circ R_{(b,B,s)})(x,A,t) = \tilde{\phi}((x,A,t) \cdot (b,B,s)) = \tilde{\phi}(x \star S_A(\mu_t(b)), AB, ts)$$

$$= (\phi(x \star S_A(\mu_t(b))), t_\phi^{-1}(x)A_\phi(AB, t_\phi(x)(ts)).$$

On the other hand,

$$(R_{(b,B,s)} \circ \tilde{\phi})(x,A,t) = (\phi(x), t_\phi^{-1}(x)A_\phi(x)A, t_\phi(x)t) \cdot (b,B,s)$$

$$= (\phi(x) \star t_\phi^{-1}(x)S_{A_\phi(x)A}(\mu_t(x), t_\phi(x)(ts)), t_\phi^{-1}(x)A_\phi(x)AB, t_\phi(x)(ts).$$

Although these two answers are in general different, when $b = 0$ they do coincide, and we record this fact in the following statement.

**Lemma 1.13.** The left action of $G_{cn}$ on $G_{cn}$ commutes with the right translations by the elements of the subgroup $\text{CSp}(n,\mathbb{R})$. 

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As in the flat case, we proceed to associate to the pseudogroup $G_{cn}$ of orientation preserving diffeomorphisms of $\mathbb{R}^{2n+1}$ a Hopf algebra $\mathcal{H}(\Pi_{cn})$, realized via its Hopf action on the crossed product algebra $\mathcal{A}(\Pi_{cn}) = C^\infty(G_{cn}) \rtimes G_{cn}$. This type of construction actually applies whenever one has a Kac decomposition of the form (1.11).

One starts with a fixed basis $\{X_i\}_{1 \leq i \leq m}$ for the Lie algebra $g_{cn}$ of $G_{cn}$. Each $X_i \in g_{cn}$ gives rise to a left-invariant vector field $X_i$ on $G_{cn}$, which is then extended to a linear operator on $\mathcal{A}(\Pi_{cn})$, $X(f U^\phi) = X(f) U^\phi$, $\phi \in G_{cn}$.

One has

$$U^\phi X_i U^\phi = \sum_{j=1}^m \Gamma_j^i(\phi) X_j,$$

$i, j = 1, \ldots, m$, (1.77)

with $\Gamma_j^i(\phi) \in C^\infty(G_{cn})$. The matrix of functions $\Gamma(\phi) = (\Gamma_j^i(\phi))_{1 \leq i,j \leq m}$ automatically satisfies the cocycle identity

$$\Gamma(\phi \circ \psi) = (\Gamma(\phi) \circ \psi) \cdot \Gamma(\psi), \quad \phi, \psi \in G_{cn}.$$ (1.78)

We next denote by $\Delta_j^i(\phi)$ the following multiplication operator on $\mathcal{A}(\Pi_{cn})$:

$$\Delta_j^i(f U^\phi) = (\Gamma(\phi)^{-1})_i^j f U^\phi, \quad i, j = 1, \ldots, m.$$

With this notation, we define $\mathcal{H}(\Pi_{cn})$ as the subalgebra of linear operators on $\mathcal{A}(\Pi_{cn})$ generated by the operators $X_k$’s and $\Delta_j^i$’s, $i, j, k = 1, \ldots, m$. In particular, $\mathcal{H}_{\Pi}$ contains all iterated commutators

$$\Delta_{j_1}^{i_1} \cdots \Delta_{j_r}^{i_r} := [X_k, \ldots, [X_{k_1}, \Delta_{j_1}^{i_1}]] \cdots],$$

i.e. the multiplication operators by the functions on $G$,

$$\Gamma_{i_1, k_1 \cdots k_r}^j(\phi) := X_k \cdots X_{k_1} (\Gamma_j^i(\phi)), \quad \phi \in G.$$

**Lemma 1.14.** For any $a, b \in \mathcal{A}(\Pi_{cn})$, one has

$$X_k(ab) = X_k(a) b + \sum_j \Delta_j^k(a) X_j(b),$$ (1.79)

$$\Delta_j^i(ab) = \sum_k \Delta_j^k(a) \Delta_k^i(b).$$ (1.80)
Proof. With \( a = f_1 U_{\phi_1}^*, b = f_2 U_{\phi_2}^* \), and assembling the \( X_k \)'s into a column vector \( X \) and the \( \Delta^j_i \)'s into a matrix \( \Delta \), one has

\[
X(a \cdot b) = X(f_1 U_{\phi_1}^* f_2 U_{\phi_2}^*) = X(f_1 U_{\phi_1}^* f_2 U_{\phi_1}) U_{\phi_2}^* = X(a) b + f_1 U_{\phi_1}^* (U_{\phi_1} X U_{\phi_1}^*) (f_2) U_{\phi_2}^* = X(a) b + f_1 U_{\phi_1}^* \Gamma(\phi_1^{-1}) X U_{\phi_2}^* = X(a) b + f_1 \Gamma(\phi_1)^{-1} U_{\phi_1} X(b) = X(a) b + \Delta(a) X(b),
\]

which proves (1.79).

The identity (1.80) is merely a reformulation of the cocycle identity (1.78).

As a consequence, by multiplicativity every \( h \in H(\Pi_{cn}) \) satisfies a Leibniz rule of the form

\[
h(ab) = \sum h_{(1)}(a) h_{(2)} b, \quad \forall a, b \in A(\Pi).
\]

(1.81)

Proposition 1.15. The operators \( \Delta_i^k \) satisfy the (Bianchi) identities

\[
\Delta_i^k - \Delta_i^k = \sum_{r,s} c_{rs}^k \Delta_i^r \Delta_j^s - \sum_{\ell} c_{ij}^\ell \Delta_\ell^k,
\]

(1.82)

where \( c_{jk}^i \) are the structure constants of \( g_{cn} \),

\[
[X_j, X_k] = \sum_i c_{jk}^i X_i.
\]

(1.83)

Proof. Applying (1.79) one has, for any \( a, b \in A(\Pi_{cn}) \),

\[
X_i X_j(a \cdot b) = X_i (X_j(a)b + \sum_s \Delta_j^s(a) X_s(b)) = X_i (X_j(a)b) + \sum_s \Delta_i^s(X_j(a)) X_s(b) + \sum_s X_i(\Delta_j^s(a)) X_s(b) + \sum_{r,s} (\Delta_i^r(\Delta_j^s(a)) X_r(X_s(b)),
\]

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and thus the commutators can be expressed as follows:

\[
[X_i, X_j](ab) = [X_i, X_j](a) b + \sum_r \Delta^r_i(X_j(a)) X_r(b) - \sum_s \Delta^s_j(X_i(a)) X_s(b) \\
+ \sum_s X_s(\Delta^s_j(a)) X_s(b) - \sum_r X_j(\Delta^r_i(a)) X_r(b) + \\
+ \sum_{r,s} (\Delta^r_i(\Delta^s_j(a))X_r(X_s(b)) - \sum_{r,s} (\Delta^s_j(\Delta^r_i(a))X_r(X_s(b)) \\
= [X_i, X_j](a) b - \sum_r (\Delta^r_i(a) - \Delta^r_j(a)) X_r(b) + \\
+ \sum_{r,s} (\Delta^r_i\Delta^s_j(a)) \sum_k c_{rs}^k X_k(b).
\]

On the other hand, by (1.83), the left hand side equals

\[
\sum_\ell c_{ij}^\ell X_\ell(a) b = \sum_\ell c_{ij}^\ell X_\ell(a) b + \sum_\ell c_{ij}^\ell \Delta^k_\ell(a) X_k(b) \\
= [X_i, X_j](a) b + \sum_\ell c_{ij}^\ell \Delta^k_\ell(a) X_k(b).
\]

Equating the two expressions one obtains after cancelation

\[
\sum_\ell c_{ij}^\ell \Delta^k_\ell(a) X_k(b) = - \sum_k (\Delta^k_i(a) - \Delta^k_j(a)) X_k(b) + \\
+ \sum_k \sum_{r,s} c_{rs}^k (\Delta^r_i\Delta^s_j)(a) X_k(b).
\]

Since \( a, b \in A(\Pi) \) are arbitrary and the \( X_k \)'s are linearly independent, this gives the claimed identity. \( \square \)

Let \( \mathcal{H}_{cn} \) be the Lie algebra generated by the operators \( X_k \) and \( \Delta^j_{i,k_1\ldots k_r}, \)

\( i, j, k_1 \ldots k_r = 1, \ldots, m, \ r \in \mathbb{N} \). Following the same line of arguments as in the proof of Corollary 1.4, one can establish its exact analog.

**Proposition 1.16.** The algebra \( \mathcal{H}(\Pi_{cn}) \) is isomorphic to the quotient of the universal enveloping algebra \( U(\mathcal{H}_{cn}) \) by the ideal \( \mathcal{B}_{cn} \) generated by the Bianchi identities (1.82).

Actually, one can be quite a bit more specific about the above cocycles as well as about the corresponding Bianchi identities, if one uses an appropriate basis of the Lie algebra \( \mathfrak{g}_{cn} \). Recalling that \( \mathfrak{g}_{cn} \) is a semidirect product of the Heisenberg Lie algebra \( \mathfrak{h}_n \) by the Lie algebra \( \mathfrak{g}_{csp} \) of the conformal symplectic group \( \text{CSp}(n, \mathbb{R}) \), one can choose the basis \( \{X_i\}_{1 \leq i \leq m} \) such that
the first \(2n+1\) vectors are the basis \(\{E_i\}_{0 \leq i \leq 2n}\) of \(\mathfrak{h}_n\), while the rest form the canonical basis \(\{Y_i, Z\}\) of \(\mathfrak{g}_{\text{csp}}\), with \(Z\) central. By Lemma 1.13, for any \(\phi \in G_{\text{cn}}\),
\[
U_\phi^* Y U_\phi = Y, \quad Y \in \mathfrak{g}_{\text{csp}}.
\] (1.84)

Thus, the elements of \(\mathfrak{g}_{\text{csp}}\) act as derivations on \(A(\Pi)\), and therefore give rise to ‘tensorial identities’. The only genuine ‘Bianchi identities’ among (1.82) are those generated by the lifts of the canonical framing \(\{E_0, E_1, \ldots, E_{2n}\}\) of \(TH_n\) to left-invariant vector fields \(\{X_0, X_1, \ldots, X_{2n}\}\) on \(G_{\text{cn}}\).

**Proposition 1.17.** The left-invariant vector fields on \(G_{\text{cn}}\) corresponding to the canonical basis of the Heisenberg Lie algebra are as follows:

\[
X_0 |_{(x, A, s)} = s^2 \frac{\partial}{\partial x^0} = s^2 E_0,
\] (1.85)

\[
X_j |_{(x, A, s)} = s \sum_{i=1}^{2n} a^i_j E_i, \quad 1 \leq j \leq 2n.
\] (1.86)

**Proof.** We start with the lift of \(E_0\). Since \(\exp(t e_0) = t e_0\), one has for any \(F \in C^\infty(G_{\text{cn}})\),
\[
X_0 F(x, A, s) = \frac{d}{dt} \bigg|_{t=0} F((x, A, s) \cdot (t e_0, \text{Id}, 1)) = \frac{d}{dt} \bigg|_{t=0} F(x^0 + t s^2, x^1, \ldots, x^{2n}, A, s) = s^2 \frac{\partial F}{\partial x^0}(x, A, s).
\]

As \(S_A(\mu_s(e_0)) = s^2 e_0\), we can continue as follows:
\[
= \frac{d}{dt} \bigg|_{t=0} F(x \ast t s^2 e_0, A, s) = \frac{d}{dt} \bigg|_{t=0} F(x^0 + t s^2, x^1, \ldots, x^{2n}, A, s) = s^2 \frac{\partial F}{\partial x^0}(x, A, s).
\]

This proves (1.85).

Next, for \(1 \leq j \leq 2n\), let \(X_j\) denote the lift of \(E_j\) to \(G_{\text{cn}}\). Again, using that \(\exp(t e_j) = t e_j\) in \(H_n\), one has
\[
X_j F(x, A, s) = \frac{d}{dt} \bigg|_{t=0} F((x, A, s) \cdot (t e_j, \text{Id}, 1)) = \frac{d}{dt} \bigg|_{t=0} F(x \ast t S_A(\mu_s(e_j)), A, s);
\]
because $S_A(\mu_s(e_j)) = sa_j$, with $a_j$ denoting the $j$th column in the matrix $A$, the above is equal to

$$\frac{d}{dt} \bigg|_{t=0} F(x + ts a_j, A, s) \frac{d}{dt} \bigg|_{t=0} F((x^0 + ts \beta(x', a_j), x' + ts a_j), A, s)$$

$$= \frac{d}{dt} \bigg|_{t=0} F(x^0 + \frac{ts}{2} \sum_{i=1}^n (x^i a_j^{n+i} - a_j^i x^{n+i}), x^1 + ts a_j^1, \ldots, x^{2n} + ts a_j^{2n}), A, s)$$

$$= \frac{s}{2} \sum_{i=1}^n (x^i a_j^{n+i} - a_j^i x^{n+i}) \frac{\partial F}{\partial x^0}(x, A, s) + s \sum_{k=1}^{2n} a_{k,j} \frac{\partial F}{\partial x^k}(x, A, s).$$

Thus,

$$X_j |_{(x, A, s)} = \frac{s}{2} \sum_{i=1}^n (x^i a_j^{n+i} - a_j^i x^{n+i}) \frac{\partial}{\partial x^0} + s \sum_{k=1}^{2n} a_{k,j} \frac{\partial}{\partial x^k} =$$

$$= \frac{s}{2} \sum_{i=1}^n (x^i a_j^{n+i} - a_j^i x^{n+i}) \frac{\partial}{\partial x^i} + \sum_{i=1}^n \left( a_j^i \frac{\partial}{\partial x^i} + a_j^{n+i} \frac{\partial}{\partial x^{n+i}} \right)$$

$$= \frac{s}{2} \sum_{i=1}^n a_j^i \left( \frac{\partial}{\partial x^i} - \frac{1}{2} x^{n+i} \frac{\partial}{\partial x^0} \right) + s \sum_{i=1}^n a_j^{n+i} \left( \frac{\partial}{\partial x^{n+i}} + \frac{1}{2} x^i \frac{\partial}{\partial x^0} \right)$$

$$= s \sum_{i=1}^n \left( a_j^i E_i + a_j^{n+i} E_{n+i} \right),$$

which is the expression in (1.86). □

**Remark 1.18.** The formulae (1.85), (1.86), which taken together are the exact analogue of the formula (1.4), simply express the fact that the transition matrix from the basis $\{E_0, E_1, \ldots, E_{2n}\}$ to the basis $\{X_0, X_1, \ldots, X_{2n}\}$ of the horizontal subspace of $T_{(x, A, s)}G_{cn} \simeq T_x H_n$ is precisely the matrix

$$\begin{pmatrix}
  s^2 & 0 & \cdots & 0 \\
  0 & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & sA
\end{pmatrix}.$$

We now give a few examples the cocycles $\Gamma_i^j(\phi) \in C^\infty(G_{cn}), \phi \in G_{cn}$, corresponding to the horizontal vector fields $\{X_0, X_1, \ldots, X_{2n}\}$. 

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Starting with $X_0$, one has

$$(U^*_\phi X_0 U_\phi) F(x, A, s) = (U^*_\phi s^2 U_\phi)(U^*_\phi \frac{\partial}{\partial x^0} U_\phi) F(x, A, s) =$$

$$= t_\phi(x)^2 s^2$$

$$\frac{\partial}{\partial y^0} F(\phi^{-1}(y), t_\phi^{-1}(y)^{-1} A_\phi^{-1}(y) B, t_\phi^{-1}(y) z) \mid_{(\phi(x), t_\phi(x)^{-1} A_\phi(x) A, t_\phi(x)s)} =$$

$$= t_\phi(x)^2 s^2 \left( \partial_x F \cdot \frac{\partial(\phi^{-1})}{\partial y^0} + \partial_A F \cdot \frac{\partial}{\partial y_0} (t_\phi^{-1}(y)^{-1} A_\phi^{-1}(y) B) + t_\phi^{-1}(y)^{-1} \partial_A F \cdot \frac{\partial}{\partial y^0} (A_\phi^{-1}(y) B) + \partial_s F \frac{\partial}{\partial y^0} (t_\phi^{-1}(y) z) \right) \mid_{(\phi(x), t_\phi(x)^{-1} A_\phi(x) A, t_\phi(x)s)}.$$

Taking into account that

$$t_\phi^{-1}(\phi(x)) t_\phi(x) = 1, \quad A_\phi^{-1}(\phi(x)) A_\phi(x) = \text{Id}, \quad (1.87)$$

and

$$\frac{\partial}{\partial x^i} (\phi(x)) = t_\phi^{-1}(\phi(x))^2 = t_\phi(x)^{-2}, \quad (1.88)$$

one obtains after evaluation at $(\phi(x), t_\phi(x)^{-1} A_\phi(x) A, t_\phi(x)s) \in G_{cn}$

$$U^*_\phi X_0 U_\phi =$$

$$= X_0 + t_\phi(x)^2 s^2 \sum_{i=1}^{2n} \frac{\partial(\phi^{-1} i)}{\partial x^i} (\phi(x)) \frac{\partial}{\partial x^i} - t_\phi(x)^3 s^2 \frac{\partial t_\phi^{-1}}{\partial x^0} (\phi(x)) \frac{\partial}{\partial A} \cdot A \quad (1.89)$$

$$+ t_\phi(x)^2 s^2 \partial_A \cdot A_\phi^{-1} (\phi(x)) A_\phi(x) A + \frac{\partial t_\phi^{-1}}{\partial x^0} (\phi(x)) t_\phi(x)^3 s^2 \frac{\partial}{\partial s}. \quad (1.90)$$

To find the cocycles of the form $\Gamma_i^0$'s, with $i = 1, \ldots, 2n$, we use (1.66) to replace the partial derivatives by the horizontal vector fields,

$$\partial_i = E_i + \frac{1}{2} x^{n+i} E_0, \quad \partial_{n+i} = E_{n+i} - \frac{1}{2} x^i E_0, \quad 1 \leq i \leq n, \quad (1.91)$$
and rewrite the second term in the right hand side of (1.90) as follows

\[ II_{\text{term}} = t_\phi(x)^2 s^2 \left( \sum_{i=1}^{n} \frac{\partial (\phi^{-1})^i}{\partial x^0}(\phi(x)) \frac{\partial}{\partial x^i} + \sum_{i=n+1}^{2n} \frac{\partial (\phi^{-1})^i}{\partial x^0}(\phi(x)) \frac{\partial}{\partial x^i} \right) = \]

\[ t_\phi(x)^2 s^2 \sum_{i=1}^{n} \left( \frac{\partial (\phi^{-1})^i}{\partial x^0}(\phi(x)) \right) (E_i + \frac{1}{2} x^{n+i} E_0) + \frac{\partial (\phi^{-1})^i}{\partial x^0}(\phi(x))(E_{n+i} - \frac{1}{2} x^i E_0) \]

\[ = t_\phi(x)^2 \beta \left( \frac{\partial (\phi^{-1})'}{\partial x^0}(\phi(x)), x' \right) X_0 + t_\phi(x)^2 s^2 \sum_{i=1}^{n} \frac{\partial (\phi^{-1})^i}{\partial x^0}(\phi(x)) E_i. \]

We next invert the formula (1.86), cf. Remark 1.18

\[ E_i |_{(x, A, s)} = s^{-1} \sum_{j=1}^{2n} a^j_i X_j, \quad 1 \leq i \leq 2n, \quad (1.92) \]

where \((a^j_i) = A^{-1}\), to obtain

\[ II_{\text{term}} = t_\phi(x)^2 s^2 \beta \left( \frac{\partial (\phi^{-1})'}{\partial x^0}(\phi(x)), x' \right) X_0 + t_\phi(x)^2 s^2 \sum_{i,j=1}^{2n} \frac{\partial (\phi^{-1})^i}{\partial x^0}(\phi(x)) a^j_i X_j. \]

We have thus shown that

\[ \Gamma^0_0(\phi)(x, A, s) = \text{Id} + t_\phi(x)^2 \beta \left( \frac{\partial (\phi^{-1})'}{\partial x^0}(\phi(x)), x' \right), \quad (1.93) \]

\[ \Gamma^i_0(\phi)(x, A, s) = t_\phi(x)^2 s \sum_{j=1}^{2n} \frac{\partial (\phi^{-1})^j}{\partial x^0}(\phi(x)) a^i_j, \quad i = 1, \ldots, 2n. \quad (1.94) \]

In particular, when restricted to \(\psi \in N_{cn}\) and evaluated at the neutral element \(e = (1, \text{Id}, 1) \in G_{cn}\), these cocycles take the simple form

\[ \Gamma^0_0(\psi)(0, \text{Id}, 1) = \text{Id}, \quad (1.95) \]

\[ \Gamma^j_0(\psi)(0, \text{Id}, 1) = \frac{\partial (\psi^{-1})^j}{\partial x^0}(0), \quad j = 1, \ldots, 2n. \quad (1.96) \]

By comparison with the flat case, these cocycles and their derivatives give the only new type of coordinate functions on the group \(\psi \in N_{cn}\), all the rest being completely analogous to the \(n^\bullet_{\bullet, \bullet}\) coordinates of (2.24).

One last ingredient needed for the construction of the Hopf algebra, is provided by the following lemma.
Lemma 1.19. The left Haar volume form of the group $G_{cn}$ is invariant under the action $\triangleright$ of $G_{cn}$.

Proof. Up to a constant factor, the left-invariant volume form of $G_{cn}$ is given, in the coordinates (1.69), by

$$\varpi_{cn} := \alpha \wedge d\alpha^n \wedge \varpi_{Sp} \wedge s^{-2(n+1)} \, \frac{ds}{s},$$

(1.97)

where $\varpi_{Sp}$ is the left-invariant volume form of $Sp(N, \mathbb{R})$. Using the formula (1.76) expressing the action of $\phi \in G_{cn}$ on $G_{cn}$, in conjunction with the left invariance of $\varpi_{Sp}$ and the fact that

$$\phi^* (\alpha \wedge d\alpha^n) = t^{2(n+1)} \phi^* (\alpha \wedge d\alpha^n),$$

one immediately sees that $\tilde{\phi}^* (\varpi_{cn}) = \varpi_{cn}$.

As a consequence, we can define an invariant trace $\tau = \tau_{cn}$ on the crossed product algebra $A(\Pi_{cn}) = C^\infty (G_{cn}) \rtimes G_{cn}$ by precisely the same formula (1.39). Furthermore, the following counterpart of Proposition 1.6 holds.

Proposition 1.20. The infinitesimal modular character $\delta(X) = \text{Tr}(\text{ad}X)$, $X \in g_{cn}$, extends uniquely to a character $\delta = \delta_{cn}$ of $H(\Pi_{cn})$, and the trace $\tau = \tau_{cn}$ is $H(\Pi_{cn})$-invariant relative to this character, i.e.

$$\tau(h(a)) = \delta(h) \, \tau(a), \quad \forall a, b \in A(\Pi_{cn}).$$

(1.98)

Proof. On the the canonical basis of $g_{cn}$, the character $\delta$ takes the values

$$\delta(E_i) = 0, \quad 0 \leq i \leq 2n, \quad \delta(Y_i^j) = 0, \quad \text{and} \quad \delta(Z) = 2n + 2;$$

indeed, $\text{ad}(E_i)$’s are nilpotent, $\text{Ad}(Y_i^j)$’s are unimodular, and

$$[Z, E_0] = 2E_0, \quad [Z, E_i] = E_i, \quad \forall 1 \leq i \leq 2n, \quad [Z, Y] = 0, \forall Y \in g_{cn}.$$  

The rest of the proof is virtually identical to that of Prop. 1.6. 

Finally, following the same line of arguments which led to Theorem 1.8, one obtains the corresponding analog.

Theorem 1.21. There exists a unique Hopf algebra structure on $H(\Pi_{cn})$, such that its tautological action makes $A(\Pi_{cn})$ a left module algebra.
2 Bicrossed product realization

In this section we reconstruct (or rather deconstruct) the Hopf algebra affiliated to a primitive Lie pseudogroup as a bicrossed product of a matched pair of Hopf algebras. In the particular case of \( \mathcal{H}_1 \), this has been proved in [15], by direct algebraic calculations that rely on the detailed knowledge of its presentation. By contrast, our method is completely geometric and for this reason applicable to the entire class of Lie pseudogroups admitting a Kac-type decomposition.

We recall below the most basic notions concerning the bicrossed product construction, referring the reader to Majid’s monograph [23] for a detailed exposition.

Let \( U \) and \( F \) be two Hopf algebras. A linear map

\[
\nabla : U \rightarrow U \otimes F, \quad \nabla u = u_{<0>} \otimes u_{<1>},
\]

defines a right coaction, and thus equips \( U \) with a right \( F \)-comodule coalgebra structure, if the following conditions are satisfied for any \( u \in U \):

\[
u_{<0>(1)} \otimes u_{<0>(2)} \otimes u_{<1>} = u_{(1)}<0> \otimes u_{(2)}<0> \otimes u_{(1)}<1> u_{(2)}<1>, \quad (2.1)
\]

\[
\epsilon(u_{<0>})u_{<1>} = \epsilon(u)1. \quad (2.2)
\]

One can then form a cocrossed product coalgebra \( F \rhd U \), that has \( F \otimes U \) as underlying vector space and the following coalgebra structure:

\[
\Delta(f \rhd u) = f_{(1)} \rhd u_{(1)}<0> \otimes f_{(2)}u_{(1)}<1> \rhd u_{(2)}, \quad (2.3)
\]

\[
\epsilon(h \rhd k) = \epsilon(h)\epsilon(k). \quad (2.4)
\]

In a dual fashion, \( F \) is called a left \( U \) module algebra, if \( U \) acts from the left on \( F \) via a left action

\[
\triangleright : F \otimes U \rightarrow F
\]

which satisfies the following condition for any \( u \in U \), and \( f, g \in F \):

\[
u \triangleright 1 = \epsilon(u)1 \quad (2.5)
\]

\[
u \triangleright (fg) = (u_{(1)} \triangleright f)(u_{(2)} \triangleright g). \quad (2.6)
\]

This time we can endow the underlying vector space \( F \otimes U \) with an algebra structure, to be denoted by \( F \rtimes U \), with \( 1 \rtimes 1 \) as its unit and the product given by

\[
(f \rtimes u)(g \rtimes v) = f u_{(1)} \triangleright g \rtimes u_{(2)}v \quad (2.7)
\]
\( \mathcal{U} \) and \( \mathcal{F} \) are said to form a \textbf{matched pair} of Hopf algebras if they are equipped, as above, with an action and a coaction which satisfy the following compatibility conditions: following conditions for any \( u \in \mathcal{U} \), and any \( f \in \mathcal{F} \).

\[
\begin{align*}
\epsilon(u \triangleright f) &= \epsilon(u)\epsilon(f), \\
\Delta(u \triangleright f) &= u_{(1)} <_{(0)} \triangleright f_{(1)} \otimes u_{(1)} ^{<_{(1)}} \triangleright f_{(2)}, \\
\nabla(1) &= 1 \otimes 1, \\
\nabla(uv) &= u_{(1)} <_{(0)} v <_{(0)} \otimes u_{(1)} ^{<_{(1)}} \triangleright v <_{(1)}, \\
u_{(2)} <_{(0)} \otimes (u_{(1)} \triangleright f)u_{(2)} ^{<_{(1)}} &= u_{(1)} <_{(0)} \otimes u_{(1)} ^{<_{(1)}} \triangleright (u_{(2)} \triangleright f).
\end{align*}
\]

(2.8) \hspace{1cm} (2.9) \hspace{1cm} (2.10) \hspace{1cm} (2.11) \hspace{1cm} (2.12)

One can then form a new Hopf algebra \( \mathcal{F} \bowtie \triangleright \mathcal{U} \), called the \textbf{bicrossed product} of the matched pair \((\mathcal{F}, \mathcal{U})\) ; it has \( \mathcal{F} \bowtie \triangleright \mathcal{U} \) as underlying coalgebra, \( \mathcal{F} \rhd \triangleright \mathcal{U} \) as underlying algebra and the antipode is defined by

\[
S(f \bowtie \triangleright u) = (1 \bowtie \triangleright S(u_{<_{(0)}}))(S(f_{u_{<_{(1)}}}) \bowtie \triangleright 1), \quad f \in \mathcal{F}, \ u \in \mathcal{U}.
\]

(2.13)

2.1 The flat case

As mentioned in the introduction, the matched pair of Hopf algebras arises from a matched pair of groups, via a splitting à la G.I. Kac [IS].

**Proposition 2.1.** Let \( \Pi \) be a flat primitive Lie pseudogroup of infinite type, \( F_{\Pi} \mathbb{R}^m \) the principal bundle of \( \Pi \)-frames on \( \mathbb{R}^m \). There is a canonical splitting of the group \( G = \text{Diff}(\mathbb{R}^m) \cap \Pi \), as a cartesian product \( G = G \cdot N \), with \( G \simeq F_{\Pi} \mathbb{R}^m \) the group of affine \( \Pi \)-motions of \( \mathbb{R}^m \), and \( N = \{ \phi \in G ; \phi(0) = 0, \phi'(0) = \text{Id} \} \).

**Proof.** Let \( \phi \in G \). Since \( G \) contains the translations, then \( \phi_0 := \phi - \phi(0) \in G \), and \( \phi_0(0) = 0 \). Moreover, the affine diffeomorphism

\[
\varphi(x) := \phi_0'(0) \cdot x + \phi(0), \quad \forall \ x \in \mathbb{R}^m
\]

(2.14)

also belongs to \( G \), and has the same 1-jet at 0 as \( \phi \). Therefore, the diffeomorphism

\[
\psi(x) := \varphi^{-1}(\phi(x)) = \phi_0'(0)^{-1}(\phi(x) - \phi(0)), \quad \forall \ x \in \mathbb{R}^m
\]

(2.15)

belongs to \( N \), and the canonical decomposition is

\[
\phi = \varphi \circ \psi, \quad \text{with} \quad \varphi \in G \quad \text{and} \quad \psi \in N
\]

(2.16)
given by (2.14) and (2.15). The two components are uniquely determined, because evidently $G \cap N = \{ e \}$.

Reversing the order in the above decomposition one simultaneously obtains two well-defined operations, of $N$ on $G$ and of $G$ on $N$:

$$\psi \circ \varphi = (\psi \triangleright \varphi) \circ (\psi \triangleleft \varphi), \quad \text{for } \varphi \in G \text{ and } \psi \in N \quad (2.17)$$

**Proposition 2.2.** The operation $\triangleright$ is a left action of $N$ on $G$, and $\triangleleft$ is a right action of $G$ on $N$. Both actions leave the neutral element fixed.

**Proof.** Let $\psi_1, \psi_2 \in N$ and $\varphi \in G$. By (2.17), on the one hand

$$(\psi_1 \circ \psi_2) \circ \varphi = ((\psi_1 \circ \psi_2) \triangleright \varphi) \circ ((\psi_1 \circ \psi_2) \triangleleft \varphi),$$

and on the other hand

$$\psi_1 \circ (\psi_2 \circ \varphi) = \psi_1 \circ (\psi_2 \triangleright \varphi) \circ (\psi_2 \triangleleft \varphi)$$

$$= (\psi_1 \triangleright (\psi_2 \triangleright \varphi)) \circ (\psi_1 \triangleleft (\psi_2 \triangleright \varphi)) \circ (\psi_2 \triangleleft \varphi).$$

Equating the respective components in $G$ and $N$ one obtains:

$$\psi_1 \triangleright (\psi_2 \circ \varphi) = \psi_1 \triangleright (\psi_2 \triangleright \varphi) \in G, \quad \text{resp.} \quad (2.18)$$

$$\psi_1 \triangleleft (\psi_2 \circ \varphi) = (\psi_1 \triangleleft (\psi_2 \triangleright \varphi)) \circ (\psi_2 \triangleleft \varphi) \in N. \quad (2.19)$$

Similarly,

$$\psi \circ (\varphi_1 \circ \varphi_2) = (\psi \triangleright (\varphi_1 \circ \varphi_2)) \circ (\psi \triangleleft (\varphi_1 \circ \varphi_2)),$$

while

$$(\psi \circ \varphi_1) \circ \varphi_2 = (\psi \triangleright \varphi_1) \circ (\psi \triangleleft \varphi_1) \circ \varphi_2$$

$$= (\psi \triangleright \varphi_1) \circ ((\psi \triangleleft \varphi_1) \triangleright \varphi_2) \circ ((\psi \triangleleft \varphi_1) \triangleleft \varphi_2),$$

whence

$$\psi \triangleleft (\varphi_1 \circ \varphi_2) = (\psi \triangleleft \varphi_1) \triangleleft \varphi_2 \in N, \quad \text{resp.} \quad (2.20)$$

$$\psi \triangleright (\varphi_1 \circ \varphi_2) = (\psi \triangleright \varphi_1) \triangleright ((\psi \triangleleft \varphi_1) \triangleright \varphi_2) \in G. \quad (2.21)$$

Specializing $\varphi = e$, resp. $\psi = e$, in the definition (2.17), one sees that $e = \text{Id}$ acts trivially, and at the same time that both actions fix $e.$

\vspace{40pt}
Via the identification $G \simeq F_1 \mathbb{R}^m$, one can recognize the action $\triangleright$ as the usual action of diffeomorphisms on the frame bundle, cf. (1.6).

**Lemma 2.3.** The left action $\triangleright$ of $N$ on $G$ coincides with the restriction of the natural action of $G$ on $F_1 \mathbb{R}^m$.

**Proof.** Let $\phi = \psi \triangleright \varphi \in G$, with $\psi \in N$ and $\varphi \in G$. The associated frame, cf. (1.1), is $(\phi(0), \phi'(0))$. By (2.17),

$$\psi(\varphi(0)) = (\psi \triangleright \varphi)((\psi \triangleright \varphi)(0)) = \phi(0),$$

since $(\psi \triangleright \varphi)(0) = 0$. On differentiating (2.17) at 0 one obtains

$$\psi'(\varphi(0)) \cdot \varphi'(0) = (\psi \triangleright \varphi)'((\psi \triangleright \varphi)(0)) \cdot (\psi \triangleright \varphi)'(0) = (\psi \triangleright \varphi)'(0) = \phi'(0),$$

since $(\psi \triangleright \varphi)'(0) = \text{Id}$. Thus, $(\phi(0), \phi'(0)) = \tilde{\psi}(\varphi(0), \varphi'(0))$, as in the definition (1.6). \hfill \Box

**Definition 2.4.** The coordinates of $\psi \in N$ are the coefficients of the Taylor expansion of $\psi$ at $0 \in \mathbb{R}^m$. The algebra of functions on $N$ generated by these coordinates will be denoted $\mathcal{F}(N)$, and its elements will be called regular functions.

Explicitly, $\mathcal{F}(N)$ is generated by the functions

$$\alpha^i_{j_1 j_2 \ldots j_r}(\psi) = \partial_{j_1} \ldots \partial_{j_r} \psi^i(x) |_{x=0}, \quad 1 \leq i, j_1, j_2, \ldots, j_r \leq m, \psi \in N;$$

note that $\alpha^i_j(\psi) = \delta^i_j$, because $\psi'(0) = \text{Id}$, while for $r \geq 1$ the coefficients $\alpha^i_{j_1 j_2 \ldots j_r}(\psi)$ are symmetric in the lower indices but otherwise arbitrary. Thus, $\mathcal{F}(N)$ can be viewed as the free commutative algebra over $\mathbb{C}$ generated by the indeterminates $\{\alpha^i_{j_1 j_2 \ldots j_r}; 1 \leq i, j, j_1, j_2, \ldots, j_r \leq m\}$.

The algebra $\mathcal{F} := \mathcal{F}(N)$ inherits from the group $N$ a canonical Hopf algebra structure, in the standard fashion.

**Proposition 2.5.** With the coproduct $\Delta : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$, the antipode $S : \mathcal{F} \rightarrow \mathcal{F}$, and the counit $\epsilon : \mathcal{F} \rightarrow \mathbb{C}$ determined by the requirements

$$\Delta(f)(\psi_1, \psi_2) = f(\psi_1 \circ \psi_2), \quad \forall \psi_1, \psi_2 \in N, \quad (2.22)$$

$$S(f)(\psi) = f(\psi^{-1}), \quad \forall \psi \in N, \quad \forall f \in \mathcal{F},$$

$$\epsilon(f) = f(e),$$

$\mathcal{F}(N)$ is a Hopf algebra.
Proof. The fact that these definitions give rise to a Hopf algebra is completely routine, once they are shown to make sense. In turn, checking that
\[ \Delta(\alpha_{j_1...j_r}^i) \in \mathcal{F} \otimes \mathcal{F} \quad \text{and} \quad S(\alpha_{j_1...j_r}^i) \in \mathcal{F} \otimes \mathcal{F}, \quad (2.23) \]
only involves elementary manipulations with the chain rule. For instance, in the case of \( \alpha_{j_k}^i \), the verification goes as follows. First, for the coproduct,
\[
\Delta(\alpha_{j_k}^i)(\psi_1, \psi_2) = \alpha_{j_k}^i(\psi_1 \circ \psi_2) = \partial_j \partial_k (\psi_1 \circ \psi_2)(x) \big|_{x=0} \\
= \partial_j ((\partial_i \psi_1^j)(\psi_2(x)) \partial_k \psi_2^j(x)) \big|_{x=0} \\
= (\partial_i \partial_j \psi_1^j)(\psi_2(x)) \big|_{x=0} \partial_j \psi_2^j(x) \big|_{x=0} + \\
+ (\partial_i \psi_1^j)(\psi_2(x)) \big|_{x=0} \partial_j \partial_k \psi_2^j(x) \big|_{x=0} \\
= \partial_j \partial_k \psi_1^j(x) \big|_{x=0} + \partial_j \partial_k \psi_2^j(x) \big|_{x=0} = (\alpha_{j,k}^i \otimes 1 + 1 \otimes \alpha_{j,k}^i)(\psi_1, \psi_2),
\]
where we have used that \( \psi_1(0) = \psi_2(0) = 0 \) and \( \psi_1'(0) = \psi_2'(0) = \text{Id} \).
To deal with the antipode, one differentiates the identity \( \psi^{-1}(\psi(x)) = x \):
\[
\delta^i_j = \partial_j ((\psi^{-1})^i(\psi(x))) = \partial^i_\lambda (\psi^{-1})^i(\psi(x)) \partial_j \psi^\lambda(x),
\]
which yields under further differentiation
\[
\partial_i \partial^i_\lambda (\psi^{-1})^i(\psi(x)) \partial_k \psi^\mu(x) \partial_j \psi^\lambda(x) + \partial^i_\lambda (\psi^{-1})^i(\psi(x)) \partial_k \partial_j \psi^\lambda(x) = 0;
\]
evaluation at \( x = 0 \) gives \( \alpha_{j,k}^i(\psi^{-1}) + \alpha_{j,k}^i(\psi) = 0 \).
Taking higher derivatives one proves \( (2.23) \) in a similar fashion. \( \square \)

We shall need an alternative description of the algebra \( \mathcal{F} \), which will be used to recognize it as being identical to the Hopf subalgebra of \( \mathcal{H}(\Pi) \) generated by the \( \delta_{j_1...j_r}^i \)'s.

**Lemma 2.6.** The coefficients of the Taylor expansion of \( \tilde{\psi} \) at \( e \in G \),
\[
\eta_{j_1...j_r}^i (\psi) := \gamma_{j_1...j_r}^i (\psi)(e), \quad \psi \in N, \quad (2.24)
\]
define regular functions on \( N \), which generate the algebra \( \mathcal{F}(N) \).

**Proof.** Evaluating the expression \( (1.27) \) at \( e = (0, \text{Id}) \in F_{\Pi \mathbb{R}^m} \) gives
\[
\gamma_{j_1...j_r}^i (\psi)(e) = \partial_{\ell_r} \ldots \partial_{\ell_1} \left((\psi^i(x)^{-1})^j \partial_j \partial_k \psi^\mu(x)\right) \big|_{x=0}.
\]
(2.25)
The derivatives of $\psi'(x)^{-1}$ are sums of terms each of which is a product of derivatives of $\psi'(x)$ interspersed with $\psi'(x)^{-1}$ itself. Since $\psi'(0) = \text{Id}$, the right hand side of (2.25) is thus seen to define a regular function on $N$.

A more careful inspection actually proves the converse as well. First, by the very definition,

\[
\eta_{ijk} = \alpha_{ijk}.
\] (2.26)

Next, one has

\[
\eta_{ijkl}(\psi) = \partial_{l}((\psi'(x)^{-1})_{\mu}^{i}\partial_{j}\partial_{k}\psi^{\nu}(x))|_{x=0}
\]

\[
= \partial_{l}((\psi'(x)^{-1})_{\mu}^{i})|_{x=0} \partial_{j}\partial_{k}\psi^{\nu}(x)|_{x=0} + \partial_{l}\partial_{k}\psi^{i}(x)|_{x=0};
\]

on differentiating $(\psi'(x)^{-1})_{\mu}^{i} \partial_{\nu}\psi^{\mu}(x) = \delta_{\nu}^{i}$ one sees that

\[
\partial_{l}((\psi'(x)^{-1})_{\mu}^{i})|_{x=0} + \partial_{l}\partial_{\nu}\psi^{i}(x)|_{x=0} = 0,
\]

and therefore

\[
\eta_{ijkl}(\psi) = \alpha_{ijkl}(\psi) - \alpha_{l\nu}^{i}(\psi) \alpha_{jk}^{\nu}(\psi).
\] (2.27)

By induction, one shows that

\[
\eta_{ijkl...r} = \alpha_{ijkl...r} + P_{ijkl...r}^{i}(\alpha_{\mu\nu}^{i}, \ldots, \alpha_{\rho\sigma\delta}^{i}, \ldots, \alpha_{\rho\sigma\delta}^{i}-1),
\] (2.28)

where $P_{ijkl...r}^{i}$ is a polynomial. The triangular form of the identities (2.26)-(2.28) allows to reverse the process and express the $\alpha_{ijkl...r}^{i}$'s in a similar fashion:

\[
\alpha_{ijkl...r}^{i} = \eta_{ijkl...r}^{i} + Q_{ijkl...r}^{i}(\eta_{\mu\nu}^{i}, \ldots, \eta_{\rho\sigma\delta}^{i}, \ldots, \eta_{\rho\sigma\delta}^{i}-1).
\] (2.29)

Let $H(\Pi)_{ab}$ denote the (abelian) Hopf subalgebra of $H(\Pi)$ generated by the operators $\{\delta_{ijkl...r}^{j}; 1 \leq i, j, k, \ell_1, \ldots, \ell_r \leq m\}$.

**Proposition 2.7.** There is a unique isomorphism of Hopf algebras $\iota: H(\Pi)_{ab}^{\text{cop}} \rightarrow F(N)$ with the property that

\[
\iota(\delta_{ijkl...r}^{j}) = \eta_{ijkl...r}^{i}, \quad \forall 1 \leq i, j, k, \ell_1, \ldots, \ell_r \leq m.
\] (2.30)
Proof. In view of (2.26)–(2.27), the generators \( \eta^i_{jk\ell_1\ldots\ell_r} \) satisfy the analogue of the Bianchi identity (1.28). Indeed,

\[
\eta^i_{jk\ell} - \eta^i_{j\ell k} = \alpha^i_{jk\ell} - \alpha^i_{j\ell k} - \alpha^i_{j\ell k} + \alpha^i_{k\rho} \delta^\rho_{j\ell} = \eta^i_{k\rho} \eta^\rho_{\ell j} - \eta^i_{\ell\rho} \eta^\rho_{k j}.
\]

From Theorem 1.9 (or rather the proof of Corollary 1.4) it then follows that the assignment (2.30) does give rise to a well-defined algebra homomorphism \( \iota : H(\Pi)_{ab} \to F(\mathcal{N}) \), which by Lemma 2.6 is automatically surjective.

To prove that \( \iota : H(\Pi)_{ab} \to F(\mathcal{N}) \) is injective, it suffices to show that the monomials \( \{ \eta_K ; K = \text{increasingly ordered multi-index} \} \), defined in the same way as the \( \delta_K \)'s of the Poincaré-Birkhoff-Witt basis of \( H(\Pi) \) (cf. Proposition 1.3), are linearly independent. This can be shown by induction on the height. In the height 0 case the statement is obvious, because of (2.26).

Next, assume

\[
\sum_{|J| \leq N-1} c_J \eta_J + \sum_{|K| = N} c_K \eta_K = 0.
\]

Using the identities (2.28) and (2.28), one can replace \( \eta_K \) by \( \alpha_K \) plus lower height. Since the \( \alpha_{\ldots} \)'s are free generators, it follows that \( c_K = 0 \) for each \( K \) of height \( N \), and thus we are reduced to

\[
\sum_{|J| \leq N-1} c_J \eta_J = 0;
\]

the induction hypothesis now implies \( c_J = 0 \), for all \( J \)’s.

It remains to prove that \( \iota : H(\Pi)^{\text{cop}}_{ab} \to F(\mathcal{N}) \) is a coalgebra map, which amounts to checking that

\[
\iota \otimes \iota(\Delta^i_{jk\ell_1\ldots\ell_r}) = \Delta^{\text{op}} \eta^i_{jk\ell_1\ldots\ell_r}.
\]

Recall, cf. (1.51), that \( \Delta : H(\Pi) \to H(\Pi) \otimes H(\Pi) \) is determined by a Leibniz rule, which for \( \delta^i_{jk\ell_1\ldots\ell_r} \) takes the form

\[
\delta^i_{jk\ell_1\ldots\ell_r}(U^*_{\phi_1}, U^*_{\phi_2}) = \sum c^{iAB}_{jA} \delta^i_{jA}(U^*_{\phi_1}) \delta^i_{jB}(U^*_{\phi_2}), \quad \phi_1, \phi_2 \in G,
\]

which is equivalent to

\[
\gamma^i_{jk\ell_1\ldots\ell_r}(\phi_2 \circ \phi_1) = \sum c^{iAB}_{jA} \gamma^i_{jA}(\phi_1) \gamma^i_{jB}(\phi_2) \circ \tilde{\phi}_1.
\]

Restricting (2.32) to \( \psi_1, \psi_2 \in \mathcal{N} \) and evaluating at \( e \in G \), one obtains

\[
\Delta^{\text{op}} \eta^i_{jk\ell_1\ldots\ell_r}(\psi_1, \psi_2) := \eta^i_{jk\ell_1\ldots\ell_r}(\psi_2 \circ \psi_1) = \sum c^{iAB}_{jA} \eta^i_{jA}(\psi_1) \eta^i_{jB}(\psi_2).
\]

\( \square \)
The right action $\triangleleft$ of $G$ on $N$ induces an action of $G$ on $\mathcal{F}(N)$, and hence a left action $\triangleright$ of $U(g)$ on $\mathcal{F}(N)$, defined by

$$(X \triangleright f)(\psi) = \frac{d}{dt} |_{t=0} f(\psi \exp tX), \quad f \in \mathcal{F}, \quad X \in g. \quad (2.33)$$

On the other hand, there is a natural action of $U(g)$ on $\mathcal{H}(\Pi)_{ab}$, induced by the adjoint action of $g$ on $\mathfrak{h}(\Pi)$, extended as action by derivations on the polynomials in $\delta_{jk\ell_1\ldots\ell_r}$. In order to relate these two actions, we need a preparatory lemma.

**Lemma 2.8.** Let $\varphi \in G$ and $\phi \in G$. Then for any $1 \leq i, j, k, \ell_1, \ldots, \ell_r \leq m$,

$$
\begin{align*}
\gamma^i_{jk\ell_1\ldots\ell_r}(\varphi \circ \phi) &= \gamma^i_{jk\ell_1\ldots\ell_r}(\phi), \\
\gamma^i_{jk\ell_1\ldots\ell_r}(\phi \circ \varphi) &= \gamma^i_{jk\ell_1\ldots\ell_r}(\phi) \circ \tilde{\varphi}.
\end{align*} \quad (2.34) \quad (2.35)
$$

**Proof.** Both identities can be verified by direct computations, using the explicit formula (1.27) for $\gamma^i_{jk\ell_1\ldots\ell_r}$, in conjunction with the fact that $\varphi$ has the simple affine expression $\varphi(x) = a \cdot x + b$, $a \in G_0(\Pi)$, $b \in \mathbb{R}^m$.

An alternative and more elegant explanation relies on the left invariance of the vector fields $X_k$, cf. Proposition 1.1. The identity (2.34) easily follows from the cocycle property (1.16) and the fact that $\varphi$ is affine,

$$
\gamma^i_{jk}(\varphi \circ \phi) = \gamma^i_{jk\ell_1\ldots\ell_r}(\varphi) \circ \tilde{\psi} + \gamma^i_{jk}(\phi) = \gamma^i_{jk}(\phi),
$$

because $\gamma^i_{jk}(\varphi) = 0$. To check the second equation one starts with

$$
\gamma^i_{jk}(\phi \circ \varphi) = \gamma^i_{jk}(\phi) \circ \tilde{\varphi} + \gamma^i_{jk}(\varphi) = \gamma^i_{jk}(\phi \circ \varphi),
$$

and notice that the invariance property $U_\varphi X U_\varphi^* = X$, for any $X \in g$, implies

$$
X(\gamma^i_{jk}(\phi \circ \varphi)) = X(\gamma^i_{jk}(\phi) \circ \tilde{\varphi}).
$$

We are now in a position to formulate the precise relation between the canonical action of $U(g)$ on $\mathcal{H}(\Pi)_{ab}$ and the action $\triangleright$ on $\mathcal{F}(N)$.

**Proposition 2.9.** The algebra isomorphism $\iota : \mathcal{H}(\Pi)_{ab} \rightarrow \mathcal{F}(N)$ identifies the $U(g)$-module $\mathcal{H}(\Pi)_{ab}$ with the $U(g)$-module $\mathcal{F}(N)$. In particular $\mathcal{F}(N)$ is a $U(g)$-module algebra.
Proof. We denote below by \( \varphi_t \) the 1-parameter subgroup \( \exp tX \) of \( G \) corresponding to \( X \in \mathfrak{g} \), and employ the abbreviated notation \( \eta = \eta^i_{j k \ell_1 \ldots \ell_r} \), \( \gamma = \gamma^i_{j k \ell_1 \ldots \ell_r} \). From (2.34) it follows that
\[
\gamma(\psi \circ \varphi_t) = \gamma(\psi \circ \varphi_t),
\]
whence
\[
(X \triangleright \eta)(\psi) = \frac{d}{dt} \bigg|_{t=0} \eta(\psi \circ \varphi_t) = \frac{d}{dt} \bigg|_{t=0} \gamma(\psi \circ \varphi_t)(e).
\]
Now using (2.35), one can continue as follows:
\[
\frac{d}{dt} \bigg|_{t=0} \gamma(\psi \circ \varphi_t)(e) = \frac{d}{dt} \bigg|_{t=0} \gamma(\psi)(\tilde{\varphi}_t(e)) = X(\gamma(\psi))(e).
\]
By iterating this argument one obtains, for any \( u \in U(\mathfrak{g}) \),
\[
(u \triangleright \eta^i_{j k \ell_1 \ldots \ell_r})(\psi) = u(\gamma^i_{j k \ell_1 \ldots \ell_r}(\psi))(e), \quad \psi \in N. \quad (2.36)
\]
The right hand side of (2.36), before evaluation at \( e \in G \), describes the effect of the action of \( u \in U(\mathfrak{g}) \) on \( \delta^i_{j k \ell_1 \ldots \ell_r} \in \mathcal{H}(\Pi)_{ab} \). In view of the defining relation (2.30) for the isomorphism \( \iota \), this achieves the proof. \( \square \)

We proceed to equip \( U(\mathfrak{g}) \) with a right \( \mathcal{F}(N) \)-comodule structure. To this end, we assign to each element \( u \in U(\mathfrak{g}) \) a \( U(\mathfrak{g}) \)-valued function on \( N \) as follows:
\[
(\nabla u)(\psi) = \tilde{u}(\psi)(e), \quad \text{where} \quad \tilde{u}(\psi) = U^*_\psi u U^*_\psi. \quad (2.37)
\]
We claim that \( \nabla u \) belongs to \( U(\mathfrak{g}) \otimes \mathcal{F}(N) \), and therefore the above assignment defines a linear map \( \nabla : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathcal{F}(N) \). Indeed, let \( \{Z_I\} \) be the PBW basis of \( U(\mathfrak{g}) \) defined in \( \S 5 \), cf. (1.31). We identify \( U(\mathfrak{g}) \) with the algebra of left-invariant differential operators on \( G \), and regard the \( Z_I \)'s as a linear basis for these operators. In particular, one can uniquely express
\[
U^*_\psi Z_I U^*_\psi = \sum_J \beta^I_J(\psi) Z_J, \quad \psi \in N, \quad (2.38)
\]
with \( \beta^I_J(\psi) \) in the algebra of functions on \( G \) generated by \( \{\gamma^i_{j K}(\psi)\} \). The definition (2.37) then takes the explicit form
\[
\nabla Z_I = \sum_J Z_J \otimes \zeta^I_J, \quad \text{where} \quad \zeta^I_J(\psi) = \beta^I_J(\psi)(e). \quad (2.39)
\]
For example, by (1.10), (1.14) and (1.16), one has
\[ \nabla Y^i_j = Y^i_j \otimes 1, \quad (2.40) \]
\[ \nabla X_k = X_k \otimes 1 + Y^j_i \otimes 1_{j,k}. \quad (2.41) \]

Thus, \( \nabla : U(g) \to U(g) \otimes \mathcal{F}(N) \) is well-defined.

**Lemma 2.10.** The map \( \nabla : U(g) \to U(g) \otimes \mathcal{F}(N) \) endows \( U(g) \) with a \( \mathcal{F}(N) \)-comodule structure.

**Proof.** On the one hand, applying (2.38) twice one obtains
\[
U \psi_1 U \psi_2 Z_I U^* \psi_1 U^* \psi_2 = \sum J \beta^I_J(\psi_2) \circ \psi_1^{-1} U \psi_1 Z_J U^*_\psi_1 \\
= \sum K \left( \sum J \beta^I_J(\psi_2) \circ \psi_1^{-1} \beta^K_J(\psi_1) \right) Z_K, \quad (2.42)
\]
while on the other hand, the same left hand side can be expressed as
\[
U \psi_1 \psi_2 Z_I U^* \psi_1 \psi_2 = \sum K \beta^K_I(\psi_1 \psi_2) Z_K; \quad (2.43)
\]
therefore
\[
\beta^K_I(\psi_1 \psi_2) = \sum J \beta^K_J(\psi_1) \beta^I_J(\psi_2) \circ \psi_1^{-1}. \quad (2.44)
\]

By the very definition (2.39), the identity (2.42) gives
\[
(\nabla \otimes \text{Id})(\nabla Z_I) = \sum K Z_K \otimes \sum J \zeta^K_J \otimes \zeta^I_J,
\]
while the definition (2.22) and (2.43) imply
\[
\Delta \zeta^K_I = \sum J \zeta^K_J \otimes \zeta^I_J.
\]

One concludes that
\[
(\nabla \otimes \text{Id})(\nabla Z_I) = \sum K Z_K \otimes \nabla \zeta^K_I = (\text{Id} \otimes \Delta)(\nabla Z_I).
\]

\( \square \)
Proposition 2.11. Equipped with the coaction $\nabla : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes F(N)$, $U(\mathfrak{g})$ is a right $F(N)$-comodule coalgebra.

Proof. It is obvious from the definition that, for any $u \in U(\mathfrak{g})$,

$$\varepsilon(u_{<0>})u_{<1>} = \varepsilon(u)1.$$ (2.45)

We just have to check that

$$u_{<0>}(1) \otimes u_{<0>}(2) \otimes u_{<1>} = u_{<1>}(1) \otimes u_{<2>}(2) \otimes u_{<1>}(1) u_{<2>}(1).$$ (2.46)

In terms of the alternative definition (2.37), this amounts to showing that

$$\Delta((\nabla u)(\psi)) = \nabla(\Delta u)(\psi), \quad \forall \psi \in N,$$ (2.47)

where

$$(\nabla(\Delta u))(\psi) := \tilde{\Delta} u(\psi)(e,e), \quad \text{with} \quad \tilde{\Delta} u(\psi) = (U_\psi \otimes U_\psi) \Delta u (U_\psi^* \otimes U_\psi^*).$$

To this end we shall use the fact that, as it follows for instance from Proposition 1.5, the decomposition $\Delta u = u_{<0>} \otimes u_{<1>}$ is equivalent to the Leibniz rule

$$u(ab) = u_{(1)}(a) u_{(2)}(b), \quad \forall a,b \in C^\infty(G).$$

Thus, since

$$\tilde{u}(\psi)(ab) = U_\psi u(U_\psi^*(a)U_\psi^*(b)) = U_\psi (u_{(1)}(U_\psi^*(a))u_{(2)}(U_\psi^*(b)))$$

$$= (U_\psi u_{(1)} U_\psi^*) (a) = (U_\psi u_{(2)} U_\psi^*) (b) = \tilde{u}_{(1)}(\psi)(a) \tilde{u}_{(2)}(\psi)(b);$$

evaluating at $e \in G$, one obtains

$$(\nabla(\Delta u))(\psi)(ab) = (\nabla(\nabla u_{(1)})(\psi)(a) (\nabla u_{(2)})(\psi)(b), \quad \forall a,b \in C^\infty(G),$$

which is tantamount to (2.47).

Lemma 2.12. For any $u,v \in U(\mathfrak{g})$ one has

$$\nabla(uv) = u_{<0>} \otimes v_{<0>} \otimes u_{<2>}(2) \otimes v_{<1>}.$$ (2.48)

Proof. Without loss of generality, we may assume $u = Z_I$, $v = Z_J$. By the definition of the coaction one has

$$\tilde{u}(\psi) = U_\psi u U_\psi^* = \sum \beta^K_I(\psi) Z_K, \quad \tilde{v}(\psi) = U_\psi v U_\psi^* = \beta^K_J(\psi) Z_K.$$
which yields
\[ \tilde{uv}(\psi) = \tilde{u}(\psi)(\sum_L \beta^L_j(\psi)Z_L) = \sum_L \tilde{u}(\psi)_1(\beta^L_j(\psi)) \tilde{u}(\psi)_2Z_L; \]
where the last equality follows from (2.47). Denoting
\[ \tilde{u}_1(\psi) = \sum_M \beta^M_1(\psi)Z_M, \quad \tilde{u}_2(\psi) = \sum_N \beta^N_2(\psi)Z_N, \]
one can continue as follows:
\[ \tilde{uv}(\psi) = \sum_{L,M,N} \beta^M_1(\psi)Z_M(\beta^L_j(\psi)) \beta^N_2(\psi)Z_NZ_L. \]
Evaluating at \( e \), one obtains
\[ \nabla(\tilde{uv}) = \sum_{L,M,N} \zeta^M_1(\psi)Z_M(\beta^L_j(\psi)) \zeta^N_2(\psi)Z_N eZ_L e; \]
taking into account that \( U(\mathfrak{g}) \) is co-commutative, this is precisely the right hand side of (2.48).

Lemma 2.13. For any \( u \in U(\mathfrak{g}) \) and any \( f \in F(N) \) one has
\[ \Delta(u \triangleright f) = u_1 \triangleright_0 f_1 \otimes u_1 \triangleright_1 (u_2 \triangleright f_2) \]  
(2.49)
Proof. By Proposition 2.7 we may assume \( f \in F(N) \) of the form
\[ f(\psi) = \tilde{f}(\psi)(e), \]
with \( \tilde{f} \) in the algebra generated by \( \{ \gamma_{j,k,l_1...l_r}^{i}; 1 \leq i, j, k, l_1, \ldots, l_r \leq m \} \). Then
\[ \Delta(u \triangleright f)(\psi_1, \psi_2) = (u \triangleright f)(\psi_1 \circ \psi_2) = u(\tilde{f}(\psi_1 \circ \psi_2))(e). \]  
(2.50)
Now \( \tilde{f} \) corresponds to an element \( \tilde{\delta} \in \mathcal{H}(\Pi)_{ab}^* \), via the \( U(\mathfrak{g})^{-}\text{equivariant isomorphism} \) \( \iota : \mathcal{H}(\Pi)_{ab}^* \to F(N) \); explicitly,
\[ \tilde{\delta}(g U^*_\psi) = \tilde{f}(\psi) g U^*_\psi. \]
Accordingly,
\[ \tilde{f}(\psi_1 \circ \psi_2) U_{\psi_2}^* U_{\psi_1}^* = \tilde{\delta}(U_{\psi_2}^* U_{\psi_1}^*) = \tilde{\delta}_1(U_{\psi_2}^*) \tilde{\delta}_2(U_{\psi_1}^*) = \]
\[ \tilde{f}_1(\psi_2) U_{\psi_2}^* \tilde{f}_2(\psi_1) U_{\psi_1}^* = \tilde{f}_1(\psi_2) \left( \tilde{f}_2(\psi_1) \circ \psi_2 \right) U_{\psi_2}^* U_{\psi_1}^*, \]
whence
\[ \tilde{f}(\psi_1 \circ \psi_2) = \tilde{f}_1(\psi_2) \left( \tilde{f}_2(\psi_1) \circ \psi_2 \right). \]

Thus, we can continue (2.50) as follows
\[ \Delta(u \triangleright f)(\psi_1, \psi_2) = u \left( \tilde{f}_1(\psi_2) \left( \tilde{f}_2(\psi_1) \circ \psi_2 \right) \right)(e) = \]
\[ u_1(\tilde{f}_1(\psi_2))(e) u_2(\tilde{f}_2(\psi_1) \circ \psi_2)(e) = \]
\[ u_1(\tilde{f}_1(\psi_2))(e) u_2(\tilde{f}_2(\psi_1) \circ \psi_2)(\psi_2^{-1}(e)) = \]
\[ u_1(\tilde{f}_1(\psi_2))(e) \left( U_{\psi_2} U_{\psi_2}^* \right)(\tilde{f}_2(\psi_1))(e). \]

Since \( \iota \) switches the antipode with its opposite, the last line is equal to
\[ (u_1 \triangleright f_2)(\psi_2) u_2(\psi_2) (u_2(\psi_2) \triangleright f_1)(\psi_1). \]

Remembering that \( U(\mathfrak{g}) \) is co-commutative, one finally obtains
\[ \Delta(u \triangleright f)(\psi_1, \psi_2) = (u_2 \triangleright f_2)(\psi_2) u_1(\psi_2) (u_1 \triangleright f_1)(\psi_1). \]

\[ \square \]

**Proposition 2.14.** The Hopf algebras \( \mathcal{U} := U(\mathfrak{g}) \) and \( \mathcal{F} := F(N) \) form a matched pair of Hopf algebras.

**Proof.** Proposition 2.9 together with Proposition 2.11 show that with the action and coaction \( d \) defined in (2.33) and (2.37) \( \mathcal{F} \) is \( \mathcal{U} \) module algebra and \( \mathcal{U} \) is a comodule coalgebra. In addition we shall show that the action and coaction satisfy (2.8) \ldots (2.12). Since \( \mathcal{U} \) is cocommutative and \( \mathcal{F} \) is commutative (2.12) is automatically satisfied. The conditions (2.9) and (2.11) are correspondingly proved in Lemma 2.13 and Lemma 2.12. Finally the conditions (2.10) and (2.8) are obviously held. \( \square \)

Now it is the time for the main result of this section.
**Theorem 2.15.** The Hopf algebras $\mathcal{H}(\Pi)^{\text{cop}}$ and $\mathcal{F} \triangleright \mathcal{U}$ are isomorphic.

**Proof.** Proposition 1.3 provides us with $\delta_K Z_I$ as a basis for the Hopf algebra $\mathcal{H} := \mathcal{H}(\Pi)^{\text{cop}}$. Let us define

$$\mathcal{I} : \mathcal{H} \to \mathcal{F} \triangleright \mathcal{U},$$

by $\mathcal{I}(\delta_K Z_I) = \iota(\delta_K) \triangleright Z_I$, where $\iota$ is defined in Proposition 2.7, and linearly extend it on $\mathcal{H}$. First let see why $\mathcal{I}$ is well-defined. It suffices to show that the $\mathcal{I}$ preserves the relations between elements of $\mathcal{U}$ and $\mathcal{F}$. Let $X \in \mathfrak{g}$ and $f \in \mathcal{F}$, by using Proposition 2.7, we have

$$\mathcal{I}(Xf - fX) = \iota(Xf - fX) \triangleright 1 = X \triangleright 1 = (\iota(f) \triangleright X) - (1 \triangleright X)(\iota(f) \triangleright 1) = \mathcal{I}(X)f - \mathcal{I}(f)X.$$

Now we show $\mathcal{I}$ is injective. This can be shown by induction on the height. In the height 0 case the statement is obvious because of $\alpha_{jk}^i \otimes Z_I$ is part of the basis of $\mathcal{F} \triangleright \mathcal{U}$. Next, assume

$$\sum_{|J| \leq N-1} c_{J,I} \eta_J \otimes Z_I + \sum_{|K| = N} c_{K,L} \eta_K \otimes Z_L = 0,$$

Using the identities (2.26) and (2.28), one can replace $\eta_K$ by $\alpha_K + \text{lower height}$. Since the $\alpha_{...,}^*$'s are free generators, it follows that $c_{K,L} = 0$ for each $K$ of height $N$, and thus we are reduced to

$$\sum_{|J| \leq N-1} c_{J,I} \eta_J \otimes Z_I = 0;$$

the induction hypothesis now implies $c_{J,I} = 0$, for all $J, I$'s.

So $\mathcal{H}$ and $\mathcal{F} \triangleright \mathcal{U}$ are isomorphic as algebras. We now show they are isomorphic as coalgebras as well. It is enough to show $\mathcal{I}$ commutes with coproducts.

$$\Delta_{\mathcal{F} \triangleright \mathcal{U}}(\mathcal{I}(u)) = \Delta_{\mathcal{F} \triangleright \mathcal{U}}((1 \triangleright u) = 1 \triangleright u_{(1)} \otimes u_{(1)} \triangleright u_{(2)}).$$

On the other hand let $u_{(1)} \otimes U_\varphi u_{(2)} U_\psi^* = u_{(1)} \otimes \sum \beta^I(\varphi) Z_I$. We have

$$u(f U_\varphi^* g U_\psi^*) =$$

$$u(f g \circ \varphi) U_\varphi^* U_\psi^* =$$

$$u_{(1)}(f) u_{(2)}(g \circ \varphi) U_\varphi^* U_\psi^* =$$

$$u_{(1)}(f) U_\varphi^* U_\varphi u_{(2)} U_\psi^*(g) U_\psi^* =$$

$$u_{(1)}(f) U_\varphi^* \beta^I(\varphi) Z_I(g) U_\psi^*,$$
which shows that $\Delta_{H \cop}(u) = u_{(1)}(u_{(2)} <_{1} ) \otimes u_{(2)} <_{0}$. Since $U(g)$ is cocommutative one has $(I \otimes I)\Delta_{H}(u) = \Delta_{F \triangleright I}(I(u))$.

\textbf{2.2 The non-flat case}

We now take up the case of the contact pseudogroup $\Pi_{cn}$, in which case the Kac decomposition is given by Lemma 1.11. As in the flat case, we define the coordinates of an element $\psi \in N_{cn}$ as being the coefficients of the Taylor expansion of $\psi$ at $0 \in \mathbb{R}^{2n+1}$,

$$\alpha_{jj_{1}j_{2}...j_{r}}^{i} (\psi) = \partial_{j_{r}}...\partial_{j_{1}}\partial_{0}\psi_{i} (x) \mid_{x=0}, \quad 0 \leq i, j, j_{1}, j_{2}, ..., j_{r} \leq 2n.$$

The algebra they generate will be denoted $\mathcal{F}(N_{cn})$. It is the free commutative algebra generated by the indeterminates $\{\alpha_{jj_{1}j_{2}...j_{r}}^{i}; 0 \leq i, j, j_{1}, j_{2}, ..., j_{r} \leq 2n, r \in \mathbb{R}\}$, that are symmetric in all lower indices.

\textbf{Proposition 2.16.} $\mathcal{F}(N_{cn})$ is a Hopf algebra, whose coproduct, antipode and counit are uniquely determined by the requirements

$$\Delta(f)(\psi_{1}, \psi_{2}) = f(\psi_{1} \circ \psi_{2}), \quad \forall \psi_{1}, \psi_{2} \in N_{cn}, \quad (2.51)$$

$$S(f)(\psi) = f(\psi^{-1}), \quad \forall \psi \in N_{cn},$$

$$\epsilon(f) = f(e), \quad \forall f \in \mathcal{F}(N_{cn}).$$

\textit{Proof.} The proof is almost identical to that of Proposition 2.5. There are $2n$ new coordinates in this case, namely $\alpha_{0}^{i}$, $i = 1, ..., 2n$. for which one checks that the coproduct is well-defined as follows:

$$\Delta(\psi_{1}, \psi_{2}) = \alpha_{0}^{i} (\psi_{1} \circ \psi_{2}) = \partial_{0}(\psi_{1} \circ \psi_{2})\psi_{i} (0) =$$

$$= \partial_{0}\psi_{1}^{i}(\psi_{2}(0))\partial_{0}\psi_{2}^{0}(0) + \sum_{j=1}^{2n} \partial_{j}\psi_{1}^{i}(\psi_{2}(0))\partial_{0}\psi_{2}^{j}(0) =$$

$$= \partial_{0}\psi_{1}^{i}(0)\partial_{0}\psi_{2}^{0}(0) + \sum_{j=1}^{2n} \partial_{j}\psi_{1}^{i}(0)\partial_{0}\psi_{2}^{j}(0) =$$

$$= \partial_{0}\psi_{1}^{i}(0) \partial_{0}\psi_{2}^{0}(0) + \sum_{j=1}^{2n} \partial_{j}\psi_{1}^{i}(0)\partial_{0}\psi_{2}^{j}(0) =$$

$$\partial_{0}\psi_{1}^{i}(0) + \sum_{j=1}^{2n} \partial_{j}\partial_{0}\psi_{2}^{j}(0) = \partial_{0}\psi_{1}^{i}(0) + \partial_{0}\psi_{2}^{0}(0) =$$

$$= (\alpha_{0}^{i} \otimes 1 + 1 \otimes \alpha_{0}^{i})(\psi_{1}, \psi_{2});$$

we have been using above the fact that, for any $\psi \in N_{cn},$

$$\psi(0) = 0, \quad \text{and} \quad \psi_{H}^{e}(0) = \text{Id}.$$
Taking $\psi_1 = \psi^{-1}$, $\psi_2 = \psi$, one obtains from the above

$$\alpha^i_0(\psi^{-1}) + \alpha^i_0(\psi) = 0, \quad \text{hence} \quad S\alpha^i_0 = -\alpha^i_0.$$  

\[ \square \]

**Lemma 2.17.** The coefficients of the Taylor expansion of $\tilde{\psi}$ at $e \in N_{cn}$,

$$\eta^i_{jk_1...k_r}(\psi) := \Gamma^i_{jk_1...k_r}(\psi)(e), \quad \psi \in N_{cn}, \quad (2.52)$$

define regular functions on $N_{cn}$, which generate the algebra $\mathcal{F}(N_{cn})$.

**Proof.** The formula (1.96) shows that

$$\eta^i_0 = -\alpha^i_0, \quad i = 1, \ldots, 2n,$$

while

$$\eta^j_0 = \delta^j_0 = \alpha^j_0, \quad i, j = 1, \ldots, 2n.$$  

To relate their higher derivatives, we observe that, in view of (1.85), (1.86)

$$X_{k_1} \ldots X_{k_r} \big|_{(0,Id,1)} = E_{k_1} \ldots E_{k_r} \big|_{(0,Id,1)};$$

on the other hand, it is obvious that the jet at 0 with respect to the frame $\{E_0, \ldots, E_{2n}\}$ is equivalent to the jet at 0 with respect to the standard frame $\{\partial_0, \ldots, \partial_{2n}\}$. This proves the statement for the ‘new’ coordinates. For the other coordinates the proof is similar to that of Lemma 2.6.  

\[ \square \]

This lemma allows to recover the analog of Proposition 2.7 by identical arguments.

**Proposition 2.18.** There is a unique isomorphism of Hopf algebras

$$\iota : H(\Pi_{cn})^{\text{cop}} \rightarrow \mathcal{F}(N_{cn})$$

with the property that

$$\iota(\Delta^i_{jk_1...k_r}) = \eta^i_{jk_1...k_r}, \quad 0 \leq i, j, k_1, k_2, \ldots, k_r \leq 2n.$$  

Next, one has the tautological counterpart of Lemma 2.8.

**Lemma 2.19.** Let $\varphi \in G_{cn}$ and $\phi \in G_{cn}$. Then

$$\Gamma^i_{jk_1...k_r}(\varphi \circ \phi) = \Gamma^i_{jk_1...k_r}(\phi),$$

$$\Gamma^i_{jk_1...k_r}(\phi \circ \varphi) = \Gamma^i_{jk_1...k_r}(\phi) \circ \tilde{\varphi}.$$  

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Proof. As is the case with its sibling result, this is simply a consequence of the left invariance the vector fields \( \{X_0, \ldots, X_{2n}\} \).

In turn, the above lemma allows to recover the analog of Proposition 2.9

**Proposition 2.20.** The algebra isomorphism \( \iota : H(\Pi)_{ab}^{\cop} \to \mathcal{F}(N) \) identifies the \( U(g) \)-module \( H(\Pi)_{ab} \) with the \( U(g) \)-module \( \mathcal{F}(N) \). In particular \( \mathcal{F}(N) \) is \( U(g) \)-module algebra.

Furthermore, \( U(g) \) can be endowed with a right \( \mathcal{F}(N) \)-comodule structure \( \nabla : U(g) \to U(g) \otimes \mathcal{F}(N) \) in exactly the same way as in the flat case, cf. (2.37), Lemma 2.10 and is in fact a right \( \mathcal{F}(N) \)-comodule coalgebra (comp. Prop. 2.11). Likewise, the analog of Proposition 2.14 holds true, establishing that \( U(g) \) and \( \mathcal{F}(N) \) form a matched pair of Hopf algebras. Finally, one concludes in a similar fashion with the bicrossed product realization theorem for the contact case.

**Theorem 2.21.** The Hopf algebras \( H(\Pi)_{ab}^{\cop} \) and \( \mathcal{F}(N) \) are canonically isomorphic.

### 3 Hopf cyclic cohomology

After reviewing some of the most basic notions in Hopf cyclic cohomology, we focus on the case of the Hopf algebras \( H(\Pi) \) constructed in the preceding section and show how their Hopf cyclic cohomology can be recovered from a bicocyclic complex manufactured out of the matched pair. We then illustrate this procedure by computing the relative periodic Hopf cyclic cohomology of \( H_n \) modulo \( gl_n \). For \( n = 1 \), we completely calculate the non-periodized Hopf cyclic cohomology as well.

#### 3.1 Quick synopsis of Hopf cyclic cohomology

Let \( H \) be a Hopf algebra, and let \( C \) be a left \( H \)-module coalgebra, such that its comultiplication and counit are \( H \)-linear, i.e.

\[
\Delta(hc) = h_{(1)}c_{(1)} \otimes h_{(2)}c_{(2)}, \quad \varepsilon(hc) = \varepsilon(h)\varepsilon(c).
\]

We recall from [16] that a right module \( M \) which is also a left comodule is called **right-left stable anti-Yetter-Drinfeld module** (SA-YD for short) over
the Hopf algebra $\mathcal{H}$ if it satisfies the following conditions, for any $h \in \mathcal{H}$, and $m \in M$:

$$m_{<\sigma>} m_{<\tau>} = m$$

$$(mh)_{<\tau>} \otimes (mh)_{<\sigma>} = S(h^{(3)}) m_{<\tau>} h^{(1)} \otimes m_{<\sigma>} h^{(2)} ,$$

where the coaction of $\mathcal{H}$ was denoted by $\nabla_M(m) = m_{<\tau>} \otimes m_{<\sigma>}$.

Having such a datum $(\mathcal{H}, \mathcal{C}, M)$, one defines (cf. [17]) a cocyclic module $\{C^n_{\mathcal{H}}(\mathcal{C}, \mathcal{M}), \partial_i, \sigma_j, \tau\}_{n \geq 0}$ as follows.

$$C^n := C^n_{\mathcal{H}}(\mathcal{C}, \mathcal{M}) = M \otimes_\mathcal{H} C^\otimes n+1, \quad n \geq 0,$$

with the cocyclic structure given by the operators

$$\partial_i : C^n \rightarrow C^{n+1}, \quad 0 \leq i \leq n + 1$$

$$\sigma_j : C^n \rightarrow C^{n-1}, \quad 0 \leq j \leq n - 1,$$

$$\tau : C^n \rightarrow C^n,$$

defined explicitly as follows:

$$\partial_i (m \otimes_\mathcal{H} c) = m \otimes_\mathcal{H} c^0 \otimes \ldots \otimes \Delta(c_i) \otimes \ldots \otimes c^n ,$$

$$\partial_{n+1} (m \otimes_\mathcal{H} c) = m_{<\sigma>} \otimes_\mathcal{H} c^0 \otimes c^1 \otimes \ldots \otimes c^n \otimes m_{<\tau>} c^0 \otimes \ldots \otimes c^n ,$$

$$\sigma_j (m \otimes_\mathcal{H} \tilde{c}) = m \otimes_\mathcal{H} c^0 \otimes \ldots \otimes \epsilon(c_{j+1}) \otimes \ldots \otimes c^n ,$$

$$\tau (m \otimes_\mathcal{H} \tilde{c}) = m_{<\sigma>} \otimes_\mathcal{H} c^1 \otimes \ldots \otimes c^n \otimes m_{<\tau>} c^0 ;$$

here we have used the abbreviation $\tilde{c} = c^0 \otimes \ldots \otimes c^n$.

One checks [17] that $\partial_i, \sigma_j$, and $\tau$ satisfy the following identities, which define the structure of a cocyclic module (cf. [5]):

$$\partial_j \partial_i = \partial_i \partial_{j-1}, \quad i < j, \quad \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad i \leq j \quad (3.1)$$

$$\sigma_j \partial_i = \begin{cases} 
\partial_i \sigma_{j-1} & i < j \\
1_n & \text{if } i = j \text{ or } i = j + 1 \\
\partial_{i-1} \sigma_j & i > j + 1;
\end{cases} \quad (3.2)$$

$$\tau_n \partial_i = \partial_{i-1} \tau_{n-1}, \quad 1 \leq i \leq n, \quad \tau_n \partial_0 = \partial_n \quad (3.3)$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \quad 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \quad (3.4)$$

$$\tau_{n+1} = 1_n . \quad (3.5)$$

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The motivating example for the above notion is the cocyclic complex associated to a Hopf algebra \( \mathcal{H} \) endowed with a modular pair in involution, (MPI for short), \((\delta, \sigma)\), which we recall from [6]. \( \delta \) is an algebra map \( \mathcal{H} \to \mathbb{C} \), and \( \sigma \in \mathcal{H} \) is a group-like element, or equivalently a coalgebra map \( \mathbb{C} \to \mathcal{H} \). The pair \((\delta, \sigma)\) is called MPI if \( \delta(\sigma) = 1 \), and \( \tilde{S}_\delta^2 = Ad\sigma \); the twisted antipode \( \tilde{S}_\delta \) is defined by

\[
\tilde{S}_\delta(h) = (\delta \ast S)(h) = \delta(h_{(1)})S(h_{(2)}).
\]

One views \( \mathcal{H} \) as a left \( \mathcal{H} \)-module coalgebra via left multiplication. On the other hand if one lets \( M = \sigma \mathbb{C} \delta \) to be the ground field \( \mathbb{C} \) endowed with the left \( \mathcal{H} \)-coaction via \( \sigma \) and right \( \mathcal{H} \)-action via the character \( \delta \), then \((\delta, \sigma)\) is a MPI if and only if \( \sigma \mathbb{C} \delta \) is a SAYD. Thanks to the multiplication and the antipode of \( \mathcal{H} \), one identifies \( \mathcal{H}(\mathcal{H}, M) \) with \( M \otimes \mathcal{H} \otimes n \) via the map

\[
I : M \otimes \mathcal{H} \otimes^{(n+1)} \to M \otimes \mathcal{H} \otimes^n,
\]

\[
I(m \otimes \mathcal{H} h^0 \otimes \ldots \otimes h^n) = mh^0_{(1)} \otimes S(h_{(2)}) \cdot (h^1 \otimes \ldots \otimes h^n).
\]

As a result, \( \partial_i \), \( \sigma_j \), and \( \tau \) acquire the simplified form of the original definition [6], namely

\[
\begin{align*}
\partial_0(h^1 \otimes \ldots \otimes h^{n-1}) & = 1 \otimes h^1 \otimes \ldots \otimes h^{n-1}, \\
\partial_j(h^1 \otimes \ldots \otimes h^{n-1}) & = h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1}, \quad 1 \leq j \leq n - 1 \\
\partial_n(h^1 \otimes \ldots \otimes h^{n-1}) & = h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma, \\
\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) & = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1}, \quad 0 \leq i \leq n, \\
\tau_n(h^1 \otimes \ldots \otimes h^n) & = (\Delta^{-1} \tilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes \sigma.
\end{align*}
\]

For completeness, we record below the bi-complex \((\mathcal{C}^*, \mathcal{C}^*; \mathcal{H}, \mathcal{H}; M)\) that computes the Hopf cyclic cohomology of a coalgebra \( C \) with coefficients in a SAYD module \( M \) under the symmetry of a Hopf algebra \( \mathcal{H} \):

\[
\mathcal{C}^{p,q}_\mathcal{H}(C, \mathcal{H}; M) = \begin{cases}
\mathcal{C}_\mathcal{H}^{q-p}(C, M), & q \geq p, \\
0, & q < p,
\end{cases}
\]

where \( b : \mathcal{C}_\mathcal{H}^n(C, M) \to \mathcal{C}_\mathcal{H}^{n+1}(C, M) \) is given by

\[
b = \sum_{i=0}^{n+1} (-1)^i \partial_i ;
\]

the operator \( B : \mathcal{C}_\mathcal{H}^n(C, M) \to \mathcal{C}_\mathcal{H}^{n-1}(C, M) \) is defined by the formula

\[
B = A \circ B_0, \quad n \geq 0,
\]
where
\[ B_0 = \sigma_{n} - ((-1)^n \tau) \]
and
\[ A = 1 + \lambda + \cdots + \lambda^n, \quad \text{with} \quad \lambda = ((-1)^n \tau). \]
The groups \( \{ HC_n(\mathcal{H}; \delta, \sigma) \}_{n \in \mathbb{N}} \) are computed from the first quadrant total complex \( (TC^*(\mathcal{H}; \delta, \sigma), b + B) \),
\[ TC^n(\mathcal{H}; \delta, \sigma) = \sum_{k=0}^{n} CC^{k,n-k}(\mathcal{H}; \delta, \sigma), \]
and the periodic groups \( \{ HP^i(\mathcal{H}; \delta, \sigma) \}_{i \in \mathbb{Z}/2} \) are computed from the full total complex \( (TP^*(\mathcal{H}; \delta, \sigma), b + B) \),
\[ TP^i(\mathcal{H}; \delta, \sigma) = \sum_{k \in \mathbb{Z}} CC^{k,i-k}(\mathcal{H}; \delta, \sigma). \]

We note that, in defining the Hopf cyclic cohomology as above, one has the option of viewing the Hopf algebra \( \mathcal{H} \) as a left \( \mathcal{H} \)-module coalgebra or as a right \( \mathcal{H} \)-module coalgebra. It was the first one which was selected as the definition in [6]. The other choice would have given to the cyclic operator the expression
\[ \tau_n(h^1 \otimes \cdots \otimes h^n) = \sigma S(h^n_{(n)}) \otimes h^1 S(h^n_{(n-1)}) \otimes \cdots \otimes h^n_{(1)} \delta(S^{-1}(h^n_{(2)})). \]

As it happens, the choice originally selected is not the best suited for the situations involving a right action. To restore the naturality of the notation, it is then convenient to pass from the Hopf algebra \( \mathcal{H} \) to the co-opposite Hopf algebra \( \mathcal{H}^\text{cop} \). This transition does not affect the Hopf cyclic cohomology, because for any \( \mathcal{H} \)-module coalgebra \( C \) and any SAYD module \( M \) one has a canonical equivalence
\[ (C^*_\mathcal{H}(C, M), b, B) \simeq (C^*_\mathcal{H}^\text{cop}(C^\text{cop}, M^\text{cop}), b, B); \quad (3.6) \]
\( M^\text{cop} := M \) is SAYD module for \( \mathcal{H}^\text{cop} \), with the action of \( \mathcal{H}^\text{cop} \) the same as the action of \( \mathcal{H} \), but with the coaction \( \nabla : M^\text{cop} \to M^\text{cop} \otimes M^\text{cop} \) given by
\[ \nabla(m) = S^{-1}(m_{<\tau>}) \otimes m_{\leq \sigma}. \quad (3.7) \]
The equivalence \( (3.6) \) is realized by the map
\[ T : C^n_{\mathcal{H}}(C, M) \to C^n_{\mathcal{H}^\text{cop}}(C^\text{cop}, M^\text{cop}), \quad (3.8) \]
\[ T(m \otimes c^0 \otimes \cdots \otimes c^n) = m_{\leq \sigma} \otimes m_{<\tau>} \otimes c^0 \otimes c^n \otimes \cdots \otimes c^1. \]
Proposition 3.1. The map $\mathcal{T}$ defines an isomorphism of mixed complexes.

Proof. The map $\mathcal{T}$ is well-defined because

\[
\begin{align*}
\mathcal{T}(mh \otimes c^0 \otimes \ldots \otimes c^n) &= mh(2) \otimes S(h(3))m_{<\pi>}h(1)c^0 \otimes c^n \otimes \ldots \otimes c^1 = \\
m_{_{\pi_1}} \otimes h_{(n+2)}S(h_{(n+3)})m_{<\pi>}h(1)c^0 \otimes h_{(n+1)}c^n \otimes \ldots \otimes h_2c^1 = \\
m_{_{\pi_1}} \otimes m_{_{\pi_1}c}h(1)c^0 \otimes h_{(n+1)}c^n \otimes \ldots \otimes h_2c^1 = \\
\mathcal{T}(m \otimes h_1c^0 \otimes \ldots \otimes h_{(n+1)}c^n).
\end{align*}
\]

We denote the cyclic structure of $C_0^\mu(C, M)$, resp. $C_{\mathcal{H}}(C^\cop, M^\cop)$ by $\partial_i$, $\sigma_i$, and $\tau$, resp. $d_i$, $s_j$, and $t$. One has in fact a stronger commutation property, namely

\[
\mathcal{T}\partial_i = d_{n+1-i}\mathcal{T}, \quad 0 \leq i \leq n + 1. \tag{3.9}
\]

Indeed,

\[
\begin{align*}
d_{n+1}\mathcal{T}(m \otimes c^0 \otimes \ldots \otimes c^n) &= \\
d_{n+1}(m_{_{\pi_1}} \otimes m_{_{\pi_1}c}c^0 \otimes c^n \otimes \ldots \otimes c^1) = \\
m_{_{\pi_1}} \otimes m_{_{\pi_1}c}c^0(1) \otimes c^n \otimes \ldots \otimes c^1 \otimes S^{-1}(m_{_{\pi_1}c}c^0(2))m_{_{\pi_1}c}c^0(3) \\
m_{_{\pi_1}} \otimes m_{_{\pi_1}c}c^0(1) \otimes c^n \otimes \ldots \otimes c^1 \otimes c^0(2) \\
\mathcal{T}(m \otimes c^0(1) \otimes c^0(2) \otimes c^1 \otimes \ldots \otimes c^n) = \\
\mathcal{T}\partial_0(m \otimes c^0 \otimes \ldots \otimes c^n)
\end{align*}
\]

\[
\begin{align*}
\mathcal{T}\partial_{n+1}(m \otimes c^0 \otimes \ldots \otimes c^n) &= \\
\mathcal{T}(m_{_{\pi_1}} \otimes c^0(2) \otimes c^1 \otimes \ldots \otimes c^n \otimes m_{_{\pi_1}c}c^0(1)) \\
m_{_{\pi_1}} \otimes m_{_{\pi_1}c}c^0(2) \otimes m_{_{\pi_1}c}c^0(1) \otimes c^n \otimes \ldots \otimes c^1 \\
d_0(m_{_{\pi_1}} \otimes m_{_{\pi_1}c}c^0 \otimes c^n \otimes \ldots \otimes c^1) = \\
d_0\mathcal{T}(m \otimes c^0 \otimes \ldots \otimes c^n);
\end{align*}
\]

on the other hand, for $1 \leq i \leq n$,

\[
\begin{align*}
\mathcal{T}\partial_i(m \otimes c^0 \otimes \ldots \otimes c^n) &= \\
\mathcal{T}(m \otimes c^0 \otimes \ldots \otimes c^{i-1} \otimes c^i(1) \otimes c^i(2) \otimes c^{i+1} \otimes \ldots \otimes c^n) = \\
m_{_{\pi_1}} \otimes m_{_{\pi_1}c}c^0 \otimes c^n \otimes \ldots \otimes c^{i+1} \otimes c^i(2) \otimes c^i(1) \otimes c^{i-1} \otimes \ldots \otimes c^1 = \\
d_{n+1-i}(m_{_{\pi_1}} \otimes m_{_{\pi_1}c}c^0) \otimes c^n \otimes \ldots \otimes c^1 = \\
d_{n+1-i}\mathcal{T}(m \otimes c^0 \otimes c^n).
\end{align*}
\]
Thus, $Tb = (-1)^{n+1}bT$.

Next, we check that $Tτ = t^{-1}T = t^nT$ as follows:

\[ Tτ(m \otimes c^0 \otimes \ldots \otimes c^n) = \]

\[ T(m \otimes c^1 \otimes c^n \otimes m \otimes c^0) = \]

\[ m \otimes m \otimes c^1 \otimes c^n \otimes \ldots \otimes c^2 = \]

\[ t^{-1}(m \otimes m \otimes c^0 \otimes c^n \otimes \ldots \otimes c^1) = \]

\[ t^{-1}T(m \otimes c^0 \otimes \ldots \otimes c^n). \]

It is easy to see that $Tσ_i = s_{n-1-i}T$, for $0 \leq i \leq n-1$. Using the above identities and $ts_0 = s_{n-1}t^2$ one obtains

\[ TB = T \sum_{j=0}^{n-1} (-1)^{(n-1)j} \tau^j \sigma_{n-1} (1 - (-1)^n T) = \]

\[ = \sum_{j=0}^{n-1} (-1)^{(n-1)j} t^j s_{n-1} t^2 (1 - (-1)^n T) = \]

\[ = (-1)^{n+1} \sum_{j=0}^{n-1} (-1)^{(n-1)j} t^{n-1-j} s_{n-1} t (1 - (-1)^n T) = \]

\[ = (-1) \sum_{k=0}^{n-1} (-1)^{(n-1)k} t^k s_{n-1} t (1 - (-1)^n T) = -BT, \]

which completes the proof.

We next recall from [9] the setting for relative Hopf cyclic cohomology. Let $\mathcal{H}$ be an arbitrary Hopf algebra and $\mathcal{K} \subset \mathcal{H}$ a Hopf subalgebra. Let

\[ \mathcal{C} = \mathcal{C}(\mathcal{H}, \mathcal{K}) := \mathcal{H} \otimes_\mathcal{K} \mathbb{C}, \]  

(3.10)

where $\mathcal{K}$ acts on $\mathcal{H}$ by right multiplication and on $\mathbb{C}$ by the counit. It is a left $\mathcal{H}$-module in the usual way, via left multiplication. As such, it can be identified with the quotient module $\mathcal{H}/\mathcal{H}\mathcal{K}^+$, $\mathcal{K}^+ = \text{Ker} \varepsilon|\mathcal{K}$, via the isomorphism induced by

\[ h \in \mathcal{H} \mapsto \hat{h} = h \otimes \varepsilon \in \mathcal{H} \otimes_\mathcal{K} \mathbb{C}. \]  

(3.11)
Moreover, thanks to the right action of $K$ on $H$, $C = C(H, K)$ is an $H$-module coalgebra. Indeed, its coalgebra structure is given by the coproduct
\[ \Delta_C (h \otimes_K 1) = (h_{(1)} \otimes_K 1) \otimes (h_{(2)} \otimes_K 1), \]
(3.12)
inherited from that on $H$, and is compatible with the action of $H$ on $C$ by left multiplication:
\[ \Delta_C (gh \otimes_K 1) = \Delta(g) \Delta_C (h \otimes_K 1); \]
similarly, there is an inherited counit
\[ \varepsilon_C (h \otimes_K 1) = \varepsilon(h), \quad \forall c \in C, \]
(3.13)
that satisfies
\[ \varepsilon_C (gh \otimes_K 1) = \varepsilon(g) \varepsilon_C (h \otimes_K 1). \]
Thus, $C$ is a $H$-module coalgebra. The relative Hopf cyclic cohomology of $H$ with respect to the $K$ and with coefficients in $M$, to be denoted by $HC(H, K; M)$, is by definition the Hopf cyclic cohomology of $C$ with coefficients in $M$. Thanks to the antipode of $H$ one simplifies the cyclic complex as follows (cf. [9, §5]):
\[ C^* (H, K; M) = \{ C^n (H, K; M) := M \otimes_K C^{\otimes n} \}_{n \geq 0}, \]
where $K$ acts diagonally on $C^{\otimes n},$
\[ \partial_0 (m \otimes_K c^1 \otimes \ldots \otimes c^{n-1}) = m \otimes_K \hat{1} \otimes c^1 \otimes \ldots \otimes c^{n-1}, \]
\[ \partial_i (m \otimes_K c^1 \otimes \ldots \otimes c^{n-1}) = m \otimes_K c^1 \otimes \ldots \otimes c_{(1)}^i \otimes c_{(2)}^i \otimes \ldots \otimes c^{n-1}, \]
\[ \forall \quad 1 \leq i \leq n - 1; \]
\[ \partial_n (m \otimes_K c^1 \otimes \ldots \otimes c^{n-1}) = m_{(0)} \otimes_K c^1 \otimes \ldots \otimes c^{n-1} \otimes \hat{m}_{(-1)}; \]
\[ \sigma_i (m \otimes_K c^1 \otimes \ldots \otimes c^{n+1}) = m \otimes_K c^1 \otimes \ldots \otimes \varepsilon(c_{i+1}^1) \otimes \ldots \otimes c^{n+1}, \]
\[ \forall \quad 0 \leq i \leq n; \]
\[ \tau_n (m \otimes_K h^1 \otimes c^2 \otimes \ldots \otimes c^n) = m_{(0)} h_{(1)}^1 \otimes_K S(h_{(2)}^1) \cdot (c^2 \otimes \ldots \otimes c^n \otimes \hat{m}_{(-1)}); \]
the above operators are well-defined and endow $C^* (H, K; M)$ with a cyclic structure.
Since
\[ k_{(1)} h S(k_{(2)}) \otimes_K 1 = k_{(1)} h \otimes_K \varepsilon(k_{(2)}) 1 = k c \otimes_K 1, \quad k \in K, \]
the restriction to \( \mathcal{K} \) of the left action of \( \mathcal{H} \) on \( \mathcal{C} = \mathcal{H} \otimes \mathcal{K} \mathcal{C} \) can also be regarded as ‘adjoint action’, induced by conjugation.

When specialized to Lie algebras this definition recovers the relative Lie algebra homology, cf. [9, Thm. 16]. Indeed, let \( \mathfrak{g} \) be a Lie algebra over the field \( F \), let \( \mathfrak{h} \subset \mathfrak{g} \) be a reductive subalgebra in \( \mathfrak{g} \), and let \( M \) be a \( \mathfrak{g} \)-module. We equip \( M \) with the \textit{trivial} \( \mathfrak{g} \)-comodule structure

\[
\nabla_M(m) = 1 \otimes m \in \mathcal{H} \otimes M, \tag{3.14}
\]

and note the stability condition is then trivially satisfied, while the AYD one follows from (3.14) and the cocommutativity of the universal enveloping algebra \( \mathfrak{A}(\mathfrak{g}) \). The relative Lie algebra homology and cohomology of the pair \( \mathfrak{h} \subset \mathfrak{g} \) with coefficients in \( M \) is computed from the Chevalley-Eilenberg complexes

\[
\{C_*(\mathfrak{g}, \mathfrak{h}; M), \delta\}, \quad C_n(\mathfrak{g}, \mathfrak{h}; M) := M \otimes_{\mathfrak{h}} \bigwedge (\mathfrak{g}/\mathfrak{h}).
\]

Here the action of \( \mathfrak{h} \) on \( \mathfrak{g}/\mathfrak{h} \) is induced by the adjoint representation and the differentials are given by the formulae

\[
\delta(m \otimes_{\mathfrak{h}} \hat{X}_1 \wedge \ldots \wedge \hat{X}_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} m X_i \otimes_{\mathfrak{h}} \hat{X}_1 \wedge \ldots \wedge \hat{X}_i \ldots \wedge \hat{X}_{n+1} \\
+ \sum_{i<j} (-1)^{i+j} m \otimes_{\mathfrak{h}} [\hat{X}_i, \hat{X}_j] \wedge \hat{X}_1 \wedge \ldots \wedge \hat{X}_i \ldots \hat{X}_j \ldots \wedge \hat{X}_{n+1} \tag{3.15}
\]

where \( \hat{X} \in \mathfrak{g}/\mathfrak{h} \) stands for the class modulo \( \mathfrak{h} \) of \( X \in \mathfrak{g} \) and the superscript \(^*\) signifies the omission of the indicated variable.

There are canonical isomorphisms between the periodic relative Hopf cyclic cohomology of the pair \( \mathfrak{A}(\mathfrak{h}) \subset \mathfrak{A}(\mathfrak{g}) \), with coefficients in any \( \mathfrak{g} \)-module \( M \), and the relative Lie algebra homology with coefficients of the pair \( \mathfrak{h} \subset \mathfrak{g} \) (see [9]):

\[
HP^*(\mathfrak{A}(\mathfrak{g}), \mathfrak{A}(\mathfrak{h}); M) \cong \bigoplus_{n \equiv \epsilon \mod 2} H_n(\mathfrak{g}, \mathfrak{h}; M). \tag{3.16}
\]

### 3.2 Reduction to diagonal mixed complex

In this section we develop an apparatus for computing Hopf cyclic cohomology of certain cocrossed coproduct coalgebras. These are made of two
coalgebras endowed with actions by a Hopf algebra. Both coalgebras are Hopf module coalgebras, and one of them is a comodule coalgebra as well, which moreover is a Yetter-Drinfeld module. Under these circumstances, one can unwind the Hopf cyclic structure of the cocrossed product coalgebra and identify it with the diagonal of a cylindrical module. We then construct a spectral sequence that computes the cohomology of the total complex of the cylindrical module, which by Eilenberg-Zilber theorem is quasi isomorphic to the diagonal of the cylindrical module.

Let $H$ be a Hopf algebra and let $C$ and $D$ be two left $H$-module coalgebras. In addition, we assume that $C$ is a left comodule coalgebra, which makes $C$ a YD module on $H$. We denote the coaction of $C$ by $\nabla_C$ and use the abbreviated notation $\nabla_C(c) = c_{<0>} \otimes c_{<1>}$. We recall that the YD condition stipulates that $\nabla_C(hc) = h(1)c_{<0>}S(h(3)) \otimes h(2)c_{<1>}$. \hfill (3.17)

Using the coaction of $H$ on $C$ and its action on $D$, one constructs a coalgebra structure on $C \otimes D$ defined by the coproduct

$$
\Delta(c \otimes d) = c_{<0>} \otimes c_{<0>} \otimes c_{<1>} \otimes d_{<1>} \otimes d_{<1>}. \hfill (3.18)
$$

We denote this coalgebra by $C \lhd D$. The import of the YD condition is revealed by the following result.

**Lemma 3.2.** Via the diagonal action of $H$, $C \lhd D$ becomes an $H$ module coalgebra.

*Proof.* One has

$$
\Delta(h_{(1)}c \lhd h_{(2)}d) =
\begin{align*}
& h_{(1)}c_{<0>}h_{(2)}d_{<0>} \\
& h_{(1)}c_{<1>}h_{(2)}c_{<2>},
\end{align*}
$$

Now let $M$ be an SAYD over $H$. We endow $M \otimes C^\otimes q$ with the following $H$ action and coaction:

$$
(m \otimes \hat{c})h = mh_{(1)} \otimes S(h_{(2)})\hat{c},

\nabla(m \otimes \hat{c}) = e^0_{<0>} \cdots e^n_{<0>} \otimes m_{<2>^<0>} \otimes e^0_{<0>} \otimes \cdots \otimes e^n_{<0>},
$$
Lemma 3.3. Let $C$ be an YD module over $\mathcal{H}$. Then, via the diagonal action and coaction $C^{\otimes q}$ is also an YD module over $\mathcal{H}$.

Proof. We verify that $\tilde{c} = c^1 \otimes \ldots \otimes c^q \in C^{\otimes n}$ and $h \in \mathcal{H}$ satisfy (3.17). Indeed,

\[
\nabla(h\tilde{c}) = \nabla(h(1)c^1 \otimes \ldots \otimes h(q)c^q) = \\
(h(1)c^1)_{<1>} \ldots (h(q)c^q)_{<1>} \otimes (h(1)c^1)_{<1>} \otimes \ldots \otimes (h(q)c^q)_{<1>} = \\
h(1)c^1_{<1>} S(h(3))h(4)c^2_{<1>} S(h(6)) \ldots h(3q-2)c^q_{<1>} S(h(3q)) \otimes \\
h(2)c^1_{<0>} \otimes h(3)c^2_{<0>} \otimes \ldots \otimes h(3q-1)c^q_{<0>} = \\
h(1)\tilde{c}_{<1>} S(h(3)) \otimes h(2)\tilde{c}_{<0>}. 
\]

$\square$

Proposition 3.4. Equipped with the above action and coaction, $M \otimes C^{\otimes q}$ is an AYD module.

Proof. Let $\tilde{c} \in C^{\otimes q}$, and $h \in \mathcal{H}$. By Lemma 3.3 we can write

\[
\nabla((m \otimes \tilde{c})h) = \nabla(mh(1) \otimes S(h(2))\tilde{c}) = \\
(S(h(2))\tilde{c})_{<1>} (mh)_{<1>} \otimes (mh(1))_{<0>} \otimes (S(h(1))\tilde{c})_{<0>} = \\
S(h(0))\tilde{c}_{<1>} S^2(h(4)) S(h(3)) m_{<1>} h(1) \otimes m_{<0>} \otimes h(2) \otimes S(h(5))\tilde{c}_{<0>} = \\
S(h(3))\tilde{c}_{<1>} m_{<1>} h(1) \otimes (m_{<0>} \otimes \tilde{c})_{<0>} h(2). 
\]

$\square$

We define the following bigraded module, inspired by [10,12,11,24], in order to obtain a cylindrical module for cocrossed product coalgebras. Set

\[
\mathfrak{X}^{p,q} := M \otimes_{\mathcal{H}} D^{\otimes_{p+1}} \otimes C^{\otimes_{q+1}}, 
\]

and endow $\mathfrak{X}$ with the operators

\[
\bar{\partial}_i : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p+1,q)}, \quad 0 \leq i \leq p + 1 \tag{3.19} \\
\bar{\sigma}_j : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p-1,q)}, \quad 0 \leq j \leq p - 1 \tag{3.20} \\
\bar{\tau} : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p,q)}, \quad \tag{3.21}
\]

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defined by
\[
\tilde{\partial}_i (m \otimes \tilde{d} \otimes \tilde{c}) = m \otimes d^0 \otimes \cdots \otimes \Delta (d^i) \otimes \cdots \otimes d^p \otimes \tilde{c},
\]
\[
\tilde{\partial}_{p+1} (m \otimes \tilde{d} \otimes \tilde{c}) = m_{<\sigma>} \otimes d^0 (2) \otimes \cdots \otimes d^p \otimes \tilde{c}_{<t>},
\]
\[
\tilde{\sigma}_j (m \otimes \tilde{d} \otimes \tilde{c}) = m \otimes d^1 \otimes \cdots \otimes \varepsilon (d^j) \otimes \cdots \otimes d^p \otimes \tilde{c},
\]
\[
\tilde{\tau} (m \otimes \tilde{d} \otimes \tilde{c}) = m_{<\sigma>} \otimes d^1 \otimes \cdots \otimes d^p \otimes \tilde{c}_{<t>},
\]
the vertical structure is just the cocyclic structure of \( \mathcal{C}(\mathcal{H}, \mathcal{K}; \mathcal{K} \otimes p + 1 \otimes M) \), with
\[
\uparrow \partial_i =: \mathcal{X}^{(p,q)} \rightarrow \mathcal{X}^{(p,q+1)}, \quad 0 \leq i \leq q + 1 \tag{3.22}
\]
\[
\uparrow \sigma_j : \mathcal{X}^{(p,q)} \rightarrow \mathcal{X}^{(p,q-1)}, \quad 0 \leq j \leq q - 1 \tag{3.23}
\]
\[
\uparrow \tau : \mathcal{X}^{(p,q)} \rightarrow \mathcal{X}^{(p,q)}, \tag{3.24}
\]
defined by
\[
\uparrow \partial_i (m \otimes \tilde{d} \otimes \tilde{c}) = m \otimes \tilde{d} \otimes c^0 \otimes \cdots \otimes \Delta (c^i) \otimes \cdots \otimes c^q,
\]
\[
\uparrow \partial_{q+1} (m \otimes \tilde{d} \otimes \tilde{c}) = m_{<\sigma>} \otimes S^{-1} (c^0 (1)_{<t>}) \tilde{d} \otimes c^0 (2) \otimes c^1 \otimes \cdots \otimes c^q \otimes m_{<\sigma>} \otimes S^{-1} (c^1 (2)_{<t>}),
\]
\[
\uparrow \sigma_j (m \otimes \tilde{d} \otimes \tilde{c}) = m \otimes \tilde{d} \otimes m \otimes c^0 \otimes \cdots \otimes \varepsilon (c^j) \otimes \cdots \otimes c^q,
\]
\[
\uparrow \tau (m \otimes \tilde{d} \otimes \tilde{c}) = m_{<\sigma>} \otimes S^{-1} (c_{<t>}) \cdot \tilde{d} \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^q \otimes m_{<\sigma>} \otimes S^{-1} (c^0_{<t>}).
\]

**Lemma 3.5.** The horizontal and vertical operators defined in (3.19), . . . , (3.24) are well-defined and the \( \tau \)-operators are invertible.

**Proof.** In view of Lemma 3.4, \( M \otimes C_{\otimes q} \) is A YD module and, since the \( q \)th row of the above bigraded complex is the Hopf cyclic complex of \( \mathcal{K} \) with coefficients in \( M \otimes C_{\otimes q+1} \), all horizontal operators are well-defined [17]. By contrast, the columns are not Hopf cyclic modules of coalgebras in general, except in some special cases such as the case in Subsection 3.4.1. Let us check that the vertical \( \tau \)-operator is well-defined, which implies that
all the others are well-defined. One has

\[ \uparrow \tau (mh \otimes \tilde{d} \otimes c^0 \otimes \ldots \otimes c^q) = \]

\[ (mh)_{<\tilde{\sigma}_{<1}>} \otimes S^{-1}(c^0_{<1>}) \tilde{d} \otimes c^1 \otimes \ldots \otimes c^n \otimes (mh)_{<\tau_{<1>}}c^0_{<0>} = \]

\[ m_{<\tilde{\sigma}_{>}}h_{(2)} \otimes S^{-1}(c^0_{<1>}) \tilde{d} \otimes c^1 \otimes \ldots \otimes c^n \otimes S(h_{(3)})m_{<\tau_{>}}h_{(1)}c^0_{<0>} = \]

\[ m_{<\tilde{\sigma}_{>}} \otimes h_{(2)}S^{-1}(c^0_{<1>}) \tilde{d} \otimes h_{(3)}c^1 \otimes \ldots \otimes h_{(n+2)}c^n \otimes m_{<\tau_{>}}h_{(1)}c^0_{<0>} = \]

\[ m_{<\tilde{\sigma}_{>}} \otimes S^{-1}((h_{(2)}c^0_{<1>})S(h_{(3)}))h_{(1)}\tilde{d} \otimes h_{(5)}c^1 \otimes \ldots \otimes h_{(n+4)}c^n \otimes \]

\[ \otimes m_{<\tau_{>}}(h_{(3)}c^0_{<0>}) = \]

\[ m_{<\tilde{\sigma}_{>}} \otimes S^{-1}((h_{(2)}c^0_{<1>})h_{(1)}\tilde{d} \otimes h_{(5)}c^1 \otimes \ldots \otimes h_{(n+2)}c^n \otimes m_{<\tau_{>}}(h_{(2)})_{<0>} = \]

\[ \uparrow \tau (m \otimes h_{(1)}\tilde{d} \otimes h_{(2)}c^0 \otimes \ldots \otimes h_{(n+2)}c^n). \]

To show that the \( \tau \)-operators are invertible, we write explicitly their inverses:

\[ \uparrow \tau^{-1}(m \otimes \tilde{d} \otimes \tilde{c}) = \]

\[ m_{<\tilde{\sigma}_{>}} \otimes m_{<\tau_{>}}c^q_{<1>}S(m_{<\tau_{>}})\tilde{d} \otimes m_{<\tau_{>}}c^q_{<0>}c^0 \otimes \ldots \otimes c^{q-1}. \]

\[ \tilde{\tau}^{-1}(m \otimes \tilde{d} \otimes \tilde{c}) = m_{<\tilde{\sigma}_{>}} \otimes S(\tilde{c}_{<1>} \otimes m_{<\tau_{>}})d^q \otimes d^0 \otimes \ldots \otimes d^{q-1} \otimes \tilde{c}_{<0>}. \]

Using the invertibility of the antipode of \( \mathcal{H} \) and also the YD module property of \( C \), one checks that \( \uparrow \tau \circ \uparrow \tau^{-1} = \uparrow \tau^{-1} \circ \uparrow \tau = \text{Id} \) and \( \tilde{\tau} \circ \tilde{\tau}^{-1} = \tilde{\tau}^{-1} \circ \tilde{\tau} = \text{Id} \).

**Proposition 3.6.** Let \( C \) be a (co)module coalgebra and \( D \) a module coalgebra over \( \mathcal{H} \). Assume that \( C \) is an YD module over \( \mathcal{H} \), and \( M \) is an AYD over \( \mathcal{H} \). Then the bigraded module \( \mathcal{X}^{p,q} \) is a cylindrical module. If in addition \( M \otimes C^{\otimes q} \) is stable, then \( \mathcal{X}^{p,q} \) is bicyclic.

**Proof.** Lemma 3.4 shows that \( M \otimes C^{\otimes q} \) is AYD module, and since the \( q \)th row of the above bigraded complex is the Hopf cyclic complex of \( C^*_{\mathcal{H}}(K; M \otimes C^{\otimes q+1}) \), it defines a paracocyclic module [17]. The columns are not necessarily Hopf cyclic modules of coalgebras though, except in some special cases. However, one can show that the columns are paracocyclic modules. The verification of the fact that \( \uparrow \tau \), \( \uparrow \partial_i \) and \( \uparrow \sigma_j \) satisfy (3.1)...(3.3) is straightforward. The only nontrivial relations are those that involve \( \uparrow \tau \) and \( \uparrow \partial_{p+1} \), the others being the same as for cocyclic

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module associates to coalgebras. One needs thus to prove that

\[ \triangleright \tau \uparrow \partial_i = \uparrow \partial_{i-1} \uparrow \tau, \quad 1 \leq i \leq q + 1, \]
\[ \triangleright \tau \uparrow \partial_0 = \uparrow \partial_{q+1}, \]
\[ \triangleright \tau \uparrow \sigma_j = \uparrow \sigma_{j-1} \uparrow \tau, \quad 1 \leq j \leq q - 1, \]
\[ \triangleright \tau \uparrow \sigma_0 = \uparrow \sigma_{q-1} \uparrow \tau^2. \]

To verify these identities, first let \( 1 \leq i \leq q \); one has

\[ \triangleright \tau \uparrow \partial_i (m \otimes \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q) = \]
\[ \triangleright \tau (m \otimes \tilde{d} \otimes e^0 \otimes \ldots \otimes c^i (1) \otimes c^i (2) \otimes \ldots \otimes e^q) = \]
\[ m_{<\overline{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{d} \otimes e^i \otimes \ldots \otimes c^i (1) \otimes c^i (2) \otimes \ldots \otimes e^q \otimes m_{<\overline{\tau}>} c^0_{<0>} = \]
\[ \triangleright \partial_{i-1} (m_{<\overline{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{d} \otimes e^1 \otimes \ldots \otimes e^q \otimes m_{<\overline{\tau}>} c^0_{<0>}) = \]
\[ \triangleright \partial_{i-1} \triangleright \tau (m \otimes \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q). \]

Next let \( i = q + 1 \); by using the fact that \( C \) is \( \mathcal{H} \) module coalgebra, one obtains

\[ \triangleright \tau \uparrow \partial_{q+1} (m \otimes \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q) = \]
\[ \triangleright \tau (m_{<\overline{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{d} \otimes e^q \otimes \ldots \otimes c^0_{<0>}) = \]
\[ m_{<\overline{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q \otimes m_{<\overline{\tau}>} c^0_{<1>} = \]
\[ m_{<\overline{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q \otimes m_{<\overline{\tau}>} c^0_{<0>} = \]
\[ \triangleright \partial_q (m_{<\overline{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q \otimes m_{<\overline{\tau}>} c^0_{<0>}) = \]
\[ \triangleright \partial_q \triangleright \tau (m \otimes \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q). \]

Finally let \( i = 0 \); one has

\[ \triangleright \tau \uparrow \partial_0 (m \otimes \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q) = \]
\[ \triangleright \tau (m \otimes \tilde{d} \otimes e^0 (1) \otimes e^0 (2) \otimes e^1 \otimes \ldots \otimes e^q) = \]
\[ m_{<\overline{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{d} \otimes e^0 (2) \otimes e^1 \otimes \ldots \otimes e^q \otimes m_{<\overline{\tau}>} c^0_{<1>} = \]
\[ \triangleright \partial_{q+1} (m \otimes \tilde{d} \otimes e^0 \otimes \ldots \otimes e^q). \]
The other identities are checked in a similar fashion. We next show that the vertical operators commute with horizontal operators. Using Lemma \([3.5]\) and since in any parcocyclic module with invertible \(\tau\) -operator one has

\[
\partial_j = \tau^{-j}\partial_0\tau^j, \quad 1 \leq j \leq n, \quad \sigma_i = \tau^{-i}\sigma_{n-i}\tau^i, \quad 1 \leq i \leq n - 1, \quad (3.25)
\]

it suffices to verify the identities

\[
\begin{align*}
\uparrow\tau\overrightarrow{\partial_0} &= \overrightarrow{\partial_0}\uparrow\tau, \quad \uparrow\tau\overrightarrow{\partial_0} = \overrightarrow{\partial_0}\uparrow\tau, \quad \tau \uparrow\partial_0 = \partial_0\tau, \\
\uparrow\tau\overrightarrow{\sigma}_{p-1} &= \overrightarrow{\sigma}_{p-1}\uparrow\tau, \quad \tau \uparrow\sigma_{q-1} = \sigma_{q-1}\tau.
\end{align*}
\]

Let us check the first; one has

\[
\begin{align*}
\uparrow\tau\overrightarrow{(m \otimes k^0 \otimes \ldots \otimes k^p \otimes c^0 \otimes \ldots \otimes c^q)} &= \\
\uparrow\tau(m_{<\overrightarrow{\sigma}>} \otimes k^1 \otimes \ldots \otimes k^p \otimes \tilde{c}_{<1>} m_{<\overrightarrow{\tau}>} k^0 \otimes \tilde{c}_{<0>}) &= m_{<\overrightarrow{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot (k^1 \otimes \ldots \otimes k^p) \otimes c^1_{<1>} \ldots c^q_{<1>} m_{<\overrightarrow{\tau}>} k^0 \otimes \\
\otimes c^1_{<0>} \ldots \otimes c^q_{<0>} \otimes m_{<\overrightarrow{\tau}>} c^0_{<\overrightarrow{\sigma}>} &= m_{<\overrightarrow{\sigma}>} \otimes S^{-1}(c^0_{<2>}) \cdot (k^1 \otimes \ldots \otimes k^p) \otimes \\
\otimes c^1_{<1>} \ldots c^q_{<1>} \cdot (m_{<\overrightarrow{\tau}>} c^0_{<0>})_{<1>} m_{<\overrightarrow{\tau}>} S^{-1}(c^0_{<2>}) k^0 \otimes \\
\otimes c^1_{<0>} \ldots \otimes c^q_{<0>} \otimes (m_{<\overrightarrow{\tau}>} c^0_{<0>})_{<0>} &= \\
\overrightarrow{\tau}(m_{<\overrightarrow{\sigma}>} \otimes S^{-1}(c^0_{<1>}) \cdot \tilde{k} \otimes c^1 \ldots \otimes c^q \otimes m_{<\overrightarrow{\tau}>} c^0) &= \\
\overrightarrow{\tau}\uparrow\tau(m \otimes k^0 \otimes \ldots \otimes k^p \otimes c^0 \otimes \ldots \otimes c^q)
\end{align*}
\]

Since \(C\) is a \(\mathcal{H}\)-module coalgebra one can write

\[
\begin{align*}
\uparrow\tau\overrightarrow{\partial_0}(m \otimes d^0 \otimes \ldots \otimes d^p \otimes c^0 \otimes \ldots \otimes c^q) &= \\
\uparrow\tau(m \otimes d^0_{(1)} \otimes d^0_{(2)} \otimes d^1 \otimes \ldots \otimes d^p \otimes c^0 \otimes \ldots \otimes c^q) &= \\
(m_{<\overrightarrow{\sigma}>} \otimes S^{-1}(c^0_{<1>})(d^0_{(1)} \otimes d^0_{(2)} \otimes d^0 \ldots \\
\ldots \otimes d^p) \otimes c^1 \otimes \ldots \otimes c^q \otimes m_{<\overrightarrow{\tau}>} c^0) &= \\
\overrightarrow{\partial_0}\uparrow\tau(m \otimes d^0 \otimes \ldots \otimes d^p \otimes c^0 \otimes \ldots \otimes c^q).
\end{align*}
\]
To show the $\tilde{\tau} \uparrow \partial_0 = \uparrow \partial_0 \tilde{\tau}$ one uses only the module coalgebra property of $C$; thus,

$$\tilde{\tau} \uparrow \partial_0 (m \otimes d \otimes c) = \tilde{\tau} (m \otimes d \otimes c^0 (1) \otimes c^0 (2) \otimes c^1 \otimes \ldots \otimes c^q) =$$

$$m_{<\tilde{\sigma}>} \otimes d^1 \otimes \ldots \otimes d^p \otimes c^0 (1)_{<1>} \> c^0 (2)_{<1>} \> c^1_{<1>} \ldots$$

$$\ldots c^q_{<1>} m_{<\tilde{\sigma}>} \otimes d^1 \otimes \ldots \otimes d^p \otimes c^0 _{<1>} \> c^0 _{<0>} \> c^1_{<0>} \otimes \ldots \otimes c^q_{<0>} =$$

$$m_{<\tilde{\sigma}>} \otimes d^1 \otimes \ldots \otimes d^p \otimes c^0 _{<1>} \> c^0 _{<1>} \> c^1_{<1>} \ldots$$

$$\ldots c^q_{<1>} m_{<\tilde{\sigma}>} \otimes d^1 \otimes c^0 _{<0>} (1) \otimes c^0 _{<0>}(2) \otimes c^1_{<0>} \otimes \ldots \otimes c^q_{<0>} =$$

$$\uparrow \partial_0 \tilde{\tau} (m \otimes d \otimes c).$$

The remaining relations, $\tilde{\tau} \uparrow \sigma_{q-1} = \uparrow \sigma_{q-1} \tilde{\tau}$ and $\uparrow \tilde{\tau} \sigma_{p-1} = \sigma_{p-1} \uparrow \tilde{\tau}$, are obviously true.

Finally, by using the stability of $M$, we verify the cylindrical condition $\tilde{\tau}^{p+1} \uparrow \tau^{q+1} = \Id$ as follows:

$$\tilde{\tau}^{p+1} \uparrow \tau^{q+1} (m \otimes d^0 \otimes \ldots \otimes d^p \otimes c^0 \otimes \ldots \otimes c^q) =$$

$$\tilde{\tau}^{p+1} (m_{<\tilde{\sigma}>} \otimes S^{-1}(c^0_{<1>} \ldots c^q_{<1>}) \cdot (d^0 \otimes \ldots \otimes d^p) \otimes$$

$$\otimes m_{<\tilde{\sigma}>} \otimes m_{<\tilde{\sigma}>} \otimes \ldots \otimes m_{<\tilde{\sigma}>} (c^0 \otimes \ldots \otimes c^q) =$$

$$m \otimes d^0 \otimes \ldots \otimes d^p \otimes c^0 \otimes \ldots \otimes c^q.$$
Proposition 3.7. The map defined in (3.29) establishes a cyclic isomorphism between the complex $\mathcal{C}_t^*(C \triangleright D; M)$ and the diagonal of $\mathcal{X}_t^*$. 

Proof. First we show that the above map is well-defined. The fact that $C$ is YD module helps in two ways: firstly, Lemma 3.2 shows that $\mathcal{H}$ is acting diagonally on $C \triangleright D$, and secondly the twisting $\top : C \otimes D \to D \otimes C$ where $\top(c \otimes d) = c_{<1_>} d \otimes c_{<0_>}$, is $\mathcal{H}$-linear; indeed,

$$
\top((h(1)c \otimes (2)d)) = (h(1)c)_{<1>} (h(2)d) \otimes (h(1))_{<0>} = 
$$

$$
h(1)c_{<1>} S((h(3)h(4)d) \otimes h(2)c_{<0>} =
$$

$$
h(1)c_{<1>} d \otimes h(2)c_{<0>} .
$$

This ensures that $\Psi$ is well-defined, because it is obtained out of $\top$ by iteration.

In order to prove that $\Psi$ is a cyclic map, it suffices to check that $\Psi$ commutes with the $\tau$-operators, the first coface, and the last codegeneracy, because the rest of the operators are made of these (3.25). One verifies that $\Psi$ commutes with cyclic operators as follows. On the one hand,

$$
\Psi\top C_{D}(m \otimes c^0 \triangleright d^0 \otimes \ldots \otimes c^n \triangleright d^n) =
$$

$$
\Psi(m \otimes c^0 \triangleright d^0 \otimes \ldots \otimes c^n \triangleright d^n m \otimes (c^0 < m_{<2>} d^0) =
$$

$$
m_{<0>} \otimes c^1 \otimes c^1 \otimes \ldots \otimes c^n \otimes d^1 \otimes \ldots \otimes c^n \otimes d^n \otimes \otimes c^1 \otimes \ldots \otimes c^n \otimes d^1 \otimes \ldots \otimes c^n \otimes d^n \otimes m_{<0>} \otimes \ldots \otimes c^n \otimes d^1 \otimes \ldots \otimes c^n \otimes d^n \otimes m_{<0>} \otimes \ldots \otimes c^n \otimes d^1 \otimes \ldots \otimes c^n \otimes d^n .
$$

On the other hand,

$$
\mathcal{T} \top \Psi(m \otimes c^0 \triangleright d^0 \otimes \ldots \otimes c^n \triangleright d^n) =
$$

$$
\mathcal{T} \top (m \otimes c^0 \otimes d^0 \otimes \ldots \otimes c^n \otimes d^n) =
$$

$$
\mathcal{T}(m \otimes c^0 \otimes d^0 \otimes \ldots \otimes c^n \otimes d^n) =
$$

$$
m_{<0>} \otimes d^1 \otimes \ldots \otimes d^n \otimes m_{<0>} \otimes \ldots \otimes d^n .
$$
The next to check is the equality $\partial_0 \Psi = \Psi \partial_0 C \triangleright D$. Using the fact that $C$ is a module coalgebra, one has

$$
\partial_0 \Psi(m \otimes m \otimes c^0 \triangleright d^0 \otimes \ldots \otimes c^n \triangleright d^n) = $$

$$
\partial_0(m \otimes c^0_{<n-1>} d^0 \otimes \ldots \otimes c^n_{<n-1>} \ldots c^0_{<1>} d^0 \otimes c^0_{<0>} \otimes \ldots \otimes c^n_{<0>}) = $$

$$m \otimes c^0_{<n-2>} d^0_{(1)} \otimes c^0_{<n-1>} d^0_{(2)} \otimes \ldots \otimes c^0_{<1>} d^0_{(n)} \otimes c^0_{<0>}(1) \otimes c^0_{<0>}(2) \otimes c^1_{<0>} \otimes \ldots \otimes c^n_{<0>} = $$

$$m \otimes c^0_{<n-2>} c^0_{(2)} c^0_{<n-2>} d^0_{(1)} \otimes c^0_{(1)} c^0_{<n-1>} c^0_{(2)} c^0_{<n-1>} d^0_{(2)} \otimes \ldots \otimes c^0_{(1)} c^0_{(2)} c^0_{<n-1>} d^0_{(1)} \otimes c^0_{(2)} c^0_{<n-1>} d^0_{(2)} \otimes \ldots \otimes c^n_{<0>}.$$

On the other hand,

$$\Psi \partial_0 C \triangleright D(m \otimes c^0 \triangleright d^0 \otimes \ldots \otimes c^n \triangleright d^n) = $$

$$\Psi(m \otimes c^0_{(1)} \triangleright c^0_{(2)}_{<n-1>} d^0_{(1)} \otimes c^0_{(2)}_{<0>} \triangleright d^0_{(2)} \otimes c^1 \triangleright d^1 \otimes \ldots \otimes c^n \triangleright d^n) = $$

$$m \otimes c^0_{(1)} c^0_{<n-2>} c^0_{(2)} c^0_{<n-2>} d^0_{(1)} \otimes c^0_{(1)} c^0_{<n-1>} c^0_{(2)} c^0_{<n-1>} d^0_{(2)} \otimes c^0_{(1)} c^0_{<n-1>} c^0_{(2)} c^0_{<n-1>} d^0_{(1)} \otimes c^0_{(2)} c^0_{<n-1>} d^0_{(2)} \otimes \ldots \otimes c^0_{(1)} c^0_{(2)} c^0_{<n-1>} d^0_{(1)} \otimes c^0_{(2)} c^0_{<n-1>} d^0_{(2)} \otimes \ldots \otimes c^n_{<0>}. $$

The equality $\sigma_{n-1} \Psi = \Psi \sigma_{n-1} C \triangleright D$ is obvious. In order to show that $\Psi$ is an isomorphism, one constructs again an iterated map made out of factors $\triangleright: D \otimes C \rightarrow C \otimes D$, defined by $\triangleright (d \otimes c) = c_{<0>} \otimes S^{-1}(c_{<n-1>}) d$. It is easy to see that $\top$ and $\triangleright$ are inverse to one another. Explicitly the inverse of $\Psi$ is defined by

$$\Psi^{-1}(m \otimes d^0 \otimes \ldots \otimes d^n \otimes c^0 \otimes \ldots \otimes c^n) = $$

$$m \otimes c^0_{<0>} \triangleright S^{-1}(c^0_{<n-1>}) d^0 \otimes \ldots \otimes c^n_{<0>} \triangleright S^{-1}(c^0_{<n-1>} c^1_{<n-1>} \ldots c^n_{<n-1>}) d^n $$

\[ \square \]

Applying now the cyclic version of the Eilenberg-Zilber theorem \[12\] one obtains the sought-for quasi-isomorphism of mixed complexes.
Proposition 3.8. The mixed complexes \((C^*_{\mathcal{H}}(D \blacktriangleright C, M), b, B)\) and \((\text{Tot}(\mathcal{X}), b_T, B_T))\) are quasi-isomorphic.

The rest of this section is devoted to showing that the Hopf algebras that make the object of this paper satisfy the conditions of the above proposition. To put this in the proper setting, we let \(\mathcal{H}\) be a Hopf algebra, and we consider a pair of \(\mathcal{H}\)-module coalgebras \(C, D\) such that \(C\) is \(\mathcal{H}\)-module coalgebra and via its action and coaction it is an YD module over \(\mathcal{H}\). Let \(\mathcal{L} \subset \mathcal{K}\) be Hopf subalgebras of \(\mathcal{H}\). One defines the coalgebra \(\mathcal{C} := \mathcal{H} \otimes_{\mathcal{K}} \mathbb{C}\), where \(\mathcal{K}\) acts on \(\mathcal{H}\) by multiplication and on \(\mathcal{C}\) via counit (cf [9, §5]). In the same fashion one defines the coalgebra \(\mathcal{K}_L := \mathcal{K} \otimes_{\mathcal{L}} \mathbb{C}\). If \(h \in \mathcal{H}\) and \(\hat{c}\) be its class in \(\mathcal{C}\), then

\[
\Delta(\hat{c}) = \hat{c}_1 \otimes \hat{c}_2 := \hat{h}(1) \otimes \hat{h}(2), \quad \varepsilon(\hat{c}) := \varepsilon(h)
\]

This coalgebra has a natural coaction from \(\mathcal{H}\).

\[
\nabla(c) = c_{<1>} \otimes c_{<0>} := h(1)_S(h(3)) \otimes \hat{h}(2).
\]

Lemma 3.9. The above action and coaction are well-defined and make \(\mathcal{C}\) a (co)module coalgebra, respectively. In addition both make \(\mathcal{C}\) an YD module over \(\mathcal{H}\).

Proof. First let us check that the action and coaction are well defined. With \(h, g \in \mathcal{H}\), and \(k \in \mathcal{K}\), one has \(h(gk \otimes 1) = hgk \otimes 1 = hg \otimes 1 \varepsilon(k) = h(g \otimes 1)\), which verifies the claim for the action. For the coaction, we write

\[
\nabla(hk \otimes 1) = h(1)_k S(k(3)) S(h(3)) \otimes (h(2) \otimes k(2) \otimes 1) = h(1)_k S(k(3)) S(h(3)) \otimes (h(2) \otimes k(2) \varepsilon(k(2)))) = h(1)_k S(k(3)) S(h(3)) \otimes (h(2) \otimes 1) = h(1) S(h(3)) \otimes (h(2) \otimes k \varepsilon(k)) = \nabla(h \otimes k).\]

It is obvious that these are indeed action, resp. coaction, and thus define a module coalgebra structure, resp. a module coalgebra structure over \(\mathcal{C}\). One checks that the action and coaction satisfy (3.17) as follows:

\[
\nabla(h(\hat{g})) = \nabla(h(g)) = h(1)_g S(g(3)) S(h(3)) \otimes h(2), g(2) = h(1) (g(1)_S(g(3))) S(h(3)) \otimes h(2), g(2).
\]

\(\square\)
Let us assume that $\mathcal{H}$ acts on $\mathcal{K}$, as well and via this action $\mathcal{K}$ is a module coalgebra. Similarly to $\mathcal{C}$ in relation to $\mathcal{H}$, $\mathcal{K}_L$ inherits in a natural way an action from $\mathcal{H}$. This makes $\mathcal{K}_L$ a module coalgebra over $\mathcal{H}$. Precisely, if $h \in \mathcal{H}$ and $k \in \mathcal{K}$, denoting $k \otimes L^1$ by $\tilde{k}$, one defines

$$h \cdot \tilde{k} = hk \otimes L^1 = \tilde{hk}.$$ 

Now $\mathcal{C}$ and $\mathcal{K}_L$ together with their action and coaction satisfy all conditions of Proposition 3.6. As a result one can form the cylindrical module $X(\mathcal{H}, \mathcal{C}, \mathcal{K}_L; M)$. Due to the special properties discussed above, we may expect some simplification. Indeed, let

$$\mathcal{Y}^{p,q} = M \otimes_{\mathcal{K}} \mathcal{K}_L^{\otimes p+1} \otimes \mathcal{C}^{\otimes q}$$

We define the following map from $\mathcal{X}$ to $\mathcal{Y}$:

$$\Phi_1 : \mathcal{X}^{p,q} \rightarrow \mathcal{Y}^{p,q},$$

(3.30)

$$\Phi_1(m \otimes_{\mathcal{H}} \tilde{k} \otimes h^0 \otimes \ldots \otimes h^n) = mh^0_{(2)} \otimes_{\mathcal{K}} S^{-1}(h^0_{(1)}) \cdot \tilde{k} \otimes S(h^0_{(3)}) \cdot (h^1 \otimes \ldots \otimes h^q).$$

Lemma 3.10. The map $\Phi_1$ defined in (3.30) is a well-defined isomorphisms of vector spaces.

Proof. To check that it is well-defined, we clarify the ambiguities in the definition of $\Phi_1$ as follows. Let $k \in \mathcal{K}$, $\tilde{k} \in \mathcal{K}_L^{\otimes (p+1)}$, and $g, h^0 \ldots h^q \in \mathcal{H}$; then

$$\Phi_1(m \otimes \tilde{k} \otimes (h^0 k \otimes_{\mathcal{K}} 1) \otimes h^1 \otimes \ldots \otimes h^q)) =$$

$$mh^0_{(2)} k_{(2)} \otimes_{\mathcal{K}} S^{-1}(h^0_{(1)} k_{(1)}) \cdot \tilde{k} \otimes S(h^0_{(3)} k_{(3)}) \cdot (h^1 \otimes \ldots \otimes h^q) =$$

$$mh^0_{(2)} \otimes_{\mathcal{K}} k_{(2)} S^{-1}(k_{(1)}) S^{-1}(h^1_{(1)}) \cdot \tilde{k} \otimes k_{(3)} S(k_{(4)}) S(h^0_{(3)}) \cdot (h^1 \otimes \ldots \otimes h^q) =$$

$$\Phi_1(m \otimes \tilde{k} \otimes (h^0 \otimes_{\mathcal{K}} \epsilon(k)) \otimes h^1 \otimes \ldots \otimes h^n)).$$

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Also,
\[ \Phi_1(m \otimes g(1) \cdot \tilde{k} \otimes g(2) \cdot (\hat{h}^0 \otimes \hat{h}^1 \otimes \ldots \otimes \hat{h}^q)) = \]
\[ m \otimes_K g(1) \cdot \tilde{k} \otimes g(2) \cdot (\hat{h}^0 \otimes \hat{h}^1 \otimes \ldots \otimes \hat{h}^q) = \]
\[ mg(3) h^0 (2) \otimes_K S^{-1}(h^0 (1)) S^{-1}(g(2)) g(1) \cdot \tilde{k} \otimes \]
\[ \otimes S(h^0 (3)) S(g(4)) g(5) \cdot (\hat{h}^1 \otimes \ldots \otimes \hat{h}^q) = \]
\[ mgh^0 (2) \otimes_K S^{-1}(h^0 (1)) : \tilde{k} \otimes S(h^0 (3)) \cdot (\hat{h}^1 \otimes \ldots \otimes \hat{h}^q) = \]
\[ \Phi_1(mg \otimes \tilde{k} \otimes \hat{h}^0 \otimes \hat{h}^1 \otimes \ldots \otimes \hat{h}^q). \]

One easily checks that the following map defines an inverse for \( \Phi_1 \):
\[ \Phi_1^{-1} : \mathcal{Y}^{p,q} \rightarrow \mathcal{X}^{p,q}, \]
\[ \Phi_1^{-1}(m \otimes_K \tilde{k} \otimes \hat{h}) = m \otimes \hat{h} \].

We next push forward the cylindrical structure of \( \mathcal{X} \) to get the following cylindrical structure on \( \mathcal{Y} \):
\[ \partial_1(m \otimes \tilde{k} \otimes \hat{h}) = m \otimes \hat{h}^0 \otimes \ldots \Delta(\hat{h}^i) \otimes \ldots \otimes \hat{h}, \]
\[ \partial_{p+1}(m \otimes \tilde{k} \otimes \hat{h}) = m_{\langle \sigma \rangle} \otimes \hat{h}^0 (2) \otimes \ldots \otimes \hat{h}^p \otimes \hat{h}_{<1 >} \cdot m_{\langle \sigma \rangle} \hat{h}^0_{(1)} \otimes \hat{h}_{<0 >}, \]
\[ \sigma_j(m \otimes \tilde{k} \otimes \hat{h}) = m \otimes \hat{h}^1 \otimes \ldots \otimes \varepsilon(\hat{h}^j) \otimes \ldots \otimes \hat{h}^p \otimes \hat{h}, \]
\[ \tau(m \otimes \tilde{k} \otimes \hat{h}) = m_{\langle \sigma \rangle} \otimes \hat{h}^1 \otimes \ldots \hat{h}^p \otimes \hat{h}_{<1 >} \cdot m_{\langle \sigma \rangle} \hat{h}^0 \otimes \hat{h}_{<0 >}. \]

To obtain a further simplification, we assume that the action of \( \mathcal{K} \subset \mathcal{H} \) on \( \mathcal{K} \) coincides with the multiplication by \( \mathcal{K} \). We define a map from \( \mathcal{Y}^{p,q} \) to
\[ \mathcal{Y}^{p,q} := \mathcal{Y}^{p,q}(\mathcal{H}, \mathcal{K}, \mathcal{L}; M) := M \otimes_{\mathcal{L}} \mathcal{K}^p \otimes C^q, \]

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as follows:

\[ \Phi_2 : \mathcal{Y}^{p,q} \rightarrow \mathcal{Z}^{p,q}, \]

\[ \Phi_2(m \otimes k \cdot (\hat{k}^0 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h})) = mk^0_1 \otimes S(k^0_2) \cdot (\hat{k}^1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}). \]

**Lemma 3.11.** The map \( \Phi_2 \) defined in (3.31) is a well-defined isomorphism of vector spaces.

**Proof.** One has

\[ \Phi_2(m \otimes k \cdot (\hat{k}^0 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h})) = \]

\[ \Phi_2(m \otimes k_1 \cdot \hat{k}^0_1 \otimes k_2 \cdot (\hat{k}^1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h})) = \]

\[ mk_1(0) \otimes S(k_2) \cdot (\hat{k}^1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}) = \]

\[ \Phi_2(mk \otimes \hat{k}^0_1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}). \]

The inverse of \( \Phi_2 \) is given by

\[ \Phi^{-1}_2 : \mathcal{Z}^{p,q} \rightarrow \mathcal{Y}^{p,q}, \]

\[ \Phi^{-1}_2(m \otimes \hat{k}^1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}^1 \otimes \ldots \otimes \hat{h}^q) = \]

\[ m \otimes k \cdot \hat{1} \otimes \hat{k}^1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}^1 \otimes \ldots \otimes \hat{h}^q. \]

\[ \square \]

We now push forward the cylindrical structure of \( \mathcal{Y} \) on \( \mathcal{Z} \) to get the following operators on \( \mathcal{Z}^{*,*} \):

\[ \varnothing (m \otimes \hat{k} \otimes \hat{h}) = m \otimes \hat{1} \otimes \hat{k}^1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}, \]

\[ \varnothing (m \otimes \hat{k} \otimes \hat{h}) = m \otimes \hat{k}^1 \otimes \ldots \otimes \Delta(\hat{k}^i) \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}, \]

\[ \varnothing (m \otimes \hat{k} \otimes \hat{h}) = m \otimes \hat{k}^1 \otimes \ldots \otimes \hat{k}^p \otimes \hat{h} \otimes \hat{h}_{=0}, \]

\[ \varnothing (m \otimes \hat{k} \otimes \hat{h}) = m \otimes k^{i,j} \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}, \]

\[ \varnothing (m \otimes \hat{k} \otimes \hat{h}) = m \otimes k^{i,j} \otimes \ldots \otimes \hat{k}^p \otimes \hat{h} \otimes \hat{h}_{=0}, \]

\[ \varnothing (m \otimes \hat{k} \otimes \hat{h}) = m \otimes \hat{k}^{i,j} \otimes \hat{k}^{i+1} \otimes \ldots \otimes \hat{k}^p \otimes \hat{h}, \]

\[ \varnothing (m \otimes \hat{k} \otimes \hat{h}) = m \otimes \hat{k}^{i,j} \otimes \hat{k}^{i+1} \otimes \ldots \otimes \hat{k}^p \otimes \hat{h} \otimes \hat{h}_{=0}. \]
\[ \uparrow \partial_0 (m \otimes \tilde{k} \otimes \tilde{h}) = m \otimes \tilde{k} \otimes \hat{1} \otimes h^1 \otimes \ldots \otimes h^q, \]
\[ \uparrow \partial_i (m \otimes \tilde{k} \otimes \tilde{h}) = m \otimes \tilde{k} \otimes \hat{h}^i_0 \otimes \ldots \otimes \Delta(h^i) \otimes \ldots \otimes \hat{h}^q, \]
\[ \uparrow \partial_{q+1} (m \otimes \tilde{k} \otimes \tilde{h}) = m_{\prec \sigma} \otimes \tilde{k} \otimes \hat{h}^1 \otimes \ldots \otimes \hat{h}^q \otimes m_{\prec \tau}, \]
\[ \uparrow \sigma_j (m \otimes \tilde{k} \otimes \tilde{h}) = m \otimes \tilde{k} \otimes \hat{h}^1 \otimes \ldots \otimes \varepsilon(\hat{h}^{j+1}) \otimes \ldots \otimes \hat{h}^q, \]
\[ \uparrow \tau (m \otimes \tilde{k} \otimes \tilde{h}) = m_{\prec \tau} h^{1,(4)} S^{-1}(h^{1,(3)} \cdot 1_{\mathcal{K}}) \otimes S(S^{-1}(h^{1,(2)} \cdot 1_{\mathcal{K}}) \cdot (\hat{h}^1 \otimes \ldots \otimes \hat{h}^q \otimes m_{\prec \tau}). \]

**Lemma 3.12.** The above operators defined on $\mathfrak{A}$ are well-defined and yield a cylindrical module.

**Proof.** The second part of the lemma holds by the very definition, in view of the fact that $\mathfrak{X}$ is cylindrical module. We check the first claim for $\uparrow \tau$ and $\tilde{\tau}$, for the other operators being obviously true. Since $M$ is AYD and $\mathcal{C}$ is YD, we have

\[ \tilde{\tau} (m \otimes l \cdot (\tilde{k} \otimes \tilde{h})) = \]
\[ m_{\prec \sigma} l^{(1)} k^{1,(1)} \otimes S(l^{(2)} k^{1,(2)}) \cdot (l^{(3)} k^2 \otimes \ldots \otimes l^{(p+1)} k^p \otimes \]
\[ (l^{(p+2)} \hat{h})_{<_{-1}>} m_{\prec \tau} \otimes (l^{(p+2)} \tilde{h})_{<_{0}>}) = \]
\[ m_{\prec \tau} l^{(1)} k^{1,(1)} \otimes S(l^{(2)} k^{1,(2)}) \cdot (l^{(3)} k^2 \otimes \ldots \otimes l^{(p+1)} k^p \otimes \]
\[ l^{(p+2)} \hat{h}^{<_{-1}>} S(l^{(p+4)} m_{\prec \tau} \otimes (l^{(p+3)} \tilde{h}^{<_{0}>}) = \]
\[ m_{\prec \sigma} l^{(1)} k^{1,(1)} \otimes S(k^{1,(2)}) \cdot (k^2 \otimes \ldots \otimes k^p \otimes \hat{h}^{<_{-1}>} S(l^{(2)} m_{\prec \tau} \otimes \tilde{h}^{<_{0}>}) = \]
\[ m_{\prec \sigma} l^{(2)} k^{1,(1)} \otimes S(k^{1,(2)}) \cdot (k^2 \otimes \ldots \otimes k^p \otimes \hat{h}^{<_{-1}>} S(l^{(3)} m_{\prec \tau} \otimes l^{(4)} \otimes \tilde{h}^{<_{0}>}) = \]
\[ \tilde{\tau} (ml \otimes \tilde{k} \otimes \tilde{h}). \]
For the vertical cyclic operator one only uses the AYD property of $M$; thus,

\[
\begin{align*}
\uparrow \tau (m \otimes l \cdot (\tilde{k} \otimes \tilde{h})) &= m_{\triangleleft \triangleright} l_+(l_1 h_{(4)}) S^{-1}(l_4 h_{(3)} \cdot 1_K) \otimes \mathcal{L} \\
S(S^{-1}(l_3 h_{(2)} \cdot 1_K) \cdot \left( S^{-1}(l_2 h_{(1)}) \cdot l_1 \tilde{k} \otimes S(l_6 h_{(5)}) \cdot (l_7 \tilde{h}^2 \otimes \\
\ldots \otimes l_{(n+g)} \tilde{h}^q \otimes m_{\triangleleft \triangleright} \right)) &= m_{\triangleleft \triangleright} l_1 h_{(4)} S^{-1}(h_{(3)} \cdot 1_K) \otimes \mathcal{L} \\
S(S^{-1}(h_{(2)} \cdot 1_K) \cdot \left( S^{-1}(h_{(1)}) \cdot \tilde{k} \otimes S(h_{(5)}) \cdot (\tilde{h}^2 \otimes \\
\ldots \otimes \tilde{h}^q \otimes S(l_3) m_{\triangleleft \triangleright} \right)) =
\end{align*}
\]

\[
\uparrow \tau (m l \cdot \tilde{k} \otimes \tilde{h}).
\]

\[\square\]

**Proposition 3.13.** The following map defines an isomorphism of cocyclic modules,

\[
\Theta := \Phi_2 \circ \Phi_1 \circ \Psi : C_{\mathcal{H}}^* (\mathcal{C} \triangleright \mathcal{K} \lhd \mathcal{L}, M) \to \mathcal{Z}^* \mathcal{L},
\]

where $\Psi, \Phi_1,$ and $\Phi_2$ are defined in (3.29), (3.30) and (3.31) respectively.

**Proof.** The maps $\Phi_2, \Phi_1,$ and $\Psi$ are isomorphisms. \(\square\)

Define now the map

\[
\Phi : \mathcal{H} \otimes \mathcal{L} 1 \to \mathcal{C} \triangleright \mathcal{K} \lhd \mathcal{L}, \quad \text{by the formula}
\]

\[
\Phi(h \otimes \tilde{1}) = \dot{h}_{(1)} \triangleright \dot{h}_{(2)} \cdot 1_K = (h_{(1)} \otimes 1_K 1) \triangleright (h_{(2)} \cdot 1_K \otimes 1_K). \tag{3.33}
\]

**Proposition 3.14.** The map $\Phi$ is a map of $\mathcal{H}$-module coalgebras.

**Proof.** Using that if $h \in \mathcal{H}$ and $l \in \mathcal{L}$ then $(hl) \cdot 1_K = (h \cdot 1_K)l$, one checks that $\Phi$ is well-defined, as follows:

\[
\begin{align*}
\Phi(hl \otimes 1) &= (h_{(1)} l_{(1)} \otimes 1_K 1) \triangleright (h_{(2)} l_{(2)} \cdot 1_K \otimes 1_K 1) = \\
(h_{(1)} \otimes 1_K c(l_{(1)})) \triangleright ((h_{(2)} \cdot 1_K) l_{(2)} \otimes 1_K 1) = \\
(h_{(1)} \otimes 1_K c(l_{(1)})) \triangleright ((h_{(2)} \cdot 1_K) \otimes 1_K c(l_{(2)})) = \Phi(h \otimes c(l)).
\end{align*}
\]

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Next, using the coalgebra structure of $\mathcal{C} \triangleright \mathcal{K}_L$, one can write

$$\Delta(\Phi(h)) = \Delta(h_{(1)} \triangleright h_{(2)} \cdot 1) =$$

$$h_{(1)} \triangleright (h_{(2)} S(h_{(4)}) \cdot (h_{(5)} \cdot 1) \otimes h_{(6)} \cdot 1) =$$

$$h_{(1)} \triangleright (h_{(2)} S(h_{(4)}) h_{(5)} \cdot 1 \otimes h_{(6)} \cdot 1) =$$

$$h_{(1)} \triangleright (h_{(2)} \cdot 1 \otimes h_{(3)} \triangleright h_{(4)} \cdot 1) =$$

$$\Phi(h_{(1)}) \otimes \Phi(h_{(2)}).$$

Finally, the $\mathcal{H}$-linearity is trivial because $\mathcal{H}$ acts on $\mathcal{C} \triangleright \mathcal{K}_L$ diagonally. \hfill $\square$

Summing up, we conclude with the following result, which applies to a many cases of interest, in particular to those which make the main object of this paper.

**Theorem 3.15.** Assuming that $\Phi$ is an isomorphism, the Hopf cyclic complex of the Hopf algebra $\mathcal{H}$ relative to the Hopf subalgebra $\mathcal{L}$ with coefficients in SAYD $\mathcal{M}$ is quasi-isomorphic with the total complex of the mixed complex of $3(H, \mathcal{K}, \mathcal{L}; M)$.

As a matter of fact, the cylindrical modules $\mathfrak{Z}$ and $\mathfrak{Y}$ are often bicocyclic modules, for example if the SAYD $\mathcal{M}$ has the property that

$$m^<_{\triangleright} \otimes m^<_{\triangleright} \in \mathcal{K} \otimes M, \quad \forall m \in M;$$

in this case $M$ will be called $\mathcal{K}$-SAYD. In this paper we compute Hopf cyclic cohomology with coefficients in $\mathbb{C}_\delta$, which is obviously $\mathcal{K}$-SAYD for any Hopf subalgebra $\mathcal{K} \subset \mathcal{H}$.

**Proposition 3.16.** If $M$ is $\mathcal{K}$-SAYD module then $M \otimes \mathbb{C}^\otimes q$ is SAYD, and hence $\mathfrak{Z}$ and $\mathfrak{Y}$ are bicocyclic modules.

**Proof.** Only the stability condition remains to be proved. We check it only for $q = 1$, but the same proof works for all $q \geq 1$. By the very definition of the coaction,

$$\Delta(m \otimes h) = h_{(3)} S(h_{(5)}) m^<_{\triangleleft} \otimes m^<_{\triangleright} \otimes h_{(2)}$$

We need to verify the identity

$$(m \otimes h)^<_{\triangleright} (m \otimes h)^<_{\triangleleft} = m \otimes h.$$
The left hand side can be expressed as follows:

\[
(m \otimes h)_{\langle \tau_0 \rangle} (m \otimes h)_{\langle \tau_1 \rangle} = \\
m_{\langle \tau_0 \rangle} h_{(1)} S(h_{(3)}) m_{\langle \tau_1 \rangle} \otimes S(m_{\langle \tau_1 \rangle}) S^2(h_{(4)}) S(h_{(2)}) h_{(3)} = \\
m_{\langle \tau_0 \rangle} h_{(1)} S(h_{(3)}) m_{\langle \tau_1 \rangle} \otimes S(m_{\langle \tau_1 \rangle}) S^2(h_{(2)}).
\]

We now recall that the cyclic operator \(\tau_1 : M \otimes H \to M \otimes H\) is given by \(\tau_1(m \otimes h) = m_{\langle \tau_0 \rangle} h_{(1)} \otimes S(h_{(2)}) m_{\langle \tau_1 \rangle}\). Since

\[
\tau_2^2(m \otimes h) = \tau_1(m_{\langle \tau_1 \rangle} h_{(1)} \otimes S(h_{(1)}) m_{\langle \tau_1 \rangle}) = \\
m_{\langle \tau_0 \rangle} h_{(2)} S(h_{(5)}) m_{\langle \tau_1 \rangle} \otimes S(m_{\langle \tau_1 \rangle}) S^2(h_{(4)}) S(h_{(3)}) m_{\langle \tau_1 \rangle} h_{(1)},
\]

the desired equality simply follows from the facts that \(\tau_2^2 = \text{Id}\) and \(M\) is \(K\)-SAYD.

\[\square\]

### 3.3 Bicocyclic complex for primitive Hopf algebras

We now proceed to show that the Hopf algebras \(H(\Pi)\) associated to a primitive pseudogroups do satisfy all the requirements of the preceding subsection, and so Theorem 3.15 applies to allow the computation of their Hopf cyclic cohomology by means of a bicocyclic complex.

First we note that, in view of Proposition 3.1, we can replace \(H(\Pi)\) by \(H(\Pi)^{\text{cop}}\). By Theorems 2.15 and 2.21, the latter can be identified to the bicrossed product \(F \rhd \triangleleft U\), where \(U := U(g)\), \(F := F(N)\), and we shall do so from now on without further warning.

Next, we need both \(F\) and \(U\) to be \(H\)-module coalgebras. Recalling that \(F\) is a left \(U\)-module, we define an action of \(H\) on \(F\) by the formula

\[
(f \rhd u) g = f(u \triangleright g), \quad f, g \in F, \quad u \in U.
\] (3.34)

**Lemma 3.17.** The above formula defines an action, which makes \(F\) an \(H\)-module coalgebra.

**Proof.** We check directly the action axiom

\[
(f^1 \rhd u^1)((f^2 \rhd u^1)g) = (f^1 \rhd u^1)(f^2 u^2 \triangleright g) = \\
f^1 u^1 \triangleright (f^2 u^1 \triangleright g) = f^1 (u^1_{(1)} \triangleright f^2)(u^1_{(2)} u^2 \triangleright g) = \\
[f^1 u^1_{(1)} \triangleright f^2 \rhd u^1_{(2)} u^2] g = [(f^1 \rhd u^1)((f^2 \rhd u^1)] g,
\]
and the property of being a Hopf action:

\[
\Delta((f \triangleright u)g) = \Delta(fu \triangleright g) = f_{(1)}u_{(1)}_{<0>,g_{(1)}} \otimes f_{(2)}u_{(1)}_{<1>}(u_{(2)} \triangleright g_{(2)}) =
\]

\[
(f_{(1)} \triangleright u_{(1)}_{<0>,})g_{(1)} \otimes (f_{(2)}u_{(1)}_{<1>}) \triangleright u_{(2)})g_{(2)}. 
\]

\[
\Box
\]

To realize \( \mathcal{U} \) as an \( \mathcal{H} \)-module and comodule coalgebra, we identify it with \( \mathcal{C} := \mathcal{H} \otimes \mathcal{F} \mathbb{C} \), via the map \( \sharp : \mathcal{H} \otimes \mathcal{F} \mathbb{C} \rightarrow \mathcal{U} \) defined by

\[
\sharp(f \triangleright u \otimes 1) = \varepsilon(f)u. \tag{3.35}
\]

**Lemma 3.18.** The map \( \sharp : \mathcal{C} \rightarrow \mathcal{U} \) is an isomorphism of coalgebras.

**Proof.** The map is well-defined, because

\[
\sharp((f \triangleright u)(g \triangleright 1) \otimes 1) = \sharp(fu \triangleright g \triangleright u \otimes 1) = \\
\varepsilon(fu \triangleright g)u_{(2)} = \varepsilon(f)\varepsilon(g)u = \sharp(f \triangleright u \otimes \varepsilon(g)).
\]

Let us check that \( \sharp^{-1}(u) = (1 \triangleright u) \otimes 1 \) is its inverse. It is easy to see that \( \sharp \circ \sharp^{-1} = \text{Id}_\mathcal{U} \). On the other hand,

\[
\sharp^{-1} \circ \sharp(f \triangleright u \otimes 1) = \sharp^{-1}(\varepsilon(f)u) = \\
(1 \triangleright u) \otimes \varepsilon(f) = (1 \triangleright u_{(2)}) \otimes \varepsilon(S^{-1}(u_{(1)})\varepsilon(f) = \\
[(1 \triangleright u_{(2)})(S^{-1}(u_{(1)}) \triangleright f \triangleright 1)] \otimes 1 = f \triangleright u \otimes 1.
\]

Finally one checks the comultiplicativity as follows:

\[
\sharp \otimes \sharp(\Delta_{\mathcal{C}}(f \triangleright u \otimes 1)) = \\
\sharp(f_{(1)} \triangleright u_{(1)} \otimes 1) \otimes \sharp(u_{(2)}_{<1>} \triangleright f_{(2)} \triangleright u_{(2)}_{<0>}) \otimes 1) = \\
\varepsilon(f_{(2)})u_{(1)} \otimes \varepsilon(u_{(2)}_{<1>}) \triangleright f_{(2)}u_{(2)}_{<0>} = \\
\varepsilon(f_{(1)})u_{(1)} \otimes \varepsilon(f_{(2)})u_{(2)} = \Delta_{\mathcal{U}}(\sharp(f \triangleright u \otimes 1)).
\]

\[
\Box
\]

By Proposition 3.14, the following is a map of \( \mathcal{H} \)-module coalgebras.

\[
\Phi : \mathcal{H} \rightarrow \mathcal{C} \triangleright \mathcal{F},
\]

\[
\Phi(h) = h_{(1)} \triangleright h_{(2)} 1.
\]

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To be able to use the Theorem 3.15 we need to show that $\Phi$ is bijective. One has

$$\Phi(f □ u) = (f_1 □ u_{<0>}) \otimes f_2 u_{<1>} =$$

$$= (1 □ (u_{<0>}(1))(S(u_{<0>}(3)) □ f_1) □ 1) \otimes f_2 u_{<1>} =$$

$$= 1 □ u_{<0>} \otimes f u_{<1>} \equiv u_{<0>} \otimes f u_{<1>} ,$$

from which it follows that $\Phi^{-1}(u \otimes f) = f S^{-1}(u_{<1>}) □ u_{<0>}$. We are now in a position to apply Theorem 3.15 to get a bicocyclic module for computing the Hopf cyclic cohomology of $\mathcal{H}$. We would like, before that, to understand the action and coaction of $\mathcal{H}$ on $\mathcal{C}$. One has

$$(f □ u) \cdot (g □ v) = f u_{<1>} □ g □ v \equiv \varepsilon(f)\varepsilon(g)uv ,$$

which coincides with the natural action of $\mathcal{H}$ on $\mathcal{U}$. Next, we take up the coaction of $\mathcal{C}$. Recall the coaction of $\mathcal{H}$ on $\mathcal{C}$,

$$\nabla : \mathcal{C} \to \mathcal{H} \otimes \mathcal{C},$$

$$\nabla(1 □ u) = (1 □ u_{(1)})S(1 □ u_{(3)}) \otimes (1 □ u_{(2)}) .$$

**Proposition 3.19.** The above coaction coincides on the original coaction of $\mathcal{F}$ on $\mathcal{U}$.

**Proof.** Using the fact that $\mathcal{U}$ is cocommutative, one has

$$(1 □ u_{(1)})S((1 □ u_{(3)}) \otimes (1 □ u_{(2)}) =$$

$$(1 □ u_{<0>})S(u_{<1>2} □ u_{<1>3} □ u_{<0>}) \otimes (1 □ u_{<1>}) □ u_{<0>2} =$$

$$(1 □ u_{<0>})(1 □ S(u_{<3>})((S(u_{<1>2} □ u_{<1>3} □ u_{<1>3} ...) □ 1) \otimes$$

$$\otimes u_{<1>1} □ u_{<0>2} =$$

$$(1 □ u_{<0>})(S(u_{<3>})(S(u_{<1>2} □ u_{<1>3} □ u_{<1>3} ...) □ 1) \otimes u_{<1>1} □ u_{<0>2} =$$

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which means that after identifying $C$ with $U$ one has

\[(1 \triangleright u_{(1)_{<0>}} S(u_{(3)_{<0>}}))(S(u_{(1)_{<1>}} u_{(2)_{<1>}} u_{(3)_{<1>}}) \triangleright 1) \otimes u_{(2)_{<0>}} =
\]
\[(1 \triangleright u_{<0>_{(1)}} S(u_{<0>_{(3)}}))(S(u_{<1>_{(1)}}) \triangleright 1) \otimes u_{<0>_{(2)}} =
\]
\[1 \triangleright S(u_{<1>_{(1)}}) \otimes u_{<0>_{(1)}} = S(u_{<1>_{(1)}}) \otimes u_{<0>_{(1)}}.
\]

With the above identifications of actions and coactions, one has the following bicyclic module $3(\mathcal{H}, \mathcal{F}; C_\delta)$, where $\delta$ is the modular character:

\[\begin{align*}
&\cdots \\
&b_f \downarrow B_f \\
&C_\delta \otimes U^\otimes 2 \\
&\cdots \\
&b_f \downarrow B_f \\
&C_\delta \otimes \mathcal{F} \otimes U^\otimes 2 \\
&\cdots \\
&b_f \downarrow B_f \\
&C_\delta \otimes \mathcal{F} \otimes C \\
&\cdots \\
&\cdots
\end{align*}\]

At this stage, we introduce the following Chevalley-Eilenberg-type bicomplex:

\[\begin{align*}
&\cdots \\
&\partial_g \\
&C_\delta \otimes \wedge^2 g \\
&\cdots \\
&\partial_g \\
&C_\delta \otimes \mathcal{F} \otimes \wedge^2 g \\
&\cdots \\
&\partial_g \\
&C_\delta \otimes g \\
&\cdots \\
&\partial_g \\
&C_\delta \otimes \mathcal{F} \otimes g \\
&\cdots \\
&\partial_g \\
&C_\delta \otimes C \\
&\cdots
\end{align*}\]
Here $\partial_g$ is the Lie algebra homology boundary of $g$ with coefficients in right module $C_\delta \otimes \mathcal{F}^\otimes p$; explicitly
\[
\partial_g(1 \otimes \tilde{f} \otimes X^0 \wedge \cdots \wedge X^{q-1}) = \\
\sum_i (-1)^i (1 \otimes \tilde{f}) \otimes X^i \otimes X^0 \wedge \cdots \wedge \hat{X}^i \wedge \cdots \wedge X^{q-1} + \\
(-1)^{i+j} \sum_{i<j} 1 \otimes \tilde{f} \otimes [X_i, X_j] \otimes X^0 \wedge \cdots \wedge \hat{X}^i \wedge \cdots \wedge \hat{X}^j \wedge \cdots \wedge X^{q-1},
\]
where
\[
(1 \otimes \tilde{f}) \otimes X = \delta(X) \otimes \tilde{f} + 1 \otimes S(1 \triangleright \otimes X) \tilde{f}. \tag{3.38}
\]
The horizontal operator is given by
\[
\beta_{\mathcal{F}}(1 \otimes \tilde{f} \otimes X^1 \wedge \cdots \wedge X^q) = \\
\sum_{i=0}^q 1 \otimes (\text{Id}^\otimes i \otimes \Delta \otimes \text{Id}^\otimes (q-i)) (\tilde{f}) \otimes X^1 \wedge \cdots \wedge X^q + \\
(-1)^{q+1} 1 \otimes \tilde{f} \otimes S(X^1 \langle_{<1>} \cdots X^q \langle_{<1>} \rangle) \otimes X^1 \langle_{<0>} \wedge \cdots \wedge X^q \langle_{<0>} \rangle.
\]
In other words, $\beta_{\mathcal{F}}$ is just the coalgebra cohomology coboundary of the coalgebra $\mathcal{F}$ with coefficients in $\wedge^q g$ induced from the coaction of $\mathcal{F}$ on $U^\otimes q$.

One notes that, since $\mathcal{F}$ is commutative, the coaction $\nabla_{\wedge g}$ is well-defined.

**Proposition 3.20.** The bicomplexes (3.36) and (3.37) have quasi-isomorphic total complexes.

**Proof.** We apply antisymmetrization map
\[
\tilde{\alpha} : C_\delta \otimes \mathcal{F}^\otimes p \otimes \wedge^q g \to C_\delta \otimes \mathcal{F}^\otimes p \otimes U^\otimes q,
\]
\[
\tilde{\alpha} = \text{Id} \otimes \alpha,
\]
where $\alpha$ is the usual antisymmetrization map
\[
\alpha(X^1 \wedge \cdots \wedge X^p) = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma X^{\sigma(1)} \otimes \cdots \otimes X^{\sigma(p)}. \tag{3.40}
\]

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Its left inverse is \( \tilde{\mu} := \text{Id} \otimes \mu \) where \( \mu \) is the natural left inverse to \( \alpha \) (see \emph{e.g.} [19] page 436). As in the proof of Proposition 7 of [6], one has the following commutative diagram:

\[
\begin{array}{c}
\delta_q \quad 0 \\
\downarrow \quad \downarrow \delta_q \\
C_\delta \otimes F \otimes \Lambda^q g \\ \\
\end{array}
\begin{array}{c}
\tilde{\alpha} \\
\alpha \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
C_\delta \otimes F \otimes \Lambda^{q+1} g \\
\end{array}
\begin{array}{c}
B_U \\
b_U \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
C_\delta \otimes F \otimes \Lambda^q g+1 \\
\end{array}
\]

Since \( \tilde{\alpha} \) does not affect \( F \otimes \mu \), it is easy to see that the following diagram also commutes:

\[
\begin{array}{c}
\delta_q \quad 0 \\
\downarrow \quad \downarrow \delta_q \\
C_\delta \otimes F \otimes \Lambda^q g \\ \\
\end{array}
\begin{array}{c}
\tilde{\alpha} \\
\alpha \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
C_\delta \otimes F \otimes \Lambda^{q+1} g \\
\end{array}
\begin{array}{c}
B_F \\
b_F \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
C_\delta \otimes F \otimes \Lambda^q g+1 \\
\end{array}
\]

Here \( B_F \) is the Connes boundary operator for the cocyclic module \( C^*_F(C_\delta \otimes \Lambda^q g) \), where \( F \) coacts on \( C_\delta \otimes \Lambda^q g \) via (3.39) and acts trivially. Since \( F \) is commutative and its action on \( C_\delta \otimes \Lambda^q g \) is trivial, Theorem 3.22 of [22] implies that \( B_F \equiv 0 \) in Hochschild cohomology. On the other hand Theorem 7 [6] (which is applied here for coefficients in a general module), proves that the columns of (3.36) with coboundary \( B_U \) and the columns of (3.37) with coboundary \( \partial_0 \) are quasi-isomorphic. This finishes the proof.

### 3.4 Applications

#### 3.4.1 Hopf cyclic Chern classes

In this section we compute the relative Hopf cyclic cohomology of \( H_n \) modulo the subalgebra \( L := U(g_{l_0}) \) with coefficients in \( C_\delta \).

To this end, we first form the Hopf subalgebra \( K := L \triangleright F \subset H \). Here \( L \) acts on \( F \) via its action inherited from \( L \subset U \) on \( F \), and \( F \) coacts on \( L \) trivially. The second coalgebra is \( C := H \otimes_K \mathbb{C} \). Letting \( \mathfrak{h} := g_{l_0} \), and \( S := S(g/\mathfrak{h}) \) be the symmetric algebra of the vector space \( g/\mathfrak{h} \), we identify \( C \) with the coalgebra \( S \), as follows. Since \( g \cong V \triangleright \mathfrak{h} \), where \( V = \mathbb{R}^n \), we can regard \( U \) as being \( U(V) \triangleright \mathfrak{h} \). The identification of \( C \) with \( S \) is achieved by the map

\[
(X \triangleright f) \mapsto f \otimes 1 \mapsto \epsilon(Y) \epsilon(f) X, \quad X \in U(V), \ Y \in U(\mathfrak{h}), \ f \in F.
\]
As in Lemma 3.18, one checks that this identification is a coalgebra isomorphism. Finally, since $g/h \cong V$ as vector spaces, $U(V) \cong S(g/h)$ as coalgebras.

Similarly, one identifies, $\mathcal{K}_L := \mathcal{K} \otimes \mathbb{C}$ with $\mathcal{F}$ as coalgebras, via

\[ Y \mapsto f \otimes 1 \mapsto \epsilon(Y)f, \quad Y \in U(\mathfrak{h}), \ f \in \mathcal{F}. \]

Let us recall the mixed complex $(\mathcal{C}_n, b, B)$, where $\mathcal{C}_n := C^n(\mathcal{H}, \mathcal{L}; C_\delta) = C_\delta \otimes \mathcal{L} C^{\otimes n}$, which computes relative Hopf cyclic cohomology of $\mathcal{H}$ modulo $\mathcal{L}$ with coefficients in $C_\delta$. We also recall that the isomorphism $\Theta$ defined in (3.32) identifies this complex with the diagonal complex $Z^*_\ast (\mathcal{H}, \mathcal{K}, \mathcal{L}; C_\delta) = C_\delta \otimes \mathcal{L} F^{\otimes p} \otimes S^{\otimes q}$. Now by using Theorem 3.15 we get the following bicomplex whose total complex is quasi isomorphic to $\mathcal{C}^n$ via the Eilenberg-Zilber Theorem.

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
C_\delta \otimes \mathcal{L} S^{\otimes 2} & C_\delta \otimes \mathcal{L} F \otimes S^{\otimes 2} & C_\delta \otimes \mathcal{L} F^{\otimes 2} \otimes S^{\otimes 2} \\
B_S & B_F & B_F \\
C_\delta \otimes \mathcal{L} S & C_\delta \otimes \mathcal{L} F \otimes S & C_\delta \otimes \mathcal{L} F^{\otimes 2} \otimes S \\
B_S & B_F & B_F \\
C_\delta \otimes \mathcal{L} C & C_\delta \otimes \mathcal{L} F \otimes C & C_\delta \otimes \mathcal{L} F^{\otimes 2} \otimes C \\
B_S & B_F & B_F \\
\end{array}
\]

(3.41)

Similar to (3.37), we introduce the following bicomplex.
Here $V$ stands for the vector space $\mathfrak{g}/\mathfrak{h}$ and $\partial_{(\mathfrak{g},\mathfrak{h})}$ is the relative Lie algebra homology boundary of $\mathfrak{g}$ relative to $\mathfrak{h}$ with coefficients in $C_\delta \otimes \mathcal{F}^p$ with the action defined in (3.38). The coboundary $\tilde{\beta}_F$ is induced by $\beta_F$ the coalgebra cohomology coboundary of $\mathcal{F}$ with trivial coefficients in $C_\delta \otimes \mathcal{V}$. One notes that the coaction of $\mathcal{F}$ on $\mathcal{S}$ is induced from the coaction of $\mathcal{F}$ on $\mathcal{U}$ via the natural projection $\pi : \mathcal{U} \rightarrow \mathcal{U} \otimes L \cong \mathcal{S}$.

**Proposition 3.21.** The total complexes of the bicomplexes (3.41) and (3.42) are quasi isomorphic.

**Proof.** The proof is similar to that of Proposition 3.20 but more delicate. One replaces the antisymentrization map by its relative version and instead of Proposition 7 [6] one uses Theorem 15 [9]. On the other hand since $L \subset H_n$ is cocommutative, every map in the bicomplex (3.41) is $L$-linear, including the homotopy maps used in Eilenberg-Zilber theorem [20], and in [19, pp. 438-442], as well as in the spectral sequence used in [22, Theorem 3.22].

To find the cohomology of the above bicomplex we first compute the subcomplex consisting of $\mathfrak{gl}_n$-invariants.

**Definition 3.22.** We say a map $\mathcal{F}^m \rightarrow \mathcal{F}_p$ is an ordered projection of order $(i_1, \ldots , i_q)$, including $\emptyset$ order, if it of the form

$$f_1 \otimes \ldots \otimes f_n \mapsto f^{i_1} \ldots f^{i_q} \otimes f^{i_{q-1}} \ldots \otimes f^{i_1} \ldots f^{i_q}. \quad (3.43)$$

We denote the set of all such projections by $\Pi^m_q$. 

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Let $S_m$ denote the symmetric group of order $m$. Let also fix $\sigma \in S_{n-q}$, and $\pi \in \Pi_{p-q}$, where $0 \leq p, q \leq n$. We define

$$\theta(\sigma, \pi) = \sum (-1)^\mu 1 \otimes \pi(\eta_{\mu(1), j_{\sigma(1)}} \otimes \cdots \otimes \eta_{\mu(n-q), j_{\sigma(n-q)}}) \otimes X_{\mu(n-q+1)} \wedge \cdots \wedge X_{\mu(n)}$$

where the summation is over all $\mu \in S_n$ and all $1 \leq j_1, j_2, \ldots, j_q \leq n$.

**Lemma 3.23.** The elements $\theta(\sigma, \pi) \in \mathbb{C}_\delta \otimes F^\otimes q \otimes \wedge^p V$, with $\sigma \in S_{n-p}$ and $\pi \in \Pi_{q-p}^n$, are $\mathfrak{h}$-invariant and span the space $\mathbb{C}_\delta \otimes F^\otimes q \otimes \wedge^p V$.

**Proof.** We identify $\mathbb{C}_\delta$ with $\wedge^n V^*$ as $\mathfrak{h}$-modules, by sending $1 \in \mathbb{C}_\delta$ into the volume element $\xi_1 \wedge \cdots \wedge \xi_n \in \wedge^n V^*$, where $\{\xi_1, \ldots, \xi_n\}^\ast$ is the dual basis to $\{X_1, \ldots, X_n\}$. One obtains thus an isomorphism

$$\mathbb{C}_\delta \otimes \mathfrak{h} F^\otimes q \otimes \wedge^p V \cong (\wedge^n V^* \otimes F^\otimes p \otimes \wedge^q V)^\mathfrak{h}.$$  

(3.45)

We now observe that the relations (2.33) and (1.24) show that the action of $\mathfrak{h}$ on the jet coordinates $\eta_{j_{j_1}, \ldots, j_{j_k}}$ is tensorial. In particular, the central element of $\mathfrak{h}$, $Z = \sum_{i=1}^n Y_i^i$, acts as a grading operator, assigning degree 1 to each $X \in V$, degree $-1$ to each $\xi \in V^*$ and degree $k$ to $\eta_{j_{j_1}, \ldots, j_{j_k}}$. As an immediate consequence, the space of invariants (3.45) is seen to be generated by monomials with the same number of upper and lower indices, more precisely

$$(\wedge^n V^* \otimes F^\otimes p \otimes \wedge^q V)^\mathfrak{h} = (\wedge^n V^* \otimes F^\otimes p[n-q] \otimes \wedge^q V)^\mathfrak{h},$$

(3.46)

where $F^\otimes p[n-q]$ designates the homogeneous component of $F^\otimes q$ of degree $n-q$.

Furthermore, using (2.28), (2.29) we can think of $F$ as generated by $\alpha_{j_{j_1}, \ldots, j_{j_k}}^i$ instead of the $\eta_{j_{j_1}, \ldots, j_{j_k}}$. The advantage is that the $\alpha_{j_{j_1}, \ldots, j_{j_k}}$ are symmetric in all lower indices and freely generate the algebra $F$. Fixing a lexicographic ordering on the set of multi-indices $j_{j_1}, \ldots, j_{j_k}$, one obtains a PBW-basis of $F$, viewed as the Lie algebra generated by the $\alpha_{j_{j_1}, \ldots, j_{j_k}}$. This allows us to extend the assignment

$$\alpha_{j_{j_1}, \ldots, j_{j_k}}^i \mapsto \xi_i \otimes X_{j_1} \otimes X_{j_2} \otimes \cdots \otimes X_{j_k} \in V^* \otimes V^\otimes k+1,$$

first to the products which define the PBW-basis and next to an $\mathfrak{h}$-equivariant embedding of $F^\otimes q[n-q]$ into a finite direct sum of tensor products of the
form \((V^*)^\otimes r \otimes V^\otimes s\). Sending the wedge product of vectors into the antisymmetrized tensor product, we thus obtain an embedding
\[
(\wedge^n V^* \otimes F \otimes p)[n-q] \otimes \wedge^q V^* \otimes \sum_{r,s}(V^*)^\otimes r \otimes V^\otimes s \otimes V \otimes q)^{\otimes} h \hookrightarrow ((V^*)^\otimes n \otimes \sum_{r,s}(V^*)^\otimes r \otimes V^\otimes s \otimes V^\otimes q)^{\otimes} h. \quad (3.47)
\]

In the right hand side we are now dealing with the classical theory of tensor invariants for \(GL(n, \mathbb{R})\), cf. [27]. All such invariants are linear combinations of the form
\[
T_\sigma = \sum \xi^{j_1} \otimes \cdots \otimes \xi^{j_n} \otimes X_{j_{\sigma(1)}} \otimes \cdots \otimes X_{j_{\sigma(p)}} \otimes \cdots \otimes \xi^{j_r} \otimes \cdots \otimes \xi^{j_m} \otimes \cdots \otimes X_{j_{\sigma(s)}} \otimes \cdots \otimes X_{j_{\sigma(m)}},
\]
where \(\sigma \in S_m\) and the sum is over all \(1 \leq j_1, j_2, \ldots, j_m \leq n\).

Because the antisymmetrization is a projection operator, such linear combinations belong to the subspace in the left hand side of (3.47) only when they are totally antisymmetric in the first \(n\)-covectors. Recalling the identification \(1 \sim \xi^{j_1} \wedge \cdots \wedge \xi^{j_n}\), and the fact that \(\alpha^{i}_{j_1, \ldots, j_k}\) are symmetric in all lower indices, one easily sees that the projected invariants belong to the linear span of those of the form (3.44). \(\square\)

For each partition \(\lambda = (\lambda_1 \geq \ldots \geq \lambda_k)\) of the set \(\{1, \ldots, p\}\), where \(1 \leq p \leq n\), we let \(\lambda \in S_p\) also denote a permutation whose cycles have lengths \(\lambda_1 \geq \ldots \geq \lambda_k\), i.e. representing the corresponding conjugacy class \([\lambda] \in [S_p]\).

We then define
\[
C_{p,\lambda} := \sum (-1)^\mu \eta_{j_{\mu}(1)}^{j_1} \wedge \cdots \wedge \eta_{j_{\mu}(p)}^{j_p} \wedge X_{\mu(p+1)} \wedge \cdots \wedge X_{\mu(n)},
\]
where the summation is over all \(\mu \in S_n\) and all \(1 \leq j_1, j_2, \ldots, j_p \leq n\).

**Theorem 3.24.** The cochains \(\{C_{p,\lambda}; 1 \leq p \leq n, [\lambda] \in [S_p]\}\) are cocycles and their classes form a basis of the group \(HP^\epsilon(H_n; U(\mathfrak{g}^n); \mathbb{C}_\delta)\), where \(\epsilon \equiv n \mod 2\). The complementary parity group \(HP^{1-\epsilon}(H_n; U(\mathfrak{g}^n); \mathbb{C}_\delta) = 0\).

**Proof.** Let \(\mathfrak{d}\) be the commutative Lie algebra generated by all \(\eta_{j,k}^i\), and let \(F = U(\mathfrak{d})\), be the polynomial algebra of \(\eta_{j,k}^i\). Obviously \(F\) is a Hopf subalgebra of \(\mathcal{F}\) and stable under the action of \(\mathfrak{h}\). By using Lemma 3.44 we have:
\[
\mathbb{C}_\delta \otimes_{\mathfrak{h}} F^{\otimes p} \otimes \wedge^q V \cong \mathbb{C}_\delta \otimes_{\mathfrak{h}} \mathcal{F}^{\otimes p} \otimes \wedge^q V. \quad (3.48)
\]
Hence we reduce the problem to compute the cohomology of \((\mathbb{C}_\delta \otimes_{\mathfrak{h}} F^{\otimes *} \otimes \wedge^q V, \beta_F)\), where \(\beta_F\) is induced by the Hochschild coboundary of the coalgebra.
F with trivial coefficients. One uses the fact that \((C_δ \otimes F^\otimes \otimes \Lambda^q V, b_F)\) and \((C_δ \otimes \Lambda^p \otimes \Lambda^q V, 0)\) are homotopy equivalent and the homotopy is \(h\)-linear to see that the \(q\)th cohomology of the complex under consideration is \(C_δ \otimes h \Lambda^p \otimes \Lambda^q V\). Similar to the proof of Lemma 3.44 we replace \(C_δ\) with \(\Lambda^p V^*\). We also replace \(\alpha_j^i\) with \(\tilde{\xi}^i \otimes X_j^i \otimes X_k^\prime\), where \(\tilde{\xi}^i\), \(X_j^i\) and \(X_k^\prime\) are basis for \(V^*, V\) and \(V\) respectively. Using the above identification one has \(C_δ \otimes \Lambda^p \otimes \Lambda^q V \cong \left( \Lambda^p V^* \otimes (\Lambda^p V^* \otimes \Lambda^p V) \otimes \Lambda^q V \right)^h\). Via the same argument as in the proof of Lemma 3.44 one concludes that the invariant space is generated by elements of the form

\[
C_{p,\sigma} := \sum (-1)^{\mu} 1 \otimes \eta^1_{\sigma(1)} \otimes \cdots \otimes \eta^j_{\sigma(p)} \otimes X_{\mu(p+1)} \otimes \cdots \otimes X_{\mu(n)} \tag{3.49}
\]

where \(\sigma \in S_p\), and the summation is taken as in (3.44). Now let \(\sigma = \sigma_1 \cdots \sigma_k\), where \(\sigma_1 = (a, \sigma(a)), \ldots, \sigma^{\alpha-1}(a), \ldots, \sigma_k = (z, \sigma(z), \ldots, \sigma^{\gamma-1}(z))\). Since distinct cyclic permutations commute among each other we may assume \(\alpha \geq \beta \cdots \geq \gamma \geq \zeta\). We defined the following two permutations.

\[
\tau := (1, 2, \ldots, \alpha)(\alpha + 1, \ldots, \beta), \ldots, (\alpha + \cdots + \gamma + 1, \ldots, \alpha + \cdots + \zeta),
\]

\[
\theta(i) = \begin{cases} 
\sigma^{i-1}(a), & 1 \leq i \leq \alpha \\
\sigma^{i-\alpha}(b), & 1 + \alpha \leq i \leq \alpha + \beta \\
\vdots & \vdots \\
\sigma^{p-1-\alpha}(z), & \alpha + \cdots + \gamma + 1 \leq i \leq p \\
i, & p + 1 \leq i \leq n 
\end{cases}
\]

We claim \(C_{p,\sigma}\) and \(C_{p,\tau}\) coincide up to a sign. Indeed,

\[
C_{p,\sigma} = \pm \sum_{\mu} (-1)^{\mu} \eta^1_{\sigma(1)} \otimes \eta^j_{\sigma(p)} \otimes \cdots \otimes \eta^{j_{\alpha-1}(a)}_{\sigma(\alpha)} \otimes X_{\mu(p+1)} \otimes \cdots \otimes X_{\mu(n)}
\]

\[
\eta^\sigma_{\mu(b), \sigma(a)} \otimes \eta^j_{\sigma(b)} \otimes \cdots \otimes \eta^{j_{\sigma-1}(b)}_{\sigma(\sigma-1)(b)} \otimes \cdots \otimes X_{\mu(p+1)} \otimes \cdots \otimes X_{\mu(n)}
\]

\[
\eta^\sigma_{\mu(1), \sigma(1)} \otimes \eta^j_{\sigma(1)} \otimes \cdots \otimes \eta^{j_{\sigma-1}(1)}_{\sigma(\sigma-1)(1)} \otimes X_{\mu(p+1)} \otimes \cdots \otimes X_{\mu(n)}
\]

\[
\pm \sum_{\mu} (-1)^{\mu} \eta^1_{\mu(a), \mu(b)} \otimes \eta^j_{\mu(1), \mu(b)} \otimes \cdots \otimes \eta^{j_{\mu(a-1)}}_{\mu(1), \mu(b)} \otimes \cdots \otimes X_{\mu(p+1)} \otimes \cdots \otimes X_{\mu(n)}
\]

\[
\eta^{j_{\mu(a)+1}}_{\mu(b), \mu(a)+2} \otimes \eta^{j_{\mu(a)+2}}_{\mu(1), \mu(b)} \otimes \cdots \otimes \eta^{j_{\mu(a)+\gamma+1}}_{\mu(1), \mu(b)} \otimes \cdots \otimes X_{\mu(p+1)} \otimes \cdots \otimes X_{\mu(n)}
\]

\[
\pm \sum_{\mu} (-1)^{\mu} \eta^1_{\mu(\theta(1))} \otimes \eta^j_{\mu(\theta(p), \theta(q))} \otimes X_{\mu(\theta)} = \pm C_{p,\tau}.
\]
Here $X(\mu)$ stands for $X_{\mu(p+1)} \wedge \cdots \wedge X_{\mu(n)}$.

The fact that $\{C_{p,\lambda} ; 1 \leq p \leq n, \quad [\lambda] \in [S_p]\}$ are linearly independent, and therefore form a basis for $\mathbb{C}_\delta \otimes_h \mathcal{F}^{\otimes p} \otimes \land^q V$, follows from the observation that, if $\sigma \in S_p$, the term

$$\sum (-1)^{\mu} 1 \otimes \eta_{\mu(1),\sigma(1)} \wedge \cdots \wedge X_{\mu(p+1)} \wedge \cdots \wedge X_{\mu(n)},$$

appears in $C_{p,\tau}$ if and only if $[\sigma] = [\tau]$. These are all the periodic cyclic classes, because all of them sit on the $n$th skew diagonal of (3.42) and there are no other invariants. Therefore $B \equiv \partial_{g,h} = 0$ on the invariant space.

In particular

$$\dim HP^*(\mathcal{H}_n, \mathcal{U}(\mathfrak{g}_n); \mathbb{C}_\delta) = p(0) + p(1) + \ldots + p(n),$$

where $p$ denotes the partition function, which is the same as the dimension of the truncated Chern ring $\mathcal{P}_n[c_1, \ldots, c_n]$. Moreover, as noted in the introduction, the assignment

$$C_{p,\lambda} \mapsto c_{p,\lambda} := c_{\lambda_1} \cdots c_{\lambda_k}, \quad \lambda_1 + \ldots + \lambda_k = p,$$

defines a linear isomorphism between $HP^*(\mathcal{H}_n, \mathcal{U}(\mathfrak{g}_n); \mathbb{C}_\delta)$ and $\mathcal{P}_n[c_1, \ldots, c_n]$.

### 3.4.2 Non-periodized Hopf cyclic cohomology of $\mathcal{H}_1$

In this section we apply the bicomplex (3.37) to compute all cyclic cohomology of $\mathcal{H}_1$ with coefficients in the MPI (1, $\delta$). Actually, invoking the equivalence (3.6), we shall compute $HC(\mathcal{H}_1^{\text{cop}}, \mathbb{C}_\delta)$ instead.

Let us recall the presentation of the Hopf algebra $\mathcal{H}_1$. As an algebra, $\mathcal{H}_1$ is generated by $X, Y, \delta_k, k \in \mathbb{N}$, subject to the relations

$$[Y, X] = X, \quad [Y, \delta_k] = k\delta_k, \quad [X, \delta_k] = \delta_{k+1}, \quad [\delta_j, \delta_k] = 0.$$

Its coalgebra structure is uniquely determined by

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y;$$
$$\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y;$$
$$\Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1;$$
$$\epsilon(X) = \epsilon(Y) = \epsilon(\delta_k) = 0.$$

By Theorem (2.15), one has $\mathcal{H}_1^{\text{cop}} \simeq \mathcal{F} \rhd \mathcal{U}$. We next apply Theorem 3.15 which gives a quasi-isomorphism between $C^*(\mathcal{F} \rhd \mathcal{U}, \mathbb{C}_\delta)$ and the
total complex of the bicomplex (3.37). One notes that $Y$ grades $H_1$ by
$[Y,X] = X$, $[Y,\delta_k] = k\delta_k$. Accordingly, $Y$ grades $U$, and also $F$ by
$$Y \triangleright \eta^1_1 \ldots 1 = k \eta^1_1 \ldots 1,$$
where $k + 1$ is the number of lower indices. Hence $Y$ grades the bicomplex (3.37). One notes that every identification, isomorphism, and homotopy which has been used to pass from the bicomplex $C^*(F \bowtie U, \mathbb{C}_g)$ to (3.37) respects this grading (cf. also [24]). As a result, one can relativize the computations to each homogeneous component. The degree of $\tilde{f} \in F^\otimes q$ will be denoted by $|\tilde{f}|$.

We next recall Gončarova’s results [14] concerning the Lie algebra cohomology of $n$. Taking as basis of $n$ the vector fields
$$e_i = x^{i+1} \frac{d}{dx} \in n, \quad i \geq 1,$$
and denoting the dual basis by $\{e^i \in n^* \mid i \geq 1\}$, one identifies $\wedge^k n^*$ with the totally antisymmetric polynomials in variables $z_1 \ldots z_k$, via the map
$$e^{i_1} \wedge \cdots \wedge e^{i_k} \mapsto \sum_{\mu \in S_k} (-1)^\mu z_{i_1}^{\mu(1)} \cdots z_{i_k}^{\mu(k)}.$$
According to [14], for each dimension $k \geq 1$, the Lie algebra cohomology group $H^k(n, \mathbb{C})$ is two-dimensional and is generated by the classes
$$\omega_k := z_1 \ldots z_k \Pi_k^2, \quad \omega'_k := z_1^2 \ldots z_k^2 \Pi_k^2,$$
where $\Pi_k := \Pi_{1 \leq i < j \leq k} (z_j - z_i)$. We shall denote by $\xi_k \in H^k(F, \mathbb{C})$, resp. $\xi'_k \in H^k(F, \mathbb{C})$ the image in Hochschild cohomology with trivial coefficients of $\omega_k$, resp. $\omega'_k$, under the van Est isomorphism of [6, Proposition 7(b)].

**Lemma 3.25.** The map $\gamma: F^\otimes i \to F^\otimes (i+1)$ defined by
$$\gamma(\tilde{f}) = \tilde{f} \otimes \eta_1,$$
where $\eta_1 := \eta^1_1$, commutes with the Hochschild coboundary and for $i \geq 1$ is null-homotopic.

**Proof.** We check that $\gamma$ is map of complexes:
$$\gamma(b(\tilde{f})) = b(\tilde{f}) \otimes \eta_1 =$$
$$b(\tilde{f}) \otimes \eta_1 + (-1)^i (\tilde{f} \otimes 1 \otimes \eta_1 - \tilde{f} \otimes 1 \otimes \eta_1 + \tilde{f} \otimes \eta_1 + 1 - \tilde{f} \otimes \eta_1 \otimes 1)$$
$$= b(\gamma(\tilde{f})).$$
Using the bicomplex (3.37), one can show that the above map is zero at the level of Hochschild cohomology. Indeed, recalling that \( \mathcal{H} \) acts on \( \mathcal{F} \) as in (3.34), and remembering that we are dealing with the normalized bicomplex, one has

\[
\partial g \beta_f (1 \otimes \tilde{f} \otimes X) = \partial g (1 \otimes b(\tilde{f}) \otimes X + (-1)^{i+1} 1 \otimes \tilde{f} \otimes \eta_1 \otimes Y =
- (-1)^{i+1} 1 \otimes Xb(\tilde{f}) + 1 \otimes \tilde{f} \otimes \eta_1 - 1 \otimes Y(\tilde{f} \otimes \eta_1) =
- (-1)^{i+1} 1 \otimes Xb(\tilde{f}) - 1 \otimes |\tilde{f}| \tilde{f} \otimes \eta_1.
\]

On the other hand,

\[
\beta_f \partial g (1 \otimes \tilde{f} \otimes X) = -\beta_f (1 \otimes X \tilde{f}).
\]

This implies that if \( b(\tilde{f}) = 0 \) and \( |\tilde{f}| \geq 1 \), then

\[
\tilde{f} \otimes \eta_1 = \frac{1}{|\tilde{f}|} b(X \tilde{f}).
\]

To compute the Hochschild cohomology of \( \mathcal{H}_1 \), we appeal again to the double complex (3.37). The \( E^{p,q}_1 \) term is \( \mathbb{C} \delta \otimes \wedge^p \mathcal{g} \otimes \mathcal{F}^{\otimes q} \) and the boundary is \( \beta_f \), as in the diagram below:

\[
\begin{align*}
\mathbb{C} \delta \otimes \wedge^2 \mathcal{g} & \overset{\beta_f}{\longrightarrow} \mathbb{C} \delta \otimes \mathcal{F} \otimes \wedge^2 \mathcal{g} \overset{\beta_f}{\longrightarrow} \mathbb{C} \delta \otimes \mathcal{F}^{\otimes 2} \otimes \wedge^2 \mathcal{g} \overset{\beta_f}{\longrightarrow} \cdots \\
\mathbb{C} \delta \otimes \mathcal{g} & \overset{\beta_f}{\longrightarrow} \mathbb{C} \delta \otimes \mathcal{F} \otimes \mathcal{g} \overset{\beta_f}{\longrightarrow} \mathbb{C} \delta \otimes \mathcal{F}^{\otimes 2} \otimes \mathcal{g} \overset{\beta_f}{\longrightarrow} \cdots \\
\mathbb{C} \delta \otimes \mathbb{C} & \overset{\beta_f}{\longrightarrow} \mathbb{C} \delta \otimes \mathcal{F} \overset{\beta_f}{\longrightarrow} \mathbb{C} \delta \otimes \mathcal{F}^{\otimes 2} \overset{\beta_f}{\longrightarrow} \cdots
\end{align*}
\]

**Lemma 3.26.**

\[
\begin{align*}
E^{0,2}_2 & = \mathbb{C}[1 \otimes 1 \otimes X \wedge Y], \\
E^{q,2}_2 & = \mathbb{C}[1 \otimes \xi_q \otimes X \wedge Y] \oplus \mathbb{C}[1 \otimes \xi'_q \otimes X \wedge Y], \\
E^{0,0}_2 & = 1 \otimes \mathbb{C}, \\
E^{q,0}_2 & = 1 \otimes \xi_q \otimes \mathbb{C} \oplus 1 \otimes \xi'_q \otimes \mathbb{C},
\end{align*}
q \geq 1
\]

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Proof. The result for the 0th row is obvious. For the second row one observes that

\[ \nabla(1 \otimes X \wedge Y) = \]
\[ 1 \otimes 1 \otimes X \otimes Y + 1 \otimes \eta_1 \otimes Y \otimes Y - 1 \otimes \eta_1 \otimes Y \otimes Y - 1 \otimes 1 \otimes Y \otimes X = \]
\[ 1 \otimes 1 \otimes X \wedge Y. \]

\[ \square \]

Lemma 3.27.

\[ E_0^{0,1} = C[1 \otimes 1 \otimes Y], \]
\[ E_1^{0,1} = C[1 \otimes \xi_1 \otimes Y] \oplus C[1 \otimes \xi_1 \otimes X - 1 \otimes X \xi_1 \otimes Y] \]
\[ \oplus C[1 \otimes \xi_1 \otimes X - \frac{1}{2} \otimes X \xi_1 \otimes Y] \]
\[ E_2^{p,1} = C[1 \otimes \xi_p \otimes Y] \oplus C[1 \otimes \xi_p \otimes X - \frac{1}{|\xi_p|} \otimes X \xi_p \otimes Y] \]
\[ \oplus C[1 \otimes \xi_p \otimes X - \frac{1}{|\xi_p|} \otimes X \xi_p \otimes Y], \quad p \geq 2. \]

Proof. We filter \( E_1^{p,1} \) by setting

\[ E_1^{p,1} = 1 \otimes \mathcal{F}^{\otimes p} \otimes X \oplus 1 \otimes \mathcal{F}^{\otimes p} \otimes Y \supset 1 \otimes \mathcal{F}^{\otimes p} \otimes Y \supset 0. \]

The spectral sequence \( \mathcal{E} \) that computes \( E_2^{p,1} \) with respect to the above filtration is:

\[ \mathcal{E}_1^{-1,q} = 1 \otimes \mathcal{F}^{\otimes q-1} \otimes X, \quad \mathcal{E}_1^{0,q} = 1 \otimes \mathcal{F}^{\otimes q} \otimes Y, \quad \text{and the rest} \]
\[ = 0, \]
\[ \ldots \]
\[ \begin{array}{ccc}
1 \otimes \mathcal{F} \otimes X & 1 \otimes \mathcal{F}^{\otimes 2} \otimes Y & \ldots \\
1 \otimes C \otimes X & 1 \otimes \mathcal{F} \otimes Y & \ldots \\
0 & 1 \otimes C \otimes Y & \ldots \\
\end{array} \]

The \( \mathcal{E}_2 \) term is then described by
Here $\tilde{\gamma}(1 \otimes \tilde{f} \otimes X) = 1 \otimes \tilde{f} \otimes \eta_1 \otimes Y$. One applies Lemma 3.25 to deduce that all maps in the above diagram are 0, except $1 \otimes H^0(\mathcal{F}, C) \otimes X \otimes \tilde{\gamma} \rightarrow 1 \otimes H^1(\mathcal{F}, C) \otimes Y$, which sends the class of $1 \otimes 1 \otimes X$ to the class of $1 \otimes \eta_1 \otimes Y = 1 \otimes \xi_1 \otimes Y$.

The spectral sequence $E$ obviously collapses at this level.

In turn, the spectral sequence $E$ collapses at $E_2$, and we obtain the following result.

**Proposition 3.28.** The Hochschild cohomology groups of of $\mathcal{H}_1$ are given by

- $H^0(\mathcal{H}_1, C) = C[1 \otimes 1 \otimes 1]$
- $H^1(\mathcal{H}_1, C) = C[1 \otimes \xi_1 \otimes 1] \oplus C[1 \otimes \xi'_1 \otimes 1] \oplus C[1 \otimes 1 \otimes Y]$
- $H^2(\mathcal{H}_1, C) = C[1 \otimes \xi_2 \otimes 1] \oplus C[1 \otimes \xi'_2 \otimes 1] \oplus C[1 \otimes \xi' \otimes Y] \oplus$
  $C[1 \otimes 1 \otimes X \wedge Y] \oplus C[1 \otimes \xi_1 \otimes X - \frac{1}{|\xi_1|} 1 \otimes X \xi_1 \otimes Y] \oplus$
  $C[1 \otimes \xi'_1 \otimes X - \frac{1}{|\xi'_1|} 1 \otimes X \xi'_1 \otimes Y]$
- $H^p(\mathcal{H}_1, C) = C[1 \otimes \xi_p \otimes 1] \oplus C[1 \otimes \xi'_p \otimes 1] \oplus C[1 \otimes \xi_{p-1} \otimes Y] \oplus$
  $C[1 \otimes \xi'_{p-1} \otimes Y] \oplus C[1 \otimes \xi_{p-1} \otimes X - \frac{1}{|\xi_{p-1}|} 1 \otimes X \xi_{p-1} \otimes Y] \oplus$
  $C[1 \otimes \xi_{p-1} \otimes X - \frac{1}{|\xi'_{p-1}|} 1 \otimes X \xi'_{p-1} \otimes Y] \oplus$
  $C[1 \otimes \xi_{p-2} \otimes X \wedge Y] \oplus C[1 \otimes \xi'_{p-2} \otimes X \wedge Y], \quad p \geq 3.$
In the following proposition we use the standard notation \( S : HC^m(\mathcal{H}_1, \mathbb{C}_\delta) \to HC^{m+2}(\mathcal{H}_1, \mathbb{C}_\delta) \) for Connes’ periodicity operator [5].

**Proposition 3.29.** The following classes form a basis of \( HC^p(\mathcal{H}_1, \mathbb{C}_\delta) \):

- For \( p = 0 \), \( \theta_0 = 1 \otimes 1 \otimes 1 \);
- For \( p = 1 \), \( GV_1 \equiv \tau_1 = 1 \otimes \xi_1 \otimes 1 \), \( \tau'_1 = 1 \otimes \xi'_1 \otimes 1 \);
- For \( p = 2q \),
  \[
  \tau_p = 1 \otimes \xi_p \otimes 1, \quad \tau'_p = 1 \otimes \xi'_p \otimes 1,
  \sigma_p = 1 \otimes \xi_{p-1} \otimes X - \frac{1}{|\xi_{p-1}|} 1 \otimes X \xi_{p-1} \otimes Y,
  \sigma'_p = 1 \otimes \xi'_{p-1} \otimes X - \frac{1}{|\xi'_{p-1}|} 1 \otimes X \xi'_{p-1} \otimes Y,
  TF_p = S^{q-1}(1 \otimes 1 \otimes X \wedge Y);
  \]
- For \( p = 2q + 1 \),
  \[
  \tau_p = 1 \otimes \xi_p \otimes 1, \quad \tau'_p = 1 \otimes \xi'_p \otimes 1,
  \sigma_p = 1 \otimes \xi_{p-1} \otimes X - \frac{1}{|\xi_{p-1}|} 1 \otimes X \xi_{p-1} \otimes Y,
  \sigma'_p = 1 \otimes \xi'_{p-1} \otimes X - \frac{1}{|\xi'_{p-1}|} 1 \otimes X \xi'_{p-1} \otimes Y,
  GV_p = S^q(1 \otimes \xi_1 \otimes 1).
  \]

**Proof.** Evidently, \( HC^0(\mathcal{H}_1, \mathbb{C}_\delta) = H^0(\mathcal{H}_1, \mathbb{C}) \). From the bicomplex [3.7], one sees that the vertical boundary \( \partial B \) is given by the Lie algebra homology boundary of \( \mathfrak{g} \) with coefficients in \( \mathbb{C}_\delta \otimes \mathcal{F}^\otimes \), where \( \mathfrak{g} \) acts by [3.9]. We thus need to compute the homology of the derived complex \( \{ H^\bullet(\mathcal{H}_1, \mathbb{C}_\delta), \partial_\mathfrak{g} \} \). Since \( \partial_\mathfrak{g} : H^1(\mathcal{H}_1, \mathbb{C}) \to H^0(\mathcal{H}_1, \mathbb{C}) \) is given by

\[
\partial_\mathfrak{g}(1 \otimes \xi_1 \otimes 1) = 0, \quad \partial_\mathfrak{g}(1 \otimes 1 \otimes Y) = 1 \otimes 1;
\]

it follows that \( HC^1(\mathcal{H}_1, \mathbb{C}_\delta) \) is 2-dimensional, with the Godbillon-Vey class \( 1 \otimes \xi_1 \otimes 1 \) and Schwarzian class \( 1 \otimes \xi'_1 \otimes 1 \) as its basis.
Next, \( \partial_g: H^2(\mathcal{H}_1, \mathbb{C}) \rightarrow H^1(\mathcal{H}_1, \mathbb{C}) \) is given by
\[
\begin{align*}
\partial_g(1 \otimes \xi_2 \otimes 1) &= \partial_g(1 \otimes \xi'_2 \otimes 1) = 0, \\
\partial_g(1 \otimes 1 \otimes Y) &= 1, \\
\partial_g(1 \otimes \xi'_1 \otimes Y) &= 1 \otimes (1 - |\xi'_1|)\xi'_1, \\
\partial_g(1 \otimes 1 \otimes X \wedge Y) &= 1 \otimes 1 \otimes X - 1 \otimes 1 \otimes X = 0, \\
\partial_g(1 \otimes \xi_1 \otimes X - \frac{1}{|\xi_1|} 1 \otimes X\xi_1 \otimes Y) &= 0 \\
\partial_g(1 \otimes \xi'_1 \otimes X - \frac{1}{|\xi_1|} 1 \otimes X\xi'_1 \otimes Y) &= 0.
\end{align*}
\]

Hence \( HC^2(\mathcal{H}_1, \mathbb{C}) \) is 5-dimensional and is generated by the transverse fundamental class \( 1 \otimes 1 \otimes X \wedge Y \), together with the classes
\[
\begin{align*}
1 \otimes \xi_2 \otimes 1, \quad 1 \otimes \xi'_2 \otimes 1, \\
1 \otimes \xi_1 \otimes X - \frac{1}{|\xi_1|} 1 \otimes X\xi_1 \otimes Y, \quad 1 \otimes \xi'_1 \otimes X - \frac{1}{|\xi_1|} 1 \otimes X\xi'_1 \otimes Y.
\end{align*}
\]

Finally, for \( p \geq 3 \), \( \partial_g: H^p(\mathcal{H}_1, \mathbb{C}) \rightarrow H^{p-1}(\mathcal{H}_1, \mathbb{C}) \) one has
\[
\begin{align*}
\partial_g(1 \otimes \xi_p \otimes 1) &= \partial_g(1 \otimes \xi'_p \otimes 1) = 0, \\
\partial_g(1 \otimes \xi_p \otimes Y) &= (1 - |\xi_p|) 1 \otimes \xi_p, \\
\partial_g(1 \otimes \xi'_p \otimes Y) &= (1 - |\xi'_p|) 1 \otimes \xi'_p, \\
\partial_g(1 \otimes \xi_{p-2} \otimes X \wedge Y) &= |\xi_{p-2}| 1 \otimes \xi_{p-2} \otimes X - 1 \otimes X\xi_{p-2} \otimes Y, \\
\partial_g(1 \otimes \xi'_{p-2} \otimes X \wedge Y) &= |\xi'_{p-2}| 1 \otimes \xi'_{p-2} \otimes X - 1 \otimes X\xi'_{p-2} \otimes Y, \\
\partial_g(1 \otimes \xi_{p-1} \otimes X - \frac{1}{|\xi_{p-1}|} 1 \otimes X\xi_{p-1} \otimes Y) &= 0 \\
\partial_g(1 \otimes \xi'_{p-1} \otimes X - \frac{1}{|\xi_{p-1}|} 1 \otimes X\xi'_{p-1} \otimes Y) &= 0,
\end{align*}
\]
whence the claimed result. \( \square \)
References

[1] Akbarpour, R., Khalkhali M., Cyclic Cohomology of Crossed Coproduct Coalgebras, arXiv:math/0107166.

[2] Bott, R. and Haefliger, A., On characteristic classes of Γ-foliations, Bull. Amer. Math. Soc., 78 (1972), 1039–1044.

[3] Cartan, E., Les groupes de transformations continus, infinis, simples, Ann. Sci. École Norm. Sup., 3e série, 26 (1909), 93–161.

[4] Connes, A., Cohomologie cyclique et foncteur Ext^n, C.R. Acad. Sci. Paris, Ser. I Math., 296 (1983), 953-958.

[5] Connes, A., Noncommutative differential geometry. Inst. Hautes Etudes Sci. Publ. Math. No. 62 (1985), 257–360.

[6] Connes, A. and Moscovici, H., Hopf algebras, cyclic cohomology and the transverse index theorem, Commun. Math. Phys. 198 (1998), 199-246.

[7] Connes, A. and Moscovici, H., Cyclic cohomology and Hopf algebras. Lett. Math. Phys. 48 (1999), no. 1, 97–108.

[8] Connes, A. and Moscovici, H., Cyclic cohomology and Hopf algebra symmetry. Conference Moshé Flato 1999 (Dijon). Lett. Math. Phys. 52 (2000), no. 1, 1–28.

[9] Connes, A. and Moscovici, H., Background independent geometry and Hopf cyclic cohomology, arXiv:math.QA/0505475.

[10] Feigin, B. L. and Tsygan, B. L., Additive $K$-theory. $K$-theory, arithmetic and geometry (Moscow, 1984–1986), 67–209, Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987.

[11] Gelfand, I. M. and Fuks, D. B., Cohomology of the Lie algebra of formal vector fields, Izv. Akad. Nauk SSSR 34 (1970), 322-337.

[12] Getzler, E. and Jones, J. D. S., The cyclic homology of crossed product algebras J. reine angew. Math., 445 (1993), 163–174.

[13] Guillemin, V., Infinite dimensional primitive Lie algebras, J. Differential Geometry 4 (1970), 257–282.

[14] Gončarova, L. V. Cohomology of Lie algebras of formal vector fields on the line. Funkcional. Anal. i Priložen. 7 (1973), no. 2, 6–14.
[15] Hadfield, T and Majid, S. Bicrossproduct approach to the Connes-Moscovici Hopf algebra. J. Algebra 312 (2007), no. 1, 228–256.

[16] Hajac, P. M., Khalkhali, M., Rangipour, B. and Sommerhäuser Y., Stable anti-Yetter-Drinfeld modules. C. R. Math. Acad. Sci. Paris 338 (2004), no. 8, 587–590.

[17] Hajac, P. M., Khalkhali, M., Rangipour, B. and Sommerhäuser Y., Hopf-cyclic homology and cohomology with coefficients. C. R. Math. Acad. Sci. Paris 338 (2004), no. 9, 667–672.

[18] Kac, G. I., Extensions of groups to ring groups, Math. USSR Sbornik, 5 (1968), 451–474.

[19] Kassel, C., Quantum groups. Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[20] Khalkhali, M. and Rangipour B., Introduction to Hopf-Cyclic Cohomology, 155–178, Aspects Math., E37, Vieweg, Wiesbaden, 2006.

[21] M. Khalkhali, and B. Rangipour, On the generalized cyclic Eilenberg-Zilber theorem, Canad. Math. Bull. 47 (2004), no. 1, 38–48.

[22] Khalkhali, M., Rangipour, B., Invariant cyclic homology. K-Theory 28 (2003), no. 2, 183–205.

[23] Majid, S., Foundations of quantum group theory. Cambridge University Press, Cambridge, 1995.

[24] Moscovici, H., Rangipour, B., Cyclic cohomology of Hopf algebras of transverse symmetries in codimension 1, Adv. Math., 210 (2007), 323–374.

[25] Ponge, R., The tangent groupoid of a Heisenberg manifold, Pacific J. Math., 227, (2006), 151–174.

[26] Singer, I. M., Sternberg, S., The infinite groups of Lie and Cartan, I. The transitive groups, J. Analyse Math. 15 (1965), 1–114.

[27] Weyl, H., The Classical Groups. Their Invariants and Representations. Princeton University Press, Princeton, N.J., 1939.