Disk Accretion onto a Magnetized Star

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Abstract—An exact solution is found for the interaction of a rotating magnetic field that is frozen into a star with a thin, highly conducting accretion disk. The disk pushes the magnetic-field lines towards the star, compressing the stellar dipole magnetic field. At the corotation radius, where the Keplerian and stellar rotational frequencies are equal, a current loop appears. Electric currents flow in the magnetosphere only along two particular magnetic surfaces, which connect the corotation region and the inner edge of the disk with the stellar surface. It is shown that a closed current surface encloses the magnetosphere. The disk rotation is stopped at some distance from the stellar surface, equal to 0.55 of the corotation radius. The accretion from the disk spins up the stellar rotation. The angular momentum transferred to the star is determined.

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1. INTRODUCTION

The interaction of an accretion disk with a magnetized star is a very important problem for our understanding of compact X-ray sources. The energy release in variable X-ray sources and X-ray pulsars is due to the accretion of matter from a companion star onto the surface of a compact star, more specifically, a neutron star. Falling onto the neutron star, a particle gains an energy per unit mass equal to the gravitational potential at the stellar surface $\varphi = 0.15 c^2 (M_*/M_\odot) (R_*/10 \text{ km})^{-1}$, where $M_*$ and $R_*$ are the stellar mass and radius and $M_\odot$ is the solar mass. The energy release depends on how the accretion is realized. The magnetic field frozen into the neutron star is sufficiently high to prevent the free fall of matter onto the star. The naive point of view is that the accretion is free outside the so called Alfvén surface, and then continues along the magnetic field lines down to the polar caps of the star [6].

In a standard approach, the Alfvén surface is the surface where the magnetic-field pressure $B^2/8\pi$ is equal to the pressure of the accreting matter. However, the pressure of the accreting gas is not clear here. Is this the total dynamical pressure $P_\text{dm}$, or only its radial part, $P_\text{dm}^r$? The Alfvén radius is estimated for the case of spherical accretion, $v = v_K$, but is then applied to disk accretion. However, in this case, the gas velocity $v$ comes mainly from the rotational velocity of the disk, and is of order the Keplerian velocity $v_K$, which is much higher than the radial velocity. The gradient of the magnetic pressure in the azimuthal direction is small, and so does not significantly oppose the rotation of the disk. In addition, it is clear that the disk should compress the magnetic field within the disk inner edge. This reasoning does not help us in answering the question of the real position of the Alfvén surface.

An accreting gas is a plasma consisting of ions and electrons, which have a quite different Larmor radii in a magnetic field. Moving across the magnetic field, the plasma becomes polarized and creates a large-scale electric field $E$. If this field is $E = -v \times B/c$, the plasma continues to drift in the crossed $B$, $E$ fields at the same velocity $v$ without slowing down. The real value of $E$ depends on the boundary conditions at the disk and stellar surface, as well as the electric currents in the plasma and the conditions for them to close. The important characteristics are the electric conductivity of the disk plasma and of the surface layers of the neutron star. The ionized plasma of the accretion disk has the conductivity $\sigma = 10^{13} (T_e/1 \text{eV})^{3/2} (\Lambda/10)^{-1} \text{s}^{-1}$, which is high enough to consider the disk to be an ideal conductor. Here, $T_e$ is the electron temperature in the disk, which is higher than 10 eV, and $\Lambda$ is the Coulomb logarithm ($\Lambda \approx 20$).

At such high conductivities $\sigma$, the skin depth $\lambda_{sk} = (\sigma e^2/\sigma)^{1/2}$ is smaller than the disk thickness $H$. The value of $\tau$ is the characteristic time for turbulent motions in the $\alpha$-disk, $\tau = (\alpha \Omega_K)^{-1}$. The condition $\sigma \gg \tau e^2/H^2$ is well satisfied in the inner parts of the disk. The conductivity of the surface layers of the neutron star $\sigma_{NS}$ is high enough (10$^{21}$ s$^{-1}$) [2,
3, 9], $\sigma_{NS} \gg \sigma$) for the neutron star likewise to be considered an ideal conductor.

As an ideal conductor, the disk tends to exclude the stellar magnetic field, pushing it toward the star. Heavy ions in the disk tend to rotate with the Keplerian velocity, but the magnetized electrons are frozen into the magnetic-field lines, which rotate with the angular velocity of the star $\omega_s$. Thus, the point $\rho = \rho_c$, where a disk rotates with the star, $\omega_s \rho_c = (GM_s/\rho_c)^{1/2}$,

$$\rho_c = \left(\frac{GM_s}{\omega_s^2}\right)^{1/3} = 1.5 \times 10^8 \left(\frac{M_s}{M_\odot}\right)^{1/3} \left(\frac{P_s}{1 \text{ s}}\right)^{2/3} \text{ cm},$$

is the point beginning from which ($\rho < \rho_c$) the motions of the ions and electrons differ substantially. Here, $G$ is the gravitational constant and $P_s = 2\pi/\omega_s$ is the period of the stellar rotation. The corotation region is the region where the interaction of the accretion disk with the stellar magnetic field begins. The corotation point ($\rho = \rho_c$) is inside the light-cylinder radius $R_L = c/\omega_s$ for all rotating neutron stars: $P_s > 3 \times 10^{-5} (M_s/M_\odot) \text{ s}$. On the other hand, the corotation radius is larger than the neutron-star radius. Therefore, we will consider the unperturbed magnetic field of the neutron star to be dipolar, with dipole axis parallel to the neutron-star rotational axis.

The paper is organized as follows. Section 2 describes an arbitrary axisymmetric magnetic field. In Section 3, we determine the motion of ions in a disk. Section 4 describes the electric currents in the stellar magnetosphere. In Section 5, we find the structure of the magnetic field. Finally, in Section 6, we find the torque acting on the star and discuss our results.

2. AN AXISYMMETRIC MAGNETIC FIELD

Since our aim is to find the electric currents and magnetic field of the stellar magnetosphere distorted by the accretion flow, we first introduce convenient variables describing the magnetic field in the simplest way. For an axisymmetric magnetosphere, when the angle between the magnetic dipole and rotational axes is zero, it is useful to introduce the flux of the poloidal magnetic field $f(\rho, z)$, where $\rho$ and $z$ are cylindrical coordinates. The components of the magnetic field are then

$$B_\rho = -\frac{1}{\rho} \frac{\partial f}{\partial z}, \quad B_z = \frac{1}{\rho} \frac{\partial f}{\partial \rho}, \quad B_\phi = \frac{1}{\rho} g.$$ \hspace{1cm} (1)

The function $g(\rho, z)$ describes the toroidal magnetic field produced by the poloidal electric currents flowing in the stellar magnetosphere. The relation $f = C$, where $C$ is the constant, is the equation for the magnetic surface on which the magnetic-field lines lie. It is also convenient to describe the poloidal magnetic-field lines not only as $f = f(\rho, z)$, but also as $\rho = \rho(\varphi, z, f)$. The poloidal magnetic field is

$$B_\rho = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial z}\right)_f \left(\frac{\partial \rho}{\partial f}\right)_z^{-1}, \quad B_z = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial f}\right)_z^{-1}.$$ \hspace{1cm} (2)

For a dipole magnetic field, the magnetic flux $f(\rho, z)$ is

$$f_d = \frac{B_s R_s^2 \rho^2}{(\rho^2 + z^2)^{3/2}},$$

where $B_s$ is the magnetic-field strength at the stellar surface at the equator, $B_s = |B_z(r = R_s)|_{eq}$. It is important for us to know the magnetic flux in the equatorial plane, $z = 0, f_0(\rho) = f(\rho, z = 0)$. The vertical magnetic field at the equator is then $B_z(z = 0) = \rho^{-1} f_0(\rho)/\partial \rho$.

It is convenient to measure the magnetic field in units of $B_s$ and the distances $\rho, z$ in terms of the stellar radius $R_s$; then, the units for the magnetic flux will be $B_s R_s^2$. The relations (1) are the same in these dimensionless variables. For a dipolar field, the dimensionless flux is $f'_d = \rho^2/(\rho^2 + z^2)^{3/2}, f'_0 = 1/\rho'$.

In the stellar magnetosphere, electric currents flow along the magnetic field lines, $j = a B$, where $a(r)$ is an arbitrary scalar. In this case, the poloidal components of the Maxwell equation, $\nabla \times B = 4\pi j/c$, lead to the relations

$$g(\rho, z) = g(f), \quad a = \frac{c}{4\pi} \frac{dg}{df}.$$ \hspace{1cm} (3)

These relations mean that the toroidal magnetic field and the electric currents in an axisymmetric magnetosphere are functions of the poloidal magnetic flux, $f$. The important case is the absence of toroidal volume electric currents, $j_\phi = 0$. Then $(\nabla \times B)_\phi = 0$, and the equation for the magnetic flux $f$ is the Laplace type

$$\rho \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial f}{\partial \rho}\right) + \frac{\partial^2 f}{\partial z^2} = 0. \hspace{1cm} (2)$$

The function $f$ is neither even nor odd with respect to the coordinate $z$ due to the electric currents flowing in accretion disk in the equatorial plane. Thus, we expand the function $f$ over the exponential functions $\exp(-\lambda z)$ for $z > 0$. The solution of the equation (2) is

$$f(\rho, z) = \int_0^\infty \exp(-\lambda z) \varphi(\lambda) \lambda J_1(\lambda \rho) d\lambda.$$ \hspace{1cm} (3)
The star acts on ions via the gravitational force, \( F_\rho = -m_i G M_s / \rho^2 \), which, neglecting other forces, should result in the rotation of the ions at the Keplerian velocity, \( v_K = (G M_s / \rho)^{1/2} \). The \( \rho \) component of (7) gives

\[
v_\rho \frac{\partial v_\rho}{\partial \rho} - \frac{v_\rho^2}{\rho} = \frac{q_i}{m_i} \left( \frac{E_\rho + q_\phi B_z}{c} \right) - \frac{v_\rho^2}{\rho}. \tag{8}
\]

We can see from (8) that, in the absence of the electromagnetic fields \( B_z \) and \( E_\rho \), the ions rotate with the Keplerian velocity, \( v_\rho = v_K \). In contrast, in a strong magnetic field \( B_z \), the ions have an electric drift velocity, \( v_\phi = -e E_\rho / B_z \). For corotation, i.e., rigid rotation at the angular velocity of the star, \( v_\phi = \omega_s \rho \), the plasma in the disk must be polarized to create a radial electric field, \( E_\rho = -\omega_s \rho B_z / c \). The corotation velocity is less than the Keplerian velocity at distances \( \rho < \rho_c \). Because the Keplerian velocity decreases with increasing distance \( \rho \), corotation of the plasma disk means that the centrifugal force in the disk becomes less than the gravitational force at \( \rho < \rho_c \), and the ions move inwards in the radial direction. However, this is not easy for them, due to the conservation of angular momentum, which is described by the \( \phi \) component of (7),

\[
v_\rho \frac{\partial v_\phi}{\partial \rho} + \frac{v_\rho v_\phi}{\rho} = -\frac{q_i}{m_i} v_\rho B_z. \tag{9}
\]

Note that there is no toroidal electric field, \( E_\phi = 0 \), because, under stationary conditions, the electric field must be potential. Since the ion radial velocity, \( v_\rho \), is not equal to zero [see (6)], the solution of (9) is

\[
\rho v_\phi + \frac{q_i}{m_i} f_0(\rho) = \text{const}. \tag{10}
\]

Equation (10) represents the conservation of the total angular momentum of an ion in the magnetic field. However, the dimensionless coefficient of the second term in (10), written \( \left( q_i B / cm_i \omega_s \right) (\omega_s f_0 / B) \), is much greater than unity. It is proportional to the ratio of the ion cyclotron frequency in the stellar magnetic field, \( \omega_{ci} = q_i B_s / cm_i \), to the frequency of the stellar rotation, \( \omega_s \). We denote this ratio \( \Omega_c = \omega_{ci} / \omega_s \). This is the cyclotron frequency of the ions in the natural units of the stellar rotation frequency

\[
\Omega_c = 1.5 \times 10^{12} \left( \frac{P_s}{1 \text{ s}} \right) \left( \frac{B_s}{10^9 \text{ G}} \right).
\]

For a dipolar magnetic field, the terms on the left-hand side of (10) become comparable only at large distances, \( \rho / B_s \approx \Omega_c^{-3/2} \). Angular-momentum conservation hinders radial motion of the ions in a magnetic field as strong as that of the star, if we assume it is (close to) dipolar. The only way to allow the ions to move radially is the expulsion of magnetic field from
the disk. Only if the magnetic flux changes slowly in the disk, \( \Delta f_0 \approx B_s \rho^2 \Omega_c^{-1} \), does the radial motion of ions become possible. However a small variation of the magnetic flux \( f_0 \) with the radial distance \( \rho \) implies a small vertical magnetic field \( B_z \) in the equatorial plane [see (1)].

The parameter \( \Omega_c \) for the electrons is at least a factor \( 10^5 \) greater than for the ions. The plasma electrons are strongly magnetized, and can move only along magnetic surfaces, \( f = \text{const} \). They generate a current toward the star at \( f = f_c = f_0 (\rho = \rho_c) \). Thus, the disk plasma becomes separated at \( \rho < \rho_c \): the ions form the plasma disk in the equatorial plane, but the electrons create the electric current \( I_c \) on the magnetic surface \( f = f_c \). This does not mean that the plasma disk is positively charged—it is neutral. Electrons from the “sea” of electrons in the stellar magnetosphere neutralize any excess electric charge.

4. MAGNETOSPHERIC AND SURFACE ELECTRIC CURRENTS

We consider the magnetospheric plasma to have infinite conductivity. Under stationary conditions, it can rotate about the \( z \) axis with angular velocity \( \omega (\rho, z) \). In a frame rotating with this velocity, the electric field \( E' \) is equal to zero. Therefore, the electric field in the laboratory frame is

\[
E = -\frac{1}{c} (\omega \times r) \times B, \quad E_\rho = -\frac{\omega}{c} \frac{\partial f}{\partial \rho},
\]

\[
E_z = -\frac{\omega}{c} \frac{\partial f}{\partial z}, \quad E_\phi = 0.
\]

The electric field must be potential \( E = -\nabla \Psi \), where \( \Psi \) is the electric potential. This means that the frequency of the rotation \( \omega (\rho, z) \) is a function of the magnetic flux \( f \), \( \omega = \omega (f) \) alone, and the electric potential is \( \Psi = \int f \omega (f') \rho f'/c \). Because the conductivity of the stellar surface is very high, \( \omega (f) = \omega_s \).

The magnetospheric current \( J_c \) flows from the stellar surface to the disk along the magnetic surface \( f = f_c \). This current is closed by the surface current \( J_s \). The current \( J_s \) flowing on the stellar surface is related to the volume magnetospheric current \( j \), by the continuity equation:

\[
\nabla_s \cdot J_s = -j_n.
\]

The divergence in this equation is taken along the surface where the currents flow, and \( j_n \) is the normal (to this surface) component of the volume current \( j \). An analogous equation relates the ion surface current of the disk \( J_d \) and the magnetospheric volume current \( j \). However, as we have already noted [see (5)], the divergence of the ion surface current is zero. The disk electrons can only rotate with the angular velocity of the magnetosphere \( \omega (f) \), and the divergence of their flux is also zero. Thus, there is no volume magnetospheric current \( j \) except on the particular magnetic surfaces \( f = f_c \) and \( f = f_s \). The value of \( f = f_s \) is the magnetic flux where the disk stops rotating, \( f_s > f_c \).

The total current \( I \) produced by the accretion flow is

\[
I = \frac{q_i \dot{M}}{m_i} = 6 \times 10^{20} \left( \frac{\dot{M}}{10^{-10} M_\odot/yr} \right) A. \tag{11}
\]

Half this current flows in the Northern and half in the Southern hemisphere. The magnetospheric currents flow along the magnetic field, on the two magnetic surfaces

\[
j_s = -B \frac{q_i \dot{M}}{4\pi m_i} \delta (f - f_s), \tag{12}
\]

\[
j_c = B \frac{q_i \dot{M}}{4\pi m_i} \delta (f - f_c).
\]

These currents are closed by the surface current flowing in the polar region of the star.

The magnetospheric electric currents \( j_s, j_c \) produce in the region \( f_s > f > f_c \) the toroidal magnetic field \( B_\phi \),

\[
B_\phi = \frac{q_i \dot{M}}{\rho c m_i}, \tag{13}
\]

The \( \phi \) component of the Maxwell equation \( \nabla \times B = 4\pi j/c \) then gives the equation for the poloidal magnetic flux

\[
\frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial f}{\partial \rho} \right) + \frac{\partial^2 f}{\partial z^2} = \frac{q_i \dot{M}}{2\pi c m_i} \left[ \delta (f - f_s) - \delta (f - f_c) \right]. \tag{14}
\]

The right hand side of (14) is non-zero only on the magnetic surfaces \( f = f_s, f_c \), where the magnetospheric electric currents flow along the magnetic field. Due to the toroidal component of the magnetic field, the longitudinal currents have a toroidal component, which distorts the poloidal magnetic field. Outside the surfaces \( f_s, f_c \), Eq. (14) coincides with (2), and its solution is given by (3), (4). However, the fields on either side of the discontinuities are different. The relationship between the magnetic fields at \( f > f_s \) and \( f < f_s \), and also at \( f > f_c \) and \( f < f_c \), can be obtained by integrating (14) over \( f \) in a small region near surfaces \( f = f_s \) and \( f = f_c \), respectively. This requires transformation to other independent variables, \( (z, f) \), considering the magnetic surfaces to be determined by the relation \( \rho = \rho (z, f) \). Equation (14) then becomes

\[
\frac{1}{2} \left[ 1 + \left( \frac{\partial \rho}{\partial z} \right)^2 \right] \frac{\partial}{\partial f} \left[ \left( \frac{\partial \rho}{\partial f} \right)^{-2} \right] \tag{15}
\]
Integrating (15) then yields the continuity:

\[ \frac{\partial \rho}{\partial f} \bigg|_{f=f_{s,c}} = \frac{\partial \rho}{\partial z} \bigg|_{f=f_{s,c}}. \]

We must take into account that the shapes of the magnetic surfaces are the same on either side of discontinuity:

\[ \frac{\partial \rho}{\partial z} \bigg|_{f=f_{s,c}+0} = \frac{\partial \rho}{\partial z} \bigg|_{f=f_{s,c}-0}. \]

Integrating (15) then yields

\[ B_\rho^2 + B_z^2 + B_\phi^2 = \text{const.} \quad (16) \]

This is the natural condition for equality of the magnetic pressures on either side of a discontinuity, which follows exactly from our equations.

5. STRUCTURE OF THE MAGNETIC FIELD

Further, we will use the dimensionless variables

\[ \omega' = \omega/\omega_s, \quad r' = r/R_s, \quad f' = f/B_s R_s^2. \]

\[ \Sigma' = 2\pi m_i \omega_s R_s^2 \Sigma/M, \quad J' = 4\pi R_s J/I, \]

omitting the \( t \) for simplicity of notation. Several dimensionless parameters arise. The ratio of the ion cyclotron frequency to the frequency of the stellar rotation, noted the Section 3, is

\[ \Omega_c = \frac{q_i B_s}{c m_i \omega_s} \quad (18) \]

The second parameter is the magnetic field created by the current \( I/2 \) (11) flowing on the stellar surface, in units of \( B_s \):

\[ B = \frac{q_i M}{c m_i R_s B_s} = \frac{M}{10^{-10} M/\text{yr}} \]

\[ \times \left( \frac{B_s}{10^{12} \text{ G}} \right)^{-1} \left( \frac{R_s}{10^6 \text{ cm}} \right)^{-1}. \]

Moving toward the star, the disk pulls out the stellar magnetic-field lines. The vertical component of the magnetic field \( B_z \) decreases, and the ions can move in the radial direction. However, the electrons remain magnetized, and must move along magnetic-field surfaces only. The single possibility for them to change their motion from Keplerian rotation to corotation with the star is to do this at the corotation point \( \rho = \rho_c \). Thus, the disk rakes up all the dipolar magnetic-field lines crossing the equatorial plane at \( \rho > \rho_c \) into the region \( \rho < \rho_c \). This must happen, because the magnetic-field lines were initially frozen into the ideally conducting matter of the disk, and had to move together with it. We therefore conclude that, under the action of the ideal accretion disk, the stellar magnetosphere is confined to the region \( \rho < \rho_c \). Thus, the magnetic flux at the point \( \rho = \rho_c \) is zero, \( f_c = 0 \). The magnetic-field line \( f = 0 \) just comes from the magnetic pole of the star. The topology of this magnetic-field line is drawn in Fig. 1.

The rotational velocity of the disk-plasma ions is given by Eq. (10):

\[ v_\phi = \frac{\rho^2}{\rho} - \frac{\Omega_c f_0(\rho)}{\rho}. \]

Here, we assume that the velocity of the ion’s rotation is equal to the Keplerian velocity \( v_K \) at the point \( \rho = \rho_c \). Equation (20) specifies the magnetic flux at the point where the disk rotation stops, \( v_\phi = 0 \), \( f = \rho_c^2/\Omega_c \). However, the disk electrons rotate with the angular rotational velocity of the magnetic-field lines, \( \omega(f) = \omega_s \). Thus, the disk plasma produces a surface toroidal electric current

\[ J_\phi = 2B\Sigma(v_\phi - \rho), \quad (21) \]

which yields a discontinuity of the radial magnetic field on the surface \( z = 0 \):

\[ B_\rho|_{z=z+0} = -B_\rho|_{z=z-0} = B\Sigma(v_\phi - \rho). \quad (22) \]
The value of \((v_\phi - \rho)\) is a function of \(\rho\), \(f_0(\rho)\) and is determined by (20):

\[
v_\phi - \rho = \frac{\rho_c^2}{\rho} - \frac{\Omega_c f_0}{\rho} - \rho. \tag{23}\]

We can see from (22) and (23) that the radial magnetic field is negative up to the stopping point \(\rho = \rho_s\), \(B_\rho(\rho, z = +0) < 0\). At the point \(\rho = \rho_c\), the radial magnetic field in the equatorial plane is zero, and the magnetic-field lines here are perpendicular to the disk. The magnetic-field lines then bend toward the disk.

The value of the radial magnetic field \(B_\rho(\rho, z = z + 0)\) determines the derivative of the magnetic field with respect to \(z\) on the equator:

\[
\frac{\partial f_0}{\partial z} \bigg|_{z=+0} = -B \Sigma \rho (v_\phi - \rho). \tag{24}\]

Let us find this derivative as a function of \(f_0(\rho)\). We obtain from (3), (4)

\[
\frac{\partial f_0}{\partial z} \bigg|_{z=+0} = -\rho \int_0^\infty d\rho' f_0(\rho') \int_0^\infty \lambda^2 J_1(\lambda \rho) J_1(\lambda \rho') d\lambda.
\]

The second integral is known \([8]\), and we obtain

\[
\frac{\partial f_0}{\partial z} \bigg|_{z=+0} = \frac{2}{\pi} \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \int_0^\infty d\rho' K(x) \frac{f_0(\rho/x)}{dx} dx 
\right.
\]

\[
\left. - \int_0^\infty d\rho' K(x) f_0(\rho x) dx \right). \tag{25}\]

Here, \(K(x)\) is a complete elliptical integral of the first kind. It equals \(\pi/2\) at \(x = 0\), but has a logarithmic singularity at \(x \to 1\), \(K(x) \to \ln[16/(1 - x^2)]/2\). Equation (25) must be supplemented with the condition that there be no radial magnetic field in the region \(\rho < \rho_s\), \(\partial f_0/\partial z = 0\). Note that substitution of a dipole magnetic field, \(f_0 = 1/\rho\), into (25) satisfies this condition for all \(\rho\). In addition, the square of the vertical magnetic field \(B_\rho^2\) behind the stopping point, \(\rho = \rho_s - 0\), must be equal to the sum of the squares of the radial, \(B_\rho^2 (B_\rho \ll B_\rho)\), and toroidal, \(B_\phi^2 = B^2 - B_\rho^2\), fields before the stopping point [see the condition (16)]. As a result, \(B_\rho = -\mathcal{B}(1 + \rho_s^2/2\rho(\rho = \rho_s))^{1/2}/\rho_s\). The magnetic field is strongly compressed here by the disk, \(|B_\rho| \gg |B_{\text{dip}}| = \rho_s^{-3}\). The equations (24), (25) close the system of equations determining the function \(f_0(\rho)\). We add also the conservation law following from (8), (9):

\[
d \left[ \frac{1}{2} \left( v_\rho^2 + v_\phi^2 - \frac{\rho_c^3}{\rho} + \Omega_c f_0(\rho) \right) \right]/d\rho = 0. \tag{26}\]

The particle energy decreases due to the work done by the radial electric field in the disk on the ions. This electric field is generated in the disk by the rotating magnetic field. Assuming that the radial velocity of the disk \(v_\rho\) is smaller than the rotational velocity \(v_\phi = v_K\) at the point \(\rho = \rho_c\), we can find the radial velocity of the disk at any point \(\rho_s < \rho < \rho_c\):

\[
v_\rho^2(\rho) = \frac{2\rho_c^3}{\rho} - \rho_c^2 - v_\phi^2 - 2\Omega_c f_0(\rho). \tag{27}\]

The radial velocity \(v_\rho\) determines the surface density \(\Sigma(\rho) = 1/(|v_\rho|)\). The radial velocity \(v_\rho\) at the stopping point \(\rho = \rho_s\) is not zero, and is negative:

\[
v_\rho(\rho = \rho_s) = -\rho_c \left[ 2 \left( \frac{\rho_c}{\rho_s} - \frac{3}{2} \right) \right]^{1/2}. \tag{28}\]

At the point \(\rho = \rho_s\), the disk ions begin to move along the magnetic surface \(f = f_s\) toward the stellar surface toward the polar cap. Particles without mechanical angular momentum \((v_\rho(\rho = 0) = 0)\) fall down toward the star freely, but along curved, fixed magnetic-field lines. They acquire the kinetic energy \(v^2/2 = \rho_c^3 - \rho_s^2\) at the stellar surface, which corresponds to falling from the height of the corotation point \(\rho_c\).

The function \(f_0(\rho)\) on the entire axis \(0 < \rho < \infty\) is different in three different regions: \(f_0 = f_0^{(1)}(\rho)\) for \(\rho < \rho_s\), \(f_0 = f_0^{(2)}(\rho)\) for \(\rho_s < \rho < \rho_c\), and \(f_0 = 0\) for \(\rho > \rho_c\). Thus, the (25) becomes

\[
\frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \int_0^{\rho_s/\rho} d\rho' K(x) f_0^{(1)}(\rho/x) dx \right)
\]

\[
+ \int_{\rho/s/\rho}^{\rho_c/\rho} d\rho' K(x) f_0^{(2)}(\rho/x) dx = 0, \quad \rho < \rho_s, \tag{27}\]

\[
\frac{2}{\pi} \frac{\partial}{\partial \rho} \left( \frac{\rho_s/\rho}{\rho} \int_0^{\rho_s/\rho} d\rho' K(x) f_0^{(1)}(\rho/x) dx \right)
\]

\[
+ \int_{\rho_s/\rho}^{\rho_c/\rho} d\rho' K(x) f_0^{(2)}(\rho/x) dx = 0, \quad \rho_s < \rho < \rho_c. \tag{28}\]
The magnetic flux \( f^{(2)}_0 \) is small. At \( \rho = \rho_s \), it is equal to \( f^{(2)}_0 = f_s \approx 1/\rho_s \), after which it decreases to zero at \( \rho = \rho_c \). On the contrary, the function \( f^{(1)}_0(\rho) \) increases strongly from the value \( f^{(1)}_0 = f_s \) at \( \rho = \rho_s \) to the dipolar value \( 1/\rho_s \) at \( \rho < \rho_s \), due to the large derivative \( \partial f^{(1)}_0 / \partial \rho = -B(1 + \rho_s^2/\rho^2)^{1/2} \). The width of this transition region \( \Delta \rho \) is small, \( \Delta \rho \approx 1/(\rho_s B) \ll \rho_s \). The magnetic field is large here, and is strongly compressed by the disk. Further, with approach toward the star, the magnetic field becomes dipolar, \( f^{(1)}_0 = 1/\rho \). The function \( f_0(\rho) \) is plotted in Fig. 2.

The position of the stopping point \( \rho_s \) can be determined from (28). At the corotation point \( \rho_c \), the right-hand side of (28) is zero. The main contribution to the left-hand side comes from the dipolar part of \( f_0 \), \( f^{(1)}_0 = 1/\rho \). Substituting this dependence, integrating, differentiating, and equating the result to zero at \( \rho = \rho_c \) yields the equation

\[
E \left( \frac{\rho_s}{\rho_c} \right) = K \left( \frac{\rho_s}{\rho_c} \right) \left( 1 - \frac{\rho_s^2}{\rho_c^2} \right).
\]

Here, \( E(x) \) is a complete elliptical function of the second kind. Apart from the trivial root, \( \rho_s = 0 \), the root of this equation is \( \rho_s = 0.55 \rho_c \). The point \( \rho = \rho_s \) can be called the Alfvén point, but it corresponds to the stopping of the disk rotation, not of the radial motion. Its position is determined by the corotation radius, and does not depend on the stellar magnetic field \( B_s \). This is due to the strong compression of the dipolar magnetic field by the disk in the region \( \rho < \rho_s \).

6. DISCUSSION

We have shown that the structure of the magnetic field frozen into the rotating neutron star and interacting with the thin accretion disk can be solved exactly, and we have obtained an analytical solution for this field. To achieve this, we used certain ideal assumptions: an axisymmetric, well conducting magnetosphere with a thin plasma disk. These conditions offer an adequate description of reality. We have demonstrated that the disk compresses the stellar magnetic field, pushing it towards the star. The magnetic field is compressed in the region \( \rho < \rho_s \), where \( \rho_s \) is the inner edge of the disk, where it stops rotating in the laboratory frame, \( \rho_s = 0.55 \rho_c \). Further, the disk material with zero mechanical-angular momentum (\( v_\phi = 0 \)) falls onto the star along the magnetic surface \( f = f_s \). An electric current loop starts at the corotation point \( \rho = \rho_c \).

The electric current \( I(11) \) flows along the disk at \( \rho_s < \rho < \rho_c \), then half of it (i.e., \( I/2 \)) flows along the magnetic surface \( f = f_s \) to the stellar surface, then along the stellar surface in the polar region, and finally back to the disk along the magnetic surface \( f = f_c = 0 \). There is a symmetric current structure in the second hemisphere. We emphasize that the magnetosphere exists only inside the magnetic surface \( f = f_c = 0 \), where it is maintained by the pinching Ampère force \( f_a = \mathbf{J}_e \times \mathbf{B}_s/\epsilon \) acting on the electron current \( \mathbf{J}_e \) flowing outside the star along the magnetic surface \( f = f_c \). This force balances the magnetic-pressure gradient \( \nabla B^2/8\pi \) inside the magnetosphere boundary.

The potential difference \( \Psi \) created by the rotating magnetic field is

\[
\Psi = \frac{\omega_s}{c} f_s.
\]

The electric current \( I \) flowing along the stellar surface perpendicular to the magnetic field creates an Ampère force and spins up the stellar rotation. The torque \( K \) is

\[
K = I \psi / \omega_s - M \rho_s^2 \omega_s.
\]

This shortens the stellar rotation period, \( \dot{\Omega} < 0 \):

\[
\dot{\Omega} = -1.5 \times 10^{-13} \left( \frac{\rho_s}{1 \text{ s}} \right)^{7/3} \left( \frac{M}{10^{-10} \odot /\text{yr}} \right) \times \left( \frac{M_s}{M_\odot} \right)^{2/3} \left( \frac{I_s}{10^{45} \text{ g cm}^2} \right)^{-1}.
\]

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Here, \( I_s \) is the moment of inertia of the star. This \( \dot{P}_s \) corresponds to the observed value for close X-ray binaries [10]. The total angular momentum possessed by the disk at the corotation point is transferred to the star. The answer does not depend on the strength of the stellar magnetic field \( B_s \). It is due to the strong magnetization of the disk plasma. The magnetization parameter \( \Omega_s \) (18) is much greater than unity, and it is not important whether it is very large \((10^{15})\), or not so large \((10^5)\). The role of the stellar magnetic field consists in stopping the disk rotation; i.e., transforming the mechanical angular momentum of the particles into electromagnetic momentum. The angular momentum of a charged particle in a magnetic field consists of two parts: mechanical and electromagnetic [see (10)]. As it approaches the star, a particle loses mechanical and gains electromagnetic angular momentum. Therefore, if an element of matter falls onto the star, \( \dot{M} > 0 \), it transfers its angular momentum to the star, \( K > 0 \), whether this angular momentum is mechanical or electromagnetic. Thus, during accretion, there cannot be a negative torque that could be attributed to an electromagnetic component [3]. The deceleration of the stellar rotation is associated with the assumptions that, first, the toroidal magnetic field \( B_\phi \) changes its direction at some radius and, second, the poloidal field \( B_z \) in the disk does not differ strongly from a dipolar field (see, e.g., [5]). The toroidal magnetic field is determined by poloidal currents, and a zero toroidal field at some distance means a closure of currents. At larger distances the field \( B_\phi \) is equal to zero. Indeed, we see that a toroidal field exists only in the region \( \rho_s < \rho < \rho_c \) and \( B_\phi > 0 \) (13).

Note that it is not possible to construct self consistent solution for the stellar magnetosphere in the case of a low accretion rate \( \dot{M} \). When the magnetic field at the point \( \rho_s \) becomes less than the dipolar field at this point, a rarification of the magnetic field arises. This is not possible for the plasma moving toward the star, because the magnetic field is frozen into the conducting material. Formally, it is not possible to match the solution for the magnetic flux \( f_0(\rho) \) with the dipolar field in this case. This takes place when \( |B_\phi(\rho = \rho_s)| < \rho_s^{-3} \). Because \( |B_\phi(\rho)| = B(1 + \rho_\phi^2/\rho_s^2)^{1/2}/\rho_s \), \( \rho_s = 0.55\rho_c \) and \( \rho_\phi = -0.8\rho_c \), this condition is \( B < 2\rho_c^{-2} \). This means that the expression (29) for the torque is valid when \( \dot{M} > \dot{M}_{\text{min}} \).

\[
\dot{M} > \dot{M}_{\text{min}} = 2\frac{c m_i B_s R_s^3}{q_i \rho_c^2} \quad (30)
\]

\[
= 3 \times 10^{-16} \frac{\mu_s}{10^{30} \text{ G cm}^3} \left( \frac{\rho_c}{10^8 \text{ cm}} \right)^{-2} M_\odot/\text{yr}.
\]

Here, \( \mu_s \) is the stellar magnetic moment, \( \mu_s = B_s R_s^3 \). For \( \dot{M} < \dot{M}_{\text{min}} \) the disk does not penetrate into the stellar magnetosphere.

In some sense, the stellar magnetic field does not hinder the accretion of ionized matter onto the star, and instead facilitates this process. A matter element cannot accrete without the presence of a magnetic field, due to angular-momentum conservation. It is necessary to introduce an anomalous viscosity represented by the \( \alpha \) parameter in \( \alpha \)-disks. A charged particle possesses two types of angular momentum in a magnetic field: mechanical and electromagnetic. In the language of magnetohydrodynamics, we have the viscous stress tensor and the magnetic stress tensor, which is proportional to the product \( B_\phi B_z \).

The appearance of a toroidal magnetic field is impossible without poloidal electric currents, which must be closed at the stellar surface, transforming the mechanical angular momentum of the disk into angular momentum of the star.

It is clear that, if the inner edge of the disk is located outside the corotation point \( (\rho_d > \rho_c) \), the disk will extract angular momentum from the star. In this case, the centrifugal force is greater than the gravitational force, and the disk should be pushed away from the star. This is the propeller regime. It represents a stationary or quasi-stationary process if we have a source of matter at the inner edge of the disk. Such a situation occurs in the magnetosphere of Jupiter, where Jupiter’s satellite Io supplies gas to a plasma disk at a point \( \rho_d > \rho_c \). The electric current in the current loop flows in the opposite direction as in the case of accretion [4]. If there is no source of plasma in the stellar magnetosphere, the disk is pushed out to the light cylinder, taking an angular momentum \( \Delta J \) from the star of the order of

\[
\Delta J \approx M_d R_L^2 \omega_s.
\]

Here, \( M_d \) is the mass of the disk and \( R_L \) is the radius of the light cylinder, \( R_L = c/\omega_s \).

Thus far, we have considered only the situation when the disk and star rotate in the same direction (a corotating disk). There also exists the interesting case when the accretion disk rotates in the opposite direction from the star. A corotating disk has the same sign of angular momentum as the star. The disk material falling onto star transfers its angular momentum to the star. The torque then becomes positive. If the disk rotates in the opposite direction from the star (it is counter-rotating), matter falling onto the stellar surface will spin down the stellar rotation, and the torque becomes negative. It is impossible to build up a stationary, self-consistent solution for a counter-rotating disk in the way we did for the corotating case. The star should destroy the inner parts of the disk that approach too close as it tries to
change the disk rotation to the direction of its own rotation. In other words, the coupling between the disk and magnetosphere tends to enforce corotation of the inner layers of the disk. The accretion becomes more quasispherical than disk-like. We can estimate with a good accuracy the torque acting on the star in this case, aided by the exact solution obtained above.

We saw that, to reduce its mechanical angular momentum to zero, a falling particle must deviate from its position on the magnetic surface $f$ by $\Delta f = \rho_m v_m / \Omega_c$ [see (20)]. Here, $\rho_m$ and $v_m$ are the radial position and azimuthal velocity of a particle at the magnetospheric boundary. For a rotating disk, the boundary of the magnetosphere is the corotation radius, $\rho = \rho_c$. In the case of counter-rotation, there is no corotation point. The magnetospheric boundary for quasispherical accretion should be close to the Alfvén radius, $\rho_m \simeq \rho_A$. Ions falling onto the star move almost along magnetic-field lines, deviating by a small amount $\Delta f$. Electrons will also deviate, but their magnetization parameter $\Omega_e$ is much larger than the value for ions, and we can neglect the electron deviation. Thus, the ions and electrons reach the stellar surface on different magnetic surfaces separated by $\Delta f$. A current $I$ (11) will flow along the surface, and will produce the torque $K$ (29). In the case of counter-rotation of the star and disk, $\Delta f$ will be negative, and the electric current on the stellar surface will have the opposite direction from the case of corotation. As a result, the torque acting on the star will be negative and equal to $K = -\dot{M} v_m \rho_m$. That is an exact relation, but the position of the magnetosphere boundary is not known exactly. If we do not take into account the compression of the magnetic field at the boundary, it is given by the traditional formula for the Alfvén radius, $\rho_m = \rho_A = (\mu_0^2 / 2^{1/2} \dot{M} \rho_c^{3/2} \omega_s)^{2/7}$ (see, for example, [1]).

In order for accretion onto the star to take place, the radius of the magnetosphere $\rho_m$ must be smaller than the corotation radius $\rho_c$. This means that the decelerating torque will always be smaller than the accelerating torque for the same accretion rate $\dot{M}$. Thus, the observed stellar spin-down with appreciable $\dot{M}$ could be the result of the accretion of matter from a counter-rotating disk. This is quite possible, because the disk formation depends on the companion star, the stellar orbital motion, and many other factors.

In conclusion we compare the results obtained here with numerical calculations carried out in the framework of an MHD approximation [11, 12]. Though our treatment was carried out using a two-fluid hydrodynamical approach, similar features were seen in the single-fluid numerical simulations: the compression of the stellar dipole magnetic field by the disk, the falling of the disk matter along fixed magnetic-field lines, the pushing out of magnetic-field lines from the disk, and the proportionality of the torque $K$ acting on the star to the mass-accretion rate $\dot{M}$.

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