Local Average and Scaling Property of $1/f^\alpha$ Random Fields

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We use the techniques developed in [1] to study the local average of random fields with spectral density $1/f^\alpha$. We study their scaling properties and show that the self-similarity of $1/f$ random fields is preserved under the local average. We study how the original random fields can be recovered from the locally averaged ones. We also study the derivative of the locally averaged random fields as a way to get the spectral density. Finally, we propose the generalization of local average by means of an arbitrary response function.

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INTRODUCTION

Random fields are associated with probabilities. If we want to study a one-dimensional random field $X(t)$, we would like to specify not only the probability density function $P(x,t)$ for this field but also the joint probability density functions $P(x_1, t_1; x_2, t_2), \ldots$, $P(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)$ and so on. Here the capital letter $X$ denotes the random field, and the lower case letter $x$ denotes the possible value it can take, and $t$ is the time. With those probabilities in hand, we can study many statistical properties of the random fields. Some are first order effects, like the mean or expectation value $E[X(t)]=\int xP(x,t)dx$, some are second order effects, like the 2-point correlation function and covariance, and some are higher orders, like the $n$-point correlation functions. The $n$-th order effects are related to the $n$-point joint probability density functions.

Most random fields have "memories", namely they have correlations of their values taken at different times. Some don’t have correlations and are thus indeed random. Those with non-zero correlations are not that random. Their covariance can be studied as deterministic and can be subtracted for simplicity of discussion. The covariance $B(t_1, t_2)$ is defined as the covariance of the fluctuation around the mean, and is associated with the joint probability functions:

$$B(t_1, t_2) = E[(X(t_1) - E[X(t_1)])(X(t_2) - E[X(t_2)])]$$

$$= \int_{-\infty}^{+\infty} x_1x_2[P(x_1, t_1; x_2, t_2) - P(x_1, t_1)P(x_2, t_2)]dx_1dx_2$$

We can see that the covariance is non-zero only if the 2-point joint probability density function is not simply the product of two probability functions, namely there is some entanglement between the two events $(x_1, t_1)$ and $(x_2, t_2)$ such that the conditional probability density function is not the same with the single probability density function: $P(x_1, t_1 \mid x_2, t_2) \neq P(x_1, t_1)$, and $P(x_2, t_2 \mid x_1, t_1) \neq P(x_2, t_2)$.

If the random field is stationary, $B(t_1, t_2)$ is only a function of $\tau = t_1 - t_2 : B(t_1, t_2) = B(\tau)$. The covariance function $B(\tau)$ can be more conveniently studied by using the Fourier transform. The Fourier transform of the covariance function is the spectral density $S(\omega)$. Without loss of generality, we can assume that the means of the random fields are zero, since the means are deterministic and can be subtracted for simplicity of discussion. The random field $X(t)$ can be decomposed into Fourier components $F(\omega)$:

$$X(t) = \operatorname{Re} \int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}d\omega$$

and thus the covariance of $X(t)$ becomes the Fourier transform of the square of $F(\omega)$:

$$E[X(t_1)X(t_2)] = \operatorname{Re} \int_{-\infty}^{+\infty} (F^2(\omega))e^{i\omega(t_1-t_2)}d\omega$$

Since $F(\omega)$ is the amplitude of each Fourier mode, the square of it should be considered as the power or energy spectral density:

$$S(\omega) = \langle F^2(\omega) \rangle$$

So we have the Wiener-Khinchine relation between $B(\tau)$ and $S(\omega)$ [1]:

$$B(\tau) = \int_{-\infty}^{+\infty} S(\omega)\cos\omega \tau d\omega$$

A class of random fields has the spectral density of the form ($\alpha$ is some positive constant):

$$S(\omega) \sim \frac{1}{\omega^\alpha}$$

They are very common in nature, especially for the case $\alpha = 1$, the $1/f$ noise, which is reported to be ubiquitous [2]. The $1/f$ noise almost present everywhere in every kinds of phenomena, from resistors [3] to traffic
from brain to number theory [3,6], etc. The microscopic origin of 1/f noise is widely studied, in e.g. 2, 7–11, 12, 13.

In real experiments, when we measure the value of a random field, we often measure an average value in a small domain. The idea of the local average [1] of the random fields is thus very practical and important. In this paper we will study the local average of the random field with spectral density $1/\omega^{\alpha}$, and their scaling properties under local average. We will also discuss the derivative and the inverse process/inverse problem of the local average.

I will first discuss their scaling properties in the covariance functions, both before and after local average. Since the spectral density is proportional to $1/\omega$, we have

$$B(\tau) \sim \int_{-\infty}^{+\infty} \frac{1}{\omega^{\alpha}} \cos \omega \tau d\omega \quad (7)$$

$$= \tau^{\alpha-1} \int_{0}^{+\infty} \frac{1}{(\omega \tau)^n} \cos \omega \tau d(\omega \tau). \quad (8)$$

So we see

$$B(\tau) \sim \tau^{\alpha-1}, \quad \text{(for} \quad \alpha > 0) \quad (9)$$

$$B(\tau) \sim \delta(\tau), \quad \text{(for} \quad \alpha = 0) \quad (10)$$

We can see that the exponent $\alpha$ determines how much correlated the $1/\omega$ random fields are. The bigger the $\alpha$, the more correlated it is.

If $\alpha > 1$, $B(\tau)$ increases with $\tau$, it is positively correlated. For example, in the Brownian motion, we know from the Einstein relation that the covariance of the position of the Brownian particle increases linearly with time ($D$ is the coefficient of diffusion): $E[X(t_1)X(t_2)] = 2D(t_1 - t_2)$, clearly $B(\tau) \sim \tau$, so for Brownian motion, $\alpha = 2$.

If $\alpha < 1$, $B(\tau)$ decreases with $\tau$, it is negatively correlated. If $\alpha = 1$, it’s the $1/f$ noise, in which $B(\tau) \sim \text{const}$. There is no a frequency scale or time scale in ideal $1/f$ noise, since the covariance function is constant everywhere, and after a rescaling of the time: $\tau \rightarrow n \tau$, it remains the same: $B(n\tau) = B(\tau)$.

If $\alpha = 0$, it’s the white noise and its spectral density $S(\omega) \sim \frac{1}{\omega}$ is constant and $B(\tau) \sim \delta(\tau)$.

Generally, the random fields with $1/\omega$ spectral density are self-similar. The curve $x$ vs. $t$ of a realization of this kind of random fields looks like a fractal, e.g. [14, 17]: For general $\alpha$, under a rescaling: $\tau \rightarrow n\tau$, $B(n\tau) = n^{\alpha-1}B(\tau)$, namely when we magnify $\tau$ times and simultaneously multiply the random field by a factor $n^{(1-\alpha)/2}$, the curve $x$ vs. $\tau$ looks the same. In other words, they are fractals with fractal dimensions $d = 2 + (1 - \alpha)/2$ [13, 19].

However, these scaling dependence should be appropriate in the large $\tau$ regime. In small $\tau$, perturbation shows $B(\tau)$ is a Gaussian with maximum $B(0) = \sigma_1^2(t)$.

Now let’s come to the local average of the random fields with spectral density $1/\omega^{\alpha}$.

**LOCAL AVERAGE OF 1/\omega^{\alpha} RANDOM FIELDS**

The local average [1] of the random field $X(t)$ with a window of length $T$ is defined as

$$X_T(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} X(t_1)dt_1. \quad (11)$$

There could be many ways to calculate the relations of the covariance functions and spectral densities between the averaged and unaveraged random fields.

One way is by defining a variance function $\frac{\sigma_T^2}{\sigma^2}$: It is shown [1] that $\gamma(T)$ is a response to the covariance function:

$$\gamma(T) \sim \frac{2}{T} \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{1}{\omega^{\alpha}} \cos \omega \tau d\omega \quad (12)$$

If $S(\omega) \sim \frac{1}{\omega}$, if we use $B(\tau)$, we have $\gamma(T) \sim T^{\alpha-1}$.

Another way is by analyzing the spectral density. It is shown [1] that:

$$S_{X_T}(\omega) = \int_{0}^{\infty} \frac{\sin(\omega T/2)}{\omega} \frac{1}{\omega^{\alpha}} \cos \omega \tau d\omega. \quad (13)$$

So we have

$$B_{X_T}(\tau) \sim \frac{1}{T} \int_{0}^{+\infty} \frac{\sin(\omega T/2)}{\omega^{\alpha+2}} \cos \omega \tau d\omega \quad (14)$$

When $T$ is very small it’s clear that $\frac{\sin(\omega T/2)}{\omega^{\alpha+2}} \approx 1$, so both the spectral density and covariance function before and after the local average are almost the same. $S_{X_T}(\omega) \approx S_X(\omega), B_{X_T}(\tau) \approx B(\tau)$. It remains a $1/\omega^{\alpha}$ spectral density when $T$ is small.

When $T$ is not small, consider the function:

$$\sin^2(\omega T/2) \cos \omega \tau = \frac{1}{2} |\cos \omega \tau - \frac{1}{2} \cos \omega (\tau + T) - \frac{1}{2} \cos \omega (\tau - T)|. \quad (15)$$

So we have

$$B_{X_T}(\tau) \sim \frac{1}{T^2} \int_{0}^{+\infty} \left[2 \cos \omega \tau - \cos \omega (\tau + T) - \cos \omega (\tau - T)\right] d\omega, \quad (16)$$

$$\sim \frac{1}{T^2} \left[2\tau^{\alpha+1} - (\tau + T)^{\alpha+1} - (|\tau - T|)^{\alpha+1} \right] \quad (17)$$
When $\alpha = 1$, $B_X(\tau) \sim \text{const}$. The local average of $1/\omega$ noise is still a $1/\omega$ noise. This is the manifestation of the self-similarity of $1/\omega$ noise.

For $\alpha \neq 1$, when $T$ is small, we can expand $T$ around $\tau$ in the above function, $B_X(\tau) \sim \tau^{\alpha - 1}$, which is the same as the unaveraged one.

If $T$ is big, we can expand $\tau$ around $T$, then

$$B_X(\tau) \sim \frac{1}{T^2} [2\tau^{\alpha + 1} - 2T^{\alpha + 1} - \alpha(\alpha + 1)\tau^2 T^{\alpha - 1}]. \quad (18)$$

**DERIVATIVES OF LOCALLY AVERAGED RANDOM FIELDS**

The spectral density of the local averaged random field can be achieved in another way in which we first calculate the spectral density of the derivative of the local averaged field and then use the relation $S_X(\omega) = \omega^2 S_X(\omega)$.

The derivative of the local averaged random field $X_T = \frac{1}{T} \int_{t - \frac{T}{2}}^{t + \frac{T}{2}} X(t_1) dt_1$ is:

$$\dot{X}_T = \frac{dX_T}{dt} = \frac{1}{T} [X(t + \frac{T}{2}) - X(t - \frac{T}{2})] \quad (19)$$

The covariance of $\dot{X}_T$ can be known from the information of the covariance of $X$, since

$$E[\dot{X}_T(t) \dot{X}_T(t + \tau)] = \frac{1}{T^2} [E[X(t + \frac{T}{2}) X(t + \frac{T}{2} + \tau)] + E[X(t - \frac{T}{2}) X(t - \frac{T}{2} + \tau)] - E[X(t - \frac{T}{2}) X(t + \frac{T}{2} + \tau)] - E[X(t + \frac{T}{2}) X(t - \frac{T}{2} + \tau)]] \quad (20)$$

If $X(t)$ is stationary, the above equation amounts to

$$B_{\dot{X}_T}(\tau) = \frac{1}{T^2} [2B_X(\tau) - B_X(\tau + T) - B_X(\tau - T)] \quad (21)$$

If we use $B(\tau) \sim \tau^{\alpha - 1}$, for $\alpha > 0$, then

$$B_{\dot{X}_T}(\tau) \sim \frac{1}{T^2} [2\tau^{\alpha - 1} - (\tau + T)^{\alpha - 1} - (|\tau - T|)^{\alpha - 1}] \quad (22)$$

It’s interesting to compare the covariance function of the local average of the $\frac{1}{\omega}$ random fields before and after taking a derivative, i.e. eqn (18) and (22).

Making a Fourier transform of both sides of (22):

$$S_{\dot{X}_T}(\omega) = \frac{1}{T^2} [2S_X(\omega) - e^{-i\omega T} S_X(\omega) - e^{i\omega T} S_X(\omega)]$$

$$= \frac{4}{T^2} S_X(\omega) \sin^2(\omega T/2) \quad (23)$$

On another hand, we have $S_X(\omega) = \omega^2 S_X(\omega)$, so we get

$$S_{\dot{X}_T}(\omega) = \left(\frac{\sin(\omega T/2)}{\omega T/2}\right)^2 S_X(\omega) \quad (24)$$

which is in agreement with eqn (18).

**INVERSE PROCESS OF LOCAL AVERAGE**

One may ask the question whether we can get the information of the original random field $X(t)$ from the information of the local averaged one $X_T(t)$. If we have two locally averaged fields with very close lengths of the window, i.e., we measure $X_{T_1}(t)$ and $X_{T_2}(t)$ in which $T_1$ and $T_2$ are very close to each other (suppose $T_1 > T_2$), since $X_{T_1}(t)$ and $X_{T_2}(t)$ are integral over range $T_1$ and $T_2$, their difference reveals the information of $X_{(t_1 - t_2)}(t)$.

$$T_1 X_{T_1}(t) - T_2 X_{T_2}(t)$$

$$= T_1 - T_2 [X_{(T_1 - T_2)}(t - \frac{T_1 + T_2}{2})]$$

By making the auto-covariance of both sides, we can get the cross-covariance of $X_{T_1}(t)$ and $X_{T_2}(t)$.

If $T_1$ and $T_2$ are so close that we can make a derivative with respect to $T$, the above equation becomes:

$$X_T(t) + T \frac{\partial X_T(t)}{\partial T} = \frac{1}{2} [X(t - \frac{T}{2}) + X(t + \frac{T}{2})]. \quad (26)$$

Together with eqn (18), we may solve $X(t)$ in terms of $X_T(t)$:

$$X(t) = X_T(t + \frac{T}{2}) + T \frac{\partial X_T(t + \frac{T}{2})}{\partial T} - \frac{T}{2} \frac{\partial X_T(t + \frac{T}{2})}{\partial t} \quad (27)$$

or

$$X(t) = X_T(t - \frac{T}{2}) + T \frac{\partial X_T(t - \frac{T}{2})}{\partial T} + \frac{T}{2} \frac{\partial X_T(t - \frac{T}{2})}{\partial t} \quad (28)$$

This recovers the original random field $X(t)$ from the locally-averaged one $X_T(t)$. The two expressions $T_1$ and $T_2$ are consistent. The derivative with respect to $T$ is understood as to compare two local averages whose lengths of windows are very close to each other.

The derivative of a local average is the local average of the derivative:

$$\frac{dX_T(t)}{dt} = \frac{1}{T} [X(t + \frac{T}{2}) - X(t - \frac{T}{2})] \quad (29)$$

$$= \frac{1}{T} \int_{t - \frac{T}{2}}^{t + \frac{T}{2}} \dot{X}(t_1) dt_1 = \left(\dot{X}(t)\right)_T \quad (30)$$
GENERALIZED LOCAL AVERAGE AND RESPONSE FUNCTION

Local average defined in [1] can be generalized to weighted local average, characterized by a normalized function $h(t)$:

$$X_h(t) = \int_{-\infty}^{+\infty} X(t_1)h(t_1-t)dt_1,$$  \hspace{1cm} (31)

where $\int_{-\infty}^{+\infty} h(t_1-t)dt_1 = 1$.

In this language, the ordinary local average (11) in [1] is the case where the function $h(t_1-t)$ is a step function:

$$h(t_1-t) = \frac{1}{T}, \text{ for } |t_1-t| \leq \frac{T}{2},$$  \hspace{1cm} (32)

$$h(t_1-t) = 0, \text{ for } |t_1-t| > \frac{T}{2}. \hspace{1cm} (33)$$

The generalization of $h(t)$ to other functions might be useful when the actual random field we measure is an uneven average, namely it is possible that in the averaging process when we measure in the real experiments, the values near the center has bigger weights than those away from the center, or inversely. The general $h$ respects this experimental detail. For example, perhaps $h(t_1-t)$ could be a Gaussian centered at $t$.

In this case, the spectral density of the weighted local average is

$$S_{X_h}(\omega) = |H(\omega)|^2 S_X(\omega),$$ \hspace{1cm} (34)

where $H(\omega)$ is the Fourier transform of the function $h(t_1-t)$.

We see that this actually means that the local average is a response to the original unaveraged random field by the response function $h$.

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[1] E. H. VanMarcke, Random Fields: Analysis and Synthesis, MIT Press (1983).