On the Number of Partitions with Designated Summands

William Y.C. Chen, Kathy Q. Ji, Hai-Tao Jin and Erin Y.Y. Shen

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

Abstract. Andrews, Lewis and Lovejoy introduced the partition function $PD(n)$ as the number of partitions of $n$ with designated summands, where we assume that among parts with equal size, exactly one is designated. They proved that $PD(3n+2)$ is divisible by 3. We obtain a Ramanujan type identity for the generating function of $PD(3n+2)$ which implies the congruence of Andrews, Lewis and Lovejoy. For $PD(3n)$, Andrews, Lewis and Lovejoy showed that the generating function can be expressed as an infinite product of powers of $(1 - q^{2n+1})$ times a function $F(q^2)$. We find an explicit formula for $F(q^2)$, which leads to a formula for the generating function of $PD(3n)$. We also obtain a formula for the generating function of $PD(3n+1)$. Our proofs rely on Chan’s identity on Ramanujan’s cubic continued fraction and some identities on cubic theta functions. By introducing a rank for the partitions with designated summands, we give a combinatorial interpretation of the congruence of Andrews, Lewis and Lovejoy.

1 Introduction

Andrews, Lewis and Lovejoy [2] investigated the number of partitions with designated summands which are defined on ordinary partitions by designating exactly one part among parts with equal size. Let $PD(n)$ denote the number of partitions of $n$ with designated summands. For example, there are ten partitions of 4 with designated summands:

\[
\begin{align*}
4', & \quad 3' + 1', & \quad 2' + 2, & \quad 2 + 2', & \quad 2' + 1' + 1, \\
2' + 1 + 1', & \quad 1' + 1 + 1 + 1, & \quad 1 + 1' + 1 + 1, & \quad 1 + 1 + 1' + 1, & \quad 1 + 1 + 1 + 1'.
\end{align*}
\]

The notion of partitions with designated summands goes back to MacMahon [10]. He considered partitions with designated summands and with exactly $k$ different sizes, see also Andrews and Rose [5]. Andrews, Lewis and Lovejoy [2] derived the following generating function of $PD(n)$.

Theorem 1.1. We have

\[
\sum_{n=0}^\infty PD(n)q^n = \frac{(q^6; q^6)_\infty}{(q; q)_\infty^2(q^2; q^2)_\infty^2(q^3; q^3)_\infty}. \tag{1.1}
\]
where \((a; q)_\infty\) stands for the \(q\)-shifted factorial
\[
(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}), \quad |q| < 1.
\]

By using modular forms and \(q\)-series identities, Andrews, Lewis and Lovejoy showed that the partition function \(PD(n)\) has many interesting divisibility properties. In particular, they obtained the following Ramanujan type congruence.

**Theorem 1.2.** ([2 Corollary 7]) For \(n \geq 0\), we have
\[
PD(3n + 2) \equiv 0 \pmod{3}. \tag{1.2}
\]

In this paper, we obtain the following Ramanujan type identity for the generating function of \(PD(3n + 2)\) which implies the above congruence.

**Theorem 1.3.** We have
\[
\sum_{n=0}^{\infty} PD(3n + 2)q^n = 3 (q_3^6; q_6^6)_\infty (q^6; q^6)_\infty^2 (q^2; q^2)_\infty^5 (q^2; q^2)_\infty^8. \tag{1.3}
\]

Andrews, Lewis and Lovejoy also obtained explicit formulas for the generating functions for \(PD(2n)\) and \(PD(2n + 1)\) by using Euler’s algorithm for infinite products [1, P. 98] and Sturm’s criterion [12]. As for \(PD(3n)\), they showed that the generating function permits the following form.

**Theorem 1.4.** ([2 Theorem 23]) Define \(c(n)\) uniquely by
\[
\sum_{n=0}^{\infty} PD(3n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-c(n)}, \tag{1.4}
\]

then for all positive \(n\),
\[
c(6n + 1) = 5, \\
c(6n + 3) = 2, \\
c(6n + 5) = 5.
\]

Equivalently, the above theorem says that there exists a series \(F(q^2)\) such that
\[
\sum_{n=0}^{\infty} PD(3n)q^n = \frac{1}{(q; q^6)_\infty^5 (q^3; q^6)_\infty^2 (q^5; q^6)_\infty^6} \times F(q^2). \tag{1.5}
\]

In this paper, we find an explicit formula for \(F(q^2)\), that is,
\[
F(q^2) = \frac{(q^4; q^6)_\infty^6 (q^6; q^6)_\infty^4}{(q^2; q^2)_\infty^{10} (q^12; q^12)_\infty^2} + 3q^2 \frac{(q^{12}; q^{12})_\infty^6}{(q^2; q^2)_\infty^6 (q^4; q^4)_\infty^2}, \tag{1.6}
\]

which leads to the following generating function of \(PD(3n)\).
Theorem 1.5. We have

\[ \sum_{n=0}^{\infty} PD(3n)q^n = \frac{1}{(q; q^6)_{\infty}^5 (q^5; q^6)_{\infty}^2 (q^5; q^6)_{\infty}^5} \times \left( \frac{(q^4; q^4)_{\infty}^6 (q^6; q^6)_{\infty}^4}{(q^2; q^2)_{\infty}^{10} (q^{12}; q^{12})_{\infty}^2} + 3q^2 \frac{(q^{12}; q^{12})_{\infty}^6}{(q^2; q^2)^{10} (q^4; q^4)_{\infty}^2} \right). \] (1.7)

In fact, we obtain explicit formulas for the 3-dissection of the generating function of \( PD(n) \), which include the following generating function for \( PD(3n + 1) \).

Theorem 1.6. We have

\[ \sum_{n=0}^{\infty} PD(3n + 1)q^n = \frac{(q^3; q^6)_{\infty}^3 (q^6; q^6)_{\infty}^6}{(q; q^2)_{\infty}^5 (q^2; q^2)_{\infty}^8} \left( 4q \frac{(q; q^2)_{\infty}^2}{(q^3; q^6)_{\infty}^6} + \frac{(q^3; q^6)_{\infty}^3}{(q; q^2)_{\infty}^2} \right). \] (1.8)

Our dissection formulas rely on the Chan’s identity on Ramanujan’s cubic continued fraction [8] and cubic theta functions [6, 9]. In Section 3, we shall give a combinatorial interpretation of the congruence \( PD(3n + 2) \equiv 0 \pmod{3} \) by introducing a rank for the partitions with designed summands.

## 2 Proofs

In this section, we give proofs of the generating functions for \( PD(3n) \), \( PD(3n + 1) \) and \( PD(3n + 2) \) by employing Chan’s identity on Ramanujan’s cubic continued fraction. It should be noted that the generating function of \( PD(3n) \) derived this way does not directly imply a formula for \( F(q^2) \). To compute \( F(q^2) \), we shall make use of some identities on cubic theta functions.

Recall that Ramanujan’s cubic continued fraction \( v(q) \) is given by

\[ v(q) := \frac{q^{\frac{3}{2}}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \ldots}}}. \]

It is known that

\[ v(q) = q^{\frac{1}{3}} \frac{(q; q^2)_{\infty}^3}{(q^3; q^6)_{\infty}^3}, \]

see Andrews and Berndt [3] P. 94. The following identity is due to Chan and will be used in our derivation of the 3-dissection formulas.

**Theorem 2.1.** ([8] Eq. (13)) We have

\[ \frac{1}{(q; q)_{\infty} (q^2; q^2)_{\infty}} = \frac{(q^9; q^9)_{\infty}^3 (q^{18}; q^{18})_{\infty}^3}{(q^3; q^3)_{\infty}^4 (q^6; q^6)_{\infty}^4} \times \left\{ \left( \frac{1}{x^2(q^3)} - 2q^3 x(q^3) \right) + q \left( \frac{1}{x(q^3)} + 4q^3 x^2(q^3) \right) + 3q^2 \right\}, \] (2.1)
where
\[ x(q) = q^{-1} v(q) = \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty}. \]

Proof of Theorems 1.3 and 1.6. Multiplying both sides of (2.1) by
\[ \frac{(q^6; q^6)_\infty}{(q; q)_\infty(q^2; q^2)_\infty(q^3; q^3)_\infty}, \]
we find
\[ \frac{(q^6; q^6)_\infty}{(q; q)_\infty(q^2; q^2)_\infty(q^3; q^3)_\infty} = \left( \frac{1}{x(q)} - 2q^3 x(q^3) \right) + q \left( \frac{1}{x(q)} + 4q^3 x(q^3) \right) + 3q^2. \]  (2.2)

Observe that the left-hand side of (2.2) is the generating function for PD(n). Extracting those terms involving the powers \( q^{3n} \), \( q^{3n+1} \) and \( q^{3n+2} \), respectively, we deduce that
\[ \sum_{n=0}^{\infty} PD(3n)q^{3n} = \frac{(q^9; q^9)_\infty(q^{18}; q^{18})_\infty}{(q^3; q^3)_\infty(q^6; q^6)_\infty}\left( -2q^3 x(q^3) + \frac{1}{x^2(q^3)} \right), \]
\[ \sum_{n=0}^{\infty} PD(3n+1)q^{3n+1} = \frac{(q^9; q^9)_\infty(q^{18}; q^{18})_\infty}{(q^3; q^3)_\infty(q^6; q^6)_\infty}\left( 4q^3 x^2(q^3) + \frac{1}{x(q^3)} \right), \]
\[ \sum_{n=0}^{\infty} PD(3n+2)q^{3n+2} = 3q^2 \frac{(q^9; q^9)_\infty(q^{18}; q^{18})_\infty}{(q^3; q^3)_\infty(q^6; q^6)_\infty}. \]  (2.3)  (2.4)  (2.5)

Thus Theorem 1.3 can be deduced from (2.5) by dividing both sides by \( q^2 \) and substituting \( q^3 \) by \( q \). Similarly, Theorem 1.6 can be deduced from (2.4) by dividing both sides by \( q \) and substituting \( q^3 \) by \( q \). This completes the proof.

It turns out that \( F(q^2) \) can be computed with the aid of some identities for cubic theta functions. These functions are introduced by Borwein, Borwein and Garvan [7] and are defined by
\[ a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \]
\[ b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n}q^{m^2+mn+n^2}, \quad \omega = e^{2\pi i/3}, \]
\[ c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}. \]

Recall that
\[ c(q) = 3 \frac{(q^3; q^3)_\infty}{(q; q)_\infty}, \]  (2.6)
see Berndt, Bhargava and Garvan [6, Eq. (5.5)]. We shall also use the following identities for \(a(q)\) and \(c(q)\)

\[
a(q) = a(q^4) + 6q \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty (q^6; q^6)_\infty}, \tag{2.7}
\]

\[
c(q) = qc(q^4) + 3q \frac{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty}, \tag{2.8}
\]

\[
a(q) = a(q^2) + 2q \frac{c^2(q^2)}{c(q)}. \tag{2.9}
\]

Identity \((2.7)\) for \(a(q)\) and identity \((2.8)\) for \(c(q)\) are due to Hirschhorn, Garvan, and Borwein [9, Eqs. (1.36) and (1.34)]. Identity \((2.9)\) for \(a(q)\) and \(c(q)\) is obtained by Berndt, Bhargava, Garvan [6, Eq. (6.3)].

We obtain the following identity on Ramanujan’s cubic continued fraction.

**Theorem 2.2.** Let

\[
x(q) = q^{-\frac{3}{8}} v(q) = \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty^3}.
\]

We have

\[
\frac{1}{x^2(q)} - 2qx(q) = 3q^2 \frac{(q^2; q^2)_\infty (q^{12}; q^{12})_\infty^5}{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^6} + \frac{(q^4; q^4)_\infty^5}{(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2 (q^{12}; q^{12})_\infty^2}. \tag{2.10}
\]

**Proof.** We first establish a connection between Ramanujan’s cubic continued fraction \(v(q)\) and the cubic theta function \(c(q)\). It is easy to check that

\[
\frac{1}{x^2(q)} = \frac{(q^3; q^6)_\infty^6}{(q; q^2)_\infty^2} = \frac{(q^2; q^2)_\infty^2 (q^4; q^4)_\infty^2 (q^6; q^6)_\infty^6}{(q^3; q^6)_\infty^3 (q; q)_\infty} \times \left( \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} \right)^2 = \frac{(q^4; q^4)_\infty^2}{9(q^6; q^6)_\infty^6} \times c^2(q), \tag{2.11}
\]

\[
2qx(q) = 2q \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty^3} = 2q \frac{(q^6; q^6)_\infty^3}{(q^2; q^2)_\infty} \times \left( \frac{(q; q)_\infty}{(q^3; q^3)_\infty^3} \right)^2 = 6q \frac{(q^6; q^6)_\infty^3}{(q^2; q^2)_\infty} \times \frac{1}{c(q)}. \tag{2.12}
\]

We now consider the 2-dissection of \(1/x^2(q)\). Identity \((2.8)\) can be viewed as the 2-dissection of \(c(q)\). Hence we deduce that

\[
c^2(q) = \left( q^2 c^2(q^4) + 9 \frac{(q^4; q^4)_\infty (q^6; q^6)_\infty^4}{(q^2; q^2)_\infty (q^{12}; q^{12})_\infty^2} \right) + q \left( 6c(q^4) \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} \right).
\]
Thus, we obtain the 2-dissection of $1/x^2(q)$,

$$
\frac{1}{x^2(q)} = \left( q^2 c^2(q^4) \frac{(q^2; q^2)\_\infty}{9(q^6; q^6)\_\infty} + \frac{(q^2; q^2)\_\infty}{(q^6; q^6)\_\infty} \frac{(q^4; q^4)\_\infty(q^6; q^6)\_\infty}{(q^2; q^2)\_\infty(q^6; q^6)\_\infty(q^{12}; q^{12})\_\infty} \right)
+ q \left( 6c(q^4) \frac{(q^2; q^2)\_\infty}{9(q^6; q^6)\_\infty} \frac{(q^4; q^4)\_\infty(q^6; q^6)\_\infty}{(q^2; q^2)\_\infty(q^{12}; q^{12})\_\infty} \right)
+ 2q \frac{(q^4; q^4)\_\infty(q^{12}; q^{12})\_\infty}{(q^6; q^6)\_\infty}. \quad (2.13)
$$

Next, we aim to derive the 2-dissection of $q/c(q)$. By (2.9), we find

$$
\frac{q}{c(q)} = \frac{a(q) - a(q^2)}{2c^2(q^2)}. \quad (2.14)
$$

Substituting (2.7) into (2.14), we arrive at

$$
\frac{q}{c(q)} = \frac{1}{2c^2(q^2)} \left( a(q^4) + 6q \frac{(q^4; q^4)\_\infty(q^{12}; q^{12})\_\infty}{(q^2; q^2)\_\infty(q^6; q^6)\_\infty} - a(q^2) \right). \quad (2.15)
$$

Using (2.9) with $q$ replaced by $q^2$, we get

$$a(q^2) - a(q^4) = 2q^2 \frac{c^2(q^4)}{c(q^2)}. \quad (2.16)
$$

Hence (2.15) can be written as

$$
\frac{q}{c(q)} = \frac{1}{2c^2(q^2)} \left( -2q^2 \frac{c^2(q^4)}{c(q^2)} + 6q \frac{(q^4; q^4)\_\infty(q^{12}; q^{12})\_\infty}{(q^2; q^2)\_\infty(q^6; q^6)\_\infty} \right)
+ 3q \frac{(q^4; q^4)\_\infty(q^{12}; q^{12})\_\infty}{c^2(q^2)(q^2; q^2)\_\infty(q^6; q^6)\_\infty}. \quad (2.17)
$$

Thus, we obtain the following 2-dissection of $2q x(q)$,

$$
2q x(q) = -6q^2 \frac{(q^6; q^6)\_\infty^3 c^2(q^4)}{(q^2; q^2)\_\infty^3 c^2(q^2)} + 18q \frac{(q^6; q^6)\_\infty^3(q^4; q^4)\_\infty(q^{12}; q^{12})\_\infty}{c^2(q^2)(q^2; q^2)\_\infty^2(q^6; q^6)\_\infty}
= -2q^2 \frac{(q^2; q^2)\_\infty^2(q^{12}; q^{12})\_\infty}{(q^6; q^6)\_\infty^2(q^4; q^4)\_\infty} + 2q \frac{(q^4; q^4)\_\infty^2(q^{12}; q^{12})\_\infty}{(q^6; q^6)\_\infty^4}. \quad (2.16)
$$

Subtracting (2.16) from (2.13), we obtain (2.10). This completes the proof. 

Proof of Theorem 1.5. Substituting $q^3$ with $q$ in (2.3), we obtain
\[
\sum_{n=0}^{\infty} PD(3n)q^n = \frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q^2)_\infty^3(q^2;q^2)_\infty^3} \left(-2qx(q) + \frac{1}{x^2(q)}\right)
\]
\[
= \frac{(q^3;q^6)_\infty^3(q^6;q^6)_\infty^6}{(q;q^2)_\infty^5(q^2;q^2)_\infty^8} \left(-2qx(q) + \frac{1}{x^2(q)}\right)
\]
\[
= \frac{1}{(q;q^6)_\infty^6(q^3;q^3)_\infty^3(q^2;q^2)_\infty^8} \times \frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^6} \left(-2qx(q) + \frac{1}{x^2(q)}\right).
\]  
(2.17)
Applying (2.10) to (2.17), we are led to the generating function for $PD(3n)$ in Theorem 1.5. This completes the proof.

3 A combinatorial interpretation

In this section, we give a combinatorial interpretation of the congruence $PD(3n+2) \equiv 0 \pmod{3}$. In doing so, we introduce a rank for partitions with designated summands. We call this rank the pd-rank which enables us to divide the set of partitions of $3n+2$ with designated summands into three equinumerous classes. The definition of the pd-rank is based on the following representation of a partition with designated summands by a pair of partitions.

Theorem 3.1. There is a bijection $\Delta$ between the set of partitions of $n$ with designated summands and the set of pairs of partitions $(\alpha, \beta)$ of $n$, where $\alpha$ is an ordinary partition and $\beta$ is a partition into parts $\not\equiv \pm 1 \pmod{6}$.

To give a proof of the above theorem, we shall use the bijective proof of the following theorem of MacMahon given by Andrews, Eriksson, Petrov and Romik [4].

Theorem 3.2. (11) The number of partitions of an integer $n$ into parts $\not\equiv \pm 1 \pmod{6}$ equals the number of partitions of $n$ not containing any part exactly once.

Proof of Andrews, Eriksson, Petrov and Romik. We construct a bijection $\Phi$ from the set $C_n$ of partitions of $n$ not containing any part exactly once to the set $B_n$ of partitions of $n$ into parts not congruent to $\pm 1 \pmod{6}$. To describe the map $\Phi$, let $\mu$ be a partition in $C_n$. Write $\mu$ as in the form of $(1^{m_1}2^{m_2} \cdots l^{m_l})$, where $m_k$ is the multiplicity of $k$ so that $n = \sum_{k=1}^{l} km_k$. Since $m_k \neq 1$ for any $k$, there is a unique way to write $m_k$ as $m_k = s_k + t_k$, where $s_k \in \{0, 3\}$ and $t_k \in \{0, 2, 4, 6, 8, \ldots\}$. Now, the partition $\lambda = \Phi(\mu) = (1^{b_1}2^{b_2} \cdots)$ is determined as follows:

\[
b_{6k+1} = 0, \quad b_{6k+5} = 0,
\]
\[
b_{6k+2} = \frac{1}{2}t_{3k+1}, \quad b_{6k+4} = \frac{1}{2}t_{3k+2},
\]
\[
b_{6k+3} = \frac{1}{3}s_{2k+1} + t_{6k+3}, \quad b_{6k+6} = \frac{1}{3}s_{2k+2} + t_{6k+6}.
\]
It is evident that $\lambda$ is a partition into parts not congruent to $\pm 1 \pmod{6}$. It is also apparent that one can recover the partition $\mu$ from $\lambda$ by reversing the above procedure. Hence $\Phi$ is a bijection. This completes the proof.

We are now in a position to present the proof of Theorem 3.1 by using the bijection $\Phi$.

**Proof of Theorem 3.1.** Let $\lambda$ be a partition of $n$ with designated summands. We wish to construct a pair of partitions $(\alpha, \beta)$ of $n$, where $\alpha$ is an ordinary partition and $\beta$ is a partition into parts $\not\equiv \pm 1 \pmod{6}$.

Suppose $t$ is a magnitude that appears in $\lambda$ and there are $m_t$ parts equal to $t$ among which the $i$-th part is designated. There are two cases.

- If $i = 1$, then move all the parts equal to $t$ (including the designated part) in $\lambda$ to the partition $\alpha$.
- If $i \neq 1$, then move $i$ parts equal to $t$ in $\lambda$ to $\gamma$ and $(m_t - i)$ parts equal to $t$ in $\lambda$ to $\alpha$.

It can be seen that each part in $\gamma$ occurs at least twice. Let $\beta = \Phi(\gamma)$. It is clear that $\beta$ is a partition into parts $\not\equiv \pm 1 \pmod{6}$ and the above procedure can be reversed. Hence $\Delta$ is a bijection. This completes the proof.

The $pd$-rank of a partition $\lambda$ with designated summands can be defined in terms of the pair of partitions $(\alpha, \beta)$ under the map $\Delta$.

**Definition 3.3.** Let $\lambda$ be a partition with designated summands and let $(\alpha, \beta) = \Delta(\lambda)$. Then the $pd$-rank of $\lambda$, denoted $r_d(\lambda)$, is defined by

$$r_d(\lambda) = l_e(\alpha) - l_e(\beta), \quad (3.1)$$

where $l_e(\alpha)$ is the number of even parts of $\alpha$ and $l_e(\beta)$ is the number of even parts of $\beta$.

The following theorem shows that the $pd$-rank can be used to divide the set of partitions of $3n + 2$ with designated summands into three equinumerous classes.

**Theorem 3.4.** For $i = 0, 1, 2$, let $N_d(i, 3; n)$ denote the number of partitions of $n$ with designated summands with $pd$-rank congruent to $i \pmod{3}$. Then we have

$$N_d(0, 3; 3n + 2) = N_d(1, 3; 3n + 2) = N_d(2, 3; 3n + 2). \quad (3.2)$$

**Proof.** Let $N_d(m; n)$ denote the number of partitions of $n$ with designated summands with $pd$-rank $m$. By the definition of the $pd$-rank, we see that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_d(m; n)z^mq^n = \frac{1}{(zq^2; q^2)_\infty(q; q^2)_\infty} \times \frac{1}{(z^{-1}q^2; q^2)_\infty(q^3; q^6)_\infty}. \quad (3.3)$$
Setting \( z = \zeta = e^{2\pi i/3} \), we find that
\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_d(m; n)\zeta^m q^n = \sum_{n=0}^{2} \sum_{i=0}^{2} N_d(i, 3; n)\zeta^i q^n
\]
\[
= \frac{1}{(\zeta q^2; q^2)_{\infty}(q; q^2)_{\infty}(\zeta^{-1}q^2; q^2)_{\infty}(q^3; q^6)_{\infty}}

= \frac{(-q^3; q^3)_{\infty}}{(q; q^2)_{\infty}(\zeta q^2; q^2)_{\infty}(\zeta^{-1}q^2; q^2)_{\infty}}.
\]
(3.4)

Multiplying the right hand side of (3.4) by
\[
\frac{(q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}},
\]
and noting that
\[
(1 - x)(1 - x\zeta)(1 - x\zeta^2) = 1 - x^3,
\]
we deduce that
\[
\sum_{n=0}^{\infty} \sum_{i=0}^{2} N_d(i, 3; n)\zeta^i q^n = \frac{(-q^3; q^3)_{\infty}}{(q; q^2)_{\infty}(\zeta q^2; q^2)_{\infty}(\zeta^{-1}q^2; q^2)_{\infty}} \times \frac{(q^2; q^2)_{\infty}}{(q^3; q^6)_{\infty}}
\]
\[
= \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \times \frac{(-q^3; q^3)_{\infty}}{(q^6; q^6)_{\infty}}.
\]

By Gauss’s identity \( \text{[1, P. 23]} \)
\[
\frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \sum_{n=0}^{\infty} q^{(n+1)}_{\frac{n+1}{2}},
\]
we get
\[
\sum_{n=0}^{\infty} \sum_{i=0}^{2} N_d(i, 3; n)\zeta^i q^n = \frac{(-q^3; q^3)_{\infty}}{(q^6; q^6)_{\infty}} \sum_{n=0}^{\infty} q^{(n+1)}_{\frac{n+1}{2}}.
\]
(3.5)

Since
\[
\binom{n+1}{2} \equiv 0 \text{ or } 1 \pmod{3},
\]
the coefficient of \( q^{3n+2} \) in (3.5) is zero. It follows that
\[
N_d(0, 3; 3n + 2) + N_d(1, 3; 3n + 2)\zeta + N_d(1, 3; 3n + 2)\zeta^2 = 0.
\]

Since \( 1 + \zeta + \zeta^2 \) is the minimal polynomial in \( \mathbb{Z}[\zeta] \), we conclude that
\[
N_d(0, 3; 3n + 2) = N_d(1, 3; 3n + 2) = N_d(2, 3; 3n + 2).
\]

This completes the proof.
For example, for \( n = 5 \), we have \( PD(5) = 15 \). The fifteen partitions of 5 with designated summands, the corresponding pairs of partitions, along with the \( pd\)-ranks modulo 3 are listed in Table 3.1. It can be checked that

\[
N_d(0, 3; 5) = N_d(1, 3; 5) = N_d(2, 3; 5) = 5.
\]

| \( \lambda \) | \((\alpha, \beta) = \Delta(\lambda)\) | \( r_d(\lambda) \) (mod 3) |
|---------------|----------------------------------|---------------------|
| \( 5' \)      | \((5, \emptyset)\)              | 0                   |
| \( 4' + 1' \)  | \((4 + 1, \emptyset)\)          | 1                   |
| \( 3' + 2' \)  | \((3 + 2, \emptyset)\)          | 1                   |
| \( 3' + 1' + 1\) | \((3 + 1 + 1, \emptyset)\)    | 0                   |
| \( 3' + 1 + 1' \) | \((3, 2)\)                   | 2                   |
| \( 2' + 2 + 1' \) | \((2 + 2 + 1, \emptyset)\)    | 2                   |
| \( 2 + 2' + 1' \) | \((1, 4)\)                  | 2                   |
| \( 2' + 1' + 1 + 1 \) | \((2 + 1 + 1 + 1, \emptyset)\) | 1                   |
| \( 2' + 1 + 1' + 1 \) | \((2 + 1, 2)\)     | 0                   |
| \( 2' + 1 + 1 + 1' \) | \((2, 3)\)                | 1                   |
| \( 1' + 1 + 1 + 1 + 1 \) | \((1 + 1 + 1 + 1 + 1, \emptyset)\) | 0                   |
| \( 1 + 1' + 1 + 1 + 1 \) | \((1 + 1 + 1, 2)\)     | 2                   |
| \( 1 + 1 + 1' + 1 + 1 \) | \((1 + 1, 3)\)         | 0                   |
| \( 1 + 1 + 1 + 1' + 1 \) | \((1, 2 + 2)\)        | 1                   |
| \( 1 + 1 + 1 + 1 + 1' \) | \((\emptyset, 3 + 2)\) | 2                   |

Table 3.1: The case for \( n = 5 \).

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

References

[1] G.E. Andrews, The Theory of Partitions, Encycl. Math. and Its Applications, Vol. 2, Addison-Wesley, Reading, 1976.
[2] G.E. Andrews, R.P. Lewis, and J. Lovejoy, Partitions with designated summands, Acta Arith. 105 (2002) 51–66.
[3] G.E. Andrews and B.C. Berndt, Ramanujan’s Lost Notebook. Part I, Springer, New York, 2005.
[4] G. Andrews, H. Eriksson, F. Petrov, and D. Romik, Integrals, partitions and MacMahon’s theorem, J. Comb. Theory A 114 (2007) 545-554.
[5] G.E. Andrews and S.C.F. Rose, MacMahon’s sum-of-divisors functions, Chebyshev polynomials, and quasi-modular forms, J. Reine Angew. Math, to appear.

[6] B.C. Berndt, S. Bhargava, and F.G. Garvan, Ramanujan’s theories of elliptic functions to alternative bases, Trans. Amer. Math. Soc. 347 (1995) 4163–4244.

[7] J.M. Borwein, P.B. Borwein, and F.G. Garvan, Some cubic modular identities of Ramanujan, Trans. Amer. Math. Soc. 343 (1994) 35–47.

[8] H.-C. Chan, Ramanujan’s cubic continued fraction and a generalization of his “most beautiful identity”, Int. J. Number Theory 6 (2010) 673–680.

[9] M.D. Hirschhorn, F. Garvan, and J. Borwein, Cubic analogues of the Jacobian theta function (z, q), Canad. J. Math. 45 (1993) 673–694.

[10] P. A. MacMahon, Divisors of numbers and their continuations in the theory of partitions, Proc. London Math. Soc. Ser. 2 19 (1919) 75–113.

[11] P.A. MacMahon, Combinatory Analysis, vols. I and II, Cambridge Univ. Press, Cambridge, 1915–1916, reissued, Chelsea, 1960.

[12] J. Sturm, On the congruence properties of modular forms, Springer Lect. Notes in Math. Vol.1240, pp. 275-280, Springer-Verlag, Berlin/New York, 1984.