ON THEOREMS OF CHERNOFF AND INGHAM
ON THE HEISENBERG GROUP

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Abstract. We prove an analogue of Chernoff’s theorem for the sublaplacian on the Heisenberg group and use it prove a version of Ingham’s theorem for the Fourier transform on the same group.

1. Introduction

Roughly speaking, the uncertainty principle for the Fourier transform on $\mathbb{R}^n$ says that a function $f$ and its Fourier transform $\hat{f}$ cannot both have rapid decay. Several manifestations of this principle are known: Heisenberg-Pauli-Weyl inequality, Paley-Wiener theorem, Hardy’s uncertainty principle are some of the most well known. But there are lesser known results such as theorems of Ingham and Levinson. The best decay a non trivial function can have is vanishing identically outside a compact set and for such functions it is well known that their Fourier transforms extend to $\mathbb{C}^n$ as entire functions and hence cannot vanish on any open set. For any such function of compact support, its Fourier transform cannot have any exponential decay for a similar reason: if $|\hat{f}(\xi)| \leq Ce^{-a|\xi|}$ for some $a > 0$, then it follows that $f$ extends to a tube domain in $\mathbb{C}^n$ as a holomorphic function and hence it cannot have compact support. So it is natural to ask the question: what is the best possible decay that is allowed of a function of compact support? An interesting answer to this question is provided by the following theorem of Ingham [10].

Theorem 1.1 (Ingham). Let $\Theta(y)$ be a nonnegative even function on $\mathbb{R}$ such that $\Theta(y)$ decreases to zero when $y \to \infty$. There exists a nonzero continuous function $f$ on $\mathbb{R}$, equal to zero outside an interval $(-a,a)$ having Fourier transform $\hat{f}$ satisfying the estimate $|\hat{f}(y)| \leq Ce^{-y^{1/\Theta(y)}}$ if and only if $\Theta$ satisfies $\int_1^{\infty} \Theta(t)t^{-1}dt < \infty$.

This theorem of Ingham and its close relatives Paley-Wiener ([20],[21]) and Levinson ([14]) theorems have received considerable attention in recent years. In [2] Bhowmik et al proved analogues of the above theorem for $\mathbb{R}^n$, the $n$-dimensional torus $\mathbb{T}^n$ and step two nilpotent
Lie groups. See also the recent work of Bhowmik-Pusti-Ray [3] for a version of Ingham’s theorem for the Fourier transform on Riemannian symmetric spaces of non-compact type. As we are interested in Ingham’s theorem on the Heisenberg group, let us recall the result proved in [2]. Let \( \mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \) be the Heisenberg group. For an integrable function \( f \) on \( \mathbb{H}^n \) let \( \hat{f}(\lambda) \) be the operator valued Fourier transform of \( f \) indexed by non-zero real \( \lambda \). Measuring the decay of the Fourier transform in terms of the Hilbert-Schmidt operator norm \( \| \hat{f}(\lambda) \|_{HS} \)

Bhowmik et al have proved the following result.

**Theorem 1.2** (Bhowmik-Ray-Sen). Let \( \Theta(\lambda) \) be a nonnegative even function on \( \mathbb{R} \) such that \( \Theta(\lambda) \) decreases to zero when \( \lambda \rightarrow \infty \). There exists a nonzero, compactly supported continuous function \( f \) on \( \mathbb{H}^n \), whose Fourier transform satisfies the estimate \( \| \hat{f}(\lambda) \|_{HS} \leq C|\lambda|^{n/2}e^{-|\lambda|\Theta(\lambda)} \) if the integral \( \int_1^{\infty} \Theta(t)t^{-1}dt < \infty \). On the other hand, if the above estimate is valid for a function \( f \) and the integral \( \int_1^{\infty} \Theta(t)t^{-1}dt \) diverges, then the vanishing of \( f \) on any set of the form \( \{ z \in \mathbb{C}^n : |z| < \delta \} \times \mathbb{R} \) forces \( f \) to be identically zero.

As the Fourier transform on the Heisenberg group is operator valued, it is natural to measure the decay of \( \hat{f}(\lambda) \) by comparing it with the Hermite semigroup \( e^{-aH(\lambda)} \) generated by \( H(\lambda) = -\Delta_{\mathbb{R}^n} + \lambda^2|x|^2 \). In this connection, let us recall the following two versions of Hardy’s uncertainty principle. Let \( p_a(z,t) \) stand for the heat kernel associated to the sublaplacian \( L \) on the Heisenberg group whose Fourier transform turns out to be the Hermite semigroup \( e^{-aH(\lambda)} \). The version in which one measures the decay of \( \hat{f}(\lambda) \) in terms of its Hilbert-Schmidt operator norm reads as follows. If

\[
|f(z,t)| \leq Ce^{-a(|z|^2+t^2)}, \quad \| \hat{f}(\lambda) \|_{HS} \leq Ce^{-b\lambda^2} \quad (1.1)
\]

then \( f = 0 \) whenever \( ab > 1/4 \). This is essentially a theorem in the \( t \)-variable and can be easily deduced from Hardy’s theorem on \( \mathbb{R} \), see Theorem 2.9.1 in [31]. Compare this with the following version, Theorem 2.9.2 in [31]. If

\[
|f(z,t)| \leq Cp_a(z,t), \quad \hat{f}(\lambda) \ast \hat{f}(\lambda) \leq Ce^{-2bH(\lambda)} \quad (1.2)
\]

then \( f = 0 \) whenever \( a < b \). This latter version is the exact analogue of Hardy’s theorem for the Heisenberg group, which we can view not merely as an uncertainty principle but also as a characterisation of the heat kernel. Hardy’s theorem in the context of semi-simple Lie groups and non-compact Riemannian symmetric spaces are also to be viewed in this perspective.

We remark that the Hermite semigroup has been used to measure the decay of the Fourier transform in connection with the heat kernel transform [12], Pfannschmidt’s theorem [33] and the extension problem for the sublaplacian [23] on the Heisenberg group. In connection with the study of Poisson integrals, it has been noted in [32] that when the Fourier transform of \( f \) satisfies an estimate of the form \( \hat{f}(\lambda) \ast \hat{f}(\lambda) \leq Ce^{-a\sqrt{H(\lambda)}} \), then the function extends to a tube domain in the complexification of \( \mathbb{H}^n \) as a holomorphic function and hence the vanishing
of $f$ on an open set forces it to vanish identically. It is therefore natural to ask if the same conclusion can be arrived at by replacing the constant $a$ in the above estimate by an operator $\Theta(\sqrt{H(\lambda)})$ for a function $\Theta$ decreasing to zero at infinity. Our investigations have led us to the following analogue of Ingham’s theorem for the Fourier transform on $\mathbb{H}^n$.

**Theorem 1.3.** Let $\Theta(\lambda)$ be a nonnegative even function on $\mathbb{R}$ such that $\Theta(\lambda)$ decreases to zero when $\lambda \to \infty$. There exists a nonzero compactly supported continuous function $f$ on $\mathbb{H}^n$ whose Fourier transform $\hat{f}$ satisfies the estimate

$$\hat{f}(\lambda)^* \hat{f}(\lambda) \leq Ce^{-2\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}$$

(1.3)

if and only if the function $\Theta$ satisfies $\int_1^\infty \Theta(t) t^{-1} dt < \infty$.

Theorem 1.1 was proved in [10] by Ingham by making use of Denjoy-Carleman theorem on quasi-analytic functions. In [2] the authors have used Radon transform and a several variable extension of Denjoy-Carleman theorem due to Bochner and Taylor [5] in order to prove the $n$-dimensional version of Theorem 1.1. An $L^2$ variant of the result of Bochner-Taylor which was proved by Chernoff in [8] has turned out to be very useful in establishing Ingham type theorems.

**Theorem 1.4.** [8, Chernoff] Let $f$ be a smooth function on $\mathbb{R}^n$. Assume that $\Delta^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^\infty \|\Delta^m f\|_2^{-\frac{1}{m}} = \infty$. If $f$ and all its partial derivatives vanish at 0, then $f$ is identically zero.

This theorem shows how partial differential operators generate the class of quasi-analytic functions. Recently, Bhowmik-Pusti-Ray [3] have established an analogue of Chernoff’s theorem for the Laplace-Beltrami operator on non-compact Riemannian symmetric spaces and use the same in proving a version of Ingham’s theorem for the Helgason Fourier transform. It is therefore natural to look for an analogue of this result for sublaplacian on the Heisenberg group. In this paper, we prove the following result.

**Theorem 1.5.** Let $\mathcal{L}$ be the sublaplacian on the Heisenberg group and let $f$ be a smooth function on $\mathbb{H}^n$ such that $\mathcal{L}^m f \in L^2(\mathbb{H}^n)$ for all $m \in \mathbb{N}$. Assume that $\sum_{m=1}^\infty \|\mathcal{L}^m f\|_2^{-\frac{1}{m}} = \infty$. If $f$ and all its partial derivatives vanish at some point, then $f$ is identically zero.

An immediate corollary of this theorem is the following, which can be seen as an $L^2$ version of the classical Denjoy-Carleman theorem on the Heisenberg group using iterates of sublaplacian.

**Corollary 1.6.** Let $\{M_k\}_k$ be a log convex sequence. Define $\mathcal{C}(\{M_k\}_k, \mathcal{L}, \mathbb{H}^n)$ to be the class of all smooth functions $f$ on $\mathbb{H}^n$ such that $\mathcal{L}^k f \in L^2(\mathbb{H}^n)$ for all $k \in \mathbb{N}$ and $\|\mathcal{L}^k f\|_2 \leq M_k \lambda^k$ for some constant $\lambda$ (may depend on $f$). Suppose that $\sum_{k=1}^\infty M_k^{-\frac{1}{k}} = \infty$. Then every member of that class is quasi-analytic.
2. Preliminaries

In this section, we collect the results which are necessary for the study of uncertainty principles for the Fourier transform on the Heisenberg group. We refer the reader to the two classical books Folland [9] and Taylor [28] for the preliminaries of harmonic analysis on the Heisenberg group. However, we will be closely following the notations of the books of Thangavelu [30] and [31].

2.1. Heisenberg group and Fourier transform. Let $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ denote the $(2n + 1)$-Heisenberg group equipped with the group law

\[(z, t)(w, s) := (z + w, t + s + \frac{1}{2}\Im(z \bar{w})), \quad \forall (z, t), (w, s) \in \mathbb{H}^n.
\]

This is a step two nilpotent Lie group where the Lebesgue measure $dzdt$ on $\mathbb{C}^n \times \mathbb{R}$ serves as the Haar measure. The representation theory of $\mathbb{H}^n$ is well-studied in the literature. In order to define Fourier transform, we use the Schrödinger representations as described below.

For each non zero real number $\lambda$ we have an infinite dimensional representation $\pi_\lambda$ realised on the Hilbert space $L^2(\mathbb{R}^n)$. These are explicitly given by

$$
\pi_\lambda(z, t) \varphi(\xi) = e^{i\lambda t} e^{i(x \cdot \xi + \frac{1}{2} x \cdot y)} \varphi(\xi + y),
$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$. These representations are known to be unitary and irreducible. Moreover, by a theorem of Stone and Von-Neumann, (see e.g., [9]) up to unitary equivalence these account for all the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ which act as $e^{i\lambda t} I$ on the center. Also there is another class of finite dimensional irreducible representations. As they do not contribute to the Plancherel measure we will not describe them here.

The Fourier transform of a function $f \in L^1(\mathbb{H}^n)$ is the operator valued function obtained by integrating $f$ against $\pi_\lambda$:

$$
\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z, t) \pi_\lambda(z, t)dz dt.
$$

Note that $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. It is known that when $f \in L^1 \cap L^2(\mathbb{H}^n)$ its Fourier transform is actually a Hilbert-Schmidt operator and one has

$$
\int_{\mathbb{H}^n} |f(z, t)|^2 dz dt = (2\pi)^{-(n+1)} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^{n} d\lambda
$$

where $\|\cdot\|_{HS}$ denote the Hilbert-Schmidt norm. The above allows us to extend the Fourier transform as a unitary operator between $L^2(\mathbb{H}^n)$ and the Hilbert space of Hilbert-Schmidt operator valued functions on $\mathbb{R}$ which are square integrable with respect to the Plancherel
measure $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$. We polarize the above identity to obtain
\[
\int_{\mathbb{H}^n} f(z, t)g(z, t)dzdt = \int_{-\infty}^{\infty} tr(\hat{f}(\lambda)\hat{g}(\lambda)^*) d\mu(\lambda).
\]
Also for suitable function $f$ on $\mathbb{H}^n$ we have the following inversion formula
\[
f(z, t) = \int_{-\infty}^{\infty} tr(\pi_\lambda(z, t)^* \hat{f}(\lambda))d\mu(\lambda).
\]
Now by definition of $\pi_\lambda$ and $\hat{f}(\lambda)$ it is easy to see that
\[
\hat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z, 0)dz
\]
where $f^\lambda$ stands for the inverse Fourier transform of $f$ in the central variable:
\[
f^\lambda(z) := \int_{-\infty}^{\infty} e^{i\lambda.t}f(z, t)dt.
\]
This motivates the following operator. Given a function $g$ on $\mathbb{C}^n$, we consider the following operator valued function defined by
\[
W_\lambda(g) := \int_{\mathbb{C}^n} g(z)\pi_\lambda(z, 0)dz.
\]
With these notations we note that $\hat{f}(\lambda) = W_\lambda(f^\lambda)$. For $\lambda = 1$, $W_1(g) := W(g)$ is called the Weyl transform of $g$. Moreover, the fourier transform behaves well with the convolution of two functions defined by
\[
f * g(x) := \int_{\mathbb{H}^n} f(xy^{-1})g(y)dy.
\]
Infact, for any $f, g \in L^1(\mathbb{H}^n)$, directly from the definition it follows that $\hat{f}^*g(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$.

2.2. Special functions and Fourier transform. For each $\lambda \neq 0$, we consider the following family of scaled Hermite functions indexed by $\alpha \in \mathbb{N}^n$:
\[
\Phi^\lambda_\alpha(x) := |\lambda|^\frac{n}{2}\Phi_\alpha(\sqrt{|\lambda|}x), \ x \in \mathbb{R}^n
\]
where $\Phi_\alpha$ denote the $n-$dimensional Hermite functions (see [29]). It is well-known that these scaled functions $\Phi^\lambda_\alpha$ are eigenfunctions of the scaled Hermite operator $H(\lambda) := -\Delta_{\mathbb{R}^n} + \lambda^2|x|^2$ with eigenvalue $(2|\alpha| + n)|\lambda|$ and $\{\Phi^\lambda_\alpha : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. As a consequence,
\[
\|\hat{f}(\lambda)\|_{\mathcal{HS}}^2 = \sum_{\alpha \in \mathbb{N}^n} \|\hat{f}(\lambda)\Phi^\lambda_\alpha\|_2^2.
\]
In view of this the Plancheral formula takes the following very useful form
\[
\int_{\mathbb{H}^n} |f(z, t)|^2dzdt = \int_{-\infty}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \|\hat{f}(\lambda)\Phi^\lambda_\alpha\|_2^2 d\mu(\lambda).
\]
Given \( \sigma \in U(n) \), we define \( R_\sigma f(z,t) = f(\sigma.z,t) \). We say that a function \( f \) on \( \mathbb{H}^n \) is radial if \( f \) is invariant under the action of \( U(n) \) i.e., \( R_\sigma f = f \) for all \( \sigma \in U(n) \). The Fourier transforms of such radial integrable functions are functions of the Hermite operator \( H(\lambda) \). In fact, if \( H(\lambda) = \sum_{k=0}^{\infty} (2k+n)|\lambda|P_k(\lambda) \) stands for the spectral decomposition of this operator, then for a radial integrable function \( f \) we have

\[
\hat{f}(\lambda) = \sum_{k=0}^{\infty} R_k(\lambda, f)P_k(\lambda).
\]

More explicitly, \( P_k(\lambda) \) stands for the orthogonal projection of \( L^2(\mathbb{R}^n) \) onto the \( k^{th} \) eigenspace spanned by scaled Hermite functions \( \Phi_\lambda^\alpha \) for \( |\alpha| = k \). The coefficients \( R_k(\lambda, f) \) are explicitly given by

\[
R_k(\lambda, f) = \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} f^\lambda(z) \varphi_{k,\lambda}^{n-1}(z) \, dz.
\]

In the above formula, \( \varphi_{k,\lambda}^{n-1} \) are the Laguerre functions of type \( (n-1) \):

\[
\varphi_{k,\lambda}^{n-1}(z) = L_k^{n-1}\left(\frac{1}{2}|\lambda||z|^2\right)e^{-\frac{1}{2}||\lambda||z|^2}
\]

where \( L_k^{n-1} \) denotes the Laguerre polynomial of type \( (n-1) \). For the purpose of estimating the Fourier transform we need good estimates for the Laguerre functions \( \varphi_{k,\lambda}^{n-1} \). In order to get such estimates, we use the available sharp estimates of standard Laguerre functions as described below in more general context.

For any \( \delta > -1 \), let \( L_k^{\delta}(r) \) denote the Laguerre polynomials of type \( \delta \). The standard Laguerre functions are defined by

\[
L_k^{\delta}(r) = \left( \frac{\Gamma(k+1)\Gamma(\delta+1)}{\Gamma(k+\delta+1)} \right)^{1/2} L_k^{\delta}(r)e^{-\frac{1}{2}r^2} r^\delta/2
\]

which form an orthonormal system in \( L^2((0, \infty), dr) \). In terms of \( L_k^{\delta}(r) \), we have

\[
\varphi_k^{\delta}(r) = 2^\delta \left( \frac{\Gamma(k+1)\Gamma(\delta+1)}{\Gamma(k+\delta+1)} \right)^{-1/2} r^{-\delta} L_k^{\delta}\left(1/2r^2\right).
\]

Asymptotic properties of \( L_k^{\delta}(r) \) are well known in the literature, see [29, Lemma 1.5.3]. The estimates in [29, Lemma 1.5.3] are sharp, see [15, Section 2] and [16, Section 7]. For our convenience, we restate the result in terms of \( \varphi_k^{n-1}(r) \).
Lemma 2.1. Let \( \nu(k) = 2(2k+n) \) and \( C_{k,n} = \left( \frac{k(n-1)!}{(k+n-1)!} \right)^{\frac{1}{2}} \). For \( \lambda \neq 0 \), we have the estimates

\[
C_{k,n} |\varphi_{k,\lambda}^{n-1}(r)| \leq C(r \sqrt{|\lambda|})^{-(n-1)} \begin{cases} 
(\frac{1}{2} \nu(k) r^2 |\lambda|)^{(n-1)/2}, & 0 \leq r \leq \frac{\sqrt{2}}{\sqrt{\nu(k)}|\lambda|} \\
(\frac{1}{2} \nu(k) r^2 |\lambda|)^{-\frac{1}{2}}, & \frac{\sqrt{2}}{\sqrt{\nu(k)}|\lambda|} \leq r \leq \frac{\nu(k)}{|\lambda|} \\
\nu(k)^{-\frac{1}{4}} (\nu(k)^{\frac{1}{2}} + |\nu(k) - \frac{1}{2} | |\lambda|^2 r^2 |\lambda|)^{-\frac{1}{4}}, & \frac{\nu(k)}{|\lambda|} \leq r \leq \frac{\sqrt{3} \nu(k)}{|\lambda|} \\
e^{-\frac{1}{2} \nu(n^2 |\lambda|)}, & r \geq \frac{\sqrt{3} \nu(k)}{|\lambda|}, 
\end{cases}
\]

where \( \gamma > 0 \) is a fixed constant and \( C \) is independent of \( k \) and \( \lambda \).

2.3. The sublaplacian and Sobolev spaces on \( \mathbb{H}^n \). We let \( \mathfrak{h}_n \) stand for the Heisenberg Lie algebra consisting of left invariant vector fields on \( \mathbb{H}^n \). A basis for \( \mathfrak{h}_n \) is provided by the \( 2n+1 \) vector fields

\[
X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, 2, ..., n
\]

and \( T = \frac{\partial}{\partial t} \). These correspond to certain one parameter subgroups of \( \mathbb{H}^n \). The sublaplacian on \( \mathbb{H}^n \) is defined by

\[
\mathcal{L} := -\sum_{j=1}^{\infty} (X_j^2 + Y_j^2)
\]

which can be explicitly calculated as

\[
\mathcal{L} = -\Delta_{\mathbb{C}^n} - \frac{1}{4} |z|^2 \frac{\partial^2}{\partial t^2} + n \frac{\partial}{\partial t}
\]

where \( \Delta_{\mathbb{C}^n} \) stands for the Laplacian on \( \mathbb{C}^n \) and \( N \) is the rotation operator defined by

\[
N = \sum_{j=1}^{n} \left( x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).
\]

This is a sub-elliptic operator and homogeneous of degree 2 with respect to the non-isotropic dilations given by \( \delta_r(z,t) = (rz, r^2 t) \). The sublaplacian is also invariant under rotation i.e., \( R_\sigma \circ \mathcal{L} = \mathcal{L} \circ R_\sigma, \sigma \in U(n) \). It is convenient for our purpose to represent the sublaplacian in terms of another set of vector fields defined as follows:

\[
Z_j := \frac{1}{2} (X_j - i Y_j) = \frac{\partial}{\partial z_j} + \frac{i}{4} \bar{z}_j \frac{\partial}{\partial t}, \quad \bar{Z}_j := \frac{1}{2} (X_j + i Y_j) = \frac{\partial}{\partial \bar{z}_j} + \frac{i}{4} \bar{z}_j \frac{\partial}{\partial t}
\]

where \( \frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \) and \( \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) \). Now an easy calculation yields

\[
\mathcal{L} = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j).
\]

The action of Fourier transform on \( Z_j f, \bar{Z}_j f \) and \( T f \) are well-known and are given by

\[
\widehat{Z_j f}(\lambda) = i \hat{f}(\lambda) A_j(\lambda), \quad \widehat{\bar{Z}_j f}(\lambda) = i \hat{f}(\lambda) A_j(\lambda)^* \quad \text{and} \quad \widehat{T f}(\lambda) = -i \lambda \hat{f}(\lambda) \quad (2.2)
\]
where $A_j(\lambda)$ and $A_j^*(\lambda)$ are the annihilation and creation operators given by

$$A_j(\lambda) = \left( -\frac{\partial}{\partial \xi_j} + i\lambda \xi_j \right), \quad A_j^*(\lambda) = \left( \frac{\partial}{\partial \xi_j} + i\lambda \xi_j \right).$$

These along with the above representation of the sublaplacian yield the relation $\hat{\mathcal{L}} f(\lambda) = \hat{f}(\lambda) H(\lambda)$.

We can define the spaces $W^{s,2}(\mathbb{H}^n)$ for any $s \in \mathbb{R}$ as the completion of $C_c^\infty(\mathbb{H}^n)$ under the norm $\| f \|_{(s)} = \|(I + \mathcal{L})^{s/2} f\|_2$ where the fractional powers $(I + \mathcal{L})^{s/2}$ are defined using spectral theorem. To study these spaces, it is better to work with the following expression of the norm $\| f \|_{(s)}$ for $f \in C_c^\infty(\mathbb{H}^n)$. In view of Plancherel theorem for the Fourier transform

$$\| f \|_{(s)}^2 = (2\pi)^{-n-1} \int_\mathbb{R}^n |\hat{f}(\lambda)(1 + H(\lambda))^{s/2}|^2 \| \lambda \|^n d\lambda$$

which is valid for any $s \in \mathbb{R}$. Here we have made use of the fact that $\hat{\mathcal{L}} f(\lambda) = \hat{f}(\lambda) H(\lambda)$.

Computing the Hilbert-Schmidt norm in terms of the Hermite basis, we have the more explicit expression:

$$\| f \|_{(s)}^2 = (2\pi)^{-n-1} \int_{-\infty}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} (1 + (2|\alpha| + n)|\lambda|)^s |\langle \hat{f}(\lambda) \Phi_\alpha^\lambda, \Phi_\beta^\lambda \rangle|^2 |\lambda|^n d\lambda.$$

Consider $\mathbb{H}^n = \mathbb{R}^* \times \mathbb{N}^n \times \mathbb{N}^n$ equipped with the measure $\mu \times \nu$ where $\nu$ is the counting measure on $\mathbb{N}^n \times \mathbb{N}^n$. The above shows that, for $f \in C_c^\infty(\mathbb{H}^n)$ the function $m(\lambda, \alpha, \beta) = |\langle \hat{f}(\lambda) \Phi_\alpha^\lambda, \Phi_\beta^\lambda \rangle|$ belongs to the weighted space

$$W^{s,2}(\mathbb{H}^n) = L^2(\mathbb{H}^n, w_s d(\mu \times \nu))$$

where $w_s(\lambda, \alpha) = (1 + (2|\alpha| + n)|\lambda|)^s$. As these weighted $L^2$ spaces are complete, we can identify $W^{s,2}(\mathbb{H}^n)$ with $W^{s,2}(\mathbb{H}^n)$. It is then clear that for any $s > 0$ we have

$$W^{s,2}(\mathbb{H}^n) \subset W^{0,2}(\mathbb{H}^n) \subset W^{-s,2}(\mathbb{H}^n)$$

and the same inclusion holds for $W^{s,2}(\mathbb{H}^n)$. It is clear that any $m \in W^{s,2}(\mathbb{H}^n)$ can be written as $m(\lambda, \alpha, \beta) = (1 + (2|\alpha| + n)|\lambda|)^{-s/2} m_0(\lambda, \alpha, \beta)$ where $m_0 \in W^{0,2}(\mathbb{H}^n) = L^2(\mathbb{H}^n)$ for any $s \in \mathbb{R}$. Consequently, any $f \in W^{s,2}(\mathbb{H}^n)$ can be written as $f = (I + \mathcal{L})^{-s/2} f_0$, where $f_0 \in L^2(\mathbb{H}^n)$ is the function which corresponds to $m_0$ which is given explicitly by

$$f_0(z, t) = \int_{\mathbb{H}^n} m_0(\lambda, \alpha, \beta) e^{-\lambda}(z, t) d\nu(\alpha, \beta) d\mu(\lambda).$$

Thus we see that $f \in W^{s,2}(\mathbb{H}^n)$ if and only if there is an $f_0 \in L^2(\mathbb{H}^n)$ such that $f = (I + \mathcal{L})^{-s/2} f_0$. The inner product on $W^{s,2}(\mathbb{H}^n)$ is given by

$$\langle f, g \rangle_s = \langle (I + \mathcal{L})^{s/2} f, (I + \mathcal{L})^{s/2} g \rangle = \langle f_0, g_0 \rangle$$
where \( \langle f, g \rangle \) is the inner product in \( L^2(\mathbb{H}^n) \). This has the following interesting consequence. Given \( f \in W^{s,2}(\mathbb{H}^n) \) and \( g \in W^{-s,2}(\mathbb{H}^n) \), let \( f_0, g_0 \in L^2(\mathbb{H}^n) \) be such that \( f = (I + \mathcal{L})^{-s/2} f_0 \) and \( g = (I + \mathcal{L})^{s/2} g_0 \). The duality bracket \( (f, g) \) defined by

\[
(f, g) = \langle (I + \mathcal{L})^{-s/2} f_0, (I + \mathcal{L})^{s/2} g_0 \rangle = \langle f_0, g_0 \rangle
\]

allows us to identify the dual of \( W^{s,2}(\mathbb{H}^n) \) with \( W^{-s,2}(\mathbb{H}^n) \). This is also clear from the identification of \( W^{s,2}(\mathbb{H}^n) \) with \( W^{s,2}(\mathbb{R}^n) \). Thus for every \( g \in W^{-s,2}(\mathbb{H}^n) \) there is a linear functional \( \Lambda_g : W^{s,2}(\mathbb{H}^n) \to \mathbb{C} \) given by \( \Lambda_g(f) = \langle f_0, g_0 \rangle \).

The following observation is also very useful in applications. For \( s > 0 \) every member \( f \in W^{s,2}(\mathbb{H}^n) \) defines a distribution on \( \mathbb{H}^n \). The same is true for every \( g \in W^{-s,2}(\mathbb{H}^n) \) as well. To see this, consider the map taking \( f \in C_c^\infty(\mathbb{H}^n) \) into the duality bracket \( (f, g) \) which satisfies

\[
|\langle f, g \rangle| \leq \|f\|_{(s)} \|g\|_{(-s)} \leq \|g\|_{(-s)} \|\langle I + \mathcal{L}\rangle^m f\|_2
\]

where \( m > s/2 \) is an integer. From the above it is clear that \( \Lambda_g(f) = \langle f, g \rangle \) is a distribution. If \( g \in W^{-s,2}(\mathbb{H}^n) \) is such a distribution, it is possible to define its Fourier transform as an unbounded operator valued function on \( \mathbb{R}^n \). Indeed, let \( g_0 \in L^2(\mathbb{H}^n) \) be such that \( g = (I + \mathcal{L})^{s/2} g_0 \) then we define \( \widehat{g}(\lambda) = \widehat{g_0}(\lambda)(1 + H(\lambda))^{s/2} \) which is a densely defined operator whose action on \( \Phi_\alpha^\lambda \) is given by

\[
\widehat{g}(\lambda) \Phi_\alpha^\lambda = (1 + (2|\alpha| + n)|\lambda|)^{s/2} \widehat{g_0}(\lambda) \Phi_\alpha^\lambda.
\]

Thus we see that when \( g \in W^{-s,2}(\mathbb{H}^n) \) we have

\[
\int_{-\infty}^\infty \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} (1 + (2|\alpha| + n)|\lambda|)^{-s} |\langle \widehat{g}(\lambda) \Phi_\alpha^\lambda, \Phi_\beta^\lambda \rangle|^2 d\mu(\lambda) = \int_{\mathbb{H}^n} |g_0(z, t)|^2 dz dt < \infty. \quad (2.3)
\]

Remark 2.1. When \( g \in W^{-s,2}(\mathbb{H}^n) \) is a compactly supported distribution, then we already have a definition of \( \widehat{g}(\lambda) \) given by \( \langle \widehat{g}(\lambda) \Phi_\alpha^\lambda, \Phi_\beta^\lambda \rangle = \langle g, e_{\alpha,\beta}^\lambda \rangle \), the action of \( g \) on the smooth function \( e_{\alpha,\beta}^\lambda(z, t) \). The two definitions agree as \( e_{\alpha,\beta}^\lambda \) are eigenfunctions of \( \mathcal{L} \) with eigenvalues \( (2|\alpha| + n)|\lambda| \).

3. Chernoff’s Theorem on the Heisenberg Group

In this section we prove Theorem 1.5 for the sublaplacian on the Heisenberg group. For the proof we need to recall some properties of the so called Stieltjes vectors for the sublaplacian.

Let \( X \) be a Banach space and \( A \), a linear operator on \( X \) with domain \( D(A) \subset X \). A vector \( x \in X \) is called a \( C^\infty \)-vector or smooth vector for \( A \) if \( x \in \cap_{n=1}^\infty D(A^n) \). A \( C^\infty \)-vector \( x \) is said to be a Stieltjes vector for \( A \) if \( \sum_{n=1}^\infty \|A^n x\|^{-\frac{1}{n}} = \infty \). These vectors were first introduced by Nussbaum [19] and independently by Masson and Mc Clary [17]. We denote the set of all Stieltjes vector for \( A \) by \( D_{St}(A) \). The following theorem summarises the interconnection
between the theory of Stieltjes vectors and the essential self adjointness of certain class of operators.

**Theorem 3.1.** Let $A$ be a semibounded symmetric operator on a Hilbert space $H$. Assume that the set $D_{St}(A)$ has a dense span. Then $A$ is essentially self adjoint.

A very nice simplified proof this theorem can be found in Simon [26]. In 1975, P.R.Chernoff used this result to prove an $L^2$-version of the classical Denjoy-Carleman theorem regarding quasi-analytic functions on $\mathbb{R}^n$.

The above theorem talks about essential self adjointness of operators. Let us quickly recall some relevant definitions from operator theory. By an operator $A$ on a Hilbert space $H$ we mean a linear mapping whose domain $D(A)$ is a subspace of $H$ and whose range $\text{Ran}(A) \subset H$. We say that an operator $S$ is an extension of $A$ if $D(A) \subset D(S)$ and $Sx = Ax$ for all $x \in D(A)$. An operator $A$ is called closed if the graph of $A$ defined by $\mathcal{G} = \{(x, Ax) : x \in D(A)\}$ is a closed subset of $H \times H$. We say that an operator $A$ is closable if it has a closed extension. Every closable operator has a smallest closed extension, called its closure, which we denote by $\overline{A}$. An operator $A$ is said to be densely defined if $D(A)$ is dense in $H$ and it is called symmetric if $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in D(A)$. A densely defined symmetric operator $A$ is called essentially self adjoint if its closure $\overline{A}$ is self adjoint. It is easy to see that an operator $A$ is essentially self adjoint if and only if $A$ has unique self adjoint extension. The following is a very important characterization of essentially self adjoint operators.

**Theorem 3.2.** ([22]) Let $A$ be a positive, densely defined symmetric operator. The followings are equivalent: (i) $A$ is essentially self adjoint (ii) $\text{Ker}(A^* + I) = \{0\}$ and (iii) $\text{Ran}(A + I)$ is dense in $H$.

We apply the above theorem to study the essential self adjointness of $\mathcal{L}$ considered on a domain inside the Sobolev space $W^{s,2}(\mathbb{H}^n), s > 0$. Let $A$ stand for the sublaplacian $\mathcal{L}$ restricted to the domain $D(A)$ consisting of all smooth functions $f$ such that for all $\alpha, \beta \in \mathbb{N}^n, j \in \mathbb{N}$ the derivatives $X^\alpha Y^\beta T^j f$ are in $L^2(\mathbb{H}^n)$ and vanish at the origin. Since $X_j, Y_j$ agree with $\partial_{x_j}, \partial_{y_j}$ at the origin, we can also define $D(A)$ in terms of ordinary derivatives $\partial_x^\alpha \partial_y^\beta \partial_t^j$.

**Proposition 3.3.** Let $A$ and $D(A)$ be defined as above where $(n - 1) < s \leq (n + 1)$. Then $A$ is not essentially self adjoint.

**Proof.** In view of Theorem 3.2 it is enough to show that for $s$ as in the statement of the proposition, $D(A)$ is dense in $W^{s,2}(\mathbb{H}^n)$ but $(I + A)D(A)$ is not. These are proved in the following lemmas. □
Lemma 3.4. \( D(A) \) is dense in \( W^{s,2} (\mathbb{H}^n) \) for any \( 0 \leq s \leq (n+1) \).

Proof. If we let \( \Omega = \mathbb{H}^n \setminus \{0\} \) so that \( C_c^\infty (\Omega) \subset D(A) \), it is enough to show that the smaller set is dense in \( W^{s,2} (\mathbb{H}^n) \). This will follow if we can show that the only linear functional that annihilates \( C_c^\infty (\Omega) \) is the zero functional (see chapter 3 of [24]). Let \( \Lambda \in (W^{s,2} (\mathbb{H}^n))^\prime \), the dual of \( W^{s,2} (\mathbb{H}^n) \), be such that \( \Lambda (C_c^\infty (\Omega)) = 0 \). Then there exists \( g \in W^{-s,2} (\mathbb{H}^n) \) such that \( \Lambda = \Lambda_g \) and hence \( \Lambda_g (\phi) = 0 \) for any \( \phi \in C_c^\infty (\Omega) \). Notice that for \( \phi \in C_c^\infty (\mathbb{H}^n) \) the linear map \( \phi \mapsto \Lambda_g (\phi) \) defines a distribution. Indeed, the estimate

\[
|\Lambda_g (\phi)| \leq \|g\|_{(-s)} \|\phi\|_{(s)} \leq \|g\|_{(-s)} \|(I + \mathcal{L})^m \phi\|_2
\]

for any integer \( m > s/2 \) shows that it is indeed a distribution. As it vanishes on \( \Omega \) it is supported at the origin. The structure theory of such distributions allow us to conclude that \( \Lambda_g \) is a finite linear combination of derivatives of Dirac \( \delta \) at the origin, \( \Lambda_g = \sum_{|a| \leq N} c_a \partial^a \delta \), see e.g Chapter 6 of [24].

Since \( X^\alpha \delta = \partial^\alpha_\delta \) and \( Y^\beta \delta = \partial^\beta_\delta \) in the above representation we can also use \( X^\alpha Y^\beta T^j \). It is even more convenient to write them in terms of the complex vector fields defined by \( Z_j = \frac{1}{2} (X_j - i Y_j) \), \( \bar{Z}_j = \frac{1}{2} (X_j + i Y_j) \). Thus we have \( g = \sum_{|a| + |b| + 2j \leq N} c_{a,b,j} Z^a \bar{Z}^b T^j \). If \( g_0 \in L^2 (\mathbb{H}^n) \) is such that \( (I + \mathcal{L})^{-s/2} g = g_0 \) then by (2.3) we have

\[
\int_{-\infty}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} (1 + (2|\alpha| + n)|\lambda|)^{-s} |\langle \hat{g}\rangle (\lambda) \Phi_\alpha, \Phi_\beta\rangle |^2 d\mu (\lambda) < \infty.
\]

Since \( g \) is compactly supported we can calculate the Fourier transform of \( g \) as in Remark 2.1. In view of the relations (2.2) we have

\[
\langle \hat{g}\rangle (\lambda) \Phi_\alpha, \Phi_\beta\rangle = \sum_{|a| + |b| + 2j \leq N} c_{a,b,j} \lambda^j \langle A(\lambda)^a (A(\lambda)^* b) \Phi_\alpha, \Phi_\beta\rangle
\]

By defining \( m(\lambda, \alpha, \beta) \) to be the expression on the right hand side of the above equation we see that

\[
\int_{-\infty}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} (1 + (2|\alpha| + n)|\lambda|)^{-s} |m(\lambda, \alpha, \beta)|^2 |\lambda|^n d\lambda < \infty. \tag{3.1}
\]

The action of \( A(\lambda)^a \) and \( (A(\lambda)^*)^b \) on \( \Phi_\alpha \) are explicitly known, see (321). It is therefore easy to see that

\[
m((2|\alpha| + n)^{-1} \lambda, \alpha, \beta) = \sum_{|a| + |b| + 2j \leq N} C_{a,b,j}(\alpha, \beta) \lambda^{j + (|a| + |b|)/2}
\]

where the coefficients \( C_{a,b,j}(\alpha, \beta) \) are uniformly bounded in both variables. We also remark that for a given \( \alpha \) the function \( C_{a,b,j}(\alpha, \beta) \) is non-zero only for a single value of \( \beta \). By making a change of variables in (3.1) we see that

\[
\sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} (2|\alpha| + n)^{-n-1} \int_{-\infty}^{\infty} \left( \sum_{|a| + |b| + 2j \leq N} C_{a,b,j}(\alpha, \beta) \lambda^{j + (|a| + |b|)/2} \right)^2 \frac{|\lambda|^n}{(1 + |\lambda|)^s} d\lambda < \infty.
\]
Proof. We first recall that the above integral is bounded by 12 BAGCHI, GANGULY, SARKAR AND THANGAVELU

\[ \int_{-\infty}^{\infty} tr(\hat{f}(\lambda))d\mu(\lambda) = f(0) = 0. \]

Let \( g \) be the functions defined by \( \hat{g}(\lambda) = (1 + H(\lambda))^{-s-1} \) we can rewrite the above as

\[ \langle(I + \mathcal{L})f, g \rangle_s = \int_{-\infty}^{\infty} tr(\hat{f}(\lambda))d\mu(\lambda) = 0. \]

So all we need to do is to check \( g \in W^{s,2}(\mathbb{H}^n) \), or equivalently

\[ \int_{-\infty}^{\infty} \left( \sum_{k=0}^{\infty} (1 + (2k + n)|\lambda|)^{-s-2} \right) \|P_k(\lambda)\|_{HS}^2 \|\lambda|^{n}d\lambda < \infty. \]

It is known that \( \|P_k(\lambda)\|_{HS}^2 = \frac{(k+n-1)!}{k!(n-1)!} \leq C(2k+n)^{n-1} \) and so by making a change of variables the above integral is bounded by

\[ \sum_{k=0}^{\infty} (2k + n)^{-2} \int_{-\infty}^{\infty} (1 + |\lambda|)^{-s-2} |\lambda|^n d\lambda. \]

As we assume that \( s > (n-1) \) the integral is finite which proves that \( g \in W^{s,2}(\mathbb{H}^n) \). Hence the lemma.

We now proceed to investigate some properties of the set \( D_{St}(A) \) of Stieltjes vectors for the operator \( A \). The following lemma about series of real numbers will be helpful in proving some properties of Stieltjes vectors for the sublaplacian (see lemma 3.2 of [7]).

Lemma 3.5. For any \( s > (n-1) \), \( (I + A)D(A) \) is not dense in \( W^{s,2}(\mathbb{H}^n) \).

Proof. For any \( f \in D(A) \) the inversion formula for the Fourier transform on \( \mathbb{H}^n \) shows that

\[ \langle(I + \mathcal{L})f, g \rangle_s = \int_{-\infty}^{\infty} tr(\hat{f}(\lambda))d\mu(\lambda) = 0. \]

We then proceed to investigate some properties of the set \( D_{St}(A) \) of Stieltjes vectors for the operator \( A \). The following lemma about series of real numbers will be helpful in proving some properties of Stieltjes vectors for the sublaplacian (see lemma 3.2 of [7]).

Lemma 3.6. If \( \{M_n\}_n \) is sequence of non-negative real numbers such that \( \sum_{n=1}^{\infty} M_n^{-\frac{1}{n}} = \infty \) and \( 0 \leq K_n \leq aM_n + b^n \), then \( \sum_{n=1}^{\infty} K_n^{-\frac{1}{n}} = \infty \).

For \( r > 0 \) the non-isotropic dilation of \( f \) is defined by \( \delta_r f(z, t) = f(rz, r^2t) \) and for \( \sigma \in U(n) \) we define the rotation \( R_\sigma f(z, t) = f(\sigma z, t) \) for all \( (z, t) \in \mathbb{H}^n \).

Lemma 3.7. Suppose \( f \in D(A) \) satisfies the condition \( \sum_{m=0}^{\infty} \|\mathcal{L}^m f\|_2^{-\frac{1}{m}} = \infty \). Then \( f \in D_{St}(A) \). Moreover, \( \delta_r f, R_\sigma f \) are also Stieltjes vectors for \( A \).

Proof. We first recall that \( \mathcal{L} \circ \delta_r = r^2 \delta_r \circ \mathcal{L} \) and \( \mathcal{L} \circ R_\sigma = R_\sigma \circ \mathcal{L} \), see e.g. [30]. Therefore, it follows that if a function satisfies \( \sum_{m=0}^{\infty} \|\mathcal{L}^m f\|_2^{-\frac{1}{m}} = \infty \) then the same is true of \( \delta_r f \) and \( R_\sigma f \). So we only need to prove our claim for \( f \); i.e., when \( f \) satisfies the above condition then we also have \( \sum_{m=0}^{\infty} \|\mathcal{L}^m f\|_2^{-\frac{1}{m}} = \infty \). To see this, we use

\[ \langle \mathcal{L}^m f, \mathcal{L}^n f \rangle_s = \langle \mathcal{L}^{2m} f, (1 + \mathcal{L})^s f \rangle \leq \|\mathcal{L}^s f\|_2 \|\mathcal{L}^{2m} f\|_2. \]
Thus we have \( \| \mathcal{L}^{m} f \|_{(s)}^{\frac{1}{m}} \geq C - \frac{1}{m} \| \mathcal{L}^{2m} f \|_{2} \frac{1}{2m} \) where \( C = \| (I + \mathcal{L})^{s} f \|_{2} \). In view of Lemma 3.6 it is enough to prove the divergence of \( \sum_{m=0}^{\infty} \| \mathcal{L}^{2m} f \|_{2}^{\frac{1}{2m}} \). Without loss of generality we can assume that \( \| f \|_{2} = 1 \). But then \( \| \mathcal{L}^{n} f \|_{2}^{\frac{1}{m}} \) is a decreasing function of \( m \), see Lemma 2.1 in [2]. Consequently, the required divergence follows from the assumption on \( f \). \( \square \)

Before stating the next lemma, let us recall some properties of the matrix coefficients \( e_{\alpha,\beta}^{\lambda}(z,t) = \langle \pi_{\lambda}(z,t) \Phi_{\alpha}^{\lambda}, \Phi_{\beta}^{\lambda} \rangle \) of the Schrödinger representations. These are eigenfunctions of the sublaplacian with eigenvalues \( (2|\alpha| + n)|\lambda| \). Moreover, they satisfy

\[
Z_{j} e_{\alpha,\beta}^{\lambda} = i(2\alpha_{j} + 2)\frac{1}{2} |\lambda|^{\frac{1}{2}} e_{\alpha+e_{j},\beta}^{\lambda}, \quad Z_{j} e_{\alpha,\beta}^{\lambda} = i(2\alpha_{j}) \frac{1}{2} |\lambda|^{\frac{1}{2}} e_{\alpha-e_{j},\beta}^{\lambda}
\]  

(3.2)

where \( e_{j} \) are the coordinate vectors in \( \mathbb{C}^{n} \). We also recall that the sublaplacian is expressed as \( \mathcal{L} = -\frac{1}{2} \sum_{j=1}^{n} (\overline{Z}_{j} Z_{j} + Z_{j} \overline{Z}_{j}) \) in terms of \( Z_{j} \) and \( \overline{Z}_{j} \).

**Lemma 3.8.** If \( f \) satisfies the hypothesis in Lemma 3.7, then \( e_{\alpha,\beta}^{\lambda} f \in D_{St}(A) \), for any \( \alpha, \beta \in \mathbb{N}^{n} \) and \( \lambda \in \mathbb{R}^{*} \).

**Proof.** As noted in the previous lemma, it suffices to show that \( \sum_{m=1}^{\infty} \| \mathcal{L}^{m} (e_{\alpha,\beta}^{\lambda} f) \|_{2}^{\frac{1}{m}} = \infty \). Since \( \mathcal{L} = -\frac{1}{2} \sum_{j=1}^{n} (\overline{Z}_{j} Z_{j} + Z_{j} \overline{Z}_{j}) \) in terms of \( Z_{j} \), a simple calculation shows that

\[
\mathcal{L}(fg) = (\mathcal{L}f)g + f(\mathcal{L}g) - \frac{1}{2} \sum_{j=1}^{n} (Z_{j} f \overline{Z}_{j} g + \overline{Z}_{j} f Z_{j} g).
\]

(3.3)

By taking \( g = e_{\alpha,\beta}^{\lambda} \) and making use of (3.2) along with the estimate \( \| e_{\alpha,\beta}^{\lambda} \|_{\infty} \leq 1 \) we infer that \( \| \mathcal{L}(e_{\alpha,\beta}^{\lambda}) \|_{2} \) is bounded by

\[
\| \mathcal{L}f \|_{2} + (2|\alpha| + n)|\lambda|\| f \|_{2} + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \left( \sqrt{(\alpha_{j} + 1)|\lambda|} \| Z_{j} f \|_{2} + \sqrt{\alpha_{j}|\lambda|} \| \overline{Z}_{j} f \|_{2} \right).
\]

As the operators \( Z_{j} \mathcal{L}^{-1/2} \) and \( \overline{Z}_{j} \mathcal{L}^{-1/2} \) are bounded on \( L^{2}(\mathbb{H}^{n}) \) with norms at most \( \sqrt{2} \), we see that the third term above can be estimated as

\[
\sum_{j=1}^{n} \left( \sqrt{(\alpha_{j} + 1)|\lambda|} + \sqrt{\alpha_{j}|\lambda|} \right) \| \mathcal{L}^{1/2} f \|_{2} \leq 2 \sum_{j=1}^{n} \left( (2\alpha_{j} + 1)|\lambda| \right) \| \mathcal{L}^{1/2} f \|_{2}.
\]

Finally using the fact that \( \| \mathcal{L}^{1/2}(1 + \mathcal{L})^{-1} f \|_{2} \leq \| f \|_{2} \) we get the estimate

\[
\| \mathcal{L}(e_{\alpha,\beta}^{\lambda}) \|_{2} \leq (2|\alpha| + n)|\lambda| \| \mathcal{L} f \|_{2} + 3 \| f \|_{2} + \| \mathcal{L} f \|_{2}.
\]

By defining \( a_{\lambda}(\alpha) = (2|\alpha| + n)|\lambda| \), \( b_{\lambda}(\alpha) = (2|\alpha| + n + 1)|\lambda| \) and \( c_{\lambda}(\alpha) = 3b_{\lambda}(\alpha) + 1 \), we rewrite the above as

\[
\| \mathcal{L}(e_{\alpha,\beta}^{\lambda}) \|_{2} \leq c_{\lambda}(\alpha)(\| \mathcal{L} f \|_{2} + \| f \|_{2}).
\]
In order to prove the lemma it is enough to show for any non-negative integer \( m \) the following estimate holds:

\[
\| \mathcal{L}^m(f e^{\lambda}_{\alpha,\beta}) \|_2 \leq 2^{m-1} c_\lambda(\alpha)^m (\| \mathcal{L}^m f \|_2 + \| f \|_2). \tag{3.4}
\]

We prove this by induction. Assuming the result for any \( m \), we write \( \mathcal{L}^{m+1}(fg) = \mathcal{L}^m(\mathcal{L}g) \) and make use of (3.3) with \( g = e^{\lambda}_{\alpha,\beta} \). The first two terms \( \mathcal{L}^m(\mathcal{L}g) \) and \( \mathcal{L}^m(f \mathcal{L}g) \) together give the estimate

\[
2^{m-1} c_\lambda(\alpha)^m(\| \mathcal{L}^{m+1} f \|_2 + \| \mathcal{L}f \|_2) + a_\lambda(\alpha)2^{m-1} c_\lambda(\alpha)^m(\| \mathcal{L}^m f \|_2 + \| f \|_2).
\]

The boundedness of \( \mathcal{L}(1 + \mathcal{L}^{m+1})^{-1} \) and \( \mathcal{L}^m(1 + \mathcal{L}^{m+1})^{-1} \) allows us to bound the above by

\[
2^m c_\lambda(\alpha)^m(1 + b_\lambda(\alpha))(\| \mathcal{L}^{m+1} f \|_2 + \| f \|_2). \tag{3.5}
\]

We now turn our attention to the estimation of the term

\[
\frac{1}{\sqrt{2}} \sum_{j=1}^n (\sqrt{(\alpha_j + 1)|\lambda|}\mathcal{L}^m(Z_j f e^{\lambda}_{\alpha+e_j,\beta}) + \sqrt{\alpha_j |\lambda|}\mathcal{L}^m(e^{\lambda}_{\alpha-e_j,\beta} Z_j f)).
\]

By using the induction hypothesis along with the fact that the operators \( \mathcal{L}^mZ_j(1 + \mathcal{L}^{m+1})^{-1} \) and \( \mathcal{L}^mZ_j(1 + \mathcal{L}^{m+1})^{-1} \) are bounded with norm at most \( \sqrt{2} \) the \( L^2 \) norm of the above is bounded by

\[
2^{m-1} \sum_{j=1}^n (c_\lambda(\alpha + e_j)^m \sqrt{(\alpha_j + 1)|\lambda|} + c_\lambda(\alpha - e_j)^m \sqrt{\alpha_j |\lambda|}) (\| \mathcal{L}^{m+1} f \|_2 + \| f \|_2).
\]

Since \( b_\lambda(\alpha + e_j) \leq 2b_\lambda(\alpha) \), we have \( c_\lambda(\alpha + e_j) \leq 2c_\lambda(\alpha) \), and so the above sum is bounded by

\[
2a_\lambda(\alpha)2^m c_\lambda(\alpha)^m(\| \mathcal{L}^{m+1} f \|_2 + \| f \|_2). \tag{3.6}
\]

Combining (3.5) and (3.6), using \( a_\lambda(\alpha) \leq b_\lambda(\alpha) \) and recalling the definition of \( c_\lambda \) we obtain (3.4), proving the lemma.

**Proposition 3.9.** Let \( A \) be as in Proposition 3.3 where we have assumed that \( (n - 1) < s \leq (n + 1) \). Assume that \( D_{St}(A) \) contains a nonzero element \( f \) such that \( \delta_r f \) and \( R_\sigma f \) are also in \( D_{St}(A) \) for all \( r > 0 \) and \( \sigma \in U(n) \). Then the linear span of \( D_{St}(A) \) is dense in \( W^{s,2}(\mathbb{H}^n) \).

**Proof.** Let \( f \) be a nonzero member of \( D_{St}(A) \). We will show that the closed linear span of \( D_{St}(A) \) equals \( W^{s,2}(\mathbb{H}^n) \). To prove this, let us take \( g \in W^{s,2}(\mathbb{H}^n) \) which is orthogonal to \( D_{St}(A) \). By Lemma 3.8 we know that \( fe^{\lambda}_{\alpha,\beta} \in D_{St}(A) \) for all \( \alpha, \beta \in \mathbb{N}^n \) and \( \lambda \in \mathbb{R}^s \). Thus,

\[
\langle (I + \mathcal{L})^s g, e^{\lambda}_{\alpha,\beta} f \rangle_{L^2} = \langle g, e^{\lambda}_{\alpha,\beta} f \rangle_{(s)} = 0.
\]

By defining \( p(z, t) = f(z, t)(I + \mathcal{L})^s g(z, t) \), the above translates into

\[
\langle \tilde{p}(\lambda) \Phi^\lambda_{\alpha}, \Phi^\lambda_{\beta} \rangle = \int_{\mathbb{H}^n} p(z, t)(\pi_\lambda(z, t) \Phi^\lambda_{\alpha}, \Phi^\lambda_{\beta})dzdt = 0.
\]
By the inversion formula for the Fourier transform on $\mathbb{H}^n$ we conclude that $p = 0$ which means $(1 + \mathcal{L})^s g$ vanishes on the support of $f$. Under the assumption on $f$ it follows that $(1 + \mathcal{L})^s g$ vanishes identically which forces $g = 0$ as the operator $(1 + \mathcal{L})^s$ is invertible. This proves the density. □

Finally, we are in a position to prove the analogue of Chernoff’s theorem for the sublaplacian on the Heisenberg group.

**Proof of Theorem 1.5** Consider the operator $A$ defined in Proposition 3.3. We have already shown that it is not essentially self adjoint. Suppose there exists a nontrivial $f$ satisfying the hypothesis of Theorem 1.5. Then by Lemma 3.7 we know that $f$ along with $\delta_r f$ and $R_\sigma f$ belong to $D_{St}(A)$. But then by Proposition 3.9 we know that the linear span of $D_{St}$ is dense in $W^{s,2}(\mathbb{H}^n)$. By Theorem 3.1 this allows us to conclude that $A$ is essentially self-adjoint. As this is not the case, $f$ has to be trivial which proves the theorem.

4. **Ingham’s theorem on the Heisenberg group**

In this section we prove Theorem 1.3 using Chernoff’s theorem for the sublaplacian. We first show the existence of a compactly supported function $f$ on $\mathbb{H}^n$ whose Fourier transform has a prescribed decay as stated in Theorem 1.3. This proves the sufficiency part of the condition on the function $\Theta$ appearing in the hypothesis. We then use this part of the theorem to prove the necessity of the condition on $\Theta$.

We begin with some preparations.

4.1. **Construction of $F$.** The Koranyi norm of $x = (z,t) \in \mathbb{H}^n$ is defined by $|x| = |(z,t)| = (|z|^4 + t^2)^{\frac{1}{4}}$. In what follows, we work with the following left invariant metric defined by $d(x,y) := |x^{-1}y|$, $x,y \in \mathbb{H}^n$. Given $a \in \mathbb{H}^n$ and $r > 0$, the open ball of radius $r$ with centre at $a$ is defined by

$$B(a,r) := \{x \in \mathbb{H}^n : |a^{-1}x| < r\}.$$ 

With this definition, we note that if $f,g : \mathbb{H}^n \to \mathbb{C}$ are such that $\text{supp}(f) \subset B(0,r_1)$ and $\text{supp}(g) \subset B(0,r_2)$, then we have

$$\text{supp}(f * g) \subset B(0,r_1).B(0,r_2) \subset B(0,r_1 + r_2),$$

where $f * g(x) = \int_{\mathbb{H}^n} f(xy^{-1})g(y)dy$ is the convolution of $f$ with $g$.

Suppose $\{\rho_j\}_j$ and $\{\tau_j\}_j$ are two sequences of positive real numbers such that both the series $\sum_{j=1}^{\infty} \rho_j$ and $\sum_{j=1}^{\infty} \tau_j$ are convergent. We let $B_{\mathbb{C}^n}(0,r)$ stand for the ball of radius $r$ centered at 0 in $\mathbb{C}^n$ and let $\chi_S$ denote the characteristic function of a set $S$. For each $j \in \mathbb{N}$, we define functions $f_j$ on $\mathbb{C}^n$ and $\tau_j$ on $\mathbb{R}$ by

$$f_j(z) := \rho_j^{-2n}\chi_{B_{\mathbb{C}^n}(0,a\rho_j)}(z), \quad \text{and} \quad \tau_j(t) := \tau_j^{-2}\chi_{[-\tau_j^2/2,\tau_j^2/2]}(t).$$
where the positive constant \( a \) is chosen so that \( \|f_j\|_{L^1(\mathbb{C}^n)} = 1 \). We now consider the functions \( F_j : \mathbb{H}^n \to \mathbb{C} \) defined by
\[
F_j(z,t) := f_j(z)g_j(t), \quad (z,t) \in \mathbb{H}^n.
\]
In the following lemma, we record some useful, easy to prove, properties of these functions.

**Lemma 4.1.** Let \( F_j \) be as above and define \( G_N = F_1 * F_2 * ... * F_N \). Then we have
\[
(1) \|F_j\|_{L^\infty(\mathbb{H}^n)} \leq \rho_j^{-2n} \tau_j^{-2}, \quad \|F_j\|_{L^1(\mathbb{H}^n)} = 1,
(2) \text{supp}(F_j) \subset B_{\mathbb{C}^n}(0,a\rho_j) \times [-\tau_j^2/2,\tau_j^2/2] \subset B(0,a\rho_j + c\tau_j), \quad \text{where } 4c^4 = 1.
(3) For any \( N \in \mathbb{N} \), \text{supp}(G_N) \subset B(0,a \sum_{j=1}^N \rho_j + c \sum_{j=1}^N \tau_j).
(4) Given \( x \in \mathbb{H}^n \) and \( N \in \mathbb{N} \), \( F_2 * F_3 * ... * F_N(x) \leq \rho_2^{-2n} \tau_2^{-2} \).

We also recall a result about Hausdorff measure which will be used in the proof of the next theorem. Let \( \mathcal{H}^n(A) \) denote the \( n \)-dimensional Hausdorff measure of \( A \subset \mathbb{R}^n \). Hausdorff measure coincides with the Lebesgue measure for Lebesgue measurable sets. For sets in \( \mathbb{R}^n \) with sufficiently nice boundaries, the \((n-1)\)-dimensional Hausdorff measure is same as the intuitive surface area. For more about this see [27, Chapter 7]. Let \( A \Delta B \) stand for the symmetric difference between any two sets \( A \) and \( B \). See [25] for a proof of the following theorem.

**Theorem 4.2.** Let \( A \subset \mathbb{R}^n \) be a bounded set. Then for any \( \xi \in \mathbb{R}^n \)
\[
\mathcal{H}^n(A \Delta (A + \xi)) \leq |\xi| \mathcal{H}^{n-1}(\partial A)
\]
where \( A + \xi \) is the translation of \( A \) by \( \xi \) and \( \partial A \) is the boundary of \( A \).

**Theorem 4.3.** The sequence defined by \( G_k = F_1 * F_2 * ... * F_k \) converges to a compactly supported \( F \in L^2(\mathbb{H}^n) \).

**Proof.** In order show that \( (G_k) \) is Cauchy in \( L^2(\mathbb{H}^n) \) we first estimate \( \|G_{k+1} - G_k\|_{L^\infty(\mathbb{H}^n)} \). As all the functions \( F_j \) have unit \( L^1 \) norm, for any \( x \in \mathbb{H}^n \) we have
\[
G_{k+1}(x) - G_k(x) = \int_{\mathbb{H}^n} G_k(xy^{-1})F_{k+1}(y)dy - G_k(x)(x) \int_{\mathbb{H}^n} F_{k+1}(y)dy
= \int_{\mathbb{H}^n} (G_k(xy^{-1}) - G_k(x)) F_{k+1}(y)dy.
\]
As \( F_j \) are even we can change \( y \) into \( y^{-1} \) in the above and estimate the same as
\[
|G_{k+1}(x) - G_k(x)| \leq \int_{\mathbb{H}^n} |G_k(xy) - G_k(x)| F_{k+1}(y)dy.
\]
(4.1)
By defining \( H_{k-1} = F_2 * F_3 * ... * F_k \) so that \( G_k = F_1 * H_{k-1} \), we have
\[
G_k(xy) - G_k(y) = \int_{\mathbb{H}^n} (F_1(xy^{-1}) - F_1(xu^{-1})) H_{k-1}(u)du.
\]
Using the estimate in Lemma 4.1 (4) we now estimate
\[ |G_k(xy) - G_k(x)| \leq \rho_2^{-2n} \tau_2^{-2} \int_{\mathbb{H}^n} |F_1(xy^{-1}) - F_1(xu^{-1})| \, du. \] (4.2)

The change of variables \( u \to ux \) transforms the integral in the right hand side of the above equation into
\[ \int_{\mathbb{H}^n} |F_1(xy^{-1}) - F_1(xu^{-1})| \, du = \int_{\mathbb{H}^n} |F_1(xyx^{-1}u^{-1}) - F_1(u^{-1})| \, du. \]

Since the group \( \mathbb{H}^n \) is unimodular, another change of variables \( u \to u^{-1} \) yields
\[ \int_{\mathbb{H}^n} |F_1(xyx^{-1}u^{-1}) - F_1(u^{-1})| \, du = \int_{\mathbb{H}^n} |F_1(xyx^{-1}u) - F_1(u)| \, du. \]

Let \( x = (z, t) = (z, 0)(0, t), \ y = (w, s) = (w, 0)(0, s) \). As \( (0, t) \) and \( (0, s) \) belong to the center of \( \mathbb{H}^n \), an easy calculation shows that \( xyx^{-1} = (w, 0)(0, s + \Im(z \cdot \bar{w})) \). With \( u = (\zeta, \tau) \) we have
\[ xyx^{-1}u = (w + \zeta, 0)(0, \tau + s + \Im(z \cdot \bar{w}) - (1/2) \Im(\zeta \cdot \bar{w})). \]

Since \( F_1(z, t) = f_1(z)g_1(t) \) we see that the integrand \( F_1(xyx^{-1}u) - F_1(u) \) in the above integral takes the form
\[ f_1(w + \zeta)g_1(\tau + s + \Im(z \cdot \bar{w}) - (1/2) \Im(\zeta \cdot \bar{w})) - f_1(\zeta)g_1(\tau). \]

By setting \( b = b(s, z, w, \zeta) = s + \Im(z \cdot \bar{w}) - (1/2) \Im(\zeta \cdot \bar{w}) \) we can rewrite the above as
\[ (f_1(w + \zeta) - f_1(\zeta))g_1(\tau + b) + f_1(\zeta)(g_1(\tau + b) - g_1(\tau)). \] (4.3)

In order to estimate the contribution of the second term to the integral under consideration we first estimate the \( \tau \) integral as follows:
\[ \int_{-\infty}^{\infty} |g_1(\tau + b) - g_1(\tau)| \, d\tau = |b + K_{\tau}| \Delta K_{\tau}, \]
where \( K_{\tau} = [-\frac{1}{2}, \frac{1}{2}]^2 \) is the support of \( g_1 \). For \( \zeta \) in the support of \( f_1 \), we have \( |\zeta| \leq a\rho_1 \) and hence
\[ |(-b + K_{\tau})\Delta K_{\tau}| \leq 2|b(z, w, \zeta)| \leq (2|s| + |z||w| + a\rho_1|w|). \]

Thus we have proved the estimate
\[ \int_{\mathbb{H}^n} f_1(\zeta)|g_1(\tau + b) - g_1(\tau)| \, d\zeta \, d\tau \leq C(2|s| + (a\rho_1 + |z|)|w|) \] (4.4)

As \( g_1 \) integrates to one, the contribution of the first term in (4.3) is given by
\[ \int_{\mathbb{H}^n} |f_1(w + \zeta) - f_1(\zeta)| \, d\zeta = \rho_1^{-2n} \mathcal{H}^{2n}((-w + B_{C^n}(0, a\rho_1)) \Delta B_{C^n}(0, a\rho_1)). \]

By appealing to Theorem 4.2 in estimating the above, we obtain
\[ \int_{\mathbb{H}^n} |f_1(w + \zeta) - f_1(\zeta)| \, g(\tau + b) \, d\zeta \, d\tau \leq C|w|. \] (4.5)
Using the estimates (4.4) and (4.5) in (4.2) we obtain
\[ |G_k(xy) - G_k(x)| \leq C \tau_2^{-2n} \| \lambda \| \left( |s| + (c_1 + c_2 |z|) |w| \right). \]

This estimate, when used in (4.1), in turn gives us
\[ |G_{k+1}(z,t) - G_k(z,t)| \leq C \int_{\mathbb{H}^n} \left( |s| + (c_1 + c_2 |z|) |w| \right) F_{k+1}(w,s) \, dw ds \tag{4.6} \]
where the constants \( c_1, c_2 \) and \( C \) depend only on \( n \). Recalling that on the support of \( F_{k+1}(w,s) = f_{k+1}(w)g_{k+1}(s), |w| \leq \rho_{k+1} \) and \( |s| \leq \tau_{k+1}^2 \), the above yields the estimate
\[ |G_{k+1}(z,t) - G_k(z,t)| \leq C \left( \tau_{k+1}^2 + (c_1 + c_2 |z|) \rho_{k+1} \right). \tag{4.7} \]

It is easily seen that the support of \( G_{k+1} - G_k \) is contained in \( B(0, a\rho + c\tau) \) where \( \rho = \sum_{j=1}^{\infty} \rho_j \) and \( \tau = \sum_{j=1}^{\infty} \tau_j \). Consequently, from the above we conclude that
\[ \|G_{k+1} - G_k\|_2 \leq \|G_{k+1} - G_k\|_\infty |B(0, a\rho + c\tau)| \leq C \left( \tau_{k+1}^2 + c_3 \rho_{k+1} \right). \]

From the above, it is clear that \( G_k \) is Cauchy in \( L^2(\mathbb{H}^n) \) and hence converges to a function \( F \in L^2(\mathbb{H}^n) \) whose support is contained in \( B(0, a\rho + c\tau) \).

4.2. Estimating the Fourier transform of \( F \). Suppose now that \( \Theta \) is an even, decreasing function on \( \mathbb{R} \) for which \( \int_1^\infty \Theta(t) t^{-1} dt < \infty \). We want to choose two sequences \( \rho_j \) and \( \tau_j \) in terms of \( \Theta \) so that the series \( \sum_{j=1}^{\infty} \rho_j \) and \( \sum_{j=1}^{\infty} \tau_j \) both converge. We can then construct a function \( F \) as in Theorem 4.3 which will be compactly supported. Having done the construction we now want to compute the Fourier transform of the constructed function \( F \) and compare it with \( e^{-\Theta(\sqrt{H(\lambda)})} \sqrt{H(\lambda)} \). This can be achieved by a judicious choice of the sequences \( \rho_j \) and \( \tau_j \). As \( \Theta \) is given to be decreasing it follows that \( \sum_{j=1}^{\infty} \Theta(j) \rho(j) / j \) is finite. It is then possible to choose a decreasing sequence \( \rho_j \) such that \( \rho_j \geq c_j e^{2 \Theta(j) / j} \) (for a constant \( c_j \) to be chosen later) and \( \sum_{j=1}^{\infty} \rho_j < \infty \). Similarly, we choose another decreasing sequence \( \tau_j \) such that \( \sum_{j=1}^{\infty} \tau_j < \infty \).

In the proof of the following theorem we require good estimates for the Laguerre coefficients of the function \( f_j(z) = \rho_j^{-2n} \chi_{B_{cn}(0,a\rho)}(z) \) where \( a \) chosen so that \( \| f_j \|_1 = 1 \). These coefficients are defined by
\[ R_{n-1}^k(\lambda, f_j) = \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} f_j(z) \varphi_{k,\lambda}^{n-1}(z) \, dz. \tag{4.8} \]

Lemma 4.4. There exists a constant \( c_n > 0 \) such that
\[ |R_{k-1}^n(\lambda, f_j)| \leq c_n \rho_j \sqrt{(2k+n)\lambda} \lambda^{-n+1/2}. \]

Proof. By abuse of notation we write \( \varphi_{k,\lambda}^{n-1}(r) \) in place of \( \varphi_{k,\lambda}^{n-1}(z) \) when \( |z| = r \). As \( f_j \) is defined as the dilation of a radial function, the Laguerre coefficients are given by the integral
\[ R_{n-1}^k(\lambda, f_j) = \frac{2\pi^n}{\Gamma(n)} \frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty \varphi_{k,\lambda}^{n-1}(\rho_j r) r^{2n-1} \, dr. \tag{4.9} \]
When $a \leq (\rho_j \sqrt{(2k + n)|\lambda|})^{-1}$ we use the bound $\frac{k!(n-1)!}{(k+n-1)!} |\phi_{k,\lambda}^n(r)| \leq 1$ to estimate

$$\frac{2\pi^n}{\Gamma(n)} \frac{k!(n-1)!}{(k+n-1)!} \int_0^a \phi_{k,\lambda}^n(\rho_j r) r^{2n-1} dr \leq \frac{\pi^n a^{n+1/2}}{\Gamma(n+1)} (\rho_j \sqrt{(2k + n)|\lambda|})^{-n+1/2}. $$

When $a > (\rho_j \sqrt{(2k + n)|\lambda|})^{-1}$ we split the integral into two parts, one of which gives the same estimate as above. To estimate the integral taken over $(\rho_j \sqrt{(2k + n)|\lambda|})^{-1} < r < a$, we use the bound stated in Lemma 2.1 which leads to the estimate

$$\frac{2\pi^n}{\Gamma(n)} \frac{k!(n-1)!}{(k+n-1)!} \int_{(\rho_j \sqrt{(2k+n)|\lambda|})^{-1}}^a \phi_{k,\lambda}^n(\rho_j r) r^{2n-1} dr \leq C_n (\rho_j \sqrt{(2k + n)|\lambda|})^{-n+1/2} \int_0^a r^{n-1/2} dr = C_n' a^{n+1/2} (\rho_j \sqrt{(2k + n)|\lambda|})^{-n+1/2}. $$

Combining the two estimates we get the lemma. \hfill \square

**Theorem 4.5.** Let $\Theta : \mathbb{R} \to [0, \infty)$ be an even, decreasing function with $\lim_{\lambda \to \infty} \Theta(\lambda) = 0$ for which $\int_0^\infty \frac{\Theta(\lambda)}{\lambda} d\lambda < \infty$. Let $\rho_j$ and $\tau_j$ be chosen as above. Then the Fourier transform of the function $F$ constructed in Theorem 4.3 satisfies the estimate

$$\hat{F}(\lambda)^* \hat{F}(\lambda) \leq e^{-2\Theta(\sqrt{H(\lambda)}) \sqrt{H(\lambda)}}, \quad \lambda \neq 0. $$

**Proof.** Observe that $F$ is radial since each $F_j$ is radial and hence the Fourier transform $\hat{F}(\lambda)$ is a function of the Hermite operator $H(\lambda)$. More precisely,

$$\hat{F}(\lambda) = \sum_{k=0}^\infty R_k^{n-1}(\lambda, F) P_k(\lambda) \quad \text{(4.10)}$$

where the Laguerre coefficients are explicitly given by (see (2.4.7) in [31]. There is a typo-the factor $|\lambda|^{n/2}$ should not be there)

$$R_k^{n-1}(\lambda, F) = \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} F^\lambda(z) \phi_{k,\lambda}^{n-1}(z) dz. $$

In the above, $F^\lambda(z)$ stands for the inverse Fourier transform of $F(z,t)$ in the $t$ variable. Expanding any $\varphi \in L^2(\mathbb{R}^n)$ in terms of $\Phi_{\lambda}^j$ it is easy to see that the conclusion $\hat{F}(\lambda)^* \hat{F}(\lambda) \leq e^{-2\Theta(\sqrt{H(\lambda)}) \sqrt{H(\lambda)}}$ follows once we show that

$$(R_k^{n-1}(\lambda, F))^2 \leq C e^{-2\Theta(\sqrt{(2k+n)|\lambda|}) \sqrt{(2k+n)|\lambda|}}$$

for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}^*$. Now note that, by definition of $g_j$ and the choice of $a$, we have

$$|\hat{g}_j(\lambda)| = \left| \frac{\sin(\frac{1}{2} \tau_j \lambda)}{\frac{1}{2} \tau_j \lambda} \right| \leq 1, \quad |R_k^{n-1}(\lambda, f_j)| \leq 1.$$

The bound on $R_k^{n-1}(\lambda, f_j)$ follows from the fact that $|\varphi_{k,\lambda}^j(z)| \leq \frac{(k+n-1)!}{k!(n-1)!}$. Since $F$ is constructed as the $L^2$ limit of the $N$-fold convolution $G_N = F_1 * F_2 \ldots \ast F_N$ we observe that for any $N$

$$(R_k^{n-1}(\lambda, F))^2 \leq (R_k^{n-1}(\lambda, G_N))^2 = (\Pi_{j=1}^N R_k^{n-1}(\lambda, F_j))^2$$
and hence it is enough to show that for a given \( k \) and \( \lambda \) one can choose \( N = N(k, \lambda) \) in such a way that
\[
(\Pi_{j=1}^{N} R_{k}^{n-1}(\lambda, F_{j}))^{2} \leq C e^{-2\Theta(\sqrt{(2k+n)|\lambda|})}.
\]  
(4.11)
where \( C \) is independent of \( N \). From the definition of \( G_{N} \) it follows that
\[
\widetilde{G}_{N}(\lambda) = \Pi_{j=1}^{N} \hat{F}_{j}(\lambda) = \Pi_{j=1}^{N} \left( \sum_{k=0}^{\infty} R_{k}^{n-1}(\lambda, F_{j}) P_{k}(\lambda) \right)
\]
and hence \( R_{k}^{n-1}(\lambda, G_{N}) = \Pi_{j=1}^{N} R_{k}^{n-1}(\lambda, F_{j}) \). As \( F_{j}(z, t) = f_{j}(z) g_{j}(t) \), we have
\[
R_{k}^{n-1}(\lambda, G_{N}) = (\Pi_{j=1}^{N} \hat{g}_{j}(\lambda))(\Pi_{j=1}^{N} R_{k}^{n-1}(\lambda, f_{j})).
\]
As the first factor is bounded by one, it is enough to consider the product \( \Pi_{j=1}^{N} R_{k}^{n-1}(\lambda, f_{j}) \).

We now choose \( \rho_{j} \) satisfying \( \rho_{j} \geq c_{n} e^{2\Theta(j)} \) where \( c_{n} \) is the same constant appearing in Lemma 4.4. We then take \( N = |\Theta(((2k+n)|\lambda|)^{\frac{1}{2}})((2k+n)|\lambda|)^{\frac{1}{2}}| \) and consider
\[
\Pi_{j=1}^{N} R_{k}^{n-1}(\lambda, f_{j}) \leq \Pi_{j=1}^{N} c_{n} \rho_{j} \sqrt{(2k+n)|\lambda|}^{-n+1/2}
\]
where we have used the estimates proved in Lemma 4.4. As \( \rho_{j} \) is decreasing
\[
\Pi_{j=1}^{N} c_{n} \rho_{j} \sqrt{(2k+n)|\lambda|}^{-n+1/2} \leq c_{n}^{N} (\rho_{N} \sqrt{(2k+n)|\lambda|})^{-N}. \tag{4.12
\]
By the choice of \( \rho_{j} \) it follows that
\[
\rho_{N}^{2}(2k+n)|\lambda| \geq c_{n} e^{4 \Theta(N)^{2}} (2k+n)|\lambda|.
\]
As \( \Theta \) is decreasing and \( N \leq \sqrt{(2k+n)|\lambda|} \) we have \( \Theta(N) \leq \Theta(\sqrt{(2k+n)|\lambda|}) \) and so
\[
\Theta(N)^{2}(2k+n)|\lambda| \geq \Theta(\sqrt{(2k+n)|\lambda|})^{2} (2k+n)|\lambda| \geq N^{2}
\]
which proves that \( \rho_{N}^{2}(2k+n)|\lambda| \geq c_{n}^{4} e^{4} \). Using this in \ref{4.12} we obtain
\[
\Pi_{j=1}^{N} c_{n} \rho_{j} \sqrt{(2k+n)|\lambda|}^{-n+1/2} \leq c_{n}^{2} e^{-2} e^{(n-1)N} e^{-N}.
\]
Finally, as \( N + 1 \geq \Theta(((2k+n)|\lambda|)^{\frac{1}{2}})((2k+n)|\lambda|)^{\frac{1}{2}}, \) we obtain the estimate \ref{4.11}.

4.3. Ingham’s theorem. We can now prove Theorem 1.3. Since half of the theorem has been already proved, we only need to prove the following.

**Theorem 4.6.** Let \( \Theta : \mathbb{R} \to [0, \infty) \) be an even, decreasing function with \( \lim_{|\lambda| \to \infty} \Theta(\lambda) = 0 \) and \( I = \int_{1}^{\infty} \Theta(\lambda) \lambda^{-1} d\lambda = \infty \). Suppose the Fourier transform of \( f \in L^{1}(\mathbb{H}^{n}) \) satisfies
\[
\hat{f}(\lambda) \hat{f}(\lambda) \leq e^{-\Theta(H(\lambda))} \sqrt{H(\lambda)}, \quad \lambda \neq 0.
\]
If \( f \) vanishes on a non-empty open set, then \( f = 0 \) a.e.
Proof. Without loss of generality we can assume that \( f \) vanishes on \( B(0, \delta) \). First we assume that \( \Theta(\lambda) \geq 2|\lambda|^{-1/2}, |\lambda| \geq 1 \). In view of Plancherel theorem for the group Fourier transform on the Heisenberg group we have

\[
\|L^m f\|_2^2 = (2\pi)^{-(n+1)} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)H(\lambda)^m\|_{HS}^2 |\lambda|^n d\lambda.
\]

Using the formula for Hilbert-Schmidt norm of an operator we have

\[
\|L^m f\|_2^2 = (2\pi)^{-(n+1)} \int_{-\infty}^{\infty} \sum_{\alpha} ((2|\alpha| + n)|\lambda|)^2m \|\hat{f}(\lambda)\Phi^\lambda_{m\alpha}\|_2^2 |\lambda|^n d\lambda
\]

Now the given condition on the Fourier transform leads to the estimate

\[
\|L^m f\|_2^2 \leq C \sum_{k=0}^{\infty} (2k + n)^{n-1} \int_{-\infty}^{\infty} ((2k + n)|\lambda|)^2m e^{-\Theta((2k+n)|\lambda|)^{1/2}} (2|\alpha| + n)|\lambda|^n d\lambda
\]

Now changing the variable from \( \lambda \) to \((2k + n)^{-1} \lambda\) we get

\[
\|L^m f\|_2^2 \leq C \sum_{k=0}^{\infty} (2k + n)^{-2} \int_{0}^{\infty} \lambda^{2m+n} e^{-\Theta(\lambda)\sqrt{\lambda}} d\lambda.
\]

The integral \( I \) appearing above can be estimated as follows. Under the extra assumption \( \Theta(\lambda) \geq 2|\lambda|^{-1/2}, \) on \( \Theta \) we have

\[
I = \int_{0}^{\infty} \lambda^{2m+n} e^{-\Theta(\lambda)\sqrt{\lambda}} d\lambda + \int_{m^4}^{\infty} \lambda^{2m+n} e^{-\Theta(\lambda)\sqrt{\lambda}} d\lambda
\]

\[
\leq 2m^{8(n+1)} \int_{0}^{\infty} \lambda^{4m-1} e^{-\Theta(\lambda)\lambda} d\lambda + 4 \int_{m^4}^{\infty} \lambda^{8m+4(n+1)-1} e^{-2\lambda} d\lambda.
\]

The above is dominated by a sum of two gamma integrals which can be evaluated to get

\[
I \leq 2m^{8(n+1)} \Gamma(4m)\Theta(m^4)^{-4m} + 4e^{-m^2\Gamma(8m + 4(n+1)).
\]

Using Stirling’s formula (see Ahlfors [1]) \( \Gamma(x) \leq \sqrt{2\pi} x^{x-1/2} e^{-x} e^{\theta(x)/12x}, 0 < \theta(x) < 1 \) valid for \( x > 0 \), we observe the the second term in the estimate for \( I \) goes to zero as \( m \) tends to infinity and the first term (and hence \( I \) itself ) is bounded by \( C(4m)^{4m}\Theta(m^4)^{-4m}\).

Thus we have proved the estimate \( \|L^m f\|_2^2 \leq C(4m)^{4m}\Theta(m^4)^{-4m}\). The hypothesis on \( \Theta \) namely, \( \int_{\infty}^{\infty} \frac{\Theta(t)}{t} dt = \infty \), by a change of variable implies that \( \int_{1}^{\infty} \frac{\Theta(y^4)}{y^4} dy = \infty \). Hence by integral test we get \( \sum_{m=1}^{\infty} \frac{\Theta(m^4)}{m} = \infty \). Therefore, it follows that \( \sum_{m=1}^{\infty} \|L^m f\|_2^{-2m} = \infty \).

Since it vanishes on \( B(0, \delta) \), \( \hat{f} \) and all its partial derivatives vanish at the origin. Therefore, by Chernoff’s theorem for the sublaplacian we conclude that \( f = 0 \). Now we consider the general case.
The function $\Psi(y) = (1 + |y|)^{-1/2}$ satisfies $\int_1^\infty \frac{\Psi(y)}{y} dy < \infty$. By Theorem 4.3 we can construct a radial function $F \in L^2(\mathbb{H}^n)$ supported in $B(0, \delta/2)$ such that

$$\hat{F}(\lambda)^* \hat{F}(\lambda) \leq e^{-\Psi(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}, \quad \lambda \neq 0.$$ 

As $f$ is assumed to vanish on $B(0, \delta)$, the function $h = f * F$ vanishes on the smaller ball $B(0, \delta/2)$. This can be easily verified by looking at

$$f * F(x) = \int_{\mathbb{H}^n} f(xy^{-1})F(y)dy = \int_{B(0, \frac{\delta}{2})} f(xy^{-1})F(y)dy.$$ 

When both $x, y \in B(0, \delta/2)$, $d(0, xy^{-1}) = |xy^{-1}| \leq |x| + |y| < \delta$ and hence $f(xy^{-1}) = 0$ proving that $f * F(x) = 0$. The same is true for all the derivatives of $h$. We now claim that

$$\hat{h}(\lambda)^* \hat{h}(\lambda) \leq e^{-2\Phi(\sqrt{H(\lambda)})}\sqrt{H(\lambda)}$$

where $\Phi(y) = \Theta(y) + \Psi(y)$. As $\hat{h}(\lambda) = \hat{f}(\lambda)\hat{F}(\lambda)$, for any $\varphi \in L^2(\mathbb{R}^n)$ we have

$$\langle \hat{h}(\lambda)^* \hat{h}(\lambda)\varphi, \varphi \rangle = \langle \hat{f}(\lambda)^* \hat{f}(\lambda)\hat{F}(\lambda)\varphi, \hat{F}(\lambda)\varphi \rangle.$$ 

The hypothesis on $f$ gives us the estimate

$$\langle \hat{f}(\lambda)^* \hat{f}(\lambda)\hat{F}(\lambda)\varphi, \hat{F}(\lambda)\varphi \rangle \leq C\langle e^{-2\Theta(\sqrt{H(\lambda)})}\sqrt{H(\lambda)}\hat{F}(\lambda)\varphi, \hat{F}(\lambda)\varphi \rangle.$$ 

As $F$ is radial, $\hat{F}(\lambda)$ commutes with any function of $H(\lambda)$ and hence the right hand side can be estimated using the decay of $\hat{F}(\lambda)$:

$$\langle \hat{F}(\lambda)^* \hat{F}(\lambda)e^{-\Theta(\sqrt{H(\lambda)})}\sqrt{H(\lambda)}\varphi, e^{-\Theta(\sqrt{H(\lambda)})}\sqrt{H(\lambda)}\varphi \rangle \leq C\langle e^{-2(\Theta+\Psi)(\sqrt{H(\lambda)})}\sqrt{H(\lambda)}\varphi, \varphi \rangle.$$ 

This proves our claim on $\hat{h}(\lambda)$ with $\Phi = \Theta + \Psi$. As $\Phi(y) \geq |y|^{-1/2}$, by the already proved part of the theorem we conclude that $h = 0$. In order to conclude that $f = 0$ we proceed as follows.

Given $F$ as above, let us consider $\delta_r F(z, t) = F(rz, r^2t)$. It has been shown elsewhere (see e.g. [13]) that

$$\delta_r \hat{F}(\lambda) = r^{-(2n+2)}d_r \circ \hat{F}(r^{-2}\lambda) \circ d_r^{-1}$$

where $d_r$ is the standard dilation on $\mathbb{R}^n$. The property of the function $F$, namely $\hat{F}(\lambda)^* \hat{F}(\lambda) \leq e^{-2\Psi(\sqrt{H(\lambda)})}\sqrt{H(\lambda)}$ gives us

$$\delta_r \hat{F}(\lambda)^* \delta_r \hat{F}(\lambda) \leq C r^{-(2n+2)}d_r \circ e^{-2\Psi(\sqrt{H(\lambda)/r^2})}\sqrt{H(\lambda)/r^2} \circ d_r^{-1}.$$ 

Testing against $\Phi_\lambda^\rho$ we can simplify the right hand side which gives us

$$\delta_r \hat{F}(\lambda)^* \delta_r \hat{F}(\lambda) \leq C r^{-2(2n+2)}e^{-2\Psi_r(\sqrt{H(\lambda)})}\sqrt{H(\lambda)}$$

where $\Psi_r(y) = \frac{1}{r}\Psi(y/r)$. If we let $F_\varepsilon(x) = \varepsilon^{-2n+2}\delta_{\varepsilon}^{-1}F(x)$ then it follows that $F_\varepsilon$ is an approximate identity. Moreover, $F_\varepsilon$ is supported in $B(0, \varepsilon\delta)$ and satisfies the same hypothesis as $F$ with $\Psi(y)$ replaced by $\varepsilon\Psi(\varepsilon y)$ which has the same integrability and decay conditions.
Hence, working with $F_\varepsilon$ we can conclude that $f \ast F_\varepsilon = 0$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ and noting that $f \ast F_\varepsilon$ converges to $f$ in $L^1(\mathbb{H}^n)$ we conclude that $f = 0$. This completes the proof. \hfill \Box

Remark 4.1. We also have a version of Ingham’s Theorem for the Weyl transform (recall the definition from subsection 2.1). With the notation $H = H(1) = -\Delta_{\mathbb{R}^n} + |x|^2$, we can prove the following theorem.

Theorem 4.7. Let $\Theta(\lambda)$ be a nonnegative even function on $\mathbb{R}$ such that $\Theta(\lambda)$ decreases to zero when $\lambda \to \infty$. There exists a nonzero compactly supported continuous function $f$ on $\mathbb{C}^n$ whose Weyl transform $W(f)$ satisfies the estimate

$$W(f) \ast W(f) \leq C e^{2\Theta(\sqrt{H})\sqrt{H}}$$

if and only if the function $\Theta$ satisfies $\int_1^\infty \Theta(t)t^{-1}dt < \infty$.

The proof of the theorem is very similar to that of Theorem 1.3. In fact, due to the absence of $t$-variable the proof of the above theorem is easier and can be obtained by doing some obvious modification in the proof of Theorem 1.3.

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