Concurrence in the inhomogeneous Tavis-Cummings model

C E López, F Lastra, G Romero and J C Retamal
Departamento de Física, Universidad de Santiago de Chile USACH, Casilla 307 Correo 2, Santiago, Chile
E-mail: calopez@usach.cl

Abstract. We study the problem of a collection of two-level atoms interacting inhomogeneously with a quantized mode of the electromagnetic field, that is, an inhomogeneous version of the Tavis-Cummings model. In this system we analyze the entanglement properties of the eigenstate spectrum and analytically we found strong dependence on the inhomogeneity.

1. Introduction
Entanglement plays a fundamental role in quantum information and quantum computation [1]. On the other hand, recent advances in the manipulation of collections of atoms, such as atomic clouds, have led to the possibility of considering inhomogeneous coupling. For instance, decoherence of collective atomic states due to the inhomogeneous coupling between the atoms and external fields has been studied by Sun et al. [2]. Also, in Ref. [3] the nonsymmetric character of the entanglement of multi-atom quantum states due the inhomogeneity has been studied. Recently, decoherence in process of quantum information storage in atomic clouds has been subject of study [4], where the phenomenon of electromagnetically induced transparency (EIT) is present.

The entanglement properties of the ground state for the Dicke model [5] has recently been studied Ref. [6, 7]. In such case, due to the symmetry of the Dicke states, the concurrence between two atoms is independent of the pair considered. A similar situation should arise in the case of the Tavis Cummings model [8]. From the point of view of the model properties, it would be interesting to study the quantum correlations between atoms under more general conditions, as is for the case of inhomogeneous coupling.

In this work we study the entanglement properties in the spectrum eigenstates of the TC model [8] by considering inhomogeneous coupling of the atoms to the quantum electromagnetic field. Specifically we study the bipartite atomic concurrence by tracing out $N - 2$ particles.

2. The Model
The interaction of a single mode of the quantum electromagnetic field with $N$ atoms in the on resonance regime is given by ($\hbar = 1$)

$$H = a^\dagger a + S_z + a J_+ + a^\dagger J_-,\quad (1)$$

where, $S_z = \sum_{j=1}^{N} \sigma_{z,j}/2$ and $J_\pm = \sum_{j=1}^{N} \kappa_j \sigma_\pm$ are the atomic collective operators and we have introduced inhomogeneous coupling constants $\kappa_j = g_j/w$. In this work, we will consider $\kappa_j = \kappa \sin(\pi x_j/L)$. We notice that due to the inhomogeneous coupling, the collective atomic operators do not satisfy a closed Lie algebra su(2), leading to a larger total Hilbert space, so that, the description in terms of symmetric Dicke states is not allowed anymore.
Figure 1. Levels scheme that involve the Hilbert space. Here is possible to see the Hilbert space’s growth due to the Inhomogeneous coupling.

An effective approach to study the dynamics of systems with inhomogeneous coupling can be developed [9]. The basic idea of this approach is to follow the Hilbert space that the coupled system will visit along the evolution and to implement a truncation criteria based on a probabilistic argument. For example, let us consider the state $|n\rangle|\bar{0}\rangle$ where $|n\rangle$ denotes the state with $n$ photons and $|\bar{0}\rangle$ denotes the collection of $N$ atoms in the ground state $|g_1g_2...g_N\rangle$. Let us see now how the state $|n\rangle|\bar{0}\rangle$ is coupled to other states under the interaction Hamiltonian (1) which conserves the number of excitations. We see that the state $|0\rangle|\bar{0}\rangle$ is not coupled by the interaction Hamiltonian to other states of the global system, that is,

$$|0\rangle|\bar{0}\rangle \rightarrow |0\rangle|\bar{0}\rangle$$

Now, if we have one excitation in the electromagnetic field $|1\rangle|\bar{0}\rangle$ this state is coupled through the term $J_+a$ so that

$$J_+a|1\rangle|\bar{0}\rangle = N_1|0\rangle |\bar{1}\rangle$$

where $|\bar{1}\rangle = (1/N_1)\sum_{i=1}^{N} \kappa_i |\bar{1}_i\rangle$, with $N_1 = (\sum_{i=1}^{N} \kappa_i^2)^{1/2}$ and $|\bar{1}_i\rangle$ represents the state where the $i$-th atom is excited.

If we increase the number of excitations $k$ in the system, for example $k = 2$, one would expect that the Hilbert space will be spanned by the vectors $\{ |2\rangle|\bar{0}\rangle, |1\rangle|\bar{1}\rangle, |0\rangle|\bar{2}\rangle \}$, where $|\bar{2}\rangle = (2/N_2)\sum_{i<j}^{N} \kappa_i \kappa_j |\bar{2}_{ij}\rangle$ and $N_2 = (4\sum_{i<j}^{N} \kappa_i^2 \kappa_j^2)^{1/2}$. However, it can be shown that the state $|0\rangle|\bar{2}\rangle$ is coupled through $a^1J_-$ to the state $|1\rangle|\Phi_1\rangle$, where $|\Phi_1\rangle = (2/N_2)\sum_{i}^{N} (\sum_{j \neq i} \kappa_i \kappa_j^2) |\bar{1}_i\rangle$, which is different from $|\bar{1}\rangle$, so we can say that $|\Phi_1\rangle$ has a component along $|\bar{1}\rangle$ and a component along a state $|\bar{1}_p\rangle$ perpendicular to the state $|\bar{1}\rangle$ [9]. Thus, if the number of excitations keep growing, then it is possible to generate a sequence of collective atomic states and its respectively perpendicular states as shown by Fig. 1, where the coupling among atomic collective states are shown.

In general, the collective atomic states $|\bar{n}\rangle$ are given by

$$|\bar{n}\rangle = \frac{1}{N_n!} \sum_{i_1<i_2<...<i_n} n! \kappa_{i_1} \kappa_{i_2}...\kappa_{i_n} |\bar{n}_{i_1i_2...i_n}\rangle,$$
Figure 2. Contribution (%) of second row of Fig. 2 to the energy as a function of the number of atoms $N$. $\Delta E^{(k)} = 100( |(E_1^{(k)} - E_2^{(k)}) / E_1^{(k)}| )$, where $E_1^{(k)}$, $E_2^{(k)}$ are the energies for $k$ excitations using 1 or 2 rows of Fig. 1.

with $N_n^2 = \sum_{i_1 < i_2 < \ldots < i_n} (n!)^2 \kappa_{i_1}^2 \kappa_{i_2}^2 \ldots \kappa_{i_n}^2$. On the other hand, the sequence of states $|\tilde{n}_{\mu}\rangle$ in the second row of Fig. 1 are given by

$$ |\tilde{n}_{\mu}\rangle = \frac{1}{\sqrt{\langle \Phi_n | \Phi_n \rangle - |\langle \tilde{n} | \Phi_n \rangle|^2}} |\Phi_n\rangle - \langle \tilde{n} | \Phi_n \rangle |\tilde{n}\rangle, \quad (5) $$

where the states $|\Phi_n\rangle$ are written as

$$ |\Phi_n\rangle = (n - 1)! \sum_{i_1 < \ldots < i_n} \kappa_{i_1} \ldots \kappa_{i_n} |\tilde{n}_{i_1 \ldots i_n}\rangle, \quad (6) $$

and $\tilde{n}_{i_1 \ldots i_n} = (N - 1) \sum_{j=1}^{N} \kappa_j^2$.

An accurate description of the model should consider all the states we have found associated with the inhomogeneity. However, based on numerical calculations we have obtained that the spectrum of the Hamiltonian is not sensitive to perpendicular states further than the first row.

In Fig. 2, we observe that the contribution of the second row in Fig. 1 is less than $1\%$. This method is also valid when the quantum dynamics of the system is studied. In that case the approximation is valid within a given window of time. The size of this window of time will depend on both $N$ and $k$ [9].

3. Eigenstate Spectrum Properties

Using the results of the previous discussion, we now study the properties of the eigenstates of the system by characterizing the entanglement behavior between a pair of particles. In the most general case, the state with $k$ excitations can be written as

$$ |\Psi^{(k)}\rangle = \sum_{s=0}^{k} A_s |\tilde{s}\rangle_A |k - s\rangle_F, \quad (7) $$
where $|\bar{s}\rangle_A$ is the collective atomic state with $s$ excitations defined in Eq. (4) and $A_s$ is a coefficient arising from the eigenvalue problem. Tracing out the quantum field, the reduced density matrix for the atomic subsystem is given by

$$
\rho^{(k)} = \sum_{s=0}^{k} |A_s|^2 |\bar{s}\rangle_A \langle \bar{s}|,
$$

where

$|\bar{s}\rangle_A = \sum_{p=1}^{N} |\bar{s}_p\rangle_A$

and

$$
\kappa_{n_1}^{(k)} = \frac{1}{N^{k}} \sum_{n_1<n_2<\ldots<n_k} \frac{(n!)}{\kappa_{n_1}^{2} \cdots \kappa_{n_k}^{2}}.
$$

Now, tracing out $N - 2$ atoms, the bipartite concurrence between atoms $i$, $j$ will be given by

$$
C_{i,j}^{(k)} = 2|\kappa_i^{(k)}\kappa_j|\left\{ \sum_{s=0}^{k} \frac{|A_s|^2}{N_s^2} M_{s-1}^2 - \left[ \sum_{s,s'=0}^{k} \frac{|A_s|^2 |A_{s'}|^2}{N_s^2 N_{s'}^2} M_{s-2}^2 M_{s'-1}^2 \right]^{1/2} \right\},
$$

where we have defined

$$
N_p = \sqrt{\sum_{n_1<n_2<\ldots<n_p} (n!)^2 \kappa_{n_1}^{2} \cdots \kappa_{n_p}^{2}},
$$

$$
M_p = \sqrt{\sum_{n_1<n_2<\ldots<n_p\neq i,j} (n!)^2 \kappa_{n_1}^{2} \cdots \kappa_{n_p}^{2}}.
$$

As we can see in Eq. (9), the entanglement between two atoms still depends directly on the coupling constant. This makes the difference with the the situation for the homogeneous case where only depends on the total number of atoms $N$ for a given number of excitations $k$ [6]. Fig. 3 shows the concurrence between the first atom and the last atom as a function of the number of atoms inside the cavity. In this picture we can observe how the concurrence decreased when the number of total excitations $k$ in the system increases. However, when $N \gg 1$ the concurrence becomes less dependent on the number of excitations tending to a fixed value. Fig. 4, shows the concurrence $C_{1,j}$ between the first and the $j$-th atom in the cavity.
4. Summary
In summary, we have studied the entanglement properties of the ITC model. We have found that, there are still some features of the eigenstates of the ITC that can be studied analytically under some suitable approximations, although the symmetry $SU(2)$ no longer exists in this case. For example, the concurrence between a pair of atoms can be obtained, leading to an explicit dependence on the inhomogeneity, number of atoms and the number of excitations.

Acknowledgments
CEL and FL acknowledge the financial support from MECESUP USA0108. GR from CONICYT Ph. D. Programmm Fellowsips, and JCR from Fondecyt 1030189 and Milenio ICM P02-049. GR, CEL and FL thank to DIGREGA.

References
[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information Cambridge University Press
[2] Sun C P, Yi S and You L 2003 Phys. Rev. A 67 063815
[3] Kuzmich A and Kennedy T A B 2004 Phys. Rev. Lett. 92 030407
[4] Liu X-J et al. 2006 Phys. Rev. A 73 013825
[5] Dicke R H 1954 Phys. Rev. 93 99
[6] Bužec V, Orszag M and Roško M 2005 Phys. Rev. Lett. 94 163601
[7] Liu R-F and Chen C-C, quant-ph/0510071.
[8] Tavis M and Cummings F W 1967 Phys. Rev. 170 379
[9] López C E, Christ H, Retamal J C and Solano E 2007 Phys. Rev. A 75 033818

Figure 4. Bipartite concurrence $C_{1,j}$ between the first and the $j$-th atom for $k = 1, 2, ..., 6$ excitations with $N = 20$ atoms.