Abstract. Existence of a perfect matching in a random bipartite digraph with bipartition \((V_1, V_2), |V_i| = n,\) is studied. The graph is generated in two rounds of random selections of a potential matching partner such that the average number of selections made by each vertex overall is below 2. More precisely, in the first round each vertex chooses a potential mate uniformly at random, and independently of all vertices. Given a fixed integer \(m,\) a vertex is classified as unpopular if it has been chosen by at most \(m\) vertices from the other side. Each unpopular vertex makes yet another uniform/independent selection of a potential mate. The expected number of selections made by a generic vertex \(v,\) i.e. its out-degree, is asymptotic to \(1 + \mathbb{P}(\text{Poisson}(1) \leq m) \in (1, 2).\) Aided by Matlab software, we prove that for \(m = 1,\) whence for all \(m \geq 1,\) the resulting bipartite graph has a perfect matching a.a.s. (asymptotically almost surely). On the other hand, for \(m = 0\) a.a.s. a perfect matching does not exist. This is a thorough revision of the joint paper (JCT(B) 88 (2003), 1-16) by the first author and the third author.

1. Introduction and main result

A standard model \(B_n(d)\) of a random bipartite (di)graph with bipartition \((V_1, V_2), |V_i| = n,\) is generated by each vertex \(v \in V_1 \cup V_2\) making \(d\) uniformly random, independent selections of a potential match from the other side. By computing the expected number of perfect matchings, Walkup [7] proved that asymptotically almost surely (a.a.s.) the graph \(B_n(1)\) has no perfect matching. In fact, Meir and Moon [6] (cf. Frieze [2]) earlier proved that the maximum matching number of \(B_n(1)\) is a.a.s. about \(0.866n.\) It is not much more difficult to show, using Hall’s Marriage Lemma, that a.a.s. the graph \(B_n(3)\) does have a perfect matching. Remarkably, Walkup managed to show that a.a.s. so does the graph \(B_n(2).\) Frieze [2] was able to prove an analogous result for a non-bipartite graph using Tutte’s criterion.

In this paper we study existence of a perfect matching in a bipartite random graph \(B_{n,m}\) which is sandwiched between \(B_n(1)\) and \(B_n(2).\) \(B_{n,m}\) is generated in two rounds of random selections of a potential match by every
vertex \( v \in V_1 \cup V_2 \). Specifically, in the first round each vertex selects a vertex from the opposite side uniformly at random, and independently of all other vertices. We call a vertex “unpopular” if it has been selected by at most \( m \) vertices. (This definition depends on the value \( m \): the larger \( m \) the larger the set of unpopular vertices.) Each unpopular vertex makes yet another uniformly random, and independent selection of a vertex from the other side. (If \( m = \infty \), then effectively all the vertices select uniformly at random and independently two vertices from the other side, so \( B_{n,\infty} = B_n(2) \).) The number of vertices that have selected a given vertex is distributed binomially with \( n \) trials and success probability \( 1/n \); thus it is Poisson(1) in the limit. It follows that the expected out-degree of a generic vertex in \( B_{n,m} \) is

\[
1 + \mathbb{P}(\text{Poisson(1)} \leq m) = 1 + e^{-1} \sum_{j=0}^{m} \frac{1}{j!} \uparrow 2, \quad m \to \infty.
\]

Thus the average out-degree of a vertex in \( B_{n,m} \) is strictly between 1 and 2. Loosely, we can interpret \( B_{n,m} \) as \( B_n(d_m) \), where \( d_m := 1 + e^{-1} \sum_{j \leq m} 1/j! \). In the joint paper [5] the first author and the third author stated and gave a proof of the claim: a.a.s. \( B_{n,0} \) (i.e. \( B_n(1 + 1/e) \)) has a perfect matching. Recently Michael Anastos and Alan Frieze [1] pointed out a simple oversight in the proof. We realized that the oversight invalidated the claim. A thorough revision of the method in [5] has allowed us to prove

**Theorem 1.1.** Let \( m \geq 1 \).

\[
\begin{align*}
&1 \mathbin{\begin{cases} \geq \quad &1 - O(n^{-c_m + o(1)}), \\ \leq \quad &1 - \frac{1.5}{1 + (m + 1) \left(1 + e^{-1} \sum_{j \leq m} 1/j! - \log 2\right)}
\end{cases}}
\end{align*}
\]

where \( c_1 \approx 0.514, \ c_m \uparrow 1(m \to \infty). \)

\[
\begin{align*}
&1 - O(n^{-c_m + o(1)}).
\end{align*}
\]

Let the reader beware that our rigorous proof techniques produced an explicit function \( H_{n,m}(t;\mathbf{r}), t \in (0, 1/2], \mathbf{r} \in \mathbb{R}^4 \), such that, to complete the proof, \( \min_{\mathbf{r}} H_{n,m}(t;\mathbf{r}) \) needed to be proved negative for all \( t \in (0, 1/2] \). This is because the minimum is the best achievable upper bound for the scaled logarithm of the expected number of Hall’s subgraphs of a given size that are present if there is no perfect matching. By upper-bounding \( \min_{\mathbf{r}} H_{n,m}(t;\mathbf{r}) \), we checked negativity “manually” for small \( t \), but deferred to Matlab algorithmic software to handle the remaining \( t \)’s.

As for \( m = 0 \), Matlab revealed that \( \min_{\mathbf{r}} H_{n,0}(t;\mathbf{r}) > 0 \) for all, but very small \( t \in (0, 1/2] \). Of course, positivity of this minimum even for all \( t \in (0, 1/2] \) does not imply almost sure non-existence of a perfect matching.
However this numerical evidence prodded us to try and prove that, contrary to our long-held belief, existence of a perfect matching is indeed highly unlikely. Using the necessity part of Hall’s Lemma we prove the opposite of the claim in [5]:

**Theorem 1.2.**

$$
\mathbb{P}(B_{n,0} \text{ has no perfect matching}) = 1 - O(n^{-1/2+o(1)}).
$$

For the proof itself, Matlab assistance was not required.

### 2. Proof of Theorem 1.1

Let $m > 0$. **Part 1.** If the graph $B_{n,m}$ has no perfect matching, than by Hall’s Marriage Lemma there exist a set $K$ of row vertices (elements of $V_1$), or of column vertices (elements of $V_2$), such that $|L| < |K|$, where $L = \Gamma(K)$ is the set of neighbors of $K$ in $B_{n,m}$. We call such $(K, L)$ “bad” pairs. We focus on the minimal pairs, minimal in a sense that there is no $K' \subset K$ such that $(K', \Gamma(K'))$ is a bad pair. For a minimal bad pair $(K, L)$, it is necessary that (i) $|L| = |K| - 1$; (ii) $|K| \in [2, \lceil n/2 \rceil]$; (iii) every vertex in $L$ has at least two neighbors in $K$. Let $E_{nk}, k \in [2, \lceil n/2 \rceil]$, denote the expected number of the minimal bad pairs $(K, L)$, with $|K| = k$. We need to prove that $\sum_k E_{nk} \to 0$. By symmetry,

$$(2.1) \quad E_{nk} \leq 2 \binom{n}{k} \binom{n}{k-1} \mathcal{P}_{nk}.$$ 

Here $\mathcal{P}_{nk}$ is the probability that $K = [k] = \{1, \ldots, k\} \subset V_1$ and $L = [k-1] = \{1, \ldots, k-1\} \subset V_2$ satisfy the conditions (ii) and (iii). In fact we broaden the condition (iii) a bit, replacing it with (iii'): in two rounds of selections every vertex in $L$ at least twice either selected a vertex in $K$ or was selected by a vertex in $K$.

#### 2.1. Case $k \leq n^{1/2}$.

On the event in question, let $Y$ ($X$ resp.) be the number of columns in $[k-1]$ (the number of rows in $[n-k]$ resp.) that selected rows in $[k]$ (columns in $[n-k+1]$ resp.) in the first round. Then the number of unpopular rows in $[k]$ (unpopular columns in $[n-k+1]$ resp.) is at least $k - Y/(m+1)$ ($n-k+1 - X/(m+1)$ resp.). (Indeed, every popular row in $[k]$ is selected by at least $m+1$ columns out of $Y$ columns.) $Y$ and $X$ are independent binomials with parameters $(k-1, k/n)$.
and \((n - k, (n - k + 1)/n)\) respectively. Therefore

\[
\mathcal{P}_{nk} \leq \left( \frac{k - 1}{n} \right)^k \left( 1 - \frac{k}{n} \right)^{n-k+1} \times \sum_{j=0}^{k-1} \binom{k-1}{j} \left( \frac{k}{n} \right)^j \left( 1 - \frac{k}{n} \right)^{k-1-j} \left( \frac{k-1}{n} \right)^{k-j/(m+1)} \\
\times \sum_{i=0}^{n-k} \left( \frac{n-k}{n} \right)^i \left( 1 - \frac{n-k+1}{n} \right)^{n-k-i} \left( 1 - \frac{n}{n} \right)^{n-k+1-i/(m+1)}.
\]

**Explanation** The first line of the RHS is the probability that the first round choices made by rows from \([k]\) (columns from \([n - k + 1]\) resp.) are columns from \([k-1]\) (rows from \([n-k]\) resp.). The second line (third line resp.) is an upper bound for the conditional probability that the second round choices made by unpopular rows from \([k]\) (unpopular columns from \([n - k + 1]\) resp.) are still columns from \([k-1]\) (rows from \([n-k]\) resp.). Further, the first sum equals

\[
\left( \frac{k - 1}{n} \right)^k \left[ 1 + \frac{k}{n} \cdot \left( \frac{k - 1}{n} \right)^{-1/(m+1)} - \frac{k}{n} \right]^{k-1} = \left( \frac{k - 1}{n} \right)^k \cdot \exp \left( O \left( \frac{k}{m+1} \right) \right).
\]

The second sum equals

\[
\left( \frac{n-k}{n} \right)^i \left( 1 - \frac{n-k+1}{n} \right)^{n-k-i} \left( 1 - \frac{n}{n} \right)^{n-k+1-i/(m+1)} = \left( \frac{1 - k}{n} \right)^{n-k+1} \exp \left( \frac{k}{m+1} + O \left( \frac{k^2}{n} \right) \right).
\]

Therefore

\[
\mathcal{P}_{nk} \leq \left( \frac{k - 1}{n} \right)^{2k} \left( 1 - \frac{k}{n} \right)^{2(n-k+1)} \exp \left( \frac{k}{m+1} + O \left( \frac{k}{m+1} + \frac{k^2}{n} \right) \right) = \left( \frac{k}{n} \right)^{2k} \exp \left( -k \frac{2m+1}{m+1} + O \left( \frac{k}{m+1} + \frac{k^2}{n} \right) \right).
\]
Consequently we have

\[(2.2) \quad E_{nk} = O\left(\frac{k}{n} \binom{n}{k}^2 \binom{k}{n}^{2k} \exp\left(-k \frac{2m+1}{m+1} + O\left(k \frac{m}{m+1} + k^2/n\right)\right)\right) = n^{-1} \exp\left(-\frac{k}{m+1} + O\left(k \frac{m}{m+1} + k^2/n\right)\right).\]

In particular, for \(\varepsilon > 0\),

\[\sum_{k \leq (m+1-\varepsilon) \log n} E_{nk} = O(n^{1/2} \log n), \text{ so that} \]

\[(2.3) \quad \mathbb{P}(\exists \text{ a bad pair } (K, L) : k \leq (m + 1 - \varepsilon) \log n) = O(n^{-\varepsilon/(m+1)}).\]

2.2. Case \(k \in [n^{1/2}, (n + 1)/2]\). We write \(P_{nk} = \mathbb{P}(A \cap B \cap C)\).

A: first round choices of the rows from \([k]\) are among the columns in \([k-1]\), and for every “unpopular” row in \([k]\) (i.e. a receiver of at most \(m\) first-round proposals), its second round choice is still one of the columns from \([k-1]\).

B: first round choices of the columns from \([n-k+1]\) are among the rows in \([n-k]\) are unpopular among the rows in \([n-k]\) in the first round, \(j\)’s second round choice—in case it is unpopular among rows in \([k]\) too—would still be a row in \([n-k]\).

C: overall, every column vertex from \([k-1]\) has taken part, as a proposer or a “proposee”, in at least two contacts with the row vertices from \([k]\).

Clearly the second round choices of rows from \([n-k]\) are irrelevant for the events \(A, B,\) and \(C\). Let \(G\) denote the (muti)graph, with labeled edges, induced by the two rounds of selections by the row set \([k]\), and the first round selections by the column set \([n]\). Let \(H\) be the graph induced by the second round choices by the column set \([n]\). Then \(G\) is independent of \(X = (X_1, \ldots, X_n)\), where \(X_j\) is the number of first round selections of column \(j\) by rows in \([n-k]\), and the distribution of \(H\) conditioned on \(\{X = x, G = G\}\) is the same no matter what the marginal distribution of \(X\) is. In the selection process \(X\) is distributed multinomially, with independent \(n-k\) trials, each having \(n\) equally likely outcomes. The Poissonization device yields that \(\mathbb{P}(X = x) \leq cn^{1/2}\mathbb{P}(Z = x)\), where \(Z = (Z_1, \ldots, Z_n)\) and \(Z_j\) are independent copies of Poisson \((1 - k/n)\).

Introduce the probability measure \(\mathbb{P}^*\) defined on the space \(S\) of triples \((x, G, H)\) by

\[\mathbb{P}^*(\{X = x\} \cap \{G = G\} \cap \{H = H\}) = \mathbb{P}(Z = x) \cdot \mathbb{P}(\{G = G\} \cap \{H = H\}|X = x).\]
Then $\mathbb{P}(E) \leq cn^{1/2}\mathbb{P}^*(E)$ for all $E \subseteq \mathcal{E}$. By switching to $\mathbb{P}^*$ we gain independence of $X_1, \ldots, X_n$ at the expense of the $cn^{1/2}$ factor. In particular, $P_{nk} \leq cn^{1/2}\mathbb{P}^*(A \cap B \cap C)$. So we turn to upper-bounding $\mathbb{P}^*(A \cap B \cap C)$. To this end, we claim first that

$$P^*(B) = (1 - t)^{n - k + 1}(1 - f(t)p_m(t))^{n - k + 1}$$

(2.4)

$$p_m(t) = \sum_{\ell=0}^{m} \frac{(1 - t)^{\ell}}{\ell!}, \quad t := \frac{k}{n}, \quad f(t) := te^{-1+t}.$$  

The first factor is the probability that none of columns from $[n - k + 1]$ selects a row from $[k]$ in the first round, and the second factor is the probability no column $j \in [n - k + 1]$, such that $X_j \leq m$ would select a vertex in $[k]$ in the second round. Here we used the independence of $X_j$ under $\mathbb{P}^*$ and

$$\mathbb{P}^*(X_j \leq m) = \sum_{\ell=0}^{m} e^{-1+t}(1 - t)^{\ell}, \quad t = \frac{k}{n},$$

So we need to estimate $P_{nk} = \mathbb{P}^*(A \cap C | B)$ the probability of $A \cap C$, conditioned on the event $B$: every column in $[n - k + 1]$ selects a row in $[n - k]$, and—if it is unpopular among those rows—would select such a row again. Let $S$ stand for the full description of selections by rows from $[k]$ in both rounds, and by columns from $[k - 1]$ in first round, compatible with $A \cap B$.

Let us specify, in four items, a generic value $T$ of $\mathcal{T}$, $\mathcal{T}$ being a partial description of $S$: (1) let $V \subseteq [k - 1]$ be the set of columns from $[k - 1]$ whose first round choice rows are in $[k]$; (2) let $U \subset [k]$ be the set of the rows each selected by at least $m + 1$ columns, and (3) so that $W := [k] \setminus U$ is the set of unpopular rows in $[k]$. Denote $u = |U|$, $v = |V|$, $w = |W|$; evidently $u(m + 1) \leq v \leq k - 1$, $w = k - u$. For $i \in [k]$, let $a_i$ be the number of columns from $V$ which selected row $i$. (4) To finish description of $T$, for $j \in [k - 1]$, let $b_j$ be the number of rows in $[k]$ whose first round selection is the column $j$, and let $\beta_j$ be the number of unpopular rows, those from $W$, whose second round selection is column $j$. On the event $A$, we have $\sum_{j \in [k-1]} b_j = k$ and

$$\sum_{j \in [k-1]} \beta_j = k - u.$$  

Let $p_j(b_j, \beta_j) = \mathbb{P}(F_j)$, $F_j$ the event that column $j$ has at least two contacts with rows in $[k]$. For $j \notin V$, we have

$$p_j(b_j, \beta_j) = \mathbb{I}\{b_j + \beta_j \geq 2\} + \mathbb{I}\{b_j = 1, \beta_j = 0\} f(t)p_{m-1}(t)$$

(2.5)

$$+ \mathbb{I}\{b_j = 0, \beta_j = 1\} p_m(t)f(t), \quad f(t) := te^{-1+t}, \quad t = k/n.$$  

**Explanation.** For $j \notin V$, the first choice of column $j$ is a row in $[n - k]$. Suppose $b_j = 1$, $\beta_j = 0$. $F_j$ holds if $X_j \leq m - 1$, making $j$ unpopular and allowing $j$ a second round selection, that happens to be a row in $[k]$, an
event of probability
\[
e^{-t \cdot \sum_{k=1}^{m-1} (1-t)^{k-1} \ell! / \ell!} \cdot t = p_{m-1}(t)f(t).
\]
Suppose \( b_j = 0, \beta_j = 1 \). This time \( F_j \) holds if \( X_j \leq m \) and, again, \( j \)'s second selection is a row in \([k]\), an event of probability \( p_m(t)f(t) \).

For \( j \in V \) (\( j \)'s first choice is a row in \([k]\)), the counterpart of (2.5) is

\[
p_j(b_j, \beta_j) = P\{b_j + \beta_j \geq 1\} + P\{b_j + \beta_j = 0\}p_m(t)f(t).
\]

Conditioned on \( \{T(S) = T\} \cap B \), the events \( F_j \) are independent, so that

\[
\mathbb{P}^{*}(C \mid B \cap \{T(S) = T\}) = \mathbb{P}^{*}\left( \bigcap_{j=1}^{k} F_j \mid B \cap \{T(S) = T\} \right) = \prod_{j=1}^{k} p_j(b_j, \beta_j).
\]

The RHS is the explicit function of \( T = T(S) \), and we need to compute its expected value to obtain \( P_{nk} = \mathbb{P}^{*}(A \cap C \mid B) \). Denoting \( \binom{n}{u} = \mu! / \bar{\nu}! = \mu! / (\nu_1! \cdots \nu_t!) \), \( (\nu_1 + \cdots + \nu_t = \mu) \), the resulting formula is

\[
P_{nk}(u, v) = \sum_{0 \leq u \leq v \leq k-1} P_{nk}(u, v),
\]

\[
P_{nk}(u, v) = \left( \begin{array}{c} k-1 \\end{array} \right) \left( \frac{n-k}{v} \right)^{k-1-v} \sum_{\sum_{i=v}^{\nu} a_i = v} \left( \begin{array}{c} v \\end{array} \right)^{n-v} \prod_{j=1}^{k} p_j(b_j, \beta_j).
\]

(In the last product we can assume that \( V = [v]\).) Given \( u \) and \( v \), the range of \( \bar{\alpha} \) is non-empty only if \( u(m + 1) \leq v \).

**Explanation.** We (1) select \( v \) columns from \([k-1]\), and note that the probability that each of the remaining \( k - 1 - v \) columns selects a row in \([n-k]\) is \( \left( \frac{n-k}{v} \right)^{k-1-v} \); (2) partition the chosen \( v \) columns into an ordered sequence of \( k \) sets of cardinalities \( a_1, \ldots, a_k \geq 0 \), with exactly \( u a_i \)'s above \( m \), so that \( a_i \) columns select row \( i \in [k] \), with overall probability \( n^{-v} \); (3) partition rows in \([k]\) (the set of \( k - u \) unpopular rows in \([k]\), i.e. those chosen by at most \( m \) columns resp.) into an ordered sequence of subsets of cardinalities \( b_1, \ldots, b_{k-1} \) (\( \beta_1, \ldots, \beta_{k-1} \) resp.), so that each column \( j \) is selected by \( b_j \) rows in the first round (by \( \beta_j \) unpopular rows in the second round resp.), with overall probability \( n^{-k}n^{-(k-u)} \); (4) add the contributions coming from each triple of partitions weighted with the factors \( \prod_{j=1}^{k} p_j(b_j, \beta_j) \).

To find a tractable upper bound for \( P_{nk}(u, v) \) we will use the generating functions and the Chernoff-type bound: for non-negative sequence \( \{g_u\} \), and \( x > 0 \), we have \( g_m \leq x^{-m}g(x) \), where \( g(x) = \sum_{\mu \geq 0} g_{\mu}x^\mu \); analogous
inequality holds for multivariate generating functions with non-negative coefficients. Needless to say, this approach is contingent on availability of an explicit formula for \( g(x) \).

The bottom sum does not depend on \( \vec{a} \). By symmetry, we have

\[
\sum_{\{i:a_i\geq m\}=u} \binom{v}{\vec{a}} = v! \binom{k}{u} \sum_{a_1,\ldots,a_{u+1},\ldots,a_{k}\leq m} \frac{1}{\vec{a}!}
\]

\[
= v! \binom{k}{u} [x^v] \left( \sum_{a>m} \frac{x^a}{a!} \right)^u \left( \sum_{a\leq m} \frac{x^a}{a!} \right)^{k-u} \leq v! \binom{k}{u} x^{-v} \exp^{u+1}(x) \cdot q^{k-u}(x),
\]

\[
\exp_s(x) := \sum_{\tau\geq s} x^\tau / \tau!, \quad q_s(x) := \sum_{\tau\leq s} x^\tau / \tau!.
\]

Similarly, by (2.5)-(2.6), the bottom sum in (2.7) equals

\[
\sum_{\sum_j b_j = k, \sum_j \beta_j = k-u} \binom{k}{b} \binom{k-u}{\vec{\beta}} \prod_j p_j(b_j, \beta_j)
\]

\[
= k! (k-u)! [y^k z^{k-u}] \prod_j \left( \sum_{b,\beta} y^b z^{\beta} \binom{b}{b' \beta'} p_j(b, \beta) \right).
\]

Denoting \( \eta := y + z \), the last product equals

\[
\left( \sum_{b,\beta} \frac{y^b z^{\beta}}{b! \beta!} \left( \mathbb{1}\{b_j + \beta_j \geq 2\} + \mathbb{1}\{b_j = 1, \beta_j = 0\} \right) p_{m-1}(t) f(t) \right)^{k-1-v}
\]

\[
+ \left( \sum_{b,\beta} \frac{y^b z^{\beta}}{b! \beta!} \left( \mathbb{1}\{b_j + \beta_j \geq 1\} + \mathbb{1}\{b_j + \beta_j = 0\} \right) p_m(t) f(t) \right)^{k-1-v}
\]

\[
\times \left( \sum_{b,\beta} \frac{(y+z)^s}{s!} + yp_{m-1}(t) f(t) + zp_m(t) f(t) \right)
\]

\[
= \left( \sum_{s\geq 1} \frac{(y+z)^s}{s!} + yp_{m-1}(t) f(t) + zp_m(t) f(t) \right)^{k-1-v}
\]

\[
= \left( \exp_1(\eta) + p_m(t) f(t) \right)^{k-1-v} \cdot \left( \exp_2(\eta) + yp_{m-1}(t) f(t) + zp_m(t) f(t) \right)^{k-1-v}.
\]
Therefore we have

\[(2.9)\quad \sum_{\sum_j b_j=k, \sum_j \beta_j=k-u} \left( \begin{array}{c} k \\ \vec{b} \end{array} \right) \left( \begin{array}{c} k-u \\ \vec{\beta} \end{array} \right) \prod_j p_j(b_j, \beta_j) \leq \frac{k!(k-u)!}{y^k z^{k-u}} \left( \exp_1(\eta) + p_m(t)f(t) \right)^v \times \left( \exp_2(\eta) + yp_{m-1}(t) + zp_m(t)f(t) \right)^{k-1-v}. \]

Combining (2.7), (2.8) and (2.9), and recalling the bound \(u(m+1) \leq v\), we conclude that

\[(2.10)\quad P_{nk} \leq \frac{(k!)^2(1-t)^{k-1}}{n^2 k y^k z^k} q_m(x) \left( \exp_2(\eta) + (yp_{m-1}(t) + zp_m(t))f(t) \right)^{k-1} \times \sum_{0 \leq u(m+1) \leq v \leq k-1} (k-1)_v \xi^u \zeta^v, \quad \xi := \frac{nz \exp_{m+1}(x)}{q_m(x)}, \quad \zeta := \frac{g}{n x (1-t)}, \quad g = g(t; y, z) := \frac{\exp_1(\eta) + p_m(t)f(t)}{\exp_2(\eta) + (yp_{m-1}(t) + zp_m(t))f(t)}. \]

Crucially, the sequence of the sums in (2.10) has a simple (exponential) generating function. Indeed if \(\zeta w < 1\), then we have

\[
\sum_{k \geq 1} \frac{w^{k-1}}{(k-1)!} \left( \sum_{0 \leq u(m+1) \leq v \leq k-1} \frac{(k-1)_v}{u!} \xi^u \zeta^v \right) = \sum_{0 \leq u(m+1) \leq v} \frac{\xi^u \zeta^v}{u!} \sum_{k \geq v} \frac{w^{k-1}(k-1)_v}{(k-1)!} = e^w \sum_{0 \leq u(m+1) \leq v} \frac{\xi^u \zeta^v w^v}{u!} = e^w \sum_{u \geq 0} \frac{(\xi(\zeta w)^{m+1}) u^{m+1}}{u!} \sum_{j \geq 0} (\zeta w)^j = \exp \left( w + \xi(\zeta w)^{m+1} \right) \frac{1}{1 - \zeta w}.
\]

Therefore we obtain: for all \(\mathbf{R} = (x, y, z, w) > 0\), such that \(w < \zeta^{-1}\),

\[(2.11)\quad P_{nk} \leq Q_{nk}(\mathbf{R}) := \frac{(k!)^2(1-t)^{k-1}}{n^2 k y^k z^k} \left( \exp_2(\eta) + (yp_{m-1}(t) + zp_m(t))f(t) \right)^{k-1} \times \frac{(k-1)! q_m(x)}{w^{k-1}} \cdot \exp \left( w + \xi(\zeta w)^{m+1} \right) \frac{1}{1 - \zeta w}; \quad (\eta = y + z).
\]

At the price of \((x, y, z, w)\), yet to be chosen, we got rid of the multi-fold summation. Denoting \(\rho = w/n\), we have \(\zeta w = \rho/[1 - (1-t)x]\). The first line
expression and the second line expression in (2.11) are respectively of orders
\[
\frac{k}{\exp_1(\eta) + p_m(t)f(t)} \cdot (1 - t)^k \left( \frac{k}{ne} \right)^{2k} \left( \frac{\exp_1(\eta) + p_m(t)f(t)}{yzg} \right)^k ;
\]
\[
\frac{k^{1/2} \rho}{t} \left( \frac{tq_m(x)}{e\rho} \right)^k \exp \left[ \frac{n}{\rho + \frac{z \exp_{m+1}(x)}{q_m(x)}} \cdot \left( \frac{g\rho}{(1-t)x} \right)^{m+1} \right] \frac{1}{1 - \frac{g\rho}{(1-t)x}},
\]
uniformly for all admissible \( R \); we used (2.10) for both expressions. So, denoting \( r = (x, y, z, \rho) \), (2.11) becomes
(2.12)
\[
H_{n,m}(t; r) := 2t \log \frac{t}{e} + t \log(1-t) + t \log \left( \frac{\exp_2(\eta) + (yp_{m-1}(t) + zp_m(t)f(t))}{yz} \right)
\]
\[+ t \log \left( \frac{tq_m(x)}{e\rho} \right) + \rho + \frac{z \exp_{m+1}(x)}{q_m(x)} \cdot \left( \frac{g\rho}{x(1-t)} \right)^{m+1}
\]
\[- n^{-1} \log \left( 1 - \frac{g\rho}{x(1-t)} \right).
\]
Recall that \( P_{nk} = \mathbb{P}^*(A \cap C|B) \) and \( \mathbb{P}^*(B) \) is given by (2.4). Therefore
(2.13) \( P_{nk} = O(n^{1/2} P_{nk} \mathbb{P}^*(B)) = O \left( n^{1/2} Q_{nk}^{*} [(1-t)(1-p_m(t)f(t))]^{n-k} \right) \),
and, by (2.1),
(2.14) \( E_{nk} = O \left( \frac{k}{n} \left( \frac{n}{k} \right)^2 P_{nk} \right). \)
Collecting the estimates (2.12)-(2.14), we arrive at
(2.15a) \( E_{nk} = O \left( \frac{(nk)^{1/2} \rho}{t(\exp_1(\eta) + p_m(t)f(t))} \cdot \exp(nH_{n,m}(t; r)) \right), \)
(2.15b) \( H_{n,m}(t; r) := - 2t + (1-t) \log \frac{1 - p_m(t)f(t)}{1-t} + t \log(1-t)
\]
\[+ t \log \left( \frac{\exp_2(\eta) + (yp_{m-1}(t) + zp_m(t)f(t))}{yz} \right) + t \log \left( \frac{tq_m(x)}{e\rho} \right)
\]
\[+ \rho + \frac{z \exp_{m+1}(x)}{q_m(x)} \cdot \left( \frac{g\rho}{x(1-t)} \right)^{m+1} - n^{-1} \log \left( 1 - \frac{g\rho}{x(1-t)} \right),
\]
where the \( H_{n,m} \) in equation (2.15b) is not the same as the \( H_{n,m} \) in (2.12) because of the inclusion of terms from equations (2.1) and (2.4). We already proved (see (2.3)) that \( \sum_{k \leq (m+1) \log n} E_{nk} = O(\epsilon^{-1/m+1}) \), if \( m > 0 \). So
the task is to establish existence of the tuple $r$ for every $t \geq (m + 1 - \varepsilon)\log \frac{n}{m}$ such that $nH_{n,m}(t; r)$ is negative enough to out-power the front factor in (2.15a), so that the sum of $E_{n,k}$ over the remaining $k$'s will go to zero as well. It is beneficial to start earlier, with $t \geq 1/n$, i.e. with $k \geq 1$.

2.3. Small t's. Our focus is on $m > 0$, but for comparison we include here $m = 0$ as well. Intuitively, for small $t$’s the search for the sub-optimal $r$ ought to be done by narrowing down the field of candidates. After some tinkering with $H_{n,m}(t; r)$ we chose $r(t) = (x(t), y(t), z(t), \rho(t))$:  

$$x(t) = at^\sigma, \quad y(t) = b_1 t^\sigma, \quad z(t) = b_2 t^\sigma, \quad \rho(t) = ct,$$

with the parameters to be determined. Then, calculating upper bounds for the various terms in (2.15b),

$$-2t + (1 - t) \log \frac{1 - p_m(t)f(t)}{1 - t} + t \log(1 - t) = -t(1 + e^{-1}p_m(0)) + O(t^2);$$

$$t \log \left( \exp_2(\eta) + \left( yp_{m-1}(t) + zp_m(t) \right) f(t) \right) = t \log \left( \frac{(b_1 + b_2)^2 t^2}{b_1 b_2} + O(t^{1+\sigma}) \right)$$

$$t \log \left( \frac{tq_m(x)}{ep} \right) + \rho = t \log \frac{1 + O(t^\sigma)}{ec} + ct = t \left( \log \frac{1}{ec} + c \right) + O(t^{1+\sigma});$$

$$z \frac{\exp_{m+1}(x)}{q_m(x)} \cdot \left( \frac{g_p}{x(1 - t)} \right)^{m+1} = \begin{cases} \frac{2b_2c}{b_1 + b_2} t + O(t^{1+\sigma}), & m = 0, \\ O(t^{2-\sigma}), & m \geq 1; \end{cases}$$

$$1 - \frac{g_p}{x(1 - t)} = 1 - (1 + O(t^\sigma)) \frac{2ct(1-t)}{a(b_1 + b_2)} = 1 - \Theta(t^{1-2\sigma});$$

since $1 - g_p/[x(1 - t)]$ is the argument of the log-function in (2.15b), we need to choose $\sigma < 1/2$. Combining the estimates we obtain:

$$H_{n,m}(t; r(t)) = -\gamma_m(b, c) t + O(t^{1+\sigma}) + \Theta(n^{-1}t^{1-2\sigma}),$$

$$\gamma_m(b, c) = 1 + e^{-1}p_m(0) - \log \frac{(b_1 + b_2)^2}{2b_1 b_2} - \left( \log \frac{1}{ec} + c \right)$$

$$- \frac{2b_2c}{b_1 + b_2} \cdot \mathbb{I}(m = 0).$$

It follows that $H_{n,m}(t; r(t))$ is continuous, but not differentiable at $t = 0$. So $\gamma_m(b, c)$ depends on $c$ and $b_2/b_1$ only, while $\sigma$ determines the behavior of the remainders. With some calculus it follows that, for $m > 0$, $\gamma_m(b, c)$ attains its maximum at $b_2/b_1 = c = 1$, while for $m = 0$ the maximum is
attained at \( b_2/b_1 = c = 1/\sqrt{3} \). Explicitly,

\[
\gamma_m := \max \gamma_m(b, c) = \begin{cases} 
1 + e^{-1} - \log(\sqrt{3} + 2), & m = 0, \\
1 + e^{-1} \sum_{j \leq m} 1/j! - \log 2, & m > 0;
\end{cases}
\]

\( \gamma_0 = 0.0509 \ldots \), and \( \gamma_1 = 1.0426 \ldots \), exceeding \( \gamma_0 \) by a \( 20+ \) factor, while \( \gamma_{\infty} = 1.3068 \ldots \) (In this regard the case \( m = 0 \) is drastically different from the case \( m > 0 \).) Therefore

\[
H_{n,m}(t) = \min_r H_{n,m}(t;r) \leq -\gamma_m t + O(t^{1+\sigma}) + \Theta(n^{-1}t^{1-2\sigma}).
\]

Consequently, for every \( m \geq 0 \), and \( n \) sufficiently large, the function \( H_{n,m}(t) \) is negative for \( t \in [n^{-\lambda}, \varepsilon_m] \), where \( \lambda \in (1, (2\sigma)^{-1}) \), and \( \varepsilon_m > 0 \) is chosen sufficiently small.

The front factor by \( \exp(nH_{n,m}(t;r)) \) in (2.15a) is of order \( (nk)^{1/2} \). So it follows from (2.15a) and (2.18) that for \( m \geq 1 \)

\[
E_{n,k} = O\left((nk)^{1/2} \exp\left(-\gamma_m k + O\left(n^{-1/2}k^{3/2}\right)\right)\right).
\]

We proved already that, for \( k = O(n^{1/2}) \) and \( m \geq 1 \),

\[
E_{nk} \leq n^{-1} \exp\left(\frac{k}{m+1} + O\left(k^{m+1}/m^{m+1} + k^2/n\right)\right).
\]

Therefore

\[
\log E_{nk} \leq \min\left\{-\log n + k/(m+1), 0.5 \log n - \gamma_m k\right\} + O\left(n^{-1/2}k^{3/2} + k^{m+1}/m^{m+1} + \log k\right).
\]

The explicit term (2.20) attains its maximum at

\[
k_n = \frac{1.5(m+1)}{(m+1)(1 + e^{-1}p_m(0) - \log 2) + 1} \cdot \log n
\]

and the maximum is \(-c_{m} \log n, c_m := 1 - \frac{1.5}{1+(m+1)\gamma_m}\), \( c_m \) increases with \( m \), \( c_1 \approx 0.514 \) and \( c_{\infty} = 1 \). It follows easily that for \( m \geq 1 \)

\[
\max\{\log E_{nk} : k \leq n^{1/2}\} \leq -c_{m} \log n + O\left((\log n)^{m+1}/m^{m+1}\right),
\]

so that

\[
\sum_{k \in [1,k_n]} E_{nk} = O\left(k_n \exp(-c_m \log n + O((\log n)^{m+1}/m^{m+1})\right) = n^{-c_m+o(1)}.
\]

And

\[
\sum_{k \in [k_n,n^{1/2}]} E_{nk} = O\left(\exp\left(-c_m \log n + O\left((\log n)^{m+1}/m^{m+1}\right)\right) \sum_{j \geq 0} e^{-\left(\gamma_m + o(1)\right)j}\right) = n^{-c_m+o(1)}.
\]
We conclude that, for $m \geq 1$, $\sum_{k \leq n^{1/2}} E_{nk} \leq n^{-c_m + o(1)}$.

2.4. Moderate and large $t$'s. At this final stage we concentrate on $m = 1$. Plugging $m = 1$ into the formula for $H_{n,m}(t; r)$ in (2.15b) we obtain

$$H_{n,1}(t; r) = (x, y, z, \rho) := -2t + (1 - t) \log \frac{1 - (2 - t)f(t)}{1 - t} + t \log (1 - t)$$

$$+ t \log \left( e^{\eta} - 1 - \eta + (y + z(2 - t))f(t) \right) + t \log \left( \frac{1}{1 - \frac{q \rho}{x(1 - t)}} \right)$$

$$+ \rho + z(e^x - 1 - x) \cdot \left( \frac{1 + x}{1 + x} \right) - n^{-1} \log \left( 1 - \frac{q \rho}{x(1 - t)} \right);$$

$$g = \frac{e^{\eta} - 1 + (2 - t)f(t)}{e^{\eta} - 1 - \eta + (y + z(2 - t))f(t)}, \quad f(t) = te^{-1 + t}, \quad \eta = y + z.$$

In light of the analysis in the previous section, we need to check that

$$H_{n,1}(t) := \min_r H_{n,1}(t; r) < 0 \text{ for } t \in [n^{-1/2}, 1/2] \text{ and } n \text{ sufficiently large.}$$

Extensive numerical analysis involving three independent Matlab minimization algorithms demonstrated that

$$\max\{H_{n,1}(t) : t \in [1/n, 1/2]\} = H_{n,1}(1/n) < 0 \text{ for all } n \geq 100.$$

In particular, $H_{n,1}(1/2) \approx -0.051$ and $\min H_{n,1}(t) \approx -0.065$. That the function $H_{n,1}(t)$ is “barely negative” for $t \in (0, 1/2]$ may be charitably interpreted as supporting our decision to use Matlab software, rather than to search for a protracted, yet uninspiring, calculus-based proof of $H_{n,1}(t)$’s negativity. See Appendix for the details.

Though negative, $\gamma_0$ in (2.17) is so close to zero, that one wonders whether $H_{n,0}(t)$ will remain negative for all $t \in (0, 1/2]$, like all other $H_{n,m}(t)$ for $m > 0$. As discussed in Appendix, once $t$ exceeds 0.0035, $H_{n,0}(t)$ becomes and remains positive for all remaining $t \leq 1/2$; see the figure. This figure shows the striking difference between $H_{n,m}$ for $m = 0$ and 1. At the very least, it means that, if true, almost sure existence of a perfect matching in

![Figure 1](image-url)
would require a different argument. More plausibly though, we need to consider a possibility that existence of a perfect matching in $B_{n,0}$ is unlikely. In the next Section we show that this is indeed the case. The proof itself will not require help of Matlab.

**Part 2.** Let us prove that for $m > 0$

$$
P_{n,m} := \mathbb{P}(B_{n,m} \text{ is connected}) = 1 - O\left(n^{-c_m + o(1)}\right).
$$

On the event $\mathcal{E}_n := \{B_{n,m} \text{ is not connected and has a perfect matching}\}$, there exist a row set $K$ and a column set $L$ such that $|K| = |L| = k$, $k \leq n/2$, that induce a component of $B_{n,m}$. Let $X_k$ denote the total number of such pairs. Denote $k_1 = k$, $k_2 = n - k$. We have

$$
\mathbb{E}[X_k] \leq \binom{n}{k} \left( \prod_{i=1}^{2} \binom{k_i}{n} \right)^{2\left(2k_i - \frac{k_i}{m+1}\right)}.
$$

**Explanation.** Let $K_1 = K$, $L_1 = L$, $K_2 = V_1 \setminus K_1$, $L_2 = V_2 \setminus L_1$. On the event “$K, L$ induce a component of $B_{n,m}$” in round 1 every column from $L_i$ selects a row in $K_i$ and every row from $K_i$ selects a column in $L_i$. The total number of unpopular columns in $L_i$ is at least $k_i - \frac{k_i}{m+1}$; so the probability that the selections by columns from $L_i$ in the two rounds are all among rows in $K_i$ is at most $(k_i/n)^{2k_i - \frac{k_i}{m+1}}$. Likewise $(k_i/n)^{2k_i - \frac{k_i}{m+1}}$ is an upper bound for the probability that the selections by rows from $K_i$ in the two rounds are all among columns in $L_i$.

Using $(\binom{n}{k}) \leq n^n/[k^k(n-k)^{n-k}]$, we obtain then

$$
\mathbb{E}[X_k] \leq \binom{n}{k}^2 \left( \frac{n^n}{k^k(n-k)^{n-k}} \right)^{-\frac{4m+2}{m+1}} \leq \binom{n}{k}^{-\frac{2m}{m+1}}.
$$

Consequently

$$
\mathbb{P}(\mathcal{E}_n) \leq \sum_{k=1}^{n/2} \mathbb{E}[X_k] = O\left(\binom{n}{1}^{-\frac{2m}{m+1}}\right) = O\left(n^{-\frac{2m}{m+1}}\right),
$$

implying that

$$
\mathbb{P}(B_{n,m} \text{ is not connected}) = O\left(n^{-\frac{2m}{m+1}}\right) + \mathbb{P}(B_{n,m} \text{ has no perfect matching})
\quad = O\left(n^{-\frac{2m}{m+1}}\right) + O(n^{-c_m + o(1)}) = O(n^{-c_m + o(1)}).
$$

□
3. Proof of Theorem 1.2

Let $k = [n^\delta]$, $\delta \in (0, 1)$ to be specified shortly, and let $Y_n$ be the number of set pairs $(K, L)$, $|K| = k$, $|L| = k - 1$ such that $L = \Gamma(K)$. This time we drop the condition that every column vertex in $L$ has at least two row neighbors in $K$. It suffices to show that a.a.s. $Y_n > 0$; indeed by Hall’s Marriage Lemma existence of such a pair rules out existence of a perfect matching. To this end we will prove that $E[Y_n] \to \infty$ and $E[Y_n^2] \sim \mathbb{E}^2[Y_n]$.

(1) For $0 \leq u \leq v \leq k - 1$, The counterpart of $P_{nk}(u, v)$ in (2.7) is given by

$$P_{nk}(u, v) = \frac{(k - 1)^{2k-u}(k)^u}{n^{2k-u+v}} \cdot \binom{k-1}{v} \left(1 - \frac{k}{n}\right)^{k-1-v} S(v, u);$$

here $S(v, u)$ is the Stirling number of the second kind, i.e. the number of partitions of $[v]$ into $u$ non-empty sets.

Explanation. We (1) choose $v$ columns from $[k - 1]$ and $u$ rows from $[k]$ in $\binom{k-1}{v}$ ways; (2) allocate $v$ columns among $u$ rows in $u!S(v, u)$ ways, so in round 1 the remaining $k - 1 - v$ columns select rows from $[n - k]$; (3) allocate $k$ rows among $k - 1$ columns, thus determining round 1 selections of columns in $L$ made by rows from $[k]$, and allocate $k - u$ unpopular rows among $k - 1$ columns, thus determining round 2 selections of columns still in $L$ made by unpopular rows from $[k]$, in $(k - 1)^{k+u}$ ways total. Finally

$$\frac{1}{\gamma 2k-u+v} \left(1 - \frac{k}{n}\right)^{k-1-v}$$

is the probability of each of the resulting outcomes. We need a sharp asymptotic formula for $P_{nk} := \sum_{0 \leq u \leq v \leq k - 1} P_{nk}(u, v)$. Notice that for $u < v$, by log-concavity of $\{S(v, u)\}_{u \leq v}$ (Harper [4], Godsil [3], Section 6.3) we have

$$P_{nk}(u + 1, v) = \frac{n(k-u)}{k-1} \cdot \frac{S(v, u+1)}{S(v, u)} \geq \frac{n(k-u)}{k-1} \cdot \frac{S(v, v)}{S(v, v - 1)} \geq \frac{n}{k} \cdot \frac{1}{v} \geq \frac{n}{k^2} \geq \frac{n}{n^{2\delta}} \to \infty,$$

if $\delta < 1/2$. So we have

$$\sum_{u=0}^{v} P_{nk}(u, v) = (1 + O(n^{-1+2\delta})) P_{nk}(v, v),$$

with the front factor absorbing $(1 - \frac{k}{n})^{k-1-v}$ from (3.1), which implies

$$P_{nk} = \sum_{0 \leq u \leq v \leq k - 1} P_{nk}(u, v) = (1 + O(n^{-1+2\delta})) \frac{(k-1)^{2k}}{n^{2k}} S_k,$$

(3.2)

$$S_k := \sum_{v \leq k-1} \frac{(k)^v}{(k-1)^v} \binom{k-1}{v}.$$
The ratio of two consecutive terms in the sum $S_k$ decreases with $v$. So the largest term corresponds to the smallest $v$ for which this ratio is at most one. It easily follows this $v$ is one of two integers closest to $v_0 = \sigma k + (-2 + 2/\sqrt{5})$, $\sigma := \frac{3 - \sqrt{5}}{2}$. The dominant contribution to $S_k$ comes from the terms with $|v - v_0| \leq k^{1/2} \log k$, and uniformly for these $v$ we have

$$
\frac{(k)_v}{(k-1)^v} \binom{k-1}{v} = (1 + O(k^{-1})) \frac{e^\sigma}{\sqrt{2\pi \sigma k}} e^{kH(v/k)},
$$

$H(z) := -z - 2(1 - z) \log(1 - z) - z \log z$.

Not surprisingly, $H(z)$ attains its maximum at $z = \sigma$. Approximating $H(v/k)$ by the quadratic Taylor polynomial around $\sigma$, we replace the sum by the Gaussian integral and obtain

$$
S_k = (1 + O(k^{-1})) \frac{\exp(\sigma + kH(\sigma))}{\sqrt{\sigma(-H''(\sigma)})},
$$

$H(\sigma) = -\sigma - 2 \log(1 - \sigma) = 0.5804576362$.

This formula for $S_k$ results in a compact estimate of $P_{nk}$ in (3.2). We hasten to add that we have not considered yet another condition a pair $(K, L)$ needs to meet: (1) in round 1 every column from $[n-k+1]$ selects a row in $[n-k]$; (2) in round 2 every column in $[n-k+1]$, which was not selected by any row from $[n-k]$ in round 1, still selects a row from $[n-k]$. The event (1) has probability $(1 - k/n)^{n-k+1} = \exp(-k + O(k^2/n))$; the event (2) is dependent on the event whose probability $P_{nk}$ we have analyzed. Its conditional probability is $(1 - k/n)^W$, where $W$ is the number of columns in $[n-k+1]$, which are unpopular among rows in $[n-k]$. Since $k \ll n$, i.e. $k = o(n)$, by the Poissonization approximation we have: for $\varepsilon \in (0, 1/2)$,

$$
\mathbb{P}\{|W - e^{-1}n| \leq n^{1/2+\varepsilon}\} \geq 1 - \exp(-\Theta(n^{2\varepsilon}))).
$$

And we observe that the total number of all pairs $(K, L)$, with $|K| = k = [n^\delta]$, is

$$
\binom{n}{k} \binom{n}{k-1} = e^{O(k \log n)} = e^{O(n^{\delta} \log n)} \ll e^{\Theta(n^{2\varepsilon})},
$$

provided that $\delta < 2\varepsilon$. Therefore, computing the moments of $Y_n$, we can—at the cost of an additive error term $e^{-\Theta(n^{2\varepsilon})}$—replace $W$ with $e^{-1}n + O(n^{1/2+\varepsilon})$, in which case $(1 - k/n)^W = \exp(-e^{-1}k + O(n^{-1/2+\delta+\varepsilon}))$. In
combination with (3.2) it follows then: for $0 < \delta < \min(1/2, \epsilon, 2\epsilon)$, (3.4)
\[
\mathbb{E}[Y_n] = (1 + O(n^{-1/2+\delta+\epsilon})) \left( \binom{n}{k} \left( \binom{n}{k-1} e^{-k(1+\epsilon-1)} P_{nk} + O(e^{-\Theta(n^{2\epsilon})}) \right) \right.
\]
\[
\left. + O(e^{-\Theta(n^{2\epsilon})}) \right)
\]
\[
= (1 + O(n^{-1/2+\delta+\epsilon})) \left( \binom{n}{k} \left( \binom{n}{k-1} e^{-k(1+\epsilon-1) \frac{(k-1)2k}{n2k}} S_k \right) \right.
\]
\[
\left. + O(e^{-\Theta(n^{2\epsilon})}) \right)
\]
\[
= \Theta(n^{-1}e^{\lambda k}), \quad \lambda := 1 - e^{-1} + H(\sigma) = 1.212578195 > 0.
\]

(2) Next we will show that $\mathbb{E}[(Y_n)_{2}] \lesssim \mathbb{E}^2[Y_n]$, where $(Y_n)_{2} = Y_n(Y_n - 1)$ is the total number of ways to select two bad pairs, $(K_1, L_1)$ and $(K_2, L_2)$, $|K_i| = k$, $|L_i| = k - 1$. Given $0 \leq \mu \leq k$, $0 \leq \nu \leq k - 1$, every $\{ (K_i, L_i) \}_{i=1,2}$ with $|K_1 \cap K_2| = \mu$, $|L_1 \cap L_2| = \nu$ has the same probability, call it $\Pi(\mu, \nu)$, that $(K_1, L_1)$ and $(K_2, L_2)$ are both bad. The contribution of all such $(\mu, \nu)$ configurations to $\mathbb{E}[(Y_n)_{2}]$ is
\[
\Pi(\mu, \nu) \left( \binom{n}{k-\mu, \mu, k-\mu} \binom{n}{k-1-\nu, \nu, k-1-\nu} \right).
\]

So we focus on $K_1 = [k]$, $L_1 = [k - 1]$, $K_2 = [k - \mu + 1, 2k - \mu]$, $L_2 = [k - \nu, 2k - 2 - \nu]$. Visually, we have two $k \times (k - 1)$ rectangles on the $n \times n$ integer lattice, the first rectangle occupying the North-West corner, and the second rectangle having its North-West corner point at $(k - \mu + 1, k - \nu)$.

Let us define $P_{nk}(u, v)$, a counterpart of $P_{nk}(u, v)$. Here $v = (v_1, v_2, \hat{v}_1, \hat{v}_2)$, $u = (u_1, u_2)$,
\[
v_1 \leq |L_1 \setminus (L_1 \cap L_2)|, \quad \hat{v}_1 + \hat{v}_2 \leq |L_1 \cap L_2|, \quad u_i \leq |K_i \setminus (K_1 \cap K_2)|.
\]

Introduce the event $A(u, v)$: $v_i$ ($\hat{v}_i$ resp.) is the number of columns belonging to $L_i \setminus (L_1 \cap L_2)$ ($L_1 \cap L_2$ resp.) that in round 1 selected a row belonging only to $K_i$, and $u_i$ is the number of those rows. $P_{nk}(u, v)$ is the probability of the event $A(u, v)$ intersected with the event that in round 1 no row from $K_i$ selected a column from $L_i^c$, and in round 2 no unpopular row belonging only to $K_i$ selected a column from $L_i^c$. $P_{nk}(u, v)$ is an upper bound for the probability that both $(K_1, L_1)$ and $(K_2, L_2)$ are bad, and the
two probabilities are equal when $\mu = \nu = 0$. Arguing like $P_{nk}(u, v)$, we have

$$P_{nk}(u, v) = \binom{\nu}{\hat{v}_1, \hat{v}_2} \left( 1 - \frac{2(k - \mu)}{n} \right)^{\nu - \hat{v}_1 - \hat{v}_2} \times \prod_{i=1}^{2} \binom{k - 1 - \nu}{v_i} \left( k - \mu \right)^{u_i} S(v_i + \hat{v}_i, u_i) (k - 1)^{2k - \mu - u_i} \times \left( 1 - \frac{k - \mu}{n} \right)^{k - 1 - \nu - v_i} \frac{1}{n^{2k - u_i + v_i + \hat{v}_i}}.$$  

**Explanation.** The trinomial coefficient and the four binomial coefficients should be clear. $u_i! S(v_i + \hat{v}_i, u_i)$ is the number of ways to assign $v_i + \hat{v}_i$ columns from $L_i$ to the already chosen $u_i$ row vertices belonging only to $K_i$, with each of these rows getting at least one column. $(k - 1)^{2k - \mu - u_i}$ is the total number of ways to assign $k$ rows from $K_i$ and $k - u_i$ unpopular rows belonging only to $K_i$ to the columns from $L_i$. $(1 - 2(k - \mu)/n)^{\nu - \hat{v}_1 - \hat{v}_2}$ is the probability that, by the definition of $\hat{v}_1$ and $\hat{v}_2$, some specific $\nu - \hat{v}_1 - \hat{v}_2$ common columns each have to select a row outside of the symmetric difference $K_1 \Delta K_2$. $(1 - (k - \mu)/n)^{k - 1 - \nu - v_i}$ is the probability that the columns from $L_i \setminus (L_1 \cap L_2)$ select a row outside $K_i \setminus (K_1 \cap K_2)$. $1/n^{2k - u_i + v_i + \hat{v}_i}$ is the probability that both first round selections by rows from $K_i$ and second round selections by unpopular rows belonging to $K_i \setminus (K_1 \cap K_2)$ are among columns from $L_i$.

To estimate $\sum_{u,v} P_{nk}(u, v)$ for guidance we use elements of the part (1). First of all, both $(1 - 2(k - \mu)/n)^{\nu - \hat{v}_1 - \hat{v}_2}$ and $(1 - (k - \mu)/n)^{k - 1 - \nu - v_i}$ are each equal to $1 + O(n^{-1+2\delta})$, since $k = \Theta(n^\delta)$. Second, by log-concavity of $(k - \mu)_u S(v_i + \hat{v}_i, u)$ (as a function of $u$) we have

$$\sum_{u_i \leq v_i + \hat{v}_i} \binom{k - \mu}{u_i} u_i! S(v_i + \hat{v}_i, u_i) \frac{(k - 1)^{2k - \mu - u_i}}{n^{2k - u_i + v_i + \hat{v}_i}} \leq (1 + O(n^{-1+2\delta}))(k - \mu)_{v_i + \hat{v}_i} \frac{(k - 1)^{2k - \mu - v_i - \hat{v}_i}}{n^{2k}};$$

the front factor on the RHS is that close to 1 because the ratio of the last term and the penultimate term of the sum is at least $n/(k - 1)^2$. So

$$\sum_{u,v} P_{nk}(u, v) \leq (1 + O(n^{-1+2\delta}))$$

$$\times \sum_{v_1, v_2} \prod_{i=1}^{2} \binom{k - 1 - \nu}{v_i} \left( k - \mu \right)_{v_i} \frac{(k - 1)^{2k - \mu - v_i}}{n^{2k}} \times \sum_{\hat{v}_1, \hat{v}_2} \binom{\nu}{\hat{v}_1, \hat{v}_2} \prod_{i=1}^{2} \left( k - \mu - v_i \right)_{\hat{v}_i} \frac{(k - 1)^{-\hat{v}_i}}{n^{2k}};$$

(3.5)
\( v_i \leq k - 1 - \nu, \quad \hat{v}_1 + \hat{v}_2 =: s \leq \nu. \) The bottom sum in (3.5) is
\[
\sum_{s \leq \nu} s!^{(k-1)-s} \sum_{\hat{v}_1 + \hat{v}_2 = s} \frac{2}{\hat{v}_i} \left( k - \mu - v_i \right) \\
= \sum_{s \leq \nu} s!^{(k-1)-s} \left( \sum_{s \leq \nu} s! \right)^s \left( k - 1 \right)^{-s} \left( 2k \right)^s \\
= \left( \frac{2k}{k-1} \right)^\nu \leq 4^\nu.
\]
The penultimate sum in (3.5) is at most
\[
\left( \frac{(k-1)^{2k}}{n^{2k}} \right)^2 \sum_{v_1, v_2} \frac{(k)^v_1}{(k-1)^{v_1}} \left( k - 1 \right) \\
\]
Therefore the equation (3.5) becomes
\[
\sum_{u,v} P_{nk}(u,v) \leq \left( 1 + O(n^{-1+2\delta}) \right) 4^\nu (k-1)^{-2\mu} \\
\]
\[
\times \left( \frac{(k-1)^{2k}}{n^{2k}} \right)^2 \sum_w \frac{(k)^w}{(k-1)^w} \left( k - 1 \right) \\
= \left( 1 + O(n^{-1+2\delta}) \right) 4^\nu (k-1)^{-2\mu} \left( \frac{(k-1)^{2k}}{n^{2k}} S_k \right)^2,
\]
according to the definition of \( S_k \) in (3.2). The LHS sum is the probability that in round 1 rows from \( K_i \) select columns from \( L_i \), and that in round 2 the unpopular rows belonging to \( K_i \setminus (K_1 \cap K_2) \) again each select column in \( L_i \). \( P(\mu, \nu) \) is, at most, the probability of this event intersected with the event: everyone of \( n - (2(k-1) - \nu) \) columns in \( (L_1 \cup L_2)^c \) selects in round 1 one of \( n - (2k - \mu) \) rows in \( (K_1 \cup K_2)^c \), and each of the \( W_1 \) columns unpopular among these rows selects a row in \( (K_1 \cup K_2)^c \) in round 2 again. Since \( k = \Theta(n^{\delta}) \), analogously to \( W \) we have
\[
\mathbb{P}\{|W_1 - e^{-1}n| \leq n^{1/2+\varepsilon}\} \geq 1 - \exp(-\Theta(n^{2\varepsilon}))
\]
see (3.3). Therefore with probability that high, the conditional probability of the event “\( W_1 \) columns stay with rows from \( (K_1 \cup K_2)^c \) in round 2” is at most
\[
\left( 1 - \frac{2k - \mu}{n} \right)^{n-(2(k-1) - \nu)} \left( 1 - \frac{k}{n} \right)^{W_1} = \left( 1 + O(n^{-1+2\delta}) \right) \left( 1 + e^{-2k(1+e^{-1})+e^{-1} \mu} \right).
\]
Therefore, by (3.6)

\[
\Pi(\mu, \nu) \leq (1 + O(n^{-1+2\delta})) 4^\nu (k-1)^{-2\mu} \left( \frac{(k-1)^{2k}}{n^{2k}} S_k \right)^2 \times \exp(-2k(1+e^{-1}) + e^{-1}\mu).
\]

So we have

\[
E[Y_n(Y_n-1)] \leq O(e^{-\Theta(n^{2\varepsilon})}) + (1 + O(n^{-1+2\delta})) e^{-2k(1+e^{-1})} \left( \frac{(k-1)^{2k}}{n^{2k}} S_k \right)^2 \times \exp(-2k(1+e^{-1}) + e^{-1}\mu).
\]

Both \( \{(n, k-\mu, \mu, k-\mu)\} \) and \( \{(n, k-1-\nu, \nu, k-1-\nu)\} \) are log-concave as functions of \( \mu \) and \( \nu \), respectively. So the sum is at most

\[
\binom{n}{k} \binom{n}{k-1} \sum_{\mu, \nu \geq 0} \binom{n}{k-\mu, \mu, k-\mu} \binom{n}{k-1-\nu, \nu, k-1-\nu} 4^\nu \left( \frac{e^{-1}}{(k-1)^2} \right)^\mu.
\]

Combining this equation and (3.4) (second line), and recalling that \( k = \Theta(n^\delta) \), we obtain

\[
E[Y_n(Y_n-1)] = O(e^{-\Theta(n^{2\varepsilon})}) + (1 + O(n^{-1/2+\delta+\varepsilon})) E^2[Y_n],
\]

implying that for \( 0 < \delta < \min(1/2 - \varepsilon, 2\varepsilon) \),

\[
\frac{E^2[Y_n]}{E[Y_n]^2} = 1 + O(n^{-1/2+\delta+\varepsilon}),
\]

since \( E[Y_n] = \exp(\Theta(n^\delta)) \), see (3.4). By Chebyshev’s inequality,

\[
P(Y_n \geq 0.5E[Y_n]) \geq 1 - O(n^{-1/2+\delta+\varepsilon}).
\]

\[\square\]

4. Components of \( B_{n,0} \).

In [5] it was asserted that a.a.s. \( B_{n,0} \) consists of a single giant component and small isolated cycles (cyclic components) with a bounded total size. The proof was based on observation that in presence of a perfect matching every isolated component \( (K, L) \) must be balanced, i.e. \( |K| = |L| \). However we know now that a.a.s. \( B_{n,0} \) has no perfect matching. Here is a sketch of the corrected proof of a close claim; the only computer aid it relies on is a surface plot.
Suppose that a pair \((K, L)\), \((|K| = k, |L| = \ell)\), induces a component of \(B_{n,0}\). We focus on smaller components, i.e. of size \(k + \ell \leq n\). Introduce 
\((K_1, L_1) = (K, L), (K_2, L_2) = (V_1 \setminus K, V_2 \setminus L)\), \(k_1 = k, k_2 = n - k; \ell_1 = \ell, \ell_2 = n - \ell\). Suppose \(\ell \leq k\), i.e. \(\ell_1 \leq k_1\); then \(\ell_2 \geq k_2\). Let us bound the probability \(P_{k,\ell}\) that none of the unpopular vertices in \(K\) and all vertices in \(L\) selects a vertex from \(K\) or with \(i \in K_2, j \in L_1\) is an edge of \(B_{n,0}\). We have

\[
P_{k,\ell} \leq P_{k,\ell}^* := \left(\frac{k_1}{n}\right)^{\ell_1} \left(\frac{\ell_2}{n}\right)^{k_2} \cdot \left(\frac{\ell_1}{n}\right)^{2k_1 - \ell_1} \cdot \left(\frac{k_2}{n}\right)^{2\ell_2 - k_2}
\]

\[
\times c_1 \sqrt{k_1} \left(1 - e^{-\frac{k_1}{n}} \frac{\ell_2}{n}\right)^{\ell_1} \cdot c_2 \sqrt{\ell_2} \left(1 - e^{-\frac{\ell_2}{n}} \frac{k_1}{n}\right)^{k_2}.
\]

**Explanation.** 1-st line: first factor is the probability that in round 1 vertices in \(L_1\) and \(K_2\) select, exclusively from their larger partner sets \(K_1\) and \(L_2\); 2-nd factor (3-rd factor resp.) is an upper bound for the probability that all vertices in \(K_1\) and all unpopular vertices in \(K_1\) (all vertices in \(L_2\) and all unpopular vertices in \(L_2\) resp.) select vertices from \(L_1\) (\(K_2\) resp.). 2-nd line: 1-st factor is an upper bound for the probability that none of the unpopular vertices in \(L_1\) selects a vertex from \(K_2\) in round 2; 2-nd factor is an upper bound for the probability that none of the unpopular vertices in \(K_2\) selects a vertex from \(L_1\) in round 2. For instance, the first bound comes from approximating the numbers of vertices in \(K_1\), which selected the vertices from \(L_1\) in round 1, by the \(\ell_1\)-long sequence of independent Poissons, each with parameter \(k_1/\ell_1\).

So, denoting the expected number of such pairs \((K, L)\) by \(E_{k,\ell}\), we have

\[
E_{k,\ell} \leq \binom{n}{k} \binom{n}{\ell} P_{k,\ell}^* \leq c n^{1/2} \exp(n \mathbb{H}(k/n, \ell/n)),
\]

\[
\mathbb{H}(x, y) := -x \log x - (1 - x) \log(1 - x) - y \log y - (1 - y) \log(1 - y)
\]

\[
+ y \log x + (1 - x) \log(1 - y)
\]

\[
+ (2x - y) \log y + (1 + x - 2y) \log(1 - x)
\]

\[
+ y \log \left(1 - e^{-\frac{x}{y}} (1 - x)\right) + (1 - x) \log \left(1 - e^{-\frac{1 - x}{1 - y}}\right).
\]

Since \(k + \ell \leq n, \ell \leq k\), we are interested at \(y \leq x, x + y \leq 1\). The 3D plot of \(\mathbb{H}(x, y)\) reveals that \(\mathbb{H}(x, y) < 0\) for all \(x + y > 0\) and \(\mathbb{H}(0+, 0+) = 0\), the latter seen directly from the formula for \(\mathbb{H}(x, y)\). Setting \(y = zx, z \in [0, 1]\),
we obtain: for \( x \) small,
\[
\mathbb{H}(x, y) = (1 - z)x \log x + x \left(2(1 - z) \log z + z(\log(1 - e^{-1/z}) - e^{-1})\right) \\
+ O(x^2)
\]
\[
\leq x \sup_{z \in [0, 1]} \left(2(1 - z) \log z + z(\log(1 - e^{-1/z}) - e^{-1})\right) + O(x^2)
\]
\[
\leq -0.648x + O(x^2).
\]

It follows that for \( \ell \leq k, \alpha > 0 \) and small \( \varepsilon > 0 \)
\[
\sum_{\alpha \log n \leq k + \ell \leq n} E_{k, \ell} \leq \sum_{0.5 \alpha \log n \leq k \leq n} E_{k, \ell} + \sum_{k \geq n, k + \ell \leq n} E_{k, \ell}
\]
\[
\leq cn^{1/2} \sum_{k \geq 0.5 \alpha \log n} k \exp\left(-k(0.648 - O(\varepsilon))\right)
\]
\[
+ O\left(n^{2.5} \exp\left(n \max\{H(x, y) : y \leq x, x + y \in [\varepsilon, 1]\}\right)\right)
\]
\[
= O\left(n^{1/2}(\log n)n^{-0.5\alpha(0.648-O(\varepsilon))}\right) \to 0,
\]
if \( \alpha > 1.55 \) and \( \varepsilon > 0 \) is sufficiently small. Thus a.a.s. all components smaller than the largest component must be of size \( 1.55\log n \) at most. The expected total size of such components is \( \sum_{k + \ell \leq 1.55\log n} E_{k, \ell} \), which is easily seen to be of order \( O(n^{1/2+o(1)}) \).

We conclude that a.a.s. \( B_{n,0} \) consists of a single giant component and some components each of size \( 1.55\log n \) at most, whose total size is a.a.s. of order \( O(n^{1/2+o(1)}) \).

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5. Appendix

We explain how the numerical calculations were carried out in Matlab to minimize $H_{n,m}(t;r)$ in (2.15b). To begin, we rewrite the equations somewhat to explicitly show the independent variables in $r$. Since $\eta = y + z$, we replace its one occurrence. Next, $p_m(t) = q_m(1-t)$, so we replace it. Finally, we have to be careful of the last term in $H_{n,m}$ to make sure its complicated argument is never non-positive, so we replace it by $u$, i.e., $-n^{-1}\log u$, and solve for $\rho$. The independent variables are now $r = (x,y,z,u)$. However, we do not remove $\rho$ completely from the equation for $H$ since in two of its four occurrences it is simpler to leave it in rather than replacing it by a complicated function of $u$. Also, from numerical evidence, $\rho$ is a much simpler function of $t$, so we can more easily estimate the asymptotic behavior of the independent variables.

Combining these modifications, we obtain

\begin{align}
    f(t) &= te^{t-1} \\
    q_k(w) &= \sum_{j=0}^{k} \frac{w^j}{j!} \quad \text{(in Matlab q(w,k))} \\
    \exp_k(w) &= e^w - q_{k-1}(w) \quad \text{(in Matlab expq(w,k))} \\
    g_m(t; y, z) &= \frac{\exp_1(y + z) + q_m(1-t)f(t) \exp_2(y + z) + [yq_{m-1}(1-t) + zq_m(1-t)]f(t)}{\exp_2(y + z) + [yq_{m-1}(1-t) + zq_m(1-t)]f(t)} \\
    \rho_m(t; x, y, z, u) &= \frac{(1-t)(1-u)x}{g_m(t; y, z)} \\
    H_{n,m}(t; x, y, z, u) &= -2t + (1-t) \log \left( \frac{1 - q_m(1-t)f(t)}{1-t} \right) + t \log(1-t) \\
    &\quad + t \log \left( \exp_2(y + z) + [yq_{m-1}(1-t) + zq_m(1-t)]f(t) \right) \\
    &\quad + t \log \left( \frac{tq_m(x)}{\exp_1(t; x, y, z, u)} \right) + \rho_m(t; x, y, z, u) \\
    &\quad + \frac{z \exp_{m+1}(x)}{q_m(x)} (1-u)^{m+1} - \frac{1}{n} \log u.
\end{align}

where we have explicitly included all the arguments in each function. In each numerical run we fix $m$ and $n$, we define the anonymous functions exactly as
written above, using precisely these arguments, and, recalling that \( t = k/n \),
calculate the minimum of \( H_{n,m}(t; r) \) for each \( t_k = k/n \) where \( k \in [1, \lfloor n/2 \rfloor] \).
Additionally, there are constraints on the independent variables that
\[
(5.2) \quad x, y, z \geq 0 \text{ and } u \in (0, 1] \text{ for all } t \in [0, 1/2].
\]
For each \( t \), we denote the location of the minimum by \( r \) and the minimum value itself by \( H_{n,m}(t) = H_{n,m}(t; r) \).

We used three independent iterative minimization functions, \texttt{fminsearch}, \texttt{fminunc}, and \texttt{fmincon}, in Matlab; the latter two are in the optimization toolbox (which costs extra). They delivered strikingly close trajectories for all \( m \geq 0 \) and \( n \). In particular, for \( m = 1 \) and \( n \geq 100 \) the trajectories are strikingly close and negative for \( t \geq 1/n \) and \( n \geq 100 \). This provides strong numerical evidence that the analytical minimum \( \min_r H_{n,m}(t; r) \) is negative for \( t \geq 1/n \).

We began with the first one which uses the Nelder-Mead simplex algorithm, that does not require the function to be differentiable, but also does not guarantee it converges to a minimum. To obtain as much accuracy as possible and to try to prevent “approximate” minima, the function and optimality tolerances were set to \( 10^{-8} \). However, it is an unconstrained minimization method. So, as it is commonly done, we added a penalty function, namely
\[
(5.3) \quad P(h(-x)x + h(-y)y + h(-z)z + h(-u)u)^2
\]
with \( P = 10^4 \), to \texttt{(5.1b)}. \( h \) is the Heaviside step function which “nudges” the iterates to stay in the constraint region \texttt{(5.2)} whenever any of the variables become negative. For each \( t_k, k > 1 \), the initial iterate is the solution at \( t_{k-1} \). The reason we start at \( t_1 = 1/n \), rather than at \( t_0 = 0 \), is that our admittedly limited analysis of the asymptotic behavior of \( H_{n,m}(t) \) as \( t \downarrow 0 \), see \texttt{(2.18)}, suggests strongly that the function is not differentiable at \( t = 0 \). Extensive numerical evidence suggests that the initial iterates can be chosen at \( t = t_1 \) from \texttt{(2.16)}: we let \( \sigma = 1/3 \), and \( a = b_2 = 1, b_1 = \sqrt{3}, c = 1/\sqrt{3} \) for \( m = 0 \), while \( a = b_1 = b_2 = c = 1 \) for \( m > 0 \), where \( u \) is obtained from \( \rho \) by using \texttt{(5.1a)}.

We are now ready to discuss the results, and we continue to focus on \texttt{fminsearch}, discussing the differences with the other minimization functions as we go along. The curves \( \Pi_{n,m}(t) \) for \( m = 0 \) and \( m = 1 \) with \( n = 100 \cdot 2^{10} \) are shown in Figure \texttt{1}.

First, we get the case \( m = 0 \) out of the way. For \( n \leq 22000 \), \( \Pi_{n,0}(t) > 0 \) for all \( t > 0 \). However, for larger values of \( n \), \( \Pi_{n,0}(t) < 0 \) for small \( t \). The values of \( t \) at which \( \Pi_{n,0}(t) \) becomes positive are \( t = 0.00215 \) for \( n = 10^5 \), \( 0.003162 \) for \( n = 10^6 \), \( 0.0033659 \) for \( n = 10^8 \), \( 0.003369374 \) for \( n = 10^9 \), and
0.0033698094 for \( n = 10^{10} \), so the switch point on \( t \)-axis certainly seems to be approaching a rather small value as \( n \to \infty \).

Next, from numerical evidence for \( m = 1 \), the trajectory is negative for all \( t > 0 \), if \( n \geq 100 \). To see that the curves are converging, we show \( \Pi_{n,1}(t) \) at \( t = 0.01 \) for \( n = 100 \cdot 2^j \) where \( j \in [0,14] \):

\[
0.0045632, -0.0055698, -0.0063425, -0.007355, \\
-0.0076648, -0.0078813, -0.0080275, -0.0081226, -0.0081822, \\
-0.0082184, -0.0082591, -0.0082630.
\]

And we show it at \( t = 0.5 \):

\[
-0.0125880, -0.028543, -0.038172, -0.043832, -0.04709, \\
-0.048934, -0.049964, -0.050533, -0.050844, -0.051014, \\
-0.051105, -0.051154, -0.051181, -0.051195, and -0.051202.
\]

Again, the numbers certainly seem to be decreasing to a limiting value < 0.

The second minimization function we used is \texttt{fminunc}, which is also unconstrained. It is based on a quasi-Newton method, specifically the Broyden-Fletcher-Goldfarb-Shanno algorithm with a cubic line search procedure, where the gradient is approximated numerically, while the Hessian is approximated by a secant-like method in higher dimensions. Over the entire numerically calculated interval \( t \in (0,1/2] \), the curves generated by \texttt{fminsearch} and \texttt{fminunc} are negative and differ by \( < 2 \times 10^{-7} \).

The third minimization function is \texttt{fmincon}, which uses interior-point optimization. It is the only function which allows constraints, so no penalty function is applied. However, the resulting curve rapidly oscillated for \( t \lesssim 10^{-3} \), repeatedly assuming positive values. These oscillations continued for \( t \lesssim 5 \times 10^{-3} \) although the curve remained negative, although for larger values of \( t \) the difference from \texttt{fminsearch}'s curve did fall below by \( 2 \times 10^{-7} \). This curve cannot be accepted; so what could have gone wrong?

These large amplitude oscillations looked like a manifestation of a numerical instability, which requires a technical explanation. Minimization algorithms often have difficulties, much more than zero-finding algorithms. The latter only require the first derivative of the function, called the Jacobian, calculated either analytically or numerically; the former require the gradient, first derivatives, and also some approximation to the Hessian, second derivatives, which introduces more errors. Also, zero-finding is inherently more accurate because, even only considering one dimension, finding the point where a curve passes through the \( x \) axis is much more accurate than finding where it attains a minimum. (As a simple example, if \( y = f(x) \) passes through the \( x \) axis with slope \( s \neq 0 \), a change in \( y \) by \( \delta y \) results in
a change in $x$ by $\delta x = \frac{\delta y}{s}$, while if $y$ has a minimum which behaves like $a(x - \xi)^2$, a change in $y$ near the minimum by $\delta y$ results in a change in $x$ by $\delta x = \sqrt{\frac{\delta y}{a}}$, i.e., $\delta y$ has an exponent of $1/2$ rather than 1, so a small error in $\delta y$ results in a much larger error in $\delta x$.) It seems that, somehow, because of the numerical approximation to the gradient, followed by a secant-like approximation to the Hessian, and in a region where the valley surrounding the minimum was very shallow, a small error in the solution at $t_j$, when used as the initial guess for $t_{j+1}$, caused a larger error. This generated a feedback loop which finally died out at $t \approx 5 \times 10^{-3}$.

To improve the accuracy of the calculations, we used alternate algorithms in \texttt{fminunc}, a trust region algorithm, and \texttt{fmincon}, a trust-region-reflective algorithm, both of which require the gradient of the function to be calculated analytically (not shown). When these more accurate algorithms were used, these two curves were always negative, and the differences between all three, i.e., including \texttt{fminsearch}'s, were always $< 10^{-7}$. We stated earlier that \texttt{fminsearch} was the most accurate of all the algorithms. This claim is supported by numerically approximating the second derivatives of all five curves using second-order centered differences. By eye, the second derivative decreased monotonically from 104 to 0.18 over the entire interval using \texttt{fminsearch}. For the other two functions, without the analytical gradient, there were fluctuations over much of the interval of magnitudes about 1000, while, with the analytical gradient, there were only fluctuations for $t \ll 1$ with magnitudes of 300 to 600.

As another, rather strong, test of the accuracy of the code, the program was only run for small $t$'s so that the slope of $H_{n,m}(t)$ at $t = t_1$ could be compared to (2.17). A straight line was fit to the first 100 points using least squares. The results for $m = 0$ and $n = 10^5$, $10^6$, $10^7$, $10^8$, $10^9$, and $10^{10}$ are $-0.008904$, $-0.03394$, $-0.04374$, $-0.04942$, $-0.04950$, and $-0.05027$ as compared to $-\gamma_0 = -0.0509$. The same calculation for $m = 1$ produces $-0.9501$, $-1.007$, $-1.028$, $-1.037$, $-1.040$, and $-1.041$ as compared to $-\gamma_1 = -1.0426$.

In conclusion we note, for readers without access to Matlab, that no modifications were required in the code to use Octave (a free software package which is mostly compatible with Matlab) with \texttt{fminsearch}.

\textit{The Matlab code is (hopefully) accessible on the journal’s website.}

We include a pseudocode showing the “guts” of the program. Most of the code is taken up in calculating the various functions and generating the plots.

$f \leftarrow (t) \cdots$;
$q \leftarrow (w,k) \cdots$;
$expq \leftarrow (w,k) \cdots$;
\[ g \leftarrow (t, y, z) \cdots ; \]
\[ \rho \leftarrow (t, x, y, z, u) \cdots ; \]
\[ H \leftarrow (t, x, y, z, u) \cdots \]
\[ + P \ast (h(-x) \ast x + h(-y) \ast y + h(-z) \ast z + h(-u) \ast u)^2; \]
\[ dH \leftarrow (t, x, y, z, u) \cdots ; \] // array containing gradient of \( H \)
\[ it \leftarrow 0; \]
\[ \text{for } t = 1/n \text{ to } 1/2 \text{ by } 1/n \]
\[ \text{it } \leftarrow \text{it } + 1; \]
\[ \text{if } t == 1/n \]
\[ \text{if } m == 0 \]
\[ x_{ic} \leftarrow t^{(1/3)}; \quad y_{ic} \leftarrow \sqrt{3} t^{(1/3)}; \quad z_{ic} \leftarrow t^{(1/3)}; \]
\[ \rho_{ic} \leftarrow t/\sqrt{3}; \]
\[ \text{else} \]
\[ x_{ic} \leftarrow t^{(1/3)}; \quad y_{ic} \leftarrow t^{(1/3)}; \quad z_{ic} \leftarrow t^{(1/3)}; \quad \rho_{ic} \leftarrow t; \]
\[ \text{end} \]
\[ u_{ic} \leftarrow 1 - g(t, y_{ic}, z_{ic}) \ast \rho_{ic}/((1 - t) \ast x_{ic}); \]
\[ \text{else} \]
\[ x_{ic} \leftarrow x_{st}(it - 1); \quad y_{ic} \leftarrow y_{st}(it - 1); \quad z_{ic} \leftarrow z_{st}(it - 1); \]
\[ u_{ic} \leftarrow u_{st}(it - 1); \]
\[ \text{end} \]
\[ \{x_{st(it)}, y_{st(it)}, z_{st(it)}, u_{st(it)}\} \]
\[ \leftarrow \minimization function(H, dH, x_{ic}, y_{ic}, z_{ic}, u_{ic}); \]
\[ \text{end} \]