Many body density of states in the edge of the spectrum: non-interacting limit

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Abstract
In noninteracting limit, the density of states (dos) of a many body system can be expressed as a convolution of the single body dos of its subunits. We use the formulation to derive, in the edge of the spectrum, a differential equation for the ensemble averaged many body dos that is relatively easier to solve. Our analysis, based on the systems in which the subunits can be modelled by a Gaussian or Wishart random matrix ensemble, indicates that a rescaling of energy by the number of subunits leaves the many body dos in a mathematically invariant form.

Keywords: random matrices, many body density of states, single body density of states, edge of spectrum, Gaussian ensembles, Wishart ensembles

(Some figures may appear in colour only in the online journal)

1. Introduction
I would like to dedicate this work to the memory of Haake, to celebrate his rich, deep and diverse contributions to Mathematical Physics, especially Quantum Chaos. It was his book [1] on the latter topic, an important resource for my graduate studies and I am deeply indebted to him for writing it.

Consider a many body system composed of independent or weakly interacting particles with mutual interaction not strong enough to perturb the single particle spectrum. The latter situation may arise, for example, for the systems in which the interactions are local within a short spatial scale, say $L$, and negligible at large scales. The system can then be described as a collection of non-interacting/weakly interacting subunits of linear size $L$, and its Hamiltonian can be expressed as a tensor sum of the local Hamiltonians (of the subunits) and its physical properties can then be determined from those of subunits (referred as ‘particles’ hereafter). The many body density of states (mdos) can then be obtained by a convolution of the single particle states (sdos) [2].

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For many-body systems with non-local interactions, the bulk density of many-body states is very large and even a small non-local interaction can in general mix the sdos, wiping out their details (due to small level spacing). The convolution route to derive mdos is therefore not valid in the bulk of the spectrum of such a system. The situation however is different in their edge: due to vanishing dos in the edge, the separation between levels is large and the many-body interactions may not be strong enough to mix the single particle energy states. The mdos in the edge of the spectrum can then be derived by the convolution of single particle spectrum.

For a system consisting of a large number of particles and with a very weak restriction on sdos, the convolution route predicts, based on central limit theorem (CLT), a Gaussian behaviour of the mdos in the bulk of the spectrum [2]. The CLT however is not applicable in the regime where the single particle spectrum is singular e.g. the spectral-edge. With growing interest in the low temperature physics, it becomes now important to know the behaviour of the mdos near the edge i.e. the region where the ground state of the many-body system lies. This motivates us to consider the following questions:

(a) assuming the theoretical form for the sdos in the single body spectral edge is known, what would be the behaviour of mdos in the same energy range,
(b) for what cases of sdos, the mdos will retain the same functional form albeit approximately?

The standard route to determine the energy states of a system is based on a matrix representation of the Hamiltonian in a physically motivated basis. Based on the nature of interactions between different subunits, referred here as ‘particles’, the Hamiltonian matrix can be of various types e.g. dense (full), sparse, banded and subjected to additional matrix constrains due to symmetry and conservation laws [1, 3, 4]. Besides, the complexity of interactions often renders an exact calculation of the matrix elements a technically challenging task. The deterministic error appears as a distribution of the matrix element around its most probable value and the Hamiltonian matrix behaves as a random matrix with some or all random elements. The physical properties of the Hamiltonian matrix are then best described by an appropriate ensemble of its replicas, with ensemble parameters determined from the system conditions. With dos as a backbone of many physical properties, it is therefore necessary to derive its general formulation applicable for a wide range of systems. This is however technically very complicated due to system dependence of the randomization of the many body Hamiltonian. The mdos also varies from one replica to the other, thus making it necessary to consider its average as well fluctuations across the ensemble. However for cases in which sdos is known a priori, the mdos can in principle be determined through convolution route.

It is now well-established, theoretically as well as experimentally, that the fluctuation properties of a one body (single particle) operator in the ergodic regime of its wavefunction dynamics can be well described by the Wigner–Dyson ensembles of Hermitian matrices if the operator is Hamiltonian and by stationary Wishart ensembles (WEs) if it is a semi-positive definite (can be described by a full Wishart random matrix) [1, 3, 4]. In case of the localized/partially localized wavefunctions or weakly violated exact symmetries of one body operator, however, the statistics is described by generalized ensembles e.g. multi-parameter dependent sparse random matrix ensembles, Brownian ensembles (BEs) etc [4–7]. The present study analyses the ensemble averaged mdos in the spectrum edge of a many particle system that consists of many non-interacting particles, with their fluctuation properties described by Gaussian or WEs of both stationary/non-stationary types [8, 9].

Contrary to standard route to derive the mdos using convolutions of sdos, we begin with the differential operator related to sdos, apply it to the definition of the two-body dos as a convolution integral and thereby derive a differential equation for the latter. Although, for the systems consisting of smaller subunits (e.g. a two-body system with known sdos), mdos can
be obtained directly from the convolution integral, the integration route becomes increasingly complicated as the number of subunits increase, motivating us to seek the alternative route. A repeated application of the differential equation for the lower order \( m_{\text{dos}} \) to convolution integral of the next order \( m_{\text{dos}} \) then leads to a non-homogeneous differential equation for \( m_{\text{dos}} \) of desired number of subunits that can be solved subjected to the boundary conditions e.g. the one derived from a smooth connection of the edge \( m_{\text{dos}} \) with that of bulk. For many cases, the \( m_{\text{dos}} \) can be approximated by the homogeneous solution of the related differential equation.

The paper is organized as follows. Section 2 reviews the basic definitions for the \( m_{\text{dos}} \) and its ensemble average as a convolution integral. Section 3, presents the derivation of the differential equation for the \( m_{\text{dos}} \), its solution and the details of the numerical analysis for the subunits modeled by a stationary Gaussian ensemble. Similar analysis for stationary WE and non-stationary ensembles is described in sections 4 and 5, respectively. The section 6 briefly summarizes our main results.

2. Average many body density of states

Consider the case in which the system Hamiltonian \( H \) with \( g \) interacting subunits can be written as

\[
H = H_0 + V
\]

with \( H_0 \) as a sum over non-interacting Hamiltonians \( H^{(s)} \), \( s = 1 \to g \),

\[
H_0 = \sum_{s=1}^{g} H^{(s)}
\]

and \( V \) as the many body interaction among subunits. Here we consider the energy limit in which \( V \) can be ignored e.g. a system generated by distributing \( g \) non-interacting particles over \( N s_{\text{dos}} \) \( E_k \) with \( k = 1 \to N \) (ignoring the occupancy constraints due to Pauli principle for cases \( g \ll N \), referred as dilute limit). Another energy range, where \( V \) often plays a negligible role, is the lower edge region of the many body spectrum and is the focus of current study.

Assuming \( E_1, E_2, \ldots, E_M \) as the eigenvalues of \( H \), with \( M = N^g \) as the size of \( H \)-matrix, the many body \( \text{dos} \ \rho_g(e) \) can be expressed as

\[
\rho_g(e) = \sum_{n=1}^{M} \delta(e - E_n).
\]

Using the same formulation for the single particle \( \text{dos} \ \rho_1(\mathcal{E}) \), (with \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_M \) as the eigenvalues of \( \mathcal{H} \)), we have

\[
\rho_1(\mathcal{E}) = \sum_{k=1}^{N} \delta(\mathcal{E} - \mathcal{E}_k).
\]

In non-interacting limit, the energy \( E_n \) of a many body levels \( \Psi_n \) can further be expressed in terms of the single single particle state as follows. Using \( n_k \) as the number operator for the \( k \)th single particle state \( \mathcal{E}_k \) (counting the number of particles with single particle energy \( \mathcal{E}_k \) in the many body state \( \Psi_n \)), \( H \) represented in the \( N^g \)-dimensional product basis of \( s_{\text{dos}} \) becomes

\[
H = \sum_{k} n_k \mathcal{E}_k.
\]
This in turn leads to \( E_n = \langle \Psi_n | H | \Psi_n \rangle = \sum \alpha_i n_i E_i \), where \( n_i = \langle \Psi_n | \hat{n}_i | \Psi_n \rangle \).

Following from the above, \( \rho_g(e) \) can then be expressed as the convolution of many body \textit{dos} of \((g - 1)\) particles and a single particle \textit{dos}

\[
\rho_g(e) = \int_{e_b}^{e_a} \rho_{g-1}(x) \rho_1(e-x) dx
\]

with notation \( \rho_m \) denoting the \textit{dos} of \( m \) particles and \( e_a, e_b \) as the many body energy range of interest.

An averaging of \( \rho_g(e) \) over all replicas of \( H \) in the ensemble at a fixed \( e \) then leads to its ensemble average at \( e \):

\[
\langle \rho_g(e) \rangle = \sum_{n=1}^{M} \langle \delta(e - E_n) \rangle.
\]

As the particles are almost mutually independent, an ensemble averaging of equation (6) can be expressed as

\[
\langle \rho_g(e) \rangle = \int_{e_a}^{e_b} \langle \rho_{g-1}(x) \rangle \langle \rho_1(e-x) \rangle dx.
\]

For later purposes, we also consider an alternative formulation for even \( g \) e.g. \( g = 2^n \), \( n = 1, 2, \ldots \); \( \rho_{2^n}(e) \) can be rewritten as

\[
\langle \rho_{2^n}(e) \rangle = \int_{e_a}^{e_b} \langle \rho_{2^n-1}(x) \rangle \langle \rho_{2^n-1}(e-x) \rangle dx.
\]

The above in turn leads to

\[
\langle \rho_{2^n+1}(e) \rangle = \int_{e_a}^{e_b} \langle \rho_{2^n}(x) \rangle \langle \rho_1(e-x) \rangle dx.
\]

Hereafter our analysis is confined to the lower edge region of the single body spectrum. Using the notation \( \langle \rho_{m,\text{edge}}(e) \rangle \) for the ensemble averaged \( m \)-body \textit{dos} in the edge region of the spectrum, we have, from equation (8),

\[
\langle \rho_{g,\text{edge}}(e) \rangle = \int_{e_s}^{e_0} \langle \rho_{g-1,\text{edge}}(x) \rangle \langle \rho_{g-1,\text{edge}}(e-x) \rangle dx
\]

with \( b = g/2 \) and \( b = 1 \) for even and odd \( g \) cases, respectively, and \( e_s < e < e_0 \) as the single body lower spectral edge. Here \( e_s \) depends on the nature of the edge e.g. \( e_s \to -\infty \) in case of the soft edge of a stationary Gaussian ensemble and \( e_s \to 0 \) in case of the hard edge of a stationary WE. Here we note that due to rapidly increasing \( mdos \), the many body spectral edge indeed lies below single body spectral edge \( e_0 \). Although the sum over contributions of single particle spectral edges of \( g \) bodies is \( g e_0 \), the number of many body levels near this energy is quite large (e.g. \( \sim g \)), thus making it part of the bulk spectrum instead of the edge.

Further mathematical steps depend on the behaviour of \( \langle \rho_{1,\text{edge}}(e-x) \rangle \) which can vary based on the complexity of the system. As mentioned in previous section, here we confine the analysis to the many body systems consisting of single body Hamiltonians which can be modelled by Gaussian and WES. Moreover, with present study focussed on the edge behaviour of the average spectral density \( \rho_{g,\text{edge}} \), hereafter the subscript ‘edge’ will be suppressed unless needed for clarity of presentation.
3. Stationary Gaussian ensembles

A stationary Gaussian ensemble, also known as Wigner–Dyson ensemble, consists of Hermitian matrices (denoted here by symbol $H$) with basis-invariant Gaussian ensemble density $\rho_H(H) \propto \exp[-\text{Tr} \, H^2]$. The underlying symmetry constraints lead to three universality classes of WDEs, labelled by the symmetry parameter $\beta$: (i) Gaussian orthogonal ensembles (GOEs) of real-symmetric matrices ($\beta = 1$), (ii) Gaussian unitary ensembles (GUEs) of complex Hermitian matrices ($\beta = 2$), (iii) Gaussian symplectic ensembles (GSE) of real-quaternion matrices ($\beta = 4$). The average density for a Wigner–Dyson ensemble follows a semicircle behaviour in the bulk of the spectrum and an Airy function type behaviour in its edge.

Following standard notation, the ensemble averaged $sdos$ is given as $\langle \rho_1(e) \rangle = KN(e, e)$, where the kernel $K$ has different scaling behaviour in the bulk and edge of the spectrum. For a stationary Gaussian ensemble, it has been shown to behave as follows [10]:

$$\lim_{N \to \infty} \frac{1}{\sqrt{2N}} K_N(e\sqrt{2N}, e\sqrt{2N}) = \frac{1}{\pi} \sqrt{1 - e^2} \quad \text{(in bulk, } e \sim 0) \quad (12)$$

and

$$\lim_{N \to \infty} \frac{1}{2^{1/2}N^{1/6}} K_N \left( -\sqrt{2N} + \frac{e}{2^{1/2}N^{1/6}}, -\sqrt{2N} + \frac{e}{2^{1/2}N^{1/6}} \right) = f_1(e) \quad \text{(in lower edge, } e \sim -\sqrt{2N}) \quad (13)$$

where $f_1(e)$ depends on the symmetry class of the ensemble.

Shifting the origin of spectrum to $e = -\sqrt{2N}$, the $sdos$ $\langle \rho_1(e) \rangle$ in the lower edge region, for a single particle Hamiltonian modelled by a stationary Gaussian ensemble, can then be given as

$$\langle \rho_1(e) \rangle = \gamma N f_1(\gamma e) \quad -\infty < e < e_0 \quad (14)$$

with $\gamma = 2^{1/2}N^{1/6}$.

As our approach is based on the convolution of $sdos$, it is useful to define the $mdos$ also in a rescaled form

$$\langle \rho_g(e) \rangle = \gamma N^g f_g(\gamma e) \quad (15)$$

From equation (11), we then have

$$f_g(y) = \int_{-\infty}^{y_0} f_{g-b}(x) f_b(y - x) \, dx \quad (16)$$

with $b = g/2$ for $g$ even, $g = 1$ for $g$ odd. The form of $f_1(e)$ depends on the symmetry class of the ensemble and can thereby lead to different $f_g(e)$. Here we consider the cases with $f_1$ given by a GOE or GUE only. The steps can however be extended directly to GSE too.

3.1. Gaussian unitary ensemble (GUE)

For a GUE, $f_1(y)$ in the lower edge is

$$f_1(y) \approx \gamma \text{Ai}^2(-y) + (\text{Ai}'(-y))^2 \quad (17)$$
with Ai(y) and Ai’(y) as the Airy function of the first kind and its derivative, respectively. We note that the exact differential equation satisfied by \( f_1(y) \) is [10]

\[
\left( \frac{d^3}{dy^3} + 4y \frac{d}{dy} - 2 \right) f_1(y) = 0. \tag{18}
\]

To derive the differential equation for \( \langle \rho_i(e) \rangle \) in the regime \( e < e_0 \) or equivalently \( f_g(y) \) (equation (15)), we use equation (16), start with \( g = 2 \) and a known differential equation for \( f_1(y) \). Proceeding iteratively and multiple partial integrations and rearrangements then lead to the differential equations of \( f_g(y) \) for higher orders in \( g \). The steps can be described as follows. For \( g = 2 \), equation (16) gives

\[
f_2(y) = \int_{-\infty}^{y_0} f_1(x) f_1(y - x) dx,
\tag{19}
\]

with \( y_0 = \gamma e_0 \) and \( f_1(x) \) given by equation (17).

We now consider the differential operator in equation (18), replace \( f_1(y) \) by \( f_2(y) \) and subsequently use equation (19). It is easy to check that \( f_2(y) \) satisfies (appendix A)

\[
\left( \frac{d^3}{dy^3} + 4y \frac{d}{dy} - 2 \right) f_2(y) = -4I_2(y),
\tag{20}
\]

with

\[
I_2(y) = \int_{-\infty}^{y_0} x f_1(x) \frac{\partial f_1(y - x)}{\partial x} dx,
\tag{21}
\]

where we have used the relation \( \frac{\partial h(y-x)}{\partial x} = - \frac{\partial h(y-x)}{\partial y} = \frac{\partial h(y-x)}{\partial y} \). Applying partial integration, \( I_2 \) can now be rewritten as (appendix A)

\[
I_2(y) = -\frac{3}{2} f_2(y) + \frac{1}{4} \frac{d^3 f_2(y)}{dy^3} + Q_2(y)
\tag{22}
\]

with

\[
Q_2(y) = \frac{1}{4} \left[ f_1(y - y_0) D_{y_0}^2 f_1(y_0) - (D_{y_0} f_1(y - y_0))(D_{y_0} f_1(y_0)) + (D_{y_0}^2 f_1(y - y_0)) f_1(y_0) + y_0 f_1(y_0) f_1(y - y_0) \right].
\tag{23}
\]

where \( D_{y_0}^2 f(y_0) \equiv \frac{d^2 f(y_0)}{dy_0^2} \).

Substitution of the above in equation (20) gives

\[
\left( 2 \frac{d^3}{dy^3} + 4y \frac{d}{dy} - 8 \right) f_2(y) = -4Q_2(y).
\tag{24}
\]

For \( g = 2^n \), equation (16) can explicitly be written as \( f_{2^n}(y) = \int_{-\infty}^{y_0} f_{2^{n-1}}(x) f_{2^{n-1}}(y - x) dx \). Using it for \( n = 2 \) and applying the differential operator \( 2 \frac{d^3}{dy^3} + 4y \frac{d}{dy} - 8 \) to both sides of the integral then leads to a differential equation for \( f_3(e) \). Repeating the steps multiple times with \( n = 3, \ldots, m \) then leads to the differential equation for \( f_g(y) \) with \( g = 2^n \) for \( n \geq 1 \).
\[
\left( a_g \frac{d^3}{dy^3} + 4y \frac{d}{dy} - b_g \right) f_g(y) = -4R_g(y),
\]  
(25)

where \( a_{2^n} = 2^n, \ b_{2^n+1} = 2(2 + b_{2^n}) \) with \( a_{2^0} = 1, \ b_{2^0} = 2 \) (i.e. for \( n = 0 \)), and, \( R_g \) can be expressed in terms of a series (derived in appendix B),

\[
R_{2^n}(y) = \sum_{n=0}^{n-1} \left[ \frac{7}{4} \left( f_{2^n-2n}(y - y_0)D_{2^n}^2 f_{2^n}(y_0) \right. \\
- (D_{2^n} f_{2^n-2n}(y - y_0))(D_{2^n} f_{2^n}(y_0)) \\
+ (D_{2^n}^2 f_{2^n-2n}(y - y_0))f_{2^n}(y_0) + y_0 f_{2^n}(y_0) f_{2^n-2n}(y - y_0) \right].
\]  
(26)

As \( f_{2^n}(y) \) becomes smaller with increasing \( a \) (can also be seen by numerical convolution as displayed in figure 1), maximum contribution to \( R_g(y) \) comes from the term with \( n = n - 1 \) which permits the approximation

\[
R_{2^n}(y) \approx 2^{n-3} \left[ (f_1(y - y_0)D_{2^n}^2 f_{2^n-1}(y_0)) - (D_{2^n} f_1(y - y_0))(D_{2^n} f_{2^n-1}(y_0)) \\
+ (D_{2^n}^2 f_1(y - y_0))f_{2^n-1}(y_0) + y_0 f_{2^n-1}(y_0) f_1(y - y_0) \right].
\]  
(27)

As discussed in appendix C, we can approximate \( f_g(y_0) \approx C_g f_1(y_0) \) with \( C_g = \int_0^\infty f_g(t)dt \) a \( g \)-dependent constant. This in turn leads to

\[
R_{2^n}(y) \approx C_{2^n-1} R_{2^n}.
\]  
(28)

Proceeding along the similar steps, the differential equation for \( f_{2^n+1}(\epsilon) \) can also be derived. This can be achieved by using the definition \( f_{2^n+1}(y) = \int_{-\infty}^\infty f_{2^n}(x) f_1(y - x)dx \) and applying the differential operator \( \left( a_{2^n} \frac{d^3}{dy^3} + 4y \frac{d}{dy} - b_{2^n} \right) \) on both sides of the above definition to express the integrant in terms of the derivatives of \( f_{2^n} \). Subsequent substitution of equation (25) with \( g = 2^n \) followed by partial integration and rearrangements, then leads to the differential equation for \( f_{2^n+1}(\epsilon) \); the latter turns out to be same as equation (25) but now \( g = 2^n + 1, \ a_{2^n+1} = 2^n + 1, \ b_{2^n+1} = 6 + b_{2^n} \) and \( R_{2^n+1} \) is given by equation (B9) in appendix A. The latter leads to an approximation similar to equation (28) for \( g = 2^n + 1 \) too. The differential equations for other cases e.g. \( f_{2^n+1}(\epsilon) \) with \( b \) arbitrary can similarly be derived.

Equation (25) is a linear non-homogeneous differential equation of the 3rd order. Writing its general solution as a sum of homogeneous and non-homogeneous parts

\[
f_g(y) = f_{g,h}(y) + f_{g,h}(y),
\]  
(29)

the three independent solutions of the homogeneous part \( f_{g,h}(y) \) can be given as

\[
f_{g,h}(y) = c_1 F \left( \frac{b_g}{12}, \frac{1}{3}, \frac{2}{3}, \frac{-4y^3}{9a_g} \right) + c_2 \left( \frac{4y^3}{9a_g} \right)^{1/3} \\
\times F \left( \frac{1}{3}, \frac{b_g}{12}, \frac{2}{3}, \frac{4}{3}, \frac{-4y^2}{9a_g} \right) \\
+ c_3 \left( \frac{4y^3}{9a_g} \right)^{2/3} F \left( \frac{2}{3}, \frac{b_g}{12}, \frac{4}{3}, \frac{5}{3}, \frac{-4y}{9a_g} \right)
\]  
(30)
Figure 1. $f_g(y)$ for a GUE for $g = 1 \rightarrow 4$. The figure depicts a comparison of $f_g(y)$ derived from (i) $g$-times numerical convolution of $f_1(y)$ given by equation (17), (points, red color online), (ii) solution $f_g(y)$ of equation (25) (solid line, blue color online) (boundary conditions given in text), (iii) solution $f_g(2y/3)$ of equation (25) (dot-dashed line, dark brown color online), (iv) equation (32) (large dashed line, green color online). The deviation near $y = -1$ seems in the cases $g > 1$ seems to be due to very small magnitudes of the quantities involved in edge calculation and numerical integration failing to capture the change due to discretization.

with $c_1, c_2, c_3$ as arbitrary constants of integration. Substitution of equation (30) in equation (15) then leads to $mdos$. Here $c_1, c_2, c_3$ can be determined by imposing the conditions that (i) $f_0(-\infty) \rightarrow 0$, (ii) $f_0(y_0)$ approaches correct bulk density smoothly, and, (iii) the normalization condition on full $mdos$; (for a $sdos$ modeled by a GUE, the $mdos$ in the bulk turns out to be a Gaussian or semicircle based on the nature of particle interactions [2]). As a check, we note that, for $g = 1$, equation (25) reduces to linear homogeneous differential equation given by equation (18) and the solution given by equation (30) reduces to equation (17) for $c_1 = f_1(0), c_2 = f'_1(0), c_3 = f''_1(0)$. Clearly $f_{1,\text{inh}} = 0$. For arbitrary $g > 1$ case, however, it is a priori not obvious whether $f_{g,\text{inh}}$ can be ignored.

Figure 1 illustrates a comparison of the solutions obtained from a direct convolution of the single body densities for four $g$-values with those given by the differential route; here $f_g(y)$, obtained from a $g$-fold numerical convolution of $f_1(y)$ given by equation (17), is displayed along with the numerical solution of the differential equation equation (25) with $y_0 = 0$; both the integration as well as differentiation analysis is performed by using Mathematica software [14]. (As mentioned above, here a theoretical solution for the inhomogeneous part is complicated, thus making it necessary to numerically solve the differential equation.) The $g = 1$ case here is displayed as a check on the numerical efficacy and the boundary conditions;
indeed the numerical solution obtained from equation (18) reduces to equation (17) for correct boundary conditions ($c_1 = f_1(0) = 0.066\,987, c_2 = f_1'(0) = 0.091\,888\,095, c_3 = f_1''(0) = 0.126\,044\,90$). For $g = 2, 3, 4$, we numerically integrate equation (16) with $f_1(y)$ given by equation (17). The results are then compared with the numerical solutions of equation (25), obtained under the conditions $f_2(-y_0) = 0, f_2'(-y_0) = 0, f_2''(-y_0) = 0$ with $y_0$ as a large positive number ($y_0 \sim 10$ used in present analysis). A technical complication in numerical analysis of equation (25) arises due to dependence of $R_0$ on the lower order solutions in $g$ thus requiring a hierarchical analysis; this can however be simplified by the approximation in equation (28).

A substitution of equation (17) gives $R_2$ explicitly as

$$R_2(y) = Q_2(y) = \frac{1}{12} \left[ \frac{2}{\Gamma(1/3)\Gamma(2/3)} y(Ai(-y))^2 + Ai'(-y))^2 \right]$$

$$+ \frac{1}{3^{1/3} \Gamma(2/3)^2} (Ai(-y))^2 - 2 \frac{3^{1/3}}{\Gamma(1/3)^2} Ai(-y)Ai'(-y). \quad (31)$$

We emphasize that the numerical results of $f_2(y)$ obtained here from the convolution integration route are not based on any approximations. A close agreement of the solutions from an exact route with a numerically approximate route for all four $g$-cases then justifies the approximation in equation (28) at least for $g \leq 4$.

It is worth noting that the small deviation seen in figure 1 near $y \approx -1$ is approximately same for $g = 2, 3, 4$. This deviation seems to originate from the significant errors involved with very small magnitude of the $mdos$ near the edge. This motivates us to compare $f_2(y)$ from equation (16) with $f_2(b)$ from equation (25) with $b = 2/3$ (hoping that the deviation originating from the numerical errors could be suppressed by the rescaling of the independent variable $y$). As can be seen from figure 1, a good agreement now follows for each $g$.

It is interesting to note that the convolutions results displayed in figure 1 also agree with the function

$$f_2(y) = N_2 Ai(-(g)^{1/3} y) \quad (32)$$

with $N_2$ a constant in $y$ but varying with $g$; $N_2 = 0.009, 0.001 \,3, 0.000 \,18$ for $g = 2, 3, 4$ respectively. This is however not surprising; as can be seen from $g = 1$ result displayed in figure 1, the function $f_1(y)$ given by equation (17) agrees well with $f_1(y) = 0.15 \, Ai(-1.8y)$ for $y < 0$. The above equation also reveals a $g$ independent form for $f_2(t)$ in terms of a rescaled spectral variable $t = g^{1/3} y$.

Further insight in the edge behavior of the $mdos$ can be derived by rewriting equation (25) in terms of $\rho_s(e)$ and $e$. Substitution of $y = \gamma e$ with $\gamma = N^{1/6}$ along with $\rho_s(e)$ from equation (15) in equation (25) leads to an approximately homogeneous equation, with $R_s(e)$ relatively negligible in large-$N$ limit: $L_y(\rho_s(e)) \approx 0$ with $L_y \equiv \left( \frac{\alpha y}{\sqrt{\pi} \, \frac{d}{dx} + 4e \frac{d}{dx} - b_y} \right) \approx (4e \frac{d}{dx} - b_y)$.

### 3.2. Gaussian orthogonal ensemble (GOE)

As in the previous case, the $sdos$ $\langle \rho_s(e) \rangle$ in the lower edge region, for a single particle Hamiltonian modeled by a GOE, can again be given by equation (14) but now

$$f_1(y) \approx x Ai^2(-y) + (Ai'(-y))^2 + \frac{1}{2} Ai(-y) \int_{-\infty}^{-y} Ai(y) dy, \quad (33)$$
with $\text{Ai}(y)$ same as in equation (17). We note that the origin of spectrum here again is shifted to $e = -\sqrt{2N}$. It is easy to check that $f_1(y)$ satisfies

$$\left( \frac{d^2}{dy^2} - y \right) f_1(y) = Q_1(y)$$

with

$$Q_1(y) = y^2 \text{Ai}^2(-y) + y(Ai'(-y))^2 - \frac{1}{2} \text{Ai}(-y)Ai'(-y).$$

(35)

To derive $\langle \rho_g(e) \rangle$ in the edge regime $e < e_0$, we again use relation (15), proceed iteratively and use equation (16) with $g = 2$, equivalently equation (19). A double differentiation of the latter along with rearrangement of the terms then leads to following differential equation for $f_2(y)$

$$\left( \frac{d^2}{dy^2} - y \right) f_2(y) = I_2(y) + Q_2(y),$$

(36)

where

$$I_2(y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_1(y - x) \, dx \, dy$$

(37)

$$Q_2(y) = \int_{-\infty}^{y} (Q_1(x)f_1(y - x) + f_1(x)Q_1(y - x)) \, dx,$$

(38)

with $y_0 = \gamma e_0$. Applying repeated partial integration, we have

$$I_2(y) = u_2(y) - \frac{d^2 f_2(x)}{dy^2},$$

(39)

where $u_2(y) = f_1(y_0) \frac{d f_2(y_0)}{dy_0} - \frac{d f_1(y_0)}{dy_0} f_1(y - y_0)$. Substitution of the above in equation (36) reduces it as

$$\left( 2 \frac{d^2}{dy^2} - y \right) f_2 = u_2(y) + Q_2(y).$$

(40)

Further using equation (9) and repeating the above steps then leads to the differential equation for $f_{2n}(y)$. The latter along with equation (10) can subsequently be used to derive the differential equation for $f_{2n+1}$. For both cases ($g = 2^n$ or $g = 2^n + 1$), we have

$$\left[ g \frac{d^2}{dy^2} - y \right] f_g(y) = u_g(y) + Q_g(y),$$

(41)

where

$$u_g(y) = \sum_{m=1}^{n-1} \left[ \frac{df_{2^{m+1}}(y_0)}{dy_0} f_{2^{m}}(y - y_0) - f_{2^{m-2}}(y_0) \frac{df_{2^{m}}(y - y_0)}{dy_0} \right] g = 2^n$$

$$= \sum_{m=1}^{n} \left[ \frac{df_{2^{m+1}}(y_0)}{dy_0} f_{2^{m}}(y - y_0) - f_{2^{m-2}+1}(y_0) \frac{df_{2^{m}}(y - y_0)}{dy_0} \right] \times g = 2^n + 1$$

(42)
and

\[ Q_g(y) = \int_{-\infty}^{y_0} \left[ Q_{g/2}(x)f_{g/2}(y-x) + f_{g/2}(x)Q_{g/2}(y-x) \right] \, dx, \quad g = 2^n \]

\[ = \int_{-\infty}^{y_0} \left[ Q_{g-1}(x)f_1(y-x) + f_{g-1}(x)Q_1(y-x) \right] \, dx \quad g = 2^n + 1. \tag{43} \]

Here again we note that, by using \( f_g(y_0) \approx C_{g-1}f_1(y) \) with \( C_{g-1} \) as a constant (appendix C), one can write \( u_d(y) \approx v_d u_1(y) \) with \( u_1(y) \equiv \left[ \frac{df(y_0)}{dy_0} f_1(y_0) - f_1(y_0) \frac{df(y_0)}{dy_0} \right] \) and \( v_d = \sum_{j=1}^{g-1} C_{g-j}c_j \) a constant.

Equation (41) is exact and its solution can be written as \( f_g(y) = f_{g,h}(y) + f_{g,ib}(y) \) with \( f_{g,h} \) and \( f_{g,ib} \) as its homogeneous and non-homogeneous solutions. Here the two independent homogeneous solutions are

\[ f_{g,h}(y) = c_1 Ai(-g^{-1/3}y) + c_2 Bi(-g^{-1/3}y) \tag{44} \]

with \( Ai(y) \) same as in equation (17) and \( Bi(y) \) as the Airy functions of the second kind. Further, as in the GUE case, the constants \( c_1, c_2 \) can be determined from the boundary conditions on \( f_g(y) \), namely, \( f_g(-\infty) \to 0 \) and \( f_g(y_0) \) approaches correct bulk density smoothly. Using the standard approach (based on the Wronskian of two independent homogeneous solutions of a second order differential equation), the inhomogeneous solution of equation (41) is then

\[ f_{g,ib}(y) = A(y)Ai(-g^{-1/3}y) + B(y)Bi(-g^{-1/3}y), \tag{45} \]

with

\[ A(y) = \int_{-\infty}^{y} (w(x))^{-1}Bi(-g^{-1/3}x) \left[ u_d(x) + Q_1(x) \right] \, dx, \tag{46} \]

\[ B(y) = \int_{-\infty}^{y} (w(x))^{-1}Ai(-g^{-1/3}x) \left[ u_d(x) + Q_1(x) \right] \, dx. \tag{47} \]

The above integrals can be calculated by a substitution of equations (42) and (43). Here the Wronskian

\[ w(x) = Ai(-g^{-1/3}x)Bi'(-g^{-1/3}x) - Bi(-g^{-1/3}x)Ai'(-g^{-1/3}x) \]

turns out to be a constant. Further addition of equations (44) and (45) and subsequently imposing the boundary condition \( f_g(-\infty) = 0 \) then leads to

\[ f_g(y) \approx (c_1 + A(y))Ai(-g^{-1/3}y). \tag{49} \]

As in the previous case, figure 2 displays the result for \( f_g(y) \), for \( g = 1 \to 4 \), obtained from a direct \( g \) convolutions of \( f_1(y) \), the latter now given by equation (33). For comparison with differential route, however we now directly use the solution given by equation (49) (instead of numerically solving the differential equation (41)); here we neglect the contribution from \( A(y) \) to simplify the numerical analysis. The constant \( c_1 \) is then numerically determined by fitting equation (49) to convolution result; \( c_1 = 0.16, 0.065, 0.03 \) for \( g = 2, 3, 4 \) respectively. Here again it is interesting to note, from the display for \( g = 1 \), that the function \( f_1(y) \) given by equation (33) can be well-mimicked by the function \( 0.5 \text{Ai}(-y) \) for \( y \leq 0 \). A close agreement
of the solutions from the two routes (i.e. equations (16) and (44)) for all four cases justifies the approximation of $A(y)$ as relatively negligible at least for $g \leq 4$; (theoretically this follows from equation (46), due to both $Q_g(y)$ and $u_g(y)$ being relatively smaller to $f_g(y)$, the former consisting of multiple products of Airy functions and the latter being proportional to $v_g$, a very small constant (defined below equation (43).)

The above along with equation (15) then leads to ensemble averaged mdos

$$\langle \rho_g(e) \rangle \approx c_1 \gamma^{N_g} \text{Ai}(\frac{g}{\gamma} e). \quad (50)$$

Here the constant $c_1$ can be determined by imposing the boundary condition that it smoothly connects with bulk $mdos$ at $e = e_0$ i.e. $\langle \rho_{g,\text{edge}}(e_0) \rangle \approx \langle \rho_{g,\text{bulk}}(e_0) \rangle$. A rescaling of $g^{1/3} e \to e$ in the above equation then gives $\langle \rho_g(e) \rangle$ in a $g$-independent functional form (with $g$ appearing only in the normalization factor).

4. Stationary Wishart ensembles (WEs)

A stationary WE consists of Hermitian matrices $L \equiv C^C$ with $C$ as a $(N + \alpha) \times N$ complex matrix. The ensemble averaged sdos for this case can again be given as $\langle \rho_1(e) \rangle = K_{g}(e, e)$ where the kernel $K$ has different scaling behavior in the bulk and edge of the spectrum. The kernel $K$ for average single particle dos in this case has following scaling
behavior: \( \lim_{N \to \infty} \frac{1}{\sqrt{2N}} K_N(e/2N, e/2N) = \frac{1}{\sqrt{2N}} \sqrt{1 - e} \) in bulk and \( \lim_{N \to \infty} \frac{1}{\sqrt{N}} K_N(e/\sqrt{N}, e/\sqrt{N}) = f_1(e) \) in the lower edge \( (e \sim 0) \) where \( f_1(e) \) again depends on the symmetry class of the WE \([10–12]\).

Following from the above, the sdos \( \langle \rho_1(e) \rangle \) in the lower edge region, for a single particle operator modeled by a stationary WE, can then be given as

\[
\langle \rho_1(e) \rangle = \gamma N f_1(\gamma e) \quad 0 < e < e_0,
\]

with \( \gamma = 4N \). Here again the scaled form of mdos is given by equation (15) but with \( \gamma \) as in equation (51).

### 4.1. Wishart unitary ensemble (WUE)

For a single particle operator modeled by a Wishart unitary ensemble (WUE), \( f_1(x) \) in the edge region can be given as (equation (2.10) of \([10]\))

\[
f_1(x) = \frac{1}{4} \left[ (J_1(\sqrt{x})^2 + J_0(\sqrt{x})J_{1/2}(\sqrt{x}) - \frac{1}{\sqrt{x}} J_0(\sqrt{x})J'_1(\sqrt{x})
+ \frac{1}{\sqrt{x}} J_0(\sqrt{x})J'_{1/2}(\sqrt{x}) \right].
\]

(52)

As discussed in appendix D, \( f_1(y) \) in this case satisfies

\[
\left( y \frac{d}{dy} - \alpha \frac{d}{dy} - 1 \right) f_1(y) = \frac{\alpha}{y} f_1(y).
\]

(53)

We note that \( f_1(x) = 0 \) here for \( x \leq 0 \).

Again using equation (16) and proceeding as in previous cases with \( g = 2 \), it can be shown that \( f_2(y) \) now satisfies

\[
\left( y \frac{d^2}{dy^2} - \alpha \frac{d}{dy} - 1 \right) f_2(y) = I_2(y) - \alpha h_2(y),
\]

(54)

where now

\[
I_2(y) = \int_0^y x f_1(x) \frac{\partial^2 f_1(y-x)}{\partial x^2} \, dx,
\]

(55)

and

\[
h_2(y) = \int_0^y f_1(x) f_1(y-x) \, dx.
\]

(56)

Applying partial integration repeatedly, \( I_2 \) can be rewritten as

\[
I_2(y) = \frac{1}{2} \left( Q_2(y) - 2 \frac{df_2(y)}{dy} + y \frac{d^2 f_2(y)}{dy^2} \right)
\]

(57)

Substitution of the above in equation (54) leads to

\[
\left( y \frac{d^2}{dy^2} - (\alpha + 1) \frac{d}{dy} - 1 \right) f_2(y) = Q_2(y) - \alpha h_2(y),
\]

(58)
where

\[
Q_2(y) = y_0 \left( f_1(y_0) \frac{\partial f_1(y - y_0)}{\partial y_0} - f_1(y - y_0) \frac{\partial f_1(y_0)}{\partial y_0} \right).
\]  

(59)

We note here that, with \( f_1(x) = 0 \) for \( x < 0 \), the terms \( f_1(e - e_0)f_1(e_0) \) and \( f_1(0)f_1(e_0) \) do not contribute i.e. \( f_1(e - e_0)f_1(e_0) = 0 \) and \( f_1(0)f_1(e_0) = 0 \).

Again using equation (15) for \( g = 4, 8, \ldots, 2^n \), \( n \) arbitrary, and iterating the above steps repeatedly (as in previous cases) leads to

\[
\left( \frac{y}{g} \frac{d^2}{dy^2} - \alpha_x \frac{d}{dy} - 1 \right) f_g(y) = R_g(y),
\]  

(60)

where \( \alpha_{2^n} = \alpha + \sum_{k=0}^{n-1} 2^{-k} = \alpha + 2^{1-n}(2^n - 1) \) and

\[
R_g(y) = \frac{1}{2} Q_g(y) + G_g(y) - \alpha h_g(y)
\]  

(61)

and

\[
Q_g(y) = y_0 \left( f_{g-\delta}(y_0) \frac{\partial f_{g-\delta}(y - y_0)}{\partial y_0} - f_{g-\delta}(y - y_0) \frac{\partial f_{g}(y_0)}{\partial y_0} \right)
\]  

(62)

\[
G_g(y) = \int_0^{y_0} f_{g-\delta}(x)(Q_1(x - y) + G_1(x - y)) dx
\]  

(63)

\[
h_g(y) = \int_0^{y_0} f_{g-\delta}(x)h_1(x - y) dx
\]  

(64)

with \( s = g/2 \) for \( g = 2^n \). We note that, with \( y_0 = 4Ne_0 \) and \( e_0 \) although small but non-zero, the integrals can be approximated by \( y_0 \to \infty \) limit for large \( N \) cases.

Further it is worth recalling that the edge region here lies within \( 0 \leq y \leq y_0 \) and both \( sdos \) and \( mdos \) vanish for \( y < 0 \); thus \( f_k(y - y_0) \) and its derivatives are zero for \( k \geq 1 \). This in turn implies, from equations (62) and (63), that \( Q_g(y) = 0 \) and \( G_g(y) = 0 \) for \( 0 \leq y \leq y_0 \). Equation (61) then gives \( R_g(y) = -\alpha h_g(y) \). Further, as can be seen by analyzing \( h_2(y), h_4(y) \ldots \) hierarchically, \( h_g(y) \) is relatively smaller as compared to the terms on left side of equation (60).

As in the previous cases, the differential equation for the present case with \( g = 2^n + 1 \) can now be derived by applying the differential operator \( \left( \frac{y}{g} \frac{d^2}{dy^2} - \alpha_x \frac{d}{dy} - 1 \right) \) to the definition \( f_{2^n+1}(y) = \int_0^{y_0} f_{2^n}(x)(f_1(x - y) \ldots) dx \) and subsequently writing the integral in terms of the derivatives of \( f_{2^n} \) by repeated partial integrations and rearrangements. The equation turns out to be same as equation (34) but now \( g = 2^n+1 \), \( \alpha_{2^n+1} = \alpha = -\sum_{k=0}^{\infty} 2^{-k} = \alpha + 2^{-n}(2^n+1 - 1) \), and, with \( s = 1 \) in equation (64). It must be noted here that while the left side of equation (60) remains same for \( g = 2^n + 1 \) and \( g = 2^{n+1} \), their right side is different.

The two independent solutions of the homogeneous part of equation (60) can be given as

\[
f_{g,h}(y) = c_1 \Gamma(1 - \eta)(gy)^{\eta/2} I_{-\eta}(2\sqrt{gy}) + c_2(-1)^\eta \Gamma(1 + \eta)(gy)^{\eta/2} I_{\eta}(2\sqrt{gy})
\]  

(65)

with \( I_\eta \) as the modified Bessel function, \( \eta = 1 + g \alpha_x \) and \( c_1, c_2 \) as the constants of integration, to be determined from the boundary conditions. The above in turn gives the Wronskian \( w(x) \)
of the two homogeneous solutions as $u(x) = (-g)^{1/(2g)} (1 + g\alpha_k)x^{\alpha_k}$. Following standard route, the solution $f_{g,ih}$ of the inhomogeneous part of equation (34) can be written as

$$f_{g,ih}(y) = -\eta_0(gy)^{\eta_0} [A_g(y)I_{\gamma/2}(2\sqrt{gy}) + B_g(y)I_{1-\gamma/2}(2\sqrt{gy})]$$  \hfill (66)

with $\eta_0 = -(1)^{\Gamma(1 + \eta)}\Gamma(1 - \eta)$ and

$$A_g(y) = \int_0^y (gx)^{\eta_0/2}I_{\gamma/2}(2\sqrt{gy})R_g(x)(w(x))^{-1} \, dx$$  \hfill (67)

$$B_g(y) = \int_0^y (gx)^{\eta_0/2}I_{1-\gamma/2}(2\sqrt{gy})R_g(x)(w(x))^{-1} \, dx.$$  \hfill (68)

Using $f_g(y) = f_{g,h}(y) + f_{g,ih}(y)$, the general solution of equation (60) can now be written as

$$f_g(y) = [c_1\Gamma(1 - \eta) - \eta_0A_g(y)] (gy)^{\eta_0/2}I_{\gamma/2}(2\sqrt{gy})$$

$$+ [c_2(-1)^{\eta_0/2}\Gamma(1 + \eta) - \eta_0B_g(y)] (gy)^{\eta_0/2}I_{1-\gamma/2}(2\sqrt{gy}).$$  \hfill (69)

Here again the constants $c_1, c_2$ can be determined by imposing the boundary condition at $y = 0$ and $y = y_0$ i.e. $f_1(0) = 0$ and $\gamma N_0^3 f_1(y_0) = \rho_{g,\text{bulk}}(e)$; this gives $c_1 = 0$. Further, as clear from the above equation, a rescaling of the variable $y$ i.e. $gy \rightarrow y$, gives $f_g(y)$ in a $g$-independent form (with $g$ now appearing through normalization of $f_g(y)$).

To verify the above results, we again use numerical analysis using Mathematica software [14]. However, due to Bessel function dependence of $f_1(x)$ in this case, the multiple convolution integration route is numerically unstable for $g > 3$ and not very reliable even for lower order $mdos$. Figure 3 displays the result for $f_{g,ih}(y)$, for $g = 1 \rightarrow 3$, obtained from a direct convolution of $f_1(y)$ given by equation (52); the solution given by equation (69) is also displayed for each case (taking $c_1 = 0$ and neglecting the contribution from $B_g(y)$, equivalently from $f_{g,ih}$, based on reasoning given below equation (64)). The constant $c_2$ is again numerically determined by fitting equation (69) to convolution result: $c_2 = 2.6 \times 10^{-55}, 8 \times 10^{-97}$ for $g = 2, 3$ respectively. Here again a close agreement of the solutions from the two routes for $g = 1, 2, 3$ cases justifies the approximation $f_g \approx f_{g,h}$ at least for $g \leq 3$. We note that a deviation near $y = 1.5$ in $g = 3$ case seems to be due to numerical error involved in calculation of very small magnitude of $f_3(y)$.

Using now equation (15) and $\gamma = 4N, \langle \rho_g(e) \rangle$ for this case can be given as

$$\langle \rho_g(e) \rangle = c_2\gamma N_0^3 \Gamma(\alpha_k + 2)(\alpha_k e)^{\alpha_k+1/2}I_{\alpha_k+1/2}(2\sqrt{\alpha_k e}),$$  \hfill (70)

with constant $c_2$ determined from the bulk $mdos$.

### 4.2. Wishart orthogonal ensemble (WOE)

As in the previous case, the $sdos$ $\langle \rho_1(e) \rangle$ in the edge region for a WOE, can still be described by equation (51) with $f_1(x)$ given as [10]

$$f_1(x) = f_{1,U}(x) - \frac{J_{\alpha_k+1}(\sqrt{\alpha_k})}{4\sqrt{\alpha_k}} \left( \int_0^{\sqrt{\alpha_k}} J_{\alpha_k+1}(v)dv - 1 \right),$$  \hfill (71)

where $f_{1,U}(x)$ is same as $f_1(x)$ of WUE case, given by equation (52).
Figure 3. $f_{g}(y)$ for a WUE for $g = 1 \rightarrow 3$ for $a = 10$. The figure depicts a comparison of $f_{g}(y)$ derived from (i) $g$-times numerical convolution of $f_{1}(y)$ given by equation (52), (points), (ii) equation (69) (with $c_{1} = 0$). Due to convergence issues related to numerical integration in this case, $f_{4}(y)$ for $a = 10$ could not be determined. The determination of $f_{g}(y)$ for lower $a$-values also is unreliable and requires sophisticated numerical integration tools.

In principle, proceeding as in previous cases, the exact differential equation for $f_{g}(y)$ for this case can again be derived. The derivation can however be simplified by noting that, near the lower edge $x \sim 0$, the 2nd term in equation (71) is negligible with respect to first term and one can approximate

$$f_{1}(x) \approx f_{1, U}(x). \quad (72)$$

The display in figure 4(a) indeed confirms the relation in equation (72); (it is worth noting that the above equality relation is not applicable for $s$dos of GOE and GUE (as can be seen from figure 3(b)).

The above approximation in turn would again lead to equation (60) for $m$dos in WUE case. This indicates the insensitivity of the behaviour in the edge of the spectrum of a WE to exact symmetry conditions.

5. Brownian ensembles

A BE in general refers to an intermediate state of perturbation of a stationary random matrix ensemble by another one of a different universality class [1, 3, 9, 13]. The type of a BE, appearing during the cross-over, depends on the nature of the stationary ensembles and their different pairs may give rise to different BEs [4, 9]. Here we consider the BEs appearing between stationary ensembles of Gaussian and Wishart type only.
Figure 4. Comparison of $f_1(y)$ for orthogonal and unitary universality classes of stationary ensembles: (a) for a WUE and WOE (equations (52) and (D1)) (b) for a GUE and GOE (equations (17) and (33)). As mentioned in the text, $f_1(y)$ for WOE in the edge is well approximated by that of WUE.

5.1. Gaussian Brownian ensembles

A BE of Hermitian matrices $H$ can be described as $H = \sqrt{\tau}(H_0 + tV)$ with $V(t)$ as a random perturbation of strength $t$, taken from a stationary Gaussian ensemble characterized by symmetry parameter $\beta$, and applied to an initial stationary state $H_0$ (see also [5]). Here $f = (1 + \gamma t^2)^{-1}$ with $\gamma$ as an arbitrary positive constant.

The $sdos$ for the above ensemble is described by a diffusion equation, referred as Dyson–Pastur equation [5, 9]:

$$\frac{1}{2\beta} \frac{\partial \langle \rho_1 \rangle}{\partial Y} = \frac{\partial}{\partial e} [\gamma e - \alpha(e)] \langle \rho_1 \rangle$$  \hspace{1cm} (73)

with $Y \propto \frac{1}{\sqrt{\gamma}} \ln(1 + \gamma; t^2)$ and

$$\alpha(e) \equiv \alpha_e \equiv P \int_{-\infty}^{\infty} \langle \rho_1(e') \rangle e - e'$$ \hspace{1cm} (74)
We note that the lower edge here corresponds to the region $-\infty < e \leq e_0$ with $e_0 = -\sqrt{2N}$, the standard edge limit for Gaussian stationary ensembles; (as equation (73) is derived for origin of the spectrum at $e = 0$).

To derive $\langle \rho_s(e) \rangle$ in the edge regime for the present case, we again proceed iteratively and start with $g = 2$. Using the definition in equation (14) for $\langle \rho_2 \rangle$ and differentiating it with respect to $Y$, we have

$$\frac{\partial \langle \rho_2(e) \rangle}{\partial Y} = J_1 + J_2 \quad (75)$$

with

$$J_1(e) = \int_{-\infty}^{\infty} \frac{\partial}{\partial Y} \langle \rho_1(x) \rangle \langle \rho_1(e - x) \rangle \, dx \quad (76)$$

and

$$J_2(e) = \int_{-\infty}^{\infty} \langle \rho_1(x) \rangle \frac{\partial}{\partial Y} \langle \rho_1(e - x) \rangle \, dx \quad (77)$$

Substitution of equation (73) followed by partial integration reduces $J_1$ as

$$J_1(e) = S_{11}(e) - \int_{-\infty}^{\infty} (\gamma x - \alpha(x)) \langle \rho_1(x) \rangle \partial_x \langle \rho_1(e - x) \rangle \, dx$$

$$= S_{11}(e) + \partial_e \int_{-\infty}^{\infty} (\gamma x - \alpha(x)) \langle \rho_1(x) \rangle \langle \rho_1(e - x) \rangle \, dx \quad (78)$$

with notation $\partial_e \equiv \frac{\partial}{\partial e}$ and

$$S_{11}(e) = (\gamma e_0 - \alpha(e_0)) \langle \rho_1(e - e_0) \rangle \rho_1(e_0). \quad (79)$$

By substitution of equation (73) in equation (76) and writing $\frac{\partial}{\partial e} \langle \rho_1(e - x) \rangle = \frac{\partial}{\partial e} \langle \rho_1(x) \rangle$, $J_2$ can similarly be rewritten as

$$J_2(e) = \int_{-\infty}^{\infty} \langle \rho_1(x) \rangle \partial_x \left[ (\gamma e - \alpha(e - x)) \langle \rho_1(e - x) \rangle \right] \langle \rho_1(e - x) \rangle \, dx \quad (80)$$

$$= \partial_e \int_{-\infty}^{\infty} \left[ (\gamma e - \alpha(e - x)) \langle \rho_1(x) \rangle \langle \rho_1(e - x) \rangle \right] \, dx. \quad (81)$$

Using equations (78) and (81) now in equation (75) gives

$$\frac{1}{2\beta} \frac{\partial \langle \rho_2(e) \rangle}{\partial Y} = S_{11}(e) + \partial_e \left[ \gamma e \langle \rho_2(e) \rangle - G_2(e) \right] \quad (82)$$

with

$$G_2(e) = \int_{-\infty}^{\infty} \left[ \alpha(x) + \alpha(e - x) \right] \langle \rho_1(x) \rangle \langle \rho_1(e - x) \rangle \, dx. \quad (83)$$

As the maximum contribution to $G_2$ comes from the neighborhood of $e_0$, it can be approximated as

$$G_2(e) \approx [\alpha(e_0) + \alpha(e - e_0)] \int_{-\infty}^{\infty} \langle \rho_1(x) \rangle \langle \rho_1(e - x) \rangle \, dx$$

$$= [\alpha(e_0) + \alpha(e - e_0)] \langle \rho_2 \rangle. \quad (84)$$
The above on substitution in equation (82) reduce it as
\[
\frac{1}{2\beta} \frac{\partial \langle p_2 \rangle}{\partial Y} = S_{11}(e) + \frac{\partial}{\partial e} [\gamma e - \alpha_2(e)] \langle p_2 \rangle
\]
(85)
with \( \alpha_2(e) = \alpha(e_0) + \alpha(e - e_0) \).

Now using equation (9) and proceeding iteratively again for \( g = 4, 6, 8, \ldots \), one can similarly derive the diffusion equation for \( \langle p_{2g} \rangle \):
\[
\frac{1}{2\beta} \frac{\partial \langle p_{2g} \rangle}{\partial Y} = Q_s(e) + \frac{\partial}{\partial e} [\gamma e - \alpha_{2g}(e)] \langle p_{2g} \rangle
\]
(86)
with
\[
Q_s(e) = \sum_{k=0}^{n-1} 2^{n-k-1} S_{2k,2g-2k}(e),
\]
(87)
where
\[
S_{ab}(e) = (\gamma e_0 - \alpha(e_0))\rho_b(e_0)p_b(e - e_0)
\]
(88)
\[
\alpha_{2g}(e) = \alpha_{2g-1}(e_0) + \alpha_{2g-1}(e - e_0)
\]
(89)
\[
= \alpha(e - ne_0) + \sum_{k=1}^{n-1} \alpha_{2k}(e_0).
\]
(90)

We note that the case \( \alpha(e_0) = 0 \) implies \( S_{ab}(e) = 0 \) and equation (86) for mdos reduces to the same form as that of sdos i.e. equation (73).

Further as \( p_b(e - e_0) \) in the edge (i) decreases with increasing \( b \) and, (ii) increases as \( e \to e_0 \), this implies \( S_{2^{g+1}-2} > S_{2^g,2^{g+1} - 2} \) for \( k > 1 \). Thus assuming the series in equation (43) is dominated by the term with \( k = n - 1 \) (notwithstanding the prefactor \( 2^{n-k-1} \) due to faster decay in \( \rho_b \) with increasing \( b \), one can approximate \( Q_s(e) \approx S_{2^{g-1}-2} \approx (\gamma e_0 - \alpha(e_0))\rho_{2^{g-1}}(e_0)p_{2^{g-1}}(e - e_0) \).

The latter along with equation (89) intuitively suggests that the mathematical form of \( \langle p_{2g} \rangle \) can be reduced to that of \( \langle p_2 \rangle \) by a rescaling of variable \( e \); this however requires a detailed numerical analysis for the confirmation.

Further using equation (10) and using equation (86), the diffusion equation for \( \langle p_{2^{g+1}} \rangle \) can similarly be derived.

5.2. Wishart Brownian ensembles

Consider an ensemble of \( N \times N \) rectangular matrices \( A(t) = \sqrt{T}(A_0 + iV(t)) \) with \( f = (1 + \gamma^2)^{-1} \) \( [8, 9] \) with \( A_0 \) as a fixed matrix and \( \gamma \) as an arbitrary positive constant. The matrices \( L = A^\dagger A \) correspond to Wishart BEs if the matrices \( V^\dagger V \) are taken from stationary WEs e.g. WOE or WUE. As clear, \( A = A_0 \) for \( t \to 0 \), \( A \to V/\sqrt{T} \) for \( t \to \infty \).

A variation of strength \( t \) of the random perturbation \( V \) leads to diffusion of the matrix elements \( A_{ij}(t) = \sqrt{T}(A_{ij0} + iV_{ij}(t)) \). For \( \rho_i(V) = \left( \frac{1}{2\pi} \right)^{\frac{N(N+1)}{2}} e^{-\frac{1}{4T} \text{tr}(VV^\dagger)} \), the diffusion equation for the \textit{sdos} for the \textit{L}-ensemble is described by a diffusion equation [5]:
\[
\frac{1}{2\beta} \frac{\partial \langle p_1 \rangle}{\partial Y} = \frac{\partial}{\partial e} (\gamma e - \alpha_w(e)) \langle p_1 \rangle
\]
(90)
with \( Y = -\frac{1}{\gamma_0} \ln f = \frac{1}{2} \ln(1 + \gamma^2) \) and \( \alpha_\omega(e) = e \alpha(e) \) same as in equation (73).

We note that the form of the equation (90) is same as that of equation (73) except for \( \alpha(e) \) in the latter replaced by \( \alpha_\omega(e) \) in the former. Thus proceeding again as in previous case, the evolution of \( \langle \rho_{2\nu} \rangle \) can again be described by equation (82) but with \( \alpha(e) \rightarrow \alpha_\omega(e) \):

\[
\frac{1}{2\beta} \frac{\partial \langle \rho_{2\nu} \rangle}{\partial Y} = Q_\nu(e) + \frac{\partial}{\partial e} \left[ e(\gamma - \alpha_\omega(e)) \right] \langle \rho_{2\nu} \rangle \tag{91}
\]

with \( Q_\nu(e) \) and \( \alpha_\omega(e) \) again given by equations (87) and (89) respectively but now

\[ S_{ab} = e_0(\gamma - \alpha(e_0))\rho_a(e_0)\rho_b(e - e_0). \tag{92} \]

5.3. Multi-parametric ensembles

Consider an operator of a complex system modeled by a multi-parametric Gaussian ensemble of Hermitian matrices with probability density

\[ \rho_H(H, v, b) = C \exp \left[ -\sum_{k \leq l} \frac{1}{2v_{kl}} (H_{kl} - b_{kl})^2 \right]; \tag{93} \]

here the variances \( v_{kl} \) and mean values \( b_{kl} \) can take arbitrary values (e.g. \( v_{kl} \rightarrow 0 \) for the non-random elements). As discussed in a series of studies \([4, 5, 7]\), the evolution of the sdos of the above ensemble is again described by equation (73) but now \( Y \) is given by a combination of ensemble parameters, \( Y = -\frac{1}{\gamma_0} \ln \left[ \prod_{k \leq l} \prod_{q=1}^{\beta} \left| b_{kl} + b_q \right|^2 \right] + \text{constant} \) and is referred as the ‘complexity parameter’. Here \( b_0 = 1 \) or 0 if \( b_{kl} = 0 \) or \( \neq 0 \) respectively.

The equivalence of the differential equation for sdos further implies that mdos for a system, consisting of subunits statistically described by equation (93), is also given by equation (82); however \( Y \) is now the complexity parameter mentioned above.

6. Conclusion

In the end, we summarize with our main results and open questions.

Based on the information about sdos i.e. the ensemble averaged dos of a single subunit of a many body system in its spectral edge, we derive the mdos i.e. the ensemble averaged mdos in noninteracting limit. Contrary to standard route to derive the mdos using convolutions of sdos, we derive here a differential equation for the mdos which is not only theoretically easier to analyze but also numerically more reliable to solve, specifically near the edge of a spectrum where integrals involving higher order convolution become increasingly complicated and more susceptible to error due to small magnitudes of the quantities involved. Our analysis clearly indicates that the mdos changes with increasing number \( g \) of subunits and differs from that of single body density of states even in non-interacting limit. A rescaling of energy however leads to a universal form for the mdos in the spectral edge, independent of number of subunits used.

While the analysis in the present study is confined to the edge of the spectrum, the derivation can directly be extended to any other energy range including bulk, within noninteracting approximation. Although the convolution route is not applicable for the interacting many body systems, it would be interesting to compare some available results for the latter with the results derived in the present study and compare the extent of their differences/deviation and sensitivity to interaction parameters.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Derivation of equation (20)

Applying the differential operator \( L(y) \equiv \left( \frac{d^3}{dy^3} + 4y \frac{d}{dy} - 2 \right) \) on equation (19) gives

\[
L(y) f_2(y) = \int_{-\infty}^{y_0} f_1(x)[L(y)f_1(y-x)]dx. \tag{A1}
\]

Further we note that

\[
L(y)f_1(y-x) = L(y-x)f_1(y-x) + 4x \frac{df_1(y-x)}{dy(x-y)}, \tag{A2}
\]

where the relation in the second equality is obtained by using (i) \( L(y-x)f_1(y-x) = 0 \) (equation (18)) and (ii) \( \frac{df_1(y-x)}{dy(x-y)} = -\frac{df_1(y-x)}{dx} = \frac{df_1(y-x)}{dy} \). Equation (A3) on substitution in equation (A1) then leads to \( L(y)f_2(y) = -4I_2(y) \) with \( I_2(y) \) given by equation (21).

Applying partial integration to the integral in \( I_2 \), we have

\[
I_2(y) = y_0f_1(y_0)f_1(y-y_0) - \int_{-\infty}^{y_0} f_1(x)f_1(y-x)dx - \int_{-\infty}^{y_0} x \frac{df_1(x)}{dx}f_1(y-x)dx,
\]

\[
= y_0f_1(y_0)f_1(y-y_0) - 3 \frac{1}{2} f_2(y) + J_2(y) \tag{A4}
\]

with \( J_2 = \int_{-\infty}^{y_0} \frac{d^3f_1(x)}{dx^3}f_1(y-x)dx \). Here the second equality is obtained from the first by using the definition, given by equation (19), in first integral and substitution of equation (18) in second integral. Applying partial integration twice repeatedly, one can write

\[
J_2(y) = f_1(y-y_0)D_{y_0}^2f_1(y_0) - D_{y_0}f_1(y-y_0)D_{y_0}f_1(y_0) + f_1(y_0)D_{y_0}^2f_1(y-y_0) + \int_{-\infty}^{y_0} \frac{d^3f_1(y-x)}{dx^3}f_1(x)dx. \tag{A5}
\]
Further using the identity \( \frac{d^3f(x-y)}{dy^3} = \frac{d^3f(x-y)}{dy^3} = \frac{d^3f(x-y)}{dy^3} \), the last integral in equation (A5) can be rewritten as \( \frac{d^3f(x-y)}{dy^3} \). The substitution of \( J_2 \) in equation (A4) now leads to equation (21).

**Appendix B. Derivation of equation (26)**

For arbitrary \( g > 1 \), the right side of equation (25) can be written as

\[
R_g(y) = -4 \left( Q_g(y) + G_g(y) \right),
\]

where for \( g = 2^n \), we have

\[
G_{2^n}(y) = \int \left[ Q_{2^n-1}(y - t) + G_{2^n-1}(y - t) \right] f_{2^n-1}(t) dt
\]

and

\[
Q_{2^n}(y) = \frac{2^{(n-1)}}{4} \left[ f_{2^n-1}(y - y_0)D_{2^n}^2 f_{2^n-1}(y_0) \right.
- (D_{2^n}f_{2^n-1}(y - y_0))(D_{2^n}f_{2^n-1}(y_0))
\]

\[ + \left. (D_{2^n}^2 f_{2^n-1}(y - y_0))f_{2^n-1}(y_0) \right] + y_0 f_{2^n-1}(y_0)f_{2^n-1}(y - y_0) \]  

(B3)

Substitution of equation (B3) in equation (B2) and subsequently using equation (16) results in reducing \( G_{2^n}(y) \) as

\[
G_{2^n}(y) = \sum_{n=0}^{n-2} \left[ \frac{2^{n-1}}{4} \left( f_{2^n-2}(y - y_0)D_{2^n}^2 f_{2^n-2}(y_0) \right. \right.
- (D_{2^n}f_{2^n-2}(y - y_0))(D_{2^n}f_{2^n-2}(y_0))
\]

\[ \left. + \left. (D_{2^n}^2 f_{2^n-2}(y - y_0))f_{2^n-2}(y_0) \right) + y_0 f_{2^n}(y_0)f_{2^n-2}(y - y_0) \right]. \]  

(B4)

Using the above along with equation (B3) in equation (26) then leads to

\[
R_{2^n}(y) = \sum_{n=0}^{n-1} \left[ \frac{2^n}{4} \left( f_{2^n-2}(y - y_0)D_{2^n}^2 f_{2^n-2}(y_0) \right. \right.
- (D_{2^n}f_{2^n-2}(y - y_0))(D_{2^n}f_{2^n-2}(y_0))
\]

\[ \left. + \left. (D_{2^n}^2 f_{2^n-2}(y - y_0))f_{2^n-2}(y_0) \right) + y_0 f_{2^n}(y_0)f_{2^n-2}(y - y_0) \right] \]. \]  

(B5)

Similarly for \( g = 2^n + 1 \), we have

\[
Q_{2^n+1}(y) = \frac{2^n}{4} \left[ f_{2^n}(y_0)D_{2^n}^2 f_1(y - y_0) - (D_{2^n}f_{2^n}(y - y_0))(D_{2^n}f_1(y_0)) \right.
\]

\[ + \left. (D_{2^n}^2 f_{2^n}(y - y_0))f_1(y_0) \right] + y_0 f_{2^n}(y_0)f_1(y - y_0) \]  

(B6)
and

\[ G_{2^n+1}(y) = \int [Q_{2^n}(y-x) + G_{2^n}(y-x)] f_1(x) dx. \]  

(B7)

Substitution of equations (B3) and (B4) in the above equation then leads to

\[
G_{2^n+1}(y) = \sum_{a=0}^{n-1} \left[ \frac{2^a}{4} (f_{2^n-2}(y-y_0)) D_{y_0}^2 f_{2^n}(y_0) 
- (D_{y_0} f_{2^n-2}(y-y_0))(D_{y_0} f_{2^n}(x)) 
+ (D_{y_0}^2 f_{2^n-2}(y-y_0)) f_{2^n}(y_0) + y_0 f_{2^n}(y_0) f_{2^n-2}(y-y_0) \right].
\]

(B8)

The above equations on substitution in equation (B1) with \( g = 2^n + 1 \) then lead to

\[
R_{2^n+1}(y) = \sum_{a=0}^{n} \left[ \frac{2^a}{4} (f_{2^n-2}(y-y_0)) D_{y_0}^2 f_{2^n}(y_0) 
- (D_{y_0} f_{2^n-2}(y-y_0))(D_{y_0} f_{2^n}(y_0)) 
+ (D_{y_0}^2 f_{2^n-2}(y-y_0)) f_{2^n}(y_0) + y_0 f_{2^n}(y_0) f_{2^n-2}(y-y_0) \right].
\]

(B9)

Appendix C. Relation between \( f_g(y_0) \) and \( f_1(y_0) \)

From equation (16), \( f_g(y_0) \) can be expressed as

\[
f_g(y_0) = \int_{-\infty}^{y_0} f_1(x) f_{g-1}(y_0-x) dx.
\]

(C1)

Noting that the maximum of the above integral comes from the neighborhood of \( x = y_0 \), the above equation can be rewritten as

\[
f_g(y_0) \approx f_1(y_0) \int_{-\infty}^{y_0} f_{g-1}(y_0-x) dx
\]

\[
\approx f_1(y_0) \int_{-\infty}^{\infty} f_{g-1}(t) dt \approx c_{g-1} f_1(y_0)
\]

(C2)

with \( c_{g-1} = \int_{0}^{\infty} f_{g-1}(t) dt \).

Appendix D. Derivation of equation (53)

The \textit{sdos} of a stationary WE of matrices \( W = C^\dagger C \), with \( C \) as a \( M \times N \) matrix and \( \alpha = M - N \), is given as

\[
\langle \rho_1(e) \rangle = K^\alpha(e, e)
\]

with \( K^\alpha(e, e') \) as the kernel for the ensemble [10].
As given by equations (2.20) and (2.21) in [10], the function \( F(x, y) = (xy)^{-\alpha/2} K_\alpha(x, y) \), with \( x = 4Ne \), \( y = 4Ne' \) satisfies following differential equation
\[
x \frac{\partial^2 F}{\partial x^2} + (\alpha + 1) \frac{\partial F}{\partial x} - F + \frac{1}{x - y} \left( x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y} \right) = 0.
\] (D2)

To determine the differential equation for \( \langle \rho_1(x) \rangle \), we now derive equation for \( F(x, x) \) as follows. For \( y = x + \epsilon \) and in the limit \( \epsilon \to 0 \), \( F(x, x + \epsilon) \) can be expanded in Taylor’s series. This leads to
\[
x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y} \approx -\epsilon \frac{\partial F}{\partial x} + \epsilon^2 \frac{\partial^2 F}{\partial x^2} + O(\epsilon^3).
\] (D3)

Substituting the above along with \( y = x + \epsilon \) in equation (D2) and taking the limit \( \epsilon \to 0 \) then leads to
\[
x \frac{\partial^2 F}{\partial x^2} + \alpha \frac{\partial F}{\partial x} - F = 0.
\] (D4)

From equation (D1), we have \( F(e, e) = e^{-\alpha} \langle \rho_1(e) \rangle \). This on substitution in equation (D4) now gives
\[
\left( \epsilon \frac{d^2}{de^2} - \alpha \frac{d}{de} - 1 \right) \langle \rho_1(e) \rangle = \frac{\alpha}{e} \langle \rho_1(e) \rangle.
\] (D5)

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