AN EXTENSION OF RUH-VILMS’ THEOREM
TO HYPERSURFACES IN SYMMETRIC SPACES
AND SOME APPLICATIONS

ÁLVARO RAMOS AND JAIME RIPOLL

Abstract. This paper has two main purposes: First, to extend a well-known
theorem of Ruh-Vilms in the Euclidean space to symmetric spaces and, sec-
ondly, to apply this result to extend the Hoffman-Osserman-Schoen theorem
(HOS theorem) to 3-dimensional symmetric spaces. Precisely, we define a
Gauss map of a hypersurface $M^{n-1}$ immersed in a symmetric space $N^n$ taking
values in the unit pseudo-sphere $S^m$ of the Lie algebra $g$ of the isometry
group of $N$, $\dim g = m + 1$, and it is proved that $M$ has CMC if and only if
its Gauss map is harmonic. As an application, it is proved that if $\dim N = 3$
and the image of the Gauss map of a CMC surface $S$ immersed in $N$ is con-
tained in a hemisphere of $S^m$ determined by a vector $X$, then $S$ is invariant by
the one-parameter subgroup of isometries of $N$ of the Killing field determined
by $X$. In particular, an extension of the HOS theorem to the 3-dimensional
hyperbolic space is obtained, which, as far as the authors know, has not been
done.

It is also shown that the holomorphic quadratic form induced by the Gauss
map coincides (up to a sign) with the Hopf quadratic form when the ambient
space is $\mathbb{H}^3$, $\mathbb{R}^3$ and $\mathbb{S}^3$, and coincides with the Abresch-Rosenberg quadratic
form when the ambient space is $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. This then provides a
unified way of relating Hopf’s and Abresch-Rosenberg’s quadratic form with
the quadratic form induced by a harmonic Gauss map of a CMC surface in
these five spaces.

1. Introduction

A well-known theorem due to Ruh-Vilms [RV] establishes that an orientable
immersed hypersurface $S$ in $\mathbb{R}^n$, $n \geq 3$, has constant mean curvature (CMC) if and
only if the Gauss map $N : S \to S^{n-1}$ of $S$ satisfies the equation
\begin{equation}
\Delta N = -\|B\|^2 N
\end{equation}
or, equivalently, that $N$ is a harmonic map, where $B$ is the second fundamental
form of $S$.

In [RR] the second author of the present paper with F. Bittencourt defined a
Gauss map of an orientable hypersurface on ambient spaces of the form $N := G/K \times
\mathbb{R}^n$, $n \geq 0$, where $G/K$ is a compact symmetric space. The Gauss map is defined
by taking the horizontal lift of the unit normal vector field of the hypersurface to
$G \times \mathbb{R}^n$ followed by a translation to the unit sphere in the Lie algebra of $G \times \mathbb{R}^n$.

Received by the editors May 16, 2014.

2010 Mathematics Subject Classification. Primary 53C42.

This research was supported by CNPq - Brasil.

\textsuperscript{1}We remark that the Ruh-Vilms result applies to submanifolds of arbitrary codimension, with
the Gauss map assuming values in a Grassmannian manifold.
Ruh-Vilms’ theorem is then extended to hypersurfaces of $N$. That is, they prove that a hypersurface of $N$ has CMC if and only if this Gauss map is harmonic (Corollary 3.4 of [BR]).

In the present paper we extend the construction of the Gauss map done in [BR] to any symmetric space, not necessarily reducible nor compact and of any dimension, obtaining an extension of Ruh-Vilms’ theorem to these spaces (Theorem 1 and Corollary 1). Our result generalizes some previous works, such as [Ma] and [EFFR].

We recall that an application of Ruh-Vilms’ theorem in the Euclidean 3-dimensional space is a theorem of Hoffman-Osserman-Shoen (HOS’ theorem for short), which reads:

**Theorem (Hoffman-Osserman-Schoen).** Let $S$ be a complete surface of constant mean curvature immersed in $\mathbb{R}^3$. If the image of the Gauss map of $S$ lies in a hemisphere, then $S$ is a plane or a cylinder.

**Sketch of the proof.** By hypothesis, there is $V \in \mathbb{S}^2$ such that $u := \langle N, V \rangle \geq 0$; from (1) it follows that the lift $\tilde{u}$ of $u$ to the universal covering $\tilde{S}$ of $S$ is a bounded superharmonic function on $\tilde{S}$. If $\tilde{S}$ has the conformal type of the plane, then $u$ must be constant and then $S$ is a plane or a cylinder. If $\tilde{S}$ has the conformal type of the disk, then, by the maximum principle, either $\tilde{u} > 0$ everywhere or $\tilde{u} \equiv 0$. But from (1) we see that $\tilde{u}$ satisfies the PDE $\Delta \tilde{u} - 2K\tilde{u} + P = 0$, where $K$ is the sectional curvature of $S$ and $P = 4H^2 \geq 0$. This contradicts Corollary 3 of [FS], which asserts that this PDE has no positive solutions if $\tilde{S}$ is conformal to the disk. □

H. Rosenberg and J. Espinar in [ER] remarked that in product spaces $M^2 \times \mathbb{R}$ the condition that the Gauss map is contained in a hemisphere can be interpreted as that the angle function $\nu = \langle \eta, \partial_t \rangle$ has a sign, where $\eta$ is a unit normal vector on the surface. They then classified all these CMC surfaces in terms of the infimum $c(S)$ of the sectional curvature at the points of $M$ that are on the projection of the surface $S$. Precisely, they proved that if $c(S) \geq 0$ and $H \neq 0$, then $S$ is a cylinder over a complete curve with curvature $2H$. If $H = 0$ and $c(S) \geq 0$, then $S$ must be either a vertical plane, a slice $M \times \{t\}$, or $M = \mathbb{R}^2$ with the flat metric. We note that when $M = \mathbb{R}^2$ these results recover HOS’ theorem. When $c(S) < 0$ and $H > \sqrt{-c(S)/2}$, then $S$ is invariant under the group of isometries generated by the Killing field $\partial_t$ and is a vertical cylinder over a complete curve on $M^2$ of constant geodesic curvature $2H$.

In [BR], using the extension of Ruh-Vilms’ theorem to $\mathbb{S}^3$ and to $\mathbb{S}^2 \times \mathbb{R}$, an extension of HOS’ theorem to these ambient spaces is obtained. In the present paper, with the extension of Ruh-Vilms’ theorem and, hence, of Corollary 3.4 of [BR] to any symmetric space, HOS’ theorem was extended to include the ambient spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$ as well. We note that an extension of HOS’ theorem to the hyperbolic space, despite all these previous results, has not been obtained via a Gauss map so far.

We think that it is important at this point to make two observations. Our paper and the one of Espinar and Rosenberg both extend HOS’ theorem to $\mathbb{H}^2(-1) \times \mathbb{R}$, and both require a lower bound for the mean curvature. In [ER] it is $H > 1/2$, which is better than the one that follows from our result, namely, $H \geq 1/\sqrt{2}$. The lower bound [ER] is in fact optimal among CMC surfaces in ambient spaces of the
form $M^2 \times \mathbb{R}$. Our case is optimal among CMC surfaces in 3-dimensional symmetric spaces since in $\mathbb{H}^3(-1)$ the lower bound is 1, which is optimal (see the last remark of the paper).

Secondly, Espinar/Rosenberg’s paper gives a description of a CMC surface in terms of the angle that the normal vector of the surface makes with the Killing field $\partial_t$. In the present paper one can replace $\partial_t$ by any Killing field of $\mathbb{H}^2(-1) \times \mathbb{R}$ (and $S^2(1) \times \mathbb{R}$): If $\mathcal{N}(S)$ is included in a hemisphere of the unit pseudo-sphere of the Lie algebra of $SO(1, 2) \times \mathbb{R}$ determined by a vector $X$ (that is, $\langle \eta, X \rangle \geq 0$), then the surface is invariant by the Killing field of $\mathbb{H}^2 \times \mathbb{R}$ induced by $X$. For example, if

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 0,$$

then the surface is rotationally symmetric around a vertical geodesic and, then, is generated by an ODE solution curve of a totally geodesic plane containing a vertical line.

Another application of Ruh-Vilms’ theorem in $\mathbb{R}^3$ is the well-known classical Hopf’s theorem ([Ho]), namely:

*The round sphere is the only CMC topological sphere in $\mathbb{R}^3$.*

*Sketch of the proof.* If $S$ is a CMC surface in $\mathbb{R}^3$, then Ruh-Vilms’ theorem implies that the Gauss map $\mathcal{N}$ of $S$ is harmonic. Then $\mathcal{N}$ induces a quadratic holomorphic $q$ form in $S$ (see 10.5 of [EL]) which coincides with the so-called Hopf differential, as is easy to see. Hence if $S$ has zero genus, $q$ must be zero everywhere, which implies that $S$ is totally umbilic and then a round sphere. $\square$

Concerning Hopf’s theorem, U. Abresch and H. Rosenberg in [AR] extended it to CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and to $S^2 \times \mathbb{R}$ defining an ad hoc quadratic form $Q$ in these spaces (presently known as the Abresch-Rosenberg quadratic form), namely:

$$Q = 2HA - T, \text{ resp. } Q = 2HA + T,$$

where $H$ is the mean curvature of the surface, $A$ is the Hopf differential and $T = (dh \otimes dh)^{2,0}$, with $h$ standing for the height function. They prove that $Q$ is holomorphic when the surface is CMC. In particular, $Q \equiv 0$ holds if $S$ is a CMC topological sphere; from this fact, they obtain that a CMC sphere is rotationally symmetric.

Abresch and Rosenberg’s result raised a natural question of whether their quadratic form $Q$ could be induced by a geometric Gauss map for surfaces in the spaces $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$, with this Gauss map having the property of being harmonic if (and only if hopefully) the surface has constant mean curvature.

This question has been answered in the affirmative first in the space $\mathbb{H}^2 \times \mathbb{R}$ and for CMC $1/2$ surfaces by I. Fernández and P. Mira in [FM]. They introduce the *hyperbolic Gauss map* $G : S \to \mathbb{H}^2$ for any surface $S \subset \mathbb{H}^2 \times \mathbb{R}$ nowhere vertical and show that if $S$ has CMC $H = 1/2$, then $G$ is harmonic. Moreover, its induced holomorphic quadratic differential in the surface coincides (up to a sign) with the Abresch-Rosenberg form. But they go further and use these previous results to
obtain another quite interesting part of their work: To prove the existence of CMC $1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with prescribed hyperbolic Gauss map and to show that any holomorphic quadratic differential on an open simply connected Riemann surface can be realized as the Abresch-Rosenberg differential of some complete surfaces with $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$.

As far as the authors know, the above question in the space $\mathbb{S}^2 \times \mathbb{R}$ was open until recently when M. L. Leite and the second author of the present article proved in [LR] that the quadratic form induced by the Gauss map defined in [BR] coincided with the Abresch-Rosenberg form on CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$. Moreover, they used this Gauss map to motivate an ad hoc construction of a Gauss map in $\mathbb{H}^2 \times \mathbb{R}$ and obtained the same result.

With the Gauss map constructed here and with Theorem 1, we have the following unifying result: The quadratic form induced by the Gauss map in a surface immersed in a space of constant sectional curvature coincides with Hopf’s quadratic form and in a surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ and in $\mathbb{S}^2 \times \mathbb{R}$ coincides with the Abresch-Rosenberg quadratic form; moreover, the surfaces have CMC and these forms are holomorphic if and only if their Gauss maps are harmonic. It seems to the authors that the converse of this equivalence is not necessarily true for the Gauss map constructed in [FM].

To close this introduction, we observe that generalizations of the Gauss map have been defined in many different spaces and in many different ways. These generalizations have been proved to be particularly useful in describing and understanding CMC surfaces in the 8 models of Thurston’s geometries and more recently in a broad class of 3-dimensional Lie groups endowed with a left invariant metric.

Quite interesting and deep results have been obtained in a series of papers by B. Daniel [Da2], by B. Daniel, I. Fernández and P. Mira [DFM], and by B. Daniel and Mira [DM] and its generalization by W. Meeks III in [Mc]. We finally mention joint works of W. Meeks III, P. Mira, J. Pérez and A. Ros [MP, MMPR, MMPR2], where, using the left invariant Gauss map on a metric Lie group (i.e. a Lie group endowed with a left invariant metric), the authors are able to show strong results concerning CMC spheres on these ambient spaces.

Since all the previous results hold in 3-dimensional ambient spaces, but do not include the hyperbolic space, we think that the main contribution of the present paper is the construction of a Gauss map in symmetric spaces of any dimension, to extend Ruh-Vilms’ theorem to these spaces, and to obtain a broader version of HOS’ theorem which includes, in particular, the hyperbolic space. Moreover, the Gauss map introduced in the present paper might be used, hopefully, to obtain similar results as those in the works mentioned in the previous paragraph.

This paper is organized as follows: In Section 2.2 a Gauss map $N$ for hypersurfaces of a symmetric space is introduced and it is proved that an orientable hypersurface $M \hookrightarrow N$ has CMC if and only if $N$ is harmonic (Corollary 1). In Section 2.3 we obtain explicit formulas for $N$ when the ambient space is $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$.

In Section 3 we study the particular case when $N$ has dimension 3 and we analyze the quadratic complex form induced by $N$, denoted by $Q_N$. We then obtain that $Q_N$ coincides with the Hopf differential when $N$ is $\mathbb{H}^3$, $\mathbb{R}^3$ or $\mathbb{S}^3$ and with the Abresch-Rosenberg quadratic form when $N$ is $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.

Finally, In Section 4 we use the Gauss map $N$ to extend HOS’ theorem when $M$ is a surface immersed in a symmetric space of dimension 3.
2. THE GENERALIZED GAUSS MAP OF A HYPERSURFACE ON A SYMMETRIC SPACE

In this section we introduce the definition and discuss some aspects of the Gauss map $N$ of a hypersurface $M^{n-1}$ immersed in a symmetric space $N$. We use the same construction as in [BR] for hypersurfaces in $G/K \times \mathbb{R}^n$ but instead of asking for a bi-invariant Riemannian metric on $G$, we show that $N = G/K$ is the quotient of a group $G$ acting transitively on $N$ via isometries, $K$ is the isotropy subgroup of $G$ at a fixed point of $N$ and $G$ admits naturally a bi-invariant pseudo-Riemannian metric. We relate the Laplacian of $N$ and the mean curvature of $M$ and as a consequence obtain that $N$ is harmonic if and only if $M$ has constant mean curvature. We finish the section by giving an explicit formula for $N$ in space forms.

Throughout the paper a hypersurface is always understood as being immersed. We will refer to the generalized Gauss map simply as the Gauss map.

2.1. Preliminaries. Let $N$ be a Riemannian symmetric space. We begin by observing that $N$ is isometric to a quotient $N = G/K$, where $G$ is endowed with a bi-invariant pseudo-Riemannian metric and the metric in $G/K$ is the one induced by the projection $\pi : G \to G/K$ such that it becomes a pseudo-Riemannian submersion.

Indeed: Assume, at first, that $N$ is an irreducible symmetric space. Let $G = \text{ISO}(N)^0$ be the connected component of the identity on the isometry group of $N$ and set $K$ as the isotropy group of some fixed point on $N$. Then $N$ is isometric to $G/K$, where the metric on $G/K$ (up to a multiple factor) is the descent of the Killing form of the Lie algebra of $G$, which is a bi-invariant pseudo-Riemannian metric (see [He]).

Now, if $N$ decomposes as the Riemannian product of irreducible symmetric spaces with an $\mathbb{R}^m$ factor

\[ N = N_1 \times N_2 \times \ldots \times N_l \times \mathbb{R}^m, \]

each $N_i = G_i/K_i$ can be written as above. Then, if we set $G = G_1 \times \ldots \times G_l \times \mathbb{R}^m$ and $K = K_1 \times \ldots \times K_l \times \{0\}$, it follows that $N$ is isometric to the quotient $G/K$. Since the metric of $\mathbb{R}^m$ is bi-invariant and the Riemannian product of bi-invariant metrics is also bi-invariant, the claim is proved.

Herein we will assume that $G$ is endowed with a bi-invariant pseudo-Riemannian metric that descends onto $G/K$ as a Riemannian metric via the projection $\pi$. We also assume that $\dim(G) = n + k$, where $n = \dim(N)$ and $k = \dim(K)$, and denote by $\mathfrak{g}$ the Lie algebra of $G$. These assumptions on $G$ and $G/K$ will be assumed throughout the paper.

Each element $g \in G$ acts on $G/K$ as an isometry via

\[ g(\pi(x)) = \pi(L_g(x)) = \pi(R_x(g)), \quad x \in G, \]

and this action is transitive, where $L_g$ and $R_x$ are the left and the right translations on $G$. Any vector $V \in \mathfrak{g}$ defines a Killing vector field on $G/K$, here denoted by $\zeta(V)$, namely

\[ \zeta(V)(p) = \left. \frac{d}{dt} (\exp tV)(p) \right|_{t=0}, \quad p \in G/K, \]

where $\exp : \mathfrak{g} \to G$ is the Lie exponential map.
Let \( p \in \mathbb{G}/\mathbb{K} \) and let \( x \in \pi^{-1}(p) \). By (2) we have
\[
\exp(tV)(p) = \exp(tV)(\pi(x)) = \pi(R_x(\exp(tV)))
\]
and then
\[
\zeta(V)(p) = d\pi_x(d(R_x)c(V)).
\]

Given \( x \in \mathbb{G} \), a vector \( u \in T_x\mathbb{G} \) is called \textit{vertical} if \( u \in T_x\mathbb{K} \) and it is called \textit{horizontal} if \( u \in (T_x\mathbb{K})^\perp \). It follows that a vector \( u \in T_x\mathbb{G} \) is vertical if and only if its projection \( d\pi_x(u) = 0 \).

We now follow the construction of [BR]. For \( x \in \mathbb{G} \), set \( \ell_x := d\pi_x|_{(T_x\mathbb{K})^\perp} \). By definition, \( \ell_x \) is a linear isometry between horizontal vectors on \( T_x\mathbb{G} \) and \( T_{\pi(x)}(\mathbb{G}/\mathbb{K}) \). We then define \( \Gamma \) on \( T(\mathbb{G}/\mathbb{K}) \) by
\[
\Gamma_p : T_p\mathbb{G}/\mathbb{K} \quad \mapsto \quad \mathfrak{g} \quad \quad u \quad \mapsto \quad d(R_{x^{-1}})_x\ell_x^{-1}(u),
\]
where \( x \) is any point on \( \pi^{-1}(p) \) and \( p \in \mathbb{G}/\mathbb{K}. \)

\textbf{Proposition 1.} For each \( p \in \mathbb{G}/\mathbb{K} \), the map \( \Gamma_p \) is well defined, is linear and preserves the metric.

\textit{Proof.} Consider \( x, y \in \pi^{-1}(p) \). There exists \( h \in \mathbb{K} \) such that \( x = R_h(y) \). Then, for any \( u \in T_{\pi_p\mathbb{G}/\mathbb{K}} \), we have
\[
u = d\pi_y\ell_y^{-1}(u) = d(\pi \circ R_h)_y\ell_y^{-1}(u) = d\pi_x d(R_h)_y\ell_y^{-1}(u).
\]

Since \( h \in \mathbb{K} \), \( R_h \) is an isometry of \( \mathbb{G} \) that additionally preserves horizontality. From the previous equation we obtain \( \ell^{-1}_x(u) = d(R_h)_y\ell_y^{-1}(u) \) and hence
\[
d(R_{x^{-1}})_x\ell_x^{-1}(u) = d(R_{x^{-1}})_x d(R_h)_y\ell_y^{-1}(u)
= d(R_{x^{-1}} \circ R_h)_y\ell_y^{-1}(u)
= d(R_{y^{-1}})_y\ell_y^{-1}(u),
\]
which proves that \( \Gamma_p \) is well defined. That it is linear and preserves the metric follows directly from the definition of \( \ell_x \) and from the fact that the projection is a pseudo-Riemannian submersion. \( \square \)

We may now define the Gauss map of an oriented hypersurface \( M \) of \( N \) by setting
\[
\mathcal{N} : \quad M \quad \mapsto \quad S^{n+k-1} \subseteq \mathfrak{g} \quad \quad p \quad \mapsto \quad \Gamma_p(\eta(p)),
\]
where \( \eta \) is a fixed unit normal vector field on \( M \).

The next result gives a characterization of the Lie subgroups of \( \mathbb{G} \) that preserve \( M \) in terms of the Gauss map of \( M \). This proposition is fundamental for the paper.

\textbf{Proposition 2.} Let \( M^{n-1} \) be an orientable hypersurface of \( \mathbb{G}/\mathbb{K} \) and let \( \mathcal{N} : M \to S^{n+k-1} \subseteq \mathfrak{g} \) be its Gauss map. Then \( \mathcal{H} := (\mathcal{N}(M))^\perp = \{ w \in \mathfrak{g} ; \langle w, \mathcal{N}(p) \rangle = 0 \forall p \in M \} \) is a Lie subalgebra of \( \mathfrak{g} \) and \( M \) is invariant under the Lie subgroup \( \mathbb{H} \) of \( \mathbb{G} \) whose Lie algebra is \( \mathcal{H} \). Conversely, if \( M \) is invariant under a Lie subgroup \( \mathbb{H} \) of \( \mathbb{G} \), then \( \mathcal{H} \subseteq (\mathcal{N}(M))^\perp \), where \( \mathcal{H} \) is the Lie algebra of \( \mathbb{H} \).
\textbf{Proof.} First we notice that if \( w \in (\mathcal{N}(M))^\perp \), then, for all \( p \in M \),
\[
0 = \langle w, \mathcal{N}(p) \rangle \\
= \langle d(R_x)_e w, \ell_x^{-1}(\eta(p)) \rangle \\
= \langle \zeta(w)(p), \eta(p) \rangle,
\]
so \( \zeta(w)(p) \in T_pM \) and therefore \( \zeta(w) \) is a vector field tangent to \( M \). Now if \( v, w \in \mathcal{N}(M)^\perp \), then \( \zeta(v), \zeta(w) \) are two vector fields on \( M \), and thus \( \langle \zeta(v), \zeta(w) \rangle \) is also a vector field on \( M \). Since \( \langle \zeta(v), \zeta(w) \rangle = \zeta([v, w]) \), for \( p \in M \) we have that
\[
0 = \langle \zeta([v, w])(p), \eta(p) \rangle \\
= \langle \ell_x^{-1}(\zeta([v, w])(p)), \ell_x^{-1}(\eta(p)) \rangle.
\]
But we also have
\[
\ell_x^{-1}(\zeta([v, w])) = \ell_x^{-1}d\pi_x d(R_x)_e[v, w] \\
= (d(R_x)_e[v, w])^h,
\]
and then
\[
0 = \langle [v, w], \mathcal{N}(p) \rangle,
\]
proving that \( [v, w] \in \mathcal{N}(M)^\perp \). Hence \( \mathcal{H} \) is a Lie subalgebra of \( \mathfrak{g} \).

Now let \( \mathbb{H} \) be a subgroup of \( \mathbb{G} \) that leaves \( M \) invariant and let \( \mathcal{H} \) be the Lie algebra of \( \mathbb{H} \). Then \( \mathcal{H} \) acts on \( M \) as Killing fields and therefore \( \langle \zeta(\mathcal{H}), \eta \rangle = 0 \). It follows that
\[
0 = \langle \zeta(\mathcal{H}), \eta \rangle = \langle \mathcal{H}, \mathcal{N} \rangle,
\]
proving that \( \mathcal{H} \subseteq \mathcal{N}(M)^\perp \).

\section*{2.2. Harmonicity of \( \mathcal{N} \) and the mean curvature of \( M \).}
It is well known that a hypersurface of \( \mathbb{R}^{n+1} \) has constant mean curvature if and only if its Gauss map is harmonic. This follows directly from the well-known formula
\[
\Delta \mathcal{N} = -\text{grad}H - \|B\|^2 \mathcal{N},
\]
where \( \|B\| \) is the norm of the second fundamental form \( B \) of \( M \).

This formula was extended to hypersurfaces in a Lie group in [EFFR] and to a homogeneous space \( \mathbb{G}/\mathbb{H} \) where \( \mathbb{G} \) has a Riemannian bi-invariant metric and \( \mathbb{H} \) is a closed subgroup in [BR]. We will now present a more general formula for the Laplacian of the Gauss map given by [G].

\textbf{Theorem 1.} Let \( M \) be an immersed orientable hypersurface of \( \mathbb{G}/\mathbb{K} \) and let \( \mathcal{N} : M \to S^{n+k-1} \subseteq \mathfrak{g} \) be the Gauss map of \( M \), where \( \mathfrak{g} \) is the Lie algebra of \( \mathbb{G} \). Then
\[
\Delta \mathcal{N}(p) = -n\Gamma_p(\text{grad}H) - (\|B\|^2 + \text{Ric}(\eta)) \mathcal{N}(p)
\]
for all \( p \in M \), where \( \eta \) is a normal vector field satisfying \( \langle \eta, \eta \rangle = 1 \), \( \text{Ric}(\eta) \) is the Ricci curvature of \( \mathbb{G}/\mathbb{K} \) with respect to \( \eta \) and \( \|B\| \) is the norm of the second fundamental form \( B \) of \( M \) in \( \mathbb{G}/\mathbb{K} \).

\textbf{Proof.} Fix \( V \in \mathfrak{g} \) and define the function
\[
f_V : M \to \mathbb{R} \\
p \mapsto \langle \mathcal{N}(p), V \rangle.
\]
For any \( p \in M \) we have \( f_V(p) = \langle \mathcal{N}(p), V \rangle = \langle \eta(p), \zeta(V)(p) \rangle \). Since \( \zeta(V) \) is a Killing field on \( \mathbb{G}/\mathbb{K} \), it follows from Proposition 1 of [FR] that
\[
\Delta f_V = -n\langle \text{grad}H, \zeta(V) \rangle - (\|B\|^2 + \text{Ric}(\eta)) f_V.
\]
But we have that \( (\text{grad}H, \zeta(V)) = \langle \Gamma_p(\text{grad}(H)), V \rangle \), and then
\[
(\Delta \mathcal{N}(p), V) = \Delta f_V = \langle -n\Gamma_p(\text{grad}H) - (\|B\|^2 + \text{Ric}(\eta)) \mathcal{N}(p), V \rangle.
\]
Since (11) holds for any \( V \in \mathfrak{g} \) we have (8), proving the theorem. \( \square \)

**Corollary 1.** Let \( M \) be an orientable hypersurface of \( \mathbb{G}/\mathbb{K} \) and let \( \mathcal{N} : M \to \mathbb{S}^{n+k-1} \subseteq \mathfrak{g} \) be the Gauss map of \( M \). Then the following alternatives are equivalent:

- i) \( M \) has constant mean curvature.
- ii) The Gauss map \( \mathcal{N} : M \to \mathbb{S}^{n+k-1} \) is harmonic.
- iii) \( \mathcal{N} \) satisfies the equation

\[
\Delta \mathcal{N}(p) = -\left( \|B\|^2 + \text{Ric}(\eta) \right) \mathcal{N}(p).
\]

### 2.3. The Gauss map on spaces of constant sectional curvature.

In the Euclidean case, our Gauss map coincides with the usual one, as the horizontal lift is simply the identity. We then pass to consider the spherical and hyperbolic cases.

**The Gauss map of \( M^{n-1} \) immersed in \( \mathbb{S}^n \).** Let \( O(n+1) \) be the orthogonal group of isometries of \( \mathbb{R}^{n+1} \) that fixes the origin. The Lie algebra \( \mathfrak{o}(n+1) \) of \( O(n+1) \) consists of the \( (n+1) \times (n+1) \) matrices \( u \) satisfying \( u + u^T = 0 \), where \( u^T \) denotes the transpose of the matrix \( u \). Consider the bi-invariant metric on \( O(n+1) \) given by

\[
\langle u, v \rangle = \frac{1}{2} \text{tr}(uv^T) = -\frac{1}{2} \text{tr}(uv), \ u, v \in \mathfrak{o}(n+1).
\]

Then \( O(n+1)/O(n) \) is isometric to the unit sphere \( \mathbb{S}^n \) centered at the origin of \( \mathbb{R}^{n+1} \), where \( O(n) \) is the subgroup of matrices \( A \) of \( O(n+1) \) such that \( Ae_1 = e_1 \), \( \{e_1, e_2, \ldots, e_{n+1}\} \) being the canonical basis of \( \mathbb{R}^{n+1} \). We obtain next an explicit expression for \( \Gamma : T\mathbb{S}^n \to \mathfrak{o}(n+1) \).

Choose \( p = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{S}^n \). Let \( \{v_2, v_3, \ldots, v_{n+1}\} \) be an orthogonal basis of \( T_p\mathbb{S}^n \) such that the matrix \( (p v_2 v_3 \ldots v_{n+1}) \in O(n+1) \). Then we define

\[
x = \begin{pmatrix}
x_1 & v_{12} & \cdots & v_{1n+1} \\
x_2 & v_{22} & \cdots & v_{2n+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n+1} & v_{n+12} & \cdots & v_{n+1n+1}
\end{pmatrix},
\]

where \( v_j = \sum_{i=1}^{n+1} v_{ij}e_i \in \mathbb{R}^{n+1} \) and it follows that \( \pi(x) = p \).

Now, let \( u = (u_1, u_2, \ldots, u_{n+1}) \in T_p\mathbb{S}^n \) and write \( u = \sum_{i=2}^{n+1} (u \cdot v_i)v_i \), where \( (\cdot) \) is the inner product of \( \mathbb{R}^{n+1} \). Let \( Z \in \mathfrak{o}(n)^\perp \) be given by

\[
Z = \begin{pmatrix}
0 & -(u \cdot v_2) & \cdots & -(u \cdot v_{n+1}) \\
(u \cdot v_2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(u \cdot v_{n+1}) & 0 & \cdots & 0
\end{pmatrix}
\]

and set \( \tilde{u} = d(L_x)_e Z \in (T_xO(n))^\perp \). In coordinates, \( \tilde{u} = x.Z \) is the usual matrix multiplication and is represented as

\[
\tilde{u} = \begin{pmatrix}
U_1 & -x_1(u \cdot v_2) & \cdots & -x_1(u \cdot v_{n+1}) \\
U_2 & -x_2(u \cdot v_2) & \cdots & -x_2(u \cdot v_{n+1}) \\
\vdots & \vdots & \ddots & \vdots \\
U_{n+1} & -x_{n+1}(u \cdot v_2) & \cdots & -x_{n+1}(u \cdot v_{n+1})
\end{pmatrix},
\]
where

\[ U_i = \sum_{j=2}^{n+1} v_{ij}(u \cdot v_j). \]

Now, we claim \( \tilde{u} \) is the horizontal lift of \( u \). To see this, just apply the projection:

\[
d\pi_x(\tilde{u}) = \sum_{i=1}^{n+1} U_i e_i = \sum_{i=1}^{n+1} \sum_{j=2}^{n+1} v_{ij}(u \cdot v_j)e_i = \sum_{j=2}^{n+1} (u \cdot v_j) \sum_{i=1}^{n+1} v_{ij}e_i = \sum_{j=2}^{n+1} (u \cdot v_j)v_j = u.
\]

This equation shows not only that \( \tilde{u} \) is the horizontal lift of \( u \) on \( T_xO(n+1) \), but also that \( U_i = (u \cdot e_i) = u_i \). Then, it becomes simple to find an expression for \( \Gamma_p(u) = d(R_{x^{-1}})_x(\tilde{u}) = \tilde{u}x^{-1} \). As \( x \in O(n+1) \) we have that \( x^{-1} = x^T \). Using again that \( U_i = u_i \), the matrix expression for \( \Gamma_p(u) \) is

\[
\Gamma_p(u) = \begin{pmatrix}
0 & u_1x_2 - u_2x_1 & \ldots & u_1x_{n+1} - u_{n+1}x_1 \\
u_2x_1 - u_1x_2 & 0 & \ldots & u_2x_{n+1} - u_{n+1}x_2 \\
\vdots & \vdots & \ddots & \vdots \\
u_{n+1}x_1 - u_1x_{n+1} & u_{n+1}x_2 - u_2x_{n+1} & \ldots & 0
\end{pmatrix}.
\]

If we let \( \Phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to M_{n+1}(\mathbb{R}) \) be given by

\[
(13) \quad \Phi(x, y) = \begin{pmatrix}
x_1 & 0 & \ldots & 0 \\
0 & x_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & x_{n+1}
\end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1
\end{pmatrix} \begin{pmatrix} y_1 & 0 & \ldots & 0 \\
y_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
y_{n+1} & 0 & \ldots & y_{n+1}
\end{pmatrix}
\]

then we can write

\[
(14) \quad \Gamma_p(u) = \Phi(u, p) - \Phi(p, u).
\]

We then obtain an explicit matrix expression for the Gauss map of a hypersurface of \( S^n \):

**Proposition 3.** Let \( M^{n-1} \) be an orientable hypersurface of \( S^n \) oriented with respect to a normal unit vector field \( \eta \). Let \( \mathcal{N} : M \to S^{(n+1)n-1} \subseteq o(n+1) \) be the Gauss map of \( M \). Then

\[
(15) \quad \mathcal{N}(p) = \Phi(\eta(p), p) - \Phi(p, \eta(p)),
\]

where \( \Phi \) is given by (13).
The Gauss map of $M^{n-1}$ immersed in $\mathbb{H}^n$. Consider the pseudo-inner product $(\ast)$ on $\mathbb{R}^{n+1}$ given by

$$(x \ast y) = -x_1y_1 + x_2y_2 + \ldots + x_{n+1}y_{n+1}.$$ 

Let us introduce the following notation: For $i = 1, 2, \ldots, n+1$, let $\xi_1 = -1$ and $\xi_i = 1$ otherwise. Then we can write $(\ast)$ as

$$(x \ast y) = \sum_{i=1}^{n+1} \xi_i x_i y_i.$$ 

In the Lorentz space $\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, (\ast))$, $H^n := \{x \in \mathbb{L}^{n+1}; (x \ast x) = -1 \text{ and } x_1 > 0\}$, endowed with the metric of $\mathbb{L}^{n+1}$, is the hyperbolic space with constant sectional curvature $-1$. Consider $O(1, n) = \{g \in M_{n+1}(\mathbb{R}); (gx \ast gy) = (x \ast y), \forall x, y \in \mathbb{L}^{n+1} \text{ and } g(H^n) = H^n\}.$

In terms of matrices, the property that characterizes $O(1, n)$ is $M \in O(1, n) \iff M^{-1} = \tilde{I}M^T\tilde{I}$, where

$$\tilde{I} = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$ 

The Lie algebra of $O(1, n)$, denoted by $\mathfrak{o}(1, n)$, can be written as

$$\mathfrak{o}(1, n) = \left\{ \begin{pmatrix} 0 & a_1 & \ldots & a_n \\ a_1 & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_A \\ a_n & \ldots & a_1 & 0 \end{pmatrix} : A \in \mathfrak{o}(n), a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}.$$ 

Note that $u = (u_{ij}) \in \mathfrak{o}(1, n) \iff u_{ij} = -\xi_i\xi_j u_{ji}$. We introduce a pseudo-Riemannian bi-invariant metric $\langle \ , \ \rangle$ on $O(1, n)$ by extending the nondegenerate bilinear form $\langle u, v \rangle = \frac{1}{2} \text{tr}(uv)$ on $\mathfrak{o}(1, n)$ to $O(1, n)$ via left translations.

With such a metric, setting $O(n) = \{x \in O(1, n); g(e_1) = e_1\}$, $\mathbb{H}^n$ is isometric to the quotient $O(1, n)/O(n).$ In the next result we obtain an explicit expression for $\Gamma : T\mathbb{H}^n \rightarrow \mathfrak{o}(1, n)$:

**Lemma 1.** Let $p \in \mathbb{H}^n$. Then, if $u \in T_p\mathbb{H}^n$, it holds that

$$\Gamma_p(u) = \Psi(p, u) - \Psi(u, p),$$

(16)
where $\Psi : \mathbb{L}^{n+1} \times \mathbb{L}^{n+1} \to M_{n+1}(\mathbb{R})$ is given by

\begin{equation}
\Psi(x, y) = \begin{pmatrix}
x_1 & 0 & \ldots & 0 \\
0 & x_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & x_{n+1}
\end{pmatrix}
\begin{pmatrix}
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 & 0 & \ldots & 0 \\
y_2 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & y_{n+1}
\end{pmatrix}
= \begin{pmatrix}
-y_1x_1 & y_2x_1 & \ldots & y_{n+1}x_1 \\
-y_1x_2 & y_2x_2 & \ldots & y_{n+1}x_2 \\
\vdots & \ddots & \ddots & \vdots \\
-y_1x_{n+1} & y_2x_{n+1} & \ldots & y_{n+1}x_{n+1}
\end{pmatrix}.
\end{equation}

Proof. The proof is similar to the spherical case. We write down some of the steps. Set $p = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{H}^n$ and $u = (u_1, u_2, \ldots, u_{n+1}) \in T_p\mathbb{H}^n$. Let $\{v_2, v_3, \ldots, v_{n+1}\}$ be an orthogonal basis of $T_p\mathbb{H}^n$ such that the matrix $(p \cdot v_2 v_3 \ldots v_{n+1}) \in O(1, n)$. Write each $v_j$ in coordinates as $v_j = (v_{1j}, v_{2j}, \ldots, v_{n+1j})$ and define

\begin{equation}
x = \begin{pmatrix}
x_1 & v_{12} & \ldots & v_{1n+1} \\
x_2 & v_{22} & \ldots & v_{2n+1} \\
\vdots & \ddots & \ddots & \vdots \\
x_{n+1} & v_{n+12} & \ldots & v_{n+1n+1}
\end{pmatrix}.
\end{equation}

Then we have $x \in O(1, n)$ and $\pi(x) = p$. As in the spherical case, define $Z \in \mathfrak{o}(n)^\perp$ by

\begin{equation}
Z = \begin{pmatrix}
0 & (u \cdot v_2) & \ldots & (u \cdot v_{n+1}) \\
(u \cdot v_2) & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
(u \cdot v_{n+1}) & 0 & \ldots & 0
\end{pmatrix}.
\end{equation}

Then $d(L_x)_x Z \in (T_x O(n))^\perp$, $d\pi_x(xZ) = u$ and hence $\ell_x^{-1}(u) = xZ$. It follows that $\Gamma_p(u) = xZx^{-1}$. In terms of matrices,

\begin{equation}
\Gamma_p(u) = \begin{pmatrix}
0 & u_2x_1 - u_1x_2 & \ldots & u_{n+1}x_1 - u_1x_{n+1} \\
-u_1x_2 + u_2x_1 & 0 & \ldots & u_{n+1}x_2 - u_2x_{n+1} \\
-u_1x_3 + u_3x_1 & u_2x_3 - u_3x_2 & \ldots & u_{n+1}x_3 - u_3x_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
-u_1x_{n+1} + u_{n+1}x_1 & u_2x_{n+1} - u_{n+1}x_2 & \ldots & 0
\end{pmatrix}
= \Psi(p, u) - \Psi(u, p).
\end{equation}

Proposition 4. Let $M$ be a hypersurface of the hyperbolic space $\mathbb{H}^n$ oriented with respect to a unitary normal vector field $\eta$. Let $\mathcal{N} : M \to S^{(n+1)n-1} \subseteq \mathfrak{o}(1, n)$ be the Gauss map of $M$. Then

\begin{equation}
\mathcal{N}(p) = \Psi(p, \eta(p)) - \Psi(\eta(p), p)
\end{equation}

holds, where $\Psi$ is given in (17).
3. The quadratic form induced by \( \mathcal{N} \) on surfaces immersed in symmetric spaces of dimension 3

It is a classic result due to Heinz Hopf \([\text{Ho}]\) that in the Euclidean three space, the Hopf differential \( \mathcal{A} \) of a surface \( M \) (that is, the complexification of the traceless part of the second fundamental form of \( M \)) is holomorphic if and only if \( M \) has constant mean curvature. This result is also true in \( \mathbb{H}^3 \) and \( S^3 \) \([\text{Ch}]\), but it is false in general. In \([\text{AR}]\) U. Abresch and H. Rosenberg “perturbed” the Hopf differential and defined a quadratic differential form \( Q = 2H\mathcal{A} - c\mathcal{T} \) of a surface \( M \) immersed in \( M^2(c) \times \mathbb{R} \) (\( H \) is the mean curvature of \( M \), \( \mathcal{A} \) is the Hopf differential and \( \mathcal{T} = (dh \otimes dh)^{2,0} \), \( h \) standing for the height function), and then extended Hopf’s theorem to CMC spheres of these ambient spaces using \( \mathcal{Q} \) instead of \( \mathcal{A} \). More generally, in any homogeneous space of dimension 3 whose isometry group has dimension at least 4, there exists a quadratic form that is holomorphic for any CMC surface \([\text{AR2}]\) \([\text{FM2}]\), and in \( \text{Sol}_3 \) there exists a quadratic form (which is holomorphic in the case of a minimal surface) that plays an important role in proving the uniqueness of CMC H-spheres \([\text{DM}]\) \([\text{Me}]\).

In \( \mathbb{R}^3 \) the differential of the Gauss map \( g : M \to S^2 \) coincides (up to a sign) with the shape operator of the surface, and the complex quadratic form induced by \( g \) is the Hopf differential \( \mathcal{A} \). In \([\text{LR}]\), the authors used the Gauss map \( \mathcal{N} \) of a surface \( M \) in \( S^2 \times \mathbb{R} \), as defined in \([\text{BR}]\), to show that the quadratic form induced by \( \mathcal{N} \) was actually the Abresch-Rosenberg quadratic form \( \mathcal{Q} \). They also defined an “ad hoc” Gauss map \( \mathcal{N} \), which they called the twisted normal map, for a surface \( M \) in \( \mathbb{H}^2 \times \mathbb{R} \) and again obtained that the quadratic form induced by \( \mathcal{N} \) was equal to the Abresch-Rosenberg quadratic form \( \mathcal{Q} \) of \( M \).

In this section we will consider a surface \( M \) immersed in a 3-dimensional symmetric space \( N := \mathbb{G}/\mathbb{K} \) satisfying the assumptions of Section 2. It will be shown that the complex quadratic form induced by \( \mathcal{N} \) on \( M \) is the Hopf differential when \( N \) is \( \mathbb{H}^3 \), \( \mathbb{R}^3 \) or \( S^3 \) and the Abresch-Rosenberg quadratic form when \( N \) is \( \mathbb{H}^2 \times \mathbb{R} \) or \( S^2 \times \mathbb{R} \). Moreover, we show that the Gauss map \( \mathcal{N} \) coincides with the twisted normal map defined in \([\text{LR}]\), when \( N = \mathbb{H}^2 \times \mathbb{R} \).

Let \( M \) be an orientable surface in \( N \) oriented with respect to a normal unitary vector field \( \eta \). Let \( p \in M \) and let \( F : U \subseteq \mathbb{C} \to M \) be a conformal structure on a neighborhood of \( p \). If \( z = x + iy \) is a complex coordinate system, then

\[
\langle F_x, F_x \rangle = \langle F_y, F_y \rangle = E > 0 \text{ and } \langle F_x, F_y \rangle = 0,
\]

which implies

\[
\langle F_z, F_z \rangle = \langle F_{\overline{z}}, F_{\overline{z}} \rangle = 0 \text{ and } \langle F_z, F_{\overline{z}} \rangle = E/2.
\]

We notice that the lower index here denotes the usual derivatives and we are considering \( 2F_z = F_x - iF_y \). Under this notation, we define a tensor field \( Q \) by \( Q(X, Y)(p) = \langle d\mathcal{N}_p(X), \Gamma_p(Y) \rangle \) and the complex quadratic form induced by \( \mathcal{N} \) as

\[
\mathcal{Q}_N = (\langle \mathcal{N}^*, \Gamma \rangle)^{2,0} = \langle \mathcal{N}_z, \Gamma(F_z) \rangle dz^2.
\]

Now, if \( A_\eta \) is the shape operator of \( M \), the Hopf differential of \( M \) (see \([\text{Ho}]\)) is defined likewise:

\[
\mathcal{A} = \langle A_\eta(F_z), F_z \rangle dz^2.
\]

3.1. The quadratic form on \( S^3 \). First, we relate the derivative of the Gauss map of a surface \( M \) in \( S^3 \) with the shape operator of \( M \).
Proposition 5. Let $M$ be an orientable surface in $\mathbb{S}^3$ oriented with respect to a normal unitary vector field $\eta$ and let $N : M \to \mathbb{S}^5 \subseteq \mathfrak{o}(4)$ be its Gauss map. Then, for any $p \in M$ and $X, Y \in T_p M$,
\[
\langle dN_p(X), \Gamma_p(Y) \rangle = -\langle A_\eta(X), Y \rangle
\]
holds, where $A_\eta$ is the shape operator of $M$.

Proof. Let $M$ be as above. Let $p \in M$ and $X, Y \in T_p M$ and let $\alpha : (-\varepsilon, \varepsilon) \to M$ be such that $\alpha(0) = p$ and $\alpha'(0) = X$. Set $N(t) = N(\alpha(t))$ and $\eta(t) = \eta(\alpha(t))$. From Proposition 3 we have
\[
N(t) = \Phi(\eta(t), \alpha(t)) - \Phi(\alpha(t), \eta(t)).
\]

Hence
\[
dN_p(X) = -\Phi(A_\eta(X), p) + \Phi(\eta(p), X) - \Phi(X, \eta(p)) + \Phi(p, A_\eta(X)),
\]
as $\eta'(0) = \nabla_X \eta = -A_\eta(X)$. On the other hand, we also have $\Gamma_p(Y) = \Phi(Y, p) - \Phi(p, Y)$. A useful (and easy to check) identity concerning $\Phi$ is
\[
\text{tr}(\Phi(x, u)\Phi(y, v)) = (x \cdot v)(y \cdot u), \forall x, y, u, v \in \mathbb{R}^4,
\]
which implies the identities:
\[
\begin{align*}
\text{tr}(\Phi(A_\eta(X), p)\Phi(Y, p)) &= 0 & \langle A_\eta(X), Y \rangle &= \text{tr}(\Phi(A_\eta(X), p)\Phi(p, Y)) \\
\text{tr}(\Phi(\eta(p), X)\Phi(Y, p)) &= 0 & 0 &= \text{tr}(\Phi(\eta(p), X)\Phi(p, Y)) \\
\text{tr}(\Phi(X, p)\Phi(Y, p)) &= 0 & 0 &= \text{tr}(\Phi(X, \eta(p))\Phi(p, Y)) \\
\text{tr}(\Phi(p, A_\eta(X))\Phi(Y, p)) &= \langle A_\eta(X), Y \rangle & 0 &= \text{tr}(\Phi(p, A_\eta(X))\Phi(p, Y)).
\end{align*}
\]

It follows that
\[
\langle dN_p(X), \Gamma_p(Y) \rangle = -\frac{1}{2} \text{tr}(dN_p(X)\Gamma_p(Y)) = -\langle A_\eta(X), Y \rangle.
\]

An immediate consequence of Proposition 5 is a generalization of the result for the classical Gauss map, whose derivative coincides – up to a sign – with the shape operator. Here it is shown that the projection of $N^*$ back to the sphere coincides with the shape operator. More precisely, we have:

Corollary 2. Let $M$ be a surface in $\mathbb{S}^3$ oriented with respect to a unitary vector field $\eta$ normal to $M$ and let $N : M \to \mathbb{S}^5 \subseteq \mathfrak{o}(4)$ be its Gauss map. Then, for any $x \in O(4)$ such that $\pi(x) \in M$, it holds that
\[
d\pi_x d(R_x) \circ dN_{\pi(x)} = -A_\eta.
\]

We then have the following theorem.

Theorem 2. Let $M$ be a surface immersed in $\mathbb{S}^3$ and let $N : M \to \mathbb{S}^5 \subseteq \mathfrak{o}(4)$ be its Gauss map. Then the following alternatives are equivalent:

i) $M$ has constant mean curvature.

ii) $N$ is harmonic.

iii) The complex quadratic form $Q_N$ induced by $N$ on $M$ is holomorphic.
Proof. Let $F : U \subseteq \mathbb{C} \to M$ be a conformal structure on a neighborhood of a point $p \in M$. The complex quadratic form induced by $\mathcal{N}$ at $p$ is given by $Q_{\mathcal{N}}(p) = \langle N_z, \Gamma_p(F_z) \rangle dz^2$.

It follows from Proposition 5 that $Q_{\mathcal{N}}$ coincides (up to a sign) with the Hopf differential $A$ of $M$ on $S^3$. Therefore, $Q_{\mathcal{N}}$ is holomorphic if and only if $M$ has constant mean curvature [Ch]. The equivalence between CMC and harmonicity of the Gauss map had already been obtained in the more general case of Corollary 1.

This proves the theorem. □

3.2. The quadratic form on $\mathbb{H}^3$. Following the steps of the last section, we first relate the derivative of the Gauss map $\mathcal{N}$ with the shape operator of $M$. Then we obtain that the complex quadratic form induced by $\mathcal{N}$, $Q_{\mathcal{N}}$, coincides with the Hopf differential $A$ of $M$.

**Proposition 6.** Let $M$ be an orientable surface in $\mathbb{H}^3$ oriented by a normal unitary vector field $\eta$ and let $\mathcal{N} : M \to S^5 \subseteq \mathfrak{o}(1,3)$ be its Gauss map. Then, for any $p \in M$ and $X, Y \in T_pM$, it holds that

$$\langle d\mathcal{N}_p(X), \Gamma_p(Y) \rangle = -\langle A_\eta(X), Y \rangle.$$  

**Proof.** The proof of this proposition is analogous to the proof of Proposition 5 with the only difference that here one uses $(p * p) = -1$ and the equation

$$\text{tr}(\Psi(x,u)\Psi(y,v)) = (x \ast v)(y \ast u)$$

instead of (21). □

As a consequence, similar to the spherical case, we obtain:

**Corollary 3.** Let $M$ be an orientable surface in $\mathbb{H}^3$ oriented by a unitary vector field $\eta$ normal to $M$ and let $\mathcal{N} : M \to S^5 \subseteq \mathfrak{o}(1,3)$ be its Gauss map. Then, for any $x \in O(1,3)$ such that $\pi(x) \in M$, it holds

$$d\pi_x d(R_x)_e d\mathcal{N}_{\pi(x)} = -A_\eta.$$  

Observing that the quadratic form induced by $\mathcal{N}$ coincides with the Hopf differential $A$, we obtain an analogue of Theorem 2 in the hyperbolic space:

**Theorem 3.** Let $M$ be a surface immersed in $\mathbb{H}^3$ and let $\mathcal{N} : M \to S^5 \subseteq \mathfrak{o}(1,3)$ be its Gauss map. Then the following alternatives are equivalent:

i) $M$ has constant mean curvature.

ii) $\mathcal{N}$ is harmonic.

iii) The complex quadratic form $Q_{\mathcal{N}}$ induced by $\mathcal{N}$ on $M$ is holomorphic.

3.3. The quadratic form on $\mathbb{H}^2 \times \mathbb{R}$ and on $S^2 \times \mathbb{R}$. In this section we prove an analogous result to Theorems 2 and 3 for a surface $M$ immersed in a product space $S^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$. We will prove that if $M$ has constant mean curvature, then the quadratic form induced by $\mathcal{N}$ is holomorphic. In order to prove this result we will show that the complex quadratic form induced by the Gauss map of $M$ coincides with the Abresch-Rosenberg quadratic form. When $\mathcal{N} = S^2 \times \mathbb{R}$, our construction of the Gauss map coincides with the one in [BR], therefore Theorem 3.1 of [LR] shows this result. Thus, we focus on the case where $M$ is a surface immersed in $\mathbb{H}^2 \times \mathbb{R}$, and we relate $\mathcal{N}$ with the twisted normal map of $M$, introduced in [LR].
For an orientable surface $M$ in $\mathbb{H}^2 \times \mathbb{R}$ oriented with a vector field $(\eta, \nu)$ normal to $M$, the twisted normal map of $M$ is defined by (see [LR]):

$$N : M \rightarrow dS^3 \subseteq \mathbb{L}^3 \times \mathbb{R}$$

(22) \hspace{1cm} (p, t) \mapsto (J(\eta(p)), \nu),

where $J$ is the operator acting on tangent planes of $\mathbb{H}^2$ as the clockwise $\pi/2$ rotation. The next proposition shows that if $p \in \mathbb{H}^2$, then $\Gamma_p = J$.

**Proposition 7.** Let $p \in \mathbb{H}^2$ and let $v \in T_p \mathbb{H}^2 \subseteq \mathbb{L}^3$. Let $\{v_2, v_3\}$ be an orthogonal basis of $T_p \mathbb{H}^2$. If $u = av_2 + bv_3$, then $\Gamma_p(u) = -bv_2 + av_3$, via the identification $$(0 \hspace{0.5cm} -r \hspace{0.5cm} s)
-r \hspace{0.5cm} 0 \hspace{0.5cm} -t
s \hspace{0.5cm} t \hspace{0.5cm} 0) \in o(1, 2) \leftrightarrow (t, s, r) \in \mathbb{L}^3.

Remark. Since we have $\Gamma_{(p, t)}(u, \nu) = (\Gamma_p(u), \nu)$ in $\mathbb{H}^2 \times \mathbb{R}$, Proposition 7 shows that the Gauss map given by the expression (6) coincides with the twisted normal map defined by (22).

**Proof.** Let $p = (x_1, x_2, x_3) \in \mathbb{H}^2$ and $u = (u_1, u_2, u_3) \in T_p \mathbb{H}^2$. Then, by equation (16), it follows that

$$\Gamma_p(u) = \begin{pmatrix}
0 & u_2x_1 - u_1x_2 & u_3x_1 - u_1x_3 \\
u_2x_1 - u_1x_2 & 0 & u_3x_2 - u_2x_3 \\
u_3x_1 - u_1x_3 & u_2x_3 - u_3x_2 & 0
\end{pmatrix}.

Writing $v_j = (v_{1j}, v_{2j}, v_{3j})$ and making the substitution $u_i = av_{i2} + bv_{i3}$, the previous equality becomes

$$\Gamma_p(u) = a \begin{pmatrix}
0 & v_{22}x_1 - v_{12}x_2 & v_{32}x_1 - v_{13}x_2 \\
v_{22}x_1 - v_{12}x_2 & 0 & v_{32}x_2 - v_{13}x_3 \\
v_{32}x_1 - v_{13}x_2 & v_{22}x_3 - v_{12}x_2 & 0
\end{pmatrix}
+b \begin{pmatrix}
0 & v_{23}x_1 - v_{13}x_2 & v_{33}x_1 - v_{13}x_3 \\
v_{23}x_1 - v_{13}x_2 & 0 & v_{33}x_2 - v_{13}x_3 \\
v_{33}x_1 - v_{13}x_3 & v_{23}x_3 - v_{13}x_2 & 0
\end{pmatrix}
=a \begin{pmatrix}
0 & -v_{33} & v_{23} \\
v_{33} & 0 & -v_{13} \\
v_{23} & v_{13} & 0
\end{pmatrix}
+b \begin{pmatrix}
0 & v_{32} & -v_{22} \\
v_{32} & 0 & v_{12} \\
-v_{22} & -v_{12} & 0
\end{pmatrix}

= av_3 - bv_2.

\[\square\]

We then obtain

**Corollary 4.** If $N = \mathbb{H}^2 \times \mathbb{R}$, then the Gauss map defined by (6) coincides with the twisted normal map of [LR] given by (22).

This corollary implies (together with Theorems 3.1 and 3.3 of [LR]) the following result.
Proposition 8. Let $M$ be an orientable surface in $\mathcal{M}_2(\kappa) \times \mathbb{R}$ oriented w.r.t. a unitary vector field $(\eta, \nu)$ normal to $M$. Let $\mathcal{N}$ be the Gauss map of $M$ and let $Q_N$ be the complex quadratic form given on $\mathbb{R}$. Then

\[ Q_N = Q, \]

where $Q$ is the Abresch-Rosenberg quadratic form of $M$ ([AR]).

Now, from our construction of the Gauss map and Theorem 1 of [AR], we have

Theorem 4. Let $M$ be a surface immersed either in $S^2 \times \mathbb{R}$ or in $H^2 \times \mathbb{R}$. If $\mathcal{N}$ is the Gauss map of $M$, then there is an equivalence between

i) $M$ has constant mean curvature;
ii) $N$ is harmonic.

Moreover, both imply

iii) $Q_N$ is holomorphic on $M$.

Remark. The converse of this theorem is false. It was shown in [FM2] that the existence of certain rotational surfaces in $H^2 \times \mathbb{R}$ with holomorphic Abresch-Rosenberg differential fails to be CMC.

4. HOS’ theorem in symmetric spaces of dimension 3

Theorem 4.9 of [BR] proves HOS’ theorem for a complete CMC surface $M$ immersed in a 3-dimensional homogeneous space $G/K$ where $G$, up to an abelian factor, is compact. In particular, this result applies for $M$ immersed in $S^3$ and in $S^2 \times \mathbb{R}$. We now extend HOS’ theorem for surfaces immersed in a symmetric space $N = G/K$ as in the preliminaries of Section 2.

Theorem 5. Let $N = G/K$ be a 3-dimensional symmetric space as in Section 2. Let $H \geq 0$ be given and assume that $2H^2 + \text{Ric}_N \geq 0$, where $\text{Ric}_N = \min_{|v|=1} \text{Ric}_N(v)$. Let $M$ be a complete orientable surface immersed with CMC $H$ in $N$. Assume that $\mathcal{N}(M)$ is contained in a hemisphere of the unit sphere in $g$ determined by a nonzero vector $V \in g$, that is, $\langle \mathcal{N}(p), V \rangle \leq 0$ for all $p \in M$. We have:

a) If $M$ has the conformal type of the disk, then $M$ is invariant under the 1-parameter subgroup of isometries of $N$ determined by $V$.
b) If $M$ has the conformal type of the plane and $\zeta(V)$ is a bounded Killing field on $M$, then $M$ is invariant under the 1-parameter subgroup of isometries of $N$ determined by $V$ or $M$ is umbilical and $\text{Ric}(\eta) = \text{Ric}_N$.

Proof. Suppose that $\mathcal{N}(M)$ is contained in a hemisphere of $g$ determined by $V$. Let $\pi : \hat{M} \to M$ be the universal covering of $M$ and consider $\hat{M}$ as an immersed surface in $N$. Write $f$ as $f \circ \pi$. Set $f(p) = \langle \zeta(V)(p), \eta(p) \rangle$, $p \in \hat{M}$, where $\zeta(V)$ is the Killing field on $N$ defined in (3). Since $\langle \zeta(V)(p), \eta(p) \rangle = \langle \mathcal{N}(p), V \rangle \leq 0$, we have $f \leq 0$. Assume first that $\hat{M}$ is conformal to the disk. We will then show that $f$ vanishes identically and thus Proposition 2 implies that $M$ is invariant under the group of isometries generated by $V$.

Using that $\langle \Gamma(\text{grad}(H)), V \rangle = 0$, we can compute the Laplacian of $f$ as in the proof of Theorem 4 and obtain that

\[ \Delta f = - (\|B\|^2 + \text{Ric}(\eta)) f \geq - (2H^2 + \text{Ric}(\eta)) f \geq 0. \]

Therefore, $f$ is a subharmonic function on $\hat{M}$. If $f$ vanishes at some point $p \in \hat{M}$, then, by the maximum principle, $f \equiv 0$ and the theorem is proved in this case. So,
let us suppose \( f < 0 \) and get a contradiction. From the Gauss equation we have \( \|B\|^2 = 4H^2 - 2(K - \bar{K}) \), where \( K \) is the sectional curvature of \( \hat{M} \) and \( \bar{K} \) is the sectional curvature of \( N \) on tangent planes of \( M \). Using this equation in (23), we obtain

\[
\Delta f - 2Kf + \left( 4H^2 + 2\bar{K} + \text{Ric}(\eta) \right) f = 0.
\]

Considering an orthonormal basis \( E_1, E_2 \) of \( T\hat{M} \) we obtain

\[
\text{Ric}(\eta) + 2\bar{K} = (\langle R(\eta, E_1)\eta, E_1 \rangle + \langle R(\eta, E_2)\eta, E_2 \rangle + 2 \langle R(E_1, E_2)E_1, E_2 \rangle \\
= \langle R(E_1, \eta)E_1, \eta \rangle + \langle R(E_1, E_2)E_1, E_2 \rangle \\
+ \langle R(E_2, \eta)E_2, \eta \rangle + \langle R(E_2, E_1)E_2, E_1 \rangle \\
= \text{Ric}(E_1) + \text{Ric}(E_2).
\]

Then, from the hypothesis,

\[
P := \text{Ric}(\eta) + 2\bar{K} + 4H^2 \geq 2\text{Ric}_N + 4H^2 \geq 0.
\]

Thus \( f \) is a negative solution to the equation \( \Delta f - 2Kf + Pf = 0 \), with \( P \geq 0 \), which contradicts Corollary 3 of [FS], since \( \hat{M} \) has the conformal type of the disk. Thus \( f \equiv 0 \) and the first part of the theorem is proved.

Assume now that \( \hat{M} \) is conformal to the plane and that \( \zeta(V) \) is bounded in \( M \). This implies that \( f \) is a bounded function on \( M \). Since by (23) \( f \) is subharmonic it follows that \( f \) is constant and then \( \Delta f = 0 \). This implies

\[
(\|B\|^2 + \text{Ric}(\eta)) f = 0.
\]

It follows that either \( f \equiv 0 \) (and then \( M \) is invariant under the 1-parameter family of isometries given by \( V \)) or \( \|B\|^2 + \text{Ric}(\eta) \equiv 0 \). In this case the inequality in (23) would be an equality; thus we would have

\[
\|B\|^2 = 2H^2 \text{ and } \text{Ric}(\eta) = \text{Ric}_N,
\]

and from \( \|B\|^2 = 2H^2 \) it follows that \( M \) is umbilical, as it is easy to see. \( \square \)

**Remark.** Since an equidistant surface of \( \mathbb{H}^3(-1) \) (that is, a surface which is at a constant distance to a totally geodesic surface of \( \mathbb{H}^3 \)) has the conformal type of the disc (since it is isometric to \( \mathbb{H}^2(c) \) for some \( c \in [-1, 0) \)) and is orthogonal to a hyperbolic Killing field (that is, the Killing field whose orbits are horospheres equidistant to a fixed geodesic) we see that the hypothesis \( 2H^2 + \text{Ric}_N \geq 0 \), which, in the hyperbolic space, is equivalent to \( H \geq 1 \), cannot be improved. Also, in the case that \( M \) has the conformal type of the plane, if \( \zeta(V) \) is not bounded, then the conclusion may not be true: any horosphere \( S \) in \( \mathbb{H}^3 \) is conformal to the complex plane (\( S \) is isometric to the Euclidean plane) and is everywhere transversal to a hyperbolic Killing field which is not bounded on \( S \).

**References**

[AR] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in \( S^2 \times \mathbb{R} \) and \( H^2 \times \mathbb{R} \)*, Acta Math. 193 (2004), no. 2, 141–174, DOI 10.1007/BF02392562. MR2134864 (2006h:53003)

[AR2] U. Abresch and H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. 28 (2005), 1–28. MR2195187 (2006h:53004)

[AL] H. Araújo and M. L. Leite, *Surfaces in \( S^2 \times \mathbb{R} \) and \( H^2 \times \mathbb{R} \) with holomorphic Abresch-Rosenberg differential*, Differential Geom. Appl. 29 (2011), no. 2, 271–278, DOI 10.1016/j.difgeo.2010.12.010. MR2784306 (2012e:53185)
A. Ramos and J. Ripoll

F. Bittencourt and J. Ripoll, "Gauss map harmonicity and mean curvature of a hypersurface in a homogeneous manifold," Pacific J. Math. 224 (2006), no. 1, 45–63. DOI 10.2140/pjm.2006.224.45. MR2231651 (2007k:53091)

S. S. Chern, "On surfaces of constant mean curvature in a three-dimensional space of constant curvature," Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 104–108. DOI 10.1007/BFb0061413. MR730266 (86b:53058)

B. Daniel, "Isometric immersions into 3-dimensional homogeneous manifolds," Comment. Math. Helv. 82 (2007), no. 1, 87–131. DOI 10.4171/CMH/86. MR2296059 (2008a:53058)

B. Daniel, "The Gauss map of minimal surfaces in the Heisenberg group," Int. Math. Res. Not. IMRN 3 (2011), 674–695. DOI 10.1093/imrn/rnq092. MR2764875 (2012b:53117)

B. Daniel and P. Mira, "Existence and uniqueness of constant mean curvature spheres in Sol3," J. Reine Angew. Math. 685 (2013), 1–32. DOI 10.1515/crelle-2012-0016. MR3181562

J. Eells and L. Lemaire, "A report on harmonic maps," Bull. London Math. Soc. 10 (1978), no. 1, 1–68. DOI 10.1112/blms/10.1.1. MR495450 (82b:58033)

N. do Espírito-Santo, S. Fornari, K. Frensel, and J. Ripoll, "Constant mean curvature hypersurfaces in a Lie group with a bi-invariant metric," Manuscripta Math. 111 (2003), no. 4, 459–470. DOI 10.1007/s00229-003-0357-5. MR1993542 (2003j:53001)

J. M. Espinar and H. Rosenberg, "Complete constant mean curvature surfaces and Bernstein type theorems in $M^2 \times \mathbb{R}$," J. Differential Geom. 82 (2009), no. 3, 611–628. MR2534989 (2010m:53015)

S. Fornari and J. Ripoll, "Killing fields, mean curvature, translation maps," Illinois J. Math. 48 (2004), no. 4, 1385–1403. MR2114163 (2005i:53037)

D. Fischer-Colbrie and R. Schoen, "The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature," Comm. Pure Appl. Math. 33 (1980), no. 2, 199–211. DOI 10.1002/cpa.3160330206. MR562550 (81i:53044)

S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original. MR1834454 (2002b:53081)

D. Hoffman, R. Osserman, and R. Schoen, "On the Gauss map of complete surfaces of constant mean curvature in $\mathbb{R}^3$ and $\mathbb{R}^4$," Comment. Math. Helv. 57 (1982), no. 4, 519–531. DOI 10.1007/BF02565874. MR694604 (84f:53004)

H. Hopf, Differential geometry in the large, Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, Berlin, 1983. Notes taken by Peter Lax and John Gray; With a preface by S. S. Chern. MR708500 (85b:53001)

M. L. Leite and J. Ripoll, "On quadratic differentials and twisted normal maps of surfaces in $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$," Results Math. 60 (2011), no. 1-4, 351–360. DOI 10.1007/s00025-011-0151-8. MR2836904 (2012i:53062)

L. A. Masal’tsev, "A version of the Ruh-Vilms theorem for surfaces of constant mean curvature in $S^3$" (Russian, with Russian summary), Mat. Zametki 73 (2003), no. 1, 92–105, DOI 10.1023/A:1022126101717; English transl., Math. Notes 73 (2003), no. 1-2, 85–96. MR1993542 (2004m:53017)
[Me] W. H. Meeks III, *Constant mean curvature spheres in Sol₃*, Amer. J. Math. **135** (2013), no. 3, 763–775, DOI 10.1353/ajm.2013.0025. MR3068401

[MMPR] W. H. Meeks III, P. Mira, J. Pérez, and A. Ros, *Constant mean curvature spheres in homogeneous three-spheres*, preprint.

[MMPR2] W. H. Meeks III, P. Mira, J. Pérez, and A. Ros, *Constant mean curvature spheres in homogeneous three-manifolds*, Work in progress.

[MP] W. H. Meeks III and J. Pérez, *Constant mean curvature surfaces in metric Lie groups*, Geometric analysis: partial differential equations and surfaces, Contemp. Math., vol. 570, Amer. Math. Soc., Providence, RI, 2012, pp. 25–110, DOI 10.1090/conm/570/11304. MR2963596

[RV] E. A. Ruh and J. Vilms, *The tension field of the Gauss map*, Trans. Amer. Math. Soc. **149** (1970), 569–573. MR0259768 (41 #4400)

Departamento de Matematica, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, 91501-970, Porto Alegre, Rio Grande do Sul, Brazil

E-mail address: alvaro.ramos@ufrgs.br

Departamento de Matematica, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, 91501-970, Porto Alegre, Rio Grande do Sul, Brazil

E-mail address: jaime.ripoll@ufrgs.br