On an exactly solvable $B_N$ type Calogero model
with nonhermitian PT invariant interaction

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Abstract

An exactly solvable many-particle quantum system is proposed by adding
some nonhermitian but PT invariant interactions to the $B_N$ Calogero model.
We have shown that such extended $B_N$ Calogero model leads to a real spec-
trum which obey generalised exclusion statistics. It is also found that the cor-
responding exchange statistics parameter exhibit ‘reflection symmetry’ pro-
vided the strength of a PT invariant interaction exceeds a critical value.
I. INTRODUCTION

As is well known, in quantum mechanics one usually chooses a hermitian Hamiltonian to ensure real energy eigenvalues of the corresponding Schrödinger equation. However, quantum mechanical systems characterised by nonhermitian Hamiltonians also play a significant role in many contexts like absorption of incident particles in nuclear physics, localisation-delocalisation transitions in superconductors and in the description of the defraction of atoms by standing light waves [1,2]. Recently, theoretical investigations on different nonhermitian Hamiltonians have received a major boost due to the remarkable observation that many such systems, whenever they are invariant under combined parity (P) and time reversal (T) symmetry, give real energy eigenvalues [3]. This seems to suggest that the condition of hermiticity on a Hamiltonian can be replaced by the weaker condition of PT symmetry to ensure that the corresponding eigenvalues would be real ones. However, till now this is merely a conjecture supported by several examples [3–15]. Moreover, in almost all of these examples, the Hamiltonians of only one particle in one space dimension have been considered.

The aim of the present work is to test the validity of the above mentioned conjecture for the cases of some exactly solvable many-particle quantum mechanical systems in one dimension. For a Hamiltonian containing $N$ number of particles, the $PT$ transformation is evidently given by

$$i \rightarrow -i, \ x_j \rightarrow -x_j, \ p_j \rightarrow p_j \quad (1.1)$$

where $j \in [1, 2, \cdots, N]$, and $x_j \ (p_j \equiv -i \frac{\partial}{\partial x_j})$ denote the coordinate (momentum) operator of the $j$-th particle. So we want to find out some exactly solvable nonhermitian Hamiltonians which remain invariant under the PT transformation (1.1) and investigate whether such systems would lead to real spectra.

In this context one may note that the well known $A_{N-1}$ Calogero model, which contains $N$ particles on a line and is described by the hermitian Hamiltonian
\[ H_A = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \omega^2 \sum_{j=1}^{N} x_j^2 + \frac{g}{2} \sum_{\substack{j,k=1 \atop j \neq k}}^{N} \frac{1}{(x_j - x_k)^2}, \] (1.2)

represents an exactly solvable system [16]. This type of exactly solvable models with long-range interaction have attracted a lot of attention due to their close connection with diverse subjects like fractional statistics, random matrix theory, level statistics for disordered systems etc. [17–24]. Recently an integrable extension of \( A_{N-1} \) Calogero model is proposed by adding some momentum dependent interaction to the Hamiltonian (1.2) and this extension of Calogero model is also solved exactly to obtain the energy eigenvalues as well as eigenfunctions [25]. It is found that these energy eigenvalues are real and bounded below, in spite of the fact that the added momentum dependent interaction given by

\[ H_p = \frac{\delta}{2} \sum_{\substack{j,k=1 \atop j \neq k}}^{N} \frac{1}{x_j - x_k} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right), \] (1.3)

where \( \delta \) is a real parameter, is not a hermitian operator.

However, we now interestingly observe that the complete Hamiltonian \( (H_A + H_p) \) of the above mentioned extended Calogero model is indeed invariant under PT transformation (1.1). Thus we find here an example of PT invariant many-particle system, which is not only exactly solvable but also leads to completely real spectrum. To obtain more examples of this type, it may be noted that the Calogero model associated with \( B_N \) root system can also be solved exactly [26,27]. Therefore, it is natural to ask whether this \( B_N \) Calogero model can be generalised in a PT invariant way so that the newly constructed model would remain exactly solvable and yield completely real spectrum. In Sec.II we try to answer this question by proposing an appropriate PT invariant extension of the \( B_N \) Calogero model and solving such model exactly. In Sec.III we study some salient features of this extended \( B_N \) Calogero model and show that its spectrum can be interpreted through generalised exclusion statistics [28] as proposed by Haldane. Sec.IV is the concluding section.
II. EXACT SPECTRUM OF A $B_N$ TYPE CALOGERO MODEL

Here we propose an extension of the well known $B_N$ Calogero model \([26,27]\) as

$$
H_B = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \omega^2 \sum_{j=1}^{N} x_j^2 + \frac{g_1}{2} \sum_{j=1}^{N} x_j^2 + \frac{g_2}{2} \sum_{j,k=1}^{N} \frac{x_j^2 + x_k^2}{(x_j^2 - x_k^2)^2} \\
+ \delta_1 \sum_{j=1}^{N} \frac{1}{x_j} \frac{\partial}{\partial x_j} + \delta_2 \sum_{j,k=1, j\neq k}^{N} \frac{1}{x_j - x_k} \left( x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right),
$$

(2.1)

where $g_1$, $g_2$, $\delta_1$, $\delta_2$ are some real coupling constants. In the special case $\delta_1 = \delta_2 = 0$, the Hamiltonian (2.1) reproduces the original $B_N$ Calogero model:

$$
H_B = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \omega^2 \sum_{j=1}^{N} x_j^2 + \frac{g_1}{2} \sum_{j=1}^{N} x_j^2 + \frac{g_2}{2} \sum_{j,k=1, j\neq k}^{N} \frac{x_j^2 + x_k^2}{(x_j^2 - x_k^2)^2}.
$$

(2.2)

It may be observed that, though the Hamiltonian (2.1) violates hermiticity property due to the presence of momentum dependent interactions like $\delta_1 \sum_{j=1}^{N} \frac{1}{x_j} \frac{\partial}{\partial x_j} + \delta_2 \sum_{j,k=1, j\neq k}^{N} \frac{1}{x_j - x_k} \left( x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right)$, it remains invariant under the combined PT transformation (1.1). Therefore, it should be interesting to solve this extended $B_N$ Calogero model and see whether it gives real energy eigenvalues.

In this context it may be noted that, both $A_{N-1}$ and $B_N$ Calogero models have been solved recently by mapping them to a system of free harmonic oscillators \([29,34]\). For the purpose of solving the Hamiltonian (2.1) through a similar procedure, we conjecture first that its ground state is given by a Laughlin type wave function

$$
\psi_{gr} = \prod_{j,k=1 \atop (j\neq k)}^{N} (x_j - x_k)^{\rho} (x_j + x_k)^{\rho} \prod_{l=1}^{N} x_l^{\sigma} e^{-\frac{\rho}{2} \sum_{j=1}^{N} x_j^2},
$$

(2.3)

$\rho$ and $\sigma$ being two real non-negative parameters which are related to the coupling constants $g_1, g_2, \delta_1, \delta_2$ as

$$
g_1 = \sigma(\sigma - 1) - 2\sigma \delta_1 ; \quad g_2 = 2\rho(2\rho - 1) - 4\rho \delta_2.
$$

(2.4)

By solving the quadratic equations (2.4) of $\sigma$ and $\rho$, it is easy to see that these two parameters take real values provided the coupling constants in the Hamiltonian (2.1) satisfy
\[ g_1 \geq -\left(\frac{1}{2} + \delta_1\right)^2; \quad g_2 \geq -\left(\frac{1}{2} + \delta_2\right)^2. \] 

(2.5)

So, in this article we shall consider the PT invariant Hamiltonian (2.1) only for the range of parameters compatible with the conditions (2.5). Now if we use the expression (2.3) for a similarity transformation to the Hamiltonian (2.1), it reduces to a simple ‘effective Hamiltonian’ like

\[ \tilde{H}_B = \psi_{gr}^{-1}(H_B - E_{gr})\psi_{gr} \]

\[ = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \omega \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} - 2\tilde{\rho} \sum_{j,k=1}^{N} x_j^2 - x_k^2 \left( x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right) - \tilde{\sigma} \sum_{j=1}^{N} \frac{1}{x_j} \frac{\partial}{\partial x_j}. \] 

(2.6)

Here

\[ E_{gr} = \frac{\omega N}{2} + \omega N\tilde{\sigma} + \omega N(N-1)2\tilde{\rho}, \] 

(2.7)

and the real valued parameters \( \tilde{\sigma}, \ 2\tilde{\rho} \) are defined as \( \tilde{\sigma} = \sigma - \delta_1 \) and \( 2\tilde{\rho} = 2\rho - \delta_2 \).

It should be noted that, at the limit \( \delta_1 = \delta_2 = 0 \), \( \tilde{H}_B \) (2.6) reduces to

\[ \tilde{H}_B = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \omega \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} - 2\rho \sum_{j,k=1}^{N} x_j^2 - x_k^2 \left( x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right) - \sigma \sum_{j=1}^{N} \frac{1}{x_j} \frac{\partial}{\partial x_j}, \] 

(2.8)

where the parameters \( \rho \) and \( \sigma \) are related with the coupling constants \( g_1, g_2 \) of the \( B_N \) Calogero model (2.2) as

\[ g_1 = \sigma(\sigma - 1) ; \quad g_2 = 2\rho(2\rho - 1). \] 

(2.9)

So, by using the expression (2.3) for a similarity transformation on the \( B_N \) Calogero model (2.2) and applying the relations (2.3), one can get the corresponding effective Hamiltonian (2.8). It has been found earlier that this effective Hamiltonian (2.8) has a complete set of polynomial eigenfunctions [35]. Such polynomial eigenfunctions, which are completely symmetric with respect to variables \( x_i^2 \), are called as generalised Laguerre polynomials. Since these symmetric polynomial eigenfunctions are evidently free from any singularity, there exists an one-to-one correspondence between the nonsingular eigenfunctions of the usual \( B_N \) Calogero Hamiltonian and the generalised Laguerre polynomials. Now it is very interesting
to observe that, after a trivial substitution of coupling constants given by $2\tilde{\rho} \to 2\rho$ and $\tilde{\sigma} \to \sigma$, the effective Hamiltonian (2.6) for our extended $B_N$ Calogero model coincides with the effective Hamiltonian (2.6) for $B_N$ Calogero model. Therefore, the generalised Laguerre polynomials again represent a complete set of polynomial eigenfunctions for the effective Hamiltonian (2.6). Consequently, there exists an one-to-one correspondence between these polynomials and the eigenfunctions of the extended $B_N$ Calogero Hamiltonian (2.1).

From the above discussion it is clear that, by following Ref. [35], one can directly solve the eigenvalue problem for $\tilde{\mathcal{H}}_B$ (2.6) through generalised Laguerre polynomials. At present, however, we want to follow a different approach which attempts to solve this eigenvalue problem by mapping $\tilde{\mathcal{H}}_B$ to a set of decoupled harmonic oscillators. To this end, we notice that this effective Hamiltonian (2.6) may be expressed as $
abla\nabla - \tilde{\sigma}$, where the $B_N$ type Lassalle operator $S^-$ and the Euler operator $S^3$ are given by [33]

$$S^- = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - 2\tilde{\rho} \sum_{j,k=1}^{N} \frac{1}{x_j^2 - x_k^2} \left( x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right) - \tilde{\sigma} \sum_{j=1}^{N} \frac{1}{x_j} \frac{\partial}{\partial x_j},$$

$$S^3 = \sum x_j \frac{\partial}{\partial x_j}. \quad (2.10)$$

It is easy to see that the above defined operators satisfy the simple commutation relation: $[S^3, S^-] = -2S^-$. By taking advantage of such commutation relation, we perform further similarity transformations to $\tilde{\mathcal{H}}_B$ and reduce it to the Hamiltonian corresponding to free oscillators as

$$\mathcal{H}_{free} = S^{-1} \tilde{\mathcal{H}}_B S = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \omega \sum_{j=1}^{N} x_j^2 - \frac{\omega N}{2}$$

(2.11)

where $S = S_0 e^{\frac{1}{2}S^-} \sum x_i^2$ and $S_0 = e^{\frac{1}{2}S^-} e^{-\frac{1}{2} \sum_{j=1}^{N} \frac{x_j^2}{\omega}}$.

Due to similarity transformations in (2.6) and (2.11), one may naively think that the eigenfunctions of the extended $B_N$ Calogero model (2.1) can be obtained from those of the free oscillators as: $\psi_{n_1,n_2,\ldots,n_N} = \psi_{gr} S_0 (H_{n_1}(x_1)H_{n_2}(x_2)\cdots H_{n_N}(x_N))$, where $H_{n_j}(x_j)$ denotes the Hermite polynomial of order $n_j$. However, similar to the case of $B_N$ Calogero model [30,33], the action of $S_0$ leads to a singularity unless all $n_j$s are chosen to be even.
integers and $H_{n_1}(x_1)H_{n_2}(x_2)\cdots H_{n_N}(x_n)$ is symmetrised with respect to all coordinates.

Therefore, nonsingular eigenfunctions of $\tilde{\mathcal{H}}_B$ \(2.1\) can be obtained from the eigenfunctions of free oscillators as

$$
\psi_{n_1,n_2,\ldots,n_N} = \psi_{gr} \mathcal{S}_0(\Lambda_+ (H_{2n_1}(x_1)H_{2n_2}(x_2)\cdots H_{2n_N}(x_n))) ,
$$

\(2.12\)

where $\Lambda_+$ completely symmetrises all coordinates and thus projects the distinguishable many-particle wave functions to the bosonic part of the Hilbert space. Eigenvalues of $\mathcal{H}_B$ corresponding to eigenfunctions \(2.12\) are given by

$$
E_{n_1,n_2,\ldots,n_N} = E_{gr} + 2\omega \sum_{j=1}^N n_j = \frac{\omega N}{2} + \omega N\tilde{\sigma} + \omega N(N-1)2\tilde{\rho} + 2\omega \sum_{j=1}^N n_j ,
$$

\(2.13\)

where the excitation numbers $n_j$s are non-negative integers obeying bosonic selection rule $n_{j+1} \geq n_j$.

Since $\tilde{\sigma}$ and $\tilde{\rho}$ are real parameters, the energy eigenvalues \(2.13\) are also real ones. Thus we interestingly find that the PT invariant nonhermitian Hamiltonian \(2.1\) generates real spectrum. However, within the above mentioned approach, it is not clear whether the corresponding eigenfunctions form a complete set. It is also evident that, apart form a constant shift for all energy levels, the spectrum \(2.13\) coincides with the spectrum of $N$ number of free bosonic oscillators with frequency $2\omega$. As the term $2\omega \sum_{j=1}^N n_j$ in eqn.\(2.13\) is always non-negative, $E_{n_1,n_2,\ldots,n_N}$ can not be less than $E_{gr}$. Consequently, $\psi_{gr} \ (2.3)$ indeed represents the ground state wave function of $\mathcal{H}_B \ (2.1)$ with energy $E_{gr}$.

Finally we want to point out an important difference between the ground state energies of $B_N$ Calogero model \(2.2\) and its $PT$ invariant extension \(2.1\). At the limit $\delta_1 = \delta_2 = 0$, eqn.\(2.7\) reproduces the ground state energy of $B_N$ Calogero model as

$$
E_{gr} = \frac{\omega N}{2} + \omega N\sigma + \omega N(N-1)2\rho ,
$$

\(2.14\)

and the form of corresponding ground state eigenfunction is given by \(2.3\). Since $\sigma$ and $\rho$ must be positive to ensure the nonsingularity of ground state eigenfunction \(2.3\), the ground state energy \(2.14\) of the $B_N$ Calogero model is always a positive quantity. However, as
evident from eqn. (2.7), the ground state energy of the extended \( B_N \) Calogero model is
dominated by the \( N^2 \) order term for large values of \( N \). The coefficient of this \( N^2 \) order
term, i.e. \( 2\bar{\rho} \), is related to the coupling constants \( \delta_2, g_2 \) through a quadratic equation given by

\[
g_2 = (2\bar{\rho})^2 - 2\bar{\rho} - \delta_2(1 + \delta_2). \tag{2.15}
\]

By using (2.15) and the second relation of (2.4), it is easy to see that the parameter \( 2\bar{\rho} \)
becomes negative (even though \( 2\rho \) remains positive) if the coupling constants \( \delta_2, g_2 \) are
chosen within the range: \( \delta_2 > 0, 0 \geq g_2 > -\delta_2(1 + \delta_2) \). Therefore, for large values of \( N \), the
ground state energy of extended \( B_N \) Calogero model (2.1) will also be a negative quantity
within the above mentioned range of two coupling constants.

III. SOME PROPERTIES OF EXTENDED \( B_N \) CALOGERO MODEL

A. Connection with fractional statistics

Generalised exclusion statistics (GES) introduced by Haldane \[28\] is believed to play
an important role in the edge excitations of fractional quantum Hall effect. Such exclusion
statistics can be realised microscopically in usual Calogero models with hermitian Hamiltonians \[22-24\]. The GES parameter for these Calogero models is a measure of ‘level repulsion’
of the quantum numbers generalising the Pauli exclusion principle \[24\]. So the partition
function and various thermodynamical quantities for these Calogero models can be derived
as a function of the corresponding GES parameters. Now for exploring GES in the case of
our PT invariant \( B_N \) type Calogero model (2.1), we observe that eqn. (2.13) can be rewritten
exactly in the form of energy spectrum for \( N \) free oscillators as

\[
E_{n_1,n_2,...,n_N} = \frac{\bar{\omega}N}{2} + \bar{\omega} \sum_{j=1}^{N} \bar{n}_j, \tag{3.1}
\]

where \( \bar{\omega} = 2\omega \) and

\[
\bar{n}_j = n_j + 2\bar{\rho}j + \frac{1}{4}(2\bar{\sigma} - 8\bar{\rho} - 1) \tag{3.2}
\]
are quasi-excitation numbers. However, from eqn.(3.2) it is evident that such quasi-excitation numbers are no longer integers and they satisfy a modified selection rule: 
\[ \bar{n}_{j+1} - \bar{n}_j \geq 2\bar{\rho} \]. Thus the minimum difference between two consecutive \( \bar{n}_j \)'s is given by 
\[ 2\bar{\rho} = 2\rho - \delta_2, \]  
(3.3)
(here we assume \( 2\rho \geq \delta_2 \)). As a consequence the spectrum of extended \( B_N \) Calogero model (2.1) satisfy GES with parameter \( 2\bar{\rho} \). It is interesting to notice that this exclusion statistics parameter depends only two coupling constants \( \delta_2 \) and \( g_2 \), which control the strength of long-range interactions in the Hamiltonian (2.1). Moreover, eqn.(2.15) describe a parabolic curve on the \( (\delta_2, g_2) \) plane for any fixed value of \( 2\bar{\rho} \). Consequently, all points on such a parabolic curve, representing extended \( B_N \) Calogero models associated with different values of \( \delta_2 \) and \( g_2 \), yield the same exclusion statistics parameter.

It may be observed that the eigenfunctions (2.3) and (2.12) pick up a phase factor \( (-1)^{2\rho} \) under the exchange of any two particles. So the exchange statistics parameter for the extended \( B_N \) Calogero model (2.1) is given by \( 2\rho \). It is clear from eqn.(3.3) that, for the case \( \delta_2 \neq 0 \), the exchange statistics parameter for the extended \( B_N \) Calogero model differs from the corresponding exclusion statistics parameter. Though the the exchange statistics parameter is not directly related to the spectrum or thermodynamics of the system, it has some interesting features. For example, in absence of confining harmonic potential, the scattering phase shift for multi-particle scattering generally depends on the exchange statistics parameter [24]. Moreover, quite similar to the case of usual Calogero models [24], the exchange statistics parameter for the extended \( B_N \) Calogero model fixes the boundary conditions on the wave functions (2.3) and (2.12) at the limits \( x_i \rightarrow x_j \).

**B. Reflection symmetry of the exchange statistics parameter**

It is well known that, for fixed values of all coupling constants, the exchange statistics parameter of the \( B_N \) Calogero model (2.2) may take two distinct values. These two distinct
values of the exchange statistics parameter are also related through a ‘reflection symmetry’ given by: $2\rho \rightarrow 1 - 2\rho$. For exploring such reflection symmetry in the case of extended $B_N$ Calogero model (2.1), we note that the second relation in (2.4) can be easily solved to obtain two solutions of the exchange statistics parameter as

$$2\rho = \frac{1}{2}(1 + 2\delta_2) \pm \frac{1}{2}\sqrt{(1 + 2\delta_2)^2 + 4g_2}. \quad (3.4)$$

Due to the existence of these two solutions, it appears that reflection symmetry is also present in the case of extended $B_N$ Calogero model (2.1). However, we have already noticed that only positive solutions of the parameter $2\rho$ lead to physically acceptable nonsingular eigenfunctions of $H_B$. So, the reflection symmetry can exist in the case of extended $B_N$ Calogero model provided both solutions of $2\rho$ in eqn. (3.4) take non-negative values. It is easy to see that both of these solution will be non-negative only within a parameter range given by $(1 + 2\delta_2) > 0$ and $g_2 \leq 0$. Thus we curiously find that a kind of ‘phase transition’ occurs at the line $\delta_2 = -\frac{1}{2}$ on the $(\delta_2, g_2)$ plane. For the case $\delta_2 \leq -\frac{1}{2}$, the reflection symmetry of the exchange statistics parameter is lost for any possible value of $g_2$. On the other hand, for the case $\delta_2 > -\frac{1}{2}$ the exchange statistics parameter shows a reflection symmetry: $2\rho \rightarrow 1 + 2\delta_2 - 2\rho$, if $g_2$ is chosen within an interval $-(\frac{1}{2} + \delta_2)^2 \leq g_2 \leq 0$.

**C. Relation with $B_N$ Calogero model**

We have seen in sec.II that, similar to the case of $B_N$ Calogero model, the extended $B_N$ Calogero model (2.1) can also be mapped to a system of free harmonic oscillators through a similarity transformation. So it is natural to enquire whether the extended $B_N$ Calogero model (2.1) is directly related to the $B_N$ Calogero model (2.2) through some similarity transformation. Investigating along this line, we find that

$$\Gamma^{-1}H_B\Gamma = H'_B \equiv \frac{1}{2} \sum j^2 + \frac{1}{2} \omega^2 \sum x_j^2 + g'_1 \sum x_j^2 + g'_2 \sum_{j,k=1, \ j\neq k}^N \frac{x_j^2 + x_k^2}{x_j^2 - x_k^2}, \quad (3.5)$$

where
\[ \Gamma = \prod_{j,k=1 \atop j \neq k}^{N} (x_j^2 - x_k^2)^{2} \prod_{l=1}^{N} x_l^{\delta_l}, \quad (3.6) \]

and \( H'_B \) denotes the Hamiltonian of \( B_N \) Calogero model with ‘renormalised’ coupling constants given by

\[ g'_1 = g_1 + \delta_1(1 + \delta_1), \quad g'_2 = g_2 + \delta_2(1 + \delta_2). \quad (3.7) \]

However it should be observed that, for any nonzero value of \( \delta_2 \), either \( \Gamma \) (3.6) or its inverse becomes singular at the limit \( x_j \to x_k \). So the relation (3.5) can not be interpreted as a similarity transformation in the usual sense and it may lead to some strange consequences. For example, due to relation (3.5), one may expect that the Hamiltonians \( \mathcal{H}_B \) and \( H'_B \) must share exactly same eigenvalues. However, we have already noticed in sec.II that the ground state energy of \( H'_B \) is always a positive quantity, while the ground state energy of \( \mathcal{H}_B \) (2.1) can be a negative quantity provided the exclusion statistics parameter \( 2\tilde{\rho} \) takes a negative value. To explain this rather unexpected result, we first recall that \( 2\tilde{\rho} \) will be negative if the coupling constants of \( \mathcal{H}_B \) (2.1) are chosen within the range: \( \delta_2 > 0 \) and \( 0 \geq g_2 > -\delta_2(1 + \delta_2) \). So, from eqn.(3.7) one finds that the renormalised coupling constant \( g'_2 \) must be a positive quantity in this case. Consequently, the corresponding exclusion statistics parameter \( 2\rho' \), which is obtained by solving the quadratic equation \( g'_2 = 2\rho'(2\rho' - 1) \), has one positive and one negative solution. The positive solution of \( 2\rho' \) leads to a ground state for \( H'_B \) with positive eigenvalue. On the other hand, for large values of \( N \), the negative solution of \( 2\rho' \) yields a ground state for \( H'_B \) with negative eigenvalue. However, one usually throws away this negative solution of \( 2\rho' \), since the corresponding ground state eigenfunction \( \psi'_gr(x_1, x_2, \cdots, x_N) \) becomes singular at the limit \( x_j \to x_k \). Nevertheless, as will be shown below, this unphysical eigenfunction of \( H'_B \) enables us to construct a physical eigenfunction of \( \mathcal{H}_B \). Due to the relation (3.5), one finds that

\[ \psi_{gr}(x_1, x_2, \cdots, x_N) = \Gamma \psi'_gr(x_1, x_2, \cdots, x_N), \quad (3.8) \]

where \( \psi_{gr}(x_1, x_2, \cdots, x_N) \) represents the ground state eigenfunction of \( \mathcal{H}_B \). It can be easily checked that the zeros of \( \Gamma \) are sufficient to cancel all singularities of \( \psi'_gr(x_1, x_2, \cdots, x_N) \) at
the limit \( x_j \to x_k \). Therefore, the ground state eigenfunction \( \psi_{gr}(x_1, x_2, \cdots, x_N) \) (3.8) is no longer singular at this limit. Thus we curiously find that a singular eigenfunction of \( H'_B \) generates a nonsingular eigenfunction of \( H_B \) through the transformation (3.8). Consequently, the ground state energy of \( H_B \) (2.1) coincides with the negative eigenvalue of \( H'_B \) associated with its unphysical eigenfunction \( \psi'_B(x_1, x_2, \cdots, x_N) \). In a similar way one can show that all singular eigenfunctions of \( H'_B \), which represent the excited states over the unphysical ground state eigenfunction \( \psi'_B(x_1, x_2, \cdots, x_N) \), generate nonsingular excited states of \( H_B \) (2.1) through a transformation by \( \Gamma \).

IV. CONCLUSION

Here we observe that a recently considered nonhermitian variant of \( A_{N-1} \) Calogero Hamiltonian with real spectrum remains invariant under the PT transformation. Being encouraged by this observation, we propose a new many-particle quantum system (2.1) by adding some nonhermitian but PT invariant interactions to the \( B_N \) Calogero model. Such PT invariant interactions depend on both coordinate and momentum variables of the particles. By using appropriate similarity transformations, we are able to map this extended \( B_N \) Calogero model to a set of free harmonic oscillators and solve this model exactly. It turns out that this many-particle system with nonhermitian Hamiltonian yields a real spectrum. This fact supports the conjecture that the condition of hermiticity on a Hamiltonian can be replaced by the weaker condition of PT symmetry to ensure that the corresponding eigenvalues would be real ones. It is also found that the spectrum of extended \( B_N \) Calogero model obeys a selection rule which leads to generalised exclusion statistics (GES). However, the extended \( B_N \) Calogero model also possess some remarkable properties which are absent in the case of usual \( B_N \) Calogero model. For example, we curiously find that the GES parameter for this extended \( B_N \) Calogero model differs from the corresponding exchange statistics parameter. Moreover a ‘reflection symmetry’ of the exchange statistics parameter, which is known to exist for \( B_N \) Calogero model, can be found in the case of extended model only if the strength
of a PT invariant interaction is chosen above a critical value. As a future study, it will be interesting to search for other exactly solvable many-particle systems with nonhermitian but PT invariant Hamiltonians and investigate the nature of their eigenvalues as well as eigenfunctions.

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