Hawking radiation of black $p$-branes via gauge and gravitational anomalies

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We investigate the Hawking radiation of black $p$-branes in superstring theories through the method of anomaly cancelation proposed by Robinson and Wilczek et al. The metrics of black $p$-branes are spherically symmetric, but not the Schwarzschild type generally. We find that in order to make the method of Robinson and Wilczek et al. effective for the calculation of Hawking fluxes for general non-Schwarzschild type spherically symmetric metrics, it is necessary to make a coordinate transformation to transform them to the Schwarzschild type. We calculate the charge and energy-momentum fluxes using the method of anomaly cancelation for the reduced two-dimensional black brane metric. We obtain that the thermodynamic temperature and chemical potential for a non-Schwarzschild type spherically symmetric black $p$-brane obtained from the method of anomaly cancelation match its Hawking temperature and chemical potential obtained from black brane thermodynamics, and the spectrum of radiation is in the Planck distribution.

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I Introduction

Hawking radiation is an important physical property of black holes. Since its original discovery by Hawking [1], many people have tried different methods for the derivation of this phenomenon [2, 3, 4, 5, 6]. The same result is obtained from different methods. Different methods for the derivation of Hawking radiation show that Hawking radiation is related with the quantum effect of a black hole’s gravitational field.

Recently, a new method for the derivation of Hawking radiation has been discovered by Robinson and Wilczek et al. which is named anomaly cancelation [7, 8]. Robinson and Wilczek et al. have pointed out that the effective field theory of quantum fields near a black hole’s horizon is a two-dimensional chiral field theory due to the fact that a black hole’s horizon is a one-way membrane. Thus there exist gauge and gravitational anomalies

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for the currents near a black hole’s horizon. However the effective action of quantum fields near a black hole’s horizon is still gauge invariant and general covariant. Then to combine the regular conditions for the covariantly anomalous currents on the horizon, gauge and energy-momentum fluxes with the Hawking temperature $T_H$ are derived to be exist outside a black hole’s horizon. In fact, the germination of this idea has been appeared in a paper of Christensen and Fulling [5] many years ago, where the Hawking radiation for a (1 + 1)-dimensional Schwarzschild black hole was derived from the conformal anomaly method. The difference lies in that the primal method of Christensen and Fulling [5] is only applicable to (1 + 1)-dimensional spacetime manifold, while the method of Robinson and Wilczek et al. [7, 8] can be applied to higher dimensional spacetime.

In [9, 10, 11], the method of anomaly cancelation for the derivation of Hawking radiation has been generalized to higher-dimensional rotating black holes. Then it has been applied to the Hawking radiation of various types of black holes [12, 13, 14]. Some recent developments and applications of this method are carried out in [15]. This method for the derivation of Hawking radiation has also been used for black holes of superfinkle-Horowitz-Strominger black hole of superstring theories [16, 17]. In [16], the authors have derived the Hawking radiation for the Garfinkle-Horowitz-Strominger black hole of superstring theories [18] from the method of anomaly cancelation. In [17], the authors have applied this method for the derivation of Hawking radiation for $D1-D5$ brane black holes of superstring theories [19].

In this paper, we will study the Hawking radiation of general spherically symmetric black $p$-branes of superstring theories [20, 21, 22, 23, 24] using the method of anomaly cancelation. The reduced two-dimensional metrics of these black branes are still spherically symmetric, but not the Schwarzschild type generally. We find that in order to make the method of anomaly cancelation effective for general non-Schwarzschild type spherically symmetric black holes and black branes, it is necessary to make a coordinate transformation to transform the metrics to the Schwarzschild type. The content of this paper contains: In Sec. II, we briefly review the metrics of black $p$-branes to make this paper self-contained. In Sec. III, we study the effective action of quantum fields near the horizon of a black brane and obtain that it is equivalent to a two-dimensional field theory. In Sec. IV, we make a coordinate transformation for the obtained two-dimensional non-Schwarzschild type spherically symmetric metric to the form of the Schwarzschild type for the purpose of the following calculations. In Sec. V, we calculate the charge flux related with the gauge anomaly for the effective two-dimensional field theory. In Sec. VI, we calculate the energy-momentum flux related with the gravitational anomaly for the effective two-dimensional field theory. We obtain that the charge and energy-momentum fluxes for a black $p$-brane deduced from the method of anomaly cancelation exactly match the Hawking radiation of a black $p$-brane. Thus the method of anomaly cancelation for the derivation of Hawking radiation can be applied to general non-Schwarzschild type spherically symmetric black holes and black branes. Sec. VII is devoted to the conclusion.

II The metric of black $p$-branes

In this section, we review the black $p$-brane solutions which appear in superstring theories
We consider the following $D$-dimensional action

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2(d+1)!} e^{-\alpha \phi} F_{d+1}^2 \right],$$

(1)

where $\phi$ denotes a dilaton and $F_{d+1}$ is a field strength for an antisymmetric tensor $A_d$ of rank $d$, i.e.,

$$F_{d+1} = dA_d.$$  

(2)

The action (1) is one part of the low energy effective actions of type IIA and type IIB superstring theories, and also one part of eleven-dimensional supergravity, which is considered as the low energy effective action of M-theory.

The field equations for the action (1) have the following black $(d-1)$-brane solution:

$$ds^2 = - \left[ 1 - \left( \frac{r_+}{r} \right)^d \right] \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\frac{4d}{3(d+1)}} dt^2 + \left[ 1 - \left( \frac{r_+}{r} \right)^d \right]^{-1} \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\frac{2\alpha^2}{3d}} dr^2 + r^2 \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^\frac{4d}{3(d+1)} d\Omega^2_{d+1} + \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{\frac{4d}{3(d+1)}} \delta_{ij} dx^i dx^j,$$

(3)

$$A_{01...d-1} = \sqrt{\frac{4}{\delta}} \left( \frac{r_+ r_-}{r^2} \right)^{\frac{d}{2}},$$

(4)

$$e^{-2\phi} = \left[ 1 - \left( \frac{r_-}{r} \right)^d \right]^{-\frac{4\phi}{\delta}},$$

(5)

where $i, j = 1, 2, \ldots, d-1$, $d\Omega^2_n$ is the metric of a unit $n$-sphere. In (3)–(5), we define

$$\tilde{d} = D - d - 2,$$

(6)

$$\alpha^2 = \delta - \frac{2d(D - d - 2)}{D - 2}.$$  

(7)

The metric (3) has an event horizon at $r = r_+$, and an inner horizon at $r = r_-$ for $r_+ > r_-$. $r_+ = r_-$ corresponds to the extremal BPS state. In the following of this paper, we study the non-extremal case of this metric. We also use $r_H$ to represent the radius of the event horizon.

The Ramond-Ramond field strength for this solution is given by

$$e^{-\alpha \phi} * F_{d+1} = \sqrt{\frac{4}{\delta}} \tilde{d}(r_+ r_-)^{\frac{d}{2}} \epsilon_{d+1},$$

(8)
where $\epsilon_{\tilde{d}+1}$ is the volume form on a unit $(\tilde{d} + 1)$-sphere. The Ramond-Ramond charge with respect to the rank $(d + 1)$ field strength that the black $(d - 1)$-brane carries is

$$Q = \frac{1}{16\pi} \int_{S^d} e^{-\alpha \phi} \star F$$

$$= \frac{1}{16\pi} \Omega_{d+1} \sqrt{\frac{4}{\delta} \tilde{d}(r_+ r_-)^{\frac{d}{2}}} ,$$

where $\Omega_n$ is the volume of a unit $n$-sphere.

The Hawking temperature of the black $(d-1)$-brane can be obtained through the standard Euclidian approach which is given by

$$T_H = \frac{1}{2\pi} \frac{\partial_r \sqrt{A(r)}}{\sqrt{B(r)}} \bigg|_{r=r_H}$$

$$= \frac{1}{4\pi} \frac{A'(r)}{\sqrt{A(r)B(r)}} \bigg|_{r=r_H}$$

$$= \frac{\tilde{d}}{4\pi r_+} \left[ 1 - \left( \frac{r_-}{r_+} \right)^{\frac{d}{2}} \right]^{\frac{d}{2}-\frac{1}{2}} .$$

The chemical potential with respect to the Ramond-Ramond charge (9) is the value of the gauge potential at the horizon

$$\mu = A_{01...d-1} \big|_{r=r_H} = \sqrt{\frac{4}{\delta} \left( \frac{r_-}{r_+} \right)^{\frac{d}{2}}} .$$

The first law of thermodynamics is satisfied by the black $(d - 1)$-branes.

### III Quantum fields near the horizon

Robinson and Wilczek et al. have shown that quantum fields near a black hole horizon can be described by a two-dimensional field theory in a curved background of the Schwarzschild type \[7,8\]. This is the primal recipe for the derivation of Hawking radiation from the method of anomaly cancelation. For the metric (3) of black $(d - 1)$-brane, we can write it in the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2C(r)d\Omega_{d+1}^2 + D(r)\delta_{ij}dx^i dx^j$$

for convenience. Its horizon is determined by $1/B(r)|_{r=r_H} = 0$. In fact, all of the metrics of spherically symmetric black branes can be cast in the form of (12). The determinant for the metric (12) reads

$$\sqrt{-g} = r^{d+1} \sqrt{A(r)B(r)C^{d+1}(r)D^{d-1}(r)} .$$

We study a free complex scalar field $\varphi$ in the background of (12). We suppose that the scalar field $\varphi$ is zero-mass for convenience. We first do not consider the coupling of the
complex scalar field with the Ramond-Ramond gauge field. Then the action is given by

\[
S_{\text{free}}(\varphi) = \int d^Dx \sqrt{-g} \ g^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi
\]

\[
= \int d^Dx \sqrt{-g} \left[ \partial_\mu (\varphi^* \partial^\mu \varphi) - \varphi^* \partial_\mu \partial^\mu \varphi \right]. \quad \text{(14)}
\]

To take care that \(\sqrt{-g}\) only depends on the coordinate \(r\), while does not depend on the other coordinates, to omit a surface term in the action, we obtain

\[
S_{\text{free}}(\varphi) = -\int d^Dx \sqrt{-g} \varphi^* \left( \partial_r \partial^r + \frac{1}{\sqrt{-g}} \partial_r \sqrt{-g} \partial^r \right) \varphi. \quad \text{(15)}
\]

To consider the near horizon limit \(r \to r_H\), the second term in (15) can be omitted compared with the first term. This fact can be made clear through transforming to the radial “tortoise” coordinate \(\tilde{r}\). On the other hand, in the near horizon limit, \(\sqrt{-g}\) tends to a constant, it can be moved outside the integral. Hence in the area near the black brane horizon, the action is dominated by

\[
S_{\text{free}}(\varphi) = -\sqrt{-g}|_{r_H} \int dt \, \sqrt{\Omega_{d+1}} \, d^{d-1}x \, \varphi^* (\partial_\ell \partial^\ell + \partial_\ell \partial^\ell + \partial_{d-2} \partial^{d-2}) \varphi, \quad \text{(16)}
\]

where \(d\Omega_{d+1}\) is the volume element of the \((d+1)\)-dimensional unit sphere, \(d^{d-1}x\) is the volume element of the \((d-1)\)-dimensional transverse space. \(\partial_{d-2} \partial^{d-2}\) represents the full Laplacian on the \((D-2)\)-dimensional space with the explicit form

\[
\partial_{d-2} \partial^{d-2} = \frac{1}{r^2 C(r)} \nabla_\Omega^2 + \frac{1}{D(r)} \nabla_X^2, \quad \text{(17)}
\]

where \(\nabla_\Omega^2\) is the Laplacian on the \((d+1)\)-dimensional unit sphere, \(\nabla_X^2\) is the Laplacian on the \((d-1)\)-dimensional transverse space.

We can expand \(\varphi(x)\) with respect to the normalized eigenfunctions for the operator \(\partial_{d-2} \partial^{d-2}\) as

\[
\varphi(x) = \sum_{l_1, \ldots, l_{d+1}, k_1, \ldots, k_{d-1}} \varphi_{l_1, \ldots, l_{d+1}, k_1, \ldots, k_{d-1}}(r, t) Y_{l_1, \ldots, l_{d+1}}(\theta_1, \ldots, \theta_{d+1}) X(k_1, \ldots, k_{d-1}). \quad \text{(18)}
\]

In (18), \(Y_{l_1, \ldots, l_{d+1}}(\theta_1, \ldots, \theta_{d+1})\) are the normalized spherical harmonics on a \((d+1)\)-dimensional unit sphere with the azimuthal angles \((\theta_1, \ldots, \theta_{d+1})\), and

\[
X(k_1, \ldots, k_{d-1}) = e^{i(k_1 x^1 + \cdots + k_{d-1} x^{d-1})}. \quad \text{(19)}
\]

They satisfy

\[
\int d\Omega_{d+1} d^{d-1}x Y_{l_1, \ldots, l_{d+1}}(\theta_1, \ldots, \theta_{d+1}) X(k_1, \ldots, k_{d-1}) \times
Y^*_{m_1, \ldots, m_{d+1}}(\theta_1, \ldots, \theta_{d+1}) X^*(p_1, \ldots, p_{d-1}) = \delta_{l_1 m_1} \cdots \delta_{l_{d+1} m_{d+1}} \delta_{k_1 p_1} \cdots \delta_{k_{d-1} p_{d-1}}, \quad \text{(20)}
\]

where the integration for the coordinates \(x^i\) takes unit volume on the transverse space of the brane.
To substitute (18) and (20) in (16), the near horizon effective action for the free complex scalar field is obtained as
\[
S_{\text{free}}(\varphi) = -\sqrt{-g} |_{r_H} \sum_{l_1, \ldots, l_{d+1}, k_1, \ldots, k_{d-1}} \int dtdr \varphi_{l_1 \ldots l_{d+1} k_1 \ldots k_{d-1}}^*(r, t) (\partial_t \partial^t + \partial_r \partial^r) \varphi_{l_1 \ldots l_{d+1} k_1 \ldots k_{d-1}}(r, t). \tag{21}
\]

Supposing that the scalar field carries Ramond-Ramond charge, then it will couple with the Ramond-Ramond gauge field of the black \((d - 1)\)-brane. Near the horizon, the Ramond-Ramond gauge field is given by (11). Thus the effective action for the complex scalar field near the black brane horizon can be written as
\[
S(\varphi) = -\sqrt{-g} |_{r_H} \sum_{l_1, \ldots, l_{d+1}, k_1, \ldots, k_{d-1}} \int dtdr \varphi_{l_1 \ldots l_{d+1} k_1 \ldots k_{d-1}}^*(r, t) (D_t D^t + \partial_r \partial^r) \varphi_{l_1 \ldots l_{d+1} k_1 \ldots k_{d-1}}(r, t), \tag{22}
\]

where
\[
D_t = \partial_t - ie A_{01 \ldots d-1}(r_H) \tag{23}
\]
is the covariant derivative, \(e\) is the Ramond-Ramond charge that the complex scalar field carries. Thus from (21) and (23) we can see that near the black brane horizon, the effective field theory for the complex scalar field is a two-dimensional field theory in a curved background whose metric is given by
\[
d s^2 = -A(r) dt^2 + B(r) dr^2. \tag{24}
\]
The metric (24) is spherically symmetric but not the Schwarzschild type generally, because \(A(r)B(r) \neq 1\) generally. For the other fields near the black brane horizon, one can also derive that their effective field theories are two-dimensional field theories in a curved background with the metric (24).

### IV Schwarzschild type coordinate transformation

We find that it is not convenience to use the metric (24) to calculate the Hawking fluxes using the method of anomaly cancelation, especially for the calculation of the energy-momentum flux. However, we can make a coordinate transformation to transform the metric (24) to the Schwarzschild type equivalently.

We choose a new radial coordinate \(r^*\) where
\[
r^* = r^*(r) \quad \text{or} \quad r = r(r^*). \tag{25}
\]
We demand that under such a coordinate transformation, the metric (24) turns to the form
\[
d s^2 = -F(r^*) dt^2 + \frac{1}{F(r^*)} dr^{*2}, \tag{26}
\]
where
\[
F(r^*) = \sqrt{\frac{A(r^*)}{B(r^*)}}. \tag{27}
\]
Here we mean that the expression for the function $A(r^*)$ is the same as the expression for the function $A(r)$, the expression for the function $B(r^*)$ is the same as the expression for the function $B(r)$. Such a coordinate transformation is exist provided

$$\frac{dr}{dr^*} = \pm \frac{1}{\sqrt{A(r)B(r)}}. \quad (28)$$

However, we need not to solve its explicit form.

For the general form of a spherically symmetric black hole or black brane metric, as given by (12) and (24) including (3), $A(r)$ and $B(r)$ can always be decomposed into the form

$$A(r) = a(r)b(r), \quad B(r) = \frac{c(r)}{a(r)}, \quad (29)$$

where

$$a(r_H) = 0 \quad (30)$$

determines the horizon’s radius, as same as the radius of the surface of infinite red-shift, but $b(r_H^*) \neq 0, c(r_H^*) \neq 0$. After the coordinate transformation (25), for the metric (26), we have

$$F(r^*) = a(r^*)\sqrt{\frac{b(r^*)}{c(r^*)}}, \quad (31)$$

where the function forms of $a(r^*)$, $b(r^*)$, and $c(r^*)$ are the same as the function forms of $a(r)$, $b(r)$, and $c(r)$ respectively. Then the condition

$$a(r_H^*) = 0 \quad (32)$$

determines the horizon’s radius for the metric (26), and $b(r_H^*) \neq 0, c(r_H^*) \neq 0$. Because the expression for the function $a(r^*)$ is the same as the expression for the function $a(r)$, from (30) and (32), we have

$$r_H^* = r_H \quad (33)$$

which means that the location of the horizon for the metric (24) is the same as the location of the horizon for the metric (26). The location of the horizon is not changed after the coordinate transformation (25).

To insert (29) in (10), the Hawking temperature for the metric (24) can be evaluated as

$$T_H = \frac{1}{4\pi} \left( a'(r)\sqrt{\frac{b(r)}{c(r)}} \right) \bigg|_{r=r_H}. \quad (34)$$

While for the metric (26), considering (31), we can obtain that its Hawking temperature is

$$T_H^* = \frac{1}{4\pi} \left( a'(r^*)\sqrt{\frac{b(r^*)}{c(r^*)}} \right) \bigg|_{r=r_H^*}. \quad (35)$$

As mentioned above, the function forms of $a(r^*)$, $b(r^*)$, and $c(r^*)$ are the same as the function forms of $a(r)$, $b(r)$, and $c(r)$ respectively, and because of (33), we have

$$T_H = T_H^*. \quad (36)$$
which means that the Hawking temperature for the metric (24) is the same as the Hawking
temperature for the metric (26). The Hawking temperature for the metric (24) is not changed
after the coordinate transformation (25).

We need to check the chemical potential in addition. We denote the Ramond-Ramond
gauge field $A_{01...d-1}$ as $A_t$ for convenience in the following. The chemical potential for the
metric (24) can be written as

$$\mu = A_t(r_H) .$$  \hspace{1cm} (37)

Under the coordinate transformation (25), the Ramond-Ramond gauge field transforms to

$$A^*_t(r^*) = A_t(r(r^*)) .$$  \hspace{1cm} (38)

Thus the chemical potential for the metric (26) is given by

$$\mu^* = A^*_t(r^*_H) = A_t(r(r^*_H)) .$$  \hspace{1cm} (39)

We write down the second formula of (25) here again

$$r = r(r^*) .$$  \hspace{1cm} (40)

Because the coordinate transformations will always transform the horizon to horizon, we have

$$r_H = r(r^*_H) .$$  \hspace{1cm} (41)

To insert (41) in (39), we have

$$\mu^* = A_t(r_H) = \mu .$$  \hspace{1cm} (42)

This means that the chemical potential for the metric (24) related with the Ramond-Ramond
gauge field is not changed after the coordinate transformation (25) to the metric (26).

From the above analysis, we have seen that to perform a radial coordinate transformation
(25), we can transform the metric (24) to the form of (26) which is the Schwarzschild type.
However, the horizon’s location, the Hawking temperature, and the chemical potential are
not changed. Thus we can use the metric (26) equivalently in the calculation of Hawking
fluxes for the non-Schwarzschild type spherically symmetric black branes using the method
of anomaly cancelation in the following. However, in order to make the notations simple, in
the following sections, we still use the notation $r$ to represent $r^*$ for the metric (26) and all
the relevant field functions. Thus we write the metric (26) as

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 ,$$  \hspace{1cm} (43)

where

$$F(r) = \sqrt{\frac{A(r)}{B(r)}} = a(r) \sqrt{\frac{b(r)}{c(r)}} ,$$  \hspace{1cm} (44)

i.e., we get rid of the notation $*$ for the coordinate $r^*$ in all of the expressions. Thus for
the metric (43) or (26), its horizon radius, Hawking temperature, and chemical potential
can be represented by $(r_H, T_H, \mu)$. From the above analysis, they are just equal to the
horizon radius, Hawking temperature, and chemical potential $(r_H, T_H, \mu)$ for the metric (24)
respectively, while all of the results for the Hawking fluxes obtained from the method of
anomaly cancelation are only related with $(r_H, T_H, \mu)$ at last.
V Gauge anomaly and the charge flux

In this section, we study the gauge anomaly and derive the Ramond-Ramond charge flux for the black \((d - 1)\)-brane from the method of anomaly cancelation. But at present, we should remember that we are discussing the problem with respect to the metric (43), or (26). At last, we will turn back to the metric (24) equivalently.

Following [7, 8], to consider the area outside the horizon, we divide it into two parts: 

\[ [r_H, r_H + \epsilon] \text{ and } [r_H + \epsilon, \infty] \]

\([r_H, r_H + \epsilon]\) is the near horizon region, where the physics has certain exotic properties. \([r_H + \epsilon, \infty]\) is region departed from the horizon, where the physics has the usual properties. The parameter \(\epsilon\) can be taken arbitrarily small, thus for the observably physical results, we can take them in the region \([r_H + \epsilon, \infty]\) always.

In the near horizon region \([r_H, r_H + \epsilon]\), to consider that a black hole's horizon is a one-way membrane, for the \((1 + 1)\)-dimensional field theory, the ingoing (left moving) modes will tend to the center singularity. Therefore they will not affect the physics of the region \([r_H, r_H + \epsilon]\). That is to say in the region \([r_H, r_H + \epsilon]\), only the outgoing (right moving) modes are responsible for the observable physics. This makes the practical field theory to be a two-dimensional chiral field theory in the near horizon region. Thus in the region \([r_H, r_H + \epsilon]\), there exist the gauge and gravitational anomalies for the currents. In the region \([r_H + \epsilon, \infty]\), the ingoing and outgoing modes are both existing, the field theory is a normal one while not chiral.

For a black hole or a black brane, it is a thermodynamical equilibrium system, all currents in the spacetime outside the horizon are static. Thus we can write the gauge current outside the horizon as

\[ J^\mu (r) = J_\text{(in)}^\mu (r) H(r) + J_\text{(out)}^\mu (r) \Theta_+(r) , \quad (45) \]

where \(\Theta_+(r) = \Theta(r - r_+ - \epsilon)\) (here we use \(r_+\) to represent the radius of the event horizon), and \(H(r) = 1 - \Theta_+(r)\). From (45), \(J_\text{(out)}^\mu (r)\) is the current in the region \([r_H + \epsilon, \infty]\), it satisfies the ordinary conservation equation. Thus we have

\[ \partial_r J_\text{(out)}^r (r) = 0 \quad (46) \]

On the other hand, \(J_\text{(in)}^\mu (r)\) is the current in the region \([r_H, r_H + \epsilon]\). As mentioned above, it is anomalous and should obey the anomalous conservation equation [25, 26]

\[ \partial_r J_\text{(in)}^r (r) = \frac{e^2}{4\pi} \partial_r A_t (r) \quad (47) \]

The solutions to (46) and (47) are given by

\[ J_\text{(in)}^r = c_o , \]

\[ J_\text{(in)}^r (H) = c_H + \frac{e^2}{4\pi} (A_t (r) - A_t (r_H)) \quad (48) \]

where \(c_o\) and \(c_H\) are two integration constants. From (45) we can see that \(c_o\) is just the charge flux for an observer to obtain outside the horizon.

The current anomaly is a purely quantum field effect. Its existence does not change the gauge invariance of the effective action. A gauge transformation for the effective action of
the two-dimensional field theory results \[7, 8\]

\[
\delta W = - \int d^2x \sqrt{-g} \lambda \nabla_\mu J^\mu \\
= - \int d^2x \lambda \left[ \partial_r \left( \frac{e^2}{4\pi} A_t H \right) + \left( J^r_{(o)} - J^r_{(H)} + \frac{e^2}{4\pi} A_t \right) \delta(r - r_+ - \epsilon) \right], \tag{49}
\]

where \( \lambda \) is the gauge parameter. The first term in the second line of (49) can be canceled by the quantum effect of the ingoing modes near the horizon. Thus gauge invariance of the effective action leads the vanishing of the coefficient of the \( \delta \)-function. To combine (48) and (49), we obtain the relation

\[
c_o = c_H - \frac{e^2}{4\pi} A_t(r_H). \tag{50}
\]

The constant \( c_H \) in (50) can be determined through introducing the covariantly anomalous current

\[
\tilde{J}^r(r) = J^r(r) + \frac{e^2}{4\pi} A_t(r) H(r), \tag{51}
\]

together with the boundary condition

\[
\tilde{J}^r(r_H) = 0, \tag{52}
\]

i.e., the covariant current vanishes on the horizon \[7, 8\]. As pointed out in \[7, 8\], such a condition makes physical quantities regular on the future horizon. To combine (45), (51), and (52), we obtain

\[
J^r_{(H)}(r_H) = - \frac{e^2}{4\pi} A_t(r_H). \tag{53}
\]

To combine (48) and (53), we obtain

\[
c_H = J^r_{(H)}(r_H) = - \frac{e^2}{4\pi} A_t(r_H). \tag{54}
\]

To substitute (54) in (50), we obtain

\[
c_o = - \frac{e^2}{2\pi} A_t(r_H) = - \frac{e^2}{2\pi} \mu, \tag{55}
\]

where \( \mu = A_t(r_H) \) is the chemical potential given by (37) or (42). Thus we have obtained the Ramond-Ramond charge flux for the two-dimensional black hole metric (43) or (26). In the following, we can see that (55) matches the charge flux of a two-dimensional black body radiation with the chemical potential \( \mu = A_t(r_H) \).

From the above derivation, we can see that the anomalous current \( c_H \) near the horizon (the outgoing chiral current) has contribution to the current \( c_o \) departed from the horizon which is a normal one. Or we can say that the existence of the outgoing chiral current makes the radiation current be a normal one and cancels its anomaly.
VI Gravitational anomaly and the energy-momentum flux

In this section, we calculate the energy-momentum flux for the black \((d-1)\)-brane from the method of anomaly cancelation. Just like that of Sec. V, at present, we are discussing the problem with respect to the metric (43), or the metric (26). At last, we will turn back to the metric (24) equivalently. Because all currents are static, like that in Sec. V, we decompose the energy-momentum tensor outside the horizon as

\[ T_{\nu}^{\mu}(r) = T_{\nu(H)}^{\mu}(r)H(r) + T_{\nu(o)}^{\mu}(r)\Theta_+(r) , \tag{56} \]

where \( T_{\nu(H)}^{\mu}(r) \) is the energy-momentum tensor in the region \([r_H, r_H + \epsilon]\), \( T_{\nu(o)}^{\mu}(r) \) is the energy-momentum tensor in the region \([r_H + \epsilon, \infty]\).

In a two-dimensional spacetime, \( T_{r}^{r}(r) \) is just the energy-momentum flux at the radial direction. For the spacetime of a charged black hole or black brane, its energy-momentum is composed by gravitational fields and gauge fields. For the current \( T_{\nu(o)}^{\mu}(r) \), because there are no gauge and gravitational anomalies, it satisfies the ordinary conservation equation. Thus we have

\[ \partial_r T_{\nu(o)}^{r}(r) = J_{\nu(o)}^{r}(r)\partial_r A_t(r) , \tag{57} \]

where the right hand side of (57) comes from the background charge current. On the other hand, for the current \( T_{\nu(H)}^{r}(r) \) near the horizon, it has gauge and gravitational anomalies. It satisfies the anomalous conservation equation \([7, 8, 27, 28]\)

\[ \partial_r T_{\nu(H)}^{r}(r) = J_{\nu(H)}^{r}(r)\partial_r A_t(r) + A_t(r)\partial_r J_{\nu(H)}^{r}(r) + \partial_r N_r^{r}(r) , \tag{58} \]

where the first term comes from the background charge current, the second term comes from the gauge anomaly, and the third term comes from the gravitational anomaly for the consistent energy-momentum tensor. For the two-dimensional metric (43), one gets \([7, 8]\)

\[ N_r^{r}(r) = \frac{1}{192\pi} \left( (F'(r))^2 + F''(r)F(r) \right) . \tag{59} \]

The integration of (57) and (58) gives

\[ T_{\nu(o)}^{r}(r) = a_o + c_o A_t(r) , \]
\[ T_{\nu(H)}^{r}(r) = a_H + \int_{r_H}^{r} dr\partial_r \left( c_o A_t(r) + \frac{e^2}{4\pi} A_t^2(r) + N_r^{r}(r) \right) , \tag{60} \]

where \( c_o = J_{\nu(o)}^{r} \) is given by (55), \( a_o, a_H \) are two integration constants. From (56) we can see that \( a_o \) is just the energy-momentum flux for an observer to measure outside the horizon. \( c_o A_t(r) \) is the potential energy of the charge current in the background gauge field \( A_t(r) \).

On the other hand, the anomaly of the energy-momentum tensor is a purely quantum field effect of the gravitational field. It does not affect the general covariance of the effective action. To perform an infinitesimal coordinate transformation for the two-dimensional field theory along the time direction with the parameter \( \xi^t \), we obtain \([7, 8]\)

\[ \delta W = - \int d^2x \sqrt{-g} \xi^t \partial_{\nu}T_{\nu}^{r} \]
\[ = - \int d^{3}x \xi^{t} \left[ c_{o} \partial_{t} A_{t} + \partial_{r} \left( \frac{e^{2}}{4\pi} A_{t}^{2} + N_{t}^{r} \right) H \right] + \left( T_{r(0)}^{r} - T_{r(H)}^{r} + \frac{e^{2}}{4\pi} A_{t}^{2} + N_{t}^{r} \right) \delta(r - r_{+} - \epsilon) \] . \tag{61}

The first term in the second line of (61) comes from the constant charge current \( c_{o} \) in the background gauge field \( A_{t}(r) \). The second term can be canceled by the quantum effect of the ingoing modes near the horizon. Thus general covariance of the effective action leads the vanishing of the coefficient of the \( \delta \)-function. This requirement leads to the following relation
\[ a_{o} = a_{H} + \frac{e^{2}}{4\pi} A_{t}^{2}(r_{H}) - N_{t}^{r}(r_{H}) . \tag{62} \]

From (44), (30), and (59), we have
\[ N_{r}^{r}(r_{H}) = \frac{1}{192\pi} \left( a'(r) \left( \frac{b(r)}{c(r)} \right) \right)^{2} \bigg|_{r=r_{H}} . \tag{63} \]

In order to determine the constant \( a_{H} \) of (62), we need to introduce the covariantly anomalous energy-momentum tensor \( \bar{T}_{\mu\nu} \) which satisfies the covariant anomaly equation \cite{25, 28}
\[ \nabla^{\mu} \bar{T}_{\mu\nu} = \frac{1}{96\pi \sqrt{-g}} \epsilon_{\mu\nu} \partial^{\mu} R , \tag{64} \]
where \( R \) is the Ricci scalar. For the component \( \bar{T}_{t}^{r} \) which is necessary for the following calculation, for the metric (43), it is obtained as \cite{7, 8}
\[ \bar{T}_{t}^{r}(r) = T_{t}^{r}(r) + \frac{1}{192\pi} \left[ F(r) F''(r) - 2(F'(r))^{2} \right] . \tag{65} \]

\( \bar{T}_{t}^{r} \) satisfies the regular boundary condition \cite{7, 8}
\[ \bar{T}_{t}^{r}(r_{H}) = 0 . \tag{66} \]

This boundary condition makes physical quantities regular on the future horizon for a free fall observer \cite{7, 8, 9}. To combine (65), (66), and (56), we obtain
\[ T_{t(H)}^{r}(r_{H}) = \frac{1}{192\pi} \left[ 2(F'(r))^{2} + F(r) F''(r) \right] \bigg|_{r=r_{H}} . \tag{67} \]

To insert (44) in (67), we obtain
\[ T_{t(H)}^{r}(r_{H}) = \frac{1}{96\pi} \left( a'(r) \left( \frac{b(r)}{c(r)} \right) \right)^{2} \bigg|_{r=r_{H}} . \tag{68} \]

To combine (68) and (60), we obtain
\[ a_{H} = \frac{1}{96\pi} \left( a'(r) \left( \frac{b(r)}{c(r)} \right) \right)^{2} \bigg|_{r=r_{H}} . \tag{69} \]
To substitute (63) and (69) in (62), we obtain
\[ a_o = \frac{e^2}{4\pi} A_t^2(r_H) + \frac{1}{192\pi} \left( d'(r) \sqrt{\frac{b(r)}{c(r)}} \right)^2 \left|_{r=r_H} \right. . \] (70)

To compare (34) and (70), we can write
\[ a_o = \frac{e^2}{4\pi} A_t^2(r_H) + \frac{\pi}{12} T_H^2, \] (71)

where \( T_H \) is just the Hawking temperature for the metric (43) or (26). As mentioned above, \( a_o \) is just the energy-momentum flux for an observer to measure departed from the horizon.

From the above derivation, we can see that the anomalous current \( a_H \) near the horizon (the outgoing chiral current) has contribution to the current \( a_o \) departed from the horizon which is a normal one. Or we can say that the existence of the outgoing chiral current makes the radiation current be a normal one and cancels its anomaly.

So far we have obtained the outgoing charge and energy-momentum fluxes (55) and (71) for the metric (43) or (26). To perform a coordinate transformation (25), we can transform the metric (26) or (43) to the form of (24), which is the original form of the non-Schwarzschild type two-dimensional spherically symmetric metric. Because the fluxes (55) and (71) are constants not related with the coordinate \( r \), they will not change after this coordinate transformation. Thus we can draw the conclusion that for the metric (24), it has the same charge and energy-momentum fluxes represented by (55) and (71). Next in the following, we can see that the fluxes (55) and (71) obtained above just match the fluxes of a two-dimensional black body radiation with the temperature \( T_H \) and chemical potential \( \mu = A_t(r_H) \).

To consider a two-dimensional black body radiation with the temperature \( T \), the distribution for a fermion field carrying charge \( e \) is given by
\[ N_e(\omega) = \frac{1}{e^{(\omega-e\Phi)/T} + 1}, \] (72)

where \( \Phi \) is the chemical potential. Here we consider the distribution of a fermion field in order to avoid the superradiance problem when applied to the case of a black hole [7, 8]. The charge flux from this two-dimensional black body radiation can be obtained as
\[ F_e = e \int_0^\infty \frac{d\omega}{2\pi}(N_e(\omega) - N_{-e}(\omega)) = -\frac{e^2}{2\pi} \Phi . \] (73)

The energy-momentum flux from this two-dimensional black body radiation can be obtained as
\[ F_E = \int_0^\infty \frac{d\omega}{2\pi} (N_e(\omega) + N_{-e}(\omega)) = \frac{e^2\Phi^2}{4\pi} + \frac{\pi}{12} T^2 . \] (74)

To compare (55) and (71) with (73) and (74), we know that (55) and (71) are just the charge flux and energy-momentum flux of a two-dimensional black body radiation with the temperature \( T_H \) and chemical potential \( \mu = A_t(r_H) \). Therefore we know that the two-dimensional black hole metric (26) or (43) has a black body radiation with the temperature \( T_H \) and chemical potential \( \mu = A_t(r_H) \). As discussed above, the original two-dimensional
metric (24) for (26) and (43) has the same black body radiation with the temperature $T_H$ and chemical potential $\mu = A_t(r_H)$. But we know that $T_H$ is just the Hawking temperature given by (34) or (10), $A_t(r_H)$ is just the chemical potential given by (37) or (11) for the two-dimensional black hole metric (24). Thus we have deduced the existence of Hawking radiation for the two-dimensional non-Schwarzschild type spherically symmetric metric (24) from the method of anomaly cancelation, with the temperature and chemical potential in accordance with its thermodynamics. For its original higher-dimensional metrics (3) and (12), from the mode decomposition for fields as the form of (18), it is not difficult to see that the distributions for the radiation fields on spectrum will not change. It is still on the temperature $T_H$ and chemical potential $A_t(r_H)$.

VII Conclusion

We have applied the method of anomaly cancelation for the calculation of Hawking radiation proposed by Robinson and Wilczec et al. [7, 8] to black p-branes of superstring theories [20, 21, 22, 23, 24]. These black p-brane metrics can represent general higher-dimensional non-Schwarzschild type spherically symmetric black holes and black branes. In [16], the method of anomaly cancelation for the calculation of Hawking radiation has been applied for the Garfinkle-Horowitz-Strominger black hole of superstring theories [18]. In [17], the method of anomaly cancelation for the calculation of Hawking radiation has been applied for the $D1-D5$ brane black holes of superstring theories [19]. Although the black hole metrics studied in [16, 17] are also non-Schwarzschild type spherically symmetric, they belong to some special cases of general non-Schwarzschild type spherically symmetric metrics. Therefore it is necessary to investigate the practicality for the method of anomaly cancelation for the calculation of Hawking radiation for general higher-dimensional non-Schwarzschild type spherically symmetric black holes and black branes. In addition, the method adopted in this paper is different from those of [16, 17].

We find that in order to make the method of Robinson and Wilczec et al. effective for the calculation of Hawking fluxes for general non-Schwarzschild type spherically symmetric black hole and black brane metrics, it is necessary to make a coordinate transformation to transform them to the Schwarzschild type. Then we derived the charge and energy-momentum fluxes using the method of anomaly cancelation for the transformed two-dimensional Schwarzschild type metric. We find that they exactly match the two-dimensional black body radiation with the temperature $T_H$ and chemical potential $\mu$, where $T_H$ and $\mu$ are just the Hawking temperature and chemical potential of the transformed two-dimensional Schwarzschild type metric. Because the charge and energy-momentum fluxes obtained are constants not related with the coordinate $r$, we deduce that the original non-Schwarzschild type spherically symmetric metric has the same charge and energy-momentum fluxes. At the same time, because the Hawking temperature $T_H$ and chemical potential $\mu$ for a non-Schwarzschild type spherically symmetric metric will not change after the Schwarzschild type coordinate transformation as shown in Sec. IV, we draw the conclusion that the original two-dimensional non-Schwarzschild type spherically symmetric metric has the two-dimensional black body radiation with the temperature $T_H$ and chemical potential $\mu$, where $T_H$ and $\mu$ match the Hawking temperature and chemical potential obtained from black hole thermodynamics. And the spectrum distributions are the same for the whole higher-dimensional object. Thus the method of anomaly cancelation for the calculation of Hawking radiation
can be used to general higher-dimensional non-Schwarzschild type spherically symmetric black holes and black branes.

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