Many Accelerating Black Holes

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March 24, 2022

Abstract

We show how the Weyl formalism allows metrics to be written down which correspond to arbitrary numbers of collinear accelerating neutral black holes in 3+1 dimensions. The black holes have arbitrary masses and different accelerations and share a common acceleration horizon. In the general case, the black holes are joined by cosmic strings or struts that provide the necessary forces that, together with the inter black hole gravitational attractions, produce the acceleration. In the cases of two and three black holes, the parameters may be chosen so that the outermost black hole is pulled along by a cosmic string and the inner black holes follow behind accelerated purely by gravitational forces. We conjecture that similar solutions exist for any number of black holes.

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QMUL-PH-01-05, HUTP-00/A039
1 Introduction

The Weyl formalism for static axisymmetric vacuum solutions in 3+1 dimensions allows one to write down a solution that represents an arbitrary number of collinear neutral black holes held in unstable equilibrium by a system of struts and strings [1]. The metric is constructed using the Newtonian potential of a number of collinear rods of differing lengths and the same mass per unit length. The thermodynamics of such configurations has been studied recently in [2]. The Weyl construction was used by Myers to produce an infinite periodic array of neutral black holes, in which case the strings and struts disappear [3]. It has also been long known that the C-metric for an accelerating black hole has an interesting Weyl form [4, 5, 6, 7]. It is constructed from the Newtonian potential of a semi-infinite-line-mass (silm) and a second, finite, rod. The silm is the position of the acceleration horizon and the finite rod is the position of the black hole horizon. There is necessarily a cosmic string or strut in the spacetime which provides the force accelerating the black hole. It is now easy to guess the form of the metric which describes a number, \( n \), of collinear black holes all accelerating in the same direction. It is constructed from the Newtonian potential of a silm and \( n \) finite non-overlapping collinear rods of arbitrary lengths.

We examine the \( n = 2 \) case in detail. We show that the interpretation as two accelerating black holes is valid. In the “Newtonian limit” where the accelerations are small compared to the inverse masses of the black holes, Newton’s law holds for each black hole with the acceleration produced by a combination of the forces due to any deficit angle cosmic strings or struts present and the gravitational attraction of the other black hole.

We show that the parameters may be chosen so that the outer black hole is pulled along by a cosmic string and the inner black hole follows behind accelerated purely by its gravitational attraction to the outer one. A similar solution can be found for the \( n = 3 \) case, in which the only deficit angle along the axis is a cosmic string attached to the outermost black hole and the other two are accelerated by gravitational forces alone. We conjecture that similar solutions exist for any number of black holes.

2 The Weyl form of the C-metric

Any static axisymmetric vacuum solution of the Einstein equations may be written in the following form

\[
ds^2 = \alpha^2(dr^2 + dz^2) + r^2\gamma^{-2}d\phi^2 - \gamma^2 dt^2
\]  

(1)

where \( \alpha, \gamma \) are functions of \( r, z \). By using the substitutions

\[
\begin{align*}
\alpha &= e^{\nu-\lambda} \\
\gamma &= e^{\lambda}
\end{align*}
\]  

(2)
the above metric is transformed into the Weyl canonical form
\[ ds^2 = e^{2(\nu - \lambda)}(dr^2 + dz^2) + r^2e^{-2\lambda}d\phi^2 - e^{2\lambda}dt^2 . \]
(3)

\( \lambda \) must satisfy Laplace’s equation
\[ \Delta^2 \lambda = \frac{d^2 \lambda}{dr^2} + \frac{1}{r} \frac{d \lambda}{dr} + \frac{d^2 \lambda}{dz^2} = 0 \]
(4)

and \( \nu \) can be found by using
\[
\begin{align*}
\frac{d\nu}{dr} &= r\left(\frac{d\lambda}{dr}\right)^2 - \left(\frac{d\lambda}{dz}\right)^2 \\
\frac{d\nu}{dz} &= 2r \frac{d\lambda}{dr} \frac{d\lambda}{dz} 
\end{align*}
\]
(5)

In [1] the case of a collinear array of arbitrarily spaced black holes of arbitrary masses was studied. In that case the Newtonian potential used for \( \lambda \) is the sum of the potentials of a number of finite nonoverlapping rods placed along the \( z \)-axis and the function \( \nu \) is found by quadrature. Explicitly, let there be \( n \) rods each centred at \( z = a_i, \ i = 1, \ldots, n \), of length \( b_i \) and with mass/unit length \( \frac{1}{2} \). The distances to the two ends of rod \( i \) are given by
\[
\begin{align*}
\rho_i^2 &= r^2 + z_i^2 \\
\rho_i'^2 &= r^2 + z_i'^2 \\
z_i &= z - (a_i + \frac{1}{2}b_i) \\
z_i' &= z - (a_i - \frac{1}{2}b_i) .
\end{align*}
\]
(6)

Then
\[
\lambda = \sum_{i=1}^{n} \lambda_i(r, z) ; \quad \nu = \sum_{i,j=1}^{n} \nu_{ij}(r, z) ,
\]
where
\[
\begin{align*}
\lambda_i &= \frac{1}{2} \ln \left( \frac{\rho_i' - z_i'}{\rho_i - z_i} \right) \\
\nu_{ij} &= \frac{1}{4} \ln \left( \frac{E(i', j)E(i, j')}{E(i, j)E(i', j')} \right)
\end{align*}
\]
(7)

where, for instance
\[ E(i', j) = \rho_i' \rho_j + z_i' z_j + r^2 . \]

As stated in the introduction, the C-metric has the Weyl form built from the potential of a semi-infinite line mass and a finite rod along the \( z \)-axis. Consider the silm to stretch
from a point $z = c_0$ to $z = +\infty$ and the finite rod to be between $z = c_2$ and $z = c_1$ so that $c_2 < c_1 < c_0$. They both have mass/unit length $\frac{1}{2}$ as before. We are clearly free to choose the origin of $z$ coordinates to be wherever we like and so we fix it by requiring that the following three equations have a solution for $A$ and $m$

$$2Ac_i^3 - c_i^2 + m^2 = 0, \quad i = 0, 1, 2.$$ 

In particular this means that $c_2 < 0 < c_1 < c_0$.

Then the following metric is the Weyl metric constructed from the corresponding Newtonian potential

$$ds^2 = -A^{-1}X_0X_1^{-1}X_2dt^2 + A^{-1}r^2[X_0X_1^{-1}X_2]^{-1}d\phi^2 + \frac{m^2}{4A^3}(c_0 - c_1)^2(c_1 - c_2)^2\frac{Y_{01}Y_{21}}{R_0R_1R_2Y_{02}}(dz^2 + dr^2)$$

where

$$\zeta_i = z - c_i, \quad R_i = (r^2 + \zeta_i^2)^{\frac{1}{2}}, \quad X_i = R_i - \zeta_i, \quad Y_{ij} = R_iR_j + \zeta_i\zeta_j + r^2.$$ 

The overall scale and normalisations of the $t$ and $\phi$ coordinates are chosen with hindsight so that the following coordinate transformation gives the C-metric in its most familiar form. Consider the coordinates $(x, y)$ related to $(z, r)$ by

$$z = A^{-1}(x - y)^{-2}(1 - mAx(x + y) - xy)$$

$$r = A^{-1}(x - y)^{-2}(-G(y))^{\frac{1}{2}}(G(x))^{\frac{1}{2}}$$

where the function $G$ is the cubic

$$G(\xi) = 1 - \xi^2 - 2mA\xi^3$$

$$= -2mA(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4).$$

The roots of $G, \xi_2 < \xi_3 < \xi_4$ are so labelled for historical reasons and there is the following relationship between them and the $c_i$

$$c_0 = -m\xi_2, \quad c_1 = -m\xi_3, \quad c_2 = -m\xi_4.$$ 

In terms of $(x, y)$ coordinates the metric \(8\) becomes the familiar

$$ds^2 = \frac{1}{A^2(x - y)^2} \left[ G(y)dt^2 - \frac{dy^2}{G(y)} + G(x)d\phi^2 + \frac{dx^2}{G(x)} \right].$$  

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3 Two accelerating black holes

In the maximally extended form of the ordinary C-metric above there are actually two
accelerating black holes, one covered by the coordinates shown above and the other on
the opposite side of the acceleration horizon, accelerating in the opposite direction and
out of causal contact with the first. It is possible however to construct metrics in which
there are two (or more) black holes on the same side of a common acceleration horizon,
accelerating in the same direction. (These then each have their “mirror black hole” on
the other side of the acceleration horizon but we will be ignoring these in what follows.)
These are formed by taking the Weyl form with a potential constructed from a silm and
two (or more) finite rods along the z-axis.

Consider the silm and first rod as in the previous section and add a second rod
between \( z = c_4 \) and \( z = c_3 \) where \( c_4 < c_3 < c_2 \) as illustrated in figure 1.

![Figure 1: The thick lines correspond to the positions of line masses along the z-axis and the dotted thick line indicates the infiniteness of the right-handmost line mass.](image)

The corresponding Weyl metric is

\[
ds^2 = -\alpha^{-1}X_0X_1^{-1}X_2X_3^{-1}X_4dt^2 + \alpha^{-1}r^2[X_0X_1^{-1}X_2X_3^{-1}X_4]^{-1}d\phi^2 + \frac{\beta^2}{4\alpha}2R_0Y_{21}Y_{03}Y_{34}Y_{14}Y_{23}(dz^2 + dr^2)
\]

where \( \alpha \) is a constant with units \( L^{-1} \) and \( \beta \) is a dimensionless constant which we discuss
further below.

Recall that we chose the origin of \( z \) so that the equations \( \phi = \alpha \) have a solution for \( A \)
and \( m \). Now let \( \hat{k} \) be such that

\[
2\hat{A}(c_i - \hat{k})^3 - (c_i - \hat{k})^2 + \hat{m}^2 = 0, \quad i = 0, 3, 4
\]

have solutions for \( \hat{A} \) and \( \hat{m} \).

We want this metric to become the C-metric in three different limits

\[
(i) \quad c_1 \to c_2 \quad (15)
(ii) \quad c_3 \to c_4 \quad (16)
(iii) \quad c_2 \to c_3 \quad (17)
\]
In limit (ii) we want to obtain the C-metric exactly as it appears in \((8)\). In limit (i) ((iii)) the C-metric would be expressed in terms of parameters \(\{c_0, c_3, c_4\}\) \(\{c_0, c_1, c_4\}\) instead of \(\{c_0, c_1, c_2\}\).

In order for the limits to work we require \(\alpha\) and \(\beta\) to behave in the following ways

\[
\begin{align*}
(i) \quad (\alpha, \beta) & \to \left(\widehat{\alpha}, \frac{m}{\widehat{A}}(c_0 - c_3)^{-1}(c_3 - c_4)^{-1}\right) \quad \text{as} \quad c_1 \to c_2 \quad (18) \\
(ii) \quad (\alpha, \beta) & \to \left(\alpha, \frac{m}{\widehat{A}}(c_0 - c_1)^{-1}(c_1 - c_2)^{-1}\right) \quad \text{as} \quad c_3 \to c_4 \quad (19) \\
(iii) \quad (\alpha, \beta) & \to \left(\widehat{\alpha}, \frac{m}{\widehat{A}}(c_0 - c_1)^{-1}(c_1 - c_4)^{-1}\right) \quad \text{as} \quad c_2 \to c_3 \quad (20)
\end{align*}
\]

where \(\widehat{A}\) and \(\ldots\) depend on \(c_0, c_1, c_4\) via

\[
2\widehat{A}(c_i - \bar{k})^3 - (c_i - \bar{k})^2 + m^2 = 0, \quad i = 0, 1, 4 \quad (21)
\]

for some \(\bar{k}\).

These conditions are not enough to fix the dependence of \(\alpha\) and \(\beta\) on \(c_0, \ldots, c_4\) but there are solutions.

Let Black Hole 1 (BH1) coordinates \((x, y)\) be defined as before

\[
\begin{align*}
z & = A^{-1}(x - y)^{-2}(1 - m\alpha x y(x + y) - xy) \\
r & = A^{-1}(x - y)^{-2}(-G(y))^{\frac{1}{2}}(G(x))^{\frac{1}{2}}. \quad (22)
\end{align*}
\]

Let Black Hole 2 (BH2) coordinates, \((\hat{x}, \hat{y})\) be given by

\[
\begin{align*}
z - \hat{k} & = \hat{A}^{-1}(\hat{x} - \hat{y})^{-2}(1 - \hat{m}\hat{A}\hat{\alpha}\hat{x}\hat{y}(\hat{x} + \hat{y}) - \hat{x}\hat{y}) \\
r & = \hat{A}^{-1}(\hat{x} - \hat{y})^{-2}(-\hat{G}(\hat{y}))^{\frac{1}{2}}(\hat{G}(\hat{x}))^{\frac{1}{2}}. \quad (23)
\end{align*}
\]

where the function \(\hat{G}\) is the cubic

\[
\hat{G}(\hat{\xi}) = 1 - \hat{\xi}^2 - 2\hat{m}\hat{A}\hat{\xi}^3
\]

\[
= -2\hat{m}\hat{A}(\hat{\xi} - \hat{\xi}_2)(\hat{\xi} - \hat{\xi}_3)(\hat{\xi} - \hat{\xi}_4) \quad (24)
\]

and

\[
c_0 - \hat{k} = -\hat{m}\hat{\xi}_2, \quad c_3 - \hat{k} = -\hat{m}\hat{\xi}_3, \quad c_4 - \hat{k} = -\hat{m}\hat{\xi}_4. \quad (25)
\]

Then we can transform the metric \((13)\) into BH1 and BH2 coordinates and it becomes, respectively

\[
\begin{align*}
ds^2 & = \frac{1}{A^2(x - y)^2} \left[ \frac{A}{\alpha} \left( \frac{X_0 G(y)}{X_3 G(x)} dt^2 + \frac{X_3 G(y)}{X_4 G(x)} d\phi^2 \right) \\
&+ \frac{\beta^2 A^3}{\alpha m^2} (c_0 - c_1)^2(c_1 - c_2)^2 \frac{Y_{03}Y_{14}Y_{23}}{2R_3R_4R_{04}Y_{31}Y_{42}} \left( -\frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} \right) \right] \quad (26)
\end{align*}
\]

\[
\begin{align*}
&+ \frac{\beta^2 A^3}{\alpha m^2} (c_0 - c_1)^2(c_1 - c_2)^2 \frac{Y_{03}Y_{14}Y_{23}}{2R_3R_4R_{04}Y_{31}Y_{42}} \left( -\frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} \right) \quad (27)
\end{align*}
\]
and

\[ ds^2 = \frac{1}{A^2(\tilde{z} - \tilde{y})^2} \left[ \frac{\hat{A}}{\alpha} \left( X_2 \tilde{G}(\tilde{y}) dt^2 + X_1 \tilde{G}(\tilde{x}) d\phi^2 \right) \right. \]

\[ + \frac{\beta^2 \hat{A}^3}{\alpha m^2} (c_0 - c_3)^2 (c_3 - c_4)^2 \frac{Y_{01} Y_{12} Y_{32} Y_{41}}{2R_1 R_2 Y_{02} Y_{13} Y_{24}} \left( -\frac{d\tilde{y}^2}{G(\tilde{y})} + \frac{d\tilde{x}^2}{G(\tilde{x})} \right) \left( \frac{\hat{A}}{\alpha} \left( X_2 \tilde{G}(\tilde{y}) dt^2 + X_1 \tilde{G}(\tilde{x}) d\phi^2 \right) \right] \quad (28) \]

Let us examine the solution close up to BH1 in BH1 coordinates. We consider the limit of (13) where

\[ A^{-2} \left[ f(x) \left( p(y - \xi_2) G'(\xi_2) dt^2 - q \frac{dy^2}{(y - \xi_2) G'(\xi_2)} + q \frac{dx^2}{G(x)} \right) + p \frac{G(x)}{f(x)} d\phi^2 \right] \quad (29) \]

where

\[ f(x) = \frac{A^{-1}(x - \xi_2)^{-2} (1 - mA x \xi_2 (x + \xi_2) - \xi_2 x) - c_3}{A^{-1}(x - \xi_2)^{-2} (1 - mA x \xi_2 (x + \xi_2) - \xi_2 x) - c_4} \]

\[ p = A \alpha^{-1} \]

\[ q = \frac{\beta^2 A^3}{\alpha m^2} (c_1 - c_4)^2 (c_0 - c_1)^2 (c_1 - c_2)^2 (c_0 - c_3)^2 (c_1 - c_2)^2 (c_0 - c_4)^2 (c_0 - c_4)^{-2} \quad (30) \]

From this we cannot read off the surface gravity of the black hole as we can in the asymptotically flat case because the timelike coordinate \( t \) is not Minkowski time at spatial infinity but, rather, a Rindler-like time. To see this we consider the limit of (33) as \( r, z \to \infty \).

\[ ds^2 \to -\zeta^2 d\eta^2 + d\xi^2 + d\rho^2 + \rho^2 d\psi^2 \quad (31) \]

where

\[ \zeta^2 = \frac{\beta^2}{\alpha} \left( \sqrt{r^2 + z^2} - z \right) \]

\[ \rho^2 = \frac{\beta^2}{\alpha} r^2 \left( \sqrt{r^2 + z^2} - z \right)^{-1} \]

\[ \eta = \frac{1}{\beta} t \]

\[ \psi = \frac{1}{\beta} \phi \quad (32) \]

and \( \beta = (c_0 - c_1)(c_1 - c_2)(c_0 - c_3)(c_3 - c_4) \). This tells us that the period, \( \Delta \tau_{\text{AH}} \), of the euclidean time coordinate \( \tau = it \) that would be required for the euclideanized solution to have no conical singularity at the acceleration horizon is \( \Delta \tau_{\text{AH}} = 2\pi \beta \).

We can, however, read off from (33) the period, \( \Delta \tau_{\text{BH1}} \), that the euclidean time coordinate \( \tau = it \) would have to have in order for the euclideanized solution to be regular on the horizon of BH1. It is

\[ \Delta \tau_{\text{BH1}} = 2\pi \beta \frac{(c_0 - c_2)(c_0 - c_3)(c_1 - c_4)}{(c_0 - c_2)(c_0 - c_4)(c_1 - c_3)} \quad (33) \]
We can also examine the smoothness conditions along the axis in either direction from BH1. Let the period of $\phi$ necessary for smoothness on the axis pointing towards the acceleration horizon, i.e. $r = 0$ and $z = c_1$, be $\Delta \phi_{1R}$ and the period of $\phi$ necessary for smoothness on the axis pointing towards the outer black hole BH2, i.e. $r = 0$, $z = c_2$ be $\Delta \phi_{1L}$. When $r$ is small, the limit $z \to c_1$ corresponds to $x \to \xi_4$ and examining (24) in that limit we can read off
\[ \Delta \phi_{1R} = 2\pi \beta \frac{(c_0 - c_1)(c_0 - c_3)}{(c_0 - c_2)(c_0 - c_4)} \] (34)

On the other side, $z \to c_2$ when $x \to \xi_3$ and
\[ \Delta \phi_{1L} = 2\pi \beta \frac{(c_1 - c_4)(c_2 - c_3)(c_0 - c_3)}{(c_1 - c_3)(c_0 - c_4)(c_2 - c_4)} \] (35)

We can repeat this all for BH2, looking at (28) near the horizon of BH2. In this limit the metric becomes
\[ \frac{\hat{A}^{-2}}{(\hat{x} - \hat{\xi}_2)^2} \left[ \hat{f}(\hat{x}) \left( \hat{p}(\hat{y} - \hat{\xi}_2)\hat{G}'(\hat{\xi}_2)dt^2 - \hat{q} \frac{d\hat{y}^2}{(\hat{y} - \hat{\xi}_2)\hat{G}'(\hat{\xi}_2)} + \hat{q} \frac{d\hat{x}^2}{\hat{G}(\hat{x})} \right) + \hat{p} \frac{\hat{G}(\hat{x})}{\hat{f}(\hat{x})} d\phi^2 \right] \] (36)
where
\[ \hat{f}(\hat{x}) = \frac{\hat{A}^{-1}(\hat{x} - \hat{\xi}_2)^{-2}(1 - \hat{m} \hat{A}^2 \hat{\xi}_2 \hat{x} + \hat{\xi}_2) - c_2}{\hat{A}^{-1}(\hat{x} - \hat{\xi}_2)^{-2}(1 - \hat{m} \hat{A}^2 \hat{\xi}_2 \hat{x} + \hat{\xi}_2) - \hat{\xi}_2 \hat{x} - c_1} \]
\[ \hat{p} = \hat{A} \alpha^{-1} \]
\[ \hat{q} = \frac{\beta^2 \hat{A}^3}{\alpha \hat{m}^2}(c_1 - c_4)^2(c_0 - c_3)^2(c_3 - c_4)^2(c_2 - c_4)^{-2} \] (37)

The period of $\tau = it$ required for regularity on the BH2 horizon is
\[ \Delta \tau_{BH2} = 2\pi \beta \frac{(c_1 - c_4)(c_3 - c_4)}{(c_2 - c_4)(c_0 - c_4)} \] (38)

The period of $\phi$ required for smoothness on the axis pointing towards BH1 and the acceleration horizon is
\[ \Delta \phi_{2R} = 2\pi \beta \frac{(c_1 - c_4)(c_2 - c_3)(c_0 - c_3)}{(c_1 - c_3)(c_0 - c_4)(c_2 - c_4)} \] (39)

Notice that this is equal to $\Delta \phi_{1L}$ as it should be because these refer to the same part of the axis, that between the black holes. The period of $\phi$ required for smoothness on the outer axis is
\[ \Delta \phi_{2L} = 2\pi \beta \] (40)
It is possible to choose the parameters $c_i$, $i = 0, \ldots, 4$ so that $\Delta \tau_{BH1} = \Delta \tau_{BH2}$. Indeed this condition implies

$$(c_0 - c_4)(c_1 - c_2) = (c_0 - c_2)(c_3 - c_4)$$  \hspace{1cm} (41)

which can be rewritten as

$$(c_1 - c_2) [(c_0 - c_3) + (c_3 - c_4)] = [(c_0 - c_1) + (c_1 - c_2)] (c_3 - c_4)$$  \hspace{1cm} (42)

and finally as

$$(c_3 - c_4) = (c_1 - c_2)(c_0 - c_3)(c_0 - c_1)^{-1} .$$  \hspace{1cm} (43)

Any choice of $c_0, c_1, c_2$ and $c_3$ can be made and $c_4$ is then fixed.

But it is not possible to have either $\Delta \tau_{BH1} = \Delta \tau_{AH}$ or $\Delta \tau_{BH2} = \Delta \tau_{AH}$, which is not a surprise given the same is true for the C-metric. Thus, on euclideanizing the solution we may choose the period of imaginary time to make both black hole horizons regular but if we do so there will be a conical singularity at the acceleration horizon.

We check that in the weak field limit, \textit{i.e.} $mA << 1$ and $\hat{m} \hat{A} << 1$ and when the distance between the black holes is large compared to their masses, the acceleration of BH2 is given by Newton’s laws. We may realise this limit by taking $c_0 \to \infty$ and keeping $c_1 - c_2$ and $c_3 - c_4$ fixed. In this limit we have the following

$$c_0 - c_1 \to \frac{1}{2A} - m + O(mA)$$  \hspace{1cm} (44)

$$c_1 - c_2 \to 2m + O((mA)^2)$$  \hspace{1cm} (45)

$$c_0 - c_3 \to \frac{1}{2A} - \hat{m} + O(\hat{m} \hat{A})$$  \hspace{1cm} (46)

$$c_3 - c_4 \to 2\hat{m} + O((\hat{m} \hat{A})^2) .$$  \hspace{1cm} (47)

The net force outwards (to the left) on BH2 due to the combination of possible strings and struts is given by

$$T_2 = \frac{1}{4} \left[ \frac{\Delta \phi_{2L}}{\Delta \phi_{2R}} - 1 \right]$$  \hspace{1cm} (48)

Expanding this out and setting $c_2 - c_3 = R$ we find

$$T_2 = \hat{m} \hat{A} + \frac{m \hat{m}}{R^2} + \text{correction}$$  \hspace{1cm} (49)

The first terms of the correction are cubic in $\frac{m}{R}$ and $\frac{\hat{m}}{R}$ and quadratic in $mA$ and $\hat{m} \hat{A}$. In order for the above to be a consistent approximation we require that $R$ not be fixed but tends to infinity in such a way that $\frac{m}{R}$ and $\frac{\hat{m}}{R}$ are larger than $mA$ and $\hat{m} \hat{A}$ in the limit. We might choose, for example, $c_0 \to \infty$, $c_1 - c_2$ and $c_3 - c_4$ fixed and $R \sim \sqrt{c_0}$.  

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Similarly the net force to the left (towards BH2) on BH1 due to the strings and/or struts is

\[ T_1 = \frac{1}{4} \left[ \frac{\Delta \phi_{1L}}{\Delta \phi_{1R}} - 1 \right] \]  

and in the same limit as above this gives

\[ T_1 = mA - \frac{m\hat{m}}{R^2} + \text{correction} \]  

Then (50) and (51) show that the acceleration of each black hole is due to the net tension on it due to the strings and/or struts together with the gravitational attraction of the other black hole.

We can ask, is it possible to choose the period of \( \phi \) so that BH1 is free of strings or struts so that its acceleration is due entirely to its gravitational attraction towards BH2? The condition for this is \( \Delta \phi_{1L} = \Delta \phi_{1R} \). This reduces to the condition

\[ (c_0 - c_1)(c_3 - c_4) = (c_1 - c_4)(c_2 - c_3) \]  

This can be satisfied by choosing \( (c_3 - c_4) \), \( (c_2 - c_3) \) and \( (c_1 - c_4) \) arbitrarily. \( (c_0 - c_1) \) is independent and can be chosen to satisfy (52). Note that in the weak field limit as defined above this condition is approximately \( A = \hat{m}/R^2 \) so indeed the acceleration of the black hole is due to its gravitational attraction towards the outer one.

4 More than two black holes

It is clear how to generalize the above results to the case of three or more black holes. In the case of three, we add a third finite rod on the \( z \)-axis between \( z = c_6 \) and \( z = c_5 \) to the left of the existing two. The metric will be constructed from (13) by multiplying \( g_{tt} \) by \( X_6/X_5 \) and \( g_{\phi\phi} \) by the inverse of this. The rest of the metric is multiplied by a factor

\[ \frac{Y_{05}Y_{56}Y_{16}Y_{52}Y_{36}Y_{45}}{2R_5R_6Y_{06}Y_{15}Y_{26}Y_{35}Y_{46}} \]  

\( \beta \) and \( \alpha \) will now be functions of the parameters \( c_0, \ldots, c_6 \) such that the metric reduces to the known C-metric in the appropriate limits. Adding more black holes proceeds similarly.

We can again calculate the various periods required for regularity. We can also ask whether the \( c_4 \) may be chosen so that there is only one cosmic string pulling the outermost black hole with the others being accelerated purely because of the gravitational attraction of the others. This is a more difficult question to answer in the general case. In the case of three black holes there are two conditions,

\[ c_{14}c_{23}c_{16}c_{25}c_{02} = c_{01}c_{13}c_{24}c_{15}c_{26} \]  
\[ c_{16}c_{25}c_{36}c_{45}c_{02}c_{04} = c_{01}c_{03}c_{15}c_{26}c_{35}c_{46} \]
where \( c_{ij} = c_i - c_j \). These do have solutions with \( c_i < c_j \) for \( i > j \), for example:

\[
\begin{align*}
c_6 &= 0, & c_5 &= a, & c_4 &= a + 1, & c_3 &= a + 2, & c_2 &= a + 3, & c_1 &= a + 4, & c_0 &= a + 4 + f \end{align*}
\]  

(56)

where \( f = (9a - 6)/19 \), and \( a \) is the positive root of \( 3z^2 - 5z - 36 \). We conjecture that such solutions do exist for any number of black holes. Then one could conceive of an accelerating version of the Myers black hole array in which there is an infinite collinear array of accelerating black holes with a common acceleration horizon and no cosmic strings or struts in sight. We also conjecture that there are solutions with any number of accelerating black holes for which it would be possible to choose the period of imaginary time to be such that the euclidean solution is regular on all the black hole horizons at once.

5 Discussion

These multi accelerating black hole solutions are classically unstable, assuming that the deficit angles of the nodal singularities are fixed as they would be if the solutions are approximations to spacetimes with field theory cosmic strings of fixed tension. A little displacement of one of the black holes will disturb the balance of forces required to maintain the solution and presumably cause some black holes to collide and/or some to be left behind. This instability prevents us from asking whether the euclideanized solutions can be interpreted as instantons for multi-pair production of black holes or the production of some pairs in the background of other accelerating pairs. This is even before we confront the problem that the uncharged solutions do not give rise to regular instantons — even if we allow cosmic strings in space — because of the impossibility of matching the acceleration and black hole horizon temperatures. (Nevertheless the euclidean action may straightforwardly be calculated as in [8] since the singularities are conical and integrable. Using the work in [8], calculating the action is just a matter or calculating the difference in the acceleration horizon areas between the instanton and background and the areas of the horizons of the black holes produced.)

These problems might be resolved were we able to give charge to the black holes. The obvious method to try is that of the charging transformation given in [3]. However this only works when the time coordinate is asymptotically Minkowski time (and in any case gives all the black holes the same charge to mass ratio and so might not be able to produce a regular instanton). Here, our asymptotic time coordinate is Rindler time and the charging transformation gives a new metric of unknown physical interpretation which has zero net charge at infinity. Finding charged versions of these solutions requires more ingenuity.
6 Acknowledgments

We would like to thank Jerome Gauntlett, David Kastor, Rob Myers and Harvey Reall for useful discussions and Gary Gibbons for telling us about the Weyl form of the C-metric. FD was supported in part by an EPSRC Advanced Fellowship.

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