Comparison of Tensor Boundary Conditions (TBCs) with Generalized Sheet Transition Conditions (GSTCs)

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Abstract—This paper compares Tensor Boundary Conditions (TBCs), which were introduced to model multilayered dielectric structures, with Generalized Sheet Transition Conditions (GSTCs), which have been recently used to model metasurfaces. It shows that TBCs, with their 3 scalar parameters, are equivalent to the direct-isotropic – cross-antiisotropic reciprocal and nongyrotropic subset of GSTCs, whose 16 tangential (particular case of zero normal polarizations) or 36 general susceptibility parameters can handle the most general bianisotropic sheet structures. It further shows that extending that TBCs scalar parameters to tensors and allowing a doubly-occurring parameter to take different values leads to a TBCs formulation that is equivalent to the tangential GSTCs, but without reflecting the polarization physics of sheet media, such as metasurfaces and two-dimensional material allotropes.

Index Terms—Generalized Sheet Transition Conditions (GSTCs), Tensor Boundary Conditions (TBCs), bianisotropy, metasurfaces, multilayer media.

I. INTRODUCTION

Recently, the Generalized Sheet Transition Conditions (GSTCs) have been abundantly and successfully used as an electromagnetic model to synthesize and analyze metasurfaces [1], [2]. GSTCs are generalizations of the conventional boundary conditions [3], relating the field differences at both sides of a sheet discontinuity not only to surface currents but also to surface polarizations [4] (Eqs. (1)-(3) and Appendix A therein). They were originally introduced by Idemen [5], [6], next expressed in terms of surface polarizability or susceptibility tensors for metasurfaces by Kuester and Holloway [7], [8], and finally extended to the most general case of bianisotropic (generally 36 parameters, and often 16 tangential parameters) metasurfaces by Achouri and Caloz [1], [9]. GSTCs may also apply to two-dimensional materials constituted of a single layer or a small number of atomic layers, such as graphene, molybdenum disulfide or black phosphorous [2]. Several extensions of the conventional boundary conditions have been reported in the literature [10], and it is important to understand their similarities and differences with GSTCs for the optimal electromagnetic modeling of sheet structures. The greatest generality of the GSTCs compared to other boundary or transition conditions is most often obvious. However, there is an exception: the Tensor Boundary Conditions (TBCs) reported by Topsakal, Volakis and Ross in [11] may be written in a form that also involves a tensor (Eq. (2) in [11]) despite their initial formulation in terms of just 3 scalar parameters (Eq. (1) in [11]). Given the quite different appearance of these TBCs compared to typical GSTCs (e.g. [1]), the degree of similarity between the two classes of conditions is far from straightforward. Are the GSTCs just an alternative formulation of TBCs, or do they indeed represent a more general description of sheet discontinuities, such as metasurfaces or two-dimensional material allotropes?

The present paper addresses this question. For this purpose, it expresses the TBCs in terms of equivalent surface tensorial susceptibilities, compares these susceptibilities with the GSTCs ones, and discusses the differences in terms of bianisotropy modeling.

II. MATHEMATICAL COMPARISON

For convenience, the parameters used in the paper are listed in Tab. 1 and Fig. 1 depicts the problem of electromagnetic scattering from a bianisotropic sheet with relevant field quantities and coordinate system. The time convention $e^{-i\omega t}$ is implicitly assumed everywhere.

A. General Sheet Transition Conditions (GSTCs)

The susceptibility-based GSTCs read [11], [4]
\[ \hat{n} \times \Delta \hat{E} = i\omega \mu_0 \hat{M}_\parallel + \nabla \| (P_\perp/\epsilon_0) - \hat{P}_{\text{imp}} \|, \] (1a)
\[ \hat{n} \times \Delta \hat{H} = -i\omega \hat{P}_\parallel - \hat{n} \times \nabla \| M_\perp + \hat{J}_{\text{imp}} \|, \] (1b)
where $\hat{n}$ is the unit vector normal the surface of the sheet, the symbols $\parallel$ and $\perp$ denote vector components tangential and normal to the metasurface, respectively, and $\Delta$ refers to the difference of the fields at both sides of the metasurface, at $z =$
TABLE I
PARAMETERS USED IN THE PAPER.

| Symbol | Quantity | Unit |
|--------|----------|------|
| \(c\) | speed of light in free-space | m/s |
| \(\omega\) | angular frequency | rad/s |
| \(k_0\) | free-space wavenumber | \(\Omega\) |
| \(\epsilon_0\) | free-space wave impedance \((\approx 120\pi)\) | F/m |
| \(Z\) | surface impedance | \(\Omega\) |
| \(\epsilon_0\) | free-space permittivity \((\approx 1/36\pi \times 10^{-9})\) | F/m |
| \(\mu_0\) | free-space permeability \((= 4\pi \times 10^{-7})\) | H/m |
| \(\sigma\) | conductivity | \(\Omega/m\) |
| \(\vec{E}\) | electric field | V/m |
| \(\vec{H}\) | magnetic field | A/m |
| \(\vec{M}\) | magnetic field polarization density | C/m |
| \(t_{\text{imp}}\) | impressed electric surface current density | A/m |
| \(R_{\text{imp}}\) | impressed magnetic surface current density | V/m |
| \(e_{\text{ee}}\) | electric-to-electric surface susceptibility | m |
| \(e_{\text{me}}\) | magnetic-to-electric surface susceptibility | m |
| \(e_{\text{mm}}\) | magnetic-to-magnetic surface susceptibility | m |
| \(R_{\text{ee}}\) | electric-to-electric surface susceptibly | m |
| \(R_{\text{me}}\) | magnetic-to-electric surface susceptibly | m |
| \(R_{\text{mm}}\) | magnetic-to-magnetic surface susceptibly | m |
| \(\rho\) | scalar resistivity | \(\Omega\) |
| \(\rho_{\text{sc}}\) | scalar conductivity | \(\Omega\) |
| \(R_{\text{cc}}\) | cross-coupling term | \(\Omega\) |

0− and z = 0+. In the sequel, we assume, for simplicity, that \(P_\perp = M_\perp = 0\), which reduces the coupled partial differential equations (11) to a simple system of algebraic linear equations.

The remaining tangential electric and magnetic surface polarizations in (11) relate to the averaged fields as

\[
\vec{M}_\parallel = \left( \begin{array}{c} M_x \\ M_y \end{array} \right) = \frac{1}{\eta_0} \left( \begin{array}{cc} \chi^{xx} & \chi^{xy} \\ \chi^{yx} & \chi^{yy} \end{array} \right) \vec{E}_{av} = \frac{1}{\eta_0} \left( \begin{array}{c} x^{yy} \\ x^{yy} \\ x^{yy} \\ x^{yy} \end{array} \right) \vec{H}_{av},
\]

(2a)

\[
\vec{P}_\parallel = \left( \begin{array}{c} P_x \\ P_y \end{array} \right) = e_{\text{ee}} \vec{E}_{av} + \frac{1}{\eta_\text{imp}} \vec{H}_{av},
\]

(2b)

where “av” refers to the average of the fields at both sides of the metasurface at \(z = -0\) and \(z = +0\), and \(\chi_{\text{me}}\), \(\chi_{\text{mm}}\), and \(\chi_{\text{em}}\) are the magnetic-to-electric, magnetic-to-magnetic, electric-to-electric and magnetic-to-electric surface susceptibility tensors, respectively (11).

Inserting Eqs. (3) into Eqs. (1) with \(P_\perp = M_\perp = 0\) while assuming that no impressed surface current densities are present \((\vec{J}_{\text{imp}}) = \vec{K}_{\text{imp}} = 0\) yields

\[
\vec{z} \times \Delta \vec{E} = i \kappa_0 \chi_{\text{me}} \vec{E}_{av} + i \kappa_0 \eta_0 \chi_{\text{mm}} \vec{H}_{av},
\]

(3a)

\[
\vec{z} \times \Delta \vec{H} = -i \kappa_0 / \eta_0 \chi_{\text{ee}} \vec{E}_{av} - i \kappa_0 \chi_{\text{em}} \vec{H}_{av},
\]

(3b)

which express the difference fields in terms of the average fields via the susceptibility tensors. The GSTCs (3), characterized by their 4 tangential susceptibility tensors (16 scalar parameters), can essentially model any transverse-biaxialisotropic metasurface that supports first-order induced polarization surface currents (6, 9).

B. Tensor Boundary Conditions (TBCs)

The TBCs, as given by Eq. (1) in (11), read

\[
\hat{n} \times \left( \vec{E}^+ + \vec{E}^- \right) = R_e \hat{n} \times \left( \hat{n} \times \vec{H}^+ - \hat{n} \times \vec{H}^- \right) - R_c \hat{n} \times \left( \hat{n} \times \vec{E}^+ - \hat{n} \times \vec{E}^- \right),
\]

(4a)

\[
\hat{n} \times \left( \vec{H}^+ + \vec{H}^- \right) = R_m \hat{n} \times \left( \hat{n} \times \vec{E}^+ - \hat{n} \times \vec{E}^- \right) + R_c \hat{n} \times \left( \hat{n} \times \vec{H}^+ - \hat{n} \times \vec{H}^- \right),
\]

(4b)

where \(\vec{E}^\pm\) and \(\vec{H}^\pm\) are the electric and magnetic fields at \(z = 0^\pm\). Using the field average and difference notation in Sec. (11) (e.g., \(\vec{E}_{av} = (\vec{E}^+ + \vec{E}^-)/2\) and \(\Delta \vec{E} = (\vec{E}^+ - \vec{E}^-)\)) and setting \(\hat{n} = \hat{z}\), these equations take the more compact form

\[
2 \hat{z} \times \vec{E}_{av} = R_e \hat{z} \times (\hat{z} \times \Delta \vec{H}) + R_c \hat{z} \times \Delta \vec{E},
\]

(5a)

\[
2 \hat{z} \times \vec{H}_{av} = R_m \hat{z} \times (\hat{z} \times \Delta \vec{E}) - R_c \hat{z} \times \Delta \vec{H},
\]

(5b)

These equations relate the average fields to the difference fields via the 3 scalar parameters \(R_e, R_m, \) and \(R_c\), whose units are respectively \(\Omega, \Omega\) and 1.

3 According to the Huygens principle, any physical fields on either side of the metasurface (i.e., everywhere outside a longitudinally (z) thin and transversally \((xy)\) infinite volume containing the metasurface) can be produced by purely tangential equivalent surface polarizations. Therefore, a metasurface involving normal polarizations can always be reduced to an equivalent metasurface with purely tangential polarizations. However, the exclusion of normal components implies restrictions in terms of design flexibility and separate transformation possibility, which are discussed in (11).

4 The assumption \(P_\perp = M_\perp = 0\) (or \(P_\perp = M_\perp\) space-wise constant) in (11) indeed reduces the effect of the most general bianisotropic \(3 \times 3 \times 3 \times 3 (4 \times 3 \times 3 \times 3 = 36\) scalar parameters\) susceptibility tensors to that of their \(2 \times 2 \times 2 \times 2 = 16\) scalar parameters\) tangential parts. If the bianisotropic sheet includes normal polarizations (e.g. in-plane rings, leading to nonzero \(M_\parallel\), the normal susceptibility components \(\chi^{ab}_{zz}, \chi^{ab}_{xy}, \chi^{ab}_{yx}, \chi^{ab}_{zy}\), and \(\chi^{ab}_{xz}\) \((a, b = e, m)\) may also be involved.
C. GSTCs Susceptibilities in Terms of TBCs Parameters

The TBCs, as given by Eqs. (4) or (5), have an obvious disadvantage compared to susceptibility-based GSTCs for handling complex sheets, such as metasurfaces or magnetized two-dimensional materials: They are not directly expressed in terms of bianisotropic medium parameters, and therefore provide little insight into the physics of the problem.

However, solving (5) for the difference fields leads to the following reformulation in terms of TBCs-equivalent susceptibilities (see Appendix A):

\[
\vec{\Delta I} = R \left( \frac{2}{\kappa_0} \chi_{\text{me}} || \cdot \vec{E}_{||}, \vec{H}_{||} + R \frac{2}{k_0} \chi_{\text{me}} || \cdot \vec{E}_{\perp}, \vec{H}_{\perp} \right),
\]

where

\[
\chi_{\text{me}} || = \frac{2}{\kappa_0} R \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad \chi_{\text{me}} || = -i \frac{2R}{k_0} \left( \begin{array}{cc} 1 \ & 0 \\ 0 & 1 \end{array} \right); \quad \chi_{\text{me}} || = \frac{2}{k_0} R \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),
\]

with

\[
D = R_c R_m - R_e^2.
\]

A. Direct Isotropy – Cross Anisotropy

Equations (7a)-(8a), and (7b)-(8b) respectively imply that

\[
\chi_{ee} || = \chi_{ee} || = 0,
\]

\[
\chi_{mm} || = \chi_{mm} || = 0,
\]

\[
\chi_{em} || = \chi_{em} || = 0,
\]

\[
\chi_{ee} || = \chi_{ee} || = 0,
\]

and

\[
\chi_{ee} || = \chi_{ee} ||, \quad \chi_{mm} || = \chi_{mm} ||, \quad \chi_{em} || = -\chi_{em} ||, \quad \chi_{me} || = -\chi_{me} ||,
\]

and

\[
\chi_{ee} || = \chi_{ee} ||, \quad \chi_{mm} || = \chi_{mm} ||, \quad \chi_{em} || = -\chi_{em} ||, \quad \chi_{me} || = -\chi_{me} ||.
\]

III. Physical Restrictions of the TBCs

The tensorial conditions (7) are associated to restrictions on the transverse bianisotropic physical properties of the sheet.
Thus, the TBCs in [11] are restricted to be nongyrotropic, and allowing the third parameter to take different values as
\( T \) where the symbol \( T \) denotes the transpose operation.

According to (10a) and (10b) (resp.), the conditions (12a) and (12b) (resp.) cannot be satisfied. Moreover, Eqs. (12c) prohibit the satisfaction of Eq. (12c).

Thus, the TBCs in [11] are restricted, from Eqs. (7), (8) and (9) (14 conditions), to reciprocal [1], [12] sheets. For instance, they cannot handle spatial-isolation metasurfaces [17].

B. Reciprocity

The condition for nonreciprocity are [11], [12], [16]
\[
\chi_{ee} \neq \chi_{ee}^T \quad (12a)
\]
or
\[
\chi_{mm} \neq \chi_{mm}^T \quad (12b)
\]
or
\[
\chi_{me} \neq -\chi_{em}^T \quad (12c)
\]
where the symbol \( T \) denotes the transpose operation.

According to (10a) and (10b) (resp.), the conditions (12a) and (12b) (resp.) cannot be satisfied. Moreover, Eqs. (12c) prohibit the satisfaction of Eq. (12c).

Thus, the TBCs in [11] are restricted, from Eqs. (7), (8) and (9) (14 conditions), to reciprocal [1], [12] sheets. For instance, they cannot handle spatial-isolation metasurfaces [17].

C. Nongyrotropy

The condition for gyrotropy is [11], [12]
\[
\chi_{xy}, \chi_{xy}^T \neq 0 \quad (13a)
\]
or
\[
\chi_{xy}, \chi_{xy}^T \neq 0 \quad (13b)
\]
or
\[
\chi_{em}, \chi_{em}^T \neq 0 \quad (13c)
\]
All of these relations are prohibited by [11] (8 conditions). Thus, the TBCs in [11] are restricted to be nongyrotropic.

For instance, they cannot handle polarization rotators [13].

IV. TENSORIAL EXTENSION OF THE TBCS

We have found in Secs. [11] and [11] that the TBCs in [11] represent a subset of the GSTCs with restriction to direct-isotropic/cross-antiisotropic susceptibilities. However, one may then legitimately ask whether making the scalar parameters in the TBCs [5] tensorial would provide the same level of generality as the GSTCs. The present section addresses this question.

Transforming the scalar parameters \( R_e, R_m \) and \( R_c \) into tensors, and allowing the third parameter to take different values at its two occurrences for maximal freedom reformulates [5] as
\[
2 \hat{z} \times \vec{E}_{||, av} = \frac{\vec{E}_{||, av}}{R_e} \left( \hat{z} \times \left( \hat{z} \times \vec{D} \right) \right) + \frac{\vec{D}_e}{R_c} \left( \hat{z} \times \vec{E} \right), \quad (14a)
\]
\[
2 \hat{z} \times \vec{H}_{||, av} = \frac{\vec{H}_{||, av}}{R_m} \left( \hat{z} \times \left( \hat{z} \times \vec{D} \right) \right) - \frac{\vec{D}_m}{R_m} \left( \hat{z} \times \vec{H} \right), \quad (14b)
\]

In order to derive the susceptibilities associated with Eq. (14), one first needs, as for the scalar-parameter case (Sec. 4), to express the difference fields in terms of the average fields for proper comparison with GSTCs. For this purpose, we vectorially pre-multiply both sides of Eqs. (14) by \( \hat{z} \), which is easily achieved after noticing that the operator \( \hat{z} \times \) for any transverse vector \( \vec{v} \) is equivalent to the operator \( \vec{N} \cdot \vec{v} \), where
\[
\vec{N} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
The result is
\[
-2 \hat{E}_{||, av} = \frac{\vec{E}_{||, av}}{R_e} \cdot \vec{N} \left( \hat{z} \times \vec{D} \right) + \frac{\vec{D}_e}{R_c} \cdot \left( \hat{z} \times \vec{E} \right), \quad (16a)
\]
\[
-2 \vec{H}_{||, av} = \frac{\vec{H}_{||, av}}{R_m} \cdot \vec{N} \left( \hat{z} \times \vec{D} \right) - \frac{\vec{D}_m}{R_m} \cdot \left( \hat{z} \times \vec{H} \right), \quad (16b)
\]
Solving these equations for the vectors \( \hat{z} \times \vec{D} \) and \( \hat{z} \times \vec{H} \) yields
\[
\hat{z} \times \vec{D} = -2 \left( \frac{\vec{E}_{||, av}}{R_e} \cdot \vec{N} \cdot \vec{D}_e \right)^{-1} \cdot \vec{E}_{||, av} - 2 \left( \frac{\vec{D}_m}{R_m} \cdot \vec{N} \cdot \vec{D}_m \right)^{-1} \cdot \vec{H}_{||, av}, \quad (17a)
\]
\[
\hat{z} \times \vec{H} = -2 \left( \frac{\vec{E}_{||, av}}{R_e} \cdot \vec{N} \cdot \vec{D}_e \right)^{-1} \cdot \vec{E}_{||, av} + 2 \left( \frac{\vec{D}_m}{R_m} \cdot \vec{N} \cdot \vec{D}_m \right)^{-1} \cdot \vec{H}_{||, av}, \quad (17b)
\]
where
\[
\vec{D}_e = \frac{1}{R_e} \cdot \frac{1}{R_m} \cdot \frac{1}{R_c} \cdot \vec{N} \cdot \vec{D}_e, \quad (17c)
\]
\[
\vec{D}_m = \frac{1}{R_m} \cdot \frac{1}{R_c} \cdot \vec{N} \cdot \vec{D}_m, \quad (17d)
\]
Comparing Eqs. (17) and (3) provides then the TBCs-equivalent susceptibility tensors
\[
\chi_{me}^{TBC} = i \frac{2}{k_0} \left( \frac{\vec{N} \cdot \vec{R}_e \cdot \vec{N} \cdot \vec{D}_e}{R_e} \right)^{-1}, \quad (18a)
\]
\[
\chi_{mm}^{TBC} = i \frac{2}{k_0} \left( \frac{\vec{N} \cdot \vec{R}_m \cdot \vec{D}_m}{R_m} \right)^{-1}, \quad (18b)
\]
\[
\chi_{ee}^{TBC} = -i \frac{2\eta_0}{k_0} \left( \frac{\vec{N} \cdot \vec{R}_c \cdot \vec{D}_m}{R_m} \right)^{-1}, \quad (18c)
\]
\[
\chi_{me}^{TBC} = i \frac{2}{k_0} \left( \frac{\vec{N} \cdot \vec{R}_e \cdot \vec{D}_m}{R_m} \right)^{-1}, \quad (18d)
\]
which may easily be verified to reduce to Eqs. (5) upon reducing the tensors \( \vec{R} \) reduce to tensors and setting \( R_c = R_m = R_e \). Since these tensors may be completely different from each other, the susceptibilities (18) involve \( 4 \times (2 \times 2) = 16 \) independent parameters, and Eqs. (14) are therefore perfectly equivalent to the GSTCs (3).

However, two important comments are here in order. First, the expressions (3) are overly complicated, and fail to properly represent the polarization physics (19) involved in the sheet media. Second, in contrast to the most general GSTCs (1), which may involve \( 4 \times (3 \times 3) = 36 \) surface susceptibility parameters through their spatial derivatives (Footnote 3), the TBCs (5) are always restricted to \( 4 \times (2 \times 2) = 16 \) susceptibility parameters.

7 In a nongyrotropic medium, the transverse electric (TE) and transverse magnetic (TM) polarizations are always decoupled.
Similarly solving Eq. (A.2b) for contrast to GSTCs, they do not reflect the polarization physics TBCs, as given in [11], are not equivalent to GSTCs, as given in [1]. They represent a subset of GSTCs which can only model sheets that are direct-isotropic – cross-antisotropic, reciprocal and nongyrotropic. Moreover, they are restricted to 16 equivalent susceptibility parameters, whereas GSTCs, in their most general form, support the 36 susceptibility parameters corresponding to the three dimensions of space. Finally, in contrast to GSTCs, they do not reflect the polarization physics of sheet media such as metasurfaces and two-dimensional material allotropes.

APPENDIX A
DERIVATION OF THE GSTCs SUSCEPTIBILITIES CORRESPONDING TO THE TBCS

The TBCs equations [3] may be recast in the matrix form

\[
\begin{align}
2 \begin{pmatrix}
-\Delta H_x \\
-\Delta H_y \\
-\Delta E_x \\
-\Delta E_y
\end{pmatrix} &= R_p \begin{pmatrix}
-\Delta E_x \\
-\Delta E_y \\
-\Delta H_x \\
-\Delta H_y
\end{pmatrix}, \\
2 \begin{pmatrix}
-\Delta E_x \\
-\Delta E_y \\
-\Delta H_x \\
-\Delta H_y
\end{pmatrix} &= R_m \begin{pmatrix}
-\Delta H_x \\
-\Delta H_y \\
-\Delta E_x \\
-\Delta E_y
\end{pmatrix},
\end{align}
\]

This system of equations may be rearranged in terms of the electromagnetically uncoupled equation pairs

\[
\begin{align}
-2E_{y,av} &= -R_e \Delta H_x - R_c \Delta E_y, \\
2H_{x,av} &= -R_m \Delta E_x - R_c \Delta H_y,
\end{align}
\]

and

\[
\begin{align}
-2E_{x,av} &= -R_e \Delta H_y + R_c \Delta E_x, \\
-2H_{y,av} &= -R_m \Delta E_y + R_c \Delta H_y.
\end{align}
\]

Solving Eq. (A.2a) for \( \Delta H_x \) and \( \Delta E_y \) yields

\[
\begin{align}
\Delta H_x &= \frac{2R_m}{D} E_{y,av} + \frac{2R_e}{D} H_{x,av}, \\
\Delta E_y &= \frac{-2R_e}{D} E_{y,av} - \frac{2R_e}{D} H_{x,av},
\end{align}
\]

with

\[
D = R_c R_m - R_e^2.
\]

Similarly solving Eq. (A.2b) for \( \Delta H_y \) and \( \Delta E_x \) yields

\[
\begin{align}
\Delta H_y &= \frac{-2R_m}{D} E_{x,av} + \frac{2R_e}{D} H_{y,av}, \\
\Delta E_x &= \frac{-2R_e}{D} E_{x,av} + \frac{2R_e}{D} H_{y,av}. \\
\end{align}
\]

Rearranging Eqs. (A.3) and (A.5) in the matrix form

\[
\begin{align}
\begin{pmatrix}
\Delta E_y \\
\Delta E_x
\end{pmatrix} &= \frac{-2R_e}{D} \begin{pmatrix}
H_{x,av} \\
H_{y,av}
\end{pmatrix} + \frac{2R_e}{D} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
E_{x,av} \\
E_{y,av}
\end{pmatrix}, \\
\begin{pmatrix}
\Delta H_y \\
\Delta H_x
\end{pmatrix} &= \frac{2R_m}{D} \begin{pmatrix}
E_{x,av} \\
E_{y,av}
\end{pmatrix} + \frac{2R_e}{D} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
H_{x,av} \\
H_{y,av}
\end{pmatrix},
\end{align}
\]

leads to the GSTCs-form equations [Eqs. (6)]

\[
\begin{align}
\dot{E} \times \Delta \vec{H} &= \frac{2R_e}{D} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} E_{||,av}, \quad (A.7a) \\
\dot{H} \times \Delta \vec{E} &= \frac{2R_e}{D} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} H_{||,av}. \quad (A.7b)
\end{align}
\]

APPENDIX B
GENERALIZED REFRACTION METASURFACE EXAMPLE

Consider a metasurface surrounded by air on both sides that is designed to refract a plane wave incident at an angle \( \theta_i \), towards an angle \( \theta_r \). The susceptibility functions characterizing such a metasurface for the \( p \) polarization, given in [14], can be written as

\[
\begin{align}
\chi_{xx} &= 2T(4 \sin \alpha \cos \theta_i + \beta T \sin(2\theta_i))/\eta_0\omega\epsilon_0(2 \cos \theta_i + \beta T \cos \alpha)^2, \\
\chi_{yy} &= 2\eta_0 T \cos \theta_i \cos \theta_t(4 \sin \alpha \cos \theta_i + \beta T \sin(2\theta_i))/\mu_0\omega(2 \cos \theta_i + \beta T \cos \alpha)^2, \\
\chi_{xy} &= -\chi_{yx} = -2(T \cos \alpha(\cos \theta_i - \cos \theta_t))/k_0(2 \cos \theta_i + \beta T \cos \alpha)^2,
\end{align}
\]

where \( \alpha = k_0 x (\sin \theta_i - \sin \theta_r) \), \( \beta = \cos \theta_i + \cos \theta_r \) and \( T = \sqrt{\cos \theta_i / \cos \theta_t} \). The susceptibility functions for the s polarization are found by duality as

\[
\begin{align}
\chi_{ee} &= 2T \cos \theta_i \cos \theta_t(4 \sin \alpha \cos \theta_i + \beta T \sin(2\theta_i))/\eta_0\omega\epsilon_0(2 \cos \theta_i + \beta T \cos \alpha)^2, \\
\chi_{mm} &= 2\eta_0 T \cos \theta_i \cos \theta_t(4 \sin \alpha \cos \theta_i + \beta T \sin(2\theta_i))/\mu_0\omega(2 \cos \theta_i + \beta T \cos \alpha)^2, \\
\chi_{me} &= -\chi_{em} = -2(T \cos \alpha(\cos \theta_i - \cos \theta_t))/k_0(2 \cos \theta_i + \beta T \cos \alpha)^2.
\end{align}
\]

Comparing Eqs (B.1) and (B.2) shows that this set of susceptibility functions violates the restrictions of Eqs. (4) of the TBCs. Hence, such a metasurface cannot be described using by the TBCs as presented in [11].

REFERENCES

[1] K. Achouri and C. Caloz, “Design, concepts, and applications of electromagnetic metasurfaces,” Nanophotonics, vol. 7, no. 6, pp. 1095-1116, Jun. 2018.
[2] Y. Vahabzadeh, N. Chamanara, K. Achouri, and C. Caloz, “Computational analysis of metasurfaces,” J. Multiscale Multiphys. Comput. Tech., vol. 3, pp. 37-49, Apr. 2018.
[3] R. F. Harrington, Time-Harmonic Electromagnetic Fields, Hoboken, USA: Wiley-IEEE Press, 2nd ed., 2001.
[4] X. Jia, Y. Vahabzadeh, C. Caloz, and F. Yang, “Synthesis of Spherical Metasurfaces based on Susceptibility Tensor GSTCs,” IEEE Trans. Antennas Propag., to be published.
[5] M. Idemen A. Hamit and Serbest, “Boundary conditions of the electromagnetic field,” Electron. Lett., vol. 23, no. 13, pp. 704–705, Jun. 1987.
[6] M. M. Idemen, Discontinuities in the Electromagnetic Field, Hoboken, IET, Wiley, 2011.
[7] E. F. Kuester, M. A. Mohamed, M. Piket-May, and C. L. Holloway, “Averaged transition conditions for electromagnetic fields at a metalfilm,” IEEE Trans. Antennas Propag., vol. 51, no. 10, pp. 2641–2651, Oct. 2003.
[8] C. L. Holloway, E. F. Kuester, J. A. Gordon, J. O. Hara, J. Booth, and D. R. Smith, “An overview of the theory and applications of metasurfaces: The two-dimensional equivalents of metamaterials,” *IEEE Antennas Propag. Mag.*, vol. 54, no. 2, pp. 10-35, Apr. 2012.

[9] K. Achouri, M. A. Salem, and C. Caloz, “General metasurface synthesis based on susceptibility tensors,” *IEEE Trans. Antennas Propag.*, vol. 63, no. 7, pp. 2977-2991, Jul. 2015.

[10] T. B. A. Senior and J. L. Volakis, *Approximate Boundary Conditions in Electromagnetics*, IET, 1995.

[11] E. Topsakal, J. L. Volakis, and D. C. Ross, “Surface integral equations for material layers modeled with tensor boundary conditions,” *Radio Science*, vol. 37, no. 4, pp. 1-6, Jul. 2002.

[12] J. A. Kong, *Electromagnetic Wave Theory*, Cambridge, USA, 2008.

[13] I. V. Lindell, A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Mixing Formulas and Applications*, Artech House, 1999.

[14] G. Lavigne, K. Achouri, V. S. Asadchy, S. A. Tretyakov, and C. Caloz, “Susceptibility derivation and experimental demonstration of refracting metasurfaces without spurious diffraction,” *IEEE Trans. Antennas Propag.*, vol. 66, no. 3, pp. 1321-1330, Mar. 2018.

[15] A. Epstein and G. V. Eleftheriades, “Arbitrary power-conserving field transformations with passive lossless omega-type bianisotropic metasurfaces,” *IEEE Trans. Antennas Propag.*, vol. 64, no. 9, pp. 3880–3895, Sep. 2016.

[16] C. Caloz, A. Ali, S. Tretyakov, D. Sounas, K. Achouri, and Z.-L. Deck-Lger, “Electromagnetic nonreciprocity,” *Phys. Rev. Appl.*, vol. 10, no. 4, pp. 047 001:1-26, Oct. 2018.

[17] T. Kodera and C. Caloz, S. Tretyakov, D. Sounas, K. Achouri, and Z.-L. Deck-Lger, “Unidirectional loop metamaterials (ULM) as magnetless artificial ferrimagnetic materials: principles and applications,” *IEEE Antennas Wirel. Propag. Lett.*, vol. 17, no. 11, pp. 1943-1947, Nov. 2018.

[18] C. Pfeiffer and A. Grbic, “Bianisotropic metasurfaces for optimal polarization control: analysis and synthesis,” *Phys. Rev. Appl.*, vol. 2, no. 044011, pp. 1-11, Feb. 2014.

[19] J. D. Jackson, *Classical Electrodynamics*, Wiley, third ed., 1998.