PARTIAL FRACTION EXPANSION BASED FREQUENCY WEIGHTED MODEL REDUCTION FOR DISCRETE-TIME SYSTEMS

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Abstract. In this paper, a partial fraction expansion based frequency weighted model reduction algorithm is developed for discrete-time systems. The proposed method is an extension to the method by Sreeram et al. [13] and it yields stable reduced order models with both single and double sided weighting functions. Effectiveness of the proposed algorithm is demonstrated by a numerical example.

1. Introduction. Since the last four decades, many methods have been proposed for the simplification of complex and high order models. Balanced realization based methods such as balanced truncation [8] and balanced singular perturbation approximation (BSPA) [7] have led to many variations of model reduction techniques. Liu and Anderson [7] described theoretical aspects of BSPA with several important properties in both the time and frequency domains. More specifically, these techniques generate reduced order models which approximate the characteristics of the original higher order models, provide frequency response a-priori error bounds and all their reduced order models are guaranteed to be stable.

Ideally, it is important that approximation error between original and reduced order systems is small for all frequencies. However, in some cases, error minimization is required only for certain frequency bands [5, 4, 1, 11, 2]. Enns [3] first introduced frequency weighted balanced truncation technique as a further development to standard balanced truncation by Moore [8] which incorporates frequency weightings. This method may use input weighting, output weighting or both. However, if both the input and output weightings are applied, stability of the reduced order model is not guaranteed. To overcome this drawback, Lin and Chiu [6] proposed a method with strictly proper weightings which yields stable reduced models in case of double-sided weightings. This technique was later generalized to include proper

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weights by Sreeram et al. [12]. Later on, Wang et al. [15] developed a modification to Enns technique to include double-sided weightings and derived a simple and elegant priori error bound. This method yields stable reduced order models by introducing fictitious input and output matrices. Another group of frequency weighted model reduction methods are based on partial fraction expansion and were originally proposed by Saggaf and Franklin [9, 10].

Gestal et al. [14] stated that many frequency weighted model reduction techniques suffer from a large frequency weighted approximation error due to presence of non-zero cross-terms. In order to address this drawback, Sreeram et al. [13] presented a combination of unweighted balancing [8] and Lin and Chius technique [6] to obtain an augmented realization with zero cross terms. Although this method yields lower approximation errors, it is also realization dependent.

In this paper, we present an extension of the work by Sreeram et al. [13] to be applicable for discrete-time systems.

2. Preliminaries. In this section, existing frequency weighted balanced truncation techniques are reviewed namely the methods by Enns [3], Lin and Chius [6] and Sreeram et al. [13]. An understanding of these techniques is required as a prerequisite for the material presented in this paper.

Let $G(z)$ be the transfer function of a stable discrete time original system with the following minimal realization

$$G(z) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Similarly let $V(z)$ and $W(z)$ be the transfer functions of stable input and output weights with the following minimal realizations

$$V(z) = \begin{bmatrix} A_V & B_V \\ C_V & D_V \end{bmatrix}$$

$$W(z) = \begin{bmatrix} A_W & B_W \\ C_W & D_W \end{bmatrix}$$

The state space realization of the augmented system $W(z)G(z)V(z)$ is then given by

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} A_W & B_W C & B_W D C_V & B_W D D_V \\ 0 & A & B C_V & B D_V \\ 0 & 0 & A_V & B_V \\ C_W & D_W C & D_W D C_V & D_W D D_V \end{bmatrix}$$

(1)

The controllability and observability of the augmented realization $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ is given as

$$\hat{P} = \begin{bmatrix} P_W & P_{12} & P_{13} \\ P_{12}^T & P_{E} & P_{23} \\ P_{13}^T & P_{23}^T & P_V \end{bmatrix}, \hat{Q} = \begin{bmatrix} Q_W & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{E} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_V \end{bmatrix}$$

(2)

such that both $\hat{P}$ and $\hat{Q}$ satisfy the following discrete time Lyapunov equations

$$\hat{A} \hat{P} \hat{A}^T - \hat{P} + \hat{B} \hat{B}^T = 0$$

$$\hat{A}^T \hat{Q} \hat{A} - \hat{Q} + \hat{C}^T \hat{C} = 0$$

(3)

(4)
2.1. Enns Technique for Discrete-Time Systems. The frequency weighted balanced truncation technique by [3] is presented in this section. Expanding the (2,2) blocks respectively of equations (3) and (4) yields the following:

\[ A_P E A^T - P_E + X_E = 0 \]  
\[ A^T Q_E A - Q_E + Y_E = 0 \]

where

\[ X_E = A_P B_{23} C_{V}^T B^T + B C_{V} P_{23}^T A^T + B C_{V} P_{V} C_{V}^T B^T + B D_{V} D_{V}^T B^T \]
\[ Y_E = C^T B_{W}^T Q_{12} A + A^T Q_{12} B_{W} C + C^T B_{W}^T Q_{W} B_{W} C + C^T D_{W}^T D_{W} C \]

Simultaneously diagonalizing the weighted gramians \( P_E \) and \( Q_E \), we get

\[ T^T Q_E T = T^{-1} P_E T^{-T} = diag\{\sigma_1, \sigma_2, ... \sigma_n\} \]

where \( \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > \sigma_{r+1} \geq ... \geq \sigma_n > 0 \). The order of the reduced order model is denoted by \( r \) whereas the order of the original model is denoted by \( n \). Transforming and partitioning the original system we get

\[ \hat{A} = T^{-1} A T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{B} = T^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]
\[ \hat{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \hat{D} = D \]

The reduced order model is given by \( G_r(z) = C_1(zI - A_{11})^{-1}B_1 + D \). The stability of the reduced order model obtained by using Enns technique is not guaranteed to be stable for the case of double sided input weighting.

2.2. Lin and Chiu’s Technique. In Lin and Chiu’s technique, the augmented system matrix in (1) is block diagonalized by the similarity transformation matrix

\[ \tilde{T} = \begin{bmatrix} I & -Q_{W}^{-1} Q_{12} & 0 \\ 0 & I & P_{23} P_{V}^{-1} \\ 0 & 0 & I \end{bmatrix} \]

which yields

\[ \begin{bmatrix} A_W & B_W C & B_W D C_V & B_W D D_V \\ 0 & A & B C_V & B D_V \\ 0 & 0 & A_V & B_V \\ C_W & D_W C_V & D_W D C_V & D_W D D_V \end{bmatrix} \]

\[ = \begin{bmatrix} \tilde{T}^{-1} \hat{A} \tilde{T} & \tilde{T}^{-1} \hat{B} \\ C T & D \end{bmatrix} \]

\[ = \begin{bmatrix} A_W & X_{12} & X_{13} & X_1 \\ 0 & A & X_{23} & X_2 \\ 0 & 0 & A_V & B_V \\ C_W & Y_1 & Y_2 & D_W D D_V \end{bmatrix} \]
In Lin and Chiu’s technique, the realization of the system \( W \) is balanced to find the balancing transformation matrix, \( T \) which is then applied to the original system \( \{A, B, C, D\} \) to obtain the reduced order model by direct truncation.

### 2.3. Sahlan and Sreeram’s Partial Fraction Expansion Based Technique.

Sreeram et al [13] had also proposed a method based on decomposing the augmented system \( W(s)G(s)V(s) \) to be equal to \( W_{\text{diag}}(s) + G_{\text{diag}}(s) + V_{\text{diag}}(s) \) by using partial fraction expansion where the system matrix is block diagonalized. The decomposed system \( (W_{\text{diag}}(s) + G_{\text{diag}}(s) + V_{\text{diag}}(s)) \) is then expressed as a new augmented system \( W(s)\tilde{G}(s)V(s) \) such that the system matrix of \( W_{\text{diag}}(s) + G_{\text{diag}}(s) + V_{\text{diag}}(s) \) is the same as the system matrix of \( W(s)\tilde{G}(s)V(s) \) and is block diagonal such that

\[
W(s)G(s)V(s) = W_{\text{diag}}(s) + G_{\text{diag}}(s) + V_{\text{diag}}(s) = W(s)\tilde{G}(s)V(s)
\]

where \( \tilde{G}(s) = \{A, \tilde{B}, \tilde{C}, \tilde{D}\} \) is the new original system, \( V(s) = \{AV, BV, \tilde{C}V, \tilde{D}V\} \) and \( W(s) = \{AW, \tilde{B}W, CW, \tilde{D}W\} \) are the input and output weights respectively where

\[
\begin{align*}
B_W &= \begin{bmatrix} B_W & A_W & Q_W^{-1} \end{bmatrix},
D_W &= \begin{bmatrix} D_W & C_W & 0 \end{bmatrix} \\
B &= \begin{bmatrix} B & -P_{23}P_V^{-1} & AP_{23} \end{bmatrix},
\tilde{C} &= \begin{bmatrix} C \\ -Q_W^{-1}Q_{12}A \\ Q_{12} \end{bmatrix},
\tilde{D} &= \begin{bmatrix} D \\ 0 \\ CP_{23} \\ Q_{12}B & -Q_{12}P_{23}P_V^{-1} & Q_{12}AP_{23} \end{bmatrix},
\tilde{C}_V &= \begin{bmatrix} C_V \\ A_V \\ P_V^{-1} \end{bmatrix}
\end{align*}
\]

The equations in (13) to (19) can then be factorized as follows:
3. **Main Work.** In this section, we present the extension of the continuous-time partial fraction expansion by frequency weighted model reduction method by Sreeram et al [13] to work with discrete time systems. The main concept of this extension is the usage of discrete-time Lyapunov equations instead of continuous-time Lyapunov equations. The error bounds of this method are also presented.

By using a similar approach to that described by Sreeram et al [13] for the continuous-time case, it can be shown that for the discrete-time case (with a specified sampling time) the following property holds true

\[ W(z)G(z)V(z) = \tilde{W}(z)\tilde{G}(z)\tilde{V}(z) \]  

(21)

where \( \tilde{G}(z) = \{A, B, \tilde{C}, \tilde{D}\} \) is the new original system, \( \tilde{V}(z) = \{A_V, B_V, \tilde{C}_V, \tilde{D}_V\} \) and \( \tilde{W}(z) = \{A_W, \tilde{B}_W, \tilde{C}_W, \tilde{D}_W\} \) are the input and output weights respectively.
where
\[
\bar{B}_W = [B_W \ A_W \ Q_{W}^{-1}], \bar{D}_W = [D_W \ C_W \ 0]
\]
\[
\bar{B} = [B \ -P_{23}P_V^{-1} \ AP_{23}], \bar{C} = \begin{bmatrix} C \\ -Q_{W}^{-1}Q_{12} \\ Q_{12}A \end{bmatrix}, \bar{D}_V = \begin{bmatrix} D_V \\ B_V \\ 0 \end{bmatrix}
\]
\[
\bar{D} = \begin{bmatrix} D \\ 0 \\ CP_{23} \\ 0 \\ 0 \\ -Q_{12}P_{23}P_V^{-1} \\ Q_{12}AP_{23} \end{bmatrix}, \bar{C}_V = \begin{bmatrix} C_V \\ A_V \\ P_V^{-1} \end{bmatrix}
\]

3.1. Proposed Algorithm. The proposed partial fraction expansion based frequency weighted model reduction algorithm for discrete-time systems is described herewith as follows

**Step 1**: Calculate the fictitious input matrix \( \bar{B}_{PF} \) and output matrix \( \bar{C}_{PF} \) by using suitable values of the free parameters \( \alpha \) and \( \beta \) as follows
\[
\bar{B}_{PF} = [B \ -\alpha P_{23}P_V^{-1} \ AP_{23}]
\]
\[
\bar{C}_{PF} = \begin{bmatrix} C \\ -\beta Q_{W}^{-1}Q_{12} \\ Q_{12}A \end{bmatrix}
\]

**Step 2**: Weighted controllability and observability gramians \( \bar{P}_{PF} \) and \( \bar{Q}_{QF} \) are obtained by solving the following discrete-time Lyapunov equations.
\[
A\bar{P}_{PF}A^T - \bar{P}_{PF} + \bar{B}_{PF}\bar{B}_{PF}^T = 0 \quad (22)
\]
\[
A^T\bar{Q}_{QF}A - \bar{Q}_{QF} + \bar{C}_{PF}\bar{C}_{PF} = 0 \quad (23)
\]

**Step 3**: Calculate the transformation matrix \( T \) which simultaneously diagonalizes the matrices \( \bar{P}_{PF} \) and \( \bar{Q}_{QF} \) such that
\[
T^T\bar{P}_{PF}T = T^{-1}\bar{P}_{PF}T^{-T} = \text{diag}\{\sigma_1, \sigma_2, ... \sigma_r, \sigma_{r+1}, ... \sigma_n\}
\]
where \( \sigma_i \geq \sigma_{i+1}, i = 1, 2, ..., n - 1 \) and \( \sigma_r > \sigma_{r+1} \)

**Step 4**: Compute the frequency weighted balanced realization.
\[
\hat{A} = T^{-1}AT, \hat{B} = T^{-1}B, \hat{C} = CT
\]

**Step 5**: Partition \( \{\hat{A}, \hat{B}, \hat{C}\} \) as follows
\[
\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \hat{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}
\]
where \( A_{11} \in \mathbb{R}^{r \times r}, B_1 \in \mathbb{R}^{r \times p}, C_1 \in \mathbb{R}^{q \times r} \) and \( r < n \)

**Step 6**: The reduced order model is given by \( G_r(z) = C_1(zI - A_{11})^{-1}B_1 + D \)

3.2. Error Bounds. In this section the error bounds for the reduced order models obtained using the proposed algorithm is derived. To establish the relationship between the input and output matrices \( (B \) and \( C) \), and the new fictitious input and output matrices \( (\bar{B}_{PF} \) and \( \bar{C}_{PF} \), we define two constant matrices, \( K_{PF} = \begin{bmatrix} I/\alpha \\ 0 \end{bmatrix} \) and \( L_{PF} = \begin{bmatrix} I/\beta & 0 \end{bmatrix} \) respectively where \( I \) is an identity matrix with the appropriate dimension. It follows that the following relationships hold true between the input
and output matrices $B$ and $C$ and the fictitious input and output matrices $\bar{B}_{PF}$ and $\bar{C}_{PF}$ as follows

$$B = B_{PF}K_{PF}$$
$$C = L_{PF}\bar{C}_{PF}$$

**Theorem 3.1.** Let $G(z)$ be a stable and strictly proper transfer function whereas $V(z)$ and $W(z)$ be the input and output weights respectively. If $G_r(z)$ is a reduced order model obtained by using the algorithm presented in Section 3.1, it follows that the following error bound holds true:

$$||W(z)(G(z) - G_r(z))V(z)||_\infty \leq \delta \sum_{k=r+1}^{n} \sigma_k$$

where $\delta = \frac{2}{\alpha \beta}||W(z)||_\infty ||V(z)||_\infty$ and $\sigma_1, \sigma_2, ... \sigma_n$ are the Hankel singular values of the system $\{A, \bar{B}_{PF}, \bar{C}_{PF}\}$.

The proof is similar to the proof of the continuous time error bounds in [13] and is therefore omitted here for brevity.

**Corollary 1.** In the case of input weighting alone, the error bound is given by

$$||G(z) - G_r(z)||_\infty \leq \frac{2}{\alpha \beta}||V(z)||_\infty \sum_{k=r+1}^{n} \sigma_k.$$
Table 1.

| Order | Enns Method | Proposed Method |
|-------|-------------|-----------------|
|       | W.E.        | α   | β   | W.E.  |
| 1     | 2.0801      | 250 | 250 | 2.1027|
| 2     | 0.2545      | 250 | 250 | 0.2411|
| 3     | 0.1072      | 250 | 250 | 0.1087|

advantage of the proposed method is the guaranteed stability of the reduced order model when double sided weightings are used and the existence of error bounds.

5. Conclusion. In conclusion, a frequency weighted model reduction method for discrete-time systems based on partial fraction expansion has been developed. The proposed method guarantees the stability of the reduced order model in the case of double sided weightings.

REFERENCES

[1] D. W. Ding, X. Du and X. Li, Finite-frequency model reduction of two-dimensional digital filters, *IEEE Trans. Autom. Control*, 60 (2015), 1624–1629.

[2] X. Du, F. Fan, D. W. Ding and F. Liu, Finite-frequency model order reduction of discrete-time linear time-delayed systems, *Nonlinear Dynamics*, 83 (2016), 2485–2496.

[3] D. Enns, Model reduction with balanced realizations: An error bound and a frequency weighted generalization, In: *Proceedings of the 23rd IEEE Conference on Decision and Control, Las Vegas, USA*, (1984), 127–132.

[4] M. Imran and A. Ghafoor, Model reduction of descriptor systems using frequency limited gramians, *J. Franklin Inst.*, 352 (2015), 33–51.

[5] X. Li, C. Yu and H. Gao, Frequency limited $H_\infty$ model reduction for positive systems, *IEEE Trans. Autom. Control*, 60 (2015), 1093–1098.

[6] C. A. Lin and T. Y. Chiu, Model reduction via frequency weighted balanced realization, *Control Theory and Advanced Technology*, 8 (1992), 341–451.

[7] Y. Liu and B. D. O. Anderson, Singular perturbation approximation of balanced systems, *International Journal of Control*, 50 (1989), 1339–1405.

[8] B. C. Moore, Principal component analysis in linear system: Controllability, observability, and model reduction, *IEEE Trans. Automat. Contr*, AC-26 (1981), 17–32.

[9] U. M. Saggaf and G. F. Franklin, On model reduction, *Proc. of the 23rd IEEE Conf. on Decision and Control*, (1986), 1064–1069.

[10] U. M. Saggaf and G. F. Franklin, Model reduction via balanced realization, *IEEE Trans. on Autom. Control*, AC-33 (1988), 687–692.

[11] H. R. Shaker and M. Tahavori, Frequency interval model reduction of bilinear systems, *IEEE Trans. Autom. Control*, 59 (2014), 1948–1953.

[12] V. Sreeram and B. D. O. Anderson, Frequency weighted balanced reduction technique: A generalization and an error bound, *Proceedings of the 34th IEEE Conference on Decision and Control*, (1995), 3576–3581.

[13] V. Sreeram, S. Sahlan, W. M. W Muda, T. Fernando and H. H. C. Iu, A generalised partial-fraction-expansion based frequency weighted balanced truncation technique, *International Journal of Control*, 86 (2013), 833–843.

[14] T. Van-Gestel, B. Anderson and P. Van-Overschee, On frequency weighted balanced truncation: Hankel singular values and error bounds, *European Journal of Control*, 7 (2001), 584–592.
[15] G. Wang, V. Sreeram and W. Q. Liu, A new frequency-weighted balanced truncation method and an error bound, *IEEE Transactions on Automatic Control*, 44 (1999), 1734–1737.

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