THE EQUIVALENCE BETWEEN DIFFERENT DARK (MATTER) ENERGY SCENARIOS

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Abstract. We have shown that the phenomenological models with a cosmological constant of the type $\Lambda = \beta \left( \frac{\dot{R}}{R} \right)$ and $\Lambda = 3\alpha H^2$, where $R$ is the scale factor of the universe and $H$ is the Hubble constant, are equivalent to a quintessence model with a scalar ($\phi$) potential of the form $V \propto \phi^{-n}$, $n$ = constant. The equation of state of the cosmic fluid is described by these parameters ($\alpha, \beta, n$) only. The equation of state of the cosmic fluid (dark energy) can be determined by any of these parameters. The actual amount of dark energy will define the equation of state of the cosmic fluid. All of the three forms can give rise to cosmic acceleration depending the amount of dark energy in the universe.

Key Words: cosmology: theory-dark energy, quintessence, cosmological constant, cosmic accelerating

1. Introduction

Recent observation of the Hubble diagram for supernovae Ia indicates that the expansion of the universe is accelerating at the present epoch (Perlmutter et al., 1998; Riess et al., 1998). This apparent acceleration is attributed to a dark energy residing in space itself, which also balances the kinetic energy of expansion so as to give the universe zero spatial curvature, as deduced from the cosmic microwave background radiation (CMBR). Cosmologists have proposed several dark energy models to explain the present cosmic acceleration. This dark energy was important in the past as it is now. It might have played a part in limiting the formation of largest gravitationally bound structures. One can model the dark energy by a cosmological constant (or a vacuum decaying energy). However, not all vacuum decaying cosmological models predict this acceleration. One way to account for such an acceleration is to propose a kind of scalar
field known as quintessence dominating the universe today. Quintessence offers a possible explanation for the observed acceleration of the universe without resorting to the cosmological constant. In essence, quintessence endeavors to replace the static cosmological constant with a dynamical negative pressure component. However, quintessence models exhibit an event horizon which poses a serious problem for string theory (Fichler et al., 2001). The quintessence is supposed to obey an equation of state of the form \( p_Q = \omega_Q \rho_Q \) and \( \omega_Q = -1 \) corresponds to vacuum energy density.

In this brief letter we show that the three forms of the cosmological constants, viz., \( \Lambda = \beta \left( \frac{\dot{R}}{R} \right) \) and \( \Lambda = 3\alpha H^2 \), where \( \beta, \alpha \) are constants, and the quintessence model with a scalar \( (\phi) \) potential of the form \( V \propto \phi^{-n}, \ n = \text{constant} \) are equivalent. We thus see that no one is more fundamental than the others. Analysis shows that \( \beta \) and \( \alpha \) determine the equation of state of the corresponding quintessence.

2. The Field Equations

The Einstein field equations with a variable cosmological constant, and energy conservation law yield

\[
\left( \frac{\ddot{R}}{R} \right)^2 + \frac{k}{R^2} = \frac{8\pi}{3} G \rho + \frac{\Lambda}{3} \tag{1}
\]

\[
\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3} \tag{2}
\]

and

\[
\dot{\rho} + 3\frac{\dot{R}}{R} (p + \rho) = -\frac{\dot{\Lambda}}{8\pi G}. \tag{3}
\]

Using the equation of the state

\[
p = (\gamma - 1)\rho, \quad 1 \leq \gamma \leq 2, \tag{4}
\]

eq(2) can be written as

\[
\frac{\ddot{R}}{R} = \frac{8\pi G}{3} \left( 1 - \frac{3}{2\gamma} \right) \rho + \frac{\Lambda}{3}. \tag{5}
\]

We see from eq.(3) that a variable \( \Lambda \) induces a term representing the rate of decay of vacuum energy into matter/radiation (or the rate of generation of entropy). It is thus apparent that for a decreasing \( \Lambda \) the entropy increases.

3. The Observable Cosmological Parameters

Following Arbab (2003, 1997), we consider following forms of \( \Lambda \)

\[
\Lambda = \beta \left( \frac{\dot{R}}{R} \right), \tag{6}
\]

and

\[
\Lambda = 3\alpha H^2, \tag{7}
\]

where \( \alpha \) and \( \beta \) are undetermined constants.
For the matter-dominated flat universe ($\gamma = 1, k = 0$). Thus eqs.(1), (2), (4) and (5) give (Arbab, 2003)

$$R(t) = Ct^{\frac{\beta - 2}{\beta - 3}}, \quad C = \text{const.}, \beta \neq 3$$

(8)

Then eq.(6) becomes

$$\Lambda(t) = \frac{\beta(\beta - 2)}{(\beta - 3)^2} t^2, \quad \beta \neq 3,$$

(9)

and eq.(3) yields

$$\rho(t) = \frac{1}{4\pi G t^2}, \quad \beta \neq 3.$$ 

(10)

The deceleration parameter is given by

$$q = -\frac{\ddot{R}R}{R^2} = \left(\frac{1}{2 - \beta}\right), \quad \beta \neq 2$$

(11)

It is clear that for a positive energy density $\beta > 3$ so that $\Lambda > 0$ and $q < 0$. Hence, we obtain a cosmic acceleration with a minimal requirement.

The mass density parameter of the universe is given by

$$\Omega_m = \frac{2}{3} \frac{\beta - 3}{(\beta - 2)^2}, \quad \beta \neq 2$$

(12)

and the the vacuum density parameter

$$\Omega_v = \frac{\beta}{3(\beta - 2)}, \quad \beta \neq 2,$$

(13)

so that $\Omega_m + \Omega_v = 1$, as preferred by inflation. Note that all other cosmological parameters depend on the constant ($\beta$). The case $\beta = 2$ defines a static universe, which is physically unacceptable for the present universe. For $\beta > 0$ eq.(13) implies that $\Omega_v > \frac{1}{3}$. We see that the universe will be ultimately driven into a de-Sitter phase of exponential expansion at the present epoch if $\Omega_v \to 1$ (or $\beta \to 3$) (Arbab, 2003).

4. The Quintessence, the Cosmological Constant and Equation of State

In general, the most important difference of a dark energy component to a cosmological constant is that its equation of state can be different form $p = -\rho$, generally implying a time-variation.

Using eq.(5) one can write eq.(14) as

$$\Lambda = \left(\frac{\beta}{3 - \beta}\right) \left(1 - \frac{3}{2} \gamma\right) 8\pi G \rho, \quad \beta \neq 3.$$ 

(14)

It is evident that when $\gamma = \frac{2}{3}$ the cosmological constant vanishes ($\Lambda = 0$). Thus if the universe is dominated by strings today, the cosmological constant must vanish!

For the matter-dominated universe ($\gamma = 1$) eq.(14) gives

$$\Lambda = \left(\frac{\beta}{\beta - 3}\right) 4\pi G \rho.$$ 

(15)
It is evident from the above equation that an empty universe \((\rho = 0)\) would imply a vanishing cosmological constant \((\Lambda = 0)\).

For the radiation-dominated epoch \((\gamma = \frac{4}{3})\) eq.(14) gives

\[
\Lambda = \left(\frac{\beta}{\beta - 3}\right) 8\pi G \rho .
\]  

(16)

Now eq.(1) and (7) give \((p = 0, k = 0)\)

\[
\Lambda = \left(\frac{\alpha}{1 - \alpha}\right) 8\pi G \rho .
\]  

(17)

Comparison of eq.(17) with eq.(15) yields

\[
\alpha = \frac{\beta}{3(\beta - 2)}.
\]  

(18)

Hence, eqs.(11) and (18) yield

\[
q = \frac{1}{2}(1 - 3\alpha).
\]  

(19)

Now a cosmic acceleration with a positive cosmological constant requires

\[
\frac{1}{3} < \alpha < 1.
\]  

(20)

Using eq.(15), eq.(5) reads

\[
\frac{\ddot{R}}{R} = \frac{4\pi G}{3} \left(\frac{3}{\beta - 3}\right) \rho ,
\]  

(21)

i.e. if \(\beta > 3\) then the cosmic acceleration is proportional to the factor \(\left(\frac{1}{\beta - 3}\right) \rho .\)

Very recently, Majerník (2001, 2002) considered a Friedmann’s model with an alternative \(\Lambda\)-part, as representing a form of the quintessence. He assumed that \(\Lambda\) is proportional to the stress-energy scalar \((T)\), viz.

\[
\Lambda = 8\pi \kappa G T ,
\]  

(22)

where \(\kappa\) is a free parameter and \(T = \rho\) in the present epoch.

For quintessence, the equation of state takes the form

\[
p_Q = \omega_Q \rho_Q , \quad -1 < \omega_Q < 0 ,
\]  

(23)

so that for eq.(22) one finds (Majerník, 2001, 2002),

\[
\omega_Q = -\frac{\kappa}{1 + \kappa}.
\]  

(24)
Unlike the ordinary ideal fluid, the equation of state of the quintessence will depend only on the amount of ordinary matter and/or dark energy involved. Thus, if one determines $\Omega_v$, then $\omega_Q$ can be calculated. Comparing eqs. (22) and (15) one reveals that

$$\kappa = \frac{\beta}{2(\beta - 3)}. \quad (25)$$

and from eq. (24) one concludes that

$$\omega_Q = -\frac{\beta}{3(\beta - 2)}, \text{ or } \omega_Q = -\Omega_v, \quad (26)$$

using eq. (13). Equation (25) can be inverted to define $\beta$ as

$$\beta = \frac{6\omega_Q}{1 + 3\omega_Q}. \quad (27)$$

Similarly, from eqs. (17), (18), (24) and (25) one finds

$$\alpha = -\omega_Q, \text{ or } \alpha = \frac{\kappa}{1 + \kappa}. \quad (28)$$

Therefore, $\beta$ (or $\alpha$) defines the equation of state of the corresponding equivalent quintessence (“dark energy”). The constraint that $\omega_Q < -0.6$ implies $\Omega_v > 0.6$. This dictates that $\beta < 4.5$. Thus one has the stringent constraint on $\beta$, viz., $3 < \beta < 4.5$ and a similar one on $\alpha$ as $0.6 < \alpha < 1$. Consequently, one has $-1 < q < -0.4$. As remarked by Majerník that when $\Omega_m \to 0$ then $\kappa \to \infty$ and $\omega_Q \to -1$; here we have as $\beta \to 3$ (or $\alpha \to 1$), $\Omega_v \to 1$ (de Sitter universe).

We would like to remark here that $|\omega_Q|$ is nothing but the vacuum energy parameter, as evident from eq. (7) and the fact that $\rho_v = \frac{\Lambda}{8\pi G}$. Measuring the dark energy equation of state will rely very much on the future observations. Thereafter, one can decide whether quintessence is indeed an acceptable explanation of the dark energy of the universe.

### 5. Cold Dark Matter Model

In cold dark matter model (CDM) the dark energy interacts only with itself and gravity. Its equation of state is given by

$$p_X = \omega_X \rho_X, \quad (29)$$

and if $\omega_X =$constant, its energy density is given by

$$\rho_X \propto R^{-3(1+\omega_X)}. \quad (30)$$

However, in a scalar field model the parameter $\omega_X$ is derived from the field model.

Ratra & Quillen (1992) and Brax & Martin (2002) considered a scalar field potential ($V$) of the form

$$V(\phi) = \frac{C}{\phi^n}. \quad (31)$$
where $n$, $C$ are some constants. They have found that

\[ \omega_X = \frac{-2}{2 + n}. \]  

(32)

Comparison of eq.(32) with eq.(26) shows that

\[ n = \frac{4(\beta - 3)}{\beta}. \]  

(33)

Consequently, an accelerated expansion ($\beta > 3$) requires $n > 0$. This constraint allows the energy density of the scalar field to roll down very slowly after inflation, but at a still high red-shift, and it had a relatively small value, so it has not disturbed the CMBR, but eventually dominates and the universe acts as if it had a cosmological constant that varying slowly with time (and possibly space)(Peebles, 2002). We observe that in the limit where $n \to 0$ the scalar field energy density mimics a cosmological constant. The cosmological consequences of the models discussed above are the same for the present era. Therefore, no one of them is more fundamental than the others.

Acknowledgements

I would like to thank Comboni College for providing a research support for this work.

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