RESEARCH ARTICLE

The trace of the affine Hecke category

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Abstract
We compare the (horizontal) trace of the affine Hecke category with the elliptic Hall algebra, thus obtaining an “affine” version of the construction of Gorsky et al. (Int. Math. Res. Not. IMRN 2022 (2022) 11304–11400). Explicitly, we show that the aforementioned trace is generated by the objects $E_d = \text{Tr}(Y_1^{d_1} \ldots Y_n^{d_n} T_1 \ldots T_{n-1})$ as $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, where $Y_i$ denote the Wakimoto objects of Elias and $T_i$ denote Rouquier complexes. We compute certain categorical commutators between the $E_d$’s and show that they match the categorical commutators between the sheaves $\mathcal{E}_d$ on the flag commuting stack that were considered in Neguț (Publ. Math. Inst. Hautes Études Sci. 135 (2022) 337–418). At the level of $K$-theory, these commutators yield a certain integral form $\tilde{A}$ of the elliptic Hall algebra, which we can thus map to the $K$-theory of the trace of the affine Hecke category.

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1 | INTRODUCTION

1.1 | Motivation

The HOMFLY-PT skein algebra $Sk(\mathbb{T})$ of the torus is generated by framed links in the thickened torus $\mathbb{T} \times [0, 1]$ modulo the skein relation (which depends on a parameter $q$), and the multiplication is given by vertical stacking:

$\mathbb{T} \times [0, 1] \cup \mathbb{T} \times [1, 2] \to \mathbb{T} \times [0, 2]$.

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Morton and Samuelson [24] proved that:

\[ \text{Sk}(\mathbb{T}) \cong E_{q,q^{-1}}, \]  

(1)

where \( E_{q_1,q_2} \) is the elliptic Hall algebra of Burban and Schiffmann [9]. Under the isomorphism (1), the \((m,n)\) torus knot goes to the generator \( P_{m,n} \in E_{q,q^{-1}} \), for any coprime integers \( m \) and \( n \). The skein algebra is naturally bigraded by \( H_1(\mathbb{T}) \cong \mathbb{Z}^2 \), and \( P_{m,n} \) has bidegree \((m,n)\). In this paper, we will mostly focus on the positive half \( \text{Sk}_+(\mathbb{T}) \) of the skein algebra generated by \( P_{m,n} \) with \( n > 0 \).

The main purpose of this paper is to propose a way to categorify (1).

In the left-hand side of (1), we recall that the affine braid group \( \text{ABr}_n \) can be defined as the mapping class group of the annulus \( \mathbb{A} \) with \( n \) marked points (i.e., automorphisms modulo isotopy of \( \mathbb{A} \setminus \{n \text{ points}\} \) that pointwise fix the two boundary circles of \( \mathbb{A} \)). The affine Hecke algebra \( \text{AH}_n = \text{AH}_n(q) \) is defined as a quotient of the group algebra \( \mathbb{C}[\text{ABr}_n] \) by local skein relations (it depends on a parameter \( q \) that we will often omit from notations). Thus, an affine braid naturally lives inside \( \mathbb{A} \times [0,1] \), and we can consider its annular closure in \( \mathbb{A} \times S^1 \cong \mathbb{T} \times [0,1] \). In other words, conjugacy classes of affine braids naturally correspond to links in the thickened torus. Furthermore, the skein multiplication corresponds to placing one annulus inside another and considering the homomorphisms

\[ \text{ABr}_n \times \text{ABr}_k \rightarrow \text{ABr}_{n+k}, \quad \text{AH}_n \otimes \text{AH}_k \rightarrow \text{AH}_{n+k}. \]

Both the affine braid group \( \text{ABr}_n \) and affine Hecke algebra \( \text{AH}_n \) are naturally graded by the rotation number \( m \) around the annulus, and correspond to the component of \( \text{Sk}_+(\mathbb{T}) \) of bidegree \((m,n)\).

The affine Hecke algebra \( \text{AH}_n \) is categorified by \( \mathcal{K}(\text{ASBim}_n) \), the homotopy category of affine Soergel bimodules [12, 25]. The operation of annular closure corresponds to the notion of (derived) horizontal trace \( \text{Tr} \) developed in [1, 2, 18, 21]. Therefore, based on the discussion above, we propose to categorify the skein of the torus by the category \( \text{Tr}(\text{ASBim}_n) \):

\[
\begin{array}{cccc}
\mathcal{K}(\text{ASBim}_n) & \longrightarrow & \text{Tr}(\text{ASBim}_n) \\
\downarrow & & \downarrow \\
\text{ABr}_n & \longrightarrow & \text{AH}_n & \longrightarrow & \text{Sk}_+(\mathbb{T})
\end{array}
\]

The diagram above is for motivational purposes only: the vertical arrows denote “categorification” and the horizontal arrows denote the natural projection and “closing annular braids”.

For the right-hand side of (1), we recall that Schiffmann and Vasserot [43] explained that (a certain integral form of) \( E_{q_1,q_2} \) can be naturally mapped to:

\[ K = \bigoplus_{n=0}^{\infty} K_{C^* \times C^*}(\text{Comm}_n), \]

where the commuting stack \( \text{Comm}_n \) is defined in (26). It is natural to categorify \( K \) by the derived category of equivariant coherent sheaves on the derived stack \( \text{Comm}_n \). Therefore, one approach to categorifying (1) would be to rigorously state and prove the following.
Problem 1.1. Construct a functor:

$$\text{Tr}(\text{ASBim}_n) \longrightarrow D^{b}_{C^* \times C^*} (\text{Coh}(\text{Comm}_n))$$

(2)

and identify its essential image.

In the following subsections, we will explain some constraints and properties that one expects from the functor (2), and perform the construction at the level of $K$-theory (see Theorem 2.10). An analogue of the functor (2) was constructed by Oblomkov and Rozansky using the language of matrix factorizations (see Subsection 1.8 for details).

1.2 | Affine Soergel bimodules

The extended affine braid group $\text{ABr}_n$ of type $\tilde{A}_{n-1}$ can be interpreted as the group of $n$-strand braids in the punctured plane or, equivalently, braids with a pole. We will denote the pole by a thick line. The generators of $\text{ABr}_n$ are denoted by $\{\sigma_i\}_{i \in \{1, \ldots, n-1\}}$ and $\{y_i\}_{i \in \{1, \ldots, n\}}$, and they are represented in Figure 1.

In [12], Elias constructed a monoidal categorification of $\text{ABr}_n$ (or rather the affine Hecke algebra $\text{AH}_n$) using the homotopy category of complexes of affine Soergel bimodules $\mathcal{K}(\text{ASBim}_n)$, see Subsection 3.4 for details. In particular, $\sigma_i$ correspond to standard Rouquier complexes $T_i$ [41], while $y_i$ correspond to the so-called Wakimoto objects $Y_i$. The objects $Y_i$ commute and so for any vector of integers $d = (d_1, \ldots, d_n)$ it is natural to consider the object $Y^d = Y_1^{d_1} \cdots Y_n^{d_n} \in \mathcal{K}(\text{ASBim}_n)$, also called a Wakimoto object in [12]. More generally, we can tensor Wakimoto objects by the Rouquier complexes of Coxeter braids, and denote the resulting object by:

$$Y^d T_{\text{cox}_{n-1}} = Y_1^{d_1} \cdots Y_n^{d_n} T_1 \cdots T_{n-1} \in \mathcal{K}(\text{ASBim}_n).$$

(3)

Consider any composition $n_1 + \cdots + n_r = n$ and any $r$-tuple of integer vectors:

$$\left\{ d^i = (d_{n_1+\cdots+n_{i-1}+1}, \ldots, d_{n_1+\cdots+n_i}) \right\}_{1 \leq i \leq r}.$$  

(4)

Then we may generalize (3) by considering the objects:

$$Y^{d^1}_{\text{cox}_{n_1}} \ast \cdots \ast Y^{d^r}_{\text{cox}_{n_r}} := Y_1^{d_1} \cdots Y_n^{d_n} \prod_{i=1}^{r} T_{n_1+\cdots+n_{i-1}+1} \cdots T_{n_1+\cdots+n_i-1} \in \mathcal{K}(\text{ASBim}_n).$$

(5)
Remark 1.2. We expect that the objects (5) can be obtained from the objects (3) by successive applications of yet undefined bifunctors:

\[ \mathcal{K}(\text{ASBim}_n) \otimes \mathcal{K}(\text{ASBim}_k) \rightarrow \mathcal{K}(\text{ASBim}_{n+k}) \]  

whose action on objects matches the homomorphisms \( \text{ABr}_n \times \text{ABr}_k \rightarrow \text{ABr}_{n+k} \) described in the previous subsection. If \( \alpha \in \mathcal{K}(\text{ASBim}_n) \) and \( \beta \in \mathcal{K}(\text{ASBim}_k) \) are images of affine braids, then the object \( \alpha \star \beta \) is well-defined in \( \mathcal{K}(\text{ASBim}_{n+k}) \) (this follows from the braid relations). However, defining \( \star \) on morphisms remains an open problem.

1.3 | Derived trace

We will use the formalism of derived horizontal traces developed in [18]. Given a monoidal pre-triangulated dg category \( \mathcal{C} \), one constructs another dg category \( \text{Tr}(\mathcal{C}) \) with the following properties.

(a) There is a well-defined dg functor \( \text{Tr} : \mathcal{C} \rightarrow \text{Tr}(\mathcal{C}) \). In particular, morphisms in \( \mathcal{C} \) correspond to morphisms in \( \text{Tr}(\mathcal{C}) \) and their cones are sent to cones.

(b) Assuming that \( \mathcal{C} \) has duals, one has isomorphisms \( \text{Tr}(XY) \cong \text{Tr}(YX) \) that are natural in both \( X \) and \( Y \).

(c) The endomorphism (dg) algebra of \( \text{Tr}(1) \) is isomorphic to the Hochschild homology of \( \mathcal{C} \) equipped with the shuffle product.

We refer to [18] and Subsection 3.6 for more details. Property (b) allows us to interpret the traces of objects corresponding to braids in \( \mathcal{K}(\text{ASBim}_n) \) as their annular closures, as in Figure 2. As the complement of the closure of the pole in \( (\mathbb{R}^2 \setminus \{s\}) \times \mathbb{R} \) is homeomorphic to the thickened torus \( \mathbb{T} \), annular closures such as Figure 2 yield elements of \( \text{Sk}(\mathbb{T}) \).

Another important feature of \( \text{Tr}(\mathcal{C}) \) is its universal trace-like property. Assume that \( \mathcal{D} \) is another dg category and \( F : \mathcal{C} \rightarrow \mathcal{D} \) is a trace-like dg functor (see [18] for the precise definitions), in particular, one has isomorphisms \( F(XY) \cong F(YX) \) that are natural in \( X \) and \( Y \). Then \( F \) factors...
through $\text{Tr}(\mathcal{C})$ and fits into the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{Tr}} & \text{Tr}(\mathcal{C}) \\
\downarrow F & & \downarrow 1_P \\
\mathbb{D} & \xrightarrow{=} & \mathbb{D}
\end{array}
$$

In this paper, we will be mostly interested in the trace of the dg category $\mathcal{C} = \mathcal{H}(\text{ASBim}_n)$ that we will abbreviate as $\text{Tr}(\text{ASBim}_n)$. The traces of the objects (3) and (5) will be denoted as follows:

$$E_d := \text{Tr}(Y_d T_{\text{cox}_n})$$

and, in the notation of (4):

$$E_{d_1} \star \cdots \star E_{d_r} := \text{Tr}\left( Y_{d_1} T_{\text{cox}_{n_1}} \star \cdots \star Y_{d_r} T_{\text{cox}_{n_r}} \right).$$

Similarly to (6), we expect that there exist bifunctors:

$$\text{Tr}(\text{ASBim}_n) \otimes \text{Tr}(\text{ASBim}_k) \xrightarrow{\star} \text{Tr}(\text{ASBim}_{n+k}).$$

Using the universal property of $\text{Tr}(\mathcal{C})$, we can rephrase Problem 1.1 as follows:

**Problem 1.3.** Construct a trace-like functor:

$$\text{ASBim}_n \rightarrow D^b_{C^* \times C^*} (\text{Coh}(\text{Comm}_n))$$

and identify its essential image.

### 1.4 The commuting and flag commuting stacks

This subsection is motivational, and will not be directly used in the present paper (in more detail, the contents herein are derived category generalizations of the principles laid out in [44]; the generalization itself is straightforward using tools of derived algebraic geometry, but it is beyond the scope on the present paper). See also [37, 38] for related constructions and computations.

As we will recall in Section 2, the commuting stack $\text{Comm}_n$ of (26) parameterizes pairs of commuting $n \times n$ matrices modulo the general linear group $GL_n$. Similarly, the flag commuting stack $\text{FComm}^*_n$ of (36) parameterizes pairs of commuting upper triangular matrices (whose diagonals are $(x, \ldots, x)$ and $(0, \ldots, 0)$, respectively, for any $x \in \mathbb{C}$) modulo the Borel subgroup $B_n$ of upper triangular matrices. There is a proper morphism:

$$\pi : \text{FComm}^*_n \rightarrow \text{Comm}_n.$$  

Any stack that is a quotient by the Borel subgroup (in this case, $\text{FComm}^*_n$) comes endowed with line bundles $L_1, \ldots, L_n$ that arise from the $n$ diagonal characters of $B_n$. Thus, for all vectors of
integers $d = (d_1, \ldots, d_n)$, we may consider:

$$E_d = \pi_*(L_1^{d_1} \cdots L_n^{d_n}) \in D^b_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Coh}(\text{Comm}_n)).$$  \hfill (12)

Let us consider the categorical version of the Hall product studied by Schiffmann–Vasserot [43], namely the bifunctor:

$$D^b_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Coh}(\text{Comm}_n)) \otimes D^b_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Coh}(\text{Comm}_k)) \xrightarrow{\star} D^b_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Coh}(\text{Comm}_{n+k}))$$  \hfill (13)

given by the formula:

$$\alpha \star \beta = R p_1^*(L p_2^*(\alpha \boxtimes \beta)).$$

Above, we let $\text{Comm}_{n,k}$ be the stack parameterizing pairs of commuting $(n, k)$-block upper triangular matrices, modulo simultaneous conjugation by the natural maximal parabolic subgroup of $GL_{n+k}$, and consider the maps:

$$\begin{align*}
\text{Comm}_{n,k} & \xrightarrow{p_1} \text{Comm}_{n+k} \\
\text{Comm}_{n} \times \text{Comm}_{k} & \xrightarrow{p_2} \text{Comm}_{n+k}
\end{align*}$$  \hfill (14)

which remember the various diagonal blocks of the matrices parameterized by $\text{Comm}_{n,k}$. We note that the derived pullback in (14) might not preserve the derived categories of bounded complexes in (13) due to the singularities of the commuting stack, but this issue can and should be solved by viewing $\text{Comm}_n$ as a quasi-smooth dg scheme (see Remark 2.1; in the framework of dg schemes, the maps $p_1$ and $p_2$ of (14) are proper and quasi-smooth, respectively). By repeated applications of the operation (13) to the objects (12), we may construct the following objects for any collection of integers (4):

$$E_d \star \cdots \star E_{d'} \in D^b_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Coh}(\text{Comm}_n)).$$  \hfill (15)

We expect the functor (2) to intertwine the products (9) and (13), and to send:

$$E_d \text{ of (7)} \mapsto E_d \text{ of (12)}.$$  \hfill (16)

As a consequence, the functor (2) would send the objects (8) to (15).

1.5 Relations for the commuting stack

Let us reinterpret [31, Theorem 1.4], which inspired the present paper. As we will explain after giving the statement of the theorem, the result below is stated in a slightly different context from the result of [31, Theorem 1.4], but we take the liberty of stating it as a “Theorem” for two reasons: its usage in the present paper is purely motivational, and adapting the proof of [31, Theorem 1.4] to the current setup is routine for specialists in the field (e.g., [39, section 2] does most of the heavy lifting necessary for this generalization).
Theorem 1.4 [31]. For any $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$, there exists a collection of objects $g_0, \ldots, g_n \in D^b_{C^* \times C^*}(\text{Coh}(\text{Comm}_n))$ with the following properties.

- $g_0 = \mathcal{E}(k) \star \mathcal{E}_d$ and $g_n = \mathcal{E}_d \star \mathcal{E}(k)$.
- For all $i \in \{1, \ldots, n\}$ there exist morphisms:

$$
\begin{cases}
g_{i-1} \to g_i & \text{if } k \geq d_i \\
g_{i-1} \leftarrow g_i & \text{if } k \leq d_i,
\end{cases}
$$

which are mutually inverse isomorphisms if $k = d_i$.
- The cones of morphisms in the previous bullet are filtered by

$$
[C_{q_2}^0 \to C_1] : \left\{ \mathcal{E}(d_1, \ldots, d_{i-1}, k-a, d_i + a, d_{i+1}, \ldots, d_n), 1 \leq a \leq k - d_i \text{ if } k > d_i \\
\mathcal{E}(d_1, \ldots, d_{i-1}, d_i - a, k + a, d_{i+1}, \ldots, d_n), 1 \leq a \leq d_i - k \text{ if } k < d_i. \right. \tag{16}
$$

In the formula above, $C_\chi$ denotes the one-dimensional vector space, viewed as a representation of $C^* \times C^*$ through the character $\chi$, and we write $q_1$ and $q_2$ for the elementary characters of the two factors of $C^*$. The asymmetry between $q_1$ and $q_2$ in (16) is due to the fact that $F\text{Comm}^*_n$ parameterizes pairs of commuting matrices, the first of which is allowed to have scalar diagonal, but the second of which is forced to have zero diagonal.

There are certain differences between the formulation of Theorem 1.4 and that of [31, Theorem 1.4], which we will now explain. In the language of [31, Theorem 1.4] (applied to the situation of the surface $S = \mathbb{A}^2$), the symbols $\mathcal{E}(d_1, \ldots, d_n)$ should be interpreted as functors:

$$
D^b_{C^* \times C^*}(\text{Coh}(\text{Hilb}_k)) \longrightarrow D^b_{C^* \times C^*}(\text{Coh}(\text{Hilb}_{k+n} \times \mathbb{A}^1))
$$

for all $k \in \mathbb{N}$, where $\text{Hilb}_k$ denotes the Hilbert scheme of $k$ points on $\mathbb{A}^2$, and $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ denotes the $x$-axis. If one composes the functor above with pushforward along $\mathbb{A}^1$, one obtains functors:

$$
D^b_{C^* \times C^*}(\text{Coh}(\text{Hilb}_k)) \longrightarrow D^b_{C^* \times C^*}(\text{Coh}(\text{Hilb}_{k+n})),
$$

which satisfy the relations postulated in the three bullets in Theorem 1.4. The match between this result and the formulation in Theorem 1.4 is completed by the correspondence:

$$
\left\{ \text{objects in } D^b_{C^* \times C^*}(\text{Coh}(\text{Comm}_n)) \right\}
$$

$$
\cong \left\{ \text{functors } D^b_{C^* \times C^*}(\text{Coh}(\text{Hilb}_k)) \to D^b_{C^* \times C^*}(\text{Coh}(\text{Hilb}_{k+n})) \text{ for all } k \in \mathbb{N} \right\} \tag{17}
$$

that underlies the philosophy of [43]. Under the correspondence above, the categorified Hall product $\star$ on the left-hand side corresponds to composition of functors on the right-hand side. Of course, the discussion above is not a proof that Theorem 1.4 follows from [31, Theorem 1.4], but it indicates the means through which to adapt the language of [31, Theorem 1.4] to establish Theorem 1.4 (see [39] for further details on this principle).

Remark 1.5. In fact, the results of [31] yield an explicit geometric construction of the objects $g_i$, and morphisms between them. In the present paper, we will mirror this construction for objects in the category $\mathcal{K}(\mathcal{A}\mathcal{S}\mathcal{B}im_n)$. 

1.6 | Results

We are ready to describe our main results. First, we describe the generators of the dg category $\text{Tr}(\text{ASBim}_n)$.

**Theorem 1.6.** The objects $E_{d_1} \star \cdots \star E_{d_r}$ of (8), as $d_1, \ldots, d_r$ run over all collections of integers (4), generate $\text{Tr}(\text{ASBim}_n)$. That is, any object of $\text{Tr}(\text{ASBim}_n)$ can be presented as a bounded twisted complex with entries given by finite direct sums of $E_{d_1} \star \cdots \star E_{d_r}$ and their shifts.

Next, we describe some interesting morphisms and relations between these generators.

**Theorem 1.7** (Theorem 5.3). For all integers $d_1, \ldots, d_n$ and all $i \in \{1, \ldots, n-1\}$, there is a map:

$$E_{(d_1, \ldots, d_i, d_i+1, \ldots, d_n)} \rightarrow E_{(d_1, \ldots, d_i-1, d_i+1+1, \ldots, d_n)}$$

(18)

with the cone filtered by two copies of:

$$E_{(d_1, \ldots, d_i)} \star E_{(d_i+1, \ldots, d_n)}$$

**Theorem 1.8** (Theorem 5.12). For any $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$, there exists a collection of objects $G_0, \ldots, G_n \in \mathcal{K}(\text{ASBim}_n)$ with the following properties.

- $\text{Tr}(G_0) = E(k) \star E_d$ and $\text{Tr}(G_n) = E_d \star E(k)$.
- For all $i \in \{1, \ldots, n\}$ there exist chain maps in $\mathcal{K}(\text{ASBim}_n)$:

$$
\begin{cases}
G_{i-1} \xrightarrow{\varphi_i} G_i & \text{if } k \geq d_i \\
G_{i-1} \xleftarrow{\bar{\varphi}_i} G_i & \text{if } k \leq d_i,
\end{cases}
$$

which are mutually inverse isomorphisms if $k = d_i$.
- $\text{Tr}(\text{Cone}(\varphi_i))$ and $\text{Tr}(\text{Cone}(\bar{\varphi}_i))$ are filtered by:

$$
[C_{q \rightarrow 0} C_{q^{-1}}] \cdot \begin{cases}
E_{(d_1, \ldots, d_{i-1}, a, d_i+a, d_i+1, \ldots, d_n)} & 1 \leq a \leq k - d_i \text{ if } k > d_i \\
E_{(d_1, \ldots, d_{i-1}, d_i-a, k+a, d_i+1, \ldots, d_n)} & 1 \leq a \leq d_i - k \text{ if } k < d_i,
\end{cases}
$$

(19)

respectively. Above, $q$ denotes the internal grading on Soergel bimodules.

By comparing Theorem 1.4 with Theorem 1.8, we may summarize our expectations about Problem 1.1 as follows. Note that the equivariant parameters $q_1$ and $q_2$ must be specialized to $q^{-2}$ and $q^2$, respectively, and we need to multiply $E_{(d_1, \ldots, d_n)}$ by $q^{1-n}$ in order for relations (19) to perfectly match (16).

**Conjecture 1.9.** The functor (2) should:

(a) send $E_{d_1} \star \cdots \star E_{d_r}$ to $E_{d_1} \star \cdots \star E_{d_r}$;
(b) send $\text{Tr}(G_i)$ to $g_i$, in the notation of Theorems 1.4 and 1.8;
(c) send the traces of the morphisms between the $G_i$’s constructed in Theorem 1.8 to the morphisms between the $g_i$’s constructed in Theorem 1.4.

Theorem 1.6 and part (a) of the conjecture immediately imply the following:

**Corollary 1.10.** Assuming Conjecture 1.9, the essential image of the functor (2) is a certain subcategory of $D_{b}^{\mathbb{C}^{*} \times \mathbb{C}^{*}}(\text{Coh}(\text{Comm}_n))$ generated by the objects $E_d \star \cdots \star E_{d'}$.

### 1.7 $K$-theory and the Elliptic Hall algebra

We will now summarize the consequences of the results and conjectures in the previous subsection to the level of Grothendieck groups (which we denote by $G$ instead of $K_0$ throughout the paper). In particular, the functor (2) should yield a linear map:

$$G(\text{Tr}(\text{ASBim}_n)) \to \mathcal{K}^{\mathbb{C}^{*}}(\text{Comm}_n),$$

where the right-hand side is $K$-theory equivariant with respect to the anti-diagonal subtorus $\mathbb{C}^{*} \hookrightarrow \mathbb{C}^{*} \times \mathbb{C}^{*}$ (in other words, $q_1 q_2 = 1$; the reason for this is that $q_1 q_2$ measures the homological degree on the category $\mathcal{K}(\text{ASBim}_n)$, and this grading must be killed in order for the functor (2) to descend to the Grothendieck groups). Corollary 1.10 would imply that the image of (20) should coincide with the linear span $\mathcal{A}_n$ of the $K$-theory classes of the complexes of sheaves (15). Moreover:

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$$

is an algebra under the $K$-theoretic version of the Hall product (13), and the map (20) should intertwine this multiplication with that of the yet-unconstructed (9). By definition, $\mathcal{A}$ is the $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-algebra generated by the elements:

$$[E_d] \in \mathcal{K}^{\mathbb{C}^{*} \times \mathbb{C}^{*}}(\text{Comm}_n)$$

as $d$ runs over $\mathbb{Z}^n$. In Section 2, we will recall a computational tool called the shuffle algebra, which could in principle allow one to describe the full set of relations between the generators (21). However, in practice these relations are quite complicated, and we will contend with defining a pre-quotient:

$$\tilde{\mathcal{A}} \to \mathcal{A},$$

which is generated by symbols $E_d$, but modulo the simpler relations (49)–(50). In particular, (50) is a $K$-theoretic version of Theorem 1.4.

As a consequence of Theorems 1.7 and 1.8, we obtain the following:

**Theorem 1.11.** There exists a $\mathcal{A}$ algebra homomorphism:

$$\tilde{\mathcal{A}}_{(q_1, q_2) \to (q^{-2}, q^2)} \biggl[ \bigoplus_{n=0}^{\infty} G(\text{Tr}(\text{ASBim}_n)) \biggr].$$
The specialization of parameters \((q_1, q_2) \rightarrow (q^{-2}, q^2)\) is a consequence of the complicated matching of gradings of both sides that was already visible in \([19]\) for finite Hecke categories. The category ASBim_{\mathfrak{n}} and its trace are naturally graded (by so-called \(q\)-grading), so the corresponding homotopy categories are \textit{bigraded} by the \(q\)-grading and the homological grading. This gives the Grothendieck group \(G(\text{Tr}(\text{ASBim}_{\mathfrak{n}}))\) the structure of a module over \(\mathbb{Z}[q^{\pm 1}]\).

On the other hand, the derived category of Comm_{\mathfrak{n}} is \textit{triply graded} by the homological and two equivariant gradings. The corresponding Grothendieck group, as explained above, is then a module over \(\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]\).

The conjectured functor between the two derived categories mixes the gradings in a complicated way: the homological grading on the affine Hecke side matches a linear combination of the homological and equivariant gradings on the geometric side. Thus, on the level of the Grothendieck group we are required to collapse one of the equivariant gradings.

\textit{Remark 1.12.} It is expected that one could equip \(\text{Tr}(\text{ASBim}_{\mathfrak{n}})\) with an additional \textit{third grading} that would map the geometric side. In this case, one should be able to state Theorem 1.11 without specialization of equivariant parameters.

It is natural to conjecture that the homomorphism in Theorem 1.11 factors through the quotient (22). If true, then we would dare to expect that the functor (20) is an isomorphism onto its image \(\mathcal{A}\).

\textit{Remark 1.13.} We note that \(\widetilde{\mathcal{A}}\) and \(\mathcal{A}\) are different only as \(\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]\)-algebras, but we will actually show in Section 2 that they are isomorphic once we tensor with \(\mathbb{Q}(q_1, q_2)\) (upon this localization, they will also be isomorphic to the elliptic Hall algebra \(E_{q_1, q_2}\) of [9], thus tying everything in with (1)). So, relations (49) and (50) describe the full ideal of relations among the elements (21) over the field \(\mathbb{Q}(q_1, q_2)\). Of course, the operation of tensoring with \(\mathbb{Q}(q_1, q_2)\) does not make much sense at the categorical level.

Alternatively, we can interpret Theorem 1.11 as a statement about the cocenter of type \(A\) affine Hecke algebras, which is defined as:

\[
\text{Tr}(\text{AH}_{\mathfrak{n}}) = \text{AH}_{\mathfrak{n}}/[[\text{AH}_{\mathfrak{n}}, \text{AH}_{\mathfrak{n}}]].
\] (23)

The results of He and Nie [22] give an explicit basis of \(\text{Tr}(\text{AH}_{\mathfrak{n}})\) (see Theorem 3.3). However, to the best of our knowledge the multiplicative structure of the algebra \(\bigoplus_{\mathfrak{n}} \text{Tr}(\text{AH}_{\mathfrak{n}})\) has not been yet described in this basis.

\textbf{Theorem 1.14 (Theorem 5.16).} There is a surjective algebra homomorphism:

\[
\widetilde{\mathcal{A}}\bigg|_{(q_1, q_2) \rightarrow (q^{-2}, q^2)} \rightarrow \bigoplus_{n=0}^{\infty} \text{Tr}(\text{AH}_{\mathfrak{n}})
\]

compatible with the natural maps:

\[
\text{Tr}(\text{AH}_{\mathfrak{n}}) = \text{Tr}(G(\text{ASBim}_{\mathfrak{n}})) \rightarrow G(\text{Tr}(\text{ASBim}_{\mathfrak{n}})).
\]
1.8 Comparison with Oblomkov–Rozansky theory

In a series of papers [34–36], Oblomkov and Rozansky constructed yet another categorification of the affine Hecke algebra using a sophisticated monoidal category of matrix factorizations that we will denote by MF₀. It is expected to be closely related to the geometric realization of the affine Hecke algebra using the derived category of the Springer resolution, constructed by Bezrukavnikov and Riche [5, 6]. In this short section, we review the relation between their results and this paper.

The main result of [34] constructs a homomorphism from $\text{ABr}_n$ to $\text{MF}_n$: to every affine braid one can associate an object of $\text{MF}_n$, the product of braids corresponds to the product in $\text{MF}_n$, and the braid relations are satisfied. In particular, there are analogues of the objects $Y_{d_1} T_{\text{cox}_{n_1}} \star \cdots \star Y_{d_r} T_{\text{cox}_{n_r}}$ in $\text{MF}_n$. Moreover, there is a natural (dg) functor:

$$F : \text{MF}_n \to D^b_{C^* \times C^*}(\text{Coh}(\text{Comm}_n)).$$

The main result of [35] shows that $F(\text{analogue of } Y_{d_1} T_{\text{cox}_{n_1}} \star \cdots \star Y_{d_r} T_{\text{cox}_{n_r}}) = \mathbb{1}_{d_1} \star \cdots \star \mathbb{1}_{d_r}$, confirming the analogue of Conjecture 1.9(a) in Oblomkov–Rozansky theory.

To the best of our knowledge, it is not known whether the categories $\text{MF}_n$ and $\text{ASBim}_n$ are equivalent (for their finite analogues see [36]), or whether the functor $F$ is trace-like in the sense of [18]. Nevertheless, one can ask whether $\text{Tr(}\text{MF}_n\text{)}$ is generated by the traces of the analogues of the objects $Y_{d_1} T_{\text{cox}_{n_1}} \star \cdots \star Y_{d_r} T_{\text{cox}_{n_r}}$, or whether the analogues of Theorems 1.7 and 1.8, or Conjectures 1.9(b,c) hold in $\text{Tr(}\text{MF}_n\text{)}$.

Finally, in [3, 4] Ben-Zvi et al. studied the trace of geometric affine Hecke category in terms of affine character sheaves, and compared it to the stack of local systems on the two-torus. It would be interesting to construct the analogues of the objects $\mathcal{E}_d$ and morphisms between them in their setup.

2 AN ALGEBRA OF MANY FACES

2.1 Motivation from geometry

The main purpose of the present paper is to consider the category of affine Soergel bimodules $\text{ASBim}_n$ (which will be defined in Section 3) and describe its horizontal trace. To get a feel of what the answer should look like, we recall that the horizontal trace of the category of finite Soergel bimodules $\text{SBim}_n$ was computed in [18], where it was shown that it is generated by the direct summands of a single object $\text{Tr}(1)$ and

$$\text{End}_{\text{Tr(SBim)}}(\text{Tr}(1)) \cong \mathbb{C}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n] \rtimes S_n.$$

Here $x_i$ are even and $\theta_i$ are odd variables, and $S_n$ permutes them diagonally. In other words,

$$\text{Tr(}\text{SBim}_n\text{)} \cong D^b(\mathbb{C}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n] \rtimes S_n - \text{grmod}) \cong D^b_{C^* \times C^*}(\text{Coh}(\text{Hilb}_n)).$$

In the right-hand side, we encounter the derived category of $C^* \times C^*$ equivariant coherent sheaves on the Hilbert scheme of $n$ points on $\mathbb{A}^2$. For general reasons that have been explored in [34–36],
one would therefore expect a connection between:

\[ \text{Tr}(\text{ASBim}_n) \quad \text{and} \quad D^b_{C^* \times C^*}(\text{Coh(Comm}_n)), \]

where in the right-hand side, we encounter the commuting stack:

\[ \text{Comm}_n = V/GL_n \]

In the formula above, we set:

\[ V = \{ X, Y \in \text{Mat}_{n \times n}, \ [X, Y] = 0 \} \]

and define the action $GL_n \curvearrowright V$ is by simultaneous conjugation. The action $C^* \times C^* \curvearrowright V$ by rescaling the $X$ and $Y$ matrices commutes with the action of $GL_n$, and thus descends to an action $C^* \times C^* \curvearrowright \text{Comm}_n$, which defines the right-hand side of (25).

Remark 2.1. The reason we do not claim (25) as an equivalence is that, as opposed to the well-understood category in the right-hand side of (24), the derived category of coherent sheaves on the commuting stack is very hard to describe. Moreover, one should not consider $V$ as a singular affine scheme, but as the dg scheme cut out by the equations:

\[ \sum_{a=1}^{n} (x_{ia} y_{aj} - y_{ia} x_{aj}) = 0, \quad \forall i, j \in \{1, ..., n\} \]

from affine space with coordinates $\{x_{ij}, y_{ij}\}_{1 \leq i, j \leq n}$. Thus, $\text{Comm}_n$ should be interpreted as a dg stack, and its derived category is defined accordingly. More precisely, the collection of equations (28) is a section of the standard $\mathfrak{gl}_n$-bundle over the $GL_n$-equivariant $2n^2$ dimensional affine space that parameterizes the entries of the matrices $X$ and $Y$.

2.2 | K-theory

Things become a bit more manageable if we pass to the Grothendieck groups of the categories in (25), so our concrete aim is to understand the relation between:

\[ G(\text{Tr}(\text{ASBim}_n)) \quad \text{and} \quad K_{C^* \times C^*}(\text{Comm}_n) \]

(we will use [11, section 5] as a reference for equivariant algebraic K-theory). We can shed some light on the right-hand side of (29) using the language of shuffle algebras, following [43]. Consider the closed embedding $j : V \hookrightarrow A^{2n^2}$ into the affine space parameterizing the coordinates of $X$ and $Y$ in (27). Letting $\mid_o$ denote the restriction from affine space to its origin, we obtain a composition of pushforward and pullback maps

\[ K_{C^* \times C^*}(\text{Comm}_n) = K_{GL_n \times C^* \times C^*}(V) \xrightarrow{j_*} K_{GL_n \times C^* \times C^*}(A^{2n^2}) \]

\[ \xrightarrow{\mid_o} K_{GL_n \times C^* \times C^*}(\text{point}) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}, z_1^{\pm 1}, ..., z_n^{\pm 1}]^\text{sym}, \]

(30)
where \(q_1, q_2\) are the inverses of the standard characters of \(\mathbb{C}^* \times \mathbb{C}^*\), \(z_1, ..., z_n\) are characters of an arbitrary maximal torus of \(GL_n\), and the superscript \(\text{sym}\) denotes symmetric functions in the variables \(z_1, ..., z_n\). The equivariant localization theorem (see, for instance, [11, Corollary 5.10.4]) implies that \(j_*\) becomes an isomorphism over the field \(\mathbb{Q}(q_1, q_2)\). Therefore, (30) yields an isomorphism:

\[
K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Comm}_n) \bigotimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2) \cong \mathbb{Q}(q_1, q_2)[z_1^{\pm 1}, ..., z_n^{\pm 1}]^{\text{sym}}.
\]

As shown in [45], \(K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Comm}_n)\) is a free \(\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]\)-module, so we have an injection:

\[
K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Comm}_n) \hookrightarrow \mathbb{Q}(q_1, q_2)[z_1^{\pm 1}, ..., z_n^{\pm 1}]^{\text{sym}}.
\]

**Proposition 2.2** [47]. The image of the map \(\iota\) lands in the vector subspace:

\[
S_n \subset \mathbb{Q}(q_1, q_2)[z_1^{\pm 1}, ..., z_n^{\pm 1}]^{\text{sym}}
\]

of symmetric Laurent polynomials \(F(z_1, ..., z_n)\) that satisfy the wheel conditions [14]:

\[
F(x, xq_1, xq_1q_2, z_4, ..., z_n) = F(x, xq_2, xq_1q_2, z_4, ..., z_n) = 0.
\]

In light of Proposition 2.2, we conclude that we have a map:

\[
K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Comm}_n) \xrightarrow{\iota} S_n.
\]

The takeaway of this subsection is that the right-hand side of (29) is a certain \(\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]\)-integral form of the \(\mathbb{Q}(q_1, q_2)\)-vector space \(S_n\), a claim that uses [28, Theorem 2.5] to establish the fact that \(\iota\) becomes an isomorphism upon tensoring with \(\mathbb{Q}(q_1, q_2)\).

**Remark 2.3.** In upcoming work, we will take the point of view that the best behaved integral form of \(S_n\) is the one studied in [33]. However, adapting the present paper to the setting of [33] will require us to slightly modify both sides of (29).

### 2.3 The shuffle algebra

There is an interesting convolution product [44] on:

\[
K = \bigoplus_{n=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Comm}_n),
\]

which is none other than the \(K\)-theoretic shadow of (13). Meanwhile, we may consider the so-called shuffle algebra:

\[
S = \bigoplus_{n=0}^{\infty} S_n,
\]
which is endowed with the following shuffle product [14]:

\[ F(z_1, \ldots, z_n) \ast F'(z_1, \ldots, z_{n'}) = \text{Sym} \left[ \frac{F(z_1, \ldots, z_n)F'(z_{n+1}, \ldots, z_{n+n'})}{n!n'} \prod_{1 \leq i < j \leq n+n'} \left( \frac{1 - \frac{z_i q_1}{z_j}}{1 - \frac{z_j q_1}{z_i}} \right) \left( 1 - \frac{z_i q_2}{z_j} \right) \left( 1 - \frac{z_j q_1 q_2}{z_i} \right) \right], \]

(34)

where \( \text{Sym} \) denotes symmetrization over all \((n + n')!\) permutations of the variables. This particular multiplication is chosen precisely so that the maps (32) give rise to an algebra homomorphism (see [44, 45]):

\[ K \overset{i}{\hookrightarrow} S. \]

It is clear from the discussion above that the shuffle algebra \( S \) provides a good computational model for the \( K \)-theory groups of commuting stacks.

### 2.4 Constructing subalgebras

The power of the shuffle algebra is that it allows one to construct many interesting elements. For example, for any \( \mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n \), the element:

\[ R_{\mathbf{d}} = (1 - q_1)^{n-1}(1 - q_2)^n \cdot \text{Sym} \left[ \frac{z_1^{d_1} \ldots z_n^{d_n}}{(1 - \frac{z_1 q_1 q_2}{z_1}) \ldots (1 - \frac{z_n q_1 q_2}{z_{n-1}})} \prod_{1 \leq i < j \leq n} \left( \frac{1 - \frac{z_i q_1}{z_j}}{1 - \frac{z_j q_1}{z_i}} \right) \right] \]

(35)

clearly lies in \( S_n \cap \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] [z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \text{sym} \).

**Definition 2.4.** Consider the \( \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \)-subalgebra \( \mathcal{A} \subset S \) generated by \( R_{\mathbf{d}} \), as \( \mathbf{d} \in \mathbb{Z}^n \).

**Proposition 2.5.** We have \( \mathcal{A} \subset \text{Im } i \), so we may think of \( \mathcal{A} \) as a subalgebra of \( K \).

**Proof.** The proof is an adaptation of the main construction of [31]. Specifically, consider the following version of the flag commuting stack:

\[ \text{FComm}_n^* = U/B_n, \]

(36)

where:

\[
U = \left\{ X = \begin{pmatrix} x & \ast & \ldots & \ast \\ 0 & x & \ast & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x \end{pmatrix}, Y = \begin{pmatrix} 0 & \ast & \ldots & \ast \\ 0 & 0 & \ast & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, [X, Y] = 0 \right\}
\]

(37)
from the affine space with coordinates \( \{x, x_{ij}, y_{ij}\}_{1 \leq i < j \leq n} \). There is a natural map:

\[
\text{FComm}_n^\ast \overset{\pi}{\longrightarrow} \text{Comm}_n
\]

induced by the inclusion of the set of pairs of upper triangular matrices in the set of pairs of all matrices. Then Proposition 2.5 follows from the claim that for all \( d_1, \ldots, d_n \in \mathbb{Z} \):

\[
R_{(d_1, \ldots, d_n)} = \iota(\pi_*(\mathcal{L}^{d_1}_1 \ldots \mathcal{L}^{d_n}_n)),
\]

(39)

where the line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_n \) on \( \text{FComm}_n^\ast \) are induced from the standard diagonal characters of the Borel subgroup \( B_n \). Equality (39) is a straightforward computation involving pushforward and pullback morphisms; we will explain the main idea of the proof, and leave the details as an exercise to the interested reader (see [32, Proposition 2.7] for a closely related computation). Consider the following diagram

\[
\begin{array}{ccc}
K_{B_n \times \mathcal{C} \times \mathcal{C}}(U) & \xrightarrow{j'_*} & K_{B_n \times \mathcal{C} \times \mathcal{C}}(\mathbb{A}^{n^2-n+1}) \\
\downarrow\pi_* & & \downarrow\pi'_* \\
K_{B_n \times \mathcal{C} \times \mathcal{C}}(V) & \xrightarrow{j_*} & K_{B_n \times \mathcal{C} \times \mathcal{C}}(\mathbb{A}^{2n^2}) \\
\end{array}
\]

(40)

where \( \mathbb{A}^{n^2-n+1} \) and \( \mathbb{A}^{2n^2} \) are the affine spaces of coordinates of the matrices \( X \) and \( Y \) in (37) and (27), respectively, and the maps \( j'_* \) and \( j_* \) denote the respective closed embeddings of \( U \) and \( V \) into these affine spaces. Using (30), we have

\[
\iota(\pi_*(\mathcal{L}^{d_1}_1 \ldots \mathcal{L}^{d_n}_n)) = \text{Ind}^{\text{GL}_n}_{B_n} \left( j_* \circ \pi_* (\mathcal{L}^{d_1}_1 \ldots \mathcal{L}^{d_n}_n) \bigg|_o \right) = \text{Ind}^{\text{GL}_n}_{B_n} \left( \pi'_* j'_* (\mathcal{L}^{d_1}_1 \ldots \mathcal{L}^{d_n}_n) \bigg|_o \right),
\]

(41)

where the induction is necessary because we consider \( B_n \) (instead of \( \text{GL}_n \)) equivariant \( K \)-theory on the bottom line of (40). By [11, Lemma 5.4.9], we have for any class \( \alpha \):

\[
\pi'_*(\alpha) \bigg|_o = \alpha \bigg|_o \cdot (1 - q_1)^{n-1} (1 - q_2)^n \prod_{1 \leq i < j \leq n} \left[ (1 - \frac{z_1}{z_j}) \left( 1 - \frac{z_1 q_1 q_2}{z_j} \right) \right]
\]

(42)

because the product of linear factors in the right-hand side of (42) is simply the exterior algebra of the vector space of matrix entries that appear in \( \mathbb{A}^{2n^2} \) but do not appear in \( \mathbb{A}^{n^2-n+1} \) (specifically, these are the lower triangular entries of the matrices \( X \) and \( Y \), and all but one of their diagonal entries). Moreover, by [11, Proposition 5.4.5], we have

\[
j'_* (\mathcal{L}^{d_1}_1 \ldots \mathcal{L}^{d_n}_n) \bigg|_o = z_1^{d_1} \ldots z_n^{d_n} \prod_{2 \leq i+1 < j \leq n} \left( 1 - \frac{z_j}{z_i q_1 q_2} \right)
\]

(43)
because the product in the right-hand side is the exterior algebra of the vector space in which the equations (38) take values. Finally, we have for any $\gamma \in K_{B_n \times \mathbb{C}^* \times \mathbb{C}^*}(\text{point})$:

$$\text{Ind}_{B_n}^{GL_n}(\gamma) = \text{Sym} \left[ \frac{\gamma}{\prod_{1 \leq i < j \leq n} \left(1 - \frac{z_i}{z_j}\right)} \right]$$ \tag{44}

by the Weyl character formula. Using (42), (43), and (44), it is clear that the right-hand side of (41) is precisely the right-hand side of (35), which implies the required (39).

\[ \square \]

2.5 | Relations in $\mathcal{A}$

The following relation in $\mathcal{S}$ is easy to prove:

$$R_{(d_1,\ldots,d_i,d_{i+1},\ldots,d_n)} - q_1 q_2 \cdot R_{(d_1,\ldots,d_{i-1},d_{i+1}+1,\ldots,d_n)} = (1 - q_1) R_{(d_1,\ldots,d_i)} R_{(d_{i+1},\ldots,d_n)}$$ \tag{45}

for all $d_1,\ldots,d_n \in \mathbb{Z}$ and $i \in \{1,\ldots,n-1\}$. If we define for all $(m,n) \in \mathbb{Z} \times \mathbb{N}$:

$$H_{m,n} = R_{(d_1,\ldots,d_n)} \text{ where } d_i = \left[ \frac{mi}{n} \right] - \left[ \frac{m(i-1)}{n} \right],$$

then we claim that $\mathcal{A}$ is generated by the $H_{m,n}$’s as an algebra (when making this claim, there is nothing special about the particular exponents $d_i$ in the definition of $H_{m,n}$; any sequence of $n$ exponents that add up to $m$ would do). Indeed, using (45) one can express any $R_d$ as a scalar multiple of $H_{|d|,n}$ (where $|d| = d_1 + \cdots + d_n$) plus products of smaller $R_{d'}$. Moreover, it was shown in [30, Theorem 2.5] that for all $(m,n),(m',n') \in \mathbb{Z} \times \mathbb{N}$ with $\frac{m}{n} \leq \frac{m'}{n'}$, we have:

$$[H_{m,n}, H_{m',n'}] \in \sum_{t \geq 1, \frac{m}{n} \leq \frac{m_1}{n_1} \leq \cdots \leq \frac{m_t}{n_t} \leq \frac{m'}{n'}} \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \cdot H_{m_1,n_1} \cdots H_{m_t,n_t}$$ \tag{46}

such that all summands that appear with nonzero coefficient in (46) have the property that at least some $\frac{m_i}{n_i}$ is contained strictly between $\frac{m}{n}$ and $\frac{m'}{n'}$. Using relation (46), one obtains the PBW theorem [30, Formula (2.12)]:

$$\mathcal{A} = \bigoplus_{t \geq 1, \frac{m}{n} \leq \frac{m_1}{n_1} \leq \cdots \leq \frac{m_t}{n_t}} \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \cdot H_{m_1,n_1} \cdots H_{m_t,n_t}$$ \tag{47}

(as $H_{m,n}$ commutes with $H_{m',n'}$ if $\frac{m}{n} = \frac{m'}{n'}$, one identifies in (47) those products of $H$’s that are obtained from each other by permuting $H_{m_i,n_i}$’s with the same slope; aside from this identification, all products of $H$’s in (47) are linearly independent). Our interest in the subalgebra $\mathcal{A}$ is motivated by the following.
Conjecture 2.6. There is an isomorphism $\mathcal{A}|_{(q_1,q_2) \to (q^{-2},q^2)} \cong \bigoplus_{n=0}^{\infty} G(\text{Tr}(\text{ASBim}_n))$.

2.6 | Understanding relations

Unless we had a way to connect $G(\text{Tr}(\text{ASBim}_n))$ to the shuffle algebra directly, it seems like one’s best bet to prove Conjecture 2.6 is to understand $\mathcal{A}$ by generators and relations. The most direct way is to think of the $H_{m,n}$’s as generators and (46) as relations, but the latter are not explicit and so difficult to check in the trace of the category ASBim$_n$. The notion below is the “next best thing”.

Definition 2.7 [31]. Consider the algebra:

$$\widetilde{\mathcal{A}} = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \left\langle E_{(d_1, \ldots, d_n)} \right\rangle_{n \in \mathbb{N}, d_1, \ldots, d_n \in \mathbb{Z}} \bigg/ \text{relations (49) \, - \, (50)} ,$$

where for all $d_1, \ldots, d_n \in \mathbb{Z}$ and $i \in \{1, \ldots, n-1\}$ we set:

$$E_{(d_1, \ldots, d_i, d_{i+1}, \ldots, d_n)} - q_1 q_2 \cdot E_{(d_1, \ldots, d_{i-1}, d_{i+1}+1, \ldots, d_n)} = (1 - q_1) E_{(d_1, \ldots, d_i)} E_{(d_{i+1}, \ldots, d_n)}$$ (49)

and for all $d_1, \ldots, d_n, k \in \mathbb{Z}$ we set:

$$\left[ E_{(k)}, E_{(d_1, \ldots, d_n)} \right] = (q_2 - 1) \sum_{i=1}^{n} \left\{ \begin{array}{ll} \sum_{a=1}^{k-d_i} E_{(d_1, \ldots, d_{i-1}, k-a, d_i+a, d_{i+1}, \ldots, d_n)} & \text{if } k \geq d_i \\ - \sum_{a=1}^{d_i-k} E_{(d_1, \ldots, d_{i-1}, d_i-a, k+a, d_{i+1}, \ldots, d_n)} & \text{if } k \leq d_i. \end{array} \right. \quad (50)$$

Proposition 2.8. There is a surjective algebra morphism $\widetilde{\mathcal{A}} \rightarrow \mathcal{A}$ induced by $E_d \mapsto R_d$.

Proof. The fact that (49) and (50) hold with $E$’s replaced by $R$’s was proved in [31] (the former of these is precisely (45), which is a simple exercise). \Box

Remark 2.9. The fact that relation (50) holds with $E$’s replaced by $R$’s was used in [8] as a step in their proof of the Extended Delta Conjecture.

With this in mind, we are ready to state the main result of the present paper.

Theorem 2.10. There is a surjective homomorphism:

$$\left. \widetilde{\mathcal{A}} \right|_{(q_1,q_2) \to (q^{-2},q^2)} \rightarrow \bigoplus_{n=0}^{\infty} G(\text{Tr}(\text{ASBim}_n)).$$

2.7 | Comparing $\mathcal{A}$ with $\widetilde{\mathcal{A}}$

In the remainder of this section, we will explain why Theorem 2.10 is a good approximation to Conjecture 2.6. After all, at first glance, it seems like the algebra $\widetilde{\mathcal{A}}$ only captures “some” of the relations that hold in the algebra $\mathcal{A}$. However, we will now show that all relations that are not
thus captured are actually contained in the $\mathbb{Q}(q_1, q_2)$-torsion. More precisely, we will prove the following result.

**Proposition 2.11.** If we consider the localized algebra:

$$A_{\text{loc}} = A \otimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2)$$

(and analogously with $\bar{A}$ replaced by $A$), then the map:

$$\bar{A}_{\text{loc}} \rightarrow A_{\text{loc}}$$

induced by Proposition 2.8 is an isomorphism.

The proof of the result above will occupy the remainder of this section. We will need to define one more algebra, which is known in the literature as the positive half of the Ding–Iohara–Miki/quantum toroidal $\mathfrak{gl}_1$ algebra.

**Definition 2.12.** Consider the algebra:

$$\mathcal{U} = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \langle E_m \rangle_{m \in \mathbb{Z}} / \text{relations (52)–(53)},$$

where we consider the formal series $E(z) = \sum_{m \in \mathbb{Z}} E_m z^m$ and let:

$$E(z)E(w)(z - wq_1)(z - wq_2) \left( z - \frac{w}{q_1 q_2} \right) = E(w)E(z)(zq_1 - w)(zq_2 - w) \left( \frac{z}{q_1 q_2} - w \right)$$

(52)

and:

$$[[E_{m+1}, E_{m-1}], E_m] = 0$$

(53)

for all $m \in \mathbb{Z}$.

**Proposition 2.13.** The assignment $\{E_m \mapsto e_{(m)}\}_{m \in \mathbb{Z}}$ yields an algebra homomorphism:

$$\mathcal{U} \rightarrow \bar{A}.$$

**Proof.** One needs to check that relations (52) and (53) hold in the algebra $\bar{A}$, which was done in [31, Proposition 4.8].

\[\square\]

2.8  Comparing localizations

Let $\mathcal{U}_{\text{loc}} = \mathcal{U} \otimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2)$. Passing all our algebras and homomorphisms to localizations, we obtain $\mathbb{Q}(q_1, q_2)$-algebra homomorphisms:

$$\mathcal{U}_{\text{loc}} \xrightarrow{f} \bar{A}_{\text{loc}} \xrightarrow{g} A_{\text{loc}} \subseteq S$$

(54)
Although localization does not usually preserve injections, the latter inclusion in (54) holds after localization because (47) implies that $\mathcal{A}$ is a free $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-module.

**Proposition 2.14.** The map $f$ is surjective.

**Proof.** We must show that for all $d_1, \ldots, d_n \in \mathbb{Z}$, the element:

$$\mathcal{E}_{(d_1, \ldots, d_n)} \in \tilde{\mathcal{A}}_{\text{loc}}$$

lies in the subalgebra $\tilde{\mathcal{A}}_{\text{loc}}^o \subseteq \tilde{\mathcal{A}}_{\text{loc}}$ generated by the elements $\{\mathcal{E}_{(m)}\}_{m \in \mathbb{Z}}$. We will prove this statement by induction of $n$, the base case being trivial. Because of (49) and the induction hypothesis, we have:

$$\mathcal{E}_{(d_1, \ldots, d_i, d_{i+1}, \ldots, d_n)} - q_1 q_2 \cdot \mathcal{E}_{(d_1, \ldots, d_{i-1}, d_{i+1}+1, \ldots, d_n)} \in \tilde{\mathcal{A}}_{\text{loc}}^o. \quad (55)$$

Iterating this relation yields:

$$\mathcal{E}_{(d_1, \ldots, d_n)} - (q_1 q_2)^i \sum_{i=1}^n (n-i)d_i \cdot \mathcal{E}_{(0, \ldots, 0, d_{i+1}+1, \ldots, d_n)} \in \tilde{\mathcal{A}}_{\text{loc}}^o. \quad (56)$$

Thus, it suffices to show that (we write $0^k$ for the $k$-tuple $(0, \ldots, 0)$):

$$\mathcal{E}_{(0^{n-1}, m)} \in \tilde{\mathcal{A}}_{\text{loc}}^o$$

for all $m \in \mathbb{Z}$. Assuming first that $m < 0$, relation (50) gives us:

$$[\mathcal{E}_{(0^{n-2}, m)}, \mathcal{E}_{(0)}] = (1 - q_2) \sum_{a = m}^{-1} \mathcal{E}_{(0^{n-2}, a, m-a)}, \quad (57)$$

which by (56) equals:

$$(1 - q_2)((q_1 q_2)^{-1} + \cdots + (q_1 q_2)^m)\mathcal{E}_{(0^{n-1}, m)} + \text{something in } \tilde{\mathcal{A}}_{\text{loc}}^o.$$

As the left-hand side of (57) lies in $\tilde{\mathcal{A}}_{\text{loc}}^o$, by the induction hypothesis, so does the right-hand side and we are done. The case $m > 0$ is treated analogously, so we leave it as an exercise to the interested reader. Finally, when $m = 0$, we use instead of (57) the relation:

$$[\mathcal{E}_{(-1,0^{n-2})}, \mathcal{E}_{(1)}] = (1 - q_2) \left( \mathcal{E}_{(0^n)} + \sum_{i=0}^{n-2} \mathcal{E}_{(-1,0^i,1,0^{n-2-i})} \right). \quad (58)$$

By (56), the right-hand side of the expression above equals:

$$(1 - q_2)((1 + (q_1 q_2)^{-1} + \cdots + (q_1 q_2)^{-n+1})\mathcal{E}_{(0, \ldots, 0)} + \text{elements in } \tilde{\mathcal{A}}_{\text{loc}}^o.$$
As the left-hand side of (58) lies in $\widetilde{A}_{\text{loc}}^\circ$ by the induction hypothesis, so does the right-hand side and we are done. 

\subsection{Extended algebras}

For any $k \in \mathbb{N}$, it is easy to show that there exist derivations:

\[ \partial_k : \mathbb{A}_{\text{loc}} \to \mathbb{A}_{\text{loc}}, \quad \partial_k\left( E_{(d_1, \ldots, d_n)} \right) = \sum_{i=1}^{n} E_{(\ldots, d_{i-1}, d_i + k, d_{i+1}, \ldots)} \]

\[ \partial_k : \mathbb{A}_{\text{loc}} \to \mathbb{A}_{\text{loc}}, \quad \partial_k\left( R_{(d_1, \ldots, d_n)} \right) = \sum_{i=1}^{n} R_{(\ldots, d_{i-1}, d_i + k, d_{i+1}, \ldots)} \]

\[ \partial_k : S \to S, \quad \partial_k(F(z_1, \ldots, z_n)) = F(z_1, \ldots, z_n) (z_1^k + \cdots + z_n^k) \]

(for the first of these, it is a matter of showing that the derivation in question preserves the ideal generated by relations (49)–(50), while for the latter two, it is a matter of showing that the derivations in question preserve the vector subspace of wheel conditions (31); both checks are elementary, and we leave them to the interested reader). Moreover, $\partial_k$ and $\partial_l$ commute for all $k, l \in \mathbb{N}$, which allows us to define algebra structures on:

\[ X_{\text{ext}} = X \otimes_{\mathbb{Q}(q_1, q_2)} \mathbb{Q}(q_1, q_2)[H_1, H_2, \ldots] \]

for all $X \in \{ \mathbb{A}_{\text{loc}}, \mathbb{A}_{\text{loc}}, S \}$, by imposing the commutation relations:

\[ [x, H_k] = \partial_k(x) \quad \forall x \in X. \]

Similarly, let us define the algebra:

\[ U_{\text{loc}}^{\text{ext}} = \mathbb{Q}(q_1, q_2)\left\langle E_m, H_k \right\rangle_{m \in \mathbb{Z}} \big/ \text{relations (52)–(53)} \quad \text{and} \quad [E_m, H_k] = E_{m+k}. \]

Because of the obvious compatibility between the various relations above involving the $H_k$’s, it is clear that the chain of homomorphisms (54) extends to:

\[ U_{\text{loc}}^{\text{ext}} \to \mathbb{A}_{\text{loc}}^{\text{ext}} \xrightarrow{h} \mathbb{A}_{\text{loc}}^{\text{ext}} \subseteq S_{\text{ext}}. \]

The composition of the maps above is an isomorphism, by combining [17, Theorem 1.3], [42, Theorem 4], and [28, Theorem 3.3]. The fact that the first two maps are surjective (due to Propositions 2.8 and 2.14) imply that the map $h$ is an isomorphism.

\textbf{Proof of Proposition 2.11.} As the algebras $\mathbb{A}_{\text{loc}}^{\text{ext}}$ and $\mathbb{A}_{\text{loc}}^{\text{ext}}$ are free right-modules over $\mathbb{Q}(q_1, q_2)[H_1, H_2, \ldots]$, the fact that $h$ is an isomorphism is preserved upon factoring by the ideal $(H_1, H_2, \ldots)$. This yields precisely the fact that the map $g$ is an isomorphism. \qed
3 | THE AFFINE HECKE CATEGORY AND ITS TRACE

3.1 | The affine braid group

Let us recall that the extended affine braid group $\text{ABr}_n$ is generated by symbols $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ and $\omega$ modulo relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| > 1), \quad \omega \sigma_i \omega^{-1} = \sigma_{i+1}$$

(indices are understood modulo $n$). Note that $\sigma_1, \ldots, \sigma_{n-1}$ and $\omega$ already generate the affine braid group. The subgroup generated by $\sigma_0, \ldots, \sigma_{n-1}$ is the braid group corresponding to the affine Coxeter group $\hat{A}_{n-1}$. We will use the following notation:

$$\sigma_{[a,b]} = \sigma_a \ldots \sigma_{b-1} \text{ for } a \leq b.$$  \hfill (59)

The affine braid group contains the lattice generated by the pairwise commuting elements:

$$y_i = \sigma_1^{-1} \ldots \sigma_{i-1}^{-1} \omega \sigma_{n-1} \ldots \sigma_i, \quad i = 1, \ldots, n.$$  

Note that $y_{i+1} = \sigma_i^{-1} y_i \sigma_i^{-1}$ and $\sigma_i, y_j$ generate $\text{ABr}_n$. For any $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, we call the monomial:

$$y^d = y_1^{d_1} \ldots y_n^{d_n}$$  \hfill (60)

a translation, and further use the term dominant translation if $d_1 \geq d_2 \geq \cdots \geq d_n$.

We will also use the product $\star : \text{ABr}_n \times \text{ABr}_k \to \text{ABr}_{n+k}$ that sends the generators $\sigma_i, y_i$ of $\text{ABr}_n$ to the namesake generators of $\text{ABr}_{n+k}$ and the generators $\sigma_j, y_j$ of $\text{ABr}_k$ to $\sigma_{j+n}$ and $y_{j+n}$, respectively.

3.2 | The affine symmetric group

Recall the extended affine symmetric group:

$$\tilde{S}_n = \text{ABr}_n / (\sigma_i^2 = 1)_{\forall i \in \mathbb{Z} / n \mathbb{Z}}.$$  

To distinguish $\tilde{S}_n$ from the braid group, we will denote its generators by $s_i$, and the image of $\omega$ by $\pi$. The group $\tilde{S}_n$ can be identified with the group of $n$-periodic permutations of $\mathbb{Z}$, that is, bijections $\nu : \mathbb{Z} \to \mathbb{Z}$ such that $\nu(i + n) = \nu(i) + n$. In this presentation $s_i$ swaps $i$ and $i + 1$ (and is extend periodically to all $i \in \mathbb{Z}$) while $\pi(i) = i + 1$ for all $i$. Also:

$$y_i(m) = \begin{cases} m + n & \text{if } m \equiv i \text{ mod } n \\ m & \text{otherwise}, \end{cases}$$  \hfill (61)

so $y_1^{d_1} \ldots y_n^{d_n}(m) = m + nd_r$, where $r = m \text{ mod } n$.  

Definition 3.1. There is a grading (on either $\text{ABr}_n$ or $\widetilde{S}_n$) defined multiplicatively via:

$$
\text{deg}(\pi) = 1 \quad \text{and} \quad \text{deg}(s_i) = 0
$$

for all $i \in \{1, \ldots, n\}$.

Note that $\text{deg}(y_i) = 1$ for all $i$, which implies that:

$$
\text{deg}(y^d) = |d| \equiv d_1 + \cdots + d_n
$$

for any $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$. In the presentation of elements of $\widetilde{S}_n$ as $n$-periodic permutations, the degree of $\nu : \mathbb{Z} \to \mathbb{Z}$ can be computed as

$$
\text{deg}(\nu) = \frac{1}{n} \sum_{i=1}^{n} (\nu(i) - i).
$$

The subgroup of elements of degree 0 is the usual affine symmetric group:

$$
\widetilde{S}_n \subset \widetilde{S}_n
$$

that is, the subgroup generated by the $s_i$’s. It is easy to see that any element $\nu \in \widetilde{S}_n$ can be uniquely written as $\nu = \pi^k \alpha$ for some $k \in \mathbb{Z}$ and some $\alpha$ of degree 0. The group $\widetilde{S}_n$ is a Coxeter group and has a natural notion of length ($s_0, \ldots, s_{n-1}$ all have length 1). We define

$$
\ell(\nu) = \ell(\pi^k \alpha) = \ell(\alpha), \ \alpha \in \widetilde{S}_n.
$$

In particular, $\ell(\pi^k) = 0$ for all $k$.

Given an element $\omega \in \widetilde{S}_n$ we can define its positive braid lift by choosing an arbitrary reduced expression and replacing each $s_i$ by $\sigma_i$ and $\pi$ by $\omega$. Clearly, for $i > 1$ the element $y_i$ is not a positive braid lift of any affine permutation. However, we have the following:

Lemma 3.2. If $d_1 \geq d_2 \geq \cdots \geq d_n$, then the dominant translation $y_{d_1}^1 \cdots y_{d_n}^n$ is a positive braid lift of an affine permutation.

Proof. Let us prove by induction that for all $k \in \{1, \ldots, n\}$, we have:

$$
y_1 \cdots y_k = (\omega \sigma_{n-1} \cdots \sigma_k)^k.
$$

For $k = 1$, we get $y_1 = \omega \sigma_{n-1} \cdots \sigma_1$ by definition. For the step of induction, we write

$$
(\omega \sigma_{n-1} \cdots \sigma_k)^k y_{k+1} = (\omega \sigma_{n-1} \cdots \sigma_k) \cdots (\omega \sigma_{n-1} \cdots \sigma_k) \sigma_k^{-1} \cdots \sigma_1^{-1} \omega \sigma_{n-1} \cdots \sigma_{k+1}.
$$

For $j < m - 1$, we have

$$
(\omega \sigma_{n-1} \cdots \sigma_m) \sigma_j^{-1} = \omega \sigma_j^{-1} \sigma_{n-1} \cdots \sigma_m = \sigma_j^{-1} (\omega \sigma_{n-1} \cdots \sigma_m),
$$

so

$$
(\omega \sigma_{n-1} \cdots \sigma_k)^k y_{k+1} = (\omega \sigma_{n-1} \cdots \sigma_k) \sigma_k^{-1} \cdots (\omega \sigma_{n-1} \cdots \sigma_k) \sigma_k^{-1} \cdot \omega \sigma_{n-1} \cdots \sigma_{k+1} =
$$

$$(\omega \sigma_{n-1} \cdots \sigma_{k+1})^{k+1}.$$
Now

\[ y_1^{d_1} \cdots y_n^{d_n} = (y_1 \cdots y_n)^{d_n} \prod_{k=1}^{n-1} (y_1 \cdots y_k)^{d_k-d_{k+1}}. \]

\[ \square \]

### 3.3 The affine Hecke algebra

The (extended) affine Hecke algebra $\mathsf{AH}_n$ is yet another quotient of the affine braid group (strictly speaking, of its group algebra). It has generators $T_i$ (for all $i \in \mathbb{Z} / n \mathbb{Z}$, which correspond to $\sigma_i$) and $\Omega$ (which corresponds to $\omega$) that satisfy the additional quadratic relation:

\[ (T_i - q)(T_i + q^{-1}) = 0. \]

This relation corresponds to the skein relation, and $\mathsf{AH}_n$ can be identified with the skein of the annulus $\mathbb{A} \times [0, 1]$ with $n$ marked points. Given $v \in \widetilde{S}_n$, we denote by $T_v$ the projection of its positive braid lift to $\mathsf{AH}_n$. It is well-known that $T_v, v \in \widetilde{S}_n$ span $\mathsf{AH}_n$. Furthermore, define the cocenter (or the trace) of $\mathsf{AH}_n$ as:

\[ \text{Tr}(\mathsf{AH}_n) = \mathsf{AH}_n / [\mathsf{AH}_n, \mathsf{AH}_n]. \]

Topologically, this is isomorphic to the degree $n$ part of the skein of the torus (where the degree of a curve is its longitudinal winding number). The following result gives an explicit basis in $\text{Tr}(\mathsf{AH}_n)$:

**Theorem 3.3** [22].

(a) Suppose that $v_1, v_2$ are conjugate in $\widetilde{S}_n$ and have minimal length in their conjugacy class. Then $[T_{v_1}] = [T_{v_2}]$ in $\text{Tr}(\mathsf{AH}_n)$.

(b) The cocenter $\text{Tr}(\mathsf{AH}_n)$ has a basis $[T_{v_i}]$ where $v_i$ are minimal length representatives of all conjugacy classes in $\widetilde{S}_n$ (by the part (a), the choice of a minimal length representative does not matter).

Thus, to get a concrete description of $\text{Tr}(\mathsf{AH}_n)$ we need to describe conjugacy classes in $\widetilde{S}_n$, which will be done in next lemmas. We will use the notation (60) in the affine symmetric group as well as the affine braid group, and further write $e_i \in \mathbb{Z}^n$ for the vector with a single 1 at position $i$, and zeros everywhere else.

**Lemma 3.4.** Two affine permutations of the form $y^{d_1} s_1 \cdots s_{n-1} \in \widetilde{S}_n$ are conjugate if and only if they have the same degree $|d|$.

**Proof.** Clearly, conjugate permutations have the same degree. Conversely, we can write:

\[
\begin{align*}
y^{d_1} s_1 \cdots s_{n-1} &\in \widetilde{S}_n \\
&\sim y^{d-e_i} s_1 \cdots s_{n-1} y_i \\
&= y^{d-e_i} s_1 \cdots s_{i-1} (s_i y_i) \cdots s_{n-1} = y^{d-e_i} s_1 \cdots s_{i-1} (y_{i+1} s_i) \cdots s_{n-1} = y^{d-e_i+e_{i+1}} s_1 \cdots s_{n-1}.
\end{align*}
\]

Therefore, we can change the vector of exponents $d$ to $d - e_i + e_{i+1}$ without changing the conjugacy class. By using these operations, we can relate any two vectors with the same degree by conjugations. \[ \square \]
Corollary 3.5. Suppose that $|d| = m$ and $\gcd(m, n) = 1$. Then $y^d s_1 \ldots s_{n-1}$ is conjugate to $\pi^m$, which is the minimal length representative in its conjugacy class.

Proof. Let $mi = q_in + r_i$, where $1 \leq i \leq n$ and $0 \leq r_i \leq n - 1$. Then $\pi^m(x) = x + m$, so for $1 \leq i < n$ we get

$$\pi^m(r_i) = \pi^m(mi - q_in) = m(i + 1) - q_in = (q_{i+1} - q_i)n + r_{i+1}.$$ Define an affine permutation $\phi$ such that $\phi(i) = r_i$ for $1 \leq i \leq n$ (it is well-defined because $\gcd(m, n) = 1$), then for $1 \leq i < n$:

$$\phi^{-1}\pi^m\phi(i) = \phi^{-1}\pi^m(r_i) = \phi^{-1}((q_{i+1} - q_i)n + r_{i+1}) = (q_{i+1} - q_i)n + (i + 1),$$

For $i = n$ we get $r_n = 0$, so $\phi^{-1}\pi^m\phi(n) = \phi^{-1}\pi^m(0) = \phi^{-1}(m) = q_1 n + 1$, and

$$\phi^{-1}\pi^m\phi = y_{q_1}^{q_1} y_{q_2}^{q_2} \ldots y_{q_n}^{q_n} \cdot s_1 \ldots s_{n-1}.$$ By Lemma 3.4, we conclude that $\pi^m$ is conjugate to $y^d s_1 \ldots s_{n-1}$ with

$$|d| = q_1 + (q_2 - q_1) + \ldots + (q_n - q_{n-1}) = m. \qed$$

Lemma 3.6. Two affine permutations $y^d w$ and $y^{d'} w'$ ($w, w' \in S_n$) are conjugate in $\tilde{S}_n$ if and only if the following two conditions hold:

(a) $w$ and $w'$ have the same cycle type, and

(b) the cycles in $w$ and $w'$ can be matched such that for each cycle $(i_1 \ldots i_k)$ in $w$ that corresponds to a cycle $(j_1 \ldots j_k)$ in $w'$, we have $d_{i_1} + \ldots + d_{i_k} = d'_{j_1} + \ldots + d'_{j_k}$.

Proof. The conjugation by a finite permutation reorders the components in $d$ and conjugates $w$, so the condition (a) is clearly necessary. On the other hand, similarly to Lemma 3.4 we can write:

\begin{align*}
y_i^{-1}(y^d w) y_i &= y_i^{-1} y^d y_{w(i)} w = y^{d - e_i + e_{w(i)}},
\end{align*}
so conjugation by $y_i$ fixes the sum $d_{i_1} + \ldots + d_{i_k}$ for any cycle $(i_1 \ldots i_k)$ of $w$, and any two affine permutations with thus matching sums are conjugate to each other. \qed

For a sequence of vectors $d_1, \ldots, d_r$, we define an affine permutation

$$e_{d_1} \star \cdots \star e_{d_r} = e_{(d_1, \ldots, d_{n_1})} \star e_{(d_{n_1+1}, \ldots, d_{n_1+n_2})} \star \cdots \star e_{(d_{n-n_r+1}, \ldots, d_n)}$$

$$= y_{d_1}^{n_1} \ldots y_{d_r}^{n_r} \cdot \sigma_{[1,n_1]} \cdot \sigma_{[n_1 + 1, n_1 + n_2]} \ldots \sigma_{[n-n_r+1,n]}.$$ Lemma 3.6 implies that any affine permutation is conjugate to some $e_{d_1} \star \cdots \star e_{d_r}$.

Definition 3.7 [22]. Let $\nu \in \tilde{S}_n$ be an affine permutation. If $\nu^k = y^d$ for some $d = (d_1, \ldots, d_n)$, we define the Newton point of $\nu$ by the equation $\nu(\nu) := \frac{1}{k} d$. 
It is easy to see that this definition does not depend on the choice of $k$. Also, conjugating $\nu$ corresponds to permuting the components of $\nu(v)$.

**Lemma 3.8.** If $v = e_{d^1} \ast \cdots \ast e_{d^r}$ where $d^i$ has length $n_i$, then:

$$
\nu(v) = \begin{pmatrix}
\frac{|d^1|}{n_1}, \ldots, \frac{|d^1|}{n_1} \\
\frac{|d^r|}{n_r}, \ldots, \frac{|d^r|}{n_r}
\end{pmatrix}.
$$

**Proof.** First observe that if $u$ and $v$ commute and $u^k = y^d, v^{k'} = y^{d'}$ then:

$$(uv)^{kk'} = y^{k'd+kd'} \implies \nu(uv) = \frac{k'd + kd'}{kk'} = \nu(u) + \nu(v),$$

so it is sufficient to prove the statement for a single $e_d$. In this case,

$$(e_d)^n = (y_d^1 \ast \cdots \ast y_d^n)^n = y_d^{|d|} \ast \cdots \ast y_d^{|d|} \implies \nu(e_d) = \left(\frac{|d|}{n}, \ldots, \frac{|d|}{n}\right).$$

**Corollary 3.9.** The conjugacy classes in $\widetilde{S}_n$ are in bijection with convex paths on the plane, that is, collections of points $(m_1, n_1), \ldots, (m_r, n_r) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\frac{m_1}{n_1} \leq \cdots \leq \frac{m_r}{n_r}$$

(considered up to reordering equal fractions). These convex paths label both the PBW basis (47) in the elliptic Hall algebra and the basis in $\text{Tr}(\text{AH}_n)$ by Theorem 3.3.

**Proof.** By conjugating $v$, we can assume that $y^d$ that appears in Definition 3.7 is dominant. For the particular $v$’s that appear in Lemma 3.8, this means that up to conjugating $v$, the sequence $\frac{|d^i|}{n_i}$ can be made nondecreasing. This corresponds to a convex path on the plane with steps $(n_i, |d^i|)$, and by Lemma 3.6 such path uniquely determines a conjugacy class in $\widetilde{S}_n$. □

### 3.4 The affine Hecke category

Following Elias and Mackaay-Thiel [12, 25], we consider the type $A$ extended affine Hecke category $ASBim_n$, which is defined as follows. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and $\overline{R} = \mathbb{C}[x_1, \ldots, x_n, \delta]$, and note that the symmetric group $S_n$ acts on both $R$ and $\overline{R}$ by permuting $x_i$ and fixing $\delta$. The rings $R$ and $\overline{R}$ are graded such that the variables $x_i$ and $\delta$ have grading 2. We have an additional endomorphism of $\overline{R}$ given by:

$$\pi(\delta) = \delta, \pi(x_n) = x_1 - \delta, \pi(x_i) = x_{i+1}, 1 \leq i \leq n - 1.$$
It is easy to see that \( \pi \) and \( S_n \) define an action of \( \widetilde{S_n} \) on \( \widetilde{R} \). We will consider \( R - R \) (respectively, \( \widetilde{R} - \widetilde{R} \)) bimodules, the simplest being \( 1 = R \) (respectively, \( \widetilde{R} \)) with the usual left and right multiplication. We will also encounter the Bott–Samelson bimodules:

\[
\widetilde{B}_i = R \otimes_{R^h} R, \quad B_i = \widetilde{R} \otimes_{\widetilde{R}^h} \widetilde{R}.
\]

There is an additional \((\widetilde{R}, \widetilde{R})\)-bimodule \( \Omega \) that is isomorphic to \( \widetilde{R} \), where the left action of \( \widetilde{R} \) is standard and the right action is twisted by \( \pi \). One can check that \( \Omega \otimes B_i \otimes \Omega^{-1} \simeq B_{i+1} \).

**Definition 3.10.** The category of finite Soergel bimodules \( \text{SBim}_n \) is defined as the smallest full subcategory of \( R - R \) bimodules containing \( R \) and \( B_i \) and closed under tensor products, direct sums and direct summands. Similarly, the category of extended affine Soergel bimodules \( \text{ASBim}_n \) is defined as a smallest full subcategory of \( \widetilde{R} - \widetilde{R} \) bimodules containing \( R, B_i \) and \( \Omega \) and closed under tensor products, direct sums and direct summands.

Note that the subcategory of \( \text{ASBim}_n \) generated by \( R \) and \( B_i \) is equivalent to \( \text{SBim}_n \) with scalars extended from \( R \) to \( \widetilde{R} \). As in [41], one can define Rouquier complexes:

\[
T_i = q^{-1}[B_i \overset{b_i}{\longrightarrow} 1], \quad T_i^{-1} = q^{-1}[q^2 1 \overset{b_i^*}{\longrightarrow} B_i],
\]

which satisfy the braid relations up to homotopy (\( q \) records a grading shift). Here the maps between \( B_i \) and \( 1 \) are given by

\[
b_i(1) = 1, \quad b_i^*(1) = x_i \otimes 1 - 1 \otimes x_{i+1}.
\]

It is easy to see that \( \Omega T_i \Omega^{-1} = T_{i+1} \), so the assignment \( \sigma_i \mapsto T_i, \omega \mapsto \Omega \) induces a homomorphism from the extended affine braid group to the homotopy category \( \mathcal{K}(\text{ASBim}_n) \).

Given an affine permutation \( \nu \), we can consider its positive braid lift and the corresponding Rouquier complex \( T_{\nu} \). Clearly, \( T_{\nu k} = \Omega^k T_{\nu} \). Elias proved in [12] that:

\[
\text{Hom}(\Omega^k T_{\alpha}, \Omega^k T_{\beta}) = \begin{cases} 
\text{Hom}(T_{\alpha}, T_{\beta}) & \text{if } k = k' \\
0 & \text{otherwise.}
\end{cases}
\]

As by Lemma 3.2 dominant translations are positive lifts of certain affine permutations, we may assign to them Rouquier complexes. One can extend these to all translations by presenting them as ratios of dominant ones in an arbitrary way.

**Definition 3.11.** For all \( i \in \mathbb{Z}/n\mathbb{Z} \), let \( Y_i \in \text{ASBim}_n \) be the Rouquier complex corresponding to the affine braid \( y_i \). For any \( d = (d_1, \ldots, d_n) \in \mathbb{Z}^n \), consider:

\[
Y^d = Y_1^{d_1} \cdots Y_n^{d_n} \in \text{ASBim}_n,
\]

which are called **Wakimoto objects** in [12].

An easy consequence of the definitions is that \( T_i^{-1} Y_i T_i^{-1} \simeq Y_{i+1} \). The automorphisms of Rouquier complexes are particularly easy.
Lemma 3.12.

(a) For any affine braid \( v \in ABr_n \) we have:

\[
\text{Hom}(T_v, T_v) \simeq \text{Hom}(1, 1) = \tilde{R}.
\]

(b) On any \( T_v \), the left action of any \( f \in \tilde{R} \) is homotopic to the right action of \( v^{-1}(f) \in \tilde{R} \).

Proof.

(a) As Rouquier complexes are invertible in \( \mathcal{K}(ASBim_n) \), we have \( \text{Hom}(T_v, T_v) \simeq \text{Hom}(1, 1) \), and the latter is clearly \( \tilde{R} \).

(b) It suffices to consider the cases \( T_v = T_i \) (where the left action of \( x_i \) is homotopic to the right action of \( x_{i+1} \), the homotopy is given by \( b_i^* \)) and \( T_v = \Omega \) (where the left and right actions are equal up to a twist by \( \pi \) by definition). \( \square \)

Corollary 3.13. The left action of \( x_i \) on \( Y_{d_1}^1 \cdots Y_{d_n}^n T_w, w \in S_n \) is homotopic to the right action of \( x_{w^{-1}(i)} - d_i \delta \).

Unless stated otherwise, we will use left action of \( \tilde{R} \) to parameterize the automorphisms of Rouquier complexes, but implicitly we will use Lemma 3.12 to translate it into the right action if necessary.

### 3.5 Generators of ASBim

We record a useful categorification of the quadratic relation in the Hecke algebra. There is a chain map from \( T_i \) to \( T_i^{-1} \), and its cone can be written as:

\[
[T_i \to T_i^{-1}] \simeq [q1 \xrightarrow{x_i-x_{i+1}} q^{-1}1].
\] (62)

Equivalently, there is a chain map from \( T_i^2 \) to \( 1 \) such that:

\[
[T_i^2 \to 1] \simeq [qT_i \xrightarrow{x_i-x_{i+1}} q^{-1}T_i].
\] (63)

We say that a (possibly infinite) collection of objects \( X_1, X_2, \ldots \) generates a pre-triangulated dg category \( \mathcal{C} \) if any object \( \mathcal{C} \) can be presented as a finite twisted complex (or iterated cone) built out of finite direct sums of the \( X_i \)'s and their grading shifts.

Theorem 3.14 [12, Theorem 3.20]. The Rouquier complexes \( T_v, v \in \tilde{S}_n \) generate \( \mathcal{K}(ASBim_n) \) as a pre-triangulated dg category. Combining this with (62), we conclude that:

\[
G(ASBim_n) = G(\mathcal{K}(ASBim_n)) = AH_n
\]

with the classes of \( T_v \) in the Grothendieck group corresponding to the namesake generators of \( AH_n \).
In more details, [12, Theorem 3.20] proves (building on results of [13]) the isomorphism $G(\text{ASBim}_n) = \text{AH}_n$ by matching the classes of indecomposable objects in $\text{ASBim}_n$ with the Kazhdan–Lusztig basis in $\text{AH}_n$, both labeled by the elements of $\tilde{S}_n$. For any reduced expression for $v \in \tilde{S}_n$ (a word in $s_i$ and $\pi$) the corresponding tensor product of $B_i$ and $\Omega$ contains a unique copy of indecomposable object $B_v$ and several copies of $B_w$ for $w < v$. Similarly, the Rouquier complex $T_v$ contains a unique copy of $B_v$ and several copies of $B_w$ for $w < v$. Therefore, any indecomposable $B_v$ can be expressed as an iterated cone of $T_v$ and the change of basis between $[T_v]$ and $[B_v]$ is uni-upper triangular.

**Proposition 3.15.** The category $\mathcal{K}(\text{ASBim}_n)$ is generated by the Rouquier complexes $Y^d T_w$, for all $d \in \mathbb{Z}^n$ and $w \in S_n$.

**Proof.** By Theorem 3.14, the category $\mathcal{K}(\text{ASBim}_n)$ is generated by the products of $\Omega$ and $T_i^{\pm 1}$, $1 \leq i \leq n-1$. We need to show that any product of this form can be resolved by $Y^d T_w$. As we can write $\Omega = Y_1 T_1^{-1} \ldots T_{n-1}^{-1}$, then any product of $\Omega$ and $T_i^{\pm 1}$ can be rewritten as a product of $Y_i$ and $T_i^{\pm 1}$. Now we use the relations:

$$T_{i-1} Y_i = Y_{i-1} T_{i-1}, \quad T_i^{-1} Y_i = Y_{i+1} T_i, \quad T_j Y_i = Y_i T_j, \quad j \neq i, i+1$$

and (62) to move all the $T_i^{\pm 1}$’s to the right (for example, we may resolve $T_i Y_i$ by $T_i^{-1} Y_i = Y_{i+1} T_i$ and $Y_i$). As a result, we can resolve any object of $\mathcal{K}(\text{ASBim}_n)$ by Rouquier complexes of the form $Y^d T_\beta$, where $T_\beta$ is the Rouquier complex for a finite braid $\beta \in \text{Br}_n$, so in particular $T_\beta \in \mathcal{K}(\text{SBim}_n)$. It remains to use the well-known fact that $\mathcal{K}(\text{SBim}_n)$ is generated by $T_w$, $w \in S_n$. □

### 3.6 The horizontal trace

Let $\mathcal{C}$ be a monoidal category.

**Definition 3.16** [1, 2]. The **horizontal trace** $\text{Tr} = \text{Tr}(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are the same as those of $\mathcal{C}$, and whose morphisms are defined as follows:

$$\text{Hom}_{\text{Tr}}(X, Y) = \bigoplus_{Z \in \text{Ob} \mathcal{C}} \text{Hom}(ZX, YZ) / \sim,$$

(64)

where the equivalence relation identifies the compositions:

$$ZX \xrightarrow{f} WX \xrightarrow{g} YZ \quad \text{and} \quad WX \xrightarrow{g} YZ \xrightarrow{f} YW$$

(65)

for arbitrary maps $f : Z \to W$ and $g : WX \to YZ$.

Although the trace has the same objects as the original category, its morphism sets are quite different: on the one hand there are more morphisms because of the direct sum in (64), on the other hand there are fewer morphisms because of the equivalence relation (65).

We will refer to (65) as the **trace relation**. Note that it does not define a dg category structure on $\text{Tr}(\mathcal{C})$, even if $\mathcal{C}$ was a dg category to begin with. In fact, (65) can be understood as the truncation
of the more complicated derived trace of \( \mathcal{C} \) from [18], we refer the reader to [18] for a precise definition and all details. In this paper, we work implicitly with the derived trace but only need some formal properties of it explained below, so treat it as a black box.

Furthermore, even if \( \mathcal{C} \) is (pre)-triangulated, \( \text{Tr}(\mathcal{C}) \) might not be closed under cones of morphisms. Below we will always take the pre-triangulated hull of \( \text{Tr}(\mathcal{C}) \) and denote it the same way, see [18] for more details. With this definition we get \( \text{Tr}(\mathcal{C}) = \text{Tr}(\mathcal{K}(\mathcal{C})) \) [18, Lemma 6.25].

There is a trace functor \( \text{Tr} : \mathcal{C} \to \text{Tr}(\mathcal{C}) \) that sends an object \( X \) to the namesake object in \( \text{Tr}(\mathcal{C}) \). If \( \mathcal{C} \) has duals then \( \text{Tr}(XY) \simeq \text{Tr}(YX) \) for all \( X, Y \in \mathcal{C} \). Therefore, we have a commutative diagram:

\[
\begin{array}{ccc}
G(\mathcal{C}) & \rightarrow & G(\text{Tr}(\mathcal{C})) \\
\downarrow & & \downarrow \\
\text{Tr}(G(\mathcal{C})) & \rightarrow & \\
\end{array}
\]  

(66)

where the bottom-most \( \text{Tr} \) is defined as in (23). We henceforth set \( \mathcal{C} = \mathcal{K}(\text{ASBim}_n) \) and abbreviate \( \text{Tr}(\mathcal{C}) = \text{Tr}(\text{ASBim}_n) \). In this case, the diagram (66) becomes:

\[
\begin{array}{ccc}
\text{AH}_n & \rightarrow & G(\text{Tr}(\text{ASBim}_n)) \\
\downarrow & & \downarrow \\
\text{Tr}(\text{AH}_n) & \rightarrow & \\
\end{array}
\]

First, we record two easy consequences of the definitions.

**Proposition 3.17.** We have \( \text{Hom}_{\text{Tr}}(X, Y) = 0 \) unless \( X \) and \( Y \) have the same degree (the grading on objects of \( \text{ASBim}_n \) is inherited from the grading of Definition 3.1).

**Proof.** The space \( \text{Hom}_{\text{Tr}}(X, Y) \) is generated by \( \text{Hom}(T_wX, YT_w) \), which vanishes unless:

\[
\deg(T_wX) = \deg(T_w) + \deg(X) = \deg(T_w) + \deg(Y) = \deg(YT_w),
\]

so \( \deg(X) = \deg(Y) \). \( \square \)

**Lemma 3.18.** For all \( X \) and \( Y \) in \( \text{ASBim}_n \) we have

\[
\text{Hom}_{\text{Tr}}(X, Y) = \text{Hom}_{\text{Tr}}(\Omega^nX, \Omega^nY).
\]

**Proof.** The object \( \Omega^n \) is in the Drinfeld center of the original category \( \text{ASBim}_n \) [12], that is, \( X\Omega^n \simeq \Omega^nX \) for all \( X \) and the isomorphism is functorial in \( X \). Tensoring by an object in the Drinfeld center of \( \mathcal{C} \) defines an endofunctor of \( \text{Tr}(\mathcal{C}) \) [18, section 6.6], and in this case it is invertible with the inverse \( \Omega^{-n} \), so the result follows.

In other words, as \( \Omega^n \) is central and invertible, it commutes with the trace relation (65) and hence preserves the morphism spaces in the trace. \( \square \)

We also record the action of \( \tilde{R} \) on the traces of Rouquier complexes.
Lemma 3.19. For any affine braid \( v \in ABr_n \), we have a natural action of the quotient:

\[ R_v := \tilde{R} / (\delta, x_1 - v(x_1), \ldots, x_n - v(x_n)) \]

on the trace \( \text{Tr}(T_v) \). Here, as above, \( v \) acts on \( \tilde{R} \) via its projection to \( \tilde{S}_n \). In particular, the action of \( \delta \) on \( \text{Tr}(T_v) \) is trivial for all \( v \).

Proof. By Lemma 3.12, the left action of \( \tilde{R} \) on \( T_v \) is homotopic to the right action twisted by \( v \). The trace relation (65) identifies the left action and right action of \( \tilde{R} \), and hence factors through \( \tilde{R} / (x_i - v(x_i)) \). It remains to prove that the action of \( \delta \) vanishes.

Assume first that the Newton point \( \nu(v) \) is nonzero. Suppose that \( v \) projects to \( y^d w \in \tilde{S}_n \). By Corollary 3.13, we have \( x_i = x_{w^{-1}(i)} - d_i \delta \), in particular, if \( (y^d w)^m = y^{m\nu(v)} \) then \( x_i = x_i - m\nu(v) \delta \). As \( \nu(v) \neq 0 \), this equation forces \( \delta = 0 \).

Finally, assume that \( \nu(v) = 0 \). Then by Lemma 3.18, we have:

\[ \text{Hom}_{\text{Tr}}(T_v, T_v) = \text{Hom}_{\text{Tr}}(\Omega^n T_v, \Omega^n T_v) \]

and \( \nu(\omega^n v) = \nu(v) + (1, \ldots, 1) \neq 0 \). As \( \delta \) acts by zero on \( \text{Tr}(\Omega^n T_v) \) (by the previous paragraph), we conclude that it also acts by zero on \( \text{Tr}(T_v) \). □

4 | TRACE OF THE AFFINE HECKE CATEGORY: GENERATORS

In the remainder of the present paper, we consider \( \text{Tr}(ASBim_n) \), the horizontal trace of the affine Hecke category. Theorem 3.14 and Proposition 3.15 yield generators of the category \( \mathcal{K}(ASBim_n) \), so their traces generate \( \text{Tr}(ASBim_n) \). In this section, we choose a subset of these generators and prove that they still generate the trace category. In the next section, we will describe certain explicit exact sequences relating these generators.

4.1 | Generation by \( E_d \)

Recall the objects \( E_d \in \text{Tr}(ASBim_n) \) of (7).

Lemma 4.1. Let \( w \) be a permutation on the first, respectively, last, \( k \) strands (that is, in the subgroup of \( S_n \) generated by \( s_1, \ldots, s_{k-1} \), respectively, \( s_{n-k}, \ldots, s_{n-1} \)). Then for any \( d \in \mathbb{Z}^n \), the object \( Y^d T_w \in \mathcal{K}(ASBim_n) \) can be written as a twisted complex of \( T_{u} Y^d' s \), where \( u \) goes over permutations on the first, respectively, last, \( k \) strands and \( d' \) goes over \( \mathbb{Z}^n \).

Proof. It suffices to prove the Lemma for the product \( Y_j T_i \) for all \( i < k \), respectively, \( i > n - k \), and \( j \in \{1, \ldots, n\} \). As \( T_i^{-1} Y_i T_i^{-1} \simeq Y_{i+1} \), there are three possible cases.

1. If \( j \neq i, i + 1 \), then \( Y_j T_i \simeq T_i Y_{i+1} \).
2. If \( j = i \), then \( Y_i T_i \simeq T_i Y_{i+1} T_i^2 \) which is an extension between two copies of \( T_i Y_{i+1} T_i \simeq Y_i \) and \( T_i Y_{i+1} \) (see (63)).
3. If \( j = i + 1 \), then \( Y_{i+1} T_i \simeq T_i^{-1} Y_i \) which is an extension between \( T_i Y_i \) and two copies of \( Y_i \) (see (62)). □
Lemma 4.2. \( \text{Tr}(\text{ASBim}_n) \) is generated by the objects \( \text{Tr}(Y^dT_w) \) where \( w \) goes over subwords of the Coxeter element \( s_1 \ldots s_{n-1} \) and \( d \) goes over \( \mathbb{Z}^n \).

Proof. By Proposition 3.15, the horizontal trace is generated by the objects \( \text{Tr}(Y^dT_w) \) as \( w \) goes over \( S_n \). It is elementary to see that we can choose a reduced expression for \( w \) that contains at most one copy of \( s_1 \). If it contains no copy of \( s_1 \), we proceed by induction, otherwise we write \( w = \alpha s_1 \beta \), where \( \alpha \) and \( \beta \) only use the last \( n-1 \) strands. Therefore, \( Y^dT_w = Y^dT\alpha T_1 T_\beta \), and by Lemma 4.1 we can resolve this object by \( T_u Y^dT_1 T_\beta \) where \( u \) only uses the last \( n-1 \) strands. The trace of \( Y^dT_w \) can thus be resolved by \( \text{Tr}(T_u Y^dT_1 T_\beta) \cong \text{Tr}(Y^dT_1 T_\beta T_u) \) that itself can be resolved by \( Y^d''T_1 T_{w'} \) where \( w' \) only uses the last \( n-1 \) strands and \( d'' \) runs over \( \mathbb{Z}^n \).

The paragraph above establishes the base case of the following inductive statement: the horizontal trace is generated by the objects \( \text{Tr}(Y^dT_cT_w) \), where \( c \) runs over subwords of \( s_1 \ldots s_k \) and \( w \) only uses the last \( n-k-1 \) strands. To prove the induction step, either \( w \) only uses the last \( n-k-2 \) strands (in which case the induction step is trivial), or we may write \( T_w = T_\alpha T_{k+1} T_\beta \) where \( \alpha \) and \( \beta \) only use the last \( n-k-2 \) strands. Then \( \alpha \) commutes with all subwords of \( s_1 \ldots s_k \), hence we can write:

\[
\text{Tr}(Y^dT_cT_w) = \text{Tr}(Y^dT_cT_\alpha T_{k+1} T_\beta) = \text{Tr}(Y^dT_\alpha T_c T_{k+1} T_\beta),
\]

which by Lemma 4.1 can be resolved by:

\[
\text{Tr}(T_u Y^dT_c T_{k+1} T_\beta) \cong \text{Tr}(Y^d T_c T_{k+1} T_\beta T_u),
\]

where \( u \) uses the last \( n-k-2 \) strands. As \( cs_{k+1} \) is a subword of \( s_1 \ldots s_{k+1} \), this establishes the induction step. Thus, the proof is complete. \[\square\]

We will now review the objects defined in (7) and (8).

Definition 4.3. For any \( d_1, \ldots, d_n \in \mathbb{Z} \), we define the object:

\[
E_{(d_1, \ldots, d_n)} := \text{Tr}(Y^{d_1}_1 \ldots Y^{d_n}_n T_1 \ldots T_{n-1}) \in \text{Tr}(\text{ASBim}_n).
\]

(67)

More generally, given a composition \( n_1 + \cdots + n_r = n \), we define the object:

\[
E_{(d_1, \ldots, d_{n_1})} \star E_{(d_{n_1+1}, \ldots, d_{n_1+n_2})} \star \cdots \star E_{(d_{n-n_r+1}, \ldots, d_n)} = \text{Tr}(Y^{d_1}_1 \ldots Y^{d_{n_1}}_{n_1} T_{[1,n_1]} \cdots Y^{d_{n_1+n_2}}_{n_1+n_2} T_{[n_1+1,n_1+n_2]} \cdots Y^{d_{n-n_r+1}}_{n-n_r+1} \cdots Y^{d_n}_{n} T_{[n-n_r+1,n]}),
\]

(68)

where \( T_{[a,b]} = T_a \ldots T_{b-1} \), by analogy with (59).

We will call \( |d| = d_1 + \cdots + d_n \) the degree of the objects (67) and (68). It is easy to see that it agrees with the degree of an affine braid of Definition 3.1.

By Lemma 4.2, the trace is generated by the objects (68). By Lemma 3.19, the action of \( \tilde{R} \) on \( E_{(d_1, \ldots, d_{n_1})} \star E_{(d_{n_1+1}, \ldots, d_{n_1+n_2})} \star \cdots \star E_{(d_{n-n_r+1}, \ldots, d_n)} \) factors through

\[
R_{n_1, \ldots, n_r} = \tilde{R}/(\delta = 0, x_1 = \cdots = x_{n_1}, x_{n_1+1} = \cdots = x_{n_1+n_2}, \ldots, x_{n-n_r+1} = \cdots = x_n).
\]
We will henceforth work only with the quotient above, and in particular, identify the left and right action of $x_i$ for all $i \in \{1, \ldots, n\}$.

Remark 4.4. Note that this quotient of $\widetilde{R}$ precisely matches the support condition for the sheaves (15) on the commuting stack. Indeed, the sheaf $\mathcal{E}_q$ arises from (37), with all the eigenvalues of the $X$ matrix being equal. Similarly, $\mathcal{E}_{q_1} \ast \ldots \ast \mathcal{E}_{q_r}$ arises from the analogue of (37) where the eigenvalues of the $X$ matrix are equal in blocks of sizes $n_1, n_2, \ldots, n_r$.

4.2 | Generation by convex paths

In this section, we prove a categorical version of Theorem 3.3. For this, we will need some notations.

Definition 4.5. We will call strong conjugation the transitive closure of the following relation on $\widetilde{S}_n$: $v \approx v'$ if $\ell(v) = \ell(v')$, $v' = xvx^{-1}$ and either $\ell(xv) = \ell(x) + \ell(v)$ or $\ell(vx^{-1}) = \ell(x) + \ell(v)$.

Lemma 4.6. Suppose that $v$ and $v'$ are strongly conjugate ($v \approx v'$) and $T_v, T_{v'}$ are the Rouquier complexes for the positive braid lifts of $v, v'$. Then $\text{Tr}(T_v) \simeq \text{Tr}(T_{v'})$ and the positive braid lifts of $v$ and $v'$ are conjugate in $\text{ABr}_n$.

Proof. Suppose that $\ell(v) = \ell(v')$, $v' = xvx^{-1}$ and $\ell(xv) = \ell(x) + \ell(v) = \ell(v'x)$. Then:

$$T_xT_v = T_{xv} = T_{v'x} = T_{v'T_x} \Rightarrow T_{v'} = T_xT_vT_x^{-1}.$$ 

Similarly, if $\ell(vx^{-1}) = \ell(x) + \ell(v)$ then:

$$T_{v'T_{x^{-1}}} = T_{v_{x^{-1}}} = T_{x^{-1}v'x} = T_{x^{-1}T_vT_{x^{-1}}} \Rightarrow T_{v'} = T_{x^{-1}T_vT_{x^{-1}}}.$$ 

See also [22, Lemma 5.1].

Theorem 4.7.

(a) The category $\text{Tr}(\text{ASBim}_n)$ is generated by $\text{Tr}(T_v)$ where $v$ runs over minimal length elements in conjugacy classes in $\widetilde{S}_n$, and $T_v$ is the Rouquier complex for its positive braid lift.

(b) If $v, v'$ are minimal length elements in the same conjugacy class then $\text{Tr}(T_v) \simeq \text{Tr}(T_{v'})$.

Proof.

(a) Recall that by Theorem 3.14, $\text{Tr}(\text{ASBim}_n)$ is generated by $\text{Tr}(T_v)$ for all $v \in \widetilde{S}_n$. Assume that $v$ is not of minimal length in its conjugacy class. Then by [22, Theorem 2.10], there exists a simple reflection $s_i$ such that $\ell(s_ivs_i) = \ell(v) - 2$. Let $v' = s_ivs_i$, then $T_v = T_iT_{v'}T_i$ and:

$$\text{Tr}(T_v) = \text{Tr}(T_iT_{v'}T_i) \simeq \text{Tr}(T_{i}^2T_{v'}),$$

which can be resolved by $\text{Tr}(T_{v'})$ and two copies of $T_iT_{v'} = T_{s_i'v'}$. As $\ell(v')$, $\ell(s_i'v') < \ell(v)$ we can proceed by induction on the length of $v$. 


(b) Suppose that \( v, v' \) are two minimal length elements in the same conjugacy class. Then by [22, Theorem 2.10] they are strongly conjugate, and by Lemma 4.6 we get \( \text{Tr}(T_v) \simeq \text{Tr}(T_{v'}) \). □

For all \((m, n) \in \mathbb{Z} \times \mathbb{N}\), define:

\[
P_{m,n} := E(d_1(m,n), \ldots, d_n(m,n)), \quad \text{where} \quad d_i(m,n) = \left\lfloor \frac{mi}{n} \right\rfloor - \left\lfloor \frac{m(i-1)}{n} \right\rfloor.
\]

**Theorem 4.8.** Let \( v \) be a minimal length element in its conjugacy class. Then there exists a sequence of pairs \((m_1, n_1), \ldots, (m_r, n_r)\) such that:

\[
\text{Tr}(T_v) = P_{m_1, n_1} \ast \cdots \ast P_{m_r, n_r} \quad \text{and} \quad \frac{m_1}{n_1} \leq \frac{m_2}{n_2} \leq \cdots \leq \frac{m_r}{n_r}.
\] (69)

Such a sequence is unique up to permutation of those \( P_{(m_i, n_i)} \) of the same slope \( \frac{m_i}{n_i} \).

**Remark 4.9.** Such sequences \((m_i, n_i)\) correspond to convex paths on the lattice \( \mathbb{Z}^2 \) that label the basis in \( \text{Tr}(AH_n) \), see Corollary 3.9.

**Proof.** The uniqueness follows from Corollary 3.9, as such \((m_i, n_i)\) are uniquely determined by the conjugacy class of \( v \) in \( \widetilde{S}_n \).

To prove existence, we use the topological interpretation of closed annular braids as in the Introduction. Let \( \beta(v) \in \text{ABr}_n \) be the positive braid lift of \( v \), then \( \ell(v) \) equals the number of crossings in \( \beta(v) \). The closure of \( \beta(v) \) is a link \( L_v \) in the thickened torus with several components (representing some homology classes \((m_i, n_i) \in H^2(\mathbb{T}, \mathbb{Z}) \simeq \mathbb{Z}^2\)), and all crossings between different components are positive. This implies \( \frac{m_i}{n_i} \leq \frac{m_j}{n_j} \) for \( i < j \).

As \( v \) has minimal length in its conjugacy class, each component of \( L_v \) has minimal possible number \( \gcd(m_i, n_i) - 1 \) of self-intersections in its homology class \((m_i, n_i)\). This can be seen by gluing the torus from a square, drawing \( \gcd(m_i, n_i) \) parallel straight lines of slope \( \frac{m_i}{n_i} \) on it and connecting them by a Coxeter braid with \( \gcd(m_i, n_i) - 1 \) crossings. Such a curve intersects the horizontal side of the square in \( n_i \) points, and hence is a closure of an annular braid on \( n_i \) strands which is conjugate to \( P_{m_i, n_i} \) in \( \text{ABr}_n \). See also [26, section 6]. □

**Remark 4.10.** As a warning to the reader, it is not true that \( P_{m,n} \) represent positive braid lifts of permutations, they are only conjugate to them. For example, suppose that \( \gcd(m, n) = 1 \), then \( \Omega^m \) is the unique positive braid lift of a minimal length element in its conjugacy class. It has length 0 and corresponds to the \((m, n)\) torus knot. Thus, while:

\[
Y_1^{d_1(m,n)} \cdots Y_n^{d_n(m,n)} T_1 \cdots T_{n-1}
\]

is not equal to \( \Omega^m \), we are claiming that they are conjugate in the affine braid group similarly to Corollary 3.5.

For \( m = 1 \) we get \( d_i(m, n) = 0 \) for \( i < n \) and \( d_n(m, n) = 1 \), so we need to consider the element

\[
Y_n T_1 \cdots T_{n-1} = T_{n-1}^{-1} \cdots T_1^{-1} \Omega T_1 \cdots T_{n-1},
\]

which is conjugate to \( \Omega \).
By combining Theorems 4.7 and 4.8, we get the following.

**Corollary 4.11.** The category $\text{Tr}(\text{ASBim}_n)$ is generated by the objects (69).

## 5 TRACE OF THE AFFINE HECKE CATEGORY: RELATIONS

### 5.1 Exact sequences

We begin by collecting several useful exact sequences in the affine Hecke category and its trace.

**Lemma 5.1.** There are chain maps $\phi : Y_i T_i \to T_i Y_{i+1}$ and $\psi : T_i Y_i \to Y_{i+1} T_i$, whose cones have the following form:

\[
\begin{align*}
[Y_i T_i \xrightarrow{\phi} T_i Y_{i+1}] & \simeq \{ q Y_i \xrightarrow{x_{i+1} - x_i - \delta} q^{-1} Y_i \}, \\
[T_i Y_i \xrightarrow{\psi} Y_{i+1} T_i] & \simeq \{ q Y_i \xrightarrow{x_i - x_{i+1}} q^{-1} Y_i \}.
\end{align*}
\]

Furthermore, $\text{Hom}(Y_i T_i, T_i Y_{i+1})$ and $\text{Hom}(Y_i T_i, T_i Y_{i+1})$ are free rank 1 $\tilde{R}$-modules spanned by $\phi$ and $\psi$, respectively.

**Proof.** We have $Y_i = T_i Y_{i+1} T_i$, so $Y_i T_i = T_i Y_{i+1} T_i^2$. Therefore,

\[
\text{Hom}(Y_i T_i, T_i Y_{i+1}) \simeq \text{Hom}(T_i Y_{i+1} T_i, T_i Y_{i+1}) \simeq \text{Hom}(T_i^2, 1) \simeq \tilde{R}.
\]

The second isomorphism follows from the fact that $T_i Y_{i+1}$ is invertible. By (63), there is a canonical map $T_i^2 \to 1$ (which spans $\text{Hom}(T_i^2, 1) \simeq \tilde{R}$) with cone $\{ q T_i \xrightarrow{x_{i+1} - x_i - \delta} q^{-1} T_i \}$. By tensoring it with $T_i Y_{i+1}$ on the left, we get the desired statement. Note that the left action of $(x_{i+1} - x_i - \delta)$ on $T_i$ is homotopic to the left action of $(x_{i+1} - x_i - \delta)$ on $T_i Y_{i+1} \cdot T_i = Y_i$, due to Corollary 3.13. We thus conclude (70).

Similarly, $T_i Y_i = T_i^2 Y_{i+1} T_i$, so by tensoring the canonical map $T_i^2 \to 1$ with $Y_{i+1} T_i$ on the right yields (71). $\square$

**Remark 5.2.** If $j \neq i, i + 1$, then $Y_j$ commutes with $Y_i, Y_{i+1}$ and $T_i$. Furthermore, Rouquier canonicity [12, Proposition 4.1] implies that there is a commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
\text{Hom}(Y_j T_i^2, Y_j) & \longrightarrow & \text{Hom}(T_i^2, 1) \\
\downarrow & & \text{Hom}(T_i^2 Y_j, Y_j) \\
\end{array}
\]

where the rightward arrows use the fact that $Y_j$ is invertible, and the vertical arrow uses $Y_j T_i \simeq T_i Y_j$.

In short, we can say that $Y_j$ commutes with the skein map $T_i^2 \to 1$, and by a similar argument, $Y_j$ commutes with the maps $\phi$ and $\psi$ from Lemma 5.1.
Theorem 5.3. For any \(1 \leq i < n\) and all integers \(d_1, \ldots, d_n\), there is a chain map:

\[
E(d_1, \ldots, d_i, d_{i+1}, \ldots, d_n) \to E(d_1, \ldots, d_i-1, d_{i+1}+1, \ldots, d_n)
\]

with cone homotopy equivalent to:

\[
[qE(d_1, \ldots, d_i) \star E(d_{i+1}, \ldots, d_n) \xrightarrow{x_i-x_n} q^{-1}E(d_1, \ldots, d_i) \star E(d_{i+1}, \ldots, d_n)].
\]

Proof. Let \(d = (d_1, \ldots, d_n)\). We have:

\[
E_d = \text{Tr}(Y^dT_1 \cdots T_{n-1}) = \text{Tr}(Y_i Y^{d-e_i} T_1 \cdots T_{n-1}) \simeq \text{Tr}(Y^{d-e_i} T_1 \cdots T_{n-1} Y_i) = \text{Tr}(Y^{d-e_i} T_1 \cdots T_{i-1}(T_i Y_i) T_{i+1} \cdots T_{n-1}).
\]

By Lemma 5.1, there is a chain map from \(E_d\) to:

\[
\text{Tr}(Y^{d-e_i} T_1 \cdots T_{i-1}(Y_{i+1} T_i) T_{i+1} \cdots T_{n-1}) = E_{d-e_i+e_{i+1}}.
\]

The cone of this map is isomorphic to two copies of:

\[
\text{Tr}(Y^{d-e_i} T_1 \cdots T_{i-1} 1 T_{i+1} \cdots T_{n-1}) = \text{Tr}(Y^{d-e_i} T_1 \cdots T_{i-1} 1 T_{i+1} \cdots T_{n-1} Y_i) \simeq \text{Tr}(Y_i Y^{d-e_i} T_1 \cdots T_{i-1} 1 T_{i+1} \cdots T_{n-1}) = E(d_1, \ldots, d_i) \star E(d_{i+1}, \ldots, d_n).
\]

By Lemma 5.1, the connecting map between the aforementioned two copies is equal to \(x_i - x_{i+1}\) on the copy of \(1\) in the middle, which is equivalent to the right action of \(x_i - x_n\) (by Corollary 3.13).

Remark 5.4. The analogous statement to Theorem 5.3 holds if the map (72) is \(\star\)-multiplied with other \(E\)'s on both the left and the right. Indeed, \(Y_i\) and \(Y_{i+1}\) commute with all other \(Y_j\) and all \(T_k, k \neq i-1, i, i+1\), so the same proof works.

Remark 5.5. Motivated by [21, Theorem 5.25], we expect that the map (72) vanishes in \(\text{Tr}(\text{ASBim}_n)\). This fact would also match the analogous vanishing that holds in \(\text{Comm}_n\), under the hypothetical functor (2). As a consequence, the objects \(E(d_1, \ldots, d_i, d_{i+1}, \ldots, d_n)\) and \(E(d_1, \ldots, d_i-1, d_{i+1}+1, \ldots, d_n)\) would be shown to be direct summands of:

\[
[qE(d_1, \ldots, d_i) \star E(d_{i+1}, \ldots, d_n) \xrightarrow{x_i-x_n} q^{-1}E(d_1, \ldots, d_i) \star E(d_{i+1}, \ldots, d_n)].
\]

5.2 Exact triangles

We introduce the following notations to keep track of more complicated exact triangles. First, we define \(\alpha_i = e_i - e_{i+1}\). Given a vector \(d = (d_1, \ldots, d_n)\), we define \(d[i, j] = (d_1, \ldots, d_j)\). Finally, for \(a = (a_1, \ldots, a_m)\) and \(b = (b_1, \ldots, b_m)\) we let:

\[
E_a \star E_b = \left[ qE_a \star E_b \xrightarrow{x_n-x_{n+m}} q^{-1}E_a \star E_b \right].
\]
Lemma 5.6. One can extend $\star$ to an associative operation on objects $E_a$.

Proof. Suppose that $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m)$ and $c = (c_1, \ldots, c_k)$. Then $(E_a \star E_b) \star E_c$ is the four term Koszul complex built of $E_a \star E_b \star E_c$ with differentials given by $(x_n - x_{n+m})$ and $(x_{n+m} - x_{n+m+k})$. The other composition $E_a \star (E_b \star E_c)$ is a similar complex with differentials given by $(x_{n+m} - x_{n+m+k})$ and $(x_n - x_{n+m+k})$. As

$$ (x_n - x_{n+m}) + (x_{n+m} - x_{n+m+k}) = (x_n - x_{n+m+k}), $$

the two complexes are isomorphic via a simple change of variables. \qed

With these notations in hand, we can compactly write Theorem 5.3 as the exact triangle:

\[
\begin{array}{ccc}
E_d & \rightarrow & E_{d-\alpha_i} \\
\downarrow & & \downarrow \\
E_{d-\alpha_j} & \rightarrow & E_{d-\alpha_i-\alpha_j}
\end{array}
\]

where $d[i, j] = (d_i, \ldots, d_j)$ for any $d = (d_1, \ldots, d_n)$ and all $i \leq j$.

Lemma 5.7.

(a) The following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
E_d & \rightarrow & E_{d-\alpha_i} \\
\downarrow & & \downarrow \\
E_{d-\alpha_j} & \rightarrow & E_{d-\alpha_i-\alpha_j}
\end{array}
\]

hence there is a well-defined (up to a homotopy) map $E_d \rightarrow E_{d-\alpha_i-\alpha_j}$.

(b) Suppose that $i < j$. The diagram in (a) fits as the central rhombus in the following diagram:

in which all oriented triangles are exact, and all oriented quadrilaterals are commutative.

Proof. Part (a) is a special case of (b). Assuming without loss of generality $i < j$, we can write:

$$ E_d \simeq \text{Tr}(Y^d-e_i-e_jT_1 \cdots (T_iY_i) \cdots (T_jY_j) \cdots T_{n-1}). $$

Clearly, the maps from skein exact sequences from Lemma 5.1 applied at positions $i$ and $j$ commute with each other, so we just need to check that various isomorphisms used in the proof of
Theorem 5.3 commute with them as well. Indeed, we can write all the entries in the diagram in (b) as follows:

1. \( E_{d[1,i]} \otimes E_{d[i+1,n]} \) is built of two copies of \( \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \);
2. \( E_{d[1,j]} \otimes E_{d[j+1,n]} \) is built of two copies of \( \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_{i+1}T_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \);
3. \( E_{d-\alpha_j} = \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_{i+1}T_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \);
4. \( E_{d[1,i]} \otimes E_{d[i+1,j]} \otimes E_{d[j+1,n]} \) is built of four copies (see Lemma 5.6) of:
   \[
   \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_i) \cdots (Y_j) \cdots (T_{n-1}) \cdots (Y_{i+1}T_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \);
5. \( E_{d-\alpha_j} = \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_{i+1}T_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \);
6. \( E_{d[1,j]} \otimes E_{d[j+1,n]} \) is built of two copies of \( \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_{i+1}T_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \);
7. \( E_{d-\alpha_j} = \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_{i+1}T_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \);
8. \( E_{d[1,i]} \otimes E_{d[i+1,n]} \) is built of two copies of \( \text{Tr}(Y^{d-e_i-e_j}T_1 \cdots (Y_i) \cdots (Y_j) \cdots (T_{n-1}) \cdots (Y_{i+1}T_i) \cdots (T_j \cdots (T_{n-1}) \cdots (Y_j) \cdots (T_{n-1})) \).

It remains to notice that \( Y_i \) commutes with \( T_j \) and \( Y_j \), and \( Y_{j+1} \) commutes with \( T_i \) and \( Y_i \). Furthermore, by Remark 5.2, \( Y_i \) commutes with the skein maps involving \( T_j \) and \( Y_j \), and \( Y_{j+1} \) commutes with the skein maps involving \( T_i \) and \( Y_i \).

\[ \square \]

Corollary 5.8. If \( d - d' \) is a nonnegative linear combination of \( \alpha_i \)'s, then there is a well-defined (up to a homotopy) map \( E_d \to E_{d'} \), and its cone is filtered by \( \star \)-products of smaller \( E_a \).

5.3 Twisted lattices and more exact sequences

Recall that the affine symmetric group has \( n! \) different lattices in it that are conjugate to the standard lattice generated by \( y_i \) under the action of \( S_n \). We can mimic this construction in the affine Hecke category.

Lemma 5.9. Define \( Y'_i = T^{-1}_i Y_i T_i \), \( Y'_{i+1} = T^{-1}_i Y_{i+1} T_i \). Then the following statements hold.

a. We have \( Y_i Y_{i+1} \simeq Y'_i Y'_{i+1} \).

b. There is a canonical map \( Y_i \to Y'_{i+1} \) with cone:

\[
[ Y_i \to Y'_{i+1} ] \simeq \left[ q Y_{i+1} T_i \xrightarrow{x_i-x_{i+1}} q^{-1} Y_{i+1} T_i \right].
\]

c. There is a canonical map \( Y'_i \to Y_{i+1} \) with cone:

\[
[ Y'_i \to Y_{i+1} ] \simeq \left[ q Y_{i+1} T_i \xrightarrow{x_i-x_{i+1}-\delta} q^{-1} Y_{i+1} T_i \right].
\]

d. The objects \( Y'_i, Y'_{i+1} \) commute with \( Y_j \) for \( j \neq i, i+1 \).

Proof.

a. The product \( Y_j \cdots Y_n = \Omega^n \) is central, so it commutes with \( T_i \). On the other hand, \( Y_j \) commutes with \( T_i \) for \( j \neq i, i+1 \), so \( Y_i Y_{i+1} \) commutes with \( T_i \) and thus:

\[
Y'_i Y'_{i+1} = T^{-1}_i Y_i Y'_{i+1} T_i = Y_i Y_{i+1}.
\]
(b) We have \( Y_i = T_i Y_{i+1} T_i \), which by (62) has a canonical map to \( Y'_{i+1} = T_i^{-1} Y_{i+1} T_i \) with cone homotopy equivalent to \([q Y_{i+1} T_i \xrightarrow{x_i - x_{i+1}} q^{-1} Y_{i+1} T_i]\) (note that we multiply 1 by \( Y_{i+1} T_i \) on the right, so the variables do not change).

(c) We have \( Y'_i = Y_{i+1} T_i^2 \), which by (63) has a canonical map to \( Y_{i+1} \) with cone homotopy equivalent to \([q Y_{i+1} T_i \xrightarrow{1} q^{-1} Y_{i+1} T_i]\) (note that we multiply \( T_i \) by \( Y_{i+1} \) on the left, so \( x_{i+1} \) is shifted by \( \delta \)). Part (d) is clear. \( \square \)

Under the functor predicted by Problem 1.1, the morphisms described in Lemma 5.9 should correspond to the morphisms induced by [31, Proposition 2.28].

**Lemma 5.10.**

(a) If \( a \geq b \), then there is a canonical map \( Y_i^a Y_{i+1}^b \rightarrow Y_i^b Y_{i+1}^a \), whose cone is filtered by objects of the form:

\[
[q Y_i^{a-k} Y_{i+1}^{b+k} T_i \xrightarrow{x_i - x_{i+1} + (a-b-k) \delta} q^{-1} Y_i^{a-k} Y_{i+1}^{b+k} T_i], \quad k = 1, \ldots, a - b.
\]

(b) If \( a \leq b \), then there is a canonical map \( Y_{i+1}^b Y_{i+1}^a \rightarrow Y_i^a Y_{i+1}^b \), whose cone is filtered by objects of the form:

\[
[q Y_i^{b-k} Y_{i+1}^{a+k} T_i \xrightarrow{x_i - x_{i+1} - k \delta} q^{-1} Y_i^{b-k} Y_{i+1}^{a+k} T_i], \quad k = 1, \ldots, b - a.
\]

**Proof.**

(a) We will first deal with the case \( a \geq 0 \) and \( b = 0 \). By Lemma 5.9(b), we have a chain of maps:

\[
Y_i^a \rightarrow Y_i^{a-1} Y_{i+1}^r \rightarrow Y_i^{a-2} Y_{i+1}^r \rightarrow \cdots \rightarrow Y_{i+1}^r.
\]

The cone of their composition is filtered by the cones of the individual maps:

\[
[Y_i^{a-k+1} Y_{i+1}^{r-k} \rightarrow Y_i^{a-k} Y_{i+1}^{r-k}] = Y_i^{a-k}[Y_i \rightarrow Y_{i+1}^r] Y_{i+1}^{r-k}
\]

\[
\approx Y_i^{a-k}[q Y_{i+1} T_i \xrightarrow{x_i - x_{i+1}} q^{-1} Y_{i+1} T_i] Y_{i+1}^{r-k}
\]

\[
= [q Y_i^{a-k} Y_{i+1}^r \xrightarrow{x_i - x_{i+1} + (a-k) \delta} q^{-1} Y_i^{a-k} Y_{i+1}^r T_i]
\]

for all \( k \in \{1, \ldots, a\} \). In the last equality above, we used the equalities:

\[
Y_{i+1} T_i Y_{i+1}^{r-k-1} = Y_{i+1} Y_{i+1}^{r-k} T_i = Y_i^{r-k} T_i.
\]

In the case of general \( a \geq b \), the discussion above for the numbers \( a - b \) and 0 implies that there is a map \( Y_i^{a-b} \rightarrow Y_{i+1}^{a-b} \) whose cone is filtered by objects of the form

\[
[q Y_i^{a-b} Y_{i+1}^k T_i \xrightarrow{x_i - x_{i+1} + (a-b-k) \delta} q^{-1} Y_i^{a-b} Y_{i+1}^k T_i], \quad k = 1, \ldots, a - b.
\]

If we tensor the objects above on the left with \((Y_i Y_{i+1})^b = (Y_i^r Y_{i+1}^r)^b\) (the equality holds due to Lemma 5.9(a)), then we conclude that there is a map \( Y_i^a Y_{i+1}^b \rightarrow Y_i^b Y_{i+1}^a \) whose cone is
filtered by maps between the objects:

\[(Y_i Y_{i+1})^b Y_i^{a-b-k} Y_{i+1}^k T_i = Y_i^{a-k} Y_{i+1}^{b+k} T_i.\]

as \(k\) goes from 1 to \(a - b\).

(b) Let us first deal with the case \(a = 0, b \geq 0\). By Lemma 5.9(c), we have a chain of maps:

\[Y_i^b \to Y_{i+1} Y_i^{b-1} \to Y_i^2 Y_{i+1}^{b-2} \to \ldots \to Y_i^b.\]

The cone of their composition is filtered by the cones of the individual maps:

\[\left[ Y_i^{k-1} Y_{i+1}^{b-k+1}, Y_i^k Y_{i+1}^{b-k} \right] = Y_i^{k-1} \left[ Y_i^k \to Y_{i+1} \right] Y_{i+1}^{b-k}\]

\[\simeq Y_i^{k-1} \left[ qY_{i+1} T_i \xrightarrow{x_i-x_{i+1}-\delta} q^{-1}Y_{i+1} T_i \right] Y_{i+1}^{b-k}\]

\[= \left[ qY_i^{b-k+1} T_i \xrightarrow{x_i-x_{i+1}-k\delta} q^{-1}Y_i^{b-k} Y_{i+1}^{b-k} T_i \right]\]

for all \(k \in \{1, \ldots, b\}\) (we used \(73\) in the last equality above). The remaining argument in part (b) is similar to that of part (a).

\[\square\]

Lemma 5.10 is the fundamental instance of the following more general result.

**Lemma 5.11.** Consider an arbitrary sequence of integers \(d = (d_1, \ldots, d_n) \in \mathbb{Z}^n\).

(a) If \(d_i \geq d_{i+1}\), then there is a chain map:

\[Y_{d_1}^1 \ldots Y_{d_i}^{d_i} Y_{i+1}^{d_{i+1}} \ldots Y_{d_n}^{d_n} \to T_i^{-1} Y_{i+1}^1 \ldots Y_i^{d_i} Y_{i+1}^{d_{i+1}} \ldots Y_{d_n}^{d_n} T_i\]

(74)

whose cone is filtered by the two-term complexes:

\[\left[ qY_i^{d_i} \ldots Y_i^{d_i-k} Y_{i+1}^{d_{i+1}+k} \ldots Y_n^{d_n} T_i \xrightarrow{x_i-x_{i+1}+(d_i-d_{i+1}-k)\delta} q^{-1}Y_i^{d_i} \ldots Y_i^{d_i-k} Y_{i+1}^{d_{i+1}+k} \ldots Y_n^{d_n} T_i \right]\]

(75)

for \(k = 1, \ldots, d_i - d_{i+1}\).

(b) If \(d_i \leq d_{i+1}\), then there is a chain map:

\[T_i^{-1} Y_{i+1}^1 \ldots Y_{i}^{d_i+1} Y_{i+1}^{d_i} \ldots Y_{d_n}^{d_n} T_i \to Y_i^1 \ldots Y_i^{d_i} Y_{i+1}^{d_{i+1}} \ldots Y_n^{d_n}\]

(76)

whose cone is filtered by the two-term complexes:

\[\left[ qY_i^{d_i} \ldots Y_i^{d_i+1-k} Y_{i+1}^{d_i+1} \ldots Y_n^{d_n} T_i \xrightarrow{x_i-x_{i+1}-k\delta} q^{-1}Y_i^{d_i} \ldots Y_i^{d_i+1-k} Y_{i+1}^{d_i+1} \ldots Y_n^{d_n} T_i \right]\]

(77)

for \(k = 1, \ldots, d_{i+1} - d_i\).

**Proof.** Let us prove case (a), and leave case (b) as an analogous exercise to the interested reader. By Lemma 5.10 in case (a) we get a map:

\[Y_{d_1}^1 \ldots Y_{i}^{d_i} Y_{i+1}^{d_{i+1}} \ldots Y_n^{d_n} \to Y_{i+1}^{d_1} \ldots Y_{i}^{d_i+1} Y_{i+1}^{d_{i+1}} \ldots Y_n^{d_n}\]
whose cone is filtered by the two-term complexes:

\[
[ qY_1^{d_1} \ldots Y_i^{d_{i-k}} Y_{i+1}^{d_{i+1-k}+k} T_i \ldots Y_n^{d_n} \xrightarrow{\gamma - \gamma_{i+1}^{(d_i - d_{i+1} - k)k}} q^{-1} Y_1^{d_1} \ldots Y_i^{d_{i-k}} Y_{i+1}^{d_{i+1-k}+k} T_i \ldots Y_n^{d_n} ]
\]

However, note that:

\[
Y_{i+1}^{a} Y_{i+1}^{b} = T_{i+1}^{-1} Y_{i+1}^{a} Y_{i+1}^{b} T_{i+1}
\]

for all integers \(a, b\). Together with the fact that \(T_j\) commutes with \(Y_j\) for \(j \neq i, i+1\), we conclude that there exists a map \((74)\) whose cone is filtered by the complexes \((75)\).

\[\square\]

### 5.4 Commutators in the trace

We will now use Lemma 5.11 to prove Theorem 1.8.

**Theorem 5.12.** For any \(\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n\) and \(k \in \mathbb{Z}\), there exists a collection of objects \(G_0, \ldots, G_n \in \text{ASBim}_n\) with the following properties.

- \(\text{Tr}(G_0) = E_{(k)} \star E_{\mathbf{d}}\) and \(\text{Tr}(G_n) = E_{\mathbf{d}} \star E_{(k)}\).
- For all \(i \in \{1, \ldots, n\}\) there exist chain maps in \(\mathcal{K}(\text{ASBim}_n)\):
  
  \[
  \begin{cases}
  G_{i-1} \xrightarrow{\varphi_i} G_i & \text{if } k \geq d_i \\
  G_{i-1} \xleftarrow{\varphi_i} G_i & \text{if } k < d_i
  \end{cases}
  \]

  which are mutually inverse isomorphisms if \(k = d_i\).
- \(\text{Tr}(\text{Cone}(\varphi_i))\) and \(\text{Tr}(\text{Cone}(\overline{\varphi_i}))\) are filtered by:

  \[
  \begin{cases}
  E_{(d_1, \ldots, d_{i-1}, k-a, d_i+a, d_{i+1}, \ldots, d_n)}, & 1 \leq a \leq k - d_i \quad \text{if } k > d_i \\
  E_{(d_1, \ldots, d_{i-1}, d_i - a, k + a, d_{i+1}, \ldots, d_n)}, & 1 \leq a \leq d_i - k \quad \text{if } k < d_i
  \end{cases}
  \]

respectively. Above, \(q\) denotes the internal grading on Soergel bimodules.

**Proof.** Define the following sequence of objects in \(\mathcal{K}(\text{ASBim}_n)\):

\[
G_0 = Y_1^{d_1} Y_2^{d_2} \ldots Y_{n+1}^{d_{n+1}} T_2 \ldots T_n,
\]

\[
G_i = T_1^{-1} \ldots T_i^{-1} Y_1^{d_1} \ldots Y_i^{d_i} Y_{i+1}^{d_{i+1}} \ldots Y_{n+1}^{d_{n+1}} T_i \ldots T_2 T_1 \ldots T_n \quad 0 < i < n,
\]

\[
G_n = T_1^{-1} \ldots T_n^{-1} Y_1^{d_1} \ldots Y_n^{d_n} Y_{n+1}^{k} T_{n+1} \ldots T_2 T_1 \ldots T_n.
\]

We have \(\text{Tr}(G_0) = E_{(k)} \star E_{\mathbf{d}}\) by definition, while the identity \(T_1 T_2 \ldots T_n \ldots T_2 T_1 = T_n \ldots T_2 T_1 T_2 \ldots T_n\) in the braid group implies that:

\[
\text{Tr}(G_n) = \text{Tr}(T_1^{-1} \ldots T_n^{-1} Y_1^{d_1} \ldots Y_n^{d_n} Y_{n+1}^{k} T_{n+1} \ldots T_2 T_1 \ldots T_n) = \text{Tr}(T_1^{-1} \ldots T_n^{-1} Y_1^{d_1} \ldots Y_n^{d_n} Y_{n+1}^{k} T_1 T_2 \ldots T_n) =
\]

\[
= \text{Tr}(T_1^{-1} \ldots T_n^{-1} Y_1^{d_1} \ldots Y_n^{d_n} Y_{n+1}^{k} T_1 T_2 \ldots T_n) =
\]


This establishes (a). As for (b), we note that Lemma 5.11 gives us chain maps \( \varphi_i : G_{i-1} \to G_i \) if \( d_i \leq k \), and \( \varphi_i : G_i \to G_{i-1} \) if \( d_i \geq k \), whose cones are filtered by two copies of:

\[
\left\{ T_{i-1}^{-1} \ldots T_1^{-1} Y_{i}^{d_i+1} \ldots Y_{n+1} T_i T_{i-1} \ldots T_1 T_2 \ldots T_n \right\}_{1 \leq a \leq k-d_i} \tag{80}
\]

if \( d_i \leq k \), and:

\[
\left\{ T_{i-1}^{-1} \ldots T_1^{-1} Y_{i}^{d_i} \ldots Y_{n+1} T_i T_{i-1} \ldots T_1 T_2 \ldots T_n \right\}_{1 \leq a \leq d_i-k} \tag{81}
\]

if \( d_i \geq k \). As \( T_1 T_{i-1} \ldots T_1 T_2 \ldots T_n T_{i-1} \ldots T_1 \) holds in the braid group, then:

\[
\text{Tr}\left[ T_{i-1}^{-1} \ldots T_1^{-1} Y_{i}^{a} \ldots Y_{n+1} T_i T_{i-1} \ldots T_1 T_2 \ldots T_n \right] = \text{Tr}\left[ Y_{i}^{a} \ldots Y_{n+1} T_i T_{i-1} \ldots T_1 T_2 \ldots T_n \right] = E_{(s_1, \ldots, s_{n+1})}
\]

for all integers \( s_1, \ldots, s_{n+1} \). Thus, the traces of the objects (80) and (81) are precisely the \( E \)'s that appear in (79). As for the differential maps between the two copies of said \( E \)'s, by Lemma 5.11 they are given by \( x_i - x_{i+1} + m \delta \) for various integers \( m \). As Lemma 3.19 ensures that \( x_i = x_{i+1} \) and \( \delta = 0 \) on \( E_d \), these differential maps vanish.

Remark 5.13. Note that the projections of \( E_{(k)} \star E_d \) and \( E_d \star E_{(k)} \) to \( \widetilde{S}_n \) are, respectively:

\[
y_1^{k} y_2^{d_1} \ldots y_n^{d_n-1} s_2 \ldots s_{n-1} \quad \text{and} \quad y_1^{d_1} \ldots y_n^{d_n-1} y_1^{k} y_2 \ldots y_{n-1} s_2 \ldots s_{n-2},
\]

which are conjugate in the affine symmetric group. The conjugating element is \( s_1 \ldots s_{n-1} \), which explains the structure of the proof of Theorem 5.12: indeed, these elements are no longer conjugate in the affine Hecke category, but at each step we conjugate by one simple reflection and apply Lemma 5.11 to collect correction terms.

Remark 5.14. A very similar proof applies for the commutation relation between \( E_{(k)} \) and a \( \star \) product of \( E_d \). The definition of \( G_i \) now reads:

\[
G_i = T_{i-1}^{-1} \ldots T_1^{-1} Y_{i}^{d_i} \ldots Y_{i+1}^{d_{i+1}} \ldots Y_{n+1} T_i T_{i-1} \ldots T_1 T_u,
\]

where \( u \) is a subword of \( s_2 \ldots s_n \). The chain maps and their cones are computed as in the proof of Theorem 5.12, and the only nontrivial computation is:

\[
T_i \ldots T_1 T_u T_{i-1} \ldots T_1^{-1} = T_i \ldots T_1 T_0 T_u T_{i-1} \ldots T_1^{-1} = T_{\nu} T_i \ldots T_1 T_{i-1} T_{i-1}^{-1} T_{i-1}^{-1} T_w = T_{\nu} T_1 T_{i-1}^{-1} T_w = T_{\nu} T_1 T_u T_{i-1} T_{i-1}^{-1} T_w,
\]

where \( u = \nu \nu \) with \( \nu \) being a product of \( s_j, 2 \leq j \leq i \), \( w \) being a product of \( s_j, j \geq i + 1 \), and \( \nu \) being obtained from \( \nu \) by decreasing all indices by 1. We leave the details as an exercise to the reader, and provide a particular instance of this computation below.

Example 5.15. For example, for \([E_{(d_1,d_2)} \star E_{(d_3,d_4)}, E_{(k)}] \), we have \( T_u = T_2 T_4 \) and thus:

\[
\text{Tr}(G_0) = \text{Tr}(y_1^{k} y_2^{d_1} y_3^{d_2} y_4^{d_3} y_5^{d_4} T_2 T_4) = E_{(k)} \star E_{(d_1,d_2)} \star E_{(d_3,d_4)}
\]

\[
\text{Tr}(G_1) = \text{Tr}(T_1^{-1} y_1^{k} y_2^{d_1} y_3^{d_2} y_4^{d_3} y_5^{d_4} T_1 T_2 T_4)
\]
\[
\text{Tr}(G_2) = \text{Tr}(T_1^{-1}T_2^{-1}Y_{d_1}Y_{d_2}Y_{k}Y_{d_3}Y_{d_4}T_2T_1T_2T_4) = \text{Tr}(Y_{d_1}Y_{d_2}Y_{k}Y_{d_3}Y_{d_4}T_2T_1T_2T_4T_1^{-1}T_2^{-1}) \\
= \text{Tr}(Y_{d_1}Y_{d_2}Y_{k}Y_{d_3}Y_{d_4}T_1T_4) = E_{(d_1,d_2)} \star E_{(k)} \star E_{(d_3,d_4)}
\]

\[
\text{Tr}(G_3) = \text{Tr}(T_1^{-1}T_2^{-1}T_3^{-1}Y_{d_1}Y_{d_2}Y_{k}Y_{d_3}Y_{d_4}T_3T_2T_4T_1^{-1}T_2^{-1}T_3^{-1}) = \text{Tr}(Y_{d_1}Y_{d_2}Y_{k}Y_{d_3}Y_{d_4}T_1T_3T_4T_1^{-1}T_2^{-1}T_3^{-1}) \\
= \text{Tr}(Y_{d_1}Y_{d_2}Y_{k}Y_{d_3}Y_{d_4}T_1T_3) = E_{(d_1,d_2)} \star E_{(d_3,d_4)} \star E_{(k)}.
\]

5.5 Application: Skein product on cocenters of \( \text{AH}_n \)

We can summarize the preceding results as follows. By (66), we have a surjective map:

\[
\text{Tr}(G(\text{ASBim}_n)) = \text{Tr}(\text{AH}_n) \rightarrow G(\text{Tr}(\text{ASBim}_n)).
\]

**Theorem 5.16.** Let \( \overline{A} \) be the algebra from Definition 2.7. There is a surjective algebra homomorphism:

\[
\overline{A}\big|_{(q_1,q_2)\mapsto(q^{-2},q^2)} \rightarrow \bigoplus_{n=0}^{\infty} \text{Tr}(\text{AH}_n),
\]

where the multiplication in the right-hand side is given by the skein product \( \text{AH}_n \times \text{AH}_k \rightarrow \text{AH}_{n+k} \).

**Proof.** The generators \( E_{d_1} \star \cdots \star E_{d_r} \) from Lemma 4.2, being the traces of objects in \( \mathcal{K}(\text{ASBim}_n) \), correspond to some explicit elements of \( \text{Tr}(\text{AH}_n) \). The proof of Lemma 4.2 and earlier Proposition 3.15 use only the skein exact sequences from \( \text{ASBim}_n \) and the relation \( \text{Tr}(XY) \simeq \text{Tr}(YX) \), so they correspond to relations in \( \text{Tr}(\text{AH}_n) \). In particular, the classes \( E_{d_1} \star \cdots \star E_{d_r} \) generate \( \text{Tr}(\text{AH}_n) \) over \( \mathbb{Z}[q^{\pm 1}] \). Similarly, the exact sequences from Theorems 5.3 and 5.12 lead to relations in \( \text{Tr}(\text{AH}_n) \) which match the relations (49) and (50) in the algebra \( \overline{A} \) after specialization \( (q_1,q_2) \mapsto (q^{-2},q^2) \), upon the substitution:

\[
E_{(d_1,\ldots,d_n)} = q^{n-1} \xi_{(d_1,\ldots,d_n)}.
\]

□

**Corollary 5.17.** There is a surjective map \( \overline{A}\big|_{(q_1,q_2)\mapsto(q^{-2},q^2)} \rightarrow \bigoplus_{n=0}^{\infty} G(\text{Tr}(\text{ASBim}_n)) \).

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