SOLVING HIGHER-ORDER INTEGRO DIFFERENTIAL EQUATIONS BY VIM AND MHPM

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Abstract: In this paper, the Variational Iteration Method (VIM) and Modified Homotopy Perturbation Method (MHPM) are applied to solve boundary value problems for higher-order Volterra integro-differential equations. The numerical results obtained with minimum amount of computation are compared with the exact solutions to show the efficiency of the methods. The results show that the variational iteration method is of high accuracy, more convenient and efficient for solving Volterra integro-differential equations. Finally, an example is included to demonstrate the validity and applicability of the proposed techniques.

AMS Subject Classification: 45J05, 65K10, 65H20
Key Words: Volterra integro-differential equation, VIM, MHPM

1. Introduction

The integro-differential equations be an important branch of modern mathematics. It arises frequently in many applied areas which include engineering,
In this work, we consider the Volterra integro-differential equation of the second kind as follows:

\[ \sum_{j=0}^{m} \xi_j(x)y^{(j)}(x) = f(x) + \int_{a}^{x} K(x,t)G(y(t))dt, \]  

(1)

with initial/boundary conditions:

\[ y^{(r)}(a) = b_r, \quad r = 0, 1, 2, \ldots, (k - 1), \]  

(2)

\[ y^{(r)}(b) = c_r, \quad r = k, (k + 1), \ldots, (m - 1), \]  

(3)

where \( y^{(j)}(x) \) is the \( j \)th derivative of the unknown function \( u(x) \) that will be determined, \( K(x, t) \) is the kernel of the equation, \( a < x \leq b \), \( f(x) \) and \( \xi_j(x) \) are analytic functions, \( G(y(t)) \), is nonlinear analytic function of \( y \). The \( b_r \) and \( c_r \) are constants.

The boundary value problems for higher-order integro-differential equations have been investigated by Morchalo [27] and Agarwal [1] among others. Recently, Hamoud (2019) presented an efficient and numerical procedure for solving boundary value problems for higher-order integro-differential equations. A variety of methods, exact, approximate and purely numerical techniques are available to solve nonlinear integro-differential equations. These methods have been of great interest to several authors and used to solve many nonlinear problems. Some of these techniques are Adomian decomposition method [9, 24], modified Adomian decomposition method [17, 18], Variational iteration method [16, 30] and homotopy perturbation method [20] and many methods for solving integro-differential equations [3, 4, 5, 15, 16, 17, 18, 19, 21, 29].

More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations. Some works based on an iterative scheme have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the VIM which is a simple and Adomian decomposition method [7, 24], and the modified decomposition method for solving Volterra-Fredholm integral and integro-differential equations which is a simple and powerful method for solving a wide class of nonlinear problems [24]. The Taylor polynomial solution of integro-differential equations has been studied in [28]. The use of Lagrange interpolation in solving integro-differential equations was investigated by Marzban [26]. The VIM has been successfully applied for solving integral and integro-differential equations [24, 20].
A variety of powerful methods has been presented, such as the homotopy analysis method [21], homotopy perturbation method [5], operational matrix with Block-Pulse functions method [4], VIM [16] and the Adomian decomposition method [9, 24]. Some fundamental works on various aspects of modifications of the Adomian’s decomposition method are given by Araghi [2]. The modified form of Laplace decomposition method has been introduced by Manafianheris [25]. Babolian et. al, [4], applied the new direct method to solve nonlinear Volterra-Fredholm integral and integro-differential equation using operational matrix with block-pulse functions. The Laplace transform method with the Adomian decomposition method to establish exact solutions or approximations of the nonlinear Volterra integro-differential equations [17]. Recently, the authors have used several methods for the numerical or the analytical solutions of linear and nonlinear Volterra and Fredholm integro-differential equations [11, 12, 13, 14, 22, 23, 24].

In this work, our aim is to solve the boundary value problems for higher-order Volterra integro-differential equations by using VIM and MHPM.

2. Description of the Methods

2.1. Variational Iteration Method (VIM)

This method has been applied to solve a large class of linear and nonlinear problems with approximations converging rapidly to exact solutions.

The main idea of this method is to construct a correction functional form using general Lagrange multipliers. These multipliers should be chosen such that its correction solution is superior to its initial approximation, called trial function. It is the best within the flexibility of trial functions. Accordingly, Lagrange multipliers can be identified by the variational theory [30]. A complete review of VIM is available in [16].

The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing boundary/initial conditions. To illustrate, we consider the following general differential equation:

\[ Ly(t) + Ny(t) = f(t), \]  

where \( L \) is a linear operator, \( N \) is a nonlinear operator and \( f(t) \) is inhomogeneous term. According to variational iteration method [24], the terms of
a sequence \( y_n \) are constructed such that this sequence converges to the exact solution. The terms \( y_n \) are calculated by a correction functional as follows:

\[
y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\tau)(Ly_n(\tau) + N\tilde{y}(\tau) - f(\tau))d\tau.
\] (5)

The successive approximation \( y_n(t), n \geq 0 \) of the solution \( y(t) \) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function \( y_0 \).

The zeroth approximation \( y_0 \) may be selected using any function that just satisfies at least the initial and boundary conditions with \( \lambda \) determined, several approximations \( y_n(t), n \geq 0 \) follow immediately. Consequently, the exact solution may be obtained by using

\[
y(t) = \lim_{n \to \infty} y_n(t).
\] (6)

### 2.2. Modified Homotopy Perturbation Method (MHPM)

The homotopy perturbation technique was proposed first by He [5]. In most cases, this method yields very rapid convergence of the series solution, usually few iterations produce very accurate approximate solution [24, 21]. To explain MHPM, we define homotopy \( H(y, P) \) by:

\[
H(y, P) = (1 - P)F(y) + PL(y),
\] (7)

where \( F(y) \) is a functional operator with known solution \( y_0 \) which generally satisfies the boundary conditions. Obviously, from Eq.(7) we have:

\[
H(y, 0) = F(y), \quad H(y, 1) = L(y),
\] (8)

where \( F(y) \) is a functional operator with known solution \( v_0 \), which can be obtained easily. In MHPM, we define

\[
v_0 = a_1 + a_2x + a_3x^2 + \cdots + a_mx^{m-1},
\]

and continuously trace an implicitly defined curve from a starting point \( H(y_0) \) to a solution \( H(\kappa, 1) \). Applying the perturbation technique Eq.(7), due to the fact that \( 0 \leq P \leq 1 \), can be considered as a small parameter. We can assume that the solution of Eq.(1) can be expressed as a series in as follows:

\[
y = y_0 + Py_1 + P^2y_2 + \ldots
\] (9)
As $P \to 1$, in Eq. (9), in most cases it converges to an approximate solution, i.e.,

$$
\kappa = \lim_{P \to 1} y = y_0 + y_1 + y_2 + \ldots
$$

(10)

3. Main Results

In this section, we shall give an existence and uniqueness results of Eq. (1), with the initial conditions (2) and prove it.

We can rewrite the equation (1) in the form of:

$$
y(x) = L^{-1} \left[ \frac{f(x)}{\xi_k(x)} \right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r
$$

$$
+ \gamma L^{-1} \left[ \int_a^x \frac{1}{\xi_k(x)} K(x,t)G(y_n(t))dt \right] - L^{-1} \left[ \sum_{j=0}^{k-1} \xi_j(x) y^{(j)}(x) \right],
$$

so that

$$
L^{-1} \left[ \int_a^x \frac{1}{\xi_k(x)} K(x,t)G(y_n(t))dt \right] = \int_a^x \frac{(x-t)^k}{k!\xi_k(x)} K(x,t)G(y_n(t))dt,
$$

and

$$
\sum_{j=0}^{k-1} L^{-1} \left[ \frac{\xi_j(x)}{\xi_k(x)} \right] y^{(j)}(x) = \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!\xi_k(t)} y^{(j)}(t)dt.
$$

We set

$$
\Psi(x) = L^{-1} \left[ \frac{f(x)}{\xi_k(x)} \right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r.
$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exist two constants $\alpha$ and $\gamma_j > 0, j = 0, 1, \ldots, k$ such that, for any $y_1, y_2 \in C(J, \mathbb{R})$

$$
|G(y_1) - G(y_2)| \leq \alpha |y_1 - y_2|
$$

and

$$
|D^j(y_1) - D^j(y_2)| \leq \gamma_j |y_1 - y_2|.
$$
We suppose that the nonlinear terms $G(y(x))$ and $D^j(y) = \left(\frac{d^j}{dx^j}\right)y(x) = \sum_{i=0}^{\infty} \gamma_{ij}$ (for $D^j$ is a derivative operator), $j = 0, 1, \cdots, k$, are Lipschitz continuous.

**(H2)** We suppose that for all $a < t \leq x \leq b$, and $j = 0, 1, \cdots, k$:

$$\left| \frac{\gamma(x-t)^k K(x,t)}{k! \xi_k(x)} \right| \leq \theta_1, \quad \left| \frac{\gamma(x-t)^k K(x,t)}{k!} \right| \leq \theta_2,$$

$$\left| \frac{(x-t)^{k-1} \xi_j(t)}{(k-1)! \xi_k(t)} \right| \leq \theta_3, \quad \left| \frac{(x-t)^{k-1} \xi_j(t)}{(k-1)!} \right| \leq \theta_4.$$

**(H3)** There exist three functions $\theta_3^*, \theta_4^*$, and $\gamma^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : a < t \leq x \leq b\}$ such that:

$$\theta_3^* = \max |\theta_3|, \quad \theta_4^* = \max |\theta_4|, \quad \gamma^* = \max |\gamma_j|.$$

**(H4)** $\Psi(x)$ is bounded function for all $x$ in $J = (a, b)$.

**Theorem 1.** Assume that (H1)–(H4) hold. If

$$0 < \psi = (\alpha \theta_1 + k \gamma^* \theta_3^*)(b - a) < 1,$$

then there exists a unique solution $y(x)$ to Eqs. (1) – (2).

**Proof.** Let $y_1$ and $y_2$ be two different solutions of Eqs. (1) – (2), then

$$\left| y_1 - y_2 \right| = \left| \int_a^x \frac{\gamma(x-t)^k K(x,t)}{\xi_k(x)k!} [G(y_1) - G(y_2)] dt \right|$$

$$- \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1} \xi_j(t)}{\xi_k(t)(k-1)!} \left[ D^j(y_1) - D^j(y_2) \right] dt$$

$$\leq \int_a^x \left| \frac{\gamma(x-t)^k K(x,t)}{\xi_k(x)k!} \right| \left| G(y_1) - G(y_2) \right| dt$$

$$- \sum_{j=0}^{k-1} \int_a^x \left| \frac{(x-t)^{k-1} \xi_j(t)}{\xi_k(t)(k-1)!} \right| \left| D^j(y_1) - D^j(y_2) \right| dt$$

$$\leq (\alpha \theta_1 + k \gamma^* \theta_3^*)(b - a) |y_1 - y_2|,$$

we get $(1 - \psi)|y_1 - y_2| \leq 0$. Since $0 < \psi < 1$, so $|y_1 - y_2| = 0$. therefore, $y_1 = y_2$ and the proof is completed.
Theorem 2. If problem (1) − (2) has a unique solution, then the solution \( y_n(x) \) obtained from the recursive relation using VIM converges when \( 0 < \phi = (\alpha \theta_2 + k \gamma^* \theta_4^*) (b - a) < 1 \).

Proof. We have recursive relation:

\[
y_{n+1}(x) - y(x) = y_n(x) - y(x) - \left( L^{-1} \left[ \sum_{j=0}^{k} \xi_j(x)[y_j^n(x) - y_j^i(x)] \right] - \left[ \gamma \int_a^x K(x,t)[G(y_n(t)) - G(y(t))] dt \right) \).
\]

If we set, \( \xi_k(x) = 1 \), and \( W_{n+1}(x) = y_{n+1}(x) - y(x) \), \( W_n(x) = y_n(x) - y(x) \) since \( W_n(a) = 0 \), then

\[
W_{n+1}(x) = W_n(x) + \int_a^x \frac{\gamma K(x,t)(x-t)^k}{k!} [G(y_n(t)) - G(y(t))] \, dt
\]

\[
- \sum_{j=0}^{k-1} \int_a^x \frac{\lambda_1 \xi_j(t)(x-t)^{k-1}}{(k-1)!} [D^j(y_n(t)) - D^j(y(t))] \, dt
\]

\[
-(W_n(x) - W_n(a)). \tag{13}
\]

Therefore,

\[
|W_{n+1}(x)| \leq \int_a^x \left| \frac{\gamma K(x,t)(x-t)^k}{k!} \right| |W_n| \, dt
\]

\[
+ \sum_{j=0}^{k-1} \int_a^x \left| \frac{\gamma \xi_j(t)(x-t)^{k-1}}{(k-1)!} \right| \max_{\gamma_j} |\gamma_j| |W_n| \, dt
\]

\[
\leq |W_n| \left[ \int_a^x \alpha \theta_5 dt + \sum_{j=0}^{k-1} \int_a^x \theta_4^* \max |\gamma_j| \right]
\]

\[
\leq |W_n| (\alpha \theta_2 + k \gamma^* \theta_4^*) (b - a) = |W_n| \phi.
\]

Hence,

\[
\|W_{n+1}\| = \max_{\forall x \in J} |W_{n+1}(x)| \leq \phi \max_{\forall x \in J} |W_n(x)| = \phi \|W_n\|.
\]

Since \( 0 < \phi < 1 \), then \( \|W_n\| \to 0 \). So, the series converges and the proof is complete.
4. Numerical Example

In this section, we present the numerical techniques based on VIM and MHPM, to solve Volterra integro-differential equations:

Example 1.

Consider the Volterra integro-differential equation as follow:

\[ y^{(4)}(x) - y(x) = x(1 + e^x) + 3e^x - \int_0^x y(t)dt, \]

with the boundary conditions: \( y(0) = 1 \), \( y(1) = 1 + e \), \( y''(0) = 2 \), \( y''(1) = 3e \).

The exact solution is \( y(x) = 1 + xe^x \).

\[
\begin{array}{cccc}
  x & \text{Exact} & \text{VIM} & \text{MHPM} \\
  0.0 & 1.00000 & 1.00000 & 1.00000 \\
  0.2 & 1.24428 & 1.24411 & 1.21187 \\
  0.4 & 1.59673 & 1.59645 & 1.54312 \\
  0.6 & 2.09327 & 2.09296 & 2.03768 \\
  0.8 & 2.78043 & 2.78022 & 2.74460 \\
  1.0 & 3.71828 & 3.71828 & 3.70279 \\
\end{array}
\]

Table 1: Numerical Results of Example 1.

5. Conclusion

In this work, the VIM and MHPM have been successfully employed to obtain the approximate solutions of a Volterra integro-differential equation. Moreover, we proved the uniqueness results and convergence of the techniques. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results in table 1 and Fig. 1 we can see these methods are similar approximately, VIM is the easiest, the most efficient and convenient.
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Figure 1: Numerical Results of Example 1

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