Braneworld Cosmological Perturbation Theory at Low Energy

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Abstract

Homogeneous cosmology in the braneworld can be studied without solving bulk equations of motion explicitly. The reason is simply because the symmetry of the spacetime restricts possible corrections in the 4-dimensional effective equations of motion. It would be great if we could analyze cosmological perturbations without solving the bulk. For this purpose, we combine the geometrical approach and the low energy gradient expansion method to derive the 4-dimensional effective action. Given our effective action, the standard procedure to obtain the cosmological perturbation theory can be utilized and the temperature anisotropy of the cosmic background radiation can be computed without solving the bulk equations of motion explicitly.

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I. INTRODUCTION

It is believed that the initial singularity problem of the cosmology does not arise in the superstring theory. It is known that a clear prediction of the superstring theory is existence of extra-dimensions. This apparently contradicts our experience. Fortunately, the superstring theory itself provides a mechanism to hide extra-dimensions, which is the so-called braneworld scenario where the standard matter lives on the brane, while only the gravity can propagate in the bulk space-time. A possible realization of the above scenario is a two-brane model proposed by Randall and Sundrum [1]. Here, we concentrate on this particular model as a playground for studying an interplay between the bulk and the brane during the cosmological evolution.

So far, no concrete prediction concerning the evolution of cosmological perturbations has been made. The difficulty in solving the bulk with moving boundary condition causes this situation. Interestingly, as to the background homogeneous universe, this difficulty can be circumvented owing to the spacetime symmetry. In this simplest case, the effect of the bulk only comes into the effective 4-dimensional effective equations as the dark radiation. The point here is that we need not to solve the bulk equations of motion to obtain the information of the bulk. Needless to say, if this point can be extended to the inhomogeneous universe, the great progress can be expected. What we want show in this paper is that, at low energy, this is indeed the case, namely, one can know the effect of the bulk geometry on the cosmological evolution of fluctuations without solving equations in the bulk. For this purpose, we combine the derivative expansion of the action and the geometrical approach to derive the 4-dimensional effective action with KK effects for the two-brane system. Our new method gives not only a simple re-derivation of known results [2], but also a new result, i.e. the effective action with KK corrections [3].

The organization of this paper is as follows: In sec.II, we review a geometrical method and discuss the homogeneous cosmology. In sec.III, we explain the gradient expansion method. In sec.IV, our strategy to obtain the effective action is illustrated. In sec.V, we present the KK corrected effective action and discuss the implications of our results on the cosmological perturbation theory. In the final section, we summarize our results.
II. GEOMETRICAL APPROACH AND HOMOGENEOUS COSMOLOGY

First, let us review the geometrical approach. Without losing the generality, we can take the Gaussian normal coordinate system in the vicinity of the brane. Then, the metric can be written as

\[ ds^2 = dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu . \]

(1)

Defining the extrinsic curvature

\[ K_{\mu\nu} = -\frac{1}{2} \frac{\partial}{\partial y} g_{\mu\nu} , \]

(2)

we can deduce the relation

\[ \left( \begin{array}{c} 5 \\ 4 \end{array} \right) G_{\mu\nu} = G_{\mu\nu} + K^\lambda_{\mu} K_{\lambda\nu} - KK_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( K^{\alpha\beta} K_{\alpha\beta} - K^2 \right) + C_{y\mu y\nu} + \frac{3}{\ell^2} g_{\mu\nu} , \]

(3)

where \( C_{y\mu y\nu} \) is components of the 5-dimensional Weyl tensor and \( \ell \) is the curvature scale set by the negative cosmological constant in the bulk.

Assuming the \( Z_2 \)-symmetry, the junction condition leads to

\[ K^\mu_{\nu} - \delta^\mu_{\nu} K = \frac{\kappa^2}{2} \left( -\sigma \delta^\mu_{\nu} + T^\mu_{\nu} \right) , \]

(4)

where \( \kappa^2, \sigma, \) and \( T_{\mu\nu} \) are the coupling constant, the tension of the brane, and the energy-momentum tensor for the matter on the brane.

It is convenient to take the unit \( \kappa^2/\ell = 8\pi G = 1 \). By eliminating the extrinsic curvature in Eq.(3) using Eq.(4), we obtain the effective equation [4]

\[ G_{\mu\nu} = T_{\mu\nu} + \ell^2 \pi_{\mu\nu} - E_{\mu\nu} , \]

(5)

where

\[ \pi_{\mu\nu} = -\frac{1}{4} T_{\mu\lambda} T^\lambda_{\nu} + \frac{1}{12} TT_{\mu\nu} + \frac{1}{8} g_{\mu\nu} \left( T^{\alpha\beta} T_{\alpha\beta} - \frac{1}{3} T^2 \right) \]

(6)

and

\[ E_{\mu\nu} = C_{y\mu y\nu} \bigg|_{y=0} . \]

(7)

Note that the projection of Weyl tensor \( E_{\mu\nu} \) represents the effect of the bulk geometry. Here, we have also assumed the relation \( \kappa^2 \sigma = 6/\ell \) so that the effective cosmological constant vanish.
The geometrical approach is useful to classify possible corrections to the conventional Einstein equations. One defect of this approach is the fact that the projected Weyl tensor cannot be determined without solving the equations in the bulk. However, the traceless property

\[ E^\mu_\mu = 0 \]  

is sufficient to determine the evolution of homogeneous universe. Indeed, the property (8) is the key to derive the effective Friedman equation

\[ H^2 = \frac{1}{3} \rho + \ell^2 \rho^2 + \frac{C}{a^4} , \]  

where the constant of integration \( C \) is referred to as the dark radiation. The effect of the bulk is encoded in this dark radiation fluid. Although its precise value of \( C \) can not be determined without solving the bulk with the appropriate boundary condition, the dynamics of the homogeneous universe can be qualitatively understood. For general spacetimes, however, this traceless condition is not sufficient to determine the evolution of the braneworld.

### III. GRADIENT EXPANSION OF EFFECTIVE ACTION

In this paper, for simplicity, we concentrate on the vacuum two-brane system. Let us start with the 5-dimensional action for this system

\[ S[\gamma_{AB}, g_{\mu\nu}, h_{\mu\nu}] \]  

where \( \gamma_{AB}, g_{\mu\nu} \) and \( h_{\mu\nu} \) are the 5-dimensional bulk metric, the induced metric on the positive and the negative tension branes, respectively. Here, \( A, B \) and \( \mu, \nu \) label the 5-dimensional and 4-dimensional coordinates, respectively. The variation with respect to \( \gamma_{AB} \) gives the bulk Einstein equations and the variation with respect to \( g_{\mu\nu} \) and \( h_{\mu\nu} \) yields the junction conditions. Now, suppose to solve the bulk equations of motion and the junction condition on the negative tension brane, then formally we get the relation

\[ \gamma_{AB} = \gamma_{AB}[g_{\mu\nu}] , \quad h_{\mu\nu} = h_{\mu\nu}[g_{\mu\nu}] . \]  

By substituting relations (11) into the original action, in principle, the 4-dimensional effective action can be obtained as

\[ S_{\text{eff}} = S[\gamma_{AB}[g_{\mu\nu}], g_{\mu\nu}, h_{\mu\nu}[g_{\mu\nu}]] . \]
It should be stressed that the above effective action is nonlocal. Moreover, the above calculation is not feasible in practice. In reality, at low energy where the most of interesting phenomena occurs, we need not to follow the above general procedure. In the following, we will give a method to obtain the effective action without doing the above calculation.

The point is that the gradient expansion approach can be used at low energy. At low energy, it is legitimate to assume that the action can be expanded by the local terms with increasing orders of derivatives if one includes all of the massless modes [2]. In the two-brane system, the relevant degrees of freedom are nothing but the metric and the radion which represents the distance between two branes [5]. Hence, we assume the general local action constructed from the metric $g_{\mu\nu}$ and the radion $\Psi$ as an ansatz. Therefore, we can write the action as

$$S_{\text{eff}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R \Psi - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right]$$

$$+ \int d^4x \sqrt{-g} \left[ A(\Psi) (\nabla^\mu \Psi \nabla_\mu \Psi)^2 + B(\Psi) (\Box \Psi)^2 + C(\Psi) \nabla^\mu \Psi \nabla_\mu \Psi \Box \Psi \\
+ D(\Psi) \Box \Psi + E(\Psi) R \nabla^\mu \Psi \nabla_\mu \Psi + F(\Psi) R^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi \\
+ G(\Psi) R^2 + H(\Psi) R^{\mu\nu} R_{\mu\nu} + I(\Psi) R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} + \cdots \right] ,$$

(13)

where $\nabla_\mu$ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$ and $\Lambda, \omega, A, \cdots$ are arbitrary coefficient functionals. Here, we have listed up all of the possible local terms which have derivatives up to fourth-order. This can be regarded as the generalization of the scalar-tensor theory including the higher derivative terms. We have the freedom to redefine the scalar field $\Psi$. In fact, we have used this freedom to fix the functional form of the coefficient of the Einstein-Hilbert term. However, we can not determine other coefficient functionals without any information about the bulk geometry.

IV. STRATEGY

In the gradient expansion approach, we have introduced the radion explicitly. While the radion never appears in the geometric approach, instead $E_{\mu\nu}$ is induced as the effective energy-momentum tensor reflecting the effects of the bulk geometry. Notice that the property (8) implies the conformal invariance of this effective matter. Clearly, both approaches should agree to each other. Hence, the radion must play a role of the conformally invariant
matter $E_{\mu \nu}$. This requirement gives a stringent constraint on the action, more precisely, the conformal symmetry (8) determines radion dependent coefficients in the action (13).

Let us illustrate our method using the following action truncated at the second order derivatives:

$$S_{\text{eff}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \Psi R - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right] ,$$

(14)

which is nothing but the scalar-tensor theory with coupling function $\omega(\Psi)$ and the potential function $\Lambda(\Psi)$. Note that this is the most general local action which contains up to the second order derivatives and has the general coordinate invariance. It should be stressed that the scalar-tensor theory is, in general, not related to the braneworld. However, we know a special type of scalar-tensor theory corresponds to the low energy braneworld [2]. Here, we will present a simple derivation of this known fact.

For the vacuum brane, i.e. $T_{\mu \nu} \propto g_{\mu \nu}$, we can put $T_{\mu \nu} + \ell^2 \pi_{\mu \nu} = -\lambda g_{\mu \nu}$. Hence, the geometrical effective equation reduces to

$$G_{\mu \nu} = -E_{\mu \nu} - \lambda g_{\mu \nu} .$$

(15)

First, we must find $E_{\mu \nu}$. The above action (14) gives the equations of motion for the metric as

$$G_{\mu \nu} = -\frac{\Lambda}{\Psi} g_{\mu \nu} + \frac{1}{\Psi} \left( \nabla_\mu \nabla_\nu \Psi - g_{\mu \nu} \Box \Psi \right) + \frac{\omega}{\Psi^2} \left( \nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{2} g_{\mu \nu} \nabla^\alpha \Psi \nabla_\alpha \Psi \right) .$$

(16)

The right hand side of this Eq. (16) should be identified with $-E_{\mu \nu} - \lambda g_{\mu \nu}$. Hence, the condition $E_{\mu \mu} = 0$ becomes

$$\Box \Psi = -\frac{\omega}{3\Psi} \nabla^\mu \Psi \nabla_\mu \Psi - \frac{4}{3} (\Lambda - \lambda \Psi) .$$

(17)

This is the equation for the radion $\Psi$. However, we also have the equation for $\Psi$ from the action (14) as

$$\Box \Psi = \left( \frac{1}{2\Psi} - \frac{\omega'}{2\omega} \right) \nabla^\alpha \Psi \nabla_\alpha \Psi - \frac{\Psi}{2\omega} R + \frac{\Psi}{\omega} \Lambda' ,$$

(18)

where the prime denotes the derivative with respect to $\Psi$. In order for these two Eqs. (17) and (18) to be compatible, $\Lambda$ and $\omega$ must satisfy

$$\frac{\omega}{3\Psi} = \frac{1}{2\Psi} - \frac{\omega'}{2\omega} ,$$

$$\frac{4}{3} (\Lambda - \lambda \Psi) = \frac{\Psi}{\omega} (2\lambda - \Lambda') ,$$

(19)

(20)
where we used $R = 4\lambda$ which comes from the trace part of Eq. (15). Eqs. (19) and (20) can be integrated as

$$
\Lambda(\Psi) = \lambda + \lambda \beta (1 - \Psi)^2, \quad \omega(\Psi) = \frac{3}{2} \frac{\Psi}{1 - \Psi}.
$$

where the constant of integration $\beta$ represents the ratio of the cosmological constant on the negative tension brane to that on the positive tension brane. Here, one of constants of integration is absorbed by rescaling of $\Psi$. In doing so, we have assumed the constant of integration is positive. We can also describe the negative tension brane if we take the negative signature.

Thus, we get the effective action

$$
S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Psi R - \frac{3}{4(1 - \Psi)} \nabla^\mu \Psi \nabla_\mu \Psi - \lambda - \lambda \beta (1 - \Psi)^2 \right].
$$

Surprisingly, this completely agrees with the previous result [2]. Our simple symmetry principle $E_{\mu \mu} = 0$ has determined the action completely.

As we have shown in [6], if $\beta < -1$ there exists a static deSitter two-brane solution which turns out to be unstable. In particular, two inflating branes can collide at $\Psi = 0$.

V. **KK CORRECTIONS AND IMPLICATIONS**

Let us apply the procedure in the previous section to the higher order case. From the linear analysis, the action in the previous section is known to come from zero modes. Hence, one can expect the other coefficients in the action (13) represent the effects of KK-modes.

Now we impose the conformal symmetry $E^\mu_{\nu \mu} = 0$ on the fourth order derivative terms in the action (13) as we did in the previous section. Starting from the action (13), one can read off the equation for the metric from which $E_{\mu \nu}$ can be identified. From the compatibility between the equations of motion for $\Psi$ and the equation $E^\mu_{\nu \mu} = 0$ determines the coefficient functionals in the action (13). The compatibility condition between $E^\mu_{\nu \mu} = 0$ and the
equation for the radion $\Psi$ leads to

\begin{align*}
(1 - \Psi)(C'' - 3A') &= C' + 3E'' + \frac{3}{2}F'' \quad (23) \\
(1 - \Psi)(2B'' - 4A) &= 2B' + C + 3D'' + 3E' + \frac{5}{2}F' \quad (24) \\
(1 - \Psi)(4C' - 8A) &= 2C + 12E' + 5F' \quad (25) \\
(1 - \Psi)(3B' - 2C) &= 2B + 3D' + F \quad (26) \\
4(1 - \Psi)B' &= 2B + 6D' + 6E + 3F \quad (27) \\
(1 - \Psi)C &= 3E + F \quad (28) \\
2(1 - \Psi)B &= 3D \quad (29) \\
(1 - \Psi)(2C - F') &= 6E + 2F + 2H'' + 4I'' \quad (30) \\
(1 - \Psi)F &= -H' - 2I' \quad (31) \\
(1 - \Psi)(D'' - E') &= D' + 6G'' + H'' \quad (32) \\
2(1 - \Psi)(D' - E) &= D + 6G' + H' \quad (33) \\
G' &= H' = I' = 0 . \quad (34)
\end{align*}

These equations seem to be over constrained. Nevertheless, one can find solutions consistently. From Eqs. (31) and (34), we see $F = 0$. Eqs. (28) and (29) can be solved as

\begin{align*}
E &= \frac{1}{3}(1 - \Psi)C , \quad D = \frac{2}{3}(1 - \Psi)B . \quad (35)
\end{align*}

Substituting these results into Eqs. (23) -(27) and Eqs. (32) and (33), we have

\begin{align*}
3(1 - \Psi)A' &= C' , \quad (36) \\
(1 - \Psi)(C' + 4A) &= 2B' , \quad (37) \\
4(1 - \Psi)A &= C , \quad (38) \\
B' - 2C &= 0 , \quad (39) \\
(1 - \Psi)C &= B , \quad (40) \\
2B &= (1 - \Psi)\left[6B' - 2(1 - \Psi)B'' - C + (1 - \Psi)C'\right] , \quad (41) \\
3B &= (1 - \Psi)(2B' - C) . \quad (42)
\end{align*}

Combining Eqs. (39) and (40), we obtain

\begin{align*}
B = \frac{\ell^2}{(1 - \Psi)^2} , \quad (43)
\end{align*}
where $\ell^2$ is the constant of integration representing the curvature scale of the bulk. Eqs. (40) and (38) give

$$C = \frac{\ell^2}{(1 - \Psi)^3}, \quad A = \frac{\ell^2}{4(1 - \Psi)^4}.\quad (44)$$

The rest of Eqs. (36), (37), (41) and (42) are identically satisfied. The coefficients $G, H$ and $I$ must be constants $g, h$ and $i$. Because of the existence of the Gauss-Bonnet topological term, we can put $i = 0$ without losing the generality.

Thus, we find the 4-dimensional effective action with KK corrections as

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Psi R - \frac{3}{4(1 - \Psi)} \nabla^\mu \Psi \nabla_\mu \Psi - \lambda - \lambda \beta (1 - \Psi)^2 \right]$$

$$+ \ell^2 \int d^4x \sqrt{-g} \left[ \frac{1}{4(1 - \Psi)^4} (\nabla^\mu \Psi \nabla_\mu \Psi)^2 + \frac{1}{(1 - \Psi)^2} (\Box \Psi)^2 + \frac{1}{(1 - \Psi)^3} \nabla^\mu \Psi \nabla_\mu \Psi \Box \Psi \right. $$

$$+ \frac{2}{3(1 - \Psi)} R \Box \Psi + \frac{1}{3(1 - \Psi)^2} R \nabla^\mu \Psi \nabla_\mu \Psi + g R^2 + h R^{\mu\nu} R_{\mu\nu} \right],$$

where constants $g$ and $h$ can be interpreted as the effects of the bulk gravitational waves.

From the point of view of the geometrical equation

$$\delta G_{\mu\nu} = \delta T_{\mu\nu} + \ell^2 \delta \pi_{\mu\nu} - \delta E_{\mu\nu}, \quad (46)$$

the key to understand the evolution of the cosmological perturbations is to understand $E_{\mu\nu}$. In the conformal Newtonian gauge

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)\delta_{ij}dx^i dx^j, \quad (47)$$

the statement can be more explicit, namely, the effective anisotropic stress $\Psi - \Phi$ should be known. Once it is determined, the anisotropy of the cosmic background radiation

$$\frac{\delta T}{T} = \zeta + \Psi - \Phi$$

(48)

can be calculated. Here, $\zeta$ denotes the curvature perturbation. Starting our action (23), we can determine $E_{\mu\nu}$ which include Kaluza-Klein effect. Thus, this anisotropic stress can be determined.

VI. CONCLUSION

We have combined the geometrical approach and the low energy gradient expansion method to derive the 4-dimensional effective action. Given our effective action, the temperature anisotropy of the cosmic background radiation can be computed without solving the
bulk equations of motion explicitly. Since the method for completing this part is standard, we have not spelled out details here [7]. We have just mentioned the importance of our result in conjunction with the anisotropy of temperature of the cosmic background radiation.

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