Fractionalization in dimerized graphene and graphene bilayer

M. V. Milovanović
Institute of Physics, P.O.Box 68, 11080 Belgrade, Serbia
(Dated: June 20, 2008)

We show that the fractional statistics of quasiparticles in dimerized graphene, in recent proposals for charge and statistics fractionalization, can have two realizations depending whether elementary objects can be considered as point-like or extended objects. Therefore, there are two phases of proposed excitations and we give their topological descriptions with their respective statistics. We propose that a natural setting for fractionalization are certain systems with excitonic instabilities and demonstrate this by an example of graphene bilayer.

I. INTRODUCTION

Recent proposals 1–3 for fractionalization in dimerized graphene and similar structures are based on the physics of charge fractionalization in polyacetylene. But dimerization is hard to achieve in monolayer graphene. Mathematically fractionalization reduces to finding zero modes in a Dirac like equation for the dimerized pattern with vortex structure. A similar problem came from finding zero modes of vortex solutions of $p$-wave and other superconducting order parameters. From this comes the idea that in the systems needed for the fractionalization previously introduced, in which number of electrons is conserved quantity, excitonic instability with its BCS like Hamiltonian and structure may induce order parameters and vortex solutions with only one zero mode necessary for fractionalization. Like quantum Hall bilayer at $\nu = 1$ also graphene bilayer represents a natural setting for an excitonic instability as found in Ref. 8. And similarly but not obviously, as fractionalization in quantum Hall bilayer into merons with half electric charge and fractional quantum statistics, we expect similar quasiparticles - vortex solutions in graphene bilayer.

In Section II we will review the recent proposals for fractionalization, where we will also clarify the situation and answer the question pertaining zero mode solutions and statistics that appeared in literature. Section III will be devoted to the bilayer graphene as a stage for charge fractionalization.

II. FRACTIONALIZATION IN DIMERIZED GRAPHENE

The effective (low-energy) Hamiltonian in the presence of the Kekule deformation of a graphene monolayer is

$$\mathcal{H} = \int d^2r \Psi^+(r) \mathcal{H}_D \Psi(r)$$

with

$$\Psi^+(r) = (u^+_a(r)v^+_a(r)v^+_b(r))$$

and

$$\mathcal{H}_D = \left( \begin{array}{ccc} 0 & -2i\partial_z & \Delta(r) & 0 \\ -2i\partial_z & 0 & 0 & \Delta(r) \\ \Delta(r) & 0 & 0 & 2i\partial_z \\ 0 & \Delta(r) & 2i\partial_z & 0 \end{array} \right)$$

$u_a, v_a$ and $u_b, v_b$ denote the electronic (effective - Dirac) variables of the two triangular sublattices, $a$ and $b$ respectively, of the honeycomb graphene lattice. For the usual Kekule texture we get with $\Delta(r) = \Delta_o$. With no texture we have two cones in the spectrum, $\epsilon_{\pm}(\vec{p}) = \pm |\vec{p}|$, and with the Kekule texture mass gaps are opening: $\epsilon_{\pm}(\vec{p}) = \pm \sqrt{\hat{p}^2 + |\Delta_o|^2}$ in the single particle spectra.

A. Charge fractionalization

Let’s assume a vortex structure in the complex parameter $\Delta(\vec{r})$:

$$\Delta(\vec{r}) = \Delta(r) \exp\{-i\theta\},$$

where polar coordinates are used. We seek solutions for electronic states of the dimerized graphene in the presence of this structure with zero energy. The equations that follow from Eq. (1), in the case of sublattice $a$ are

$$\partial_z u + i\Delta(\vec{r}) v = 0$$

$$i\Delta(\vec{r}) u - \partial_z v = 0$$

and with the exchange $z \leftrightarrow \bar{z}$ they are also valid in the case of sublattice $b$.

In the polar coordinates we have

$$\exp\{-i\theta\} \left( \partial_r - \frac{i}{r} \partial_\theta \right) u(\vec{r}) + i \exp\{-i\theta\} \Delta(r) v(\vec{r}) = 0$$

$$i \exp\{i\theta\} \Delta(r) u(\vec{r}) - \exp\{i\theta\} \left( \partial_r + \frac{i}{r} \partial_\theta \right) v(\vec{r}) = 0$$

In order to separate angular dependence we substitute $u(\vec{r}) = u_o \exp\{-i\theta\} u(r)$ and $v(\vec{r}) = v_o \exp\{-i\theta\} v(r)$ to have

$$\left( \partial_r - \frac{m}{r} \right) u(r) + i \Delta(r) \frac{v_o}{u_o} v(r) = 0$$

$$\left( \partial_r - \frac{1}{r} \right) v(r) + i \Delta(r) \frac{u_o}{v_o} u(r) = 0$$

with

$$\Delta(r) = \Delta_o \exp\{-i\theta\}.$$
if \( l = n - 1 - m \). Further if we take \( i\Delta(r) \frac{\partial}{\partial \phi} = f(r) \equiv |\Delta(r)| \) that fixes the ratio \( \frac{\partial}{\partial \phi} \), the radial problem is reduced to solving

\[
\begin{align*}
(\partial_r - \frac{m}{r})u(r) + f(r)v(r) &= 0 \\
(\partial_r - \frac{n - 1 - m}{r})v(r) + f(r)u(r) &= 0
\end{align*}
\]

There are two linearly independent solutions to the equations. The behavior at large \( r \) is, apart from powers of \( r, \exp\{\mp \mu r\} \). Since the solution must be normalizable only one is acceptable. At the origin, the asymptotics that follow, with \( f(r) \sim f(r^{|n|}) \),

\[
\begin{align*}
u(r) &= u_1 r^m + u_2 r^{|n|+n-m} \\
v(r) &= v_1 r^{|n|+1+m} + v_2 r^{n-1-m}
\end{align*}
\]

To have single-valued, non-singular solutions at the origin we have to demand that \( m \) is integer and

\[
n - 1 \geq m \geq 0.
\]

The solutions that we get are similar but not the same as in Ref. [3]. There superconducting couplings in the Dirac lagrangian induce different signs in the angular dependence \( u \sim \exp\{-im\theta\}, v \sim \exp\{i(n - 1 - m)\theta\} \) which then guarantees one angular momentum eigenstate per any odd value of the vorticity, \( v = -n \). In our case only \( v = -1 \) vorticity solution is an angular momentum eigenvalue: \( m = 0 \). In this case the radial problem is simplified and the explicit solution is

\[
\Psi(r) = C \begin{bmatrix} 0 & i \exp\{i\alpha\} \\ i \exp\{i\alpha\} & 0 \end{bmatrix} \exp\{-\int_0^r f(r')dr'\}
\]

where \( \alpha \) is a constant defined by \( \Delta(r) = |\Delta(r)|\exp\{i\alpha\} = f(r)\exp\{i\alpha\} \) and \( C \) is a normalization constant. If we do not demand that the solutions have to be eigenstates of angular momentum the condition, Eq. (5) ensures that we have \( n \) zero mode solutions in the case of the vortex with vorticity \( v = -n \) (\( n \) positive). Thus negative vorticity vortex states exist only on sublattice \( a \), and very similar analysis shows that only positive vorticity vortex states exist on sublattice \( b \) in which case again there are as many zero modes as the value of vorticity \( m \).

It can be argued that the charge bound to a vortex of vorticity \( v = 1 \) is \( \frac{\pi}{2} \). We have to study the change in the local density of states of the Dirac Hamiltonian \( \text{[1]} \) in the presence of mass twist in Eq. (2). Because of the sublattice symmetry, to any negative eigenstate of the Dirac kernel \( \Psi_-(r) \) corresponds to a positive energy state \( \Psi_+(r) \), related to \( \Psi_-(r) \) by a unitary transformation. Hence the local density of states

\[
\nu(r, \epsilon) = \sum_{\epsilon'} \Psi_+^{\dagger}(r)\Psi_-(r)\delta(\epsilon - \epsilon')
\]

is symmetric with respect to zero energy. Demanding the conservation of the total number of states after the inclusion of the mass twist, we get

\[
\int d\vec{r}\left\{ 2 \int_{-\infty}^{0-} \delta \nu(\vec{r}, \epsilon) \; d\epsilon + |\Psi_0(\vec{r})|^2 \right\} = 0,
\]

where \( \Psi_0(\vec{r}) \) stands for the single zero mode. Its normalization to one leads to

\[
\int d\vec{r} \int_{-\infty}^{0-} \delta \nu(\vec{r}, \epsilon) \; d\epsilon = -\frac{1}{2}
\]

so the net charge difference is \( -\frac{\pi}{2} \).

B. Statistics

The introduced system can be in short described as Dirac electrons in the presence of a twisted mass \( \omega \).

\[
\mathcal{L} = \Psi(i\gamma_\mu \partial_\mu + \Delta \exp\{i\gamma_5 \phi\})\Psi
\]

where \( \gamma_\mu, \gamma_5 = 0, 1, 2 \) are \( 4 \times 4 \) Dirac matrices in the Weyl representation. The problem may be reformulated by dividing the vortex excitations into two groups, + and -, according to + and - value of vorticity corresponding to singularities in \( \phi_+ \) and \( \phi_- \) respectively, where \( \phi = \phi_+ + \phi_- \) and introducing gauge fields \( \omega \).

\[
\begin{align*}
\alpha_\mu &= \frac{1}{2}(\partial_\mu \phi_+ - \partial_\mu \phi_-) \\
b_\mu &= \frac{1}{2}(\partial_\mu \phi_+ + \partial_\mu \phi_-)
\end{align*}
\]

We denote by \( sgn(m_3) \) a quantity that at the singular point of any vortex takes + or - depending whether the charge of the vortex is of sublattice \( a \) or \( b \) kind. \( sgn(m_3)\frac{\partial}{\partial x} \) represents vortex excitation electric charge current:

\[
\tilde{j}_\mu = \frac{sgn(m_3)}{2\pi} e^{\mu \lambda \gamma} \partial_\mu \partial_\lambda \phi.
\]

\( sgn(m_3) \) is necessary as we found out that vortex excitations with both positive (+1) and negative (-1) vorticity that live on sublattice \( b \) and sublattice \( a \) respectively, possess the zero mode and unoccupied by electron represent \( -\frac{\pi}{2} \) absence of charge.

\( sgn(m_3)\frac{\partial}{\partial x} \) represents current of axial charge (valley index) associated with vortex excitations (normalized by vortex axial charge) - \( \tilde{j}_5 \). The sign of axial charge in the density-current \( \tilde{j}_5 \) comes from \( sgn(m_3) \). This comes from the correspondence - see Appendix, in the sign of the sublattice density difference \( \gamma_0 \gamma_5 \Psi \) (our states are eigenstates of \( \gamma_0 \gamma_5 \tilde{B} \)) and the sign of the expectation value of \( \bar{\Psi} \gamma_0 \gamma_5 \Psi \) - axial charge density. The expectation value of the axial charge is \( \pm \frac{\pi}{2} \) like of ordinary charge.

Then a simple statement follows for the topological part of the effective action in the dual representation of
the theory i.e. in terms of vortices instead of Dirac particles (electrons). (In the dual picture elementary \(2\pi\) fluxes of gauge fields are particles and the gauge fields represent particle background.) The form of the topological part of the action is

\[
\frac{\text{sgn}(m_3)}{2\pi} a(\partial \times b) - \frac{1}{2} a j^\mu - \frac{1}{2} b j^5_\mu. \tag{16}
\]

We just encoded in Eq.\((\ref{10})\) the expressions of our currents found previously. There is overall \(\frac{1}{2}\) factor because of the value of the charge of the vortices that must couple as \(-\frac{1}{2} A^\mu \tilde{j}_\mu\) to the external field if we introduce it \((a \rightarrow a + A)\).

If we introduce gauge fields

\[
R = a + b, \quad L = a - b
\]

we arrive at the following form of the Lagrangian,

\[
\frac{\text{sgn}(m_3)}{8\pi} (R \partial R - L \partial L) - R \tilde{j}^+ - L \tilde{j}^-.
\tag{17}
\]

where

\[
\tilde{j}^+ = \frac{\tilde{j}^+ + \tilde{j}^-}{2}, \tag{18}
\]

and

\[
\tilde{j}^- = \frac{\tilde{j}^+ - \tilde{j}^-}{2}.
\tag{19}
\]

the currents of the good quantum number - vorticity as opposed to charge \(j\) and axial \(j^5\) currents. Considering the Aharonov-Bohm phases for encircling quasiparticles around each other we easily and clearly get that quasiparticles, \(\tilde{j}^+\) and \(\tilde{j}^--\), have semionic statistics among themselves and their mutual statistics is trivial.

Therefore through a straightforward analysis of quasiparticles as point particles i.e. singularities of phase - vortex solutions we come to the conclusion that they obey semionic statistics and their theory is the doubled Chern-Simons i.e. \(U(1)_2 \times U(1)_2\) - see Ref.\([\ref{10}]\), as found in Ref.\([\ref{2}]\). On the other hand in Ref.\([\ref{4}]\) the quasiparticles were viewed as extended objects - meron configurations of field \(n\), \(n^2 = 1\), and the conclusion was that they possess quarton statistics. If we apply a simplification that vector \(n\) is always in the \(x-y\) plane except at the center of excitation we will find following the arguments of Ref.\([\ref{4}]\) that the excitation possess semionic statistics. Considering this we may ask ourselves what modifications of our approach are necessary to account for extended vortices.

The electric charge current \(j^\mu\) and axial charge current \(j^5_\mu\) can be introduced in the topological part of the action by simply taking \(j^\mu = \frac{1}{2} j^\mu\) and \(j^5_\mu = \frac{1}{2} j^5_\mu\) because the point charges carry half of the unit of electric and axial charge:

\[
\frac{\text{sgn}(m_3)}{2\pi} a(\partial \times b) - a_{\mu} j^\mu - b_{\mu} j^5_\mu
\]

\[
= \frac{\text{sgn}(m_3)}{8\pi} (R \partial R - L \partial L) - \tilde{j}^+ - \tilde{j}^-.
\tag{20}
\]

Now we can see what principle can guide us to modify the theory. \(j^+\) and \(j^-\) should, in principle, correspond to the charged fermions, electrons that may appear even in low-energy theory (they should certainly exist in a complete theory). To have that we will add appropriate Chern-Simons term as additional dynamics that is allowed:

\[
\frac{\text{sgn}(m_3)}{2\pi} a(\partial \times b) + \frac{\text{sgn}(m_3)}{2\pi} a(\partial \times b) - a_{\mu} j^\mu - b_{\mu} j^5_\mu. \tag{21}
\]

Now fractionalized excitations, \(\tilde{j}^\mu = 2 j^\mu\) and \(\tilde{j}^5_\mu = 2 j^5_\mu\), have quarton statistics i.e.

\[
\frac{\text{sgn}(m_3)}{4\pi} (R \partial R - L \partial L) - \frac{1}{2} R \tilde{j}^+ - \frac{1}{2} L \tilde{j}^-.
\tag{22}
\]

This theory alone (as a topological one - so-called BF Chern-Simons field theory and without \(\text{sgn}(m_3)\) which can be absorbed by simple redefinitions) was investigated in Ref.\([\ref{11}]\) as the theory of 2d superconductors with vortices and quasiparticles as excitations, and was proposed as the description of the topological part of a phase for the QH bilayer\([\ref{12}]\). These considerations also imply that quasiparticles can be only found confined in pairs.

The beginning Lagrangian in (13) can be restated by a gauge transformation: \(\Psi_\pm \rightarrow e^{ie\gamma_5} \Psi_\pm\), where \(\Psi_\pm\) are chiral components of Dirac field, \(\Psi_\pm = \frac{1}{2} (1 \pm \gamma_5) \Psi\) as

\[
\mathcal{L} = \overline{\Psi} i \gamma^\mu \partial_\mu - \gamma_5 (\hat{\gamma} + \Delta) \Psi.
\tag{23}
\]

After integrating out Dirac fermions, the total \(\mathcal{L}\) can be expressed also as

\[
\mathcal{L} = -\frac{\pi}{12\Delta} (\partial \times a)^2 + \frac{\Delta}{2\pi} b^2 + \frac{\text{sgn}(m_3)}{2\pi} a(\partial \times b) - \frac{1}{2} j^\mu a^\mu - \frac{1}{2} j^5_\mu b^\mu.
\tag{24}
\]

When \(\Delta\) or screening charge (in the Maxwell term) is large we may expect that the point-like description (via semions) of vortices is appropriate, but when \(\Delta\) is small we are in a superfluid phase where presumably quartons, but certainly some extended objects, are confined and appear in pairs. This latter physics remind us of the quantum Hall bilayer physics where merons in the superfluid phase for the bilayer are extended objects, appear in pairs, and have quarton statistics\([\ref{13}]\). And indeed the action (after integrating out fermions in the presence of an additional field - a staggered chemical potential i.e. a third component of the \(\tilde{n}\) vector) that was found in Ref.\([\ref{4}]\) is similar to the effective action for the quantum Hall bilayer (as in Ref.\([\ref{13}]\) where also the fermionic current is equal to the \(O(3)\) topological current\([\ref{13}]\). The theory in Eq.\((\ref{22})\) maybe a crude oversimplification, but it tells us what is the feature of any phase which includes both, fractionalized excitations (merons) and fermions - the excitations must be bound in pairs (compare Ref.\([\ref{4}]\). In other words when merons are deconfined, they can be viewed as point-like objects with semionic statistics. That was also a result of the numerical study in Ref.\([\ref{14}]\).
of small (quantum dot) systems with spin in which deconfinement of moners was proposed and mapped to a spinon (semon) gas of Haldane-Shastry chain.

The conclusion is that the difference between Ref. 4 and Ref. 2 in assigning the exchange statistics comes as a difference in how one considers $\vec{n}$: (1) as a continuous vector field as in the quantum Hall bilayer where quarton statistics (meron - extended description) will survive even with no bias that is, in Ref. 4, with no uniform $\mu_s$ - staggered chemical potential but with adjustments of $\mu_s$ at places of excitations, or (2) as a field that has to take values only in the $x-y$ plane as in Ref. 2 (and only uniform, constant value of $\mu_s$ is allowed) with singular behavior on vortices and hence semionic statistics. So the question is whether we are promoting $\mu_s$ into a dynamical variable. Both possibilities seem allowed but lead to different phases in general, in which fractional objects have same charge but different statistics, and are confined and deconfined respectively.

The derivation of the quarton statistics (Eqs. [14] - [16][20]) that we gave is very general and still valid even with inclusion of a time reversal breaking term $-\eta\dot{\Psi}\dot{\gamma}_3\Psi$ that was included and discussed in Ref. 4. The only assumption is the value of the charge of the excitation for that case, which we can safely take to be $\frac{1}{2}$ in accordance with the $\vec{n}$ formalism in Ref. 2 (for $\mu_s = 0$). A related question or comment may be that the gauge transformation of Ref. 2 of $\mathcal{L}$ in Eq. (13) will not lead to a simple transformed form with only fields: $\Psi, a_{\mu},$ and $b_{\mu}$ (see Eq. (21)) in this case. That is true but it is very unlikely that (small) $\eta$ perturbation will lead to an effective Lagrangian of $\Psi, a_{\mu},$ and $b_{\mu}$ fields with a change of the coefficient of the minimal coupling of field $a_{\mu}$ from one to two that is needed for quarton statistics if we follow the same steps as in the derivation of the quarton statistics. Therefore, the time reversal breaking term alone cannot lead to a statistical transmutation of semions into quartons, although it seems its presence is the only way to recover and demonstrate (Ref. 4) quarton statistics in the $\vec{n}$ formalism (when $\Delta > |\eta|$). The inclusion of the uniform staggered chemical potential will change the overall factor $\frac{1}{2}$ (the charge of the excitation) that multiplies Eq. (10), but still the statistical angle will be given by $\pi = Q_s,$ where $Q_s$ is the charge of the excitation (and not by $\frac{\theta}{\pi} = Q_s^2$).

III. FRACTIONALIZATION IN BILAYER GRAPHENE

The question is what are physical systems that may support fractionalization - obviously dimerized graphene is a hypothetical system. The bosonic degrees of freedom that we need can come as a result of electronic correlations, most notably excitonic and superconducting. From these, only excitonic conserve charge and may produce in their defects charged zero modes (as opposed to neutral zero modes in superconductors) that we need to have fractionalization.

In order to facilitate the discussion of the excitonic instability and its zero modes in graphene bilayer, we will first discuss zero modes in the case of $p$-wave superconducting and excitonic system.

(1) The effective BCS Hamiltonian for the quasiparticles is

$$H = \sum_k \xi_k c_k^+ c_k + \frac{1}{2}(\Delta_k^c c_{-k}^+ c_k + \Delta_k^c c_k^+ c_{-k}).$$

Introducing the Bogoliubov transformation:

$$\alpha_k = u_k c_k - v_k c_{-k}^+ \quad (27)$$

that should diagonalize the Hamiltonian into $H = \sum E_k \alpha_k^\dagger \alpha_k + \text{const},$ implies Bogoliubov-de Gennes equations

$$E_k u_k = \xi_k u_k - \Delta_k^c v_k, \quad (28)$$
$$E_k v_k = -\xi_k v_k - \Delta_k u_k. \quad (29)$$

If $\Delta_k = \Delta(k_x - ik_y)$ and $\xi_k \approx -\mu(\mu > 0),$ in the long-distance approximation the equations for zero mode(s) become:

$$-\mu u - \Delta(-i)\partial v = 0, \quad (30)$$
$$\mu v - (-i)\Delta\partial u = 0. \quad (31)$$

We may ask for zero modes that exist in vortex solutions for which we demand $v(\theta + 2\pi) = -v(\theta)$ and $u(\theta + 2\pi) = -u(\theta)$ in polar coordinates. By solving with $\Delta = \text{const}$ we are neglecting the short (small radial) distance details of the solution. We seek the solution in the following form

$$u = \frac{u(r)}{z^k}, \quad v = \frac{v(r)}{z^k} \quad (32)$$

and the equations that we get are

$$-\mu \frac{u(r)}{z^k} + i\Delta \frac{1}{z^k} e^{i\theta} \partial_r v(r) = 0, \quad (33)$$
$$\mu \frac{v(r)}{z^k} + i\Delta \frac{1}{z^k} e^{-i\theta} \partial_r v(r) = 0. \quad (34)$$

To separate angular and radial dependence we must have $k = l = \frac{1}{2}$ and the equations are reduced to

$$-\mu u(r) + i\Delta \partial_r v(r) = 0, \quad (35)$$
$$\mu v(r) + i\Delta \partial_r u(r) = 0. \quad (36)$$

If $u(r) = u_0 f(r)$ and $v(r) = v_0 f(r),$ the equations reduce to a single one,
\[ \mu f(r) + \Delta \partial_r f(r) = 0, \quad (37) \]

if \( \frac{\mu v}{v_0} = -i \). Therefore our solutions can be cast in the following form:

\[ u = ie^{i\frac{\pi}{4}} f(r) \sqrt{r}, \quad v = e^{i\frac{\pi}{4}} f(r) \sqrt{r}, \quad (38) \]

where \( f(r) \) is of the simple radial dependence \( \sim e^{-\frac{\mu}{r}} \) for \( \mu = \text{const} \).

The usual approach to model order parameter with the vortex singularity i.e. to take, instead of \( \Delta_k \), \( e^{\frac{i\pi}{4}} \Delta_k e^{\frac{i\pi}{4}} \) in our case. This symmetrized with respect to phase expression is used to ensure the antisymmetry of the order parameter i.e. that the term \( \int d^2\Psi^+(\vec{r}) \Delta_k \Psi^+(\vec{r}) \) in the Bogoliubov Hamiltonian is well-defined and consistent with the anticommutativity of fermi operators when phase is coordinate-dependent.

From the approach we used in getting the zero mode we can turn to the usual approach by the simple phase transformation \( u \rightarrow e^{-i\frac{\pi}{4}} u, v \rightarrow e^{i\frac{\pi}{4}} v \), so that at the end our zero mode solution has the components:

\[ u = ie^{i\frac{\pi}{4}} f(r) \sqrt{r}, \quad v = e^{i\frac{\pi}{4}} f(r) \sqrt{r} \quad (39) \]

Now the quasiparticle operator for the zero energy state can be written as

\[ \gamma_0^+ = \int d^2\vec{r} (u(\vec{r}) c^+(\vec{r}) + v(\vec{r}) c(\vec{r})) \quad (40) \]

and immediately we can conclude that \( \gamma_0^+ = \gamma_0 \) for our solution i.e. it represents neutral Majorana mode.

We should notice that with respect to the Dirac problem in the dimerized graphene here, in the latter approach, momentum operators are together, in the same term, with order parameter and phase singularity in the Hamiltonian.

On the other hand, in the case of the excitonic problem which may be related and become a physical realization of dimerized graphene lattice, we have a different basic Hamiltonian:

\[ H = \sum_k E_k (b_k^+ \beta_k - \gamma_k^+ \gamma_k) - \sum_k (\Delta_k \beta_k^+ \gamma_k + \Delta_k^* \gamma_k^+ \beta_k) \quad (41) \]

Introducing

\[ B_k = u_k \beta_k - v_k \gamma_k, \quad C_k = v_k \beta_k + u_k \gamma_k \quad (42) \]

that should diagonalize the Hamiltonian into \( H = \sum_k \epsilon_k (B_k^* B_k - C_k^* C_k) \), we get

\[ \epsilon_k u_k = E_k u_k + v_k \Delta_k^*, \quad (44) \]
\[ \epsilon_k v_k = E_k v_k - u_k \Delta_k. \quad (45) \]

For the zero modes \( (\epsilon_k = 0) \) simple redefinition \( v_k \rightarrow -v_k \) transforms the equations into the same as for the superconducting problem.

We will assume \( \Delta_k = \Delta (k_x - ik_y) \) and \( E_k = \epsilon = \text{const} > 0 \). Then for the zero modes we have

\[ \epsilon u + \Delta (-i) \partial_\vec{r} v = 0, \quad (46) \]
\[ \epsilon v - \Delta (-i) \partial_\vec{r} u = 0. \quad (47) \]

The equations are the same as in Eq. (41) and Eq. (41).

Again, we may ask for the zero modes that exist in vortex solutions for which we demand \( v(\theta + 2\pi) = -v(\theta) \) and \( u(\theta + 2\pi) = -u(\theta) \) in polar coordinates and the answer would be the same. But if we want to stay in the language of the order parameter, we may model it, in the presence of a vortex, as \( \Delta_k e^{i\theta} \), with no symmetrization as was necessary in the superconducting problem. In this case the solutions become

\[ u = ie^{i\frac{\pi}{4}} e^{i\theta} f(r) \sqrt{r}, \quad v = e^{i\frac{\pi}{4}} e^{i\theta} f(r) \sqrt{r} \quad (48) \]

which are not satisfactory for the electronic wave functions because they are not single-valued while the singularity is borne by the order parameter. If we choose in the order parameter \( e^{-i\frac{\pi}{4}} \Delta_k e^{\frac{i\pi}{4}} \), our solutions are represented by

\[ u = ie^{i\frac{\pi}{4}} e^{i\theta} f(r) \sqrt{r}, \quad v = e^{i\frac{\pi}{4}} f(r) \sqrt{r} \quad (49) \]

and this, although not an angular momentum eigenstate, is the solution that can describe a charged mode because \( u^* \neq v \) as a crucial difference with respect to the superconducting case. The excitonic Bogoliubov transformation mixes the same kind of charged operators, leading to charged zero modes as in the dimerized graphene problem. Although the precise form of the equations for the zero modes is not the same in the two cases (excitonic p-wave system and dimerized graphene), they are very similar in the long-distance regime and should lead to the same conclusion about the charge fractionalization.

(2) For the case of bilayer graphene excitonic instability described in Ref. 5, we have in the limit of small interlayer hopping

\[ E_k \approx V \left[ 1 - \frac{1}{2} \left( \frac{\epsilon_k}{t_\perp} \right)^2 \right] \quad (50) \]

for the energy in Eqs. (41). Here \( t_\perp \) is the interlayer hopping parameter, \( \pm V \) is the bias that causes excess
electrons and holes in the lower and upper graphene layers, respectively, and $\epsilon_k$ is the bare kinetic energy for which $\epsilon_k \sim k$. The gap function is of the following form, near two nodal points,\footnote{32}:

$$\Delta_k = i(k_x \mp ik_y)|k|\Delta,$$  

(51)

where $\Delta$ is a positive constant. We will write $E_k$ as $E_k = V - \delta k^2$ with also $\delta$ being a positive constant. Then the equations (44) with $\epsilon = \epsilon_k$ becomes

$$\left(V + \delta \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}\right)u + \Delta(-i)(-i)\partial_z v = 0,$$  

(52)

$$\left(V + \delta \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}\right) v - \Delta(i)(i)\partial_z u = 0.$$  

(53)

To consider the vortex solution we have to set $e^{-i\frac{\pi}{4}} \Delta_k e^{i\frac{\pi}{4}}$ instead of $\Delta_k$, which if we want to absorb the phase into $v$ and $u$ leads to slightly different equations due to the presence of the $k^2$ terms that we also must keep in this long-distance analysis. Because

$$e^{i\frac{\pi}{4}} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} e^{-i\frac{\pi}{4}} = \left(\frac{\partial}{\partial z} - \frac{1}{2\pi}i \frac{\partial}{\partial \bar{z}} + \frac{1}{2\pi}\right),$$  

(54)

the equations for the vortex solution become

$$\left[V + \delta \left(\frac{\partial}{\partial z} - \frac{1}{2\pi}i \frac{\partial}{\partial \bar{z}} + \frac{1}{2\pi}\right)\right]u$$  

$$+ \Delta(i)(-i)\partial_z v = 0,$$  

(55)

$$\left[V + \delta \left(\frac{\partial}{\partial z} - \frac{1}{2\pi}i \frac{\partial}{\partial \bar{z}} + \frac{1}{2\pi}\right)\right] v$$  

$$+ \Delta(-i)(i)\partial_z u = 0,$$  

(56)

with the requirement $u(\theta + 2\pi) = -u(\theta)$ and $v(\theta + 2\pi) = -v(\theta)$. If we seek solutions in the following form (see Eq. (32))

$$u = \frac{u(r)}{z^k}, v = \frac{v(r)}{z^k}$$  

(57)

we face the problem of properly performing the operations with the square root operator, $\sqrt{\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}}$. Its action we will define on the space of monomials in $z$ and $\bar{z}$, for which we have

$$\sqrt{z} \frac{\partial}{\partial z} \sqrt{z} z^m \bar{z}^n = n \sqrt{m} z^{m} \bar{z}^n.$$  

(58)

We will assume the asymptotic behavior of $u(r)$ and $v(r)$ as $u(r) \sim e^{-\lambda r}$ and $v(r) \sim e^{-\lambda r}$, and justify it in the end. Then, for example, in the case of $v(r)$ we have

$$\sqrt{z} \frac{\partial}{\partial z} \sqrt{z} z^k e^{i\theta} \partial_r v = -\lambda v_0 \sqrt{z} \frac{\partial}{\partial z} \left(\frac{1}{2\pi} \frac{z^l}{z^l} + \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda)^n z^{n/2} \frac{z^n}{z^n} \right)$$  

$$= -\lambda v_0 \sqrt{z} \frac{\partial}{\partial z} \left(\frac{1}{2\pi} \frac{z^l}{z^l} + \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda)^n \right)$$  

$$= -\lambda v_0 \sqrt{z} \frac{\partial}{\partial z} \left(\frac{1}{2\pi} \frac{z^l}{z^l} + \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda)^n \right)$$  

$$\times \sqrt{n+1} - k \sqrt{n-1} \frac{1}{2} - k \sqrt{n+1} - k \frac{1}{2} = 0.$$  

(59)

Therefore the Eq. (56) becomes

$$\left[\left[V + \delta \left(\frac{\partial}{\partial z} - \frac{1}{2\pi}i \frac{\partial}{\partial \bar{z}} + \frac{1}{2\pi}\right)\right] u(r)$$  

$$+ \Delta(i)(-i)\partial_z v(r) = 0,$$  

(60)

$$\left[V + \delta \left(\frac{\partial}{\partial z} - \frac{1}{2\pi}i \frac{\partial}{\partial \bar{z}} + \frac{1}{2\pi}\right)\right] v(r)$$  

$$+ \Delta(-i)(i)\partial_z u(r) = 0.$$  

(61)

To factor the angular dependence we choose $k = l = \frac{1}{2}$ and in the long distance approximation, for which we take also $\sqrt{n(n-1)} \approx n$ in the series expansion, we have

$$\left[V + \delta \partial_r^2 \right] u + i\Delta(-\lambda)^2 v(r) = 0,$$  

(62)

$$\left[V + \delta \partial_r^2 \right] v - i\Delta(-\lambda)^2 u(r) = 0.$$  

(63)

This means

$$u_0 \left[ V + \delta \lambda^2 \right] + i\Delta\lambda^2 v_0 = 0,$$  

(64)

$$v_0 \left[ V + \delta \lambda^2 \right] - i\Delta \lambda^2 u_0 = 0.$$  

(65)

i.e. $\lambda_1^2 = \frac{V}{(\sigma + \Delta)}$ or $\lambda_2^2 = \frac{V}{\Delta - \delta}$. Both, $\lambda_1, \lambda_2 = \pm i \sqrt{\frac{V}{\Delta + \Delta}}$, represent delocalized zero modes. In the case of $\lambda_2$ that would be if $\delta > \Delta$, otherwise we have only one physical localized state with $e^{-\lambda r}$, $\lambda_2 = \sqrt{\frac{V}{\Delta - \Delta}}$, decay function. Comparing with solutions in Ref. [9] for $\Delta$ for fixed parameters for the graphene bilayer, we find that for large enough $V$, bias parameter, we can have the bound solution. Then the three zero modes carry fractional charge, $\frac{3}{2}$, that we get applying the same arguments that were
given in Ref. [1] for the dimerized graphene. Essential in these arguments is that Dirac kernel has the correspondence of positive and negative energy solutions. In this case we also have the Dirac structure of the problem, with the nodal points, and the same correspondence.

In this Hartree-Fock treatment of the excitonic problem we get three zero modes (for large enough \( V \)) and the question comes whether they will stay with further inclusion of interactions. In the case of three zero modes, the charge that they carry is mostly smeared out through the system. If the interactions were able to split the two delocalized zero modes, we could have localized charge. The conclusions about the topological terms and fractional statistics that we reached in the case of dimerized graphene will still hold. The interaction term that we have in mind would be \( V \rho_{b+}\rho_{b-} \) where \( k_o = |\lambda_1| \) and an excitonic coupling between the mode \( k_o \) and \( -k_o \) would produce necessary splitting. Otherwise (with no splitting) the situation is less clear but if we assume that the localized mode describes a missing of \( \frac{1}{2} \) charge the delocalized modes will describe additional degrees of freedom (that may be occupied or unoccupied) that may further decrease the value of the statistical angle of excitations or equivalently increase the coupling of the doubled Abelian Chern-Simons term.

The recent work, Ref. [13] that appeared while we were finishing the writing, concerns fractionalization in excitonic bilayer graphene that is not naturally (Bernal) stacked but consists of two parallel layers at some larger distance that leads to even number of zero modes due to the valley degeneracy and, therefore, no charge fractionalization. Still we can not rule out, on the basis of the long distance analysis of vortex solutions in the excitonic condensate that we presented, the same doubling in our case. This important question can be resolved only by a detailed, numerical analysis of the bilayer graphene.

### IV. ACKNOWLEDGMENT

The author thanks Chang-Yu Hou for correspondence. The work was supported by Grant No. 141035 of the Ministry of Science of the Republic of Serbia.

### APPENDIX

In the second quantized formalism the zero mode solution in Eq. (9), of definite vorticity \( v = -1 \), contributes to the expansion of the Dirac field as a term equal to

\[
\Psi_o(\vec{r}) = \begin{pmatrix} 0 \\ i\exp\{i\alpha\} \\ 0 \end{pmatrix} c_0 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} C\exp\{-|\Delta|\vec{r}\},
\]

(A.1)

where we simplified the decay function by taking \( |\Delta(\vec{r})| = |\Delta| = \text{const.} \), and again the \( C \) is the normalization constant. Notice the absence of two different operators (one for particle, the other for hole) that we would have for a non-zero energy level. For the sake of the argument, going in reverse, we can fix the normalization constant by demanding that the charge associated with the zero mode is \((-1)^\frac{1}{2}\) i.e.

\[
\int d^2\vec{r} \Psi_o^+(\vec{r})\Psi_o(\vec{r}) = -\frac{1}{2}.
\]

(A.2)

That would imply

\[
\int d^2\vec{r} \Psi_o^+(\vec{r})\gamma_5\Psi_o(\vec{r}) = \frac{1}{2},
\]

(A.3)

for the expectation value of the axial charge, where we took

\[
\gamma_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\]

in the Weyl representation. On the other hand our state is an eigenstate of the sublattice charge difference operator:

\[
R = \alpha_3 = \gamma_0\gamma_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

so that

\[
\int d^2\vec{r} \Psi_o^+(\vec{r})\alpha_3\Psi_o(\vec{r}) = \frac{1}{2},
\]

(A.4)

of the same sign as the expectation value of the axial charge. Both signs would reverse if the solution were with vorticity \( v = 1 \), on sublattice \( b \), although the sign of the electric charge would remain the same.
With respect to the Ref. 2 we fix that partitioning is into the different vorticity groups. The vorticity is a good number and allows this partitioning.

\[ \text{sgn}(m_3) \] is another good quantum number, sign of the charge carried by the zero mode (occupied or unoccupied). There are four kinds of vortices depending on the \( \text{sgn}(m_3) \) and vorticity.

1. C.-Y. Hou, C. Chamon, and C. Mudry, Phys. Rev. Lett. 98, 186809 (2007).
2. B. Seradjeh, C. Weeks, and M. Franz, Phys. Rev. B 77, 033104 (2008); B. Seradjeh and M. Franz, arXiv:0709.4258
3. R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. 98, 266402 (2007).
4. C. Chamon, C.-Y. Hou, R. Jackiw, C. Mudry, S.-Y. Pi, and A.P. Schnyder, Phys. Rev. Lett. 100, 110405 (2008).
5. W.P. Su, J.R. Schrieffer, and A.J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).
6. With respect to the Ref. 2 we fix that partitioning is into the different vorticity groups. The vorticity is a good number and allows this partitioning.

\[ \text{sgn}(m_3) \] is another good quantum number, sign of the charge carried by the zero mode (occupied or unoccupied). There are four kinds of vortices depending on the \( \text{sgn}(m_3) \) and vorticity.
8. R. Dillenschneider and Jung Hoon Han, arXiv:0709.1230
9. R. Jackiw and P. Rossi, Nucl. Phys. B 190, 681 (1981).
10. M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Ann. Phys. 310, 428 (2004).
11. T.H. Hansson, V. Oganesyan, and S.L. Sondhi, Ann. Phys. 313, 497 (2004).
12. M.V. Milovanović and Z. Papić, arXiv:0710.0478
13. K. Moon, H. Mori, Kun Yang, S.M. Girvin, A.H. MacDonald, L. Zheng, D. Yoshioka, and S.-C. Zhang, Phys. Rev. B 51, 5138 (1995).
14. A. Petković and M.V. Milovanović, Phys. Rev. Lett. 98, 066808 (2007).
15. N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
16. M. Stone and R. Roy, Phys. Rev. B 69, 184511 (2004); M. Stone and S.-B. Chung, Phys. Rev. B 73, 014505 (2006).
17. S. Tewari, S. Das Sarma, C. Nayak, C. Zhang, and P. Zoller, Phys. Rev. Lett. 98, 010506 (2007).
18. B. Seradjeh, H. Weber, and M. Franz, arXiv:0806.0849