DOUBLE VARIATIONAL PRINCIPLE FOR MEAN DIMENSION

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Abstract. We develop a variational principle between mean dimension theory and rate distortion theory. We consider a minimax problem about the rate distortion dimension with respect to two variables (metrics and measures). We prove that the minimax value is equal to the mean dimension for a dynamical system with the marker property. The proof exhibits a new combination of ergodic theory, rate distortion theory and geometric measure theory. Along the way of the proof, we also show that if a dynamical system has the marker property then it has a metric for which the upper metric mean dimension is equal to the mean dimension.

1 Introduction

1.1 Statement of the main result. The purpose of this paper is to develop a new variational principle in dynamical systems theory. We first quickly prepare the terminologies and state the main result. Backgrounds will be explained in Section 1.2.

A pair $(\mathcal{X}, T)$ is called a dynamical system if $\mathcal{X}$ is a compact metrizable space and $T: \mathcal{X} \to \mathcal{X}$ is a homeomorphism. We denote by $\mathcal{M}^T(\mathcal{X})$ the set of $T$-invariant Borel probability measures on $\mathcal{X}$. The standard variational principle [Goodw69, D70, Goodm71] states that the topological entropy $h_{\text{top}}(T)$ is equal to the supremum of the ergodic-theoretic entropy $h_\mu(T)$ over $\mu \in \mathcal{M}^T(\mathcal{X})$:

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} h_\mu(T).$$

(1.1)

Our main result below is an analogous formula in mean dimension theory.

Mean dimension (denoted by mdim$(\mathcal{X}, T)$) is a topological invariant of dynamical systems introduced by Gromov [Gro99]. It counts how many parameters per iterate we need to describe an orbit in $(\mathcal{X}, T)$. We review its definition in Section 2.1. We would like to connect mean dimension to some information-theoretic quantity as in

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An appropriate notion turns out to be rate distortion dimension, which was first introduced by Kawabata and Dembo [KD94].

Let $D(X)$ be the set of metrics (i.e. distance functions) on $X$ compatible with the topology. Take $d \in D(X)$ and $\mu \in \mathcal{M}^T(X)$. Consider a stochastic process $\{T^n x\}_{n \in \mathbb{Z}}$ where $x \in X$ is chosen randomly according to $\mu$. We denote by $R(d, \mu, \varepsilon)$, $\varepsilon > 0$, the rate distortion function of this process with respect to the distortion measure $d$. This evaluates how many bits per iterate we need to describe the process within the distortion (w.r.t. $d$) bound by $\varepsilon$. We review its definition in Section 2.3. We define the upper/lower rate distortion dimensions by

$$\overline{\text{rdim}}(X, T, d, \mu) = \limsup_{\varepsilon \to 0} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)},$$

$$\underline{\text{rdim}}(X, T, d, \mu) = \liminf_{\varepsilon \to 0} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)}.$$  \hfill (1.2)

When the upper and lower limits coincide, we denote their common value by $\text{rdim}(X, T, d, \mu)$.

A dynamical system $(X, T)$ is said to have the marker property if for any $N > 0$ there exists an open set $U \subset X$ satisfying

$$X = \bigcup_{n \in \mathbb{Z}} T^{-n} U, \quad U \cap T^{-n} U = \emptyset \quad (\forall 1 \leq n \leq N).$$

This property implies that $(X, T)$ is free (i.e. it has no periodic points). Free minimal systems and their extensions have the marker property. The marker property has been intensively used in the context of the embedding problem (see Section 1.2) and related issues [Lin99, Gut15, GLT16, GT, GQT].

Now we can state our main result.

**Theorem 1.1 (Double Variational Principle).** If a dynamical system $(X, T)$ has the marker property, then

$$\text{mdim}(X, T) = \min_{d \in D(X)} \sup_{\mu \in \mathcal{M}^T(X)} \overline{\text{rdim}}(X, T, d, \mu) = \min_{d \in D(X)} \sup_{\mu \in \mathcal{M}^T(X)} \underline{\text{rdim}}(X, T, d, \mu).$$  \hfill (1.3)

Here “min” indicates that the minimum is attained by some $d$.

A fundamental difference between the standard variational principle (1.1) and our new one (1.3) is that (1.1) is a maximization problem with respect to the single variable $\mu$ whereas (1.3) is a minimax problem with respect to the two variables $d$ and $\mu$. We have used the word “double” in order to emphasize that there exist two variables playing different roles.

\footnote{Throughout the paper we assume that the base of the logarithm is two. The natural logarithm (i.e. the logarithm of base $e$) is written as $\ln(\cdot)$.}
Remark 1.2. (1) We have adopted the minimax approach in (1.3). It might also look interesting to consider a maximin approach:

\[
\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \inf_{d \in \mathcal{D}(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu),
\]

\[
\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \inf_{d \in \mathcal{D}(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).
\]

(1.4)

However this turns out to be fruitless. Indeed both the quantities in (1.4) are always zero. More strongly, we can prove that for any dynamical system \((\mathcal{X}, T)\) and \(\mu \in \mathcal{M}^T(\mathcal{X})\) there exists \(d \in \mathcal{D}(\mathcal{X})\) satisfying \(\text{rdim}(\mathcal{X}, T, d, \mu) = 0\).

(2) The rate distortion dimension depends on both metrics \(d\) and measures \(\mu\). It might look more satisfactory to define a certain “ergodic-theoretic mean dimension” in a purely measure-theoretic way and prove a corresponding “variational principle” for mean dimension. But this naive approach is impossible: Let us consider an arbitrary ergodic measure-preserving system. By the Jewett–Krieger theorem [J70, Kri70] we can find a dynamical system \((\mathcal{X}, T)\) which has only one invariant probability measure (say, \(\mu\)) and that \((\mathcal{X}, \mu, T)\) is measure-theoretically isomorphic to the given system. It is known that uniquely ergodic systems have zero mean dimension [LW00, Theorem 5.4]. So, if we have a “variational principle”, the given measure-preserving system must have zero “ergodic-theoretic mean dimension”.

(3) We conjecture that the marker property assumption in Theorem 1.1 is, in fact, unnecessary. The proof of Theorem 1.1 shows that the inequality

\[ \text{mdim}(\mathcal{X}, T) \leq \inf_{d \in \mathcal{D}(\mathcal{X})} \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu) \]

holds true for any dynamical system \((\mathcal{X}, T)\). So the remaining problem is how to prove the reverse inequality. See Section 1.4 for further discussions.

1.2 Backgrounds. Mean dimension provides a nontrivial information for infinite dimensional dynamical systems of infinite topological entropy. It has several applications which cannot be touched within the framework of topological entropy [LW00, Lin99, GLT16, GT, GQT, MT]. As an illustration, we explain an application to the problem of embedding dynamical systems into shift actions.

Consider the \(N\)-dimensional cube \(C_N := [0, 1]^N\) and let \(\sigma: (C_N)^\mathbb{Z} \to (C_N)^\mathbb{Z}\) be the shift on the alphabet \(C_N\) where \((C_N)^\mathbb{Z}\) is endowed with the standard product topology. The mean dimension of \(((C_N)^\mathbb{Z}, \sigma)\) is \(N\). Given a dynamical system \((\mathcal{X}, T)\), we are interested in whether we can embed it into \(((C_N)^\mathbb{Z}, \sigma)\) or not.

Periodic points are an obvious obstruction: If \((\mathcal{X}, T)\) has too many periodic points (e.g. if the set of fixed points has dimension greater than \(N\)) then it cannot be embedded into \((C_N)^\mathbb{Z}\). Mean dimension provides another obstruction: If we can

\[ f: \mathcal{X} \to (C_N)^\mathbb{Z} \]

is called an embedding of a dynamical system if it is a topological embedding and satisfies \(f \circ T = \sigma \circ f\).
embed \((\mathcal{X}, T)\) into \((C_N)^\mathbb{Z}\) then \(\text{mdim}(\mathcal{X}, T) \leq \text{mdim}((C_N)^\mathbb{Z}, \sigma) = N\). We can construct free (and, moreover, minimal) dynamical systems of arbitrary mean dimension [LW00, Proposition 3.5]. So there exist plenty of examples which are free but cannot be embedded into \((C_N)^\mathbb{Z}\). (This observation by [LW00] solved a question posed by Auslander in 1970s.)

Somehow surprisingly, a partial converse also holds. Based on the work [Lin99], the papers [GT, GQT] proved that if \((\mathcal{X}, T)\) has the marker property and satisfies \(\text{mdim}(\mathcal{X}, T) < N/2\) then we can embed it into \((C_N)^\mathbb{Z}\). The example in [LT14] shows that the condition \(\text{mdim}(\mathcal{X}, T) < N/2\) is optimal. These results demonstrate that mean dimension is certainly a reasonable measure of the “size” of dynamical systems.

It is classically known that the concepts of entropy and dimension are closely connected (Rényi [Rén59], Kolmogorov and Tihomirov [KT63] and Kawabata and Dembo [KD94]). So it is natural to expect that we can approach to mean dimension from the entropy theory viewpoint. The first attempt of such an approach was made by Weiss and the first named author [LW00] by introducing the notion of metric mean dimension. This is a dynamical analogue of Minkowski dimension defined as follows. Let \((\mathcal{X}, T)\) be a dynamical system with a metric \(d\).\(^3\) Let \(S(\mathcal{X}, T, d, \varepsilon)\) be its entropy detected at the resolution \(\varepsilon > 0\). (See Section 2.1 for the precise definition.) The topological entropy is given by \(h_{\text{top}}(T) = \lim_{\varepsilon \to 0} S(\mathcal{X}, T, d, \varepsilon)\). We define the upper/lower metric mean dimensions by

\[
\overline{\text{mdim}}_{M}(\mathcal{X}, T, d) = \limsup_{\varepsilon \to 0} \frac{S(\mathcal{X}, T, d, \varepsilon)}{\log(1/\varepsilon)},
\]
\[
\underline{\text{mdim}}_{M}(\mathcal{X}, T, d) = \liminf_{\varepsilon \to 0} \frac{S(\mathcal{X}, T, d, \varepsilon)}{\log(1/\varepsilon)}.
\]

When the upper and lower limits coincide, we denote their common value by \(\text{mdim}_{M}(\mathcal{X}, T, d)\). In analogy with the well-known fact that Minkowski dimension bounds topological dimension, we have [LW00, Theorem 4.2]

\[
\text{mdim}(\mathcal{X}, T) \leq \text{mdim}_{M}(\mathcal{X}, T, d) \leq \overline{\text{mdim}}_{M}(\mathcal{X}, T, d).
\]

It was also proved in [Lin99, Theorem 4.3] that if \((\mathcal{X}, T)\) has the marker property then there exists a metric \(d\) on \(\mathcal{X}\) satisfying \(\text{mdim}(\mathcal{X}, T) = \text{mdim}_{M}(\mathcal{X}, T, d)\). The same statement for upper metric mean dimension remained open since [Lin99]. We will establish it as a part of the proof of Theorem 1.1. See Theorem 3.12 below.

Metric mean dimension seems to be quite useful. In particular, it provides a powerful method to obtain upper bounds on mean dimension via (1.6). This was used for example in [T18a] for solving a problem of Gromov [Gro99] to estimate the mean dimension of a dynamical system in holomorphic curve theory. It also has an application to the study of expansive group actions [MT].

\(^3\) The papers [GT, GQT] used the ideas of communication theory and signal processing. This is another manifestation of the intimate connections between mean dimension and information theory.

\(^4\) It seems that these attract new interests of information theory researchers in the context of compressed sensing; see, e.g. [WV10, RJEP].
It seems desirable to inject ergodic theory and in particular invariant measures into mean dimension theory in order to broaden the scope of applications. This motivated the authors to begin the study of our previous paper [LT18]. In [LT18] we proved the following variational principle between metric mean dimension and rate distortion function under a mild condition on \( d \) (called tame growth of covering numbers; see Definition 3.8):

\[
\overline{\text{mdim}}_M(\mathcal{X}, T, d) = \limsup_{\varepsilon \to 0} \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)},
\]

\[
\underline{\text{mdim}}_M(\mathcal{X}, T, d) = \liminf_{\varepsilon \to 0} \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)}.
\]

We proved this by developing a rate distortion theory version of Misiurewicz’s proof [Mis76] of the standard variational principle (1.1). This is an initial step of our program to inject measure into mean dimension theory. However it is still not completely satisfactory. The equation (1.7) implies that we can construct \( \mu \in \mathcal{M}^T(\mathcal{X}) \) capturing (most of) dynamical complexity of \((\mathcal{X}, T)\) at each fixed resolution \( \varepsilon > 0 \). It would be nicer if we could find \( \mu \) capturing the dynamical complexity over all resolutions. In other words, we would like to exchange the order of the limit and supremum in (1.7). This naturally leads us to the following question (this was also posed by Velozo–Velozo [VV, Section 6]):

**Problem 1.3.** When do the following equalities hold?

\[
\overline{\text{mdim}}_M(\mathcal{X}, T, d) = \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \overline{\text{rdim}}(\mathcal{X}, T, d, \mu),
\]

\[
\underline{\text{mdim}}_M(\mathcal{X}, T, d) = \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \underline{\text{rdim}}(\mathcal{X}, T, d, \mu).
\]

Of course, (1.8) does not hold in general. The following example clarifies the situation:

**Example 1.4.** Let \( A = \{1, 1/2, 1/3, \ldots \} \cup \{0\} \subset [0, 1] \) and \( \mathcal{X} = \mathcal{X}^\mathbb{Z} \) with the shift \( \sigma \). Define a metric \( d \) on \( \mathcal{X} \) by \( d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} |x_n - y_n| \). Then it is straightforward to check that \( \overline{\text{mdim}}_M(\mathcal{X}, \sigma, d) = 1/2 \) and that \( \overline{\text{rdim}}(\mathcal{X}, \sigma, d, \mu) = 0 \) for any \( \sigma \)-invariant probability measure \( \mu \) (cf. [KD94, Lemma 3.1]). So (1.8) does not hold even for this simple example. However we can push our consideration further. Let \( B = \{1, 2^{-1}, 2^{-2}, \ldots \} \cup \{0\} \) and consider a homeomorphism \( f: A \to B \) defined by \( f(1/n) = 2^{-n} \) and \( f(0) = 0 \). Define a new metric \( d' \) on \( \mathcal{X} = \mathcal{X}^\mathbb{Z} \) by \( d'(x, y) = \sum_{n \in \mathbb{Z}} 2^{-n} |f(x_n) - f(y_n)| \). Then we can check that

\[
\overline{\text{mdim}}_M(\mathcal{X}, \sigma, d') = \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \overline{\text{rdim}}(\mathcal{X}, \sigma, d', \mu) = 0.
\]

In particular (1.8) holds true for \( d' \).
The above example shows that we can expect the equality (1.8) only for well-chosen metrics $d$. This suggests a new viewpoint: We cannot stick to a fixed metric $d$. We should regard $d$ as a variable and move both $d$ and $\mu$. The double variational principle (Theorem 1.1) is a crystallization of this idea.

The proof of Theorem 1.1 also provides a partial answer to Problem 1.3 (see Corollary 3.14): If $(\mathcal{X}, T)$ has the marker property, then there exists a metric $d$ on $\mathcal{X}$ such that all the following quantities are equal to each other:

$$\text{mdim}(\mathcal{X}, T), \quad \overline{\text{mdim}}_{\mathcal{M}}(\mathcal{X}, T, d), \quad \underline{\text{mdim}}_{\mathcal{M}}(\mathcal{X}, T, d),$$

$$\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \overline{\text{rdim}}(\mathcal{X}, T, d, \mu), \quad \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \underline{\text{rdim}}(\mathcal{X}, T, d, \mu).$$

1.3 Outline of the proof of Theorem 1.1. Let $(\mathcal{X}, T)$ be a dynamical system. The proof of Theorem 1.1 consists of the following three steps:

**Step 1 (Metric mean dimension bounds rate distortion dimension):** For all $d \in \mathcal{D}(\mathcal{X})$ and $\mu \in \mathcal{M}^T(\mathcal{X})$

$$\overline{\text{rdim}}(\mathcal{X}, T, d, \mu) \leq \overline{\text{mdim}}_{\mathcal{M}}(\mathcal{X}, T, d),$$

$$\underline{\text{rdim}}(\mathcal{X}, T, d, \mu) \leq \underline{\text{mdim}}_{\mathcal{M}}(\mathcal{X}, T, d).$$

**Step 2 (Constructing invariant measures encoding dynamical complexity):** For all $d \in \mathcal{D}(\mathcal{X})$

$$\text{mdim}(\mathcal{X}, T) \leq \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).$$

**Step 3 (Constructing nice metrics):** Under the marker property assumption

$$\exists d \in \mathcal{D}(\mathcal{X}): \quad \text{mdim}(\mathcal{X}, T) = \overline{\text{mdim}}_{\mathcal{M}}(\mathcal{X}, T, d).$$

We emphasize that the marker property is used only in this step.

Combining the above three steps, we get Theorem 1.1. Step 1 is easy to prove. (See Section 3.2.) So the main issues are Steps 2 and 3.

**About Step 2:** Let $d$ be a metric on $\mathcal{X}$. In the proof of Step 2, we introduce a new notion called mean Hausdorff dimension (denoted by $\text{mdim}_H(\mathcal{X}, T, d)$). This is a dynamical version of Hausdorff dimension. As is well known in geometric measure theory, Hausdorff dimension is more closely related to measure theory than Minkowski dimension. So it is natural to expect that its dynamical analogue is helpful to connect measure theory to mean dimension.\(^5\) We decompose Step 2 into two smaller steps\(^6\):

\(^5\) The idea of introducing mean Hausdorff dimension was partly motivated by the study of Kawabata–Dembo [KD94, Proposition 3.2]. Roughly speaking, their result [KD94, Proposition 3.2] corresponds to Step 2.2 for $(\mathcal{X}, T) = (A^Z, \text{shift})$ with $A \subset \mathbb{R}^n$. In other words, Step 2.2 is a generalization of their result to arbitrary dynamical systems.

\(^6\) There also exists a small issue about the tame growth of covering numbers condition. But we ignore it here.
Step 2.1 (Mean Hausdorff dimension bounds mean dimension):

$$\text{mdim}(\mathcal{X}, T) \leq \text{mdim}_H(\mathcal{X}, T, d).$$

Step 2.2 (Dynamical analogue of Frostman’s lemma): Under a mild condition on $d$ (the tame growth of covering numbers; see Definition 3.8)

$$\text{mdim}_H(\mathcal{X}, T, d) \leq \sup_{\mu \in \mathcal{M}_T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).$$

Step 2.1 is a dynamical analogue of the fact that Hausdorff dimension bounds topological dimension. Its proof is given in Section 3.2. Step 2.2 is the main part of Step 2. Frostman’s lemma is a classical result in geometric measure theory. Roughly speaking, it claims that we can construct a probability measure which obeys the scaling law corresponding to the Hausdorff dimension. We establish Step 2.2 by combining Frostman’s lemma with the techniques of our previous variational principle (1.7). It roughly goes as follows. For $n \geq 1$ we set $d_n(x, y) = \max_{0 \leq k < n} d(T^k x, T^k y)$. By using the geometric measure theory around Frostman’s lemma, for each $n \geq 1$, we construct a (non-invariant) probability measure $\nu_n$ on $\mathcal{X}$ which captures the geometric complexity of $(\mathcal{X}, d_n)$ over all resolutions. Consider

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k \nu_n.$$

From the compactness we can choose a subsequence $\mu_{n_k}$ which converges to some invariant probability measure (say, $\mu$). We apply to $\mu$ the rate distortion theory version of Misiurewicz’s technique [Mis76] (developed in [LT18]) and prove that $\mu$ captures most of the dynamical complexity of $(\mathcal{X}, T)$ over all resolutions.

About Step 3: Step 3 is technically hard. As we briefly noted in Section 1.2, it was already proved in [Lin99, Theorem 4.3] that if $(\mathcal{X}, T)$ has the marker property then

$$\exists d \in \mathcal{D}(\mathcal{X}): \text{mdim}(\mathcal{X}, T) = \text{mdim}_M(\mathcal{X}, T, d). \quad (1.9)$$

The claim of Step 3 looks very similar. But, in fact, it is much subtler and remained to be an open problem for about 20 years since [Lin99]. It is difficult to briefly explain the ideas of the proof. (See Section 5.1 for more background.) Here we just remark that the above (1.9) (with Steps 1 and 2) are already enough for proving the equality for the lower rate distortion dimension:

$$\text{mdim}(\mathcal{X}, T) = \min_{d \in \mathcal{D}(\mathcal{X})} \sup_{\mu \in \mathcal{M}(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu). \quad (1.10)$$
1.4 Open problems and future directions. The most important open problem is to remove the marker property assumption in Theorem 1.1:

**Problem 1.5.** Prove the double variational principle (1.3) for all dynamical systems.

As we explained in Section 1.3, the marker property is used only in Step 3 of the proof of Theorem 1.1. So Problem 1.5 reduces to

**Problem 1.6.** Prove that for any dynamical system \((X, T)\)

\[
\exists d \in \mathcal{D}(X): \text{mdim}(X, T) = \underline{\text{mdim}_M}(X, T, d).
\]

We emphasize that the same problem for lower metric mean dimension is also open.

Problems 1.5 and 1.6 are certainly the central open problems. But there also exists a different interesting direction. Step 2 of the proof of Theorem 1.1 does not use the marker property assumption. So we always have the inequality

\[
\text{mdim}(X, T) \leq \inf_{d \in \mathcal{D}(X)} \sup_{\mu \in \mathcal{M}^T(X)} \text{rdim}(X, T, d, \mu),
\]

although we don’t know whether the equality holds or not. This implies that we can always find a “sufficiently rich” invariant measure \(\mu\). Study of these measures for concrete examples seems very interesting. We begin such study in Section 6. Although our investigation in this direction has just started, the result in Section 6 seems to suggest a high potential of this research direction. It is desirable to study geometric examples in [Gro99, T18a, T18b] from the viewpoint of the double variational principle.

1.5 Organization of the paper and how to read it. Section 2 is a preparation of basics of mean dimension and rate distortion function. In Section 3 we introduce mean Hausdorff dimension and establish Step 1 and Step 2.1 of the proof of Theorem 1.1. In Section 4 we prepare some basics of geometric measure theory and establish Step 2.2. We establish Step 3 and complete the proof of Theorem 1.1 in Section 5. We study a concrete example in Section 6. Although the result in Section 6 is not used in the proof of Theorem 1.1, hopefully it will help readers to understand various concepts in the paper.

This paper is rather lengthy. We would like to suggest readers how to read it. Section 5 is technically hard. So it may be reasonable to concentrate on Sections 3 and 4 at the first reading. Section 2 is a preparation for these two sections. So, after reading only the main definitions in Section 2 (topological/metric mean dimensions, mutual information and rate distortion function), readers may skip to Section 3 and return to Section 2 when they need the results there. Section 6 might help readers to improve the understanding. So it may be nice to briefly look at it in the midst of reading Sections 3 and 4.
2 Preliminaries

2.1 Topological and metric mean dimensions. We review basics of topological and metric mean dimensions in this subsection [Gro99, LW00]. Throughout this paper we assume that all simplicial complexes are finite (i.e. they have only finitely many faces).

Let \((X,d)\) be a compact metric space. We introduce some metric invariants of \((X,d)\). Take a positive number \(\varepsilon\). Let \(f: X \to Y\) be a continuous map from \(X\) to some topological space \(Y\). The map \(f\) is said to be an \(\varepsilon\)-embedding if \(\text{diam} f^{-1}(y) < \varepsilon\) for every \(y \in Y\). We define the \(\varepsilon\)-width dimension \(\text{Widim}_{\varepsilon}(X,d)\) as the minimum \(n \geq 0\) such that there exists an \(\varepsilon\)-embedding \(f: X \to P\) from \(X\) to some \(n\)-dimensional simplicial complex \(P\). The topological dimension of \(X\) is given by \(\dim X = \lim_{\varepsilon \to 0} \text{Widim}_{\varepsilon}(X,d)\).

We define the \(\varepsilon\)-covering number \(#(X,d,\varepsilon)\) as the minimum \(n \geq 1\) such that there exists an open cover \(\{U_1, \ldots, U_n\}\) of \(X\) satisfying \(\text{diam}\ U_i < \varepsilon\) for all \(1 \leq i \leq n\). We also define the \(\varepsilon\)-separating number \(#\text{sep}(X,d,\varepsilon)\) as the maximum \(n \geq 1\) such that there exist \(x_1, \ldots, x_n \in X\) satisfying \(d(x_i, x_j) \geq \varepsilon\) for all \(i \neq j\). For \(0 < \delta < \varepsilon / 2\)

\[ #\text{sep}(X,d,\varepsilon) \leq #(X,d,\varepsilon) \leq #\text{sep}(X,d,\delta). \] (2.1)

The upper and lower Minkowski dimensions (or box dimensions) of \((X,d)\) are given by

\[ \overline{\text{dim}}_M(X,d) = \limsup_{\varepsilon \to 0} \frac{\log #(X,d,\varepsilon)}{\log(1/\varepsilon)}, \]

\[ \underline{\text{dim}}_M(X,d) = \liminf_{\varepsilon \to 0} \frac{\log #(X,d,\varepsilon)}{\log(1/\varepsilon)}. \]

Example 2.1. Let \((V,\|\cdot\|)\) be a finite dimensional Banach space and \(B_r(V)\) the closed \(r\)-ball around the origin \((r > 0)\). Then for \(0 < \varepsilon < r\)

\[ \text{Widim}_\varepsilon(B_r(V),\|\cdot\|) = \dim V, \] (2.2)

\[ #(B_r(V),\|\cdot\|,\varepsilon) \geq (r/\varepsilon)^\dim V. \] (2.3)

(2.2) is due to Gromov [Gro99, §1.1.2]. See [T09, Appendix] for a simple proof. The proof of (2.3) is easy: Take the Lebesgue measure \(\mu\) on \(V\) normalized by \(\mu(B_r(V)) = 1\). Let \(B_r(V) = U_1 \cup \cdots \cup U_n\) with \(\text{diam}\ U_i < \varepsilon\). Pick \(x_i \in U_i\). Then \(B_r(V) \subset B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n)\) (\(B_\varepsilon(x_i)\) is the closed \(\varepsilon\)-ball centered at \(x_i\)). It follows that

\[ 1 = \mu(B_r(V)) \leq \sum_{i=1}^n \mu(B_\varepsilon(x_i)) = n(\varepsilon/r)^\dim V. \]

This shows \(n \geq (r/\varepsilon)^\dim V\).
Let \((\mathcal{X}, T)\) be a dynamical system with a metric \(d\). For \(N \geq 1\) we define a new metric on \(\mathcal{X}\) by

\[
d_N(x, y) = \max_{0 \leq n < N} d(T^n x, T^n y).
\]

We define the (topological) **mean dimension** by

\[
\text{mdim}(\mathcal{X}, T) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\text{Widim}_\varepsilon(\mathcal{X}, d_N)}{N} \right).
\]

The limit always exists because \(\text{Widim}_\varepsilon(\mathcal{X}, d_N)\) is subadditive in \(N\). The value of \(\text{mdim}(\mathcal{X}, T)\) is independent of the choice of \(d\), namely it becomes a topological invariant of \((\mathcal{X}, T)\). We define the **entropy at the resolution** \(\varepsilon > 0\) by

\[
S(\mathcal{X}, T, d, \varepsilon) = \lim_{N \to \infty} \frac{\log \#(\mathcal{X}, d_N, \varepsilon)}{N},
\]

where the limit exists because \(\log \#(\mathcal{X}, d_N, \varepsilon)\) is subadditive in \(N\). We define the upper and lower metric mean dimensions by (1.5) in Section 1.2.

The following two theorems were proved in [LW00, Theorem 4.2] and [Lin99, Theorem 4.3] respectively.

**Theorem 2.2.**

\[
\text{mdim}(\mathcal{X}, T) \leq \text{mdim}_M(\mathcal{X}, T, d) \leq \overline{\text{mdim}}_M(\mathcal{X}, T, d).
\]

**Theorem 2.3.** If \((\mathcal{X}, T)\) has the marker property then there exists a metric \(d\) on \(\mathcal{X}\) compatible with the topology satisfying

\[
\text{mdim}_M(\mathcal{X}, T, d) = \text{mdim}(\mathcal{X}, T).
\]

**Example 2.4.** Let \(\sigma : [0, 1]^Z \to [0, 1]^Z\) be the shift on the alphabet \([0, 1]\) (the unit interval). We define a metric \(d\) on it by \(d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} |x_n - y_n|\). Then

\[
\text{mdim}([0, 1]^Z, \sigma) = \text{mdim}_M([0, 1]^Z, \sigma, d) = 1.
\]

The only nontrivial point is the lower bound \(\text{mdim}([0, 1]^Z, \sigma) \geq 1\), which follows from (2.2); cf. also [LW00, Proposition 3.3].

### 2.2 Mutual information.

Here we prepare some basics of mutual information [CT06, Chapter 2]. Throughout this subsection we fix a probability space \((\Omega, \mathcal{P})\) and assume that all random variables are defined on it.

Let \(X\) and \(Y\) be two random variables taking values in some measurable spaces \(\mathcal{X}\) and \(\mathcal{Y}\) respectively. We want to define their **mutual information** \(I(X; Y)\), which measures the amount of information shared by both \(X\) and \(Y\). If \(\mathcal{X}\) and \(\mathcal{Y}\) are finite sets,\(^7\) then we set

\[
I(X; Y) = H(X) + H(Y) - H(X, Y) = H(X) - H(X|Y),
\]

\(^7\) We always assume that the \(\sigma\)-algebra of a finite set is the largest one (the set of all subsets).
where \( H(X|Y) \) is the conditional entropy of \( X \) given \( Y \). With the convention that \( 0 \log(0/a) = 0 \) for all \( a \geq 0 \), we can also write this as
\[
I(X; Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P(X = x, Y = y) \log \frac{P(X = x, Y = y)}{P(X = x)P(Y = y)}.
\]

In general we proceed as follows. Take finite measurable partitions \( \mathcal{P} = \{P_1, \ldots, P_M\} \) and \( \mathcal{Q} = \{Q_1, \ldots, Q_N\} \) of \( \mathcal{X} \) and \( \mathcal{Y} \) respectively. For \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \) we set \( \mathcal{P}(x) = P_m \) and \( \mathcal{Q}(y) = Q_n \) where \( x \in P_m \) and \( y \in Q_n \). Then we can consider the mutual information \( I(\mathcal{P} \circ X; \mathcal{Q} \circ Y) \) defined by (2.4) because \( \mathcal{P} \circ X \) and \( \mathcal{Q} \circ Y \) take only finitely many values. We define \( I(X; Y) \) as the supremum of \( I(\mathcal{P} \circ X; \mathcal{Q} \circ Y) \) over all finite measurable partitions \( \mathcal{P} \) and \( \mathcal{Q} \) of \( \mathcal{X} \) and \( \mathcal{Y} \). This definition is compatible with (2.4) when \( \mathcal{X} \) and \( \mathcal{Y} \) are finite sets.\(^8\)

We gather properties of mutual information required in the proof of the double variational principle (Theorem 1.1) below. They are not used in Section 3. So readers may postpone to read the rest of this subsection until they come to Section 4.

**Lemma 2.5** (Data-Processing inequality). Let \( X \) and \( Y \) be random variables taking values in measurable spaces \( \mathcal{X} \) and \( \mathcal{Y} \). If \( f: \mathcal{Y} \to \mathcal{Z} \) is a measurable map, then
\[
I(X; f(Y)) \leq I(X; Y).
\]

*Proof.* This immediately follows from the definition. A nontrivial point is that the above definition is compatible with (2.4) for discrete random variables. \( \square \)

**Lemma 2.6.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be finite sets and \( (X_n, Y_n) \) a sequence of random variables taking values in \( \mathcal{X} \times \mathcal{Y} \). If \( (X_n, Y_n) \) converges to some \( (X, Y) \) in law, then \( I(X_n; Y_n) \) converges to \( I(X; Y) \).

*Proof.* This follows from (2.4). \( \square \)

**Lemma 2.7** (Subadditivity of mutual information). Let \( X, Y, Z \) be random variables taking values in finite sets \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) respectively. Suppose \( X \) and \( Y \) are conditionally independent given \( Z \), namely for every \( x \in \mathcal{X}, y \in \mathcal{Y} \) and \( z \in \mathcal{Z} \) with \( P(Z = z) \neq 0 \) we have
\[
P(X = x, Y = y|Z = z) = P(X = x|Z = z)P(Y = y|Z = z).
\]
Then
\[
I(X, Y; Z) \leq I(X; Z) + I(Y; Z).
\]

*Proof.* \( I(X, Y; Z) = H(X, Y) - H(X, Y|Z). \) From the conditional independence \( H(X, Y|Z) = H(X|Z) + H(Y|Z) \). Hence
\[
I(X, Y; Z) = H(X, Y) - H(X|Z) - H(Y|Z)
\]
\[
\leq H(X) + H(Y) - H(X|Z) - H(Y|Z)
\]
\[
= I(X; Z) + I(Y; Z).
\]
Here we have used \( H(X, Y) \leq H(X) + H(Y) \). \( \square \)

---

\(^8\) We can show this by proving the data-processing inequality (Lemma 2.5) for the quantity defined by (2.4) in the case that \( \mathcal{X} \) and \( \mathcal{Y} \) are finite sets. See [CT06, Section 2.8].
Let $X$ and $Y$ be random variables taking values in finite sets $\mathcal{X}$ and $\mathcal{Y}$. We set $\mu(x) = P(X = x)$ and $\nu(y|x) = P(Y = y|X = x)$ for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. (The conditional probability mass function $\nu(y|x)$ is defined only for $x \in \mathcal{X}$ with $P(X = x) \neq 0$.) The mutual information $I(X; Y)$ is determined by the distribution of $(X, Y)$, which is given by $\mu(x)\nu(y|x)$. It will be convenient for us to write $I(X; Y)$ sometimes as $I(\mu, \nu).

**Lemma 2.8** (Concavity/convexity of mutual information). In the above setting $I(\mu, \nu)$ is a concave function of $\mu(x)$ for fixed $\nu(y|x)$ and a convex function of $\nu(y|x)$ for fixed $\mu(x)$. Namely for $0 \leq t \leq 1$

\[
I \left( (1-t)\mu_1 + t\mu_2, \nu \right) \geq (1-t)I(\mu_1, \nu) + tI(\mu_2, \nu),
\]

\[
I \left( \mu, (1-t)\nu_1 + t\nu_2 \right) \leq (1-t)I(\mu, \nu_1) + tI(\mu, \nu_2).
\]

**Proof.** See [CT06, Theorem 2.7.4] for the detailed proof. First we prove the concavity.

\[
I(\mu, \nu) = I(X; Y) = H(Y) - H(Y|X).
\]

$H(Y)$ is a concave function of $\mu(x)$ for fixed $\nu(y|x)$ (since the Shannon entropy is a concave function of distribution) and $H(Y|X)$ is a linear function of $\mu(x)$. So $I(\mu, \nu)$ is a concave function of $\mu(x)$.

Next we prove the convexity. From the convexity of $\phi(t) := t \log t$

\[
\phi \left( \frac{a + a'}{b + b'} \right) \leq \frac{b}{b + b'} \phi \left( \frac{a}{b} \right) + \frac{b'}{b + b'} \phi \left( \frac{a'}{b'} \right)
\]

for positive $a, a', b, b'$. This leads to the *log sum inequality*:

\[
(a + a') \log \frac{a + a'}{b + b'} \leq a \log \frac{a}{b} + a' \log \frac{a'}{b'}.
\] (2.6)

Set $\sigma_i(y) = \sum_{x \in \mathcal{X}} \mu(x)\nu_i(y|x)$ for $i = 1, 2$.

\[
I \left( \mu, (1-t)\nu_1 + t\nu_2 \right) = \sum_{x, y} \left\{ (1-t)\mu(x)\nu_1(y|x) + t\mu(x)\nu_2(y|x) \right\}
\]

\[
\times \log \frac{(1-t)\mu(x)\nu_1(y|x) + t\mu(x)\nu_2(y|x)}{(1-t)\mu(x)\sigma_1(y) + t\mu(x)\sigma_2(y)}.
\]

Apply (2.6) to each summand: This is bounded by

\[
\sum_{x, y} (1-t)\mu(x)\nu_1(y|x) \log \frac{\mu(x)\nu_1(y|x)}{\mu(x)\sigma_1(y)} + \sum_{x, y} t\mu(x)\nu_2(y|x) \log \frac{\mu(x)\nu_2(y|x)}{\mu(x)\sigma_2(y)},
\]

which is equal to $(1-t)I(\mu, \nu_1) + tI(\mu, \nu_2)$.

$\square$
We borrow the next lemma from [KD94, Lemma A.1]. This is a duality of convex programming. (See Section 2.5 of [B71], specifically [B71, Theorem 2.5.3], for further information.) Recall that the base of the logarithm is two and the natural logarithm is written as ln(·).

**Lemma 2.9.** Let \( X \) and \( Y \) be compact metric spaces and \( \rho: X \times Y \to [0, \infty) \) a continuous function. Let \( \mu \) be a Borel probability measure on \( X \), \( \varepsilon > 0 \) and \( a \geq 0 \) real numbers. Suppose a continuous function \( \lambda: X \to [0, \infty) \) satisfies
\[
\forall y \in Y: \quad \int_X \lambda(x) 2^{-\alpha \rho(x, y)} d\mu(x) \leq 1. \tag{2.7}
\]
If \( X \) and \( Y \) are random variables taking values in \( X \) and \( Y \) respectively and satisfying Law\((X) = \mu \) and \( \mathbb{E} \rho(X, Y) < \varepsilon \) then
\[
I(X; Y) \geq -a\varepsilon + \int_X \log(\lambda(x)) d\mu(x). \tag{2.8}
\]

**Proof.** We divide the proof into two steps.

**Step 1: Assume \( X \) and \( Y \) are finite sets.** Let \( \nu = \text{Law}(Y) \) be the distribution of \( Y \). We define a function \( f(x, y) \) by \( \mathbb{P}(X = x, Y = y) = f(x, y) \mathbb{P}(X = x) \mathbb{P}(Y = y) \). (We do not need to define the value \( f(x, y) \) if \( \mathbb{P}(X = x) \mathbb{P}(Y = y) = 0 \).) It follows from (2.5) that
\[
I(X; Y) = \sum_{x \in X, y \in Y} \mathbb{P}(X = x) \mathbb{P}(Y = y) f(x, y) \log f(x, y) = \int_{X \times Y} f(x, y) \log f(x, y) d\mu(x) d\nu(y).
\]
Set \( g(x, y) = \lambda(x) 2^{-\alpha \rho(x, y)} \). The right-hand side of (2.8) is equal to
\[
- a\varepsilon + \int_{X \times Y} \log(\lambda) d\text{Law}(X, Y) = -a\varepsilon + \int_{X \times Y} f(x, y) \log(\lambda(x)) d\mu(x) d\nu(y).
\]
Since \( -\varepsilon < -\mathbb{E} \rho(X, Y) = - \int_{X \times Y} \rho(x, y) f(x, y) d\mu(x) d\nu(y) \), this is less than
\[
\int_{X \times Y} f(x, y) \log g(x, y) d\mu(x) d\nu(y).
\]

\[
I(X; Y) = \int_{X \times Y} f(x, y) \log g(x, y) d\mu(x) d\nu(y) - \int_{X \times Y} f(x, y) \log(g(x, y)) d\mu(x) d\nu(y) = \int_{X \times Y} f \log(f/g) d\mu d\nu.
\]
As \( \ln 2 \cdot \log(1/u) = \ln(1/u) \geq 1 - u \), we have \( \ln 2 \cdot f \log(f/g) \geq f(1 - g/f) = f - g \) and hence
\[
\ln 2 \cdot \int_{X \times Y} f \log(f/g) d\mu d\nu \geq \int_{X \times Y} (f - g) d\mu d\nu
\]
\[
= 1 - \int_Y \left( \int_X g(x, y) d\mu(x) \right) d\nu(y) \geq 0.
\]

\footnote{The continuity of \( \rho \) and \( \lambda \) is inessential. But we assume it for simplicity. Indeed in our applications, \( X = Y \), \( \rho \) is a distance function and \( \lambda \) is a constant.}
Here we have used the assumption (2.7) in the last inequality.

**Step 2: General case.** Let $\delta > 0$. Take finite partitions $P = \{P_1, \ldots, P_M\}$ and $Q = \{Q_1, \ldots, Q_N\}$ of $X$ and $Y$ respectively. For each $P_m$ we take a point $x_m \in P_m$ satisfying $\lambda(x_m) \geq (1 + \delta)^{-1} \sup_{P_m} \lambda$. We pick arbitrary $y_n \in Q_n$ for each $Q_n$. We set $X' = \{x_1, \ldots, x_M\}$ and $Y' = \{y_1, \ldots, y_N\}$ and define maps $P: X' \to X'$ and $Q: Y' \to Y'$ by $P(P_m) = \{x_m\}$ and $Q(Q_n) = \{y_n\}$. Set $X' = P \circ X, Y' = Q \circ Y$ and $\mu' = P_* \mu = \text{Law}(X')$.

From the continuity of $\rho$ and $\lambda$, by taking $P$ and $Q$ sufficiently fine, we can assume $E\rho(X', Y') < \varepsilon$ and

$$\forall y \in Y', \int_{X'} \lambda(x) 2^{-a \rho(x, y)} d\mu'(x) \leq 1 + \delta.$$  

Then we can apply Step 1 to $X', Y'$ and the function $\lambda'(x_m) := (1 + \delta)^{-1}\lambda(x_m)$. This yields

$$I(X'; Y') \geq -a\varepsilon + \int_{X'} \log \lambda' d\mu' = -a\varepsilon - \log(1 + \delta) + \int_{X'} \log \lambda d\mu'.$$

It follows from the choice of $x_m$ that

$$\int_{X'} \log \lambda' d\mu' \geq -\log(1 + \delta) + \int_{X'} \log \lambda d\mu.$$  

As $I(X; Y) \geq I(X'; Y')$ by the definition of mutual information,

$$I(X; Y) \geq -a\varepsilon - 2\log(1 + \delta) + \int_{X'} \log \lambda d\mu.$$

Let $\delta \to 0$. This shows the statement. $\square$

The next lemma is essentially due to [KD94, Proposition 3.2]. This is a key to connect geometric measure theory to rate distortion theory.

**Lemma 2.10.** Let $\varepsilon$ and $\delta$ be positive numbers with $2\varepsilon \log(1/\varepsilon) \leq \delta$. Let $0 \leq \tau \leq \min(\varepsilon/3, \delta/2)$ and $s \geq 0$ be real numbers. Let $(X, d)$ be a compact metric space with a Borel probability measure $\mu$ satisfying

$$\mu(E) \leq (\tau + \text{diam} E)^s, \quad \forall E \subset X \text{ with diam} E < \delta. \quad (2.9)$$

Let $X$ and $Y$ be random variables taking values in $X$ with $\text{Law}(X) = \mu$ and $\mathbb{E}d(X, Y) < \varepsilon$. Then

$$I(X; Y) \geq s \log(1/\varepsilon) - C(s + 1),$$

where $C$ is a universal positive constant independent of $\varepsilon, \delta, \tau, s, (X, d), \mu$. 

Proof. We apply Lemma 2.9 with \( a = s/\varepsilon \). Set \( b = a \ln 2 \) and estimate
\[
\int_X e^{-bd(x,y)} d\mu(x) = \int_X e^{-bd(x,y)} d\mu(x) \text{ for each } y \in X:
\]
\[
\int_X e^{-bd(x,y)} d\mu(x) = \int_0^1 \mu\{ x \mid e^{-bd(x,y)} \geq u \} du
\]
\[
= \int_0^{\infty} \mu\{ x \mid d(x,y) \leq v \} be^{-bv} dv \quad \text{(set } u = e^{-bv})
\]
\[
= \left( \int_0^\tau + \int_{\tau}^{\delta/2} + \int_{\delta/2}^{\infty} \right) \mu\{ x \mid d(x,y) \leq v \} be^{-bv} dv.
\]
In the last line we have used \( \tau \leq \delta/2 \). From (2.9)
\[
\int_0^{\tau} \mu\{ x \mid d(x,y) \leq v \} be^{-bv} dv \leq (3\tau)^s \int_0^\infty be^{-bv} dv = (3\tau)^s \leq \varepsilon^s,
\]
where we have used \( \tau \leq \varepsilon/3 \).
\[
\int_{\tau}^{\delta/2} \mu\{ x \mid d(x,y) \leq v \} be^{-bv} dv \leq \int_{\tau}^{\delta/2} (\tau + 2v)^s be^{-bv} dv
\]
\[
\leq 3^s \int_{\tau}^{\delta/2} v^s be^{-bv} dv
\]
\[
\leq (3/b)^s \int_0^\infty t^s e^{-t} dt \quad \text{(set } t = bv)
\]
\[
= \varepsilon^s (3 \log e)^s s^{-s} \Gamma(s+1).
\]
In the last step we have used \( b = s \ln 2/\varepsilon = s/ (\varepsilon \log e) \).
\[
\int_{\delta/2}^\infty \mu\{ x \mid d(x,y) \leq v \} be^{-bv} dv \leq \int_{\delta/2}^\infty be^{-bv} dv
\]
\[
= e^{-b\delta/2} = \left( 2^{-\delta/(2\varepsilon)} \right)^s \leq \varepsilon^s.
\]
In the last inequality we have used \( 2\varepsilon \log(1/\varepsilon) \leq \delta \). Summing the above estimates, we get
\[
\int_X 2^{-ad(x,y)} d\mu(x) \leq \varepsilon^s \left\{ 2 + (3 \log e)^s s^{-s} \Gamma(s+1) \right\}.
\]
Thus the constant function \( \lambda(x) := \varepsilon^{-s} \left\{ 2 + (3 \log e)^s s^{-s} \Gamma(s+1) \right\}^{-1} \) satisfies
\[
\forall y \in X: \int_X \lambda(x) 2^{-ad(x,y)} d\mu(x) \leq 1.
\]
From Lemma 2.9
\[
I(X;Y) \geq -a\varepsilon + \int_X \log \lambda d\mu
\]
\[
= s \log(1/\varepsilon) - s - \log \left\{ 2 + (3 \log e)^s s^{-s} \Gamma(s+1) \right\}.
\]
Recalling Stirling’s formula $\Gamma(s + 1) \sim s^s e^{-s} \sqrt{2\pi s}$, we can find a universal constant $C > 0$ satisfying

$$s + \log \left\{ 2 + (3 \log e)^s s^{-s} \Gamma(s + 1) \right\} \leq C + Cs.$$  

This proves the statement. \(\Box\)

### 2.3 Rate distortion theory

Here we review rate distortion function ([Sh48, Sh59], [CT06, Chapter 10]). The Shannon entropy is the fundamental limit in lossless data compression of discrete random variables and processes. For a stationary stochastic process $X_1, X_2, \ldots$, its entropy is equal to the minimum expected number of bits per symbol for describing the process. But if random variables $X_n$ take continuously many values, the entropy is simply infinite (namely, we cannot describe continuous variables perfectly within finitely many bits). For continuous random variables and processes (e.g. audio signals, images, etc.) we have to consider lossy data compression method\(^{10}\) achieving some distortion constraint. This is the primary object of rate distortion theory. Rate distortion function is the fundamental limit of data compression in this context.

Let $(\mathcal{X}, T)$ be a dynamical system with a metric $d$ and an invariant probability measure $\mu$. For $\varepsilon > 0$ we define the rate distortion function $R(d, \mu, \varepsilon)$ as the infimum of

$$\frac{I(X; Y)}{N},$$

where $N > 0$ is a natural number, $X$ and $Y = (Y_0, \ldots, Y_{N-1})$ are random variables defined on some probability space $(\Omega, \mathbb{P})$ such that all $X$ and $Y_n$ take values in $\mathcal{X}$ and satisfy

$$\text{Law}(X) = \mu, \quad \mathbb{E} \left( \frac{1}{N} \sum_{n=0}^{N-1} d(T^n X, Y_n) \right) < \varepsilon.$$  

We define the lower and upper rate distortion dimensions by (1.2) in Section 1.1.

**Remark 2.11.** In the above definition of rate distortion function we can assume that $Y$ takes only finitely many values, namely its distribution is supported on a finite set: Take a finite partition $\mathcal{P}$ of $\mathcal{X}$ and pick a point $x_P \in P$ for each $P \in \mathcal{P}$. Define $f: \mathcal{X} \to \mathcal{X}$ by $f(P) = \{x_P\}$ for $P \in \mathcal{P}$ and set $Z = (Z_0, \ldots, Z_{N-1}) = (f(Y_0), \ldots, f(Y_{N-1}))$. If $\mathcal{P}$ is sufficiently fine then

$$\mathbb{E} \left( \frac{1}{N} \sum_{n=0}^{N-1} d(T^n X, Z_n) \right) < \varepsilon.$$  

\(^{10}\) E.g. expanding signals in a wavelet basis, discarding small terms and quantizing the remaining terms.
From the definition of mutual information (or the data-processing inequality; Lemma 2.5),

\[ I(X; Z) \leq I(X; Y). \]

The random variable \( Z \) takes only finitely many values.

The rate distortion function \( R(d, \mu, \varepsilon) \) is the minimum rate when we try to quantize the process \( \{T^n X\}_{n \in \mathbb{Z}} \) within the average distortion bound by \( \varepsilon \) [Gra90, Chapter 11]: For simplicity,\(^{11}\) suppose \( \mu \) is ergodic. For any \( \delta > 0 \), if \( N \) is sufficiently large, there exists a map \( f = (f_0, \ldots, f_{N-1}) : X \to X^N \) which has a finite range (i.e. it takes only finitely many values) and satisfies

\[
\frac{\log |f(X)|}{N} < R(d, \mu, \varepsilon) + \delta, \quad \mathbb{E} \left( \frac{1}{N} \sum_{n=0}^{N-1} d(T^n X, f_n(X)) \right) < \varepsilon.
\]

Namely we can approximate the process \( X, TX, \ldots, T^{N-1} X \) by the quantization \( f_0(X), f_1(X), \ldots, f_{N-1}(X) \) within the average distortion bound by \( \varepsilon \). The bits per iterate required for this description is less than \( R(d, \mu, \varepsilon) + \delta \).

**Example 2.12.** Consider the shift \( \sigma : [0,1]^\mathbb{Z} \to [0,1]^\mathbb{Z} \) with a metric \( d(x,y) = \sum_{n \in \mathbb{Z}} 2^{-|n|}|x_n - y_n| \) and an invariant probability measure \( \mu = (\text{Lebesgue measure})^{\otimes \mathbb{Z}}. \)

Then [LT18, Example 22]

\[ \text{rdim} ([0,1]^\mathbb{Z}, \sigma, d, \mu) = 1. \]

### 3 Mean Hausdorff Dimension and the Proof of the Double Variational Principle

In this section we introduce the key concept of the paper—**mean Hausdorff dimension**. We develop various comparison estimates between topological/metric mean dimensions, mean Hausdorff dimension and rate distortion dimension. Some of the proofs are postponed to later sections. We prove the double variational principle (Theorem 1.1) by using these comparison estimates at the end of Section 3.2.

#### 3.1 Definition of mean Hausdorff dimension.

Let \( (\mathcal{X}, d) \) be a compact metric space. For \( s \geq 0 \) and \( \varepsilon > 0 \) we define \( \mathcal{H}^s_\varepsilon(\mathcal{X}, d) \) as

\[
\inf \left\{ \sum_{n=1}^{\infty} (\text{diam } E_n)^s \left| \mathcal{X} = \bigcup_{n=1}^{\infty} E_n \right. \text{ with diam } E_n < \varepsilon \text{ for all } n \geq 1 \right\}.
\]

\(^{11}\) Although the “operational meaning” of rate distortion function is important for the understanding, we do not use it in the paper. So we do not give a complete explanation. See [LDN79, ECG94, Gra90] for the non-ergodic case.
Here we use the convention that $0^0 = 1$ and $\text{diam}(\emptyset)^s = 0$. We also define

$$\mathcal{H}^s_\infty(\mathcal{X}, d) = \inf \left\{ \sum_{n=1}^{\infty} (\text{diam}E_n)^s \left| \mathcal{X} = \bigcup_{n=1}^{\infty} E_n \right. \right\}.$$

We set

$$\dim_H(\mathcal{X}, d, \varepsilon) = \sup \{ s \geq 0 | \mathcal{H}^s_\varepsilon(\mathcal{X}, d) \geq 1 \}.$$

The **Hausdorff dimension** $\dim_H(\mathcal{X}, d)$ is given by

$$\dim_H(\mathcal{X}, d) = \lim_{\varepsilon \to 0} \dim_H(\mathcal{X}, d, \varepsilon).$$

Let $(\mathcal{X}, T)$ be a dynamical system with a metric $d$. As in Section 2.1 we set $d_N(x, y) = \max_{0 \leq n < N} d(T^nx, T^ny)$. We define the **mean Hausdorff dimension** by

$$\text{mdim}_H(\mathcal{X}, T, d) = \lim_{\varepsilon \to 0} \left( \limsup_{N \to \infty} \frac{1}{N} \dim_H(\mathcal{X}, d_N, \varepsilon) \right). \quad (3.1)$$

**Remark 3.1.** We can also define the **lower mean Hausdorff dimension** $\text{mdim}_H(\mathcal{X}, T, d)$ by replacing $\limsup_N$ in (3.1) with $\liminf_N$. But we do not seriously use this concept in the paper.

### 3.2 Comparison between various dynamical dimensions.

The following proposition extends Theorem 2.2 to mean Hausdorff dimension and rate distortion dimension.

**Proposition 3.2.** Let $(\mathcal{X}, T)$ be a dynamical system with a metric $d$ and an invariant probability measure $\mu$.

$$\text{mdim}(\mathcal{X}, T, d) \leq \text{mdim}_H(\mathcal{X}, T, d) \leq \text{mdim}_M(\mathcal{X}, T, d). \quad (3.2)$$

$$\text{rdim}(\mathcal{X}, T, d, \mu) \leq \text{mdim}_M(\mathcal{X}, T, d),$$

$$\text{rdim}(\mathcal{X}, T, d, \mu) \leq \text{mdim}_M(\mathcal{X}, T, d). \quad (3.3)$$

**Proof.** The nontrivial result is only $\text{mdim}(\mathcal{X}, T) \leq \text{mdim}_H(\mathcal{X}, T, d)$. The rest of the statement is easy. We first prove easy estimates. Let $N \geq 1$ and $\varepsilon > 0$. Consider an open cover $\mathcal{X} = U_1 \cup \cdots \cup U_n$ with $\text{diam}(U_i, d_N) < \varepsilon$ and $n = \#(\mathcal{X}, d_N, \varepsilon)$.

We have $\mathcal{H}_\varepsilon^s(\mathcal{X}, d_N) \leq n\varepsilon^s$. If $s > \log n / \log(1/\varepsilon)$ then $\mathcal{H}^s(\mathcal{X}, d_N) < 1$. This shows

$$\dim_H(\mathcal{X}, d_N, \varepsilon) \leq \frac{\log \#(\mathcal{X}, d_N, \varepsilon)}{\log(1/\varepsilon)}.$$

Divide this by $N$ and take limits with respect to $N$ and then $\varepsilon$. It follows that $\text{mdim}_H(\mathcal{X}, T, d) \leq \text{mdim}_M(\mathcal{X}, T, d)$. 


Next we consider (3.3). Let \( X \) be a random variable obeying \( \mu \). Choose a point \( x_i \) from each \( U_i \). We define \( f: \mathcal{X} \to \{x_1, \ldots, x_n\} \) by \( f(x) = x_i \) where \( i \) is the smallest number with \( x \in U_i \). Set \( Y = (f(X), Tf(X), \ldots, T^{N-1}f(X)) \). Since \( d(T^kx, T^kf(x)) < \varepsilon \) for all \( x \in \mathcal{X} \) and \( 0 \leq k < N \),

\[
\frac{1}{N} \sum_{k=0}^{N-1} d(T^kX, T^kf(X)) < \varepsilon.
\]

Since \( Y \) takes at most \( n = \#(\mathcal{X}, d_{N}, \varepsilon) \) values,

\[
I(X; Y) \leq H(Y) \leq \log n.
\]

This shows

\[
R(d, \mu, \varepsilon) \leq \frac{\log \#(\mathcal{X}, d_{N}, \varepsilon)}{N}.
\]

Letting \( N \to \infty \), we get \( R(d, \mu, \varepsilon) \leq S(\mathcal{X}, T, d, \varepsilon) \). Divide this by \( \log(1/\varepsilon) \) and take the upper/lower limits with respect to \( \varepsilon \). This proves (3.3).

Now we come to the main point; the comparison between mean dimension and mean Hausdorff dimension. We use the idea of the proof of Theorem 2.2 (comparison between topological/metric mean dimensions). We need some preliminary claims. In the sequel we denote by \( \nu_N \) and \( ||\cdot||_\infty \) the standard Lebesgue measure and \( \ell^\infty \)-norm on \( \mathbb{R}^N \). For \( A \subset \{1, 2, \ldots, N\} \) we denote by \( \pi_A: \mathbb{R}^N \to \mathbb{R}^A \) the projection to the \( A \)-coordinates. For \( 0 \leq n \leq N \) we define \( P_n \) as the \( n \)-skeleton of the cube \([0,1]^N\), i.e. the set of \( x \in [0,1]^N \) satisfying \( |\{k|x_k = 0 \text{ or } 1\}| \geq N - n \).

**Claim 3.3.** Let \( K \subset [0,1]^N \) be a closed subset and \( 1 \leq n \leq N \).

1. \( \nu_N(K) \leq 2^N H^n_\infty (K, ||\cdot||_\infty) \).
2. \( \nu_N \left( \bigcup_{|A| \geq n} \pi_A^{-1}(\pi_A K) \right) \leq 4^N H^n_\infty (K, ||\cdot||_\infty) \).
3. If \( H^{n+1}_\infty (K, ||\cdot||_\infty) < 4^{-N} \) then there exists a 1-embedding \( f: K \to P_n \), i.e. a continuous map satisfying \( f(x) \neq f(y) \) for any \( x, y \in K \) with \( ||x - y||_\infty = 1 \).

**Proof.** (1) Let \( K = \bigcup_{k \geq 1} E_k \) and set \( l_k = \text{diam}(E_k, ||\cdot||_\infty) \). Take \( x_k \in E_k \). Since \( E_k \subset x_k + [-l_k, l_k]^N \),

\[
\nu_N(K) \leq \sum_{k=1}^{\infty} (2l_k)^N = 2^N \sum_{k=1}^{\infty} l_k^N.
\]

(2) \( \nu_N \left( \bigcup_{|A| \geq n} \pi_A^{-1}(\pi_A K) \right) \) is bounded by

\[
\sum_{|A| \geq n} \nu_N (\pi_A^{-1}(\pi_A K)) = \sum_{|A| \geq n} \nu_{|A|}(\pi_A K).
\]
Apply the above (1) to $\pi_A K \subset [0, 1]^A$:

$$
\nu_{|A|} (\pi_A K) \leq 2^{|A|} \mathcal{H}^{|A|}_\infty (\pi_A K, \| \cdot \|_\infty) \leq 2^N \mathcal{H}^{|A|}_\infty (\pi_A K, \| \cdot \|_\infty).
$$

Since $\pi_A$ is one-Lipschitz, $\mathcal{H}^{|A|}_\infty (\pi_A K, \| \cdot \|_\infty) \leq \mathcal{H}^{|A|}_\infty (K, \| \cdot \|_\infty)$. Thus

$$
\nu_n \left( \bigcup_{|A| \geq n+1} \pi_A^{-1} (\pi_A K) \right) \leq 2^N \sum_{|A| \geq n} \mathcal{H}^{|A|}_\infty (K, \| \cdot \|_\infty)
$$

$$
\leq 2^N \sum_{|A| \geq n} \mathcal{H}^n_\infty (K, \| \cdot \|_\infty)
$$

$$
\leq 4^N \mathcal{H}^n_\infty (K, \| \cdot \|_\infty).
$$

(3) If $n = N$ then the statement is trivial. So we assume $n < N$. It follows from the above (2) that

$$
\nu_n \left( \bigcup_{|A| \geq n+1} \pi_A^{-1} (\pi_A K) \right) < 1.
$$

In particular we can find $q \in (0, 1)^N$ outside of $\bigcup_{|A| \geq n+1} \pi_A^{-1} (\pi_A K)$. For $1 \leq m \leq N$ we set

$$
C_m = P_m \cap \bigcup_{|A| = m} \pi_A^{-1} (\pi_A (q)).
$$

This is a finite set. (Each facet of $P_m$ contains exactly one point of $C_m$.) By using the central projection from each point of $C_m$, we define a continuous map $g_m : P_m \setminus C_m \to P_{m-1}$. This map has the following properties:

- $\| g_m (x) - g_m (y) \|_\infty = 1$ for $x, y \in P_m \setminus C_m$ with $\| x - y \|_\infty = 1$.
- For $1 \leq l < m$

$$
g_m \left( \bigcup_{|A| = l} \pi_A^{-1} (\pi_A (q)) \right) = P_{m-1} \setminus \bigcup_{|A| = l} \pi_A^{-1} (\pi_A (q)).
$$

Since $K \cap \bigcup_{|A| \geq n+1+1} \pi_A^{-1} (\pi_A (q)) = \emptyset$, we can define $f = g_{n+1} \circ g_{n+2} \circ \cdots \circ g_N : K \to P_n$. If $x, y \in K$ satisfy $\| x - y \|_\infty = 1$ then $\| f(x) - f(y) \|_\infty = 1$. In particular $f$ is a 1-embedding.

**Claim 3.4.** Let $N$ be a positive integer and $\varepsilon, \delta, s, \tau, L$ positive numbers with $4^N L^{s+\tau} < 1$. Let $(K, d)$ be a compact metric space with $\text{dim}_H (K, d, \delta) < s$. Suppose there exists an $L$-Lipschitz map $\varphi : (K, d) \to ([0, 1]^N, \| \cdot \|_\infty)$ such that if $x, y \in K$ satisfy $\| \varphi(x) - \varphi(y) \|_\infty < 1$ then $d(x, y) < \varepsilon$. Then $\text{Widim}_\varepsilon (K, d) \leq s + \tau$. 

\[\square\]
Proof. It follows from $\dim_H(K, d, \delta) < s$ that there exists a covering $K = \bigcup_{n=1}^{\infty} E_n$ satisfying $\operatorname{diam} E_n < \delta$ and $\sum_{n=1}^{\infty}(\operatorname{diam} E_n)^s < 1$. Then

$$\mathcal{H}_{\infty}^{s+\tau}(K, d) \leq \sum_{n=1}^{\infty}(\operatorname{diam} E_n)^{s+\tau} < \delta^\tau.$$  

Since $\varphi$ is $L$-Lipschitz,

$$\mathcal{H}_{\infty}^{s+\tau}(\varphi(K), \|\cdot\|_\infty) < L^{s+\tau}\delta^\tau < 4^{-N}. $$

Hence $\mathcal{H}_{\infty}^{[s+\tau]+1}(\varphi(K), \|\cdot\|_\infty) < 4^{-N}$. Apply Claim 3.3 (3) to $\varphi(K)$: There exists a 1-embedding $f: \varphi(K) \to P_{[s+\tau]}$. Then $f \circ \varphi: K \to P_{[s+\tau]}$ becomes an $\varepsilon$-embedding. The skeleton $P_{[s+\tau]}$ admits a structure of a $[s + \tau]$-dimensional simplicial complex. So $\operatorname{Widim}_\varepsilon(K, d) \leq [s + \tau]$.

We start the proof of $\operatorname{mdim}(\mathcal{X}, T) \leq \operatorname{mdim}_H(\mathcal{X}, T, d)$. We can assume $\operatorname{mdim}_H(\mathcal{X}, T, d) < \infty$. We take $\tau > 0$ and $s > \operatorname{mdim}_H(\mathcal{X}, T, d)$. Let $\varepsilon > 0$.

**Claim 3.5.** There exist a positive number $L$, a positive integer $M$ and an $L$-Lipschitz map $\varphi: (\mathcal{X}, d) \to ([0,1]^{M}, \|\cdot\|_\infty)$ such that if $x, y \in \mathcal{X}$ satisfy $\|\varphi(x) - \varphi(y)\|_\infty < 1$ then $d(x, y) < \varepsilon$.

**Proof.** Choose a Lipschitz function $\psi: \mathbb{R} \to [0,1]$ satisfying $\psi(t) = 1$ for $t \leq \varepsilon/4$ and $\psi(t) = 0$ for $t \geq \varepsilon/2$. Take an $\varepsilon/4$-spanning subset $\{x_1, \ldots, x_M\} \subset \mathcal{X}$, i.e. so that for any $x \in \mathcal{X}$ there exists $x_i$ with $d(x, x_i) < \varepsilon/4$. We define $\varphi: \mathcal{X} \to [0,1]^{M}$ by

$$\varphi(x) = (\psi(d(x, x_1)), \ldots, \psi(d(x, x_M))).$$

For $N \geq 1$ we define an $L$-Lipschitz map

$$\varphi_N: (\mathcal{X}, d_N) \to ([0,1]^{M})^N,$$

by $\varphi_N(x) = (\varphi(x), \varphi(Tx), \ldots, \varphi(T^{N-1}x))$. This has the property that if $x, y \in \mathcal{X}$ satisfy $\|\varphi_N(x) - \varphi_N(y)\|_\infty < 1$ then $d_N(x, y) < \varepsilon$.

Choose a sufficiently small $\delta > 0$ satisfying $4^M L^{s+\tau}\delta^\tau < 1$. It follows from $\operatorname{mdim}_H(\mathcal{X}, T, d) < s$ that there exists $N_1 < N_2 < N_3 < \cdots \to \infty$ satisfying $\dim_H(\mathcal{X}, d_{N_k}, \delta) < s N_k$. Note

$$4^{MN_k} L^{sN_k + \tau N_k} \delta^\tau N_k = (4^M L^{s+\tau}\delta^\tau)^{N_k} < 1.$$  

Then we can apply Claim 3.4 to the space $(\mathcal{X}, d_{N_k})$ and the map $\varphi_{N_k}$ with the parameters $MN_k, \varepsilon, \delta, sN_k, \tau N_k, L$. This provides

$$\operatorname{Widim}_\varepsilon(\mathcal{X}, d_{N_k}) \leq sN_k + \tau N_k,$$







12 Indeed here we use only $\operatorname{mdim}_H(\mathcal{X}, T, d) < s$. 


and hence
\[ \lim_{N \to \infty} \frac{1}{N} \text{Widim}_\varepsilon(\mathcal{X}, d_N) \leq s \tau. \]
The right-hand side is independent of $\varepsilon$. Thus $\text{mdim}(\mathcal{X}, T) \leq s + \tau$. Let $\tau \to 0$ and $s \to \text{mdim}_H(\mathcal{X}, T, d)$. This proves the statement. \qed

**Remark 3.6.** The above proof actually shows
\[ \text{mdim}(\mathcal{X}, T) \leq \text{mdim}_H(\mathcal{X}, T, d), \]
where the right-hand side is the lower mean Hausdorff dimension (Remark 3.1).

**Example 3.7.** Let $\sigma: [0, 1]^\mathbb{Z} \to [0, 1]^\mathbb{Z}$ be the shift on the alphabet $[0, 1]$ with a metric $d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|}|x_n - y_n|$ as in Example 2.4. Then $\text{mdim}_H([0, 1]^\mathbb{Z}, \sigma, d) = 1$ because
\[ 1 = \text{mdim}([0, 1]^\mathbb{Z}, \sigma) \leq \text{mdim}_H([0, 1]^\mathbb{Z}, \sigma, d) \leq \text{mdim}_M([0, 1]^\mathbb{Z}, \sigma, d) = 1. \]

The next two theorems are the most crucial ingredients of the proof of the double variational principle. Their proofs are postponed to later sections. Before stating the results we need to introduce a concept expressing some regularity of metrics:

**Definition 3.8.** Let $(\mathcal{X}, d)$ be a compact metric space. It is said to have the **tame growth of covering numbers** if for every $\delta > 0$
\[ \lim_{\varepsilon \to 0} \varepsilon^\delta \log \#(\mathcal{X}, d, \varepsilon) = 0. \]
Notice that this is purely a condition on metrics and does not involve dynamics.

**Example 3.9.** (1) If $\mathcal{X}$ is a compact subset of a finite dimensional Banach space $(V, \|\|)$, then $(\mathcal{X}, \|\|)$ has the tame growth of covering numbers because
\[ \#(\mathcal{X}, \|\|, \varepsilon) = O(\varepsilon^{-\dim V}). \]
(2) If a compact metric space $(K, \rho)$ has the tame growth of covering numbers, then the following metric $d$ on the shift space $K^\mathbb{Z}$ also has the tame growth of covering numbers:
\[ d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|}\rho(x_n, y_n). \]
(3) It follows from (1) and (2) that the metric $d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|}|x_n - y_n|$ on $[0, 1]^\mathbb{Z}$ has the tame growth of covering numbers.

The next lemma shows that the tame growth of covering numbers is a fairly mild condition.
**Lemma 3.10.** Let \((\mathcal{X}, d)\) be a compact metric space. There exists a metric \(d'\) on \(\mathcal{X}\) such that \(d'(x, y) \leq d(x, y)\) and \((\mathcal{X}, d')\) has the tame growth of covering numbers. In particular every compact metrizable space admits a metric having the tame growth of covering numbers.

**Proof.** We can assume \(\text{diam}(\mathcal{X}, d) \leq 1\). Let \(K = [0, 1]^{\mathbb{N}}\) be the one-sided infinite product of the unit interval. We define a metric \(\rho\) on it by \(\rho(x, y) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|\).

As in Example 3.9, \((K, \rho)\) has the tame growth of covering numbers. Take a countable dense subset \(\{x_i\}_{i=1}^{\infty} \subset \mathcal{X}\). We define \(f: \mathcal{X} \to K\) by \(f(x) = \left(d(x, x_i)\right)_{i=1}^{\infty}\). \(f\) is a topological embedding and it is one-Lipschitz:

\[
\rho(f(x), f(y)) = \sum_{i=1}^{\infty} 2^{-i} |d(x, x_i) - d(y, x_i)| \leq \sum_{i=1}^{\infty} 2^{-i} d(x, y) = d(x, y).
\]

The metric \(d'(x, y) := \rho(f(x), f(y))\) satisfies the requirements. \(\square\)

Recall that we have denoted by \(\mathcal{M}^T(\mathcal{X})\) the set of all \(T\)-invariant Borel probability measures on \(\mathcal{X}\).

**Theorem 3.11** (Existence of nice measures). Let \((\mathcal{X}, T)\) be a dynamical system with a metric \(d\) such that \((\mathcal{X}, d)\) has the tame growth of covering numbers. Then

\[
\text{mdim}_H(\mathcal{X}, T, d) \leq \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).
\]

**Theorem 3.12** (Existence of nice metrics). If a dynamical system \((\mathcal{X}, T)\) has the marker property, then there exists a metric \(d\) on \(\mathcal{X}\) compatible with the topology such that

\[
\text{mdim}_M(\mathcal{X}, T, d) = \text{mdim}(\mathcal{X}, T).
\]

The inequalities \(\text{mdim}(\mathcal{X}, T) \leq \text{mdim}_M(\mathcal{X}, T, d) \leq \text{mdim}(\mathcal{X}, T, d)\) always hold true. So \(\text{mdim}(\mathcal{X}, T) = \text{mdim}_M(\mathcal{X}, T, d)\) implies that \(\text{mdim}_M(\mathcal{X}, T, d)\) exists and is equal to \(\text{mdim}(\mathcal{X}, T)\).

**Corollary 3.13.** Let \((\mathcal{X}, T)\) be a dynamical system with a metric \(d\). Then

\[
\text{mdim}(\mathcal{X}, T) \leq \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).
\]

**Proof.** Notice that if \(d\) has the tame growth of covering numbers then the statement immediately follows from Proposition 3.2 and Theorem 3.11. Hence the problem is how to reduce the general case to this case.

Let \(d'\) be a metric given by Lemma 3.10. It has the tame growth of covering numbers. So for any \(\varepsilon > 0\) there exists an invariant probability measure \(\mu\) on \(\mathcal{X}\) satisfying

\[
\text{mdim}(\mathcal{X}, \sigma) \leq \text{rdim}(\mathcal{X}, T, d', \mu) + \varepsilon.
\]
Since \( d' \leq d \),
\[
\text{rdim}(\mathcal{X}, T, d', \mu) \leq \text{rdim}(\mathcal{X}, T, d, \mu).
\]

Since \( \varepsilon > 0 \) is arbitrary, this proves the claim. \( \square \)

**Corollary 3.14.** If a dynamical system \((\mathcal{X}, T)\) has the marker property, then there exists a metric \( d \) on \( \mathcal{X} \) such that all the following quantities are equal to each other:

\[
\begin{align*}
\text{mdim}(\mathcal{X}, T), & \quad \text{mdim}_H(\mathcal{X}, T, d), \\
\text{mdim}_M(\mathcal{X}, T, d), & \quad \text{mdim}_M(\mathcal{X}, T, d), \\
\sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu), & \quad \sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).
\end{align*}
\]  

(3.4)

**Proof.** All the quantities in (3.4) are bounded between \( \text{mdim}(\mathcal{X}, T) \) and \( \text{mdim}_M(\mathcal{X}, T, d) \) by Proposition 3.2 and Corollary 3.13. Take a metric \( d \) given in Theorem 3.12. Then \( \text{mdim}(\mathcal{X}, T) = \text{mdim}_M(\mathcal{X}, T, d) \) and hence all the quantities in (3.4) coincide with each other. \( \square \)

Now we can prove the double variational principle (Theorem 1.1).

**Proof of Theorem 1.1.** Let \((\mathcal{X}, T)\) be a dynamical system having the marker property. From Corollary 3.13

\[
\text{mdim}(\mathcal{X}, T) \leq \inf_{d \in \mathcal{D}(\mathcal{X})} \sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu),
\]

\[
\leq \inf_{d \in \mathcal{D}(\mathcal{X})} \sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).
\]

On the other hand we already know (Corollary 3.14) that there exists \( d \in \mathcal{D}(\mathcal{X}) \) satisfying

\[
\text{mdim}(\mathcal{X}, T) = \sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu) = \sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).
\]

\( \square \)

**Remark 3.15.** It follows from Proposition 3.2 and Theorem 3.11 that for a metric \( d \) having the tame growth of covering numbers

\[
\text{mdim}_H(\mathcal{X}, T, d) \leq \sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu) \leq \text{mdim}_M(\mathcal{X}, T, d),
\]

\[
\text{mdim}_H(\mathcal{X}, T, d) \leq \sup_{\mu \in \mathbb{M}^T(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu) \leq \text{mdim}_M(\mathcal{X}, T, d).
\]

Hence we have a sufficient criterion (under the assumption of the tame growth of covering numbers) for the equality \( (1.8) \) in Problem 1.3: If mean Hausdorff dimension is equal to metric mean dimension, then they also coincide with the supremum of rate distortion dimensions.
4 Proof of Theorem 3.11: Geometric Measure Theory and Misiurewicz’s Technique

We prove Theorem 3.11 in this section. The proof is a combination of geometric measure theory and the rate distortion theory version of Misiurewicz’s technique [Mis76, LT18].

4.1 Geometric measure theory around Frostman’s lemma. The purpose of this subsection is to prepare some basics of geometric measure theory around Frostman’s lemma. Frostman’s lemma is a fundamental result in geometric measure theory. It states that a Borel subset \( A \subset \mathbb{R}^n \) has positive (possibly infinite) \( s \)-dimensional Hausdorff measure if and only if there exists a nonzero Radon measure \( \mu \) on \( \mathbb{R}^n \) supported on \( A \) and satisfying \( \mu(B_r(x)) \leq r^s \) for all \( x \in \mathbb{R}^n \) and \( r > 0 \) (see [Mat95, 8.8 Theorem]). We need a generalization of this result to compact metric spaces, which is due to Howroyd [H95]. Our presentation follows the book of Mattila [Mat95, Sections 8.14-8.17].

Let \((\mathcal{X}, d)\) be a compact metric space. For \( \delta > 0 \) and \( s \geq 0 \) we define \( \lambda_s^\delta(\mathcal{X}, d) \) as

\[
\inf \sum_{n=1}^{\infty} c_n (\text{diam} E_n)^s
\]

where the infimum is taken over all countable families \( \{(E_n, c_n)\} \) such that \( 0 < c_n < \infty \), \( E_n \subset \mathcal{X} \) with \( \text{diam} E_n < \delta \) and

\[
\forall x \in \mathcal{X} : \sum_{n=1}^{\infty} c_n 1_{E_n}(x) \geq 1.
\]

Obviously \( \lambda_s^\delta(\mathcal{X}, d) \leq \mathcal{H}_s^\delta(\mathcal{X}, d) \).

**Lemma 4.1**.

\[
\mathcal{H}_s^\delta(\mathcal{X}, d) \leq 6^s \lambda_s^\delta(\mathcal{X}, d).
\]

*Proof.* The proof is essentially the same as [Mat95, 8.16 Lemma], but the above statement is a bit different\(^{13}\) (at least formally) from [Mat95, 8.16 Lemma]. So we include a proof. \(\square\)

**Claim 4.2.** Let \( a_1, \ldots, a_N \) and \( m \) be positive integers. Let \( B_n^\circ = B_{r_n}^\circ(x_n) \) \((1 \leq n \leq N)\) be open balls in \( \mathcal{X} \) of radius \( r_n < \delta \). If

\[
\mathcal{X} = \left\{ \sum_{n=1}^{N} a_n 1_{B_n^\circ} \geq m \right\}
\]

then

\[
\mathcal{H}_s^\delta(\mathcal{X}, d) \leq m^{-1} 6^s \sum_{n=1}^{N} a_n r_n^s.
\]

\(^{13}\) An important point for us is that the statement is valid for each fixed \( \delta \) (not only the limits of \( \delta \to 0 \)).
Proof. The induction on \( m \): If \( m = 1 \) then \( \mathcal{X} = \bigcup_{n=1}^{N} B_n^o \) and \( \text{diam}B_n^o < 2\delta \). Hence

\[
\mathcal{H}_{2\delta}^s(\mathcal{X}, d) \leq \sum_{n=1}^{N} (\text{diam}B_n^o)^s \leq 2^s \sum_{n=1}^{N} r_n^s.
\]

Suppose \( m \geq 2 \). By Finite Vitali’s covering lemma (see [EW11, Lemma 2.27]) there exists a disjoint family \( \mathcal{B} \subset \{ B_1^o, \ldots, B_N^o \} \) satisfying \( \mathcal{X} = \bigcup_{B_n^o \in \mathcal{B}} B_n^o (x_n) \). We have \( \text{diam}(3B_n^o) \leq 6r_n < 6\delta \) and hence

\[
\mathcal{H}_{6\delta}^s(\mathcal{X}, d) \leq 6^s \sum_{B_n^o \in \mathcal{B}} r_n^s.
\]

We set

\[
a'_n = \begin{cases} 
a_n & \text{if } B_n^o \notin \mathcal{B} \\
a_n - 1 & \text{if } B_n^o \in \mathcal{B}. \end{cases}
\]

Since \( \mathcal{B} \) is a disjoint family, we have \( \sum a'_n r_n^s (x_n) \geq m - 1 \) for all \( x \in \mathcal{X} \). By the induction hypothesis,

\[
(m - 1) \mathcal{H}_{6\delta}^s(\mathcal{X}, d) \leq 6^s \sum_{n=1}^{N} a'_n r_n^s.
\]

Thus

\[
m \mathcal{H}_{6\delta}^s(\mathcal{X}, d) \leq 6^s \sum_{n=1}^{N} a'_n r_n^s + 6^s \sum_{B_n^o \in \mathcal{B}} r_n^s = 6^s \sum_{n=1}^{N} a_n r_n^s. \tag{4.1}
\]

Let \( 0 < c_n < \infty, E_n \subset \mathcal{X} \) such that \( \text{diam}E_n < \delta \) and \( \sum_{n=1}^{\infty} c_n 1_{E_n}(x) \geq 1 \) for all \( x \in \mathcal{X} \). Let \( \varepsilon > 0 \) and \( 0 < t < 1 \). We choose \( \text{diam}E_n < r_n < \delta \) satisfying

\[
\sum_{n=1}^{\infty} c_n r_n^s < \varepsilon + \sum_{n=1}^{\infty} c_n (\text{diam}E_n)^s. \tag{4.1}
\]

We pick \( x_n \in E_n \). The open balls \( B_n^o := B_{r_n}^o (x_n) \) contain \( E_n \) and hence

\[
\mathcal{X} = \bigcup_{N=1}^{\infty} \left\{ \sum_{n=1}^{N} c_n 1_{B_n^o} > t \right\}.
\]

Each set \( \{ \ldots \} \) here is open. Since \( \mathcal{X} \) is compact, we can find \( N \) such that

\[
\mathcal{X} = \left\{ \sum_{n=1}^{N} c_n 1_{B_n^o} > t \right\}. \tag{4.1}
\]

We choose rational numbers \( 0 < b_n \leq c_n \) so that

\[
\mathcal{X} = \left\{ \sum_{n=1}^{N} b_n 1_{B_n^o} > t \right\}. \tag{4.1}
\]

Take a positive integer \( p \) such that all \( a_n := pb_n \) become integers. Set \( m = \lceil pt \rceil \). Then \( \mathcal{X} = \left\{ \sum_{n=1}^{N} a_n 1_{B_n^o} \geq m \right\} \). By Claim 4.2

\[
\mathcal{H}_{6\delta}^s(\mathcal{X}, d) \leq m^{-1} 6^s \sum_{n=1}^{N} a_n r_n^s \leq pm^{-1} 6^s \sum_{n=1}^{N} b_n r_n^s \leq pm^{-1} 6^s \sum_{n=1}^{N} c_n r_n^s.
\]
It follows from $m \geq pt$ and (4.1) that

$$\mathcal{H}^s_{\delta/6}(\mathcal{X}, d) \leq t^{-1/6^s} \left( \varepsilon + \sum_{n=1}^{\infty} c_n (\text{diam} E_n)^s \right).$$

Let $\varepsilon \to 0$ and $t \to 1$. This proves the statement.

\begin{lemma}
There exists a Borel measure $\mu$ on $\mathcal{X}$ satisfying $\mu(\mathcal{X}) = \lambda^s_{\delta}(\mathcal{X}, d)$ and $\mu(E) \leq (\text{diam} E)^s$ for all $E \subset \mathcal{X}$ with $\text{diam} E < \delta$.
\end{lemma}

\textbf{Sketch of the proof.} See [Mat95, 8.17 Theorem] for the details. We define a sublinear functional $p(f)$ for continuous functions $f : \mathcal{X} \to \mathbb{R}$ by

$$p(f) = \inf \sum_{n=1}^{\infty} c_n (\text{diam}(E_n))^s,$$

where the infimum is taken over all countable families $\{(E_n, c_n)\}$ such that $0 < c_n < \infty$, $E_n \subset \mathcal{X}$ with $\text{diam} E_n < \delta$ and

$$\forall x \in \mathcal{X} : \sum_{n=1}^{\infty} c_n 1_{E_n}(x) \geq f(x).$$

We have $p(1) = \lambda^s_{\delta}(\mathcal{X}, d)$. By using the Hahn–Banach theorem, we can find a linear functional $L$ defined on the space of continuous functions in $\mathcal{X}$ such that $L(1) = p(1)$ and for any continuous function $f$ on $\mathcal{X}$

$$-p(-f) \leq L(f) \leq p(f).$$

If $f \geq 0$ then $L(f) \geq -p(-f) = 0$. So $L$ is a positive functional. It follows from the Riesz representation theorem that there exists a Borel measure $\mu$ on $\mathcal{X}$ satisfying $L(f) = \int_{\mathcal{X}} f d\mu$. We can easily check that $\mu$ satisfies the statement.

\begin{corollary}
Let $0 < c < 1$. We can choose $\delta_0 = \delta_0(c) \in (0, 1)$ independent of $(\mathcal{X}, d)$ so that for any $0 < \delta \leq \delta_0$ there exists a Borel probability measure $\mu$ on $\mathcal{X}$ satisfying

$$\mu(E) \leq (\text{diam} E)^{c \dim_H(\mathcal{X}, d, \delta)}$$

for all $E \subset \mathcal{X}$ with $\text{diam} E < \delta/6$.
\end{corollary}

\textbf{Proof.} Take $0 < \delta_0 < 1$ satisfying

$$\left( \frac{1}{\delta_0} \right)^{1-c} \geq 6.$$

Let $0 < \delta \leq \delta_0$. If $\dim_H(\mathcal{X}, d, \delta) = 0$ then the statement is trivial (the delta measure satisfies the claim; recall that we promised $0^0 = 1$). So we assume $\dim_H(\mathcal{X}, d, \delta) > 0$. From Lemma 4.3 it is enough to prove $\lambda^s_{\delta/6}(\mathcal{X}, d) \geq 1$ for $s := c \dim_H(\mathcal{X}, d, \delta)$. 

\textit{Proof.} Take $0 < \delta_0 < 1$ satisfying

$$\left( \frac{1}{\delta_0} \right)^{1-c} \geq 6.$$
Set \( t = \frac{1+c}{2} \dim_H(\mathcal{X}, d, \delta) \). We have \( t - s = \frac{1-c}{2c} s \) and hence

\[
\mathcal{H}_\delta^s(\mathcal{X}, d) \geq \left( \frac{1}{\delta} \right)^{t-s} \mathcal{H}_\delta^t(\mathcal{X}, d) \\
\geq \left( \frac{1}{\delta} \right)^{t-s} \quad \text{(by } t < \dim_H(\mathcal{X}, d, \delta)) \\
= \left( \frac{1}{\delta} \right)^{1-s} \geq 6^s.
\]

Then \( \lambda_{\delta/6}^s(\mathcal{X}, d) \geq 1 \) by Lemma 4.1.

4.2 \textbf{\( L^1 \)-mean Hausdorff dimension and the condition of tame growth of covering numbers.} We need a modification of mean Hausdorff dimension. Let \((\mathcal{X}, T)\) be a dynamical system with a metric \( d \). For \( N \geq 1 \) we define a new metric \( \bar{d}_N \) on \( \mathcal{X} \) by

\[
\bar{d}_N(x, y) = \frac{1}{N} \sum_{n=0}^{N-1} d(T^n x, T^n y).
\]

This metric is more closely connected to the distortion condition

\[
\mathbb{E} \left( \frac{1}{N} \sum_{n=0}^{N-1} d(T^n X, Y_n) \right) < \varepsilon
\]

in the definition of rate distortion function (see Section 2.3) than \( d_N(x, y) = \max_{0 \leq n < N} d(T^n x, T^n y) \). The next lemma is a manifestation of this connection. (This will be used only in Section 6. But it is conceptually a toy model of the proof of Theorem 3.11.)

**Lemma 4.5.** Let \( \mu \) be a \( T \)-invariant probability measure on \( \mathcal{X} \). Suppose that there exist \( s \geq 0 \) and \( \delta > 0 \) such that for any \( N \geq 1 \)

\[
\mu(E) \leq \left( \text{diam}(E, \bar{d}_N) \right)^{sN} \quad \text{for all } E \subset \mathcal{X} \text{ with } \text{diam}(E, \bar{d}_N) < \delta.
\]

Then \( \text{rdim}(\mathcal{X}, T, d, \mu) \geq s \).

**Proof.** Define \( \Delta: \mathcal{X} \to \mathcal{X}^N \) by \( \Delta(x) = (x, Tx, \ldots, T^{N-1} x) \). We define a metric \( \bar{d}_N \) on \( \mathcal{X}^N \) by

\[
\bar{d}_N ((x_0, \ldots, x_{N-1}), (y_0, \ldots, y_{N-1})) = \frac{1}{N} \sum_{n=0}^{N-1} d(x_n, y_n).
\]

The push-forward measure \( \Delta_* \mu \) on \( \mathcal{X}^N \) satisfies

\[
\Delta_* \mu(E) \leq \left( \text{diam}(E, \bar{d}_N) \right)^{sN} \quad \text{for all } E \subset \mathcal{X}^N \text{ with } \text{diam}(E, \bar{d}_N) < \delta.
\]
Let $\varepsilon > 0$ with $2\varepsilon \log(1/\varepsilon) \leq \delta$. Let $X$ and $Y = (Y_0, \ldots, Y_{N-1})$ be random variables such that all $X$ and $Y_n$ take values in $\mathcal{X}$ and satisfy

$$\text{Law}\, X = \mu, \quad \mathbb{E}\left(\frac{1}{N} \sum_{n=0}^{N-1} d(T^n X, Y_n)\right) < \varepsilon.$$ 

This condition is equivalent to $\text{Law}\Delta(X) = \Delta_* \mu$ and $\mathbb{E}\bar{d}_N(\Delta(X), Y) < \varepsilon$. So we apply Lemma 2.10 to $(\Delta(X), Y)$ and get

$$\frac{I(X; Y)}{N} = \frac{I(\Delta(X); Y)}{N} \geq s \log(1/\varepsilon) - C(s + 1),$$

where $C > 0$ is a universal constant. Therefore

$$R(d, \mu, \varepsilon) \geq s \log(1/\varepsilon) - C(s + 1),$$

$$\text{rdim}(X, T, d, \mu) = \liminf_{\varepsilon \to 0} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)} \geq s. \quad \Box$$

For a dynamical system $(\mathcal{X}, T)$ with a metric $d$, we define the $L^1$-mean Hausdorff dimension by

$$\text{mdim}_{L^1}(\mathcal{X}, T, d) = \lim_{\varepsilon \to 0} \left(\limsup_{N \to \infty} \frac{\dim_{H}(\mathcal{X}, \bar{d}_N, \varepsilon)}{N}\right).$$

Since $\bar{d}_N \leq d_N$, this always satisfies

$$\text{mdim}_{L^1}(\mathcal{X}, T, d) \leq \text{mdim}(\mathcal{X}, T, d).$$

These two quantities actually coincide under the tame growth of covering numbers condition (this is the only place where we use the tame growth of covering numbers in the proof of Theorem 3.11):

**Lemma 4.6.** If $(\mathcal{X}, d)$ has the tame growth of covering numbers, then $\text{mdim}_{L^1}(\mathcal{X}, T, d) = \text{mdim}(\mathcal{X}, T, d)$.

**Proof.** We assume $\text{mdim}_{L^1}(\mathcal{X}, T, d) < \infty$ and prove $\text{mdim}(\mathcal{X}, T, d) \leq \text{mdim}_{L^1}(\mathcal{X}, T, d)$. We set $[N] = \{0, 1, 2, \ldots, N-1\}$ and define a metric $d_A$ on $\mathcal{X}$ for $A \subset [N]$ by $d_A(x, y) = \max_{a \in A} d(T^a x, T^a y)$. In particular $d_N = d_{[N]}$.

For $\tau > 0$ we set $M(\tau) = \#(\mathcal{X}, d, \tau)$. We can find a covering $\mathcal{X} = W_1^\tau \cup \cdots \cup W_{M(\tau)}$ with $\text{diam}(W_m^\tau, d) < \tau$ for all $1 \leq m \leq M(\tau)$. Take any $0 < \delta < 1/2$ and $s > \text{mdim}_{L^1}(\mathcal{X}, T, d)$. By the tame growth of covering numbers, we can choose $0 < \varepsilon_0 < 1$ satisfying

- $\tau^6 \log M(\tau) < 1$ for $0 < \tau < \varepsilon_0$.
- $4 \cdot 2^{s/(1-2\delta)} \cdot \varepsilon_0^{\delta s/(1-2\delta)} < 1$. 


Let \( 0 < \varepsilon < \varepsilon_0 \). Let \( N \) be a sufficiently large number. From \( \text{mdim}_{H, L^1}(\mathcal{X}, T, d) < s \) we can find a covering \( \mathcal{X} = \bigcup_{n=1}^{\infty} E_n \) satisfying \( \tau_n := \text{diam}(E_n, d_N) < \varepsilon \) for all \( n \) and

\[
\sum_{n=1}^{\infty} \tau_n^{sN} < 1.
\]

Set \( L_n = (1/\tau_n)^{\delta} \). Pick a point \( x_n \) from each \( E_n \). Every point \( x \in E_n \) satisfies \( d_N(x, x_n) \leq \tau_n \) and hence

\[
|\{ k \in [N] \mid d(T^k x, T^k x_n) \geq L_n \tau_n \}| \leq \frac{N}{L_n}.
\]

Thus there exists \( A \subset [N] \) (depending on \( x \in E_n \)) satisfying \( |A| \leq N/L_n \) and \( d_{[N]}(x, x_n) < L_n \tau_n \). This implies

\[
E_n \subset \bigcup_{A \subset [N], \mid A \mid \leq N/L_n} B_{L_n \tau_n}^0(x_n, d_{[N]\setminus A}).
\]

Here \( B_{L_n \tau_n}^0(x_n, d_{[N]\setminus A}) \) is the open ball of radius \( L_n \tau_n \) with respect to \( d_{[N]\setminus A} \) around \( x_n \), which for \( A = \{a_1, \ldots, a_r\} \) we can write as

\[
\bigcup_{1 \leq i_1, \ldots, i_r \leq M(\tau_n)} B_{L_n \tau_n}^0(x_n, d_{[N]\setminus A}) \cap T^{-a_1} W_{i_1}^{\tau_n} \cap \cdots \cap T^{-a_r} W_{i_r}^{\tau_n}.
\]

Therefore \( \mathcal{X} \) can be written as a union of

\[
B_{L_n \tau_n}^0(x_n, d_{[N]\setminus A}) \cap T^{-a_1} W_{i_1}^{\tau_n} \cap \cdots \cap T^{-a_r} W_{i_r}^{\tau_n}, \quad (4.2)
\]

where \( n \geq 1 \), \( A = \{a_1, \ldots, a_r\} \subset [N] \) with \( r \leq N/L_n \) and \( 1 \leq i_1, \ldots, i_r \leq M(\tau_n) \). The diameter of (4.2) with respect to \( d_N \) is bounded by \( 2L_n \tau_n = 2\tau_n^{1-\delta} < 2\varepsilon^{1-\delta} \). Thus

\[
\mathcal{H}_{2 \varepsilon^{1-\delta}}^{sN/(1-2\delta)}(\mathcal{X}, d_N) \leq \sum_{n=1}^{\infty} 2^N M(\tau_n)^{N/L_n} (2\tau_n^{1-\delta})^{sN/(1-2\delta)}.
\]

Here \( 2^N \) comes from the choice of \( A \subset [N] \).

\[
2^N M(\tau_n)^{N/L_n} (2\tau_n^{1-\delta})^{sN/(1-2\delta)} = \left\{ 2^{1+\frac{s}{1-2\delta}} M(\tau_n)^{sN} \tau_n^{-\frac{s\delta}{1-2\delta}} \right\}^N \tau_n^{sN}.
\]

Recall \( \tau_n < \varepsilon < \varepsilon_0 \) and the choice of \( \varepsilon_0 \) above. We have

\[
2^{1+\frac{s}{1-2\delta}} M(\tau_n)^{sN} \tau_n^{-\frac{s\delta}{1-2\delta}} < 4 \cdot 2^{1-\delta} \varepsilon_0^{-\frac{s\delta}{1-2\delta}} < 1.
\]

Therefore

\[
\mathcal{H}_{2 \varepsilon^{1-\delta}}^{sN/(1-2\delta)}(\mathcal{X}, d_N) < \sum_{n=1}^{\infty} \tau_n^{sN} < 1.
\]
So we get
\[
\dim_H(\mathcal{X}, d_N, 2\varepsilon^{1-\delta}) \leq \frac{sN}{1-2\delta}.
\]
Since this holds for any sufficiently large \( N \),
\[
\limsup_{N \to \infty} \left( \frac{1}{N} \dim_H(\mathcal{X}, d_N, 2\varepsilon^{1-\delta}) \right) \leq \frac{s}{1-2\delta}.
\]
Here \( 0 < \varepsilon < \varepsilon_0 \) is arbitrary. Thus
\[
\text{mdim}_H(\mathcal{X}, T, d) \leq \frac{s}{1-2\delta}.
\]
Letting \( \delta \to 0 \) and \( s \to \text{mdim}_H, L_1(\mathcal{X}, T, d) \), we get the statement. \( \square \)

**Remark 4.7.** The same argument also proves that the lower mean Hausdorff dimension \( \text{mdim}_H(\mathcal{X}, T, d) \) (see Remark 3.1) coincides with
\[
\lim_{\varepsilon \to 0} \left( \liminf_{N \to \infty} \frac{\dim_H(\mathcal{X}, \tilde{d}_N, \varepsilon)}{N} \right)
\]
if \((\mathcal{X}, d)\) has the tame growth of covering numbers.

**Remark 4.8.** In [LT18, Lemmas 25 and 26] we showed a similar result for metric mean dimension. In [LT18, Lemma 25] we proved the following statement: Let \((\mathcal{X}, T)\) be a dynamical system with a metric \( d \). For any integer \( N \geq 1 \) and real numbers \( \varepsilon > 0 \) and \( L > 1 \), we have
\[
\log \#(\mathcal{X}, \tilde{d}_N, \varepsilon) \geq \log \#(\mathcal{X}, d_N, 2L\varepsilon) - N - \frac{N}{L} \log \#(\mathcal{X}, d, \varepsilon).
\]
This will be used in Section 6.

### 4.3 Proof of Theorem 3.11

**Theorem 3.11** follows from Lemma 4.6 and the next theorem.

**Theorem 4.9.** For any dynamical system \((\mathcal{X}, T)\) with a metric \( d \)
\[
\text{mdim}_{H, L_1}(\mathcal{X}, T, d) \leq \sup_{\mu \in \mathcal{M}(\mathcal{X})} \text{rdim}(\mathcal{X}, T, d, \mu).
\]

Notice that here we do not assume the condition of the tame growth of covering numbers.

We use the following elementary lemma.

**Lemma 4.10.** Let \( A \) be a finite set and \( \{\mu_n\} \) a sequence of probability measures on \( A \). Suppose \( \mu_n \) converges to some \( \mu \) in the weak* topology.\(^{14}\) Then there exists a sequence of probability measures \( \pi_n \) on \( A \times A \) such that

\(^{14}\) Since we assume that \( A \) is a finite set, this just means that \( \mu_n(x) \to \mu(x) \) at each \( x \in A \).
• \(\pi_n\) is a coupling between \(\mu_n\) and \(\mu\), namely its first and second marginals are equal to \(\mu_n\) and \(\mu\) respectively.
• \(\pi_n\) converges to \((id \times id)_*\mu\) in the weak* topology, namely

\[
\pi_n(a, b) \to \begin{cases} 
\mu(a) & \text{if } a = b \\
0 & \text{if } a \neq b.
\end{cases}
\]

**Proof.** This follows from a much more general fact on optimal transport that the Wasserstein distance metrizes the weak* topology [Vil09, Theorem 6.9]. See [LT18, Appendix] for an elementary self-contained proof. \(\square\)

As in the proof of Lemma 4.5, we extend the definition of \(\bar{d}_n\). For \(x = (x_0, \ldots, x_{n-1})\) and \(y = (y_0, \ldots, y_{n-1})\) in \(X^n\) we set

\[
\bar{d}_n(x, y) = \frac{1}{n} \sum_{k=0}^{n-1} d(x_k, y_k).
\]

**Proof of Theorem 4.9.** We can assume \(\text{mdim}_{H,L^1}(\mathcal{X}, T, d) > 0\). Let \(0 < s < \text{mdim}_{H,L^1}(\mathcal{X}, T, d)\). We will prove that there exists \(\mu \in \mathcal{M}^T(\mathcal{X})\) satisfying \(\text{rdim}(\mathcal{X}, T, d, \mu) \geq s\).

Take \(0 < c < 1\) with \(c \cdot \text{mdim}_{H,L^1}(\mathcal{X}, T, d) > s\) and let \(\delta_0 = \delta_0(c) \in (0, 1)\) be the constant given by Corollary 4.4. There exist \(0 < \delta < \delta_0\) and \(n_1 < n_2 < n_3 < \ldots\) satisfying \(c \cdot \text{dim}_H(\mathcal{X}, \bar{d}_{n_k}, \delta) > sn_k\). By Corollary 4.4 we can find Borel probability measures \(\nu_k\) on \(\mathcal{X}\) satisfying

\[
\nu_k(E) \leq (\text{diam}(E, \bar{d}_{n_k}))^{sn_k}, \quad \forall E \subset \mathcal{X} \text{ with } \text{diam}(E, \bar{d}_{n_k}) < \frac{\delta}{6}. \quad (4.3)
\]

Set

\[
\mu_k = \frac{1}{n_k} \sum_{n=0}^{n_k-1} T_n^\mu \nu_k.
\]

By choosing a subsequence (also denoted by \(\mu_k\)), we can assume that \(\mu_k\) converges to some \(T\)-invariant probability measure \(\mu\) on \(\mathcal{X}\) in the weak* topology. We will show \(\text{rdim}(\mathcal{X}, T, d, \mu) \geq s\).

Let \(\varepsilon\) be any positive number with \(2\varepsilon \log(1/\varepsilon) \leq \delta/10\). We would like to prove an estimate such as

\[
R(d, \mu, \varepsilon) \geq s \log(1/\varepsilon) + \text{small error terms}.
\]

For this purpose, let \(X\) and \(Y = (Y_0, \ldots, Y_{m-1})\) be coupled random variables such that \(X, Y_0, \ldots, Y_{m-1}\) take values in \(\mathcal{X}\) and satisfy

\[
\text{Law} X = \mu, \quad \mathbb{E} \left( \frac{1}{m} \sum_{j=0}^{m-1} d(T^j X, Y_j) \right) < \varepsilon.
\]
We need to show $I(X; Y) \geq sm \log(1/\varepsilon) + \text{small error terms}$. As we remarked in Remark 2.11, we can assume that $Y$ takes only finitely many values. Let $\mathcal{Y} \subset \mathcal{X}^m$ be the set of possible values of $Y$.

**Idea of the proof.** Here we roughly explain the idea of the proof, assuming $m = 1$. That is to say, we give a lower bound on $I(X; Y)$ if Law $X = \mu$ and $Y$ is any $\mathcal{X}$-valued random variable coupled to it with $\mathbb{E}d(X, Y) < \varepsilon$.

Since $\mu_k \to \mu$, we can find random variables $X(k)$ coupled to $X$ such that $X(k)$ take values in $\mathcal{X}$ with Law $X(k) = \mu_k$ and

$$\mathbb{E}d(X, X(k)) \to 0, \quad I(X(k); Y) \to I(X; Y).$$

(Here we have ignored a technical problem. The above convergence $I(X(k); Y) \to I(X; Y)$ essentially follows from Lemma 2.6. But Lemma 2.6 is valid only if the underlying space is a finite set. We will address this issue by introducing an appropriate finite partition $\mathcal{P}$ below. At the moment we pretend that $X$ and $X(k)$ take values in a finite set.)

Using this coupling between $X(k)$ and $Y$, we will construct a new coupling between $Z(k) = (X'(k), TX'(k), \ldots, T^{m-1}X'(k))$ with $X'(k)$ a random variable taking values in $\mathcal{X}$ and obeying $\nu_k$, and a new random variable $W(k) = (W(k)_0, \ldots, W(k)_{n_k-1})$ so that

$$\mathbb{E}d_{n_k}(Z(k), W(k)) < \varepsilon$$

and $I(Z(k); W(k))$ is bounded from above using $I(X; Y)$. As we can bound $I(Z(k); W(k))$ from below using Lemma 2.10 this will give us the desired lower bound on $I(X; Y)$.

We define the random variable $W(k) = (W(k)_0, \ldots, W(k)_{n_k-1})$, coupled to $Z(k)$ and taking values in $\mathcal{X}^{n_k}$, as follows. Let $\rho_k(y|x) = \mathbb{P}(Y = y|X(k) = x)$. Then the law of $W(k) = (W(k)_0, \ldots, W(k)_{n_k-1})$ is determined by requiring that $W(k)_0, \ldots, W(k)_{n_k-1}$ are conditionally independent given $Z(k)$ and

$$\mathbb{P}(W(k)_n = y|T^nX'(k) = x) = \rho_k(y|x).$$

By (4.4) we have that

$$\mathbb{E}d_{n_k}(Z(k), W(k)) = \mathbb{E}d(X(k), Y) \to \mathbb{E}d(X, Y) < \varepsilon,$$

so that indeed $\mathbb{E}d_{n_k}(Z(k), W(k)) < \varepsilon$ for large $k$.

Since $W(k)_0, \ldots, W(k)_{n_k-1}$ are conditionally independent given $Z(k)$, we use the subadditivity of mutual information (Lemma 2.7) and get

$$I(Z(k); W(k)) \leq \sum_{n=0}^{n_k-1} I(Z(k); W(k)_n) = \sum_{n=0}^{n_k-1} I(T^nX'(k); W(k)_n).$$
By definition of $W(k)_n$, we have $I(T^n X'(k); W(k)_n) = I(T^n_* \nu_k, \rho_k)$ in the notation of Lemma 2.8. Recall $\mu_k = (1/n_k) \sum_{n=0}^{n_k-1} T^n_* \nu_k$. By using the concavity property of mutual information given in Lemma 2.8 it follows that
\[
\frac{1}{n_k} \sum_{n=0}^{n_k-1} I(T^n_* \nu_k, \rho_k) \leq I(\mu_k, \rho_k) = I(X(k); Y).
\]
Thus we get
\[
I(Z(k); W(k)) \leq n_k I(X(k); Y) \quad (4.5)
\]
Since Law $X'(k) = \nu_k$, it follows from (4.3) and Lemma 2.10 that
\[
I(Z(k); W(k)) \geq s n_k \log(1/\varepsilon) + \text{small error terms},
\]
thus by (4.5) we see that
\[
I(X(k); Y) \geq s \log(1/\varepsilon) + \text{small error terms}.
\]
As $I(X(k); Y) \to I(X; Y)$ by (4.4)
\[
I(X; Y) \geq s \log(1/\varepsilon) + \text{small error terms}.
\]
This is what we want to prove. \(\Box\)

Now we return to the proof. Recall our situation: The random variables $X$ and $Y = (Y_0, \ldots, Y_{m-1})$ take values in $\mathcal{X}$ and a finite set $\mathcal{Y} \subset \mathcal{X}^m$ respectively. They satisfy Law $X = \mu$ and
\[
\frac{1}{m} \mathbb{E} \left( \sum_{j=0}^{m-1} d(T^j X, Y_j) \right) < \varepsilon.
\]
We want to estimate $I(X; Y)$ from below.

We choose a positive number $\tau$ satisfying
\[
\tau \leq \min \left( \frac{\varepsilon}{3}, \frac{\delta}{20} \right), \quad \tau + \mathbb{E} \left( \frac{1}{m} \sum_{j=0}^{m-1} d(T^j X, Y_j) \right) < \varepsilon. \quad (4.6)
\]
We take a finite partition $\mathcal{P} = \{P_1, \ldots, P_L\}$ of $\mathcal{X}$ such that for all $1 \leq l \leq L$
\[
diam(P_l, d) < \frac{\tau}{2}, \quad \mu(\partial P_l) = 0.
\]
Pick a point $p_l$ from each $P_l$ and set $A = \{p_1, \ldots, p_L\}$. We define a map $\mathcal{P}: \mathcal{X} \to A$ by $\mathcal{P}(P_l) = \{p_l\}$. For $n \geq 1$ we define a map $\mathcal{P}^n: \mathcal{X} \to A^n$ by $\mathcal{P}^n(x) = (\mathcal{P}(x), \mathcal{P}(Tx), \ldots, \mathcal{P}(T^{n-1}x))$.

Claim 4.11. $\mathcal{P}^n_* \nu_k(E) \leq (\tau + \text{diam} (E, \bar{d}_{nk}))^{sn_k}$ for all $E \subset A^n$ with diam $(E, \bar{d}_{nk}) < \delta/10$. 

Proof. We have $P_{*}^{n_{k}} \nu_{k}(E) = \nu_{k}\\((P_{*}^{n_{k}})^{-1} E\\). Since $\tau \leq \delta/20$

$$\text{diam}\\((P_{*}^{n_{k}})^{-1} E, \bar{d}_{n_{k}}\\) < \tau + \text{diam}\\((E, \bar{d}_{n_{k}}\\) < \delta/6.$$ Then by (4.3)

$$\nu_{k}\\((P_{*}^{n_{k}})^{-1} E\\) \leq (\text{diam}\\((P_{*}^{n_{k}})^{-1} E, \bar{d}_{n_{k}}\\)^{s_{n_{k}}} < (\tau + \text{diam}\\((E, \bar{d}_{n_{k}}\\)^{s_{n_{k}}}.

\[\square\] It follows from $\mu_{k} \to \mu$ and $\mu(\partial P_{l}) = 0$ that $P_{*}^{m} \mu_{k} \to P_{*}^{m} \mu$ as $k \to \infty$. Then by Lemma 4.10 there exists a sequence of couplings $\pi_{k}$ between $P_{*}^{m} \mu_{k}$ and $P_{*}^{m} \mu$ converging to $(\text{id} \times \text{id})_{*} P_{*}^{m} \mu$. We take a random variable $X(k)$ coupled to $P_{*}^{m}(X)$ such that $X(k)$ takes values in $A^{m}$ with Law $(X(k), P_{*}^{m}(X)) = \pi_{k}$. (In particular $X(k)$ obeys $P_{*}^{m} \mu_{k}$. This satisfies

$$\mathbb{E} \bar{d}_{m}(X(k), P_{*}^{m}(X)) \to 0, \quad I(X(k); Y) \to I(P_{*}^{m}(X); Y).$$ (4.7)

The latter condition\textsuperscript{15} follows from Lemma 2.6. Since $\text{diam}(P_{l}, d) < \tau/2$,\n
$$\mathbb{E} \bar{d}_{m}(X(k), Y) \leq \mathbb{E} \bar{d}_{m}(X(k), P_{*}^{m}(X)) + \frac{\tau}{2} + \mathbb{E} \left(\frac{1}{m} \sum_{j=0}^{m-1} d(T^{j} X, Y_{j})\right)$$ (4.8)\n
$$\to \frac{\tau}{2} + \mathbb{E} \left(\frac{1}{m} \sum_{j=0}^{m-1} d(T^{j} X, Y_{j})\right) < \varepsilon \quad \text{by (4.6)}.$$

For $x = (x_{0}, \ldots, x_{n-1}) \in X^{m}$ and $0 \leq a \leq b < n$ we denote $x_{a}^{b} = (x_{a}, x_{a+1}, \ldots, x_{b})$. For $x, y \in X^{m}$ with $P_{*}^{m} \mu_{k}(x) > 0$ we consider a conditional probability mass function

$$\rho_{k}(y|x) = \mathbb{P}(Y = y|X(k) = x).$$

Fix a point $a \in X$. Let $n_{k} = mq + r$ with $m \leq r < 2m$. For $x, y \in X^{n_{k}}$ and $0 \leq j < m$ we define a conditional probability mass function

$$\sigma_{k,j}(y|x) = \prod_{i=0}^{q-1} \rho_{k}(y^{j+im+m-1}_{j+im}|x^{j+im+m-1}_{j+im}) \cdot \prod_{n \in [0,j] \cup [mq+j,n_{k})} \delta_{a}(y_{n}).$$

\textsuperscript{15} The coupling between $X(k)$ and $Y$ is given by the probability mass function

$$\sum_{x' \in A^{m}} \pi_{k}(x, x') \mathbb{P}(Y = y|P_{*}^{m}(X) = x'),$$

which converges to $\mathbb{P}(P_{*}^{m}(X) = x, Y = y)$.
Here $\delta_a(\cdot)$ is the delta probability measure at $a$ on $\mathcal{X}$. We set
\[
\sigma_k(y|x) = \frac{\sigma_{k,0}(y|x) + \sigma_{k,1}(y|x) + \cdots + \sigma_{k,m-1}(y|x)}{m}.
\]
This is defined for $x \in \mathcal{X}^{n_k}$ with $\mathcal{P}_{*}^{n_k} \nu_k(x) > 0$.

Let $\mathcal{X}'(k)$ be a random variable taking values in $\mathcal{X}$ and obeying $\nu_k$. We set $Z(k) = \mathcal{P}_{*}^{n_k} (X'(k))$. We take a random variable $W(k)$ coupled to $Z(k)$ and taking values in $\mathcal{X}^{n_k}$ with
\[
\mathbb{P}(W(k) = y|Z(k) = x) = \sigma_k(y|x).
\]
For $0 \leq j < m$ we also take a random variable $W(k,j)$ coupled to $Z(k)$ and taking values in $\mathcal{X}^{n_k}$ with
\[
\mathbb{P}(W(k,j) = y|Z(k) = x) = \sigma_{k,j}(y|x).
\]

**Claim 4.12.** $\mathbb{E} \bar{d}_{n_k}(Z(k), W(k)) < \varepsilon$ for large $k$.

**Proof.**
\[
\mathbb{E} \bar{d}_{n_k}(Z(k), W(k)) = \frac{1}{m} \sum_{j=0}^{m-1} \mathbb{E} \bar{d}_{n_k}(Z(k), W(k,j)).
\]

$\bar{d}_{n_k}(Z(k), W(k,j))$ is bounded by
\[
\frac{r \cdot \text{diam}(\mathcal{X}, d)}{n_k} + \frac{m}{n_k} \sum_{i=0}^{q-1} \bar{d}_m(\mathcal{P}_m^{i}(T^j + im X'(k)), W(k,j)_{j+im}^{j+im+m-1}).
\]

$\mathbb{E} \bar{d}_m(\mathcal{P}_m^{i}(T^j + im X'(k)), W(k,j)_{j+im}^{j+im+m-1})$ is equal to
\[
\sum_{x,y \in \mathcal{X}^m} \bar{d}_m(x,y) \rho_k(y|x) \mathcal{P}_m^j \nu_k(x).
\]

Therefore $\mathbb{E} \bar{d}_{n_k}(Z(k), W(k))$ is bounded by
\[
\frac{r \cdot \text{diam}(\mathcal{X}, d)}{n_k} + \sum_{x,y \in \mathcal{X}^m} \bar{d}_m(x,y) \rho_k(y|x) \left( \frac{1}{n_k} \sum_{\substack{0 \leq i < q \quad 0 \leq j < m}} \mathcal{P}_m^i T^j + im \nu_k(x) \right)
\]
\[
\leq \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n_k} + \sum_{x,y \in \mathcal{X}^m} \bar{d}_m(x,y) \rho_k(y|x) \left( \frac{1}{n_k} \sum_{n=0}^{m-1} \mathcal{P}_m^{n} T^j + im \nu_k(x) \right)
\]
\[
= \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n_k} + \sum_{x,y \in \mathcal{X}^m} \bar{d}_m(x,y) \rho_k(y|x) \mathcal{P}_m^j \mu_k(x)
\]
\[
= \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n_k} + \mathbb{E} \bar{d}_m(X(k), Y).
\]
From $r \leq 2m$ and (4.8), this is less than $\varepsilon$ for large $k$. $\square$

Claim 4.13.

$$\frac{1}{n_k} I(Z(k); W(k)) \leq \frac{1}{m} I(X(k); Y).$$

Proof. By the convexity of mutual information (Lemma 2.8)

$$I(Z(k); W(k)) \leq \frac{1}{m} \sum_{j=0}^{m-1} I(Z(k); W(k, j)).$$

By the subadditivity of mutual information under conditional independence (Lemma 2.7)

$$I(Z(k); W(k, j)) \leq \sum_{i=0}^{q-1} I(Z(k); W(k, j)^{j+im+m-1}).$$

The term $I(Z(k); W(k, j)^{j+im+m-1})$ is equal to

$$I(\mathcal{P}^m(T^{j+im}X'(k)); W(k, j)^{j+im+m-1}) = I(\mathcal{P}^m T^{j+im}_*\nu_k, \rho_k).$$

Hence

$$\frac{m}{n_k} I(Z(k); W(k)) \leq \frac{1}{n_k} \sum_{0 \leq j < m} I(\mathcal{P}^m(T^{j+im}_*\nu_k), \rho_k)$$

$$\leq \frac{1}{n_k} \sum_{n=0}^{n_k-1} I(\mathcal{P}^n T^{j+im}_*\nu_k, \rho_k)$$

$$\leq I\left(\frac{1}{n_k} \sum_{n=0}^{n_k-1} \mathcal{P}^n T^{j+im}_*\nu_k, \rho_k\right)$$

by the concavity in Lemma 2.8

$$= I(\mathcal{P}^*_*\nu_k, \rho_k)$$

$$= I(X(k); Y).\square$$

Recall $2\varepsilon \log(1/\varepsilon) \leq \delta/10$, $\tau \leq \min(\varepsilon/3, \delta/20)$ and $\text{Law}Z(k) = \mathcal{P}^{n_k}_*\nu_k$. We apply Lemma 2.10 with Claims 4.11 and 4.12 to $(Z(k), W(k))$:

$$I(Z(k); W(k)) \geq sn_k \log(1/\varepsilon) - C(sn_k + 1) \quad \text{for large } k.$$  

Here $C$ is a universal positive constant. From Claim 4.13

$$\frac{1}{m} I(X(k); Y) \geq s \log(1/\varepsilon) - C\left(s + \frac{1}{n_k}\right).$$
We know $I(X(k); Y) \to I(\mathcal{P}^m(X); Y)$ as $k \to \infty$ in (4.7). Hence

$$\frac{1}{m} I(\mathcal{P}^m(X); Y) \geq s \log(1/\varepsilon) - Cs.$$  

By the data-processing inequality (Lemma 2.5)

$$\frac{1}{m} I(X; Y) \geq \frac{1}{m} I(\mathcal{P}^m(X); Y) \geq s \log(1/\varepsilon) - Cs.$$  

Therefore for any positive number $\varepsilon$ with $2\varepsilon \log(1/\varepsilon) \leq \delta/10$

$$R(d, \mu, \varepsilon) \geq s \log(1/\varepsilon) - Cs.$$  

Thus

$$\text{rdim}(\mathcal{X}, T, d, \mu) = \liminf_{\varepsilon \to 0} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)} \geq s. \quad \Box$$

5 Proof of Theorem 3.12

The purpose of this section is to prove Theorem 3.12. The proof does not involve rate distortion theory, and in particular it is independent of Section 4.

5.1 Background: Pontrjagin–Schnirelmann’s theorem. Theorems 2.3 and 3.12 look quite similar. Theorem 2.3 claims the existence of $d$ satisfying $\text{mdim}_m(\mathcal{X}, T, d) = \text{mdim}(\mathcal{X}, T)$ whereas Theorem 3.12 claims the existence of $d$ satisfying $\text{mdim}_m(\mathcal{X}, T, d) = \text{mdim}(\mathcal{X}, T)$. But indeed the upper case (Theorem 3.12) is substantially subtler. The difficulty is already visible in a classical, non-dynamical setting. Let $\mathcal{X}$ be a compact metrizable space. Pontrjagin and Schnirelmann [PS32] proved$^{16}$

**Theorem 5.1.** There exists a metric $d$ on $\mathcal{X}$ satisfying $\text{dim}_M(\mathcal{X}, d) = \dim \mathcal{X}$.

Compare this statement with

**Theorem 5.2.** There exists a metric $d$ on $\mathcal{X}$ satisfying $\text{dim}_M(\mathcal{X}, d) = \dim \mathcal{X}$.

They look similar. But their natures are different. A (now) standard approach to Theorem 5.2 is to use the Baire category theorem as follows. Fix an arbitrary metric $d$ on $\mathcal{X}$ and consider an infinite dimensional Banach space $(V, \| \cdot \|)$. We denote by $C(\mathcal{X}, V)$ the space of continuous maps from $\mathcal{X}$ to $V$ endowed with the norm topology. For each $n \geq 1$ we consider $A_n \subset C(\mathcal{X}, V)$ consisting of $f: \mathcal{X} \to V$ such that $f$ is a $(1/n)$-embedding with respect to $d$ and satisfies

$$\exists \varepsilon < 1/n: \frac{\log \#(f(\mathcal{X}), \| \cdot \|, \varepsilon)}{\log(1/\varepsilon)} < \dim \mathcal{X} + \frac{1}{n}.$$  

$^{16}$ Indeed we cannot find this statement in [PS32]. The main theorem of their paper states that $\dim \mathcal{X}$ is equal to the infimum of $\text{dim}_M(\mathcal{X}, d)$ over $d \in \mathcal{D}(\mathcal{X})$. But their argument actually proves Theorem 5.1.
It is not hard to show that \( A_n \) are open and dense.\(^{17}\) Then \( \bigcap_{n=1}^{\infty} A_n \) is a residual (i.e. dense and \( G_\delta \)) subset of \( C(\mathcal{X}, V) \) (in particular, non-empty). On the other hand this is equal to
\[
\{ f \in C(\mathcal{X}, V) \mid f \text{ is an embedding and } \dim_M(f(\mathcal{X}), \|\cdot\|) = \dim \mathcal{X} \}.
\]
Pick \( f \) in this set. Then the metric \( \|f(x) - f(y)\| \) \( (x, y \in \mathcal{X}) \) has the lower Minkowski dimension equal to \( \dim \mathcal{X} \). This proves Theorem 5.2.

Let's try a similar approach to Theorem 5.1. It is natural to consider
\[
\{ f \in C(\mathcal{X}, V) \mid f \text{ is an embedding and } \dim_M(f(\mathcal{X}), \|\cdot\|) = \dim \mathcal{X} \}.
\]
One might hope that this is also a residual subset of \( C(\mathcal{X}, V) \). But this does not hold true. We can prove that if \( \mathcal{X} \) is an infinite set then
\[
\{ f \in C(\mathcal{X}, V) \mid \dim_M(f(\mathcal{X}), \|\cdot\|) = \infty \}
\]
is a residual subset of \( C(\mathcal{X}, V) \). This implies that the set (5.1) is never residual in a nontrivial situation (namely if \( \mathcal{X} \) is infinite and \( \dim \mathcal{X} < \infty \)). It is a very thin set. So it is more delicate to find an element in (5.1). We note that the set (5.1) is however dense; see Section 5.3.

5.2 Preparations on combinatorial topology. As we promised in Section 2.1, all simplicial complexes are assumed to have only finitely many vertices.

Let \( P \) be a simplicial complex. We denote the set of vertices of \( P \) by \( \text{Ver}(P) \). For \( v \in \text{Ver}(P) \) we define the open star \( O_P(v) \) as the union of the open simplices of \( P \) one of whose vertex is \( v \). (We declare that \( \{v\} \) itself is an open simplex.) The open star \( O_P(v) \) is an open neighborhood of \( v \) and \( \{O_P(v)\}_{v \in \text{Ver}(P)} \) is an open cover of \( P \). When vertices \( v_0, \ldots, v_n \) span a simplex \( \Delta \) in \( P \), we set \( O_P(\Delta) = \bigcup_{i=0}^{n} O_P(v_i) \).

Let \( P \) and \( Q \) be simplicial complexes. A map \( f: P \to Q \) is said to be simplicial if it satisfies the following two conditions:

- For every simplex \( \Delta \subset P \), \( f(\Delta) \) is a simplex in \( Q \). (In particular \( f(v) \in \text{Ver}(Q) \) for every \( v \in \text{Ver}(P) \).
- If \( v_0, \ldots, v_n \in \text{Ver}(P) \) span a simplex in \( P \) then
  \[
f \left( \sum_{i=0}^{n} \lambda_i v_i \right) = \sum_{i=0}^{n} \lambda_i f(v_i),
\]
  where \( 0 \leq \lambda_i \leq 1 \) and \( \sum_{i=0}^{n} \lambda_i = 1 \).

\(^{17}\) “Open” is easy. To show “dense”, take arbitrary \( f \in C(\mathcal{X}, V) \) and \( \delta > 0 \). Choose \( 0 < \varepsilon < 1/n \) such that \( d(x, y) < \varepsilon \) implies \( \|f(x) - f(y)\| < \delta \). There exists an \( \varepsilon \)-embedding \( \pi: \mathcal{X} \to P \) in a simplicial complex \( P \) of dimension \( \leq \dim \mathcal{X} \). From Lemma 5.3 (2) and (3) in Section 5.2 we can find a linear embedding \( g: P \to V \) with \( \|g(\pi(x)) - f(x)\| < \delta \). From Lemma 5.3 (1), \( \log \#(g(P), \|\cdot\|, \varepsilon')/\log(1/\varepsilon') \) is less than \( \dim \mathcal{X} + 1/n \) for sufficiently small \( \varepsilon' \). This shows \( g \circ \pi \in A_n \). So \( A_n \) is dense. Therefore the main point of the proof of Theorem 5.2 is a “polyhedral approximation”. The basic idea of the proof of Theorem 5.1 is also a polyhedral approximation, but in a much more accurate way. See Section 5.3.
Let $V$ be a vector space over real numbers. A map $f : P \to V$ is said to be linear if for any $v_0, \ldots, v_n \in \text{Ver}(P)$ spanning a simplex in $P$

$$f \left( \sum_{i=0}^{n} \lambda_i v_i \right) = \sum_{i=0}^{n} \lambda_i f(v_i),$$

where $0 \leq \lambda_i \leq 1$ and $\sum_{i=0}^{n} \lambda_i = 1$. We denote the space of linear maps $f : P \to V$ by $\text{Hom}(P,V)$.

**Lemma 5.3.** Let $(V, \|\cdot\|)$ be a Banach space and $P$ a simplicial complex.

1. If $f : P \to V$ is a linear map with $\text{diam} f(P) \leq 2$, then for any $0 < \varepsilon \leq 1$

$$\#(f(P), \|\cdot\|, \varepsilon) \leq C(P) \cdot (1/\varepsilon)^{\dim P},$$

where $C(P)$ is a positive constant depending only on $\dim P$ and the number of simplices of $P$.

2. Suppose $V$ is infinite dimensional. Then the set

$$\{ f \in \text{Hom}(P,V) \mid f \text{ is injective} \}$$

is dense in $\text{Hom}(P,V)$. (Indeed it is also open; but we do not need this.) Here $\text{Hom}(P,V) \cong V^{\text{Ver}(P)}$ is endowed with the product topology.

3. Let $(X, d)$ be a compact metric space and $\varepsilon, \delta > 0$. Let $\pi : X \to P$ be a continuous map satisfying $\text{diam} \pi^{-1}(O_P(v)) < \varepsilon$ for all $v \in \text{Ver}(P)$. Let $f : X \to V$ be a continuous map such that

$$d(x, y) < \varepsilon \implies \|f(x) - f(y)\| < \delta.$$

Then there exists a linear map $g : P \to V$ satisfying

$$\|f(x) - g(\pi(x))\| < \delta \quad (x \in X).$$

Moreover if $f(X) \subset B_1^0(V)$ (the open unit ball) then it can be chosen so that $g(P) \subset B_1^0(V)$.

**Proof.** (1) We can assume that $P$ is a simplex (we denote its vertices by $v_0, \ldots, v_n$) and that $f(v_0) = 0$ and $\|f(v_i)\| \leq 2$. Set

$$\Delta = \{(\lambda_1, \ldots, \lambda_n) \in [0,1]^n \mid \lambda_1 + \cdots + \lambda_n \leq 1\}.$$

Then $f(P)$ is covered by the $(\varepsilon/3)$-open balls

$$B^{\circ}_{\varepsilon/3}(\lambda_1 f(v_1) + \cdots + \lambda_n f(v_n))$$

where

$$(\lambda_1, \ldots, \lambda_n) \in \Delta \cap \left( \frac{\varepsilon}{6n} \mathbb{Z} \right)^n.$$
(2) Let $\text{Ver}(P) = \{v_0, \ldots, v_n\}$. The set (5.2) contains
\[ \{ f \in \text{Hom}(P, V) | f(v_0), \ldots, f(v_n) \text{ are affinely independent} \}, \]
which is dense because $V$ is infinite dimensional.

(3) Let $v \in \text{Ver}(P)$. Pick $x_v \in \pi^{-1}(O_P(v))$ and set $g(v) = f(x_v)$. If $\pi^{-1}(O_P(v)) = \emptyset$ then $g(v)$ may be an arbitrary point in $B_1^o(V)$. We extend $g$ to a linear map $g: P \to V$. Let $x \in \mathcal{X}$ and $\Delta^0$ be the open simplex of $P$ containing $\pi(x)$. Let $v_0, \ldots, v_n$ be the vertices of $\Delta^0$. Then $\pi(x) = \sum_{i=0}^n \lambda_i v_i$ with $0 < \lambda_i \leq 1$ and $\sum_{i=0}^n \lambda_i = 1$. $g(\pi(x)) = \sum_{i=0}^n \lambda_i f(x_{v_i})$.

Since $\pi(x) \in O_P(v_i)$, $x \in \pi^{-1}(O_P(v_i))$ and hence $d(x, x_{v_i}) < \varepsilon$. Then $\|f(x) - f(x_{v_i})\| < \delta$. It follows that
\[ \|f(x) - g(\pi(x))\| \leq \sum_{i=0}^n \lambda_i \|f(x) - f(x_{v_i})\| < \delta. \]

If $f(\mathcal{X}) \subset B_1^o(V)$ then $g(v) \in B_1^o(V)$ for all $v \in \text{Ver}(P)$ and hence $g(P) \subset B_1^o(V)$. □

Let $f: \mathcal{X} \to P$ be a continuous map from a topological space $\mathcal{X}$ to a simplicial complex $P$. It is said to be essential if for any $v_0, \ldots, v_n \in \text{Ver}(P)$ spanning a simplex in $P$
\[ f^{-1}(O_P(v_0) \cap \cdots \cap O_P(v_n)) \neq \emptyset. \]

**Lemma 5.4.** Let $f: \mathcal{X} \to P$ be a continuous map from a topological space $\mathcal{X}$ to a simplicial complex $P$. There exists a subcomplex $P' \subset P$ such that $f(\mathcal{X}) \subset P'$ and $f: \mathcal{X} \to P'$ is essential.

**Proof.** It is easy to check that $f: \mathcal{X} \to P$ is essential if and only if $f(\mathcal{X}) \not\subset P'$ for any proper subcomplex $P' \subset P$. Then the statement is trivial; just take the minimal subcomplex $P' \subset P$ containing $f(\mathcal{X})$. □

The next lemma is one of the central ingredients of the proof of Theorem 3.12. It has a spirit similar to Lemma 5.3 (3), though its statement is rather technical (see Corollary 5.7 for a simplified version). Its rough idea is as follows: Let $\mathcal{X}$ be a topological space and $P, Q$ simplicial complexes. Let $\pi: \mathcal{X} \to P$ and $q: \mathcal{X} \to Q$ be continuous maps. We would like to formulate a condition which guarantees the existence of a simplicial map $h: P \to Q$ such that $h \circ \pi$ is approximately equal to $q$.

For two open covers $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{X}$, we denote by $\mathcal{U} \prec \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$, namely for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ satisfying $V \subset U$.

**Lemma 5.5.** Let $\mathcal{X}$ be a topological space and $P, Q$ simplicial complexes. Let $\pi: \mathcal{X} \to P$ and $q_i: \mathcal{X} \to Q$, $1 \leq i \leq N$, be continuous maps. We suppose that $\pi: \mathcal{X} \to P$ is essential and satisfies
\[ \{ q_i^{-1}(O_Q(w)) \}_{w \in \text{Ver}(Q)} \prec \{ \pi^{-1}(O_P(v)) \}_{v \in \text{Ver}(P)} \]
for every $1 \leq i \leq N$. (Here both sides are open covers of $\mathcal{X}$.) Then there exist simplicial maps $h_i : P \to Q$, $1 \leq i \leq N$, such that

1. For every $1 \leq i \leq N$ and $x \in \mathcal{X}$ the two points $q_i(x)$ and $h_i \circ \pi(x)$ belong to the same simplex of $Q$.
2. Let $1 \leq i \leq N$. Let $\Delta \subset P$ be a simplex and $Q' \subset Q$ a subcomplex. If $\pi^{-1}(O_P(\Delta)) \subset q_i^{-1}(Q')$ then $h_i(\Delta) \subset Q'$.
3. For $1 \leq i < j \leq N$ and a simplex $\Delta \subset P$, if $q_i = q_j$ on $\pi^{-1}(O_P(\Delta))$ then $h_i = h_j$ on $\Delta$.

Proof. Let $v \in \text{Ver}(P)$. We can choose $h_i(v) \in \text{Ver}(Q)$, $1 \leq i \leq N$, satisfying

- $\pi^{-1}(O_P(v)) \subset q_i^{-1}(O_Q(h_i(v)))$.
- If $q_i = q_j$ on $\pi^{-1}(O_P(v))$ then $h_i(v) = h_j(v)$.

Suppose $v_0, \ldots, v_n \in \text{Ver}(P)$ span a simplex in $P$. Since $\pi$ is essential,

$$
\emptyset \neq \pi^{-1}(O_P(v_0) \cap \cdots \cap O_P(v_n))
\subset q_i^{-1}(O_Q(h_i(v_0)) \cap \cdots \cap O_Q(h_i(v_n))).
$$

In particular $O_Q(h_i(v_0)) \cap \cdots \cap O_Q(h_i(v_n)) \neq \emptyset$. So $h_i(v_0), \ldots, h_i(v_n)$ span a simplex in $Q$. This implies that we can extend $h_i$ to a simplicial map from $P$ to $Q$. The condition (3) immediately follows from the choices of $h_i(v)$.

For the proof of (1), take $x \in \mathcal{X}$ and let $v_0, \ldots, v_n \in \text{Ver}(P)$ be the vertices of the open simplex of $P$ containing $\pi(x)$. Then $h_i(\pi(x))$ belongs to the simplex spanned by $h_i(v_0), \ldots, h_i(v_n)$. On the other hand

$$
x \in \pi^{-1}(O_P(v_0) \cap \cdots \cap O_P(v_n))
\subset q_i^{-1}(O_Q(h_i(v_0)) \cap \cdots \cap O_Q(h_i(v_n))).
$$

Hence $q_i(x) \in O_Q(h_i(v_0)) \cap \cdots \cap O_Q(h_i(v_n))$ and there exists a simplex $\Delta \subset Q$ containing $q_i(x)$ and $h_i(v_0), \ldots, h_i(v_n)$. Then $\Delta$ contains both $q_i(x)$ and $h_i(\pi(x))$.

For the proof of (2), take a subcomplex $Q' \subset Q$. Then

Claim 5.6. Let $v \in \text{Ver}(P)$ and $1 \leq i \leq N$. If $\pi^{-1}(O_P(v)) \subset q_i^{-1}(Q')$ then $h_i(v) \in Q'$ and $\pi^{-1}(O_P(v)) \subset q_i^{-1}(O_Q(h_i(v)))$.

Proof.

$$
q_i(\pi^{-1}(O_P(v))) \subset Q' \cap O_Q(h_i(v)).
$$

If $h_i(v) \notin Q'$ then the right-hand side is empty. So $h_i(v) \in Q'$ and hence $Q' \cap O_Q(h_i(v)) = O_Q(h_i(v))$. 

\[\square\]
Let $\Delta \subset P$ be a simplex with vertices $v_0, \ldots, v_n$. If $\pi^{-1}(O_P(\Delta)) \subset q_i^{-1}(Q')$ then $h_i(v_0), \ldots, h_i(v_n) \in Q'$ and (since $\pi$ is essential)
$$\emptyset \neq \pi^{-1}(O_P(v_0) \cap \cdots \cap O_P(v_n))$$
$$\subset q_i^{-1}(O_Q(h_i(v_0)) \cap \cdots \cap O_Q(h_i(v_n))).$$
In particular $O_Q(h_i(v_0)) \cap \cdots \cap O_Q(h_i(v_n)) \neq \emptyset$. So $h_i(v_0), \ldots, h_i(v_n)$ span a simplex in $Q'$. Then $h_i(\Delta) \subset Q'$.

Letting $N = 1$ in Lemma 5.5, we get the following corollary. This is used in Section 5.3.

**Corollary 5.7.** Let $\mathcal{X}$ be a topological space and $P, Q$ simplicial complexes. Let $\pi: \mathcal{X} \to P$ and $q: \mathcal{X} \to Q$ be continuous maps. If $\pi$ is essential and
$$\{q^{-1}(O_Q(w))\}_{w \in \mathcal{V}} < \{\pi^{-1}(O_P(v))\}_{v \in \mathcal{V}},$$
then there exists a simplicial map $h: P \to Q$ such that for every $x \in \mathcal{X}$ the two points $q(x)$ and $h(\pi(x))$ belong to the same simplex in $Q$.

We need to introduce a notation for Lebesgue number. Let $(\mathcal{X}, d)$ be a compact metric space and $\mathcal{U}$ its open cover. We denote the **Lebesgue number** of $\mathcal{U}$ by $\text{LN}(\mathcal{X}, d, \mathcal{U})$, namely it is the supremum of $\varepsilon > 0$ such that if a subset $A \subset \mathcal{X}$ satisfies $\text{diam} A < \varepsilon$ then there exists $U \in \mathcal{U}$ containing $A$.

### 5.3 Warmup: the proof of Pontrjagin–Schnirelmann’s theorem.

Here we prove Pontrjagin–Schnirelmann’s theorem (Theorem 5.1) by using the preparations of Section 5.2. This is a toy model of the proof of Theorem 3.12. (This subsection is logically independent of the proof of Theorem 3.12.) Our proof of Theorem 5.1 roughly follows the line of ideas of [PS32]. Our purpose here is to help readers to get acquainted with how to use lemmas in the previous subsection. Theorem 5.1 follows from

**Theorem 5.8.** Let $(V, \|\cdot\|)$ be an infinite dimensional Banach space and $\mathcal{X}$ a compact metrizable space. For a dense subset of $f$ in $C(\mathcal{X}, V)$ (the space of continuous maps from $\mathcal{X}$ to $V$ endowed with the norm topology), $f$ is a topological embedding and satisfies
$$\overline{\dim}_M(f(\mathcal{X}), \|\cdot\|) = \dim \mathcal{X}.$$  

**Proof.** We can assume that $D := \dim \mathcal{X} < \infty$. Fix a metric $d$ on $\mathcal{X}$ and take arbitrary $f \in C(\mathcal{X}, V)$ and $\eta > 0$. We want to construct a topological embedding $f': \mathcal{X} \to V$ satisfying $\|f(x) - f'(x)\| < \eta$ and $\overline{\dim}_M(f'(\mathcal{X}), \|\cdot\|) = D$. (It is enough to prove $\overline{\dim}_M(f'(\mathcal{X}), \|\cdot\|) \leq D$ because Minkowski dimension always dominates topological dimension.) We may assume that $f(\mathcal{X})$ is contained in the open unit ball $B_1^2(V)$. We will inductively construct the following data ($n \geq 1$):

**Data 5.9.** (1) Positive numbers $\varepsilon_n$ and $\delta_n$ with $\varepsilon_{n+1} < \varepsilon_n/2$, $\delta_{n+1} < \delta_n/2$ and $\delta_1 < \eta/2$. 

Hence (Since Condition 5.10. (1)

$$P_\pi$$ implies that $$f$$ is a simplicial complex of dimension $$\leq D$$.

(3) A linear embedding $$g_n: P_n \to B_1^2(V)$$. We assume the following conditions:

**Condition 5.10. (1)**

$$\# (g_n(P_n), \|\|, \varepsilon) < \begin{cases} \left( \frac{2}{\varepsilon} \right)^{D+\frac{1}{n-1}} & (\varepsilon < \varepsilon_{n-1}) \\ \left( \frac{1}{\varepsilon} \right)^{D+\frac{1}{n}} & (\varepsilon < \varepsilon_n) \end{cases}$$

Here the former condition is empty for $$n = 1$$. (2) Set $$f_n = g_n \circ \pi_n: \mathcal{X} \to V$$. If a continuous map $$f': \mathcal{X} \to V$$ satisfies $$\|f'(x) - f_n(x)\| < \delta_n$$ then $$f'$$ is a $$(1/n)$$-embedding with respect to $$d$$. (3)

$$\|f_1(x) - f(x)\| < \frac{\eta}{2},$$

$$\|f_n(x) - f_{n+1}(x)\| < \min \left( \frac{\varepsilon_n}{8}, \frac{\delta_n}{2} \right).$$

Suppose we have constructed the above data. Then we can define $$f' \in C(\mathcal{X}, V)$$ by $$f'(x) = \lim_{n \to \infty} f_n(x)$$. It follows from Condition 5.10 (3) that $$\|f'(x) - f_n(x)\| < \delta_n$$ for all $$n \geq 1$$. So by Condition 5.10 (2) $$f'$$ is a $$(1/n)$$-embedding for all $$n \geq 1$$, which implies that $$f'$$ is a topological embedding. It also satisfies $$\|f'(x) - f(x)\| < \eta$$.

We want to prove $$\overline{\dim}_M (f'(\mathcal{X}), \|\|) \leq D$$. Let $$0 < \varepsilon < \varepsilon_1$$. Take $$n > 1$$ with $$\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}$$. It follows from Condition 5.10 (3) that $$\|f'(x) - f_n(x)\| < \varepsilon_n/4$$. Hence

$$\# \left( f'(\mathcal{X}), \|\|, \varepsilon \right) \leq \# \left( f_n(\mathcal{X}), \|\|, \varepsilon - \frac{\varepsilon_n}{2} \right)$$

$$\leq \# \left( g_n(P_n), \|\|, \varepsilon - \frac{\varepsilon_n}{2} \right) \quad \text{(by } f_n(\mathcal{X}) \subset g_n(P_n))$$

$$< \left( \frac{2}{\varepsilon - \frac{\varepsilon_n}{2}} \right)^{D+\frac{1}{n-1}} \quad \text{(by Condition 5.10(1))}$$

$$\leq \left( \frac{4}{\varepsilon} \right)^{D+\frac{1}{n-1}} \quad \text{(by } \varepsilon \geq \varepsilon_n).$$

Since $$n \to \infty$$ as $$\varepsilon \to 0$$, this shows $$\overline{\dim}_M (f'(\mathcal{X}), \|\|) \leq D$$.

Now we start to construct the data. We choose $$0 < \tau_1 < 1$$ so that

$$d(x, y) < \tau_1 \implies \|f(x) - f(y)\| < \eta/2.$$

Let $$\pi_1(\mathcal{X}, d) \to P_1$$ be a $$\tau_1$$-embedding in a simplicial complex $$P_1$$ with $$\dim P_1 \leq D$$. (Since $$D = \dim \mathcal{X} = \lim_{\varepsilon \to 0} \text{Widim}_\varepsilon(\mathcal{X}, d)$$, we can find such a map.) By subdividing $$P_1$$ sufficiently fine, we can assume $$\text{diam} \pi_1^{-1}(O(v)) < \tau_1$$ for all $$v \in \text{Ver}(P_1)$$.
Then by Lemma 5.3 (3) we can find a linear map $\tilde{g}_1: P_1 \to B^*_1(V)$ satisfying $\|\tilde{g}_1(\pi_1(x)) - f(x)\| < \eta/2$. Since linear embeddings are dense in $\text{Hom}(P_1, V)$ (Lemma 5.3 (2)), we can also find a linear embedding $g_1: P_1 \to B^*_1(V)$ satisfying $\|g_1(\pi_1(x)) - f(x)\| < \eta/2$. By Lemma 5.3 (1), we can find $\varepsilon_1 > 0$ satisfying Condition 5.10 (1). The map $f_1 = g_1 \circ \pi_1$ is a 1-embedding and “1-embedding” is an open condition. So there exists $\delta_1 > 0$ such that Condition 5.10 (2) holds true. This finishes the construction for $n = 1$.

Suppose that we have already done the construction for the $n$th step. We try to construct the data for the $(n + 1)$th step. We subdivide $P_n$ sufficiently fine so that every simplex $\Delta \subset P_n$ satisfies $\text{diam}(g_n(\Delta), \|\cdot\|) < \min(\varepsilon_n/8, \delta_n/2)$. We take $0 < \tau_{n+1} < 1/(n + 1)$ with

$$\tau_{n+1} < \text{LN}(\mathcal{X}, d, \{\pi_n^{-1}(O_{P_n}(v))\}_{v \in \text{Ver}(P_n)})\,.$$  

Take a $\tau_{n+1}$-embedding $\pi_{n+1}: (\mathcal{X}, d) \to P_{n+1}$ with a simplicial complex $P_{n+1}$ of dimension $\leq D$. By subdividing $P_{n+1}$ sufficiently fine, we can assume $\text{diam}(\pi_{n+1}^{-1}(O_{P_n+1}(v))) < \tau_{n+1}$ for all $v \in \text{Ver}(P_{n+1})$. Moreover, by replacing $P_{n+1}$ with a subcomplex (if necessarily), we can assume that $\pi_{n+1}: \mathcal{X} \to P_{n+1}$ is essential (Lemma 5.4). The open cover $\{\pi_{n+1}^{-1}(O_{P_n+1}(v))\}_{v \in \text{Ver}(P_{n+1})}$ of $\mathcal{X}$ becomes a refinement of $\{\pi_n^{-1}(O_{P_n}(v))\}_{v \in \text{Ver}(P_n)}$ because of the Lebesgue number condition above. Then by applying Corollary 5.7 to $\pi_{n+1}: \mathcal{X} \to P_{n+1}$ and $\pi_n: \mathcal{X} \to P_n$ (with $P = P_{n+1}$ and $Q = P_n$), we can find a simplicial map $h: P_{n+1} \to P_n$ such that for every $x \in \mathcal{X}$ the two points $\pi_n(x)$ and $h(\pi_{n+1}(x))$ belong to the same simplex of $P_n$. Set $\bar{g}_{n+1} = g_n \circ h: P_{n+1} \to B^*_1(V)$. This satisfies (recall $f_n = g_n \circ \pi_n$)

$$\|\bar{g}_{n+1}(\pi_{n+1}(x)) - f_n(x)\| < \min\left(\frac{\varepsilon_n}{8}, \frac{\delta_n}{2}\right).$$

Since $\bar{g}_{n+1}(P_{n+1}) \subset g_n(P_n)$, the induction hypothesis implies

$$\#(\bar{g}_{n+1}(P_{n+1}), \|\cdot\|, \varepsilon) < \left(\frac{1}{\varepsilon}\right)^{D_1 + \frac{1}{\varepsilon}} \quad (\varepsilon < \varepsilon_n).$$  

(5.3)

By Lemma 5.3 (1), there exists $0 < \varepsilon_{n+1} < \varepsilon_n/2$ such that for all linear maps $g: P_{n+1} \to V$ with $g(P_{n+1}) \subset B^*_1(V)$

$$\#(g(P_{n+1}), \|\cdot\|, \varepsilon) < \left(\frac{1}{\varepsilon}\right)^{D_1 + \frac{1}{\varepsilon}} \quad (\varepsilon < \varepsilon_{n+1}).$$

(5.4)

We slightly perturb $\bar{g}_{n+1}$ by Lemma 5.3 (2): There exists a linear embedding $g_{n+1}: P_{n+1} \to B^*_1(V)$ such that

$$\|g_{n+1}(\pi_{n+1}(x)) - f_n(x)\| < \min\left(\frac{\varepsilon_n}{8}, \frac{\delta_n}{2}\right),$$

$$\|g_{n+1}(u) - \bar{g}_{n+1}(u)\| < \frac{\varepsilon_{n+1}}{4} \quad (u \in P_{n+1}).$$

(5.4)
By the choice of $\varepsilon_{n+1}$ we have
\[
\# (g_{n+1}(P_{n+1}), \|\cdot\|, \varepsilon) < \left( \frac{1}{\varepsilon} \right)^{D + \frac{1}{n+1}} (\varepsilon < \varepsilon_{n+1}).
\]
For $\varepsilon_{n+1} \leq \varepsilon < \varepsilon_n$
\[
\# (g_{n+1}(P_{n+1}), \|\cdot\|, \varepsilon) \leq \# (\tilde{g}_{n+1}(P_{n+1}), \|\cdot\|, \varepsilon - \frac{\varepsilon_{n+1}}{2}) \quad \text{(by (5.4))}
\leq \# (\tilde{g}_{n+1}(P_{n+1}), \|\cdot\|, \frac{\varepsilon}{2}) \quad \text{(by } \varepsilon - \frac{\varepsilon_{n+1}}{2} \geq \frac{\varepsilon}{2})
\leq \left( \frac{2}{\varepsilon} \right)^{D + \frac{1}{n}} \quad \text{(by (5.3))}.
\]

$f_{n+1} = g_{n+1} \circ \pi_{n+1}$ is a $1/(n + 1)$-embedding. So we can find $0 < \delta_{n+1} < \delta_n/2$ satisfying Condition 5.10 (2). This has completed the construction for the $(n+1)$th step. \hfill \Box

5.4 Dynamical tiling construction. Here we review a construction introduced in [GLT16]. Let $(\mathcal{X}, T)$ be a dynamical system and $\varphi: \mathcal{X} \to [0, 1]$ a continuous function. For $x \in \mathcal{X}$ we consider
\[
\left\{ \left( a, \frac{1}{\varphi(T^a x)} \right) \mid a \in \mathbb{Z} \text{ with } \varphi(T^a x) > 0 \right\} \subset \mathbb{R}^2. \tag{5.5}
\]
We assume that this is nonempty for every $x \in \mathcal{X}$. (Namely, for every $x \in \mathcal{X}$, there exists $a \in \mathbb{Z}$ with $\varphi(T^a x) > 0$.) Let $\mathbb{R}^2 = \bigcup_{a \in \mathbb{Z}} V_{\varphi}(x, a)$ be the associated Voronoi diagram, namely $V_{\varphi}(x, a)$ is the set of $u \in \mathbb{R}^2$ satisfying
\[
\left| u - \left( a, \frac{1}{\varphi(T^a x)} \right) \right| \leq \left| u - \left( b, \frac{1}{\varphi(T^b x)} \right) \right|
\]
for any $b \in \mathbb{Z}$ with $\varphi(T^b x) > 0$. This is a convex subset of the plane. We set
\[
I_{\varphi}(x, a) = V_{\varphi}(x, a) \cap (\mathbb{R} \times \{0\})
\]
See Figure 1. If $\varphi(T^a x) = 0$ then $V_{\varphi}(x, a) = I_{\varphi}(x, a) = \emptyset$.

We naturally identify $\mathbb{R} \times \{0\}$ with $\mathbb{R}$. Then this construction gives a decomposition of $\mathbb{R}$:
\[
\mathbb{R} = \bigcup_{a \in \mathbb{Z}} I_{\varphi}(x, a).
\]
$I_{\varphi}(x, a)$ are closed intervals. We set
\[
\partial_{\varphi}(x) = \bigcup_{a \in \mathbb{Z}} \partial I_{\varphi}(x, a) \subset \mathbb{R},
\]
where $\partial I_{\varphi}(x, a)$ is the boundary of the interval $I_{\varphi}(x, a)$ (e.g. $\partial[0, 1] = \{0, 1\}$). This construction is equivariant, namely
\[
I_{\varphi}(T^n x, a) = -n + I_{\varphi}(x, a + n), \quad \partial_{\varphi}(T^n x) = -n + \partial_{\varphi}(x).
\]
Lemma 5.11. Suppose \((\mathcal{X}, T)\) has the marker property. Then for any \(\varepsilon > 0\) we can find a continuous function \(\varphi: \mathcal{X} \to [0, 1]\) such that (5.5) is nonempty for every \(x \in \mathcal{X}\) and that it satisfies the following conditions.

1. There exists \(M > 0\) such that \(I_{\varphi}(x, a) \subset (a - M, a + M)\) for all \(x \in \mathcal{X}\) and \(a \in \mathbb{Z}\) (in particular, all \(I_{\varphi}(x, a)\) are finite length intervals).

2. \(\lim_{R \to \infty} \sup_{x \in \mathcal{X}} |\partial_{\varphi}(x) \cap [0, R]| / R < \varepsilon.\)

Here \(|\partial_{\varphi}(x) \cap [0, R]|\) is the cardinality of \(\partial_{\varphi}(x) \cap [0, R]\). Notice that the above (1) implies that \(\partial_{\varphi}(x)\) is a discrete set in the real line.

3. The intervals \(I_{\varphi}(x, a)\) continuously depend on \(x \in \mathcal{X}\): i.e. if \(x_k \to x\) in \(\mathcal{X}\) and \(I_{\varphi}(x_k, a)\) has positive length then \(I_{\varphi}(x_k, a)\) converges to \(I_{\varphi}(x, a)\) in the Hausdorff topology, and if \(I_{\varphi}(x, a) = \emptyset\) then for all \(k\) large enough \(I_{\varphi}(x_k, a)\) is also empty.

Proof. Take \(N > 1/\varepsilon\). From the marker property, there exists an open set \(U \subset \mathcal{X}\) such that \(U \cap T^{-n}U = \emptyset\) for \(1 \leq n \leq N\) and \(\mathcal{X} = \bigcup_{n=1}^{N} T^{-n}U\). We can find \(M > N\) and a compact subset \(K \subset U\) with \(\mathcal{X} = \bigcup_{n=0}^{M-1} T^{-n}K\). Let \(\varphi: \mathcal{X} \to [0, 1]\) be a continuous function such that \(\varphi = 1\) on \(K\) and \(\text{supp } \varphi \subset U\).

Let \(x \in \mathcal{X}\) and consider

\[ \Lambda_x = \{a \in \mathbb{Z} | \varphi(T^n x) > 0\}, \quad \Lambda'_x = \{a \in \mathbb{Z} | \varphi(T^n x) = 1\}. \]

Any gap of \(\Lambda_x\) (i.e. the difference between two successive numbers in \(\Lambda_x\)) is larger than \(N\), and any gap of \(\Lambda'_x\) is smaller than or equal to \(M\). For \(a \in \Lambda'_x\) the interval \(I_{\varphi}(x, a)\) contains \(a\) as an interior point.

Let \(a \in \Lambda_x\). There exist \(s, t \in \Lambda'_x\) such that \(s < a < t\) and \(a - s, t - a \leq M\). Then \(I_{\varphi}(x, a) \subset (s, t) \subset (a - M, a + M)\). The continuity of \(I_{\varphi}(x, a)\) is an immediate consequence of the definition. The condition (2) follows from

\[ \lim_{R \to \infty} \sup_{x \in \mathcal{X}} |\partial_{\varphi}(x) \cap [0, R]| / R \leq \lim_{R \to \infty} \sup_{x \in \mathcal{X}} |\Lambda_x \cap [0, R]| / R \leq \frac{1}{N} < \varepsilon. \]
5.5 Proof of Theorem 3.12. Let $(V, \| \cdot \|)$ be an infinite dimensional Banach space and $(\mathcal{X}, T)$ a dynamical system. As in Section 5.3 we denote by $C(\mathcal{X}, V)$ the space of continuous maps from $\mathcal{X}$ to $V$ endowed with the norm topology. Theorem 3.12 follows from

**Theorem 5.12.** Suppose $(\mathcal{X}, T)$ has the marker property. For a dense subset of $f \in C(\mathcal{X}, V)$, $f$ is a topological embedding and satisfies

$$\overline{\text{indim}}(\mathcal{X}, T, f^* \| \cdot \|) = \text{mdim}(\mathcal{X}, T).$$

Here $f^* \| \cdot \|$ is the metric $\| f(x) - f(y) \|$ $(x, y \in \mathcal{X})$.

**Proof.** We can assume $D := \text{mdim}(\mathcal{X}, T) < \infty$. Fix a metric $d$ on $\mathcal{X}$ and take an arbitrary $f \in C(\mathcal{X}, V)$ and $\eta > 0$. We want to construct a topological embedding $f' : \mathcal{X} \to V$ satisfying $\| f(x) - f'(x) \| < \eta$ for all $x \in \mathcal{X}$ and $\overline{\text{indim}}(\mathcal{X}, T, (f')^* \| \cdot \|) = D$. (It is enough to show $\overline{\text{indim}}(\mathcal{X}, T, (f')^* \| \cdot \|) \leq D$ because the reverse inequality is always true by Theorem 2.2). We may assume that $f(\mathcal{X})$ is contained in the open unit ball $B_{\eta}^*(V)$.

We prepare some notations. For a natural number $N$ we set $[N] = \{0, 1, 2, \ldots, N - 1\}$. We define a norm $\| \cdot \|_N$ on $V^N$ (the $N$th power of $V$) by

$$\|(x_0, x_1, \ldots, x_{N-1})\|_N = \max(\|x_0\|, \|x_1\|, \ldots, \|x_{N-1}\|).$$

For a simplicial complex $P$ we define the number $A(P)$ as the minimum of $A \geq 1$ such that for any linear map $g : P \to B_1(V)$ we have $\#(g(P), \| \cdot \|, \varepsilon) \leq (1/\varepsilon)^A$ for all $0 < \varepsilon \leq 1/2$. (Such a number always exists by Lemma 5.3 (1).) For simplicial complexes $P$ and $Q$, we denote their join by $P \ast Q$, namely it is the quotient of $[0, 1] \times P \times Q$ by the equivalence relation

$$(0, p, q) \sim (0, p, q'), \quad (1, p, q) \sim (1, p', q), \quad (p, p' \in P, q, q' \in Q).$$

We denote the equivalence class of $(t, p, q)$ by $(1 - t)p \oplus tq$. We identify $P$ and $Q$ with $\{(0, p, *)| p \in P\}$ and $\{(1, *, q)| q \in Q\}$ in $P \ast Q$ respectively. If $g : P \to B_1(V)$ and $g' : Q \to B_1(V)$ we define the map $g \ast g' : P \ast Q \to B_1(V)$ by sending $(1 - t)p \oplus tq$ to $(1 - t)g(p) + tq'(q)$.

For a continuous map $f' : \mathcal{X} \to V$ and $I \subset \mathbb{R}$ we define $\Phi_{f', I} : \mathcal{X} \to V^{I \cap \mathbb{Z}}$ by

$$\Phi_{f', I}(x) = (f'(T^ax))_{a \in I \cap \mathbb{Z}}.$$ 

For a natural number $R$ we set $\Phi_{f', R} := \Phi_{f', [0, R]} : \mathcal{X} \to V^R$. We fix a continuous function $\alpha : \mathbb{R} \to [0, 1]$ such that $\alpha(t) = 1$ for $t \leq 1/2$ and $\alpha(t) = 0$ for $t \geq 3/4$.

We will inductively construct the following data for $n \geq 1$.

**Data 5.13.** (1) $1/2 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$ with $\varepsilon_{n+1} < \varepsilon_n/2$ and $\eta/2 > \delta_1 > \delta_2 > \cdots > 0$ with $\delta_{n+1} < \delta_n/2$.

(2) A natural number $N_n$. 


(3) A continuous function $\varphi_n: \mathcal{X} \to [0, 1]$ such that for every $x \in \mathcal{X}$ there exists $a \in \mathbb{Z}$ with $\varphi_n(T^ax) > 0$. We apply the dynamical tiling construction of Section 5.4 to this function and get the decomposition $\mathbb{R} = \bigcup_{a \in \mathbb{Z}} I_{\varphi_n}(x, a)$ for each $x \in \mathcal{X}$.

(4) A $(1/n)$-embedding $\pi_n: (\mathcal{X}, d_{N_n}) \to P_n$ with a simplicial complex $P_n$ of dimension less than $(D + \frac{1}{n})N_n$.

(5) A $(1/n)$-embedding $\pi'_n: (\mathcal{X}, d) \to Q_n$ with a simplicial complex $Q_n$.

(6) For each $\lambda \in [N_n]$, a linear embedding $g_{n, \lambda}: P_n \to B_1^\varepsilon(V)$.

(7) A linear embedding $g'_n: Q_n \to B_1^\varepsilon(V)$.

We assume the following conditions.

**Condition 5.14.** (1) For each $\lambda \in [N_n]$, the join $g_{n, \lambda} * g'_n: P_n * Q_n \to B_1^\varepsilon(V)$ is a linear embedding. For $\lambda_1 \neq \lambda_2$

$$g_{n, \lambda_1} * g'_n(P_n * Q_n) \cap g_{n, \lambda_2} * g'_n(P_n * Q_n) = g'_n(Q_n).$$

(2) Set $g_n = (g_{n, 0}, g_{n, 1}, \ldots, g_{n, N_n - 1}): P_n \to V^{N_n}$. Then

$$\#(g_n(P_n), \|\cdot\|_{N_n}, \varepsilon) < \begin{cases} 4^{N_n} \left(\frac{2}{\varepsilon}\right)^{(D + \frac{2}{n})N_n} & (0 < \varepsilon < \varepsilon_{n-1}) \\ \left(\frac{1}{\varepsilon}\right)^{(D + \frac{1}{n})N_n} & (0 < \varepsilon < \varepsilon_n). \end{cases}$$

Here the former condition is empty for $n = 1$.

(3) There exists $M_n > 0$ such that $I_{\varphi_n}(x, a) \subset (a - M_n, a + M_n)$ for all $x \in \mathcal{X}$ and $a \in \mathbb{Z}$. The sets $\partial_{\varphi_n}(x)$ are discrete in $\mathbb{R}$ and satisfy

$$\lim_{R \to \infty} \sup_{x \in \mathcal{X}} \|\partial_{\varphi_n}(x) \cap [0, R]\| \leq \frac{1}{2nN_n^2 \cdot A(P_n * Q_n)}.$$

(4) We define a continuous map $f_n: \mathcal{X} \to B_1^\varepsilon(V)$ as follows: Let $x \in \mathcal{X}$ and take $a \in \mathbb{Z}$ with $0 \in I_{\varphi_n}(x, a)$. Take $b \in \mathbb{Z}$ such that $b \equiv a \pmod{N_n}$ and $0 \in b + [N_n]$. Set

$$f_n(x) = \{1 - \alpha (\text{dist}(0, \partial_{\varphi_n}(x)))\} g_{n, -b} \left(\pi_n(T^bx)\right)$$

$$+ \alpha (\text{dist}(0, \partial_{\varphi_n}(x))) g'_n \left(\pi'_n(x)\right),$$

where $\text{dist}(0, \partial_{\varphi_n}(x)) := \min_{t \in \partial_{\varphi_n}(x)} |t|$. Then we assume that if a continuous map $f': \mathcal{X} \to V$ satisfies $\|f'(x) - f_n(x)\| < \delta_n$ for all $x \in \mathcal{X}$ then it is a $(1/n)$-embedding with respect to $d$.

(5) For all $x \in \mathcal{X}$

$$\|f(x) - f_1(x)\| < \frac{\eta}{2},$$

$$\|f_n(x) - f_{n+1}(x)\| < \min\left(\frac{\varepsilon_n}{8}, \frac{\delta_n}{2}\right).$$
Suppose we have constructed the above data. Then we can define $f' \in C(\mathcal{X}, V)$ by $f'(x) = \lim_{n \to \infty} f_n(x)$. It satisfies $\|f'(x) - f(x)\| < \eta$ and $\|f'(x) - f_n(x)\| < \min(\varepsilon_n/4, \delta_n)$ for all $n \geq 1$. From Condition 5.14 (4), $f'$ is a $(1/n)$-embedding for all $n \geq 1$. So it is a topological embedding. Set $d'(x, y) = \|f'(x) - f'(y)\|$. We want to show $\text{mdim}_M(\mathcal{X}, T, d') \leq D$. Notice that $\text{mdim}_M(\mathcal{X}, T, d')$ is equal to

$$\lim_{\varepsilon \to 0} \left\{ \lim_{R \to \infty} \frac{\log \#(\Phi_{f', R}(\mathcal{X}), \| \cdot \|_R, \varepsilon)}{\log(1/\varepsilon)} \right\}.$$  \hspace{1cm} (5.6)

**Claim 5.15.** Let $0 < \varepsilon < \varepsilon_{n-1}$ $(n \geq 2)$. For sufficiently large natural numbers $R$

$$\#(\Phi_{f_n, R}(\mathcal{X}), \| \cdot \|_R, \varepsilon) < 2^{4R} \left( \frac{2}{\varepsilon} \right)^{(D + \frac{2}{n-1})R + \frac{2}{n}}.$$  

**Proof.** In this proof $n \geq 2$ is fixed and $R$ is a sufficiently large natural number. Let $x \in \mathcal{X}$. We call $J = \{b, b + 1, \ldots, b + N_n - 1\}$ $(b \in \mathbb{Z})$ **good for** $x$ if there is $a \in \mathbb{Z}$ such that $b \equiv a \pmod{N_n}$ and $(b - 1, b + N_n) \subset I_{\varepsilon_n}(x, a)$. If $J$ is good for $x$ then $\Phi_{f_n, J}(x)$ is contained in $g_n(P_n)$ in $V^{N_n}$. We denote by $J_x$ the union of $J \subset [R]$ which are good for $x$. The number of possibilities of $J_x$ (when $x \in \mathcal{X}$ varies) is bounded by $2^R$. Then $\#(\Phi_{f_n, R}(\mathcal{X}), \| \cdot \|_R, \varepsilon)$ is bounded by

$$2^R \cdot \#(g_n(P_n), \| \cdot \|_n, \varepsilon) \cdot \#(f_n(\mathcal{X}), \| \cdot \|, \varepsilon) \cdot \sup_{x \in \mathcal{X}} |\partial_{\varepsilon_n}(x) \cap [0, R]| + 2N_n.$$  

Here $+2N_n$ is the edge effect. $f_n(\mathcal{X})$ is contained in the union of $g_n, f_n'(P_n * Q_n)$ over $\lambda \in [N_n]$. So

$$\#(f_n(\mathcal{X}), \| \cdot \|, \varepsilon) \leq N_n \left( \frac{1}{\varepsilon} \right)^{A(P_n * Q_n)}.$$  

Using Condition 5.14 (2) and (3), we get the statement of the claim. \hfill $\square$

We now return to the proof of Theorem 5.12.

Let $0 < \varepsilon < \varepsilon_1$. Take $n > 1$ with $\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}$. Recall that $\|f'(x) - f_n(x)\| < \varepsilon_n/4$. Hence

$$\#(\Phi_{f', R}(\mathcal{X}), \| \cdot \|_R, \varepsilon) \leq \#(\Phi_{f_n, R}(\mathcal{X}), \| \cdot \|_R, \varepsilon - \frac{\varepsilon_n}{2}) \leq \#(\Phi_{f_n, R}(\mathcal{X}), \| \cdot \|_R, \varepsilon - \frac{\varepsilon_n}{2}).$$  

From Claim 5.15,

$$\lim_{R \to \infty} \frac{\log \#(\Phi_{f', R}(\mathcal{X}), \| \cdot \|_R, \varepsilon)}{R} \leq 4 + \left( D + \frac{2}{n-1} + \frac{1}{n} \right) \log(4/\varepsilon).$$

Notice that $n \to \infty$ as $\varepsilon \to 0$. Using (5.6) we get $\text{mdim}_M(\mathcal{X}, T, d') \leq D$.
Induction: Step 1. Now we start to construct the data. Take $0 < \tau_1 < 1$ such that

$$d(x, y) < \tau_1 \implies \|f(x) - f(y)\| < \frac{\eta}{2}. \tag{5.7}$$

From the definition of mean dimension, we can find $N_1 > 0$ and $\tau_1$-embeddings $\pi_1: (\mathcal{X}, d_{N_1}) \to P_1$ and $\pi'_1: (\mathcal{X}, d) \to Q_1$ such that $P_1$ and $Q_1$ are simplicial complexes with $\dim P_1 < N_1(D + 1)$. By subdividing $P_1$ and $Q_1$, we can assume that

$$\diam(\pi_1^{-1}(O_{P_1}(v), d_{N_1})) < \tau_1, \quad \diam((\pi'_1)^{-1}(O_{Q_1}(w), d)) < \tau_1$$

for all $v \in \Ver(P_1)$ and $w \in \Ver(Q_1)$. By Lemma 5.3 (3), we can find linear maps $\tilde{g}_{1, \lambda}: P_1 \to B^1_8(V)$ for $\lambda \in [N_1]$ and $\tilde{g}'_1: Q_1 \to B^2_8(V)$ satisfying

$$\|f(T^\lambda x) - \tilde{g}_{1, \lambda}(\pi_1(x))\| < \frac{\eta}{2}, \quad \|f(x) - \tilde{g}'_1(\pi'_1(x))\| < \frac{\eta}{2}$$

for all $x \in \mathcal{X}$. By Lemma 5.3 (2), we can replace $\tilde{g}_{1, \lambda}$ and $\tilde{g}'_1$ with linear embeddings $g_{1, \lambda}: P_1 \to B^1_8(V)$ and $g'_1: Q_1 \to B^2_8(V)$ satisfying Condition 5.14 (1) and

$$\|f(T^\lambda x) - g_{1, \lambda}(\pi_1(x))\| < \frac{\eta}{2}, \quad \|f(x) - g'_1(\pi'_1(x))\| < \frac{\eta}{2}. \tag{5.8}$$

By Lemma 5.3 (1), we can find $0 < \varepsilon_1 < 1/2$ satisfying Condition 5.14 (2).

By Lemma 5.11, we can take a continuous function $\varphi_1: \mathcal{X} \to [0, 1]$ satisfying Condition 5.14 (3). By Condition 5.14 (1) (which has been already established for $n = 1$), the map $f_1: \mathcal{X} \to V$ becomes a 1-embedding. It also satisfies Condition 5.14 (5) by (5.7). Since “1-embedding” is an open condition, we can find $0 < \delta_1 < \eta/2$ satisfying Condition 5.14 (4). The first step of the induction has been completed.

Induction: Step $n \Rightarrow$ Step $n + 1$. Next we suppose that we have constructed the data for $n$. We will construct the data for $n + 1$.

We subdivide $P_n \ast Q_n$ sufficiently fine so that

$$\diam(\pi_{n, \lambda} \ast g'_n(\Delta, \|\cdot\|)) < \min \left(\frac{\varepsilon_n}{8}, \frac{\delta_n}{2}\right) \tag{5.9}$$

for any simplex $\Delta \subset P_n \ast Q_n$ and $\lambda \in [N_n]$.

We define a continuous map $q_n: \mathcal{X} \to P_n \ast Q_n$ as follows. Let $x \in \mathcal{X}$ and take $a \in \mathbb{Z}$ with $0 \in I_{\varphi_n}(x, a)$. We take $b \in \mathbb{Z}$ such that $b \equiv a \mod N_n$ and $0 \in b + [N_n]$. Set

$$q_n(x) = \left\{1 - \alpha \left(\dist(0, \partial_{\varphi_n}(x))\right)\right\} \pi_n(T^b x) \oplus \alpha \left(\dist(0, \partial_{\varphi_n}(x))\right) \pi'_n(x).$$

We take $0 < \tau_{n+1} < 1/(n + 1)$ such that

- If $d(x, y) < \tau_{n+1}$ then $\|f_n(x) - f_n(y)\| < \min(\varepsilon_n/8, \delta_n/2)$ and

$$|\dist(0, \partial_{\varphi_n}(x)) - \dist(0, \partial_{\varphi_n}(y))| < \frac{1}{4}. \tag{5.10}$$
If \( d(x, y) < \tau_{n+1} \) and \((-1/4, 1/4) \subset I_{\varphi_n}(x, a) \) then \( I_{\varphi_n}(y, a) \) contains 0 as an interior point.

- \( \tau_{n+1} \) is smaller than the Lebesgue number of the open cover \( \{ q_n^{-1}(O_{P_n * Q_n}(v)) \} \) \( v \in \text{Ver}(P_n * Q_n) \):

\[
\tau_{n+1} < LN(\mathcal{X}, d, \{ q_n^{-1}(O_{P_n * Q_n}(v)) \} \subseteq \text{Ver}(P_n * Q_n)).
\]

Take a \( \tau_{n+1} \)-embedding \( \pi_{n+1}^\prime : (\mathcal{X}, d) \to Q_{n+1} \) with a simplicial complex \( Q_{n+1} \).

We can assume \( \text{diam}((\pi_{n+1}^\prime)^{-1}(O_{Q_{n+1}}(w)), d) < \tau_{n+1} \) for every \( w \in \text{Ver}(Q_{n+1}) \).

By Lemma 5.3 (3), we can take a linear map \( \tilde{g}_{n+1} : Q_{n+1} \to B^2(V) \) satisfying

\[
\| \tilde{g}_{n+1}(\pi_{n+1}(x)) - f_n(x) \| < \min\left( \frac{\varepsilon_n}{8}, \frac{\delta_n}{2} \right).
\]

We can find \( N_{n+1} > N_n \) such that

- There exists a \( \tau_{n+1} \)-embedding \( \pi_{n+1} : (\mathcal{X}, d_{N_{n+1}}) \to P_{n+1} \) with a simplicial complex \( P_{n+1} \) of dimension less than \( N_{n+1} \)

\[
\left( D + \frac{1}{n+1} \right).
\]

Moreover by Lemma 5.4 we can assume that \( \pi_{n+1} : \mathcal{X} \to P_{n+1} \) is essential.

We apply Lemma 5.5 (with \( P = P_{n+1}, Q = P_n * Q_n, N = N_{n+1}, \) and \( Q' = P_n \) or \( Q_n \)) to continuous maps \( \pi_{n+1} : \mathcal{X} \to P_{n+1} \) and \( q_n \circ T^\lambda : \mathcal{X} \to P_n * Q_n, \lambda \in [N_{n+1}] \).

(The assumption of Lemma 5.5 is satisfied because of the above Lebesgue number condition.) Then we get simplicial maps \( h_\lambda : P_{n+1} \to P_n * Q_n, \lambda \in [N_{n+1}], \) so that

- For every \( \lambda \in [N_{n+1}] \) and \( x \in \mathcal{X} \), the two points \( h_\lambda(\pi_{n+1}(x)) \) and \( q_n(T^\lambda x) \) belong to the same simplex of \( P_n * Q_n \).
- Let \( \Delta \subset P_{n+1} \) be a simplex and \( \lambda \in [N_{n+1}] \). If \( \pi_{n+1}^{-1}(O_{P_{n+1}}(\Delta)) \subset T^{-\lambda} q_n^{-1}(P_n) \) then \( h_\lambda(\Delta) \subset P_n \). Similarly, if \( \pi_{n+1}^{-1}(O_{P_{n+1}}(\Delta)) \subset T^{-\lambda} q_n^{-1}(Q_n) \), then \( h_\lambda(\Delta) \subset Q_n \).
- For \( \lambda, \lambda' \in [N_{n+1}] \) and a simplex \( \Delta \subset P_{n+1} \), if \( q_n \circ T^{\lambda} = q_n \circ T^{\lambda'} \) on \( \pi_{n+1}^{-1}(O_{P_{n+1}}(\Delta)) \) then \( h_\lambda = h_{\lambda'} \) on \( \Delta \).

\[\text{Here is a technical point. The number } A(P_n * Q_n) \text{ is defined by using the simplicial complex structure of } P_n * Q_n. \] We use the natural simplicial complex structure of the join \( P_n * Q_n \) here, not its subdivision introduced in (5.8).
Let \( u \in P_{n+1} \), and let \( \Delta \subset P_{n+1} \) be a simplex containing \( u \). Since \( \pi_{n+1} : \mathcal{X} \to \mathcal{P}_{n+1} \) is essential, there exists \( x \in \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta) \right) \). Let \( \lambda \in [N_{n+1}] \). We take \( a, b \in \mathbb{Z} \) such that \( \lambda \in I_{\varphi_n}(x, a) \), \( b \equiv a \pmod{N_n} \) and \( \lambda \in b + [N_n] \). We set

\[
\tilde{g}_{n+1,\lambda}(u) = g_{n,\lambda - b} \ast g'_n \left( h_{\lambda}(u) \right) \in B_0^\circ(V).
\]

We will check that this is independent of the choices of \( x \) and \( a \); see Claim 5.16 below. In this setting we have \( f_n(T^\lambda x) = g_{n,\lambda - b} \ast g'_n(q_n(T^\lambda x)) \). It follows from (5.8) and the first condition of \( h_\lambda \) that

\[
\left\| \tilde{g}_{n+1,\lambda}(\pi_{n+1}(x)) - f_n(T^\lambda x) \right\| < \min \left( \epsilon_n, \frac{\delta_n}{2} \right). \tag{5.12}
\]

**Claim 5.16.** The map \( \tilde{g}_{n+1,\lambda} : P_{n+1} \to V \) is a linear map.

**Proof.** The point is that the above definition of \( \tilde{g}_{n+1,\lambda}(u) \) is independent of the choices of \( x \) and \( a \). Let \( \Delta' \subset P_{n+1} \) be another simplex containing \( u \) and pick \( x' \in \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta') \right) \). We take \( a', b' \in \mathbb{Z} \) such that \( \lambda \in I_{\varphi_n}(x', a') \), \( b' \equiv a' \pmod{N_n} \) and \( \lambda \in b' + [N_n] \).

**Case 1:** Suppose \( \text{dist}(\lambda, \partial \varphi_n(x)) > 1/4 \). We have \( d(T^\lambda x, T^\lambda x') < \tau_{n+1} \) by (5.11). From the second condition of the choice of \( \tau_{n+1} \), we have \( a = a' \) and \( b = b' \). Hence

\[
g_{n,\lambda - b} \ast g'_n(h_\lambda(u)) = g_{n,\lambda - b'} \ast g'_n(h_\lambda(u)).
\]

**Case 2:** Suppose \( \text{dist}(\lambda, \partial \varphi_n(x)) \leq 1/4 \). Take an arbitrary \( y \in \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta) \right) \cup \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta') \right) \). We have \( d(T^\lambda x, T^\lambda y) < \tau_{n+1} \) by (5.11) and hence \( \text{dist}(\lambda, \partial \varphi_n(y)) < 1/2 \) by the condition (5.9) of the choice of \( \tau_{n+1} \). Then \( q_n(T^\lambda y) = \pi_n'(T^\lambda y) \in Q_n \). Hence \( \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta) \right) \cup \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta') \right) \subset T^\lambda q_n^{-1}(Q_n) \). So \( h_\Delta(\Delta) \cup h_\Delta(\Delta') \subset Q_n \) by the second condition of \( h_\lambda \). This implies

\[
g_{n,\lambda - b} \ast g'_n(h_\lambda(u)) = g'_n(h_\lambda(u)) = g_{n,\lambda - b'} \ast g'_n(h_\lambda(u)). \tag{\*}
\]

**Claim 5.17.** Set \( \tilde{g}_{n+1}(u) = (\tilde{g}_{n+1,0}(u), \tilde{g}_{n+1,1}(u), \ldots, \tilde{g}_{n+1,N_{n+1}-1}(u)) \). Then for \( 0 < \varepsilon < \varepsilon_n \)

\[
\# \left( \tilde{g}_{n+1}(P_{n+1}), \| \cdot \|_{N_{n+1}}, \varepsilon \right) < 4^{N_{n+1}} \left( \frac{1}{\varepsilon} \right)^{(D+2)N_{n+1}}.
\]

**Proof.** This is similar to the proof of Claim 5.15. Let \( \Delta \subset P_{n+1} \) be a simplex. For \( b \in \mathbb{Z} \cap [0, N_{n+1} - N_n] \), a discrete interval \( J = \{ b, b+1, \ldots, b+N_n - 1 \} \) is said to be **good for** \( \Delta \) if there exist \( x \in \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta) \right) \) and \( a \in \mathbb{Z} \) such that \( b \equiv a \pmod{N_n} \) and \( (b-1, b+N_n) \subset I_{\varphi_n}(x, a) \). This condition implies that every \( y \in \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta) \right) \) satisfies \( (b-3/4, b+N_n - 1/4) \subset I_{\varphi_n}(y, a) \) by (5.11) and the first and second conditions of \( \tau_{n+1} \). In particular for every \( \lambda \in J \) and \( y \in \pi_{n+1}^{-1} \left( O_{P_{n+1}}(\Delta) \right) \)

\[
q_n(T^\lambda y) = q_n(T^b y) = \pi_n(T^b y) \in P_n.
\]
Then the second and third conditions of \( h_\lambda \) imply \( h_\lambda(u) = h_b(u) \in P_n \) on \( u \in \Delta \) and \( \lambda \in J \). Hence for \( u \in \Delta \)

\[
(\bar{g}_{n+1, \lambda}(u))_{\lambda \in J} = g_n(h_b(u)) \in g_n(P_n) \subset V^{N_n}.
\]

As a conclusion, if \( J \) is good for \( \Delta \) then \( (\bar{g}_{n+1, \lambda}(u))_{\lambda \in J} \in g_n(P_n) \) for all \( u \in \Delta \).

Let \( J_\Delta \) be the union of \( J = \{b, b+1, \ldots, b+N_n-1\} \subset [N_{n+1}] \) which are good for \( \Delta \). The number of possibilities of \( J_\Delta \) (when \( \Delta \subset P_{n+1} \) varies) is bounded by \( 2^{N_{n+1}} \).

Then \( \#(\bar{g}_{n+1}(P_{n+1}), \|\cdot\|_{N_{n+1}} : \varepsilon) \) is bounded by

\[
2^{N_{n+1}} \times \#(g_n(P_n), \|\cdot\|_{N_n} : \varepsilon)^{\frac{N_{n+1}}{N_n}} \times \# \left( \bigcup_{\lambda \in [N_n]} g_{n, \lambda} \ast g'_n(P_n \ast Q_n), \|\cdot\| : \varepsilon \right) \text{ contribution over } [N_{n+1}] \setminus J_\Delta.
\]

We have

\[
\# \left( \bigcup_{\lambda \in [N_n]} g_{n, \lambda} \ast g'_n(P_n \ast Q_n), \|\cdot\| : \varepsilon \right) \leq N_n \left( \frac{1}{\varepsilon} \right)^{A(P_n \ast Q_n)}.
\]

Then we get the claim by Condition 5.14 (2) and (3) for \( g_n \) and \( \varphi_n \).

By Lemma 5.3 (1) and \( \dim P_{n+1} < \left( D + \frac{1}{n+1} \right) N_{n+1} \), we can take \( 0 < \varepsilon_{n+1} < \varepsilon_n/2 \) such that for any linear map \( g: P_{n+1} \to B^2_1(V)^{N_{n+1}} \)

\[
\#(g(P_{n+1}), \|\cdot\|_{N_{n+1}} : \varepsilon) < \left( \frac{1}{\varepsilon} \right)^{(D + \frac{1}{n+1}) N_{n+1}} (0 < \varepsilon < \varepsilon_{n+1}).
\]

By Lemma 5.3 (2) and the above (5.10) and (5.12), we can find linear embeddings \( g'_{n+1}, Q_{n+1} \to B^2_1(V) \) and \( g_{n+1, \lambda}: P_{n+1} \to B^2_1(V), \lambda \in [N_{n+1}] \), such that they satisfy Condition 5.14 (1) and for any \( x \in X \) and \( u \in P_{n+1} \)

\[
\|g_{n+1}(\pi_{n+1}(x)) - f_n(x)\| < \min \left( \frac{\varepsilon_n}{8}, \frac{\delta_n}{2} \right),
\]

\[
\|g_{n+1, \lambda}(\pi_{n+1}(x)) - f_n(T^\lambda x)\| < \min \left( \frac{\varepsilon_n}{8}, \frac{\delta_n}{2} \right),
\]

\[
\|g_{n+1, \lambda}(u) - \tilde{g}_{n+1, \lambda}(u)\| < \frac{\varepsilon_{n+1}}{4}. \tag{5.14}
\]
It follows from (5.14) and Claim 5.17 that for \( \varepsilon_{n+1} \leq \varepsilon < \varepsilon_n \)

\[
\# \left( g_{n+1}(P_{n+1}), \| \cdot \|_{N_{n+1}}, \varepsilon \right) \leq \# \left( \tilde{g}_{n+1}(P_{n+1}), \| \cdot \|_{N_{n+1}}, \varepsilon - \frac{\varepsilon_{n+1}}{2} \right)
\leq \# \left( \tilde{g}_{n+1}(P_{n+1}), \| \cdot \|_{N_{n+1}}, \frac{\varepsilon}{2} \right)
\leq 4^{N_{n+1}} \left( \frac{2}{\varepsilon} \right)^{(D + \frac{r}{n+1})N_{n+1}}.
\]

From the choice of \( \varepsilon_{n+1} \), for \( 0 < \varepsilon < \varepsilon_{n+1} \)

\[
\# \left( g_{n+1}(P_{n+1}), \| \cdot \|_{N_{n+1}}, \varepsilon \right) < \left( \frac{1}{\varepsilon} \right)^{(D + \frac{r}{n+1})N_{n+1}}.
\]

Hence \( g_{n+1, \lambda} \) satisfy Condition 5.14 (2).

From Lemma 5.11 we can choose a continuous function \( \varphi_{n+1}: \mathcal{X} \to [0, 1] \) satisfying Condition 5.14 (3). From (5.13), \( f_{n+1} \) satisfies Condition 5.14 (5). Since \( g_{n+1, \lambda} \) and \( g'_{n+1} \) satisfy Condition 5.14 (1), \( f_{n+1} \) is a \( 1/(n+1) \)-embedding with respect to \( d \). Since “1/(n+1)-embedding” is an open condition, we can choose \( \delta_{n+1} > 0 \) satisfying Condition 5.14 (4).

We have established all the data for the \((n+1)\)th step. \( \square \)

### 6 Example: Algebraic Actions

We study an example in this section. Probably the example below can be more generalized (e.g. more general group actions), but we restrict ourselves to a simple case because our purpose here is just to illustrate the concepts studied in the paper. We plan to study more examples in future works.

Set \( T = \mathbb{R}/\mathbb{Z} \). Let \( r > 0 \) be an integer and consider the shift \( \sigma: (\mathbb{T}^r)^\mathbb{Z} \to (\mathbb{T}^r)^\mathbb{Z} \) on the alphabet \( \mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r \). This becomes a compact Abelian group under the component-wise addition. A subset \( \mathcal{X} \subset (\mathbb{T}^r)^\mathbb{Z} \) is called an algebraic action if it is a \( \sigma \)-invariant closed subgroup.\(^{19}\) Equivalently [Sch95, Definitions 3.7 and 4.1, Theorems 3.8 and 4.2], a subset \( \mathcal{X} \subset (\mathbb{T}^r)^\mathbb{Z} \) is an algebraic action if and only if there exist a positive integer \( a \) and a closed subgroup \( H \subset (\mathbb{T}^r)^a \) such that

\[
\mathcal{X} = \left\{ (x_n)_{n \in \mathbb{Z}} \in (\mathbb{T}^r)^\mathbb{Z} \mid (x_n, x_{n+1}, \ldots, x_{n+a-1}) \in H \ (\forall n \in \mathbb{Z}) \right\}.
\]

We define metrics \( \rho \) and \( \rho_r \) on \( \mathbb{T} \) and \( \mathbb{T}^r \) respectively by

\[
\rho(t, t') = \min_{n \in \mathbb{Z}} |t - t' - n|,
\rho_r \left( (t_1, \ldots, t_r), (t'_1, \ldots, t'_r) \right) = \max_{1 \leq i \leq r} \rho(t_i, t'_i).
\]

\(^{19}\) Since \( r \) is finite, this is more restricted than in the literatures [Sch95, LL18]. They consider automorphisms of general compact Abelian groups. But we study only the restricted class here for simplicity.
We define a metric $d$ on $(\mathbb{T}^r)^\mathbb{Z}$ by
\[ d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} \rho_r(x_n, y_n), \quad (x_n, y_n \in \mathbb{T}^r). \]

Later we will use the fact that $d$ is homogeneous, namely it is invariant under the addition
\[ d(x + z, y + z) = d(x, y), \quad (x, y, z \in (\mathbb{T}^r)^\mathbb{Z}). \]

For $N > 0$ we denote by $\pi_N: (\mathbb{T}^r)^\mathbb{Z} \to (\mathbb{T}^r)^N$ the projection to the $\{0, 1, 2, \ldots, N - 1\}$-coordinates:
\[ \pi_N(x) = (x_0, \ldots, x_{N-1}). \]

Let $X \subset (\mathbb{T}^r)^\mathbb{Z}$ be an algebraic action. Following Gromov [Gro99, §1.9] we define the projective dimension of $X$ by
\[ \text{prodim}(X) = \lim_{N \to \infty} \frac{\dim \pi_N(X)}{N}. \]

Here $\dim \pi_N(X)$ is the topological dimension of $\pi_N(X)$. This limit always exists because $\dim \pi_N(X)$ is subadditive in $N$. (Note that, a priori, the projective dimension may depend on the way of the embedding $X \subset (\mathbb{T}^r)^\mathbb{Z}$. So the notation $\text{prodim}(X)$ might be misleading. But we use it for simplicity.)

Li and Liang [LL18, Theorem 4.1, Theorem 7.2] proved:
\[ \text{mdim}(X, \sigma) = \text{mdim}_M(X, \sigma, d) = \text{prodim}(X). \] (6.1)

Indeed they proved more general results. But we stick to this simple case. Since mean Hausdorff dimension is bounded between mean dimension and metric mean dimension (Proposition 3.2), we also have
\[ \text{mdim}_H(X, \sigma, d) = \text{prodim}(X). \]

The purpose of this section is to show:

**Proposition 6.1.** Let $X \subset (\mathbb{T}^r)^\mathbb{Z}$ be an algebraic action and $\mu$ the normalized Haar measure on it ($\mu(X) = 1$). Then
\[ \text{rdim}(X, \sigma, d, \mu) = \text{prodim}(X). \]

Hence the rate distortion dimension with respect to the Haar measure coincides with the mean dimension, mean Hausdorff dimension and metric mean dimension.

Therefore algebraic actions provide natural examples where all the dynamical dimensions studied in this paper coincide with each other.

In the sequel, we also include the proof of (6.1) for the completeness. The idea of the proof is the same as [LL18].

The following lemma is a key estimate [LL18, Lemma 4.2].
Lemma 6.2. Let $A$ be a compact Abelian group with a metric $d$ and $f: A \to \mathbb{T}^r$ a continuous homomorphism satisfying $\rho_r(f(x), f(y)) \leq d(x, y)$. Then for any $0 < \varepsilon < 1/4$

\[
\text{Widim}_\varepsilon(A, d) \geq \dim f(A), \tag{6.2}
\]
\[
\#(A, d, \varepsilon) \geq \left(\frac{1}{4 \varepsilon}\right)^{\dim f(A)}. \tag{6.3}
\]

Here $\dim f(A)$ is the topological dimension of $f(A)$.

Proof. We can assume that $f(A)$ is connected. (If it is not, we replace $A$ with the inverse by $f$ of the connected component of $f(A)$ through the origin.) Let $\pi: \mathbb{R}^r \to \mathbb{T}^r$ be the natural covering map and set $V = \pi^{-1}(f(A))$. $V$ is a subvector space of $\mathbb{R}^r$ of dimension $\dim f(A)$. We consider the $\ell^\infty$-norm $|| \cdot ||_{\infty}$ on $\mathbb{R}^r$.

Claim 6.3. There exists a continuous homomorphism $g: V \to A$ satisfying $f \circ g = \pi|_V$.

Proof. Let $M$ be the Pontrjagin dual of $f(A)$. The dual group $\hat{V}$ of $V$ (we denote Pontrjagin duality by hat) is identified with $M \otimes \mathbb{R}$ and $(\pi|_V)(m) = m \otimes 1$ for $m \in M$. It is enough to construct a homomorphism $h: \hat{A} \to M \otimes \mathbb{R}$ satisfying $h \circ \hat{f}(m) = m \otimes 1$ for $m \in M$. (Note that every homomorphism defined on $\hat{A}$ automatically becomes continuous because its topology is discrete.)

Since the map $\hat{f}: M \to \hat{A}$ is injective,

\[
\hat{f} \otimes \text{id}: M \otimes \mathbb{R} \to \hat{A} \otimes \mathbb{R}
\]

is also injective. Take an $\mathbb{R}$-linear map $\varphi: \hat{A} \otimes \mathbb{R} \to M \otimes \mathbb{R}$ satisfying $\varphi \circ (\hat{f} \otimes \text{id}) = \text{id}_{M \otimes \mathbb{R}}$. Then the map $h: \hat{A} \to M \otimes \mathbb{R}$ defined by $h(a) = \varphi(a \otimes 1)$ satisfies $h \circ \hat{f}(m) = m \otimes 1$ for $m \in M$. \hfill \Box

Let $B_{1/4}(V)$ be the closed $1/4$-ball of $V$ around the origin. Note that the map $\pi: (B_{1/4}(\mathbb{R}^r), ||\cdot||_{\infty}) \to (\mathbb{T}^r, \rho_r)$ is an isometry. So for $x, y \in B_{1/4}(V)$

\[
||x - y||_{\infty} = \rho_r(f(g(x)), f(g(y)) \leq d(g(x), g(y)).
\]

This implies

\[
\text{Widim}_\varepsilon(B_{1/4}(V), ||\cdot||_{\infty}) \leq \text{Widim}_\varepsilon(A, d).
\]

We have $\text{Widim}_\varepsilon(B_{1/4}(V), ||\cdot||_{\infty}) = \dim V = \dim f(A)$ for $0 < \varepsilon < 1/4$ by (2.2) in Example 2.1 in Section 2.1. This shows (6.2). We can prove (6.3) in the same way by using (2.3) in Example 2.1. \hfill \Box
Proof of (6.1). Let $\mathcal{X} \subset (\mathbb{T}^r)^{\mathbb{Z}}$ be an algebraic action. First we prove $\text{mdim}(\mathcal{X}, \sigma) \geq \text{prodim}(\mathcal{X})$. Consider the projection $\pi_N: \mathcal{X} \to (\mathbb{T}^r)^N = \mathbb{T}^{rN}$. This satisfies $\rho_{rN}(\pi_N(x), \pi_N(y)) \leq d_N(x, y)$. So we can use Lemma 6.2 and get

\[ \text{Widim}_\varepsilon(\mathcal{X}, d_N) \geq \dim \pi_N(\mathcal{X}), \quad (0 < \varepsilon < 1/4). \]

Divide this by $N$ and take the limits with respect to $N$ and then $\varepsilon$. We get $\text{mdim}(\mathcal{X}, \sigma) \geq \text{prodim}(\mathcal{X})$.

Next we prove $\text{mdim}_M(\mathcal{X}, \sigma, d) \leq \text{prodim}(\mathcal{X})$. This completes the proof of (6.1) because metric mean dimension always dominates mean dimension (Theorem 2.2).

Notice that the following map is an isometric embedding

\[ (\pi_{M+N}(\mathcal{X}), \rho_{r(M+N)}) \to (\pi_M(\mathcal{X}) \times \pi_N(\mathcal{X}), \rho_M \times \rho_N), \]

where the metric of the right-hand side is given by

\[ \rho_M \times \rho_N((x, y), (z, w)) = \max(\rho_M(x, z), \rho_N(y, w)). \]

It follows that $\#(\pi_N(\mathcal{X}), \rho_N, \varepsilon)$ is subadditive in $N$ and hence for any $\varepsilon > 0$

\[ \lim_{N \to \infty} \frac{\log \#(\pi_N(\mathcal{X}), \rho_N, \varepsilon)}{N} = \inf_{N > 0} \frac{\log \#(\pi_N(\mathcal{X}), \rho_N, \varepsilon)}{N}. \quad (6.4) \]

For $A \subset \mathbb{R}$, let $\pi_A: \mathcal{X} \to (\mathbb{T}^r)^{A \cap \mathbb{Z}}$ be the projection to $A \cap \mathbb{Z}$-coordinates. Let $\varepsilon > 0$ and take $L = L(\varepsilon) > 0$ satisfying $\sum_{|n| > L} 2^{-|n|} < \varepsilon/4$. Then

\[ \#(\mathcal{X}, d_N, \varepsilon) \leq \#(\pi_{[-L, N+L]}(\mathcal{X}), \rho_{r(N+2L+1)}, \varepsilon/4) \]

\[ = \#(\pi_{N+2L+1}(\mathcal{X}), \rho_{r(N+2L+1)}, \varepsilon/4). \]

Noting the above (6.4), we get

\[ S(\mathcal{X}, \sigma, d, \varepsilon) = \lim_{N \to \infty} \frac{\log \#(\mathcal{X}, d_N, \varepsilon)}{N} \]

\[ \leq \lim_{N \to \infty} \frac{\log \#(\pi_{N+2L+1}(\mathcal{X}), \rho_{r(N+2L+1)}, \varepsilon/4)}{N} \]

\[ = \inf_{N > 0} \frac{\log \#(\pi_N(\mathcal{X}), \rho_N, \varepsilon/4)}{N}. \]

Thus for any $N > 0$

\[ \overline{\text{mdim}}_M(\mathcal{X}, \sigma, d) = \limsup_{\varepsilon \to 0} \frac{S(\mathcal{X}, \sigma, d, \varepsilon)}{\log(1/\varepsilon)} \]

\[ \leq \frac{1}{N} \limsup_{\varepsilon \to 0} \frac{\log \#(\pi_N(\mathcal{X}), \rho_N, \varepsilon/4)}{\log(1/\varepsilon)}. \]
\( \pi_N(\mathcal{X}) \) is a closed subgroup of \( \mathbb{T}^r N \) and hence a smooth submanifold. Then the upper Minkowski dimension

\[
\overline{\dim}_M(\pi_N(\mathcal{X}), \rho_N) = \limsup_{\varepsilon \to 0} \frac{\log \#(\pi_N(\mathcal{X}), \rho_N, \varepsilon)}{\log(1/\varepsilon)}
\]

is equal to the topological dimension \( \dim \pi_N(\mathcal{X}) \). Thus for any \( \mathcal{X} \)

\[
\overline{\dim}_M(\mathcal{X}, \sigma, d) \leq \dim \pi_N(\mathcal{X}).
\]

Let \( N \to \infty \). This proves \( \overline{\dim}_M(\mathcal{X}, \sigma, d) \leq \text{prodim}(\mathcal{X}) \).

For any \( N > 0 \) we define a distance \( \bar{d}_N \) on \((\mathbb{T}^r)^N\) by (see Section 4.2)

\[
\bar{d}_N(x, y) = \frac{1}{N} \sum_{n=0}^{N-1} d(\sigma^n x, \sigma^n y).
\]

**Lemma 6.4.** Let \( \mathcal{X} \subset (\mathbb{T}^r)^N \) be an algebraic action. For any \( 0 < \delta < 1 \) there exists \( \varepsilon_0 = \varepsilon_0(\delta) > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) and \( N > 0 \)

\[
\#_{\text{sep}}(\mathcal{X}, \bar{d}_N, \varepsilon) \geq 4^{-N} (1/\varepsilon)^{(1-\delta) \dim \pi_N(\mathcal{X})}.
\]

Recall that \( \#_{\text{sep}}(\mathcal{X}, \bar{d}_N, \varepsilon) \) is the maximum cardinality of \( \{x_1, \ldots, x_n\} \subset \mathcal{X} \) satisfying \( \bar{d}_N(x_i, x_j) \geq \varepsilon \) for \( i \neq j \).

**Proof.** We use (2.1) in Section 2.1 and the estimate in Remark 4.8: For any \( L > 1 \)

\[
\log \#_{\text{sep}}(\mathcal{X}, \bar{d}_N, \varepsilon) \geq \log \#(\mathcal{X}, \bar{d}_N, 3\varepsilon) \geq \log \#(\mathcal{X}, d_N, 6L\varepsilon) - N - \frac{N}{L} \log \#(\mathcal{X}, d, 3\varepsilon).
\]

Let \( L = (1/24)(1/\varepsilon)^\delta \). Then

\[
\log \#_{\text{sep}}(\mathcal{X}, \bar{d}_N, \varepsilon) \geq \log \left( \mathcal{X}, d_N, \frac{\varepsilon^{1-\delta}}{4} \right) - N - 24N \varepsilon^\delta \log \#(\mathcal{X}, d, 3\varepsilon).
\]

\((\mathcal{X}, d)\) has the tame growth of covering numbers (see Example 3.9). So there exists \( \varepsilon_0 > 0 \) so that for any \( 0 < \varepsilon < \varepsilon_0 \)

\[
\frac{\varepsilon^{1-\delta}}{4} < \frac{1}{4}, \quad \varepsilon^\delta \log \#(\mathcal{X}, d, 3\varepsilon) < \frac{1}{24}.
\]

By applying Lemma 6.2 to \( \pi_N: \mathcal{X} \to (\mathbb{T}^r)^N \),

\[
\# \left( \mathcal{X}, d_N, \frac{\varepsilon^{1-\delta}}{4} \right) \geq \left( \frac{1}{\varepsilon^{1-\delta}} \right)^{\dim \pi_N(\mathcal{X})} (0 < \varepsilon < \varepsilon_0).
\]

Combining these estimates we get

\[
\log \#_{\text{sep}}(\mathcal{X}, \bar{d}_N, \varepsilon) \geq (1 - \delta) \dim \pi_N(\mathcal{X}) \log(1/\varepsilon) - 2N.
\]

This is equivalent to the statement. (Recall that the base of the logarithm is two.) \( \square \)
Proof of Proposition 6.1. We know from Proposition 3.2 and (6.1) that

\[\bar{\text{rdim}}(\mathcal{X}, \sigma, d, \mu) \leq \text{mdim}_M(\mathcal{X}, \sigma, d) = \text{prodim}(\mathcal{X}).\]

So it is enough to prove \(\bar{\text{rdim}}(\mathcal{X}, \sigma, d, \mu) \geq \text{prodim}(\mathcal{X}).\) We can assume \(\text{prodim}(\mathcal{X}) > 0.\)

Recall that the distance \(d\) is homogeneous. In particular the measure \(\mu(B^\circ(x, \bar{d}_N))\) of the open ball around \(x \in \mathcal{X}\) is independent of \(x\). So we denote it by \(\mu(B^\circ(\bar{d}_N))\).

Let \(\{x_1, \ldots, x_K\}\) be a separated set of \((\mathcal{X}, \bar{d}_N)\) with \(K = \#_{\text{sep}}(\mathcal{X}, \bar{d}_N, \varepsilon)\). Since the balls \(B^\circ_{\varepsilon/2}(x_i, \bar{d}_N)\) are disjoint with each other, \(K \mu(B^\circ_{\varepsilon/2}(\bar{d}_N)) \leq 1\). Let \(0 < \delta < 1/2\). It follows from Lemma 6.4 that for \(0 < \varepsilon < \varepsilon_0(\delta)\)

\[\mu\left(B^\circ_{\varepsilon/2}(\bar{d}_N)\right) \leq K^{-1} \leq 4^N \varepsilon^{(1-\delta)} \text{dim} \pi_N(\mathcal{X}).\]

Since \(\text{dim} \pi_N(\mathcal{X})\) is subadditive in \(N\),

\[\text{prodim}(\mathcal{X}) = \lim_{N \to \infty} \frac{\text{dim} \pi_N(\mathcal{X})}{N} = \inf_{N > 0} \frac{\text{dim} \pi_N(\mathcal{X})}{N}\]

and we assumed that this is positive. Therefore there exists \(\varepsilon_1 = \varepsilon_1(\delta) > 0\) such that for any \(0 < \varepsilon < \varepsilon_1\)

\[\mu\left(B^\circ_{\varepsilon/2}(\bar{d}_N)\right) \leq \left(\varepsilon/2\right)^{N(1-2\delta)} \text{prodim}(\mathcal{X}).\]

This implies that for all \(E \subset \mathcal{X}\) with \(\text{diam}(E, \bar{d}_N) < \varepsilon_1/2\)

\[\mu(E) \leq \left(\text{diam}(E, \bar{d}_N)\right)^{N(1-2\delta)} \text{prodim}(\mathcal{X}).\]

We use Lemma 4.5 in Section 4.2 and get

\[\bar{\text{rdim}}(\mathcal{X}, \sigma, d, \mu) \geq (1 - 2\delta) \text{prodim}(\mathcal{X}).\]

Let \(\delta \to 0.\) This proves \(\bar{\text{rdim}}(\mathcal{X}, \sigma, d, \mu) \geq \text{prodim}(\mathcal{X}).\) \(\square\)

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