Stress analysis of orthotropic plate based on complex variable method

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Abstract. Typical stress boundary problem for a wedge with orthotropic materials is considered. The complex variable functions including a material parameter are fully analyzed for solving the partial derivation equation. By constructing new stress function, the mechanic analysis of the wedge subjected to a concentrated force is carried out. The stress boundary problem and the governing equation are resolved. The formulae of stress fields in rectangular and polar coordinates are derived for the wedge.

1. Introduction

The complex variable theory provides a very powerful tool for the solution of many boundary value problems in the elastic body. Such theory was originally found by some researchers for solving general boundary problems in isotropic materials [1–3]. Furthermore, the complex variable technique has also been expanded to use for anisotropic materials [4–6]. The orthotropic plate may have been the base of common engineering use. The feasible method to solve stress-field problems in anisotropic composites is to use the analytic function theory, and the results have been reported [7, 8]. The plane stress state of composite sheets is common and very importance for the application. It is the key point to solve stress-field problems in orthotropic materials. In plane elasticity, the equilibrium equations of three stress components are (no body forces):

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0
\end{align*}
\]

(1)

Usually, the method of solving the equations is by introducing a real function \( U(x, y) \), called the stress function [1, 2]. It is easily checked that the equilibrium equations can be satisfied by using the following expressions for the stress components, namely
For solving two-dimensional elastic problems of orthotropic materials, we suppose the principal elastic directions of the plate coincide with the coordinate directions, and let the directions 1, 2 parallel to the axes x, y, respectively. The governing equation of the strain compatibility condition can be expressed by the stress function $U(x, y)$, and it has the form

$$\frac{\partial^4 U}{\partial y^4} + 2B \frac{\partial^4 U}{\partial x^2 \partial y^2} + C \frac{\partial^4 U}{\partial x^4} = 0$$

(3)

where

$$B = \frac{E_1}{2G_{12}} - \nu_{12}, \quad C = \frac{E_1}{E_2}.$$  

They are called the engineering elastic constants in the principal axes for the orthotropic materials. Above basic equations are often appeared in hand books [3~5]. The key point may be to find the stress function $U$ in terms of the stress boundary conditions and to be suitable for the partial differential equation (3).

2. Complex variable method

In the solution of the partial differential equation and also in the construction of suitable stress functions, it is very advantageous to use complex variables. Generally, Two real numbers $x$ and $y$ form the complex number, $z = x + iy$, (with $i^2 = -1$). And often the conjugate complex number $\bar{z} = x - iy$ must be used together. For the convenience of general investigation, now another complex variable ($w$) and its conjugate ($\bar{w}$) are also be introduced and defined by

$$w = x + iy, \quad \bar{w} = x - iy$$

(4)

where $h$ is a real arbitrary constant. And we suppose the constant $h$ to be positive, $h > 0$, also it can be called tensile or compressive ratio for the coordinate system. The partial derivative relation must be given by

$$\frac{\partial w}{\partial x} = \frac{\partial \bar{w}}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = -\frac{\partial \bar{w}}{\partial y} = ih$$

In terms of the relationship between rectangular and polar coordinates, the complex variables can be written as the general form

$$\begin{cases} w = x + iy = r \cos \theta + ihr \sin \theta \\
\bar{w} = x - iy = r \cos \theta - ihr \sin \theta \end{cases}$$

(5)

On the basis of above definition, the stress function $U$ can be expressed by the complex variables. The partial derivation of $U$ with $x$ or $y$ can be transformed into other expressions with $w$ or $\bar{w}$, namely

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial w} \frac{\partial w}{\partial x} + \frac{\partial U}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial x} = \frac{\partial U}{\partial w} + \frac{\partial U}{\partial \bar{w}}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial w} \frac{\partial w}{\partial y} + \frac{\partial U}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial y} = ih(\frac{\partial U}{\partial w} - \frac{\partial U}{\partial \bar{w}})$$

Then we obtain
Furthermore we have

$$\left\{ \begin{align*}
\frac{\partial^2 U}{\partial x^2} &= \frac{\partial^2 U}{\partial w^2} + 2 \frac{\partial^2 U}{\partial w \partial \overline{w}} \frac{\partial^2 U}{\partial \overline{w}^2} \\
\frac{\partial^2 U}{\partial y^2} &= -h^2 \left( \frac{\partial^2 U}{\partial w^2} - 2 \frac{\partial^2 U}{\partial w \partial \overline{w}} + \frac{\partial^2 U}{\partial \overline{w}^2} \right) \\
\frac{\partial^2 U}{\partial x \partial y} &= i h \left( \frac{\partial^2 U}{\partial w^2} - \frac{\partial^2 U}{\partial \overline{w}^2} \right)
\end{align*} \right. \quad (6)$$

By substituting above equations into the governing equation (3), we obtain the partial derivative equation with complex variables in the following way

$$\left\{ \begin{align*}
\frac{\partial^2 U}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 U}{\partial y^2} &= 4 \frac{\partial^4 U}{\partial w \partial \overline{w}} \\
\left( \frac{\partial^2 U}{\partial w^2} + \frac{\partial^2 U}{\partial \overline{w}^2} \right)^2 &= 16 \frac{\partial^4 U}{\partial w^2 \partial \overline{w}^2}
\end{align*} \right. \quad (7)$$

Next, we shall solve this equation. We may divide it into the following two cases.

**Case I** $\frac{\partial^2 U}{\partial w \partial \overline{w}} \neq 0$, $\frac{\partial^4 U}{\partial w \partial \overline{w}^2} = 0$.

In this case, the coefficient parts of other terms in equation (8) must become zero. Thus, two characteristic equations are given as follows

$$h^4 - 2Bh^2 + C = 0, \quad C - h^4 = 0.$$  

Obviously, the solution ought be

$$h^4 = C = B^2.$$  

Denote $B = \frac{E_1}{2G_{12}} - \nu_{12}$, $C = \frac{E_1}{E_2}$. Leads to

$$h = \sqrt{\frac{E_1}{2G_{12}} - \nu_{12}} = \sqrt{\frac{E_1}{E_2}}$$ \quad (9)

And again by the equation $\frac{\partial^4 U}{\partial w^2 \partial \overline{w}^2} = 0$, we can obtain the stress function $U$ of the form

$$U = A_1 \text{Re} \Psi + A_2 \text{Im} \Psi$$

$$+ A_3 \text{Re} (\overline{w} \Phi) + A_4 \text{Im} (\overline{w} \Phi)$$ \quad (10)

where $\Psi = \Psi(w)$ and $\Phi = \Phi(w)$ are the analytic functions with one complex number $w = x + ihy$, as having partial derivatives.

**Case II** $\frac{\partial^2 U}{\partial w \partial \overline{w}} = 0$. 

In this case, the coefficient part of the first terms in equation (8) must become zero. Thus, the characteristic equation is given as

\[ h^4 - 2Bh^2 + C = 0 \]

Obviously, the solution may be as

\[ h^2 = B \pm \sqrt{B^2 - C} \quad (\text{for } B^2 > C) \]

Let \( h_1 > h_2 > 0 \), then we have

\[
\begin{aligned}
& h_1 = \sqrt{B + \sqrt{B^2 - C}} \\
& h_2 = \sqrt{B - \sqrt{B^2 - C}}
\end{aligned}
\] (11)

So that

\[
\begin{aligned}
& h_1 = \sqrt{\frac{E_1}{2G_{12}} - \nu_{12} + \sqrt{(\frac{E_1}{2G_{12}} - \nu_{12})^2 - \frac{E_1}{E_2}}} \\
& h_2 = \sqrt{\frac{E_1}{2G_{12}} - \nu_{12} - \sqrt{(\frac{E_1}{2G_{12}} - \nu_{12})^2 - \frac{E_1}{E_2}}}
\end{aligned}
\]

And again by the equation \( \frac{\partial^2 U}{\partial w \partial \bar{w}} = 0 \), we can obtain the stress function \( U \) in the following form

\[ U = A_1 \text{Re} \Psi_1 + A_2 \text{Re} \Psi_2 + A_3 \text{Im} \Psi_1 + A_4 \text{Im} \Psi_2 \] (12)

where \( \Psi_1 = \Psi_1(w_1) \) and \( \Psi_2 = \Psi_2(w_2) \). Both are the analytic functions with two complex variables, i.e.

\[ w_1 = x + ih_1 y, \quad w_2 = x + ih_2 y. \]

Up to now, we complete the construction of the stress function \( U \) with complex variable method. In the following, we shall give an example to show the application of complex variable method.

3. Stress analysis of orthotropic plate

Consider a wedge of orthotropic materials in the \( x-y \) plane subjected to a concentrated force \( P \) at its top angle as shown in Figure 1, the top angle is of \( 2\alpha \) degrees. The force \( P \) is parallel to \( y \) direction.

![Figure 1. Scheme of the wedge and loading case.](image)

The thickness of the wedge is taken as unity in the direction perpendicular to the \( x-y \) plane. So that \( P \) is the load per unit thickness, and the distribution of the load is uniform. Thus, this is a typical problem in plane stress state. The conditions along two edge faces of the wedge are free ( \( y = \pm x \tan \alpha \), or
\( \theta = \pm \alpha \). That can be satisfied by taking the boundary values as zero for the stress components. The stress boundary conditions are

\[
\sigma_\theta = 0, \quad \tau_{r\theta} = 0 \quad \text{for} \quad \theta = \pm \alpha
\]

In order to solve this problem, now we can suppose the complex functions to be that

\[
\Phi = \Phi(w) = \ln w \\
\Psi = w\Phi = w\ln w
\]

Then we have

\[
\Phi = \ln \bar{w}, \quad \Psi = \bar{w}\ln \bar{w}.
\]

3.1. Solution for first case

For case I, we may take the stress function \( U \) as

\[
U = -A \operatorname{Im} \Psi - A \operatorname{Im}(\bar{w}\Phi)
\]

Then we have

\[
U = \frac{Ai}{2}(w\Phi - \bar{w}\Phi) + \frac{Ai}{2}(w\Phi - \bar{w}\Phi)
\]

\[
= \frac{Ai}{2}(w + \bar{w})(\Phi - \bar{\Phi})
\]

Namely,

\[
U = Aix(\Phi - \bar{\Phi}) = Aix(\ln w - \ln \bar{w})
\]

Thus the first order partial derivatives can be easily obtained as follows

\[
\frac{\partial U}{\partial x} = Aih\left[\ln w - \ln \bar{w}\right] + x\left(\frac{1}{w} - \frac{1}{\bar{w}}\right)
\]

\[
\frac{\partial U}{\partial y} = -Ahx\left(\frac{1}{w} + \frac{1}{\bar{w}}\right)
\]

The second order partial derivatives are

\[
\frac{\partial^2 U}{\partial x^2} = 4A \frac{h^3 y^3}{(w\bar{w})^2}
\]

\[
\frac{\partial^2 U}{\partial y^2} = 4A \frac{h^3 x^3 y}{(w\bar{w})^2}
\]

\[
\frac{\partial^2 U}{\partial x\partial y} = -4A \frac{h^3 xy^2}{(w\bar{w})^2}
\]

Therefore, the stress components can be given by

\[
\begin{align*}
\sigma_x &= 4A \frac{h^3 x^3 y}{(w\bar{w})^2} \\
\sigma_y &= 4A \frac{h^3 y^3}{(w\bar{w})^2} \\
\tau_{xy} &= 4A \frac{h^3 xy^2}{(w\bar{w})^2}
\end{align*}
\]

where \( w\bar{w} = x^2 + h^2 y^2 \).
In order to solve the stress boundary problem of the wedge, we can consider to taking the relations between stresses in the two coordinate systems. It is common knowledge that the transformation of stresses can be expressed by the following relations:

\[
\begin{align*}
\sigma_r &= \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + \tau_{xy} \sin 2\theta \\
\sigma_\theta &= \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - \tau_{xy} \sin 2\theta \\
\tau_{r\theta} &= (\sigma_y - \sigma_x) \sin \theta \cos \theta + \tau_{xy} \cos 2\theta
\end{align*}
\]

(17)

Substituting the stresses of the expression (16) into the above equations, and to simplify them, then the stress components \(\sigma_r, \sigma_\theta, \tau_{r\theta}\) in the polar coordinate system can be obtained as

\[
\begin{align*}
\sigma_r &= \frac{4Ah^3y}{(wh)^2} [x^2 \cos^2 \theta + y^2 \sin^2 \theta + xy \sin 2\theta] \\
\sigma_\theta &= \frac{4Ah^3y}{(wh)^2} [x^2 \sin^2 \theta + y^2 \cos^2 \theta - xy \sin 2\theta] \\
\tau_{r\theta} &= \frac{4Ah^3y}{(wh)^2} [(y^2 - x^2) \sin \theta \cos \theta + xy \cos 2\theta]
\end{align*}
\]

In terms of the coordinate transformation relations, \(x = r \cos \theta, y = r \sin \theta\), the above stresses can be simplified as following expressions

\[
\begin{align*}
\sigma_r &= -\frac{4Ah^3 \sin \theta}{r(\cos^2 \theta + h^2 \sin^2 \theta)^2} \\
\sigma_\theta &= \tau_{r\theta} = 0
\end{align*}
\]

(18)

Obviously, the stress fields can satisfy the boundary conditions of the wedge in Figure 1. Next, the constant \(A\) ought to be determined. By the loading condition and the coordinate systems, the force must be in equilibrium. In a simple and clear method, let \(x = 1\), that is to take a regular cross section. Then both boundary points are \(y = \tan \alpha\) and \(y = -\tan \alpha\). Thus, the equilibrant equation along \(y\) direction is given by

\[
\int_{-\tan \alpha}^{\tan \alpha} \tau_{xy} dy = \int_{-\tan \alpha}^{\tan \alpha} \frac{4Ah^3 y^2}{(1 + h^2 y^2)^2} dy = P
\]

(19)

By integrating, we have the solution

\[
4A[\arctan(h \tan \alpha) - \frac{h \tan \alpha}{1 + (h \tan \alpha)^2}] = P
\]

And again we introduce another letter \(\beta\) to simplify the solution, which is defined as \(\beta = \arctan(h \tan \alpha)\) or \(\tan \beta = h \tan \alpha\). Hence it can be seen that the solution becomes

\[
4A = \frac{P}{\beta - \sin \beta \cos \beta}
\]

(20)

From this, the radial stress can be in the form

\[
\sigma_r = \frac{P}{\beta'} \frac{h^3 \sin \theta}{r(\cos^2 \theta + h^2 \sin^2 \theta)^2}
\]

(21)

where \(\beta' = \beta - \sin \beta \cos \beta\).

For the isotropic materials \((h = 1)\), then \(\beta = \alpha\). The radial stress can be converted into
\[
\sigma_r = \frac{P \sin \theta}{(\alpha - \sin \alpha \cos \alpha)r}
\]  
(22)

This is a given result in elastic books.

3.2. Solution for second case

For case II, we can select the stress function \( U \) as

\[
U = \frac{A}{h_1} \text{Im} \Psi_1 - \frac{A}{h_2} \text{Im} \Psi_2
\]  
(23)

Where \( \Psi_1 = w_1 \ln w_1 = (x + ih_1y) \ln(x + ih_1y) \),

and \( \Psi_2 = w_2 \ln w_2 = (x + ih_2y) \ln(x + ih_2y) \).

Then we have

\[
U = -\frac{Ai}{2h_1} (w_1 \ln w_1 - \overline{w}_1 \ln \overline{w}_1) + \frac{Ai}{2h_2} (w_2 \ln w_2 - \overline{w}_2 \ln \overline{w}_2)
\]  
(24)

The partial derivatives can be found out

\[
\frac{\partial U}{\partial x} = -\frac{Ai}{2h_1} (\ln w_1 - \ln \overline{w}_1)
\]

\[
+ \frac{Ai}{2h_2} (\ln w_2 - \ln \overline{w}_2)
\]

\[
\frac{\partial U}{\partial y} = \frac{A}{2} (\ln w_1 + \ln \overline{w}_1) - \frac{A}{2} (\ln w_2 + \ln \overline{w}_2)
\]

The second order partial derivatives are

\[
\frac{\partial^2 U}{\partial x^2} = -\frac{Ai}{2h_1} \left( \frac{1}{w_1} - \frac{1}{\overline{w}_1} \right) + \frac{Ai}{2h_2} \left( \frac{1}{w_2} - \frac{1}{\overline{w}_2} \right)
\]

\[
\frac{\partial^2 U}{\partial y^2} = \frac{A}{2} \left( \frac{ih_1}{w_1} - \frac{ih_1}{\overline{w}_1} \right) - \frac{A}{2} \left( \frac{ih_2}{w_2} - \frac{ih_2}{\overline{w}_2} \right)
\]

\[
\frac{\partial^2 U}{\partial x \partial y} = \frac{A}{2} \left( \frac{1}{w_1} + \frac{1}{\overline{w}_1} \right) - \frac{A}{2} \left( \frac{1}{w_2} + \frac{1}{\overline{w}_2} \right)
\]

So the stress components can be given by

\[
\begin{align*}
\sigma_x &= \frac{\partial^2 U}{\partial y^2} \frac{A h_1 y}{w_1 \overline{w}_1} - \frac{A h_2 y}{w_2 \overline{w}_2} \\
\sigma_y &= \frac{\partial^2 U}{\partial x^2} = -\frac{A}{w_1 \overline{w}_1} + \frac{A}{w_2 \overline{w}_2} \\
\tau_{xy} &= -\frac{\partial^2 U}{\partial x \partial y} = -\frac{A x}{w_1 \overline{w}_1} + \frac{A x}{w_2 \overline{w}_2}
\end{align*}
\]  
(25)

Where \( w_1 \overline{w}_1 = x^2 + h_1^2 y^2 \), \( w_2 \overline{w}_2 = x^2 + h_2^2 y^2 \)

Now we can substitute above stresses into the equation (17) and simplify them. And also we can use the coordinate transformation relations, \( x = r \cos \theta \), \( y = r \sin \theta \). Then the stress components in the polar coordinate system can be obtained as follows
\[
\begin{aligned}
\sigma_r &= \frac{A \sin \theta (-3\cos^2 \theta + h_1^2 \sin^2 \theta)}{r(\cos^2 \theta + h_1^2 \sin^2 \theta)} \\
&\quad + \frac{A \sin \theta (-3\cos^2 \theta + h_2^2 \sin^2 \theta)}{r(\cos^2 \theta + h_2^2 \sin^2 \theta)} \\
\sigma_\theta &= 0 \\
\tau_{r\theta} &= 0
\end{aligned}
\]

(26)

It is evident that the stress fields can satisfy the boundary conditions of the wedge in Figure 1. Next, the constant \( A \) ought to be determined by the loading condition. And also we select a section \((x = 1)\). Then both boundary points are \( y = \tan \alpha \) and \( y = -\tan \alpha \). Therefore, the equilibrant equation along \( y \) direction is given by

\[
\int_{-\tan \alpha}^{\tan \alpha} \tau_{y} \, dy = \int_{-\tan \alpha}^{\tan \alpha} \left( -\frac{Ax}{w_1 w_1} + \frac{Ax}{w_2 w_2} \right) dy = P
\]

By making the limit integration, we have the solution

\[
\frac{i}{h_1} \ln \left( 1 - ih_1 \tan \alpha \right) + \frac{i}{h_2} \ln \left( 1 + ih_2 \tan \alpha \right) = -\frac{P}{A}
\]

To simplify the solution, we introduce other letters, \( \beta_1, \beta_2 \). They are defined as follows

\[
\begin{aligned}
\tan \beta_1 &= h_1 \tan \alpha \\
\tan \beta_2 &= h_2 \tan \alpha
\end{aligned}
\]

Hence it can be seen that the solution becomes

\[
2A = \frac{P h_2}{h_1 \beta_2 - h_2 \beta_1}
\]

(27)

Thus, the radial stress can be given by

\[
\sigma_r = \frac{Ph_2 h_2 \sin \theta [1 - f_1(\theta) - f_2(\theta)]}{(h_1 \beta_2 - h_2 \beta_1) r}
\]

(28)

Where \( f_1(\theta) = \frac{2 \cos^2 \theta}{\cos^2 \theta + h_1^2 \sin^2 \theta} \)

\[
f_2(\theta) = \frac{2 \cos^2 \theta}{\cos^2 \theta + h_2^2 \sin^2 \theta}
\]

Under the rectangular coordinates, we can write the stress components for

\[
\begin{aligned}
\sigma_x &= \frac{Ph_2 h_2 y}{2(h_1 \beta_2 - h_2 \beta_1)} \left( \frac{h_1^2}{w_1 w_1} - \frac{h_2^2}{w_2 w_2} \right) \\
\sigma_y &= \frac{Ph_2 h_2 y}{2(h_1 \beta_2 - h_2 \beta_1)} \left( \frac{1}{w_1 w_1} - \frac{1}{w_2 w_2} \right) \\
\tau_{xy} &= \frac{Ph_2 h_2 x}{2(h_1 \beta_2 - h_2 \beta_1)} \left( \frac{1}{w_1 w_1} - \frac{1}{w_2 w_2} \right)
\end{aligned}
\]

(29)

4. Conclusion
By constructing new stress function, the mechanic analysis of the wedge subjected to a concentrated force is carried out. The stress boundary problem and the governing equation are resolved. The formulae of stress fields in rectangular and polar coordinates are derived for the wedge.

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