ON A STRATIFICATION OF KONTSEVICH’S MODULI SPACE $\overline{M}_{0,n}(G(2,4),d)$ AND ENUMERATIVE GEOMETRY.

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Abstract. We consider a particular stratification of the moduli space $\overline{M}_{0,n}(G(2,4),d)$ of stable maps to $G(2,4)$. As an application we compute the degree of the variety parametrizing rational ruled surfaces with a minimal directrix of degree $\frac{d}{2} - 1$ by studying divisors in this moduli space of stable maps. For example, there are 128054031872040 rational ruled sextics passing through 25 points in $\mathbb{P}^3$ with a minimal directrix of degree 2.

1. Introduction

The geometry of the Kontsevich’s moduli space $\overline{M}_{g,n}(\mathbb{P}^r, d)$ of degree $d$ stable maps from $n$-pointed, genus $g$ curves to $\mathbb{P}^r$ has been studied intensively in the literature. R. Vakil studied its connection with the enumerative geometry of rational and elliptic curves in projective space in [Vak]. R. Pandharipande studied the theory of $\mathbb{Q}$–Cartier divisors on this space in [Pan1] and proved an algorithm for computing all characteristic numbers of rational curves in $\mathbb{P}^r$. R. Pandharipande also computed the degree of the 1-cuspidal rational locus in the linear system of degree $d$ plane curves.

The enumerative geometry of rational ruled surfaces of degree $d$ in $\mathbb{P}^3$ is closely related to the intrinsic geometry of the moduli space $\overline{M}_{0,n}(G(2,4),d)$, of Kontsevich stable maps from $n$–pointed genus 0 curves to the Grassmannian $G(2,4)$ of lines in $\mathbb{P}^3$, representing $d$ times the positive generator of the homology group $H_2(G(2,4),\mathbb{Z})$, (see [Mar1]).

In [Mar1] we solved the enumerative problem of computing the degree of the Severi variety of degree $d$ rational ruled surfaces in the ambient projective space of surfaces of degree $d$ in $\mathbb{P}^3$. Here we study the more refined problem of enumerating rational ruled surfaces with a fixed directrix of minimum degree. In particular we compute the degree of the codimension one subvariety of rational ruled surfaces with a minimal directrix of degree $\frac{d}{2} - 1$ by intersecting divisors in the moduli space of stable maps $\overline{M}_{0,0}(G(2,4),d)$.

In a first step, we use the variety $R_d^0$ of degree $d$ morphisms from $\mathbb{P}^1$ to the Grassmannian $G(2,4)$, as a parameter space for parametrized rational ruled surfaces in $\mathbb{P}^3$. We consider a certain stratification of the variety $R_d^0$ defined as follows. If $Q$ is the universal bundle on $G(2,4)$, the stratum is defined by the following locally closed condition:

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\[ R_{d,a}^0 = \{ f \in R_d^0 | \dim H^1(\mathbb{P}^1, f^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^1}(-a-2)) \geq 1 \text{ and } \dim H^0(\mathbb{P}^1, f_q^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^1}(-a-1)) = 0 \}. \]

This has a geometrical interpretation as rational ruled surfaces with a minimal directrix of degree \(a\). First, we consider the closure \( R_{d,a} \) of the stratum \( R_{d,a}^0 \) in the Quot scheme compactification \( R_d \) of the space of morphisms \( R_{d,a}^0 \). We prove that the subschemes \( R_{d,a} \) are irreducible and have the expected codimension as determinantal varieties (Theorem 4.1). We give an explicit formula for their Poincaré duals derived from Porteous formula (Proposition 4.3). In the last section, we consider their closure in the Kontsevich compactification of stable maps \( \overline{M}_{0,3}(G(2,4), d) \). We study its Picard group (Theorem 5.1) and we show that the given stratification of the space \( R_{d,a}^0 \) can be extended to \( \overline{M}_{0,3}(G(2,4), d) \), but in this case the codimension of the stratum is different, that is, not all boundary points correspond to closure points of the open set \( R_{d,a}^0 \), except for the big stratum of rational ruled surfaces with a minimal directrix of degree \( \frac{d}{2} - 1 \). In this case we compute its degree by interpreting it as a tautological intersection on the space of stable maps (Theorem 5.3). We give explicit enumerative formulas. We show, for example, that there are exactly 128054031872040 rational ruled sextic surfaces passing through 25 points with a minimal directrix of degree 2.

I. Coskun has approached this problem in [Cos1] by considering Gromov-Witten invariants for bundles with a fixed splitting type that correspond to ordinary Gromov-Witten invariants of certain flag varieties.

We work over the field of complex numbers \( \mathbb{C} \). By a scheme we mean a scheme of finite type over \( \mathbb{C} \). By a variety, we mean a separated integral scheme. Curves are assumed to be complete and reduced. For a variety \( X \), \( A_d X \) and \( A^{d} X \) can be taken to be the Chow homology and cohomology groups. \( A^d X \) and \( A_{-d} X \) are identified by the Poincaré duality isomorphism. For \( \beta \in A_k X \), \( \int_\beta c \) is the degree of the zero cycle obtained by evaluating \( c_k \) on \( \beta \), where \( c_k \) is the component of \( c \) in \( A^k X \). If \( F \) is a sheaf over a curve \( C \), \( h^0(C, F) \) and \( h^1(C, F) \) denote the dimensions of the cohomology groups \( H^0(C, F) \) and \( H^1(C, F) \).

For any formal series \( a(t) = \sum_{k=-\infty}^{k=+\infty} a_k t^k \), we set

\[ \triangle_{p,q}(a) = \text{det} \begin{pmatrix} a_p & \ldots & a_{p+q-1} \\ \vdots & & \vdots \\ a_{p-q+1} & \ldots & a_p \end{pmatrix}. \]

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2. A Stratification of the Space $R^0_d$ and Enumerative Geometry

Let $R^0_d$ be the variety of degree $d$ morphisms from $\mathbb{P}^1$ to $G(2, 4)$. We will denote $R^0_d$ the Zariski open set in $R^1_d$ corresponding to morphisms which are birational onto their images. The Grothendieck Quot scheme $R_d$ parametrizing rank 2 and degree $d$ quotients of a trivial vector bundle $\mathcal{O}_{\mathbb{P}^1}^1$, is a projective scheme which provides an algebro-geometric compactification of $R^0_d$. We consider the evaluation map from $R^0_d \times \mathbb{P}^1$ to the Grassmannian $G(2, 4)$, and the projection map over the first component.

$$R^0_d \times \mathbb{P}^1 \xrightarrow{\pi_1} G(2, 4)$$

(1)

The pull back of the universal exact sequence on $G(2, 4)$,

$$(2) \quad 0 \to N \to \mathcal{O}_{G(2, 4)}^4 \to \mathcal{Q} \to 0$$

gives a universal exact sequence on $R^0_d$ which is the restriction of the universal quotient over $R_d \times \mathbb{P}^1$.

For $f \in R^0_d$, the pull-back $f^*\mathcal{Q}$ under $f$ of the universal quotient on the Grassmannian, is a bundle of rank 2 on $\mathbb{P}^1$ isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$ where $0 \leq a \leq \frac{d}{2}$, in particular $a \leq d-a$. There is a canonical exact sequence

$$0 \to N \to \mathcal{O}_{\mathbb{P}^1}^4 \to \mathcal{O}(a) \oplus \mathcal{O}(d-a) \to 0$$

where $N$ is a rank 2 and degree $-d$ bundle. Thus, there is a morphism from $\mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(d-a))$ to $\mathbb{P}^3$. The image of this morphism is a scroll. We will denote it by $X_{a,d-a}$.

**Definition 2.1.** For each $0 \leq a \leq \frac{d}{2}$, $R^0_{d,a}$ is the subvariety of $R^0_d$ consisting of those $f \in R^0_d$ such that $f^*\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$.

**Proposition 2.2.** $R^0_{d,a}$ is locally closed in $R^0_d$ and irreducible. If $d > 2a$ then it has dimension $3d + 5 + 2a$ and if $d = 2a$, it has dimension $4d + 4$.

**Proof.** The set $R^0_{d,a}$ is locally closed by Proposition 10 of [Sha].

Let $F_a \subset \mathbb{P}\text{Hom}(\mathcal{O}^4, \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a))$ be the open set of epimorphisms

$$\mathcal{O}^4 \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a).$$

$F_a$ is in bijection with the set of locally free rank 2 quotients of $\mathcal{O}_{\mathbb{P}^1}^4$ isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$ modulo $\mathbb{C}^*$. Let $F_a \xrightarrow{\psi} R^0_{d,a}$ be the surjective morphism such that the image by $\psi$ of each quotient is its class of isomorphism in $R^0_{d,a}$. The fibers are isomorphic to $\mathbb{P}(\text{Aut}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)))$.

A dimension calculation yields:

1. If $a = d-a$,
   $$\dim \mathbb{P}(\text{Aut}(\mathcal{O}(a) \oplus \mathcal{O}(d-a))) = 3.$$  

2. If $d-a > a$,
   $$\dim \mathbb{P}(\text{Aut}(\mathcal{O}(a) \oplus \mathcal{O}(d-a))) = d - 2a + 2.$$
The dimension of $F_a$ is given by

$$\dim \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)) = \dim F_a = 4d + 7.$$ 

By the fiber dimension theorem ([Sha], §I.6.3),

$$\dim R^0_{d,a} = \begin{cases} 
4d + 4 & \text{if } d = 2a \\
3d + 5 + 2a & \text{if } d > 2a 
\end{cases}$$

In particular this proves that the subvarieties $R^0_{d,a}$ are irreducible, because $F_a$ is irreducible.

\[\square\]

**Lemma 2.3.** $q \in R^0_{d,a}$ if and only if $h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a)) \geq 1$ and $h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a-1)) = 0$.

*Proof.* Let $q$ be a point in $R^0_{d,a}$, and $f_q : \mathbb{P}^1 \to G(2,4)$ the morphism corresponding to that point. Suppose $f_q^* Q$ decomposes as $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$.

Tensoring with the line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$,

$$f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a) = \mathcal{O}_{\mathbb{P}^1}(2a-d) \oplus \mathcal{O}_{\mathbb{P}^1},$$

and tensoring with $\mathcal{O}_{\mathbb{P}^1}(a-1)$, we have

$$f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a-1) = \mathcal{O}_{\mathbb{P}^1}(2a-d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Since $a \leq d-a$, it follows that $h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a)) \geq 1$, and

$$h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a-1)) = 0.$$

Conversely, if $f_q^* Q^\vee \cong \mathcal{O}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(n-d)$ with $n \leq d-n$, then

$$f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a) = \mathcal{O}_{\mathbb{P}^1}(n-d+a) \oplus \mathcal{O}_{\mathbb{P}^1}(a-n),$$

then $h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a)) \geq 1$ implies $a \geq n$ and $h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a-1)) = 0$ implies $a \leq n$, therefore $n = a$ and $q \in R^0_{d,a}$.

\[\square\]

**Remark 1.** We observe that the cases $d = 2a$ and $d = 2a + 1$ are special.

In these cases $W^0_{d,a}$ has the maximum dimension. This means that $R^0_{d,a}$ is an open set in $R^0_{d,a}$ and therefore the corresponding degree has already been computed in ([Mar1]).

**Remark 2.** By the Serre duality theorem and Lemma 2.3 the subvarieties $R^0_{d,a}$ can also be defined as

$$R^0_{d,a} = \{ f \in R^0_{d,a} | h^1(\mathbb{P}^1, f^* Q \otimes \mathcal{O}_{\mathbb{P}^1}(-a-2)) \geq 1 \}$$

$$h^1(\mathbb{P}^1, f^* Q \otimes \mathcal{O}_{\mathbb{P}^1}(-(a-1)-2)) = 0 \}.$$ 

Note that when $d > 2a$ a slightly stronger result is true:

$q \in R^0_{d,a}$ if and only if

$$h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a)) = 1 \text{ and } h^0(\mathbb{P}^1, f_q^* Q^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a-1)) = 0.$$

These sets are locally closed by the Semicontinuity Theorem.
Lemma 2.4. The Zariski closure $\overline{R}_{d,a}^0$ of the stratum $R_{d,a}^0$ in $R_d^0$ coincides with
\begin{equation}
\{ f \in R_d^0 | h^1(\mathbb{P}^1, f^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \geq 1 \}.
\end{equation}

Proof. By definition, the closure $\overline{R}_{d,a}^0$ is the minimum closed set containing $R_{d,a}^0$. We observe that $R_{d,a}^0$ is contained in the set given in (3) that it is closed, therefore it also contains $\overline{R}_{d,a}^0$. Now we notice that every closed set containing $R_{d,a}^0$ contains the closed set
\begin{equation}
\{ f \in R_d^0 | h^0(\mathbb{P}^1, f^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^1}) \geq 1, \quad h^0(\mathbb{P}^1, f^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^1}(a - 1)) \geq 0 \},
\end{equation}
but the second closed condition is satisfied trivially, therefore the closure $\overline{R}_{d,a}^0$ coincides with the set (3). □

Remark 3. We observe that if $f^* \mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(n - d)$ with $n \leq d - n$, $h^0(\mathbb{P}^1, f^* \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^1}(a)) \geq 1$ implies $a \geq n$ and therefore
$\overline{R}_{d,a}^0 = R_{d,a}^0 \cup R_{d,a-1}^0 \cup \ldots$

Lemma 2.5. The subvariety $R_{d,a}^0$ parametrizes rational ruled surfaces with a minimal directrix of degree $a$.

Proof. The projection
\[ \pi_a : \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d - a) \to \mathcal{O}_{\mathbb{P}^1}(a), \]
gives a section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)) \to \mathbb{P}(f^* \mathcal{Q})$. This section is mapped to a rational curve $C^a$ of degree $a$. A study of the map
\begin{equation}
\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d - a) \to \mathcal{O}_{\mathbb{P}^1}(k),
\end{equation}
shows that the map is surjective for $k \geq a$, therefore $C^a$ is a directrix of minimal degree and (4) has a geometrical interpretation as the rational ruled surfaces with a minimal directrix of degree $a$. □

3. Change of basis in Pic($R_d$).

In this section we describe geometric generators for the Picard group of the Quot scheme $R_d$.

We consider the cohomology ring of the Grassmannian $G(2, 4)$ and the universal sequence over it:
\begin{equation}
0 \to \mathcal{N} \to \mathcal{O}_{G(2,4)}^4 \to \mathcal{Q} \to 0.
\end{equation}
The special Schubert cycles on $G(2, 4)$ can be represented as Chern classes of the universal quotient bundle and the universal subbundle:
\[ T_1 = c_1(\mathcal{Q}) = c_1(\mathcal{N}^\vee), \quad T_b = c_2(\mathcal{N}), \quad T_a = c_3(\mathcal{Q}). \]
Also, $T_3 \in H_2(G(2, 4), \mathbb{Z})$ will stand for the class of a line and $T_4$ for the class of a point in $G(2, 4)$. The Poincare dual of the hyperplane class $c_1(\mathcal{N}^\vee)$ in $H^2(G(2, 4), \mathbb{Z})$ determines the Plucker embedding of the Grassmannian $G(2, 4)$ as a quadric in $\mathbb{P}^5$, that corresponds to a variety of lines in $\mathbb{P}^3$. Then the cycles defined above, have an interpretation as lines in $\mathbb{P}^3$. The cycle $T_1$ corresponds to lines in $\mathbb{P}^3$ meeting a given line, $T_b$ corresponds to lines contained in a given plane and $T_a$ to lines containing a given point.
Let \( e : R^0_d \times \mathbb{P}^1 \to G(2,4) \) be the evaluation map. The following sets of morphisms define Weil divisors on \( R^0_d \):

(A) The locus of morphisms whose image meets an \( a \)-plane \( T_a \) in the Grassmannian associated to a point \( P \in \mathbb{P}^3 \). We denote it by:
\[
D := \{ \varphi \in R^0_d \mid e(t, \varphi) \cap T_a \neq \emptyset \}.
\]

(B) The set of morphisms \( Y \) sending a fixed point \( t \in \mathbb{P}^1 \) to a hyperplane \( T_1 \) on the Grassmannian:
\[
Y := \{ \varphi \in R^0_d \mid e(t, \varphi) \in T_1 \text{ for a fixed } t \in \mathbb{P}^1 \}.
\]

Since the boundary of \( R_d \) is of codimension greater than 1 (see Theorem 1.4 of [Ber2]), these divisors extend to divisors in \( R_d \), that is \( A^1(R^0_d) = A^1(R_d) \). Let \( Y, D \) also denote the divisors on \( R_d \). It is clear by the definition that when we move the point \( t \in \mathbb{P}^1 \) we get a linearly equivalent divisor. For each index \( i \in \mathbb{Z}^+ \), we associate a point \( t_i \) in \( \mathbb{P}^1 \), \( Y_i \) will denote the corresponding associated divisor. In the same way, every \( a \)-plane \( T_a \) in \( G(2,4) \) is associated to a point \( P \) in \( \mathbb{P}^3 \) and if we move the point we also get an equivalent divisor with the same index. The divisors \( Y_i \) were already considered by Bertram in [Ber1], (see Lemma 1.1 and Corollary 1.3) where he proved a moving lemma stating that these varieties can be made to intersect transversally. S.A. Stromme (Theorem 6.2, [Str]) gives a basis for the Picard group \( A^1(R_d) \), formed by the divisors:
\[
\alpha = c_1(\pi_1^*(\mathcal{E} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)) - c_1(\pi_1^*(\mathcal{E} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1))),
\]
\[
\beta = c_1(\pi_1^*(\mathcal{E} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(d-1)),
\]
where \( \mathcal{E} \) is the universal quotient over \( R_d \times \mathbb{P}^1 \), and \( \pi_1, \pi_2 \) are the projections maps over the first and second factors respectively.

Let \( h \) be the positive generator of the Picard group of \( \mathbb{P}^1 \) as well as its pull-back to \( R_d \times \mathbb{P}^1 \).

The Chern classes of the universal subbundle \( \mathcal{K} \) are described in the proof of theorem 5.3 of [Str], that is
\[
A(R_d \times \mathbb{P}^1) = \{ A(R_d)[h] \mid h^2 = 0 \}.
\]

Every class \( z \in A(R_d \times \mathbb{P}^1) \) can be written in the form \( z = x + hy \), with \( y = \pi_{1*}(h) \in A(R_d) \), and \( x = \pi_{1*}(hz) \in A(R_d) \),
\[
c_i(\mathcal{K}) = t_i + hu_{i-1} \quad (1 \leq i \leq 2),
\]
\[
t_i \in A^i(R_d), \quad u_i \in A^{i-1}(R_d), \quad i = 1,2 \text{ as in [Str], in particular } u_0 = -d.
\]

We express the classes \( t_1, u_1 \) in terms of the generators \( \alpha \) and \( \beta \).

**Lemma 3.1.** The change of basis is
\[
\alpha = -t_1,
\]
\[
\beta = u_1.
\]

**Proof.** We need to compute the first Chern class of the bundle,
\[
B_m = \pi_{1*}(\mathcal{E} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(m)).
\]
By Riemann Roch, we have that
\[ \text{ch}(B_m) = \pi_{1*}(1 + h) \cdot \text{ch}(\mathcal{E} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(m)). \]
The Chern classes of \( \mathcal{E} \) can be computed from those of \( \mathcal{K} \):
\begin{align*}
c_1(\mathcal{E}) &= -c_1(\mathcal{K}) = -t_1 + d h, \\
c_2(\mathcal{E}) &= c_1^2(\mathcal{K}) - c_2(\mathcal{K}) = t_1^2 - 2 d h t_1 - t_2 - u_1 h.
\end{align*}
It follows that,
\[ c_1(B_m) = (d - 1 - m) t_1 + u_1. \]
\[ \square \]

**Lemma 3.2.** The change of basis from the \( \alpha, \beta \) divisors to the geometric divisors \( Y, D \) is:
\[ \alpha = Y, \]
\[ \beta = -2dY + D. \]

**Proof.** By the universal property of the Grassmannian, the pull back under the evaluation map \( e \) of the universal exact sequence \([7]\) on \( G(2, 4) \) gives us a universal exact sequence on \( R_d^1 \times \mathbb{P}^1 \) which is the restriction of the universal exact sequence on \( R_d \times \mathbb{P}^1 \). Therefore, by the description of the special Schubert cycles given, the class \([Y]\) of the divisor \( Y \) in the Chow ring \( A(R_d) \) can be identified with the pushforward of the projection map over the first factor \( \pi_{1*}(e^*(T_1)) = \pi_{1*}(e^*(c_1(Q)) \cdot h) = \pi_{1*}((h c_1(E)), \) where \( E \) is the universal quotient over \( R_d \times \mathbb{P}^1 \). Similarly, the class of the divisor \( D \) can be identified with \( \pi_{1*}(c_2(\mathcal{E})) \).

The Grothendieck-Riemann-Roch theorem shows that the geometric divisor \( Y \) is just the divisor \( \alpha \). Now, we see that \([D] = \pi_{1*}(t_1^2 - 2t_1 d h - t_2 + h + h u_1) = -2dt_1 + u_1 = 2d \alpha + \beta \), where \( \beta \) is the other generator of the Picard group of \( R_d \). In particular, this also proves that the divisors \( D \) and \( Y \) constitute a basis for the Picard group \( A^1(R_d) \).
\[ \square \]

Let \( P_d \) be the degree of \( R_d \) by the morphism induced by the hyperplane class in \( R_d \). By lemma 3.2 \( P_d \) is the degree of the top codimensional cohomology class given by the self-intersection,
\[ P_d = \int_{[R_d]} [Y]^{4d+4} \cap [R_d] = \int_{[R_d]} \alpha^{4d+4} \cap [R_d]. \]
This intersection number is computed in [RRW] via Quantum cohomology. It is a Gromov-Witten invariant and can be obtained too by using the formulas of Vafa and Intriligator, (see section 5 of [BerII]).

In Theorem 4.1 of [Mar1] we prove that the intersection \( Y_1 \cap Y_2 \cap Y_3 \cap D_1 \cap \ldots \cap D_{4d+1} \) has an excess intersection component contained in the boundary of \( R_d \), and therefore the divisors cannot be moved to make the intersection transversal. We compute it by reinterpreting it as a Gromov-Witten invariant in the Kontsevich moduli space of stable maps \( \overline{M}_{0,4d+4}(G(2, 4), d) \). For \( d \geq 3 \), this intersection is \( d^3 \cdot Q_d \), where \( Q_d \) is the Severi degree of the variety of rational ruled surfaces in \( \mathbb{P}^3 \), and the factor \( d^3 \) comes from the multiple covers.
4. Classes of strata in the Chow ring of $R_d$

In this section we prove that the subschemes $R_{d,a}^0$ extend to a projective subscheme representing a Chern class and we compute the classes of the closure of the strata $R_{d,a}^0$ in the Quot scheme compactification $R_d$.

We consider the universal exact sequence over $R_d \times \mathbb{P}^1$,

\begin{equation}
0 \rightarrow K \rightarrow \mathcal{O}_{R_d \times \mathbb{P}^1}^4 \rightarrow \mathcal{E} \rightarrow 0.
\end{equation}

Tensoring (10) with $\pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)$, where $\pi_2$ is the projection map over the second factor, we obtain the exact sequence:

\begin{equation}
0 \rightarrow K \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2) \rightarrow \mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2) \rightarrow \mathcal{E} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2) \rightarrow 0.
\end{equation}

Let $\pi_1$ be the projection map over the first factor. The $\pi_1$ direct image yields an exact sequence over $R_d$:

\begin{equation}
0 \rightarrow \pi_1^*(K \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \rightarrow \pi_1^*(\mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \rightarrow \\
\rightarrow \pi_1^*(\mathcal{E} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \rightarrow R^1 \pi_1^*(K \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \rightarrow \\
\rightarrow R^1 \pi_1^*(\mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \rightarrow R^1 \pi_1^*(\mathcal{E} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \rightarrow 0.
\end{equation}

By applying the Serre duality theorem, it follows that

\begin{equation}
R^1 \pi_1^*(K \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \cong \pi_1^*(K^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(a))^\vee.
\end{equation}

We want to see that the sheaf (11) is a bundle. First, by the base change theorem, their fibers are isomorphic to

\begin{equation}
H^1(\mathbb{P}^1, K \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)|_\{q\} \times \mathbb{P}^1) \cong H^0(\mathbb{P}^1, K^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(a)|_\{q\} \times \mathbb{P}^1)^\vee.
\end{equation}

It is enough to check that the cohomology group

\begin{equation}
H^1(\mathbb{P}^1, K^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(a)|_\{q\} \times \mathbb{P}^1),
\end{equation}

vanishes or that the group $H^0(\mathbb{P}^1, K \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)|_\{q\} \times \mathbb{P}^1)$ vanishes, by the Serre duality theorem.

We can assume that $K|_\{q\} \times \mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(n - d)$ with $n \geq 0$ and $n \leq d - n$, therefore $K \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)|_\{q\} \times \mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(-n - a - 2) \oplus \mathcal{O}_{\mathbb{P}^1}(n - d - a - 2)$. Since the integers $-n - a - 2$ and $n - d - a - 2$ are both negative, the result follows.

For similar reasons, the sheaf

\begin{equation}
R^1 \pi_1^*(\mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a - 2)),
\end{equation}

is a bundle over $R_d$.

**Remark 4.** Since the universal quotient sheaf $\mathcal{E}$ is flat over the Quot scheme $R_d$, we have for each $q \in R_d$, that $E_q := \mathcal{E}|_\{q\} \times \mathbb{P}^1$ is a coherent sheaf over $\mathbb{P}^1$ and therefore it can be written as a direct sum of its locally free part plus its torsion. By the Riemann-Roch theorem, we have that

\begin{equation}
h^0(E_q) - h^1(E_q) = \text{deg}(E_q) + \text{rank}(E_q)(1 - g) = d + 2.
\end{equation}

In particular, the degree of $E_q$ is constant for every $q \in R_d$. The sets $h^1(E_q) \geq r$,
where \( r \) is an integer, are closed by the Semicontinuity theorem and by the identity \([13]\), they can also be defined as:
\[
h^0(E_q) \geq d + 2 + r.
\]

Let us consider the Zariski closure \( R_{d,a} \) of the sets \( R_{d,a}^0 \) inside the Quot scheme compactification of the space of morphisms:
\[
\{ q \in R_d \mid h^0(\mathbb{P}^1, E_q^* \otimes \mathcal{O}_{\mathbb{P}^1}(a)) \geq 1 \},
\]
or equivalently by Serre duality:
\[
\{ q \in R_d \mid h^1(\mathbb{P}^1, E_q \otimes \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \geq 1 \}.
\]

**Theorem 4.1.** \( R_{d,a} \) is the locus where the map
\[(14) \quad R^1 \pi_{1*}(\mathcal{K} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-a - 2)) \rightarrow R^1 \pi_{1*}(\mathcal{O}_{R_d \times \mathbb{P}^1}^1 \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-a - 2)).
\]
is not surjective. It is non-empty and has the expected codimension \( d - 2a - 1 \) as a determinantal variety.

**Proof.** The map \([14]\) is not surjective in the support of the sheaf
\[(15) \quad R^1 \pi_{1*}(R_d \times \mathbb{P}^1, \mathcal{E} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-a - 2)),
\]
that is, in the points \( q \in R_d \) such that
\[
h^1(\mathbb{P}^1, E_q \otimes \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \geq 1,
\]
or equivalently by Serre duality in \( R_{d,a} \), or by the observation in Remark \([4]\) in
\[
\{ q \in R_{C,d} \mid h^0(\mathbb{P}^1, E_q \otimes \mathcal{O}_{\mathbb{P}^1}(-a - 2)) \geq d - 2a - 1 \}.
\]

The non-emptiness follows from the fact that given a vector bundle \( E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d - a) \), it is generated by global sections since by assumption \( a, d - a \geq 0 \) and therefore there exists \( f \in R_d^0 \) such that \( f^*\mathcal{Q} \cong E \).

By applying the Serre duality theorem we get that
\[(16) \quad R^1 \pi_{1*}(\mathcal{K} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-a - 2)) \cong \pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a))^\vee.
\]

We apply Riemann-Roch in \( \mathbb{P}^1 \) in order to compute the dimension of their fibers:
\[
\dim H^0(\mathbb{P}^1, \mathcal{K}^\vee \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a)|_{\{p\} \times \mathbb{P}^1}) = d + 2a + 2, \quad p \in R_d.
\]

As a consequence \( \pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \) is a bundle of rank \( d + 2a + 2 \) over \( R_d \). Again we see by the Serre duality theorem that
\[(17) \quad R^1 \pi_{1*}(\mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^*\mathcal{O}(-a - 2)) \cong \pi_{1*}(\mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^*\mathcal{O}(-a - 2))^\vee.
\]

Since \( \mathcal{O}_{\mathbb{P}^1} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a)|_{\{p\} \times \mathbb{P}^1} \) is a locally free sheaf over \( \mathbb{P}^1 \), it can be decomposed over \( \mathbb{P}^1 \), so that
\[
\dim H^0(\mathbb{P}^1, \mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a)|_{\mathbb{P}^1 \times \{p\}}) = 4(a + 1).
\]

As a consequence \( \pi_{1*}(\mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \) is a trivial bundle of rank \( 4a + 4 \) over \( R_d \), that is,
\[
\pi_{1*}(\mathcal{O}_{R_d \times \mathbb{P}^1}^4 \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(a)) \cong \mathcal{O}_{R_d}^{4a + 4}.
\]
Finally, all these facts give us the following morphism of bundles:

$$
\pi_{1*}(K^\vee \otimes \pi_2^*O_{\mathbb{P}^1}(a)) \xrightarrow{\phi_2} O_{R_d}^{4a+4}.
$$

The expected codimension of $R_{d,a}$ as a determinantal variety is:

$$
((d + 2a + 2) - (4a + 3)) \cdot ((4a + 4) - (4a + 3)) = d - 2a - 1 \ (\S \ II.4, \ [ACGH]).
$$

This codimension coincides with the computed codimension of $R_{d,a}^0$ in \ref{22}.

4.0.1. Class of $R_{d,a}$ in $A(R_d)$.

**Lemma 4.2.** $R_{d,a}$ is irreducible and does not have components contained in the boundary $R_{d,a} - R_{d,a}^0$.

**Proof.** Since the subschemes $R_{d,a}$ are the closure of the irreducible sets $R_{d,a}^0$, it follows that they are also irreducible, and consequently cannot have components contained at infinity. \hfill \Box

**Proposition 4.3.** If $R_{d,a}$ is either empty or has the expected codimension $d - 2a - 1$, its fundamental class is given by the formula:

$$
[R_{d,a}] = -c_{d-2a-1}(\pi_{1*}(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a)))).
$$

**Proof.** By Lemma \ref{22} and Theorem \ref{41}, the subschemes $R_{d,a}$ are irreducible and have the expected codimension as determinantal varieties. The formula for their Poincare duals is derived from Porteous formula:

$$
[R_{d,a}] = \Delta_{d-2a-1,1}(c_l(-\pi_{1*}(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a)))))
$$

\hfill \Box

**Computation of Chern classes.** In order to compute the Chern classes of the bundle $\pi_{1*}(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a)))$ we apply the Grothendieck-Riemann-Roch theorem, so that

$$
(18) \ \ ch(\pi_{1*}(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a)))) = \pi_{1*}(Td(R_d \times \mathbb{P}^1/R_d) \cdot ch(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))))
$$

$$
(19) \ \ c_i(K) = t_i + h u_{i-1} \ (1 \leq i \leq 2),
$$

$t_i, u_i \in A^i(R_d), i = 1, \ldots, n - 1$ as in \[Str\], in particular $u_0 = -d$.

First we compute the Chern classes of $K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))$:

$$
c_1(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))) = c_1(K^\vee) + c_1(\pi_2^*(O_{\mathbb{P}^1}(a))) \cdot c_0(K^\vee)
$$

$$
= h(2a + d) - t_1,
$$

$$
c_2(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))) = c_2(K^\vee) + c_1(K^\vee) \cdot c_1(\pi_2^*(O_{\mathbb{P}^1}(a))) + c_1(\pi_2^*(O_{\mathbb{P}^1}(a)))^2 \cdot c_0(K^\vee)
$$

$$
= -a t_1 h - t_2 - h u_1,
$$

$$
c_3(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))) = a h t_2.
$$

Let us denote by $ch_i$, the $i$–homogeneous part of the Chern character of a bundle.

$$
ch_0(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))) = 2,
$$

$$
ch_1(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))) = 0,
$$

$$
ch_2(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))) = 0,
$$

$$
ch_3(K^\vee \otimes \pi_2^*(O_{\mathbb{P}^1}(a))) = 0.
$$
\[ \text{ch}_1 \left( \mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a) \right) = -t_1 + h (d + 2a), \]

\[ \text{ch}_2 \left( \mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a) \right) = \frac{1}{2} \left[ t_1^2 - 2t_1 h (2a + d) + 2at_1h + 2t_2 + 2hu_1 - 3u_2 h - 3a h t_2 \right], \]

**Lemma 4.4.**

\[ \text{ch}_n \left( \mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a) \right) = \text{coeff}_{t^n} \left( \sum \frac{(-1)^{n-1}}{n} \sum c_j (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a)) t^j \right)^n. \]

**Proof.** The Chern character of a bundle \( E \) is a polynomial in the Chern classes \( x_i = c_i(E) \) defined by the formula:

\[ \text{ch}(E) = \sum_i e^{x_i} = \sum_n \sum_i \frac{(x_i)^n}{n!} \in H^*(X, \mathbb{Q}). \]

If we call \( \sigma_i \) to the symmetric functions, that is,

\[ \sum_{r=0}^n \sigma_r t^r = \prod_{i=1}^n (1 + x_i t), \]

we get that

\[ \log (1 + \sigma_1 t + \sigma_2 t^2 + \ldots) = \log \prod_i (1 + x_i t) = \sum_i \sum_n \frac{(-1)^{n-1}}{n} (x_i t)^n = \sum_n \frac{(-1)^{n-1}}{n} (\sum x_i^n) t^n, \]

and from here the formula above follows. \( \square \)

Now applying (18), we have

\[ \text{ch}(\pi_1^* (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a))) = \pi_1^* \left( (1 + h) \left( \text{ch} \left( \mathcal{K}^\vee \otimes \mathcal{O}_{\mathbb{P}_1}(a) \right) \right) \right). \]

Therefore the Chern classes of \( \pi_1^* (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a)) \) are:

\[ c_1(\pi_1^* (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a))) = -t_1 (a + d + 1) + u_1, \]

\[ c_2(\pi_1^* (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a))) = \frac{1}{2} c_1^2 - \left[ \frac{1}{2} (2a + d) t_1^2 - \frac{1}{2} a t_1^2 + \frac{1}{2} (2a + d) t_2 - \frac{1}{2} u_1 t_1 \right], \]

\[ \vdots \]

**Lemma 4.5.**

\[ c_n(\pi_1^* (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a))) = \]

\[ -\sum_{r=1}^n \frac{(-1)^{r-1}}{n} r! \text{ch}_r(\pi_1^* (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a))) c_{n-r}(\pi_1^* (\mathcal{K}^\vee \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_1}(a))). \]
Proof. For each $r \geq 1$ the $r^{th}$ power sum is:

$$p_r = \sum x_i^r = m_r.$$ 

The generating function for the $p_r$ is:

$$p(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{i \geq 1} \sum_{r \geq 1} x_i^r t^{r-1} = \sum_{i \geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}.$$ 

$$P(t) = \frac{d}{dt} \prod_{i \geq 1} (1 - x_i t)^{-1} = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)}$$ 

(21)

$$P(-t) = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)}.$$

From (20) and (4.0.1) we get that

$$n \sigma_n = \sum_{r=1}^{n} (-1)^{r-1} p_r \sigma_{n-r},$$

and the formula above follows. \hfill \Box

5. Another compactification of the space $R_d^0$

Now we will consider the coarse moduli space $\overline{M}_{0,n}(G(2,4),d)$ of degree $d$ Kontsevich stable maps from $n$-pointed, genus 0 curves to $G(2,4)$. It is smooth, has dimension $4d + 1 + n$ and there is a universal family over it:

$$0 \to \mathcal{N} \to \mathcal{O}^4 \to \mathcal{Q} \to 0$$

$$\overline{M}_{0,n+1}(G(2,4),d) \xrightarrow{e_i} G(2,4)$$

$$\xrightarrow{\pi_i} \sigma_i$$

$$\overline{M}_{0,n}(G(2,4),d)$$

Here $\pi_n : \overline{M}_{0,n+1}(G(2,4),d) \to \overline{M}_{0,n}(G(2,4),d)$ is the forgetful morphism, which consists of forgetting the mark $p_1$ and $\sigma_i$ is the section corresponding to the mark $p_i$. The image of $\sigma_i$ is the (closure of the) locus of maps whose source curve has two components, one of which is rational, carries just the two marks $p_1$ and $p_{n+1}$ and is contracted by $\mu$. For $n = 3$, the space $\overline{M} := \overline{M}_{0,3}(G(2,4),d)$ provides a compactification of the space $R_d^0$. The marked points yield canonical line bundles $\mathcal{L}_i = e_i^*(T_1)$ on $\overline{M}$ via the $i$-evaluation map, where $T_1$ is the hyperplane class in $G(2,4)$. Since the boundary is of pure codimension 1 in $\overline{M}_{0,n}(G(2,4),d)$, each irreducible component is a Weil divisor. Let $\Delta$ be the set of components of the boundary. For $d \geq 1$, Weil divisors are obtained on $\overline{M}_{0,n}(G(2,4),d)$ by considering the locus corresponding to maps meeting an $a$–plane and a $b$–plane on the Grassmannian. We denote the corresponding classes in $Pic(\overline{M}) \otimes \mathcal{Q}$ by $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $w_\pi$ denote the relative dualizing sheaf, then a map $\mu \in \overline{M}_{0,n}(G(2,4),d)$ is stable if and only if $w_\pi(p_1 + \ldots + p_n) \otimes \mu^*(T_3)$ is ample. In particular $\text{deg}(w_\pi(p_1 + \ldots + p_n) \otimes \mu^*(T_3)) > 0$, so that if $E \subset C$ such that $\text{dim} \mu(E) = 0$, then $\text{deg}(w_\pi(p_1 + \ldots + p_n)) > 0$. 

Theorem 5.1. If $d \geq 1$, then $\text{Pic}(\overline{M})$ is generated by $\triangle \cup \{A\} \cup \{L_i\}_{i=1,2,3}$.

Proof. Consider the universal family of stable degree $d$ maps of 3-pointed curves $R^0_d \times \mathbb{P}^1 \to G(2,4)$. By the universal property, there is an injection

$$R^0_d \hookrightarrow \overline{M}.$$ 

The complement of $R^0_d$ is the boundary of $\overline{N}$ and it is also the locus corresponding to maps with reducible domain. Since the complement of $R^0_d$ in $R^0_d$ is of codimension greater or equal to 2 (see Lemma 2.5 of [Mar1]), the Picard group of $R^0_d$ coincides with the Picard group of $R^0_d$. The divisor $A$ restricted to $R^0_d$, i.e. to the moduli points corresponding to stable maps with irreducible source, is linearly equivalent to $D_i$ and $L_i$ is linearly equivalent to the corresponding divisor $Y_i$. In these points, markings are not necessary since the maps are already stable. In section 5.1 we argued that the divisors $Y_i$ and $D_i$ constitute a basis of $A^1(R^0_d)$, so that $A_{4d+3}(\overline{M})$ is generated by the boundary $\triangle$ and the Weil divisors $L_i$ ($i = 1, 2, 3$), and $A$. We note that since $Y_1 \sim Y_2 \sim Y_3$ in $R^0_d$, each one of the $L_i$ generates the Picard group. $\square$

If $d = 0$, then $\overline{M}_{0,n}(G(2,4),0) \cong \overline{M}_{0,n} \times G(2,4)$ and $L_i$ is the pull-back of $\mathcal{O}_{G(2,4)}(1)$ from the second factor.

Boundary of $\overline{M}_{0,n}(G(2,4),d)$. The irreducible components of the boundary are in bijective correspondence with the data of weighted partitions $(A \cup B, d_A, d_B)$, where

- $A \cup B$ is a partition of the set of marked points. There are only 2 possible partitions: $1+2$, $3+0$.
- $d_A + d_B = d$, $d_A \geq 0$, $d_B \geq 0$.
- if $d_A = 0$ (resp. $d_B$), then $|A| \geq 2$ (resp. $|B| \geq 2$).

A component $(A \cup B, d_A, d_B)$ consists generically of maps with a union of $\mathbb{P}^1$'s as domains. Each one gives a curve in $G(2,4)$ of degrees $d_A$ and $d_B$, respectively.

5.1. Stratification of $\overline{M}$. The stratification of the space $R^0_d$ given in [265] can be extended to $\overline{M}$, we can even compute the classes of the strata in the Chow ring $A(\overline{M})$, but they have no enumerative meaning.

5.1.1. Closure of the stratum in $\overline{M}$. The closure of the stratum has already been studied by I. Coskun in [Cos1] by taking flat limits of surfaces in one parameter families.

Let $V_{d,a}$ denote the closure of $R^0_{d,a}$ in $\overline{M}$. Let $K'$ be the boundary component corresponding to the degree partition $i + (d - i) = d$. Geometrically these boundary points correspond to reducible curves in the Grassmannian $G(2,4)$ of degree $i$, $d - i$ respectively, or equivalently a union of irreducible rational ruled surfaces of degree $i$ and $d - i$ in $\mathbb{P}^3$.

A generic intersection point of the boundary with $V_{d,a}$ corresponds to a moduli point, $[C, p_1, p_2, p_3, \mu : C \to G(2,4)]$, where $C = C_1 \cup C_2$ is a union of two $\mathbb{P}^1$'s with $\mu^* \mathcal{Q}|_{C_1} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_1)$, $a_1 + b_1 = i$ and $\mu^* \mathcal{Q}|_{C_2} \cong \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)$, $a_2 + b_2 = d - i$. The integers $a_j, b_j$ satisfy $a_j \geq 0$, $b_j \geq 0$, $\sum_j (a_j + b_j) = d$ and $a_1 + a_2 \leq a$. This last condition means that the flat limit of the directrices is a connected curve of degree $a$ whose restriction to
each of the surfaces is in a section class. Conversely, given any connected
curve \( C \) of degree \( a \leq \sum (a_j + b_j)/2 \) whose restriction to each component is
in a section class, there exists a one parameter family of \( X_{a,b} \) specializing to
the reducible surface such that the limit of the directrices is \( C \).

The stratification given in section 2 is extended to \( \mathcal{M} \) by defining the
stratum as

\[
\mathcal{M}_{d,a} := \{ \mu \in \mathcal{M} | h^1(\mu^*Q \otimes w_{\pi}(a \cdot p_i)) \geq 1 \}.
\]

This set is a closed subset of \( \mathcal{M} \) by the Semicontinuity Theorem.

**Proposition 5.2.** \( \mathcal{M}_{d,a} \) is the locus where the map

\[
R^1\pi_*(e^*_i(N) \otimes w_{\pi}(a \cdot p_i)) \to R^1\pi_*(e^*_i(O^4) \otimes w_{\pi}(a \cdot p_i))
\]

is not surjective.

**Proof.** We proceed as in section 4, considering the long exact sequence:

\[
0 \to \pi_*(e^*_i(N) \otimes w_{\pi}(a \cdot p_i)) \to \pi_*(e^*_i(O^4) \otimes w_{\pi}(a \cdot p_i)) \to \pi_*(e^*_i(Q \otimes w_{\pi}(a \cdot p_i)) \to \pi_*(e^*_i(Q) \otimes w_{\pi}(a \cdot p_i)) \to 0.
\]

The map (23) is not surjective in the support of the sheaf \( R^1\pi_*(e^*_i(Q) \otimes w_{\pi}(a \cdot p_i)) \), that is at the moduli points \( [C, p_1, p_2, p_3, \mu : C \to G(2, 4)] \) where
\( h^1(C, \mu^*Q \otimes w_{\pi}(a \cdot p_i)) \geq 1 \), and these are the points defining the stratum (22).

\[\square\]

5.1.2. **Enumerative geometry of** \( \mathcal{M}_{d,a} \). We would like to know which points
in the boundary of \( \mathcal{M} \) satisfy the condition:

\( h^1(C, \mu^*Q \otimes w_{\pi}(a \cdot p_i)) \geq 1. \)

The next theorem will address this question.

Recalling the notation of [Mar1], \( Q_d \) will be the number of rational ruled surfaces
through \( 4d + 1 \) points in \( \mathbb{P}^3 \). The number \( Q_d \) is the zero-cycle \( A^{4d+1} \)
in the Chow ring of \( \mathcal{M}_{0,0}(G(2, 4), d) \). The number \( Q^b_d \) will denote the zero-cycle
\( A^{4d}B \), or equivalently the Gromov-Witten invariant, \( I_{0,4d+1,d}(T_a, T_b, \ldots, T_a, T_b) \). The following table lists the numbers \( Q^b_d \) for each degree \( 1 \leq d \leq 9 \).

To compute these numbers we have applied Farsta, a program due to Andrew Kresch. These numbers correspond geometrically to rational ruled surfaces
of degree \( d \) through \( 4d \) points in \( \mathbb{P}^3 \) and tangent to a fixed plane.
Theorem 5.3. \( V_{d,a} \subset \overline{M}_{d,a} \) and \( V_{d,a} = \overline{M}_{d,a} \) only when \( a = \frac{d}{2} - 1 \). In this case, the stratum \( R^3_{d,a} \) is of pure codimension 1 and its degree is given by the formula
\[
\frac{1}{2} d^3 \cdot (Q_d + Q_d^b).
\]

Proof. By semicontinuity (see [Har], III. 12.8), the points \([C, p_1, p_2, p_3, \mu : C \to G(2, 4)]\) in the closure of \( R^3_{d,a} \) in \( \overline{M} \) satisfy the condition \( \text{(5.1.2)} \). The converse is false. We will construct some families satisfying the condition but which are not in the closure of the stratum.

Grothendieck duality implies
\[
h^1(C, \mu^* Q \otimes w_a(-a p_i)) = h^0(C, \mu^* Q^\vee \otimes \mathcal{O}(a p_i)).
\]

For a general boundary point, the source curve \( C = C_1 \cup C_2 \) is a nodal curve. There is a canonical exact sequence:
\[
0 \to \mu^* Q^\vee|_C \otimes \mathcal{O}(ap_i) \to \mu^* Q^\vee|_{C_1} \otimes \mathcal{O}(ap_i) \oplus \mu^* Q^\vee|_{C_2} \otimes \mathcal{O}(ap_i) \to C^2 \to 0.
\]

This gives the following injections:
\[
H^0(C_1, \mu^* Q^\vee|_{C_1} \otimes \mathcal{O}(ap_i)) \to H^0(C, \mu^* Q^\vee|_C \otimes \mathcal{O}(ap_i))
\]
\[
\uparrow
\]
\[
H^0(C_2, \mu^* Q^\vee|_{C_2} \otimes \mathcal{O}(ap_i)).
\]

Since \( C_1 \cong \mathbb{P}^1 \), it can be assumed that \( \mu^* Q|_{C_1} \cong \mathcal{O}(a_1) \oplus \mathcal{O}(b_1) \) and \( \mu^* Q|_{C_2} \cong \mathcal{O}(a_2) \oplus \mathcal{O}(b_2) \) with \( a_1, a_2, b_1, b_2 \) being integers greater than or equal to 0 and satisfying \( \sum a_i + b_i \leq d \). Also we assume that \( P_i \in C_1 \), so that \( h^0(C_2, \mu^* Q^\vee|_{C_2} \otimes \mathcal{O}(ap_i)) = h^0(C_2, \mathcal{O}(a_2) \oplus \mathcal{O}(b_2)) \), and \( h^0(C_1, \mu^* Q^\vee|_{C_1} \otimes \mathcal{O}(ap_i)) = h^0(C_1, \mathcal{O}(a_1) \oplus \mathcal{O}(a-b_1)) \).

1. If \( a < a_1 \) and \( a = b_1 \), condition \( \text{(5.1.2)} \) is still satisfied, but these points are not in the closure \( V_{d,a} \), (see [Cos1]).
2. If \( a = a_1 \) and \( a_2 = 0 \), then there exists a family of surfaces degenerating to a union of a cone of degree \( b_2 \) and a scroll \( X_{a_1, b_1} \), so that these points are in the closure.
3. If \( a = a_1 \) and \( a_2 > 0 \), then by [Cos1] the union \( X_{a_1, b_1} \cup X_{a_2, b_2} \) cannot be a flat limit of scrolls.

By degree we mean, the number of rational ruled surfaces of this specifying type through \( 4d + 1 \) points, which we studied in [Mar1].
For \( a = \frac{d}{2} - 1 \), proposition 2.2 shows that \( R_0^a \) is a one codimensional subvariety. Its class \((a - d + 1)Y + D\) in the Chow ring \( A(R_0^a)\), is given by Porteous formula in section 2 and Lemma 3.2. Because the divisors in the Picard group \( A^1(\mathbb{N}) \) intersect the boundary transversally, it follows by (5.1) that this divisor extends in the moduli compactification \( M_{d-1} \), to the divisor \((a + d - 1)L_1 + A\). Therefore, the degree of the stratum \( M_{\frac{d}{2}-1} \), that is, the degree of the variety parametrizing rational ruled surfaces with a minimal directrix of degree \( \frac{d}{2} - 1 \), is the degree of the top codimensional class \((a + d - 1)L_1 + A\). Now since \( L_1^2 \cdot A^{4d} = Q_d + Q_b^d \) in the Chow ring \( A(M_{0,1}(G(2,4), d)) \), the result follows.

\[ \square \]

Remarks and Conclusions. Note that when we say that \( a \) takes the value \( \frac{d}{2} - 1 \) it is implicitly understood that \( d \) is even, therefore we are only interested in the numbers \( Q_b^d \) listed in the table above for \( d \) even. The case \( d = 2 \) corresponds to the cone \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \). It is known that there are 4 cones through 8 general points. This is a degenerate case that was excluded from the beginning (see 2.1 of [Mar1]) for the degree computation of the variety of rational ruled surfaces. As an example we mention that according to Theorem 5.3 there are 128054031870240 ruled surfaces with fixed parameters \( d = 6 \) and \( a = 2 \).

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