Abstract

We study the geometric structure of Lorentzian spin manifolds, which admit imaginary Killing spinors. The discussion is based on the cone construction and a normal form classification of skew-adjoint operators in signature $(2, n-2)$. Derived geometries include Brinkmann spaces, Lorentzian Einstein-Sasaki spaces and certain warped product structures. Exceptional cases with decomposable holonomy of the cone are possible.

1 Introduction

A classical object of interest in differential geometry are Killing vector fields. These are by definition infinitesimal isometries, which means that the flow of such a vector field preserves the metric. A spinorial analog are the so-called Killing spinor fields, which occur on spin manifolds and are defined as solutions of the field equation

$$\nabla_X^S \varphi = \lambda X \cdot \varphi$$

for all vector fields $X$ and some fixed $\lambda \in \mathbb{C}$, where $\nabla^S$ denotes the spinor derivative and $\cdot$ the Clifford multiplication.

In Riemannian geometry, it was proved in [Fri80] that real Killing spinors realize the lower bound of the eigenvalue estimation for the Dirac equation on compact spaces with positive scalar curvature. In the sequel Riemannian spaces admitting Killing spinors were intensively studied (cf. [DNP86], [BFGK91]) and a complete geometric description of such spaces was established. For the case of imaginary Killing spinors this was done by H. Baum in [Bau89] and then for real Killing spinors by Ch. Bär using the cone construction and the holonomy classification of Riemannian spaces with parallel spinors (cf. [Bar93]). Both results characterize Riemannian spaces with Killing spinors by the Einstein condition and the existence of certain differential forms, which can be understood as generalized Killing vectors. Real Killing spinors in Lorentzian geometry were first studied in [Boh98].

In this paper we will tread the Killing spinor equation to an imaginary Killing number $\lambda$ on a pseudo-Riemannian space with Lorentzian signature. As technical tool we will use again the cone construction for the investigation. Contrary to the Riemannian case, a holonomy description of the cone can not be used, since there is no classification of indecomposable holonomy groups for pseudo-Riemannian manifolds. Moreover, the geodesical completeness of a Lorentzian manifold does not imply that a cone with decomposable holonomy is flat. Instead, our geometrical description is mainly based on a normal form classification of skew-adjoint
operators in signature \((2, n-2)\), which is more rich then in the Euclidean case (cf. \[Bou00\]). The derived Lorenztian geometries are then described by the causal properties of the corresponding Dirac current and the existence of parallel spinors or certain Killing forms. Thereby, we will use the knowledge of structure results for Lorentzian manifolds admitting conformal gradient fields (cf. \[KR97\]) and twistor spinors with lightlike Dirac current (cf. \[BL02\]). Examples of geometries that occur are the Brinkmann spaces with parallel spinors and the Lorenztian Einstein-Sasaki manifolds.

The order of the paper is as follows. In the next section we introduce the basic notations and definitions appropriate for the study of Killing spinors and state basic curvature conditions for their existence (cf. Proposition 2.1 and 2.2). In section 3 we recall the cone construction over a Lorenztian base manifold and the correspondence of 'Killing objects' on the base and parallel objects on its cone (Theorem 3.1). We present the normal form classification of skew-adjoint operators in signature \((2, n-2)\) due to the work of Ch. Boubel in section 4. It turns out that there are exactly four generic types of normal forms for skew-adjoint operators coming from a spinor (cf. Corollary 4.7). The cone of a Lorenztian manifold admitting imaginary Killing spinors is furnished with at least one parallel 2-form, which corresponds to one of the generic types (Proposition 5.1). According to this type of a parallel 2-form on the cone we undertake in three of the four generic cases a discussion of the geometry of Lorenztian manifolds with imaginary Killing spinors in the last section. The results of the discussion are summerized in Theorem 5.3.

2 Basic facts on Killing spinors

In this section we recall the definition of Killing spinors on a spin manifold and fix some notations. A basic integrability condition for Killing spinors is stated. For more details we refer to \[BFGK91\]. Moreover, we will come across special Killing forms as they were introduced in \[Sem02\].

Let \((M^{n,k}, g)\) be a semi-Riemannian spin manifold of dimension \(n \geq 3\) and signature \((k, n-k)\) \((k\) is the number of timelike vectors in an orthonormal basis at a point). We denote by \(S\) the complex spinor bundle and by \(\cdot\) the Clifford multiplication on spinors. The Dirac operator \(D : \Gamma(S) \to \Gamma(S)\) acting on smooth spinor fields is defined as superposition of spinor derivative \(\nabla^S\) and Clifford multiplication. A spinor field \(\varphi \in \Gamma(S)\) is called Killing spinor to the Killing number \(\lambda \in \mathbb{C}\) if it satisfies the equation

\[
\nabla^S_X \varphi = \lambda X \cdot \varphi \quad \text{for all vector fields } X.
\]

It follows immediately from this definition that a Killing spinor \(\varphi\) is an eigenspinor of the Dirac operator \(D\) to the eigenvalue \(-n\lambda\) and \(\varphi\) is obviously a parallel spinor field with respect to the modified spinor derivative \(\nabla^\lambda\) defined by

\[
\nabla^\lambda := \nabla^S - \lambda id_{TM}.
\]

In particular, this implies that a Killing spinor \(\varphi\) admits no zeros. It holds the following basic integrability condition.
Proposition 2.1 ([BFGK91]) Let $\varphi \in \Gamma(S)$ be a Killing spinor to the Killing number $\lambda \in \mathbb{C}$.

1. It is $W(\eta) \cdot \varphi = 0$ for any 2-form $\eta$, where $W$ denotes the Weyl tensor.

2. $(\text{Ric}(X) - 4\lambda^2(n-1)X) \cdot \varphi = 0$, i.e. the image of the map $\text{Ric} - 4\lambda^2(n-1)\text{id}_{TM}$ is totally lightlike or trivial.

3. The scalar curvature is constant and given by $\text{scal} = 4n(n-1)\lambda^2$. The Killing number $\lambda$ is real or purely imaginary.

If the Killing number $\lambda$ is zero ($\text{scal} = 0$), $\varphi$ is a parallel spinor, in case that $\lambda$ is real and non-zero ($\text{scal} > 0$), $\varphi$ is called real Killing spinor, and in case that $\lambda$ is purely imaginary ($\text{scal} < 0$), $\varphi$ is called imaginary Killing spinor. We will treat in this paper the Killing spinor equation with imaginary Killing number on a space of Lorentzian signature $(- + \ldots +)$.

Let $(M^{n,1}, g)$ be a connected, oriented and time-oriented Lorentzian spin manifold. There exists an indefinite non-degenerate inner product $\langle \cdot, \cdot \rangle$ on the spinor bundle $S$ such that

$$
\langle X \cdot \varphi, \psi \rangle = \langle \varphi, X \cdot \psi \rangle \quad \text{and} \\
X(\langle \varphi, \psi \rangle) = \langle \nabla^S_X \varphi, \psi \rangle + \langle \varphi, \nabla^S_X \psi \rangle
$$

for all vector fields $X$ and all spinor fields $\varphi, \psi$. Each spinor field $\varphi \in \Gamma(S)$ defines a vector field $V_\varphi$ on $M$, the so-called Dirac current, by the relation $g(V_\varphi, X) := -\langle X \cdot \varphi, \varphi \rangle$ for all vector fields $X$. The Dirac current satisfies the following pointwise properties.

Lemma 2.1 ([Lei01]) Let $(M^{n,1}, g)$ be a Lorentzian spin manifold and let $\varphi(p) \neq 0$ be a spinor in a point $p \in M^{n,1}$. Then

1. $V_\varphi(p) \neq 0$ and $V_\varphi(p)$ is causal (i.e. $g_p(V_\varphi, V_\varphi) \leq 0$).

2. If $X \cdot \varphi(p) = \rho\varphi(p)$ for some $0 \neq X \in T_pM$ and $\rho \in \mathbb{R}$ then the vector $X$ is parallel to $V_\varphi(p)$.

The lemma makes clear that the Dirac current to a Killing spinor on a Lorentzian manifold is everywhere causal. Moreover, it is now possible to prove a stronger curvature condition for the existence of Killing spinors.

Proposition 2.2 Let $(M^{n,1}, g)$ be a Lorentzian spin manifold admitting a Killing spinor $\varphi$, whose Dirac current $V_\varphi$ is timelike. Then $(M, g)$ is an Einstein space.

**Proof:** Let us assume that $M^{n,1}$ is a non-Einstein space. Then there is an open set $U$ in $M$, where $H := \text{Ric}(X) - \frac{\text{scal}}{n} X \neq 0$ is lightlike for some vector field $X$. The Clifford product $H \cdot \varphi$ vanishes and, by Lemma 2.1, this implies that $H$ and $V_\varphi$ are parallel, which is a contradiction to the assumption. \qed

Especially, for imaginary Killing spinors it holds

Proposition 2.3 ([BFGK91]) Let $\varphi$ be an imaginary Killing spinor on a Lorentzian spin manifold $(M^{n,1}, g)$. Then the length $\langle \varphi, \varphi \rangle$ is constant on $M^{n,1}$ and if $\langle \varphi, \varphi \rangle \neq 0$ the space $M^{n,1}$ is Einstein.
Proof: It is \( X(\langle \varphi, \varphi \rangle) = \langle \lambda X \cdot \varphi, \varphi \rangle + \langle \varphi, \lambda X \cdot \varphi \rangle = 0 \) and with Proposition 2.1 we calculate

\[
\frac{1}{4(n-1)} \text{Ric}(X,Y)(\varphi,\varphi) = -\frac{1}{4(n-1)} \text{Re}\langle \text{Ric}(X) \cdot \varphi, Y \cdot \varphi \rangle = -\text{Re}\langle \lambda^2 X \cdot \varphi, Y \cdot \varphi \rangle
\]

\[
= \lambda^2 g(X,Y)(\varphi,\varphi)
\]

for all vector fields \( X \) and \( Y \), which shows that \( \text{Ric}(X) = \text{scal} \frac{n}{n} X \) in case that \( \langle \varphi, \varphi \rangle \neq 0 \).

Proposition 2.2 and 2.3 imply that an imaginary Killing spin or \( \varphi \) on a Lorentzian non-Einstein space \( M^{n,1} \) must have vanishing length \( \langle \varphi, \varphi \rangle \equiv 0 \) and the Dirac current \( V_{\varphi} \) to \( \varphi \) must be lightlike on an open subset of \( M^{n,1} \). We will see later that in this case \( V_{\varphi} \) is even lightlike everywhere on \( M^{n,1} \). Moreover, the Dirac current satisfies

Proposition 2.4

Let \( \varphi \) be an imaginary Killing spinor on a Lorentzian spin manifold \( (M^{n,1},g) \). The Dirac current \( V_{\varphi} \) is a Killing vector field, which in addition satisfies \( \nabla_X dV_{\varphi} = -4\lambda^2 X^\flat \wedge V_{\varphi}^\flat \).

Proof: It holds

\[
g(\nabla_{e_i} V_{\varphi}, e_j) = -\langle \lambda e_j e_i \cdot \varphi, \varphi \rangle - \langle e_j \varphi, \lambda e_i \cdot \varphi \rangle = -g(\nabla_{e_j} V_{\varphi}, e_i)
\]

for all \( i,j \in \{1,\ldots,n\} \), where \((e_1,\ldots,e_n)\) is an arbitrary orthonormal basis on \( M^{n,1} \). This proves that \( V_{\varphi} \) is a Killing vector field. Moreover,

\[
dV_{\varphi}^b = 4\lambda \sum_{i<j} \varepsilon_i \varepsilon_j \langle e_i e_j \cdot \varphi, \varphi \rangle e_i^b \wedge e_j^b \quad \text{and}
\]

\[
\nabla_X dV_{\varphi}^b = 4\lambda^2 \sum_{i<j} \varepsilon_i \varepsilon_j ((\langle e_i e_j X \cdot \varphi, \varphi \rangle - \langle X e_i e_j \cdot \varphi, \varphi \rangle)) e_i^b \wedge e_j^b = -4\lambda^2 X^\flat \wedge V_{\varphi}^\flat .
\]

In general, a \( p \)-form \( \alpha^p \), which solves the equation

\[
\nabla_X d\alpha^p = c X^\flat \wedge \alpha^p \quad \text{for all vectors } X
\]

and some fixed \( c \in \mathbb{R} \), is called a special Killing \( p \)-form (cf. [Sem02]). Proposition 2.4 states that the dual of the Dirac current to an imaginary Killing spinor is a special Killing 1-form. Killing spinors also produce special Killing forms of other degree then 1. For this, we observe that one constructs a \( p \)-form \( \alpha^p_{\varphi} \) to a spinor \( \varphi \) by the rule

\[
g(\alpha^p_{\varphi}, X^p) := -i \frac{(p+1)}{2} \langle X^p \cdot \varphi, \varphi \rangle \quad \text{for all } p \text{-forms } X^p
\]

and, in fact, if \( p \) is odd and the Killing number \( \lambda \) of a Killing spinor \( \varphi \) is imaginary (or if \( p \) is even and \( \lambda \) is real) then the associated \( p \)-form \( \alpha^p_{\varphi} \) to \( \varphi \) is special Killing.

3 The cone \( \hat{M} \)

We defined in the last section Killing spinors and special Killing \( p \)-forms on a Lorentzian manifold. In this section we will interpret these as parallel objects on the cone manifold. The cone construction was originally applied in order to describe Riemannian geometries admitting real Killing spinors (see [Bar93]) and can be modified here for our requirements.
Let \((M^{n,1}, g)\) be a Lorentzian manifold. We consider the cone \(\hat{M}\) of signature \((2, n-1)\) on \(M^{n,1}\), which is defined as
\[
\hat{M} := (M \times \mathbb{R}_+, \hat{g} := r^2 g - dr^2).
\]
The vector \(r \cdot \partial_r\) is called the Euler vector of \(\hat{M}\). The 1-level \(M \times \{1\}\) of the cone \(\hat{M}\) is naturally isometric to the base manifold \(M^{n,1}\) itself. We denote by \(\hat{X}\) the pullback of an arbitrary base vector field \(X \in \Gamma(M)\) to \(\hat{M}\) through the projection \(\pi\). Then we have the following rules for the Levi-Civita connection \(\hat{\nabla}\) on the cone
\[
\hat{\nabla}_{\partial_r} \partial_r = 0, \quad \hat{\nabla}_{\partial_r} \hat{X} = \hat{\nabla}_X \partial_r = \frac{1}{r} \hat{X},
\]
\[
\hat{\nabla}_X \hat{Y} = \nabla_X Y - r g(X,Y) \partial_r.
\]
In case that \(M^{n,1}\) is a spin manifold the cone \(\hat{M}\) is a spin manifold, too. Then we denote the spinor bundle of the cone with \(\hat{S}\). For \(n\) even the restriction of \(\hat{S}\) to the 1-level \(M \times \{1\}\) of the cone is naturally isomorphic to the spinor bundle \(S\) on the base manifold \(M^{n,1}\) by a map
\[
\Phi : S \cong \hat{S}\big|_{M \times \{1\}}
\]
with \(\Phi(X \cdot \varphi) = X \cdot \Phi(\varphi)\) for all \(X \in TM^{n,1}\). Similar, if \(n\) is odd, there are isomorphisms \(\Phi_{\pm} : S \cong \hat{S}_{\pm}\big|_{M \times \{1\}}\) for the restricted half spinor bundles such that
\[
-i X \cdot \Phi_{\pm}(\varphi) = \Phi_{\mp}(X \cdot \varphi).
\]
for all tangent vectors \(X \in TM^{n,1}\). With respect to the metric \(\hat{g}\) the projection \(\pi\) gives rise to a pullback \(\pi^{*} : \Gamma(\hat{S}\big|_{M \times \{1\}}) \to \Gamma(\hat{S})\) of spinor fields on the 1-level to the cone. Eventually, we denote by \(K_{\lambda}(M)\) the space of Killing spinors on \((M^{n,1}, g)\) to the Killing number \(\lambda\).

**Theorem 3.1** (cf. [Bar93] and [Sem02]) Let \((M^{n,1}, g)\) be a Lorentzian manifold and \(\hat{M}\) its cone with signature \((2, n-1)\). The following correspondences exist.

1. The special Killing \(p\)-forms on \(M^{n,1}\) to the positive constant \(c = p + 1\) are in 1-to-1-correspondence with the parallel \((p+1)\)-forms on the cone \(\hat{M}\). The correspondence is given by
\[
\alpha \in \Omega^p(M) \quad \mapsto \quad r^p dr \wedge \alpha - \frac{r^{p+1}}{p+1} d\alpha \in \Omega^{p+1}(\hat{M})
\]
2. If \(M^{n,1}\) is spin and \(\text{scal} = -n(n-1)\) then there are natural isomorphisms
\[
K_{\frac{1}{2}}(M) \oplus K_{-\frac{1}{2}}(M) \cong K_{0}(\hat{M})
\]
\[
\varphi \quad \mapsto \quad \hat{\varphi} := \pi^{*} \circ \Phi(\varphi)
\]
for \(n\) even and
\[
K_{\pm\frac{1}{2}}(M) \cong K_{\pm\frac{1}{2}}(\hat{M})
\]
\[
\varphi \quad \mapsto \quad \hat{\varphi} := \pi^{*} \circ \Phi_{\pm}(\varphi)
\]
for \(n\) odd,

where \(K_{0}^{\pm}(\hat{M})\) is the space of parallel \(\pm\)-half spinors on the cone.
The Riemannian version of Theorem 3.1 is classical for the application to the case of real Killing spinors. The result for Killing \(p\)-forms on Riemannian manifolds was established in [Sem02]. The proof for the correspondence here in case of imaginary Killing spinors \(\varphi\) in Lorentzian geometry is based on the observation that \(\varphi\) is parallel with respect to the modified spinor connection \(\tilde{\nabla}_\lambda\) coming from an affine connection, which takes values in

\[ i\mathbb{R}^{1,n-1} \oplus \text{spin}(1,n-1) \cong \text{spin}(2,n-1) \subset \text{Cliff}^C_{1,n-1}. \]

We remark for the application of Theorem 3.1 that the metric \(g\) on \(M^{n,1}\) can be rescaled by a positive constant such that the positive constant \(c\) to an arbitrary special Killing \(p\)-form equals \(p+1\) and the Killing number \(\lambda\) to an arbitrary imaginary Killing spinor satisfies \(\lambda^2 = -\frac{1}{4}\).

The spinor bundle \(S^{n,2}\) on a time-oriented pseudo-Riemannian spin manifold \((N^{n,2}, h)\) of signature \((2,n-2)\) is equipped with an invariant inner product \(\langle \cdot, \cdot \rangle_{2,n-2}\) (cf. [Bau81]). Similar to the induced Dirac current of a spinor in Lorentzian geometry, a spinor \(\gamma \in \Gamma(S^{n,2})\) induces a 2-form \(\alpha^2\) on \(N^{n,2}\) by the rule

\[ h(\alpha^2, X^2) := -i \langle X^2 \cdot \gamma, \gamma \rangle_{2,n-2} \quad \text{for all 2-forms } X^2. \]

In case that \(\hat{M}\) is the cone over a Lorenztian spin manifold \(M^{n,1}\) the inner product \(\langle \cdot, \cdot \rangle_{2,n-1}\) admits the property

\[ \langle \varphi, \psi \rangle = -\langle \partial_r \cdot \Phi_-(\varphi), \Phi_+(\psi) \rangle_{2,n-1} \quad \text{for } n \text{ odd} \quad \text{and} \quad \langle \varphi, \psi \rangle = i \langle \partial_r \cdot \Phi_+(\varphi), \Phi_-(\psi) \rangle_{2,n-1} \quad \text{for } n \text{ even} \]

on the 1-level of \(\hat{M}\), where \(\varphi, \psi\) are spinor fields on \(M^{n,1}\). Then the following relation is true.

**Lemma 3.2** Let \(\varphi \in \Gamma(S)\) be a spinor with Dirac current \(V_\varphi\) on a Lorentzian spin manifold \(M^{n,1}\) and let \(\hat{\varphi}\) be the corresponding (±-half) spinor with associated 2-form \(\alpha^2_{\hat{\varphi}}\) on the cone \(\hat{M}\). It holds \(V^b_\varphi = \partial_r \cdot 1 \alpha^2_{\hat{\varphi}}\) on the 1-level \(M^{n,1} \subset \hat{M}\).

**Proof:** With respect to an orthonormal basis \(e = (e_0, e_1, \ldots, e_n)\) with \(e_0 = \partial_r\) in an arbitrary point of the 1-level it holds

\[ \partial_r \cdot 1 \alpha^2_{\hat{\varphi}} = -i \sum_{i<j} \langle e_i e_j \cdot \hat{\varphi}, \hat{\varphi} \rangle_{2,n-1} \cdot \partial_r \cdot 1 \ e_i^* \wedge e_j^* = -i \sum_{j=1}^{n} \langle \partial_r e_j \cdot \hat{\varphi}, \hat{\varphi} \rangle_{2,n-1} e_j^* = -i \sum_{j=1}^{n} \langle e_j \varphi, \varphi \rangle e_j^* = V^b_\varphi. \]

\(\square\)

The lemma also shows that \(\alpha^2_{\hat{\varphi}}\) is non-trivial for all (half) spinors \(\hat{\varphi} \neq 0\) on the cone \(\hat{M}\), since the corresponding Dirac current \(V^b_\varphi\) on \(M^{n,1}\) is non-trivial.
4 Normal forms for skew-adjoint operators in signature \((2, n - 2)\)

In this section we present a complete list of normal forms for skew-adjoint endomorphisms acting on the pseudo-Euclidean space \(\mathbb{R}^{2,n-2}\) of dimension \(n\) and signature \((2, n - 2)\). This list was established in [Bou00]. Parallel 2-forms on the cone of signature \((2, n - 1)\) over a Lorentzian manifold correspond to parallel skew-adjoint operators and are therefore distinguished by normal forms of the list. This observation will be the crucial point in our description of Lorentzian geometries admitting imaginary Killing spinors in the last section.

**Theorem 4.1** (cf. [Bou00]) Let \(\beta\) be an arbitrary 2-form on the pseudo-Euclidean space \(\mathbb{R}^{2,n-2}\). Then there exist vector spaces \(V_i\) such that \(\mathbb{R}^{2,n-2} = \bigoplus V_i\) is an orthogonal direct sum and the skew-adjoint endomorphism \(b\), which corresponds to \(\beta\), satisfies \(b(V_i) \subset V_i\) for all \(i\). Moreover, there is a basis \((e_{i_1}, \ldots, e_{i_{r(i)}})\) for every \(V_i\) such that the corresponding matrix for the inner product and for \(b\) is one pair of blocks as it occurs in the lines of Table 1 below.

The basis of \(\mathbb{R}^{2,n-2},\) in which a skew-adjoint operator takes a normal form, is called an adapted basis. There is always an orthogonal decomposition \(\mathbb{R}^{2,n-2} = E \oplus P\) to a skew-adjoint operator \(b\) such that \(E\) is Euclidean and \(b\) preserves the decomposition. We call the normal form to \(b\) on \(E\) an Euclidean block and the normal form to \(b\) on \(P\) a pseudo-Euclidean block.

**Example 4.2**

a) Let \(\omega_\nu := \sum_{i=1}^{m} e_{2i-1}^* \wedge e_{2i}^*\) be the standard (pseudo)-Kähler form on \(\mathbb{R}^{2,n-2}\), where \((e_1, \ldots, e_{2m})\) is the standard basis. The normal form of the skew-adjoint operator corresponding to a multiple \(\nu \cdot \omega_\nu\) of the Kähler form with respect to the adapted basis \((e_1, \ldots, e_{2m})\) is built up by one block of the form \(B_1(\nu)\) (pseudo-Euclidean block) and \((m-1)\) blocks of the form \(B(\nu)\) (Euclidean block).

b) A 2-form \(\omega = \ell_1^* \wedge \ell_2^*\) on \(\mathbb{R}^{2,n-2}\), where \(\ell_1\) and \(\ell_2\) are lightlike vectors, which span a totally lightlike plane, corresponds as skew-adjoint operator with respect to some adapted basis to a composition of a pseudo-Euclidean block of the form \(B_{1a}\) and an Euclidean 0-block of length \(n-4\).

c) A 2-form \(\omega = t_1^* \wedge t_1^*\) on \(\mathbb{R}^{2,n-2}\), where \(t_1\) is lightlike, \(t_1\) is timelike and both vectors are orthogonal, corresponds as skew-adjoint operator with respect to some adapted basis to a composition of a block \(B_{1b}\) and a 0-block of length \(n-3\).

Let \(\hat{\varphi}\) be a spinor on the pseudo-Euclidean space \(\mathbb{R}^{2,n-1}\). There corresponds a 2-form \(\alpha_{\varphi}^2\) to \(\hat{\varphi}\) on \(\mathbb{R}^{2,n-1}\) defined by the rule

\[
(\alpha_{\varphi}^2, x^2) := -i \langle x^2 \cdot \varphi, \varphi \rangle_{2,n-2}\quad \text{for all } x^2 \in \Lambda^2(\mathbb{R}^{2,n-1}^*),
\]

where \((\cdot, \cdot)\) denotes the induced inner product on \(\Lambda^2(\mathbb{R}^{2,n-1}^*)\) (cf. section 3). The following statement is a version of Lemma 3.3 considered in a single point only and the proof for it works the same as before.

**Lemma 4.3** Let \(\hat{\varphi}\) be a \((\pm\)half\) spinor on \(\mathbb{R}^{2,n-1}\) and \(T \in \mathbb{R}^{2,n-1}\) an arbitrary unit timelike vector. The 1-form \(\alpha_{T,\varphi} := T \cdot \alpha_{\varphi}^2\) is dual to the associated vector induced by the spinor \(\varphi\) on the Minkowski space \(T^\perp \subset \mathbb{R}^{2,n-1}\), which corresponds naturally to \(\hat{\varphi}\).
| signature \((p, q)\) | \(A = \) inner product | \(B = \) skew-adjoint operator |
|------------------|------------------|------------------|
| \((0, 1)\)      | \[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\] | \[
B(\nu) = \left( \begin{array}{cc}
0 & -\nu \\
\nu & 0
\end{array} \right)
\] \(\nu \neq 0\) |
| \((0, 2)\)      | \[
\left( \begin{array}{cccc}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array} \right)
\] | \[
B_{Ia} = \left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right)
\] |
| \((1, 0)\)      | \(
-1
\) | \(\lambda \neq 0\) |
| \((1, 2)\)      | \[
\left( \begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array} \right)
\] | \[
B_{Ib} = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array} \right)
\] |
| \((1, 1)\)      | \[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\] | \[
B_{II}(\nu) = \left( \begin{array}{cc}
0 & -\nu \\
\nu & 0
\end{array} \right)
\] \(\nu \neq 0\) |
| \((2, 2)\)      | \[
\pm \left( \begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array} \right)
\] | \[
B_{II}^\pm_{Ia} = \left( \begin{array}{cccc}
0 & -\nu \pm & 1 & 0 \\
\nu \pm & 0 & 0 & -\nu \pm \\
0 & 0 & 0 & -\nu \pm \\
0 & 0 & \nu \pm & 0
\end{array} \right)
\] \(\nu \pm \neq 0\) |
| \((2, 0)\)      | \[
\left( \begin{array}{cc}
1 & 0 \\
0 & -1
\end{array} \right)
\] | \[
\left( \begin{array}{cccc}
0 & -\nu & 0 & I_2 \\
\nu & 0 & 0 & -\nu
\end{array} \right)
\] \(\nu \neq 0\) |
| \((2, 4)\)      | \[
\left( \begin{array}{cccc}
0 & 0 & 0 & -I_2 \\
0 & I_2 & 0 & 0 \\
-I_2 & 0 & 0 & 0
\end{array} \right)
\] | \[
\left( \begin{array}{cccc}
\lambda I_2 & 0 & 0 & -\lambda I_2 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0
\end{array} \right)
\] \(\lambda \neq 0\) |
| \((2, 2)\)      | \[
\left( \begin{array}{cc}
0 & I_2 \\
I_2 & 0
\end{array} \right)
\] | \[
\left( \begin{array}{cc}
0 & I_2 \\
0 & \lambda I_2 & 0 & -\lambda I_2 \\
0 & 0 & \lambda & 0
\end{array} \right)
\] \(\lambda \neq 0\) |
| \((2, 2)\)      | \[
\left( \begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array} \right)
\] | \[
B_{IIb} = \left( \begin{array}{cccc}
\xi & -\nu & 0 & 0 \\
\nu & \xi & 0 & 0 \\
0 & 0 & -\xi & \nu \\
0 & 0 & -\nu & -\xi
\end{array} \right)
\] \(\xi, \nu \neq 0\) |

Table 1: These are the building blocks for the normal forms of skew-adjoint operators in signature \((2, n - 2)\). The matrices in the first column (denoted by \(A\)) indicate an inner product (of index \(s \leq 2\)) with respect to some basis and the matrices in the second column (denoted by \(B\)) are skew-adjoint endomorphisms with respect to the inner product in column \(A\) and the chosen basis.
The lemma imposes a condition on the nature of a 2-form induced by a spinor in signature \((2, n - 2)\), since the associated vector to a non-trivial spinor on the Minkowski space is not arbitrary, but causal. With some simple calculations we can sort out the normal forms for skew-adjoint operators corresponding to 2-forms, which do not satisfy the condition imposed by Lemma 4.6 and therefore can not be induced by a spinor.

**Corollary 4.4** Let \(\omega\) be a 2-form in signature \((2, n - 2)\) such that the covector \(T \cdot \omega\) is causal for every timelike vector \(T \in \mathbb{R}^{2, n-2}\).

1. If there is a timelike \(T\) such that \(T \cdot \omega\) is lightlike then the normal form corresponding to \(\omega\) is a composition of a pseudo-Euclidean block of the form \(B_{1a}\) or \(B_{1b}\) with an Euclidean 0-block.

2. If \(T \cdot \omega\) is timelike for all timelike \(T\) then the normal form of \(\omega\) is a composition of \(B_{II}\), \(B^\pm_{IIa}(\nu_+ > 0, \nu_- < 0)\) or \(B_{IIb}(\nu^2 \geq \xi^2)\) with an Euclidean block consisting of blocks of the form \(B(\nu)\) and/or a 0-block.

With stabilizer of a skew-adjoint operator (resp. 2-form) we mean in the following the subgroup of the (pseudo)-orthogonal group, which leaves the operator (resp. the 2-form) invariant under conjugated action. A simple consideration shows the following fact.

**Lemma 4.5** The stabilizer of a normal form, which is built from a pseudo-Euclidean block of the form \(B_{IIa}(\nu_\pm)\) or \(B_{IIb}(\nu, \xi)\) and some Euclidean block of length \(n - 4\) is included in \(U(1,1) \times SO(n-4)\) for all eigenvalues \(\nu_\pm, \nu\) and \(\xi \neq 0\).

**Definition 4.6** Let \(\mathbb{R}^{2, n-2}\) be the pseudo-Euclidean space of signature \((2, n - 2)\) and \(\omega \in \Lambda^2 \mathbb{R}^{2, n-2}\) be a non-trivial 2-form. We say that \(\omega\) is of

- **Type** \((I_a)\) if \(\omega = l_1 \wedge l_2\) for some vectors \(l_1\) and \(l_2\), which span a totally lightlike plane.
- **Type** \((I_b)\) if \(\omega = l_1 \wedge l_2\) for some lightlike vector \(l_1\) and a \(l_1\)-orthogonal timelike vector \(t_1\).
- **Type** \((II_a)\) or Kähler Type if \(\omega\) is a non-trivial multiple of the Kähler form.
- **Type** \((II_b)\) if there exists a non-trivial Euclidean subspace \(E\) in \(\mathbb{R}^{2, n-2}\) such that \(\omega\) restricted to \(E\) vanishes and \(\omega\) is a (pseudo)-Kähler form on the orthogonal complement of \(E\) in \(\mathbb{R}^{2, n-2}\).

Lemma 4.5 makes clear, which stabilizers of the normal forms occuring in Corollary 4.4 are maximal.

**Corollary 4.7** A 2-form \(\omega\) on \(\mathbb{R}^{2, n-2}\), which is of Type \((I_a)\), \((I_b)\), \((II_a)\) or \((II_b)\), is exclusively distinguished by its properties that

1. the covector \(T \cdot \omega\) is causal for every timelike vector \(T \in \mathbb{R}^{2, n-2}\) and

2. its stabilizer \(S_\omega\) in \(SO(2, n-2)\) is maximal, in the sence that there is no non-trivial 2-form satisfying the first property, whose stabilizer properly contains \(S_\omega\).

We observe that the stabilizer of a 2-form of Type \((I_a)\) and Type \((I_b)\) acts indecomposable but reducible on \(\mathbb{R}^{2, n-2}\), i.e. there exist non-trivial and invariant subspaces of \(\mathbb{R}^{2, n-2}\), but the inner product is degenerate on all of them. The stabilizer of a Kähler type form is \(U(1, m - 1)\) and acts irreducible on \(\mathbb{R}^{2, 2m-2}\). The stabilizer of a form of Type \((II_b)\) acts decomposable on \(\mathbb{R}^{2, n-2}\).
5 Imaginary Killing spinors

With the construction of the cone in section 3 and the normal form classification for skew-adjoint operators in signature \((2, n - 2)\) (coming from a spinor) in the last section we are now in the position to discuss a geometric description of Lorentzian manifolds admitting imaginary Killing spinors.

In the following, the metric \(g\) on the Lorentzian spin manifold \(M_{n,1}\) will be scaled such that the Killing number \(\lambda\) to any Killing spinor satisfies \(\lambda^2 = -\frac{1}{4}\). We start with a proposition, which characterizes the cone \(\hat{M}\) of a Lorentzian manifold \(M_{n,1}\) with imaginary Killing spinor and indicates the different cases that are to be considered for the geometry of the base manifold \(M_{n,1}\). We remark that the normal form corresponding to a parallel 2-form on the cone is in every point the same.

**Proposition 5.1** Let \((M^{n,1}, g)\) be a Lorentzian spin manifold admitting an imaginary Killing spinor. Either there exists a parallel 2-form \(\omega\) of Type \((IIb)\) on the cone \(\hat{M}\) or there exists at least one parallel (half) spinor \(\hat{\phi}\) on \(\hat{M}\) such that the induced parallel 2-form \(\omega = \alpha_\phi^2\) is of Type \((Ia),(Ib)\) or \((IIa)\).

**Proof:** Let \(\psi\) be an imaginary Killing spinor on \(M^{n,1}\). According to Corollary 4.4 the normal form of the skew-adjoint endomorphism corresponding to \(\alpha_\psi^2\) is a composition with one block of the form \(B_{Ia}, B_{Ib}, B_{II}, B_{IIa}^\pm\) or \(B_{IIb}\). In case that the normal form is built with a block of the form \(B_{Ia}\) or \(B_{Ib}\) the parallel 2-form \(\omega = \alpha_\phi^2\) is of Type \((Ia)\) or \((Ib)\).

In the other cases there exists a biggest number \(s > 0\) such that the stabilizer of the normal form to \(\alpha_\psi^2\) is included in \(U(1, s - 1) \times SO(n - 2s)\). This group includes the holonomy group of the cone \(\hat{M}\). In case that \(2s = n\) the cone \(\hat{M}\) is a Kähler spin manifold. Moreover, since \(V_\psi = \partial_r \cdot \alpha_\psi^2\) is everywhere timelike (Corollary 4.3), the base \(M^{n,1}\) is Einstein and hence the cone is Ricci-flat. This implies that there exists a parallel (half) spinor \(\hat{\phi}\), which induces a Kähler form on the cone. If \(2s < n\) there exists a parallel 2-form \(\omega\) of Type \((IIb)\). \(\square\)

We discuss now a description of the Lorentzian geometries on the base manifold \(M^{n,1}\) with imaginary Killing spinor according to the cases \((Ia),(Ib)\) and \((IIa)\) that occur in Proposition 5.1.

5.1 Type \((Ia)\)

In this case there exists a parallel (half) spinor \(\hat{\phi}\) on the cone \(\hat{M}\), which induces a parallel 2-form \(\omega \neq 0\) that is locally of the form \(l_1^2 \wedge l_2^2\) for some lightlike vector fields \(l_1\) and \(l_2\), which span a totally lightlike plane. The dual \(V_\phi^2\) of the Dirac current of the imaginary Killing spinor \(\varphi\), which corresponds to \(\hat{\phi}\) on \(\hat{M}\), is equal to \(\partial_r \cdot l_1 \omega\), which shows that the Dirac current \(V_\phi\) is everywhere lightlike.

There is a known description of Lorentzian metrics admitting *twistor spinors* with lightlike Dirac current. We call a Lorentzian space admitting a lightlike parallel vector field a *Brinkmann space*. 
Two spinor fields on (pseudo)-Riemannian spaces are said to be \textit{conformally equivalent} if there exists a conformal diffeomorphism, which identifies both spinor fields. In particular, it holds

**Proposition 5.2** (see [BL02]) Let \( \varphi \) be a spinor field, which satisfies the twistor equation \( \nabla_X \varphi + \frac{1}{n} X \cdot D \varphi = 0 \) for all vector fields \( X \), such that the Dirac current \( V_\varphi \) is a lightlike Killing vector field on \( M^{n,1} \). If \( \text{Ric}(V_\varphi, V_\varphi) = 0 \) then \( \varphi \) is locally conformally equivalent to a parallel spinor on a Brinkmann space.

This gives rise to

**Proposition 5.3** Let \( \varphi \) be an imaginary Killing spinor on \( M^{n,1} \) such that \( \alpha_\varphi^2 \) on \( M \) is of Type \((I_a)\). Then \( \varphi \) is locally conformally equivalent to a parallel spinor on a Brinkmann space.

**Proof:** From Proposition 2.1 and Lemma 2.1 we know that \( \text{Ric}(V_\varphi) = \rho V_\varphi \) for some real function \( \rho \). Then we can apply Proposition 5.2 to prove the result. \( \square \)

### 5.2 Type \((I_b)\)

There exists a parallel (half) spinor \( \hat{\varphi} \) on the cone \( \hat{M} \), which induces a parallel 2-form \( \omega \) of Type \((I_b)\). In this situation it holds

**Lemma 5.1** The function \( f_\varphi := \sqrt{-g(V_\varphi, V_\varphi)} \) to the imaginary Killing spinor \( \varphi \) on \( M^{n,1} \) satisfies

1. \( \text{Hess}(f_\varphi) = f_\varphi \cdot g \), i.e. \( \text{grad} f_\varphi \) is a conformal gradient field, and \( f_\varphi^2 = g(\text{grad} f_\varphi, \text{grad} f_\varphi) \),

2. \( \text{grad} f_\varphi \neq 0 \) and \( f_\varphi \neq 0 \) on disjoint dense subspaces in \( M^{n,1} \).

**Proof:** The 2-form \( \omega \) can be written as \( rdr \wedge V_\varphi^2 - \frac{t^2}{2} dV_\varphi^2 \) (Theorem 3.1). The 2-dimensional parallel subbundle \( E_\omega \subset TM^{n,1} \), which corresponds to the indecomposable \( \omega \) is degenerate and there is a unique parallel lightlike direction in \( E_\omega \). In particular, there exists a parallel lightlike vector field \( l_1 \) on \( M \). Moreover, we can find locally a timelike vector \( t_1 \) of constant length such that \( \omega = l_1^2 \wedge t_1^2 \). We choose the parallel lightlike field \( l_1 \) with the scaling \( \tilde{g}(t_1, t_1) = -1 \). Since \( l_1 \) is parallel, there is a unique function \( f \) on \( M^{n,1} \) such that \( l_1^2 = fdr - rdf \) and \( f^2 = g(\text{grad} f, \text{grad} f) \). The function \( f \) is a special Killing 0-form on \( M^{n,1} \), i.e. \( \text{grad} f \) is a conformal gradient field. Since neither \( dr \) nor the lift of \( df \) to the cone are parallel, it is \( f \neq 0 \) and \( \text{grad} f \neq 0 \) on a dense subset of \( M^{n,1} \).

We calculate the function \( f \) with respect to \( \varphi \). The local field \( l_1^2 \) is given by \( t_1^2 = A dr + u \), where \( A \) is a function and \( u \) a 1-form on \( M^{n,1} \). It follows that \( V_\varphi^2 = fu + Adf \)

\[
g(V_\varphi, V_\varphi) = f^2 g(u, u) + A^2 g(df, df) + 2f A \cdot g(u, df).
\]

Since \( g(u, u) = -1 + A^2 \) and \( g(u, df) = -Af \), we can conclude \( f^2 = -g(V_\varphi, V_\varphi) \), which shows that \( f_\varphi \) has the claimed properties. \( \square \)

The assertions of Lemma 5.1 imply together with Proposition 2.2 that \( M^{n,1} \) is Einstein and \( \text{grad} f_\varphi \) is a non-homothetic conformal gradient field. There is a known description of (pseudo)-Riemannian Einstein metrics admitting such conformal fields. In particular, there is
Proposition 5.4 (cf. [KR97]) Let \((M^{n,1}, g)\) be a Lorentzian Einstein space admitting a non-constant solution \(f\) of the equation \(\text{Hess}(f) = l \cdot g\) for some function \(l\). Then, in a neighborhood of any point with \(v := g(\text{grad}(f), \text{grad}(f)) \neq 0\), the metric \(g\) is a warped product \(\varepsilon \cdot dt^2 + f^2(t)k\), where \(\varepsilon := \text{sign}(v)\), \(k\) is an Einstein metric and \(f\) satisfies \(f'' + \frac{\varepsilon \text{scal}_k}{n(n-1)} f^2 = 0\).

This leads to

Proposition 5.5 Let \(\varphi\) be an imaginary Killing spinor on \(M^{n,1}\) such that \(\alpha^2 \varphi\) on \(\hat{M}\) is of Type (I_b). Then, in a neighborhood of any point with \(V\varphi\) timelike, the metric \(g\) is a warped product of the form \(dt^2 + f^2 k\), where \(k\) is a Lorentzian Einstein metric admitting a Killing spinor to the Killing number

1. \(\lambda_k = 0\) and \(f = \exp t\)
2. \(\lambda_k = \frac{i}{2}\) and \(f = \sinh t\) or
3. \(\lambda_k = \frac{i}{2}\) and \(f = \cosh t\).

Proof: The function \(f_\varphi = \sqrt{-g(V, V_\varphi)}\) satisfies the assumptions of Proposition 5.4. Since \(f_\varphi^2 > 0\), the warping function \(f = f_\varphi\) must solve the ordinary differential equation \(f'' - n(n-1)f^2 = \text{scal}_k\). There are three different solutions \(f = \exp t, \cosh t, \sinh t\) according to the values \(\text{scal}_k = 0, \pm (n-1)(n-2)\). In each case the imaginary Killing spinor \(\varphi\) induces a Killing spinor to the Killing number \(\frac{\text{scal}_k}{n(n-1)(n-2)}\) on the space with Einstein metric \(k\) (cf. [Boh98]).

5.3 Type (II_a)

Lemma 5.2 Let \((M^{n,1}, g)\) be a Lorentzian Einstein manifold with a Killing vector \(V\) such that \(g(V, V) = -1\) is constant and \(\nabla W = 0\) (\(W\) Weyl tensor). Then the operator \(J\) defined by \(J(X) := \nabla_X V\) on \(TM\) satisfies

1. \(J(V) = 0\) and \(J^2(X) = \frac{\text{scal}}{n(n-1)}(X + g(V, X)V)\)
2. \((\nabla_X J)(Y) = \frac{\text{scal}}{n(n-1)}(g(V, Y)X - g(X, Y)V)\).

Proof: Because \(V\) is Killing with constant length, it follows \(\nabla V = 0\) and \(g(\nabla X V, \nabla Y V) = \mathcal{R}(V, X, Y, V)\), where \(\mathcal{R}\) denotes the Riemannian curvature tensor. It is \(\mathcal{R} = \mathcal{W} + g \ast L\), where \(L = \frac{1}{n-2}(\frac{\text{scal}}{2(n-1)}g - \mathcal{R}c)\) is the Schouten tensor and \(\ast\) denotes the Kulkarni-Nomizu product (cf. [Bes87]). Then from \(\nabla J W = 0\) we obtain

\[
g(J^2(X), Y) = -g(J(X), J(Y))
\]

\[
= -g(V, Y)L(X, V) - g(V, X)L(Y, V) - L(X, Y) + g(X, Y)L(V, V)
\]

The relation for \(J^2\) follows immediately, since for \(M^{n,1}\) an Einstein space it holds \(L = -\frac{\text{scal}}{2(n-1)} g\).

Moreover, it is \(g(\nabla_{e_i} e_j, V) = \mathcal{R}(e_i, e_j, e_k, V)\) for all \(i, j, k \in \{1, \ldots, n\}\) in \(p \in M^{n,1}\) arbitrary, where \((e_1, \ldots, e_n)\) is a local parallel frame in \(p\). Then

\[
g((\nabla_{e_i} J)(e_k), e_l) = \mathcal{R}(e_k, e_l, e_i, V)
\]

\[
= g(e_k, e_l) L(e_l, V) + g(e_l, V) L(e_k, e_l) - g(e_k, V) L(e_l, e_i) - g(e_l, V) L(e_k, e_i)
\]

which shows the identity for \(\nabla J\) in an arbitrary point \(p\) of \(M^{n,1}\).
A Lorentzian manifold \((M^{n,1}, g, V)\) with \(V\) a timelike Killing vector of constant length such that the operator \(J = \nabla V\) satisfies the both properties of Lemma 5.2 is called a Lorentzian Sasaki manifold. It is well-known that a Sasaki structure \((V, J)\) on \(M^{n,1}\) corresponds to a Kähler structure on the cone \(\hat{M}\) (cf. [Bar93] and [Bau00]).

**Proposition 5.6** A Lorentzian spin manifold \((M^{n,1}, g)\) with an imaginary Killing spinor \(\varphi\), whose Dirac current \(V_{\varphi}\) is timelike and has constant length, is a Lorentzian Einstein-Sasaki manifold. This is exactly the case when the lift \(\hat{\varphi}\) induces a Kähler form on the cone \(\hat{M}\).

**Proof:** We have only to show that \(V_{\varphi} \cdot W = 0\) on \(M^{n,1}\) and then apply Lemma 5.2. With the identity \(W(\eta) \cdot \varphi = 0\) (Proposition 2.1) and the relation \(X \cdot \eta = -X \cdot \eta + X^b \eta\) in the Clifford algebra, where \(X\) denotes a vector and \(\eta\) a 2-form, we obtain

\[
W(\varphi, X, Y, Z) = \langle \varphi, W(X, Y, Z) \cdot \varphi \rangle = \langle \varphi, Z^b \wedge W(X, Y) \cdot \varphi \rangle \in \mathbb{R} \text{ for all } X, Y, Z \in TM.
\]

But \(\langle \varphi, \rho^3 \cdot \varphi \rangle \in i\mathbb{R}\) for all 3-forms \(\rho^3\), and therefore \(V_{\varphi} \cdot W = 0\).

We summarize the different cases to

**Theorem 5.3** Let \((M^{n,1}, g)\) be a Lorentzian spin manifold with imaginary Killing spinor \(\varphi\).

1. If \(M^{n,1}\) is not Einstein then \(M^{n,1}\) is locally conformally equivalent to a Brinkmann space with parallel spinor.

2. If \(g(V_{\varphi}, V_{\varphi})\) is constant then
   
   i) \(g(V_{\varphi}, V_{\varphi}) = 0\) and \(M^{n,1}\) is locally conformally equivalent to a Brinkmann space with parallel spinor or
   
   ii) \(g(V_{\varphi}, V_{\varphi}) < 0\) and \(M^{n,1}\) is a Lorentzian Einstein-Sasaki manifold.

3. If the cone \(\hat{M}\) is indecomposable and \(V_{\varphi}\) is timelike then \(M^{n,1}\) is either
   
   i) locally conformally equivalent to a Brinkmann space with parallel spinor,
   
   ii) locally a warped product of the form \(dt^2 + f^2 k\), where \(k\) is a Lorentzian Einstein metric admitting a Killing spinor and \(f = \exp t, \cosh t\) or \(\sinh t\) or
   
   iii) a Lorentzian Einstein-Sasaki space (and the cone \(\hat{M}\) is irreducible).

4. If \(V_{\varphi}\) changes the causal type then the set \(Z_{\varphi} \subset M^{n,1}\), where \(V_{\varphi}\) is lightlike, is a hypersurface and \(M^{n,1} \setminus Z_{\varphi}\) admits locally a warped product structure as in 3. ii).

In case that the metric \(g\) does not belong to one of those listed in 3. then either \(V_{\varphi}\) changes the causal type or there is a parallel 2-form of Type (IIb) on the cone \(\hat{M}\).

**Example 5.4** Let \(H^{n,1} := \{x \in \mathbb{R}^{2,n-1} : ||x||^2 = -1\} \subset \mathbb{R}^{2,n-1}\) be the pseudo-hyperbolic space of signature \((1, n - 1)\) with negative scalar curvature \(\text{scal} = -n(n-1)\). The space \(H^{n,1}\) is geodesically complete, time-orientable and spin. The cone over \(H^{n,1}\) is an open subset of \(\mathbb{R}^{2,n-1}\). Each parallel (half) spinor on \(\mathbb{R}^{2,n-1}\) restricted to \(H^{n,1}\) gives rise to an imaginary Killing spinor. It is not difficult to see that every generic type \((I_a), (I_b), (II_a)\) and \((II_b)\) is realized by a 2-form, which comes from a parallel (half) spinor on \(\mathbb{R}^{2,n-1}\) and thus belongs to an imaginary Killing spinor on \(H^{n,1}\). This means that there are examples of imaginary Killing spinors on \(H^{n,1}\), \(n \geq 3\), for each case where the Dirac current is everywhere lightlike \((I_a)\), changes the causal type \((I_b)\), timelike with constant length \((II_a)\) or everywhere timelike with non-constant length.
Remark 5.5

1. In fact, there exist examples of imaginary Killing spinors on non-Einstein spaces, which are generated by an appropriate conformal change of certain non-Einstein Brinkmann spaces with parallel spinors (cf. [Boh98]).

2. Partial structure results and examples for Lorentzian metrics with parallel or real Killing spinors are known (cf. e.g. [Bry00], [Boh98]). The warped product structure in case of Type (I_b) then provides a concrete construction principle for imaginary Killing spinors on Lorentzian spaces.

The complete result in [KR97] for the description of Einstein spaces with conformal gradient fields does not apply when the field changes the causal type. As consequence, Theorem 5.4 does not describe Lorentzian metrics with imaginary Killing spinors when the Dirac current changes the causal type.

3. There is a construction principle for Lorentzian Einstein-Sasaki spin spaces. They appear as $S^1$-fiber bundles over Riemannian Kähler-Einstein spin spaces of negative scalar curvature (cf. [Bau00]).

4. In case that there exists a parallel 2-form of Type (II_b) the cone $\hat{M}$ is decomposable. Different from the Riemannian case, this does not imply that the cone is flat, even if the base $M^{n,1}$ is geodesically complete. The geometry of the base $M^{n,1}$ with imaginary Killing spinor in this case remains to be investigated and is subject of a forthcoming paper.

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