SINGULAR SUPPORT OF THE GLOBAL ATTRACTOR FOR A DAMPED BBM EQUATION

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(Communicated by Chunyou Sun)

Abstract. The singular support of the global attractor is introduced. It is shown that the singular support of the global attractor for a damped BBM equation equals to the singular support of the force term. This gives a delicate description of the local regularity, which roughly says that the attractor is smooth exactly where the force is smooth.

1. Introduction. In this work, we aim to consider the singular support theory of the global attractor. First, we recall the singular support of a function, see [14, p.42].

Definition 1.1. Let $u$ be a locally integrable function on an open set $X \subset \mathbb{R}^n$. Then the singular support of $u$, denoted by $\text{sing supp } u$, is the set of points in $X$ having no open neighborhood to which the restriction of $u$ is a $C^\infty$ function.

For example, if $u(x) = |x|, x \in \mathbb{R}$, then $\text{sing supp } u = \{0\}$. Clearly, every point in $X \setminus \text{sing supp } u$ has a neighborhood where $u$ is a $C^\infty$ function. Thus the restriction of $u$ to $X \setminus \text{sing supp } u$ is a $C^\infty$ function. A famous theorem on the singular support due to Hörmander [15, p.61] is,

$$\text{sing supp } Q(D)u = \text{sing supp } u$$

if and only if the differential operator $Q(D)$ is hypoelliptic. In particular, this implies that $\text{sing supp } u = \text{sing supp } f$ if $u$ is a locally integrable function solving

$$\Delta u = f \quad \text{in } \mathbb{R}^n, \quad \Delta = \partial^2_{x_1} + \partial^2_{x_2} + \cdots + \partial^2_{x_n}.$$
Here we used the fact that the Laplacian $\Delta$ is elliptic, and thus hypoelliptic. Inspired from this, after some calculations we can show that, if $Q \in H^1(\mathbb{R})$ solves
\begin{equation}
-\gamma Q_{xx} + \delta Q + Q_x + QQ_x = f(x) \quad \text{in } \mathbb{R},
\end{equation}
then $\text{sing supp } Q = \text{sing supp } f$. Throughout this paper, the coefficients $\gamma$ and $\delta$ are positive constants.

On the other hand, the Benjamin-Bona-Mahony (BBM) equation
\begin{equation}
\begin{cases}
    u_t - u_{txx} + u_x + uu_x = 0,
\end{cases}
\end{equation}
is used to model the propagation of unidirectional, one-dimensional, small-amplitude long waves in nonlinear dispersive media, see [2]. If the dissipative effect is considered, then one obtains the damped BBM equation [3, 4]
\begin{equation}
\begin{cases}
    u_t - u_{txx} - \gamma u_{xx} + \delta u + u_x + uu_x = f(x), \\
    u(0, x) = u_0(x).
\end{cases}
\end{equation}
The damping term $\delta u$ is added here since the Poincâré inequality does not hold on the real line $\mathbb{R}$.

There are many works denoted to the global attractor theory for damped BBM equations, see e.g. [7, 8, 6, 16, 19, 20, 22, 27, 28, 29, 30, 35, 36]. Recall that the global attractor is a compact invariant set, attracting every bounded sets in the phase space. Clearly, all stationary solutions are included in the global attractor. Therefore, if $\mathcal{A}$ is the global attractor of (2), and $Q$ is a function satisfying (1)(thus a stationary point of (2)), then $Q \in \mathcal{A}$. Moreover, as mentioned above, we are able to show that $\text{sing supp } Q = \text{sing supp } f$. Thus it is natural to ask, whether $\text{sing supp } \mathcal{A} = \text{sing supp } f$ holds? Here the singular support of the global attractor $\mathcal{A}$, denoted by $\text{sing supp } \mathcal{A}$, is understood as follows.

**Definition 1.2.** Let $\mathcal{A}$ be a set of locally integrable functions on $X \subset \mathbb{R}^n$. Then the singular support of $\mathcal{A}$ is defined by
\begin{equation}
\text{sing supp } \mathcal{A} = \bigcup_{u \in \mathcal{A}} \text{sing supp } u.
\end{equation}

The main result is as follows, which gives a positive answer to the above mentioned question.

**Theorem 1.3.** Let $s \geq 1$ and $f \in H^{s-2}(\mathbb{R})$. Then problem (2) has a global attractor $\mathcal{A}$ in $H^s(\mathbb{R})$. Moreover, we have
\begin{equation}
\text{sing supp } \mathcal{A} = \text{sing supp } f.
\end{equation}

The most interesting part in Theorem 1.3 is that, it gives a more precise characteristic on the local regularity of the global attractor. In particular, Theorem 1.3 shows that the global attractor is smooth at some point $x_0 \in \mathbb{R}$ if and only if the force is also smooth at the point $x_0$. This seems to be new for us. Most results on the regularity of the global attractor are global. In other words, if the force belongs to some Sobolev spaces, then the attractor is also bounded in some Sobolev spaces, see e.g. [12, 21, 24, 25, 29, 33].

Another point in Theorem 1.3 in that, in order to prove the existence of the global attractor in $H^s(\mathbb{R})$, we only assume that $f \in H^{s-2}(\mathbb{R})$. Similar results have been proved in [31] for the gKdV equation with a slight stronger assumption on the force term.

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1This fact follows from our main result directly, so we omit the the proof.
Finally, for further results on the well posedness for the BBM equation, we refer to \[1, 5, 32, 34].

Notations. We use \( A \lesssim B \) to denote \( A \leq CB \) with a constant \( C \) depending on \( c_1, c_2, \ldots \). If \( C \) is an absolute constant or \( C \) depends on some unimportant quantities, we use \( A \lesssim B \) instead. The Fourier transform \( f \mapsto \widehat{f} \) is defined by

\[
\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx.
\]

For every \( s \in \mathbb{R} \), the Sobolev spaces \( H^s(\mathbb{R}) \) is defined by the norm

\[
\|f\|_{H^s(\mathbb{R})} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s|\hat{f}(\xi)|^2d\xi \right)^{\frac{1}{2}} = \|J^s u\|_{L^2(\mathbb{R})},
\]

where \( J = (1 - \partial_x^2)^{s/2} \) is the Fourier multiplier operator defined by the symbol \( (1 + |\xi|^2)^{s/2} \), namely

\[
\widehat{Jf} = (1 + |\xi|^2)^{s/2}\hat{f}(\xi), \quad f \in \mathcal{S}.
\]

2. The global attractor.

2.1. Well posedness and absorbing sets. We start with the local well posedness of the equation (2) in \( H^s(\mathbb{R}) \). To this end, acting \( (1 - \partial_x^2)^{-1} \) on both sides of (2), we obtain

\[
u_t + (-\gamma \partial_x^2 + \delta)(1 - \partial_x^2)^{-1}u + (1 - \partial_x^2)^{-1}(u_x + uu_x) = (1 - \partial_x^2)^{-1}f, \quad t > 0.
\]

Moreover, it is convenient to rewrite the equation (4) into an integral form as

\[
u(t) = e^{-tP(D)}u_0 + \int_0^t e^{-(t-\tau)P(D)}(1 - \partial_x^2)^{-1}(f - u_x - uu_x)d\tau,
\]

where \( P(D) = (-\gamma \partial_x^2 + \delta)(1 - \partial_x^2)^{-1} \), the corresponding strongly continuous semigroup \( e^{-tP(D)} \) is understood as a Fourier multiplier operator, defined by

\[
e^{-tP(D)}\varphi = e^{-t(\gamma \xi^2 + \delta)(1 + \xi^2)^{-1}}\hat{\varphi}(\xi), \quad t \geq 0.
\]

Then we claim that

\[
\|e^{-tP(D)}\|_{H^s(\mathbb{R}), H^s(\mathbb{R})} \leq e^{-\lambda t}, \quad \text{for all } s \in \mathbb{R}, t \geq 0.
\]

Here and in the sequel, we set

\[
\lambda := \min\{\gamma, \delta\} > 0.
\]

In fact, by Plancherel theorem, it is easy to see that

\[
\|e^{-tP(D)}\varphi\|_{H^s(\mathbb{R})} = \|(1 + \xi^2)^{\frac{s}{2}}e^{-t(\gamma \xi^2 + \delta)(1 + \xi^2)^{-1}}\hat{\varphi}(\xi)\|_{L^2(\mathbb{R})} \leq e^{-\lambda t}\|(1 + \xi^2)^{\frac{s}{2}}\hat{\varphi}(\xi)\|_{L^2(\mathbb{R})} = e^{-\lambda t}\|\varphi\|_{H^s(\mathbb{R})}.
\]

This proves (7).

**Theorem 2.1.** Let \( s \geq 1 \) and \( f \in H^{s-2}(\mathbb{R}) \). Then for every \( u_0 \in H^s(\mathbb{R}) \), there exists a time \( T = T(\|u_0\|_{H^s(\mathbb{R})}, \|f\|_{H^{s-2}(\mathbb{R})}) > 0 \) and a unique solution \( u \in C([0, T], H^s(\mathbb{R})) \) of (2). Moreover, for every \( t \in [0, T] \), the mapping \( u_0 \mapsto u(t) \) is continuous from \( H^s(\mathbb{R}) \) to \( H^s(\mathbb{R}) \).
Proof. Define a mapping $u \mapsto \Gamma u$ as

$$\Gamma u = e^{-tP(D)u_0} + \int_0^t e^{-(t-\tau)P(D)}(1 - \partial_x^2)^{-1}(f - u_x - uu_x)d\tau. \quad (9)$$

Let $\mathcal{B}$ be a ball in $C([0, T], H^s(\mathbb{R}))$ defined by

$$\mathcal{B} = \left\{ v \in C([0, T], H^s(\mathbb{R})) : \sup_{t \in [0, T]} \|v\|_{H^s(\mathbb{R})} \leq 2(\|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})}) \right\}.$$

Then for $u \in \mathcal{B}$, using (7), $\|1 - \partial_x^2\|^{-1}_{H^s(\mathbb{R})} \leq \|u\|_{H^s(\mathbb{R})}$ and the algebra property $\|u^2\|_{H^s(\mathbb{R})} \lesssim \|u\|^2_{H^s(\mathbb{R})}$, we have for some $C \geq 1$

$$\sup_{t \in [0, T]} \|\Gamma u\|_{H^s(\mathbb{R})}$$

$$\leq \|u_0\|_{H^s(\mathbb{R})} + \int_0^T (C\|u\|_{H^s(\mathbb{R})}^2 + C\|u\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})})d\tau$$

$$\leq \|u_0\|_{H^s(\mathbb{R})} + T\|\partial_x f\|_{H^{s-2}(\mathbb{R})}$$

$$+ 4CT(\|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})})(\|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})} + 1). \quad (10)$$

Moreover, for different $u_1, u_2 \in \mathcal{B}$, we have

$$\sup_{t \in [0, T]} \|\Gamma u_1 - \Gamma u_2\|_{H^s(\mathbb{R})}$$

$$\leq \int_0^T \left\|e^{-(t-\tau)P(D)}(1 - \partial_x^2)^{-1}(u_1u_{1x} - u_2u_{2x} + u_{1x} - u_{2x})\right\|_{H^s(\mathbb{R})} d\tau$$

$$\leq CT \sup_{t \in [0, T]} \left(\frac{1}{2}\|u_1 + u_2\|_{H^s(\mathbb{R})} + 1\right)\|u_1 - u_2\|_{H^s(\mathbb{R})}$$

$$\leq 2CT(\|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})} + 1) \sup_{t \in [0, T]} \|u_1 - u_2\|_{H^s(\mathbb{R})}. \quad (11)$$

Set

$$T = \frac{1}{4C(\|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-1}(\mathbb{R})} + 1)} \leq 1.$$ 

Then it follows from (10) and (11) that

$$\sup_{t \in [0, T]} \|\Gamma u\|_{H^s(\mathbb{R})} \leq 2(\|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})}), \quad \forall u \in \mathcal{B},$$

$$\sup_{t \in [0, T]} \|\Gamma u_1 - \Gamma u_2\|_{H^s(\mathbb{R})} \leq \frac{1}{2} \sup_{t \in [0, T]} \|u_1 - u_2\|_{H^s(\mathbb{R})}, \forall u_1, u_2 \in \mathcal{B}.$$ 

Thus $\Gamma$ is contraction mapping on $\mathcal{B}$ and we obtain a unique fixed point $u \in \mathcal{B}$ so that $\Gamma u = u$. This gives the existence and uniqueness of the solution for (2).

Let $u, v \in C([0, T], H^s(\mathbb{R}))$ be the solutions of (2) with data $u_0$ and $v_0$, respectively. Then by (5), we have

$$(u - v)(t) = e^{-tP(D)(u_0 - v_0)} + \int_0^t e^{-(t-\tau)P(D)}(1 - \partial_x^2)^{-1}(vv_x - uu_x + v_x - u_x)d\tau.$$
This, together with (7) and the local existence in \( \mathbb{R} \), implies that for \( t \in [0, T] \)
\[
\|(u - v)(t)\|_{H^{s}(\mathbb{R})} 
\leq \|u_0 - v_0\|_{H^{s}(\mathbb{R})} + \int_0^t e^{-\lambda(t - \tau)} C(\|u + v\|_{H^{s}} + 1)\|u - v\|_{H^{s}} d\tau 
\leq \|u_0 - v_0\|_{H^{s}(\mathbb{R})} + 4C(\|u_0\|_{H^{s}} + \|v_0\|_{H^{s}} + \|f\|_{H^{s-2}} + 1) \int_0^t \|u - v\|_{H^{s}} d\tau.
\]
Applying Gronwall lemma to (12), we obtain for all \( t \in [0, T] \)
\[
\|(u - v)(t)\|_{H^{s}(\mathbb{R})} \leq e^{4Ct(\|u_0\|_{H^{s}} + \|v_0\|_{H^{s}} + \|f\|_{H^{s-2}} + 1)} \|u_0 - v_0\|_{H^{s}(\mathbb{R})}.
\]
This gives the continuity of \( u_0 \mapsto u(t) \) in \( H^{s}(\mathbb{R}) \).

**Lemma 2.2 (\( H^1 \) bound).** Let \( s \geq 1 \) and \( f \in H^{s-2}(\mathbb{R}) \). Then for every \( u_0 \in H^1(\mathbb{R}) \), there exists a time \( T = T(\|u_0\|_{H^1(\mathbb{R})}, \|f\|_{H^{s-2}}) > 0 \) so that the solutions of (2) satisfies
\[
\|u(t)\|_{H^1(\mathbb{R})} \lesssim \|f\|_{H^{s-2}} 1, \quad \text{for all } t \geq T.
\]

**Proof.** Multiplying (24) with \( u \) and integrating over \( \mathbb{R} \), we find
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) dx + \int_{\mathbb{R}} (\gamma u_x^2 + \delta u^2) dx = \int_{\mathbb{R}} fudx.
\]
For the right hand side of (13), using the duality we find
\[
\left| \int_{\mathbb{R}} fudx \right| \leq \|f\|_{H^{-1}(\mathbb{R})} \|u\|_{H^1(\mathbb{R})} \leq \frac{\lambda}{2} \int_{\mathbb{R}} (u^2 + u_x^2) dx + C\|f\|_{H^{-1}(\mathbb{R})}^2.
\]
It follows from (13)-(14) and (8) that
\[
\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) dx + \frac{\lambda}{2} \int_{\mathbb{R}} (u^2 + u_x^2) dx = 2C\|f\|_{H^{-1}(\mathbb{R})}^2.
\]
Since \( s \geq 1 \), we have \( \|f\|_{H^{-1}(\mathbb{R})} \leq \|f\|_{H^{s-2}(\mathbb{R})} \). Applying Gronwall lemma to (15), we conclude the desired bound.

**Lemma 2.3 (\( H^s \) bound).** Let \( s \geq 1 \) and \( f \in H^{s-2}(\mathbb{R}) \). Then for every \( u_0 \in H^s(\mathbb{R}) \), there exists a time \( T = T(\|u_0\|_{H^s(\mathbb{R})}, \|f\|_{H^{s-2}}) > 0 \) so that the solutions of (2) satisfies
\[
\|u(t)\|_{H^s(\mathbb{R})} + \|u(t)\|_{H^{s-3}(\mathbb{R})} \lesssim \|f\|_{H^{s-2}} 1, \quad \text{for all } t \geq T.
\]

**Proof.** Multiplying both sides of (2) with \( J^{2s-2}u \) and integrating over \( \mathbb{R} \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |J^sJ^{s-2}u|^2 dx + \lambda \int_{\mathbb{R}} |J^sJ^{s-2}u|^2 dx + \int_{\mathbb{R}} u_{xx} J^{2s-2}udx \leq \int_{\mathbb{R}} f J^{2s-2}udx,
\]
where \( \lambda \) is given by (8), \( J \) is defined by (3). Since \( J \) is adjoint, we have on one hand
\[
\left| \int_{\mathbb{R}} f J^{2s-2}udx \right| = \int_{\mathbb{R}} J^{s-2} J^s u dx \leq \frac{\lambda}{4} \int_{\mathbb{R}} |J^s u|^2 dx + C \|f\|_{H^{s-2}(\mathbb{R})}^2.
\]
On the other hand, since \( s - 1 \geq 0 \), it follows from Lemma 4.1 that
\[
\left| \int_{\mathbb{R}} u_{xx} J^{2s-2}udx \right| \lesssim \|u\|_{H^1(\mathbb{R})} \|J^{s-\frac{3}{2}} u\|_{L^2(\mathbb{R})} \|J^s u\|_{L^2(\mathbb{R})}.
\]
Using interpolation inequalities in \( H^s \) spaces, we deduce from (18) that
\[
RHS(18) \leq \|u\|_{H^1(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} \|J^s u\|_{L^2(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})} \|J^s u\|_{L^2(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})} \|J^s u\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|J^s u\|_{L^2(\mathbb{R})}^\frac{1}{2},
\]
Combining (18)-(19), since $s \geq 1$, using Young's inequality, we get

$$\left| \int_{\mathbb{R}} u_x J^{2s-2} u dx \right| \leq \frac{\lambda}{4} \int_{\mathbb{R}} |J^s u|^2 dx + Q(\|u\|_{H^s(\mathbb{R})}),$$

(20)

where $Q(\cdot)$ is a continuous function. Thanks to Lemma 2.2, there exists $T_1 > 0$ so that $\|u(t)\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-2}} 1$ for all $t \geq T_1$. This, together with (16), (17) and (20), implies that for $t \geq T_1$,

$$\frac{d}{dt} \int_{\mathbb{R}} |J^s u|^2 dx + \lambda \int_{\mathbb{R}} |J^s u|^2 dx \lesssim \|f\|_{H^{s-2}} 1.$$

(21)

Then applying Gronwall lemma to (21), we conclude that for some $T > 0$

$$\|u(t)\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-2}} 1, \quad t \geq T.$$

This, together with (4), implies that

$$\|u(t)\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-2}} 1, \quad t \geq T.$$

Thus the proof is complete.

\[\square\]

**Remark 1.** It follows from Lemma 2.3 that, start with an arbitrary $u_0$ in a bounded set in $H^s(\mathbb{R})$, after some time $T = T(\|u_0\|_{H^s(\mathbb{R})}, \|f\|_{H^{s-2}}) > 0$, then the solution of (2) will get into the ball

$$B := \left\{ u \in H^s(\mathbb{R}) : \|u\|_{H^s(\mathbb{R})} \leq C \right\}.$$

Moreover, the ball is an absorbing set. Namely, if $u_0 \in B$, then the solution of (2) $u(t) \in B$ for all $t > 0$.

### 2.2. Asymptotic compactness

In this subsection, we shall show that the solution semigroup associated with the equation (2) is asymptotically compact. Since the force term $f$ only belongs to $H^{s-2}(\mathbb{R})$, which is much less regular than the phase space $H^s(\mathbb{R})$, we need to make a decomposition of the solution of (2). To this end, let $Q$ be the unique solution of the elliptic equation

$$-\gamma Q_{xx} + \delta Q + Q_x = f(x), \quad x \in \mathbb{R}.$$  

(22)

Then the Fourier transform $\hat{Q}$ satisfies that

$$\hat{Q}(\xi) = \frac{\hat{f}(\xi)}{\gamma \xi^2 + \delta + i \xi},$$

from which we deduce that for all $s \geq 1$

$$\|Q\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-2}}.$$  

(23)

Clearly, a function $u$ solves (2) if and only if $v = u - Q$ solves

$$\begin{cases}
    v_t - v_{txx} - \gamma v_{xx} + \delta v + v_x + v v_x + (Q v)_x = -Q Q_x, & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
    v(0, x) = u_0(x) - Q(x).
\end{cases}$$  

(24)

**Lemma 2.4** (Tailed estimates). Assume that $u_0 \in B, B$ is the absorbing set in Remark 1. For any $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0, k = k(\varepsilon) > 0$ so that the solutions of (24) satisfies

$$\int_{|x| \geq k} v^2(t) + v_x^2(t) dx \leq \varepsilon, \quad t \geq T.$$
Proof. Let $0 \leq \phi \leq 1$ be a smooth function on $\mathbb{R}$ such that $\phi(x) = 0$ if $|x| \leq 1/2$ and $\phi(x) = 1$ if $|x| \geq 1$. For every $k > 0$, we let $\phi_k(\cdot) = \phi(\cdot/k)$. Let $v$ be a solution of (24), set $v_k = \phi_k v$, then $v_k$ satisfies

$$
\begin{aligned}
&v_{kt} - v_{kxx} + 2\phi_k v_t - \gamma v_{kxx} + \phi_k v + v_k v_x + (Qv)_x = F, \quad t \in \mathbb{R}^+,
&v_k(0, x) = \phi_k(u_0(x) - Q(x)),
\end{aligned}
$$

where

$$
F = -\phi_{kxx} v_t - 2\phi_k v_t - \gamma(\phi_{kxx} v + 2\phi_k v_x) + v_k v_x + Qv\phi_k + QQ\phi_k.
$$

Thanks to Remark 1, Lemma 2.3 and the bound (23), we have

$$
\|F(t, \cdot)\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^{s-2}} \frac{1}{k} + \frac{1}{k^2} + \|Q\|_{H^1(|x| \geq k/2)}^2, \quad t \geq 0.
$$

Multiplying (25) with $v_k$ and integrating over $\mathbb{R}$, we obtain that

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (v_k^2 + v_{kx}^2) \, dx + \lambda \int_{\mathbb{R}} (v_{kx}^2 + v_{kxx}^2) \, dx + \int_{\mathbb{R}} (Qv)_k v_k \, dx \leq \int_{\mathbb{R}} Fv_k \, dx.
$$

On one hand, by (26) and Hölder inequality, we have

$$
\left| \int_{\mathbb{R}} Fv_k \, dx \right| \leq C \left( \frac{1}{k^2} + \frac{1}{k^4} + \|Q\|^2_{H^1(|x| \geq k/2)} \right) + \frac{\lambda}{4} \int_{\mathbb{R}} |v_k|^2 \, dx.
$$

On the other hand, using the support property of $v_k$, Hölder inequality and Sobolev embedding, we have

$$
\left| \int_{\mathbb{R}} (Qv)_k v_k \, dx \right| \leq \|Q\|_{L^2(|x| \geq k/2)} \|v_k\|_{L^4(\mathbb{R})} \|v_{kx}\|_{L^2(\mathbb{R})} \leq \frac{\lambda}{4} \|v_k\|^2_{H^1(\mathbb{R})}
$$

if $k$ large enough. Here we use the fact that $\|Q\|_{L^4(|x| \geq k/2)} \lesssim \|Q\|_{H^1(|x| \geq k/2)}$ goes to 0 if $k \to \infty$. It follows from (27)-(29) that

$$
\frac{d}{dt} \int_{\mathbb{R}} (v_k^2 + v_{kx}^2) \, dx + \lambda \int_{\mathbb{R}} (v_{kx}^2 + v_{kxx}^2) \, dx \leq C \left( \frac{1}{k^2} + \frac{1}{k^4} + \|Q\|^2_{H^1(|x| \geq k/2)} \right).
$$

Then applying Gronwall lemma to (30), we conclude that, for every $\varepsilon > 0$, there exist $T = T(\varepsilon) > 0, k = k(\varepsilon) > 0$ so that

$$
\int_{\mathbb{R}} (v_k^2(t) + v_{kx}^2(t)) \, dx \leq \varepsilon, \quad t \geq T.
$$

Thanks to the definition of $v_k$ and $\phi_k$, we completes the proof. \qed

Lemma 2.5 (Asymptotic smoothing effect). Assume that $u_0 \in \mathcal{B}$. Then there exists a bounded set $\mathcal{B}_1$ in $H^{s+1}(\mathbb{R})$ so that

$$
\lim_{t \to \infty} \text{dist}_{H^s(\mathbb{R})}(v(t), \mathcal{B}_1) = 0,
$$

where the distance $\text{dist}_{H^s(\mathbb{R})}(v, \mathcal{B}_1) = \inf_{\varphi \in \mathcal{B}_1} \|v - \varphi\|_{H^s(\mathbb{R})}$.

Proof. Similar to (4), one can rewrite (24) as

$$
v(t) = e^{-tP(D)}(u_0 - Q) - \int_0^t e^{-(t-\tau)P(D)} \partial_x^2 (1 - \partial_x^2)^{-1} \left( \frac{u_0^2}{2} + u \right) \, d\tau,
$$

where $e^{-tP(D)}$ is defined by (6). Since $u_0$ and $Q$ is bounded in $H^s(\mathbb{R})$, thanks to (7), we have

$$
\|e^{-tP(D)}(u_0 - Q)\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-2}} e^{-\lambda t}, \quad t \geq 0.
$$
Moreover, since \( u(t) \) is uniformly bounded in \( H^s(\mathbb{R}) \) for all \( t \geq 0 \), by (7) again, we find
\[
\left\| \int_0^t e^{-(t-\tau)P(D)} \partial_x (1 - \partial_x^2)^{-1} \left( \frac{u^2}{2} + u \right) \, d\tau \right\|_{H^{s+1}(\mathbb{R})}
\lesssim \int_0^t e^{-\lambda t} \left\| \partial_x (1 - \partial_x^2)^{-1} \left( \frac{u^2}{2} + u \right) \right\|_{H^{s+1}(\mathbb{R})} \, d\tau
\lesssim \int_0^t e^{-\lambda t} \left( \|u(\tau)\|_{H^s(\mathbb{R})}^2 + \|u(\tau)\|_{H^s(\mathbb{R})} \right) d\tau \lesssim \|f\|_{H^{s-2}} 1, \quad \text{for all } t \geq 0.
\](33)
Combining (31)-(33), we conclude the result. \( \square \)

Now we are ready to state the main result in this section.

**Theorem 2.6.** Assume that \( f \in H^{s-2}(\mathbb{R}) \) with \( s \geq 1 \). Then for every \( u_0 \in H^s(\mathbb{R}) \), there exists a unique global solution \( u \in C([0, \infty), H^s(\mathbb{R})) \) of the equation (2). Moreover, problem (2) has a global attractor \( \mathcal{A} \) in \( H^s(\mathbb{R}) \), which is a compact, invariant set in \( H^s(\mathbb{R}) \) and attracting every bounded set in \( H^s(\mathbb{R}) \).

**Proof.** The global existence follows from the local existence in Theorem 2.1 and global bounds in Lemma 2.3. Moreover, since the solution mapping \( S(t) : u_0 \rightarrow u(t) \) is continuous in \( H^s(\mathbb{R}) \), thus the mapping defines a dynamical system in \( H^s(\mathbb{R}) \). Note that problem (2) has a bounded absorbing set \( \mathcal{B} \), the existence of a global attractor follows if one can show that \( S(t) \) is asymptotically compact, see e.g. [13, 23]. In other words, it suffices to show that
\[
\lim_{t \rightarrow \infty} \kappa_{H^s(\mathbb{R})}(S(t)\mathcal{B}) = 0,
\](34)
where \( \kappa_{H^s(\mathbb{R})}(E) \) denotes the Kuratowski measure of non-compactness of \( E \), given by
\[
\kappa_{H^s(\mathbb{R})}(E) = \inf \left\{ \delta > 0 \mid E \text{ has a finite open cover of sets of diameter } < \delta \right\}.
\]
Note that the diameter is defined by the norm of \( H^s(\mathbb{R}) \). Since \( v = S(t)u_0 - Q \), by Lemma 2.5, we have
\[
S(t)\mathcal{B} - Q = K(t) + \mathcal{B}_1,
\](35)
where \( \mathcal{B}_1 \) is bounded in \( H^{s+1}(\mathbb{R}) \) and
\[
\lim_{t \rightarrow \infty} \|K(t)\|_{H^s(\mathbb{R})} = 0.
\](36)
Thanks to (35)-(36) and Lemma 2.4, we infer that for any \( \varepsilon > 0 \), there exists \( k = k(\varepsilon) > 0 \)
\[
\lim_{t \rightarrow \infty} \|\phi_k \mathcal{B}_1\|_{H^s(\mathbb{R})} \leq \varepsilon,
\](37)
where \( \phi_k \) is the same cutoff function as that in the proof of Lemma 2.4. Moreover, by the compact Sobolev embedding \( H^{s+1}(|x| \leq k) \rightarrow H^s(|x| \leq k) \), we have
\[
\kappa_{H^s(\mathbb{R})}((1 - \phi_k)\mathcal{B}_1) = 0, \quad \text{for all } t \geq 0.
\](38)
Finally, by (35)-(38) and the properties of Kuratowski measure of non-compactness, we conclude (34). \( \square \)
3. **Singular support.** In this section, we prove the singular support result in Theorem 1.3, namely

\[ \text{sing supp } \mathcal{A} = \text{sing supp } f. \]

This will follows from the facts sing supp \( \mathcal{A} \subset \text{sing supp } f \) and sing supp \( f \subset \text{sing supp } \mathcal{A} \), which are proved in Theorem 3.4 and Theorem 3.6, respectively.

3.1. **sing supp \( \mathcal{A} \subset \text{sing supp } f.** In this subsection, we show that if \( f \) is smooth in a neighborhood of some point \( x_0 \in \mathbb{R} \), then the global attractor \( \mathcal{A} \) is also smooth in a neighborhood of some point \( x_0 \). To this end, we need a high-low frequency decomposition, introduced in [10, 11]. Fix \( N > 0 \). Let \( P_{\leq N} \) and \( P_{> N} \) be Fourier projectors on lower frequency and higher frequency, respectively. Precisely,

\[ P_{\leq N} \varphi = 1_{|\xi| \leq N} \hat{\varphi}(\xi), \quad \varphi \in \mathcal{A} \]

and \( P_{> N} = 1 - P_{\leq N} \), where \( 1_{|\xi| \leq N} \) is the characteristic function of the set \( \{ \xi : |\xi| \leq N \} \). Let \( u \) be a solution of the BBM equation (2). We split \( u = q + w \) such that \( v \) satisfies

\[ q_t - q_{txx} - \gamma q_{xx} + \delta q + q_x + P_{> N}(qq_x) = -P_{> N}(qw)_x, \quad q(0) = P_{> N}u_0, \tag{39} \]

while the remainder \( w \) satisfies

\[ w_t - w_{txx} - \gamma w_{xx} + \delta w + w_x + qw_x = f - P_{\leq N}(qq_x + (qw)_x), \quad w(0) = P_{\leq N}u_0. \tag{40} \]

**Lemma 3.1** (Decay of \( q \)). Let \( s \geq 1 \) and \( u_0 \in H^s(\mathbb{R}), f \in H^{s-2}(\mathbb{R}) \). If \( N \geq N_0(\|u_0\|_{H^s(\mathbb{R})}, \|f\|_{H^{s-2}(\mathbb{R})}) \), then problem (39) has a global solution \( q \in C([0, \infty), H^s(\mathbb{R})) \) satisfying that, with \( \lambda \) given by (8),

\[ \|q(t)\|_{H^s(\mathbb{R})} \leq e^{-\lambda t}\|q(0)\|_{H^s(\mathbb{R})}, \quad \text{for all } t \geq 0. \]

**Proof.** Since \( u = q + w \), (39) can be rewritten as

\[ q_t - q_{txx} - \gamma q_{xx} + \delta q + q_x + P_{> N}\partial_x (q u - \frac{1}{2} q^2) = 0, \quad q(0) = P_{> N}u_0. \tag{41} \]

Using the bound of \( u \) in Lemma 2.3, by the contraction mapping principle, we get the local existence of \( q \). To obtain a global bound, similar to Lemma 2.2, multiplying (41) with \( q \) and integrating over \( \mathbb{R} \), we find

\[ \frac{1}{2} \frac{d}{dt} \|q(t)\|_{H^1(\mathbb{R})}^2 + \lambda \|q(t)\|_{H^1(\mathbb{R})}^2 + \int_{\mathbb{R}} q P_{> N} \partial_x (q u - \frac{1}{2} q^2) dx \leq 0. \tag{42} \]

Since \( P_{> N} q = q \), using integration by parts and the Sobolev embedding, we infer that

\[ \left| \int_{\mathbb{R}} q P_{> N} \partial_x (q u - \frac{1}{2} q^2) dx \right| \leq \left| \int_{\mathbb{R}} q^2 \partial_x u dx \right| \leq \|u\|_{H^1(\mathbb{R})} \|q\|_{L^1(\mathbb{R})}^2 \leq N^{-\frac{1}{2}} \|u\|_{H^1(\mathbb{R})} \|q\|_{H^1(\mathbb{R})}^2. \tag{43} \]

One can choose \( N \geq N_1(\|u_0\|_{H^s(\mathbb{R})}, \|f\|_{H^{s-2}(\mathbb{R})}) \) so that the LHS of (43) is less than \( \frac{1}{2} \|q(t)\|_{H^1(\mathbb{R})}^2 \). Then it follows from (42) that,

\[ \frac{d}{dt} \|q(t)\|_{H^1(\mathbb{R})}^2 + \lambda \|q(t)\|_{H^1(\mathbb{R})}^2 \leq 0. \tag{44} \]

This proves that

\[ \|q(t)\|_{H^1(\mathbb{R})}^2 \leq e^{-\lambda t}\|q(0)\|_{H^1(\mathbb{R})}^2, \quad t \geq 0. \tag{45} \]
Moreover, multiplying (41) with $J^{2s-2}q$ and integrating over $\mathbb{R}$, we find
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |J^s q|^2 dx + \lambda \int_{\mathbb{R}} |J^s q|^2 dx + \int_{\mathbb{R}} J^{2s-2}q P_N \partial_x (qu - \frac{1}{2} q^2) dx \leq 0.
\]
(46)

By Lemma 4.1 and the bound (45), we have
\[
\left| \int_{\mathbb{R}} J^{2s-2}q P_N \partial_x \left( \frac{1}{2} q^2 \right) dx \right| \lesssim \|q\|_{H^1(\mathbb{R})} \|J^{s-\frac{1}{2}}q\|_{L^2(\mathbb{R})} \|J^s q\|_{L^2(\mathbb{R})} \\
\leq \|u_0\|_{H^s(\mathbb{R})} N^{-\frac{1}{2}} \|J^s q\|_{L^2(\mathbb{R})}^2.
\]
(47)

Also, by the algebra property of $H^s$ and the global bound of $u$ in Lemma 2.3,
\[
\left| \int_{\mathbb{R}} J^{2s-2}q P_N \partial_x (qu) dx \right| \leq \|J^{s-1}q\|_{L^2(\mathbb{R})} \|J^s q\|_{L^2(\mathbb{R})} \\
\lesssim \|J^{s-1}q\|_{L^2(\mathbb{R})} \|J^s u\|_{L^2(\mathbb{R})} \|J^s q\|_{L^2(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})} \|f\|_{H^{s-2}(\mathbb{R})} N^{-1} \|J^s q\|_{L^2(\mathbb{R})}^2.
\]
(48)

Thanks to (47)-(48), one can choose $N \geq N_2(\|u_0\|_{H^s(\mathbb{R})}, \|f\|_{H^{s-2}(\mathbb{R})})$ so that
\[
\left| \int_{\mathbb{R}} J^{2s-2}q P_N \partial_x (qu - \frac{1}{2} q^2) dx \right| \leq \frac{\lambda}{2} \|J^s q(t)\|_{L^2(\mathbb{R})}^2.
\]
Then (46) becomes
\[
\frac{d}{dt} \int_{\mathbb{R}} |J^s q|^2 dx + \lambda \int_{\mathbb{R}} |J^s q|^2 dx \leq 0, \quad t \geq 0.
\]
(49)

Applying Gronwall lemma to (49), we conclude the result. \qed

In the sequel, we shall need Sobolev spaces on an interval. Let $I \subset \mathbb{R}$ be an interval and let $k$ be a positive integer. The Sobolev space $H^k(I)$ is defined by the norm
\[
\|u\|_{H^k(I)} = \left( \int_I |u|^2 dx + \int_I |\partial_x^k u|^2 dx \right)^{\frac{1}{2}}.
\]

For $r > 0$ and $x_0 \in \mathbb{R}$, we shall use $I_r(x_0)$ to denote the interval $(x_0 - r, x_0 + r)$. Let $r_2 > r_1 > 0, x_0 \in \mathbb{R}$. Let $\psi_{x_0, r_1, r_2} : \mathbb{R} \to \mathbb{R}^+$ be a smooth cutoff function such that $\psi_{x_0, r_1, r_2}(x) = 1$ if $|x - x_0| \leq r_1$ and $\psi_{x_0, r_1, r_2}(x) = 0$ if $|x - x_0| \geq r_2$. Then it is easy to see that
\[
\|u\|_{H^k(I_{r_1}(x_0))} \lesssim \|\psi_{x_0, r_1, r_2} u\|_{H^k(\mathbb{R})} \lesssim \|u\|_{H^k(I_{r_2}(x_0))}.
\]
(50)

**Lemma 3.2.** For all $k \in \mathbb{N}$, $r_2 > r_1 > 0, x_0 \in \mathbb{R}$, we have
\[
\|(1 - \partial_x^2)^{-1} f\|_{H^{k+2}(I_{r_1}(x_0))} \lesssim \|\psi_{x_0, r_1, r_2} f\|_{H^k(\mathbb{R})} + \|(1 - \partial_x^2)^{-1} f\|_{L^2(I_{r_2}(x_0))}.
\]

**Proof.** Let $u = (1 - \partial_x^2)^{-1} f$. Then it suffices to show that
\[
\|u\|_{H^{k+2}(I_{r_1}(x_0))} \lesssim \|\psi_{x_0, r_1, r_2} f\|_{H^k(\mathbb{R})} + \|u\|_{L^2(I_{r_2}(x_0))}.
\]
(51)

Since $(1 - \partial_x^2)u = f$, (51) follows from the interior regularity of the elliptic equation, see e.g. [9, Theorem 2, p.314]. \qed

**Lemma 3.3 (Local regularity of $w$).** Assume that $f \in H^k(I_{r}(x_0))$ for some $k \in \mathbb{N}$, $r > 0, x_0 \in \mathbb{R}$. Let $N = N_0$ so that Lemma 3.1 holds and $w$ be the solution of (40). Then $w$ is bounded in $H^{k+2}(I_{\frac{1}{2}}(x_0))$ for all $t \geq 0$. 

Proof. By the global bound of $u$ in Lemma 2.3 and the bound of $q$ in Lemma 3.1, we find that $\|w(t)\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})}$ for all $t \geq 0$. It follows that for all $t \geq 0$

$$\|P_{\leq N}w(t)\|_{H^{s+2}(I^2_2(x_0))} \lesssim (1 + N)^{k+1}\|P_{\leq N}w(t)\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})}$$

Thus it remains to bound $\|P_{>N}w(t)\|_{H^{s+2}(I^2_2(x_0))}$. To this end, acting both sides of (40) with $P_{>N}$, denoting $w^N = P_{>N}w$, we obtain

$$w_t^N - w_{txx}^N - \gamma w_{xx}^N + \delta w^N + P_{>N}(ww_x) = P_{>N}f, \quad w^N(0) = 0. \quad (52)$$

Here we have used $P_{>N}P_{\leq N} = 0$. Acting $(1 - \partial_x^2)^{-1}$ on both sides of (52), we obtain

$$w_t^N + w^N = (1 - \partial_x^2)^{-1}F, \quad w^N(0) = 0, \quad (53)$$

where $F = P_{>N}f - w_{xx}^N - P_{>N}(ww_x) + (\gamma - \delta)w^N$. By (53) and Duhamel principle,

$$w^N(t) = \int_0^t e^{-\gamma(t-\tau)}(1 - \partial_x^2)^{-1}F d\tau. \quad (54)$$

Applying Lemma 3.2, we deduce from (54) that for all $r_2 > r_1 > 0$ and $j \in \mathbb{N}$

$$\sup_{\tau \in [0,t]} \|w^N(\tau)\|_{H^{s+2}(J_{r_1}(x_0))} \lesssim_{j,r_1,r_2} \int_0^t e^{-\gamma(t-\tau)} \left( \|F\|_{H^s(J_{r_2}(x_0))} + \|(1 - \partial_x^2)^{-1}F\|_{L^2(J_{r_2}(x_0))} \right) d\tau. \quad (55)$$

Now we give bounds for the terms on RHS of (55). On one hand, since $f$ is bounded in $H^{s-2}(\mathbb{R})$, $\|w(t)\|_{H^s} \lesssim \|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})}$ for all $t \geq 0$, we infer that

$$\|(1 - \partial_x^2)^{-1}F\|_{L^2(J_{r_2}(x_0))} \lesssim \|F\|_{H^{s-2}(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^{s-2}(\mathbb{R})} \quad (56)$$

On the other hand, since $H^m(I)$ is an algebra if $m \geq 1$, we have

$$\|F\|_{H^1(J_{r_2}(x_0))} \lesssim \|w^N\|_{H^1(J_{r_2}(x_0))} + \|w\|_{H^{s+1}(J_{r_2}(x_0))} + \|w\|^2_{H^{s+1}(J_{r_2}(x_0))} + \|f\|_{H^1(J_{r_2}(x_0))} \lesssim_{r_2,N} \|f\|_{H^1(J_{r_2}(x_0))} + (1 + \|w\|_{H^{s+1}(J_{r_2}(x_0))})^2. \quad (57)$$

It follows from (55)-(57) that

$$\sup_{\tau \in [0,t]} \|w^N(\tau)\|_{H^{s+2}(J_{r_1}(x_0))} \lesssim_{r_1,r_2,N} \|u_0\|_{H^s(\mathbb{R})} + \|f\|_{H^1(\mathbb{R})} + \|w(\tau)\|^2_{H^{s+1}(J_{r_2}(x_0))} \quad (58)$$

The inequality (58) tells that, if $\|w(\tau)\|_{H^{s+1}(J_{r_2}(x_0))}$ is bounded and $r_1 < r_2$, then $\sup_{\tau} \|w(\tau)\|_{H^{s+2}(J_{r_2}(x_0))}$ is bounded, thus $\sup_{\tau} \|w(\tau)\|_{H^{s+2}(J_{r_2}(x_0))}$ is bounded. Then repeating the iteration process, after finite steps, we conclude the desired conclusion. \hfill \square

**Theorem 3.4.** Let $\mathcal{A}$ be the global attractor of (2). Then

$$\text{sing supp } \mathcal{A} \subset \text{sing supp } f.$$ 

**Proof.** By the definition of the singularity support, it suffices to show that, if $f \in C^\infty(J, I(x_0))$ for some $r > 0$, $x_0 \in \mathbb{R}$, then every element of $\mathcal{A}$ belongs to $C^\infty(I^2_2(x_0))$. To show this, arbitrarily choose $u_0 \in \mathcal{A}$. Let $S(t)$ be the solution map of (2).
as before. By the invariance of the global attractor, there exists \{u_{0t}\} uniformly bounded in \(H^s(\mathbb{R})\) so that
\[
 u_0 = S(t)u_{0t}, \quad \text{for all } t \geq 0. \tag{59}
\]
According to Lemma 3.1, we can decompose \(S(t)u_{0t} = q(t) + w(t)\) so that
\[
 q(t) \to 0 \quad \text{in } H^s(\mathbb{R}) \text{ as } t \to \infty. \tag{60}
\]
It follows from (59) and (60) that \(w(t) \to u_0\) in \(H^s(\mathbb{R})\) as \(t \to \infty\). Moreover, by Lemma 3.3, if \(f \in H^k(I_r(x_0))\), we find that \(\sup_{t>0}\|w(t)\|_{H^{k+2}(I^2_r(x_0))}\) is bounded. Thus \(\{w(t)\}_{t>0}\) has a subsequence, denoted by \(\{w(t_n)\}\), so that \(w(t_n)\) converges weakly in \(H^{k+2}(I^2_r(x_0))\). By the uniqueness of the limit, we find that \(w(t_n)\) converges weakly to \(u_0\). Thus \(u_0\) is bounded in \(H^{k+2}(I^2_r(x_0))\). Note that \(k\) can be arbitrarily large, we find \(u_0 \in C^\infty(I^2_r(x_0))\).

### 3.2. sing supp \(f \subset \text{sing supp } \mathcal{A}\)

In this subsection, we show that if \(\mathcal{A}\) is of \(C^\infty\) in a neighborhood of \(x_0 \in \mathbb{R}\), then \(f\) is of \(C^\infty\) in a neighborhood of \(x_0 \in \mathbb{R}\). To this end, we first present an estimate of \(\partial_t u\).

#### Lemma 3.5

Let \(\mathcal{A}\) be the global attractor of (2) in \(H^s(\mathbb{R})\) with \(s \geq 1\). Let \(u(t), t \in \mathbb{R}\), be a complete orbit on the attractor \(\mathcal{A}\). Assume that \(u\) is bounded in \(L^\infty(\mathbb{R}, H^k(I_r(x_0)))\) for some \(k > s, r > 0\) and \(x_0 \in \mathbb{R}\). Then \(\partial_t u\) is bounded in \(L^\infty(\mathbb{R}, H^k(I^2_r(x_0)))\).

**Proof.** Since \(u\) is a complete orbit on the attractor, \(u \in L^\infty(\mathbb{R}, H^s(\mathbb{R}))\) and solves
\[
 u_t - u_{txx} - \gamma u_{xx} + \delta u + u_x + uu_x = f(x), \quad (t, x) \in \mathbb{R}^2. \tag{61}
\]
Moreover, according to (4), one has that \(\partial_t u\) is also bounded in \(L^\infty(\mathbb{R}, H^s(\mathbb{R}))\). Taking the derivative with respect to time variable \(t\) on both sides of (61), letting \(v = \partial_t u\), noting \(f\) is independent of time, we obtain
\[
 v_t - v_{txx} - \gamma v_{xx} + \delta v + v_x + (uv)_x = 0, \quad (t, x) \in \mathbb{R}^2. \tag{62}
\]
Without loss of generality, it suffices to show that \(v(0)\) is bounded in \(H^k(I^2_r(x_0))\). By Duhamel principle, we can deduce from (62) that for all \(t \geq s\)
\[
 v(t) = e^{-\gamma(t-s)}v(s) + \int_s^t e^{-\lambda(t-\tau)}(1 - \partial^2_x)^{-1}\left((\gamma - \delta)u - v_x - \partial_x(vu)\right)d\tau. \tag{63}
\]
In particular, letting \(t = 0\) and \(s = -t\) in (63), one finds for \(t \geq 0\)
\[
 v(0) = e^{-\gamma t}v(-t) + \int_{-t}^0 e^{-\lambda(t-\tau)}(1 - \partial^2_x)^{-1}\left((\gamma - \delta)u - v_x - \partial_x(vu)\right)d\tau. \tag{64}
\]
Since \(v(-t)\) is bounded in \(L^\infty([0, \infty), H^s(\mathbb{R}))\) and \(\lambda > 0\), we find
\[
 e^{-\gamma t}v(-t) \to 0 \quad \text{in } H^s(\mathbb{R}) \quad \text{as } t \to \infty. \tag{65}
\]
It follows from (62) and (65) that
\[
 \int_{-t}^0 e^{-\lambda(t-\tau)}(1 - \partial^2_x)^{-1}\left(\partial_x(vu)(\tau) + (\gamma - \delta)u(\tau)\right)d\tau \to v(0) \quad \text{in } H^s(\mathbb{R}) \quad \text{as } t \to \infty.
\]
From this, we argue as in the proof of Theorem 3.4 to find that for all \(1 \leq j \leq k\) and \(r_1 > 0\)
\[
\|v(0)\|_{H^{j+1}(I_{r_1}(x_0))} \lesssim \int_{-t}^{0} e^{-\lambda(t-\tau)} \left\| \left(1 - \partial_{x}^2 \right)^{-1} \left(\partial_{x}(vu)(\tau) + (\gamma - \delta)u(\tau)\right) \right\|_{H^{j}(I_{r_1}(x_0))} d\tau. \tag{66}
\]
Applying Lemma 3.2, we deduce from (66) that for all \(r > r_2 > r_1 > 0\)
\[
\|v(0)\|_{H^{j+1}(I_{r_1}(x_0))} \lesssim_{r_1,r_2,j} \sup_{t \in \mathbb{R}} \left(\|u\|_{H^{j}(I_{r_2}(x_0))}\|v\|_{H^{j}(I_{r_2}(x_0))} + \|u\|_{H^{j}(I_{r_2}(x_0))}\right)
\lesssim_{r_1,r_2,j} \sup_{t \in \mathbb{R}} \|u\|_{H^{k}(I_{r_2}(x_0))} + \sup_{t \in \mathbb{R}} \|v\|_{H^{j}(I_{r_2}(x_0))}. \tag{67}
\]
Since \(\sup_{t \in \mathbb{R}} \|u\|_{H^{k}(I_{r_2}(x_0))}\) is finite, we deduce from (67) that \(v(0)\) is bounded in \(H^{j+1}(I_{r_1}(x_0))\) if \(\sup_{t \in \mathbb{R}} \|v\|_{H^{j}(I_{r_2}(x_0))}\) is bounded. By a translation with respect to the time \(t\), we obtain in fact that \(v(t)\) is bounded in \(H^{j+1}(I_{r_1}(x_0))\) uniformly for all \(t \in \mathbb{R}\). Also note that the right hand side of (67) is finite if \(j = 1\). Using these and an induction argument, after finite steps, we find that \(v(t)\) is bounded in \(L^{\infty}(\mathbb{R},H^{k}(I_{r_2}(x_0)))\).

**Theorem 3.6.** Let \(\mathcal{A}\) be the global attractor of (2) in \(H^s(\mathbb{R})\) with \(s \geq 1\). Then
\[\text{sing supp } f \subset \text{sing supp } \mathcal{A}.\]

**Proof.** Assume that the global attractor \(\mathcal{A}\) is bounded in \(H^{k+2}(I_{r}(x_0))\) for some \(k \in \mathbb{N}, r > 0, x_0 \in \mathbb{R}\). Let \(u(t), t \in \mathbb{R}\), be a complete orbit on the attractor. Then \(u(t)\) is bounded in \(L^{\infty}(\mathbb{R},H^{k+2}(I_{r}(x_0)))\). Moreover, according to Lemma 3.5, we find that \(\partial u\) is bounded in \(L^{\infty}(\mathbb{R},H^{k+2}(I_{r}(x_0)))\). The complete orbit \(u\) satisfies the equation
\[u_t - u_{txx} - \gamma u_{xx} + \delta u + u_x + uu_x = f(x), \quad (t, x) \in \mathbb{R}^2. \tag{68}\]
It is easy to check that every term on the left hand side of (68) is bounded in \(H^{k}(I_{r}(x_0))\). Thus we find that \(f\) is bounded in \(H^{k}(I_{r}(x_0))\).

Now assume that the attractor \(\mathcal{A}\) is of \(C^\infty\) in \(I_{r}(x_0)\), then \(\mathcal{A}\) is bounded in \(H^{k+2}(I_{r}(x_0))\) for arbitrarily large \(k\), and we obtain that \(f\) is bounded in \(H^{k}(I_{r}(x_0))\) for the same \(k\). Therefore, \(f\) is of \(C^\infty\) in \(I_{r}(x_0)\). This completes the proof.

**4. Appendix.** Here we present a technical lemma used in Section 2 and 3. Let \(J\) be defined by (3). Then according to [17, Lemma 2.10] or [18, Theorem 1.9], we have for all \(f, g \in \mathcal{S}\)
\[
\|J^s(fg)\|_{L^p(\mathbb{R})} \lesssim_{s,p,p_1,p_2} \|J^s f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{q_1}(\mathbb{R})} + \|f\|_{L^{p_2}(\mathbb{R})} \|J^s g\|_{L^{q_2}(\mathbb{R})}, \tag{69}
\]
where \(s > 0, 1 < p, p_1, p_2 < \infty\) and
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.
\]
In particular, letting \(g = f\) and \(p = 2\) in (69), we find for all \(f \in \mathcal{S}\)
\[
\|J^s(f^2)\|_{L^2(\mathbb{R})} \lesssim_{s,p_1} \|J^s f\|_{L^{p_1}(\mathbb{R})} \|f\|_{L^{q_1}(\mathbb{R})} \tag{70}
\]
with \(1 < p_1, q_1 < \infty\) and \(\frac{1}{2} = \frac{1}{p_1} + \frac{1}{q_1}.

**Lemma 4.1.** Assume that \(s \geq 0\). Then for all \(u \in \mathcal{S}\)
\[
\left| \int_{\mathbb{R}} uu_x J^{s+\frac{1}{2}} u \right| \lesssim \|u\|_{H^{s+1}(\mathbb{R})} \|J^{s+\frac{1}{2}} u\|_{L^2(\mathbb{R})} \|J^{s+1} u\|_{L^2(\mathbb{R})}.
\]
Proof. In the case $s = 0$, the lemma holds clearly. So we assume $s > 0$ now. Since $J$ is adjoint, we write
\[
\left| \int_{\mathbb{R}} uu_x J^{2s} u dx \right| = \frac{1}{2} \left| \int_{\mathbb{R}} \partial_x (u^2) J^{2s} u dx \right| = \frac{1}{2} \left| \int_{\mathbb{R}} J^s (u^2) \cdot \partial_x J^s u dx \right|.
\] (71)
Using Hölder inequality in (71) we get
\[
\left| \int_{\mathbb{R}} uu_x J^{2s} u dx \right| \leq \| J^s (u^2) \|_{L^2(\mathbb{R})} \| \partial_x J^s u \|_{L^2(\mathbb{R})}.
\] (72)
Now we estimate the two terms on RHS of (72). On one hand, by Plancherel theorem,
\[
\| \partial_x J^s u \|_{L^2(\mathbb{R})} \leq \| J^{s+1} u \|_{L^2(\mathbb{R})}.
\] (73)
On the other hand, using (70) with $p_1 = q_1 = 4$, we find
\[
\| J^s (u^2) \|_{L^2(\mathbb{R})} \lesssim \| J^s u \|_{L^4(\mathbb{R})} \| u \|_{L^4(\mathbb{R})}.
\] (74)
By the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$, we deduce from (74) that
\[
\| J^s (u^2) \|_{L^2(\mathbb{R})} \lesssim \| J^{s+\frac{1}{2}} u \|_{L^2(\mathbb{R})} \| u \|_{H^1(\mathbb{R})}.
\] (75)
Finally, it follows from (72)-(73) and (75) that
\[
\left| \int_{\mathbb{R}} uu_x J^{2s} u dx \right| \lesssim \| J^{s+1} f \|_{L^2(\mathbb{R})} \| J^{s+\frac{1}{2}} u \|_{L^2(\mathbb{R})} \| u \|_{H^1(\mathbb{R})}.
\]
This completes the proof. \qed

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Received for publication June 2020.

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