Time-symmetric optimal stochastic control problems in space-time domains

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ABSTRACT
We present a pair of adjoint optimal control problems characterizing a class of time-symmetric stochastic processes defined on random time intervals. The associated PDEs are of free-boundary type. The particularity of our approach is that it involves two adjoint optimal stopping times adapted to a pair of filtrations, the traditional increasing one and another, decreasing. They are the keys of the time symmetry of the construction, which can be regarded as a generalization of ‘Schrödinger’s problem’ (1931–1932) to space-time domains. The relation with the notion of ‘Hidden diffusions’ is also described.

1. Introduction

The notion of Bernstein or reciprocal stochastic processes (BPs, for short) dates back to 1932 (see [1]) and followed from a probabilistic interpretation of a suggestion made by E. Schrödinger, one year before (Schrödinger [2]). During decades this line of ideas attracted very little attention. In 1986, it was shown ([3] and references therein) that, behind it, there is a quantum-like regularization method for classical dynamical systems but, in contrast with quantum theory, using well defined probability measures on appropriate path spaces.

More recently, starting with Mikami in 2004 (cf. [4] and references therein), the community of Mass Transportation theory adopted part of the resulting framework under the logo of ‘Schrödinger’s problem’ [5]. It is regarded as a regularization of Monge foundational problem, separate from the one of Kantorovich. It allows, in particular, to construct very efficient regularizations in numerical approaches to optimal transport problems of interest in image processing, natural sciences and Economics (cf., for instance [6–10]). Nevertheless, the relationship
between Schrödinger’s problem and mass transportation is not limited to computational aspects, but covers for instance geometric and functional inequalities, ergodicity results, among others.

Schrödinger’s original (one dimensional) problem was to construct random processes interpolating optimally between two ‘arbitrary’ probability densities, associated with the heat equation, but given at the boundaries of any fixed time interval $I$. In particular the given future probability had, a priori, nothing to do with the traditional probabilistic interpretation of this parabolic equation.

The answer to this problem, suggested by Schrödinger himself, is a class of ('Bernstein') diffusions, generally time inhomogeneous but enjoying a time reversibility property more general than the one known by most probabilists. The probability density of those optimal diffusions has an (integrable) product form of a positive solution of a forward heat equation and a positive solution of a backward heat equation, both defined on the fixed time interval $I$, respectively, with (positive) initial and final boundary conditions.

If we adopt the terminology of Mathematical Physics that calls ‘Euclidean’ to any approach of quantum physics where Schrödinger’s type of equations are replaced by parabolic ones, the probability density of Bernstein diffusions (BD or BPs, for short) expresses nothing but the Euclidean version of Born’s fundamental interpretation of the wave function or, more precisely, of the $L^2$-scalar product of two wave functions. After [3], the program inspired by Schrödinger was developed in various directions, illustrating the generality of its starting idea, in no way limited to the situation considered initially [11].

In Schrödinger’s problem, the fact that the time interval of BPs existence was fixed is not a necessary or even natural restriction of the method. A natural construction would be to define BPs in space-time domains, which is the aim of this paper. Through the paper, we will show that BPs in space-time domains can be constructed as a result of the minimization of both a forward and a backward action functionals, respectively, $J_{t,x}$ and $J^*_{t,x}$, of the type

$$J_{t,x}(Z; \tau, b) = E_{t,x} \left[ \int_t^{\tau \wedge T/2} \left( \frac{1}{2} |b(u, Z_u)|^2 + V(Z_u) \right) du + S(Z_{\tau \wedge T/2}) \right], \quad (1)$$

$$\begin{cases} dZ_u = b(u, Z_u) \, du + \hbar^{1/2} \, dW_u \\ Z_t = x \text{ and } -\frac{T}{2} \leq t \leq u \leq \frac{T}{2} \end{cases} \quad (2)$$

and

$$J^*_{t,x}(Z; \tau^*, b^*) = E_{t,x} \left[ \int_{-T/2 \vee \tau^*}^t \left( \frac{1}{2} |b^*(s, Z_s)|^2 + V(Z_s) \right) ds + S^*(Z_{-T/2 \vee \tau^*}) \right], \quad (3)$$

$$\begin{cases} d_s Z_s = b^*(s, Z_s) \, ds + \hbar^{1/2} \, d_s W_s^* \\ Z_t = x \text{ and } -\frac{T}{2} \leq s \leq t \leq \frac{T}{2} \end{cases} \quad (4)$$
by finding the controls $b$ and $b^*$ and the stopping times $\tau$ and $\tau^*$ on specific sets to be defined in the next section. In the previous functionals, $h$ is a positive constant, $S$ is the terminal boundary condition in functional $J_{t,x}$ and $S^*$ is the initial boundary condition in functional $J^*_{t,x}$. As stated in Theorem 5.2 we prove that the critical stochastic process $Z$ satisfies a local Markov property that takes into account the randomness of the time-space domain where the process is defined.

From the physical side of Schrödinger’s idea, it is clear that his motivation was rooted in the status of time in quantum theory. He described the fact that time is the same parameter as in classical mechanics as a major weakness of this theory. The present paper extends the analogy he suggested (cf. the program of ‘Stochastic deformation’ [11] and references therein) in a direction where the status of time becomes compatible with what is needed for a consistent relativistic quantum theory. This aspect will be addressed in another publication. A distinct recent approach of stochastic optimization on space-time can be found in [12].

Apart from the mathematical interest of the results presented in this paper, we highlight the importance of the developments of Schrödinger theory, BPs, and control theory for engineering applications. For instance, in ‘target tracking’, Bernstein processes are used to extend the classical Gauss-Markov state-space models since the latter models do not properly recover the properties of the system when one considers long trajectories and the destination is known [13].

The stochastic control problems studied in this paper consider the drift’s expected squared magnitude as part of the cost-to-go function. Although the quadratic state is usually different from the drift vector field’s magnitude in stochastic control problems of engineering, there are still applications where the formulation here proposed is important. In robotic, often, feedback strategies are proposed to control mini-robots. Due to external conditions, such vehicles may exhibit stochastic behaviour, which makes deterministic control feedback unappropriated to manage their navigation. Contrarily, stochastic control may be more appropriated to provide control feedback strategies. The cost-to-go function may contain the drift’s squared magnitude, as one can see in Section V.A of [14].

The development of stochastic control theory in discrete and continuous time has been boosted not only by engineering applications (cf. [15,16] and references therein) but also by economic and financial applications such as American options and portfolio optimization (cf. [17–19] and references therein).

The organization of the paper is the following. Section 2 summarizes the original construction of a large class of BDs, on a given (deterministic) time interval. Their particularity is to solve simultaneously two Itô’s stochastic differential equations, one with an initial boundary condition, the other with a final one. Their relation with the notion of ‘hidden diffusions’ is indicated. Section 3 provides a characterization of BDs defined on random time intervals as solutions of two adjoint optimal control problems where pairs of random times and drifts should be optimized. In PDE terms, these problems are of a free-boundary type.
In Section 4, viscosity solutions of the adjoint Hamilton-Jacobi-Bellman (HJB, for short) equations underlying our construction are described. Section 5 shows that the solutions of these two boundary value problems are unique. In addition, their relations with Schrödinger’s original problem and the associated dynamics of Bernstein optimal drifts are given. The characterization of the distributions of the two adjoint optimal stopping times used in the construction is the subject of Section 6. It amounts to the construction of a forward and a backward martingale of the process. Along the paper, an illustration is provided.

2. Bernstein and hidden diffusions stochastic processes

Let \( Z_t \in \mathbb{R}^n \) be a stochastic process defined on a filtered probability space \((\Omega, \Sigma, \{\mathcal{F}_t\}_{t \in I}, P)\), where \( \mathcal{F}_t \) is an increasing and \( \mathcal{P}_t \) is a decreasing filtration for the process \( Z_t \), respectively. We say that \( Z \equiv \{Z_t \in \mathbb{R}^n : t \in I \equiv [-T/2, T/2]\} \) is a Bernstein process (cf. [1]) if, for any bounded measurable function \( f \),

\[
E\left[ f(Z_t) | \mathcal{P}_s \cup \mathcal{F}_u \right] = E\left[ f(Z_t) | Z_s, Z_u \right],
\]

for all \( s \leq t \leq u \), \([s,u] \subset I\). This is known today as the 'two-sided' Markov property and represents a ‘reciprocity’ property of the process \( Z \) in time. We stress that (5) is weaker than the Markov property. The construction of a Bernstein process relies on the definition of its transition probability \( Q \equiv Q(s, x, t, B, u, z) \) that verifies:

(i) for all \( x, z \in \mathbb{R}^n \) and \( s < t < u \) in \( I \), \( B \rightarrow Q(s, x, t, B, u, z) \) is a probability measure in the Borel \( \sigma \)-algebra \( \mathcal{B}^n \) of \( \mathbb{R}^n \);

(ii) for a fixed \( B \in \mathcal{B}^n \) and \( s < t < u \) in \( I \), \( (x, z) \rightarrow Q(s, x, t, B, u, z) \) is a measurable function;

(iii) for all \( B_1, B_2 \in \mathcal{B}^n \), and \( s < t < u < r \) in \( I \),

\[
\int_{B_2} Q(s, x, t, B_1, u, w) \, Q(s, x, u, d\omega, r, z) = \int_{B_1} Q(s, x, t, dy, r, z) \, Q(t, y, u, B_2, r, z).
\]

Additionally, it can be found in [20] a proof of the following theorem:

**Theorem 2.1:** Let \( Q \) be a Bernstein transition probability and \( m \) a probability measure on \( \mathcal{B}^n \times \mathcal{B}^n \). Then, there is a unique probability measure \( P = P_m \), such that

1. Property (5) is satisfied;
2. \( P_m(Z_{-T/2} \in A, Z_{T/2} \in C) = m(A \times C) \), for all \( A, C \in \mathcal{B}^n \);
3. \( P_m(Z_t \in B | Z_s, Z_u) = Q(s, Z_s, t, B, u, Z_u) \), for all \( -T/2 < s \leq t \leq u < T/2 \) and \( B \in \mathcal{B}^n \).
(4) \( P_m(Z_{-T/2} \in A, Z_{t_1} \in B_{t_1}, \ldots, Z_{t_n} \in B_{t_n}, Z_{T/2} \in C) = \int_{A \times C} dm(x, z) \times \int_{B_{t_1}} Q(-T/2, x, t_1, dy_1, T/2, z) \times \int_{B_{t_2}} \cdots \int_{B_{t_n}} Q(t_{n-1}, y_{n-1}, t_n, dy_n, T/2, z) \)

Let \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) be a bounded below measurable potential function and \( h \) be a positive constant such that the integral kernel \( h(s, x, t, y) = e^{-\frac{(t-s)}{h}H(x, y)} \), defined on \( L^2(\mathbb{R}^n) \), is positive and jointly continuous in \( x, y \in \mathbb{R}^n \), where \( H \) is a parabolic operator (the Hamiltonian) of the form \( H = -\frac{\hbar^2}{2} \Delta + V \). Then an appropriate density of Bernstein transition probability takes the form (cf. [3]),

\[ Q(s, x, t, dy, u, z) = \frac{h(s, x, t, y)h(t, y, u, z)}{h(s, x, u, z)} dy. \]

After choosing a joint probability measure \( m \) of the following (Markovian [20]) form

\[ m(B_{-T/2} \times B_{T/2}) = \int_{B_{-T/2} \times B_{T/2}} \eta_{-T/2}(x)h(-T/2, x, T/2, y)\eta_{T/2}(y) \, dx \, dy, \]

for \( \eta_{-T/2}, \eta_{T/2} : \mathbb{R}^n \rightarrow \mathbb{R} \) two measurable positive functions, it follows from the marginals of \( m \) that they must solve a nonlinear system of integral equations

\[ \begin{align*}
\eta_{-T/2}(x) \int_{\mathbb{R}^n} h(-T/2, x, T/2, z)\eta_{T/2}(z) \, dz &= p_{-T/2}(x) \\
\eta_{T/2}(z) \int_{\mathbb{R}^n} \eta_{-T/2}(x)h(-T/2, x, T/2, z) \, dx &= p_{T/2}(z),
\end{align*} \]

for \( p_{-T/2} \) and \( p_{T/2} \) any given pair of (strictly positive) boundary probability densities. The existence and uniqueness of solution of the above nonlinear system was shown by Beurling [21]. Finally, if \( \rho(t, x) \) is the density of the process at time \( t \), the probability of the process \( Z_t \) being in \( B \in \mathcal{B}^n \) is of the product form

\[ P(Z_t \in B) = \int_B \rho(t, x) \, dx = \int_B \eta^*(t, x)\eta(t, x) \, dx \]

where

\[ \eta(t, x) = \int h(t, x, T/2, z)\eta_{T/2}(z) \, dz \quad \text{and} \]

\[ \eta^*(t, x) = \int \eta_{-T/2}(y)h^*(-T/2, y, -t, x) \, dy \]

and \( h^* \) is, more generally, the integral kernel of \( e^{-(t+T/2)H^*} \). For self-adjoint \( H \) as before \( H^* = H \). One can prove that, in this case, the functions \( \eta_t^* \) and \( \eta_t \) are two
positive solutions of the initial and terminal problems on \([-\frac{T}{2}, \frac{T}{2}]\),

\[
\begin{align*}
-\hbar \frac{\partial \eta^*}{\partial t} &= H \eta^* \\
\eta^*(-T/2, x) &= \eta^*_{-T/2}(x)
\end{align*}
\]  \quad \text{and} \quad \begin{align*}
\hbar \frac{\partial \eta}{\partial t} &= H \eta \\
\eta(T/2, x) &= \eta_{T/2}(x)
\end{align*}
\]

This construction, for a given \(H\), was done initially in [3]. For ulterior versions see, for instance [5,22]. Afterwards we are going to focus on this Markovian framework.

As a forward Markovian diffusion with initial probability density \(p_{-\frac{T}{2}}\), the transition probability of \(Z_t\) is of the form

\[
p\left(-\frac{T}{2}, x, t, dy\right) = h\left(-\frac{T}{2}, x, t, y\right) \frac{\eta(t, y)}{\eta\left(-\frac{T}{2}, x\right)} \, dy, \quad t \in I
\]

and, as a backward Markovian diffusion with final density \(p_{\frac{T}{2}}\), its transition probability becomes

\[
p_*\left(t, dy, \frac{T}{2}, z\right) = \frac{\eta^*(t, y)}{\eta\left(\frac{T}{2}, z\right)} h(t, y, \frac{T}{2}, z) \, dy, \quad t \in I
\]

As a consequence, its forward drift \(B^Z\) and diffusion coefficient \(C\) (resp. backward \(B^*_Z, C^*_Z\)) are given by (cf. [3,11] and references therein).

\[
B^Z(s, x) = h \nabla \log \eta(s, x), \quad B^*_Z(s, x) = -h \nabla \log \eta^*(s, x)
\]

\[
C^Z(s, x) = C^*_Z(s, x) = h I_{n \times n},
\]

where \(I_{n \times n}\) is the identity matrix of dimension \(n\). Equivalently, for smooth drifts, the Markov BP solve the forward and backward SDE’s:

\[
dZ_u = B^Z(u, Z_u) \, du + h^{1/2} \, dW_u,
\]

\[
d_*Z_s = B^*_Z(s, Z_s) \, ds + h^{1/2} \, d_*W^*_s,
\]

where \(W\) represents a Brownian motion adapted to the usual past filtration, \(W^*\) denotes a Brownian motion adapted to the future filtration and \(d_*\) should be understood as the backward differential. Backward Itô’s calculus and associated notions of stochastic integrals and martingales can be found in [23,24]. Additionally, it is straightforward to observe that

\[
B^*_Z(s, x) = B^Z(s, x) - h \nabla \log \rho(s, x).
\]

For sufficiently smooth functions the operators \(\mathcal{L}\) and \(\mathcal{L}^*\), defined by

\[
\mathcal{L} = \partial_t + B^Z \cdot \nabla + \frac{h}{2} \Delta \quad \text{and} \quad \mathcal{L}^* = \partial_t + B^*_Z \cdot \nabla - \frac{h}{2} \Delta,
\]
coindece, respectively, with the (time dependent) forward and backward infinitesimal generators of the process $Z$:

$$(\mathcal{L}v)(t, x) = \lim_{\Delta t \downarrow 0} E_{t, x} \left[ \frac{v(t + \Delta t, Z(t + \Delta t)) - v(t, Z(t))}{\Delta t} \right],$$

$$(\mathcal{L}^*v)(t, x) = \lim_{\Delta t \downarrow 0} E_{t, x} \left[ \frac{v(t, Z(t)) - v(t - \Delta t, Z(t - \Delta t))}{\Delta t} \right],$$

where $E_{t, x} [\cdot]$ represents the expected value conditioned on the information $Z_t = x$. Let us stress that $\mathcal{L}$ and $\mathcal{L}^*$ involve indeed the same forward and backward increments as in SDEs (9) and (10). Those generators can be regarded as ‘stochastic deformations’ of the (absolute) derivative along classical vector fields (cf. [11]). For a recent review of properties of Bernstein reciprocal processes cf. [25]. They can be interpreted as non-trivial generalizations of the Brownian Bridge (our case $V = 0$, $p_\frac{T}{2} = \delta_x$, $p_\frac{T}{2} = \delta_x$). The relation between forward/backward Itô calculus and a more recent framework using only one filtration (also inspired by Nelson [24]) can be found in [26].

In a quantum-like context, BPs are usually seen as critical points of forward and backward action functionals, cf. [11,22], among others. In fact, one may see that $\eta(t, x) = e^{-\frac{1}{h}F_t(x)}$ and $\eta^*(t, x) = e^{-\frac{1}{h}F^*_t(x)}$ where $F$ and $F^*$ can be obtained as solutions (or ‘value functions’) of optimal control problems. Let $\mathcal{U}$ (resp., $\mathcal{U}^*$) be the set of functions $b$ (resp., $b^*$) such that the processes $b(s, Z_s)$ (resp. $b^*(s, Z_s)$) are progressively measurable processes valued in a compact metric separable space $M$ (resp., $M^*$), with respect to the increasing filtration $\{\mathcal{P}_t\}_{t \in I}$ (resp., decreasing $\{\mathcal{F}_t\}_{t \in I}$) and $T_t$ (resp., $T_t^*$) be the set of all stopping times adapted to the filtration $\{\mathcal{P}_t\}_{t \in I}$ (resp.,$\{\mathcal{F}_t\}_{t \in I}$) that are greater (resp., smaller) than or equal to $t$. If the action functionals $I_{t, x}(Z; \tau, b)$ and $I^*_{t, x}(Z; \tau^*, b^*)$ are defined as in (1) and (3), for $H$ as before, then $F(t, x) = \inf_{b \in \mathcal{U}} I_{t, x}(Z; T/2, b)$ and $F^*(t, x) = \inf_{b^* \in \mathcal{U}^*} I^*_{t, x}(Z; -T/2, b^*)$. Notice that the square drifts in (1) and (3) are natural regularizations of the kinetic energy term in the least Action principle associated with classical Hamilton–Jacobi Equations ((12) and (13) when $h = 0$). According to a time-symmetrized version of Fleming and Soner result [27], Example 8.2, Chap. 3, we have:

**Proposition 2.2:** Let $v(t, x)$ and $v^*(t, x)$ be classical solutions of HJB equations

$$\frac{\partial v}{\partial t} - \frac{1}{2} |\nabla v|^2 + \frac{h}{2} \Delta v + V(x) = 0, \quad (12)$$

for $t < T/2$ and $x \in \mathbb{R}^n$ and

$$\frac{\partial v^*}{\partial t} + \frac{1}{2} |\nabla v^*|^2 - \frac{h}{2} \Delta v^* - V(x) = 0, \quad (13)$$

for $t > -T/2$ and $x \in \mathbb{R}^n$, satisfying the boundary conditions $v(T/2, x) = S(x)$ and $v^*(-T/2, x) = S^*(x)$. Then, $v = F$ and $v^* = F^*$ and $b(t, x) = B^Z(t, x)$ and $b^*(t, x) = B^{Z^*}_x(t, x).$
Below, we introduce an example of Proposition 2.2 that will be used along the paper to clarify our results.

**Example 2.3:** Assuming that \( n = 1, V(x) = 0, S(x) = |x| \) and \( S^*(x) = \log(|x| + 1) \), for all \( x \in \mathbb{R} \), we can construct a BP by solving the pair of adjoint PDE’s (12) and (13) with the boundary conditions \( F(T/2, x) = S(x) \) and \( F^*(-T/2, x) = S^*(x) \). Taking into account the above change of variables \( \eta(t, x) = e^{-\frac{1}{2\hbar}F(t,x)} \) and \( \eta^*(t, x) = e^{-\frac{1}{2\hbar}F^*(t,x)} \), we notice that the functions \( \eta(t, x) \) and \( \eta^*(t, x) \) are unique positive solutions of the adjoint boundary problems

\[
\begin{aligned}
\frac{\partial \eta}{\partial t}(t, x) + \frac{\hbar}{2} \frac{\partial^2 \eta}{\partial x^2}(t, x) &= 0, \quad (t, x) \in [-T/2, T/2] \times \mathbb{R} \\
\eta(T/2, x) &= e^{-\frac{|x|}{\hbar}}, \quad x \in \mathbb{R},
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{\partial \eta^*}{\partial t}(t, x) - \frac{\hbar}{2} \frac{\partial^2 \eta^*}{\partial x^2}(t, x) &= 0, \quad (t, x) \in (-T/2, T/2] \times \mathbb{R} \\
\eta^*(-T/2, x) &= (1 + |x|)^{-\frac{1}{\hbar}}, \quad x \in \mathbb{R},
\end{aligned}
\]

namely,

\[
\begin{aligned}
\eta(t, x) &= \frac{1}{\sqrt{2\pi \hbar(T/2 - t)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\hbar(T/2 - t)}} \frac{e^{\frac{|y|}{\hbar}}}{\pi} \, dy, \\
\eta^*(t, x) &= \frac{1}{\sqrt{2\pi \hbar(t + T/2)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\hbar(t+T/2)}} (1 + |y|)^{-\frac{1}{\hbar}} \, dy.
\end{aligned}
\]

Additionally, the BD, \( Z \), verifies the forward and backward SDE’s

\[
\begin{aligned}
dZ_t &= \hat{b}(t, Z_t) \, dt + \hbar^{1/2} \, dW_t \\
d_\ast Z_t &= \hat{b}^*(t, Z_t) \, dt + \hbar^{1/2} \, dW_t^*,
\end{aligned}
\]

where the control functions \( \hat{b} \) and \( \hat{b}^* \) are given by

\[
\begin{aligned}
\hat{b}(t, x) &= 2e^{-\frac{T/2 - t - x}{2\hbar}} \int_{-\infty}^{\infty} e^{-\frac{(y-(T/2-t)^2)}{2\hbar(t+T/2)}} \eta(t, x) \sqrt{2\pi \hbar(T/2 - t)} \, dy - 1 \\
\hat{b}^*(t, x) &= -\frac{1}{\sqrt{2\pi \hbar}} \frac{e^{\frac{(x-y)^2}{2\hbar(t+T/2)}}}{\eta^*(t, x)(t + T/2)^{3/2}} \frac{((1 - y)^{-1/\hbar} - 1)}{\sqrt{2\pi \hbar}} \, dy,
\end{aligned}
\]

for every \( (t, x) \in (-T/2, T/2) \times \mathbb{R} \).

Following this approach, originated in Schrödinger [2], long before the advent of the theory of stochastic differential equations, the BP, \( Z \), is by well-defined
in the domain $[-T/2, T/2] \times \mathbb{R}^n$, by construction. We are interested in constructing a time reversible process defined in a time varying domain contained in $[-T/2, T/2]$. We shall use results derived from ‘hidden diffusions’ (cf. [28]).

Without taking into account the preceding construction, let us only assume that $Z$ satisfies the forward and backward SDEs (9)–(10), where $B^Z(t,x)$ is bounded and uniformly Lipschitz continuous for $(t,x) \in [-T/2, T/2] \times \mathbb{R}^n$, $B^Z_*(t,x)$ satisfies (11) and $\rho(t,x)$ is the solution of the forward Kolmogorov equation for the process $Z_t$. An auxiliary process $Y_t$ taking values in $\mathbb{R}^n \cup \text{‘hidden’}$ is defined as

$$Y_t = \begin{cases} Z_t, & \text{if } (t, Z_t) \notin A \\ \text{‘hidden’}, & \text{if } (t, Z_t) \in A \end{cases}$$

where $A$ is a Borel set in $[-T/2, T/2] \times \mathbb{R}^n$. If $Y_{[-T/2,T/2]}$ represents the $\sigma-$algebra $\sigma\{Y_t : -T/2 \leq t \leq T/2\}$, $\tau_{t,A} = \inf\{u > t : (u, Z_u) \notin A\}$ and $\tau^*_{t,A} = \sup\{s < t : (s, Z_s) \notin A\}$ then it is straightforward to see that

$$E[f(Z_t) | Y_{[-T/2,T/2]}] = E_{t,x} \left[ f(Z_t) | \mathcal{F}_{\tau_{t,A}} \cup \mathcal{F}_{\tau^*_{t,A}} \right].$$

Combining the decomposition

$$E[f(Z_t) | \mathcal{F}_{\tau_{t,A}} \cup \mathcal{F}_{\tau^*_{t,A}}] = E \left[ f(Z_t) 1_{Z_t \notin A} | \mathcal{F}_{\tau_{t,A}} \cup \mathcal{F}_{\tau^*_{t,A}} \right]$$

$$+ E \left[ f(Z_t) 1_{Z_t \in A} | \mathcal{F}_{\tau_{t,A}} \cup \mathcal{F}_{\tau^*_{t,A}} \right]$$

with the strong Markov property and Theorem 4 in [28], one obtains the following result:

**Theorem 2.4:** Assume that $B^Z(t,x)$ and $\nabla B^Z(t,x)$ are bounded and uniformly Lipschitz continuous function for $(t,x) \in [-T/2, T/2] \times \mathbb{R}^n$, $B^Z_*(t,x)$ satisfies (11). If $Z$ satisfies the forward and backward SDEs (9)–(10) then, for any bounded Borel function $f$, we have

$$E[f(Z_t) | Y_{[-T/2,T/2]}] = E \left[ f(Z_t) | \mathcal{F}_{\tau_{t,A}} \cup \mathcal{F}_{\tau^*_{t,A}} \right] = E[f(Z_t) | \tau_{t,A}, Z_{\tau_{t,A}}, \tau^*_{t,A}, Z_{\tau^*_{t,A}}].$$

This property generalizes indeed the local Markov property (5) since one may consider the domain

$$A = ((s,u) \times \mathbb{R}^n) \cup (\infty, s] \cup [u, +\infty \times (\infty)),$$

For this particular case, it is straightforward to recover equality (1):

$$E \left[ f(Z_t) | \mathcal{F}_s \cup \mathcal{F}_u \right] = E[f(Z_t) | Z_s, Z_u].$$

Let $B$ be a Borel set contained in $A$ and $\tilde{Q}$ be the transition probability of $Z$ conditional to the event $\{Z_t \in A\}$ that is defined as $\tilde{Q}(\tau_{t,A}, Z_{\tau_{t,A}}, t, B, \tau^*_{t,A}, Z_{\tau^*_{t,A}}) =$
According to Choi and Nam [28], the conditional density is given by
\[ \tilde{Q}(s, x, t, dy, u, z) = \frac{\tilde{h}(s, x, t, y, u, z)}{\int_A \tilde{h}(s, x, t, y, u, z) dy}, \]
where \( \tilde{h}(s, x, t, y, u, z) \) represents the joint density of \((\tau_{tA}^*, Z_{tA}^*, Z_t, \tau^*, Z_{tA}^* ) \). Additionally, \( \tilde{h}(s, x, t, y, u, z) \) admits the following decomposition
\[ \tilde{h}(s, x, t, y, u, z) = L_{tA}(s, x, t, y) \rho(t, y) L_{tA}^*(t, y, u, z), \]
where \( L_{tA} \) (resp., \( L_{tA}^* \)) is the joint density of \((\tau_{tA}, Z_{tA}, Z_t) \) (resp., \((\tau^*, Z_{tA}^*, Z_t)\)) and \( \rho \) is the product density of the process \( Z \). Furthermore, according to Lemma 2 in [28], and Lemma 3.2, in [29], the density functions \( M_{s,x}(t, y) = L_{tA}(s, x, t, y) \) and \( M_{u,z}^*(t, y, u, z) \) are the solutions to the martingale boundary problems:
\[
\begin{align*}
\frac{\partial M_{s,x}}{\partial t} (t, y) + B^Z(t, y) \cdot \nabla M_{s,x}(t, y) + \frac{\theta}{2} &\Delta M_{s,x}(t, y) = 0, \quad (t, y) \in A \\
M_{s,x}(t, y) = \delta_{(s,x)}(t, y), &\quad (t, y) \in \partial A \\
M_{s,x}(T/2, y) = 0, &\quad (T/2, y) \in A
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial M_{u,z}^*}{\partial t} (t, y) + B_{*}^Z(t, y) \cdot \nabla M_{u,z}^*(t, y) - \frac{\theta}{2} &\Delta M_{u,z}^*(t, y) = 0, \quad (t, y) \in A \\
M_{u,z}^*(t, y) = \delta_{(u,z)}(t, y), &\quad (t, y) \in \partial A \\
M_{u,z}^*(-T/2, y) = 0, &\quad (-T/2, y) \in A
\end{align*}
\]

### 3. Stochastic optimal control problems

In this work, we will argue that the above generalized concept of BPs may be introduced using stochastic control and optimal stopping theories. Below, we introduce the control problems.

Consider the functionals \( J \) and \( J^* \), and the controlled stochastic process \( Z \) defined by Equations (1)–(4).

The stochastic optimal control problems consist in finding \( \hat{\tau} \) and \( \hat{b} \) (resp., \( \hat{\tau}^* \) and \( \hat{b}^* \)) that minimize the functional \( J \) (resp., \( J^* \)). Equivalently, we can look for the ‘value functions’ \( U \) and \( U^* \) given by
\[
U(t, x) = \inf_{(b, \tau) \in \mathcal{L} \times \mathcal{T}_t} J_{t,x}(Z; \tau, b) = J_{t,x}(Z; \hat{\tau}, \hat{b}) \quad (14)
\]
\[
U^*(t, x) = \inf_{(b^*, \tau^*) \in \mathcal{L}^* \times \mathcal{T}^*_t} J^*_{t,x}(Z; \tau^*, b^*) = J^*_{t,x}(Z; \hat{\tau}^*, \hat{b}^*). \quad (15)
\]
These control problems differ from the ones in Section 2, since in the latter we optimize the functionals only with the respect to \( \hat{b} \) and \( \hat{b}^* \) while in the former we optimize the functionals with respect to \((b, \hat{\tau})\) and \((\hat{b}^*, \hat{\tau}^*)\).
Throughout the paper, we will refer to the set
\[ C = \{(t,x) \in [-T/2, T/2] \times \mathbb{R}^n : U(t,x) < S(x)\} \]
as forward continuation region\(^2\) and as forward stopping region, the set
\[ S = \{(t,x) \in [-T/2, T/2] \times \mathbb{R}^n : (t,x) \notin C\}. \]
Additionally, the backward continuation and stopping regions will be denoted, respectively, by
\[ C^* = \{(t,x) \in (-T/2, T/2] \times \mathbb{R}^n : U^*(t,x) < S^*(x)\} \quad \text{and} \quad S^* = \{(t,x) \in [-T/2, T/2] \times \mathbb{R}^n : (t,x) \notin C^*\}. \]
In \( C \) and \( C^* \), the functions \( S: \mathbb{R}^n \mapsto \mathbb{R} \) and \( S^*: \mathbb{R}^n \mapsto \mathbb{R} \) are those of Equations (1) and (3). Henceforth, we make the following assumption.

**Assumption 3.1:** Let the potential \( V \), the terminal condition \( S \) and the initial condition \( S^* \) be such that

1. \( V, S \) and \( S^* \) are Lipschitz continuous;
2. \( V \) is a lower-bounded function and \( S \) and \( S^* \) are such that \( \{S(Z_{\tau})\}_{\tau \in T} \) and \( \{S^*(Z_{\tau})\}_{\tau \in T^*} \) are two uniformly integrable families of random variables.

Taking into account Assumption 3.1, one can easily prove that the functions \( (x,t) \mapsto U(x,t) \) and \( (x,t) \mapsto U^*(x,t) \) are continuous. A proof of the next result, for the forward version \( U \), can be found in [30], Proposition 3.3. The continuity of \( U^* \) can be obtained by using the same type of arguments.

**Proposition 3.1:** Consider the value functions \( U \) and \( U^* \) defined in (14) and (15). Then \( U^* \in C^0([-T/2, T/2] \times \mathbb{R}^n) \) and \( U \in C^0([-T/2, T/2] \times \mathbb{R}^n) \). Additionally, \( x \mapsto U(t,x) \) and \( x \mapsto U^*(t,x) \) are Lipschitz continuous, uniformly in \( t \).

Condition (1) in Assumption 3.1 is imposed in order to prove continuity of \( U \) and \( U^* \). If we assume continuity of \( U \) and \( U^* \), then we can drop Lipschitz continuity, but the uniqueness result in Section 5 will depend on the continuity of \( U \) and \( U^* \).

Before we finish this section, one may notice that, in light of point (2) in Assumption 3.1, \( U \) and \( U^* \) satisfy the following.

**Proposition 3.2:** Let \( U \) and \( U^* \) be defined as in Equations (14) and (15). Then \( \{U(\tau, Z_{\tau})\}_{\tau \in T} \) and \( \{U^*(\tau^*, Z_{\tau^*})\}_{\tau^* \in T^*} \) are two uniformly integrable families of random variables.
Proof: To prove this proposition, one should observe that for a single random variable uniform integrability means that its expected value is finite. Additionally, taking into account that
\[ -\infty < U(t,x) \leq S(x) \quad \text{and} \quad -\infty < U^*(t,x) \leq S^*(x), \]
the first inequality following from Assumption 3.1, we deduce that
\[
\int_t^{\hat{\tau}} \frac{1}{2} |b(s,Z_s)|^2 + V(Z_s) \, ds + S(Z_{\hat{\tau}}) \quad \text{and} \quad \int_{\hat{\tau}}^{T} \frac{1}{2} |b^*(s,Z_s)|^2 + V(Z_s) \, ds + S^*(Z_{\hat{\tau}^*})
\]
are two uniformly integrable random variables. In addition, this is equivalent to say that there is a uniformly integrable test function \( f : [0, \infty) \to [0, \infty) \) (see Definition C.2 and Theorem C.3 in Øksendal [31]) such that
\[
E_{t,x} \left[ f \left( \left| \int_t^{\hat{\tau}} \frac{1}{2} |b(s,Z_s)|^2 + V(Z_s) \, ds + S(Z_{\hat{\tau}}) \right| \right) \right] < \infty
\]
\[
E_{t,x} \left[ f \left( \left| \int_{\hat{\tau}}^{T} \frac{1}{2} |b^*(s,Z_s)|^2 + V(Z_s) \, ds + S^*(Z_{\hat{\tau}^*}) \right| \right) \right] < \infty
\]
Thus,
\[
E_{t,x} \left[ f \left( |U(\tau,Z_{\tau})| \right) \right] \leq E_{t,x} \left[ f \left( \left| \int_t^{\hat{\tau}} \frac{1}{2} |b(s,Z_s)|^2 + V(Z_s) \, ds + S(Z_{\hat{\tau}}) \right| \right) \right] < \infty,
\]
for any \( \tau \in \mathcal{T}_t \), where the first inequality follows from the strong Markov property and Jensen’s inequality, while the second inequality follows from the comments above.

In this paper, our main goal is to show that stochastic processes obtained as solutions of such stochastic control problems will be BPs. To do this, in the next sections we will show that the value functions \( U \) and \( U^* \) are viscosity solutions for the correspondent HJB equations.

### 4. Dynamic programming principle and viscosity solutions

The relationship between stochastic control problems and PDE’s is well known: usually the value function can be recovered as a solution to a suitable HJB equation (see, for instance Fleming and Soner [27]). As, often, the value function associated with the control problem is not \( C^2 \), in this section, we will prove that the value functions \( U \) and \( U^* \) are, respectively, viscosity solutions to the HJB
where \( v \) and \( v^* \), and the HJB equations reduce to

\[
\max \left\{ -\frac{\partial v}{\partial t} + \frac{1}{2} |\nabla v|^2 - \frac{\hbar}{2} \Delta v - V, v - S \right\} = 0
\]

\[
\max \left\{ \frac{\partial v^*}{\partial t} + \frac{1}{2} |\nabla v^*|^2 - \frac{\hbar}{2} \Delta v^* - V, v^* - S^* \right\} = 0,
\]

where \( v \) and \( v^* \) are functions of \( (t, x) \), \( V, S \) and \( S^* \) are functions of \( x \), and the maximum is taken for each \( (t, x) \).\(^3\) Additionally, \( v \) and \( v^* \) satisfy the following terminal and initial conditions for all \( x \in \mathbb{R}^n \)

\[
v \left( \frac{T}{2}, x \right) = S(x) \quad \text{and} \quad v^* \left( -\frac{T}{2}, x \right) = S^*(x).
\]

**Definition 4.1:** Consider a continuous function \( v : [-T/2, T/2] \times \mathbb{R}^n \to \mathbb{R} \). The function \( v \) is a viscosity subsolution (resp., supersolution) to (17) if whenever \( \psi \in C^2([-T/2, T/2] \times \mathbb{R}^n) \) and \( v - \psi \) has a local maximum (resp., minimum) at \( (t, x) \in [-T/2, T/2] \times \mathbb{R}^n \), such that \( v(t, x) = \psi(t, x) \), then

\[
\max \left\{ -\frac{\partial \psi}{\partial t}(t, x) + \frac{1}{2} |\nabla \psi|^2(t, x) - \frac{\hbar}{2} \Delta \psi(t, x) - V(t, x), v(t, x) - S(t, x) \right\} \leq 0
\]

\[
\left( \text{resp.,} \quad \max \left\{ -\frac{\partial \psi}{\partial t}(t, x) + \frac{1}{2} |\nabla \psi|^2(t, x) - \frac{\hbar}{2} \Delta \psi(t, x) - V(t, x), v(t, x) - S(t, x) \right\} \geq 0 \right) \]
for each \((t, x)\). The function \(v\) is viscosity solution to (17) if it is simultaneously a viscosity subsolution and a viscosity supersolution to (17). A viscosity solution for the HJB Equation (18) can be defined analogously.

To reach the main result of this section, one needs to state a suitable Bellman’s principle for the control problems (14) and (15). Further details about Bellman’s principle for these control problems are given in [32], Section 3.1 (see also Proposition 3.1 and 3.2 in [30]). Fixing \(\epsilon > 0, b \in \mathcal{U}\) and \(b^* \in \mathcal{U}^*\), one can define the stopping times \(\tau_{t,x,b,\epsilon} = \inf\{t \leq s \leq T/2 : U(s, Z_s) \geq S(Z_s) - \epsilon\}\) that is a \(\mathcal{P}_t\) stopping time, and and \(\tau^*_{t,x,b^*,\epsilon} = \sup\{t \leq s \leq T/2 : U^*(s, Z_s) \geq S^*(Z_s) - \epsilon\}\) that is a \(\mathcal{F}_t\) stopping time.

**Proposition 4.2:** Let \((t, x) \in [-T/2, T/2] \times \mathbb{R}^n\) and \(\epsilon > 0\). Then, if \(\tau_b \leq \tau_{t,x,b,\epsilon}\), for all \(b \in \mathcal{U}\),

\[
U(t, x) = \inf_{b \in \mathcal{U}} E_{t,x} \left[ \int_t^{\tau_b} \left( \frac{1}{2} |b(s, Z_s)|^2 + V(Z_s) \right) \, ds + U(\tau_b, Z_{\tau_b}) \right].
\]

Similarly, if \((t, x) \in [-T/2, T/2] \times \mathbb{R}^n, \epsilon > 0\) and \(\tau^*_{b} \geq \tau^*_{t,x,b^*,\epsilon}\), then

\[
U^*(t, x) = \inf_{b^* \in \mathcal{U}^*} E_{t,x} \left[ \int_t^{\tau^*_{b^*}} \left( \frac{1}{2} |b^*(s, Z_s)|^2 + V(Z_s) \right) \, ds + U^*(\tau^*_{b^*}, Z_{\tau^*_{b^*}}) \right].
\]

It is now possible to state the existence of solution for the adjoint boundary problems defined above.

**Proposition 4.3:** Consider the forward and backward stochastic optimal control problems defined, respectively, by (1)–(2)–(14) and (3)–(4)–(15) and Assumption 3.1. Then \(U\) and \(U^*\) are, respectively, viscosity solutions to Equations (17) and (18). Additionally, the following conditions are satisfied for all \(x \in \mathbb{R}^n\)

\[
U \left( \frac{T}{2}, x \right) = S(x) \quad \text{and} \quad U^* \left( -\frac{T}{2}, x \right) = S^*(x).
\]

**Proof:** The proof of the results for the forward and backward cases are similar. Therefore, we will focus our attention on the backward case, which is less common. However, we also present some details about the forward case for the sake of completeness of the proof.

**Proof of the backward case:** To prove that \(U^*\) is a viscosity solution to the HJB Equation (18) and satisfies the boundary condition \(U^*(-T/2, x) = S^*(x)\), we will split the proof in three steps.

**Supersolution property:** Let \((t, x) \in [-T/2, T/2] \times \mathbb{R}\) and \(\psi \in C^2([-T/2,T/2] \times \mathbb{R})\) be such that \((x, t)\) is a local minimizer of \(U^* - \psi\) and \(U^*(t, x) = \psi(t, x)\). We start by noticing that for every \((t, x) \in S^*\), the backward stopping region as defined of Section 3, we have \(U^*(t, x) = S^*(x)\). Fix \((t, x) \in C^*\),
the backward continuation region, and let \( \theta_{b^*} \in (\tau, t, b^*, t) \) be such that \( Z_s \) starts at \( x \) and stays in a neighbourhood \( N(x) \) for \( \theta_{b^*} \leq s \leq t \). Therefore, from the dynamical programming principle, we have

\[
U^*(t, x) = \inf_{b^* \in \mathcal{U}^*} E_{t,x} \left[ \int_t^t \left( \frac{1}{2} |b^*(s, Z_s)|^2 + V(Z_s) \right) ds + U^*(\theta_{b^*}, Z_{\theta_{b^*}}) \right]
\]

\[
\geq \inf_{b^* \in \mathcal{U}^*} E_{t,x} \left[ \int_t^t \left( \frac{1}{2} |b^*(s, Z_s)|^2 + V(Z_s) \right) ds + \psi^*(\theta_{b^*}, Z_{\theta_{b^*}}) \right].
\]

(20)

Applying Dynkin's formula to \( E_{t,x} [\psi(\theta_{b^*}, Z_{\theta_{b^*}})] \), we get

\[
U^*(t, x) - E_{t,x} \left[ \psi(\theta_{b^*}, Z_{\theta_{b^*}}) \right] = E_{t,x} \left[ \int_t^t \left( \frac{\partial \psi}{\partial t}(s, Z_s) + b^*(s, Z_s) \cdot \nabla \psi - \frac{h}{2} \Delta \psi(s, Z_s) \right) ds \right].
\]

(21)

Consequently, combining (20) and (21), it follows that

\[
0 \leq \inf_{b^* \in \mathcal{U}^*} E_{t,x} \left[ \int_t^t \left( (\mathcal{L}^* \psi)(s, Z_s) - \frac{1}{2} |b^*(s, Z_s)|^2 - V(Z_s) \right) ds \right].
\]

By letting \( \theta_{b^*} \rightarrow t \) and dividing by \( E[t - \theta_{b^*}] \), we get

\[
0 \leq \inf_{b^* \in \mathcal{U}^*} \left\{ (\mathcal{L}^* \psi)(t, x) - \frac{1}{2} |b^*(t, x)|^2 - V(x) \right\},
\]

and, consequently, for all \( (t, x) \in (-T/2, T/2) \times \mathbb{R}^n \)

\[
\frac{\partial \psi}{\partial t} + \frac{1}{2} |\nabla \psi|^2 - \frac{h}{2} \Delta \psi - V(x) \geq 0.
\]

**Subsolution property:** Let \( (t, x) \in [-T/2, T/2] \times \mathbb{R} \) and \( \psi \in C^2([-T/2, T/2] \times \mathbb{R}) \) be such that \( (x, t) \) is a local maximizer of \( U^* - \psi \) and \( U^*(x, t) = \psi(x, t) \). From the dynamical programming principle, we have that, for any \( s \leq t \),

\[
U^*(t, x) \leq E_{t,x} \left[ \int_s^t \left( \frac{1}{2} |b^*(u, Z_u)|^2 + V(Z_u) \right) du + U^*(s, Z_s) \right]
\]

\[
\leq E_{t,x} \left[ \int_s^t \left( \frac{1}{2} |b^*(u, Z_u)|^2 + V(Z_u) \right) du + \psi(s, Z_s) \right].
\]

(22)

Consequently, combining (21) and (22) and using a similar argument to the one used for the supersolution property, it follows that

\[
0 \geq \frac{1}{t-s} E_{t,x} \left[ \int_s^t \left( \mathcal{L}^* \psi(u, Z_u) - \frac{1}{2} |b^*(u, Z_u)|^2 - V(Z_u) \right) du \right].
\]
Therefore, letting \( s \not= t \) and using the dominated convergence theorem, we get that
\[
0 \geq (\mathcal{L}^* \psi)(t, x) - \frac{1}{2} |b^*(t, x)|^2 - V(x).
\]
Since \( b^* \) is an arbitrary control, we have the required result:
\[
0 \geq \frac{\partial \psi}{\partial t} + \frac{1}{2} |\nabla \psi|^2 - \frac{\hbar}{2} \Delta \psi - V(x). \tag{23}
\]
Combining (23) with the fact that \( U^*(t, x) \leq J_{t,x}^*(Z; t, b) = S^*(x) \), this implies
\[
\max \left\{ \frac{\partial \psi}{\partial t} + \frac{1}{2} |\nabla \psi|^2 - \frac{\hbar}{2} \Delta \psi - V(x), \psi - S^*(x) \right\} \leq 0.
\]

**Boundary condition:** By construction, if \( \tau^* \in T_{-T/2}^* \) (the set of all \( \mathcal{F}_{-\frac{T}{2}} \) stopping times), then \( \tau^* \leq -T/2 \). Thus, \( U^*(-T/2, x) = \)
\[
\inf_{(b^*, \tau^*) \in T_{-T/2}^*} \mathbb{E}_{-T/2, x} \left[ \int_{-T/2}^{-\tau^*} \left( \frac{1}{2} |b^*(s, Z_s)|^2 + V(Z_s) \right) ds + S^*(Z_{-T/2 \vee \tau^*}) \right] = S^*(x)
\]

**Proof of the forward case:** The proof that \( U \) is a viscosity solution to the HJB equation (17) is based on the same arguments as for the backward case. The main difference relies on the fact that one has to use the forward Itô formula instead of the backward one.

**Supersolution property:** Let \( (t, x) \in (-T/2, T/2] \times \mathbb{R} \) and \( \psi \in C^2([-T/2 T/2] \times \mathbb{R}) \) be such that \( (x, t) \) is a local minimizer of \( U - \psi \) and \( U(t, x) = \psi(t, x) \). As in the previous case, we fix \( (t, x) \in \mathcal{C} \), the forward continuation region, because for every \( (t, x) \in \mathcal{S} \), the forward stopping region, we have \( U(t, x) = S(x) \). Now we let \( \theta_b \in (t, \tau_{x, b, \epsilon}) \) be such that \( Z_s \) starts at \( x \) and stays in a neighbourhood \( N(x) \) for \( t \leq s \leq \theta_b \). From Dynkin’s formula we get
\[
\mathbb{E}_{t,x} \left[ \psi(\theta_b, Z_{\theta_b}) \right] - U(t, x)
\]
\[
= \mathbb{E}_{t,x} \left[ \int_t^{\theta_b} \frac{\partial \psi}{\partial t} (s, Z_s) + b(s, Z_s) \cdot \nabla \psi + \frac{\hbar}{2} \Delta \psi (s, Z_s) \, ds \right],
\]
which, combined with the dynamical programming principle and the definition of \( \psi \), allows us to obtain
\[
0 \geq \inf_{b \in \mathcal{B}} \mathbb{E}_{t,x} \left[ \int_t^{\theta_b} \frac{1}{2} |b(s, Z_s)|^2 + V(Z_s) \right.
\]
\[
+ \frac{\partial \psi}{\partial t} (s, Z_s) + b(s, Z_s) \cdot \nabla \psi + \frac{\hbar}{2} \Delta \psi (s, Z_s) \, ds \right].
\]
Dividing the inequality by \( E[\theta_b - t] \) and letting \( \theta_b \searrow t \), we get the desired inequality.
**Subsolution property:** Let \((t, x) \in (-T/2, T/2) \times \mathbb{R}\) and \(\psi \in C^2((-T/2T/2) \times \mathbb{R})\) be such that \((x, t)\) is a local maximizer of \(U - \psi\) and \(U(x, t) = \psi(x, t)\). The same kind of argument permit us to obtain:

\[
0 \geq \frac{1}{s - t} E_{t, x} \left[ \int_t^s (L\psi)(u, Z_u) + \frac{1}{2} |b(u, Z_u)|^2 + V(Z_u) \, du \right].
\]

Letting \(s \searrow t\) and using the dominated convergence theorem in the previous inequality, together with the fact that \(U(t, x) \leq J(Z; t, b) = S(x)\), we get the desired inequality.

**Boundary condition:** Similarly to the backward case, we have that if \(\tau \in \mathcal{T}_{T/2}\) (the set of all \(\mathcal{P}^\tau T\) - stopping times) then \(\tau \geq T/2\). Thus \(U(T/2, x) = S(x)\). \(\blacksquare\)

Free-boundary problems appear naturally in optimal stopping (see, for example, [33]). In our case, the boundary problems (17)–(19) and (18)–(19) are also of this type. In fact, from a classical point of view, a function \(v\) is a solution to Equation (17) if

\[
-\frac{\partial v}{\partial t} + \frac{1}{2} |\nabla v|^2 - \frac{h}{2} \Delta v - V \leq 0 \quad \text{and} \quad v - S \leq 0, \text{ for all } (t, x) \in [-T/2, T/2) \times \mathbb{R}^n
\]

with at least one of these terms being strictly equal to zero. Similarly, a function \(v^*\) is a solution to Equation (18) if

\[
\frac{\partial v^*}{\partial t} + \frac{1}{2} |\nabla v^*|^2 - \frac{h}{2} \Delta v^* - V \leq 0 \quad \text{and} \quad v^* - S^* \leq 0, \text{ for all } (t, x) \in (-T/2, T/2] \times \mathbb{R}^n
\]

with at least one of these terms being strictly equal to zero. Thus, defining the sets \(C_v = \{(t, x) \in [-T/2, T/2) \times \mathbb{R}^n : v(t, x) \leq S(x)\}\) and \(C_{v^*} = \{(t, x) \in (-T/2, T/2] \times \mathbb{R}^n : v^*(t, x) < S^*(x)\}\), we get that the functions \(v\) and \(v^*\) satisfy the following boundary problems:

\[
-\frac{\partial v}{\partial t} + \frac{1}{2} |\nabla v|^2 - \frac{h}{2} \Delta v - V = 0, \quad (t, x) \in C_v \quad \text{and} \quad v = S, \quad (t, x) \in \partial C_v_v,
\]

and

\[
\frac{\partial v^*}{\partial t} + \frac{1}{2} |\nabla v^*|^2 - \frac{h}{2} \Delta v^* - V = 0, \quad (t, x) \in C_{v^*} \quad \text{and} \quad v^* = S^*, \quad (t, x) \in \partial C_{v^*}.\]

It is now clear that the sets \(C_v\) and \(C_{v^*}\) in (24) and (25) are themselves unknown since they depend on the solutions of the PDEs. A discussion on this type of problems is provided in [34].
In the next proposition, we prove that the value functions $U$ and $U^*$ are, respectively, viscosity solutions to the boundary problems (24) and (25), replacing $C_v$ by $C$ and $C^*_v$ by $C^*$. 

**Proposition 4.4:** Consider the forward and backward stochastic optimal control problems defined, respectively, by (1)–(2)–(14) and (3)–(4)–(15) and Assumption 3.1. Then $U$ and $U^*$ are, respectively, viscosity solutions to the boundary problems (24) and (25), replacing $C_v$ by $C$ and $C^*_v$ by $C^*$. 

The proof of Proposition 4.4 follows immediately from the next auxiliary results. To present these results we let 

\[ \tilde{S}: [-T/2, T/2] \times \mathbb{R}^n \to \mathbb{R} \text{ and } \tilde{S}^*: [-T/2, T/2] \times \mathbb{R}^n \to \mathbb{R} \text{ be two continuous functions,} \]

be an open bounded set, $\tau_A = \inf\{u > t : (u, Z_u) \notin A\}$ and $\tau^*_A = \sup\{s < t : (s, Z_s) \notin A\}$. 

**Lemma 4.5:** Consider the next two modified control problems 

\[
\tilde{U}(x, t) = \inf_{(b, \tau) \in U \times T} E_{t,x} \left[ \int_t^T \left( \frac{1}{2} |b(s, Z_s)|^2 + V(Z_s) \right) ds + \tilde{S}(\tau \wedge \tau_A, Z_{\tau \wedge \tau_A}) \right],
\]

\[
\tilde{U}^*(x, t) = \inf_{(b^*, \tau^*) \in U^* \times T^*} E_{t,x} \left[ \int_{\tau^* \vee T}^T \left( \frac{1}{2} |b^*(s, Z_s)|^2 + V(Z_s) \right) ds + \tilde{S}^*(\tau^* \vee \delta, Z_{\tau^* \vee \delta}) \right].
\]

Then, the value functions $\tilde{U} : A \to \mathbb{R}$ and $\tilde{U}^* : A \to \mathbb{R}$ are, respectively, viscosity solutions of the following adjoint HJB equations, for $(t, x) \in A$, 

\[
\max \left\{ -\frac{\partial v}{\partial t} + \frac{1}{2} |\nabla v|^2 - \frac{h}{2} \Delta v - V, v - \tilde{S} \right\} = 0 \quad (26)
\]

\[
\max \left\{ \frac{\partial v^*}{\partial t} + \frac{1}{2} |\nabla v^*|^2 - \frac{h}{2} \Delta v^* - V, v^* - \tilde{S}^* \right\} = 0 \quad (27)
\]

and satisfy the following boundary conditions 

\[ v = \tilde{S}, \quad \text{and} \quad v^* = \tilde{S}^*, \ (t, x) \in \partial A. \]

This result can be proved by similar arguments to the ones of Proposition 4.3. In the next lemma, we prove that solutions $v$ and $v^*$ of (26) and (27) are also
solutions to the boundary problems

\[-\frac{\partial v}{\partial t} + \frac{1}{2}\|
abla v\|^2 - \frac{h}{2}\Delta v - V = 0, \quad (t, x) \in A_v \quad \text{and} \quad v = \tilde{S}, \quad (t, x) \in \partial A_v, \tag{28}\]

\[-\frac{\partial v^*}{\partial t} + \frac{1}{2}\|
abla v^*\|^2 - \frac{h}{2}\Delta v^* - V = 0, \quad (t, x) \in A_v^* \quad \text{and} \quad v^* = \tilde{S}^*, \quad (t, x) \in \partial A_v^*, \tag{29}\]

where

\[A_v = \left\{(t, x) : v(t, x) < \tilde{S}(t, x)\right\} \quad \text{and} \quad A_v^* = \left\{(t, x) : v^*(t, x) < \tilde{S}^*(t, x)\right\}.\]

**Lemma 4.6:** Let \(v : A \to \mathbb{R}\) and \(v^* : A \to \mathbb{R}\) be viscosity solutions to (26) and (27). Then, \(v\) and \(v^*\) are viscosity solutions of the boundary problems (28) and (29). Additionally, \(v(t, x) \leq \tilde{S}(t, x)\) and \(v^*(t, x) \leq \tilde{S}^*(t, x)\).

**Proof:** Since the function \(v\) is continuous, for any ball \(B_\epsilon(t, x)\) with radius \(\epsilon > 0\) and centre \((x, t)\), there exists \((t_0, x_0)\) such that

\[v(t_0, x_0) = \max_{(t', x') \in B_\epsilon(t, x)} v(t', x').\]

By choosing \(\psi(t, x) = v(t_0, x_0)\), we have that \(\psi(t_0, x_0) = v(t_0, x_0)\) and \((t_0, x_0)\) is a local minimum for the function \(v - \psi\). Since the function \(v\) is a subsolution to (26), we have \(v(t_0, x_0) = \psi(t_0, x_0) \leq \tilde{S}(t_0, x_0)\). Letting \(\epsilon\) go to 0, we obtain \(v(t, x) \leq \tilde{S}(t, x)\) for all \((t, x) \in A\). In particular \(v(t, x) < \tilde{S}(t, x)\) when \((t, x) \in A_v\) and \(v(t, x) = \tilde{S}(t, x)\) when \((t, x) \notin A_v\).

Since \(v(t, x) < \tilde{S}(t, x)\) when \((t, x) \in A_v\) and \(v\) is a viscosity solution to (26), we deduce that: (i) if \(\psi \in C^2([-T/2, T/2] \times \mathbb{R}^n)\) and \((t, x) \in [-T/2, T/2] \times \mathbb{R}^n\) are such that \(v - \psi\) has a local maximum at \((t, x)\) and \(v(t, x) = \psi(t, x)\), then

\[-\frac{\partial \psi}{\partial t} + \frac{1}{2}\|
abla \psi\|^2 - \frac{h}{2}\Delta \psi - V(x) \leq 0;\]

(ii) if \(\psi \in C^2([-T/2, T/2] \times \mathbb{R}^n)\) and \((t, x) \in [-T/2, T/2] \times \mathbb{R}^n\) are such that \(v - \psi\) has a local minimum at \((t, x)\) and \(v(t, x) = \psi(t, x)\), then

\[-\frac{\partial \psi}{\partial t} + \frac{1}{2}\|
abla \psi\|^2 - \frac{h}{2}\Delta \psi - V(x) \geq 0;\]

which means that \(v\) is a viscosity solution (28). A similar argument can be used to prove the statements for \(v^*\). \(\blacksquare\)

To finalize this section, we notice that, in light of stochastic optimal control theory, (cf. [27], Theorem 4.3 Chap IV), the value functions \(S_{\tau_A}\) and \(S_{\tau_A^*}^*\), defined
by
\begin{align}
S_{\tau_A}(t, x) &= \inf_{b \in \mathcal{U}} J_{t, x}(Z; \tau_A, b) \\
S^{*}_{\tau_{A}^{*}}(t, x) &= \inf_{b^{*} \in \mathcal{U}^{*}} J^{*}_{t, x}(Z; \tau_{A}^{*}, b^{*})
\end{align}

satisfy, respectively, the boundary problems (24) and (25) replacing \( C \) by \( A \) and \( C^{*} \) by \( A^{*} \). In addition the optimal strategy is given by \( \hat{b} = -\nabla S_{\tau_A} \) and \( \hat{b}^{*} = \nabla S^{*}_{\tau_{A}^{*}} \).

5. A uniqueness result

In this section, we prove that our value functions are, indeed, unique solutions for the boundary problems presented in (17)–(18)–(19). A similar uniqueness result in the field of optimal stopping can be found in [35]. The authors of the latter paper proved a verification theorem for an optimal stopping problem adapted to the increasing filtration, in which the diffusion cannot be controlled, because both the drift and volatility are defined as known Lipschitz continuous functions. Thus, the authors neither formalize the backward optimal stopping problem nor prove the existence and uniqueness of solution to the associated HJB equation.

**Theorem 5.1:** Consider the forward and backward stochastic optimal control problems defined, respectively, by (1)–(2)–(14) and (3)–(4)–(15). Then:

1. The value function \( U \) is the unique viscosity solution to the HJB equation (17) that satisfies both the left boundary condition in (19) and the condition

\[ \{U(\tau, X_{\tau})\}_{\tau \in \mathcal{P}} \text{ is a uniformly integrable family of random variables.} \]

Additionally the optimal strategy is given by

\[ \hat{\tau} = \inf\{-T/2 \leq t \leq u \leq T/2 : U(u, Z_u) \geq S(Z_u)\} \]

\[ \hat{b}(t, x) = -\nabla U(t, x). \]

2. The value function \( U^{*} \) is the unique viscosity solution to the HJB equation (18) that satisfies both the right-hand side right boundary condition in (19) and the condition

\[ \{U^{*}(\tau, X_{\tau})\}_{\tau \in \mathcal{F}} \text{ is a uniformly integrable family of random variables.} \]

Additionally the optimal strategy is given by

\[ \hat{\tau}^{*} = \sup\{-T/2 \leq s \leq t \leq T/2 : U^{*}(s, Z_s) \geq S^{*}(Z_s)\} \]

\[ \hat{b}^{*}(t, x) = \nabla U^{*}(t, x). \]
**Proof:** Consider an open bounded set $A_N \subset [-T/2, T/2] \times \mathbb{R}^n$ such that $A_N \not
rightarrow [-T/2, T/2] \times \mathbb{R}^n$, as $N \to \infty$, and function $v_N^*(t, x)$ with $v_N^*(t, x) = v^*(t, x)$, for all $(t, x) \in \bar{A}_N$, where $v^*$ is a viscosity solution to (18) such that the right-hand side of (19) is satisfied and $\{v^*(X_t)\}_{t \in \mathcal{F}}$ is a uniformly integrable family of random variables. By construction, we know that $v_N^*$ is a viscosity solution of (27), when one fixes $\tilde{S}^* = v_N^*$ that is unique according to the comparison principle for bounded domains, presented by Crandall et al. [36]. Therefore, from Proposition 4.3 and Lemma 4.5, we have

$$v_N^*(t, x) = \inf_{(b^*, \tau^*) \in \mathcal{U} \times \mathcal{T}_t^*} E_{t, x} \left[ \int_{\tau_N^* \vee \tau^*}^t \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) ds \right] + v_N^*(\tau \vee \tau_N^*, Z_{\tau \vee \tau_N^*})$$

where, $\tau_N^* = \sup \{-T/2 \leq s \leq t \leq T/2 : (s, Z_s) \notin A_N\}$. By construction, $b^*$ is already chosen (as one can see in (16)), i.e. $b^* = \tilde{b}^* \equiv -\nabla v^*$. Therefore,

$$v_N^*(t, x) = \inf_{\tau^* \in \mathcal{T}_t^*} E_{t, x} \left[ \int_{\tau_N^* \vee \tau^*}^t \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) ds \right] + v_N^*(\tau \vee \tau_N^*, Z_{\tau \vee \tau_N^*})$$

the last inequality being a consequence of Lemma 4.6. Since $A_N \not
rightarrow I \times [0, \infty)$ as $N \to \infty$, then $\tau_N^* \vee \tau^* \not
rightarrow -T/2 \vee \tau^*$. Additionally,

$$0 \leq \int_{\tau_N^* \vee \tau^*}^t \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) \pm ds$$

$$\not
rightarrow \int_{\tau_N^* \vee \tau^*}^t \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) \pm ds$$

where $(\frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s))^+ = \max(0, \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s))$ and $(\frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s))^− = \max(0, -(\frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s)))$. From the monotone convergence theorem, we obtain

$$\lim_{n \to \infty} E_{t, x} \left[ \int_{\tau_N^* \vee \tau^*}^t \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) ds \right] = E_{t, x} \left[ \int_{-T/2 \vee \tau^*}^t \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) ds \right]$$

Furthermore, $\{S^*(Z_{\tau^*})\}_{\tau^*}$ is a uniformly integrable family of random variables, which implies that

$$\lim_{N \to \infty} E_{t, x} \left[ S^*(Z_{\tau_N^* \vee \tau^*}) \right] = E_{t, x} \left[ S^*(Z_{-T/2 \vee \tau^*}) \right]$$.
Since this holds true for every $\tau^* \in T^*$, we have

$$v^*(t, x) = \lim_{N \to +\infty} v^*_N(t, x) \leq \lim_{N \to +\infty} U^*_N(t, x) = U^*(t, x).$$

To prove the result, one still needs to show that $v^*(t, x) \geq U^*(t, x)$. Let $A^N = \{(t, x) \in A_N : v^*(t, x) < S^*(x)\}$ and $\tilde{\tau}^*_N = \sup\{-T/2 \leq s \leq t \leq T/2 : (s, Z_s) \notin A^N\}$. Combining the first part of this proof with the results in Lemma 4.6 it follows that $v^*_N$ is the unique viscosity solution of (29), provided that $A^N = A^N_s$ and $\tilde{S}^*(t, x) = v^*_N(t, x)$. Additionally, according to our discussion regarding the representation of the value function for the control problems (30) and (31), we obtain

$$v^*_N(t, x) = E_{t,x} \left[ \int_{\tilde{\tau}^*_N}^{t} \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) ds + v^*_N \left( \tilde{\tau}^*_N, Z_{\tilde{\tau}^*_N} \right) \right].$$

Noticing that $A_N \not\supset \{(t, x) \in [-T/2, T/2] \times \mathbb{R}^n : v^*(t, x) < S^*(x)\}$ as $N \to \infty$, then $\tilde{\tau}^*_N = \tau^*_N \vee \tilde{\tau}^*_N \wedge -T/2 \vee \tau^*_N$, where $\tau^*_N = \sup\{s < t : (s, Z_s) \notin [-T/2T/2] \times \mathbb{R}^n\}$. Therefore, using a similar argument to the previous one, we get $v^*(t, x) = \lim_{N \to +\infty} v^*_N(t, x)$, and, consequently,

$$v^*(t, x) = E_{t,x} \left[ \int_{-T/2 \vee \tau^*_N}^{t} \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) ds + v^*_N (-T/2 \vee \tau^*_N, Z_{-T/2 \vee \tau^*_N}) \right]$$

$$= E_{t,x} \left[ \int_{-T/2 \vee \tau^*_N}^{t} \left( \frac{1}{2} |\tilde{b}^*(s, Z_s)|^2 + V(Z_s) \right) ds + S^*_N (Z_{-T/2 \vee \tau^*_N}) \right]$$

$$\geq U^*(t, x).$$

From this argument, it follows that $U^* = v^*$, that is unique, $\hat{\tau}^* = \tau^*_N$ and $\hat{b} = \nabla v^* = \nabla U^*$. The argument to prove the result for $U$ would be very similar to the one presented here. 

The usual construction of BPs relies on the solutions of forward and backward heat equations with respective positive (not necessarily integrable) final and initial conditions. In our case, we can construct this class of diffusion processes following a similar strategy. Let $\eta$ and $\eta^*$ be two functions defined as follows: $\eta(t, x) = e^{-\frac{1}{\overline{\eta}} U(t, x)}$ and $\eta^*(t, x) = e^{-\frac{1}{\overline{\eta}} U^*(t, x)}$. One can check that $\eta$ and $\eta^*$ satisfy, respectively, the PDEs

$$\min \left\{ -\h \frac{\partial \eta}{\partial t} - \frac{h^2}{2} \Delta \eta + V \eta, \eta - e^{-\frac{1}{\overline{\eta}} \eta^s} \right\} = 0$$

$$\min \left\{ h \frac{\partial \eta^*}{\partial t} - \frac{h^2}{2} \Delta \eta^* + V \eta^*, \eta^* - e^{-\frac{1}{\overline{\eta}} \eta^s} \right\} = 0,$$
on \((-\frac{T}{2}, \frac{T}{2})\), and the boundary conditions
\[
\eta \left( \frac{T}{2}, x \right) = e^{-\frac{1}{\hbar}S}, \quad \eta^* \left( -\frac{T}{2}, x \right) = e^{-\frac{1}{\hbar}S^*}(x).
\]
The reverse is also true in the sense that if \(\eta\) and \(\eta^*\) are positive functions then \(v(t, x) = -\hbar(\log \, \eta)(t, x)\) and \(v^*(t, x) = -\hbar(\log \, \eta^*)(t, x)\) solve Equations (17) and (18). Furthermore, as a consequence of Lemma 4.6, we also know that \(\eta\) and \(\eta^*\) satisfy, respectively, the following boundary problems:

\[
\begin{cases}
-\frac{\partial \eta}{\partial t} - \frac{\hbar}{2} \Delta \eta + V \eta = 0, & (t, x) \in C \\
\eta = e^{-\frac{1}{\hbar}S^*}, & (t, x) \in \partial C
\end{cases}
\] (32)

\[
\begin{cases}
\hbar \frac{\partial \eta^*}{\partial t} - \frac{\hbar}{2} \Delta \eta^* + V \eta^* = 0, & (t, x) \in C^* \\
\eta^* = e^{-\frac{1}{\hbar}S^*}, & (t, x) \in \partial C^*.
\end{cases}
\] (33)

Next result characterizes the stochastic process \(Z\) when one knows the functions \(U\) and \(U^*\) (or equivalently \(\eta\) and \(\eta^*\)).

**Theorem 5.2:** Assume that \(\hat{b}\) and \(\hat{b}^*\), defined as in (16), are such that the forward and backward diffusions (2) and (4) have unique strong solutions with densities given by the solutions to the respective forward and backward Fokker–Planck equations. Additionally, assume that \(\hat{b}\) and \(\hat{b}^*\) are bounded and uniformly Lipschitz continuous functions for \((t, x) \in [-T/2, T/2] \times \mathbb{R}^n\). Then, there is a stochastic process \(Z\) satisfying simultaneously (2) and (4), its density is given \(\rho(t, x) = \eta(t, x)\eta^*(t, x)\) and \(Z_t\) satisfies the property

\[
E[f(Z_t)|\mathcal{P}_t \cup \mathcal{F}_{\bar{t}^*}] = E[f(Z_t)|\bar{t}, Z_{\bar{t}}, \bar{t}^*, Z_{\bar{t}^*}].
\]

**Proof:** Let \(Z\) and \(\tilde{Z}\) be the unique solutions to (2) and (4), respectively. Additionally, \(\rho\) and \(\rho^*\) are the unique solutions to the respective forward and backward Fokker–Planck Equations ([37]). We will prove that for each time \(t\) the processes \(Z\) and \(\tilde{Z}\) have the same density. Indeed, \(\rho\) verifies the boundary problem

\[
\begin{cases}
\frac{\partial \rho}{\partial s}(s, y) = -\nabla \cdot (\hat{b}(s, y)\rho(s, y)) + \frac{\hbar}{2} \Delta \rho(s, y), & s \geq t \\
\rho(t, x) = \eta(t, x)\eta^*(t, x), & (t, x) \in C
\end{cases}
\]

To prove that \(\rho(s, y) = \eta(s, y)\eta^*(s, y)\), one only has to verify that

\[
\frac{\partial \eta(s, y)\eta^*(s, y)}{\partial s} = -\nabla \cdot (\hat{b}(s, y)\eta(s, y)\eta^*(s, y)) + \frac{\hbar}{2} \Delta (\eta(s, y)\eta^*(s, y)).
\]
Taking into account $(16)$ and $(32)$–$(33)$, we get
\[
\frac{\partial \eta(s,y) \eta^*(s,y)}{\partial s} + \nabla \cdot (\hat{b}(s,y) \eta(s,y) \eta^*(s,y)) - \frac{h}{2} \Delta \eta(s,y) \eta^*(s,y)
\]
\[
= \frac{\partial \eta(s,y) \eta^*(s,y)}{\partial s} + h \nabla \cdot ((\nabla \eta) \eta^*) - \frac{h}{2} \Delta (\eta(s,y) \eta^*(s,y))
\]

It is a matter of calculations to see that the expression above can be written as

\[
\eta^*(s,y) \left( \frac{\partial \eta(s,y)}{\partial s} + \frac{h}{2} \nabla \eta(s,y) - V(y) \eta(s,y) \right)
\]

\[
+ \eta(s,y) \left( \frac{\partial \eta^*(s,y)}{\partial s} - \frac{h}{2} \nabla \eta^*(s,y) + V(y) \eta(s,y) \right) = 0.
\]

Additionally, using similar arguments, we can prove that the backward Fokker–Plank equation

\[
\begin{align*}
-\frac{\partial \rho^*(s,y)}{\partial s} &= \nabla \cdot (\hat{b}^*(s,y) \rho^*(s,y)) + \frac{h}{2} \Delta \rho^*(s,y), \quad s \leq t \\
\rho^*(t,x) &= \eta(t,x) \eta^*(t,x), \quad (t,x) \in C^* 
\end{align*}
\]

as a unique solution $\rho^*(s,y) = \eta(s,y) \eta^*(s,y)$. This proves that there is a version of the process $Z$ satisfying simultaneously $(2)$ and $(4)$ and its density is $\rho(s,y) = \eta(s,y) \eta^*(s,y)$. Moreover, in light of Theorem 2.4, the local Markov property is satisfied.

To end this section, we note that when $U$ and $U^*$ are smooth enough then the controlled drifts are, respectively, $\hat{b}(t,x) = h \nabla \log(\eta(t,x))$ and $\hat{b}^*(t,x) = -h \nabla \log(\eta^*(t,x))$. Additionally, $\hat{b}$ and $\hat{b}^*$ solve the boundary problems:

\[
\begin{align*}
\frac{\partial \hat{b}}{\partial t} + (\hat{b} \cdot \nabla) \hat{b} + \frac{h}{2} \Delta \hat{b} - \nabla V(t,x) &= 0, \quad (t,x) \in C \\
\hat{b}(t,x) &= -\nabla S(x), \quad (t,x) \in \partial C \\
\frac{\partial \hat{b}^*}{\partial t} + (\hat{b}^* \cdot \nabla) \hat{b}^* - \frac{h}{2} \Delta \hat{b}^* - \nabla V(t,x) &= 0, \quad (t,x) \in C^* \\
\hat{b}^*(t,x) &= \nabla S^*(x), \quad (t,x) \in \partial C^*
\end{align*}
\]

These equations can be written in terms of the infinitesimal generators $L, L^*$ of $Z$ as $L \hat{b} = \nabla V$ and $L^* \hat{b}^* = \nabla V$.

**Example 5.3:** We continue with the data of Example 2.3. To find the value function $U$ associated with the forward control problem, one has to solve the free-boundary problem $(17)$ with boundary condition $U(T/2,x) = S(x)$, (equivalently, to solve $(24)$). This is a free-boundary problem because the continuation region $C$ is unknown. Therefore, one of the first steps to solve the control problem
In this case, the optimal strategy, \( (\hat{b}, \hat{\tau}) \), is the following:

\[
\hat{b}(t, x) = -\text{sign}(x)\frac{e^{-\frac{T}{2} - t}}{\eta(t, x)\sqrt{2\pi h(T/2 - t)}} \times \left( e^{\frac{x}{\hat{\eta}}} \int_{-\infty}^{-x} e^{-\frac{(y-T/2)^2}{2h(T/2-t)}} \, dy + e^{-\frac{x}{\hat{\eta}}} \int_{-\infty}^{x} e^{-\frac{(y-T/2)^2}{2h(T/2-t)}} \, dy \right)
\]
where or is given by for all these stopping times, one has to provide a characterization of the following (the level 0). Therefore, the process is stopped once it attains the forward stochastic control problem. In fact, one can check that the stopping region is \( S^* = [-T/2, T/2] \times \{0\} \) and the function \( U^*(t, x) = -h \log(\eta^*(t, x)) \), where \( \eta^*(t, x) \) is given by

\[
\eta^*(t, x) = \begin{cases} 
1 + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{2h(t+t/2)}} - e^{-\frac{(x+y)^2}{2h(t+t/2)}} \left((1-y)^{-\frac{1}{h}} - 1\right) dy, & x < 0 \\
1 + \int_{0}^{\infty} e^{-\frac{(x-y)^2}{2h(t+t/2)}} - e^{-\frac{(x+y)^2}{2h(t+t/2)}} \left((y+1)^{-\frac{1}{h}} - 1\right) dy, & x > 0 
\end{cases}
\]

for all \( t \in [-T/2, T/2] \). The optimal strategy for the backward control problem is given by

\[
\hat{b}^*(t, x) = A \int_{-\infty}^{0} \left((x-y)e^{-\frac{(x-y)^2}{2h(t+t/2)}} - (x+y)e^{-\frac{(x+y)^2}{2h(t+t/2)}}\right) \times \left((1-y)^{-1/h} - 1\right) dy, \quad x < 0
\]

or

\[
\hat{b}^*(t, x) = A \int_{0}^{\infty} \left((x-y)e^{-\frac{(x-y)^2}{2h(t+t/2)}} - (x+y)e^{-\frac{(x+y)^2}{2h(t+t/2)}}\right) \times \left((y+1)^{-1/h} - 1\right) dy, \quad x > 0
\]

where \( A = \frac{1}{\eta^*(t,x)(t+T/2)^2 \sqrt{2\pi h}} \) and \( \hat{\tau}^* = \sup\{-T/2 \leq s \leq t : Z_s = 0\} \).

Given the structure of the terminal cost, the process is stopped once it attains the level 0. Therefore, the process \((t, Z_t)\) is well defined in the space-time domain \([\hat{\tau}^*, \hat{\tau}] \times \mathbb{R} \setminus \{0\}\).

By comparing the process \( Z \) constructed in Example 2.3 with the one obtained in the present section, we conclude that the presence of random times in the action functionals changes effectively the optimal stochastic process. Indeed, the BP constructed in a random interval of time is different from the one constructed in a deterministic interval of time, although they minimize the same action functionals.

6. Characterization of the optimal times

In this section, we are interested in obtaining a full characterization of the optimal stopping times \( \hat{\tau} \) and \( \hat{\tau}^* \) used in the previous section. To find the distribution of these stopping times, one has to provide a characterization of the following
The functions:
\[
q(t, x) = P_{t,x}(\hat{t} > \tilde{T}) \quad \text{and} \quad q^*(t, x) = P_{t,x}(\hat{t}^* < \tilde{T}),
\]
with \((t, x) \in [-T/2, T/2] \times \mathbb{R}^n\) and any \(-T/2 \leq \tilde{T} \leq T/2\). One observes that the functions \(q\) and \(q^*\) can be written in the following way:
\[
q(t, x) = E_{t,x}\left[ g\left(\hat{t} \land \tilde{T}, Z_{\hat{t} \land \tilde{T}}\right) \right] \quad \text{and} \quad q^*(t, x) = E_{t,x}\left[ g^*\left(\hat{t}^* \land \tilde{T}, Z_{\hat{t}^* \land \tilde{T}}\right) \right],
\]
with \((t, x) \in [-T/2, T/2] \times \mathbb{R}^n\) and \(-T/2 \leq \tilde{T} \leq T/2\), where functions \(g\) and \(g^*\) are defined by:
\[
g(s, x) = 1_{\{U(s, x) < S(x)\}} \quad \text{and} \quad g^*(s, x) = 1_{\{U^*(s, x) < S^*(x)\}}.
\]
This construction follows immediately from the definition of \(\hat{t}\) and \(\hat{t}^*\), and the fact that black
\[
\{\hat{t} > \tilde{T}\} = \left\{ \omega \in \Omega : U(u, Z_u(\omega)) < S(Z_u(\omega)), \forall u \in [t, \tilde{T}] \right\},
\]
\[
\{\hat{t}^* < \tilde{T}\} = \left\{ \omega \in \Omega : U^*(s, Z_s(\omega)) < S^*(Z_s(\omega)), \forall s \in [\tilde{T}, t] \right\}.
\]
The results derived below require to assume some regularity on the controls.

**Assumption 6.1:** The functions \(U : [-T/2, T/2] \times \mathbb{R}^n \to \mathbb{R}\) and \(U^* : [-T/2, T/2] \times \mathbb{R}^n \to \mathbb{R}\), defined in (14) and (15), are such that the following stochastic differential equations have a unique strong solution
\[
dZ_u = -\nabla U(u, Z_u) \, du + h^{1/2} \, dW_u, \quad \text{with} \quad Z_t = x, \quad \text{and} \quad -\frac{T}{2} \leq t \leq u \leq \frac{T}{2}
\]
\[
d_sZ_s = \nabla U^*(s, Z_s) \, ds + h^{1/2} d_sW_s^*, \quad \text{with} \quad Z_t = x, \quad \text{and} \quad -\frac{T}{2} \leq s \leq t \leq \frac{T}{2}
\]
Additionally, there are constants \(K > 0\) and \(K^* > 0\) such that
\[
|\nabla U(s, x) - \nabla U(s, y)| \leq K|x - y|, \quad \text{for all} \quad (s, x) \text{ and } (s, y) \in C,
\]
\[
|\nabla U^*(s, x) - \nabla U^*(s, y)| \leq K^*|x - y|, \quad \text{for all} \quad (s, x) \text{ and } (s, y) \in C^*.
\]
In some cases the above distribution probabilities are easy to evaluate, as one may see in the next lemma, stated without proof. We introduce the following notation
\[
\tilde{t} = \sup_t \{t \in [-T/2, T/2] : (t, x) \in C\} \quad \text{and} \quad t = \inf_t \{t \in [-T/2, T/2] : (t, x) \in C^*\},
\]
which will be useful for the next lemma.
Lemma 6.1: Let \((t,x) \in [-T/2, T/2] \times \mathbb{R}^n\) and \(-T/2 \leq \tilde{T} \leq T/2\). Then function \(q\) verifies

(i) \(q(t,x) = 1\) if \(((t,x) \in \mathcal{S} \text{ and } t > \tilde{T})\) or \(((t,x) \in \mathcal{C} \text{ and } t \geq \tilde{T})\);

(ii) \(q(t,x) = 0\) if \(((t,x) \in \mathcal{S} \text{ and } t \leq \tilde{T})\) or \(((t,x) \in \mathcal{C} \text{ and } \tilde{T} \geq t)\).

Regarding function \(q^*\), symmetric statements can be obtained:

(iii) \(q^*(t,x) = 1\) if \(((t,x) \in \mathcal{S}^* \text{ and } t < \tilde{T})\) or \(((t,x) \in \mathcal{C}^* \text{ and } t \leq \tilde{T})\);

(iv) \(q^*(t,x) = 0\) if \(((t,x) \in \mathcal{S}^* \text{ and } t \geq \tilde{T})\) or \(((t,x) \in \mathcal{C}^* \text{ and } \tilde{T} \leq t)\).

In the remaining cases, we will show that, under additional conditions, functions \(q\) and \(q^*\) are unique continuous viscosity solutions of the following boundary problems:

\[
\begin{align*}
\frac{\partial q}{\partial t} - \nabla U(t,x) \cdot \nabla q + \frac{h}{2} \Delta q &= 0, \quad (t,x) \in \mathcal{C} \\
q(\tilde{T},x) &= 1, \quad (\tilde{T},x) \in \mathcal{C}, \quad (i) \\
q(i,\tilde{x}) &= 0, \quad (i,\tilde{x}) \in \partial \mathcal{C}
\end{align*}
\]  

(40) with \(-T/2 \leq t < \tilde{T} < \bar{t}\), and

\[
\begin{align*}
\frac{\partial q^*}{\partial t} + \nabla U^*(t,x) \cdot \nabla q^* - \frac{h}{2} \Delta q^* &= 0, \quad (t,x) \in \mathcal{C}^* \\
q^*(\tilde{T},x) &= 1, \quad (\tilde{T},x) \in \mathcal{C}^*, \quad (ii) \\
q^*(i,\tilde{x}) &= 0, \quad (i,\tilde{x}) \in \partial \mathcal{C}^*
\end{align*}
\]  

(41) with \(t < \tilde{T} < t \leq T/2\). Note that there is an implicit relationship between \(q\) and \(q^*\) since \(\nabla U^*(t,x) = -\nabla U(t,x) - h\nabla \log \rho(t,x)\), where \(\rho\) represents the density of the process. Let us also observe that, by definition of the optimal drifts \(\hat{b}\) and \(\hat{b}^*\) in (16), as well as \(q\) and \(q^*\) are, respectively, a \(\mathcal{P}_t\) – martingale and a \(\mathcal{F}_t\) – martingale of the process \(Z\).

Proposition 6.2: Let \(q\) and \(q^*\) be the functions defined in (36). Then, \(q\) is continuous in the domain \((t,x) \in \mathcal{C} \text{ and } -T/2 \leq t < \tilde{T} < \bar{t}\) and \(q^*\) in \((t,x) \in \mathcal{C}^* \text{ and } t < \tilde{T}^* < t \leq T/2\).

To prove this proposition, we will first state two auxiliary results. In the next result we present some estimates on the moments of the process \(Z\). This allows us to prove the continuity of the map \((s,t,x) \rightarrow Z_{s,t}^{(s,x)}(\omega)\). We prove this result by using standard arguments and, consequently, we will simply draft the proof highlighting the more relevant steps.
**Lemma 6.3:** Let $Z$ be the Bernstein process satisfying the forward and backward stochastic differential Equations (38) and (39). Then, for each fixed $\omega \in \Omega$, the application $(s, t, x) \rightarrow Z_{s}^{t,x}(\omega)$ is continuous for each $(t, x) \in C$ and $t \leq s < \tau$.

**Proof:** We start the proof by noticing that

$$E \left[ |Z_{s}^{t,x} - Z_{s'}^{t,x}|^p \right] \leq 3^{p-1} \left( E \left[ |Z_{s}^{t,x} - Z_{s'}^{t,x}|^p \right] + E \left[ |Z_{s'}^{t,x} - Z_{s'}^{t,x'}|^p \right] + E \left[ |Z_{s'}^{t,x'} - Z_{s'}^{t,x'}|^p \right] \right).$$

Firstly we will prove that $E[|Z_{s}^{t,x} - Z_{s'}^{t,x}|^p] \leq 2^{p-1}(s' - s)^\frac{p}{2} (K^p T^\frac{p}{2} + (\frac{p(p-1)}{2})^\frac{p}{2} h^p)$. To prove this estimate, one can notice that

$$E \left[ |Z_{s}^{t,x} - Z_{s'}^{t,x}|^p \right] \leq 2^{p-1} E \left[ \int_s^{s'} \nabla U(u, Z_{u}^{t,x}) \, du \right] + \int_s^{s'} h \, dW_u \right]$$

$$\leq 2^{p-1} \left( L^p (s' - s)^p + \left( \frac{p(p-1)}{2} \right)^\frac{p}{2} h^p (s' - s)^\frac{p}{2} \right)$$

$$\leq 2^{p-1} (s' - s)^\frac{p}{2} \left( L^p T^\frac{p}{2} + \left( \frac{p(p-1)}{2} \right)^\frac{p}{2} h^p \right),$$

the first inequality following from Hölder’s inequality, Theorem 1.7.1 in Mao [38] and the fact that $U$ is Lipschitz in $x$ uniformly in $t$, which implies that $|\nabla U(s, x)|$ is bounded by a constant $L > 0$, for every $(s, t) \in C$.

To find the estimate $E[|Z_{s}^{t,x} - Z_{s'}^{t,x'}|^p] \leq 2^{p-1}|x - x'|^p e^{K^p(t - s')^p} \leq 2^{p-1}|x - x'|^p e^{(2KT)^p}$ one may notice that

$$E \left[ |Z_{s'}^{t,x} - Z_{s'}^{t,x'}|^p \right] \leq 2^{p-1} \left( |x - x'|^p + E \left[ \int_s^{t} \nabla U(u, Z_{u}^{t,x}) - \nabla U(u, Z_{u}^{t,x'}) \, du \right] \right)$$

$$\leq 2^{p-1} \left( |x - x'|^p + K^p (t - s')^{p-1} E \left[ \int_s^{t} |Z_{u}^{t,x} - Z_{u}^{t,x'}|^p \, du \right] \right),$$

where Hölder’s inequality has been used as well as Theorem 1.7.1 in Mao [38] and Assumption 6.1. By Gronwall’s inequality, this shows the last estimate.

Finally, along the same lines of the previous estimates, we may prove that

$$E \left[ |Z_{s'}^{t,x} - Z_{s'}^{t,x'}|^p \right] \leq 2^{p-1} L^p (t' - t) e^{(2KT)^p}.$$
Lemma 6.4: Let \( \hat{\tau} \) and \( \hat{\tau}^* \) be the stopping times defined in Theorem 5.1. Then, for fixed \( \omega \in \Omega \), \( (t, x) \rightarrow \tau_{t,x}(\omega) \) is a continuous map in the domain \( (t, x) \in C \) and \(-T/2 \leq t < \bar{T} < t\) and \( (t, x) \rightarrow \tau_{t,x}^*(\omega) \) is a continuous map in \( (t, x) \in C^* \) and \( \hat{t} < \bar{T} < t \leq T/2 \).

Proof: To prove that the map \( (t, x) \rightarrow \tau_{t,x}(\omega) \) is continuous for a fixed \( \omega \in \Omega \), we notice that, due to the continuity of \( x \rightarrow U(s, x) - S(x) \) and \( (t, x) \rightarrow X_s^{t,x} \), for any \( s \in [-T/2, T/2] \), we get that for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|t - t'| + |x - x'| < \delta \Rightarrow |U(s, X_{s}^{t,x}) - S(X_{s}^{t,x}) - U(s, X_{s}^{t',x'}) - S(X_{s}^{t',x'})| < \epsilon. \tag{42}
\]

According to the definition of \( \hat{\tau} \), for all \( \gamma > 0 \) there is \( \zeta > 0 \) such that

\[
U(s, X_{s}^{t,x}) - S(X_{s}^{t,x}) < -\zeta, \quad \text{for all} \ s \in [t, \hat{\tau}_{t,x} - \gamma] \tag{43}
\]

\[
U(s, X_{s}^{t',x'}) - S(X_{s}^{t',x'}) < -\zeta, \quad \text{for all} \ s \in [t, \hat{\tau}_{t',x'} - \gamma]. \tag{44}
\]

Thus, combining (42) with (43) and choosing \( \epsilon = \frac{\zeta}{2} \), we have

\[
U(s, X_{s}^{t',x'}) - S(X_{s}^{t',x'}) - \frac{\zeta}{2} < U(s, X_{s}^{t,x}) - S(X_{s}^{t,x}) < -\zeta.
\]

Therefore, \( U(s, X_{s}^{t',x'}) - S(X_{s}^{t',x'}) < -\frac{\zeta}{2} \) for all \( s \in [t', \hat{\tau}_{t',x'} - \gamma] \) which implies that \( \tau^{t,x} - \gamma < \tau^{t',x'} \). By combining (42) with (44) and using a similar argument, we conclude that \( \tau^{t',x'} - \gamma < \tau^{t,x} \). Therefore, we have proved that for all \( \gamma > 0 \) there is \( \delta > 0 \) such that

\[
|t - t'| + |x - x'| < \delta \Rightarrow |\tau_{t,x} - \tau_{t',x'}| < \gamma,
\]

as required. The proof that \( (t, x) \rightarrow \tau_{t,x}^*(\omega) \) is a continuous application in \( (t, x) \in C^* \) and \( \hat{t} < \bar{T} < t \leq T/2 \) is analogous. \( \blacksquare \)

Next, we present the proof of Proposition 6.2.

Proof of Proposition 6.2: In what follows, we prove that \( (t, x) \rightarrow q(t, x) \) is a continuous function in the domain \(-T/2 \leq t < \bar{T} < t\) and \( (t, x) \in C \). A similar argument can be established for the remaining case.

Fix \( \omega \in \Omega \) such that \( \hat{\tau}_{t',x'} > \bar{T} \) or \( \hat{\tau}_{t',x'} < \bar{T} \). Due to the continuity of the functions \( (t, x) \rightarrow \hat{\tau}_{t,x} \) and \( (s, t, x) \rightarrow Z_{s}^{t,x} \), one has for \( g \) of Equation (37),

\[
\lim_{(t,x) \rightarrow (t',x')} g(\hat{\tau}_{t,x} \land \bar{T}, Z_{\hat{\tau}_{t,x} \land \bar{T}}^{t,x}) = g(\hat{\tau}_{t',x'} \land \bar{T}, Z_{\hat{\tau}_{t',x'} \land \bar{T}}^{t',x'}). \]

Additionally, since the drifts of the process \( Z \) are bounded, as noticed in the proof of Lemma 6.4, Girsanov theorem holds true, and consequently the law of \( Z \) is
absolutely continuous with respect to law of the Brownian motion. Therefore, combining this fact with the continuity of $U$ and $S$, we get that

$$P(\hat{\tau}_{(t',x')} = \bar{T}) \leq P(U(\bar{T}, Z_{\bar{T}}) = S(Z_{\bar{T}})) = 0.$$  

Since by definition $0 \leq g(t, x) \leq 1$ for all $(t, x) \in [-T/2, T/2] \times \mathbb{R}^n$,

$$\lim_{(t, x) \to (t', x')} E\left[g(\hat{\tau}_{t,x} \wedge \tilde{T}, Z_{\hat{\tau}_{t,x} \wedge \tilde{T}}^t)\right] = E\left[g(\hat{\tau}_{t',x'}' \wedge \tilde{T}, Z_{\hat{\tau}_{t',x'}' \wedge \tilde{T}}^t)\right],$$

follows from the dominated convergence theorem. \hfill \blacksquare

**Theorem 6.5:** Let $q$ and $q^*$ be the functions defined in (36). Then, $q$ is the unique continuous viscosity solution of the boundary problem (40) in the domain $(t, x) \in C$ with $-T/2 \leq t < \tilde{T} < \bar{T}$ and $q^*$ is the unique continuous viscosity solution to the boundary problem (41) in $(t, x) \in \mathcal{C}$ with $\hat{t} < \tilde{T}^* < t \leq T/2$. Outside of this domain, functions $q$ and $q^*$ are characterized according to Lemma 6.1.

**Proof:** Although the proof of Theorem 6.5 relies on the same type of arguments used in the proof of Proposition 4.3 and Theorem 5.1, we will prove the result for $q$. Regarding $q^*$, the statement can be shown along the same lines noticing that the problem is adapted to the decreasing filtration. Thus, one has to use the backward Itô calculus instead of the forward one. In fact, if $\psi \in C^2([−T/2, T/2] \times \mathbb{R}^n)$ and $\tau^* \in \mathcal{T}$ such that $\tau^* \geq \hat{\tau}^*$ we get

$$\psi(t, x) - E_{t,x} [\psi(\tau, X_\tau)] = \int_{\tau^*}^{\tau} \mathcal{L}^* \psi(s, X_s) \, ds.$$  

We split the proof regarding the statement of $q$ in two steps: (i) existence of solution to (40) and (ii) uniqueness of solution to (40).

**Proof of (i):** Let $(t, x) \in C$ and $-T/2 \leq t < \tilde{T} < \bar{T}$ and $\psi \in C^2([-T/2, T/2] \times \mathbb{R}^n)$ be such that $(t, x)$ is a local maximizer of $q - \psi$ and $q(t, x) - \psi(t, x) = 0$. Let $\tau \in \mathcal{T}_t$ be a stopping time satisfying $\tau \leq \hat{\tau}$. Then, by the strong Markov property and (37) we get

$$\psi(t, x) = q(t, x) = E_{t,x} \left[\psi(\tau, X_\tau)\right] = E_{t,x} \left[\int_{\tau}^{\tau} \mathcal{L}^* \psi(s, X_s) \, ds\right].$$

Dividing the inequality $0 \leq E_{t,x} \left[\int_{\tau}^{\tau} \mathcal{L}^* \psi(s, X_s) \, ds\right]$ by $E_{t,x} [\tau - t]$ and letting $\tau \searrow t$ we obtain that $\mathcal{L}^* \psi(t, x) \geq 0$, which allows us to conclude that $q$ is a viscosity subsolution to the PDE (40). To prove the viscosity supersolution a similar argument may be used, namely, pic $\psi \in C^2([-T/2, T/2] \times \mathbb{R}^n)$ and $(t, x) \in C$ with
$-T/2 \leq t < \bar{T} < \hat{t}$ such that $(t, x)$ is a local minimizer of $u - \psi$ and $q(t, x) - \psi(t, x) = 0$. Then,
\[
0 \geq E_{t,x} \left[ \int_{t}^{\bar{T}} L\psi(s, X_s)ds \right],
\]
for $\tau \in T_t$ with $\tau \leq \hat{\tau}$. Dividing the last expression by $E_{t,x}[\tau]$ and letting $\tau \downarrow t$ we obtain that $L\psi(t, x) \leq 0$. Therefore, $q$ is a viscosity supersolution to the PDE (40).

Finally, to prove that $q$ is a viscosity solution of the boundary problem (40), one can see that, according to Lemma 6.1, $q(\bar{T}, x) = 1$ for all $(T, x) \in C \cup \{\bar{T}, \bar{\bar{T}}\}$. Additionally, it is straightforward that, if $(\bar{t}, \bar{x}) \in \{(t, x) : U(t, x) = S(x) \wedge -T/2 \leq t < \bar{T} < \bar{\bar{T}}\}$, then $q(\bar{t}, \bar{x}) = P(\bar{\bar{T}} > \bar{T}) = 0$.

Proof of (ii): Let $A_N$ be an open bounded set such that
\[
A_N \subset \{(t, x) \in C : -T/2 \leq t < \bar{T} < \bar{\bar{T}}\} \quad \text{and} \quad A_N \not\supset \{(t, x) \in C : -T/2 \leq t < \bar{T} < \bar{\bar{T}}\}
\]
and $\tau_N = \inf\{-T/2 \leq t \leq u : Z_u \notin A_N\}$. Additionally, let $q_N$ be given by the function $q_N(t, x) = q(t, x)$ for all $(t, x) \in \overline{A_N}$, where $q$ is a viscosity solution to (40). By construction, $q_N$ is a viscosity solution of the boundary problem
\[
L\nu = 0 \quad \text{with} \quad \nu = q_N.
\]
Additionally, by using the comparison principle for bounded domains, presented by Crandall et al. [36], one may conclude that $q_N$ is the unique viscosity solution of (46).

Along the same lines as the first part of this proof, we have
\[
q_N(t, x) = E\left[q_N\left(\tau_N \wedge \bar{T}, Z^{t,x}_{\tau_N \wedge \bar{T}}\right)\right].
\]
Since $0 \leq q_N(t, x) \leq 1$ for all $(t, x) \in \{(t, x) \in C : -T/2 \leq t < \bar{T} < T/2\}$, the dominated convergence theorem allows us to conclude that
\[
q(t, x) = \lim_{N \to +\infty} q_N(t, x) = \lim_{N \to +\infty} E\left[q_N\left(\hat{\tau}_N \wedge \bar{T}, Z^{t,x}_{\hat{\tau}_N \wedge \bar{T}}\right)\right]
\]
\[
= E\left[q\left(\hat{\tau}_{t,x} \wedge \bar{T}, Z^{t,x}_{\hat{\tau}_{t,x} \wedge \bar{T}}\right)\right],
\]
the last equality following from the continuity of $q$ and the fact that $\tau_N \not\supset \hat{\tau}_{t,x}$. Therefore, it is straightforward that $q$ is given by (36).

Among many problems left open, one will be to look if there are dynamical invariants (first integrals) associated with action functionals (1) and (3). For deterministic lifetimes of the processes the answer is known to be positive (cf. Stochastic Noether Theorem in [11]).
A more conceptual problem will be to understand if the two (Euclidean) Markov times, fundamental in the present construction, correspond in regular quantum theory to some new kind of physical observable in Hilbert spaces. More generally, one should verify systematically in what way our space-time formulation transforms the formulation of the program of Stochastic Deformation described in [11].

Notes

1. $\tau_{t,A}$ is a $\mathcal{P}_t$-stopping time and $\tau_{t,A}^*$ is a $\mathcal{F}_t$-stopping time.
2. The continuation region is a space-time domain where the process is active.
3. If there is no confusion, we may drop the argument of the functions when stating the equations, to short the notation.

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