SUBLUMINAL AND SUPERLUMINAL ELECTROMAGNETIC WAVES AND THE LEPTON MASS SPECTRUM

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Abstract. Maxwell equation $\partial F = 0$ for $F \in \sec \mathcal{L}^2 M \subset \sec \mathcal{O}(M)$, where $\mathcal{O}(M)$ is the Clifford bundle of differential forms, have subluminal and superluminal solutions characterized by $F^2 \neq 0$. We can write $F = \psi \gamma_{21} \tilde{\psi}$ where $\psi \in \sec \mathcal{O}^+(M)$. We can show that $\psi$ satisfies a non-linear Dirac-Hestenes Equation (NLDHE). Under reasonable assumptions we can reduce the NLDHE to the linear Dirac-Hestenes Equation (DHE). This happens for constant values of the Takabayasi angle ($0$ or $\pi$). The massless Dirac equation $\partial \psi = 0, \psi \in \sec \mathcal{O}^+(M)$, is equivalent to a generalized Maxwell equation $\partial F = J_e - \gamma_5 J_m = J$. For $\psi = \psi^\dagger$ a positive parity eigenstate, $j_e = 0$. Calling $\psi_e$ the solution corresponding to the electron, coming from $\partial F_e = 0$, we show that the NLDHE for $\psi$ such that $\psi \gamma_{21} \tilde{\psi} = F_e + F^\dagger$ gives a linear DHE for Takabayasi angles $\pi/2$ and $3\pi/2$ with the muon mass. The Tau mass can also be obtained with additional hypothesis.

1. Introduction

In section 1 we briefly recall how to write Maxwell and Dirac equations in the Clifford and Spin-Clifford bundle formalisms. In section 2 we present some mathematical preliminaries. Then in section 3 we prove that the free Maxwell equations $\partial F = 0, F \in \sec \mathcal{L}^2 M \subset \sec \mathcal{O}(M)$, have subluminal and superluminal solutions characterized by $F^2 \neq 0$. In particular, we show that there are some interesting solutions of $\partial F = 0$ which are equivalent to a non-homogeneous Maxwell equation $\partial F' = j$. In section 4 we prove that the solutions of $\partial F = 0$ for which $F^2 \neq 0$ are equivalent to solutions of a non-linear Dirac-Hestenes equation (NLDHE) for $\psi \in \sec \mathcal{O}^+(M)$ such that $F = \psi \gamma_{21} \psi$. Under reasonable assumptions, namely that $\psi$ has only six degrees of freedom, the NLDHE gives a linear Dirac-Hestenes equation (DHE) for a constant mass. We may identify this solution $\psi_e$ with the DHE for the electron (or positron) depending on the value of the Takabayasi angle. In section 5 we study some properties of the linear and nonlinear Dirac-Hestenes equations. In this formalism the Weyl equation is written $\partial \psi_W = 0$, with $\psi_W \in \sec \mathcal{O}^+(M)$ and $\psi_W \gamma_{21} \psi_W = 0$. The expression $\partial \psi_W = 0$ is equivalent to a generalized Maxwell

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equation $\partial F = J_x - \gamma_5 J_m = J$. If $\psi^\dagger$ is a Dirac-Hestenes spinor which is a positive parity eigenstate (see eq.(2.16)) and $\psi^\dagger$ satisfies $\partial \psi^\dagger = 0$, then the equivalent Maxwell equation reads $\partial F^\dagger = -\gamma_5 J_m$ (see eq.(5.42)). In section 6 we study the NLDHE associated to $\partial (F_x + F^\dagger) = \partial F = J$, which yields, for certain values of the Takabayasi angle associated to $\psi$ such that $F = \psi\gamma_2\psi$, a DHE with the correct value of the muon mass. Under additional hypothesis this theory gives also the values of the Tau lepton mass.

2. Mathematical Preliminaries

Here we briefly recall how to write Maxwell and Dirac equations in the Clifford and Spin-Clifford bundles over Minkowski spacetime. Details concerning these theories can be found in (Rodrigues and Souza, 1993; Rodrigues and Souza, 1994; Rodrigues et al., 1995a; Rodrigues et al., 1995b).

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime. $(M, g)$ is a four dimensional time oriented and space oriented Lorentzian manifold, with $M \simeq \mathbb{R}^4$ and $g \in \mathrm{sec}(T^*M \times T^*M)$ being a Lorentzian metric of signature $(1,3)$. $T^*M$ is the cotangent bundle. $T^*M = \bigcup_{x \in M} T^*_x M$ and $TM = \bigcup_{x \in M} T_x M$, and $T_x M \simeq T^*_x M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space (Sachs and Wu, 1977; Rodrigues and Rosa, 1989). $D$ is the Levi-Civita connection of $g$, i.e., $Dg = 0$, $T(D) = 0$. Also $\mathbb{R}(D) = 0$, $T$ and $R$ being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{C}(M)$ is the bundle of algebras $\mathcal{C}(M) = \bigcup_{x \in M} \mathcal{C}(T_x^* M)$, where $\forall x \in M, \mathcal{C}(T_x^* M) = \mathcal{C}_{1,3}$, the so called spacetime algebra (Lounesto, 1993; Hestenes and Sobczyk, 1987; Hestenes, 1966). Locally as a linear space over the real field $\mathbb{R}$, $\mathcal{C}(T_x^* M)$ is isomorphic to the Cartan algebra $\bigwedge(T_x^* M)$ of the cotangent space and let $\bigwedge(T_x^* M) = \bigwedge^k(T_x^* M)$, where $\bigwedge^k(T_x^* M)$ is the $(4^k)$-dimensional space of $k$-forms. The Cartan bundle $\bigwedge(M) = \bigcup_{x \in M} \bigwedge(T_x^* M)$ can then be thought as “imbbeded” in $\mathcal{C}(M)$. In this way sections of $\mathcal{C}(M)$ can be represented as a sum of inhomogeneous differential forms. Let $\{ e_\mu = \frac{\partial}{\partial x^\mu} \} \in \mathrm{sec}TM, (\mu = 0,1,2,3)$ be an orthonormal basis $g(e_\mu, e_\nu) = \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and let $\{ \gamma^\nu = dx^\nu \} \in \mathrm{sec} \bigwedge^1(M) \subset \mathrm{sec}\mathcal{C}(M)$ be the dual basis. Then, the fundamental Clifford product (in what follows to be denoted by juxtaposition of symbols) is generated by $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$ and if $\mathcal{C} \in \mathrm{sec}\mathcal{C}(M)$ we have

$$
\mathcal{C} = s + v_\mu \gamma^\mu + \frac{1}{2!} b_{\mu\nu} \gamma^\mu \gamma^\nu + \frac{1}{3!} a_{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho + p \gamma^5, \tag{2.1}
$$

where $\gamma^5 = \gamma_0 \gamma^1 \gamma^2 \gamma^3 = dx^0 dx^1 dx^2 dx^3$ is the volume element and $s, v_\mu, b_{\mu\nu}, a_{\mu\nu\rho}, p \in \mathrm{sec} \bigwedge^0(M) \subset \mathrm{sec}\mathcal{C}(M)$. For $A_r \in \mathrm{sec} \bigwedge^r(M) \subset \mathrm{sec}\mathcal{C}(M), B_s \in \mathrm{sec} \bigwedge^s(M) \subset \mathrm{sec}\mathcal{C}(M)$ we define (Lounesto, 1993; Lounesto, 1994; Hestenes and Sobczyk, 1987) $A_r \cdot B_s = \langle A_r, B_s \rangle_{r+s}$, where $\langle \cdot \rangle_k$ is the component in $\bigwedge^k(M)$ of the Clifford field.

Besides the vector bundle $\mathcal{C}(M)$ we need also to introduce another vector bundle $\mathcal{C}_{\mathrm{Spin}^+(1,3)}(M) \simeq \mathrm{SL}(2,\mathbb{C})$ called the Spin-Clifford bundle (Rodrigues and Figueiredo, 1990). We can show that $\mathcal{C}_{\mathrm{Spin}^+(1,3)}(M) \simeq \mathcal{C}(M)/\mathcal{R}$, i.e., it is a quotient bundle. This means that sections of $\mathcal{C}_{\mathrm{Spin}^+(1,3)}(M)$ are some special
equivalence classes of sections of the Clifford bundle, i.e., they are equivalence sections of non-homogeneous differential forms (see eqs. (2.2), (2.3) below).

Now, as is well known, an electromagnetic field is represented by $F \in \text{sec} \bigwedge^2(M) \subset \text{sec}\mathbb{C}ℓ(M)$. How to represent the Dirac spinor fields in this formalism? We can show that even sections of $\mathbb{C}ℓ\text{Spin}^+(1,3)(M)$, called Dirac-Hestenes spinor fields, do the job. If we fix two orthonormal basis, $\Sigma = \{\gamma^\mu\}$ as before, and $\dot{\Sigma} = \{\dot{\gamma}^\mu = R^{\mu}_\nu \dot{R} = \Lambda^\mu_\nu \gamma^\nu\}$ with $\Lambda^\mu_\nu \in \text{SO}_+(1,3)$ and $R \in \text{Spin}^+(1,3)$ $\forall x \in M$, $\dot{R} = \ddot{R} = 1$, and where $\sim$ is the reversion operator in $\mathbb{C}ℓ(1,3)$ (Lounesto, 1993; Lounesto, 1994; Hestenes and Sobczyk, 1987), then the representations of an even section $\psi \in \text{sec}\mathbb{C}ℓ\text{Spin}^+(1,3)(M)$ are the sections $\psi_{\Sigma}$ and $\psi_{\dot{\Sigma}}$ of $\mathbb{C}ℓ(M)$ related by

$$\psi_{\dot{\Sigma}} = \psi_{\Sigma} R \quad (2.2)$$

and

$$\psi_{\Sigma} = s + \frac{1}{2!} b_{\mu\nu} \gamma^\mu \gamma^\nu + p \gamma^5 \quad (2.3)$$

Note that $\psi_{\Sigma}$ has the correct number of degrees of freedom in order to represent a Dirac spinor field, which is not the case with the so called Dirac-Kähler spinor field.

Let $\star$ be the Hodge star operator $\star : \bigwedge^k(M) \to \bigwedge^{4-k}(M)$. Then we can show that if $A_p \in \text{sec} \bigwedge^p(M) \subset \text{sec}\mathbb{C}ℓ(M)$ we have $\star A = A \gamma^5$. Let $d$ and $\delta$ be respectively the differential and Hodge codifferential operators acting on sections of $\bigwedge(M)$. If $\omega_p \in \text{sec} \bigwedge^p(M) \subset \text{sec}\mathbb{C}ℓ(M)$, then $\delta \omega_p = (-)^p \star^{-1} d \star \omega_p$, with $\star^{-1} \star = \text{identity}$.

The Dirac operator acting on sections of $\mathbb{C}ℓ(M)$ is the invariant first order differential operator

$$\partial = \gamma^\mu D_{\mu}, \quad (2.4)$$

and we can show the very important result (Maia et al., 1990):

$$\partial = \partial \wedge + \partial \cdot = d - \delta. \quad (2.5)$$

With these preliminaries we can write Maxwell and Dirac equations as follows (Hestenes, 1966; Rodrigues and Oliveira, 1990):

$$\partial F = 0 \quad (2.6)$$

$$\partial \psi_{\Sigma} \gamma^1 \gamma^2 + m \psi_{\Sigma} \gamma^0 = 0 \quad (2.7)$$

If $m = 0$ we have the massless Dirac equation

$$\partial \psi_{\Sigma} = 0, \quad (2.8)$$

which is Weyl’s equation (see eq. (2.12) below) when $\psi_{\Sigma}$ is reduced to a Weyl spinor field (Lounesto, 1993; Lounesto, 1994). Note that in this formalism Maxwell equations condensed in a single equation! Also, the specification of $\psi_{\Sigma}$ depends on the frame $\Sigma$. When no confusion arises we represent $\psi_{\Sigma}$ simply by $\psi$.

When $\psi_{\Sigma} \sim \psi_{\Sigma} \neq 0$, where $\sim$ is the reversion operator, then $\psi_{\Sigma}$ has the following cannonical decomposition:

$$\psi_{\Sigma} = \sqrt{\rho} e^{\delta \gamma_5 / 2} R, \quad (2.9)$$
where \( \rho, \beta \in \sec \Lambda^0(M) \subset \sec \mathcal{O}(M) \) and \( R \in \text{Spin}_+(1,3) \subset \mathcal{O}^+_{1,3}, \forall x \in M. \beta \) is called the Takabayasi angle.

If one wants to work in terms of the usual spinor field formalism, we can translate our results by choosing, for example, the standard matrix representation of \( \{\gamma^\mu\} \), and for \( \psi_\Sigma \) given by eq.\( (2.3) \) we have the following (standard) matrix representation (Rodrigues et al., 1995b):

\[
\Psi = \begin{pmatrix} \phi_1 - \phi_2^* \\ \phi_2 \phi_1^* \end{pmatrix},
\]

(2.10)

where

\[
\phi_1 = \begin{pmatrix} s - ib_{12} & b_{13} - ib_{23} \\ -b_{13} - ib_{23} & s + ib_{12} \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} -b_{03} + ip & -b_{01} + ib_{02} \\ -b_{01} - ib_{02} & b_{03} + ip \end{pmatrix}.
\]

(2.11)

with \( s, b_{12}, \ldots \) real functions. Right multiplication by

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

gives the usual Dirac spinor field.

We need also the concept of Weyl spinors. By definition, \( \psi \in \sec \mathcal{O}^+(M) \) is a Weyl spinor if (Lounesto 1993,1994)

\[
\gamma_5 \psi = \pm \gamma_2 \psi.
\]

(2.12)

The positive (negative) “eigenstate” of \( \gamma_5 \) will be denoted \( \psi_+ (\psi_-) \). For a general \( \psi \in \sec \mathcal{O}^+(M) \) we can write

\[
\psi_\pm = \frac{1}{2} [\psi \mp \gamma_5 \psi \gamma_2] \quad (2.13)
\]

and then

\[
\psi = \psi_+ + \psi_- \quad (2.14)
\]

The parity operator \( P \) in this formalism is represented in such a way (Lounesto 1993,1994) that for \( \psi \in \sec \mathcal{O}^+(M) \),

\[
P \psi = -\gamma_0 \psi_0 \quad (2.15)
\]

The following Dirac-Hestenes spinors are eigenstates of the parity operator with eigenvalues \( \pm 1 \):

\[
P \psi^+ = +\psi^+, \quad \psi^+ = \gamma_0 \psi_+ - \gamma_0 \psi_- \quad (2.16)
\]

We recall that the even subbundle \( \mathcal{O}^+(M) \) of \( \mathcal{O}(M) \) is such that its typical fiber is the Pauli algebra \( \mathcal{O}_{3,0} \equiv \mathcal{O}^+_{1,3} \). The isomorphism \( \mathcal{O}_{3,0} \equiv \mathcal{O}^+_{1,3} \) is exhibited by
putting $\sigma_i = \gamma_i \gamma_0$, whence $\sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij}$. Then if $F = \frac{1}{2} F^{\mu \nu} \gamma_\mu \gamma_\nu \in \text{sec} \wedge^2 M \subset \text{sec} \mathcal{O}(M)$ with

$$F^{\mu \nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B_3 & B_2 \\ E^2 & B_3 & 0 & -B_1 \\ E^3 & -B_2 & B_1 & 0 \end{pmatrix},$$

(2.17)

we can write $F = \vec{E} + i \vec{B}$ with $i = \sigma_1 \sigma_2 \sigma_3 = \gamma_5$, $\vec{E} = E^i \sigma_i$, $\vec{B} = B^i \sigma_i$.

Before we present the subluminal and superluminal solutions $F_<$ and $F_>$ of Maxwell equations we shall define precisely an inertial reference frame (irf) (Sachs and Wu, 1977; Rodrigues and Rosa, 1989). An irf $I \in \text{sec}TM$ is a timelike vector field pointing into the future such that $g(I, I) = 1$ and $DI = 0$. Each integral line of $I$ is called an observer. The coordinate functions $\langle x^\mu \rangle$ of a chart of the maximal atlas of $M$ are called naturally adapted to $I$ if $I = \partial / \partial x^0$. Putting $I = e_0$, we can find $e_i = \partial / \partial x^i$ such that $g(e_\mu, e_\nu) = \eta_{\mu \nu}$ and the coordinate functions $\langle x^\mu \rangle$ are the usual Einstein-Lorentz ones and have a precise operational meaning (Rodrigues and Tionno, 1985). $x^0$ is measured by “ideal clocks” at rest synchronized “à la Einstein” and $x^i$ are measured with “ideal rulers”.

3. Subluminal and Superluminal Undistorted Progressive Waves (UPWs) Solutions of Maxwell Equations (ME)

We start by reanalyzing in section 3.1 the plane wave solutions (PWS) of ME with the Clifford bundle formalism. We clarify some misconceptions and explain the fundamental role of the duality operator $\gamma_5$ and the meaning of $i = \sqrt{-1}$ in standard formulations of electromagnetic theory.

Next, in section 3.2 we discuss subluminal UPWs solutions of ME and an unexpected relation between these solutions and the possibility of the existence of purely electromagnetic particles (PEPs) envisaged by (Einstein, 1919; Poincaré, 1906; Ehrenfest, 1907) and recently discussed by Waite, Barut and Zeni (Waite, 1995; Waite et al., 1996).

In section 3.3 we discuss in detail the theory of superluminal electromagnetic X-waves (SEXWs). In (Rodrigues and Lu, 1996) it is discussed how to produce these waves with appropriate physical devices.

3.1. Plane Wave Solutions of Maxwell Equations

We recall that ME in vacuum can be written as

$$\nabla F = 0,$$

(3.1)

where $F \in \text{sec} \wedge^2 (M) \subset \text{sec} \mathcal{O}(M)$. The well known PWS of eq.(3.1) are obtained as follows. We write in a given Lorentzian chart $\langle x^\mu \rangle$ of the maximal atlas of $M$ a PWS moving in the $z$-direction

$$F = f e^{x^0 k^0}$$

$$k = k^\mu \gamma_\mu, k^1 = k^2 = 0, x = x^\mu \gamma_\mu,$$

(3.3)
where $k, x \in \sec \Lambda^1(M) \subset \sec \mathcal{C}(M)$ and where $f$ is a constant 2-form. From eqs (3.2) and (3.3) we obtain

$$kF = 0 \quad (3.4)$$

Multiplying eq (3.4) by $k$ we get

$$k^2F = 0 \quad (3.5)$$

and since $k \in \sec \Lambda^1(M) \subset \sec \mathcal{C}(M)$ then

$$k^2 = 0 \leftrightarrow k_0 = \pm |k| = k^3, \quad (3.6)$$

i.e., the propagation vector is light-like. Also

$$F^2 = F.F + F \wedge F = 0 \quad (3.7)$$

as can be easily seen by multiplying both members of eq (3.4) by $F$ and taking into account that $k \neq 0$. Eq (3.7) says that the field invariants are null.

It is interesting to understand the fundamental role of the volume element $\gamma_5$ (duality operator) in electromagnetic theory. In particular since $e^{\gamma_5 kx} = \cos kx + \gamma_5 \sin kx$, we see that

$$F = f \cos kx + \gamma_5 f \sin kx, \quad (3.8)$$

i.e., in a PWS the electric and magnetic fields are oscillating out of phase by $90^\circ$. Writing $F = \vec{E} + i\vec{B}$, (see eq.(2.17)) with $i \equiv \gamma_5$ and choosing $f = \tilde{e}$, eq (3.8) becomes

$$(\vec{E} + i\vec{B}) = \tilde{e} \cos kx + i\tilde{e} \sin kx. \quad (3.9)$$

This equation is important because it shows that we must take care with the $i = \sqrt{-1}$ that appears in usual formulations of Maxwell Theory using complex electric and magnetic fields. The $i = \sqrt{-1}$ in many cases unfolds a secret that can only be known through eq (3.9).

From eq (3.4) we can also easily show that $\vec{k}.\vec{E} = \vec{k}.\vec{B} = 0$, i.e., PWS of ME are transverse waves.

We can rewrite eq (3.4) as

$$k\gamma_0 \gamma_0 F \gamma_0 = 0 \quad (3.10)$$

and since $k\gamma_0 = k_0 + \vec{k}$, $\gamma_0 F \gamma_0 = -\vec{E} + i\vec{B}$ we have

$$\vec{k}f = k_0 f. \quad (3.11)$$

Now, we recall that in $\mathcal{C}^+(M)$ (where the typical fiber is isomorphic to the Pauli algebra $\mathcal{C}_{3,0}$) we can introduce the operator of space conjugation denoted by $\ast$ (Hestenes, 1966) such that writing $f = \tilde{e} + i\vec{b}$ we have

$$f^* = -f; \quad k_0^* = k_0; \quad \vec{k}^* = -\vec{k}. \quad (3.12)$$

We can now interpret the two solutions of $k^2 = 0$, i.e., $k_0 = |\vec{k}|$ and $k_0 = -|\vec{k}|$ as corresponding to the solutions $k_0 f = \vec{k}f$ and $k_0 f^* = -\vec{k}f^*$: $f$ and $f^*$ correspond in quantum theory to “photons” which are of positive or negative helicities. We can interpret $k_0 = |\vec{k}|$ as a particle and $k_0 = -|\vec{k}|$ as an antiparticle.

Summarizing we have the following important facts concerning PWS of ME: (i) the propagation vector is light-like, $k^2 = 0$; (ii) the field invariants are null, $F^2 = 0$; (iii) the PWS are transverse waves, i.e., $\vec{k}.\vec{E} = \vec{k}.\vec{B} = 0$. 
3.2. Subluminal Solutions of Maxwell Equations and Purely Electromagnetic Particles (PEPs)

In order to present the subluminal and superluminals solutions of Maxwell equations we need the following result (Rodrigues and Vaz, 1995):

Let \( A \in \text{sec} \mathcal{A}^1(M) \subset \text{sec} \mathcal{A}(M) \) be the vector potential. We fix the Lorentz gauge, i.e., \( \partial \cdot A = -\delta A = 0 \) such that \( F = \partial A = (d - \delta)A = dA \). We have the following theorem:

**Theorem:** Let \( \pi \in \text{sec} \mathcal{A}^2(M) \subset \text{sec} \mathcal{A}(M) \) be the so-called Hertz potential. If \( \pi \) satisfies the wave equation, i.e., \( \partial^2 \pi = \eta^{\mu\nu} \partial_\mu \partial_\nu \pi = -((d\delta + d\delta)\pi = 0, \) and if \( A = -\pi \), then \( F = \partial A \) satisfies the Maxwell equations \( \partial F = 0 \).

The proof is trivial. Indeed \( A = -\pi \), then \( \delta A = -\delta^2 \pi = 0 \) and \( F = \partial A = dA \). Now \( \partial F = (d - \delta)(d - \delta)A = -(d\delta + d\delta)A = \delta d(\delta \pi) = -\delta^2 d\pi = 0 \) since \( \delta d\pi = -\delta d\pi \) from \( \partial^2 \pi = 0 \).

We take \( \Phi \in \text{sec} \mathcal{A}^0(M) \cap \mathcal{A}^4(M) \subset \text{sec} \mathcal{A}(M) \) and consider (Vaz and Rodrigues, 1995) the following Hertz potential \( \pi \in \text{sec} \mathcal{A}^2(M) \subset \text{sec} \mathcal{A}(M) \):

\[
\pi = \Phi \gamma^1 \gamma^2. \tag{3.13}
\]

We now write

\[
\Phi(t, \vec{x}) = \phi(\vec{x}) e^{\gamma_5 \Omega t}. \tag{3.14}
\]

Since \( \pi \) satisfies the wave equation, we have

\[
\nabla^2 \phi(\vec{x}) + \Omega^2 \phi(\vec{x}) = 0. \tag{3.15}
\]

Solutions of eq.(3.15) (the Helmholtz equation) are well known. Here, we consider the simplest solution in spherical coordinates,

\[
\phi(\vec{x}) = C \frac{\sin \Omega r}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}, \tag{3.16}
\]

where \( C \) is an arbitrary real constant. We obtain the following stationary electromagnetic field, which is at rest in the reference frame \( I \) where \( (x^\mu) \) are naturally adapted coordinates:

\[
F_0 = \frac{C}{r^3} [\sin \Omega t (\alpha \Omega r \sin \theta \sin \rho - \beta \cos \theta \sin \rho \cos \rho) \gamma^0 \gamma^1
- \sin \Omega t (\alpha \Omega r \sin \theta \cos \rho + \beta \sin \theta \cos \rho \sin \rho) \gamma^0 \gamma^2
+ \sin \Omega t (\beta \sin^2 \theta - 2\alpha) \gamma^0 \gamma^3 + \cos \Omega t (\beta \sin^2 \theta - 2\alpha) \gamma^1 \gamma^2
+ \cos \Omega t (-\beta \sin \theta \cos \rho \sin \rho + \alpha \Omega r \sin \theta \cos \rho) \gamma^1 \gamma^3
+ \cos \Omega t (-\beta \sin \theta \cos \rho \cos \rho + \alpha \Omega \sin \theta \sin \rho) \gamma^2 \gamma^3]
\]

with \( \alpha = \Omega r \cos \Omega r - \sin \Omega r \) and \( \beta = 3\alpha + \Omega^2 r^2 \sin \Omega r \). Observe that \( F_0 \) is regular at the origin and vanishes at infinity. Let us rewrite the solution using the Pauli-algebra in \( \mathcal{A}^+(M) \). Writing \( (i \equiv \gamma_5) \)

\[
F_0 = \vec{E}_0 + i \vec{B}_0 \tag{3.18}
\]
we get
\[ \vec{E}_0 = -\vec{W} \sin \Omega t, \quad \vec{B}_0 = \vec{W} \cos \Omega t \]  
(3.19)

with
\[ \vec{W} = C \left( \frac{\alpha \Omega y}{r^3} - \frac{\beta x z}{r^5}, -\frac{\alpha \Omega x}{r^3} - \frac{\beta y z}{r^5}, \frac{\beta (x^2 + y^2)}{r^5} - \frac{2\alpha}{r^3} \right). \]  
(3.20)

We verify that \( \text{div}\vec{W} = 0, \ \text{div}\vec{E}_0 = \text{div}\vec{B}_0 = 0, \ \text{rot}\vec{E}_0 + \partial \vec{B}_0/\partial t = 0, \ \text{rot}\vec{B}_0 - \partial \vec{E}_0/\partial t = 0, \) and
\[ \text{rot}\vec{W} = -\Omega \vec{W}. \]  
(3.21)

Now, it is well known that \( T_0 = \frac{1}{2} \tilde{\vec{F}} \gamma_0 F \) is the 1-form representing the energy density and the Poynting vector (Maia et al., 1990; Hestenes, 1966). It follows that \( \vec{E}_0 \times \vec{B}_0 = 0, \) i.e., the solution has zero angular momentum. The energy density \( u = S^{00} \) is given by
\[ u = \frac{1}{r^6} \left[ \sin^2 \theta (\Omega^2 r^2 \alpha^2 + \cos^2 \theta \beta^2) + (\beta \sin^2 \theta - 2\alpha)^2 \right] \]  
(3.22)

Then \( \iiint_{\mathbb{R}^3} u \, d\nu = \infty. \) In (Rodrigues and Lu, 1996) it is discussed how to generate finite energy solutions. It can be constructed by considering “wave packets” with a distribution of intrinsic frequencies \( F(\Omega) \) satisfying appropriate conditions. Many possibilities exist, but they will not be discussed here. Instead, we prefer to direct our attention to eq.(3.21). As it is well known, this is a very important equation (called the force free equation (Waite, 1995)) that appears e.g. in hydrodynamics and in several different situations in plasmaphysics (Reed, 1994). The following considerations are more important.

Einstein, among others, (see (Waite, 1995) for a review) studied the possibility of constructing PEPs. He started from Maxwell equations for a PEP configuration described by an electromagnetic field \( \vec{F}_p \) and a current density \( \vec{J}_p = \rho_p \gamma_0 + \vec{j}_p \gamma_1, \) where
\[ \Theta \vec{F}_p = \vec{J}_p \]  
(3.23)

and rightly concluded that the condition for existence of PEPs is
\[ \vec{J}_p . \vec{F}_p = 0. \]  
(3.24)

This condition implies, in vector notation,
\[ \rho_p \vec{E}_p = 0, \quad \vec{j}_p . \vec{E}_p = 0, \quad \vec{j}_p \times \vec{B}_p = 0. \]  
(3.25)

From eq.(3.25) Einstein concluded that the only possible solution of eq.(3.23) with the subsidiary condition given by eq.(3.24) is \( \vec{J}_p = 0. \) However, this conclusion is correct only if \( J^2_p > 0, \) i.e., if \( \vec{J}_p \) is a time-like current density. However, if we suppose that \( \vec{J}_p \) can be spacelike, i.e., \( J^2_p < 0, \) there exists a reference frame where \( \rho_p = 0 \) and a possible solution of eq.(3.24) is
\[ \rho_p = 0, \quad \vec{E}_p . \vec{B}_p = 0, \quad \vec{j}_p = k C \vec{B}_p, \]  
(3.26)
where $k = \pm 1$ is called the chirality of the solution and $C$ is a real constant. In (Waite, 1995; Waite et al., 1996) static solutions of eqs. (3.23) and (3.24) are exhibited where $\vec{E}_p = 0$. In this case we can verify that $\vec{B}_p$ satisfies

$$\nabla \times \vec{B}_p = kC\vec{B}_p. \quad (3.27)$$

Now, if we choose $F \in \sec \wedge^2(M) \subset \sec \mathcal{O}(M)$ such that

$$F_0 = \vec{E}_0 + i\vec{B}_0, \quad \vec{E}_0 = -\vec{B}_p \cos \Omega t, \quad \vec{B}_0 = \vec{B}_p \sin \Omega t, \quad (3.28)$$

we immediately realize that

$$\partial F_0 = 0. \quad (3.29)$$

This is an amazing result, since it means that the free Maxwell equations may have stationary solutions that model PEPs. In such solutions the structure of the field $F_0$ is such that we can write, e.g.,

$$F_0' = \vec{F}'_p + \vec{F} = i\vec{W} \cos \omega t - \vec{W} \sin \Omega t, \quad \partial F_0' = -\partial \vec{F} = J'_p, \quad (3.30)$$

i.e., $\partial F_0 = 0$ is equivalent to a field plus a current. This opens several interesting possibilities for modelling PEPs (see also (Waite, 1995; Waite et al., 1996)) and we discuss more this issue in another publication.

We observe that moving subluminal solutions of ME can be easily obtained choosing as Hertz potential, e.g.,

$$\pi^< (t, \vec{x}) = C \frac{\sin \Omega \xi_<}{\xi_<} e^{[\gamma_1(\omega_< t - k_< z)]} \gamma_1 \gamma_2 \quad (3.31)$$

$$\omega^2_< - k^2_< = \Omega^2_<$$

$$\xi_< = [x^2 + y^2 + \gamma^2_< (z - vt)^2]$$

$$\gamma_< = \frac{1}{\sqrt{1 - v^2_<}}, \quad v_<= d\omega_>/dk_<$$

We are not going to write explicitly the expression for $F^<$ corresponding to $\pi^<$ because it is very long and will not be used in what follows.

We end this section with the following observations: (i) In general for subluminal solutions of ME (SSME) the propagation vector satisfies an equation like eq(3.32). (ii) As can be easily verified, for a SSME the field invariants are non-null. (iii) A SSME is not a tranverse wave. This can be seen explicitly from eqs(3.19) and (3.20) and a special Lorentz transformation with parameter $v_<$. Conditions (i), (ii), (iii) are in contrast with the case of the PWS of ME. In (Vaz and Rodrigues, 1993; Vaz and Rodrigues, 1995) it is shown that for free electromagnetic fields ($\partial F = 0$) such that $F^2 \neq 0$, there exists a Dirac-Hestenes equation for $\psi \in \sec \wedge^0(M) + \wedge^2(M) + \wedge^4(M) \subset \sec \mathcal{O}(M)$ where $F = \psi \gamma_1 \gamma_2 \psi$. This was the reason why (Vaz and Rodrigues, 1995) discovered subluminal and superluminal solutions of Maxwell equations (and also of Weyl equations) which solve the Dirac-Hestenes equation.
3.3. The Superluminal Electromagnetic X-Wave (SEXW)

To simplify the matter in what follows we now suppose that the functions $\Phi_{X_n}$ and $\Phi_{XBB_n}$ below, which are superluminal solutions of the scalar wave equation (Lu and Greenleaf, 1992; Rodrigues and Lu, 1996), are 0-forms sections of the complexified Clifford bundle $\mathcal{AC}(M) = \mathcal{C} \otimes \mathcal{C}(M)$. Sections of $\mathcal{AC}(M)$ are like in eq. (2.1) with the coefficients $s, v_\mu, b_{\mu\nu} \ldots$ being complex functions. We have

$$\Phi_{X_n}(t, \vec{x}) = e^{in\theta} \int_0^\infty B(\vec{k}) J_n(\vec{k} \rho \sin \eta) e^{-i[a_0 - i(z \cos \eta - t)]} d\vec{k}$$  (3.33)

where $n = 0, 1, 2, \ldots$ and $\eta$ is a constant, called the axicon angle. Choosing $B(\vec{k}) = a_0$, we have

$$\Phi_{XBB_n}(t, \vec{x}) = a_0(\rho \sin \eta)^n e^{in\theta} \sqrt{M(\tau + \sqrt{M})^n}$$  (3.34)

$$M = (\rho \sin \eta)^2 + \tau^2; \quad \tau = [a_0 - i(z \cos \eta - t)].$$  (3.35)

Further, we suppose now that the Hertz potential $\pi$, the vector potential $A$ and the corresponding electromagnetic field $F$ are appropriate sections of $\mathcal{AC}(M)$. We take

$$\pi = \Phi_{XBB_n} \in \sec \mathcal{C} \otimes \bigwedge^2(\mathcal{M}) \subset \sec \mathcal{AC}(\mathcal{M}),$$  (3.36)

where $\Phi$ can be $\Phi_{X_n}$ or $\Phi_{XBB_n}$. Let us start by giving the explicit form of the $F_{XBB_n}$ i.e., the SEXWs. In this case writing $\pi = \pi_e + \mathbf{i} \pi_m$ eq. (3.36) gives $\pi = \pi_m$ and

$$\pi_m = \Phi_{XBB_n} \mathbf{z},$$  (3.37)

where $\mathbf{z}$ is the versor of the $z$-axis. Also, let $\rho, \theta$ be respectively the versors of the $\rho$ and $\theta$ directions where $(\rho, \theta, z)$ are the usual cylindrical coordinates.

Writing

$$F_{XBB_n} = \vec{E}_{XBB_n} + \gamma_5 \vec{B}_{XBB_n}$$  (3.38)

we obtain

$$\vec{E}_{XBB_n} = -\frac{\rho}{\rho \partial \rho \partial z} \Phi_{XBB_n} + \theta \frac{\partial^2}{\partial t \partial \rho} \Phi_{XBB_n}$$  (3.39a)

$$\vec{B}_{XBB_n} = \rho \frac{\partial^2}{\partial \rho \partial z} \Phi_{XBB_n} + \theta \frac{1}{\rho \partial \theta \partial z} \Phi_{XBB_n} + \frac{\partial^2}{\partial z \partial z} \Phi_{XBB_n} - \frac{\partial^2}{\partial t \partial t} \Phi_{XBB_n}$$  (3.39b)

Explicitly we get for the components in cylindrical coordinates,

$$(\vec{E}_{XBB_n})_\rho = -\frac{1}{\rho} \frac{n \rho}{\sqrt{M}} \Phi_{XBB_n}$$  (3.41a)

$$(\vec{E}_{XBB_n})_\theta = \frac{1}{\rho} \frac{M_3}{\sqrt{M}} \Phi_{XBB_n}$$  (3.41b)

$$(\vec{B}_{XBB_n})_\rho = \cos \eta (\vec{E}_{XBB_n})_\theta$$  (3.41c)

$$(\vec{B}_{XBB_n})_\theta = -\cos \eta (\vec{E}_{XBB_n})_\rho$$  (3.41d)

$$(\vec{B}_{XBB_n})_z = -\sin^2 \eta \frac{M_7}{\sqrt{M}} \Phi_{XBB_n}.$$  (3.41e)
The functions $M_i (i = 2, \ldots, 7)$ in (3.41) are

$$M_2 = \tau + \sqrt{M}$$  \hspace{1cm} (3.42a)

$$M_3 = n + \frac{1}{\sqrt{M}} \tau$$  \hspace{1cm} (3.42b)

$$M_4 = 2n + \frac{\sqrt{M}}{3} \tau$$  \hspace{1cm} (3.42c)

$$M_5 = \tau + n\sqrt{M}$$  \hspace{1cm} (3.42d)

$$M_6 = (\rho^2 \sin^2 \eta \frac{M_4}{M} - nM_3)M_2 + n\rho^2 \sin^2 \rho \frac{M_5}{M}$$  \hspace{1cm} (3.42e)

$$M_7 = (n^2 - 1) \frac{1}{\sqrt{M}} + 3n \frac{1}{M} \tau + 3 \frac{1}{\sqrt{M^3}} \tau^2$$  \hspace{1cm} (3.42f)

We immediately see from eq.(3.41) that the $F\times_{BB} n$ are indeed superluminal UPWs solutions of ME, propagating with speed $1/\cos \eta$ in the $z$-direction. That $F\times_{BB} n$ are undistorted progressive waves is trivial and that they propagate with speed $c_1 = 1/\cos \eta$ follows because $F\times_{BB} n$ depends only on the combination of variables $(z - c_1 t)$ and any derivatives of $\Phi_{BB} n$ will keep the $(z - c_1 t)$ dependence structure. More details can be found in (Rodrigues and Lu, 1996) where we show how to generate finite aperture approximations to the SEXWs. In particular, the name $X$-wave is due to the shape of these waves.

### 4. The Equivalence Between Maxwell and Dirac Equations

Let us consider the generalized Maxwell equations

$$\partial F = J,$$  \hspace{1cm} (4.1)

where $\partial = \gamma^\mu \partial_\mu$ is the Dirac operator and $J$ is the electromagnetic current (an electric current $J_e$ plus a magnetic monopole current $-\gamma^5 J_m$, where $J_c, J_m \in \sec \Lambda^1 M \subset \mathcal{O}(M)$). In (Vaz and Rodrigues, 1993; Vaz and Rodrigues, 1995) we proved that $F$ can be written as

$$F = \psi_{\gamma_{21}} \tilde{\psi},$$  \hspace{1cm} (4.2)

where $\psi \in \sec \mathcal{O}^+(M)$ is a Dirac-Hestenes spinor field. If we use eq.(4.2) in eq.(4.1) we get

$$\partial (\psi_{\gamma_{21}} \tilde{\psi}) = \gamma^\mu \partial_\mu (\psi_{\gamma_{21}} \tilde{\psi}) = \gamma^\mu (\partial_\mu \psi_{\gamma_{21}} \tilde{\psi} + \psi_{\gamma_{21}} \partial_\mu \tilde{\psi}) = J.$$  \hspace{1cm} (4.3)

But $\psi_{\gamma_{21}} \partial_\mu \tilde{\psi} = - (\partial_\mu \psi_{\gamma_{21}} \tilde{\psi})^\dagger$, and since reversion does not change the sign of scalars and of pseudo-scalars (4-vectors), we have that

$$2\gamma^\mu (\partial_\mu \psi_{\gamma_{21}} \tilde{\psi})_2 = J.$$  \hspace{1cm} (4.4)

There is a more convenient way of rewriting the above equation. Note that

$$\gamma^\mu (\partial_\mu \psi_{\gamma_{21}} \tilde{\psi})_2 = \partial \psi_{\gamma_{21}} \tilde{\psi} - \gamma^\mu (\partial_\mu \psi_{\gamma_{21}} \tilde{\psi})_0 - \gamma^4 (\partial_\mu \psi_{\gamma_{21}} \tilde{\psi}) - 4,$$  \hspace{1cm} (4.5)

and if we define the vectors

$$j = \gamma^\mu (\partial_\mu \psi_{\gamma_{21}} \tilde{\psi})_0,$$  \hspace{1cm} (4.6)
\[ g = \gamma^\mu (\partial_\mu \psi \gamma_5 \gamma_{21} \tilde{\psi})_0, \quad (4.7) \]

we can rewrite eq.(4.4) as
\[ \partial \psi \gamma_{21} \tilde{\psi} = \left[ \frac{1}{2} J + (j + \gamma_5 g) \right] \psi. \quad (4.8) \]

Eq.(4.8) is a spinorial representation of Maxwell equations. In the case where \( \psi \) is non-singular (which corresponds to non-null electromagnetic fields) we have
\[ \partial \psi \gamma_{21} = \frac{e^{\gamma_5 \beta}}{\rho} \left[ \frac{1}{2} J + (j + \gamma_5 g) \right] \psi. \quad (4.9) \]

Eq.(4.9) has been proved (Vaz and Rodrigues, 1993) to be equivalent to the spinorial representation of Maxwell equations obtained originally by (Campolattaro, 1990) in terms of the usual covariant Dirac spinor field.

The spinorial equation (4.3) representing Maxwell equations, written in that form, does not appear to have any relationship with Dirac-Hestenes equation \( 2.7 \).

However, we shall make some modifications on it in such a way as to put it in a form that suggests a very interesting and intriguing relationship between them, and consequently between electromagnetism and quantum mechanics.

Since \( \psi \) is supposed to be non-singular (\( F \) non-null) we can use the canonical decomposition of \( \psi \) and write \( \psi = \rho e^{\beta \gamma_5/2} R \), with \( \rho, \beta \in \text{sec} \bigwedge^0 M \subset \text{sec} \mathcal{O}(M) \) and \( R \in \text{Spin}_+(1,3) \forall x \in M \). Then
\[ \partial_\mu \psi = \frac{1}{2} (\partial_\mu \ln \rho + \gamma_5 \partial_\mu \beta + \Omega_\mu) \psi, \quad (4.10) \]

where we defined
\[ \Omega_\mu = 2(\partial_\mu R) \tilde{R}. \quad (4.11) \]

Using this expression for \( \partial_\mu \psi \) into the definitions of the vectors \( j \) and \( g \) (eqs.(4.6,4.7)) we obtain that
\[ j = \gamma^\mu (\Omega_\mu \cdot S) \rho \cos \beta + \gamma_\mu \Omega_\mu \cdot (\gamma_5 S) \rho \sin \beta, \quad (4.12) \]
\[ g = \gamma^\mu [(\Omega_\mu \cdot (\gamma_5 S)] \rho \cos \beta - \gamma_\mu (\Omega_\mu \cdot S) \rho \sin \beta, \quad (4.13) \]

where we defined the bivector \( S \) by
\[ S = \frac{1}{2} \psi \gamma_{21} \psi^{-1} = \frac{1}{2} R \gamma_{21} \tilde{R}. \quad (4.14) \]

A more convenient expression can be written. Let \( v \) be given by \( \rho v = J = \psi \gamma_0 \tilde{\psi} \), and \( v_\mu = v \cdot \gamma_\mu \). Define the bivector \( \Omega = v^\mu \Omega_\mu \) and the scalars \( \Lambda \) and \( K \) by
\[ \Lambda = \Omega \cdot S, \quad (4.15) \]
\[ K = \Omega \cdot (\gamma_5 S). \quad (4.16) \]

Using these definitions we have that
\[ \Omega_\mu \cdot S = \Lambda v_\mu, \quad (4.17) \]
\[ \Omega_\mu \cdot (\gamma_5 S) = Kv_\mu, \]  
(4.18)

and for the vectors \( j \) and \( g \):

\[ j = \Lambda v \rho \cos \beta + K v \rho \sin \beta = \lambda v \rho, \]  
(4.19)

\[ g = K v \rho \cos \beta - \Lambda v \rho \sin \beta = \kappa v \rho, \]  
(4.20)

where we defined

\[ \lambda = \Lambda \cos \beta + K \sin \beta, \]  
(4.21)

\[ \kappa = K \cos \beta - \Lambda \sin \beta. \]  
(4.22)

The spinorial representation of Maxwell equations is written now as

\[ \partial \psi_{\gamma 21} = \frac{e^{\gamma_5 \beta}}{2 \rho} J \psi + \lambda \psi_{\gamma 0} + \gamma_5 \kappa \psi_{\gamma 0}. \]  
(4.23)

If \( J = 0 \) (free case) we have that

\[ \partial \psi_{\gamma 21} = \lambda \psi_{\gamma 0} + \gamma_5 \kappa \psi_{\gamma 0}, \]  
(4.24)

which is very similar to the Dirac-Hestenes equation (2.7).

In order to go a step further into the relationship between those equations, we remember that the electromagnetic field has six degrees of freedom, while a Dirac-Hestenes spinor field has eight degrees of freedom; we are free therefore to impose two constraints on \( \psi \) if it is to represent an electromagnetic field. We choose these two constraints as

\[ \partial \cdot j = 0 \quad \text{and} \quad \partial \cdot g = 0. \]  
(4.25)

Using eqs. (4.19, 4.20) these two constraints become

\[ \partial \cdot j = \rho \dot{\lambda} + \lambda \partial \cdot J = 0, \]  
(4.26)

\[ \partial \cdot g = \rho \dot{\kappa} + \kappa \partial \cdot J = 0, \]  
(4.27)

where \( J = \rho v \) and \( \dot{\lambda} = (v \cdot \partial) \lambda, \ \dot{\kappa} = (v \cdot \partial) \kappa. \) These conditions imply that

\[ \kappa \dot{\lambda} = \lambda \dot{\kappa}, \]  
(4.28)

which gives \( \lambda \neq 0 \):

\[ \frac{\kappa}{\lambda} = \text{const.} = - \tan \beta_0, \]  
(4.29)

or from eqs. (4.21, 4.22):

\[ \frac{K}{\Lambda} = \tan (\beta - \beta_0). \]  
(4.30)

Now we observe that \( \beta \) is the angle of the duality rotation from \( F \) to \( F' = e^{\gamma_5 \beta} F \). If we perform another duality rotation by \( \beta_0 \) we have \( F \mapsto e^{\gamma_5 (\beta + \beta_0)} F \), and for the Yvon-Takabayasi angle \( \beta \mapsto \beta + \beta_0 \). If we work therefore with an electromagnetic field duality rotated by an additional angle \( \beta_0 \), the above relationship becomes

\[ \frac{K}{\Lambda} = \tan \beta. \]  
(4.31)
This is, of course, just a way to say that we can choose the constant $\beta_0$ in eq. (4.29) to be zero. Now, this expression gives

$$\lambda = \Lambda \cos \beta + \Lambda \tan \beta \sin \beta = \frac{\Lambda}{\cos \beta}, \quad (4.32)$$

$$\kappa = \Lambda \tan \beta \cos \beta - \Lambda \sin \beta = 0, \quad (4.33)$$

and the spinorial representation (4.24) of the free Maxwell equations becomes

$$\partial \psi_{\gamma 21} = \lambda \psi_{\gamma 0}. \quad (4.34)$$

Note that $\lambda$ is such that

$$\rho \dot{\lambda} = -\lambda \partial \cdot J. \quad (4.35)$$

The current $J = \psi_{\gamma 0} \tilde{\psi}$ is not conserved unless $\lambda$ is constant. If we suppose also that

$$\partial \cdot J = 0 \quad (4.36)$$

we must have

$$\lambda = \text{const.} \quad (4.37)$$

Now, throughout these calculations we have assumed $\hbar = c = 1$. We observe that in eq. (4.34) $\lambda$ has the units of $(\text{length})^{-1}$, and if we introduce the constants $\hbar$ and $c$ we have to introduce another constant with unit of mass. If we denote this constant by $m$ such that

$$\lambda = mc \hbar, \quad (4.38)$$

then eq. (4.34) assumes a form which is identical to Dirac equation:

$$\partial \psi_{\gamma 21} = mc \hbar \psi_{\gamma 0}. \quad (4.39)$$

It is true that we didn’t prove that eq. (4.39) is really Dirac equation since the constant $m$ has to be identified in this case with the electron’s mass. However in (Vaz and Rodrigues, 1995) we present several arguments based on the stochastic interpretation of quantum mechanics which suggest that we must take the identification seriously.

5. Some Properties of the Linear and Non-Linear Dirac-Hestenes Equation

5.1. Projection Operators and Energy-Momentum Tensors

We recall that the free DHE

$$\partial \psi_{\gamma 12} + m \psi_{\gamma 0} = 0 \quad (5.1)$$

has the following plane wave solutions:
\[ \psi_{\uparrow}^{(+)} = \sqrt{\rho} e^{-\gamma_{21}m t} ; \]
\[ \psi_{\downarrow}^{(+)} = \sqrt{\rho} \gamma_{31} e^{-\gamma_{21}m t} ; \]
\[ \psi_{\uparrow}^{(-)} = \sqrt{\rho} \gamma (\gamma_{12} e^{\gamma_{21}m t}) ; \]
\[ \psi_{\downarrow}^{(-)} = \sqrt{\rho} \gamma \gamma_{31} (\gamma_{12} e^{\gamma_{21}m t}) . \]

(5.2)

Observe that for the solutions (+) we have \( \beta = 0 \), and for the solutions (−) we have \( \beta = \pi \).

The solutions (+) [(−)] are solutions of positive [negative] energy and the arrows indicate spin-up (↑) and spin-down (↓). To see this we consider the energy projection operators \( \Lambda_\pm \) (Lounesto, 1993; Lounesto, 1994):

\[ \Lambda_\pm (\psi) = \frac{1}{2} [\psi \pm \gamma_0 \psi \gamma_0] \]

(5.3)

and the spin projector operators \( \Sigma_\pm \):

\[ \Sigma_\pm (\psi) = \frac{1}{2} [\psi \pm \gamma_{21} \psi \gamma_{21}] . \]

(5.4)

It is easy to see that \( \Lambda_\pm^2 = \Lambda_\pm \), \( \Lambda_\pm \Lambda_\pm = 0 \), \( \Lambda_+ + \Lambda_- = 1 \) and \( \Sigma_\pm^2 = \Sigma_\pm \), \( \Sigma_\pm \Sigma_\pm = 0 \), \( \Sigma_+ + \Sigma_- = 1 \). Then,

\[ \Lambda_+ \Sigma_+ \psi_{\uparrow}^{(+)} = \Sigma_+ \Lambda_+ \psi_{\uparrow}^{(+)} = \psi_{\uparrow}^{(+)} ; \]
\[ \Lambda_+ \Sigma_- \psi_{\downarrow}^{(+)} = \Sigma_- \Lambda_+ \psi_{\uparrow}^{(+)} = \psi_{\downarrow}^{(+)} ; \]
\[ \Lambda_- \Sigma_+ \psi_{\uparrow}^{(-)} = \Sigma_+ \Lambda_- \psi_{\downarrow}^{(-)} = \psi_{\uparrow}^{(-)} ; \]
\[ \Lambda_- \Sigma_- \psi_{\downarrow}^{(-)} = \Sigma_- \Lambda_- \psi_{\downarrow}^{(-)} = \psi_{\downarrow}^{(-)} . \]

(5.5)

The fact that the Takabayasi angle \( \beta \) is 0 or \( \pi \) for these solutions is very interesting and a mystery. We already saw that the Takabayasi angle appears in the NLDHE, which is equivalent to Maxwell equation, and we are going to disclose some of the secrets of the variable \( \beta \) in what follows.

Before going on we recall that the solutions of the Dirac equation (coupled with the electromagnetic field) for the hydrogen atom are solutions with variable \( \beta \), i.e. \( \beta = \beta(x) \) (Quilichini, 1971). But it is very interesting that (Krüger, 1991) obtained solutions for the hydrogen atom with \( \beta = 0 \) or \( \pi \).

We return now to our NLDHE (eq.(4.9)) which we can also write as

\[ \partial \psi \gamma_{21} = \Lambda \psi \gamma_0 e^{\beta \gamma_5} + \gamma_5 K \psi \gamma_0 e^{\beta \gamma_5} + \frac{1}{2 \rho} e^{\beta \gamma_5} \mathcal{J} \psi . \]

(5.6)

For \( \mathcal{J} = 0 \) and \( K = 0 \) we have

\[ \partial \psi \gamma_{21} = \Lambda \psi \gamma_0 e^{\beta \gamma_5} , \]

(5.7)

which has been studied extensively by (Daviau, 1993), but in a context different from the present one. Note that this result is consistent with eq.(4.39) if \( \beta = 0 \) or \( \beta = \pi \).

Equation (5.7) is non linear. Except for the term \( e^{\beta \gamma_5} \), which introduces the non-linearity, this equation is like the DHE if \( \Lambda = m \). We briefly discuss, before
arriving at the main topics of this section, the fact that eq. (5.7) is more satisfactory than the DHE.

To start with, we know (Hestenes, 1973) that the energy-momentum operator in the Clifford bundle formalism is given by

$$\hat{p}\psi = \partial\psi\gamma_{21}. \quad (5.8)$$

Since \(\gamma_0\partial = \partial_0 + \nabla\), \(\gamma_0p = p_0 - \vec{p}\), we have

$$\hat{E}\psi = i\partial_0\psi, \quad \hat{p}\psi = -i\nabla\psi, \quad (5.9)$$

which are the usual definitions of the operators \(\hat{E}\) and \(\hat{p}\). We can then write eq. (5.7) as

$$\hat{p}\psi = mc\psi\gamma_0e^{\beta\gamma_5} \quad (5.10)$$

and if \(\psi\) is a nonsingular eigenvector of \(\hat{p}\) we have, after multiplying eq. (5.10) by \(\psi^{-1}\), that

$$p = mv, \quad (5.11)$$

with \(v = \frac{1}{\rho}\psi\gamma_0\tilde{\psi}\).

On the other hand, writing the DHE (eq. (5.1)) as

$$\hat{p}\psi = m\psi\gamma_0 \quad (5.12)$$

we get for the nonsingular eigenvectors of \(\hat{p}\) that

$$p = e^{\beta\gamma_5}mv. \quad (5.13)$$

Since \(p\) and \(v\) are 1-forms, eq. (5.1) implies that \(\beta = 0\) or \(\beta = \pi\), and we get

$$p = \pm mv, \quad (5.14)$$

i.e., the case \(\beta = \pi\) introduces a negative sign without any obvious physical meaning. Equation (5.13) implies that in order to make sense the DHE for a free particle must have \(\beta = 0\) or \(\beta = \pi\), and we have the problem of interpreting the negative energy. The known solution is to say that such solutions correspond to antiparticles.

Now, the NLDHE (5.7) does not present any problem, because\( p = mv\) for whatever angle \(\beta\). The following point concerning eq. (5.7) is important. The superposition principle of quantum mechanics remains valid if we only superpose solutions with the same \(\beta\). This means that fixed values of \(\beta\) imply in a superselection rule. To understand this fact we write \(\psi = \phi e^{\beta\gamma_5}\) and supposing \(\beta = \text{constant}\) eq. (5.7) becomes

$$\partial\phi\gamma_{21} = m\phi\gamma_0, \quad (5.15)$$

which is the DHE for \(\phi = \sqrt{p}R\). Then, the solutions of eq. (5.15) for \(\phi\) are identical to the solutions of the DHE for \(\beta = 0\). We have the solutions
\[ \psi^\uparrow = \sqrt{\rho e^{\beta \gamma_5/2} e^{-\gamma_21 mt}} ; \]  
\[ \psi^\downarrow = \sqrt{\rho e^{\beta \gamma_5/2} \gamma_{31} e^{-\gamma_21 mt}} ; \]  

(5.16)  
(5.17)

where \( \beta \) is arbitrary but constant. Since \( \gamma_5 \) commutes with \( \gamma_{21} \), it is obvious that these solutions are eigenvectors of the spin projector operators \( \Sigma_{\pm} \) (eq.(5.4)). On the other hand, equations (5.16) and (5.17) are not eigenvectors of the energy projector operators \( \Lambda_{\pm} \) (eq.(5.3)). Indeed, we have

\[ \Lambda_+ (\psi^\uparrow \downarrow) = \left[ \cos^2 \frac{\beta}{2} - \frac{1}{2} \gamma_5 \sin \beta \right] \psi^\uparrow \downarrow ; \]  
(5.18)

\[ \Lambda_- (\psi^\uparrow \downarrow) = \left[ \gamma_5 \sin \beta + \sin^2 \frac{\beta}{2} \right] \psi^\uparrow \downarrow . \]  
(5.19)

It follows that \( \psi^\uparrow \downarrow \) are eigenvectors of \( \Lambda_{\pm} \) only for \( \beta = 0 \) or \( \beta = \pi \) and in these cases we have

\[ \Lambda_+ (\psi^\uparrow \downarrow) = \begin{cases} \psi^\uparrow \downarrow (\beta = 0) \\ 0 (\beta = \pi) \end{cases} \]  
(5.20)

\[ \Lambda_- (\psi^\uparrow \downarrow) = \begin{cases} 0 (\beta = 0) \\ \psi^\uparrow \downarrow (\beta = \pi) \end{cases} \]  
(5.21)

and we then can write

\[ \psi^{(+)} \uparrow = \sqrt{\rho e^{-\gamma_21 mt}} ; \]  
\[ \psi^{(+)} \downarrow = \sqrt{\rho \gamma_{31} e^{-\gamma_21 mt}} ; \]  
\[ \psi^{(-)} \uparrow = \sqrt{\rho \gamma_5 e^{-\gamma_21 mt}} ; \]  
\[ \psi^{(-)} \downarrow = \sqrt{\rho \gamma_5 \gamma_{31} e^{-\gamma_21 mt}} . \]  
(5.22)

The interesting point concerning these solutions is that here the interpretation of \( \Lambda_{\pm} \) as energy projector operators has no meaning. Indeed, if we recall that the Tetrode (energy-momentum 1-form) for the NLDHE is equal to the same tensor for the DHE (Vaz, 1993), i.e.

\[ T_\mu = \langle \partial \psi \gamma_210 \tilde{\psi} \gamma_\mu \rangle , \]  
(5.23)

we have for all solutions (5.22) that \( T = T_\mu^\mu = T_\mu \gamma^\mu > 0 \). Indeed,

\[ T = \langle \partial \psi \gamma_210 \tilde{\psi} \rangle_0 , \]  
(5.24)

and using eq.(5.23) we find

\[ T = m \langle \psi \tilde{\psi} e^{-\gamma_5 \beta} \rangle_0 = m \rho . \]  
(5.25)

On the other hand, in the case of the DHE the trace of the Tetrode tensor is

\[ T_{\text{Dirac}} = m \langle \psi \tilde{\psi} \rangle = m \rho \cos \beta \]  
(5.26)

and we have
\( T_{\text{Dirac}} = T \cos \beta. \) (5.27)

We see that according to the NLDHE all solutions given by eq.5.22 have positive energy and so in this theory it is a nonsense to interpret \( \Lambda_{\pm} \) as energy projector operators. In (Vaz, 1993) it is shown that \( T_{\mu} \) can be interpreted as the "extremal square root" of the Maxwell energy-momentum 1-form, \( S_{\mu} = \frac{1}{2} F_{\mu
u} F^{\nu} \), which is a very interesting result.

Since we cannot interpret \( \Lambda_{\pm} \) as energy projector operators for the NLDHE we propose to interpret them as operators of particle (\( \Lambda_{+} \)) and antiparticle (\( \Lambda_{-} \)). In this way the superposition principle continues to hold good only for solutions with the same \( \beta \). Explicitly this means that the superposition principle holds for particles and for antiparticles separately, where \( \beta = 0 \) refers to particles and \( \beta = \pi \) refers to antiparticles. In this way the transformation \( \beta \mapsto \beta + \pi \) can be interpreted as transforming particle in antiparticle and vice-versa.

To sum up, we saw that for consistence the DHE implies \( \beta = 0 \) or \( \beta = \pi \) for a free particle. On the other hand in the NLDHE \( \beta = 0 \) and \( \beta = \pi \) appear as conditions for \( \psi \) to be an eigenfunction of \( \Lambda_{\pm}(\psi) = 1/2(\psi + \gamma_{0}\psi\gamma_{0}) \). This suggests to us to search for a new projection operator which imply in other possible values for \( \beta \) and in this case these different values may eventually be associated with the other known leptons. The new projection operator \( \Lambda_{\beta} \) is introduced by

\[
\Lambda_{\beta}(\psi) = \frac{1}{2}(\psi + e^{\gamma_{5}\beta\gamma_{0}\psi\gamma_{0}}). \tag{5.28}
\]

Then

\[
\Lambda_{+} = \Lambda_{\beta=0} \text{ and } \Lambda_{-} = \Lambda_{\beta=\pi}. \tag{5.29}
\]

It is easy to verify that \( \Lambda_{\beta} \) is indeed a projection operator for all \( \beta \) because

\[
\Lambda_{\beta}^{2} = \Lambda_{\beta} , \quad \Lambda_{\beta} = \Lambda_{\beta+\pi} , \quad \Lambda_{\beta} \Lambda_{\beta+\pi} = \Lambda_{\beta+\pi} \Lambda_{\beta} = 0. \tag{5.30}
\]

5.2. Gauge Invariance of the Dirac-Hestenes Equation

Consider the Dirac-Hestenes field in interaction with the electromagnetic field \( A \in \text{sec} \bigwedge^{1}(M) \subset \text{sec} \mathcal{O}(M) \), which is described by

\[
\partial \psi\gamma_{21} + m\psi\gamma_{0} + eA\psi = 0. \tag{5.31}
\]

It is well known that this equation is invariant under the following gauge transformations:

\[
\psi \mapsto \psi \exp(e^{\gamma_{21}\theta}) ; \quad A \mapsto A + \partial \theta ; \tag{5.32}
\]

where \( \theta \in \text{sec} \bigwedge^{0}(M) \subset \text{sec} \mathcal{O}(M) \).

Now let \( \psi_{+} \) and \( \psi_{-} \) be Weyl spinor fields, i.e. \( \gamma_{5}\psi_{+} = +\psi_{+}\gamma_{21} , \gamma_{5}\psi_{-} = -\psi_{-}\gamma_{21} \). Then \( \psi_{+} \) and \( \psi_{-} \) satisfy the Weyl equation

\[
\partial \psi_{\pm} = 0. \tag{5.33}
\]
Consider the equation for $\psi_+$ coupled with an electromagnetic potential $B$, i.e.

$$\partial \psi_+ \gamma_{21} + gB \psi_+ = 0 \ .$$  \hspace{1cm} (5.34)

This equation is invariant under the gauge transformations

$$\psi_+ \mapsto \psi_+ \exp(g\gamma_5 \theta) \ ; \ B \mapsto B + \partial \theta \ .$$ \hspace{1cm} (5.35)

On the other hand, the equation for $\psi_-$,

$$\partial \psi_- \gamma_{21} + gB \psi_- = 0 \ ,$$ \hspace{1cm} (5.36)

is invariant under the gauge transformations

$$\psi_- \mapsto \psi_- \exp(-g\gamma_5 \theta) \ ; \ B \mapsto B + \partial \theta \ .$$ \hspace{1cm} (5.37)

Consider now the Dirac-Hestenes spinor fields (eqs. 2.16)

$$\psi^\uparrow = \gamma_0 \psi_0 - \psi^- = \psi^\uparrow_+ - \psi^\uparrow^- \ ;$$
$$\psi^\downarrow = \psi_+ + \gamma_0 \psi_0 = \psi^\downarrow_+ - \psi^\downarrow^- \ .$$ \hspace{1cm} (2.16')

which satisfy $P \psi^\uparrow = \psi^\uparrow$, $P \psi^\downarrow = -\psi^\downarrow$. When $\psi^\uparrow$ is coupled with an electromagnetic potential $B$ we have

$$\partial \psi^\uparrow \gamma_{21} + gB \psi^\uparrow = 0 \ ,$$ \hspace{1cm} (5.38)

which decouples in

$$\partial \psi^\uparrow_+ + g\gamma_5 B \psi^\uparrow_+ = 0 \ ; \ \partial \psi^\uparrow_- - g\gamma_5 B \psi^\uparrow_- = 0 \ .$$ \hspace{1cm} (5.39)

These results show clearly that $\psi^\uparrow$ describes a pair of particles with charges $+g$ ($\psi^\uparrow_+$) and $-g$ ($\psi^\uparrow_-$). We will call such particles magnetic monopoles following the important work of (Lochak, 1985), where similar ideas have been introduced.

Now, writing $\psi_- = -S + F - \gamma_5 P$ we have

$$\psi^\uparrow = \gamma_0 (-S + F - \gamma_5 P) \gamma_0 - (-S + F - \gamma_5 P) \ .$$ \hspace{1cm} (5.40)

Then from the fact that $\partial \psi^\uparrow = 0$ we get

$$\partial F^\uparrow = -\gamma_5 \partial P \ ; \ F^\uparrow = \frac{1}{2} (\gamma_0 F \gamma_0 - F) \ ;$$ \hspace{1cm} (5.41)

which means that $\partial \psi^\uparrow = 0$ is equivalent to a Maxwell equation with magnetic current $J_m = \partial P$, i.e.

$$\partial F^\uparrow = -\gamma_5 J_m \ .$$ \hspace{1cm} (5.42)
6. The Lepton Mass Spectrum

We can now present our theory of the Lepton Mass Spectrum. We start by supposing that the electron corresponds to a free electromagnetic field configuration \( F_e \), \( \partial F_e = 0 \) and \( F_e^2 \neq 0 \), such that for

\[
F_e = \psi_e \gamma_{21} \psi_e
\]  

(6.1)

we have

\[
\partial \psi_e \gamma_{21} + m \psi_e \gamma_{21} = 0,
\]  

(6.2)

which follows from the NLDHE once \( \beta = 0 \). (For the positron we have \( \beta = \pi \).)

We now imagine that the other leptons are also electromagnetic excitations. Since it is a known fact that, e.g. the muon decays according to \( \mu \rightarrow e + \nu + \bar{\nu} \) and that the \( \mu \) has the same intrinsic parity as \( e^- \), we develop the following idea. We regard the muon as the electromagnetic configuration \( F_e + F_\uparrow = F \). From now on we must use correct gaussian units as discussed above and we must pay attention to the dimensions of the physical quantities. We use \([X]\) as meaning the physical dimension of a given quantity \( X \). \( L \) means the length unit, \( T \) the time unit, \( M \) the mass unit and \( Q \) the charge unit. In gaussian units \([F] = QL^2\). Then, when we write \( F = \psi \gamma_{21} \psi \) we are taking \([\psi] = (QL^2)^{1/2}\). But usually \([\psi] = L^{-3/2}\). Now, \([F] = QL^{-2} = Q(ML^2T^{-1})M^{-1}(LT^{-1})^{-1}L^{-3}\). Since \([e] = Q, [h] = (ML^2T^{-1}), [m] = M \) and \([c] = LT^{-1}\) we see that if we take

\[
F = k \frac{\hbar}{mc} \psi \gamma_{21} \tilde{\psi},
\]  

(6.3)

where \( k \) is a numerical constant and \( m \) is the electron mass, the units of \( \psi \) result equal to \( L^{-3/2} \). We show below that fixing \( k = \frac{2\pi}{3} \) we can get the muon mass spectrum. We take \( \psi_\uparrow \) with the dimensions of an electromagnetic field.

If gaussian units are used we must write \( \partial F = (4\pi/c)J \). Then using eq.(5.42)

we have

\[
\partial F = \frac{4\pi}{c} J = -\frac{4\pi}{c} \gamma_5 J_m.
\]  

(6.4)

The corresponding NLDHE for \( \psi \) is obtained as in section 4. We get

\[
\partial \psi \gamma_{21} = \frac{K_1 c}{h} \psi_\gamma_0 e^{\beta \gamma_5} + \gamma_5 \frac{K_2 c}{h} \psi_\gamma_0 e^{\beta \gamma_5} + \frac{e^{\beta \gamma_5}}{\rho} \left( \frac{3m}{\epsilon \hbar} \right) J \psi.
\]  

(6.5)

Writing

\[
\psi = e^{\gamma_5 \beta / 2} \phi,
\]  

(6.6)

where \( \phi = \sqrt{\rho} R \) and supposing \( \beta = \) constant we have

\[
\partial \phi \gamma_{21} = \frac{K_1 c}{h} \phi \gamma_0 + \gamma_5 \frac{K_2 c}{h} \phi \gamma_0 + \frac{e^{\gamma_5 \beta}}{\rho} \left( \frac{3m}{\epsilon \hbar} \right) J \phi
\]  

(6.7)

and differently from the case \( J = 0 \) the factor \( e^{\beta \gamma_5} \) is not factored out. Now
$e^{\gamma \beta} J = \sin \beta J_m - \gamma_5 \cos \beta J_m$. \hfill (6.8)

For the free solution $\partial F = 0$ we arrived at $\partial \phi \gamma_{21} + (mc/\hbar)\gamma_0 = 0$, which in (6.7) means to make $K_2 = 0$ (as already explained) and to put $K_1 = m$, the electron mass.

We look now for solutions of (6.7) such that

$J_m = cq_m \phi_0 \tilde{\phi}$ \hfill (6.9)

where $q_m$ denotes the “magnetic” charge. Using eqs. (6.8) and (6.9) in (6.5) we get

$\partial \phi \gamma_{21} = \frac{Mc}{\hbar} \phi_0 + \frac{Nc}{\hbar} \phi_0$, \hfill (6.10)

which is analogous to eq.(4.24) but where now

$M = m + \frac{3m}{e} q_m \sin \beta$; \hfill (6.11)

$N = -\frac{3m}{e} q_m \cos \beta$. \hfill (6.12)

This equation reduces to a DHE if $N = 0$ and we have

$\partial \phi \gamma_{21} = \frac{Mc}{\hbar} \phi_0$. \hfill (6.13)

If $N = 0$ we must have

$q_m \cos \beta = 0 \Rightarrow \beta = \frac{\pi}{2}, \beta = \frac{3\pi}{2}$. \hfill (6.14)

(Before proceeding we suppose that to $\beta = \pi/2$ is associated $+q_m$ and to $\beta = 3\pi/2$ is associated $-q_m$.)

With eq.(6.14) we get

$M = m + 3m \frac{q_m}{e}$. \hfill (6.15)

It can be shown (Lochak, 1985) that the monopole theory developed in section 5.2 implies in Dirac’s quantization condition:

$e q_n m = \frac{n}{2}$, $n$ integer. \hfill (6.16)

where $e$ is the electric charge of the electron. Then

$q_m^n = \frac{e}{2\alpha} n$ \hfill (6.17)

where $\alpha = e^2/\hbar c$ is the fine structure constant. Taking $n = 1$ we have

$M = \left(1 + \frac{3}{2\alpha}\right) m$, \hfill (6.18)
which gives $M = 206.55 m = 105.5 \text{ MeV}$. The muon mass is $m_\mu = 206.70 m$. We see that our theory gives an excellent result.

For $\beta = \pi/2$ or $\beta = 3\pi/2$ we have the following solutions:

\begin{align*}
\psi^{(+)}_{\uparrow} &= \sqrt{\rho} e^{\gamma \pi/4} e^{-\gamma_2 Mct/\hbar}; \\
\psi^{(+)}_{\downarrow} &= \sqrt{\rho} e^{\gamma \pi/4} \gamma e^{-\gamma_2 Mct/\hbar}; \\
\psi^{(-)}_{\uparrow} &= \sqrt{\rho} e^{\gamma 3\pi/4} e^{-\gamma_2 Mct/\hbar}; \\
\psi^{(-)}_{\downarrow} &= \sqrt{\rho} e^{\gamma 3\pi/4} (\gamma e^{-\gamma_2 Mct/\hbar}).
\end{align*}

(6.19)

where the solutions with index $ (+ )$ are eigenspinors of $\Lambda_{\beta=\pi/2}$ and the solutions with index $ (- )$ are eigenspinors of $\Lambda_{\beta=3\pi/2}$, i.e.

\begin{align*}
\Lambda_{\beta=\pi/2}(\psi^{(+)}_{\uparrow\downarrow}) &= \psi^{(+)}_{\uparrow\downarrow}; \\
\Lambda_{\beta=3\pi/2}(\psi^{(-)}_{\uparrow\downarrow}) &= \psi^{(-)}_{\uparrow\downarrow}.
\end{align*}

(6.20)

In this way we may take the projector operators $\Lambda_\beta$ with the appropriate values of $\beta$ as the projectors of electron, positron, muon and antimuon, i.e.

\begin{align*}
\Lambda_{\beta=0} &= \Lambda_e^-; \\
\Lambda_{\beta=\pi} &= \Lambda_e^+; \\
\Lambda_{\beta=\pi/2} &= \Lambda_\mu^-; \\
\Lambda_{\beta=3\pi/2} &= \Lambda_\mu^+.
\end{align*}

(6.21)

At this point the question that naturally arises is the following: May the above theory give the mass of other leptons, in particular the mass of the Tau? This can be done using an *ad hoc* hypothesis. We suppose that the Tau is made of an excited state of the muon. Proceeding as before we arrive at a new equation for $M$, which we now write $\overline{M}$:

\begin{equation}
\overline{M} = \left(1 + \frac{3}{2\alpha}\right) m + 3m \frac{q^n}{e}.
\end{equation}

(6.22)

In this equation, supposing $n = 2^4 = 16$ in the Dirac quantization condition, we have

\begin{equation}
\overline{M} = \left(1 + \frac{3}{2\alpha}\right) m + \frac{3}{2\alpha} 2^4 m = \left(1 + 17 \frac{3}{2\alpha}\right) m = 3845m = 1785 \text{ MeV}/c^2,
\end{equation}

(6.23)

which is a good approximation for the Tau mass. The hypothesis $n = p^4$ ($p$ integer) in the formula for the possible values of $q^n_m$ leads to the spectrum

\begin{equation}
M_p = m + \frac{3}{2\alpha} m \sum_{l=0}^{p} l^4,
\end{equation}

(6.24)

which corresponds to a formula found by (Barut, 1980) based on arguments completely different from the ones presented above. From eq. (6.24) we obtain $M_0 = m$, $M_1 = m_\mu$, $M_2 = m_\tau$ and $M_3 = 20090m = 10291 \text{ MeV}/c^2$ and such a lepton has not been found as yet.
7. Conclusions

We saw in this paper that there are many news from Maxwell and Dirac. The existence of subluminal and superluminal UPW solutions of the free Maxwell equations looks, at least to the authors, an extraordinary fact with implications in all branches of physics. For a discussion of these implications see (Rodrigues and Lu, 1996). We saw that these solutions correspond to solutions with field invariants different from zero and that they have also longitudinal components. In this respect we must say that (Evans, 1994) presents some evidences that there are electromagnetic waves with longitudinal electromagnetic fields. Also important is the fact that Maxwell equations are equivalent to a NLDHE which, for particular values of the Takabayasi angle, gives the lepton mass spectrum through the construction of section 5.

We finish by commenting that Weyl equations also have subluminal and superluminal solutions (Vaz and Rodrigues, 1995; Rodrigues and Lu, 1996). This may eventually explain some of the mysteries associated to neutrinos, because in both cases they move as massive particles with momenta satisfying \( p_x^2 = \Omega^2 > 0 \) and \( p_z^2 = -\Omega^2 < 0 \), the symbols < and > corresponding to the subluminal and superluminal solutions.

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