NONLINEAR STABILITY OF HIGHER ORDER MKDV BREATHERS

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ABSTRACT. In this work, we present stability results for breather solutions of the 5th and 7th order mKdV equations. We show that these higher order mKdV breathers are stable in $H^2$, in the same way than classical mKdV breathers. We also show that breather solutions of the 5th and 7th order mKdV equations satisfy the same stationary fourth order nonlinear elliptic equation than the mKdV breather, independently of the order, 5th or 7th, considered.

1. Introduction

In this paper we continue our work, started in [4], on stability properties for breather solutions of the modified Korteweg-de Vries (mKdV) equation,

$$u_t + (u_{xx} + 2u^3)_x = 0,$$

(1.1)

but in this case focusing respectively on the 5th-order mKdV equation

$$u_t + (u_{4x} + 10uu_x^2 + 10u^2u_{xx} + 6u^5)_x = 0,$$

(1.2)

and the 7th-order mKdV equation

$$u_t + (u_{6x} + 14u^2u_{4x} + 56uu_{x}u_{3x} + 42uu_x^2 + 70u_x^2u_{xx} + 70u_x^4u_{xx} + 140u_x^3u_x^2 + 20u^7)_x = 0,$$

(1.3)

and which we will call them higher order mKdV equations and we will denote them as 5th and 7th-mKdV hereafter. Note that other higher order cases, like the 9th and 11-mKdV cases will not be treated here, since they read three and five lines and tens of terms (see Appendix A) but this approach will be sufficient for our purposes. Here $u = u(t, x)$ is a real valued function.

These higher order mKdV equations are a well-known completely integrable set of models [11, 12, 21], with infinitely many conservation laws and well-known (long-time) asymptotic behavior of its solutions obtained with the help of the inverse scattering transform [13]. As a physical model, (1.2) and (1.3) describe large-amplitude internal solitary waves, showing a dynamics which can look rather different from the KdV form. On the other hand, solutions of the inverse scattering transform [13]. As a physical model, (1.2) and (1.3) describe large-amplitude internal solitary waves, showing a dynamics which can look rather different from the KdV form. On the other hand, solutions of (1.2) and (1.3) are invariant under space and time translations. Indeed, for any $t_0, x_0 \in \mathbb{R}$, $u(t - t_0, x - x_0)$ is also a solution of both equations. Linares by using a contraction mapping argument, in [23], has showed that the Cauchy problem for the 5th-mKdV equation is locally well-posed at $H^2(\mathbb{R})$. Kwon, [20], obtained a better result: the 5th-mKdV equation is locally well-posed at $H^4(\mathbb{R})$, $s \geq \frac{5}{4}$. Finally, Grünrock, [14], deduced well-possedness results to other higher-order mKdV equations, establishing for instance that the 7th-mKdV equation is locally well-posed at $H^s(\mathbb{R})$, $s \geq \frac{5}{4}$. The Cauchy problem for the 5th-mKdV equation is globally well-posed at $H^s(\mathbb{R})$, $s \geq 1$ and in the case of the 7th-mKdV equation at $H^s(\mathbb{R})$, $s \geq 2$. See e.g. Linares [23], Kwon [20] and Grünrock [14] for further details.

Note moreover that we have the following inner relation between mKdV

$$u_t = -\partial_x(u_{xx} + 2u^3),$$

(1.4)

and its higher order versions, the 5th-mKdV,

$$u_t = -\partial_x\left(\partial_x^2(u_{xx} + 2u^3) - (2uu_x^2 - 4u^2u_{xx} - 6u^5)\right),$$

(1.5)

and the 7th-mKdV

$$u_t = -\partial_x\left(\partial_x^2(u_{xx} + 2u^3) - \partial_x^2(2uu_x^2 - 4u^2u_{xx} - 6u^5)\right)$$

$$- (4uu_xu_{3x} - 4u^2u_{4x} - 2uu_x^2 - 40u_x^4u_{xx} - 20u^3u_x^2 - 20u^7),$$

(1.6)
In the case of the 5th and 7th-mKdV equations (1.2)-(1.3), the profile of their soliton solutions is completely similar to the Sech mKdV soliton profile, and it is explicitly given by the formula (we denote by \(v_5, v_7\) the speeds of 5th and 7th order solitons)

\[
\begin{align*}
  u(t, x) &:= Q_c(x - v_it)|_{i=5,7}, \quad v_5 = c^2, \quad v_7 = c^3, \\
  Q_c(s) &:= \sqrt{c} \operatorname{sech}(\sqrt{c}s), \quad c > 0.
\end{align*}
\]

Moreover, it is easy to see, by substitution that both 5th and 7th-mKdV soliton solutions \(Q_c\) satisfy (1.7) satisfy the nonlinear stationary elliptic equation

\[
Q''_c - c Q_c + 2Q^3_c = 0, \quad Q_c > 0, \quad Q_c \in H^1(\mathbb{R}).
\]

Note that this second order ODE is precisely the one satisfied by the mKdV soliton. Moreover, note that the soliton solution (1.7) of the 5th and 7th-mKdV equations also satisfy the 4th and 6th order elliptic ODEs coming naturally from integration in space of the 5th and 7th mKdV equations (1.2)-(1.3) respectively. Namely, 5th and 7th higher order mKdV solitons satisfy respectively the following nonlinear stationary elliptic equations:

\[
Q^{(iv)}_c - c^2 Q_c + 10(Q'_c)^2 Q_c + 10Q^2_c Q''_c + 6Q^5_c = 0,
\]

and

\[
Q^{(vi)}_c - c^3 Q_c + 42Q_c(Q''_c)^2 + 56Q'_c Q'_c Q''''_c + 14Q^2_c Q'''_c + 70(Q'_c)^2 Q''_c + 70Q^2_c Q''_c + 14Q^3_c(Q'_c)^2 + 20Q^7_c = 0.
\]

Instead integrating directly in (1.2) and (1.3), another way to check the validity of (1.9) and (1.10) is by using the lowest order nonlinear stationary elliptic equation (1.8) satisfied by all higher order mKdV solitons. For instance, in the case of (1.9), just substitute and obtain:

\[
Q^{(iv)}_c - c^2 Q_c + 10(Q'_c)^2 Q_c + 10Q^2_c Q''_c + 6Q^5_c = 0.
\]

Note that the second order elliptic equation (1.8) satisfied by all higher order mKdV solitons is deeply related to the so-called variational structure of the soliton solution. To be more precise, it is well-known that some of the (first) standard conservation laws of both 5th and 7th-mKdV equations are the mass

\[
M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x) dx = M[u](0),
\]

the energy

\[
E[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x) dx - \frac{1}{2} \int_{\mathbb{R}} u^4(t, x) dx = E[u](0),
\]

and the higher order energies, defined respectively in \(H^2(\mathbb{R})\)

\[
E_5[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u^2_{xx} - 5u^2 u_x^2 + u^6 \right) (t, x) dx = E_5[u](0),
\]

and \(H^3(\mathbb{R})\)

\[
E_7[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u^2_{xxx} + \frac{7}{2} u^2_x - 7u^2 u_{xx} + 35u^4 u_x^2 - \frac{5}{2} u^8 \right) (t, x) dx = E_7[u](0).
\]

Using the lowest order conserved quantities (i.e., mass and energy (1.12)-(1.13)), the variational structure of any higher order mKdV soliton (1.7) can be characterized as follows: there exists a suitable Lyapunov functional, 

\[
\mathcal{H}_0[u](t) = E[u](t) + c M[u](t),
\]

where \(c > 0\) is the scaling of the solitary wave, and \(M[u], E[u]\) are given in (1.12) and (1.13). Indeed, it is easy to see that for any \(z(t) \in H^1(\mathbb{R})\) small,

\[
\mathcal{H}_0[Q_c + z][t] = \mathcal{H}_0[Q_c] + \int_{\mathbb{R}} z(Q''_c - cQ_c + Q^5_c) + O(\|z(t)\|^2_{H^1}).
\]
The zero order term above is independent of time, whilst the first order term in \( z \) is zero from (1.8), proving the critical character of \( Q_c \).

Note that by using higher order conservation laws (1.14) and (1.15), and therefore higher order Lyapunov functionals, we are also able to characterize 5th and 7th-mKdV solitons (1.7) as extremal points of these higher order functionals. More precisely, for instance, in the 5th-mKdV case, and using the quantities \( M[u] \), \( E_5[u] \) given in (1.12) and (1.14), this functional is explicitly given by

\[
\mathcal{H}_5[u](t) = E_5[u](t) + c^2 M[u](t), \quad c > 0,
\]

and for the 7th-mKdV case, using the quantities \( M[u] \), \( E_7[u] \) given in (1.12) and (1.15), we get

\[
\mathcal{H}_7[u](t) = E_7[u](t) + c^3 M[u](t), \quad c > 0.
\]

Indeed, in both cases it is easy to see that for any \( z(t) \in H^2(\mathbb{R}) \) small,

\[
\mathcal{H}_5[Q_c + z](t) = \mathcal{H}_5[Q_c] - \int_{\mathbb{R}} z \left( Q_c^{(iv)} - c^2 Q_c + 10(Q_c')^2 Q_c + 10Q_c^2Q_c'' + 6Q_c^5 \right) + O(\|z(t)\|_{H^2}^2),
\]

and

\[
\mathcal{H}_7[Q_c + z](t) = \mathcal{H}_7[Q_c] - \int_{\mathbb{R}} z \left( Q_c^{(iv)} - c^3 Q_c + 42Q_c(Q_c'')^2 + 56Q_cQ_c'Q_c''' + 14Q_c^2Q_c^{(iv)} + 70(Q_c')^2Q_c'' + 70Q_c^4Q_c'' \right.

\]

\[
+ 140Q_c^3(Q_c'')^2 + 20Q_c^5) + O(\|z(t)\|_{H^2}^2).
\]

In both cases, the zero order term above is independent of time, whilst the first order term in \( z \) is zero from (1.9) and (1.10).

1.1. Breathers in 5th and 7th order mKdV equations. Besides these soliton solutions of 5th and 7th-mKdV equations (1.2)-(1.3), it is possible to find another big set of explicit and oscillatory solutions, known in the physical and mathematical literature as the breather solution, and which is a periodic in time, spatially localized real function. Although there is no universal definition for a breather, we remember here the following convention introduced in [7].

Definition 1.1 (Aperiodic breather). We say that \( B = B(t, x) \) is a breather solution for a particular one-dimensional dispersive equation if there are \( T > 0 \) and \( L = L(T) \in \mathbb{R} \) such that, for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \), one has

\[
B(t + T, x) = B(t, x - L),
\]

and moreover, the infimum among times \( T > 0 \) such that property (1.22) is satisfied for such a time \( T \) is uniformly positive in space.

Remark 1.1. Observe that the last condition ensures that solitons (and multisolitons) are not breathers, since e.g. \( Q_c(x - c(t + T)) = Q_c(x - L - ct) \) for \( L := cT \) but \( T \) can be any real-valued time.

For the 5th and 7th-mKdV equations (1.2)-(1.3), the breather solution in the line can be obtained by using different methods (e.g. Inverse Scattering, Hirota method. See [25] [26] for further details). Particularly we use here a matching method to find these breather solutions, i.e. proposing a well known ansatz, with speeds as free parameters to be determined in order to define a solution. Note that the same procedure can be used to obtain periodic breather solutions of the 5th and 7th-mKdV equations. See appendix [14] for further details.

Definition 1.2 (5th and 7th-mKdV breathers). Let \( \alpha, \beta > 0 \) and \( x_1, x_2 \in \mathbb{R} \). The real-valued breather solution of the 5th and 7th-mKdV equations (1.2)-(1.3) is given explicitly by the formula

\[
B = B_{\alpha, \beta}(t, x; x_1, x_2) := 2\partial_x \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right),
\]

with \( y_1 \) and \( y_2 \)

\[
y_1 = x + \delta_i t + x_1, \quad y_2 = x + \gamma_i t + x_2, \quad i = 5, 7,
\]

and with velocities \( (\delta_5, \gamma_5) \) in the 5th order case

\[
\delta_5 := -\alpha^4 + 10\alpha^2 \beta^2 - 5\beta^4, \quad \gamma_5 := -\beta^4 + 10\alpha^2 \beta^2 - 5\alpha^4,
\]
and \((\delta_7, \gamma_7)\) in the 7th order case

\[
\delta_7 := \alpha^6 - 21\alpha^4\beta^2 + 35\alpha^2\beta^4 - 7\beta^6, \quad \gamma_7 := -\beta^6 + 21\alpha^2\beta^4 - 35\alpha^4\beta^2 + 7\alpha^6. \tag{1.26}
\]

**Remark 1.2.** This is a four-parametric solution, with two scalings \((\alpha, \beta)\) and two shift translations \((x_1, x_2)\). Note moreover that from this formula one has, for any \(k \in \mathbb{Z}\),

\[
B_{\alpha, \beta}(t, x; x_1 + \frac{k\pi}{\alpha}, x_2) = (-1)^k B_{\alpha, \beta}(t, x; x_1, x_2), \tag{1.27}
\]

which are also solutions of (1.2) - (1.3). This identity reveals the periodic character of the first translation parameter \(x_1\).

**Remark 1.3.** Observe that the breather solution for 5th and 7th order mKdV equations has the same functional expression that the classical mKdV breather solution [4 Def.1.1]

\[
B \equiv B_{\alpha, \beta}(t, x; x_1, x_2) := 2\partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right], \tag{1.28}
\]

with \(y_1 = x + \delta t + x_1, \ y_2 = x + \gamma t + x_2\), and velocities \(\delta = \alpha^2 - 3\beta^2, \ \gamma = 3\alpha^2 - \beta^2\), and in fact only differing in speeds (1.25).

**Remark 1.4.** Finally be aware that these 5th and 7th breather solutions (1.23) in the line could be used to re-approach the ill-posedness of the Cauchy problem for both 5th and 7th-mKdV equations (1.2) and (1.3), in the same way they were used by Kenig-Ponce and Vega [17] and Alejo [3], to show a failure of the flow map associated to some nonlinear dispersive equations to be uniformly continuous. This procedure could afford a complementary proof to the previous works on the ill-posedness of these higher order equations presented by Kwon [20] and Grünrock [14]. Note finally, that in the periodic setting for the 5th and 7th-mKdV equations (1.2) and (1.3), we could follow the same reasoning, but instead using the periodic breathers (1.1) to handle ill-posedness questions on these equations. See appendix B for further reading on these periodic breathers of higher order 5th and 7th equations.

One of the main results of this work will be to prove that, exactly as it happens with all higher order soliton solutions (1.7) of (1.2) and (1.3), these breather solutions (1.23) satisfy the same fourth order stationary elliptic equation than the classical mKdV breather solution

**Theorem 1.3.** 5th and 7th mKdV breathers \(B\) satisfy the same fourth order stationary elliptic equation than the classical mKdV breather, namely

\[
B_{4x} + 10BB_x^2 + 10B^2B_{xx} + 6B_5 - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B = 0.
\]

This fact can be interpreted as if all mKdV breathers and higher order mKdV breathers are characterized by the same elliptic equation, in a similar way as it was showed for the KdV equation by Lax [22]. Moreover, and as second main result in this paper, we give a positive answer to the question of breathers stability for higher order mKdV equations.

**Theorem 1.4.** 5th and 7th mKdV breathers are orbitally stable in the \(H^2\)-topology.

A more detailed version of this result is given in Theorem 5.1. As we will see, the space \(H^2\) is required by a regularity argument and through the variational characterization that we obtain of these breather solutions of higher order mKdV equations. Even more, this space comes from the fact that breather structures are bound states, which means that there is no mass decoupling as time evolves.

### 1.2. Organization of this paper.

In Sect 2 we study generalized Weinstein conditions and we present some nonlinear identities and stability tests satisfied by 5th and 7th-mKdV breathers. Furthermore, we prove that any 5th or 7th-mKdV breather solutions satisfy a fourth order nonlinear ODE, which characterizes them. Sect 3 is devoted to the study of a linear operator associated to the breather solutions. In Sect 4 we introduce a suitable \(H^2\)-Lyapunov functional for both higher order mKdV equations (1.2) and (1.3). Finally, in Sect 5 we present a detailed version of Theorem 5.1.

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2. **Nonlinear higher order identities**

The aim of this section is to get stability tests by computing generalized Weinstein conditions for any 5th or 7th-mKdV breathers \( B \). We begin with the simpler case of the 5th and 7th-mKdV soliton solutions \( Q_c \), where the mass (1.12) and the energy (1.13) are given by the quantities

\[
M[Q_c](t) := 2\sqrt{c},
\]

and

\[
E[Q_c](t) = -\frac{2}{3}c^{3/2},
\]

which are the same as the well known mass and energy of the mKdV soliton solution \([4, eqn. (2.1)]\). Note that the Weinstein condition \([27]\) is, for \( c > 0 \),

\[
\partial_c M[Q_c] = \frac{1}{\sqrt{c}} > 0.
\]

This inequality guarantees the nonlinear stability of the 5th and 7th-mKdV solitons (1.7). Moreover note that the same condition for the energy of the 5th and 7th-mKdV solitons \( \partial_c E[Q_c] = -\sqrt{c} \), does not vanish either.

Now, we approach the case of 5th and 7th-mKdV breathers. Firstly we present the following identity for solutions of mKdV, 5th or 7th-mKdV equations (1.1 - 1.3). Then \([3]\)

\[
u^2 = \frac{\partial^2}{\partial x^2} \log(G^2 + F^2).
\]

**Proof.** See Appendix C. \(\square\)

This result allows us to compute easily the mass of any 5th and 7th-mKdV breather:

**Lemma 2.2.** Let \( B = B_{\alpha,\beta} \) be any 5th or 7th order mKdV breather, for \( \alpha, \beta > 0 \) as in Definition 1.2. Then, the mass of \( B \) is

\[
M[B](t) = 2\beta.
\]

**Proof.** It follows directly by using the above identity (2.4) and substitution in the definition (1.12). In fact, we obtain:

\[
M[B](t) = \frac{1}{2} \int_{\mathbb{R}} B^2 dt = 2\beta.
\]

**Remark 2.1.** Note that as we could expect, after the mKdV breather case (see \([4, Lemma 2.1]\)), the mass of any 5th and 7th-mKdV breather solution depends only on the scaling \( \beta \) and indeed is equal to the mass of the classical mKdV breather solution (see \([4]\)).

From the involved integral in (2.5), we can define the **partial** mass of any 5th and 7th-mKdV breather as follows:

**Definition 2.3.** Let \( B = B_{\alpha,\beta} \) be any 5th or 7th-mKdV breather solution, for \( \alpha, \beta > 0 \) as in definition (1.12). Then, we define the **partial** mass associated to any 7th-mKdV breather as:

\[
M(t, x) \equiv M_{\alpha,\beta}(t, x) := \frac{1}{2} \int_{\mathbb{R}} B^2(t, x; s; x_1, x_2) ds
= \beta + \frac{1}{2} \partial_x \log(G^2 + F^2)(t, x).
\]

A direct consequence of the above results are the following generalized Weinstein conditions:

1\(^1\)Following this notation, in this work \( G \) and \( F \) will correspond to \( \beta \sin(\alpha y_1) \) and \( \alpha \cosh(\beta y_2) \) respectively.
Corollary 2.4. Let $B = B_{\alpha, \beta}$ be any 5th or 7th-mKdV breather solutions of the form (1.2). Given $t \in \mathbb{R}$ fixed, let

$$\Lambda_\alpha B := \partial_x B, \quad \text{and} \quad \Lambda_\beta B := \partial_x B. \quad (2.7)$$

Then both functions $\Lambda_\alpha B$ and $\Lambda_\beta B$ are in the Schwartz class for the spatial variable, and satisfy the identities

$$\partial_\alpha M[B] = \int_{\mathbb{R}} B\Lambda_\alpha B = 0, \quad (2.8)$$

and

$$\partial_\beta M[B] = \int_{\mathbb{R}} B\Lambda_\beta B = 2 > 0, \quad (2.9)$$

independently of time.

Proof. By simple inspection, one can see that, given $t$ fixed, $\Lambda_\alpha B_{\alpha, \beta}$ and $\Lambda_\beta B_{\alpha, \beta}$ are well-defined Schwartz functions. The proof of (2.8) and (2.9) is consequence of definition (2.3), and the definition of mass (1.12). \qed

Remark 2.2. Comparing (2.8) and (2.9) with the Weinstein condition (2.3), we may think that the second scaling parameter $\alpha$ is $L^2$-critical. On the opposite side, the first scaling $\beta$ can be seen as a stable parameter.

Consider now the two directions associated to spatial translations. Let $B_{\alpha, \beta}$ as introduced in (1.2). We define

$$B_1(t; x_1, x_2) := \partial_{x_1} B_{\alpha, \beta}(t; x_1, x_2) \quad \text{and} \quad B_2(t; x_1, x_2) := \partial_{x_2} B_{\alpha, \beta}(t; x_1, x_2). \quad (2.10)$$

It is clear that, for all $t \in \mathbb{R}$, and $\alpha, \beta$ as in definition (1.2) and $x_1, x_2 \in \mathbb{R}$, both $B_1$ and $B_2$ are real-valued functions in the Schwartz class, exponentially decreasing in space. Moreover, it is not difficult to see that they are linearly independent as functions of the $x$-variable, for all time $t$ fixed. Let consider $\tilde{B} = \tilde{B}_{\alpha, \beta}$ as the following $L^\infty$-function defined for the 5th and 7th-mKdV equations:

$$\tilde{B}(t, x) := 2 \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right). \quad (2.11)$$

Now, we show the following nonlinear identities for 5th-mKdV breathers:

Lemma 2.5. Let $B = B_{\alpha, \beta}$ be any 5th-mKdV breather solution of the form (1.23), and $\tilde{B}$ as in (2.11). Then

1. For any fixed $t \in \mathbb{R}$, we have $\tilde{B}_t$ well-defined in the Schwartz class, satisfying

$$\tilde{B}_t + B_{4x} + 10BB_x^2 + 10B^2B_{xx} + 6B^5 = 0. \quad (2.12)$$

2. Let $M$ be defined by (2.6). Then

$$B_{xx}^2 - 2B^6 - 2B_xB_{xxx} - 2BB_t + 2(M)_t - 10B^2B_x^2 = 0. \quad (2.13)$$

Proof. The first item above (2.12) is a consequence of (2.11) and a convenient integration in space (from $-\infty$ to $x$). To obtain (2.13) we multiply (2.12) by $B_x$ and integrate in space in the same region. \qed

And the corresponding nonlinear identities for 7th-mKdV breathers are:

Lemma 2.6. Let $B = B_{\alpha, \beta}$ be any 7th-mKdV breather solution of the form (1.2). Then

1. For any fixed $t \in \mathbb{R}$, we have $\tilde{B}_t$ well-defined in the Schwartz class, satisfying

$$\tilde{B}_t + B_{6x} + 14B^2B_{4x} + 56BB_xB_{3x} + 42B^2B_{xx} + 70B^2B_{xx} + 70B^4B_{xx} + 140B^3B_x^2 + 20B^7 = 0. \quad (2.14)$$

2. Let $M$ be defined by (2.6). Then

$$B_{x}^2 + 5B^8 + 2B^6B_{5x} - 2B_{xx}B_{4x} + 2BB_t - 2(M)_t - 28BB_xB_{3x} - 14B^2B_{xx}^2 + 56BB_x^2B_{xx} + 7B_x^4 + 70B^4B_x^2 = 0. \quad (2.15)$$

Proof. It follows in a similar way than the proof of Lemma 2.5. \qed

Corollary 2.7. Let $B = B_{\alpha, \beta}$ be any 5th or 7th-mKdV breather solution of the form (1.23), and $\tilde{B}$ as in (2.11). Then if $B_1$ and $B_2$ are as in (2.10) and $\tilde{B}_i \equiv \partial_{x_i} \tilde{B}$. Then we have that

$$\int_{-\infty}^{x} (\tilde{B}_{12}^2 - \tilde{B}_{11}\tilde{B}_{22}) = -BB_{11} + \frac{1}{2}\partial_{x_i}^2 \partial_{x_j} \log(G^2 + F^2). \quad (2.16)$$
Lemma 2.8. Let $B = B_{\alpha, \beta}$ be any 5th or 7th order $mKdV$ breather solutions respectively, for $\alpha, \beta$ as in definition (1.2). Then the higher order energies (1.14) and (1.15) of a 5th and 7th-mKdV breather $B$ are respectively

$$E_5[B] := -\frac{2}{5} \beta \gamma_5 \quad \text{and} \quad E_7[B] := \frac{2}{7} \beta \gamma_7,$$

with $\gamma_5, \gamma_7$ given in (1.20) - (1.26).

Remark 2.3. Note that as in comparison with the case of a classical $mKdV$ breather solution $B$, where $E[B] := \frac{2}{7} \beta \gamma$ (see [4, Lemma 2.4]), the sign of the higher order energies $E_5, E_7$ is driven by a nonlinear balance among the different terms depending on scalings $\alpha, \beta$.

Remark 2.4. From the above Lemma, we conjecture that for any $(2n+1)$-order $mKdV$ breather $B$, its $(2n+1)$-order energy is given by

$$E_{2n+1}[B](t) = (-1)^{n+1} \frac{2\beta}{2n+1} \gamma_{2n+1}, \quad n \in \mathbb{N},$$

and with

$$\gamma_{2n+1} := \sum_{j=0}^{n} (-1)^{j} \frac{(2n+1)!}{(2j)!(2n+1-2j)!} \alpha^2 \beta^2(n-j), \quad n \in \mathbb{N}.$$

Proof. (of Lemma 2.8)

We start with the 5th order case. First of all, let us prove the following reduction

$$E_5[B](t) = -\frac{1}{5} \int_{\mathbb{R}} \left((M)_t(t, x)\right) dx.$$
Indeed, we multiply (2.12) by $B$ and integrate in space: we get
\[
\int_{\mathbb{R}} B^2_{xx} = \int_{\mathbb{R}} 20B^2 B_x^2 - 6B^6 - B\dot{B}_t.
\]
On the other hand, integrating (2.13),
\[
\int_{\mathbb{R}} B^2_{xx} = \frac{2}{3} \int_{\mathbb{R}} B^6 + \frac{2}{3} \int_{\mathbb{R}} B\dot{B}_t - \frac{2}{3} \int_{\mathbb{R}} (M)_t + \frac{10}{3} \int_{\mathbb{R}} B^2 B_x^2.
\]
From these two identities, we get
\[
\int_{\mathbb{R}} B^6 = \frac{1}{10} \int_{\mathbb{R}} (M)_t - \frac{1}{4} \int_{\mathbb{R}} B\dot{B}_t + \frac{5}{2} \int_{\mathbb{R}} B^2 B_x^2,
\]
and therefore
\[
\int_{\mathbb{R}} B^2_{xx} = -\frac{3}{5} \int_{\mathbb{R}} (M)_t + \frac{15}{3} \int_{\mathbb{R}} B^2 B_x^2 + \frac{1}{2} \int_{\mathbb{R}} B\dot{B}_t.
\]
Finally, substituting the last two identities into (1.14), we get (2.21), as desired. Proceeding in the same way, in the 7th order case we obtain the corresponding reduction
\[
E_7[B](t) = \frac{1}{7} \int_{\mathbb{R}} (M)(t, x) dx.
\]
Now we prove (2.6). From (2.6), we have that
\[
M(t, x) = \beta + \frac{1}{2} \partial_x \log(G^2 + F^2),
\]
and hence,
\[
M_t(t, x) = \frac{1}{2} \partial_x \partial_t \log(G^2 + F^2).
\]
Now substituting in the energy (2.21), remembering the identity (2.12) and the explicit expression for $M[B]$ in (2.6), we get
\[
E_5[B](t) = -\frac{1}{5} \int_{\mathbb{R}} (M)_t(t, x) dx = -\frac{11}{52} \int_{\mathbb{R}} \partial_x \partial_t \log(G^2 + F^2) dx
\]
\[
= -\left(\frac{11}{52} \partial_t \log(G^2 + F^2)\right)|^{+\infty}_{-\infty} = -\frac{2}{5} \beta \gamma_5.
\]
For the 7th order case, we proceed as above, but now using (2.14) and (2.15), and we get
\[
E_7[B] = \frac{2}{7} \beta \gamma_7.
\]

Note that since the profiles of both 5th and 7th order mKdV breathers (solitons) agree with the form of the classical mKdV breather (soliton), and since the energy $E$ (1.13) is a conserved quantity for the mKdV and 5th and 7th higher order equations, when the lowest energy $E$ is evaluated in these 5th and 7th higher order breathers we obtain in both cases the same value than the mKdV breather energy, $\frac{2}{7} \beta \gamma$. For the sake of simplicity and to understand that property, we remember here the relation [6, (4.2),(4.4)] in the case of low order conserved quantities at breather solutions $B$ and at soliton solutions $Q_c$:
\[
M[B] = 2\text{Re}[M(Q_c)|_{\sqrt{c}=\beta+i\alpha}] \quad \text{and} \quad E[B] = 2\text{Re}[E(Q_c)|_{\sqrt{c}=\beta+i\alpha}].
\]

**Corollary 2.9.** Let $B = B_{\alpha, \beta}$ be any 5th or 7th order mKdV breather. Then
\[
\partial_\alpha E[B] = 4\alpha \beta, \quad \partial_\beta E[B] = 2(\alpha^2 - \beta^2).
\]

**Remark 2.5.** Note that condition $\alpha = \beta$ is equivalent to the identity $\partial_\beta E[B] = 0$. Higher order solitons do not satisfy this last identity.

The next nontrivial identity for 5th-mKdV breathers (1.2) will be useful in the proof of the nonlinear stationary equation that they satisfy.
Lemma 2.10. Let \( B = B_{\alpha, \beta} \) be any 5th-mKdV breather. Then, for all \( t \in \mathbb{R} \),

\[
\dot{B}_t = (\alpha^2 + \beta^2)B - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3). \tag{2.25}
\]

Proof. We will use the following notation:

\[
\dot{B}_t = 2\partial_x \left[ \text{arctan} \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right] = \frac{H(t, x)}{N(t, x)},
\]

\[
H := H(t, x) = 2 \left( \beta \alpha^2 \cosh(\beta y_2) \cos(\alpha y_1) - \beta^2 \alpha \sinh(\beta y_2) \sin(\alpha y_1) \right),
\]

\[
N := N(t, x) = \alpha^2 \cosh^2(\beta y_2) + \beta^2 \sin^2(\alpha y_1),
\]

and from \( \dot{B} \) (2.11),

\[
\dot{B}_t := 2\partial_t \left[ \text{arctan} \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right] = \frac{P(t, x)}{N(t, x)},
\]

\[
P := P(t, x) = 2 \left( \beta \alpha \delta_5 \cosh(\beta y_2) \cos(\alpha y_1) - \beta \alpha \gamma_5 \sinh(\beta y_2) \sin(\alpha y_1) \right), \tag{2.26}
\]

with \( \delta_5, \gamma_5 \) as in (1.25). For the sake of simplicity, we are going to use the following notation:

\[
N_1 := N_x = 2\alpha \beta^2 \cos(\alpha y_1) \sin(\alpha y_1) + 2\alpha^2 \beta \cosh(\beta y_2) \sinh(\beta y_2), \tag{2.27}
\]

\[
N_2 := N_{xx} = 2\alpha^2 \beta^2 \cos^2(\alpha y_1) - \sin^2(\alpha y_1) + \cosh^2(\beta y_2) + \sinh^2(\beta y_2), \tag{2.28}
\]

and

\[
H_1 := H_x = -2\alpha \beta (\alpha^2 + \beta^2) \cosh(\beta y_2) \sin(\alpha y_1). \tag{2.29}
\]

\[
H_2 := H_{xx} = -2\alpha \beta (\beta^2 + \alpha^2)(\alpha \cosh(\beta y_2) \cos(\alpha y_1) + \beta \sin(\alpha y_1) \sinh(\beta y_2)). \tag{2.30}
\]

First of all, we start rewriting the following terms of the l.h.s. of (2.25):

\[
B_{xx} + 2B^3 = \frac{1}{N^3} \left( 2H^3 + H_2N^2 - 2H_1NN_1 + 2HN_1^2 - HNN_2 \right), \tag{2.31}
\]

and hence, we have that

\[
- \dot{B}_t - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B = \frac{M_0}{N^3}, \tag{2.32}
\]

with

\[
M_0 := -PN^2 + (\alpha^2 + \beta^2)^2HN^2 - 2(\beta^2 - \alpha^2) \left( 2H^3 - 2HN_1N_1 + 2HN_1^2 + N^2H_2 - HNN_2 \right). \tag{2.33}
\]

Indeed, we verify, after substituting \( P \) and \( H' \)'s and \( N' \)'s terms explicitly in (2.33) and having in mind basic trigonometric and hyperbolic identities, that

\[
M_0 = 0, \tag{2.34}
\]

and we conclude. \( \Box \)

We are ready now to present one of the most important results of this work, namely, we are going to show that in fact, breather solutions (1.23) of 5th and 7th-mKdV equations satisfy the same fourth order ODE satisfied by the classical mKdV breather solutions and that indeed characterizes it. This result means that this ODE identifies breather functions at different levels in the mKdV hierarchy, i.e. at the mKdV level and at 5th and 7th mKdV levels, as being solutions of the same stationary fourth order ODE.

Theorem 2.11. Let \( B = B_{\alpha, \beta} \) be any 5th or 7th-mKdV breather solution given in (1.23). Then, for any fixed \( t \in \mathbb{R} \), \( B \) satisfies the same nonlinear stationary equation than the classical mKdV breather solution (1.25), namely

\[
G[B] := B_{4x} + 10BB_x^2 + 10B^2B_{xx} + 6B^5 - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B = 0. \tag{2.35}
\]
Proof. In the case of the 5th order breather, since by (2.12) the first four terms in (2.35) equal \( -\dot{B}_t \) and using the above identity (2.25), we simply get

\[
G[B] = -\dot{B}_t - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B = 0.
\]

The 7th order case is more involved since we do not have at hand any identity like (2.25). Therefore, we first recast the l.h.s. of (2.35). Taking into account the r.h.s. of (1.5), we rewrite the first four terms in (2.35) and simplify the l.h.s. of (2.35), as follows:

\[
B_{4x} + 10BB_x^2 + 10B^2B_{xx} + 6B^5 - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B
\]

We get

\[
\frac{\partial^2}{\partial t^2}(B_{xx} + 2B^3) = \frac{1}{N^5} \left( 2H^3 + H_2N^2 - 2H_1NN_1 + 2HN_1^2 - HNN_2 \right),
\]

and

\[
H_3 := H_{xxx} = 2\alpha\beta((\alpha^4 - \beta^4)\cosh(\beta y_2) - 2\alpha(\alpha_2 + \beta^2)\cosh(\alpha y_1)\sinh(\beta y_2)),
\]

\[
H_4 := H_{4x} = 2\alpha\beta((\alpha^5 - 3\alpha_2 \beta^2 - 3\alpha \beta^4)\cosh(\beta y_2)\sinh(\alpha y_1) + (3\alpha_4 \beta + 2\alpha^2 \beta^3 - \beta^5)\sin(\alpha_2)\sinh(\beta y_2)).
\]

First of all, remembering from (2.31) that

\[
B_{xx} + 2B^3 = \frac{1}{N^3} \left( 2H^3 + H_2N^2 - 2H_1NN_1 + 2HN_1^2 - HNN_2 \right),
\]

we get

\[
\frac{\partial^2}{\partial t^2}(B_{xx} + 2B^3) = \frac{1}{N^5} \left( 6H^2N(H_2N - 6H_1N_1) + 6H^3(4N_1^2 - NN_2) + N(N(H_4N^2 - 4H_3NN_1 + 12H_2N_2^2 - 6H_2N_2N_2) - 4H_1(6N_1^3 - 6NN_1N_2 + N^2N_2)) + H(24N_1^4 - 36NN_1^2N_2 + 2N^2(6H_1^2 + 3N_2^2 + 4N_1N_3) - HNN_4) \right).
\]

Hence, we have that

\[
\frac{\partial^2}{\partial t^2}(B_{xx} + 2B^3) + (4B^2 - 2(\beta^2 - \alpha^2))(B_{xx} + 2B^3) = \frac{M_1}{N^5},
\]

with

\[
M_1 := \left( 8H^5 + 2H^2N(5H_2N - 2H_1N_1) + 2H^3(16N_1^2 - 5NN_2 + 2(\alpha^2 - \beta^2)N^2) + H \left[ 24N_1^4 - 36NN_1^2N_2 + 2N^2(6H_1^2 + 3N_2^2 + 4N_1N_3) + 2(\alpha^2 - \beta^2)N_1^2 \right] - N^3(N_1 + 2(\alpha^2 - \beta^2)N_2) \
+ H \left[ -24H_1N_1^3 + 12N_1N_1(H_2N_1 + 2H_1N_2) + N^3(H_1 + 2(\alpha^2 - \beta^2)H_2) \
- 2N^2(2H_3N_1 + 3H_2N_2 + 2H_1(N_3 + (\alpha^2 - \beta^2)N_1)) \right] \right).
\]

Moreover, we have that

\[
-2B[B_x^2 + B^4] = \frac{-2H}{N^5} \left( H^4 + (H_1N - HNN_1)^2 \right),
\]

and therefore,

\[
-2B[B_x^2 + B^4] + (\alpha^2 + \beta^2)^2 B = \frac{M_2}{N^5},
\]

with

\[
M_2 := \left( H(-2(H^4 + (H_1N - HNN_1)^2) + (\alpha^2 + \beta^2)^2 N_1) \right).
\]

Hence, we get the following simplification of (2.36):
\[ G[B] = \partial^2_x(B_{xx} + 2B^3) + (4B^2 - 2(\beta^2 - \alpha^2))(B_{xx} + 2B^3) - 2B[B_x^2 + B^4] + (\alpha^2 + \beta^2)^2 B \]
\[ = \frac{M_1 + M_2}{N^3}. \quad (2.46) \]

with \(M_1, M_2\) in (2.43) and (2.45) respectively. In fact, we verify, using the symbolic software Mathematica, that after substituting \(H's\) and \(N's\) terms explicitly in (2.46) and lengthy rearrangements, that

\[ M_1 + M_2 = 0, \quad (2.47) \]

and we conclude. \(\Box\)

A direct consequence from Theorem 2.11 and identity (2.14), implies that for the 7th order case, we are able to obtain a new identity relating \(\tilde{B}_t\) and lower order spatial derivatives of the 7th-mKdV breather (see (2.14) for comparison):

**Corollary 2.12.** Let \(B = B_{\alpha, \beta}\) be any 7th-mKdV breather solution (1.2). Then, for any fixed \(t \in \mathbb{R}\), \(\tilde{B}\) satisfies the following nonlinear identity

\[ \tilde{B}_t - 2(\beta^2 - \alpha^2)(\alpha^2 + \beta^2)^2 B + 4(\alpha^4 - 6\alpha^2 \beta^2 + \beta^4)B^3 + 4(\beta^2 - \alpha^2)B^5 - 4B^7 \]
\[ + (3\alpha^4 - 10\alpha^2 \beta^2 + 3\beta^4)B_{xx} + 4(\beta^2 - \alpha^2)BB_x^2 - 20B^3B_x^2 + 2BB_x^2 - 4BB_xB_{3x} = 0. \quad (2.48) \]

**Proof.** Substituting \(B_{4x}\) in (2.45) and simplifying, we get (2.48). \(\Box\)

### 3. Spectral analysis

Let \(z = z(x)\) be a function to be specified in the following lines. Let \(B = B_{\alpha, \beta}\) be any 5th or 7th-mKdV breather solution, with shift parameters \(x_1, x_2\). Let us introduce the following fourth order linear operator:

\[ \mathcal{L}[z](x; t) := z(4x)(x) - 2(\beta^2 - \alpha^2)z_{xx}(x) + (\alpha^2 + \beta^2)^2 z(x) + 10B^2z_{xx}(x) + 20BB_xz_x(x) \]
\[ + \left[ 10B_x^2 + 20BB_x + 30B^4 - 12(\beta^2 - \alpha^2)B^2 \right] z(x). \quad (3.1) \]

In this section we describe the spectrum of this operator associated to 5th and 7th-mKdV breathers. More precisely, our main purpose is to find a suitable coercivity property, independently of the nature of scaling parameters. The main result of this section is contained in Proposition 3.10. Part of the analysis carried out in this section has been previously introduced by Lax [22], and Maddocks and Sachs [24], so we follow their arguments adapted to the breather case, sketching several proofs.

**Lemma 3.1.** \(\mathcal{L}\) is a linear, unbounded operator in \(L^2(\mathbb{R})\), with dense domain \(H^4(\mathbb{R})\). Moreover, \(\mathcal{L}\) is self-adjoint, and is a compact perturbation of the constant coefficients operator

\[ \mathcal{L}_0[z] := z(4x) - 2(\beta^2 - \alpha^2)z_{xx} + (\alpha^2 + \beta^2)^2 z. \]

In particular, the continuous spectrum of \(\mathcal{L}\) is the closed interval \([\alpha^2 + \beta^2, +\infty)\) in the case \(\beta \geq \alpha\), and \([4\alpha^2 \beta^2, +\infty)\) in the case \(\beta < \alpha\), with no embedded eigenvalues are contained in this region.

**Proof.** Let \(w, \tilde{w} \in H^4(\mathbb{R})\). Integrating by parts, one has

\[ \int_{\mathbb{R}} w \mathcal{L}[z] = \int_{\mathbb{R}} w \left[ z(4x) - 2(\beta^2 - \alpha^2)z_{xx} + (\alpha^2 + \beta^2)^2 z + 10B^2z_{xx} + 20BB_xz_x \right] \]
\[ + \int_{\mathbb{R}} \left[ 10B_x^2 + 20BB_x + 30B^4 - 12(\beta^2 - \alpha^2)B^2 \right] zw \]
\[ = \int_{\mathbb{R}} \left[ w(4x) - 2(\beta^2 - \alpha^2)w_{xx} + (\alpha^2 + \beta^2)^2 w + 10B^2w_{xx} - (20BB_xw_x) \right] z \]
\[ + \int_{\mathbb{R}} \left[ 10B_x^2 + 20BB_x + 30B^4 - 12(\beta^2 - \alpha^2)B^2 \right] wz = \int_{\mathbb{R}} \mathcal{L}[w]z. \]

Finally, it is clear that \(D(\mathcal{L}^*)\) can be identified with \(D(\mathcal{L}) = H^4(\mathbb{R})\). The description of the continuous spectrum is a consequence of the Weyl Theorem. Furthermore, note that the nonexistence of embedded eigenvalues (or resonances) is consequence of the rapidly decreasing character of the potentials involved in the definition of \(\mathcal{L}\). \(\Box\)
We introduce now two directions associated to spatial translations. Let $B_{\alpha,\beta}$ as defined in (1.23). We defined in (2.10) the associated functions

$$B_1(t; x_1, x_2) := \partial_{x_1} B_{\alpha,\beta}(t; x_1, x_2), \quad \text{and} \quad B_2(t; x_1, x_2) := \partial_{x_2} B_{\alpha,\beta}(t; x_1, x_2).$$

It is clear that, for all $t \in \mathbb{R}$, $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$, both $B_1$ and $B_2$ are real-valued functions in the Schwartz class, exponentially decreasing in space. Moreover, it is not difficult to see that they are linearly independent as functions of the $x$-variable, for all time $t$ fixed.

**Lemma 3.2.** For each $t \in \mathbb{R}$, one has

$$\ker \mathcal{L} = \text{span} \{ B_1(t; x_1, x_2), B_2(t; x_1, x_2) \}.$$  

*Proof.* The proof follows the same steps than in the case of the classical mKdV breather and therefore we refer to [4, Lem. 4.3]. \hfill $\Box$

We consider now the natural modes associated to the scaling parameters $\alpha, \beta$, which are the best candidates to generate negative directions for the related quadratic form defined by $\mathcal{L}$. Recall the definitions of $\Lambda_\alpha B_{\alpha,\beta}$ and $\Lambda_\beta B_{\alpha,\beta}$ introduced in (2.7). We have the following:

**Lemma 3.3.** Let $B = B_{\alpha,\beta}$ be any 5th or 7th-mKdV breather solution. Consider the scaling directions $\Lambda_\alpha B$ and $\Lambda_\beta B$ introduced in (2.7). Then

$$\int_\mathbb{R} \Lambda_\alpha B \mathcal{L}[\Lambda_\alpha B] = 16\alpha^2 \beta > 0,$$

and

$$\int_\mathbb{R} \Lambda_\beta B \mathcal{L}[\Lambda_\beta B] = -16\alpha^2 \beta < 0.$$  

*Proof.* From Theorem (2.11), we get after derivation with respect to $\alpha$ and $\beta$,

$$\mathcal{L}[\Lambda_\alpha B] = -4\alpha [B_{xx} + 2B^3 + (\alpha^2 + \beta^2)B], \quad \mathcal{L}[\Lambda_\beta B] = 4\beta [B_{xx} + 2B^3 - (\alpha^2 + \beta^2)B].$$

We deal with the first identity above. Note that from (2.8), (1.13) and Remark (2.3),

$$\int_\mathbb{R} \Lambda_\alpha B \mathcal{L}[\Lambda_\alpha B] = -4\alpha \int_\mathbb{R} [B_{xx} + 2B^3 + (\alpha^2 + \beta^2)B] \Lambda_\alpha B = 4\alpha \partial_\alpha E[B] = 16\alpha^2 \beta > 0.$$

This last identity proves (3.2). Following a similar analysis, one has from (2.9) and (2.24),

$$\int_\mathbb{R} \Lambda_\beta B \mathcal{L}[\Lambda_\beta B] = 4\beta \int_\mathbb{R} [B_{xx} + 2B^3 - (\alpha^2 + \beta^2)B] \Lambda_\beta B$$

$$= -4\beta \partial_\beta E[B] - 8\beta (\alpha^2 + \beta^2) = -16\alpha^2 \beta < 0.$$  

Therefore, (3.3) is proved. \hfill $\Box$

A direct consequence of the identities above and Corollary (2.4) is the following result:

**Corollary 3.4.** With the notation of Lemma 3.3 let

$$B_0 := \frac{\alpha \Lambda_\beta B + \beta \Lambda_\alpha B}{8\alpha \beta (\alpha^2 + \beta^2)}.$$  

Then $B_0$ is Schwartz and satisfies $\mathcal{L}[B_0] = -B$,

$$\int_\mathbb{R} B_0 B = \frac{1}{4\beta (\alpha^2 + \beta^2)} > 0, \quad \text{and} \quad \frac{1}{2} \int_\mathbb{R} B_0 \mathcal{L}[B_0] = -\frac{1}{8\beta (\alpha^2 + \beta^2)} < 0.$$  

*Remark 3.1.* In other words, $B_0$ is also a negative direction, is not orthogonal to the breather itself, and the constant involved in (3.5) is time independent.

It turns out that the most important consequence of (3.2) is the fact that $\mathcal{L}$ possesses, for all time and even for 5th and 7th mKdV breathers, only one negative eigenvalue. Indeed, in order to prove that result, we follow the Greenberg and Maddocks-Sachs strategy [12, 24], applied this time to the linear, oscillatory operator $\mathcal{L}$. More specifically, we will use the following:
Hence, we get that

\[ \sum_{x \in \mathbb{R}} \dim \ker W[B_1, B_2](t; x) \]

negative eigenvalues, counting multiplicity. Here, \( W \) is the Wronskian matrix of the functions \( B_1 \) and \( B_2 \),

\[ W[B_1, B_2](t; x) := \begin{bmatrix} B_1 & B_2 \\ (B_1)_x & (B_2)_x \end{bmatrix}, \quad (t, x). \quad (3.6) \]

**Proof.** This result is essentially contained in [12, Theorem 2.2], where the finite interval case was considered. As shown in several articles (see e.g. [24, 15]), the extension to the real line is direct and does not require additional efforts. We skip the details. \( \square \)

In what follows, we compute the Wronskian (3.6), and we obtain this simple expression for the determinant of (3.6) in the case of 5th and 7th mKdV breathers.

**Lemma 3.6.** Let \( B = B_{\alpha, \beta} \) be any 5th or 7th mKdV breather, and \( B_1, B_2 \) the corresponding kernel elements defined in (2.10). Then

\[ \det W[B_1, B_2](t; x) = -\frac{8\alpha^3\beta^3(\alpha^2 + \beta^2)[\alpha \sinh(2\beta y_2) - \beta \sin(2\alpha y_1)]}{(\alpha^2 + \beta^2 + \alpha^2 \cosh(2\beta y_2) - \beta^2 \cos(2\alpha y_1))^2}. \quad (3.7) \]

**Proof.** We split the proof in two cases, depending on whether we deal with a 5th or 7th order breather. In the first case, let \( B \) a 5th order mKdV breather. We start with a very useful simplification. We claim that

\[ \det W[B_1, B_2](x) = 2(\alpha^2 + \beta^2) \left[ -BB_{11} + \frac{1}{2} \partial_x^2 \partial_x \log \left( G^2 + F^2 \right) \right] \quad (3.8) \]

with \( \tilde{B} = \tilde{B}(t; x; x_1, x_2) \) defined in (2.11), and \( \tilde{B}_1, \tilde{B}_1, i, j = 1, 2, \) as in (2.4). In order to prove the above simplification, we start from (2.25), and taking derivative with respect to \( x_1 \) and \( x_2 \), we get

\[ \begin{align*}
(B_1)_t - (\alpha^2 + \beta^2)^2 B_1 + 2(\beta^2 - \alpha^2) [(B_1)_{xx} + 6B^2 B_1] &= 0, \\
(B_2)_t - (\alpha^2 + \beta^2)^2 B_2 + 2(\beta^2 - \alpha^2) [(B_2)_{xx} + 6B^2 B_2] &= 0.
\end{align*} \quad (3.9) \]

Multiplying the first equation above by \( B_2 \) and the second by \( -B_1 \), and adding both equations, we obtain

\[ \tilde{B}_1 B_2 - (\tilde{B}_2) B_1 + 2(\beta^2 - \alpha^2) [(B_1)_{xx} B_2 - (B_2)_{xx} B_1] = 0, \]

that is,

\[ ((B_1)_x B_2 - (B_2)_x B_1)_x = \frac{1}{2(\beta^2 - \alpha^2)} [(\tilde{B}_2)_t B_1 - (\tilde{B}_1)_t B_2]. \quad (3.10) \]

On the other hand, since we are working with smooth functions, one has \( B = \tilde{B}_1 + \tilde{B}_2 \),

\[ B_1 = \tilde{B}_1 + \tilde{B}_1, \quad B_2 = \tilde{B}_2 + \tilde{B}_2, \]

and

\[ (\tilde{B}_1)_t = \delta_5 \tilde{B}_1 + \gamma_5 \tilde{B}_1, \quad (\tilde{B}_2)_t = \delta_5 \tilde{B}_2 + \gamma_5 \tilde{B}_2. \]

Replacing in (3.10), we get

\[ ((B_1)_x B_2 - (B_2)_{xx} B_1)_x = \frac{(\delta_5 - \gamma_5)}{2(\beta^2 - \alpha^2)} (\tilde{B}_2)_t - \tilde{B}_1 \tilde{B}_2). \]

Now, integrating in \( x \), using \( \delta_5, \gamma_5 \) in (2.25) and the nonlinear identity (2.16), we get the identity showed in (3.8). Now, let see how to get (3.7) from (3.8). With notation,

\[ G = \beta \sin(\alpha y_1) \quad \text{and} \quad F = \alpha \cosh(\beta y_2), \quad (3.11) \]

we define the following functions:

\[ G := G, G_1 := G_x, G_2 := G_{xx}, G_3 := G_{xxx}, G_4 := G_{x_{1x_1}}, G_5 := G_{x_{1x_1x_1}}, \]

\[ F := F, F_1 := F_x, F_2 := F_{xx}, F_3 := F_{xxx}, F_4 := F_{x_{1x_1}}, F_5 := F_{x_{1x_1x_1}}. \]

Hence, we get that

\[ B = (\tilde{B})_x = \frac{2(FG_1 - F_1 G)}{D}, \quad (3.12) \]

\[ \sum_{x \in \mathbb{R}} \dim \ker W[B_1, B_2](t; x) \]
with \( D = F^2 + G^2 \). Moreover,
\[
\hat{B}_{11} = \frac{2}{D^2} \left[ -G^2(F_4G - 2F_2G_2) - F^2(F_4G + 2F_2G_2) + F^3G_4 + FG(2F_2^2 - 2G^2 + GG_4) \right].
\tag{3.13}
\]
Then, by combining (3.12) and (3.13),
\[
-B\hat{B}_{11} = \frac{4M_1}{D^3},
\tag{3.14}
\]
where
\[
M_1 = (FG_1 - F_1G)[G^2(F_4G - 2F_2G_2) + F^2(F_4G + 2F_2G_2) - F^3G_4 - FG(2F_2^2 - 2G^2 + GG_4)].
\]

For the other hand,
\[
\partial_{x_1}^2 \partial_{x} \log (G^2 + F^2) = \frac{2M_2}{D^3},
\tag{3.15}
\]
where
\[
M_2 := 8(FF_1 + GG_1)(FF_2 + GG_2)^2 - 2D(FF_1 + GG_1)(FF_4 + F_2^2 + GG_4 + G^2) - 4D(FF_2 + GG_2)(F_1F_2 + FF_3 + G_1G_2 + GG_3) + D^2(2F_2F_3 + F_1F_4 + FF_5 + 2G_2G_3 + G_1G_4 + GG_5).
\]

So, putting (3.14) and (3.15) together, we get that (3.8) becomes
\[
-B\hat{B}_{11} + \frac{1}{2} \partial_{x_1}^2 \partial_{x} \log (G^2 + F^2) = \frac{4M_1 + M_2}{D^3}.
\]
Indeed, it is possible to see that the above numerator reduces to
\[
4M_1 + M_2 = M_3 D,
\]
where
\[
M_3 := 4(FG_1 - F_1G)(F_4G - 2F_2G_2) - 4F^2G_1G_4 + 8GG_1G_2^2 + 4FF_2GG_4 + 8FF_1F_2^2 - 2(FF_1 + GG_1)(FF_4 + F_2^2 + GG_4 + G^2) - 4(FF_2 + GG_2)(F_1F_2 + FF_3 + G_1G_2 + GG_3) + D(2F_2F_3 + F_1F_4 + FF_5 + 2G_2G_3 + G_1G_4 + GG_5).
\]

We verify, using the symbolic software Mathematica, that after substituting \( F \)'s and \( G \)'s terms explicitly in \( M_3 \), the following identity
\[
M_3 = -2\alpha^3 \beta^3 \left[ \alpha \cosh(\beta y_2) \sinh(\beta y_2) - \beta \sin(\alpha y_1) \cos(\alpha y_1) \right] = -\alpha^3 \beta^3 \left[ \alpha \sinh(2\beta y_2) - \beta \sin(2\alpha y_1) \right].
\]
Finally,
\[
-B\hat{B}_{11} + \frac{1}{2} \partial_{x_1}^2 \partial_{x} \log (G^2 + F^2) = \frac{-\alpha^3 \beta^3 \left[ \alpha \sinh(2\beta y_2) - \beta \sin(2\alpha y_1) \right]}{D^2} = \frac{-4\alpha^3 \beta^3 \left[ \alpha \sinh(2\beta y_2) - \beta \sin(2\alpha y_1) \right]}{[\alpha^2 + \beta^2 + \alpha^2 \cosh(2\beta y_2) - \beta^2 \cos(2\alpha y_1)]^2}.
\tag{3.16}
\]
Therefore, from (3.8) and (3.16),
\[
\det W[B_1, B_2](t, x) = -\frac{8\alpha^3 \beta^3 (\alpha^2 + \beta^2) \cdot [\alpha \sinh(2\beta y_2) - \beta \sin(2\alpha y_1)]}{[\alpha^2 + \beta^2 + \alpha^2 \cosh(2\beta y_2) - \beta^2 \cos(2\alpha y_1)]^2}.
\]

Consider now the case of a 7th order mKdV breather \( B \). Note first that we do not have at hand a nonlinear identity like (3.8) since 7th order mKdV breathers does not satisfy simple identities like the ones (2.10) for the 5th breather. Therefore, we will prove (3.7) directly. With notation (2.27), we write the Wronskian’s terms as follows:

\[
B_1 := \frac{G_{11}}{N^2},
\]
\[
G_{11} := 2\alpha^2 \beta \left( \alpha^3 \sin(\alpha y_1) \cosh^3(\beta y_2) + \alpha^2 \beta \cos(\alpha y_1) \sinh(\beta y_2) \cosh^2(\beta y_2) + \beta^3 \sin^2(\alpha y_1)(-\cos(\alpha y_1)) \sinh(\beta y_2) + \frac{1}{4} \alpha \beta^2 \cosh(\beta y_2) (\sin^3(\alpha y_1) + 5 \sin(\alpha y_1) + 3 \sin(\alpha y_1) \cos^2(\alpha y_1)) \right).
\tag{3.17}
\]
\[ (B_1)_x := \frac{G_{12}}{N^3}, \]
\[ G_{12} := -\alpha^2 \beta \left( -4 \alpha^6 \beta \sin(\alpha y_1) \sinh(\beta y_2) \cosh(\beta y_2) \right) \]
\[ - 2 \alpha^4 \beta^2 \cos(\alpha y_1) \sinh^3(\beta y_2) \left( 3 \sin^2(\alpha y_1) - 3 \cos^2(\alpha y_1) + \sinh^2(\beta y_2) + \cosh^2(\beta y_2) \right) \]
\[ - 24 \alpha^3 \beta^3 \sin(\alpha y_1) \cos^2(\alpha y_1) \sinh(\beta y_2) \cosh(\beta y_2) + 24 \beta^4 \sin^2(\alpha y_1) \cos(\alpha y_1) \cosh(\beta y_2) \left[ -17 \alpha^2 - \beta^2 \right. \]
\[ + \left( \beta^2 - \alpha^2 \right) \left( \cos^2(\alpha y_1) - \sin^2(\alpha y_1) \right) + 6 \alpha^2 \cosh(2 \beta y_2) \right] + 24 \left( \alpha^2 + \beta^2 \right) \]
\[ \cos(\alpha y_1) \cosh^5(\beta y_2) + 2 \alpha^3 \beta^3 \sin^3(\alpha y_1) \sinh(\beta y_2) \left( -\sin^2(\alpha y_1) + \cos^2(\alpha y_1) + 3 \right) \right) ; \]
\[ B_2 := \frac{G_{22}}{N^3}, \]
\[ G_{22} := -2 \alpha^2 \beta \left( 3 \cos(\alpha y_1) \sinh(\beta y_2) \cosh(\beta y_2) + 2 \beta \sin(\alpha y_1) \cosh(\beta y_2) \right) \]
\[ - \frac{1}{2} \alpha \beta \sin(\alpha y_1) \sinh(\beta y_2) \left( 2 \beta \sin(\alpha y_1) \cos(\alpha y_1) + 4 \alpha \sinh(\beta y_2) \cosh(\beta y_2) \right) \; \]
\[ (B_2)_x := \frac{G_{21}}{N^3}, \]
\[ G_{21} := 2 \alpha^2 \beta \left( -2 \alpha^5 \beta \cos(\alpha y_1) \sinh^5(\beta y_2) + 4 \alpha^5 \beta \cos(\alpha y_1) \sinh^3(\beta y_2) \cosh(\beta y_2) + \alpha^2 \beta^2 \sin(\alpha y_1) \cosh(\beta y_2) \right) \]
\[ \sinh(\beta y_2) \cosh(\beta y_2) \left( 2 \left( 3 \alpha^2 + \beta^2 \right) + \left( 3 \alpha^2 - 2 \beta^2 \right) \left( \cos^2(\alpha y_1) - \sin^2(\alpha y_1) \right) - 3 \alpha^2 \left( \sinh^2(\beta y_2) + \cosh^2(\beta y_2) \right) \right) \]
\[ + \alpha^4 \left( \alpha^2 + 5 \beta^2 \right) \sin(\alpha y_1) \sinh(\beta y_2) \cosh(\beta y_2) - \beta \sin(\alpha y_1) \sinh(\beta y_2) \left[ 12 \alpha^3 \sin(\alpha y_1) \cos(\alpha y_1) \sinh(\beta y_2) \cosh(\beta y_2) \right] \]
\[ + \alpha^2 \beta \sin^4(\alpha y_1) - 2 \alpha^2 \beta^2 \sin^2(\alpha y_1) \sinh^2(\beta y_2) + \frac{1}{2} \alpha^2 \beta \left[ - \frac{1}{2} \sin^4(\alpha y_1) - \frac{1}{2} \cos^4(\alpha y_1) + \sin^2(\alpha y_1) \cosh(\beta y_2) + \frac{1}{2} \right] \]
\[ + \beta^3 \sin^4(\alpha y_1) \right) + 2 \alpha^3 \beta^3 \sin^3(\alpha y_1) \cos(\alpha y_1) \cosh(\beta y_2) \right) \).

Hence, we re-write the Wronskian (3.6) as:
\[
(B_2)_xB_1 - (B_1)_xB_2 = \frac{\bar{G}}{N^2}, \quad \bar{G} := N(G_{11}G_{21} + G_{22}G_{12}). \tag{3.21}
\]
Expanding $G_{11}$, $G_{12}$, $G_{21}$, $G_{22}$ terms, it is possible to see that, after lengthy but direct computations,
\[
N(G_{11}G_{21} + G_{22}G_{12}) = N \left( 8 \alpha^3 \beta^3 \left( \alpha^2 + \beta^2 \right) \left( \alpha \sinh(2 \beta y_2) - \beta \sin(2 \alpha y_1) \right) \right) .
\]
Finally, substituting in (3.21) the expression above, we obtain:
\[
(B_2)_xB_1 - (B_1)_xB_2 = - \frac{8 \alpha^3 \beta^3 \left( \alpha^2 + \beta^2 \right) \left( \alpha \sinh(2 \beta y_2) - \beta \sin(2 \alpha y_1) \right)}{\left( \alpha^2 + \beta^2 + \alpha^2 \cosh(2 \beta y_2) - \beta^2 \cosh(2 \alpha y_1) \right)^2}. \tag{3.22}
\]
\[ \square \]

Proposition 3.7. The operator $\mathcal{L}$ defined in 3.14 (associated with 5th and 7th $mKdV$ equations) has a unique negative eigenvalue $-\lambda_0^2 < 0$, of multiplicity one, and $\lambda_0 = \lambda_0(\alpha, \beta, x_1, x_2, t)$.

Proof. We compute the determinant (3.7) required by Lemma 3.6. From Lemma 3.6 after a standard translation argument, we just need to consider the behavior of the function
\[
f(y_2) = f_{\alpha, \beta, \delta}(y_2) := \alpha \sinh(2 \beta y_2) - \beta \sin(2 \alpha y_2 + (\delta_1 - \gamma_i)t + \bar{x}_2) \),
\]
for $\bar{x}_2 := x_1 - x_2 \in \mathbb{R}$ and
\[
\delta_1 - \gamma_i = \begin{cases} 
4(\alpha^4 - \beta^4), & \text{if } i = 5, \\
-6\alpha^6 + 14\alpha^4 \beta^2 + 14\alpha^2 \beta^4 - 6\beta^6, & \text{if } i = 7.
\end{cases}
\]
A simple argument shows that for $y_2 \in \mathbb{R}$ such that $|\sinh(2 \beta y_2)| > \frac{\beta}{\alpha}$, $f$ has no root. Moreover, there exists $R_0 = R_0(\alpha, \beta) > 0$ such that, for all $y_2 > R_0$, one has $f(y_2) > 0$ and for all $y_2 < -R_0$, $f(y_2) < 0$. Therefore, since $f$ is continuous, there is a root $y_0 = y_0(t, \alpha, \beta, \bar{x}_2) \in [-R_0, R_0]$ for $f$. Additionally, if $y_2 \neq 0$,}
\[ f'(y_2) = 2\alpha \beta \left[ \cosh(2\beta y_2) - \cos(2\alpha (y_2 - (\delta_i - \gamma_i) t + \tilde{x}_2)) \right] > 0. \]

Therefore, if \( y_0 \neq 0 \) then
\[
\dim \ker W[B_1, B_2](t; x) = \dim \ker W[B_1, B_2](t; y_0 - \gamma_i t - x_2) = 1.
\]

\[ \square \]

We consider now some standard remarks. We can reduce the spectral problem to another independent of time. Indeed, from (1.27) and after translation and redefinition of the parameters \( x_1 \) and \( x_2 \), we can assume that
\[
B = B_{\alpha, \beta}(0, x, x_1, 0), \quad x_1 \in \left[ 0, \frac{2\pi}{\alpha} \right].
\]

**Corollary 3.8.** There exists a continuous function \( f_0 = f_0(\alpha, \beta) \), well-defined for all \( \alpha, \beta > 0 \), and such that
\[
-\lambda_0^2 < f_0(\alpha, \beta) < 0,
\]
for all \( \alpha, \beta > 0 \) and \( t, x_1, x_2 \in \mathbb{R} \).

**Proof.** From the variational characterization of the first eigenvalue via minimization problem we have
\[
-\lambda_0^2 \leq \frac{\int_{\mathbb{R}} \mathcal{L}[\Lambda_{\alpha \beta} B] \Lambda_{\alpha \beta} B}{\int_{\mathbb{R}} (\Lambda_{\alpha \beta} B)^2} = \frac{-16\alpha^2 \beta}{\int_{\mathbb{R}} (\Lambda_{\alpha \beta} B)^2}.
\]

Since \( \int_{\mathbb{R}} (\Lambda_{\alpha \beta} B)^2 \) is continuous on \( x_1 \in \left[ 0, \frac{2\pi}{\alpha} \right] \), we conclude this proof. \[ \square \]

Let \( z \in H^2(\mathbb{R}) \) and \( B = B_{\alpha, \beta} \) be any 5th or 7th mKdV breather. Let us consider the quadratic from associated to \( \mathcal{L} \):
\[
\mathcal{Q}[z] := \int_{\mathbb{R}} z \mathcal{L}[z] = \int_{\mathbb{R}} z_{xx}^2 + 2(\beta^2 - \alpha^2) \int_{\mathbb{R}} z_x^2 + (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} z^2 - 10 \int_{\mathbb{R}} B^2 z_x^2 - 10 \int_{\mathbb{R}} B^2 z^2 - 40 \int_{\mathbb{R}} B B_x z z_x + 30 \int_{\mathbb{R}} B^4 z^2 - 12(\beta^2 - \alpha^2) \int_{\mathbb{R}} B^2 z^2.
\]

**Remark 3.2.** From the definition of \( \mathcal{Q} \) and Lemma 3.2 it is clear that \( \mathcal{Q}[B_1] = \mathcal{Q}[B_2] = 0 \). Moreover, inequality (3.2) means that \( \Lambda_{\alpha \beta} B \) is actually a positive direction for \( \mathcal{Q} \), a completely unexpected result. Additionally, from (3.24) \( \mathcal{Q} \) is bounded below, namely
\[
\mathcal{Q}[z] \geq -c_{\alpha, \beta} \|z\|^2_{H^2(\mathbb{R})},
\]
for some positive constant \( c_{\alpha, \beta} \) depending on \( \alpha \) and \( \beta \) only.

Let \( B_{-1} \in \mathcal{S} \setminus \{0\} \) be an eigenfunction associated to the unique negative eigenvalue of the operator \( \mathcal{L} \), as stated in Proposition 3.1. We assume that \( B_{-1} \) has unit \( L^2 \)-norm, so \( B_{-1} \) is now unique. In particular, one has \( \mathcal{L}[B_{-1}] = -\lambda_0^2 B_{-1} \). It is clear from Proposition 3.7 and Lemma 3.2 that the following result holds.

**Lemma 3.9.** The eigenvalue zero is isolated. Moreover, there exists a continuous function \( \nu_0 = \nu_0(\alpha, \beta) \), well-defined and positive for all \( \alpha, \beta > 0 \) and such that, for all \( z_0 \in H^2(\mathbb{R}) \) satisfying
\[
\int_{\mathbb{R}} z_0 B_{-1} = \int_{\mathbb{R}} z_0 B_1 = \int_{\mathbb{R}} z_0 B_2 = 0,
\]
then
\[
\mathcal{Q}[z_0] \geq \nu_0 \|z_0\|^2_{H^2(\mathbb{R})}.
\]

**Proof.** The isolatedness of the zero eigenvalue is a direct consequence of standard elliptic estimates for the eigenvalue problem associated to \( \mathcal{L} \), corresponding uniform convergence on compact subsets of \( \mathbb{R} \), and the non-degeneracy of the kernel associated to \( \mathcal{L} \).

On the other hand, the existence of a positive constant \( \nu_0 = \nu_0(\alpha, \beta, x_1) \) such that (3.26) is satisfied is now clear. Moreover, thanks to the periodic character of the variable \( x_1 \), and the non-degeneracy of the kernel, we obtain a uniform, positive bound independent of \( x_1, x_2 \) and \( t \), still denoted \( \nu_0 \). The proof is complete. \( \square \)

It turns out that \( B_{-1} \) is hard to manipulate; we need a more tractable version of the previous result.
Proposition 3.10. Let $B = B_{\alpha, \beta}$ be any 5th or 7th-mKdV breather, and $B_1, B_2$ the corresponding kernel of the associated operator $L$. There exists $\mu_0 > 0$, depending on $\alpha, \beta$ only, such that, for any $z \in H^2(\mathbb{R})$ satisfying
\[ \int_{\mathbb{R}} B_1 z = \int_{\mathbb{R}} B_2 z = 0, \]
(3.27)
one has
\[ Q[z] \geq \mu_0 \|z\|_{H^2(\mathbb{R})}^2 - \frac{1}{\mu_0} \left( \int_{\mathbb{R}} z B \right)^2. \]
(3.28)
Proof. This result is contained in [4, Prop. 4.11] and therefore we do not include it here. □

4. Variational characterization of higher order mKdV breathers

In this section we introduce a suitable $H^2$-Lyapunov functional for both 5th and 7th-mKdV equations (1.2) and (1.3). Consider $u_0 \in H^2(\mathbb{R})$ and let $u = u(t) \in H^2(\mathbb{R})$ be the associated local in time solution of the Cauchy problem associated to (1.2) or (1.3), with initial condition $u(0) = u_0$ (cf. [23], [20], [14]).

Using the functional $E_3[u]$ given in (1.14) which is a conserved quantity for both 5th and 7th order mKdV equations, we build a new $H^2$-Lyapunov functional specifically associated to the breather solution. Let $B = B_{\alpha, \beta}$ be any 5th or 7th-mKdV breather and $t \in \mathbb{R}$. Consider $M[u]$ and $E[u]$ given in (1.12) and (1.13) respectively. We define
\[ \mathcal{H}[u(t)] := E_3[u](t) + 2(\beta^2 - \alpha^2)E[u](t) + (\alpha^2 + \beta^2)^2 M[u](t), \]
(4.1)

Therefore, $\mathcal{H}[u]$ is a real-valued conserved quantity, well-defined for $H^2$-solutions of (1.2) and (1.3). Moreover, one has the following:

Lemma 4.1. 5th and 7th-mKdV breathers (1.23) are critical points of the Lyapunov functional $\mathcal{H}$ (4.1). In fact, for any $z \in H^2(\mathbb{R})$ with sufficiently small $H^2$-norm, and $B = B_{\alpha, \beta}$ any 5th and 7th-mKdV breather solution, then, for all $t \in \mathbb{R}$, one has
\[ \mathcal{H}[B + z] - \mathcal{H}[B] = \frac{1}{2} Q[z] + \mathcal{N}[z], \]
(4.2)
with $Q$ being the quadratic form defined in (3.24), and $\mathcal{N}[z]$ satisfying $|\mathcal{N}[z]| \leq Kz^3_{H^2(\mathbb{R})}$.

Proof. Considering any 5th or 7th-mKdV breather $B$, we compute
\[
\begin{align*}
\mathcal{H}[B + z] &= \frac{1}{2} \int_{\mathbb{R}} (B + z)^2_{xx} - 5 \int_{\mathbb{R}} (B + z)^2(B + z)^2 + \int_{\mathbb{R}} (B + z)^6 \\
&\quad + (\beta^2 - \alpha^2)^2 \left[ \frac{1}{2} \int_{\mathbb{R}} (B + z)^2 - (\beta^2 - \alpha^2) \int_{\mathbb{R}} (B + z)^4 + \frac{1}{2} (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} (B + z)^2 \right] \\
&= \frac{1}{2} \int_{\mathbb{R}} B_{xx}^2 - 5 \int_{\mathbb{R}} B^2 B_{xx} + \int_{\mathbb{R}} B^6 + (\beta^2 - \alpha^2)^2 \left[ \frac{1}{2} \int_{\mathbb{R}} B_{xx}^2 - \frac{1}{2} (\beta^2 - \alpha^2) \right] \int_{\mathbb{R}} B^4 + \frac{1}{2} (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} B^2 \\
&\quad + \int_{\mathbb{R}} \left[ B_{4x} - 2(\beta^2 - \alpha^2)B_{xx} + 2B^3 \right] + (\alpha^2 + \beta^2)^2 B + 10BB_{x}^2 + 10B^2B_{xx} + 6B^5 \\
&+ \frac{1}{2} \left[ \int_{\mathbb{R}} z B_{xx}^2 + 2(\beta^2 - \alpha^2) \int_{\mathbb{R}} z^2_{xx} + (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} z^2 + 10 \int_{\mathbb{R}} z B_{zz} \right] \\
&\quad + 20 \int_{\mathbb{R}} B B_{zz} + \int_{\mathbb{R}} (30B^4 - 10B_{x}^2 - 12(\beta^2 - \alpha^2)B^2)^2 \\
&\quad - 5 \int_{\mathbb{R}} (z^2_{x}^2 + 2B_x z_{xx} + 2B_{zz} z_{x}^2) + \int_{\mathbb{R}} 5B^3 z_{x}^3 + \frac{15}{4} B^2 z_{x}^2 + \frac{3}{2} B_{x} z_{x}^2 + \frac{1}{4} \int_{\mathbb{R}} z^6 \\
&\quad - 2(\beta^2 - \alpha^2) \int_{\mathbb{R}} B z^3 - \frac{1}{2} (\beta^2 - \alpha^2) \int_{\mathbb{R}} z^4.
\end{align*}
\]

We finally obtain:
\[ \mathcal{H}[B + z] = \mathcal{H}[B] + \int_{\mathbb{R}} G[B] z(t) \, dx + \frac{1}{2} Q[z] + \mathcal{N}[z], \]
where $Q$ is defined in (3.24) and
\[ G[B] := B_{4x} + 10BB_{x}^2 + 10B^2B_{xx} + 6B^5 - 2(\beta^2 - \alpha^2)B_{xx} + 2B^3 + (\alpha^2 + \beta^2)^2 B. \]
From Theorem (2.1), one has \(G[B] = 0\). Finally, the term \(\mathcal{N}[z]\) is given by

\[
\mathcal{N}[z] := -10 \int_{\mathbb{R}} B_{zz}^2 + \frac{10}{3} \int_{\mathbb{R}} [B_{xzz} z^3 - 5z^2 z_x^2 + 20B^2 z^3 + 15B^2 z^4 + 6Bz^5 + z^6] - 4(\beta^2 - \alpha^2) \int_{\mathbb{R}} Bz^3 - (\beta^2 - \alpha^2) \int_{\mathbb{R}} z^4.
\]

(4.3)

Therefore, from direct estimates one has \(|\mathcal{N}[z]| \leq \mathcal{O}(\|z\|_{H^2(\mathbb{R})}^3)\) as desired.

\[\square\]

Using the previous Lemma, we are able to prove the main result of the paper.

5. MAIN THEOREM

**Theorem 5.1 (H^2-stability of 5th and 7th Order mKdV breathers).** Let \(\alpha, \beta \in \mathbb{R}\setminus\{0\}\) and \(B = B_{\alpha,\beta}\) any 5th or 7th order mKdV breather. There exist positive parameters \(\eta_0, A_0\), depending on \(\alpha\) and \(\beta\), such that the following holds. Consider \(w_0 \in H^2(\mathbb{R})\), and assume that there exists \(\eta \in (0, \eta_0)\) such that

\[
\|w_0 - B(t = 0, 0, 0)\|_{H^2(\mathbb{R})} \leq \eta.
\]

(5.1)

Then there exist \(x_1(t), x_2(t) \in \mathbb{R}\) such that the solution \(w(t)\) of the Cauchy problem for the 5th-mKdV (1.2) or for the 7th-mKdV (1.3) equations, with initial data \(w_0 \in H^2(\mathbb{R})\), satisfies

\[
\sup_{t \in \mathbb{R}} \|w(t) - B(t; x_1(t), x_2(t))\|_{H^2(\mathbb{R})} \leq A_0 \eta,
\]

(5.2)

with

\[
\sup_{t \in \mathbb{R}} |x_1'(t)| + |x_2'(t)| \leq K A_0 \eta,
\]

(5.3)

for a constant \(K > 0\).

**Remark 5.1.** The initial condition (5.1) can be replaced by any initial breather profile of the form \(\tilde{B} := B_{\alpha,\beta}(t_0; x_1^0, x_2^0)\), with \(t_0, x_1^0, x_2^0 \in \mathbb{R}\), thanks to the invariance of the equation under translations in time and space. In addition, a similar result is available for the negative breather \(-B_{\alpha,\beta}\) which is also a solution of (1.2) or (1.3).

**Proof of Theorem 5.1.** Once we guaranteed for the case of 5th and 7th-mKdV breathers, that they satisfy the same 4th order stationary ODE (2.35) than the classical mKdV breather, that a suitable coercivity property holds for the bilinear form \(Q\) associated to any of these higher order breathers (see Proposition 3.10 and Lemma 3.9), and the existence of a unique negative eigenvalue (3.7) of the linearized operator \(L\) associated again to these higher order breathers, the stability proof follows the same steps and is completely similar to the proof of the \(H^2\)-stability of mKdV breathers [3, Theorem (6.1)].

\[\square\]

APPENDIX A. 9TH AND 11TH-MKDV EQUATIONS

The 9th order mKdV equation is written as follows

\[
u_t + \partial_x \left( u_{8x} + 18u^2 u_{6x} + 108u u_{x} u_{5x} + 228u u_{2x} u_{4x} + 210(u_x)^2 u_{4x} + 126u^4 u_{4x} + 138u(u_{3x})^2 \right) + 750u u_{x} u_{x} u_{3x} + 108u^3 u_{x} u_{3x} + 182(u_{2x})^3 + 756u^3 (u_{2x})^2 + 3108u^2 u_{2x}^2 + 420u^6 u_{2x} + 798u(u_x)^4 + 1260u^5(u_x)^2 + 79u^9 = 0.
\]

(A.3)

**Definition A.1 (9th-mKdV breather).** Let \(\alpha, \beta > 0\) and \(x_1, x_2 \in \mathbb{R}\). The real-valued breather solution of the 9th-mKdV equation (A) is given explicitly by the formula

\[
B(t, x; x_1, x_2) := 2\partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right],
\]

(A.4)

with \(y_1\) and \(y_2\)

\[
y_1 = x + \delta_1 t + x_1, \quad y_2 = x + \gamma_2 t + x_2,
\]

(A.5)

and with velocities

\[
\delta_1 := -\alpha^8 + 36\alpha^6 \beta^6 - 126\alpha^4 \beta^4 + 84\alpha^3 \beta^6 - 9\beta^8, \quad \gamma_2 := -\beta^8 + 36\alpha^2 \beta^6 - 126\alpha^4 \beta^4 + 84\alpha^6 \beta^2 - 9\alpha^8.
\]

(A.6)
The 11th order mKdV equation is written as follows

\[ u_t + \partial_x \left( u_{10x} + 22u^2 u_{8x} + 198u^4 u_{6x} + 924u^6 u_{4x} + 506u(u_{4x})^2 + 3036u^3 (u_{3x})^2 + 2310u^8 u_{xx} \right. \\
+ 8316u^5 (u_{5x})^2 + 9372u^2 (u_{6x})^2 + 9240u^7 (u_{7x})^2 + 26796u^3 (u_{3x})^4 + 176uu_{7x} u_{7x} + 484uu_{xx} u_{6x} + 462 (u_x)^2 u_{6x} \\
+ 836uu_{3x} u_{5x} + 2376u^3 u_x u_{5x} + 5016u^3 u_{xx} u_{4x} + 2706(u_{xx})^2 u_{4x} + 11220u^2 (u_x)^2 u_{4x} + 3498u_{xx}(u_{3x})^2 \\
\left. + 11088u_5 u_{3x} u_{3x} + 21120(u_x)^3 u_{3x} + 54516u^4 (u_x)^2 u_{xx} + 44748(u_x)^2 (u_{xx})^2 + 13398 (u_x)^4 u_{xx} + 2376u_x u_{xx} u_{5x} \\
+ 3696u_x u_{3x} u_{4x} + 39336u^4 u_{xx} u_{3x} + 252u^{11} \right) = 0. \]

(A.7)

**Definition A.2 (11th-mKdV breather).** Let \( \alpha, \beta > 0 \) and \( x_1, x_2 \in \mathbb{R} \). The real-valued breather solution of the 11th-mKdV equation \( (A) \) is given explicitly by the formula

\[ B = B_{\alpha, \beta}(t, x; x_1, x_2) := 2\partial_x \left[ \arctan \left( \frac{\alpha \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right], \quad \text{(A.8)} \]

with \( y_1 \) and \( y_2 \)

\[ y_1 = x + \delta_{11} t + x_1, \quad y_2 = x + \gamma_{11} t + x_2, \quad \text{(A.9)} \]

and with velocities

\[ \delta_{11} = \alpha^{10} - 55\alpha^8 \beta^2 + 330\alpha^6 \beta^4 - 462\alpha^4 \beta^6 + 165\alpha^2 \beta^8 - 11\beta^{10}, \]

\[ \gamma_{11} = 11\alpha^{10} - 165\alpha^8 \beta^2 + 462\alpha^6 \beta^4 - 330\alpha^4 \beta^6 + 55\alpha^2 \beta^8 - \beta^{10}. \]

(A.10)

**Appendix B. Periodic breathers of 5th and 7th order mKdV equations**

**B.1. Definitions.** We consider now the case of the higher order, 5th and 7th periodic (in space) mKdV equations. A kind of periodic mKdV breathers (named KKSH breathers), were found by Kevrekidis, Khare, Saxena and Herring [18][19] by using elliptic functions and a matching of free parameters. See [18][19][18] for further reading on these classical periodic mKdV breathers. For the higher order mKdV equations (1.2) and (1.3), an equivalent expression of periodic breathers is available, by following a similar matching of parameters. Namely, we consider the 5th and 7th mKdV equations (1.2) (1.3) where now

\[ u : \mathbb{R}_t \times \mathbb{T}_x \mapsto \mathbb{R}_x, \]

is periodic in space, and \( \mathbb{T}_x = \mathbb{T} = \mathbb{R}/L\mathbb{Z} = (0, L) \) denotes a torus with period \( L \), to be fixed later.

**Definition B.1 (Periodic breathers of higher order mKdV equations).** Given \( \alpha, \beta > 0 \), \( x_1, x_2 \in \mathbb{R} \) and \( k, m \in [0,1] \), periodic breather solutions of the 5th and 7th mKdV equations (1.2) (1.3), are given by the explicit formula (see [18] and [8] for checking and comparison reasons)

\[ B = B(t, x; \alpha, \beta, k, m, x_1, x_2) := \partial_x B := 2\partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1,k)}{\alpha \cosh(\beta y_2, m)} \right) \right], \quad \text{(B.1)} \]

with \( \sin(\cdot, k) \) and \( \cosh(\cdot, m) \) the standard Jacobi elliptic functions of elliptic modulus \( k \) and \( m \), respectively, but now

\[ y_i := x + \delta_i t + x_i, \quad y_i := x + \gamma_i t + x_i, \quad i = 5, 7, \quad \text{(B.2)} \]

with velocities \( (\delta_5, \gamma_5) \) in the 5th order case and \( (\delta_7, \gamma_7) \) in the 7th order case given by, respectively:

\[ \delta_5 := -\alpha^4 (k^2 - 26k + 1) + 10\alpha^2 \beta^2 (1 + k)(2 - m) - 5\beta^4 (m^2 - 16m + 16), \]

\[ \gamma_5 := -\beta^4 (m^2 + 24m - 24) + 10\alpha^2 \beta^2 (1 + k)(2 - m) - 5\alpha^4 (k^2 + 14k + 1), \]

\[ \delta_7 := \alpha^6 (k^3 + 135k^2 + 135k + 1) + 21\alpha^4 \beta^2 (-2 + k^2 (m - 2) + m + 2m (7m - 6)) \]

\[ + 7\alpha^2 \beta^4 (1 + k)(5m^2 - 24m + 24) + 7\beta^6 (m^3 - 2m^2 + 48m - 48), \]

\[ \gamma_7 := -\beta^6 (-m^3 - 254m^2 - 2256m + 2512) + 7\alpha^2 \beta^4 (1 + k)(3m^2 + 88m - 88) \]

\[ + 7\alpha^4 \beta^2 (5k^2 + 1)(m - 2) + k(70m + 292)) + 7\alpha^6 (k^3 + 135k^2 + 135k + 1). \]
See [2, 10] for a more detailed account on the Jacobi elliptic functions $sn$ and $nd$ presented in (1.23). Additionally, in order to be a periodic solution of 5th and 7th-mKdV equations (and for mKdV too), the parameters $m, k, \alpha$ and $\beta$ must satisfy the following commensurability conditions on the spatial periods

$$\frac{\beta^4}{\alpha^4} = \frac{k}{1 - m}, \quad K(k) = \frac{\alpha}{2\beta} K(m),$$

where $K$ is the complete elliptic integral of the first kind, defined as [10]

$$K(r) := \int_0^{\pi/2} (1 - r \sin^2(s))^{-1/2} ds = \int_0^1 ((1 - t^2)(1 - rt^2))^{-1/2} dt,$$

and which satisfies

$$K(0) = \frac{\pi}{2} \quad \text{and} \quad \lim_{k \to 1^-} K(k) = \infty.$$

So defined, the spatial period is given by

$$L := 4\alpha K(k) = 2\beta K(m).$$

Note that conditions (B.3) formally imply that $B$ has only four independent parameters (e.g. $\beta, k$ and translations $x_1, x_2$). Additionally, if we assume that the ratio $\beta/\alpha$ stays bounded, we have that $k$ approaches 0 as $m$ is close to 1. Using this information, the standard non periodic 5th and 7th-mKdV breathers (1.2) can be formally recovered as the limit of very large spatial period $L \to +\infty$, obtained e.g. if $k \to 0$.

**Remark B.1.** First of all, note that these breathers can be written using only two parametric variables, say $\beta$ and $k$, and have a characteristic period $L = L(\beta, k)$, with $L \to +\infty$ as $k \to 0$. Moreover, compare the periodic higher order velocities $(\delta_i, \gamma_i), \ i = 5, 7$ above, with the equivalent periodic ones in the simpler mKdV case [8, Def.1.1]:

$$\delta := \alpha^2(1 + k) + 3\beta^2(m - 2), \quad \text{and} \quad \gamma := 3\alpha^2(1 + k) + \beta^2(m - 2),$$

and with velocities $(\delta_i, \gamma_i), \ i = 5, 7$ (1.25) in the non periodic case.

**Appendix C. Proof of Lemma 2.1**

This Lemma is a direct consequence of a result appeared in [26, p.792]. Namely, for (1.1), (1.2) and (1.3) equations, we select an ansatz for $u$ as

$$u(t, x) := \phi_x, \ \phi(t, x) = i \log \left( \frac{G(t, x)}{F(t, x)} \right), \ \text{where} \ \ F := F + iG, \ \ G := F - iG = F^*.$$

Substituting the above expression in (1.1), (1.2) and (1.3) and using Hirota’s bilinear operators (see more details in [24]), the following bilinearization holds

$$D_t(GF) + D_x^{(2n+1)}(GF) = 0,$$

$$D_x^2(GF) = 0,$$

where the index $n$ corresponds to the different equations considered, i.e. mKdV ($n=1$), 5th-mKdV ($n=2$) and 7th-mKdV ($n=3$). To prove Lemma 2.1 we will only need the second identity (C.1) satisfied by mKdV, 5th-mKdV and 7th-mKdV equations. In fact, dividing (C.1) by $GF$, we get the following identity:

$$\frac{D_x^2(GF)}{GF} = \partial^2_x [\log(GF)] + \left( \partial_x \log \left( \frac{G}{F} \right) \right)^2 = 0$$

and finally

$$u^2 = \frac{\partial^2}{\partial x^2} [\log(P^2 + G^2)].$$
REFERENCES

[1] M. Ablowitz and P. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, London Mathematical Society Lecture Note Series, 149. Cambridge University Press, Cambridge, 1991.

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York. Dover, 1972.

[3] M.A. Alejo, *On the ill-posedness of the Gardner equation*, J. Math. Anal. Appl., 396 no. 1, 256-260 (2012).

[4] M.A. Alejo and C. Muñoz, *Nonlinear stability of mKdV breathers*, Comm. Math. Phys., 37 (2013), 2050–2080.

[5] M.A. Alejo and C. Muñoz, *On the nonlinear stability of mKdV breathers*, J. Phys. A: Math. Theor. 45 432001 (2012).

[6] M.A. Alejo, *Nonlinear stability of Gardner breathers*, Jour. Diff. Equat. 264, n.2, 1192-1230 (2018).

[7] M.A. Alejo, C. Muñoz and José M. Palacios, *On the variational structure of breather solutions I: the Sine-Gordon case*, Jour. Math. Anal. Appl. Vol.453/2 (2017) pp. 1111-1138.

[8] M.A. Alejo, C. Muñoz and José M. Palacios, *On the variational structure of breather solutions II: periodic mKdV case*, Elect. J. Diff. Eq., Vol. 2017 (2017), No. 56, pp. 1-26.

[9] T.B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London A 328, 153–183 (1972).

[10] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd ed., Springer-Verlag, New York and Heidelberg, 1971.

[11] C.S. Gardner, M.D. Kruskal and R. Miura, *Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion*, J. Math. Phys. 9, no. 8, 1204–1209 (1968).

[12] L. Greenberg, *An oscillation method for fourth order, self-adjoint, two-point boundary value problems with nonlinear eigenvalues*, SIAM J. Math. Anal. 22 (1991), no. 2, 1021–1042.

[13] R. Grimshaw, A. Slunyaev and E. Pelinovsky, *Generation of solitons and breathers in the extended Korteweg-de Vries equation with positive cubic nonlinearity*, Chaos 20 (2010), n.1, 01310201–01310210.

[14] A. Grünrock, *On the hierarchies of higher order mKdV and KdV equations with positive cubic nonlinearity*, Chaos 20 (2010), n.1, 01310201–01310210.

[15] J. Holmer, G. Perelman and M. Zworski, *Effective dynamics of double solitons for perturbed mKdV*, to appear in Comm. Math. Phys.

[16] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. 46, 527–620 (1993).

[17] C.E. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. 106, no. 3, 617–633 (2001).

[18] P.G. Kevrekidis, A. Khare, A. Saxena and G. Herring, *On some classes of mKdV periodic solutions*, Journal of Physics A: Mathematical and General, 37, 10959-10965 (2004).

[19] P.G. Kevrekidis, A. Khare and A. Saxena, *Breather lattice and its stabilization for the modified Korteweg-de Vries equation*, Phys.Rev. E, 68, 0477011-0477014 (2003).

[20] S. Kwon, *Well posedness and Ill-posedness of the Fifth-order modified KdV equation*, Electr. Journal Diff. Equations. vol. 2008, n.1, 1–15 (2008).

[21] G.L. Lamb, *Elements of Soliton Theory*, Pure Appl. Math., Wiley, New York, 1980.

[22] P.D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. 21, 467–490 (1968).

[23] F. Linares, *A higher order modified Korteweg-de Vries equation*, Comp. Appl. Math. 14, n.3, 253-267, (1995).

[24] J.H. Maddocks and R.L. Sachs, *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math. 46, 867–901 (1993).

[25] Y. Matsuno, *Bilinearization of Nonlinear Evolution Equations: Higher Order mKdV*, Jour. Phys. Soc.Japan, 49, n.2 (1980).

[26] Y. Matsuno, *Bilinearization of Nonlinear Evolution Equations: Higher Order mKdV*, Jour. Phys. Soc.Japan, 49, n.2 (1980).

[27] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. 39, 51–68 (1986).

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