Electromagnetic Wave Propagation in a Quasi-1D Rhombic Linear Optical Waveguide Array

Andrey I. Maimistov\textsuperscript{1,2}, Viktor A. Patrikeev\textsuperscript{1}

\textsuperscript{1}: Department of General Physics, Moscow Institute of Physics and Technology, Dolgoprudny, Moscow region, 141700 Russia
\textsuperscript{2}: Department of Solid State Physics and Nanostructures, National Nuclear Research University, Moscow Engineering Physics Institute, Moscow, 115409
E-mails: aimaimistov@gmail.com, sugrobs@yandex.ru

(Dated: April 11, 2018)

The quasi-one-dimensional rhombic array of the waveguides is considered. System of equations describing coupled waves in the waveguide in the linear limit is solved exactly. The electric field distribution was found both for the diffractionless (or dispersionless) flat band modes and for the dispersive modes.

PACS numbers: 42.81.Qb, 42.70.Qs, 42.79.Gn

I. INTRODUCTION

Recently the optical simulations of the different phenomena of solid state have been developed. There is one interested example. The investigation of two dimensional electron systems demonstrated that presence of the third atom in the elementary cell as well as long range interaction in the lattice leads to emerging of a flat sheet (flat band) between conventional zones. Similar optical lattices can be realized by means of waveguides as nodes of the lattice. Some kinds of the optical lattices that demonstrate the photonic spectrum with flat band have been discussed in [1–3].

Let us consider wave guide array consisting from three parallel linear chain of waveguides. The central waveguide chain is shifted according to either chains at half lattice period. Resulting configuration seams as linear chain of the rhombus. This array of waveguides was named as the quasi-one-dimensional rhombic array [1, 4, 5].

In this article the electromagnetic field distribution in the quasi-one-dimensional rhombic array of the waveguides is obtained. All waveguides are supposed as linear waveguides. It allows using the standard technique of solution of the differential-difference equations. The exact solution of the coupled mode equations is found. These solutions demonstrate the discrete diffraction phenomenon that takes place in general case of the boundary (or initial) conditions. But in the special case of electromagnetic field excitation the flat band modes will be existed.

II. BASE EQUATIONS AND DISPERSION RELATION OF PROPAGATING MODES

Let us consider the quasi-one-dimensional rhombic waveguide array of waveguides [1, 4]. The system of equations describing coupled waves in the array has the following form [4–6]

\[ i \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \xi} \right) A_n = (B_n + B_{n-1}) + (C_n + C_{n-1}), \]
\[ i \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \xi} \right) B_n = (A_{n+1} + A_n), \]
\[ i \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \xi} \right) C_n = (A_{n+1} + A_n). \] (1)

Here \( A_n \), \( B_n \) and \( C_n \) are dimensionless slowly varying amplitudes of the electric fields in the waveguide of the \( n \)-th elementary cell, the sub-indices \( n \) are markers of the elementary sells. Phase matching condition is assumed to be satisfied, and the coupling constants between waveguides are equal to unit. Since system of equation (1) is linear, the dispersion relation can be found in a standard way.
If new variable $\zeta$ so that $\partial/\partial \tau + \partial/\partial \xi = \partial/\partial \zeta$ is introduced, then the system of equation (1) can be written as

$$i \frac{\partial A_n}{\partial \zeta} = (B_n + B_{n-1}) + (C_n + C_{n-1}),$$
$$i \frac{\partial B_n}{\partial \zeta} = (A_{n+1} + A_n),$$
$$i \frac{\partial C_n}{\partial \zeta} = (A_{n+1} + A_n).$$

(2)

Since system of equation (2) is linear, the dispersion relation can be found in a standard way:

$$q = \pm 2 \sqrt{2} |\cos(\pi s/M)|, \quad q_0(s) = 0.$$  

(3)

Here $q(s)$ is transversal wave number of the mode with index $s$, $s = -M, \ldots, -1, 0, 1, \ldots, M N = 2M + 1$ is number of the elementary sells of waveguide array. As noted in [1–3], the wave numbers $q_\pm(s)$ are correspond to waves propagating between waveguides in array. The modes with $q_0(s)$ form the flat band.

III. THE SOLUTION OF THE BASE EQUATIONS

To obtain the solution of the system of equation (2) we can use the generation function method. Let us introduce the following functions

$$P_A(\zeta, y) = \sum_{n=-\infty}^{\infty} A_n(\zeta)e^{iny}, \quad P_B(\zeta, y) = \sum_{n=-\infty}^{\infty} B_n(\zeta)e^{iny}, \quad P_C(\zeta, y) = \sum_{n=-\infty}^{\infty} C_n(\zeta)e^{iny}.$$  

From (2) the following system of equations can be obtained

$$i \frac{\partial}{\partial \zeta} P_A = \kappa^*(P_B + P_C), \quad i \frac{\partial}{\partial \zeta} P_B = \kappa P_A, \quad i \frac{\partial}{\partial \zeta} P_C = \kappa P_A,$$  

(4)

where

$$\kappa(y) = (1 + e^{-iy}) = 2 \cos(y/2)e^{-iy/2}.$$  

These equations can be reduced to one second order differential equation

$$\frac{\partial^2}{\partial \zeta^2} P_A + 2|\kappa|^2 P_A = 0.$$  

The general solution of this equation is

$$P_A(\zeta, y) = C_1(y)e^{i\Omega y} + C_2(y)e^{-i\Omega y},$$

where $\Omega^2 = 8 \cos^2(y/2)$. The constants of integration $C_1$ and $C_2$ can be found from the boundary conditions

$$\zeta = 0: \quad P_A(0, y) = P_{A0}, \quad \left. \frac{\partial P_A}{\partial \zeta} \right|_{\zeta=0} = \kappa^*(P_{B0} + P_{C0}),$$

where

$$P_{A0} = \sum_{n=-\infty}^{\infty} A_n(0)e^{iny}, \quad P_{B0} = \sum_{n=-\infty}^{\infty} B_n(0)e^{iny}, \quad P_{C0} = \sum_{n=-\infty}^{\infty} C_n(0)e^{iny}.$$  

Taking into account these conditions the equations for constants of integration can be written as

$$C_1 + C_2 = P_{A0},$$
$$C_1 - C_2 = -\frac{\kappa^*}{\Omega}(P_{B0} + P_{C0}).$$
That results in
\[ P_A(\zeta, y) = P_{A0} \cos \Omega \zeta - i \beta (P_{B0} + P_{C0}) \sin \Omega \zeta, \] (5)

where
\[ \beta = \frac{\kappa^*}{\Omega} = \frac{1}{\sqrt{2}} e^{iy/2}. \]

The second and third equations from (4) can be rewritten as following ones
\[ i \frac{\partial}{\partial \zeta} (P_B + P_C) = 2 \kappa P_A, \]
\[ i \frac{\partial}{\partial \zeta} (P_B - P_C) = 0. \]

It follows that the function \( S = P_B - P_C \) is constant \( S_0 = P_{B0} - P_{C0} \). Either function \( R = P_B + P_C \) is obtained from the first equation of (4), which is rewritten as:
\[ i \frac{\partial P_A}{\partial \zeta} = \kappa^* R. \]

From this equation it follows that
\[ R(\zeta) = R_0 \cos \Omega \zeta - \frac{1}{i \beta} P_{A0} \sin \Omega \zeta, \quad R_0 = P_{B0} + P_{C0}. \]

Taking into account the definitions \( S = P_B - P_C \) \( R = P_B + P_C \), the generation functions can be expressed by following equations
\[ P_B(\zeta, y) = \frac{1}{2} \left[ P_{B0} - P_{C0} + (P_{B0} + P_{C0}) \cos \Omega \zeta - \frac{i}{\beta} P_{A0} \sin \Omega \zeta \right], \] (6)
\[ P_C(\zeta, y) = \frac{1}{2} \left[ P_{C0} - P_{B0} + (P_{B0} + P_{C0}) \cos \Omega \zeta - \frac{i}{\beta} P_{A0} \sin \Omega \zeta \right], \] (7)

If the boundary conditions are choosing in the form \( A_n(0) = 0, B_n(0) = -C_n(0) = 0 \) then \( P_{B0} = -P_{C0} \) and \( P_{A0} = 0 \). From (5), (6) and (7) it follows that
\[ P_A(\zeta, y) = 0, \quad P_B(\zeta, y) = P_{B0}, \quad P_C(\zeta, y) = -P_{B0} \]
at \( \zeta > 0 \). It means that the electromagnetic fields are localized in waveguides without spreading to neighbor waveguides. This kind of diffractionless propagation has been discussed and observed in [4, 5].

By using the orthogonality condition
\[ \int_{-\pi}^{\pi} e^{i(y-n-m)} dy = 2\pi \delta_{nm}, \]
the amplitudes \( A_n(\zeta), B_n(\zeta) \) and \( C_n(\zeta) \) can be determined from (5), (6) and (7):
\[ 2\pi A_n(\zeta) = \int_{-\pi}^{\pi} P_A(\zeta, y)e^{-iyn} dy, \quad 2\pi B_n(\zeta) = \int_{-\pi}^{\pi} P_B(\zeta, y)e^{-iyn} dy, \quad 2\pi C_n(\zeta) = \int_{-\pi}^{\pi} P_C(\zeta, y)e^{-iyn} dy. \]

IV. SOME PARTICULAR SOLUTION OF THE BASE EQUATIONS

Let us consider the particular case of the boundary conditions for problem under consideration. The simplest case is
\[ A_n(0) = A_0 \delta_{n0}, \quad B_n(0) = B_0 \delta_{n0}, \quad C_n(0) = C_0 \delta_{n0}. \]

This conditions are correlated to situation where the radiation is initially input only to waveguides of one elementary sell in array. Thus,
\[ P_{A0} = A_0, \quad P_{B0} = B_0, \quad P_{C0} = C_0. \]
By using the expressions (5), (6), (7), the amplitudes $A_n(\zeta)$, $B_n(\zeta)$ and $C_n(\zeta)$ can be represented by the following equations:

$$2\pi A_n(\zeta) = A_0 \int_{-\pi}^{\pi} \cos \Omega \zeta e^{-iy_n} dy - \frac{R_0}{\sqrt{2}} \int_{-\pi}^{\pi} \sin \Omega \zeta e^{-iy_n} dy,$$

$$2\pi B_n(\zeta) = \frac{1}{2} S_0 \int_{-\pi}^{\pi} e^{-iy_n} dy + \frac{1}{2} R_0 \int_{-\pi}^{\pi} \cos \Omega \zeta e^{-iy_n} dy - \frac{iA_0}{\sqrt{2}} \int_{-\pi}^{\pi} \sin \Omega \zeta e^{-iy_n} dy,$$

$$2\pi C_n(\zeta) = \frac{1}{2} S_0 \int_{-\pi}^{\pi} e^{-iy_n} dy + \frac{1}{2} R_0 \int_{-\pi}^{\pi} \cos \Omega \zeta e^{-iy_n} dy - \frac{iA_0}{\sqrt{2}} \int_{-\pi}^{\pi} \sin \Omega \zeta e^{-iy_n} dy.$$  

Here the constants $S_0 = B_0 - C_0$ and $R_0 = B_0 + C_0$ are introduced. Expressions (8)–(10) demonstrate the principal role of the following integrals

$$I_n^{(1)} = \int_{-\pi}^{\pi} \cos \Omega \zeta e^{-iy_n} dy = \int_{-\pi}^{\pi} \cos \left[2\sqrt{2}\zeta \cos(y/2)\right] e^{-iy_n} dy,$$

$$I_n^{(2)} = \int_{-\pi}^{\pi} \sin \Omega \zeta e^{-iy_n} dy = \int_{-\pi}^{\pi} \sin \left[2\sqrt{2}\zeta \cos(y/2)\right] e^{-iy_n} dy.$$

To determine these integrals the Anger’s formula can be used [8]. In the particular case this formula takes the following form

$$\cos(z \cos \varphi) = J_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(z) \cos(2k\varphi),$$

$$\sin(z \cos \varphi) = 2 \sum_{k=1}^{\infty} J_{2k-1}(z) \cos((2k-1)\varphi).$$

In this formula substitutions $\varphi = y/2$ and $z = 2\sqrt{2}\zeta = \eta$ must be done.

Integral $I_n^{(1)}$ can be determined by using (11). Thus,

$$I_n^{(1)} = \int_{-\pi}^{\pi} J_0(\eta) e^{-iy_n} dy + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(\eta) \int_{-\pi}^{\pi} \cos(ky) e^{-iy_n} dy =$$

$$= J_0(\eta) \delta_{n0} + \sum_{k=1}^{\infty} (-1)^k J_{2k}(\eta) \int_{-\pi}^{\pi} \left( e^{iy(k-n)} + e^{-iy(k+n)} \right) dy =$$

$$= 2\pi J_0(\eta) \delta_{n0} + 2\pi \sum_{k=1}^{\infty} (-1)^k J_{2k}(\eta) (\delta_{kn} + \delta_{-kn}).$$

As only positive integer numbers we take into attention, the second term in brackets is equal to zero. Thus,

$$I_0^{(1)} = 2\pi J_0(\eta), \quad I_n^{(1)} = 2\pi (-1)^n J_{2n}(\eta), \quad n \geq 1.$$  

As $J_k(z) = (-1)^k J_k(z)$ is held, the expression (13) will be correct at negative integer subindexes $n$.

Integral $I_n^{(2)}$ can be determined by using (12).

$$I_n^{(2)} = -2 \sum_{k=1}^{\infty} (-1)^k J_{2k-1}(\eta) \int_{-\pi}^{\pi} \cos((2k-1)y/2) e^{-iy(n+1/2)} dy =$$
\[
\begin{align*}
&= -\sum_{k=1}^{\infty} (-1)^k J_{2k-1}(\eta) \int_{-\pi}^{\pi} \left( e^{iy(k-1/2)} + e^{-iy(k-1/2)} \right) e^{-iy(n+1/2)} dy = \\
&= -\sum_{k=1}^{\infty} (-1)^k J_{2k-1}(\eta) \int_{-\pi}^{\pi} \left( e^{iy(k-n-1)} + e^{-iy(k+n)} \right) dy = \\
&= -\sum_{k=1}^{\infty} (-1)^k J_{2k-1}(\eta) 2\pi (\delta_{n+1} + \delta_{-n}).
\end{align*}
\]

The second term in brackets results in zero contribution. Thus,

\[
I_n^{(2)} = -2\pi (-1)^{n+1} J_{2n+1}(\eta) = 2\pi (-1)^n J_{2n+1}(\eta), \quad n \geq 0. \tag{14}
\]

With taking into account the expressions (13) and (14) the amplitudes \(A_n(\zeta), B_n(\zeta)\) and \(C_n(\zeta)\) can be written as

\[
A_n(\zeta) = A_0 (-1)^n J_{2n}(\eta) - \frac{iR_0}{\sqrt{2}} (-1)^n J_{2n+1}(\eta), \quad n \geq 0, \tag{15}
\]

\[
B_n(\zeta) = \frac{1}{2} S_0 \delta_{n0} + \frac{1}{2} R_0 (-1)^n J_{2n}(\eta) - \frac{iA_0}{\sqrt{2}} (-1)^n J_{2n+1}(\eta), \tag{16}
\]

\[
C_n(\zeta) = -\frac{1}{2} S_0 \delta_{n0} + \frac{1}{2} R_0 (-1)^n J_{2n}(\eta) - \frac{iA_0}{\sqrt{2}} (-1)^n J_{2n+1}(\eta). \tag{17}
\]

In the case of boundary conditions

\[
A_n(0) = A_0 \delta_{n0}, \quad B_n(0) = C_n(0) = 0,
\]

the expressions (15)–(17) result in

\[
A_n(\zeta) = A_0 (-1)^n J_{2n}(\eta), \quad B_n(\zeta) = -\frac{iA_0}{\sqrt{2}} (-1)^n J_{2n+1}(\eta), \tag{18}
\]

\[
C_n(\zeta) = -\frac{iA_0}{\sqrt{2}} (-1)^n J_{2n+1}(\eta). \tag{19}
\]

The expressions (18)–(20) describe the discrete diffraction (i.e., the electromagnetic radiation spreading along array).

In the case of boundary conditions

\[
A_n(0) = 0, \quad B_n(0) = -C_n(0) = B_0 \delta_{n0},
\]

we have \(R_0 = 0\), but \(S_0 = 2B_0\). The distribution of the electromagnetic fields in waveguide array is

\[
A_n(\zeta) = 0, \quad B_n(\zeta) = B_0 \delta_{n0}, \quad C_n(\zeta) = -B_0 \delta_{n0}. \tag{21}
\]

In this case the discrete diffraction is absent. It corresponds for the flat-band. However, if the radiation will input into one of the waveguide of the central part of array, i.e., to use the boundary condition

\[
A_n(0) = A_0 \delta_{n0}, \quad B_n(0) = -C_n(0) = B_0 \delta_{n0},
\]

than the distribution of the amplitudes \(A_n(\zeta), B_n(\zeta)\) and \(C_n(\zeta)\) takes the following form

\[
A_n(\zeta) = A_0 (-1)^n J_{2n}(\eta), \tag{22}
\]

\[
B_n(\zeta) = B_0 \delta_{n0} - \frac{iA_0}{\sqrt{2}} (-1)^n J_{2n+1}(\eta), \tag{23}
\]

\[
C_n(\zeta) = -B_0 \delta_{n0} - \frac{iA_0}{\sqrt{2}} (-1)^n J_{2n+1}(\eta). \tag{24}
\]

The discrete diffraction takes place in this case.
V. CONCLUSION

The propagation of the electromagnetic continues wave in the quasi-one-dimensional rhombic array of the waveguides is investigated. The exact solution of the coupled mode equations \[2\] is found by the use of generation function method. In general case of the boundary conditions the discrete diffraction is described by these solutions. However, the without diffraction regime of the wave propagation at the particular boundary condition exists. Taking into account the system of equations \[5\]–\[7\], the interference of two discrete beams could be studied.

Acknowledgement

We are grateful to Prof. I. Gabitov and Dr. C. Bayun for enlightening discussions. This investigation is funded by Russian Science Foundation (project 14-22-00098).

[1] St. Longhi, Aharonov-Bohm photonic cages in waveguide and coupled resonator lattices by synthetic magnetic fields. Opt. Lett. 39(20), 5892-5895 (2014).

[2] A. Maimistov, Quasi-flat bands in waveguide arrays. J.Phys. Conference Series 613, 012011 (2015)

[3] A.I. Maimistov, I.R. Gabitov, Optical flat bands in 2D waveguide arrays with alternating sign of refraction index, J.Phys. Conference Series 714, 012013 (4 pp) (2016).

[4] S. Mukherjee and R.R. Thomson. Observation of localized flat-band modes in a quasi-one-dimensional photonic rhombic lattice. Opt. Lett. 40(23), 5443-5446 (2015).

[5] S. Mukherjee, A. Spracklen, D. Choudhury, N. Goldman, P. Ohberg, E. Andersson, and R.R. Thomson, Observation of a Localized Flat-Band State in a Photonic Lieb Lattice Phys.Rev.Lett. 114, 245504 (2015).

[6] R.A. Vicencio, C. Cantillano, L. Morales-Inostroza, B. Real, C. Mejia-Cortes, St. Weimann, Al. Szameit, and M.I. Molina. Observation of Localized States in Lieb Photonic Lattices. Phys.Rev.Lett. 114, 245503 (2015).

[7] Yuanyuan Zong, Shiqiang Xia, Liqin Tang, Daohong Song, Yi Hu, Yumiao Pei, Jing Su, Yigang Li, and Zhigang Chen. Observation of localized flat-band states in Kagome photonic lattices Opt.Express 24(8), 8877-8885 (2016).

[8] H. Bateman, and A. Erdelyi (eds). Higher Transcendental Functions, v.2. Mc Graw-Hill Book Company, Inc., New York, Toronto, London, 1953.