Problem solution of the elasticity theory for the half-plane

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Abstract. The classical problem of equilibrium of elastic half-plane loaded along boundary contour can be resolved with different methods. In this paper, the solution is received with a new method that is founded on Cauchy-type integral and Plemelj identity. First, we solve with the general half-plane. Then we consider four special half-planes \( x_1 \leq 0, \ x_2 \geq 0, x_1 \leq 0, x_1 \geq 0 \).

1. Introduction

The problem of an elastic half-plane loaded along a boundary contour by a self-balanced distributed load is one of the main planar problems of the theory of elasticity [1,9,10]. Its solution is not only of independent interest but also enters as a component into the solution of more complex problems, in particular, the problem of an edge crack in a half-plane [2-4].

To solve this problem, various methods were used. A solution is known based on the integral Fourier transform [5]. The solution is also found by constructing the Green's function, which is used as the solution to the Flaman problem [6]. Methods of the theory of functions of a complex variable are also used [7]. Various paths are possible here. In particular, the problem for the half-plane is reduced to the conjugation problem [1,9,10]. The theory of Cauchy type integrals is also used [1, 7]. The same theory is the basis of this study. In contrast to [1], the boundary equation is not transformed into a functional equation but is solved directly taking into account the holomorphy of the desired functions. A similar approach was applied in [8] when deriving the generalized Muskhelishvili equation, which allows solving plane problems of the theory of elasticity for regions with boundary cracks.

The problem has some applications such as the solution to the problem of an edge cohesive crack in a half-plane and others [11, 12, 13].

Our contribution in this paper includes the solution that is received with a new method that is founded on Cauchy-type integral and Plemelj identity.

2. Problem solution of the elasticity theory for the half-plane

Stresses in the plane problem of the theory of elasticity are determined by Kolosov's formulas [1]

\[
\begin{align*}
\sigma_{11} + \sigma_{22} &= 2 \left( \Phi(z) + \overline{\Phi(z)} \right) \\
\sigma_{22} - \sigma_{11} + 2\imath \sigma_{12} &= 2 \left( \overline{\Phi(z)} + \Psi(z) \right)
\end{align*}
\]

(1)
where $\sigma_{km}$ is the stress in Cartesian coordinates $x_k$; $z = x_1 + ix_2$ is a complex variable ($i$ is an imaginary unit); $\Phi(z)$, $\Psi(z)$ are holomorphic functions determined from the boundary conditions of the problem; the line above the symbol denotes complex conjugation, the comma denotes the derivative.

Let the considered half-plane (Fig. 1) be located below the straight line

$$L: \begin{cases} x_1 = \tau \cos \alpha; & \tau \in (-\infty, \infty) \\ x_2 = \tau \sin \alpha; & \end{cases}$$

**Figure 1** - The design scheme.

$n$ is the unit external normal to the boundary of the half-plane $D$ - circuit $L \ (AC)$. The arrow indicates the bypass direction of the circuit $\Gamma$.

The boundary condition [1] is set on the contour

$$\left[ \Phi(t) + \Phi(t) \right] \bar{n} - \left[ \bar{\tau} \Phi'(t) + \Psi(t) \right] n = \bar{p};$$

$$t \in L; \ n = n_1 + in_2; \ |n| = 1; \ p = p_1 + ip_2$$

(2)

where $n_1, n_2$ are the components of the unit external normal to the contour $L$ (Fig. 1); $p_1, p_2$ are the components of the distributed external load applied to the circuit $L$. About this task we get:

$n_1 = -\sin \alpha; \ n_2 = \cos \alpha; \ n = i e^{i\alpha}; \ t = \tau e^{i\alpha}$.

In this case, the boundary condition (2) is transformed into the form:

$$\left[ \Phi(t) + \Phi(t) \right] e^{-2i\alpha} + \tau e^{-i\alpha} \Phi'(t) + \Psi(t) = i e^{-i\alpha} \bar{p}$$

(3)

The boundary condition (3) must be supplemented by the boundary conditions at the infinitely remote point $r = |z| \to \infty$. Due to the self-balancing of the load $p$ (the main vector and the main moment are equal to zero), the functions $\Phi(z)$, $\Psi(z)$ for $r \to \infty$ are infinitely small of a higher order than the function $r^{-1}$ [1]:

$$\lim_{r \to \infty} r \Phi(z) = 0; \ \lim_{r \to \infty} r \Psi(z) = 0$$

(4)

We exclude the derivative $\Phi'(t)$ from equation (3). For this purpose, we introduce a new holomorphic function $\Omega(z)$ related to the functions $\Phi(z)$, $\Psi(z)$ by the equality:
\[ \Psi(z) = -e^{-2\alpha z} \left[ \Omega(z) + z\Phi'(z) + \Phi(z) \right] \]  

(5)

As a result of the substitution of expression (4) in the boundary condition (3), the latter takes the form:

\[ \Phi(t) - \Omega(t) = i e^{\alpha t} \overline{p} \]

(6)

Equality (6) is an equation for determining the functions \( \Phi(z) \) and \( \Omega(z) \) holomorphic in the half-plane \( D \) lying below the line \( AC \) (Fig. 1) and satisfying the boundary conditions at infinity. To solve equation (6), we choose a closed contour \( \Gamma \), consisting of a segment \( CA \) of the boundary of the half-plane and semicircle \( ABC \) centered at the origin (Fig. 1).

With an unlimited increase in the radius of the semicircle, that is, for \( R \to \infty \), the contour \( \Gamma \) covers the entire half-plane \( D \).

Let us denote \( [7] \) by the sign \( \langle + \rangle \) (plus) the neighborhood of the contour \( \Gamma \) remaining on the left when traversing it counterclockwise and by the sign \( \langle - \rangle \) (minus) the neighborhood remaining to the right. Since \( \Phi(t) \) and \( \Omega(t) \) are the boundary values of functions holomorphic in the domain \( D \), they must satisfy the relations:

\[ \Phi(t_+) = \Phi(t); \quad \Phi(t_-) = 0; \quad \Omega(t_+) = \Omega(t); \quad \Omega(t_-) = 0 \]

For this, it is necessary and sufficient to fulfill the Plemelj identity [1]:

\[ \Phi(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\Phi(\xi) d\xi}{\xi - t}; \quad \Omega(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\Omega(\xi) d\xi}{\xi - t}, \]

(7)

where \( t, \xi \in \Gamma \). The integrals in these formulas are singular [7].

We express the function from equality (6) and substitute it into Plemelj's identity (the second of formulas (7)). We get:

\[ \Phi(t) - i e^{\alpha t} \overline{p} = \frac{1}{\pi i} \int_{\Gamma} \frac{\Phi(\xi) d\xi}{\xi - t} - e^{\alpha t} \frac{1}{\pi i} \int_{\Gamma} \frac{pd\xi}{\xi - t} \]

We pass to complex conjugate quantities and take into account that

\[ \frac{d\overline{\xi}}{\overline{\xi - T}} = \frac{e^{-\alpha t} d\eta}{e^{-\alpha t} \eta - e^{-\alpha t} \tau} \]

\[ = \frac{d\eta}{\eta - \tau} = \frac{e^{\alpha t} d\eta}{e^{\alpha t} \eta - e^{\alpha t} \tau} = \frac{d\overline{\xi}}{\overline{\xi - T}} \]

We get:

\[ \Phi(t) + i e^{-\alpha t} p = -\frac{1}{\pi i} \int_{\Gamma} \frac{\Phi(\xi) d\xi}{\xi - t} e^{-\alpha t} \frac{1}{\pi} \int_{\Gamma} \frac{pd\xi}{\xi - t} \]

Taking into account Plemelj's identity (the first of equalities (7)), we find:

\[ \Phi(t) = \Phi(t_+) = -i e^{-\alpha t} \left[ P + \frac{1}{2 \pi i} \int_{\Gamma} \frac{pd\xi}{\xi - t} \right] \]

(8)

The right-hand side is the boundary value of the Cauchy-type integral [7]. From the Sokhotskyi-Plemelj formulas [7] it turns out:
It follows from the boundary conditions at infinity that the integral along the section of the contour $ABC$ is equal to zero. Consequently, the integral along the contour $\Gamma$ turns out to be equal to the integral along the contour $L$ with the opposite sign (due to the change in the direction of the traversal to the opposite). We come to the expression:

$$\Phi(z) = \frac{e^{-ia}}{2\pi} \int_{\Gamma} \frac{p d\xi}{\xi - z}$$

Let us now find the function $\Omega(z)$. From equalities (6) and (8) it follows:

$$\Omega(t) = \Omega(t_+) = \Phi(t) - i e^{ia} \bar{p} = -i e^{ia} \left[ \bar{p} + \frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{p} d\xi}{\xi - t} \right]$$

Arguing in the same way as above, we get:

$$\Omega(z) = -\frac{e^{-ia}}{2\pi} \int_{\Gamma} \frac{\bar{p} d\xi}{\xi - z} = e^{-ia} \int_{L} \frac{\bar{p} d\xi}{\xi - z}$$

Substitution of relations (9), (10) into equality (4), and further into formulas (1) makes it possible to calculate the stresses at any point of the half-plane.

3. The special cases

3.1. The problem for a half-plane $x_2 \leq 0$

Let the material body occupy the lower half-plane. In this case, the contour $L$ is the abscissa axis, the unit external normal is written in the form

$$n_1 = 0; \quad n_2 = 1; \quad n = i$$

Equation (2) is then transformed into the form

$$\overline{\Phi(t)} + \Phi(t) + i \Phi'(t) + \Psi(t) = i \bar{p}$$

Since on the abscissa axis $\bar{t} = t$, we obtain

$$\overline{\Phi(t)} + \Phi(t) + i \Phi'(t) + \Psi(t) = i \bar{p}$$

Let us introduce a new holomorphic function

$$\Omega(z) = \Phi(z) + z \Phi'(z) + \Psi(z)$$

and instead of equation (5), we will have

$$\overline{\Phi(t)} + \Omega(t) = i \bar{p}$$

Let us choose a closed contour $\Gamma$ consisting of the boundary of the half-plane - the contour $\Gamma$ and a semicircle of the infinite radius $R \to \infty$ (Fig. 2).
Figure 2. Contours $L$ and $\Gamma$ for the problem 3.1.

The plus signifies the left neighborhood of the contour $L$, the minus sign denotes the right one. $ABC$ - the semicircle of infinite radius; $CA$ - the segment of the abscissa axis; the arrow indicates the direction of bypassing contour $\Gamma$.

Since the function $\Omega(z)$ is holomorphic in the domain bounded by the contour $\Gamma$ (in the lower half-plane), it satisfies the Plemelj identity [1]

$$\Omega(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\Omega(\xi)d\xi}{\xi - t}$$

(14)

where $t, \xi \in \Gamma$.

Eliminating function $\Omega(t)$ from equality (14) using expression (13), we obtain

$$ip(t) - \Phi(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{[ip(\xi) - \Phi(\xi)]d\xi}{\xi - t}$$

Let's move on to complex conjugate quantities. In this case, we take into account that the integral along the section of the contour $ABC$ is equal to zero by relations (4) and on the real axis $t = \bar{T}, \xi = \bar{\xi}$. We arrive at the equality

$$\Phi(t) + \frac{1}{\pi i} \int_{\Gamma} \frac{\Phi(\bar{\xi})d\bar{\xi}}{\bar{\xi} - t} = -\left[ip(t) + \frac{1}{\pi i} \int_{\Gamma} ip(\bar{\xi})d\bar{\xi}\right]$$

We use the Plemelj formula again, now concerning the function. We get

$$\Phi(t) = -i \left[\frac{p(t)}{2} + \frac{1}{2\pi i} \int_{\Gamma} \frac{p(\xi)d\xi}{\xi - t}\right] = -i \left[-\frac{p(t)}{2} + \frac{1}{2\pi i} \int_{\Gamma} \frac{p(\xi)d\xi}{\xi - t}\right]$$

(15)

The expression in square brackets is the boundary value of the Cauchy type integral [7] for the right boundary of the contour $L$ (denoted by the minus sign [7]). In this case, this is the lower half-plane. We obtain [7] for its interior points

$$\Phi(z) = i \left[\frac{1}{2\pi i} \int_{\Gamma} \frac{p(\xi)d\xi}{\xi - z}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\xi)d\xi}{\xi - z}$$

(16)
Expression (16) coincides with the known solution [1]. Equation (13) using equality (14) yields a formula for the function $\Omega(t)$

$$\Omega(t) = -i \left[ - \bar{p}(t) \frac{1}{2\pi i} \int_{L} \frac{\bar{p}(\xi)d\xi}{\xi - t} \right]$$

whence follows

$$\Omega(z) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\bar{p}(\xi)d\xi}{\xi - z}$$

(17)

The substitution of expressions (16), (17) into equality (12) makes it possible to determine the function $\Psi(z)$ and further from formulas (1) - the stress field.

3.2. The problem for a half-plane $x_2 \geq 0$

The difference from the previous problem is that now $n = -i$. In this case, the right-hand side in formulas (11), (13) changes sign. Contour $\Gamma$ now runs along the upper half-plane, and the direction of its traversal coincides with the direction of traversing contour $L$ (Fig. 3)

$$\Phi(t) = i \left[ \frac{p(t)}{2} + \frac{1}{2\pi i} \int_{L} \frac{p(\xi)d\xi}{\xi - t} \right]$$

(18)

The expression in square brackets is the boundary value of the Cauchy type integral [7] for the left boundary of the contour $L$ (denoted by the plus sign [7]). In this case, this is the upper half-plane. We obtain for its interior points a formula that coincides with the formula (16), with the only difference that now the complex coordinate lies in the upper half-plane.

From expression (13) with a changed sign in front of the right-hand side and expression (18) it follows
\[
\Omega(t) = -i \left[ \bar{p}(t) \frac{1}{2} + \frac{1}{2\pi i} \oint_{L} \bar{p}(\xi) d\xi \right]
\]
whence, for lying in the upper half-plane, we obtain formula (17).

3.3. The problem for a half-plane \( x_1 \geq 0 \)

In this case, the contour \( L \) is the ordinate axis and \( n = -1 \). Equation (1) is written in the form

\[
\Phi(t) + \Phi(t) - i\Phi'(t) - \Psi(t) = -\bar{p}
\]

![Figure 4. Contours L and Γ for the problem 3.3.](image)

The plus signifies the left neighborhood of the contour \( L \), the minus signifies the right one. \( ABC \) - the semicircle of infinite radius; \( CA \) - the segment of the abscissa axis; the arrow indicates the direction of bypassing contour \( \Gamma \).

Since on the imaginary axis \( t = -t \), we get

\[
\Phi(t) + \Phi(t) - i\Phi'(t) - \Psi(t) = -\bar{p}
\] (19)

Function \( \Omega(z) \), in this case, is written as

\[
\Omega(z) = \Phi(z) + z\Phi'(z) - \Psi(z)
\] (20)

In this case, equation (19) takes the form

\[
\Phi(t) + \Omega(t) = -\bar{p}
\] (21)

Further transformations are similar to those described above. Eliminating \( \Omega(t) \) using Plemelj's identity, we obtain

\[
\bar{p}(t) + \Phi(t) = \frac{1}{2\pi i} \oint_{L} \frac{\bar{p}(\xi) + \Phi(\xi)}{\xi - t} d\xi
\]
Passing to the conjugate quantities, taking into account the fact that $\bar{\xi} = -\xi$, $\bar{t} = -t$, we find

$$\Phi(t) = -\frac{p(t)}{2} + \frac{1}{2\pi i} \int_L \frac{p(\xi) d\xi}{\bar{\xi} - \bar{t}}$$

(22)

Here it is taken into account that the directions of integration along the contours $\Gamma$ and $L$ are the opposite. The resulting expression is the boundary value of the Cauchy-type integral for the right boundary of the contour $L$ - the right half-plane. We obtain for the interior points of the half-plane

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{p(\xi) d\xi}{\xi - z} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\eta) d\eta}{i\eta - z}$$

(23)

where $\eta \in (-\infty, \infty)$ is a real variable of integration.

It follows from relations (21) and (22) that

$$\Omega(t) = -\frac{p(t)}{2} + \frac{1}{2\pi i} \int_L \frac{p(\xi) d\xi}{\xi - t}$$

and further

$$\Omega(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\eta) d\eta}{i\eta - z}$$

(24)

Formulas (23), (24) together with expressions (20), and (1) make it possible to find stresses at any point of the half-plane.

3.4. The problem for a half-plane $x_1 \leq 0$

In this case, the contour $L$ is still the ordinate axis, but $n = 1$. In this case, the right-hand sides of equalities (19) and (21) change sign. In this case, the directions of traversing contours $\Gamma$ and $L$ coincide. Therefore, instead of equality (22), we have

$$\Phi(t) = \frac{p(t)}{2} + \frac{1}{2\pi i} \int_L \frac{p(\xi) d\xi}{\xi - t}$$

(25)

This is the boundary value of the Cauchy-type integral for the left boundary of the contour $L$ - the left half-plane. We arrive at the formula $\Phi(z)$ that coincides with the formula (23). But the point with the coordinate is in the left half-plane.
Figure 5. Contours $L$ and $\Gamma$ for the problem 3.4.

The plus signifies the left neighborhood of the contour $L$, the minus signifies the right one. $ABC$ - the semicircle of infinite radius; $CA$ - the segment of the abscissa axis; the arrow indicates the direction of bypassing contour $\Gamma$.

Further, similarly to the previous one, is

$$\Omega(t) = \frac{p(t)}{2} + \frac{1}{2\pi i} \int_{\Gamma} \frac{p(\xi)d\xi}{\xi - t}$$

Hence follows the formula for $\Omega(z)$, which coincides with the formula (24).

4. Conclusions

The problem of determining stress is one of the fundamental problems in mechanical mathematics. In this paper, we study the stress on the equilibrium state of the elastic half-plane affected by boundary contours in elastic theory. The solution we have obtained with a new method is based on using Cauchy integrals and the Plemeli equation. This solution coincides with the results of previous studies. With this new method, we can solve for all semi-plane.

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