Exclusive decay of $J/\Psi$ into a lepton pair combined with light hadrons

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Abstract

We study the exclusive decay of $J/\Psi$ into a lepton pair combined with light hadrons in the kinematic region, specified by that the total energy of the light hadrons is much smaller than $m_c$, the mass of the $c$-quark. In this region, the nonperturbative effect related to $J/\Psi$ and that related to the light hadrons can be separated, the former is represented by a NRQCD matrix element, while the later is represented by a matrix element of a correlator of electric chromofields. The results are obtained in a axial gauge by assumming that contributions from two-gluon emission are dominant. But we can show that these results can be obtained without the assumption in arbitrary gauges. A discussion of the results are presented.

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1. Introduction

Several ten millions $J/\Psi$ events will be collected with Beijing Spectrometer (BES), this provides many opportunities to study different decay modes of $J/\Psi$ with high statistics. In this work we propose to study the exclusive decay into a lepton pair combined light hadrons with a total energy much smaller than the mass of $J/\Psi$, this decay will be studied experimentally at BES [1]. Many interesting properties of QCD can be learned from studies of $J/\Psi$ or a quarkonium system. $J/\Psi$ mainly consists of a $c$-quark and its antiquark and those quarks move with a small velocity $v_c$ inside $J/\Psi$ in its rest-frame. Hence an expansion in $v_c$ can be employed to describe properties of $J/\Psi$. The expansion can be systematically performed in the framework of nonrelativistic QCD (NRQCD) [2] for inclusive decays and for production rates. In this framework the inclusive decay of $J/\Psi$ can be imagined at the leading order of $v_c$ as the following: The $c$- and $\bar{c}$- quark in $J/\Psi$ has certain probability to be freed at the same space point and this $c\bar{c}$ pair decays subsequently. The probability has a nonperturbative nature and is at order of $v_c^0$, while the decay of the $c\bar{c}$ pair can be treated with perturbative QCD. Various effects at higher order of $v_c$ can be taken into account, e.g., relativistic effect, and the effect of that the freed $c\bar{c}$ pair does not possess the same quantum numbers as those of $J/\Psi$.

For the exclusive decay we will study, this interpretation still can be given at the leading order of $v_c$. Then the decay is mainly of the decay of the $c\bar{c}$ pair and it can be thought that the pair emit some soft gluons and annihilates into a virtual photon. The soft gluons will be transmitted into the light hadrons with a total energy which is small. This interpretation implies that the nonperturbative effect related to $J/\Psi$ and that related to the light hadrons can be separated, a factorization of the $S$-matrix element for the decay can be performed. It should be noted that the factorization here is performed at the amplitude level for the exclusive decay. It is not like the factorization for inclusive decays proposed in [2]. We will assume that the factorization can be performed and then try to obtain a factorized form for the decay amplitude. A proof for the factorization in our case may be done as done for other exclusive process [4]. In our case one may also take the $J/\Psi$ as a nonrelativistic bound-state of a $c\bar{c}$ pair and use a wave-function for the bound-state. Then the wave-function at the origin can be written as NRQCD matrix elements defined in [2], and our results can be obtained.

For the light hadrons we will first assume that the soft gluons consists only of two gluons and take the gauge $G^0(x) = 0$, where $G^\mu(x)$ is the gluon field. Because the soft gluons have smaller momenta compared with the $c$-quark mass $m_c$, an expansion in $m_c^{-1}$ can be used. At the leading order, the $S$-matrix element for the decay is suppressed by $m_c^{-1}$, this may be understood in the framework of HQET [3]. In the axial gauge the leading term of HQET does not contain gluons, hence an emission of one or two gluons is suppressed by $m_c^{-1}$. At the leading order of $m_c^{-1}$ the decay amplitude can be factorized into three parts: The first part is a NRQCD matrix element representing the bound-state effect of $J/\Psi$, the second part is a matrix element of a correlator of electric chromofields, which characterize the soft-gluon transition into the light hadrons. The third part consists of some coefficient.
However, the assumption of two-gluon-emission is not justified, because it sounds that we use perturbative theory for soft gluons. But we can show that without the assumption one can also obtain the same results only by using the expansion in $m_c^{-1}$. Hence the results are nonperturbative and the number of emitted gluons is not restricted in an arbitrary gauge. In our work we only consider the emission of soft gluons which are transmitted into the light hadrons. Effects of gluons exchanged between quarks, between gluons and between quarks and gluons are not considered. If the exchanged gluons are hard, then their effect can be calculated with perturbative theory and it results in a correction to our results at higher orders of $\alpha_s$. If the exchanged gluons are soft, their effect is nonperturbative and may be factorized into nonperturbative matrix elements provided that the factorization can be proved. It is beyond this work to consider the effect of exchanging soft gluons.

Typically, the light hadrons are two pions. Hence the decay studied here has similarities comparing with the decay $\Psi' \to J/\Psi + \pi + \pi$, which is studied in [3, 4], where the soft-gluon transition can be described by matrix elements of local fields in the decay width. In our case the transition is characterized by a matrix element of nonlocal fields. However, they are related and we will show this in detail.

Our work is organized as the following: In Sect. 2 we derive our results for the decay with the assumption of the two-gluon emission in the axial gauge and discuss the relation to the decay $\Psi' \to J/\Psi + \pi + \pi$. In Sect. 3 we will derive the results in arbitrary gauge and without the assumption, i.e., the number of emitted gluons is not restricted. In Sect. 4 we summarize our work and discuss our approach.

Throughout of our work we take nonrelativistic normalization for the $J/\Psi$ state and for $c$-quark.

2. The Decay with Two-Gluon Emission

We consider the exclusive decay of $J/\Psi$ in its rest-frame:

$$J/\Psi(P) \to \ell^+(p_1) + \ell^-(p_2) + \text{light hadrons},$$

(1)

where the momenta are given in the brackets and the light hadrons are specified hadrons with the total momentum $k$, whose component is much smaller than $m_c$. Typically, they are two soft pions. At the leading order of QED the $S$-matrix element for the decay is

$$\langle f | S | i \rangle = -i e^2 Q_c L_\mu \cdot \frac{1}{q^2} \int d^4 z e^{i q \cdot z} \langle LH | \bar{c}(z) \gamma^\mu c(z) | J/\Psi \rangle,$$

(2)

where $Q_c$ is the electric charge of $c$-quark in unit of $e$, $c(x)$ is the Dirac field for $c$-quark, $\langle LH \rangle$ stands for the final state of the light hadrons and

$$q = p_1 + p_2,$$

$$L_\mu = \bar{u}(p_2) \gamma_\mu v(p_1).$$

(3)

In Eq.(2) we only take the part of $c$-quark in the electric current into account. Assuming that only two gluons are emitted by the $c$- or $\bar{c}$-quark, we obtain
\[ \langle f | S | i \rangle = -\frac{1}{2} e^2 Q c g_s^2 L_{\mu} \cdot \frac{1}{q^2} \int d^4 x d^4 y d^4 z e^{iqz} \langle LH | T [\bar{c}(x) \gamma \cdot G(x) c(x) \bar{c}(y) \gamma \cdot G(y) c(y) \bar{c}(z) \gamma^\mu c(z)] | J/\Psi \rangle, \]  

where \( G(x) \) is the gluon field. Using Wick-theorem we can calculate the \( T \)-ordered product and we only keep those terms in which one \( c \)-field and one \( \bar{c} \)-field remain uncontracted. Then the matrix element takes a complicated form and can be written in a short notation:

\[ \langle f | S | i \rangle = -\frac{1}{2} e^2 Q c g_s^2 L_{\rho} \cdot \frac{1}{q^2} \int d^4 x d^4 y d^4 z \langle LH | G_{\mu}^a(x) G_{\nu}^b(y) | 0 \rangle \]

\[ \langle 0 | \bar{c}_j(x_1) c_i(y_1) | J/\Psi \rangle \cdot M_{ji}^{\mu \nu, ab}(x, y, x_1, y_1, z), \]

where \( M_{ji}^{\mu \nu, ab}(x, y, x_1, y_1, z) \) is a known function, \( i \) and \( j \) stand for Dirac- and color indices, \( a \) and \( b \) is the color of gluon field. The above equation can be generalized to emission of arbitrary number of soft gluons, then the \( S \)-matrix element is the sum of the contributions with 2, 3, \( \cdots \) soft gluons and in each contribution there is the same matrix element \( \langle 0 | \bar{c}_j(x) c_i(y) | J/\Psi \rangle \). For this matrix element the expansion in \( v_c \) can be now performed, the result is:

\[ \langle 0 | \bar{c}_j(x) c_i(y) | J/\Psi \rangle = -\frac{1}{6} (P_+ \gamma^\ell P_-)_{ij} \langle 0 | \chi^\dagger \gamma^\ell \psi | J/\Psi \rangle e^{-ip \cdot (x+y)} + O(v_c^2), \]

where \( \chi^\dagger(\psi) \) is the NRQCD field for \( \bar{c}(c) \) quark and

\[ \begin{align*}
P_\pm &= (1 \pm \gamma^0)/2, \\
p^\mu &= (m_c, 0, 0, 0).
\end{align*} \]

The leading order of the matrix element is \( O(v_c^0) \), we will neglect the contribution from higher orders and the momentum of \( J/\Psi \) is then approximated by \( 2p \). It should be noted that effects at higher order of \( v_c \) can be added with the expansion in Eq.(6). Taking the result in Eq.(6) we can write the \( S \)-matrix element as:

\[ \langle f | S | i \rangle = \frac{1}{24} e^2 Q c g_s^2 (2\pi)^4 \delta^4(2p - k - q)L_{\rho} \cdot \frac{1}{q^2} \int d^4 x \int \frac{d^4 q_1}{(2\pi)^4} e^{iq_1 \cdot x} \langle LH | G_{\mu}^a(x) G_{\nu}^b(-x) | 0 \rangle R^{\mu \nu, \rho \phi}(p, k, q_1). \]

To obtain the equation we have used the color-symmetry and the translational covariance. The quantity \( R \) takes the form

\[ R^{\mu \nu, \rho \phi}(p, k, q_1) = \operatorname{Tr}(P_+ \gamma^\ell P_-) \{(\gamma^\rho \gamma^\mu \gamma^\nu \gamma^\phi) \}
\]

\[ \{ \begin{align*}
&\frac{1}{\gamma \cdot (p - k) - m_c + i0^+} \gamma \cdot (p - k - q_1) - m_c + i0^+ \gamma^\mu \\
&\frac{1}{\gamma \cdot (-p + \frac{1}{2}k - q_1) - m_c + i0^+} \gamma \cdot (k - p) - m_c + i0^+ \gamma^\nu \\
&\frac{1}{\gamma \cdot (-p + \frac{1}{2}k - q_1) - m_c + i0^+} \gamma \cdot (p - \frac{1}{2}k - q_1) - m_c + i0^+ \gamma^\rho \\
&\frac{1}{\gamma \cdot (-p + \frac{1}{2}k - q_1) - m_c + i0^+} \gamma \cdot (p - \frac{1}{2}k - q_1) - m_c + i0^+ \gamma^\phi
\end{align*} \]

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The physical interpretation of $R$ is that it is the amplitude for a $^3S_1$ $c\bar{c}$ pair emitting two soft gluons and a virtual photon, and $c$ and $\bar{c}$ have the same momentum $p$. We take the gauge

$$G^0(x) = 0,$$

and in this gauge the electric chromofield is given by:

$$E(x) = E^a(x)T^a = \partial_0 G(x).$$

The quantity $R$ depends on $p, k$ and $q_1$. In the kinematic region we consider, $k$ is a small vector. The dominant region for $q_1$ integration is characterized by $k$ and by $\Lambda_{QCD}$, because the $x$-dependence of the matrix element $\langle LH \cdots |0\rangle$ is characterized by $k$ and by $\Lambda_{QCD}$. Therefore one can expand $R(p, k, q_1)$ in $m_c^{-1}$. Keeping only the leading order we obtain

$$R_{\mu
u\rho\ell}^{\mu
u\rho\ell}(p, k, q_1)\langle LH |G^a_{\mu}(x)G^a_{\nu}(x)|0\rangle = \frac{4g^{\mu\rho}}{m_c k^0} \frac{1}{k^0 + 2q_1^0 - i0^+} \cdot \frac{1}{k^0 - 2q_1^0 - i0^+} \cdot ((k^0)^2 - 4(q_1^0)^2)$$

$$\langle LH |G^a(x) \cdot G^a(-x)|0\rangle + \mathcal{O}\left(\frac{1}{m_c^2}\right)$$

Substituting the above expression into the $S$-matrix element, the integration over $q_1$ and over $x$ can be performed. Then we use the following equations

$$\int d^4x e^{2iq\cdot x} (k + 2q)_{\rho}\langle LH|G^a_{\mu}(x)G^a_{\nu}(x)|0\rangle = -2i \int d^4x e^{2iq\cdot x} \langle LH|G^a_{\mu}(x)\partial_\rho G^a_{\nu}(x)|0\rangle$$

$$\int d^4x e^{2iq\cdot x} (k - 2q)_{\rho}\langle LH|G^a_{\mu}(x)G^a_{\nu}(x)|0\rangle = -2i \int d^4x e^{2iq\cdot x} \langle LH|\partial_\rho G^a_{\mu}(x)G^a_{\nu}(x)|0\rangle,$$

and we obtain

$$\langle f|S|i\rangle = i\frac{2}{3}e^2 Q_c g_s^2 (2\pi)^4 \delta^4(2p - k - q)L_{\rho} \cdot \frac{g^{\mu\rho}}{q^2} \langle 0|\chi^\dagger \sigma^\ell \psi|J/\Psi\rangle \cdot \frac{1}{m_c} \cdot \frac{1}{(k^0)^2}$$

$$\cdot \int \frac{d\tau}{2\pi} \frac{1}{1 + \tau - i0^+} \cdot \frac{1}{1 - \tau - i0^+} \int dt e^{i\tau k^0 t} \langle LH|E^a(t, 0) \cdot E^a(-t, 0)|0\rangle$$

$$+ \mathcal{O}\left(\frac{1}{m_c^2}\right) + \mathcal{O}(v_c^2),$$

where $\tau$ is related to $q_1^0$ by $2q_1^0 = \tau k^0$. The term with $g^{\mu\rho}$ is expected in the heavy quark limit. In this limit gluons will not change the spin of $c$- or $\bar{c}$ quark. The matrix element $\langle 0|\chi^\dagger \sigma^\ell \psi|J/\Psi\rangle$ is proportional to the spin of $J/\Psi$, hence the spin of $J/\Psi$ is transferred to the virtual photon. The result is obtained in the axial gauge defined in Eq.(10), to make the $S$-matrix element gauge invariant we must add a gauge link between the two operators of the electric chromofield. We define a distribution amplitude for the gluon conversion into the light hadrons:

$$h(\tau) = \frac{g_s^2}{2\pi} \int dt e^{i\tau k^0 t} \langle LH|E^a(t, 0) \cdot [P \exp\{-ig_s \int_{-t}^{t} dx \cdot G^0_{\mu}(x, 0)\tau^c\}]_{ab} E^b(-t, 0)|0\rangle,$$

where $P$ means path-ordering and $\tau^c$ is the generator of $SU(3)$ in adjoint representation.
The function \( h(\tau) \) is gauge invariant, it represents the transition of soft gluons into the light hadrons. With the gauge link the transition is of two space-like gluon plus many time-like gluons. It also depends on each individual momentum of each hadron, we denote the dependence as \( h(\tau, k) \). Because of the energy conservation \( h(\tau, k) = 0 \) if \(|\tau| > 1\). With this the \( S \)-matrix element can be written:

\[
\langle f | S | i \rangle \equiv \frac{i}{2} e^2 Q_c g_s^2 (2\pi)^4 \delta^4(2p - k - q) \lambda^\mu \cdot \frac{g^{\rho\ell}}{q^2} \langle 0 | \chi^\dagger \sigma^\rho \psi | J / \Psi \rangle \cdot \frac{1}{m_c} \cdot \frac{1}{(k^0)^3}
\]

\[\cdot T_{LH}(k) + \mathcal{O}(\frac{1}{m^2_c}) + \mathcal{O}(v^2_c),\]

\[
T_{LH}(k) = \int d\tau \frac{1}{1 + \tau - i0^+} \cdot \frac{1}{1 - \tau - i0^+} h(\tau, k).
\tag{17}
\]

Eq.(15) and Eq.(17) are our main results, derived in the axial gauge with the assumption of two-gluon emission, then we add a gauge link to make the \( S \)-matrix element gauge invariant. This may be unsatisfied, because the assumption of two-gluon emission sounds that we performed an expansion in \( g_s \) for soft-gluons and we add the gauge link by hand. It is possible that the \( \bar{c}c \) pair emits soft gluons which are all time-like and form the light hadrons, and this emission is not suppressed by \( m_{\bar{c}}^{-1} \). This type of contributions is excluded with the gauge. Therefore, it is important to have the results derived in an arbitrary gauge without the assumption. In the next section we will show that the results can be obtained without the assumption and the gauge-link is automatically supplied.

Before ending this section we would like to discuss the relation between the decay studied here and the decay \( \Psi' \to J / \Psi + \pi + \pi \). If we expand the integrand in \( q^0_1 \) in Eq.(12) and then perform the \( q^0_1 \) integration we obtain:

\[
\langle f | S | i \rangle \equiv \frac{i}{3} e^2 Q_c g_s^2 (2\pi)^4 \delta^4(2p - k - q) L^\mu \cdot \frac{g^{\rho\ell}}{q^2} \langle 0 | \chi^\dagger \sigma^\rho \psi | J / \Psi \rangle \cdot \frac{1}{m_c} \cdot \frac{1}{(k^0)^3}
\]

\[
\sum_{n=0}^{1} \frac{1}{(k^0)^{2n}} \langle LH | O_{2n} | 0 \rangle + \mathcal{O}(\frac{1}{m^2_c}) + \mathcal{O}(v^2_c),
\tag{18}
\]

where \( O_{2n} \) are well known twist-2 operators

\[
O_{2n} = t^{2n} G^{\alpha,0,\mu} \cdot \tilde{D}_0^{2n} G^{\alpha,0,\nu} g_{\mu\nu}
\tag{19}
\]

in the axial gauge. With this we can also obtain a series for \( T_{LH}(k) \):

\[
T_{LH}(k) = \frac{g_s^2}{k^0} \cdot \sum_{n=0}^{1} \frac{1}{(k^0)^{2n}} \langle LH | O_{2n} | 0 \rangle.
\tag{20}
\]

It should be noted that \( T_{LH} \) is defined at a renormalization scale \( \mu \), the evolution of the operators \( O_{2n} \) with \( \mu \) is well known, hence one can also obtain the evolution of \( T_{LH} \). The same operators appear in the decay \( \Psi' \to J / \Psi + \pi + \pi \), the \( S \)-matrix element for the decay can be written:
\[ \langle J/\Psi + LH|S|\Psi' \rangle = \sum_{n=0} c_{2n} \langle LH|O_{2n}|0 \rangle, \] (21)

where we denote the state of the two pions as \( \langle LH \rangle \). In this decay the coefficient \( c_n \) is proportional to \( m_c^{-n} \). Hence one can truncate the series as an approximation. In our case, the corresponding coefficients are not suppressed by the power of \( m_c^{-1} \), we can only sum the series into a function \( T_{LH} \). This also prevents us from giving numerical predictions by knowing several leading terms in the series.

3. An exact derivation of the results

In this section we work in an arbitrary gauge and do not assume of two-gluon emission. The number of emitted soft-gluons is not restricted. With the discussion after Eq.(5) in the last section and after performing the expansion in \( v_c \), the problem can be formulated as a transition of the \( c\bar{c} \) pair into a virtual photon and the light hadrons. The \( S \)-matrix element can be written as

\[ \langle f|S|i \rangle = -ie^2 Q_c L_\mu \cdot \frac{1}{q^2} \frac{1}{6} (0|\gamma^\mu \sigma^\ell \psi|J/\Psi) \]

\[ \sum_{s_1 s_2} \bar{v}(p, s_2) \gamma^\ell u(p, s_1) \int d^4z e^{i q \cdot z} \langle LH|c(z)\gamma^\mu c(z)|c(p, s_1), \bar{c}(p, s_2) \rangle + O(v_c^2) \]

\[ = -ie^2 Q_c (2\pi)^4 \delta^4(2p - k - q) L_\mu \cdot \frac{1}{q^2} \frac{1}{6} (0|\gamma^\mu \sigma^\ell \psi|J/\Psi) \]

\[ \sum_{s_1 s_2} \bar{v}(p, s_2) \gamma^\ell u(p, s_1) \langle LH|\bar{c}(0)\gamma^\mu c(0)|c(p, s_1), \bar{c}(p, s_2) \rangle + O(v_c^2), \] (22)

where the \( c- \) and \( \bar{c} \)-quark has same color and the summation over the color is implied, \( s_1 \) or \( s_2 \) is the spin of the \( c \)-quark or of the \( \bar{c} \)-quark, respectively. Now we need to calculate the matrix element \( \langle LH|\bar{c}(0)\gamma^\mu c(0)|c(p, s_1), \bar{c}(p, s_2) \rangle \). For this we use LSZ reduction formula and obtain:

\[ \langle LH|\bar{c}(0)\gamma^\mu c(0)|c(p, s_1), \bar{c}(p, s_2) \rangle = -\frac{1}{Z} \int dx dy e^{-ip \cdot (x+y)} \bar{v}(p, s_2)(i\gamma \cdot \partial_x - m_c) \]

\[ \langle LH|T \{ c(x)\bar{c}(0)\gamma^\mu c(0)\bar{c}(y) \} |0 \rangle (i\gamma \cdot \overleftarrow{\partial} y + m_c) u(p, s_1), \] (23)

where \( Z \) is the renormalization constant for the field \( c(x) \). We can evaluate the matrix element with help of the QCD path-integral, and perform first the integration over \( c \)-quark field we obtain:

\[ \langle LH|\bar{c}(0)\gamma^\mu c(0)|c(p, s_1), \bar{c}(p, s_2) \rangle = \frac{1}{Z} \int dx dy e^{-ip \cdot (x+y)} \bar{v}(p, s_2)(i\gamma \cdot \partial_x - m_c) \]

\[ \langle LH|S(x, \overrightarrow{x})\gamma^\mu S(0, y)|0 \rangle (i\gamma \cdot \overleftarrow{\partial} y + m_c) u(p, s_1). \] (24)

In the above equation the average over gluon fields, i.e., the integration over gluon fields, will be taken later, and \( S(x, y) \) is the \( c \)-quark propagator in a background of gluon fields:

\[ \frac{1}{T} S(x, y) = \langle 0|T \{ c(x)\bar{c}(y) \} |0 \rangle. \] (25)
We can define wave-function $\psi_c(x)$ and $\bar{\psi}_c(x)$ for the $c$- and $\bar{c}$-quark respectively in a background of gluon fields:

$$
\psi_c(x) = \int d^4y e^{-ip\cdot y} \left\{ S(x, y)(i\gamma \cdot \hat{D}_y + m_c) \right\} u(p, s_1),
$$

$$
\bar{\psi}_c(x) = \int d^4y e^{-ip\cdot y} \bar{v}(p, s_2)(i\gamma \cdot \partial_y - m_c)S(y, x)
$$

These wave functions satisfy the Dirac equations

$$(i\gamma \cdot D - m_c)\psi_c(x) = 0,$$

$$\bar{\psi}_c(x)(i\gamma \cdot \hat{D} + m_c) = 0,$$

and the boundary conditions

$$\psi_c(x) \to e^{-ip\cdot x}u(p, s_1) \quad \text{for} \quad x^0 \to -\infty,$$

$$\bar{\psi}_c(x) \to e^{-ip\cdot x}\bar{v}(p, s_2) \quad \text{for} \quad x^0 \to -\infty,$$

where $D^\mu = \partial^\mu + ig_\Lambda G^\mu(x)$, $G^\mu(x)$ is the background field of gluons and $\bar{\psi}_c D^\mu \psi_c = (D^\mu \psi_c)\gamma^\mu$. It should be noted that the propagator in Eq.(25) is a Feynman propagator, wave-functions defined in Eq.(26) do satisfy the Dirac equation with a background field of gluons, but they do not have a simple boundary condition like in Eq.(28). However, it was shown that the Feynman propagator in Eq.(26) can be replaced with a retarded propagator, if the background field varies enough slowly with the space-time. Then one can have solutions of Eq.(27) and they satisfy the boundary conditions in Eq.(28). In our case we will make the expansion in $m_c^{-1}$ and in this expansion the $c$-quark and $\bar{c}$ quark are decoupled, and it results in that the Feynman propagator will automatically be a retarded propagator for $c$ and for $\bar{c}$. Therefore one can obtain a solution from the Dirac equation and the solution can satisfy the boundary condition in Eq.(28). It should be also noted that the boundary condition in Eq.(28) implies $G^\mu(x) \to 0$ for $x^0 \to -\infty$.

With the wave-functions we have

$$
\langle LH|\bar{c}(0)\gamma^\mu c(0)|c(p, s_1), \bar{c}(p, s_2)\rangle = \frac{1}{Z} \langle LH|\bar{\psi}_c(0)\gamma^\mu \psi_c(0)|0\rangle
$$

and it is gauge-invariant. If we take c-quark as a heavy quark, we can expand the wave-functions in $m_c^{-1}$. The expansion is the same as that in HQET, we write the wave-functions as:

$$
\psi_c(x) = e^{-ip\cdot x} \left\{ h_c(x) + \frac{i}{2m_c} \gamma \cdot D_T h_c + \mathcal{O}(m_c^{-2}) \right\},
$$

$$
\bar{\psi}_c(x) = e^{-ip\cdot x} \left\{ \bar{h}_c(x) - \frac{i}{2m_c} h_c(x)\gamma \cdot \hat{D}_T + \mathcal{O}(m_c^{-2}) \right\},
$$

where $D^\mu_T = (0, \mathbf{D})$. For the functions $h_c(x)$ and $\bar{h}_c(x)$ we have

$$
(iD_0 - \frac{1}{2m_c}(\gamma \cdot D_T)^2)h_c(x) = 0 + \mathcal{O}(m_c^{-2}),
$$

$$
\bar{h}_c(x)(iD_0 - \frac{1}{2m_c}(\gamma \cdot \hat{D}_T)^2) = 0 + \mathcal{O}(m_c^{-2}),
$$

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and the boundary condition for these functions can be read from Eq.(28). Giving a background field of gluons we solve the above equations by expanding these functions in $m_c^{-1}$. At the leading order c-quark will only interact with time-like gluons and the solution is simple. We define

$$V(x) = P \exp\{-ig_s \int_{-\infty}^{x^0} dt G^0(t, x)\}$$  \hspace{1cm} (32)$$

and with the solution at the leading order the wave-functions are

$$\psi_c(x) = e^{-ip \cdot x} V(x) u(p) + \mathcal{O}(m_c^{-1}),$$

$$\bar{\psi}_c(x) = e^{-ip \cdot x} \bar{v}(p) V^\dagger(x) + \mathcal{O}(m_c^{-1}).$$  \hspace{1cm} (33)

Because $V^\dagger(x)V(x) = 1$ we obtain after averaging the gluon fields:

$$\langle LH | \bar{c}(0) \gamma^\mu c(0) | c(p, s_1), \bar{c}(p, s_2) \rangle = 0 + \mathcal{O}(m_c^{-1}).$$  \hspace{1cm} (34)

That means that the leading order of the S-matrix element is at $m_c^{-1}$ and only time-like gluons will not lead to a contribution to the S-matrix element at order of $m_c^0$. This is in consistency with the results in the last section, where the axial gauge and perturbative theory is used. Here we derive this and the results at order of $m_c^{-1}$ in an arbitrary gauge and without using perturbative theory.

To calculate the matrix element at order of $m_c^{-1}$, we first make a gauge transformation:

$$\psi'_c(x) = V^\dagger(x) \psi_c(x), \quad \bar{\psi}'_c(x) = \bar{\psi}_c(x) V(x),$$

$$h'_c(x) = V^\dagger(x) h'_c(x), \quad \bar{h}'_c(x) = \bar{h}_c(x) V(x),$$

$$G'_\mu(x) = V^\dagger(x) G_\mu(x) V(x) - \frac{i}{g_s} V^\dagger(x) \partial_\mu V(x).$$  \hspace{1cm} (35)

After the transformation we have

$$\langle LH | \bar{c}(0) \gamma^\mu c(0) | c(p, s_1), \bar{c}(p, s_2) \rangle = \frac{1}{Z} \langle LH | \bar{\psi}'_c(0) \gamma^\mu \psi'_c(0) | 0 \rangle. \hspace{1cm} (36)$$

With this transformation one can show $G'^0(x) = 0$. The wave functions $\bar{\psi}'_c$ and $\psi'_c$ have the same expansion in $m_c^{-1}$ as those in Eq.(30), with the replacement that all functions are replaced by functions with a prime. The functions $h'_c(x)$ and $\bar{h}'_c(x)$ satisfy the same equations in Eq.(31), where the covariant derivative is $D_\mu = \partial_\mu + ig_s G'_\mu(x)$. Solving these equations with the boundary condition we have

$$h'_c(x) = \left\{ 1 - \frac{i}{2m_c} \int_{-\infty}^{x^0} dt (\gamma \cdot D_T(t, x))^2 \right\} u(p) + \mathcal{O}(m_c^{-2}),$$

$$\bar{h}'_c(x) = \bar{v}(p) \left\{ 1 - \frac{i}{2m_c} \int_{-\infty}^{x^0} dt (\gamma \cdot D_T(t, x))^2 \right\} + \mathcal{O}(m_c^{-2}).$$  \hspace{1cm} (37)

With these solutions we have
\[
\psi_c'(x) = e^{-ip\cdot x} \left\{ 1 - \frac{i}{2m_c} \int_{-\infty}^{x} dt G'(t, x) \cdot G'(t, x) + \cdots \right\} u(p) + \mathcal{O}(m_c^{-2}),
\]

\[
\bar{\psi}_c(x) = e^{-ip\cdot x} \bar{v}(p) \left\{ 1 - \frac{i}{2m_c} \int_{-\infty}^{x} dt G'(t, x) \cdot G'(t, x) + \cdots \right\} + \mathcal{O}(m_c^{-2}).
\] (38)

In the above equations \( \cdots \) in the bracket \( \{ \} \) will not lead to contributions at the order we consider. Using these results we have

\[
B^{\mu\nu} = \sum_{s_1, s_2} \bar{v}(p, s_2) \gamma^\mu u(p, s_1) \langle LH | \bar{\psi}_c(0) \gamma^\nu \psi_c(0) | 0 \rangle
\]

\[
= -ig_s^g \frac{g_{\mu\nu}}{Z m_c} \langle LH | G''\nu(0) G''\mu(0) \rangle \int_{-\infty}^{0} dt e^{i k^0 t} + \mathcal{O}(m_c^{-2}),
\] (39)

where we have averaged the gluon fields. It should be noted that \( k^0 \) is the total energy of the light hadrons, which forms an asymptotic final state, hence \( k^0 \) should be understood as \( k^0 - i0^+ \). With this in mind the integration over \( t \) can be performed. One can also use \( \theta \)-function

\[
\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i0^+} = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}
\] (40)

to perform the integral. For the transformed gauge fields \( G''\mu \) we have

\[
E'(x) = E''\alpha(x) T^\alpha = \partial_0 G'(x),
\] (41)

then the gluon field \( \mathbf{G}' \) can be related to \( \mathbf{E}' \) by

\[
\mathbf{G}'(t, x) = \int_{t}^{\infty} d\xi \mathbf{E}'(\xi, x) = \int_{-\infty}^{\infty} \theta(t-\xi) \mathbf{E}'(\xi, x).
\] (42)

Using this relation and translational covariance we have

\[
\langle LH | G''\alpha(0) G''\alpha(0) | 0 \rangle = 2 \int_{-\infty}^{\infty} dt dT e^{i k^0 T} \theta(-t - T) \theta(t - T) \langle LH | E''\alpha(t) E''\alpha(-t) | 0 \rangle
\]

\[
= 2 \int_{-\infty}^{\infty} dt \frac{d\omega}{2\pi} e^{2 i \omega t} \frac{1}{\omega + \frac{1}{2} k^0 + i0^+} \cdot \frac{1}{\omega - \frac{1}{2} k^0 - i0^+} \cdot \langle LH | E''\alpha(t) E''\alpha(-t) | 0 \rangle.
\] (43)

Changing the variable \( 2\omega = \tau \) and we obtain:

\[
\langle f | S | i \rangle = \frac{2}{3} e^2 Q_c g_s^2 (2\pi)^4 \delta^4(2p - k - q) L_\rho \cdot \frac{g_{\rho\nu}}{q^2} \langle 0 | \bar{\chi} \gamma^\nu \sigma^\rho | J/\Psi \rangle \cdot \frac{1}{m_c} \cdot \frac{1}{(k^0)^2}
\]

\[
\cdot \int \frac{dt}{2\pi} \frac{1}{1 - \tau - i0^+} \cdot \frac{1}{1 - \tau + i0^+} \int dt e^{i k^0 t} \langle LH | E''\alpha(t, 0) \cdot E''\alpha(-t, 0) | 0 \rangle + \mathcal{O}(m_c^2) + \mathcal{O}(v_c^2),
\] (44)
Expressing $E'$ with $E$ we realize that the gauge link in Eq.(15) is automatically generated,
\[
\langle LH|E'^a(t,0) \cdot E'^a(-t,0)|0\rangle = \\
\langle LH|E^a(t,0) \cdot [P \exp\{-ig_s \int_{-t}^{t} dx^0 G^{0,c}(x^0,0) \tau^c]\}_{ab} E^b(-t,0)|0\rangle
\]
while in Eq.(15) and Eq.(17) we sandwich it by hand.

The last factor which needs to be considered is the renormalization constant $Z$ for wave-function. Following [8] we consider the current
\[
J^\mu(x) = \bar{c}(x) \gamma^\mu c(x).
\]
The current $J^\mu(x)$ is a conserved current and it will be not renormalized. Hence we have:
\[
\langle c(p)|J^\mu(0)|c(p)\rangle = \bar{u}(p)\gamma^\mu u(p).
\]
Using the same techniques in this section, one can derive
\[
Z = 1 + O(m_c^{-1}).
\]
With the results here we find that the $S$-matrix element obtained here is exactly the same obtained in the last section. We emphasize here that it is derived here in arbitrary gauge and without the assumption of two-gluon emission. Hence the results are nonperturbative.

With the given $S$-matrix element, one can calculate the differential decay width. We obtain for unpolarized $J/\Psi$:
\[
\frac{d\Gamma}{dq^2} = \alpha Q_c^2 \frac{8\pi}{9m_c^4} \langle J/\Psi|O_1^{J/\Psi}(3S_1)|J/\Psi\rangle \int d\Gamma_{LH} \delta((2p - k)^2 - q^2) \frac{1}{(k^0)^4} |T_{LH}(k)|^2,
\]
where $\Gamma_{LH}$ is the phase-space integral for the light hadrons and we only keep the leading dependence on momenta of the light hadrons. The formula is only for $q^2$ close to $4m_c^2$ and the predicted differential width is measurable in experiment. In the calculation we used
\[
\langle J/\Psi|\psi^i\sigma^i\chi|0\rangle\langle 0|\psi^\dagger \sigma^i\psi|J/\Psi\rangle = \langle J/\Psi|O_1^{J/\Psi}(3S_1)|J/\Psi\rangle \varepsilon^i(\varepsilon^1)^\dagger.
\]
In the equation $\varepsilon$ is the polarization vector of $J/\Psi$, the matrix element $\langle J/\Psi|O_1^{J/\Psi}(3S_1)|J/\Psi\rangle$ is defined in [8] and the average over the spin is implied in the matrix element.

In this section we mainly have considered emission of gluons by a $c\bar{c}$ pair, and these gluons are transmitted into light hadrons. Because the kinematic region considered here, these gluons must be soft. That is why we have defined the nonperturbative object $T_{LH}(k)$ which includes $g_s^2$. However, emission of hard gluons can happen, this can only happen as exchange of hard gluons between $c$, $\bar{c}$- quark and gluons. The effect of the exchange can be calculated with perturbative QCD, it will result in that the coefficient in Eq.(17) will be modified. For example, $Z$ in Eq.(48) will be smaller than 1 by neglecting the effect at $m_c^{-1}$. For exchange of soft gluons, the effect may be factorized into the NRQCD matrix element and $T_{LH}$.

4. Summary and Discussion
In this work we have studied the exclusive $J/\Psi$-decay into a lepton pair combined with light hadrons, in which the total energy of the light hadrons is much smaller than $M_{J/\Psi}$. The light hadrons are formed from soft gluons emitted by the $c$-flavored quarks. In the limit $m_c \to \infty$ the nonperturbative effects related to $J/\Psi$ and to the light hadrons can be separated and they are represented by a NRQCD matrix element and by a correlator of electric chromofields. Corrections to the limit and those due to hard gluons may systematically added. We emphasize that our results are obtained without perturbative theory and are gauge invariant, although we firstly have derived them in the axial gauge with perturbative theory. But we have also derived them in an arbitrary gauge without using perturbative theory.

In our approach an expansion in $v_c$, the velocity of the $c$-quark inside $J/\Psi$ in its rest-frame, is used. We have only taken the leading order contributions at $v_c^0$. At this order, $J/\Psi$ can be considered as a bound state of the $c\bar{c}$ pair in color singlet. The corrections to the leading-order results may also added. However, problems arise at order of $v_c^4$. At this order $J/\Psi$ has a component in which the $c\bar{c}$ pair is in color octet and the component is a bound state of the $c\bar{c}$ pair with soft gluons. It is unclear how to add corrections from this component at order of $v_c^4$. It deserves a further study of these problems to understand a bound state of many dynamical freedoms of QCD.

Our approach applies for a class of decays. The results given here can be directly used for the $\Upsilon$ decay and can be generalized to $B_c^* \to \ell\bar{\nu} + \text{light hadrons}$, where the light hadrons also have a total energy which is small. The same correlator appears in these decays to characterize the transition of soft gluons into light hadrons. In that sense the correlator is universal. Our results do not apply for $B_c$, because it is a spin-0 meson. However, with the approach presented here one may obtain corresponding predictions for $B_c$. Our results do not apply for decays of excited quarkonium $^3S_1$ states like $\Psi' \to \ell^+\ell^- + \pi + \pi$, in this decay after emission of soft gluons, which forms the light hadrons, the $c\bar{c}$ pair can be formed into $J/\Psi$, then the $J/\Psi$ decays into a lepton pair. This type of contributions is not included in our results. But this type of contributions may be excluded by experimental cuts, through requiring the two leptons are not exactly back-to-back in the $J/\Psi$ rest-frame. With this requirement our results may apply.

In this work we are unable to give any numerical result. In our prediction, the nonperturbative effect related to $J/\Psi$ is known, e.g., from the leptonic decay of $J/\Psi$, while the nonperturbative effect related to the light hadrons is unknown. We currently develop a model for the correlator and will make an attempt to give numerical predictions.

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