Renormalization of Nielsen Identities

Adrian Lewandowski

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Hoża 69, 00-681, Warsaw, Poland

Abstract

We study renormalization of identities governing the dependence of 1PI Green’s functions on gauge-fixing parameters. For general dimensionally regularized Yang-Mills theories with gauge groups being direct products of arbitrary compact simple Lie groups and $U(1)$ groups coupled to scalar fields, we extend the well known analysis in Fermi gauges to the class of generalized ’t Hooft gauges $R_{\xi,u}$, in which also symmetry under global gauge transformations is broken by the gauge-fixing procedure. We also discuss conditions ensuring homogeneity of the Nielsen identity satisfied by the effective potential.

1 Introduction

Renormalizability, unitarity and gauge-independence of Yang-Mills gauge theories (without the Adler-Bardeen anomaly) were proved by Becchi, Rouet, Stora [1] and Tyutin [3]. The case of a general algebra of a compact gauge group was considered in [2]. While unitarity of the S-matrix, owing to the Kugo-Ojima quartet mechanism [4], is an immediate consequence of the BRST symmetry, gauge-fixing independence follows only from an extended BRST symmetry [5, 6], which acts also on gauge-fixing parameters. Slavnov-Taylor identities of this symmetry are usually called Nielsen identities [7, 13]. They were originally used by Nielsen [5] in the study of gauge-independence of spontaneous symmetry breaking. Since then, Nielsen identities constitute an efficient tool for studying virtually all problem related to gauge-independence. Under the extended BRST symmetry, gauge-fixing parameters are transformed into anticommuting classical fields (‘Nielsen sources’) coupled to composite operators. Renormalization of Green’s functions with insertions of these operators requires additional counterterms, which at the same time control gauge-dependence of ordinary counterterms. For pure Yang-Mills theories quantized in the so-called Fermi gauges with a single gauge-fixing parameter $\xi$ this was demonstrated in the work [12] of Piguet and Sibold in which the problem of renormalization of the Nielsen identities in such theories was worked out.

If spontaneous gauge symmetry breaking is anticipated, the quantization scheme should in general be more complex and involve several gauge-fixing parameters (a very general class of such schemes will be considered in this paper). Yet, in theories in which gauge symmetries are broken spontaneously by a vacuum expectation value (VEV) of a scalar field existing already at the tree level (as in the Standard Model) one commonly uses the ’t Hooft $R_{\xi}$ gauge [23] which is effectively a one parameter scheme. Using it one avoids non-diagonal scalar-vector propagators and (except for the Landau $\xi=0$ gauge) infrared divergences which are typical of Fermi gauges. In this class of theories the perturbative expansion is constructed around the tree level vacuum and in practical computations of Green’s functions there is no need to investigate the effective potential. Hence the fact that it is well defined only in the Landau $\xi=0$ gauge does not preclude the possibility of checking gauge-independence of the calculated physical quantities. Nielsen identities applied to this case allow e.g. to simplify the proof of gauge-independence of the S-matrix [9, 10] and to analyze gauge-independence of masses and widths of unstable particles in the Standard Model [9].

Minimization of the effective potential becomes an important ingredient of the perturbative expansion in theories in which symmetry breaking is triggered only by radiative corrections [21] (see [19] for a recent proposal). In order to obtain a well defined effective potential for $\xi \neq 0$, another gauge-fixing parameter $u$ has to be introduced [24]. This leads to the class of generalized ’t Hooft $R_{\xi,u}$ gauges (in theories like the Standard Model, identifying $u$ with the VEV of the scalar field, one recovers the ’t Hooft $R_{\xi}$ gauges). In this class of gauges the Nielsen identities were derived in [13] and checked in various one-loop calculations mainly in the abelian Higgs model [7, 16, 11] for both bare (regularized) and renormalized Green’s functions (see also [8] for the proof of the gauge-independence of the false vacuum decay rate in abelian theories with radiative symmetry breaking). General considerations of necessary additional counterterms based on power-counting arguments for abelian models quantized in non-linear and background field gauges can be found in [28] and [29], respectively. Although ordinary counterterms necessary for renormalization of Green’s functions in the $R_{\xi,u}$ gauge in general non-abelian theories are well known [2, 24], to

\*E-mail: lewandow@fuw.edu.pl
the best of our knowledge, based on power-counting arguments determination of all possible additional counterterms necessary to renormalize the action with operators coupled to the Nielsen sources has never been presented in the literature.

In this paper we find all these additional counterterms for the action with Nielsen sources of a general Yang-Mills theory coupled to bosonic matter fields in the generalized ’t Hooft $R_{\xi,u}$ gauges. We work in the Dimensional Regularization which (in the considered class of theories) is consistent with the usual BRST symmetry as well as with the extended one. Thus, the counterterms in the MS-scheme are directly constrained by the symmetry requirements. Renormalization of the Nielsen identities is indispensable to obtain equations that govern gauge-dependence of the renormalized effective potential, which partly motivated the analysis presented here.

We are particularly interested in the $u$-dependence, which was not studied in [12]. We will show that the Nielsen identities governing the $u$-dependence of the renormalized action allow to determine (in the MS-scheme) the additive VEV counterterm $\delta v$ in terms of a two-point function of composite operators, which elucidates the origin of $\delta v$. This method of determination of $\delta v$ turns out to be very convenient in the one-loop approximation. It reproduces the well-known one-loop results found in the Standard Model [14] and in the MSSM [20] and can be used in any model irrespectively of whether the tree-level VEV exists or not.

Furthermore we extend the results of [12] by allowing for the $\xi$ parameters to be a general matrix in the space of the gauge group generators. We find that in the most general case, when also the symmetry of the action with respect to global transformations is broken by such $\xi$ parameters (as happens e.g. in the Standard Model quantized with a separate $\xi$ parameter for each mass eigenstate), an additional superficially divergent three-point function of composite operators may appear. Its renormalization would therefore require a new counterterm (which we call ‘the curvature’). At the one-loop level, as we have checked by explicit calculation, this three-point function is however finite owing to a cancellation between two diagrams.

Finally we discuss the Nielsen identity satisfied by the effective potential. In the generalized ’t Hooft $R_{\xi,u}$ gauge this identity is homogeneous only if the scalar fields are restricted to the subspace on which the gauge-fixing function vanishes. The condition of no spontaneous breaking of the BRST symmetry requires that the scalar fields VEVs belong precisely to this subspace. We will show that in the class of theories considered in this paper, a stationary point $\phi_0$ of the effective potential restricted to this subspace is also a stationary point of the full effective potential, provided that $\phi_0$ obeys a certain condition which does not rely on invariance of the action under additional discrete transformations (like CP) which may be not exact or can be broken spontaneously.

The results presented in this paper can immediately be extended to theories with fermions in nonchiral representations of the gauge group. They have been omitted for the sake of simplicity - the relevant formulae are analogous to the ones presented here. Inclusion of chiral fermions (in nonanomalous representations) is possible but requires a dedicated analysis of the necessary counterterms.

The paper is organized in the following way. Section 2 contains, for the reader’s convenience, the derivation of the Nielsen identities based on the method of Piguet and Sibold [12]. In Section 3 the complete action with all possible counterterms is presented together with constraints imposed on it by the Nielsen identities; a detailed derivation is given in appendices. In Section 4 the gauge-independence of bare coupling constants is proved with the help of the Nielsen identities. In Section 5 some of the formal results are checked by explicit one-loop calculations. Section 6 contains explicit computation of the counterterm $\delta v$ for the effective potential in the $R_{\xi,u}$-scheme and a discussion of the homogeneity of the Nielsen identity satisfied by the effective potential. Section 7 is devoted to our conclusions.

2 Nielsen Identities

We begin with the gauge-fixed action of general Yang-Mills fields coupled to scalar fields in an arbitrary representation of the gauge group. In terms of parameters and fields which after inclusion of counterterms will acquire the interpretation of renormalized field and parameters the Lagrangian density reads

$$L^b(x) = -\frac{1}{4} \delta_{\alpha\beta} F^b_{\mu\alpha}(x) F^{\beta\mu}(x) + \frac{1}{2} \delta_{ab} (D_{\mu}\phi)^a(x) (D_{\mu}\phi)^b(x) - V(\phi(x)) + s \left( \bar{\phi}(x) F^a(x) + \frac{1}{2} \bar{\phi}(x) \xi^{\alpha\beta} h_{\beta}(x) \right),$$

(1)

in which

$$F^b_{\mu\nu}(x) = \partial_{\mu} A^b_{\nu}(x) - \partial_{\nu} A^b_{\mu}(x) + e_{\nu}^{\alpha\beta\gamma} A^b_{\mu}(x) A^\gamma_{\alpha}(x),$$

(2)

$$(D_{\mu}\phi)^a(x) = \partial_{\mu} \phi^a(x) + A^a_{\mu}(x) [T_{R_{\sigma_{\alpha}}}^{b}(\phi^b(x) + v^b)],$$

(3)

1This is why these parameters carry the subscript $R$; on the other hand to keep the notation manageable on renormalized fields this subscript is omitted.
and $s$ denotes the BRST operator (see e.g. [14, 18])

$$s (\phi^a (x)) = \omega^a (x) (T_{\alpha a} (\phi (x) + v_n))^a,$$

$$s (A^a_{\mu} (x)) = - \partial_{\mu} \omega^a (x) + e^n_{\gamma \alpha \beta} \omega^\gamma (x) A^b_{\mu} (x),$$

$$s (\omega^a (x)) = \frac{1}{2} e^n_{\alpha \beta} \omega^b (x) \omega^\gamma (x),$$

$$s ((\omega^a (x)) = h_n (x),$$

$$s (h_n (x)) = 0.$$  

The matrices $[T_{\alpha a}]^a$, are antisymmetric and span a representation of the gauge Lie algebra (which is the direct sum of simple compact Lie algebras and $u(1)$ algebras) with totally antisymmetric structure constants $e^n_{\alpha \beta \gamma}$ (we prefer nevertheless to distinguish the upper and lower indices). We assume that $[T_{\alpha a}]^a$ and $e^n_{\alpha \beta \gamma}$ have been brought into the usual block-diagonal form. The scalar potential has the form $V (\phi) \equiv V_{\text{sym}} (\phi + v_n)$ and satisfies the following symmetry conditions

$$[T_{\alpha a}, (\phi^b + v_n^b)] \frac{\partial V (\phi)}{\partial \phi^a} = 0.$$  

We work in the class of linear $R_{\xi, u}$ gauges defined by the functions

$$f^a (x) = - \partial_{\mu} A^{\alpha \mu} (x) = \xi^{\alpha \beta} \delta_{a c} [T_{n\beta}]^c_b (\phi^b (x) + v_n^b).$$  

in which $u^n$ are additional gauge-fixing parameters. For greater generality we allow for the parameters $\xi^{\alpha \beta}$ which are arbitrary matrices in the space of the gauge Lie algebra generators. The Lagrangian (1) depends on the constant background $v_n$ only through the sum $\phi + v_n$ and the effective potential may be calculated by the usual methods [22]. In the derivations presented below it is convenient to treat the background $v_n$ (similarly as other renormalized parameters) as independent of gauge-fixing parameters.  

The simplest way to get the Nielsen identities [12] is to replace the $s$ operator in (1) with its extended counterpart $s_{\text{ext}}$ defined so that $s_{\text{ext}} = s$ on quantum fields and

$$s_{\text{ext}} (u^n (x)) = q^a (x), \quad s_{\text{ext}} (\xi^{\alpha \beta} (x)) = q^{\alpha \beta} (x),$$  

where $q^a (x)$ and $q^{\alpha \beta} (x)$ are fermionic external fields, called ‘Nielsen sources’ in the rest of the paper ($s_{\text{ext}} (q (x)) = 0$ to ensure nilpotency of $s_{\text{ext}}$). Unlike [5, 6, 12], we treat $\xi$, $u$ and consequently $q$, as $x$-dependent. (Of course $\xi$ and $u$ should be eventually restricted to constant configurations). This approach will allow us to avoid some of the IR divergences in explicit one-loop calculations presented in section 5. We choose to work without the Nakanishi-Lautrup multipliers $h_n (x)$, what seems to make the perturbative calculations easier. After elimination of $h_n (x)$ by using their equations we get the following Lagrangian density

$$\mathcal{L}^N (x) = - \frac{1}{4} \delta_{\alpha \beta} F^a_{\mu \nu} (x) F^{a \mu \nu} (x) + \frac{1}{2} \delta_{ab} (D_{\mu} \phi)^a (x) (D^{\mu} \phi)^b (x) - V (\phi (x))$$

$$- \int d^4 y \int d^4 z \bar{\omega}_a (x) \frac{\delta f^a (x)}{\delta A^b_{\gamma} (z)} D^i_{\alpha \gamma} (z, y) \omega^\gamma (y)$$

$$- \frac{1}{2} (\xi^{-1})_{\alpha \beta} (x) \frac{\delta f^a (x)}{\delta A^b_{\gamma} (x)} \left( f^a (x) + \frac{1}{2} q^{\alpha \beta} (x) \bar{\omega}_a (x) \right)$$

$$+ q^{\alpha \beta} (x) \bar{\omega}_a (x) (\phi^a (x) + v_n^a) \delta_{ac} [T_{n\beta}]^c_b (\phi^b (x) + v_n^b)$$

$$+ \frac{1}{2} L_{\alpha} (x) e^{n}_{\beta \gamma} \omega^\beta (x) \omega^\gamma (x) + \mathcal{K}_i (x) \int d^4 y \omega^\alpha (y) D^i_{\alpha \gamma} (x, y).$$  

In the last line of (12) we have added the usual BRST contributions (see e.g. 18). We use the notation

$$A^i (z) = (\phi^a (z), A^a_{\mu} (z)), \quad \mathcal{K}_i (z) = (K_a (z), K^a_{\mu} (z)),$$

and

$$D^i_{\alpha \gamma} (z, y) \equiv \frac{\delta}{\delta \omega^\gamma (y)} s (A^i (z)).$$

The action $\mathcal{T}^N$ corresponding to (12) satisfies the following Nielsen identity

$$\frac{\delta \mathcal{T}^N}{\delta K_{\alpha i}} - \frac{\delta \mathcal{T}^N}{\delta A^i} + \frac{\delta \mathcal{T}^N}{\delta L_{\alpha}} \cdot \frac{\delta \mathcal{T}^N}{\delta \omega^\alpha} - (\xi^{-1} (x))_{\beta a} \left( f^b + \frac{1}{2} q^{\beta \gamma} \bar{\omega}_a \right)$$

$$\cdot \frac{\delta \mathcal{T}^N}{\delta \omega^\beta} + q^{\alpha \beta} \frac{\delta \mathcal{T}^N}{\delta \xi^{\alpha \beta}} + q^a \frac{\delta \mathcal{T}^N}{\delta v^a} = 0.$$  

\[\text{If studying the effective potential is not needed, instead of treating } v_n \text{ as the constant background, one can determine } v_n \text{ from the condition } (\phi) = 0, \text{ that is from the requirement that the loop corrections cancel the tree level tadpole diagrams. This is equivalent to the minimization of the effective potential, and the gauge-dependence inherited by } v_n \text{ is then controlled by the appropriate Nielsen identity. For } u = v_n \text{ and } \xi^{\alpha \beta} \propto \delta^{\alpha \beta} \text{ this choice of } v_n \text{ reduces the } R_{\xi, u} \text{ gauges to the ordinary 't Hooft } R_{\xi} \text{ gauges.} \]
and the ghost equation
\[ \frac{\delta I^N}{\delta \omega_\alpha} (x) + \frac{\delta f^\alpha (x)}{\delta A^\mu} + \frac{\delta x^N}{\delta \delta_i} = \frac{1}{2} q^{\alpha \delta} (x) (\xi^{-1} (x))_{\delta \beta} \left( f^\beta (x) + \frac{1}{2} q^{\beta \gamma} (x) \omega_\gamma (x) \right) - q^{\alpha \beta} (x) (\phi^\beta (x) + v_\beta^a) \delta_{ac} \left[ T_{\beta \delta} \right]^c_b u^b (x) 
+ q^{\alpha} (x) \xi^{\alpha \beta} (x) \delta_{ac} \left[ T_{\beta \delta} \right]^c_b (\phi^b (x) + v_\beta^a), \]

in which \( F \cdot G \equiv \int d^4x \: F (x) G (x) \). For \( q = 0 \), the formula \((13)\) reduces to the Slavnov-Taylor identity of the BRST symmetry, the so-called Zinn-Justin equation \((17)\). The right-hand side of \((15)\) is at most linear in the quantum fields. The same is true for the coefficients multiplying the functional derivatives in \((14)\). Putting \((14)\) and \((15)\) under the path integral and integrating by parts, we obtain the corresponding identities satisfied by the functional generating all Green’s functions. Converting them into identities for the functional generating connected Green’s functions and, finally, performing the Legendre transform, we find that the regularized effective action also satisfies \((14)\) and \((15)\).

This is true because the dimensional regularization, which we implicitly use, preserves the (extended) BRST symmetry of the Yang-Mills theories coupled to scalars and vector-like fermions. On the other hand, if there are chiral fermions additional counterterms are needed to restore the identities \((32)\) \((36)\).

3 Renormalized Action

Since the regularized effective action respects the extended BRST symmetry, the standard Zinn-Justin arguments (see e.g. \((17)\) \((18)\) and Appendix \(A.1)\) imply that the action \(\tilde{I}^N\) which includes counterterms (as well as the regularized effective action \(I^N\)) also satisfy the equations \((14)\) and \((15)\). Therefore finding the most general form of \(\tilde{I}^N\), which is the purpose of this paper, reduces to writing down the most general, local dimension four function of the fields and sources with zero ghost number and to extract the constraints imposed on the coefficients of \(\tilde{I}^N\) by the Nielsen identity \((14)\) and the ghost equation of motion \((15)\). Details of the derivation of these constraints (i.e. to the equations \((14)\) and \((15)\)) are given in Appendix \(A.1.\) Here we present only the final result.

The Lagrangian with all possible counterterms has the form
\[ \tilde{L}^N (x) = - \frac{1}{4} \left[ Z_A (\xi (x)) \right]_{\alpha \beta} \tilde{F}^\alpha_{\mu \nu} (x) \tilde{F}^\beta_{\mu \nu} (x) + \frac{1}{2} \left[ Z_b (\xi (x)) \right]_{ab} \left( \tilde{D}^\alpha \phi \right)^a (x) \left( \tilde{D}^\beta \phi \right)^b (x) - \tilde{V} (\phi (x), \xi (x)) 
- \int d^d y \int d^d z \omega_\alpha (x) \frac{\delta f^\alpha (x)}{\delta A^\mu (y)} D^\gamma [A, u, \xi, y, \omega^\gamma (y)] 
- \frac{1}{2} \left( \xi^{-1} (x) \right)_{\alpha \beta} \left( f^\alpha (x) + \frac{1}{2} q^{\beta \gamma} (x) \omega_\gamma (x) \right) \left( f^\beta (x) + \frac{1}{2} q^{\gamma \delta} (x) \omega_\delta (x) \right) 
+ q^{\alpha \beta} (x) \omega_\alpha (x) \left( \phi^\beta (x) + v_\beta^a \right) \delta_{ac} \left[ T_{\beta \delta} \right]^c_b u^b (x) - q^{\alpha} (x) \omega_\alpha (x) \xi^{\alpha \beta} (x) \delta_{ac} \left[ T_{\beta \delta} \right]^c_b (\phi^b (x) + v_\beta^a) 
+ \frac{1}{2} \left( \xi^{-1} (x) \right) C^{\alpha}_{\beta \gamma} (\xi (x)) \omega^\beta (x) \omega^\gamma (x) + K_i (x) \int d^d y \omega^\alpha (y) D^i [A, u, \xi, y] 
+ \Delta \tilde{L}^N (x). \] (16)

Each line (except the last one) of the Lagrangian \((16)\) is a counterpart with counterterms included of the corresponding line of the Lagrangian \((12)\). The gauge-fixing function \(f^\alpha (x)\) is the same in both cases because we restrict ourselves to linear gauges \((10)\). For the same reason the fourth line of \((16)\) does not change after renormalization (see Appendix \(A.1)\). \(\Delta \tilde{L}^N (x)\) denotes additional counterterms required in the renormalization of Green’s functions with insertions of composite operators, which are coupled to Nielsen sources \(q (x)\):
\[ \Delta \tilde{L}^N (x) = L_{\alpha \beta \gamma \delta} (x) q^{\alpha \beta} (x) \omega^\gamma (x) \omega^\delta (x) + L_{\alpha \beta} (x) H^\alpha_{\beta \gamma \delta} (\xi (x)) q^{\beta \gamma} (x) q^{\delta \mu} (x) 
+ K_i (x) \int d^d y \int d^d z \omega_\alpha (x) \frac{\delta f^\alpha (x)}{\delta \omega^\gamma (y)} \delta_{ac} \left[ T_{\beta \delta} \right]^c_b u^b (x) 
+ K_c (x) b^a_{\alpha} (\xi (x)) q^a (x) - \int d^d z \omega_\alpha (x) \frac{\delta f^\alpha (x)}{\delta \phi^\gamma (z)} b^c_{\alpha} (\xi (x)) q^a (z). \]

The gauge invariant part of \((16)\) depends on the combinations
\[ \tilde{F}^\alpha_{\mu \nu} (x) = \partial_\mu A^\alpha_\nu (x) - \partial_\nu A^\alpha_\mu (x) + e^{\alpha \beta \gamma} (\xi (x)) A^\beta_\mu (x) A^\gamma_\nu (x) + \ldots, \]
\[ \left( \tilde{D}^\alpha \phi \right) (x) = \partial_\mu \phi(x) + A^\alpha_\mu (x) \left[ T_\alpha (\xi (x)) \phi (x) + V_{\alpha \beta} (\xi (x)) \right] + \ldots, \]
where the ellipses stand for contributions which vanish for \(x\)-independent \(\xi\) configurations; explicit form of these terms is given in Appendix \(A.1)\) (formulæ following eq. \((20)\)). Notice that the gauge invariant part of the Lagrangian depends on \(u\) only through the combination \(\phi (x) \equiv \phi (x) - b (\xi (x)) u(x)\),
\[ \phi (x) \equiv \phi (x) - b (\xi (x)) u(x), \] (20)
in which $b(\xi(x))$ is a matrix which determines the counterterms to Green’s functions with insertions of the composite operators coupled to the external sources (the third line of (17)). This follows immediately from the Nielsen identity (see Appendix A.1). The kernels $D^\alpha_\alpha[A,u,\xi|x,y]$ of the gauge transformations are now given by

$$D^\alpha_\alpha[A,u,\xi|x,y] = N^\beta_\alpha(\xi(x)) \left[ -\delta^\beta_\gamma \partial_\delta (x-y) + e^\beta_\gamma(\xi(x)) A^\gamma_\delta(x-y) \right] + \ldots,$$

$$D^\alpha_\alpha[\phi,u,\xi|x,y] = N^\beta_\alpha(\xi(x)) \left\{ T_\delta(\xi(x)) A^\gamma_\delta + V^\alpha_\beta(\xi(x)) \right\} \delta(x-y).$$

Finally, the kernels $k^\alpha_\beta\gamma[A,u,\xi|x,y]$ correspond to the extended gauge transformations

$$k^\alpha_\beta\gamma[A,u,\xi|x,y] = -\Omega^\beta_\alpha(\xi(x)) \partial_\gamma (x-y) + e^\beta_\gamma(\xi(x)) A^\gamma_\delta(x-y) + \ldots,$$

whose dimensionful parameters $V^\alpha_\beta(\xi)$ and $e^\beta_\gamma(\xi)$ are independent of $u$.

The Nielsen identity imposes a number of conditions on the coefficients of the Lagrangian (16). Among them one finds of course the ordinary BRST constraints [34]. These imply firstly that the “bare” parameters $e^\gamma_\alpha\beta$ must satisfy the Jacobi identity whereas the matrices $T_\alpha$ must obey the commutation relations

$$[T_\alpha, T_\beta] = e^\gamma_{\alpha\beta} T_\gamma.$$  

(25)

Secondly, that the structure constants $C^\kappa_\beta\gamma$ are related to $e^\gamma_{\alpha\beta}$ by the change of the Lie algebra basis

$$C^\kappa_\beta\gamma = N^\alpha_\beta N^\gamma_\kappa [N^{-1}]^\delta_\alpha e^\delta_{\alpha\epsilon},$$

and that the usual antisymmetry conditions must hold

$$[Z_A]_\alpha e^\alpha_\beta\gamma = -[Z_A]_\gamma e^\gamma_\alpha\beta, \quad [Z_\phi]_\alpha [T_\alpha]_b = -[Z_\phi]_b [T_\alpha]_\alpha,$$

(26)

(27)

together with the following equation for the function $\bar{V}(\dot{\phi}, \xi)$

$$([T_\alpha(\xi)]^a_\beta \Phi^b + V^\alpha_\beta(\xi)) \frac{\partial \bar{V}(\Phi, \xi)}{\partial \Phi^a} = 0.$$  

(28)

Finally, that the matrices $V^\beta_\alpha(\xi)$ must satisfy the standard cocycle equation [11]

$$[T_\alpha]_b^a V^b_\beta(\xi) - [T_b]_a^a V^b_\alpha(\xi) = e^\gamma_{\alpha\beta} V^\alpha_\gamma.$$  

(29)

Moreover, $V^\beta_\alpha(\xi)$ are independent of $u$, because the scalar field has been shifted as specified in (20).

The remaining requirements of the Nielsen identity can most concisely be expressed in terms of the differential forms

$$\hat{\theta}^\alpha_\delta = \theta^\alpha_\beta \delta^\beta_\gamma(\xi) \delta\xi^\gamma, \quad \hat{H}^\alpha = H^\alpha_\beta\gamma(\xi) \delta\xi^\beta \wedge \delta\xi^\gamma,$$

(30)

and

$$\hat{\theta}^\beta_\epsilon = \theta^\beta_\alpha \epsilon^\gamma(\xi) \delta\xi^\gamma, \quad \hat{\zeta}_b = \zeta^a_\alpha \delta\xi^\alpha, \quad \hat{\Sigma}_b = \Sigma^a_\alpha(\xi) \delta\xi^\alpha.$$  

(31)

In addition it is convenient to define

$$\hat{\psi}^\kappa = N^\kappa_\alpha \hat{H}^\alpha.$$  

(32)

In this language the $\xi$-dependence of the matrix Lie algebra generators acting on vector and scalar fields, respectively is governed by the following equations

$$\text{d} e^\gamma_\alpha = \left[ \bar{\theta}, e^\gamma_\alpha \right] - \hat{\theta}^\beta_\epsilon e^\epsilon_\delta, \quad \text{d} T_\gamma = \left[ \hat{\zeta}_b, T_\gamma \right] - \hat{\theta}^\beta_\delta T_\delta.$$  

(33)

(34)

We also get the relation

$$\text{d} V^\alpha_\beta(\xi) = \hat{\zeta}_b V^\alpha_\beta(\xi) - [T_\gamma]_b^a \hat{\Sigma}_b - V^\alpha_\gamma(\xi) \delta\xi^\alpha.$$  

(35)

Furthermore, the 1-forms $\hat{\theta}$, $\hat{\zeta}$ and $\hat{\Sigma}$ satisfy the equations

$$\text{d} \hat{\theta} = \hat{\theta} \wedge \hat{\theta} - \hat{\psi}^\epsilon e^\epsilon_\alpha.$$  

(36)

---

3Coefficients $H^\alpha_\beta\gamma$ can be treated as antisymmetric with respect to the interchange $(\beta\gamma) \leftrightarrow (\delta\epsilon)$.

4In our notation $e^\alpha_\beta = [e^\alpha_\beta]_\gamma^\beta \equiv e^\beta_\gamma$. 

---

5
Finally, we find that the gauge and scalar field renormalization constants \( \rho \) for any forms \( \hat{\sigma} \) we obtain the relation
\[
d\hat{\Psi} = \hat{\theta}_\alpha \wedge \hat{\psi}^\alpha.
\] (39)

\( \xi \)-dependence of the potential \( \tilde{V}(\Phi, \xi) \) with the counterterms included obeys
\[
\frac{\partial \tilde{V}(\Phi, \xi)}{\partial \xi_{\alpha\beta}} + \left( \zeta_{\alpha\beta}(\xi) \Phi^b + \Sigma_{\alpha\beta}(\xi) \right) \frac{\partial \tilde{V}(\Phi, \xi)}{\partial \phi^a} \equiv 0.
\] (40)

Finally, we find that the gauge and scalar field renormalization constants \( Z_A \) and \( Z_\phi \) satisfy the conditions
\[
d[Z_A]_{\alpha\beta} = -[Z_A]_{\alpha\xi} \tilde{\theta}^\xi \delta_{\beta\xi} - [Z_A]_{\beta\xi} \tilde{\theta}^\xi \delta_{\alpha\xi}, \quad d[Z_\phi]_{ab} = -[Z_\phi]_{a\xi} \tilde{\psi}_{a\xi} - [Z_\phi]_{b\xi} \tilde{\psi}^\xi_{\alpha\xi},
\] (41)
while \( \xi \)-dependence of the factors \( N \), which in linear gauges (like (10)) have the interpretation of the ghost fields renormalization constant, is constrained by the condition
\[
dN = \tilde{\theta}_N - N \tilde{\phi}.
\] (42)

More information can be obtained by exploiting invariance of the action (12) under global gauge transformations which remain symmetries of the action if \( \xi_{\alpha\beta}(x) \) and \( u^\alpha(x) \) are treated as external fields, which also undergo transformations. Details are presented in Appendix A2. Introducing the vector fields \( \Lambda_\alpha \) generating transformations of \( \xi_{\alpha\beta} \)
\[
\Lambda_\alpha = \left( e_R^\alpha e^\xi_{\beta\xi} + e_R^\alpha e^\xi_{\beta\xi} \right) \frac{\partial}{\partial \xi_{\alpha\beta}}, \quad [\Lambda_\alpha, \Lambda_\beta] = -\Lambda_\gamma \epsilon_{\alpha\beta\gamma}.
\] (43)
we obtain the relation
\[
(\Lambda_\alpha e_\kappa)(\xi) = \left[ e_\alpha_{\alpha\beta} - e_\kappa(\xi) e_R^\alpha e_{\alpha\kappa} \right], \quad (\Lambda_\alpha T_\kappa)(\xi) = \left[ T_{\alpha\beta} - T_\kappa(\xi) e_R^\alpha e_{\alpha\kappa} \right],
\] (44)
and similar equations for \( V^a_{\alpha\beta} \)
\[
(\Lambda_\alpha V_\kappa)(\xi) = T_{\alpha\beta} V_\kappa(\xi) - T_\kappa(\xi) T_{\alpha\beta} e_R^\alpha - V_\kappa(\xi) e_R^\alpha e_{\alpha\kappa}.
\] (46)

The matrix valued field renormalization constants \( Z_A, Z_\phi \) and \( N \) obey
\[
\left( \Lambda_\alpha [Z_A]_{\beta\rho} \right)(\xi) = -[Z_A(\xi)]_{\beta\rho} \delta_{\alpha\xi} - [Z_A(\xi)]_{\beta\rho} e_{\alpha\xi}, \quad (\Lambda_\alpha [Z_\phi]_{\beta\rho} \zeta)(\xi) = -[Z_\phi(\xi)]_{\beta\rho} T_{\alpha\beta} e_R^\alpha - [Z_\phi(\xi)]_{\beta\rho} T_{\alpha\beta} e_R^\alpha,
\] (47)
\[
(\Lambda_\alpha N)(\xi) = \left[ e_R^\alpha N(\xi) \right],
\] (49)
while the matrix \( b \) appearing in (20) satisfies the condition
\[
(\Lambda_\alpha b)(\xi) = \left[ T_{\alpha\beta} b(\xi) \right].
\] (50)
The corresponding relations satisfied by the differential forms (31) can be compactly expressed with the help of the Lie derivatives with respect to vector fields (19)
\[
\mathcal{L}_{\Lambda_\alpha} \hat{\theta} = \left[ e_R^\alpha, \hat{\theta} \right], \quad \mathcal{L}_{\Lambda_\alpha} \hat{\xi} = \left[ T_{\alpha\beta}, \hat{\xi} \right],
\] (51)
\[
\mathcal{L}_{\Lambda_\alpha} \hat{\Sigma} = \left[ T_{\alpha\beta} \Sigma - \hat{\xi} T_{\alpha\beta} v_R \right].
\] (53)

Similarly, the form \( \hat{\Psi} \) defined in (32) satisfies
\[
\mathcal{L}_{\Lambda_\alpha} \hat{\Psi} = e_R^\alpha \hat{\Psi}.
\] (54)
Defining \( \hat{\Omega}^\alpha \equiv \Omega^\alpha_{\beta \gamma} (\xi) \, d\xi^{\beta \gamma} \) for the coefficient of \( k^{\alpha} \) distribution we find

\[ \mathcal{L}_{\Lambda_\alpha} \hat{\Omega} = e_{\alpha \alpha} \hat{\Omega}. \]  

(55)

The analogous equation for \( \hat{g} \) follows now from (32) (see Appendix A.2). Finally, invariance with respect to global transformations implies the relation

\[ \left( A_\alpha \hat{V} \right) (\Phi, \xi) + [T_{\alpha \alpha}]^a_b (\Phi^b + v^b_r) \frac{\partial \hat{V} (\Phi, \xi)}{\partial \Phi^a} = 0. \]

(56)

In Appendix A.3 we consider the case in which the gauge Lie algebra contains an abelian ideal. For any abelian gauge field \( A^\mu_\alpha \) the Ward-Takahashi identity gives

\[ T_{\alpha \alpha} (\xi) = T_{\alpha \alpha}^a, \quad V_{N\alpha \alpha} (\xi) = T_{\alpha \alpha} v^a_r. \]

(57) (58)

Equation (57) is a QED-like ‘\( Z_1 = Z_2 \)’ identity. It is also shown in Appendix A.3 that the factors \( e^{\alpha \beta \gamma}, \hat{\beta}_{\alpha \beta} \) and \( \hat{\psi}^a \) vanish, when any of their indices corresponds to an abelian field. (In particular, the first equation (51) tells us that the abelian field renormalization constants \( [Z_A]_{\alpha \alpha \beta \gamma} \) are \( \xi \)-independent, what is well known.) Similar non-renormalization theorems hold for any gauge-singlet scalar field \( \phi^a \):

\[ [T_{\alpha \alpha}]^a_b = V_{N\alpha \alpha}^a = \hat{\xi}_{\alpha \alpha}^a = \hat{\Sigma}_{\alpha \alpha} = b^a r_c = 0. \]

(59)

In this way we have exhausted general information coming from the Nielsen identities as well as from the symmetry under global gauge transformations. (In specific models other global symmetries can of course provide additional constraints). The relations (33), (34) and (40) show that the 1-forms \( \hat{\xi} \) under global gauge transformations. (In specific models other global symmetries can of course provide additional contributions carrying the indices respectively of the vector and scalar fields. On the other hand, the remaining equations that govern the \( \xi \)-dependence, i.e. (33), (37) and (39), can be treated as the consistency conditions which ensure that \( d^2 = 0 \). In particular, the 2-form \( R \equiv \psi^a e_r \) is the curvature associated with the extended gauge invariance. Comparing with the case of a single \( \xi \) parameter considered in [12], \( \hat{\psi}^a \) is an additional counterterm a priori necessary to make finite Green’s functions of three composite operators; it is shown in Appendix A.3 that this counterterm is required only if global gauge invariance is broken by \( \xi^\alpha \beta \). Moreover, \( \hat{\psi}^a \) vanishes in the one-loop order because divergences of two graphs cancel each other (see Section 5). As was noticed in [12], in case of Fermi gauges with a single \( \xi \) parameter the equations controlling the \( \xi \)-dependence of the counterterms ensure that the bare gauge coupling constant of bare gauge fields is \( \xi \)-independent. We will show in the next section that occurrence of \( \hat{\psi}^a \) does not spoil this property in the case of the general \( R_{\xi \alpha} \) gauges (10) and that the formula (10) leads to the similar conclusion for all coupling constants of bare scalar fields.

Since the gauge-fixing parameter \( u \) has positive dimension, the dependence of counterterms on \( u \) is even more constrained. The gauge transformations, the covariant derivative of the scalar fields and the potential with counterterms depend on \( u \) only through the shifted field (20). It is therefore natural to check, whether this shift makes the entire contribution to the infinite VEV counterterm. Thus, we are interested in the relation between \( V_{N\alpha \alpha} \) and the background \( v^a_r \). Comparing the action (12) with its renormalized counterpart (16), we see that to the lowest order \( V_{N\alpha \alpha} = T_{\alpha \alpha} v^a_r \). For an abelian index \( \alpha \) the equality \( V_{N\alpha \alpha} = T_{\alpha \alpha} v^a_r \) is exact, as follows from (57) and (58). Moreover, \( \Lambda_{\alpha \alpha} \equiv 0 \equiv \mathcal{L}_{\Lambda_{\alpha \alpha}} \), hence the equations (55) and (52) read

\[ [T_{\alpha \alpha}, \, T_{\alpha \alpha} (\xi)] = 0, \quad \hat{T} \equiv \hat{\xi} = 0, \]

(60)

while (55) and (58) reduce to

\[ T_{\alpha \alpha} V_{N\kappa} (\xi) - T_{\kappa} (\xi) T_{\alpha \alpha} v^a_r = 0, \quad T_{\alpha \alpha} \hat{\Sigma} - \hat{\xi} T_{\alpha \alpha} v^a_r = 0, \]

(61)

so that the commutativity (60) leads to the relations

\[ T_{\alpha \alpha} (V_{N\kappa} (\xi) - T_{\kappa} (\xi) v^a_r) = 0, \quad T_{\alpha \alpha} (\hat{\Sigma} - \hat{\xi} v^a_r) = 0. \]

(62)

Let us first consider the class of theories (containing the Standard Model), in which the gauge algebra is not semisimple and there are no scalar singlets with respect to the abelian gauge ideal. In such cases equations (62) yield

\[ V_{N\kappa} (\xi) = T_{\kappa} (\xi) v^a_r, \quad \hat{\Sigma} = \hat{\xi} v^a_r, \]

(63)

for all gauge indices \( \kappa \), so that the gauge invariant part of the renormalized action depends only on the sum

\[ \hat{\phi} + v^a_r \equiv \phi + v^a_r - b (\xi) u, \]

(64)
and the parameter
\[ v_\mu \equiv v_R + \delta v = v_R - b(\xi) u, \] (65)
can be interpreted as the bare background with
\[ \delta v = -b(\xi) u, \] (66)
being the scalar field VEV counterterm. Considering more general theories, we know only that \( V_{N\alpha} \) are \( u \)-independent on account of the Nielsen identities. On the other hand the tree level action (12) depends on the background \( v_\mu \) only through the sum \( \phi + v_\mu \), and the same has to be true for the renormalized action (15), because the Dimensional Regularization respects formal invariance of the path integral under translations. This implies that
\[ V_{N\alpha}(\xi) = T_\alpha (\xi) v_\mu + W_\alpha(\xi), \] (67)
with \( W_\alpha(\xi) \) independent of \( u \) and \( v_\mu \). As argued above, \( W_{\alpha_0}(\xi) = 0 \) for any abelian index \( \alpha_0 \). For arbitrary indices the equation (29) yields
\[ [T_\alpha]^a_b W^b_\beta - [T_\beta]^a_b W^b_\alpha = \epsilon^{\gamma}_{\alpha\beta} W^a_{\gamma}. \] (68)
As we have seen, the coefficients \( \epsilon^{\gamma}_{\alpha\beta} \) are non-vanishing only for non-abelian indices \( \gamma_1, \alpha_1 \) and \( \beta_1 \) which means that the matrices \( T_{\alpha_1} \) form a representation of a semisimple Lie algebra. The solution to the equation (68) must therefore have the form
\[ W_{\gamma_1} = T_{\gamma_1} w, \] (69)
because the first cohomology space is trivial for any representation of a semisimple Lie algebra. In general one should not expect that \( w = 0 \) because global symmetry breaking by \( \xi \) can produce a scalar-vector mixing even in the symmetric phase, if the Lagrangian involves trilinear scalar couplings. On the other hand \( \Lambda_{\alpha} = 0 \), if the parameters \( \xi \) are introduced without spoiling the invariance with respect to global gauge transformations. Hence, comparing equations (46) and (45) one finds in this case that
\[ T_{\mu} W_{\kappa} = W_{\kappa} \epsilon_{\mu}^{\epsilon} \epsilon_{\alpha\kappa}, \] (70)
Using the solution (69) in (46) and comparing with (45) one obtains
\[ T_{\mu} T_{\mu} w = 0. \] (71)
It is well known that if \( \xi \) preserves global invariance, \( T_{\gamma_1} \) differ from \( T_{\mu \gamma_1} \) only by the separate renormalizations of gauge couplings of each simple ideal (see e.g. [34]). As a result, owing to the block-diagonal form of \( T_{\mu \gamma_1} \), the equality (71) implies that \( T_{\mu \beta_1} w = 0 \) and, finally, the formula (69) yields
\[ W_{\gamma_1} = 0, \] (72)
so that (63) remains true. Furthermore, in the presence of singlets under the entire gauge group, additional global symmetries can be used (like in models of spontaneous lepton number violation [9]) to ensure that singlets do not acquire an infinite VEV, while for singlets neutral with respect to global symmetries any \( \delta v \) can be absorbed into a linear term in the scalar potential. Therefore, if the global gauge symmetry is broken only by the \( u \) parameters, the equation (66) is satisfied and the Nielsen identity allows to completely determine the VEV counterterm \( \delta v \) in terms of \( b(\xi) \). To our knowledge this relation has never been presented in the literature. At one-loop the relation (66) offers a simple way to compute \( \delta v \) (see section 5).

4 Gauge Independence of Bare Coupling Constants

In this section we will show that the Nielsen identities for counterterms express the \( \xi \)-independence of bare coupling constants of properly defined bare fields. Since the Nielsen identity (14) does not involve the tree level representations, \( e_{\epsilon\alpha} \) and \( T_{\mu\alpha} \), the general solution of (14) depends on arbitrary generators \( e_{\epsilon} \) and \( T_{\alpha} \). The relation between these two sets of generators follows from the linearized Nielsen identity (22) and reads
\[ e^{\kappa}_{\beta, \gamma} = [Z_{e}]^\gamma_{\beta} [Z_\gamma]^-_{\epsilon} [Z_e]^-_{\delta} \epsilon^{\delta}_{\alpha\kappa}, \] (73)
\[ T_{\beta} = [Z_{e}]^\gamma_{\beta} [Z_{\gamma}]^-_{\kappa} T_{\kappa} [Z_T] \] (74)
(see [2] and discussion in appendix C), where \( Z_{e} = 1 + \Omega(\hbar) \) and \( Z_{T} = 1 + \Omega(\hbar) \) are arbitrary matrices, which eventually have to be determined from Feynman diagrams and all indices are restricted to the semisimple ideal.

We first focus on the \( \xi \)-dependence of the structure constants. Differentiating the formula (73) and comparing the outcome with (65) one finds that a matrix \( \hat{\epsilon} \) \( = \delta + Z_{e}^{-1} dZ_{e} \) obeys
\[ [\hat{\epsilon}, e_{\gamma}] = e_{\beta} \hat{\epsilon}^{\beta}_{\gamma}, \] (75)
8
where all indices (including those hidden in the matrix multiplication) are effectively restricted to non-abelian ones. Thus $\tilde{\hat{E}}$ is a linear combination of generators $e_\alpha$ and $\hat{E} = i\hat{\theta} e_\sigma$, and

$$\hat{\theta} = -Z^{-1}_e dZ_e + i\hat{\eta} e_\sigma.$$  \hfill (76)

Equation (76) is also true for abelian indices provided that $Z_e$ is extended to a block-diagonal matrix with the identity matrix in the abelian sector (1-forms $i\hat{\eta}$ are nonzero only for non-abelian indices). Computing $\hat{\theta} \wedge \hat{\theta} - d\hat{\theta}$ and using (34), one gets

$$\hat{\Psi}^\sigma = -d\hat{\eta}^\sigma + \hat{\theta}^\sigma \wedge \hat{\eta}^\delta - \frac{1}{2} \hat{\sigma}^\kappa \hat{\eta}^\kappa \wedge \hat{\eta}^\lambda.$$  \hfill (77)

Equations (41) and (76) yield

$$dZ_A = Z_A Z_e^{-1} dZ_e + [dZ_e]^T [Z_e^{-1}]^T Z_A,$$  \hfill (78)

($\hat{\eta}$-terms cancel each other owing to (27)). For a symmetric matrix

$$\mathcal{R}_A = [Z_e^{-1}]^T Z_A Z_e^{-1},$$  \hfill (79)

formula (80) gives

$$d\mathcal{R}_A = 0,$$  \hfill (80)

thus $\mathcal{R}_A$ is gauge-independent. Furthermore, using the parametrization (83) in (27) and taking into account the antisymmetry of generators $e_{\alpha \rho}$, we find

$$[\mathcal{R}_A, e_{\alpha \rho}] = 0.$$  \hfill (81)

The above condition holds also for a (symmetric) matrix $\sqrt{\mathcal{R}_A}$. Thus $\sqrt{\mathcal{R}_A}$ is a block-diagonal matrix with blocks corresponding to the entire abelian ideal and different simple ideals, moreover blocks corresponding to simple ideals are proportional to the identity matrix. Defining an orthogonal matrix

$$\mathcal{U}_A = \sqrt{Z_A} Z_e^{-1} \sqrt{\mathcal{R}_A^{-1}}; \quad \mathcal{U}_A \mathcal{U}_A^T = 1,$$  \hfill (82)

we can introduce bare gauge fields

$$A_\alpha^\rho \equiv [\mathcal{U}_A^{-1} \sqrt{Z_A}]^\alpha_\delta A_\delta^\rho.$$  \hfill (83)

Rewriting (18) in terms of $A_\alpha^\rho$, we obtain

$$\tilde{F}_{\mu \nu}^\delta = \left[ \sqrt{Z_A^{-1}} \mathcal{U}_A \right]_\alpha^\delta \left( \partial_\mu A_\alpha^\rho - \partial_\nu A_\rho^\alpha + e_{\beta \gamma} A_\beta^\rho A_\gamma^\alpha + \ldots \right),$$  \hfill (84)

where the bare structure constants read

$$e_{\alpha \beta \gamma} = [\mathcal{U}_A^{-1} \sqrt{Z_A}]_\alpha^\beta [\sqrt{Z_A^{-1}} \mathcal{U}_A]_\beta^\gamma.$$  \hfill (85)

Taking into account the relation $\mathcal{U}_A^{-1} \sqrt{Z_A} = \sqrt{\mathcal{R}_A Z_e}$ and equation (73), one finds

$$e_{\rho \alpha \beta \gamma} = [\sqrt{\mathcal{R}_A}]_\alpha^\rho [\mathcal{U}_A]_\rho^\alpha [\sqrt{\mathcal{R}_A^{-1}}]_\gamma^\beta.$$  \hfill (86)

Finally, using (81) we get

$$e_{\rho \alpha \beta \gamma} = e_{\rho \alpha \beta \gamma} [\sqrt{\mathcal{R}_A}]_\beta^\rho,$$  \hfill (87)

thus the bare structure constants differ from the renormalized ones only by the separate renormalization of coupling constants of each simple ideal. The Nielsen identity (80) ensures that these bare couplings are gauge-independent.

Consider now the $\xi$-dependence of generators $T_{\alpha \rho}$ in the space of scalar fields. Since the formula (14) is correct only for non-abelian indices, we denote them as $\alpha_1, \beta_1$, etc. Differentiating (14) and eliminating $dT_{\gamma \rho}$ with the aid of (81) one finds

$$[T_{\gamma \rho}, \hat{\tilde{\theta}}] = 0,$$  \hfill (88)

with $\hat{\tilde{\theta}}$ defined by

$$\hat{\tilde{\theta}} = \hat{\tilde{\theta}} - Z_T^{-1} dZ_T + i\hat{\eta} T_{\sigma \rho}.$$  \hfill (89)

\footnote{After some manipulations (75) yields $\bar{\hat{E}}_{\gamma \delta}^\rho = \bar{\hat{E}}_{\gamma \delta}^\rho [\mathcal{R}_T^{-1}]^\rho_{\gamma \delta} \text{tr} (e_{\alpha \beta} \bar{\hat{E}})$, where $\mathcal{R}_{\alpha \beta} = \text{tr} (e_{\alpha \beta})$ is invertible as a formal series (assuming a restriction to non-abelian indices) since $e_{\alpha \beta} = e_{\alpha \beta} + O(h)$.}
In terms of parametrization
\[ \dot{\rho} = Z^{-1}_T \dot{r} Z_T, \] (90)
the condition \([T_{\gamma_1}, \dot{\rho}] = 0\) reads
\[ [T_{\eta \gamma_1}, \dot{r}] = 0. \] (91)
Computing \(\dot{\zeta} \wedge \dot{\zeta} - d\dot{\zeta}\), one finds
\[ \dot{\zeta} \wedge \dot{\zeta} - d\dot{\zeta} = Z^{-1}_T (\dot{r} \wedge \dot{r} - d\dot{r}) Z_T + T_\sigma \left( -d\hat{\eta}^\sigma + \hat{\theta}_\sigma \wedge \hat{\eta}^\delta - \frac{1}{2} \hat{c}_{\kappa\lambda} \hat{\eta}^\kappa \wedge \hat{\eta}^\lambda \right). \] (92)
Owing to (37) and (77) the above formula yields the Maurer-Cartan equation
\[ d\dot{r} - \dot{r} \wedge \dot{r} = 0. \] (93)
Any 1-form obeying (93) can be represented as (see e.g. [35])
\[ \dot{r} = -M^{-1} dM, \] (94)
moreover, for a given \(\dot{r}\), equation (93) determines \(M\) uniquely up to a constant of integration \(M \rightarrow cM\). This freedom allows us to choose \(M(\xi_0) = 1\), which together with (91) ensures
\[ [T_{\eta \gamma_1}, M] = 0. \] (95)
With the aid of (94) we can rewrite \(\dot{\zeta}\) in the form
\[ \dot{\zeta} = -\tilde{Z}^{-1}_T d\tilde{Z}_T + \tilde{\eta}^\sigma T_\sigma. \] (96)
where
\[ \tilde{Z}_T \equiv M Z_T, \] (97)
and (74) takes the form (owing to (93))
\[ T_{\beta_1} = [Z_c]^{\alpha_1}_{\beta_1} \tilde{Z}^{-1}_T T_{\alpha_1 \alpha_1} \tilde{\zeta}_T. \] (98)
Defining
\[ R_\phi = \left[ \tilde{Z}^{-1}_T \right]^T Z_\phi \tilde{Z}^{-1}_T, \] (99)
we find (similarly to the case of \(R_A\))
\[ dr_\phi = 0, \] (100)
\[ [R_\phi, T_{\alpha_1}] = 0. \] (101)
We need also the following matrix
\[ U_\phi = \sqrt{Z_\phi} \tilde{Z}^{-1}_T \sqrt{R^{-1}_\phi}, \quad U_\phi U_\phi^T = 1, \] (102)
which allows us to define the bare scalar field
\[ \phi^a_B \equiv \left[ U_\phi^{-1} \sqrt{Z_\phi} \right]^a_d \phi^d. \] (103)
In order to rewrite the covariant derivative in terms of bare fields we have to compute \(A^a_B T_{\alpha_1} \phi\) and \(A^a_B T_{\alpha_0} \phi\) separately:
\[ A^a_B T_{\alpha_1} \phi = A^a_B \left[ Z_c^{-1} \sqrt{R_A^{-1}} \right]^{\alpha_1}_{\beta_1} T_{\alpha_1} \tilde{Z}^{-1}_T \sqrt{R^{-1}_\phi} \phi_B = A^a_B \left[ \sqrt{R_A^{-1}} \right]^{\alpha_1}_{\beta_1} \tilde{Z}^{-1}_T T_{\alpha_1} \sqrt{R^{-1}_\phi} \phi_B = \sqrt{Z_\phi^{-1}} U_\phi \left( A^a_B \left[ \sqrt{R_A^{-1}} \right]^{\alpha_1}_{\beta_1} T_{\alpha_1} \phi_B \right), \] (104)
hence bare generators have the form
\[ T_{B \beta_1} = \left[ \sqrt{R_A^{-1}} \right]^{\alpha_1}_{\beta_1} T_{\alpha_1}, \] (105)
in agreement with (87). On the other hand (due to \(T_{\alpha \alpha} = T_{\alpha \alpha}^T = 0\) and \([T_{\rho \alpha}, Z_\phi] = 0\) )
\[ A^a_B T_{\alpha_0} \phi = A^a_B \left[ Z_c^{-1} \sqrt{R_A^{-1}} \right]^{\alpha_0}_{\beta_0} T_{\alpha_0} \tilde{Z}^{-1}_T \sqrt{Z_\phi^{-1}} U_\phi \phi_B = A^a_B \left[ \sqrt{R_A^{-1}} \right]^{\alpha_0}_{\beta_0} \tilde{Z}^{-1}_T T_{\alpha_0} \sqrt{Z_\phi^{-1}} U_\phi \phi_B = \sqrt{Z_\phi^{-1}} U_\phi \left( A^a_B \left[ \sqrt{R_A^{-1}} \right]^{\alpha_0}_{\beta_0} U_\phi^T T_{\alpha_0} U_\phi \phi_B \right), \] (106)
thus
\[
T_{b;0} = \left[ \sqrt{R_A^{-1}} \right]_{\alpha_0}^{\beta_0} U_{\Phi}^T T_{R_{\alpha 0}} U_{\Phi},
\]
with
\[
U_{\Phi}^T T_{R_{\alpha 0}} U_{\Phi} = \sqrt{R_{\alpha}^{-1}} \left[ \hat{Z}_T^{-1} \right]_T Z_{\alpha} T_{R_{\alpha 0}} \hat{Z}_T^{-1} \sqrt{R_{\alpha}^{-1}} = \sqrt{R_{\alpha}^{-1}} \hat{Z}_T T_{R_{\alpha 0}} \hat{Z}_T^{-1} \sqrt{R_{\alpha}^{-1}}.
\]
Differentiating the above equation and eliminating \( d \tilde{\zeta} \),
\[
d \left[ U_{\Phi}^T T_{R_{\alpha 0}} U_{\Phi} \right] = \sqrt{R_{\alpha}^{-1}} \left[ T_{R_{\alpha 0}} ; \right] \hat{\zeta} - \hat{\eta}^T T_{\sigma} \hat{Z}_T^{-1} \sqrt{R_{\alpha}^{-1}} = 0,
\]
since the commutator vanishes according to (60). Hence the bare generators (107) are gauge-independent, and one can calculate them for \( \xi \)'s preserving the symmetry under global gauge transformations. In this case \( Z_T \equiv 1 \) (see e.g. 34), so that
\[
\left[ T_{R_{\alpha 0}} ; \right] = 0,
\]
and thus \( [T_{R_{\alpha 0}} , \mathcal{M}] = 0, \) yielding \( [T_{R_{\alpha 0}} , R_{\alpha}] = 0. \) Finally, equation (108) gives
\[
T_{b;0} = \left[ \sqrt{R_A^{-1}} \right]_{\alpha_0}^{\beta_0} T_{R_{\alpha 0}}.
\]

Since \( [R_A]_{\alpha_0,\beta_0} = [Z_A]_{\alpha_0,\beta_0} \), the above equation is yet another form of the \( \cdot Z_1 = Z_2 \) identity.

Hence the bare field \( \hat{\xi} \) is defined as the solution of the equation (62) in the presence of the Nielsen sources \( q(x) \). External lines of the diagrams correspond therefore either to 'quantum' fields \( (\phi, A_{\mu}, \omega, \varphi) \) or to classical sources \( (q, K, L) \). Solid and dotted lines represent respectively matter fields and the Faddeev-Popov ghosts. The necessary Feynman rules can be read off directly from the Lagrangian (10).

\[ K_c(x) \rightarrow y \rightarrow q^d(y) \]

**Figure 1:** One-loop contribution to \( b'_d \).

Thus, taking into account (66) and (112), \( \tilde{V}_{sym}^B (\varphi, \xi) \) is \( \xi \)-independent.

5 **Explicit Calculation of Counterterms**

In this section we compute (at the one-loop order) some of the counterterms in order to check validity of our results. We are interested in one-particle-irreducible (1PI) diagrams in the presence of the Nielsen sources \( q(x) \). External lines of the diagrams correspond therefore either to 'quantum' fields \( (\phi, A_{\mu}, \omega, \varphi) \) or to classical sources \( (q, K, L) \). Solid and dotted lines represent respectively matter fields and the Faddeev-Popov ghosts. The necessary Feynman rules can be read off directly from the Lagrangian (10).
The one-loop diagram shown in figure 1 after including the $b_d^c$ counterterm, gives the renormalized two-point function $F^c_d(x, y)$ of composite operators coupled to the external sources $K_c(x)$ and $q^d(y)$:

$$F^c_d(x, y) = b_d^c \delta(4)(x - y) + i \hbar \mathcal{N}_\alpha [T^\alpha]_a \xi_{\beta\gamma} \delta_{db} \left(T_{\mu\gamma}\right)^b \langle \phi^\alpha(x) \phi^\beta(y) \rangle_0 \times \langle \omega^\alpha(x) \omega^\beta(y) \rangle_0 + O(\hbar^2).$$  \hspace{1cm} (116)

In the dimensional regularization ($d = 4 - 2\epsilon$) the products of the tree level propagators gives

$$\langle \phi^\alpha(x) \phi^\beta(y) \rangle_0 \times \langle \omega^\alpha(x) \omega^\beta(y) \rangle_0 = \frac{i}{(4\pi)^d} \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{k^2 - M_\phi^2 + i\epsilon} + \ldots \right]^{ac} \left[ \frac{1}{(k-p)^2 - M_\omega^2 + i\epsilon} \right]^{\delta \beta}$$

$$= \frac{1}{(4\pi)^d} \delta_{\alpha\beta} \delta(4)(x - y) + O(\epsilon^0).$$  \hspace{1cm} (117)

The ellipses in the first square bracket stand for terms arising from the mixing of the scalar and vector fields. These terms do not change the leading UV behavior of the propagator and can, therefore, be omitted in the present calculation. Since to the order we are working $N^{\alpha}_\alpha = \delta^{\alpha}_\alpha$, and $T^\alpha = T^\alpha_{\mu\delta}$, we get (in the $MS$ scheme)

$$b_d^c = \frac{\hbar}{(4\pi)^2} \xi^{\alpha\gamma} \left[ T^{\alpha\gamma}_{\mu\alpha} T_{\alpha\gamma}\right]^{c}_{d} + O(\hbar^2).$$  \hspace{1cm} (118)

The above matrix clearly respects the symmetry requirements \[54\]. If $(\xi^{-1})_{\alpha\gamma}$ is an invariant form on the gauge Lie algebra, the formula \[118\] tells us that $b_d^c$ is proportional to the Casimir operator of the representation $T_{\mu\alpha}^{\alpha}$.

Other dimensionless counterterms such as $\tilde{\theta}, \tilde{g}$ etc., can be calculated in the restricted 't Hooft gauge ($u = v_\mu$ with $\langle \phi \rangle = 0$). This choice removes the tree-level mixing between scalar and vector fields and leads to the standard form of the propagator \[54\]:

$$\langle A^\alpha_{\mu} A^\beta_{\nu}\rangle_0(k) = -i \left[ \eta_{\mu\nu} \frac{1}{k^2 - M^2 + i\epsilon} + k_\mu k_\nu \frac{1}{k^2 - M^2 + i\epsilon} \right]^{(\xi - 1)} \left[ \frac{1}{k^2 - M^2 + i\epsilon} \right]^{\alpha\beta}. $$  \hspace{1cm} (119)

As long as we are interested in dimensionless parameters, the non-diagonal form of the mass matrices is immaterial and calculations with general $\xi^{\alpha\beta}$ parameters can be easily performed. Computing divergent parts of the diagrams shown in figure 2 we find (\{,\} denotes the anticommutator)

$$\theta^\delta_{\alpha\epsilon\beta} = -\frac{1}{(4\pi)^2} \frac{1}{8} \{e_{\epsilon\alpha}, e_{\epsilon\beta}\}^\delta_{\beta},$$  \hspace{1cm} (120)

while for $g^\delta_{\alpha\epsilon\beta}$ we obtain (see figure 3) the result:

$$g^\delta_{\alpha\epsilon\beta} = -\frac{1}{(4\pi)^2} \frac{1}{4} \{e_{\epsilon\alpha}, e_{\beta}\}^\delta_{\beta}. $$  \hspace{1cm} (121)
Figure 4: One-loop diagrams contributing to $H^{\delta}_{\alpha\beta\kappa\epsilon}$. 

Figure 5: One-loop contribution to $N^{\alpha}_{\delta}$. 

The sum of two divergent diagrams shown in figure 4 is finite, hence $\hat{H}^{\epsilon} = O(h^2)$. This result agrees with (32) and (36), since $d\hat{\theta} = 0$ while $\hat{\theta} \wedge \hat{\theta}$ is of the order of $h^2$. The one-loop correction to the ghost propagator, which is relevant for the computation of $dN$, is shown in figure 5. It gives

$$dN^{\alpha}_{\delta} = \frac{1}{(4\pi)^2\epsilon} \frac{1}{4} \delta^\alpha_{\delta} \{ e^\epsilon_{\nu} e^\epsilon_{\mu\kappa} \}^\epsilon_{\beta} d\xi^{\kappa\epsilon}. \tag{122}$$

The results (120), (121) and (122) are consistent with the Nielsen identity requirements (42). Among various corrections to the gauge field propagator, only the diagram of figure 6 contributes to $dZ_A$. A short calculation gives:

$$d[Z_A]^{\alpha}_{\beta} = \frac{1}{(4\pi)^2\epsilon} \frac{1}{4} \delta^\alpha_{\beta} \{ e^\epsilon_{\nu} e^\epsilon_{\mu\kappa} \}^\epsilon_{\delta} d\xi^{\kappa\epsilon}. \tag{123}$$

This agrees with (120) and (111). Finally, the diagram shown in figure 7 determines renormalization of the structure constants $C^{\alpha}_{\beta\gamma}$ yielding

$$C^{\alpha}_{\sigma\delta} = e^\alpha_{\nu} e^\sigma_{\epsilon} e^\delta_{\epsilon} = [Z_C]^{\alpha}_{\nu} e^\nu_{\sigma} e^\sigma_{\gamma} e^\gamma_{\delta} = [Z_C]^{\alpha}_{\nu} e^\nu_{\sigma} e^\sigma_{\gamma} e^\gamma_{\delta} = [Z_C]^{\alpha}_{\nu} e^\nu_{\sigma} e^\sigma_{\gamma} e^\gamma_{\delta} = O(h^2), \tag{124}$$

with

$$[Z_C]^{\alpha}_{\nu} = \delta^\alpha_{\nu} - \frac{h}{(4\pi)^2\epsilon} \frac{1}{4} \{ e^\epsilon_{\nu} e^\epsilon_{\mu\kappa} \}^\epsilon_{\beta} \xi^{\kappa\epsilon}, \tag{125}$$

so that $\hat{\psi} = [dB_C] Z_C^{-1} + O(h^2)$. The $\xi$-dependent part of the structure constants $e^\alpha_{\beta\gamma}$ can be then obtained with the aid of (29) and (122) - the resulting expression is consistent with (33) and (120).

We end this section by rederiving, with the help of the Nielsen identities, the well known equation for the gauge-dependence of the electron field renormalization constant in QED (see e.g. [18]). (As we have shown in the preceding section, in abelian theories gauge-independence of the gauge field renormalization constant follows immediately from the Nielsen identities.) In QED the renormalized Lagrangian includes the terms

$$\tilde{\mathcal{L}}^N(x) = ieR \overline{K}_A(x) \omega(x) \psi^A(x) + \overline{\psi}(x) \gamma^4 \gamma^3 \psi^A(x), \tag{126}$$

(and analogous couplings of the $\overline{\psi}_A(x)$ field), in which $e_R$ denotes the renormalized charge. Instead of the second equation (111) we now have

$$\frac{1}{Z_\psi} \frac{\partial Z_\psi}{\partial \xi} = -\zeta - \zeta. \tag{127}$$

Owing to the decoupling of ghosts there is only one (‘dressed’) diagram contributing to $\zeta$. It is shown in figure 8 in which double lines represent the full (renormalized) propagators of photon and electrons. The blob stands for the
A_\mu(x) \rightarrow A_\mu^B(y)

Figure 6: One-loop contribution to d [Z_A]_{\alpha\beta}.

L_\delta(z) \rightarrow \omega^\nu(y)

Figure 7: One-loop contribution to C_{\delta \kappa \beta}.

(renormalized) 1PI vertex $A_\mu \overline{\psi}_B \psi^C$. The vertex $q \overline{\psi} A_\mu$ comes from the third line of the Lagrangian (16), and depends only on $\partial^\mu A_\mu$. Therefore, to compute this diagram we need only the transverse part of the photon propagator, which is unaffected by radiative corrections. Consequently, we can eliminate the divergence of the $A_\mu \overline{\psi}_B \psi^C$ vertex, by using the Ward-Takahashi identity. In this way we get the following contribution to the effective action

$$\Gamma^N \supset \frac{1}{2} \epsilon^2 R \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \overline{K}_A(-p - l)q(l)\psi^B(p) \int \frac{d^d k}{(2\pi)^d} \frac{i}{(k + l)^2 k^2} \left[1 + i \tilde{G}^{(2)}_R (p - k) \tilde{\Gamma}^{(2)}_R (-p) \right]^{A}{B}.$$ (128)

The electron propagator $\tilde{G}^{(2)}_R (p - k)$ makes the integral of the second component convergent. Hence,

$$\zeta = \frac{1}{(4\pi)^2} \frac{1}{2} \epsilon^2 R.$$ (129)

Computing an analogous diagram with $K$ and $\overline{\psi}$ external lines instead of $K$ and $\psi$ we find $\zeta = \zeta$, and finally

$$\frac{1}{Z_\psi} \frac{\partial Z_\psi}{\partial \xi} = -\frac{1}{(4\pi)^2} \epsilon^2 R.$$ (130)

The derivation presented here should be compared with the standard one based on the Ward-Takahashi identity, which can be found e.g. in [18].

6 Determination of $\delta \psi$

One-loop checks of the Nielsen identity for the effective potential in the abelian Higgs model in $R_\xi, u$-gauges can be found in [17, 16]. In order to verify the relation (66), as well as other requirements of the Nielsen identity, we have computed the effective potential in a simplified version of the Standard Model, with $g_t \approx 1$ as the only non-vanishing Yukawa coupling. In this calculation known problems with $\gamma_5$ do not play any role, and one expects that (66) should hold at the one-loop order. Compared to its Landau gauge form the effective potential in the $R_\xi, u$ gauge has some unusual features which deserve special discussion. In particular, the vacuum direction depends on the gauge already at tree level.

On the quartet of the real scalar fields $\phi \in \mathbb{R}^4$ the generators of the $u(1)_Y \times su(2)_L$ algebra are represented by the

---

8In the Landau gauge the effective potential has been computed in a general renormalizable theory up to two-loops [30, 31].
The following four matrices:

\[
T_{n0} = \frac{g_v}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad T_{n1} = \frac{g}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_{n2} = \frac{g}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_{n3} = \frac{g}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

The structure constants read

\[
e_{\alpha \beta \gamma} = \begin{cases} \frac{g \epsilon_{\alpha \beta \gamma}}{2} & \text{for } \alpha, \beta, \gamma \in \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}
\]

For simplicity, we take

\[
\xi_{\alpha \beta} = \xi \delta_{\alpha \beta},
\]

and

\[
u = (0, \bar{u}, 0, 0)^T.
\]

In the presence of a constant background \(v_n\), the scalar potential with all counterterms allowed by (56) has the form

\[
\tilde{V}(\phi, \xi) = \frac{1}{2} Z_\alpha (m^2 + \delta m^2) \left( \phi + v_n \right)^T (\phi + v_n) + \frac{1}{4!} Z_\alpha^2 (\lambda + \delta \lambda) \left( (\phi + v_n)^T (\phi + v_n) \right)^2,
\]

in which (see (20))

\[
\phi = \phi + \delta \nu.
\]

The tree level effective potential written in terms of the background field \(v_n\) includes also the contributions of the gauge-fixing term and reads

\[
V_{eff}^{(0-loop)}(v_n) = \frac{1}{2} m^2 v_n^T v_n + \frac{1}{4!} \lambda (v_n^T v_n)^2 + \frac{\xi}{2} \delta_{\alpha \beta} (v_n^T T_{n \alpha} u) (v_n^T T_{n \beta} u).
\]

For \(\xi > 0\), \(V_{eff}^{(0-loop)}\) has a minimum at

\[
v_n = (0, \bar{v}_n, 0, 0)^T,
\]

with (assuming \(m^2 < 0\))

\[
\bar{v}_n^2 = \frac{6 m^2}{\lambda}.
\]

While the occurrence of spontaneous symmetry breaking (i.e. the existence of the solution (139)) depends only on the parameters of the gauge invariant part of the Lagrangian, the form (135) of \(v_n\) indicates that the vacuum alignment depends on the gauge (i.e. on the direction of \(\bar{u}\)) already at the tree level. This is reminiscent of the well-known Dashen vacuum alignment condition [33]. In the \(R_{\xi, u}\) gauge with the choice (134), the vacuum degeneracy is entirely removed - we have to choose the solution to (139) which has the same sign as \(\bar{u}\). Otherwise mass squares of unphysical ‘particles’ would become negative and the usual interpretation of the Cutkosky rules in terms of the (pseudo)unitarity would be destroyed. The solution (138) implies the following identification of the electromagnetic \(u(1)_{EM}\) generator

\[
Q = e \frac{g_v}{g_v} T_{n0} + \frac{g}{g} T_{n3}, \quad e = \frac{g_y}{g_x}, \quad g_x = \sqrt{g^2 + g_y^2}.
\]

\(^{5}\)Is is worth stressing here again that in the Nielsen identity (13) we differentiate with respect to \(\xi\) and \(u\) keeping the background \(v_n\) fixed. From the Nielsen identity satisfied by \(F^N\) it then follows that the gauge-dependence of the VEV of the scalar field (i.e. of the minimum of \(V_{eff}\)) cancels with the explicit gauge-dependence of 1PI functions, ensuring that physical masses and couplings expressed as functions of parameters of the tree level action (12) do not depend on \(\xi\) and \(u\) (see [3]).
and leads to the usual parametrization of the scalar field
\[ \phi = (G^1, h, G^2, G^3)^T. \] (141)

Computing the one-point function of \( h(x) \), we obtain\(^{10}\)
\[ -\frac{\partial V_{\text{eff}}^{(1-\text{loop})}}{\partial v_n} (v_n) = -\frac{h}{(4\pi)^2} \epsilon \left[ -3g_t^2 v_n^3 + \lambda v_n \left( m^2 + \frac{\lambda}{3} v_n^2 \right) + \frac{3}{16} (2g^4 + g_c^2) v_n^3 + \frac{1}{4} \xi v_n (2g^2 + g_c^2) \left( m^2 + \frac{\lambda}{3} v_n^2 \right) + \frac{1}{4} \xi u (2g^2 + g_c^2) \left( m^2 + \frac{\lambda}{2} v_n^2 \right) \right] + \mathcal{O}(\epsilon^0) + \frac{-Z_v (m^2 + \delta m^2) (v_n + \delta v) - \frac{1}{3!} Z_v^2 (\lambda + \delta \lambda) (v_n + \delta v)^3. \] (142)

The above function is indeed finite (at \( \mathcal{O}(\epsilon) \) order) for any value of \( v_n \), provided that
\[ \delta v = \frac{h}{(4\pi)^2} \epsilon \left[ \frac{1}{4} \xi (2g^2 + g_c^2) u + \mathcal{O}(\epsilon^2) \right], \] (143)
\[ \delta m^2 = \frac{h}{(4\pi)^2} \epsilon \left[ \frac{1}{4} \lambda (2g^2 + g_c^2) \right] - m^2 \delta Z_v + \mathcal{O}(\epsilon^2), \] (144)
\[ \delta \lambda = \frac{h}{(4\pi)^2} \epsilon \left[ 2\lambda^2 - 18g_t^4 + \frac{9}{8} (2g^4 + g_c^4) - \frac{1}{2} \lambda \xi (2g^2 + g_c^2) \right] - 2\lambda \delta Z_v + \mathcal{O}(\epsilon^2), \] (145)
where \( \delta Z_v = Z_v - 1 \). For the generators \( B_e \), the formula \( \mathbf{118} \) yields
\[ b^a_e = -\frac{h}{(4\pi)^2} \epsilon \left[ \frac{1}{4} \xi (2g^2 + g_c^2) \delta^a_e + \mathcal{O}(\epsilon^2) \right], \] (146)
so that \( \mathbf{65} \) holds true as required by the Nielsen identity. Furthermore, the mass counterterm \( \mathbf{143} \) is independent of \( u \), as it should be - the scalar potential in \( \mathbf{10} \) can depend on \( u \) only through \( \phi \). Since \( Z_v \) cannot depend on \( u \) and \( v_n \), we have computed the two-point function of \( h(x) \) in the restricted \( \text{'t Hooft gauge} \) (i.e. setting \( u = v_n \) with the background \( v_n \) chosen so that \( \langle \phi \rangle = 0 \)). This gives
\[ \delta Z_v = \frac{h}{(4\pi)^2} \epsilon \left[ \frac{1}{4} \xi (\xi - 3) (2g^2 + g_c^2) \right] + \mathcal{O}(\epsilon^2). \] (147)

The explicit \( \xi \)-dependence of \( \delta m^2 \) and \( \delta \lambda \) given by \( \mathbf{145} \) and \( \mathbf{144} \), respectively is canceled by that of \( Z_v \). This also agrees with the Nielsen identity, since the relations \( \mathbf{40} \) and \( \mathbf{11} \) allow the scalar potential \( \mathbf{135} \) to depend on \( \xi \) only through \( Z_v \), as we have shown in Section 4.

We end this section by deriving a condition which ensures homogeneity of the Nielsen identity satisfied by the (renormalized) effective potential. To this end we set \( v_n = 0 \), so that now the field \( \phi \) implicitly includes its VEV and differentiate the Nielsen identity \( \mathbf{14} \) for the effective action \( \Gamma^\text{eff} \) with respect to \( \phi^{\text{ext}}(x) \). In this way we obtain the equation originally derived by Nielsen \( \mathbf{5} \)
\[ \left[ \left( \frac{\delta}{\delta q^{\text{ext}}(x)} \frac{\delta \Gamma^N}{\delta \kappa} \right) \cdot \frac{\delta \Gamma^N}{\delta \Lambda^e} - \left( \frac{\delta}{\delta q^{\text{ext}}(x)} \frac{\delta \Gamma^N}{\delta \omega^a} \right) \frac{\delta \Gamma^N}{\delta \omega^a} \right]_{\gamma h.n. = 0} = 0, \] (148)
where \( gh.n. = 0 \) indicates restriction to terms having zero the ghost number. For spacetime-independent configurations of \( \phi, \xi, u \) and vanishing vector fields \( \mathbf{148} \) reduces to
\[ \frac{\partial V_{\text{eff}}}{\partial \xi^{\text{ext}}} + \frac{\partial V_{\text{eff}}}{\partial \phi} C^{\alpha}_{\kappa \xi}(\phi, \xi) = \left( u^T T_{\alpha \xi} \phi \right) B^\alpha_{\kappa \xi}(\phi, \xi), \] (149)
with
\[ C^{\alpha}_{\kappa \xi}(\phi, \xi) \equiv \int d^4 y \left( \frac{\delta \Gamma^N}{\delta q^{\text{ext}}(x)} \frac{\delta K^\gamma}{\delta \Lambda^e(y)} \right) \bigg|_{\phi = \text{const. rest = 0}}, \quad B^\alpha_{\kappa \xi}(\phi, \xi) \equiv \int d^4 y \left( \frac{\delta \Gamma^N}{\delta q^{\text{ext}}(x)} \frac{\delta \omega^a}{\delta \omega^a(y)} \right) \bigg|_{\phi = \text{const. rest = 0}}. \] (150)

Nielsen worked in the \( u = 0 \) gauge, for which the right-hand side of \( \mathbf{150} \) is zero and the resulting identity has the form analogous to the renormalization group equation satisfied by \( V_{\text{eff}} \). In this case \( \xi \)-independence of the occurrence of spontaneous symmetry breaking is ensured (see \( \mathbf{5} \) for details). As pointed out in \( \mathbf{17} \), to reach the same conclusion\(^{10}\) the background \( v_n \) is restricted to the \( h(x) \)-direction. Under our assumptions, the vanishing of other tadpoles is then ensured by the \( CP \) symmetry and the \( U(1)_{\text{EM}} \) symmetry.
for $u \neq 0$, the effective potential should be restricted to field configurations obeying $u^T T_{R\alpha} \phi = 0$. The physical minimum of the full effective potential must belong to this subspace anyway - for the gauge-fixing function \( f(x) \) this condition is equivalent to the requirement that
\[
\langle f(x) \rangle = 0. \tag{151}
\]
Violation of (151) would mean spontaneous BRST symmetry breaking \cite{25,26} which would spoil the Kugo-Ojima quartet mechanism and, consequently, unitarity of the physical $S$-matrix. Still, one should check whether the minima of the potential restricted to this subspace are indeed stationary points of the full effective action. Usually this is guaranteed by discrete symmetries \cite{26} what in most cases requires invariance of the theory under $CP$ which however need not always be an exact symmetry of the theory of interest. As we now show, even in the absence of requisite discrete symmetries, the Slavnov-Taylor identity of the ordinary BRST symmetry itself (which always holds true if there are no anomalies) ensures that at the minimum of the restricted potential the remaining tadpoles vanish automatically provided a certain condition is satisfied at the tree level. This generalizes the observation made in \cite{9} for the Standard Model quantized in the ordinary ’t Hooft $R_g$ gauge.

Differentiating the identity \cite{13} written for $\Gamma^N$ with respect to $\omega^\gamma(x)$ one finds that only one term on the left-hand side contributes when $\phi$ is spacetime-independent and satisfies $u^T T_{R\gamma} \phi = 0$ while all other fields are taken to zero. This gives rise to the relation
\[
\frac{\partial V_{\text{eff}}(\phi)}{\partial \phi^a} \chi^a_{\gamma}(\phi) = 0, \tag{152}
\]
in which
\[
\chi^a_{\gamma}(\phi) \equiv \left[ \int d^4y \left. \frac{\delta}{\delta \omega^\gamma(x)} \delta \Gamma^N \right|_{\text{accum}} \right]_{\text{rel}} = [T_{R\gamma}]^a_b \phi^b + \mathcal{O}(\hbar). \tag{153}
\]
Defining a matrix
\[
\Omega_{BA} \equiv (u^T T_{R\gamma})_a, \tag{154}
\]
one gets
\[
\frac{\partial V_{\text{eff}}(\phi)}{\partial \phi^a} \bigg|_{\phi = \phi_0} = \lambda^B \Omega_{Ba}, \tag{155}
\]
where $\phi_0$ is a stationary point of the effective potential restricted to $\ker \Omega$, $\lambda^B$ are Lagrange multipliers and the index $B$ runs over a set $R$ of linearly independent rows of the matrix $\Omega_{BA}$. Comparing (152) and (155) gives
\[
\lambda^B \Omega_{Ba} \chi^a_{\gamma}(\phi_0) = 0. \tag{156}
\]
Finally, taking into account the expansion (155), one finds
\[
\Omega_{Ba} \chi^a_{\gamma}(\phi_0) = \frac{1}{2} u^T \{ T_{R\gamma}, T_{R\beta} \} \phi_0 + \mathcal{O}(\hbar). \tag{157}
\]
Thus, if the quadratic form
\[
\mu_{BG} \equiv u^T \{ T_{R\gamma}, T_{R\beta} \} \phi_0, \quad B, G \in R, \tag{158}
\]
is nondegenerate, then from the relation (156) one recursively infers that $\lambda^B$ vanishes to all orders. Hence, according to (156), $\phi_0$ is the stationary point of the full effective potential. \footnote{It is easy to see that the same conclusion readily follows if $\phi_0$ in the definition (155) is replaced by its tree approximation. However, in theories in which a nonzero VEV is generated only radiatively, it is better to treat $\phi_0$ as the minimum of the potential calculated to a given order in the loop expansion.

\footnote{We disagree with the suggestions made in \cite{27}, that no relation between $\langle \phi \rangle$ and $u$ is required, and that one can construct a quantum theory around any minimum of the modified effective potential $V_{\text{eff}}^{\text{mod}}(\phi)$, which in our notation reads
\[
V_{\text{eff}}^{\text{mod}}(\phi) \equiv V_{\text{eff}}(\phi) - \frac{\mu_{BG}}{2} \left( u^T T_{R\alpha} \phi \right) \left( u^T T_{R\beta} \phi \right).
\]
$V_{\text{eff}}^{\text{mod}}$ defined in this way is gauge-independent at the tree level and satisfies the homogeneous Nielsen identity (this follows immediately from (159) for $\Gamma^N$), since the functions defined in (160) are related to each other by the ghost equation (155) for $\Gamma^N$. The modified potential was obtained in \cite{27} from the effective action by setting to zero the Nakanishi-Lautrup multipliers. This is however an off-shell configuration of fields if the scalar fields do not satisfy the condition $u^T T_{R\alpha} \phi = 0$. Thus, the potential $V_{\text{eff}}$, in which the Nakanishi-Lautrup fields are always on-shell (with respect to a given configuration of the scalar fields) seems more physical despite its gauge-dependence. Of course, this is not the case in the model considered in the first part of this section, where $\phi_0$ is the VEV associated with the Nakanishi-Lautrup fields. Nevertheless, the present discussion remains valid for the off-shell configuration of fields, if only the physical fields are taken into account.}}
The above reasoning shows that in theories without chiral fermions in which the requisite discrete symmetries are not exact (e.g. CP can be explicitly and/or spontaneously broken in the extended Higgs sector) the effective potential can be restricted to configurations satisfying the condition $u^T T_{R, \alpha} \phi = 0$ under relatively mild requirements. On the other hand, in theories with chiral fermions in which (as in the Standard Model) CP is not an exact symmetry, our reasoning formally can still be applied, since it is based only on the Slavnov-Taylor identity satisfied by the effective action $\Gamma^N$. However, in this case the renormalized action $\tilde{\Gamma}^N$ must include counterterms explicitly violating its BRST invariance in order to restore the invariance of $\Gamma^N$ \cite{16}. It appears that as such BRST-noninvariant counterterms appropriate terms linear in the scalar fields (linear in the would-be Goldstone bosons) can be indispensable to ensure restoration of the identity \cite{132} which has been the starting point of our arguments.

7 Conclusions

In this paper we have found all counterterms required to render finite the effective action with Nielsen sources included of a general Yang-Mills theory coupled to arbitrary scalar fields and quantized in linear $R_{\alpha \mu}$ gauges. We have shown that the dependence of all counterterms on $u$ is controlled by a single matrix $b(\xi)$ that acts as a counterterm for a two-point function of certain composite operators. In particular, the gauge-invariant part of the action with counterterms depends on $u$ only through the shifted scalar field $\phi = \phi - b(\xi) u$. Assuming that the parameters $\xi$ are consistent with the symmetry under global gauge transformations, we have proved that this shift constitutes the only contribution to the well known VEV counterterm $\delta v$. This is our main new result that allows a simple calculation of $\delta v$ at one-loop and clarifies its origin. The resulting expression for $\delta v$ agrees with explicit computations in Section 5 as well as those of \cite{14,20,15}.

We have also considered the case of multiple parameters $\xi$. We have shown that an additional counterterm (the 'curvature') $\Psi$, which trivially vanishes in the situation studied in \cite{12}, can be generated only if the matrix $\xi^{-1}$ breaks the symmetry under global gauge transformations. We have also shown that the interpretation of the Nielsen identities for counterterms in terms of $\xi$-independence of coupling constants of bare fields is unaffected by $\Psi$.

Finally, we have considered the problem of homogeneity of the Nielsen identities that control the gauge-dependence of the effective potential. There has been much discussion in the literature of this issue, e.g. \cite{26,7,27}. The effective potential restricted to configurations which preserve the BRST symmetry satisfies the homogeneous Nielsen identities. We have introduced a condition which allows to check whether a minimum of such a restricted potential is a stationary point of the full effective action.

Acknowledgments. I am indebted to Professors P. H. Chankowski and K. A. Meissner for enlightening conversations and comments on an early version of this paper.

A Determination of $\tilde{\Gamma}^N$

A.1 Nielsen Identities for $\tilde{\Gamma}^N$

The renormalized action $\tilde{\Gamma}^N$ which includes all possible counterterms is a general local solution to the Nielsen identities and the ghost equation, constrained by the power-counting and the ghost number conservation. Scalar and vector fields have dimension 1. We treat parameters $u^\alpha(x)$ and $\xi(x)$ as external fields of vanishing ghost number and dimension 1 and 0, respectively. If, as is customary, we ascribe dimension 1 and ghost number +1 to the ghost field $\xi$, then for the other fields dimension 4 and zero ghost number of the Lagrangian \cite{12} implies:

$$\nabla_\alpha \sim (1, -1), \quad q^a \sim (2, +1), \quad q^{a\beta} \sim (1, +1), \quad K_i \sim (2, -1), \quad L_\alpha(x) \sim (2, -2).$$

(159)

The renormalized action functional $\tilde{\Gamma}^N$ must satisfy the Nielsen identity

$$\frac{\delta \tilde{\Gamma}^N}{\delta K_i} = \frac{\delta \tilde{\Gamma}^N}{\delta A^4} + \frac{\delta \tilde{\Gamma}^N}{\delta L_\alpha} - (\xi^{-1})_{\beta a} \left( f^\beta + \frac{1}{2} q^{\beta\gamma} \nabla_\gamma \right) \cdot \frac{\delta \tilde{\Gamma}^N}{\delta \omega^a} + q^a \frac{\delta \tilde{\Gamma}^N}{\delta \omega^a} = 0,$$

(160)

and the ghost equation

$$\frac{\delta \tilde{\Gamma}^N}{\delta \omega^a(x)} + \frac{\delta f^\alpha(x)}{\delta A^4} = \frac{1}{2} \frac{\delta}{\delta \omega^a(x)} (\xi^{-1})_{\beta a} \left( f^\beta(x) + \frac{1}{2} q^{\beta\gamma}(x) \nabla_\gamma(x) \right) - q^{a\beta}(x) \left( \delta_\alpha^\beta(x) + v^a_\alpha \right) \delta_\beta^c \left[ T_{R}^\beta \right]^c_b u^b(x)$$

$$+ q^a(x) \xi^{a\beta}(x) \delta_\alpha^c \left[ T_{R}^\beta \right]^c_b \left( v^b(x) + v^b_R \right).$$

(161)

if one restricts the space of scalar fields to configurations obeying $u^T T_{R, \alpha} \phi = 0$, then stationary points of $\mathcal{V}_{\text{eff}}$ are the same as those of $\mathcal{V}_{\text{eff}}$, however - contrary to the conclusions of \cite{27} - by replacing $\mathcal{V}_{\text{eff}}$ with $\mathcal{V}_{\text{eff}}$ one cannot avoid the condition $u^T T_{R, \alpha} \phi = 0$, since it is necessary on physical grounds. On the other hand, the usual potential $\mathcal{V}_{\text{eff}}$ naturally ‘feels’ this conditions, owing to the Dashen mechanism mentioned above.
The most general functional of dimension 4 allowed by the ghost number conservation has the following dependence on the external sources $q^\alpha(x)$, $q^{a\beta}(x)$, $K_i(x)$ and $L_\alpha(x)$

$$\tilde{L}^N = \int d^4x \, \tilde{L}^N [\mathcal{A}, \mathcal{F}, \omega; K; L; q, u, \xi|x],$$

where

$$\tilde{L}^N(x) = \frac{1}{2} L_\alpha(x) C^\alpha_{\beta\gamma} (\xi(x)) \omega^\beta(x) \omega^\gamma(x) + L_\alpha(x) g^{a\alpha\beta}(\xi(x)) q^{a\beta}(x) \omega^\delta(x) + L_\alpha(x) H^\alpha_{\beta\gamma\delta\epsilon}(\xi(x)) q^{\beta\gamma}(x) q^{\delta\epsilon}(x) + K_i(x) \int d^4y \, \omega^\alpha(y) D^i_\alpha [\mathcal{A}, u, \xi|x,y] + K_i(x) \int d^4y \, k^{i\alpha}_{\beta\alpha\gamma}(\mathcal{A}, u, \xi|x,y) q^{\alpha\beta}(y) + K_i(x) b_i^\alpha(\xi(x)) q^\alpha(x) + q^\alpha(x) \int d^4y \, d^a_\alpha [\mathcal{A}, u, \xi|x,y] \omega^\alpha(y) + q^{a\beta}(x) \int d^4y \, r_{a\beta}^\alpha(\mathcal{A}, u, \xi|x,y) \omega^\alpha(y) + \tilde{\omega}_0(\mathcal{A}, \mathcal{F}, \omega; u, \xi|x)\right].$$

Terms of the identity which are linear in $\omega$ and $q^{a\beta}$ have all dimension 0. $H^\alpha_{\beta\gamma\delta\epsilon}$ and $k^{i\alpha}_{\beta\alpha\gamma}$ are antisymmetric under the interchange $\beta \leftrightarrow \gamma$ and symmetric with respect to interchanges $\beta \leftrightarrow \epsilon$ and $\sigma \leftrightarrow \epsilon$. The kernels $D^i_\alpha [\mathcal{A}, u, \xi|x,y]$, $k^{i\alpha}_{\beta\alpha\gamma}[\mathcal{A}, u, \xi|x,y]$ and $d^a_\alpha [\mathcal{A}, u, \xi|x,y]$ have dimension 1, while $r_{a\beta}^\alpha [\mathcal{A}, u, \xi|x,y]$ has dimension 2.

Constraints imposed by the Nielsen identity can be conveniently expressed in terms of the differential forms

$$\tilde{g}^\alpha = g^{a\beta\gamma}(\xi) \, d\xi^{\beta\gamma}, \quad \tilde{H}^\alpha = H^\alpha_{\beta\gamma\delta\epsilon}(\xi) \, d\xi^{\beta\gamma} \wedge d\xi^{\delta\epsilon},$$

and

$$l^{\kappa\sigma} = l^{\kappa\sigma}_{\beta\gamma\delta\epsilon}(\xi) \, d\xi^{\beta\gamma} \wedge d\xi^{\delta\epsilon}.$$ 

Terms of the identity which are linear in $L_\alpha$ and proportional to different powers of $q^{a\beta}$ impose respectively the relations

$$[C_\beta, C_\gamma] = C^\alpha_{\beta\gamma} C_\alpha,$$

and

$$dC^\alpha_{\beta\gamma} = \tilde{g}^\alpha C^\beta_{\gamma\delta} - C^\alpha_{\beta\gamma} \tilde{g}^\alpha - C^\alpha_{\gamma\delta} \tilde{g}^\alpha - C^\alpha_{\delta\epsilon} \tilde{g}^\alpha,$$

$$d\tilde{g}^\alpha = \tilde{g}^\alpha \wedge \tilde{g}^\alpha - C^\alpha_{\gamma\delta} H^\alpha,$$

$$dH^\alpha = \tilde{g}^\alpha \wedge H^\alpha.$$ 

In turn, terms involving products $q^a \times K_i \times \omega^\beta$ and $q^a \times K_i \times q^{a\beta}$ give respectively

$$\left( b^i_\alpha(\xi(x)) \frac{\delta}{\delta A^i}\right) + \frac{\delta}{\delta u^a}(z) D^i_\alpha [\mathcal{A}, u, \xi|x,y] = 0,$$

$$\left( b^i_\alpha(\xi(x)) \frac{\delta}{\delta A^i}\right) + \frac{\delta}{\delta u^a}(z) \right) k^{i\alpha}_{\beta\alpha\gamma}[\mathcal{A}, u, \xi|x,y] D^i_\alpha [\mathcal{A}, u, \xi|x,y] = D^i_\alpha [\mathcal{A} - b(\xi) u, 0, \xi|x,y].$$

Hence

$$D^i_\alpha [\mathcal{A}, u, \xi|x,y] = D^i_\alpha [\mathcal{A} - b(\xi) u, 0, \xi|x,y],$$

and

$$k^{i\alpha}_{\beta\alpha\gamma}[\mathcal{A}, u, \xi|x,y] = k^{i\alpha}_{\beta\alpha\gamma}[\mathcal{A} - b(\xi) u, 0, \xi|x,y] + \frac{\partial b^i_\alpha}{\partial \xi^{\alpha\beta}}(\xi(x)) u^a(x) \delta^{(4)}(x - y).$$

Terms of the Nielsen identity involving products $q^{a\beta} \times K_i \times \omega^\gamma$ and $q^{a\beta} \times q^{a\gamma} \times K_i$ yield the relations

$$\int d^4x \left( k^{i\alpha}_{\beta\alpha\gamma}[\mathcal{A}, u, \xi|x,y] \frac{\delta D^i_\alpha [\mathcal{A}, u, \xi|x,w]}{\delta A^i}\right) - D^i_\gamma [\mathcal{A}, u, \xi|x,w] \frac{\delta k^{i\alpha}_{\beta\alpha\gamma}[\mathcal{A}, u, \xi|x,y]}{\delta A^i}\right) = -g^{\delta\beta}_{\alpha\gamma}(\xi(w)) \delta^{(4)}(y - w) D^i_\delta [\mathcal{A}, u, \xi|z,w] - \frac{\delta D^{i\alpha}_{\beta\alpha\gamma}[\mathcal{A}, u, \xi|z,w]}{\delta \xi^{\alpha\beta}}(y),$$

13 Assigning dimensions to these kernels we can treat $\delta^{(4)}(x - y)$ as dimensionless since in the lagrangian it is accompanied by the measure $\int d^4x$.

14 The coefficients $b^i_\alpha$ are treated here as vanishing if the index $i$ corresponds to vector fields.
Returning to the implications of the identity \(160\), we find that its terms cubic in equations \(180\), \(175\). Terms involving products \(q\) the terms of \(160\) which are linear in \(\omega\) automatically satisfied by virtue of the equations \(177\) and \(180\) respectively. Using \(181\) and \(177\), we find that identity \(160\) linear in \(\omega\) is an arbitrary functional of its arguments. After taking \(180\) and \(174\) into account, the terms of the Nielsen identity which are independent of \(L_\alpha\) and \(K_\gamma\), it will be convenient to extract information from the ghost equation \(161\). Its terms linear in \(q^a\) give the relation

\[
d_a^\alpha \left[ A, u, \xi | z, x \right] = -\xi^{q\gamma} (z) \delta \omega \left[ T_{\gamma \beta} \right]_{b}^{c} \left( \phi^b (x) + v^b_k \right) \delta \gamma (z - x) + \hat{\delta}^a (x) \delta A^\gamma (z) b_a^\gamma (z). \tag{177} \]

Furthermore, \(161\) implies also that

\[
m^{a\varphi}_{\gamma \beta \gamma} (\xi) = 0, \tag{178} \]

\[
l^{a\varphi} = \frac{1}{8} (\xi^{-1})_{a\varphi} d\xi^a \wedge d\xi^\varphi, \tag{179} \]

and that

\[
\int d^4 y \ q^{\gamma \gamma} (y) r_{\gamma \beta}^\alpha \left[ A, u, \xi | y, x \right] = \int d^4 y \ \delta f^a (x) \delta A^\beta (y) \int d^4 z \ k_{\beta \gamma} \left[ A, u, \xi | y, z \right] q^{\beta \gamma} (z) \]

\[
- \frac{1}{2} q^{\beta \gamma} (x) (\xi^{-1})_{\beta \gamma} f^\beta (x) + q^{\alpha \beta} (x) (\phi^\alpha (x) + v^\alpha_k) \delta \omega \left[ T_{\gamma \beta} \right]_{b}^{c} u^b (x). \tag{180} \]

Finally, those terms of \(161\) which are independent of the Nielsen sources lead to an equation for \(\delta \tilde{\mathcal{L}}_{FP}^N / \delta \omega_a (x)\) (i.e. for the derivative of the last term in \(163\)) whose solution has the general form

\[
\tilde{\mathcal{L}}_{FP}^N \left[ A, \omega; u, \xi \right] = - \int d^4 x \int d^4 y \ \int d^4 z \ \omega_a (x) \delta \tilde{\mathcal{L}}_{FP}^N \left[ A, u | y, z \right] \delta \omega_a (y) + \tilde{\mathcal{L}}_{test}^N \left[ A; u, \xi \right]. \tag{181} \]

Returning to the implications of the identity \(160\), we find that its terms cubic in \(q^{\gamma \beta}\) vanish automatically after taking into account the form \(179\) of \(l^{a\varphi}\). Similarly, terms quadratic in \(q^{\alpha \beta}\) vanish due to form of \(l^{a\varphi}\) and the equations \(180\), \(175\). Terms involving products \(q^a \times q^a\) and \(q^a \times q^\beta\) do not give any new information either, being automatically satisfied by virtue of the equations \(177\) and \(180\) respectively. Using \(181\) and \(177\), we find that the terms of \(160\) which are linear in \(q^a\) impose

\[
b_a^\gamma (x) \delta_{\delta \tilde{\mathcal{L}}_{test}^N} \frac{\delta \tilde{\mathcal{L}}_{test}^N}{\delta A^\gamma (x)} + \delta_{\delta \tilde{\mathcal{L}}_{test}^N} \frac{\delta \tilde{\mathcal{L}}_{test}^N}{\delta u^a (x)} = - \int d^4 y \ (\xi^{-1})_{a\beta} f^\beta (y) d_a^\alpha \left[ A, u | x, y \right]. \tag{182} \]

The above equation can be solved with the help of \(177\):

\[
\tilde{\mathcal{L}}_{test}^N = - \frac{1}{2} (\xi^{-1})_{a\beta} f^\alpha \cdot f^\beta + \tilde{\mathcal{L}}_{G1}^N \left[ A^\alpha_a, \phi^a - b^a_\alpha u^\epsilon, \xi^{bg} \right], \tag{183} \]

where \(\tilde{\mathcal{L}}_{G1}^N\) is an arbitrary functional of its arguments. After taking \(180\) and \(174\) into account, the terms of the identity \(160\) linear in \(q^{\alpha \beta}\) yield

\[
\left( \delta \frac{\delta}{\delta \omega_a (x)} \right) + \int d^4 z \ k_{\beta \gamma} \left[ A, u, \xi | z, x \right] \delta \delta \tilde{\mathcal{L}}_{G1}^N \left[ A^\alpha_a, \phi^a - b^a_\alpha u^\epsilon, \xi^{bg} \right] = 0. \tag{184} \]

Finally, vanishing of the terms of \(160\) independent of external sources gives leads to the constraint

\[
\int d^4 z \ D^\gamma \left[ A, u, \xi | z, x \right] \delta \tilde{\mathcal{L}}_{G1}^N \left[ A^\alpha_a, \phi^a - b^a_\alpha u^\epsilon, \xi^{bg} \right] = 0, \tag{185} \]

using \(175\).
which is the ordinary condition of gauge invariance, whereas the condition \(183\) expresses an ‘additional gauge invariance’ connected with the extended BRST symmetry.

The most general form \(D^\beta_{\mu \alpha} [A, u, \xi | x, y]\) of the gauge transformations allowed by power-counting, Lorentz invariance and their general structure \(172\) is

\[
D^\beta_{\mu \alpha} \left[ A, u, \xi | x, y \right] = \left\{ -N^\alpha_{\mu} (\xi (x)) \partial^\mu (x) + Q^\beta_{\alpha \gamma \kappa \lambda} (\xi (x)) \partial^\mu \xi^{\gamma \kappa \lambda} (x) + \tilde{c}^\beta_{\alpha \gamma} (\xi (x)) A^\gamma_{\mu} (x) \right\} \delta (x - y), \tag{186}
\]

\[
D^\alpha_{\mu} [\phi, u, \xi | x, y] = \left\{ \left[ \tilde{T}_\alpha (\xi (x)) \right] \right\}^\alpha_\gamma \delta (x - y), \tag{187}
\]

where \(Q^\beta_{\alpha \gamma \kappa \lambda} (\xi)\) are dimensionless functions. Similarly, taking into account the general form \(173\) one obtains the following formula for the extended gauge transformations \(k^\alpha_{\mu \gamma} [A, u, \xi | x, y]::

\[
k^\beta_{\alpha \gamma} \left[ A, u, \xi | x, y \right] = \left\{ -\Omega^\beta_{\alpha \gamma} (\xi (x)) \partial^\mu (x) + \mathcal{P}^\beta_{\alpha \gamma \kappa \lambda} (\xi (x)) \partial^\mu \xi^{\gamma \kappa \lambda} (x) + g^\beta_{\alpha \gamma} (\xi (x)) A^\gamma_{\mu} (x) \right\} \delta (x - y), \tag{188}
\]

\[
k^\alpha_{\alpha \gamma} [\phi, u, \xi | x, y] = \left[ \zeta^a_{\alpha \gamma b} (\xi (x)) \phi_b (x) + \partial^\beta_{\xi^a_{\alpha \gamma}} (\xi (x)) u^d (x) + \Sigma^a_{\alpha \gamma \beta} (\xi (x)) \right] \delta (x - y). \tag{189}
\]

We have introduced here the notation

\[
\tilde{c}^\beta_{\alpha \gamma} (\xi (x)) = \phi_b (x) - b^b_a (\xi (x)) u^d (x). \tag{190}
\]

For the functions of \(\xi\) appearing in the formulae \(189\) and \(187\) it is convenient to introduce the following parametrization:

\[
\tilde{c}^\beta_{\alpha \gamma} (\xi) = N^\gamma_{\alpha \beta} (\xi) e^\beta_{\delta \epsilon}, \tag{191}
\]

This can be done without loss of generality because in the perturbation theory \(N^\gamma_{\alpha \beta} = \delta^\gamma_{\alpha \beta} + \text{O} (h)\). The commutation relations \(170\) yield

\[
C^\kappa_{\beta \gamma} = N^\alpha_{\beta \gamma} \left[ N^{-1} \right]_\delta^\kappa \delta^\epsilon_{\alpha \epsilon}, \tag{192}
\]

and

\[
[T_{\alpha \gamma}, T_{\beta \delta}] = e^\gamma_{\alpha \beta} T_{\gamma \delta}, \quad [e_{\alpha \beta}, e_{\gamma \delta}] = e^\gamma_{\alpha \beta} e_{\gamma \delta}. \tag{193}
\]

Moreover \(170\) implies also that

\[
[T_{\alpha \gamma}]_b V^b_{\lambda \beta} = [T_{\beta \delta}]_a V^a_{\lambda \beta} = e^\gamma_{\alpha \beta} V^a_{\lambda \gamma}, \tag{194}
\]

and imposes the following constraints on the functions \(Q^\beta_{\alpha \gamma \kappa \lambda} (\xi)\):

\[
\tilde{c}^\alpha_{\beta \delta} Q^\beta_{\gamma \kappa \lambda} - \tilde{c}^\gamma_{\gamma \kappa} Q^\gamma_{\delta \kappa \lambda} = C^\delta_{\beta \gamma} Q^\beta_{\delta \kappa \lambda} - \delta^\gamma_{\delta \gamma} \frac{\partial C^\beta_{\beta \gamma}}{\partial \xi^\gamma_{\delta \gamma}}. \tag{195}
\]

Equations \(182\) are the standard requirements of the BRST symmetry. However, the information about \(u\)-independence of \(V^a_{\lambda \gamma}\) follows only from the Nielsen identity. Constraints imposed by \(173\) can be compactly expressed by the 1-forms

\[
d^\beta = \theta^\beta_{\alpha \gamma} (\xi) d\xi^\alpha \gamma, \quad \tilde{\zeta}^a_{\beta \gamma} = \zeta^a_{\alpha \gamma b} (\xi) d\xi^\alpha \gamma, \quad \tilde{\Sigma}^a = \Sigma^a_{\alpha \gamma \beta} (\xi) d\xi^\alpha \gamma, \tag{196}
\]

and read

\[
d\bar{\gamma} = \left[ \tilde{\theta}, \tilde{\zeta} \right] - \tilde{g}^\beta \gamma \delta \bar{\beta}, \quad d\tilde{T}_\gamma = \left[ \tilde{\zeta}, \tilde{T}_\gamma \right] - \tilde{g}^\beta \gamma \tilde{T}_\beta, \tag{197}
\]

\[
dN = \tilde{\theta} N - \tilde{N} \tilde{\theta}, \tag{198}
\]

\[
Q^\beta_{\gamma \kappa \lambda} = \tilde{c}^\beta_{\gamma \kappa \lambda} Q^\beta_{\gamma \kappa \lambda} + \delta^\beta_{\gamma \kappa \lambda} g^\alpha_{\alpha \beta \gamma \kappa \lambda}, \tag{199}
\]

\[
d\tilde{V}^a_{\lambda \gamma} = \tilde{\zeta}^b_{\lambda \gamma} V^b_{\lambda \gamma} - \left[ \tilde{T}_a \right]^a_{\beta \gamma} \Sigma^b_{\beta \gamma} + \delta^a_{\beta \gamma} \frac{\partial \tilde{V}^b_{\lambda \gamma}}{\partial \xi^\beta \gamma} + \frac{\partial Q^b_{\beta \gamma}}{\partial \xi^\beta \gamma}. \tag{200}
\]

In turn, the relation \(175\) leads to

\[
\bar{d} \theta = \bar{\theta} \wedge \theta - \bar{H}^\epsilon \epsilon, \quad \bar{d} \zeta = \bar{\zeta} \wedge \zeta - \bar{H}^\epsilon \epsilon, \quad \bar{d} \Sigma = \bar{\zeta} \wedge \Sigma - \bar{H}^\epsilon \tilde{V}_{\lambda \epsilon}, \tag{201}
\]

and

\[
\mathcal{P}^\rho_{\delta \gamma \alpha \beta} = \frac{\partial \Omega^\rho_{\alpha \beta}}{\partial \xi^\delta \gamma} + \theta^\rho_{\delta \gamma \alpha \beta} \Omega^\rho_{\gamma \alpha \beta} + 2N^\rho_{\alpha \beta} \bar{H}^\epsilon \alpha \beta \gamma, \tag{202}
\]

and

\[
\mathcal{P}^\rho_{\delta \gamma \alpha \beta} = \frac{\partial \Omega^\rho_{\alpha \beta}}{\partial \xi^\delta \gamma} + \theta^\rho_{\delta \gamma \alpha \beta} \Omega^\rho_{\gamma \alpha \beta} + 2N^\rho_{\alpha \beta} \bar{H}^\epsilon \alpha \beta \gamma, \tag{203}
\]

\[
\mathcal{P}^\rho_{\delta \gamma \alpha \beta} = \frac{\partial \Omega^\rho_{\alpha \beta}}{\partial \xi^\delta \gamma} + \theta^\rho_{\delta \gamma \alpha \beta} \Omega^\rho_{\gamma \alpha \beta} + 2N^\rho_{\alpha \beta} \bar{H}^\epsilon \alpha \beta \gamma, \tag{204}
\]

and

\[
21
\]
\[ \frac{1}{2} \left( \theta^\rho_{\delta\gamma} P^\sigma_{\alpha\beta\kappa} - \theta^\rho_{\alpha\beta\sigma} P^\sigma_{\delta\gamma\kappa} \right) = N^\rho_{\sigma} \frac{\partial H^\sigma_{\alpha\beta\kappa}}{\partial \xi^\kappa} + Q^\rho_{\alpha\beta\kappa} H^\sigma_{\delta\gamma\beta} - \frac{1}{2} \left( \frac{\partial P^\rho_{\delta\gamma\kappa}}{\partial \xi^\kappa} - \frac{\partial P^\rho_{\alpha\beta\kappa}}{\partial \xi^\kappa} \right). \]  

There is also one additional condition on the antisymmetric component of \( P^\rho_{\delta\gamma\beta} \) which holds automatically due to \( \mathcal{P} \). The coefficients \( Q \) and \( P \) describing the dependence of gauge transformations on derivatives of \( \xi \) are unambiguously determined by the remaining parameters according to equations \( \mathcal{O} \) and \( \mathcal{P} \). Constraints on them (represented by the eqs. \( \mathcal{O1} \) and \( \mathcal{P1} \)) are also automatically ensured by \( \mathcal{O2} \), \( \mathcal{P2} \), \( \mathcal{P3} \) and \( \mathcal{P4} \).

Finally, we need the general solution to gauge invariance conditions \( \mathcal{O3} \) and \( \mathcal{P4} \). The ‘ordinary’ BRST symmetry (e.g. \( \mathcal{O4} \)) suggests that the functional \( \bar{\xi} \) by the eqs. \( \mathcal{O5} \), \( \mathcal{O6} \) and \( \mathcal{P7} \) are also automatically ensured by \( \mathcal{O8} \), \( \mathcal{O9} \), \( \mathcal{O10} \) and \( \mathcal{O11} \).

\[ L_{\text{GI}}^{\mathcal{O}} [A, u, \varphi | x] = -\frac{1}{4} [Z_\varphi (\xi (x))]_{\alpha\beta} \tilde{F}_\mu^\alpha (x) \tilde{F}^\nu_\mu (x) + \frac{1}{2} [Z_\varphi (\xi (x))]_{ab} \left( \tilde{D}_\mu \tilde{\varphi} \right)^a \left( x \right) \left( \tilde{D}^\mu \tilde{\varphi} \right)^b \left( x \right), \]  

(205)

where, in terms of parameters introduced in \( \mathcal{P11} \),

\[ \tilde{F}_\mu^\alpha (x) = \partial_\mu A^\alpha (x) - \partial_\nu A^\mu_\nu (x) + \epsilon^{\alpha}_\beta_\gamma (\xi (x)) A^\beta_\gamma (x) \]  

(206)

\[ \left( \tilde{D}_\mu \tilde{\varphi} \right) (x) = \partial_\mu \tilde{\varphi} (x) + A^\mu_\nu (x) \tilde{\varphi} (x) \]  

(207)

The additional terms with derivatives of \( \xi \) have been included in these formulae in agreement with the power-counting and the Lorentz invariance. The constraint \( \mathcal{O3} \) yields the usual conditions of the BRST symmetry

\[ [Z_{\mathcal{O}}]_{ab} e^\alpha_{\beta \gamma} = - [Z_{\mathcal{O}}]_{\alpha \beta} e^\alpha_{\gamma \epsilon}, \]  

(208)

and leads to the following conditions on the scalar potential \( \tilde{V} (\Phi, \xi) \):

\[ \left( [T_\alpha (\xi)]_{ab} \Phi^b + V_\alpha (\xi) \right) \frac{\partial \tilde{V} (\Phi, \xi)}{\partial \Phi^a} \equiv 0. \]  

(209)

Similarly, \( \mathcal{O4} \) gives

\[ d [Z_{\mathcal{O}}]_{ab} = - [Z_{\mathcal{O}}]_{\gamma \epsilon} \tilde{\varphi}^\gamma_{\beta \gamma} - [Z_{\mathcal{O}}]_{ab} \tilde{\varphi}^\gamma_{\epsilon \gamma}, \]  

(210)

and

\[ \frac{\partial \tilde{V} (\Phi, \xi)}{\partial \xi^\alpha} + \epsilon_{\alpha \beta \gamma} (\xi) \frac{\partial \tilde{V} (\Phi, \xi)}{\partial \Phi^b} \equiv 0. \]  

(211)

Once again we find that the terms dependent on derivatives of \( \xi \) are entirely determined by the remaining ones. Defining:

\[ \dot{\lambda}^\alpha = \lambda^\alpha_{\gamma \epsilon} (\xi) \text{d} \xi^\gamma_{\epsilon}, \]  

(212)

\[ \dot{\lambda}^\gamma_{\alpha \beta} = \lambda^\gamma_{\alpha \beta} (\xi) \text{d} \xi^\gamma_{\alpha \beta}, \]  

(213)

we conclude that the relations \( \mathcal{O3} \)–\( \mathcal{O4} \) require

\[ \dot{\lambda} = \tilde{\varphi} - \Omega^\gamma_{\epsilon \gamma} \]  

(214)

\[ \dot{\lambda}^\gamma_{\alpha \beta} = \dot{\lambda}^\gamma_{\alpha \beta} + \Omega^\gamma_{\epsilon \gamma} \]  

(215)

It will be convenient to rewrite the constraint \( \mathcal{O7} \) in terms of the structure constants \( e^\alpha_{\beta \gamma} \) and the generators \( T_\alpha \). Using \( \mathcal{O5} \) and \( \mathcal{P1} \) one obtains

\[ \text{d} e_\gamma = \left[ \theta, e_\gamma \right] = \tilde{\varphi}^\gamma_{\epsilon \delta} e_\delta, \]  

(216)

\[ \text{d} T_\gamma = \left[ \tilde{\varphi}^\gamma_{\epsilon \gamma}, T_\gamma \right] = \tilde{\varphi}^\gamma_{\epsilon \delta} T_\delta. \]  

(217)

Moreover, the condition \( \mathcal{O7} \) follows immediately from \( \mathcal{O10} \) and \( \mathcal{O13} \). Similarly, the condition \( \mathcal{O8} \) is equivalent to the first relation in \( \mathcal{O12} \) and can be rewritten as

\[ \text{d} \tilde{\varphi} = \tilde{\varphi} \theta - \Omega^\gamma_{\epsilon \gamma} e_\epsilon, \]  

(218)
where
\[ \hat{\Psi}^\kappa \equiv N^\kappa, \hat{H}^\kappa. \] (219)

Instead of (169), we have
\[ d\hat{\Psi}^\sigma = \theta^\sigma_{\alpha} \wedge \hat{\Psi}^\alpha, \] (220)

while the last two equations in (202) read respectively
\[ d\hat{\zeta} = \hat{\zeta} \wedge \hat{\zeta} - \hat{\Psi}^\gamma T_\gamma, \] (221)
\[ d\hat{\Sigma} = \hat{\zeta} \wedge \hat{\Sigma} - \hat{\Psi}^\gamma V_{N\gamma}. \] (222)

Finally, (200) takes the form
\[ dV_{N\gamma} = \hat{\zeta}^\gamma V_{N\gamma} - [T_\gamma]_b \hat{\Sigma}^b - V_{N\delta} \theta^\delta_{\gamma}. \] (223)

The relations (210)-(224), supplemented with (193), (210) and (211) govern the \( \xi \)-dependence of the counterterms. Since \( u \) has dimension 1, the \( u \)-dependence is even more restricted: the form (183) of the solution to the constraint (202) shows that the gauge-invariant part of the action depends on \( u \) only through the shifted field \( \hat{\phi} \) defined in (190). Putting all this together, we obtain the renormalized action presented in Section 3.

### A.2 Global gauge invariance

Aside from the Nielsen identity and the ghost equation, the action (12) also satisfies the Ward-Takahashi identity
\[ W^N_\alpha \bar{T}^N \equiv 0, \] (224)

where \( W^N_\alpha \) are the following differential operators
\[ W^N_\alpha = [T_\alpha b]_b q^b \frac{\delta}{\delta q^b} + [T_\alpha c]_c q^c \frac{\delta}{\delta q^c} + [T_\alpha b]_b (\phi^b + e^b_{\alpha}) \frac{\delta}{\delta \phi^b} + e^b_{\alpha} A^b_{\alpha} \frac{\delta}{\delta A^b_{\alpha}} + e^b_{\alpha} \omega^b_{\alpha} \frac{\delta}{\delta \omega^b_{\alpha}} \]
\[ + \delta_{\beta\gamma} e^b_{\alpha} \frac{\delta}{\delta \phi^b} + \delta_{\beta\gamma} e^b_{\alpha} \frac{\delta}{\delta \phi^b} + \delta_{\beta\gamma} e^b_{\alpha} \frac{\delta}{\delta \phi^b} + \delta_{\beta\gamma} e^b_{\alpha} \frac{\delta}{\delta \phi^b} + \delta_{\beta\gamma} e^b_{\alpha} \frac{\delta}{\delta \phi^b} + \delta_{\beta\gamma} e^b_{\alpha} \frac{\delta}{\delta \phi^b} \]

The operators \( -W^N_\alpha \) form a representation of the Lie algebra with the structure constants \( e_{\alpha}^{\beta\gamma} \). Standard arguments (e.g. [18]) are unaffected by the presence of Nielsen sources, and one can conclude that the renormalized action \( \tilde{Z}^N \) satisfies the same identity
\[ W^N_\alpha \tilde{Z}^N \equiv 0. \] (226)

It is convenient to introduce the vector fields
\[ \Lambda_\alpha = \left( e_{\alpha}^{\beta\kappa} \xi^\beta - e_{\alpha}^{\kappa} \xi^\beta \right) \frac{\partial}{\partial \xi^\alpha}, \quad [\Lambda_\alpha, \Lambda_\beta] = -\Lambda_\gamma e_{\alpha}^{\gamma\beta}. \] (227)

In the identity (226), vanishing of the terms proportional to the products \( K_i \times q^a \) and \( L_\alpha \times \omega^\beta \omega^\gamma \) implies respectively the equalities
\[ (\Lambda_\alpha b)(\xi) = [T_{\alpha b}, b(\xi)], \] (228)
\[ (\Lambda_\alpha C_\gamma)(\xi) = [e_{\alpha b}, C_\gamma(\xi)] - C_\gamma(\xi) e_{\alpha}^{\epsilon \gamma}, \] (229)

while vanishing of the coefficients of the products \( L_\alpha \times q^{\beta\gamma} \omega^\delta \) and \( L_\alpha \times q^{\beta\gamma} \delta^\epsilon \) give the following equations for the Lie derivatives
\[ \mathcal{L}_{\Lambda_\alpha} \hat{\beta} = [e_{\alpha b}, \hat{\beta}], \] (230)
\[ \mathcal{L}_{\Lambda_\alpha} \hat{H} = e_{\alpha b} \hat{H}, \] (231)

Vanishing of the terms involving the product \( K_i^\mu \times \omega^\alpha \) gives
\[ e_{\alpha}^{\beta \epsilon} D^{\epsilon}_{\mu \kappa} [A, u, \xi; x, y] - D^{\epsilon}_{\mu \kappa} [A, u, \xi; x, y] e_{\alpha}^{\epsilon \kappa} = W^N_\alpha D^{\beta}_{\mu \kappa} [A, u, \xi; x, y]. \] (232)

Hence
\[ (\Lambda_\alpha N)(\xi) = [e_{\alpha b}, N(\xi)]. \] (233)
\[ (\Lambda_\alpha e_\kappa)(\xi) = [e_{\alpha b}, e_\kappa(\xi)] - e_\kappa(\xi) e_{\alpha}^{\epsilon \kappa}. \] (234)
and \((Q^\rho_\mu)_\kappa \equiv Q^\beta_{\kappa \mu \nu}\)

\[
\left(\Lambda_\alpha Q^\rho_\mu\right)(\xi) = \left[e_{\mu \alpha}; Q^\rho_\mu\right](\xi) - \left(e^\epsilon_{\alpha \beta}Q^\epsilon_\omega(\xi) + e^\epsilon_{\rho \alpha}Q^\rho_\omega(\xi)\right).
\]

The condition (229) is automatically satisfied, provided (233) and (234) hold. The equation for gauge transformation of scalars which is analogous to (226) yields

\[
(\Lambda_\alpha T_\kappa)(\xi) = [T^\rho_\alpha, T_\kappa(\xi)] - T_\epsilon(\xi)e^\epsilon_{\kappa \alpha},
\]

(236)

\[
(\Lambda_\alpha V_{N\kappa})(\xi) = T^\rho_{\alpha N}(\xi) - T_\alpha(\xi)T^\rho_{\kappa}v_\rho - V_{N\kappa}(\xi)e^\epsilon_{\rho \alpha}.
\]

(237)

The terms of the form \(K^\mu_\beta \times e^\kappa_\lambda\) in (226) yield

\[
e_{\mu \alpha \beta}k^\kappa_\mu \kappa e^\epsilon_{\alpha \lambda}[A, u, \xi|x, y] - \left(k^\beta_\mu \kappa [A, u, \xi|x, y] e^\epsilon_{\alpha \kappa} + k^\beta_\kappa \alpha [A, u, \xi|x, y] e^\epsilon_{\alpha \lambda}\right) = \hat{W}^a_{\alpha} k^\beta_\mu \kappa [A, u, \xi|x, y],
\]

(238)

whence

\[
\mathcal{L}_{\Lambda_\alpha} \dot{\theta} = \left[e_{\mu \alpha} \theta, \hat{\theta}\right],
\]

(239)

\[
\mathcal{L}_{\Lambda_\alpha} \hat{\phi} = e_{\alpha \beta} \hat{\phi}.
\]

(240)

The constraint (238) gives also a condition on \(\mathcal{P}(\xi)\), which is however automatically satisfied by \(\mathcal{P}(\xi)\) determined by (203) owing to the relations (230), (240), (231) and (233). The scalar counterpart leads to

\[
\mathcal{L}_{\Lambda_\alpha} \dot{\xi} = \left[T^\rho_{\alpha}, \xi\right],
\]

(241)

\[
\mathcal{L}_{\Lambda_\alpha} \dot{\Delta} = T_{\alpha \kappa} \Delta - \xi T_{\alpha \kappa}v_\kappa.
\]

(242)

Taking into account the above equations one can check that the operators \(\hat{F}^a_{\mu \nu}\) and \(\tilde{D}_\mu \hat{\phi}\) transform covariantly

\[
\hat{W}^a_{\alpha} \hat{F}^b_{\mu \nu}(x) = e_{\alpha \beta} \hat{F}^b_{\mu \nu},
\]

(243)

\[
\hat{W}^a_{\alpha} \left(\tilde{D}_\mu \phi\right)^b(x) = \left[T^a_{\mu \alpha} \nu_b, \nu_\kappa \right] \phi^b(x).
\]

(244)

Thus one obtains the relations

\[
(\Lambda_\alpha [ZA]_{\beta \gamma})(\xi) = \left[Z\phi, \xi\right]_{\beta \gamma} e^\epsilon_{\alpha \kappa} - [ZA(\xi)]_{\beta \kappa} e^\epsilon_{\alpha \gamma},
\]

(245)

\[
(\Lambda_\alpha [Z\phi]_{bc})(\xi) = \left[Z\phi, \xi\right]_{bc} e^\epsilon_{\alpha \kappa} - [Z\phi(\xi)]_{bc} e^\epsilon_{\alpha \kappa}.
\]

(246)

Similarly, using the formula

\[
\hat{W}^a_{\alpha} \hat{\phi}^a(x) = \left[T^a_{\alpha} \nu_b, \nu^b \right] \phi^b(x),
\]

(247)

we obtain the following constraint on the counterterms for the potential of the scalar fields:

\[
\left[\left(\Lambda_\alpha \tilde{V}\right)\Phi, \xi\right]_{\beta \gamma} + \left[T^a_{\alpha} \nu_b, \nu^b \right] \phi^b(x) = 0.
\]

(248)

The form (183) of the solution to the constraint (182) shows that the gauge-fixing function \(f^\beta(x)\) is not altered by renormalization. From its explicit form (10) it is easy to find that

\[
\hat{W}^a_{\alpha} f^\beta(x) = e_{\alpha \beta} f^\gamma(x),
\]

(249)

and the gauge-fixing part of the renormalized action (i.e. the first term of (183)) automatically satisfies the identity (229). All other terms in the renormalized action are also consistent with the global invariance (226), provided the conditions listed in this appendix are fulfilled. Furthermore, the relations (230) and (234) do not give any new information: (231) follows from (239), (233) and (193), while (234) is ensured by (199), (233), (241), (240) and (244). It will be convenient to rewrite (231) in terms of the 2-form \(\hat{\Psi}\) defined in (219). With the help of (233) one finds

\[
\mathcal{L}_{\Lambda_\alpha} \hat{\Psi} = e_{\alpha \beta} \hat{\Psi}.
\]

(250)

Finally, let us notice that if \(\xi^{-1}\) is an invariant form on the gauge Lie algebra, then the vector fields \(\Lambda_\alpha\) vanish and the counterterms are subject to the ordinary algebraic constraints.
A.3 Singlets

If the gauge Lie algebra is a direct sum of a semisimple Lie algebra (of a compact group) \( g \) and an abelian Lie algebra \( h \), then additional constraints are available. The structure constants \( e_r^{\gamma \alpha \beta} \) corresponding to the basis \( \{ \tau_{\alpha_1} \} \cup \{ \tau_{\alpha_0} \} \) whose generators \( \tau_{\alpha_1} \) span \( g \) and \( \tau_{\alpha_0} \) span \( h \) are such that \( e_r^{\gamma \alpha \beta} = 0 \), if any of the the indices \( \alpha, \beta \) or \( \gamma \) correspond to a generator of \( h \). Moreover, as follows from \eqref{A.27}, \( \Lambda_{\alpha_0} \equiv 0 \) and the relations found in Appendix A.2 become algebraic. For instance one has

\[
[T_{\alpha_0}, T_{\gamma}] = 0, \quad [T_{\alpha_0}, b] = 0, \quad [T_{\alpha_0}, \xi] = 0. \tag{251}
\]

For any abelian generator \( \tau_{\alpha_0} \) we define

\[
\mathcal{W}^N_{\alpha_0} (x) = \frac{\delta}{\delta x^\alpha} \delta A^\alpha_{\alpha_0} (x) + [T_{R_{\alpha_0}}, b] \left( \phi^b (x) + v^b \right) + \delta [T_{R_{\alpha_0}}, c] \delta K_b (x) \frac{\delta}{\delta x^\alpha} + [T_{R_{\alpha_0}}, a] u^a (x) + \delta [T_{R_{\alpha_0}}, a] \frac{\delta}{\delta x^\alpha} (x). \tag{252}
\]

The operator \( \mathcal{W}^N_{\alpha_0} (x) \) is obviously connected with the one introduced in \eqref{A.25}: \( \mathcal{W}^N_{\alpha_0} = \int d x \, \mathcal{W}^N_{\alpha_0} (x) \). The tree level action \eqref{12} satisfies the Ward-Takahashi identity

\[
\mathcal{W}^N_{\alpha_0} (x) \mathcal{I}^N = -\partial x^\alpha \left\{ \left( \xi^{-1} (x) \right)_{\alpha_0} \beta \left( f^\beta (x) + \frac{1}{2} q^{\gamma \kappa} (x) \varpi^\kappa (x) \right) \right\}. \tag{253}
\]

The right-hand side of \eqref{253} is linear in quantum fields and therefore the renormalized effective action \( \Gamma^N \) as well as the action with counterterms \( \tilde{\mathcal{I}}^N \) also obeys \eqref{253}. For this reason we have

\[
T_{\alpha_0} (\xi) = T_{R_{\alpha_0}}, \tag{254}
\]

\[
V_{N\alpha_0} (\xi) = T_{R_{\alpha_0} V_{R}}, \tag{255}
\]

\[
e^{\gamma \alpha_0 \beta} (\xi) = d e^{\gamma \alpha_0 \beta} = \theta^{\gamma \kappa_0 \alpha_0} (\xi) = \frac{\partial \theta^{\gamma \kappa_0 \alpha_0}}{\partial \xi^\kappa} = 0, \tag{256}
\]

in addition to the global invariance conditions of the form \eqref{251}. Moreover, since \( e^{\alpha_0 \beta \gamma} = 0 \), the following identities are also satisfied

\[
\frac{\delta \mathcal{I}^N}{\delta L_{\alpha_0} (x)} = 0, \quad \frac{\delta \mathcal{I}^N}{\delta K^\alpha_{\alpha_0} (x)} = -\partial x^\alpha \left( c_{\alpha_0} \right). \tag{257}
\]

The corresponding equations for the renormalized action functional \( \tilde{\mathcal{I}}^N \) yield

\[
C_{\alpha_0 \beta \gamma} = \hat{\gamma}_{\alpha_0 \gamma} = \hat{H}^\alpha_0 = 0, \tag{258}
\]

and

\[
N^{\alpha_0 \beta} = \delta^{\alpha_0 \beta}, \tag{259}
\]

\[
e^{\alpha_0 \beta \gamma} = \hat{\theta}^{\alpha_0 \gamma} = \hat{\Omega}^{\alpha_0} = 0. \tag{260}
\]

It is worth noting that these equations agree with \eqref{102} and \eqref{103}. Expressed in terms of the form \( \hat{\Psi}^c \), the relations \eqref{250} and \eqref{258} read

\[
\hat{\Psi}^{\alpha_0} = 0. \tag{261}
\]

Thus we see that coefficients \( e^{\alpha_0 \beta \gamma}, \hat{\theta}^{\alpha_0 \beta} \) and \( \hat{\Psi}^{\alpha_0} \) vanish, if any of their indices corresponds to an abelian generator. The summations in the formulae \eqref{216}, \eqref{218}, \eqref{220} etc. can be, therefore, restricted to semisimple indices only.

If \( \phi^{\alpha_0} \) is a gauge singlet, then \( [T_{\alpha_0}]^{\alpha_0}_{\beta} = 0 \) since \( T_{\alpha_0} \) is completely reducible. This gives another identity

\[
\frac{\delta \mathcal{I}^N}{\delta K^\alpha_{\alpha_0} (x)} = 0. \tag{262}
\]

Applied to the functional \( \tilde{\mathcal{I}}^N \), this gives

\[
[T_{\alpha}]^{\alpha_0}_{\beta} = V^{\alpha_0}_{N \alpha} = \hat{\sigma}^{\alpha_0}_{\beta} = S^{\alpha_0}_{\alpha} = b^{\alpha_0}_{\beta} = 0. \tag{263}
\]
B The case of $\xi$ being an invariant form

Here we show that $\hat{\Psi}^\epsilon = 0$ if $(\xi^{-1})_{\alpha\beta}$ is an invariant form. As we have seen in Appendix A.3 the coefficients $e^\alpha_{\beta\gamma}$, $\hat{\theta}^\alpha$, and $\hat{\Psi}^\alpha$ are non-vanishing only if all their indices correspond to non-abelian generators. Limiting ourselves to these indices we can assume that the gauge Lie algebra is semisimple. Contracting the 1-forms in (210) with a vector field $\Lambda_\alpha$ we get

$$\Lambda_\alpha e_\omega = \left[ \hat{\theta} (\Lambda_\alpha), e_\omega \right] - e_\rho \left( \hat{\theta} (\Lambda_\alpha) \right) e_\omega .$$  (264)

Comparing this with (251) we find that the matrices $\mathcal{E}_\alpha = e_{R\alpha} - \hat{\theta} (\Lambda_\alpha)$ satisfy the rule

$$[\mathcal{E}_\alpha, e_\gamma] = e_\beta \mathcal{E}_\alpha^{\beta\gamma}$$  (265)

In the case of semisimple Lie algebras the rule (265) can be satisfied only if $\mathcal{E}_\alpha$ are linear combinations of the generators $e_\alpha$ (see section 4)

$$e_{R\alpha} - \hat{\theta} (\Lambda_\alpha) = \mathcal{M}^\beta_\alpha e_\beta .$$  (266)

Differentiating both sides of (266) we get the relation

$$\mathcal{L}_{\Lambda_\alpha} \hat{\theta} - \Lambda_\alpha a d \hat{\theta} = - (d \mathcal{M}^\beta_\alpha) e_\beta - \mathcal{M}^\beta_\alpha d e_\beta .$$  (267)

which rewritten with the help of (249) and (245) takes the form

$$\mathcal{L}_{\Lambda_\alpha} \hat{\theta} = \left[ \Lambda_\alpha q \hat{\theta} + \mathcal{M}^\beta_\alpha e_\beta , \hat{\theta} \right] - \Lambda_\alpha \hat{\psi}^\beta e_\beta - \left( d \mathcal{M}^\beta_\alpha \right) e_\beta + \hat{\theta}^{\beta\gamma} \mathcal{M}^\gamma_\alpha e_\beta .$$  (268)

Using (266) once more, one obtains finally

$$\mathcal{L}_{\Lambda_\alpha} \hat{\theta} - \left[ e_{R\alpha}, \hat{\theta} \right] = - \Lambda_\alpha \hat{\psi}^\beta e_\beta - \left( d \mathcal{M}^\beta_\alpha \right) e_\beta + \hat{\theta}^{\beta\gamma} \mathcal{M}^\gamma_\alpha e_\beta .$$  (269)

The left-hand side of (269) vanishes on account of (259), so the linear independence of $\{ e_\beta \}$ leads to

$$d \mathcal{M}^\beta_\alpha = \hat{\theta}^{\gamma\beta} \mathcal{M}^\gamma_\alpha - \Lambda_\alpha \hat{\psi}^{\beta} .$$  (270)

Since $\Lambda_\alpha \equiv 0$ on the submanifold specified by the invariance of $\xi^{-1}$, this means that

$$\hat{\theta}^{\gamma\beta} = (d \mathcal{M}^\beta_\alpha) \left( \mathcal{M}^{-1} \right)^\alpha_\gamma ,$$  (271)

and the equation (213) shows then that on this submanifold ‘the curvature’ $\hat{\psi}^\epsilon$ vanishes.

C Stability of the Action $\mathcal{I}^N$

In this appendix we present more detailed arguments that the renormalized action $\mathcal{I}^N$ obeys the Nielsen identity (14). We begin by repeating the standard Zinn-Justin arguments [17, 18]. For simplicity we omit the superscript $N$ on functionals $\mathcal{I}^N$, $\hat{\mathcal{I}}^N$, $\Gamma^N$. Let $S (\mathcal{I})$ be the left-hand side of the Nielsen identity (14). Since $S (\cdot)$ is a nonlinear differential operator, one needs also its linearized counterpart $S \mathcal{F}$ defined (for arbitrary functionals $\mathcal{F}$ and $\mathcal{G}$) by

$$S (\mathcal{F} + \varepsilon \mathcal{G}) = S (\mathcal{F}) + \varepsilon S \mathcal{F} \mathcal{G} + O (\varepsilon^2) .$$

The renormalized action $\hat{\mathcal{I}} (n)$ generates the 1PI effective action $\Gamma (n) = \sum h^k \Gamma (n) (k)$, which is finite up to the order $h^n$. If $\hat{\mathcal{I}} (n)$ satisfies (14), then (assuming the Dimensional Regularization is used) so does $\Gamma (n)$, hence the divergent part of $\Gamma (n)$ obeys

$$S \mathcal{F} \Gamma (n+1, \text{div}) = 0 ,$$  (272)

with $\mathcal{I}$ being the tree level action (and $\hat{\mathcal{I}} (0) \equiv \mathcal{I}$). This equation ensures that in the $\text{MS}$-scheme the renormalized action at the order $h^{n+1}$, i.e.

$$\hat{\mathcal{I}} (n+1) = \hat{\mathcal{I}} (n) - h^{n+1} \Gamma (n+1, \text{div}) + O (h^{n+2}) ,$$  (273)

satisfies

$$S \left( \hat{\mathcal{I}} (n+1) \right) = O (h^{n+2}) .$$

To extend the Nielsen identity to the next order, the terms denoted $O (h^{n+2})$ in (272) have to be chosen so that $S \left( \hat{\mathcal{I}} (n+1) \right) \equiv 0$. For $q = 0$ (i.e. for the ordinary Zinn-Justin equation) a proof that this can be done was given in [2]; it provides additional constraints on possible counterterms, giving rise to the equations (73) and (74). Since the
The equation (280) is a counterpart of the relation (73) for structure constants. It is clear that

\[ C^{\alpha} \beta \gamma = C^{\alpha} \beta \gamma + h^{\alpha + 1} C^{\alpha} \beta \gamma + O(h^{\alpha + 2}) = e^{\alpha} \beta \gamma + O(h), \]

while (272) requires

\[ [e_{n \beta}, e_{n \gamma}] - [e_{n \beta}, e_{n \gamma}] = e_{n \alpha} e^{\alpha} \beta \gamma + c_{n \alpha} e_{n \beta} \gamma. \]

Taking into account identity (257), which gives \( C^{\alpha} \beta \gamma = 0 \), and performing some manipulations on (276) (see e.g. [2]) one finds

\[ e^{\eta} \beta \gamma = z^{\eta} e_{n \beta} \gamma - z^{\eta} e_{n \beta} \epsilon - z^{\epsilon} e_{n \eta} \gamma, \]

where

\[ z^{\eta} \beta = e_{n \eta} \alpha_{1} e_{n_{1} \alpha_{1}} \gamma \alpha_{1}, \]

with \( R^{\delta 1} \alpha \gamma \) being the inverse of the Killing form \( R_{n \delta 1} \alpha \gamma \equiv tr[e_{n \delta 1} e_{n \alpha_{1}}] \). Let us define

\[ (n+1) \eta \beta = \delta^{\eta} \beta + h^{n+1} z^{\eta} \beta, \]

and

\[ (n+1) \beta \gamma = \left[ (n+1) \right]^{\kappa} \delta^{\beta}_{\eta} C^{\alpha} \left[ (n+1) \right]^{-1} \beta \eta Z^{-1} \gamma, \]

It is clear that \( C^{\kappa} \beta \gamma \) are consistent with (276) and obey the Jacobi identity (109) exactly. Moreover (279) yields

\[ (n+1) \beta \gamma = \left[ (n+1) \right]^{\kappa} \delta^{\beta}_{\eta} Z\kappa C^{\alpha} \left[ (n+1) \right]^{-1} \beta \eta Z^{-1} \gamma, \]

where

\[ \left[ (n+1) \right]^{\kappa} \beta \gamma ZC = \left[ (n+1) \right]^{-1} ZC = Z = 1. \]

The equation (280) is a counterpart of the relation (73) for structure constants \( C^{\kappa} \beta \gamma \).

We now check stability of relation (107) under radiative corrections. Considerations similar to ones leading to (73) suggest the inductive hypothesis

\[ \hat{g} = \frac{\partial C^{\alpha} \beta \gamma}{\partial Z} \hat{C}^{\alpha} + \hat{p}^{\alpha} \beta \gamma, \]

which ensures (107) and holds at the tree level with \( \hat{p}^{\alpha} \beta \gamma = 0 \). At the \( (n+1) \)-th order we have

\[ \hat{g} = \hat{g} + h^{n+1} \hat{G} + O(h^{n+2}) = O(h), \]

and linear constraints (272) read

\[ \partial e_{n \beta} = \left[ \hat{G}, e_{n \beta} \right] - e_{n \alpha} \hat{G}^{\alpha} \beta. \]

Comparing the above equation with the derivative of (277) and defining \( \hat{E} = \hat{G} - dz \) one gets

\[ \hat{E}, e_{n \beta} = e_{n \beta} \hat{E}^{\beta} \gamma. \]
Since $e_{R\beta_{1}}$ are linearly independent, equation (285) requires
\[ \hat{E}_{\beta_{1}\gamma_{0}} = 0. \] (286)

For non-abelian indices $\beta$ and $\gamma$, (285) yields
\[ \hat{E}^{\beta}_{\gamma} = e_{R \sigma_{1}} \hat{R}_{n}^{\sigma_{1}} \delta_{1} \text{tr} \{ e_{R \delta_{1}} \hat{E} \}. \] (287)

Finally, identity (257) requires $\hat{G}_{\beta_{0} \epsilon^{0}} = 0$, and thus (see (273))
\[ \hat{E}_{\beta_{0} \epsilon} = 0. \] (288)

Equations (286) and (288) show that (287) holds for arbitrary indices $\beta$ and $\gamma$. Defining
\[ (n+1)^{\hat{p}_{\sigma_{0}}} = \hat{p}_{\sigma_{1}} \delta_{1} + 2(n+1)^{\hat{p}_{\sigma_{1}}} \delta_{1} \text{tr} \{ e_{R \delta_{1}} \hat{G} \}, \] (289)
and
\[ \hat{g} = \hat{d} Z_{C}^{-1} \hat{p}_{\sigma_{1}} C_{\sigma_{1}}, \] (290)

it is easy to convince oneself that the expansion (283) is correct. Thus, equation (167) has been established.

Formula (168) holds for $n$th-order parameters provided that (see (77))
\[ (n)^{\hat{H}_{\sigma}} = - \hat{p}_{\sigma} + \hat{g}_{\sigma \delta} \hat{p}_{\delta} - \frac{1}{2} (n)^{C_{\sigma \lambda}} (n)^{\hat{p}_{\kappa}} (n)^{\hat{p}_{\lambda}}, \] (291)
with $\hat{p}_{\sigma_{0}} \equiv 0$. The relevant part of (279) has the form
\[ (n+1)^{\hat{H}} = (n+1)^{\hat{h}} + O \left( h^{n+2} \right) = O \left( h \right), \] (292)
while (272) gives restrictions on the counterterm $\hat{h}$
\[ (n)^{\hat{h}_{\alpha_{1}}} e_{R \alpha_{1}} = - d \hat{G}. \] (293)

We have
\[ h^{n+1} \hat{G} = h^{n+1} \left( dz + \hat{E} \right) = h^{n+1} dz + e_{R \alpha_{1}} \left( (n+1)^{\hat{p}_{\alpha_{1}}} - (n)^{\hat{p}_{\alpha_{1}}} \right), \] (294)

hence
\[ h^{n+1} \hat{h} e_{R \alpha_{1}} = d \hat{p}_{\alpha_{1}} - d \hat{p}_{\alpha_{1}}, \] (295)

thus the formula (291) can be extended to the next order without violating (292). Finally, the ‘Bianchi identity’ (169) is automatically satisfied if equations (280), (282) and (291) hold.

References
[1] C. Becchi, A. Rouet, R. Stora; Commun.Math.Phys. 42 (1975) 127-162; Annals Phys. 98 (1976) 287-321
[2] G. Bandelloni, C. Becchi, A. Blassi, R. Collina; Ann.Inst. Henri Poincare, A 28, 225 (1978); A 28, 255 (1978)
[3] I. V. Tyutin; Lebedev Institute preprint N39 (1975),
[4] T. Kugo, I. Ojima; Suppl.Progr.Theor.Phys. 66, 1 (1979)
[5] N. K. Nielsen; Nucl. Phys. B101, 173 (1975)
[6] H. Kluberg-Stern and J. B. Zuber; Phys.Rev. D12, 467 (1975)
[7] I. Aitchison, C. Fraser; Ann.Phys., 156, 1 (1984),
[8] D. Metaxas, E. J. Weinberg; Phys.Rev. D53 (1996) 836-843, D. Metaxas, Phys.Rev. D63 (2001) 085009,
[9] P. Gambino, P. A. Grassi; Phys.Rev. D62 (2000) 076002, P. A. Grassi, B. A. Kniehl, A. Sirlin; Phys.Rev. D65 (2002) 085001,
[10] W. Kummer; Eur.Phys.J. C21 (2001) 175-179
[11] J. C. Breckenridge, M. J. Lavelle, T. G. Steele; Z.Phys. C65 (1995) 155-164
[12] O. Piguet, K. Sibold; Nucl. Phys., B253, 517 (1985)
[13] D. A. Johnston; Nucl.Phys., B253, 687 (1985); B283, 317 (1987)
[14] M. Bohm, H. Spiesberger, W. Hollik; Fortsch.Phys. 34 (1986) 687-751
[15] W. Loinaz, R.S. Willey; Phys.Rev. D56 (1997) 7416-7426
[16] A. F. de Lima, D. Bazeia; Z.Phys. C45 (1990) 471; J.R.S. Do Nascimento, D. Bazeia; Phys.Rev. D35 (1987) 2490-2494,
[17] J. Zinn-Justin; in Trends in Elementary Particle Theory, International Summer Institute on Theoretical Physics in Bonn 1974, Springer-Verlag, Berlin, (1975)
[18] J. Zinn-Justin; Quantum field theory and critical phenomena, Oxford : Clarendon Press, (1993),
[19] K. A. Meissner, H. Nicolai; Eur.Phys.J. C57, 493, (2008); A. Latosinski, K. A. Meissner, H. Nicolai; Nucl.Phys. B868 (2013) 596
[20] P. H. Chankowski, S. Pokorski, J. Rosiek; Phys.Lett. B286, 307 (1992); Nucl.Phys. B423 (1994) 437-496
[21] S. Coleman, E. Weinberg; Phys. Rev., D7, 1888, (1973)
[22] R. Jackiw; Phys. Rev., D9, 1686 (1974)
[23] G. ’t Hooft; Nucl.Phys., B35, 167, (1971); K. Fujikawa, B. W. Lee, A. Sanda; Phys.Rev., D6, 2923 (1972)
[24] L.Dolan, R. Jackiw; Phys. Rev., D10, 2904 (1974)
[25] B. de Wit, N. Papanicolou; Nucl.Phys. B113 (1976) 261
[26] R. Fukuda, S. Kugo; Phys.Rev., D13, 3469
[27] O. M. Del Cima, D. H. T. Franco, Olivier Piguet; Nucl.Phys. B551 (1999) 813-825
[28] D. Binosi, J. Papavassiliou, A. Pilaftsis; Phys.Rev. D71 (2005) 085007
[29] E. Kraus, K. Sibold; Z.Phys. C68, 331, (1995)
[30] C. Ford, I. Jack, D.R.T. Jones; Nucl.Phys. B387 (1992) 373-390, Erratum-ibid. B504 (1997) 551-552
[31] S. P. Martin; Phys.Rev. D65 (2002) 116003
[32] C. P. Martin, D. Sanchez-Ruiz; Nucl.Phys. B572 (2000) 387-477
[33] R. Dashen; Phys.Rev. 183 (1969) 1245
[34] S. Weinberg; The Quantum Theory of Fields, Vol. 2, Cambridge University Press (1996)
[35] R. W. Sharpe; Differential Geometry, Springer-Verlag (1997)
[36] D. Bardin, B. Passarino; The Standard Model in the Making, Oxford University Press (1999)