Knots of Constant Curvature

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Abstract

In this paper we show how to realize all knot (and link) types as $C^2$ smooth curves of constant curvature. Our proof is constructive: we build the knots with copies of a fixed finite number of "building blocks" that are particular segments of helices and circles. We use these building blocks to construct all closed braids.

1 Introduction

Circles and helices are standard examples of smooth curves of constant curvature. To construct other $C^2$ curves with constant curvature we may splice together pieces of helices and circles with the same curvature in such a way that the resulting curve is $C^2$ (see below). In [2], the authors Koch and Engelhardt integrated piecewise circular curves on $S^2$ to obtain nonplanar unknots of constant curvature. By manipulating several helix and circle segments, we found the granny knot shown in figure one. In this paper we develop a systematic method for constructing knots and show that all knot and link types can be realized as $C^2$ smooth curves with constant curvature. In a subsequent paper, we develop a different technique for realizing many $C^2$ constant curvature knots using the method of Koch and Engelhardt.

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2 The splicing method

Suppose that $\Gamma_1$ and $\Gamma_2$ are $C^2$ smooth curves of the same constant non-vanishing curvature $\kappa$. Let $r_1 : [0, \mathcal{L}_1] \rightarrow \mathbb{R}^3$ and $r_2 : [0, \mathcal{L}_2] \rightarrow \mathbb{R}^3$ be $C^2$ arclength parameterizations of $\Gamma_1$ and $\Gamma_2$ respectively. For $i \in \{1, 2\}$, let $\{T_i(\cdot), N_i(\cdot), B_i(\cdot)\}$ denote the Frenet frame of $r_i$ at the point $r_i(\cdot)$. Since the Frenet frame is an orthonormal set of vectors, there exists a rotation $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $A(T_2(0)) = T_1(\mathcal{L}_1)$, $A(N_2(0)) = N_1(\mathcal{L}_1)$, and $A(B_2(0)) = B_1(\mathcal{L}_1)$. We will define $u(s)$ to be the curve obtained by rotating $\Gamma_2$ so that its initial Frenet frame coincides with the terminal Frenet frame of $\Gamma_1$ and translating the rotated curve so that its initial point is $r_1(\mathcal{L}_1)$. In particular let $w = r_1(\mathcal{L}_1) - A(r_2(0))$ and define $u(s)$ by $u(s) = A(r_2(s - \mathcal{L}_1)) + w$ for $s \in [\mathcal{L}_1, \mathcal{L}_1 + \mathcal{L}_2]$. Consider the curve $\Gamma_1 * \Gamma_2$ given by the parameterization

$$r(s) = \begin{cases} r_1(s) & \text{if } s \in [0, \mathcal{L}_1], \\ u(s) & \text{if } s \in [\mathcal{L}_1, \mathcal{L}_1 + \mathcal{L}_2] \end{cases}$$

where $s$ is the arclength parameter of $\Gamma_1$. For a discussion of Frenet frames and curvature please see [1].

Since $A$ is linear, it follows that $\frac{du}{ds}(s) = A(r_2'(s - \mathcal{L}_1))$, and consequently $u'(\mathcal{L}_1) = A(r_2'(\mathcal{L}_1 - \mathcal{L}_1)) = A(r_2'(0)) = A(T_2(0)) = T(\mathcal{L}_1) = r_1'(\mathcal{L}_1)$. Since $r$ is defined piecewise by the $C^2$ functions $r_1$ and $u$, it follows from this that $r$ is differentiable and $C^1$. Similarly since $\frac{d^2u}{ds^2}(s) = A(r_2''(s - \mathcal{L}_1)) = A(T_2'(s - \mathcal{L}_1)) = (\kappa A(N_2(s - \mathcal{L}_1))) = \kappa A(N_2(s - \mathcal{L}_1))$ and since $u''(\mathcal{L}_1) = A(N_2(0)) = \kappa A(N_2(0)) =\kappa N_1(\mathcal{L}_1) = T_1'(\mathcal{L}_1) = r_1''(\mathcal{L}_1)$, it follows that $r$ is $C^2$. Furthermore since $r$ is an arclength parameterization of $\Gamma_1 * \Gamma_2$ and $r''$ is given by

$$r''(s) = \begin{cases} \kappa N_1(s) & \text{if } s \in [0, \mathcal{L}_1], \\ \kappa A(N_2(s - \mathcal{L}_1)) & \text{if } s \in [\mathcal{L}_1, \mathcal{L}_1 + \mathcal{L}_2] \end{cases}$$

it follows that $\Gamma_1 * \Gamma_2$ has constant curvature $\kappa$.

Essentially $\Gamma_1 * \Gamma_2$ is obtained by rotating $\Gamma_2$ so that its initial Frenet frame coincides with the terminal Frenet frame of $\Gamma_1$ and then attaching the beginning of the rotated $\Gamma_2$ to the end of $\Gamma_1$. This process of joining $\Gamma_2$ to $\Gamma_1$ to achieve $\Gamma_1 * \Gamma_2$, a $C^2$ smooth curve, will be refered to as splicing $\Gamma_2$ to $\Gamma_1$. Note that the splicing operation is associative. Hence for the sake of simplicity, $\Gamma_1 \Gamma_2$ will denote $\Gamma_1 * \Gamma_2$ and parenthesis will be ignored.

The curvature $\kappa$ of a helix parameterized by $\alpha(t) = [\pm r \cos(t), \pm r \sin(t), \pm h t]$ is given by $\kappa = \frac{r}{r^2 + h^2}$. Hence for any fixed $\kappa \neq 0$, there is an infinite number
of parameter values for $r$ and $h$ which determine a helix with constant curvature $\kappa$. Consequently there are infinitely many helices which can be used to form $C^2$ curves of constant curvature $\kappa$ (see figure 1). In this paper, we focus on two curves of curvature one: the unit circle and the helix with $r = h = \frac{1}{2}$.

Figure 1: a $C^2$ granny knot with constant curvature constructed by splicing together pieces of helicities

3 Fundamental Building Blocks

Let the vector notation $[r, h, t_i, t_e]$ denote the piece of the helix $\alpha(t) = [r\cos(t), r\sin(t), ht]$ with initial point $\alpha(t_i)$ and terminal point $\alpha(t_e)$. We
define the following elementary pieces:

\[
\begin{align*}
a &= \left[ \frac{1}{2}, \frac{1}{2}, 0, \frac{\pi}{2} \right] \\
b &= \left[ 1, 0, 0, \pi \right] \\
c &= \left[ \frac{1}{2}, \frac{1}{2}, 0, 2\pi \right] \\
d &= \left[ \frac{1}{2}, -\frac{1}{2}, \frac{3\pi}{2}, 2\pi \right] \\
e &= \left[ \frac{1}{2}, -\frac{1}{2}, 0, 2\pi \right] \\
f &= \left[ \frac{1}{2}, -\frac{1}{2}, 0, \frac{3\pi}{2} \right] \\
g &= \left[ \frac{1}{2}, -\frac{1}{2}, 0, \frac{\pi}{2} \right] \\
l &= \left[ \frac{1}{2}, \frac{1}{2}, 0, 8\pi \right]
\end{align*}
\]

All of these helices have curvature equal to one. In fact, the elementary pieces are either part of the unit circle or part of a helix with \( r = h = \frac{1}{2} \).

After reparametrizing by arclength we may splice together the elementary pieces to form the following family of curves \( \mathcal{B} = \{ i_+, i_-, j_+, j_-, k_+, k_- \} \) where \( i_+ = (abcd)^4, i_- = (aebd)^4, j_+ = (adbe)^4, j_- = (adbc)^4, k_+ = l, \) and \( k_- = abfdgbabfdgb \). We will call elements of \( \mathcal{B} \) sticks.

Let \( F \) denote the Frenet frame at the initial point of \( a \). The sticks are constructed so that, as parameterized curves, the Frenet frame at the initial and terminal point of each stick is \( F \). This property affects the geometry of the sticks. For example, if we subtract the initial point of \( k_- \) from its terminal point, we get the displacement vector \( [0, 0, -4\pi] \). Hence \( k_- \) accomplishes a net movement in the negative \( z \)-direction. However since we require the Frenet frame of \( k_- \) to be \( F \) at its initial and terminal points and since the \( z \)-coordinate of the unit tangent in \( F \) is positive, \( k_- \) must travel in the positive \( z \)-direction before turning downwards (see figure 2). Similarly, each of the remaining sticks must twist about in space in order for the initial and terminal Frenet frame to be \( F \). Because of this twisting and because we will eventually construct knots, we have to be careful to make sure that the spliced sticks are simple curves (see figures 2 and 3). Furthermore, since every element of \( \mathcal{B} \) has its initial and terminal Frenet frame equal to \( F \), any two sticks can be spliced together without rotation. Hence the resulting curve also has the attribute that its initial and terminal Frenet frame is \( F \).
Figure 2: The stick $k_-$ has been thickened so it can be viewed more easily. The piece of the curve in the box has been redrawn as a spacecurve and magnified to demonstrate that the stick $k_-$ does not intersect itself.

In addition to requiring that every stick have $F$ as its initial and terminal Frenet frame, the sticks have been constructed so that the distance from the initial point of each stick to its terminal point is the same. If we call the vector from the initial point of the stick to its terminal point the displacement vector of the stick, then the displacement vectors of $i_+, i_-, j_+, j_-, k_+$, and $k_-$ are $(4\pi, 0, 0)$, $(-4\pi, 0, 0)$, $(0, 4\pi, 0)$, $(0, -4\pi, 0)$, $(0, 0, 4\pi)$, and $(0, 0, -4\pi)$ respectively. Therefore since sticks and curves created from elements of $\mathcal{B}$ are spliced by simply translating the beginning of one stick to the end of another, it follows that given any path on a square lattice where the unit length is $4\pi$, we can imitate the path with a $\mathcal{C}^2$ curve of constant curvature by splicing together elements of $\mathcal{B}$.

4 Knots

In this section we will show how to splice together elementary pieces to realize a knot type as a $\mathcal{C}^2$ curve of constant curvature. We will realize the knot types by forming braid closures.
While any number of elementary pieces can be spliced together to form a \( C^2 \) curve of constant curvature, not every such curve is simple. Each stick has been constructed to be a simple curve. Figure 3 shows the sticks \( i_+ \), \( i_- \), \( j_+ \), and \( j_- \). A view of \( k_- \) is shown in figure 2 along with enlargements of parts of the curve to demonstrate that there are indeed no self-intersections. The stick \( k_+ \) is simply a piece of a single helix, and therefore clearly does not intersect itself.

![Figure 3: The sticks have all been thickened so they can be viewed more easily. If the sticks are rotated in \( \mathbb{R}^3 \), then it is clear that they are simple curves.](image)

Though each stick in \( B \) is simple, not every word made from elements of \( B \) represents a simple curve. For example, \( i_+i_- \), \( i_-i_+ \), \( j_+j_- \), \( j_-j_+ \), \( k_+k_- \), \( k_-k_+ \), \( k_-j_+ \), and \( i_-k_- \) are all curves with self intersections. We will consider these letter pairs as \emph{not allowable} because they lead to self-intersection. All other letter pairs represent simple curves and consequently these remaining
letter pairs will be called *allowable letter pairs*. As an example of a curve represented by an allowable letter pair, $k_-k_-$ is shown in figure 4.

![Figure 4](image)

Figure 4: Here we have close views of the join after a $k_-$ stick has been spliced to another $k_-$ stick. In the first drawing, the curve has been thickened for ease of viewing and this thickening accounts for the self-intersections. The second drawing shows the join as a spacecurve, and it is clear that after the join the curve remains simple.

Let $I_+, I_-, J_+, J_-, K_+, K_-$ denote $(4\pi, 0, 0)$, $(-4\pi, 0, 0)$, $(0, 4\pi, 0)$, $(0, -4\pi, 0)$, $(0, 0, 4\pi)$, and $(0, 0, -4\pi)$ respectively. We call any word in $I_\pm$, $J_\pm$, and $K_\pm$ a lattice polygon, and we identify the word with an actual polygon on a lattice (with unit length equal to $4\pi$) in $\mathbb{R}^3$. As before we will call $I_+I_-$, $I_-I_+$, $J_+J_-$, $J_-J_+$, $K_+K_-$, $K_-K_+$, $K_-J_+$, and $I_-K_-$ unallowable letter pairs. We have the following lemma:

**Lemma 0.1.** If $P$ is a lattice polygon without self-intersection and without unallowable letter pairs, then the same word written with elements of $B$ represents a simple curve.

**Proof.** Let $P = E_1E_2...E_n$ where $E_m \in \{I_\pm, J_\pm, K_\pm\}$, and let $p = e_1e_2...e_n$ where $e_m \in \{i_\pm, j_\pm, j_\pm\}$, $e_m$ is the lowercase of $E_m$, and $e_m$ has the same subscript as $E_m$. Then $p$ realizes the polygon $P$ as a $C^2$ curve of constant...
curvature. Since $P$ does not contain any unallowable letter pairs, $p$ also does not contain any unallowable letter pairs. Therefore adjacent sticks in $p$ do not intersect. It is easy to show that each stick $i_{\pm}$, $j_{\pm}$, $k_{\pm}$ is contained in a tube of radius $\pi$ centered about the line containing the initial and terminal points of the stick. Therefore since $P$ is simple and since nonconsecutive sticks in $p$ are separated by a distance of at least $4\pi$, it follows that the nonconsecutive sticks in $p$ do not intersect. Thus we have that $p$ is a simple curve.

4.1 Braids

Braid components can be formed from the sticks in such a way that if each strand in the braid component is expressed as a word in elements of $B$, the words contain only allowable word pairs. It follows from the previous lemma that each strand in the braid component is a simple $C^2$ curve of constant curvature. In figure 5, we have views of a braid component that contains a positive crossing. The $n^{th}$ strand is formed by translating the curve $k_+ k_- i_+ k_+ k_- k_+ k_+$ so that its initial point is $(-4\pi,0,0)$. We translate the curve without rotation so that the Frenet frame at the initial and terminal points of the translated curve is still $F$. The $(n+1)^{th}$ strand is formed by translating (without rotation) the curve $k_+ j_- k_+ i_- k_+ j_- k_+$ so that its initial point is the origin. The remaining strands are simply the curve $(k_+)^5$ translated (without rotation) so that the initial point of each strand is an integer multiple of $4\pi$. Views of a braid component containing a negative crossing are given in figure 6. The $n^{th}$ strand is the curve $k_+ j_+ k_+ i_+ k_+ j_- k_+$ translated (without rotation) so that its initial point is $(4\pi,0,0)$, and the $(n+1)^{th}$ strand is the curve $k_+ k_+ i_- k_+ k_- k_+$ translated (without rotation) to have its initial point at the origin. As before the remaining strands are the curves $(k_+)^5$ translated so that the initial points are all integer multiples of $4\pi$. 

\[\Box\]
Figure 5: a braid component with a positive crossing
Figure 6: a braid component with a negative crossing

The braid components have been constructed so that the initial and terminal points of each strand are on equally spaced lines contained in the \((x, z)\)-plane and perpendicular to the \(x\)-axis. This allows the braid components to be nicely stacked. Furthermore since the braid components are constructed from elements of \(B\), the Frenet frame at the initial and terminal points of each strand is \(F\). Hence the braid components can be stacked in a \(C^2\) fit, and each strand in the resulting braid remains a \(C^2\) curve of constant curvature (see figure 7).
Figure 7: A six strand braid component containing a negative crossing is spliced to the top of another braid component containing a negative crossing to form a $C^2$ braid with two crossings.

4.2 Closing pieces

In addition to the braid components, closing pieces may also be fashioned from elements of $\mathcal{B}$. Again these closing pieces are constructed so that when they are expressed as words made from the elements of $\mathcal{B}$, only allowable letter pairs appear. In figure 8 we have constructed a closing piece. This is the curve $k_+i_-k_-k_-k_-i_+k_+$ which has been translated without rotation so that its initial point is $(0, 0, 12\pi)$. By adding and deleting an appropriate number of sticks, the lengths in the various directions of the closing pieces can be adjusted.
Since we can adjust the dimensions of the closing pieces and since the initial and terminal points of each strand are on equally spaced lines contained in the $(x, z)$-plane and perpendicular to the $x$-axis, the closing pieces can be attached to the braids to form the braid closure. As before since the closing pieces are formed from elements of $B$, the initial and terminal Frenet frames of the closure pieces are $F$, and consequently the closure of the braid is a $C^2$ curve of constant curvature. Since Alexander’s Theorem (see [3]) states that every knot (and link) can be realized as the closure of a braid, we have the following theorem:
Theorem 1. Every knot (and link) can be represented by a $C^2$ curve of constant curvature.

In figure 9 we show a $C^2$ trefoil of constant curvature constructed from sticks as the closure of a braid.

Figure 9: a $C^2$ trefoil of constant curvature
References

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