ACCELERATED, OPTIMAL, AND PARALLEL: SOME RESULTS ON MODEL-BASED STOCHASTIC OPTIMIZATION

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Abstract. We extend the Approximate-Proximal Point (aProx) family of model-based methods for solving stochastic convex optimization problems, including stochastic subgradient, proximal point, and bundle methods, to the minibatch and accelerated setting. To do so, we propose specific model-based algorithms and an acceleration scheme for which we provide non-asymptotic convergence guarantees, which are order-optimal in all problem-dependent constants and provide linear speedup in minibatch size, while maintaining the desirable robustness traits (e.g. to stepsize) of the aProx family. Additionally, we show improved convergence rates and matching lower bounds identifying new fundamental constants for “interpolation” problems, whose importance in statistical machine learning is growing; this, for example, gives a parallelization strategy for alternating projections. We corroborate our theoretical results with empirical testing to demonstrate the gains accurate modeling, acceleration, and minibatching provide.

1. Introduction

We move beyond stochastic and “minibatch”-gradient methods for stochastic optimization problems to develop parallelizable and minibatch aware model-based and (approximate) proximal point methods for the problem

\[
\text{minimize } f(x) := \mathbb{E}_P[F(x; S)] = \int_S F(x; s) dP(s)
\]

subject to \(x \in \mathcal{X}\).

Here, \(\mathcal{S}\) denotes the sample space, and \(S \sim P\) is an \(\mathcal{S}\)-valued random variable, where for each sample \(s \in \mathcal{S}\), \(F(\cdot; s) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is a closed convex function, subdifferentiable on the closed convex domain \(\mathcal{X}\).

While stochastic gradient methods are the de facto choice for problem (1)—enjoying several convergence guarantees [46, 31, 10, 41] with straightforward parallel extensions that make them practically attractive [24, 15, 17]—they are sensitive to the objective \(f\), noise, and hyperparameter tuning [26, 2, 3]. They may even diverge for objectives that do not satisfy their convergence criteria or with slightly mis-specified stepsizes [3, 31]. Motivated by these limitations of gradient methods, researchers [9, 22, 13, 19, 3] have developed stochastic (approximate) proximal-point (aProx) and model-based methods as a more robust alternative. These aProx methods, as we explain in Section 1.1, construct a model of the function and iterate by minimizing regularized versions of the model. They improve over standard stochastic gradient methods, as they are robust to stepsize choice, adaptive to problem difficulty, and converge on a broader range of problems than stochastic gradient methods [19, 3]. Yet these aProx methods are inherently sequential, and as we hit physical limits on processor speeds, it is becoming clear that opportunities for improvements in large-scale computation and energy use must focus on parallelization [20]; it is not immediately apparent how to efficiently parallelize stochastic model-based methods.

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We study methods to parallelize the APROX family via minibatched samples $S^{1:m} \in S^m$, that is, where each iteration of the method receives an independent batch $S^{1:m} \overset{\text{iid}}{\sim} P$, developing several new results for model-based methods the problem (1) more generally along the way. We provide the following:

1. **Non-asymptotic rates and accelerated convergence:** In Section 3, we develop nonasymptotic convergence guarantees that depend on the *variance* of sample gradient estimates, in distinction to previous analyses [13, 3] that depend only on their magnitude, showing that model-based methods enjoy linear speedup in minibatch size $m$ (from standard $1/\sqrt{k}$ convergence rates to $1/\sqrt{km}$), analogous to standard speedup guarantees for gradient methods [24, 15]. These also allow us to develop an order-optimal accelerated method for the APROX family in Section 3.2.

2. **Optimal convergence and interpolation problems:** In Sections 4 and 5, we consider interpolation problems, that is, problems for which there exists $x^* \in \mathcal{X}$ minimizing $F(\cdot; s)$ with $P$-probability 1. Such problems arise in numerous modern machine learning applications [7, 8]—where one can achieve zero training error—or, for example, in finding a point in the intersection of convex sets $\cap_{i=1}^N C_i$, where one takes $\mathcal{S} = \{1, \ldots, N\}$ and $F(x; i) = \text{dist}(x, C_i)$. For these problems, we both develop new optimality results, characterizing (worst-case) problem difficulty based on a particular growth condition Asi and Duchi [3] introduce, which is (by these results) evidently fundamental; we also give some sufficient conditions for minibatching to yield improved convergence.

3. **Experimental evaluation:** We conclude with an experimental evaluation in Section 6, where we study the robustness and acceleration properties of the methods; performance profiles highlight the benefits of using these better models.

1.1. **Preliminaries.** The starting point of our methods is the model-based approximate proximal-point (APROX) framework [13, 19, 3], which approximates the functions $F$ via *models* $F_x$ of $F$ localized at $x$, which satisfy the following conditions:

(C.i) **Convexity:** The function $y \mapsto F_x(y; s)$ is convex and subdifferentiable on $\mathcal{X}$.

(C.ii) **Lower bounds and local accuracy:** For all $y \in \mathcal{X}$,

$$F_x(y; s) \leq F(y; s) \quad \text{and} \quad F_x(x; s) = F(x; s).$$

Note that Condition (C.ii) immediately implies that $\partial F_x(y; s)|_{y=x} \subset \partial F(x; s)$.

With such a model, APROX algorithms iteratively sample $S_k \overset{\text{iid}}{\sim} P$ and update

$$x_{k+1} := \arg\min_{x \in \mathcal{X}} \left\{ F_x(x; S_k) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}.$$

Typical choices for the models include the following three:

- **Stochastic gradient methods:** for some $F'(x; s) \in \partial F(x; s)$, use the linear model

  $$(3) \quad F_x(y; s) := F(x; s) + \langle F'(x; s), y - x \rangle.$$

- **Stochastic proximal point methods:** use the full proximal model

  $$(4) \quad F_x(y; s) := F(y; s).$$

- **Truncated methods:** for some $F'(x; s) \in \partial F(x; s)$, use

  $$(5) \quad F_x(y; s) := \max \left\{ F(x; s) + \langle F'(x; s), y - x \rangle, \inf_{z \in \mathcal{X}} F(z; s) \right\}.$$

The model (5) is often simple to apply: in many applications, the objective is non-negative, so $\inf_{z \in \mathcal{X}} F(z; s) = 0$ and the model is simply the positive part of the linear approximation (3).
Notation. For a convex function $f$, $\partial f(x)$ denotes its subgradient set at $x$, and $f'(x) \in \partial f(x)$ denotes an arbitrary element of the subdifferential. We let $X^* = \text{argmin}_{x \in X} f(x)$ denote the optimal set of problem (1) and $x^* \in X^*$ denote a single minimizer. We let $F_k := \sigma(S_1, \ldots, S_k)$ be the $\sigma$-field generated by the first $k$ random variables $S_i$, so $x_k \in F_{k-1}$ for all $k$ under iteration (2).

1.2. Related work. Stochastic gradient methods [39] are the most widely used method for solving stochastic minimization problems; an enormous literature gives numerous convergence results [37, 38, 46, 31, 45, 23, 4]. The growth of parallel computing has motivated the development of “minibatch” methods that use multiple samples $S$ in each iteration, where researchers have shown how stochastic gradient-like methods enjoy linear speedups as batch sizes increase [24, 15, 17, 34, 12]. Other work proposes accelerated stochastic optimization methods, showing faster (worst-case optimal) associated convergence rates [27, 24]. In spite of their successes, stochastic gradient methods still suffer a number of drawbacks. For example, they are sensitive to problem parameters, where mis-specified stepsizes may force slow (even order sub-optimal or exponentially slower) convergence [31]; objective functions without appropriate scaling or that grow too quickly may cause divergence [2, 3]; they can fail to adapt to problem geometry [18, 25]. This motivates work to make stochastic gradient methods more robust [31] and adaptive [18, 36] as well as research on stochastic proximal-point and model-based methods [22, 21, 9, 19, 13]. In this vein, Asi and Duchi [3, 2] show how better models in stochastic optimization yield improved stability, robustness, and convergence guarantees over classical stochastic subgradient methods.

A second line of work studies acceleration, mini-batching, and parallelism in stochastic optimization [15, 24, 14, 35, 12, 40]. Here, the key insights typically show that mini-batching—averaging stochastic gradients—yields reduced variance and hence improved convergence [15, 24]. Other key insights show how in large-scale communication-limited problems, the noise inherent to sampling dominates deterministic components of convergence rates and errors due to delay or communication [12, 29, 40]. For model-based methods, appropriate notions of variance are less immediate, and in interpolation problems (recall item 2 above) there is essentially no noise, so that an important part of our development is to extend accelerated and variance-dependent rates of convergence (as available for gradient-based methods [24]) to model-based methods. An important component of accelerated and parallel methods is their (worst-case) optimality [33, 24, 1]; as one of the major successes for model-based methods is in interpolation problems, it is also of interest to develop corresponding optimality results, which (to our knowledge) do not exist.

2. Methods

While at some level, the extension of standard stochastic gradient methods to parallel settings—average gradients to reduce noise—is clear, such extension is less immediate for proximal and model-based methods. To that end, we identify several different possibilities for extending the APROX framework—which coincide for linear models (stochastic gradient methods)—but can exhibit different optimization behavior. Given a batch $S^1_k \in S^n$ of samples, we consider the following:

**Iterate averaging (IA):** The naive extension of APROX to use minibatches is to perform an individual update for each sample $S^i_k$, then average the updates:

$$x_{k+1} := \frac{1}{m} \sum_{i=1}^m x^i_{k+1} \quad \text{where} \quad x^i_{k+1} = \text{argmin}_{x \in X} \left\{ F_{x_k}(x; S^i_k) + \frac{1}{2\alpha_k} \| x - x_k \|_2^2 \right\}. \tag{6}$$

This method’s simplicity and (near) full parallelization makes it attractive, and when $X = \mathbb{R}^n$ and each of the models $F_x$ is the subgradient model (3), it coincides with the mini-batch stochastic gradient method. Unfortunately, in general it does not enjoy the same acceleration properties of our other methods.
A method that more naturally dovetails with the model-based perspective is to minimize a model of the average

\[
\overline{F}(x; S^{1:m}_k) := \frac{1}{m} \sum_{i=1}^m F(x; S^i_k)
\]

at every iteration. In particular, with any model \(\overline{F}_{x_k}(x; S^{1:m}_k)\) of the average satisfying Conditions (C.i) and (C.ii), we can perform the update

\[
x_{k+1} := \arg\min_{x \in \mathcal{X}} \left\{ \overline{F}_{x_k}(x; S^{1:m}_k) + \frac{1}{2\alpha_k} \|x - x_k\|_2^2 \right\}.
\]

While our theorems hold for any model-based algorithm satisfying Conditions (C.i)–(C.ii) (and Condition (C.iii) to come), we find two instantiations of the approach (8) of particular interest.

**Truncated Average** (TruncAv): The first such model extends the truncated model (5). Let \(\Lambda(s^{1:m})\) be any lower bound on \(\overline{F}(\cdot, s^{1:m})\); for example, \(\Lambda(s^{1:m}) = \frac{1}{m} \sum_{i=1}^m \inf_{z \in \mathcal{X}} F(z; s^i)\) suffices. Then set

\[
\overline{F}_x(y; s^{1:m}) := \max \left\{ \overline{F}(x; s^{1:m}) + \langle \overline{F}'(x; s^{1:m}), y - x \rangle, \Lambda(s^{1:m}) \right\}.
\]

In the standard case that the functions \(F\) are nonnegative and \(\mathcal{X} = \mathbb{R}^n\), the update (8) corresponds to (stochastic) Polyak stepping [37], and becomes

\[
x_{k+1} = x_k - \min \left\{ \alpha_k, \frac{\overline{F}(x_k; S^{1:m}_k) - \Lambda(S^{1:m}_k)}{\|\overline{F}'(x_k; S^{1:m}_k)\|_2^2} \right\} \overline{F}'(x_k; S^{1:m}_k).
\]

The update (9) for the truncated models thus yields an embarrassingly parallelizable scheme: each worker computes \(F(x_k; S^i_k)\) and \(\nabla F(x_k; S^i_k)\), which need only be averaged to apply the update (9).

**Average of Truncated Models** (AvTrunc): The update (9) ignores some structural aspects of the objectives \(F\); it is natural to consider a more accurate averaging of models. Letting \(F_x(y; s^i) = \max\{F(x; s^i) + \langle F'(x; s^i), y - x \rangle, \inf_{z \in \mathcal{X}} F(z; s^i)\}\), the average \(\frac{1}{m} \sum_{i=1}^m F_x(\cdot; s^i)\) satisfies conditions (C.i) and (C.ii), and we consider the update

\[
x_{k+1} := \arg\min_{x \in \mathcal{X}} \left\{ \frac{1}{m} \sum_{i=1}^m F_{x_k}(x; S^i_k) + \frac{1}{2\alpha_k} \|x - x_k\|_2^2 \right\}.
\]

When \(m\) is not too large, problem (10) is relatively easy to solve. Indeed, define \(g_i = F'(x_k; S^i_k)\) and let \(G = [g_1 \ldots g_m] \in \mathbb{R}^{n \times m}\) and \(v = [F(x_k; S^1_k) \cdots F(x_k; S^m_k)]^T \in \mathbb{R}^m\). Then the dual to problem (10) is

\[
\text{maximize } -\frac{\alpha}{2} \lambda^T G^T G \lambda + \lambda^T v \text{ subject to } 0 \preceq \lambda \preceq \frac{1}{m},
\]

and letting \(\lambda_k\) be the solution, we update \(x_{k+1} = x_k - \alpha_k G \lambda_k\). In situations where computing the (sub)gradients \(F'(x_k; S^i_k)\) or infima \(\inf_{z \in \mathcal{X}} F(z; S^i_k)\) are more expensive than solving the dual—reasonable when \(m\) is small—one can parallelize efficiently.

**Remark** The preceding two methods provide two natural approaches to mini-batching model-based stochastic methods; any approach that guarantees the model \(\overline{F}_x(y; s^{1:m})\) satisfies conditions (C.i)–(C.ii) for the average \(\overline{F}'(\cdot; s^{1:m})\) will similarly suffice for our results.

Before proceeding to our theoretical guarantees, we provide a simple example to help illustrate the methods. Consider the problem of finding a point in the intersection \(C_1 \cap C_2\) of convex sets \(C_1\) and \(C_2\) by minimizing \(f(x) = \frac{1}{2}(\text{dist}(x, C_1) + \text{dist}(x, C_2))\). Figure 1 illustrates the IA, TruncAv, and AvTrunc updates given infinite stepsize \(\alpha\) (which still guarantees convergence if \(\cap C_i\) is non-empty [3]). Let \(\pi_i(x)\) denote the projection of \(x\) onto \(C_i\), so that the hyperplane tangent to \(C_i\) at \(\pi_i(x)\) is \(x - \pi_i(x)\). In this case, iterate averaging (6) projects the current iterate \(x_k\) to the two sets in the batch and averages these updates (Fig. 1(a)). The TruncAv update (9) constructs the average
of the hyperplanes \( v = \frac{1}{2} \sum_{i=1}^{2} (x - \pi_i(x)) \) and distances \( d = \frac{1}{2} \sum_{i=1}^{2} \text{dist}(x, C_i) \) and projects to the halfspace \{ \( x | v^T(x - x_k) \leq -d \) \}, which yields more progress (Fig. 1(b)). The AvTrunc update (10) projects to the set defined by the intersection of the tangent halfspaces at \( \pi_i(x) \) (Fig. 1(c)). We expect generally—and our theory and experiments will confirm—that the TruncAv and AvTrunc updates are more effective than naive averaging.

![Figure 1](image_url)

**Figure 1.** Updates for the truncated model using (a) iterate averaging (6), (b) truncated averaging (9), and (c) averaging of models (10).

### 3. Non-Asymptotic Convergence Results

Our first set of theoretical results extends the familiar non-asymptotic rates of convergence for smooth convex stochastic optimization [24] to model-based methods. Here, we show that model-based methods for problem (1) enjoy optimal dependence on the variance of stochastic gradients, and, building off of Tseng [43] and Lan [24], can be accelerated to achieve worst-case optimal complexity. To present our results in the most generality, we allow non-Euclidean geometries to generalize mirror descent [6, 31].

To that end, recall that a differentiable convex function \( h \) is a distance generating function for \( \mathcal{X} \) if it is strongly convex with respect to a norm \( \| \cdot \| \) over \( \mathcal{X} \), meaning \( h(y) \geq h(x) + \langle \nabla h(x), y-x \rangle + \frac{1}{2} \| x-y \|^2 \) for \( x, y \in \mathcal{X} \). The associated Bregman divergence is then \( D_h(x,y) := h(x) - h(y) - \langle \nabla h(y), x-y \rangle \), which evidently satisfies \( D_h(x,y) \geq \frac{1}{2} \| x-y \|^2 \). Recalling the dual norm \( \| z \|_* = \sup_{\| x \| \leq 1} \langle z, x \rangle \), throughout this section, we will work with the following standard assumption [24, 15].

**Assumption 1.** The function \( f \) has \( L \)-Lipschitz gradient with respect to the norm \( \| \cdot \| \), meaning that
\[
\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|,
\]
and there exists \( \sigma_0^2 < \infty \) such that for each \( x \in \mathcal{X} \),
\[
\mathbb{E}[\| \nabla f(x) - \nabla F(x; S) \|_*^2] \leq \sigma_0^2.
\]

When \( D_h(x,y) \leq R^2 \) for all \( x, y \in \mathcal{X} \) and Assumption 1 holds, mirror descent methods achieve convergence guarantees of the form \( \frac{LR^2}{k} + \frac{\sigma_0 R}{\sqrt{k}} \), while accelerated methods [24] can achieve \( \frac{LR^2}{k^2} + \frac{\sigma_0 R}{\sqrt{k}} \). The latter is worst-case optimal [32]. By considering the natural generalization
\[
(11) \quad x_{k+1} = \text{argmin}_{x \in \mathcal{X}} \left\{ F_{x_k}(x; S_k) + \frac{1}{\alpha_k} D_h(x, x_k) \right\}
\]
of the model-based iteration (2), we achieve the same (optimal) rates here for the model-based mirror method (11); combined with the results of the paper [3], these show that model-based methods offer the same benefits (efficiency, parallelizability, and worst-case optimality) that stochastic gradient and mirror descent methods do while simultaneously guaranteeing more robustness.
3.1. A basic non-asymptotic convergence guarantee. Our first result gives convergence of the basic iteration (11).

\textbf{Theorem 1.} Let Assumption 1 hold, and assume that \( D_h(x,y) \leq R^2 \) for all \( x, y \in \mathcal{X} \). Let \( x_k \) follow the model-based iteration (2) for any model satisfying Conditions (C.i) and (C.ii). Define the stepsizes \( \alpha_k = \frac{1}{L + \eta_k} \), where \( \eta_k \geq 0 \) is non-decreasing. Then

\[
\sum_{i=1}^{k} \mathbb{E}[f(x_{i+1}) - f(x^*)] \leq \frac{LR^2}{2} + \frac{R^2 \eta_k}{2k} + \frac{\sigma_0^2}{2\eta_k}.
\]

The proof of this result, while not completely standard as we cannot rely on linearity in the updates or gradients to achieve the variance bound, builds off of several well-established techniques, so we defer it to Appendix A.1.

Having established a convergence result that depends on the noise of the gradient estimates, convergence guarantees for the average \( \bar{x}_k = \frac{1}{k} \sum_{i=1}^{k} x_{i+1} \) are immediate. First, under the conditions of Theorem 1 we have

\[
\mathbb{E}[f(\bar{x}_k)] - f(x^*) \leq \frac{LR^2}{k} + \frac{R^2 \eta_k}{k} + \frac{\sigma_0^2}{2\eta_k},
\]

and with the choice \( \eta_k = \eta_0 \sigma_0 \sqrt{k} / R \) we obtain the rate

\[
\mathbb{E}[f(\bar{x}_k)] - f(x^*) \leq \frac{LR^2}{k} + \frac{R \sigma_0}{\sqrt{k}} \left( \eta_0 + \frac{1}{2\eta_0} \right).
\]

When we use the standard Euclidean choice \( h(x) = \frac{1}{2} \|x\|_2^2 \), we see an immediate speedup guarantee for the minibatched aPROM methods:

\textbf{Corollary 3.1.} Let the conditions of Theorem 1 hold, let \( \eta_k = \eta_0 \sqrt{k} \) with \( \eta_0 = \frac{\sigma_0}{\sqrt{mR}} \), and let \( x_k \) be generated by the iteration (8) with any model \( F_x(y; S^{1:m}) \) satisfying conditions (C.i) and (C.ii) and minibatch size \( m \). Then

\[
\mathbb{E}[f(\bar{x}_k)] - f(x^*) \leq \frac{LR^2}{k} + \frac{3R \sigma_0}{2\sqrt{km}}.
\]

When the iteration count \( k \gg \frac{L^2 R^2 m}{\sigma_0^2} \), the second term dominates the rate of convergence. Letting \( T(\epsilon) \) denote the number of iterations to achieve \( \mathbb{E}[f(\bar{x}_{T(\epsilon)})] - f(x^*)] \leq \epsilon \), we obtain that

\[
T(\epsilon) \leq \frac{R^2 \sigma_0^2}{\epsilon^2 m},
\]

that is, there is a linear speedup as a function of the minibatch of size \( m \). This is similar to the speedup that standard stochastic gradient methods achieve [24, 15] and is minimax optimal.

3.2. Accelerated model-based methods. We now develop an accelerated analogue of the iteration (2), which gives a leading minimax-optimal \( O(1/k^2) \) rate, by building off of the ideas of Lan [24] and Tseng [43]. We consider a modified iteration, which augments the model-based update (2) with two auxiliary sequences whose momentum allows accelerated convergence. For full generality and completeness, we consider an augmented version of problem (1), where we wish to minimize

\[
f(x) + r(x) = \mathbb{E}_P[F(x; S)] + r(x),
\]

where \( r \) is a known convex function (typically a regularizer of some type). We require a non-increasing sequence \( \theta_k \in [0,1] \) of stepsizes and consider the three term iteration

\[
y_k = (1 - \theta_k) x_k + \theta_k z_k
\]

\[
z_{k+1} = \arg\min_{z \in \mathcal{X}} \left\{ F_{y_k}(x; S_k) + r(x) + \frac{1}{\alpha_k} D_h(x, z_k) \right\}
\]

\[
x_{k+1} = (1 - \theta_k) x_k + \theta_k z_{k+1}.
\]

3.3. Accelerated, optimal, and parallel.
All our analysis requires is that the additional stepsizes \( \theta_k \) satisfy \( \theta_0 = 1, \frac{1-\theta_k}{\eta_k} \leq \frac{1}{\eta_{k-1}} \) for all \( k \), and are non-increasing; for example, our choice \( \theta_k = \frac{2}{k+1} \) satisfies these desiderata, as does taking any constant stepsize. We then have the following theorem.

**Theorem 2.** Let Assumption 1 hold, and assume that \( D_h(x, x) \leq R^2 \) for all \( x \in X \). Let \((y_k, z_k, x_k)\) follow the three term iteration (13) for any model satisfying Conditions (C.i) and (C.ii). Take stepsizes \( \theta_k = \frac{2}{k+1} \) and \( \alpha_k = \frac{1}{L+R} \) for \( \eta_k = \eta_0 \sqrt{k+1} \), where \( \eta_0 \geq 0 \). Then

\[
\mathbb{E}[f(x_{k+1}) + r(x_{k+1}) - f(x^*) - r(x^*)] \leq \frac{4LR^2}{(k+2)^2} + 2 \frac{R^2}{\sqrt{k}} \left[ \frac{\sigma_0^2}{\eta_0} + \eta_0 \right].
\]

See Appendix A.2 for a proof.

Specializing to the “minibatch” setting with \( h(x) = \frac{1}{2} \|x\|_2^2 \) again yields a minimax optimal algorithm for the class of problems we consider.

**Corollary 3.2.** Let the conditions of Theorem 2 hold, except that we use a minibatch \( S_k^{1:m} \iidsim P \) of size \( m \) at each iteration, and \( F_{y_k}\) is a model of \( \frac{1}{m} \sum_{i=1}^m F() \) satisfying Conditions (C.i) and (C.ii). Set \( \eta_0 = \frac{\sigma_0 \sqrt{m}}{R} \). Then

\[
\mathbb{E}[f(x_{k+1}) + r(x_{k+1}) - f(x^*) - r(x^*)] \leq \frac{4LR^2}{(k+2)^2} + 3 \frac{R \sigma_0}{\sqrt{km}}.
\]

The error rate \( O(1/k^2 + 1/\sqrt{km}) \) is faster than the \( O(1/k + 1/\sqrt{km}) \) rate we showed for the basic minibatched APROX algorithm (2), and it is minimax rate optimal.

## 4. Interpolation Problems

In **interpolation problems**, there exists a consistent solution \( x^* \in X \) satisfying \( F(x^*; S) = \inf_{x \in X} F(x; S) \) with probability 1. While this is a strong assumption, it holds in numerous practical scenarios: in machine learning problems, where a perfect predictor (at least on training data) exists [7, 8, 28]; in problems of finding a point in the intersection \( C^* = \cap_{i=1}^N C_i \) of convex sets \( C_i \), assuming \( C^* \neq \emptyset \), where we may take \( F(x; i) = \text{dist}(x, C_i) \) (e.g. [5]); or in least-squares problems with consistent solutions [30, 42]. We show a few results in this section, first that model-based methods (often) enjoy linear convergence on these problems—in analogy to the results available for stochastic gradient methods [28]—while also demonstrating improvement via mini-batching and reducing variance. Second, we revisit the convergence guarantees that Asi and Duchi [3] provide, giving a unified treatment and some discussion of the possibilities of parallelism. These conditions appear on their face to be somewhat non-standard, but as we show, they capture the essential difficulties of interpolation problems, and we can provide sharp (matching to within numerical constants) lower bounds for optimization using them.

**Definition 4.1.** Let \( X^* := \arg\min_{x \in X} f(x) \). Then problem (1) is an **interpolation problem** if there exists \( x^* \in X^* \) such that for \( P \)-almost all \( s \in S \), we have \( \inf_{x \in X} F(x; s) = F(x^*; s) \).

We develop two sets of upper bounds for such interpolation problems. The first applies to any model-based method, while the second relies on the models having more fidelity to the functions \( F \).

### 4.1. Upper bounds under smoothness and quadratic growth.

Our first set of upper bounds relies on two assumptions about the growth of the function \( f \) at the optimum—which is weaker than typical strong convexity assumptions [28] that require quadratic growth everywhere—and the noise in its gradients.

**Assumption 2** (Quadratic Population Growth). There exist \( \lambda > 0 \) such that for all \( x \in X \),

\[
f(x) - f(x^*) \geq \lambda \text{dist}(x, X^*)^2.
\]
Assumption 3. There exists $\sigma_2^2 < \infty$ such that for every $x \in X$, we have $\mathbb{E}[\|f'(x) - F'(x; S)\|_2^2] \leq \sigma_2^2 \text{dist}(x, \mathcal{X}^*)^2$.

It is straightforward to give examples satisfying the assumptions; noiseless linear regression problems provide the simplest such approach.

**Example 1:** Consider a linear regression problem with data $s = (a, b) \in \mathbb{R}^n \times \mathbb{R}$, where $a^T x^* = b$ for all $(a, b)$, and set $F(x; (a, b)) = \frac{1}{2}(a^T x - b)^2$. If the data $a$ belong to a subspace $V \subset \mathbb{R}^n$ (which may be $V = \mathbb{R}^n$), then Assumption 2 holds with $\lambda = \inf_{\|v\|_2 = 1} \{v^T \mathbb{E}[aa^T]v / 2 \mid v \in V\}$, and it is immediate that $\text{Var}(F'(x; S)) \leq \mathbb{E}[\|a\|^2_2 (a, x - x^*)^2]$, so Assumption 3 holds with $\sigma_2^2 = \lambda_{\max}(\mathbb{E}[\|a\|^2_2 aa^T])$. For example, if $a$ is uniform on the scaled sphere $\sqrt{n}S^{n-1}$, then $\lambda = 1$ and $\sigma_2^2 = n$. ◇

Alternatively, we may follow Ma et al. [28] by considering a problem where the functions $F$ have Lipschitz gradients:

**Example 2:** If $F(\cdot; s)$ has $L(s)$-Lipschitz gradient and problem (1) is an interpolation problem with $x^* \in \text{int } X$, then $\nabla F(x^*; S) = 0$ with probability 1, and so

$$\mathbb{E}[\|\nabla f(x) - \nabla F(x; S)\|_2^2] \leq \mathbb{E}[\|\nabla f(x) - \nabla f(x^*) - (\nabla F(x^*; S) - \nabla F(x; S))\|_2^2]$$

$$\leq 2 \|\nabla f(x) - \nabla f(x^*)\|_2^2 + 2\mathbb{E}[\|\nabla F(x^*; S) - \nabla F(x; S)\|_2^2]$$

$$\leq 4\mathbb{E}[L(S)^2] \|x - x^*\|_2^2.$$

We may thus take $\sigma_2^2 \leq \mathbb{E}[L(S)^2]$. ◇

Under these assumptions, model-based methods enjoy linear (or nearly linear) convergence with constant and decaying stepsize choices.

**Theorem 3.** Assume problem (1) is an interpolation problem (Definition 4.1) and let $f$ have $L$-Lipschitz gradient and satisfy Assumptions 2 and 3, where $L \geq \lambda$. Let $x_k$ follow the model-based iteration (8) with any model $\overline{F}_x(y; S^{1:m})$ satisfying conditions (C.i) and (C.ii) with minibatch size $m$. Then

(i) Let $\alpha_k = \frac{1}{L + \eta_k}$ for $\eta_k \geq 0$. Then

$$\mathbb{E}[\text{dist}(x_k, \mathcal{X}^*)^2] \leq \exp \left( - \frac{1}{2} \sum_{i=1}^k \frac{\lambda \alpha_k}{m \eta_i} \sum_{i=1}^k \frac{\sigma_2^2 \alpha_i}{m \eta_i} \right) \text{dist}(x_0, \mathcal{X}^*)^2.$$

(ii) With the constant stepsize choice $\alpha_k = (L + \eta)^{-1}$ and $\eta = \max \{L; \frac{8\sigma_2^2}{m \lambda}\}$,

$$\mathbb{E}[\text{dist}(x_k, \mathcal{X}^*)^2] \leq \exp \left( -k \min \left\{ \frac{\lambda}{8L}, \frac{m \lambda^2}{64\sigma_2^2} \right\} \right) \mathbb{E}[\text{dist}(x_0, \mathcal{X}^*)^2].$$

**Proof** We assume without loss of generality that $f(x^*) = 0 = F(x^*; s)$ for notational simplicity. We begin with the single step guarantee of Lemma A.1. Let $D_k = \text{dist}(x_k, \mathcal{X}^*)$ for shorthand, and recall our notations $e_k = (F(x^*; S_k) - f(x^*)) - (F(x_k; S_k) - f(x_k)) = f(x_k) - F(x_k; S_k)$ (in this case) and $\xi_k = \nabla F(x_k; S_k) - \nabla f(x_k)$. Then Lemma A.1 implies

$$\frac{1}{2} D_{k+1}^2 \leq \frac{1}{2} D_k^2 - \alpha_k f(x_{k+1}) + \alpha_k e_k + \frac{\alpha_k}{2\eta_k} \|\xi_k\|_2^2$$

$$\leq \frac{1}{2} D_k^2 - \frac{\alpha_k \lambda}{2} D_{k+1}^2 + \alpha e_k + \frac{\alpha_k}{2\eta_k} \|\xi_k\|_2^2,$$
where the second inequality follows from Assumption 2 that $f(x_{k+1}) \geq \frac{1}{2} D_{k+1}^2$. Noting that $E[\|\xi_k\|^2 | x_k] \leq \frac{\sigma^2_k}{m} D_k^2$ by Assumption 3, we rearrange and take expectations on both sides to obtain

$$E[D_{k+1}^2] \leq \frac{1}{\alpha_k \lambda + 1} \left( 1 + \frac{\alpha_k \sigma^2_k}{\eta m} \right) E[D_k^2] \leq \exp \left( -\frac{\lambda \alpha_k}{2} + \frac{\sigma^2_k \alpha_k}{\eta k m} \right) E[D_k^2].$$

Iterate this inequality to achieve the result (i) in the theorem.

For result (ii), we simply note that if $\alpha_k = \frac{1}{L + \eta}$, then using $2 \max\{L, \eta\} > L + \eta > \eta$, we have

$$E[D_{k+1}^2] \leq \exp \left( -\frac{\lambda}{4 \max\{L, \eta\}} + \frac{\sigma^2_0}{\eta^2 m} \right) E[D_k^2].$$

Substituting $\eta = \max\{L; \frac{8 \sigma^2_0}{m^2} \}$ gives the result.

The results in Theorem 3 imply that when the batch size is large enough that $m \gtrsim \frac{\sigma^2_0}{\lambda L}$, we achieve convergence rate $E[\text{dist}(x_k, x^*)] \lesssim (1 - c^2 \frac{1}{L})^k E[\text{dist}(x_0, x^*)]$, where $c > 0$ is a numerical constant, which is the rate of convergence for (deterministic) gradient methods with optimal stepsize choices [33]. More generally, we see a roughly linear speedup in the batch size $m$ to achieve a given accuracy until $m \geq \frac{\sigma^2_0}{\lambda^2}$: to obtain $E[\text{dist}(x_k, x^*)]^2 \leq \epsilon$ takes

$$k = O(1) \max \left\{ \frac{L, \sigma^2_0}{\lambda \lambda^2 m} \right\} \log \frac{1}{\epsilon}$$

iterations with appropriately chosen stepsize $\alpha$. That is, we expect to see a linear improvement in the number of iterations to achieve a given accuracy $\epsilon$ until the condition number $\frac{L}{\lambda}$ dominates the variance of the gradient estimates.

4.2. Upper bounds under an expected growth condition. In Theorem 3 above, we restrict the stepsizes to have the form $\alpha_k = \frac{1}{L + \eta k}$. With more accurate models and an alternative growth assumption on the functions $F$ and $f$, we can remove this weakness, highlighting the robustness of more accurate models. To that end, we revisit a few results of Asi and Duchi [3], beginning with a slight generalization of their growth assumption (which corresponds to the choices $\gamma \in \{0, 1\}$ below):

**Assumption 4** ($\gamma$-Growth). There exist constants $\lambda_0, \lambda_1 > 0$ and $\gamma \in [0, 1]$, such that for all $\alpha \in \mathbb{R}_+, x \in \mathcal{X}, x^* \in \mathcal{X}^*$, we have

$$E \left[ (F(x; S) - F(x^*; S)) \min \left\{ \alpha, \frac{F(x; S) - F(x^*; S)}{\|F'(x; S)\|^2} \right\} \right] \geq \min\{\lambda_0 \alpha, \lambda_1 \text{dist}(x, x^*)^{1-\gamma}\} \text{dist}(x, x^*)^{1+\gamma}.$$

As we will show in the coming section, while Assumption 4 looks like a technical assumption, it actually fairly closely governs the complexity of solving interpolation problems, in that the $\lambda_1$ parameter describes lower bounds on the convergence of any method. Essentially, the assumption states that the functions $F$ must grow relative to the magnitude of their gradients at a particular rate, so that it provides a type of stochastic growth condition. We shall revisit this in the next section when we prove our lower bounds, for now focusing on algorithms and their convergence under the assumption. First, however, we may again rely on linear regression-type objectives for an example satisfying Assumption 4.

**Example 3:** Consider a problem with data $s = (a, b) \in \mathbb{R}^n \times \mathbb{R}$, where $b = \langle a, x^* \rangle$ for all $(a, b)$, and set $F(x; (a, b)) = \frac{1}{1+\gamma} |\langle a, x - x^* \rangle|^{1+\gamma}$, so $\|F'(x; (a, b))\|^2 = \|a\|^2 + \|a\|^2 |\langle a, x - x^* \rangle|^{2\gamma}$. If $a \sim \mathcal{N}(0, I_n)$, then
\( |\langle a, x - x^* \rangle| \geq \frac{1}{2} \|x - x^*\|_2 \) with probability at least \( \frac{3}{5} \), and similarly \( \|a\|_2^2 \leq 2n \) with probability at least \( \frac{2}{5} \), so that both occur with probability at least \( \frac{1}{5} \). We then obtain
\[
\mathbb{E} \left[ F(x; S) \min \left\{ \alpha, \frac{F(x; S)}{\|F(x; S)\|_2^2} \right\} \right] \geq 1 \|x - x^*\|_2^2 \min \left\{ \alpha, \|x - x^*\|_2^{-\gamma} \right\} 2^{1+\gamma(1+\gamma)/2n} \min \left\{ 1, \|x - x^*\|_2^{-\gamma} \right\} ,
\]
so that Assumption 4 holds with \( \lambda_0 \geq \frac{1}{5(1+\gamma)2^{1+\gamma}} \) and \( \lambda_1 \geq \frac{1}{2^{1+\gamma}(1+\gamma)n} \).

To give stronger convergence results under Assumption 4, we require one additional condition on our models, which Asi and Duchi [3] introduce:

\( \text{(C.iii) For all } s \in S, \text{ the models } F_x(\cdot; s) \text{ satisfy: } F_x(y; s) \geq \inf_{z \in \mathcal{X}} F(z; s). \)

In minibatch settings, where one considers a batch \( S^{1:m} \) of samples in each model, the condition (C.iii) can be somewhat challenging to verify, as it requires accuracy for the average \( \inf_{z} F(z; s^{1:m}) \), though (obviously) proximal methods (4) satisfy this condition, and in typical situations (e.g. linear regression) where the batch size \( m \leq n \), the average of truncated models (10) will be similarly accurate.

**Corollary 4.1.** Let Assumption 4 hold, and let \( x_k \) be generated by the stochastic iteration (2) for a model satisfying conditions (C.i)–(C.iii). Take stepsizes \( \alpha_k = \alpha_0 k^{-\beta} \) for some \( \beta \in [0, 1] \). Define \( K_0 := \lceil (\lambda_0 \alpha_0 / (\lambda_1 \text{dist}(x_1, \mathcal{X}^{1-\gamma})) \rceil^{1/\beta} \). Then
\[
\mathbb{E}[\text{dist}(x_{k+1}, \mathcal{X}^*)^2] \leq \exp \left( -\lambda_1 \min \{k, K_0\} - \frac{\lambda_0}{\text{dist}(x_1, \mathcal{X}^{1-\gamma})} \sum_{i=K_0+1}^k \alpha_i \right) \text{dist}(x_1, \mathcal{X}^*)^2.
\]

**Proof** 
Let \( D_k = \text{dist}(x_k, \mathcal{X}^*) \) and \( F_k = \sigma(S_1, \ldots, S_k) \) be the \( \sigma \)-field generated by the first \( k \) samples \( S_i \). Then Lemma 4.1 of the paper [3] immediately yields
\[
\mathbb{E}[D_{k+1}^2 | F_k] \leq D_k^2 - \min \{\lambda_0 \alpha_k D_{k+1}^{1+\gamma}, \lambda_1 D_k^2 \}.
\]
As \( D_1 \geq D_k \) (again, by [3], Lemma 4.1), we in turn obtain
\[
\mathbb{E}[D_{k+1}^2 | F_k] \leq \max \left\{ 1 - \lambda_1, 1 - \lambda_0 \alpha_k / D_k^{1-\gamma} \right\} D_k^2.
\]
The remainder of the argument is algebraic manipulations, as in the proof of Proposition 2 from [3].

In the best case—when the stepsizes \( \alpha_k \uparrow \infty \) in Corollary 4.1—we achieve convergence scaling as \( \mathbb{E}[\text{dist}(x_k, \mathcal{X}^*)^2] \leq \exp(-\lambda_1 k) \text{dist}(x_1, \mathcal{X}^*)^2 \), and moreover (as we show in the next section) this dependence on the growth constant \( \lambda_1 \) is unimprovable. With this as motivation, one might hope that increased sampling (minibatching) might increase the growth constant \( \lambda_1 \) in Assumption 4; here we provide a sketch of such a result, which also makes it somewhat easier to check the conditions of Assumption 4, by giving three growth conditions.

\( \text{(G.i) There exists } \mu > 0 \text{ and a probability } p > 0 \text{ such that for all } x \in \mathcal{X}, \text{ we have } \mathbb{P}(F(x; S) - F(x^*; S) \geq \mu \text{dist}(x, \mathcal{X}^*)^{1+\gamma}) \geq p. \)
\( \text{(G.ii) The (sub)gradient } f' \text{ is } (L, \gamma)-\text{Holder continuous, meaning } \|f'(x) - f'(y)\|_2 \leq L \|x - y\|_2^2, \text{ and } 0 \in \partial f(x^*). \)
\( \text{(G.iii) There exists } \rho \text{ such that } \rho \geq \frac{\text{Var}(F(x; S))}{\|f'(x)\|_2^2} \text{ for all } x \in \mathcal{X}. \)

Our typical situation is to think of \( \mu \) and \( p \) numerical constants, where the scaling \( \rho \) measures the noise inherent to the problem. In any case, a short calculation shows how Conditions (G.i)–(G.iii) suffice to give Assumption 4.
Let conditions (G.i)–(G.iii) hold. Then the average $\mathcal{F}(x; s_{1:m}) = \frac{1}{m} \sum_{i=1}^{m} F(x; s^i)$ satisfies the $\gamma$-growth condition of Assumption 4 with

$$\lambda_0 = \frac{|mp|}{4m} \mu$$

and

$$\lambda_1 = \frac{(|mp|/m)^2 \mu^2}{16L^2(1 + \frac{2}{m})}.$$

**Proof** For shorthand, we assume w.l.o.g. that $F(x^*; S) = 0$ with probability 1. The event that $F(x; S^i) \geq \mu \text{dist}(x, x^*)^{1+\gamma}$ has probability at least $p$, and as the median of a Binomial$(m, p)$ distribution lies in $\{|mp|, |mp|\}$, we have

$$\mathbb{P}\left(\mathcal{F}(x; S^{1:m}) \geq \frac{|mp|}{m} \mu \text{dist}(x, x^*)^{1+\gamma}\right) \geq \frac{1}{2},$$

Thus, the event

$$A := \left\{\|\mathcal{F}'(x; S^{1:m})\|_2^2 \leq 4\mathbb{E}\left[\|\mathcal{F}'(x; S^{1:m})\|_2^2\right], \mathcal{F}(x; S^{1:m}) \geq \frac{|mp|}{m} \mu \text{dist}(x, x^*)^{1+\gamma}\right\}$$

satisfies

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) \geq 1 - \mathbb{P}\left(\|\mathcal{F}'(x; S^{1:m})\|_2^2 \geq 4\mathbb{E}\left[\|\mathcal{F}'(x; S^{1:m})\|_2^2\right]\right) - \frac{1}{2} \geq \frac{1}{4},$$

where we use inequality (14). We also have

$$\mathbb{E}\left[\|\mathcal{F}'(x; S^{1:m})\|_2^2\right] = \|f'(x)\|_2^2 \left(1 + \frac{\text{Var}(F'(x; S))}{\|f'(x)\|_2^2}\right) \leq \left(1 + \frac{\rho}{m}\right) \|f'(x)\|_2^2$$

$$\leq \left(1 + \frac{\rho}{m}\right) L^2 \text{dist}(x, x^*)^{2\gamma},$$

where we have used Conditions (G.iii) and (G.ii). Applying these observations gives

$$\mathbb{E}\left[\min\left\{\alpha\mathcal{F}(x; S^{1:m}), \frac{\mathcal{F}(x; S^{1:m})^2}{\|\mathcal{F}(x; S^{1:m})\|_2^2}\right\}\right]$$

$$\geq \frac{1}{4} \min\left\{\alpha \frac{|mp|}{m} \mu \text{dist}(x, x^*)^{1+\gamma}, \frac{(|mp|/m)^2 \mu^2 \text{dist}(x, x^*)^{2+2\gamma}}{4\mathbb{E}[\|\mathcal{F}'(x; S^{1:m})\|_2^2]}\right\}$$

$$\geq \frac{1}{4} \min\left\{\alpha \frac{|mp|}{m} \mu \text{dist}(x, x^*)^{1+\gamma}, \frac{(|mp|/m)^2 \mu^2 \text{dist}(x, x^*)^{2}}{4L^2(1 + \frac{2}{m})}\right\},$$

as desired. \hfill \Box

In brief, we see that mini-batches of size $m$ suggest improved convergence related to the noise-to-signal ratio $\rho := \sup_x \frac{\text{Var}(F'(x; S))}{\|f'(x)\|_2^2}$; once the sample size $m$ is large enough that $\rho / m \ll 1$, we expect relatively little improvement, though we do see a linear improvement in the growth constant $\lambda_1$ as $m$ grows whenever $m \ll \rho$. To see this, let us for simplicity assume that in Conditions (G.i)–(G.iii) we have $p \gtrsim 1$ and $L / \mu \lesssim 1$ (that is, the problem is well-conditioned). Then applying Corollary 4.1, we see that for large enough stepsizes $\alpha$,

$$k = O(1) \left(1 + \frac{\rho}{m}\right) \log \frac{1}{\epsilon}$$

iterations of any model-based method (2) with minibatches of size $m$—assuming that Conditions (C.i)–(C.iii) hold for the models $\mathcal{F}_x$—are sufficient to guarantee $\mathbb{E}[	ext{dist}(x_k, x^*)^2] \leq \epsilon$. 

\(\text{(15)}\)
5. Optimality in Interpolation Problems

We conclude the theoretical portion of this paper by developing several new optimality results for interpolation problems, that is, those satisfying Definition 4.1. In brief, we shall show that the dependence of Corollary 4.1 on the growth constant \( \lambda_1 \) is sharp and unimprovable, and that in some cases, the dependence on the signal-to-noise ratio \( \rho^{-1} := \inf_x \frac{\|f'(x)\|_2}{\text{Var}(F(x;S))} \) is essentially sharp as well. We do so via information-theoretic lower bounds on estimation of optimal points, the first in a stylized \( n = 1 \) dimensional problem that gives the correct dependence on the growth constants in Assumption 4, the second in standard regression problems but where we choose the dimension \( n \in \mathbb{N} \) more carefully.

We define our minimax risk as follows. Let \( \mathcal{P} \) be a family of problems, where a problem is a pair \((F, \mathcal{P})\) consisting of a probability distribution \( \mathcal{P} \) supported on \( \mathcal{S} \) and function \( F \) as defined in the introduction. We let \( \mathcal{X}^*(F, \mathcal{P}) = \arg\min_{x \in \mathcal{X}} \mathbb{E}_{\mathcal{P}}[F(x; S)] \) be the collection of minimizers, and define the minimax squared error

\[
\mathcal{M}_k(\mathcal{P}, \mathcal{X}) := \inf_{\tilde{x}^k} \sup_{(F, \mathcal{P}) \in \mathcal{P}} \mathbb{E}_{\mathcal{P}} \left( \text{dist}(\tilde{x}^k, \mathcal{X}^*(F, \mathcal{P}))^2 \right),
\]

where the infimum is over all measurable \( \tilde{x}^k : \mathcal{S}^k \to \mathbb{R}^n \), the supremum is over problems \((F, \mathcal{P}) \in \mathcal{P}\), and the inner expectation is over the samples \( S_1, \ldots, S_k \overset{iid}{\sim} \mathcal{P} \).

5.1. A lower bound for one-dimensional problems. We first focus on problems for which we can isolate the contributions of the growth constant \( \lambda_1 \) in Assumption 4, letting the dimension \( n = 1 \) to show that our complexity bounds hold independent of dimension; higher dimensions can only yield increased complexity. We consider a collection of well-conditioned problems, where we analogize the typical condition number of \( f \) by defining

\[
\lambda_\gamma(f) := \inf_{x \not\in \mathcal{X}^*} \frac{\lambda}{{(1 + \gamma)}^{\frac{1}{1 + \gamma}}} \text{dist}(x, \mathcal{X}^*)^{1 + \gamma} \quad \text{and} \quad L_\gamma(f) := \sup_{x \not= y} \frac{|f'(x) - f'(y)|}{|x - y|^{\gamma}},
\]

calling \( \kappa_\gamma(f) := \frac{\lambda_\gamma}{L_\gamma} \) the condition number. We also note in passing that the constant \( \lambda_1 \leq 1 \) in Assumption 4, as by convexity we have

\[
\frac{(F(x; S) - F(x^*; S))^2}{F'(x; S)^2} \leq \frac{F'(x; S), x - x^*)^2}{F'(x; S)^2} \leq |x - x^*|^2,
\]

so taking \( \alpha \uparrow \infty \) in Assumption 4 guarantees \( \lambda_1 \in [0, 1] \). Thus, for our first collection of problems, we let

\[
\mathcal{P}_\gamma(\lambda_1)
\]

be those problems satisfying Assumption 4 with a given \( \gamma, \lambda_1 \in [0, 1] \), any \( \lambda_0 \geq \lambda_1 \), our standing assumption of the interpolation condition in Definition 4.1, and condition number \( \kappa_\gamma(f) = 1 \).

The choice of the condition number serves to highlight the difficulties from stochasticity in the problem, eliminating the contributions of hardness from the population (deterministic) objective \( f \); an identical lower bound will of course hold in the coming theorem for more poorly conditioned problems with \( \kappa_\gamma(f) \geq 1 \), as this is simply a larger collection.

**Theorem 4.** Let \( \mathcal{P}_\gamma(\lambda_1) \) be the collection \((17)\), assume that \( \mathcal{X} \) contains an \( \ell_2 \)-ball of radius \( R \geq 0 \). Then

\[
\mathcal{M}_k(\mathcal{P}_\gamma(\lambda_1), \mathcal{X}) \geq \frac{R^2}{2} \left[ 1 - (1 + \gamma)^2 \lambda_1 \right]^{\frac{k}{2}}.
\]

We make a few remarks before proceeding to the proof. First, the convergence guarantees in Section 4.2 show that appropriate model-based methods converge to \( \epsilon \) accuracy in \( O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \) iterations, which by the theorem is optimal. Thus, in a strong sense, the *a priori* esoteric-seeming growth condition in Assumption 4 is indeed fundamental.
Proof. Let $P = P_\gamma(\lambda_1)$ for short, and assume w.l.o.g. that $\lambda_1 \leq 1/(1+\gamma)^2$, as the result is trivial otherwise. We base our argument on Le Cam’s two point method (see, e.g., [44], Eq. (15.14)).

We consider two probability distributions $P_1, P_{-1}$, and let $X^*_v$ be (for now) arbitrary sets indexed by $v \in \{-1, 1\}$. Then recall the variation distance $\|P - Q\|_{TV} = \sup_A |P(A) - Q(A)|$ between distributions $P$ and $Q$, we have Le Cam’s two-point method:

**Lemma 5.1 (Le Cam).** Let $\hat{x}^k$ be an arbitrary function of $S_1, \ldots, S_k$. Then

$$\max_{v \in \{-1, 1\}} \mathbb{E}_{P_k^v} \left[ \text{dist}(\hat{x}^k, X^*_v)^2 \right] \geq \frac{1}{8} \text{dist}(X^*_{-1}, X^*_1)^2 \left( 1 - \|P_{-1}^k - P_1^k\|_{TV} \right).$$

To use Lemma 5.1 to lower bound the minimax risk it suffices to choose a pair of problems $(F, P_v) \in \mathcal{P}$ whose optimal sets are well-separated and apply the lemma. To that end, let $\delta \in (0, 1)$ to be chosen later, and consider the choices

$$(18) \quad P_{-1} : \begin{cases} S = 0 & \text{w.p. 1} - \frac{\delta}{2} \\ S = -1 & \text{w.p. } \frac{\delta}{2} \end{cases} \quad P_1 : \begin{cases} S = 0 & \text{w.p. 1} - \frac{\delta}{2} \\ S = 1 & \text{w.p. } \frac{\delta}{2}. \end{cases}$$

Our functions $F$ are trivial to construct: given the radius $R$, we define

$$(19) \quad F(x; 1) = \frac{1}{1 + \gamma} |x - R|^{1+\gamma}, \quad F(x; -1) = \frac{1}{1 + \gamma} |x + R|^{1+\gamma}, \quad F(x; 0) = 0.$$ 

The intuition here is that given a sample $S \in \{-1, 0, 1\}$, we either completely identify the distribution or receive no information.

It remains to show that the pairs $(F, P_v) \in \mathcal{P}$ and to bound the variation distance $\|P_{-1}^k - P_1^k\|_{TV}$. For the latter, we have

**Lemma 5.2.** Let $P_{-1}, P_1$ be as in Eq. (18). Then $\|P_{-1}^k - P_1^k\|_{TV} = 1 - (1 - \delta)^k$.

Proof. For any distributions $P, Q$, with densities $p, q$ w.r.t. a base measure $\mu$, we have $\|P - Q\|_{TV} = P(p > q) - Q(p > q)$. For $P_{-1}, P_1$ as above, we thus have

$$\|P_{-1}^k - P_1^k\|_{TV} = P_1^k(\text{there exists } i \in [k] \text{ s.t. } S_i = 1)$$

$$= 1 - P_1(S_1 = 0, \ldots, S_k = 0) = 1 - (1 - \delta)^k.$$ 

Now, consider the functions

$$f_v(x) := \mathbb{E}_{P_v}[F(x; S)] = \frac{\delta}{1 + \gamma} |x - vR|^{1+\gamma}.$$ 

We have $\kappa_\gamma(f) = 1$, so that the problem is well-conditioned, and the optimal sets $X^*_v := \arg\min_{x \in X} f_v(x)$ are the singletons $X^*_v = \{x^*_v = vR\}$. Additionally, we have

$$\mathbb{E}_v \left[ (F(x; S) - F(x^*_v; S)) \min \left\{ \alpha, \frac{F(x; S) - F(x^*_v; S)}{\|F(x; S)\|_2} \right\} \right]$$

$$= \frac{\delta}{1 + \gamma} |x - vR|^{1+\gamma} \min \left\{ \alpha, \frac{|x - vR|^{1+\gamma}}{(1 + \gamma)^2 |x - vR|^{2\gamma}} \right\}$$

$$= \min \left\{ \frac{\delta \alpha}{1 + \gamma}, \frac{\delta}{(1 + \gamma)^2} \text{dist}(x, X^*_v)^{1-\gamma} \right\} \text{dist}(x, X^*_v)^{1+\gamma},$$

so by choosing $\delta = (1 + \gamma)^2 \lambda_1 \leq 1$, our problems problems $(F, P_v)$ belong to $\mathcal{P}_{\gamma}(\lambda_1)$. Le Cam’s Lemma 5.1 and the variation distance bound in Lemma 5.2 imply that

$$\max_{v \in \{\pm 1\}} \mathbb{E}_{P_v} \left[ |\hat{x}^k - x^*_v|^2 \right] \geq \frac{1}{8} |x_1^* - x_{-1}^*|^2 (1 - \delta)^k \geq \frac{R^2}{2} (1 - \delta)^k.$$
Substituting δ = (1 + γ)2λ1 gives the result. □

5.2. A lower bound for well-conditioned regression problems. The proof of Theorem 4 relies on constructing certain power functions and a very careful choice of growth and probability. An alternative approach is to mimic those ideas in proving complexity results for deterministic problems [32, 33, 11], where one takes the dimension larger. By allowing high-dimensional problems, we can show that the noise-to-signal ratio ρ := sup_x Var(F(x; S)) / ∥∇f(x)∥^2 and growth constant λ1 from Assumption 4 remain fundamental, even in noiseless linear regression.

To make the proof cleaner we make a slight modification to the class of problems we consider: instead of assuming a bounded domain X, we instead assume X = R^n, but now we consider a randomized (instead of minimax/worst case) adversary that chooses a problem (F, P) ∈ P according to a measure π on the space of problems; in particular, we assume that E[∥x_0 − x^*∥^2] ≤ R^2, that is, the expected distance of x_0 to x^* is at most R. Letting X^*(F, P) = argmin_x E_P[F(x; S)] be the optimal set for a given problem (F, P), we then define the minimum average risk

$$\mathcal{M}_k(P, π) := \inf_{x^k} \int E_P[\text{dist}(x^k, X^*(F, P))^2] dπ(F, P).$$

We note that the minimum average risk defined here naturally lower bounds the minimax risk (16), redefined analogously for our problem.

We specialize this randomized risk for each n ∈ N, letting P_n be a collection of noiseless linear regression problems on R^n, where we identify the prior measure π with x^* ∼ N(0, R^2/n I_n×n). Then certainly E[∥x^*∥^2] = R^2. We consider samples s consisting of a pair A ∈ R^{m×n} and b = Ax^*, considering the quadratic loss

$$E(x; s) = E(F(x; S)) = \frac{1}{2} ∥Ax - b∥^2_2,$$

and we call the resulting objective f(x) = E[F(x; S)] perfectly conditioned if f(x) = c ∥x − x^*∥^2 for a constant c ∈ R_+. We have the following theorem.

**Theorem 5.** Let λ_1 ∈ [0, 1/4] and γ = 1. Then there exists a collection P of perfectly conditioned interpolating problems with squared error (20), satisfying Assumption 4 and E[∥x^*∥^2] = R^2, such that

$$\mathcal{M}_k(P, π) \geq R^2 \left(1 - 4λ_1\right)^k.$$  

Alternatively, let ρ ∈ [1, ∞]. There exists a collection P of perfectly conditioned interpolating problems with squared error (20), with noise-to-signal ratio satisfying sup_x Var(∇F(x; S)) / ∥∇f(x)∥^2 ≤ ρ, such that

$$\mathcal{M}_k(P, π) \geq R^2 \left(1 - \frac{1}{ρ}\right)^k.$$  

Thus, one cannot hope to achieve (much) better convergence even for quadratics than that we have outlined: the dependence on either the growth λ_1 or the signal-to-noise ρ^{-1} is unavoidable, and one must collect at least k ≥ 1/M log 1/λ or k ≥ ρ log 1/ε samples S to achieve accuracy ε, again highlighting that these quantities—as we (inspired by Asi and Duchi [3]) identify in Corollary 4.1 and the iteration bound (15)—are fundamental for interpolation problems.

**Proof** Let U = [u_1 · · · u_n] ∈ R^{m×n} be an arbitrary orthogonal matrix, so U^TU = UU^T = I_n. Let P_n be the collection of linear regression problems with data matrices A ∈ R^{m×n} chosen by taking m ≤ n columns (u_{i(1)}, · · · , u_{i(m)}) of U uniformly at random and setting A = \sqrt{n/m}[u_{i(1)} · · · u_{i(m)}]^T, so that E[A^TA] = I_n and (A^TA)^2 = (n/m)A^TA, and let b = Ax^*, where x^* ∼ π = N(0, R^2/n I_n) follows a Gaussian prior. Each observation S_t corresponds to releasing (perfectly) a random linear
projection of $x^*$, so that given the $k$ observations, if we let $C_k = [A_1 \cdots A_k] \in \mathbb{R}^{n \times mk}$ denote the concatenated data matrix after $k$ observations, the posterior on $x^*$ is

$$x^* \mid (S_1, \ldots, S_k) \sim \mathcal{N} \left( \mathbb{E}[x^* \mid S_1, \ldots, S_k], \frac{R^2}{n} (I_n - C_k(C_k^T C_k)^{-1} C_k^T) \right),$$

that is, the covariance projects out $C_k$. By a standard Bayesian argument,

\begin{equation}
\inf_{\tilde{x}^k} \mathbb{E} \left[ \|\tilde{x}^k - x^*\|^2_2 \right] = \mathbb{E} \left[ \|\mathbb{E}[x^* \mid S_k^\dagger] - x^*\|^2_2 \right] = R^2 \mathbb{E} \left[ \frac{n - \text{rank}(C_k)}{n} \right],
\end{equation}

as $I_n - C_k(C_k^T C_k)^{-1} C_k^T$ is a rank $n - \text{rank}(C_k)$ projection matrix. Let $r_k = \text{rank}(C_k)$ for shorthand. Then we may compute $\mathbb{E}[r_k]$ exactly by noting that

$$\mathbb{E}[r_k \mid r_{k-1}] = r_{k-1} + m \frac{n - r_{k-1}}{n} = \left(1 - \frac{m}{n}\right) r_{k-1} + m,$$

so that with $r_1 = m$ we obtain

$$\mathbb{E}[r_k] = m \sum_{i=1}^{k} \left(1 - \frac{m}{n}\right)^{k-i} = m \frac{1 - (1 - m/n)^k}{1 - (1 - m/n)} = n - n \left(1 - \frac{m}{n}\right)^k,$$

and substituting this into expression (21) gives

\begin{equation}
\inf_{\tilde{x}^k} \mathbb{E} \left[ \|\tilde{x}^k - x^*\|^2_2 \right] = R^2 \left(1 - \frac{m}{n}\right)^k.
\end{equation}

We now use expression (22) to prove the two results in the theorem. For the first, we note that for $s = (A, b)$, we have $\nabla F(x; s) = A^T(Ax - b) = A^TA(x - x^*)$, and as $(A^TA)^2 = \frac{n}{m} A^TA$ by construction and $\mathbb{E}[A^TA] = I_n$,

$$\mathbb{E} \left[ (F(x; S) - F(x^*; S)) \min \left\{ \alpha, \frac{F(x; S) - F(x^*; S)}{\|\nabla F(x; S)\|^2_2} \right\} \right] = \mathbb{E} \left[ \min \left\{ \frac{\alpha}{2} \|A(x - x^*)\|^2_2, \frac{\|A(x - x^*)\|^4_2}{4 \|A^TA(x - x^*)\|^2_2} \right\} \right] = \min \left\{ \frac{\alpha}{2}, \frac{m}{4n} \right\} \|x - x^*\|^2_2.$$

In particular, we can choose $m, n$ so that $\frac{m}{n} \geq \lambda_1$ the problem satisfies Assumption 4 with $\gamma = 1$ and $\lambda_0 = \frac{1}{2}$. This gives the first result by substituting into expression (22) and taking $m, n$ so that $\frac{m}{n}$ is arbitrarily close to $4\lambda_1$.

For the second result, we recognize the noise-to-signal ratio

$$\frac{\text{Var}(\nabla F(x; S))}{\|\nabla f(x)\|^2_2} \leq \frac{n}{m} \frac{\|x - x^*\|^2_2}{\|x - x^*\|^2_2} = \frac{n}{m},$$

Making appropriate substitutions by taking $\frac{n}{m} \leq \rho$ gives the second lower bound. \qed

6. Experiments

Our goal now is to study and demonstrate the speedup and robustness of AProx methods with minibatches, comparing the relative performance of the proposed methods on several benchmark stochastic optimization problems. We consider the following five methods in our experiments, where we use both single sample ($m = 1$) and minibatch ($m > 1$) versions:

(1) SGM: stochastic gradient methods, i.e., the linear model (3).
(2) Proximal: full proximal model (4) with averaged function (7).
(3) IA: truncated model (5) with naive iterate averaging (6).
(4) TruncAv: iterates via truncating the averaged linear model, update (9).
(5) AvTrunc: iterates defined an average of truncated models, using updates (10).
For our experiments, we use stepsizes $\alpha_k = \alpha_0 k^{-1/2}$, varying $\alpha_0$, and for each algorithm $a$ report the number $T_{a,m}(\alpha_0)$ of total samples consumed—as a proxy for time—to reach $\varepsilon$ accuracy using minibatches of size $m$; that is, $T_{a,m}(\alpha_0) = km$ where $k$ is the first iteration to satisfy $f(x_k) - f(x^*) \leq \varepsilon$. We also let $T_{a,m}^* = \min_{\alpha_0} T_{a,m}(\alpha_0)$ denote the smallest time to convergence for a method $a$ using batch size $m$. Each of our experiments involves data $(A,b) \in \mathbb{R}^{N \times n} \times \mathbb{R}^N$, where $f_{A,b}(x) = \frac{1}{N} \sum_{i=1}^N F(x; a_i, b_i)$ for a given loss $F$, and we vary the condition number of $A$, taking $N = 10^3$ and $n = 40$. With these values identified, we present three types of results, focusing on results that allow a more careful accounting for the robustness of the various methods:

1. **Performance profiles** [16]: For each method $a$, we evaluate for each $r \geq 1$ the fraction of the total executed experiments for which the $T_{a,m}(\alpha_0) \leq r T_{a,m}^*(\alpha_0)$, where $a^*$ is the best performing method in each experiment, giving $r$ on the horizontal axis and the proportion on the vertical. Here, to evaluate robustness, we define a single experiment as one execution of each of the 5 methods for a particular step size $\alpha_0$, minibatch size $m$, and condition number combination. We discard the experiments where more than 3 of the methods fail to complete before the max number of iterations.

2. **Best speedups for minibatching**: For each method $a$, we plot $T_{a,m}^*/T_{a,m}^*$ against the minibatch size $m$ to show the speedup minibatching provides using the best step sizes. This shows the best possible speedup obtained by minibatching through tuning the initial step size $\alpha_0$.

3. **Time to solution w.r.t. step-size**: For each method $a$ and minibatch size $m$, we plot $T_{a,m}(\alpha_0)$ against the initial step size $\alpha_0$.

We use minibatch sizes $m \in \{1, 4, 8, 16, 32, 64\}$ and initial steps $\alpha_0 \in \{10^{i/2}, i \in \{-4, -3, \ldots, 5\}\}$. For all experiments we run 30 trials with different seeds and plot the 95% confidence sets. We describe the objective function and noise mechanism for each problem in the respective subsections.

### 6.1. Linear Regression

We have $f(x) = \frac{1}{2N} \|Ax - b\|_2^2$. For each experiment we generate rows of $A$ and $x^*$ i.i.d. $\mathcal{N}(0, I_n)$ and, setting $b = Ax^* + \sigma v$ with $v \sim \mathcal{N}(0, I_N)$. In the noisy setting for our experiments, we set $\sigma = 0.5$. Figure 2 outlines the performance profiles for the linear regression experiments. The fully proximal, AvTrunc, and TruncAv methods are noticeably better than IA and SGM. Fig. 3 also reflects this behavior, where the accelerated fully proximal, AvTrunc, and TruncAv methods are more robust to initial step size choice.

![Performance profiles for linear regression.](image)

**Figure 2**: Performance profiles for linear regression.

### 6.2. Absolute loss regression

We have $f(x) = \frac{1}{2N} \|Ax - b\|_1$. Again we generate rows of $A$ and $x^*$ i.i.d. $\mathcal{N}(0, I_n)$, setting $b = Ax^* + \sigma v$ and drawing $v \sim \text{Lap}(1)^N$. In the noisy setting for our experiments, we set $\sigma = 0.5$. We provide performance profiles for the non-accelerated
and accelerated algorithms in fig. 4. Similar to the linear regression setting, we see that AvTrunc, TruncAv, and full-prox, outperform IA and SGM. In fig. 5, we plot the speedup of each algorithm (relative to minibatch size $m = 1$) against minibatch size in the noiseless setting. Here, we see the linear improvement in convergence rate our theoretical results predict, but there is a superlinear region for large minibatches $m > 32$; while our theory does not predict this, this is because once $m \geq n$, a single step of the stochastic proximal point method can perfectly solve the problem.

6.3. Logistic Regression. We have $f(x) = \frac{1}{2N} \sum_{i=1}^{N} \log(1+\exp(-b_i\langle a_i, x \rangle))$. We generate rows of $A$ and $x^\star$ i.i.d. $N(0, I_n)$, setting $b_i = \text{sign}(\langle a_i, x^\star \rangle)$. To add noise, we flip each label $b_i$ independently with probability $p = .01$. We again plot performance profiles in fig. 6. The fully proximal, AvTrunc, and TruncAv methods are noticeably more robust than IA and SGM. In the non-accelerated case,
AvTrunc and TruncAv even outperform the fully proximal method, whereas the fully proximal method outperforms AvTrunc and TruncAv in the accelerated case. This performance boost from acceleration especially for the stochastic proximal-point methods may be worthy of further investigation.

![Performance profiles for logistic regression.](image1)

**Figure 6:** Performance profiles for logistic regression.

![Time to convergence of the non-accelerated methods vs. initial stepsizes for noiseless logistic regression](image2)

**Figure 7:** Time to convergence of the non-accelerated methods vs. initial stepsizes for noiseless logistic regression.

### Appendix A. Proofs of non-asymptotic upper bounds

We collect our proofs of Theorem 1 and 2 in this section. Both rely on a standard claim on minimizers of sums of convex functions, which we state and prove here for convenience.

**Claim A.1.** Let $u$ and $\psi$ be convex, $\psi$ be differentiable on $\mathcal{X}$, and $D_\psi(x,y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle$. If $x^+$ minimizes $u(x) + \psi(x)$ over $x \in \mathcal{X}$, then

$$u(x^+) + \psi(x^+) \leq u(x) + \psi(x) - D_\psi(x, x^+) \text{ for all } x \in \mathcal{X}.$$  

**Proof.** By convexity and the optimality of $x^+$, there exists $u'(x^+) \in \partial u(x^+)$ such that $\langle u'(x^+) + \nabla \psi(x^+), x - x^+ \rangle \geq 0$ for all $x \in \mathcal{X}$. Using the standard first-order convexity inequality, we thus obtain

$$u(x) \geq u(x^+) + \langle u'(x^+), x - x^+ \rangle$$

$$= u(x^+) + \langle u'(x^+) + \nabla \psi(x^+), x - x^+ \rangle - \langle \nabla \psi(x^+), x - x^+ \rangle$$

$$\geq u(x^+) - \langle \nabla \psi(x^+), x - x^+ \rangle$$

$$= u(x^+) + \psi(x^+) - \psi(x) + D_\psi(x, x^+),$$

as desired. \qed
A.1. Proof of Theorem 1. The key to the proof, as is familiar from other analyses of such methods [46, 31, 24, 13, 3], is a one-step progress bound.

Lemma A.1. Let the conditions of Theorem 1 hold, and define the function value errors $e_k = [F(x^*; S_k) - f(x^*)] - [F(x^*_k; S_k) - f(x_k)]$. Then

$$f(x_{k+1}) - f(x^*) \leq \frac{1}{\alpha_k} \left[ D_h(x^*, x_k) - D_h(x^*, x_{k+1}) \right] + e_k + \frac{1}{2\eta_k} \| \nabla F(x_k; S_k) - \nabla f(x_k) \|^2.$$  

Proof Setting $u(\cdot) = F_{x_k}(\cdot; S_k)$ and $\psi(x) = \frac{1}{\alpha_k} D_h(x, x_k)$ in Claim A.1, and taking $x^+ = x_{k+1}$ and $x = x^*$, we have the progress bound

$$F_{x_k}(x_{k+1}; S_k) + \frac{1}{\alpha_k} D_h(x_{k+1}, x_k) \leq F_{x_k}(x^*; S_k) + \frac{1}{\alpha_k} \left[ D_h(x^*, x_k) - D_h(x^*, x_{k+1}) \right].$$  

We turn to bounding the difference $F_{x_k}(x^*; S_k) - F_{x_k}(x_{k+1}; S_k)$. Let $g_k = \nabla F(x_k; S_k)$ and define the gradient error $e_k := g_k - \nabla f(x_k)$. Using the convexity of $F_{x_k}(\cdot; S_k)$ and recalling that $g_k \in \partial F_{x_k}(x_k; S_k)$ as in our discussion following Condition (C.ii), we have $F_{x_k}(x_{k+1}; S_k) \geq F_{x_k}(x_k; S_k) + \langle g_k, x_{k+1} - x_k \rangle$. As a consequence, we have

$$F_{x_k}(x^*; S_k) - F_{x_k}(x_{k+1}; S_k) \leq F_{x_k}(x^*; S_k) - F(x_k; S_k) + \langle g_k, x_k - x_{k+1} \rangle$$

$$= F_{x_k}(x^*; S_k) - F(x_k; S_k) + \langle \nabla f(x_k), x_k - x_{k+1} \rangle + \langle \xi_k, x_k - x_{k+1} \rangle$$

$$\leq f(x^*) - f(x_k) + \langle \nabla f(x_k), x_k - x_{k+1} \rangle + e_k + \langle \xi_k, x_k - x_{k+1} \rangle,$$

where we used the error $e_k = [F(x^*; S_k) - f(x^*)] - [F(x_k; S_k) - f(x_k)]$. Finally, the smoothness of $f$ implies $f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \| x_k - x_{k+1} \|^2$, so

$$F_{x_k}(x^*; S_k) - F_{x_k}(x_{k+1}; S_k) \leq f(x^*) - f(x_k) + \frac{L}{2} \| x_k - x_{k+1} \|^2 + e_k + \langle \xi_k, x_k - x_{k+1} \rangle.$$

Substituting this into inequality (23) and rearranging, we obtain

$$f(x_{k+1}) - f(x^*) \leq \frac{1}{\alpha_k} \left[ D_h(x^*, x_k) - D_h(x^*, x_{k+1}) - D_h(x_k, x_{k+1}) \right] + e_k + \langle \xi_k, x_k - x_{k+1} \rangle + \frac{L}{2} \| x_k - x_{k+1} \|^2.$$  

We apply the Fenchel-Young inequality to control the error $\langle \xi_k, x_k - x_{k+1} \rangle$: we have $\langle \xi_k, x_k - x_{k+1} \rangle \leq \frac{1}{2\eta_k} \| \xi_k \|^2 + \frac{\eta_k}{2} \| x_k - x_{k+1} \|^2$, so

$$f(x_{k+1}) - f(x^*) \leq \frac{1}{\alpha_k} \left[ D_h(x^*, x_k) - D_h(x^*, x_{k+1}) \right] + e_k + \frac{1}{2\eta_k} \| \xi_k \|^2 + \frac{L + \eta_k}{2} \| x_k - x_{k+1} \|^2 - \frac{1}{\alpha_k} D_h(x_k, x_{k+1}),$$

which with $\alpha_k = \frac{1}{L + \eta_k}$ gives the lemma once we apply the strong convexity of $h$, that is, that $D_h(x_k, x_{k+1}) \geq \frac{1}{2} \| x_k - x_{k+1} \|^2$. □
Lemma A.2. We have the one-step progress bound

\[
\sum_{i=1}^{k} [f(x_{i+1}) - f(x^*)] \leq \sum_{i=2}^{k} \left( \frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}} \right) D_h(x^*, x_i) - \frac{1}{\alpha_{k+1}} D_h(x^*, x_{k+1})
+ \frac{1}{\alpha_1} D_h(x^*, x_1) + \sum_{i=1}^{k} e_i + \sum_{i=1}^{k} \frac{1}{2\eta_i} \|\nabla F(x_i; S_i) - \nabla f(x_i)\|^2.
\]

Taking expectations and using that \(E[e_k] = 0\) and \(\alpha_k = \frac{1}{L+\eta_k}\) gives the theorem.

A.2. Proof of Theorem 2. The proof is somewhat analogous to that of Theorem 1, in that we begin with a deterministic one-step progress bound and then iterate the bound. In analogy to Lemma A.1, we rely on the conditionally mean-zero function and gradient errors

\[ e_k := F(x^*; S_k) - f(x^*) + f(y_k) - F(y_k; S_k) \text{ and } \xi_k := \nabla f(y_k) - \nabla F(y_k; S_k). \]

We have the one-step progress bound

Lemma A.2. Let \(\alpha_k \leq \frac{1}{L\eta_k + \eta_k}\) and \(\Delta_k = f(x_k) + r(x_k) - f(x^*) - r(x^*)\). Then

\[
\Delta_{k+1} \leq (1 - \theta_k) \Delta_k + \theta_k \left[ e_k + \langle \xi_k, z_k - y_k \rangle + \frac{\|\xi_k\|^2}{2\eta_k} + \frac{1}{\alpha_k} (D_h(x^*, z_k) - D_h(x^*, z_{k+1})) \right].
\]

Proof. We follow the proof of Tseng, Proposition 1 [43]. For shorthand, let

\[ \text{lin}_f(x, y) := f(y) + \langle \nabla f(y), x - y \rangle + r(x), \]

which linearly approximates \(f\) and does not approximate the additive component \(r\). Then by the \(L\)-smoothness of \(\nabla f\), we obtain

\[
f(x_{k+1}) + r(x_{k+1}) \leq \text{lin}_f(x_{k+1}, y_k) + \frac{L}{2} \|x_{k+1} - y_k\|^2 \\
= \text{lin}_f((1 - \theta_k) x_k + \theta_k z_{k+1}, y_k) + \frac{L\theta_k^2}{2} \|z_{k+1} - z_k\|^2 \\
\overset{(i)}{\leq} (1 - \theta_k) \text{lin}_f(x_k, y_k) + \theta_k \text{lin}_f(z_{k+1}, y_k) + \frac{L\theta_k^2}{2} \|z_{k+1} - z_k\|^2 \\
\overset{(ii)}{\leq} (1 - \theta_k) (f(x_k) + r(x_k)) + \theta_k \left[ \text{lin}_f(z_{k+1}, y_k) + \frac{L\theta_k}{2} \|z_{k+1} - z_k\|^2 \right],
\]

where the inequality \((i)\) used that \(r\) is convex and \((ii)\) that \(f\) is convex.

We consider the final two terms in the bound \((25)\), and with function and gradient errors \(e_k^{(1)} := f(y_k) - F(y_k; S_k)\) and \(\xi_k := \nabla f(y_k) - \nabla F(y_k; S_k)\), we expand the first in terms of the random samples to write

\[
\text{lin}_f(z_{k+1}, y_k) = F(y_k; S_k) + \langle \nabla F(y_k; S_k), z_{k+1} - y_k \rangle + r(z_{k+1}) + e_k^{(1)} + \langle \xi_k, z_{k+1} - y_k \rangle \\
\leq F_{y_k}(z_{k+1}; S_k) + r(z_{k+1}) + e_k^{(1)} + \langle \xi_k, z_{k+1} - y_k \rangle,
\]

where the inequality uses that the models \(F_{y_k}\) necessarily upper bound the first-order (linear) approximation to \(F\) at \(y_k\) (recall the discussion following Condition \((C.ii)\)). To control term \((26)\), we
apply Claim A.1 with \( u(x) = F_{yk}(x;S_k) \), \( \psi(x) = D_h(x,z_k) \), and \( x^+ = z_{k+1} \), so that inequality (26) implies

\[
\ln f(z_{k+1}, y_k) \leq F_{yk}(x;S_k) + r(x) + \frac{1}{\alpha_k} \left[ D_h(x, z_k) - D_h(x, z_{k+1}) - D_h(z_{k+1}, z_k) \right] \\
+ e_k^{(1)} + \langle \xi, z_{k+1} - y_k \rangle
\]

for any \( x \in X \). Rearranging terms and using the Fenchel-Young inequality to see that

\[
\langle \xi, z_{k+1} - y_k \rangle = \langle \xi, z_k - y_k \rangle + \langle \xi, z_{k+1} - y_k \rangle \leq \langle \xi, z_k - y_k \rangle + \frac{\|\xi\|^2}{2\eta_k} + \frac{\eta_k}{2} \|z_{k+1} - z_k\|^2
\]

and using the strong convexity bound \( D_h(z_{k+1}, z_k) \geq \frac{1}{2} \|z_{k+1} - z_k\|^2 \) then implies

\[
\ln f(z_{k+1}, y_k) \leq F_{yk}(x;S_k) + r(x) + \frac{1}{\alpha_k} \left[ D_h(x, z_k) - D_h(x, z_{k+1}) \right] + e_k^{(1)} \\
+ \langle \xi, z_k - y_k \rangle + \frac{\|\xi\|^2}{2\eta_k} + \frac{\eta_k}{2} \|z_{k+1} - z_k\|^2 - \frac{1}{2\alpha_k} \|z_{k+1} - z_k\|^2.
\]

Our modeling assumptions guarantee that \( F_{yk}(x;S_k) \leq F(x;S_k) \), so writing the function error \( e_k = F(x;S_k) - f(x) + f(y_k) - F(y_k;S_k) \) and substituting this upper bound on \( \ln f(z_{k+1}, y_k) \) into the bound (25) gives the single-step progress guarantee

\[
f(x_{k+1}) + r(x_{k+1}) \leq (1 - \theta_k)(f(x_k) + r(x_k)) + \theta_k(f(x) + r(x)) \\
+ \theta_k \left[ e_k + \langle \xi, z_k - y_k \rangle + \frac{\|\xi\|^2}{2\eta_k} + \frac{1}{\alpha_k} \left[ D_h(x, z_k) - D_h(x, z_{k+1}) \right] \\
+ \frac{L\theta_k + \eta_k}{2} \|z_{k+1} - z_k\|^2 + \frac{1}{\alpha_k} \|z_{k+1} - z_k\|^2 \right].
\]

Any stepsize \( \alpha_k \leq \frac{1}{2\theta_k + \eta_k} \) cancels the the \( \|z_{k+1} - z_k\|^2 \) terms, and setting \( x = x^* \) gives the lemma.

Iterating Lemma A.2 with \( \Delta_k = f(x_k) + r(x_k) - f(x^*) - r(x^*) \geq 0 \) yields the following deterministic convergence guarantee.

**Lemma A.3.** Let the conditions of Theorem 2 hold. Define the error terms \( \zeta_k := e_k + \langle \xi, z_k - y_k \rangle + \frac{\|\xi\|^2 - \sigma_0^2}{2\alpha_k} \). Then

\[
\frac{1}{\theta_k^2} \left[ f(x_{k+1}) + r(x_{k+1}) - f(x^*) - r(x^*) \right] \leq \sum_{i=0}^{k} \frac{\sigma_0^2}{2\theta_i\eta_i} + \left( L + \frac{\eta_k}{\theta_k} \right) R^2 + \sum_{i=0}^{k} \frac{1}{\theta_i} \zeta_i.
\]

**Proof.** Lemma A.2 yields

\[
\frac{1}{\theta_k^2} \Delta_{k+1} \leq \frac{1 - \theta_k}{\theta_k^2} \Delta_k + \frac{1}{\theta_k \alpha_k} \left[ D_h(x^*, z_k) - D_h(x^*, z_{k+1}) \right] + \frac{\sigma_0^2}{2\eta_k \theta_k} \\
+ \frac{1}{\theta_k} \left[ e_k + \langle \xi, z_k - y_k \rangle + \frac{\|\xi\|^2 - \sigma_0^2}{2\alpha_k} \right]_{=\zeta_k}
\]

\[
\leq \frac{1}{\theta_k^2} \Delta_k + \frac{1}{\theta_k \alpha_k} \left[ D_h(x^*, z_k) - D_h(x^*, z_{k+1}) \right] + \frac{\sigma_0^2}{2\eta_k \theta_k} + \frac{1}{\theta_k} \zeta_k.
\]
where we recalled that $(1 - \theta_k) / \theta_k^2 \leq 1 / \theta_{k-1}^2$. Iterating the inequality and using $\frac{1}{\theta_0^2} = 0$, we find that

$$\frac{1}{\theta_k^2} \Delta_{k+1} \leq \sum_{i=0}^{k} \frac{\sigma_i^2}{2\eta_i\theta_i} + \sum_{i=0}^{k} \frac{1}{\theta_i\alpha_i} (D_h(x^*, z_i) - D_h(x^*, z_{i+1})) + \sum_{i=0}^{k} \frac{1}{\theta_i} \zeta_i.$$  

Rearranging the middle summation above as in the proof of Theorem 1 and noting that $\frac{1}{\theta_i\alpha_i} = L + \frac{\eta_i}{\theta_i}$ gives $\sum_{i=0}^{k} \frac{1}{\theta_i\alpha_i} (D_h(x^*, z_i) - D_h(x^*, z_{i+1}) \leq LR^2 + \frac{\eta_k}{\theta_k} R^2$, as desired. \[\square\]

Now take expectations in Lemma A.3. We have $E[\zeta_k] \leq 0$, and

$$\sum_{i=0}^{k} \frac{i + 2}{\sqrt{i + 1}} \leq \sum_{i=1}^{k+1} \sqrt{i} + \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$$

$$\leq \int_{1}^{k+2} \sqrt{t} dt + \int_{0}^{k+1} \frac{1}{\sqrt{t}} dt = \frac{2}{3} (k + 2)^{3/2} - 1) + 2\sqrt{k+1} \leq (k+2)^{3/2},$$

where inequality $(i)$ holds for $k > 2$. Multiplying by $\theta_k^2 = 4/(k + 2)^2$ and using $\eta_k\theta_k = \eta_0 \frac{2\sqrt{k}}{k+2} \leq 2\eta_0 / \sqrt{k}$ gives the deterministic bound

$$\theta_k^2 \sum_{i=0}^{k} \frac{\sigma_i^2}{2\theta_i\eta_i} + \theta_k^2 \left( L + \frac{\eta_k}{\theta_k} \right) R^2 \leq \frac{4LR^2}{(k + 2)^2} + \frac{2R^2\eta_0}{\sqrt{k}} + \frac{2\sigma^2}{\eta_0 \sqrt{k + 2}},$$

as desired.

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