Integrability test for evolutionary lattice equations of higher order

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Abstract

A generalized summation by parts algorithm is presented for solving of difference equations of the form $T^m(y) - a[u]y = b[u]$ where $T$ denotes the shift $u_j \rightarrow u_{j+1}$. Solvability of such type of equations with respect to coefficients of formal symmetry (or formal recursion operator) provides a convenient integrability test for evolutionary differential-difference equations $u_t = f(u_{-m}, \ldots, u_m)$. The algorithm is implemented in Mathematica.

Key words: Volterra type lattice, higher symmetry, conservation law, integrability test, summation by parts, computer algebra

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1 Introduction

Existence of an infinite set of higher symmetries is a characteristic property of integrable equations. For a given evolutionary equation $\partial_t(u) = f[u]$, it implies solvability of the Lax equation

$$D_t(G) = [f_*, G]$$

(1)

where $D_t$ denotes evolutionary derivative corresponding to the equation and $f_*(v) = df[u + \epsilon v]/d\epsilon|_{\epsilon=0}$ is the linearization operator. The unknown $G$ (called formal symmetry or formal recursion operator) is a power series with respect to differentiation $D$ in the continuous case or to automorphism $T$ in the difference case. Solvability of equation (1) with respect to the coefficients of $G$

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provides a sequence of necessary integrability conditions which can be applied both for testing of a given equation and for classification of integrable cases among a whole set of equations under consideration. Additionally, one can use the conditions which follow from existence of an infinite set of higher order conservation laws. This approach allowed to solve a number of classification problems for integrable partial differential equations of the Korteweg–de Vries and the nonlinear Schrödinger type, see e.g. Sokolov and Shabat (1984); Mikhailov et al. (1987, 1991); Mikhailov and Shabat (1993); Meshkov and Sokolov (2012) and for differential-difference equations of the Volterra and the Toda lattice type (Yamilov, 1983, 2006; Shabat and Yamilov, 1991; Adler et al., 2000; Adler, 2008). Problems of symbolic computation of higher symmetries, conservation laws, recursion operators and Lax pairs were discussed in many papers, see e.g. Göktaş and Hereman (1999); Hickman and Hereman (2003); Hereman et al. (2005); Sokolov and Wolf (2001); Tsuchida and Wolf (2005); integrability tests based on these notions were developed e.g. in Gerdt et al. (1985); Gerdt (1993); Hereman et al. (1998).

The goal of this article is to describe an algorithm which allows to check the solvability of equation (1) for a given scalar lattice equation of the form

$$\partial_t(u_n) = f(u_{n-m}, \ldots, u_{n+m}), \quad n \in \mathbb{Z}. \quad (2)$$

Recall, that the case $m = 1$ (equations of Volterra lattice type) was classified by Yamilov (1983). At $m > 1$, only few examples of integrable equations are known at the moment, the Bogoyavlensky lattices (Bogoyavlensky, 1991) being the most well studied ones.

The operator $f_*$ corresponding to equation (2) is of the form $f_* = \sum f^{(j)} T^j$ where $f^{(j)} = \partial_j (f(u_{n-m}, \ldots, u_m))$, $\partial_j = \partial / \partial u_j$ and solution of equation (1) is sought as a power series $G = g_k T^k + g_{k-1} T^{k-1} + \ldots$. One can easily see that equation (1) in each order of $T$ is equivalent to a relation of the form

$$f^{(m)} T^m (g_j) - g_j T^j (f^{(m)}) = b_j, \quad j = k, k-1, \ldots \quad (3)$$

where $b_j$ is computed explicitly if the coefficients $g_k, \ldots, g_{j+1}$ are already known. Therefore, the integrability test for the lattice equation under consideration amounts to stepwise checking of whether equation (3) is solvable with respect to $g_j$; if not then it is not integrable, if yes then one have to compute $g_j$ and to go to the next condition. In practice, such a test turns out to be very effective, although, formally, checking of infinite number of conditions is needed in order to prove the integrability.

Although this scheme is rather standard, two technical issues should be mentioned in the case $m > 1$ which were not paid enough attention till now. Firstly, the form of equations (3) depends on the degree $k$ of the series $G$ which is not known in advance. This question does not stand at all in the continuous case.
(for the KdV type equations), because the operation of root extraction $G^{1/k}$ is defined for generic pseudodifferential operator $G = g_k D^k + g_{k-1} D^{k-1} + \ldots$ and it allows us to reduce the study of formal symmetries to the case $\deg G = 1$. In the difference case ($m = 1$), it is also possible to refrain from this question, since, according to Levi and Yamilov (1997); Yamilov (2006), the exhaustive classification here is based on just few simple conditions which can be easily derived under nonrestrictive assumptions about the orders of higher symmetries and conservation laws. In the case $m > 1$, this issue was settled in Adler (2014) where it was proved that if equation (1) admits a solution of any degree $k \neq 0$ then it admits as well a solution $G$ of degree $m$, moreover, one can assume without loss of generality that the positive parts of $G$ and $f_*$ coincide.

Another issue is related with the algorithm of solving of equation (3) itself. If $m = 1$ then this equation can be brought, by substitution $g_j = T^{-j-1}(f^{(m)}) \cdot \ldots f^{(m)} y_j$, to the standard form

$$(T - 1)(y_j) = \tilde{b}_j.$$  

The problem of inversion of the total difference operator $T - 1$ was addressed by many authors, both in the theory of integrable equations and in the context of discrete calculus of variations (Kupershmidt, 1985; Hydon and Mansfield, 2004; Mansfield and Quispel, 2005), see also Olver (1993) for the parallel theory in the continuous case. In particular, it is well known that $\mathbb{K} \oplus \text{Im}(T - 1) = \ker E$ where $\mathbb{K}$ denotes the field of constants and $E = \sum T^{-j} \partial_j$ is the difference Euler operator (or the variational derivative). The preimage of $T - 1$ can be computed by use of the so-called summation by parts algorithm or by use of the discrete homotopy operator (Hereman et al., 2006).

At $m > 1$ we arrive at the inversion problem for slightly more general operators $T^m(a[u])$. Although the setting is quite natural, I was not able to find any discussion of this problem in literature. The main result of the article is the description of an algorithm which allows either to solve an equation

$$T^m(y) - ay = b$$

with given functions $a[u], b[u]$ or to prove that solution does not exist.

An approach which makes use of the formal identity $(T^m - a)^{-1} = T^{-m}(1 + aT^{-m} + (aT^{-m})^2 + \ldots)$ is considered in section 3.1. This method is quite simple, but, in practice, it is applicable only if the coefficients $a, b$ are not too complicated.

Section 3.2 contains a more effective ‘generalized summation by parts algorithm’ based on simplification of $a, b$ by a sequence of suitable substitutions. An implementation of this algorithm in the Mathematica programming lan-
guage is presented in appendix A.1.

A variational meaning of operators $T^m - a$ and generalization of the discrete homotopy operators remain open questions, very interesting from the theoretical standpoint, however this approach can hardly give an effective computation scheme for practical applications.

Section 4 contains few basic notions and facts from the symmetry approach which are necessary to describe the procedure of testing of a given lattice equation (2). Several simple examples are given in Section 5 accompanied with a sample code in appendix A.2.

2 Notations

Let $\mathcal{F}$ be a differential field of functions depending on finite number of dynamical variables $u_j, j \in \mathbb{Z}$ and let the shift operator $T$ act on elements of $\mathcal{F}$ according to the rule

$$T^k(f(u_i, \ldots, u_j)) = f(u_{i+k}, \ldots, u_{j+k}).$$

We will assume that the field of constants $\mathbb{K}$ is equal to $\mathbb{R}$ or $\mathbb{C}$. The partial derivatives with respect to dynamical variables will be denoted $\partial_j = \partial/\partial u_j$, $f^{(j)} = \partial_j(f)$. Note the identity $T^k\partial_j = \partial_{j+k}T^k$.

The orders of a function $f \in \mathcal{F}$ are defined as follows:

$$\text{ord } f = \min\{j : f^{(j)} \neq 0\}, \quad \text{ord } f = \max\{j : f^{(j)} \neq 0\}, \quad f \neq \text{const}, \quad (4)$$

$$\text{ord } f = +\infty, \quad \text{ord } f = -\infty, \quad f = \text{const}. \quad (5)$$

We will also use the notation

$$J(f) = [\text{ord}(f), \text{ord}(f)], \quad J(\text{const}) = \emptyset.$$ 

The following operation (extraction) will be used in the summation by parts algorithm

$$g = X_k(f) : \quad g^{(k)} = f^{(k)}, \quad J(g) = J(f^{(k)}). \quad (6)$$

Informally, it can be described as erasing of additive terms in $f$ which do not depend on the distinguished variable $u_k$. Function $g$ is defined up to addition of an arbitrary function of variables $u_j, j \in J(f^{(k)}) \setminus \{k\}$. In practice, we will define this operation as follows:

$$X_k(f) = \int f^{(k)} du_k \quad (7)$$
where it is assumed that the integrand is cast in a form which does not contain variables \( u_j \) at \( j \not\in J(f^{(k)}) \) explicitly and the integration constant does not depend on these variables as well (which is natural). Alternatively, one can accept the definition

\[
X_k(f) = f|_{u_j=c_j, \ j \notin J(f^{(k)})}
\]

where \( c_j \) are any constants such that expression in the right hand side makes sense (for instance, if \( f \) is a polynomial then we can just set to zero all unnecessary \( u_j \)).

3 Solving of equation \( T^m(y) - ay = b \)

Let us consider a difference equation of the form

\[
T^m(y) - ay = b
\]

where \( m > 0 \), functions \( a, b \in \mathcal{F} \) are given, function \( y \in \mathcal{F} \) is unknown. Our goal is to obtain an algorithm which allows either to construct the solution \( y \) or to prove that some relation between coefficients \( a, b \) does not hold which is necessary for the existence of \( y \).

The uniqueness of solution (if it exists) depends on the form of the coefficient \( a \). If \( a \neq T^m(h)/h \) then the solution is unique (indeed, let \( \tilde{y} \) be another solution, then \( a = T^m(y - \tilde{y})/(y - \tilde{y}) \)). If \( a = T^m(h)/h \) (\( a = 1 \), in particular) then the solution is defined up to addition of a term \( \text{const} \), that is \( \ker(T^m - a) = \mathbb{K}h \).

It follows from equations below that if a solution of equation (9) exists then it is a rational function of coefficients \( a, b \) and functions obtained from \( a, b \) by means of the operators \( T, \partial_j \) and substitutions \( u_j = \text{const} \). In particular, if \( a, b \) are rational functions of the variables \( u_j \) then the solution is rational as well. If \( a = \text{const} \) and \( b \) is a polynomial then the solution is polynomial as well.

3.1 Formal inversion of operator \( T^m - a \)

1) Case \( a = \text{const} \neq 0 \). It is easy to see that if also \( b = \text{const} \) then

\[
\begin{align*}
y &= c & \text{at } a = 1, \ b = 0, \\
\tilde{y} &\neq y & \text{at } a = 1, \ b \neq 0, \\
y &= b/(1 - a) & \text{at } a \neq 1
\end{align*}
\]

where \( c \) is an arbitrary constant.
Let $b \neq \text{const}$, ord $b = p_1$ and ord $b = p_2$. If a solution $y$ exists then ord $y = p_1$ and ord $y = p_2 - m$ (in particular, this implies $p_2 \geq p_1 + m$). Let us consider the equation

$$y - ar^{-m}(y) = T^{-m}(b) + \cdots + a^{r-1}T^{-m}(b), \quad r \geq 1$$

which follows from equation (9). If $r$ is large enough, such that the inequality $p_1 > p_2 - m - rm$ holds, then the arguments of two functions in the left hand side of the equation belong to disjoint sets. This implies the formula

$$y = c + \sum_{s=1}^{r} a^{s-1}T^{-sm}(b) \bigg|_{uj=c_j}, \quad r = \left\lfloor \frac{p_2 - p_1}{m} \right\rfloor$$

(10)

where $c_j$ are any constants such that the sum in the right hand side is well defined (for instance, if $b$ is a polynomial then one can just set $c_j = 0$) and $c$ is an undetermined constant (arbitrary if $a = 1$). Thus, if a solution exists then it is of the form (10) and we only have to make a direct check by substitution into equation (9). Notice, that if $a \neq 1$ then the problem of choice of constants can be avoided by use of the formula

$$y = 1/(1 - ar) \sum_{s=1}^{r} a^{s-1}T^{-sm}(b) \bigg|_{uj=u_{j+rm}, \quad j < p_1}$$

instead of (10).

2) Case $a = \alpha T^m(h)/h$ is brought to the previous one by the change $y = h\tilde{y}$. The question, whether the given coefficient $a$ is of such form, is answered by investigation of auxiliary equation $T^m(z) - z = \log a - \lambda$ with unknown parameter $\lambda$.

3) Case $a \neq \alpha T^m(h)/h$. Let us differentiate equation (9) with respect to $u_j$:

$$T^m(y^{(j-m)}) - ay^{(j)} = a^{(j)}y + b^{(j)}.$$ 

Elimination of derivatives of $y$ brings to equation

$$\sum s a_s T^{sm}(a^{(j-sm)}y + b^{(j-sm)}) = 0 \quad (11)$$

where the sum contains only finite number of nonvanishing terms corresponding to the values of $s$ from the interval

$$\left\lfloor \frac{j - \max(\text{ord } a, \text{ord } b)}{m} \right\rfloor \leq s \leq \left\lfloor \frac{j - \min(\text{ord } a, \text{ord } b)}{m} \right\rfloor.$$ 

The coefficients are defined by the recurrent relation $a_{s-1} = a_s T^{sm}(a), \quad a_0 = 1,$
that is

\[ a_s = \prod_{k=s+1}^{0} T^{km}(a), \quad s \leq 0, \quad a_s = 1/ \prod_{k=1}^{s} T^{km}(a), \quad s \geq 0. \]

The equation corresponding to \( j = j + m \) differs from (11) just by a factor. Elimination of \( T^{sm}(y) \) by use of equation (9) allows us to bring (11) to the form

\[ A_j y = B_j, \quad j = 0, 1, \ldots, m - 1 \]

where coefficient \( A_j = \sum_s T^{sm}(\partial_j - sm(\log a)) \) does not vanish at least for one value of \( j \). So, function \( y \) is found explicitly and we only have to substitute it into (9) in order to check whether it is a solution.

**Remark 1.** Equation (11) makes sense for \( a = \text{const} \) as well. In this case it turns into the set of equations

\[ \sum_s a^{-s} T^{sm}(b^{(j-sm)}) = 0, \quad j = 0, 1, \ldots, m - 1 \]  

(12)

which serve as necessary solvability conditions for equation (9). In particular, at \( a = 1, m = 1 \) this is the usual condition \( E(b) = 0 \). One can prove that if \( a \neq 1 \) then conditions (12) are sufficient as well, that is, this is an exact definition of \( \text{Im}(T^m - a) \); if \( a = 1 \) then conditions (12) characterize the set \( \mathbb{K} \oplus \text{Im}(T^m - 1) \).

### 3.2 Generalized summation by parts algorithm

The above approach is rather clear and can be easily realized in the computer algebra systems. Unfortunately, computing of sums in (10) or (11) is not too effective in practice. An alternative algorithm makes use of a sequence of suitable substitutions of the form

\[ y = \tilde{y}/A, \quad \tilde{a} = aT^m(A)/A, \quad \tilde{b} = T^m(A)b \]

(13)

or

\[ y = \tilde{y} - B, \quad \tilde{a} = a, \quad \tilde{b} = b + (T^m - a)(B) \]

(14)

which bring (9) to equivalent equations of the same type, but with coefficients which depend on a reduced set of variables. As a result of a finite number of steps, one can either construct the solution \( y \) explicitly or prove that it does not exist. The substitutions can be applied in different ways, but the final answer does not depend on this, because of their reversibility.

The flow of control can be organized by use of inequalities involving the orders

\[ \text{ord} a = q_1, \quad \text{ord} a = q_2, \quad \text{ord} b = p_1, \quad \text{ord} b = p_2. \]
Some conditions imply that substitutions (13), (14) with required properties exist, with functions $A, B$ defined from $a, b$ by the extraction operation (6). Other conditions mean that equation (9) is unsolvable, in such a case the algorithm should return some nonzero expression which plays the role of an obstacle for existence of a solution. Analysis of such obstacles is important in a situation when the equation coefficients contain arbitrary parameters.

1) Case $a = \text{const} \neq 0$. If a solution $y$ exists then $J(y) = [p_1, p_2 - m]$. If the inequality

\[ p_1 > p_2 - m \]

holds then solution may be only constant, that is $y = b/(1 - a)$ if $a \neq 1$ and $y = c$ if $a = 1$, and we only have to check it by inspection. Notice, that the arbitrary constant $c$ here is the only source of possible nonuniqueness for the whole algorithm, upon all substitutions below.

Let $p_1 \leq p_2 - m$, then differentiation of equation (9) with respect to $u_{p_2}$ yields the relation $b^{(p_2)} = T^m(y^{(p_2-m)})$ which implies the inequality

\[ r = \text{ord} b^{(p_2)} \geq p_1 + m. \]

If it fails then the solution does not exist and expression $b^{(r,p_2)}$ is returned as an obstacle for its existence. If the inequality is fulfilled then $J(b^{(p_2)}) \subseteq [p_1 + m, p_2]$ and the change

\[ B = X_{p_2}(b), \quad y = \tilde{y} + T^{-m}(B), \quad \tilde{b} = b - B + aT^{-m}(B) \]

brings (9) to an equivalent equation with $J(\tilde{b}) \subseteq [p_1, p_2 - 1]$ (possibly, with $\tilde{b} = \text{const}$).

By repeating this argument while possible (no more than $p_2 - p_1 + 1$ times), we will either construct the solution as a finite sum $y = T^{-m}(B + \tilde{B} + \ldots)$ or prove that it does not exist.

2) Reduction of the coefficient $a$. Now let $a \neq \text{const}$. If

\[ q_1 \leq q_2 - m \quad \text{and} \quad \text{ord} \partial_{q_1} (\log a) \leq q_2 - m \]

(possibly $\partial_{q_1} (\log a) = \text{const}$) then $a$ is of the form

\[ a = A(u_{q_1}, \ldots, u_{q_2-m})\hat{a}(u_{q_1+1}, \ldots, u_{q_2}), \quad A = \exp(X_{q_1} (\log a)). \]

Then the substitution

\[ y = \tilde{y}/A, \quad \tilde{a} = aT^m(A)/A, \quad \tilde{b} = T^m(A)b \]

reduces (9) to an equivalent equation with $J(\tilde{a}) \subseteq [q_1 + 1, q_2] \quad$ (possibly with $\tilde{a} = \text{const}$). Iteration of this transformation leads either to the case 1) or to
the case when one of inequalities (15) fails, that is
\[
\max(q_1, \text{ord } \partial_{q_1}(\log a)) > q_2 - m. \tag{16}
\]
We will assume that this condition is fulfilled from now on. Further substitutions will not change \(a\).

3) **Reduction of the coefficient \(b\).** First, if \(p_2 > q_2\) then iteration of the substitution
\[
B = X_{p_2}(b), \quad y = \tilde{y} + T^{−m}(B), \quad \tilde{b} = b - B + aT^{-m}(B) \tag{17}
\]
brings the problem to the case \(p_2 \leq q_2\). Notice, that instead of this change we can apply a simpler one
\[
y = \tilde{y} + T^{-m}(b), \quad \tilde{b} = aT^{-m}(b)
\]
with the same effect. However, (17) turns out to be more effective, because it is desirable to drop the lower order \(p_1\) not too much.

Next, if \(p_1 < q_1\) and a solution \(y\) exists then \(\text{ord } y \leq q_2 - m\) and \(-y^{(p_1)} = b^{(p_1)}/a\) from where it follows
\[
r = \text{ord}(b^{(p_1)}/a) \leq q_2 - m.
\]
If this inequality fails then the equation does not have a solution and expression \(\partial_r(b^{(p_1)}/a)\) is returned as an obstacle. If the inequality holds then \(J(b^{(p_1)}/a) \subseteq [p_1, q_2 - m]\) and therefore the substitution
\[
B = X_{p_1}(b/a), \quad y = \tilde{y} - B, \quad \tilde{b} = b + (T^m - a)(B)
\]
brings to an equivalent equation with \(J(\tilde{b}) \subseteq [p_1 + 1, q_2]\). Iterating of this change brings the problem to the following case 4).

4) **Solving of a linear system.** The problem is reduced now to the case \(J(b) \subseteq J(a) = [q_1, q_2]\). If a solution \(y\) exists then \(J(y) \subseteq [q_1, q_2 - m]\). Therefore, if \(q_1 > q_2 - m\) then equation (9) may admit only a constant solution \(y = b/(1 - a)\) which can be checked by inspection. Let \(q_1 \leq q_2 - m\). Then, according to (16),
\[
r = \text{ord } \partial_{q_1}(\log a) > q_2 - m \geq \text{ord } y
\]
and differentiation of equation (9) yields a system of linear equations with nonzero determinant with respect to \(y, y^{(q_1)}\):
\[
a^{(q_1)} y + a y^{(q_1)} = -b^{(q_1)},
\]
\[
a^{(q_1, r)} y + a^{(r)} y^{(q_1)} = -b^{(q_1, r)}.
\]
From here, function $y$ is uniquely determined and, again, we only have to make a direct check whether it solves equation (9).

4 Formal symmetry test

Let us recall some basic notions of the symmetry approach in application to the scalar evolutionary lattice equations

$$
\partial_t(u_n) = f(u_{n-m}, \ldots, u_{n+m}), \quad n \in \mathbb{Z}
$$

or, in a shorthand notation,

$$
u, t = f(u_{-m}, \ldots, u_m).
$$

A detailed exposition can be found in Mikhailov et al. (1987, 1991); Levi and Yamilov (1997); Yamilov (2006). For any function $f \in \mathcal{F}$, the infinite-dimensional vector field

$$D_t = \nabla_f = \sum_{j \in \mathbb{Z}} T^j(f)\partial_j$$

is called evolutionary derivative and the difference operator

$$f_* = \sum_{j \in \mathbb{Z}} f^{(j)} T^j$$

is called linearization operator. Differentiation $D_t(g)$ in virtue of equation (18) is defined, for a function $g \in \mathcal{F}$, by two equivalent formulas $D_t(g) = \nabla_f(g) = g_*(f)$.

A lattice equation

$$u, \tau = g(u_{-t}, \ldots, u_t)
$$

is called (generalized) symmetry of equation (18) if differentiations $D_t, D_\tau$ commute, that is the equality

$$\nabla_f(g) = \nabla_g(f)
$$

holds identically with respect to $u_j$. Equation (18) is considered integrable if it admits symmetries of order $l$ arbitrarily large. Equation (20) yields, upon the linearization, a more convenient operator equation

$$\nabla_f(g_*) = \nabla_g(f_*) + [f_*, g_*].$$

The degree $m$ of the operator $f_*$ is fixed and this allows us to consider $g_*$ as an approximate solution of the Lax equation

$$D_t(G) = [f_*, G].
$$
More precisely, it can be proved that existence of a sequence of symmetries of arbitrarily large orders implies that (21) admits a solution in the form of power series

\[ G = g_k T^k + \cdots + g_1 T + g_0 + g_{-1} T^{-1} + \ldots, \quad g_j \in \mathcal{F}, \quad k > 0 \]

which is called formal symmetry, or formal recursion operator, of lattice equation (18). Conditions of solvability of equation (21) with respect to the coefficients \( g_j \) serve therefore as necessary integrability conditions for equation (18) under consideration. A weak point here is that the degree \( k \) is not known in advance. In the continuous setting, we can assume that \( k = 1 \) without loss of generality, due to the extraction of root \( G \rightarrow G^{1/k} \) which is correctly defined for generic pseudodifferential operators \( G = g_k D^k + \cdots + g_1 D + g_0 + g_{-1} D^{-1} + \ldots, \)

but in the difference situation this argument does not work. Nevertheless, it turns out that the degree \( k \) can always be chosen equal to the order \( m \) of equation (18) itself (this degree may be not minimal).

**Theorem 2** (Adler, 2014). If lattice equation (18) admits symmetries (19) of arbitrarily large order then the Lax equation (21) admits a solution of the form

\[ G = f^{(m)} T^m + \cdots + f^{(1)} T + g_0 + g_{-1} T^{-1} + \ldots \in \mathcal{F}(\mathcal{T}^{-1}). \] (22)

Now, equation (21) turns into a convenient and effective test, since the resulting necessary integrability conditions do not depend on actual orders of higher symmetries and can be written down immediately from the right hand side of equation (18). It is easy to see that collecting of terms with \( T^j \) in (21) brings to a sequence of recurrent equations of type (9) with respect to \( g_j \):

\[ T^m(g_j) - a_j g_j = b_j, \quad a_j = \frac{T^j(f^{(m)})}{f^{(m)}}, \quad j = 0, -1, -2, \ldots \] (23)

where expression

\[ b_j = \frac{1}{f^{(m)}} \left( D_l(g_{j+m}) - \sum_{s=-m}^{m-1} f^{(s)} T^s(g_{j+m-s}) - g_{j+m-s} T^{j+m-s}(f^{(s)}) \right) \] (24)

involves only coefficients \( g_m = f^{(m)}, \ldots, g_1 = f^{(1)} \) which play the role of initial conditions and coefficients \( g_0, \ldots, g_{j+1} \) which are already computed. Thus, the integrability test amounts to step by step checking of solvability of equations (23).

It can be proved that existence of a symmetry of order \( l \geq m + r \) implies that first \( r \) equations (23) can be resolved with respect to \( g_0, \ldots, g_{-r+1} \). Symmetries of orders \( l \leq m \) give no conditions in this approach, being lost on the background of the trivial symmetry \( u_\tau = f \). Concerning the sufficiency, the
fulfilment of first \( r \) conditions (23) does not formally guarantee existence of even one generalized symmetry, however, if \( r \) is large enough then it is a very strong evidence of integrability.

**Remark 3.** In addition to conditions (23), (24) there is a complementary sequence corresponding to the formal symmetry of the form

\[
\bar{G} = f^{-m} T^{-m} + \cdots + f^{(-1)} T^{-1} + \bar{g}_0 + \bar{g}_1 T + \cdots \in \mathcal{F}(T). 
\]

Solutions \( G, \bar{G} \) turn out to be equivalent if equation (18) admits a sequence of conservation laws \( D_t(\rho) = (T - 1)\sigma \) of orders arbitrarily large. In this case, equation

\[
D_t(R) + f^\dagger R + R f_* = 0 
\]

is solvable and admits a solution of the form

\[
R = r_l T^l + r_{l-1} T^{l-1} + \cdots \in \mathcal{F}(T^{-1}), \quad 0 \leq l < m, 
\]

such that \( \bar{G}^\dagger = -RGR^{-1} \) where \( \dagger \) denotes the conjugation \( (aT^j)^\dagger = T^{-j} a \). Formal symmetries \( G, \bar{G} \) can be considered in more general situation for equations with different negative and positive orders

\[
u_{u,t} = f(u_{-\bar{m}}, \ldots, u_m),
\]

however equation (25) may admit nonzero solutions only in the symmetric case \( \bar{m} = m \). It is clear that equations for \( \bar{g}_j \) and \( r_j \) are similar to (23) and can be checked analogously, so we will not discuss these additional conditions any more.

It is worth to notice that integrability conditions become especially simple at \( m = 1 \), that is for the Volterra type lattice equations. It was already mentioned in Introduction that equations (23) can be brought in this case to the standard form \( (T - 1)(y_j) = \tilde{b}_j \). Moreover, there exists a more complicated, but still invertible substitution which allows to rewrite these conditions in the form of conservation laws (possibly, trivial)

\[
D_t(\rho_j) = (T - 1)\sigma_j, \quad j \geq 0 
\]

where the so-called canonical densities \( \rho_j, j > 0 \) are equivalent to \( j^{-1} \text{coef}_{T^0} \bar{G}^j \) modulo \( \text{Im}(T - 1) \). Although this form makes no essential advantage when testing a given equation, it clarifies a general structure of the integrability conditions.

**Proposition 4.** If \( m = 1 \) then solvability of equations (23), (24) is equivalent to solvability with respect to \( \sigma_j \in \mathcal{F} \) of conservation laws (26) where densities
\( \rho_j \) are defined by recurrent relations

\[
\begin{align*}
\rho_0 &= \log f^{(1)}, \\
\rho_1 &= f^{(0)} + \sigma_0, \\
T^{-1}(f^{(1)}_{j-1} \rho) + \sigma_j &= 0, \\
T^{-1}(f^{(1)}_{j} \rho) + \sigma_j &= 0, \\
&\quad j > 0
\end{align*}
\]

with polynomials \( P_j \) defined by the generating function

\[
P_0[\rho] + P_1[\rho] \lambda + P_2[\rho] \lambda^2 + \ldots = \exp(\rho_1 \lambda + \rho_2 \lambda^2 + \rho_3 \lambda^3 + \ldots).
\]

The proof can be found in Adler (2014). Several first polynomials \( P_j \) are

\[
\begin{align*}
P_0 &= 1, \\
P_1 &= \rho_1, \\
P_2 &= \rho_2 + \frac{\rho_1^2}{2}, \\
P_3 &= \rho_3 + \rho_1 \rho_2 + \frac{\rho_1^3}{6}, \\
&\quad \ldots
\end{align*}
\]

and the corresponding conserved densities are

\[
\begin{align*}
\rho_2 &= f_{-1}^{-1}(f_1) + \frac{1}{2} \rho_1^2 + \sigma_1, \\
\rho_3 &= f_{-1}^{-1}(f_1 \rho_1) + \rho_1 \rho_2 - \frac{1}{6} \rho_1^3 + \sigma_2, \\
\rho_4 &= f_{-1}^{-1}(f_1(\rho_2 + \frac{1}{2} \rho_1^2)) + \rho_1 \rho_3 + \frac{1}{2} \rho_2^2 - \frac{1}{2} \rho_1^2 \rho_2 + \frac{1}{24} \rho_1^4 + \sigma_3.
\end{align*}
\]

In the general case \( m > 1 \), only part of conditions (23) can be rewritten as conservation laws. For instance, it is easy to prove that if equations (23) are solvable till \( j = -m \) then functions \( \sigma, \sigma_1 \in F \) exist such that

\[
\begin{align*}
D_t(\log f^{(m)}) &= (T^m - 1)(\sigma), \\
D_t(f^{(0)} + \sigma) &= (T - 1)(\sigma_1).
\end{align*}
\]

5 Examples

Here we present several simple examples, in order to clarify various computational aspects rather than to obtain new results.

**Example 5.** Solving of equations (23), (24) for the Volterra lattice

\[
u_{,t} = u(u_1 - u_{-1})
\]

and setting all integration constants to zero yields

\[
\begin{align*}
g_1 &= u, \\
g_0 &= u + u_1, \\
g_{-1} &= \frac{uu_1}{u_{-1}}, \\
g_j &= \frac{u(u_1 - u_{-1})}{u_j}, \\
&\quad j < -1.
\end{align*}
\]
It is easy to see that the series \( G = \sum g_j T^j \) can be rewritten in a closed form

\[
G = uT + u + u_1 + uT^{-1} + u(u_1 - u_{-1})(T - 1)^{-1} \frac{1}{u}
\]

which is the well known recursion operator for the Volterra lattice. However, in most cases expressions for \( g_j \) are much more complicated and search of corresponding recursion operators is a very nontrivial problem. For instance, in the case of the second order Bogoyavlensky lattice

\[
u_t = u(u_2 + u_1 - u_{-1} - u_{-2}),
\]

we get

\[
g_2 = u, \quad g_1 = u, \quad g_0 = u + u_1 + u_2, \quad g_{-1} = 0,
\]

\[
g_{-2} = \frac{1}{u-2}(u_{-1}u_1 + uu_1 + uu_2), \quad g_{-3} = -\frac{1}{u_{-3}}(u_{-2}u + u_{-1}u + u_{-1}u_1),
\]

\[
g_{-4} = \frac{1}{u_{-4}u_{-2}}((u_{-3} + u_{-2})u_{-1}u_1 + u_{-2}u(u_1 + u_2)), \ldots
\]

which gives little hint on the factored form of \( G \)

\[
G = u(1 + T^{-1} + T^{-2})(T^2u - uT^{-1})(Tu - uT^{-1})^{-1}(Tu - uT^{-2})(u - uT^{-2})^{-1}.
\]

This is a particular example of recursion operators found by Wang (2012) for the Bogoyavlensky lattices of any order \( m \). Notice, that in these operators all inverse factors are binomial and therefore computation of \( G(f) \) for a given function \( f \) amounts to solving of a sequence of equations of the type (9).

**Example 6.** As a sample classification problem, consider a Bogoyavlensky type equation

\[
u_t = u(u_2 + k_1u_1 + k_2u + k_3u_{-1} + k_4u_{-2})
\]

with undetermined coefficients. Application of the formal symmetry test yields on the first step the obstacle

\[
(1 + k_2 + k_4)u + (k_1 + k_3)u_1 = 0 \quad \Rightarrow \quad k_2 = -1 - k_4, \quad k_3 = -k_1.
\]

After the substitutions, \( g_0 \) is successfully found, but computing of \( g_{-1} \) encounters the next obstacle

\[
k_1(k_1 + k_4)(u - u_2) + k_1(k_1 - 1)(u_{-1} - u_1) = 0.
\]

If \( k_1 \neq 0 \) then \( k_1 = 1, k_4 = -1 \) and we arrive to the Bogoyavlensky lattice. If \( k_1 = 0 \) then computation of \( g_{-2} \) brings to the obstacle

\[
k_4(1 + k_4)/u_{-2} = 0.
\]
and we get two more integrable (albeit disappointing) cases: a linearizable equation \( u_t = u(u_2 - u) \) and the stretched Volterra lattice \( u_t = u(u_2 - u_{-2}) \).

**Example 7.** According to Yamilov (2006), the lattice equation

\[
u_t = h(u_1 - u) + h(u - u_{-1})
\]

is integrable if \( h \) satisfies equation

\[
h' = \alpha h^2 + \beta h + \gamma
\]

with arbitrary constant coefficients. Equation (27) can be solved in elementary functions, but this leads to consideration of several cases corresponding to different parameter sets and special solutions. In order to handle the whole family in a uniform manner we only have to compute the coefficients \( b_j \) (24) modulo a rule which replaces first and second derivatives of \( h \) in virtue of (27). After this, equations (23) are solved as usual by the summation by parts algorithm.

**Example 8.** In the above examples, equations pass the test for any choice of integration constants, but this is not always the case. Consider the modified Bogoyavlensky lattice

\[
u_t = u(u_2 u_1 - u_{-1} u_{-2})
\]

with \( f^{(2)} = u u_1 \). It is easy to see that operator \( T^2 - T^j(f^{(2)})/f^{(2)} \) possesses nontrivial kernel for any \( j \), so that the general solution of equation (23) contains an arbitrary constant \( c_j \) on each step. However, it turns out that \( c_{-2k+1} \) becomes an obstacle when we proceed to computing of \( g_{-2k} \) and, as a result, the test passes only if we set to zero every second integration constant. This indicates that the minimal degree of the formal symmetry \( G \) is equal to 2, so that (28) cannot be a symmetry of an equation of order 1.

The same is true for equation

\[
u_t = (u^2 + 1)((u_1^2 + 1)(u_2 - u) + (u_{-1}^2 + 1)(u - u_{-2}))
\]

which is related by the non-autonomous change \( u_n = (-1)^n u_n \) to the second order symmetry of the modified Volterra lattice

\[
u_{\tau} = (u^2 + 1)(u_1 - u_{-1})
\]

6 Conclusion

The presented algorithm is designed for straightforward computation of the formal symmetry for a scalar evolutionary lattice equation of any order. It
is suitable mainly for testing integrability of a single equation or a family depending on several parameters.

Further generalizations may include equations with two or more components such as the Toda or the Ablowitz–Ladik type lattices and their ‘hungry’ analogs. In this case the formal symmetry coefficients are matrices and the inversion of difference operators becomes a more difficult problem. The vectorial case (see e.g. Adler, 2008) can also be handled by enlarging the set of dynamical variables.

Concerning the classification problem in general, a lot of results were obtained in the continuous case for scalar evolutionary equations of orders 3, 5, 7, see Meshkov and Sokolov (2012) and references therein. Moreover, there is a conjecture that all (or at least all polynomial) integrable equations of higher orders are symmetries of equations of orders 3 and 5, so that there is just a finite set of integrable hierarchies. In the difference case the classification is much more difficult and the Yamilov (1983) list of the first order lattices remains the only rigorous result obtained so far. The known examples show that there are primitive integrable lattice equations of any orders, and description of the set of integrable hierarchies is a challenging problem, even in the polynomial case.

A Mathematica implementation of algorithms

A.1 Generalized summation by parts

Let \( f \) be an expression depending on the variables \( u_j \). The following lines define the shift \( T^k(f) \), a list of variables involved in \( f \), and orders of (unsimplified form of) \( f \):

\[
\begin{align*}
T[f_, k_] & := f /. u[j_] :> u[j + k] \\
vars[f_] & := \text{Union}[\text{Cases}[f, _u, \{0, \infty\}]] \\
ords[f_] & := \text{If}[\# == \{\}, \{\infty, -\infty\}, \\
&\{\#[[1, 1]], \#[[-1, 1]]\}] & \text{&}[\text{vars}[f]]
\end{align*}
\]

For the sake of simplicity, we will assume that all expressions under consideration are rational, then the command \( \text{ords}[\text{Together}[f]] \) returns correct orders \((4), (5)\) of \( f \).

Function \( \text{psum}[m, a, b] \) defined below solves the equation \( T^m(y) - ay = b \). It returns a pair of expressions \((y, z)\) where \( z \) (obstacle) vanishes if and only if the equation is solvable. If this is the case then \( y \) is the general solution.
of equation, with possible integration constant denoted by the symbol \( \text{const} \). The computation is performed according to the algorithm described in section 3.2, with substitutions (13), (14) realized as recursive calls (function \( \text{psum} \) just blocks the default limitation on the recursion depth and calls another function which makes all job). The computation stops either when some necessary condition for existence of solution fails or when the solution can be found immediately.

\[
p\text{sum}[m_-, a_-, b_-] /; m > 0 := 
\text{Block}[[\text{$\text{RecursionLimit}$} = \infty], \text{psu}[m, a, b]]
\]

\[
p\text{su}[m_-, aa_-, bb_-] := \text{Module}[
\{a = \text{Together}[aa], b = \text{Together}[bb],
A, B, p1, p2, q1, q2, r, y\},
q2 = \text{ords}[a]; q1 = q2[[1]]; q2 = q2[[2]];
p2 = \text{ords}[b]; p1 = p2[[1]]; p2 = p2[[2]];

\text{Catch}[
\text{If}[a === 0, \text{Throw}[\{T[b, -m], 0\}]];
(* Case \(a = \text{const}\) *)
\text{If}[q1 == \infty,
\text{If}[p2 < p1 + m,
y = \text{If}[a === 1, \text{const}, b/(1 - a)];
\text{Throw}[\{y, T[y, m] - a y - b\}];
B = \text{Together}[\text{D}[b, u[p2]]];
r = \text{ords}[B][[1]]; 
\text{If}[r < p1 + m, \text{Throw}[\{0, \text{D}[B, u[r]]\}]];
B = \text{Integrate}[B, u[p2]]; 
\text{Throw}[\text{psu}[m, a T[B, -m]] + \{T[B, -m], 0\}];
]
(* Reduction of \(a\) *)
A = \text{Together}[\text{D}[a, u[q1]]]/a;
r = \text{ords}[A][[2]];
\text{If}[\text{And}[q1 <= q2 - m, r <= q2 - m],
A = \text{Exp}[\text{Together}[\text{Integrate}[A, u[q1]]]];
\text{Throw}[\text{Together}[\text{psu}[m, a T[A, m]/A, b T[A, m]]/A]];
(* Reduction of \(b\) *)
\text{If}[p2 > q2, 
B = \text{Integrate}[\text{D}[b, u[p2]], u[p2]]; 
\text{Throw}[\text{psu}[m, a, b - B + a T[B, -m]] + \{T[B, -m], 0\}];
]
If[p1 < q1, 
  B = Together[D[b, u[p1]]/a]; 
  r = ords[B][[2]]; 
  If[r > q2 - m, Throw[{0, D[B, u[r]]}]]; 
  B = Integrate[B, u[p1]]; 
  Throw[psu[m, a, b + T[B, m] - a B] - {B, 0}]; ];

(* Solving of a linear system *)
y = If[q1 <= q2 - m, 
  -(D[b, u[q1]] D[a, u[r2]] - D[b, u[q1], u[r2]] a) / 
  (D[a, u[q1]] D[a, u[r2]] - D[a, u[q1], u[r2]] a), 
  b/(1 - a)]; 
  Throw[Together[{y, T[y, m] - a y - b}]] ]

A.2 Computation of formal symmetry

The following cell defines the differential \(\sum_j f^{(i)} du_j\) and the evolutionary derivative \(\nabla_g f = f_*(g)\):

\[
df[f_] := \text{Plus} \@@ (D[f, #]dif[#] & /\@ vars[f]) \\
dt[f_, g_] := df[f] /. dif[u[j_]] :> T[g, j] \\
dt[f_] := dt[f, F]
\]

The global variables \(m, F\) will be used to denote the order and the right hand side of the lattice equation \(u_t = f[u]\) under consideration. Commands in the next cell define partial derivatives \(f^{(i)}\), the positive part of formal symmetry \(G > 0 = (f_*) > 0\) as initial conditions for further computation and coefficients of equation (23). Procedure mytest[k] computes coefficients \(g_0, \ldots, g_{-k}\) while it is possible and stops if an obstacle occurs.

Clear[m, a, b, c, f, g]
f[j_] := D[F, u[j]]
g[j_] /; j > 0 := f[j]
a[j_] := T[f[m], j]/f[m]
b[j_] := 1/f[m](dt[g[j + m]] - Sum[f[s]T[g[j + m - s], s] - 
  g[j + m - s]T[f[s], j + m - s], {s, -m, m - 1}])
mytest[k_] := Do[
  ps = Factor[psum[m, a[j], b[j]]];
  obst = ps[[2]]; 
  Print[obst]; 
  If[Not[obst === 0], Break[]];
]
\[ g[j] = ps[[1]] /. \text{const} \rightarrow c[j], \{j, 0, -k, -1\} \]

**Example 5.** Next cell demonstrates the basic usage of the above commands by the examples of the Volterra lattice, its second order symmetry and the Bogoyavlensky lattice.

\[
F = u[0](u[1] - u[-1]);
F2 = u[0](u[1](u[2] + u[1] + u[0]) - u[-1](u[0] + u[-1] + u[-2]));
\text{Expand}[dt[F, F2] - dt[F2, F]]
\]

\[ m = 1; \]
\text{mytest}[4]
\text{Table}[\text{Factor}[g[j] /. c[j_] :> 0], \{j, m, -4, -1\}]

\[ m = 2; \]
\text{F} = F2;
\text{mytest}[4]
\text{Table}[\text{Factor}[g[j] /. c[j_] :> 0], \{j, m, -4, -1\}]

\[ F = u[0](u[2] + u[1] - u[-1] - u[-2]); \]
\text{mytest}[6]
\text{Table}[\text{Factor}[g[j] /. c[j_] :> 0], \{j, m, -6, -1\}]

The output of \text{mytest} consists here from a sequence of zeroes which means that the computation encounters no obstacles. The actual coefficients of the formal symmetry are stored as the variables \( g[j] \).

**Example 6.** A sample classification problem solved by analyzing of the obstacles to the test.

\[ m = 2; \]
\text{F} = u[0](u[2] + k1 u[1] + k2 u[0] + k3 u[-1] + k4 u[-2]);
\text{mytest}[6]
\text{Collect}[-\text{obst}, \_u]

Out: \((1 + k2 + k4) u[0] + (k1 + k3) u[1]\)

\[ F = u[0](u[2] + k1 u[1] - (1 + k4) u[0] - k1 u[-1] + k4 u[-2]); \]
\text{mytest}[6]
\text{Collect}[-\text{obst}, \_u]

Out: \(-(-1 + k1) k1 u[-1] - k1 (k1 + k4) u[0] - (1 - k1) k1 u[1] - k1 (-k1 - k4) u[2] \)

\[ F = u[0](u[2] - (1 + k4) u[0] + k4 u[-2]); \]
Example 7. The following modification of the test includes an additional transformation rule. It is applied to the lattice equation with function $h$ in the r.h.s. which is defined as a solution of an ODE.

\[
\text{mytest1}[k_, ru_] := \text{Do[}
\begin{align*}
\text{ps} &= \text{Factor}[
\text{psum}[m, a[j] /. ru, b[j] /. ru]];
\text{obst} &= \text{ps}[[2]];
\text{Print}[\text{obst}];
\text{If}[(\text{Not}[\text{obst} === 0], \text{Break[]}];
\text{g[j]} &= \text{ps}[[1]] /. \text{const} \rightarrow \text{c[j]},
\{j, 0, -k, -1\}]
\end{align*}
\]

\[
\text{m} = 1;
\text{F} = h[u[1] - u[0]] + h[u[0] - u[-1]]; \\
\text{P}[x_] := \alpha x^2 + \beta x + \gamma
\]

Example 8. First two calls of the test for the modified Bogoyavlensky lattice show that integration constants $c_{-1}, c_{-3}$ are obstacles; test passes after setting to zero all constants with odd numbers. The same is true for equation (29). In contrast, second order symmetry of the modified Volterra lattice passes the test for arbitrary integration constants.

\[
\text{m} = 2; \\
\text{F} = u[0](u[2]u[1] - u[-1]u[-2]);
\text{mytest}[6] \\
\text{c[-1]} := 0 \\
\text{mytest}[6] \\
\text{c[j_]} /. \text{OddQ}[j] := 0 \\
\text{mytest}[6] \\
\text{Clear}[\text{c}]
\]

\[
\text{F} = (u[0]^2 + 1)((u[1]^2 + 1)(u[2] - u[0]) + (u[-1]^2 + 1)(u[0] - u[-2]));
\text{mytest}[4] \\
\text{c[j_]} /. \text{OddQ}[j] := 0 \\
\text{mytest}[4] \\
\text{Clear}[\text{c}]
\]

\[
\text{F} = (u[0]^2 + 1)((u[1]^2 + 1)(u[2] + u[0]) - (u[-1]^2 + 1)(u[0] + u[-2]));
\text{mytest}[4]\
\]

Out: \(-2 k_4 \frac{(1 + k_4)}{u^{-2}}\)
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