On non-standard graded algebras

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Abstract
Positively graded algebras are fairly natural objects which are arduous to be studied. In this article we query quotients of non-standard graded polynomial rings with combinatorial and commutative algebra methods.

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Introduction
It has been shown in many works of the last three decades that combinatorial methods applied to Commutative Algebra and Algebraic Geometry are very effective. Most of these, though, deal with standard-graded polynomial rings, i.e. with polynomial rings where the weight or degree of all the variables involved is 1.

Other than the work [A], which is concerned with positively graded algebras having a specific Hilbert function (see [S], Chapter 10 for a nice survey) and the introductory work [BR], there is a small amount of literature about non-standard graded (or weighted) algebras, where the weight of a variable can be any positive integer.

This lack persuaded us to explore the realm of weighted graded algebras with algebraic and combinatorial tools.
Our work is divided in four sections, each centred on one topic and inspired by some of the more significant results in the standard case: generic initial ideals, Prime Avoidance, Castelnuovo-Mumford regularity and lexicographic ideals.

The first section is dedicated to the definition and main properties of generic initial ideals. We first study and describe the automorphisms of a weighted polynomial ring, which are necessary for the definition, the existence of generic initial ideals is discussed and a Borel-fixed type of property, i.e. fixedness under the action of a subgroup of the group of the automorphisms is proved. Moreover, we define a combinatorial counterpart of this property, i.e. being “weighted strongly stable”, and prove that generic initial ideals are enriched with it.

In the second section we recover an analogue of the homogeneous prime avoidance Lemma (Lemma 2.1), which grants the existence of an almost-regular form of degree equal to the least common multiples of the weights. This weaker statement is though enough to evince some conclusions which generalize the known fact that depth does not change after taking generic initial ideals with respect to the degree reverse-lexicographic order (Proposition 2.7). This is performed under the assumption that each weight is divisible by the previous ones.

The third section takes into consideration the Castelnuovo-Mumford regularity, which is defined in terms of local cohomology. In the first part we report some technical lemmata, which are useful in what follows as computational tools. Then we prove that the regularity of an ideal can be calculated using its graded Betti numbers, as it can be done in the standard case but with a correction due to the weights and the number of their occurrences. This is achieved in Theorem 3.5 and is essentially a paraphrase of what was performed in [Be] for pseudograded algebras. The section concludes with a result which predicts that Castelnuovo-Mumford regularity does not change when taking generic initial ideal (Proposition 3.6). Again, this is proved under the hypothesis that each weight is divisible by the previous ones, and is false in general, as shown in Example 3.7.

In the last section we deal with lexicographic ideals. In a standard graded polynomial ring lexicographic ideals are enhanced with many features and are well understood. As a consequence a certain amount of information about an arbitrary homogeneous ideal can be gathered by studying the associated lexicographic ideal. In other contexts though, the generalization of this notion turned out to be rather complex (cf. [ADK], [DH] and [MP]). In our situation there is a natural way to define lexicographic ideals, but it is difficult to describe them and to give a criterion to decide whether a homogeneous ideal in a given weighted polynomial ring is lexicifiable, i.e. admits an associated lexicographic ideal with the same Hilbert function.

First we recollect some known results about Hilbert functions of a positively graded algebra and underline which are the factors which make our analysis difficult by means of some examples. In particular the shadow of a lexsegment is a lexsegment in the standard graded case, but this fact does not hold in general in the non-standard setting. We thus proceed by proving Proposition 4.9, which yields a method to verify whether a given ideal is lexicographic. It is indeed enough to check if finitely many graded components are generated by
lexsegments. This is accomplished by means of an invariant $G(w)$, which was introduced in [D]. From a computational point of view the test is not optimal yet, since $G(w)$ not only depends on the weights but grows rapidly with the number of variables having the same weight.

Next, we are interested in describing the Hilbert function of lexicographic ideals, but to give a complete solution (cf. Problem 4.12) is an hard task. Theorem 4.11 provides an exhaustive answer for polynomial rings in two variables. The next topic we handle is expressed by Question 4.15. One would like to know which sets of weights make a polynomial ring *Macaulay-Lex*, i.e. such all of its ideals are lexifiable. Theorem 4.16 and the subsequent examples provide partial answers to this issue. We would like to observe that Theorem 4.11 and Theorem 4.16 provide a complete description of lexicographic ideals in two variables. Still, even in two variables it is not clear which ideals are lexifiable.

As a final remark, which completes and concludes this survey, we also mention the technique of polarization. In this setting admissible numerical functions need not to be Hilbert functions of lexicographic ideals. Thus completely polarized ideals (which in the standard case characterize lexicographic ideals), might be the right tool for a Theorem `a la Macaulay` [M].

The calculations underlying many of the examples and the material of the last section were carried out using [CoCoA]. We implemented some procedures (for the computation of Hilbert functions, generic initial ideals, polarization and associated lexicographic ideals) which can be obtained by any of the two authors.

**Notation**

In this paper we use some non-standard notation (!) which we illustrate here. When we consider polynomial rings with a non-standard grading, we mean that we work over an infinite field $K$ of characteristic 0 and assume the degrees of the variables to be positive integers with no further restriction.

We order the variables by increasing degree or *weight* and often group together those with the same degree. Therefore we denote the polynomial ring by $R = K[X_1, \ldots, X_n]$, where $X_i = (X_{i1}, \ldots, X_{il_i})$, $\deg X_{ij} = q_i$ for any $j = 1, \ldots, l_i$, and $q_1 < q_2 < \ldots < q_n$.

It is convenient to denote by $R[i]$ the polynomial ring $K[X_1, \ldots, X_i]$. We let $w$ be the weight vector $(\deg X_{11}, \ldots, \deg X_{nl_n})$ so that $(R, w)$ stands for a polynomial ring with the graduation given by $w$. If $w$ does not play an explicit role we denote $(R, w)$ simply by $R$.

If not elsewhere specified we consider term orderings $>$ which are degree compatible and assume $X_{ij} > X_{ik}$ if $j < k$, $i = 1, \ldots, n$.

Since they are often used, it may be convenient to fix some notation for the total numbers of variables and the least common multiple of the weights. Thus we let $l = \sum_{i=1}^n l_i$ and $q = \lcm(q_1, \ldots, q_n)$.

Finally, given a set $A \subseteq R_d$, $\langle A \rangle$ denotes the $K$-vector space spanned by $A$. If $V \subseteq R_d$ is a $K$-vector space, $\{V\}$ denotes its monomial basis.
1 Weighted generic initial ideals

Generic initial ideals are monomial ideals attached to homogeneous ideals. It has been shown in many works that generic initial ideals, although simpler in nature, still contain a considerable amount of information about the original geometrical object. In order to extend their definition in our setting we have first of all to understand which are the graded automorphisms of \( R \).

**Proposition 1.1.** The assignment

\[
\varphi(X_{ij}) = \sum_{h=1}^{l_i} a^i_{jh} X_{ih} + \psi_{ij}(X_1, \ldots, X_{i-1}),
\]

where, for all \( i, j \), \( \psi_{ij} \) are homogeneous polynomials in \( R_{i-1} \) of degree \( q_i \) and \( A_1 = (a^i_{jh})_{j,h=1,\ldots,l_i} \in M_{l_i}(K) \) are invertible matrices, defines a graded automorphism of \( R \). Vice versa, any graded automorphism of \( R \) is of this kind.

**Proof.** Since \( q_i < q_j \) if \( i < j \), the requirement that \( \varphi \) is a graded homomorphism forces \( \psi_{ij} \) to be polynomials in the first \( i-1 \) sets of variables. Thus it is sufficient to show that \( \varphi \) is surjective if and only if \( A_i \) are invertible for all \( i = 1, \ldots, n \).

With some abuse of notation we write \( \varphi(X_i) = A_i X_i + B_{i-1} \), where \( B_{i-1} \) is a \( l_i \times 1 \) matrix with entries in \( R_{i-1} \). If \( \varphi \) is surjective then for all \( i = 1, \ldots, n \) there exists a \( l_i \times 1 \) matrix \( C_i \) with entries in \( R_{i-1} \) such that \( X_i = \varphi(C_i) \). If we write \( C_i = D_i X_i + E_{i-1}, \) where \( D_i \in M_{l_i}(K) \) and \( E_{i-1} \) is a \( l_i \times 1 \) matrix with entries in \( R_{i-1} \), we get that \( X_i = D_i A_i X_i + E_{i-1} \) where \( E_{i-1} \) consists of polynomials in \( X_1, \ldots, X_{i-1} \). Therefore \( E_{i-1} = 0 \) and \( D_i A_i = I \). Vice versa, suppose that \( A_i \) is invertible for \( i = 1, \ldots, n \). Since \( A_1 \) is invertible, \( \varphi(A_1^{-1} X_1) = X_1 \). If \( i > 1 \), we have \( \varphi(A_i^{-1} X_i) = X_i + A_i^{-1} E_{i-1}, \) where \( A_i^{-1} E_{i-1} \) has entries in \( R_{i-1} \). Thus there exists a \( l_i \times 1 \) matrix \( C_{i-1} \) with entries in \( R_{i-1} \) such that \( \varphi(C_{i-1}) = A_i^{-1} E_{i-1} \) and \( X_i = \varphi(A_i^{-1} X_i) - \varphi(C_{i-1}) = \varphi(A_i^{-1} X_i - C_{i-1}) \), as required. ▶

Let \( T \) be the subset of upper triangular automorphisms consisting of those graded automorphisms of \( R \) such that, for all \( i = 1, \ldots, n \), \( A_i \) is an upper triangular invertible matrix. By the previous proposition it is clear that \( T \) is a group, since upper triangular invertible matrices form a group (which is called Borel group). Let \( U \) be the set of the elementary upper triangular automorphisms \( \tau^e_{ij} \), where \( r < j \) and \( c \in K \), determined by the assignment \( \tau^e_{ij}(X_{ij}) = X_{ij} + c X_{ir} \) and \( \tau^e_{ij}(X_{hk}) = X_{hk} \) if \( (h, k) \neq (i, j) \). Finally, let \( N \) be the set of elementary non-linear automorphisms \( \eta^n_{ij} \), where \( m \) is a term of degree \( q_i \) in \( R_{[i-1]} \) defined by \( \eta^n_{ij}(X_{ij}) = X_{ij} + m \) and \( \eta^n_{ij}(X_{hk}) = X_{hk} \) if \( (h, k) \neq (i, j) \).

**Proposition 1.2.** \( T \) is generated by the diagonal subgroup, by \( U \) and by \( N \).

**Proof.** The proof is an easy induction on the number of variables \( l = l_1 + \ldots + l_n \).

If \( l = 1 \), the only graded automorphisms are the diagonal automorphisms. Let \( \varphi \in T \), we say \( \varphi(X_{ij}) = \sum_{h=1}^{l_i} a^i_{jh} X_{ih} + \psi_{ij} \), and \( A_i = (a^i_{jh})_{j,h=1,\ldots,l_i} \) are upper triangular invertible matrices; also, let \( \varphi' \) be defined by \( \varphi'(X_{nl_n}) = X_{nl_n} \) and...
Moreover, since \( \varphi' \) is an automorphism and that therefore belongs to \( T \). We now write the polynomial \( \psi_{nl} \) as sum of monomials \( m_{nl}^{i} \in K[X_1, \ldots, X_{n-1}], h = 1, \ldots, s \), of degree \( q_h \). We want to find a decomposition of \( \varphi \) by means of elementary non-linear, upper triangular, diagonal automorphisms and of \( \varphi' \). This leads to the conclusion by induction, since \( \varphi' \) fixes the last variable and can be thought of as an automorphism of a polynomial ring in \( l - 1 \) variables. We denote by \( \delta_{ij}^{c} \), with \( c \in K \), the diagonal automorphism defined by \( \delta_{ij}^{c}(X_{ij}) = cX_{ij} \) and \( \delta_{ij}^{c}(X_{hk}) = X_{hk} \) if \((h, k) \neq (i, j)\) Moreover, since \( a_{ln}^n T_{ln} \neq 0 \), we may write \( u_h \doteq m_{nl}^{h}/a_{ln}^{n} \) for all \( h = 1, \ldots, s \) and \( b_k \doteq a_{ln}^{n}/a_{ln}^{n} \), for \( k = 1, \ldots, l_n - 1 \). It is now easy to see that

\[
\varphi = \eta_{nl}^{u_1} \circ \cdots \circ \eta_{nl}^{u_s} \circ \tau_{nl}^{b_1} \circ \cdots \circ \tau_{nl}^{l_n - 1 - b_{l_n - 1}} \circ \delta_{nl}^{a_{ln}} \circ \varphi',
\]

as desired. ▲

1.1 Existence of the generic initial ideal

In the standard graded case the generic initial ideal \( \text{Gin}(I) \) of an ideal \( I \subset \) \( K[X_1, \ldots, X_{n}] \) plays a central role in problems regarding Hilbert functions and free resolutions of graded ideals. Since \( \text{Gin}(I) \) with respect to some assigned term order is defined as the initial ideal of \( gI \), where \( g \) is a generic change of coordinates, i.e. that \( g \) is a matrix chosen out of a Zariski non-empty open set of \( \text{GL}_{n}(K) \), one way of computing it is the following. Let the \( n^2 \) entries of \( g \) be new indeterminates, we say \( Y_{ij} \), with \( i, j = 1, \ldots, n \). Write \( gI \) explicitly and apply the Buchsberger’s Algorithm to compute \( \text{in}(I) \) as an ideal of \( K[Y_{ij}]/[X_1, \ldots, X_{n}] \). After finitely many computations of the so-called \( S \)-pairs the process finishes, the output result is the sought after monomial ideal - in the variables \( X_i \) only - and the Zariski open set consists of all those matrices for which the finitely many polynomial denominators of the \( S \)-pairs are non-zero. If one considers this point of view, it is evident that weights do not play any role in the construction, which is thus also possible in the weighted case. Thus we can talk of generic initial ideals of homogeneous ideals in a non-standard graded algebra.

In the standard case it is well-known that generic initial ideals are \( \text{Borel-fixed} \), i.e. fixed under the action of the Borel subgroup of \( \text{GL}_{n}(K) \) consisting of the upper triangular invertible matrices.

Theorem 1.3. Let \( I \) be a homogeneous ideal of a weighted polynomial algebra \( R \). Then \( \text{Gin}(I) \) is \( T \)-fixed, i.e. \( \varphi(\text{Gin}(I)) = \text{Gin}(I) \) for all \( \varphi \in T \).

Proof. We only need to observe that by applying a non-linear automorphism to a monomial \( u \) one obtains a polynomial of the form \( u + v \) where \( v \) is bigger than \( u \) in the chosen term order. By Proposition 1.2, this is enough to argue as in the standard case, see for instance the proof of Theorem 15.20 in [E]. ▲

1.2 Weighted strongly stable ideals

Generic initial ideals in a standard graded polynomial ring are characterized combinatorially, the simplicity of this description depending on the characteris-
tic of the base field. In a weighted polynomial ring over a base field of characteristic 0 the same can be performed, via the following definition.

**Definition 1.4.** Let $I$ be a monomial ideal. $I$ is called (strongly) stable if the following holds: for every $u \in I$, if $X_{ij} | u$ then $\frac{u}{X_{ij}} \in I$, for every $h < j$ and $\frac{u}{X_{ij}} \in I$ for all monomials $v$ of degree $q_i$ in $R_{[j-1]}$.

It is not difficult to prove that weighted generic initial ideals are stable according to this definition.

**Proposition 1.5.** Let $I$ be an homogeneous ideal. $I$ is $T$-fixed if and only if $I$ is strongly stable.

**Proof.** One begins by observing that $I$ is fixed by the subgroup of diagonal matrices if and only if $I$ is monomial. Let $m = X_{ij}^t m'$, where $X_{ij} \not| m'$. The images $\tau_{ij}^t(m), \eta_{ij}^s(m)$, with deg $s = q_i$, can be written as $(X_{ij} + cX_{ir})^t m'$ and $(X_{ij} + s)^t m'$ respectively. If $I$ is $T$-fixed both polynomials, and so each of their monomials, belong to $I$. In particular the conditions which define strongly stable ideals are verified. Conversely, if $I$ is strongly stable the same argument shows that $I$ is fixed by the action of the generators of $U$ and $N$, and is therefore $T$-fixed. ▲

One of the key properties of strongly stable ideals in the standard graded polynomial ring $K[X_1, \ldots, X_n]$ is that $I : (X_1, \ldots, X_n) = I : X_n$. In fact, beside the trivial inclusion $I : (X_1, \ldots, X_n) \subseteq I : X_n$ one has that $m \in I : x_n$ iff $mX_n \in I$, which, because of the stability property, implies $mX_i \in I$ for all $i$. It is quite clear how this property is weakened in the more general case where variables might have different weights. In particular the good property of stable ideals with respect to taking colons with the last variables plays a central role in the construction of the Eliahou-Kervaire resolution [EK] of such an ideal. This is a completely described minimal graded free resolution of such an ideal in terms of its minimal set of monomial generators. On the other hand being able to construct such a resolution having no restriction on the weight vector would mean to know how to describe a minimal resolution of any monomial ideal, since given any such ideal $I$, one can choose weights so that in the corresponding polynomial ring $I$ is stable, as the next example shows.

**Example 1.6.** Let $A$ be a set of monomials in $n$ variables. Then, there exist non-negative integers $q_1, \ldots, q_n$ such that in the weighted polynomial ring $(K[X_1, \ldots, X_n], (q_1, \ldots, q_n))$ the ideal generated by $A$ is strongly stable. In fact, one can choose weights in such a way that none of the exchanges which were described in Definition 1.4 is possible. For instance, it is enough to choose $q_1 < q_2 < \ldots < q_n$ so that $2q_1 > q_n$, we say $q_1 = n + 1, q_2 = n + 2, \ldots, q_n = 2n$.

2 Prime Avoidance

A simple fact of linear algebra gives rise to a powerful tool when combined with techniques dealing with generic forms. This is known as Homogeneous
Prime Avoidance: If $p_1, \ldots, p_n$ are prime ideals strictly contained in the graded maximal ideal of a standard graded algebra over an infinite field then there exists a homogeneous form of degree 1 in $m \setminus \cup_i p_i$. It turns out to be essential in many proofs, since avoiding a finite number of primes is an open property.

**Lemma 2.1 (Weighted Prime Avoidance).** Let $q = \text{lcm}(q_1, \ldots, q_n)$ and let $p_1, \ldots, p_n$ be prime ideals with $p_i \subseteq m$. Then $m_q \setminus \cup_i (p_i)_q \neq \emptyset$.

**Proof.** Since the prime ideals are strictly contained in the maximal ideal, we have that $(p_i)_q \neq m_q$ for all $i$. Else, one would have that $(p_i)_q = m_q$ and $X_{jk} \in p_i$, for all $j$ and $k$, since $X_{jk}^{q_i/q_j} \in m_q$ and $p$ is prime. But the infinite vector space $m_q$ cannot be written as a finite union of proper subspaces $p_q$, and the claim follows. ▲

The next example shows that in general it is not possible to find such a form in a smaller degree.

**Example 2.2.** Let $(R, w) = (K[X,Y], (2,3))$. If $p_1 = (X)$ and $p_2 = (Y)$ then the smallest degree $d$ such that $(X,Y)_d \supseteq (p_1)_d \cup (p_2)_d$ is 6.

In the following we recover some results which are known in the standard case, provided that some condition on the weights is assumed. It may be convenient to state one of these conditions here.

**Condition 2.3.** $(R, w)$ is a weighted polynomial ring with $q_i | q_i + 1$ for $i = 1, \ldots, n - 1$.

**Lemma 2.4.** Let $(R, w)$ be a ring for which Condition 2.3 is satisfied, and let $I \subseteq R$ be a strongly stable ideal. For any $i = 1, \ldots, n$ and $j = 1, \ldots, t_i$, one has $I : X_{ij}^{\infty} = I : (X_{11}, \ldots, X_{ij})^{\infty}$.

**Proof.** We only have to prove the inclusion $\subseteq$ since the other one is obvious. Let $m$ be a monomial such that $mX_{ij}^s \in I$ for some $s \in \mathbb{N}$. Since $I$ is strongly stable, $mX_{ih}^s \in I$ for any $1 \leq h \leq j$; furthermore the assumption on the degrees of the indeterminates implies that $\deg X_{ij} = q_i = q_h = \deg X_{ih}^{q_i/q_h}$, and consequently $m(X_{hk}^{q_i/q_h})^s \in I$ for any $1 \leq h \leq i - 1$ and $1 \leq k \leq l_h$, as desired. ▲

As a consequence we obtain the following proposition.

**Proposition 2.5.** Let $(R, w)$ be a ring for which Condition 2.3 is satisfied, let $I$ be a strongly stable ideal and let $X_{hk}$ be the (lex-)smallest variable which divides some minimal generator of $I$. Then $X_{hk} + 1, \ldots, X_{nl}$ form a maximal regular sequence on $R/I$.

**Proof.** Clearly the elements $X_{hk+1}, \ldots, X_{nl}$ form a regular sequence on $R/I$. Since in the quotient ring $\overline{R} = K[X_{11}, \ldots, X_{hk}]$ the ideal $\overline{I}$ is strongly stable, by the previous lemma $\overline{I}^{\text{sat}} = \overline{I} : X_{hk}^{\infty} \neq \overline{I}$, which implies that depth $\overline{R}/\overline{I} = 0$. ▲
We recall the following theorem [BS], which holds independently of the given weights and is needed for the proof of the final result of this section.

**Theorem 2.6.** Let $F$ be a free $R$-module with basis and consider the degree reverse lexicographic monomial order. Let $M$ be a graded submodule of $F$. The elements $X_{nl}, X_{nl-1}, \ldots, X_{ij}, X_{ij+1}$ form a regular sequence on $F/M$ if and only if they form a regular sequence on $F/\in(M)$.

**Proof.** See that of Theorem 15.13 in [E]. ▲

**Theorem 2.7.** Let $(R,w)$ be a ring for which Condition 2.3 is satisfied, and consider the degree reverse lexicographic order. Then, for any homogeneous ideal $I \subseteq R$,

$$\text{depth } R/I = \text{depth } R/\text{Gin}(I).$$

**Proof.** Since depth $R/I \geq \text{depth } R/\text{Gin}(I)$, we may assume depth $R/I > 0$. By Lemma 2.1, a generic form of degree $q_n$ is a non-zerodivisor on $R/I$. Thus, after a generic change of coordinates, we may assume that $X_{nl}, X_{nl-1}, \ldots, X_{ij}$ is a maximal $R/I$-regular sequence and $\text{Gin}(I) = \in(I)$. By Theorem 1.3 and Proposition 1.5 $\text{Gin}(I)$ is strongly stable, and consequently, by Proposition 2.5, there is a maximal $R/\text{Gin}(I)$-regular sequence $X_{nl}, X_{nl-1}, \ldots, X_{hk}$. Now Theorem 2.6 yields that $(h,k) = (i,j)$, from which the conclusion is straightforward. ▲

## 3 Regularity

Local cohomology modules of a graded module over a weighted polynomial ring have a graded structure arising from resolutions by graded injective modules or equivalently from the construction of the Čech complex. The usual definition of Castelnuovo-Mumford regularity by means of local cohomology still works in this context and we recall it here. Let $H^i_m(M)$ denote the $i$-th graded local cohomology module of the graded $R$-module $M$ with support on the graded maximal ideal $m$.

**Definition 3.1.** Let $R$ be a weighted polynomial ring with graded maximal ideal $m$. We let

$$a^i(M) = \begin{cases} \max \{ j \in \mathbb{Z}: H^i_m(M)_j \neq 0 \} & \text{if } H^i_m(M) \neq 0 \\ -\infty & \text{otherwise} \end{cases}$$

denote the end of the $i$-th local cohomology module of $M$. The *Castelnuovo-Mumford regularity* of $M$ is then $\text{reg } M = \max_{1 \leq i \leq \dim M} \{ a^i(M) + i \}$.

However, one of the aspects that made the Castelnuovo-Mumford regularity interesting, i.e. its direct interpretation through the Betti numbers of the minimal free resolution by means of the formula

$$\text{reg } M = \max_{i \geq 0} \{ b_i(M) - i \},$$

(3.1)
where \( b_i(M) = \max_{j \in \mathbb{Z}} \{ \beta_{ij}(M) \neq 0 \} \), fails in the general weighted case. In this section we re-prove some results about regularity which still hold in the weighted case, in order to give in Theorem 3.5 a formula that generalizes (3.1). In the last part, we consider the regularity of a generic initial ideal \( \text{Gin}(I) \) and prove in Proposition 3.6 that under some assumption on weights it does not differ from that of \( I \). Also, we provide a counterexample that shows that in general there is no analogue of the well-known theorem [BS] valid in the standard case.

We start by recalling some lemmata which are useful in order to control regularity in the non-standard case.

**Lemma 3.2.** Let \( 0 \to N \to M \to Q \to 0 \) be a short exact sequence of finitely generated graded \( R \)-modules. Then

(i) \( \text{reg } N \leq \max \{ \text{reg } M, \text{reg } Q + 1 \} \);

(ii) \( \text{reg } M \leq \max \{ \text{reg } N, \text{reg } Q \} \);

(iii) \( \text{reg } Q \leq \max \{ \text{reg } N - 1, \text{reg } M \} \);

(iv) If \( N \) has finite length, then \( \text{reg } M = \max \{ \text{reg } N, \text{reg } Q \} \).

**Proof.** The proofs of (i) – (iii) are easy and descend from the use of the long exact sequence in cohomology...

As for the proof of (iv), it is clear that \( \text{reg } N = a^0(N) \) and \( a^0(M) = \max \{ a^0(N), a^0(Q) \} \). Thus,

\[
\text{reg } M \doteq \max \{ a^0(M), \max_{i > 0} \{ a^i(M) + i \} \}
= \max \{ a^0(N), a^0(Q), \max_{i > 0} \{ a^i(Q) + i \} \},
\]

as desired. ▲

**Lemma 3.3.** Let \( M \) be a finitely generated graded \( R \)-module and let \( x \in R_d \). If \( x \) is a non-zerodivisor on \( M \) then \( \text{reg } M/\langle x \rangle M = \text{reg } M + (d-1) \). More generally, if \( x \) is such that \( (0 :_M x) \) has finite length, then

\[
\text{reg } M = \max \{ \text{reg } 0 :_M x, \text{reg } M/\langle x \rangle M - (d-1) \}.
\]

**Proof.** From the exact sequence \( 0 \to (0 :_M x)(-d) \to M(-d) \to M \to M/\langle x \rangle M \to 0 \) one obtains the two short exact sequences \( 0 \to (0 :_M x)(-d) \to M(-d) \to xM \to 0 \) and \( 0 \to xM \to M \to M/\langle x \rangle M \to 0 \) so that the proof follows easily as an application of Lemma 3.2. ▲

**Lemma 3.4.** Let \( x \in R_d \) such that \( 0 :_M x \) is of finite length. Then for all \( i \geq 0 \)

\[
a_{m+1}^i(M) + d \leq a_m^i(M/\langle x \rangle M) \leq \max \{ a_m^i(M), a_{m+1}^i(M) + d \}.
\]
**Proof.** From the two short exact sequences contained in the proof of the last lemma we deduce that $H^i_m(M(-d)) \simeq H^i_m(xM)$ for all $i > 0$ and obtain the long exact sequence in cohomology $\ldots \rightarrow H^i_m(M) \rightarrow H^i_m(M/xM) \rightarrow H^{i+1}_m(xM) \rightarrow \ldots$. To prove the first inequality, it is enough to observe that, if $a^i_m(M/xM) < a^{i+1}_m(M) + d$ the above long exact sequence in degree $a^{i+1}_m(M) + d$ would deliver a contradiction. The proof of the second inequality is analogous. ▲

Let $M$ be a finitely generated $R$-module of finite projective dimension $s$. For $i = 1, \ldots, s$ let as before $b_i(M) = \max_{j \in \mathbb{Z}} \{ \beta_{ij}(M) \neq 0 \}$.

**Theorem 3.5.** Let $R = K[X_1, \ldots, X_n]$ be a graded polynomial ring and let $M$ be a finitely generated $R$-module with proj dim $M < \infty$. Then

$$\text{reg } M = \max_{i \geq 0} \{ a^i_m(M) + i \} = \max_{i \geq 0} \{ b_i(M) - i \} - \sum_{j=1}^{n} l_j(q_j - 1).$$

**Proof.** By virtue of the previous lemma and induction on the number of variables one first proves that

$$b_0(M) \leq \max_{i \geq 0} \{ a^i_m(M) + i \} + \sum_{j=1}^{n} l_j(q_j - 1). \quad (3.2)$$

Moreover, it is easy to verify that, if $F$ is a free $R$-module,

$$a^n_m(F) = b_0(F) - \sum_{i=1}^{n} l_i q_i. \quad (3.3)$$

The assertion follows by the use of (3.2) and (3.3) combined with an induction argument on the projective dimension of $M$. The proof is an adaptation of that of Theorem 5.5 in [Be] (to which the interested reader is referred) and, therefore, the details are omitted here. ▲

**Proposition 3.6.** Let $(R, w)$ be a weighted polynomial ring for which Condition 2.3 is satisfied and consider the degree reverse lexicographic order. If $I$ an homogeneous ideal of $R$ then

$$\text{reg } R/I = \text{reg } R/ \text{Gin}(I).$$

**Proof.** Since $q_n = \text{lcm}(q_1, \ldots, q_n)$, by Lemma 2.1 a generic form in $m q_n$ does not belong to any associated prime $p \neq m$ of $I$. By applying a generic automorphism we may assume that $\text{Gin}(I) = \text{in}(I)$ and that $X_{nl_n}$ is almost-regular. Therefore $(I : X_{nl_n})/I$ and, consequently, $\text{in}(I : X_{nl_n})/\text{in}(I) \simeq (\text{in}(I) : X_{nl_n})/\text{in}(I)$ have finite length. By Lemma 3.3, it is enough to verify that $\text{reg}(I : X_{nl_n})/I = \text{reg}(\text{in}(I) : X_{nl_n})/\text{in}(I)$ in order to apply induction on the numbers of the variables, since the assumption on the weights still holds for $R/(X_{nl_n})$. But this is clear because the above modules coincide with their 0th local cohomology module and they have the same Hilbert function. ▲
Notice that the assumption on the weights is essential in order to have a generic form of the right degree for the induction. The following example shows that the above result cannot be extended for any choice of weights.

**Example 3.7.** Let \((R, w) = (K[X, Y, Z], (2, 4, 5))\) and \(I = (XY, YZ, X^5)\) and consider the degree reverse lexicographic order. A computation with [CoCoA] shows that \(\text{Gin}(I) = (X^3, X^2Z, XY^2, Y^3Z)\). By Theorem 3.5, the regularity of \(I\) and \(\text{Gin}(I)\) can be computed by the use of the resolutions

\[
0 \rightarrow R(-14) \oplus R(-11) \rightarrow R(-10) \oplus R(-9) \oplus R(-6) \rightarrow I \rightarrow 0
\]

and

\[
0 \rightarrow R(-19) \rightarrow R(-17) \oplus R(-14) \oplus R(-11) \rightarrow R(-17) \oplus R(-10) \oplus R(-9) \oplus R(-6) \rightarrow \text{Gin}(I) \rightarrow 0
\]

of \(I\) and \(\text{Gin}(I)\) respectively.

This example points out that also Proposition 2.7 is not valid without Condition 2.3.

### 4 Lexicographic ideals

Although the definition and some of the main properties of Hilbert functions are still valid in a non-standard setting, a great deal is still unknown about them. In particular Macaulay’s Theorem, which provides a necessary and sufficient condition for a numerical function to be the Hilbert function of a finitely generated standard graded algebra has no counterpart in the weighted case. The main tool which is involved in this context, lexicographic ideals, can be easily defined in the non-standard case, but they are not so easily investigated, as the following analysis shows.

#### 4.1 Hilbert functions

Here we point out some facts about non-standard graded algebras which are relevant for our purposes. We start by recalling the well-known Hilbert-Serre Theorem: Let \(I\) be a homogeneous ideal in \((R, w)\). The Poincare series \(P(R/I, t)\) of \(R/I\) is a rational function in \(t\) of the form \(g(t)/\prod_{i=1}^{n}(1-t^{q_i})^{l_i}\), where \(g(t) \in \mathbb{Z}[t]\). It is known that the Hilbert function of \(R/I\) is quasi-polynomial. Some more information is provided by the following result to be found in [B], Theorem 2.2.

**Proposition 4.1.** Let \(I\) be a homogeneous ideal in \((R, w)\) and let \(d\) be the order of the pole of \(P(R/I, t)\) at the point \(t = 1\). Then there exist \(q = \text{lcm}(q_1, \ldots, q_n)\) polynomials \(p_0, \ldots, p_{q-1} \in \mathbb{Q}[t]\) of degree at most \(d - 1\) with coefficients in \([q^{d-1}(d-1)!]^{-1}\mathbb{Z}\) such that, for all \(l \gg 0\),

\[
H_{R/I}(l) = p_j(l) \text{ for } l \equiv j \mod q
\]
It is also worth observing that in general some of the Hilbert polynomials described in the above proposition can be 0. Also in the case \( \gcd(q_1, \ldots, q_n) = 1 \) the vanishing of the Hilbert function of \( R/I \) in \( t = t_0 \) does not imply that \( H_{R/I}(t) = 0 \) for all \( t > t_0 \). However, this is true for \( H_R(t) \) if \( t_0 \) is bigger than the Frobenius number of \( q_1, \ldots, q_n \) (cf. [SS], Chapter 1, Section 3 for more details about this subject).

**Remark 4.2.** Let \((R, w)\) be a weighted polynomial ring. If \( w_i = q \) for all \( i \), then the Hilbert function \( H_{(R,w)}(t) \) is equal to \( H_{(R,\langle 1,\ldots,1 \rangle)}(t/q) \) if \( q \mid t \) and 0 otherwise; this case is thus essentially equivalent to the standard case. The same observation shows that one may assume without loss of generality that the gcd of the weights is 1.

Another pathology of the weighted case is shown in the following example [BR].

**Example 4.3.** Let \((R, w) = (K[X,Y,Z,T], \langle 1, 6, 10, 15 \rangle)\).

The monomial \( X^4Y^2ZT \) has degree 60, but it is not multiple of any monomial of degree 30.

However, one can show that this can only occur in low degrees, as it is shown in [BR], Proposition 4B.5. One makes use of an invariant introduced in [D], which we denote by \( G(w) \). For the reader’s sake we recall here the result, omitting the definition of \( G(w) \) since it is not essential in what follows.

**Proposition 4.4.** Let \((R, w)\) be a weighted polynomial ring and let \( n > G(w) \).

Then every monomial of \( R_{n+hq} \) is divisible by a monomial in \( R_{hq} \), for any \( h \in \mathbb{N} \).

One might wonder if the same holds for an arbitrary ideal generated in more than one degree, i.e. if there exists \( l \in \mathbb{N} \) such that for all \( r \gg 0 \) one has \( I_r = I_lR_{r-l} \). Unfortunately this is false and it partially explains why the study of lexicographic ideals is complicated.

**Example 4.5.** Let \((R, w) = (K[X,Y,Z], \langle 2, 2, 3 \rangle)\) and \( I = (X^\alpha, XYZ) \) for some integer \( \alpha > 1 \). Let us suppose that there exists some \( l \in \mathbb{N} \) such that \( I_r = I_lR_{r-l} \) for all \( r \gg 0 \). Then, for all \( k \gg 0 \), \( X^k \in I \) and this implies that \( l \) is even. On the other hand \( XY^kZ \in I \) for all \( k \gg 0 \); thus there exists \( k_0 \) such that \( XY^{k_0}Z \in I_l \) and \( l = 2 + 2k_0 + 3 \), which means that \( l \) is odd.

### 4.2 Lexifiable ideals

Let us consider a standard graded polynomial ring \( K[X_1, \ldots, X_n] \) with the degree lexicographic order. We recall that a **lexsegment (of degree d)** is a set \( L \) of monomials of degree \( d \) with the property: if \( u \in L \) and \( v > u \) with \( \deg v = \deg u \) then \( v \in L \). A homogeneous monomial ideal is said to be lexigraphic if all its graded components are spanned as a \( K \)-vector space by lexsegments. It is clear that these definitions can be overtaken and used in the weighted case. Lexigraphic ideals play a central role in many results in commutative algebra because of their well understood structure. It would be of great interest to
Example 4.6. Let \((R, w) = (K[X, Y], (2, 3))\), and consider the monomial \(XY\). This is the only monomial of degree 5, and \(\{XY\}\) is obviously a lexsegment. Its shadow in degree 8 is \(\{XY^2\}\), which is not lexsegment, since \(X^4\) does not belong to it.

It is thus quite clear that there are strong restrictions also on Hilbert functions of lexsegment ideals generated in one degree. For instance, an ideal \(I\) generated in degree \(d\) is a lexicographic ideal only if contains \(X_{11}^k\) for some \(k > d\); this is possible only if \(d = \alpha q_1\).

Example 4.7. Let \(I \subseteq R = K[X_1, \ldots, X_n]\) with \(n > 1\) be an ideal generated by a lexsegment in degree \(d = \alpha q_1\) and \(H_1(d) \leq l_1\). Then \(I\) is a lexicographic ideal.

In fact \(I\) is generated by the lexsegment \(\{X_{11}^d, X_{12}^{d-1}, \ldots, X_{11}^{d-1}X_{1h}\}\), with \(h \leq l_1\); if we let \(m \in I_r\) be a monomial with \(r \geq d\) then, since \(m\) belongs to \(I\), \(m = X_{11}^{d-1}X_{1j}m'\) for some \(j = 1, \ldots, h\) and a monomial \(m' \in R_{r-d}\). If \(s\) is a monomial with \(\deg s = \deg m\) and \(s \geq m\), then \(s\) must be \(X_{11}^{d-1}X_{1j}^l s'\) with \(j' < j\) or \(j' = j\) and \(s' \geq m'\). It is thus clear that \(s \in I_r\).

Example 4.8. Let \((R, w) = (K[X, Y, Z], (2, 2, 3))\). By the previous example, the ideal \((X^3, X^2Y)\) is lexicographic, whereas \((X^3, X^2Y, XY^2)\) is not. However the ideal \((X^3, X^2Y, XY^2, X^2Z^2)\) is lexicographic, as an easy verification shows.

At this point it is still not evident if it is possible to determine whether an ideal is lexicographic by looking at finitely many of its graded components. The following proposition yields that this is indeed the case.

Proposition 4.9. Let \(I \subset (R, w)\) be a homogeneous ideal generated in degree \(\leq d\) and let \(q = \text{lcm}(q_1, \ldots, q_n)\). If \(I_i\) is spanned (as a \(K\)-vector space) by a lexsegment for all \(i \leq d + q + G(w)\), then \(I\) is a lexicographic ideal.

Proof. The proof is by induction on \(i\). We only have to prove that \(I_i\) is spanned by a lexsegment if \(i > d + q + G(w)\), provided that this is true for \(I_r\) with \(r < i\). If \(I_i = \emptyset\) there is nothing to prove. Else, let \(v_i\) be the smallest monomial in \(I_i\) and let \(u > v_i\), with \(\deg u = \deg v_i\), be a monomial not in \(I\). Finally let \(X_{jh}\) denote the (lex-)smallest variable which divides \(u\). Now we write \(v_i\) as \(vm\), where \(v\) is a minimal generator of \(I\), we say of degree \(d' \leq d\), and \(m\) is the smallest monomial in \(R_{i-d'}\). Since \(i - d' > q + G(w)\), by Proposition 4.4, we may write \(m = m'm''\), where \(m'\) is the smallest monomial of \(R_q\), which is \(X_{nln}^{\alpha q_n}\).

Thus, \(v_i = vX_{nln}^{\alpha q_n} m''\). If we now let \(w = \frac{v_iX_{nln}^{\alpha q_n}}{X_{nln}^{\alpha q_n}}\), it is clear that \(u \geq w \geq v_i\).
We start by proving that if \( w \in I_i \) and \( w/X_{jh} \in I_{r-q_j} \). But this is a contradiction, since \( u/X_{jh} \geq w/X_{jh} \) and \( I_{r-q_j} \) is spanned by a lexsegment.

In a polynomial ring with two variables and coprime weights, one can expect to have a description of lexicographic ideals, because of the following observation. Given a polynomial ring \((R,w)\) according to [CL] we call any set of consecutive monomials of the same degree a block. If \( R \) is a polynomial ring in two variables, any shadow of a block is clearly a block. With this notation, a lexsegment is an initial block, and Example 4.6 shows that in general the shadow of an initial block needs not to be such.

Before proceeding with the characterization of lexicographic ideals of \( K[X,Y] \), we need to fix some notation. Given any set \( A \) of monomials of degree \( d \), we let \( \text{Shad}_d(A) \subseteq R_{d+i} \) denote the set of the elements \( um \), where \( u \in A \) and \( m \) is a monomial in \( R_1 \). Clearly, the cardinality of \( \text{Shad}_d(A) \) equals that of \( A \). Moreover, \( |\text{Shad}_{q_2}(A)| = |A| \) if \( q_1 \neq 1 \) and \( |\text{Shad}_{q_2}(A)| = |A| + 1 \) if \( q_1 = 1 \) and \( A \) is a block.

Finally, if \( d \in \mathbb{N} \) and \( q_1 \nmid d \), we let \( \delta \) denote the smallest integer \( d + \beta q_2 \), \( \beta \in \mathbb{N} \), divisible by \( q_1 \). It is not difficult to see that such a number exists and it is such that \( \beta < q_1 \) since \( q_1 \) and \( q_2 \) are assumed to be coprime.

**Lemma 4.10.** Let \( L \) be a lexsegment of degree \( d \).

(i) If \( q_1 \mid d \) then \( \text{Shad}_d(L) \) is a lexsegment for all \( i \).

(ii) If \( q_1 \nmid d \) then \( \{X^\delta/q_1\} \cup \text{Shad}_{\delta-d}(L) \) is a lexsegment (of degree \( \delta \)).

**Proof.** The proof of (i) is obvious.

To prove (ii), let \( X^aY^b \) be the largest monomial of \( L \) and \( \delta = d + \beta q_2 \). First observe that \( b < q_1 \). Secondly, notice that the largest monomial of \( \text{Shad}_{\delta-d}(L) \) is \( X^aY^{b+\beta} \). Since \( b + \beta < 2q_1 \) and is a multiple of \( q_1 \), we have \( b + \beta = q_1 \), so that the only monomial of \( R_\delta \) which is larger is \( X^\delta/q_1 \).\( \square \)

**Theorem 4.11.** Let \( I \) be a monomial ideal minimally generated in degrees \( d_1 < d_2 < \ldots < d_r \) such that the monomials of \( I_{d_i} \) form a lexsegment for all \( i = 1, \ldots, r \). Then \( I \) is a lexicographic ideal if and only if \( q_1 \mid d_1 \) or \( q_1 \nmid d_1 \) and there exists \( 1 < s \leq r \) such that \( q_1 \mid d_s \) and \( d_s \leq \min\{\delta_i\} \).

**Proof.** We start by proving that if \( q_1 \mid d_1 \) then \( I \) is a lexicographic ideal. For this purpose it is enough to observe that the shadow of a block is a block and use Lemma 4.10.

We show now that if \( q_1 \nmid d_1 \), the conditions on \( d_s \) imply that \( I \) is lexicographic. In fact it is sufficient to verify that \( \text{Shad}_d(\{I_{d_i}\}) \) are lexsegments for all \( i \) and \( j = 1, \ldots, r \). Since the generator \( X^{d_s/w} \) occurs in degree \( \leq \min\{\delta_i\} \), the conclusion follows again by Lemma 4.10.

Finally, if \( I \) is lexicographic and \( q_1 \nmid d_1 \), then \( X^k \in I_{kq_1} \) with \( kq_1 = d_s \) for some \( 1 < s \leq r \). By Lemma 4.10 (ii) it is thus clear that \( d_s \leq \min\{\delta_i\} \).\( \square \)
The conditions in the previous proposition can be easily reformulated in terms of Hilbert series. In general, it would be interesting to have a solution for the following problem.

**Problem 4.12.** Find a combinatorial characterization for the Hilbert series of lexicographic ideals.

In the same spirit of [MP], we say that an ideal \( I \subseteq (R, w) \) is **lexifiable** if there exists a lexicographic ideal \( L \) with the same Hilbert function as \( I \).  

Given a subset \( A \) of monomials in \( R_t, \) \( i \in \mathbb{N} \), we let \( \text{Lex}(A) \) denote the set of the \( |A| \) lexicographic largest elements of \( R_t \). We also let \( L \doteq \oplus_{i \in \mathbb{N}} (\text{Lex}(\{I_i\})) \).

Thus, \( I \) is lexifiable iff \( L \) is an ideal of \( (R, w) \). To establish which ideals are lexifiable is not an easy task. The following example shows an ideal which is not lexifiable in any lex-order.

**Example 4.13.** Let \( (R, w) = (K[X, Y], (2, 3)) \). The ideal \( (Y) \) provides an easy example of an ideal which is lexifiable if \( Y \succ_{\text{Lex}} X \) and not lexifiable if \( X \succ_{\text{Lex}} Y \).  

Let \( I = (X^3Y^3, X^2Y^4) \). \( I \) is not lexifiable in both cases \( X \succ_{\text{Lex}} Y \) and \( Y \succ_{\text{Lex}} X \). If \( X \succ_{\text{Lex}} Y \) then the candidate to be the associated lexicographic ideal with \( I \) is the ideal \( L = (X^8, X^6Y) \) but \( H_I(18) = 1 \) and \( H_L(18) = 2 \). If \( Y \succ_{\text{Lex}} X \) the candidate is \( L = (Y^5, Y^4X^2) \), but again \( H_L(18) = 2 \).

**Example 4.14.** Let \( (R, w) = (K[X, Y], (2, 7)) \). The monomials of degree 28 and 35 are \( X^{14}, X^7Y^2, Y^4 \) and \( X^{14}Y, X^7Y^3, Y^5 \) respectively. Let us consider the ideals which have exactly one minimal generator in these two degrees. These are \( I_1 = (X^{14}, X^7Y^3), I_2 = (X^7Y^2, Y^5), I_3 = (X^{14}, Y^5), I_4 = (X^7Y^2, X^{14}Y), I_5 = (Y^4, X^7Y^3) \) and \( I_6 = (Y^4, X^{14}Y) \). According to our definitions \( I_1 \) is a lexicographic ideal, \( I_2 \) is lexifiable and \( I_1 \) is the lexicographic ideal associated with it, \( I_3 \) is lexifiable associated with \( (X^{14}, X^7Y^3, Y^7) \). The ideals \( I_4, I_5 \) and \( I_6 \) are not lexifiable, as a computation of the Hilbert function in degree 42, 42 and 56 shows.

Again according to [MP], we say that a graded polynomial ring \( (R, w) \) is **Macaulay-Lex** if every homogeneous ideal in \( (R, w) \) is lexifiable. Macaulay’s Theorem together with Remark 4.2 says that \( (R, w) \) with \( w = (a, \ldots, a) \) is Macaulay-Lex, whereas for a general choice of \( w \) there are many ideals which are not lexifiable. Thus it is natural to ask the following question.

**Question 4.15.** Which polynomial rings \( (R, w) \) are Macaulay-Lex?

The results of the last part of this section shed some light on the problem and provide partial answers to the above question.

**Theorem 4.16.** Let \( I \) be a homogeneous ideal in \( (R, w) = (K[X, Y], (1, q_2)) \). There exists a unique lexicographic ideal \( L \) such that \( H_{R/I}(t) = H_{R/L}(t) \) for any \( t \in \mathbb{N} \).
Proof. Taking in consideration what we have said before Example 4.13, it is sufficient to prove that

\[ \text{Shad}_1(\text{Lex}([I_d])) \subseteq \text{Lex}([I_{d+1}]) \quad \text{and} \quad \text{Shad}_q(\text{Lex}([I_d])) \subseteq \text{Lex}([I_{d+q}]). \]

Since \( q_1 = 1 \), \( \text{Shad}_1 \) of an initial block is still an initial block, and therefore we can reason on cardinalities.

The first inclusion is immediate since, for any \( A \), \( |\text{Shad}_1(A)| = |A| \), and consequently \( |\text{Shad}_1(\text{Lex}([I_d]))| = |\text{Lex}([I_d])| = |[I_d]| = |\text{Shad}_1([I_d])| \) which is equal to \( |\text{Lex}([\text{Shad}_1([I_d])])| \).

For the second inclusion, we write \( [I_d] \) as \( \sqcup_{i=1}^{s} B_i \), where \( B_i \) are maximal blocks. It is easy to see that \( \text{Shad}_q(B_i) \cap \text{Shad}_q(B_j) = \emptyset \). Therefore

\[
|\text{Lex}(\text{Shad}_q([I_d]))| - |\text{Lex}([I_d])| = |\text{Shad}_q([I_d])| - |[I_d]|
= \sum_{i=1}^{s} |\text{Shad}_q(B_i)| - |B_i|
\geq 1
= |\text{Shad}_q(\text{Lex}([I_d]))| - |\text{Lex}([I_d])|,
\]

and the proof is concluded. \(\blacksquare\)

**Example 4.17.** This is an example of an ideal \( I \) which is not lexifiable in a ring for which Condition 2.3 is satisfied.

Consider \( (R, w) = (K[X, Y, Z], (1, 2, 4)) \) and let \( I = (X^4, Y^2, X^3Y) \), for which we have that \( H_I(4) = 2, H_I(5) = 3, H_I(6) = 4, H_I(7) = 4 \). \( I \) is not lexifiable, in fact, if we try to construct the associated lexicographic ideal \( L \), we shall have \( L_4 = \{X^4, X^2Y\}, L_5 = \{X^5, X^3Y, XY^2\} \) and \( L_6 = \{X^6, X^4Y, X^2Y^2, X^2Z\} \), so that \( H_L(7) \geq 5 \).

**Example 4.18.** Let \( (R, w) = (K[X, Y, Z], (1, a, ab)) \). The ideal

\[ I = (X^{ab}, Y^b, X^{a+1}Y^{b-1}) \]

is not lexifiable, therefore \( (R, w) \) is not Macaulay-Lex. Let \( b > 2 \) and suppose that \( I \) is lexifiable with associated lexicographic ideal \( L \). We first observe that \( I_j \) does not contain any monomial divisible by \( Z \) for all \( j < 2ab \). Secondly, we show that \( X^{ab-2a}Z \) is a monomial of \( L \). Since \( H_I(ab + (a + 1)a) \geq H_I(ab + aa) + 1 \) for all positive \( a \in \mathbb{Z} \), and \( H(ab + a) = 5 \), one gets that \( H_I(ab + (b - 2)a) \geq H_I(ab + (a - 1)a) + b - 3 = b + 2 \). Since the first \( b + 2 \) monomials in degree \( ab + aa \) are \( X^{ab+aa}, X^{ab+(a-1)a}Y, \ldots, X^{aa}Y^b, X^{aa}Z \), this proves our claim. It is convenient now to write down all the monomials of degree \( 2ab - 2a \) which are \( \geq Y^{2b-2} \), which is the smallest monomial of \( I \) in this degree. These are

\[
X^{2ab-2a}, X^{2ab-3a}Y, \ldots, X^{ab-2}Y^b, X^{ab-2a}Z, \\
X^{ab-3a}Y^{b+1}, X^{ab-3a}Y^bZ, \ldots, X^aY^{2b-3}, X^aY^{b-3}Z, Y^{2b-2}.
\]

As a consequence, it is easy to compute that \( H_I(2ab - 2a) \) is \( (b+1) + (b-3+1) = 2b - 1 \). Analogously one gets that \( H_I(2ab - 1) = 2b \) and furthermore the monomials

\[ \ldots, X^{ab-2}Y^b, X^{ab-2a}Z, X^{ab-2a}Y^{b-1}, X^{ab-2a}Y^bZ, \ldots, X^aY^{2b-3}, X^aY^{b-3}Z, Y^{2b-2}, \ldots, \]

\[ \ldots, X^{ab-2}Y^b, X^{ab-2a}Z, X^{ab-2a}Y^{b+1}, X^{ab-2a}Y^bZ, \ldots, X^aY^{2b-3}, X^aY^{b-3}Z, Y^{2b-2}, \ldots, \]

\[ \ldots, X^{ab-2}Y^b, X^{ab-2a}Z, X^{ab-2a}Y^{b+1}, X^{ab-2a}Y^bZ, \ldots, X^aY^{2b-3}, X^aY^{b-3}Z, Y^{2b-2}, \ldots, \]

\[ \ldots, X^{ab-2}Y^b, X^{ab-2a}Z, X^{ab-2a}Y^{b+1}, X^{ab-2a}Y^bZ, \ldots, X^aY^{2b-3}, X^aY^{b-3}Z, Y^{2b-2}, \ldots, \]

\[ \ldots, X^{ab-2}Y^b, X^{ab-2a}Z, X^{ab-2a}Y^{b+1}, X^{ab-2a}Y^bZ, \ldots, X^aY^{2b-3}, X^aY^{b-3}Z, Y^{2b-2}, \ldots, \]
of $L_{2ab-2}$ have a certain number of multiples in degree $2ab-1$, which we can count. We can thus estimate the cardinality of $L_{2ab-1}$ as follows:

$$|L_{2ab-1}| \geq \left| \{ u \in L_{2ab-1} : Z \mid u \} \right| + \left| \{ u \in L_{2ab-1} : Z \notdivides u \} \right|$$

$$= \left( b + 1 \right) + \left\lceil \frac{b - 3}{2} \right\rceil + 1 + \left( \left\lfloor \frac{b - 3}{2} \right\rfloor + 1 \right) + 1$$

$$= 2b + 1.$$

This is a contradiction since $I$ and $L$ have by definition the same Hilbert function.

Finally, if $b = 2$, an easy computation of the Hilbert function in degree $3a + 1$ shows that $I$ is not lexifiable.

### 4.3 Polarization

In [P] it is shown how in the standard case the lexicographic ideal $L$ associated with an ideal $I \subseteq R$ can be also obtained as the result of a finite process which consists of three fundamental steps, which are a) polarizing a monomial ideal b) modding out by a sequence of generic linear forms and c) taking initial ideals (with respect to the lexicographic order). In the non-standard case, following step-by-step the original proof and using generic sequences of homogeneous forms (which are not necessarily linear) it is not difficult to prove that the same procedure also terminates and leads to an ideal $I^p$, which we call completely polarized. Since, as we have already seen, not all ideals are lexifiable, one might make use of the $I^p$, which is a strongly stable monomial ideal with the same Hilbert function as $I$, instead.

**Example 4.19.** Let $(R, w) = (K[X, Y, Z], (1, 2, 4))$ and

$$I = (X^8, X^6Y, X^4Y^2, X^2Y^3, Y^4, X^2YZ, X^6Z).$$

One can verify that the ideal

$$L = (X^8, X^6Y, X^4Y^2, X^4Z, X^2Y^3, X^2YZ, X^2Z^2, Y^6)$$

is the lexicographic ideal associated with $I$ and, thus, $I$ is lexifiable. On the other hand

$$I^p = (X^8, X^6Y, X^4Y^2, X^4Z, X^2Y^3, X^2Y^2Z, Y^4).$$

This shows that, even in the favourable case when Condition 2.3 is satisfied, one might have $I^p \neq L$.

We conclude by posing the following question.

**Question 4.20.** Is there a combinatorial characterization of completely polarized ideals?
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