SPECTRAL INVARIANTS IN LAGRANGIAN FLOER HOMOLOGY OF OPEN SUBSET

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ABSTRACT. We define and investigate spectral invariants for Floer homology $HF(H,U:M)$ of an open subset $U \subset M$ in $T^*M$, defined by Kasturirangan and Oh as a direct limit of Floer homologies of approximations. We define a module structure product on $HF(H,U:M)$ and prove the triangle inequality for invariants with respect to this product. We also prove the continuity of these invariants and compare them with spectral invariants for periodic orbit case in $T^*M$.

Keywords: Lagrangian submanifolds, Floer homology, spectral invariants

1. Introduction

Lagrangian Floer homology for open subsets in cotangent bundles was introduced by Kasturirangan and Oh in [8] as a part of a project of “quantization of Eilenberg–Steenrod axioms” (see [7]). The construction goes as follows. Let $U \subset M$ be an open subset of a compact smooth manifold $M$, with a smooth compact boundary $\partial U$. The conormal bundle, $\nu^*(\partial U)$, defined as

$$\nu^*(\partial U) = \{(q,p) \in T^*M \mid q \in \partial U, p|_{T_q\partial U} = 0\},$$

is a Lagrangian submanifold of the cotangent bundle $T^*M$. Define

$$\nu^*_{\epsilon}(\partial U) := \{\alpha \in \nu^*(\partial U) \mid \alpha(n) \leq 0, \text{ for } n \text{ outward normal to } \partial U\}$$

and

$$\nu^* \overline{U} := O_U \cup \nu^*_{\epsilon}(\partial U).$$

The set $\nu^* \overline{U}$, called the conormal to $U$, is a singular Lagrangian submanifold, but it allows a smooth approximation by exact Lagrangian submanifolds. Following [8], we denote these approximations by $\Upsilon_{\epsilon}$. It holds $\Upsilon_{\epsilon} \rightarrow \nu^* \overline{U}$ as $\epsilon \rightarrow 0$ in Lipschitz topology (see [8] for the details).

Floer homology for the open set $U$ is defined to be a direct limit of Floer homologies of approximations. In order to have the latter well defined, one needs to consider a Hamiltonian $H : T^*M \times [0,1] \rightarrow \mathbb{R}$ such that

$$\phi_H^1(O_M) \cap O_M$$

and, for a fixed $H$, an open set $U$ such that

$$\phi_H^1(O_M) \cap O_M|_{\partial U} = \emptyset, \quad \phi_H^1(O_M) \cap \nu^* \overline{U}$$

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which can be obtained by generic choice of $U$. Floer homology for the pair $(O_M, \Upsilon_\varepsilon)$ is now defined in a standard way, the set of the generators $CF(O_M, \Upsilon_\varepsilon : H)$ consists of the Hamiltonian paths

\begin{equation}
\dot{x} = X_H(x), \quad x(0) \in O_M, \quad x(1) \in \Upsilon_\varepsilon,
\end{equation}

which are critical points of the effective action functional:

\begin{equation}
A^\Upsilon_\varepsilon_H(g) := \int \gamma^* \theta - \int_0^1 H(\gamma(t), t) dt - h_{\Upsilon_\varepsilon}(\gamma(1)).
\end{equation}

Here $\theta$ is a canonical Liouville form on $T^* M$ and $h_{\Upsilon_\varepsilon}$ is a function satisfying $i^* \theta = dh_{\Upsilon_\varepsilon}$ which exists since $\Upsilon_\varepsilon$ is exact. The boundary map $\partial_{J,H}$ is defined as the number of perturbed holomorphic discs with boundary on $O_M$ and $\Upsilon_\varepsilon$:

\begin{equation}
u : \mathbb{R} \times [0, 1] \to T^* M, \quad \frac{\partial u}{\partial t} + J_\varepsilon \left( \frac{\partial u}{\partial s} - X_H(u) \right) = 0, \quad u(s, 0) \in O_M, \quad u(s, 1) \in \Upsilon_\varepsilon.
\end{equation}

Here $J_\varepsilon$ is an almost complex structure compatible to the standard symplectic form $\omega = -d\theta$, which coincides with the canonical almost complex structure $J_0$ on $T^* M$ at infinity. By the canonical almost complex structure $J_0$ we assume the one induced by the Levi-Civita connection for a fixed metric $g_0$.

Denote by $HF(O_M, \Upsilon_\varepsilon : H, J_\varepsilon)$ the corresponding Floer homology.

Floer homology of the open subset $U$ is defined as a direct limit of above Floer homologies for the approximations $\Upsilon_\varepsilon$:

\begin{equation}
HF(H, U : M) := \lim_{\varepsilon \to 0} HF(O_M, \Upsilon_\varepsilon : H, J_\varepsilon),
\end{equation}

after defining an appropriate partial ordering to the set of pairs $(\Upsilon_\varepsilon, J_\varepsilon)$ (see below or [8] for more details).

In this paper we construct and investigate spectral invariants for the Floer homology of the open subset $HF(H, U : M)$ and establish their compatibility with the direct limit (5).

The first step in this direction is the construction of PSS-type isomorphisms $\Psi : HM_* (f, U) \to HF_* (H, U : M)$ and singular homology of $U$ (modelled by Morse homology). More precisely, we prove the following theorem. Let $HM_* (f, U)$ denotes the Morse homology of $U$ (see Section 2 below).

**Theorem 1.** There exist PSS-type isomorphisms

\[ \Psi : HM_* (f, U) \to HF_* (H, U : M), \quad \Phi : HF_* (H, U : M) \to HM_* (f, U) \]

which are inverse to each other and natural with respect to canonical isomorphisms in Morse and Floer theory. More precisely, if

\[ S_{\alpha \beta} : HF_* (H_\alpha, U : M) \to HF_* (H_\beta, U : M), \quad T_{\alpha \beta} : HM_* (f_\alpha, U) \to HM_* (f_\beta, U) \]

denote the canonical isomorphisms in Floer and Morse theory respectively, and $\Psi_\alpha$ and $\Psi_\beta$ the corresponding PSS isomorphisms, then the diagram

\[
\begin{array}{ccc}
HF_* (H_\alpha, U : M) & \xrightarrow{S_{\alpha \beta}} & HF_* (H_\beta, U : M) \\
\Psi_\alpha \downarrow & & \downarrow \Psi_\beta \\
HM_* (f_\alpha, U) & \xrightarrow{T_{\alpha \beta}} & HM_* (f_\beta, U)
\end{array}
\]

commutes.
We construct PSS isomorphisms and prove Theorem 1 in Section 2. First we construct the corresponding isomorphisms for approximations $HF(O_M, \mathcal{Y}_z : H, J_z)$ and prove that they commute with the homomorphisms that define the direct limit [5].

Next, we construct three pair-of-pants type products in Morse and Floer theory for open sets. Products in Morse and Floer theory were studied by various authors: Abbondandolo and Schwarz [2], Auroux [4], Oh [19] and also in [10].

Here we establish the following products for open subset.

**Theorem 2.** There exist a pair-of-pants type products:

- $\circ : HF_*(H_1, U : M) \otimes HF_*(H_2, U : M) \to HF_*(H_3, U : M)$
- $\cdot : HM_*(f_1, U) \otimes HM_*(f_2, U) \to HM_*(f_3, U)$
- $\ast : HM_*(f, U) \otimes HF_*(H, U : M) \to HF_*(H, U : M)$

that turns Floer homology for an open set $HF_*(H, U : M)$ into a $HM_*(f, U)$-module.

Theorem 2 is proven in Section 3. Since Floer homology for the open set is defined as a direct limit, the key step is to prove that the products defined on approximation commute with the homomorphisms that define the direct limit.

Finally, using the above PSS isomorphism, we construct the spectral invariants for Lagrangian Floer homology of the open subset $HF$. Spectral invariants were defined by Oh in [18] in Lagrangian Floer theory, and by Schwarz [25] for periodic orbit case, following the work of Viterbo [20]. They were further studied by Leclercq [11], by Monzner, Vichery and Zapolsky in [17] and also in [12, 13, 14, 15].

We prove the following properties of these spectral invariants: their continuity with respect to $H$ and their subadditivity with respect to the products from Theorem 2. We also compare the above spectral invariants with the invariants for periodic orbits case, using the homomorphisms defined via "chimneys" introduced by Albers [3] and Abbondandolo and Schwarz [1]. Further, we prove the inequality of spectral invariants between two open sets $U \hookrightarrow V$ and a specific singular homology class (see Subsection 4.1). This slightly generalizes a result by Oh [20] for a spectral invariant

$$c_+(H, U) := \inf \{ \lambda \in \mathbb{R} \mid \lambda_k : HF^\lambda_k (H, U : M) \to HF_k (H, U : M) \text{ is surjective} \}.$$ 

More precisely, in Section 4 we prove the following theorem.

**Theorem 3.** For given singular or Morse homology class $\alpha \in HM_*(f, U) \setminus \{0\}$, the spectral invariant $c_U(\alpha, H)$ defined via PSS isomorphism from Theorem 2 has the following properties:

- **triangle inequality.** $c_U(\alpha \cdot \beta, H_1 \sharp H_2) \leq c_U(\alpha, H_1) + c_U(\beta, H_2)$, for $\alpha \cdot \beta \neq 0$
- **continuity.** relative spectral invariant $C_U(\alpha, H) := c_U(\alpha, H) - c_U(1, H)$ is continuous with respect to $H$
- **comparison with periodic orbit invariants.** If $\rho(\cdot, H)$ stands for a spectral invariants for periodic orbit case in $T^*M$ and $\iota_*$, $\iota_!$ are homomorphisms in homology induced by the inclusion map, we have:
  $$\rho(\alpha, H) \geq c_U(\iota_!(\alpha), H), \quad c_U(\alpha, H) \geq \rho(\iota_*(\alpha), H)$$
- **invariants for subsets.** Let $U \hookrightarrow V$ be two open subset of $M$ and let $j_{*UV} : HM_k(f, U) \to HM_k(f, V)$

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(the homomorphism induced by inclusion \( j : U \to V \)) be surjective. For \( \alpha \in HM_k(f, U) \setminus \{0\} \) it holds:
\[
c_V(j_*U,V(\alpha), H) \leq c_U(\alpha, H).
\]

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## 2. PSS isomorphism

PSS type isomorphism was originally constructed by Piunikhin, Salamon and Schwarz [21] for periodic orbit case, and later adapted in [9, 3] for Lagrangian case.

### 2.1. Isomorphism for approximations.

We first establish the PSS isomorphism for approximations. It follows from (1) that all solutions of Hamiltonian equation \( \dot{x} = X_H(x) \) with \( x(0) \in O_M \) satisfy \( x(1) \notin O_M\partial U \), so by choosing \( \Upsilon \) to be close enough to \( \nu^*U \), we may assume that all solutions of (2) satisfy \((6) x(0), x(1) \in O_M \) or \( x(0) \in O_M \), \( x(1) \in \nu^*\partial U \).

The grading for \( x \in CF(O_M, \Upsilon : H) \) is defined to be
\[
\mu(x) := \mu_M(x) + \frac{1}{2} \dim M, \quad \text{for } x(1) \in O_U,
\]
\[
\mu(x) := \mu_{\partial U}(x) + \frac{1}{2} \dim M, \quad \text{for } x(1) \in \nu^*(\partial U),
\]
where \( \mu_S \) is a canonically assigned Maslov index, defined for any smooth closed submanifold \( S \subset M \) (see Definition 5.9 in [18]). The dimension of the space \( \mathcal{M}(x, y, O_M, \Upsilon : H, J) \) of perturbed holomorphic discs the satisfy (4) and the infinity boundary conditions:
\[
u(-\infty, t) = x(t), \quad u(+\infty, t) = y(t)
\]
is
\[
\dim \mathcal{M}(x, y, O_M, \Upsilon : H, J) = \mu(y) - \mu(x)
\]
for all \( x, y \in CF(O_M, \Upsilon : H) \) (see [8]).

Consider now a Morse function \( f : U \to \mathbb{R} \) such that the set of critical points of \( f \), \( \text{Crit}(f) \) is contained in a closed set \( A \), \( \text{Crit}(f) \subset A \subset U \), and therefore, is finite. For \( p \in \text{Crit}(f) \) and \( x \in CF(O_M, \Upsilon : H) \) define the space of mixed objects (see figure below):
\[
\mathcal{M}(p, x) := \mathcal{M}(p, x, O_M, \Upsilon : f, H, J, g) := \left\{ \gamma : (-\infty, 0] \to U, \ u : [0, +\infty) \times [0, 1] \to T^*M, \right. \\
\left. \gamma(s) = -\nabla_g f(\gamma(s)), \ \frac{\partial u}{\partial t} + J(\frac{\partial u}{\partial s} - X_{\rho H}(u)) = 0, \right. \\
\left. u(s, 0), u(0, t) \in O_M, \ u(s, 1) \in \Upsilon, \right. \\
\left. \gamma(-\infty) = p, \ u(+\infty, t) = x(t), \right. \\
\left. u(0, 1) = \gamma(0), \right. \\
\left. (\gamma, u) \right\}
\]
where \( \rho_R : [0, +\infty) \to \mathbb{R} \) is a smooth function such that
\[
\rho_R(t) = \begin{cases} 
1, & t \geq R \\
0, & t \leq R - 1.
\end{cases}
\]

Let \( m_f(p) \) denotes the Morse index of a critical point \( p \) and \( n = \dim M \).

**Proposition 4.** For generic choices the set \( M(p, x) \) is a smooth manifold of dimension \( m_f(p) - \left( \mu(x) + \frac{n}{2} \right) \).

**Proof:** Denote by \( W^u(p, f) \) the unstable manifold of \( p \) and by \( W^s(x, O_M, \Upsilon : H, J) \) the set of solutions of
\[
\begin{cases}
\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_{\tilde{\rho}_RH}(u) \right) = 0,
\gamma(s) = -\nabla_g f(\gamma(s)),
u(s, 0), u(0, t) \in O_M, u(s, 1) \in \Upsilon,
u(-\infty, t) = x(t).
\end{cases}
\]

For a generic choice of parameters the evaluation map
\[
ev : W^u(p, f) \times W^s(x, H) \to U \times U, \quad (\gamma, u) \mapsto (\gamma(0), u(0, 1))
\]
is transversal to the diagonal, so
\[
M(p, x) = ev^{-1}(\Delta)
\]
is a smooth manifold of codimension \( n \) in \( W^u(p, f) \times W^s(x, H) \). Since \( \dim W^u(p, f) = m_f(p) \) (see [16]) and \( \dim W^s(x, H) = -\mu(x) + \frac{n}{2} \) (see Appendix in [19] or [2] or [5]), the dimension of \( M(p, x) \) is
\[
\dim M(p, x) = m_f(p) - \mu(x) + \frac{n}{2} = m_f(p) - \left( \mu(x) + \frac{n}{2} \right).
\]

\( \square \)

In the same way we conclude that the set
\[
M(x, p) := M(x, p, O_M, \Upsilon : f, H, J, g) :=
\]
\[
\begin{cases}
u : (-\infty, 0] \times [0, 1] \to T^*M, \quad \gamma : [0, +\infty) \to U, \\
\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_{\tilde{\rho}_RH}(u) \right) = 0,
\gamma(s) = -\nabla_g f(\gamma(s)),
u(s, 0), u(0, t) \in O_M, u(s, 1) \in \Upsilon,
u(-\infty, t) = x(t), \gamma(+\infty) = p,
\gamma(0) = u(0, 1)
\end{cases}
\]

![Figure 1. Mixed object \( M(p, x) \) that defines PSS isomorphism](image.png)
is a smooth manifold of dimension $\mu(x) + \frac{n}{2} - m_f(p)$. Here $\rho_R(s) := \rho_R(-s)$.

Define $\hat{\mathcal{M}}(p,q)$ to be the set of all solutions of the differential equation

\[
\begin{cases}
\gamma : \mathbb{R} \to U, \\
\dot{\gamma}(s) = -\nabla g f(\gamma(s)), \\
\gamma(-\infty) = p,
\gamma(+\infty) = q
\end{cases}
\]

modulo $\mathbb{R}$ action and, similarly, denote by

\[
\hat{\mathcal{M}}(x,y) := \mathcal{M}(x,y,o_M,\mathcal{Y} : H,J)/\mathbb{R}.
\]

By standard arguments using the Arzela-Ascoli and Gromov compactness theorems, one can prove the following proposition. Bubbling cannot occur due to exactness of $\omega$ and Lagrangian boundary conditions.

**Proposition 5.** (1) If $m_f(p) = \mu(x) + \frac{n}{2}$, then the zero-dimensional manifolds $\mathcal{M}(p,x)$ and $\mathcal{M}(x,p)$ are compact, and hence, finite sets.

(2) For $m_f(p) = \mu(x) + \frac{n}{2} + 1$, the topological boundary of the one-dimensional manifold $\mathcal{M}(p,x)$ is

\[
\partial \mathcal{M}(p,x) = \bigcup_q \hat{\mathcal{M}}(p,q) \times \mathcal{M}(q,x) \cup \bigcup_y \mathcal{M}(p,y) \times \hat{\mathcal{M}}(y,x)
\]

where the first union is taken over all $q \in \text{Crit}(f)$, with $m_f(q) = m_f(p) - 1$, and the second over all $y \in CF(o_M,\mathcal{Y} : H)$, such that $\mu(y) = \mu(x) + 1$. Similarly, when $\mu(x) + \frac{n}{2} = m_f(p) + 1$, we have

\[
\partial \mathcal{M}(x,p) = \bigcup_y \hat{\mathcal{M}}(x,y) \times \mathcal{M}(y,p) \cup \bigcup_q \mathcal{M}(x,q) \times \hat{\mathcal{M}}(q,p).
\]

\[\square\]

The part (1) in the previous proposition enables us to define the homomorphism between Morse and Floer homology. Denote by

\[
CM_k(f,U) := \mathbb{Z}_2 \langle p \in \text{Crit}(f) \mid m_f(p) = k \rangle,
\]

\[
CF_k(o_M,\mathcal{Y} : H) := \mathbb{Z}_2 \langle x \in CF(o_M,\mathcal{Y} : H) \mid \mu(x) = k - \frac{n}{2} \rangle
\]

(i.e. $\mathbb{Z}_2$-vector spaces over the sets of generators of corresponding indices). Let $HM_k(f,U : g)$ and $HF_k(o_M,\mathcal{Y} : H,J)$ denote the corresponding Morse and Floer homology groups.

Denote:

\[
n(p,x) := \sharp \mathcal{M}(p,x) \pmod{2}
\]

\[
n(x,p) := \sharp \mathcal{M}(x,p) \pmod{2}
\]

and define

\[
\phi^\mathcal{Y} : CM_k(f,U) \to CF_k(o_M,\mathcal{Y} : H), \quad \phi^\mathcal{Y} : p \mapsto \sum_{x \in CF_k(o_M,\mathcal{Y} : H)} n(p,x)x
\]

\[
\psi^\mathcal{Y} : CF_k(o_M,\mathcal{Y} : H) \to CM_k(f,U), \quad \psi^\mathcal{Y} : x \mapsto \sum_{p \in CM_k(f,U)} n(x,p)p.
\]

It follows from the standard cobordism arguments and the part (2) of the Proposition 5 that the homomorphisms $\phi^\mathcal{Y}$ and $\psi^\mathcal{Y}$ define the homomorphisms

\[
\Phi^\mathcal{Y} : HM_k(f,U : g) \to HF_k(o_M,\mathcal{Y} : H,J)
\]

\[
\Psi^\mathcal{Y} : HF_k(o_M,\mathcal{Y} : H,J) \to HM_k(f,U : g)
\]

(8)
on the homology level.

**Theorem 6.** The homomorphisms $\Phi^\gamma$ and $\Psi^\gamma$ are isomorphisms and it holds

$$\Phi^\gamma \circ \Psi^\gamma = \text{Id}_{HF_k(O_M, \gamma ; H,J)}, \quad \Psi^\gamma \circ \Phi^\gamma = \text{Id}_{HM_k(f;U ; g)}.$$ 

**Proof:** The proof relies on standard cobordism arguments. The auxiliary one-dimensional manifolds are:

$$\overline{\mathcal{M}}(p, q, O_M, \gamma; f, H, J) := \left\{ (\gamma^-, \gamma^+, u) \right\},$$

$$\left\{ \begin{array}{l}
\gamma^- : (-\infty, 0] \to U, \\
\gamma^+ : [0, +\infty) \to U, \\
u : \mathbb{R} \times [0, 1] \to T^* M \\
d\gamma^\pm = -\nabla f(\gamma^\pm), \\
d\gamma^- + J(\partial_u - X_{pR}(u)) = 0, \\
u(-\infty) = p, \gamma^+(+\infty) = q, \\
u(s, 0) \in O_M, \quad u(s, 1) \in \gamma, \\
u(\pm\infty) = \gamma^\pm(0)
\end{array} \right\}.$$ \hfill (9)

and

$$\overline{\mathcal{M}}(x, y, O_M, \gamma; f, H, J) := \left\{ (u^-, u^+, u^\gamma) \right\},$$

$$\left\{ \begin{array}{l}
u^- : (-\infty, 0] \times [0, 1] \to T^* M, \\
u^+ : [0, +\infty) \times [0, 1] \to T^* M, \\
g^- : [-\varepsilon, \varepsilon] \to U \\
d\gamma = -\nabla f(\gamma), \\
d\gamma^- + J(\partial_u - X_{pR}(u)) = 0, \\
u_\pm(s, 0), \nu_\pm(0, t) \in O_M, \quad u_\pm(s, 1) \in \gamma, \\
u_\pm(0, 1) = \gamma(\pm\varepsilon), \\
u^-(\infty, t) = x(t), \\
u^+(\infty, t) = y(t),
\end{array} \right\}.$$ \hfill (10)

Unlike before, in the equations (9) and (10), the smooth cut-off function $\rho_R : \mathbb{R} \to [0, 1]$ is symmetric:

$$\rho_R(t) = \begin{cases} 1, & |t| \leq R - 1 \\ 0, & |t| \geq R. \end{cases} \hfill (11)$$

The rest of the proof is similar to the proof of Theorem 6 in [9]. \hfill $\Box$

For two Morse functions $f_\alpha$ and $f_\beta$, Morse homologies $HM(f_\alpha, U : g)$ and $HM(f_\beta, U : g)$ are canonically isomorphic (see [24]). Similarly, for two Hamiltonians $H_\alpha$ and $H_\beta$, the corresponding Floer homologies $HF(O_M, \gamma : H_\alpha, J)$ and $HF(O_M, \gamma : H_\beta, J)$ are isomorphic. Denote these canonical isomorphisms by

$$T_{\alpha\beta} : HM(f_\alpha, U : g) \xrightarrow{\cong} HM(f_\beta, U : g)$$

$$S^\gamma_{\alpha\beta} : HF(O_M, \gamma : H_\alpha, J) \xrightarrow{\cong} HF(O_M, \gamma : H_\beta, J).$$

Denote by

$$\Phi^\gamma_{\alpha} : HM_k(f_\alpha, U : g) \to HF_k(O_M, \gamma : H_\alpha, J)$$

$$\Psi^\gamma_{\beta} : HF_k(O_M, \gamma : H_\alpha, J) \to HM_k(f_\beta, U : g)$$

the isomorphisms defined in [8].
Theorem 7. The diagram

\[ HF_k(O, \mathcal{Y} : H, J) \xrightarrow{S_{\alpha \beta}^\mathcal{Y}} HF_k(O, \mathcal{Y} : H, J) \xrightarrow{\Psi_{\alpha \beta}^\mathcal{Y}} HF_k(O, \mathcal{Y} : H, J) \xrightarrow{\Psi_{\beta}^\mathcal{Y}} HF_k(O, \mathcal{Y} : H, J) \]

commutes.

Proof: The proof is also based on cobordism arguments, so we will only give the description of the auxiliary one-dimensional manifold that we use here:

\[ \widehat{M}(x, p, O, \mathcal{Y} : H, J, f_{\alpha \beta}) := \begin{cases} \gamma, u, \lambda, & u : (-\infty, 0] \times [0, 1] \rightarrow T^*M, \\ \gamma : [0, +\infty) \rightarrow U, \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho R H_{\alpha \beta}}(u)) = 0, \\ u(-\infty, t) = x_{\alpha}(t), \\ u(0, 0), u(0, t) \in O, u(s, 1) \in \mathcal{Y}, \\ \frac{\partial \gamma}{\partial t} = -\nabla f_{\alpha \beta}(\gamma), \\ \gamma(+\infty) = p_{\beta}, \\ \gamma(0) = u(0, 1), \end{cases} \]

where \((f_{\alpha \beta}(H_{\alpha \beta})), 0 \leq \lambda \leq 1\) is a homotopy connecting \((f_{\alpha \beta}(H_{\alpha \beta})), (f_{\alpha \beta}(H_{\alpha \beta})), 0 \leq \lambda = 0 = (f_{\alpha \beta}(H_{\alpha \beta})), (f_{\alpha \beta}(H_{\alpha \beta})), 0 \leq \lambda = 1\). Here \(H_{\alpha \beta}\) is a smooth homotopy connecting \(H_{\alpha}\) and \(H_{\beta}\), and similarly for \(f_{\alpha \beta}\), while \(\rho R\) is as in (11).

2.2. Isomorphism for Floer homology of open set. In order to define Floer homology for the open set as a direct limit of Floer homologies for the approximations, Kasturirangan and Oh defined a partial ordering on the set of approximations as:

\[ \mathcal{Y}_a \leq \mathcal{Y}_b \iff \varphi_a \leq \varphi_b \text{ on } U. \]

The function \(\varphi_a\) is defined by \(h_a = \varphi_a \circ \pi\) on \(U\), where \(h_a : \mathcal{Y}_a \rightarrow \mathbb{R}\) is a smooth function such that

\[ \nu^\ast \theta = dh_a \]

(recall that \(\mathcal{Y}_a\) is exact) and \(\pi : T^*M \rightarrow M\) is a canonical projection. Since \(H\) is fixed, one has to vary the almost complex structure \(J\) to obtain a transversal position. Denote by \(J_a\) an almost complex structure corresponding to \(\mathcal{Y}_a\) and denote by

\[ F_{ab} : HF_k(O, \mathcal{Y}_a : H, J_a) \rightarrow HF_k(O, \mathcal{Y}_b : H, J_b) \]

a canonical homomorphism that satisfies:

\[ F_{ac} = F_{bc} \circ F_{ab} \]

for given triple \(\mathcal{Y}_c \leq \mathcal{Y}_b \leq \mathcal{Y}_a\) sufficiently close to \(\nu^\ast \mathcal{U}\) (see [8]).

Since we want to establish an isomorphism between Floer homology and Morse homology for a fixed Morse function, we will vary Riemannian metric, so the term "generic choices" in the Proposition refers to an almost complex structure \(J\) and Riemannian metric \(g\).

Fix a Hamiltonian function \(H\) and a Morse function \(f\). For a Lagrangian approximation \(\mathcal{Y}_a\), choose an almost complex structure \(J_a\) and a Riemannian metric \(g_a\) such that all the transversality conditions are fulfilled, i.e. the sets \(\widehat{M}(p, q),\)
$\widehat{\mathcal{M}}(x, y), \mathcal{M}(p, x)$ and $\mathcal{M}(x, p)$ are manifolds for all $p, q \in \text{Crit}(f)$ and all Hamiltonian paths $x, y$ with boundaries on $O_M$ and $\Upsilon_a$. Since for two Riemannian metric $g_a$ and $g_b$ there is a canonical isomorphism $G_{ab} : H M_k(f, U : g_a) \to H M_k(f, U : g_b)$ satisfying

$$G_{ac} = G_{bc} \circ G_{ab},$$

we can define Morse homology $H M_k(f, U)$ as a direct limit:

$$H M_k(f, U) := \lim_{\rightarrow} H M_k(f, U : g_a).$$

Consider a diagram:

$$\begin{array}{c}
\cdots 
\quad \Rightarrow \quad H M_k(g_a) 
\quad \downarrow \Phi^a 
\quad \Rightarrow \quad H M_k(g_b) 
\quad \downarrow \Phi^b 
\quad \Rightarrow \quad H M_k(g_c) 
\quad \downarrow \Phi^c 
\quad \Rightarrow \quad \cdots \\
\!
\cdots 
\quad \Rightarrow \quad H F_k(\Upsilon_a) 
\quad \downarrow \Phi^a 
\quad \Rightarrow \quad H F_k(\Upsilon_b) 
\quad \downarrow \Phi^b 
\quad \Rightarrow \quad H F_k(\Upsilon_c) 
\quad \downarrow \Phi^c 
\quad \Rightarrow \quad \cdots 
\end{array} \tag{14}$$

where we use the abbreviations

$$H M_k(g_a) := H M_k(f, U : g_a),$$

$$\Phi^a := \Phi_{\Upsilon_a},$$

$$H F_k(\Upsilon_a) = H F_k(O_M, \Upsilon_a : H, J_a),$$

and so on.

**Proposition 8.** The diagram (14) commutes.

**Proof:** The commutativity of (14) is equivalent to the identity

$$G_{ab} = (\Phi^b)^{-1} \circ F_{ab} \circ \Phi^a. \tag{15}$$

In order to prove (15), let us fix $R, T > 0$ and define

- $\overline{\Upsilon}_s$ to be a monotone homotopy for $s \in \mathbb{R}$ such that

$$\overline{\Upsilon}_s = \begin{cases} \Upsilon_a, & s \leq -R, \\ \Upsilon_b, & s \geq R; \end{cases} \tag{16}$$

and $\overline{J}_s$ the corresponding family of almost complex structures such that all transversality conditions hold;

- $\overline{g}_s$ to be a homotopy of Riemannian metrics such that

$$\overline{g}_s = \begin{cases} g_a, & s \leq -T, \\ g_b, & s \geq T; \end{cases}$$

Recall that the homomorphism $G_{ab}$ at the chain level (we denoted by $G_{ab}$ the induced homomorphism in homology) is defined via the number of the set

$$\mathcal{M}(p, q : \overline{g}_s) := \left\{ \gamma : \mathbb{R} \to U, \begin{array}{l} \frac{d\gamma}{ds} = -\nabla_{\overline{g}_s} f(\gamma), \\ \gamma(-\infty) = p, \gamma(+\infty) = q, \end{array} \right\}$$

and the homomorphism $F_{ab}$ via the number of

$$\mathcal{M}(x, y : \overline{\Upsilon}_s) := \left\{ u : \mathbb{R} \times [0, 1] \to T^* M, \begin{array}{l} \frac{\partial u}{\partial s} + J_s \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0, \\ u(s, 0) \in O_M, u(s, 1) \in \overline{\Upsilon}_s \\ u(-\infty, t) = x(t), u(+\infty, t) = y(t). \end{array} \right\} \tag{17}$$
For two critical points \( p, q \in U \) of Morse function \( f \) define the following auxiliary manifolds

\[
\widetilde{M}_R(p, q : \tilde{T}_s, \tilde{g}_s) := \left\{ (\gamma_1, u, \gamma_2) \mid u : \mathbb{R} \times [0, 1] \to T^*M, \right. \\
\left. \gamma_1 : (-\infty, 0] \to U, \gamma_2 : [0, +\infty) \to U, \right. \\
\left. \frac{\partial u}{\partial \gamma_1} + J_s(\frac{\partial u}{\partial \gamma_2}) - X_H(u) = 0, \right. \\
\left. \frac{\partial u}{\partial \gamma_2} = -\nabla_{\tilde{g}_s} f(\gamma), \ j = 1, 2, \right. \\
\left. u(s, 0), u(\pm \infty, t) \in O_M, \ u(s, 1) \in \tilde{T}_s, \right. \\
\left. \gamma_1(0) = u(-\infty, 1), \ \gamma_2(0) = u(+\infty, 1) \right\}
\]

and

\[
\widetilde{\mathcal{M}}(p, q : \tilde{T}_s, \tilde{g}_s) := \{(R, \gamma_1, u, \gamma_2) \mid R \in [R_0, +\infty), (\gamma_1, u, \gamma_2) \in \widetilde{M}_R(p, q : \tilde{T}_s, \tilde{g}_s)\}
\]

For \( m_f(p) + 1 = \mu(x) + \frac{d}{d\gamma} \), the set \( \widetilde{M}_R(p, q : \tilde{T}_s, \tilde{g}_s) \) is a zero-dimensional and \( \widetilde{\mathcal{M}}(p, q : \tilde{T}_s, \tilde{g}_s) \) is an one-dimensional smooth manifold. The topological boundary of \( \mathcal{M}(p, q : \tilde{T}_s, \tilde{g}_s) \) consists of the union of the following components:

\[
B_1 = \bigcup_{\mu(r) = \mu(p) - 1} \widetilde{M}(p, r) \times \widetilde{\mathcal{M}}(r, q : \tilde{T}_s, \tilde{g}_s)
\]

\[
B_2 = \bigcup_{m_f(r) = m_f(p) + 1} \widetilde{\mathcal{M}}(p, r : \tilde{T}_s, \tilde{g}_s) \times \widetilde{\mathcal{M}}(r, q)
\]

\[
B_3 = \bigcup_{x,y} \mathcal{M}(p, x; \tilde{T}_a : f, \tilde{g}_s) \times \mathcal{M}(x, y : \tilde{T}_s) \times \mathcal{M}(y, q; \tilde{T}_b : f, \tilde{g}_s)
\]

\[
B_4 = \widetilde{\mathcal{M}}_{R_0}(p, q : \tilde{T}_s, \tilde{g}_s).
\]

In \( B_3 \), \( x \) and \( y \) are Hamiltonian paths with \( x(0), y(0) \in O_M, x(1) \in \tilde{T}_a, y(1) \in \tilde{T}_b \). The loss of compactness corresponding to the boundary points from \( B_1, B_2 \) and \( B_4 \) happens when \( R \) is bounded, and \( B_3 \) occurs when \( R \to \infty \). Counting the objects from \( B_1 \) and \( B_2 \) gives us the maps of the type \( K \circ \partial \) and \( \partial \circ K \), for a map \( K \) defined via the number of zero component of manifold \( \widetilde{M}(p, q : \tilde{T}_s, \tilde{g}_s) \). Therefore, the maps \( A \) and \( B \), defined by the number of zero-dimensional boundary components \( B_3 \) and \( B_4 \) respectively, are the same in the homology.

The map \( A \) is obviously the same as the map \( (\Phi^b)^{-1} \circ \Phi_{ab} \circ \Phi^a \), i.e. the right hand side in [15], although the Riemannian metrics defining the gradient lines in mixed objects are not the same. This follows from standard parameter independence arguments.

So we need to prove that that \( B = G_{ab} \) in the homology level. By standard cobordism arguments, we can conclude that the map \( B \) does not depend on the choice of \( R_0 \). By letting \( R_0 \to 0 \), we conclude that, in the holomoly level, the map \( B \) is equal to the map \( C \), defined by the number of the set

\[
\left\{ \begin{aligned}
& u : \mathbb{R} \times [0, 1] \to T^*M, \\
& \gamma_1 : (-\infty, 0] \to U, \gamma_2 : [0, +\infty) \to U, \\
& \frac{\partial u}{\partial \gamma_1} + J_s(\frac{\partial u}{\partial \gamma_2}) - X_H(u) = 0, \\
& \frac{\partial u}{\partial \gamma_2} = -\nabla_{\tilde{g}_s} f(\gamma), \ j = 1, 2, \\
& u(s, 0), u(\pm \infty, t) \in O_M, \ u(s, 1) \in \tilde{T}_s, \\
& \gamma_1(0) = u(-\infty, 1), \ \gamma_2(0) = u(+\infty, 1)
\end{aligned} \right\}
\]
But since the above \( u \) is holomorphic, it holds
\[
\int \int \|u\|^2 = \int \int_\mathbb{R} \times [0, 1] u^* \omega = \int \int_{\beta(\mathbb{R} \times [0, 1])} u^* \theta \quad (i)
\]
\[
\int_{-\infty}^{+\infty} u(1, s)^* dh_{\gamma} = h_{\gamma}(u(+\infty, 1)) - h_{\gamma}(u(-\infty, 1)) \quad (ii) = 0.
\]
The equality (i) follows from (13) and from the fact that \( u \) maps \( \{-\infty\} \times [0, 1], \{+\infty\} \times [0, 1] \) and \( \mathbb{R} \times \{0\} \) to \( O_M \). The equality (ii) holds since the function \( h_{\gamma} \) are zero on \( O_M \) (see [8]).

We conclude that the maps \( C \) and \( G_{ab} \) are the same, therefore, (15) holds. □

From Proposition [8] and the fact that all the maps in (14) commute, it follows that there exists a direct limit isomorphism
\[
(19) \quad \Phi : HM_k(f, U) \xrightarrow{\cong} HF_k(H, U : M).
\]
Similarly, by considering the commutative diagram
\[
\cdots \rightarrow HF_k(\gamma_a) \xrightarrow{F_{a\beta}} HF_k(\gamma_b) \xrightarrow{F_{b\gamma}} HF_k(\gamma_c) \rightarrow \cdots \\
\downarrow \psi^a \downarrow \psi^b \downarrow \psi^c
\]
\[
\cdots \rightarrow HM_k(g_a) \xrightarrow{G_{a\beta}} HM_k(g_b) \xrightarrow{G_{b\gamma}} HM_k(g_c) \rightarrow \cdots
\]
we obtain the direct limit isomorphism
\[
\Psi : HF_k(H, U : M) \xrightarrow{\cong} HM_k(f, U).
\]

**Theorem 9.** The induced maps \( \Phi \) and \( \Psi \) are isomorphisms and it holds
\[
\Psi \circ \Phi = \text{Id}_{HM_k(f, U)} \quad \Phi \circ \Psi = \text{Id}_{HF_k(H, U : M)}.
\]

**Proof:** From Theorem [6] we have
\[
\Psi^a \circ \Phi^a = \text{Id}_{HM_k(f, U : g_a)} \quad \Phi^a \circ \Psi^a = \text{Id}_{HF_k(O_M, \gamma_a : H, J_a)}.
\]
Let \( p_a \in HM_k(f, U : g_a) \) be the representative of the class \([p_a] \in HM_k(f, U)\). We have
\[
\Psi \circ \Phi([p_a]) = \Psi(\Phi^a(p_a)) = [\Psi^a(\Phi^a(p_a))] = [p_a],
\]
and similarly for the second equality. □

From the canonical isomorphisms
\[
S^a_{\alpha\beta} := S^T_{\alpha\beta} : HF(O_M, \gamma_a : H, J_a) \xrightarrow{\cong} HF(O_M, \gamma_a : H, J_a)
\]
and the commutativity of
\[
\cdots \rightarrow HF_k(\gamma_a : H_{\alpha}) \xrightarrow{F^a_{a\beta}} HF_k(\gamma_b : H_{\alpha}) \xrightarrow{F^a_{b\gamma}} HF_k(\gamma_c : H_{\alpha}) \rightarrow \cdots \\
\downarrow S^a_{\alpha\beta} \downarrow S^b_{\alpha\beta} \downarrow S^c_{\alpha\beta}
\]
\[
\cdots \rightarrow HF_k(\gamma_a : H_{\beta}) \xrightarrow{F^a_{a\beta}} HF_k(\gamma_b : H_{\beta}) \xrightarrow{F^a_{b\gamma}} HF_k(\gamma_c : H_{\beta}) \rightarrow \cdots
\]
one obtains an isomorphism:
\[
S_{\alpha\beta} : HF_k(H_{\alpha}, U : M) \xrightarrow{\cong} HF_k(H_{\beta}, U : M).
\]
Similarly, we have
\[ T_{\alpha \beta} : HM_k(f, U) \xrightarrow{\cong} HM_k(f, U). \]

**Theorem 10.** The diagram
\[
\begin{array}{ccc}
HF_k(H_\alpha, U : M) & \xrightarrow{S_{\alpha \beta}} & HF_k(H_\beta, U : M) \\
\Psi_\alpha \downarrow & & \downarrow \Psi_\beta \\
HM_k(f, U) & \xrightarrow{T_{\alpha \beta}} & HM_k(f, U)
\end{array}
\]
commutes.

**Proof:** Recall that the diagram \([\mathbb{12}]\) commutes for all approximations close enough to \(\nu^* U\) and for generic choices. So we have
\[
T_{\alpha \beta} \circ \Psi_\alpha ([x_a]) = T_{\alpha \beta} ([\Psi_\alpha^a (x_a)]) = [T_{\alpha \beta}^a (\Psi_\alpha^a (x_a))] = [\Psi_\beta^a (S_{\alpha \beta}^a (x_a))] = \Psi_\beta \circ S_{\alpha \beta} ([x_a]),
\]
for every \([x_a] \in HF_k(H_\alpha, U : M)\).

This proves Theorem \([\mathbb{1}]\).

### 3. Product on homology and module structure

#### 3.1. Product on homology

In this section we construct a product on Floer homology for an open subset
\[ \circ : HF_*(H_1, U : M) \otimes HF_*(H_2, U : M) \rightarrow HF_*(H_3, U : M). \]

First, we define \(\circ\)
\[ (20) \circ : HF_*(O_M, \Upsilon : H_1, J_\Upsilon) \otimes HF_*(O_M, \Upsilon : H_2, J_\Upsilon) \rightarrow HF_*(O_M, \Upsilon : H_3, J_\Upsilon), \]
for every approximation \(\Upsilon\). Product \((20)\) is defined by counting pair–of–pants objects. We define a Riemannian surface (with a boundary) \(\Sigma^\Upsilon\) as a disjoint union
\[ \mathbb{R} \times [-1, 0] \sqcup \mathbb{R} \times [0, 1] \]
with the identification \((s, 0^-) \sim (s, 0^+)\) for \(s \geq 0\) (see Figure \([\mathbb{2}]\)).

Denote by \(\Sigma_1, \Sigma_2, \Sigma_3\) the two ”incoming” and one ”outgoing” ends, such that
\[
\Sigma_1, \Sigma_2 \approx [0, 1] \times (-\infty, 0], \\
\Sigma_3 \approx [0, 1] \times [0, +\infty),
\]
and by \(u_j := u|_{\Sigma_j}, j = 1, 2, 3\). Let \(\rho_j : \mathbb{R} \rightarrow [0, 1]\) denote the smooth cut–off functions such that
\[
\rho_1(s) = \rho_2(s) = \begin{cases} 1, & s \leq -2, \\
0, & s \geq -1 \end{cases} \rho_3(s) := \rho_1(-s).
\]
For \( x \in CF_*(O_M, \Upsilon : H_1) \), \( y \in CF_*(O_M, \Upsilon : H_2) \) and \( z \in CF_*(O_M, \Upsilon : H_3) \) we define a moduli space \( \mathcal{M} \) (see Figure 2).

\[ \mathcal{M}(x, y; z) = \left\{ u : \Sigma^T \to T^*M \right\} \]

\[ \mathcal{M}(x, y; z) = \left\{ u : \Sigma^T \to T^*M \right\} \]

\[ \partial_s u_j + J_T(\partial_t u_j - X_{H_j} \circ u_j) = 0, \quad j = 1, 2, 3, \]

\[ \partial_s u + J_T \partial_t u = 0, \quad \text{on } \Sigma_0 := \Sigma^T \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3), \]

\[ \begin{cases} u(s, -1) \in O_M, \quad u(s, 1) \in \Upsilon, \quad s \in \mathbb{R}, \\ u(s, 0^-) \in \Upsilon, \quad u(s, 0^+) \in O_M, \quad s \leq 0, \\ u_1(-\infty, t) = x(t), \\ u_2(-\infty, t) = y(t), \\ u_3(+\infty, t) = z(t) \end{cases} \]

(see Figure 2).

For generic choices, \( \mathcal{M}(x, y; z) \) is a smooth \( \mu(x) + \mu(y) - \mu(z) \)-dimensional manifold. We define a map \( \circ \) at the chain level as

\[ x \circ y = \sum_z \tilde{z}_2 \mathcal{M}(x, y; z) z, \]

for the generators \( x \) and \( y \). Here, \( \tilde{z}_2 \mathcal{M}(x, y; z) \) denotes the (modulo 2) number of elements of a zero-dimensional component of \( \mathcal{M}(x, y; z) \). We extend the product \( \circ \) to \( CF_*(O_M, \Upsilon : H_1) \otimes CF_*(O_M, \Upsilon : H_2) \) by bilinearity. One can show that \( \circ \) commutes with boundary maps and induces a product in homology (20).

**Proposition 11.** The product \( \circ \) defines a product on Floer homology for open subset:

\[ \circ : HF_*(H_1, U : M) \otimes HF_*(H_2, U : M) \to HF_*(H_3, U : M). \]

In order to prove Proposition 11 we need the following lemma. Recall that we denote by \( F_{ab} \) the homomorphism

\[ F_{ab} : HF_*(O_M, \Upsilon_a : H, J_a) \to HF_*(O_M, \Upsilon_b : H, J_b) \]

defined by (17). Here \( J_a = J_{\Upsilon_a} \), etc. To emphasize the Hamiltonian we are considering, we will sometimes write \( F_{ab}^H \).

**Lemma 12.** For \( x_a \in HF_*(O_M, \Upsilon_a : H_1, J_a) \), \( y_a \in HF_*(O_M, \Upsilon_a : H_2, J_a) \) it holds

\[ (21) \quad F_{ab}^H(x_a \circ y_a) = F_{ab}^H(x_a) \circ F_{ab}^H(y_a). \]

**Proof of Proposition 11.** Let us take \([x] \in HF_*(H_1, U : M)\) and \([y] \in HF_*(H_2, U : M)\). The elements \( x \) and \( y \) belong to some spaces \( HF_*(O_M, \Upsilon_a : H_1, J_a) \) and \( HF_*(O_M, \Upsilon_{a'} : H_2, J_{a'}) \). In general, \( a \) and \( a' \) are not the same. Since \( x \) and \( F_{aa}(x) \) represent the same element in \( HF_*(H_1, U : M) \) we can take \( F_{a \max\{a, a'\}}(x) \) and \( F_{a' \max\{a, a'\}}(y) \) as representatives of \([x]\) and \([y]\) respectively. So we can assume that
\( x \) and \( y \) belongs to some \( HF_*(O_M, \Upsilon : H_1, J) \) and \( HF_*(O_M, \Upsilon : H_2, J) \), for the same \( \Upsilon \). We now define a product \( \circ \) on homology as
\[
[x] \circ [y] := [x \circ y]
\]
We need to show that a product does not depend on representatives of a class. Let \( x_a \) and \( x_b \) represent the same element in \( HF_*(H_1, U : M) \) and similarly \( y_a \) and \( y_b \) in \( HF_*(H_2, U : M) \). This means that there exist
\[
F_{ac} : (O_M, \Upsilon_a : H_1, J_a) \to (O_M, \Upsilon_c : H_1, J_c)
\]
\[
F_{ad} : (O_M, \Upsilon_a : H_2, J_a) \to (O_M, \Upsilon_d : H_2, J_d)
\]
such that
\[
F_{ac}(x_a) = F_{bc}(x_b), \quad F_{ad}(y_a) = F_{bd}(y_b).
\]
Let \( e = \max\{c, d\} \). We have
\[
F_{ac}(x_a \circ y_a) = F_{ac}(x_a) \circ F_{ac}(y_a) = F_{ce}(F_{ac}(x_a)) \circ F_{de}(F_{ad}(y_a)) = F_{ce}(F_{bc}(x_b)) \circ F_{de}(F_{bd}(y_b)) = F_{be}(x_b) \circ F_{be}(y_b)
\]
which means that \( x_a \circ y_a \) and \( x_b \circ y_b \) represent the same element in \( HF_*(H_3, U : M) \), i.e. the product \( \circ \) is well defined on homology for the open set. \( \square \)

**Proof of Lemma 12** The homomorphism \( F_{ab} \) is an isomorphism for \( a, b \) large enough. The inverse homomorphism is actually \( F_{ba} \). This can be proved using the cobordism arguments similar to ones in the proof of the independence of Floer homology with respect to the parameters (Hamiltonian, almost complex structure). Therefore, (22) is equivalent to
\[
x_a \circ y_a = \left(F_{ab}^{-1}\right)^{-1} \left(F_{ab}(x_a) \circ F_{ab}(y_a)\right).
\]
The proof of (22) is similar to the proof of the Proposition 8 so let us just specify the auxiliary one-dimensional manifold whose boundaries provide the corresponding homology maps. Let \( \tilde{\Upsilon}_a \) be as in (16) and \( x_a, y_a, z_a \) be the solutions of
\[
\dot{x}_a(t) = X_{H_1}(x_a(t)), \quad x_a(0) \in O_M, \quad x_a(1) \in \Upsilon_a,
\]
\[
\dot{y}_a(t) = X_{H_2}(y_a(t)), \quad y_a(0) \in O_M, \quad y_a(1) \in \Upsilon_a,
\]
\[
\dot{z}_a(t) = X_{H_3}(z_a(t)), \quad z_a(0) \in O_M, \quad z_a(1) \in \Upsilon_a.
\]
For appropriate Maslov indices of \( x_a, y_a, z_a \) we define the auxiliary one-dimensional manifold \( \mathcal{M}(x_a, y_a, z_a : \tilde{\Upsilon}_a) \) to be the set of all pairs \( (R, u) \), where \( R \in [R_0, +\infty) \) and
\[
u : \Sigma \Upsilon \to T^* M
\]
is the solution of the equation
\[
\bar{\partial}_{\tilde{J}, \tilde{H}} u = 0
\]
where \( \tilde{H} \) is depicted in the Figure 3 as well as corresponding boundary conditions. The almost complex structure \( \tilde{J} \) is chosen to satisfy all the regularity conditions. \( \square \)

The following proposition establishes the ring structure PSS isomorphism.

**Proposition 13.** For \( \alpha, \beta \in HM_k(f, U) \) it holds
\[
\Phi(\alpha \cdot \beta) = \Phi(\alpha) \circ \Phi(\beta),
\]
where \( \Phi \) is the PSS isomorphism (19).
**Figure 3.** Manifold $\mathcal{M}(x_a, y_a, z_a : \tilde{\Upsilon}_s)$

**Figure 4.** Auxiliary one-dimensional manifold from the proof of Proposition 13

**Proof:** It follows from the definition of $\Phi$ and Proposition 11 that it is enough to show that

$$(23) \quad \Phi^{\Upsilon} (\alpha \cdot \beta) = \Phi^{\Upsilon} (\alpha) \circ \Phi^{\Upsilon} (\beta)$$

for a fixed approximation $\Upsilon$ and fixed Riemannian metric defining the product $\cdot$. The equality (23) is equivalent to

$$(\Phi^{\Upsilon})^{-1} (\Phi^{\Upsilon} (\alpha) \circ \Phi^{\Upsilon} (\beta)) = \alpha \cdot \beta.$$ 

The latter equality follows from cobordism arguments similar to ones used in the proof of Proposition 8 and Lemma 12. The auxiliary one-dimensional manifold we use here is explained by the Figure 4.

$\square$
3.2. Module structure. Let us recall the construction of the homology product on $HM_*(f, U : g_T)$. Let $f_1, f_2, f_3$ be Morse functions defined on $U$ such that $W_{f_i}(p_i)$ and $W_{f_j}(p_j)$ intersect transversally for every critical point $p_k$ of $f_k$. For $p_i \in CM_*(f_i, U), i = 1, 2, 3$, we define the moduli space $\mathcal{M}(p_1, p_2; p_3)$ to be the set of all trees $\gamma := (\gamma_1, \gamma_2, \gamma_3)$ such that

\[
\begin{aligned}
\gamma_1, \gamma_2 : (-\infty, 0] \to U, & \quad \gamma_3 : [0, +\infty) \to U, \\
\dot{\gamma}_j = -\nabla f_j(\gamma_j), & \quad j = 1, 2, 3, \\
\gamma_i(-\infty) = p_i, & \quad i = 1, 2, \\
\gamma_3(+\infty) = p_3, & \\
\gamma_1(0) = \gamma_2(0) = \gamma_3(0)
\end{aligned}
\]

For generic choices these spaces are manifolds of dimension

$m_{f_1}(p_1) + m_{f_2}(p_2) - m_{f_3}(p_3) - \dim U.$

If $n(p_1, p_2; p_3)$ denotes the mod 2 number of a zero–dimensional component, then the product $\cdot$ is defined at the chain level:

$\cdot : CM_*(f_1, U) \otimes CM_*(f_2, U) \to CM_*(f_3, U),$

as:

$p_1 \cdot p_2 = \sum_{p_3} n(p_1, p_2; p_3)p_3$

on generators. Since it commutes with the Morse boundary operator, it is well defined at the homology level. By standard combinatorial and cobordism arguments one can show that it defines a ring structure on $HM_*(f, U : g_T)$.

Now, we note that for every approximation $\Upsilon$, $HF_*(OM, \Upsilon : H, J_T)$ is a $HM_*(f, U : g_T)$–module. As before, the external product is defined at the chain level

$\star : CM_*(f, U) \otimes CF_*(OM, \Upsilon : H, J_T) \to CF_*(OM, \Upsilon : H, J_T).$

Let $H^s$ denotes a smooth family of Hamiltonians such that

$H^s(\cdot, t) = \begin{cases} H(\cdot, 1-t), & s \leq -2, \\ 0, & -1 \leq s \leq 1, \\ H(\cdot, t), & s \geq 2. \end{cases}$
For $p \in CM_*(f, U)$, $x, y \in CF_*(O_M, \Upsilon : H, J_T)$ we define the moduli space $\mathcal{M}(p, x; y)$ as a set of pairs $(\gamma, u)$ such that

$$
\begin{align*}
\gamma : (-\infty, 0) &\to U, u : \mathbb{R} \times [0, 1] \to T^*M, \\
\dot{\gamma} &= -\nabla f(\gamma(t)), \\
\partial_s u + J_T(\partial_t u - X_{H^s}(u)) &= 0, \\
u(s, 0) &\in \Upsilon, u(s, 1) \in O_M, s \leq 0, \\
u(s, 0) &\in O_M, u(s, 1) \in \Upsilon, s \geq 0, \\
\gamma(-\infty) &= p, \\
u(-\infty, t) &= x(t), \nu(+\infty, t) = y(t), \\
\gamma(0) &= u(0, 0).
\end{align*}
$$

These spaces are $\mu(x) - \mu(y) + m_f(p) - \dim M$ dimensional manifolds and their zero-dimensional component is compact. We define the product $\star$ on the set of the generators of chain complexes as:

$$p \star x = \sum_y \sharp_2 \mathcal{M}(p, x; y)y,$$

where $\sharp_2 \mathcal{M}(p, x; y)$ denotes the cardinality of the zero-dimensional component of $\mathcal{M}(p, x; y)$. Using standard cobordism arguments, as above, one can show that $\star$ induces a product in homology. Similarly to [11] it follows that $HF^*(O_M, \Upsilon : H, J_T)$ is a $HM_*(f, U; g_T)$–module, i.e.

$$(24) \quad (p \cdot q) \star x = p \star (q \star x),$$

for all $p, q \in HM_*(f, U; g_T)$ and $x \in HF_*(O_M, \Upsilon : H, J_T)$.

In order to have the products $\cdot$ and $\star$ well defined on a direct limit of Morse and Floer homology groups, one needs to check their compatibilities with homomorphisms $G$ and $F$.

**Lemma 14.** Let $F_{ab}$ and $G_{ab}$ be the homomorphisms that define the direct limit Morse and Floer homology groups, obtained by the number of (17) and (18) respectively. Then it holds

$$(25) \quad G_{ab}(p_a \cdot q_a) = G_{ab}(p_a) \cdot G_{ab}(q_a)$$

for all $p_a, q_a \in HM_*(f, U; g_{T_a})$ and $x_a \in HF_*(O_M, \Upsilon_a : H, J_{T_a})$.

The proof is similar to the proof of the Lemma [12]. From (24) and (25) it follows that $\cdot$ and $\star$ are well defined operations on $HM_*(f, U)$ and $HF_*(H, U; M)$ and that $HF_*(H, U; M)$ is a $HM_*(f, U)$–module.

### 4. Spectral invariants

#### 4.1. Invariants for the open subset.

From now on, we will denote the PSS isomorphisms defined in the previous section by

$$
PSS_U^1 := \Psi : HF_k(H, U : M) \xrightarrow{\cong} HM_k(f, U),$$

$$
PSS_U := \Phi : HM_k(f, U) \xrightarrow{\cong} HF_k(H, U : M).$$
and the isomorphisms for the approximations by

\[ PSS: \Phi^T : HM_k(f, U : g_T) \xrightarrow{\sim} HF_k(O_M, \Upsilon : H, J) \]

\[ PSS^{-1} := \Psi^T : HF_k(O_M, \Upsilon : H, J_T) \xrightarrow{\sim} HM_k(f, U : g_T). \]

If we consider \( A^T_H \) restricted to \( \Omega(O_M, \Upsilon) := \{ \gamma \in C^\infty([0, 1], T^* M) \mid \gamma(0) \in O_M, \gamma(1) \in \Upsilon \} \), we have

\[ dA^T_H(\gamma)(\xi) = \int_0^1 \omega(\dot{\gamma}, \xi) - dH(\gamma)(\xi) \, dt. \]

Recall that the filtered Floer homology groups for approximations are defined as homology groups of the filtered chain complex

\[ CF^\lambda_k(O_M, \Upsilon : H) := \{ x \in CF_k(O_M, \Upsilon : H) \mid A^T_H(x) < \lambda \}. \]

Since the action functional decreases along the strips that define the boundary operator

\[ \partial_{J,H} : CF_k(O_M, \Upsilon : H) \to CF_{k-1}(O_M, \Upsilon : H), \]

the boundary operator descends to \( CF^\lambda_k(O_M, \Upsilon : H) \) and defines

\[ \partial^\lambda_{J,H} : CF^\lambda_k(O_M, \Upsilon : H) \to CF^\lambda_{k-1}(O_M, \Upsilon : H). \]

Denote the corresponding homology groups by \( HF^\lambda_k(O_M, \Upsilon : H, J) \).

Now denote by

\[ \nu^\lambda_{\Upsilon} : HF^\lambda_k(O_M, \Upsilon : H, J_T) \to HF_k(O_M, \Upsilon : H, J_T) \]

the homomorphism induced by the inclusion map \( \nu^\lambda_{\Upsilon} \) and, for \( \alpha \in HM_k(f, U : g_T) \setminus \{ 0 \} \) define

\[ c_{\Upsilon}(\alpha, H) := \inf \{ \lambda \mid PSS_{\Upsilon}(\alpha) \in \text{Im}(\nu^\lambda_{\Upsilon}) \}. \]

The filtered Floer homology for an open set is defined similarly, as a direct limit:

\[ HF^\lambda_k(H, U : M) := \lim_{s \to 0} HF^\lambda_k(O_M, \Upsilon_s : H, J_s) \]

and the induced inclusion maps

\[ \nu^\lambda : HF^\lambda_k(H, U : M) \to HF_k(H, U : M) \]

are naturally and well defined (see [8]).

**Definition 15.** Let \( \alpha \in HM_k(f, U) \setminus \{ 0 \} \). A spectral invariant for an open set is defined as

\[ c_U(\alpha, H) := \inf \{ \lambda \mid PSS_U(\alpha) \in \text{Im}(\nu^\lambda_{\Upsilon}) \}. \]

The next step is to verify that, for a fixed Hamiltonian \( H \), \( c_{\Upsilon}(\cdot, H) \) tends to \( c_U(\cdot, H) \) as \( \Upsilon \to \nu^*U \). Actually, stronger property holds.

**Lemma 16.** Let \( \alpha \in HM_*(f, U : g_T) \setminus \{ 0 \} \). Then there exists an approximation \( \Upsilon \) such that

\[ c_U([\alpha], H) = c_{\Upsilon}(G_{\Upsilon}(\alpha), H) \]

for all \( \Upsilon \leq \Upsilon \).
Proof: We have the following commutative diagram
\[
\begin{array}{ccccccccc}
\cdots & \to & HM_k(f, U : g_{T^\lambda}) & \overset{G_{\lambda}}{\to} & HM_k(f, U : g_{T^\rho}) & \overset{G_{\lambda}}{\to} & HM_k(f, U : g_{T^\varepsilon}) & \to & \cdots \\
\downarrow PSS_{T^\rho} & & \downarrow PSS_{T^\rho} & & \downarrow PSS_{T^\varepsilon} & & \downarrow PSS_{T^\varepsilon} & & \\
\cdots & \to & HF_k(\Upsilon : H, J_a) & \overset{F_{\lambda}}{\to} & HF_k(\Upsilon : H, J_b) & \overset{F_{\lambda}}{\to} & HF_k(\Upsilon : H, J_c) & \to & \cdots \\
\uparrow i_{T^\rho*} & & \uparrow i_{T^\rho*} & & \uparrow i_{T^\varepsilon*} & & \uparrow i_{T^\varepsilon*} & & \\
\cdots & \to & HF_k(\Upsilon : H, J_a) & \overset{F_{\lambda}}{\to} & HF_k(\Upsilon : H, J_b) & \overset{F_{\lambda}}{\to} & HF_k(\Upsilon : H, J_c) & \to & \cdots 
\end{array}
\]

Let us take \([\alpha] \in HM_k(f, U) \setminus \{0\}\) and \(\lambda \in \mathbb{R}\) such that
\[PSS_U([\alpha]) \in \text{Im}(i^\lambda_{\Upsilon}).\]

Then, there exists \([\tilde{\alpha}] \in HF_k(\Upsilon, U : M)\) such that
\[PSS_U([\alpha]) = i^\lambda_{\Upsilon}([\tilde{\alpha}]).\]

From the definition of a direct limit we conclude that
\[\alpha \in HM_k(f, U : g_{T}), \quad \tilde{\alpha} \in HF_k(\Upsilon, U : M)\]
for some \(\Upsilon\) and \(\Upsilon'.\)
Since
\[PSS_U([\alpha]) = [PSS_{\Upsilon}(\alpha)] = [i^\lambda_{\Upsilon*}(\tilde{\alpha})] = i^\lambda_{\Upsilon*}([\tilde{\alpha}]),\]
we find that
\[F_{\Upsilon}(PSS_{\Upsilon}(\alpha)) = F_{\Upsilon}(i^\lambda_{\Upsilon*}(\tilde{\alpha}))\]
for some \(\tilde{\Upsilon}\) which is closer to \(\nu^*U\) than \(\Upsilon\) and \(\Upsilon\) and \(\Upsilon'\). Using the commutativity of the above diagram we get
\[PSS_{\tilde{\Upsilon}}(G_{\Upsilon\tilde{\Upsilon}}(\alpha)) = F_{\tilde{\Upsilon}}(PSS_{\Upsilon}(\alpha)) = F_{\tilde{\Upsilon}}(i^\lambda_{\Upsilon*}(\tilde{\alpha})) = i^\lambda_{\Upsilon*}(F_{\tilde{\Upsilon}}(\tilde{\alpha})).\]

Thus,
\[PSS_{\tilde{\Upsilon}}(G_{\Upsilon\tilde{\Upsilon}}(\alpha)) \in \text{Im}(i^\lambda_{\tilde{\Upsilon}*}).\]

We find that
\[(27) \quad c_{\tilde{\Upsilon}}(G_{\Upsilon\tilde{\Upsilon}}(\alpha), H) \leq c_{\tilde{\Upsilon}}([\alpha], H).\]

If we take \(\alpha \in HM_k(f, U : g_{T}) \setminus \{0\}\) and \(\lambda \in \mathbb{R}\) such that
\[PSS_{\Upsilon}(\alpha) \in \text{Im}(i^\lambda_{\Upsilon*})\]
then \(PSS_{\Upsilon}(\alpha) = i^\lambda_{\Upsilon*}(\tilde{\alpha})\) for some \(\tilde{\alpha} \in HF_k(\Upsilon, H, J_{\Upsilon}).\)
Again, from the definition of a direct limit we obtain the equalities
\[PSS_U([\alpha]) = [PSS_{\Upsilon}(\alpha)] = [i^\lambda_{\Upsilon*}(\tilde{\alpha})] = i^\lambda_{\Upsilon*}([\tilde{\alpha}]).\]

This gives us the inequality
\[(28) \quad c_{\Upsilon}([\alpha], H) \leq c_{\Upsilon}(\alpha, H).\]

In the quotient space \(HM_k(f, U)\) elements \(\alpha\) and \(G_{\Upsilon\tilde{\Upsilon}}(\alpha)\) represent the same element. From (27) and (28) we derive
\[c_{\tilde{\Upsilon}}(G_{\Upsilon\tilde{\Upsilon}}(\alpha), H) \leq c_{\tilde{\Upsilon}}([\alpha], H) = c_{\Upsilon}([G_{\Upsilon\Upsilon}(\alpha)], H) \leq c_{\tilde{\Upsilon}}(G_{\Upsilon\tilde{\Upsilon}}(\alpha), H),\]
and all inequalities become equalities. Since spectral invariants decrease as \(\Upsilon \to \nu^*U\), i.e.
\[c_{\tilde{\Upsilon}}(G_{\Upsilon\tilde{\Upsilon}}(\alpha), H) \leq c_{\Upsilon}(\alpha, H),\]
for every \(\tilde{\Upsilon} \leq \Upsilon\), they all become equal to \(c_{\Upsilon}([\alpha], H)\), starting from some \(\tilde{\Upsilon}\). \(\square\)
4.2. Continuity of spectral invariants.

**Theorem 17.** Relative spectral invariants for an open set $U$

$$C_U(\alpha, H) := c_U(\alpha, H) - c_U(1, H)$$

are continuous with respect to the Hamiltonian $H$,

$$|C_U(\alpha, H) - C_U(\alpha, H')| \leq \|H - H'\|,$$

( here $1$ denotes the generator of zero homology group, $HM_0(f, U : g_T)$).

**Proof:** First, we prove that

$$C_T(\alpha, H) := c_T(\alpha, H) - c_T(1, H)$$

is continuous with respect to Hamiltonian $H$. Let us fix a good approximation $\Upsilon$ and let $H$ and $H'$ be two Hamiltonians satisfying $[1]$. Consider the linear homotopy

$$H(s) = (1 - s)H + sH' = H + \sigma(s)(H' - H),$$

(we can approximate this linear homotopy with a regular one). The isomorphism $S_{H,H'}^\Upsilon$ is defined by a number of the holomorphic strips that connect $x \in CF_*(O_M, \Upsilon : H, J_T)$ and $y \in CF_*(O_M, \Upsilon : H', J_T)$:

$$\mathcal{M}(x, y, O_M, \Upsilon : H, H', J_T) :=$$

$$\{ u : \mathbb{R} \times [0, 1] \to T^*M \mid \frac{\partial u}{\partial s} + J_T(\frac{\partial u}{\partial t} - X_{H'}(u)) = 0, u(s, 0) \in O_M, u(s, 1) \in \Upsilon, u(-\infty, t) = x(t), u(+\infty, t) = y(t) \}.$$

If there exists $u \in \mathcal{M}(x, y, O_M, \Upsilon : H, H', J_T)$ that connects $x$ and $y$ then it holds

$$A_{H'}^\Upsilon(y) - A_{H'}^\Upsilon(x) = \int_{-\infty}^{+\infty} \frac{d}{ds} A_{H'}^\Upsilon(u(s, \cdot)) ds$$

$$\leq E_+(H - H') := \int_0^1 \max_x (H - H') dt$$

(see $[8]$ for details). It follows

$$A_{H'}^\Upsilon(S_{H,H'}^\Upsilon(x)) \leq A_{H'}^\Upsilon(x) + E_+(H - H').$$

For $x \in HF_*(O_M, \Upsilon : H, J_T)$ we can define

$$\overline{c}_T(x, H) := \inf\{\lambda \in \mathbb{R} \mid x \in \text{Im}(i_{T,H_*}^\lambda)\}.$$

It is obvious that $PSS_T^H$ relates $c_T$ and $\overline{c}_T$, i.e.

$$c_T(\alpha, H) = \overline{c}_T(PSS_T^H(\alpha), H).$$

From $[29]$ it follows

$$\overline{c}_T(S_{H,H'}^\Upsilon(x), H') \leq \overline{c}_T(x, H) + E_+(H - H').$$

Since $S_{H,H'}^\Upsilon \circ PSS_T^H = PSS_T^{H'}$ we get the inequality

$$c_T(\alpha, H') = \overline{c}_T(PSS_T^H(\alpha), H') = \overline{c}_T(S_{H,H'}^\Upsilon(\alpha), H')$$

$$\leq \overline{c}_T(PSS_T^H(\alpha), H) + E_+(H - H')$$

$$= c_T(\alpha, H) + E_+(H - H'),$$

(see $[8]$ for details).
that holds for all $\alpha \in HM_*(f, U : g_T)$. If we write the same inequality for the generator of zero homology group, we derive the continuity of relative spectral invariants for approximations:

$$|C_T(\alpha, H') - C_T(\alpha, H)| \leq \|H - H'\|.$$ 

From this inequality and from the Lemma it easily follows that relative spectral invariants for an open subset are continuous with respect to the Hamiltonian $H$. □

4.3. Triangle inequality.

**Proposition 18.** For an approximation $\Upsilon$ and for $\alpha, \beta \in HM_*(f, U)$ such that $\alpha \cdot \beta \neq 0$ it holds

$$c_T(\alpha \cdot \beta, H_1 \sharp H_2) \leq c_T(\alpha, H_1) + c_T(\beta, H_2).$$

**Proof:** Choose a Hamiltonian $H_3$ that is regular, smooth and close enough to $H_1 \sharp H_2$:

$$\|H_3 - H_1 \sharp H_2\|_{C^0} < \varepsilon.$$

We prove that a product $\circ$ descends to a product on filtered homologies

$$\circ : HF^\lambda_*(\Upsilon : H_1) \otimes HF^\sigma_*(\Upsilon : H_2) \rightarrow HF^{\lambda + \sigma + 4\varepsilon}_*(\Upsilon : H_3).$$

Let us take a smooth family of Hamiltonians $K : \mathbb{R} \times [-1, 1] \times T^*M \rightarrow \mathbb{R}$ such that

$$K(s, t, \cdot) = \begin{cases} 
    H_1(t + 1, \cdot), & s \leq -1, -1 \leq t \leq 0, \\
    H_2(t, \cdot), & s \leq -1, 0 \leq t \leq 1, \\
    \frac{1}{2} H_3(t + \frac{1}{2}, \cdot), & s \geq 1 
\end{cases}.$$

We can choose $K$ such that

$$\left\| \frac{\partial K}{\partial s} \right\| \leq \varepsilon, \quad s \in [-1, 1],$$

and

$$\frac{\partial K}{\partial s} = 0,$$

elsewhere. Assume that for $x \in CF^\lambda_*(\Upsilon : H_1)$ and $y \in CF^\sigma_*(\Upsilon : H_2)$ there exists $u \in \mathcal{M}(x, y; z)$ for some $z \in CF_*(\Upsilon : H_3)$. Here, $u$ are pseudo–holomorphic pants for a Hamiltonian $\tilde{K}$

$$\tilde{K}_{\Upsilon, \gamma}(u) = 0.$$

Using the relations

$$\int_\Sigma \left\| \frac{\partial u}{\partial s} \right\|^2 ds dt \geq 0,$$

$$\int_\Sigma u^* \omega = - \int x^* \theta + h_T(x(1)) - \int y^* \theta + h_T(y(1)) + \int z^* \theta - h_T(z(1)),$$

Stoke's formula and properties of a Hamiltonian $K$ it follows

$$A_{H_3}^\Upsilon(z) \leq A_{H_1}^\Upsilon(x) + A_{H_2}^\Upsilon(y) + 4\varepsilon.$$

Now, from Proposition we obtain the inequality

$$c_T(\alpha \cdot \beta, H_3) \leq c_T(\alpha, H_1) + c_T(\beta, H_2) + 4\varepsilon.$$

Since spectral invariants are continuous with respect to the Hamiltonian we get the claimed inequality. □
Theorem 19. For $\alpha, \beta \in HM_\ast(f, U)$ such that $\alpha \cdot \beta \neq 0$ it holds
\[ c_U(\alpha \cdot \beta, H_1 \sharp H_2) \leq c_U(\alpha, H_1) + c_U(\beta, H_2). \]

Remark 20. Using Proposition 13 we can restate the previous theorem as
\[ c_U(PSS^{-1}_U(a \circ b), H_1 \sharp H_2) \leq c_U(PSS^{-1}_U(a), H_1) + c_U(PSS^{-1}_U(b), H_2), \]
for all $a, b \in HF_\ast(H, U : M)$ such that $a \circ b \neq 0$.

4.4. Invariants for periodic orbits. Recall the definition of spectral invariants for periodic orbit Floer homology. Denote by $HF_k(T^* M : H, J)$ and $HM_k(f, T^* M)$ Floer homology for periodic orbits in $T^* M$ and Morse homology for the Morse function $f : T^* M \to \mathbb{R}$ respectively. Denote by $H^k_T(T^* M : H, J)$ the corresponding filtered group (with respect to the standard action functional) and, again, by $j_\lambda$ the map induced by inclusion. Let PSS stands for PSS isomorphism for periodic orbits, defined in [21]
\[ \text{PSS} : HM_k(f, T^* M : g) \xrightarrow{\cong} HF_k(T^* M : H, J) \]
and let $\alpha \in HM_k(f, T^* M : g)$.

The filtration in Floer homology for periodic orbits is given by the standard action functional
\[ a_H(\gamma) := \int \gamma^* \theta - \int_0^1 H dt \]
which is well defined in the cotangent bundle setting. Filtered Floer homology groups are homology groups of a chain complex generated by
\[ CF_k(T^* M : H) := \{ a \in CF_k(T^* M : H) \mid a_H(a) < \lambda \}, \]
where $CF_k(T^* M : H)$ denotes the $\mathbb{Z}_2$–vector space over the set of periodic Hamiltonian $H$–orbits in $T^* M$ of Conley–Zehnder index $k$.

Define
\[ \rho(\alpha, H) := \inf \{ \lambda \mid \text{PSS}(\alpha) \in \text{Im}(j_\lambda^\ast) \}, \]
where $\alpha \in HM_\ast(f, T^* M : g) \setminus \{0\}$.

4.5. Chimneys and relation between the two invariants. The homomorphisms defined using ”chimneys” are considered by Abbondandolo and Schwarz in [2] (in the context of Floer homology of cotangent bundles and the ring-isomorphism with the homology of the loop space) and Albers in [3] (in the construction of the comparison homomorphisms between Lagrangian and Hamiltonian Floer homology). The construction of a chimney is different in our situation, due to the boundary conditions.

Let
\[ \Sigma := \mathbb{R} \times [0, 1] / \sim, \quad \text{where } (s, 0) \sim (s, 1) \text{ for } s \geq 0. \]

For $x \in CF_\ast(O_M, \Upsilon : H)$ and $a \in CF_\ast(T^* M : H)$ define the manifold of chimneys as:
\[ M(x, a, O_M, \Upsilon : H, J) := \left\{ u : \Sigma \to T^* M \mid \begin{array}{l} \partial_s u + J(\partial_t u - X_H \circ u) = 0 \\ u(s, 0) \in O_M, \ u(s, 1) \in \Upsilon \text{ for } s \leq 0 \\ u(-\infty, t) = x(t), \ u(+\infty, t) = a(t) \end{array} \right\}. \]
For generic choices, \( \mathcal{M}(x, a) \) is a smooth manifold of dimension \( \mu_{CZ}(a) - \mu(x) - \frac{n}{2} \), where \( \mu_{CZ}(a) \) denotes the Conley-Zehnder index of a loop \( a \).

Define
\[
\chi : CF_k(O_M, \Upsilon : H) \to CF_k(T^* M : H) \\
\chi(x) := \sum (\sharp \mathcal{M}(x, a, O_M, \Upsilon : H, J) \mod 2) a.
\]
(30) 

It holds \( \chi \circ \partial = \partial \circ \chi \), hence \( \chi \) is well defined on the homology level:
\[
\chi : HF_k(O_M, \Upsilon : H, J) \to HF_k(T^* M : H, J).
\]

Let \( a \) be a periodic orbit. If there exists \( u \in \mathcal{M}(x, a, O_M, \Upsilon : H, J) \), let \( y \) be a periodic orbit defined as
\[
y(t) := u(0, t).
\]
Since \( y(0) = y(1) \in O_M \), we have \( h_T(y(1)) = 0 \). Therefore, we have
\[
A_T^y(y) = a_H(y),
\]
so
\[
a_H(a) - A_T^y(x) = a_H(a) - a_H(y) + A_T^y(y) - A_T^y(x) = \\
\int_{-\infty}^{0} \frac{d}{ds} a_H(u(s, t))ds + \int_{0}^{+\infty} \frac{d}{ds} A_T^y(u(s, t))ds + \\
- \int_{-\infty}^{0} \int_{0}^{1} \omega(\partial_s u, \partial_t u - X_H \circ u) dt ds = - \int_{-\infty}^{\infty} \int_{0}^{1} \left\| \frac{\partial u}{\partial s} \right\|^2 dt ds \leq 0.
\]

It follows that \( \chi \) defines the mapping
\[
\chi^\lambda := \chi|_{CF_k^\lambda(O_M, \Upsilon : H)} : CF_k^\lambda(O_M, \Upsilon : H) \to CF_k^\lambda(T^* M : H)
\]
which also descends to the homology level:
\[
\chi^\lambda : HF_k^\lambda(O_M, \Upsilon : H, J_T) \to HF_k^\lambda(T^* M : H, J_T)
\]
(see also [6]). The diagram
\[
\begin{array}{ccc}
HF_k^\lambda(O_M, \Upsilon : H, J_T) & \xrightarrow{\chi^\lambda} & HF_k^\lambda(T^* M : H, J_T) \\
\downarrow \iota^\lambda & & \downarrow \jmath^\lambda \\
HF_k(O_M, \Upsilon : H, J_T) & \xrightarrow{\chi} & HF_k(T^* M : H, J_T)
\end{array}
\]
commutes.

Similarly, set
\[
\Delta = \mathbb{R} \times [0, 1]/\sim, \quad \text{where } (s, 0) \sim (s, 1) \text{ for } s \leq 0,
\]
and define
\[
\mathcal{M}(a, x, O_M, \Upsilon : H, J) := \left\{ u : \Delta \to T^* M \left| \begin{array}{l}
\partial_s u + J(\partial_t u - X_H \circ u) = 0 \\
u(s, 0) \in O_M, u(s, 1) \in \Upsilon \text{ for } s \geq 0 \\
u(-\infty, t) = a(t), u(+\infty, t) = x(t).
\end{array} \right. \right\}
\]

For generic choices, \( \mathcal{M}(a, x) \) is a smooth manifold of dimension \( \mu(x) - \frac{n}{2} - \mu_{CZ}(a) \).

Define
\[
\tau : CF_k(T^* M : H) \to CF_{k-n}(O_M, \Upsilon : H) \\
\tau(a) := \sum (\sharp \mathcal{M}(a, x, O_M, \Upsilon : H, J) \mod 2) x.
\]
This homomorphism also descends to the homology level
\[ \tau : HF_k(T^* M : H, J) \to HF_{k-n}(O_M, \Upsilon : H, J) \]
since it commutes with the boundary operators. As above, one can show that it also
induces a homomorphism on the filtered homology level, and that the corresponding
diagram (analogous to (31)) commutes.

**Proposition 21.** Let \( f \) be Morse function on \( U \) as above and \( \tilde{f} \) Morse function on \( T^* M \). Let \( H : T^* M \to \mathbb{R} \) be a Hamiltonian function and \( \iota_* : HM_k(f, U : g_\Upsilon) \to HM_k(\tilde{f}, T^* M : g_\Upsilon) \) a homomorphism induced by inclusion. Suppose all the choices
are generic. The diagram

\[
\begin{array}{c}
\xymatrix{
HF_k(O_M, \Upsilon : H, J_\Upsilon) \ar[r]^{\chi_\Upsilon} \ar[d]_{\iota_*} & HF_k(T^* M : H, J_\Upsilon) \ar[d]^{\mathcal{PSS}_\Upsilon^{-1}} \ar[r]_{j_\Upsilon^*} & \ar[d]_{\mathcal{PSS}^{-1}} \ar[r] & \ar[d] \ar[r] & \\
HF_k(O_M, \Upsilon : H, J_\Upsilon) \ar[r]^{\chi} \ar[d]_{\mathcal{PSS}_\Upsilon^{-1}} & HF_k(T^* M : H, J_\Upsilon) \ar[d]^{\mathcal{PSS}^{-1}} \ar[r]_{j_*} & \ar[d]_{\mathcal{PSS}^{-1}} \ar[r] & \ar[d] \ar[r] & \\
HM_k(f, U : g_\Upsilon) \ar[r]^{\iota_*} & HM_k(\tilde{f}, T^* M : g_\Upsilon) \ar[r]_{T_{F, \tilde{f}}} & HM_k(\tilde{f}, T^* M : g_\Upsilon) \ar[r]_{\mathcal{PSS}^{-1}} & \\
HM_k(f, U : g_\Upsilon) \ar[r]^{\iota_*} & HM_k(F, T^* M : g_\Upsilon) \ar[r]_{T_{F, \tilde{f}}} & HM_k(\tilde{f}, T^* M : g_\Upsilon) & \mathcal{PSS}^{-1} \circ \mathcal{X} \circ \mathcal{PSS}_\Upsilon = \iota_0.
\end{array}
\]

commutes.

**Remark 22.** In the statement of the proposition \( \iota_* \) denotes the composition of the
inclusion
\[ \iota_0 : HM_k(f, U : g_\Upsilon) \to HM_k(F, T^* M : g_\Upsilon) \]
and the canonical isomorphism
\[ T_{F, \tilde{f}} : HM_k(F, T^* M : g_\Upsilon) \to HM_k(\tilde{f}, T^* M : g_\Upsilon) \]
where a Morse function \( F \) is the following appropriate extension of \( f \) to \( T^* M \).
Consider a tubular neighbourhood \( W \subseteq T^* M \) of \( U \). First, we extend \( f \) to the vector
bundle \( W \) over \( U \), and obtain the Morse function \( f_W \) such that
\( f_W|_U = f \) and \( \text{Crit}_k(f_W) = \text{Crit}(f) \).

Then we extend the Morse function \( f_W \) defined on the open subset \( W \) to the Morse
function \( F \) on \( T^* M \) (see [24] for details). We chose the extension \( F \) in such way
that there are no trajectories for the negative gradient flow of \( F \) leaving \( U \). Under
this assumptions the Morse complex \( CM_k(f) \) is a subset of the Morse complex
\( CM_k(F) \) and the inclusion of these complexes becomes the homomorphism on the
homology level \( \iota_0 \).

**Proof:** The upper diagram is (31). The lower diagram is

\[
\begin{array}{c}
\xymatrix{
HF_k(O_M, \Upsilon : H, J_\Upsilon) \ar[r]^\chi \ar[d]_{\mathcal{PSS}_\Upsilon^{-1}} & HF_k(T^* M : H, J_\Upsilon) \ar[d]_{\mathcal{PSS}^{-1}} \ar[r]^\text{id} & HF_k(T^* M : H, J_\Upsilon) \ar[d]_{\mathcal{PSS}^{-1}} \ar[r]_{\mathcal{PSS}^{-1}} & \\
HM_k(f, U : g_\Upsilon) \ar[r]^\iota_* & HM_k(F, T^* M : g_\Upsilon) \ar[r]^{T_{F, \tilde{f}}} & HM_k(\tilde{f}, T^* M : g_\Upsilon) \ar[r]_{\mathcal{PSS}^{-1}} & \\
HM_k(f, U : g_\Upsilon) \ar[r]^\iota_* & HM_k(F, T^* M : g_\Upsilon) \ar[r]^{T_{F, \tilde{f}}} & HM_k(\tilde{f}, T^* M : g_\Upsilon) & \mathcal{PSS}^{-1} \circ \mathcal{X} \circ \mathcal{PSS}_\Upsilon = \iota_0.
\end{array}
\]

The commutativity of the diagram on the right side was resolved in [21]. To prove
the commutativity of the diagram on the left side we will show that
\[ \mathcal{PSS}^{-1} \circ \mathcal{X} \circ \mathcal{PSS}_\Upsilon = \iota_0. \]
In order to do that using the usual cobordism arguments, we consider the following two auxiliary manifolds. Let $\rho_R$ be the symmetric cut-off function (11). Let $p$ be the critical point of a Morse function $f$ and $Q$ the critical point of a Morse function $F$. Fix $R > 0$ and define

$$\tilde{\mathcal{M}}_R(p, Q) := \tilde{\mathcal{M}}_R(p, Q, O_M, \Upsilon : H, J_\Upsilon) := \left\{ (\gamma_1, u, \gamma_2) \mid \begin{aligned} \gamma_1 : (-\infty, 0] &\to U, \quad \gamma_1' = -\nabla f(\gamma_1), \\
\gamma_2 : [0, +\infty) &\to T^*M, \quad \gamma_2' = -\nabla F(\gamma_2), \\
\gamma_1(-\infty) &= p, \quad \gamma_1(0) = u(-\infty, t), \\
\gamma_2(+\infty) &= Q, 
\end{aligned} \right\}$$

(see Figure 6).

Define also

$$\tilde{\mathcal{M}}(p, Q) := \left\{ (\gamma_1, u, \gamma_2, R) \mid R \in [R_0, \infty), \ (\gamma_1, u, \gamma_2) \in \tilde{\mathcal{M}}_R(p, Q) \right\}.$$  

For $m_f(p) = m_F(Q)$ and generic choices, $\tilde{\mathcal{M}}(p, Q)$ is a smooth one-dimensional manifold with topological boundary that can be identified with

$$\partial \left( \tilde{\mathcal{M}}(p, Q) \right) = B_1 \cup B_2 \cup B_3 \cup B_4$$

where

$$B_1 = \tilde{\mathcal{M}}_{R_0}(p, Q);$$
$$B_2 = \bigcup_{s \in \text{Crit}(f)} \tilde{\mathcal{M}}(p, s) \times \tilde{\mathcal{M}}_R(s, Q);$$
$$B_3 = \bigcup_{S \in \text{Crit}(F)} \tilde{\mathcal{M}}_R(p, S) \times \tilde{\mathcal{M}}(S, Q);$$
$$B_4 = \bigcup_{\begin{subarray}{l} x \in CF_k(O_M, \Upsilon : H) \\
a \in CF_k(T^*M : H) \end{subarray}} \mathcal{M}(p, x) \times \mathcal{M}(x, a) \times \mathcal{M}(a, Q).$$
Here
\[ M(a, Q) := \begin{cases} 
(u, \gamma) & \text{if } u: (-\infty, 0] \times [0, 1] \to T^*M, \\
\partial_u u + J(\partial_k u - X_{\rho H} \circ u) = 0, \\
u(s, 0) = u(s, 1), \\
\gamma: [0, +\infty) \to T^*M, \quad \dot{\gamma} = -\nabla F(\gamma), \\
u(0, t) = \gamma(0), \quad u(-\infty, t) = a(t), \quad \gamma(+\infty) = Q,
\end{cases} \]

i.e. it is the space of combined object defining a PSS isomorphism for periodic orbits and \( \tilde{\rho}(s) = \begin{cases} 1, & s \leq -R - 1, \\
0, & s \geq -R \end{cases} \).

Denote by \( F_j \) the homomorphism obtained by counting the elements of the zero dimensional manifold \( B_j \). Since the number of the boundary of the one-dimensional manifold \( \tilde{M}(p, Q) \) is even, i.e. zero in \( \mathbb{Z}_2 \), and the maps \( F_2 \) and \( F_3 \) are of the form
\[ F_2 = \partial \circ K, \quad F_3 = K \circ \partial, \]
the homomorphisms \( F_1 \) and \( F_4 \) are equal in the homology. By standard cobordism argument one can show that the mapping \( F_1 \) does not depend on \( R_0 \). Letting \( R_0 \to 0 \), we obtain that, at the homology level, the mapping \( F_1 \) counts the objects \( (\gamma_1, u, \gamma_2) \) where \( u \) is a holomorphic map from \( \Sigma \) to \( T^*M \). The boundary of \( u \) is on \( O_M \cup \Upsilon \), so it must be constant due to the exactness of both Lagrangian manifold and the fact that \( h_T|_{O_M} = 0 \) (see the proof of Lemma \[\text{12}\] for the details). Hence \( F_1 \) is chain homotopic to the map obtained by counting the pairs \( (\gamma_1, \gamma_2) \) with properties:
\[ \begin{aligned}
\gamma_1: (-\infty, 0] & \to U, \quad \dot{\gamma}_1 = -\nabla f(\gamma_1), \quad \gamma_1(-\infty) = p, \\
\gamma_2: [0, +\infty) & \to T^*M, \quad \dot{\gamma}_2 = -\nabla F(\gamma_2), \quad \gamma_2(+\infty) = Q \\
\gamma_1(0) & = \gamma_2(0).
\end{aligned} \]
Since we know that \( F|_{U} = f, \gamma_1, \gamma_2 \) is a negative gradient trajectory of \( F \) connecting two critical points of the same Morse index. Number of such pairs is equal 1 in case \( p = Q \) and 0 otherwise. Thus, \( F_1 \) is chain homotopic to the homomorphism \( u_0 \). On the other hand, the mapping \( F_4 \) is exactly the homomorphism \( \text{PSS}^{-1} \circ \chi \circ \text{PSS}_\Upsilon \), so the claim follows. \( \square \)

Now, for two generic almost complex structures \( J_a \) and \( J_b \), denote by \( D_{ab} \) a canonical isomorphism of Floer homologies for periodic orbits:
\[ D_{ab}: HF_k(T^*M : H, J_a) \to HF_k(T^*M : H, J_b) \]
that satisfies
\[ D_{bc} \circ D_{ab} = D_{ac}. \]

As before, define Floer homology for periodic orbits as a direct limit
\[ HF_k(T^*M : H) := \lim_{k \to \infty} HF_k(T^*M : H, J_s). \]
The filtered Floer homology is defined as:
\[ HF_k^\lambda(T^*M : H) := \lim_{k \to \infty} HF_k^\lambda(T^*M : H, J_s). \]

**Proposition 23.** Let \( \chi^a \) stands for a homomorphism \( \text{PSS}^{-1} \circ \chi \circ \text{PSS}_\Upsilon \) for the almost complex structure \( J_a \). We use the abbreviations
\[ HF_k^\lambda(\Upsilon_a) := HF_k^\lambda(O_M, \Upsilon_a : H, J_a), \quad HF_k^\lambda(J_a) := HF_k^\lambda(T^*M : H, J_a). \]
The diagram:

\[
\cdots \rightarrow HF_k^\lambda(Y_a) \xrightarrow{F_{ab}} HF_k^\lambda(Y_b) \xrightarrow{F_{bc}} HF_k^\lambda(Y_c) \rightarrow \cdots
\]

commutes.

\[\chi^a = D^{-1}_{ab} \circ \chi^b \circ F_{ab} = D_{ba} \circ \chi^b \circ F_{ab}.\]

The proof of the above equality is similar to the proofs of the Proposition 8 and Lemma 12. The auxiliary one-dimensional manifold will be the set of the pairs \((R, u), R \in [R_0, +\infty), u\) is a chimney with the properties depicted in Figure 7.

**Corollary 24.** The homomorphism (30) induces the homomorphism
\[\chi^\lambda : HF_k^\lambda(H, U : M) \rightarrow HF_k^\lambda(T^*M : H),\]
and the homomorphism
\[\chi : HF_k(H, U : M) \rightarrow HF_k(T^*M : H).\]

From the commutativity of (32) for all approximations, we derive the following corollary.

**Corollary 25.** The diagram

\[
\begin{array}{ccc}
HF_k^\lambda(H, U : M) & \stackrel{\chi^\lambda}{\longrightarrow} & HF_k^\lambda(T^*M : H) \\
\uparrow \iota_k^* & & \uparrow \jmath_k^* \\
HM_k(f, U) & \stackrel{\chi}{\longrightarrow} & HM_k(f, T^*M) \\
\downarrow \text{PSS}^{-1} & & \downarrow \text{PSS}^{-1} \\
HM_k(f, U) & \stackrel{\iota}{\longrightarrow} & HM_k(f, T^*M)
\end{array}
\]

commutes.
Theorem 26. Let \( \alpha \in HM_k(f, U) \setminus \{0\} \). Then
\[
c_U(\alpha, H) \geq \rho(\iota_*(\alpha), H).
\]
Proof: From the commutativity of \([35]\) ones easily gets
\[
\{ \lambda \mid PSS_U(\alpha) \in \text{Im}(\iota_{*}^{\lambda}) \} \subseteq \{ \lambda \mid PSS(\iota_*(\alpha)) \in \text{Im}(\iota_{*}^{\lambda}) \},
\]
(see the proof of Theorem 28 below or \([6]\) for more details) so the claim follows. \(\square\)

One can obtain the inequality of similar type by using the homomorphism \(\tau\).
The corresponding commutative diagram is
\[
\begin{array}{ccc}
HF_k(T^*M : H) & \xrightarrow{\tau^\lambda} & HF_{k-n}(H, U : M) \\
\downarrow{\lambda^\lambda} & & \downarrow{\iota^\lambda} \\
HF_k(T^*M : H) & \xrightarrow{\tau} & HF_{k-n}(H, U : M) \\
\downarrow{\text{PSS}^{-1}} & & \downarrow{\text{PSS}^{-1}} \\
HM_k(\tilde{f}, T^*M) & \xrightarrow{\iota} & HM_{k-n}(\tilde{f}, U)
\end{array}
\]
where \(\iota\) is the map obtained by inclusion map and Poincaré duality map:
\[
\iota! := \text{PD}^{-1} \circ \iota_* \circ \text{PD}.
\]
From this commutativity, we have the following

Theorem 27. Let \( \alpha \in HM_k(\tilde{f}, T^*M) \setminus \{0\} \), then
\[
\rho(\alpha, H) \geq c_U(\iota!(\alpha), H).
\]

4.6. Invariants for subsets. In \([20]\) Oh considered a spectral invariant
\[
c_+(H, U) := \inf \{ \lambda \in \mathbb{R} \mid \iota_{*}^{\lambda} : HF_k^\lambda(H, U : M) \to HF_k(H, U : M) \text{ is surjective} \}
\]
(the notions are the same as in the Subsection 4.1). If \( U \xhookrightarrow{j} V \) are two open subset of \( M \) and
\[
\iota_{*UV} : H_k^{\text{sing}}(U, \mathbb{Z}) \to H_k^{\text{sing}}(V, \mathbb{Z})
\]
is surjective, Oh proved that
\[
c_+(H, V) \leq c_+(H, U).
\]
We can prove slightly more precise statement, the inequality for any homology class (with \(\mathbb{Z}_2\) coefficients), using the PPS isomorphism for an open subset in the proof.

Theorem 28. Let \( U \xhookrightarrow{j} V \) be two open subset of \( M \) and let
\[
j_{*UV} : HM_k(f, U) \to HM_k(f, V)
\]
(the homomorphism induced by inclusion \(j : U \hookrightarrow V\) is surjective. Let \( c_U(\alpha, H) \) be as in \([26]\). For \( \alpha \in HM_k(f, U) \setminus \{0\} \) it holds:
\[
c_V(j_{*UV}(\alpha), H) \leq c_U(\alpha, H).
\]
Proof: Let
\[ \nu_{UV} : HF_k(H, U : M) \to HF_k(H, V : M), \]
\[ \iota_{UV}^\lambda : HF_k^\lambda(H, U : M) \to HF_k^\lambda(H, V : M), \]
\[ \iota_U^\lambda : HF_k^\lambda(H, U : M) \to HF_k(H, U : M), \]
\[ \iota_V^\lambda : HF_k^\lambda(H, V : M) \to HF_k(H, V : M) \]
denote the inclusion homomorphisms defined by Oh in [20]. The following diagram is commutative:
\[ (36) \quad HF_k^\lambda(H, U : M) \xrightarrow{\iota_{UV}^\lambda} HF_k^\lambda(H, V : M) \]
\[ \downarrow \iota_U^\lambda \quad \downarrow \iota_V^\lambda \]
\[ HF_k(H, U : M) \xrightarrow{\nu_{UV}} HF_k(H, V : M) \]
\[ \xrightarrow{\text{PSS}_U^{-1}} \quad \xrightarrow{\text{PSS}_V^{-1}} \]
\[ HM_k(f, U) \xrightarrow{J_{UV}} HM_k(f, V). \]
The commutativity of the upper diagram is proven in [20]. To prove the commutativity of the lower one, it is enough to prove the commutativity of
\[ HF_k(O_M, \Upsilon^U : H, J) \xrightarrow{\iota_{UV}^{(H, J)}} HF_k(O_M, \Upsilon^V : H, J) \]
\[ \xrightarrow{\text{PSS}^{-1}_U} \quad \xrightarrow{\text{PSS}^{-1}_V} \]
\[ HM_k(f, U) \xrightarrow{J_{UV}} HM_k(f, V), \]
for all \( \Upsilon^U \) close enough to \( \nu^* \Upsilon \) and \( \Upsilon^V \) close enough to \( \nu^* \Upsilon \). Here \( \iota_{UV}^{(H, J)} \) is the inclusion map also defined in [20]. Take \( [x] \) in \( HF_k(O_M, \Upsilon^U : H, J) \). It holds
\[ (37) \quad \text{PSS}^{-1}_V(\iota_{UV}^{(H, J)}([x])) = \sum_{p \in CM_k(V)} n(x, p)[p]. \]

On the other hand, we have
\[ J_{UV}(\text{PSS}^{-1}_U([x])) = \sum_{p \in CM_k(U)} n(x, p)J_{UV}([p]), \]
which is the same as (37) if \( J_{UV} \) is surjective.

Let
\[ A^U_\alpha := \{ \lambda \in \mathbb{R} \mid \text{PSS}_U(\alpha) \in \text{Im}(\iota_U^\lambda) \}. \]
If \( \lambda \in A^U_\alpha \), then \( \text{PSS}_U(\alpha) = \iota_U^\lambda(\beta) \), for \( \beta \in HF_k^\lambda(H, U : M) \), so, from the commutativity of (36) we have
\[ \iota_V^\lambda(\iota_U^\lambda(\beta)) = \nu_{UV}(\iota_U^\lambda(\beta)) = \nu_{UV}(\text{PSS}_U(\alpha)) = \text{PSS}_V(J_{UV}(\alpha)). \]
We conclude that \( \lambda \in A^V_{J_{UV}(\alpha)} \), therefore
\[ A^U_\alpha \subset A^V_{J_{UV}(\alpha)}, \]
so by taking an infimum over \( \lambda \), we obtain
\[ c_V(J_{UV}(\alpha), H) \leq c_U(\alpha, H). \]
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