Good formal structures for flat meromorphic connections, III: Irregularity is nef

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Abstract

Given a formal flat meromorphic connection over an excellent scheme over a field of characteristic zero, in a previous paper we established existence of good formal structures and a good Deligne-Malgrange lattice after suitably blowing up; however, no control was achieved on the structure of the blowup. In this paper, we show that the blowups that achieve good formal structures are precisely those that resolve a certain nef Cartier b-divisor, the irregularity b-divisor. This suggests the possibility of finding functorial blowups to produce good formal structures, which would allow our results to be transferred globally (not just locally) to complex analytic spaces.

Introduction

The Hukuhara-Levelt-Turrittin decomposition theorem gives a classification of differential modules over the field $\mathbb{C}((z))$ of formal Laurent series resembling the decomposition of a finite-dimensional vector space equipped with a linear endomorphism into generalized eigenspaces. It implies that after adjoining a suitable root of $z$, one can express any differential module as a successive extension of one-dimensional modules. This classification serves as the basis for the asymptotic analysis of meromorphic connections around a (not necessarily regular) singular point. In particular, it leads to a coherent description of the Stokes phenomenon, i.e., the fact that the asymptotic growth of horizontal sections near a singularity must be described using different asymptotic series depending on the direction along which one approaches the singularity. (See [29] for a beautiful exposition of this material.)

This is the third in a series of papers, starting with [14, 15], in which we give some higher-dimensional analogues of the Hukuhara-Levelt-Turrittin decomposition for irregular flat formal meromorphic connections on complex analytic or algebraic varieties. (The regular case is already well understood by work of Deligne [6].) We do not discuss asymptotic analysis or the Stokes phenomenon; these have been treated in the two-dimensional case by Sabbah [24] (building on work of Majima [18]), and one expects the higher-dimensional case to behave similarly.

In the remainder of this introduction, we recall what was established in [14 and 15], then explain what is added in this paper.
0.1 Resolution of turning points

In [14], we developed a numerical criterion for the existence of a good decomposition (in the sense of Malgrange [19]) of a formal flat meromorphic connection at a point where the polar divisor has normal crossings. This criterion is inspired by the treatment of the original decomposition theorem given by Robba [23] using spectral properties of differential operators on nonarchimedean rings; our treatment depends heavily on joint work with Xiao [17] concerning differential modules on some nonarchimedean analytic spaces.

We then applied this criterion to prove a conjecture of Sabbah [24, Conjecture 2.5.1] concerning formal flat meromorphic connections on a two-dimensional complex algebraic or analytic variety. We say that such a connection has a good formal structure at some point if it acquires a good decomposition after pullback along a finite cover ramified only over the polar divisor. In general, even if the polar divisor has normal crossings, one only has good formal structures away from some discrete set, the set of turning points. However, Sabbah conjectured that one can replace the given surface with a suitable blowup in such a way that the pullback connection admits good formal structures everywhere. Such a blowup might be called a resolution of turning points; we constructed it using the numerical criterion plus some analysis on a certain space of valuations (called the valuative tree by Favre and Jonsson [7]).

In [15], we constructed resolutions of turning points for formal flat meromorphic connections on excellent schemes of characteristic zero, which include algebraic varieties of all dimensions over any field of characteristic zero. This combined the numerical criterion of [14] with a more intricate valuation-theoretic argument, based on the properties of one-dimensional Berkovich nonarchimedean analytic spaces.

We also obtained a partial result for complex analytic varieties, using the fact that the local ring of a complex analytic variety at a point is an excellent ring. Namely, we obtained local resolution of turning points, i.e., we only construct a good modification in a neighborhood of a fixed starting point. For excellent schemes, one can always extend the resulting local modifications, by taking the Zariski closure of the graph of a certain rational map, then take a global modification dominating these. However, this approach is not available for analytic varieties.

0.2 Irregularity b-divisors

The resolutions of turning points obtained in [15] are highly uncontrolled; for instance, they may involve blowing up outside of the space of turning points. This is because their construction is highly local in nature, depending on some intricate valuation-theoretic analysis. In this paper, we take a step towards bringing these blowups under control by relating them to a problem of pure birational geometry: the determination of Cartier b-divisors.

Consider a meromorphic differential module $\mathcal{E}$ on an excellent $\mathbb{Q}$-scheme $X$. (More precisely, we insist that $X$ is a nondegenerate differential scheme in the sense of [15].) Following Malgrange, we construct from the differential module a canonical function, the irregularity, on the set of exceptional divisors on local modifications of $X$. One may view this function
as a *Weil divisor on the Riemann-Zariski space* in the language of Boucksom-Favre-Jonsson [4], or as a *b-divisor* in the language of Shokurov [25]; we adopt the latter terminology here.

In this language, we establish the following facts. On one hand, the irregularity function of $E$ is computed by a certain *nef Cartier b-divisor*, i.e., the function which measures multiplicities in pullbacks of a certain Cartier divisor $D$ on a certain blowup $f : Y \to X$ such that $D$ is nef relative to $f$ (i.e., has nonnegative degree on curves contracted by $f$); we call this b-divisor the *irregularity b-divisor* of $E$. On the other hand, a blowup $f : Y \to X$ with $Y$ regular is a resolution of turning points if and only if the irregularity b-divisors of both $f^*E$ and $\text{End}(f^*E) = f^* \text{End}(E)$ correspond to Cartier divisors on $Y$ itself, rather than a further blowup. These two statements are largely a reinterpretation of our prior results aside from the nef property, which requires a significant new argument.

Note that we are unable to establish that the irregularity b-divisor of $E$ is *semiample*, i.e., that some positive integer multiple of it is globally generated. Rather, the nef condition can be interpreted (as in [3, 4]) as saying that the irregularity b-divisor belongs to the closure of the space of semiample b-divisors. This causes some complications in our program, as described below.

### 0.3 Functorial determination and resolution of turning points

We next describe what remains to be done in this series and why the aforementioned discussion is insufficient for these purposes.

We would like to obtain resolutions of turning points which satisfy *functoriality* for regular morphisms on the base space. Here the adjective *regular* does not perform its colloquial function of distinguishing true morphisms of schemes from *rational morphisms*, which are only defined on a Zariski open dense subspace of the domain. Rather, a *regular morphism* is one which is flat with geometrically regular fibres; for instance, any smooth morphism is regular. Even more specifically, open immersions are regular, so functoriality for regular morphisms implies locality for the Zariski topology. This formalism is modeled on the formalism of functorial (nonembedded and embedded) resolution of singularities for quasiecellent schemes over a field of characteristic zero, as established by Temkin [27, 28] using the resolution algorithm for complex algebraic varieties given by Bierstone and Milman [1, 2].

To obtain functorial resolution of turning points, it would suffice to obtain *functorial determination* of Cartier b-divisors on excellent $\mathbb{Q}$-schemes, i.e., a recipe that given an excellent $\mathbb{Q}$-scheme $X$ and a Cartier b-divisor $D$ specifies a modification $f : Y \to X$ for which $D$ occurs as a Cartier divisor on $Y$, in a manner functorial with respect to regular morphisms. Unfortunately, it is easy to produce examples which show that no such recipe can exist (see for instance Example 1.3.3). On the other hand, if one considers only *semiample* Cartier b-divisors, then functorial determination is straightforward: any sufficiently divisible multi-

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1 This is sometimes misleading called *canonical* resolution of singularities, but there is no unique or even distinguished choice involved. For instance, Temkin’s proofs can in principle be adapted to other functorial resolution algorithms for complex analytic varieties (several of which are described in [10]): this should lead to different (but still functorial) resolutions of singularities for quasiecellent schemes over a field of characteristic zero.
ple of the b-divisor corresponds to a coherent fractional ideal sheaf of $X$, the blowup along which does the job. Since we have only so far shown that irregularity b-divisors are nef, we are left with the task of affirmatively resolving one of the following two problems.

1. Establish that all irregularity b-divisors are semiample.

2. Construct a functorial determination for nef Cartier b-divisors.

A stronger version of the first problem would be to construct *logarithmic characteristic cycles* for algebraic $D$-modules, as described in the rank 1 case by Kato [12]. However, it is unclear whether even the stated problem should have an affirmative answer; the ordinary characteristic cycle does not seem to help much with this. As for the second problem, it should be possible to reduce to the case of varieties over fields of characteristic 0 by an algebraization argument, but even this case seems to be quite mysterious.

### 0.4 Transfer to the analytic category

We conclude with some remarks about resolution of turning points on complex analytic spaces. As in [15], functorial resolution of turning points on excellent schemes would imply an analogous local result on complex analytic varieties, using the fact that the localization of a complex analytic variety at a suitably small compact subset (one contained in a Stein subspace) is an excellent ring. The difference is that in this case, the resulting local resolutions of turning points would be compatible with open immersions, and would thus glue to give global blowups.

If one is willing to forgo functoriality, an alternate approach to global resolution of turning points on analytic spaces may be drawn from the work of Mochizuki [21, 22] using algebraic and analytic properties of Deligne-Malgrange lattices (i.e., Malgrange's *canonical lattices*). However, this approach seems to be limited to true meromorphic connections, whereas our proposed method would apply also to *formal* meromorphic connections. The latter might be treatable using a purely analytic variant of the arguments in [15] replacing the Riemann-Zariski space with its analytic analogue, the *voûte étoilée* of Hironaka [11].

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### 1 Birational geometry of excellent schemes

We begin with some statements and results concerning the birational geometry of excellent schemes, especially over $\mathbb{Q}$. We begin by recalling the functorial versions of resolutions of
singularities. We then introduce Shokurov’s language of b-divisors. Finally, we make a closer analysis of determinations of nef Cartier b-divisors.

**Definition 1.0.1.** By a *schematic pair*, we will mean a pair \((X, Z)\) in which \(X\) is a scheme and \(Z\) is a closed subscheme of \(X\). We say such a pair is *regular* (and describe it for short as a *regular pair*) if \(X\) is regular and \(Z\) is a divisor of simple normal crossings on \(X\). By a *morphism* \(f : (X', Z') \rightarrow (X, Z)\) of schematic pairs, we will mean a morphism \(f : X' \rightarrow X\) of schemes for which \(f^{-1}(Z) = Z'\). In other words, the inverse image ideal sheaf under \(f\) of the ideal sheaf \(I_Z\) defining \(Z\) should be the ideal sheaf \(I_{Z'}\) defining \(Z'\).

### 1.1 Functorial resolution of singularities

In [15], extensive use was made of the fact that quasiexcellent \(\mathbb{Q}\)-schemes admit nonembedded and embedded desingularization; this was originally proposed by Grothendieck, but only recently verified by Temkin [26]. In this paper, we need results of this form with the additional feature that the final modification is functorial for morphisms which are regular (flat with geometrically regular fibres), such as smooth morphisms. Such results can be obtained by approximation arguments from a resolution algorithm for varieties over a field in which one repeatedly blows up so as to reduce some local invariant. One suitable algorithm is that of Bierstone and Milman [1] as refined by Bierstone, Milman, and Temkin [2]; using this algorithm, Temkin has established the following functorial desingularization theorems. (Temkin also obtains some control over the sequence of blowups used; we have not attempted to exert such control in the following statements.)

**Definition 1.1.1.** A morphism of locally ringed spaces \(f : Y \rightarrow X\) is regular if it is flat with geometrically regular fibres. In other words, for each \(y \in Y\), for \(x = f(y)\), the homomorphism \(\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}\) is flat, and for each finite extension \(\ell\) of the residue field \(\kappa_x\) of \(\mathcal{O}_{X,x}\), \(\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} \ell\) is a regular semilocal ring. See [20, §33] or [8, Définition 6.8.2] for further discussion.

**Remark 1.1.2.** A morphism of schemes which is locally of finite presentation is smooth if and only if it is regular. See [9, Théorème 17.5.1].

**Remark 1.1.3.** The definition of a regular morphism arises naturally as the relative version of regularity for individual schemes. Unfortunately, it contravenes a well-established convention in algebraic geometry in which the term *regular morphism* is used as an emphatic term for a *morphism*, to contrast it with a *rational morphism* which is not really a morphism at all (being defined only on a dense open subset of the domain). We will make no use of this convention.

**Theorem 1.1.4 (Temkin).** Let \(\mathbf{Sch}\) be the category of schemes. Let \(\mathbf{C}\) be the subcategory of \(\mathbf{Sch}\) whose objects are the reduced integral noetherian quasiexcellent schemes over \(\text{Spec}(\mathbb{Q})\) and whose morphisms are the regular morphisms of schemes. Let \(\iota : \mathbf{C} \rightarrow \mathbf{Sch}\) denote the inclusion. There then exist a covariant functor \(Y : \mathbf{C} \rightarrow \mathbf{Sch}\) and a natural transformation \(F : Y \rightarrow \iota\) satisfying the following conditions.
(a) For each $X \in \mathcal{C}$, the scheme $Y(X)$ is regular, and the morphism $F(X) : Y(X) \to X$ of schemes is a projective modification.

(b) For each regular $X \in \mathcal{C}$, $F(X)$ is an isomorphism.

(c) For each morphism $f : X' \to X$ in $\mathcal{C}$, the square

$$
\begin{array}{ccc}
Y(X') & \xrightarrow{Y(f)} & Y(X) \\
\downarrow{F(X')} & & \downarrow{F(X)} \\
X' & \xrightarrow{f} & X
\end{array}
$$

is cartesian in $\text{Sch}$.

Proof. See [27, Theorem 1.2.1].

**Theorem 1.1.5** (Temkin). Let $\text{Sch}'$ be the category of schematic pairs. Let $\mathcal{C}'$ be the subcategory of $\text{Sch}'$ whose objects are the pairs for which the underlying schemes are regular integral noetherian quasiexcellent schemes over $\text{Spec}(\mathbb{Q})$, and whose morphisms are those for which the underlying morphisms of schemes are regular. Let $\iota' : \mathcal{C}' \to \text{Sch}'$ denote the inclusion. Then there exist a covariant functor $(Y, W) : \mathcal{C}' \to \text{Sch}'$ and a natural transformation $F : (Y, W) \to \text{id}_{\mathcal{C}'}$ satisfying the following conditions.

(a) For each $(X, Z) \in \mathcal{C}'$, the pair $(Y, W)(X, Z)$ is regular and the morphism $F(X, Z) : Y(X, Z) \to X$ is a projective modification.

(b) For each regular $(X, Z) \in \mathcal{C}'$, $F(X, Z)$ is an isomorphism.

(c) For each morphism $f : (X', Z') \to (X, Z)$ in $\mathcal{C}$, the square

$$
\begin{array}{ccc}
(Y, W)(X', Z') & \xrightarrow{(Y, W)(f)} & (Y, W)(X, Z) \\
\downarrow{F(X', Z')} & & \downarrow{F(X, Z)} \\
(X', Z') & \xrightarrow{f} & (X, Z)
\end{array}
$$

is cartesian in $\text{Sch}'$.

Proof. See [28, Theorem 1.1.7].

**Remark 1.1.6.** A key special case of functoriality in both Theorem 1.1.4 and Theorem 1.1.5 is that of open immersions. This case implies that the modifications in question blow up in the smallest possible centers. Namely, in Theorem 1.1.4 $F(X)$ is an isomorphism over the maximal regular open subscheme of $F(X)$; in Theorem 1.1.5 $F(X, Z)$ is an isomorphism over the maximal open subscheme of $X$ on which $Z$ is a divisor of simple normal crossings.
1.2 The language of b-divisors

We next introduce the language of b-divisors (birational divisors), following [3, §1] but working in the context of excellent \( \mathbb{Q} \)-schemes rather than varieties over a field.

**Hypothesis 1.2.1.** For the remainder of §1 let \( X \) be a reduced noetherian separated excellent \( \mathbb{Q} \)-scheme.

**Definition 1.2.2.** The Riemann-Zariski space of \( X \), denoted \( \text{RZ}(X) \), is the inverse limit \( \text{RZ}(X) = \varprojlim Y \) of underlying topological spaces for \( f : Y \to X \) running over all modifications of \( X \). For \( x \in \text{RZ}(X) \), write \( x(Y) \) for the image of \( x \) in \( Y \).

We say a point \( x \in \text{RZ}(X) \) is divisorial if for some modification \( f : Y \to X \), \( x(Y) \) is the generic point of some prime divisor of \( Y \). Let \( \text{RZ}^{\text{divis}}(X) \) be the subset of \( \text{RZ}(X) \) consisting of divisorial points.

**Remark 1.2.3.** One may of course also define \( \text{RZ}(X) \) using any cofinal set of modifications of \( X \). For example, by Theorem 1.1.4, it suffices to consider projective modifications for which \( Y \) is regular. Also, the normalization of \( X \) is a disjoint union of integral schemes \( Y_i \), so \( \text{RZ}(X) \) is isomorphic to the disjoint union of the \( \text{RZ}(Y_i) \).

**Remark 1.2.4.** One can also define \( \text{RZ}(X) \) for \( X \) nonreduced, but this adds no real generality: for \( X^{\text{red}} \) the underlying reduced closed subscheme of \( X \), there is a natural homeomorphism \( \text{RZ}(X) \cong \text{RZ}(X^{\text{red}}) \).

**Remark 1.2.5.** In case \( X \) is integral, we may identify \( \text{RZ}(X) \) with the set of equivalence classes of Krull valuations \( v \) on the function field \( K(X) \) of \( X \) such that for some \( x \in X \), the local ring \( \mathcal{O}_{X,x} \) is contained in the valuation ring \( \mathfrak{o}_v \) (we say that such valuations are centered on \( X \)). Under this identification, \( \text{RZ}^{\text{divis}}(X) \) corresponds to the equivalence classes of divisorial valuations, i.e., those valuations measuring order of vanishing along some prime divisor on some modification of \( X \).

**Definition 1.2.6.** The group of integral b-divisors on \( X \), denoted \( \text{Div}_\mathbb{Z} X \), is the inverse limit of \( \text{Div}_\mathbb{Z} Y \) (the group of integral Weil divisors on \( Y \)) over all modifications \( f : Y \to X \), in which transition maps are pushforwards. Define similarly the groups \( \text{Div}_\mathbb{Q} X, \text{Div}_\mathbb{R} X \) of rational b-divisors and b-divisors on \( X \). Note that for any modification \( f : Y \to X \), the restriction maps \( \text{Div}_* X \to \text{Div}_* Y \) are isomorphisms. For \( D \in \text{Div}_\mathbb{R} X \), the component \( D(Y) \) of \( D \) on a modification \( Y \) of \( X \) is called the trace of \( D \) on \( Y \); it is a closed proper subset of \( Y \).

**Remark 1.2.7.** One may identify b-divisors on \( X \) with certain real-valued functions on \( \text{RZ}^{\text{divis}}(X) \). The functions that occur are those having the following finiteness property: for any modification \( Y \to X \), the support of the function must only include finitely many divisorial valuations corresponding to prime divisors on \( Y \). Similarly, integral and rational b-divisors on \( X \) correspond to functions on \( \text{RZ}^{\text{divis}}(X) \) with values in \( \mathbb{Z} \) and \( \mathbb{Q} \), respectively, satisfying the same finiteness condition.
Definition 1.2.8. The group of integral Cartier b-divisors on $X$, denoted $\text{CDiv}_Z X$, is the direct limit of $\text{CDiv}_Z Y$ (the group of integral Cartier divisors on $Y$) over all modifications $f: Y \to X$, in which transition maps are pullbacks. Define similarly the groups $\text{CDiv}_Q X, \text{CDiv}_R X$ of rational Cartier b-divisors and Cartier b-divisors on $X$. Note that for any modification $f: Y \to X$, the transition maps $\text{CDiv}_* X \to \text{CDiv}_* Y$ are isomorphisms.

For any $X$, we have a map $\text{CDiv}_* X \to \text{Div}_* X$ which is injective when $X$ is normal and bijective when $X$ is locally factorial. Thanks to the compatibility between pullback and pushforward for Cartier divisors, we also get a map $\text{CDiv}_* X \to \text{Div}_* X$; this map is injective because every modification of $X$ is dominated by a normal modification (because $X$ is excellent). We may thus view Cartier b-divisors as a subclass of all b-divisors.

Remark 1.2.9. The term $b$-divisor was introduced by Shokurov [25] in his construction of 3-fold and 4-fold flips, but has since become standard in birational geometry. See [5] for further discussion. A very similar notion appears in the work of Boucksom-Favre-Jonsson [4], under the guise of Weil divisors on Riemann-Zariski spaces, and is further developed in [3]. (In that language, Cartier b-divisors correspond to Cartier divisors on Riemann-Zariski spaces.)

The distinction between $b$-divisors and Cartier b-divisors has nothing to do with the distinction between Weil and Cartier divisors on an individual space; after all, one can define b-divisors using only modifications $f: Y \to X$ with $Y$ regular, in which case $\text{Div}_R Y = \text{CDiv}_R Y$. Rather, the terminology refers to the distinction between pushforward functoriality for Weil divisors and pullback functoriality for Cartier divisors.

1.3 More on Cartier b-divisors

We continue with some further discussion of Cartier b-divisors, including the semiample and nef properties.

Definition 1.3.1. For $D \in \text{CDiv}_R X$, a determination of $D$ is a modification $f: Y \to X$ such that $D$ belongs to the image of $\text{CDiv}_R Y$ in $\text{CDiv}_R X$. For any determination $f$ of $D$, the image $f(D(Y))$ in $X$ is a closed proper subset of $X$ which does not depend on $f$; we call this set the support of $D$.

The Cartier locus of $D$ is the maximal open subset $U$ of $X$ such that the restriction of $D$ to $\text{CDiv}_R U$ belongs also to $\text{CDiv}_R U$. The complement of this set is the non-Cartier locus of $D$. Note that the non-Cartier locus is contained in the support, and therefore is nowhere dense. Also, if $X$ is normal, then the non-Cartier locus has codimension at least 2.

For $f: Y \to X$, we refer to the Cartier locus and the non-Cartier locus of $f^* D \in \text{CDiv}_R Y$ also as the Cartier locus and non-Cartier locus of $D$ on $Y$.

Definition 1.3.2. For $f: Y \to X$ a modification, the center of $f$ is the complement of the maximal open subscheme $U$ of $X$ over which $f$ is an isomorphism. If $f$ is a determination of $D \in \text{CDiv}_R X$, then the center of $f$ must contain the non-Cartier locus of $D$. In general, it is not possible to choose a determination of $D$ with center equal to the non-Cartier locus of $D$; this is shown by the following example of Fulger taken from [3, Example 4.2].
Example 1.3.3. Let $X$ be the affine 3-space over $\mathbb{C}$ with origin $O$. Let $f_1 : Y_1 \to X$ be the blowup along a line $L$ through $O$. Let $f_2 : Y \to Y_1$ be the blowup at a closed point $P$ in $f_1^{-1}(O)$. Put $f = f_1 \circ f_2$. Then the exceptional divisor of $f_2$ may be viewed as an element $D \in \text{CDiv}_\mathbb{Z}(Y)$ and hence as an element of $\text{CDiv}_\mathbb{Z}(X)$ with support and non-Cartier locus both equal to $O$.

Suppose that $f' : Y' \to X$ were a determination of $D$ with center $O$. The exceptional fibre $E$ of $f_1$ may be viewed as a $\mathbb{P}^1$-bundle over $L$. Let $C_0$ (resp. $C_1$) be the strict transform in $Y$ of a section of $E \to L$ not passing through (resp. passing through) the point $P$. The intersection number $D(Y) \cdot C_i$ is then equal to $i$. On the other hand, if we write $L'$ for the strict transform of $L$ in $Y'$, then $D(Y) \cdot C_i = D(Y') \cdot L'$ for $i = 0, 1$, a contradiction.

Remark 1.3.5. If $D \in \text{CDiv}_\mathbb{Q} X$ is semiample, then for $n, f$ as above, we may view $f_* L^\otimes n$ as a fractional ideal sheaf, along which we may blow up to obtain a new modification $f' : Y' \to X$. Then $f'$ is a determination of $D$ with center equal to the non-Cartier locus of $D$.

Unfortunately, we will see that this condition is too strong for our current work; we are thus led to the following related condition and conjecture.

Definition 1.3.6. For $D \in \text{CDiv}_\mathbb{R} X$, we say that $D$ is nef if there exists a determination $f : Y \to X$ of $D$ such that the pullback of $\mathcal{O}(-D(Y))$ to each fibre of $f$ is nef; that is, for any commutative diagram

\[
\begin{array}{ccc}
C & \rightarrow & Y \\
\downarrow & & \downarrow f \\
\text{Spec}(k) & \rightarrow & X
\end{array}
\]

(1.3.6.1)

in which $k$ is an algebraically closed field and $C$ is a smooth proper connected curve over $k$, the pullback of $\mathcal{O}(-D(Y))$ to $C$ has nonnegative degree. The same is then true for any other determination of $D$. For example, if $D \in \text{CDiv}_\mathbb{Q} X$ is semiample, then it is also nef. On the other hand, for $D$ as in Example 1.3.3, neither $D$ nor $-D$ is nef.

Conjecture 1.3.7. If $D \in \text{CDiv}_\mathbb{Q} X$ is nef, then there exists a determination of $D$ with center equal to the Cartier locus of $f$.

Remark 1.3.8. There is also a version of the nef condition for general b-divisors, using which one can assert that a limit (for the locally convex direct limit topology) of nef b-divisors is again nef. See [3, 4].

While there is no meaningful notion of ampleness for Cartier b-divisors, one can make the following definition.

Definition 1.3.4. For $D \in \text{CDiv}_\mathbb{Q} X$, we say that $D$ is semiample if there exist a positive integer $n$ and a determination $f : Y \to X$ of $D$ such that the adjunction map $f^* f_* \mathcal{O}(-nD(Y)) \to \mathcal{O}(-nD(Y))$ is an isomorphism. Note that the same is then true for any other determination.

Remark 1.3.5. If $D \in \text{CDiv}_\mathbb{Q} X$ is semiample, then for $n, f$ as above, we may view $f_* L^\otimes n$ as a fractional ideal sheaf, along which we may blow up to obtain a new modification $f' : Y' \to X$. Then $f'$ is a determination of $D$ with center equal to the non-Cartier locus of $D$.

Unfortunately, we will see that this condition is too strong for our current work; we are thus led to the following related condition and conjecture.
2 Irregularity b-divisors

We next establish the existence and properties of irregularity b-divisors, then explain how functorial determination for nef Cartier b-divisors stands between us and our target results on resolution of turning points.

2.1 Nefness of irregularity b-divisors

We first define the irregularity b-divisor associated to a differential module on a nondegenerate differential scheme and show that it is Cartier and nef. Note that while this approach gives us some control on resolutions of turning points, it requires the prior knowledge of the existence of such resolutions, so we do not obtain any shortcut around [15].

Hypothesis 2.1.1. Throughout §2.1 let $X$ be a nondegenerate differential scheme in the sense of [15, Definition 3.2.2]. In addition, let $Z$ be a closed subscheme of $X$ containing no connected component of $X$ and let $E$ be a $\nabla$-module over $\mathcal{O}_X(\ast Z)$.

Definition 2.1.2. By a regular modification, we will mean a modification $f : Y \to X$ such that $(Y, W)$ is a regular pair for $W = f^{-1}(Z)$.

Definition 2.1.3. By Remark 1.2.7 there exists a unique $\operatorname{Irr}(E) \in \operatorname{Div}_Z X$ characterized as follows: for any determination $f : Y \to X$ of $\operatorname{Irr}(E)$ with $Y$ normal and any prime divisor $E$ of $Y$, the irregularity of $f^*E$ along $E$ equals the multiplicity of $\operatorname{Irr}(E)$ along $E$. We will see shortly that in fact $\operatorname{Irr}(E) \in \operatorname{CDiv}_Z X$ (Corollary 2.1.6); we call $\operatorname{Irr}(E)$ the irregularity (Cartier) b-divisor of $E$.

Definition 2.1.4. The turning locus of $E$ is the set of points $y \in X$ at which either $(X, Z)$ fails to be a regular pair or $E$ fails to admit a good formal structure in the sense of [15, Definition 5.1.1]. By [15, Proposition 5.1.4], this set is Zariski closed.

For $f : Y \to X$ a regular modification, the turning locus of $f^*E$ is contained in the inverse image of the turning locus of $E$. If the turning locus of $f^*E$ is empty, we say that $f$ is a resolution of turning points of $E$.

Proposition 2.1.5. Let $f : Y \to X$ be a regular modification. Then $f$ is a resolution of turning points if and only if both $\operatorname{Irr}(f^*E)$ and $\operatorname{Irr}(f^*\operatorname{End}(E))$ belong to the image of $\operatorname{CDiv}_Z Y \to \operatorname{CDiv}_Z X$.

Proof. This is immediate from [15, Proposition 5.2.3].

Corollary 2.1.6. The irregularity b-divisor $\operatorname{Irr}(E)$ is an integral Cartier b-divisor.

\[^2\text{As noted in [15] Remark 8.1.4], Mochizuki [21, 22] works with a more restrictive definition of good formal structures, so any turning point in our definition would be a turning point in Mochizuki’s definition but not vice versa. However, at any given point, if both $E$ and $\operatorname{End}(E)$ have good formal structures in our sense, then $E$ also has a good formal structure in Mochizuki’s sense.}\]
Proof. This is immediate from Proposition 2.1.5 plus the existence of a resolution of turning points [15, Theorem 8.1.3].

Since \( \text{Irr}(E) \) is Cartier, we may formally restate Proposition 2.1.5 as follows.

**Theorem 2.1.7.** For \( f : Y \to X \) a regular modification, the turning locus of \( f \) is the union of the non-Cartier loci of \( \text{Irr}(f^*E) \) and \( \text{Irr}(f^*\text{End}(E)) \). Consequently, \( f \) is a resolution of turning points if and only if \( f \) is a determination of both \( \text{Irr}(E) \) and \( \text{Irr}(\text{End}(E)) \).

**Corollary 2.1.8.** Under Conjecture 1.3.7, there exists a resolution of turning points \( f : Y \to X \) whose center is the turning locus of \( E \). (Note that this is not given by [15, Theorem 8.1.3].)

We now come to the main new result of the present paper, which gives some further control on the irregularity b-divisor. The proof is analogous to that of a similar statement about \( p \)-adic connections [16, Proposition 4.1.3].

**Theorem 2.1.9.** Suppose that \( Z \) is the support of an effective Cartier divisor on \( X \). Then the Cartier b-divisor \( \text{Irr}(E) \) is nef.

**Proof.** Let \( f : Y \to X \) be a regular modification which is a determination of both \( \text{Irr}(E) \) and \( \text{Irr}(\text{End}(E)) \), put \( W = f^{-1}(Z) \), and set notation as in (1.3.6.1). Let \( \eta \) be the image in \( Y \) of the generic point of \( C \). Let \( D_1, \ldots, D_m \) be the components of \( W \) passing through \( \eta \), and let \( i_1, \ldots, i_m \) be their respective multiplicities in \( \text{Irr}(E) \). By blowing up \( Y \) further, we may assume that \( C \) is embedded in \( Y \) and meets \( W \setminus (D_1 \cup \cdots \cup D_m) \) transversely.

Choose local coordinates \( x_1, \ldots, x_n \) of \( Y \) at \( \eta \) such that \( x_1, \ldots, x_m \) locally cut out \( D_1, \ldots, D_m \) and \( x_1, \ldots, x_{n-1} \) locally cut out the image of \( C \) at \( \eta \). Then the divisor \( E = \text{Irr}(E) - (x_1^{i_1} \cdots x_m^{i_m}) \) has support not containing \( C \), so we may pull it back to \( C \); the degree of this pullback then computes the degree of \( \text{Irr}(E) \) on \( C \). It thus suffices to show that the degree of the pullback is nonnegative.

Identify the local ring \( \mathcal{O}_{C, \eta} \) with the function field \( K(C) \) of \( C \) and the completed local ring \( \widehat{\mathcal{O}}_{Y, \eta} \) with \( K(C)[[x_1, \ldots, x_{n-1}]] \). By [15, Proposition 3.4.8], for some finite Galois extension \( L \) of \( K(C) \) and some positive integer \( h \), there exists a minimal admissible decomposition

\[
f^*E \otimes_{\mathcal{O}_Y} L\left[x_1, \ldots, x_{n-1}\right]\left[x_1^{1/h}, \ldots, x_m^{1/h}\right] \cong \bigoplus_{\alpha \in I} E(\phi_\alpha) \otimes \mathcal{R}_\alpha
\]

(2.1.9.1)
of differential modules over \( L\left[x_1, \ldots, x_{n-1}\right]\left[x_1^{-1/h}, \ldots, x_m^{-1/h}\right] \) such that for each \( \alpha \in I \), there exist some nonnegative integers \( k_{\alpha,1}, \ldots, k_{\alpha,m} \) so that \( u_\alpha = x_1^{k_{\alpha,1}/h} \cdots x_m^{k_{\alpha,m}/h} \phi_\alpha \) is a unit in the integral closure of \( \mathcal{O}_{Y, \eta} \) in \( L\left[x_1, \ldots, x_{n-1}\right]\left[x_1^{1/h}, \ldots, x_m^{1/h}\right] \). Let \( I' \) be the set of \( \alpha \in I \) for which \( k_{\alpha,1}, \ldots, k_{\alpha,m} \) are not all zero. Let \( g \) be the product of \( u_\alpha^{-\text{rank}(\mathcal{R}_\alpha)} \) over all \( \alpha \in I' \), viewed as an element of \( L \).

Let \( z \in C \) be an arbitrary closed point. Choose a second set \( y_1, \ldots, y_n \) of local coordinates of \( Y \) at \( z \) such that \( y_1, \ldots, y_m \) locally cut out \( D_1, \ldots, D_m \) and, if \( C \) meets \( W \setminus (D_1 \cup \cdots \cup D_m) \...
at \( z \), then the unique component of \( W - (D_1 \cup \cdots \cup D_m) \) meeting \( C \) at \( z \) is locally cut out by \( y_n \). For a suitable positive integer \( h \), we obtain another admissible decomposition
\[
f^*E \otimes_{O_Y} k[[y_1, \ldots, y_n]][y_1^{1/h}, \ldots, y_m^{1/h}, y_n^{1/h}] \cong \bigoplus_{\beta \in J} E(\phi_\beta') \otimes R'_\beta \tag{2.1.9.2}
\]
with \( \phi_\beta' \in O_{Y,y}[y_1^{1/h}, \ldots, y_m^{1/h}, y_n^{1/h}] \).

Let \( C' \) be the smooth finite cover of \( C \) with function field \( L \). Let \( C'' \) be a finite cover of \( C' \) such that \( K(C'') \) contains an \( h \)-th root of the image of \( y_n \) in \( K(C) \). Let \( z'' \in C'' \) be any lift of \( z \). Let \( L_{z''} \) denote the completion of \( K(C'') \) at \( z'' \). Let \( v_z \) denote the valuation on \( L_{z''} \) with the normalization that a uniformizer of \( C'' \) at \( z \) has valuation 1. Then both of the decompositions \( \text{(2.1.9.1)} \) and \( \text{(2.1.9.2)} \) can be base-extended to
\[
R = L_{z''}[x_1, \ldots, x_{n-1}][x_1^{1/h}, \ldots, x_m^{1/h}].
\]
By the uniqueness of minimal admissible decompositions over \( R \) \[15\] Remark 3.4.7], we can partition \( J \) into subsets \( \{ J_\alpha : \alpha \in I \} \) so that over \( R \),
\[
E(\phi_\alpha) \otimes R_\alpha \cong \bigoplus_{\beta \in J_\alpha} E(\phi'_\beta) \otimes R'_\beta. \tag{2.1.9.3}
\]
In particular, for \( R_0 = L_{z''}[x_1, \ldots, x_{n-1}][x_1^{1/h}, \ldots, x_m^{1/h}] \), we have
\[
\phi_\alpha - \phi'_\beta \in R_0^x \quad (\alpha \in I, \beta \in J_\alpha). \tag{2.1.9.4}
\]
Let \( (E \cdot C)_z \) denote the multiplicity at \( z \) of the pullback of \( E \) to \( C \). For each \( \beta \in J_\alpha \), put \( u'_\beta = y_1^{k_{\alpha,1}/h} \cdots y_m^{k_{\alpha,m}/h} \phi'_\beta \); by \( \text{(2.1.9.4)} \), \( u'_\beta \) is a unit in \( R_0 \) unless \( k_{\alpha,1} = \cdots = k_{\alpha,m} = 0 \), in which case it is still an element of \( R_0 \). We then have
\[
(E \cdot C)_z = \sum_{\alpha \in I} \sum_{\beta \in J_\alpha} \text{rank}(R'_\beta) \times \begin{cases} \max\{0, -v_z(u'_\beta)\} & k_{\alpha,1} = \cdots = k_{\alpha,m} = 0 \\ -v_z(u'_\beta) & \text{otherwise} \end{cases}
\]
\[
\geq \sum_{\alpha \in I'} \sum_{\beta \in J_\alpha} -v_z(u'_\beta) \text{rank}(R'_\beta)
\]
\[
= \sum_{\alpha \in I'} \sum_{\beta \in J_\alpha} -v_z(y_1^{k_{\alpha,1}/h} \cdots y_m^{k_{\alpha,m}/h} \phi_\alpha) \text{rank}(R'_\beta) \quad \text{(by \text{(2.1.9.4))}}
\]
\[
= \sum_{\alpha \in I'} -v_z(y_1^{k_{\alpha,1}/h} \cdots y_m^{k_{\alpha,m}/h} \phi_\alpha) \text{rank}(R_\alpha)
\]
\[
= v_z(g) - \sum_{\alpha \in I'} \text{rank}(R_\alpha) \sum_{i=1}^m \frac{k_{\alpha,i}}{h} v_z(y_i/x_i),
\]
with equality if \( I' = I \). Since this inequality holds for any choice of \( z'' \), we may average over these choices and put \( g_0 = \text{Norm}_{L/K(C)}(g) \) to obtain
\[
(E \cdot C)_z \geq [L : K(C)]^{-1} v_z(g_0) - \sum_{\alpha \in I'} \text{rank}(R_\alpha) \sum_{i=1}^m \frac{k_{\alpha,i}}{h} v_z(y_i/x_i),
\]
12
again with equality if $I' = I$. If we now sum over closed points $z$ of $C$, the terms $v_z(g_0)$ cancel out because their sum is the degree of a principal divisor on $C$; we thus have

$$E \cdot C \geq -\sum_{a \in I'} \text{rank}(R_a) \sum_{i=1}^m \frac{k_{a,i}}{h} \sum_z v_z(y_i/x_i),$$

(2.1.9.5)

with equality if $I' = I$.

Since we assumed $Z$ is the support of an effective Cartier divisor on $X$, by shrinking $X$ we may reduce to the case where $Z$ is the zero locus of some section $s \in \mathcal{O}(X)$. If we take $\mathcal{E} = E(s^{-1})$, then the left side of (2.1.9.3) equals 0 (because $\text{Irr}(\mathcal{E}) = (s)$ is already a Cartier divisor on $X$) while the right side becomes $\sum_z v_z(y_i/x_i)$. Consequently, this sum must equal 0, so in the general case we have $E \cdot C \geq 0$ as desired.

\[\square\]

2.2 Functorial determinations

As observed in Corollary [2.1.8] knowledge about determinations of nef integral Cartier b-divisors has consequences for the resolution of turning points. We now introduce a refinement of Conjecture [1.3.7] modeled on Theorems [1.1.4] and Theorem [1.1.5] then indicate its consequences for our work.

**Conjecture 2.2.1.** Let $\mathcal{C}$ denote the category of pairs $(X, D)$ where $X$ is a reduced integral noetherian excellent $\mathbb{Q}$-scheme and $D$ is a nef integral Cartier b-divisor on $X$, in which a morphism $(Y, D') \to (X, D)$ consists of a morphism $f : Y \to X$ and an isomorphism $D' \cong f^*D$. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ on the same objects whose morphisms are those morphisms in $\mathcal{C}$ whose underlying morphisms of schemes are regular. Let $\iota : \mathcal{C}' \to \mathcal{C}$ denote the natural inclusion. Then there exist a covariant functor $Y : \mathcal{C}' \to \mathcal{C}$ and a natural transformation $F : Y \to \iota$ satisfying the following conditions.

(a) For each $(X, D) \in \mathcal{C}$, the morphism $F(X, D) : Y(X, D) \to X$ of schemes is a modification which is a determination of $D$.

(b) For each $(X, D) \in \mathcal{C}$ for which $D \in \text{CDiv}_R X$, $F(X, D)$ is an isomorphism.

(c) For each morphism $f : (X', D') \to (X, D)$ in $\mathcal{C}$, the square

$$\begin{array}{ccc}
Y(X', D') & \xrightarrow{Y(f)} & Y(X, D) \\
\downarrow F(X', D') & & \downarrow F(X, D) \\
X' & \xrightarrow{f} & X
\end{array}$$

is cartesian in $\mathcal{C}$.

**Remark 2.2.2.** Note that Conjecture 2.2.1 implies Conjecture 1.3.7 by functoriality along the open immersion of the Cartier locus into $X$. Also note that the analogue of Conjecture 2.2.1 for semiample integral Cartier b-divisors is true by Remark 1.3.5.
Remark 2.2.3. One possible approach to Conjecture 2.2.1 would be to establish that the fractional ideal sheaves $\mathcal{O}_X(nD)$ (as in [3, Definition 1.7]) for $n = 1, 2, \ldots$ give rise to a sequence $D_n$ in $\text{CDiv}_\mathbb{Q}(X)$ for $n = 1, 2, \ldots$ spanning a finite-dimensional $\mathbb{Q}$-vector space $V$. This space would also contain $D$ because the sequence $n^{-1}D_n$ converges to $D$ thanks to the nef condition (as in [3, Corollary 2.13]); blowing up the $\mathcal{O}_X(nD)$ for $n = 1, 2, \ldots, N$ for sufficiently large $N$ gives a determination of $D$ which does not depend on $N$. (Note that the finite-dimensionality of $V$ is weaker than finite generation of the graded ring $\bigoplus_n \mathcal{O}_X(nD)$, which can easily fail.)

An analogous approach would use multiplier ideals and the subadditivity property (see for instance [3, §3]). Such an approach would apply to algebraic varieties and complex analytic spaces, but not to excellent $\mathbb{Q}$-schemes due to the lack of a suitable version of the relative Kawamata-Viehweg vanishing theorem. However, one could then try to handle excellent $\mathbb{Q}$-schemes via algebraization arguments, as in [27, 28].

Assuming a functorial determination, one gets a functorial resolution of turning points as follows.

Theorem 2.2.4. Assume Conjecture 2.2.1. Let $X$ be a nondegenerate differential scheme, let $Z$ be a closed subscheme containing no connected component of $X$, and let $\mathcal{E}$ be a $\nabla$-module on $\mathcal{O}_X(\ast Z)$. Then there exists a projective modification $f : Y \to X$ such that $(Y, W)$ is a regular pair for $W = f^{-1}(Z)$ and $f^* \mathcal{E}$ admits a good formal structure at each point of $Y$ (i.e., the turning locus of $f^* \mathcal{E}$ is empty). Moreover, the construction of $f$ is functorial for regular morphisms to $X$.

Proof. Apply Theorem 1.1.5 to construct a modification $f_1 : X_1 \to X$ such that $(X_1, Z_1)$ is a regular pair for $Z_1 = f_1^{-1}(Z)$. Since $(X_1, Z_1)$ is a regular pair, $Z_1$ is automatically a Cartier divisor on $X_1$, so by Theorem 2.1.9 $\text{Irr}(\mathcal{E})$ and $\text{Irr}(\text{End}(\mathcal{E}))$ are nef integral Cartier b-divisors on $X_1$. We may thus apply Conjecture 2.2.1 twice, then take a fibred product, to construct a modification $f_2 : X_2 \to X_1$ which is a determination of both $\text{Irr}(\mathcal{E})$ and $\text{Irr}(\text{End}(\mathcal{E}))$. Apply Theorem 1.1.4 and then Theorem 1.1.5 to construct a modification $f_3 : X_3 \to X_2$ with $(X_3, Z_2)$ a regular pair for $f = f_1 \circ f_2 \circ f_3$ and $Z_3 = f^{-1}(Z)$. By Theorem 2.1.7 $f$ is a resolution of turning points, and its construction is evidently functorial for regular morphisms.

This would then transfer to the formal algebraic and complex analytic categories as follows.

Theorem 2.2.5. Assume Conjecture 2.2.1. Let $X$ be a nondegenerate differential scheme, let $Z$ be a closed subscheme of $X$ containing no connected component of $X$, and let $\hat{X} \mid Z$ be the formal completion of $X$ along $Z$. Let $\mathcal{E}$ be a $\nabla$-module over $\mathcal{O}_{\hat{X} \mid Z}(\ast Z)$. Then there exists a projective modification $f : Y \to X$ such that $(Y, W)$ is a regular pair for $W = f^{-1}(Z)$ and $f^* \mathcal{E}$ admits a good formal structure at each point of $W$. Moreover, the formation of $f$ is functorial for regular morphisms to $X$.
Proof. In case $X$ is affine, we may argue as in [15, Theorem 8.2.1]. Put $X = \text{Spec}(R)$ and $Z = \text{Spec}(R/I)$. Let $\widehat{R}$ be the $I$-adic completion of $R$, and put $\widehat{I} = I\widehat{R}$. Put $\widehat{X} = \text{Spec}(\widehat{R})$ and $\widehat{Z} = \text{Spec}(\widehat{R}/\widehat{I})$. We can then view $\mathcal{E}$ as a $\nabla$-module on $\mathcal{O}_{\widehat{X}}(\ast\widehat{Z})$, and apply Theorem 2.2.4 to deduce the claim. Since Theorem 2.2.4 is functorial for regular morphisms, the construction glues over a Zariski open cover of $X$ to prove the desired result.

**Theorem 2.2.6.** Assume Conjecture 2.2.1. Let $X$ be a smooth (separated) complex analytic space. Let $Z$ be a closed subspace of $X$ containing no connected component of $X$. Let $\widehat{X}|_Z$ be the formal completion of $X$ along $Z$. Let $\mathcal{E}$ be a $\nabla$-module over $\mathcal{O}_{\widehat{X}|_Z}(\ast Z)$. Then there exists a projective modification $f : Y \to X$ such that $(Y, W)$ is a regular pair for $W = f^{-1}(Z)$ and $f^*\mathcal{E}$ admits a good formal structure at each point of $W$. Moreover, the formation of $f$ is functorial for smooth morphisms to $X$.

**Proof.** In a neighborhood of a given point of $Z$, we obtain the desired modification by reducing to Theorem 2.2.4 as in the proof of [15, Proposition 5.1.4]. Since Theorem 2.2.4 is functorial for regular morphisms, the construction glues over a topological open cover of $X$ to prove the desired result.

**Remark 2.2.7.** We mention in passing an alternate approach for constructing global resolutions of turning points for flat meromorphic connections on analytic spaces. Beware that this method does not guarantee functoriality for smooth morphisms, nor does it apply to formal flat meromorphic connections (as in Theorem 2.2.6).

The approach in question is to modify the proof of [22, Theorem 19.5], which asserts this conclusion for algebraic connections. In fact, most of the proof takes place in the analytic category; the only use of algebraicity is to invoke the corresponding result for surfaces, which Mochizuki proved in [21] using reduction to positive characteristic. Since our prior result for surfaces [14, Theorem 6.4.1] applies to the analytic category, one may substitute it for the use of [21] and then proceed as in [22] to obtain the claimed result.

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