On the Perturbative Expansion around a Lifshitz Point

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Abstract

The quantum Lifshitz model provides an effective description of a quantum critical point. It has been shown that even though non–Lorentz invariant, the action admits a natural supersymmetrization. In this note we introduce a perturbative framework and show how the supersymmetric structure can be used to greatly simplify the Feynman rules and thus the study of the model.
1 Introduction

In 1941, Lifshitz [1] introduced models with anisotropic scaling between space and time in the context of tri–critical models. Since then, such models have been studied in the context of solid state physics. Materials with strongly correlated electrons, such as copper oxides, show this type of critical behaviour, and also the smectic phase of liquid crystals for example can be described this way. Our treatment is based on quantum Lifshitz models as were studied in [2]. Quantum Lifshitz points are especially interesting, since they are quantum critical points [3], i.e. points at which a continuous phase transition happens at $T = 0$ which is driven by zero point quantum fluctuations. A quantum Lifshitz point is characterized by the vanishing of the term $(\nabla \phi)^2$ in the effective Hamiltonian. While scale invariance is conserved, this gives rise to an anisotropy between space and time. This anisotropy is quantified by the dynamical critical exponent $z$,

$$t \rightarrow \lambda^z t, \; x \rightarrow \lambda x. \quad (1.1)$$

For models in $2+1$ dimensions at a Lifshitz point, $z = 2$, as opposed to the Lorentz invariant $z = 1$.

Models at a Lifshitz point have recently met with a large amount of interest beyond their original field of application. A $3+1$ dimensional theory of gravity with $z = 3$ put forward by Hořava [4] has generated a big echo. But also in the context of the AdS/CFT correspondence, interest in gravity duals of non–Lorentz invariant models has arisen, see e.g. [5, 6, 7, 8]. In [7] in particular, a gravity dual for a Lifshitz type model with $z = 2$ was proposed. As discussed recently in [9], it seems difficult to find string theory embeddings for gravity duals of Lifshitz–type points.

While often, calculations are easier to do on the gravity side of the correspondence, we are able to perform a number of calculations directly on the field theory side, which are presented in this article. Apart from being of interest directly for statistical physics, our results can serve as a point of reference for comparison to results derived on the gravity side.

In [10] it was shown that systems of Lifshitz type in $(d + 1)$ dimensions admit a natural supersymmetrization, a property which results from their relation to $d$–dimensional models via a Langevin equation. The quantum Lifshitz model in [2], described by the action

$$S[\phi] = \int dt \, dx \left[ \dot{\phi}^2 + (\partial^i \phi)^2 \right], \quad x = x_i, \; i = 1, 2, \quad (1.2)$$

We cannot give an extensive account of the existing literature here and thus content us to only mention a few main examples.
can be thought of as descending from a free boson in two dimensions with action
\[ W[\phi] = \int dx \left[ \partial_i \phi \partial^i \phi \right]. \tag{1.3} \]

This formulation allows the generalization of the quantum Lifshitz model to massive and interacting cases. It becomes possible to consider the class of models satisfying the detailed balance condition whose (bosonic part of the) action takes the form
\[ S[\phi] = \int dt dx \left[ \dot{\phi}^2 + \left( \frac{\delta W[\phi]}{\delta \phi} \right)^2 \right], \tag{1.4} \]

where \( W[\phi] \) is a local functional of the field \( \phi(t, x) \). The structure due to the Langevin equation implies supersymmetry in the time direction, so that the complete action includes also a fermionic field. It is given by
\[ S[\phi, \psi, \bar{\psi}] = \int dt dx \left[ \dot{\phi}^2 + \left( \frac{\delta W[\phi]}{\delta \phi} \right)^2 - \bar{\psi} \left( \frac{d}{dt} + \frac{\delta^2 W[\phi]}{\delta \dot{\phi}^2} \right) \psi \right]. \tag{1.5} \]

This is the supersymmetric theory we focus on in this work.

A major advantage of models with this structure is that they can be studied very efficiently by using a perturbative expansion of the underlying Langevin equation, as proposed in [11]. In this way, the cancellation of bosonic and fermionic terms in the perturbative expansion becomes automatic. In consequence, there is a great simplification of the Feynman diagrams of the theory in \((d + 1)\) dimensions which are reformulated in terms of those of the \(d\)–dimensional system described by \( W[\phi] \), plus a set of additional rules. If we consider only \(n\)–point functions for the bosonic field \( \phi \), all the fermionic contributions are automatically accounted for, so that it is not even necessary to introduce a fermionic propagator.

For relativistic theories, this construction is possible only for \(d = 0\) and \(d = 1\). Giving up Lorentz invariance, we concentrate on \(d = 2\), which – as we show in the following – is the critical case. The generalization to any \(d\) is however clear.

In the following we derive
- the expression for the propagator of the free Lifshitz scalar (Sec. 3.1);
- the Feynman rules for the simplest generalization to the interacting case (Sec. 3.2);
- a scheme for UV regularization (Sec. 3.4).

As examples, the three–point function (Sec. 3.3.1) and the one–loop propagator (Sec. 3.3.2) are discussed.
2 The Langevin equation and the Nicolai map

Having chosen to study the supersymmetric extension of the quantum Lifshitz model, we can make use of the Nicolai map \([12]\). In a supersymmetric field theory, a Nicolai map is a transformation of the bosonic fields

\[
\phi(t, x) \mapsto \eta(t, x),
\]

such that the bosonic part of the Lagrangian is quadratic in \(\eta\) and the Jacobian for the transformation is given by the determinant of the fermionic part:

\[
S_B = \int dt \, dx \, \left[ \frac{1}{2} \eta(t, x)^2 \right],
\]

\[
\det \left[ \frac{\delta \eta}{\delta \phi} \right] = \int D\psi D\bar{\psi} \exp[-S_F].
\]

Following \([11]\), we would like to interpret the mapping in Eq. (2.1) as a Langevin equation for the field \(\phi(t, x)\) with noise \(\eta(t, x)\). More precisely, we want to show the equivalence of the action in Eq. (1.5) to the Langevin equation

\[
\frac{\partial \phi(t, x)}{\partial t} = -\frac{\delta W}{\delta \phi} + \eta(t, x).
\]

The correlations of \(\eta\), which is a white Gaussian noise (as in Eq. (2.2)), are given by

\[
\langle \eta(t, x) \rangle = 0, \quad \langle \eta(t, x) \eta(t', x') \rangle = 2 \delta(t - t') \delta(x - x').
\]

A stochastic equation of this type, where the dissipation term depends on the gradient of a function of the field is said to satisfy the detailed balance condition. Equation (2.4) has to be solved given an initial condition, leading to an \(\eta\)-dependent solution \(\phi_{\eta}(t, x)\). As a consequence, \(\phi_{\eta}(t, x)\) becomes a stochastic variable. Its correlation functions are defined by

\[
\langle \phi_{\eta}(t_1, x_1) \ldots \phi_{\eta}(t_k, x_k) \rangle_{\eta} = \frac{\int D\eta \exp \left[ -\frac{1}{4} \int dt \, dx \, \eta(t, x)^2 \right] \phi_{\eta}(t_1, x_1) \ldots \phi_{\eta}(t_k, x_k) \int D\eta \exp \left[ -\frac{1}{4} \int dt \, dx \, \eta(t, x)^2 \right]}{\int D\eta \exp \left[ -\frac{1}{4} \int dt \, dx \, \eta(t, x)^2 \right]}. \tag{2.6}
\]

A necessary condition for this approach to work is that thermal equilibrium is reached for \(t \to \infty\), and that

\[
\lim_{t \to \infty} \langle \phi_{\eta}(t, x_1) \ldots \phi_{\eta}(t, x_k) \rangle_{\eta} = \langle \phi(x_1) \ldots \phi(x_k) \rangle, \tag{2.7}
\]

i.e. the equal time correlators for \(\phi_{\eta}\) tend to the corresponding quantum Green’s functions for the \(d\)-dimensional theory described by \(W[\phi]\). Since \(\phi(t, x)\) is a stochastic variable,
the expectation value of any functional $F[\phi]$ is obtained by averaging over the noise:

$$\langle F[\phi]\rangle_\eta = \frac{1}{Z} \int D\eta \, F[\phi] e^{-\frac{1}{2} \int dt \, dx \, \eta(t,x)^2},$$

(2.8)

where the partition function is defined by

$$Z = \int D\eta \, e^{-\frac{1}{2} \int dt \, dx \, \eta(t,x)^2}.$$

(2.9)

It is convenient to change the integration variable from $\eta$ to $\phi$. The expression becomes

$$Z = \int D\phi \, \det \left[ \frac{\delta \eta}{\delta \phi} \right]_{\eta=\phi+\frac{\delta W}{\delta \phi}} \exp \left[ -\frac{1}{2} \int dt \, dx \, \left( \phi + \frac{\delta W}{\delta \phi} \right)^2 \right].$$

(2.10)

The Jacobian can be expressed by introducing two fermionic fields $\psi(t,x)$ and $\bar{\psi}(t,x)$ such that (as in Eq. (2.3))

$$\det \left[ \frac{\delta \eta}{\delta \phi} \right]_{\eta=\phi+\frac{\delta W}{\delta \phi}} = \int D\psi D\bar{\psi} \exp \left[ -\int dt \, dx \, \left( \bar{\psi}(t,x) \left( \partial_t - \frac{\delta^2 W}{\delta \phi^2} \right) \psi(t,x) \right) \right].$$

(2.11)

In this way we can directly read off the $(d+1)$–dimensional action that, up to a boundary term, reproduces Eq. (1.5):

$$S[\phi, \psi, \bar{\psi}] = \int dt \, dx \left[ \phi^2 + \left( \frac{\delta W[\phi]}{\delta \phi} \right)^2 - \bar{\psi} \left( \frac{d}{dt} + \frac{\delta^2 W[\phi]}{\delta \phi^2} \right) \psi \right].$$

(2.12)

Finally, one can show [10] that in a Hamiltonian formulation, the supersymmetric ground state is the bosonic state

$$|\Psi_0\rangle = e^{-W[\phi]},$$

(2.13)

as is already the case in the standard quantum Lifshitz model.

This construction bears an obvious resemblance with the stochastic quantization of the $d$–dimensional theory described by $W[\phi]$. We would however like to stress a fundamental difference. In the case of stochastic quantization, one is only interested in the $t \to \infty$ limit and hence in the ground state. This means that the action in Eq. (1.5) is seen as a topological theory. Here, on the other hand, we wish to study the finite–time behaviour of the system, and the same action is taken to describe a conventional supersymmetric model.

In the following, we will work in $d = 2$, but in principle, all calculations are equally valid for general $d$ and $z = 2$. The case $d = 2$ is arguably the most interesting, since it corresponds to the Lifshitz point with its quantum critical behaviour.
3 Perturbative Solution of the Langevin equation

In the following, we will show how to perturbatively solve the Langevin equation (2.4), which gives rise to the dynamics of the Lifshitz model. As we will see, the main advantage of this approach is that the perturbative expansion is realized in terms of the Feynman diagrams for the theory in \(d\) dimensions which does not include fermionic contributions.

To solve a transport equation like the Langevin equation in Eq.(2.4), it is convenient to consider an integral transform. The choice of transform depends on the choice of boundary conditions. In space, the natural choice is given by requiring the field to vanish at infinity,

\[
\phi(t, x) \xrightarrow{|x| \to \infty} 0. \tag{3.1}
\]

This means that the Fourier transform is well defined:

\[
\phi(t, k) = \int dx \left[ e^{-ik \cdot x} \phi(t, x) \right]. \tag{3.2}
\]

For the time direction, we have two possible choices:

(a) The field vanishes at negative infinity. If we impose \(\phi(t, x) \xrightarrow{t \to -\infty} 0\), we can define a Fourier transform in time and use

\[
\phi(\omega, k) = \int_{-\infty}^{\infty} dt \left[ e^{-i\omega t} \phi(t, k) \right]. \tag{3.3}
\]

(b) Initial condition at \(t = 0\). If we impose \(\phi(0, x) = \phi_0(x)\), it is convenient to define a Laplace transform in the time direction:

\[
\phi(s, k) = \int_{0}^{\infty} dt \left[ e^{-st} \phi(t, k) \right]. \tag{3.4}
\]

Note that the first choice preserves time–translation invariance which in case (b) is broken by an extra mode that describes the evolution of the initial condition. On the other hand, if the kernel of the Langevin equation \(\delta W / \delta \phi\) is positive definite (as it is for the cases we are considering here), this extra mode decays exponentially, and the large–time behaviours of both choices coincide. In other words, one can without loss of generality choose to impose the initial condition (b) and then take the large–time limit to recover the finite–time Fourier transform behaviour of case (a)\(^2\). From now on, we will consider the Fourier–Laplace transform (i.e. a Fourier transform in space and Laplace transform in the time direction):

\[
\phi(s, k) = \int_{0}^{\infty} dt \int dx \left[ e^{-ik \cdot x - st} \phi(t, x) \right]. \tag{3.5}
\]

\(^2\)An analogous problem was solved by Landau in [13] in the study of oscillations in plasma.
3.1 Free propagator

As a first application, let us consider the action obtained by adding a relevant perturbation
to the quantum Lifshitz model, described by

\[
S[\phi, \psi, \bar{\psi}] = \int dt \, dx \left[ \frac{1}{2} \dot{\phi}^2 + (\partial_i \dot{\phi})^2 + m^2 \partial_i \phi \partial^i \phi + m^4 \phi^2 + \text{fermions} \right].
\] (3.6)

According to the argument in Sec. 2, this is equivalent to the Langevin equation corresponding to the massive boson in 2 dimensions described by the functional

\[
W[\phi] = \int dx \left[ \frac{1}{2} \partial_i \phi \partial^i \phi + \frac{1}{2} m^2 \phi^2 \right].
\] (3.7)

After the integral transform, the Langevin equation (2.4) takes the form

\[
s \phi(s, k) - \phi_0(k) = -\Omega^2 \phi(s, k) + \eta(s, k),
\] (3.8)

where we introduced \( \Omega^2 = (k^2 + m^2) \). The Gaussian noise \( \eta(s, k) \) has the two-point function

\[
\langle \eta(s, k) \eta(s', k') \rangle = \frac{2 (2\pi)^2 \delta(k + k')}{s + s'}. \] (3.9)

The retarded Green’s function for this problem is the solution to the equation

\[
s \, G(s, k) = -\Omega^2 G(s, k) + 1.
\] (3.10)

It follows that the solution to the Langevin equation is

\[
\phi(s, k) = G(s, k) \eta(s, k) + G(s, k) \phi_0(k), \tag{3.11}
\]

and since the Laplace transform exchanges point-wise products and convolution products, one finds that the field \( \phi \) as a function of time can be written as

\[
\phi(t, k) = G(t, k) * \eta(t, k) + \phi_0(k) G(t, k) = \int_0^t d\tau \left[ G(t - \tau, k) \eta(\tau, k) \right] + \phi_0(k) G(t, k).
\] (3.12)

More explicitly, using the fact that

\[
G(t, k) = e^{-t\Omega^2},
\] (3.13)

we find the solution

\[
\phi(t, k) = \int_0^t d\tau \left[ e^{-(t - \tau)\Omega^2} \eta(\tau, k) \right] + \phi_0(k) e^{-t\Omega^2}. \tag{3.14}
\]
Having expressed $\phi$ as a function of the noise, we are now in a position to evaluate the two-point function:

$$D(s, k; s', k') = \langle \phi(s, k) \phi(s', k') \rangle = G(s, k) G(s', k') \langle \eta(s, k) \eta(s', k') \rangle = \frac{2 (2\pi)^2 \delta(k + k')}{(s + \Omega^2) (s' + \Omega^2) (s + s')}.$$ \hspace{1cm} (3.15)

Taking the (bidimensional) inverse Laplace transform, this becomes

$$D(t, k; t', k') = \int_{\sigma - i\infty}^{\sigma + i\infty} ds \int_{\sigma - i\infty}^{\sigma + i\infty} ds' \frac{2 (2\pi)^2 \delta(k + k')}{(s + \Omega^2) (s' + \Omega^2) (s + s')} \delta(k + k').$$ \hspace{1cm} (3.16)

Two limits are interesting:

- For $t, t' \to \infty$, the second exponential is suppressed and the two-point function becomes

$$D(t, k; t', k') \overset{t, t' \to \infty}{\longrightarrow} (2\pi)^2 \frac{e^{-\Omega^2|t-t'|} - e^{-\Omega^2(t+t')}}{\Omega^2} \delta(k + k').$$ \hspace{1cm} (3.17)

Its Fourier transform is given by

$$D(\omega, k; \omega', k') = \frac{\delta(\omega + \omega')\delta(k + k')}{\omega^2 + \Omega^2},$$ \hspace{1cm} as found in [2].

- For $t = t' \to \infty$, the two-point function reproduces the usual bosonic propagator in $d$ dimensions (as expected from the general structure of the Langevin equation and shown in Eq. (2.7)):

$$D(t, k; t, k') \overset{t \to \infty}{\longrightarrow} (2\pi)^2 \frac{\delta(k + k')}{\Omega^2}.$$ \hspace{1cm} (3.19)

Note that for $m = 0$, this means that for large times, the propagator will behave polynomially and the theory is critical, just like in the case of a two-dimensional bosonic field.

### 3.2 Interacting theory

As an example of an interacting theory, let us consider the action descending from the theory of a massive boson with a $\phi^3$ interaction. This is described by a Langevin equation with functional

$$W[\phi] = \int dx \left[ \frac{1}{2} \partial_i \phi \partial^i \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} \delta^2 \phi^3 \right].$$ \hspace{1cm} (3.20)
The action in \((2 + 1)\) dimensions in Eq. (1.5) is immediately found to be

\[
S[\phi, \psi, \tilde{\psi}] = \int dt \, dx \left[ \frac{1}{2} \phi^2 + (\partial_i \partial_i^t \phi)^2 + m^2 \partial_i \phi \phi^t + g^2 \phi \partial_i \phi \phi^t + m^4 \phi^2 + g^2 m^2 \phi^3 + g^4 \phi^4 + \text{fermions} \right], \quad (3.21)
\]

and in particular, for the critical \(m = 0\) case,

\[
S[\phi] = \int dt \, dx \left[ \frac{1}{2} \phi^2 + (\partial_i \partial_i^t \phi)^2 + g^2 \phi \partial_i \phi \phi^t + g^4 \phi^4 + \text{fermions} \right]. \quad (3.22)
\]

Note that the coefficients of the three– and four–point interactions are not independent, but fixed by the detailed balance condition. It is in fact the detailed balance, which here plays the role of a symmetry the theory has to fulfill, which keeps terms other than those given in Eq. (3.21) from appearing. This relation between the different coupling constants is a property which is accessible to experimental checks in materials which are described by a Lifshitz point effective action and thus is a testable prediction of this framework.

The most effective way of making use of this symmetry consists once more in starting from the corresponding Langevin equation, which takes the simple form

\[
\partial_t \phi(t, x) = - ( - \nabla^2 \phi(t, x) + m^2 \phi(t, x) + g^2 \phi^2(t, x)) + \eta(t, x), \quad (3.23)
\]

where \(\eta(t, x)\) is the same Gaussian noise as in the previous example. Taking the Fourier–Laplace transform as above, one has to deal with the quadratic term that can be recast in the following form:

\[
- g^2 \int_0^\infty dt \int dx \, e^{-i k \cdot x - i t} \phi(t, x)^2 = \int dt \, dx \frac{d k_1}{(2 \pi)^2} \frac{d k_2}{(2 \pi)^2} \frac{d s_1}{2 \pi i} \frac{d s_2}{2 \pi i} \left[ e^{-i (s_1 k_1 - s_2 k_2) - (s_1 s_1 - s_2 s_2)} \phi(s_1, k_1) \phi(s_2, k_2) \right] = \int d[k_1, k_2, s_1, s_2] \frac{\delta(k - k_1 - k_2)}{s_1 - s_2} \phi(s_1, k_1) \phi(s_2, k_2). \quad (3.24)
\]

The Langevin equation becomes an integral equation:

\[
\phi(s, k) = G(s, k) \eta(s, k) - g^2 G(s, k) \int d[k_1, k_2, s_1, s_2] \frac{\delta(k - k_1 - k_2)}{s_1 - s_2} \phi(s_1, k_1) \phi(s_2, k_2) \]

\[
+ G(s, k) \phi_0(k). \quad (3.25)
\]

This type of equation can be solved perturbatively in \(g^2\), using the usual Feynman diagram techniques by denoting \(G(s, k)\) with an arrow and the noise \(\eta(s, k)\) with a cross. In particular,
The field \( \phi \) can be expanded as

\[
\phi = \ldots \quad (3.26)
\]

The Feynman rules for this model are summarized in Table 1. Note that even if the action in Eq. (3.21) has two cubic and a quartic interaction, the Feynman diagrams obtained from the Langevin equation only have one cubic vertex. This might at first seem surprising, but is again a simplifying consequence of the detailed balance condition.

### 3.3 Examples

#### 3.3.1 Three–point function

As a first example, let us consider the three–point function \( \langle \phi(s_1, k_1) \phi(s_2, k_2) \phi(s_3, k_3) \rangle \) at tree level. Expanding the field as in Eq. (3.26), we find that at tree level the function is given by the sum of three contributions:

\[
\langle \phi(s_1, k_1) \phi(s_2, k_2) \phi(s_3, k_3) \rangle = \quad (3.27)
\]

Using the Feynman rules in Table 1 it is immediate to find that each diagram gives a contribution of

\[
- \int \frac{ds_1 ds_2}{4\pi^2} \left[ \frac{4\delta^2}{(s_1 + \Omega_i^2)(s_2 + \Omega_i^2)(s_3 + \Omega_j^2)} \right], \quad (3.28)
\]
where \( \{i,j,k\} \) take the values \( \{1,2,3\} \) and their cyclic permutations. After an inverse Laplace transform in the time direction and taking the large time limit (which together is equivalent to taking the Fourier transform), the three–point function becomes:

\[
\langle \phi(t_1, k_1)\phi(t_2, k_2)\phi(t_3, k_3) \rangle = g^4 e^{-\frac{\Omega_1^2 t_1 - \Omega_3^2 t_1}{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}} e^{-\frac{\Omega_1^2 t_2 - \Omega_3^2 t_2}{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}} e^{-\frac{\Omega_1^2 t_3 - \Omega_3^2 t_3}{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}} \frac{\Omega_1^2 \Omega_2^2 \Omega_3^2}{(\Omega_1^2 - \Omega_2^2)^2 - \Omega_1^2}, \tag{3.29}
\]

where \( t_{ij} = t_i - t_j \) and \( t_1 \leq t_2 \leq t_3 \). A consistency check for this expression is obtained by considering the equal time case \( t_1 = t_2 = t_3 \to \infty \), which reproduces, as expected, the usual bosonic result (as in Eq. (2.7)):

\[
\langle \phi(t, k_1)\phi(t, k_2)\phi(t, k_3) \rangle \xrightarrow{t \to \infty} \frac{g^2}{\Omega_1^2 \Omega_2^2 \Omega_3^2}. \tag{3.30}
\]

### 3.3.2 One–loop propagator

As a next example, let us consider the one–loop correction to the two–point function \( \langle \phi(s_1, k_1)\phi(s_2, k_2) \rangle \). Using the expansion in Eq. (3.26), we find that

\[
\langle \phi(s_1, k_1)\phi(s_2, k) \rangle = \begin{array}{c}
\begin{array}{c}
\text{(a)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{(b)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{(c)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{(d)}
\end{array}
\end{array}
\end{array}
\]

The first term is just the usual propagator \( D(s_1, k_1; s_2, k_2) \). The contributions from the diagrams (b), (c), (d) are as follows:

\[
\begin{align*}
(b) & = g^4 G(s_1, k)D(s_1; k_1; s_2, -k_1)D(s_2, k_2; s_2, -k_2)G(s_2, k), \tag{3.32} \\
(c) & = g^4 D(s_1, k_1; s_2, -k)D(s_2, k_1; s_2, -k_1)G(s_2, k_2)G(s_2, k), \tag{3.33} \\
(d) & = g^4 G(s_1, k)D(s_1; k_1; s_2, -k_1)G(s_2, k_2)D(s_2, k; s_2, -k). \tag{3.34}
\end{align*}
\]

The two–point function is obtained by summing up all the contributions and integrating over the internal momenta. Once more, the result is more transparent if we take the inverse
Laplace transform and consider the large time limit $t_1, t_2 \to \infty$:

$$
\langle \phi(t_1, k)\phi(t_2, k) \rangle \xrightarrow{t_1, t_2 \to \infty} \frac{e^{-\Omega^2 t_{21}}}{\Omega^2} + g^4 \int \frac{dk_1 dk_2}{(2\pi)^4} \left[ \frac{\Omega^2 e^{-(\Omega_1^2 + \Omega_2^2) t_{21}} - (\Omega_1^2 + \Omega_2^2) e^{-\Omega^2 t_{21}}}{\Omega^4 \Omega_1^2 \Omega_2^2 \left( \Omega^2 - \Omega_1^2 - \Omega_2^2 \right)} \delta(k - k_1 - k_2) \right], \quad (3.35)
$$

where $t_{21} = |t_2 - t_1|$. Taking the $t_{21} \to 0$ limit, one reproduces the usual bosonic two–point function at one loop (as in Eq. (2.7)):

$$
\langle \phi(t, k)\phi(t, k) \rangle \xrightarrow{t \to \infty} \frac{1}{\Omega^2} + g^4 \int \frac{dk_1 dk_2}{(2\pi)^4} \frac{\delta(k - k_1 - k_2)}{\Omega^4 \Omega_1^2 \Omega_2^2}, \quad (3.36)
$$

### 3.4 UV Regularization

The two–point function we derived above suffers from an ultraviolet divergence. In order to regularize it, one can either change the Langevin equation (Eq. (2.4)), or the noise correlation function in Eq. (2.5). Here, we follow the latter approach and show how smearing out the delta function in time by introducing a cut–off $\Lambda$ results in an ultraviolet regularization in space which, in the large time limit, reproduces the usual Pauli–Villars result.

Consider the noise function $\eta_\Lambda(t, x)$ with the following two–point function:

$$
\langle \eta_\Lambda(t, x)\eta_\Lambda(t', x') \rangle = \delta(x - x') \Lambda^2 e^{-\Lambda^2 |t - t'|}. \quad (3.37)
$$

For $\Lambda \to \infty$, it converges to the two–point function of the usual noise:

$$
\langle \eta_\Lambda(t, x)\eta_\Lambda(t', x') \rangle \xrightarrow{\Lambda \to \infty} 2\delta(x - x') \delta(t - t'), \quad (3.38)
$$

$$
\eta_\Lambda(t, x) \xrightarrow{\Lambda \to \infty} \eta(t, x). \quad (3.39)
$$

Applying the Fourier–Laplace transform, we get

$$
\langle \eta_\Lambda(t, x)\eta_\Lambda(t', x') \rangle = \frac{(2\pi)^2 \delta(k + k')} {s + s'} \frac{\Lambda^2 (s + s' + 2\Lambda^2)}{(s + \Lambda^2)(s' + \Lambda^2)}. \quad (3.40)
$$

In this scheme, the Langevin equation remains unchanged, which implies that the retarded Green’s function remains the same,

$$
G(s, k) = \frac{1}{s + \Omega^2}, \quad (3.41)
$$
while the field receives a $\Lambda^2$ correction. In particular, for the free case:

$$\phi_\Lambda(s, k) = G(s, k)\eta_\Lambda(s, k) + G(s, k)\phi_0(k).$$  \hfill (3.42)

We are now in a position to calculate the $\Lambda^2$ correction to the propagator in Eq. (3.15):

$$D_\Lambda(s, k; s', k') = \langle \phi_\Lambda(s, k)\phi_\Lambda(s', k') \rangle = \left(\frac{2\pi}{s + \Omega^2}\right)^2 \frac{\Lambda^2 (s + s' + 2\Lambda^2)}{(s + \Omega^2) (s + s') (s + \Lambda^2) (s' + \Lambda^2)}. \hfill (3.43)$$

To see how this corresponds to an ultraviolet regularization, let us perform the inverse Laplace transform:

$$D_\Lambda(t, k; t', k') = \left(\frac{2\pi}{t + \Omega^2}\right)^2 \frac{\Lambda^2 \Omega^2 (\Lambda^2 + \Omega^2)}{\Lambda^2 (\Lambda^2 - \Omega^2)} \left(\Lambda^2 e^{-\Omega^2|t-t'|} - \Omega^2 e^{-\Lambda^2|t-t'|} - (\Lambda^2 + \Omega^2) e^{-\Omega^2(t+t')} + \Omega^2 e^{-\Omega^2 t - \Lambda^2 t'} + \Omega^2 e^{-\Omega^2 t' - \Lambda^2 t'}\right). \hfill (3.44)$$

For large times, only the first two terms in the sum contribute, and for $t = t' \to \infty$, we find the usual Pauli–Villars propagator for the 2–dimensional boson:

$$D(t, k; t, k') \xrightarrow{t \to \infty} \left(\frac{2\pi}{t + \Omega^2}\right)^2 \frac{\Lambda^2}{\Omega^2 (\Lambda^2 + \Omega^2)}. \hfill (3.45)$$

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