Independence Tests Without Ground Truth for Noisy Learners
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Abstract

Exact ground truth invariant polynomial systems can be written for arbitrarily correlated binary classifiers. Their solutions give estimates for sample statistics that require knowledge of the ground truth of the correct labels in the sample. Of these polynomial systems, only a few have been solved in closed form. Here we discuss the exact solution for independent binary classifiers - resolving an outstanding problem that has been presented at this conference and others. Its practical applicability is hampered by its sole remaining assumption - the classifiers need to be independent in their sample errors. We discuss how to use the closed form solution to create a self-consistent test that can validate the independence assumption itself absent the correct labels ground truth. It can be cast as an algebraic geometry conjecture for binary classifiers that remains unsolved. A similar conjecture for the ground truth invariant algebraic system for scalar regressors is solvable, and we present the solution here. We also discuss experiments on the Penn ML Benchmark classification tasks that provide further evidence that the conjecture may be true for the polynomial system of binary classifiers.

1 Introduction

Self-assessment problems are everywhere in science and technology. For AI algorithms the topic appears with various names. This conference has the subject heading "Unsupervised Ensemble Learning". In this paper we will call it ground truth inference (GTI) to highlight the evaluation or monitoring aspect of the task - infer sample statistics that require knowledge of the correct answers given by an ensemble of algorithms. Since this methodology is non-parametric in the inference or recovery process, it cannot generalize its findings but merely evaluate or monitor specific samples.

The problem of GTI for independent binary classifiers was first discussed by Dawid and Skene [1979]. Their work further motivated Raykar et al. [2010] to develop a Bayesian approach to the problem. Spectral methods have also been developed and, in this conference, the work of Jaffe et al. [2015] gave a partial solution to the independent binary classifiers problem. This paper will discuss an exact, closed solution obtained by an algebraic approach (Theorem 1) - construct ground-truth invariant polynomial systems relating the unknown ground truth statistics - the sample prevalence of one label $\phi_\alpha$, and the sample label accuracies ($\{\phi_{\alpha,i}\}_{i=1}^3$, and $\{\phi_{\beta,i}\}_{i=1}^3$) to the label decision event counts, $\{m_{\ell_1,\ell_2,\ell_3}\}$. Given how many times the classifiers voted with one of the eight possible ways they could carry out binary classification, the $\{m_{\ell_1,\ell_2,\ell_3}\}$, we obtain, in a fully non-parametric fashion, two algebraic point solutions for the unknown ground truth statistics with one of them guaranteed to be the true one.

We will also discuss and use another polynomial system - the exact polynomial description of the $\{m_{\ell_1,\ell_2,\ell_3}\}$ in terms of the prevalence and the label accuracies of the classifiers that we have mentioned but also the decision error correlations on the sample (Theorem 2). No one can solve this larger system, but it can be used to develop detection tests that can verify if, in fact, the classifiers are independent in their errors.

The non-parametric character of these polynomial systems has strengths and weaknesses. Unlike the Bayesian approach to GTI, for example, it makes no assumptions about the functional form needed to explain the sample errors. This expands the practical application of GTI algorithms by eliminating theoretical assumptions that may bound the applicability of a particular Bayesian formulation, a deficiency that has been observed in experiments, Zheng et al. [2017]. And, as we have noted, we obtain an exact polynomial formulation of the problem for arbitrarily correlated classifiers (Theorem 2). This allows us to study the case of practical interest - classifiers that are weakly or sparsely correlated. These are the operating points of well designed engineering systems. We may not be able to solve this more general formulation, but the ground truth values that we seek to estimate will be members of the algebraic variety that solves this full polynomial formulation. This neatly sidesteps the problem of representation in
AI and allows for strong mathematical conclusions to be made without knowledge of the underlying ground truth or the processes that could have generated the errors we now see in a sample. The drawback of this non-parametric approach is clear. By focusing on sample statistics alone, we lose any ability to generalize the sample estimates to other samples.

We construct two independence tests for binary classifiers when the ground truth is absent. The first shows the utility of the algebraic approach by constructing a nearly perfect detector of non-independence when we have just three classifiers. Applying Theorem 1 to non-independent classifiers results in an estimate for the prevalence of either label that contains an irreducible square root. Since for any finite sample the sample prevalence is always an integer ratio, this can only happen if the classifiers are non-independent.

The second independence test considers how we could use the consistency or inconsistency of solutions when Theorem 1 is used on all possible trios of an ensemble of \( n \geq 4 \) classifiers as a way to detect nearly independent classifiers. We demonstrate, with experiments on a set of binary classification datasets, that the use of four classifiers and repeated use of Theorem 1 seems to be related to the label error correlations and that they do not "mix" - poor recovery of the accuracies for one label that has large correlations is accompanied by good recovery for the other label where the error correlations are much smaller. This too may have an algebraic explanation we briefly note later.

### 1.1 Ground truth statistics as polynomial systems

Our first theorem gives the solution to the independent binary classifiers problem.

**Theorem 1.** Given the set of eight integer counts \( \{m_{t_1,t_2,t_3}\} \) for all possible ways that three binary classifiers independent in their errors can output the two labels \( \alpha \) and \( \beta \) on a sample they have classified, if \( \{m_{t_1,t_2,t_3} \geq 1\} \), and \( M = \sum m_{t_1,t_2,t_3} \), the following set of quartic equations has only two point solutions and one of them is the ground truth prevalence of the \( \alpha \) label, \( \phi_\alpha \), and the classifiers accuracies on each of the labels, \( \{\phi_{\alpha,i}\}_{i=1}^3 \) and \( \{\phi_{\beta,i}\}_{i=1}^3 \):

\[
\begin{align*}
\frac{m_{\alpha,\alpha,\alpha}}{M} &= \phi_\alpha \phi_{1,\alpha} \phi_{2,\alpha} \phi_{3,\alpha} + (1 - \phi_\alpha)(1 - \phi_{1,\beta})(1 - \phi_{2,\beta})(1 - \phi_{3,\beta}) \\
\frac{m_{\alpha,\alpha,\beta}}{M} &= \phi_\alpha \phi_{1,\alpha} \phi_{2,\alpha}(1 - \phi_{3,\alpha}) + (1 - \phi_\alpha)(1 - \phi_{1,\beta})(1 - \phi_{2,\beta})\phi_{3,\beta} \\
\frac{m_{\alpha,\beta,\alpha}}{M} &= \phi_\alpha \phi_{1,\alpha}(1 - \phi_{2,\alpha})\phi_{3,\alpha} + (1 - \phi_\alpha)(1 - \phi_{1,\beta})\phi_{2,\beta}(1 - \phi_{3,\beta}) \\
\frac{m_{\beta,\alpha,\alpha}}{M} &= \phi_\alpha(1 - \phi_{1,\alpha})\phi_{2,\alpha}\phi_{3,\alpha} + (1 - \phi_\alpha)\phi_{1,\beta}(1 - \phi_{2,\beta})(1 - \phi_{3,\beta}) \\
\frac{m_{\beta,\alpha,\beta}}{M} &= \phi_\alpha(1 - \phi_{1,\alpha})(1 - \phi_{2,\alpha})\phi_{3,\alpha} + (1 - \phi_\alpha)\phi_{1,\beta}\phi_{2,\beta}(1 - \phi_{3,\beta}) \\
\frac{m_{\beta,\beta,\alpha}}{M} &= \phi_\alpha(1 - \phi_{1,\alpha})(1 - \phi_{2,\alpha})(1 - \phi_{3,\alpha}) + (1 - \phi_\alpha)\phi_{1,\beta}\phi_{2,\beta}\phi_{3,\beta}.
\end{align*}
\]

The second theorem highlights that sample statistics of the decisions on a sample are complete, and we can write down the exact polynomial system that can compute the decision event counts in terms of a finite set of unknown ground truth statistics.

**Theorem 2.** Given the set \( \{m_{t_1,t_2,...,t_n}\} \) of decision event counts done by an ensemble of \( n \) binary classifiers that are arbitrarily correlated in their sample errors, we can write an exact polynomial system that relates the counts to \( \phi_\alpha \), \( \{\phi_{\alpha,i}\}_{i=1}^n \) and the error correlations on the sample, \( \{\Gamma_{i,j,t}\} \) up to \( \{\Gamma_{i,j,k,...,t}\} \).

The existence of multiple solutions for the polynomial systems, such as the ones in Theorems 1 and 2, follows from their invariance under ground truth transformations. This is evident for the polynomial system in Theorem 1 because it is invariant under the following transformations of the ground truth statistics we are solving for:

\[
\begin{align*}
\phi_\alpha &\rightarrow 1 - \phi_\alpha \\
\{\phi_{\alpha,i}\} &\rightarrow \{\phi_{\beta,i}\} \\
\{\phi_{\beta,i}\} &\rightarrow \{\phi_{\alpha,i}\}.
\end{align*}
\]
So if one set of values for the ground truth statistics solves the polynomial system, another set of values - the one obtained by applying the transformations above - will also solve it.

The practical use of this algebraic approach to GTI is that the polynomial systems are neither trivial and may have interesting point solutions that greatly simplify the use of GTI for nearly unsupervised online evaluation of noisy learners as Theorem 1 does. In this way, algebraic GTI is no different than the use of error-correcting codes to fix bit flips in modern computers. All codes have multiple solutions to the decoding problem. Yet they are widely used in modern computers because their engineering context - well-manufactured hardware is reliable enough to make a few bit flips much more likely than many - makes them practical as a way to fix occasional, stray bit flips.

Following the intuition that nearly independent algorithms would produce consistent results when different trios of them are used to apply Theorem 1, we explore here a proposition we call the four consistent trio solutions conjecture.

**Conjecture 1.** Given the decisions of four binary classifiers on a sample, if the 4 possible ways to use Theorem 1 on their decision event counts yield consistent estimates for \( \phi_\alpha \), \( \{ \phi_{\alpha,i} \}_{i=1}^4 \), and \( \{ \phi_{\beta,i} \}_{i=1}^4 \) then their 2 and 3 way error correlations on the sample are identically zero,

\[
\begin{align*}
\{ \Gamma_{\alpha,i,j} = 0 \} & \quad \{ \Gamma_{\beta,i,j} = 0 \} \quad (12) \\
\{ \Gamma_{\alpha,i,j,k} = 0 \} & \quad \{ \Gamma_{\beta,i,j,k} = 0 \} \quad (13)
\end{align*}
\]

### 1.2 Organization of the paper

We finish the introduction placing algebraic GTI in relation to data sketching, compressed sensing, and algebraic statistics. Section 2 formalizes ground truth, ground truth statistics, and observable statistics in the context of two GTI tasks - estimating the accuracy of binary classifiers as we have been discussing and recovering the precision error covariance matrix for scalar regressors. Section 3 constructs an independence test with three classifiers based on the algebraic nature to the solutions found in Theorem 1. This highlights the relevant practical problem: given unknown error correlations, can we use the variations in independent model solution to bound them? Section 4 tests the four consistent trios conjecture on three Penn ML binary classification benchmarks. Section 5 partially resolves the conjecture for a similar formulation on a GTI task for scalar regressors. Finally, a closing section summarizes the work and advocates for more research into these algebraic methods for measuring AI errors in a non-parametric fashion.

### 1.3 Similarities of algebraic GTI to other fields

Algebraic GTI has similarities and differences with other areas of research that may be known by the reader - data sketching, compressed sensing, and algebraic statistics.

Like data sketching, any specific algebraic GTI polynomial system solves for a specific statistic. There is no recovery of the whole ground truth with algebraic GTI. Only statistics of the ground truth can be recovered. Algebraic GTI is data sketching for AI errors. Also, similar to data sketching, the recovery of a ground truth statistic only requires small memory footprints. Theorem 1 shows how eight integer counters are enough to recover the sample label accuracies for three independent classifiers. Algebraic GTI is non-parametric in the same way that data sketching algorithms are non-parametric.

It may seem strange to the reader that statistics of an unknown ground truth could be computed robustly in the absence of said ground truth, or that we could do it non-parametrically with minimal theoretical assumptions. But this has been done before. The Good-Turing estimate is able to estimate the number of unseen species or types in a finite survey by using the frequency of species observed one time, two times and so on. It then shifts counts in these observed frequency bins to estimate the count that should be in the zero count bin. The Good-Turing algorithm can be restated as a series of polynomial equations that push the observed counts to the zero count bin.

The under-determined nature of ground truth invariant polynomials is also similar to the problems that Compressed Sensing tries to solve. Indeed, the ground truth invariant linear system for the precision error covariance matrix was solved for sparsely correlated scalar regressors in Corrada-Emmanuel and Schultz [2008] by using \( \ell_1 \) minimization - a favorite for CS recovery of a signal. There is an important difference between CS and algebraic GTI. In GTI you are recovering the error in the signal, not the signal itself. By focusing on the error of the signal, not the signal itself, we can get away with having no knowledge of what produced the signal. Algebraic GTI is about error recovery, not signal recovery.

In the 1990s, Pistone et al. [2001] pioneered the use of algebraic geometry in statistics by recasting traditional statistical tasks, such as experimental design, as polynomial problems. Algebraic GTI also uses the math of algebraic geometry.
geometry for a statistical task. The major difference between them is the object of study. Algebraic statistics has been developed focusing solely on parametric statistics problems. Algebraic GTI is an example of non-parametric statistics. Perhaps future editions of Algebraic Statistics books will include a chapter on this non-parametric approach to estimate the statistics of errors made by AI algorithms.

2 Terminology and definitions for algebraic GTI

2.1 What is ground truth?

Ground truth in non-parametric estimation of AI errors refers to the correct responses the noisy learners should have returned for the sample. In the case of binary classifiers this refers to the true label for each item in the sample that was classified. For regressors, it refers to the correct scalar, or vector of each item in the sample. The ground truth may be hidden, unavailable or unknown.

2.2 What are ground truth statistics?

*Ground truth statistics* are then statistics of the sample that require knowledge of the ground truth to be computed. Examples of this for binary classifiers are the true prevalence of the labels in the sample, the accuracies of each classifier on a label, their correlations, etc. Examples for regressors are their precision error covariance tensors.

These should be contrasted with *observable statistics* - statistics of the sample that require no knowledge of the ground truth. Examples of these are the counts with which the classifiers agreed or disagreed on labels of the sample, and moments of the predictions for regressors.

Ground truth inference via algebraic methods is based on establishing a polynomial system relating the ground truth statistics - to be solved for - with the observable statistics - readily computed from the sample of noisy decisions. By imposing invariance to ground truth values, the polynomials eliminate the need for theoretical assumptions about the process that led to the errors in the sample. No assumptions are needed besides generic ones like the independence of their errors. Algebraic GTI is trying to do error recovery, not signal recovery.

3 Detecting non-zero error correlations

The algebraic nature of the solution to Theorem 1 and the full polynomial system of Theorem 2 allows one to immediately create a nearly perfect detector for non-independent classifiers on the given sample. It is based on the following theorem

**Theorem 3.** Given the decision event counts, \( \{ m_{\ell_1, \ell_2, \ell_3} \} \), for three binary classifiers, the independent polynomial system solution for the sample prevalence \( \phi_\alpha \) will contain the square root of an irreducible polynomial that is not functionally a perfect square except for the case of zero error correlations.

Because of this theorem, whenever we use the independent polynomial system to estimate \( \phi_\alpha \), we are almost guaranteed to obtain an irreducible square root in the solution for \( \phi_\alpha \). But for any finite size sample the prevalence is an integer ratio, whether the classifiers are correlated or not in their sample errors. The appearance of this irreducible square root is almost surely a sign that the sample does not have all error correlations are zero. We now outline the algebraic proof for this.

One way to solve a multi-variable polynomial systems is to use Buchberger’s algorithm to obtain an Elimination Ideal, see [Cox et al. 2015]. This Ideal consists of a series of polynomial equations that have, at its start, a polynomial with a single unknown variable. Solving this polynomial then allows us to proceed down a chain of other polynomials to resolve the other unknown variables. This is the constructive algorithm that we use to prove Theorem 1. The existence of just two point solutions for the independent classifiers polynomial system is based on the appearance of a quadratic for the unknown prevalence \( \phi_\alpha \) at the base of Elimination Ideal. This quadratic is enormous but it can be represented easily in the following symbolic form,

\[
a(\{ m_{\ell_1, \ell_2, \ell_3} \}) \phi_\alpha^2 + b(\{ m_{\ell_1, \ell_2, \ell_3} \}) \phi_\alpha + c(\{ m_{\ell_1, \ell_2, \ell_3} \}) = 0, \tag{14}
\]

where the coefficients \( a(\{ m_{\ell_1, \ell_2, \ell_3} \}) \), \( b(\{ m_{\ell_1, \ell_2, \ell_3} \}) \), and \( c(\{ m_{\ell_1, \ell_2, \ell_3} \}) \) are themselves polynomials in the \( \{ m_{\ell_1, \ell_2, \ell_3} \} \) variables. So the independent polynomial solution for \( \phi_\alpha \) is immediately given by the quadratic formula and will contain a square root of the form \( \sqrt{b^2 - 4ac} \).
But the ground truth value for $\phi_\alpha$ is an integer ratio, whether they are correlated or not. So the appearance of a non-reducible square root is a clear signal that the independent model cannot be the correct description of the correlations between the classifiers. But can correlated classifiers return a solution to the independent model quadratic that is an integer ratio?

Here is where the theoretical advantages of having a full, exact polynomial description for the observed $\{m_{\ell_1,\ell_2,\ell_3}\}$ counts. Theorem 2 becomes useful. Start by noting that the independent polynomial system can be fed back into the $\phi_\alpha$ quadratic so we can rewrite $a(\{m_{\ell_1,\ell_2,\ell_3}\}$, $b(\{m_{\ell_1,\ell_2,\ell_3}\}$, and $c(\{m_{\ell_1,\ell_2,\ell_3}\}$ as polynomials of the unknown statistics. When we do that in the square root term in the quadratic formula solution becomes,

$$\sqrt{b(\{m_{\ell_1,\ell_2,\ell_3}\})^2 - 4ac(\{m_{\ell_1,\ell_2,\ell_3}\})} = \sqrt{(1 - 2\phi_\alpha)^2 (\phi_\alpha - 1)^4 \phi_\alpha^4 (\phi_1,\beta - 1)^4 \sqrt{(\phi_2,\alpha + \phi_2,\beta - 1)^4 (\phi_3,\alpha + \phi_3,\beta - 1)^4}}.$$

Since this is the square root of a perfect square, we obtain a self-consistent result - independent classifiers have integer ratio solutions for the unknown $\phi_\alpha$.

If the classifiers are not independent, then we must put in the full polynomial equations assuming non-zero correlations between the classifiers. Having done so, further algebraic manipulations show that the square root term in the quadratic formula becomes,

$$(g_1(\{\phi_\alpha\}) + g_2(\{\phi_\alpha\}, \{\Gamma_{i,j,l}\}, \{\Gamma_{i,j,k,l}\}))^2 (g_3(\{\phi_\alpha\}) + g_4(\{\phi_\alpha\}, \{\Gamma_{i,j,l}\}, \{\Gamma_{i,j,k,l}\})).$$

The $g$ polynomials, $(g_1, g_2, g_3, g_4)$, are polynomials in the variables of their arguments. The $g_1$ and $g_4$ terms are reducible, and when the error correlations are identical zero, this product term reduces to the one in Equation 15. However, the sum $g_3 + g_4$ is not reducible in general and thus cannot be guaranteed to become a perfect square. Indeed, to make the general polynomial reducible, you have to assume all correlations are equal and the classifiers have equivalent accuracies.

Thus the appearance of an irreducible square root term in the solution for the sample prevalence when we use the independent classifiers polynomial is a nearly perfect detector for the existence of non-zero correlations. A complete understanding of this result theoretically would then require enunciating theorems that can pronounce what conditions guarantee, almost surely, no non-zero correlations can produce a perfect square in the independent model prevalence solution.

This satisfying theoretical result has limited practical use since we expect real samples to have some correlations, albeit maybe small ones. Must we not use the independent system solutions even when the correlations are small? What if the solutions have small disagreements? Hence our interest in Conjecture 1. Can we start by saying that no correlated system would yield consistent independent solutions? If so, can we then proceed further by using the inevitable small disagreements between the trio solutions to understand the size of the correlations between the classifiers? In this paper we do not fully work out this approach but we can offer further experimental and theoretical results to motivate the possibility of an affirmative answer.

4 Testing the conjecture experimentally

We now provide experimental evidence for the four consistent trios conjecture in the case of binary classifiers. We use three of the classification benchmarks in the Penn ML Benchmarks suite, Olson et al. [2017]: twonorm, spambase, and mushroom. Our goal is to exhibit cases where the algebraic method returns consistent and inconsistent results. We are not asserting the general applicability of the independent polynomial system solutions, but rather the identification of settings where it does seem to work reasonably well.

In addition, we will note an interesting phenomenon in the experimental results - using the consistency of trios seems to detect label correlations somewhat independently. In our experiments we were able to consistently get good estimates for one label whenever that label had small error correlations but inconsistent estimates for the other label when it had larger error correlations. A possible algebraic explanation for this is that the Elimination Ideal contains two polynomial chains, one for each label, starting at the base quadratic for the prevalence $\phi_\alpha$. 
4.1 Label error correlations

Label error correlations are also examples of ground truth statistics. Given the true label for the items in the sample of size \( m \), we can define the n-way error correlation for a label, \( \ell \), as,

\[
\Gamma_{\ell,i,j,...} = \frac{1}{m} \sum_{d=1}^{m} (x_{d,i} - \bar{x}_i)(x_{d,j} - \bar{x}_j) \ldots
\]

(17)

Here \( x_{d,i} \) is an indicator variable that is one if the classifier \( i \) correctly labeled item \( d \) in the instances of label \( \ell \) in the data. The term \( \bar{x}_i \) is the accuracy of the classifier on the \( \ell \) items in the sample and is the definition of the \( \phi_{i,\ell} \) ground truth statistic.

The assumption of independence of errors on a given sample is equivalent to saying that all these correlations are zero for all labels as shown in Conjecture 1. This is an assertion about the sample correlations not about any underlying process that may be causing the ensemble members to agree or disagree on their decisions.

For ease of illustration, we will be quoting the average, and standard deviations of 2-way error correlations for each label in our experiments. The experimental verification of the conjecture then consists of observing that whenever the trio solutions are consistent for all 4 classifiers, the correlations are small and whenever they disagree, the correlations are large. In other words, we expect the size of the disagreements to be monotonically related to the correlation magnitudes. Elucidating how the discrepancy relates to the size of the error correlations is a subject for future work.

4.2 Methodology for a single experiment

We tried our best to create nearly independent classifiers for the three examples shown in this experimental section. The reader is seeing the experiments for which we got closest to that independence condition. Our approach for inducing as much independence in the errors as possible was to use these three basic technique: using different classification algorithms, reducing the overlap in training data, and having no or little intersection between classifier feature sets.

We trained four classifiers for an experimental run and then apply the independent errors algebraic solution for each of the 4 trios possible. Thus, each classifier will get three estimates for each of the labels.

The alert reader will note the practical value for schemes like AutoML in having a monitoring algorithm like this. It can be used to error correct bad classifiers. Independence of errors, not accuracy, is enough to create reliable ensemble algorithms once you have solved the ground truth inference problem for a task.

The Penn ML Benchmarks use "0" and "1" for the two binary labels. To maintain consistency with our notation and avoid confusing the labels with the classifier indices, we use \( \alpha \) for the "0" label and \( \beta \) for the "1" label.

4.3 twonorm experiment

The twonorm binary classification benchmark consists of 7,400 items (3703/3697) with 20 features. We divided the features randomly into 4 disjoint sets with 5 features each. Four classifiers were trained, using “off-the-shelf” algorithms provided by the Mathematica system: NeuralNetwork, GradientBoostedTrees, NaiveBayes, and LogisticRegression. Each classifier was trained on 1500 items randomly selected from a training set of 2000 (1000/1000). We then assembled their decisions on the remaining items in the benchmark and used these as sole input into the repeated application of Theorem 1.

Our exemplar experiment (Figure 1) shows that the consistency between the recovered values is about 1 percent. As noted, each classifier gets three estimates for each of its label accuracies. On the x-axis we plot the ground truth value and on the y-axis, the recovered value using the independent polynomial system solution. For ease of reference, we include the diagonal line so the reader can see when the recovered value is close to the ground truth one. For this experiment the average label correlations are shown in Table 1. This experiment is an example where both labels had small error correlations and both recovered estimates were close to the ground truth values for the sample.

| \( \Gamma_{\alpha,i,j} \) | \( \sigma_{\Gamma_{\alpha,i,j}} \) | \( \Gamma_{\beta,i,j} \) | \( \sigma_{\Gamma_{\beta,i,j}} \) |
|-----------------|-----------------|-----------------|-----------------|
| -0.00000048     | 0.0023          | -0.0021         | 0.0024          |

Table 1: Mean and standard deviation for 2-way sample error correlations in the twonorm experiment
4.4 spambase experiment

The spambase binary classification benchmark consists of 4601 \((2788/1813)\) items with 57 features. We divided the features randomly into 4 sets with 10 features each. Four classifiers were trained using the Mathematica system algorithms: NeuralNetwork, SupportVectorMachine, DecisionTree and NaiveBayes. Each classifier was trained on \((200/100)\) items randomly selected from a training set of \((279/181)\). These results are an examplar of how we can recover good estimates for one label with small correlations even in the presence of larger correlations (and inconsistent estimates) for the other label.

Table 2: Mean and standard deviation for 2-way sample error correlations in the spambase experiment

| \(\Gamma_{\alpha, i, j}\) | \(\sigma_{\Gamma_{\alpha, i, j}}\) | \(\Gamma_{\beta, i, j}\) | \(\sigma_{\Gamma_{\beta, i, j}}\) |
|--------------------------|-----------------|-----------------|-----------------|
| 0.0056                   | 0.0036          | 0.067           | 0.020           |

4.5 mushroom experiment

The mushroom binary classification benchmark consists of 8124 \((4208/3916)\) items with 22 features. We divided the features randomly into 4 sets with \((6,6,6,5)\) features each. Four classifiers were trained using the Mathematica system algorithms: DecisionTree, NaiveBayes, NeuralNetwork, SupportVectorMachine. Each classifier was trained on \((100/100)\) items randomly selected from a training set of \((421/392)\). These results are an examplar of roughly equally noisy recovery for both labels.

Table 3: Mean and standard deviation for 2-way sample error correlations in the spambase experiment

| \(\Gamma_{\alpha, i, j}\) | \(\sigma_{\Gamma_{\alpha, i, j}}\) | \(\Gamma_{\beta, i, j}\) | \(\sigma_{\Gamma_{\beta, i, j}}\) |
|--------------------------|-----------------|-----------------|-----------------|
| 0.012                   | 0.011           | 0.017           | 0.025           |
5 The four consistent trios theorem for regressors

The four consistent trios conjecture remains unsolved for binary classifiers. We now discuss and resolve the conjecture for noisy regressors. The ground truth inference problem we will be solving consists of recovering the precision error covariance matrix for scalar regressors. This was solved by Corrada and Schults (2008) in the context of fusing multiple Digital Elevation Models (DEMs). Here we are interested in the task for theoretical purposes. If we did not have such a sparsely-correlated recovery method, could we use something like the four consistent trios to convince ourselves that the regressors were nearly independent on their sample errors?

5.1 A ground truth invariant equation for a pair of regressors

For ease of illustration we will be discussing scalar regressors, but the same methodology can be applied to the multi-dimensional case. The ground truth for this task is the set \( \{y_{m}\}_{m=1}^{M} \). We also have the set \( \{y_{m,i}\}_{m=1}^{M} \) for any member of the ensemble of noisy regressors. The ground truth inference problem we are trying to solve is the recovery of the error covariances defined by,

\[
\epsilon_{i,j} = \frac{1}{M} \sum_{m=1}^{M} (y_{m} - y_{m,i})(y_{m} - y_{m,j}).
\]

The following theorem is easily proved and forms the basis for constructing a ground-truth invariant linear system of equations that can be used to recover the error covariances.

**Theorem 4.** Given the estimates \( \{y_{m,i}\} \) and \( \{y_{m,j}\} \) by two regressors \( i \) and \( j \), the following equation holds,

\[
\epsilon_{i,i} + \epsilon_{j,j} - 2\epsilon_{i,j} = \frac{1}{M} \sum_{m=1}^{M} y_{m,i}^2 + \frac{1}{M} \sum_{m=1}^{M} y_{m,j}^2 - 2 \frac{1}{M} \sum_{m=1}^{M} y_{m,i}y_{m,j}.
\]

The rhs of equation [19] is independent of any knowledge of the ground truth as it only involves moments of the estimates given by the two regressors. Furthermore, it is invariant to arbitrary global shifts for each regressor, \( \{y_{m,i}\} \rightarrow \{y_{m,i} + a\} \).

5.2 The under-determined linear system for solving the regressors GTI problem

Equations of the form in Equation [19] give us \( n(n - 1)/2 \) equations for the \( n(n + 1)/2 \) terms needed to recover the precision error covariance matrix. Why do we say precision error and not just error? The constructed linear system
is invariant to arbitrary global shifts. So its use cannot recover the full error covariance matrix but instead, recovering the precision error covariance matrix. This is equivalent to saying that we are computing with de-meaned signals for all of the regressors.

5.3 The GTI solution for three independent regressors

For regressors, we have the same phenomena as for binary classifiers. At some large enough ensemble, an independent errors model becomes solvable.

Consider independent regressors, at \( n = 3 \) the number of unknowns is equal to the number of pairs. We just have to solve the system,

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\epsilon_{1,1} \\
\epsilon_{2,2} \\
\epsilon_{3,3}
\end{pmatrix}
= 
\begin{pmatrix}
\Sigma_{1,2} \\
\Sigma_{1,3} \\
\Sigma_{2,3}
\end{pmatrix}
\]  

(20)

5.4 All trios consistent for four scalar regressors implies homogeneous error correlations or vanishing data moments

Now suppose that we are given 4 regressors and we pretend they are independent in their sample errors so we go ahead and solve four times a system of the form given in Equation 20. The following theorem is easy to prove as a linear system enforcing the consistency between the four independent model solutions,

**Theorem 5.** Four trio solutions for the independent GTI system 20 are consistent only if,

\[
\epsilon_{i,j} + \epsilon_{k,l} = \epsilon_{i,k} + \epsilon_{j,l}.
\]

(21)

The above theorem does not resolve the conjecture for regressors but it makes it plausible. Certainly homogeneous solutions, including zero, satisfy this condition. But can non-homogeneous cross-correlations satisfy this equation? The answer is yes for consistent trios when 4 or more regressors are involved.

**Theorem 6.** Aside from homogeneously constant cross-correlations, scalar regressors are consistent if all possible 4 distinct classifiers \( \{i, j, k, l\} \) for \( n \geq 4 \) classifiers satisfy,

\[
\sum_{d=1}^{D} (y_{d,i} - y_{d,k})(y_{d,j} - y_{d,l}) = 0
\]

(22)
The immediate consequence of this is that we can quickly check, using data alone, if Theorem 6 applies. If not, it requires further theoretical investigation to ascertain when consistent solutions are more likely to imply zero or small correlations than some large homogeneous value.

6 Conclusions

We have shown an algebraic way of solving the problem of self-assessing three independent binary classifiers quickly. Due to its algebraic nature, it can detect non-independent classifiers. And by using it on all four trios of an ensemble of four, we showed how it could detect when it was working on three Penn ML classification benchmarks. A similar consistency conjecture for scalar regressors was shown to require perfectly homogeneous cross-correlations, an unlikely balance point that requires further theoretical elucidation. For both classifiers and regressors we were able to build tests that can confirm the independence of the algorithms without the ground truth for the correct answers.

We intend to continue this work by exploring further the prevalence quadratic to see how it can be used to develop functional forms that measure the error correlations.

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A Proofs

A.1 Proof of Theorem 1

Theorem 1, Whenever three independent binary classifiers have \( \{m_1, e_2, e_3\} \) not all equal to each other and all non-zero, they satisfy a solvable quartic polynomial system with two solutions, one of them, with the ground truth values.

The proof has two parts. The first part is purely algebraic - the proof that the system of eight quartic polynomials has only two point solutions. This is done with Buchberger’s algorithm and inspection of the Gröbner Basis we get as its output. The second part consists of showing that the ground truth values for the statistics of the sample we seek to estimate, the prevalence, and classifier accuracies of the ensemble of three classifiers, is one of these two solutions.
A.1.1 The quartic system has two point solutions

The condition that not all \( \{m_{\ell_1, \ell_2, \ell_3}\} \) are equal prevents the degenerate case of all classifiers equally good on both labels. This is a clearly unrealistic finite sample - any prevalence would satisfy it. Similarly, loss of information due to few counts would also lead to degenerate cases that are easy to detect. For example, they could just agree on everything. Those are of mathematical, not practical interest. We exclude them so we can talk about the case of practical interest - not all counts are equal.

This proof is by construction, an application of Buchberger’s algorithm for computing the Gröbner Basis to the quartic system. This can be done, for example, with the command `GroebnerBasis` in the Mathematica software system. we obtain a quadratic equation in the unknown prevalence \( \phi_{\alpha} \) by using the following order of the variables,

\[
\{\phi_{3,\beta}, \phi_{3,\alpha}, \phi_{2,\beta}, \phi_{2,\alpha}, \phi_{1,\beta}, \phi_{1,\alpha}, \phi_{\alpha}\}
\]

and a lexicographic order. This single quadratic is the start of two distinct chains of polynomials; one for the \( \alpha \) label, the other for \( \beta \). The \( \alpha \) chain allows us to sequentially solve for the \( \{\phi_{\alpha, i}\}_{i=1}^{3} \), the other solves for the \( \{\phi_{\beta, i}\}_{i=1}^{3} \). All the polynomials in these two chains are linear equations in the unknown variable one being solved for. The variables involved are detailed in Table 4. The conclusion is immediate. There are only two point solutions - each starting with one of the two values that solve the prevalence quadratic formula.

| \( \alpha \) | \( \beta \) |
|--------------|--------------|
| \( \phi_{\alpha}, \phi_{3,\alpha} \) | \( \phi_{\alpha}, \phi_{3,\beta} \) |
| \( \phi_{3,\alpha}, \phi_{2,\alpha} \) | \( \phi_{3,\beta}, \phi_{2,\beta} \) |
| \( \phi_{2,\alpha}, \phi_{1,\alpha} \) | \( \phi_{2,\beta}, \phi_{1,\beta} \) |

Table 4: Variables in the two elimination chains

A.1.2 For independent classifiers, the ground truth values are one of those two solutions to the prevalence quadratic

If the classifiers are independent in their sample errors, then the observed counts of any decision event can be factored in the following form,

\[
\frac{n(\ell_1, \ell_2, \ell_3 | \ell_{\text{true}})}{n(\ell_{\text{true}})} = \frac{n(\ell_1 | \ell_{\text{true}})}{n(\ell_{\text{true}})} \frac{n(\ell_2 | \ell_{\text{true}})}{n(\ell_{\text{true}})} \frac{n(\ell_3 | \ell_{\text{true}})}{n(\ell_{\text{true}})}
\]

But these can all be expressed as the \( \phi_{i,\ell} \) terms in the quartic polynomials. Since we are assuming the binary labels are disjoint for any item in the sample, every decision count observed in the sample is equal to

\[
m_{\ell_1, \ell_2, \ell_3} = n(\alpha)\phi_{1,\ell_1,\alpha}\phi_{2,\ell_2,\alpha}\phi_{3,\ell_3,\alpha} + n(\beta)\phi_{1,\ell_1,\beta}\phi_{2,\ell_2,\beta}\phi_{3,\ell_3,\beta}.
\]

Dividing by \( M \) and rewriting all the \( \phi_{i,\ell_1,\ell_2} \) terms of the classifier accuracies,

\[
\phi_{i,\beta,\alpha} = 1 - \phi_{i,\alpha}
\]

\[
\phi_{i,\alpha,\beta} = 1 - \phi_{i,\beta}
\]

yields the quartic system of Theorem 1. Hence the ground truth values solve the system and it must be one of the two solutions to the polynomial system.

A.1.3 What this theorem is not saying

The assumption stated in Equation 24 gives the whole game away. If we are allowed to assume this, it removes any finite sample, however it was produced, that does not meet this factorization assumption. The interesting part in practical terms is that, for small enough samples, this will be hard to do if, say, we start with an independent generative process. Nonetheless, independence theorems like this are always interesting because they can start as a foundation for a sparsely or weakly correlated approach. For this more practical task, one needs Theorem 2.

One can get a feel for how easy it is to create a small sample for it by considering random accuracies of either 1/3 or 2/3 and pretend the prevalence is 1/2. On one simulation of this procedure using Theorem 1 to produce
the decision event counts, $M = 18$ was enough to create a finite sample that obeys the Theorem. But most of the $8^{18} = 18,014,398,509,481,984$ ways three binary classifiers could have labeled the sample would not satisfy Theorem 1.

Hence our interest in handling non-zero correlations. They can occur due to small sample size or because asymptotically the classifiers are correlated. In either case, a useful non-parametric method has to handle non-zero error correlations.

A.2 Proof of Theorem 2

Theorem 2, A complete polynomial description of the $L^n$ decision event counts for $n$ arbitrarily correlated $L$-label classifiers is possible. The system is of polynomial order $n + 1$ and the ground truth values are in the algebraic variety of this complete description.

By the disjointness of labels, any of the $L^n$ decision even counts can be expressed as,

$$m_{\ell_1, \ell_2, \ldots, \ell_n} = \sum_{l=1}^{L} m(\ell_1, \ell_2, \ldots, \ell_n | \ell_l).$$

(28)

This can be rewritten as,

$$m_{\ell_1, \ell_2, \ldots, \ell_n} = \sum_{l=1}^{L} \frac{m(\ell_1, \ell_2, \ldots, \ell_n | \ell_l)}{m(\ell_l)} m(\ell_l).$$

(29)

The rest of the proof can now proceed by focusing solely on how to write any $m(\ell_1, \ell_2, \ldots, \ell_n | \ell_l)$ in terms of the prevalences, marginal accuracies, and the error moments for that label alone.

A.2.1 Error moments for arbitrary number of labels

For arbitrary number of labels it makes more sense to talk about error moments instead of correlations as in the binary case ($L = 2$). These error moments are numerous since we must take into account $L^n$ decision event counts for a given true label. The general definition of the error moments is,

$$\Gamma_{\ell_i, \ell_j, \ldots; \ell_{\text{true}}} = \frac{1}{m(\ell_{\text{true}})} \sum_{d=1}^{m(\ell_{\text{true}})} (x_{d,\ell_i,\ell_{\text{true}}} - \phi_{\ell_i,\ell_{\text{true}}})(x_{d,\ell_j,\ell_{\text{true}}} - \phi_{\ell_j,\ell_{\text{true}}})$$

(30)

Note that any decision even count for a label can be written as the sum, over the sample items, of evaluating a monomial of the form,

$$x_{d,\ell_1,\ell_{\text{true}}} \cdots x_{d,\ell_n,\ell_{\text{true}}},$$

(31)

where the $x_{d,\ell_i,\ell_{\text{true}}}$ are indicator variables for making the $\ell_i$ decision on an item of the true label, $\ell_{\text{true}}$. These per label indicator variables for each classifier satisfy,

$$\sum_{\ell_i} x_{d,\ell_i,\ell_{\text{true}}} = 1.$$  

(32)

This condition and the normalization condition for marginal accuracies of each classifier,

$$\sum_{\ell_i} \phi_{\ell_i,\ell_{\text{true}}} = 1$$

(33)

means that these error moments are related. For example, in the binary label case a pair of classifiers has four error moments for each true label. But they can all be reduced to a single one as shown below for the case of the $\alpha$ label

$$\Gamma_{\alpha,\beta;\alpha} = -\Gamma_{\alpha,\alpha;\alpha}$$

(34)

$$\Gamma_{\beta,\alpha;\alpha} = -\Gamma_{\alpha,\alpha;\alpha}$$

(35)

$$\Gamma_{\beta,\beta;\alpha} = \Gamma_{\alpha,\alpha;\alpha}$$

(36)
A.2.2 All decision event counts can be written in terms of the error moments and marginal accuracies

The crucial point is that all decision event counts for a true label can be rewritten in terms of these error moments and the marginal accuracies of the classifiers. Starting with the full monomial shown in Equation [31] we can rewrite a count in terms of the n-way correlation moment and lower order monomials of the indicator functions. But these lower order monomials can be, in turn, rewritten in terms of lower order correlations and so on. This descent ends at the marginal accuracies of the classifiers.

The rest is algebra.

By construction, the ground truth values for all these unknown sample statistics will satisfy these polynomial equations exactly. Therefore, the ground truth values must reside in the set of points that satisfy the polynomial system. This is called the algebraic variety of a polynomial system. QED.

A.2.3 The binary label case

To illustrate the above generic observation we will illustrate how the computation proceeds for the binary label case. To simplify the heavy notation that is required in the general case we use,

\[ x_{d,i} = e_{true,i} = x_{d,i}, \]  

(37)

when discussing binary classifiers.

Any decision event count for a label in binary classification can be written as products of \( x_{d,i} \) and \( (1 - x_{d,i}) \) depending on whether a classifier has the label correct or not. So any of these label ensemble decision counts can be written as monomials involving, at most, \( n \) powers of the \( x_{d,i} \). We can use all the error correlations to rewrite any of these monomials. We sketch how this can be done. To simplify the notation further, we will omit the sum over the sample items. The presence of the subscript \( d \) is to be taken as implying that omitted sum.

Consider the case of two classifiers, \( n = 2 \), and the \( m(\alpha, \beta | \alpha) \) decision event count. This is given by,

\[ x_{d,1}(1 - x_{d,2}) = m_1 \phi_1 \alpha - x_{d,1} x_{d,2}. \]  

(38)

But the sample sum of \( x_{d,i} x_{d,j} \) can be re-written using the 2-way correlation term,

\[ m_1 \Gamma_{i,j;\alpha} = (x_{d,i} - \phi_1 \alpha)(x_{d,j} - \phi_2 \alpha) \]

(39)

to give us,

\[ x_{d,i} x_{d,j} = m_1 (\Gamma_{i,j;\alpha} - \phi_2 \phi_1 \alpha) \]

(40)

Putting together these equations we can write the label ensemble decision count for the \( \alpha \) items in the sample as,

\[ m(\alpha, \beta | \alpha) = m_1 (\phi_1 \alpha (1 - \phi_2 \alpha) - \Gamma_{1,2;\alpha}). \]

(41)

We can immediately generalize this to obtain the exact polynomial description of the \( m(\alpha, \beta) \) count for any two arbitrarily correlated classifiers,

\[ m(\alpha, \beta) = (\phi_1 \alpha (1 - \phi_2 \alpha) - \Gamma_{1,2;\alpha}) m(\alpha) + ((1 - \phi_1 \beta)\phi_2 \beta - \Gamma_{1,2;\beta}) m(\beta). \]

(42)

Proceeding to any number of classifiers is just the repeated application of this process. For illustration purposes, we give here a single one of the eight polynomials that comprise the three arbitrarily correlated binary classifiers,

\[ \phi_\alpha (1 - \phi_2 \alpha)\phi_2 \alpha (1 - \phi_3 \alpha) - \Gamma_{1,2,\alpha} (1 - \phi_3 \alpha) + \Gamma_{1,3,\alpha} \phi_2 \alpha - \Gamma_{2,3,\alpha} (1 - \phi_1 \alpha) + \Gamma_{1,2,3,\alpha} + (1 - \phi_\alpha) (\phi_1 \beta (1 - \phi_2 \beta) \phi_3 \beta - \Gamma_{1,2,\beta} \phi_3 \beta + \Gamma_{1,3,\beta} (1 - \phi_2 \beta) - \Gamma_{2,3,\beta} \phi_1 \beta - \Gamma_{1,2,3,\beta}) \]

(43)

We leave it as an exercise for the reader to guess which ensemble decision event count is described by this polynomial.

A.3 Proof of Theorem 3, correlated classifiers have an irreducible square root in the independent solution for the prevalence

The Gröbner basis for the independent classifiers polynomial system contains a quadratic that solves for the unknown \( \phi_\alpha \), as detailed in the proof of Theorem 1. It can be represented as follows,

\[ a \{ m_{\ell_1, \ell_2, \ell_3} \} \phi_\alpha^2 + b \{ m_{\ell_1, \ell_2, \ell_3} \} \phi_\alpha + c \{ m_{\ell_1, \ell_2, \ell_3} \} = 0, \]

(44)
Table 5: Some algebraic properties of the polynomial coefficients of the prevalence quadratic

| coefficient | number of terms | irreducible? |
|-------------|-----------------|--------------|
| \(a\{\{m_{\ell_1,\ell_2,\ell_3}\}\}\) | 72 | yes |
| \(b\{\{m_{\ell_1,\ell_2,\ell_3}\}\}\) | 72 | yes |
| \(c\{\{m_{\ell_1,\ell_2,\ell_3}\}\}\) | 17/17/17 | no |

The \(a\{\{m_{\ell_1,\ell_2,\ell_3}\}\}\), \(b\{\{m_{\ell_1,\ell_2,\ell_3}\}\}\), and \(c\{\{m_{\ell_1,\ell_2,\ell_3}\}\}\) polynomials are quite large so it is not illustrative to write them out. Table 5 shows some of their properties. \(c\{\{m_{\ell_1,\ell_2,\ell_3}\}\}\) is the only term that is factorizable when using the \(m\) variables.

The proof then rests on the irreducibility of the square root term in the quadratic formula,

\[
\sqrt{b^2 - 4ac}. \tag{45}
\]

If the classifiers are independent on the sample, then the independent polynomial system is an exact expression of the observed ensemble decision event counts. This allows us to re-express the counts as polynomials of the \(\phi_{\alpha}\), \(\{\phi_{\alpha,i}\}_{i=1}^{3}\), and \(\{\phi_{i,j}\}_{i=1}^{3}\). The resulting expression for the quadratic square root is then a perfect square as described in the paper. The factor is precisely the one needed to obtain either \(\phi_{\alpha}\) or \(1 - \phi_{\alpha}\) as the solution depending on whether we use the plus or minus sign for the square root term.

But Theorem 2 gives us a set of eight polynomials that allow us to express the observed counts for any amount of correlation between the classifiers, not just the independent case. This transforms the square root term into a polynomial expression in the variables \(\phi_{\alpha}\), \(\{\phi_{\alpha,i}\}_{i=1}^{3}\), \(\{\phi_{i,j}\}_{i=1}^{3}\), \(\{\Gamma_{i,j,\ell}\}\), and \(\{\Gamma_{i,j,k,\ell}\}\). As described in the paper, the resulting factorization of \(b^2 - 4ac\) leads to this very general form under the square root operation,

\[
p_1(\phi_{\alpha}, \{\phi_{\alpha,i}\}_{i=1}^{3}, \{\phi_{i,j}\}_{i=1}^{3}, \{\Gamma_{i,j,\ell}\}, \{\Gamma_{i,j,k,\ell}\})^2 p_2(\phi_{\alpha}, \{\phi_{\alpha,i}\}_{i=1}^{3}, \{\phi_{i,j}\}_{i=1}^{3}, \{\Gamma_{i,j,\ell}\}, \{\Gamma_{i,j,k,\ell}\}). \tag{46}
\]

where \(p_1\) and \(p_2\) are irreducible polynomials in their arguments. \(p_2\) is an irreducible polynomial with 72 terms, monomials of the \(\{m_{\ell_1,\ell_2,\ell_3}\}\). Substitution of the full polynomial system for three classifiers transform this polynomial into an irreducible polynomial with 2,411 terms (it can be written as the sum of factorizable factors to make it more manageable). Since the irreducibility is due to the correlation terms, one can explore what transformations would make it reducible by making simplifying assumptions. We show some of these in Table 6. Some of the assumptions led to further factorization but still retained a large irreducible polynomial component. We must conclude that obtaining an integer ratio solution for correlated classifiers is rare and further theorems are needed to quantify this unlikely event. All of our experiments with the Penn ML benchmarks contained an irreducible square root in the independent solution to the prevalence.

| transformation | irreducible? | length |
|----------------|-------------|--------|
| all label pair correlations equal | yes | 1830 |
| all pair correlations equal | yes | 1206 |
| all 2-way and 3-way correlations equal | yes | 1122 |
| one label correlations zero | no | 2/516 |
| one label zero, other constant | no | 2/436 |
| all correlations equal | yes | 1039 |
| all correlations and accuracies equal | yes | 48 |

Table 6: Simplifying assumptions for unknown correlations and their effect on the irreducibility of the square root in the quadratic formula for the independent prevalence solution.

A.3.1 Addendum: the square root can be imaginary or lead to unphysical values

We did not discuss in the paper, because of space considerations, another mathematical possibility for the independent system solution. It could result in complex solutions or values that lie outside the (0,1) range. None of these solutions make any sense as values for the accuracies or prevalence since these must all lie within the (0,1) interval. Both of these cases were regularly encountered during the Penn ML Benchmark experiments whenever the classifiers were too correlated in their errors.
A.3.2 Addendum: Taylor expanding the independent solution about the unknown error correlations

Future work along this algebraic approach will focus on the obvious application of Theorem 1 and Theorem 2 - Taylor expand the independent solution about zero error correlations so that we can use the inconsistencies between independent polynomial solutions as a way of estimating the correlations themselves.

A.4 Proof of Theorem 3

Theorem 3, There exists a linear equation for a pair of scalar regressors that is independent of the ground truth values.

Consider two scalar regressors. We can write their error correlations as

\[ \epsilon_{i,i} = \sum_{d=1}^{D} (y_d - y_{d,i})^2 \]  \hspace{1cm} (47)

\[ \epsilon_{j,j} = \sum_{d=1}^{D} (y_d - y_{d,j})^2 \]  \hspace{1cm} (48)

\[ \epsilon_{i,j} = \sum_{d=1}^{D} (y_d - y_{d,i})(y_d - y_{d,j}). \]  \hspace{1cm} (49)

Abusing notation, we can see that these three equations contain three terms involving the unknown ground truth: \( y_{d,i}^2 \), \( y_{d,j} \), and \( y_{d,j} \). We can eliminate these three unknowns using the three equations above to obtain the required linear equation that is independent of ground truth,

\[ \epsilon_{i,i} + \epsilon_{j,j} - 2\epsilon_{i,j} = \sum_{d=1}^{D} y_{d,i}^2 + \sum_{d=1}^{D} y_{d,j}^2 - \sum_{d=1}^{D} y_{d,i}y_{d,j} \]  \hspace{1cm} (50)

A.5 Proof of Theorem 4, purely algebraic part

Theorem 4, Consistency of independent solutions for four regressors implies,

\[ \epsilon_{i,j} + \epsilon_{k,l} = \epsilon_{i,k} + \epsilon_{j,l}, \] \hspace{1cm} (51)

for all permutations of the four indices \( \{i, j, k, l\} \).

Consistency of four trios is equivalent to the following linear system of six equations for the six possible pairs for four regressors,

\[ \epsilon_{i,i} + \epsilon_{j,j} - 2\epsilon_{i,j} = \epsilon_{i,i}^{(\text{ind})} + \epsilon_{j,j}^{(\text{ind})} \] \hspace{1cm} (52)

\[ \epsilon_{i,i} + \epsilon_{k,k} - 2\epsilon_{i,k} = \epsilon_{i,i}^{(\text{ind})} + \epsilon_{k,k}^{(\text{ind})} \] \hspace{1cm} (53)

\[ \epsilon_{i,i} + \epsilon_{l,l} - 2\epsilon_{i,l} = \epsilon_{i,i}^{(\text{ind})} + \epsilon_{l,l}^{(\text{ind})} \] \hspace{1cm} (54)

\[ \epsilon_{j,j} + \epsilon_{k,k} - 2\epsilon_{j,k} = \epsilon_{j,j}^{(\text{ind})} + \epsilon_{k,k}^{(\text{ind})} \] \hspace{1cm} (55)

\[ \epsilon_{j,j} + \epsilon_{l,l} - 2\epsilon_{j,l} = \epsilon_{j,j}^{(\text{ind})} + \epsilon_{l,l}^{(\text{ind})} \] \hspace{1cm} (56)

\[ \epsilon_{k,k} + \epsilon_{l,l} - 2\epsilon_{k,l} = \epsilon_{k,k}^{(\text{ind})} + \epsilon_{l,l}^{(\text{ind})} \] \hspace{1cm} (57)

where the \( \epsilon_{i,i}^{(\text{ind})} \) are the consistent independent system solutions. Straightforward linear algebra leads to the elimination of all of the \( \epsilon_{i,i}^{(\text{ind})} \) as well as the \( \epsilon_{i,i} \). This yields the consistency constraint equations,

\[ \epsilon_{i,j} + \epsilon_{k,l} = \epsilon_{i,k} + \epsilon_{j,l} = \epsilon_{i,l} + \epsilon_{j,k} \] \hspace{1cm} (58)

A.6 Proof of Theorem 5

Theorem 5, The solutions to Theorem 4 are either all cross-correlations are equal or all the following data moments are zero

\[ \sum_{d=1}^{D} (y_{d,j} - y_{d,k})(y_{d,i} - y_{d,l}) = 0 \] \hspace{1cm} (59)
As shown in Theorem 4, for all \( \{i, j, k, l\} \) the consistency of independent solutions requires

\[
\epsilon_{i,j} + \epsilon_{k,l} = \epsilon_{i,k} + \epsilon_{j,l}.
\]

You can turn the above equation into a statement of data moments alone by using the definition of cross-correlation. All the terms in the cross-correlations involving the unknown ground truth cancel. The final result can be expressed solely in terms of the estimates by the four regressors as,

\[
\sum_d (y_{d,j} - y_{d,k})(y_{d,i} - y_{d,l}) = 0
\]

Therefore we can check, without knowledge of the ground truth, if all these relations hold for any permutation of the \( i, j, k, \) and \( l \). This is clearly a very particular condition for situations other than constant cross-correlations. Therefore, we must conclude that we are left with a situation similar to other theorems in the paper - consistency is possible with non-zero correlations but highly unlikely for most practical cases. This conclusion must be clarified by further theoretical analysis.

As we have pointed out in the paper, the case of sparsely correlated regressors is solvable by using Compressed Sensing algorithms so the solution for practical cases near independence already exists.

### A.6.1 The conjecture for binary classifiers

This approach can also be followed for the conjecture as it applies to binary classifiers. In that case we have 4 sets of eight polynomial equations each. On the LHS are the full polynomial systems of Theorem 2. On the RHS are the solutions obtained by pretending that the classifiers are independent. We have been unable to resolve the consequences of this system of 32 quartic equations using a brute-force application of Buchberger’s algorithm.

### B General comment about ground truth statistics yet to be described

It is clear by this algebraic formulation that we have shown how to derive a particular set of ground truth statistics. The two GTI problems discussed here are just exemplar of a similar algebraic approach that can be applied to other ground truth statistics. We mention one only - sequential error statistics. In applications such as DNA sequences, the accuracy of pairs or triplets of bases may be an important consideration for some task requiring high precision.

### C Details of the Penn ML Benchmarks experiments

Here we provide more detail about the Penn ML Benchmarks experiments and include the ground truth counts for each of the experiments. These counts are a sufficient statistic for all relevant quantities needed for an experiment. From these tables one can calculate any of the ground truth statistics we discussed in this paper - the prevalence, the label accuracies of the classifiers, and their sample error correlations up 2 to 4-way.

All experiments used the Mathematica v12.1.1.0 software package, a natural choice for us given the combination of algebraic and classification computations needed to carry out the theoretical and experimental work we have presented. Unless otherwise stated, we used default versions for the classifiers provided by Mathematica.

#### C.1 The two-norm experiment

The classifiers used were,

1. NeuralNetwork, with NetworkDepth = 5
2. GradientBoostedTrees
3. NaiveBayes
4. LogisticRegression

The ground truth counts are shown in Table 7.
Table 7: Observed decision event counts by true label for the ensemble of four classifiers in the \texttt{twonorm} exemplar experiment

C.2 The \texttt{spambase} experiment

The classifiers used were,

1. \texttt{NeuralNetwork}, with NetworkDepth = 4
2. \texttt{SupportVectorMachine}, with KernelType = Polynomial, and PolynomialDegree = 3
3. \texttt{DecisionTree}, with DistributionSmoothing = 5
4. \texttt{NaiveBayes}

The ground truth counts are shown in Table 8

Table 8: Observed decision event counts by true label for the ensemble of four classifiers in the \texttt{spambase} exemplar experiment
C.3 The mushroom experiment

The classifiers used were,

1. **DecisionTree**, with DistributionSmoothing = 5
2. **NaiveBayes**
3. **NeuralNetwork**, with NetworkDepth = 4
4. **SupportVectorMachine**, with KernelType = Polynomial, and PolynomialDegree = 3

The ground truth counts are shown in Table 9.

| decision event | 0 label | 1 label |
|----------------|---------|---------|
| {0,0,0,0}      | 2929    | 0       |
| {0,0,0,1}      | 75      | 0       |
| {0,0,1,0}      | 70      | 28      |
| {0,0,1,1}      | 45      | 266     |
| {0,1,0,0}      | 135     | 35      |
| {0,1,0,1}      | 0       | 0       |
| {0,1,1,0}      | 16      | 14      |
| {0,1,1,1}      | 42      | 174     |
| {1,0,0,0}      | 310     | 0       |
| {1,0,0,1}      | 5       | 0       |
| {1,0,1,0}      | 110     | 29      |
| {1,0,1,1}      | 10      | 106     |
| {1,1,0,0}      | 20      | 29      |
| {1,1,0,1}      | 0       | 129     |
| {1,1,1,0}      | 20      | 0       |
| {1,1,1,1}      | 0       | 2714    |

Table 9: Observed decision event counts by true label for the ensemble of four classifiers in the mushroom exemplar experiment