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Gap solitons in a dipolar Bose–Einstein condensate on a three-dimensional optical lattice

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Abstract

We suggest and study the stable disc- and cigar-shaped gap solitons of a dipolar Bose–Einstein condensate of ⁵²Cr atoms localized in the lowest band gap by three optical-lattice (OL) potentials along orthogonal directions. The one-dimensional version of these solitons of experimental interest confined by an OL along the dipole moment direction and harmonic traps in transverse directions is also considered. Important dynamics of (i) the breathing oscillation of a gap soliton upon perturbation and (ii) the dragging of a gap soliton by a moving lattice along the axial \( z \) direction demonstrates the stability of gap solitons. A movie clip (available at stacks.iop.org/JPhysB/44/121001/mmedia) of the dragging of the three-dimensional gap soliton is included.

Online supplementary data available from stacks.iop.org/JPhysB/44/121001/mmedia

(Some figures in this article are in colour only in the electronic version)
The controllable short-range interaction together with the dipolar interaction makes the DBEC an attractive system for experimental soliton generation and a challenging system for theoretical investigation. There have already been studies of bright solitons in the DBEC [17].

Using the numerical and variational solution of the GP equation, we predict and study stationary and dynamical properties of three-dimensional (3D) DBEC gap solitons, of both cigar and disc shapes, localized in the lowest band gap by three orthogonal OL potentials. The BEC gap solitons are only realizable for all repulsive atomic interactions above a maximum value so that the chemical potential falls in the band gap. The DBEC gap solitons in the disc shape can also be formed for the weakly attractive short-range interaction. The cigar-shaped DBEC gap solitons are formed only for short-range repulsion above a limiting value. We also consider the one-dimensional gap solitons of direct experimental interest confined by an OL along the dipole moment direction and harmonic traps in transverse directions.

We numerically explore the breathing oscillation of the gap solitons upon a small perturbation. We illustrate the dragging of the gap solitons by an OL moving along the axial z direction [18, 19]. These solitons can be dragged without deformation for a reasonably large velocity.

Here we study the DBEC gap soliton of N atoms, each of mass m, using the dimensionless GP equation [12]

\[
\frac{i}{\hbar}\frac{\partial \phi(r, t)}{\partial t} = -\frac{1}{2}\nabla^2 + V(r) + 4\pi aN|\phi(r, t)|^2 + N \int U_{dd}(r - r')|\phi(r', t)|^2 \, dr'.
\]

with the dipolar interaction \(U_{dd}(R) = 3a_{dd}(1 - 3\cos^2 \theta) / R^3\), \(R = r - r'\). Here \(V(r)\) is the confining potential, \(\phi(r, t)\) is the wavefunction at time \(t\) with the normalization \(\int |\phi(r, t)|^2 \, dr\) = 1, \(a\) is the atomic scattering length, and \(\theta\) is the angle between \(R\) and the polarization direction. The constant \(a_{dd} = \mu_0\beta^2m/(12\pi\hbar^2)\) is a length characterizing the strength of the dipolar interaction and its experimental value for \(^{52}\text{Cr}\) is 15\(\beta\alpha\) [12], with \(\alpha\) the Bohr radius, \(\beta\) the (magnetic) dipole moment of a single atom, and \(\mu_0\) the permeability of free space. The OL potential in a specific direction, say \(\zeta\), is \(V(\zeta) = s_\zeta E_\zeta \sin^2(2\beta_\zeta/\lambda)\), where \(E_\zeta = h^2/(2m\lambda^2)\) is the recoil energy of an atom, \(\lambda\) is the wavelength of the laser and \(s_\zeta\) is the strength of the OL. In (1), the length is measured in units of \(\lambda/2\pi\) (taken here as 1 \(\mu\)m for a far infrared laser), time \(t\) in units of \(m\lambda^2/2\pi\hbar\), and energy in units of \(h^2/m\lambda^2\).

The dimensionless 3D periodic OL trap can now be written as

\[
V(r) = -[V_\rho \cos(2x) + \cos(2y)] + V_z \cos(2z),
\]

where the parameters \(V_\rho\) and \(V_z\) are the strengths of the OLs in the radial and axial directions and can be varied to achieve the disc- (\(V_z > V_\rho\)) and cigar-shaped (\(V_\rho > V_z\)) gap solitons in the DBEC.

The gap solitons considered here predominantly have a Gaussian shape and for these solitons a Gaussian variational solution is known to lead to a good description [7]. The variational approximation provides an analytical understanding and also yields interesting results when the numerical procedure is difficult to implement. A better but more complicated description can be obtained with a different ansatz of variational functions [20]. The Lagrangian density of (1) is given by

\[
\mathcal{L} = \frac{i}{2}(\phi^*\dot{\phi} - \phi\dot{\phi}^*) + \frac{1}{2}(|\nabla \phi|^2 + V(r)|\phi|^2 + 2\pi aN|\phi|^4 + N^2|\phi|^2 \int U_{dd}(r - r')|\phi(r')|^2 \, dr'.
\]

We use the Gaussian ansatz [21]

\[
\phi(r, t) = \frac{\pi^{-3/4}}{w_\rho \sqrt{w_z}} \exp\left(-\frac{\rho^2}{2w_\rho^2} - \frac{z^2}{2w_z^2} + i\alpha\rho^2 + i\beta z^2\right)
\]

for a variational calculation, where \(w_\rho\) and \(w_z\) are the time-dependent radial and axial widths, and \(\alpha\) and \(\beta\) are the time-dependent phases. The effective Lagrangian \(\mathcal{L} = \int \mathcal{L} \, dr\) (per particle) becomes

\[
L = \left(w_\rho^2 \alpha + w_z^2 \beta / 2\right) - 2V_\rho \exp\left(-w_\rho^2\right) - V_z \exp\left(-w_z^2\right) + 1/(2w_\rho^2) + 1/(4w_z^2) + w_\rho^2\alpha^2 + w_z^2\beta^2 + N a_{dd} / \left(32\pi^2 w_\rho^2 w_z^2\right)\left[1/a_{add} - f(\kappa)\right],
\]

with

\[
f(\kappa) = \frac{1}{\sqrt{1 - \kappa^2}}, \quad d(\kappa) = \frac{\alpha + \beta}{\sqrt{1 - \kappa^2}}, \quad \kappa = \frac{w_\rho}{w_z}.
\]

The Euler–Lagrange equations for the parameters \(w_\rho\), \(w_z\), \(\alpha\) and \(\beta\) yield the following equations for widths \(w_\rho\) and \(w_z\):

\[
\frac{d}{dt}w_\rho = \frac{4V_\rho w_\rho}{\sqrt{2\pi}} - \frac{N}{2\sqrt{2\pi}} \left[2a - a_{dd}g(\kappa)\right],
\]

\[
\frac{d}{dt}w_z = \frac{4V_z w_z}{\sqrt{2\pi}} - \frac{N}{2\sqrt{2\pi}} \left[2a - a_{dd}h(\kappa)\right],
\]

with

\[
g(\kappa) = \frac{1}{\sqrt{1 - \kappa^2}}, \quad h(\kappa) = \frac{1}{\sqrt{1 - \kappa^2}}.
\]

The chemical potential \(\mu\) is given by

\[
\mu = \frac{1}{2w_\rho^2} + \frac{1}{4w_z^2} + \frac{2N\left[a - a_{dd}f(\kappa)\right]}{\sqrt{2\pi} w_\rho^2 w_z^2} - 2V_\rho \exp\left(-w_\rho^2\right) - V_z \exp\left(-w_z^2\right).
\]

For gap solitons the system must be repulsive. For a normal BEC (\(a_{dd} = 0\)), attraction corresponds to \(a < 0\) and repulsion to \(a > 0\) and gap solitons are formed below a limiting repulsive scattering length \(\kappa < a_{cr}\) for \(a > 0\) (\(a_{cr} > 0\), in the disc shape, the dipole moment contributes repulsively and due to this extra repulsion a DBEC gap soliton can be formed in a window of scattering lengths \(-a_1 < a < a_2\) between the limiting attractive (\(-a_1\)) and repulsive (\(a_2\)) limits. However, in a weak cigar shape, the dipole moment contributes attractively and due to the extra attraction a DBEC gap soliton...
Figure 1. Band (shaded area), gap (white area) and the numerical (N) and variational (V) $a-\mu$ plots of 3D DBEC gap solitons for $a_{dd} = 15a_0$, $N = 500$ and $V_r = 5$, (a) $V_z = 50$ (disc) and (b) $V_z = 4$ (cigar).

can be formed in a window of repulsive scattering lengths $a_3 < a < a_5$. The limiting values $a_1$, $a_2$, $a_3$ and $a_4$ can be obtained from a solution of the GP equation. In a strong cigar regime, the dipole moment contributes to strong attraction and no gap soliton can be formed due to collapse instability.

The cigar-shaped quasi-1D gap solitons with a radial harmonic trap of frequency $\Omega_r$ satisfy (1) with $V(r) = \Omega_r^2 \rho^2 / 2 - V_z \cos(2z)$. In this case, it is convenient to solve the 1D GP equation [22, 23] with the reduced dipolar interaction $U_{dd}^{1D}(Z) = 6\delta_{dd}[4\delta(\sqrt{r})/3 + 2\sqrt{r} - \sqrt{3}(1 + 2t)] \times e^{i(1 - \text{erf}[(\sqrt{r})]/(\sqrt{2}d_r)^2)}$, (13)

where $Z = |z - z'|$, $d_r = 1/\sqrt{\Omega_r}$, $t = [Z/(\sqrt{2}d_r)]^2$ and where we have included the proper $\delta$-function term mentioned in [22]. The 1D GP equation is given by

$$
\frac{\partial \phi(z, t)}{\partial t} = \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} - V(z) \cos(2z) + \frac{2aN}{d_r^2} |\phi(z, t)|^2 \right] \phi(z, t) + N \int U_{dd}^{1D}(Z) |\phi(z', t)|^2 d\rho' |\phi(z, t)|^2.
$$

Using the Gaussian ansatz

$$
\phi(z) = \frac{\pi^{-1/4}}{\sqrt{w_z}} \exp \left[ - \frac{z^2}{2w_z^2} + i\beta z^2 \right],
$$
the variational Lagrangian becomes

$$
L = \frac{w_z^2 \beta}{2} - V_z \exp \left[ - \frac{w_z^2}{2w_z^2} \right] + \frac{1}{4w_z^2} + w_z^2 \beta^2 + \frac{N a_{dd}}{\sqrt{2\pi d_r^2 w_z^2}} \left[ \frac{a}{a_{dd}} - f(\kappa_0) \right], \quad \kappa_0 = \frac{d_r}{w_z},
$$

(16)

The variational equation for $w_z$ becomes

$$
\ddot{w}_z + \frac{4V_z w_z}{\exp (w_z^2)} = \frac{1}{w_z^2} + \frac{2N[a - a_{dd} h(\kappa_0)]}{\sqrt{2\pi w_z^2 d_r^2}}.
$$

(17)

The corresponding chemical potential is

$$
\mu = \frac{1}{4w_z^2} + \frac{2N[a - a_{dd} h(\kappa_0)]}{\sqrt{2\pi w_z^2 d_r^2}} - \frac{V_z}{\exp (w_z^2)}.
$$

(18)

To study the gap solitons, we perform a 3D numerical simulation employing real-time propagation with the Crank–Nicolson method [24]. The numerical solutions are obtained by averaging the wavefunction over iterations with the variational solution as input. The dipolar interaction is evaluated by the fast Fourier transform [14].

The stationary gap solitons appear in the band gap of the OL and a prior knowledge of the band and gap of the periodic OL is of advantage. The dipolar interaction is most prominent in the cigar (attractive) and disc (repulsive) shapes and hence we study the gap solitons in these two shapes. The OL parameters in these cases are taken as (a) $V_r = 5$, $V_z = 50$ (disc shape) and (b) $V_r = 5$, $V_z = 4$ (cigar shape). The band and gap in these cases are shown in figure 1 together with the variational and numerical chemical potential $\mu$ for the stable gap solitons in the lowest band gap as a function of the scattering length $a$ for a DBEC ($a_{dd} = 15a_0$). The gap solitons in the higher band gaps are found to be dynamically unstable. In figure 1, the agreement between variational and numerical results worsens for large $a$. The system is more repulsive for large $a$ and tends to occupy multiple OL sites and a single-peak Gaussian variational ansatz may not provide a good approximation to reality. In the cigar shape ($V_z = 4$), the dipolar interaction is attractive and hence DBEC gap solitons are possible for $a \equiv a_3 > 0.1343$ nm as in figure 1(b). In the disc shape ($V_z = 50$), the dipolar interaction is repulsive, and hence DBEC gap solitons are possible for $a \equiv -a_1 > -0.2635$ nm as in figure 1(a).

In figure 2 we show the 3D contour density profiles of some typical stationary gap solitons with $N = 500$, $V_r = 5$, $a_{dd} = 15a_0$ for variable $a$ and $V_z$. We show disc-shaped profiles for $V_z = 50$ and (a) $a = 0.5$ and (b) $a = -0.1$ and cigar-shaped profiles for $V_z = 4$ and (c) $a = 1$ and (d) $a = 0.5$. For both disc and cigar shapes, the increased scattering length implies more nonlinear repulsion and one can have more atoms in adjacent OL sites. Although the gap solitons in figure 2 occupy multiple OL sites, more than 95% of the matter is contained in the central site and an approximate Gaussian distribution is valid.

Next we study 1D gap solitons trapped by the OL $-V_z \cos(2z)$ along the $z$ direction and harmonic trap $\Omega_r^2 \rho^2$ in transverse directions as calculated from numerical and variational solutions of the reduced 1D GP equation [22] with dipolar interaction [13]. In figure 3(a), we show the bands and
Figure 2. 3D contour density plot of gap solitons for $N = 500, V_x = 5, a_{dd} = 15a_0$. (a) $V_x = 50, a = 0.5$ (disc), (b) $V_x = 50, a = -0.1$ (disc), (c) $V_x = 4, a = 1$ (cigar), (d) $V_x = 4, a = 0.5$ (cigar). The density at contour is 0.002.

Figure 3. (a) Band (shaded area), gap (white area) and the numerical (N) and variational (V) $a-\mu$ plot of 1D DBEC gap solitons for $\Omega_x = 1$ (disc) and $\Omega_x = 4$ (cigar). Variational and numerical densities of the 1D DBEC for (b) $\Omega_x = 4, a = 0.75$ nm, (c) $\Omega_x = 4, a = 2.75$ nm, (d) $\Omega_x = 1, a = 1$ nm, and (e) $\Omega_x = 1, a = 3$ nm. In all cases $a_{dd} = 15a_0, N = 500$ and $V_x = 5$.

Gaps as well as the $a-\mu$ plot of the gap solitons. For $\Omega_x = 1$, the DBEC is of disc shape and the contribution of the dipolar term is repulsive, and for $\Omega_x = 4$, the DBEC is of cigar shape and the dipolar term contributes attractively. In figures 3(b)–(e), we plot the typical numerical 1D density profiles of some gap solitons and compare with variational results. Although variational results exist across the bands, stable DBEC 3D gap solitons exist in the lowest band gap away from the bands. In 1D some stable gap solitons are found in the first excited gap. Gap solitons cannot be stabilized close to the bands, e.g., near $\mu = -45$ and $-49$ in figure 1(a), near $\mu = -4.5$ and $-8$ in figure 1(b), and near $\mu = 1$ and $\mu = 4$ in figure 3. In figures 1(a) and 3(a), we clearly see that gap solitons are possible for small negative scattering lengths in the disc-shaped DBEC and that, in the second band gap, 1D gap solitons can be stabilized only in a small domain of $\mu$ values between two bands.

To test the stability of the 3D DBEC gap solitons, we now consider the breathing oscillation of a disc-shaped soliton initiated by slightly changing the scattering length $a$. Such a change in $a$ can be made by varying a background magnetic field near a Feshbach resonance [13]. In figure 4, we plot the variational and numerical rms sizes $\langle \rho \rangle, \langle z \rangle$ versus time $t$. We find, using Fourier analysis, that the principal axial frequencies $1.0137$ (variational) and $1.0812$ (numerical) and the principal radial frequencies $0.2421$ (variational) and $0.2469$ (numerical) are in good agreement with each other.

Finally, we study the stability of a cigar-shaped 3D DBEC when dragged by an OL moving in the axial $z$ direction. It has been experimentally found that a BEC confined by a transverse harmonic trap remains stable [19] when dragged by a moving 1D OL along the axial $z$ direction below a critical velocity $v_c$ of half the recoil velocity $v_R = h/(m\lambda)$, which in present units ($\lambda = 2\pi$ and $m = \hbar = 1$) is $v_c = v_R/2 = 0.5$. The similar result is found to be true in the case of present 3D DBEC gap solitons. Steady dragging is possible in this case even for velocities slightly larger than
Figure 4. Numerical (num) and variational (var) rms sizes $\langle \rho \rangle$, $\langle z \rangle$ during the breathing oscillation of a disc-shaped 3D DBEC for $N = 500$, $a_{ad} = 15a_0$, $V_c = 50$, $V_p = 5$ as $a$ is changed from 0 to 0.1.

Figure 5. Snapshots of 3D contour density profiles of a cigar-shaped gap soliton for $V_z = 4$, $V_p = 5$, $N = 500$, $a_{ad} = 15a_0$, and $a = 0.5$ nm during dragging by the moving OL $-V_c[\cos(2x) + \cos(2y)] - 2V_c \cos(2(z - vt))]$, $v = 0.75$ at times $t = 0, 4, 8, 12, 16$ and 20.

To conclude, we suggested the possibility of 3D DBEC gap solitons of about 1000 Cr atoms confined in the lowest band gap by three OL in orthogonal directions and studied their statics (shape and chemical potential) and dynamics (breathing oscillation and dragging by an OL). In addition, we studied 1D DBEC gap solitons using reduced 1D GP equations with a transverse harmonic trap and an axial OL along the polarization direction. The 3D DBEC gap solitons are found to be dynamically stable during breathing oscillation and dragging for a long enough time for experiments and, with available technology, these solitons could be created and studied in laboratory. The present study opens up new directions of research that include, among others, excited states of gap solitons [4] and vortex gap solitons [25] in the DBEC.

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