A MACROSCOPICALLY FRUSTRATED ISING MODEL

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Abstract

A disordered spin glass model where both static and dynamical properties depend on macroscopic magnetizations is presented. These magnetizations interact via random couplings and, therefore, the typical quenched realization of the system exhibit a macroscopic frustration. The model is solved by using a revisited replica approach, and the broken symmetry solution turns out to coincide with the symmetric solution. Some dynamical aspects of the model are also discussed, showing how it could be a useful tool for describing some properties of real systems as, for example, natural ecosystems or human social systems.
I. INTRODUCTION

Macroscopic frustration can be found in different domains, from interpersonal relationships to granular matter or natural ecosystems. All these systems are characterized by frustrated components with a thermodynamically macroscopic size. In other words, in all these systems, there are components whose size is comparable with that of the whole system and which underly to the action of opposite forces. The classical example is the case of a man A who desires to spend some time with a dear friend B, which, unfortunately, wants to bring his wife C, which is really detested by A.

Dozen of examples can be found in nature. Consider the antler of a deer, it is known that it represents a frustrated phenotype. In fact, sexual selection tends to prefer its growth in order to increment the chance of reproduction but antler is an obstacle in some situations, such a predator pursuit in a forest and, therefore, natural selection pressure is for its reduction.

From a more strict physical point of view, systems which exhibit frustration are very common (see [1] for a general view). For a disordered spin system Toulouse [2] has introduced the definition of frustration for an elementary plaquette of bonds, consisting in the product of the corresponding couplings. Nevertheless, systems where frustration appears on macroscopic scales are less ordinary and not yet investigated as far as we know.

In this paper we present a spin glass model where spins are organized in macroscopic sets, with the corresponding macroscopic magnetizations interacting via random couplings. For a typical random realization of the couplings, the system is an ensemble of interacting frustrated macroscopic entities and, therefore, it could be a natural candidate for mathematical modeling of phenomena where macroscopic frustration plays a central role.

Let us briefly sum up the contents of the paper.

In sect. II the model is introduced. The model becomes self-averaging when the number of components is large, nevertheless, some considerations about its finite size version are also written down.
In sect. III we look for a solution of the model using a revisited version of the replica trick. This revised version could be applied in a more general context to a large class of models, as it will be explained.

In sect. IV and in sect. V, respectively, the replica symmetry solution and the broken symmetry solution a la Parisi are derived in detail. The two solutions turn out to coincide, vanishing the benefits that the Parisi ansatz has in other spin glass models.

In sect. VI the symmetric solutions is studied in detail from a numerical point of view showing that, at variance with the S-K model, it keeps its physical meaning even at very low temperature.

In sect. VII some final remarks are contained, in particular some dynamical aspects are illustrated. Dynamics could be a profitable argument of future investigations especially for its possible applications to ecosystems and natural selection modeling.

II. THE MODEL

Let us consider an Hamiltonian where \( N \) spins are divided in \( L \) sets, each set consisting of exactly \( M = N/L \) spins. Each spin interacts with all other spins, but the coupling does not depend on the sites of the spins, but only on the sets of the spins involved. In other words, two spins of different sets interact via a coupling which depends only on the coordinates of the two sets of membership. Then, we can speak of coupling between sets rather than between spins. We also assume that spins of the same set do not interact.

This Hamiltonian can be written as

\[
H_{M,L}(J, \sigma) = -\frac{1}{M\sqrt{L}} \sum_{k>l} J_{k,l} \sigma_k \sigma_l ,
\]

where \( J \) is a \( N \times N \) symmetric matrix, consisting of \( L^2 \) blocks of \( M^2 \) entries each, being \( M \) the linear size of a block. All the \( M^2 \) entries of a given block take the same value and, in particular, the diagonal blocks consist of null entries. The free energy of the system is
\[ f_{M,L}(J) = -\frac{1}{\beta ML} \ln \sum_{\{\sigma\}} \exp \left[ -\beta H_{M,L}(J,\sigma) \right] , \] (2)

where the sum is intended over all the spin configurations.

The thermodynamic limit \( N \to \infty \) can be obtained in two different ways since \( N \) is the product of two variables \( (N = LM) \). In fact, the limit \( L \to \infty \) would mean to consider a system whose properties and characteristics are the same of those of the S-K model [3]. On the contrary, the limit \( M \to \infty \) leads to a mean field model with a macroscopic frustration. The self-average properties are obtained by also performing the limit \( L \to \infty \) after the limit \( M \to \infty \). Nevertheless, non self-averaging macroscopic frustration also is exhibited for finite \( L \) as we will show later with an example.

We thus perform the limit \( M \to \infty \), keeping \( L \) finite. After some algebra, the free energy reads

\[ f_L(J) = -\frac{1}{\beta L} \max_m \Gamma(J,m) , \]

where \( m = (m_1, \ldots, m_L) \), having defined the \( i \)th set magnetization \( m_i \) as

\[ m_i = \lim_{M \to \infty} \frac{1}{M} \sum_{k \in \text{ith set}} \sigma_k , \]

and where

\[ \Gamma(J,m) = \frac{\beta}{\sqrt{L}} \sum_{i,j} J_{i,j} m_i m_j + \sum_i \Phi(m_i) . \] (3)

The indices \( i \) and \( j \) run over the spin sets, and \( J \) is now a symmetric \( L \times L \) matrix, obtained from matrix in (1) substituting each block with a single entry, being \( J_{i,j} \) the value of the coupling connecting a spin of set \( i \) with a spin of set \( j \), with \( J_{i,i} = 0 \ \forall i \). Furthermore, \( \Phi(m_i) \) represents the entropic term of spin set \( i \)

\[ \Phi(m_i) = -\frac{1 + m_i}{2} \ln \frac{1 + m_i}{2} - \frac{1 - m_i}{2} \ln \frac{1 - m_i}{2} . \]

Let us suppose that the non diagonal elements of \( J \) are independent identically distributed random quenched variables. For the sake of simplicity, we restrict ourselves to consider normal Gaussian variables with vanishing average and unitary variance. Our aim is to compute the quenched free energy
\[ f = \lim_{L \to \infty} f_L(J) = -\lim_{L \to \infty} \frac{1}{\beta L} \max_m \Gamma(J, m), \quad (4) \]

where the last equality is due to the self-averaging property of the free energy which holds in the large \( L \) limit. The max in (4) is reached for \( m^* = (m^*_1, \ldots, m^*_L) \), which obey to the following \( L \) self-consistent equations

\[ m^*_i = \tanh \left[ \frac{\beta}{\sqrt{L}} \sum_j J_{i,j} m^*_j \right], \quad 1 \leq i \leq L. \quad (5) \]

We consider the large \( L \) limit, because we have in mind a system with many macroscopic frustrated components, nevertheless the glassy characteristics (except self-averaging) can be also found for finite \( L \). Consider, for instance \( L = 3 \) with the product of the three couplings with negative sign. At low temperature (temperature below transition, not vanishing!) the system is degenerated since it has six different pure states with the same free energy and with non trivial and non all equal values of the three magnetizations involved.

When \( L \) increases, frustration increases and also the number of pure states corresponding to the same free energy. We hope to find in this way an interesting spin glass model with new peculiarities.

**III. REPLICA TRICK REVISITED**

In order to perform the limit \( L \to \infty \) we need to compute \( \max_m \Gamma(J, m) \). We will accomplish this task by means of replica trick with a slight but crucial variant. Let us stress from the beginning that this way of applying replica trick is not restricted to our model, but it is more general and, in principle, could be of some help in solving many other models with macroscopic variables. In fact, what we propose here is a useful technique for computing quantities of the type \( \max_m \Gamma(J, m) \), i.e. an average whose argument is a maximum over an expression which depends on random variables \((J)\) and on variables to be maximized \((m)\).

It is easy to check that

\[ \max_m \Gamma(J, m) = \lim_{\mu \to \infty} \lim_{n \to \infty} \frac{1}{\mu n} \ln \left[ \int d\mathbf{m} \exp(\mu \Gamma(J, \mathbf{m})) \right]^n, \quad (6) \]
where \( dm = \prod_i dm_i \). In fact, after having performed the limit \( n \to 0 \) as in ordinary replica trick in right hand side of (6), the saddle point method allows to compute the limit \( \mu \to \infty \), giving equality (6). The variable \( \mu \) is here only an auxiliary one.

Making explicit the \( n \) replicas, the average in right hand side of (6) can be written as

\[
\left[ \int dm \exp (\mu \Gamma(J, m)) \right]^n = \int \prod_\alpha dm^\alpha \exp G_n(\mu, m^1, \ldots, m^n) ,
\]

having defined

\[
G_n(\mu, m^1, \ldots, m^n) \equiv \ln \exp \sum_\alpha \mu \Gamma(J, m^\alpha) ,
\]

where the index \( \alpha \) runs over the \( n \) replicas. Finally this leads to the following expression for the free energy \( f \)

\[
f = - \lim_{L \to \infty} \lim_{\mu \to \infty} \lim_{n \to 0} \frac{1}{\beta L \mu n} \ln \int \prod_\alpha dm^\alpha \exp G_n(\mu, m^1, \ldots, m^n) . \tag{7}
\]

In our case, taking in mind equation (3), we can give an explicit expression for \( G_n \). For the sake of simplicity we do not write in the following the argument of \( G_n \). After have taking the averages over the Gaussian \( J \) variables, and after some algebra, one has

\[
G_n = \frac{\mu^2 \beta^2}{4L} \sum_{\alpha,\alpha'} (\sum_i m_i^\alpha m_i^{\alpha'})^2 + \mu \sum_{i,\alpha} \Phi(m_i^\alpha) ,
\]

where \( \alpha \) and \( \alpha' \) run over the replicas, and where terms not diverging with \( L \) have been neglected since they would disappear in the successive limit \( L \to \infty \). By means of the parabolic maximum trick, the above expression can be rewritten as

\[
G_n = \max_{\{q_{\alpha,\alpha'}\}} \left[ \frac{\mu^2 \beta^2}{2} \sum_{\alpha,\alpha'} (q_{\alpha,\alpha'} \sum_i m_i^\alpha m_i^{\alpha'} - \frac{L}{2} q_{\alpha,\alpha'}^2) + \mu \sum_{i,\alpha} \Phi(m_i^\alpha) \right] ,
\]

where \( \{q_{\alpha,\alpha'}\} \) is a \( n \times n \) matrix, which represents from a physical point of view the overlap between replicas in spin glass theory.

Now the integral in (7) can be fully factorized among the different spin sets, individuated by the index \( i \). This fact allows us to perform the limit \( L \to \infty \) which gives the final expression for the free energy in the replica context:
\[ f = - \max_{\{q_{\alpha,\alpha'}\}} \lim_{\mu \to \infty} \lim_{n \to 0} \beta \mu n \ln \int \prod_{\alpha} dm^\alpha \exp \tilde{G}_n \] (8)

where

\[ \tilde{G}_n = \frac{\mu^2 \beta^2}{2} \sum_{\alpha,\alpha'} \left( q_{\alpha,\alpha'} m^\alpha m^{\alpha'} - \frac{1}{2} q_{\alpha,\alpha'}^2 \right) + \mu \sum_\alpha \Phi(m^\alpha) , \]

and now \( m_1, \ldots, m_n \) are \( n \) replicas of a scalar magnetization. Notice that interchange of the position between the \( \max_{\{q_{\alpha,\alpha'}\}} \) and the integration is allowed since in the limit \( L \to \infty \) this maximum corresponds to a saddle point approximation of an integration with respect to the same variables \( \{q_{\alpha,\alpha'}\} \).

**IV. REPLICA SYMMETRIC SOLUTION**

In order to find a solution, i.e. to compute the quenched free energy (8), we start by trying the usual symmetry unbroken strategy. Let us stress that the diagonal terms of matrix \( q \) are relevant for this model, at variance with the celebrated replica solution of the S-K model [3]. Therefore, in spite of assuming that the diagonal terms vanish as in symmetry unbroken solution of S-K, we assume

\[ q_{\alpha,\alpha'} = q + \frac{x}{\beta \mu} \delta_{\alpha,\alpha'} , \]

where \( \delta_{\alpha,\alpha'} \) is the Kroenkeker delta. Notice that elements on the diagonal differ only for a quantity of the order of \( \mu^{-1} \) from the other entries, otherwise one would have diverging terms in the limit \( \mu \to \infty \). This fact implies that overlap turns out to be a constant only once the limit \( \mu \to \infty \) has been performed. With this choice one gets

\[ \tilde{G}_n = \left[ \frac{\mu^2 \beta^2}{2} q \left( \sum_\alpha m^\alpha \right)^2 + \frac{\mu \beta x}{2} \sum_\alpha \left( (m^\alpha)^2 - q \right) \right] + \mu \sum_\alpha \Phi(m^\alpha) , \]

where terms which vanish in the two limits \( n \to 0 \) and \( \mu \to \infty \) have been neglected. By means of the standard Gaussian trick we have

\[ \exp \left[ \frac{\mu^2 \beta^2}{2} q \left( \sum_\alpha m^\alpha \right)^2 \right] = \exp \left( \mu \beta x \sqrt{q} \sum_\alpha m^\alpha \right) , \]

(9)
where the average $\langle \rangle$ is on an independent normal Gaussian variable $\omega$. The above equality allows for writing

\[
\exp \tilde{G}_n = \left\langle \prod_{\alpha} \exp \left[ \mu_\beta \omega \sqrt{q m^\alpha} + \frac{\mu_\beta x}{2} \left( (m^\alpha)^2 - q \right) + \mu \Phi(m^\alpha) \right] \right\rangle .
\]

Notice that the argument inside the $\langle \rangle$ average in the previous expression is fully factorized among the $n$ replicas. For this reason the integral in (8) becomes the $n$th power of a single integral, and therefore the limit $n \to 0$ can be performed:

\[
f = - \max_{q,x} \lim_{\mu \to \infty} \frac{1}{\mu} \left\langle \ln \int dm \exp \left[ \mu_\beta \omega \sqrt{q m} + \frac{\mu_\beta x}{2} (m^2 - q) + \mu \Phi(m) \right] \right\rangle .
\]

Finally, the limit $\mu \to \infty$ can be performed by means of the saddle point technique, obtaining

\[
f = - \max_{q,x} \left\langle \max_m \left[ \omega \sqrt{q m} + \frac{x}{2} (m^2 - q) + \frac{\Phi(m)}{\beta} \right] \right\rangle .
\]

(10)

Let us stress once again the important role played by the small symmetry breaking (non vanishing $x$) introduced in the overlap. In fact, if we fix $x = 0$ choosing in this way a pure unbroken solution, the extremization with respect to $q$ would be impossible, since the argument in (10) would diverge for $q \to \infty$. It also should be noticed that at least one of the maximum with respect to $q$ and $x$ could has become a minimum after having performed the limit $n \to 0$.

V. FAILURE OF BREAKING

Trying to apply the ordinary approach to spin glass models, the following step consists in introducing an asymmetry in the overlap matrix. Assume now that

\[
q_{\alpha,\alpha'} = q + \frac{x}{\beta \mu} \delta_{\alpha,\alpha'} + \frac{y}{\beta \mu} \gamma_{\alpha,\alpha'} .
\]

Following Parisi parameterization [4–8], $\gamma_{\alpha,\alpha'}$ is a matrix whose entries vanish except in $n/l$ quadratic blocks of $l^2$ elements along the diagonal, where all entries are equal one. Notice that we have made explicit once again a factor $\mu^{-1}$, otherwise we would have divergent terms. In this case the maximum has to be taken with respect to $q,x,y$ and $l$.
With this ansatz and neglecting terms vanishing in the successive limits $n \to 0$ and $\mu \to \infty$, $\tilde{G}_n$ turns out to be

$$
\tilde{G}_n = \frac{\mu^2 \beta^2 q}{2} \left( \sum_{\alpha} m^\alpha \right)^2 + \frac{\mu \beta x}{2} \sum_{\alpha} \left( (m^\alpha)^2 - q \right) + \frac{\mu \beta y}{2} \sum_{k} \left[ \left( \sum_{\alpha \in k} m^\alpha \right)^2 - q l^2 \right] + \mu \sum_{\alpha} \Phi(m^\alpha),
$$

where the index $k$ runs over the $n/l$ blocks on the diagonal of $\gamma_{\alpha,\alpha'}$ and the sum on $\alpha \in k$ goes on the $l$ values of $\alpha$ corresponding to the $k$th block.

By means of the parabolic maximum trick it is possible to write

$$
\left[ \frac{\mu \beta y}{2} \left( \sum_{\alpha \in k} m^\alpha \right)^2 \right] = \max_{\rho_k} \left[ \sqrt{\mu \beta y} \rho_k \sum_{\alpha \in k} m^\alpha - \frac{\rho_k^2}{2} \right].
$$

In this way, repeating also the trick in (9), we have factorized $\tilde{G}_n$ with respect to the $n/l$ blocks, and, therefore, the limit $n \to 0$ can be performed. One gets

$$
f = - \max_{q,x,y,l} \lim_{\mu \to \infty} \frac{1}{\mu \beta l} \left( \ln \int \prod_{\alpha} dm^\alpha \max_{\rho} \tilde{G}_n \right) , \quad (11)
$$

with

$$
\tilde{G}_n = \sum_{\alpha} \left[ \mu \beta \omega \sqrt{q} m^\alpha + \frac{\mu \beta x}{2} \left( (m^\alpha)^2 - q \right) + \sqrt{\mu \beta y} \rho \sum_{\alpha \in k} m^\alpha - \frac{\rho^2}{2l} - \frac{\mu \beta y}{2} q l + \mu \Phi(m^\alpha) \right],
$$

where now the index $\alpha$ runs over only a single block, whose corresponds the scalar variable $\rho$, and where $\langle \rangle$ means the average over the normal Gaussian $\omega$.

The max$_\rho$ in (11) can be put outside the integration. This change is allowed and can be understood by the same argument used after equation (8). As a consequence, the integral in the previous expression is factorized among the $l$ replicas of a block, and reduces to a single integral because of the factor $l$ in the denominator. Moreover, this integral can be computed by means of the saddle point method in the limit $\mu \to \infty$, obtaining

$$
f = - \max_{q,x,y,l} \lim_{\mu \to \infty} \frac{1}{\mu \beta} \left( \max_{\rho,m} \left[ \mu \beta \omega \sqrt{q} m + \frac{\mu \beta x}{2} (m^2 - q) + \sqrt{\mu \beta y} \rho m - \rho^2 \frac{l}{2} - \frac{\mu \beta y}{2} q l + \mu \Phi(m) \right] \right). \quad (11)
$$

The maximum with respect to $\rho$ can be computed, and then performing the limit $\mu \to \infty$ one finally has

$$
f = - \max_{q,x,y,l} \left( \max_{m} \left[ \omega \sqrt{q} m + \frac{x + y l}{2} (m^2 - q) + \mu \Phi(m) \right] \right). \quad (11)
$$
Unfortunately, this final result is exactly the same of the unbroken case (10), the only difference being that the variable $x$ is substituted by $x + yl$, which is irrelevant when the maximum is taken.

This result could imply that the model simply has a constant overlap which depends only on the temperature; otherwise one should admit that the Parisi ansatz for replica symmetry breaking is inappropriate in this context.

VI. UNDERSTANDING REPLICA SYMMETRIC SOLUTION

The unlucky result of the replica broken solution allows us to suppose that the symmetric solution (10) could be the exact solution of the model. For this reason we have to study it in detail in order to get more evidences for supporting this hypothesis.

The extremization with respect to $q$, $x$ and $m$ (this last inside the average and, therefore, for any different $\omega$) leads to a system of self-consistent equations:

$$m_\omega = \tanh(\beta \sqrt{q} \omega + \beta x m_\omega)$$

$$q = \langle m_\omega^2 \rangle$$

$$x = \frac{1}{\sqrt{q}} \langle \omega m_\omega \rangle$$

(12)

this system of equations is solved by $q^*$, $x^*$ and $m^*_\omega$ and the free energy may be written as

$$f = -x^* q^* - \frac{1}{\beta} \langle \Phi(m^*_\omega) \rangle$$

Let us stress that $q^*$ corresponds to a maximum with respect to $q$ while the limit $n \to 0$ has transformed $x^*$ in a minimum with respect to $x$.

For a given $\omega$, the first equations (12) which refer to the $m_\omega$ could have a single solution (a maximum) or three different solutions, depending on the temperature. At low temperature we have a single solution for $\omega \gtrsim x/\sqrt{q}$ and three solutions for $\omega \lesssim x/\sqrt{q}$. Two of these correspond to a maximum and the third to a minimum, and this introduce an element of uncertainty. We follow the rule of taking the solution $m^*_\omega$ of the first equation which corresponds to the larger of the two maxima for every given $\omega$. 
In fig. 1 we plot the overlap $q^*$ and the parameter $x^*$ as functions of the temperature $T \equiv 1/\beta$. The spin glass transition occurs at the critical temperature $T_c = 2$, the same of the S-K model.

In fig. 2 the free energy $f$ and the entropy $S = \langle \Phi(m^*_\omega) \rangle$ are plotted as functions of the temperature $T$. At $T = 0$ the free energy is $f_0 = \sqrt{2/\pi} \simeq 0.798$, which is very close to the value of the S-K symmetric solution. On the contrary, the entropy simply vanishes at $T = 0$ at variance with the S-K case, where the negative entropy proves the unphysical nature of the solution in that case.

Let us stress that how to take the right extreme point with respect to $q$, $x$ and $m$ is a crucial step of the solution, and our choice, previously described, could be inappropriate. In fact, with the limit $n \to 0$ the maximum with respect to $x$ has become a minimum, and this could also has happened for some of the $m_\omega$. In this case one should look for the minimum with respect to the $m_\omega$ (or for the second maximum), at least for a subset of $\omega$. Indeed, at this stage, we are not able to give a sure answer on this point, which should be argument of future deep investigations.

**VII. CONCLUSIONS**

A dynamical approach to our spin glass model could be of some help in deciding for the correct solution. Following equation (5), the deterministic dynamics of $L$ magnetizations is

$$m_i(t + 1) = \tanh \left[ \frac{\beta}{\sqrt{L}} \sum_j J_{i,j} m_j(t) \right] \quad 1 \leq i \leq L .$$

Let us remind that matrix $J$ has vanishing diagonal entries $J_{i,i} = 0$, so that at each step the new value of the individual magnetization $m_i$ does not depend on its previous one. The above dynamics takes advantage of peculiar features. For instance, at each updating it moves $m_i$ in a value corresponding to a minimum free energy with respect to $m_i$ itself keeping fixed the other magnetizations. Moreover, the free energy always decreases at each updating of a single magnetization.
The dynamics makes the system evolve toward a fixed point, which is a relative minimum of the free energy (not, in general, a global minimum). Repeating many times this evolution, starting from different initial values for the magnetizations, allows to find the global minimum corresponding to the solution of the static spin glass model. Preliminary results seem to suggest that the theoretical symmetric solution of section VI is slightly different from dynamic solution only for very low temperatures. This not necessarily implies that the symmetric solution is not the correct one. In fact, in order to avoid finite size effects, one has to deal with large lattices (large \( L \)) in numerical simulations, so that the basin of attraction of the global minimum tends reasonably to become so small that one never uses correct initial conditions in spite of the large number of attempts.

The above mentioned features make such a dynamics for magnetization versatile and very fast from a numerical point of view. Furthermore, not only it is useful for understanding the associated static model, but it is also interesting in itself. In fact, it describes a dynamical system which monotonically relaxes towards a stable point corresponding to a local minimum of the free energy.

For this reason it is the ideal candidate for modeling some complex systems, such as natural ecosystems, where each agent or species try to maximize its own fitness in a given context of other active agents. The fitness corresponds to the individual free energy with changed sign (the part of the free energy which depends on a given magnetization \( m_i \)), and the magnetization \( m_i \) to the species degree of specialization. The individual attempts to improve its own condition and it happens to push the whole systems to maximize the total fitness. This is the very peculiar feature of many real systems which is reproduced by our dynamical model, which also exhibit other realistic peculiarities, such as the fact that the phase space is a landscape of a large number of local maxima of the fitness at low temperature. In case of catastrophe (even a small change of the couplings) the system is not anymore in a state of maximal fitness and the evolution restarts towards a different local maximum (a new period of stability in evolution story), which is not necessary higher than the previous.
In conclusion, this model seems to be very versatile, since its dynamics could become both a powerful benchmark where to test general hypothesis about spin glasses, and a paradigmatic model for evolving complex systems.

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FIG. 1. Overlap $q^*$ and parameter $x^*$ as functions of temperature $T$ for the symmetric solution. The critical temperature below that we have a spin glass phase ($q^* > 0$) turns out to be $T_c = 2$.

FIG. 2. Free energy $f$ and entropy $S$ as functions of temperature $T$ for the symmetric solution. In the limit $T \to 0$ the solution keeps a physical meaning since the entropy never becomes negative.