On the homogeneity of global minimizers for the Mumford-Shah functional when $K$ is a smooth cone.

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Abstract. We show that if $(u, K)$ is a global minimizer for the Mumford-Shah functional in $\mathbb{R}^N$, and if $K$ is a smooth enough cone, then (modulo constants) $u$ is a homogenous function of degree $\frac{1}{2}$. We deduce some applications in $\mathbb{R}^3$ as for instance that an angular sector cannot be the singular set of a global minimizer, that if $K$ is a half-plane then $u$ is the corresponding cracktip function of two variables, or that if $K$ is a cone that meets $S^2$ with an union of $C^\infty$ curvilinear convex polygones, then it is a $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$.

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Introduction

The functional of D. Mumford and J. Shah [18] was introduced to solve an image segmentation problem. If $\Omega$ is an open subset of $\mathbb{R}^2$, for example a rectangle, and $g \in L^\infty(\Omega)$ is an image, one can get a segmentation by minimizing

$$J(K, u) := \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_{\Omega \setminus K} (u - g)^2 dx + H^1(K)$$

over all the admissible pairs $(u, K) \in A$ defined by

$$A := \{(u, K); \ K \subset \Omega \text{ is closed} , \ u \in W^{1,2}_{\text{loc}}(\Omega \setminus K)\}.$$  

Any solution $(u, K)$ that minimizes $J$ represents a “smoother” version of the image and the set $K$ represents the edges of the image.

Existence of minimizers is a well known result (see for instance [11]) using SBV theory.

The question of regularity for the singular set $K$ of a minimizer is more difficult. The following conjecture is currently still open.

**Conjecture 1 (Mumford-Shah).** [18] Let $(u, K)$ be a reduced minimizer for the functional $J$. Then $K$ is the finite union of $C^1$ arcs.

The term “reduced” just means that we cannot find another pair $(\tilde{u}, \tilde{K})$ such that $K \subset \tilde{K}$ and $\tilde{u}$ is an extension of $u$ in $\Omega \setminus \tilde{K}$.

Some partial results are true for the conjecture. For instance it is known that $K$ is $C^1$ almost everywhere (see [7], [4] and [2]). The closest to the conjecture is probably the result of A. Bonnet [4]. He proves that if $(u, K)$ is a minimizer, then every isolated connected component of $K$ is a finite union of $C^1$-arcs. The approach of A. Bonnet is to use blow up limits. If $(u, K)$ is a minimizer in $\Omega$ and $y$ is a fixed point, consider the sequences $(u_k, K_k)$ defined by

$$u_k(x) = \frac{1}{\sqrt{t_k}} u(y + t_k x), \quad K_k = \frac{1}{t_k} (K - y), \quad \Omega_k = \frac{1}{t_k} (\Omega - y).$$

When $\{t_k\}$ tends to infinity, the sequence $(u_k, K_k)$ may tend to a pair $(u_\infty, K_\infty)$, and then $(u_\infty, K_\infty)$ is called a Global Minimizer. Moreover, A. Bonnet proves that if $K_\infty$ is connected, then $(u_\infty, K_\infty)$ is one of the list below:

- **1st case:** $K_\infty = \emptyset$ and $u_\infty$ is a constant.
- **2nd case:** $K_\infty$ is a line and $u_\infty$ is locally constant.
3rd case: “Propeller”: \( K_\infty \) is the union of 3 half-lines meeting with 120 degrees and \( u_\infty \) is locally constant.

4th case: “Cracktip”: \( K_\infty = \{(x, 0); x \leq 0\} \) and \( u_\infty (r \cos(\theta), r \sin(\theta)) = \pm \sqrt{\frac{2}{\pi}} r^{1/2} \sin \frac{\theta}{2} + C \), for \( r > 0 \) and \( |\theta| < \pi \) (\( C \) is a constant), or a similar pair obtained by translation and rotation.

We don’t know whether the list is complete without the hypothesis that \( K_\infty \) is connected. This would give a positive answer to the Mumford-Shah conjecture.

The Mumford-Shah functional was initially given in dimension 2 but there is no restriction to define Minimizers for the analogous functional in \( \mathbb{R}^N \). Then we can also do some blow-up limits and try to think about what should be a global minimizer in \( \mathbb{R}^N \). Almost nothing is known in this direction and this paper can be seen as a very preliminary step. Let state some definitions.

Definition 2. Let \( \Omega \subset \mathbb{R}^N \), \((u, K) \in \mathcal{A}\) and \( B \) be a ball such that \( B \subset \Omega \). A competitor for the pair \((u, K)\) in the ball \( B \) is a pair \((v, L) \in \mathcal{A}\) such that
\[
\begin{align*}
u &= v \\
K &= L
\end{align*}
\]
in \( \Omega \setminus B \)
and in addition if \( x \) and \( y \) are two points in \( \Omega \setminus (B \cup K) \) that are separated by \( K \) then they are also separated by \( L \).

The expression “be separated by \( K \)” means that \( x \) and \( y \) lie in different connected components of \( \Omega \setminus K \).

Definition 3. A global minimizer in \( \mathbb{R}^N \) is a pair \((u, K) \in \mathcal{A}\) (with \( \Omega = \mathbb{R}^N \)) such that for every ball \( B \) in \( \mathbb{R}^N \) and every competitor \((v, L)\) in \( B \) we have
\[
\int_{B \setminus K} |\nabla u|^2 \, dx + H^{N-1}(K \cap B) \leq \int_{B \setminus L} |\nabla v|^2 \, dx + H^{N-1}(L \cap B)
\]
where \( H^{N-1} \) denotes the Hausdorff measure of dimension \( N - 1 \).

Proposition 9 on page 267 of [8] ensures that any blow up limit of a minimizer for the Mumford-Shah functional in \( \mathbb{R}^N \), is a global minimizer in the sense of Definition 3. As a beginning for the description of global minimizers in \( \mathbb{R}^N \), we can firstly think about what should be a global minimizer in \( \mathbb{R}^3 \). If \( u \) is locally constant, then \( K \) is a minimal cone, that is, a set that locally minimizes the Hausdorff measure of dimension 2 in \( \mathbb{R}^3 \). Then by [9] we know that \( K \) is a cone of type \( \mathbb{P} \) (hyperplane), \( \mathbb{V} \) (three half-planes meeting with 120 degrees angles) or of type \( \mathbb{T} \) (cone over the edges of a regular tetraedron centered at the origin). Those cones became famous by the theorem of J. Taylor [20] which says that any minimal surface in \( \mathbb{R}^3 \) is locally \( C^1 \) equivalent to a cone of type \( \mathbb{P}, \mathbb{V} \) or \( \mathbb{T} \).
Cones of type $Y$ and $T$ in $\mathbb{R}^3$.

To be clearer, this is a more precise definition of $Y$ and $T$, as in [10].

**Definition 4.** Define $\text{Prop} \subset \mathbb{R}^2$ by

$$\text{Prop} = \{(x_1, x_2); x_1 \geq 0, x_2 = 0\} \cup \{(x_1, x_2); x_1 \leq 0, x_2 = -\sqrt{3}x_1\} \cup \{(x_1, x_2); x_1 \leq 0, x_2 = \sqrt{3}x_1\}.$$  

Then let $Y_0 = \text{Prop} \times \mathbb{R} \subset \mathbb{R}^3$. The spine of $Y_0$ is the line $L_0 = \{x_1 = x_2 = 0\}$. A cone of type $Y$ is a set $Y = R(Y_0)$ where $R$ is the composition of a translation and a rotation. The spine of $Y$ is then the line $R(L_0)$.

**Definition 5.** Let $A_1 = (1, 0, 0)$, $A_2 = (-\frac{1}{3}, 2\sqrt{3}, 0)$, $A_3 = (-\frac{1}{3}, -\sqrt{2}, \frac{2\sqrt{3}}{3})$, and $A_4 = (-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{3}}{3})$ the four vertices of a regular tetrahedron centered at 0. Let $T_0$ be the cone over the union of the 6 edges $[A_i, A_j]$ $i \neq j$. The spine of $T_0$ is the union of the four half lines $[0, A_j]$. A cone of type $T$ is a set $T = R(T_0)$ where $R$ is the composition of a translation and a rotation. The spine of $T$ is the image by $R$ of the spine of $T_0$.

So the pairs $(u, Z)$ where $u$ is locally constant and $Z$ is a minimal cone, are examples of global minimizers in $\mathbb{R}^3$. Another global minimizer can be obtained with $K_\infty$ a half-plane, by setting $u := \text{Craktip} \times \mathbb{R}$ (see [8] section 76). These examples are the only global minimizers in $\mathbb{R}^3$ that we know.

Note that if $(u, K)$ is a global minimizer in $\mathbb{R}^N$, then $u$ locally minimizes the Dirichlet integral in $\mathbb{R}^N\setminus K$. As a consequence, $u$ is harmonic in $\mathbb{R}^N\setminus K$. Moreover, if $B$ is a ball such that $K \cap B$ is regular enough, then the normal derivative of $u$ vanishes on $K \cap B$.

In this paper we wish to study global minimizers $(u, K)$ for which $K$ is a cone. It seems natural to think that any singular set of a global minimizer is a cone. But even if all known examples are cones, there is no proof of this fact. In addition, we will add some regularity on $K$. We denote by $S^{N-1}$ the unit sphere in $\mathbb{R}^N$ and, if $\Omega$ is a open set, $W^{1,2} (\Omega)$ is the Sobolev space. We will say that a domain $\Omega$ on $S^{N-1}$ has a piecewise $C^2$ boundary, if the topological boundary of $\Omega$, defined by $\partial \Omega = \Omega \setminus \Omega$, consists of an union of $N-2$ dimensional
hypersurfaces of class $C^2$. This allows some cracks, i.e. when $\Omega$ lies in each sides of its boundary. We will denote by $\tilde{\Sigma}$ the set of all the singular points of the boundary, that is

$$\tilde{\Sigma} := \{ x \in \partial \Omega; \forall r > 0, B(x, r) \cap \partial \Omega \text{ is not a } C^2 \text{ hypersurface} \}.$$ 

**Definition 6.** A smooth cone is a set $K$ of dimension $N - 1$ in $\mathbb{R}^N$ such that $K$ is conical, centered at the origin, and such that $S^{N-1} \setminus K$ is a domain with piecewise $C^2$ boundary. Moreover we assume that the embedding $L^2(S^{N-1} \setminus K) \to W^{1,2}(S^{N-1} \setminus K)$ is compact. Finally we suppose that we can strongly integrate by parts in $B(0,1) \setminus K$. More precisely, denoting by $\Sigma$ the set of singularities

$$\Sigma := \{ tx; (t, x) \in \mathbb{R}^+ \times \tilde{\Sigma} \},$$

we want that

$$\int_{B(0,1) \setminus K} \langle \nabla u, \nabla \phi \rangle = 0$$

for every harmonic function $u$ in $B(0,1) \setminus K$ with $\frac{\partial u}{\partial n} = 0$ on $K \setminus \Sigma$, and for all $\phi \in W^{1,2}(B(0,1) \setminus K)$ with vanishing trace on $S^{N-1} \setminus K$.

**Remark 7.** For instance, the cone over a finite union of $C^2$-arcs on $S^2$ is a smooth cone in $\mathbb{R}^3$. Another example in $\mathbb{R}^N$ is given by the union of admissible set of faces (as in Definition (22.2) of [5]).

Now this is the main result.

**Theorem 15.** Let $(u, K)$ be a global minimizer in $\mathbb{R}^N$. Assume that $K$ is a smooth cone. Then there is a $\frac{1}{2}$-homogenous function $u_1$ such that $u - u_1$ is locally constant.

As we shall see, this result implies that if $(u, K)$ is a global minimizer in $\mathbb{R}^N$, and if $K$ is a smooth cone other than a minimal cone, then $\frac{3-2N}{4}$ is an eigenvalue for the spherical Laplacian in $S^{N-1} \setminus K$ with Neumann boundary conditions. In section 2 we will give some applications about global minimizers in $\mathbb{R}^3$, using the estimates on the first eigenvalue that can be found in [6], [5] and [14]. More precisely, we have:

**Proposition 17** Let $(u, K)$ be a global Mumford-Shah minimizer in $\mathbb{R}^3$ such that $K$ is a smooth cone. Moreover, assume that $S^2 \cap K$ is a union of convex curvilinear polygons with $C^\infty$ sides. Then $u$ is locally constant and $K$ is a cone of type $P$, $Y$ or $T$.

Another consequence of the main result is the following.
Proposition 19. Let \((u,K)\) be a global Mumford-Shah minimizer in \(\mathbb{R}^3\) such that \(K\) is a half plane. Then \(u\) is equal to a function of type cracktip \(\times \mathbb{R}\), that is, in cylindrical coordinates,
\[
u(r,\theta,z) = \pm \sqrt{\frac{2}{\pi}} r^\frac{1}{2} \sin \frac{\theta}{2} + C
\]
for \(0 < r < +\infty, -\pi < \theta < \pi\) where \(C\) is a constant.

Finally, we deduce two other consequences from Theorem 15. Let \((r,\theta,z)\) be the cylindrical coordinates in \(\mathbb{R}^3\). For all \(\omega \in [0,\pi]\) set
\[
\partial \Gamma_\omega := \{(r,\theta,z) \in \mathbb{R}^3; \theta = -\omega \text{ or } \theta = \omega \}.
\]
and
\[
S_\omega := \{(r,\theta,z) \in \mathbb{R}^3; z = 0, r > 0, \theta \in [-\omega,\omega]\}
\]
(1)

Observe that \(S_0\) is a half line, \(S_{\frac{\pi}{2}}\), \(\partial \Gamma_0\) and \(\partial \Gamma_{\pi}\) are half-planes, and that \(S_{\pi}\) and \(\partial \Gamma_{\frac{\pi}{2}}\) are planes.

Proposition 18. There is no global Mumford-Shah minimizer in \(\mathbb{R}^3\) such that \(K\) is wing of type \(\partial \Gamma_\omega\) with \(\omega \notin \{0,\frac{\pi}{2},\pi\}\).

Proposition 23. There is no global Mumford-Shah minimizer in \(\mathbb{R}^3\) such that \(K\) is an angular sector of type \((u,S_\omega)\) for \(\omega \notin \{\frac{\pi}{2},\pi\}\).

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1 If \(K\) is a cone then \(u\) is homogenous

In this section we want to prove Theorem 15. Notice that this result is only useful if the dimension \(N \geq 3\). Indeed, in dimension 2, if \(K\) is a cone then it is connected thus it is in the list described in the introduction.
1.1 Preliminary

Let us recall a standard uniqueness result about energy minimizers.

**Proposition 8.** Let $\Omega$ be an open and connected set of $\mathbb{R}^N$ and let $I \subset \partial \Omega$ be a hypersurface of class $C^\infty$. Suppose that $u$ and $v$ are two functions in $W^{1,2}(\Omega)$ such that $u = v$ a.e. on $I$ (in terms of trace), solving the minimizing problem

$$\min E(w) := \int_\Omega |\nabla w(x)|^2 dx$$

over all the functions $w \in W^{1,2}(\Omega)$ that are equal to $u$ and $v$ on $I$. Then

$$u = v.$$

**Proof:** This comes from a simple convexity argument which can be found for instance in [8], but let us write the proof since it is very short. By the parallelogram identity we have

$$E\left(\frac{u + v}{2}\right) = \frac{1}{2} E(u) + \frac{1}{2} E(v) - \frac{1}{4} E(u - v). \quad (2)$$

On the other hand, since $\frac{u + v}{2}$ is equal to $u$ and $v$ on $I$, and by minimality of $u$ and $v$ we have

$$E\left(\frac{u + v}{2}\right) \geq E(u) = E(v).$$

Now by (2) we deduce that $E(u - v) = 0$ and since $\Omega$ is connexe, this implies that $u - v$ is a constant. But $u - v$ is equal to 0 on $I$ thus $u = v$. \[\Box\]

**Remark 9.** The existence of a minimizer can also be proved using the convexity of $E(v)$.

1.2 Spectral decomposition

The key ingredient to obtain the main result will be the spectral theory of the Laplacian on the unit sphere. Since $u$ is harmonic, we will decompose $u$ as a sum of homogeneous harmonic functions just like we usually use the classical spherical harmonics. The difficulty here comes from the lack of regularity of $\mathbb{R}^N \setminus K$.

It will be convenient to work with connected sets. So let $\Omega$ be a connected component of $S^{N-1} \setminus K$, and let $A(r)$ be

$$A(r) := \{tx; (x, t) \in \Omega \times [0, r[ \}. \quad \text{We also set} \quad A(\infty) := \{tx; (x, t) \in \Omega \times \mathbb{R}^+ \}. $$
All the following results are using that the embedding $W^{1,2}(\Omega)$ in $L^2(\Omega)$ is compact. Recall that this is the case by definition, since $K$ is a smooth cone. Notice that for instance the cone property insures that the embedding is compact (see Theorem 6.2, p 144 of [1]).

Consider the quadratic form

$$Q(u) = \int_\Omega |\nabla u(x)|^2 dx$$

of domain $W^{1,2}(\Omega)$ dense into the Hilbert space $L^2(\Omega)$. Since $Q$ is a positive and closed quadratic form (see for instance Proposition 10.61, p.129 of [16]) there exists a unique selfadjoint operator denoted by $-\Delta_n$ of domain $D(-\Delta_n) \subset W^{1,2}(\Omega)$ such that

$$\forall u \in D(-\Delta_n), \forall v \in W^{1,2}(\Omega), \int_\Omega \langle \nabla u, \nabla v \rangle = \int_\Omega \langle -\Delta_n u, v \rangle.$$

**Proposition 10.** The operator $-\Delta_n$ has a countably infinite discrete set of eigenvalues, whose eigenfunctions span $L^2(\Omega)$.

**Proof:** The proof is the same as if $\Omega$ was a regular domain. Consider the new quadratic form

$$\tilde{Q}(u) := Q(u) + \|u\|_2^2$$

with the same domain $W^{1,2}(\Omega)$. The form $\tilde{Q}$ has the same properties than $Q$ and the associated operator is $\text{Id} - \Delta_n$. Moreover $\tilde{Q}$ is coercive. As a result, the operator $\text{Id} - \Delta_n$ is bijective and its inverse goes from $L^2(\Omega)$ to $D(-\Delta_n) \subset W^{1,2}(\Omega)$. By hypothesis the embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact. Thus the resolvant $(\text{Id} - \Delta_n)^{-1}$ is a compact operator, and we conclude using the spectral theory of operators with a compact resolvant (see [19], Theorem XIII.64, p.245).

**Remark 11.** The domain of $-\Delta_n$ is not known in general. If $\Omega$ was smooth, then we could show that the domain is exactly $D(-\Delta_n) = \{u \in W^{2,2}(\Omega); \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. Here, the boundary of $\Omega$ has some singularities so this result doesn’t apply directly. But knowing exactly the domain of $-\Delta_n$ will not be necessary for us.

Now we want to study the link between the abstract operator $\Delta_n$ and the classical spherical Laplacian $\Delta_S$ on the unit sphere. Recall that if we compute the Laplacian in spherical coordinates, we obtain the following equality

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_S.$$

**Proposition 12.** For every function $f \in D(-\Delta_n)$ such that $-\Delta_n f = \lambda f$ we have

\begin{enumerate}
  \item $f \in C^\infty(\Omega)$
  \item $-\Delta_S f = -\Delta_n f = \lambda f \text{ in } \Omega$
  \item $\frac{\partial f}{\partial n}$ exists and is equal to 0 on $K \cap \overline{\Omega} \setminus \Sigma$
\end{enumerate}
**Proof:** Let \( \varphi \) be a \( C^\infty \) function with compact support in \( \Omega \) and \( f \in D(-\Delta_n) \). Then the Green formula in the distributional sense gives

\[
\int_{\Omega} \nabla f \cdot \nabla \varphi = \langle -\Delta_S f, \varphi \rangle
\]

where the left and right brackets mean the duality in the distributional sense. On the other hand, by definition of \(-\Delta_n\) and since \( f \) is in the domain \( D(-\Delta_n) \), we also have

\[
\int_{\Omega} \nabla f \cdot \nabla \varphi = \langle -\Delta_n f, \varphi \rangle
\]

where this time the brackets mean the scalar product in \( L^2 \). Therefore

\[
\Delta_n f = \Delta_S f \quad \text{in} \quad D'(\Omega).
\]

In other words, \(-\Delta_S f = \lambda f \) in \( D'(\Omega) \). But now since \( f \in W^{1,2}(\Omega) \), by hypoellipticity of the Laplacian we know that \( f \) is \( C^\infty \) and that \(-\Delta_S f = \lambda f \) in the classical sense. That proves i) and ii). We even know by the elliptic theory that, since \( K\setminus\Sigma \) is regular, \( f \) is regular at the boundary on \( K\setminus\Sigma \).

Now consider a ball \( B \) such that the intersection with \( K\cap\overline{\Omega} \) does not meet \( \Sigma \). Assume that \( B \) is cut in two parts \( B^+ \) and \( B^- \) by \( K \), and that \( B^+ \) is one part in \( \Omega \). Possibly by modifying \( B \) in a neighborhood of the intersection with \( K \), we can assume that the boundary of \( B^+ \) and \( B^- \) is \( C^2 \). The definition of \( \Delta_n \) implies that for all function \( \varphi \in C^2(\overline{\Omega}) \) that vanishes out of \( B^+ \) we have

\[
\int_{B^+} \langle \nabla f, \nabla \varphi \rangle dx = \int_{B^+} \langle -\Delta_n f, \varphi \rangle dx = \lambda \int_{B^+} \langle f, \varphi \rangle dx.
\]

On the other hand, integrating by parts,

\[
\int_{B^+} \langle \nabla f, \nabla \varphi \rangle dx = \int_{B^+} \langle -\Delta_S f, \varphi \rangle + \int_{\partial B^+} \frac{\partial u}{\partial n} \varphi
\]

\[
= \lambda \int_{\partial B^+} \langle f, \varphi \rangle + \int_{\partial B^+} \frac{\partial f}{\partial n} \varphi
\]

thus

\[
\int_{\partial B^+} \frac{\partial f}{\partial n} \varphi = 0.
\]

In other words the function \( f \) is a weak solution of the mixed boundary value problem

\[
-\Delta_S u = \lambda f \quad \text{in} \quad B^+
\]

\[
u = f \quad \text{on} \quad \partial B^+ \setminus K
\]

\[
\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad K \cap \partial B^+
\]
Therefore, some results from the elliptic theory imply that \( f \) is smooth in \( B \) and is a strong solution (see \([21]\)).

Let us recapitulate what we have obtained. For all function \( f \in L^2(\Omega) \), there is a sequence of numbers \( a_i \) such that

\[
f = \sum_{i=0}^{+\infty} a_i f_i
\]

where the sum converges in \( L^2 \). The functions \( f_i \) are in \( C^\infty(\Omega) \cap W^{1,2}(\Omega) \), verify \(-\Delta_S f_i = \lambda_i f_i \) and \( \partial f_i / \partial n = 0 \) on \( K \cap \overline{\Omega} \setminus \Sigma \). Moreover, we can normalize the \( f_i \) in order to obtain an orthonormal basis on \( L^2(\Omega) \), in particular we have the following Parseval formula

\[
\|f\|_2^2 = \sum_{i=0}^{+\infty} |a_i|^2.
\]

Note that if \( f \) belongs to the kernel of \(-\Delta_n\) (i.e. is an eigenfunction with eigenvalue 0), then

\[
\langle \nabla f, \nabla f \rangle = \langle -\Delta_n f, f \rangle = 0
\]

and since \( \Omega \) is connected that means that \( f \) is a constant. Thus 0 is the first eigenvalue and the associated eigenspace has dimension 1. Then we can suppose that \( \lambda_0 = 0 \) and that all the \( \lambda_i \) for \( i > 0 \) are positive.

We define the scalar product in \( W^{1,2}(\Omega) \) by

\[
\langle u, v \rangle_{W^{1,2}} := \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2}.
\]

**Proposition 13.** The family \( \{f_i\} \) is orthogonal in \( W^{1,2}(\Omega) \). Moreover if \( f \in W^{1,2}(\Omega) \) and if its decomposition in \( L^2(\Omega) \) is \( f = \sum_{i=0}^{+\infty} a_i f_i \), then the sum \( \sum_{i=0}^{+\infty} |a_i|^2 \|\nabla f_i\|_2^2 \) converges and

\[
\sum_{i=0}^{+\infty} |a_i|^2 \|\nabla f_i\|_2^2 = \|\nabla f\|_2^2.
\]

**Proof:** We know that \( \{f_i\} \) is an orthogonal family in \( L^2(\Omega) \). In addition if \( i \neq j \) then

\[
\int_\Omega \nabla f_i \nabla f_j = \int_\Omega -\Delta_n f_i f_j = \lambda_i \int_\Omega f_i f_j = 0
\]

thus \( \{f_i\} \) is also orthogonal in \( W^{1,2}(\Omega) \) and

\[
\|f_i\|_{W^{1,2}}^2 := \|f_i\|_2^2 + \|\nabla f_i\|_2^2 = 1 + \lambda_i.
\]
Consider now the orthogonal projection (for the scalar product of $L^2$)

$$P_k : f \mapsto \sum_{i=0}^k a_i f_i.$$ 

The operator $P_k$ is the orthogonal projection on the closed subspace $A_k$ generated by \{ $f_0, ..., f_k$ \}. More precisely, we are interested in the restriction of $P_k$ to the subspace $W^{1,2}(\Omega) \subset L^2(\Omega)$. Also denote by $\tilde{P}_k : W^{1,2} \to A_k$ the orthogonal projection on the same subspace but for the scalar product of $W^{1,2}$. We want to show that $P_k = \tilde{P}_k$. To prove this, it suffice to show that for all sets of coefficients \{ $a_i$ \}_{i=1..k} and \{ $b_i$ \}_{i=1..k},

$$\langle f - \sum_{i=0}^k a_i f_i, \sum_{i=0}^k b_i f_i \rangle_{W^{1,2}} = 0.$$ 

Since we already have

$$\langle f - \sum_{i=0}^k a_i f_i, \sum_{i=0}^k b_i f_i \rangle_{L^2} = 0,$$

all we have to show is that

$$\int_{\Omega} \langle \nabla f - \sum_{i=0}^k a_i \nabla f_i, \sum_{i=0}^k b_i \nabla f_i \rangle dx = 0.$$

Now

$$\int_{\Omega} \langle \nabla f - \sum_{i=0}^k a_i \nabla f_i, \sum_{i=0}^k b_i \nabla f_i \rangle = \int_{\Omega} \langle \nabla f, \sum_{i=0}^k b_i \nabla f_i \rangle - \sum_{i=0}^k a_i b_i \| \nabla f_i \|^2_2$$

$$= \sum_{i=0}^k b_i \langle -\Delta_n f_i, f \rangle_{L^2} - \sum_{i=0}^k a_i b_i \lambda_i$$

$$= \sum_{i=0}^k a_i b_i \lambda_i - \sum_{i=0}^k a_i b_i \lambda_i$$

$$= 0$$

thus $P_k = \tilde{P}_k$ and therefore, by Pythagoras

$$\| P_k(f) \|^2_{W^{1,2}} \leq \| f \|^2_{W^{1,2}}.$$ 

By letting $k$ tend to infinity we obtain

$$\sum_{i=0}^{+\infty} a_i^2 \| \nabla f_i \|^2_2 \leq \| \nabla f \|^2_2.$$  

(6)
From this inequality we deduce that the sum is absolutely converging in $W^{1,2}(\Omega)$. Therefore, the sequence of partial sum $\sum_{i=0}^{K} a_i f_i$ is a Cauchy sequence for the norm $W^{1,2}(\Omega)$. Thus, since the sum $\sum a_i f_i$ already converges to $f$ in $L^2(\Omega)$, by uniqueness of the limit the sum converges to $f$ in $W^{1,2}(\Omega)$, so we deduce that (6) is an equality and the prove is over.

Once we have a basis $\{f_i\}$ on $\Omega \subset S^{N-1}$, we consider for a certain $r_0 > 0$, the functions

$$h_i(x) = r_0^{\alpha_i} f_i \left( \frac{x}{r_0} \right)$$

defined on $r_0 \Omega$. The exponent $\alpha_i$ is defined by

$$\alpha_i = - (N - 2) + \sqrt{(N - 2)^2 + 4\lambda_i}. \quad (7)$$

The functions $h_i$ form a basis of $W^{1,2}(r_0 \Omega)$. Indeed, if $f \in W^{1,2}(r_0 \Omega)$, then $f(r_0 x) \in W^{1,2}(\Omega)$ thus applying the decomposition on $\Omega$ we obtain

$$f(r_0 x) = \sum_{i=0}^{+\infty} b_i f_i(x)$$

thus

$$f(x) = \sum_{i=0}^{+\infty} a_i h_i(x)$$

with

$$a_i = b_i r_0^{-\alpha_i}. \quad (8)$$

Notice that since $\|h_i\|_2^2 = r_0^{2\alpha_i + N - 1}$ we also have

$$\sum_{i=0}^{\infty} a_i^2 \|h_i\|_2^2 = \sum_{i=0}^{\infty} a_i^2 r_0^{2\alpha_i + N - 1} = \|f\|_{L^2(r_0 \Omega)}^2 < +\infty. \quad (9)$$

Moreover, applying Proposition [13] we have that

$$\sum_{i=0}^{\infty} b_i^2 \|\nabla f_i\|_2^2 = \|\nabla f(r_0 x)\|_2^2 < +\infty. \quad (10)$$

We are now able to state our decomposition in $A(r_0)$. 

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Proposition 14. Let $K$ be a smooth cone in $\mathbb{R}^N$, centered at the origin and let $\Omega$ be a connected component of $S^{N-1}\setminus K$. Then there exist some harmonic homogeneous functions $g_i$, orthogonal in $W^{1,2}(A(1))$, such that for every function $u \in W^{1,2}(A(1))$ harmonic in $A(1)$ with $\frac{\partial u}{\partial n} = 0$ on $K \cap A(1) \setminus \Sigma$, and for every $r_0 \in ]0, 1[$, we have that

$$u = \sum_{i=0}^{+\infty} a_ig_i \quad \text{in } A(r_0)$$

where the $a_i$ do not depend on radius $r_0$ and are unique. The sum converges in $W^{1,2}(A(r_0))$ and uniformly on all compact sets of $A(1)$. Moreover

$$\|u\|^2_{W^{1,2}(A(r_0))} = \sum_{i=0}^{+\infty} a_i^2\|g_i\|^2_{W^{1,2}(A(r_0))}. \quad (11)$$

Proof: Since $u \in W^{1,2}(A(1))$ then for almost every $r_0$ in $]0, 1[$ we have that $u|_{r_0\Omega} \in W^{1,2}(r_0\Omega)$. Thus we can apply the decomposition on $r_0\Omega$ and say that

$$u = \sum_{i=0}^{+\infty} a_i h_i \quad \text{on } r_0\Omega.$$ 

Define $g_i$ by

$$g_i(x) := \|x\|^{\alpha_i} f_i \left( \frac{x}{\|x\|} \right)$$

where $\alpha_i$ is defined by (7). Since the $f_i$ are eigenfunctions for $-\Delta_S$, we deduce from (3) that

$$\Delta g_i = \frac{\partial^2}{\partial r^2} g_i + \frac{N-1}{r} \frac{\partial}{\partial r} g_i + \frac{1}{r^2} \Delta_S g_i$$

$$= \alpha_i (\alpha_i - 1) r^{\alpha_i - 2} f_i + \frac{N-1}{r} \alpha_i r^{\alpha_i - 1} f_i - r^{\alpha_i - 2} \lambda_i f_i$$

$$= (\alpha_i^2 + (N-2)\alpha_i - \lambda_i) r^{\alpha_i - 2} f_i$$

$$= 0$$

by definition of $\alpha_i$, thus the $g_i$ are harmonic in $A(+\infty)$. Notice that the $g_i$ are orthogonal in $L^2(A(1))$ because they are homogeneous and orthogonal in $L^2(\Omega)$. Note also that $h_i$ is equal to $g_i$ on $r_0\Omega$. Moreover for all $0 < r \leq 1$ we have

$$\|g_i\|^2_{L^2(A(r))} = \int_{A(r)} |g_i|^2 = \int_0^r \int_{\partial B(t) \cap A(1)} |g_i(w)|^2dwdt$$

$$= \int_0^r \int_\Omega t^{N-1} |g_i(ty)|^2dydt = \int_0^r t^{2\alpha_i + N - 1} \int_{\Omega} |g_i(y)|^2dydt$$

$$= \frac{r^{2\alpha_i+N}}{2\alpha_i + N} \|f_i\|^2_{L^2(\Omega)} = \frac{r^{2\alpha_i+N}}{2\alpha_i + N} \leq 1. \quad (12)$$
In the other hand, since the $f_i$ and their tangential gradients are orthogonal in $L^2(\Omega)$, we deduce that the gradients of $g_i$ are orthogonal in $A(1)$. Then, by a computation similar to \cite{12} we obtain for all $0 < r \leq 1$

$$\|\nabla g_i\|_{L^2(A(r))}^2 = \int_0^r \int_{\partial B(t) \cap A(1)} \left| \frac{\partial g_i}{\partial r} \right|^2 + |\nabla_r g_i|^2 \, dw \, dt$$

$$= \int_0^r \int_{\partial B(t) \cap A(1)} \left| \alpha_i t^{\alpha_i - 1} f_i \left( \frac{w}{t} \right) \right|^2 + \left| t^{\alpha_i} \nabla_r f_i \left( \frac{w}{t} \right) \right|^2 \, dw \, dt$$

$$= \alpha_i^2 \int_0^r t^{2(\alpha_i - 1)} \int_{\partial B(t) \cap A(1)} \left| f_i \left( \frac{w}{t} \right) \right|^2 \, dw \, dt + \int_0^r t^{2(\alpha_i - 1)} \int_{\partial B(t) \cap A(1)} \left| \nabla_r f_i \left( \frac{w}{t} \right) \right|^2 \, dw \, dt$$

$$= \alpha_i^2 \int_0^r t^{2(\alpha_i - 1)} \int_{\Omega} \left| f_i(w) \right|^2 t^{N-1} \, dw \, dt + \int_0^r t^{2(\alpha_i - 1)} \int_{\Omega} \left| \nabla_r f_i(w) \right|^2 t^{N-1} \, dw \, dt$$

$$= \frac{\alpha_i^2}{2(\alpha_i - 1) + N} \left( \left\| f_i \right\|_{L^2(\Omega)}^2 + \frac{r^{2(\alpha_i - 1) + N}}{2(\alpha_i - 1) + N} \left\| \nabla_r f_i \right\|_{L^2(\Omega)}^2 \right)$$

$$\leq C r^{2\alpha_i} (\alpha_i^2 + \lambda_i)$$

(13)

because $\|\nabla_r f_i\|_2^2 = \lambda_i \|f_i\|^2_2$, $r \leq 1$ and $\alpha_i \geq 0$. Moreover the constant $C$ depends on the dimension $N$ but does not depend on $i$.

We denote by $g$ the function defined in $A(\infty)$ by

$$g := \sum_{i=0}^{+\infty} a_i g_i.$$

Then $g$ lies in $L^2(A(r_0))$ because using \cite{12} and \cite{9}

$$\|g\|_{L^2(A(r_0))} = \sum_{i=0}^{+\infty} |a_i|^2 \|g_i\|_{L^2(A(r_0))}^2 \leq \sum_{i=0}^{+\infty} |a_i|^2 r_0^{2\alpha_i + N} < +\infty.$$

We want now to show that $g = u$.

- **First step**: We claim that $g$ is harmonic in $A(r_0)$. Indeed, since the $g_i$ are all harmonic in $A(r_0)$, the sequence of partial sums $s_k := \sum_{i=0}^k a_i g_i$ is a sequence of harmonic functions, uniformly bounded for the $L^2$ norm in each compact set of $A(r_0)$. By the Harnack inequality we deduce that the sequence of partial sums is uniformly bounded for the uniform norm in each compact set. Thus there is a subsequence that converges uniformly to a harmonic function, which in fact is equal to $g$ by uniqueness of the limit. Therefore, $g$ is harmonic in $A(r_0)$. 

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• **Second step**: We claim that \( g \) belongs to \( W^{1,2}(A(r_0)) \). Firstly, since \( u \in W^{1,2}(r_0\Omega) \), by (8) and (10) we have that
\[
\sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \|\nabla_{\tau}f_i\|^2_{L^2(\partial B(0,1)\setminus K)} < +\infty. \tag{14}
\]
In addition, since \( \|\nabla_{\tau}f_i\|^2 = \lambda_i \|f_i\|^2 \) and \( \|f_i\|^2 = 1 \), we deduce
\[
\sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \lambda_i < +\infty \tag{15}
\]
and since \( \alpha_i \) and \( \lambda_i \) are linked by the formula (7) we also have that
\[
\sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \alpha_i^2 < +\infty. \tag{16}
\]
Now, since \( \sum a_i g_i \) converges absolutely on every compact set, we can say that
\[\nabla g = \sum_{i=0}^{+\infty} a_i \nabla g_i \]
thus using (13), (15), (16), and orthogonality,
\[
\|\nabla g\|^2_{L^2(A(r_0))} = \sum_{i=0}^{+\infty} a_i^2 \|\nabla g_i\|^2_{L^2} \\
\leq C \sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} (\alpha_i^2 + \lambda_i) < +\infty.
\]
Therefore, \( g \in W^{1,2}(A(r_0)) \).

• **Third step**: We claim that \( \frac{\partial g}{\partial n} = 0 \) on \( K \cap \overline{A(r_0)} \setminus \Sigma \). We already know that \( \frac{\partial g}{\partial n} = 0 \) on \( K \setminus \Sigma \) (because the \( f_i \) have this property). We want to show that \( g \) is so regular that we can exchange the order of \( \frac{\partial}{\partial n} \) and \( \sum \). So let \( x_0 \) be a point of \( K \cap \overline{A(r_0)} \setminus \Sigma \) and let \( B \) be a neighborhood of \( x_0 \) in \( \mathbb{R}^N \) that doesn’t meet \( \Sigma \) and such that \( K \) separates \( B \) in two parts \( B^+ \) and \( B^- \). Assume that \( B^+ \) is a part in \( A(r_0) \). The sequence of partial sums \( s_k := \sum_{i=0}^{k} a_i g_i \) is a sequence of harmonic functions in \( B^+ \). Since \( \partial B^+ \cap K \) is \( C^2 \) we can do a reflection to extend \( s_k \) in \( B^- \). For all \( k \), this new function \( s_k \) is the solution of a certain elliptic equation whose operator become from the composition of the Laplacian with the application that makes \( \partial B^+ \cap K \) flat. Thus since \( \sum a_i g_i \) converges absolutely for the \( L^2 \) norm, by the Harnack inequality \( \sum a_i g_i \) converges absolutely for the uniform norm in a smaller neighborhood \( B' \subset B \) that still contains \( x_0 \). Thus \( s_k \) converges to a \( C^1 \)
function denoted by \( s \), which is equal to \( g \) on \( B^+ \). And since \( \frac{\partial s}{\partial n}(x_0) = 0 \), by the absolute convergence of the sum we can exchange the order of the derivative and the symbol \( \sum \) so we deduce that \( \frac{\partial s}{\partial n}(x_0) = 0 \). Finally, since \( s \) is equal to \( g \) on \( B^+ \) we deduce that \( g \) is \( C^1 \) at the boundary and \( \frac{\partial g}{\partial n} = 0 \) at \( x_0 \).

**Fourth step**: we claim that \( g \) is equal to \( u \) on \( r_0 \Omega \). Let \( r \) be a radius such that \( r < r_0 \). Then the function \( x \mapsto g_r(x) := g(r \frac{x}{r_0}) \) is well defined for \( x \in r_0 \Omega \), and since the \( g_i \) are homogeneous we have

\[
g(r \frac{x}{r_0}) = \sum_{i=0}^{+\infty} a_i g_i(r \frac{x}{r_0}) = \sum_{i=0}^{+\infty} \left( \frac{r}{r_0} \right)^{\alpha_i} a_i g_i(x) = \sum_{i=0}^{+\infty} \left( \frac{r}{r_0} \right)^{\alpha_i} a_i h_i(x).
\]

We deduce that the function \( x \mapsto g_r(x) \) is in \( L^2(r_0 \Omega) \) and its coefficients in the basis \( \{h_i\} \) are \( \{(\frac{r}{r_0})^{\alpha_i} a_i\} \). We want to show that \( \|g_r - u\|_{L^2(r_0 \Omega)} \) tend to 0. Indeed, writing \( u \) in the basis \( \{h_i\} \)

\[
u = \sum_{i=0}^{+\infty} a_i h_i,
\]

we obtain

\[
\|g_r - u\|_2^2 = \sum_{i=0}^{+\infty} \left( \left( \frac{r}{r_0} \right)^{\alpha_i} - 1 \right)^2 a_i^2 \|h_i\|_2^2
\]

which tends to zero when \( r \) tends to \( r_0 \) by the dominated convergence theorem because \( \left( \left( \frac{r}{r_0} \right)^{\alpha_i} - 1 \right)^2 \leq 1 \). Therefore, there is a subsequence for which \( g_r \) tends to \( u \) almost everywhere. On the other hand, since \( g \) is harmonic, the limit of \( g_r \) exists and is equal to \( g \). That means that \( g \) tends to \( u \) radially at almost every point of \( r_0 \Omega \).

**Fifth step**: The functions \( u \) and \( g \) are harmonic functions in \( A(r_0) \), with finite energy, with a normal derivative equal to zero on \( K \cap A(r_0) \setminus \Sigma \) and that coincide on \( \partial A(r_0) \setminus \Sigma \). To show that \( u = g \) in \( A(r_0) \) we shall prove that \( g \) is an energy minimizer. Proposition \( 8 \) will then give the uniqueness.

Let \( \varphi \in W^{1,2}(A(r_0) \setminus K) \) have a vanishing trace on \( \partial B(0, r_0) \). Then, setting \( J(v) := \int_{A(r_0)} |\nabla v|^2 \) for \( v \in W^{1,2}(A(r_0)) \) we have

\[
J(g + \varphi) = J(g) + \int_{A(r_0)} \nabla g \nabla \varphi + J(\varphi).
\]

Now since \( g \) is harmonic with Neumann condition on \( K \setminus \Sigma \) and since \( \varphi \) vanishes on \( r_0 \Omega \), integrating by parts we obtain

\[
J(g + \varphi) = J(g) + J(\varphi).
\]
Since $J$ is non negative and $g + \varphi$ describes all the functions in $W^{1,2}(A(r_0))$ with trace equal to $u$ on $r_0\Omega$, we deduce that $g$ minimizes $J$. We can do the same with $u$ thus $u$ and $g$ are two energy minimizers with same boundary conditions. Therefore, by Proposition 8 we know that $g = u$.

- **Sixth step**: The decomposition do not depends on $r_0$. Indeed, let $r_1$ be a second choice of radius. Then we can do the same work as before to obtain a decomposition

$$u(x) := \sum_{i=0}^{+\infty} b_i g_i(x) \quad \text{in } B(0, r_1) \setminus K.$$ 

Now by uniqueness of the decomposition in $B(0, \min(r_0, r_1))$ we deduce that $b_i = a_i$ for all $i$.

In addition, $r_0$ was initially chosen almost everywhere in $[0,1[$. But since the decomposition does not depend on the choice of radius, $r_0$ can be chosen anywhere in $[0,1[$, by choosing a radius almost everywhere in $]r_0,1[$. \qed

**Theorem 15.** Let $(u, K)$ be a global minimizer in $\mathbb{R}^N$ such that $K$ is a smooth cone. Then for each connected component of $\mathbb{R}^N \setminus K$ there is a constant $u_k$ such that $u - u_k$ is $\frac{1}{2}$-homogenous.

**Proof**: Let $\Omega$ be a connected component of $\mathbb{R}^N \setminus K$. We apply the preceding proposition to $u$. Thus

$$u(x) = \sum_{i=0}^{+\infty} a_i g_i(x) \quad \text{in } A(r_0).$$

for a certain radius $r_0$ chosen in $]0,1[$. Let us prove that the same decomposition is true in $A(\infty)$. Applying Proposition 14 to the function $u_R(x) = u(Rx)$ we know that there are some coefficients $a_i(R)$ such that

$$u_R(x) = \sum_{i=0}^{+\infty} a_i(R) g_i(x) \quad \text{in } A(r_0).$$

Now since $u_R(\frac{2}{R}) = u(x)$ we can use the homogeneity of the $g_i$ to identify the terms in $B(0, r_0)$ thus $a_i(R) = a_i R^\alpha$. Now we fix $y = Rx$ and we obtain that

$$u(y) = \sum_{i=0}^{+\infty} a_i g_i(y) \quad \text{in } A(Rr_0).$$

Since $R$ is arbitrary the decomposition is true in $A(\infty)$. 

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In addition for every radius $R$ we know that
\[ \| \nabla u \|_{L^2(A(R))}^2 = \sum_{i=0}^{+\infty} a_i^2 \| \nabla g_i \|_{L^2(A(R))}^2 \] (17)
and since $g_i$ is $\alpha_i$-homogenous,
\[ \| \nabla g_i \|_{L^2(A(R))}^2 = R^{2(\alpha_i - 1) + N} \| \nabla g_i \|_{L^2(A(1))}^2. \]

Now, since $u$ is a global minimizer, a classical estimate on the gradient obtained by comparing $(u, K)$ with $(v, L)$ where $v = 1_{B(0, R)^C} u$ and $L = \partial B(0, R) \cup (K \setminus B(0, R))$ gives that there is a constant $C$ such that for all radius $R$
\[ \| \nabla u \|_{L^2(B(0, R) \setminus K)}^2 \leq CR^{N-1}. \]
We deduce
\[ \sum_{i=0}^{+\infty} a_i^2 R^{2(\alpha_i - 1) + N} \| \nabla g_i \|_{L^2(A(1))}^2 \leq CR^{N-1}. \]
Thus
\[ \sum_{i=0}^{+\infty} a_i^2 R^{2\alpha_i - 1} \| \nabla g_i \|_{L^2(A(1))}^2 \leq C. \]

This last quantity is bounded when $R$ goes to infinity if and only if $a_i = 0$ whenever $\alpha_i > 1/2$. On the other hand, this quantity is bounded when $R$ goes to 0, if and only if $a_i = 0$ whenever $0 < \alpha_i < 1/2$. Therefore, $u - a_0$ is a finite sum of terms of degree $\frac{1}{2}$.

**Remark 16.** In Chapter 65 of [8], we can find a variational argument that leads to a formula in dimension 2 that links the radial and tangential derivatives of $u$. For all $\xi \in K \cap \partial B(0, r)$, we call $\theta_\xi \in [0, \frac{\pi}{2}]$ the non oriented angle between the tangent to $K$ at point $\xi$ and the radius $[0, \xi]$. Then we have the following formula
\[
\int_{\partial B(0,r) \setminus K} \left( \frac{\partial u}{\partial r} \right)^2 dH^1 = \int_{\partial B(0,r) \setminus K} \left( \frac{\partial u}{\partial \tau} \right)^2 dH^1 + \sum_{\xi \in K \cap \partial B(0,r)} \cos \theta_\xi - \frac{1}{r} H^1(K \cap B(0, r)).
\]
Notice that for a global minimizer in $\mathbb{R}^2$ with $K$ a centered cone we find
\[ \int_{\partial B(0,r) \setminus K} \left( \frac{\partial u}{\partial r} \right)^2 dH^1 = \int_{\partial B(0,r) \setminus K} \left( \frac{\partial u}{\partial \tau} \right)^2 dH^1. \] (18)

Now suppose that $(u, K)$ is a global minimizer in $\mathbb{R}^N$ with $K$ a smooth cone centered at 0. Then by Theorem 15 we know that $u$ is harmonic and $\frac{1}{2}$-homogenous. Its restriction to the unit sphere is an eigenfunction for the spherical Laplacian with Neumann boundary condition and associated to the eigenvalue $\frac{2N - 3}{4}$. We deduce that
\[ \| \nabla_{\tau} u \|_{L^2(\partial B(0,1))}^2 = \frac{2N - 3}{4} \| u \|_{L^2(\partial B(0,1))}^2. \]
On the other hand
\[
\frac{\partial u}{\partial r}(x) = \frac{1}{2} \|x\|^{-\frac{1}{2}} u \left( \frac{x}{\|x\|} \right)
\]
thus
\[
\| \frac{\partial u}{\partial r} \|_{L^2(\partial B(0,1))}^2 = \frac{1}{4} \| u \|_{L^2(\partial B(0,1))}^2.
\]
So
\[
\| \nabla_\tau u \|_{L^2(\partial B(0,1))}^2 = (2N - 3) \| \frac{\partial u}{\partial r} \|_{L^2(\partial B(0,1))}^2.
\]
In particular, for \(N = 2\) we have the same formula as \((18)\).

2 Some applications

As it was claimed in the introduction, here is some few applications of Theorem 15.

**Proposition 17.** Let \((u, K)\) be a global minimizer in \(\mathbb{R}^3\) such that \(K\) is a smooth cone. Moreover, assume that \(S^2 \cap K\) is a union of convex curvilinear polygons with \(C^\infty\) sides. Then \(u\) is locally constant and \(K\) is a cone of type \(P, Y\) or \(T\).

**Proof:** In each polygon we know by Proposition 4.5. of \([6]\) that the smallest positive eigenvalue for the operator minus Laplacian with Neumann boundary conditions is greater than or equal to 1. Thus it cannot be \(\frac{3}{4}\) and \(u\) is locally constant. Then \(K\) is a minimal cone in \(\mathbb{R}^3\) and we know from \([9]\) that it is a cone of type \(P, Y\) or \(T\). \(\square\)

Let \((r, \theta, z) \in \mathbb{R}^+ \times [-\pi, \pi] \times \mathbb{R}\) be the cylindrical coordinates in \(\mathbb{R}^3\). For every \(\omega \in [0, \pi]\) set
\[
\Gamma_\omega := \{(r, \theta, z) \in \mathbb{R}^3; -\omega < \theta < \omega\}
\]
of boundary
\[
\partial \Gamma_\omega := \{(r, \theta, z) \in \mathbb{R}^3; \theta = -\omega \text{ or } \theta = \omega\}.
\]
Consider \(\Omega_\omega = \Gamma_\omega \cap S^2\) and let \(\lambda_1\) be the smallest positive eigenvalue of \(-\Delta_S\) in \(\Omega_\omega\) with Neumann conditions on \(\partial \Omega_\omega\). Then by Lemma 4.1. of \([6]\) we have that
\[
\lambda_1 = \min(2, \lambda_\omega)
\]
where
\[
\lambda_\omega = \left( \frac{\pi}{2\omega} + \frac{1}{2} \right)^2 - \frac{1}{4}.
\]
In particular for the cone of type \(Y\), \(\omega = \frac{\pi}{3}\) thus \(\lambda_1 = 2\).

Observe that for \(\omega \neq \pi\), \(\lambda_\omega \neq \frac{3}{4}\). So we get this following proposition.
Proposition 18. There is no global Mumford-Shah minimizer in $\mathbb{R}^3$ such that $K$ is wing of type $\partial \Gamma_\omega$ with $\omega \not\in \{0, \frac{\pi}{2}, \pi\}$.

Another consequence of Theorem 15 is the following. Let $P$ be the half plane

$$P := \{(r, \theta, z) \in \mathbb{R}^3; \theta = \pi\}.$$

Proposition 19. Let $(u, K)$ be a global Mumford-Shah minimizer in $\mathbb{R}^3$ such that $K = P$. Then $u$ is equal to $\text{cracktip} \times \mathbb{R}$, that is in cylindrical coordinates

$$u(r, \theta, z) = \pm \sqrt{\frac{2}{\pi}} r^{\frac{1}{2}} \sin \frac{\theta}{2} + C$$

for $0 < r < +\infty$ and $-\pi < \theta < \pi$.

Remark 20. In Section 3 we will give a second proof of Proposition 19.

Remark 21. We already know that $u = \text{cracktip} \times \mathbb{R}$ is a global minimizer in $\mathbb{R}^3$ (see [8]).

To prove Proposition 19 we will use the following well known result.

Proposition 22 ([5], [13]). The smallest positive eigenvalue for $-\Delta_n$ in $S^2 \setminus P$ is $\frac{3}{4}$, the corresponding eigenspace is of dimension 1 generated by the restriction on $S^2$ of the following function in cylindrical coordinates

$$u(r, \theta, z) = r^{\frac{1}{2}} \sin \frac{\theta}{2}$$

for $0 < r < +\infty$ and $-\pi < \theta < \pi$.

Now the proof of Proposition 19 can be easily deduce from Proposition 22 and Theorem 15.

Proof of Proposition 19: If $(u, P)$ is a global minimizer, we know that after removing a constant the restriction of $u$ to the unit sphere is an eigenfunction for $-\Delta_n$ in $S^2 \setminus P$ associated to the eigenvalue $\frac{3}{4}$. Therefore, from Proposition 22 we know that

$$u(r, \theta, z) = Cr^{\frac{1}{2}} \sin \frac{\theta}{2}$$

so we just have to determinate the constant $C$. But by a well known argument about Mumford-Shah minimizers we prove that $C$ must be equal to $\pm \sqrt{\frac{2}{\pi}}$ (see [8] Section 61 for more details).

Now set

$$S_\omega := \{(r, \theta, 0); r > 0, \theta \in [-\omega, \omega]\}$$
Proposition 23. There is no global Mumford-Shah minimizer in \( \mathbb{R}^3 \) such that \( K \) is an angular sector of type \((u, S_\omega)\) for \( 0 < \omega < \frac{\pi}{2} \) or \( \frac{\pi}{2} < \omega < \pi \).

Proof: According to Theorem 15, if \((u, S_\omega)\) is a global minimizer, then \(u - u_0\) is a homogenous harmonic function of degree \( \frac{1}{2} \), thus its restriction to \( S^2 \setminus S_\omega \) is an eigenfunction for \(-\Delta_n\) associated to the eigenvalue \( \frac{3}{4} \). Now if \( \lambda(\omega) \) denotes the smallest eigenvalue on \( \partial B(0,1) \setminus S_\omega \), we know by Theorem 2.3.2. p.47 of [14] that \( \lambda(\omega) \) is non decreasing with respect to \( \omega \). Since \( \lambda(\frac{\pi}{2}) = \frac{3}{4} \), we deduce that for \( \omega < \frac{\pi}{2} \), we have

\[
\lambda(\omega) \geq \frac{3}{4}.
\] (19)

In [14] page 53 we can find the following asymptotic formula near \( \omega = \frac{\pi}{2} \)

\[
\lambda(\omega) = \frac{3}{4} + \frac{2}{\pi} \cos \omega + O(\cos^2 \omega).
\] (20)

this proves that the case when (19) is a equality only arises when \( \omega = \frac{\pi}{2} \). Thus such eigenfunction \( u \) doesn’t exist.

Consider now the case \( \omega > \frac{\pi}{2} \). For \( \omega = \pi \) there are tow connected components. Thus 0 is an eigenvalue of multiplicity 2. The second eigenvalue is equal to 2. Therefore, for \( \omega = \pi \) the spectrum is

\[
0 \leq 0 \leq 2 \leq \lambda_3 \leq ... \quad \omega = \pi
\]

By monotonicity, when \( \omega \) decreases, the eigenvalues increase. Since the domain becomes connexe, 0 become of multiplicity 1 thus the second eigenvalue become positive. The spectrum is now

\[
0 \leq \lambda_1 \leq \lambda_2 \leq ... \quad \omega < \pi
\]

with \( \lambda_i \geq 2 \) for \( i \geq 2 \). Thus the only eigenvalue that could be equal to \( \frac{3}{4} \) is \( \lambda_2 \) which is increasing from from 0 to \( \frac{3}{4} \), reached for \( \omega = \frac{\pi}{2} \). Now (20) says that the increasing is strict near \( \omega = \frac{\pi}{2} \). Therefore there is no eigenvalue equal to \( 3/4 \) for \( \omega > \frac{\pi}{2} \) and there is no possible global minimizer.

\[ \square \]

3 Second proof of Propositions [19] and [22]

Here we want to give a second proof of Proposition [19] without using Theorem [15] and which do not use Proposition [22]. In a remark at the end of this section, we will briefly explain how to use this proof of Proposition [19] in order to obtain a new proof of Proposition [22] as well.

Let assume that \( K \) is a half plane in \( \mathbb{R}^3 \). We can suppose for instance that

\[
K = P := \{x_2 = 0\} \cap \{x_1 \leq 0\}
\] (21)
We begin by studying the harmonic measure in $\mathbb{R}^3 \setminus P$.

Let $B := B(0, R)$ be a ball of radius $R$ and let $\gamma$ be the trace operator on $\partial B(0, R) \setminus P$. We denote by $T$ the image of $W^{1,2}(B \setminus K)$ by $\gamma$. We also denote by $C^0_b(\partial B \setminus K)$ the set of continuous and bounded functions on $\partial B(0, 1) \setminus P$. Finally set $A := T \cap C^0_b$. Obviously $A$ is not empty. To every function $f \in A$, Proposition 15.6. of [8] associates a unique energy minimizing function $u \in W^{1,2}(B \setminus K)$ such that $\gamma(u) = f$ on $\partial B \setminus P$. Since $u$ is harmonic we know that it is $C^\infty$ in $B \setminus K$. Let $y \in B \setminus K$ be a fixed point and consider the linear form $\mu_y$ defined by

$$
\mu_y : A \to \mathbb{R}, \quad f \mapsto u(y).$$

By the maximum principle for energy minimizers, we know that for all $f \in A$ we have

$$|\mu_y(f)| \leq \|f\|_\infty$$

thus $\mu_y$ is a continuous linear form on $A$ for the norm $\| \|_\infty$. We identify $\mu_y$ with its representant in the dual space of $A$ and we call it harmonic measure.

Moreover, the harmonic measure is positive. That is, if $f \in A$ is a non negative function, then (by the maximum principle) $\mu_y(f)$ is non negative. By positivity of $\mu_y$, if $f \in A$ is a non negative function and $g \in A$ is such that $fg \in A$, then since $(\|g\|_\infty + g)f$ and $(\|g\|_\infty - g)f$ are two non negative functions of $A$ we deduce that

$$|\langle fg, \mu_y \rangle| \leq \|g\|_\infty \langle f, \mu_y \rangle.$$

Now here is an estimate on the measure $\mu_y^R$.

**Lemma 24.** There is a dimensional constant $C_N$ such that the following holds. Let $R$ be a positive radius. For $0 < \lambda < \frac{R}{2}$ consider the spherical domain

$$C_\lambda := \{ x \in \mathbb{R}^3 ; |x| = R \text{ and } d(x, P) \leq \lambda \}.$$

Let $\varphi_\lambda \in C^\infty(\partial B(0, R))$ be a function between 0 and 1, that is equal to 1 on $C_\lambda$ and 0 on $\partial B(0, R) \setminus C_2\lambda$ and that is symmetrical with respect to $P$. Then for every $y \in B(0, \frac{R}{2}) \setminus P$ we have

$$\mu_y^R(\varphi_\lambda) \leq C_\lambda \frac{\lambda}{R}.$$

**Proof:** Since $\varphi_\lambda$ is continuous and symmetrical with respect to $P$, by the reflection principle, its harmonic extension $\varphi$ in $B(0, R)$ has a normal derivative equal to zero on $P$ in the interior of $B(0, R)$. Moreover $\varphi_\lambda$ is clearly in the space $A$. Thus by definition of $\mu_y$,

$$\varphi(y) = \langle \varphi_\lambda, \mu_y^R \rangle.$$
On the other hand, since $\varphi$ is continuous on the entire sphere, we also have the formula

$$\varphi(y) = \frac{R^2 - |y|^2}{N\omega_N R} \int_{\partial B_R} \frac{\varphi(x)}{|x - y|^3} ds(x)$$

with $\omega_N$ equal to the measure of the unit sphere. In other words

$$\mu_y^R(\varphi) = \frac{R^2 - |y|^2}{N\omega_N R} \int_{\partial B_R} \frac{\varphi(x)}{|x - y|^3} ds(x).$$

For $x \in \partial B_R$ we have

$$\frac{1}{2}R \leq |x| - |y| \leq |x - y| \leq |x| + |y| \leq \frac{3}{2}R.$$

We deduce that

$$\mu_y^R(\varphi) \leq C_N \frac{1}{R^2} \int_{C_2\lambda} ds.$$

Now integrating by parts,

$$\int_{C_\lambda} ds = 2 \int_0^\lambda 2\pi \sqrt{R^2 - w^2} dw$$

$$= 4\pi \frac{\lambda}{2} \sqrt{R^2 - \lambda^2} + R^2 \arcsin\left(\frac{\lambda}{R}\right)$$

$$\leq CR\lambda$$

because $\arcsin(x) \leq \frac{\pi}{2}x$. The proposition follows.

Now we can prove the uniqueness of cracktip $\times \mathbb{R}$.

**Second Proof of Proposition 19**: Let us show that $u$ is vertically constant. Let $t$ be a positive real. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ set $x_t := (x_1, x_2, x_3 + t)$. We also set

$$u_t(x) := u(x) - u(x_t).$$

Since $u$ is a function associated to a global minimizer, and since $K$ is regular, we know that for all $R > 0$, the restriction of $u$ to the sphere $\partial B(0, R) \setminus K$ is continuous and bounded on $\partial B(0, R) \setminus K$ with finite limits on each sides of $K$. It is the same for $u_t$. Thus for all $x \in \mathbb{R}^3 \setminus P$ and for all $R > 2||x||$ we can write

$$u_t(x) := \langle u_t|_{\partial B(0, R) \setminus P}, \mu_x^R \rangle$$

where $\mu_x$ is the harmonic measure defined in [22]. We want to prove that for $x \in \mathbb{R}^3 \setminus P$, $\langle u_t|_{\partial B(0, R) \setminus P}, \mu_x^R \rangle$ tends to 0 when $R$ goes to infinity. This will prove that $u_t = 0$. 

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So let $x \in \mathbb{R}^3 \setminus P$ be fixed. We can suppose that $R > 100(\|x\| + t)$. Let $C_\lambda$ and $\varphi_\lambda$ be as in Lemma 24. Then write

$$u_t(x) = \langle u_t|_{\partial B(0,R) \setminus P}, \varphi_\lambda, \mu_x^R \rangle + \langle u_t|_{\partial B(0,R) \setminus P}(1 - \varphi_\lambda), \mu_x^R \rangle.$$ 

Now by a standard estimate on Mumford-Shah minimizers (that comes from Campanato’s Theorem, see [3] p. 371) we have for all $x \in \mathbb{R}^N \setminus P$,

$$|u_t(x)| \leq C \sqrt{t}.$$ 

Then, using Lemma 24 we obtain

$$\left| \langle u_t|_{\partial B(0,R) \setminus P}, \varphi_\lambda, \mu_x^R \rangle \right| \leq C \sqrt{t} \lambda R.$$ 

On the other hand, for the points $y$ such that $d(y, P) \geq \lambda$, since $\bar{u}: u(\cdot) - u(y)$ is harmonic in $B(y, d(y, P))$ we have, by a classical estimation on harmonic functions (see the introduction of [12])

$$|\nabla \bar{u}(y)| \leq C \frac{1}{d(y, P)} \|\bar{u}\|_{L^\infty(\partial B(y, \frac{3}{2}d(y, P)))}.$$ 

Now using Campanato’s Theorem again we know that

$$\|\bar{u}\|_{L^\infty(\partial B(y, \frac{3}{2}d(y, P)))} \leq C d(y, P)^{\frac{1}{2}}$$

thus

$$|\nabla u(y)| \leq C \frac{1}{d(y, P)^{\frac{1}{2}}}$$

and finally by the mean value theorem we deduce that for all the points $y$ such that $d(y, P) \geq \lambda$,

$$|u_t(y)| \leq C \sup_{z \in [y, y_t]} |\nabla u(z)| \cdot |y - y_t| \leq \frac{t}{\lambda^2}.$$ 

Therefore,

$$\left| \langle u_t|_{\partial B(0,R) \setminus P}(1 - \varphi_\lambda), \mu_x^R \rangle \right| \leq Ct \frac{1}{\lambda^2}.$$ 

So

$$|u_t(x)| \leq C \sqrt{t} \frac{\lambda}{R} + Ct \frac{1}{\lambda^2}$$

thus by setting $\lambda = R^{\frac{1}{2}}$ and by letting $R$ go to $+\infty$ we deduce that $u_t(x) = 0$ thus $z \mapsto u(x, y, z)$ is constant.

Now we fix $z_0 = 0$ and we introduce $P_0 := P \cap \{z = 0\}$. We want to show that $(u(x, y, 0), P_0)$ is a global minimizer in $\mathbb{R}^2$. Let $(v(x, y), \Gamma)$ be a competitor for $u(x, y, 0)$ in...
the 2-dimensional ball $B$ of radius $\rho$. Let $C$ be the cylinder $C := B \times [-R,R]$. Define $\tilde{v}$ and $\tilde{\Gamma}$ in $\mathbb{R}^3$ by

$$\tilde{v}(x,y,z) = \begin{cases} v(x,y) & \text{if } (x,y,z) \in C \\ u(x,y,z) & \text{if } (x,y,z) \notin C \end{cases}$$

$$\tilde{\Gamma} := (C \cap [\Gamma \times [-R,R]]) \cup (P \setminus C) \cup (B \times \{\pm R\}).$$

It is a topological competitor because $\mathbb{R}^3 \setminus P$ is connected (thus $P$ doesn’t separate any points). Now finally let $\tilde{B}$ be a ball that contains $C$. Then $(\tilde{v}, \tilde{\Gamma})$ is a competitor for $(u,P)$ in $\tilde{B}$. By minimality we have:

$$\int_{\tilde{B}} |\nabla u|^2 + H^2(P \cap \tilde{B}) \leq \int_{\tilde{B}} |\nabla \tilde{v}|^2 + H^2(\tilde{\Gamma} \cap \tilde{B}).$$

In the other hand $u$ is equal to $\tilde{v}$ in $\tilde{B} \setminus C$ and it is the same for $\Gamma$ and $\tilde{\Gamma}$. We deduce

$$\int_C |\nabla u|^2 dxdydz + H^2(P \cap C) \leq \int_C |\nabla \tilde{v}|^2 dxdydz + H^2(\tilde{\Gamma} \cap C).$$

Now, since $u$ and $\tilde{v}$ are vertically constant, $\nabla_z u = \nabla_z \tilde{v} = 0$, and $\nabla_x u$, $\nabla_y u$ are also constant with respect to the variable $z$ (as for $\tilde{v}$). Thus

$$2R \int_B |\nabla u(x,y,0)|^2 dxdy + H^2(P \cap C) \leq 2R \int_B |\nabla v(x,y)|^2 dxdy + H^2(\tilde{\Gamma} \cap C).$$

To conclude we will use the following lemma.

**Lemma 25.** If $\Gamma$ is rectifiable and contained in a plane $Q$ then

$$H^2(\Gamma \times [-R,R]) = 2RH^1(\Gamma).$$

**Proof:** We will use the coarea formula (see Theorem 2.93 of [3]). We take $f : \mathbb{R}^3 \to \mathbb{R}$ the orthogonal projection on the coordinate orthogonal to $Q$. By this way, if $E := \Gamma \times [-R,R]$, we have $E \cap f^{-1}(t) = \Gamma$ for all $t \in [-R,R]$. $E$ is rectifiable (because $\Gamma$ is by hypothesis). So we can apply the coarea formula. To do this we have to calculate the jacobian $c_kd^Ef_x$. By construction, the approximate tangente plane in each point of $E$ is orthogonal to $Q$. We deduce that if $T_x$ is a tangent plane, then there is a basis of $T_x (\vec{b}_1, \vec{b}_2)$ such that $\vec{b}_1$ is orthogonal to $Q$. Since the function $f$ is the projection on $\vec{b}_1$, and its derivative as well (because $f$ is linear ) we obtain that the matrix of $d^Ef_x : T_x \to \mathbb{R}$ in the basis $(\vec{b}_1, \vec{b}_2)$ is

$$d^Ef_x = (1,0)$$

thus

$$c_kd^Ef_x = \sqrt{det[(1,0)^t(1,0)]} = 1.$$
Therefore

\[ H^2(E) = \int_{-R}^{R} H^1(\Gamma) = 2R H^1(\Gamma). \]

Here we can suppose that \( \Gamma \) is rectifiable. Indeed, the definition of Mumford-Shah minimizers is equivalent if we only allow rectifiable competitors. This is because the jump set of a SBV function is rectifiable and in [11] it is proved that the relaxed functional on the SBV space has same minimizers.

So we have

\[
2R \int_B |\nabla u(x,y,0)|^2 dx dy + 2R H^1(P \cap B) \leq 2R \int_B |\nabla v(x,y)|^2 dx dy + 2R H^1(\Gamma \cap B) + H^2(B \times \{ \pm R \}).
\]

Then, dividing by \( 2R \),

\[
\int_B |\nabla u(x,y,0)|^2 dx dy + H^1(P \cap B) \leq \int_B |\nabla v(x,y)|^2 dx dy + H^1(\Gamma \cap B) + \pi \rho^2 R/2
\]

thus, letting \( R \) go to infinity,

\[
\int_B |\nabla u(x,y,0)|^2 dx dy + H^1(P \cap B) \leq \int_B |\nabla v(x,y)|^2 dx dy + H^1(\Gamma \cap B).
\]

This last inequality proves that \((u(x,y,0), P_0)\) is a global minimizer in \( \mathbb{R}^2 \), and since \( P_0 \) is a half-line, \( u \) is a cracktip.

\[ \square \]

Remark 26. Using a similar argument as the preceding proof, we can show that the first eigenvalue for \(-\Delta\) in \( S^2 \setminus P \) with Neumann boundary conditions (where \( P \) is still a half-plane), is equal to \( \frac{3}{4} \). Moreover we can prove that the eigenspace is of dimension 1, generated by a function of type cracktip \( \times \mathbb{R} \), thus we have a new proof of Proposition 22. The argument is to take an eigenfunction \( f \) in \( S^2 \setminus P \), then to consider \( u(x) := \|x\|^\alpha f(\frac{x}{\|x\|}) \) with a good coefficient \( \alpha \in [0, \frac{1}{2}] \) that makes \( u \) harmonic. Finally we use the same sort of estimates on the harmonic measure to prove that \( u \) is vertically constant. Thus we have reduced the problem in dimension 2 and we conclude using that we know the eigenfunctions on the circle. A detailed proof is done in [15].

4 Open questions

As it is said in the introduction, this paper is a very short step in the discovering of all the global minimizers in \( \mathbb{R}^N \). This final goal seems rather far but nevertheless some open questions might be accessible in a more reasonable time. All the following questions were pointed out by Guy David in [8], and unfortunately they are still open after this paper.

- Suppose that \((u, K)\) is a global minimizer in \( \mathbb{R}^N \). Is it true that \( K \) is conical?
• Suppose that \((u, K)\) is a global minimizer in \(\mathbb{R}^N\), and \(K\) is a cone. Is it true that \(\frac{3-2N}{4}\) is the smallest eigenvalue of the Laplacian on \(S^{N-1}\setminus K\) ?

• Suppose that \((u, K)\) is a global minimizer in \(\mathbb{R}^3\), and suppose that \(K\) is contained in a plan (and not empty). Is it true that \(K\) is a plane or a half-plane ?

• Could one found an extra global minimizer in \(\mathbb{R}^3\) by blowing up the minimizer described in section 76.c. of [8] (see also [17])?

One can find other open questions on global minimizers in the last page of [8].

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