Quantum corrections to mass and central charge of supersymmetric solitons

Alfred Scharff Goldhaber

C. N. Yang Institute for Theoretical Physics,
SUNY Stony Brook, NY 11794-3840 USA

Anton Rebhan

Institute for Theoretical Physics, Vienna University of Technology,
Wiedner Hauptstr. 8–10, A-1040, Vienna, Austria

Peter van Nieuwenhuizen

C. N. Yang Institute for Theoretical Physics,
SUNY Stony Brook, NY 11794-3840 USA

Robert Wimmer

Institute for Theoretical Physics,
University of Hannover,
Appelstr. 2, D-30167, Germany

We review some recent developments in the subject of quantum corrections to soliton mass and central charge. We consider in particular approaches which use local densities for these corrections, as first discussed by Hidenaga Yamagishi. We then consider dimensional regularization of the supersymmetric kink in 1+1 dimensions and an extension of this method to a 2+1-dimensional gauge theory with supersymmetric abelian Higgs vortices as the solitons.
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I. INTRODUCTION: LOCAL DENSITIES FOR QUANTUM ENERGY AND CENTRAL CHARGE

A characteristic theme in the work of Hidenaga Yamagishi is the exploration of quantum field contributions to the structure of solitons. The first effort was his masterful study of the effect on the Dirac electron vacuum of the choice of chiral boundary condition on the electron wave function at the location of a Dirac monopole [1]. This problem can be viewed as an appropriate limit of the case of Dirac isospinor fermions interacting with an 't Hooft-Polyakov monopole, in which one member of the isospinor doublet becomes extremely light. He showed that there is a fractional electric charge at the monopole (i.e., dyon charge), in the form of a polarization charge of the electron vacuum. Thus the boundary condition in this context expresses the vacuum angle earlier shown by Witten to imply fractional dyon charge in a theory without fermions [2].

An equally remarkable and original contribution was Yamagishi’s introduction, in the context of solitons in one space and one time dimension, of the concept of local quantum energy density [3]. He was led to investigate this subject by the fact that different calculations of the quantum energy for a soliton in $\mathcal{N} = 1$ supersymmetric theory gave different results, both zero and nonzero. He realized that there could be energies associated with the choice of boundary conditions on the wave functions for field oscillations, governing their behavior at the edges of a large region containing the soliton. Computing local densities would eliminate this ambiguity, because those contingent boundary energies would be localized far from the soliton center.

Yamagishi recognized the necessity of proper renormalization of the quantum perturbation theory for obtaining correct results, and described how to do this renormalization and make the resulting calculation systematic. His conclusion was that there is a nontrivial local energy density in the examples he studied, but that the integrated energy density vanishes. This conclusion had considerable appeal, because it implemented an identity between the energy and the central charge (whose quantum correction was expected to vanish [4, 5]). Indeed Yamagishi calculated also the local central charge density, verifying the equality of the two integrated densities. The aforementioned equality arises because in such theories, of the two supersymmetry generators (corresponding to the two components of a fermion wave function), one remains unbroken even in the presence of the soliton. The difference between energy and central charge is proportional to the norm of a state generated by action of a supersymmetry operator on the soliton ground state. Thus, if one such operator is unbroken, i.e., annihilates the ground state, the equality holds.

Well over a decade passed after this work, during which there was little further development towards consensus on the correct value of the quantum mass correction for solitons in such 2D supersymmetric theories. The next stage began when Rehan and van Nieuwenhuizen [RV] realized that a naive energy cutoff used in much of the susy kink literature was inconsistent with other, more carefully defined regularization methods. They used mode number regularization (described below) to compute the total quantum correction for a supersymmetric kink with periodic boundary conditions on the fermion as well as boson

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1 Yamagishi’s analysis contained as a special case the known fact that in the presence of strictly massless fermions chiral rotation undoes the effect of vacuum angle, so that the latter becomes unobservable. Also, in light of the main topic of the present work, it is worth noting that Yamagishi computed the (logarithmically divergent) energy associated with the boundary condition.
fluctuation wave functions [6].\textsuperscript{2} Of course, this nonzero result, differing from some prior results, was open to the objection that it might include a boundary energy contribution. Not long after, the same two authors joined by Nastase and Stephanov [NRSV] imposed ‘natural’ or ‘invisible’ boundary conditions on the fermion wave functions both in the trivial or vacuum background and in the kink background (so that there could be no extra energy localized near the boundary), and reproduced one of the finite answers from the earlier period [8]. Graham and Jaffe [GJ] did a calculation in terms of scattering phase shifts, with no explicit dependence on boundary conditions, and obtained the same result [9].

While this new activity agreed with the result first found by Schonfeld in his consideration of a kink-antikink system with the two objects well separated [10], a complete understanding required reconciliation of the global result with the local calculation of Yamagishi. Shifman, Vainshtein, and Voloshin [SVV] [11], stimulated by the work of the Stony Brook–Vienna group, decided to tackle the question head-on by returning to the calculation of the local central charge density pioneered by Yamagishi. They identified an ambiguity hidden in the earlier work. Yamagishi had been explicit in his computation of renormalization effects, but had not discussed explicitly the other ‘r’ required for a one loop calculation: regularization. The point here, which Yamagishi clearly understood, is that in calculating an energy sum one must subtract from the sum in the presence of the soliton the equally (quadratically) divergent sum in a flat, or vacuum background.\textsuperscript{3} The subtracted quantity is only logarithmically divergent, and that is the divergence compensated by renormalization. Given the regularization scheme, one still has to fix the finite parts in the renormalization by suitable renormalization conditions. Lacking an explicit regularization scheme (possibly he had simple energy cutoff in mind), Yamagishi was not in a position to determine uniquely the finite part of the renormalization. By implementing higher-derivative regularization for the central charge density, SVV were able to accomplish this while explicitly maintaining supersymmetry. They obtained a nonzero integrated correction to the central charge, precisely equal to the energy shift previously calculated by Schonfeld, as well as NRSV and GJ.

Thus the equality of mass and central charge which Yamagishi had found was preserved, but both quantities were shifted from zero by an extra finite piece which was not expected. NRSV, on the basis of their nonzero result for the quantum correction to the energy, suspected the existence of an anomaly, given the naive expectation that the quantum correction to the central charge should vanish. GJ did not discuss anomalies at all, but did notice that the terms corresponding to each state in the unregulated, unrenormalized energy-density sum are equal one to one with the corresponding terms in the central charge density. They accomplished regulation and renormalization for energy by subtracting from the divergent sum the divergent contribution from the first Born approximation to the phase shift, and in the case of fermions also the divergent part of the second Born approximation. They justified this choice by noting that the no-tadpole condition in the trivial sector requires a counterterm for the soliton mass which is precisely equal to the integral over momentum of the contribution from the subtracted divergent phase shifts. They took the momentum cut-

\textsuperscript{2} They made this choice of boundary conditions following the original work for a kink in purely bosonic theory by Dashen, Hasslacher, and Neveu [DHN] [7]. Amusingly, in the bosonic theory, the choice of boundary conditions makes no difference to the result.

\textsuperscript{3} Actually in the supersymmetric system the quadratic divergence in the trivial sector cancels between bosons and fermions, but this just replaces the previous regulation problem by one of correctly matching the energy sums for the bosons and fermions in the kink sector.
off used to define the counterterm equal to the momentum cutoff in the sum over momenta of contributions from continuum states to the energy shift. Having already determined the counterterm (using naive energy cutoff), by this choice they implicitly defined a regularization scheme. To implement the equality of central charge with energy, they used the identical subtraction to define central charge as they had used for energy. This certainly was consistent, but it left open the question whether and if so where an anomalous contribution to the central charge might be identified, as is remarked at the end of the paper of SVV. In any case, what remained to be demonstrated was that one could deduce the regularization and renormalization of the central charge directly by analyzing its field-theoretic structure. SVV did exactly that, showing explicitly that there is an anomaly, which preserves supersymmetry by providing a shift in the central charge equal to that in the energy. Like the famous chiral anomaly this one is an automatic consequence of using a regularization scheme that preserves a relevant symmetry, in the present case supersymmetry.

In an important respect the line of development begun by Yamagishi still was incomplete, because the local energy density corresponding to the established global energy and local central charge density was not computed directly. That was accomplished by Goldhaber, Litvintsev, and van Nieuwenhuizen [GLV2] [12] and independently by Wimmer [13]. They demonstrated by explicit checks the validity of a local regularization scheme called local mode regularization. This scheme makes computation of energy density of a soliton in 2D theory very clear and simple. Not surprisingly, it confirms directly for the supersymmetric kink the equality of energy density and central charge density proposed by Yamagishi and demonstrated by SVV. At the same time, local energy density is an interesting quantity in non-supersymmetric theories, where the definition of central charge is not clear.

Even with the above results, there remains an open frontier in the computation of quantum energy densities, especially in the direction of higher dimensions. Clearly, the tree of local energy determination which Hidenaga Yamagishi planted continues to grow and bear fruit, a tribute to his insight and originality.

In section II we focus on a development which was essential to the new understanding of local energy and central charge densities, namely, the careful regularization by the older, global methods of the total mass of a soliton, as discussed in GLV1 [14]. This discussion is followed by an explicit demonstration of the universal equivalence, in results for the anomalous or high-energy contribution to quantum energy of a field in the presence of some background potential, between higher-derivative regularization and local mode regularization.

In section III and IV, we describe some recent work in (slightly) higher dimensions. We begin by embedding the supersymmetric kink as a domain wall in 2+1 dimensions and use this to set up a supersymmetry-preserving dimensional regularization scheme. This allows us to derive the anomaly in the central charge in a particularly transparent manner, which is found to be made possible by parity violation and a corresponding nonvanishing expectation value for the momentum operator in the extra dimension. In section IV we apply this method to the supersymmetric abelian vortex 2+1 dimension and determine its quantum corrections to mass and central charge.
II. SOLITONS IN 1+1 DIMENSIONS

A. Fermion zero modes in global mode regularization

One method of regularization, utilized by DHN in their investigation of the quantum correction to the mass of a kink or a sine-Gordon soliton in a purely bosonic theory, is called \textit{mode number regularization}. There is a very concrete picture behind this scheme. Imagine introducing, in what initially is vacuum, some background potential which influences the motion of a fermion obeying the Schrödinger equation. Suppose further that at the start one had a free Fermi sea filled to some level, so that all possible states up to a maximum number $N$ were occupied. Then, provided the boundary conditions on the wave functions at large positive and negative values of the spatial coordinate $x$ prevent any probability leakage through the boundaries, the total number $N$ will not change when the potential is introduced. Further, if the Fermi level is high enough compared to the magnitude of the potential, then there will be no level crossings near that level as the potential is gradually altered. Thus, just as before the potential was introduced, the first $N$ states will be filled, though the maximum energy may be changed slightly, and the low-energy spectrum may be altered dramatically, for example, to include bound states. To find the regulated quantum shift in energy one may subtract the sums of energies of the first $N$ states with and without the potential present. The net sum might still be divergent with $N$, in which case renormalization would be required as well, but in any case the effect of this regularization would be well defined.

One now can adopt the same maneuver for a relativistic Bose field in the presence of a potential. Before the potential is introduced, one uses an energy regularization to say that all solutions of the wave equation up to a maximum energy $\epsilon_{\text{max}}$ are counted. There are $N$ pairs of these solutions, each pair with one positive and one (equal) negative frequency. After introducing the potential, we again require $N$ pairs of solutions, and those solutions are all the ones having energy $\epsilon'_i \leq \epsilon'_{\text{max}}(N)$. Thus we find a regulated, but not yet renormalized, energy,

$$E_{\text{reg}} = \frac{1}{2} \sum_{i=1}^{N} (\epsilon'_i - \epsilon_i),$$

(1)

where the factor $\frac{1}{2}$ is the familiar coefficient in the zero-point energy contribution of each mode.

A subtlety arises in this calculation if the presence of a potential happens to introduce a zero-frequency solution, which would occur either accidentally or as a result of some particular symmetry obeyed by the potential. For example, if the potential for Bose fluctuations arises from the presence of a nontrivial classical soliton solution of the equations of motion of the Bose field, then there is clearly a zero frequency deformation corresponding to uniform translation of the soliton. As mentioned above, solutions of (equal) positive and negative frequency come in pairs, and one pair defines one state. At zero frequency there is also one state, because the ‘momentum’ conjugate to the Bose field for the translation solution becomes the momentum of the soliton moving with some definite velocity. Each pair of solutions of the bosonic field equations (including any at zero frequency) corresponds to one term in (1).

For fermions the situation is less familiar. Again, there are paired positive and negative frequency solutions, each corresponding to a term in (1). However, if there is only one zero energy solution of the fermionic fluctuation equations, this does not correspond to a term
in (1). The coefficient of this solution in the expansion of the Fermi field is an operator $c_0$ satisfying $c_0^2 = 1$, and that $c_0$ appears in the expression for one of the two supersymmetry operators (the one that does not annihilate the soliton ground state). Several remarkable facts follow from these observations. First, such a wave function does not correspond to a zero mode of the soliton, because any mode requires two independent, noncommuting nilpotent creation and annihilation operators, and here one has instead one hermitean, idempotent operator (which may be represented as a Pauli $\sigma$ matrix). This immediately introduces a serious difficulty for the standard recipe of mode regularization, because the spectrum of nonnegative frequency solutions no longer is in one-to-one correspondence with the number of terms in (1). Secondly, because the one supersymmetry operator associated with the zero-frequency solution does not annihilate the soliton ground state, it follows that the ground state spontaneously breaks supersymmetry.

There is another remarkable aspect of this phenomenon. The ground state not only is not annihilated by a supersymmetry operator, but becomes an eigenstate of that operator, if one decomposes the kink vacua $|K\rangle$ and $c_0|K\rangle$ into $\frac{1}{\sqrt{2}}(1 + c_0)|K\rangle$ and $\frac{1}{\sqrt{2}}(1 - c_0)|K\rangle$ [15]. This is remarkable because there is a discrete $Z_2$ operator which takes the Fermi field into its negative, and which in other contexts could be used to prove that one cannot build a coherent superposition of a boson and a fermion. Now we find that an operator anticommuting with this $Z_2$ operator leaves the ground state unchanged. Therefore two states with equal and opposite eigenvalues for the supersymmetry generator must be identified. In other words, the Hilbert space splits into two noncommunicating parts, which are gauge copies of each other [14]. This is a discrete analogue to the Higgs mechanism, in which a continuous local gauge symmetry is hidden because a scalar field in a nontrivial representation of that gauge symmetry has a nonzero expectation value.

Something not considered in GLV1 is the generalization of this discussion about the $Z_2$ gauge symmetry to the case of many solitons, which would be relevant for sine-Gordon solitons if not for the kink (the focus of that study). The $Z_2$ symmetry is not a local gauge symmetry, so that for a multisoliton system the relative signs of eigenvalues of the supersymmetry operator acting on different solitons might be significant. This may be related to a discovery of Moore and Read long before, in a 2+1-dimensional system which appears to describe the 5/2 state of the fractional quantum Hall effect [16]. They discussed ‘nonabelian statistics’ for objects carrying electric charge $\mp e/4$. This kind of statistics seems to us interpretable in terms of the eigenvalues of the supersymmetry operator, which would reverse sign when one such object made a full circle around another. Specifically, for a system with $n$ solitons separated from each other by a finite distance, charge conjugation symmetry guarantees $[n/2]$ pairs of equal positive and negative frequency solutions, and for odd $n$ one zero frequency solution. To each of the paired solutions corresponds a fermion creation operator, and hence there are $2^{[n/2]}$ nearly degenerate states of the n-soliton system. For infinite separation, there are $n$ operators $c_0$ which each have equal positive and negative eigenvalues. Superficially this might imply $2^n$ distinct states, but as we already have argued there is a gauge redundancy which reduces this number. The calculation for finite separation shows that the reduction must bring that number down to $2^{[n/2]}$.

While these ideas are surprising and beautiful, one would like to have a clear idea what has happened to the principle of mode number regularization, now that the counting of

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4 Because a kink cannot be adjacent to another kink, pure multi-kink systems, with no anti-kinks, are not possible.
fermion modes suddenly has a lacuna. GLV1 found two different ways to deal with this question. First, following NRSV, one may use locally invisible (periodic or antiperiodic) boundary conditions in vacuum, and locally invisible (twisted periodic (TP) or twisted antiperiodic (TAP), where the twist refers to a chiral rotation of a wave function by $\pi/2$ at the one boundary with respect to that at the other) conditions in the presence of a kink. Now an assumption necessary to derive the equality of numbers of modes before and after the background is introduced no longer holds, because the twisting at the boundary allows ‘leakage’ of mode number at the boundary. This is a concept that goes back to an article of Goldstone and Wilczek describing adiabatic flow of fermion number out of a magnetic monopole as the Yukawa coupling of the fermions to a Higgs field is chirally rotated [17]. In their discussion the leakage is out to infinity rather than into the boundary. The final result for localized charge is the same, whether the chiral rotation is applied to the mass, while the boundary condition is fixed, or applied to the boundary condition, while the mass remains constant (as in Yamagishi’s work on monopole dyon charge [1]). The leakage in our example involves exactly half a unit of mode number, and this shift reconciles the count of fermion modes with the principle of mode number regularization.

A second approach is to use fixed boundary conditions, which insures no leakage of mode number, but to average over a set of boundary conditions, so that in the average any contributions to the energy localized near the boundary cancel out. In this case, the mode number regularization is used for each set of fixed boundary conditions, but in some cases there are two zero-frequency solutions in the kink sector, one localized near the kink and one localized near the boundary, and in some cases there is one in the kink sector localized at the kink and one in the vacuum sector localized near the boundary. When the difference between the sectors is two solutions in the kink sector, this implies one (delocalized) zero mode in the kink sector. Thus the average over all boundary conditions amounts to half a zero mode excess in the kink sector.

We see that the two different approaches agree: Effectively there is a half-mode at zero energy in the kink sector. This was discussed already by GJ (who analyzed half-modes at zero energy as well as nonlocalized ‘half-bound’ modes at the continuum threshold). What GLV1 added was a precise specification of the boundary conditions which make this notion well defined. The half-mode has the consequence for the mode sums used to compute the energy that at the cutoff one needs to include an excess of one half-mode in the sum for the vacuum energy compared to the sum for the kink energy. One may recognize this fact directly by matching modes so as to insure that no difference between the two sums linear in the energy will occur. Thus, as with many other examples of anomalies and associated regulation prescriptions, one may find the correct prescription and its implications either by focusing on the lowest energies (in this case zero modes), or on the highest energies.

The half mode bears an interesting relation to the original discovery by Jackiw and Rebbi [18] that a soliton can polarize a charged fermion vacuum to localize a half unit of fermion number around the soliton. For neutral excitations, there is no directly observable charge or charge density that corresponds to a mode or to the probability density in its wave function. Nevertheless modes of nonzero energy contribute to vacuum zero point energy and energy density. Thus mode number and mode density are mathematically well defined and indirectly observable, making them also meaningful, even if not so tangible as the directly observable charge and charge density of a charged field. In other words, mode number $1/2$ has operational significance just as does charge $1/2$. A further indication of this significance is additivity: Two configurations each possessing a zero frequency solution corresponding to
mode number 1/2, when brought within a finite distance of each other, accommodate one observable excitation with near-zero energy. Thus, even though it is not possible to define directly a “half-excitation” it is possible to count a half-mode, and to combine two of them to make not only a full mode but also a full excitation or fermion state.

Although the invisible boundary conditions guarantee that the quantum correction is precisely the mass shift of the soliton, there remains even in this case a strong reason for averaging over each of the two types of condition, namely to enforce time-reversal symmetry. A chiral or twisted boundary condition produces different spectra for right-moving and left-moving waves. Time reversal symmetry interchanges TP and TAP boundary conditions, so that by averaging over the two one restores the symmetry. In the vacuum or trivial sector, periodic and antiperiodic conditions (being real) each separately produce spectra invariant under the symmetry.

B. Local mode regularization from higher-derivative regularization

SVV in their discussion emphasized the value of using a local regularization scheme which enforces supersymmetry, in particular, higher-derivative regularization. In GLV2 it was shown that a familiar scheme, point–splitting regularization, implies local mode regularization for energy density. Here we show directly that higher-derivative regularization also implies local mode regularization for energy density, thus verifying explicitly its consistency with supersymmetry.

In higher derivative regularization, extra terms are added to the Lagrangian involving extra factors of the square of the spatial gradient of the field. This gives rise to an equation for small fluctuations of a Bose field, including a possible background ‘potential’ $V(x)$,

$$\omega^2(1-\partial_x^2/\Lambda^2)\phi = -\partial_x^2\phi + m^2\phi + V(x)\phi + \partial_x^4\phi/\Lambda^2 \ .$$

(2)

Our notation uses $\Lambda$ as the regulator mass called $M_r$ by SVV. The effect of the higher-derivative term on the classical kink solution and the resulting potential $V(x)$ for fluctuations tends to zero for large $\Lambda$, and therefore is ignored in our discussion, i.e., we use the $V$ obtained from the classical solution of the unregulated equations. Here as in GLV2, we may use the JWKB approximation to estimate the wave functions at large $\omega$. This allows one to determine with sufficient accuracy the phase space density of these wave functions both for the trivial or vacuum background with $V=0$ and the kink background with $V = -3m^2/2 \cosh^2(mx/2)$,

(3)

where $m$ is the mass for small fluctuations about the trivial vacuum. Then one may obtain a subtracted density which can be integrated to give the high-energy contribution to the regulated net local mode density, as we shall now see.

Concentrate on a high-energy regime, with wave vectors $|k| > K \gg \sqrt{V}$, with $\omega^2 \equiv k^2 + m^2/(1 + k^2/\Lambda^2)$, and take $\Lambda \gg K$. Write the wave function as $\phi(x) = e^{ikx}e^{i\phi(x)}$, and make a gradient expansion, $f(x) = f^{(0)}(x) + f^{(1)}(x) + \ldots$ :

$$2k f^{(0)}(1 + k^2/\Lambda^2) = -V(x) \ .$$

(4)

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5 For simplicity we focus here only on the pure Bose field case, but this time there is no difficulty in generalizing the procedure to include Fermi fields as well.
and

\[ 2k f^{(1)}(1 + \frac{k^2}{\Lambda^2}) = -\partial_x^2 i f^{(0)}(1 + 5\frac{k^2}{\Lambda^2}) = -i \partial_x V(x)(1 + 5\frac{k^2}{\Lambda^2})/2k(1 + \frac{k^2}{\Lambda^2}). \tag{5} \]

Because \( f^{(1)} \) is imaginary, it alters the modulus of \( \phi \), and because \( f^{(1)} \) and \( V \) both vanish at spatial infinity, this equality relating first derivatives of these two functions implies an equality relating the functions themselves.

We need now to compute the mode density difference between the case with and without nonzero \( V(x) \). Define normalization of the wave function \( \phi_k(x) \) so that \( \phi_k^2 \) is equal to unity far from the region of the potential. The regulated mode density before subtraction is then

\[ d\rho(x)/dk| = \phi_k(x)^* \frac{1}{1 - \partial_x^2/\Lambda^2} \phi_k(x)/\pi. \tag{6} \]

To lowest order in \( V/K^2 \) and \( V/\Lambda^2 \) we now compute the mode density difference

\[ d\delta\rho/d|k| = \frac{V(x)}{2\pi} \left\{ \frac{1}{K^2} \left[ \frac{k^2}{\Lambda^2} + 1 \right] + \frac{1}{\Lambda^2} \left[ \frac{k^2}{\Lambda^2} + 1 \right] \right\} \left[ \frac{k^2}{\Lambda^2} + 1 \right]^3. \tag{7} \]

Here the first term in the large braces comes from the influence of \( V(x) \) on the absolute square of the wave functions, while the second term comes from the influence of \( V \) on the regulating denominator in Eq. (6). To first order in \( V \), this denominator factor is

\[ [1 - \partial_x^2/\Lambda^2]^{-1} \approx \left\{ \frac{1}{K^2} - \frac{V(x)}{\Lambda^2} \left[ 1 + \frac{k^2}{\Lambda^2} \right] \right\}^{-1}. \tag{8} \]

Omitting terms of order \( 1/\Lambda \), the integral of (7) over the range \( |k| = K \) to \( |k| = \infty \) gives the extra contribution at high energies to the local mode density (number of modes per unit length)

\[ \rho_{\text{mode}}(x) = V(x)/2\pi K. \tag{9} \]

Note that this result is insensitive to \( \Lambda \gg K \).

In the original local mode regularization approach, this extra density was treated as all associated with momentum \( |k| = K \), so that the resulting ‘anomalous’ extra energy density resulting from the regularization is

\[ E_{\text{anom}}(x) = \frac{K}{2} \rho_{\text{mode}}(x) = V(x)/4\pi, \tag{10} \]

including a correction by a factor -2 to Eq.(57) of GLV2. Now, for the full calculation with higher-derivative regularization, we must compute \( \int_{|k| = K}^{\infty} d|k|(d\delta\rho/dk)(|k|/2) \), where the factor \( |k|/2 \) at the end is the zero-point energy corresponding to a given mode (neglecting quantities of \( O(m^2/k^2) \)). This integral is easily evaluated, yielding a total high-energy contribution to the regulated (but not yet renormalized) energy density

\[ E_{\text{unrenormalized}}(x) = \frac{V(x)}{8\pi} \left[ 2 + \ell n \frac{\Lambda^2}{K^2} \right]. \tag{11} \]
Here the anomaly term 2 in Eq. (11) bears exactly the same ratio to the term (compensated by mass renormalization) \( \ell n \Lambda^2 \) as found by SVV in their analysis of the supersymmetric kink, e.g., their Eq. (4.5). This ratio of course is trivial in local mode regularization, as is easy to see from the description above. However, in Eq. (11), the term 2 comes half from the first term in Eq. (7) and half from the second, while the logarithm clearly comes entirely from the first.

Looking at this straightforward but still somewhat intricate computation, one may understand why SVV focused on calculation of the anomaly in the central charge density, and did not attack directly the anomaly in the energy density. However, Eq. (11) shows that the result of local mode regularization for the anomaly is universally correct in higher-derivative regularization, and thus completes a circle:

DHN used (global) mode regularization for the mass of the bosonic kink, but refrained from tackling the problem with fermions. Yamagishi introduced local energy and central charge density calculations, but did not fully address the role of regularization in these calculations. A variety of approaches using global methods managed to respect constraints of regulation and still circumvent the difficulty of contributions to quantum energy from boundary conditions on fermions.

SVV made a complete calculation of the central charge density (both for the anomaly and for other contributions) but did not compute energy density directly, so that their method was not applicable in its original form for purely bosonic structures (or indeed any structure without supersymmetry). GLV2 introduced an efficient method, local mode regularization, for computing the anomaly in the energy density of any system, but previous history of regularization schemes implies that one should check whether a new scheme preserves symmetries which we want keep at the quantum level, in this case supersymmetry. The present calculation, verifying local mode regularization for arbitrary background potentials starting from higher-derivative regularization (which does preserve supersymmetry), brings us back to the beginning with a reliable local scheme for doing what DHN did globally.

There is another interesting aspect of local mode regularization. The derivation from point-splitting regularization in GLV2 involves a shift in the real part of the effective wave number at a given value of \( k \). However, the original method, and also the computation done here, involves an integration over \( k \) of the imaginary part of the wave number. This has a suggestive resemblance to dispersion relations connecting the real part of a scattering amplitude to an integral over the imaginary part, related by unitarity to a cross section.

C. Why is rigorous, manifest supersymmetry unnecessary for correct calculation?

We have seen that a completely rigorous and systematic way to assure consistency with supersymmetry in computing the quantum correction to the mass and central charge of a soliton is to insist on explicit supersymmetry at every step. Thus SVV adopted the approach of higher-derivative regularization in the ultraviolet and supersymmetric boundary conditions discretizing the spectrum to provide infrared regularization. They then computed a local central charge density in the vicinity of the kink, and its spatial integral in a large but finite region around the soliton, which gives the desired quantum correction. While this is the safest method, and also fairly straightforward because the central charge density has a weaker superficial divergence than the energy density, a number of other methods all yield the same result.

It seems to us that this not only demonstrates that the result is correct, but also suggests
that there is a deeper principle behind the robustness of that result. The principle, we suggest, is cluster decomposition. In GLV1 it was deduced erroneously from an incomplete calculation that besides localized energy in the vicinity of the kink and possible localized energy near the boundary, there can be a delocalized energy, giving a finite contribution. They imposed the principle of cluster decomposition to sum over different boundary conditions, so that the delocalized energy violating cluster decomposition would cancel out. It was observed by Wimmer [13], and argued in detail in the light of the discrete C,P, and T symmetries by Goldhaber, Rebhan, van Nieuwenhuizen, and Wimmer [19] that such delocalized energy does not appear in mode number regularization for known boundary conditions. Cluster decomposition is satisfied automatically. Thus, a boundary condition which violates the supersymmetry preserved by the kink can alter the energy localized near the boundary, but cannot influence the energy localized near the soliton, i.e., the quantum correction to the kink mass. This means that if one uses boundary conditions which do not result in boundary energy, or if one averages over conditions so that there is no net boundary energy, then the resulting global calculation is reliable for the mass of the kink. This is true even if the boundary conditions do not preserve supersymmetry – that violation does not propagate into the region of the kink.

As pointed out in GLV2, even methods whose consistency with supersymmetry has not been explicitly established, such as point-splitting regularization, give the same answer not only for the energy density but also for the central charge density. Thus these local densities are quite robust with respect to the choice of (local) regularization methods. This suggests that, despite the long history of discrepant calculations, one must make quite a large error to get the wrong answer for the local densities. For example, failure to specify carefully the regularization method can make the result indeterminate, and therefore most likely wrong. The history of incorrect global calculations draws attention in particular to regulation by energy cutoff, which was used in many of those calculations. As local mode regularization makes manifest, using a fixed and identical energy cutoff for kink and trivial backgrounds is incorrect. Litvintsev and van Nieuwenhuizen [20] proposed a method to repair energy cutoff, which is mathematically equivalent to replacing it with mode number regularization.

### III. DIMENSIONAL REGULARIZATION OF THE SUSY KINK

Usually dimensional regularization breaks susy. However, the 1+1 dimensional \( \mathcal{N} = 1 \) susy kink can be embedded as a domain wall in 2+1 dimensions with the same field content while keeping \( \mathcal{N} = 1 \) susy invariance.

For the corresponding classically BPS saturated domain wall (a 1+1 dimensional object by itself), [21] has also found a nontrivial quantum correction to the energy density. In order to have BPS saturation at the quantum level in 2+1 dimensions, there has to be a matching correction to the momentum in the extra dimension which corresponds to the central charge of the 1+1 dimensional case.

In this section we show that if one uses susy-preserving dimensional regularization by means of dimensional reduction from 2+1 dimensions, one indeed finds the required correction to the extra momentum. Such a nonvanishing correction turns out to be possible because the 2+1 dimensional theory spontaneously breaks parity.

By dimensionally reducing to 1+1 dimensions, the parity-violating contributions to the extra momentum turn out to provide an anomalous contribution to the central charge as obtained in Ref. [11], thereby giving a novel physical explanation of the latter [22]. This is
in line with the well-known fact that central charges of susy theories can be reinterpreted as "momenta" in higher dimensions.

The latter statement has to be handled with care, though. The classical central charge stems entirely from the classical antisymmetric part of the energy momentum tensor of the 2 + 1 dimensional theory and thus would be missed by dimensional reduction if one were to start in 2 + 1 dimensions with the gravitational energy momentum tensor, which is always symmetric on-shell (in the absence of local Lorentz anomalies) and which contains the genuine momentum operator. However, it is the symmetric part of the 2 + 1 dimensional EM-tensor which gives the anomalous contribution to the quantum central charge. This anomalous contribution can be reduced to a surface term and is thus completely determined by the topology of the soliton background, independent of the precise field profile in the bulk. Therefore when we refer to the $\varphi^4$ kink in the following, this is just a special case of a more general situation.

In the case of the susy kink, standard ('t Hooft-Veltman) dimensional regularization is seen to be compatible with susy invariance only at the expense of a spontaneous parity violation, which in turn allows non-vanishing quantum corrections to the extra momentum in one higher spatial dimension. On the other hand, as we shall see, the surface term that provides the classical central charge does not receive quantum corrections in dimensional regularization, by the same reason that led to null results previously in other schemes [5, 6, 8]. In dimensional regularization (by going up in the number of dimensions), the nontrivial anomalous quantum correction to the central charge operator is thus seen to be entirely the remnant of the spontaneous parity violation in the higher-dimensional theory in which a susy kink can be embedded by preserving minimal susy.

Alternatively, we shall consider dimensional regularization by dimensional reduction from 1 to 1-\(\epsilon\) spatial dimensions, which also preserves supersymmetry. In this case we show that an anomalous contribution to the central charge arises from the necessity to add an evanescent counterterm to the susy current [22]. This counterterm preserves susy but produces an anomaly in the conformal-susy current. We also construct the conformal central-charge current [22] whose divergence is proportional to the ordinary central-charge current and thus contains the central-charge anomaly as superpartner of the conformal-susy anomaly [11].

A. The model

A real scalar field model in 1+1 dimensions with spontaneously broken \(Z_2\) symmetry (\(\varphi \rightarrow -\varphi\)) has topologically nontrivial finite-energy solutions called "kinks" which interpolate between the two neighboring degenerate vacuum states, as for example \(\varphi = \pm v\). If the potential is of the form \(V(\varphi) = \frac{1}{2} U^2(\varphi)\) it has a minimal, \(\mathcal{N} = (1, 1)\), supersymmetric extension [23]

\[
\mathcal{L} = -\frac{1}{2} \left[ (\partial_\mu \varphi)^2 + U(\varphi)^2 + \bar{\psi} \gamma^\mu \partial_\mu \psi + U'(\varphi) \bar{\psi} \psi \right]
\]  

(12)

where \(\psi\) is a Majorana spinor, \(\bar{\psi} = \psi^T C\) with \(C \gamma^\mu = -(\gamma^\mu)^T C\). We use a Majorana representation of the Dirac matrices with \(\gamma^0 = -i\sigma^2, \gamma^1 = \sigma^3\), and \(C = \sigma^2\) in terms of the standard Pauli matrices \(\sigma^k\) so that \(\psi = (\psi^+)^T\) with real \(\psi^+(x,t)\) and \(\psi^-(x,t)\). (The reason for choosing \(\gamma^1 = \sigma^3\), rather than \(\gamma^1 = \sigma^1\), is that it diagonalizes the Dirac equation.)
The $\phi^4$ model is defined as the special case

$$U(\phi) = \sqrt{\frac{\lambda}{2}} (\phi^2 - v_0^2), \quad v_0^2 \equiv \mu_0^2 / \lambda$$

where the $Z_2$ symmetry of the susy action also involves the fermions according to $\phi \rightarrow -\phi, \psi \rightarrow \gamma^5 \psi$ with $\gamma^5 = \gamma^0 \gamma^1$. A classical kink at rest at $x = 0$ which interpolates between the two vacua is given by [24]

$$\varphi_K = v_0 \tanh \left( \frac{\mu_0 x}{\sqrt{2}} \right).$$

At the quantum level we have to renormalize, and we shall employ the simplest possible scheme\textsuperscript{6} which consists of putting all renormalization constants to unity except for a mass counter term chosen such that tadpole diagrams cancel completely in the trivial vacuum. So we set $Z_\phi = Z_\psi = Z_\lambda = 1$ and $\mu_0^2 = \mu^2 + \delta \mu^2$, for which at the one-loop level and using dimensional regularization we find

$$\delta \mu^2 = \lambda \delta v^2 = \lambda \int \frac{dk_0 dk d^d k}{(2\pi)^{d+1}} \frac{-i}{k^2 + m^2 - i\epsilon} = \lambda \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 + m^2]^{1/2}},$$

where $m = U'(v) = \sqrt{2} \mu$ is the tree-level mass of elementary bosons and fermions, and $k^2 = \tilde{k}^2 - k_0^2$.

The susy invariance of the model (12) under

$$\delta \phi = \bar{\epsilon} \psi, \quad \delta \psi = (\bar{\partial} \phi - U) \epsilon,$$

(with $\mu_0^2$ replaced by $\mu^2 + \delta \mu^2$) leads to the on-shell conserved Noether current

$$j_\mu = - (\bar{\partial} \phi + U(\phi)) \gamma_\mu \psi$$

and two conserved charges $Q^\pm = \int dx \, j^{\pm}$. The model (12) is equally supersymmetric in 2+1 dimensions, where we use $\gamma^2 = \sigma^1$, and the Noether current and charges are unchanged in form. The same renormalization scheme can be used, only the renormalization constant (15) has to be evaluated for $d = 2 - \epsilon$ in place of $d = 1 - \epsilon$ spatial dimensions.

While classical kinks in 1+1 dimensions have finite energy (rest mass) $M = m^3 / \lambda$, in 2+1 dimensions they yield domain walls with a profile given by (14) and finite surface (string) tension $M / L = m^3 / \lambda$. With a compact extra dimension one can use these configurations to form “domain strings” of finite total energy proportional to the length $L$ of the string when wrapped around the extra dimension.

The 2+1 dimensional case is different also with respect to the discrete symmetries of (12). In 2+1 dimensions, $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 = \pm 1$ corresponding to the two inequivalent irreducible representations for $\gamma^2 = \pm \sigma^1$. Therefore, the sign of the fermion mass (Yukawa) term can no longer be reversed by $\psi \rightarrow \gamma^5 \psi$ and there is no longer the $Z_2$ symmetry $\phi \rightarrow -\phi, \psi \rightarrow \gamma^5 \psi$.

What the 2+1 dimensional model does break spontaneously instead is parity, which corresponds to changing the sign of one of the spatial coordinates. The Lagrangian is

\textsuperscript{6} See [21] for a detailed discussion of more general renormalization schemes in this context.
invariant under $x^m \rightarrow -x^m$ for a given spatial index $m = 1$ or $m = 2$ together with $\varphi \rightarrow -\varphi$ (which thus is a pseudoscalar) and $\psi \rightarrow \gamma^m \psi$. Each of the trivial vacua breaks these invariances spontaneously, whereas a kink background in the $x^1$-direction with $\varphi_K(-x^1) = -\varphi_K(x^1)$ preserves $x^1 = x$ reflection symmetry, but breaks it with respect to $x^2 = y$.

This is to be contrasted with the 1+1 dimensional case, where parity ($x^1 \rightarrow -x^1$) can be represented either by $\psi \rightarrow \gamma^0 \psi$ and a true scalar $\varphi \rightarrow \varphi$ or by $\psi \rightarrow \gamma^1 \psi$ and a pseudoscalar $\varphi \rightarrow -\varphi$. The former leaves the trivial vacua invariant, and the latter the ground state of the kink sector.

B. Susy algebra

The susy algebra for the 1+1 and the 2+1 dimensional cases can both be covered by starting from 2+1 dimensions, the 1+1 dimensional case following from reduction by one spatial dimension.

In 2+1 dimensions one obtains [25]

$$\{Q^\alpha, Q_\beta\} = 2i (\gamma^M)^\alpha_\beta P_M, \quad (M = 0, 1, 2)$$

$$= 2i(\gamma^0 H + \gamma^1(\tilde{P}_x + \tilde{Z}_y) + \gamma^2(\tilde{P}_y - \tilde{Z}_x))^{\alpha}_\beta,$$

where we separated off two surface terms $\tilde{Z}_m$ in defining

$$\tilde{P}_m = \int d^d x \tilde{P}_m, \quad \tilde{P}_m = \tilde{\varphi} \partial_m \varphi - \frac{1}{2} (\bar{\psi} \gamma^0 \partial_m \psi),$$

$$\tilde{Z}_m = \int d^d x \tilde{Z}_m, \quad \tilde{Z}_m = U(\varphi) \partial_m \varphi = \partial_m \mathcal{W}(\varphi)$$

with $\mathcal{W}(\varphi) \equiv \int d\varphi U(\varphi)$. Note that the usual central charge density of the two-dimensional model, $\tilde{Z}_m$, is obtained by dimensional reduction of the antisymmetric part of the three-dimensional energy momentum tensor. The local version of the susy algebra (18) is obtained by a susy variation (16) of the supercurrent (17) as follows

$$T^{MN} \sim Tr(\gamma^M \delta j^N) = Tr(\gamma^M \gamma^N \gamma^P) \partial_P \varphi U(\varphi) + \text{symm. part}$$

$$\sim \varepsilon^{MNP} \partial_P \varphi U(\varphi) + \text{symm. part},$$

and the central charge density is then the momentum density $T^{02}$ in the reduced extra dimension.

With our choice of Dirac matrices the supercharges and the superalgebra they generate are given by

$$Q^\pm = \int d^2 x [(\varphi \mp \partial_y \varphi) \psi^\pm + (\partial_x \varphi \pm U(\varphi)) \psi^\mp],$$

$$\{Q^\pm, Q^\mp\} = 2(H \pm (\tilde{Z}_x - \tilde{P}_y)), \quad \{Q^+, Q^-\} = 2(\tilde{P}_x + \tilde{Z}_y).$$

Having a kink profile in the $x$-direction, which satisfies the Bogomol’nyi equation $\partial_x \varphi_K = -U(\varphi_K)$, one finds that the charge $Q^+$ (corresponding to the terms in (16) with $\varepsilon^-$) leaves the classical topological (domain-wall) vacuum ($\varphi = \varphi_K$, $\psi = 0$) invariant. This corresponds to classical BPS saturation, since with $P_x = 0$ and $\tilde{P}_y = 0$ one has $\{Q^+, Q^+\} = 2(H + \tilde{Z}_x)$ and, indeed, with a kink domain wall $\tilde{Z}_x/L^{d-1} = \mathcal{W}(+v) - \mathcal{W}(-v) = -M/L^{d-1}$. 
At the quantum level, hermiticity of $Q^\pm$ and positivity of the Hilbert space norm imply a lower bound for the energy(density):

$$\langle \Sigma | H | \Sigma \rangle \geq | \langle \Sigma | P_y | \Sigma \rangle | \equiv | \langle \Sigma | (\tilde{P}_y - \tilde{Z}_x) | \Sigma \rangle |,$$

where $| \Sigma \rangle$ denotes any state in the Hilbert space. This inequality is saturated when

$$Q^+ | \Sigma \rangle = 0.$$

Massive BPS states in 1+1 dimensions correspond to massless states in 2+1 dimensions, since with $[H, P_m] = 0$ one has

$$\langle P_M P^M \rangle = -\frac{1}{4}((Q^+)^2 Q^2 - \{Q^+, Q^-\}^2) = 0$$

for BPS saturated states (25) with $\langle P_y \rangle = M$ for a kink domain wall with kink profile in the $x$-direction. An anti-kink domain wall has instead $Q^- | \Sigma \rangle = 0$. In both cases, half of the supersymmetry is spontaneously broken. To take into account that there is infinite momentum and energy unless the $y$-direction is compact with finite length $L$, one can formulate the above identities for energy and central charge per unit length or for energy and central charge densities.

Omitting regularization the susy algebra in 1+1 dimensions is obtained from (18) simply by dropping $\tilde{P}_y$ as well as $\tilde{Z}_y$ so that $P_x \equiv \tilde{P}_x$. The term $\gamma^2 \tilde{Z}_x$ remains, however, with $\gamma^2$ being the nontrivial $\gamma^5$ of 1+1 dimensions. Identifying $\tilde{Z}_x$ with $Z$, the susy algebra simplifies to

$$\{Q^\pm, Q^\pm\} = 2(H \pm Z), \quad \{Q^+, Q^-\} = 2P_x$$

and one obtains the quantum BPS bound

$$\langle \psi | H | \psi \rangle \geq | \langle \psi | Z | \psi \rangle |$$

for any state $| \psi \rangle$. BPS saturated states have $Q^+ | \psi \rangle = 0$ or $Q^- | \psi \rangle = 0$, corresponding to kink and anti-kink, respectively, and break half of the supersymmetry.

**C. Fluctuations**

In a kink (or kink domain wall) background one spatial direction is singled out and we choose this to be along $x$. The direction orthogonal to the kink direction (parallel to the domain wall) will be denoted by $y$.

The quantum fields can then be expanded in the eigenfunctions, which are known analytically for the $\varphi^4$ and sine-Gordon soliton [24], times plane waves in the extra dimensions. For the bosonic fluctuations we have

$$[-\Box + (U'^2 +UU'')]\eta = 0$$

which is solved by

$$\eta = \int \frac{d^{d-1} \ell}{(2\pi)^{d-2} \omega} \sum_{k,\ell} \frac{dk}{\sqrt{4\pi}} \left( a_{k,\ell} e^{-i(\omega t - \ell y)} \phi_k(x) + a_{k,\ell}^\dagger e^{i(\omega t - \ell y)} \phi_k^*(x) \right).$$
The kink eigenfunctions $\phi_k$ are normalized according to $\int dx |\phi|^2 = 1$ for the discrete states and to Dirac distributions for the continuum states according to $\int dx \phi_k^* \phi_{k'} = 2\pi \delta(k - k')$. The latter are deformed plane waves because there is no reflection in the case of the kink. The mode energies are $\omega = \sqrt{\omega_k^2 + \ell^2}$ where $\omega_k$ is the energy in the 1+1-dimensional case.

The canonical equal-time commutation relations $[\eta(\vec{x}), \dot{\eta}(\vec{x}')] = i\delta(\vec{x} - \vec{x}')$ are fulfilled with

$$[a_{k,\ell}, a^\dagger_{k',\ell'}] = \delta_{kk'}\delta(\ell - \ell'),$$

where for the continuum states $\delta_{kk'}$ becomes a Dirac delta function.

For the fermionic modes which satisfy the Dirac equation $[\slashed{\partial} + U']\psi = 0$, i.e. explicitly

$$(\partial_x + U')\psi^+ + i(\omega + \ell)\psi^- = 0,$$

$$(\partial_x - U')\psi^- + i(\omega - \ell)\psi^+ = 0,$$

one finds

$$\psi = \psi_0 + \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum_{\ell'} \frac{dk}{4\pi\omega_k} \left[ b_{k,\ell} e^{-i(\omega t - 2y)\eta_k(x)} + b^\dagger_{k,\ell} (c.c.) \right],$$

$$\psi_0 = \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} b_{0,\ell} e^{-i(\omega t - 2y)} \left( \begin{array}{c} \phi_0 \\ 0 \end{array} \right), \quad b^\dagger_{0,\ell} = b_{0,(-\ell)}.$$

The fermionic zero mode\(^7\) of the susy kink turns into massless modes located on the domain wall, which have only one chirality, forming a Majorana-Weyl domain wall fermion [21, 26–28].\(^8\)

For the massive modes the Dirac equation relates the eigenfunctions appearing in the upper and the lower components of the spinors as follows:

$$s_k = \frac{1}{\omega_k}(\partial_x + U')\phi_k = \frac{1}{\sqrt{\omega_k^2 + \ell^2}}(\partial_x + U')\phi_k,$$

(33)

so that the function $s_k$ is the SUSY-quantum mechanical [29] partner of $\phi_k$ and thus coincides with the eigen modes of the sine-Gordon model if $\phi_k$ belongs to the $\varphi^4$-kink (hence the notation) [30]. With (33), their normalization is the same as that of the $\phi_k$. It is the relation (33) and the fact that it relates bosonic to fermionic modes, as well as different components of the fermionic modes to each other, which makes it possible to compute the one loop-corrections to energy and central charge without explicit knowledge of the mode functions.

The canonical equal-time anti-commutation relations $\{\psi^\alpha(\vec{x}), \psi^\beta(\vec{x}')\} = \delta^{\alpha\beta}\delta(\vec{x} - \vec{x}')$ are satisfied if (using that $\phi_{-k}(x) = \phi_k^*(x)$ and $s_{-k}(x) = s_k^*(x)$)

$$\{b_{0,\ell}, b^\dagger_{0,\ell'}\} = \{b_{0,\ell}, b_{0,(-\ell')}\} = \delta(\ell - \ell'),$$

$$\{b_{k,\ell}, b^\dagger_{k',\ell'}\} = \delta_{kk'}\delta(\ell - \ell'),$$

(34)

\(^7\) By a slight abuse of notation we shall always label this by a subscript 0, but this should not be confused with the threshold mode $k = 0$ (which does not appear explicitly anywhere below).

\(^8\) The mode with $\ell = 0$ corresponds in 1+1 dimensions to the fermionic zero mode of the susy kink. If there are no other fermionic zero modes, it has to be counted as half a degree of freedom in mode regularization [14]. For dimensional regularization such subtleties do not play a role because the zero mode only gives scale-less integrals and these vanish.
and again the $\delta_{k,k'}$ becomes a Dirac delta for the continuum states. The algebra (34) and the solution for the massless mode (32) show that the operator $b_0(\ell)$ creates right-moving massless states on the wall when $\ell$ is negative and annihilates them for positive momentum $\ell$. Thus only massless states with momentum in the positive $y$-direction can be created.

Changing the representation of the gamma matrices by $\gamma_2 \rightarrow -\gamma_2$, which is inequivalent to the original one, reverses the situation. Now only massless states with momenta in the negative $y$-direction exist. Thus depending on the representation of the Clifford algebra one chirality of the domain wall fermions is singled out. This is a reflection of the spontaneous violation of parity when embedding the susy kink as a domain wall in 2+1 dimensions.

Notice that in (32) $d$ can be only 2 or 1, for which $\ell$ has 1 or 0 components, so for strictly $d = 1$ one has $\ell \equiv 0$. In order to have a susy-preserving dimensional regularization scheme by dimensional reduction, we shall start from $d = 2$ spatial dimensions, and then make $d$ continuous and smaller than 2.

D. Energy corrections

Before turning to a direct calculation of the anomalous contributions to central charge and momentum, we derive the one-loop energy density of the susy kink (domain wall) in dimensional regularization.

Expanding the Hamiltonian density of the model (12),

$$\mathcal{H} = \frac{1}{2} [\dot{\varphi} + (\nabla \varphi)^2 + U^2(\varphi)] + \frac{1}{2} \psi^\dagger i\gamma^0 [\vec{\gamma} \vec{\nabla} + U'(\varphi)] \psi,$$

around the kink/domain wall, using $\varphi = \varphi_K + \eta$, one obtains

$$\mathcal{H} = \frac{1}{2} [(\partial_x \varphi_K)^2 + U^2] - \frac{\delta \mu^2}{\sqrt{2}\lambda} U - \partial_x (U\eta) +$$

$$+ \frac{1}{2} [\eta^2 + (\nabla \eta)^2] + \frac{1}{2} (U^2)'\eta^2] + \frac{1}{2} \psi^\dagger i\gamma^0 [\vec{\gamma} \vec{\nabla} + U'] \psi + O(\hbar^2),$$

(36)

where $U$ without an explicit argument implies evaluation at $\varphi = \varphi_K$ and use of the renormalized $\mu^2$. The first two terms on the r.h.s. are the classical energy density and the counterterm contribution. The terms quadratic in the fluctuations are the only ones contributing to the total energy. They give [22]

$$\langle \mathcal{H}^{(2)} \rangle = -\partial_x \left( \frac{1}{2} \int \frac{d^{d-1} \ell}{(2\pi)^{d-1}} \sum \int \frac{dk}{2\pi} \frac{U'}{2\omega} |\phi_k|^2 \right)$$

$$+ \frac{1}{2} \int \frac{d^{d-1} \ell}{(2\pi)^{d-1}} \sum \int \frac{dk}{2\pi} \frac{\ell^2}{2\omega} (|\phi_k^2| - |s_k|^2).$$

(37)

When integrated, the first term, which is a pure surface term, cancels exactly the counterterm (see (15)), because

$$\int dx \langle \frac{1}{2} \partial_x (U'\eta^2) \rangle = \frac{1}{2} U' \langle \eta^2 \rangle \bigg|_{-\infty}^{\infty} = m \int \frac{d^{d-1} \ell}{(2\pi)^{d-1}} \int \frac{dk}{2\pi} \frac{1}{2\omega} \equiv m\delta v^2,$$

(38)

9 The third term in (36) is of relevance when calculating the energy profile [11, 12].
where we have used that \( U'(x = \pm \infty) = \pm m \).

In these expressions, the massless modes (which correspond to the zero mode of the 1+1 dimensional kink) can be dropped in dimensional regularization as scale-less and thus vanishing contributions, and the massive discrete modes cancel between bosons and fermions.\(^\text{10}\) Using the explicit form of \( \phi_k(x), s_k(x) \) as given e.g. in [12], the \( x \)-integration over the continuous mode functions gives the required difference of spectral densities as

\[
\int dx (|\phi_k(x)|^2 - |s_k(x)|^2) = -\theta'(k) = -\frac{2m}{k^2 + m^2},
\]

where \( \theta(k) \) is the additional phase shift of the mode functions \( s_k \) compared to \( \phi_k \).

With the help of the susy-quantum mechanical relation (33) for the fermionic modes in the BPS background the integral (39) can also be computed without detailed knowledge of the mode functions [31]. Denoting the operator in (33) by \( A = \partial_x + U' \) the fluctuation equation above (30) of the "bosonic" modes \( \phi_k \) factorizes as

\[
A^\dagger A \phi_k = \omega^2_k \phi_k,
\]

where \( A^\dagger = -\partial_x + U' \) is the adjoint operator. Using (33) the spectral density (39) can be written as

\[
-\theta'(k) = \int dx \left[ |\phi_k(x)|^2 - \frac{1}{\omega_k^2} (A\phi_k)^*(A\phi_k) \right] = \int dx \left[ |\phi_k(x)|^2 - \frac{1}{\omega_k^2} \phi_k^* A^\dagger A \phi_k \right] + \text{surface term}.
\]

The first term simply vanishes because of (40). The surface term results from the fact that in the above expression the operator \( A^\dagger \) is only formally the hermitian conjugate of \( A \). The difference of the spectral densities is therefore given by

\[
-\theta'(k) = -\frac{1}{\omega_k^2} \int dx \partial_x \left( U'|\phi_k(x)|^2 \right) = -\frac{2m}{k^2 + m^2},
\]

where we have omitted a term \( \phi^* \partial_x \phi \) because it is is even in \( x \) for large \( |x| \) and thus only needed the asymptotic values \( U'|\varphi_K(x = \pm \infty)| = U'(\pm v) = \pm m \) and that the mode functions are plane waves asymptotically, i.e. \(|\phi_k|^2 \to 1\). This result coincides with the one from a direct evaluation of (39).

Thus we obtain for the remaining correction from (37)

\[
\frac{\Delta M^{(1)}}{L^{d-1}} = -\frac{1}{4} \int \frac{dk \, d^{d-1}\ell}{(2\pi)^d} \omega^2 \theta'(k) = -\frac{2}{d} \frac{\Gamma\left(\frac{3-d}{2}\right)}{(4\pi)^{\frac{d+1}{2}}} m^d.
\]

This reproduces the correct, known result for the susy kink mass correction \( \Delta M^{(1)} = -m/(2\pi) \) (for \( d = 1 \)) and the surface (string) tension of the 2+1 dimensional susy kink domain wall \( \Delta M^{(1)}/L = -m^2/(8\pi) \) (for \( d = 2 \)) [21].

Notice that the entire result is produced by an integrand proportional to the extra momentum component \( \ell^2 \), which for strictly \( d = 1 \) would not exist.

\(^\text{10}\) The zero mode contributions in fact do not cancel by themselves between bosons and fermions, because the latter are chiral. This non-cancellation is in fact crucial in energy cutoff regularization (see Ref. [21]).
E. Anomalous contributions to the central charge and extra momentum

In a kink (domain wall) background with only nontrivial $x$ dependence, the central charge density $\tilde{Z}_x$ receives nontrivial contributions. Expanding $\tilde{Z}_x$ around the kink background gives

$$\tilde{Z}_x = U \partial_x \varphi_K - \frac{\delta \mu^2}{\sqrt{2} \lambda} \partial_x \varphi_K + \partial_x (U \eta) + \frac{1}{2} \partial_x (U' \eta^2) + O(\eta^3).$$ (44)

Again only the part quadratic in the fluctuations contributes to the integrated quantity at one-loop order\(^1\). However, this leads just to the contribution shown in (38), which matches precisely the counterterm $m \delta v^2$ from requiring vanishing tadpoles. Straightforward application of the rules of dimensional regularization thus leads to a null result for the net one-loop correction to $\langle \tilde{Z}_x \rangle$ in the same way as found in Refs. [5, 6, 8] in other schemes.

On the other hand, by considering the less singular combination $\langle H + \tilde{Z}_x \rangle$ and showing that it vanishes exactly, it was concluded in Ref. [9] that $\langle \tilde{Z}_x \rangle$ has to compensate any nontrivial result for $\langle H \rangle$, which in Ref. [9] was obtained by subtracting successive Born approximations for scattering phase shifts. In fact, Ref. [9] explicitly demonstrates how to rewrite $\langle \tilde{Z}_x \rangle$ into $-\langle H \rangle$, apparently without the need for the anomalous terms in the quantum central charge operator postulated in Ref. [11].

Because the authors of Ref. [9] did not discuss regularization of $\langle \tilde{Z}_x \rangle$, the manipulations needed to rewrite it as $-\langle H \rangle$ (which eventually is regularized and renormalized) are not defined in their work. If we choose to use dimensional regularization, $\langle \tilde{Z}_x \rangle$ contains the mode energies $\omega = \sqrt{k^2 + m^2 + \ell^2}$ instead of $\omega_k = \sqrt{k^2 + m^2}$ and so the manipulations carried through in Ref. [9] (eq. 56) are no longer possible. Using dimensional regularization one in fact obtains a nonzero result for $\langle H + \tilde{Z}_x \rangle$, apparently in violation of susy. However, dimensional regularization by embedding the kink as a domain wall in (up to) one higher dimension, which preserves susy, instead leads to

$$\langle H + \tilde{Z}_x - \tilde{P}_y \rangle = 0,$$ (45)

i.e. the saturation of (24), as we shall now verify.

The bosonic contribution to $\langle \tilde{P}_y \rangle$ involves an $\ell$ integral which is scale-less and odd and thus vanishes. Only the fermions turn out to give interesting contributions:

$$\langle \tilde{P}_y \rangle = \frac{i}{2} \langle \psi^\dagger \partial_y \psi \rangle$$

$$= \frac{1}{2} \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum_k \frac{dk}{2\pi} \left( \ell \langle |\phi_k|^2 + |s_k|^2 \rangle + \frac{\ell^2}{2\omega} (|\phi_k|^2 - |s_k|^2) \right).$$ (46)

We have already omitted the contributions which vanish either by symmetric integration or due to scale-less integrals, which are zero in dimensional regularization. The remaining $\ell$-integration no longer factorizes because $\omega = \sqrt{k^2 + \ell^2 + m^2}$, and is in fact identical to the finite contribution in $\langle H \rangle$ obtained already in (37):

$$-\Delta Z = \int dx \langle \tilde{P}_y \rangle = \frac{1}{4} \int dk d^{d-1}\ell \frac{\ell^2}{(2\pi)^d} \theta^d(k) = \frac{2}{d} \frac{\Gamma\left(\frac{3-d}{2}\right)}{(4\pi)^{d/2}} m^d.$$ (47)

\(^1\) Again, this does not hold for the central charge density locally [11, 12].
So for all \( d \leq 2 \) we have BPS saturation, \( \langle H \rangle = |\langle \hat{Z}_x - \hat{P}_y \rangle| \), which in the limit \( d \to 1 \), the susy kink, is made possible by a non-vanishing \( \langle \hat{P}_y \rangle \). The anomaly in the central charge is seen to arise from a parity-violating contribution in \( d = 1 + \epsilon \) dimensions which is the price to be paid for preserving supersymmetry when going up in dimensions to embed the susy kink as a domain wall.

It is again the difference in the spectral densities, \( \theta' \), which determines the one-loop corrections, which thus depend only on the derivative of the pre-potential (or equivalently the second derivative of super-potential \( \mathcal{W} = \int d\varphi U(\varphi) \)) at the critical points \( \pm v \). In general the spectral density difference for a model with spontaneously broken \( \mathbb{Z}_2 \) symmetry is given by

\[
\theta'(k) = \frac{\mathcal{W}''(v) - \mathcal{W}''(-v)}{k^2 + \mathcal{W}''(v)^2},
\]

which has an obvious generalization for \( \mathbb{Z}_N \) symmetric models like the sine-Gordon model. From (40,41) one can see that this quantity is closely related to the index of the operator \( AA^\dagger \). For the simple models considered here, where only one spatial direction is nontrivial, \( \theta'(k) \) is easily obtained from the Dirac equation in the asymptotic regions \( x \to \pm \infty \), far away from the kink \([6]\). But as we will see below, in case of a less trivial background like the vortex, the formulation as surface term will provide essential simplifications \([31]\).

### F. Dimensional reduction and evanescent counterterms

In the above, we have effectively used the ’t Hooft-Veltman version of dimensional regularization \([32]\) in which the space-time dimensionality \( n \) is made larger than the dimension of interest. This is possible in a supersymmetric way if the model of interest can be obtained from a higher dimensional supersymmetric model by dimensional reduction. The nontrivial corrections to the central charge of the kink come from the “genuine” momentum operator \( \hat{P}_y \), and are due to a spontaneous breaking of parity.

We now comment on how the central charge anomaly can be recovered from Siegel’s version of dimensional regularization \([33–35]\) where \( n \) is smaller than the dimension of space-time and where one keeps the number of field components fixed, but lowers the number of coordinates and momenta from \( 2 \) to \( n < 2 \). At the one-loop level one encounters 2-dimensional \( \delta^\nu_\mu \) coming from Dirac matrices, and \( n \)-dimensional \( \hat{\delta}^\nu_\mu \) from loop momenta. An important concept which is going to play a role are the evanescent counterterms \([36–38]\) involving the factor \( \frac{1}{\epsilon} \hat{\delta}^\nu_\mu \gamma_\nu \psi \), where \( \hat{\delta}^\nu_\mu \equiv \delta^\nu_\mu - \delta^\nu_\mu \) has only \( \epsilon = 2 - n \) nonvanishing components.

The supercurrent is given by

\[
\langle 0 | j_\mu | p \rangle^{\text{div}} = i U''(v) \int_0^1 dx \int d^3 \kappa \frac{\kappa \gamma_\mu \kappa}{(2\pi)^n [\kappa^2 + p^2 x(1 - x) + m^2]^2} u(p).
\]

Only matrix elements with one external fermion are divergent. The term involving \( U''(v) \eta^2 \) in (49) gives rise to a divergent scalar tadpole that is cancelled completely by the counterterm \( \delta \mu^2 \) (which itself is due to an \( \eta \) and a \( \psi \) loop). The only other divergent diagram is due to the term involving \( \hat{\delta}_\mu \eta \) in (49) and has the form of a \( \psi \)-selfenergy. Its singular part reads
Using $\hat{\delta}_\mu^\nu \equiv \delta^\nu_\mu - \hat{\delta}_\mu^\nu$ we find that under the integral

$$ k^\nu k^\rho \gamma_\mu = -\kappa^2 (\delta^\lambda_\mu - \frac{2}{n} \delta^\lambda_\mu) \gamma_\lambda = \frac{\epsilon}{n} \kappa^2 \gamma_\mu - \frac{2}{n} \kappa^2 \hat{\delta}_\mu^\lambda \gamma_\lambda $$

so that

$$ \langle 0 | j_\mu | p \rangle \ \text{div} = \frac{U''(v)}{2\pi} \hat{\delta}_\mu^\lambda \gamma_\lambda u(p). \tag{51} $$

Hence, the regularized one-loop contribution to the susy current contains the evanescent operator

$$ j_\mu^{\text{div}} = \frac{U''(\varphi)}{2\pi} \hat{\delta}_\mu^\lambda \gamma_\lambda \psi. \tag{52} $$

It is called evanescent because the numerator vanishes in strictly $n = 2$; for $n \neq 2$ it has a pole, but in matrix elements this composite operator gives a finite contribution. $j_\mu^{\text{div}}$ is by itself a conserved quantity, because all fields depend only on the $n$-dimensional coordinates, but it has a nonvanishing contraction with $\gamma^\mu$. The latter gives rise to an anomalous contribution to the renormalized conformal-susy current $\hat{x} j_\mu^{\text{ren}}$, where $j_\mu^{\text{ren}} = j_\mu - j_\mu^{\text{div}}$

$$ \partial^\mu (\hat{x} j_\mu^{\text{ren}})_{\text{anom.}} = -\gamma^\mu j_\mu^{\text{div}} = -\frac{U''}{2\pi} \psi. \tag{53} $$

(There are also nonvanishing nonanomalous contributions to $\partial^\mu (\hat{x} j_\mu)$ because our model is not conformal-susy invariant at the classical level [39, 40].)

Ordinary susy on the other hand is unbroken; there is no anomaly in the divergence of $j_\mu^{\text{ren}}$. A susy variation of $j_\mu$ involves the energy-momentum tensor and the topological central-charge current $\zeta_\mu$ according to

$$ \delta j_\mu = -2 T_\mu^{\nu \gamma} \gamma_\nu - 2 \zeta_\mu \gamma^5 \epsilon, \tag{54} $$

where classically $\zeta_\mu = \epsilon_{\mu\nu} U \partial^\nu \varphi$.

At the quantum level, the counter-term $j_\mu^{\text{ct}} = -j_\mu^{\text{div}}$ induces an additional contribution to the central charge current

$$ \zeta_\mu^{\text{anom}} = \frac{1}{4\pi} \hat{\delta}_\mu^\nu \epsilon_{\nu\rho} \partial^\rho U' \tag{55} $$

which despite appearances is a finite quantity: using that total antisymmetrization of the three lower indices has to vanish in two dimensions gives

$$ \hat{\delta}_\mu^\nu \epsilon_{\nu\rho} = \epsilon \epsilon_{\mu\rho} + \hat{\delta}_\mu^\nu \epsilon_{\nu\rho} \tag{56} $$

and together with the fact the $U'$ only depends on $n$-dimensional coordinates this finally yields

$$ \zeta_\mu^{\text{anom}} = \frac{1}{4\pi} \epsilon_{\mu\nu} \partial^\nu U' \tag{57} $$

in agreement with the anomaly in the central charge as obtained previously.

We emphasize that $\zeta_\mu$ itself does not require the subtraction of an evanescent counterterm. The latter only appears in the susy current $j_\mu$, which gives rise to a conformal-susy anomaly in $\hat{x} j_\mu$. A susy variation of the latter shows that it forms a conformal current multiplet
involving besides the dilatation current $T_{\mu\nu}x^\nu$ and the Lorentz current $T_{\mu\nu}x^\rho\epsilon_{\nu\rho}$ also a current $j^{(C)}_{(\nu)} = x^\mu\epsilon_{\rho\nu}\zeta^\mu$. We identify this with the conformal central-charge current, which is to be distinguished from the ordinary central-charge current $\zeta^\mu$.

Since $\partial_\mu j^{(C)}_{(\nu)} = \epsilon_{\mu\nu}\zeta^\mu$, and $\epsilon_{\mu\nu}$ is invertible, the entire central-charge current $\zeta^\mu$ enters in the divergence of the conformal central-charge current, whereas in the case of the conformal-susy current it was the contraction $\gamma^\mu j^\mu$.

The current $j^{(C)}$ thus has the curious property of being completely determined by its own divergence. For this reason it is in fact not associated with any continuous symmetry (as is also the case for the ordinary central-charge current, which is of topological origin). In the absence of classical breaking of conformal invariance it is conserved trivially by its complete disappearance and then there is no symmetry generating charge operator. Nevertheless, in the conformally noninvariant susy kink model this current is nonvanishing and has in addition to its nonanomalous divergence an anomalous one, namely the anomalous contribution to the central charge current inherited from the evanescent counterterm in the renormalized susy current.

G. Multiplet shortening, BPS saturation and fermion parity

We construct representations of the strictly two dimensional algebra

$$Q^{+2} = H + Z , \quad Q^{-2} = H - Z , \quad \{Q^+, Q^-\} = 2P,$$

(58)

where $Q^{\pm \dagger} = Q^\pm$ are hermitian, by going to the rest frame ($P = 0$) in the topological sector, i.e. with non-vanishing central charge. $M$ and $Z$ are then ordinary numbers. In the general case, $M \neq |Z|$, the irreducible representations are two-dimensional $\{|\Sigma^-\rangle, \Sigma^+\rangle\}$. The supercharges can be represented as

$$\hat{Q}^+ = \sqrt{M + Z} \sigma_1 , \quad \hat{Q}^- = \sqrt{M - Z} \sigma_2.$$

(59)

For BPS states the absolute value of the central charge is per definition equal to the energy of this state, i.e. for the eigen values in (58) we have

$$M - |Z| = 0.$$

(60)

We choose $Z = -M$ which corresponds in our convention to the kink. The algebra (58) in such a BPS representation becomes now

$$\hat{Q}^{+2} = 0 , \quad \hat{Q}^{-2} = 2M , \quad \{\hat{Q}^+, \hat{Q}^-\} = 0.$$

(61)

Because of the hermiticity of $\hat{Q}^+$, (61) implies $||\hat{Q}^+|\Sigma\rangle||^2 = 0$ and thus

$$\hat{Q}^+|\Sigma\rangle = 0.$$

(62)

This equation is equivalent to (60) for the definition of BPS states and means that the BPS states are left invariant by half of the supersymmetry, namely $Q^+$ in our case. Operators and states can be characterized by the cohomology of the operator $Q^+$. Analogous to BRST exact operators which have vanishing matrix elements for physical states we can say that each operator which is $Q^+$– exact has vanishing expectation value for BPS states:

$$\mathcal{O} = \{Q^+, \mathcal{O}'\} \Rightarrow \langle BPS|\mathcal{O}| BPS\rangle = 0.$$

(63)
For the other supercharge in (61) $\hat{Q}^-$, which acts nontrivially, there exist two inequivalent irreducible representations,

$$\hat{Q}^-|\Sigma\rangle = \pm \sqrt{2M}|\Sigma\rangle,$$

which are connected by a $\mathbb{Z}_2$ transformation $\psi \rightarrow -\psi$ for all fermions, which is clearly a symmetry for each fermionic action. Thus the irreducible representation is one-dimensional and the fermionic operator is diagonal [41]. This is the reason why it was originally thought that multiplet shortening does not occur in two dimensions [4, 6, 11]. Therefore a reducible two dimensional representation for the soliton states was assumed such that the fermion parity operator $(-1)^F$ is still defined. For a reducible two dimensional representation $\{|\Sigma_b\rangle, |\Sigma_f\rangle\}$ we may choose:

$$\hat{Q}^- = \sqrt{2M} \sigma_1, \quad (-1)^F = \sigma_3,$$

so that $(-1)^F$ is diagonal in this representation and $\hat{Q}^-$ has fermion parity $-1$, i.e. $\{\hat{Q}^-, (-1)^F\} = 0$. Note that this is the direct sum of the two inequivalent irreducible representations (64), which are obtained as $\frac{1}{2}(|\Sigma_b\rangle \pm |\Sigma_f\rangle)$.

Witten and Olive [4] argued that in four dimensional susy gauge theories the number of particle states is not changed in the Higgs phase, although massive representations have $2^N$ times as many states than massless one. Thus, they concluded that the Higgs phase corresponds to a BPS saturated representation which has the same number of physical states as the massless representation. Because of this multiplet shortening the BPS saturation should be protected against perturbative corrections since they cannot change the number of particle states.

The counting of susy soliton states in two dimensions is somewhat peculiar (see below) and the loss of fermion parity (64) suggested a two dimensional representation, as for the non-susy soliton [18], and thus no multiplet shortening would occur. In [11], nevertheless BPS-saturation was assumed, to match the central charge correction to the mass correction obtained in [8]. The crucial relation for BPS saturation is the annihilation by one supercharge (62). It was stated that this relation is protected without multiplet shortening, by analogous arguments that constrain supersymmetry breaking [29]. A simple argument shows that this is not sufficient. Assume that in some approximation a reducible multiplet is BPS saturated, i.e. $\hat{Q}^+/|\Sigma_i\rangle = 0$. Since the operator $Q^+$ is hermitian its representation is of the form (because of the reducibility the $\mathbb{Z}_2$-grading through $(-1)^F$ exists)

$$\hat{Q}^+ = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix},$$

and the BPS states can be separated in zero-eigen states of $M$ and $M^\dagger$. To answer the question if the multiplet remains BPS saturated under perturbations (corrections) we consider the quantity

$$\lim_{\beta \to \infty} \text{Tr} \left[ (-1)^F e^{-\beta Q^+^2} \right] = n^0_b - n^0_f = \text{Ind}(M),$$

which is the difference between the number of zero-eigenvalue eigen states (singlets) of $M$ and $M^\dagger$. That this index is invariant under perturbative corrections can be seen quite analogously to the arguments of [29]. If a state is no longer annihilated by $Q^+$, say for

\[ \text{Note that BPS states } (Q^{+2} = 0) \text{ contribute } \text{Tr}_{BPS}(-1)^F = n^0_b - n^0_f, \text{ whereas for non-BPS states is } Q^{+2} = \langle H + Z \rangle > 0 \text{ such that their contribution vanish for } \beta \to \infty. \]
example one with fermionic parity, then also the bosonic super-partner state is no longer annihilated:

\( 0 \neq Q^+ |f \rangle \sim Q^+ Q^- |b \rangle = -Q^- Q^+ |b \rangle \to Q^+ |b \rangle \neq 0 \). (68)

So the difference in the number of singlet states is unchanged under perturbative corrections, and thus it can be calculated in a semi-classical approximation. So what can this index tell us? In case that it would be nonzero, there would exist, at least one, BPS saturated state, which is then protected against quantum corrections. But in this case \((-1)^F\) is no longer defined as we will see immediately. If the index vanishes in an approximation, \(n_0^b - n_0^f = 0\), the number of fermionic and bosonic singlets coincides. The trivial case is of course that they both vanish already in the approximation and there are no BPS states. In the nontrivial case there exist susy pairs of BPS singlets in the approximation, but susy does not protect them from being lifted pairwise above the BPS bound as described in (68). But this is exactly the case of the \(\mathbb{Z}_2\) symmetric two-dimensional multiplet (65). So the equality between the mass correction and the anomalous contribution to the central charge needs a different explanation. In fact, it was found that one has to give up the usual fermion parity for the topological soliton state which is then a single-state short super-multiplet (64) [41]. If we look now again on the BPS saturation equation (62), we see immediately that a lift above the BPS bound would give a twice as long irreducible multiplet (59) which cannot be caused by perturbative corrections. So, in the absence of other mechanisms as for example a difference in a conserved quantum number, multiplet shortening is a necessary condition for BPS saturation being protected.

Up to now we have only discussed abstract representations of the susy algebra (58). In a quantum field theory the operators in the algebra (58) and their representations correspond to Heisenberg operators of conserved, i.e. time-independent, charges and Heisenberg states. In general neither operators nor states in the Heisenberg picture are known explicitly. Instead one quantizes the field operators in the interaction picture in terms of creation/annihilation operators which are defined w.r.t. a perturbative ground state. The canonical commutation relations imply an algebra for the mode coefficients which usually has to be represented in an irreducible manner. This determines which kind of the above representations is realized in the (perturbative) quantum field theory. From the fermionic creation/annihilation operators (34) one obtains an infinite dimensional Clifford algebra of pairs of generators \(\gamma^+_k = (b_k + b_k^\dagger)\) and \(\gamma^-_k = i(b_k - b_k^\dagger)\) and the single generator \(b_0\) corresponding to the zero-mode. This is quite analogous to an odd dimensional Clifford algebra, and the operator \(b_0\) plays the role of the \(\gamma_5\) of the even-dimensional algebra \(\gamma^\pm\). So there are two inequivalent representations of the full algebra governed by the sign of the “gamma five” operator \(b_0\). Because \(b_0\) has to anti-commute with the \(b_k\)’s it cannot be represented as a number, as in the quasi-classical approximation [41]. The \(b_k\)-algebra can be represented as usual on a Fock space, constructed from the Clifford vacuum \(|\Omega\rangle\) with \(b_k |\Omega\rangle = 0\). The whole algebra, including \(b_0\) can then be realized by two inequivalent irreducible representations [14]:

\[ |s_\pm \rangle = \frac{1}{2}(1 \pm b_0) |\Omega\rangle \quad b_0 |s_\pm \rangle = \pm |s_\pm \rangle \quad b_k |s_\pm \rangle = 0. \] (69)

According to the usual fermion-parity counting \(b_0\) is an odd, i.e. fermionic operator, and thus the ground states \(|s_\pm \rangle\) are half fermionic and half bosonic. But in two dimensions there is less distinction between fermions and bosons. In fact, since there are no rotations in one spatial dimension, the definition of fermion parity is more abstract. Indeed, the assignment of fermion number to different vacua depends on the sign of the eigenvalue of the fermion
mass matrix at the considered vacuum [29]. In the case of the kink this means that if the vacuum with \( \varphi = +v \) is defined to be bosonic the vacuum at \( -v \) is automatically fermionic. Now a topological state like the kink connects these two vacua with opposite fermion parity which heuristically explains that this state cannot have a definite fermion parity in the usual sense.

Let us now check if also semi-classically the BPS saturation condition is satisfied. With the regularized mode expansion for the quantum fields (30,32) and the BPS equation \( \partial_x \varphi_K + U = 0 \) one obtains

\[
Q^+|s_\pm\rangle = \int dx\left[(\dot{\eta} - \partial_y \eta)\psi^\dagger + (\partial_x \eta + U' \eta)\psi\right]|s_\pm\rangle + O(h)
= i \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum \frac{d}{2\pi} \left(\sqrt{\omega - \ell} - \sqrt{\omega + \ell}\right) a_k^\dagger b_k^\dagger |s_\pm\rangle = 0.
\]

So both states are BPS saturated semi-classically.

**IV. SUPERSYMMETRIC VORTICES IN 2+1 DIMENSIONS**

Following [42] we next consider the Abrikosov-Nielsen-Olesen [43–46] vortex solution of the abelian Higgs model in 2+1 dimensions which has a supersymmetric extension [47, 48] (for related models see [49, 50]) such that classically the Bogomolnyi bound [51] is saturated. We implement dimensional regularization of the 2 + 1-dimensional \( \mathcal{N} = 2 \) vortex by dimensionally reducing the \( \mathcal{N} = 1 \) abelian Higgs model in 3 + 1 dimensions. We confirm the results of [47, 52, 53] that in a particular gauge (background-covariant Feynman-’t Hooft) the sums over zero-point energies of fluctuations in the vortex background cancel completely, but contrary to [47, 52] we find a nonvanishing quantum correction to the vortex mass coming from a finite renormalization of the expectation value of the Higgs field in this gauge [31, 53]. In contrast to [47], where a null result for the quantum corrections to the central charge was stated, we show that the central charge receives also a net nonvanishing quantum correction, namely from a nontrivial phase in the fluctuations of the Higgs field in the vortex background, which contributes to the central charge even though the latter is a surface term that can be evaluated far away from the vortex. The correction to the central charge exactly matches the correction to the mass of the vortex.

In Ref. [52], it was claimed that the usual multiplet shortening arguments which prove BPS saturation would not be applicable to the \( \mathcal{N} = 2 \) vortex since in the vortex background there would be two rather than one fermionic zero modes [54], leading to two short multiplets which have the same number of states as one long multiplet.\(^{13}\) We show however that the extra zero mode postulated in [52] has to be discarded because its gaugino component is singular, and that only after doing so there is agreement with the results from index theorems [54–56]. For this reason, standard multiplet shortening arguments do apply, explaining the BPS saturation at the quantum level that we observe in our explicit one-loop calculations.

\(^{13}\) Incidentally, Refs. [52, 54] considered the supersymmetric abelian Higgs model augmented by a Chern-Simons term.
A. The model

The superspace action for the vortex in terms of 3+1-dimensional superfields contains an \( \mathcal{N} = 1 \) abelian vector multiplet and an \( \mathcal{N} = 1 \) scalar multiplet, coupled as usual, together with a Fayet-Iliopoulos term but without superpotential,

\[
\mathcal{L} = \int d^2 \theta W^\alpha W_\alpha + \int d^4 \Phi e^{-\epsilon \Phi} \Phi + \kappa \int d^4 V.
\]  

(71)

In terms of 2-component spinors in 3+1 dimensions, the action reads\(^ {14} \)

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu}^2 + \bar{\chi}^A i \sigma^\mu_{\alpha \beta} \partial_\mu \chi^\beta + \frac{1}{2} D^2 + (\kappa - e|\Phi|^2) D^2 - D_{\mu} \phi |^2 + \bar{\psi}^\dagger i \sigma^\mu_{\alpha \beta} D_{\mu} \psi^\beta + |F|^2 + \sqrt{2} e \left[ \phi^\dagger \chi^\alpha \psi^\alpha + \phi \bar{\chi}^\dagger \bar{\psi}^\dagger \right],
\]

(72)

where \( D_{\mu} = \partial_{\mu} - ieA_{\mu} \) when acting on \( \phi \) and \( \bar{\psi} \), and \( F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). Elimination of the auxiliary field \( D \) yields the scalar potential \( V = \frac{1}{2} D^2 = \frac{1}{2} e^2 (|\phi|^2 - \nu^2)^2 \) with \( \nu^2 \equiv \kappa/e \).

This model is invariant under the following transformation rules:

\[
\begin{align*}
\delta A_{\mu} &= \epsilon_{\alpha} \sigma^\mu_{\alpha \beta} \bar{\chi}_{\beta} - \epsilon^\dagger \sigma_{\beta \alpha} \chi^\alpha, & \delta D &= i(\epsilon_{\alpha} \sigma^{\mu \alpha \beta} \partial_\mu \bar{\chi}_{\alpha} + \epsilon^\dagger \sigma^{\mu \alpha \beta} \partial_\mu \chi^\alpha) \\
\delta \chi^\alpha &= -i F_{\mu \nu} \sigma^{\mu \nu \alpha \beta} \beta + D \epsilon^\alpha, & \delta \bar{\chi}_{\alpha} &= i F_{\mu \nu} \sigma^{\mu \nu \alpha \beta} \beta - D \bar{\epsilon}_{\alpha},
\end{align*}
\]

(73)\hspace{1cm}(74)

for the gauge multiplet; the matter multiplet transforms as

\[
\begin{align*}
\delta \phi &= -\sqrt{2} \epsilon_\alpha \psi^\alpha, & \delta \bar{\psi}_{\alpha} &= -i \sqrt{2} \epsilon^\dagger \sigma^\mu_{\alpha \beta} D_{\mu} \phi + \sqrt{2} F^{*} \epsilon_{\alpha} \\
\delta \phi^* &= \sqrt{2} \epsilon^\dagger \bar{\psi}_{\alpha}, & \delta \bar{\psi}_{\alpha} &= -i \sqrt{2} (D_{\mu} \phi)^* \sigma^{\mu \alpha \beta} \epsilon^\beta + \sqrt{2} F \epsilon_{\alpha} \\
\delta F &= -i \sqrt{2} \epsilon_{\alpha} \sigma^{\mu \alpha \beta} D_{\mu} \bar{\psi}_{\alpha} + 2 e \epsilon_{\alpha} \chi, & \delta F^* &= -i \sqrt{2} \epsilon^\dagger \sigma^{\mu \alpha \beta} D_{\mu} \psi^\alpha + 2 e \epsilon^\dagger \bar{\chi}_{\alpha},
\end{align*}
\]

(75)

where

\[
\sigma^{\mu \nu \alpha \beta} = \frac{1}{4} (\sigma^{\mu \alpha \beta} \sigma^{\nu \beta \alpha} - \sigma^{\nu \alpha \beta} \sigma^{\mu \beta \alpha}) \quad \text{and} \quad \sigma^{\nu \beta \alpha} = \frac{1}{4} (\sigma^{\mu \alpha \beta} \sigma^{\nu \beta \alpha} - \sigma^{\nu \alpha \beta} \sigma^{\mu \alpha \beta}).
\]

(76)

As always, before eliminating auxiliary fields, the transformation rules of the gauge multiplet do not depend on the matter fields, and the transformation rules themselves are gauge covariant and lead to a closed superalgebra.

In 2+1 dimensions, dimensional reduction gives an \( \mathcal{N} = 2 \) model involving, in the notation of \([52]\), a real scalar \( N = A_3 \) and two Dirac spinors \( \psi^\alpha, \chi^\alpha \).

Completing squares in the bosonic part of the classical Hamiltonian density for time-independent fields one finds the Bogomolnyi equations and the central charge

\[
H = \frac{1}{4} F_{kl}^2 + |D_k \phi|^2 + \frac{1}{2} e^2 (|\phi|^2 - \nu^2)^2
\]

\[
+ \frac{1}{2} |D_k \phi + i \epsilon_k D_l \phi|^2 + \frac{1}{2} (F_{12} + e (|\phi|^2 - \nu^2))^2
\]

\[
+ \frac{e}{2} \nu^2 \epsilon_{kl} F_{kl} - i \partial_k (\epsilon_{kl} \phi^* D_l \phi) \quad \text{with} \ k, l = 1, 2.
\]

(77)

\(^{14}\) Our conventions are \( \eta^{\mu \nu} = (-1, +1, +1, +1) \), \( \chi^\alpha = \epsilon^{\alpha \beta} \chi^\beta \) and \( \bar{\chi}^\dagger = \epsilon^{\dagger \beta} \bar{\chi}^\dagger \bar{\chi} \). With \( \epsilon^{\alpha \beta} = \epsilon_{\alpha \beta} = -\epsilon^{\dagger \beta} = -\epsilon_{\dagger \beta} \) and \( \epsilon^{12} = +1 \). In particular we have \( \bar{\psi}_{\alpha} = \psi_{\alpha}^* \) but \( \bar{\psi}^\dagger = -(\psi^\alpha)^* \). Furthermore, \( \bar{\sigma}_{\alpha \beta} = -(1, \sigma) \) with the usual representation for the Pauli matrices \( \sigma \), and \( \sigma^{\mu \alpha \beta} = \sigma^\mu \bar{\sigma}^\dagger \bar{\sigma}^\alpha \) with \( \sigma^{\mu \alpha \beta} = (1, \sigma) \).
The classical central charge reads
\[
Z = \int d^2x \epsilon_{kl} \partial_k \left( ev^2 A_l - i\phi^* D_l \phi \right),
\]
where asymptotically $D_l \phi$ tends to zero exponentially fast. Classically, BPS saturation $E = |Z| = 2\pi v^2 |n|$ holds when the BPS equations $(D_1 \pm iD_2)\phi \equiv D_\pm \phi = 0$ and $F_{12} \pm e(|\phi|^2 - v^2) = 0$ are satisfied, where the upper and lower sign corresponds to vortex and antivortex, respectively. In this paper we use the vortex solution. For winding number $n$, it is given by
\[
\phi_V = e^{in\theta} f(r), \quad eA^V_+ = -ie^{ia(r)}(r - n), \quad A^V_\pm \equiv A_1^V \pm iA_2^V
\]
or, alternatively,
\[
\phi_V = \phi^1_V + i\phi^2_V = \left( \frac{x^1 + ix^2}{r} \right)^n f(r), \quad eA^V_k = \frac{\epsilon_{kl}^a a(r) - n}{r},
\]
where $f'(r) = \frac{2}{r} f(r)$ and $a'(r) = re^2(f(r)^2 - v^2)$ with boundary conditions [46]
\[
a(r \to \infty) = 0, \quad f(r \to \infty) = v, \quad a(r \to 0) = n + O(r^2), \quad f(r \to 0) \to r^n + O(r^{n+2}).
\]

B. Fluctuation equations

For the calculation of quantum corrections to a vortex solution we decompose $\phi$ into a classical background part $\phi_V$ and a quantum part $\eta$. Similarly, $A_\mu$ is decomposed as $A^V_\mu + a_\mu$, where only $A^V_\mu$ with $\mu = 1, 2$ is nonvanishing. We use a background $R_\xi$ gauge fixing term [57, 58] which is quadratic in the quantum fields,
\[
L_{g, \text{fix}} = -\frac{1}{2\xi}(\partial_\mu a^\mu - ie\xi(\phi_V \eta^* - \phi^*_V \eta))^2.
\]
The corresponding Faddeev-Popov Lagrangian reads
\[
L_{\text{ghost}} = b \left( \partial_\mu^2 - e^2 \xi \{ 2 |\phi_V|^2 + \phi_V \eta^* + \phi^*_V \eta \} \right) c.
\]

The fluctuation equations in 2+1 dimensions have been given in [47, 52] for the choice $\xi = 1$ (Feynman-'t Hooft gauge) which leads to important simplifications. We shall mostly use this gauge choice when considering fluctuations in the solitonic background, but will carry out renormalization in the trivial vacuum for general $\xi$ to highlight some of the gauge dependences.

Because we are going to consider dimensional regularization by dimensional reduction of the 3+1 dimensional model, we shall need the form of the fluctuation equations with derivatives in the $x^3$ direction included. (This one trivial extra dimension will eventually be turned into $\epsilon \to 0$ dimensions.)

In the 't Hooft-Feynman gauge, the part of the bosonic action quadratic in the quantum fields reads ($m = 0, 1, 2$ but $\mu = 0, 1, 2, 3$)
\[
L^{(2)}_{\text{bos}} = -\frac{1}{2}(\partial_\mu a_\mu)^2 - \frac{1}{2}(\partial_\mu a_3)^2 - e^2|\phi_V|^2 a_\mu^2 - |D^Y_\mu \eta|^2 - e^2(3|\phi_V|^2 - v^2)|\eta|^2 - 2iea_\mu \left[ \eta^* D^Y_\mu \phi_V - \eta(D^V_\mu \phi_V)^* \right].
\]
In the trivial vacuum, which corresponds to $\phi_V \to v$ and $A^\mu_V \to 0$, the last term vanishes, but in the solitonic vacuum it couples the linearized field equations for the fluctuations $B \equiv (\eta, a_+ / \sqrt{2})$ with $a_+ = a_1 + i a_2$ to each other according to ($k = 1, 2$)

$$
(\partial_3^2 - \partial_t^2) B = \left( \begin{array}{cc}
-(D^V_k)^2 + e^2(3|\phi_V|^2 - v^2) & i \sqrt{2} e (D_+ \phi_V) \\
-i \sqrt{2} e (D_- \phi_V)^* & -\partial_k^2 + 2 e^2 |\phi_V|^2
\end{array} \right) B.
$$

The quartet $(a_3, a_0, b, c)$ with $b, c$ the Faddeev-Popov ghost fields has diagonal field equations at the linearized level

$$
(\partial_\mu^2 - 2 e^2 |\phi_V|^2) Q = 0, \quad Q \equiv (a_3, a_0, b, c).
$$

For the fermionic fluctuations, which we group as $U = (\psi_1 \bar{\chi}_1, V = (\psi_2 \bar{\chi}_2)$, the linearized field equations read

$$
LU = i (\partial_t + \partial_3) V, \quad L^\dagger V = i (\partial_t - \partial_3) U,
$$

with

$$
L = \left( \begin{array}{cc}
i D^V_+ & \sqrt{2} e \phi_V \\
-\sqrt{2} e \phi_V^* & i \partial_-
\end{array} \right), \quad L^\dagger = \left( \begin{array}{cc}
i D^V_- & -\sqrt{2} e \phi_V \\
\sqrt{2} e \phi_V^* & i \partial_+
\end{array} \right).
$$

Iteration shows that $U$ satisfies the same second order equations as the bosonic fluctuations $B$,

$$
L^\dagger L U = (\partial_3^2 - \partial_t^2) U, \quad L^\dagger L B = (\partial_3^2 - \partial_t^2) B
$$

with $L^\dagger L$ given by (85), whereas $V$ is governed by a diagonal equation with

$$
L L^\dagger = \left( \begin{array}{cc}
-(D^V_k)^2 + e^2|\phi_V|^2 + e^2 v^2 & 0 \\
0 & -\partial_k^2 + 2 e^2 |\phi_V|^2
\end{array} \right).
$$

(In deriving these fluctuation equations we used the BPS equations throughout.)

C. Renormalization

At the classical level, the energy and central charge of vortices are multiples of $2\pi v^2$ with $v^2 = \kappa / e$. Renormalization of $v^2$, even when only by finite amounts, will therefore contribute directly to the quantum mass and central charge of the \(N = 2\) vortex, a fact that has been neglected in the original literature [47, 52] on quantum corrections to the \(N = 2\) vortex.

In 2+1 dimensions, it is possible, just as in the case of the 1+1-dimensional supersymmetric kink, to adopt a “minimal” renormalization scheme where the wave function renormalization constants are set to unity, and only $v^2$ is renormalized. The renormalization of $v^2$ is then fixed by the requirement of vanishing tadpoles in the trivial sector of the 2+1 dimensional model. The calculation can be conveniently performed by using dimensional regularization of the 3+1 dimensional $N = 1$ model. By going down in the number of spatial dimensions, $3 \rightarrow 3 - (1 - \epsilon)$, and setting $\epsilon \rightarrow 0$ eventually, we have a supersymmetry preserving regularization method in analogy to the embedding of the supersymmetric kink as a domain wall in up to 2+1 dimensions.
For the calculation of the tadpoles we decompose $\phi = v + \eta \equiv v + (\sigma + i\rho)/\sqrt{2}$, where $\sigma$ is the Higgs field and $\rho$ the would-be Goldstone boson. With the gauge fixing term (82) and $\xi = 1$ all fields (including the Faddeev-Popov fields $b$ and $c$) have the same mass $|m| = \sqrt{2}ev$. This gauge choice also avoids mixed $a_{\mu}\rho$ propagators, but there are mixed $\chi-\psi$ propagators, which can be diagonalized by introducing new spinors $s = (\psi + i\chi)/\sqrt{2}$ and $d = (\psi - i\chi)/\sqrt{2}$ with mass terms $\frac{im}{2}(s_{\alpha}s^{\alpha} - d_{\alpha}d^{\alpha}) + h.c.$

The part of the interaction Lagrangian which is relevant for $\sigma$ tadpoles to one-loop order is given by

$$L_{\sigma-\text{tadpoles}}^{\text{int}} = e(\chi_\alpha \psi^\alpha + \bar{\chi}_\alpha \bar{\psi}^\alpha) \sigma - \frac{em}{2}(\sigma^2 + \rho^2) \sigma - em(a_{\mu}^2 + \xi bc - \delta v^2) \sigma,$$

where $b$ and $c$ are the Faddeev-Popov fields.

The one-loop contributions to the $\sigma$ tadpole thus read

$$\{ -2\text{tr} I(m) + \frac{3}{2} I(m) + \frac{1}{2} I(\xi^2 m) + [3I(m) + \xi I(\xi^2 m)] - \xi I(\xi^2 m) - \delta v^2 \},$$

where

$$I(m) = \int \frac{d^{3+\epsilon} k}{(2\pi)^{3+\epsilon} k^2 + m^2} = -\frac{m^{1+\epsilon}}{(4\pi)^{1+\epsilon/2} \Gamma(-\frac{1}{2} - \frac{\epsilon}{2})} = -\frac{m}{4\pi} + O(\epsilon).$$

Note that because we use dimensional reduction the component $a_3$ is kept also in the limit of 2+1 dimensions. In fact, the result equals that of ordinary dimensional regularization in 2+1 dimensions, where $a_3$ appears as an additional scalar field.

Requiring that the sum of tadpole diagrams (93) vanishes fixes $\delta v^2$,

$$\delta v^2 = \left. \frac{1}{2} \left( I(m) + I(\xi^2 m) \right) \right|_{D=3} = -\frac{1 + \xi^2}{8\pi} m.$$
of (95) is replaced by $\delta v^2 - v^2 \delta Z_\phi$. While the residue of the scalar propagator is finite and does not enforce a nontrivial $\delta Z_\phi$, to dispose of a UV divergence, one can nevertheless choose a finite value of $\delta Z_\phi$ precisely such that $\delta v^2 = 0$. By allowing for a nontrivial wave function renormalization of the vector boson field, $Z_1^{1/2} = Z_\phi^{-1}$, one can accommodate further renormalization conditions (for example an on-shell definition of the elementary masses, residues, and coupling constants), which generically involve a nontrivial $\delta v^2$. In the following we shall however stick to the “minimal” scheme with $Z_\phi = 1$, which has been used predominantly also in the case of the 1+1 dimensional kink (for a detailed discussion of other renormalization schemes in the latter context see [21]).

Since in general $\delta v^2$ is gauge-parameter dependent, the remaining contributions to vortex mass and central charge must be gauge dependent, too, so that the final result is gauge independent when expressed in terms of physical parameters. (Notice that $v^2$ is not such a physical parameter, but in any well-defined renormalization scheme it can be related to the physical mass and coupling constant.)

The gauge breaking term in fact breaks susy, but because the final result should be gauge-choice independent, we should get the correct result for the mass and central charge from this $x$-space susy-breaking approach. One could also use a superspace approach and fix the U(1) gauge symmetry without breaking rigid supersymmetry, but in the presence of solitons the background superspace formalism leads to some problems, as we sketch in the appendix.

To gain further insight into the occurrence of tadpoles and their contribution to the quantum mass of the $\mathcal{N} = 2$ vortex, we briefly consider an extension of the vortex model in which the U(1) anomaly in 3+1 dimensions cancels. In the 3+1 dimensional model, the U(1) coupling of the vector multiplet to the chiral multiplet is chiral (in terms of 4-component Majorana spinors it contains the matrix $\gamma_5$), and in order to cancel the chiral U(1) anomaly, additional scalar multiplets would be needed such that the sum over charges vanishes, $\sum_i e_i = 0$. As it turns out, this anomaly is also of concern in 2+1 dimensions. There it does not entail a local gauge anomaly when $\sum_i e_i \neq 0$, but leads to a parity-violating Chern-Simon term [59–61].

The simplest extension fulfilling this condition is massless super-QED, consisting of a vector multiplet coupled to two chiral multiplets with opposite electric charges and still without superpotential [62]. In such a theory, the Fayet-Iliopoulos term is not generated in any order of perturbation theory if it is not present classically [63]. As we now briefly show, this does not imply the absence of quantum corrections to $v^2 = \kappa/e$ if $v^2$ is nonzero.

In massless super-QED, the kinetic term, Yukawa coupling and the scalar coupling to $D$ for the new terms are the same as for $\psi$ and $\phi$, except for an opposite overall sign. Eliminating $D$, the potential becomes

$$V = \frac{1}{2} e^2 \left( |\psi|^2 - |\bar{\psi}|^2 - \frac{\kappa}{e} \right)^2$$

(96)

where $\bar{\phi}$ is the new complex scalar. We locate the vortex solution again in $\phi$, and use the same gauge fixing term as in (82).\footnote{In fact, it has been shown in [64] that the only solutions with nonvanishing winding number that are time-independent are the original vortex solutions, with $\bar{\phi} = 0$ and the vortex located in $\phi$. An analogous result holds also for the kink [65].} We find then two new couplings which produce
\( \sigma \)-tadpoles

\[
\mathcal{L}_{\text{int, extra} \, \sigma \text{-tadpoles}} = \frac{1}{2} e m \sigma (\tilde{\sigma}^2 + \tilde{\rho}^2)
\]  

(97)

However, these new fields are massless and therefore the additional tadpole diagrams vanish in dimensional regularization.

On the other hand, expanding about \( \tilde{\phi} = 0 \) the \( \tilde{\sigma} \) field does not have tadpole diagrams—there is only a \( \chi \tilde{\psi} \tilde{\sigma} \) vertex left, but this does not give rise to a tadpole diagram because \( \tilde{\psi} \) does not mix with \( \chi \) and \( \psi \) (or \( s \) and \( d \)) when \( \tilde{\phi} \) vanishes.

Hence, the finite renormalization of \( v^2 \) in this extension of the 2+1-dimensional susy Higgs models is identical to that of the simpler model we have introduced above, provided the minimal renormalization scheme with trivial wave function renormalization constants is employed.\(^{17}\)

In the following we shall show that the nonzero result for \( \delta v^2 \) leads to a nonvanishing mass correction for the \( \mathcal{N} = 2 \) vortex in 2+1 dimensions and is in fact required to match an equally nonzero correction to the central charge in order that the BPS bound remains saturated.

### D. Quantum corrections to mass and central charge

The expressions for the central charge and stress tensor can be constructed from the classical action without any gauge artefacts. However, when one evaluates one-loop corrections, one uses the gauge-fixing term to obtain propagators and well-defined fluctuation equations, and then one should use the quantum expression for \( H \). In that case one should consider sums over zero-point energies including unphysical degrees of freedom and Faddeev-Popov ghosts. This can be done in a well-defined manner by using dimensional regularization by dimensional reduction from the 3+1 dimensional model. Using this method, the central charge contains the standard 2+1 dimensional terms and, as a potential anomalous contribution, a remainder from the momentum operator in the extra spatial dimension.

#### 1. Mass

At the one-loop level, the quantum mass of a solitonic state is given by

\[
M = M_{\text{cl}} + \frac{1}{2} \sum \omega_{\text{bos}} - \frac{1}{2} \sum \omega_{\text{term}} + \delta M
\]

(98)

where \( M_{\text{cl}} \) is the classical mass expressed in terms of renormalized parameters, \( \delta M \) represents the effects of the counter-terms to these renormalized parameters, and the sums are over zero-point energies in the soliton background (the zero-point energies in the trivial vacuum, which one should subtract in principle, cancel in a susy theory).

\(^{17}\) It should be noted, however, that the situation is more complicated in 3+1 dimensions, since there one cannot do without wave function renormalization.
In the $\xi = 1$ gauge the sum over zero-point energies is formally

$$\frac{1}{2} \sum \omega_{\text{bos}} - \frac{1}{2} \sum \omega_{\text{form}} = \sum \omega_\eta + \sum \omega_{\alpha^+} - \sum \omega_U - \sum \omega_V = \sum \omega_U - \sum \omega_V, \quad (99)$$

where the quartet $(a_3, a_0, b, c)$ cancels separately. (Note that in (99) all frequencies appear twice because all fields are complex.)

Using dimensional regularization for models with solitons as developed in [21], these sums can be made well defined by replacing all eigen frequencies $\omega_k$ in 2+1 dimensions by $\omega_{k,\ell} = (\omega_k^2 + \ell^2)^{1/2}$ where $\ell$ are the extra momenta, and integrating over $\ell$:

$$\sum \omega_U - \sum \omega_V = \int \frac{d^2k}{(2\pi)^2} \int \frac{d\ell}{(2\pi)^\epsilon} \omega_{k,\ell} \int d^2x \left[ |u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 \right](x;k) \quad (100)$$

where we have written out only the contributions from the continuous part of the spectrum, using that $L^\dagger L$ and $LL^\dagger$, which govern $U$ and $V$, respectively, are isospectral up to zero modes. In dimensional regularization, the zero modes of $L^\dagger L$ and $LL^\dagger$ become massless modes (with energy $\sqrt{\ell^2}$), but they can be dropped because the $\ell$-integration is scaleless, and thus vanishes in dimensional regularization.

The existing literature [47, 52] also proves that the spectral densities of the continuous spectrum is equal for $U$ and $V$, so there is a complete cancellation of the sums over zero-point energies in the $\xi = 1$ gauge. In analogy to the calculation performed in eqs. (41) and (42), we can verify this result explicitly as follows.

Using that $LU_k = \omega V_k$ and $L^\dagger V_k = \omega_k U_k$, we can write the difference in the spectral densities appearing in (100) as

$$\Delta \rho(k) = \int d^2x [U_k^\dagger U_k - V_k^\dagger V_k] = \int d^2x [U_k^\dagger U_k - \omega_k^{-2}(LU)_k^\dagger LU_k] \quad (101)$$

and partially integrate to obtain a surface term of the form

$$\Delta \rho(k) = i\omega_k^{-2} \int d^2x \left[ \partial_- [u_1^* (iD_+ u_1 + \sqrt{2}e\phi\nu u_2)] + \partial_+ [u_2^* (i\partial_- u_2 - \sqrt{2}e\phi^* \nu u_1)] \right]$$

$$= i\omega_k^{-2} \lim_{r \to \infty} \int d\theta [u_1^* (-\partial_\theta + in)u_1 + u_2^* \partial_\theta u_2] = 0, \quad (102)$$

which vanishes because cylindrical waves decay like $|u_{1,2}| \sim r^{-1/2}$.

Hence, $\sum \omega_U - \sum \omega_V = 0$, and all that remains is a possible renormalization of $\nu^2$. As we discussed, in the minimal renormalization scheme where all wave functions renormalizations are trivial there is a finite result for $\delta \nu^2$, and this leads to a nonvanishing quantum correction of the vortex mass according to

$$E = 2\pi |n|(\nu^2 + \delta \nu^2)_{\xi = 1} = 2\pi |n|((\nu^2 - m)/4\pi) \equiv |n| \left( \frac{\pi m^2}{c^2} - \frac{m}{2} \right), \quad (103)$$

where $\xi = 1$, since in other gauges the fluctuation equations for the $B$ fields, i.e. $\eta, a_\pm$, no longer match those of the $U$ fermions. As always in quantum field theory, the explicit form of the quantum corrections depends on the definition of the parameters in the Lagrangian, i.e.
on the renormalization conditions employed, which we have chosen in the simplest possible manner. There are of course renormalization schemes which are more physical such as on-shell renormalization of the parameters in the trivial vacuum. In the case of kink and kink domain walls, an extensive analysis of renormalization schemes other than the minimal one can be found in [21].

The above result for the vortex mass in the minimal renormalization scheme agrees with [53], where however a careful analysis of boundary conditions in the heat-kernel approach was needed because the vortex had to be put in a box to discretize the spectrum. In dimensional regularization one does not need to put the system in a box, and as a consequence there is no need to study the contributions from these artificial boundaries.

In the super-QED model considered at the end of the previous section we have found that the additional (tilde) fields do not change $\delta v^2$ as given in (95). Concerning the additional fluctuation equations for these fields, these are even simpler than that of the minimal model: The $\tilde{\eta}$ field is governed by

$$ (\partial_3^2 - \partial_t^2)\tilde{\eta} = [- (D_k^V)^2 + e^2 (v^2 - |\phi_V|^2)]\tilde{\eta} = - D_{+}^V D_+^V \tilde{\eta} $$

(104)

where $D_+^V$ differs from $D$ only in the sign in front of $e$. The fermionic tilde field equations read

$$ i D_{-}^V \tilde{\psi}^2 = i (\partial_t + \partial_3) \tilde{\psi}^2, \quad i D_{+}^V \tilde{\psi}^2 = i (\partial_t - \partial_3) \tilde{\psi}^1. $$

(105)

Iteration shows that $\tilde{\psi}^2$ has the same field equation as $\tilde{\eta}$, so that in the mode sum we have

$$ \sum \omega (D_{-}^V D_+^V) - \sum \omega (D_+^V D_{+}^V). $$

(106)

The same arguments that prove $\sum \omega U - \sum \omega V = \sum \omega (L^\dagger L) - \sum \omega (L L^\dagger) = 0$ can now be repeated for these simpler operators, and so the above result for the mass correction to the $N = 2$ vortex in 2+1 dimensions remains unchanged when considering the vortex of the anomaly-free super-QED model.

2. Central charge

By starting from the susy algebra in 3+1 dimensions one can derive the central charge in 2+1 dimensions as the component $T^{03}$ of

$$ T^{\mu\nu} = -\frac{i}{4} \text{Tr} \sigma^{\mu\dot{\alpha}} \{ \bar{Q}_\dot{\alpha}, J_{\alpha}^\nu \} $$

(107)

where $J_{\alpha}^\nu$ is the susy Noether current.

The antisymmetric part of $T^{\mu\nu}$ gives the standard expression for the central charge density, while the symmetric part is a genuine momentum in the extra dimension:

$$ \langle Z \rangle = \int d^2 x \langle T^{03} \rangle = \langle \tilde{Z} + \tilde{P}_3 \rangle. $$

(108)

(A similar decomposition is valid for the kink [22].)

$\tilde{Z}$ corresponds to the classical expression for the central charge. Being a surface term, its quantum corrections can be evaluated at infinity:

$$ \langle \tilde{Z} \rangle = \int d^2 x \partial_k \epsilon_{kl} \langle \tilde{\zeta}_l \rangle = \int_0^{2\pi} d\theta \langle \tilde{\zeta}_\theta \rangle |_{r\to\infty} $$

(109)
with \( \tilde{\zeta} = ev_0^2 A_l - \imath \phi^\dagger D_l \phi \) and \( v_0^2 = v^2 + \delta v^2 \).

Expanding in quantum fields \( \phi = \phi_V + \eta, A = A^V + a \) and using that the classical fields \( \phi_V \rightarrow ve^{\imath \theta}, A^V_{\theta} \rightarrow n/e, D_{\theta}^V \phi_V \rightarrow 0 \) as \( r \rightarrow \infty \), we obtain to one-loop order

\[
\langle \tilde{Z} \rangle = 2\pi n v_0^2 - \imath \int_0^{2\pi} d\theta \langle (\phi^\dagger V + \eta^\dagger) (D^\dagger_\theta - ie a_\theta) (\phi_V + \eta) \rangle |_{r \rightarrow \infty}
\]

\[
= 2\pi n \{ v_0^2 - \langle \eta^\dagger \eta \rangle |_{r \rightarrow \infty} \} - \imath \int_0^{2\pi} d\theta \{ \langle \eta^\dagger \partial_\theta \eta \rangle - ie \phi^\dagger_V \langle a_\theta \eta \rangle - ie \phi_V \langle a_\theta \eta^\dagger \rangle \} |_{r \rightarrow \infty}
\]

\[
\equiv Z_a + Z_b
\]  

(110)

where we have used \( \langle \eta (r \rightarrow \infty) \rangle \rightarrow 0 \) (which determines \( \delta v^2 \) at infinity, but at finite \( r \), the vacuum expectation value does not vanish [11, 12]), \( \langle a_\theta \rangle = 0 \), and \( \langle \eta^\dagger \eta a_\theta \rangle = O(\hbar^2) \).

The first contribution, \( Z_a \), can be easily evaluated for arbitrary gauge parameter \( \xi \), yielding

\[
Z_a = 2\pi n \{ v_0^2 - \frac{1}{2} \langle (\sigma \sigma) + (\rho \rho) \rangle |_{r \rightarrow \infty} \}
\]

\[
= 2\pi n \{ v_0^2 - \frac{1}{2} [I(m) + I(\xi^\dagger m)] \}
\]

\[
= 2\pi n (v_0^2 - \delta v^2) = 2\pi n v^2.
\]  

(111)

If this was all, the quantum corrections to \( Z \) would cancel, just as in the naive calculation of \( Z \) in the susy kink [5, 6].

The second contribution in (110), however, does not vanish when taking the limit \( r \rightarrow \infty \). This contribution is simplest in the \( \xi = 1 \) gauge, where the \( \eta \) and \( a_\theta \) fluctuations are governed by the fluctuation equations (85). In the limit \( r \rightarrow \infty \) one has \( |\phi_V| \rightarrow v \) and \( D_\phi \phi_V \rightarrow 0 \) exponentially. This eliminates the contributions from \( \langle a_\theta \eta \rangle \) (note that at finite \( r \) there is a cross-term in the kinetic terms for \( a_\theta \) and \( \eta \)). However, \( D^2 \xi \), which governs the \( \eta \) fluctuations, contains long-range contributions from the vector potential. Making a separation of variables in \( r \) and \( \theta \) one finds that asymptotically

\[
|D_k^V \eta|^2 \rightarrow |\partial_r \eta|^2 + \frac{1}{r^2} |(\partial_\theta - in) \eta|^2
\]  

(112)

so that the \( \eta \) fluctuations have an extra phase factor \( e^{in\theta} \) compared to those in the trivial vacuum. We thus have, in the \( \xi = 1 \) gauge,

\[
Z_b = -i \int_0^{2\pi} d\theta \{ \langle \eta^\dagger \partial_\theta \eta \rangle - ie \phi^\dagger_V \langle a_\theta \eta \rangle - ie \phi_V \langle a_\theta \eta^\dagger \rangle \} |_{r \rightarrow \infty}
\]

\[
= -i \int_0^{2\pi} d\theta \langle \eta^\dagger \partial_\theta \eta \rangle_{\xi = 1} = 2\pi n \langle \eta^\dagger \eta \rangle_{\xi = 1, r \rightarrow \infty} = 2\pi n \delta v^2|_{\xi = 1}.
\]  

(113)

This is exactly the result for the one-loop correction to the mass of the vortex in eq. (103), implying saturation of the BPS bound provided that there are now no anomalous contributions to the central charge operator as there are in the case in the \( \mathcal{N} = 1 \) susy kink [22].

In dimensional regularization by dimensional reduction from a higher-dimensional model such anomalous contributions to the central charge operator come from a finite remainder
of the extra momentum operator \[ [22] \]. The latter is given by \[ [52] \]

\[
Z_c = \langle \tilde{P}_3 \rangle = \int d^2x \left\langle F_{0i} F_{3i} + (D_0 \phi)^\dagger D_3 \phi + (D_3 \phi)^\dagger D_0 \phi - i \bar{\chi} \bar{\sigma}_0 \partial_3 \chi - i \bar{\psi} \bar{\sigma}_0 D_3 \psi \right\rangle .
\] (114)

Inserting mode expansions for the quantum fields one immediately finds that the bosonic contributions vanish because of symmetry in the extra trivial dimension. However, this is not the case for the fermionic fields, which have a mode expansion of the form

\[
\left( \begin{array}{c} U \\ V \end{array} \right) = \int \frac{d\ell}{(2\pi)^{c/2}} \int_k \frac{1}{\sqrt{2\omega}} \left\{ b_{k,\ell} e^{-i(\omega t - \ell z)} \left( \begin{array}{c} \sqrt{\omega - \ell} u_1 \\ \sqrt{\omega + \ell} u_2 \\ \sqrt{\omega - \ell} v_1 \\ \sqrt{\omega + \ell} v_2 \end{array} \right) + d_{k,\ell}^\dagger \times (c.c.) \right\},
\] (115)

where we have not written out explicitly the zero-modes (for which \( \omega^2 = \ell^2 \)). The fermionic contribution to \( Z_c \) reads

\[
Z_c = \langle \tilde{P}_3 \rangle = \int \frac{d^2k}{(2\pi)^c} \int \frac{d\ell}{2\omega} \int d^2x \left[ |u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 \right] (x;k)
\] (116)

where \( \omega = \sqrt{\omega_k + \ell^2} \), so that the \( \ell \) integral is nontrivial in dimensional regularization. Only the continuous spectrum can contribute because zero modes give scaleless integrals which vanish in dimensional regularization and if there were other discrete states, they would cancel between \( U \) and \( V \). However, the \( x \)-integration over the mode functions \( u_{1,2} \) and \( v_{1,2} \) produces their spectral densities, and we find

\[
Z_c = \int \frac{d^2k}{(2\pi)^c} \frac{d\ell}{(2\pi)^c} \frac{\ell^2}{2\omega} \Delta \rho(k) = 0
\] (117)

because \( \Delta \rho(k) = 0 \) as we have seen in (102). Hence, \( |Z| = |Z_a + Z_b| = E \), so that the BPS bound is saturated at the (one-loop) quantum level.

E. Fermionic zero modes and multiplet shortening

Massive representations of the Poincaré supersymmetry algebra for which the absolute value of the central charge equals the energy, i.e. when the BPS bound is saturated, contain as many states as massless representations, which is half of that of massive representations for which the BPS bound is not saturated. These results also apply in 2+1 dimensions for the \( \mathcal{N} = 2 \) super-Poincaré algebra [52].

A particular multiplet of states is obtained by taking the vortex solution, and acting on it with the susy generators of the \( \mathcal{N} = 2 \) susy algebra, which contains two complex charges \( Q^+ \), \( Q^- \), and their hermitian conjugates \( (Q^+)\dagger \) and \( (Q^-)\dagger \). One of these charges, \( Q^+ \), annihilates the vortex solution, while the other one, \( Q^- \), is to linear order in quantum operators proportional to the annihilation operator of a fermionic zero mode.
However, if there indeed is a second fermionic zero mode in the model as claimed in [52]\(^{18}\), in second quantization it would be present in the mode expansion of the fermionic quartet \(U\) and \(V\),

\[
\begin{pmatrix} U \\ V \end{pmatrix} = a_1 \begin{pmatrix} \psi^\dagger_1 \\ \bar{\chi}^\dagger_1 \\ 0 \\ 0 \end{pmatrix} + a_\# \begin{pmatrix} \psi^\dagger_\# \\ \bar{\chi}^\dagger_\# \\ 0 \\ 0 \end{pmatrix} + \text{non-zero modes.} \tag{118}
\]

As a result, there would then be a quartet of BPS states

\[
|v\rangle, \hspace{1em} a^\dagger_1|v\rangle, \hspace{1em} a^\dagger_\#|v\rangle, \hspace{1em} a^\dagger_1a^\dagger_\#|v\rangle
\]

comprising two short multiplets of \(\mathcal{N} = 2\) susy, which are degenerate and together have as many states as one long multiplet without BPS saturation. As stressed in [52], the standard argument for stability of BPS saturation under quantum corrections from multiplet shortening [42] thus would not be applicable.

However, we shall now show that there is in fact only a single fermionic zero mode in a vortex background with winding number \(n = 1\) [42]. To this end, we first observe that the zero modes must lie in \(U\), because \(V\) is governed by the operator \(LL^1\) of Eq. (91), whose only zero mode solution is \(V_0 \equiv 0\). A zero mode for \(U\) must satisfy \(LU = 0\), and to analyse this equation we follow [52] and set \(\psi^1(x,y) = -ie^{i(\bar{\theta} - \frac{1}{2} + n)}u(r)\) and \(\bar{\chi}^1 = e^{i(\bar{\theta} + \frac{1}{2})}d(r)\). The equation \(LU = 0\) reduces then to

\[
\begin{pmatrix} \partial_r - (a + j - \frac{1}{2})/r & \sqrt{2}ef \\ \sqrt{2}ef & \partial_r + (j + \frac{1}{2})/r \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = 0, \tag{120}
\]

where \(f = f(r)\) and \(a = a(r)\) satisfy \(f' = \frac{a^2}{r^2}f\) and \(a' = re^2(f^2 - v^2)\). Iterating this equation yields

\[
\begin{pmatrix} \partial_r^2 + \frac{1}{r}\partial_r - (j - \frac{1}{2})^2 \frac{1}{r^2} - 2e^2f^2 \\ \partial_r \end{pmatrix} u \bigg/ f = 0. \tag{121}
\]

Given a solution for \(u\), the corresponding solution for \(d\) follows from \(LU = 0\).

For given \(j\), this equation has two independent solutions, a linear combination of which yields solutions which decrease exponentially fast as \(r \rightarrow \infty\). Hence, both solutions should be regular at \(r = 0\). For \(j \neq \frac{1}{2}\), one has, using \(f(r \rightarrow 0) \sim r^n\),

\[
\psi^1 \sim u \sim r^n(C_1r^{j - \frac{1}{2}} + C_2r^{-(j - \frac{1}{2})}) \quad \text{for } r \rightarrow 0 \tag{122}
\]

which selects for \(n = 1\) only \(j = -\frac{1}{2}\). This solution is the zero mode that is obtained by acting with \(Q^-\) on the background solutions, which gives \(\psi^1 = -iD_\phi/\sqrt{2} = -i\sqrt{2}f'\), \(\bar{\chi}^1 = F_{12} = -e(f^2 - v^2)\). For \(j = \frac{1}{2}\), one finds for \(n = 1\) near \(r = 0\)

\[
\psi^1 \sim C_1 (x + iy) + C_2 (x + iy) \ln r. \tag{123}
\]

\(^{18}\) In the literature one can in fact find two different conventions for indicating the number of fermionic zero modes. Like Refs. [52, 56] we only count the number of zero modes in the fermionic quartet \((U, V)\) and not additionally those in the corresponding conjugated fields \((U^\dagger, V^\dagger)\). One zero mode in this way of counting then corresponds to a pair of creation/annihilation operators. Alternatively one may count the zero modes in both \((U, V)\) and \((U^\dagger, V^\dagger)\) and thus ascribe one zero mode to each creation or annihilation operator. The latter way of counting is employed for instance in Ref. [50].
For large $r$, $\psi_1 \sim e^{-mr}e^{i\theta}$, as follows from (121). This solution corresponds to the second fermionic zero mode postulated in Ref. [52].

However, while (123) is finite at the origin, the associated gaugino component is not: (120) implies that
\[ \tilde{\chi}^i \sim C_2 e^{i\theta} \frac{1}{r}, \] (124)
so this solution has to be discarded when $C_2 \neq 0$.

Similarly, one can show that for winding number $n > 1$ regularity of the gaugino component generically requires that $j \leq -\frac{1}{2}$ so that the correct quantization condition for normalizable fermionic zero modes is $-n + \frac{1}{2} \leq j \leq -\frac{1}{2}$. Hence, there are $n$ independent fermionic zero modes, not $2n$ as concluded in [52]. It is in fact only the former value that agrees with the results [54, 56] obtained from the index theorem [55].

As has been proved rigorously in [66], in the bosonic sector there are $2n$ zero modes, which are related to the above $n$ independent fermionic zero modes by supersymmetry. In the $R_{\xi=1}$ background gauge $\partial \mu a^\mu - ie(\phi_V \eta^* - \phi^*_V \eta) = 0$, the bosonic zero modes satisfy a set of equations completely equivalent to those for the fermionic zero modes [54]. But the linearly dependent solutions $(-U_0)$ and $i(U_0)$ correspond to linearly independent solutions for the bosonic zero modes $a$ and $\eta$. In particular, for $n = 1$, the $j = -\frac{1}{2}$ solution $(\psi^1 = -iu(r), \tilde{\chi}^1 = d(r))$ with real $u(r)$ and $d(r)$ corresponds to the bosonic zero mode $\eta(r) = -iu(r), (a_1, a_2) = (\sqrt{2}d(r), 0)$, while multiplying the fermionic solution by $i$ corresponds to the bosonic zero mode $\eta(r) = u(r), (a_1, a_2) = (0, \sqrt{2}d(r))$, which is evidently linearly independent of the former. For both solutions the $R_{\xi=1}$ gauge condition is satisfied due to the lower component of the field equation (120). Conversely, one can start from the classical vortex solution and find two independent bosonic zero modes by considering their derivatives with respect to the $x$ and $y$ coordinates. Performing a gauge transformation to satisfy the $R_{\xi=1}$ gauge condition leads one back to the above solutions. This additionally confirms that the above analysis has identified all fermionic zero modes in the quartet $(U, V)$.

We thus conclude that for the basic vortex (winding number $n = 1$) there is exactly one fermionic zero mode (corresponding to one pair of fermionic creation/annihilation operators) and this gives rise to a single short multiplet at the quantum level. Standard multiplet shortening arguments therefore do apply and explain the preservation of BPS saturation that we verified at one-loop order.

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APPENDIX A: $R_{\xi}$ GAUGE IN SUPERSPACE

It seems natural to use superfields in the calculation of quantum corrections, since in this way we manifestly preserve rigid susy. Since we want to describe quantum fluctuations about a nontrivial background, we use the background formalism for superspace. The action reads $\bar{\Phi}e^V\Phi$, where $\Phi = \phi_V(x) + \psi(x, \theta)$ and $V = V_V(x) + w(x, \theta)$ for abelian gauge theories.

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\[ \text{\textsuperscript{19} For an analogous case see eq. (3.8) of Ref. [67].} \]
Expanding in terms of quantum fields, the classical action contains again off-diagonal terms quadratic in quantum fields,

\[ \mathcal{L}_{\text{kin.}} = \int d^4 \theta \left[ \phi \dot{\psi} \psi + \bar{\psi} \dot{\phi} \phi + \ldots \right]. \]  

To cancel these, we try to modify the usual \( D = 4 \) gauge fixing term \( D^2 V \bar{D}^2 V \) into a superspace \( R_\xi \) gauge fixing term

\[ \mathcal{L}_{\text{fix}} = - (\bar{D}^2 w + \bar{D}^2 \frac{1}{\Box} \bar{\psi} \phi_V)(D^2 w + D^2 \frac{1}{\Box} \psi \phi_V^*). \]  

In a trivial (constant) vacuum with \( \phi_V = v \), the terms \( \bar{\psi} \phi_V \) and \( \phi_V^* \psi \) are antichiral and chiral, respectively. So to maintain these chirality properties we extend \( \phi_V(x) \) in the soliton sector to an anti-chiral superfield \( \phi_V(x, \theta) = \phi_V(x^\mu - i \bar{\theta} \sigma^\mu \theta) \). Then \( \phi_V^* \) is chiral, and one has the identities

\[ D^2 D^2 \frac{1}{\Box} (\psi \phi_V^*) = \psi \phi_V^*; \quad \bar{D}^2 D^2 \frac{1}{\Box} (\bar{\psi} \phi_V) = \bar{\psi} \phi_V. \]  

The terms \( - \int d^4 \theta w (\psi \phi_V^* + \bar{\psi} \phi_V) \) in the gauge-fixing term cancel the classical off-diagonal kinetic terms just as in \( x \)-space.

However, one is left with the following diagonal terms in the gauge-fixing term

\[ \int d^4 \theta \left[ \bar{D}^2 \frac{1}{\Box} (\bar{\psi} \phi_V) \right] \left[ D^2 \frac{1}{\Box} (\psi \phi_V^*) \right] = \int d^4 \theta \bar{\psi} \phi_V \frac{1}{\Box} (\psi \phi_V^*) = D^2 \left[ (\bar{D}^2 (\bar{\psi} \phi_V)) \frac{1}{\Box} (\psi \phi_V^*) \right]. \]  

Although \( D^2 \bar{D}^2 (\bar{\psi} \phi_V) \) produces a factor \( \Box \) which cancels the nonlocal \( 1/\Box \), further terms with one or two \( D_\alpha \) acting on \( \psi \phi_V^* \) yield nonlocal interactions. Thus it seems that one cannot construct a local \( R_\xi \) gauge with \( \xi = 1 \) in superspace.\(^{20}\)

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