ON A SHARP ESTIMATE FOR HANKEL OPERATORS AND PUTNAM’S INEQUALITY

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Abstract. We obtain a sharp norm estimate for Hankel operators with anti-analytic symbol. The estimate improves the corresponding classical Putnam inequality for commutators of Toeplitz operators with analytic symbol by a factor of $1/2$, answering a recent conjecture by Bell, Ferguson and Lundberg. As an application, this yields a new proof of the de St. Venant inequality, which relates the torsional rigidity of a domain with its area. We also obtain the corresponding sharp estimates for weighted Bergman spaces.

1. Introduction

Guided by applications to classical isoperimetric inequalities, we are concerned with finding sharp estimates for certain Hankel operators acting on the Bergman space $A^2(D)$. More precisely, we are looking for sharp estimates in terms of the derivative of the symbol, which we assume to be anti-analytic.

We recall some definitions. Let $dA$ denote Lebesgue area measure on the unit disc $D$, normalized so that the measure of $D$ equals 1. The Bergman space $A^2(D)$ is defined to be the closed subspace of analytic functions in the Hilbert space $L^2(D) = L^2(D, dA)$. The orthogonal projection from $L^2(D)$ into $A^2(D)$ is known as the Bergman projection and has the integral representation

$$Pf(z) = \int_D \frac{f(w)}{(1 - wz)^2} dA(w).$$

The Hankel operator on $A^2(D)$, with symbol $\varphi \in L^2(D)$, is defined as

$$H_\varphi(f) := (I - P)(\varphi f), \quad f \in A^2(D),$$

where $I$ denotes the identity map. We will also need the Toeplitz operator on $A^2(D)$, with symbol $\varphi \in L^2(D)$, which is defined by

$$T_\varphi(f) := P(\varphi f), \quad f \in A^2(D).$$

We note that the Hankel operator sends $A^2(D)$ into its orthogonal complement in $L^2(D)$, and that $H_\varphi + T_\varphi = M_\varphi$, the multiplication operator on $L^2(D)$ with symbol $\varphi$. For more background on the theory of Hankel operators on Bergman spaces, we refer the reader to [3].

Our main result is the following:
**Theorem 1.** Let \( \psi \in A^2(\mathbb{D}) \) be such that \( \psi' \in A^2(\mathbb{D}) \). Then
\[
\|H_\psi\|_{A^2(\mathbb{D}) \to L^2(\mathbb{D})} \leq \frac{1}{\sqrt{2}} \|\psi'\|_{A^2(\mathbb{D})}.
\] (1)

Moreover, this inequality is sharp.

The estimate in (1) can be used to obtain a Putnam type inequality. We recall that Putnam’s inequality [6] asserts the following: let \( H \) be a Hilbert space and let \( T : H \to H \) be a bounded linear operator whose spectrum is denoted by \( \sigma(T) \). If \( \langle T^*T - TT^* f, f \rangle \geq 0 \) for all \( f \in H \) then
\[
\|T^*T - TT^*\| \leq \frac{\text{Area}(\sigma(T))}{\pi}.
\] (2)

As an application of Putnam’s inequality, Khavinson [4] deduced the classical isoperimetric inequality. Indeed, in the case of the Hardy space \( H_2(\Omega) \) where \( \Omega \) is a reasonable bounded domain and \( T \) is the Toeplitz operator \( T_z \), Khavinson obtained the following lower bound for the commutator
\[
\|T_z^*T_z - T_zT_z^*\|_{H_2(\Omega) \to H_2(\Omega)} \geq \frac{4\text{Area}^2(\Omega)}{\text{Per}(\Omega)},
\]
where \( \text{Per}(\Omega) \) denotes the perimeter of the region \( \Omega \). He noted that when combined with Putnam’s inequality, this lower bound immediately yields the isoperimetric inequality of the region \( \Omega \).

Bell, Ferguson and Lundberg [1] considered the problem of obtaining a lower bound for the commutators of Toeplitz operators on the Bergman space. In particular, they were able to prove the inequality
\[
\|T_z^*T_z - T_zT_z^*\|_{A^2(\Omega) \to A^2(\Omega)} \geq \frac{\rho(\Omega)}{\text{Area}(\Omega)}.
\] (3)
Here, \( \rho(\Omega) \) is a constant from mechanics known as the torsional rigidity. It is geometric in the sense that it only depends on the shape of the domain \( \Omega \), and can be said to measure the resistance an object with cross-sectional \( \Omega \) has to rotation (see Section [4] for a precise definition).

Bell, Ferguson and Lundberg observed that the estimate (3) combined with Putnam’s inequality yields
\[
\rho(\Omega) \leq \frac{\text{Area}(\Omega)^2}{\pi}.
\]
As they note, this is close to the classical de St. Venant inequality
\[
\rho(\Omega) \leq \frac{\text{Area}(\Omega)^2}{2\pi}.
\]

Consequently, it was conjectured that in this setting it should be possible to improve Putnam’s inequality by a factor of \( 1/2 \). This conjecture is answered in the positive by Theorem [1]. To be precise, we obtain the following corollary.
Corollary 1. Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $\psi$ an analytic function, and put $T = T_\psi$. Then

$$\|T^*T - TT^*\|_{A^2(\Omega) \to A^2(\Omega)} \leq \|\psi'\|^2_{A^2(\Omega)}.$$ 

Hence, by choosing $\psi = z$, we immediately get the following improvement of Putnam’s inequality for the shift operators $T = T_z$ on $\Omega$. Indeed, in this case

$$\|T^*T - TT^*\|_{A^2(\Omega) \to A^2(\Omega)} \leq \frac{\text{Area}(\Omega)}{2\pi}.$$ 

(Recall that in the measure $dA$ is normalized on $\mathbb{D}$, whence the factor $1/\pi$.)

By the arguments of Bell, Ferguson and Lundberg, a new proof of the de St. Venant inequality now follows. For completeness, we include in this paper a proof of their lower bound (4) using d-bar methods. The following observation is immediate from the d-bar approach to the lower bound.

Corollary 2. Let $h$ be an analytic function on a simply connected domain $\Omega$. Then there exists a function $u \in L^2(\Omega, dA)$ such that $\overline{\partial} u = h$ with

$$\|u\|^2_{L^2(\Omega)} \leq \frac{\text{Area}(\Omega)}{2}\|h\|^2_{A^2(\Omega)}.$$ 

The structure of the paper is as follows. We begin by proving Theorem 1 in Section 2. In Section 3 we show how to obtain Corollary 1 from the theorem. In Section 4, we give a d-bar proof of inequality (3). We end the paper in Section 5, where we extend Theorem 1 to weighted Bergman spaces.

2. Proof of Theorem 1

The strategy of the proof is as follows. For $f \in A^2(\mathbb{D})$, write $f(z) = \sum_{n \geq 0} a_n z^n$, and set $\psi(z) = \sum_{k \geq 1} c_k z^k$ (note that we can assume $\psi(0) = 0$ without loss of generality). We then express the function $H_{\overline{\partial}} f$ in terms of these Taylor coefficients, and obtain the desired norm estimate by working directly with the coefficients. Essentially, the only inequality we use is $ab \leq (1/2)(a^2 + b^2)$.

As a first step, we observe that

$$P(\overline{\psi} z^n) = \sum_{k \geq 1} \overline{c_k} P(z^k z^n) = \sum_{k=1}^n \frac{n - k + 1}{n + 1} \overline{c_k} z^{n-k}$$

From the previous computation, for $n \geq 1$, it follows that

$$P(\overline{\psi} z^n) = \sum_{k \geq 1} \overline{c_k} P(z^k z^n) = \sum_{k=1}^n \frac{n - k + 1}{n + 1} \overline{c_k} z^{n-k}$$
whence we get the expression

\[ H(\bar{\psi} f)(z) = \bar{\psi}(z) f(z) - P(\bar{\psi} f)(z) \]

\[ = \sum_{\ell \geq 1} \sum_{n \geq 0} \bar{c}_\ell a_n z^\ell z^n - \sum_{n \geq 1} \sum_{k=0}^{n-1} \frac{k + 1}{n + 1} a_n \bar{c}_{n-k} z^k. \]

Our next objective is to integrate with respect to the radial part of the measure. Before we do this, we rewrite this expression in order to more easily use the orthogonal structure. First, we rewrite the above expression as

\[ \sum_{l \geq 1} a_0 \bar{c}_l z^l + \sum_{n \geq 1} a_n \left( \sum_{\ell \geq 1} \bar{c}_\ell z^\ell z^n - \sum_{0 \leq k \leq n-1} \frac{k + 1}{n + 1} \bar{c}_{n-k} z^k \right). \] (4)

Now,

\[(*) = \sum_{1 \leq \ell \leq n} \bar{c}_\ell |z|^{2\ell} + \sum_{\ell \geq n+1} \bar{c}_\ell |z|^{2\ell} - \sum_{0 \leq k \leq n-1} \frac{k + 1}{n + 1} \bar{c}_{n-k} z^k \]

\[ = \sum_{1 \leq k \leq n} \bar{c}_{n-k} |z|^{2(n-k)} + \sum_{k \geq 1} \bar{c}_{k+n} |z|^{2n} \bar{c}_{n-k} z^k - \sum_{0 \leq k \leq n-1} \frac{k + 1}{n + 1} \bar{c}_{n-k} z^k. \]

Plugging this back in to equation (4), we get

\[ \sum_{k \geq 1} z^k \left( a_0 \bar{c}_k + \sum_{n \geq 1} a_n \bar{c}_{k+n} |z|^{2n} \right) + \sum_{n \geq 1} a_n \sum_{0 \leq k \leq n-1} \bar{c}_{n-k} z^k \left( |z|^{2(n-k)} - \frac{k + 1}{n + 1} \right) \]

\[ = \sum_{k \geq 1} z^k \left( \sum_{n \geq 0} a_n \bar{c}_{k+n} |z|^{2n} \right) + \sum_{k \geq 0} \sum_{n \geq k+1} a_n \bar{c}_{n-k} \left( |z|^{2(n-k)} - \frac{k + 1}{n + 1} \right). \]

Taking the modulus squared of this at \( z = re^{i\theta} \) and integrating with respect to \( \int_0^{2\pi} d\theta/\pi \) yields

\[ 2 \sum_{k \geq 1} r^{2k} \left| \sum_{n \geq 0} a_n \bar{c}_{k+n} r^{2n} \right|^2 + 2 \sum_{k \geq 0} r^{2k} \left| \sum_{n \geq k+1} a_n \bar{c}_{n-k} \left( r^{2(n-k)} - \frac{k + 1}{n + 1} \right) \right|^2. \] (I)

Before integrating with respect to the radial part of the measure \( dA = rdrd\theta/\pi \), we multiply out and rewrite these expressions slightly:

\[ (I) = 2 \sum_{n,m \geq 0} a_n \bar{a}_m c_{k+m+n} |z|^{2n+2m+2k}, \]

and
Integration against \( f_0^1 r \, dr \) now yields

\[
(I) = \sum_{n,m \geq 0, k \geq 1} \frac{a_n \bar{a}_m c_{k+m} \bar{c}_{k+n}}{n + m + k + 1} \cdot \frac{2}{(n+1)(m+1)(n+m-k+1)}.
\]

At this point, we make a change of coefficients by setting \( a_n = b_{n+1}(n+1) \). With this notation,

\[
(I) = \sum_{n,m \geq 0, k \geq 1} b_{n+1} \bar{b}_{m+1} c_{k+m} \bar{c}_{k+n} \frac{(n+1)(m+1)}{n + m + k + 1},
\]

\[
(II) = \sum_{k \geq 0, n,m \geq k+1} b_{n+1} \bar{b}_{m+1} c_{m-k} \bar{c}_{n-k} \frac{(m-k)(n-k)}{n + m - k + 1}.
\]

Adjusting the indices of summation slightly,

\[
(I) = \sum_{n,m \geq 1, k \geq 0} b_n \bar{b}_m c_{k+m} \bar{c}_{k+n} \frac{nm}{n + m + k},
\]

\[
(II) = \sum_{n,m \geq 1, k \geq 1} b_{n+k} \bar{b}_{m+k} c_m \bar{c}_n \frac{nm}{n + m + k}.
\]

By the symmetry in \( m, n \) we may interpret each term as being half that of its real part, so that the inequality \( 2 \Re(ab) \leq |a|^2 + |b|^2 \) applied to each of these expression, yields

\[
(I) \leq \sum_{n,m \geq 1, k \geq 0} \left( |b_n c_{k+m}|^2 + |b_m c_{k+n}|^2 \right) \frac{nm}{2(n + m + k)} = \sum_{n,m \geq 1, k \geq 0} |b_n c_{k+m}|^2 \frac{nm}{n + m + k} =: (I_s),
\]

\[
(II) \leq \sum_{n,m \geq 1, k \geq 1} \left( |b_{n+k} c_m|^2 + |b_{m+k} c_n|^2 \right) \frac{nm}{2(n + m + k)} = \sum_{n,m \geq 1, k \geq 1} |b_{n+k} c_m|^2 \frac{nm}{n + m + k} =: (II_s).
\]

Here, we again used the symmetry in \( m, n \). The next step is to isolate all unique pairs of indices. We change the order of summation as follows,

\[
(I_s) = \sum_{n \geq 1} \sum_{m \geq 1} \sum_{k \geq m} \frac{b_n c_k}{n + k} 2 \frac{nm}{n + m + k} = \sum_{n,k \geq 1} \frac{b_n c_k}{n + k} 2 \sum_{m=1}^{k} \frac{nm}{n + k} = \sum_{n,k \geq 1} \frac{b_n c_k}{n + k} \frac{nk(k+1)}{2(n+k)}.
\]
\[(II_*) = \sum_{m \geq 1} \sum_{n \geq 1} \sum_{k \geq n+1} |b_k c_m|^2 \frac{nm}{m+k} = \sum_{m \geq 1} \sum_{n \geq 1} |b_k c_m|^2 \frac{k-1}{m+k} = \sum_{m \geq 1} |b_k c_m|^2 \frac{mn}{2(m+k)}\]

We notice that we can add the index \(k = 1\) to the second sum without changing its value. Finally, taking this into account, we add these terms together to get

\[(I_*) + (II_*) = \sum_{n,m \geq 1} |b_n|^2 |c_m|^2 \left( \frac{nm(m+1)}{2(n+m)} + \frac{mn(n-1)}{2(m+n)} \right)\]

Replacing \(a_n = b_{n+1}(n+1)\), we now see that the right-hand side exactly equals

\[\frac{1}{2} \sum_{n \geq 0} |b_n|^2 |c_m|^2 \frac{m}{n+1} = \frac{1}{2} \left( \sum_{n \geq 0} \frac{|a_n|^2}{n+1} \right) \left( \sum_{m \geq 1} |c_m|^2 m \right) = \frac{1}{2} \frac{\|f\|_{A^2(\Omega)}^2}{\|\psi\|_{A^2(\Omega)}^2},\]

which was to be shown.

3. Proof of Corollary

We now turn to proving the corollary. Recall that \(T = T_\psi\) for an analytic map on \(\Omega\). The strategy is first to relate the commutator to a Hankel operator on \(A^2(\Omega)\) and then to pass to the unit disk. There, we apply Theorem which yields the result. Indeed, by a straight-forward computation, we get

\[\|T^*T - TT^*\|_{A^2(\Omega) \to A^2(\Omega)} = \sup_{h \in A^2(\Omega)} \langle (T^*T - TT^*)h, h \rangle = \sup_{h \in A^2(\Omega)} \left( \|Th\|_{A^2(\Omega)}^2 - \|T^*h\|_{A^2(\Omega)}^2 \right)\]

To relate this to a Hankel operator on the unit disc, we recall the formula for the orthogonal projection from \(L^2(\Omega)\) to \(A^2(\Omega)\). By, e.g. [2], it is given by

\[P : f \mapsto Pf(\xi) = \int_{\Omega} \frac{\phi'(\xi)\overline{\phi'(\eta)}}{(1 - \phi(\xi)\overline{\phi(\eta)})^2} f(\eta) dA(\eta),\]

where \(\phi\) is a Riemann map from \(\Omega\) to \(\mathbb{D}\). This yields the formula

\[H_\xi f(\xi) = \overline{\psi(\xi)}f(\xi) - P_\Omega(\psi f)(\xi)\]
\[ \int_{\Omega} \frac{\phi'(\xi)\phi'(\eta)}{(1 - \phi(\xi)\phi(\eta))^2} (\overline{\psi(\xi)} - \overline{\psi(\eta)}) f(\eta) dA(\eta). \]

We now make the change of variables \( z = \phi(\xi) \) and \( w = \phi(\eta) \). Passing the integration from \( \Omega \) to \( \mathbb{D} \), we get the Jacobian \( |(\phi^{-1})'(w)|^2 \). Thus, the above expression is equal to

\[ \int_{\mathbb{D}} \frac{\phi'(\phi^{-1}(z))\phi'(\phi^{-1}(w))}{(1 - zw)^2} (\psi \circ \phi^{-1}(z) - \psi \circ \phi^{-1}(w)) \left((\phi^{-1})'(w)\right)^2 dA(w) \]

\[ = \int_{\mathbb{D}} \frac{\psi \circ \phi^{-1}(z) - \psi \circ \phi^{-1}(w)}{(1 - zw)^2} \frac{(\phi^{-1})'(w)}{(\phi^{-1})'(z)} (f \circ \phi^{-1}(w)) dA(w). \]

Changing notations, this yields the formula

\[ (\phi^{-1})'(z)\left((H_{\psi}f) \circ \phi^{-1}\right)(z) = H_{\psi \circ \phi^{-1}} g(z), \]

where \( g(z) = (\phi^{-1})'(w) (f \circ \phi^{-1})(w) \) and \( H_{\psi \circ \phi^{-1}} \) is a Hankel operator on the disk. Taking norms, it is clear that

\[ \| H_{\psi} \|_{A^2(\Omega) \to L^2(\Omega)} = \| H_{\psi \circ \phi^{-1}} \|_{A^2(\mathbb{D}) \to L^2(\mathbb{D})}. \]

Applying Theorem 1, it now follows that

\[ \| H_{\psi \circ \phi^{-1}} \|_{A^2(\mathbb{D}) \to L^2(\mathbb{D})}^2 \leq \frac{\| (\psi \circ \phi^{-1})' \|_{A^2(\mathbb{D})}^2}{2} = \frac{\| \psi' \|_{A^2(\mathbb{D})}^2}{2}. \]

### 4. Proof of Inequality \( \Xi \)

Let \( \Omega \subset \mathbb{C} \) be a simply connected domain. Let \( T = T_z \) be the shift operator on the Bergman space \( A^2(\Omega) \). In this section we provide a new proof of the lower bound

\[ \| T^* T - TT^* \|_{A^2(\Omega) \to A^2(\Omega)} \geq \frac{\rho}{\text{Area}(\Omega)}, \]

where \( \rho \) is the torsional rigidity of \( \Omega \).

As we noted in the introduction, the torsional rigidity is a constant from mechanics which quantifies the resistance to rotation of a cylindrical object, imagined as being perpendicular to the complex plane, with cross-section equal to \( \Omega \) at all heights. We refer the reader to [5, p. 2] for a more accurate physical description. Mathematically, this quantity has several equivalent definitions. See, e.g. [3, p.87–89] for a discussion of this. In particular, the following definition, which we use, is discussed.

**Definition 1.** Let \( \Omega \subset \mathbb{C} \) be a simply connected domain. We define the torsional rigidity \( \rho = \rho(\Omega) \) to be

\[ \rho = 2 \int_{\Omega} v, \]

\[ \Xi. \]
where \( v \) is the unique solution to the Dirichlet problem
\[
\begin{align*}
\Delta v &= -2 \\
v |_{\partial \Omega} &= 0
\end{align*}
\] (7)

With this definition in hand, we turn to proving inequality (5). As in Section 3, we start by rewriting the norm of the commutator, but this time we formulate it in terms of the norm of the minimal solution to a d-bar problem:
\[
\| TT^* - T^* T \|_{L^2(\Omega) \to L^2(\Omega)} = \sup_{\| h \|_{A^2(\Omega)} = 1} \| \bar{\partial} h \|_{L^2(\Omega)}^2 - \| P(\bar{\partial} h) \|_{L^2(\Omega)}^2
\]
\[
= \sup_{\| h \|_{A^2(\Omega)} = 1} \| u_h \|_{L^2(\Omega)}^2,
\]
where \( u_h \) is the solution with minimal norm to the following d-bar problem:
\[
\bar{\partial} u_h = h.
\] (8)

Indeed there exist a well-known expression for \( u_h \) in terms of the Green function for the Laplacian operator, i.e.,
\[
\begin{align*}
u_h(z) &= \int_{\Omega} K_\Omega(z, \xi) h(\xi) d\mu(\xi),
\end{align*}
\] (9)
where \( K_\Omega(z, \xi) = -4 \frac{\partial^2}{\partial \xi^2} G_\Omega(z, \xi) \) and \( G_\Omega \) is the Green’s function associated to the Laplacian operator.

We now calculate \( \| u_h \|_{L^2(\Omega)}^2 \) using (9):
\[
\begin{align*}
\| u_h \|_{L^2(\Omega)}^2 &= 16 \int_{\Omega} \int_{\Omega} \int_{\Omega} \frac{\partial}{\partial \xi} G_\Omega(z, \xi) h(\xi) \frac{\partial}{\partial \eta} G_\Omega(z, \eta) \frac{\partial}{\partial \eta} G_\Omega(z, \eta) d\xi d\eta dz \\
&= 16 \int_{\Omega} \int_{\Omega} \int_{\Omega} \frac{\partial}{\partial \xi} G_\Omega(z, \xi) h(\xi) \frac{\partial}{\partial \eta} G_\Omega(z, \eta) d\xi d\eta dz \\
&= 16 \int_{\Omega} \int_{\Omega} \int_{\Omega} \frac{\partial}{\partial \xi} G_\Omega(z, \xi) d\xi d\eta dz \\
&= -16 \int_{\Omega} \int_{\Omega} \int_{\Omega} \frac{\partial}{\partial \xi} G_\Omega(z, \xi) d\xi d\eta dz \\
&= -4 \int_{\Omega} \int_{\Omega} G(\xi, \eta) h(\xi) h(\eta) d\xi d\eta,
\end{align*}
\]
where we have used integration by parts and standard properties of the Green’s function. Moreover, the Green function for the Laplacian is real valued so we can conclude
\[
\| u_h \|_{L^2(\Omega)}^2 = -4 \int_{\Omega} \int_{\Omega} G(\xi, \eta) h(\xi) h(\eta) d\xi d\eta.
\] (10)

By choosing \( h \equiv \text{Area}(\Omega)^{-1/2} \), we get
\[
\| TT^* - T^* T \|_{L^2(\Omega) \to L^2(\Omega)} \geq \frac{4}{\text{Area}(\Omega)} \int_{\Omega} \int_{\Omega} G(\xi, \eta) d\xi d\eta.
\] (11)
The point is now to recognize that the right-hand side of this inequality is exactly equal to $\rho/\text{Area}(\Omega)$. But this is clear from Definition 1 since by the properties of the Green’s function,

$$v(\eta) = -2 \int_{\Omega} G(\xi, \eta) d\xi$$

satisfies condition (7). This ends our proof of inequality (5).

5. Weighted Bergman spaces

We now explain how to extend Theorem 1 to the setting of weighted Bergman spaces (see below for definitions). This yields an interpolation of the sharp upper bound in the Putnam inequality, which cannot be improved for the Hardy space, but which we improved by a factor of $1/2$ in the Bergman space. Indeed, we state the result which is as follows.

**Theorem 2.** Let $\alpha > -1$ and suppose that $\psi$ be an analytic function on $D$. Then

$$\|H_\psi\|_{A_2^\alpha(D) \to L^2(D, dA_\alpha)} \leq \frac{\|\psi'\|_{A_2^\alpha(D)}}{\sqrt{2 + \alpha}}.$$ 

Moreover, this inequality is sharp.

The spaces $A_2^\alpha$ are defined for $\alpha > -1$ to be the spaces of holomorphic functions $f(z)$ on the unit disc finite in the norm

$$\|f\|_{A_2^\alpha} = \|f\|_{A_2^\alpha} = \int_D |f(z)|^2 dA_\alpha(z),$$

where $dA_\alpha(z) = (1 + \alpha)(1 - |z|^2)^\alpha dA(z)$. For $\alpha = 0$, this is the usual Bergman space studied above, while for $\alpha \to -1$ the space, in a sense that can be made precise through scaling, tends to the usual Hardy space on the unit disc. As a consequence, the estimate of the theorem matches precisely the previous estimates in these two extremal cases.

We recall some facts about these spaces. These can be checked by consulting the introductory chapter of the reference [3]. First, the monomials $z^n$ form an orthogonal base for the space and have norm

$$\|z^n\|_2^\alpha = \frac{n! \Gamma(n + 2)}{\Gamma(n + \alpha + 2)},$$

where $\Gamma(z)$ is the usual gamma-function satisfying the functional relation $\Gamma(z + 1) = z\Gamma(z)$. To keep the notation simple, we introduce the coefficients

$$D^\alpha_n = \frac{n! \Gamma(n + 2)}{\Gamma(n + \alpha + 2)}.$$

In particular, we have

$$D^0_n = \frac{n! \Gamma(2)}{\Gamma(n + 2)} = \frac{n!}{(n + 1)!} = \frac{1}{n + 1} \quad \text{and} \quad D^{-1}_n = \frac{n! \Gamma(1)}{\Gamma(n + 1)} = 1.$$
Furthermore, it is well-known that
\[ \sum_{n \geq 0} \left( \frac{z}{w} \right)^n D_n^\alpha = \frac{1}{(1 - \overline{w} z)^{2+\alpha}}. \]
From this, and the orthogonality of monomials, the orthogonal projection from \( L^2(\mathbb{D}, dA_\alpha) \) to \( A_\alpha(\mathbb{D}) \) is given by
\[ f \mapsto P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w} z)^{2+\alpha}} dA_\alpha(w), \]
and so we have an explicit expression for the associated Hankel operator
\[ H\psi f(z) = \psi(z) f - P_\alpha(\psi f)(z). \]
Repeating the same type of arguments as above, we get an expression in terms of the Taylor coefficients of \( f = \sum_{n \geq 0} a_n z^n \) and \( \psi = \sum_{k \geq 1} c_k z^k \).
\[ H(\overline{\psi} f)(z) = \sum_{\ell \geq 1} \sum_{n \geq 0} \overline{c_\ell} a_n z^n z^\ell - \sum_{n \geq 1} \sum_{k=0}^{n-1} D_n^\alpha D_k^\alpha a_n \overline{c}_{n-k} z^k. \]
As before, the next steps consist of reorganizing these sums and taking its norm. Doing this, we get
\[ \|H\overline{\psi} f\|^2_\alpha = \sum_{n,m \geq 0} \sum_{k \geq 1} a_n \overline{a}_m c_{k+n} \overline{c}_{k+n} D_{n+m+k}^\alpha \]
\[ + \sum_{k \geq 0} \sum_{n,m \geq k+1} a_n \overline{a}_m c_{m-k} \overline{c}_{n-k} \left( D_{n+m-k}^\alpha - \frac{D^n_{n-k} D_{m+k-1}^\alpha}{D_{k-1}^\alpha} \right). \]
(12)
Here, the labeling into (I) and (II) exactly match the previous case. After this, we make the change of coefficients \( D_n^\alpha a_n = b_{n+1} \). Some algebra then yields
\[ (I) + (II) = \sum_{n,m \geq 1} \sum_{k \geq 1} b_n \overline{b}_{m+k} c_{k+m} \overline{c}_{k+m} \]
\[ \times \frac{D_{n+m+k-1}^\alpha}{D_{n-1}^\alpha D_{m-1}^\alpha} \]
\[ + \sum_{n,m \geq 1} \sum_{k \geq 1} b_{n+k} \overline{b}_{m+k} c_{m} \overline{c}_{n} \left( \frac{D_{n+m+k-1}^\alpha}{D_{n+k-1}^\alpha D_{m+k-1}^\alpha} - \frac{1}{D_{k-1}^\alpha} \right). \]
Using the symmetry in the indices \( m, n \), we apply the inequality \( 2 \Re ab \leq |a|^2 + |b|^2 \) in the same way as in the unweighted case, to obtain
\[ (I) + (II) \leq \sum_{n,m \geq 1} \sum_{k \geq 0} |b_n c_{k+m}|^2 \frac{D_{n+m+k-1}^\alpha}{D_{n-1}^\alpha D_{m-1}^\alpha}. \]
The next step is to isolate all unique pairs of indices. Changing the indices and the order of summation, in exactly the same way as before, we get that the above expression is equal to

\[
\sum_{n,m \geq 1} |b_n c_m|^2 \left( \frac{D_{n+m}^\alpha}{D_{n-1}^\alpha D_m^\alpha} - \frac{1}{D_k^\alpha} \right).
\]

(13)

The proof of the desired inequality is complete once we show that this expression is smaller than

\[
\|f\|_a^2 \|\psi\|_a^2 = \frac{1}{2 + \alpha} \sum_{m \geq 1} \sum_{n \geq 0} |c_m|^2 |b_n|^2 m \sum_{\ell = m} D_{n-1}^\alpha D_{\ell}^\alpha - 1 \sum_{\ell = m} D_{n-1}^\alpha D_{\ell-m}^\alpha.
\]

This follows exactly by applying the following lemma to the sums in the expression (13).

**Lemma 1.** Suppose \( v \in \mathbb{N} \), then

\[
\sum_{\ell = 0}^v \frac{1}{D_{\ell}^\alpha} = \frac{v + 1}{2 + \alpha} \frac{1}{D_{v+1}^\alpha}.
\]

**Proof.** The relation holds for \( v = 0 \). Indeed,

\[
D_1^\alpha = \frac{1! \Gamma(\alpha + 2)}{\Gamma(3 + \alpha)} = \frac{\Gamma(2 + \alpha)}{(2 + \alpha) \Gamma(2 + \alpha)} = \frac{1}{2 + \alpha} D_0^\alpha.
\]

The induction step is easily verified from the observation

\[
D_v^\alpha = \frac{v! \Gamma(2 + \alpha)}{\Gamma(v + \alpha + 2)} = \frac{v}{v + \alpha + 1} D_{v-1}^\alpha.
\]

Finally, we note that the sharpness is seen from formula (12) by choosing \( \Omega = \mathbb{D} \), \( \psi = z \) and \( f \equiv 1 \). This ends the proof.

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