DiProber: Using Dual Probing to Estimate Tor Relay Capacities in Underloaded Networks

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Abstract—Tor is the most popular anonymous communication network. It has millions of daily users seeking privacy while browsing the internet. It has thousands of relays to route and anonymize the source and destinations of the users packets. To create a path, Tor authorities generate a probability distribution over relays based on the estimates of the capacities of the relays. An incoming user will then sample this probability distribution and choose three relays for their paths. The estimates are based on the bandwidths of observation probes the authority assigns to each relay in the network. Thus, in order to achieve better load balancing between users, accurate estimates are necessary. Unfortunately, the currently implemented estimation algorithm generate inaccurate estimates causing the network to be under utilized and its capacities unfairly distributed between the users paths. We propose DiProber, a new relay capacity estimation algorithm. The algorithm proposes a new measurement scheme in Tor consisting of two probes per relay and uses maximum likelihood to estimate their capacities. We show that the new technique works better in the case of under-utilized networks where users tend to have very low demand on the Tor network.

I. INTRODUCTION

Tor [7] is the most popular anonymous communication network, with several million estimated daily users [17]. It offers users a way to communicate online while preserving their identity and relationship to third parties. Tor operates by using a network of volunteer relays to forward an encrypted version of users’ traffic in order to obscure the source and/or the destination of network traffic. To ensure consistent performance, users’ traffic needs to be load-balanced across the relays. The network capacities of the relays are quite heterogeneous, spanning many orders of magnitude; an accurate estimate of these capacities is an important input to the load-balancing process.

Initially Tor relied on relays’ own measurements of their own capacities to generate estimates; this, however, created the possibility of low-resource attacks on the Tor network [2]. This motivated the development of TorFlow, a bandwidth monitoring system [12]. TorFlow uses external probes to monitor the performance of individual relays and uses this value to adjust the bandwidth value reported by the relay itself. The capacity estimates produced by TorFlow, however, vary considerably over time and between different TorFlow instance, impacting the traffic allocation algorithm’s ability to properly balance load across relays. A new estimation algorithm based on maximum likelihood estimation was also developed, MLEFlow [5]. While the estimation accuracy of this new algorithm showed a lot of promises, the probabilistic model used to derive the estimator had many simplifying assumptions that are not necessarily true in practise. Notably, the model used assumed that users utilize all the bandwidth allocated to them while in practise users’ demand fluctuate and is generally lower than the available bandwidth. Moreover, while the estimation algorithm used had significantly lower estimation error for exit relays, it still led to relatively high error for guard and middle relays.

We propose a new maximum likelihood estimator based on the work of MLEFlow, where we relax the assumption aforementioned about clients usage of the available bandwidth. The developed algorithm proposes a new measurement scheme of two probes per relay and uses the results of both measurements to identify whether a relay is bottlenecked or not during each epoch. Then, depending on the case, the algorithm uses a distinct probabilistic model relating measurements to actual capacity and performs maximum likelihood estimation. We derive analytical bounds of convergence for the estimates. We then validate the results of our analysis in flow-based Python simulations, where we simulate both loaded and under-loaded networks and show the benefits of using the new measurements scheme.

II. PATH ALLOCATION IN TOR

The current Tor network consists of around 6000 relays [16] that are used to forward user traffic. To create a connection, a user chooses a path of three different relays to construct a circuit that forwards traffic in both directions. Only the user knows the entire path; the relays know only their predecessor and successor, obscuring the relationship between clients and destinations. The traffic is also encrypted / decrypted at each node to hide the correspondence between incoming and outgoing traffic from a network observer.

Relays in Tor have heterogeneous capabilities and have network capacity sizes that differ by orders of magnitude (see Figure 5). Relays also have different capabilities and can be divided into three classes: exits, which can be used in the last position of the path, guards, which can be used in the first or second position, and middles which can only be used in the second position [15]. We denote the corresponding sets of relays by E, G, and M, respectively. To create a

\[\text{By “capacity” we refer to the smaller of upload and download bandwidth limit on the relay. This may be imposed by the ISP, the network configuration, or manually configured by the relay operator. In some cases, there may exist other bottlenecks on the path between two relays but a per-node bandwidth limit is a common and useful model of network capacity constraints.}\]
path, nodes are sampled from these sets with a probability proportional to their estimated capacity. For example, if we define $C[j]$ to be the estimated capacity of relay $j$, then the probability of choosing relay $j$ in a path as the last node in a path is $w^m[j] = C[j] / \left( \sum_{j' \in E} C[j'] \right)$; likewise for guard nodes being chosen in the first position. The middle position can be chosen from both guard and middle nodes; to balance bandwidth among classes, guard node capacity is adjusted by a multiplier $W_{mg}$; i.e., a guard node $j \in G$ is chosen for the middle position with probability:

$$w^m[j] = \frac{W_{mg} C[j]}{\sum_{j' \in G} W_{mg} C[j']} \frac{1}{\sum_{j' \in M} C[j']}$$

The multiplier is computed as:

$$W_{mg} = \frac{\sum_{j' \in G} C[j'] - \sum_{j' \in M} C[j']}{2 \sum_{j' \in G} C[j']}$$

This is a somewhat simplified presentation that describes the scenario where exit bandwidth is scarce and there is more guard bandwidth than middle bandwidth, as is the case in the actual Tor network. (See the Tor Directory Specification for more details on how other cases would be handled.)

It is easy to see that, in this scenario, if the estimated capacities are equal to the true relay capacities, which we will call $C^*[j]$, the expected number of paths using each exit relay will be proportional to its bandwidth; likewise, the expected number of paths using each guard and middle node will be proportional to their bandwidth. Using $X[j]$ to denote the number of paths on relay $j$, we have:

$$E[X[j]]/C^*[j] = E[X[j']]/C^*[j'] \quad \text{for } j, j' \in E$$

$$E[X[j]]/C^*[j] = E[X[j']]/C^*[j'] \quad \text{for } j, j' \in G \cup M$$

Thus, in expectation, each path would have the same bandwidth—$C^*[j]/E[X[j]]$ for $j \in E$. Our goal is therefore to estimate these capacities as accurately as possible.

A. Capacity Estimation

Each relay estimates its own network capacity by computing the maximum sustained download and upload bandwidth over a 5-second period over the last 5 days. It reports this value (called the *observed bandwidth*) to directory authorities, who then compile it across all relays and distribute the information to the clients in a consensus document, published every hour.

We will use $b_k[j]$ to refer to the observed bandwidth of relay $j$ in the consensus document published at time $t$.

Using the observed bandwidth directly for load-balancing creates the opportunity for a low-resource attack on the Tor network. In particular, a relay can publish a high observed bandwidth for itself, which will cause more clients to choose it, and create more chances for it to break users’ anonymity. This motivated the design of TorFlow, which uses observations of actual relay performance to estimate capacities, rather than simply trusting the value reported by the relay itself. In TorFlow, a bandwidth authority creates probe circuits through each relay and downloads a file of a certain size, measuring the realized bandwidth. We will call this the *measured bandwidth*, $m_k[j]$. Note that in a perfectly load-balanced network, all of these observations should be equal, regardless of the chosen relay. The design of TorFlow uses a PID controller to attempt to bring these observations into balance.

More specifically, let $C_{t+1}^{TF}[j]$ be the capacity of relay $j$, as estimated by TorFlow, at time $t$. TorFlow computes an error term, $e_t[j]$ as the difference of the measured bandwidth and the average measured bandwidth, normalized by the average bandwidth:

$$e_t[j] = (m_t[j] - \bar{m}_t)/\bar{m}_t$$

where

$$\bar{m}_t = \sum_{j \in G \cup M \cup E} m_t[j]/(|G| + |M| + |E|)$$

It then computes the new estimate as $C_{t+1}^{TF}[j]$:

$$C_{t+1}^{TF}[j] = C_{t}^{TF}[j] \left( 1 + K_p e_t[j] + K_i \int_0^t e_v[j] dt' + K_d \frac{de_t[j]}{dt} \right)$$

Note that some research suggests allocation other than proportional to bandwidth results in better performance; nevertheless, an accurate capacity estimate is still needed for these alternative path allocation strategies. Since Tor does not allow one-hop circuits, these circuits use two relays. The relay under measurement and a high-bandwidth relay.
The constants $K_p$, $K_i$, and $K_d$ control the proportional, integral, and derivative components of the PID controller. In the default configuration of TorFlow, $K_i = K_d = 0$ and $K_p = 1$, so we will call this version of TorFlow TorFlow-$P$. In this case the update equation can be simplified as:

$$C_{t+1}^{TF}[j] = C_t^{TF}[j]m_t[j]/\bar{m}_t$$

Since the update equation does not have a normalization step, when TorFlow-$P$ was enabled in late 2011 in the actual Tor network, the absolute values of estimated bandwidth grew without bound. This caused the Tor network to turn off TorFlow-$P$ and switch to using a version of TorFlow that uses adjusted observed bandwidth instead, called $sbws$. We will call estimates produced by this version $C^A$, with:

$$C_{t+1}^{A}[j] = b_t[j]m_t[j]/\bar{m}_t$$

This version uses the observed bandwidth published by the relay itself, but adjusts it down if the observed bandwidth is below average or up if it is above. It has been in use in Tor since 2012; however, it has a number of disadvantages. A relay that is not sufficiently loaded may underestimate its observed bandwidth; this leads to a well-documented ramp-up period of new relays, where their low observed bandwidth leads to a small estimated capacity and low load, which in turn leads to low observed bandwidth. But even established relays see their observed bandwidth change. Figure 2 shows the observed bandwidth of 10 randomly selected relays over the month of May 2020, demonstrating that the observed values vary significantly over time.

A new method for estimating the capacity of relays based on maximum likelihood estimation was proposed with MLEFlow. To develop MLEFlow, a probabilistic model relating the actual relay capacities $C^*[j]$’s and the bandwidth measurements $m[j]$’s was derived. In order to derive this relationship, a number of assumptions was made in [5]:

1) Relays fall into a single category and each user path goes through only a single relay.
2) A synchronized model where time is divided into epochs and user connections all terminate at the end of each epoch. At the end of the epoch the weight vector is updated and the new vector is used by all users in the next epoch.
3) Users arrive to the network randomly following a Poisson process with rate $\lambda_s$, denoted by Pois($\lambda_s$).
4) Clients circuits use all the bandwidth allocated to them and are only bottlenecked by the relays.

The estimates produced by this algorithm, denoted $C^H$ are then computed using:

$$C_{t+1}^{H}[j] = \arg\max_{\kappa \in C} f(\kappa, m_t[j], w_t[j]),$$

where

$$f(\kappa, m_t[j], w_t[j]) = \prod_{i=0}^{\kappa} \frac{e^{-\lambda_s w_t[j]}}{(\lambda_s w_t[j])^i} \left(1 - \frac{e^{-\lambda_s w_t[j]}}{\lambda_s w_t[j]}\right)^{\kappa - i}$$

While the algorithm derived showed a lot of promises, the model used to derive the estimator assumed that users can utilize arbitrary amounts of bandwidth and are only bottlenecked on the Tor network. In practice, at times the client demand on Tor is lower than the overall available bandwidth. When simulated in a low-load scenario, MLEFlow tended to misestimate relays capacities. Also, due to this assumption, the probabilistic model in [5] is expected to produce useful results for exit relays since those relays are generally scarce in the Tor network and are expected to be paths bottlenecked. This is not true for guard and middle relays. As shown in both Python and Shadow simulations, guard and middle relays had larger estimation errors when using MLEFlow.

III. DiProber: Maximum Likelihood Estimation of Relays Capacities using Dual Probing

We propose a new method DiProber for estimating the capacity of relays based on maximum likelihood estimation (MLE). We use the same assumptions used to derive MLEFlow while relaxing the last assumption made. The paths in our model consist of single relays. We assume that time is divided into epochs and connections terminate at the end of each epoch. We denote the number of paths passing through the $j^{th}$ relay in the $i^{th}$ epoch by $x_i[j]$. Hence, given $w_t[i]$, the number of paths using the $j^{th}$ relay during the $i^{th}$ epoch is a random variable $X_i[j]$ with distribution Pois($\lambda_s w_t[j]$). We design an updated measurement mechanism based on two measurement probes assigned by the authorities to each relay instead of just one. The result of the two measurements is used to group the relays in two different groups, then derive a probabilistic model for each case and apply MLE. In this section, we will present the new measurement scheme proposed as well as the probabilistic models derived to relate the actual capacity of a relay to the measurements obtained. We then derive analytical guarantees to the new estimation algorithm proposed.

A. Measurement mechanism

Currently, Tor authorities assign a measurement probe to each relay in the network, download a file of a certain size and measure the realized bandwidth of the probe. Instead of having only one measurement probe for each relay, we propose having two measurement probes added sequentially to each relay. The authorities start by activating the first probe and measure its realized bandwidth, denoted $m_1^k[j]$. Then, while the first probe is still active, the authority starts the second probe circuit and measure its bandwidth, denoted $m_2^k[j]$. Using both non-noisy measurements of a relay $j$ at a given epoch $t$, we can divide the relays in two groups and derive a 4

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4 As can be seen on this graph: https://metrics.torproject.org/totalew.html?start=2011-06-01&end=2011-12-31
probabilistic model for each one. First, we let $C_{client,i}[j]$ be the total bandwidth used by clients using relay $j$ during epoch $i$.

Case 1: Relay is not bottlenecked by clients and when adding the two probes.

When the clients using relay $j$, during the $i^{th}$ epoch, are not using all the available bandwidth of the relay we have $C_{client,i}[j] < C^*[j]$. After adding the first probe, in the case where the capacity of the relay was equally divided between the users and the probe, each path using relay $j$ will have a bandwidth of $C^*[j] / x_{i,j} + 1$. Hence all the clients using this relay will have a total bandwidth of $x_{i,j} C^*[j] / x_{i,j} + 1$. If $C_{client,i}[j] < x_{i,j} C^*[j] / x_{i,j} + 1$, then client utilization is not affected by adding the first probe and the probe will use all the remaining unused capacity of the relay, notably $m_1^2[j] = C^*[j] - C_{client,i}[j]$. The same logic is true when adding the second probe. If $C_{client,i}[j] < x_{i,j} C^*[j] / x_{i,j} + 1$, then client utilization is not affected by adding the second probe. The remaining unused capacity will then be equally divided between the two probes, notably $m_2^1[j] = C^*[j] - C_{client,i}[j] / 2 = m_1^2[j] / 2$. Figure 3 illustrates the idea aforementioned.

Case 2: Relay bottlenecked by clients or when adding any of the two probes.

(a) If the clients are using all the available capacity of the relay there will be no remaining unused capacity. Hence, as in $[5]$, the capacity of the relay will be divided equally between all the paths going through it, notably $m_1^1[j] = C^*[j] / x_{i,j} + 1$ and $m_2^2[j] = C^*[j] / x_{i,j} + 2$. Figure 4 illustrates this case.

(b) In this case, the clients are not using all the available capacity of the relay. However, when adding the first probe, the client utilization can be affected if $C_{client,i}[j] > x_{i,j} C^*[j] / x_{i,j} + 1$. In other words, clients will then be using all the capacity available to them after the first probe was added and there will be no remaining unused capacity.

Thus as case (a), the capacity of the relay will be divided equally between all the paths going through it, notably $m_1^1[j] = C^*[j] / x_{i,j} + 1$ and $m_2^2[j] = C^*[j] / x_{i,j} + 2$.

(c) In this case, clients are not using all the available capacity of the relay, nor adding the first probe will affect their utilization. Thus, the first probe will be first assigned all the remaining unused capacity, $m_1^1[j] = C^*[j] - C_{client,i}[j]$. However, when adding the second probe, the clients utilization is affected if $C_{client,i}[j] > x_{i,j} C^*[j] / x_{i,j} + 2$ and there will be no remaining capacity. Thus, $m_2^2[j] = C^*[j] / x_{i,j} + 2$.

Thus, if the relay falls under case 1, the second observation of the relay will be equal to half the first observation obtained and the relationship between the actual capacity of the relay $C^*[j]$ and the observation is $M^2_2[j] = C^*[j] - C_{client,i}[j]$. While for relays that fall under the second case, we can’t make the same conclusion about $m_1^1[j]$ and $m_2^2[j]$. However, for all subcases of case 2, we have a relationship between $C^*[j]$ and $M^2_2[j]$ with $M^2_2[j] = C^*[j] / x_{i,j} + 2$.

B. MLE capacities estimation

In this section, we will show the method used to compute the maximum likelihood estimation of relays capacities using the sequence of pairs of non-noisy measurements and the weights published by the Tor authority. More specifically, for any relay $j \in [n]$, the MLE estimate of its actual capacity $C^*[j]$ is the maximizer in $C \in \mathbb{R}^n_{>0}$ of the probability of observing the full history of measurements $(m_1^1[j],m_2^2[j])$, given the published weights $w_{i,j}[j]$ over the first $t + 1$ periods. First, at each epoch $i \in [t]$, and in order to use the correct probabilistic model, the algorithm compares the values of $m_1^1[j]$ and $m_2^2[j]$ and determine which case of the two cases discussed in III-A is true for the relay $j$ at the $i^{th}$ epoch.

- If $m_1^1[j] = 2m_2^2[j]$, then $M^2_2[j] = \frac{C^*[j] - C_{client,i}[j]}{2}$.
- If $m_1^1[j] \neq 2m_2^2[j]$, then $M^2_2[j] = \frac{C^*[j]}{X_{i,j} + 2}$.
Assuming that we know total client utilization of each relay during each epoch is not a practical assumption. In this paper, we assume that the average utilization of a client on the Tor network is known and denoted \( C_{\text{client}}^{\text{avg}} \). Thus the total utilization of a relay \( j \) during the \( i \)-th epoch will be \( X_i[j] C_{\text{client}}^{\text{avg}} \).

We add the superscript \( D \) to the capacity estimate to denote that only the last result of the two probes is considered.

**Theorem 1** [MLE estimates using dual probing]. For any \( j \in [n] \) and \( t \in \mathbb{N} \), the MLE estimate of \( C^*[j] \) given the weight and observation pairs vectors \( w_i[j] \) and \( (m_i^1[j], m_i^2[j]) \) is

\[
C_{t+1}^D[j] = \arg\max_{\kappa \in C} \prod_{i=0}^t f(\kappa, m_i[j], w_i[j]), \text{ where } (1)
\]

\[
f(\kappa, m_i[j], w_i[j]) = \begin{cases} 
\Pr_{X_i[j] \sim \text{DiProber}(\kappa, w_i[j])} \left( \frac{-X_i[j] C_{\text{client}}^{\text{avg}}}{\lambda} = m_i^1[j] \right) \text{ if } m_i^1[j] = 2m_i^2[j] \\
\Pr_{X_i[j] \sim \text{DiProber}(\kappa, w_i[j])} \left( \frac{-X_i[j] C_{\text{client}}^{\text{avg}}}{\lambda} = m_i^2[j] \right) \text{ if } m_i^1[j] \neq 2m_i^2[j] 
\end{cases}
\]

We now write the probability in equation (1) in terms of the known quantities: \( \lambda, w_i[j] \), and \( m_i[j] \). For any \( j \in [n] \) and \( t \in \mathbb{N} \), the function \( f(\kappa, m_i[j], w_i[j]) \) can be written as follows:

\[
f(\kappa, m_i[j], w_i[j]) = \begin{cases} 
\exp \left( -\lambda w_i[j] \right) \frac{1}{\lambda} \frac{2m_i^2[j]}{m_i^1[j] C_{\text{client}}^{\text{avg}}} \cdot \frac{1}{\lambda} \frac{m_i^1[j]}{m_i^2[j]} \text{ if } m_i^1[j] = 2m_i^2[j] \\
\exp \left( -\lambda w_i[j] \right) \frac{1}{\lambda} \frac{m_i^1[j]}{m_i^2[j]} \frac{1}{m_i^1[j]} \text{ if } m_i^1[j] \neq 2m_i^2[j]
\end{cases}
\]

**Definition 1 [DiProber-WH].** For any relay \( j \in [n] \) and \( t \in \mathbb{N} \), **DiProber-WH** updates the weight vector by:

\[
w_{t+1}^{\text{WH}}[j] = \frac{C_{t}^{D}[j]}{\sum_{k=0}^{n} C_{t}^{D}[k]}, \text{ where we use the superscript } \text{WH} \text{ in } w_{t}^{\text{WH}} \text{ to identify } \text{DiProber-WH}.
\]

Equation (1) does not account for non-modeled noise in the measurements. Once noise affects \( m_i^1[j] \) and \( m_i^2[j] \), we will not be able to accurately discern between the two cases.

### C. One step maximum likelihood

In this section, we look at the special case of only considering the last relay measurement at each epoch instead of the whole history of measurements. We add the superscript \( D1 \) to the capacity estimate to denote that only the last result of the two probes is considered. Equation (1) can now be written as:

\[
C_{t+1}^{D1}[j] = \arg\max_{\kappa \in C} f(\kappa, m_i[j], w_i[j]), \text{ where } (4)
\]

\[
f(\kappa, m_i[j], w_i[j]) \text{ is defined in Equation (2)}
\]

To approximate the maximizer of equation (4), we find \( \kappa \) that sets the derivative of \( f \) to zero. The result is given in the following theorem.

**Theorem 2** [One step MLE closed form]. For any \( j \in [n] \) and \( t \in \mathbb{N} \), the MLE estimate of \( C^*[j] \) given the last weight and observations pair, \( w_i[j] \) and \( (m_i^1[j], m_i^2[j]) \) is

\[
C_{t+1}^{D1}[j] = \begin{cases} 
\lambda_s w_i[j] C_{\text{client}}^{\text{avg}} + 2m_i^2[j], \text{ if } m_i^1[j] = 2m_i^2[j] \\
m_i^2[j] (\lambda_s w_i[j] + 2), \text{ if } m_i^1[j] \neq 2m_i^2[j]
\end{cases}
\]

From Theorem 2 in the case where the relay is not bottlenecked, i.e. \( m_i^1[j] = 2m_i^2[j] \), the estimated capacity is equal to the sum of the unused capacity, \( 2m_i^2[j] \), and the expected capacity used by clients, \( \lambda_s w_i[j] C_{\text{client}}^{\text{avg}} \).

\[
C_{t+1}^{D1}[j] = \lambda_s w_i[j] C_{\text{client}}^{\text{avg}} + m_i^1[j]
\]

While in the case where the relay is bottlenecked, the estimated capacity is the result of the product of the observation, \( m_i^2[j] \), and the expected number of users, \( \lambda_s w_i[j] \). The added two in the denominator refers to the two probes added.

\[
C_{t+1}^{D1}[j] = \lambda_s w_i[j] C_{\text{client}}^{\text{avg}} + 2
\]

**Definition 2 [DiProber-O].** For any relay \( j \in [n] \) and \( t \in \mathbb{N} \), **DiProber-O** updates the weight vector by:

\[
w_{t+1}^{O}[j] = \frac{C_{t}^{D1}[j]}{\sum_{k=0}^{n} C_{t}^{D1}[k]}, \text{ where we use the superscript } O \text{ in } w_{t}^{O} \text{ to identify } \text{DiProber-O}.
\]

### D. Convergence of DiProber-WH and DiProber-O estimates

In this section we show that, starting with any initial weight vector, the mean of the dual probing algorithm estimates for any relay capacity, whether considering the full history in every update as in **DiProber-WH** (Definition 1) or only the most recent measurement in every update as in **DiProber-O** (Definition 2), converges to the actual relay capacity. We also show that the variance of the estimates when using **DiProber-WH** estimation goes to zero as the number of epochs increases.

**Theorem 3** [Estimates of both methods converge]. For any \( j \in [n] \), \( t \in \mathbb{N} \), and a method \( y \in \{ \text{TorFlow-P, MLEFlow-CP} \text{ DiProber-WH} \} \),

\[
E[C_{t}^{y}[j]] \leq C^*[j].
\]

Moreover, as \( t \to \infty \), \( E[C_{t}^{y}[j]] \geq C^*[j] \left( 1 - \frac{1}{\lambda_s w^*[j]} \right).
\]
Theorem 5 [Convergence of variance of DiProber-WH]. As $t \to \infty$, $\text{Var}[C_t^{WH}[j]] \to 0$.

Hence the variances of DiProber estimates when considering the full history of weights and measurements are close to the actual capacities.

We will show this experimentally in the next section.

E. DiProber: Implementation

In order to solve the maximization of Theorem 1, we discretize the bounded capacity set $C$ and iteratively find the maximizer of (1). Note that quantization requires knowing a lower and upper bound on the relay capacity, which can be estimated based on past observations.

Consider a partition $\bar{C}$ of $C$ into bins. The set $\bar{C}$ contains the centers of the bins of $C$. For any $j \in [n]$, $t \in \mathbb{N}$, and $\kappa \in \bar{C}$, we define $L_t(j, \kappa)$ to be:

\[
\begin{align*}
-\lambda_j w_t[j] - \log \left( \left( \frac{m_t[j]}{w_t[j]} \right) \right) + \frac{m_t[j]}{w_t[j]} \log (\lambda_j w_t[j]), & \text{if } m_t[j] = 2w_t^2[j] \\
-\lambda_j w_t[j] - \log \left( \left( \frac{m_t[j]}{w_t[j]} - 2 \right) \right) + \frac{m_t[j]}{w_t[j]} \log (\lambda_j w_t[j]), & \text{if } m_t[j] \neq 2w_t^2[j]
\end{align*}
\]

(9)

the $t$th term of the sum when taking the log of $\mathbb{E}$. For DiProber-O, since we are only considering the last measurement, the estimate of the capacity of relay $j$ is computed by iteratively searching for the maximizer $\kappa$ over the discretized capacity set $\bar{C}$ in the following equation:

\[
C_t^O[j] := \max_{\kappa \in \bar{C}} L_t(j, \kappa).
\]

(10)

For DiProber-WH, the sum of $L_t(j, \kappa)$ over measurement periods is stored in a variable $S_t^{WH}[j]$

\[
S_{t+1}(j, \kappa) = S_t(j, \kappa) + L_t(j, \kappa),
\]

with $S_0(j, \kappa) = 0$. Then, the maximum likelihood estimate is computed by iteratively searching for the maximizer $\kappa$ over the discretized capacity set $\bar{C}$ in the following equation:

\[
C_{t+1}^{WH}[j] := \max_{\kappa \in \bar{C}} S_{t+1}(j, \kappa).
\]

IV. ANALYTICAL MODEL SIMULATIONS: LOW FIDELITY EXPERIMENTS

To better understand the properties and performance of TorFlow-P, MLEFlow, sbus, and DiProber, we evaluated them using simulations of our analytical model of the Tor network. These simulations are implemented in Python; we will make our implementation publicly available at publication time.

We evaluate the performance of the different methods using two metrics: (a) the accuracy of the relay capacity estimates in a network where clients use all the bandwidth allocated to them, (b) the accuracy of the relay capacity estimates in an underloaded network and (c) the amount of bandwidth allocated to the user paths resulting from the weight vectors generated using the capacity estimates.

The simulation algorithm we have used is shown in Algorithm 1. The algorithm takes as input: the Poisson arrival rate of users $\lambda_s$, the total number of measurement periods $T$ to be simulated, a method $\in \{\text{Actual, TorFlow-P, MLEFlow, DiProber-O, DiProber-WH}\}$ to compute the capacities of the relays from measurements. The algorithm outputs the bandwidths allocated for users paths and the weight vectors published over all periods between 0 and $T$.

The simulation algorithm iterates over measurement periods. In each period $i$, it generates the total number of users $N_i$ that will join the network by sampling a Poisson distribution with rate $\lambda_s$ in line 3. Then, it uses the weight vector $w_i$ computed in the previous period as a probability distribution for the users to choose the relays of their paths from in line 4. In line 5, it adds the first probe to each relay and uses the max-min fairness bandwidth allocation algorithm to get the bandwidth allocated for each path, and thus generate the observation vector $m_i$. In line 6, it adds the second probe and generates $m_i^2$. After that, it computes $w_{i+1}$ using the given method in line 7. Finally, it deletes all the paths for a fresh start of the next period.

Algorithm 1: Low fidelity simulation

1: input: $\lambda_s, T$, method $\in \{\text{Actual, TorFlow-P, MLEFlow, DiProber-WH, DiProber-O}\}$.
2: for $i \in [0, ..., T]$ do
3:     Pick the number of users $N_i \sim \text{Poi}(\lambda_s)$.
4:     Construct users paths of 3 relays using $w_i$.
5:     Compute $m_i$ using max-min bandwidth allocation.
6:     Compute $m_i^2$ using max-min bandwidth allocation.
7:     Compute $w_{i+1}$ based on $m_i$ and $w_i$ using method.
8:     Delete all paths in the network.
9: return: $m_{0:T}, w_{0:T}$

We consider a network analogous to the current Tor network with 6037 relays as of June 23rd 2020. The relays are distributed as follows: 2351 are guard relays, 2576 are middle relays, and 1110 are exit relays (this includes any relays that have both the Exit and Guard flags set). Lacking a ground truth, we used the measured capacity in the Tor consensus document as the actual capacity of the Tor relays in our simulation. The maximum capacity of all relays was 169,000 kb/s, while the total capacity of the guard, middle and exit relays in the network are around $42.6 \times 10^6, 6.7 \times 10^6$ and $17.7 \times 10^6$ kb/s respectively. Hence, the capacity set is $C = [0, 169000]$. The capacity distributions of the guard, middle, and exit relays are shown in Figure 5.

The max-min bandwidth allocation algorithm [4], [3], in Python assumes that clients will use the full bandwidth allocated to them. Thus we simulated two types of networks: (1) a full utilization network using the max-min bandwidth allocation algorithm where each client path consists of three relays and (2) an under loaded network scenario taking into account client side bottlenecks. To simulate this idea, we adjusted the simulation to add a bandwidth cap to each client flow. This bandwidth cap is enforced by a fourth relay added to each client flow, with a capacity selected uniformly at random from the interval [8,18] KB/s. This interval was chosen based on the full utilization scenario bandwidth distribution. Since the average bandwidth of flows in the full utilization scenario
was approximately 22 KB/s, the cap means that the flows can utilize at most about 60% of the Tor network capacity.

To use DiProber, we need to partition C into bins. From Figure 5, we use the same technique used in [5]. We choose the bins to be intervals of the form $[a^{b-1}, a^b]$ where $a$ is a strictly positive real number and $b \in [1, ..., b_{max}]$ where $b_{max} = \lceil \log(\max(C[j])) \rceil$ for $j \in [n]$.

A. Low-fidelity sims results and analysis

In all of the simulation runs, we chose the number of periods $t$ to be 50, the initial weight vector $w_0$ to be uniform, the rate of arrival $\lambda$ to be $10^6$.

a) DiProber-WH performs better than DiProber-O: While both algorithms of DiProber outperformed the other algorithms, DiProber-WH had a lower average error than DiProber-O for all classes of relays. The results are shown in Figure 6.

b) DiProber leads to better guard and middle relays estimates in fully loaded networks: We tested the different estimation algorithms on three-relays paths networks. Both DiProber algorithms and MLEFlow had lower average error than TorFlow-P and sbus. The results are shown in Figure 7. While the errors for exit relays when using DiProber and MLEFlow were close, DiProber leads to lower average error for guard and middle relays. The average estimation errors for DiProber and MLEFlow stayed below 10% for exit relays, while it was higher for TorFlow-P with 72%. For guard and middle relays, the error was lowest for DiProber-WH at 18% and 14% respectively. It was higher for MLEFlow at around 25% error and even higher for TorFlow-P at 75%.

c) The dual probing algorithm advantage extend to the under-loaded network scenario: We simulate the case of under-loaded networks as was done in [5] and as described above by adding a forth relay to each path created. DiProber outperforms all other algorithms. The error for all classes of relays was below 25% for DiProber. While the error for exit relays remained relatively low when using MLEFlow, other classes estimates all had an average error above 45%. The results are shown in Figure 8.
DiProber-\textit{WH} and DiProber-\textit{O} give better and fairer bandwidth allocation than than the other algorithms: The means of the bandwidths allocated for paths using \textit{MLEFlow} and DiProber are equal to that of the \textit{Actual} scenario, while that of \textit{TorFlow-P} is slightly smaller. The advantage of both those methods is the stability of their estimates. The standard deviation of DiProber is a maximum of 1.4, while \textit{MLEFlow} was 1.7, when that of \textit{Actual} is around 0.6 and \textit{TorFlow-P} is around 165, orders of magnitude larger. Moreover, the maximum and minimum bandwidths allocated of our methods are similar to that of \textit{Actual} while \textit{TorFlow-P} had orders of magnitude larger maximum. That means that our methods distribute bandwidths more fairly than \textit{TorFlow-P}.

V. RELATED WORK

Improving the performance of the Tor network has been the subject of much research; we refer the reader to the survey by AlSabah and Goldberg for an overview [1]. Here we summarize related work specifically focusing on relay capacity estimation.

Snader and Borisov proposed using opportunistic measurements, where each relay measures the bandwidth of each other relay it communicates with as part of normal operation, and designed EigenSpeed [13], which combines these measurements using principal component analysis to derive a single relay capacity. EigenSpeed was designed to avoid certain types of collusion and misreporting attacks; however, Johnson et al. [10] discovered that it is subject to a number of other attacks that allow colluding adversaries to inflate their bandwidth. They also designed PeerFlow, which is a more robust mechanism to combine opportunistic measurements from relays with provable limits on inflation attacks. These bounds, however, depend on having a fraction of bandwidth being on trusted nodes, and it has slow convergence properties due to its limitations on changing bandwidth values.

\textit{FlashFlow} is a new proposal to replace TorFlow. FlashFlow uses several servers that measure a relay simultaneously, generating a large network load intended to max out its capacity. FlashFlow has a guaranteed inflation bound of only 33% but it is based on the assumption that a relay capacity is based on a hard limit that cannot be exceeded, as TorFlow uses traffic that is explicitly labeled for for bandwidth probing. In practice, it is often easier and cheaper to obtain high peak bandwidth capability than sustaining the same bandwidth continuously.

At the same time, TorFlow probes, though not explicitly labeled, can also be identified by their distinct characteristics, and can be used to preferentially forward probe traffic to inflate bandwidth estimates, or to perform sophisticated denial-of-service attacks [9]. A more stealthy approach for bandwidth measurement probing remains an open research question. The simplest and most effective attack on TorFlow, however, is to inflate the observed bandwidth published by the relay [10]: this attack does not apply to DiProber as it does not use the observed bandwidth.

VI. CONCLUSION

We have developed a new method for estimating the relay capacities in the Tor network, DiProber. We show that \textit{MLEFlow} fails to accurately estimate relays capacities in under-loaded networks. Our detailed mathematical analysis showed that DiProber capacity estimates converge to their true value, while the estimate variance converges to 0, as the number of observations grows. We validated the performance of \textit{MLEFlow} with extensive simulations using a our custom flow-based simulator. Our results show that DiProber produces much more accurate estimates of relay capacities, which in turn results in much better load balancing of user traffic across the network, as compared with current methods.

REFERENCES

[1] M. AlSabah and I. Goldberg, “Performance and security improvements for Tor: A survey,” ACM Computing Surveys (CSUR), vol. 49, no. 2, p. 32, 2016.
[2] K. Bauer, D. McCoy, D. Grunwald, T. Kohno, and D. Sicker, “Low-resource routing attacks against Tor,” in Proceedings of the 2007 ACM Workshop on Privacy in Electronic Society (WPES), 2007, pp. 11–20.
[3] H. Darir, “Privacy-preserving network congestion control,” 2019.
[4] H. Darir, H. Sibai, N. Borisov, G. Dullerud, and S. Mitra, “Tightrope: Towards optimal load-balancing of paths in anonymous networks,” in Proceedings of the 2018 Workshop on Privacy in the Electronic Society, 2018, pp. 76–85.
[5] H. Darir, H. Sibai, C.-Y. Cheng, N. Borisov, G. Dullerud, and S. Mitra, “MFlow: Learning from history to improve load balancing in tor,” Proceedings on Privacy Enhancing Technologies, vol. 2022, no. 1, pp. 75–104, 2022. [Online]. Available: https://doi.org/10.2478/poptech-2022-0005
[6] R. Dingledine, “The lifecycle of a new relay,” The Tor Project Blog. https://blog.torproject.org/lifecycle-new-relay Sep. 2013.
[7] R. Dingledine, N. Mathewson, and P. F. Syverson, “Tor: The second-generation onion router,” in USENIX Security Symposium. USENIX, 2004, pp. 303–320.
[8] S. Herbert, S. J. Murdoch, and E. Punskaya, “Optimising node selection probabilities in multi-hop m/d/1 queuing networks to reduce latency of tor,” Electronics letters, vol. 50, no. 17, pp. 1205–1207, 2014.
[9] R. Jansen, T. Vaidya, and M. Sherr, “Point break: a study of bandwidth denial-of-service attacks against Tor,” in 28th USENIX Security Symposium, 2019, pp. 1823–1840.
[10] A. Johnson, R. Jansen, N. Hopper, A. Segal, and P. Syverson, “PeerFlow: Secure load balancing in Tor,” Proceedings on Privacy Enhancing Technologies, vol. 2017, no. 2, pp. 74–94, 2017.
[11] K. Loesing, M. Perry, and A. Gibson, “Bandwidth scanner specification,” https://gitweb.torproject.org/torflow.git/tree/NetworkScanners/BwAuthority/README.spec.txt 2011.
[12] M. Perry, “TorFlow: Tor network analysis,” in Proceedings of the 2nd Workshop on Hot Topics in Privacy Enhancing Technologies (HotPETS), 2009, pp. 1–14.
[13] R. Snader and N. Borisov, “EigenSpeed: Secure peer-to-peer bandwidth evaluation,” in 8th International Workshop on Peer-To-Peer Systems, R. Rodrigues and K. Ross, Eds. Berkeley, CA, USA: USENIX Association, Apr. 2009.
[14] ———, “Improving security and performance in the Tor network through tunable path selection,” IEEE Transactions on Dependable and Secure Computing, vol. 8, no. 5, pp. 728–741, 2011.
[15] The Tor Project, “Tor directory protocol, version 3,” https://gitweb.torproject.org/torspec.git/tree/dir-spec.txt, 2020.
[16] ———, “Tor metrics: Servers,” https://metrics.torproject.org/networksize.html, 2020.
[17] ———, “Tor metrics: Users,” https://metrics.torproject.org/userstats-relay-country.html, 2020.
[18] M. Traudt, R. Jansen, and A. Johnson, “Flashflow: A secure speed test for tor,” 2020.
APPENDIX

A. Proofs

Theorem 1 [MLE estimates using dual probing]. For any $j \in [n]$ and $t \in \mathbb{N}$, the MLE estimate of $C^*_j$ given the weight and observation pairs vectors $w_i[t]$ and $(m^2_i[j], m^2_i[j])$ is

$$C^H_{t+1}[j] = \arg\max_{i \in C} \prod_{i=0}^t f(\kappa, m_i[j], w_i[j]),$$

where

$$f(\kappa, m_i[j], w_i[j]) = \begin{cases} 
\Pr_{X_i[j] \sim \text{Pois}(\lambda_i w_i[j])} \left( \frac{\exp(-\lambda_i w_i[j])}{\frac{\kappa - 2m_i^2[j]}{C_{\text{Client}}}} \right), & \text{if } m_i^2[j] = 2m_i^2[j] \\
\Pr_{X_i[j] \sim \text{Pois}(\lambda_i w_i[j])} \left( \frac{\exp(-\lambda_i w_i[j])}{\left(\frac{\kappa - 1}{C_{\text{Client}}}\right)^2} \right), & \text{if } m_i^2[j] \neq 2m_i^2[j]
\end{cases}$$

(1)

Using the Poisson distribution probability mass function, we can write:

$$C^H_{t+1}[j] = \arg\max_{i \in C} \prod_{i=0}^t e^{-\lambda_i w_i[j]} \frac{1}{(x_i[j])!(\lambda_i w_i[j])^{x_i[j]}},$$

$$= \arg\max_{i \in C} \prod_{i=0}^t e^{-\lambda_i w_i[j]} \frac{1}{(\kappa - 2m_i^2[j])},$$

(15)

Considering Case 2, the objective function of the MLE, the observation random variable of the $j^{th}$ relay $M_i[j]$ can be written as a function of $\kappa$, as if we are assuming $\kappa = C^*_j$, and the random variable $X_i[j]$ for $i \in [t]$:

$$M_i^2[j] = \frac{\kappa}{X_i[j] + 2}. $$

(16)

Recall that we assume that the random variable $X_i[j]$ follows a Poisson distribution with parameter $\lambda_i w_i[j]$ and all users leave at the end of each epoch. Hence, given $w_i[j]$ for $j \in [n]$, the $M_i^2[j]$’s at different iterations are independent random variables. Thus eq. (16) can be written as the product of the probability of the independent random variables $[M_i^2[j], ..., M_i^2[j]]$:

$$C_{t+1}^H[j] = \arg\max_{i \in C} \prod_{i=0}^t \Pr_{X_i[j] \sim \text{Pois}(\lambda_i w_i[j])} \left( \frac{(\lambda_i w_i[j])^{x_i[j]}}{x_i[j]!(\kappa - 2m_i^2[j])} \right),$$

(17)

When the measurement is made and the observation is fixed, i.e. $M_i^2[j] = m_i^2[j]$, the probability in eq. (17) can be expressed in terms of the random variable $X_i[j]$:

$$C_{t+1}^H[j] = \arg\max_{i \in C} \prod_{i=0}^t \Pr(X_i[j] = x_i[j] | W_i[j] = w_i[j]).$$

(18)

Using the Poisson distribution probability mass function, we can write:

$$C_{t+1}^H[j] = \arg\max_{i \in C} \prod_{i=0}^t e^{-\lambda_i w_i[j]} \frac{1}{(x_i[j])!(\lambda_i w_i[j])^{x_i[j]}},$$

$$= \arg\max_{i \in C} \prod_{i=0}^t e^{-\lambda_i w_i[j]} \frac{1}{(\kappa - 2m_i^2[j])},$$

(19)

Theorem 2 [One step MLE closed form]. For any $j \in [n]$ and $t \in \mathbb{N}$, the MLE estimate of $C^*_j$ given the weight and observations, $w_i[j]$ and $(m^2_i[j], m^2_i[j])$ is

$$C_{t+1}^{D1}[j] = \begin{cases} 
\lambda_i w_i[j](\kappa + m^2_i[j] + 2), & \text{if } m^2_i[j] = 2m^2_i[j] \\
\lambda_i w_i[j](m_i^2[j] + 2), & \text{if } m^2_i[j] \neq 2m^2_i[j]
\end{cases}$$

(20)
Proof: Case 1: As discussed previously, we know that in this case if \( m_1^i[j] = 2m_2^i[j] \). We denote \( C_{avg,i} \) the average bandwidth used by each client in the network during the \( j^{th} \) epoch. Hence since the relay is not bottlenecked, we can say that \( m_1^i[j] = C^*[j] - x_i[j]C_{avg,i} \), hence, given \( w_i \), the measurement of the \( i^{th} \) relay at the \( t^{th} \) iteration is a random variable \( M_1^i[j] = C^*[j] - x_i[j]C_{avg,i} \).

W will use superscript D1 to indicate that we are only using the most recent value of \( m_1^i[j] \) and \( w_t[j] \). Using maximum likelihood estimation, we have

\[
C_{t+1}^{D1} = \arg\max_{\kappa \in \mathbb{C}} f(\kappa, m_1^i[j], w_t[j]), \quad \text{where} \quad f(\kappa, m_1^i[j], w_t[j]) = \frac{\Pr[X_t[j] \sim \text{Pois}(\kappa w_t[j])]}{\Pr[X_t[j] \sim \text{Pois}(\kappa w_t[j])]} (X_t[j] = \frac{\kappa - m_1^i[j]}{C_{avg,t}}),
\]

Using Stirling approximation for the second term, we get

\[
C_{t+1}^{D1} = \arg\max_{\kappa \in \mathbb{C}} \Pr[X_t[j] \sim \text{Pois}(\kappa w_t[j])] (X_t[j] = \frac{\kappa - m_1^i[j]}{C_{avg,t}})
\]

Using maximum likelihood estimation, we have

\[
f(\kappa, m_t^i[j], w_t[j]) = \Pr[X_t[j] \sim \text{Pois}(\kappa w_t[j]) \left( \frac{\kappa}{X_t[j] + 2} = m_1^i[j] \right),
\]

Using Stirling approximation for the second term, we get

\[
C_{t+1}^{D1} = \arg\max_{\kappa \in \mathbb{C}} \Pr[X_t[j] \sim \text{Pois}(\kappa w_t[j]) \left( X_t[j] = \frac{\kappa - m_1^i[j]}{C_{avg,t}} \right)]
\]

In order to find the optimum, we differentiate the right side with respect to \( \kappa \) and equate it to zero:

\[
- \frac{1}{m_1^i[j]} \log \left( \frac{\kappa - m_1^i[j]}{C_{avg,t}} \right) - \frac{1}{m_t^i[j]} + \frac{1}{m_t^i[j]} \log (\lambda w_t[j]) = 0
\]

Thus \( \kappa = \frac{\lambda w_t[j]}{m_1^i[j]} + \frac{m_1^i[j]}{m_t^i[j]} \)

Case 2: We know that we are in this case if \( m_1^i[j] \neq 2m_2^i[j] \). Since the relay is bottlenecked when we add the second probe in all cases described previously, we can say that

\[
m_2^i[j] = \frac{C^*[j]}{x_i[j]+2};
\]

hence, given \( w_i \), the measurement of the \( i^{th} \) relay at the \( t^{th} \) iteration is a random variable \( M_2^i[j] = \frac{C^*[j]}{x_i[j]+2} \).

Using maximum likelihood estimation, we have

\[
f(\kappa, m_t^i[j], w_t[j]) = \Pr[X_t[j] \sim \text{Pois}(\kappa w_t[j]) \left( \frac{\kappa}{X_t[j] + 2} = m_2^i[j] \right).
\]

In order to find the optimum, we differentiate the right side with respect to \( \kappa \) and equate it to zero:

\[
- \frac{1}{m_1^i[j]} \log \left( \frac{\kappa - m_1^i[j]}{C_{avg,t}} \right) - \frac{1}{m_t^i[j]} - \frac{1}{m_t^i[j]} \log (\lambda w_t[j]) = 0
\]

Thus \( \kappa = \frac{\lambda w_t[j]}{m_1^i[j]} + \frac{m_1^i[j]}{m_t^i[j]} \)

\[\lambda w_t[j] \] expected number of users

Thus we have achieved the expected number of users.
We know that \( C_i = 0 \) for all \( i \in C \). Hence the expected value of the estimate \( E(\kappa_t) = C^* \), since the right hand side of Equation (24) is equal to the actual capacity of the relay if it was never bottlenecked on average.

Now we consider case 2, where the relay is always bottlenecked for its whole measurement history. In this proof we will drop the superscript 2 from the measurement since we are only dealing with the second measurement of a relay. As we derived in eq. (19), we know that for any \( j \in [n] \), the weight at iteration \((t + 1)\) should satisfy the following equation:

\[
C_{t+1}^D[j] = \arg\max_{\kappa \in C} \prod_{i=0}^{t} e^{-\lambda_s w_i[j]} \frac{1}{(\frac{\kappa}{m_i[j]} - 2)!} (\lambda_s w_i[j])^{\frac{\kappa n}{m_i[j]} - 2}
\]

Since the logarithm function is a strictly increasing function, the maximum likelihood estimate of the capacity of a relay \( j \in [n] \) using full history can be found:

\[
\log(\lambda_s w_i[j] C_{avg}) - \log(\kappa - m_i[j])
\]

Hence the expected value of the estimate \( E(\kappa_t) = C^* \) for the steady state convergence of the above iterative formulation:

\[
\kappa_t = \frac{\log (\lambda_s w_t[j] C_{avg}) - \log (\kappa - m_t[j])}{\sum_{i=0}^{t} \frac{1}{\kappa_{t-1} - m_t[j]}} + \kappa_{t-1}
\]

Thus we can find an iterative solution of the optimization.

\[
C^D_{t+1}[j] = \arg\max_{\kappa \in C} \prod_{i=0}^{t} e^{-\lambda_s w_i[j]} \frac{1}{(\frac{\kappa}{m_i[j]} - 2)!} (\lambda_s w_i[j])^{\frac{\kappa n}{m_i[j]} - 2}
\]

Using Stirling’s approximation, we have \( \log(x!) \approx x \log(x) - x \). Thus substituting in eq. (26):

\[
\log(\lambda_s w_i[j] C_{avg}) - \log(\kappa - m_i[j])
\]
for which the derivative is zero.

\[
\sum_{i=0}^{t} \frac{2}{m_i[j]} \frac{1}{m_i[j]} \log \left( \frac{\kappa}{m_i[j]} \right) - \frac{1}{m_i[j]} \log(\lambda_s w_i[j]) = 0
\]

\[
\sum_{i=0}^{t} \frac{2}{m_i[j]} \log(e) + \frac{1}{m_i[j]} \log(\lambda_s w_i[j]) = 0
\]

\[
\sum_{i=0}^{t} \frac{1}{m_i[j]} \log(\kappa) = \sum_{i=0}^{t} \frac{2}{m_i[j]} \log(\lambda_s w_i[j])
\]

\[
\log(e) = \sum_{i=0}^{t} \frac{1}{m_i[j]} \log(\lambda_s w_i[j])
\]

\[
\log(e) = \sum_{i=0}^{t} \frac{1}{m_i[j]} \log(\lambda_s w_i[j])
\]

\[
\frac{\sum_{i=0}^{t} \frac{1}{m_i[j]} \log(m_i[j] \lambda_s w_i[j])}{\sum_{i=0}^{t} \frac{1}{m_i[j]}} = \frac{\sum_{i=0}^{t} \frac{1}{m_i[j]} \log(m_i[j] \lambda_s w_i[j])}{\sum_{i=0}^{t} \frac{1}{m_i[j]}}
\]

Letting \( z = \frac{2(t+1)}{\kappa(\sum_{i=0}^{t} \frac{1}{m_i[j]})} \) in eq. 28, we have:

\[
\frac{1}{z} e^{-z} = \frac{\sum_{i=0}^{t} \frac{1}{m_i[j]} \log(m_i[j] \lambda_s w_i[j])}{2(t+1)} e^{\sum_{i=0}^{t} \frac{1}{m_i[j]}}
\]

\[
1 e^{-z} = \sum_{i=0}^{t} \frac{1}{m_i[j]} e^{\sum_{i=0}^{t} \frac{1}{m_i[j]}}
\]

\[
2 e^{-z} = \frac{2(t+1)}{\sum_{i=0}^{t} \frac{1}{m_i[j]}} e^{\sum_{i=0}^{t} \frac{1}{m_i[j]}}
\]

where \( W_0 \) is the Lambert \( W \) function along the principal branch. The Lambert \( W \) function is the multi-valued complex function \((ze^n)^{-1}\) and \( W_0 \) is the unique-valued real function that takes the unique real value of \( W \) when \( z > \frac{1}{e} \). Implementations of Lambert function exist in multiple software libraries.

\( W_0 \) has the following Taylor series expansion for \( z \) in the neighborhood of 0: \( W_0(z) = z + o(z^2) \). Moreover, the argument of \( W_0 \) is small if the rate of users arrival to the network \( \lambda_s \) is large enough. Hence, the Taylor expansion around zero is valid and therefore:

\[
C_{t+1}^{D}[j] = \frac{\sum_{i=0}^{t} \frac{1}{m_i[j]} \log(m_i[j] \lambda_s w_i[j])}{\sum_{i=0}^{t} \frac{1}{m_i[j]}} e^{\sum_{i=0}^{t} \frac{1}{m_i[j]}} = \frac{\sum_{i=0}^{t} \frac{1}{m_i[j]} \log(m_i[j] \lambda_s w_i[j])}{\sum_{i=0}^{t} \frac{1}{m_i[j]}} e^{\sum_{i=0}^{t} \frac{1}{m_i[j]}}
\]

We refer the reader to the Appendix of [5] for a proof of the expected value of the closed form found above since this form exactly matches the form derived for MLEFlow-CF.

\[\textbf{Theorem 5} \text{ [Convergence of variance of DiProber-WH]. As } t \to \infty, \text{ Var}[C_{t+1}^{WH}[j]] \to 0.\]

\[\text{\textbf{Proof:}} \text{ We refer the reader to the Appendix of [5] for a complete proof of the theorem.}\]

\[\text{[https://kite.com/python/docs/mpmath.lambertw]}\]