On the Isoperimetric Inequality in Finitely Generated Groups

Bruno Luiz Santos Correia and Marc Troyanov

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Abstract

We present a sharp version of the isoperimetric inequality for finitely generated groups due to T. Couhlon and L. Saloff-Coste based on the proof suggested by M. Gromov.

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1 Introduction and statement of the Main Results

Let $\Gamma$ be a group generated by the finite symmetric set $S = S^{-1} \subset \Gamma$. We denote by $B_S(n) \subset \Gamma$ the subset of those elements in $\Gamma$ that can be written as a product of at most $n$ generators in $S$. The right invariant word metric on $\Gamma$ with respect to the generating set $S$ is defined as

$$d_S(x, y) = \min\{n \in \mathbb{N} \mid xy^{-1} \in B_S(n)\}.$$ 

We shall denote by $\|x\|_S = d_S(e, x) = \min\{n \in \mathbb{N} \mid x \in B_S(n)\}$ and we define the growth function $\gamma_S : \mathbb{N} \to \mathbb{N}$ with respect to $S$ as

$$\gamma_S(n) = \text{Card}(B_S(n)).$$

The boundary of a non empty finite subset $D \subset \Gamma$ is defined as

$$\partial_S D = \{x \in D \mid \text{dist}_S(x, \Gamma \setminus D) = 1\} = \{x \in D \mid \exists s \in S \text{ such that } sx \notin D\}.$$ 

Following T. Couhlon and L. Saloff Coste in [6], we define the “reverse growth function” of $(\Gamma, S)$ to be the following function $\phi_S : \mathbb{R}_+ \to \mathbb{N}$:

$$\phi_S(t) = \min\{n \in \mathbb{N} \mid \gamma_S(n) \geq t\}. \tag{1}$$

With this preparation, we state the following isoperimetric inequality:

**Theorem 1.** Let $\Gamma$ be a finitely generated group. For any non empty finite subset $D \subset \Gamma$ and any real number $\lambda$ such that $1 < \lambda \leq \frac{\text{Card}(\Gamma)}{\text{Card}(D)}$, we have

$$\text{Card}(\partial D) \geq \left(1 - \frac{1}{\lambda}\right) \frac{\text{Card}(D)}{\phi_S(\lambda \text{Card}(D))}. \tag{2}$$

This is the main result of the paper and the proof will be given in the next section. It is well known that the growth $\gamma_S(n)$ of an infinite finitely generated group is at least a linear and at most an exponential function of $n$. We refer to the survey [3] for more on the growth function of finitely generated groups. Optimizing the inequality (2) over $\lambda$ allows us to prove in §3 the following explicit isoperimetric inequalities for groups with a given lower bound on the growth:
Corollary 2 Let $\Gamma$ be an infinite finitely generated group with growth function $\gamma_S$.

(I) If $\gamma_S(n - 1) \geq Cn^d$ for some constants $C > 0$ and $d \geq 1$ and any integer $n \geq 1$, then

$$\text{Card}(\partial D) \geq \frac{C^{\frac{1}{d}}d^{\frac{n-1}{d}}}{(d+1)^{\frac{n-1}{d}}} \text{Card}(D)^{\frac{n-1}{d}}$$

(3)

for any finite subset $D \subset \Gamma$.

(II) If $\gamma_S(n - 1) \geq Ce^{bn^\alpha}$ for some constants $C > 0$, $b > 0$ and $0 < \alpha \leq 1$, and any integer $n \geq 1$, then

$$\text{Card}(\partial D) \geq \mu(\text{Card}(D)) \cdot \frac{\text{Card}(D)}{\left(\frac{1}{\alpha}(\log(\text{Card}(D)))^{1/\alpha}\right)}$$

(4)

for any finite subset $D \subset \Gamma$, where $\mu(v)$ is an explicit function such that $\mu(v) \to 1$ as $v \to \infty$.

Remarks. ○ Theorem 1 is a refinement of a result by T. Coulhon and L. Saloff-Coste, see [8] th. 1 and [5] Theorem 3.2. However the first version of such an isoperimetric inequality, at least for groups of polynomial growth, is usually attributed to N. Varopoulos (the reader may compare with Property 3 page 388 and its proof page 391 in [5]; note that the discussion in this paper is about nilpotent Lie groups.)

○ In Section E.6 of Chapter 6 in the book [4], M. Gromov stated the inequality (2) for $\lambda = 2$ and, with essentially the same proof, see also [7]. We obtain here the general case.

○ The explicit expression for the function $\mu$ appearing in the isoperimetric inequality (4) is given in [19] below.

2 Proof of Theorem 1.

Our proof is based on the strategy given by Gromov in [4]; we divide the argument in five steps.

Step (i). For any $y \in \Gamma$, we define the following subsets of $D$:

$$E_y = \{x \in D \mid xy \notin D\} \quad \text{and} \quad I_y = D \setminus E_y = y^{-1}D \cap D.$$  

The points in $E_y$ are the “exit points”, they are transported outside $D$ by the left translation by $y$, while the points in $I_y$ remain inside $D$. We claim that

$$\text{Card}(E_y) \leq \|y\| \text{Card}(\partial D).$$

(5)

Indeed, suppose that $y = s_n \cdot s_{n-1} \cdots s_1$, with $s_j \in S$ and $n = \|y\|$, and define inductively $y_k \in \Gamma$ by $y_0 = e$ and $y_j = s_jy_{j-1}$ for $1 \leq k \leq n$. We then define a map $f : E_y \to \Gamma$ as

$$f(x) = y_mx, \quad \text{where} \quad m = \max\{k \in \mathbb{N} \mid k \leq (n-1) \text{ and } y_kx \in D\}.$$  

(6)

Observe that $f(x) \in \partial D$ for any $x \in E_y$ since $yx = y_nx \notin D$. Furthermore, for any $z \in \partial D$ we have

$$f^{-1}(z) \subset \{y_0^{-1}z, \ldots, y_{n-1}^{-1}z\}.$$  

(7)

We thus have $\text{Card}(f^{-1}(z)) \leq n$ for any $z \in \partial D$ and therefore

$$\text{Card}(E_y) = \sum_{z \in \partial D} \text{Card}(f^{-1}(z)) \leq n \text{Card}(\partial D) = \|y\| \text{Card}(\partial D),$$

this proves (5).
Step (ii). We have
\[ \sum_{y \in \Gamma} \text{Card}(I_y) = \text{Card}(D)^2. \] (8)
This follows from the identities:
\[ \sum_{y \in \Gamma} \text{Card}(I_y) = \sum_{y \in \Gamma} \sum_{x \in D} \chi_D(xy) = \sum_{x \in D} \sum_{y \in \Gamma} \chi_D(yx) = \sum_{x \in D} \text{Card}\{y \in \Gamma \mid y \in Dx^{-1}\}, \]
where \( \chi_D \) is the characteristic function of \( D \).

Step (iii). For any \( n \) we can find \( y \in B_S(n) \) such that
\[ \gamma_S(n) \text{Card}(I_y) \leq \text{Card}(D)^2. \] (9)
Indeed, choosing \( y \in B_S(n) \) such that \( \text{Card}(I_y) = \min\{\text{Card}(I_w) \mid w \in B_S(n)\} \) and applying (8) yields
\[ \gamma_S(n) \text{Card}(I_y) = \text{Card}(B_S(n)) \text{Card}(I_y) \leq \sum_{z \in B_S(n)} \text{Card}(I_z) \leq \sum_{z \in \Gamma} \text{Card}(I_z) = \text{Card}(D)^2. \]

Step (iv). Let us now set \( n = \phi_S(\lambda \text{Card}(D)) \), then \( n \) is the smallest integer such that \( \lambda \text{Card}(D) \leq \gamma_S(n) \). From step (iii), we find \( y \in B_S(n) \) such that
\[ \lambda \text{Card}(D) \text{Card}(I_y) \leq \gamma_S(n) \text{Card}(I_y) \leq \text{Card}(D)^2. \]
Therefore
\[ \text{Card}(I_y) \leq \frac{1}{\lambda} \text{Card}(D). \] (10)

Step (v). Using (6) and (10), we obtain
\[ \left(1 - \frac{1}{\lambda}\right) \text{Card}(D) \leq \text{Card}(D) - \text{Card}(I_y) = \text{Card}(E_y) \leq \|y\|_S \text{Card}(\partial D) \leq n \text{Card}(\partial D) = \phi_S(\lambda \text{Card}(D)) \text{Card}(\partial D). \]
The proof of Theorem 1 is complete.

\[ \square \]

3 Proof of Corollary 2
To prove Corollary 2, it will be useful to first look at some properties of the function \( \phi_S \):

Lemma 3 The function \( \phi_S \) satisfies the following properties

(i) \( \phi_S \) is a non decreasing function.

(ii) \( \phi_S \) is a left inverse of \( \gamma_S \), that is \( \phi_S(\gamma_S(n)) = n \) for any integer \( n \in \mathbb{N} \).

(iii) We have \( \gamma_S(\phi_S(m)) \geq m \) for any \( m \in \mathbb{N} \) and \( \gamma_S(\phi_S(m)) = m \) for any \( m \in \gamma_S(\mathbb{N}) \).

(iv) Suppose that \( \gamma_S(n) \geq g(n + 1) \) for any \( n \in \mathbb{N} \), where \( g : [0, \infty) \to [\beta, \infty) \) is a homeomorphism, then \( \phi_S(t) \leq g^{-1}(t) \) for any \( t \in \mathbb{R}_+ \).

1Permuting the sum in the equality \( (*) \) is allowed since the set of \( (x, y) \in \Gamma \times \Gamma \) such that \( x \in D \) and \( \chi_D(xy) \neq 0 \) is finite.
Remark. Simple examples show that the inequality $\gamma_S(n) \geq g(n)$ for all $n \in \mathbb{N}$ does not generally imply $\phi_S(m) \leq g^{-1}(m)$ for all $m \in \mathbb{N}$. This is the reason we work with the hypothesis $\gamma_S(n) \geq g(n+1)$.

Proof. The first three statements are elementary, let us prove the last one. The condition $g(n + 1) \leq \gamma_S(n)$ implies
\[
\{ n \in \mathbb{N} \mid g(n + 1) \geq t \} \subset \{ n \in \mathbb{N} \mid \gamma_S(n) \geq t \},
\]
therefore
\[
\phi_S(t) = \min \{ n \in \mathbb{N} \mid \gamma_S(n) \geq t \} \leq \min \{ n \in \mathbb{N} \mid g(n + 1) \geq t \}.
\]
Because $g(r)$ is monotone increasing, we have $g(n) < g(r) \leq g(n + 1)$ whenever $n < r \leq n + 1$. Therefore
\[
\phi_S(t) \leq \min \{ n \in \mathbb{N} \mid g(n + 1) \geq t \} \leq \min \{ r \in \mathbb{R}_+ \mid g(r) \geq t \} = g^{-1}(t).
\]

\hfill \Box

Using the previous Lemma and Theorem 1 we immediately obtain the following Corollary:

Corollary 4 Let $\Gamma$ be an infinite group generated by a finite symmetric set $S = S^{-1}$. Assume that $\gamma_S(n) \geq g(n + 1)$ for any $n \in \mathbb{N}$, where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a homeomorphism. Then the following isoperimetric inequality holds for any non-empty finite subset $D \subset \Gamma$:
\[
\text{Card}(\partial D) \geq F(\text{Card}(D)),
\]
where $F(v)$ is defined as
\[
F(v) = \sup_{1 < \lambda < \infty} \left( \frac{(1 - \frac{1}{\lambda}) v}{g^{-1}(\lambda v)} \right).
\]

\hfill \Box

Proof of Corollary 2. We first consider the general situation as in Corollary 4. Let us denote by $h(v) = g^{-1}(v)$ the inverse of $g$ and set
\[
H(\lambda, v) = \frac{(1 - \frac{1}{\lambda}) v}{h(\lambda v)}.
\]
Observe that $H(\lambda, v)$ is a continuous positive function that converges to 0 as $\lambda \to 1$ or $\lambda \to \infty$. If $h$ is everywhere differentiable the maximum of $H(\lambda, v)$ for a fixed value of $v$ is therefore attained for some $\lambda \in (1, \infty)$ satisfying
\[
\frac{\partial}{\partial \lambda} \left( \frac{(\lambda - 1)v}{\lambda h(\lambda v)} \right) = \frac{h(\lambda v) + (\lambda - \lambda^2)vh'(\lambda v)}{(\lambda h(\lambda v))^2} = 0,
\]
that is
\[
h(\lambda v) = \lambda (\lambda - 1) vh'(\lambda v).
\]
If this equation has a unique solution $\lambda = \lambda(v)$, then (11) holds for the function $F(v) = (1 - \frac{1}{\lambda(v)}) v h(\lambda(v))$.

With this preparation, it is easy to prove the first statement in Corollary 2. Assume that $\gamma_S(n) \geq g(n + 1)$ with $g(r) = Cr^d$. Then $h(v) = g^{-1}(v) = h(v) = (v/C)^{1/d}$ and $h'(v) = \frac{1}{v/d} h(v)$. Equation (13) is then the following equality:
\[
0 = h(\lambda v) + (\lambda - \lambda^2) vh'(\lambda v) = h(\lambda v) \left( 1 + (\lambda - \lambda^2) \frac{1}{\lambda d} \right).
\]
The unique solution is $\lambda = d + 1$, therefore [11] holds with the function
\[
F(v) = (1 - \frac{1}{\lambda(v)}) \frac{v}{h(\lambda(v)v)} = C^\frac{1}{\alpha} v \frac{d v^{-\frac{1}{\alpha}}}{(d + 1) v^{\frac{1}{\alpha}}},
\]
which is equivalent to the inequality [3].

We now prove the second statement in Corollary 2. Suppose that $\gamma_\leq(n) \geq g(n + 1)$ with $g(r) = C e^{b r^\alpha}$, then
\[
h(v) = g^{-1}(v) = \left(\frac{1}{b} \log \left( \frac{v}{C} \right) \right)^{1/\alpha},
\]
and we have
\[
h'(v) = \frac{1}{\alpha} b^{1/\alpha} \log \left( \frac{v}{C} \right) \frac{1}{\alpha - 1} = \frac{h(v)}{\alpha v \log \left( \frac{v}{C} \right)}.
\]
Equation [13] is then equivalent to
\[
\alpha \log \left( \frac{\lambda v}{C} \right) = (\lambda - 1), \quad \text{equivalently} \quad v = \frac{C}{\lambda} e^{\frac{\lambda}{\alpha}}.
\]
We claim that for any $v \geq C \cdot \frac{1}{\alpha} e^{\frac{\lambda}{\alpha}}$, there exists a unique solution $\lambda = \lambda(v)$ satisfying [14]. It is indeed easy to check that the function
\[
f(x) = C \cdot \frac{1}{x} e^{\frac{\lambda}{\alpha}}
\]
has its minimum value at $x = \alpha$ and defines a diffeomorphism $f : [\alpha, \infty) \to [C \cdot \frac{1}{\alpha} e^{\frac{\lambda}{\alpha}}, \infty)$. Clearly $\lambda = f^{-1}(v)$ is the unique solution of [15]. The function in [12] is then given by
\[
F(v) = \left(1 - \frac{1}{\lambda(v)}\right) \frac{v}{h(\lambda(v) \cdot v)},
\]
where $h(v)$ is defined in [14]. This function can also be written as
\[
F(v) = \left(1 - \frac{1}{\lambda(v)}\right) \frac{v}{\left(\frac{1}{b} (\log(v) + \log(\lambda(v)) - \log(C))\right)^{1/\alpha}} = \frac{\mu(v)v}{\left(\frac{1}{b} (\log(v))\right)^{1/\alpha}},
\]
with
\[
\mu(v) = \left(1 - \frac{1}{\lambda(v)}\right) \left(1 + \frac{\log(\lambda(v))}{\log(v)} - \frac{\log(C)}{\log(v)}\right)^{-1/\alpha}.
\]
Applying Corollary 4 we then have
\[
\text{Card}(\partial D) \geq F(\text{Card}(D)) = \mu(\text{Card}(D)) \cdot \frac{\text{Card}(D)}{\left(\frac{1}{b} (\log(\text{Card}(D)))\right)^{1/\alpha}}.
\]
It remains to study the limit of $\mu(v)$ as $v \to \infty$. To see this, observe first that equation [15] can also be written as
\[
\log(v) = \frac{1}{\alpha} (\lambda - 1) - \log(\lambda) + \log(C).
\]
From this equality, we see that $\lim_{v \to \infty} \lambda(v) = \infty$ and therefore.
\[
\lim_{v \to \infty} \frac{\log(\lambda(v))}{\log(v)} = \lim_{\lambda \to \infty} \frac{\log(\lambda)}{(\frac{1}{\alpha} (\lambda - 1) - \log(\lambda) + \log(C))} = 0.
\]
Therefore
\[
\lim_{v \to \infty} \frac{\lambda(v)}{\log(v)} = \alpha \quad \text{and} \quad \lim_{v \to \infty} \frac{\log(\lambda(v))}{\log(v)} = 0.
\]
Applying [18] to [16], we find that $\lim_{v \to \infty} \mu(v) = 1$, which completes the proof of the Corollary. \(\Box\)
4 Final Remarks

Remarks 1. The function $\lambda(v)$ in the proof of the second part of the Corollary is defined implicitly to be the solution of (15). We can also define $\lambda(v)$ explicitly using the Lambert $W$-function. The Lambert function is multivaluated and in our case we consider the branch $W_{-1} : [-1/e, 0) \to (-\infty, -1]$ defined by the condition $W_{-1}(x) \exp(W_{-1}(x)) = x$, see [1, 2] for more on this function. It is then not difficult to check that

$$
\lambda(v) = -\alpha W_{-1} \left( -\frac{Ce^{-\frac{1}{\alpha v}}}{\alpha v} \right).
$$

Combining this formula with (16), we see that $\mu(v)$ as the following explicit, albeit complex, expression

$$
\mu(v) = \left( 1 + \frac{1}{\alpha W_{-1} \left( -\frac{Ce^{-\frac{1}{\alpha v}}}{\alpha v} \right)} \right) \left( 1 + \frac{\log \left( -\alpha C \cdot W_{-1} \left( -\frac{Ce^{-\frac{1}{\alpha v}}}{\alpha v} \right) \right)}{\log(v)} \right).
$$

2. We have worked with the interior definition of the boundary of a subset $D \subset \Gamma$. We could have also chosen the exterior boundary

$$
\partial_S D = \{ x \in \Gamma \setminus D \mid \text{dist}_S(x, D) = 1 \} = \{ x \in \Gamma \setminus D \mid \exists s \in S \text{ such that } sx \in D \}.
$$

Because every point in $\partial_S D'$ is at distance 1 from a point in $\partial_S D$ and vice-versa, we clearly have

$$
\frac{1}{\text{Card}(S)} \text{Card}(\partial_S D) \leq \text{Card}(\partial_S D) \leq \text{Card}(S) \text{Card}(\partial_S D).
$$

The proof of Theorem 1, and therefore all the estimates in the paper, also holds if we replace the interior boundary $\partial_S D$ with the exterior boundary $\partial_S' D$. Only minor changes have to be made in step (i) of the proof; namely the function $f$ in (6) has to be modified to

$$
f(x) = y_m x, \quad \text{where } m = \max\{ k \in \mathbb{N} \mid k \leq n \text{ and } y_k x \in \partial_S' D \}.
$$

In that case the inverse image of a point $z \in \partial_S' D$ satisfies

$$
f^{-1}(z) \subseteq \{ y_1^{-1} z, \ldots, y_n^{-1} z \},
$$

and we still have $\text{Card}(f^{-1}(z)) \leq n$ for any $z \in \partial_S' D$.

3. The function $\phi_S$ has a slightly different definition in [5] and [6]. These authors use instead the function

$$
\tilde{\phi}_S(t) = \min\{ n \in \mathbb{N} \mid \gamma_S(n) > t \}.
$$

Observe that $\tilde{\phi}_S(t) \geq \phi_S(t)$, therefore the isoperimetric inequality (2) still holds if one uses $\tilde{\phi}_S$ instead of $\phi_S$, and it is in fact slightly weaker in that case.

4. The isoperimetric inequality in Corollary [4] is not optimal, but by construction, it is the best possible inequality that can be proved for a general finitely generated group based on the arguments we have used. It seems that proving sharper isoperimetric inequalities for general groups would require new and possibly very different ideas and techniques.
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EPFL, Institut de Mathématiques, Station 8, 1015 Lausanne, Switzerland.
bruno.santoscorreia@epfl.ch, marc.troyanov@epfl.ch