FEJÉR MEANS OF VILENKIN-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to prove that there exist a martingale \( f \in H_{1/2} \) such that Fejér means of Vilenkin-Fourier series of the martingale \( f \) is not uniformly bounded in the space \( L_{1/2} \).

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1. INTRODUCTION

In one-dimensional case the weak type inequality

\[
\mu (\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1, \quad (\lambda > 0)
\]

can be found in Zygmund [14] for the trigonometric series, in Schipp [7] for Walsh series and in Pál, Simon [6] for bounded Vilenkin series. Again in one-dimensional, Fujii [5] and Simon [9] verified that \( \sigma^* \) is bounded from \( H_1 \) to \( L_1 \). Weisz [12] generalized this result and proved the boundedness of \( \sigma^* \) from the martingale space \( H_p \) to the space \( L_p \) for \( p > 1/2 \). Simon [8] gave a counterexample, which shows that boundedness does not hold for \( 0 < p < 1/2 \). The counterexample for \( p = 1/2 \) due to Goginava ( [4], see also [3]).

In [3] the following is proved:

For any bounded Vilenkin system the maximal operator of the Fejér means is not bounded from the martingale Hardy space \( H_{1/2} \) to the space \( L_{1/2} \).

In this paper we shall prove a stronger result than the unboundedness of the maximal operator from the Hardy space \( H_{1/2} \) to the space \( L_{1/2} \), in particular, we shall prove that there exists a martingale \( f \in H_{1/2} \) such that Fejér means of Vilenkin-Fourier series of the martingale \( f \) is not uniformly bounded in the the space \( L_{1/2} \).

2. DEFINITIONS AND NOTATIONS

Let \( N_+ \) denote the set of the positive integers, \( N := N_+ \cup \{0\} \). Let \( m := (m_0, m_1, \ldots) \) denote a sequence of the positive integers, not less than 2. Denote by \( Z_{m_k} := \{0, 1, \ldots, m_k - 1\} \) the addition group of integers modulo \( m_k \).

Define the group \( G_m \) as the complete direct product of the groups \( Z_{m_i} \), with the product of the discrete topologies of \( Z_{m_i} \).

The direct product \( \mu \) of the measures

\[
\mu_k (\{j\}) := 1/m_k, \quad (j \in Z_{m_k})
\]

is the Haar measure on \( G_{m_k} \) with \( \mu (G_m) = 1 \).
If \( \sup m_n < \infty \), then we call \( G_m \) a bounded Vilenkin group. If the generating sequence \( m \) is not bounded, then \( G_m \) is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.

The elements of \( G_m \) represented by sequences

\[ x := (x_0, x_1, \ldots, x_j, \ldots), \quad (x_i \in \mathbb{Z}_{m_j}). \]

It is easy to give a base for the neighborhood of \( G_m \):

\[ I_0(x) := G_m, \]

\[ I_n(x) := \{ y \in G_m \mid y_0 = x_0, \ldots y_{n-1} = x_{n-1}\}, \quad (x \in G_m, n \in \mathbb{N}). \]

Denote \( I_n := I_n(0) \), for \( n \in \mathbb{N}_+ \).

If we define the so-called generalized number system based on \( m \) in the following way:

\[ M_0 := 1, \quad M_{k+1} := m_k M_k, \quad (k \in \mathbb{N}), \]

then every \( n \in \mathbb{N} \) can be uniquely expressed as \( n = \sum_{j=0}^{\infty} n_j M_j \), where \( n_j \in \mathbb{Z}_{m_j}, \quad (j \in \mathbb{N}_+) \) and only a finite number of \( n_j \)'s differ from zero.

Next, we introduce on \( G_m \) an orthonormal system, which is called the Vilenkin system. At first define the complex valued function \( r_k(x) : G_m \to \mathbb{C} \), the generalized Rademacher functions as

\[ r_k(x) := \exp(2\pi i x/m_k), \quad (i^2 = -1, \quad x \in G_m, k \in \mathbb{N}). \]

Now define the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) as:

\[ \psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}). \]

Specifically, we call this system the Walsh-Paley one if \( m \equiv 2 \).

The Vilenkin system is orthonormal and complete in \( L_2(G_m) \) \([11][10]\).

Now we introduce analogues of the usual definitions in Fourier-analysis. If \( f \in L_1(G_m) \) we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system \( \psi \) in the usual manner:
\( \hat{f}(k) := \int_{G_m} f \overline{\psi_k} \, d\mu, \quad (k \in \mathbb{N}), \)

\( S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, \ S_0 f := 0), \)

\( \sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f, \quad (n \in \mathbb{N}_+), \)

\( D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+). \)

Recall that

\( D_{M_n}(x) = \begin{cases} 
M_n, & \text{if } x \in I_n, \\
0, & \text{if } x \notin I_n.
\end{cases} \)

The norm (or quasinorm) of the space \( L_p(G_m) \) is defined by

\[ \|f\|_p := \left( \int_{G_m} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}, \quad (0 < p < \infty). \]

The σ-algebra generated by the intervals \( \{I_n(x) : x \in G_m\} \) will be denoted by \( F_n \ (n \in \mathbb{N}). \)

Denote by \( f = (f^{(n)}, n \in \mathbb{N}) \) a martingale with respect to \( F_n \ (n \in \mathbb{N}) \). (for details see e.g. [11].)

The maximal function of a martingale \( f \) is defined by

\[ f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|. \]

In case \( f \in L_1(G_m) \), the maximal functions are also be given by

\[ f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) \, d\mu(u) \right|. \]

For \( 0 < p < \infty \), the Hardy martingale spaces \( H_p(G_m) \) consist of all martingale, for which

\[ \|f\|_{H_p} := \|f^*\|_{L_p} < \infty. \]

If \( f \in L_1(G_m) \), then it is easy to show that the sequence \( (S_{M_n} f : n \in \mathbb{N}) \) is a martingale.

If \( f = (f^{(n)}, n \in \mathbb{N}) \) is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

\[ \hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \overline{\Psi_i}(x) \, d\mu(x). \]

The Vilenkin-Fourier coefficients of \( f \in L_1(G_m) \) are the same as those of the martingale \( (S_{M_n} f : n \in \mathbb{N}) \) obtained from \( f \).
For a martingale $f$ the maximal operators of the Fejér means are defined by

$$\sigma^* f(x) = \sup_{n \in \mathbb{N}} |\sigma_n f(x)|.$$ 

A bounded measurable function $a$ is p-atom, if there exists an interval $I$, such that

$$\begin{cases} 
  a) & \int_I ad\mu = 0, \\
  b) & \|a\|_\infty \leq \mu(I)^{1/p}, \\
  c) & \text{supp}(a) \subset I.
\end{cases}$$

3. FORMULATION OF MAIN RESULT

**Theorem 1.** There exist a martingale $f \in H_{1/2}$ such that

$$\sup_n \|\sigma_n f\|_{1/2} = +\infty.$$ 

**Corollary 1.** There exist a martingale $f \in H_{1/2}$ such that

$$\|\sigma^* f\|_{1/2} = +\infty.$$ 

4. AUXILIARY PROPOSITIONS

**Lemma 1.** [13] A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p \ (0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$, of a real numbers, such that for every $n \in \mathbb{N}$:

$$(1) \quad \sum_{k=0}^\infty \mu_k S_{M_n} a_k = f^{(n)},$$

Moreover, $\|f\|_{H_p} \sim \inf (\sum_{k=0}^\infty |\mu_k|^p)^{1/p}$, where the infimum is taken over all decomposition of $f$ of the form (1).

**Lemma 2.** [2] Let $2 < A \in \mathbb{N}_+, \ k \leq s < A$ and $q_A = M_{2A} + M_{2A-2} + \ldots + M_2 + M_0$, then

$$q_{A-1} |K_{q_{A-1}}(x)| \geq \frac{M_{2k} M_{2s}}{4}.$$

for $x \in I_{2A} (0, \ldots, x_{2k} \neq 0, 0, \ldots, 0, x_{2s} \neq 0, x_{2s+1}, \ldots x_{2A-1})$

$$k = 0, 1, \ldots, A - 3, \quad s = k + 2, k + 3, \ldots, A - 1.$$
5. PROOF OF THE THEOREM

Let \( \{ \alpha_k : k \in N \} \) be an increasing sequence of the positive integers such that:

\[
(2) \quad \sum_{k=0}^{\infty} \alpha_k^{-1/2} < \infty,
\]

\[
(3) \quad \sum_{\eta=0}^{k-1} \frac{(M_{2\alpha_\eta})^2}{\alpha_\eta} < \frac{(M_{2\alpha_k})^2}{\alpha_k},
\]

\[
(4) \quad \frac{32M(M_{2\alpha_{k-1}})^2}{\alpha_{k-1}} < \frac{M_{\alpha_k}}{\alpha_k},
\]

where \( M = \sup \{ m_0, m_1 \ldots \} , \ (2 \leq M < \infty) \).

We note that such an increasing sequence \( \{ \alpha_k : k \in N \} \) which satisfies conditions (2-4) can be constructed.

Let

\[
f^{(A)}(x) = \sum_{\{k; 2\alpha_k < A\}} \lambda_k a_k,
\]

where

\[
\lambda_k = \frac{1}{\alpha_k},
\]

and

\[
a_k(x) = \frac{M_{2\alpha_k}}{M} \left( D_{M(2\alpha_{k+1})}(x) - D_{M_{2\alpha_k}}(x) \right).
\]

It is easy to show that the martingale \( f = (f^{(1)}, f^{(2)}, \ldots, f^{(A)}, \ldots) \in H_{1/2}. \)

Indeed, since

\[
S_{M_A}a_k(x) = \begin{cases} a_k(x), & 2\alpha_k < A, \\ 0, & 2\alpha_k \geq A, \end{cases}
\]

\[
\text{supp}(a_k) = I_{2\alpha_k},
\]

\[
\int_{I_{2\alpha_k}} a_k d\mu = 0
\]

and

\[
\|a_k\|_\infty \leq \frac{M_{2\alpha_k}M_{2\alpha_k+1}}{M} \leq (M_{2\alpha_k})^2 = (\text{supp } a_k)^{-2}.
\]

if we apply lemma 1 and (2) we conclude that \( f \in H_{1/2}. \)

It is easy to show that
\[
\hat{f}(j) = \begin{cases} 
\frac{1}{M^{2\alpha_k}}, & \text{if } j \in \{M^{2\alpha_k}, \ldots, M^{2\alpha_k}+1 \}, \ k = 0, 1, 2, \ldots, \\
0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M^{2\alpha_k}, \ldots, M^{2\alpha_k+1} - 1 \}.
\end{cases}
\]

We can write
\[
\sigma_{q_{\alpha_k}} f(x) = \frac{1}{q_{\alpha_k}} \sum_{j=0}^{M^{2\alpha_k} - 1} S_j f(x) + \frac{1}{q_{\alpha_k}} \sum_{j=M^{2\alpha_k}}^{q_{\alpha_k} - 1} S_j f(x) = I + II.
\]

Let \(M^{2\alpha_k} \leq j < q_{\alpha_k}\). Then applying (6) we have
\[
S_j f(x) = \sum_{v=0}^{M^{2\alpha_k} - 1} \hat{f}(v) \psi_v(x) + \sum_{v=M^{2\alpha_k}}^{q_{\alpha_k} - 1} \hat{f}(v) \psi_v(x)
= \sum_{\eta=0}^{k-1} \sum_{v=M^{2\alpha_k}}^{M^{2\alpha_{\eta+1}} - 1} \hat{f}(v) \psi_v(x) + \sum_{v=M^{2\alpha_k}}^{q_{\alpha_k} - 1} \hat{f}(v) \psi_v(x)
= \frac{1}{M} \sum_{\eta=0}^{k-1} \sum_{v=M^{2\alpha_k}}^{M^{2\alpha_{\eta+1}} - 1} M^{2\alpha_{\eta}} \alpha_{\eta} \psi_v(x) + \frac{1}{M} \sum_{\eta=0}^{k-1} \sum_{v=M^{2\alpha_k}}^{q_{\alpha_k} - 1} \alpha_{\eta} \psi_v(x)
= \frac{1}{M} \sum_{\eta=0}^{k-1} \frac{M^{2\alpha_{\eta}}}{\alpha_{\eta}} \left(D_{M^{2\alpha_{\eta+1}}} - D_{M^{2\alpha_{\eta}}} \right)
+ \frac{1}{M} \frac{M^{2\alpha_k}}{\alpha_k} \left(D_j - D_{M^{2\alpha_k}} \right).
\]

Applying (8) in II we have
\[
II = \frac{1}{M} \frac{q_{\alpha_k} - M^{2\alpha_k}}{q_{\alpha_k}} \sum_{\eta=0}^{k-1} \frac{M^{2\alpha_{\eta}}}{\alpha_{\eta}} \left(D_{M^{2\alpha_{\eta+1}}} - D_{M^{2\alpha_{\eta}}} \right)
+ \frac{1}{M} \frac{M^{2\alpha_k}}{\alpha_k q_{\alpha_k}} \sum_{j=M^{2\alpha_k}}^{q_{\alpha_k} - 1} \left(D_j - D_{M^{2\alpha_k}} \right)
= II_1 + II_2.
\]

It is evident
\[
\left| \frac{q_{\alpha_k} - M^{2\alpha_k}}{q_{\alpha_k}} \right| < 1
\]
and
\[
\left| \left( D_{M_{2^{\alpha q+1}}} (x) - D_{M_{2^{\alpha q}}} (x) \right) \right| \\
\leq M_{2^{\alpha q+1}} = M_{2^{\alpha q}} M_{2^{\alpha q}} \leq M \cdot M_{2^{\alpha q}}.
\]

Applying (3) we have

(9) \[ |II_1| \leq \sum_{\eta=0}^{k-1} \frac{M_{2^{\alpha q}}}{\alpha q} \frac{1}{M} M \cdot M_{2^{\alpha q}} \leq \frac{2 (M_{2^{\alpha k-1}})^2}{\alpha k-1}. \]

Since

\[ D_{j+M_{2^{\alpha k}}} (x) = D_{M_{2^{\alpha k}}} (x) + \psi_{M_{2^{\alpha k}}} (x) D_j (x), \quad \text{when } j < M_{2^{\alpha k}}, \]

for \(II_2\) we have:

\[
|II_2| = \frac{1}{M} \frac{M_{2^{\alpha k}}}{\alpha_k} q_{\alpha_k-1} \left| \sum_{j=0}^{q_{\alpha_k-1}-1} D_{j+M_{2^{\alpha k}}} (x) - D_{M_{2^{\alpha k}}} (x) \right| \\
= \frac{1}{M} \frac{M_{2^{\alpha k}}}{\alpha_k} q_{\alpha_k-1} \left| \psi_{M_{2^{\alpha k}}} (x) \sum_{j=0}^{q_{\alpha_k-1}-1} D_j (x) \right| \\
= \frac{1}{M} \frac{M_{2^{\alpha k}}}{\alpha_k} q_{\alpha_k-1} \left| K_{q_{\alpha_k-1}} (x) \right| \\
\geq \frac{1}{2M} \frac{q_{\alpha_k-1}}{\alpha_k} \left| K_{q_{\alpha_k-1}} (x) \right|.
\]

Since

\[ q_{\alpha_k} \leq M_{2^{\alpha_k}} \left( 1 + \frac{1}{4} + \ldots + \frac{1}{4^n} \right) \leq 2M_{2^{\alpha_k}}, \]

for \(II_2\) we obtain

\[ |II_2| \geq \frac{1}{2M} \frac{q_{\alpha_k-1}}{\alpha_k} \left| K_{q_{\alpha_k-1}} (x) \right|. \]

Let \( M_{2^{\alpha_{k-1}+1}} - 1 \leq j < M_{2^{\alpha_k}} \). Then from (8) we have
\[
|S_j f(x)| = \left| \sum_{v=0}^{j-1} \hat{f}(v) \psi_v(x) \right|
\]

\[
= \left| \sum_{v=M_{2\alpha_{k-1}+1}}^{M_{2\alpha_{k-1}+1}-1} \hat{f}(v) \psi_v(x) \right|
\]

\[
= \left| \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_{\eta}}}^{M_{2\alpha_{\eta}+1}-1} \frac{M_{2\alpha_{\eta}} \psi_v(x)}{M \cdot \alpha_{\eta}} \right|
\]

\[
= \left| \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_{\eta}}}{M \cdot \alpha_{\eta}} \left( D_{M_{2\alpha_{\eta}+1}}(x) - D_{M_{2\alpha_{\eta}}}(x) \right) \right|
\]

\[
\leq \frac{2 \left( M_{2\alpha_{k-1}} \right)^2}{\alpha_{k-1}}.
\]

Hence

\[(10) \quad |I| \leq \frac{1}{q_{\alpha_k}} \sum_{j=0}^{M_{2\alpha_k}-1} |S_j f(x)|
\]

\[
\leq \frac{2M_{2\alpha_k} \left( M_{2\alpha_{k-1}} \right)^2}{q_{\alpha_k}} \frac{1}{\alpha_{k-1}}
\]

\[
\leq \frac{2 \left( M_{2\alpha_{k-1}} \right)^2}{\alpha_{k-1}}.
\]

Applying (11) we have

\[
|I|, |II_1| \leq \frac{2 \left( M_{2\alpha_{k-1}} \right)^2}{\alpha_{k-1}} \leq \frac{1}{16M} M_{\alpha_k}.
\]

Consequently,

\[(11) \quad |\sigma_{q_{\alpha_k}} f(x)| \geq |II_2| - (|I| + |II_1|)
\]

\[
\geq \frac{1}{8M \cdot \alpha_k} \left( 4q_{\alpha_k-1} \left| K_{q_{\alpha_k-1}}(x) \right| - M_{\alpha_k} \right).
\]

Denote

\[
I_{2\alpha_k}(0, ..., x_{2\eta} \neq 0, 0, ..., 0, x_{2s} \neq 0, x_{2s+1}, ..., x_{2\alpha_k-1}) = I_{2\alpha_k}^{\eta,s}.
\]

Let

\[
x \in I_{2\alpha_k}^{\eta,s}, \quad \eta = \left[ \frac{\alpha_k}{2} \right], \left[ \frac{\alpha_k}{2} \right] + 1, ..., \alpha_k - 3, \ s = \eta + 2, \eta + 3, \alpha_k - 1.
\]

Applying lemma 2 we have:
\[ 4q_{\alpha_k-1} \left| K_{q_{\alpha_k-1}} (x) \right| \geq M_{2\eta}M_2. \]

Since
\[ 2s \geq 2 \left\lceil \frac{\alpha_k}{2} \right\rceil + 4 > \alpha_k + 1, \]
we have
\[ M_{2s} > M_{\alpha_k+1} \geq m_{\alpha_k}M_{\alpha_k} \geq 2M_{\alpha_k}. \]

Hence
\[ (12) \quad M_{2s}M_{2\eta} - M_{\alpha_k} \geq \frac{1}{M}M_{2s}M_{2\eta}. \]

From (11-12) we have
\[ \left| \sigma_{q_{\alpha_k}} f (x) \right| \geq \frac{1}{8M^2 \cdot \alpha_k} M_{2s}M_{2\eta}, \quad x \in I_{2\alpha_k}^{\eta,s}, \]

where
\[ \eta = \left\lceil \frac{\alpha_k}{2} \right\rceil, \left\lceil \frac{\alpha_k}{2} \right\rceil + 1, ..., \alpha_k - 3, \quad s = \eta + 2, \eta + 3, \alpha_k - 1. \]

Hence we can write
\[
\int_{G_m} \left| \sigma_{q_{\alpha_k}} f (x) \right|^{\frac{1}{2}} d\mu (x) \\
\geq \sum_{s=\eta+2}^{\alpha_k-1} \sum_{s=\eta+2}^{m_{2\alpha_k}-1} \sum_{x=0}^{m_{2\alpha_k}-1} \int_{I_{2\alpha_k}^{\eta,s}} \left| \sigma_{q_{\alpha_k}} f (x) \right|^{\frac{1}{2}} d\mu (x) \\
\geq \frac{1}{8M^2 \cdot \alpha_k} \sum_{s=\eta+2}^{\alpha_k-3} \sum_{s=\eta+2}^{\alpha_k-1} \frac{M_{2s+1} \cdot M_{2\alpha_k-1}}{M_{2\alpha_k}} \sqrt{M_{2s}M_{2\eta}} \\
\geq \frac{1}{8 \sqrt{M_k} \cdot \alpha_k} \sum_{s=\eta+2}^{\alpha_k-3} \sum_{s=\eta+2}^{\alpha_k-1} \sqrt{M_{2s}M_{2\eta}} \cdot \sqrt{M_{2s+1}} \\
\geq \frac{1}{8M^2 \sqrt{\alpha_k}} \sum_{s=\eta+2}^{\alpha_k-3} \sum_{s=\eta+2}^{\alpha_k-1} \frac{M_{2\eta}}{M_{2s}}. \]

It is easy to show that
\[ \sum_{s=\eta+2}^{\alpha_k-3} \sqrt{\frac{M_{2\eta}}{M_{2s}}} \geq \sqrt{\frac{M_{2\eta}}{M_{2\eta+4}}} \geq \frac{1}{M^2}. \]

Consequently,
\[ \int_G \left| \sigma_{q_k} f(x) \right|^{\frac{1}{2}} \, d\mu(x) \geq \frac{1}{8M^2/\sqrt{\alpha_k}} \sum_{\eta=[\alpha_k/2]}^{\alpha_k-3} \left( \sum_{s=\eta+2}^{\alpha_k-1} \sqrt{\frac{M_{2s}}{M_{2\eta}}} \right) \]
\[ \geq \frac{1}{8M^4/\sqrt{\alpha_k}} \sum_{\eta=[\alpha_k/2]}^{\alpha_k-3} 1 \]
\[ \geq c\sqrt{\alpha_k} \rightarrow \infty, \quad \text{as } k \rightarrow \infty. \]

Theorem 1 is proved.

**References**

[1] G. N. AGAEV, N. Ya. VILENKIN, G. M. DZHAFA RLY and A. I. RUBINSHTEIN, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, Baku, Ehim, 1981 (in Russian).

[2] I. BLAHO T A, G. GÁT and U. GOGINAV A, maximal operators of Fejér means of double Vilenkin-Fourier series, Colloq. Math. 107 (2007), no. 2, 287–296.

[3] I. BLAHO T A, G. GÁT and U. GOGINAV A, Maximal operators of Fejér means of Vilenkin-Fourier series. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), 1-7.

[4] U. GOGINAVA, The maximal operator of Marcinkiewicz-Fejér means of the d-dimensional Walsh-Fourier series. East J. Approx. 12 (2006), no. 3, 295–302.

[5] N. J. FUJII, A maximal inequality for $H_1$ functions on the generalized Walsh-Paley group, Proc. Amer. Math. Soc. 77 (1979), 111-116.

[6] J. PÁL and P. SIMON, On a generalization of the concept of derivative, Acta Math. Hung., 29 (1977), 155-164.

[7] F. SCHIPP, Certain rearrangements of series in the Walsh series, Mat. Zametki, 18 (1975), 193-201.

[8] P. SIMON, Cesáro summability with respect to two-parameter Walsh systems, Monatsh. Math., 131 (2000), 321-334.

[9] P. SIMON, Investigations with respect to the Vilenkin sistem, Annales Univ. Sci. Budapest Eotv., Sect. Mat., 28 (1985), 87-101.

[10] N. Ya. VILENKIN, Aclass of complete orthonormal systems, Izv. Akad. Nauk. U.S.S.R., Ser. Mat., 11 (1947), 363-400.

[11] F. WEISZ, Martingale Hardy spaces and their applications in Fourier Analysis, Springer, Berlin-Heidelberg-New York, 1994.

[12] F. WEISZ, Cesáro summability of one and two-dimensional Fourier series, Anal. Math. Studies, 5 (1996), 353-367.

[13] F. WEISZ, Hardy spaces and Cesáro means of two-dimensional Fourier series, Bolyai Soc. math. Studies, (1996), 353-367.

[14] A. ZYGMUND, Trigonometric Series, Vol. 1, Cambridge Univ. Press, 1959.

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