The Effective Lorentzian and Teleparallel Spacetimes Generated by a Free Electromagnetic Field

E. Notte-Cuello\(^{(1)}\), R. da Rocha\(^{(2)}\) and W. A. Rodrigues Jr.\(^{(3)}\)

\(^{(1)}\)Departamento de Matemáticas, Universidad de La Serena
Av. Cisternas 1200, La Serena-Chile

\(^{(2)}\)Instituto de Física Teórica, UNESP, Rua Pamplona 145, 01405-900, São Paulo, SP, Brazil.
and
DRCC - Institute of Physics Gleb Wataghin, UNICAMP CP 6165
13083-970 Campinas, SP, Brazil

\(^{(3)}\)Institute of Mathematics, Statistics and Scientific Computation
IMECC-UNICAMP CP 6065
13083-859 Campinas, SP, Brazil

walrod@ime.unicamp.br; enotte@userena.cl; roldao@ifi.unicamp.br

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Abstract

In this paper we show that a free electromagnetic field living in Minkowski spacetime generates an effective Weitzenböck or an effective Lorentzian spacetime whose properties are determined in details. These results are possible because we found using the Clifford bundle formalism the noticeable result that the energy-momentum densities of a free electromagnetic field are sources of the Hodge duals of exact 2-form fields which satisfy Maxwell like equations.

1 Introduction

In this paper we determine an effective Weitzenböck \([(M, g, \nabla, \uparrow, \tau_g)]\) spacetime and an effective Lorentzian \([(M, g, \bar{D}, \uparrow, \tau_g)]\) spacetime determined by a free electromagnetic field \(F \in \text{sec} \bigwedge^2 T^*M\) configuration living on Minkowski spacetime \([(M, \eta, D, \uparrow, \tau_\eta)]\) and satisfying Maxwell equation \(\not\! \partial F = 0\). The determination of these effective spacetimes become easily feasible because using the Clifford bundle formalism we found that the energy-momentum densi-
ties $T_a \in \text{sec} \wedge^1 T^* M$ of the Maxwell field are sources of exact 2-forms fields $W^a \in \text{sec} \wedge^2 T^* M$ (which we explicitly identify) and which satisfy Maxwell like equations $dW^a = 0$, $\delta W^a = -T^a$ in Minkowski spacetime. Our results depend crucially on the Riesz formula $T_a = \frac{1}{2} F \vartheta_a \tilde{F}$, which, of course has meaning only in the Clifford bundle formalism. We identify $g = h^* \eta$ as a metric field on $M$ generated by a diffeomorphism $h : M \rightarrow M$ associated with a distortion in the Minkowski vacuum caused by $A \in \text{sec} \wedge^1 T^* M$ ($F = dA$). In the case of the effective Weitzenböck (teleparallel) spacetime we give the explicit form for the torsion 2-forms $\Theta^a$ and show that their sources are $T^a = h^* T^a$, more precisely, $d \star g \Theta^a = -\star g T^a$. In the case of the effective Lorentzian spacetime we give the explicit forms of the Ricci 1-form fields $R^a \in \text{sec} \wedge^1 T^* M$ which can also be expressed in terms of the Hodge Laplacian and covariant D'Alembertian of the $g$-orthonormal cotetrads $\{\theta^a\}$. The explicit formulas for the $R^a$ for our problem gives in one sense an unified theory of the gravitational and electromagnetic field in the sense of the Rainich-Misner-Wheeler theory. The details of the present paper are as follows. In Section 2 we present Maxwell equation in the Clifford bundle $\mathcal{A}(M, \eta)$, prove Riesz formula $T_a = \frac{1}{2} F \vartheta_a \tilde{F}$ and obtain the Maxwell like equations $\partial T^a = J$. In Section 3 we derive the effective Weitzenböck (teleparallel) spacetime generated by $F$ and in Section 4 we derive the effective Lorentzian spacetime generated by $F$. In Section 5 we present our conclusions.

The paper has several Appendices. Appendix A recalls the definition of Clifford bundles. In Appendix A.1 the Clifford product is introduced and we present several Clifford algebra identities necessary to follows the calculations. Next in Appendix A.2 we introduce the definition of the Hodge star operator associated with the metrics $\eta$ and $g = h^* \eta$ and show to calculate each one it using Clifford algebra methods and also how they are related. The important notion of the Dirac operator associated with a given Levi-Civita connection of a metric field is introduced in Appendix A.5.

2 Maxwell Equation

We start this section by recalling that Eq. (77) of Appendix permits us to write the Maxwell equations

$$dF = 0, \quad \delta F = -J \quad (1)$$

for $F \in \text{sec} \wedge^2 T^* M \hookrightarrow \text{sec} \mathcal{A}(M, \eta)$ living in Minkowski spacetime as a single equation (Maxwell equation),

$$\partial F = J. \quad (2)$$

Next we investigate a noticeable formula which can be written only in the Clifford bundle formalism and which is essential for all our developments.

\[\text{If they are not enough for completely intelligibility of the paper, please consult [8].}\]
2.1 The Noticeable Riesz Formula $T_a = \frac{1}{2} F \partial_a \tilde{F}$

We now prove that the energy-momentum densities $\star T_a$ of the Maxwell field can be written in the Clifford bundle formalism as

$$\star T_a = \frac{1}{2} \star (F \partial_a \tilde{F}) \in \sec \bigwedge^3 T^* M \hookrightarrow \sec \mathcal{C}(M, \eta).$$

(3)

To derive Eq. (3) we start from the Maxwell Lagrangian

$$L_m = \frac{1}{2} F \wedge \star \eta F,$$

(4)

where $F = \frac{1}{2} F_{ab} \partial^a \wedge \partial^b := \frac{1}{2} F_{ab} \partial^a \partial^b \in \sec \bigwedge^2 TM \hookrightarrow \sec \mathcal{C}(M, \eta)$ is the electromagnetic field. Now, denoting by $\delta$ the variational symbol we can easily verify that

$$\delta \star \eta \partial^a = \delta \theta^c \wedge [\partial^c \star \eta \partial^a].$$

Moreover, in general $\delta$ and $\star$ do not commute. Indeed, for any $A_p \in \sec \bigwedge^p T^* M \hookrightarrow \sec \mathcal{C}(M, \eta)$ we have

$$[\delta, \star] A_p = \delta \star A_p - \star \delta A_p$$

$$= \delta \partial^a \wedge \left( \partial^a \star A_p \right) - \star \left[ \delta \partial^a \wedge \left( \partial^a \star A_p \right) \right].$$

(5)

Multiplying both members of Eq. (5) with $A_p = F$ on the right by $F \wedge$ we get

$$F \wedge \delta \star F = F \wedge \star \delta F + F \wedge \{ \delta \partial^a \wedge (\partial^a \star F) - \star [\delta \partial^a \wedge (\partial^a \star F)] \}.$$

Next we sum $\delta F \wedge \star F$ to both members of the above equation obtaining

$$\delta \left( \frac{F \wedge \star F}{\eta} \right) = 2 \delta F \wedge \star F + \delta \partial^a \wedge [F \wedge (\partial^a \star F) \wedge F].$$

Then, it follows (see, [8, 7] for some details) that $\delta \partial^a = - \mathcal{L}_\xi \partial^a$, for some diffeomorphism generated by the vector field $\xi$ that

$$\star T_a = \frac{\partial L_m}{\partial \partial^a} = \frac{1}{2} \left[ F \wedge (\partial^a \star F) - (\partial^a F) \wedge \star F \right].$$

2 The formula $T_a = \frac{1}{2} F \partial_a \tilde{F}$ has been first obtained (but, not using the algebraic derivatives of the Lagrangian density) by M. Riesz in 1947 [6] and it has been rediscovered by Hestenes in 1996 [1] (which also does not use the algebraic derivatives of the Lagrangian density). Algebraic derivatives of homogenous form fields has been described, e.g., in Thirring's book [9].

3 Please, do not confuse the variational symbol $\delta$ with the symbol $\delta$ of the Hodge coderivative.

4 $\mathcal{L}_\xi$ denotes the Lie derivative in the direction of the vector field $\xi$. 

3
Now,
\[(\partial_{a,F} \eta \eta) \wedge \ast F = - \ast [((\partial_{a,F} \eta \eta) \eta F] - [(\partial_{a,F} \eta \eta) \eta F] \tau_{\eta} \]
and using also the identity \(8\)
\[(\partial_{a,F} \eta \eta) \wedge \ast F = \partial_{a}(F \cdot F) \tau_{\eta} - F \wedge (\partial_{a,F} \eta \eta) \ast F),\]
we get
\[
\frac{1}{2} \left[ F \wedge (\partial_{a,F} \eta \eta) \ast F) - (\partial_{a,F} \eta \eta) \wedge \ast F \right] = \frac{1}{2} \left\{ \partial_{a}(F \cdot F) \tau_{\eta} - (\partial_{a,F} \eta \eta) \wedge \ast F - (\partial_{a,F} \eta \eta) \wedge \ast F \right\} \\
= \frac{1}{2} \left\{ \partial_{a}(F \cdot F) \tau_{\eta} - 2(\partial_{a,F} \eta \eta) \wedge \ast F \right\} \\
= \frac{1}{2} \left\{ \partial_{a}(F \cdot F) \tau_{\eta} + 2[((\partial_{a,F} \eta \eta) \eta F] \tau_{\eta} \right\} \\
= \ast \left( \frac{1}{2} \partial_{a}(F \cdot F) + (\partial_{a,F} \eta \eta) \eta F \right) = \frac{1}{2} \ast (F \partial_{a,F}),
\]
where in writing the last line we used the identity
\[
\frac{1}{2} F n \tilde{F} = (n \eta F) \eta F + \frac{1}{2} n(F \cdot F),
\]
whose proof is as follows:
\[
(n \eta F) \eta F + \frac{1}{2} n(F \cdot F) = \frac{1}{2} \left[ (n \eta F) F - F(n \eta F) \right] + \frac{1}{2} n(F \cdot F) \\
= \frac{1}{4} \left[ n F F - F n F - F n F + n F F \right] + \frac{1}{2} n(F \cdot F) \\
= - \frac{1}{2} F n F + \frac{1}{4} \left[ -2 n(F \cdot F) + n(F \wedge F) + (F \wedge F) n \right] \\
+ \frac{1}{2} n(F \cdot F) \\
= - \frac{1}{2} F n F - \frac{1}{2} n(F \cdot F) + \frac{1}{2} n \wedge (F \wedge F) + \frac{1}{2} n(F \cdot F) \\
= - \frac{1}{2} F n F = \frac{1}{2} F n \tilde{F}.
\]
valid for any \(n \in \sec T^* M \hookrightarrow \sec \mathcal{C}(M, \eta)\) and \(F \in \sec (2 T^* M \hookrightarrow \sec \mathcal{C}(M, \eta))\).

For completeness and presentation of some more tricks of the trade we detail
the proof that $\mathcal{T}_a \cdot \vartheta_b = \mathcal{T}_b \cdot \vartheta_a$.

\[
\mathcal{T}_a \cdot \vartheta_b = -\frac{1}{2} \langle F \vartheta_a F \vartheta_b \rangle_0 = -\frac{1}{2} \langle (F \vartheta_a) (F \vartheta_b) \rangle_0 - \frac{1}{2} \langle (\vartheta_a \wedge F) F \vartheta_b \rangle_0
\]

\[
= -\frac{1}{2} \langle (F \vartheta_a) F \vartheta_b \rangle_0 - \frac{1}{2} \langle (\vartheta_a FF \vartheta_b) \rangle_0
\]

\[
= -\frac{1}{2} \langle (F \vartheta_a) (F \vartheta_b) \rangle_0 + \frac{1}{2} \langle (F \vartheta_a) (F \wedge \vartheta_b) \rangle_0
\]

\[
+ \frac{1}{2} \langle (\vartheta_a (F \cdot F) \vartheta_b) \rangle_0 - \frac{1}{2} \langle (\vartheta_a (F \wedge F) \vartheta_b) \rangle_0
\]

\[
= -\frac{1}{2} \langle (F \vartheta_a) (F \vartheta_b) \rangle_0 + \frac{1}{2} \langle (F \cdot F) (\vartheta_a \cdot \vartheta_b) \rangle_0
\]

\[
= -\frac{1}{2} \langle (F \vartheta_a) \cdot (F \vartheta_b) \rangle_0 + \frac{1}{2} \langle (F \cdot F) (\vartheta_b \cdot \vartheta_a) \rangle_0 = \mathcal{T}_b \cdot \vartheta_a.
\]

Note moreover that

\[
\mathcal{T}_{ab} = \mathcal{T}_a \cdot \vartheta_b = -\eta^{cl} F_{ac} F_{b1} + \frac{1}{4} F_{cd} F^{cd} \eta_{ab}, \quad (7)
\]

a well known result.

Of course, for the free electromagnetic field we have that $d \mathcal{T} = 0$, which is equivalent to $\delta \mathcal{T} = -\mathcal{T}_a \cdot \vartheta_a = 0$. Indeed, observe that

\[
\vartheta_a \mathcal{T}^a = \vartheta_a \frac{1}{2} \langle F \vartheta^a \tilde{F} \rangle
\]

\[
= \frac{1}{2} \langle \vartheta (F \vartheta^a \tilde{F}) \rangle_0
\]

\[
= \frac{1}{2} \langle (\vartheta F) \vartheta^a \tilde{F} + \vartheta b \left(F \vartheta^a D_{cb} \tilde{F}\right) \rangle_0
\]

\[
= \frac{1}{2} \langle (\vartheta b \left(F \vartheta^a D_{cb} \tilde{F}\right) \rangle_0, \quad (8)
\]

where we used that $\vartheta F = 0$. Now,

\[
\vartheta b \left(F \vartheta^a D_{cb} \tilde{F}\right) = \vartheta b \left\langle F \vartheta^a D_{cb} \tilde{F}\right\rangle_1 + \vartheta b \left\langle F \vartheta^a D_{cb} \tilde{F}\right\rangle_3
\]

\[
= \vartheta b \left\langle F \vartheta^a D_{cb} \tilde{F}\right\rangle_1 + \vartheta b \wedge \left\langle F \vartheta^a D_{cb} \tilde{F}\right\rangle_3, \quad (9)
\]

Then

\[
\left\langle \vartheta b \left(F \vartheta^a D_{cb} \tilde{F}\right) \right\rangle_0 = \vartheta b \left\langle F \vartheta^a D_{cb} \tilde{F}\right\rangle_1 = \left\langle F \vartheta^a D_{cb} \tilde{F}\right\rangle_1 \wedge \vartheta b
\]

\[
= \left\langle F \vartheta^a D_{cb} \tilde{F}\tilde{\vartheta} b \right\rangle_0
\]

\[
= \left\langle F \vartheta^a (\vartheta F) \right\rangle_0 = 0,
\]

where we used the symbol $(\vartheta F) := D_{cb} \tilde{F} \vartheta^b$ and the fact that $(\vartheta F) = 0$.  

2.2 Enter New Maxwell Like Equations

Let $\star T^a = \frac{1}{2} \eta (F^a \tilde{F}) \in \sec \wedge^3 T^* M \hookrightarrow \sec C\ell(M, \eta)$ be the energy-momentum densities of a free electromagnetic field configuration $F \in \sec \wedge^2 T^* M \hookrightarrow \sec C\ell(M, \eta)$ ($\tilde{\theta} F = 0$). As we already know, we have

$$-\delta \eta T^a = \partial | \eta T^a = 0.$$  \hspace{1cm} (10)

Eq. (10) which is equivalent to $d \star \eta T^a = 0$ and since we are in Minkowski space-time there must exist $W^a \in \sec \wedge^2 T^* M \hookrightarrow \sec C\ell(M, \eta)$ such that

$$- T^a = \delta \eta W^a.$$  \hspace{1cm} (11)

We claim that

**Proposition 1**

$$W^a = d \Gamma^a,$$  \hspace{1cm} (12)

$$\Gamma^a = -\frac{1}{4} (A \eta A + X^a)$$  \hspace{1cm} (13)

where $A$ is the electromagnetic potential ($dA = F$), and $X^a \in \sec \wedge^1 T^* M \hookrightarrow \sec C\ell(M, \eta)$ and

$$dX^a = -2 \partial^B \eta (A \eta D_a A) + 2 B^a,$$  \hspace{1cm} (14)

with

$$\tilde{\theta} (A \eta F) = \tilde{\partial} B^a.$$  \hspace{1cm} (15)

**Proof:** To prove the proposition we first note that

$$A \eta A = \left( \partial^a \cdot A \right) A + A (\eta A \wedge A)$$

$$= \left( \partial^a \cdot A \right) A + A \eta (\partial^a \wedge A) + (\partial^a \wedge A),$$

and since $A \wedge \partial^a \wedge A = 0$ we have

$$A \eta A = A \eta \tilde{A} = \langle A \eta \tilde{A} \rangle_1.$$  \hspace{1cm} (16)

Then

$$d \left( A \eta \tilde{A} \right)_1 = d \left( A \eta \tilde{A} \right)_1 = \tilde{\theta} \wedge \langle A \eta \tilde{A} \rangle_1 = \tilde{\partial} D_{\eta} (A \eta A)$$

$$= \left( \partial^b D_{\eta} (A \eta A) \right)_1 \eta A + \partial^b A \eta A D_{\eta} \tilde{A}.$$  \hspace{1cm} (17)
But
\[ \partial^b \wedge \left< A \partial^a D_{cb} \tilde{A} \right>_1 = \left< \partial^b A \partial^a D_{cb} \tilde{A} \right>_2 - \partial^b \left< \partial^a A D_{cb} \tilde{A} \right>_3 = - \left< A \partial^a D_{cb} \tilde{A} \right>_1 \wedge \partial^b \]
\[ = - \left< A \partial^a D_{c} \tilde{A} \right>_2 + \left< A \partial^a D_{c} \tilde{A} \right>_3 \wedge \partial^b \]
\[ = - \left< A \partial^a \tilde{A} \right>_2 + \left< A \partial^a D_{c} \tilde{A} \right>_3 \wedge \partial^b \]
\[ = - \left< A \partial^a F \right>_2 + \left< A \partial^a D_{c} \tilde{A} \right>_3 \wedge \partial^b = \langle A \partial^a F \rangle_2 + \langle A \partial^a D_{c} \tilde{A} \rangle_3 \wedge \partial^b \]

Then from Eq. (18) we can write Eq. (17) as
\[ d (A \partial^a A) = 2 \langle (A \partial^a) F \rangle_2 + 2 \partial^b \left< A \partial^a D_{cb} \tilde{A} \right>_3 \]
and then
\[ \mathcal{W}^a = - \frac{1}{4} d (A \partial^a A + X^a) = - \frac{1}{2} \langle A \partial^a F \rangle_2 - \frac{1}{2} B^a \]

We now verify that \( \partial \mathcal{W}^a = - \mathcal{T}^a \). Indeed, since \( \langle \partial^b \langle A \partial^a (D_{cb} F) \rangle_4 \rangle_1 = 0 = \langle \partial^b \langle (D_{cb} F) \partial^a A \rangle_4 \rangle_1 \), taking into account the last identity in Eq. (16) we can write
\[ \delta \mathcal{W}^a = - \frac{1}{2} \partial_\eta \mathcal{W}^a = \frac{1}{2} \partial_\eta \langle (A \partial^a) F \rangle_2 + \frac{1}{2} \partial_\eta B^a = \frac{1}{2} \langle \partial (A \partial^a F) \rangle_1 - \frac{1}{2} \partial \langle (A \partial^a F) \rangle_0 + \frac{1}{2} \partial_\eta B^a \]
\[ = \frac{1}{2} \langle F \partial^a F \rangle_1 + \frac{1}{2} \langle \partial^b A \partial^a (D_{cb} F) \rangle_1 \]
\[ = \frac{1}{2} \langle F \partial^a F \rangle_1 + \frac{1}{2} \langle \partial^b (A \partial^a (D_{cb} F))_0 + \partial^b (A \partial^a (D_{cb} F))_2 + \partial^b (A \partial^a (D_{cb} F))_4 \rangle_1 \]
\[ = - \frac{1}{2} \langle F \partial^a F \rangle_1 + \frac{1}{2} \langle \partial^b ((D_{cb} F) \partial^a A)_0 + \partial^b ((D_{cb} F) \partial^a A)_2 + \partial^b ((D_{cb} F) \partial^a A)_4 \rangle_1 \]
\[ = - \frac{1}{2} F \partial^a F + \frac{1}{2} \langle \partial^b (D_{cb} F) \partial^a A \rangle_1 = - \frac{1}{2} F \partial^a F + \frac{1}{2} \langle \partial (F) \partial^a A \rangle_1 \]
\[ = - \frac{1}{2} F \partial^a F. \]

Finally note that from Eqs. (22) and (21) we can write \( d \mathcal{W}^a = d \mathcal{T}^a = - \frac{1}{4} d (A \partial^a A + X^a) \) and \( F = dA \)
\[ d \mathcal{W}^a = 0, \delta \mathcal{W}^a = - \mathcal{T}^a, \]
\[ \partial \mathcal{W}^a = - \mathcal{T}^a. \]

and we get the non trivial result that the 2-form fields \( \mathcal{W}^a \) (\( a = 0, 1, 2, 3 \)) describing the energy-momentum propagation satisfy Maxwell like equations \( \partial \mathcal{W}^a = \mathcal{T}^a \) with sources being the energy-momentum 1-form fields \( \mathcal{T}^a \).
3 The Effective Weitzenböck Spacetime Generated by $F$

First we recall some results of [3, 7, 8] where we showed how effective Riemann-Cartan spacetimes can be generated by the presence of distortion fields which arises from diffeomorphisms $h: M \to M$. Here we investigate a particular diffeomorphism $h$ associated with the electromagnetic potential $A \in \sec\bigwedge^1 T^* M \hookrightarrow \sec\mathcal{Cl}(M, \eta) \ (F = dA)$ by

$$\theta^a = h^* \Gamma^a, \quad (23)$$

$$g = h^* \eta, \quad (24)$$

$$g = \eta_{ab} \theta^a \otimes \theta^b \in \sec T^2_2 M. \quad (25)$$

Now, the metric of the cotangent bundle associated with $g$ is $g \in \sec T^2_2 M$ and we have

$$g(\theta^a, \theta^b) = \eta_{ab}. \quad (26)$$

Calling $T^a = h^* T^a \in \sec \bigwedge^1 T^* M$ (33)

we can write the last equation as

$$d \theta^a = \Theta^a. \quad (30)$$

Now, recall that for any $\omega_p \in \sec \bigwedge^p T^* M$, if $g = h^* \eta$ it holds that (see, e.g., [8])

$$\star g(h^* \omega_p) = h^* (\star \omega_p), \quad (31)$$

then, returning to Eq. (11), equivalent to $d \star W^a = - \star T^a$ we can write

$$d \star g(h^* W^a) = - h^* (\star g(T^a)) = - \star (h^* T^a). \quad (32)$$

Calling

$$T^a = h^* T^a \in \sec \bigwedge^1 T^* M \quad (33)$$
the deformed energy-momentum 1-form field we have \( d \ast \Theta^a = - \ast T^a \) or equivalently
\[
\delta g \Theta^a = - T^a. 
\] (34)

We summarize the above results in the following:

**Proposition 2** A free electromagnetic field \( F \in \text{sec} \wedge^2 T^* M \) living in Minkowski spacetime structure \((M, \eta, \nabla, \uparrow, \tau_\eta)\) and satisfying Maxwell equation \( \partial F = 0 \) generates an effective Weitzenböck (or teleparallel) geometry, i.e., a teleparallel spacetime \((M, g, \nabla, \uparrow, \tau_g)\) where \( g = h^* \eta, \nabla g = 0 \), and the torsion 2-forms \( \Theta^a \) are given by Eq. (30) in the teleparallel g-orthonormal cobasis \( \{\theta^a\} \). Moreover the torsion 2-form fields in this theory propagate, i.e., satisfy Maxwell equations
\[
d \Theta^a = 0, \delta g \Theta^a = - T^a, 
\] (35)

with the surprising result that the sources of the \( \Theta^a \) are the energy-momentum 1-form fields.

### 4 The Effective Lorentzian Spacetime Generated by a Free Electromagnetic Field \( F \)

We now introduce an effective Lorentzian spacetime \((M, g, \hat{D}, \uparrow, \tau_g)\) as follows. We know that for an electromagnetic field the trace of the energy-momentum tensor is null, i.e., \( T^a_a = 0 \). It follows that with \( g = h^* \eta \) as defined above we have that the trace of \( T^a = h^* T^a \) is also null. Then since the Einstein equations can be written (see, e.g., [8, 3]) as
\[
\ast R^a - \frac{1}{2} R \ast \theta^a = - \ast T^a, 
\] (36)

where \( R^a = R^a_b \theta^b \in \text{sec} \wedge^1 T^* M \to \mathcal{C}(M, g) \) are the Ricci 1-form fields and \( R = R^a_a \) is the scalar curvature, which for the present case is null since \( R = - T^a_a = 0 \).

Then, we have that
\[
\ast \Theta^a = - \ast T^a, 
\] (37)

Using the fact that \( d \ast \Theta^a = - \ast T^a \) (where now, of course, we are not interpreting the \( \Theta^a \) as 2-forms of torsion of a teleparallel connection) we get
\[
\ast \Theta^a = d \ast \Theta^a, 
\] (38)

or equivalently
\[
\delta \Theta^a = R^a 
\] (39)
Moreover recalling that the Hodge Laplacian is $\Box = -(d\delta + \delta d)$ we have:

$$R^a = -\Box \theta^a - d\delta \theta^a$$  \hspace{1cm} (40)$$

or taking into account Eqs. (23), (37), (33) we end with

$$\Box \theta^a + d\delta \theta^a = \frac{1}{2g}\star (h^*(F\theta^aF)).$$

We summarize the above results in the following

**Proposition 3** A free electromagnetic field $F \in \text{sec} \text{\bigwedge}^2 T^*M$ living in Minkowski spacetime structure $(M, \eta, D, \uparrow, \tau_\eta)$ and satisfying Maxwell equation $\partial|F = 0$ generates an effective Lorentzian $(M, g, D, \uparrow, \tau_g)$ where $g = h^*\eta$, $Dg = 0$, such that the dual of the Ricci 1-forms are exact differentials, i.e., $\star R^a = -d\star \Theta^a$. Moreover, we have

$$R^a = -\Box \theta^a - d\delta \theta^a.$$  \hspace{1cm} (41)$$

**Remark 4** Eq. (41) is the condition that the Ricci 1-form fields in a Lorentzian spacetime modelling a gravitational field must satisfy in order to describe an electromagnetic field propagating in a Minkowski spacetime. We then arrive in a kind of already unified theory as in the Rainich-Misner-Wheeler theory.

**Remark 5** Recalling that we can write

$$-\star R^a = d\star S^a + \star t^a = \star T^a,$$

where

$$\star t^c = -\frac{1}{2}\omega_{ab} \wedge [\omega^c_d \wedge \star (\theta^a \wedge \theta^b \wedge \theta^d) + \omega^b_d \wedge \star (\theta^a \wedge \theta^d \wedge \theta^c)],$$

$$\star S^c = \frac{1}{2}\omega_{ab} \wedge \star (\theta^a \wedge \theta^b \wedge \theta^c).$$  \hspace{1cm} (43)$$

with $\omega^{cd} \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} \text{Cl}(M, g)$ given by

$$\omega^{cd} = \frac{1}{2} \left[ \theta^d_c \wedge d\theta^c - \theta^c_c \wedge d\theta^d + \theta^c_d \left( g^d_{\wedge} \right) \theta^a \right].$$  \hspace{1cm} (44)$$

which recalling that $\theta^a = h^*\Gamma^a$ and that $h^*\omega_p, h^*\omega_k = h^*(\omega_p, \omega_k)$ for any $\omega_p \in \text{sec} \bigwedge^p T^*M$, $\omega_k \in \text{sec} \bigwedge^k T^*M$ can also be written as

$$\omega^{cd} = \frac{1}{2} h^* \left[ \Gamma^d_c \wedge d\Gamma^c_d - \Gamma^e_c \wedge d\Gamma^d_e + \Gamma^e_d \left( \Gamma^d_{\wedge} \right) \Gamma^a \right].$$  \hspace{1cm} (45)$$

\footnote{We mention also that the Ricci 1-forms may be written in terms of the Ricci operator \(\Box\)}

\footnote{See, e.g., [10, 8].}
Using Eq. (23) we can rewrite Eqs. (43) as

\[
\begin{align*}
\star t^c_e &= \frac{1}{2} \omega_{ab} \wedge [\omega_t^a \wedge \star (\theta^a \wedge \theta^b \wedge \theta^d) + \omega_d^b \wedge \star (\theta^a \wedge \theta^d \wedge \theta^c)] \\
&= \frac{1}{4} h^e \left[ \Gamma_{b \eta} d \Gamma_a - \Gamma_{a \eta} d \Gamma_b + \Gamma_{a \eta} \left( \Gamma_{b \eta} d \Gamma_p \right) \Gamma^p \right] \\
&\quad \wedge h^e \left[ \Gamma_{d \eta} d \Gamma^e - \Gamma^e \eta d \Gamma_d + \Gamma^e \eta \left( \Gamma_d \eta d \Gamma_p \right) \Gamma^p \right] \\
&\quad \wedge h^e \star (\Gamma_a \wedge \Gamma_b \wedge \Gamma^d),
\end{align*}
\]

\[
\begin{align*}
\star S^c_e &= \frac{1}{4} h^e \left[ \Gamma_{b \eta} d \Gamma_a - \Gamma_{a \eta} d \Gamma_b + \Gamma_{a \eta} \left( \Gamma_{b \eta} d \Gamma_p \right) \Gamma^p \right] \\
&\quad \wedge h^e \star (\Gamma_a \wedge \Gamma_b \wedge \Gamma^d).
\end{align*}
\]

Finally, taking into account that \( \dashv t^c_e = h^e \star t^c_M = h^e \star \tau^c_M \) and \( \dashv S^c_e = h^e S^c_M \) we get

\[
\begin{align*}
\dashv t^c_{M \eta} &= \frac{1}{4} \left[ \Gamma_{b \eta} d \Gamma_a - \Gamma_{a \eta} d \Gamma_b + \Gamma_{a \eta} \left( \Gamma_{b \eta} d \Gamma_p \right) \Gamma^p \right] \\
&\quad \wedge \left[ \Gamma_{d \eta} d \Gamma^e - \Gamma^e \eta d \Gamma_d + \Gamma^e \eta \left( \Gamma_d \eta d \Gamma_p \right) \Gamma^p \right] \\
&\quad \wedge \star (\Gamma_a \wedge \Gamma_b \wedge \Gamma^d) \\
&\quad + \frac{1}{2} \left[ \Gamma_{d \eta} d \Gamma^b - \Gamma^b \eta d \Gamma_d + \Gamma^b \eta \left( \Gamma_d \eta d \Gamma_p \right) \Gamma^p \right] \\
&\quad \wedge \star (\Gamma^a \wedge \Gamma^b \wedge \Gamma^d),
\end{align*}
\]

\[
\begin{align*}
\dashv S^c_{M \eta} &= \frac{1}{4} \left[ \Gamma_{b \eta} d \Gamma_a - \Gamma_{a \eta} d \Gamma_b + \Gamma_{a \eta} \left( \Gamma_{b \eta} d \Gamma_p \right) \Gamma^p \right] \\
&\quad \wedge \star (\Gamma_a \wedge \Gamma_b \wedge \Gamma^d). \tag{47}
\end{align*}
\]

To end we recall that since \( d \star \tau^a = 0 \) we also have \( d \star \tau^a = 0 \). Then, in our theory the \( \Theta^a \) are also superpotentials for the gravitational field described by the tetrad fields \( \{ \theta^a \} \)!

5 Conclusions

In this paper we proved using the Clifford bundle formalism that the energy-momentum densities \( \star \tau^a = \frac{1}{2} \star (F \theta^a F) \) of a free electromagnetic field \( F \) living on Minkowski spacetime are sources of exact 2-forms \( \mathcal{W}^a \) which satisfy Maxwell like equations \( d \mathcal{W}^a = 0, \ d \star \mathcal{W}^a = \star \tau^a \). Which this noticeable result we show that the free electromagnetic field may be interpreted as generating a (teleparallel) Weitzenböck spacetime \( \{(M, g, \nabla, \tau_g)\} \) or an effective Lorentzian spacetime.
\([\{M, g, D, \tau, \nu\}], \) whose properties are determined with details. In both structures the metric \(g = h^* \eta\) where \(h : M \to M\) is a conformal diffeomorphism given by Eq. (24).

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## A Clifford Bundles

Let \((M, \eta, D, \tau, \nu)\) be Minkowski spacetime. \((M, \eta)\) is a four dimensional space oriented (by the volume form \(\tau\)) and time oriented (by the equivalence relation \(\uparrow\), see \([2]\)) Lorentzian manifold, with \(M \simeq \mathbb{R}^4\) and \(\eta \in \sec T_0^1 M\) is a Lorentzian metric of signature \((1, 3)\). \(T^* M\) is the cotangent \(\langle\text{tangent}\rangle\) bundle. \(T^* M = \cup_{x \in M} T^*_x M, TM = \cup_{x \in M} T_x M, \) and \(T_2 M \simeq T^*_x M \simeq \mathbb{R}^{1,3}\), where \(\mathbb{R}^{1,3}\) is the Minkowski vector space. \(D\) is the Levi-Civita connection of \(\eta\), i.e., \(D \eta = 0, R(\eta) = 0\). Also \(\Theta(D) = 0, R\) and \(\Theta\) being respectively the torsion and curvature tensors. Let \(\eta \in \sec T^*_0 M\) be the metric on the cotangent bundle associated with \(\eta \in \sec T^*_0 M\). The Clifford bundle of differential forms \(\mathcal{C}(M, \eta)\) is the bundle of algebras, i.e., \(\mathcal{C}(M, \eta) = \cup_{x \in M} \mathcal{C}(T^*_x M)\), where \(\forall x \in M, \mathcal{C}(T^*_x M) = \mathbb{R}_{1,3}\), the so called spacetime algebra. Recall also that \(\mathcal{C}(M, \eta)\) is a vector bundle associated with the \(\eta\)-orthonormal frame bundle \(\mathbf{P}_{SO^{+}_{1,3}}(M, \eta)\), i.e., \(\mathcal{C}(M, \eta) = \mathbf{P}_{SO^{+}_{1,3}}(M) \times_{\text{ad}} \mathbb{R}_{1,3}\) (see more details in, e.g., [5] [8]). For any \(x \in M\), \(\mathcal{C}(T^*_x M)\) is a linear space over the real field \(\mathbb{R}\). Moreover, \(\mathcal{C}(T^*_x M)\) is isomorphic to the Cartan algebra \(\wedge T^*_x M\) of the cotangent space and \(\wedge T^*_x M = \bigwedge^k T^*_x M\), where \(\bigwedge^k T^*_x M\) is the \(k\)-dimensional space of \(k\)-forms. Then, sections of \(\mathcal{C}(M, \eta)\) can be represented as a sum of non homogeneous differential forms. Let \(\{x^\mu\}\) be coordinates in Einstein-Lorentz-Poincaré gauge for \(M\) and let \(\{e_\mu = \partial / \partial x^\mu\} \in \sec FM\) (the frame bundle) be an orthonormal basis for \(TM\), i.e., \(\eta(e_\mu, e_\nu) = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). Let \(\gamma^\nu = dx^\nu \in \sec \bigwedge^1 T^* M \rightarrow \sec \mathcal{C}(M, \eta) \ (\nu = 0, 1, 2, 3)\) such that the set \(\{\gamma^\nu\}\) is the dual basis of \(\{e_\mu\}\) and of course, \(\eta(\gamma^\mu, \gamma^\nu) = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)\). We introduce moreover the notations \(\vartheta^a = \delta^a_\mu dx^\mu\) and \(e_a = \delta^a_\mu \frac{\partial}{\partial x^\mu}\). We say that \(\{e_a\}\) is a section of the orthonormal frame bundle \(\mathbf{P}_{SO^+_{1,3}}(M, \eta)\) and its dual basis \(\{\vartheta^a\}\) is a section of the orthonormal coframe bundle (denoted \(\mathbf{P}_{SO^+_{1,3}}(M, \eta)\)).

### A.1 Clifford Product

The fundamental Clifford product (in what follows to be denoted by juxtaposition of symbols) is generated by \(\vartheta^a \vartheta^b + \vartheta^b \vartheta^a = 2\vartheta^{ab}\) and if \(C \in \mathcal{C}(M, \eta)\) we have...
\[ C = s + v_a \theta^a + \frac{1}{2!} b_{ab} \theta^a \theta^b + \frac{1}{3!} a_{abc} \theta^a \theta^b \theta^c + p \theta^5, \]  

where \( \tau_\eta := \theta^5 = \theta^0 \theta^1 \theta^2 \theta^3 = dx^0 dx^1 dx^2 dx^3 \) is the volume element and \( s, v_a, b_{ab}, a_{abc}, p \in \sec \bigwedge^0 T^*M \mapsto \sec \mathcal{O}(M, \eta). \)

Let \( \mathcal{A}_r, \in \sec \bigwedge^r T^*M \mapsto \sec \mathcal{O}(M, \eta), \mathcal{B}_s \in \sec \bigwedge^s T^*M \mapsto \sec \mathcal{O}(M, \eta). \) For \( r = s = 1, \) we define the scalar product as follows:

For \( a, b \in \sec \bigwedge^1 T^*M \mapsto \sec \mathcal{O}(M, \eta), \)

\[ a \cdot b = \frac{1}{2} (ab + ba) = \eta(a, b). \]

We define also the exterior product \( (\forall r, s = 0, 1, 2, 3) \) by

\[ \mathcal{A}_r \wedge \mathcal{B}_s = (\mathcal{A}_r \mathcal{B}_s)_{r+s}, \]
\[ \mathcal{A}_r \wedge \mathcal{B}_s = (-1)^{r \cdot s} \mathcal{B}_s \wedge \mathcal{A}_r, \]

where \( \langle \rangle_k \) is the component in \( \bigwedge^k T^*M \) (projection) of the Clifford field. The exterior product is extended by linearity to all sections of \( \mathcal{O}(M, \eta). \)

For \( \mathcal{A}_r = a_1 \wedge \ldots \wedge a_r, \mathcal{B}_r = b_1 \wedge \ldots \wedge b_r, \) the scalar product is defined here as follows,

\[ \mathcal{A}_r \cdot \mathcal{B}_r = (a_1 \wedge \ldots \wedge a_r) \cdot (b_1 \wedge \ldots \wedge b_r) = \begin{vmatrix} a_1 \cdot b_1 & \ldots & a_1 \cdot b_r \\ \ldots & \ldots & \ldots \\ a_r \cdot b_1 & \ldots & a_r \cdot b_r \end{vmatrix}. \]

We agree that if \( r = s = 0, \) the scalar product is simple the ordinary product in the real field.

Also, if \( r \neq s, \) then \( \mathcal{A}_r \cdot \mathcal{B}_s = 0. \) Finally, the scalar product is extended by linearity for all sections of \( \mathcal{O}(M, \eta). \)

For \( r \leq s, \mathcal{A}_r = a_1 \wedge \ldots \wedge a_r, \mathcal{B}_s = b_1 \wedge \ldots \wedge b_s \) we define the left contraction by

\[ \eta : (\mathcal{A}_r, \mathcal{B}_s) \mapsto \mathcal{A}_r \cdot \mathcal{B}_s = \sum_{i_1 < \ldots < i_r} \epsilon^{i_1 \ldots i_r} (a_1 \wedge \ldots \wedge a_r) \cdot (b_{i_1} \wedge \ldots \wedge b_{i_s}) \sim b_{i_r+1} \wedge \ldots \wedge b_{i_s}. \]

And extended by linearity to all sections of \( \mathcal{O}(M, \eta). \) We agree that for \( \alpha, \beta \in \sec \bigwedge^0 T^*M \) the contraction is the ordinary (pointwise) product in the real field and that if \( \alpha \in \sec \bigwedge^0 T^*M \mapsto \mathcal{O}(M, \eta), \mathcal{A}_r \in \sec \bigwedge^r T^*M \mapsto \mathcal{O}(M, \eta), \mathcal{B}_s \in \sec \bigwedge^s T^*M \mapsto \mathcal{O}(M, \eta) \) then \( (\alpha \mathcal{A}_r) \cdot \mathcal{B}_s = \mathcal{A}_r \cdot (\alpha \mathcal{B}_s). \) Left contraction
is extended by linearity to all pairs of elements of sections of \( \mathcal{O}(M, \eta) \), i.e., for \( A, B \in \text{sec} \mathcal{O}(M, \eta) \)

\[
A \cdot B = \sum_{r,s} \langle A \rangle_{r,\eta} \langle B \rangle_{s, \eta}, \quad r \leq s.
\] (54)

It is also necessary to introduce the operator of right contraction denoted by \( \langle \rangle \). The definition is obtained from the one presenting the left contraction with the imposition that \( r \geq s \) and taking into account that now if \( A_r \in \text{sec} \bigwedge^r T^*M \hookrightarrow \mathcal{O}(M, \eta), B_s \in \text{sec} \bigwedge^s T^*M \hookrightarrow \mathcal{O}(M, \eta) \) then \( A_r \cdot \langle B_s \rangle = (\langle a A_r \rangle)_{\eta, B_s} \).

The main formulas used in the present paper can be obtained (details may be found in \([S]\)) from the following ones (where \( a \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} \mathcal{O}(M, \eta) \)):

\[
a B_s = a \cdot B_s + a \wedge B_s, \quad B_s a = B_s \cdot a + B_s \wedge a,
\]

\[
a \cdot B_s = \frac{1}{2} (a B_s - (-1)^s B_s a),
\]

\[
A_r \cdot B_s = (-1)^{(r-s)B_s} B_s \cdot A_r,
\]

\[
a \wedge B_s = \frac{1}{2} (a B_s + (-1)^s B_s a),
\]

\[
A_r \cdot B_s = \langle A_r B_s \rangle_{[r-s]} + \langle A_r B_s \rangle_{[r-s]+2} + \ldots + \langle A_r B_s \rangle_{[r+s]}
\]

\[
= \sum_{k=0}^{m} \langle A_r B_s \rangle_{[r-s]+2k}
\]

\[
A_r \cdot B_r = B_r \cdot A_r = \tilde{A}_r \cdot B_r = A_r \cdot \tilde{B}_r = \langle A_r B_r \rangle_0 = \langle A_r \tilde{B}_r \rangle_0 = 0.
\] (55)

\[
\langle AB \rangle_r = (-1)^{(r-1)/2} \langle B \tilde{A} \rangle_r,
\]

\[
\langle A_r B_s \rangle_r = \langle \tilde{B}_s A_r \rangle_r = (-1)^{(s-1)/2} \langle B_s A_r \rangle_r,
\]

\[
\langle A_r B_s C_l \rangle_q = (-1)^{(l-1)/2} \langle C_l B_s A_r \rangle_q,
\]

\[
\epsilon = \frac{1}{2} (q^2 + r^2 + s^2 + t^2 - q - r - s - t)
\] (56)

A.2 Hodge Star Operator \( \star \)

Let \( \star \) be the Hodge star operator, i.e., the mapping

\[
\star : \bigwedge^k T^*M \rightarrow \bigwedge^{4-k} T^*M, \quad A_k \mapsto \star A_k
\]

where for \( A_k \in \text{sec} \bigwedge^k T^*M \hookrightarrow \mathcal{O}(M, \eta) \)

\[
[B_k \cdot A_k]_{\eta} = B_k \wedge \star A_k, \quad \forall B_k \in \text{sec} \bigwedge^k T^*M \hookrightarrow \text{sec} \mathcal{O}(M, \eta).
\] (57)
\( \tau_\eta = \theta^5 \in \bigwedge^4 T^*M \) is a standard volume element. Then we can verify that
\[
\star_{\eta} A_k = \bar{A}_k \tau_\eta = \bar{A}_k \theta^5. 
\]  

(58)

## A.3 \( \mathcal{C}(M, \eta), \mathcal{C}(M, g), h \) and \( h \)

### A.3.1 \( h \)

In this section \( h \) is a diffeomorphism \( h : M \rightarrow M, e \mapsto h e \) such that if \( h^* \Gamma^a = \theta^a \) and if \( \eta \in \text{sec} T^0_2 M \) then
\[
g = h^* \eta = \eta_{ab} \theta^a \otimes \theta^b, 
\]
where \( \{\theta^a\} \) is the \( g \)-orthonormal cobasis used in the main text.

### A.3.2 \( h \)

Consider the Clifford bundles of nonhomogeneous multiform fields \( \mathcal{C}(M, \eta) \) and \( \mathcal{C}(M, g) \). In \( \mathcal{C}(M, \eta) \) we denoted the Clifford product by juxtaposition of symbols, the scalar product by \( \cdot \) and the contractions by \( \lrcorner \) and \( \rhd \) we denote the Hodge dual. The Clifford product in \( \mathcal{C}(M, g) \) will be denoted by the symbol \( \lor \), the scalar product will be denoted by \( \triangleq \equiv \cdot g \) and the contractions by \( \lrcorner g \) and \( \rhd g \) while by \( \star g \) we denote the Hodge dual operator associated with \( g \).

Let \( \{e_a\} \) be a non coordinate basis of \( TM \) dual to the cobasis \( \{\theta^a\} \). We take the \( \theta^a \) as sections of the Clifford bundle \( \mathcal{C}(M, \eta) \), i.e., \( \theta^a \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, \eta) \). In this basis taking into account Eq.\((59)\) that \( g \in \text{sec} T^0_2 M \) is given by
\[
g = \eta_{ab} e_a \otimes e_b. 
\]
(60)

The cobasis \( \{\theta^a\} \) defines a Clifford product in \( \mathcal{C}(M, \eta) \) by
\[
\theta^a \theta^b + \theta^b \theta^a = 2 \eta_{ab}, 
\]
(61)  
and taking into account that the cobasis \( \{\theta^a\} \) defines a deformed Clifford product \( \lor \) in \( \mathcal{C}(M, \eta) \) (see details in [S]) generating a representation of the Clifford bundle \( \mathcal{C}(M, g) \) we can write
\[
\theta^a \lor \theta^b = \theta^a \circ \theta^b + \theta^a \land \theta^b, 
\]
\[
\theta^a \lor \theta^b + \theta^b \lor \theta^a = 2 \eta_{ab}. 
\]
(62)

Then, as proved, e.g., in [S] there exist \((1, 1)\)-extensor fields \( g \) and \( h^{-1} \) such that
\[
g(\theta^a, \theta^b) = \theta^a \circ \theta^b = \theta^a \cdot g(\theta^b) = h(\theta^a) \cdot h(\theta^b) = \eta_{ab}. 
\]
(63)

The gauge metric extensor \( h : \text{sec} \bigwedge^1 T^*M \rightarrow \text{sec} \bigwedge^1 T^*M \) is defined (modulo a Lorentz transformation) by
\[
h(\theta^a) = \theta^a. 
\]
(64)
A.3.3 Relation Between $h$ and $h^*$

Introduce coordinates functions $\{y^\mu\}$ in the Einstein-Lorentz-Poincaré gauge for $M$ such that $y^\mu = x^\mu$ (where the $\{x^\mu\}$ are the coordinate functions in the Einstein -Lorentz-Poincaré gauge already introduced above). Put

\[
x^\mu(\epsilon) = x^\mu, \quad y^\mu(h\epsilon) = y^\mu.
\]

If $y^\mu = h^\mu(x^\nu)$ is the coordinate expression\(^7\) for $h$, the coordinate expressions of $\eta$ at $h\epsilon$ and $g$ at $\epsilon$ can be written as:

\[
\eta|_{h\epsilon} = \eta|_\epsilon \delta^\alpha_a \delta^\beta_b dx^\alpha \otimes dy^\beta, \quad g|_{\epsilon} = g_{\mu\nu} dx^\mu \otimes dx^\nu.
\]

Since $h^* \eta(\delta^a_\alpha \partial/\partial x^\mu, \delta^b_\beta \partial/\partial x^\nu)|_\epsilon = \eta(\delta^a_\alpha h, \partial/\partial x^\mu, \delta^b_\beta (h, \partial/\partial x^\nu)|_h \epsilon$, we have

\[
g = h^* \eta = \eta|_\epsilon \delta^a_\alpha \delta^b_\beta \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} dx^\mu \otimes dx^\nu
\]

with

\[
g_{\mu\nu} = \eta|_\epsilon \delta^a_\alpha \delta^b_\beta \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}.
\]

Now, take notice that at $\epsilon$, $\{f_a\}$, $f_a = \delta^a_\alpha h^{-1} \partial/\partial y^\mu = \delta^a_\alpha \frac{\partial y^\mu}{\partial x^\nu} \frac{\partial}{\partial x^\nu}$ is not (in general) a coordinate basis for $TM$. It is also not $\eta$-orthonormal\(^8\) $\sigma^a$. The dual basis of $\{f_a\}$ at $\epsilon$ is $\{\sigma^a\}_\epsilon$, with $\sigma^a|_\epsilon = \delta^a_\alpha \frac{\partial y^\mu}{\partial x^\nu} dx^\mu \otimes dx^\nu = h^* (\delta^a_\alpha dy^\mu|_h \epsilon)$, Then it exists an extensor field $\hat{h}$ differing from $h$ by a Lorentz extensor, i.e., $\hat{h} = h\Lambda$ such that $\sigma^a|_\epsilon = \hat{h}^{-1}(\delta^a_\alpha dy^\mu)|_\epsilon = \hat{h}^{-1a}(dy^\mu)|_\epsilon$, where, we have for any $\epsilon \in M$,

\[
\delta^a_\alpha \frac{\partial y^\alpha}{\partial x^\mu} = \hat{h}^{-1a}_\mu.
\]

To determine $\hat{h}$ we proceed as follows. Suppose $g = \eta|_\epsilon \sigma^a \otimes \sigma^b$ is known. Let $(v_i, \lambda_i)$ be respectively the eigen-covectors and the eigenvalues of $g$, i.e., $g(v_i) = \lambda_i v_i$ (no sum in $i$) and $\{\sigma^a\}$ the $\eta$-orthonormal coordinate basis for $TM$ introduced above. Then, since $g = \hat{h}^2 = \hat{h}^2$ we immediately have

\[
\hat{h}(v_i) = \sqrt{\lambda_i} \eta(v_i, \sigma_a) \sigma^a,
\]

which then determines the extensor field $\hat{h}$ (modulus a local Lorentz rotation) at any spacetime point, and thus the diffeomorphism $h$ (modulus a local Lorentz rotation).

A.3.4 Relation Between $*$ and $*_{\eta}$

If $g = h^2$ we have that $\eta|_\epsilon \sigma^a \otimes \sigma^b$ for any $\omega_p \in \sec \wedge TM$

\[
*_{\eta} \omega_p = \Lambda \frac{\hat{h} \ast h^\ast \omega_p},
\]

where $\Lambda$ is the exterior power extension of $h$.

\(^7\)Recall that the $h^\mu$ are invertible differentiable functions.

\(^8\)Indeed, $\eta(e_a, e_b) = \delta^a_\alpha \delta^b_\beta \frac{\partial y^\mu}{\partial x^\nu} \frac{\partial y^\nu}{\partial x^\alpha} \lambda_{a\beta}$. 

16
A.4 Dirac Operator acting on Sections of a General Clifford Bundle $\mathcal{C}(M, g)$

Let $d$ and $\delta$ be respectively the differential and Hodge codifferential operators acting on sections of $\text{sec} \bigwedge^k T^*M \hookrightarrow \text{sec} \mathcal{C}(M, \eta)$. If $A_p \in \text{sec} \bigwedge^p T^*M \hookrightarrow \text{sec} \mathcal{C}(M, g)$, then $\delta A_p = (-1)^p \ast^{-1} d \ast A_p$, with $\ast^{-1} \ast = \text{identity}$.

The Dirac operator acting on sections of $\mathcal{C}(M, g)$ is the invariant first order differential operator

$$\partial = \theta^a D_e_a,$$

where $D_e_a$ is the Levi-Civita connection of $g$.

A.4.1 Useful Formula for Calculation of $D_e_a A$

The reciprocal basis of $\{\theta^b\}$ is denoted $\{\theta_a\}$ and we have $\theta_a \cdot \eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1)$. Also,

$$D_e_a \theta^b = -\omega_{ac}^b \theta^c = -\omega_{ac}^b \theta^c,$$

with $\omega_{ac}^b = -\omega_{ca}^b$, and $\omega_{ac}^b = \eta^{bk} \omega_{ak} \eta_{cl}$, $\omega_{abc} = \eta_{ad} \omega_{bc}^d = -\omega_{cba}$. Defining

$$\omega_a = \frac{1}{2} \omega_{ac}^b \theta^c \in \text{sec} \bigwedge^2 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, g),$$

we have (by linearity) that for any $A \in \text{sec} \bigwedge^p T^*M \hookrightarrow \text{sec} \mathcal{C}(M, g)$

$$D_e_a A = \partial_{e_a} A + \frac{1}{2} [\omega_a, A],$$

where $\partial_{e_a}$ is the Pfaff derivative.\footnote{E.g., if $A = \frac{1}{p!} A_{i_1 \ldots i_p} \theta^{i_1} \ldots \theta^{i_p}$ then $\partial_{e_a} A = \frac{1}{p!} [e_a (A_{i_1 \ldots i_p})] \theta^{i_1} \ldots \theta^{i_p}$.}

A.5 Dirac Operator $\partial = d - \delta$

Using Eq. (76) we can easily show the very important result:

$$\partial \wedge A = \partial \wedge A + \partial_{e_a} A = dA - \delta A,$$
$$\partial \wedge A = dA, \quad \partial_{e_a} A = -\delta A.$$

Remark 6 We will use the symbol $\partial$ for the Dirac operator acting on sections of $\mathcal{C}(M, \eta)$ over Minkowski spacetime. In this case we have with $\{e_a\}$ a $\eta$-orthonormal basis and $\{\theta^a\}$ its dual basis (as defined above)

$$\partial = \theta^a D_e_a$$
\[ \partial A = \partial \wedge A + \partial_\eta A = dA - \delta A, \]
\[ \partial \wedge A = dA, \quad \partial_\eta A = -\delta A. \]  
(77)

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