Quantum mechanics and low energy nucleon dynamics

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We discuss the problem of consistency of quantum mechanics as applied to low energy nucleon dynamics with the symmetries of QCD. It is shown that the dynamics consistent with these symmetries is not governed by the Schrödinger equation. We present a new way to formulate the effective theory of nuclear forces as an inevitable consequence of the basic principles of quantum mechanics and the symmetries of strong interactions. We show that being formulated in this way the effective theory of nuclear forces can be put on the same firm theoretical grounds as the quantum mechanics of atomic phenomena. In this case the effective theory allows one to describe with a given accuracy not only two-nucleon scattering, but also the evolution of nucleon systems, and places the constraints on the off-shell behavior of the two-nucleon interaction. In this way we predict the off-shell behavior of the S wave two-nucleon $T$-matrix at very low energies when the pionless theory is applicable. Further extensions and applications of this approach are discussed.

I. INTRODUCTION

Ideas from the foundations of quantum mechanics are being applied now to many branches of physics. In the quantum mechanics of particles interacting through the Coulomb potential one deals with a well-defined interaction Hamiltonian and the Schrödinger equation governing the dynamics of the theory. This theory is perfectly consistent and provides an excellent description of atomic phenomena at low energies. It is natural to expect that low energy nuclear physics can be described in the same way. However, one has not yet constructed a fundamental nucleon-nucleon (NN) potential. Nowadays there exist phenomenological NN potentials which successfully describe scattering data to high precision, but they do not emerge from QCD and contain ad hoc form factors. A first attempt to systematically solve the problem of low energy nucleon dynamics and construct a bridge to QCD was made by Weinberg. He suggested to derive a NN potential in time-ordered chiral perturbation theory (ChPT). However, such a potential is singular and the Schrödinger (Lippmann-Schwinger) equation does not make sense without regularization and renormalization. This means that in the effective field theory (EFT) of nuclear forces, which following the pioneering work of Weinberg has become very popular in nuclear physics (for a review, see Ref. 2), the Schrödinger equation is not valid. On the other hand, the whole formalism of fields and particles can be considered as an inevitable consequence of quantum mechanics, Lorentz invariance, and the cluster decomposition principle 3. Thus in the nonrelativistic limit QCD must produce low energy nuclear physics consistent with the basic principles of quantum mechanics. However, as it follows from the Weinberg analysis, QCD leads through ChPT to the low energy theory in which the Schrödinger equation is not valid. This means that either there is something wrong with QCD and ChPT or the Schrödinger equation is not the basic dynamical equation of quantum theory. Meanwhile, in Ref. 4 it has been shown that the Schrödinger equation is not the most general equation consistent with the current concepts of quantum physics, and a more general equation of motion has been derived as a consequence of the basic postulates of the Feynman 4 and canonical approaches to quantum theory. Being equivalent to the Schrödinger equation in the case of instantaneous interactions, this generalized dynamical equation permits the generalization to the case where the dynamics of a system is generated by a nonlocal-in-time interaction. The generalized quantum dynamics (GQD) developed in this way has proved an useful tool for solving various problems in quantum theory 5, 6.

The formalism of the GQD allows one to consider the problem of consistency of quantum mechanics with the low energy predictions of QCD from a new point of view. From this viewpoint, in investigating the consequences of ChPT we must not restrict ourselves to the assumption that the NN interaction can be parameterized by a NN potential, and low energy nucleon dynamics is governed by the Schrödinger equation. This dynamics may be governed by the generalized dynamical equation with a nonlocal-in-time interaction operator when this equation is not equivalent to the Schrödinger equation, and hence the above divergence problems may be the cost of trying to describe low energy nucleon dynamics in terms of Hamiltonian formalism while this dynamics is really non-Hamiltonian. In the present paper, by using the example of the $^{1}\text{S}_0$ channel, we will show that from the analysis of time-ordered diagrams for the $T$-matrix in ChPT it really follows that nucleon dynamics at very low energies is governed by the generalized dynamical equation with a nonlocal-in-time interaction operator. The GQD provides a new way to formulate the effective theory of nuclear forces as an inevitable consequence of the basic principles of quantum mechanics and the symmetries of QCD, and, as it will be shown, in this way one can construct the theory that is perfectly consistent and free from ultraviolet (UV) divergences.

The aim of the present paper is twofold. From the one hand, we intend to show that the low energy predictions of QCD are consistent with the basic principles of quantum mechanics, but in this case a new insight
into these principles provided by the formalism of the GQD is needed. On the other hand, it is our intention to demonstrate that this insight into the basic principles of quantum mechanics opens new possibilities for describing low energy nucleon dynamics. In Sec.II we briefly consider the main features of the formalism of the GQD developed by one of the authors (R.G.) in Ref. [4]. We mainly focus on the physical meaning of the generalized dynamical equation which is a direct consequence of the principle of the superposition of the probability amplitudes and the requirement that the evolution operator be unitary. This equation plays a key role in the present work.

In Sec.III we consider the pionless theory of nuclear forces. By focusing the attention on the $^1S_0$ channel, we show that the requirement that the two-nucleon ($2N$) $T$-matrix satisfy the generalized dynamical equation and be consistent with the symmetries of QCD completely determines its form. This means that the pionless effective theory can be formulated as an inevitable consequence of the symmetries of QCD and the basic principles of quantum mechanics, if this principles are entered into the theory via the generalized dynamical equation. In Sec.IV we show that, starting from the expression for the $2N$ $T$-matrix obtained in this way one can organize calculations of observables in the spirit of the effective theory, i.e., by expanding amplitudes in powers of observables in the spirit of the effective theory, i.e., by matrix obtained in this way one can organize calculations and the requirement that the evolution operator be unitary. This equation plays a key role in the present work.

In Sec.IV we show that the two-nucleon ($2N$) $T$-matrix satisfy the generalized dynamical equation and be consistent with the symmetries of QCD completely determines its form. This means that the pionless effective theory can be formulated as an inevitable consequence of the symmetries of QCD and the basic principles of quantum mechanics, if this principles are entered into the theory via the generalized dynamical equation. In Sec.IV we show that, starting from the expression for the $2N$ $T$-matrix obtained in this way one can organize calculations of observables in the spirit of the effective theory, i.e., by expanding amplitudes in powers of $Q/\Lambda$ where $Q$ being some low energy scale, and $\Lambda$ is the scale at which the theory is expected to break down. This makes it possible to calculate amplitudes up to some order in $Q/\Lambda$ keeping the terms in this expansion up to the corresponding order. In this way we reproduce all results for the $NN$ scattering amplitudes obtained in the standard EFT of nuclear forces. It is important that these results are reproduced starting with a well defined $2N$ $T$-matrix without resorting to the regularization and renormalization procedures. This shows that really the ChPT does not lead, in the nonrelativistic limit, to a low energy theory with UV divergences, i.e., low energy predictions of QCD are consistent with the principles of quantum mechanics.

In Sec.V we show that the use of the generalized dynamical equation allows one to formulate the effective theory as a perfectly consistent theory free from UV divergences. Being formulated in this way, the effective theory keeps all advantages of the traditional nuclear physics approach. At the same time, its advantage over the traditional approach is that it allows one to find constraints on the off-shell behavior of the $2N$ $T$-matrix that are placed by the symmetries of QCD. In contrast, the realistic potentials that has been developed within the traditional approach cannot guarantee a reliable off-shell extrapolation of the $2N$ $T$-matrix since they are all constrained by the $2N$ phase shift analysis.

In Sec.VI we show that, in despite of the wide-spread opinion, from the first principles of quantum mechanics it follows that in general renormalization is not necessary to separate the low energy physics that is described by the effective theory of nuclear forces from the underlying high energy physics. An important feature of the generalized dynamical equation is that it provides such a separation without renormalization. The only thing that is needed for this is that at low energies nucleons emerge as the only effective degrees of freedom.

In Sec.VII the proposed formalism is considered from the point of view of the Weinberg program for physics of the two-nucleon systems. We show that our formulation of the effective theory of nuclear forces can be considered as a new way of the realization of the Weinberg program. This way leads to a perfectly consistent effective theory that allows one not only to calculate the $2N$ scattering amplitudes but $2N$ $T$-matrix and hence the evolution and Green operators. Moreover, the effective theory developed in this way allows one to determine the off-shell behavior of the $2N$ $T$-matrix as an inevitable consequence of the first principles of quantum mechanics and the symmetries of QCD. This is very important, because, as is well known, the off-shell behavior of the $2N$ $T$-matrix may play a crucial role in solving the many-nucleon problem and is an important factor in calculating in-medium observables [8] and in microscopic nuclear structure calculations. This results, for example, in the fact that the predictions by the Bonn potential for nuclear structure problems differ in a characteristic way from the ones obtained with local realistic potentials [9]. The off-shell ambiguities of realistic $NN$ potentials are argued to be one of the main causes of many problems in describing three-nucleon systems. One of such problems is the $A_y$ puzzle which refers to the inability to explain the nucleon vector analyzing power $A_y$ in elastic nucleon-deuteron ($Nd$) scattering at low energy. As has been shown in Ref. [10], it is not possible with reasonable changes in realistic $NN$ potentials to increase the $Nd$ $A_y$ and at the same time to keep the $2N$ observables unchanged. The same situation also takes place in the case of chiral potentials [11]. This means that the introduction of three-nucleon forces (3NF) is needed for resolving the problem. However, as it turned out, conventional 3NF’s change the predictions for $Nd$ $A_y$ only slightly and do not improve them [12-14]. This motivated new types of 3NF’s [15-17]. On the other hand, reliable 3N calculations and even testing for the presence of 3N forces require to constrain the off-shell properties of the $2N$. Moreover, these properties play a crucial role in a new 3NF proposed by Canton and Schadow [18]. These aspects of the three-nucleon problem are discussed in Sec.VI. We show that the fact that the GQD allows one to predict the off-shell behavior of the $2N$ $T$-matrix, by using the symmetry constrains placed by a QCD, may open new possibilities for solving three-body problem in nuclear physics.

II. GENERALIZED QUANTUM DYNAMICS

The basic concept of the canonical formalism of quantum theory is that it can be formulated in terms of vectors of a Hilbert space and operators acting on this
space. In this formalism the postulates that establish the connection between the vectors and operators and states of a quantum system and observables are used together with the dynamical postulate according to which the time evolution of a quantum system is governed by the Schrödinger equation. In the Feynman formalism quantum theory is formulated in terms of probability amplitudes without resorting to vectors and operators acting on a Hilbert space. Feynman’s theory starts with an analysis of the phenomenon of quantum interference. The results of this analysis which leads directly to the concept of the superposition of probability amplitudes are summarized by the following postulate [3]:

\[ \langle \psi | U(t, t_0) | \varphi \rangle = \langle \psi | \varphi \rangle + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \langle \psi | \tilde{S}(t_2, t_1) | \varphi \rangle. \]

Here \( \langle \psi | \tilde{S}(t_2, t_1) | \varphi \rangle \) is the probability amplitude that, if at time \( t_1 \) the system was in the state \( | \varphi \rangle \), then the interaction in the system will begin at time \( t_1 \) and end at time \( t_2 \) and at this time the system will be in the state \( | \psi \rangle \). By using the operator formalism, one can represent amplitudes \( \langle \psi | U(t, t_0) | \varphi \rangle \) by the matrix elements of the unitary evolution operator \( U(t, t_0) \) in the interaction picture. The operator \( \tilde{S}(t_2, t_1) \) represents the contribution to the evolution operator from the process in which the interaction in the system begins at time \( t_1 \) and ends at time \( t_2 \). As has been shown in Ref. [4], for the evolution operator \( \tilde{S} \) to be unitary for any \( t \) and \( t_0 \) the operator \( \tilde{S}(t_2, t_1) \) must satisfy the equation

\[ (t_2 - t_1) \tilde{S}(t_2, t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 (t_4 - t_3) \tilde{S}(t_2, t_4) \tilde{S}(t_3, t_1). \]

A remarkable feature of this equation is that it works as a recurrence relation and allows one to obtain the operators \( \tilde{S}(t_2, t_1) \) for any \( t_1 \) and \( t_2 \), if \( \tilde{S}(t_2', t_1') \) corresponding to infinitesimal duration times \( \tau = t_2' - t_1' \) of interaction are known. It is natural to assume that most of the contribution to the evolution operator in the limit \( t_2 \to t_1 \) comes from the processes associated with the fundamental interaction in the system under study. Denoting this contribution by \( H_{int}(t_2, t_1) \) we can write

\[ \tilde{S}(t_2, t_1) \xrightarrow{t_2 \to t_1} H_{int}(t_2, t_1) + o(\tau^2). \]

where \( \tau = t_2 - t_1 \). The parameter \( \varepsilon \) is determined by demanding that \( H_{int}(t_2, t_1) \) called the generalized interaction operator must be so close to the solution of Eq. [2] in the limit \( t_2 \to t_1 \) that this equation has a unique solution having the behavior [3] near the point \( t_2 = t_1 \). If \( H_{int}(t_2, t_1) \) is specified, Eq. [2] allows one to find the operator \( \tilde{S}(t_2, t_1) \), and hence the evolution operator. Thus Eq. [2] which is a direct consequence of the principle of the superposition can be regarded as an equation of motion for states of a quantum system. This equation allows one to construct the evolution operator by using the contributions from fundamental processes as building blocks. In the case of Hamiltonian dynamics the fundamental interaction is instantaneous. The generalized interaction operator describing such an interaction is of the form

\[ H_{int}(t_2, t_1) = -2i \delta(t_2 - t_1) H_{1}(t_1) \]

(4)

(the delta function \( \delta(t_2 - t_1) \) emphasizes that the interaction is instantaneous). In this case Eq. [2] is equivalent to the Schrödinger equation [4] and the operator \( H_{1}(t) \) is an interaction Hamiltonian (in the interaction picture).
At the same time, Eq. (2) permits the generalization to the quantum system where the fundamental interaction in a quantum system is nonlocal in time, and hence the dynamics is non-Hamiltonian.

By using Eq. (1), for \( U(t, t_0) \), we can write

\[
U(t, t_0) = 1 + i \int_{-\infty}^{\infty} dx \exp[-i(z - H_0)t] \times (z - H_0)^{-1}T(z)(z - H_0)^{-1} \exp[i(z - H_0)t_0],
\]

where \( z = x + iy, y > 0 \), and \( H_0 \) is the free Hamiltonian. The operator \( T(z) \) is defined by

\[
T(z) = i \int_0^\infty d\tau \exp(i\tau \hat{T}(\tau)),
\]

where \( \hat{T}(\tau) = \exp(-iH_0t_2)\hat{S}(t_2, t_1)\exp(iH_0t_1) \), and \( \tau = t_2 - t_1 \). In terms of the \( T \)-matrix defined by Eq. (6) the equation of motion (2) can be rewritten in the form (3)

\[
\frac{d\langle \psi_2 | T(z) | \psi_1 \rangle}{dz} = - \sum_n \frac{\langle \psi_2 | T(z) | n \rangle \langle n | T(z) | \psi_1 \rangle}{(z - E_n)^2},
\]

where \( n \) stands for the entire set of discrete and continuous variables that characterize the system in full, and \( | n \rangle \) are the eigenvectors of \( H_0 \). As it follows from Eq. (4), the boundary condition on this equation is of the form

\[
\langle \psi_2 | T(z) | \psi_1 \rangle \bigg|_{z \rightarrow \infty} = \langle \psi_2 | B(z) | \psi_1 \rangle + o(|z|^{-\beta}) = \langle \psi_2 | B(z) | \psi_1 \rangle + O \{ h(z) \}, \quad \beta = \varepsilon + 1.
\]

Equation (8) with this boundary condition is equivalent to the Lippmann-Schwinger (LS) equation with the interaction Hamiltonian \( H_I \). By definition, the operator \( B(z) \) represents the contribution which \( H_{int}^{(s)}(\tau) \) gives to the operator \( T(z) \), and is the interaction operator in the energy representation. This operator must be so close to the relevant solution of Eq. (7) in the limit \( |z| \rightarrow \infty \) that this differential equation has a unique solution having the asymptotic behavior \( \delta \). For this the operator \( B(z) \) must satisfy the condition

\[
\frac{d\langle \psi_2 | B(z) | \psi_1 \rangle}{dz} = - \sum_n \frac{\langle \psi_2 | B(z) | n \rangle \langle n | B(z) | \psi_1 \rangle}{(z - E_n)^2} + o(|z|^{-\beta-1}), \quad |z| \rightarrow \infty.
\]

From Eq. (5) it follows that the evolution operator in the Schrödinger picture can be represented in the form

\[
U_s(t, 0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt)G(z),
\]

where \( z = x + iy, y > 0 \), and

\[
G(z) = G_0(z) + G_0(z)T(z)G_0(z)
\]

with \( G_0(z) = \frac{1}{z - H} \). Being equivalent to representation (1), Eqs. (12) and (13) express the principle of the superposition of probability amplitudes. In the ordinary quantum mechanics the similar equation plays an important role and establishes the connection between the evolution operator and the Green operator \( G(z) \) which is defined by

\[
G(z) = \frac{1}{z - H},
\]

with \( H \) being the total Hamiltonian. Such a form of the Green operator follows from the fact that in the Hamiltonian formalism the evolution operator satisfies the Schrödinger equation. In the canonical formalism the \( T \)-matrix is defined by Eq. (13) starting with the Green operator of the form (14). In the formalism of the GQD the \( T \)-matrix plays a more fundamental role. It is defined by Eq. (10), and in this case the starting point is representation (1) with \( \hat{S}(t_2, t_1) \) being the contribution to the evolution operator from the processes in which the interaction begins at time \( t_1 \) and ends at time \( t_2 \). In this case the operator \( G(z) \) itself is defined by Eq. (13) via the \( T \)-matrix. This is a more general definition of the Green operator, since representation (1) is a consequence of the principle of the superposition of probability amplitudes and must be valid in any case while the evolution operator can be represented in the form (12) with the operator \( G(z) \) given by Eq. (13) only in the case where the interaction in a system is instantaneous.

As has been shown in Ref. [3], there is a one-to-one correspondence between the character of the dynamics and the large-momentum behavior of the \( T \)-matrix. If this behavior satisfies the requirements of ordinary quantum mechanics, then the interaction in a system is instantaneous and the dynamics is Hamiltonian. In the case where this behavior is "bad", i.e., does not meet the requirements of the ordinary quantum mechanics, the interaction generating the dynamics of the system must necessarily be nonlocal-in-time. Let us now illustrate this point by using the model developed in Refs. [4, 11] as a test model demonstrating the possibility of the extension of quantum dynamics provided by the GQD. This model describes the evolution of the system of two identical non-relativistic particles whose interaction is separable, and hence the interaction operator has the form

\[
< P_2 | H_{int}^{(s)}(\tau) | P_1 > = \varphi^*(P_2)\varphi(P_1)f(\tau),
\]
where $p$ is the relative momentum of the particles, and $f(\tau)$ is some function of the duration time $\tau$ of the interaction in the system. It is assumed that in the limit $|p| \to \infty$ the form factor $\varphi(p)$ behave as

$$\varphi(p) \sim |p|^{-\alpha}, \quad (|p| \to \infty).$$

(16)

In this case the general solution of Eq. (7) is

$$\langle p_2|T(z)|p_1 \rangle = \frac{\varphi^*(p_2)\varphi(p_1)}{g_1^{-1} + (z - a)} \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{|z - E_k|},$$

(17)

where $g_1 = t(a)$, and $a \in (-\infty, 0]$. In the case $\alpha > \frac{1}{2}$, the amplitude $\langle p_2|T(z)|p_1 \rangle$ given by Eq. (17), tends to $\lambda \varphi^*(p_2)\varphi(p_1)$, where

$$\lambda = \left( g_1^{-1} + \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{a - E_k} \right)^{-1}.$$ 

From Eqs. (8) and (9) it follows that in this case the interaction operator $H_{int}^{(s)}(\tau)$ should be of the form

$$< p_2|H_{int}^{(s)}(\tau)|p_1 > = -2i\delta(\tau)\varphi^*(p_2)\varphi(p_1),$$

and hence the dynamics of the system is Hamiltonian and is governed by the Schrödinger equation.

In the case $\alpha < \frac{1}{2}$, the T-matrix (17) tends to zero as $|z| \to \infty$

$$\langle p_2|T(z)|p_1 \rangle \underset{|z| \to \infty}{\to} \varphi^*(p_2)\varphi(p_1)$$

$$\times \left( b_1(-z)^{-\alpha} + b_2(-z)^{2\alpha - 1} \right) + O(|z|^{2\alpha - 1}),$$

(18)

where $b_1 = -4\pi \cos(\alpha\pi)m^{-\frac{2}{3}}$ and $b_2 = b_1 + a_2(\hat{M}(a) + g_1^{-1})$ with

$$\hat{M}(a) = \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2 - |k|^{-2\alpha}}{a - E_k}.$$ 

In this case the dynamics is non-Hamiltonian and, as it follows from Eqs. (8) and (9), is generated by the nonlocal-in-time interaction operator

$$< p_2|H_{int}^{(s)}(\tau)|p_1 > = \varphi^*(p_2)\varphi(p_1)$$

$$\times \left( a_1(-z)^{-\alpha} + a_2(-z)^{2\alpha - 2\alpha} \right),$$

(19)

where $a_1 = 4\pi \cos(\alpha\pi)m^{-\frac{2}{3}}\Gamma^{-1}(\alpha - \frac{1}{2})\exp[i(-\frac{1}{2} + \frac{1}{2})\pi]$, and $a_2$ is a free parameter of the theory. The solution of Eq. (7) with the interaction operator (19) is of the form

$$\langle p_2|T(z)|p_1 \rangle = \frac{b_1^2 \varphi^*(p_2)\varphi(p_1)}{-b_2 + b_1(-z)^{2\alpha} - \hat{M}(z)b_1^2}.$$ 

(20)

### III. THE PIONLESS THEORY OF NUCLEAR FORCES

The Weinberg program for low energy nucleon physics employs the analysis of time-ordered diagrams for the $2N$ T-matrix in ChPT to derive a $NN$ potential and then to use it in the LS equation for constructing the full $NN$ T-matrix. Obviously the starting point for this program is the assumption that in the nonrelativistic limit ChPT leads to low energy nucleon dynamics which is Hamiltonian and is governed by the Schrödinger equation. However, the fact that the chiral potentials constructed in this way are singular and lead to UV divergences means that this assumption has not corroborated. At the same time, the GQD allows one to analyze the predictions of ChPT without making a priori assumptions about the character of low energy nucleon dynamics: This character should results from the analysis. Let us consider, for example, the low energy predictions of ChPT for the $2N$ system in the $1S_0$ channel. At very low energy, even the pion field can be integrated out, and the diagrams of the ChPT take the form of the diagrams being produced by the effective Lagrangian containing only contact interactions among nucleons and derivatives thereof [24]. From the analysis of time-ordered diagrams of this theory it follows that the $2N$ T-matrix in the $1S_0$ channel must be of the form

$$\langle p_2|T(z)|p_1 \rangle = \sum_{n,m=0}^{\infty} p_2^n p_1^m t_{nm}(z),$$

(21)

where $p_i$ is relative momentum of nucleons, and the terms $t_{nm}(z)$ are of order $|t_{nm}(z)| \sim O \{\Lambda^{-2(n+m)}\}$, with the expansion parameter $\Lambda$ being set by the pion mass. Obviously, chiral symmetry play no role in this case. However, for our analysis it is important that at extreme low energies ChPT gives rise to this result.

From Eq. (21) it follows that at leading order the $2N$ T-matrix is of the form

$$\langle p_2|T^{(0)}(z)|p_1 \rangle = t_{00}(z).$$

(22)

On the other hand, this T-matrix must satisfy the generalized equation of motion, and this requirement determines its form up to one arbitrary parameter $i$

$$\langle p_2|T^{(0)}(z)|p_1 \rangle = \left( C_0^{-1} - \frac{m \sqrt{-zm}}{4\pi} \right)^{-1},$$

(23)

where $m$ being nucleon mass. Note that the standard EFT approach yields the same expression for the leading order T-matrix [21, 22]. It is remarkable that the T-matrix [23], which in the EFT approach is obtained by performing regularization and renormalization of the solution of the Schrödinger (LS) equation, is not a solution of this equation. At the same time, it is a solution of Eq. (7) corresponding to the interaction operator $\varphi(p)$.
where \( \gamma = 4 \pi \exp(-i \pi/4)m^{-3/2}C_0^{-1} \). This operator is nonlocal in time and this is the only reason why in this case the generalized dynamical equation cannot be reduced to the LS equation, and hence the T-matrix is not its solution.

Let us now consider the full \( 2N \) T-matrix including all the terms in Eq. (21). From Eqs. (7) and (21) it follows that the functions \( t_{nm}(z) \) satisfy the equations

\[
\frac{dt_{nm}(z)}{dz} = -t^2(z) \int \frac{d^3k}{(2\pi)^3} \frac{\psi_2(k\Lambda)\psi_1(k\Lambda)}{(z - E_k)^2},
\]

with

\[
\psi_1(k\Lambda) = 1 + \sum_{n=1}^{\infty} c_{2n} k^{2n},
\]

\[
\psi_2(k\Lambda) = 1 + \sum_{n=1}^{\infty} c_{2n} k^{2n}.
\]

The form of the functions \( \psi_1(k\Lambda) \) and \( \psi_2(k\Lambda) \) manifests the fact that the constants \( c_{2n} \) and \( c_{2n}' \) are of order \( O(\Lambda^{-2n}) \), and hence the relevant domain of the definition of these functions corresponds to the scale \( \Lambda \). As has been shown in Ref. [11] for the T-matrix being a solution of Eq. (20) to be unitary, it must satisfy the condition

\[
\langle p_2|T^+(z)|p_1 \rangle = \langle p_2|T(z)|p_1 \rangle, \quad z \in (-\infty, 0).
\]

From this and Eq. (27) it follows that

\[
\psi_2(k\Lambda) = \psi_1^*(k\Lambda), \quad c_{2n} = c_{2n}'^*.
\]

Now we can rewrite Eq. (27) in the form

\[
\frac{dt(z)}{dz} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{(z - E_k)^2} + \int \frac{d^3k}{(2\pi)^3} \frac{|\psi(k\Lambda)|^2 - 1}{(z - E_k)^2},
\]

where \( \tilde{t}(z) = t^{-1}(z) \) and

\[
\psi(k\Lambda) \equiv \psi_1(k\Lambda) = \sum_{n=0}^{\infty} c_{2n} k^{2n},
\]

with \( c_0 = 1 \). The solution of this equation with the boundary condition \( \tilde{t}(z = 0) = t_0 \) is

\[
\tilde{t}(z) = \tilde{t}_0 - zm^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{(zm - k^2)k^2} - M(z),
\]

where

\[
M(z) = zm^2 \int \frac{d^3k}{(2\pi)^3} \frac{F_1(k\Lambda)}{(zm - k^2)k^2},
\]

with \( F_1(k\Lambda) = |\psi(k\Lambda)|^2 - 1 \). Hence

\[
t(z) = \left[ C_0^{-1} - \frac{m}{4\pi} \sqrt{z/2m - M(z)} \right]^{-1},
\]

where \( C_0 = \tilde{t}_0^{-1} \). Thus from the requirement that the \( 2N \) T-matrix having the structure (21) satisfy the generalized dynamical equation it follows that this T-matrix should be of the form

\[
\langle p_2|T(z)|p_1 \rangle = \frac{\psi^*(p_2\Lambda)\psi(p_1\Lambda)}{C_0^{-1} - \frac{m}{4\pi} \sqrt{z/2m - M(z)}}.
\]

IV. THE EXPANSION OF THE EFFECTIVE THEORY.

As we have shown, the requirement that low energy nucleon dynamics be governed by the generalized dynamical equation places additional constraints on the form of the \( 2N \) T-matrix. This requirement together with the requirement that in the \( ^1S_0 \) channel the \( 2N \) T-matrix have the structure (21) yield the expressions (24) for this T-matrix which can be used for constructing the scattering amplitude and the evolution operator. The only problem is that we do not know the values of the parameters \( c_{2n} \). One may hope that in future it will be possible to derive them from QCD. Another way is to obtain these parameters from low energy experiment. Of course, in this way one has to restrict oneself to a few parameters that should be fitted to experiment. In other words, one has to use Eq. (29) in the spirit of the effective theory of nuclear forces, keeping only terms in the theory up to some order in \( Q/\Lambda \) in calculations of observables up to the corresponding order.

In the EFT approach the scattering amplitude, which is given by the equation

\[
A(p) = \frac{4\pi}{m} \frac{1}{p \cot \delta - i\rho}
\]

(here and below we focus on the \( ^1S_0 \) partial wave), is described in terms of an expansion in powers of momentum over the characteristic short-distance scale \( \Lambda \). It is well known that the quantity \( p \cot \delta \) has a nice momentum expansion for \( p << \Lambda \) known as the The Effective Range Expansion (ERE):

\[
p \cot \delta = -\frac{1}{a} + \frac{1}{2} \sum_{n=0}^{\infty} r_n p^{2(n+1)}.
\]

The scattering length \( a \) can be arbitrary large, while the other length scales \( r_0, r_1, \ldots \) are assumed to all have natural sizes, \( r_n \sim 1/\Lambda^{(2n+1)} \). Substituting Eq. (31) in Eq. (30) we can expand the scattering amplitude about \( p = 0 \):

\[
A(p) = -\frac{4\pi a}{m} \left[ 1 - i\rho + (ar_0/2 - a^2)p^2 \right] + O(p^3/\Lambda^3).
\]
The radius of convergence of this expansion is about $1/|a|$, and hence in the case $|a| > > 1/\Lambda$, which is of relevance to realistic $NN$ scattering, the expansion shown in Eq. (31) breaks down for momenta far below $\Lambda$. In this case, one needs to expand the scattering amplitude in powers of $p/\Lambda$ while retaining $ap$ to all orders:

$$A(p) = -\frac{4\pi}{m} \frac{1}{a-1+ip} \left[ 1 + \frac{r_0/2}{a-1+ip} p^2 + \frac{(r_0/2)^2}{(a-1+ip)^2} p^4 + \frac{r_1/2}{a-1+ip} p^4 + \ldots \right].$$

This is the Kaplan-Savage-Wise (KSW) expansion of the effective theory (KSW) expansion of the effective theory. It is the expansion that we wish to reproduce starting with the expression for the $T$-matrix given by Eq. (29). Using Eq. (29), for the scattering amplitude

$$A(p) = -(p | T(z = E_p + i\epsilon) | p),$$

we can write

$$A(p) = -\frac{|\psi(p/\Lambda)|^2}{C_0^{-1} + \frac{m}{4\pi} ip - M(E_p)}.$$

In order to perform the momentum expansion of this amplitude, let us represent $M(z)$ as follows

$$M(z) = zm^2 \int \frac{3d^3k}{(2\pi)^3} \left( zm - k^2 \right)f_k = -zm^2 \int \frac{3d^3k}{(2\pi)^3} \left( k^4 \right) + z^2m^3 \int \frac{3d^3k}{(2\pi)^3} \left( zm - k^2 \right)^2 f_k = \ldots = -\sum_{n=1}^{\infty} \left( zm \right)^n \mathcal{J}_n - \frac{m4\pi}{\sqrt{-zm}} \sum_{i=0}^{n} C_{2i} C_{2(n-i)} \left( zm \right)^n = \sum_{n=1}^{\infty} M_n(z),$$

with

$$M_n(z) = -(zm)^n \mathcal{J}_n + \frac{m4\pi}{\sqrt{-zm}} \left( zm \right)^n \sum_{i=0}^{n} C_{2i} C_{2(n-i)}.$$

Equation (35) allows one to expand the scattering amplitude as

$$A(p) = -\frac{C_0}{1 + \frac{4\pi a}{m} C_0} - \frac{C_2 p^2}{\left( 1 + \frac{4\pi a}{m} \right) C_0} + \frac{m}{4\pi} \left( C_2 \right)^2 p^5 \left( 1 + \frac{4\pi a}{m} C_0 \right) + \ldots$$

Comparing with Eq. (32), this expression relate the couplings $C_{2n}$ to the low energy scattering data $a, r_n$:

$$C_0 = \frac{4\pi a}{m}, \quad C_2 = \frac{2\pi}{m} a^2 r_0, \quad C_4 = \frac{4\pi a^3}{m} \left( \frac{r_0^2}{4} + \frac{r_1}{2a} \right), \ldots$$

By using the expression

$$\frac{1}{\infty} \sum_{n=0}^{\infty} C_{2n}(\mu) p^{2n} + \frac{m}{4\pi} \mu = \frac{1}{\infty} \sum_{n=0}^{\infty} C_{2n} p^{2n},$$

which relates couplings $C_{2n}(\mu)$ to $C_{2n}(0) = C_{2n}$ as

$$C_{2n}(\mu) = \frac{C_{2n} + \frac{m}{4\pi} \mu \sum_{i=0}^{n-1} C_{2i}(\mu) C_{2(n-i)}}{1 - \frac{m}{4\pi} \mu \mu_{C_0}},$$

Eq. (36) can be rewritten in the form

$$A(p) = -\frac{\sum_{n=0}^{\infty} C_{2n}(\mu) p^{2n}}{1 + \frac{m}{4\pi} (ip + \mu) \sum_{n=0}^{\infty} C_{2n} p^{2n}}.$$

By using Eq. (34), we can represent the scattering amplitude in the form

$$A(p) = -\frac{\sum_{n=0}^{\infty} C_{2n} p^{2n}}{1 + \frac{m}{4\pi} ip + \sum_{n=0}^{\infty} C_{2n} p^{2n}},$$

where $C_{2n}$ are determined by the recurrence relation

$$C_{2n} = C_0 \left( \sum_{i=0}^{n} C_{2i} C_{2(n-i)} - \sum_{j=1}^{n} C_{2(n-j)} \mathcal{J}_j \right).$$
This equation is exactly what Kaplan, Savage, and Wise have derived in the pionless theory by using the PDS subtraction scheme with the subtraction point \( p^2 = -\mu^2 \), and Eq. (41) yields the following expressions for \( C_0(\mu) \) and \( C_2(\mu) \):

\[
C_0(\mu) = \left( \frac{1}{C_0} - \frac{m}{4\pi \mu} \right)^{-1} = \frac{4\pi/m}{1/a - \mu},
\]

\[
C_2(\mu) = C_2 \left( \frac{C_0(\mu)}{C_0} \right)^2 = \frac{4\pi}{m} \frac{r_0/2}{(1/a - \mu)^2}.
\]

As for the couplings \( C_{2n}(\mu) \) for \( n > 2 \), from Eq. (41) it follows that they satisfy the equation

\[
\mu \frac{d}{d\mu} C_{2n}(\mu) = \frac{m\mu}{4\pi} \sum_{m=0}^{n} C_{2m}(\mu)C_{2(n-m)}(\mu),
\]

i.e., satisfy the complete, coupled RG equation.

We have demonstrated that the expression for the 2N \( T \)-matrix shown in Eq. (24) can be used for calculating the scattering amplitude up to some order in \( Q/\Lambda \). As we have seen, Eq. (24) reproduces the same expansion of the scattering amplitude that in the standard pionless EFT is derived by summing the bubble diagrams and performing regularization and renormalization. On the other hand, in general one cannot restrict oneself only to the scattering amplitudes. In the next section we will show that the formalism of the GQD provides a new way to formulate the effective theory of nuclear forces as a perfectly consistent theory that allows one to construct not only scattering amplitudes but also the off-shell 2N \( T \)-matrix, and the Green and evolution operators.

V. THE EFFECTIVE THEORY OF NUCLEAR FORCES.

As follows from Eq. (49), in order to describe the low energy dynamics we have to obtain the \( T \)-matrix for any \( z \) relevant for the low energy theory. Equation (49) allows one to obtain the operator \( T(z) \) for any \( z \) provided that some boundary condition on this equation is specified. The boundary condition (49) means that the most of the contribution to the operator \( T(z) \) in the limit \( |z| \to \infty \) comes from the processes associated with the fundamental interaction in the system under study that are described by the operator \( B(z) \). On the other hand, for the low energy theory to be consistent, the operator \( B(z) \) must be determined in terms of low energy degrees of freedom. This means that, despite the boundary condition (49) formally implies that \( z \) must be let to infinity, really one has to restrict oneself to a "high" energy region which is much above the low energy scale but is well below the scale of the underlying high energy physics. We will use \( D \) to denote this "high" energy region. The fact that the energy region \( D \) lies much above the scale of the low energy dynamics implies that at such "infinite" \( z \) the most of the contribution to the operator \( T(z) \) comes from processes that can be thought of as a "fundamental" interaction in the low energy theory, and, as a result, the interaction operator \( B(z) \) is close enough to the true 2N \( T \)-matrix. Thus, in order to take into account the fact that any theory has a range of validity, instead of the boundary condition (49), we have to use the following boundary condition on Eq. (47):

\[
\langle \psi_2| T(z) |\psi_1 \rangle = \langle \psi_2| B(z) |\psi_1 \rangle + O \left( |h(z)| \right), \quad z \in D.
\]

For Eq. (47) with the boundary condition (49) has a unique solution the interaction operator must be close enough inside domain \( D \) to the true 2N \( T \)-matrix. This means that it must satisfy the equation

\[
\frac{d}{dz} \langle \psi_2| B(z) |\psi_1 \rangle = - \sum_n \frac{\langle \psi_2| B(z) |n \rangle \langle n| B(z) |\psi_1 \rangle}{(z - E_n)^2} + O \left( \{|z|^{-1} h(z)| \right) \right), \quad z \in D.
\]

In the low energy theory of nuclear forces all processes in ChPT are described by irreducible diagrams involving only two external nucleons can be considered as such "fundamental" processes. Here irreducible diagrams are 2N irreducible: Any intermediate state contains at least one pion or isobar. It is natural to expect that the relative contribution of reducible 2N diagrams tends to zero as \( z \) increases, and, in a region that lies much above the scale of low energy nucleon dynamics, the main contribution comes from the "fundamental" processes that are described by the 2N irreducible diagrams, i.e., the operator \( B(z) \) becomes very close to the true 2N \( T \)-matrix. On the other hand, because of the separation of scales provided by QCD the above "high" energy region of the low energy theory lies still much below the scale of the underlying physics. In other words, the interaction operator and amplitudes \( \langle \psi_2| B(z) |\psi_1 \rangle \) that constitute the interaction operator and generate low energy nucleon dynamics really are low energy (in the scale of the underlying theory) amplitudes that in ChPT are described by the irreducible 2N diagrams. In principle they can be obtained within the underlying high energy theory in some low energy limit. It is important that the matrix elements of the \( NN \) interaction operator are just the amplitudes that are described by these diagrams, and hence there is no a need to make a priori assumption that what we have to extract from the analysis of the diagrams in ChPT is the \( NN \) potential. The amplitudes that are described by these diagrams can be directly used as building blocks for constructing the low energy theory. Thus the GQD allows one to build a bridge between QCD and low energy nucleon dynamics. It is hoped that in the future it will be possible to obtain the amplitudes \( \langle \psi_2| B(z) |\psi_1 \rangle \) in terms of QCD with such accuracy that the corresponding operator \( B(z) \) will determine a unique low energy 2N \( T \)-matrix. For this the operator \( B(z) \) obtained in this way must be close enough to the true \( T \)-matrix inside the domain \( D \). There is no reason to believe that this operator must necessarily generate the Hamiltonian low energy
dynamics. In fact, as it follows from Eq. (10), for this expansion, \( \langle p_2 | B(z) | p_1 \rangle \) must have a negligible dependence on \( z \) inside the domain \( \mathcal{D} \). However, as we will show below, such a behavior of \( \langle p_2 | B(z) | p_1 \rangle \) is at variance with the symmetries of QCD. On the other hand, QCD must produce the low energy 2N \( T \)-matrix that satisfies the generalized dynamical equation. From this in turn it follows that QCD must give rise to the NN interaction operator \( B(z) \) satisfying the condition (43).

In order to demonstrate how Eq. (43) and the analysis of diagrams in ChPT can be used for constructing the interaction operator, let us turn to the pionless theory. As has been noted, from this analysis it follows that the 2N \( T \)-matrix in the \( ^1S_0 \) channel must be of the form (44). Obviously the interaction operator \( B(z) \) that reproduces this \( T \)-matrix inside the domain \( \mathcal{D} \) must have the same structure

\[
\langle p_2 | B(z) | p_1 \rangle = \sum_{n, m=0}^{\infty} p_2^{2n} p_1^{2m} b_{nm}(z). \tag{44}
\]

Correspondingly from the requirement that the operator \( B(z) \) have such a structure it follows that this operator should be of the form

\[
\langle p_2 | B(z) | p_1 \rangle = \psi^*(p_2/\Lambda) \psi(p_1/\Lambda) f(z), \tag{45}
\]

where \( \psi(p/\Lambda) \) is given by Eq. (28), and the function \( f(z) \) is a solution of the equation

\[
\frac{df(z)}{dz} = -f^2(z) J(z) + O \{ |z|^{-1} h(z) \} , \quad z \in \mathcal{D}, \tag{46}
\]

with

\[
J(z) = \int \frac{d^3k}{(2\pi)^3} \frac{|\psi(k/\Lambda)|^2}{(z - E_k)^2}.
\]

Solving Eq. (46) yields

\[
f(z) = -\mathcal{M}^{-1}(z) - C_0^{-1} \mathcal{M}^{-2}(z) - C_0^{-2} \mathcal{M}^{-3}(z) \ldots + O \{ h(z) \}, \quad z \in \mathcal{D}, \tag{47}
\]

where \( \mathcal{M}(z) = M_0(z) + M(z) \), with \( M_0(z) = \frac{\beta}{4\pi} \sqrt{-z_m} \). Obviously the number of the terms we have to keep in this expansion depends on the value of the parameter \( \beta \), i.e., on the accuracy with which the operator \( B(z) \) of the form (44) reproduces the relevant solution of Eq. (7) in the limit \( |z| \to \infty \). Since the function \( \mathcal{M}(z) \) is completely determined by the form factor \( \psi(p/\Lambda) \), all solutions of Eq. (46) for given form factor have the same first term in the expansion (47), and only the second term distinguishes the different solutions (they are fixed by the free parameter \( C_0 \)). Thus in order that Eq. (7) with the interaction operator of the form (44) has a unique solution, we have to keep the first two terms in the expansion of the function \( f(z) \) shown in Eq. (47). This means that the function \( h(z) \) in Eq. (46) is determined by the equation

\[
h(z) = O \{ C^{-2}_0 \mathcal{M}^{-3}(z) \} = O \{ C^{-2}_0 M^{-3}(z) \}. \tag{48}
\]

From the above it follows that the operator describing the NN interaction in the \( ^1S_0 \) channel can be written as

\[
\langle p_2 | B(z) | p_1 \rangle = -\psi^*(p_2/\Lambda) \psi(p_1/\Lambda) \times (\mathcal{M}^{-1}(z) + C^{-1}_0 \mathcal{M}^{-2}(z)). \tag{49}
\]

Correspondingly, for the interaction operator \( H_{int}(\tau) \), we have

\[
\langle p_2 | H_{int}(\tau) | p_1 \rangle = \psi^*(p_2/\Lambda) \psi(p_1/\Lambda) \times (f_1(\tau) + C^{-1}_0 f_2(\tau)),
\]

where

\[
f_n(\tau) = \frac{i}{2\pi} \int dx \exp(-iz\tau) \mathcal{M}^{-n}(z).
\]

As we have shown, the knowledge of the form factor in Eq. (43) completely determines the form of the NN interaction operator which in turn determines the dynamics of the system. Equation (46) with this interaction operator has a unique solution, and this solution is the \( T \)-matrix shown in Eq. (28). However, this implies that we know the details of the dynamics at high energies, because the relevant domain of definition of the function \( \psi(p/\Lambda) \) spreads to such energies. At the same time, the basic premise of effective theories is that low energy dynamics can be described in terms of a few parameters without any knowledge of the details of high energy interactions. As we have seen in Sec.IV, the functions \( \psi(p/\Lambda) \) and \( M(z) \) that enter the expression for the operator \( B(z) \) can be expanded as is shown in Eqs. (25) and (41), and in order to calculate these functions and hence \( \langle p_2 | B(z) | p_1 \rangle \) with accuracy up to the order \( (Q/\Lambda)^{2N} \) it is sufficient to know the first \( N \) terms in these expansions. In other words, for determining the interaction operator with the accuracy up to the order \( (Q/\Lambda)^{2N} \), one need only to know the properties of the form factor that are parameterized by the constants \( c_{2n}, \ldots, c_{2N} \) and \( J_1, \ldots, J_{2N} \). Let us now show that, starting with the interaction operator (49), where only the first \( N \) terms in the expansion of \( \psi(p/\Lambda) \) and the same in the expansion of \( M(z) \) are known, one can calculate the \( T \)-matrix with the accuracy up to the order \( (Q/\Lambda)^{2N} \) for all relevant \( z \). For calculating with this accuracy, instead of the interaction operator (49), one can use the following effective interaction operator.
\[ \langle p_2 | B^{(N)}_{\text{eff}}(z) | p_1 \rangle = - \left( 1 + c_2 p_2^2 + \ldots + c_{2N} p_2^{2N} + \varphi_N(p_2/\Lambda) \right) \left( 1 + c_2 p_1^2 + \ldots + c_{2N} p_1^{2N} + \varphi_N(p_1/\Lambda) \right) \times (\mathcal{M}_N^{-1}(z) + C_0^{-1} \mathcal{M}_N^2(z)), \] (50)

with

\[ \mathcal{M}_N(z) = \sum_{n=0}^{N} M_n(z) \]
\[ = \frac{m}{4\pi} \sqrt{-2m} \sum_{n=0}^{N} (zm)^n n \sum_{i=0}^{n} c_i^* c_{2(n-i)} - \sum_{n=1}^{N} (zm)^n \mathcal{J}_n, \]

where \( \varphi_N(p/\Lambda) \) is an arbitrary function that satisfies the condition

\[ \varphi_N(p/\Lambda) = O \left\{ (p/\Lambda)^{2(N+1)} \right\}. \] (51)

The form of this effective interaction operator manifests uncertainty in the details of the \( NN \) interaction at high energies. In contrast with the operator \( B(z) \) which uniquely determines the relevant solution of Eq. 7, the effective interaction operator \( B^{(N)}_{\text{eff}}(z) \) determines the set of the solutions that coincide inside the domain \( D \) with the true \( 2N \) \( T \)-matrix with the relative accuracy up to the order \( (Q/\Lambda)^{2(N+1)} \)

\[ \langle p_2 | T(z) | p_1 \rangle = \langle p_2 | B^{(N)}_{\text{eff}}(z) | p_1 \rangle \]
\[ \times \left( 1 + O(Q/\Lambda)^{2(N+1)} \right) + O \left\{ |C_0^{-2} z^{-1} M_0^{-3}(z)| \right\}, \quad z \in \mathcal{D}. \] (52)

This implies that the operator \( B^{(N)}_{\text{eff}}(z) \) must satisfy Eq. 7 with the same accuracy

\[ \frac{d\langle p_2 | B^{(N)}_{\text{eff}}(z) | p_1 \rangle}{dz} = - \int\frac{d^3k}{(2\pi)^3} \frac{\langle p_2 | B^{(N)}_{\text{eff}}(z) | k \rangle \langle k | B^{(N)}_{\text{eff}}(z) | p_1 \rangle}{(z - E_k)^2} \times \left( 1 + O(Q/\Lambda)^{2(N+1)} \right) + O \left\{ |C_0^{-2} z^{-1} M_0^{-3}(z)| \right\}, \quad z \in \mathcal{D}. \] (53)

By taking into account Eqs. (35-37), one can easily verify that this condition is really satisfied, provided the function \( \varphi_N(p/\Lambda) \) is such that

\[ m \int \frac{d^3k}{(2\pi)^3} \frac{|1 + c_2 k^2 + \ldots + c_{2N} k^{2N} + \varphi_N(k/\Lambda)|^2}{k^{2(n+1)}} = \mathcal{J}_n, \] (54)

where \( 1 \leq n \leq N \). Thus the requirement that inside the domain \( \mathcal{D} \) the effective interaction operator is close enough to the true \( 2N \) \( T \)-matrix determines some properties of the function \( \varphi_N(p/\Lambda) \) that is the unknown part of the form factor \( \psi(p/\Lambda) \). The functions \( \varphi_N(p/\Lambda) \) mainly characterize the form factor at high energies. For this reason, for describing low energy nucleon dynamics with the accuracy up to the order \( O(Q/\Lambda)^{2(N+1)} \) one need not to know the function \( \varphi_N(p/\Lambda) \) exactly. This is because the details of the high energy physics must not affect on the low energy dynamics. It is sufficient to know the integration properties of this function shown in Eq. 54. Thus the constants \( \mathcal{J}_1, \ldots, \mathcal{J}_N \) in the effective interaction operator \( B^{(N)}_{\text{eff}}(z) \) parameterize the effects of high energy physics on low energy nucleon dynamics.

By definition, the effective interaction operator \( B^{(N)}_{\text{eff}}(z) \) is not so close inside the domain \( \mathcal{D} \) to the true \( 2N \) \( T \)-matrix to determine a unique solution of Eq. 7. However, it determines a set of the solutions \( T^{(N)}(z) \) of Eq. 7 that coincide with the true \( 2N \) \( T \)-matrix, with the relative accuracy up to the order \( O(Q/\Lambda)^{2(N+1)} \) (we will denote this set \( \Omega_N \)). Each of these solutions corresponds to some definite function \( \varphi_N(p/\Lambda) \) satisfying the conditions (51) and (54) and can be represented as

\[ \langle p_2 | T^{(N)}(z) | p_1 \rangle = t_N(z) \left( \sum_{n=0}^{N} c_{2n} p_2^{2n} + \varphi_N(p_2/\Lambda) \right) \]
\[ \times \left( \sum_{n=0}^{N} c_{2n} p_1^{2n} + \varphi_N(p_1/\Lambda) \right), \]

where \( t_N(z) \) is the solution of the equation

\[ \frac{dt_N(z)}{dz} = -t_N^2(z) \mathcal{J}(z) = t_N^2(z) \frac{dM(z)}{dz}, \] (55)

\[ \mathcal{J}(z) = \sum_{n=1}^{N} \mathcal{J}_n, \quad \mathcal{J}_n = m \int \frac{d^3k}{(2\pi)^3} \frac{|1 + c_2 k^2 + \ldots + c_{2n} k^{2n} + \varphi_N(k/\Lambda)|^2}{k^{2(n+1)}}. \]
with the boundary condition
\[ t_N(z) = - \left( M_N^{-1}(z) + C_0^{-1}M_N^{-2}(z) \right) \times \left( 1 + O \left\{ \frac{(Q/A)^2(N+1)}{2} \right\} \right), \]
\[ z \in D. \]

The general solution of Eq. \( 55 \) is
\[ t_N(z) = - \frac{1}{C^{-1} - M(z)}, \]
where \( C \) is some arbitrary constant. Equation \( 57 \) may be rewritten in the form
\[ t_N(z) = - \frac{1}{C^{-1} - M_N(z) - M_N(z)}, \]
where \( \tilde{M}_N(z) \equiv \sum_{n=0}^{N} M_n(z) \) is the part of \( M(z) \) that depends on the choice of the function \( \varphi_N(p/A) \). From this equation it follows that
\[ t_N(z) = - \left( M_N^{-1}(z) + C^{-1}M_N^{-2}(z) \right) \times \left( 1 + O \left\{ \frac{(Q/A)^2(N+1)}{2} \right\} \right), \]
\[ z \in D. \]

Here we have used the fact that \( \tilde{M}_N(z) \) is of order \( O \left\{ \frac{(Q/A)^2(N+1)}{2} \right\} \). On the other hand the relevant solution of Eq. \( 55 \) must satisfy the boundary condition \( 56 \). From this it follows that the parameter \( C \) in Eq. \( 57 \) must be equal to \( C_0 \). Thus the solution of Eq. \( 54 \) with the boundary condition given by Eqs. \( 57 \) and \( 58 \) where \( \varphi_N(p/A) \) is specified in some way is

\[ (p_2|T^{(N)}(z)|p_1) = \frac{\left( \sum_{n=0}^{N} c_n^2 p_2^{2n} + \varphi_N(p_2/A) \right) \left( \sum_{n=0}^{N} c_n p_1^{2n} + \varphi_N(p_1/A) \right)}{C_0^{-1} - M_N(z) - M_N(z)} \times \left( 1 + O \left\{ \frac{(Q/A)^2(N+1)}{2} \right\} \right). \]

However, as has been noted, the function \( \varphi_N(p/A) \) that enters the effective interaction operator is not known exactly. At a given order, only its integral properties shown in Eq. \( 55 \) are known. For this reason Eq. \( 57 \) actually represents the set \( \Omega_N \) of the solutions of Eq. \( 54 \) which coincide each with other with accuracy up to the order \( O \left\{ \frac{(Q/A)^2(N+1)}{2} \right\} \), and the true \( 2N \) \( T \)-matrix belongs to this set. This is because the ”operator” \( B^{(N)}_{\text{eff}}(z) \) actually is a set of the interaction operators \( B(\omega) \) which coincide with the true \( 2N \) interaction operator with the same accuracy. This uncertainty in specifying the boundary condition on the generalized dynamical equation is only difference of the effective theory from a ”fundamental” one whose predictions are assumed to be exact. It is important that in both of these cases we deal with the same well defined equation of motion which can be used not only for analytic, but also for numerical calculations.

As an example, let us consider the effective pionless theory at next-to-leading order (\( N = 1 \)). In this case, for the effective interaction operator, we have
\[ (p_2|B^{(1)}_{\text{eff}}(z)|p_1) = - \left( 1 + c_2 p_2^2 + \varphi_1(p_2/A) \right) \times \left( 1 + c_2 p_1^2 + \varphi_1(p_1/A) \right) \left( M_1^{-1}(z) + C_0^{-1}M_1^{-2}(z) \right), \]
where
\[ M_1(z) = \frac{m}{4\pi} \sqrt{-zm} - zmJ_1 - \frac{m}{2\pi} (-zm)^{3/2} \text{Re}c_2, \]
and \( \varphi_1(p/A) \) is some function satisfying the condition shown in Eq. \( 54 \). The solution of Eq. \( 54 \) with this effective interaction operator yields
\[ (p_2|T^{(1)}(z)|p_1) = \left( 1 + c_2 p_2^2 + \varphi_1(p_2/A) \right) \times \left( 1 + c_2 p_1^2 + \varphi_1(p_1/A) \right) \left( M_1^{-1}(z) + C_0^{-1}M_1^{-2}(z) \right) \left( 1 + O \left\{ \frac{(Q/A)^4}{2} \right\} \right). \]

This equation represents the set of the solutions of the generalized dynamical equation that coincide with the true \( 2N \) \( T \)-matrix with the relative accuracy up to the order \( (Q/A)^4 \). At this order one may use any of these solutions. Correspondingly the operator \( B^{(1)}_{\text{eff}}(z) \) with any function \( \varphi_1(p/A) \) satisfying the conditions \( 51 \) and \( 52 \) may be used as the interaction operator \( B(z) \) that determines a unique solution of Eq. \( 54 \). For example, one may chose this function as follows \( \varphi_1(p/A) = c_2 p^2 \exp(-p^2/\Lambda_1^2) \), with \( \Lambda_1 = 2\pi \sqrt{\text{Re}c_2} \). It should be noted that the expression \( 55 \) which represents the form of the effective \( 2N \) \( T \)-matrix in the \( {1 \over 2} S_0 \) channel of the pionless theory can also be derived from...
Eq. (29) by keeping only the first \( N \) terms in both the expansions of \( \psi(p/\Lambda) \) and \( M(z) \) that enter in this equation. This is because in the case of the pionless theory the generalised dynamical equation can be solved exactly, and as we have shown, the unique solution consistent with symmetries of QCD is the \( T \)-matrix \( \psi_2 \). However, this is not the case when pions must be included as explicit degrees of freedom. In the theory with pions, one cannot solve the problem exactly, and hence the way of constructing the \( 2N \) \( T \)-matrix presented in Sec.III is inapplicable. Above we have presented the way of constructing the effective theory that prescribes to start with the analysis of the theory at "high" energies belonging to the domain \( \mathcal{D} \). At these energies the theory has the most simple structure. For example, in the case where the dynamics in the theory is Hamiltonian, \(|\psi_2|T(z)\rangle \) have a negligible dependence on \( z \) inside the domain \( \mathcal{D} \). This means that the interaction generating the dynamics can be assumed to be instantaneous and hence the low energy dynamics can be described by an interaction Hamiltonian that is just the value of the \( T \)-matrix in this domain. If we deal with several different interactions, then the interaction Hamiltonian is the sum of the corresponding interaction Hamiltonians. This is because processes that involve several different interactions cannot occur instantaneously, and hence give a negligible contribution to the \( T \)-matrix inside the domain \( \mathcal{D} \). In the general case, however, the operator \( T(z) \) have significantly depend on \( z \) even inside this domain. As it follows from Eqs. (29) and (58), this takes place in the case of the contact \( NN \) interaction.

As an illustration of the advantages of the above formulation of the effective theory, let us consider the dynamics of the \( 2N \) system in the presence of an external potential. For reason of physical transparency, we will consider nucleons as spinless particles whose interaction is described by the interaction operator \( B \), i.e., is the same as in the \( ^1S_0 \) channel of the pionless theory. The interaction operator generating the dynamics of such a system should be of the form

\[
\langle p_2; p_2 | B_{\text{eff}}^{\text{tot}}(z) | p_1; p_1 \rangle = \langle p_2; p_2 | B(z) | p_1; p_1 \rangle + \langle p_2; p_2 | V | p_1; p_1 \rangle,
\]

where the vector \(| p; p \rangle \) describes the state of the \( 2N \) system with the momentum of the center of mass of the system \( p \) and the relative momentum of nucleons \( p \). The operator \( B(z) \) that describes the interaction of nucleons with themselves and, in the center of mass frame, is shown in Eq. (49). The second term on the right hand side of Eq. (61) describes the instantaneous interaction with the external potential. We will assume that this potential is local, i.e., has the form

\[
\langle p_2; p_2 | V | p_1; p_1 \rangle = V(q)(2\pi)^3 \delta^{(3)}(p_2 - p_1 - q)
\]

with \( q = p_2 - p_1 \), and is weak enough to be considered in the Born approximation.

Let the details of the \( NN \) interaction are known only up to next-to-leading order. In this case we have to solve the problem in the spirit of the effective theory starting with the effective interaction operator of the form

\[
\langle p_2; p_2 | B_{\text{eff}}^{\text{tot}}(z) | p_1; p_1 \rangle = \langle p_2; p_2 | B_{\text{eff}}^{(1)}(z) | p_1; p_1 \rangle + V(q) \delta^{(3)}(p_2 - p_1 - q),
\]

where \( B_{\text{eff}}^{(1)}(z) \) is given by Eq. (59). This operator determines a set of the solutions of Eq. (7) that coincide with the true \( T \)-matrix describing the dynamics of the system with accuracy up to the order \((Q/\Lambda)^4\). It can be checked that in the Born approximation these solutions are given by the equation

\[
T_{\text{pot}}^{(1)}(z) = T^{(1)}(z) + \left(1 + T^{(1)}(z)G_0(z) \right) V \left(1 + G_0(z)T^{(1)}(z) \right),
\]

where \( T^{(1)}(z) \) is one of the solutions of Eq. (7) shown in Eq. (60). By using Eq. (60), one can construct, with a given accuracy, the scattering amplitude and the evolution operator. For example, for the amplitude of \( 2N \) scattering in the presence of the external potential, we can write

\[
A(p_2, p_2; p_1, p_1) = -\langle p_2; p_2 \left| 1 + T^{(1)}(z)G_0(z) \right) V \left(1 + G_0(z)T^{(1)}(z) \right) | p_1; p_1 \rangle,
\]

where \( z = \frac{p_2^2}{m_1^2} + \frac{p_1^2}{m_1^2} + i\delta = \frac{p_2^2}{m_1^2} + \frac{p_1^2}{m_1^2} + i0 \), and \( p_1 \neq p_2 \). By using Eq. (60), for the scattering amplitude \( A(p_2, p_2; p_1, p_1) \) we get

\[
A(p_2, p_2; p_1, p_1) = A_{00}(p_2, p_2; p_1, p_1) + A_{01}(p_2, p_2; p_1, p_1) + A_{10}(p_2, p_2; p_1, p_1) + A_{11}(p_2, p_2; p_1, p_1),
\]

where

\[
A_{00}(p_2, p_2; p_1, p_1) = -V(q)(2\pi)^3 \delta^{(3)}(p_2 - p_1 - q),
\]

\[
A_{01}(p_2, p_2; p_1, p_1) = -\frac{V(q)}{z_1 - E_{p_2} - E_{p_1}} (1 + O(Q/\Lambda)^4)
\]
\[
V(q) = -\frac{V(q)(1 + c_2^2p^2 + c_2p'_2)}{(E_{p_1} - E_p)} \left( C_0^{-1} + \frac{m}{4\pi}ip_1 + J_1p'_1 + \frac{m}{2\pi}ip_2^2\Re c_2 \right), \quad p = p_2 - q.
\]

\[
A_{10}(p_2; p_1) = -\frac{V(q)p_2 |T^{(1)}(z - E_{p_2})|^p'}{z - E_{p'} - E_{p_2}} (1 + O(Q/\Lambda)^4)
\]

\[
A_{11}(p_2; p_1) = -V(q) \int \frac{d^3p}{(2\pi)^3} \frac{\langle p_2 | T^{(1)}(z - E_{p_2}) | p' \rangle \langle p | T^{(1)}(z - E_{p_1}) | p_1 \rangle}{(z - E_{p'} - E_{p_2} + i0)(z - E_{p'} - E_{p_1} + i0)} (1 + O(Q/\Lambda)^4)
\]

\[
\times \int \frac{d^3p}{(2\pi)^3} \frac{1}{(E_{p_2} - E_{p'} + i0)(E_{p_1} - E_{p'} + i0)} + \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p - E_{p'} + E_{p_2} + i0)} \left( c_2^2p^2 + c_2p'_2 \right)
\]

\[
- E_{p_2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{p}'(E_{p_1} - E_{p'} + i0)} + mJ_1 \right) (1 + O(Q/\Lambda)^4), \quad p' = p + q.
\]

(66)

Here we have used the fact that the function \( \varphi_1(p/\Lambda) \) satisfies the condition \( \text{[24]} \). Thus the generalized dynamical equation with the effective \( NN \) interaction operator \( \text{[54]} \) allows one to calculate, with the desired accuracy, not only \( 2N \) data but also processes in the presence of an external field. As we have seen, in this way regularization and renormalization are not required, and hence one need not to introduce new parameters: The solution depends only on the constants \( C_0, c_2, \) and \( J_1 \) that are contained in the effective interaction operator. As we have noted, the coupling \( C_0 \) is fixed by the scattering length \( C_0 = \frac{2m}{\alpha} \alpha \). However, knowing the \( 2N \) scattering data is not sufficient to obtain the parameter \( c_2 \) and \( J_1 \). Only the coupling \( C_2 = C_0(c_2 + C_1) = C_0 \frac{2m}{\alpha} \alpha \) is fixed by the scattering data at this order, but not the couplings \( c_2 \) and \( J_1 \) separately. At the same time, as it follows from Eq. \( \text{[24]} \), these parameters manifest themselves in \( 2N \) scattering in the presence of an external potential. This is because the \( 2N \) \( T \)-matrices enter half-of-the-energy shell in the scattering amplitude \( \text{[24]} \) describing such processes. In other words, such processes can be obtained for the above couplings.

VI. THE PROBABILISTIC FRAME OF QUANTUM MECHANICS AND THE CHARACTER OF LOW ENERGY NUCLEON DYNAMICS.

In an EFT renormalization allows one to describe low energy physics using an effective Lagrangian that contains only a few degrees of freedom, because after renormalization that consists in absorbing infinities in a re-definition of constants in the Lagrangian the remaining integrals are effectively cut off at internal momenta larger than \( Q \). This might lead us to the conclusion that the definition of any effective theory must necessarily include regularization and renormalization. However, as we have shown, by using the example of the pionless theory, regularization and renormalization are needed when we describe the theory in terms of an effective Lagrangian. But, if we consider the problem from the more general point of view provided by the GQD and do not restrict ourselves to the assumption that the effective action is instantaneous, we see that the effective theory of nuclear forces manifests itself as a perfectly consistent theory free from the UV divergences. As we have shown, in the pionless theory the \( 2N \) \( T \)-matrix in the \( S_0 \) channel is given by Eq. \( \text{[24]} \). This \( T \)-matrix satisfies the generalized dynamical equation, i.e., the solution of this equation does not require regularization and renormalization. In other words, in the pionless theory the generalized dynamical equation allows one to separate the low energy physics from the underlying high energy physics without renormalization. Let us now show that this directly follows from the first principles of quantum mechanics.

The fact that because of the separation of scales, at low energies nucleons emerge as the only effective degrees of freedom means that the temporal evolution of the system can be described in terms of the \( 2N \) Hilbert subspace \( \mathcal{H}_{2N} \) and the evolution operator defined on this space

\[
U_{2N}(t, 0) = P_{2N}U_S(t, 0)P_{2N},
\]

where \( P_{2N} \) being the projection operator on the subspace \( \mathcal{H}_{2N} \). Here we use the Schrödinger picture. Since in this case the \( 2N \) system is considered to be closed, the evolution operator \( U_{2N}(t, 0) \) should be unitary

\[
U_{2N}^+(t, 0)U_{2N}(t, 0) = 1.
\]

(68)
This is one of the main requirements that quantum mechanics imposes on such a theory. It expresses the fact that, if at initial time the system was in the state \( |\psi\rangle \in \mathcal{H}_{2N} \), then in a measurement at time \( t \) the system will be necessarily found in one of the \( 2N \) states. Here, of course, it is assumed that \( \mathcal{H}_{2N} \) describes \( 2N \) states at low energies. The above means that the sum of the probabilities to find the system in all the possible \( 2N \) states must be equal to unite

\[
\int \frac{d^3k}{(2\pi)^3} |\langle k | U_{2N}(t,0) | \psi \rangle|^2 = 1, \tag{69}
\]

with \( |\psi\rangle \) being a normalized vector belonging to \( \mathcal{H}_{2N} \). Here we consider nucleons as spinless particles, and \( |\langle k | U_{2N}(t_2,t_1) | \psi \rangle|^2 \frac{d^3k}{(2\pi)^3} \) is the probability of finding the quantum system in the \( 2N \) state with the relative momentum \( p \) in the volume of the momentum space \( k_i \leq p_i \leq k_i + dk_i \) \((i = 1, 2, 3)\). Finally, for Eq. \((69)\) to be valid for any normalized vector \( |\psi\rangle \in \mathcal{H}_{2N} \), the operator \( U_{S}(t,0) \) must be unitary.

Another basic principle of quantum mechanics is the principle of the superposition of the probability amplitudes from which it follows that the evolution operator is defined by Eq. \((12)\) which in this case reads

\[
\langle p_2 | U_{2N}(t,0) | p_1 \rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt) G_{2N}(z), \tag{70}
\]

with

\[
G_{2N}(z) = G_0'(z) + G_0'(z) T_{2N}(z) G_0'(z), \tag{71}
\]

where

\[
G_0'(z) = P_{2N} G_0(z) P_{2N} \tag{72}
\]

and \( T_{2N}(z) \) is defined by the equation

\[
\langle p_2 | T_{2N}(z) | p_1 \rangle = \int_0^{\infty} dt (t_2 - t_1) \exp[izt(t_2 - t_1)] \\
\times \langle p_2 | T_{2N}(t_2 - t_1) | p_1 \rangle, \tag{73}
\]

Thus, in the case of the reduced \( 2N \) system, we deal with the same equations as in describing the complete system. The only difference is that the amplitudes \( \langle p_2 | T_{2N}(t_2 - t_1) | p_1 \rangle \) describes the contributions to the evolution operator from the processes in which the interaction in the \( 2N \) system (not in the complete system) begins at time \( t_1 \) and ends at time \( t_2 \). By using Eq. \((70)\), the unitarity condition \((68)\) may be rewritten as

\[
\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \exp[iz_2 - z_1] \times \langle \psi_2 | G_{2N}^+(z_2) G_{2N}(z_1) | \psi_1 \rangle = \langle \psi_2 | \psi_1 \rangle. \tag{74}
\]

For Eq. \((74)\) to be valid for any time \( t \), the \( 2N \) \( T \)-matrix must satisfy Eq. \((7)\). Note that in Ref. \((2)\) the generalized dynamical equation has been derived just in this way. At the same time, the above does not mean that \( T_{2N}(z) \) satisfies the LS equation. In other words, from the existence of the separation of scales which results in the unitarity of the \( 2N \) evolution operator \( U_{2N}(t,0) \) and the superposition principle it follows that the \( 2N \) \( T \)-matrix must necessarily satisfy the generalized dynamical equation but not the LS equation. Thus the dynamics of the \( 2N \) system may be non-Hamiltonian.

Let us now show that the above is true even if the dynamics in the underlying theory is governed by the Schrödinger equation. As we have noted, in the case where the dynamics of a quantum system is Hamiltonian the evolution operator \( U_{S}(t,0) \) can be represented in the form \((12)\) with the Green operator given by Eq. \((14)\). From this it follows that

\[
U_{2N}(t,0) = P_{2N} U_{S}(t,0) P_{2N} \tag{75}
\]

\[
= \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt) P_{2N}(z - H)^{-1} P_{2N} \tag{76}
\]

\[
= \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt) G_{2N}(z). \tag{77}
\]

It can be shown, that the Green operator \( G_{2N} = P_{2N}(z - H)^{-1} P_{2N} \) may be represented in the form

\[
G_{2N}(z) = G_0'(z) + G_0'(z) T_{2N}(z) G_0'(z), \tag{78}
\]

where \( G_0'(z) \) is defined by Eq. \((22)\). The above means that, if the underlying dynamics is governed by the Schrödinger equation, then the \( 2N \) evolution operator \( U_{2N}(t,0) \) can be represented in the form

\[
U_{2N}(t,0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt) \times (G_0'(z) + G_0'(z) T_{2N}(z) G_0'(z)). \tag{79}
\]

From this it follows that \( T_{2N}(z) \) must satisfy Eq. \((7)\), because the operator \((74)\) is assumed to be unitary. At the same time, the above does not mean that the amplitudes \( \langle k_2 | T(z) | k_1 \rangle \) fall off rapidly enough above the scale of the effective theory for the dynamics in the theory to be governed by the Schrödinger equation. In fact, the Hamiltonian character of the underlying dynamics implies only that the \( 2N \) \( T \)-matrix must fall off rapidly above the scale of the underlying physics. Thus, the dynamics in the effective low energy theory may be non-Hamiltonian even if the underlying high energy dynamics is governed by the Schrödinger equation.

An important lesson to be learned from the above analysis is that low energy nucleon dynamics can be described by the generalized dynamical equation reduced to the \( 2N \) Hilbert space without regularization and renormalization. The divergence problems may arise only in describing the dynamics of the effective theory by using
the Schrödinger (LS) equation, because such a description is based on the additional assumption that the effective $NN$ interaction is instantaneous and, as a consequence, the generalized dynamical equation is equivalent to the Schrödinger equation. Note, in this connection, that from the physical point of view the fact that the chiral potentials lead to UV divergences means that the LS equation with such potentials does not provide a well separation of the low energy physics from the underlying high energy physics. In other words, the amplitudes $\langle \mathbf{k}_2 | T(z) | \mathbf{k}_1 \rangle$ does not fall off rapidly enough above the scale of the effective theory for integrals in the LS equation to be effectively cut off at internal momenta larger than $Q$. In this case, in order to integrate out the high energy degrees of freedom, one need to use regularization and renormalization. In contrast, the generalized dynamical equation should separate the low energy physics that is described by the effective theory of nuclear forces from the underlying high energy physics provided that at very low energies nucleons may emerge as the only explicit degrees of freedom. In fact, the letter means that the matrix elements of the operator $U_{2N}(t,0)$ must fall off rapidly enough at momenta larger than $Q$. This is because the probability of finding, for a measurement at time $t$, the system in the $2N$ state with momenta larger than $Q$, if at time $t = 0$ the system was in the low energy state $|\psi\rangle$, must be negligible. If this is not the case, one cannot also ignore other states with such momenta in which the high energy degrees of freedom manifest themselves, and hence the unitarity condition breaks down. As it follows from Eqs. (6), (12) and (13), the fact that $\langle \mathbf{k}_0 | U(t,t_0) | \mathbf{k}_1 \rangle$ fall off rapidly at momenta larger than $Q$ turn means that the integrals in the generalized dynamical equation effectively cut off at such momenta, and hence, in this case the high energy degrees of freedom can be integrated out without renormalization.

VII. A NEW LOOK AT THE WEINBERG PROGRAM

Our formulation of the effective theory of nuclear forces may be regarded as a new way of realizing the Weinberg program for physics of the two-nucleon systems. The implementation of this program has three stages. Firstly, one must consider the case where the $NN$ system is not subject to any external probes and employ ChPT to generate a nonrelativistic particle-number-conserving Hamiltonian for the nuclear system. At the second stage this Hamiltonian should be employed for constructing the full $2N$ $T$-matrix via the LS (the Schrödinger) equation. The third step of the Weinberg program is to employ the $2N$ $T$-matrix obtained in this way in the case where $2N$ system is subject some external probe: a pion, a photon, or some weakly-interacting particle. Provided this probe carries momentum of order $m_\pi$ its interaction with the $2N$ system can be represented as a sum of irreducible diagrams which forms a kernel $K_{\text{probe}}$ for the process of interest. The full amplitude for this process is then found by multiplying this kernel $K_{\text{probe}}$ by the factors describing the interaction of the $NN$ pair in the initial and final states

$$A = (1 + TG_0)K_{\text{probe}}(1 + G_0T).$$

The Weinberg proposal was based on the assumption that the only equation that can govern low energy nucleon dynamics is the LS (the Schrödinger) equation, and hence what one has to derive from the analysis of diagrams in ChPT is an $NN$ potential. However, there is no reason to consider that low energy nucleon dynamics is necessarily governed by the Schrödinger equation. In principle this dynamics may be governed by the generalized dynamical equation with a nonlocal-in-time interaction operator when this equation is not equivalent to the Schrödinger equation. In fact, as has been shown in Ref. 4, only the generalized dynamical equation must be satisfied in any case, not the Schrödinger equation. In the light of this fact the Weinberg program can be considered from a new point of view: Instead of the Schrödinger (LS) equation, one should use the generalized dynamical equation. In this case one need not to use a prioriy assumption that from the analysis of diagrams for the $2N$ $T$-matrix it follows that the an effective $NN$ interaction is instantaneous. Such a modification of the Weinberg program is natural and does not change its character. Indeed, the fact that the analysis of the diagrams in ChPT leads to singular chiral potentials in the case of which the LS equation makes no sense without renormalization means that really this equation is not sufficient for constructing the $2N$ $T$-matrix. In addition one need to specify a renormalization scheme to make these predictions finite. But the LS equation plus a renormalization scheme is something more general than this equation itself; and, after renormalization we deal with the dynamics that is governed by another equation which, as has been shown in Ref. 4, may be only the generalized dynamical equation. In order to illustrate this point let us consider the effective theory at leading order of the Weinberg power counting. The $NN$ effective potential that has been derived from the Weinberg analysis of diagrams in ChPT is

$$V(p_2,p_1) = - \left( \frac{g_A^2}{2f_\pi^2} \right) \frac{q \cdot \sigma_1 q \cdot \sigma_2 \tau_1 \cdot \tau_2}{q^2 + m_\pi^2} + C_S + C_T \sigma \cdot \sigma_2,$$

with $q \equiv p_2 - p_1$. The coupling constant $g_A$ is the axial coupling constant, $m_\pi$ is the pion mass, $f_\pi$ is the pion decay constant, and $\sigma(\tau)$ are the Pauli matrices acting in spin (isospin) space. At very low energies the one-pion-exchange part of the $NN$ interaction may be included into the contact term, and the $NN$ potential takes the form

$$\langle p_2 | V | p_1 \rangle = C,$$
where \( C_0 = C_S - 3C_T + g_A^2/f^2_N \). Obviously, the potential \( \Sigma_1 \) is singular, and the LS equation with this potential makes no sense without regularization and renormalization. Let us perform regularization by using a momentum cut-off. In this case the singular potential \( \Sigma_1 \) is replaced by the regularized one \( \langle p_2|V_\lambda(p_1)\rangle = f^*(p_2/\Lambda)C_0(\Lambda)f(p_1/\Lambda) \), where the form factor \( f(p/\Lambda) \) satisfies \( f(0) = 1 \) and falls off rapidly for \( p/\Lambda > 1 \). The solution of the LS equation with this potential is

\[
\langle p_2|T_\lambda(z)|p_1 \rangle = f^*(p_2/\Lambda)f(p_1/\Lambda) \times \left( C_0^{-1}(\Lambda) - \int \frac{d^3k}{(2\pi)^3} \frac{|f(k/\Lambda)|^2}{z - E_k} \right)^{-1}
\]

Defining the renormalized value \( C_R \) of \( C_0 \) as the value of the\( T \)-matrix \( \langle p_2|T_\lambda(z)|p_1 \rangle \) at \( z = \frac{p_1^2}{m^2} = \frac{p_2^2}{m^2} = 0 \)

\[
C_R^{-1} = C_0^{-1}(\Lambda) + \int \frac{d^3k}{(2\pi)^3} \frac{|f(k/\Lambda)|^2}{E_k}, \quad (80)
\]

we may rewrite this equation as

\[
\langle p_2|T_\lambda(z)|p_1 \rangle = \frac{f^*(p_2/\Lambda)f(p_1/\Lambda)}{C_R^{-1} - z \int \frac{d^3k}{(2\pi)^3} \frac{|f(k/\Lambda)|^2}{(z - E_k)E_k}}
\]

where the renormalized value is fixed by the scattering length \( a = mC_R/4\pi \). Thus, after renormalization the integral in the expression for the leading order 2N\( T \)-matrix is effectively cut off, and hence at this stage the regularization may be removed by letting \( \Lambda \to \infty \), and we get

\[
\langle p_2|T(z)|p_1 \rangle = \left( C_R^{-1}(\Lambda) - \frac{m^{3/2}}{4\pi} \sqrt{-z} \right)^{-1}. \quad (81)
\]

Correspondingly, for the half-of-the-energy shell \( T \)-matrix, we have

\[
\langle p'|T(E_p + i0)|p \rangle = \left( C_R^{-1} + \frac{im|p|}{4\pi} \right)^{-1}
\]

This is just what has been obtained by Weinberg in Ref. 1. Obviously the\( T \)-matrix \( \Sigma_1 \) is not a solution of the LS equation. At the same time, as we have seen, this\( T \)-matrix is the well defined solution of the generalized dynamical equation with the interaction operator \( \Sigma_1 \) describing a nonlocal in time interaction. The dependence of this interaction operator on \( z \) expresses the fact that the instantaneous interaction which is described by the contact potential \( \Sigma_1 \) does not contribute to the 2N\( T \)-matrix separately. In fact, as it follows from Eq. \( \Sigma_1 \), after removing regularization, this potential becomes equal to zero. A nonzero contribution to this\( T \)-matrix comes only from the sum of diagrams involving infinite number of such contact interactions. In other words renormalization spreads effective contact \( NN \) interaction in time.

Thus even if we start with the LS equation and try to extract an\( NN \) potential from the analysis of diagrams in ChPT, despite it is singular and makes no sense, after renormalization we come to low energy nucleon dynamics that is governed by the generalized dynamical equation with a nonlocal-in-time interaction operator. This means that in order to realize the Weinberg program in a consistent way one has to use this equation instead of the LS equation. As we have shown in the previous section, in contrast with the LS equation, the generalized dynamical equation separates the low energy physics from the high energy physics without renormalization. This manifests itself in the fact that, being reduced to describing the 2N system, this equation is free from UV divergences. In Sec.V we have demonstrated the advantages of this way of realizing the Weinberg program by using the example of the pionless theory. Firstly, by using the analysis of diagrams in ChPT, we have obtained the effective\( NN \) interaction operator. In the \( ^1S_0 \) channel it is of the form \( \Sigma_1 \). By solving the generalized dynamical equation with this interaction operator we have constructed the \( ^1S_0 \) channel 2N\( T \)-matrix shown in Eq. \( \Sigma_1 \). It is important that in this way we have obtained not only the 2N scattering amplitude but also the off-shell\( T \)-matrix. Equation \( \Sigma_1 \) expresses the constraints that the symmetries of QCD place on the off-shell behavior of the 2N\( T \)-matrix. At a given order there are only a few free parameters in Eq. \( \Sigma_1 \) that can be derived from low energy experiment. Once they are obtained in any way, the off-shell 2N\( T \)-matrix that in this case is completely determined by Eq. \( \Sigma_1 \) can be used in Eq. \( \Sigma_1 \) for describing the interaction of the 2N system with an external probe. In Sec.V we have demonstrated this fact by using the example of the 2N scattering in the presence of an external potential. As we have seen, with the 2N\( T \)-matrix \( \Sigma_1 \) in hand the full amplitudes of the processes involving the interaction with external probes can be calculated without regularization and renormalization.

The key point of the above way of the realization of the Weinberg program is the derivation of the\( NN \) interaction operator from the analysis of the time-ordered diagrams in the time-ordered diagrams in the energy region \( D \). This region is sufficiently above the scale of the low energy physics for the processes that are described by the 2N\( T \)-matrix at \( z \in D \) may be considered as a "fundamental" interaction in this theory. At the same time, this energy region is much below the scale of the underlying high energy physics, and hence the effective\( NN \) interaction operator is really determined by the low energy in the scale of QCD behavior of the time-ordered diagram for 2N\( T \)-matrix. Moreover, one need not to know this behavior exactly. It is sufficient to know the structure of this\( T \)-matrix, i.e., its dependence on momenta of nucleons, that is predicted by ChPT. For example, in the pionless theory the \( ^1S_0 \) channel 2N\( T \)-matrix has the structure shown in Eq. \( \Sigma_1 \). Correspondingly the structure of the interaction operator is given by Eq. \( \Sigma_1 \). On the other hand, this interaction must be close enough in the energy region \( D \) to the dynamical equation, i.e., must satisfy Eq. \( \Sigma_1 \). This requirement yields the expression
for the effective interaction operator shown in Eq. (15). This interaction operator can then be used for constructing the $2N$ $T$-matrix. At the same time, in the pionless theory the requirement that the $2N$ $T$-matrix having the structure (24) satisfy the generalized dynamical equation directly yields the expression for the $2N$ $T$-matrix shown in Eq. (28), and this expression can be used for organizing the calculations of the effective $2N$ $T$-matrix. This is because the $2N$ $T$-matrix has the same simple structure at all relevant $z$ as in the "high" energy region $\mathcal{D}$. However, this is not the case in the theory with pions or (and) when one must take into account the Coulomb interaction between protons. In such theories the structure of the $2N$ $T$-matrix is much simpler in the "high" energy region than at low energies where the nonperturbative character of nucleon dynamics manifests itself. In this case only in the region $\mathcal{D}$ the $2N$ $T$-matrix, with the accuracy needed for constructing the interaction operator, can be represented as a sum of contributions from the contact, pion-exchange, and Coulomb interactions. The fact that starting from this high energy representation one can construct the $NN$ interaction operator and then use them for obtaining the $2N$ $T$-matrix has been demonstrated in Ref. [8] by using the example of the $2N$ dynamics at leading order of the Weinberg power counting. As we have seen, the same situation takes place in the case when one consider the $2N$ dynamics in presence of an external potential. In this case interaction operator generating the dynamics in the system is given by Eq. (61), and its form manifests the fact that the "high" energy region $\mathcal{D}$ the main contribution to the $2N$ $T$-matrix can be represented as a sum of the contributions from the interaction of nucleons with themselves and the external potential. By using the interaction operator (28) one can describe the dynamics of this system.

VIII. THE OFF-SHELL BEHAVIOR OF THE $2N$ $T$-MATRIX AND THE THREE-NUCLEON PROBLEM

As we have shown, the formulation of the effective theory of nuclear forces presented in Sec. V allows one to construct not only the $2N$ scattering amplitude (in this case we reproduce all results of the standard EFT approach), but also the off-shell $T$-matrix, and the evolution and Green operators. This is very important because the $S$-matrix is not everything. For example, at finite temperature there is no $S$-matrix because particles cannot get out to infinite distances from a collision without bumping into things. This means that the off-shell structure of the $2N$ $T$-matrix should influence on in-medium observables. In Ref. [8] it has been shown that the off-shell behavior of the in-medium nucleon-nucleon $T$-matrix and hence the off-shell properties of nuclear forces have substantial effects on the transition amplitudes and cross sections at large nuclear matter densities. These properties are crucial for solving the many-nucleon problem. For example, the $2N$ amplitudes enter off-the-energy-shell in the $3N$ equations.

The realistic $NN$ potentials that describe $2N$ scattering data to high precision can not guarantee that a similar precision will be achieved in the description of larger nuclear systems. In fact, the simplest observable in the $3N$ system, the binding energy of the triton, is under predicted by the realistic $NN$ potentials which are so successful in describing the $2N$ observables. The energy deficit ranges from 0.5 to 0.9 MeV and depends on the off-shell and short-range parameterization of the $2N$ force [14]. In order to resolve this problem one has to take into account three-nucleon force (3NF) contributions to the $3N$ binding energy. The common way of solving the $3N$ bound state problem is to use in the Schrödinger equation phenomenological $NN$ potentials and then to introduce a 3NF to provide supplementary binding. However, from the point of view of the three-nucleon problem, it is not sufficient to generate a phenomenological $NN$ potential that perfectly reproduces the $2N$ scattering amplitudes. One must also generate a $NN$ potential by using theoretical insight as much as possible in order to constrain the off-shell properties of the $2N$ $T$-matrix. If this is not the case, a $NN$ potential which fits precisely the $2N$ phase shifts but produces the erroneous off-shell behavior of the $T$-matrix would not provide reliable results for the $3N$ system, nor can be used to test for the presence of $3N$ forces. It is important, in this context, that the requirement that the $2N$ $T$-matrix satisfy the generalized dynamical equation and has the form consistent with the symmetries of QCD places a constrain on the off-shell behavior of this $T$-matrix. As we have shown, in the pionless theory the $2N$ $T$-matrix in the $^1S_0$ channel should be of the form (29). The expression shown in Eq. (29) contains the infinite set of parameters $c_2$, that in principle could be obtained in terms of QCD. At the same time, as we have seen, the generalized dynamical equation allows one to organize calculations of the two-nucleon $T$-matrix in the spirit of the effective theory by parameterizing the effects of the underlying physics in a few parameters that can be derived from a low energy experiment. In this way we have obtained the expression (29) that determines $2N$ $T$-matrix in the $^1S_0$ channel with the accuracy up to the order $(Q/\Lambda)^{2(N+1)}$. For example, at next-to-leading order the $T$-matrix is given by Eq. (60) containing only three free parameters $\alpha_1$, $\alpha_2$ and $i\omega_c$ (the parameter $C_0$ is fixed by the scattering length at leading order). One of them is fixed by the $2N$ phase shifts analysis while other two can be derived from the in-medium scattering data or from the low energy data corresponding to the processes of the interaction of the two-nucleon system with an external probe. In other words, the requirement that the $2N$ $T$-matrix be consistent with basic principles of quantum mechanics and the symmetries of QCD removes the off-shell ambiguities. This may provide a better understanding of the many-nucleon problem.

The models for the 3NF that are usually used for solv-
ing the problem with the triton understanding are based on two-pion exchange with intermediate \( \Delta \)-isobar excitation. However, these 3NF models cannot explain the \( A_y \) puzzle. In Ref. \[24\] a three-nucleon force generated by the exchange of one pion in the presence of a 2\( N \) correlation \[18\] has been suggested as a possible candidate to explain the \( A_y \) puzzle. This 3NF contribution is fixed by the 2\( N \) \( T \)-matrix describing the underlying 2\( N \) interaction while the pion is "in flight". The expression for this force derived in Ref. \[18\] contains the off-shell 2\( N \) \( T \)-matrix, more precisely its subtracted part

\[
\hat{t}_{12}(\mathbf{p}_2, \mathbf{p}_1, z) = t_{12}(\mathbf{p}_2, \mathbf{p}_1, z) - v_{12}(\mathbf{p}_2, \mathbf{p}_1),
\]

where \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) are Jacobi momenta of nucleons 1 and 2, and the potential - like term \( v_{12}(\mathbf{p}_2, \mathbf{p}_1) \) contains only OPE/OBE - type diagrams. The subtraction in Eq. \[82\] is needed to take into account a cancellation effect which has been observed in Refs. \[18\] - \[24\]. This cancellation involves meson retardation effects of the iterated Born term, and the irreducible diagrams generated by subsumming all time ordering diagrams describing the combined exchange of two mesons amongst the three nucleons. In principle there are no free parameters to adjust, and the 3\( N \) force is completely determined by the 2\( N \) \( T \)-matrix. However, as has been shown in Ref. \[18\] in order to explain the \( A_y \) puzzle, instead of the subtracted \( T \)-matrix shown in Eq. \[82\], one has to use the amplitude defined according to the prescription

\[
\hat{t}_{12}(\mathbf{p}_2, \mathbf{p}_1, z) = c(z)\hat{t}_{12}(\mathbf{p}_2, \mathbf{p}_1, z) - v_{12}(\mathbf{p}_2, \mathbf{p}_1),
\]

with the effective parameter \( c(z) \), which represents an overall correction factor for the far-off-the-energy-shell 2\( N \) \( T \)-matrix. Ideally, this parameter should be one for the 2\( N \) potential to provide a reliable extrapolation of the 2\( N \) \( T \)-matrix down to \( z \approx 160 \) MeV. However, as has been shown in Ref. \[24\] none of the existing 2\( N \) \( T \)-matrices can guarantee the off-shell behavior that is needed for the explanation of the \( A_y \) puzzle with the parameter \( c(z) \) set to one. For example, in order to reproduce the \( nd \) experimental data with the Bonn B potential, the factor \( c(z) \) must be set to 0.73 for the energy 3 MeV.

The lesson one must learn from the above is that the off-shell behavior of the 2\( N \) \( T \)-matrix may play a crucial role in explaining the \( A_y \) puzzle, and the existing realistic potentials do not provide the off-shell behavior that is needed for the correct reproduction for \( A_y \) with the 3\( NF \) suggested in Ref. \[18\]. Note in this connection that, being fitted to the 2\( N \) scattering data, the existing realistic potentials lead to ambiguities in the off-shell behavior of the 2\( N \) \( T \)-matrix. In order to remove the off-shell ambiguities one has to find the way of constructing the 2\( N \) \( T \)-matrix as an inevitable consequence of the basic principles of quantum mechanics and the symmetries of QCD. The remarkable feature of the EFT approach is that it implies to build the theory of nuclear forces as a consequence of these principles. However, the standard EFT of nuclear forces does not predict the off-shell behavior of the 2\( N \) \( T \)-matrix. The cause of this is that in this theory the Schrödinger (LS) equation makes no sense without regularization and renormalization. On the other hand, there is no reason to restrict ourselves to the assumption that low energy nucleon dynamics is governed by the Schrödinger equation: Only the generalized dynamical equation must be satisfied in any case, and, as we have seen, the requirement that the solutions of this equation that are consistent with the symmetries of QCD correspond to a nonlocal-in-time interaction operator when the generalized dynamical equation cannot be reduced to the Schrödinger equation. At the same time, as has been shown in Ref. \[18\], the nonlocality in time of the interaction results in an anomalous off-shell behavior of the \( T \)-matrix. In other words, the formalism of the QCD predicts that in order that the 2\( N \) \( T \)-matrix to be consistent with the symmetries of QCD it must have the off-shell behavior that cannot take place in the case of the ordinary potentials. From this point of view, the "anomalous" off-shell behavior that the 2\( N \) \( T \)-matrix must have for the 3\( NF \) be able to explain the \( A_y \) puzzle may be considered as a manifestation of the fact that low energy nucleon dynamics is really non-Hamiltonian.

IX. SUMMARY AND DISCUSSION

We have shown that from the Weinberg analysis of time-ordered diagrams for the 2\( N \) \( T \)-matrix in ChPT it follows that nucleon dynamics at low energies is governed by the generalized dynamical equation with a nonlocal-in-time interaction operator. A remarkable feature of the generalized dynamical equation which follows straightforwardly from the first principles of quantum mechanics is that it allows one to construct all physical amplitudes relevant for the theory under consideration by using the amplitudes describing processes in which the duration of interaction is infinitesimal. It is natural to assume that the most of contribution to these amplitudes comes from the processes associated with a fundamental interaction in a quantum system. This point manifests itself in the boundary condition \[4\]. If we do not consider a theory that is valid up to infinitely high energies (infinitesimal times), then the above infinitesimal of the duration times of interaction should mean that being much smaller than the scale of the theory these times may be much larger than the time scale of the underlying high energy physics. This in turn means that the amplitudes describing the "fundamental" interaction in the low energy theory can be computed in terms of the underlying high energy physics. Thus the generalized dynamical equation allows one to take into account that every theory with which we deal is a low energy approximation to a more fundamental one and provides a bridge between them.

The generalized dynamical equation can be represented in the form of the differential equation \[17\] for the
operator $T(z)$. The boundary condition on the generalized dynamical equation of this form is shown in Eq. \( \mathcal{S} \), where the operator $B(z)$ describes the fundamental interaction in this system. By definition, this operator must be so close to the true $T$-matrix in the limit $|z| \to \infty$ that the generalized dynamical equation with the boundary condition \( \mathcal{S} \) have a unique solution. The above means that really this region of "infinite" energies with which we have to deal is the domain $\mathcal{D}$ that lies much above the scale of the low energy dynamics but much below the scale of the underlying high energy physics. Correspondingly for describing the low energy dynamics we have to start with the boundary condition \( \mathcal{D} \) that implies that the interaction operator $B(z)$ is so close to the relevant $T$-matrix inside the domain $\mathcal{D}$ that Eq. \( \mathcal{D} \) with this initial condition has a unique solution. In the theory of nuclear forces this domain is a region of energies that are high enough for the most of contribution to the $2N$ $T$-matrix to come from processes that are described by the irreducible $2N$ diagrams for the $2N$ $T$-matrix (these processes are associated with a "fundamental" interaction), but not so high for the heavy degrees of freedom manifest themselves explicitly. By using the analysis of the time-ordered diagrams for the $2N$ $T$-matrix in ChPT inside the domain $\mathcal{D}$ where the structure of the theory is much simpler than in the low energy region, we can obtain the $NN$ interaction operator which should be used in the boundary condition \( \mathcal{D} \) on Eq. \( \mathcal{D} \). We have shown that the dynamics which is generated by the interaction operator obtained in this way is non-Hamiltonian. This is because, for low energy nucleon dynamics to be Hamiltonian, the operator $T(z)$ must have a negligible dependence on $z$ inside the domain $\mathcal{D}$, while, as we have shown, this is not the case. In other words, in the nonrelativistic limit QCD leads through ChPT to low energy nucleon dynamics that is not governed by the Schrödinger equation. However, this does not mean that the low energy predictions of QCD are not consistent with quantum mechanics. This means only that in this case we deal with a nonlocal-in-time interaction when the generalized dynamical equation cannot be reduced to the Schrödinger equation.

We have shown that from the fact that at extreme low energies nucleons emerge as the only effective degrees of freedom it follows that the evolution operator $U_{2N}(t, t_0)$ defined on the $2N$ subspace $\mathcal{H}_{2N}$ is unitary and can be represented in the form \( \mathcal{D} \). This in turn means that the $2N$ $T$-matrix that enters Eq. \( \mathcal{D} \) must satisfy the generalized dynamical equation. In other words the dynamics in a consistent effective theory of nuclear forces must be governed by the generalized dynamical equation, and the problems with UV divergences can arise only if we restrict ourselves to the assumption that the effective $NN$ interaction is instantaneous and hence the generalized dynamical equation is equivalent to the Schrödinger equation. What we have to do then is to obtain the interaction operator $B(z)$, i.e., the operator that is so close to the true solution that the generalized dynamical equation with the boundary condition \( \mathcal{S} \) has a unique solution. Ideally, this operator should be derived from QCD. However, for describing low energy nucleon dynamics with a given accuracy one need not to know the operator $B(z)$ exactly. It is sufficient to know the effective interaction operator $B^{(N)}_{\text{eff}}(z)$ that determines the solution of Eq. \( \mathcal{D} \) with the same accuracy. The advantage of using such an interaction operator whose form manifests the symmetries of QCD is that in this case the effects of high energy physics on low energy nucleon dynamics are parameterized by a few constants. The effective interaction operator $B^{(N)}_{\text{eff}}(z)$ determines the set $\Omega_N$ of the solutions of Eq. \( \mathcal{D} \) that coincide with the true two-nucleon $T$-matrix with the accuracy up to the order $(Q/\Lambda)^{2N}$, i.e., it represents the set of the interaction operators $B(z)$ that determine the above solutions. This uncertainty in specifying the initial condition for the generalized dynamical equation manifests the fact that we do not know the details of the underlying physics. The effects of this physics on the low energy dynamics are parameterized by the constants $c_2$, ..., $c_{2N}$ and $J_1$, ..., $J_N$ containing in the operator $B^{(N)}_{\text{eff}}(z)$ that is shown in Eq. \( \mathcal{D} \). As $N$ increases the set $\Omega_N$ of the solutions of Eq. \( \mathcal{D} \) that are determined by the effective interaction operator $B^{(N)}_{\text{eff}}(z)$ become smaller and smaller. This means that in this case one approaches to the true solution closer and closer, because the true $2N$ $T$-matrix enters each of these sets. Each of the $T$-matrices belonging to the set $\Omega_N$ corresponds to some function $\varphi_N(p/\Lambda)$, which must be such that the effective interaction operator satisfies Eq. \( \mathcal{D} \). This requirement puts the constraints on the function $\varphi_N(p/\Lambda)$: it must satisfy the condition \( \mathcal{D} \) where the parameters $J_n$ enter the effective interaction operator through the function $M_n(z)$. In other words, the effective interaction operator contains all parameters that appear in the theory at a given order, and these parameters need not be redefined in the process of calculations. From this point of view the only difference of the effective theory from a full one is uncertainty in specifying the boundary condition on the generalized dynamical equation, and the boundary condition \( \mathcal{D} \) with the effective interaction operator of the form \( \mathcal{D} \) means that one may choose any function $\varphi_N(p/\Lambda)$ satisfying the conditions \( \mathcal{D} \) and \( \mathcal{D} \). This is equivalent to the choice of the interaction operator $B(z)$ that formally determines a unique solution of Eq. \( \mathcal{D} \). Thus being formulated in terms of the GQD the effective theory of nuclear forces can be put on the same firm theoretical grounds as the quantum mechanics of atomic phenomena. In this case we deal with a well defined equation of motion that allows one to construct not only the scattering amplitudes but also the off-shell $T$-matrix, and the evolution and Green operators.

Our formalism may be regarded as a new way of realizing the Weinberg program for physics of the two-nucleon systems based on the use of the generalized dynamical equation instead the LS equation. In this case at the
first stage of the program we derive from the analysis of the diagrams ChPT an effective interaction operator. In the $^1S_0$ channel of the pionless theory this operator is of the form $a$. Solving the generalized dynamical equation with this interaction operator yields the expression for the $2N T$-matrix shown in Eq. (58). This $T$-matrix leads to the KSW expansion of the scattering amplitude shown in Eq. (10). However, in contrast with standard approach to the EFT of nuclear forces where this expansion is obtained by summing the bubble diagrams and performing regularization and renormalization, we reproduce the same result in a consistent way free from UV divergences. At the same time, Eq. (55) determines the off-shell $2N T$-matrix in the $^1S_0$ channel up to a few constants that parameterize the effects of the underlying physics on low energy nucleon dynamics. In other words, the requirement that low energy nucleon dynamics be consistent with the symmetries of QCD leads to a constrain on the off-shell behavior of the $2N T$-matrix, and one of the lessons that one must learn from the results of our work is that at very low energies (in the pionless theory) the $^1S_0$ off-shell $2N T$-matrix consistent with the basic principles of quantum mechanics and the symmetries of QCD must, be of the form $a$. With this $T$-matrix in hand implementation of the third step of the Weinberg program can proceed without resorting to regularization and renormalization. For example, formula (65) we have derived allows one to calculate, in a consistent way free from UV divergences, the $2N$ scattering amplitudes in the presence of an external potential to subleading order. This potential, for example, may be the Coulomb potential. Of course, in this case an additional term describing the contribution from gauge-invariant, four-nucleon, one-photon operator that arise at next-to-leading order must be included. However, in our case this term does not play a role of a regulator, like the counterterms that are used in the standard EFT approach for calculating such observables as the electric-quadrupole moment of the deuteron. It should be emphasized that, if the expression shown in Eq. (55) where divergent, then this term could not absorb divergences for all momenta of the nucleons. Moreover, the same off-shell $2N T$-matrix can be used in describing the interaction of the $2N$ system with other probes, in the three-body continuum calculations, and in solving the microscopic nuclear structure problems. The advantage of the formulation of the effective theory of nuclear forces based on the GQD is that it allows one to construct the $2N$ interaction operator and hence the off-shell $2N T$-matrix as inevitable consequence of the basic principles of quantum mechanics and the symmetries of QCD. This opens new possibilities for solving many problems in nuclear physics, and, in particular, for explaining the $A_g$ puzzle.

In this paper we focused on the pionless theory because in this simple case the proposed way to formulate the effective theory of nuclear forces can be investigated in detail, and its ideas can be verified exactly. At the same time, the approach should be applicable to the theory with pions. The only problem is that in this case the definition of a consistent power counting scheme and the derivation of the effective operator of the $NN$ interaction from the analysis of the diagrams in ChPT are more complicated. However, once this operator is constructed the generalized dynamical equation can be used for performing not only analytic but also numerical calculations of any observables without regularization and renormalization.