Tight Bounds for Maximal Identifiability of Failure Nodes in Boolean Network Tomography

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Abstract—We study maximal identifiability, a measure recently introduced in Boolean Network Tomography to characterize networks’ capability to localize failure nodes in end-to-end path measurements. Under standard assumptions on topologies and on monitors placement, we prove tight upper and lower bounds on the maximal identifiability of failure nodes for specific classes of network topologies, such as trees, bounded-degree graphs, $d$-dimensional grids, in both directed and undirected cases. Among other results we prove that directed $d$-dimensional grids with support $n$ have maximal identifiability $d$ using $nd$ monitors; and in the undirected case we show that $2d$ monitors suffice to get identifiability of $d - 1$. We then study identifiability under embeddings: we establish relations between maximal identifiability, embeddability and dimension when network topologies are modelled as DAGs. Through our analysis we also refine and generalize results on limits of maximal identifiability recently obtained in [11] and [1]. Our results suggest the design of networks over $N$ nodes with maximal identifiability $Ω(\sqrt{\log N})$ using $2\sqrt{\log N}$ monitors and heuristics to place monitors and edges in a network to boost maximal identifiability.

Index Terms—Boolean Network Tomography, Node Failure Localization, Maximal Identifiability, Hypergrids, Embeddings, Dimension

I. INTRODUCTION

Monitoring a network to localize corrupted components is essential to guarantee a correct behaviour and the reliability of a network. In many real networks direct access and direct monitoring of the individual components is not possible (for instance because of limited access to the network) or unfeasible in terms of available resources (protocols, communications, response-time etc.). A well-studied approach to localization of failing components is network tomography. Network tomography focuses on detecting the state of single components in the network by running a measurement process along the network. The process starts by sending packets (containing suitable data to capture interesting failures) from specific source-monitor nodes and terminates receiving another data packet on other specific target-monitor nodes. Measurement is done along a set of end-to-end paths, each one starting and ending with a monitor node. In this work we focus on the problem of detecting node states (failing/working), using a boolean network tomography approach [4], [5] where the received data at each monitor is one bit (failure (1) /working (0)), capturing the presence or the absence of a failure along a path. We are interested in identifying (uniquely) failure nodes. Receiving a 0 (working state) at an end monitor of a path means that each node in the path is working properly. Then the localization of failing nodes in a set of paths $\mathcal{P}$ (or a network view as a set of paths) is captured by the solutions to the following boolean system:

\[ \bigwedge_{p \in \mathcal{P}} \left( \bigvee_{v \in p} x_v \equiv b_p \right) \]

where $\vec{b}$ is a vector of boolean values (corresponding to final measurement in the paths) and $x_v$ are boolean variables, one for each node $v$. Any solution to this system is a possible placement of node-failures satisfying the measurements. Since working paths ($b_p = 0$) entail $x_v = 0$ for all $v \in p$, we can assume that the system is (in fact equivalent to) a boolean formula in Conjunctive Normal Form on positive variables.

A. The Problem

A set of (non monitor) nodes failing simultaneously is a failure set. Each solution to Eq. [1] captures a failure set that can occur in the network according to the measurements. But as readily seen solutions to Eq. [1] are often multiple. Ma et al. in [13] proposed a parameter measuring the ability of a network of capturing the maximum number of simultaneous failure nodes which are uniquely identifiable. This is called maximal identifiability (Definition [13]). Recently the study of maximal identifiability received a lot of attention due to its importance in applications in boolean network tomography [1], [9]–[13], [16].

In this work we focus on this measure and we study upper and lower bounds for maximal identifiability of specific classes of network topologies.

B. Previous Works and Contributions

Identifiability as defined in [13] (Definition 1) captures the combinatorial property that to separate two sets $U$ and $W$ (of failure nodes) one wants to exhibit a measurement path in $\mathcal{P}$ touching nodes of exactly one of the two sets. The maximal size of sets in $\mathcal{P}$ of failure nodes one can guarantee identifiability for, is then a measure of the ability of $\mathcal{P}$ to identify failure sets uniquely. For link failure identifiability [11] there are structural results on graphs proving necessary and sufficient conditions for identifiability metrics based on links. In the work [1] the authors studied upper bounds on the maximum identifiability of failure nodes with a given number

1 Notice we are not including monitor nodes in the boolean system.
of monitoring paths in several concrete and applied network scenarios (i.e. consistency routing see [1] and Definition [V.1] with the aim of design optimal networks for node failure identifiability.

We explore what maximal identifiability of node failure requires and implies in terms of network topologies. Under standard assumptions on topologies and on monitors placement, we prove upper bounds on the maximal identifiability of specific classes of network topologies, such as trees, bounded-degree graphs, d-dimensional grids, in both directed and undirected cases. Instead of checking experimentally the optimality of the upper bounds, we give an algorithmic/combinatorial analysis of lower bounds for maximal identifiability obtaining tight bounds. This approach naturally leads to algorithms to devise network topologies with a guarantee of reaching a precise maximal identifiability of failure nodes. Since d-dimensional grids play an important role in our results, we start the study of maximal identifiability of node failure under embeddings of DAGs. We establish relations between maximal identifiability, embeddability and dimension of networks modelled by DAGs. Results which also entail algorithmic approaches to approximate the maximal identifiability of failure nodes in any (directed acyclic) network topology.

C. Results and Organization

We study maximal identifiability µ for several classes of network topologies both directed and undirected. First we reformulate an equivalent definition of maximal identifiability (Definition [II.3]) in terms of set operations, which entails a clean combinatorial approach to prove lower and upper bounds for µ. After some preliminaries in Section II on basic notions, in Subsection II-C we explain our network models and some assumptions we work with. The two assumptions simple-path-free and balanced-monitor trees allow to avoid trivial and uninteresting cases in studying maximal identifiability on the network topology associated to a set of measurement paths (see Lemmas [II.6], [II.7], [II.10], [II.11]).

In Subsection II-D we start proving a structural result: namely that independently from the topology, µ is always (strictly) upper bounded by the max between the number of input and output monitors. In particular this implies that with 2 monitors there is no identifiability of failures at all. This result generalizes (but also simplifies the proof of) the analogous result of [1] stating that link-failure identifiability needs strictly more than two monitors independently of the topology.

In Section III we consider directed topologies. We prove that for trees maximal identifiability is exactly 1 (Theorem III.3). After looking to the case of bounded degree graphs and proving that µ is upper bounded by the minimal in-degree (Lemma III.4), we consider the question whether there are (directed) topologies with maximal identifiability strictly greater than 1. Theorem III.9 gives a positive answer proving that for directed n x n grids, maximal identifiability is exactly 2. In Theorem III.10 we lift to d-dimensional directed grids for d > 2 and generalizing previous theorem we prove that maximal identifiability is exactly d. According to our models, there are nd monitors in such grids.

Section IV is focused on undirected topologies. Since we loose directions, monitor placement and assumptions on the models here are slightly different from the directed case (see Subsection IV-C). After obtaining (in Theorem IV.3 and Lemma [IV.4]) the same bounds for the directed case for trees and bounded-degree graphs, we look at the case of grids. We can exploit the greater number of paths in the undirected hypergrid and prove in Theorem IV.6 that reducing the number of monitors from nd to 2d, maximal identifiability in d-dimensional grids remains at least d − 1 and at most d.

Section V starts the study of the relations between identifiability, embeddings and dimension of DAG and posets. We observe that if a DAG G is embeddable into a d-hypergrid then its dimension (see Preliminaries) is bounded by the maximal identifiability of the hypergrid. We explore other possible connections: we present a simple example proving that is not possible to relate precisely maximal identifiability with embeddability. Nevertheless we explore two directions: (1) restricting the class of topologies we want to embed and (2) restricting the mapping that defines the embedding. In the first case we prove in Theorem V.2 that the maximal identifiability of a (directed) network G which satisfies routing consistency (see [1] and Definition [V.1]) is upper bounded by the maximal identifiability of any other DAG G′ where G is embeddable to. In the second case we consider embeddings which preserve some distance properties (Definition [V.1]). We prove (Theorem V.4) that if a DAG G is embeddable into a DAG G′ by a distance preserving embedding then they have same maximal identifiability. Finally we obtain a general result (Theorem V.7) proving that if a DAG G is closed under transitivity and embeddable into another DAG G′, maximal identifiability can only increase. This is sufficient to prove that for any DAG G its maximal identifiability is lower bounded by the dimension of its transitive closure.

In Section VI we discuss applications of our results. In Subsection VI-A we propose an algorithm to build a network on N nodes where maximal identifiability is \(\sqrt{\log N}\) using only \(2\sqrt{\log N}\) monitors. In Subsection VI-B we propose an heuristic to boost maximal failure identifiability of a network \(\mathcal{P}\) already built. The idea is to “simulate” in \(\mathcal{P}\) an hypergrid of dimension \(d \leq \log N\). In the final version we will present some experiments validating the heuristics. Finally in Subsection VI-C we refine and simplify (in Theorem VI.1 and Corollary VI.2) results on limits on maximal identifiability recently showed in [1].

In the last Section VII we point out some directions raising from this approach we think worthy to be further explored. Due to page limitations some proofs are omitted in this draft. Omitted proofs can be found in the appendix.

II. Preliminaries

A. Sets, Graphs, Paths, Posets, Embeddings

For a set \(U\) let \(\chi_U\) be its indicator variable, i.e. \(\chi_U(x) = 1\) iff \(x \in U\). \(U \Delta V = (U \setminus V) \cup (V \setminus U)\) is the symmetric
difference between $U$ and $V$. Notice that $\chi_{U \triangle V} = \chi_U \oplus \chi_V$.

In a graph $G = (V, E)$, $V$ is a set of nodes and $E \subseteq V \times V$. $G$ is undirected if pairs in $E$ are unordered, i.e. $(uv) \in E$ iff $(vu) \in E$. Otherwise $G$ is directed. $G$ is DAG if a directed graph with no cycle. A path $p$ in $G$ from a node $u$ to a node $v$ is a sequence of edges $p = (u_1u_2), (u_2u_3) \ldots (u_{k-1}u_k)$ such that $u_1 = u$ and $u_k = v$ and $(u_iu_{i+1}) \in E$ for all $i \in [k-1]$. If $G$ is a DAG, then we identify the path $p$ also with sequence of nodes $u_1 \ldots u_k$.

For a node $u$ in $G$, $N(u)$ is the set of neighbourhood of $G$, i.e. $\{v \in V \mid (vu) \in E\}$. The degree of $u$, $\deg(u)$, is the cardinality of $N(u)$. The degree of $G$ is $\Delta(G) = \max_{u \in V} \deg(u)$. We also consider the minimal degree $\delta(G)$ of $G$.

If $G$ is directed then we distinguish $N_{\text{i}}(u)$, the set of neighbourhood $v$ of $u$ s.t. $(vu) \in E$, from $N_{\text{o}}(u)$, the neighbourhood $v$ of $u$ s.t. $(uv) \in E$. For all degree measures on $G$ we distinguish the in-degree $\Delta_{\text{i}}(G)$ and $\delta_{\text{i}}(G)$ and the out-degree $\Delta_{\text{o}}(G)$, and $\delta_{\text{o}}(G)$. In a DAG $G$, we denote by $S$, the source nodes, i.e. nodes $u$ in with $\deg(u)$ 0, and by $T$ the target nodes $v$ with out-degree 0.

**Definition II.1 (Directed hypergrids).** Let $d \in \mathbb{N}^+$ and $n \in \mathbb{N}$, $n \geq 4$. The (directed) hypergrid of dimension $d$ over support $[n]$, $\mathcal{H}_{n,d}$, is the graph with vertex set $[n]^d$ and where there is a directed edge from a node $x = (x_1, x_2, \ldots, x_d)$ to a node $y = (y_1, y_2, \ldots, y_d)$ if for some $i \in [d]$ we have $y_i = x_i \neq 1$ and $x_j = y_j$ for all $j \neq i$.

In the case of undirected hypergrid in $\mathcal{H}_{n,d}$ there is an edge between a node $x$ and a node $y$ if for some $i \in [d]$ we have $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. In the case of simple grids over $n$ nodes, i.e. $d = 2$, we use the notation $\mathcal{H}_n$, $\delta_{\text{i}}$ is the set of nodes $x = (x_1, x_2, \ldots, x_d)$ such that $x_1 = 1$.

A border node is a node of $\mathcal{H}_{n,d}$ which is also in some $\delta_i$.

We consider directed root trees $\mathcal{T}_n$ over $n \in \mathbb{N}^+$ nodes (from now on we omit that $n \in \mathbb{N}^+$). An (undirected) tree is an acyclic graph with no cycle when any two nodes $u$ and $v$ are connected by a path. We consider: (1) downward directed trees $\mathcal{T}_n$ where the root of $\mathcal{T}_n$ is the only source node and the leaves of $\mathcal{T}_n$ are the only target nodes (i.e. $\Delta_{\text{i}}(\mathcal{T}_n) \leq 1$) and (2) upward directed trees $\mathcal{T}_n$, where the root is the only target node and the leaves are source nodes (i.e. $\Delta_{\text{o}}(\mathcal{T}_n) \leq 1$).

Let $\mathcal{P}$ be a set of paths over nodes $V$. For a node $v \in V$, let $\mathbb{P}_\mathcal{P}(v)$ be the set of paths in $\mathcal{P}$ passing through $v$ (we will omit index $\mathcal{P}$ from now on since it will always be clear). For a set of nodes $U$, $\mathbb{P}(U) = \bigcup_{u \in U} \mathbb{P}(u)$. Hence if $U \subseteq V$, $\mathbb{P}(U) \subseteq \mathbb{P}(V)$.

Each DAG $G = (V, E)$ is equivalent to a poset with elements $V$ and partial order $\preceq_G$. Here $u \preceq_G v$ if $v$ is reachable from $u$ in $G$. Elements $u$ and $v$ are comparable if $u \preceq_G v$ or $v \preceq_G u$, and incomparable otherwise. We write $u \prec_G v$ if $u \preceq_G v$ and $u \neq v$.

A mapping $f$ from a poset $G = (V, E)$ to a poset $G' = (V', E')$ is called an embedding if it respects the partial order, that is, all $u, v \in V$ are mapped to $u', v' \in V'$ such that $u \preceq_G v$ if $u' \preceq_{G'} v'$. If $G$ is embeddable into $G'$ we write $G \rightarrow G'$. It is immediate to see that embeddings are one-to-one (i.e. injective) mappings.

**Definition II.2 (Poset dimension [1]).** Let $G$ be a poset with $n$ elements. The dimension of $G$, $\dim(G)$, is the smallest integer $d$ such that $G \rightarrow \mathcal{H}_{n,d}$.

Dushnik and Miller [2] proved that for any $n > 1$, the hypergrid $\mathcal{H}_{n,d}$ has dimension exactly $d$.

**B. Identifiability**

A preliminary definition of identifiability was introduced in [13] and refined and studied in [1], [11], [12] among others. We formulate identifiability in terms of set operations. Let $\mathcal{P}$ be a set of paths over $n$ nodes $V$.

**Definition II.3 ($k$-identifiability [11]).** $\mathcal{P}$ is $k$-identifiable iff the following property holds: for all $U, W \subseteq V$, with $U \Delta W \neq \emptyset$ and $|U|, |W| \leq k$, it holds that $\mathbb{P}(U) \Delta \mathbb{P}(W) \neq \emptyset$.

Monotonicity of identifiability (a property noticed in several works [1], [11], i.e. that $k$-identifiability implies $k'$-identifiability for $k' < k$, is trivial from our definition. Notice moreover that the definition (given in [1], [11]) of $k$-identifiability with respect to a node set $S$, is equivalent to our definition refining $U \Delta W \neq \emptyset$ with $U \cap S \Delta(V \cap S) \neq \emptyset$.

**Definition II.4 (Maximal identifiability).** Let $\mathcal{P}$ be a set of paths. The maximal identifiability $\mu(\mathcal{P})$ of $\mathcal{P}$ is the $\max_{k \geq 0}$ such that $\mathcal{P}$ is $k$-identifiable.

We want to compute, possibly exactly, the value for the maximum identifiability for sets of paths forming specific topologies. Following Definition II.3, to prove that a set $\mathcal{P}$ is not $k$-identifiable it is sufficient to find two distinct node sets $U$ and $W$ of cardinality at most $k$, and prove that $\mathbb{P}(U) \Delta \mathbb{P}(W) = \emptyset$. Notice that by the monotonicity property of identifiability, this implies that $\mu(\mathcal{P}) \leq k - 1$, hence upper bounds on $\mu(\mathcal{P})$. Lower bounds on $\mu$ can be proved by arguing that for all distinct node sets $U$ and $W$ of cardinality at most $\mu(\mathcal{P})$, $\mathbb{P}(U) \Delta \mathbb{P}(W) \neq \emptyset$, that is one can always build a path touching exactly one between $U$ and $W$. Lower bounds on $\mu(\mathcal{P})$ are hence interesting since to prove them we have to provide algorithms to build paths that distinguish between any two (failure) node sets of cardinality at most $\mu(\mathcal{P})$.

**C. Physical Networks and Topology Models**

Boolean Network Tomography concerns with localizing failure nodes in a set of end-to-end measurement paths. We discuss the assumptions we do modelling a physical network or a set of measurement paths and then with a graph topology. We assume the paths $\mathcal{P}$ are defined on a set of nodes $V$ and a set of monitor nodes. We assume the paths to be either all directed or all undirected. Monitor nodes are considered different from nodes in $V$. Associated to $\mathcal{P}$ we consider a graph $G_P = (V, E)$, where the edges $E$ encode the wires between nodes $V$ in $\mathcal{P}$. We do not admit multi-edges in $E$. i.e. if two different paths pass through the same edge $(uv)$, we count it only once in $E$. $G$ will be directed or undirected according to $\mathcal{P}$.
1) MONITORS: We assume to know if each monitor is an input-monitor (denoted by m) where the measurement signal is routed out and output-monitors (denoted by M) where the boolean failure/correct message is received. Monitor nodes will be not part of the topology and hence not counted in V. But in order to measure identifiability in some cases we want to count in the degree of a node \( u \in V \) the number of monitors it was linked to in \( P \). Analogously for target nodes. Notice that this model works also if we want to design a real network \( P \) from a logical model \( G \).

2) DIRECTED TOPOLOGIES: In the case of a directed set \( P \) the information about the monitor nodes will be explicit in \( G_P \). We consider two subsets \( S \) (sources) and \( T \) (targets) of \( V \), including respectively the nodes in \( P \) linked to input-monitors and output-monitors. For directed topologies the (in)-degree of a source node \( u \in S \) is exactly the number of monitors \( w \) was linked to in \( P \). Analogously for target nodes. Notice that this is a path starting and ending in \( u \), hence among the paths in \( G \) to-end measurement paths, in such a way that all and only the paths in \( G_P \) correspond to paths in \( P \). But since we do not keep monitor nodes explicitly in \( G_P \) and we can have simple measurement paths like \((m_u)(u_M)\) we generalize the definition of path in graphs. A degenerate path in \( G_P \) is a path made by only one node \( u \), iff \( u \in P \) is linked to both an input- and an output-monitor.

3) UNDIRECTED TOPOLOGIES: The difference with the directed case is that we no longer have source and target nodes in \( G \). So formally in this case our graph \( G_P = (V,E) \) is completed by a set of nodes \( Z \subseteq V \). \( Z \) are the nodes in \( G_P \) linked to a monitor in \( P \). In the undirected case we choose not to save in the degree of a node \( u \) the info about the number of monitors it was linked to in \( P \) (though we might do that). The reason is that in an undirected network messages can flow in each direction, while the link monitor-nodes are always only in one direction. We assume that in graph \( G \), from each node \( u \) it is possible to reach two different nodes in \( Z \). As said, the degree is relevant only for real nodes, so we merge different monitors linked to the same node in the physical network into one monitor in the logical topology. For instance if \( P \) was made by two paths \( p_1 = (m_1u)(uw)(wM_1) \) and \( p_2 = (m_2u)(uw)(wM_2) \), its logical model is a graph made by the only edge \((uw)\). We are interested in finding a relation between the topology of a network and its maximal identifiability. For this reason we assume that our graph (both directed and undirected) do not contain nodes with degree less than or equal to 2. Notice that by our assumption on monitors a node linked to a monitor might have degree 2. The simple path free assumption for undirected graphs \( G_P = (V,E) \) and \( Z \subseteq V \) associated to a set of measurement paths \( P \) is hence as follows: all nodes \( u \in V \setminus Z \) have \( \deg(u) \geq 3 \). In the case of directed graphs, the simple path free assumption requires that there is no node \( u \) in \( G \) such that both \( \deg_i(u) \leq 1 \) and \( \deg_o(u) \leq 1 \). Previous Lemma has an analogous for the case of directed graphs, that justify this assumption, which can be proved the same way.

**Definition II.5.** The hitting number of \( G_P \), \( h(G_P) \) is defined as follows: for an edge \( e \) in \( G_P \), \( h(e) = |\{ p \in P : e \in p \} | \) and \( h(G_P) = \max_{e \in G_P} h(e) \).

**Definition II.8.** Let \( T_n \) be a SPF undirected tree. We say that \( T_n \) is an input tree if there is a node in \( T_n \) linked to a source-monitor. We say that \( T_n \) is an output tree if there a node in \( T_n \) linked to a target-monitor.
A tree can be both an input and an output tree. Given a tree $T_n$ and one of its edges $e = (w, v)$, let $T^e(w)$ (resp.$T^e(v)$) be the subtree of $T_n$ obtained from cutting the edge $(u, v)$ and taking the tree rooted at $u$ (resp. $v$).

**Definition II.9.** A SPF undirected tree $T_n$ is balanced if for each node $u$ in $T_n$

1) there are $w, z \in N(u)$ such that $T^{(uw)}(w)$ and $T^{(uz)}(z)$ are both input trees;
2) there are $w, z \in N(u)$ such that $T^{(uw)}(w)$ and $T^{(uz)}(z)$ are both output trees.

Previous definition is motivated by the fact non-balanced trees are very weak from identifiability perspective:

**Lemma II.10.** If $T_n$ is SPF tree but not MB, then $\mu(T_n) < 1$.

**Proof.** If $T_n$ is not balanced there will always be a node $u$ in $T_n$ and a neighbour $w$ of $u$ where exactly the same paths are passing through. Since $T_n$ is SPF, then for all $v \in V \setminus Z$, $\deg(u) \geq 3$. Assume you have a node $u$ with three neighbours $w_1, w_2, w_3$ such that $T_1 = T^{(uw_1)}(w_1)$, $T_2 = T^{(uw_2)}(w_2)$ are input trees and $T_3 = T^{(uw_3)}(w_3)$ is an output tree. Then from $u$ and $w_3$ pass exactly the same paths. Analogously if $T_1$ and $T_2$ are output and $T_3$ input trees. Fix $U = \{u\}$ and $W = \{w_3\}$. We have found two disjoint sets of cardinality 1 where exactly the same paths are passing through. Hence $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. Hence $\mu(T_n) < 1$.

Moreover in SPF-MB trees, there are at least two different paths passing through each edge.

**Lemma II.11.** If $T_n$ is SPF-MB undirected tree, then each edge $e$ belongs to at least two different paths.

**Proof.** Since $T_n$ is SPF, then $\deg(u) \geq 3$ for all $u \in V \setminus Z$. Fix 3 neighbours $w_i, i = 1, 2, 3$ of $u$ and let $e_i = (uw_i)$ and $T_i = T^{e_i}(w_i)$. It is easy to argue that by MB assumption at least two of them have the same label (input/output) and the other has the opposite label. Hence there is at least an $e_i$ such that two different (end-to-end) paths are touching $e_i$.

D. Node Failure Identifiability needs more than 2 monitors

As mentioned above we distinguish between input and output monitors. Only for the following result (where we do no talk specifically about directed or undirected graphs), we do not consider any of the assumptions we made before.

**Theorem II.12.** Let $G$ be a (connected) graph modelling a set of measurement paths $\mathcal{P}$. Let $m$ be the number of input monitors and $M$ the number of output monitors in $\mathcal{P}$. Then independently of the topology $G$, $\mu(G) < \max(m, M)$.

**Proof.** If every monitor is linked to one and only one node in $G$, then we fix $U$ to be the set of nodes in $G$ linked to the input monitors and $W$ to be the set of nodes in $G$ linked to the output monitors. Now, it is obvious that $|U|, |W| \leq \max(m, M)$. Since our topology is connected, there is no way of separating $U$ from $W$ with a path going from an input monitor to an output monitor. We will always touch both. Thus $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$ and $\mu(G) < \max(m, M)$. Assume now monitors are linked to more than one node in $G$. For each monitor $m_i$, we select one node $n$ among the one in $G$ $m_i$ is linked to and add it to the set $U$, and we select another node $w$ linked to an output monitor and add it to the set $W$. We know there is a path from $u$ to $w$ since $G$ is connected. Now as above there exist $U$ and $W$ of cardinality at most $\max(m, M)$ such that $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. Therefore we have $\mu(G) < \max(m, M)$.

Since to run measurements in a set of paths we need at least one input and one output monitor, previous theorem immediately implies that two monitors in a set of measurement give no identifiability at all.

**Corollary II.13.** If a set of measurement paths $\mathcal{P}$ has only two monitors, then maximal identifiability is 0.

III. IDENTIFIABILITY FOR DIRECTED TOPOLOGIES

A. Trees

We start proving that in terms of identifiability directed trees are very weak, even assuming the SPF hypothesis that they do not contain simple paths.

**Lemma III.1.** Let $T_n$ a SPF directed tree. Then $\mu(T_n) \leq 1$.

**Proof.** Consider a node $u$ in $T_n$. Since $T_n$ is SPF $u$ has either in-degree $\geq 2$ or out-degree $\geq 2$. According to whether the tree is downward (paths from root to leaves) or upward (path from leaves to root), one of the two cases can happen:

- For the downward case, consider the node $u$ and its two children $v_1$ and $v_2$. Since $u$ is a leaf, its in-degree is 1. By SPF, at least one of $v_1$ or $v_2$ must have an edge pointing to $u$. Without loss of generality, assume it is $v_1$. Then, if $v_1$ is an input node, we can choose $U = \{u\}$ and $W = \{v_1\}$, and we have $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. Therefore $\mu(T_n) \leq 1$.

- For the upward case, consider the node $u$ with its two children $v_1$ and $v_2$. By SPF, at least one of $v_1$ or $v_2$ must have an edge pointing to $u$. Without loss of generality, assume it is $v_1$. Then, if $v_1$ is an input node, we can choose $U = \{u\}$ and $W = \{v_1\}$, and we have $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. Therefore $\mu(T_n) \leq 1$.

Lower bounds for maximal identifiability for directed trees follows by:

**Lemma III.2.** Let $T_n$ be a SPF tree. Then $\mu(T_n) \geq 1$.

**Proof.** Let $u$ and $w$ two distinct nodes in $T_n$. Let $U = \{u\}$ and $W = \{w\}$. Each node in $T_n$ is on some (end-to-end) path from the root to a leaf. If $u$ and $w$ lie on different paths, then clearly there are paths in $\mathbb{P}(U)$ but not in $\mathbb{P}(W)$. Hence $\mathbb{P}(U) \triangle \mathbb{P}(W) \neq \emptyset$. Assume there is a root-to-leaf path $p$ touching both $u$ and $w$ and $w$ and $p$ meets before $u$. Let $p_w$ be the subpath of $p$ truncated at node $w$. Let $w_1 \in N_e(w)$
be the neighbour of \( w \) lying on \( p \). Since \( \mathcal{T}_n \) is SPF there is necessarily another node \( w_2 \in N_\delta(w) \) and in \( \mathcal{T}_n \) there is a path \( q \) from \( w_2 \) to a leaf. Hence the concatenation of \( p_w \) with \( q \) is a path from the root to a leaf touching \( w \) but not \( u \). Hence in \( \mathbb{P}(W) \) but not in \( \mathbb{P}(U) \). Hence \( \mathbb{P}(U) \triangle \mathbb{P}(W) \neq \emptyset \). \( \square \)

**Theorem III.3.** Let \( \mathcal{T}_n \) be a directed SPF tree. Then \( \mu(\mathcal{T}_n) = 1 \).

**B. Bounded in-degree graphs**

The proofs given for trees can be extended to bounded degree graphs where maximal identifiability is upper bounded by the minimal in-degree of \( G \). Remember that \( \delta_1(G) \) and \( \Delta_1(G) \) are respectively the minimal and the maximal in-degree of \( G \).

**Lemma III.4.** Let \( G = (V, E) \) be a SLF directed graph. Then \( \mu(P) \leq \delta_1(G) \).

**Proof.** Let \( w \) be a node in \( G \) with \( \text{deg}_G(w) = \delta_1(G) \). Define \( W = N_1(w) \) and \( U = N_1(w) \cup \{w\} \). It is clear that each path passing through \( w \) is passing through a node in \( N_1(w) \), hence \( \mathbb{P}(\{w\}) \subseteq \mathbb{P}(N_1(w)) \). Therefore \( \mathbb{P}(U) = \mathbb{P}(W) \), which proves the claim since \( |U| = \delta_1(G) + 1 \). \( \square \)

**C. Grids**

We have seen in the previous Subsection that directed trees have maximal identifiability exactly 1. Can we find topologies whose maximal identifiability is strictly greater than 1? We start with 2-dimensional grid \( \mathcal{H}_n \). Since directed grids \( \mathcal{H}_n \) have minimal in-degree bounded by 2 by Lemma III.4 we have.

**Lemma III.5.** Let \( n \geq 3 \). Then \( \mu(\mathcal{H}_n) \leq 2 \).

Let \( S \) be the set of sources nodes in \( \mathcal{H}_n \) and \( T \) is the set of target nodes in \( \mathcal{H}_n \). Given a node \( u \in \mathcal{H}_n \), we also define \( T(v) \) the set of nodes in \( \mathcal{H}_n \) reachable from \( v \) and \( S(u) \), the set of nodes in \( \mathcal{H}_n \) \( u \) is reachable from.

\[
\begin{align*}
S(u) & = \{u\}, u \in S \\
S(u) & = \bigcup_{(v,u) \in \mathcal{H}_n} S(v), u \in V \setminus S \\
T(v) & = \{v\}, v \in T \\
T(v) & = \bigcup_{(v,u) \in \mathcal{H}_n} T(u), v \in V \setminus T
\end{align*}
\]

The following Lemmas give a way to build paths avoiding specific nodes.

**Lemma III.6.** Let \( n \geq 3 \). Let \( u \) be a node in \( \mathcal{H}_n \) and \( w \in S(u) \) with \( w \neq u \). There is path \( p_{w}^u \) from a node in \( S \) to \( u \) not touching \( w \).

**Proof.** By induction on \( m = |S(u)| \). If \( m = 1 \), then \( S(u) = \{u\} \) and \( u \) is in \( S \) and since \( w \neq u \), then \( p_{w}^u \) is the (degenerate) path made by the only node \( u \).

Assume \( |S(u)| = m > 1 \). Then \( |N_i(u)| = 2 \). Hence there is \( w_1 \in N_i(u) \) such that \( w_1 \neq w \). Since \( |N_i(u)| = 2 \), then \( S(w_1) \subseteq S(u) \) and hence \( |S(w_1)| < m \). By induction we have a path \( p_{w_1}^u \) from \( S \) to \( w_1 \) avoiding \( w \). Then define as \( p_{w}^u \), the path concatenating \( p_{w_1}^u \) with \( u \). \( \square \)

A similar proof holds also for the nodes reachable from \( u \). We omit details in this version.

**Lemma III.7.** Let \( n \geq 3 \). Let \( u \) be a node in \( \mathcal{H}_n \) and \( w \in T(u) \) with \( w \neq u \). There is path \( q_{w}^u \) from \( u \) to a node in \( T \) not touching \( w \).

**Lemma III.8.** Let \( n \geq 3 \). \( \mu(\mathcal{H}_n) \geq 2 \).

**Proof.** Let \( V \) be the set of nodes of \( \mathcal{H}_n \). We have to prove that for any \( U, W \subseteq V \) with \( U \neq W \) and such that \( |U|, |W| \leq 2 \), \( \mathbb{P}(U) \triangle \mathbb{P}(W) \neq \emptyset \). It is sufficient to find a path \( p \in \mathcal{H}_n \) from \( S \) to \( T \) touching exactly one between \( U \) and \( W \). We split in the following cases:

1) at least one between \( U \) and \( W \) has cardinality 1;
2) both \( U \) and \( W \) have cardinality 2.

**Case 1**

Assume wlog that \( W = \{w\} \). Since \( U \triangle W \neq \emptyset \), then there is a node \( u \in U \setminus W \), such that \( w \neq u \), \( w \) can be either in: (1) \( S(u) \); or (2) in \( T(u) \); or (3) in \( V \setminus (S(u) \cup T(u)) \). In case (3) any path \( p \) from \( S \) to \( T \) passing through \( u \) is not touching \( w \) and proves the claim. In case (1) we use Lemma III.6 to have a path \( p_{u}^w \) from \( S \) to \( u \) avoiding \( W \). Moreover, any path \( p \) from \( u \) to \( T \) is avoiding \( w \), then the composition of \( p_{u}^w \) with \( p \) proves the claim. In Case (2) any path \( p \) from \( S \) to \( u \) avoids \( w \), and Lemma III.7 guarantees a path \( q_{u}^w \) from \( u \) to \( T \) avoiding \( w \). Hence the composition of the paths \( p \) and \( q_{u}^w \) proves the claim.

**Case 2**

Observe that though \( U \triangle W \neq \emptyset \), they might share a node. So there might be two cases: (A) \( |U \cap W| = 1 \) and (B) \( |U \cap W| = 0 \). In case (A) we fix \( u \) to be the node of \( U \) not in \( W \). In case (B) say \( U = \{u_0, u_1\} \) we fix \( u \) to be the node in \( U \) not reachable in \( \mathcal{H}_n \) by the other node in \( U \), i.e. the \( u_i \) such that \( u_i \notin S(u_{i-1}) \). Notice that this node always exists since the nodes in \( U \) cannot reach each other in \( \mathcal{H}_n \). As in case (1) we divide in three cases according to the position of \( W \) wrt \( u \).

i. \( W \subseteq S(u) \);
ii. \( W \subseteq T(u) \);
iii. \( |S(u) \cap W| \leq 1 \) and \( |T(u) \cap W| \leq 1 \);

In case (iii) a similar argument as above works. Since \( |S(u) \cap W| \leq 1 \), then either if \( |S(u) \cap W| = 0 \) any path from \( S \) to \( u \) avoids \( W \), or (if \( |S(u) \cap W| = 1 \) we can apply Lemma III.6 to find a path \( p_u \) from \( S \) to \( u \) avoiding \( W \). Using \( |T(u) \cap W| \leq 1 \), a similar argument works for finding a path \( q_u \) from \( u \) to \( T \) avoiding \( W \). Hence the composition of \( p_u \) and \( q_u \) is a path from \( S \) to \( T \) passing from \( u \) but avoiding \( W \).
In case (i) we further distinguish two cases and fix the w as follows:

(A) \( |U \cap W| = 1 \), w is the only node in \( U \cap W \).

(B) \( w \) is any node in \( W \). Denote by \( v \) the other node in \( W \).

Since \( u \neq w \), then by Lemma \[III.6\] there is a path \( p_u^w \) from \( S \) to \( u \) avoiding \( w \). Moreover, since \( W \subseteq S(u) \) any path \( q_u \) from \( u \) to \( T \) avoids \( W \). Hence the path \( p_u \) concatenation of \( p_u^w \) with \( q_u \) touches \( U \) and, avoids \( W \), unless \( v \in p_u \). If \( v \in p_u \) then we modify \( p_u \) into a new path \( p_v \) touching \( W \) but avoiding \( U \). We first identify a node \( z \) on \( p_u \). Assume \( u \) to be the node \( u = (x_1, x_2) \) with \( x_1, x_2 \in [n] \). Since \( p_u^w \) is ending at \( u \) and \( P \) is directed, there is a first node \( z \) in \( p_u^w \) such that starting from \( z \) all the nodes in \( p_u^w \) lie either on the same row \( (x_1) \) or on the same column \( (x_2) \) of \( u \). Let \( p_v^w \) be the subpath obtained from truncating \( p_u^w \) to the node \( z \). Notice that by the definition of \( z \), \( z \neq w \). Clearly \( u \in T(z) \), then we can use Lemma \[III.7\] on \( z \) and \( u \) to find a path \( q_u^w \) from \( z \) to \( T \) avoiding \( u \). Since \( w \in S(u) \), then \( u \notin T(z) \) and by the construction of \( z \), \( S(u) \cap T(z) \) is the set of nodes (a path) in \( p_u^w \) from \( z \) to \( u \), moreover recall that \( p_u^w \) avoids \( w \). Since in Case (A) \( w \in U \), the path \( p_v \), concatenating \( p_v^w \) with \( q_u^w \) goes from \( S \) to \( T \), touches \( W \) but avoids \( U \).

![Diagram](image)

Case (i).A

In case (B), i.e. when \( w \notin U \) the other node of \( U \), say \( u \), might belong to \( p_v \) and hence in this case \( p_v \) does not avoid \( U \). To handle this last case we build another path \( p_v' \) from \( S \) to \( T \) touching \( v \) and avoiding the whole \( U \). Let \( t_z \) be the subpath of \( p_v \) from \( S \) to \( z \). Let \( q \) be the subpath of \( p_v \) starting at \( z \) and ending in \( u_1 \). Assume that \( u_1 \) has coordinates \((x_2, y_2)\). Similarly as done for \( z \), there is a first node \( z_1 \) in \( q \) such that starting from \( z_1 \) all the nodes in \( q \) lie either on the same row \( (x_2) \) or on the same column \( (y_2) \) of \( u_1 \). Let \( z_1 \neq u \) since \( q \) being a subpath \( p_v \) avoids \( u \). \( z_1 \neq u_1 \) by the definition of \( z_1 \). Let \( q_{z_1} \) be the subpath obtained from truncating \( q \) at \( z_1 \). Using Lemma \[III.7\] on \( z_1 \) and \( u_1 \) there is a path \( q_{z_1}^w \) from \( z_1 \) to \( T \) avoiding \( u_1 \). Hence the path \( p_v' \) obtained from concatenating \( t_z \) with \( q_{z_1} \), and then with \( q_{z_1}^w \) goes from \( S \) to \( T \), touches \( v \) but avoids both \( u \) and \( u_1 \), hence \( U \). Claim is proved.

In case (ii) a symmetric argument of case (i) works. We omit the details in this extended abstract.

Together previous theorem and Lemma \[III.5\] implies the following

\section*{D. Hypergrids}

In this section we extend previous results to hypergrids. Remember that our model of \( H_{n,d} \) assumes that in the physical network each border node is linked to a monitor (actually one monitor for each dimension \( i \in [d] \) such that \( x \in \partial_i \)).

\begin{thm}
Let \( d, n \in \mathbb{N} \), \( d > 2 \) and \( n \geq 3 \). Then \( \mu(H_{n,d}) = d \).
\end{thm}

Since \( H_{n,d} \) has in-degree exactly \( d \), then Lemma \[III.4\] implies immediately that \( \mu(H_{n,d}) \leq d \).

First observe that a node \( u = (u_1, \ldots, u_d) \) is in \( S(u) \) if \( u_i = 1 \) for some \( i \in [d] \) and is in \( T \) if \( u_i = n \) for some \( i \in [d] \).

\[
S(u) = \{u\}, u \in S \quad S(u) = \bigcup_{v \in N_i(u)} S(v), u \in V \setminus S \\
T(v) = \{v\}, v \in T \quad T(v) = \bigcup_{u \in N_i} T(u), v \in V \setminus T
\]

For the lower bound, we lift Lemma \[III.6\] and Lemma \[III.7\] to dimension \( d > 2 \). Their proof is similar to the previous case. We prove the first one.

\begin{thm}
Let \( d, n \in \mathbb{N} \), \( d > 2 \) and \( n \geq 3 \). Let \( u \) be a node in \( H_{n,d} \) and \( X = \{w_1, \ldots, w_j\} \subseteq S(u) \) be a set of \( j \leq d \) pairwise distinct nodes all distinct from \( u \). Then there is a path \( p_X^{w_j} \) from a node in \( S(u) \) to \( u \) avoiding (any node of) \( X \).
\end{thm}

\begin{prf}
Let \( u \) be a node in \( H_{n,d} \). Let \( X = \{w_1, \ldots, w_j\} \subseteq S(u) \) fulfilling the hypothesis. By induction on \( m = |S(u)| \).

If \( m = 1 \), then \( S(u) = \{u\} \) and \( u \) is in \( S \) and since \( u \) is different from all the nodes in \( X \), then the (degenerate) path \( p_u^{w_j} \) made by the only node \( u \) proves the claim.

Assume \( |S(u)| = m > 1 \). Then \( u \) is neither in \( S \) nor in \( T \). Hence \( |N_i(u)| = d \). Then , since \( j < d \), there is \( w \in N_i(u) \) such that \( w \notin X \). Since \( w \in N_i(u) \), then \( S(w) \subseteq S(u) \), and hence \( |S(w)| < m \). By induction we have a path \( p_w^X \) from \( S \) to \( w \) avoiding \( X \). Then define as \( p_u^X \), the path concatenating \( p_w^X \) with \( u \) through \( w \).
\end{prf}

A similar proof holds also for the nodes reachable from \( u \) in \( H_{n,d} \) we omit the proof in this version.

\begin{thm}
Let \( d, n \in \mathbb{N} \), \( d > 2 \) and \( n \geq 3 \). Let \( \mathcal{P} \) be a \( H_{n,d} \). Let \( u \) be a node in \( \mathcal{P} \) and \( X = \{w_1, \ldots, w_j\} \subseteq T(u) \) be a set of \( j < d \) pairwise distinct nodes all distinct from \( u \). Then there is path \( p_u^X \) from \( u \) to a node in \( T \) avoiding (any node of) \( X \).
\end{thm}

\begin{prf}
(Thmorem \[III.10\]) Let \( V = [n]^d \). We have to prove that for any \( U, W \subseteq V \) with \( U \triangle W \neq \emptyset \) and such that \( |U|, |W| \leq d \), \( P(U) \triangle P(W) \neq 1 \). It is sufficient to find a path \( p \in H_{n,d} \) from \( S \) to \( T \) touching exactly one between \( U \) and \( W \). As for the case of simple grids we split in two cases:

1) at least one between \( U \) and \( W \) has cardinality at most \( d - 1 \);

2) both \( U \) and \( W \) have cardinality \( d \).

Using previous Lemmas the proof is similar to the case \( d = 2 \) and we defer its sketch to the appendix.
\end{prf}
E. Minimizing the number monitors on directed hypergrids

Our model of $\mathcal{H}_{n,d}$ assumes that each border node $x$ is linked to one monitor in the physical network for each dimension $i \in [d]$ such that $x \in \partial_i$. Is it possible to reduce the number of monitors but still get same lower bounds? An ideal placement of monitors in $\mathcal{H}_{n,d}$ would have only $2d$ monitors on the nodes $(1, \ldots, 1)$ and $(n, \ldots, n)$. Notice however that in the directed case Lemma III.4 applies. Hence if we remove monitors from the border nodes, the minimal in-degree drop-off to 1 and then $\mu(\mathcal{H}_{n,d}) \leq 1$. In the next subsection, when we analyze undirected grids, we show that maximal identifiability can be $d-1$ with only $2d$ monitors.

IV. IDENTIFIABILITY FOR UNDIRECTED TOPOLOGIES

A. Trees

For the case of undirected trees even assuming monitor-balancing, we can prove that SPF trees have maximal identifiability exactly 1.

Lemma IV.1. Let $T_n$ an undirected SPF-MB tree. Then $\mu(T_n) \leq 1$.

Proof. We prove that $T_n$ is not 2-identifiable. We have to show two sets $W$ and $U$ of cardinality at most 2 such that $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. Let $w$ a node in $T_n$, and $v \in N(w)$ such that $T(v)$ is an input-tree. There must exists another neighbour of $w$, $z \in N(w)$, with $z \neq v$ such that $T(z)$ is an output-tree. Fix $U = \{v\}$, $W = \{v, w\}$. $\mathbb{P}(U) \subseteq \mathbb{P}(W)$. Moreover each path passing through $w$ and $z$ is also touching $v$, hence $\mathbb{P}\{w\} \subseteq \mathbb{P}\{v\}$. Therefore $\mathbb{P}(W) \subseteq \mathbb{P}(U)$. Hence $\mathbb{P}(U) = \mathbb{P}(W)$ and therefore $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. \hfill $\Box$

Lemma IV.2. Let $T_n$ be a balanced a SPF-MB tree. Then $\mu(T_n) \geq 1$.

Proof. (sketch) Let $u$ and $w$ two distinct nodes in $T_n$. Fix $U = \{u\}$ and $W = \{w\}$. If $w$ and $u$ lie on different (end-to-end) paths, the claim is trivially proved. The BMT assumption implies (see Lemma [I.11]) that from each node there are at least two different (end-to-end) paths passing through. Hence from $u$ there is another path which is not touching $w$ (otherwise there would be a cycle in a tree). \hfill $\Box$

Theorem IV.3. Let $T_n$ be a balanced a SPF-MB tree. Then $\mu(T_n) = 1$.

B. Bounded degree

Recall Subsection II-C3. Let $G = (V,E)$ be an undirected graph and $Z \subseteq V$. We assume each node in $G$ reachable from two distinct nodes in $Z$. $1 \leq \delta(G) \leq \Delta(G) \leq |V|$. As for the directed case maximal identifiability only depends on the minimal degree of $G$ and not where monitor are placed.

Lemma IV.4. Let $G = (V,E)$ undirected. Then $\mu(G) \leq \delta(G)$ 1

Proof. Let $u \in V$ be such that $\text{deg}(u) = \delta(G)$. Fix $U = N(u)$ and $W = \{u\} \cup N(u)$. Each path touching $u$ is passing through at least a node in $N(u)$. Hence $\mathbb{P}\{u\} \subseteq \mathbb{P}(N(u))$. Hence $\mathbb{P}(W) = \mathbb{P}\{u\} \cup \mathbb{P}(N(u)) = \mathbb{P}(N(u)) = \mathbb{P}(U)$ and then $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. We have found two sets $U,W$ of cardinality at most $\delta(G) + 1$ such that $\mathbb{P}(U) \triangle \mathbb{P}(W) = \emptyset$. Hence $\mu(G) \leq \delta(G)$. \hfill $\Box$

C. Hypergrids

Recall Subsection II-C3 and notice that in the undirected case monitors are no longer counting in the degree of nodes of the graph modelling the network. Hence if $\mathcal{H}_{n,d}$ is undirected, $\delta(\mathcal{H}_{n,d}) = d$ (exactly on the corner nodes). Using Lemma IV.4 we then have the upper bound $\mu(\mathcal{H}_{n,d}) \leq d$. If we assume that all border nodes are linked to monitors that is $Z = \bigcup_{i \in [d]} \partial_i$, then each (end-to-end) path in the directed $\mathcal{H}_{n,d}$ is also a path for the undirected $\mathcal{H}_{n,d}$. Hence Theorem III.10 gives a lower bound also in this case. Therefore:

Theorem IV.5. Let $\mathcal{H}_{n,d}$ be undirected and $Z = \bigcup_{i \in [d]} \partial_i$. Then $\mu(\mathcal{H}_{n,d}) = d$.

Clearly in the undirected case we have many more paths in $\mathcal{H}_{n,d}$ than in the directed case. All these paths are becoming relevant to obtain a maximal identifiability of $d$ greatly minimizing the number of monitor nodes needed. While previous theorem assumes $dn$ monitors, we show in the next Subsection that only $2d$ monitors suffice to get maximal identifiability $d-1$ in the case of undirected $d$-dimensional grids.

D. Minimizing the number of monitors on undirected hypergrids

We prove that $2d$ monitors are sufficient to reach $\mu \geq d-1$ in undirected $\mathcal{H}_{n,d}$

Theorem IV.6. Let $\mathcal{H}_{n,d}$ be undirected and $n \geq 2^{d+1}$. Let $S = \{(1,\ldots,1)|1 \leq y_i \leq x_i, i \in [2]\}$ and $T = \{(n,\ldots,n)|1 \leq y_i \leq n, i \in [2]\}$ be the set of nodes linked to input and output monitors. Then $\mu(\mathcal{H}_{n,d}) \geq d-1$.

The section is devoted to the proof of the theorem for the case $d = 2$. The proof of Theorem IV.6 is along the same lines and its proof can be found in the appendix

Definition IV.7. Let $x = (x_1, x_2)$ be a node in in $\mathcal{H}_n$.

$S(x) = \{y = (y_1,y_2)|1 \leq y_i \leq x_i, i \in [2]\}$

$T(x) = \{y = (y_1,y_2)|x_i \leq y_i \leq n, i \in [2]\}$

Theorem IV.8. Let $\mathcal{H}_{n,2}$ be undirected and $n \in \mathbb{N}$, s.t. $n \geq 2^3$. Let $S = \{(1,1),(n,1)\}$ the set of input monitors and let $T = \{(n,n),(1,n)\}$ be the set of output monitors. Then $\mu(\mathcal{H}_{n,2}) \geq 1$.

Proof. We have to prove that for any $U,W \subseteq V$ with $U \Delta W \neq \emptyset$ such that $|U|, |W| \leq 1$, then $\mathbb{P}(U) \triangle \mathbb{P}(W) \neq \emptyset$. 

\hfill $\Box$
It is sufficient to find a path $p$ from $S$ to $T$ touching exactly one between $U$ and $W$. Assume $W = \{w\}$, $U = \{u\}$. Since $U \cap W \neq \emptyset$, then $u \neq w$. According to the position of $W$, we divide in the three following cases:

1. $w \in V \setminus (S(u) \cup T(u))$
2. $w \in S(u)$
3. $w \in T(u)$;

In case 1, any path from $S$ to $T$ passing through $u$ is not touching $w$ and proves the claim.

In case 2, if $w \in N((1,1))$, any path from the other input monitor $((n,1))$ passing through $u$ to an output monitor is not touching $w$.

If $w \notin N(u)$, we prove it by induction on $|S(u)| = m$. If $|S(u)| = 1$, then $S(u) = \{u\}$ and the path $p$ made by $u$ from $S$ to $u$ avoids $w$. Notice that this case implies $u = (1,1)$. Now consider $|S(u)| = m$. Then we fix any $u' \in N(u) \cap S(u)$. Therefore $S(u') \subseteq S(u)$ and $|S(u')| < m$. By induction, there is a path $p_{u'}$ from $S$ to $u'$ avoiding $w$. So the path $p$, the composition of $p_{u'}$ and $u'u$, from $S$ to $u$ avoids $w$ as well. Moreover, any path $p'$ from $u$ to $T$ is avoiding $W$. Therefore the composition of $p$ and $p'$ is touching $u$ but avoiding $W$ and the claim is proved.

If $w \in N(u)$, then we fix $u' \in N(u)$ such that $u' \neq u$. If $w \notin S(u')$, then any path $p_{u'}$ from $S$ to $u'$ avoid $w$, the path $p$, the composition of $p_{u'}$ and $u'u$, from $S$ to $u$ avoid $w$ as well. If $w \in S(u')$, then any path $p_{u'}$ from $S$ to $u'$ avoid $w$, the path $p$, the composition of $p_{u'}$ and $u'u$, from $S$ to $u$ avoid $w$. Moreover, any path $p'$ from $u$ to $T$ is avoiding $W$. So the composition of $p$ and $p'$ touches $u$ but avoids $W$.

In case 3, a symmetric argument of case 2 works.

V. IDENTIFIABILITY, EMBEDDINGS, DIMENSION

Recall that the dimension dim($G$) of a DAG $G$ is the smallest $d \in \mathbb{N}$ such that $G \hookrightarrow \mathcal{H}_{n,d}$ i.e. $G$ is embeddable in $\mathcal{H}_{n,d}$. Hence if $G \hookrightarrow \mathcal{H}_{n,d}$, then dim $G \leq d$. Since $\mu(\mathcal{H}_{n,d}) \geq d$, then dim $G \leq \mu(\mathcal{H}_{n,d})$. We explore possible relationships between the dimension of directed graphs and maximal identifiability. The next example shows that in the most general form embeddability can destroy identifiability. Consider the following two graphs.

$G_1$ is embeddable in $G_2$ (i.e. $f(x) = x \hat{}$). But while in $G_1$ there is a path (an edge) from $u_1$ to $u_2$ avoiding $W = \{w_1, w_2\}$, in $G_2$ no path connecting $u'_1$ to $u'_2$ is avoiding $f(W)$ in $G_2$.

Still in some cases we can use embeddability to say something on identifiability. Consider the following Definition of routing consistency given in [1].

**Definition V.1.** ([1]) A set of paths $\mathcal{P}$ is routing consistent if any two distinct paths $p$ and $p'$ in $\mathcal{P}$ and any distinct nodes $u$ and $w$ traversed by both paths (if any) $p$ and $p'$ follow the same subpath between $u$ and $w$.

Though a weak class of topologies (any edge-cut disconnect the graph into two components) these graphs are employed in many practical routing protocols (see [1]). For these topologies (in the directed case) we can prove the following result.

**Theorem V.2.** Assume that $G$ is routing consistent and $G \hookrightarrow G'$, then $\mu(G) \leq \mu(G')$.

**Proof.** Assume that $\mu(G') \leq k$. We prove that $\mu(G) \leq k$. Since $\mu(G') \leq k$, there are two sets $U', W' \subseteq V'$ such that $U' \cap W' \neq \emptyset$, at least one of them, wlog say $U'$, has cardinality $k + 1$, and $\mathbb{P}_G(U') \triangle \mathbb{P}_G(W') = \emptyset$. Fix $U = f^{-1}(U')$ and $W = f^{-1}(W')$. By injectivity of $f$, $U$ has cardinality $k + 1$ and $U \cap W \neq \emptyset$ (since otherwise $U' \cap W' = \emptyset$). Assume by contradiction that $\mathbb{P}_G(U) \triangle \mathbb{P}_G(W) \neq \emptyset$. That is, there exists a path $p$ in $G$ from $S$ to $T$ touching nodes in only one between $U$ and $W$, say $U$. Let $p = (u_1 u_2 \ldots u_k)$. Since $U$ is an embedding (i.e. $x \leq y$ iff $f(x) = f(y)$), then $u_1 \leq u_{i+1}, u_i \in S'$ and $u_{i+1} \in T'$. Hence there are paths $p_i'$ in $G'$ from $u_{i+1}$ to $u_i$. Fix a path $p_i'$ is a path from $S'$ to $T'$ in $G'$. If all nodes in $p_i'$ are in $V'$, this is a contradiction with the fact $\mathbb{P}_G(U') \triangle \mathbb{P}_G(W') = \emptyset$. Then there is a path $p_i'$ such that in $p_i'$ is a node $w' \in W'$. Hence we have that in $G'$, $u_1 \leq u_{i+1}$. Since $f$ is an embedding and since $u_i = f^{-1}(u'_i)$, this means that in $G$, $u_i \leq f^{-1}(w') \leq u_{i+1}$. Then in $G$ there is a path from $u_i$ to $u_{i+1}$ passing through $f^{-1}(w')$. This contradicts the routing consistency of $G$ since between $u_1$ and $u_2$ there is another path, the edge that is in $p$.

The previous examples shows that restricting the class of graphs one can still hope to bound identifiability using embeddability. In the next two results we restrict embeddings, obtaining similar relationships but for broader classes of topologies. Assume that $f$ is an embedding between two DAGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let us say that $f$ is distance-reducing (d.r.) if for all $x, y \in V_1$, $d_{G_1}(x, y) \leq d_{G_2}(f(x), f(y))$. Here $d_{G}(x, y)$ is the length of the shortest path between $x$ and $y$ in $G$. We call $f$ distance preserving (d.p.) if $d_{G_1}(x, y) = d_{G_2}(f(x), f(y))$.

**Theorem V.3.** Let $G$ and $G'$ be two DAGs such that $G \hookrightarrow G'$, where $f$ is a (d.r.)-embedding. Then $\mu(G) \geq \mu(G')$.

**Proof.** Assume $\mu(G) \leq k$, we prove that $\mu(G') \leq k$. Since $\mu(G) \leq k$, there are two sets $U, W \subseteq V$ such that $U \cap W \neq \emptyset$, at least one of them, wlog say $U$, has cardinality $k + 1$, and $\mathbb{P}_G(U) \triangle \mathbb{P}_G(W) = \emptyset$. Fix $U' = f(U)$ and $W' = f(W)$. By injectivity of $f$, $U'$ has cardinality $k + 1$ and clearly $U' \cap W'$ is $\emptyset$ (since otherwise $\mathbb{P}_G(U) \triangle \mathbb{P}_G(W) = \emptyset$). Assume by contradiction that $\mathbb{P}_G(U') \triangle \mathbb{P}_G(W') \neq \emptyset$. That means that there exists a path $p$ from $S'$ to $T'$ touching node of only one between $U'$ and $W'$, say $U'$. Let $p_i' = (u'_1, u'_2 \ldots u'_{i+1})$, $r \geq 1$. Hence $u'_i \leq u'_{i+1}$ for all $i \in [r]$. Let $u_1 = f^{-1}(u'_1)$. Since $f$ is an embedding $u_i \in U$, $u_i \leq u_{i+1}$ and $u_1 \in S$ and $u_{r+1}$
Moreover since $f$ distance reducing and $(u_i, u_{i+1})$ is an edge in $G'$, then $(u_i, u_{i+1})$ is an edge in $G$. But then path in $G$ $p = (u_1, u_2) \ldots (u_r, u_{r+1})$ is a path from $S$ to $T$ touching only nodes in $U$. This is a contradiction with the fact $P_G(U) \Delta P_G(W) = \emptyset$.

It is straightforward to see that if $f$ is distance-preserving, then equality holds.

**Theorem V.4.** Let $G$ and $G'$ be two DAGs such that $G \hookrightarrow f G'$, where $f$ is a (d.p.)-embedding. Then $\mu(G) = \mu(G')$.

For a general DAG $G$, we cannot obtain a direct relation between identifiability and dimension and, as previous example shows, we cannot relate identifiability of two graphs which are embeddable one into the other. Nevertheless we can lower bound identifiability of a DAG $G$ using embeddability.

Let $G^*$ be the transitive closure of a DAG $G$. Since $G$ and $G^*$ are defined on the same set of nodes and each path in $G$ is a path in $G^*$, it follows:

**Lemma V.5.** $\mu(G) \geq \mu(G^*)$.

Now following the idea of Theorem V.3 it is not difficult to prove the following theorem (see appendix)

**Theorem V.6.** Let $G$ be a DAG closed under transitivity. Assume that $G \hookrightarrow G'$. Then $\mu(G) \geq \mu(G')$.

All together these result imply:

**Theorem V.7.** Let $G$ be a DAG and assume that $G^* \hookrightarrow G'$. Then $\mu(G) \geq \mu(G'^*)$.

Hence by definition of dimension of a DAG $G$ we get

**Corollary V.8.** Let $G$ be a DAG. Then $\mu(G) \geq \dim G^*$.

VI. APPLICATIONS AND EXPERIMENTS

A. Designing an optimal network wrt failure maximal identifiability

Assume we have to design a network over $N \geq 4$ nodes and we aim to have maximal identifiability of failure nodes. Theorem IV.6 suggests how to set edges between the nodes in the network and how to place monitors in such a way to reach an identifiability of at least $\sqrt{\log N}$. Fix $d = \sqrt{\log N}$ and $n = N^{1/d}$. Then $N = n^d$ and it is easy to see that for $N \geq 4$ it holds that $n \geq 2^{d+1}$ (as required by the theorem). Assume wlog that all values are integers (otherwise approximate them to smallest integer values such that $N \leq n^d$). Assign an address to each node in $N$ as a $d$-dimensional vector in $[n]$ and place edges between nodes in $N$ following the construction of $\mathcal{H}_{n,d}$. Finally place input and output monitors $(2\sqrt{\log N})$ linked to those nodes of $N$, as required by Theorem IV.6.

B. Adding edges to boost node failure identifiability

Assume we have a network with very low maximal identifiability of failure nodes (for instance due to a small minimal in-degree). We explore the idea to add edges to get better max identifiability. We propose the following algorithm whose main idea is that of trying to approximate a hypergrid of dimension $d$ (a parameter to be tuned), choosing the appropriate input and output monitors and adding edges (according to certain heuristic) in order to increase the $\delta$. Experiments on concrete networks will appear on the full version.

**Algorithm 1 AGrid**

**Input:** $G = (V, E)$

**Output:** $G' = (V, E')$, $I_m \subseteq V$, $I_M \subseteq V$

/* Select input and output monitors*/

1. for $i = 1 \ldots d$
2. $(x, y)$ nodes at max shorter distance in $G$
3. $I_m = I_m \cup \{x\}; I_M = I_M \cup \{y\}$
4. $V = V \cup \{x, y\}$
5. /* Boost minimal degree to $d$ */
6. choose $u \in V$ do $|N(v)|$ nodes $w_i$ in $V \setminus N(v)$
7. for all $w_i$
8. $E = E \cup (v, w_i)$

C. Upper bounding $\mu$ in terms on number of nodes and paths

Our bounds can be used to prove more refined upper bounds on maximal identifiability than those obtained in [1] (Theorems IV.1 and IV.2). We consider the undirected case but similar results can be proved for the directed case.

**Theorem VI.1.** Let $\mathcal{G}$ be a network defined over $n$ nodes and $m$ paths $p_i$ each of length (i.e. number of edges) $\ell_i$. Let $\ell = \sum \ell_i$. Then $\mu(G^\ell) \leq \min\{n, \lfloor \frac{2m}{n\ell} \rfloor\}$.

**Proof.** Assume a graph $G$ has $n$ nodes and minimal degree $\delta(G) \geq d$. Then there are at least $nd/2$ paths in $G$. This is because we count at least the paths of length 1 in $G$. Now assume that $G$ is defined over $n$ nodes and $m$ paths and $\ell$ edges in total. Let $d$ be the minimal degree in $G$. Since we have $m\ell$ nodes in $G$, it follows by previous observation that $m \geq nd\ell/2$. Hence it must be that $d \leq 2m/n\ell$. Since moreover $d \leq n$, the result follows by Lemma IV.4.

This result can be lifted to a set $\mathcal{P}$ of measurement paths using the hitting number of $G_\mathcal{P}$ (see Definition II.5).

**Corollary VI.2.** Let $\mathcal{P}$ be a set of measurement paths over $n$ nodes and $m$ paths of total length $\ell$. Let $h$ be the hitting number of $G_\mathcal{P}$. Then $\mu(G_\mathcal{P}) \leq \min\{n, \lfloor \frac{2mh}{n\ell} \rfloor\}$.

VII. FURTHER RESEARCH

We shortly address (see the appendix for a complete discussion) three directions related to our approach.

What can we say on boolean network tomography and failure identifiability using algorithm for computing dimension of DAGs? It is a well known result that planar graphs over $n$ nodes can be embedded into a $(n-2) \times (n-2)$ grid. Our results implies a lower bound. What else can be said on upper bounds for failure maximal identifiability in planar graphs? A $k$-TC-Scanner of a graph $G$ is a graph $H$ with a small diameter that preserves the connectivity of the
original graph. Are k-TC-Spanners useful to identify maximal failure identifiability of a network?

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APPENDIX

VIII. APPENDIX

Proof. (of Theorem III.10)

Case 1

Let \( W = \{w_1, \ldots, w_{d-1}\} \). Since \( U \neq W \), there is a node \( u \in U \setminus W \), such that \( u_i \neq u, \forall i \in [d-1] \). Now according to the position of \( W \), we have four cases:

i. \( W \) is neither in \( S(u) \) nor in \( T(u) \)

ii. \( W \subseteq S(u) \)

iii. \( W \subseteq T(u) \)

iv. \( |W \cap S(u)| \leq d - 1 \) and \( |W \cap T(u)| \leq d - 1 \);

In case i, any path from \( S \) to \( T \) passing through \( u \) is not touching \( W \) and proves the claim.

In case ii, we use lemma III.11. Then there is a path \( p^u_w \) from \( S \) to \( u \) avoiding \( W \). Furthermore, any path \( p \) from \( u \) to \( T \) is avoiding \( W \). Therefore, the composition of \( p^u_w \) with \( p \) proves the claim.

In case iii, by lemma III.12 there is a path \( p^u_w \) from \( u \) to \( T \) avoiding \( W \) and of course any path \( p \) from \( S \) to \( u \) avoids \( W \). Then the composition of \( p \) and \( p^u_w \) proves the claim.

In case iv, we use lemma III.11. Then there is a path \( p^u_w \) from \( S \) to \( u \) avoiding \( W \). Moreover by lemma III.12 there is a path \( q^u_w \) from \( u \) to \( T \) avoiding \( W \). Hence the composition of \( p^u_w \) and \( q^u_w \) proves the claim.

Case 2

Although \( U \neq W \), they might share some nodes. So there are two cases: (A) \( |U \cap W| = 0 \) and (B) \( |U \cap W| \leq d - 1 \). In case (A), let \( U = \{u_1, \ldots, u_d\} \). We fix \( u \) to be the node in \( U \) not reachable in \( H_{n,d} \) by other nodes in \( U \). In case (B), we fix \( u \) be the node of \( U \) not in \( W \). Like in case 1, we divide in three cases as following:

i. \( W \subseteq S(u) \)

ii. \( W \subseteq T(u) \)

iii. \( |S(u) \cap W| \leq d - 1 \) and \( |T(u) \cap W| \leq d - 1 \);

In case (iii), since \( |S(u) \cap W| \leq d - 1 \), then we can apply lemma III.11 to find a path \( p_u \) from \( S \) to \( u \) avoiding \( W \). Using \( \{T(u) \cap W\} \leq d - 1 \) a similar argument works for finding a path \( p^u_w \) from \( u \) to \( T \) avoiding \( W \). Then the composition of \( p_u \) and \( p^u_w \) is a path from \( S \) to \( T \) passing through \( u \) but avoiding \( W \).

In case (i), first we fix \( w_1, \ldots, w_{d-1} \in W \) this way. For case (A), let them be any \( d - 1 \) nodes in \( W \). For case (B), if \( |U \cap W| = d - 1 \), \( w_1, \ldots, w_{d-1} \) are the nodes in \( U \cap W \) and let \( X = \{w_1, \ldots, w_{d-1}\} \). If \( |U \cap W| < d - 1 \), let \( w_1, \ldots, w_i \) \((i < d - 1)\) be the nodes in \( U \cap W \). The rest of \( d - i \) nodes that we need are any nodes in \( W \setminus (U \cap W) \). Now we have \( w_1, \ldots, w_{d-1} \) and let \( X = \{w_1, \ldots, w_{d-1}\} \).

Now let \( v \) be the other node in \( W \). Since \( u \neq w_i, \forall i \in [d - 1] \), then by lemma III.11 there is a path \( p^X_u \) from \( S \) to \( u \) avoiding \( X \). Furthermore, since \( X \subseteq S(u) \) any path \( p^X_u \) from \( u \) to \( T \) avoids \( X \). Hence the path \( p_u \) concatenation \( p^X_u \) of with \( p_u \) touches \( U \) and avoids \( W \) unless \( v \in p_u \).

If \( v \in p_u \), then we modify \( p_u \) into a new path \( p_{uv} \) touching \( W \) but avoiding \( U \). We first identify a node \( z \) on \( p_{uv} \). Let \( u = (x_1, \ldots, x_d) \) with \( x_i \in [n], \forall i \in [d] \). Since \( p^X_u \) is ending at \( u \) and \( H_{n,d} \) is directed, there is a first node \( z \) in \( p^X_u \) such
that starting from \( z \), all the nodes in \( p_u^X \) lie on one of the \( x_i; s \) (of \( u \)). Let \( p_u^X \) be the subpath obtained from truncating \( p_u^X \) to the node \( z \). Notice that by the definition of \( z, z \neq u \).

Since \( u \in T(z) \), by lemma [III.12] we can find a path \( q_z \) from \( z \) to \( T \) avoiding \( u \). In case (B), we have \( X \not\subseteq T(z) \) because \( X \subseteq S(u) \) and by the construction of \( z, S(u) \cap T(z) \) is the set of nodes in \( p_u^X \) from \( z \) to \( u \) and \( p_u^X \) avoids \( X \). In case (B) when \( |U \cap W| = d - 1 \), the path \( p_v \), concatenating \( p_u^X \) with \( q_z \) goes from \( S \) to \( T \), touches \( W \) but avoids \( U \). In case (B), when \( |U \cap W| < d - 1 \) and in case (A), we have \( X \not\subseteq U \).

Therefore, the other node of \( U \), might belong to \( p_u \) and hence in this case \( p_v \) does not avoid \( U \). To handle this last case we do as the following:

If another node of \( U \), say \( u_1 \), belongs \( p_v \), we do this way: Let \( t_2 \) be the subpath of \( p_v \) from \( z \) to \( u_1 \). Let \( q \) be the subpath from \( u_1 \) starting at \( u_1 \) and ending in \( z \). Assume that \( u_1 = (y_1, \ldots, y_d) \). Similarly as done for \( z \), there is a first node \( z_1 \) in \( q \) such that before \( z_1 \), all the nodes in \( q \) lie on one of the \( y_i; s \) (of \( u_1 \)). Since \( z_1 \neq u \) since \( q \) is a subpath of \( p_v \) which avoids \( u \). Also \( z_1 \neq u_1 \) by the definition of \( z_1 \). Let \( q_{z_1} \) be the subpath obtained from truncating \( q \) at \( z_1 \). Since \( u_1 \in T(z_1) \), by lemma [III.12] there is a path \( q_{z_1} \) from \( z_1 \) to \( T \) avoiding \( u_1 \). Hence the path \( p_v \) obtained from concatenating \( t_2 \) with \( q_{z_1} \) and then with \( q_{z_1} \) goes from \( S \) to \( T \), touches \( v \) but avoids both \( u \) and \( u_1 \). If still another node of \( U \), say \( u_2 \), belongs to \( p_v \), then we do the same argument (as above) until our obtained path avoids the whole \( U \). Claim is proved.

Proof. (of Theorem [IV.6]) Let \( V \) be the set of nodes of \( H_{n,d} \). We have to prove that for any \( U \subseteq V \) with \( U \triangle W \neq \emptyset \) such that \( |U|, |W| \leq d - 1 \), then \( P(U) \cup P(W) \neq \emptyset \). It is sufficient to find a path \( p \) from \( S \) to \( T \) touching \( W \) but avoiding \( V \). We split in two cases:

1. \( |U \cap W| = 0 \)
2. \( |U \cap W| < d - 1 \)

For case 1, since \( U \neq V \) and \( |U \cap W| = 0 \), then we fix \( u \) be any node in \( U \). For case 2, we fix \( u \) be the node of \( U \) not in \( W \). Now, it is obvious that \( u \notin W, \forall \nu \in W \).

First let \( u = (u_1, \ldots, u_d) \) be a node in \( H_{n,d} \), we define \( S(u) \) and \( T(u) \) as below:

\[
S(u) = \{ v = (v_1, \ldots, v_d) | 1 \leq v_i \leq u_i, i \in [d] \} \\
T(u) = \{ v = (v_1, \ldots, v_d) | u_i \leq v_i \leq n, i \in [d] \}
\]

According to the position of \( W \), we have four following cases:

i. \( W \) is neither in \( S(u) \) nor in \( T(u) \)
ii. \( W \subseteq S(u) \)
iii. \( W \subseteq T(u) \)
iv. \( W \cap S(u) \leq d - 1 \) and \( |W \cap T(u)| \leq d - 1 \)

In case i, any path \( p \) from \( S \) to \( T \) passing through \( u \) is not touching \( W \) and proves the claim.

In case ii, if \( W \) is linked to input monitors, since \( |W| \leq d - 1 \), then any path \( p \) from the other input monitor passing through \( u \) to an output monitor is avoiding \( W \) and the claim is proved. If \( |W \cap N(u)| \leq d - 1 \), then there is \( u' \in N(u) \) such that \( u' \notin W \) and \( u \notin N(u') \), \( \forall u \in W \). Now, we prove that there exists a path \( p \) from \( S \) to \( u' \) avoiding \( W \) by induction on \( |S(u')| = m \).

If \( |S(u')| = 1 \), then \( S(u') = \{ u' \} \) and the path \( p \) made by \( u' \) avoids \( W \).

If \( |S(u')| = m \), then there is \( v \in N(u') \) \( \cap S(u') \). Therefore \( S(u') \subseteq S(u') \) \( \cap |S(u)| < m \). By induction, there is a path \( p \) from \( S \) to \( u \) avoiding \( W, W \) and \( u \notin S(u') \) avoiding \( W \).

Hence the composition of \( p \) and \( u' \) from \( S \) to \( u \) is avoiding \( W, W \) and \( u \) avoiding \( W \). Moreover, any path \( p \) from \( u \) to \( T \) is avoiding
W. Therefore the composition of \( p_{u',u} \) and \( p \) is touching \( u \) but avoiding \( W \) and the claim is proved.

For the case \( iii \), a symmetric argument of case \( ii \) works.

For the case \( iv \), let \( X = \{x_1, ..., x_i\} = W \cap S(u) \) such that \( i \leq d - 1 \) and \( V = \{v_1, ..., v_j\} = W \cap T(u) \) such that \( j \leq d - 1 \). If \( X \) and \( V \) are both (or one of them) linked to input and output monitors, since \( i, j \leq d - 1 \), then any path from the other \( d - 1 - i \) input monitors passing through \( u \) to any other \( d - 1 - j \) output monitors is not touching \( W \) and proves the claim.

Now assume that \( |(X \cup V) \cap N(u)| \leq d - 1 \). There exists \( u' \in N(u) \) such that \( u' \notin W \) and \( w \notin N(u'), \forall w \in W \). Now by the same argument in the case \( ii \), there is a path \( p_{u'} \) from \( S \) to \( u' \) avoiding \( W \). Thus the path \( p \) from \( S \) to \( u \), the composition of \( p_{u'} \) and \( u' u \), avoids \( W \). By the symmetric argument, there is a path \( p' \) from \( u \) to \( T \) avoiding \( W \). Hence the composition of \( p \) and \( p' \) touches \( U \) but avoids \( W \) and proves the claim.

\[
\text{Proof. (of Theorem VI.6, sketch)} \text{ Upper bounds on } \mu(G^*) \text{ imply upper bounds on } \mu(G). \text{ From } \mu(G^*) \leq k \text{ get } U, W \text{ distinct set of nodes in } G \text{ of cardinality } k + 1 \text{ s.t. } \mathbb{P}(U) \mathbb{P}(W) = \emptyset \text{ Define } U' = f(U) \text{ and } W' = f(W). \text{ If by contradiction there is a path touching nodes of only one between } U' \text{ and } W', \text{ then since } f \text{ is an embedding (and is injective) and } G \text{ is closed under transitivity, there is path in } G \text{ touching nodes only in } U. \text{ Contradiction.}
\]

IX. Experiments of Subsection VI-B

We explain some measurements we have run to try to validate the idea that to boost maximal identifiability one can “approximate” a grid adding a number of edges in such a way that the minimal degree of the graph underlying the network topologies increases with some suitable function \( f(N) \) of the number nodes \( N \) of the network (we consider only \( f = \sqrt{\log N} \) and \( f = \log N \)). According to Section V where we explore embeddings another idea we experiment is that of performing some steps (we start by only 2) of the transitive closure of the incidence matrix of the network graph.

Our networks examples are networks whose topology is publicly available. We consider the AGIS Network for North America and ATT Network See Figure 2 and 1 as available on the Internet Topology Zoo.

First we introduce some notations to present the data which are collected in Table I and Table II. First we transform the two topologies in two graphs through a numbering of the nodes. To avoid trivial case for identifiability in both networks we first perform a removal of the simple paths substituting them by one edge and recursively repeating this operation if needed.

For each network in the Figures then we collect data in next the Tables for three instantiation of the original network:

1) the original network, \( G \);
2) the network where we add a certain number of random edges \( G^* \);
3) the network obtained from the original graph boolean squaring its incidence matrix \( G^2 \).

For each of the networks we have to decide where to place the monitors (both input and output) and how many of them to add. For the second case we also have to decide how many edges to add to each node in order to increase minimal degree. In these experiments we follow the idea to approximate a grid of dimension \( d = f(N) \) inside the original network: hence we decide to

- adding random to each node a number of edges in such a way to reach degree \( d \)
- Considering \( d \) input e \( d \) output monitors that we choose at random

In order to compare each other all the instantiations of the networks considered, we decide of choosing in cases (1) and (3) the same number of input and output monitors than in the random case. In this case however we do no choose random the monitors but we follow the strategy of considered as monitor all the nodes with smaller degrees and including always nodes with degree 1.

| Table I: \( P \) Measures |
|--------------------------|
| \( \mu \) | maximal identifiability |
| | \( |P| \) | n. measurment paths |
| | \( L \) | length of longest path |
| | \( l \) | length of shortest path |
| | \( |m| + |M| \) | n. monitors |
| \( d \) | simulated dimension |
| | \( U \) | Certificate for \( \mu \) |
| | \( W \) | Certificate for \( \mu \) |
TABLE II: $G_P$ Measures

| $|E|$ | n. of edges | $|V|$ | n. nodes | $\Delta$ | max degree | $\delta$ | min degree | $m$ | input monitors | $M$ | output monitors |
|-----|-----------|-----|--------|------|---------|------|---------|-----|----------|-----|----------------|
| 4   |           | 6   |        | 2    |         | 2    |         | 29  |           | 69  |                |
| 6   |           | 2   |        | 2    |         | 2    |         | 10  |           | 56  |                |
| 2   |           | 2   |        | 4    |         | 4    |         | 29  |           | 25  |                |
| 29  |           | 4   |        | 2    |         | 2    |         | 2   |           | 19  |                |

TABLE III: AGIS - $x \sqrt{\log N}$

| $\sqrt{\log N}$ | $G$ | $G^*$ | $G^2$ |
|-----------------|-----|------|------|
| $\mu$ | [1, 2] | [1, 2] | [1, 2] |
| $|P|$ | 9 | 12 | 14 |
| $|E|$ | 25 | 26 | 69 |
| $|V|$ | 18 | 18 | 18 |
| $L$ | 14 | 11 | 16 |
| $t$ | 3 | 6 | 2 |
| dim | 2 | 2 | 2 |
| $\Delta$ | 4 | 4 | 11 |
| $\delta$ | 2 | 2 | 3 |
| $m$ | (17, 15) | (2, 7) | (16, 17) |
| $M$ | (5, 12) | (11, 15) | (5, 11) |
| $U$ | (16, 17) | (16, 17) | (17) |
| $W$ | (15, 17) | (15, 17) | (17) |

TABLE IV: AGIS - $\log N$

| $\log N$ | $G$ | $G^*$ | $G^2$ |
|----------|-----|------|------|
| $\mu$ | [1, 2] | [1, 2] | [1, 2] |
| $|P|$ | 9 | 12 | 14 |
| $|E|$ | 25 | 41 | 69 |
| $|V|$ | 18 | 18 | 18 |
| $L$ | 10 | 8 | 12 |
| $t$ | 2 | 2 | 2 |
| dim | 4 | 4 | 4 |
| $\Delta$ | 4 | 7 | 11 |
| $\delta$ | 2 | 2 | 3 |
| $m$ | (17, 15, 5, 11) | (16, 13, 19, 4) | (16, 17, 5, 6) |
| $M$ | (6, 14, 1, 18) | (14, 2, 12, 10) | (15, 11, 15, 1) |
| $U$ | (16, 17) | (16, 17) | (16, 17) |
| $W$ | (15, 17) | (15, 17) | (15, 17) |

TABLE V: ATT-North America - $\sqrt{\log N}$

| $\sqrt{\log N}$ | $G$ | $G^*$ | $G^2$ |
|-----------------|-----|------|------|
| $\mu$ | [0, 1] | [1, 2] | [1, 2] |
| $|P|$ | 8 | 118 | 29 |
| $|E|$ | 56 | 56 | 192 |
| $|V|$ | 25 | 25 | 25 |
| $L$ | 19 | 18 | 23 |
| $t$ | 3 | 3 | 2 |
| dim | 2 | 2 | 2 |
| $\Delta$ | 19 | 10 | 22 |
| $\delta$ | 2 | 2 | 6 |
| $m$ | (10, 2) | (5, 25) | (2, 24) |
| $M$ | (24, 20) | (1, 18) | (25, 25) |
| $U$ | (24) | (23, 24) | (23, 24) |
| $W$ | (13) | (22, 24) | (22, 24) |

TABLE VI: ATT-North America - $\log N$

| $\log N$ | $G$ | $G^*$ | $G^2$ |
|----------|-----|------|------|
| $\mu$ | [1, 2] | [1, 2] | [1, 2] |
| $|P|$ | 69 | 86 | 123 |
| $|E|$ | 56 | 69 | 192 |
| $|V|$ | 25 | 25 | 25 |
| $L$ | 17 | 6 | 19 |
| $t$ | 2 | 2 | 2 |
| dim | 4 | 4 | 4 |
| $\Delta$ | 10 | 10 | 22 |
| $\delta$ | 2 | 4 | 6 |
| $m$ | (19, 2, 24, 20) | (15, 7, 16) | (2, 24, 25, 15) |
| $M$ | (5, 25, 11, 13) | (20, 6, 3, 2) | (5, 20, 19, 8) |
| $U$ | (23, 24) | (24) | (23, 24) |
| $W$ | (22, 24) | (32) | (22, 24) |

Full Section [VII] We shortly address three directions related to our approach we consider interesting to be further explored, in the analysis of identifiability of failure nodes. In 1982 [18] showed that for $k \geq 3$ to test if a partial order has dimension $\leq k$ is $NP$-complete. Nevertheless there are some algorithms to compute the dimension of poset [17, 19] which are practically used. It would be interesting to explore connections between boolean network tomography and poset dimension theory for instance to get better estimates on the maximal identifiability for DAG network topologies. It is a well known result [15] that planar graphs over $n$ nodes can be embedded through a straight line embedding into a $(n-2) \times (n-2)$ 2-dimensional grid. It is not difficult to see (details omitted in this version) that our results on embeddability can be generalized to obtain a lower bound of 2 for the maximal identifiability when a network is a planar graph. What can be said on upper bounds for failure maximal identifiability in planar graphs? Connections with dimension might also be explored in the case of planar networks [6]. A $k$-Transitive-Closure-Spanner of a graph $G$ is a graph $H$ with a small diameter - $k$- that preserves the connectivity of the original graph. The edges of the transitive closure of $G$, added to $G$ to obtain a TC-spanner, are called shortcuts and the parameter $k$ is called the stretch. These graphs and their relations with dimension of poset was recently studied in [2]. From our results it is clear that adding edges to a graph $G$ can strength the potential of failure identifiability. Are $k$-TC-Spanners and in particular Steiner-$k$-TC-Spanners(see [2]) useful to identify maximal failure identifiability of a network.

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