The invariant based investigation of Shortcut to Adiabaticity for Quantum Harmonic Oscillators under time varying frictional force

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Abstract:
We investigate the Shortcuts To Adiabaticity (STA) of a quantum harmonic oscillator under time-dependent frictional force, using invariant based inverse engineering method with a class of invariants characterized by a time-dependent frictional coefficient. We discuss the implementation of shortcut protocol in a generalized framework and study the STA for the harmonic oscillator with time varying mass as a special case. For an illustration, we consider the coupled photonic lattice as a harmonic oscillator with time-varying mass and frequency and discuss the implementation of the above protocol.

1 Introduction
Shortcuts To Adiabaticity (STA) protocols are nonadiabatic processes that reproduce in finite time the same initial and final states as that of an infinitely slow adiabatic process [1, 2]. These protocols can be used as an alternate driving of the system to implement the adiabatic process. The path of the transition will be different and decided by the various factors involved in the specified technique of STA. The interesting factor is that there is no need for the complete suppression of the unwanted transition throughout the path. But the initial and final states of the overall process need to be adiabatic in all sense. In other words, the STA process will mimic the dynamics of very slow adiabatic process within a finite time by allowing transitions at intermediate times [3, 4, 5, 6]. Experiments confirmed the feasibility of such process on various grounds, noticeably for the frictionless transport of trapped ions [7, 8, 9], cold atoms [10, 11], fast equilibration of a Brownian particle [12] and high-fidelity driving of a Bose-Einstein condensate [13]. Different kinds of methods are developed so far to establish the adiabaticity through non-adiabatic transitions. Some of them are Counterdiabatic Driving process by incorporating a global Hamiltonian to surpass the non-adiabatic transitions [14, 15, 16, 17], Local-Counterdiabatic Driving, where the local potential take charge of counterdiabatic contribution [4]. Fast-forward approach [18, 19], and Invariant based Inverse Engineering (IE) method by using the Lewis-Reisenfield (LR) Invariants to connect the initial and final states through a non-adiabatic path [20, 21, 22, 23, 24, 25]. IE method is found useful in many applications and recently considered it in the context of cost of the shortcut process [26, 27, 28].

Invariant method is employed extensively to find solution for a given system under frictional contact [31]. Time-dependancy of the frictional force is also considered, which brings the concept of parametric variation of frictional force or equivalent change in the mass of the system. The mass varying Quantum Harmonic Oscillator (QHO) is considered in the context of STA using IE method and connected it with the construction of photonic lattice [29]. The mass varying QHO Hamiltonian is mathematically equivalent to the QHO experiencing frictional force, which is a relevant topic since the works of Caldirola [30] and Kanai [31], discussing the idea of quantization of the systems experiencing certain types of non-conservative forces. Physical description of such systems were debated for years and still these two interpretations are valid, one with dissipating energy and another with exponentially varying mass [30, 31, 32, 33]. We can generate a class of invariants by following the similar methods in ref. [29] for the Hamiltonian of a QHO experiencing time dependent frictional force in both the aspects [35, 36]. The class of invariants for the above system are characterised by the different solutions of Ermakov equation [37]. Among these invariants, a particular choice of invariant can be used to implement STA for QHO by using the time-dependent control of frictional force, which is not explored in the context of STA.
As the invariant method found to be the most efficient way to implement shortcut process for investigating the thermodynamic engines [29], studying oscillator under frictional force using invariant will be useful in quantum thermal engine studies. Also the arbitrary time dependancy of the frictional force can be used to improve the protocols to drive adiabatic strokes of quantum thermal engines.

In this paper, we investigate a class of Invariants for the Hamiltonian for a QHO under time-dependent frictional force in a generalized framework. In section two, we discuss such a class of invariants and corresponding Ermokov equation. We also discuss the necessary boundary conditions to establish the generalized framework of STA in the third section. Following the general formalism, in section four, an STA protocol is illustrated by choosing an appropriate solution for Ermokov equation called scaling factor and designed the time dependent frictional force to drive the system to achieve STA for QHO. We analyze the characteristics of such an STA protocol including cost of implementation. In section five, we consider a QHO with time dependent mass as an illustration to prove that the STA protocol developed for QHO under time dependent frictional force can be applied for QHO with time dependent mass. We use the same shortcut protocol to get desired output in a photonic lattice described by the differential set similar to the time dependent Schrodinger equation with mass varying QHO Hamiltonian. STA is achieved by arbitrarily controlling its lattice parameters as a function of propagation distance. Finally we summarize our results in conclusion section.

2 Quantum harmonic oscillator with time varying friction

We start with an harmonic oscillator experiencing a frictional force ($\gamma \dot{x}$) and a time dependent perturbative force ($F(t)$). The force equation corresponding to such an oscillator of unit mass is [34]

$$\ddot{x} + \gamma \dot{x} + \omega^2(t)x = F(t),$$  \hspace{1cm} (1)

where $\gamma$ is the constant damping coefficient, $\omega(t)$ is the time dependent frequency of the oscillator, $x$, $\dot{x}$ and $\ddot{x}$ are the position, velocity and acceleration of the oscillator respectively. From the above equation, it is evident that the motion of the oscillator continuously decreased at a constant rate $\gamma$ and the time dependent force can drive the system in any arbitrary rate. Assuming the damping coefficient as a function of time $\gamma(t)$ instead of a constant and in an unperturbed environment ($F(t) = 0$), the force equation will be modified as [35]

$$\ddot{x} + \gamma(t) \dot{x} + \omega^2(t)x = 0. \hspace{1cm} (2)$$

The rate of variation of motion of the above oscillator is also determined by the time varying function $\gamma(t)$ and it is termed as Time Dependent Coefficient of Friction (TDCF). The Lagrangian corresponding to the above system is given as [38]

$$\mathcal{L} = e^{-\Gamma(t)} \left( \frac{\dot{x}^2}{2} - \frac{1}{2} \omega^2(t)x^2 \right), \hspace{1cm} (3)$$

where the force equation [2] is obtained from the Lagrangian equation of motion $\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$ and the exponential term contains the time dependent function $\Gamma(t)$ related to TDCF as $\gamma(t) = \Gamma(t)$. Varying the TDCF of the Oscillator with respect to time will alter both the potential and kinetic energy of the system. This will allow us to control the oscillator by varying the TDCF in a pre-determined fashion. In other words, the system is now parameterized with $\Gamma(t)$ and we have the authority to vary its motion as a function of time. In this context, TDCF is the only physically relevant quantity related to a time-varying frictional force in terms of the dimensionless quantity $\Gamma(t)$ (time varying number), which helps to analyze the system dynamics. The variation of total energy with this type of parameterization can be understood by obtaining the Hamiltonian as an operator of the total energy content of the system using the equation [3] and by replacing the momentum and position variables with the corresponding operators $\hat{p} = -i \frac{\partial}{\partial \hat{x}}$ and $\hat{x}$ respectively with $\hbar = 1$, which is given as [34].

$$\hat{H} = e^{-\Gamma(t)} \hat{p}^2/2 + e^{\Gamma(t)} \omega^2(t) \hat{x}^2/2. \hspace{1cm} (4)$$

We can observe that the Caldirola-Kanai Hamiltonian can be sorted out from the above equation for a constant TDCF [39, 41]. Also, it is evident that, at any point in the oscillator path, both the kinetic and potential energy varies respectively with the negative and positive exponential factors of the order of $\Gamma(t)$, which is entirely a matter of specific functions of TDCF which alter the total energy of the oscillator. The implementation of STA differs from other STA methods as we use a frictional control in addition to frequency control.
The Lewis-Reisenfield method of invariants allows to cook up the invariant \( \hat{I} \) for any arbitrary Hamiltonian \( \hat{H} \), by imposing the condition of invariance using the formula

\[
\frac{\partial \hat{I}}{\partial t} + \frac{1}{i} [\hat{I}, \hat{H}] = 0.
\]

Applying this to the Hamiltonian \( \hat{H} \) in equation (4), we obtain the corresponding Invariant \( \hat{I} \) of the form

\[
\hat{I} = a(t) \hat{x}^2 + b(t) [\hat{x}, \hat{p}] + c(t) \hat{p}^2
\]

where, \( a, b \) and \( c \) are functions of time and \( [\hat{x}, \hat{p}] \) is the anticommutator of position and momentum operators.

Solving for the time-dependent coefficients by using equation (5) we will be able to deduce the invariant explicitly as, (here after we represent \( \Gamma(t) \) simply as \( \Gamma \)

\[
\hat{I} = \frac{1}{2} \left\{ \left( \frac{x}{\rho} \right)^2 \frac{\omega_0^2}{\rho^2} + \left( \rho \rho_e \frac{\Gamma}{\rho} - \left\{ \dot{\rho} - \frac{\Gamma}{2} \right\} e^{\frac{\rho}{\Gamma}} \right)^2 \right\}
\]

with time-dependent functions,

\[
a(t) = 2 \left\{ \left( \frac{\dot{\rho} - \frac{\Gamma}{2}}{\rho} \right)^2 + \frac{1}{\rho^2} \right\} e^{\frac{\rho}{\Gamma}}
\]

\[
b(t) = -2 \left\{ \rho \rho - \frac{\rho^2 \Gamma}{2} \right\}
\]

\[
c(t) = 2\rho^2 e^{-\frac{\rho}{\Gamma}}.
\]

It can be observed that the formulated \( \hat{I} \) is in the same form as in the ref [34], but in the absence of time-dependent perturbative force. The variable \( \rho \) is a function of time is generally called as scaling factor, first introduced by Lewis and Reisenfield to scale the invariant equation and later used to control STA using inverse engineering approach. The necessary condition to be satisfied by the \( \rho \) is

\[
\ddot{\rho} + \Omega^2 \rho = \frac{\omega_0^2}{\rho^3},
\]

such that the \( \hat{I} \) will obey equation (5) with \( \hat{H} \). The above equation is in the form of Ermokov equations [37].

The so called shifted frequency \( \Omega = \sqrt{\omega^2 - \frac{\Gamma^2}{2}} - \frac{\Gamma}{2} \) is influenced by both the TDCF of the oscillation and its first derivative. The same shift in the frequency is obtained in ref [35] using canonical transformation. In following sections, we will utilize \( \hat{I} \) to implement STA protocols.

### 3 STA Protocol

Shortcut protocols corresponding to the Hamiltonian of equation (4) for some frequency modulation obeying equation (5) can be obtained by using appropriate boundary conditions, which generate exact initial and final states of equilibrium adiabatic process. The boundary conditions are very important in STA and it decides the form of the scaling factor could be considered for inverse engineering. In our system, apart from the scaling factor \( \rho(t) \), the mathematical structure of \( \Gamma(t) \) is also important, which provides the behavioral change of controlled frictional force \( \dot{\Gamma} \) to assure STA. The initial and final state of the system under observation is specified through boundary condition and it will construct the initial and final structure of invariant. As we have two time-dependent functions to drive the system in the required fast path, the boundary condition is expected to fix both the functions at the starting and ending of the process. Among these two functions, \( \Gamma \) will be used to design the interaction of the system with controlled frictional force and \( \rho \) will be used to inverse engineer the Hamiltonian to establish the STA. This inverse engineering is done by finding the expression for frequency \( \omega(t) \) from the Ermokov equation using \( \rho(t) \) and \( \Gamma(t) \) for the necessary boundary conditions. We rewrite the Hamiltonian \( \hat{H} \) (Eq. [3]) with the corresponding expression of \( \omega(t) \) obtained from Ermokov equation and represent it as \( \hat{H}^{IE} \) where \( IE \) represents 'Inverse Engineered'.

3
The eigenstates of the Hamiltonian $\hat{H}$ and the Invariant $\hat{I}$ are related to each other with a time-dependent factor $e^{i\alpha t}$, where $\alpha$ is the phase factor [39]. But there could be some eigenstates shared by both $\hat{H}$ and $\hat{I}$ and these eigenstates can be obtained using the commutation relation [40]

$$[\hat{H}, \hat{I}] = 0.$$ (9)

The system should be in the corresponding eigenstate of $\hat{H}$, before and after the Inverse engineered process if it satisfies the equation (9) and make sure that the adiabatic transition should be within the time scale of invariant dynamics. Using the expressions for $\hat{H}$ and $\hat{I}$ (Eq. 4 & 7) we can solve for the commutation relation in equation (9) resulting,

$$\left(\dot{\rho} - \frac{\hat{I}\rho}{2}\right)^2 + \frac{\omega_0^2}{\rho^2} - \omega^2\rho^2 = 0$$ (10)

$$\rho\dot{\rho} - \frac{\hat{I}\rho^2}{2} = 0.$$ (11)

Solution for the above equations for the initial and final instants of time can be found as

$$\rho(0) = 1$$ (12)

$$\rho(\tau) = \sqrt{\frac{\omega_0}{\omega_\tau}}$$ (13)

at time 0 (initial) and $\tau$ (final) respectively. By fixing the initial and final values of the frequency $\omega_0$ and $\omega_\tau$ respectively, the boundary conditions for the $\Gamma(t)$ can be obtained from Ermokov equation [8] for the specific form of the scaling factor.

In an adiabatic process, the system is isolated and any change in the energy levels of the system is considered as work done by or on the system. We can consider the Harmonic oscillator under time-dependent frictional force as an isolated entity during the adiabatic process and the STA process is achieved by the evolution of such an isolated entity under the inverse engineered Hamiltonian $\hat{H}^{IE}$ with modified frequency as resulting from equation [8]. The expectation value $\langle \hat{H}^{IE} \rangle$ in the STA path is obtained by operating the instantaneous eigenstates of $\hat{I}$ with $\hat{H}^{IE}$. We can simplify the above mathematical process by rewriting $\hat{H}^{IE}$ in terms of

$$\hat{X} = \frac{\sqrt{\omega_0}}{\rho} \hat{x}$$ (14)

and

$$\hat{P} = \frac{\rho e^{-i\pi \frac{1}{\omega_0}}}{\sqrt{\omega_0}} \hat{p} + \left(\dot{\rho} - \frac{\hat{I}\rho}{\omega_0}\right) e^{-i\pi \frac{1}{\omega_0}} \hat{x},$$ (15)

where $\hat{X}$ and $\hat{P}$ are the position and momentum operators of the invariant $\hat{I}$. By using the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ defined as

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(\hat{X} - i\hat{P}\right)$$

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\hat{X} + i\hat{P}\right),$$

with properties

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

on the instantaneous eigenstates $|n\rangle$ of invariant $\hat{I}$, the expectation value $\langle \hat{H}^{IE} \rangle$ is obtained as

$$\langle \hat{H}^{IE} \rangle = \frac{(2n+1)}{4\omega_0} \left(\frac{\omega_0^2}{\rho^2} + \left[\dot{\rho} - \frac{\hat{I}\rho}{2}\right]^2 + \omega^2\rho^2\right).$$ (16)
4 STA for Harmonic Oscillator

The invariant constructed for the Hamiltonian $\hat{H}$ with controlled frictional force can be used to control the STA dynamics of a QHO potential with Hamiltonian

$$\hat{H}_{ho} = \frac{\hat{p}^2}{2} + \frac{\omega_0^2(t)\hat{x}^2}{2}$$

and its expectation value at any instant of time $[21]$

$$\langle \hat{H}_{ho} \rangle = \left( n + \frac{1}{2} \right) \omega(t).$$  

Such a control is possible by identifying that the QHO Hamiltonian can be deduced from the Hamiltonian $\hat{H}$ given in equation (4) at the initial and final times by assigning the value of $\Gamma(t)$ as zero at $t = 0$ and $t = \tau$. We can obtain a logical criteria by applying equations (12) and (13) to the expression $\langle \hat{H}^E \rangle$ that, $\dot{\rho}(t) = \dot{\Gamma}(t) = 0$ at $t = 0$ and $t = \tau$ to converge the expectation value to that of QHO. Thus the Hamiltonian $\hat{H}$ with TDCF can be used to implement STA, physically depending on the specific form of the TDCF and the Scaling factor.

STA of QHO can be implemented using the well known scaling factor $[21]$

$$\rho(t) = 6 \left( \sqrt{\frac{\omega_0}{\omega_\tau}} - 1 \right) s^5 - 15 \left( \sqrt{\frac{\omega_0}{\omega_\tau}} - 1 \right) s^4 + 10 \left( \sqrt{\frac{\omega_0}{\omega_\tau}} - 1 \right) s^3 + 1.$$  

This specific function is used in most of the applications (Quantum Otto engines, Atomic transport etc.) of STA protocols to drive the QHO using invariant method. We use $\rho(t)$ to design a specific function for $\Gamma(t)$ to achieve STA for QHO under time dependent frictional force in the intermediate times. In the above equation, $s = \frac{t}{\tau}$ and $\omega_0$ and $\omega_\tau$ are the initial ($t = 0$) and final ($t = \tau$) frequencies of the oscillator respectively. This specific choice of scaling factor satisfies the boundary conditions found so far,

$$\rho(0) = 1, \dot{\rho}(0) = 0$$

$$\rho(\tau) = \sqrt{\frac{\omega_0}{\omega_\tau}}, \dot{\rho}(\tau) = 0.$$

In addition to that, it also satisfies,

$$\ddot{\rho}(0) = \ddot{\rho}(\tau) = 0,$$

applying which to the Ermokov equation $[8]$ gives the specific boundary conditions for the function $\Gamma(t)$ and its first and second order derivatives,

$$\Gamma(0) = \dot{\Gamma}(0) = \ddot{\Gamma}(0) = 0$$

$$\Gamma(\tau) = \dot{\Gamma}(\tau) = \ddot{\Gamma}(\tau) = 0.$$

One of the easiest solution for $\Gamma(t)$ obeying the above boundary conditions could be

$$\Gamma(t) = s^3(s - 1)^3,$$  

where $s = \frac{t}{\tau}$ makes the function dimensionless. Numerical computation of expectation values of energy for various final times can be done using the above form of $\Gamma(t)$ and equation (19). Figure 1 shows the variation of the expectation value of Inverse engineered Hamiltonian in the setting for STA of Harmonic Oscillator. We have plotted the dimensionless ratio of the expectation value of $\langle \hat{H}_{ho} \rangle$ to initial energy of the Harmonic Oscillator against $s = \frac{t}{\tau}$ for various final times $\tau$. It is assumed that the system was thermalized to ground state before the shortcut process, where the system expands within a short time $\tau$. During this expansion process, the frequency of the oscillator will change from a higher value to a lower value. We have selected an experimentally executable frequency change $250 \times 2\pi$ Hz to $2.5 \times 2\pi$ Hz for numerical calculation $[21]$.

Adiabaticity parameter $Q^*$ is a measure of adiabaticity of the shortcut process and defined as the ratio of average energy of the shortcut process $\langle \hat{H}_{\text{ho}} \rangle_{\text{ave}}$ to the average adiabatic energy $\langle \hat{H}_{\text{ho}} \rangle_{\text{ave}} [41][42]$

$$Q^* = \frac{\langle \hat{H}_{\text{ho}} \rangle_{\text{ave}}}{\langle \hat{H}_{\text{ho}} \rangle_{\text{ave}}} = \frac{\int_0^\tau \langle \hat{H}_{\text{ho}}(t') \rangle dt'}{\int_0^\tau \langle \hat{H}_{\text{ho}}(t') \rangle dt'} = \frac{\int_0^\tau \langle \hat{H}_{\text{ho}}(t') \rangle dt'}{\int_0^\tau \langle \hat{H}_{\text{ho}}(t') \rangle dt'}, \quad (21)$$
where $\langle \rangle_{\text{ave}}$ represents the time average of expectation values of corresponding Hamiltonians given in equation (16) and (18). Instantaneous behaviour of Adiabaticity parameter can be analysed numerically using

$$Q^*(t) = \frac{\langle \hat{H}^{IE} \rangle}{\langle \hat{H}_{ho} \rangle} = \frac{1}{2\omega_0\omega} \left( \frac{\omega_0^2}{\rho^2} + \left( \dot{\rho} - \frac{\Gamma \rho}{2} \right)^2 + \omega^2 \rho^2 \right).$$

(22)

Adiabaticity parameter is plotted in Figure 2a and its instantaneous behavior is plotted in Figure 2b for all the other variables specified in Figure 1. It is evident from the plot 2a that the adiabaticity parameter tends to 1 for large times scales, thus the process tends to be completely adiabatic as $\tau$ increases. Instantaneous behaviour of Adiabaticity parameter varies from the initial value 1 to the final value 1 to make sure the adiabatic final states and the value deviates from the adiabatic trajectory at intermediate times. This deviation can be made negligible by appropriate control of the dynamics with proper designing of TDCF.

The whole process of shortcut is done by modifying the Hamiltonian thus it is able to bring back the adiabatic final states within a short duration of time. There must be a cost for such deviation in the dynamics of the process from the actual adiabatic dynamics and this cost can be measured as the difference between the expectation value of energy for inverse engineered Hamiltonian and that of harmonic oscillator Hamiltonian at any instant of time and it is given by the formula (23)

$$\langle \hat{H}^{STA} \rangle = \langle \hat{H}^{IE} \rangle - \langle \hat{H}_{ho} \rangle = \frac{(2n+1)}{4\omega_0} \left( \frac{\omega_0^2}{\rho^2} + \left( \dot{\rho} - \frac{\Gamma \rho}{2} \right)^2 + \omega^2 \rho^2 - 2\omega_0^2 \rho \right),$$

(23)

which is plotted in Figure 2d. This implementation cost is very high for small time scales of the shortcut, which makes difficult to achieve very short processes. As the initial and final energy of the shortcut process coincides with that of the actual adiabatic process, the cost value is zero for both the endpoints of the process. The average value of implementation cost can be found for any final time $\tau$ by time averaging the expectation value as

$$\langle \hat{H}^{STA} \rangle_{\text{ave}} = \left( \frac{1}{\tau} \right) \int_0^\tau \langle \hat{H}^{STA} \rangle dt$$

(24)
and it is plotted in Figure 2c, which shows a gradual decrease in implementation cost and it tends to zero for long time scales implying that the process is equivalent to the equilibrium adiabatic process without any control over the system for large $\tau$.

5 General Approach to Mass Variation

The Hamiltonian $\hat{H}$ (Eq. 4) considered so far is worth studying as it stands for yet another physically relevant and distinct situation, where the mass of the observed system varies with time. Controlling the dynamics of any system by arbitrarily varying its mass is found to be unrealistic but controlling the system with inherent mass variation is a realistic problem. The interpretation of mass variation in harmonic oscillator Hamiltonian is also found useful to model the optical lattices, identifying the mass as a function of propagation distance [43, 44]. Shortcut mechanism for such model is proposed by using invariants for Hamiltonian of forced oscillators with varying mass and frequency [29]. Below, we will discuss the usefulness of our general approach in the particular case of harmonic oscillator with mass $M(\xi)$ and frequency $\omega(\xi)$ for some parameter $\xi$ as,

$$\hat{H}' = \frac{\hat{p}^2}{2M(\xi)} + \frac{M(\xi)\omega^2(\xi)\hat{x}^2}{2}. \quad (25)$$
Comparing Equation (25) with (4) gives (26),

$$\dot{\Gamma}(\xi) = \frac{d}{d\xi} \ln \frac{M(\xi)}{\xi}.$$  

(26)

From equation (7), the invariant in terms of $M(\xi)$ is

$$\mathcal{F}' = \frac{1}{2} \left( \left( \frac{\rho}{\rho'} \right)^2 M(\xi) \omega_0^2 + \frac{1}{M(\xi)} \left( \rho' p - \frac{M(\xi) \rho'}{2} \right) x^2 \right).$$

(27)

This invariant is the exact invariant for $\mathcal{H}'$, which is different from the one considered in ref [36] and the existence of this invariant depends on the Ermokov equation

$$\rho'' + \Omega^2 \rho' = \frac{\omega_0^2}{\rho^3},$$

(28)

where the new shift in the frequency is $\Omega' = \sqrt{\omega(\xi)^2 + \left( \frac{M(\xi)}{2M(\xi)} \right)^2 - \frac{M(\xi)}{2M(\xi)}}$ and a general STA protocol can be formulated as discussed in section 3. Implementation of the protocol requires the knowledge of variation of mass with respect to corresponding parameter (time, length, etc.), which decides the particular form of scaling factor $\rho'$ for appropriate STA boundary conditions. We used the Ermokov equation (28) to inverse engineer the frequency to drive the system in a shortcut path. The mass variation directly influence the frequency of the oscillator, which will be evident on inversion of the Ermokov equation to construct frequency variation $\omega(\xi)$, which is in good agreement with some of existing interpretations of shortcuts [18]. On comparison with equation (16), the expectation value of energy on the inverse engineered shortcut path is,

$$\langle \mathcal{H}'_{\xi\epsilon} \rangle = \frac{(2n + 1)}{4\omega_0} \left( \frac{\omega_0^2}{\rho^2} + \left( \rho - \frac{M(\xi) \rho}{2M(\xi)} \right)^2 + \omega^2 \rho^2 \right).$$

(29)

5.1 Photonic Lattice as Mass Varying Hamiltonian

The photonic lattice model proposed in ref [43] is semi-infinite and composed of individual waveguides, whose index of refraction vary linearly. It can be modelled as a harmonic oscillator with mass $M(z)$ and frequency $\Omega(z)$, where $z$ is the propagation distance [13-29]. Considering the field amplitude at $n^{th}$ waveguide as $C_n(z)$, $a_0(z)$ to modulate linear variation of the refractive index and $a_1(z), a_2(z)$ as first and second coupling functions. The lattice is described by the differential set,

$$i \frac{\partial C_n(z)}{\partial z} + a_0(z) n C_n(z) + a_1(z) [f_{n+1} C_{n+1}(z) + f_{n} C_{n-1}(z)] + a_2(z) [g_{n+2} C_{n+2}(z) + g_n C_{n-2}(z)] = 0,$$

(30)

where $f_n = \sqrt{n}$ and $g_n = \sqrt{n(n-1)}$ are the functions of the positions $n = 0, 1, 2...$ of the waveguides in the array and $C_n(z) = 0$ for $n < 0$. If we define a wavefunction $|\Psi(z)\rangle = \sum_{j=0}^{n} C_j(z)|j\rangle$ using the field amplitude $C_j$ at $j^{th}$ waveguide, we can rewrite the equation (30) as Schrodinger-like equation [33],

$$\mathcal{H}'|\Psi(z)\rangle = i \frac{\partial |\Psi(z)\rangle}{\partial z},$$

(31)

and the corresponding Hamiltonian in terms of annihilation ($\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$) and creation ($\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$) operators is

$$\mathcal{H}' = - [a_0(z) \hat{a} \hat{a}^\dagger + a_1(z) (\hat{a} + \hat{a}^\dagger) + a_2(z) (\hat{a}^2 + \hat{a}^4)].$$

(32)

Using the form of $\hat{a}$ and $\hat{a}^\dagger$ in terms of normalized position and momentum operators

$$\hat{a} = \frac{1}{\sqrt{2}} \left( \hat{X} + i \hat{P} \right),$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( \hat{X} - i \hat{P} \right),$$

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Figure 3: Variation of the parameters $\Omega(z)$, $a_0(z)$, $a_1(z)$ and $a_2(z)$ against the propagation distance $z$ in arbitrary units is similar to one given in ref [43]. $\epsilon = 0.5$ and $z_s = 5$

the Hamiltonian becomes,

$$H' = -\left[\frac{\hat{P}^2}{2M(z)} + \frac{M(z)\Omega^2(z)\hat{X}^2}{2} + \sqrt{2}a_1(z)\hat{X} - \frac{a_0(z)}{2}\right], \quad (33)$$

where

$$M(z) = \frac{1}{a_0(z) - 2a_2(z)}$$

$$\Omega^2(z) = a_0^2(z) - 4a_2^2(z).$$

We can simplify the problem by considering the solution with a displacement and an overall phase factor as

$$|\Psi(z)\rangle = e^{-i\int\Phi(z)dz}e^{-i[u(z)\hat{P} + M(z)\hat{u}(z)\hat{X}]}|\psi(z)\rangle,$$

where the role of first coupling function $a_1(z)$ is only by defining the auxiliary function $u(z)$ (see ref [43] for complete expressions of $\Phi(z)$ and $u(z)$). Thus the differential equation (31) will be modified as,

$$\left[\frac{\hat{P}^2}{2m(t)} + \frac{m(t)\omega^2(t)\hat{X}^2}{2}\right]|\psi(z)\rangle = i\frac{\partial|\psi(z)\rangle}{\partial z}, \quad (34)$$

where $m(t) = M(-z)$ and $\omega(t) = \Omega(-z)$. The above equation expresses the differential equation for a photonic lattice as a mass varying harmonic oscillator Hamiltonian. This is a right point to consider the usage of STA for this Hamiltonian as we developed in section 4. Unlike the work done by Dionisis Stefanatos [29], we fixed the lattice parameters for a desired output and try to produce the same output for various propagation distances using the class of invariants given in equation (27). To illustrate the control on propagation distance to get a desired output, we can take the example given in ref [43] with parameters,

$$\Omega(z) = \left[3 + \epsilon \tanh(z - z_s)\right], \quad a_0(z) = \frac{[M^2(z)\Omega^2(z) + 1]}{2M(z)}, \quad a_1(z) = 1, \quad a_2(z) = \frac{[M^2(z)\Omega^2(z) - 1]}{4M(z)}. \quad (35)$$
Figure 4: a-c; Variation of parameters $\Omega(z), a_0(z), a_1(z), a_2(z)$ using STA for various final propagation distances plotted against $s = \frac{z}{z_\tau}$. d; Mass variation in STA process given by equation (38). Initial and final parameters are $\Omega(0) = M(0) = 1, \Omega(z_\tau) = 2$ and $M(z_\tau) = 1$.

All the above parameters are plotted in figure (3) for $M(z) = 1$, where the frequency is a smooth step function and the steepness of the curve is decided by the constant $\epsilon$. We have considered the frequency function for $\epsilon = 0.5$, where the initial and final required frequency is tending close to 1 and 2 respectively. This is equivalent to a Glauber-Fock oscillator lattice that make transitions smoothly from just first-neighbor couplings to first- and second-neighbor couplings [43, 45]. We consider the desired output as the one corresponding to $a_0(z_\tau) = \frac{5}{2}, a_1(z_\tau) = 1$ and $a_2(z_\tau) = \frac{3}{4}$ while the initial parameters fixed as $a_0(0) = a_1(0) = 1$ and $a_2(0) = 0$. Here we have the freedom to decide the arbitrary selection of the mass function, during the shortcut process, assigning initial and final values as 1. Considering the equation (26) connecting mass variation and $\Gamma$, the above boundary conditions for mass variations will be in good agreement with the boundary conditions for the specific form of $\Gamma$ of the shortcut process for the harmonic oscillator in section 4. Changing the variables of the both the equations (20) and (26) in terms of propagation distance, we will obtain similar functions as

$$\Gamma(z) = s^3(s - 1)^3,$$

(36)

and

$$\Gamma(z) = \ln M(z),$$

(37)

where $s = \frac{z}{z_\tau}$ and $z_\tau$ is the location where we need to get the final values of parameters. From the above equations, we get

$$M(z) = e^{s^3(s - 1)^3}.$$  

(38)
A protocol similar to the shortcut protocol in section 4 will redefine the lattice parameters (index of refraction, first- and second-couplings parameters) through the new propagation distance dependent functions $\Omega(z)$ and $M(z)$ to control the location of output in the array of the waveguides. A propagation distance dependent scaling factor similar to the one in equation (19),

$$\rho(z) = \left( \frac{\Omega(0)}{\Omega(z_\tau)} - 1 \right) s^5 - 15 \left( \sqrt{\frac{\Omega(0)}{\Omega(z_\tau)}} - 1 \right) s^4 + 10 \left( \sqrt{\frac{\Omega(0)}{\Omega(z_\tau)}} - 1 \right) s^3 + 1,$$

(39)
can be used to construct such protocol. In the above equation $\Omega(0)$ and $\Omega(z_\tau)$ are the boundary values of frequency $\Omega(z)$. Variation of parameters (index of refraction and coupling parameters) resulting from the set of equations (35) is plotted in figure (4a-4c), and the variation in mass $M(z)$ is plotted in figure (4d) with initial parameters $\Omega(0) = M(0) = 1$ and final parameters $\Omega(z_\tau) = 2, M(z_\tau) = 1$. Irrespective of the manner in which the parameters vary over the propagation distance, we could drive it from the desired initial to the final values. This mechanism can be used to construct output at necessary locations by arbitrarily controlling the lattice parameters. However, the cost for implementation of the protocol is not measurable with the methods explained in section 4 since the working of photonic lattice is different from that of a single harmonic oscillator.

6 Conclusion

We have successfully derived a class of invariants for the harmonic oscillator under time-dependent frictional force. In the Ermokov equation, the frequency is shifted by the terms with TDCF and its first derivative. The scope of a general approach to STA using the invariant with TDCF is studied and found it is feasible but the specific form of the scaling factor at the boundaries decides the boundary conditions for time-dependent frictional force. An interesting case of STA protocol for the quantum harmonic oscillator is framed by allowing the frictional control only at the intermediate times, such that the TDCF should be zero at both the ends of the shortcut process. We have analyzed the variation of adiabaticity parameter $Q^*$, the expectation value of energy $\langle \hat{H}^{1E} \rangle$ and the cost of the shortcut process $\langle \hat{H}^{STA} \rangle$ for various time scales. Interpreting the harmonic oscillator system under time-dependent frictional force as a harmonic oscillator with time varying mass, which make use of the same shortcut protocol to control the dynamics of the harmonic oscillator with inbuilt variation in mass (without taking control over the mass). We have illustrated the case of coupled photonic lattice by identifying the propagation of light through the array of waveguides as the evolution of harmonic oscillator wavefunction. The protocol can be improved by formulating some other intelligent TDCF obeying the corresponding boundary conditions. We have left space for such works with different TDCF for improved characteristics of shortcut protocol. Also, it is possible to incorporate such improved shortcut protocols to Quantum thermal machines for enhanced performance. Further, our study might be useful for the applications analogous to the case of photonic lattice which can be studied using harmonic oscillator hamiltonian with variation in mass.

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