NEW ESTIMATES ON INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper, we obtain some inequalities by using a kernel and an inequality which is a result of Young inequality. Besides we give some applications to special means.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The inequality
\[
(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]
is well known in the literature as Hermite-Hadamard’s inequality for convex functions [1].

In [2], Dragomir and Agarwal proved one lemma and some Hadamard’s type inequalities for convex functions as following:

**Lemma 1.** Let \( f : I^0 \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \). If \( f' \in L[a,b] \), then the following equality holds:
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b) \, dt.
\]

**Theorem 1.** Let \( f : I^0 \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \). If \( |f'| \) is convex on \( [a,b] \), then the following inequality holds:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.
\]

**Theorem 2.** Let \( f : I^0 \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \), and let \( p > 1 \). If the new mapping \( |f'|^{p/(p-1)} \) is convex on \( [a,b] \), then the following inequality holds:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)}{2(p+1)^{1/p}} \left[ \left( |f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right) \right]^{(p-1)/p}.
\]

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We recall the well-known Young’s inequality which can be stated as follows.

**Theorem 3. (Young’s inequality, see [3], p. 117)** If \(a, b > 0\) and \(p, q > 1\) satisfy \(\frac{1}{p} + \frac{1}{q} = 1\), then

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}
\]

Equality holds if and only if \(a^p = b^q\).

**Remark 1.** [4] If we take \(a = t^{\frac{1-p}{2}}\) and \(b = t^{\frac{1}{pq}}\) in (1.2), we have

\[
1 \leq \frac{1}{p}t^{\frac{1-p}{2}} + \left(1 - \frac{1}{p}\right)t^{\frac{1}{pq}}
\]

for all \(t \in (0, 1)\).

Chebyshev inequality is given in the following theorem.

**Theorem 4.** Let \(f, g : [a, b] \to \mathbb{R}\) be integrable functions, both increasing or both decreasing. Furthermore, let \(p : [a, b] \to \mathbb{R}_0^+\) be an integrable function. Then

\[
\int_a^b p(x) \, dx \int_a^b p(x) f(x) g(x) \, dx \geq \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx.
\]

If one of the functions \(f\) and \(g\) is nonincreasing and the other is nondecreasing, then the inequality in (1.4) is reversed. Inequality (1.4) is known in the literature as Chebyshev inequality and so are the following special cases of (1.4):

\[
\int_a^b f(x) g(x) \, dx \geq \frac{1}{b-a} \int_a^b f(x) \, dx \int_a^b g(x) \, dx
\]

and

\[
\int_0^1 f(x) g(x) \, dx \geq \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx.
\]

In the following sections our main results are given.

2. New Results

**Theorem 5.** Let \(f : I^o \subseteq \mathbb{R} \to \mathbb{R}\) be a differentiable convex mapping on \(I^o\), \(a, b \in I^o\) with \(a < b\). If \(f, f' \in L[a, b]\), for \(p, q > 1\), \(\frac{1}{p} + \frac{1}{q} = 1\), the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^{\frac{p}{2}}}{(p+1)^{\frac{p}{2}} \left( \int_a^b |f'(x)|^q \, dx \right)^{\frac{1}{q}}}
\]
Proof. By Lemma 1 in [2] we have

(2.1) \[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx = \frac{b - a}{2} \int_0^1 (1 - 2t) f'(ta + (1 - t)b) \, dt.
\]

As we choose \( f \) is convex on \( I^o \) by using Hadamard’s inequality, we can see that both sides are positive of Lemma 2.1.

On the other hand by using Young’s inequality we have (\( t \in [0, 1], \ p > 1 \))

\[ 1 \leq \frac{1}{p} t^{\frac{1}{p} - 1} + \left( 1 - \frac{1}{p} \right) t^\frac{1}{p} \]

which is proved in [4]. If we integrate both sides of above inequality respect to \( t \) over \([0,1]\) we get

(2.2) \[ 1 \leq \int_0^1 \left( \frac{1}{p} t^{\frac{1}{p} - 1} + \left( 1 - \frac{1}{p} \right) t^\frac{1}{p} \right) \, dt \]

By multiplying both sides of (2.1) and (2.2) we have

(2.3) \[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{2} \int_0^1 \left( \frac{1}{p} t^{\frac{1}{p} - 1} + \left( 1 - \frac{1}{p} \right) t^\frac{1}{p} \right) \, dt \times \int_0^1 |(1 - 2t) f'(ta + (1 - t)b)| \, dt
\]

To use Hölder’s inequality we apply properties of absolute value as

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2} \int_0^1 \left( \frac{1}{p} t^{\frac{1}{p} - 1} + \left( 1 - \frac{1}{p} \right) t^\frac{1}{p} \right) \, dt \times \int_0^1 |(1 - 2t) f'(ta + (1 - t)b)| \, dt
\]

By using Hölder’s inequality we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2} \int_0^1 \left( \frac{1}{p} t^{\frac{1}{p} - 1} + \left( 1 - \frac{1}{p} \right) t^\frac{1}{p} \right) \, dt \times \left( \int_0^1 |1 - 2t|^p \, dt \right)^\frac{1}{p} \left( \int_0^1 |f'(ta + (1 - t)b)|^q \, dt \right)^\frac{1}{q}
\]

\[ = \frac{b - a}{2} \left( \frac{2p}{p + 1} \right) \left( \int_0^1 (1 - 2t)^p \, dt + \int_{\frac{1}{2}}^1 (2t - 1)^p \, dt \right)^\frac{1}{p} \times \left( \int_0^1 |f'(ta + (1 - t)b)|^q \, dt \right)^\frac{1}{q}
\]

\[ = \frac{b - a}{2} \left( \frac{2p}{p + 1} \right) \left( \frac{1}{p + 1} \right) \left( \int_0^1 |f'(ta + (1 - t)b)|^q \, dt \right)^\frac{1}{q}.
\]

By simple calculation we get the desired result. \( \square \)
Corollary 1. If we choose \( p = q = 2 \) in Theorem 5, we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{2(b-a)^{\frac{1}{2}}}{3^{\frac{3}{2}}} \left( \int_{a}^{b} |f'(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

Theorem 6. Let \( f : I^o \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable convex mapping on \( I^o \), \( a, b \in I^o \) with \( a < b \). If \( f, f' \in L[a, b] \), for \( p, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), the following inequality holds:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{p(p-1)}{2(2p+1)} [f'(b) - f'(a)].
\]

Proof. The same steps are followed as in Theorem 5 until (2.3). Then we know that the function
\[
f(t) = 1 - 2t
\]
is nonincreasing on \([0, 1]\), and since \( f \) is convex, \( f'' \) is positive on \( I^o \). So \( f' \) is nondecreasing. By using these phrases we can use Chebyshev inequality as:
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{b-a}{2} \int_{0}^{1} \left( \frac{1}{p^{\frac{1}{p}} - 1} + \left( 1 - \frac{1}{p} \right) t^{\frac{1}{p}} \right) dt \\
\times \int_{0}^{1} (1 - 2t) f'(ta + (1 - t)b) \, dt
\leq \frac{b-a}{2} \int_{0}^{1} \left( \frac{1}{p^{\frac{1}{p}} - 1} + \left( 1 - \frac{1}{p} \right) t^{\frac{1}{p}} \right) dt \\
\times \int_{0}^{1} (1 - 2t) dt \int_{0}^{1} f'(ta + (1 - t)b) \, dt
= \frac{f(b) - f(a)}{2} \int_{0}^{1} \left( \frac{1}{p^{\frac{1}{p}} - 1} + \left( 1 - \frac{1}{p} \right) t^{\frac{1}{p}} \right) dt \\
\times \int_{0}^{1} (1 - 2t) dt
\]
Since both of the functions \( \left( \frac{1}{p^{\frac{1}{p}} - 1} + \left( 1 - \frac{1}{p} \right) t^{\frac{1}{p}} \right) \) and \( (1 - 2t) \) are nonincreasing, we can use Chebyshev inequality again as:
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(b) - f(a)}{2} \int_{0}^{1} \left( \frac{1}{p^{\frac{1}{p}} - 1} + \left( 1 - \frac{1}{p} \right) t^{\frac{1}{p}} \right) (1 - 2t) dt
\]
By simple calculation, the proof is completed. \( \square \)

Corollary 2. If we choose \( p = 1, 1 \) in Theorem 5, we have
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{11}{483} [f(b) - f(a)].
\]

Theorem 7. Let \( f : I^o \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable convex mapping on \( I^o \), \( a, b \in I^o \) with \( a < b \). If \( f, f' \in L[a, b] \), for \( p, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), the following
inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{2^{\frac{p}{q}}}{(p+1) (b-a)} \left( \int_a^b \left| f'(x) \right|^q \, dx \right)^{\frac{1}{q}}.
\]

**Proof.** The same steps are followed as in Theorem 4 until (2.3). Then by using convexity of \( f \) and properties of absolute value we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \int_0^1 \left( \frac{1}{p} t^{\frac{p}{p}-1} + \left(1 - \frac{1}{p} \right) t^{\frac{q}{p}} \right) dt
\]

\[
\times \int_0^1 \left| (1-2t) f'(ta + (1-t) b) \right| dt.
\]

By using Power-mean inequality we have

\[
(2.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| 
\]

\[
\leq \frac{b-a}{2} \int_0^1 \left( \frac{1}{p} t^{\frac{p}{p}-1} + \left(1 - \frac{1}{p} \right) t^{\frac{q}{p}} \right) dt
\]

\[
\times \left( \int_0^1 \left| (1-2t) \right| dt \right)^{\frac{q}{p}} \left( \int_0^1 \left| f'(ta + (1-t) b) \right|^q dt \right)^{\frac{1}{q}}
\]

\[
= \frac{2^{\frac{p}{q}}}{p+1} \left( \int_0^1 \left| (1-2t) \right| \left| f'(ta + (1-t) b) \right|^q dt \right)^{\frac{1}{q}}.
\]

And using the change of the variable \( x = ta + (1-t) b, t \in [0,1] \), inequality (2.4) can be written as

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{2^{\frac{p}{q}}}{(p+1) (b-a)} \left( \int_a^b \left| x - \frac{a+b}{2} \right| f'(x) \, dx \right)^{\frac{1}{q}}.
\]

Then the proof is completed. \( \square \)

**Corollary 3.** If we choose \( p = q = 2 \) in Theorem 7, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{2^{\frac{p}{q}}}{3} \left( \int_0^1 \left| x - \frac{a+b}{2} \right| f'(x) \, dx \right)^{\frac{1}{q}}.
\]

3. Applications to special means

We now consider the applications of our Theorems to the following special means

The arithmetic mean: \( A = A(a,b) := \frac{a+b}{2}, \ a, b \geq 0, \)

The geometric mean: \( G = G(a,b) := \sqrt{ab}, \ a, b \geq 0, \)

The harmonic mean: \( H = H(a,b) := \frac{2ab}{a+b}, \ a, b \geq 0, \)

The logarithmic mean: \( L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b, \ a, b \geq 0, \end{cases} \)

The Identric mean: \( I = I(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left( \frac{ab}{e^a} \right)^{\frac{1}{b}} & \text{if } a \neq b, \ a, b \geq 0, \end{cases} \)
The p-logarithmic mean: 

\[ L_p = L_p(a, b) := \begin{cases} \frac{(b^{p+1} - a^{p+1})}{(p+1)(b-a)}^{1/p} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases} \]

\( p \in \mathbb{R}\setminus \{-1, 0\}; \ a, b > 0. \)

The following inequality is well known in the literature:

\[ H \leq G \leq L \leq I \leq A \leq K \]

It is also known that \( L_p \) is monotonically increasing over \( p \in \mathbb{R} \), denoting \( L_1 = A \), \( L_0 = I \) and \( L_{-1} = L \).

The following propositions holds:

**Proposition 1.** Let \( a, b \in \mathbb{R}^+ \), \( a < b \) and \( n \in \mathbb{N} \), \( n \geq 2 \). Then, we have

\[ |A(a^n, b^n) - L_n(a, b)| \leq \frac{n.p.(b-a)}{(p+1)^{1+\frac{1}{p}}} A^{n-1}(a, b). \]

*Proof.* The proof is immediate from Theorem 5 applied for \( f(x) = x^n, x \in \mathbb{R} \). \( \square \)

**Proposition 2.** Let \( a, b \in \mathbb{R}^+ \), \( a < b \) and \( n \in \mathbb{N} \), \( n \geq 2 \). Then, for all \( p > 1 \), the following inequality holds:

\[ A(a^n, b^n) - L_n(a, b) \leq \frac{p^2}{(p+1)(2p+1)} (b^n - a^n). \]

*Proof.* The proof is immediate from Theorem 6 applied for \( f(x) = x^n, x \in \mathbb{R} \). \( \square \)

**Proposition 3.** Let \( a, b \in \mathbb{R}^+ \), \( a < b \) and \( n \in \mathbb{N} \), \( n \geq 2 \). Then, for all \( p > 1 \), the following inequality holds:

\[ |A(a^n, b^n) - L_n(a, b)| \leq \frac{2^p}{(p+1)(b-a)} \left( \int_a^b \left| x - \frac{a+b}{2} \right| |f'(x)|^q \, dx \right)^{\frac{1}{q}}. \]

*Proof.* The proof is immediate from Theorem 7 applied for \( f(x) = x^n, x \in \mathbb{R} \). \( \square \)

**Proposition 4.** Let \( a, b \in \mathbb{R}^+ \), \( a < b \). Then, we have

\[ |A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq \frac{p^2}{(p+1)^{1+\frac{1}{p}}} \left( \int_a^b |x|^{-2q} \, dx \right)^{\frac{1}{q}}. \]

*Proof.* The proof is obvious from Theorem 5 applied for \( f(x) = 1/x, x \in [a, b] \). \( \square \)

**Proposition 5.** Let \( a, b \in \mathbb{R}^+ \), \( a < b \). Then, we have

\[ A(a^{-1}, b^{-1}) - L^{-1}(a, b) \leq \frac{2p(p-1)}{(p+1)(2p+1)} H^{-1}(a, b). \]

*Proof.* The proof is obvious from Theorem 6 applied for \( f(x) = 1/x, x \in [a, b] \). \( \square \)
Proposition 6. Let $a, b \in \mathbb{R}^+$, $a < b$. Then, we have

$$|A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq \frac{2^{\frac{1}{q}}p}{(p+1)(b-a)} \left( \int_a^b \left| x - \frac{a+b}{2} \right|^\frac{p}{2} \left( x^2 - 2a \right)^{\frac{1}{q}} \right).$$

Proof. The proof is obvious from Theorem 7 applied for $f(x) = 1/x$, $x \in [a, b]$. □

References

[1] S.S. Dragomir, C. E. M. Pearce, Selected Topic on Hermite–Hadamard Inequalities and Applications, URL: http://www.maths.adelaide.edu.au/App lied/staff /cpearce.html
[2] S.S. Dragomir, R.P. Agarwal, Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula, Appl. Math. Lett., Vol. 11, No. 5, pp. 91-95, 1998.
[3] S.S. Dragomir, R.P. Agarwal and N.S. Barnett, Inequalities for beta and gamma functions via some classical and new integral inequalities, J. Inequal. & Appl., 5 (2000), 103-165.
[4] M. Tunç, Two New Definitions on Convexity and Related Inequalities, arXiv:1205.5189v1 [math.CA], 23 May 2012.

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