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Abstract. We prove the existence of ground state solutions to critical growth $p$-Laplacian and fractional $p$-Laplacian problems that are nonresonant at zero.

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Consider the problem

\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{p-2} u + |u|^{p^*-2} u & \text{in } \Omega \\
 u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $1 < p < N$, $\Delta_p u = \text{div}(\nabla |\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian of $u$, $\lambda \in \mathbb{R}$, and $p^* = Np/(N - p)$ is the critical Sobolev exponent. Solutions of this problem coincide with critical points of the $C^1$-functional

\[
E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\Omega} |u|^p \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx, \quad u \in W^{1,p}_0(\Omega).
\]

Let $K = \{ u \in W^{1,p}_0(\Omega) \setminus \{0\} : E'(u) = 0 \}$ be the set of nontrivial critical points of $E$ and set

\[
c = \inf_{u \in K} E(u).
\]

Recall that $u_0 \in K$ is called a ground state solution if $E(u_0) = c$. For each $u \in K$,

\[
E(u) = E(u) - \frac{1}{p^*} E'(u) u = \frac{1}{N} \int_{\Omega} |u|^{p^*} \, dx > 0,
\]

so $c \geq 0$, and $c > 0$ if there is a ground state solution. Let

\[
S = \inf_{u \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left( \int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{p/p^*}}
\]

be the best Sobolev constant. Denote by $\sigma(-\Delta_p)$ the Dirichlet spectrum of $-\Delta_p$ in $\Omega$ consisting of those $\lambda \in \mathbb{R}$ for which the eigenvalue problem

\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{p-2} u & \text{in } \Omega \\
 u &= 0 & \text{on } \partial \Omega
\end{aligned}
\]

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has a nontrivial solution. We have the following theorem.

**Theorem 1.** If problem (1) has a nontrivial solution $u$ with

$$E(u) < \frac{1}{N} S^{N/p}$$

and $\lambda \notin \sigma(-\Delta_p)$, then it has a ground state solution.

**Proof.** Let $(u_j) \subset K$ be a minimizing sequence for $c$. Then $(u_j)$ is a (PS)$_c$ sequence for $E$. Since problem (1) has a nontrivial solution satisfying (3), $c < S^{N/p}/N$. So $E$ satisfies the (PS)$_c$ condition (see Guedda and Véron [6, Theorem 3.4]). Hence a renamed subsequence of $(u_j)$ converges to a critical point $u_0$ of $E$ with $E(u_0) = c$. We claim that $u_0$ is nontrivial and hence a ground state solution of problem (1). To see this, suppose $u_0 = 0$. Then $\rho_j := \|u_j\| \to 0$. Let $\tilde{u}_j = u_j/\rho_j$. Since $\|\tilde{u}_j\| = 1$, a renamed subsequence of $(\tilde{u}_j)$ converges to some $\tilde{u}$ weakly in $W^{1,p}_0(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in $\Omega$. Since $E'(u_j) = 0$,

$$\int_\Omega |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v \, dx = \lambda \int_\Omega |u_j|^{p-2} u_j v \, dx + \int_\Omega |u_j|^{p^*-2} u_j v \, dx \quad \forall \, v \in W^{1,p}_0(\Omega),$$

and dividing this by $\rho_j^{p-1}$ gives

$$\int_\Omega |\nabla \tilde{u}_j|^{p-2} \nabla \tilde{u}_j \cdot \nabla v \, dx = \lambda \int_\Omega |\tilde{u}_j|^{p-2} \tilde{u}_j v \, dx + o(\|v\|) \quad \forall \, v \in W^{1,p}_0(\Omega).$$

Passing to the limit in (4) gives

$$\int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla v \, dx = \lambda \int_\Omega |	ilde{u}|^{p-2} \tilde{u} v \, dx \quad \forall \, v \in W^{1,p}_0(\Omega),$$

so $\tilde{u}$ is a weak solution of (2). Taking $v = \tilde{u}_j$ in (4) and passing to the limit shows that $\lambda \int_\Omega |\tilde{u}|^p \, dx = 1$, so $\tilde{u}$ is nontrivial. This contradicts the assumption that $\lambda \notin \sigma(-\Delta_p)$ and completes the proof. \qed

Combining this theorem with the existence results in García Azorero and Peral Alonso [5], Egnell [4], Guedda and Véron [6], Arioli and Gazzola [1], and Degiovanni and Lancelotti [3] gives us the following theorem for the case $N \geq p^2$.

**Theorem 2.** If $N \geq p^2$ and $\lambda \in (0, \infty) \setminus \sigma(-\Delta_p)$, then problem (1) has a ground state solution.

For $N < p^2$, combining Theorem 1 with Perera et al. [10, Corollary 1.2] gives the following theorem, where $(\lambda_k) \subset \sigma(-\Delta_p)$ is the sequence of eigenvalues based on the $\mathbb{Z}_2$-cohomological index introduced in Perera [8] and $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$.

**Theorem 3.** If $N < p^2$ and

$$\lambda \in \bigcup_{k=1}^{\infty} \left( \lambda_k - \frac{S}{|\Omega|^{p/N}}, \lambda_k \right) \setminus \sigma(-\Delta_p),$$

then problem (1) has a ground state solution.

**Remark 4.** In the semilinear case $p = 2$, Theorem 2 was proved in Szulkin et al. [11] using a Nehari–Pankov manifold approach, and Theorems 1 and 3 were proved in Chen et al. [2] using a more direct approach. Moreover, they allow $\lambda$ to be an eigenvalue when $N \geq 5$. However, their proofs are strongly dependent on the fact that $H^1_0(\Omega)$ splits into the direct sum of its subspaces spanned by the eigenfunctions of the Laplacian that correspond to eigenvalues that are less than or equal to $\lambda$ and those that are greater than $\lambda$. Those proofs do not extend to the $p$-Laplacian since it is a nonlinear operator and hence has no linear eigenspaces.

**Remark 5.** We conjecture that the assumption $\lambda \notin \sigma(-\Delta_p)$ can be removed from Theorems 1 and 2 when $N^2/(N+1) > p^2$. 
Our argument can be easily adapted to obtain ground state solutions of other types of critical growth problems as well. For example, consider the nonlocal problem
\[
\begin{cases}
(-\Delta)^s_p u = \lambda |u|^{p-2} u + |u|^{p^*_s - \gamma} u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\] (5)
where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with Lipschitz boundary, \(s \in (0, 1), 1 < p < N/s, (-\Delta)^s_p\) is the fractional \(p\)-Laplacian operator defined on smooth functions by
\[
(-\Delta)^s_p u(x) = 2 \lim_{r \to 0} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,
\]
\(\lambda \in \mathbb{R}\), and \(p^*_s = Np/(N-sp)\) is the fractional critical Sobolev exponent. Let \(\| \cdot \|_p\) denote the norm in \(L^p(\mathbb{R}^N)\), let
\[
\|u\|_{s,p} = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{1/p}
\]
be the Gagliardo seminorm of a measurable function \(u : \mathbb{R}^N \to \mathbb{R}\), and let
\[W^{s,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) : \|u\|_{s,p} < \infty \}\]
be the fractional Sobolev space endowed with the norm
\[
\|u\|_{s,p} = (\|u\|_p^p + \|u\|_{s,p}^p)^{1/p}.
\]
We work in the closed linear subspace
\[W^{s,p}_0(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \ \text{a.e. in } \mathbb{R}^N \setminus \Omega \}
\]
equivalently renormed by setting \(\| \cdot \| = \| \cdot \|_{s,p}\). Solutions of problem (5) coincide with critical points of the \(C^1\)-functional
\[
E_s(u) = \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy - \frac{\lambda}{p} \int_{\Omega} |u|^p \, dx - \frac{1}{p^*_s} \int_{\Omega} |u|^{p^*_s} \, dx, \quad u \in W^{s,p}_0(\Omega).
\]
As before, a ground state is a least energy nontrivial solution. Let
\[\dot{W}^{s,p}(\mathbb{R}^N) = \{ u \in L^{p^*_s}(\mathbb{R}^N) : \|u\|_{s,p} < \infty \}\]
edowed with the norm \(\| \cdot \|\) and let
\[
S = \inf_{u \in \dot{W}^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy}{\left( \int_{\mathbb{R}^N} |u|^{p^*_s} \, dx \right)^{p/p^*_s}}
\]
be the best fractional Sobolev constant. Denote by \(\sigma((-\Delta)^s_p)\) the Dirichlet spectrum of \((-\Delta)^s_p\) in \(\Omega\) consisting of those \(\lambda \in \mathbb{R}\) for which the eigenvalue problem
\[
\begin{cases}
(-\Delta)^s_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\]
has a nontrivial solution. Following theorem can be proved arguing as in the proof of Theorem 1.

**Theorem 6.** If problem (5) has a nontrivial solution \(u\) with
\[
E_s(u) < \frac{s}{N} S^{N/s}
\]
and \(\lambda \in \sigma((-\Delta)^s_p)\), then it has a ground state solution.

Combining this theorem with the existence results in Mosconi et al. [7] and Perera et al. [9] gives us the following theorem, where \((\lambda_k) \subset \sigma((-\Delta)^s_p)\) is the sequence of eigenvalues based on the \(\mathbb{Z}_2\)-cohomological index.
Theorem 7. Problem (5) has a ground state solution in each of the following cases:

(i) $N > sp^2$ and $\lambda \in (0, \infty) \setminus \sigma((-\Delta)_p^s)$,
(ii) $N = sp^2$ and $\lambda \in (0, \lambda_1)$,
(iii) $N \leq sp^2$ and

$$\lambda \in \bigcup_{k=1}^{\infty} \left( \Lambda_k - \frac{S}{|\Omega|^{sp/N}}, \lambda_k \right) \setminus \sigma((-\Delta)_p^s).$$

Remark 8. Theorems 6 and 7 are new even in the semilinear case $p = 2$.

Remark 9. We conjecture that problem (5) has a ground state solution for all $\lambda > 0$ when $N^2/(N + s) > sp^2$.

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