Instantons and fermion condensate in adjoint $QCD_2$

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Abstract

We show that $QCD_2$ with adjoint fermions involves instantons due to nontrivial
$\pi_1[SU(N)/Z_N] = Z_N$. At high temperatures, quasiclassical approximation works
and the action and the form of effective (with account of quantum corrections) in-
stanton solution can be evaluated. Instanton presents a localized configuration with
the size $\propto g^{-1}$. At $N = 2$, it involves exactly 2 zero fermion modes and gives rise
to fermion condensate $<\bar{\lambda}^a\lambda^a>_T$ which falls off $\propto \exp\{-\pi^3/2T/g\}$ at high $T$ but
remains finite. At low temperatures, both instanton and bosonization arguments
also exhibit the appearance of fermion condensate $<\bar{\lambda}^a\lambda^a>_{T=0} \sim g$. For $N > 2$,
the situation is paradoxical. There are $2(N-1)$ fermion zero modes in the instanton
background which implies the absence of the condensate in the massless limit. From
the other hand, bosonization arguments suggest the appearance of the condensate
for any $N$. Possible ways to resolve this paradox (which occurs also in some 4-dim
gauge theories) are discussed.

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1 Introduction

Two-dimensional $QCD$ with fermions belonging to adjoint representation of $SU(N)$ group attracted lately a considerable attention. In very interesting recent works [1], the spectrum of the theory in the large $N$ limit has been determined. It displayed the features which are strikingly analogous to the spectrum of 4-dimensional $QCD$. In contrast to $QCD_2$ with fundamental fermions where the meson states lie on one Regge-like trajectory $^2$ so that

$$M_n^2 \sim g^2 N_c n$$  \hspace{1cm} (1.1) 

and the density of states rises linearly with mass $dn/dM \sim M$, here the number of such trajectories is infinite, and the density of states grows exponentially with mass $^3$.

Of course, it is exactly the same behaviour as in large $N$ $QCD_4$ where the number of infinitely narrow resonances also rises exponentially with energy so that the Hagedorn phenomenon — the appearance of limiting temperature above which the system cannot be heated — takes place. $^3$.

In this paper, we show that the adjoint $QCD_2$ bears much resemblance with 4-dim $QCD$ describing the real world also for finite $N$. The situation is clear and the analogy is straightforward for $N = 2$. In particular, we show that, in contrast to what happens in $QCD_2$ with fundamental quarks, fermion condensate is generated here which falls down rapidly at high $T$. The physical picture is the same as in $QCD_4$ with only one light quark flavor $^4$ and in the Schwinger model $^5, 6, 7$.

The main effect leading to appearance of the fermion condensate is the presence of instantons. They are specific for the theory with adjoint matter and were absent in $QCD_2^{fund}$. The topological reason for their existence is the nontrivial $\pi_1(G)$.

$^2$The notion of trajectories makes sense only for few first states with small enough mass. At larger masses, the trajectories begin to overlap, and the spectrum becomes stochastic $^3$.
where the gauge group $G$ is $SU(N)/Z_N$ rather than just $SU(N)$ (adjoint fields are not transformed under the action of the elements of the center), so that there are $N$ topologically nonequivalent sectors.

Instantons appear by the same token as in the Schwinger model $[8, 6, 9]$. In the latter, the topological reason for existence of instantons is the nontrivial $\pi_1[U(1)] = \mathbb{Z}$. The difference with nonabelian case is that, in Schwinger model, topological charge can be written as an integral invariant

$$\nu = \frac{g}{4\pi} \int d^2 x \, F_{\mu\nu} \varepsilon_{\mu\nu}$$

(1.2)

(it is the two-dimensional analog of the 4-dim Pontryagin class $\propto \int d^4 x \text{Tr} \{ F_{\mu\nu} \tilde{F}_{\mu\nu} \}$).

$\nu$ is an arbitrary integer which labels different topological sectors. In nonabelian theory, no such integral invariant can be written ($\text{Tr} \{ F_{\mu\nu}^a t^a \} = 0$). That is understandable, of course. If such an integral invariant would exist, the number of topologically nontrivial sectors would be infinite, but it is finite in nonabelian case.

These new instantons which are specific for theories involving only adjoint fields occur also in 4 dimensions. Actually, they have been known for a long time by the nickname of ’t Hooft fluxes $[10]$. The difference with two dimensions is that, for $d = 4$, the corresponding configurations are not localized (they do not depend on two transverse directions), and their action is infinite. For high $T$, these ”planar instantons” have been studied in $[11]$ and also earlier in $[12]$ (where they were, however, misinterpreted as real walls in Minkowsky space separating different thermal vacua — we refer an interested reader to $[11]$ for detailed discussion of this question).

Topologically nontrivial fields appear in $QCD_{2}^{adj.}$ both at low and at high temperatures. However, high-$T$ case is more ”clean” because quantum fluctuations are small here, quasiclassical approximation works, and a quantitative calculation for the instanton contribution in the partition function is possible.
One immediate effect related to instantons is the generation of the fermion condensate due to the presence of fermion zero modes in the instanton background. Recall the situation in $QCD_4$. Instantons involve there one complex zero mode for each light fermion flavor (one for $\psi$ and one for $\bar{\psi}$). If $N_f = 1$, these zero modes are "absorbed" by external $\psi$-operators in the Euclidean functional integral

$$<\bar{\psi}\psi> \sim \int \prod dA d\bar{\psi} d\psi \bar{\psi} \psi \exp \left\{ \int d^4x \left[ -\frac{1}{2} Tr(F_{\mu\nu}^2) + i\bar{\psi} D_{\mu} \gamma_\mu \psi \right] \right\}$$ (1.3)

and we get a finite result even for very large $T$. If $N_f \geq 2$, there are extra zero modes for extra flavors, and $<\bar{\psi}\psi>_{T \gg \Lambda_{QCD}}$ is zero for massless quarks. For small $T$, $<\bar{\psi}\psi>$ is nonzero (It is the experimental fact. Theoretically, its appearance can also be related to instanton zero modes but not in a direct way [13]) which means that the phase transition occurs.

The main observation of this paper is that the physics of $QCD_2^{adj}$ with $N = 2$ and one Majorana adjoint fermion flavor, is essentially the same as that of $QCD_4$ with $N_f = 1$. High-$T$ instanton (the topologically nontrivial configuration which minimizes the effective action) involves exactly two zero modes which are absorbed by external fermion operators in the functional integral for $<\bar{\lambda}^a \lambda^a>$ and leads to exponentially suppressed $\propto \exp\{-\pi^{3/2} T/g\}$ but nonzero result.

What happens at low temperatures? Quantum fluctuations are large there and only dimensional estimates can be done. Still, these estimates display the presence of the condensate. Its value is of order $g$. The appearance of the condensate is also very clearly seen in the framework of bosonization approach. It is very essential that, in contrast to $QCD_2^{fund}$, the bosonized version of $QCD_2^{adj}$ does not involve a massless field which smears away the condensate $<\bar{\psi}\psi>$ in the former for any finite $N$.

Whereas, for $N = 2$, the picture is rather clear and self-consistent, it is not so for $N \geq 3$. High-$T$ instantons involve here $2(N - 1)$ fermion zero modes which
is "larger than necessary". Similar to what happens in $QCD_4$ with $N_f \geq 2$, the extra zero modes lead to the suppression of the condensate in the massless limit. In $QCD_4$, the statement of the absence of the condensate at high $T$ could not be extrapolated to low temperature region due to the presence of Goldstone bosons which display themselves in the low temperature partition function [14]. But in $QCD_{2_{adj}}$, golstones are absent. They cannot appear in two dimensions due to the Coleman theorem [15] and they do not as the generation of the fermion condensate is not associated with spontaneous breaking of a continuous symmetry.

Assuming that any topologically nontrivial background involves exactly $2(N-1)$ fermion zero modes and the absence of massless modes in the spectrum, we have to conclude that the condensate is absent also at low $T$. From the other hand, bosonization arguments display the presence of the condensate universally for any $N$. This is a clear paradox. A possible way to resolve it which we suggest will be discussed later in this paper.

The structure of the paper is the following. In the next section, we fix notations and discuss the symmetries of the theory considered. In Sect. 3, the explicit form of the high-$T$ instanton for $N = 2$ is obtained and the estimate for the fermion condensate is done. In Sect.4, we discuss the low temperature region and show that both the instanton arguments and the bosonization arguments imply the appearance of fermion condensate. In Sect.5, we discuss characteristic field configurations contributing to the partition function of the theory and show that the instantons are in some sense "confined" for strictly massless case and are "liberated" for any small but nonzero fermion mass. In Sect.6, we analyze the case $N \geq 3$ and display the paradox. The paradox and possible ways for its resolution are discussed further in Sect.7. Conclusive remarks are given in the last section.
2 QCD$_2$ with real adjoint fermions

The lagrangian of the model reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{i}{2} \bar{\lambda}^a \left[ \delta^{ab} \partial_{\mu} - g \epsilon^{abc} A_{\mu}^c \right] \gamma_{\mu} \lambda^b$$  \hspace{1cm} (2.1)

where $\lambda^a_{\alpha}$ is the 2-dimensional Majorana (real) spinor, $\alpha = 1, 2$; $\bar{\lambda}^a \equiv \lambda^a \gamma^0$.

It is convenient to choose the representation $\gamma_0 = \sigma^2$, $\gamma_1 = i\sigma^1$. In that case, $\gamma_5 = \gamma_0 \gamma_1 = \sigma^3$ and the left $\lambda_L = \frac{1}{2}(1 + \gamma_5)\lambda$ and the right $\lambda_R = \frac{1}{2}(1 - \gamma_5)\lambda$ components of the fermion field are described by the upper and lower components of the spinor $\lambda_{\alpha}$, respectively. The fermion part of the lagrangian can be written in terms of $\lambda_L$ and $\lambda_R$ as

$$\mathcal{L}_{\text{ferm}} = \frac{i}{2} \left\{ \lambda_L^a \left[ \delta^{ab} \partial_{-} - g \epsilon^{abc} A_{\mu}^c \right] \lambda_L^b + \lambda_R^a \left[ \delta^{ab} \partial_{+} - g \epsilon^{abc} A_{\mu}^c \right] \lambda_R^b \right\}$$  \hspace{1cm} (2.2)

with $\partial_{\pm} = \partial_0 \pm \partial_1$, $A_{\mu}^c = A_{\mu}^0 \pm A_{\mu}^1$ (Left fermions are the left movers and right fermions are the right movers).

Note (and this is very important) that, in contrast to the theory with fundamental Dirac fermion, the lagrangian \textit{(2.1)} does not enjoy any continuous global symmetry. The phase transformations $\lambda \rightarrow \exp\{i\alpha\}\lambda$ or $\lambda \rightarrow \exp\{i\beta\gamma_5\}\lambda$ are not allowed as they destroy the reality condition. The would-be currents corresponding to these transformations $\bar{\lambda} \gamma_{\mu} \lambda$ and $\bar{\lambda} \gamma_{\mu} \gamma_5 \lambda$ are just zero for Majorana fermions. One cannot also mix left and right components $\lambda_L \equiv \lambda_1$ and $\lambda_R \equiv \lambda_2$ — the lagrangian \textit{(2.2)} is not invariant under such a transformation.

In this respect, the situation in two dimensions differs essentially from the 4-dim case. The 4-dimensional Majorana spinor can be expressed in terms of a complex 2-component Weyl spinor $w_\alpha$, and the chiral phase transformation $w_\alpha \rightarrow e^{i\phi} w_\alpha$ is the symmetry of tree lagrangian.

There is, however, a discrete two-dimensional remnant of this 4-dim chiral sym-
Either of the transformations

\[ \lambda_L \rightarrow -\lambda_L \]  \hspace{1cm} (2.3)

\[ \lambda_R \rightarrow -\lambda_R \]

leaves the lagrangian \((2.2)\) invariant. The mass term

\[ m\bar{\lambda}\lambda = -2im\lambda_L\lambda_R \]  \hspace{1cm} (2.4)

would break this \(Z_2 \otimes Z_2\) symmetry down to \(Z_2\) (only \textit{simultaneous} change of sign of \(\lambda_L\) and \(\lambda_R\) is now allowed). We shall see later that, even in the massless case, the symmetry \((\lambda_L \rightarrow -\lambda_L, \lambda_R \rightarrow \lambda_R)\) is actually not there in the full quantum theory due to \textit{anomaly} (this is true, at least, for \(N = 2\) theory which we understand well).

3 \quad \textit{N = 2: Instantons and condensate at high T.}

A. Preliminaries

Let us consider the theory \((2.1)\) with two colors. As was already mentioned, the gauge-symmetry group of this theory \(G\) is \(SU(2)/Z_2 = SO(3)\) with nontrivial \(\pi_1[G] = Z_2\). It admits therefore noncontractible topologically nontrivial field configurations \(\equiv\) instantons. All nontrivial configurations belong to one and the same topological class. In this section, we are interested only with high temperature case where quasiclassical description works and quantitative estimates are possible. Euclidean path integrals are defined on the asymmetrical 2-dim torus which is very long in spatial direction, \(L \gg g^{-1}\), and narrow in Euclidean time direction, \(\beta = 1/T \ll g^{-1}\).

To understand better how instantons appear, let us write down the high-\(T\) effective potential on the constant \(A_0\) - background in this theory. The evaluation of the one-loop fermion determinant \(^3\)

\(^3\)I am indebted to I.Klebanov who pointed my attention to this point.
freedom associated with gauge fields, and the latter do not contribute. Technically,
the contribution of longitudinal degrees of freedom $A_0^a$ cancels out the contribution
of the ghosts) gives the answer [16, 17].

$$V_{eff}^T (A_0^3) = \frac{g^2}{2\pi} \left[ (A_0^3 + \frac{\pi T}{g}) \mod \frac{2\pi T}{g} - \frac{\pi T}{g} \right]^2 \tag{3.1}$$

where we directed $A_0^a$ along the third isotopic axis for definiteness. The potential
(3.1) is plotted in Fig.1. It has exactly the same form as in Schwinger model [9] and
is much analogous to the similar potential $V_{eff}^T (A_0^3)$ for pure Yang-Mills theory in
four dimensions [18].

The potential (3.1) is periodic with the period $2\pi T/g$. The periodicity is not
causal. Really, the variable $A_0^a$ is canonically conjugate to the Gauss law constraint,
and the matrix

$$O^{ab} = \exp\{\beta g f^{abc} A_0^c\} \tag{3.2}$$

($f^{abc} \equiv \epsilon^{abc}$ for $N = 2$) may and should be thought of as the gauge transformation
matrix acting on the dynamic variables $A_i^a, \lambda_0^a$. Now, the points $A_0^a = 0$ and
$A_0^a = \delta^{a3} \frac{2\pi T}{g}$ correspond to one and the same matrix $O^{ab} = \delta^{ab}$ and are
therefore physically equivalent (see Ref. [11] for more detailed discussion).

One can consider, however, a field configuration which is $x$-dependent and interpolates
between the values $A_0^a = 0$ at $x = -\infty$ and $A_0^a = \delta^{a3} \frac{2\pi T}{g}$ at $x = \infty$
. It presents a noncontractible loop in $SO(3)$ and cannot be trivialized. Instanton
is the configuration belonging to this class with the minimal action. Usually, e.g.
in 4-dim Yang-Mills theory, the term ”instanton” applies to solution of classical
equations of motion, i.e. to the configuration which minimizes the tree action. In
two dimensions, this definition is not convenient by two reasons. First, such a classical solution does not have nice properties — it is just a constant field strength
configuration which is smeared out over the whole volume $V = \beta L$ with the very
small field strength $F_{01}^3 = 2\pi T/g L$ ($A_0^3$ interpolates between 0 at $x = -L/2$ and $2\pi T/g$ at $x = L/2$ with constant slope). Second, quantum corrections can be taken into account explicitly here — at high $T$, higher loop corrections to the potential \[ (3.1) \] are small (In the exactly soluble Schwinger model, they are just absent at any temperature). And if so, why not doing it? Thus, our definition of instanton is the configuration which minimizes the effective action, quantum corrections being taken into account.

How to do that? One may be tempted to allow the argument $A_0^3$ in Eq.\[ (3.1) \] to be $x$-dependent, add the tree-level kinetic term $\frac{1}{2}(\partial_x A_0^a)^2$ and solve the equations of motion for the effective lagrangian thus obtained. Though this naive procedure gives even the correct answer for the profile of the instanton, it cannot be justified — the expansion over derivatives of $A_0(x)$ breaks down at the point $A_0^3 = \pi T/g$ due to severe infrared singularities \[ [11] \], and the true effective lagrangian is highly nonlocal. One should proceed more accurately.

B. Fermion determinant and zero modes.

As we have seen, instanton presents a noncontractable loop $O^{ab}(x)$ in $SO(3)$ group. In the covering $SU(2) \equiv S^3$, it corresponds to a path which goes from the north pole $U \in SU(2) = 1$ at $x = -\infty$ to the south pole $U \in SU(2) = -1$ at $x = \infty$. By symmetry considerations, the path which minimizes the action should go along one of the meridians. Each such meridian corresponds to the Ansatz

\[
A_0^a(x) = n^a a(x),
\]

\[
a(-\infty) = 0, \quad a(\infty) = 2\pi T/g, \quad (3.3)
\]

where $n^a$ is the unit color vector. Let us choose for definiteness $n^a = \delta^a_3$ and calculate the fermion determinant on this background. Minimizing the effective action thus obtained, we will find the profile of the instanton $a(x)$ and evaluate its
contribution to the partition function.

Right from the beginning, however, we run into the problem. The matter is, the lagrangian (2.1) is well defined in Minkowski space but not in the Euclidean space. In Euclidean space, we cannot keep the fermion fields real — if we try to do so, the Euclidean counterpart of (2.2) with $\partial_0 \rightarrow i\partial_0$ becomes complex. This problem is well know in 4-dimensions [19] and its resolution is also known [20, 14]. One should define the integral over Majorana fields as the square root of the determinant of the full Dirac operator. The latter is well defined also in Euclidean space. The extraction of square root also does not present problems here. The matter is, the spectrum of the eigenvalue equation for the Euclidean Dirac operator for complex adjoint fields

$$
\gamma^E_\mu (\partial_\mu \delta^{ab} - g e^{abc} A^c_\mu ) \psi_n^b (x, \tau) = \mu_n \psi_n^b (x, \tau)
$$

has a double degenerate spectrum ($\gamma^E_\mu$ are antihermitian Euclidean $\gamma$-matrices). If $\psi_n(x, \tau)$ is a complex solution to (3.4), the function

$$
\tilde{\psi}_n(x, \tau) = C \psi_n^*(x, \tau)
$$

is also a solution with the same eigenvalue $\mu_n$. (C is the charge conjugation matrix defined by $(\gamma^E_\mu)^* = - (\gamma^E_\mu)^T = C \gamma^E_\mu C^{-1}$. In 2-dimensions, $C = \sigma^2$, under the particular choice $\gamma^E_0 = i\sigma^2, \gamma^E_1 = i\sigma^1$). In view of $C^*C = -1$, the two solutions are linearly independent. Hence, the square root is taken without pain:

$$
[Det \|\hat{D}\|]^{1/2} = \left[ \prod_n \mu_n^2 \right]^{1/2} = \prod_n \mu_n
$$

where only one of the double degenerate eigenvalues $\mu_n$ is accounted for in the product. Let us write the equation (3.4) on the abelian background (3.3). It splits apart in two:

$$
\gamma^E_\mu (\partial_\mu - ig \delta_{\mu 0} a ) \psi_n^- = \mu_n \psi_n^-
$$

$$
\gamma^E_\mu (\partial_\mu + ig \delta_{\mu 0} a ) \psi_n^+ = \mu_n \psi_n^+
$$

10
where $\psi^\pm = \psi^1 \pm i\psi^2$. [there is also the third equation for $\psi^3$, but it is just a free one — the component $\psi^3$ decouples from the background (3.3)]. It is explicitly seen that the solutions to these two equations are related by the transformation (3.5). The equations are exactly the same as for 2-dim QED on the instanton background $A_\mu(x) = \delta_{\mu0}a(x)$ for the fermions with the charges $g$ and $-g$, respectively. Thus, we need not calculate the determinant anew, but rather use the results of [6, 9] where the instanton Dirac determinant has been calculated for the abelian theory:

$$\left[\text{Det}_{Ab.Ans.}\|i\hat{D}\right]\right]^{1/2} = \left\{\left[\text{Det}_{QED}\|i\hat{D}\right]\right\}^{1/2} = \text{Det}_{QED}\|i\hat{D}\|$$

(3.8)

Now, it is a proper time to note that all these determinants calculated on the instanton background just turn to zero for strictly massless fermions due to the presence of fermion zero modes in the spectrum! Each of the equations in (3.7) has exactly one normalizable solution with $\mu = 0$, the left one for $\psi^-$ and the right one for $\psi^+$:

$$\psi^-(0)_\alpha(x, \tau) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-g\phi(x)} e^{i\pi T \tau}$$

$$\psi^+(0)_\alpha(x, \tau) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-g\phi(x)} e^{-i\pi T \tau}$$

(3.9)

where

$$\partial \phi / \partial x = a(x) - \frac{\pi T}{g}$$

(3.10)

(the $\tau$-dependence provides the correct antiperiodic boundary conditions $\psi(\beta) = -\psi(0)$ for the fermion fields in Euclidean time direction). We show in the Appendix that zero mode solutions are still there also for configurations involving small fluctuations around the abelian Ansatz (3.3).
The presence of fermion zero modes suppresses the contribution of topologically nontrivial sectors to the partition function exactly in the same way as it does in QCD. To get a nontrivial result, one should introduce a small but finite fermion mass $m \ll g$. In that case, the partition function involves $\text{Det} \| \dot{\mathcal{D}} - m \|$ rather than just $\text{Det} \| \dot{\mathcal{D}} \|$, and the whole result (3.8) will be proportional to $m$.

The accurate calculation of the determinant gives the result [6]

$$
\left[ \text{Det}_{\text{Ab.Ans.}} \| \dot{i} \mathcal{D} - m \| \right]^{1/2} = \text{Det}_{\text{QED}} \| \dot{i} \mathcal{D} - m \| \\
\propto m \int dx \, e^{-2g\phi(x)} \exp \left\{ -\frac{\beta g^2}{2\pi} \int dy \, a^2(y) \right\} 
$$

(3.11)

The second factor in the determinant comes from nonzero modes. In Schwinger model, it was responsible for generating the photon mass. The first factor is due to the zero modes. The proportionality coefficient in (3.11) can be explicitly determined (in finite box which provides infrared regularization) if choosing a particular convention for $\phi(x)$ (the equation (3.10) defines $\phi(x)$ only up to an arbitrary constant). We refer the reader to Ref.[6] for the detailed and accurate analysis.

If we substitute now the result (3.11) in the bosonic functional integral, calculate it, differentiate over mass and divide over the similar functional integral for the partition function $Z_0$ in the topologically trivial sector, we obtain the expectation value for the operator $\bar{\lambda}\lambda$, i.e. the fermion condensate.

Let us recall how it has been done in the Schwinger model. The functional integral in the one-instanton topological sector had the form

$$
Z_1 \propto Z_0 m \prod_y d\phi(y) \int dx \, e^{-2g\phi(x)} \\
\exp \left\{ -\frac{\beta}{2} \int dy \, \phi(y) \left[ \frac{\partial^4}{\partial y^4} - \frac{g^2}{\pi} \frac{\partial^2}{\partial y^2} \right] \phi(y) \right\} 
$$

(3.12)

The saddle points of this integral were determined from the equation

$$
\left[ \frac{\partial^4}{\partial y^4} - \frac{g^2}{\pi} \frac{\partial^2}{\partial y^2} \right] \phi(y) = -2gT\delta(y - x) 
$$

(3.13)
The parameter $x$ has the meaning of the collective coordinate describing the position of the instanton. Substituting the solution of this equation in Eq. (3.10), we got the result

$$a(y) = \begin{cases} \frac{\pi T}{g} \exp\{\mu(y - x)\}, & y \leq x \\ \frac{\pi T}{g} \left[2 - \exp\{\mu(x - y)\}\right], & y \geq x \end{cases} \quad (3.14)$$

where $\mu = g/\sqrt{\pi}$. The function $a(y)$ is plotted in Fig.2. The field density $E(y) = -\partial a(y)/\partial y$ is localized at $|y - x| \sim \mu^{-1}$ so that the topological charge (1.2) is equal to -1 as expected. The characteristic quantum fluctuations determined by the integral (3.12) are $a^{q\alpha} \sim \sqrt{T/g}$ which is much less than the characteristic amplitude of the solution (3.14) $a^{cl} \sim T/g$ so that the quasiclassical picture works.

Calculating the whole integral (3.12) and adding the equal contribution from the one-antiinstanton sector (in the abelian case, the relevant topology is $\pi_1[U(1)] = Z$ and instanton and antiinstanton configurations are topologically nonequivalent), one obtains the following result for the fermion condensate

$$<\bar{\psi}\psi>_{T \gg g} = -\frac{1}{\beta L Z_0} \frac{\partial}{\partial m}(Z_1 + Z_{-1}) = -2T \exp\left\{-\frac{\pi^{3/2}T}{g}\right\} \quad (3.15)$$

[the large factor $L$ in the denominator cancels out the large factor $L$ coming from the integration over translational zero mode of the instanton solution (3.14)]

Let us turn now to the nonabelian case. In the framework of the Ansatz (3.3), the functional integral for $Z_1$ is basically the same as in the Schwinger model, and its saddle point is given by the same expression (3.14). However, two novel features appear. First of all, besides integrating over $\prod da(y)$ and $dx$, we should integrate also over $dn^a$ in the Ansatz (3.3). $n^a$ is the new collective coordinate describing the orientation of the instanton in color space. Naturally, rotation of $n^a$ does not change action and corresponds to zero modes in bosonic determinant. Let us make an estimate for the contribution of these zero modes. The general method for such calculation is presenting the integral over quantum fluctuations over the classical
solution (3.3), (3.14) which do not change the action as the integral over collective coordinates \( n^a \): \([21, 22, 23]\). To this end, one should express \( A_0^{qu(0)}(y) \) as a sum of two independent normalized zero modes

\[
A_0^{qu(0)}(y) = c_a^{(0)} \frac{\partial A_0^{cl}(y)}{\partial n^b} \left[ \int dy \left( \frac{\partial A_0^{cl}(y)}{\partial n^c} \right)^2 / 2 \right]^{-1/2} (1 - n^a n^b) \quad (3.16)
\]

and then write

\[
d^{(0)}A_0^{qu}(y) \sim d^a d^b (1 - n^a n^b) \sim d\vec{n} \int dy \left( \frac{\partial A_0^{cl}(y)}{\partial n^a} \right)^2 \quad (3.17)
\]

The representation (3.3) is, however, not convenient for this purpose because the zero modes \( \partial A_0^{cl}/\partial n^a \) appear to be not normalizable (this difficulty is also well known in 4-dim theories \([21]\)). The paradox can be resolved by noting that the proper measure in the functional integral is \( \prod_y d^{inv.} O^{ab}(y) \), \( O^{ab}(y) \) being given by Eq. (3.2), rather than just \( \prod_y dA^{a}_0(y) \). Thus, we have

\[
\int d^{(0)}O^{qu}_{ab}(y) \sim \int d\vec{n} \int dy \left( \frac{\partial O^{cl}_{ab}(y)}{\partial n^a} \right)^2 \sim \int dy \sin^2 \left[ g a^{cl}(y) / 2T \right] \sim \frac{1}{g} \quad (3.18)
\]

To find the condensate, we should divide \( Z_1 \) by \( Z_0 \). The latter may be estimated in the one loop approximation (see, however, the discussion of the validity of this approximation later in the paper) in which case the fermion condensate depends on the ratio of two bosonic determinants — one calculated on the background (3.3), (3.14), and the other — on the trivial background \( O^{ab}(y) = \delta^{ab} \). Thus, one should divide the result (3.18) by the corresponding integral in the topologically trivial sector where only the constant harmonics of \( A_0^{1}(y) \) and \( A_0^{2}(y) \) should be taken into account (the integrals over \( y \)-dependent parts of \( A_0^{1}(y) \) and \( A_0^{2}(y) \) cancel out the contribution of \( \text{nonzero} \) modes in the bosonic determinant in the topologically trivial sector \([18, 16, 17, 9]\)). The range of \( y \) where this constant harmonic mode should be normalized is the characteristic size of the instanton \( \propto g^{-1} (O^{ab} \sim \delta^{ab}) \).
far away from the instanton center and the contributions of these distances in $Z_1$ and $Z_0$ cancel out). Hence, the denominator over which the integral (3.18) should be divided is

$$\sim \int_{|y-x| \sim g^{-1}} dy \left( \frac{\partial O^{ab}}{\partial A_0^b} \right)^2 \int dA_0^1 dA_0^2$$

(3.19)

$$\exp \left\{ -\frac{\beta g^2}{2\pi} \int_{|y-x| \sim g^{-1}} dy \left[ (A_0^1)^2 + (A_0^2)^2 \right] \right\} \sim \frac{1}{g} \left( \frac{\beta g}{2} \right)^2 \frac{1}{\beta g} \sim \frac{1}{T}$$

(3.20)

Thus, rotational zero modes provide the factor $\propto T/g$ in the condensate. In fact, this estimate could be obtained immediately using the rule of thumb coined in [22] (see also [23]): each bosonic zero mode provides the factor $\sqrt{S_0}$ in the measure where $S_0$ is the instanton action. In our case, $S_0 = \pi^{3/2} T/g$ [see Eq.(3.15)], and there are two rotational zero modes. Our final result for the fermion condensate in $QCD_{2}^{adj.}$ with $N = 2$ at high $T$ is

$$< \bar{\lambda} \lambda >_{T \gg g} = - \frac{1}{\beta L Z_0} \frac{\partial}{\partial m} (Z_1) = C \frac{T^2}{g} \exp \left\{ -\frac{\pi^{3/2} T}{g} \right\}$$

(3.21)

Unfortunately, the numerical coefficient $C$ cannot be fixed here and a more accurate calculation for the ratio of determinants which could, in principle, be done would not help. The matter is (and this is the second and more serious nuisance which distinguishes the nonabelian case compared to the exactly soluble Schwinger model) that the partition function $Z_0$ in the topologically trivial sector by which the integral for $\bar{\lambda} \lambda$ should be divided cannot be determined analytically here — one-loop approximation is not justified and higher-loop effect provide a comparable contribution in the free energy. We return to the discussion of this point in Sect.5.

But it convenient for us to adjourn now for a while the discussion of high-$T$ instanton physics and look first what happens in the low temperature region.
4 Low temperatures.

Consider now $QCD_2^{adj.}$ with $N = 2$ at $T = 0$. Let us assume that the fields contributing to the Euclidean path integral tend to pure gauge at spatial infinity:

$$i\epsilon^{abc} A^a_\mu(x) \xrightarrow{r \to \infty} i\Omega^{-1}(x)\partial_\mu \Omega(x)$$

(4.1)

with $\Omega(x) \in SO(3)$. All fields belong to one of two topological classes: the trivial class consisting of the fields which can be continuously deformed to zero and the instanton class for which $\Omega(x)$ presents a noncontractible loop in $SO(3)$ group when $x$ goes around the large spatial circle. Another way to look at the problem is to define the theory on large 2-dimensional sphere. A topologically nontrivial field cannot be written as a uniform regular expression on the whole sphere. Such a field can be described by use of two different regular expressions defined on two patches, the northern and the southern hemispheres, which are glued together (related by a gauge transformation) on the equator. The transition matrix $\Omega(\phi)$ presents then a nontrivial loop in the $SO(3)$ group (cf. the analogous description of the Schwinger model in Ref.[8]).

High-$T$ analysis has taught us that the fields belonging to the instanton class involve two fermion zero modes related to each other by the transformation (3.5)\footnote{We return to the discussion of this point in sect.7.}. That means that the partition function in the topologically nontrivial sector $Z_1$ involves a factor $m$ and the fermion condensate is generated

$$<\bar{\lambda}\lambda>_{T=0} = \pm \lim_{m \to 0} \frac{1}{VZ_0} \frac{\partial}{\partial m} (Z_1) \sim g$$

(4.2)

where $V$ is the volume of our 2-dim sphere. In contrast to the high-$T$ case, one-loop calculation for $Z_1$ has no sense here as quantum fluctuations are out of control. The estimate (4.2) has been done purely on dimensional grounds. Two signs in Eq.(4.2)
correspond to two possible choices for the partition function:

\[ Z_\pm = Z_0 \pm Z_1 \]  

(4.3)

The freedom in choosing the sign is exactly analogous to the freedom of choice of the vacuum angle \( \theta \) in \( QCD_4 \) or in the Schwinger model. The difference is that here we have only two topologically distinct sectors and ”vacuum angle” can acquire only two values: 0 or \( \pi \). On the hamiltonian language, the choices \[[4.3]\] correspond to two possible superselection rules imposed on the wave functionals. The plus and minus sectors of the theory do not talk to each other: the matrix elements of all physical operators between the states from different sectors are zero.

The spectrum of the theory does not include massless states, the lowest excited state having the mass \( M_{\text{gap}} \) of order \( g \) \[\text{[1]}\]. That means that, for large volumes \( V g^2 \gg 1 \), the partition functions \( Z_\pm \) enjoy the extensive property \[[24]\]:

\[ Z_\pm \propto \exp\{ -\epsilon_{\pm}^{\text{vac}}(m, g) V \} \]  

(4.4)

and the finite size corrections to the vacuum energy are exponentially small \( \propto \exp\{ -M_{\text{gap}} R \} \). The presence of the condensate \[[4.2]\] implies that the function \( \epsilon^{\text{vac}}(m, g) \) involves a nonzero first order term of the Taylor expansion in \( m \), and we can write for \( m \ll g \):

\[ Z_\pm \propto \exp\{ -m < \bar{\lambda} \lambda >_\pm V \} \]  

(4.5)

with \( < \bar{\lambda} \lambda >_- = - < \bar{\lambda} \lambda >_+ \), and hence

\[ Z_0 \propto \cosh\{ m < \bar{\lambda} \lambda >_+ V \} \]

\[ Z_1 \propto - \sinh\{ m < \bar{\lambda} \lambda >_+ V \} \]  

(4.6)

The result \( [4.6] \) is the analog of the result \( Z_\nu \propto I_\nu(m | \bar{\psi} \psi | V) \) for the partition function in the sector with a given topological charge \( \nu \) for \( QCD_4 \) with one light fermion flavor derived in \[\text{[14]}\].
Note that the representations (4.5) and (4.6) are valid as long as \( m \ll g, V g^2 \gg 1 \); the dimensionless combination \( x = m|\bar{\lambda}\lambda|_\pm V \) may be either large or small. The instanton zero modes are responsible for the formation of the condensate only in the limit when \( x \) is small and \( Z_1 \propto x \). In the physical large volume limit (large \( x \)), the value of the condensate is the same but the mechanism for its formation is quite different being related to small \( \sim 1/|\bar{\lambda}\lambda|_\pm V \) but nonzero modes of the Dirac operator (see [14] for detailed explanations and discussions).

The presence of 2 fermion zero modes in the instanton background gives rise to the 't Hooft term \( \sim \bar{\lambda}^0\lambda^0 \sim \bar{\lambda}_L^0\lambda_R^0 \) in the effective lagrangian. That means that the \( Z_2 \otimes Z_2 \) symmetry (2.3) is in fact anomalous - quantum corrections break it down to \( Z_2 \) explicitly. And that means that the condensate \( \langle \bar{\lambda}^a\lambda^a \rangle \) does not break spontaneously any symmetry of the full quantum theory. The appearance of two sectors of the theory (4.3) with opposite signs of the condensate should not be interpreted as a spontaneous breaking because, as we have already mentioned, these two sectors correspond to different superselection rules which should be imposed uniformly in the whole physical space and the formation of the "domains" separated by the "walls" so that \( \langle \bar{\lambda}\lambda \rangle \) is negative to the left and positive to the right is not possible.

Again, the situation is exactly the same as in QCD with \( N_f = 1 \) — the presence of the sectors with different \( \theta \) in the theory should not be interpreted as a spontaneous breaking of \( U(1) \) -symmetry. \( \theta \) is one and the same in the whole physical space and the spatial fluctuations of \( \theta \) (which would give rise to Goldstone bosons) are not possible.

The existence of the condensate is also clearly seen in the framework of bosoniza-
tion approach. Since [23], it is known that a theory involving Majorana fermion fields $\lambda^a$ is dual to some other theory involving the bosonic field presenting an orthogonal matrix $\Phi^{ab}$. The correlators of all fermion bilinears in the original theory coincide identically with the correlators of their bosonized counterparts in the bosonized theory. For the scalar bilinear $\bar{\lambda}^a \lambda^b$, the correspondence rule is just

$$\bar{\lambda}^a \lambda^b \equiv \mu \Phi^{ab}$$  \hspace{1cm} (4.7)

where $\mu$ depends on the normalization procedure for the operator $\Phi^{ab}$. $\mu$ is of order $g$ if the normalization convention $\langle \Phi^{ab} \rangle_{\text{vac}} = \delta^{ab}$ is chosen. It is obvious then that

$$\langle \bar{\lambda}^a \lambda^a \rangle_{\text{vac}} \sim \mu \sim g$$  \hspace{1cm} (4.8)

Note the difference with the theory involving fundamental Dirac fermions. For $\text{QCD}^2_{\text{fund}}$, the bosonization rule is not (4.7) but rather

$$\bar{\psi}^i \psi^j \equiv \mu U^{ij} \exp \left( i \sqrt{\frac{4\pi}{N}} \phi \right)$$  \hspace{1cm} (4.9)

where $U$ is the unitary $SU(N)$ matrix, and $\phi$ is a light color singlet. In that case, the normalization mass $\mu$ is not $g$ but depends on the mass of the scalar singlet which in turn depends on the fermion mass $m$. Both $\mu$ and the light singlet mass tend to zero in the limit $m \rightarrow 0$ for any finite $N$ (and the singlet becomes sterile) [26]. The condensate $\langle \bar{\psi}^i \psi^i \rangle_{\text{vac}}$ also tends to zero in the massless limit. One can say that the light singlet $\phi$ smears the condensate away \footnote{The absence of the condensate in $\text{QCD}^2_{\text{fund}}$ is, of course, natural. The condensate would break spontaneously the global chiral symmetry, and such a breaking is not allowed in two dimensions [15].}

But in the adjoint theory, all fields in the spectrum are massive and the condensate (4.8) survives.

The rapid fall-off of the condensate at high temperature as given by Eq. (3.21) is also naturally explained in the bosonization language. Taking into account finite
$T$ effects — namely, the presence of excited states in the heat bath, the thermal average $\langle \Phi^{ab} \rangle_T >$ is no longer $\delta^{ab}$, but can acquire any value on the $SO(3)$ group with almost equal (at high $T \gg g$) probability, and

$$\langle \Phi^{aa} \rangle_T \to \infty \to \int d^{inv} \Phi \chi^{adj}(\Phi) = 0 \quad (4.10)$$

As follows from Eq.(3.21), for high but finite $T$ the direction $\delta^{ab}$ in the group is still a little bit preferred, and the condensate is still nonzero though exponentially small. The physical picture is exactly the same as in the Schwinger model where the quantitative calculation is possible at any temperature.[7]

5 High-$T$ partition function.

All the arguments of the previous section which have led to the results (4.5) and (4.6) can be repeated without change also for high temperatures. We only have to substitute $\beta L$ for $V$ and $\langle \bar{\lambda} \lambda \rangle_T$ for $\langle \bar{\lambda} \lambda \rangle_{vac}$. Let us look how the partition functions (4.6) behave when the spatial volume $L$ is very large,

$$x_T = m\beta L |\langle \bar{\lambda} \lambda \rangle_T| \gg 1 \quad (5.1)$$

The $cosh$ and $sinh$ functions in Eq.(4.6) can be expanded in the series and, if $x_T$ is large, the number of the terms in the series to be taken into account is also large. Each such term is

$$Z^{(k)} = \frac{(-m\beta L |\langle \bar{\lambda} \lambda \rangle_T|)^k}{k!} \quad (5.2)$$

where $k$ is even for $Z_0$ and odd for $Z_1$. The series converge at $k \sim x_T$. The contribution (5.2) in the partition function can be interpreted as being due to $k$ instantons (3.3). Each instanton brings about the factor $m$ from the fermion zero mode and the factor $L$ from the translational bosonic zero mode in the partition function. The instantons are very well spatially separated, the characteristic inter-instanton distance being of order $L/k_{\text{char}} \sim 1/(m\beta |\langle \bar{\lambda} \lambda \rangle_T|) \gg g^{-1}$. Thus,
we see that the characteristic field configurations in the high-\(T\) partition function present a rarefied noninteracting instanton gas. Naturally, the total number of instantons is even for \(Z_0\) (the configuration is topologically trivial) and odd for \(Z_1\).

The same picture is valid in high-\(T\) Schwinger model \([1]\) and in high-\(T\) \(QCD_4\) \([13]\). Note that we cannot extrapolate it to low temperatures. When \(T < g\), the characteristic separation between instantons is of the same order as their size \(\sim g^{-1}\), and their interaction (as well as distortion of their form due to quantum fluctuations) cannot be neglected. Instead of a rarefied instanton gas, we have a dense strongly interacting instanton liquid \([13, 27]\).

It is interesting to look also at the limit when \(L\) is kept large but finite and \(m\) tends to zero. In strictly massless theory, \(Z_1\) vanishes and \(Z_0\) has no trace of instantons at all. The explanation is simple. Consider the contribution of two well separated instantons to \(Z_0\). The zero modes of individual instantons are now shifted from zero, but the shift is tiny:

\[
\mu_{\text{quasizero}}(R) \sim \exp\{-\pi TR\} \tag{5.3}
\]

where \(R\) is the inter-instanton separation. Thus, the large \(R\) configurations provide exponentially small contribution to the path integral, instantons are "confined" and cannot be separated from each other. (The same phenomenon occurs in the Schwinger model \([11, 9]\). For \(QCD_2^{\text{adj}}\), it has been actually observed in Ref.\([17]\).

If \(m \neq 0\), the contribution of the two-instanton contribution to the path integral ceases to depend on \(R\) as soon as \(\mu_{\text{quasizero}}(R) \ll m\). If \(m\) is large enough (the condition \((5.1)\) is fulfilled), two-instanton contribution dominates over zero-instanton one: instantons are "liberated".

Now, the time has come to pay our old debt and to discuss nonperturbative effects in \(Z_0\) for the massless theory (one can forget about instantons till the end of this section). Let us estimate free energy density \(F = -TL^{-1}\ln Z_0\) of the theory at
high temperature. In the leading order, it is given just by the free fermion loop and is of order $T^2$. But what are preasymptotic effects? It is instructive to consider first Schwinger model. In the bosonized language, it is just a theory of free scalars with the mass $\mu = g/\sqrt{\pi}$. At finite $T$, they are excited and the exact expression for $F$ is

$$F_{SM}(T) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \ln \left[ 1 - e^{-\beta \sqrt{p^2 + \mu^2}} \right] \quad (5.4)$$

Its high-$T$ asymptotics is

$$F_{SM}(T \gg \mu) = -\frac{\pi T^2}{6} \left[ 1 - \frac{3\mu}{\pi T} + o\left(\frac{\mu}{T}\right) \right] \quad (5.5)$$

When $T \sim \mu$, subleading effects are essential.

For $QCD^2_{adj}$, the qualitative estimate is the same, but we cannot determine now the coefficient of preasymptotic term exactly: the mass of bosons in the spectrum cannot be determined analytically, and their interaction cannot be neglected. Thus, we can only write

$$\Delta F_{adj}^{nonpert}(T) \sim gT \quad (5.6)$$

That is the same uncertainty which prevented us to determine the exact coefficient in Eq.(3.21): the uncertainty in $Z_0$ is

$$\sim \exp\{-\beta \Delta F_{nonpert}(T) g^{-1}\} \sim 1 \quad (5.7)$$

($g^{-1}$ is the instanton size where the background field (3.3) differs essentially from zero and the determinants of fluctuations in $Z_1$ and $Z_0$ are different). ☞

6 $N \geq 3$.

8Note that uncertainties of essentially the same kind in the determination of the instanton measure appear also in $QCD_4$ when the size of the instanton $\rho$ becomes comparable with the characteristic scale of the theory. The corrections to the measure are of order $\rho^4 \epsilon_{vac} \sim \rho^4 \Lambda_{QCD}^4 \frac{28}{28}$. When $\rho \Lambda_{QCD} \sim 1$, the situation is out of control.
6.1 A. High T.

Let us repeat the analysis of Sect.3 for higher color groups. Consider first the case \( N = 3 \). The effective potential on the constant \( A_0 \) background has been calculated in Ref.\[16\]. For \( N = 3 \), it depends on 2 group invariants: \( A_0^a A_0^a \) and \( d^{abc} d^{ade} A_0^b A_0^c A_0^d A_0^e \). It is convenient to choose the matrix \( A_0^a t^a \) in the diagonal form

\[
A_0^a t^a = \text{diag} \left( a_1, a_2, a_3 \right), \quad \sum_i a_i = 0 \quad (6.1)
\]

and write the effective potential as a function of \( a_i \) (or, if you will, as a function of \( A_0^3 \) and \( A_0^8 \)). The result is

\[
V(a_i) = \frac{g^2}{2\pi} \sum_{i>j}^3 \left[ \left( a_i - a_j + \frac{\pi T}{g} \right) \mod \frac{2\pi T}{g} - \frac{\pi T}{g} \right]^2 \quad (6.2)
\]

This potential has a hexagonal symmetry. The structure of its minima is shown in Fig.3.

What is the proper range of integration over \( A_0^a \) in the functional integral? As was discussed earlier, the proper integration variable is not \( A_0^a \) but rather the adjoint gauge transformation matrix \((3.2)\) (which was the orthogonal matrix in the case \( N = 2 \)). To count each such matrix only once, we should restrict the range of integration by the "small" Weyl cell (marked out by the dashed lines inside the solid triangle in Fig.3) which is spread out over the whole \( SU(3)/Z_3 \) group by the transformations from the torus of the group with nonzero \( A_0^1, A_0^2, A_0^4, A_0^5, A_0^6, A_0^7 \).

Note that in the general case where the theory involves also fundamental matter fields, the integration goes over unitary matrices \( U = \exp \{ i\beta g A_0^a t^a \} \) which are different in the three different classes of minima:

\[
U_0 = 1, \quad U_\Box = e^{2\pi i/3}, \quad U_\triangle = e^{-2\pi i/3} \quad (6.3)
\]

In a theory with fundamental matter, these three sets of points mark out physically different gauge transformations, the counterpart of Eq.\((3.2)\) would also be
different at these points: \( V_{\text{fund.}}(U_0) \neq V_{\text{fund.}}(U_\square) \neq V_{\text{fund.}}(U_\triangle), \) and the proper integration region would be the standard Weyl cell (solid triangle in Fig.3) + transformations from the torus.

But in QCD\(^{adj.}\), the proper gauge group is \( SU(3)/Z_3 \) rather than \( SU(3) \), and all minima of the potential (6.2) [which occur at the points (6.3)] should be identified. There are, however, noncontractible Euclidean configurations which interpolate between different center elements (6.3) of the unitary group so that, say,

\[
U(x = -\infty) = 1, \quad U(x = \infty) = e^{2\pi i/3}
\]  

(6.4)

The configuration (6.4) presents a nontrivial loop in the \( SU(3)/Z_3 \) - space. For \( N = 3 \), there are two different topologically nontrivial classes: the configurations (6.4) which may be called instantons and the configurations interpolating between 1 and \( e^{-2\pi i/3} \) which may be called antiinstantons (double instanton configurations are topologically equivalent to antiinstantons).

Consider a representative of the instanton class which has the form

\[
t^a A_0^a(x) = \frac{1}{3} a(x) \text{diag}(1, 1, -2)
\]

\[
a(-\infty) = 0, \quad a(\infty) = \frac{2\pi T}{g}
\]  

(6.5)

It corresponds to going upwards along the vertical side of the solid triangle in Fig.3 with the transformation from the torus being fixed to be trivial (so that different points on the side correspond to all different elements of the group \( SU(3)/Z_3 \)).

Let us estimate the fermion determinant in this background field. The eigenvalue equation for the Euclidean Dirac operator [the analog of (3.7)] on the background (6.5) has the form

\[
\gamma_\mu^E (\partial_\mu \pm ig a(x) \delta_{\mu 0} ) \psi_n^{4 \pm i5} = \mu_n \psi_n^{4 \pm i5}
\]

\[
\gamma_\mu^E (\partial_\mu \pm ig a(x) \delta_{\mu 0} ) \psi_n^{6 \pm i7} = \mu_n \psi_n^{6 \pm i7}
\]  

(6.6)

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and the components $\psi^{1,2,3,8}$ decouple from the background.

We see that the Dirac equation admits now not one but two pairs of zero modes \((3.9)\). That means that the partition function in the instanton sector involves now the factor $m^2$ rather than just $m$. And that means that the contribution of topologically nontrivial sectors in the condensate is

$$< \bar{\lambda} \lambda >_{T \gg g}^{N=3} = - \frac{1}{\beta L Z_0} \frac{\partial}{\partial m} (Z_I + Z_A) \propto m \quad (6.7)$$

and turns to zero in the massless limit. The situation looks the same as in $QED_2$ with two Dirac charged fermions where the fermion condensate is zero by the same reason.

Similar analysis can be done also for larger $N$. The generalization of the Ansatzes \((3.3), (6.5)\) for any $N$ is

$$t^a A^a_0(x) = \frac{1}{N} a(x) \text{ diag}(1,1,\ldots,1-N),$$

$$a(-\infty) = 0, \quad a(\infty) = \frac{2\pi T}{g} \quad (6.8)$$

which supports $N-1$ pairs of fermion zero modes. The determinant has the same structure as in the Schwinger model with $N-1$ flavors, and the contribution to the condensate is

$$< \bar{\lambda} \lambda >_{T \gg g}^{N} \propto m^{N-2} \quad (6.9)$$

which vanishes in the massless limit. \footnote{For $N > 3$, the leading contribution to the condensate comes not from instantons but just from the topologically trivial sector. The latter gives $< \bar{\lambda} \lambda > \propto m$ for any $N$ (cf. Eq.(8.22) in Ref.[14])}

Thus, at high temperatures, fermion condensate seems not to be formed in $QCD_2^{adj}$ with $N \geq 3$.

### 6.2 Low T: The paradox.

The bosonization arguments of Sect.4 which have led to the conclusion of existence of the fermion condensate for $N = 2$ can be transferred without essential change to
larger $N$. The theory involves now the set of $N^2 - 1$ Majorana fermion fields. Staying in the framework of the original Witten’s paper [25] where only free fermions were discussed, we would have to put such a set of field into correspondence to the boson fields presenting orthogonal $SO(N^2 - 1)$ matrices. In the case when the fermions interact with gauge fields, it is more convenient, however, to write the bosonized theory in terms of the fields

$$\Phi^{ab} = Tr\{t^a U t^b U^+\}$$

(6.10)

where $U \in SU(N)$ and $\Phi \in SU(N)/Z_N$. This modified bosonization procedure has been worked out in [29]. $\Phi^{ab}$ is dual to the scalar bilinear $\bar{\lambda}^a \lambda^b$ as written in Eq.(4.7). As earlier, $\mu \sim g$ and the estimate (4.8) for the fermion condensate is valid.

Again, the spectrum of the theory involves a gap and, in the limit $m \ll g, V g^2 \gg 1$, the partition function can be written in the form

$$Z \propto \exp\{-m < \bar{\lambda} \lambda > V\}$$

(6.11)

both for large and for small values of $m|<\bar{\lambda}\lambda>|V$.

But that contradicts instanton arguments.

Let us consider for simplicity the case $N = 3$. There are 3 topological classes: the trivial, the instanton and the antiinstanton. In the topologically trivial sector, the partition function

$$Z_0 \propto \left< \prod_n (\lambda^2_n + m^2) \right>, \quad \lambda_n \neq 0$$

(6.12)

is expanded in the even powers of $m$. The expansion of $Z_I$ and $Z_A$ in $m$ also starts from the term $\propto m^2$ due to the presence of 2 pairs of fermion zero modes. It is absolutely not clear how the linear term in the expansion of the exponential (6.11) can appear.
Thus, bosonization arguments tell that the condensate is formed whereas the instanton arguments tell that it is not formed.

7 Confronting the controversy.

The paradox appeared when putting together the following premises:

1. Validity of topological classification.

2. The presence of $2(N-1)$ zero modes in the instanton sector.

3. Bosonization arguments displaying the presence of condensate.

4. Absence of massless states which allowed us to write the partition function in the extensive form (6.11) also for small values of exponent.

The only way to resolve the paradox is to invalidate one of these premises.

For example, in the conventional $QCD_4$ with several flavors where instantons involve $N_f$ zero modes, their contribution to the partition function is $\propto m^{N_f}$ but the condensate is still generated without any paradox because the premise 4 is false. There are Goldstone states in the spectrum which lead to finite volume effects which are essential in the region of small $m|\langle \bar{\psi}\psi \rangle |V$ and the partition function cannot be written in the extensive form (6.11) but has a more complicated structure [14].

But in our case, no continuous symmetry is broken spontaneously and there are no goldstones.

At first sight, the weakest point is the second premise. We have obtained $2(N-1)$ zero modes by solving explicitly the Dirac equation in a particular background. We have also checked that the zero modes are stable with respect to small deformations of background (see Appendix). But we cannot write down an index theorem which would enforce the presence of $2(N-1)$ zero modes for any background belonging to the instanton class. The "normal" index $n^0_L - n^0_R \propto \int Tr \{ F_{\mu\nu} t^a \} \epsilon_{\mu\nu} d^2 x$ is just
zero in $QCD_2$ [indeed, we have established the presence of $N - 1$ left-handed and $N - 1$ right-handed zero modes related to each other by the transformation (3.5)], and we do not know of any other relevant integral invariant.

Thus, we cannot rule out that, for some fields belonging to the instanton class and located at some finite distance from the abelian Ansatz in the Hilbert space, the number of zero modes is less which would allow the generation of the condensate.

We think, however, that it is not the case, and there is some index theorem prescribing the existence of exactly $2(N - 1)$ zero modes, only we are not clever enough to unravel it. The reason why we believe it is the following.

The paradox discovered is actually not specific for $QCD_2^{adj}$. The same paradox appears also in some 4-dim gauge theories where the conventional Atiah-Singer theorem works and the analog of our premise 2 is certainly valid.

As we have already mentioned, there is no paradox in the conventional $QCD$. Consider, however, supersymmetric $d = 4$, $N = 1$ nonabelian Yang-Mills theories involving a Majorana fermion in the adjoint representation of the gauge group. The paradox does not arise when the group is unitary. Let us understand why.

At first sight, it does. The fields belonging to the instanton class involve $2N_c$ fermion zero modes (the trace $Tr\{T^aT^a\}$ which enters the index theorem differs, for the generators $T^a$ in the adjoint representation, by the factor $2N_c$ from the analogous trace for the fundamental representation). That means that the instanton contribution to the partition function involves a factor $m^{N_c}$. From the other hand, exact supersymmetric Ward identities tell us that the correlator $<\bar{\lambda}\lambda(x_1)\ldots\bar{\lambda}\lambda(x_{N_c})>$ does not depend on $x_i$. The computation in the instanton background gives nonzero result which implies that the correlator does not vanish also when all $|x_i - x_j|$ tend to $\infty$ [30]. And that implies the presence of the condensate $<\bar{\lambda}\lambda>$. As it does not break spontaneously any exact symmetry of the quantum theory, no massless
states appear, the extensive representation \((6.11)\) for the partition function is valid, and we cannot reproduce the linear in mass term in the expansion of \(Z\) when taking into consideration only the fields with integer winding number.

The paradox is resolved in this case by noting that, for a theory involving only adjoint fields, the fields carrying \(\text{fractional}\) winding numbers \(\nu = \pm 1/N_c, \pm 2/N_c, \ldots\) are equally admissible \([31, 32, 33, 14]\). The reason is, again, that the gauge group here is actually \(SU(N)/Z_N\) rather than \(SU(N)\) and the gauge transformation matrices differing by an element of the center are undistinguishable. The configurations with \(\nu = \pm 1/N_c\) involve only 2 fermion zero modes and are responsible for the formation of fermion condensate for small \(m|\bar{\lambda}\lambda > |V|\).

The situation is much worse, however, for higher orthogonal and exceptional groups. The simplest example where the problem appears is the SYM theory with \(SO(7)\) gauge group \([14]\). The instantons involve here \(7-2 = 5\) pairs of zero modes and the corresponding contribution to the partition function is \(\propto m^5\). The group \(SO(7)\) does not have a nontrivial center and, in contrast to what we had for \(SU(N)\) groups, we cannot pinpoint a topological field configuration with winding number \(\nu = 1/5\). Things are not better when \(N > 7\). Thus, \(SO(N \geq 7)\) 4-dim SYM theories are as paradoxical as \(QCD_2^{\text{adj}}\) with \(N \geq 3\).

We present here another very simple 4-dimensional example where the paradox also appears. Consider the \(SU(2)\) Yang-Mills theory involving a Dirac fermion \(\psi\) belonging to the color representation with isospin \(I = 3/2\). Suppose that the fermion condensate \(<\bar{\psi}\psi>\) is formed here. Like in the conventional \(QCD_4\) with \(N_f = 1\), it breaks only \(U_A(1)\) subgroup of the chiral symmetry group which is anyway anomalous, and no massless states appear. From the other hand, comparing \(Tr(T^aT^a) = I(I + 1)(2I + 1)\) in the representation with \(I = 3/2\) with the same trace for \(I = 1/2\), we see that the instantons involve here 10 fermion zero modes.
and provide the contribution $\propto m^{10}$ to the partition function. There is no way to get the fermion condensate in the path integral framework.

Of course, the paradox here is not so prominent as in two other theories considered above. It appeared when assuming that the condensate is generated. The assumption looks natural — the dynamics of the theory is rather similar to that of conventional $QCD$ with $N_f = 1$ where the condensate is formed, but there is also a distinction. The first coefficient in the Gell-Mann-Low function

$$b = \frac{22}{3} - \frac{2}{3} \times 10 = \frac{2}{3}$$

(7.1)

is comparatively small here (though the theory is still asymptotically free) which may after all prevent the formation of fermion condensate. And, in contrast to two previous cases, we cannot present solid independent theoretical arguments that the condensate is formed. Thus, this theory may serve only as an additional indication that something is grossly wrong in our understanding; we could not claim that solely on its basis.

But, for $SO(N \geq 7)\ SYM$ and for $QCD_{2}^{adj}$ with $N \geq 3$, the situation is really mysterious.

We cannot say that we understand how this mystery is resolved. But, if there is a universal reason which resolves it in both theories, the only one we can think of is that the premise 1 in the list in the beginning of this section is false. Perhaps, there are some singular field configurations which contribute to the path integral and which cannot be classified by topological considerations. If these unspecified configurations have only one pair of fermion zero modes, the condensate may be generated. One argument in favor of this guess comes from the observation that, in strong coupling theory, fields fluctuate wildly and the topological classification which is based on the assumption that the fields are smooth and regular may be not true.
Suggestions that this may happen can be found in the literature. In particular, Crewther [35] and Zhitnitsky [33] argued that, for the conventional $QCD_4$ with $N_f$ light flavors with equal mass, field configurations carrying winding number $1/N_f$ (obviously, such fields cannot be described in topological terms) can be relevant. Actually, we do not see compelling reasons to assume this for standard $QCD$ — the usual description including only the fields with integer winding numbers works perfectly well there. But for $QCD_2^{adj}$ with $N \geq 3$, for $SO(7)$ 4-dim SYM, and may be for $SU(2)$ 4-dim gauge theory with Dirac fermions belonging to the representation $I = 3/2$ of the color group, we are kind of forced to think in this direction. What is absolutely unclear by now is in what respects path integral dynamics of these paradoxical theories differs from that in standard QCD and other well-studied theories where no need of invoking exotic nontopological fields arises.

8 Conclusions.

The $SO(3) \times QCD_2^{adj}$ which we analyzed first in this paper presents no problems. The picture is self consistent: the instantons which are present there due to nontrivial $\pi_1[SO(3)] = Z_2$ involve two fermion zero modes and lead to the formation of the fermion condensate. This condensate falls down as the temperature increases [see Eq. (3.21)] but never turns to zero. Qualitatively, the same follows from bosonization arguments. This model can serve as a remarkably good playground which may allow us to understand better the physics of $QCD$ (in particular, of $QCD$ with only one quark flavor). For example, lattice simulations of this theory would be very interesting. One could try to calculate the fermion condensate on the lattice at zero and at high temperature and compare the numerical results with theoretical prediction (3.21). Such simulations are much simpler than in 4 dimensions and could provide an independent test for the whole lattice technology.

For $N \geq 3$, we encountered an explicit paradox: the existence of the condensate
follows from bosonization arguments but we could not get it in the path integral approach. As was discussed in details in sect.7 of this paper, a similar paradox displays itself also in some 4-dimensional gauge theories. Its satisfactory resolution could bring about a progress in our understanding of quantum field theory in general.

In conclusion, we note that, if we would believe in the bosonization arguments at low temperatures and in the instanton arguments at high temperatures (at high $T$, quasiclassical approximation works and one could think that it still suffices to consider only smooth topological field configurations), the conclusion of the existence of the phase transition in the theories with $N \geq 3$ would follow — at some temperature $T_c$, the condensate would vanish and stay zero beyond it. But, at the present level of understanding, we cannot really claim it is true.

If nontopological fields contribute to the path integral also at high temperatures, there is no phase transition but only a crossover where the condensate falls down but never turns to zero (as it is the case for $N = 2$). As $N$ grows, the crossover is expected to become more and more sharp. Its temperature is estimated as

$$T^* \sim g\sqrt{N} \quad (8.1)$$

(a natural mass scale of the theory). In the limit $N \to \infty$, $T^* \to T_H$, the Hagedorn limiting temperature.

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Appendix.

We want to show here that the zero modes (3.9) and their counterparts for larger $N$ are stable under small deformations of the abelian high-$T$ instanton background (3.3), (6.8). Consider first the case $N = 2$. Choose as earlier $n^a = \delta^a_3$ in Eq. (3.3) and deform it in the transverse direction in the color space so that

$$A^a_0(x) = \delta^a_3 a(x) + (1 - \delta^a_3) b^a(x)$$  \hspace{1cm} (A.1)

with $b^a(-\infty) = b^a(\infty) = 0$ and $b \ll a$ for all $x$. Then the deformation $b^a(x)$ has no projection on the global gauge rotation modes discussed at length in sect. 3.

For $b^a(x) = 0$, the Dirac eigenvalue equation (3.7) had two zero mode solutions (3.9). With $b \neq 0$, the solutions are modified. Unfortunately, in contrast to the more simple abelian case [8, 1], we cannot solve the zero mode equation explicitly for any gauge field background. What we can do is to develop a perturbation theory in the small parameter $b/a$ and find the solution as the series in this parameter:

$$\psi^a(\text{zero}) = \psi^a_0(\text{zero}) + \psi^a_1(\text{zero}) + \psi^a_2(\text{zero}) + \ldots$$  \hspace{1cm} (A.2)

Let us start, for definiteness, from the solution $\psi_0^{-(\text{zero})}(x, \tau)$ and find the corresponding $\psi_1^{a(\text{zero})}(x, \tau)$. It satisfies the equation

$$\left[(\partial_0 \sigma_2 + \partial_x \sigma_1) \delta^{ab} - g \epsilon^{a\beta\lambda} a(x) \sigma_2\right] \psi_1^{b(\text{zero})}(x, \tau) = -i \frac{g}{2} \delta^{a3} b^+(x) \sigma_2 \psi_0^{-(\text{zero})}(x, \tau)$$  \hspace{1cm} (A.3)

We see that only the component $\psi_1^{3(\text{zero})}$ appears. It is left-handed as $\psi_0^{-(\text{zero})}$ was and also has the same $\tau$-dependence $\propto \exp(i\pi T \tau)$. The solution of (A.3) is

$$\psi_1^{3(\text{zero})}(x, \tau) = -\frac{g}{2} e^{i\pi T x} \int_x^\infty b^+(y) \psi_0^{-(\text{zero})}(y, \tau) e^{-\pi T y} dy$$  \hspace{1cm} (A.4)

It is easy to see that $\psi_1^{3(\text{zero})}$ has the same asymptotics $\propto \exp\{-\pi T |x|\}$ at $|x| \to \infty$ and is normalizable.
Generally, the n-th term of the series (A.2) $\psi_n^{a(zero)}(x)$ is related to $\psi_{n-1}^{a(zero)}(x)$ by a similar integral kernel which provides the asymptotics $\propto \exp\{-\pi T|x|\}$ for $\psi_n$ if $\psi_{n-1}$ had such, and the normalizability of the deformed zero mode is proven by induction.

For larger $N$, the analysis is quite similar. The integral kernels are a little bit different for different $\psi_0^{a(zero)}$ — the different color components of the deformation $b^a(x)$ enter, but the result is the same: if the perturbation is small, all $2(N-1)$ different zero modes remain normalizable and are there in the spectrum.

Certainly, this analysis cannot rule out bifurcations in the space of zero modes when the perturbation is large enough so that the number of zero modes would be less than $2(N-1)$ for some $b$, but we do not think that this possibility is realized (see the main text for more detailed discussion).

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Figure captions.

Fig.1. Effective potential in adjoint $SU(2)$ theory at high $T$.

Fig.2. High-$T$ instanton.

Fig.3 Geometry of effective potential for high-$T$ $QCD_2^{adj}$ with $N = 3$.

The minima occur at the points 0, □, and △ which are related to each other by $Z_3$ transformations and are physically undistinguishable in the adjoint theory. The solid triangle marks out the standard "fundamental" Weyl cell and the dashed lines inside — the "adjoint" Weyl cell.
This figure "fig1-1.png" is available in "png" format from:

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This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9402066v2
Fig. 1

\[ V_{T}^{\text{eff}}(A_{0}^{3}) \]

Fig. 2

\[ a(y) \]

\[ 2 \pi T/g \]

\[ x \quad y \]
Fig. 3