Differential function spectra, the differential Becker-Gottlieb transfer, and applications to differential algebraic $K$-theory

Ulrich Bunke* and David Gepner†

May 7, 2014

Abstract

We develop differential algebraic $K$-theory for rings of integers in number fields and we construct a cycle map from geometrized bundles of modules over such a ring to the differential algebraic $K$-theory. We also treat some of the foundational aspects of differential cohomology, including differential function spectra and the differential Becker-Gottlieb transfer. We then state a transfer index conjecture about the equality of the Becker-Gottlieb transfer and the analytic transfer defined by Lott. In support of this conjecture, we derive some non-trivial consequences which are provable by independent means.

Contents

1 Introduction 3

2 Differential function spectra 6
  2.1 Differential cohomology – the axioms . . . . . . . . . . . . . . . . . . . . . 6
  2.2 The construction of the differential function spectrum . . . . . . . . . . . . 10
  2.3 Homotopy groups and long exact sequences . . . . . . . . . . . . . . . . . . . 12
  2.4 Differential Data and Transformations . . . . . . . . . . . . . . . . . . . . 16

3 Cycle maps 19
  3.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
  3.2 Complex $K$-theory – a warm-up . . . . . . . . . . . . . . . . . . . . . . . 19
  3.3 Algebraic and differential algebraic $K$-theory . . . . . . . . . . . . . . . . 21

* NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, GERMANY, ulrich.bunke@mathematik.uni-regensburg.de
† NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, GERMANY, david.gepner@mathematik.uni-regensburg.de
3.4 $KR$-classes of locally constant sheaves of $R$-modules .................................. 23
3.5 Geometric local systems and characteristic forms .............................................. 25
3.6 The cycle map for geometric locally constant sheaves ....................................... 30
3.7 Extension from free to projective modules ......................................................... 32
3.8 Proof of Proposition 3.13 ...................................................................................... 37
3.8.1 Overview ........................................................................................................... 37
3.8.2 Construction of $\hat{I}$ ..................................................................................... 39
3.8.3 Construction of $\hat{R}$ ..................................................................................... 42
3.8.4 The commutative diagram (56) ........................................................................ 45
3.9 $\widehat{KR}^j(\ast)$ .................................................................................................. 47
3.10 Torsion sheaves and torsion ............................................................................... 50
3.11 Functoriality of the cycle map .............................................................................. 57
3.12 Smoothness of the cycle map ............................................................................... 60

4 The transfer in differential cohomology .................................................................. 63
4.1 Introduction ........................................................................................................... 63
4.2 Differential Becker-Gottlieb transfer ................................................................... 63
4.3 Geometric bundles and integration of forms ......................................................... 66
4.4 Transfer structures ............................................................................................... 68
4.5 The left square in (98) and the end of proof of Theorem 4.2 ............................... 71
4.6 Proof of Proposition 4.4 ....................................................................................... 75
4.7 Functoriality of the transfer .................................................................................. 76
4.8 Proof of Proposition 4.5 ....................................................................................... 80

5 A transfer index conjecture ...................................................................................... 80
5.1 Introduction ........................................................................................................... 80
5.2 The statement of the transfer index conjecture ..................................................... 81
5.3 The analytic index ............................................................................................... 84
5.4 Discussion of the transfer index conjecture .......................................................... 86
5.4.1 The relation with the work of Lott .................................................................. 86
5.4.2 The relation with the Cheeger-Müller theorem .............................................. 90
5.4.3 $S^1$-bundles .................................................................................................. 92
5.4.4 Higher analytic and Igusa-Klein torsion ........................................................ 97
5.5 Discussion of Lott’s relation (Theorem 5.7) ........................................................ 103

6 Technicalities .......................................................................................................... 109
6.1 Categories with weak equivalences and $\infty$-categories ....................................... 109
6.2 $\infty$-categories of spaces, spectra, and chain complexes ...................................... 110
6.3 Commutative and non-commutative group completion ........................................ 111
6.4 Smooth objects .................................................................................................... 114
6.5 Constant sheaves ................................................................................................. 116
6.6 Smooth function objects ..................................................................................... 118
6.7 The de Rham complex ........................................................................................ 122
1 Introduction

The study of differential extensions of generalized cohomology theories merge the fields of homotopy theory and differential geometry. Roughly speaking, a differential cohomology class combines the information on the underlying homotopy theoretic cohomology class and related characteristic forms with secondary data. It is this secondary information which makes differential cohomology theory relevant in applications.

Historically, the first example of a differential cohomology theory was the differential extension of integral cohomology defined in terms of Cheeger-Simons characters \[CS_{1v}\]. For a complex vector bundle with connection on a smooth manifold Cheeger and Simons constructed Chern classes in differential integral cohomology which combine the information on the underlying topological Chern classes with the characteristic forms given by Chern-Weyl theory. The differential Chern classes contain non-trivial secondary invariants for flat bundles.

Much later, motivated by developments in mathematical physics, the differential extension of topological complex \(K\)-theory attracted a lot of attention and was popularized among mathematicians in particular by the work of Freed and Hopkins \[FH00\], \[Fre00\], \[BS09\], \[SS10\], \[BM06\], \[FMS07\]. Differential topological complex \(K\)-theory captures the information on the underlying topological \(K\)-theory class of a vector bundle with connection in combination with the Chern character form, again given by Chern-Weyl theory. Differential extensions of bordism theories and Landweber exact cohomology theories have been constructed in \[BSSW09\]. The construction of the differential extension in all these examples starts off from a geometric cycle/relation model of the underlying generalized cohomology theory. A homotopy theoretic construction of a differential extension for an arbitrary cohomology theory was first given by Hopkins and Singer in \[HS05\].

The main example of the present paper is the differential extension \(\hat{KR}\) of the cohomology theory \(KR^*\) represented by the connective algebraic \(K\)-theory spectrum \(KR\) of a number ring \(R\). This differential extension will be defined by a homotopy theoretic construction. The construction which associates a differential cohomology class to a geometric object (e.g. differential Chern or \(K\)-theory class to a complex vector bundle with connection in the above examples) will be called a cycle map. Usually, a cycle map is natural transformation of set-valued contravariant functors on the category of smooth manifolds

\[
\{\text{geometric objects on } M\} \xrightarrow{\text{cycl}} \{\text{differential cohomology classes on } M\}
\]

In most cases where the differential cohomology theory is build from a geometric model the construction of a cycle map is easy, sometimes even tautological. For example, in the model \[\text{[BS09]}\] of differential topological $K$-theory the cycle of complex vector bundle $V$ with connection $\nabla^V$ is simply the class $[V, \nabla^V]$ represented by the pair $(V, \nabla^V)$. For geometric models of differential extensions of generalized cohomology theories it turned out to be difficult to obtain the functorial properties, e.g. differential extensions of natural transformations between different cohomology theories.

In this respect, the homotopy theoretic constructions like \[\text{[HS05]}\] or the one presented in the present paper are much better behaved. On the other hand, it is turned out to be difficult to construct cycle maps into the homotopy theoretic version of differential cohomology theory. This also applies to the case considered in the present paper. We are going to invest a great effort to construct a cycle map (Section 3) which associates a differential algebraic $K$-theory class

\[
\text{cycl}(V, h^V) \in \widehat{K^R}^0(M)
\]

to a locally constant sheaf $V$ of finitely generated projective $R$-modules with a geometry $h^V$ (Definition 3.10) on a smooth manifold $M$.

This discrepancy raises the question of comparison of different models. Partial answers are given by the uniqueness theorems \[\text{[BS10]}\]. The main assumption there is that the coefficients of the cohomology theory are torsion groups in odd degree. Therefore these uniqueness results apply well to ordinary cohomology, topological $K$-theory and bordism theories (see \[\text{[Bun10a]}\] and \[\text{[Bun10b]}\] for applications) but not to the algebraic $K$-theory of a number rings where the non-torsion coefficients all live in degree zero and odd degrees.

A differential cohomology theory is in particular a contravariant functor from the category of smooth manifolds to $\mathbb{Z}$-graded abelian groups. Its topological counterpart has wrong-way (Umkehr, integration) maps for suitably oriented fibre bundles $\pi: W \to B$. The notion of a differential orientation and the refinement of the wrong-way map to differential cohomology has been discussed in general in \[\text{[HS05]}\], \[\text{[Bun]}\] and precise versions for geometric models have been given in \[\text{[BS09]}\], \[\text{[BSSW09]}\], \[\text{[BKS10]}\], \[\text{[FL10]}\].

If one also has a descent of geometric objects along $\pi$ one can consider the compatibility with the cycle map. A (differential) index theorem is the statement that the following diagram commutes

\[
\begin{array}{ccc}
\{(\text{geometric objects on } W)\} & \xrightarrow{\text{cycl}} & \{(\text{differential cohomology classes on } W)\} \\
\downarrow{\text{descent}} & & \downarrow{\text{wrong way map}} \\
\{(\text{geometric objects on } B)\} & \xrightarrow{\text{cycl}} & \{(\text{differential cohomology classes on } B)\}
\end{array}
\]

The most prominent example of an index theorem is the index theorem of Atiyah-Singer which compares the descent of vector bundles along $K$-oriented maps with the wrong-way map in topological $K$-theory. Its differential refinement has been given in \[\text{[FL10]}\].

All wrong-way maps for generalized cohomology theories in topology use the suspension isomorphisms (i.e. the fact that a cohomology theory is represented by a spectrum) in an
essential way. From this it is clear that for a differential refinement of the wrong-way map one has to take into account properly the spectrum aspect of a differential cohomology theory. A first construction of a differential cohomology spectrum has been given in [HS05]. In the present paper we take the spectrum aspect of differential cohomology as a starting point and give in Section 2 a direct and straight-forward homotopy theoretic construction of a differential function spectrum. The differential cohomology will then be defined in terms of homotopy groups of the differential function spectrum. We consider our construction here is a short-cut to the construction of [HS05].

A particularly simple example of a wrong-way map is the Becker-Gottlieb transfer [BG75] which does not require any kind of orientation. It is defined for every smooth fibre bundle with closed fibres and thus applies to the cohomology theory $KR^*$ represented by the algebraic $K$-theory spectrum of a number ring $R$. On the geometric side one can descend a locally constant sheaf $\mathcal{V}$ of $R$-modules from $W$ to $B$ using the sheaf theoretic higher-derived images $R^i\pi_*(\mathcal{V})$. In this case the corresponding index theorem (the Dwyer-Weiss-Williams index theorem) has been proven in [DWW03] and [BL95] (a characteristic class version). In order to define a differential Becker-Gottlieb transfer we must choose a Riemannian structure (Definition 4.1) on the bundle $\pi : W \to B$. One of our main results (presented in Section 4) is the construction of a differential Becker-Gottlieb transfer (and the verification of all its expected functorial properties) for smooth fibre bundles with closed fibres equipped with Riemannian structures.

On the level of geometric objects one can descend the metric $h^V$ to metrics on the higher-derived images $R^i\pi_*(\mathcal{V})$ using fibrewise Hodge theory and $L^2$-metrics on harmonic forms. In this way we obtain a descent for geometric objects from $W$ to $B$ which will also involve the Bismut-Lott analytic torsion form [BL95]. This geometric construction has first been introduced by [Lot00].

Our main result here is the development of all ingredients needed to state a differential refinement of the Dwyer-Weiss-Williams index theorem, Conjecture 5.3, which we call the transfer index conjecture (TIC). We are far from having a proof of this transfer index conjecture in differential algebraic $K$-theory. Instead, in Subsection 5.4 we are going to discuss special cases and consequences which can be proved by different means. Some of these consequences are deep theorems in different fields like arithmetic geometry or global analysis. Therefore we expect that a proof of Conjecture 5.3 will be quite complicated.

To finish this introduction let us explain the structure of the present paper and its main achievements.

1. In Section 2 we review the axioms for a differential extension of a generalized cohomology theory. Our first main result is the construction of the differential function spectrum. We define differential cohomology in terms of the homotopy groups of the differential function spectrum and verify that this definition satisfies the required axioms. We further discuss the dependence on data.

2. In Section 3 we review the definition and rational calculation of the algebraic $K$-theory for number rings $R$ and define the differential algebraic $K$-theory $\hat{KR}^0$ (Definition 3.4). Its main result is the construction of a cycle map which associates
classes \( \text{cyc}(\mathcal{V}, h^V) \in KR^0(M) \) to locally constant sheaves of projective \( R \)-modules with geometry on \( M \).

3. In Section 4 we construct the differential Becker-Gottlieb transfer and verify its functorial properties.

4. In Section 5 we state the transfer index conjecture 5.3. Our main contribution here is to work out links between this conjecture and interesting deep results in other branches of mathematics.

5. Our homotopy theoretic construction of the differential function spectrum takes place in an \( \infty \)-category of sheaves of spectra on smooth manifolds. In Section 6 develop the necessary background and language which will be used throughout the paper. We have also shifted the proofs of some of the technical details to this section.

The theory of differential cohomology developed in this paper can be refined to theories of multiplicative and twisted differential cohomology, and will be treated in future work.

2 Differential function spectra

2.1 Differential cohomology – the axioms

A generalized cohomology theory is a homotopy invariant contravariant functor

\[ E^* : \text{Top}^{op} \to \text{Ab}_{\mathbb{Z} - \text{gr}} \]

from the category of topological spaces \( \text{Top} \) to the category of \( \mathbb{Z} \)-graded abelian groups \( \text{Ab}_{\mathbb{Z} - \text{gr}} \) with some additional structures like suspension isomorphisms or Mayer-Vietoris sequences. Examples which are relevant for the present paper are ordinary integral cohomology \( H^*_\mathbb{Z} \), topological complex \( K \)-theory \( KU^* \), and the algebraic \( K \)-theory \( KR^* \) of a ring \( R \). Under suitable hypotheses a generalized cohomology theory \( E^* \) can be represented by a spectrum \( E \in \text{Sp} \) (see [Swi02, Ch.9] on the Brown representability theorem). In the present paper we will always assume that a generalized cohomology theory is represented by a spectrum.

Given a cohomology theory \( E^* \) represented by a spectrum \( E \) we have the notion of a differential extension \( \hat{E}^* \) of \( E^* \). It is a contravariant functor defined on the category of smooth manifolds

\[ \hat{E}^* : \text{Mfd}^{op} \to \text{Ab}_{\mathbb{Z} - \text{gr}}. \]

The differential \( E \)-cohomology \( \hat{E}^* \) extends the restriction of \( E^* \) from the category of topological spaces to the category of smooth manifolds by differential forms. Note that a generalized cohomology \( E^* \) is completely determined by its restriction to smooth manifolds [KS10], which implies that we can recover the cohomology theory \( E \) from the functor \( \hat{E}^* \) together with the structure maps \( R, I, a \) of differential cohomology, see Definition 2.2.
The goal of the present subsection is to explain the axiomatic characterization of differential cohomology \cite{BS10, SS08}. We start with a description of the data and basic constructions on which the differential cohomology will depend. We consider the Eilenberg-MacLane spectrum $H \mathbb{Z}$. It is a strictly commutative ring spectrum \cite{EKMM97} and gives rise to an $\infty$-category $\text{Mod}(H \mathbb{Z})$ of $H \mathbb{Z}$-modules (see Subsection 6.1 for some basics on $\infty$-categories). The Eilenberg-MacLane spectrum functor \cite{Shi07} (see Subsection 6.8)

$$H : \mathbb{N}(\text{Ch})[W^{-1}] \to \text{Mod}(H \mathbb{Z})$$ (2)

extends the construction of $H \mathbb{Z}$ from the group $\mathbb{Z}$, which is considered as a chain complex concentrated in degree zero, to general chain complexes. For a (homological) chain complex $A$ the homotopy groups of the associated Eilenberg-MacLane spectrum $H(A)$ are naturally isomorphic to the homology groups of $A$, i.e. we have an isomorphism

$$\pi_*(H(A)) \cong H_*(A).$$

For an abelian group $G$ we can define a Moore spectrum $M G$ which is characterized uniquely up to equivalence by the property that

$$H_\mathbb{Z}^\ast(M G) \cong \begin{cases} G & \ast = 0 \\ 0 & \ast \neq 0 \end{cases}.$$  We write $E \mathbb{R} := E \wedge M \mathbb{R}$ for the smash product of $E$ with the Moore spectrum $M \mathbb{R}$ of $\mathbb{R}$. The effect of smashing with $M \mathbb{R}$ on the level of homotopy groups is tensoring with $\mathbb{R}$

$$\pi_*(E \mathbb{R}) \cong \pi_*(E) \otimes \mathbb{R},$$

see \cite{Bou79}. There is a natural map $\epsilon_\mathbb{R} : E \to E \mathbb{R}$ of spectra which induces the map

$$\pi_*(E) \to \pi_*(E \mathbb{R}) \cong \pi_*(E) \otimes \mathbb{R}, \quad x \mapsto x \otimes 1.$$  In order to define the notion of a differential extension of $E$ we choose a chain complex $A \in \text{Ch}$ of real vector spaces together with an equivalence of spectra

$$c : E \mathbb{R} \xrightarrow{\sim} H(A).$$

**Definition 2.1** The triple $(E, A, c)$ will be called the differential data for the differential extension of $E$.

The complex $A$ is, of course, not arbitrary but necessarily satisfies

$$\pi_*(E) \otimes \mathbb{R} \cong H_*(A).$$

From the data $(E, A, c)$ we obtain the composition

$$E^* \xrightarrow{c} E \mathbb{R}^* \xrightarrow{\epsilon_\mathbb{R}} H(A)^*$$ (3)
of natural transformations between functors from manifolds to \( \mathbb{Z} \)-graded abelian groups. For a smooth manifold \( M \) we can define the complex \( \Omega A(M) \) of smooth \( A \)-valued differential forms; see Definition 6.14 for technical details. Here we will consider \( A \) and \( \Omega A \) as cohomological chain complexes using the convention \( A^n := A_{-n} \).

We have a de Rham isomorphism (203)

\[
j : H^*(\Omega A(M)) \xrightarrow{\sim} H(A)^*(M)
\]

which is natural in the manifold \( M \) and the chain complex \( A \).

If \( B \) is a cohomological chain complex, then we let

\[
Z^k(B) := \ker(d : B^k \to B^{k+1}) \subseteq B^k
\]
denote the group of cycles in degree \( k \). We have a natural map

\[
[\ldots] : Z^k(B) \to H^k(B)
\]

which associates to a cycle the cohomology class it represents. The de Rham isomorphism thus induces a natural transformation

\[
Z^*(\Omega A) \xrightarrow{[\ldots]} H^*(\Omega A) \xrightarrow{j} H(A)^*.
\]

The notion of a differential extension of \( E^* \) depends on the choice of the complex \( A \) and the equivalence \( c \).

**Definition 2.2** A differential extension \( \hat{E}^* \) of the cohomology theory \( E^* \) is a tuple \( (\hat{E}^*, I, R, a) \) consisting of

1. a functor \( \hat{E}^* : \text{Mf}^{op} \to \text{Ab}_{\mathbb{Z}-gr} \) from smooth manifolds to \( \mathbb{Z} \)-graded abelian groups,
2. a natural transformation \( I : \hat{E}^* \to E^* \),
3. a natural transformation \( R : \hat{E}^* \to Z^*(\Omega A) \),
4. and a natural transformation \( a : \Omega A[1]/\text{im}(d) \to \hat{E}^* \).

These objects must satisfy the following axioms:

i. \( R \circ a = d \).

ii. The diagram

\[
\begin{array}{ccc}
\hat{E}^* & \xrightarrow{I} & E^* \\
\downarrow R & & \downarrow j \\
Z^*(\Omega A) & \xrightarrow{[\ldots]} & H(A)^*
\end{array}
\]

commutes.
iii. The sequence

\[ E[1]^* \xrightarrow{j^{-1}_{\text{cocor}}} \Omega A[1]^*/\text{im}(d) \xrightarrow{a} \hat{E}^* \xrightarrow{\text{I}} E^* \to 0 \]  

(6)

is exact.

The kernel

\[ \hat{E}^\flat := \ker \left( R : \hat{E}^* \to Z^*(\Omega A) \right) \subseteq \hat{E}^* \]

is called the flat part of \( \hat{E}^* \).

As explained in \[\text{BS10}\], in general these axioms do not characterize the differential extension of \( E^* \) uniquely. For uniqueness one needs additional structures like an integration. We will not discuss these structures in the present paper, but see \[\text{Bum}\].

The differential cohomology functor \( \hat{E}^* \) is not homotopy invariant. Its deviation from homotopy invariance is measured by the homotopy formula \[\text{BS10}, (1)\]. Let

\[ x \in \hat{E}^*([0,1] \times M) \]

and \( f_i : M \to [0,1] \times M \) be the inclusions corresponding to the endpoints of the interval. Then we have

\[ f_1^*x - f_0^*x = a \left( \int_{I \times M/M} R(x) \right) . \]

(7)

The first example of a differential extension of a cohomology theory was the differential extension of integral cohomology represented by the Eilenberg-MacLane spectrum \( H\mathbb{Z} \).

In this example constructed by Cheeger-Simons \[\text{CS1v}\] the group \( H\mathbb{Z}(M) \) was realized as the group of differential characters. We refer to \[\text{BS09}, \text{BSSW09}\] and the literature cited therein for further examples constructed using geometric cycles. A general construction of a differential extension for an arbitrary cohomology theory has been given by Hopkins-Singer in \[\text{HS05}\].

In the present paper we give a general construction of a differential extension \( \hat{E}^* \) in terms of the differential function spectrum, Definition 2.4. The evaluation on \( M \) of the differential function spectrum for \( E \) is a spectrum \( \text{Diff}(E)(M) \) which refines the function spectrum \( E^{\Sigma^\infty M_+} \) by Eilenberg-MacLane spectra associated with \( \Lambda \)-valued differential forms on \( M \) such that

\[ \hat{E}^0(M) := \pi_0(\text{Diff}(E)(M)) \]

is the underlying functor of the differential extension. In order to get the differential cohomology in all degrees we use shifts

\[ \hat{E}^n(M) := \pi_0(\text{Diff}(\Sigma^n E)(M)) . \]

Using a slightly different language, the differential function spectrum has first been constructed in \[\text{HS05}\].

An interesting aspect of differential cohomology is that it provides a natural notion of a smooth map \( f : N \to \hat{E}^0(M) \) from a smooth manifold \( N \) to the differential \( E \)-cohomology of another manifold \( M \).
Definition 2.3 A map \( f : N \to \hat{E}^0(M) \) is called smooth if there exists an open covering \( \{ U_\alpha \}_{\alpha \in I} \) of \( N \) and elements \( f_\alpha \in \hat{E}^0(M \times U_\alpha) \) such that \( i_{\alpha,n}^* f_\alpha = f(n) \) for all \( n \in U_\alpha \) and \( \alpha \in I \), where \( i_{\alpha,n} : M \to M \times U_\alpha \) is the embedding given by \( i_{\alpha,n}(m) := (m, n) \).

The group \( \hat{E}^0(M) \) can be considered as a possibly infinite-dimensional manifold modeled on the vector space \( \Omega A^{-1}(M)/\text{im}(d) \). This manifold structure has played an important role e.g. in [FMS07]. Using the abelian group structure we can identify \( T_u \hat{E}^0(M) \sim \Omega A^{-1}(M)/\text{im}(d) \) for all \( u \in \hat{E}^0(M) \) in a canonical way. The derivative of the smooth map \( f \) at \( n \in N \) is given by

\[
\left. df(n)(X) = i_{\alpha,n}^*[i_X R(f_\alpha)] \right|_{\Omega A^{-1}(M)/\text{im}(d)} \quad X \in T_n N ,
\]

where \( i_X \) denotes the insertion of \( X \) and \( \alpha \in I \) is such that \( n \in U_\alpha \). This is the infinitesimal version of the homotopy formula (7). One checks that the right-hand side of (8) is independent of the choice of \( \alpha \in I \) such that \( n \in U_\alpha \).

In many cases, such as in the model considered in the present paper, we have an identification of the flat part with the \( \mathbb{R}/\mathbb{Z} \)-version of \( E \) (see (19)):

\[
\hat{E}^0_{\text{flat}}(M) \sim \mathbb{R}/\mathbb{Z}^{-1}(M) .
\]

If \( \tilde{f} \in \hat{E}^0(M \times N) \) is such that \( f(n) := i_n^* \tilde{f} \) is flat for all \( n \in N \), then we get a smooth map

\[
f : N \to \hat{E}^0_{\text{flat}}(M) \sim \mathbb{R}/\mathbb{Z}^{-1}(M) .
\]

A frequent application of differential cohomology is the construction of secondary invariants which live in \( \mathbb{R}/\mathbb{Z}^{-1}(M) \), and the reasoning above can be considered as the formalism which allows to show the smooth dependence of the secondary invariant on the data and to derive a formula for its variation. We refer to Subsection 3.12 for an application of these ideas to differential algebraic \( K \)-theory.

2.2 The construction of the differential function spectrum

In this section we construct for every choice of data \( (E, A, c) \) as in Definition 2.1 a differential function spectrum

\[
\text{Diff}(E, A, c) \in \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Sp})[W^{-1}]) ,
\]

see Section 6.4 for notation. Often we will use the shorter notation \( \text{Diff}(E) \) instead of \( \text{Diff}(E, A, c) \).

We have a functor

\[
\text{Sm}_\infty : \mathbb{N}(\text{Sp})[W^{-1}] \to \text{Sm}(\mathbb{N}(\text{Sp})[W^{-1}])
\]

which associates to a spectrum \( E \in \text{Sp} \) the smooth spectrum \( \text{Sm}_\infty(E) \in \text{Sm}(\mathbb{N}(\text{Sp})[W^{-1}]) \). Roughly speaking, the smooth function spectrum \( \text{Sm}_\infty(E) \) maps a manifold \( M \) to the
spectrum of functions from $M$ to $E$. In detail, this construction will be given in Subsection 6.6. The main point of the construction of $S_{\infty}$ is to fix precisely the transition from the context of topological spaces, e.g. the underlying spaces of smooth manifolds, to the simplicial world in which our spectra live.

For a chain complex $A \in \text{Ch}$ of real vector spaces we can form the de Rham complex

$$\Omega A \in \text{Sm}(\mathbb{N}(\text{Ch})),$$

see Definition 6.14. If

$$C : \cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$$

is a cohomological chain complex, then we let

$$\sigma C : \cdots \to 0 \to C^0 \to C^1 \to \cdots$$

be its truncation to non-negative degrees. This truncation functor $\sigma : \text{Ch} \to \text{Ch}$ extends to smooth chain complexes, so we may form the truncated de Rham complex $\sigma \Omega A$. The inclusion morphism of smooth complexes $\sigma \Omega A \to \Omega A$ will eventually give the left vertical arrow in the square (10) below. By Lemma 6.16 the images $\Omega A_{\infty}$ and $\sigma \Omega A_{\infty}$ of $\Omega A$ and $\sigma \Omega A$ under the functor

$$\text{Sm}(\mathbb{N}(\text{Ch})) \to \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}])$$

satisfy descent, i.e. we have

$$\Omega A_{\infty}, \sigma \Omega A_{\infty} \in \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})[W^{-1}]).$$

One reason to invert the weak equivalences is that we want to post-compose with the composition of the Eilenberg-MacLane spectrum functor and the forgetful functor from $H\mathbb{Z}$-modules to spectra

$$\mathbb{N}(\text{Ch})[W^{-1}] \xrightarrow{\boxtimes} \text{Mod}(H\mathbb{Z}) \to \mathbb{N}(\text{Sp})[W^{-1}]$$

(see Subsection 6.8 for further details). We thus get a functor (also denoted by $H$)

$$H : \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})[W^{-1}]) \to \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Sp})[W^{-1}]).$$

Given a chain complex of real vector spaces $A \in \text{Ch}$ we can now form two smooth spectra with descent, namely

$$S_{\infty}(H(A)) \ , \ H(\Omega A_{\infty}) \in \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Sp})[W^{-1}]).$$

They are related by a canonical equivalence

$$j : H(\Omega A_{\infty}) \xrightarrow{\sim} S_{\infty}(H(A))$$

of smooth spectra, a spectrum level version of the de Rham isomorphism (4), which is described in Proposition 6.20.
The input for the construction of the differential function spectrum is the datum \((E, A, c)\) as in Definition 2.1. We form the map of smooth spectra

\[
\text{rat} : \text{Sm}_\infty(E) \xrightarrow{\rho} \text{Sm}_\infty(E\mathbb{R}) \xrightarrow{c} \text{Sm}_\infty(H(A)) \xleftarrow{j} H(\Omega A_\infty). \tag{9}
\]

The fact that this map involves the inverse of \(j\) gives a second reason for inverting the weak equivalences.

**Definition 2.4** We define the differential function spectrum

\[
\text{Diff}(E, A, c) \in \text{Sm}^{\text{desc}}(N(\text{Sp})[W^{-1}])
\]

as the pull-back

\[
\begin{array}{ccc}
\text{Diff}(E, A, c) & \longrightarrow & H(\sigma \Omega A_\infty) \\
\downarrow & & \downarrow \\
\text{Sm}_\infty(E) & \xrightarrow{\text{rat}} & H(\Omega A_\infty)
\end{array}
\tag{10}
\]

The Eilenberg-MacLane spectrum functor is stable, which implies that for each \(n \in \mathbb{Z}\) we have an equivalence \(\Sigma^n H(A) \cong H(\Sigma^n A)\). The datum \((E, A, c)\) thus induces a datum \(\Sigma^n(E, A, c) := (\Sigma^n E, \Sigma^n A, c_n)\), where

\[
c_n : \Sigma^n E\mathbb{R} \xrightarrow{\Sigma^n c} \Sigma^n H(A) \cong H(\Sigma^n A).
\]

Also, the functor \(\pi_0 : N(\text{Sp})[W^{-1}] \to \text{Ab}\) induces a map

\[
\pi_0 : \text{Sm}(N(\text{Sp})[W^{-1}]) \to \text{Sm}(N(\text{Ab}))
\]

**Definition 2.5** The differential \(E\)-cohomology in degree \(n \in \mathbb{Z}\) is the smooth abelian group

\[
\hat{E}^n := \pi_0(\text{Diff}(\Sigma^n(E, A, c))) \in \text{Sm}(N(\text{Ab})).
\]

Note that a smooth abelian group is nothing else than a functor \(\text{Mf}^{\text{op}} \to \text{Ab}\). In the next subsection we will construct the natural transformations \(R, I,\) and \(a\) and verify the axioms stated in Definition 2.2.

**2.3 Homotopy groups and long exact sequences**

In this section we calculate the smooth abelian groups \(\pi_n(\text{Diff}(E)) \in \text{Sm}(N(\text{Ab}))\). By \(Z^0(\Omega A) \in \text{Sm}(N(\text{Ab}))\) we denote the smooth abelian group which associates to a manifold \(M\) the abelian group of cycles of degree zero in \(\Omega A(M)\). We consider \(Z^0(\Omega A)\) as a smooth chain complex \(Z^0(\Omega A)^\bullet \in \text{Sm}(N(\text{Ch}))\) which is concentrated in degree zero. We have an
inclusion map of smooth complexes $i : Z^0(\Omega A)^\bullet \to \Omega A$ which induces an isomorphism of smooth abelian groups

$$
Z^0(\Omega A) \cong H^0(Z^0(\Omega A)^\bullet) \xrightarrow{\cong H(i)} H^0(\sigma \Omega A) \cong \pi_0(H(\sigma \Omega A_\infty)) .
$$

We furthermore get a map of smooth spectra

$$
H(Z^0(\Omega A)_\infty^\bullet) \to H(\sigma \Omega A_\infty) \to H(\Omega A_\infty) \xrightarrow{\jmath} \mathcal{S}_\infty(H(A)) .
$$

These maps give the first two maps in the composition

$$
Z^0(\Omega A) \xrightarrow{(11)} \pi_0(H(Z^0(\Omega A)_\infty)) \xrightarrow{(12)} \pi_0(\mathcal{S}_\infty(H(A))) \xrightarrow{(181) and c} E \mathcal{R}^0
$$

used to define the fibre product in the second equation of the following proposition.

**Proposition 2.6** We have isomorphisms of smooth abelian groups

$$
\pi_n(\text{Diff}(E)) \cong \begin{cases} 
E^{-n} & n \leq -1 \\
E/R[\mathbb{Z}^{-n-1}] & n \geq 1 \\
\text{see below} & n = 0 
\end{cases} .
$$

The smooth abelian group $\hat{E}_0^0$ fits into the natural exact sequence

$$
E^{-1} \to E/R^{-1} \to \hat{E}_0^0 \xrightarrow{(I,R)} E^0 \times_{E/R} Z^0(\Omega A) \to 0 .
$$

**Proof.** Let $\tau^{\geq 1} : \mathbb{N}(\mathcal{S}p)[W^{-1}] \to \mathbb{N}(\mathcal{S}p)[W^{-1}]$ be functor which takes $(0)$-connective covers. It induces a functor denoted by the same symbol

$$
\tau^{\geq 1} : \text{Sm}(\mathbb{N}(\mathcal{S}p)[W^{-1}]) \to \text{Sm}(\mathbb{N}(\mathcal{S}p)[W^{-1}]) .
$$

If we define the intermediate smooth spectrum $X$ by the pull-back in $\text{Sm}(\mathbb{N}(\mathcal{S}p)[W^{-1}])$

$$
X \to \tau^{\geq 1} H(\sigma \Omega A_\infty) ,
$$

then we get a map $X \to \text{Diff}(E)$ induced by the natural map of pull-back diagrams $\xrightarrow{(14)} \to \xrightarrow{(10)}$ coming from the morphism $\tau^{\geq 1} \to \text{id}$ . By definition the spectrum $E/R[\mathbb{Z}]$ fits into a pull-back square

$$
\Sigma^{-1}E/R[\mathbb{Z}] \to 0 \\
\downarrow \quad \downarrow \\
E \xrightarrow{\epsilon} E \mathcal{R}
$$

13
in \( N(\text{Sp})[W^{-1}] \). Since the smooth function spectrum construction

\[
\text{Sm}_\infty : N(\text{Sp})[W^{-1}] \to \text{Sm}(N(\text{Sp})[W^{-1}])
\]

preserves pull-back squares and commutes with shifts we have a pull-back square of smooth spectra

\[
\begin{array}{ccc}
\Sigma^{-1}\text{Sm}_\infty(E_{\mathbb{R}/\mathbb{Z}}) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\text{Sm}_\infty(E) & \xrightarrow{e_{\mathbb{R}}} & \text{Sm}_\infty(E_{\mathbb{R}})
\end{array}
\]

This can be extended to a diagram of squares

\[
\begin{array}{ccc}
\Sigma^{-1}\text{Sm}_\infty(E_{\mathbb{R}/\mathbb{Z}}) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\text{Sm}_\infty(E) & \xrightarrow{e_{\mathbb{R}}} & \text{Sm}_\infty(E_{\mathbb{R}}) \\
\downarrow & & \downarrow \\
\text{Sm}_\infty(E) & \xrightarrow{j^{-1}_{\text{rat}}} & H(\Omega A_{\infty})
\end{array}
\]

Since \( \pi_*(\tau^{\geq 1}H(\sigma\Omega A_{\infty})) \cong H^{-*}(\tau^{\geq 1}H(\sigma\Omega A_{\infty})) \) is concentrated in negative degrees equation (190) implies that \( \tau^{\geq 1}H(\sigma\Omega A_{\infty}) \cong 0 \). Thus we also have a map \( X \to \Sigma^{-1}\text{Sm}_\infty(E_{\mathbb{R}/\mathbb{Z}}) \) induced by the natural map from the pull-back diagram (14) to the outer square of (16). Each of these pull-back diagrams provides a long exact sequence of homotopy groups (similar to (17)), and these sequences are related by corresponding maps. It now follows from the Five Lemma that the induced maps of smooth abelian groups

\[
\pi_n(\text{Diff}(E)) \xleftarrow{\sim} \pi_n(X) \xrightarrow{\sim} \pi_{n+1}(\text{Sm}_\infty(E_{\mathbb{R}/\mathbb{Z}}))
\]

are isomorphisms for \( n \geq 1 \). Using (181) in order to identify the group on the right with \( E_{\mathbb{R}/\mathbb{Z}}^{-n-1}(M) \) we get the assertion in the case \( n \geq 1 \).

We let

\[
\tau^{\leq -1} : \text{Sm}(N(\text{Sp})[W^{-1}]) \to \text{Sm}(N(\text{Sp})[W^{-1}])
\]

be the cofibre of the natural transformation \( \tau^{\geq 0} \to \text{id} \). We get a map \( \text{Diff}(E) \to Y \), where the intermediate smooth spectrum \( Y \in \text{Sm}(N(\text{Sp})[W^{-1}]) \) is defined by the pull-back

\[
\begin{array}{ccc}
Y & \longrightarrow & \tau^{\leq -1}H(\sigma\Omega A_{\infty}) \\
\downarrow & & \downarrow \\
\tau^{\leq -1}\text{Sm}_\infty(E) & \longrightarrow & \tau^{\leq -1}H(\Omega A_{\infty})
\end{array}
\]

Since we have an equivalence

\[
\tau^{\leq -1}H(\sigma\Omega A_{\infty}) \xrightarrow{\sim} \tau^{\leq -1}H(\Omega A_{\infty})
\]

we conclude that \( Y \to \tau^{\leq -1}\text{Sm}_\infty(E) \) is an equivalence. It again follows from the Five-Lemma that

\[
\pi_n(\text{Diff}(E)) \xrightarrow{\sim} \pi_n(Y)
\]
for $n \leq -1$. This gives the assertion for the case $n \leq -1$.

Finally, in order to see the case $n = 0$, we use the long exact sequence associated to the fibre sequence

$$\cdots \to \Sigma^{-1}H(\Omega A_\infty) \xrightarrow{a} \text{Diff}(E) \to \text{Sm}_\infty(E) \sqcup H(\sigma \Omega A_\infty) \to H(\Omega A_\infty) \to \cdots \ .$$

(17)

The homotopy groups of all entries different from $\text{Diff}(E)$ are known. In particular,

$$\pi_0(H(\sigma \Omega A_\infty)) \cong Z^0(\Omega A) .$$

This gives the desired result. \[\square\]

The following proposition justifies calling the smooth abelian group $\hat{E}^n = \pi_0(\text{Diff}(\Sigma^n E))$ the $n$th differential $E$-cohomology. We verify that it satisfies the axioms for a differential extension Definition 2.2 (see [SS08] and [BS10]). It suffices to consider the case $n = 0$.

**Proposition 2.7** The smooth abelian groups $E^0$ and $\hat{E}^0$ fit into the following diagram

$$\begin{array}{ccc}
E^0 & \xrightarrow{\partial} & E^0 \\
\downarrow{R} & & \downarrow{R} \\
\hat{E}^0 & \xrightarrow{a} & E^0 \\
\Omega A^{-1}/\text{im}(d) & \xrightarrow{d} & Z^0(\Omega A)
\end{array}$$

(18)

Moreover, we have an isomorphism

$$E^0 \cong \hat{E}^0 \_{\text{flat}}$$

(19)

identifying the flat part (Definition 2.2) with a homotopy theoretic object.

**Proof.** The maps $R$ and $I$ have been constructed in Proposition 2.6. The map $a$ is induced by the map in (17) denoted by the same symbol. There is a natural map of pull-back diagrams (16) \rightarrow (10) which induces the maps of smooth spectra and their zeroth homotopy groups

$$\Sigma^{-1}\text{Sm}_\infty(E^0) \to \text{Diff}(E), \ E^0 \to \hat{E}^0 .$$

We get the commutativity of (18) and the isomorphism (19) by an analysis of the induced maps of long-exact sequences of homotopy groups. \[\square\]

Let us now derive, as a consequence of the descent property, a Mayer-Vietoris sequence for differential cohomology. We consider a smooth manifold $M$ and assume that $U, V \subseteq M$ are open subsets such that $M = U \cup V$. 
Proposition 2.8 We have a long exact sequence of groups
\[ \cdots \to E_{\mathbb{R}/\mathbb{Z}}^{-2}(U \cap V) \to \hat{E}^0(M) \to \hat{E}^0(U) \oplus \hat{E}^0(V) \to \hat{E}^0(U \cap V) \to E^1(M) \to \cdots \]
which extends to the left and right with the long exact sequences for \( E_{\mathbb{R}/\mathbb{Z}}^* \) and \( E^* \).

Proof. Since \( \text{Diff}(E) \) satisfies descent we have a pull-back square
\[
\begin{array}{ccc}
\text{Diff}(E)(M) & \longrightarrow & \text{Diff}(U) \\
\downarrow & & \downarrow \\
\text{Diff}(E)(V) & \longrightarrow & \text{Diff}(E)(U \cap V)
\end{array}
\]
It induces the Mayer-Vietoris sequence in view of the calculation of homotopy groups of Proposition 2.6.

\[ \square \]

2.4 Differential Data and Transformations

In this subsection we explain how \( \text{Diff}(E, A, c) \) depends on the data \((E, A, c)\). To this end we introduce an \( \infty \)-category \( \hat{\text{Sp}} \) of such data and describe \( \text{Diff}(E, A, c) \) as a functor.

Definition 2.9 We define the \( \infty \)-category of data \( \hat{\text{Sp}} \) as the pullback
\[
\begin{array}{ccc}
\hat{\text{Sp}} & \longrightarrow & \mathbb{N}(\text{Ch}) \\
\downarrow & & \downarrow H \\
\mathbb{N}(\text{Sp})[W^{-1}] \cdot \wedge \mathcal{M}_R & \longrightarrow & \mathbb{N}(\text{Sp})[W^{-1}]
\end{array}
\]
More concretely, a an object of \( \hat{\text{Sp}} \) consists of a spectrum \( E \), a chain complex \( A \), and a specified homotopy equivalence \( c : E_{\mathbb{R}} \simeq H(A) \). Similarly, a map of data \((f, g, \phi) : (E, A, c) \rightarrow (E', A', c')\) consists of a map of spectra \( f : E \rightarrow E' \), a map of chain complexes \( g : A \rightarrow A' \), and a homotopy \( \phi \) from \( H(g) \circ c \) to \( c' \circ f \wedge M_{\mathbb{R}} \).

Let \( \Lambda_2^2 \) be the nerve of the category of the shape
\[
\bullet \to \bullet \leftarrow \bullet .
\]
Then we have a natural functor
\[ P : \hat{\text{Sp}} \rightarrow \text{Fun}(\Lambda_2^2, \text{Sm}(\mathbb{N}(\text{Sp})[W^{-1}])) \]
which maps the triple \((E, A, c)\) to the diagram
\[ \text{Sm}_\infty(E) \overset{\text{rat}}{\longrightarrow} H(\Omega A_\infty) \leftarrow H(\sigma \Omega A_\infty) . \]
If we compose with 

$$\lim_{A_2^2} : \text{Fun}(A_2^2, \text{Sm}(\text{N}(\text{Sp})[W^{-1}]))) \to \text{Sm}(\text{N}(\text{Sp})[W^{-1}])$$

then we obtain the differential function spectrum functor (Definition 2.4)

$$\text{Diff} = \lim_{A_2^2} \circ P : \hat{\text{Sp}} \to \text{Sm}(\text{N}(\text{Sp})[W^{-1}])$$

This construction makes clear how the differential function spectrum depends on the differential data \((E, A, c)\).

We now discuss the possibility of the choice of canonical differential data. There is a version of the Eilenberg-MacLane equivalence (Lemma 6.17)

$$H : \text{N}(\text{Ch}_Q)[W^{-1}] \xrightarrow{\sim} \text{Mod}(HQ)$$

where \(\text{Ch}_Q\) denote the category of chain complexes over \(Q\). We call a spectrum \(E\) rational if it is in the essential image of the forgetful functor

$$\text{Mod}(HQ) \to \text{N}(\text{Sp})[W^{-1}]$$

For a chain complex of rational vector spaces \(A_Q \in \text{N}(\text{Ch}_Q)[W^{-1}]\) we can choose an equivalence \(A_Q \xrightarrow{\sim} H_*(A_Q)\), where we consider the homology \(H_*(A_Q)\) of \(A_Q\) as a complex with zero differential. After application of \(H\) we get an equivalence

$$H(A_Q) \xrightarrow{\sim} H(H_*(A_Q))$$

such that

$$\pi_*(H(A_Q)) \xrightarrow{\cong} \pi_*(H(H_*(A_Q)))$$

$$\cong$$

$$H_*(A_Q) \xrightarrow{\cong} H_*(A_Q)$$

commutes.

Since we have an equivalence \(M_Q \xrightarrow{\sim} HQ\) of the Moore and the Eilenberg-MacLane spectra associated to \(Q\) the spectrum \(EQ := E \wedge MQ\) is rational for any spectrum \(E \in \text{N}(\text{Sp})[W^{-1}]\). If we define the complex \(A_Q\) by \(A_Q := \pi_*(E) \otimes Q \in \text{N}(\text{Ch}_Q)[W^{-1}]\) with trivial differential, then there exists an equivalence

$$c_Q : EQ \xrightarrow{\sim} H(A_Q)$$

which induces the canonical identification in homotopy groups. We define \(A := \pi_*(E) \otimes \mathbb{R}\). Using the canonical equivalences \(EQ \mathbb{R} \xrightarrow{\sim} E \mathbb{R}\) and \(H(A_Q) \mathbb{R} \xrightarrow{\sim} H(A)\) the equivalence \(c_Q\) induces an equivalence

$$c : E \mathbb{R} \xrightarrow{\sim} H(A)$$

Definition 2.10 Any differential data \((E, A, c) \in \hat{\text{Sp}}\) which arises in the manner described above will be referred to as canonical differential data of \(E\).
For \(A_Q, B_Q \in \mathbb{N}(\text{Ch}_Q)[W^{-1}]\) we know that a map \(A_Q \to B_Q\) which induces the zero map in homology is equivalent to the zero map. This easily follows from the fact that \(A_Q\) and \(B_Q\) are equivalent to their homology complexes with trivial differential. This fact implies via the equivalence \(H\) that a map between rational spectra is equivalent to the zero map in homotopy. It follows that the canonical differential data \((E, A, c) \in \widehat{\text{Sp}}\) of \(E\) is unique up to homotopy. But one should be careful as the construction which associates to \(E\) the canonical differential data \((E, A, c)\) can not be turned into a functor even when considered with values in the homotopy category \(\text{Ho}(\widehat{\text{Sp}})\) of \(\widehat{\text{Sp}}\).

Let \(f : E \to E'\) be a map of spectra and \((E, A, c)\) and \((E', A', c')\) be the canonical data. Then we get an induced map \(f_* : A \to A'\). By the above there exists a homotopy \(H(f_*) \circ c_Q \sim c'_Q \circ (f \wedge M_Q)\). A choice of this homotopy \(\phi_Q\) induces a homotopy \(\phi : H(f_*) \circ c \sim c' \circ (f \wedge M_R)\) and therefore produces a map

\[
(f, f_*, \phi) : (E, A, c) \to (E', A', c')
\]

in \(\text{Ho}(\widehat{\text{Sp}})\).

But in general there is no way to choose this homotopy \(\phi_Q\) naturally. In general the group

\[
\pi_1(\text{map}(E_Q, H(A'_Q))) \cong \text{hom}(A_Q, A'_Q[-1]) \cong \prod_{n \in \mathbb{Z}} \text{hom}(A_{Q,n}, A'_{Q,n+1})
\]

acts simply transitively on the set of choices of this homotopy. If we apply this to \(\text{id} : E \to E\) we see that we can define the canonical datum only uniquely up to the automorphism group

\[
\prod_{n \in \mathbb{Z}} \text{hom}(A_{Q,n}, A_{Q,n+1})
\]

After fixing the choice of a canonical differential data this group acts by automorphisms on the functor \(\widehat{E^0}\). In the case of differential algebraic \(K\)-theory we describe the action in (75).

Let \(\mathbb{N}(\text{Sp}^{Q-\text{ev}})[W^{-1}] \subset \mathbb{N}(\text{Sp})[W^{-1}]\) be the full subcategory of rationally even spectra, i.e. spectra satisfying \(\pi_i(E) \otimes \mathbb{Q} = 0\) for all odd \(i \in \mathbb{Z}\). If \(E\) is rationally even, then \(\prod_{n \in \mathbb{Z}} \text{hom}(\pi_n(E_Q), \pi_{n+1}(E_RQ)) = 0\) and therefore the canonical differential data is well-defined in \(\text{Ho}(\widehat{\text{Sp}})\) up to unique isomorphism.

**Corollary 2.11** We have a functor

\[
\mathbb{N}(\text{Sp}^{Q-\text{ev}})[W^{-1}] \to \text{Ho}(\widehat{\text{Sp}})
\]

which associates to a rationally even spectrum the class of its canonical differential data. A similar statement holds true for rationally odd spectra.

This corollary complies with the results of [BS10]. Examples of rationally even spectra are \(H\mathbb{Z}\), the bordism spectra \(MBO(n)\) for \(n \geq 0\), \(MU\) and the topological \(K\)-theory...
spectra $KU$ and $KO$. The connected cover $KR(1)$ of the algebraic $K$-theory spectrum of a number ring is rationally odd by Theorem 3.3. Since $\pi_0(KRQ) = Q$ and, for example, $\dim_Q \pi_5(KRQ) > 0$ (Theorem 3.3), the algebraic $K$-theory spectrum of a number ring itself is neither rationally even nor odd. The group 

$$\text{hom}(\pi_0(KRQ), \pi_1(KRQ)) \cong K_1(R) \otimes Q$$

acts on $\widehat{KR}^0$ by automorphisms. An explicit formula for this action will be given in (75).

## 3 Cycle maps

### 3.1 Introduction

It often occurs in topology or differential topology that various structures are classified by invariants in a cohomology theory $E^*$. In this case one motivation to consider the differential extension $\widehat{E}^*$ is because it may capture invariants of additional geometric structures.

Complex vector bundles present a typical example. They are classified by their $KU$-theory classes, whereas differential $KU$-theory can be used to capture additional information about connections. As our definition of differential $KU$-theory in the present paper is homotopy theoretic, it requires some work to construct the refined geometric invariants. The result will be encoded into a cycle map.

As a warm-up, we begin with a discussion of the easy case of complex vector bundles and $KU$-theory. We then move on to one of the main results of this paper; namely, the construction of a cycle map for locally constant sheaves of finitely generated projective $R$-modules with geometry taking values in the differential version of the algebraic $K$-theory of $R$, where $R$ is the ring of integers in a number field (Definition 3.12 and Theorem 3.14).

### 3.2 Complex $K$-theory – a warm-up

We let $KU$ denote the complex $K$-theory spectrum. We use the canonical differential data (Definition 2.10) with

$$A := \pi_*(KU) \otimes \mathbb{R} \cong \mathbb{R}[b, b^{-1}]$$

where $b \in \pi_{-2}(KU)$ denotes the Bott element) in order to define the differential function spectrum $\text{Diff}(KU)$, see Definition 2.4. We then define the differential $K$-theory $\widehat{KU}^0(M)$ of a smooth manifold $M$ as in Definition 2.5.

A geometric vector bundle $V = (V, h^V, \nabla^V)$ over a manifold $M$ is by definition a complex vector bundle $V \to M$ together with a hermitean metric $h^V$ and a hermitean (i.e. metric preserving, see (33)) connection $\nabla^V$. We let

$$\text{Vect}_{h, \nabla} : \text{Mf}^{op} \to \text{Set}$$
be the smooth set which associates to a manifold $M$ the set of isomorphism classes of geometric vector bundles. We have maps of smooth sets

$$\hat{I} : \text{Vect}_{h,\nabla} \to KU^0, \quad \hat{R} : \text{Vect}_{h,\nabla} \to Z^0(\Omega A)$$

which sends a geometric vector bundle $V$ on $M$ to the $KU$-theory class $[V] \in KU^0(M)$ of the underlying complex vector bundle, and to the Chern character form

$$\text{ch}(\nabla V) := \text{Tr} \exp \left( -\frac{R^\nabla}{b 2\pi i} \right) \in Z^0(\Omega A(M)) , \quad (21)$$

where $R^\nabla \in \Omega^2(M, \text{End}(V))$ denotes the curvature of the connection $\nabla V$. In this situation we define the notion of a cycle map as follows.

**Definition 3.1** A cycle map is a map of smooth sets

$$\text{cycl} : \text{Vect}_{h,\nabla} \to \hat{KU}^0$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\text{Vect}_{h,\nabla} & \xrightarrow{\text{cycl}} & \hat{KU}^0 \\
\uparrow{\hat{I}} & & \downarrow{\hat{R}} \\
K^0 & & Z^0(\Omega A)
\end{array}$$

In order to connect with geometric approach to differential $KU$-theory developed in [BS09] we assume for the remainder of the present subsection that all smooth objects are defined on, or are restricted to the subcategory of compact manifolds $M_{f,\text{comp}} \subseteq M_{f}$. We will not write the restrictions explicitly. We have the following theorem.

**Theorem 3.2** There exists a unique cycle map $\text{cycl} : \text{Vect}_{h,\nabla} \to \hat{KU}^0$.

**Proof.** For the moment let $\hat{KU}_{BS}$ be the differential extension of complex $K$-theory defined by the model [BS09]. By construction, in this model differential $KU$-theory classes are represented by geometric families. In particular a geometric vector bundle, as a special kind of geometric family, represents a differential $KU$-theory class.

We now use the result of [BS10] which states that the functor $\hat{KU}^0$ is uniquely characterized by the axioms for a differential extension $(KU, I, R, a)$. We thus have a unique comparison isomorphism

$$\text{comp} : \hat{KU} \xrightarrow{\sim} \hat{KU}_{BS}.$$
For the model $\widehat{KU}_{BS}$ the cycle map is easy. It maps a geometric vector bundle $V$ on a manifold $M$ to the class $\text{cycl}_{BS}(V) := [V] \in \widehat{KU}_{BS}(M)$ it represents. Therefore

$$\text{cycl} := \text{comp}^{-1} \circ \text{cycl}_{BS} : \text{Vect}_{h,V} \to \widehat{KU}^0$$

is a cycle map.

We now show uniqueness. If $\text{cycl}'$ is a second cycle map, then in view of the exact sequence (13) the difference $\text{cycl} - \text{cycl}'$ factorizes as the composite

$$\delta : \text{Vect}_{h,V} \to KU_1 / KU^1 \to \widehat{KU}^0.$$

For every geometric vector bundle $V$ on a manifold $M$ there exists a map $f : M \to M'$ and a geometric vector bundle $V'$ on $M'$ such that $f^*V' \cong V$ and the map $f^* : H^{odd}(M'; \mathbb{Q}) \to H^{odd}(M; \mathbb{Q})$ vanishes. This implies that $f^* : KU_1(M') \to KU_1(M)$ vanishes and therefore

$$\delta(V) = \delta(f^*V') = f^*\delta(V') = 0.$$

For $M'$ one can take an approximation of the $\dim(M) + 1$-skeleton of the classifying space $BU$ such that a classifying map of $V$ factorizes over an embedding $f : M \to M'$. We refer to [Bun10a] for similar arguments.

It follows from [SS10] that the extension of the cycle map to virtual geometric vector bundles is surjective.

### 3.3 Algebraic and differential algebraic K-theory

Our main concern in this section is a cycle map for differential algebraic $K$-theory. In the present subsection we review some basic definitions and results about algebraic $K$-theory and fix the data for its differential extension.

We consider a number field $k$, i.e. a finite field extension $\mathbb{Q} \subseteq k$ of the field of rational numbers. We let $R \subseteq k$ denote the ring of integers in $k$, defined as the integral closure of $\mathbb{Z}$ in $k$. We refer to the comment at the end of Subsection 3.5 for the reason to stick to this special case. A typical example, relevant in Subsection 5.4.3, is the cyclotomic field $k := \mathbb{Q}(\xi)$ for an $r^{th}$ root of unity $\xi$ whose ring of integers is given by

$$R \cong \mathbb{Z}[\xi]/(1 + \xi + \cdots + \xi^{r-1}).$$

To a ring $R$ one can associate its $-1$-connective algebraic $K$-theory spectrum $KR$. There are various constructions of the algebraic $K$-theory spectrum $KR$. For example, one can form the spectrum associated to an infinite loop space $BGL(R)^+$ obtained by Quillen’s $+$-construction or group completion from the collection of classifying spaces $BGL(n,R)$. Alternatively one can apply Waldhausen’s $S_\bullet$-construction [Wal85] to the Waldhausen category of finitely generated projective $R$-modules. In the present paper we will take the approach via group completion. More details will be given in Subsection 3.4.
The algebraic $K$-groups of $R$ are then defined as the homotopy groups

$$K_i(R) := \pi_i(KR), \quad i \geq 0$$

of this spectrum. The first two groups, $K_0(R)$ and $K_1(R)$, have a simple algebraic description which is explained e.g. in the classical book of Milnor [Mil71]. The higher algebraic $K$-groups of a ring were first defined by Quillen [Qui73].

In the case of the ring of integers in a number field we have

$$K_0(R) \cong \mathbb{Z} \oplus \mathfrak{C}(R), \quad K_1(R) \cong R^*,$$

where $\mathfrak{C}(R)$ denotes the ideal class group of $R$ and $R^* \subset R$ is the group of multiplicative units. The ideal class group $\mathfrak{C}(R)$ is finite (see [Neu99, Thm. 6.3]), and the summand $\mathbb{Z}$ measures the rank of free $R$-modules. Dirichlet’s unit theorem describes the group of units $R^* \subset R$ as an extension of $\mathbb{Z}^{r_R + r_C - 1}$ by the finite group of roots of unity in $R$, where the integers $r_R$ and $2r_C$ denote the number of real and complex embeddings of $k$, respectively.

The higher algebraic $K$-groups are much more complicated. For the ring of integers of a number field the ranks of the abelian groups $K_i(R)$ are finite and have been calculated by Borel [Bor74, Prop. 12.2]. Recall that $\pi_i(KR) \cong \pi_i(KR) \otimes \mathbb{R}$. Together with the classical calculations for $i = 0, 1$ [Mil71] we have:

**Theorem 3.3 (Borel)** If $R$ is the ring of integers in a number field $k$, then $\dim_{\mathbb{R}}(\pi_i(KR))$ is four-periodic for $i \geq 2$ and given by the following table:

| $i$  | $j \geq 1$ | $4j - 2$ | $4j - 1$ | $4j$ | $4j + 1$ |
|------|------------|----------|----------|------|----------|
| $\dim_{\mathbb{R}}(\pi_i(KR))$ | 0         | $r_C$    | 0        | $r_R + r_C$ |

where $r_R$ and $2r_C$ are the numbers of real and complex embeddings of $k$. Furthermore, we have $\pi_0(KR) \cong \mathbb{R}$ and $\pi_1(KR) \cong R^* \otimes \mathbb{R}$, where $R^*$ is the finitely generated group of units of $R$ which has rank $r_R + r_C - 1$ by Dirichlet’s unit theorem.

A more precise statement (choice of a basis) will be given in Proposition 3.9. It is known by a result of Quillen that the groups $K_*(R)$ for the ring of integers of a number field are finitely generated.

**Definition 3.4** We define the differential algebraic $K$-theory $\widehat{KR}^0$ of a number ring $R$ using the canonical differential data $(KR, A, c)$ (Definition 2.10).

In greater detail, we take for $A \in \text{Ch}$ the complex of real vector spaces with trivial differentials and

$$A^{-i} := \pi_i(KR), \quad i \in \mathbb{Z},$$

and we let

$$c : KR \xrightarrow{\sim} H(A)$$

be a representative of the canonical map. With these choices the differential algebraic $K$-theory $\widehat{KR}^0(M)$ of a smooth manifold $M$ is defined according to Definition 2.5 up to the action of $K_1(R) \otimes \mathbb{Q}$ as discussed in Subsection 2.4 and Lemma 3.25.
3.4 KR-classes of locally constant sheaves of R-modules

A complex vector bundle $V \to X$ on a space $X$ can be considered as a cycle for a complex $K$-theory class $[V] \in KU^0(X)$. The cycle map Definition 3.1 refines this correspondence to geometric vector bundles and differential $K$-theory. Likewise, locally constant sheaves of finitely generated $R$-modules on $X$ provide cycles for algebraic $K$-theory classes of $M$. Again, we are looking for a cycle map which refines this to geometric locally constant sheaves and differential algebraic $K$-theory. Our notion of a geometry on a local system will be introduced in Definition 3.10.

There is a striking difference between topological complex $K$-theory $KU^*$ and the cohomology theory $KR^*$ represented by the algebraic $K$-theory spectrum $KR$. For a compact space $X$ the group $KU^0(X)$ can be defined in terms of complex vector bundles $V \to X$, and if one adopts this model of topological complex $K$-theory, the definition of the class $[V] \in KU^0(X)$ is a tautology. Likewise, one can define a model of differential $KU$-theory $\hat{K}U^0(M)$ of a compact manifold $M$ based on geometric vector bundles [SS10], and the cycle map is again tautological, see Subsection 3.2.

It is not possible to define $KR^0(X)$ as a Grothendieck group of locally constant sheaves of finitely generated $R$-modules on $X$ (but see Karoubi, [Kar87, Sec.3] for an approach using locally constant sheaves on spaces $Y$ with a homology equivalence $Y \to X$). Because of this, cycle maps for algebraic $K$-theory are far from being tautological.

We start with a description of the algebraic $K$-theory class $[V] \in KR^0(X)$ associated to a locally constant sheaf $V$ of finitely generated projective $R$-modules on a space $X$. Later in Subsection 3.10 we will drop the assumption “projective”.

To be precise note that, on the one hand, our $K$-theory spectrum $KR$ is a spectrum in simplicial sets. On the other hand, the locally constant sheaves live on topological spaces, e.g. the underlying spaces of smooth manifolds. We use the pairs of adjoint functors $\vdash : N(sSet)[W^{-1}] \leftrightarrows N(Top)[W^{-1}] : sing$, $\Sigma^\infty : N(sSet_*)[W^{-1}] \leftrightarrows N(Sp)[W^{-1}] : \Omega^\infty$ on the $\infty$-category level (see Subsection 6.4) to connect these categories. Here $\vdash$ and $\Sigma^\infty$ and $\Omega^\infty$ are the functors which map a pointed simplicial set to its suspension spectrum and a spectrum to its infinite loop space. By definition we have

$$KR^0(X) := \pi_0(\text{Map}(X, |\Omega^\infty KR|)) .$$

In order to relate this with locally constant sheaves of finitely generated projective $R$-modules on $X$ we will use the following explicit description of the algebraic $K$-theory spectrum $KR$.

For any ring $R$ the category $\text{Mod}(R)$ has a symmetric monodical structure given by the categorial product. We consider the symmetric monoidal subcategory $iP(R) \subseteq \text{Mod}(R)$ of finitely generated projective $R$-modules and isomorphisms (this is indicated by the letter $i$ in front of $P$). It is a symmetric monoidal groupoid, and hence its nerve is a commutative monoid in the $\infty$-category of simplicial sets:

$$N(iP(R)) \in \text{CommMon}(N(sSet)[W^{-1}]) .$$
The inclusion of the full subcategory of commutative groups into commutative monoids has as left-adjoint the group completion functor (see Subsection 6.3)

$$\Omega B : \text{CommMon}(\mathbb{N}(\text{sSet})[W^{-1}]) \leftrightarrows \text{CommGroups}(\mathbb{N}(\text{sSet})[W^{-1}]) . \quad (23)$$

Finally, we have the well-known equivalence (170)

$$\text{sp} : \text{CommGroups}(\mathbb{N}(\text{sSet})[W^{-1}]) \leftrightarrows \mathbb{N}(\text{Sp}_{\geq 0})[W^1] : \Omega^\infty$$

between the $\infty$-categories of commutative groups in $\mathbb{N}(\text{sSet})[W^{-1}]$ and connective spectra.

**Definition 3.5** The algebraic $K$-theory spectrum of the ring $R$ is defined as

$$KR := \text{sp}(\Omega B(\mathbb{N}(i\mathbb{P}(R)))) . \quad (25)$$

We refer to [Wei, Sec. IV] for a verification that $\pi_*(KR)$ is Quillen’s higher $K$-theory of the ring $R$.

The classifying space of the category $i\mathbb{P}(R)$ is given by

$$\mathbb{N}(i\mathbb{P}(R)) \in \text{Top} .$$

The set $\pi_0(\mathbb{N}(i\mathbb{P}(R)))$ is in canonical bijection with the set of isomorphism classes $[P]$ of objects of $\mathbb{P}(R)$, and we have an equivalence

$$\mathbb{N}(i\mathbb{P}(R)) \cong \bigsqcup_{[P] \in \pi_0(\mathbb{N}(\mathbb{P}(R)))} B\text{Aut}(P) , \quad (26)$$

where for a discrete group $G$ we write $BG := |\mathbb{N}(G)|$ for the classifying space of the associated groupoid $\tilde{G}$ with one object and automorphism group $G$.

We consider the functor $i\mathbb{P}(R) \to \text{Set}$ which maps each finitely generated projective $R$-module to its underlying set and form the Grothendieck construction $\mathbb{P}(R) \to i\mathbb{P}(R)$. Recall that an object of $\mathbb{P}(R)$ is a pair $(x \in P)$ of an object $P \in \mathbb{P}(R)$ and the choice of an element $x \in P$. The sheaf of sections of the induced map $|\mathbb{N}(\tilde{\mathbb{P}}(R))| \to |\mathbb{N}(i\mathbb{P}(R))|$ is a locally constant sheaf

$$\mathcal{V}_{\text{univ}} \in \text{Sh}_{\text{Mod}(R)}(|\mathbb{N}(i\mathbb{P}(R))|)$$

of finitely generated projective $R$-modules.

The space $|\mathbb{N}(i\mathbb{P}(R))|$ is the classifying space for locally constant sheaves of finitely generated projective $R$-modules. Since geometric realization preserves products we can consider

$$|\mathbb{N}(i\mathbb{P}(R))| \in \text{CommMon}(\mathbb{N}(\text{Top})[W^{-1}]) .$$

Let $\text{Loc}^{\text{proj}}(X)$ denote the of isomorphism classes of locally constant sheaves of finitely generated projective $R$-modules on a $CW$-complex $X$. Then we have a natural isomorphism

$$\pi_0(|\text{Map}(X,|\mathbb{N}(i\mathbb{P}(R))|)|) \xrightarrow{\sim} \text{Loc}^{\text{proj}}(X) , \quad [f] \mapsto [f^*\mathcal{V}_{\text{univ}}] . \quad (28)$$

Note that $\text{Loc}^{\text{proj}}(X)$ is a monoid with respect to the direct sum and (28) is additive.
Definition 3.6  We define the smooth monoids
\[ \Loc^{\text{free}} \subseteq \Loc^{\text{proj}} \subseteq \Loc \subseteq \Sm(N(\Mon)) \]
as the functors which associates to a manifold \( M \) the monoids of isomorphism classes of locally constant sheaves of finitely generated free (respectively, finitely generated projective or just finitely generated) \( R \)-modules on \( M \), and to \( f : M' \to M \) the pull-back \( \mathcal{V} \mapsto f^*\mathcal{V} \).

In particular, from (28) we get an isomorphism of smooth monoids
\[ \pi_0(\Sm_\infty(|N(i\mathcal{P}(R))|)) \xrightarrow{\sim} \Loc^{\text{proj}} . \] (29)
The composition of the units of the adjunctions (23) and (24) in view of the definition (25) provides a map of topological monoids
\[ |N(i\mathcal{P}(R))| \to |\Omega B(N(i\mathcal{P}(R)))| \to |\Omega^\infty KR| . \] (30)

Definition 3.7  The composition of the inverse of (29) with the map of smooth sets induced by (30) defines the transformation of smooth monoids
\[ \hat{I} : \Loc^{\text{proj}} \to \pi_0(\Sm_\infty(|\Omega^\infty KR|)) \cong KR^0 . \] (31)

Often we write \([\mathcal{V}] := \hat{I}(\mathcal{V})\). In Subsection 3.10 we extend the transformation \( \hat{I} \) to all of \( \Loc \).

3.5 Geometric local systems and characteristic forms

Given a complex vector bundle \( V \to M \) on a smooth manifold, a hermitean metric \( h^V \) and a hermitean connection \( \nabla^V \) can be used to define a Chern character form \( \text{ch}(\nabla^V) \in Z^0(\Omega A(M)) \) (see (21) and \( A \) is as in (20)). The Chern character form \( \text{ch}(\nabla^V) \) represents the image of the class \([V] \in K^0(M)\) under the map
\[ K^0(M) \to KR^0(M) \xrightarrow{\sim} H(A)^0(M) \xrightarrow{j_!} H^0(\Omega A(M)) . \]
The main goal of the present subsection is the analog for algebraic \( K \)-theory. In Definition 3.10 we introduce the notion of a geometry on a locally constant sheaf \( \mathcal{V} \) of finitely generated \( R \)-modules. A geometry determines a characteristic form (42) which represents the image of \([\mathcal{V}] \in KR^0(M)\) under
\[ KR^0(M) \to KR^0(M) \xrightarrow{\sim} H(A)^0(M) \xleftarrow{j_!} H^0(\Omega A(M)) , \] (32)
where \( A \) is from now on as in (22).

Let \( \Sigma \) denote the set of embeddings \( R \hookrightarrow \mathbb{C} \). The group \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \Sigma \) by complex conjugation. The set \( \Sigma \) equivariantly decomposes as the disjoint union \( \Sigma = \Sigma_\mathbb{R} \sqcup \Sigma_\mathbb{C} \) of
the subsets of real and complex embeddings, where $\Sigma_\mathbb{R} = \Sigma_{\mathbb{Z}/2\mathbb{Z}} \subseteq \Sigma$ is the subset of fixed points. Note that $r_\mathbb{R} = |\Sigma_\mathbb{R}|$, $r_\mathbb{C} = |\Sigma_\mathbb{C}/(\mathbb{Z}/2\mathbb{Z})|$.

The set of orbits $\bar{\Sigma} := \Sigma/(\mathbb{Z}/2\mathbb{Z})$ is called the set of places of $R$. We fix a set of representatives $\Sigma^* \subseteq \Sigma$ of the set of places.

Let $M$ be a smooth manifold. We consider a locally constant sheaf $\mathcal{V}$ of finitely generated $R$-modules. For an embedding $\sigma \in \Sigma$ we can form the locally constant sheaf of finite-dimensional complex vector spaces $\mathcal{V}_\sigma := \mathcal{V} \otimes_\sigma \mathbb{C}$. This sheaf is the sheaf of parallel sections of a uniquely determined complex vector bundle $V_\sigma \to M$ with a flat connection $\nabla^{V_\sigma}$. This vector bundle will be referred to as the complex vector bundle associated to $\mathcal{V}$ and $\sigma$.

We fix a complex vector bundle with a flat connection $(V, \nabla^V)$ on a smooth manifold $M$. A reference for the following constructions is [BL95]. Let us choose a hermitean metric $h^V$ on $V$. Using the metric we define the adjoint connection $\nabla^{V,*hV}$. It is characterized by the identity

$$dh^V(\phi, \psi) = h^V(\nabla^V \phi, \psi) + h^V(\phi, \nabla^{V,*hV} \psi)$$

for sections $\phi, \psi$ of $V$. The metric $h^V$ is called flat (or equivalently, $\nabla^V$ is hermitean with respect to $h^V$) if

$$\nabla^{V,*hV} = \nabla^V.$$  \tag{33}

In general it is not possible to choose a flat metric on $V$. The deviation from flatness of $h^V$ is measured by the form

$$\omega(h^V) := \nabla^{V,*hV} - \nabla^V \in \Omega^1(M, \text{End}(V)).$$  \tag{34}

This form can be used to define the Kamber-Tondeur forms by

$$\omega_{2j+1}(h^V) := \frac{1}{(2\pi i)^{j+1}} \text{Tr} \omega(h^V)^{2j+1} \in Z^{2j+1}(\Omega(M)) $$  \tag{35}

for all $j \geq 0$. One can check that the Kamber-Tondeur forms are real and closed. The cohomology class

$$[\omega_{2j+1}(h^V)] \in H^{2j+1}(M; \mathbb{R})$$

does not depend on the choice of the metric $h^V$. The formula (35) defines Kamber-Tondeur forms in the Bismut-Lott normalization which is dictated by the local index theory methods of [BL95] culminating in the construction of higher analytic torsion forms. The relation of the Kamber-Tondeur forms with the Borel regulator will be explained in Subsection 6.14. In Subsection 6.15 we compare various different normalizations appearing in the literature.

Let $h^{V_\sigma}$ be a hermitean metric on the flat bundle $V_\sigma$ associated to $\mathcal{V}$ and $\sigma$. Then we define

$$\omega_{2j+1}(\mathcal{V}, \sigma) := [\omega_{2j+1}(h^{V_\sigma})] \in H^{2j+1}(M; \mathbb{R}).$$

In the following we record some properties of these classes which follow immediately from the definition. For a finitely generated $R$-module $L$ we let $L^M$ denote the associated constant sheaf on $M$.  

26
Proposition 3.8  

1. Naturality: If \( f : M' \to M \) is a smooth map, then we have
\[
f^* \omega_{2j+1}(V, \sigma) = \omega_{2j+1}(f^*V, \sigma).
\]

2. Additivity: For two locally constant sheaves \( V, V' \) of finitely generated \( R \)-modules on \( M \) we have the equality
\[
\omega_{2j+1}(V \oplus V', \sigma) = \omega_{2j+1}(V, \sigma) + \omega_{2j+1}(V', \sigma).
\]

3. Stability: If \( L \) is a finitely generated \( R \)-module, then
\[
\omega_{2j+1}(L, \sigma) = 0
\]
since we can choose a flat metric on the associated bundle.

We call two locally constant sheaves \( V_0, V_1 \) of finitely generated \( R \)-modules on a connected manifold \( M \) stably equivalent if there exist finitely generated projective \( R \)-modules \( L_0, L_1 \) and an isomorphism
\[
V_0 \oplus L_{0M} \cong V_1 \oplus L_{1M}.
\]
For a general manifold we define the notion componentwise. It follows that the association
\[
V \mapsto \omega_{2j+1}(V, \sigma)
\]
is a characteristic class of stable equivalence classes of locally constant sheaves of finitely generated \( R \)-modules.

Note that \( GL(n, R) = \text{Aut}(R^n) \). We identify \( BGL(n, R) \) with the component in the disjoint union [26] corresponding to \( P = R^n \). We write
\[
V_{\text{univ}}(n) := (V_{\text{univ}})|_{BGL(n, R)}
\]
for the restriction of the universal locally constant sheaf [27]. The space \( BGL(n, R) \) gives rise to the smooth set
\[
\pi_0(\text{Sm}_\infty(BGL(n, R))) \in \text{Sm}(\mathbb{N}(\text{Set}))
\]
which associates to a manifold \( M \) the set of homotopy classes \( \pi_0(\text{Map}(M, BGL(n, R))) \) of continuous maps from \( M \) to \( BGL(n, R) \).

Let \( \text{Loc}^{\text{free}}(n) \in \text{Sm}(\mathbb{N}(\text{Set})) \) denote the smooth subset of \( \text{Loc}^{\text{proj}} \) which associates to a manifold \( M \) the set isomorphism classes of locally constant sheaves of free \( R \)-modules of rank \( n \). By restriction of [29] we get an isomorphism of smooth sets
\[
\pi_0(\text{Sm}_\infty(BGL(n, R))) \simto \text{Loc}^{\text{free}}(n).
\]

We have another smooth set \( H^{2j+1} \in \text{Sm}(\mathbb{N}(\text{Set})) \) which associates to a manifold \( M \in \text{Mf} \) the real cohomology \( H^{2j+1}(M; \mathbb{R}) \) of \( M \) in degree \( 2j+1 \). The restriction of our
characteristic class to locally constant sheaves of free $R$-modules of rank $n$ thus induces a map of smooth sets

$$\omega_{2j+1}(?, \sigma) : \pi_0(\text{Sm}_\infty(BGL(n, R))) \to H^R_{2j+1}.$$ 

By Proposition 6.27 it determines a universal class

$$\omega'_{2j+1}(n, \sigma) \in H^{2j+1}(BGL(n, R); R).$$

Stability (37) and additivity (36) implies that

$$\omega'_{2j+1}(n + 1, \sigma)|_{BGL(n, R)} = \omega'_{2j+1}(n + 1, \sigma).$$

Since by the stability result of Borel [Bor74] we have

$$H^{2j+1}(BGL(R); R) \cong \lim_{n} H^{2j+1}(BGL(n, R); R)$$

we get a class

$$\omega'_{2j+1}(\sigma) \in H^{2j+1}(BGL(R); R)$$

which restricts to $\omega'_{2j+1}(n, \sigma)$ for all $n$.

As a consequence of (36) the class $\omega'_{2j+1}(\sigma)$ is primitive, i.e. it satisfies the identity

$$\mu^* \omega'_{2j+1}(\sigma) = \omega'_{2j+1}(\sigma) \boxtimes 1 + 1 \boxtimes \omega'_{2j+1}(\sigma)$$

where

$$\mu : BGL(R) \times BGL(R) \to BGL(R)$$

is the $h$-space structure corresponding to the direct sum locally constant sheaves, and $\boxtimes$ denotes the exterior product in cohomology. Since the natural map

$$p : BGL(R) \to BGL(R)^+ \cong |\Omega^\infty KR(0)|$$

(here $X\langle n \rangle$ denotes the $n$-connected cover of a space $X$) is a cohomology equivalence of $h$-spaces we get a corresponding primitive class in

$$\omega''_{2j+1}(\sigma) \in H^{2j+1}(|\Omega^\infty KR(0)|; R)$$

such that $p^* \omega''_{2j+1}(\sigma) = \omega'_{2j+1}(\sigma)$. We let

$$\omega''_{2j+1}(\sigma) \in H^{2j+1}(|\Omega^\infty KR|; R)$$

denote the pull-back of $\omega''_{2j+1}(\sigma)$ along the map

$$|\Omega^\infty KR| \cong K_0(R) \times |\Omega^\infty KR(0)| \to |\Omega^\infty KR(0)|$$

given by the projection to the second factor. The restriction map

$$H^R_* (KR) \to H^*(|\Omega^\infty KR|; R)$$
from the real cohomology of the spectrum $KR$ to the real cohomology of its infinite loop space is injective, and its image consists exactly of the primitive elements. Since $\omega''_{2j+1}(\sigma)$ is primitive it is the restriction of a well-defined spectrum cohomology class

$$\omega_{2j+1}(\sigma) \in H\mathbb{R}^{2j+1}(KR). \quad (39)$$

A more precise version of the calculation of Borel, Theorem 3.3 is now given by the following proposition.

**Proposition 3.9** Let $R$ be the ring of Borel, Theorem 3.3 is now given by the following proposition.

**Proposition 3.9** Let $R$ be the ring of Borel, Theorem 3.3 is now given by the following proposition.

**Proposition 3.9** Let $R$ be the ring of Borel, Theorem 3.3 is now given by the following proposition.
Definition 3.10 A geometry on a local system \( V \) of finitely generated \( R \)-modules is the choice of a family \( h^V := (h^V_\sigma)_{\sigma \in \Sigma} \) of hermitean metrics on the bundles \( V_\sigma \) such that
\[
h^V_\sigma = \overline{h^V_\sigma}
\]
for the complex places of \( k \). Given a geometry on \( V \) we define the characteristic form
\[
\omega(h^V) := b_0 \dim_V + \sum_{j \geq 0} \sum_{\sigma \in \Sigma} \frac{1}{\text{Stab}_{\mathbb{Z}/2\mathbb{Z}}(\sigma)} b_{2j+1}(\sigma) \omega_{2j+1}(h^V_\sigma) \in \mathbb{Z}^h(\Omega A(M))
\]
where \( \dim_V : M \rightarrow \mathbb{Z} \) is the locally constant dimension function of \( V \).

Let us now explain the reason why we restrict our attention to the ring of integers of a number field. As we are interested in the cycle map we want a geometric description of the generators of \( H^R_\ast(KR) \) as presented above. Therefore we should look at rings \( R \) with the property that these classes can be pulled back via a collection of maps \( R \rightarrow \mathbb{C} \) from secondary characteristic classes of flat complex vector bundles. Most of the theory would work for rings \( R' \) such that \( R \subseteq R' \subseteq k \) as long as the induced maps \( K_i(R) \rightarrow K_i(R') \) are rational isomorphism. If \( R' \) is a localization of \( R \), then this is indeed the case for \( i = 0 \) and \( i \geq 2 \) by the localization sequence for higher \( K \)-theory (see Weibel’s \( K \)-book project [Wei], Ch.V.6). The problem is caused by the units, i.e. the case \( i = 1 \). In general the rank of \( K_1(R') \) is too large so that we do not have a nice geometric description of the generators of \( H^R_1(KR') \). An exception is a ring of the form \( R' = R^\ell \), the localization of \( R \) away from some prime ideal \( \ell \subset R \). This case will be used in the proof of Proposition 5.24.

3.6 The cycle map for geometric locally constant sheaves

The goal of the present Subsection is to define the notion of a cycle map for geometric locally constant sheaves of finitely generated projective \( R \)-modules with values differential algebraic \( K \)-theory.

Definition 3.11 We define
\[
\underline{\text{Loc}}_{\text{geom}}^{\text{free}} \subseteq \underline{\text{Loc}}_{\text{geom}}^{\text{proj}} \subseteq \underline{\text{Loc}}_{\text{geom}} \in \text{Sm}(\mathbb{N}(\text{Mon}(\text{Set})))
\]
to be the smooth monoids which associate to a manifold \( M \) the monoids (with respect to direct sum) of isomorphism classes of pairs \((\mathcal{V}, h^\mathcal{V})\) of locally constant sheaves of finitely generated free (projective or without any condition, respectively) \( R \)-modules with geometry (Definition 3.10).

We have maps of smooth monoids
\[
\tilde{I} : \underline{\text{Loc}}_{\text{geom}} \rightarrow KR^0, \quad \tilde{R} : \underline{\text{Loc}}_{\text{geom}} \rightarrow Z^0(\Omega A)
\]
which associate to \((\mathcal{V}, h^\mathcal{V}) \in \underline{\text{Loc}}_{\text{geom}}(M)\) the underlying class \( \tilde{I}(\mathcal{V}) \in KR^0(M) \) (see Definition 3.7 for the projective and Subsection 3.10 for the general case) and the characteristic form \( \omega(h^\mathcal{V}) \) given in (42).

30
Definition 3.12 A cycle map for geometric locally constant sheaves of finitely generated projective \( R \)-modules with values differential algebraic \( K \)-theory is a map of smooth monoids

\[
\text{cycl} : \overline{\text{Loc}_{\text{geom}}} \to \widehat{KR}^0
\]

such that the diagram

\[
\begin{array}{ccc}
\overline{\text{Loc}_{\text{geom}}} & \xrightarrow{\text{cycl}} & \overline{KR}^0 \\
\downarrow i & & \downarrow i \\
R \ & \to \ & Z^0(\Omega A)
\end{array}
\]

commutes.

There are analogous notions in the cases where \( \overline{\text{Loc}_{\text{geom}}} \) replaced by its smooth submonoids \( \overline{\text{Loc}_{\text{free}}} \) or \( \overline{\text{Loc}_{\text{proj}}} \).

We will construct the cycle map in three stages. We start with the free case.

Proposition 3.13 There exists a cycle map for geometric locally constant sheaves of finitely generated free \( R \)-modules with values in differential algebraic \( K \)-theory.

The proof of this proposition will be given in Subsection 3.8. It turns out to be quite technical. In Subsection 3.7 we construct an extension of Proposition 3.13 from the free to the projective case. Finally, in Subsection 3.10 we provide the general case. The existence of this cycle map is one of the major results of the present paper.

Theorem 3.14 There exists a cycle map for geometric locally constant sheaves of finitely generated \( R \)-modules with values in differential algebraic \( K \)-theory.

Note that Theorem 3.14 only asserts the existence of a cycle map. The construction of the cycle map in the proof of Proposition 3.13 actually depends on the choice of the homotopy \( \sigma_\mathbb{Q} \) in (73). It is clear that the cycle map can be altered using the action of \( K_1(R) \otimes \mathbb{Q} \) by automorphisms of \( \widehat{KR}^0 \) (see Lemma 3.25). For a similar reason, the extension of the cycle map from the free to the projective case is not unique in general. It can be changed by changing the choices of the lifts (49).

In [BT] we provide, using different methods, a canonical cycle map which is functorial in the ring. But note that the version of differential algebraic \( K \)-theory in [BT] contains the differential algebraic \( K \)-theory of the present paper as a subgroup which is, in general, proper. The reason is that [BT] uses a much larger complex of forms.
3.7 Extension from free to projective modules

In this subsection we extend the cycle map from the free to the projective case assuming the existence of a cycle map Proposition 3.13 and the Lemmas 3.26 and 3.27 in the free case. The idea for the extension of the cycle map from the free to the projective case is to compose the cycle map in the free case with a left-inverse of the inclusion \( \text{Loc}_{\text{geom}}^{\text{free}} \hookrightarrow \text{Loc}_{\text{geom}}^{\text{proj}} \) and to correct the result suitably.

If \( R \) is a ring of class number one, then every finitely generated projective \( R \)-module is free and the natural map \( \mathbb{Z} \rightarrow K_0(R) \), given by sending the generator \( 1 \in \mathbb{Z} \) to \( [R] \in K_0(R) \), is an isomorphism. In general this map is not surjective and we have a canonical decomposition \([\text{Mil71, Cor. 1.11}]\)

\[
K_0(R) \cong \mathbb{Z} \oplus \text{Cl}(R), \quad x \mapsto \dim(x) + \text{cl}(x),
\]

(44)

where \( \text{Cl}(R) \) is called the ideal class group of \( R \). It is known that the ideal class group is finite \([\text{Neu99, Ch.1.6}]\). The ideal class group of \( R \) coincides with the Picard group \( \text{Pic}(R) \) of the ring \( R \), which can be defined in more general situations.

The class group \( \text{Cl}(R) \) considered as a discrete space represents a smooth set

\[
\text{Sm}(\text{Cl}(R)) \in \text{Sm}(\mathbb{N}(\text{Set}))
\]

such that

\[
\text{Sm}(\text{Cl}(R))(M) = \{ f : M \rightarrow \text{Cl}(R) \}
\]

is the group of continuous (hence locally constant) maps from \( M \) to \( \text{Cl}(R) \). We can also consider \( \text{Cl}(R) \) as a zero-dimensional manifold. Then \( \text{id}_{\text{Cl}(R)} \in \text{Sm}(\text{Cl}(R))(\text{Cl}(R)) \) is a preferred element.

We have a map of smooth sets

\[
\text{cl} : KR^0 \rightarrow \text{Sm}(\text{Cl}(R))
\]

defined as follows. If \( x \in KR^0(M) \), then \( \text{cl}(x) : M \rightarrow \text{Cl}(R) \) is given by \( \text{cl}(x)(m) := \text{Cl}(x|_{\{m\}}) \), where \( x|_{\{m\}} \in KR^0(\{\ast\}) \cong K_0(R) \) is the restriction of \( x \) along the embedding of the point \( m : \ast \rightarrow M \), and \( \text{cl}(x|_{\{m\}}) \in \text{Cl}(R) \) is the second component of \( x|_{\{m\}} \) as in \((44)\).

Similarly, given a locally constant sheaf of finitely generated projective \( R \)-modules \( \mathcal{V} \in \text{Loc}^{\text{proj}}(M) \), we get a smooth map \( \text{cl}(\mathcal{V}) : M \rightarrow \text{Cl}(R) \) which maps the point \( m \in M \) to the component \( \text{cl}(\mathcal{V})(m) \in \text{Cl}(R) \) of the \( K \)-theory class \( [\mathcal{V}_m] \in K_0(R) \) of the fibre \( \mathcal{V}_m \).

We can interpret this construction as a map of smooth spaces:

\[
\text{cl} : \text{Loc}^{\text{proj}} \rightarrow \text{Sm}(\text{Cl}(R)).
\]

(45)

Let us consider \( \text{Cl}(R) \) as a zero dimensional manifold and group with multiplication \( \mu : \text{Cl}(R) \times \text{Cl}(R) \rightarrow \text{Cl}(R) \).
Lemma 3.15  There exists a class $\hat{x}_\bullet \in \hat{KR}_0^0(\text{Cl}(R))$ such that

$$\mu^* \hat{x}_\bullet = \text{pr}_1^* \hat{x}_\bullet + \text{pr}_2^* \hat{x}_\bullet$$  \hspace{1cm} (46)

in $\hat{KR}_0^0(\text{Cl}(R) \times \text{Cl}(R))$ and

$$c_1(I(\hat{x}_\bullet)) = \text{id}_{\text{Cl}(R)}$$

in $\text{Sm}(\text{Cl}(R))(\text{Cl}(R))$.

Proof. We first observe that there is a class $x_\bullet \in KR_0(\text{Cl}(R))$ such that $\mu^* x_\bullet = \text{pr}_1^* x_\bullet + \text{pr}_2^* x_\bullet$ and $c_1(x_\bullet) = \text{id}_{\text{Cl}(R)}$. The class $x_\bullet$ is the class represented by the map

$$\text{Cl}(R) \xrightarrow{x \mapsto 0} \mathbb{Z} \oplus \text{Cl}(R) \cong K_0(R) \overset{\text{bp}}{\to} |\Omega^\infty KR|,$$

where the last map $\text{bp}$ is a section of the canonical map $|\Omega^\infty KR| \to \pi_0(|\Omega^\infty KR|) = K_0(R)$. The differential algebraic $K$-theory of the zero-dimensional manifold $\text{Cl}(R)$ fits into an exact sequence (a special case of (13))

$$0 \to K_1(\text{Cl}(R)) \otimes \mathbb{R}/\mathbb{Z} \to \hat{KR}_0^0(\text{Cl}(R)) \overset{I}{\to} K_0(\text{Cl}(R)) \to 0.$$

Therefore we can find a lift of $x_\bullet$ to $\hat{KR}_0^0(\text{Cl}(R))$. But we must choose this lift such that (46) is satisfied. This is done as follows. Since $\text{Cl}(R)$ is a finite abelian group, we can choose a collection of generators $(g_i)_{i=1}^l \in \text{Cl}(R)$ of orders $n_i := \text{ord}(g_i) \in \mathbb{N}$ such that

$$\text{Cl}(R) \cong \bigoplus_{i=1}^l \mathbb{Z}g_i \cong \bigoplus_{i=1}^l \mathbb{Z}/n_i \mathbb{Z}.$$

We have a canonical decomposition

$$\hat{KR}_0^0(\text{Cl}(R)) \cong \bigoplus_{g \in \text{Cl}(R)} \hat{KR}_0^0(\{g\}).$$

We first choose lifts

$$\hat{x}_{g_i} \in \hat{KR}_0^0(\{g_i\})$$

of the restrictions $(x_\bullet)_{\{g_i\}}$ of $x_\bullet$ to the generators for all $i = 1, \ldots, l$ such that $n_i \hat{x}_{g_i} = 0$. This is possible by adjusting an arbitrary choice of a lift by an element in the torus $K_1(R) \otimes \mathbb{R}/\mathbb{Z} \subset \hat{KR}_0^0(\{g_i\})$. Let now $g \in \text{Cl}(R)$ be presented as $g = \bigoplus_{i=1}^l p_i g_i$. Then we define

$$\hat{x}_g := \sum_{i=1}^l p_i \hat{x}_{g_i}$$

using the canonical identifications

$$\hat{KR}_0^0(\{g\}) \cong \hat{KR}_0^0(\ast), \hspace{1cm} g \in \text{Cl}(R).$$
This element is well-defined independent of the choice of the presentation of \( g \). The sum of these elements

\[
\hat{x}_* := \bigoplus_{g \in \text{Cl}(R)} \hat{x}_g \in \bigoplus_{g \in \text{Cl}(R)} \hat{K}R^0(\{g\}) \cong \hat{K}R^0(\text{Cl}(R))
\]

then has the required properties. \( \square \)

Because of (46) the universal classes \( \hat{x}_* \in \hat{K}R^0(\text{Cl}(R)) \) and \( x_* = I(\hat{x}_*) \in K R^0(\text{Cl}(R)) \) constructed in Lemma 3.15 induce homomorphisms of smooth groups

\[
U : \text{Sm}(\text{Cl}(R)) \to K R^0, \quad \hat{U} : \text{Sm}(\text{Cl}(R)) \to \hat{K} R^0
\]

which associate to the smooth map \( f : M \to \text{Cl}(R) \) the classes

\[
U(f) := f^* x_* \in K R^0(M), \quad \hat{U}(f) := f^* \hat{x}_* \in \hat{K} R^0(M).
\]

Again, since \( \text{Cl}(R) \) is a zero-dimensional manifold, we have canonical decompositions

\[
\text{Loc}_{\text{proj}}(\text{Cl}(R)) \cong \prod_{g \in \text{Cl}(R)} \text{Loc}_{\text{proj}}(\{g\}), \quad \text{Loc}_{\text{geom}}(\text{Cl}(R)) \cong \prod_{g \in \text{Cl}(R)} \text{Loc}_{\text{geom}}(\{g\}).
\]

For each element \( g \in \text{Cl}(R) \) we choose a representative \( V_g \subset R \) of the ideal class \(-g\) (note the minus sign). This gives a universal element

\[
\mathcal{V}_* := (V_g)_{g \in \text{Cl}(R)} \in \prod_{g \in \text{Cl}(R)} \text{Loc}_{\text{proj}}(\{g\}) \cong \text{Loc}_{\text{proj}}(\text{Cl}(R)).
\]

Note that

\[
\hat{I}(\mathcal{V}_*) - 1 = -x_* . \tag{50}
\]

For every generator \( g_i \in \text{Cl}(R), i = 1, \ldots, l \) we equip the \( R \)-module \( V_{g_i} \) with a geometry \( h^{V_{g_i}} \) such that

\[
\text{cycl} \left( (V_{g_i}, h^{V_{g_i}}) \otimes n_i \right) = 1_*, \quad \tag{51}
\]

where \( 1_* : = \text{cycl}(R, h^R) \) with the canonical geometry \( h^R \) on \( R \). Note that \( V_{g_i} \otimes n_i \) is free so that \( \text{cycl} \left( (V_{g_i}, h^{V_{g_i}}) \otimes n_i \right) \) is well-defined. The vanishing condition (51) can be matched by scaling the standard geometry on \( V_{g_i} \) appropriately, using Lemma 3.27 in the free case.

A general element \( g \in \text{Cl}(R) \) can be represented uniquely in the form \( g = \sum_{i=1}^l p_i g_i \) with \( p_i \in \{0, \ldots, n_i - 1\} \). We define

\[
(V_g, h^{V_g}) := \bigotimes_{i=1}^l (V_{g_i}, h^{V_{g_i}}) \otimes p_i \in \text{Loc}_{\text{geom}}(\{g\}) \tag{52}
\]

for all \( g \in \text{Cl}(R) \) and hence an element

\[
(\mathcal{V}_*, h^{\mathcal{V}_*}) := (V_g, h^{V_g})_{g \in \text{Cl}(R)} \in \prod_{g \in \text{Cl}(R)} \text{Loc}_{\text{geom}}(\{g\}) \cong \text{Loc}_{\text{geom}}(\text{Cl}(R)).
\]
The universal elements $\mathcal{V}_\bullet$ and $(\mathcal{V}_\bullet, h^{\mathcal{V}_\bullet})$ define maps of smooth sets

$$T : \text{Sm}(\text{Cl}(R)) \to \text{Loc}_{\text{geom}}^{\text{proj}}, \quad \hat{T} : \text{Sm}(\text{Cl}(R)) \to \text{Loc}_{\text{geom}}^{\text{proj}}$$

which associate to $f \in \text{Sm}(\text{Cl}(R))(M)$ given by a continuous (and hence smooth map) $f : M \to \text{Cl}(R)$ the elements

$$T(f) := f^*\mathcal{V}_\bullet \in \text{Loc}_{\text{geom}}^{\text{proj}}(M), \quad \hat{T}(f) := f^*(\mathcal{V}_\bullet, h^{\mathcal{V}_\bullet}) \in \text{Loc}_{\text{geom}}^{\text{proj}}(M),$$

respectively. The relation $V_{g+h} \oplus V_0 \cong V_g \oplus V_h$ for $g, h \in \text{Cl}(R)$ yields

$$\mu^*(\mathcal{V}_\bullet) \oplus R_{\text{Cl}(R)} \cong \text{pr}_1^*\mathcal{V}_\bullet + \text{pr}_2^*\mathcal{V}_\bullet.$$

This implies that

$$T(f + g) \oplus T(0_M) = T(f) \oplus T(g), \quad (53)$$

where $0_M \in \text{Sm}(\text{Cl}(R))(M)$ is the zero element. For $f, g \in \text{Sm}(\text{Cl}(R))(M)$ the elements

$$\hat{T}(f) \oplus \hat{T}(-f), \quad \hat{T}(f + g) \oplus \hat{T}(-f) \oplus \hat{T}(-g), \quad \hat{T}(0_M)$$

belong to the subset $\text{Loc}_{\text{geom}}^{\text{free}}(M) \subseteq \text{Loc}_{\text{geom}}^{\text{proj}}(M)$ so that we can apply to them the cycle map whose existence is claimed in Proposition 3.13. We define $1_M := \text{cycl}(T(0_M)) = \text{cycl}(R_M, h^{\text{R}M})$ to be the differential algebraic $K$-theory class given by the constant sheaf generated by $R$ with its canonical geometry $h^{\text{R}M}$.

**Lemma 3.16** For $f, g \in \text{Sm}(\text{Cl}(R))(M)$ we have the relations

$$\text{cycl} \left( \hat{T}(f) \oplus \hat{T}(-f) \right) = 2 \cdot 1_M, \quad \text{cycl} \left( \hat{T}(f + g) \oplus \hat{T}(-f) \oplus \hat{T}(-g) \right) = 3 \cdot 1_M.$$

**Proof.** The first assertion follows from the second by setting $g := -f$. Since the cycle map is natural it suffices to show the second equality in the universal case. We thus must verify that

$$\text{cycl} \left( \mu^*(\mathcal{V}_\bullet, h^{\mathcal{V}_\bullet}) \oplus -\text{pr}_1^*(\mathcal{V}_\bullet, h^{\mathcal{V}_\bullet}) \oplus -\text{pr}_2^*(\mathcal{V}_\bullet, h^{\mathcal{V}_\bullet}) \right) = 3 \cdot 1_{\text{Cl}(R) \times \text{Cl}(R)} \in \widetilde{K}^0(\text{Cl}(R) \times \text{Cl}(R)).$$

Using the decomposition [48] this equality boils down to

$$\text{cycl} \left( (V_{f+g}, h^{V_{f+g}}) \oplus (V_{-f}, h^{V_{-f}}) \oplus (V_{-g}, h^{V_{-g}}) \right) = 3 \cdot 1, \in \widetilde{K}^0(\text{Cl}(R) \times \text{Cl}(R)) \quad (54)$$

for all $f, g \in \text{Cl}(R)$. Note that $V_{f+g} \oplus V_{-f} \oplus V_{-g}$ is free so that we can assume as already proven in the case where the underlying $R$-modules are free. In view of Lemma 3.26 the Equation (54) is equivalent to

$$\text{cycl} \left( V_{f+g} \otimes V_{-f} \otimes V_{-g}, h^{V_{f+g}} \otimes h^{V_{-f}} \otimes h^{V_{-g}} \right) = 1.$$

Using the definition [52] we see that the left-hand side is equal to

$$\text{cycl} \left( \bigotimes_{i=1}^{l} (V_{y_i}, h^{V_{y_i}}) \otimes c_{i_M} \right),$$

35
where $c_i \in \{0, 1, 2\}$ are chosen appropriately. Using again Lemma 3.26 and Theorem 3.14 in the free case we can transform this expression further to

$$
\text{cycl} \left( \bigotimes_{i=1}^{l} (V_{g_i}, h_{V_i})^c \right) = \text{cycl} \left( \bigoplus_{i=1}^{l} c_i (V_{g_i}, h_{V_i})^c \right) + (1 - l) 1_*
$$

$$
= \sum_{i=1}^{l} c_i \text{cycl} \left( (V_{g_i}, h_{V_i})^c \right) + (1 - l) 1_*
$$

(51)

Let now $\mathcal{V} \in \overline{\text{Loc}}_{\text{proj}}(M)$ be a locally constant sheaf of finitely generated projective $R$-modules. Recall the definition (45) of $c_1(\mathcal{V}) \in \text{Sm}(\text{Cl}(R))$. Then we can form the local system of finitely generated free $R$-modules

$$
\Phi(\mathcal{V}) := \mathcal{V} \oplus T(c_1(\mathcal{V})) \in \overline{\text{Loc}}_{\text{free}}(M).
$$

Similarly, for a local system with geometry $(\mathcal{V}, h^{\mathcal{V}}) \in \overline{\text{Loc}}_{\text{proj geom}}(M)$ we define

$$
\hat{\Phi}(\mathcal{V}, h^{\mathcal{V}}) := (\mathcal{V}, h^{\mathcal{V}}) \oplus \hat{T}(c_1(\mathcal{V})) \in \overline{\text{Loc}}_{\text{free geom}}(M).
$$

This construction defines maps of smooth sets

$$
\Phi : \overline{\text{Loc}}_{\text{proj}} \rightarrow \overline{\text{Loc}}_{\text{free}}, \quad \hat{\Phi} : \overline{\text{Loc}}_{\text{proj geom}} \rightarrow \overline{\text{Loc}}_{\text{free geom}}.
$$

Recall the Definition 3.7 of the homomorphism of smooth monoids $\hat{I} : \overline{\text{Loc}}_{\text{proj}} \rightarrow \overline{\text{KR}}^0$.

**Lemma 3.17** The map $\hat{I} : \overline{\text{Loc}}_{\text{proj}} \rightarrow \overline{\text{KR}}^0$ satisfies

$$
\hat{I}(\mathcal{V}) := \hat{I}(\Phi(\mathcal{V})) + U(c_1(\mathcal{V})) - 1.
$$

**Proof.** We calculate, using (50) in the marked step,

$$
\hat{I}(\Phi(\mathcal{V})) + U(c_1(\mathcal{V})) - 1 = \hat{I}(\mathcal{V} \oplus T(c_1(\mathcal{V}))) + U(c_1(\mathcal{V})) - 1
$$

$$
= \hat{I}(\mathcal{V}) + \hat{I}(T(c_1(\mathcal{V}))) + U(c_1(\mathcal{V})) - 1
$$

$$
= \hat{I}(\mathcal{V}) + c_1(\mathcal{V})^*(\hat{I}(\mathcal{V})) + c_1(\mathcal{V})^*x_\bullet - 1
$$

$$
= \hat{I}(\mathcal{V}) .
$$

(51)

**Definition 3.18** We extend the cycle map to local systems of finitely generated projective $R$-modules with geometry as

$$
\text{cycl} : \overline{\text{Loc}}_{\text{geom}} \xrightarrow{\Phi, c_1} \overline{\text{Loc}}_{\text{geom free}} \times \text{Sm}(\text{Cl}(R)) \xrightarrow{\text{cycl} + \hat{U} - 1} \overline{\text{KR}}^0,
$$

where $1 := \text{cycl}(\hat{T}(0))$.  

36
In order to finish the proof of Theorem 3.14 we must verify that the diagram (43) commutes, and that the cycle map defined in 3.18 is additive. We start with the compatibility of curvatures. We use that by construction $\hat{R} \circ \hat{T} = b_0$ and $R \circ \hat{U} = 0$, and that (43) commutes in the free case. Let $\hat{x} \in \Loc^{\text{proj}}(M)$.

$$R(\text{cyc}l(\hat{x})) = R(\text{cyc}l(\hat{\Phi}(\hat{x}))) + R(\hat{U}(\text{cyc}l(\hat{x}))) - R(1_M)$$

$$= \hat{R}(\hat{x} \oplus \hat{T}(\text{cyc}l(\hat{x}))) - b_0$$

$$= \hat{R}(\hat{x}) + \hat{R}(T(\text{cyc}l(\hat{x}))) - b_0$$

$$= \hat{R}(\hat{x}) .$$

Furthermore, using $I \circ \hat{U}(\hat{x}) = U(x)$, $I \circ \text{cyc}l \circ \hat{\Phi}(\hat{x}) = \hat{I} \circ \Phi(x)$ and Lemma 3.17 we get

$$I(\text{cyc}l(\hat{x})) = I(\text{cyc}l(\hat{\Phi}(\hat{x}))) + I(\hat{U}(\text{cyc}l(\hat{x}))) - I(1_M)$$

$$= \hat{I}(\Phi(x)) + \hat{I}(U(\text{cyc}l(x))) - 1$$

$$= \hat{I}(x) ,$$

where $x \in \Loc^{\text{proj}}(M)$ denotes the image of $\hat{x}$ under the map which forgets the geometry. Finally we show the additivity. Let $\hat{x}, \hat{y} \in \Loc^{\text{proj}}(M)$. Then we have using the additivity of the cycle map in the free case:

$$\text{cyc}l(\hat{x} \oplus \hat{y}) = \text{cyc}l(\hat{\Phi}(\hat{x} \oplus \hat{y})) + \hat{U}(\hat{x} \oplus \hat{y}) - 1_M$$

Using Lemma 3.16

$$= \text{cyc}l(\hat{x} \oplus \hat{y} \oplus \hat{T}(\text{cyc}l(\hat{x} \oplus \hat{y}))) + \text{cyc}l(\hat{T}(\text{cyc}l(\hat{x}))) \oplus \hat{T}(\text{cyc}l(\hat{x}))$$

$$= \text{cyc}l(\hat{x} \oplus \hat{y} \oplus \hat{T}(\text{cyc}l(\hat{x}))) + \text{cyc}l(\hat{T}(\text{cyc}l(\hat{x}))) \oplus \hat{T}(\text{cyc}l(\hat{x}))$$

$$= \text{cyc}l(\hat{x} \oplus \hat{y} \oplus \hat{T}(\text{cyc}l(\hat{x}))) + \hat{U}(\hat{x}) + \hat{U}(\hat{y}) - 5 \cdot 1_M$$

This finishes the derivation of Theorem 3.14 in the projective case under the assumption of the free case.

\[\square\]

### 3.8 Proof of Proposition 3.13

#### 3.8.1 Overview

We have transformations

$$\Omega^\infty : \text{Sm}(\mathcal{N}(\text{Sp})[W^{-1}]) \rightarrow \text{Sm}(\text{Mon}(\mathcal{N}(\text{sSet})[W^{-1}]))$$

and

$$\pi_0 : \text{Sm}(\text{Mon}(\mathcal{N}(\text{sSet})[W^{-1}])) \rightarrow \text{Sm}(\text{Mon}(\text{Set}))$$
which take the smooth infinite loop space of a smooth spectrum (and forgets commutativity and inverses) or the smooth monoid of connected components of a smooth monoid space. The domain $\overline{\text{Loc}}^\text{free}_{\text{geom}}$ of the cycle map will be written as the smooth monoid $\pi_0(\infty \text{Loc}^b)$ of connected components of a smooth monoid space of locally constant sheaves with geometry. In general, it seems to be a complicated task to write down all data for an object of $\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}])$. One would have to define not only the multiplication map, but also all higher associator homotopies. We found it much easier to induce such an object from a strictly associative monoid space. We were able to find strictly associative models for the space of locally constant sheaves of free $R$-modules with geometry. For this technical reason we start to construct the cycle map in the free case first. Hence we consider a free version $KR^\text{free}$ of the algebraic $K$-theory spectrum which will be defined in (57). It comes with a natural map $KR^\text{free} \to KR$ which induces an isomorphism of homotopy groups in non-zero degrees. Moreover, $\pi_0(KR^\text{free}) \cong \mathbb{Z}$ is the free part of $\pi_0(KR) \cong \mathbb{Z} \oplus \text{Cl}(R)$. In particular, we can use the same complex $A \in \text{Ch}$ as for $KR$ and the equivalence

$$c^\text{free} : KR^\text{free} \to KR \cong H(A)$$

in order to define the differential function spectrum $\text{Diff}(KR^\text{free})$. The homotopy class of this map is fixed uniquely, but in order talk about elements in $\overline{KR^\text{free}}^0$ we need an actual spectrum map (defined up to at most contractible choice). In the following we fix a choice of such a map.

By Definition 2.5, the differential free algebraic $K$-theory of $R$ is the smooth monoid (actually a group)

$$\overline{KR^\text{free}}^0 := \pi_0(\Omega^\infty \text{Diff}(KR^\text{free})) .$$

In view of the definition of the smooth spectrum $\text{Diff}(KR^\text{free})$ by a pull-back (10) and since $\Omega^\infty$ preserves pull-backs, the smooth monoid space $\Omega^\infty \text{Diff}(KR^\text{free})$ can also be obtained as the pull-back in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$

$$\Omega^\infty \text{Diff}(KR^\text{free}) \cong \Omega^\infty \text{Sm}_\infty(KR^\text{free}) \times_{\Omega^\infty H(\Omega \sigma A_\infty)} \Omega^\infty H(\sigma \Omega A_\infty) .$$

(55)

By definition the free version of the cycle map is a map of smooth monoids

$$\text{cycl}^\text{free} : \overline{\text{Loc}}^\text{free}_{\text{geom}} \to \overline{KR^\text{free}}^0 .$$

In order to construct it we will actually construct a smooth monoid space

$$\infty \text{Loc}^b \in \text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$$

such that

$$\pi_0(\infty \text{Loc}^b) \cong \overline{\text{Loc}}^\text{free}_{\text{geom}}$$

38
and commutative squares in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$

\[
\begin{array}{ccc}
\Omega^\infty \text{Sm}_\infty(KR^{free}) & \xrightarrow{i} & \Omega^\infty \text{Loc}^\flat \\
\Omega^\infty \text{rat} & \xrightarrow{\hat{I}} & \Omega^\infty H(\sigma \Omega A_\infty) \\
& \xleftarrow{\hat{R}} & \Omega^\infty H(\Omega A_\infty)
\end{array}
\] (56)

such that the maps $\hat{I}$ and $\hat{R}$ induce the corresponding maps in (43) (denoted by the same symbols) on the level of $\pi_0$. By the universal property of the pull-back (55) it induces a map

$$\text{cycl}^{free} : \infty \text{Loc}^\flat \to \Omega^\infty \text{Diff}(KR^{free})$$

which in turn gives the cycle map after applying $\pi_0$:

$$\pi_0(\infty \text{Loc}^\flat) \xrightarrow{\pi_0(\text{cycl})} \pi_0(\Omega^\infty \text{Diff}(KR^{free})) \cong KR^{free}_0.$$ 

A map of data

$$(KR^{free}, A, c^{free}) \to (KR, A, c)$$

induces a map $KR^{free}_0 \to KR^0$, and we get a cycle map (whose existence was asserted in Proposition 3.13) as the composition

$$\text{cycl} : \text{Loc}^{free}_\text{geom} \xrightarrow{\text{cycl}^{free}} \text{Loc}^{free}_\text{geom} \to KR^{free}_0 \to KR^0.$$ 

We now start with the technical details.

### 3.8.2 Construction of $\hat{I}$

The first ingredient of the construction of the map $\hat{I}$ in (56) is the construction of smooth monoids

$$\text{Loc}^{free}_\text{geom}, \text{Loc}^{free} \in \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet})))$$

which associate to a manifold $M$ the spaces of locally constant sheaves of finitely generated free $R$-modules with and without geometry, and of the smooth submonoid $\text{Loc}^\flat \subseteq \text{Loc}^{free}_\text{geom}$ such that $\pi_0(\text{Loc}^\flat) \cong \text{Loc}^{free}_\text{geom}$. The second ingredient is the precise connection between these smooth spaces and the infinite loop space $\Omega^\infty KR^{free}$.

As explained above, it is easier to work with strictly associative monoids. We therefore consider the category $\text{F}(R)$ with objects the free $R$-modules $R^n$ for all $n \geq 0$. By $i\text{F}(R) \subseteq \text{F}(R)$ we denote its underlying groupoid. The category $\text{F}(R)$ can be given a symmetric monoidal structure such that associators are identities. It follows that $\mathbb{N}(i\text{F}(R)) \in$
\textbf{Mon(sSet)} and therefore \(|\mathbb{N}(i\mathcal{F}(R))| \in \text{Mon}(\text{Top})\) is a strictly associative monoid. Using the commutative structure, i.e. viewing \(\mathbb{N}(i\mathcal{F}(R)) \in \text{CommMon}(\mathbb{N}(\text{sSet})[W^{-1}])\) we define the free \(K\)-theory spectrum (compare Subsection 3.4)

\[ \text{KR}_\text{free} := \text{sp}(\Omega B(\mathbb{N}(i\mathcal{F}(R)))) \]  

We have a symmetric monoidal functor \(F(R) \rightarrow \mathcal{P}(R)\) which induces under \(\text{sp}(\Omega B(\mathbb{N}(\ldots )))\) a map

\[ \text{KR}_\text{free} \rightarrow \text{KR} . \]

We let \(\mathcal{V}_{\text{univ}}^\text{free} \in \text{Sh}_{\text{Mod}(R)}(\mathbb{N}(i\mathcal{F}(R)))\) be the universal locally constant sheaf constructed similarly as in the projective case, see Subsection 3.4. It comes with canonical isomorphisms

\[ s^* : \mathcal{V}_{\text{univ}}^\text{free} \simeq \text{pr}_0^* \mathcal{V}_{\text{univ}}^\text{free} \oplus \text{pr}_1^* \mathcal{V}_{\text{univ}}^\text{free} \]  

satisfying the usual associator relation, where

\[ s : |\mathbb{N}(i\mathcal{F}(R))| \times |\mathbb{N}(i\mathcal{F}(R))| \rightarrow |\mathbb{N}(i\mathcal{F}(R))| \]

is the monoid structure.

We consider the function space

\[ \text{Loc}^\text{free} := \text{Sm}(\mathbb{N}(i\mathcal{F}(R))) \in \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet}))) \]  

A \(p\)-simplex \(u \in \text{Sm}(\mathbb{N}(i\mathcal{F}(R)))(M)\) is by definition a continuous map \(u : M \times \Delta^p \rightarrow |\mathbb{N}(i\mathcal{F}(R))|\). We form the smooth space of geometric locally constant sheaves of free \(R\)-modules of rank \(n\)

\[ \text{Loc}_{\text{geom}}^\text{free} \in \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet}))) \]

whose \(p\)-simplices on a manifold \(M\) are pairs \((u, h^u \mathcal{V}_{\text{univ}}^\text{free})\) of a map \(u : M \times \Delta^p \rightarrow |\mathbb{N}(i\mathcal{F}(R))|\) and a geometry \(h^u \mathcal{V}_{\text{univ}}^\text{free}\) on the pull-back \(u^* \mathcal{V}_{\text{univ}}^\text{free}\). The monoid structure is given by

\[ (u_0, h_0) + (u_1, h_1) := (s \circ (u_0, u_1), h_0 \oplus h_1) , \]

where we define the metric \(h_0 \oplus h_1\) on \((s \circ (u_0, u_1))^* \mathcal{V}_{\text{univ}}^\text{free}\) using the identification

\[ (s \circ (u_0, u_1))^* \mathcal{V}_{\text{univ}}^\text{free} \simeq u_0^* \mathcal{V}_{\text{univ}}^\text{free} \oplus u_1^* \mathcal{V}_{\text{univ}}^\text{free} \]

induced by \([58]\).

Let \((\mathcal{V}, h^\mathcal{V})\) be a locally constant sheaf of finitely generated projective \(R\)-modules on a product \(M \times N\) of manifolds. We say that the geometry \(h^\mathcal{V}\) is constant along \(N\) if for all complex embeddings \(\sigma \in \Sigma\) the form \(\omega(h^\mathcal{V}_\sigma)\) defined in \([34]\) vanishes if one inserts any vector which is tangential to \(N\). If \((\mathcal{V}, h^\mathcal{V})\) on \(M \times \mathbb{R}\) is constant along \(\mathbb{R}\), then parallel transport along \(\mathbb{R}\) induces isomorphisms between the geometric locally constant sheaves \((\mathcal{V}, h^\mathcal{V})|_{M \times \{t\}}\) for all \(t \in \mathbb{R}\).
Definition 3.19 We define the smooth submonoid

\[ \text{Loc}^b \subseteq \text{Loc}_{geom}^{free} \]  

whose \( p \)-simplices on \( M \) are those pairs \( (u, h^* \nu^{free}_{\text{univ}}) \in \text{Loc}_{geom}^{free}(M)[p] \) where the geometry has the additional property that it is constant along the simplex direction.

We have maps of smooth simplicial monoids in \( \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet}))) \)

\[ \text{Loc}^b \to \text{Loc}_{geom}^{free} \to \text{Loc}^{free} \]

where the first map is the inclusion, and the second map forgets the geometry.

We let

\[ \infty\text{Loc}^b, \infty\text{Loc}_{geom}^{free}, \infty\text{Loc}^{free} \in \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet})[W^{-1}]))) \]

be the functors induced by

\[ \text{Loc}^b, \text{Loc}_{geom}^{free}, \text{Loc}^{free} \in \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet}))) \]

under the transformation

\[ \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet}))) \to \text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet})[W^{-1}]))) . \]

Lemma 3.20 The natural map

\[ \infty\text{Loc}_{geom}^{free} \to \infty\text{Loc}^{free} \]  

(61)

is an equivalence.

Proof. Using the flexibility of geometries one can show that for each manifold \( M \) the map of simplicial sets \( \text{Loc}_{geom}^{free}(M) \to \text{Loc}^{free}(M) \) is a trivial Kan fibration. Indeed, given a simplex \( u : M \times \Delta^p \to |\mathbb{N}(\text{F}(R))| \) in \( \text{Loc}_{geom}^{free}(M)[p] \) and a lift to \( \text{Loc}_{geom}^{free}(M) \) of its restriction to a horn \( \Lambda^p_k \subseteq \Delta^p \) (resp. boundary \( \partial \Delta^p \subseteq \Delta^p \) ), i.e. a geometry on \( u^*|_{M \times \Lambda^p_k} \nu^{free}_{\text{univ}} \) (resp. \( u^*|_{M \times \partial \Delta^p} \nu^{free}_{\text{univ}} \)), then one can find a lift \( (u, h^* \nu^{free}_{\text{univ}}) \in \text{Loc}_{geom}^{free}(M)[p] \). One can define \( h^*\nu^{free}_{\text{univ}} \) by glueing the geometry already given on the horn or boundary with an arbitrary metric given on the interior of the simplex using a partition of unity. A map between smooth spaces which is an objectwise trivial Kan fibration is an equivalence of smooth spaces (Lemma 6.3). \( \square \)

Lemma 3.21 The canonical map \( \text{Loc}^b([0]) \to \text{Loc}_{geom}^{free} \) induces an isomorphism of smooth monoids

\[ \pi_0(\infty\text{Loc}^b) \cong \text{Loc}_{geom}^{free} . \]  

(62)

41
Proof. We use that $|N(iF(R))|$ classifies isomorphism classes of locally constant sheaves of free $R$-modules of rank $n$, and loops in this space the corresponding automorphisms. If two zero simplices in $\text{Loc}^\circ(M)([0])$ are connected by a path, then the corresponding locally constant sheaves with geometry are isomorphic by parallel transport. Hence they are mapped to the same point in $\text{Loc}^\circ_{\text{geom}}(M)$. This shows that the map \((62)\) is well-defined and surjective. We now show injectivity. If two zero simplices $\left( u_i, h_i^*V^\text{free}_{\text{univ}} \right) \in \text{Loc}^\circ(M)([0])$, $i = 0, 1$, are mapped to the same point in $\text{Loc}^\circ_{\text{geom}}(M)$, then we can connect them by a path which we consider as a one-simplex in $\text{Loc}^\circ(M)([1])$. Indeed, since the maps $u_0, u_1 : M \rightarrow |N(iF(R))|$ classify isomorphic locally constant sheaves they can be connected by a path. The parallel transport of the geometry $h_0^*V^\text{free}_{\text{univ}}$ along this path differs from $h_1^*V^\text{free}_{\text{univ}}$ by an automorphism of $u_1^*V^\text{free}_{\text{univ}}$. This automorphism can be turned into the identity by replacing the initially chosen path from $u_0$ to $u_1$ by its composition with an appropriate loop. \(\square\)

**Definition 3.22** We define the map of smooth monoids in $\text{Sm}(\text{Mon}(N(sSet)[W^{-1}]))$

\[ \tilde{I} : \infty\text{Loc}^\circ \rightarrow \Omega^\infty\text{Sm}_\infty(KR^{\text{free}}) \]

as the composition of $\infty\text{Loc}^\circ \rightarrow \infty\text{Loc}^{\text{free}}$ with the map

\[ \tilde{I} : \infty\text{Loc}^{\text{free}} \rightarrow \Omega^\infty\text{Sm}_\infty(KR^{\text{free}}) \]

defined by

\[
\begin{align*}
\infty\text{Loc}^{\text{free}} & = \text{Sm}_\infty(|N(iF(R))|) \\
\rightarrow & \text{Sm}_\infty(|\Omega^\infty KR^{\text{free}}|) \\
\cong & \text{Sm}_\infty(\Omega^\infty KR^{\text{free}}) \\
\cong & \Omega^\infty\text{Sm}_\infty(KR^{\text{free}}),
\end{align*}
\]

where the marked arrow is a free version of \((30)\).

3.8.3 Construction of $\hat{R}$

Given a locally constant sheaf of free $R$-modules with geometry $(\mathcal{V}, h^\mathcal{V})$ on a manifold $M$ we can define a characteristic form $\omega(h^\mathcal{V}) \in Z^0(\Omega A(M))$ according to \((42)\). While $(\mathcal{V}, h^\mathcal{V})$ determines a connected component of $\text{Loc}^\circ(M)$ by Lemma\(3.21\), the form $\omega(h^\mathcal{V})$ determines a connected component of the space $\Omega^\infty H(\sigma\Omega A_\infty)(M)$ by \((11)\). The map $\hat{R}$ is the space level refinement of the above correspondence on the level of connected components. It is constructed by the following technical procedure.

We consider the category $\text{sCh}$ of simplicial chain complexes. We define the smooth simplicial chain complex

$$s\Omega A \in \text{Sm}(\text{sCh})$$
which associates to \( M \) the simplicial chain complex \([p] \mapsto \Omega A(M \times \Delta^p)\). By \( s\Omega A^p \subseteq s\Omega A \) we denote the smooth simplicial subcomplex of those \( \omega \in \Omega A(M \times \Delta^p) \) which vanish after insertion of any vector which is tangential to the simplex.

**Proposition 3.23** We consider a smooth simplicial monoid \( L \in \text{Sm}(\mathbb{N}(\text{Mon}(\mathbb{sSet}))) \) and its image \( \infty L \in \text{Sm}(\text{Mon}(\mathbb{N}(\mathbb{sSet})[W^{-1}]))) \). There exists a construction (explained in the proof) which associates to a map

\[
\omega : L \to Z^0(s\Omega A)
\]

in \( \text{Sm}(\mathbb{N}(\text{Mon}(\mathbb{sSet}))) \) a map

\[
\hat{\omega} : \infty L \to \Omega^\infty H(\Omega A_{\infty})
\]

in \( \text{Sm}(\text{Mon}(\mathbb{N}(\mathbb{sSet})[W^{-1}]))) \). If \( \hat{\omega} \) factors over \( Z^0(s\Omega A^p) \), then we obtain a refinement

\[
\hat{\omega} : \infty L \to \Omega^\infty H(\sigma\Omega A_{\infty}) .
\]

**Proof.** We consider \( Z^0(s\sigma\Omega A) \in \text{Sm}(\mathbb{N}(\mathbb{sAb})) \). Then we have a canonical map of smooth chain complexes

\[
N(Z^0(s\sigma\Omega A)) \to N(s\sigma\Omega A)_{\text{tot}} , \tag{63}
\]

where \( N : \text{Sm}(\mathbb{N}(\mathbb{sAb})) \to \text{Sm}(\mathbb{N}(\mathbb{Ch})) \) denotes the normalization. Integration over the simplex direction provides a map of smooth chain complexes.

\[
\int : N(s\sigma\Omega A)_{\text{tot}} \to \Omega A . \tag{64}
\]

In detail, if \( \alpha \in \sigma\Omega A^q(M \times \Delta^p) \subseteq N(s\sigma\Omega A)_{\text{tot}}^{q-p} \), then its integral is given by

\[
\int \alpha := \int_{M \times \Delta^p/M} \alpha \in \Omega A^{q-p}(M) .
\]

The integration thus preserves degree and is a morphism of complexes by Stokes' theorem. It does not preserve the truncation, but its restriction to the flat part does, yielding a map

\[
\int : N(s\sigma\Omega A^p)_{\text{tot}} \to \sigma\Omega A . \tag{65}
\]

The composition of (63) and (64) induces maps

\[
N(Z^0(s\sigma\Omega A))_{\infty} \to \Omega A_{\infty} , \quad N(Z^0(s\sigma\Omega A^p))_{\infty} \to \sigma\Omega A_{\infty}
\]

in \( \text{Sm}(\mathbb{N}(\mathbb{Ch})[W^{-1}]) \). We finally apply

\[
\Omega^\infty \circ H : \text{Sm}(\mathbb{N}(\mathbb{Ch})[W^{-1}]) \to \text{Sm}(\text{CommGroups}(\mathbb{N}(\mathbb{sSet})[W^{-1}]))
\]
in order to get maps

\[ \Omega^\infty H(N(Z^0(s\sigma\Omega A)))_\infty \rightarrow \Omega^\infty H(\Omega A_\infty), \quad \Omega^\infty H(N(Z^0(s\sigma\Omega A^v)))_\infty \rightarrow \Omega^\infty H(\sigma\Omega A_\infty). \tag{66} \]

The equivalence of categories $\mathbb{N}(\text{sAb}) \simeq \text{CommGroups}(\text{sSet})$ gives a functor

\[ i : \mathbb{N}(\text{sAb}) \rightarrow \mathbb{N}(\text{sAb})[W^{-1}] \rightarrow \text{CommGroups}(\mathbb{N}(\text{sSet})[W^{-1}]) \]

which fits into a commutative diagram of $\infty$-categories

\[
\begin{array}{ccc}
\mathbb{N}(\text{sAb}) & \longrightarrow & \mathbb{N}(\text{sAb})[W^{-1}] \\
\downarrow \quad N & & \downarrow \quad \Omega^\infty H(N(\ldots))_\infty & \longrightarrow & \text{CommGroups}(\mathbb{N}(\text{sSet})[W^{-1}]) \\
\mathbb{N}(\text{Ch}) & \longrightarrow & \mathbb{N}(\text{Ch})[W^{-1}] & \overset{H}{\longrightarrow} & \mathbb{N}(\text{Sp})[W^{-1}]
\end{array}
\]

in which the vertical maps are equivalent to the inclusions of the full subcategories of connective objects. Restricting to the bottom row to the full subcategories of connective objects results in a commutative diagram of $\infty$-categories

\[
\begin{array}{ccc}
\mathbb{N}(\text{sAb}) & \longrightarrow & \mathbb{N}(\text{sAb})[W^{-1}] & \longrightarrow & \text{CommGroups}(\mathbb{N}(\text{sSet})[W^{-1}]) \\
\downarrow \quad N & & \downarrow \quad \Omega^\infty H(N(\ldots))_\infty & \longrightarrow & \Omega^\infty \\
\mathbb{N}(\text{Ch}_{\geq 0}) & \longrightarrow & \mathbb{N}(\text{Ch}_{\geq 0})[W^{-1}] & \overset{H}{\longrightarrow} & \mathbb{N}(\text{Sp}_{\geq 0})[W^{-1}]
\end{array}
\]

The natural equivalence between the inclusion $i$ and the composition $\Omega^\infty H(N(\ldots))_\infty$ gives the maps

\[ i(Z^0(s\sigma\Omega A)) \rightarrow \Omega^\infty H(N(Z^0(s\sigma\Omega A)))_\infty, \quad i(Z^0(s\sigma\Omega A^v)) \rightarrow \Omega^\infty H(N(Z^0(s\sigma\Omega A)))_\infty \tag{67} \]

in $\text{Sm}(\text{CommGroups}(\mathbb{N}(\text{sSet})[W^{-1}]))$. Forgetting part of the structure, we now consider the maps \[ \text{(66)} \] and \[ \text{(67)} \] in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$. The map \( \omega : L \rightarrow Z^0(s\Omega A) \) in $\text{Sm}(\mathbb{N}(\text{Mon}(\text{sSet}))[W^{-1}]))$ induces a map \( \infty L \rightarrow i(Z^0(s\Omega A)) \) in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$ so that we can define

\[ \hat{\omega} : \infty L \rightarrow i(Z^0(s\Omega A)) \overset{\text{(67)}}{\rightarrow} \Omega^\infty H(N(Z^0(s\Omega A)))_\infty \overset{\text{(66)}}{\rightarrow} \Omega^\infty H(\Omega A_\infty) \]

in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$. If we start with \( \omega : L \rightarrow Z^0(s\Omega A^v) \), then we get

\[ \hat{\omega} : \infty L \rightarrow i(Z^0(s\Omega A^v)) \overset{\text{(67)}}{\rightarrow} \Omega^\infty H(N(Z^0(s\Omega A^v)))_\infty \overset{\text{(66)}}{\rightarrow} \Omega^\infty H(\sigma\Omega A_\infty). \]

This finishes the proof of Proposition 3.23. \( \square \)

Since the characteristic form given in \( \text{(42)} \) is natural and additive it induces a map of smooth monoid spaces

\[ \omega : \text{Loc}^\text{free}_{\text{geom}} \rightarrow Z^0(s\Omega A). \tag{68} \]
Its restriction to the subspace $\text{Loc}^b \subseteq \text{Loc}^\text{free}_{\text{geom}}$ has a factorization

$$\omega : \text{Loc}^b \to Z^0(s\Omega A^b) .$$  \hfill (69)

**Definition 3.24** We define the map

$$\hat{R} : \infty\text{Loc}^b \to \Omega^\infty H(\sigma\Omega A^\infty)$$

in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[[W^{-1}]]))$ to be the result of the construction Proposition 3.23 applied to (69).

### 3.8.4 The commutative diagram (56)

If we apply the construction Proposition 3.23 to the map $\omega : \text{Loc}^\text{free}_{\text{geom}} \to Z^0(\Omega A)$, then we obtain a map

$$\hat{\omega} : \infty\text{Loc}^\text{free}_{\text{geom}} \to \Omega^\infty H(\Omega A^\infty)$$

in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[[W^{-1}]]))$. The map of smooth monoid spaces

$$\Omega^\infty H(\sigma\Omega A^\infty) \to \Omega^\infty H(\Omega A^\infty)$$

is induced by the inclusion

$$\sigma\Omega A \hookrightarrow \Omega A .$$

By construction of the maps all solid triangles and squares of the following diagram in $\text{Sm}(\text{Mon}(\mathbb{N}(\text{sSet})[[W^{-1}]]))$ commute.

In order to finish the construction of the diagram (56) we therefore only have to construct the dotted map such that the diagram of smooth monoid spaces

$$\omega : \infty\text{Loc}^\text{free}_{\text{geom}} \to \Omega^\infty H(\Omega A^\infty)$$

commutes, as well as a homotopy between $g$ and $\Omega^\infty \text{rat}$.  \hfill (70)
We consider the following diagram of monoids in $\text{Sm}^{desc}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}])))$:

\[
\begin{array}{ccc}
\infty \text{Loc}^{free}_{\text{geom}} & \xrightarrow{\hat{\omega}} & \Omega^\infty H(\Omega A) \\
\downarrow \cong \quad (i) \quad \downarrow \cong \quad (iv) \quad \downarrow \cong \quad (v) \\
\infty \text{Loc}^{free} & \xrightarrow{f} & \Omega^\infty \text{Sm}_{\infty}(H(A)) \\
\cong \downarrow \cong \downarrow \cong \downarrow \cong \\
\text{Sm}_{\infty}(\mathbb{N}(iF(R))) & \xrightarrow{\psi} & \text{Sm}_{\infty}(\Omega^\infty H(A)) \\
\cong \downarrow \cong \downarrow \cong \downarrow \cong \\
\text{Sm}_{\infty}(\Omega B^{nc}(\mathbb{N}(iF(R)))) & \xrightarrow{g} & \text{Sm}_{\infty}(\Omega^\infty K^F) \\
\end{array}
\]

The equivalences of smooth monoid spaces $(i)$ is shown in Lemma (3.20). The equivalence $(iv)$ is obtained from the de Rham equivalence (6.20) by applying $\Omega^\infty$. The maps $(v)$ and $(vii)$ are instances of Lemma 6.13. We therefore get a morphism of smooth monoid spaces $f$ such that the corresponding upper hexagon commutes. By Lemma 6.8 (or rather its refinement to monoids (??)) there exists a map in $\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}])$

$$\phi : \mathbb{N}(iF(R)) \to \Omega^\infty H(A)$$

which induces the map

$$f = \text{Sm}_{\infty}(\phi) \ .$$

We now apply the non-commutative version of group completion

$$\Omega B^{nc} : \text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]) \Rightarrow \text{Groups}(\mathbb{N}(\text{sSet})[W^{-1}])$$

to $\phi$. Since $\Omega^\infty H(A)$ is already grouplike, the natural map $\Omega^\infty H(A) \to \Omega B^{nc}(\Omega^\infty H(A))$ is an equivalence so that we get a unique factorization

$$
\begin{array}{c}
\Omega B^{nc}(\mathbb{N}(iF(R))) \xrightarrow{\psi} \Omega^\infty H(A) \\
\downarrow \cong \\
\Omega B^{nc}(\Omega^\infty H(A)) \\
\end{array}
\]

of maps in $\text{Groups}(\mathbb{N}(\text{sSet})[W^{-1}])$. For the map $(vi)$ we use Lemma 6.2 which states that the non-commutative group completion of a commutative monoid naturally coincides with its commutative group completion. This gives the second equivalence in

$$\Omega^\infty KR^F = \Omega B(\mathbb{N}(iF(R))) \xrightarrow{\cong} \Omega B^{nc}(\mathbb{N}(iF(R)))$$

46
in Mon(N(sSet)[W−1]) which induces (vi).
This determines the map of smooth monoid spaces $g$ such that diagram (70) commutes. In order to finish the construction of the diagram (56) we must choose a homotopy

$$g \sim^{\kappa} \Omega^\infty(\text{rat}) : \Omega^\infty \mathbf{Sm}_\infty(KR^{free}) \to \Omega^\infty H(\Omega A_\infty).$$  \hspace{1cm} (72)

It follows from (40) that we can choose a homotopy

$$\Omega^\infty KR^{free} \xrightarrow{\psi} \Omega^\infty H(A).$$ \hspace{1cm} (73)

The reason to consider the rational instead of real coefficients in this intermediate step is to minimize the set of choices for the cycle map. The choice of $\sigma_Q$ determines a homotopy $\sigma$ in

$$\Omega^\infty KR^{free}Q \xrightarrow{c} \Omega^\infty H(A_Q) \xrightarrow{\sigma} \Omega^\infty H(A).$$

In view of the definition of $g$ and rat the latter homotopy $\sigma$ determines the required homotopy $\kappa$ in (72)

This finishes the construction of the diagram (56). Note that we have constructed the maps $\hat{I}$ and $\hat{R}$ such that (43) commutes. This completes the proof of Proposition 3.13.

3.9 $\widehat{KR}^j(\ast)$

In this subsection we discuss the differential algebraic K-theory $\widehat{KR}^j(\ast)$ of the number ring $R$ evaluated on a point. The interesting case is $j = 0$. Recall that $A$ is the chain complex of real vector spaces with trivial differential determined by $A^{-j} := K_j(R) \otimes \mathbb{R}$ (compare (22)). We have canonical isomorphisms

$$KR^{-1}(\ast) \cong K_1(R), \quad \Omega A^{-1}(\ast) \cong A^{-1}.$$

With these identifications the map $c : KR^{-1}(\ast) \to \Omega A^{-1}(\ast)$ identifies with the canonical map $K_1(R) \to K_1(R) \otimes \mathbb{R}$ given by $x \mapsto x \otimes 1$. In particular,

$$\Omega A^{-1}(\ast)/\text{im}(c) \cong K_1(R) \otimes \mathbb{R}/\mathbb{Z}.$$  \hspace{1cm} (74)

The exact sequence (13) specializes to the exact sequence

$$0 \to K_1(R) \otimes \mathbb{R}/\mathbb{Z} \to \widehat{KR}^0(\ast) \xrightarrow{I} K_0(R) \to 0.$$

47
We can now describe the action of the group
\[ \text{hom}(\pi_0(KR\mathbb{Q}), \pi_1(KR\mathbb{Q})) \cong K_1(R) \otimes \mathbb{Q} \]
on the functor on \( \hat{KR}^0 \). The automorphism corresponding to \( x \in K_1(R) \otimes \mathbb{Q} \) will be denoted by \( \Phi_x : \hat{KR}^0 \to \hat{KR}^0 \). The element \( x \in K_1(R) \otimes \mathbb{Q} \) gives naturally a class \( [x] \in K_1(R) \otimes \mathbb{R}/\mathbb{Z} \subset \hat{KR}^0(*) \).

We now define the class \( u_x \in \hat{KR}^0(\mathbb{Z}) \) such that it restricts to \( n[x] \) at the point \( n \in \mathbb{Z} \). We have a natural homomorphism \( d : \hat{KR}^0 \to \pi_0(\text{Sm}(\mathbb{Z})) \) of smooth groups whose evaluation at \( M \) is
\[ d : \hat{KR}^0(M) \to KR^0(M) \xrightarrow{\dim} \pi_0(\text{map}(M, \mathbb{Z})) . \]

**Lemma 3.25** For \( x \in K_1(R) \otimes \mathbb{Q} \) the homomorphism \( \Phi_x : \hat{KR}^0(M) \to \hat{KR}^0(M) \) is given by
\[ \Phi_x(y) = y + d(y)^*u_x . \] (75)

The one-dimensional free \( R \)-module \( R \) has a canonical metric \( h_R \) given by the standard euclidean metric \( h_R^\sigma := h^\mathbb{C} \) on \( R_\sigma = R \otimes_\sigma \mathbb{C} \cong \mathbb{C} \) for all places \( \sigma \in \Sigma \). We let
\[ 1 := \text{cycl}(R,h^R) \in \hat{KR}^0(*) . \] (76)

For a finitely generated projective \( R \)-module \( V \) we consider its top exterior power \( \det(V) \). In \( K_0(R) \) have the relation
\[ [V] - \dim(V) = [\det(V)] - 1 . \]

We are going to refine this relation to \( \hat{KR}^0(*) \). We consider the finitely generated projective \( R \)-module \( V \) as a locally constant sheaf on the point \( * \). In this way we have a well-defined notion of a geometry, Definition 3.10. For each complex embedding \( \sigma \in \Sigma \) of \( R \) we have a canonical isomorphism
\[ \det(V)_\sigma \cong \det(V_\sigma) . \]

Therefore, if we are given a geometry \( h^V \) on \( V \), then we have an induced geometry \( h^{\det(V)} \) such that \( h^{\det(V)_\sigma} \) is the metric on \( \det(V_\sigma) \) induced by \( h^V_\sigma \) for each complex embedding \( \sigma \in \Sigma \).

**Lemma 3.26** In \( \hat{KR}^0(*) \) we have the relation
\[ \text{cycl}(V,h^V) - \dim(V) \ 1 = \text{cycl}(\det(V),h^{\det(V)}) - 1 . \] (77)
Proof. We first assume that
\[
(V, h^V) \cong \bigoplus_{i=1}^{\dim(V)-1} (R, h^R) \oplus (\det(V), h^{\det(V)}) .
\] (78)

The additivity of the cycle map, Theorem 3.14, gives
\[
\text{cycl}(V, h^V) \cong (\dim(V) - 1) \mathbf{1} + \text{cycl}(\det(V), h^{\det(V)}) .
\]

Hence the assertion of the Lemma holds true in this special case. We now use the fact (the Steinitz theorem, see [Mil71, Thm. 1.6]) that every finitely generated projective \( R \)-module can be written in the form
\[
V \cong R^{\dim(V)-1} \oplus \det(V) .
\]

Hence we must compare the variations of both sides of (77) with respect to the choice of the geometry.
We consider now a finitely generated projective \( R \)-module \( V \) with two geometries \( h_i^V, i = 0, 1 \). Then we can form the locally constant sheaf \( V \) on the unit interval with fibre \( V \) and choose a geometry \( h^V \) which interpolates between the \( h_i^V \) on the ends. By the homotopy formula (see [BS10, Equation (1)]) we have
\[
\text{cycl}(V, h_1^V) - \text{cycl}(V, h_0^V) = a \left( \int_0^1 R(\text{cycl}(V, h^V)) \right)
\]
and
\[
\text{cycl}(\det(V), h_1^{\det(V)}) - \text{cycl}(\det(V), h_0^{\det(V)}) = a \left( \int_0^1 R(\text{cycl}(\det(V), h^{\det(V)})) \right).
\]

We now observe that
\[
R(\text{cycl}(V, h^V)) = \sum_{\sigma \in \Sigma^*} b_1(\sigma)\omega_1(h^V_\sigma) = \sum_{\sigma \in \Sigma^*} b_1(\sigma)\omega_1(h^{\det(V)_\sigma}) = R(\text{cycl}(\det(V), h^{\det(V)})).
\]

We get
\[
\text{cycl}(V, h_1^V) - \text{cycl}(V, h_0^V) = \text{cycl}(\det(V), h_1^{\det(V)}) - \text{cycl}(\det(V), h_0^{\det(V)}) .
\]

This implies the desired result. \( \Box \)

Next we analyse the effect of scaling the geometry. Let \((V, h^V) \in \text{Loc}_{\text{proj geom}}(*)\) be a projective \( R \)-module of rank one with geometry \( h^V = (h^V_\sigma)_{\sigma \in \Sigma^*} \). Given a collection \( \lambda = (\lambda_\sigma)_{\sigma \in \Sigma^*} \) of non-zero real numbers we can define the rescaled metric \( \lambda h^V = (\lambda_\sigma h^V_\sigma)_{\sigma \in \Sigma^*} \). The formula of the following lemma is closely related with [Lot00, Eq. (28)], see also Subsubsection 5.4.1).
Lemma 3.27 In $\widehat{KR}^0(\ast)$ we have the following identity:

$$\text{cycl}(V, \lambda h^V) - \text{cycl}(V, h^V) = a \left( \frac{1}{2} \sum_{\sigma \in \Sigma^*} \ln(|\lambda_\sigma|) b_1(\sigma) \right).$$

Proof. We consider the constant sheaf of $R$-modules $V$ on the unit interval $[0, 1]$ generated by $V$. We define a geometry $h^V$ on $V$ by linearly interpolating between the geometries $h$ and $\lambda h$. Hence $h^V_*(t) = (1 + t(\lambda_\sigma^2 - 1))h^V$. This gives by (35)

$$\omega_1(h^V_*)(t) = \frac{1}{2} \frac{(\lambda_\sigma^2 - 1)dt}{1 + t(\lambda_\sigma^2 - 1)}.$$

We have

$$R(\text{cycl}(V, h^V)) = b_0 + \frac{1}{4} \sum_{\sigma \in \Sigma^*} \frac{(\lambda_\sigma^2 - 1)dt}{1 + t(\lambda_\sigma^2 - 1)} b_1(\sigma).$$

By the homotopy formula [BS10, Equation (1)] we have

$$\text{cycl}(V, \lambda h^V) - \text{cycl}(V, h^V) = a \left( \int_0^1 R(\text{cycl}(V, h^V)) \right)$$

$$= a \left( \sum_{\sigma \in \Sigma^*} \frac{1}{4} \int_0^1 \frac{(\lambda_\sigma^2 - 1)}{1 + t(\lambda_\sigma^2 - 1)} dt b_1(\sigma) \right)$$

$$= a \left( \frac{1}{2} \sum_{\sigma \in \Sigma^*} \ln(|\lambda_\sigma|) b_1(\sigma) \right).$$

\[\square\]

3.10 Torsion sheaves and torsion

Let $M$ be a manifold and $T$ be a locally constant sheaf of finitely generated torsion $R$-modules. Then we can construct a canonical resolution

$$0 \rightarrow E(T) \rightarrow F(T) \rightarrow T \rightarrow 0,$$

where $E(T)$ and $F(T)$ are locally constant sheaves of finitely generated projective $R$-modules. The locally constant sheaf $F(T)$ is determined by the condition that on every contractible open subset $U \subset M$ we have $F(T)(U) = R[T(U)]$, the free $R$-module generated by $R[T(U)]$. There is a canonical augmentation $F(T) \rightarrow T$, and $E(T)$ is defined as its kernel. As a subsheaf of a locally constant sheaf of finitely generated projective $R$-modules it is again of this type.

We consider the difference

$$Z(T) := [F(T)] - [E(T)] \in KR^0(M).$$

(79)
In the following we show that this class does not depend on the choice of the resolution of $T$. Let

$$0 \to A \to B \to T \to 0$$

be any resolution of $T$ by locally constant sheaves of finitely generated projective $R$-modules.

**Lemma 3.28** In $KR^0(M)$ we have

$$[B] - [A] = \mathcal{Z}(T).$$

**Proof.** We consider a degree-wise surjective morphism of resolutions. It extends to a diagram

$$
\begin{array}{c}
0 & 0 & 0 \\
K & L & \\
\downarrow & \downarrow & \\
0 & U & V \\
\downarrow & \downarrow & \downarrow \\
0 & A & B \\
\downarrow & \downarrow & \\
0 & 0 & 0
\end{array}
$$

We conclude that

$$[V] - [U] = [B] - [A]$$

in $KR^0(M)$. In order to finish the argument we consider a second resolution

$$0 \to A' \to B' \to T \to 0.$$  

We must find a resolution which maps degree-wise surjectively to both. To this end we consider the sum

$$0 \to A \oplus A' \to B \oplus B' \to T \oplus T \to 0$$

and form the pull-back along the diagonal $T \to T \oplus T$. We get the resolution

$$0 \to A \oplus A' \to V \to T \to 0.$$  

The natural projections now induce the required degree-wise surjective maps. \qed

Let $\mathcal{V}$ be a locally constant sheaf of finitely generated $R$-modules on a manifold $M$. Then there exists a canonical exact sequence

$$0 \to \text{Tors}(\mathcal{V}) \to \mathcal{V} \to \text{Proj}(\mathcal{V}) \to 0,$$  

(81)
where $\text{Tors}(\mathcal{V})$ is a locally constant sheaf of finitely generated torsion $R$-modules and $\text{Proj}(\mathcal{V})$ is a locally constant sheaf of finitely generated projective $R$-modules. By Definition 3.7 we have a map of smooth monoids

$$\hat{I} : \overline{\text{Loc}^{\text{proj}}} \to KR^0.$$ 

**Definition 3.29** We define its extension to a map

$$[\ldots] = \hat{I} : \overline{\text{Loc}} \to KR^0$$

by

$$[\mathcal{V}] := [\text{Proj}(\mathcal{V})] + \mathcal{Z}(\text{Tors}(\mathcal{V})) , \quad \mathcal{V} \in \overline{\text{Loc}}(M).$$

Let

$$\mathcal{V} : 0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$$

be a (not necessarily exact) complex of locally constant sheaves of finitely generated $R$-modules. Then the cohomology sheaves $H^i(\mathcal{V})$ are locally constant sheaves of finitely generated $R$-modules.

**Lemma 3.30** In $KR^0(M)$ we have the identity

$$\sum_{i=0}^{n} (-1)^i [V^i] = \sum_{i=0}^{n} (-1)^n [H^i(\mathcal{V})].$$

**Proof.** We consider the difference of the left- and right hand sides

$$\delta := \sum_{i=0}^{n} (-1)^i [V^i] - \sum_{i=0}^{n} (-1)^i [\text{Proj}(H^i(\mathcal{V}))] - \sum_{n=0}^{n} (-1)^n \mathcal{Z}(\text{Tors}(H^i(\mathcal{V}))).$$

We argue by induction on $n$ that $\delta = 0$. The lemma is obvious in the case $n = 0$. We consider the locally constant sheaf of finitely generated projective modules

$$Z^1(\mathcal{V}) := \ker(V^1 \to V^2)$$

and define $Z^1_{\text{tors}}(\mathcal{V})$ as the pull-back

$$Z^1_{\text{tors}}(\mathcal{V}) \longrightarrow Z^1(\mathcal{V}) .$$

Then we have the exact sequences

$$0 \to V^0/Z^0(\mathcal{V}) \to Z^1_{\text{tors}}(\mathcal{V}) \to \text{Tors}(H^1(\mathcal{V})) \to 0.$$
\[ 0 \to Z^1_{\text{tors}}(V) \to Z^1(V) \to \text{Proj}(H^1(V)) \to 0 \]

and
\[ 0 \to Z^1(V) \to V^1 \to V^1/Z^1(V) \to 0 . \]

Note that in \( KR^0(M) \) we have, using Lemma 3.28
\[ Z(Tors(H^1(V))) = [Z^1_{\text{tors}}(V)] - [V^0/Z^0(V)], \quad (83) \]
\[ [Z^1_{\text{tors}}(V)] + [\text{Proj}(H^1(V))] = [Z^1(V)] \]
\[ [V^0] = [\text{Proj}(H^0(V))] + [V^0/Z^0(V)] \]
\[ [Z^1(V)] + [V^1/Z^1(V)] = [V^1] \quad (84) \]

We therefore can replace \( V \) by
\[ 0 \to V^1/Z^1(V) \to \cdots \to V^n \to 0 \]
without changing \( \delta \). By induction we now conclude that \( \delta = 0 \).

Let \( T \) be again a locally constant sheaf of finitely generated torsion \( R \)-modules. The canonical geometry on \( R \) induces induces a geometry on the free sheaf of \( R \)-modules \( F(T) \) generated by \( T \). Note that the induced maps \( E(T)_\sigma \to F(T)_\sigma \) are isomorphisms of complex vector bundles for all embeddings \( \sigma \in \Sigma \). Using these isomorphisms we obtain a geometry \( h^{E(T)} \) on \( E(T) \) by restriction. We consider the difference
\[ \hat{Z}(T) := \text{cycl}(F(T), h^{F(T)}) - \text{cycl}(E(T), h^{E(T)}) \in \hat{KR}^0(M) . \quad (85) \]

We again show that this class is actually independent of the choice of the resolution if the geometries are related as above. Let
\[ 0 \to A \to B \to T \to 0 \]
be any resolution and \( h^B \) a metric on \( B \). Then again we define a metric \( h^A \) by restriction.

**Lemma 3.31** We have
\[ \hat{Z}(T) = \text{cycl}(B, h^B) - \text{cycl}(A, h^A) . \]

**Proof.** Since \( (A_\sigma, h^{A_\sigma}) \) and \( (B_\sigma, h^{B_\sigma}) \) are unitarily equivalent for all \( \sigma \in \Sigma \) we conclude that \( \text{cycl}(B, h^B) - \text{cycl}(A, h^A) \) is flat. It follows from a homotopy argument that this difference does not depend on the choice of \( h^B \). We now consider a diagram (80). A metric \( h^V \) determines metrics on \( U, K, L \) by restriction, and a quotient metric on \( B \). The quotient metric on \( A \) coincides with the metric obtained by restriction from \( B \). Furthermore, \( K \) and \( L \) become isometric, and the complexifications of the two vertical sequences are isometric for every \( \sigma \in \Sigma \). We now apply Lott’s relation (139) to the two
vertical sequences of sheaves of projective $R$-modules which have equal analytic torsions in order to conclude that

$$\text{cycl}(B, h^B) - \text{cycl}(A, h^A) = \text{cycl}(V, h^V) - \text{cycl}(U, h^U).$$

We now finish the argument for the independence of the choice of the resolution similarly as in the proof of Lemma 3.28.

In order to avoid a circular argument note that in the proof of Lemma 3.31 we use Lott’s relation for complexes sheaves of projective $R$-modules which is a Theorem by [BT]. The Lemma will be employed to verify Lott’s relation in general (without projectivity assumption) in the proof of Lemma 5.20.

Here is an example which shows that $\hat{Z}(T)$ can be non-trivial. We consider the ring

$$R := \mathbb{Z}[a]/(a^2 - 2)$$

and the torsion module defined by

$$0 \to R^5 \to R \to T \to 0.$$  

We calculate $Z(T) \in \mathbb{K}R^0(*)$. We first observe that $I(\hat{Z}(T)) = [R] - [R] = 0$. Hence there exists $f \in \Omega A^{-1}(*) \cong A^{-1}$ such that $a(f) = \hat{Z}(T)$. We have

$$\hat{Z}(T) = \text{cycl}(R, h_{\text{can}}) - \text{cycl}(R, h),$$

where $h_{\text{can}}$ is the canonical geometry and the geometry $h$ has the following description. We have two real embeddings $\sigma_\pm : R \to \mathbb{C}$ given by $a \mapsto \pm \sqrt{2}$. Then $h_{\sigma_\pm} = \frac{1}{|5 \pm \sqrt{2}|} h_{\text{can}, \sigma_\pm}$.

We conclude with Lemma 3.27 and using the relation $b_1(\sigma_+) = -b_1(\sigma_-)$ that

$$f = -\frac{1}{2} \ln \frac{1}{5 + \sqrt{2}} b_1(\sigma_+) - \frac{1}{2} \ln \frac{1}{5 - \sqrt{2}} b_1(\sigma_-) = -\frac{1}{2} \ln \frac{5 + \sqrt{2}}{5 - \sqrt{2}} b_1(\sigma_+).$$

The units of $R$ are spanned by $-1$ and $1 + \frac{a}{4}$. We have by a similar calculation as above

$$c(R^*) = \frac{1}{2} \ln \frac{1 + \sqrt{2}}{1 - \sqrt{2}} \mathbb{Z} b_1(\sigma_+) \subset A^{-1}.$$  

We see that $f \notin c(R^*)$ and hence $\hat{Z}(T) = a(f) \neq 0$.

If $V$ is a locally constant sheaf of finitely generated $R$-modules, then for every place $\sigma$ of $R$ the canonical map $V_\sigma \to \text{Proj}(V)_\sigma$ of complex vector bundles is an isomorphism which is used to transfer a geometry $h^V$ on $V$ to a geometry on $\text{Proj}(V)$ denoted by the same symbol.

1Note that $(1 + \frac{a}{2})(2 - a) = 1$.
2Here $\text{Proj}(V)_\sigma$ denotes the complex vector bundle associated to the locally constant sheaf $\text{Proj}(V) \otimes_{R, \sigma} \mathbb{C}$.  

54
Definition 3.32 We extend the cycle map Definition 3.18 to a map
\[ \text{cyc1}: \text{Loc}_{\text{geom}} \to \hat{KR}^0, \quad \text{cyc1}(\mathcal{V}, h^\mathcal{V}) := \text{cyc1}(\text{Proj}(\mathcal{V}), h^\mathcal{V}) + \hat{Z}(\text{Tors}(\mathcal{V})). \] (86)

We finish this subsection with some calculations in $\hat{KR}^0(\ast)$ which play a role in Subsection 5.4.2. We consider a complex (not necessarily exact)
\[ \mathcal{V}: 0 \to V^0 \to V^1 \to \cdots \to V^n \to 0 \]
of finitely generated $R$-modules and a collection $h^\mathcal{V} := (h^{V^i})$ of geometries. For each embedding $\sigma \in \Sigma$ we define
\[ \det(\mathcal{V}_\sigma) := \prod_{i=0}^{n} \det(V^i_\sigma)^{(-1)^i}, \quad \det(H(\mathcal{V}_\sigma)) := \prod_{i=0}^{n} \det(H^i(\mathcal{V}_\sigma))^{(-1)^i}. \]

There exists a canonical isomorphism (compare [Mül93, Sec. 1])
\[ \phi_\sigma(\mathcal{V}): \det(\mathcal{V}_\sigma) \sim \rightarrow \det(H(\mathcal{V}_\sigma)). \]

The collection of metrics $h^{V^i}_\sigma = (h^{V^i})_{i=0, \ldots, n}$ induces a metric on $\det(\mathcal{V}_\sigma)$, and therefore via $\phi_\sigma(\mathcal{V})$ a metric $h^{\det(H(\mathcal{V}_\sigma))}_{RM}$ on $\det(H(\mathcal{V}_\sigma))$ called the Reidemeister metric.

Let $h^H = (h^{H^i})_{i=1, \ldots, n}$ be a choice of geometries on the cohomology modules. We get an induced geometry $h^{\det(H(\mathcal{V}))}_{RM}$.

Definition 3.33 The Reidemeister torsion $\tau_\sigma(\mathcal{V}, h^\mathcal{V}, h^H) \in \mathbb{R}^*$ is the unique positive real number such that
\[ \tau_\sigma(\mathcal{V}, h^\mathcal{V}, h^H)^2 h^{\det(H(\mathcal{V}_\sigma))}_{RM} = h^{\det(H(\mathcal{V}_\sigma))}_{RM}. \] (87)

We define
\[ \ln \tau(\mathcal{V}, h^\mathcal{V}, h^H) := \sum_{\sigma \in \Sigma^*} \ln \tau_\sigma(\mathcal{V}, h^\mathcal{V}, h^H) b_1(\sigma) \in A^{-1}. \]

Proposition 3.34 In $\hat{KR}^0(\ast)$ we have
\[ \sum_{i=0}^{n} (-1)^i \text{cyc1}(V^i, h^{V^i}) = \sum_{i=0}^{n} (-1)^i \text{cyc1}(H^i(\mathcal{V}), h^{H^i}(\mathcal{V})) + a(2 \ln \tau(\mathcal{V}, h^\mathcal{V}, h^H)). \]

Proof. We define
\[ \delta := \sum_{i=0}^{n} (-1)^i \text{cyc1}(V^i, h^{V^i}) - \sum_{i=0}^{n} (-1)^i \text{cyc1}(H^i(\mathcal{V}), h^{H^i}(\mathcal{V})) \]
\[ -a(2 \ln \tau(\mathcal{V}, h^\mathcal{V}, h^H)). \]
We first show that $\delta$ is independent of the geometries $h^V$ and $h^H$. To this end we consider families of such geometries $\tilde{h}^V$ and $\tilde{h}^H$ parametrized by $\mathbb{R}$ and let $\tilde{\delta} \in \tilde{KR}^0(\mathbb{R})$ be defined by

$$\tilde{\delta} := \sum_{i=0}^{n} (-1)^n \text{cycl}(pr^*V_i, \tilde{h}^V_i) - \sum_{i=0}^{n} (-1)^n \text{cycl}(pr^*H^i(V), \tilde{h}^H(V)) - a(2 \ln \tau(V, \tilde{h}^V, \tilde{h}^H)),$$

where $pr : \mathbb{R} \to \ast$ is the projection. We now observe that $R(\tilde{\delta}) = 0$. First of all note that the torsion part of the cohomology does not contribute since classes of the form $\hat{Z}(T)$ are flat (see the proof of Lemma 3.31). Let us now write

$$R \left( \sum_{i=0}^{n} (-1)^n \text{cycl}(pr^*V_i, \tilde{h}^V_i) - \sum_{i=0}^{n} (-1)^n \text{cycl}(pr^*\text{Proj}(H^i(V)), \tilde{h}^H(V)) \right) = a \ dt$$

for some $a \in \Omega A^{-1}(\mathbb{R})$ (note that there is no contribution of $b_0$). Using Lemma 3.26 we reduce the modules $V^i$ and $\text{Proj}(H^i(V))$ to their determinants and observe, using a similar calculation as in the proof of Lemma 3.27, that

$$adt = 2 d \ln \tau(V, \tilde{h}^V, \tilde{h}^H).$$

Since $R(\tilde{\delta}) = 0$ it now follows from a homotopy argument that $\delta$ does not depend on the geometry. In order to show that $\delta = 0$ we can choose a very special geometry. We essentially repeat the proof of Lemma 3.30. A short exact sequence of $R$-modules with projective quotient splits. We first choose decompositions

$$V^i \cong Z^i_{\text{tors}}(V) \oplus \text{Proj}(H^i(V)) \oplus V^i/Z^i(V)$$

as $R$-modules for all $i \in \{i = 0, \ldots, n\}$. Then we choose (by downwards induction starting with $i = n$) metrics preserving these compositions and such that the inclusions

$$V^i/Z^i(V) \hookrightarrow Z^i_{\text{tors}}(V)$$

induce isometries. The complex $V$ splits geometrically into a direct sum of geometrically split exact complexes

$$0 \to Z^i_{\text{tors}}(V) \to Z^i(V) \to \text{Proj}(H^i(V)) \to 0$$

and the in general non-exact non-split complexes

$$0 \to V^i/Z^i(V) \to Z^i_{\text{tors}}(V) \to 0.$$

Both summands have trivial Reidemeister torsions. Since the Reidemeister torsion is multiplicative for direct sums we see that for this special choice of the geometry on $V$ we have $\tau(V, h^V, h^H) = 0$. The proof now goes as in 3.30 using the additivity of the cycle map and the differential versions of the relations (83) and (84). □


3.11 Functoriality of the cycle map

In this subsection we discuss some aspects of the compatibility of the cycle map with a change of rings $\phi : R \to R'$. We will be concerned with the case where this is the homomorphism between the rings of integers or localizations of rings of integers induced by a finite extension $\phi : k \to k'$ of number fields.

In order to indicate the dependence on $R$ we add an argument $R$ to the notation of various objects. E.g. we now write $A^{-\ast}(R) := K_\ast(R) \otimes \mathbb{R}$ instead of $A^{-\ast}$, and $\text{Loc}^{\text{proj}}_{\text{geom}}(R)$ instead of $\text{Loc}^{\text{proj}}_{\text{geom}}$.

The homomorphism $\phi : R \to R'$ induces a symmetric monoidal functor $P(R) \to P(R')$, $M \mapsto M \otimes_R R'$ and by the functoriality of the construction of the algebraic $K$-theory spectrum (Definition 3.5) a map of spectra $\phi_* : KR \to KR'$. We choose equivalences $c_Q : KRQ \sim H(A_Q)$ and $c'_{Q'} : K'R'Q \sim H(A'_{Q'})$ which determine canonical differential data $(KR, A, c)$ and $(KR', A', c')$ (see Subsection 2.4). Then we can further choose a map of canonical differential data

$$(KR, A(R), c) \to (KR', A(R'), c')$$

by choosing a homotopy

$$H(\phi_*) \circ c \sim c' \circ \phi \wedge M \mathbb{R}. \quad (88)$$

This map of differential data induces a map of differential function spectra and smooth sets

$$\text{Diff}(KR) \to \text{Diff}(KR'), \quad \hat{\phi}_* : \widehat{KR}^0 \to \widehat{KR'}^0 .$$

Let $(\mathcal{V}, h^\mathcal{V})$ be a locally constant sheaf of finitely generated projective $R$-modules with geometry on a smooth manifold $M$, and define the locally constant sheaf of finitely generated projective $R$-modules $\mathcal{V}' := \mathcal{V} \otimes_R R'$ on $M$.

The homomorphism $\phi : R \to R'$ induces a $\mathbb{Z}/2\mathbb{Z}$-equivariant pull-back map between the sets of complex embeddings $\phi^* : \Sigma(R') \to \Sigma(R)$. Since $k'$ is algebraic over $k$ every complex embedding $k \hookrightarrow \mathbb{C}$ extends to $k'$. Hence the map $\phi^* : \Sigma(R') \to \Sigma(R)$ is surjective.

Note that we have a canonical isomorphism of sheaves

$$(\mathcal{V} \otimes_R R') \otimes_{R',\sigma} \mathbb{C} \cong \mathcal{V} \otimes_{R,\phi^*\sigma} \mathbb{C} ,$$

and therefore a canonical isomorphism of complex vector bundles

$$V'_\sigma \cong V_{\phi^*\sigma} .$$

Hence we can define a geometry $h^{\mathcal{V}'}$ on $\mathcal{V}'$ by setting

$$h^{\mathcal{V}'_\sigma} := h^{V_{\phi^*\sigma}}$$

for all $\sigma \in \Sigma(R')$. In this way we get a map of smooth sets

$$\phi_* : \text{Loc}^{\text{proj}}_{\text{geom}}(R) \to \text{Loc}^{\text{proj}}_{\text{geom}}(R') , \quad (\mathcal{V}, h^\mathcal{V}) \mapsto (\mathcal{V}', h^{\mathcal{V}'}) . \quad (89)$$

These change of rings maps are compatible with the cycle maps in the free case:
Theorem 3.35 There is a canonical choice for the homotopy \( \kappa \) in (88). With this choice the diagram

\[
\begin{array}{ccc}
\text{Loc}_{geom}^{free}(R) & \xrightarrow{\text{cycl}} & K^R \\
\downarrow \phi^* & & \downarrow \hat{\phi}^* \\
\text{Loc}_{geom}^{free}(R') & \xrightarrow{\text{cycl}} & K^{R'}
\end{array}
\]

commutes.

Proof. The idea is to provide a functorial construction of the cycle map. To this end we consider the \( \infty \)-category

\[ \Delta^1 := N(\bullet \to \bullet) . \]

For an \( \infty \)-category \( C \) we form the \( \infty \)-categories of \( I \)-smooth objects in \( C \)

\[ I \text{ Sm}(C) := \text{Fun}(N(Mf^{op}) \times \Delta^1, C) \cong \text{Fun}(\Delta^1, \text{Fun}(N(Mf^{op}), C)) . \]

We thus can consider the map \( \text{Diff}(K^R) \to \text{Diff}(K^{R'}) \) as an element of \( I \text{ Sm}(\mathbb{N}(Sp)[W^{-1}]) \). Similarly, we get \( I \)-smooth sets and groups

\[ \text{Loc}_{geom}^{free}(R) \to \text{Loc}_{geom}^{free}(R') \in I \text{ Sm}(\mathbb{N}(Set)) , \quad K^R \text{ free} \to K^{R'} \text{ free} \in I \text{ Sm}(\mathbb{N}(Ab)) . \]

By an inspection of the proof Proposition 3.13 given in Subsection 3.8 we observe that it can be generalized by introducing \( I \)-objects at the appropriate places. Most of this argument extends word by word. The argument gives in particular the following diagram

\[
\begin{array}{ccccccc}
\Omega^\infty K R^\text{free} & \xleftarrow{\sigma_Q} & \Omega^\infty K R^\text{free} & \xrightarrow{\psi} & \Omega^\infty K R^{\prime \text{free}} & \xleftarrow{\sigma_Q'} & \Omega^\infty K R'^{\prime \text{free}} \\
\downarrow c_Q & & \downarrow \psi & & \downarrow c_Q' & & \downarrow c_Q' \\
\Omega^\infty H(A_Q) & \xrightarrow{\phi} & \Omega^\infty H(A) & \xrightarrow{\psi'} & \Omega^\infty H(A') & \xrightarrow{\phi'} & \Omega^\infty H(A'_Q)
\end{array}
\] (90)

The middle square is the \( I \)-analog of the map \( \psi \) in (71). The homotopies \( \sigma_Q \) and \( \sigma'_Q \) determine the cycle maps for \( R \) and \( R' \) as explained in Subsubsection 3.8.4. After passing to real coefficients the composition of the three homotopies in (90) determines the canonical choice of the homotopy in (88). With this canonical choice we get a map of \( I \)-smooth sets

\[
\begin{array}{ccc}
\text{Loc}_{geom}^{free}(R) & \xrightarrow{\text{cycl}} & K^R \text{ free} \\
\downarrow \phi^* & & \downarrow \hat{\phi}^* \\
\text{Loc}_{geom}^{free}(R') & \xrightarrow{\text{cycl}} & K^{R'} \text{ free}
\end{array}
\]

i.e. the square commutes. \( \square \)
Since the cycle map in the projective case depends on further choices we can not expect a direct extension of Theorem 3.35 from the free to the projective case. Nevertheless it is possible to construct a version of the cycle map which is completely natural in the manifold and the ring variable by specializing a more general construction whose details are given in the paper [BT]. In the present paper the extension of cycle map to the projective case may depend on choices. This implies that the comparison map between the two versions of differential algebraic $K$-theory of the present paper and [BT] in general is not compatible with the cycle maps, see [BT, Lemma 5.14].

We now discuss the transfer. If $V'$ is a locally constant sheaf of projective $R'$-modules on a manifold $M$, then we can form the locally constant sheaf of projective $R$-modules $V'_R|$. For every projective $R$-module $W'$ the natural map

$$W'_R \otimes_{R,\sigma} \mathbb{C} \to \bigoplus_{\sigma' \in \phi^*, \varepsilon=1}(\sigma) M'_R \otimes_{R',\sigma'} \mathbb{C}, \quad w' \otimes z \mapsto \bigoplus_{\sigma' \in \phi^*, \varepsilon=1}(\sigma) w' \otimes z$$

is an isomorphism. Hence we get a canonical isomorphism of the vector bundle $V'_{R,\sigma}$ associated to $V'_R$ and $\sigma \in \Sigma(R)$ with the sum

$$V'_{R,\sigma} \cong \bigoplus_{\sigma' \in \phi^*, \varepsilon=1}(\sigma) V'_{\sigma}.$$ 

Given a geometry $h_V$ we can define a geometry $h_{V'_R}$ by setting

$$h'_{V'_R,\sigma} := \bigoplus_{\sigma' \in \phi^*, \varepsilon=1}(\sigma) h'_{V'_{\sigma}}$$

for all $\sigma \in \Sigma(R)$. In this way we define a restriction map between smooth sets

$$\phi^* : \text{Loc}^{\text{proj}}_{\text{geom}}(R') \to \text{Loc}^{\text{proj}}_{\text{geom}}(R), \quad (V', h_V) \mapsto (V'_R, h'_{V'_R}).$$

If $R'$ is a free $R$-module, then its rank is given by the degree of the field extension $d := [k' : k]$. In this case we easily see that

$$\phi^* \circ \phi_* = d \text{id}.$$ 

The finite extension of rings $R \to R'$ induces a transfer map of $K$-theory spectra

$$\text{trans} : KR' \to KR$$

and therefore a map

$$\overset{\text{trans}}{\text{trans}} : \overset{\wedge}{KR'} \to \overset{\wedge}{KR}$$

of differential $K$-theory associated to some choice of induced map of canonical data. We think the following holds true:

59
**Conjecture 3.36** If $k \to k'$ is an extension of number fields, then for an appropriate choice of the map of canonical differential data we have a commutative diagram of smooth sets

\[
\begin{array}{ccc}
\text{Loc}_{\text{geom}}^\text{free}(R') & \xrightarrow{\text{cycl}} & \hat{KR}'^0 \\
\phi^* & & \downarrow \text{trans} \\
\text{Loc}_{\text{geom}}^\text{free}(R) & \xrightarrow{\text{cycl}} & \hat{KR}^0
\end{array}
\]

### 3.12 Smoothness of the cycle map

In this subsection we discuss the notion of a smooth family of geometric locally constant sheaves of finitely generated projective $R$-modules on $M$. The obvious smoothness condition for a map

\[ f : N \to \text{Loc}_{\text{geom}}^{\text{proj}}(M) \]

is that there exists an open covering $\{U_\alpha\}_{\alpha \in I}$ of $N$ and elements $x_\alpha \in \text{Loc}_{\text{geom}}^{\text{proj}}(M \times U_\alpha)$ with the property that $i_{\alpha,n}*x_\alpha = f(n)$ for all $n \in U_\alpha$ and $\alpha \in I$. It is clear that for a smooth map $f : N \to \text{Loc}_{\text{geom}}^{\text{proj}}(M)$ the composition

\[ \text{cycl} \circ f : N \to \hat{KR}^0(M) \]

is smooth in the sense of Definition 2.3. The underlying family of isomorphism classes of locally constant sheaves of a flat family is constant. It is only the geometry which can vary non-trivially in a smooth family of geometric locally constant sheaves.

In the following we describe a more general notion of a smooth family of geometric locally constant sheaves where the isomorphism class of the locally constant sheaf itself can vary while its $KR$-theory class is still kept fixed.

Let $(V, \nabla^V)$ be a complex vector bundle with connection on a smooth manifold $M$. Following a suggestion of S. Goette we make the following definition. Recall that $R^{\nabla V} \in \Omega^2(M, \text{End}(V))$ denotes the curvature tensor of $\nabla^V$.

**Definition 3.37** We say that $\nabla^V$ is locally nilpotent, if for every $m \in M$ there exists a nilpotent subalgebra $N_m \subseteq \text{End}(V_m)$ such that $R^{\nabla^V}(X,Y) \in N_m$ for all $X,Y \in T_m M$.

Let $h^V$ be a hermitean metric and $\nabla^{V,h^V}$ be the adjoint connection of $\nabla^V$ with respect to $h^V$ (see Subsection 3.5). We define the transgression Chern forms

\[ \omega_{2j-1}(h^V) := \text{N Chern}(2j-1)^{-1}\text{ch}_{2j-1}(\nabla^{V,h^V}, \nabla^V) \in \Omega^{2j-1}(M) \]

(see (238) for the normalizing constant). If $\nabla^V$ is locally nilpotent, then for all $j \geq 1$ we have $\text{ch}_{2j}(\nabla^V) = 0$ and $\text{ch}_{2j}(\nabla^{V,h^V}) = 0$. Hence $d\omega_{2j-1}(h^V) = 0$. This construction of the form $\omega_{2j-1}(h^V)$ extends the definition (35) for flat bundles to the locally nilpotent case.
We call two vector bundles with connection stably equivalent if they become isomorphic as vector bundles with connection by adding suitable trivial bundles with trivial connections to both. We let
\[ \text{Vect}_{\nabla,\text{flat}}^{s} \subseteq \text{Vect}_{\nabla,\text{locnil}}^{s} \subseteq \text{Sm}(\mathbb{N}(\text{sSet})) \]
denote the smooth spaces which associate to each smooth manifold \( M \) the simplicial sets with \( p \)-simplices the stable equivalence classes of pairs \((V, \nabla V)\) of complex vector bundles with flat or locally nilpotent connections on \( M \times \Delta^p \), respectively.
For a topological space \( X \) we define the smooth simplicial set \( \widetilde{\text{Sm}}(X) \) such that its evaluations are given by
\[ \widetilde{\text{Sm}}(X)(M)([p]) := \text{Map}(M \times \Delta^p, X) \, . \]
For each embedding \( \sigma : R \to \mathbb{C} \) we consider the following diagram
\[ \widetilde{\text{Sm}}(BGL(R)) \xrightarrow{\phi_{\sigma}} \text{Vect}_{\nabla,\text{flat}}^{s} \xrightarrow{\phi_{\sigma}^{+}} \text{Vect}_{\nabla,\text{locnil}}^{s} \]

The upper horizontal map \( \phi_{\sigma} \) is defined as follows. Let \( (f : M \times \Delta^p \to BGL(R)) \in \text{Sm}(BGL(R))(M)([p]) \). Then we have a factorization of \( f \) over \( BGL(m, R) \) for a suitable \( m \geq 0 \). We define \( \phi_{\sigma}(f) \) to be the stable equivalence class of the bundle associated to the locally constant sheaf \( f^* \mathcal{V}_{univ}(m) \) and \( \sigma \) (see \( (38) \) for \( \mathcal{V}_{univ}(m) \)). The existence of the map \( \phi_{\sigma}^{+} \) is a consequence of the fact announced by Goette and proved in \( [\text{Sch11}, \text{Sec.2}] \), that the universal bundle associated to \( \mathcal{V}_{univ} \) on \( BGL(R) \) extends as a locally nilpotent bundle over \( BGL(R)^{+} \).

We set
\[ \text{Loc}^{+} := \widetilde{\text{Sm}}(BGL(R)^{+}) \subseteq \text{Sm}(\mathbb{N}(\text{sSet})) \, . \] (91)
We define \( \text{Loc}_{\text{geom}}^{+} \subseteq \text{Sm}(\mathbb{N}(\text{sSet})) \) as the smooth spaces whose \( p \)-simplices in \( \text{Loc}_{\text{geom}}^{+}(M) \) are maps \( f : M \times \Delta^p \to BGL(R)^{+} \) together with a choice of metrics on the bundles \( \phi_{\sigma}^{+}(f) \) for all \( \sigma \in \Sigma^{*} \). Then we have a forgetful map
\[ \text{Loc}_{\text{geom}}^{+} \to \text{Loc}^{+} \, . \] (92)

An element
\[ x = (f : M \times \Delta^p \to BGL(R)^{+}, (h_{\phi^+_\sigma}(f))_{\sigma \in \Sigma^{*}}) \in \text{Loc}_{\text{geom}}^{+}(M)([p]) \]
gives rise to a characteristic form (extending Definition \( [3,10] \))
\[ \omega(x) := \sum_{\sigma \in \Sigma^{*}} \sum_{j \geq 1} \omega_{2j-1}(\nabla^0_{\phi^+_\sigma}(f), h_{\phi^+_\sigma}(f))_{b_{2j-1}(\sigma)} \in Z^0(\sigma \Omega A(M))(\langle p \rangle) . \]

Note that we leave out the zero-dimensional part since we consider \( x \) as having dimension zero. We define the smooth subspace \( \text{Loc}_{\text{geom}}^{++} \subseteq \text{Loc}_{\text{geom}}^{+} \) similar to \( (60) \) as the subspace...
of objects \( x \) with the property that \( \omega(x) \) is annihilated by all vectors which are tangential to the simplex.

As before we indicate the images of these smooth spaces under the localization map

\[
\text{Fun}(\mathbb{N}(s\text{Set})) \rightarrow \text{Fun}(\mathbb{N}(s\text{Set})[W^{-1}])
\]

by an upper left index \( \infty \). The argument given in Lemma 3.20 shows that the induced map

\[
\phi : \infty \text{Loc}^+_{\text{geom}} \rightarrow \infty \text{Loc}^+
\]

is an equivalence in \( \text{Sm}(\mathbb{N}(s\text{Set})[W^{-1}]) \).

The characteristic form \( x \mapsto \omega(x) \) induces maps

\[
\omega : \text{Loc}^+_{\text{geom}} \rightarrow Z^0(\sigma \Omega A), \quad \omega : \text{Loc}^{b+}_{\text{geom}} \rightarrow Z^0(\sigma \Omega A^b).
\]

The construction Proposition 3.23 produces maps

\[
\hat{\omega} : \infty \text{Loc}^+_{\text{geom}} \rightarrow \Omega^\infty \text{H}(\Omega A_{\infty}), \quad \hat{R} : \infty \text{Loc}^{b+}_{\text{geom}} \rightarrow \Omega^\infty \text{H}(\sigma \Omega A_{\infty}).
\]

We construct a commutative diagram

\[
\begin{array}{ccc}
\infty \text{Loc}^+ & \xrightarrow{\phi} & \infty \text{Loc}^{b+}_{\text{geom}} \\
pr \downarrow & & \downarrow i \\
\Omega^\infty \text{Sm}_{\infty}(KR) & \xrightarrow{\rho} & \infty \text{Loc}^{b+}_{\text{geom}} \\
\downarrow g & & \downarrow \hat{R} \\
\Omega^\infty \text{H}(\Omega A_{\infty}) & \xrightarrow{\hat{\omega}} & \Omega^\infty \text{H}(\sigma \Omega A_{\infty}) \\
\end{array}
\]

where

\[
\text{pr} : \Omega^\infty \text{Sm}_{\infty}(KR) \cong \text{Sm}_{\infty}(K_0(R) \times BGL(R)^+) \rightarrow \text{Sm}_{\infty}(BGL(R)^+) \cong \infty \text{Loc}^+
\]

is induced by the projection to the second factor. The construction of \( g \) is now simpler than in 3.8.4. It is given as the composition

\[
\hat{\omega} \circ \phi^{-1} \circ \text{pr} =: g : \Omega^\infty \text{Sm}_{\infty}(KR) \rightarrow \Omega^\infty \text{H}(\Omega A_{\infty}).
\]

We define the smooth set

\[
\overline{\text{Loc}}^+_{\text{geom}} := \pi_0(\infty \text{Loc}^{b+}_{\text{geom}}) \in \text{Sm}(\mathbb{N}(\text{Set})).
\]

Then we get a cycle map

\[
\text{cycl}^+ : \overline{\text{Loc}}^+_{\text{geom}} \rightarrow \tilde{K}R^0
\]

which extends our original cycle map under the natural map \( j : \overline{\text{Loc}}_{\text{geom}} \rightarrow \overline{\text{Loc}}^+_\text{geom} \).

Let now \( N \) be a smooth manifold and

\[
f : N \rightarrow \overline{\text{Loc}}^+_{\text{geom}}(M)
\]

be a map of sets.
Definition 3.38 We say that \( f \) is nil-smooth if \( N \) admits an open covering \( \{ U_\alpha \}_{\alpha \in I} \) such that there exist elements \( x_\alpha \in \text{Loc}^+_\text{geom}(M \times U_\alpha) \) with \( i_{\alpha, n}^* x_\alpha = j(f(n)) \) for all \( n \in U_\alpha \) and \( \alpha \in I \) (see Definition 2.3 for notation).

The existence of the cycle map \( \text{cycl}^+ \) now implies that

\[
\text{cycl} : \text{Loc}^\text{free}_{\text{geom}} \to \hat{KR}^0
\]

maps nil-smooth maps to smooth maps. The notion of nil-smooth maps will be employed in Subsection 5.5.

4 The transfer in differential cohomology

4.1 Introduction

We consider a proper submersion \( \pi : W \to B \) between smooth manifolds. Equivalently, \( \pi \) is a locally trivial fibre bundle of manifolds with closed fibres. The Becker-Gottlieb transfer associated to \( \pi \) is a map of spectra in topological spaces (well-defined up to contractible choice)

\[
\text{tr} : \Sigma^\infty B_+ \to \Sigma^\infty W_+ \tag{94}
\]

first constructed in [BG75] (this construction will be recalled below, see (106)). It induces for any spectrum \( E \) the cohomological Becker-Gottlieb transfer

\[
\text{tr}^* : E^*(W) \to E^*(B). \tag{95}
\]

The cohomological Becker-Gottlieb transfer behaves naturally with respect to cartesian diagrams of the form (100), iterated fibre bundles \( W \to B \to A \), and spectrum maps \( E \to E' \). Moreover, it can be characterized in an axiomatic way, see [BS98].

The goal of the present section is to refine the cohomological Becker-Gottlieb transfer to a transfer in differential cohomology

\[
\check{\text{tr}} : \hat{E}^*(W) \to \hat{E}^*(B). 
\]

This differential transfer depends on the choice of a Riemannian structure on the submersion \( \pi \). In Subsection 4.2 we introduce this notion and formulate our statements about the existence and functorial properties of the differential Becker-Gottlieb transfer. Proofs will be deferred to subsequent subsections.

4.2 Differential Becker-Gottlieb transfer

We start with the notion of a Riemannian structure on a submersion \( \pi : W \to B \).

Definition 4.1 A Riemannian structure \( g := (g^{T\pi}, T^h \pi) \) on a submersion \( \pi : W \to B \) consists of
1. a metric \( g^{Tv} \) on the vertical tangent bundle \( Tv\pi := \ker(d\pi) \subseteq TW \),

2. a horizontal subbundle \( Th\pi \subseteq TW \) (i.e. vector subbundle such that \( Tv\pi \oplus Th\pi = TW \)).

Note that \( d\pi|_{Th\pi} : Th\pi \to \pi^*TB \) is an isomorphism of vector bundles. A Riemannian structure \( g \) on \( \pi \) induces a connection \( \nabla^{Tv} \) on the vertical tangent bundle \( Tv\pi \) as follows. We choose a Riemannian metric \( g^{TB} \) on the base \( B \). It induces a metric \( h^{Th\pi} \) on the horizontal subbundle \( Th\pi \subseteq TW \) such that \( d\pi|_{Th\pi} : Th\pi \to \pi^*TB \) becomes an isometry. The orthogonal sum \( g^{TW} := g^{Tv\pi} \oplus g^{Th\pi} \) is a Riemannian metric on the total space \( W \) of the submersion and induces a Levi-Civita connection \( \nabla^{LC,TW} \) on the tangent bundle \( TW \). Its projection \( \nabla^{Tv} \) to the vertical tangent bundle turns out to be independent of the choice of the metric \( g^{TB} \) (see [BGV92, Ch. 9]). The connection \( \nabla^{Tv} \) is the desired connection on the vertical tangent bundle \( Tv\pi \) induced by the Riemannian structure \( g \).

For a submersion \( \pi : W \to B \) we let \( \Lambda \to W \) denote the orientation bundle associated to the real vector bundle \( Tv\pi \to W \), see (222). The Euler form of \( Tv\pi \) associated to the Riemannian structure \( g \) on \( W \to B \) is an element

\[
e(g) := e(\nabla^{Tv}) \in Z^n(\Omega(W, \Lambda))
\]  

(see (224)), where \( n := \dim(W) - \dim(B) \) is the fibre dimension.

We now assume that the submersion \( \pi : W \to B \) is proper. Then we have an integration map of chain complexes

\[
\int_{W/B} : \Omega(W, \Lambda)[n] \to \Omega(B).
\]

Since the Euler form \( e(g) \) is closed we obtain an induced morphism of chain complexes

\[
\int_{W/B} \cdots \wedge e(g) : \Omega(W) \to \Omega(B).
\]

For differential data \((E, A, c)\) consisting of a spectrum \( E \in \text{Sp} \), a chain complex \( A \in \text{Ch} \) of real vector spaces, and an equivalence \( c : E^R \xrightarrow{\sim} HA \) let \( \hat{E}^* \) denote the associated differential extension of \( E \) introduced in Definition 2.5.

The main goal of the present section is to prove:

**Theorem 4.2** For every proper submersion \( W \to B \) with a Riemannian structure \( g \), there exists a canonical differential transfer \( \hat{\text{tr}} : \hat{E}^0(W) \to \hat{E}^0(B) \) such that

\[
\begin{align*}
\Omega A^{-1}(W) & \xrightarrow{\alpha} \hat{E}^0(W) \xrightarrow{I} E^0(W) \xrightarrow{Z^0(\Omega A(W))} \\
\int_{W/B} \cdots \wedge e(g) & \xrightarrow{\hat{\text{tr}}} \hat{E}^0(B) \xrightarrow{I} E^0(B) \xrightarrow{Z^0(\Omega A(B))} \\
\Omega A^{-1}(B) & \xrightarrow{\alpha} \hat{E}^0(B) \xrightarrow{\hat{\text{tr}}} E^0(B) \xrightarrow{Z^0(\Omega A(B))} \\
\int_{W/B} \cdots \wedge e(g) & \xrightarrow{\hat{\text{tr}}} \hat{E}^0(W) \xrightarrow{I} E^0(W) \xrightarrow{Z^0(\Omega A(W))}
\end{align*}
\]
commutes. The differential transfer behaves naturally with respect to pull-back in cartesian diagrams (100).

By proving Theorem 4.2 we construct a particular choice of a differential transfer map. We do not know whether a differential transfer map is characterized uniquely by naturality and the diagram (97).

The differential transfer will be induced by a map of differential function spectra

$$\text{Diff}(E)(W) \to \text{Diff}(E)(B).$$

In view of the definition of the differential function spectrum (as the pull-back in Definition 2.4), in order to obtain such a map we must construct a commutative diagram

$$
\begin{array}{ccc}
\text{Sm}_\infty(E)(W) & \xrightarrow{\text{rat}} & H(\Omega A_\infty(W)) \\
\downarrow^{\text{tr}} & & \downarrow^{H(f_{W/B} \cdots \wedge e(g))} \\
\text{Sm}_\infty(E)(B) & \xrightarrow{\text{rat}} & H(\Omega A_\infty(B))
\end{array}
$$

in the $\infty$-category of spectra $\mathbb{N}(\text{Sp})[[W^{-1}]]$ which is natural in the bundle $\pi : W \to B$ and its Riemannian structure. The main effort lies in the left square. In order to formulate the naturality of the diagram (98) properly we consider it as a diagram in the $\infty$-category of $\text{Bundle}_{\text{geom,trans}}^\delta$-spectra, where $\text{Bundle}_{\text{geom,trans}}^\delta$ is the category of triples $(\pi, g, b)$ consisting of a proper submersion $\pi$ equipped with Riemannian and transfer structures $g$ and $b$ (precise definitions will be given below). In Subsection 4.3 we construct the right square. In Subsection 4.4 we construct the left vertical arrow. Finally, in Subsection 4.5 we provide the left square. The end of the proof of Theorem 4.2 now goes as follows. Given $\pi : W \to B$ with Riemannian structure $g$ we choose a transfer datum $b$. If we evaluate (98) at $(\pi, g, b)$, then we get an induced map of differential function spectra $\text{Diff}(E)(W) \to \text{Diff}(E)(B)$ which in turn induces the differential transfer

$$\hat{\text{tr}} : \hat{E}_0(W) \to \hat{E}_0(B)$$

on the level of zeroth homotopy groups. We must finally show:

**Lemma 4.3** The differential transfer $\hat{\text{tr}}$ does not depend on the choice of the transfer datum $b$.

The proof of this Lemma will be given at the end of Subsection 4.5. Note that by Proposition 2.7 we have an identification of smooth sets

$$E\mathbb{R}/\mathbb{Z}^{-1} \cong \hat{E}_0^{\text{flat}} := \ker(R).$$

Since the transfer $\hat{\text{tr}}$ is compatible with the curvature transformation $R$ it preserves the flat subfunctors.
Proposition 4.4 If \( W \to B \) is a proper submersion with a Riemannian structure \( g \), then we have a commutative diagram

\[
\begin{array}{c}
\hat{E}_{flat}^0(W) \xrightarrow{\hat{\kappa}} E\mathbb{R}/\mathbb{Z}^{-1}(W) \\
\downarrow \hat{\text{tr}} \quad \downarrow \hat{\text{tr}}^* \\
\hat{E}_{flat}^0(B) \xrightarrow{\hat{\kappa}} E\mathbb{R}/\mathbb{Z}^{-1}(B) 
\end{array}
\]

This proposition will be shown in Subsection 4.6.

The functoriality of the differential transfer with respect to the composition of two proper submersions will be discussed in Proposition 4.12.

A map of differential data \( \phi : (E, A, c) \to (E', A', c') \) induces a map of differential function spectra \( \phi : \text{Diff}(E) \to \text{Diff}(E') \) (see Subsection 2.4). The differential transfer is compatible with these maps:

Proposition 4.5 We have a commutative diagram

\[
\begin{array}{c}
\hat{E}_{flat}^0(W) \xrightarrow{\hat{\kappa}} E\mathbb{R}/\mathbb{Z}^{-1}(W) \\
\downarrow \hat{\text{tr}} \quad \downarrow \hat{\text{tr}}^* \\
\hat{E}_{flat}^0(B) \xrightarrow{\phi} \hat{E}_{flat}^0(W) \\
\phi \\
\end{array}
\]

The proof will be sketched in Subsection 4.8.

4.3 Geometric bundles and integration of forms

In this subsection we make the right square of (98) precise. We consider the category \( \text{Bundle} \) whose objects are proper submersions \( \pi : W \to B \) between manifolds, and whose morphisms \( (\pi' : W' \to B') \to (\pi : W \to B) \) are cartesian squares (that is, pairs of smooth maps \( (f, F) \) such that the resulting square

\[
\begin{array}{c}
W' \xrightarrow{F} W \\
\downarrow \pi' \quad \downarrow \pi \\
B' \xrightarrow{f} B
\end{array}
\]

is cartesian). Adopting a similar convention as for smooth objects (see Subsection 6.4), for an \( \infty \)-category \( C \) we write

\[
\text{Bundle}(C) := \text{Fun}(\mathbb{N}(\text{Bundle}^{op}), C)
\]

for the \( \infty \)-category of bundle objects in \( C \). Similar conventions apply to \( \text{Bundle}_{geom}^\delta \)-objects below.

We consider the bundle set

\[
\text{Riem}^\delta \in \text{Bundle}(\mathbb{N}(\text{Set}))
\]
which associates to each bundle $\pi : W \to B$ the set of Riemannian structures $\text{Riem}^\delta(\pi)$ on $\pi$, see Definition 4.1. Given a morphism $(f,F) : \pi' \to \pi$ as in diagram (100) and $g = (g^{T^v\pi}, T^h\pi) \in \text{Riem}^\delta(\pi)$ we define the induced Riemannian structure $(f,F)^* g = (g^{T^v\pi'}, T^h\pi') \in \text{Riem}^\delta(\pi')$ as follows. The differential $dF$ restricts to an isomorphism $dF_{|T^v\pi'} : T^v\pi' \to T^v\pi$. We define the induced vertical metric as a pull-back

$$g^{T^v\pi} = (dF_{|T^v\pi'})^* g^{T^v\pi}.$$  

Furthermore, we define the induced horizontal subbundle by taking the preimage

$$T^h\pi' := dF^{-1}(T^h\pi).$$

We let

$$q : \text{Bundle}^\delta_{\text{geom}} \to \text{Bundle} \tag{101}$$

denote the Grothendieck construction of $\text{Riem}^\delta$. The objects of the category $\text{Bundle}^\delta_{\text{geom}}$ are pairs $(\pi : W \to B, g)$ consisting of a bundle together with a Riemannian structure $g \in \text{Riem}^\delta(\pi)$. We refer to the objects of $\text{Bundle}^\delta_{\text{geom}}$ as bundles with geometry. The functor $q$ forgets the Riemannian structure.

We have two functors

$$\text{dom, ran} : \text{Bundle} \to \text{Mf}$$

given by the evaluation at the domain and range

$$\text{dom}(\pi : W \to B) := W, \quad \text{ran}(\pi : W \to B) := B.$$  

They induce corresponding pull-back functors between the categories of smooth objects, $\text{Bundle}^\delta_{\text{geom}}$-objects, and their $\infty$-categorical analogs. Given a smooth space (smooth chain complex, ...) $X$, we define (in order to shorten the notation) the $\text{Bundle}^\delta_{\text{geom}}$-spaces (bundle chain complexes, ...)

$$Xd := X \circ \text{dom} \circ q, \quad Xr := X \circ \text{ran} \circ q. \tag{102}$$

The usage of this notation is as follows. Assume that $X$ is a smooth set and we have a natural way to define a push-forward $X(W) \to X(B)$ for a bundle $\pi : W \to B$ which depends on the choice of a Riemannian structure on $\pi$. Using the notation above we can formalize this by saying that we define a map of $\text{Bundle}^\delta_{\text{geom}}$-spaces $Xd \to Xr$.

As a first instance of this reasoning, we are going to define morphisms of $\text{Bundle}^\delta_{\text{geom}}$-chain complexes

$$\int : \Omega Ad \to \Omega Ar, \quad \int_\sigma : \sigma\Omega Ad \to \sigma\Omega Ar$$

as follows. If $(\pi : W \to B, g) \in \text{Bundle}^\delta_{\text{geom}}$, then we define the morphism of chain complexes by

$$\int : \Omega A(W) \to \Omega A(B)$$
\[ \Omega A(W) \ni \alpha \mapsto \int_{W/B} \alpha \wedge e(g) \in \Omega A(B) \]

It depends on the Riemannian structure on \( \pi \) via the Euler form \( e(g) \in Z^n(\Omega(W, \Lambda)) \) defined in (96). The transformation \( \int_{\sigma} \) is obtained from \( \int \) by restriction. We get a commutative diagram in \( \text{Bundle}^\delta_{\text{geom}}(N(Ch)) \):

\[
\begin{array}{cccc}
\Omega A d & \xleftarrow{\sigma} & \sigma \Omega A d \\
\downarrow f & & \downarrow f_{\sigma} \\
\Omega A r & \xleftarrow{\sigma} & \sigma \Omega A r
\end{array}
\]

We indicate the image of a bundle chain complex under

\( \text{Bundle}^\delta_{\text{geom}}(N(Ch)) \to \text{Bundle}^\delta_{\text{geom}}(N(Ch)[W^{-1}]) \)

by a subscript \( \infty \) and observe that

\[ \Omega A_{d \infty} \cong \Omega A_{\infty d} \]

Then (103) induces the diagram

\[
\begin{array}{cccc}
\Omega A_{\infty d} & \xleftarrow{\sigma} & \sigma \Omega A_{\infty d} \\
\downarrow f & & \downarrow f_{\sigma} \\
\Omega A_{\infty r} & \xleftarrow{\sigma} & \sigma \Omega A_{\infty r}
\end{array}
\]

in the \( \infty \)-category \( \text{Bundle}^\delta_{\text{geom}}(N(Ch)[W^{-1}]) \). Applying the functor between \( \infty \)-categories

\[ H : \text{Bundle}^\delta_{\text{geom}}(N(Ch)[W^{-1}]) \to \text{Bundle}^\delta_{\text{geom}}(N(Sp)[W^{-1}]) \]

we obtain the commutative diagram

\[
\begin{array}{cccc}
H(\Omega A_{\infty d}) & \xleftarrow{H(\sigma \Omega A_{\infty d})} & H(\sigma \Omega A_{\infty d}) \\
\downarrow H(f) & & \downarrow H(f_{\sigma}) \\
H(\Omega A_{\infty r}) & \xleftarrow{H(\sigma \Omega A_{\infty r})} & H(\sigma \Omega A_{\infty r})
\end{array}
\]

in the \( \infty \)-category \( \text{Bundle}^\delta_{\text{geom}}(N(Sp)[W^{-1}]) \). This is the right square of (98).

### 4.4 Transfer structures

In this subsection we discuss the details of the construction of the Becker-Gottlieb transfer \[\text{[BG75]}\]. Its main goal is the construction of the left vertical arrow in (98). We first observe that the map (94) depends on choices. In order to formalize this we will subsume these choices under the notion of transfer data.
Definition 4.6  Transfer data on a proper submersion \( \pi : W \to B \) consists of
1. an integer \( k \geq 0 \) (the dimension of the transfer data),
2. a fibrewise embedding
\[
\begin{array}{ccc}
W & \hookrightarrow & B \times \mathbb{R}^k \\
\downarrow & & \downarrow \\
B & & 
\end{array}
\]
3. an extension of the embedding to an open embedding \( \text{emb} : N \to B \times \mathbb{R}^k \) of the fibrewise normal bundle \( N \to W \) of this inclusion.

We will simply write \( b \) for a given choice of transfer data on \( \pi : W \to B \). There is an obvious construction of a stabilization \( b \mapsto S^l(b) \) which increases the dimension of \( b \) from \( k \) to \( k + l \). Stabilization introduces an equivalence relation amongst transfer data. The equivalence classes are referred to as stable transfer data.

If \( V \to X \) is a vector bundle, then \( X^V \) denotes its Thom space. A transfer datum \( b \) gives rise to a map
\[
\text{tr}(b) : \Sigma^k B_+ \cong B^{B \times \mathbb{R}^k} \xrightarrow{\text{clps}} W^N \xrightarrow{z} W^{T^v \pi \oplus N} \cong W^{W \times \mathbb{R}^k} \cong \Sigma^k W_+ .
\]

Here \( \text{clps} : B^{B \times \mathbb{R}^k} \to W^N \) is the collapse map which is given by \( \text{emb}^{-1} \) on the image of the embedding \( \text{emb} : N \to B \times \mathbb{R}^k \), and which maps the complement of that image to the base point of \( W^N \). The map \( z : W^N \to W^{T^v \pi \oplus N} \) is induced by the zero section of the vertical tangent bundle \( T^v \pi \). We use the same symbol
\[
\text{tr}(b) : \Sigma^\infty B_+ \to \Sigma^\infty W_+ \quad (106)
\]
to denote the associated stable transfer map between spectra in topological spaces (this is the map \( (94) \), though we add the argument in order to indicate the dependence on the transfer data). The construction of the stable transfer map is compatible with stabilization. Indeed, for \( l \geq 0 \), we have \( \text{tr}(S^l(b)) = \text{tr}(b) \), so that the stable transfer map only depends on the stable transfer data represented by \( b \). The space of choices for the stable transfer data \( b \) is contractible so that \( (94) \) is well-defined up to contractible choice. We will only need the fact that any two choices of stable transfer data can be connected by a path for the proof of Lemma 4.3; compare \([BG75\text{ Sec. 3}]\).

We now consider a morphism \( \phi : \pi' \to \pi \) (i.e. a diagram of the shape \((100)\)) in \textbf{Bundle} and a choice of transfer data \( b \) on \( \pi \). Then there is a natural and functorial (with respect to compositions of morphisms in \textbf{Bundle}) way to define transfer data \( b' := \phi^* b \) on \( \pi' \). It has the same dimension as \( b \). The second component of the fibrewise embedding of \( b' \) is given by the composition
\[
W' \to W \to B \times \mathbb{R}^k \xrightarrow{pr} \mathbb{R}^k ,
\]
where the second map is the fibrewise embedding for \( b \). We get a canonical identification of the fibrewise normal bundles \( N' \to F^* N \), and the second component of the embedding for \( b' \) is given by
\[
N' \to F^* N \xrightarrow{\text{emb}} B \times \mathbb{R}^k \xrightarrow{pr} \mathbb{R}^k .
\]
The first component of these maps is the natural map to $B'$ in both cases. On checks that the following diagram in $\textbf{Sp}_\text{Top}$ strictly commutes

$$\begin{array}{c}
\Sigma^\infty W'_+ \ar[r] & \Sigma^\infty W_+ \\
\Sigma^\infty B'_+ \ar[u]^{\text{tr}(b')} & \ar[l] \Sigma^\infty B_+ \ar[u]^{\text{tr}(b)}
\end{array} \quad (107)
$$

We consider the smooth set

$$\text{Trans}^\delta \in \text{Bundle}(\text{N}(\text{Set}))$$

which associates to a bundle $\pi : W \to B$ the set of stable transfer data, and we let

$$q : \text{Bundle}^\delta_{\text{trans}} \to \text{Bundle}$$

denote the associated Grothendieck construction. As before we write

$$\text{Sm}_\infty(E)d := \text{Sm}_\infty(E) \circ \text{dom} \circ q, \quad \text{Sm}_\infty(E)r := \text{Sm}_\infty(E) \circ \text{ran} \circ q.$$  

**Lemma 4.7** We have a map

$$\text{tr} : \text{Sm}_\infty(E)d \to \text{Sm}_\infty(E)r \quad (108)$$

in $\text{Bundle}^\delta_{\text{trans}}(\text{N}(\text{Sp})[W^{-1}])$.

**Proof.** We consider the functors $D, R : \text{Bundle} \to \text{Sp}_\text{Top}$ given by

$$D : (\pi : W \to B) \mapsto \Sigma^\infty W_+, \quad R : (\pi : W \to B) \mapsto \Sigma^\infty B_+.$$  

The discussion leading to the diagram \([107]\) shows that we have a natural transformation $\tilde{\text{tr}} : D \circ q \to R \circ q$ whose evaluation at the pair $(\pi, b)$ of a bundle $\pi$ with transfer datum $b$ is the map $\tilde{\text{tr}}(\pi, b) := \text{tr}(b)$ (see \([106]\)). Recall the definition \([184]\)

$$\tilde{\text{Sm}}_\infty(E)(M) := \text{Map}(\text{sing}(\Sigma^\infty M_+), E).$$

We thus get a natural map

$$\tilde{\text{tr}} : \tilde{\text{Sm}}_\infty(E)d \to \tilde{\text{Sm}}_\infty(E)r \quad (109)$$

between objects of $\text{Bundle}^\delta_{\text{trans}}(\text{N}(\text{Sp})[W^{-1}])$. We now define the map

$$\text{tr} : \text{Sm}_\infty(E)d \to \text{Sm}_\infty(E)r$$

70
such that the diagram
\[
\begin{array}{ccc}
\widetilde{\text{Sm}}_\infty(E)d & \xrightarrow{\text{tr}} & \widetilde{\text{Sm}}_\infty(E)r \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{Sm}_\infty(E)d & \xrightarrow{\text{tr}} & \text{Sm}_\infty(E)r
\end{array}
\]
commutes in \(\text{Bundle}^\delta_{\text{trans}}(\mathbb{N}(\text{Sp})[W^{-1}])\), where the upper line is the map \((109)\) and the vertical maps are the equivalences induced by \((185)\).

If \(f : E \to F\) is a map of spectra, then by construction and the naturality of the equivalence \((185)\) in the spectrum we have a commutative diagram of \(\text{Bundle}^\delta_{\text{trans}}\)-spectra
\[
\begin{array}{ccc}
\text{Sm}_\infty(E)d & \xrightarrow{\text{Sm}_\infty(f)d} & \text{Sm}_\infty(F)d \\
\downarrow{\text{tr}} & & \downarrow{\text{tr}} \\
\text{Sm}_\infty(E)r & \xrightarrow{\text{Sm}_\infty(f)r} & \text{Sm}_\infty(F)r
\end{array}
\]

4.5 The left square in \((98)\) and the end of proof of Theorem 4.2

We now let \(q : \text{Bundle}^\delta_{\text{geom,trans}} \to \text{Bundle}\) be the Grothendieck construction on the smooth set
\[
\text{Riem}^\delta \times \text{Trans}^\delta \in \text{Bundle}(\mathbb{N}((\text{Set}))).
\]
The constructions given in Subsections 4.3 and 4.4 provide maps \((108)\) and diagrams of the shape \((105)\) in
\[
\text{Bundle}^\delta_{\text{geom,trans}}(\mathbb{N}(\text{Sp})[W^{-1}])
\]
via the obvious pull-backs along the functors which forget either the Riemannian or the transfer structures.

We now recall the steps of construction of the map \((9)\)
\[
\text{rat} : \text{Sm}_\infty(E) \to H(\Omega A_\infty)
\]
in \(\text{Sm}(\mathbb{N}(\text{Sp})[W^{-1}])\). It depends on the choice of an equivalence of spectra \(c : E \mathbb{R} \to H(A)\) in \(\mathbb{N}(\text{Sp})[W^{-1}]\) which induces the second map in
\[
\text{Sm}_\infty(E) \xrightarrow{\epsilon_c} \text{Sm}_\infty(E \mathbb{R}) \xrightarrow{c} \text{Sm}_\infty(H(A))
\]

It furthermore involves the spectrum level de Rham equivalence
\[
\text{rat} : \text{Sm}_\infty(E) \to \text{Sm}_\infty(H(A)) \xrightarrow{\sim} H(\Omega A_\infty)
\]
constructed in Proposition \((6.20)\). The map \(\text{rat}\) is now defined as the composition
\[
\text{rat} : \text{Sm}_\infty(E) \xrightarrow{\epsilon_c} \text{Sm}_\infty(H(A)) \xleftarrow{\sim} H(\Omega A_\infty)
\]
Because of \((111)\) we have a commutative diagram
\[
\begin{array}{ccc}
\text{Sm}_\infty(E)d & \xrightarrow{\text{tr}} & \text{Sm}_\infty(H(A))d \\
\downarrow & & \downarrow \\
\text{Sm}_\infty(E)r & \xrightarrow{\text{tr}} & \text{Sm}_\infty(H(A))r
\end{array}
\] (112)

in \(\text{Bundle}^\delta_{\text{geom,trans}}(\mathbb{N}(\text{Sp})[W^{-1}])\). Thus in order to complete the construction of the left square in \((98)\) we must show the following proposition.

**Proposition 4.8** There exists a preferred commutative square
\[
\begin{array}{ccc}
H(\Omega A_\infty)d & \xrightarrow{j} & \text{Sm}_\infty(H(A))d \\
\downarrow & & \downarrow \\
H(\Omega A_\infty)r & \xrightarrow{j} & \text{Sm}_\infty(H(A))r
\end{array}
\] (113)

in \(\text{Bundle}^\delta_{\text{geom,trans}}(\mathbb{N}(\text{Sp})[W^{-1}])\).

**Proof.** This proposition is non-trivial since it compares transfers in two different frameworks. On the one hand, the left vertical arrow in \((113)\) is defined via integration of differential forms. On the other hand, the right vertical arrow is defined using homotopy theory. In order to understand the transition between analysis and homotopy theory we decompose the square \((113)\) into a composite \((114)\) of many smaller squares. Note that, although we are actually constructing a diagram in \(\text{Bundle}^\delta_{\text{geom,trans}}(\mathbb{N}(\text{Sp})[W^{-1}])\), in order to make the argument readable we write down the construction evaluated at a proper submersion \(\pi : W \to B\) with a geometry \(g\) and transfer data \(b\).

The decomposition of \((113)\) is as follows.

\[
\begin{array}{ccc}
H(\Omega A_\infty(W)) & \xrightarrow{j} & \text{Sm}_\infty(H(A))(W) \\
\downarrow & & \downarrow \\
H(\Omega A_{c,\infty}(W \times \mathbb{R}^k)[k]) & \xrightarrow{j} & \text{Sm}_{c,\infty}(H(A[k])(W \times \mathbb{R}^k) \\
\downarrow \cong & & \downarrow \cong \\
H(\Omega A_{c,\infty}(T^v\pi \oplus N)[k])) & \xrightarrow{j} & \text{Sm}_{c,\infty}(H(A[k])(T^v\pi \oplus N) \\
\downarrow z^* & & \downarrow z^* \\
H(\Omega A_{c,\infty}(B \times \mathbb{R}^k)[k]) & \xrightarrow{j} & \text{Sm}_{c,\infty}(H(A[k])(B \times \mathbb{R}^k) \\
\downarrow \text{emb} & & \downarrow \text{emb} \\
H(\Omega A_\infty(B)) & \xrightarrow{j} & \text{Sm}_\infty(H(A))(B) \\
\downarrow \text{desusp} & & \downarrow \text{desusp}
\end{array}
\] (114)
The normal bundle $N \to W$ and the map $\text{emb} : N \to B \times \mathbb{R}^k$ are part of the transfer data $b$. The map $z^*$ is induced by the zero section of $T^v \pi$. The diagram involves forms and function spectra with proper support for vector bundles as well as the de Rham map with proper support [205], which are introduced in Subsection 6.10. Furthermore, it involves the Thom forms for euclidean vector bundles with connection introduced in Subsection 6.13. The choice of connections will be made precise during the proof of the next Lemma.

Here and in the text below the symbol $\Lambda$ stands for an appropriate orientation bundle; the precise object will be clear from the context in each case. The commutativity of the right hexagon is the definition of $\text{tr}(b)$, written in terms of function spectra with proper support. The outer square is the desired commutative square (113). We get it as the composite of the smaller cells, all of which commute. However, there are three cells which do not obviously commute, namely the upper left and middle, and the lower middle. These will be discussed in the subsequent lemmas.

**Lemma 4.9** The upper left cell

\[
\begin{array}{c}
H(\Omega A_c(W))[n] \xrightarrow{\text{pr}^*(\ldots \wedge e(g))} H(\Omega A_{c,\infty}(W \times \mathbb{R}^k)[k]) \xrightarrow{\approx} H(\Omega A_{c,\infty}(T^v \pi \oplus N)[k]) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow z^* \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
H(\Omega A_{c,\infty}(W, \Lambda)[n]) \xrightarrow{\text{pr}^*(\ldots) \cup U(\nabla N)} H(\Omega A_{c,\infty}(N)[k])
\end{array}
\]

in (114) commutes.

**Proof.** We have a natural isomorphism of euclidean vector bundles

\[W \times \mathbb{R}^k \cong T^v \pi \oplus N.\]

The bundle on the left-hand side is equipped with the trivial connection $\nabla^{W \times \mathbb{R}^k}$. The right-hand side has the connection $\nabla^{T^v \pi \oplus \nabla N}$. Here $\nabla^{T^v \pi}$ is the connection induced by the Riemannian structure $g$, and $\nabla^N$ is the connection obtained by the projection of $\nabla^{W \times \mathbb{R}^k}$ to $N$. For a euclidean vector bundle $V \to M$ with connection $\nabla^V$ we use the Thom form $U(\nabla^V) \in \Omega_c(V, \Lambda)$ with proper support introduced in 6.13. Then we have a natural transgression form

\[
\tilde{U}(\nabla^{W \times \mathbb{R}^k}, \nabla^{T^v \pi \oplus \nabla N}) \in \Omega_c(W \times \mathbb{R}^k, \Lambda)
\]

such that

\[
d\tilde{U}(\nabla^{W \times \mathbb{R}^k}, \nabla^{T^v \pi \oplus \nabla N}) = U(\nabla^{W \times \mathbb{R}^k}) - U(\nabla^{T^v \pi \oplus \nabla N}).
\]

The pull-back by the inclusion $z : N \to T^v \pi \oplus N$ of the natural transgression form provides the chain homotopy for the version of (115) obtained before application of the Eilenberg-MacLane spectrum functor $H$. Here we use that $e(g) = 0_{T^v \pi} U(\nabla^{T^v \pi})$. The chain homotopy induces the required homotopy after application of $H$, and this construction is natural in the bundle $\pi$. \qed
Lemma 4.10 The upper and lower middle cells in (114) commute.

Proof. It suffices to show that the lower right square commutes. This is because, up to replacing $W$ with $B$, the vertical maps in the lower right square are the inverses of the vertical maps of the upper right square. Commutativity of the lower right cell is provided by Proposition 6.23.

This finishes the construction of the diagram (the functorial version of (98))

\[
\begin{align*}
\text{Sm}_\infty(E)d & \xrightarrow{\text{rat}} H(\Omega A_{\infty})d \xleftarrow{\text{H}(f)} H(\sigma\Omega A_{\infty})d \\
\text{Sm}_\infty(E)r & \xrightarrow{\text{rat}} H(\Omega A_{\infty})r \xleftarrow{\text{H}(f_\ast)} H(\sigma\Omega A_{\infty})r
\end{align*}
\]

in $\text{Bundle}^{\delta}_{\text{geom,trans}}(N(\text{Sp})[W^{-1}])$ which induces the transfer map of bundle spectra

\[\hat{t}_r : \text{Diff}(E)d \to \text{Diff}(E)r.\]

Its evaluation at a proper submersion $W \to B$ with Riemannian structure $g$ and transfer data $b$ provides, upon application of $\pi_0$, the differential transfer

\[\hat{t}_r : \hat{E}^0(W) \to \hat{E}^0(B)\]

of Theorem 4.2.

The diagram (97) commutes by construction. In order to finish the proof of Theorem 4.2 we must prove Lemma 4.3.

Proof. [of Lemma 4.3] Given another choice of stable transfer data $b'$ on $\pi$, we can choose stable transfer data $\tilde{b}$ on $\tilde{\pi} := \pi \times \text{id} : W \times [0, 1] \to B \times [0, 1]$ which interpolates between $b$ and $b'$. We equip the bundle $\tilde{\pi}$ with the Riemannian structure $\tilde{g}$ induced from $g$. We let $\text{pr}_W : W \times [0, 1] \to W$, $\text{pr}_B : B \times [0, 1] \to B$ be the projections and $i_j : B \to B \times [0, 1]$, $j = 0, 1$ be the inclusions at the endpoints of the interval. For the moment we let $t'_r$ and $\hat{t}_r$ denote the differential transfers for $(\pi, g, b')$ and $(\tilde{\pi}, \tilde{g}, \tilde{b})$. Using the naturality of the diagram (98) we get

\[t'_r = i_0^* \circ t_r \circ \text{pr}_W^*, \quad t'_r = i_1^* \circ t_r \circ \text{pr}_W^*.\]

We now apply the homotopy formula (7) which for $x \in \hat{E}^0(W)$ gives

\[t'_r(x) - t_r(x) = a\left(\int_{[0,1] \times B/B} R(\text{pr}_W^*x)\right).\]

Since $e(\tilde{g}) = \text{pr}_W^*e(g)$ we see that

\[\int_{[0,1] \times B/B} R(\text{pr}_W^*x) = \int_{[0,1] \times B/B} \text{pr}_B^* \int_{[0,1] \times W/[0,1] \times B} R(x) \wedge e(g) = 0\]

and therefore

\[t_r(x) = t'_r(x).\]
4.6 Proof of Proposition 4.4

By Equation (181) we have a natural isomorphism

\[ \mathbb{E} \mathbb{R}/\mathbb{Z}^{-1}(M) \cong \pi_0(\Sigma^{-1}\mathbb{S}_{\infty}(\mathbb{E}\mathbb{R}/\mathbb{Z})(M)) , \]  

(116)

where the spectrum \( \Sigma^{-1}\mathbb{S}_{\infty}(\mathbb{E}\mathbb{R}/\mathbb{Z})(M) \) fits into the homotopy pull-back (15).

Now let \( W \to B \) be a proper submersion with Riemannian structure \( g \) and transfer data \( b \). Under the identification (116), the Becker-Gottlieb transfer

\[ \text{tr}^* : \mathbb{E} \mathbb{R}/\mathbb{Z}^{-1}(W) \to \mathbb{E} \mathbb{R}/\mathbb{Z}^{-1}(B) \]  

(117)

corresponds to the map induced on \( \pi_0 \) by the map

\[ \text{tr} : \Sigma^{-1}\mathbb{S}_{\infty}(\mathbb{E}\mathbb{R}/\mathbb{Z})(W) \to \Sigma^{-1}\mathbb{S}_{\infty}(\mathbb{E}\mathbb{R}/\mathbb{Z})(B), \]

which in turn is induced by the map of pull-back squares

Using Proposition 4.8 and the diagrams (112) we see that the same map (117) is induced by the map of pull-back squares

\[ \text{Sm}_{\infty}(E)(W) \xrightarrow{\text{rat}} H(\Omega A_{\infty}(W)) \leftarrow 0 . \]  

(118)

By (16) the homotopy limit of upper line in (118) is \( \Sigma^{-1}\mathbb{S}_{\infty}(\mathbb{E}\mathbb{R}/\mathbb{Z})(W) \), while the homotopy limit of the lower is \( \Sigma^{-1}\mathbb{S}_{\infty}(\mathbb{E}\mathbb{R}/\mathbb{Z})(B) \). The map \( 0 \to H(\sigma \Omega A_{\infty}(W)) \) induces a map of pull-back squares

\[ \text{Sm}_{\infty}(E)(W) \xrightarrow{\text{rat}} H(\Omega A_{\infty}(W)) \leftarrow 0 \]

which induces the map

\[ \Sigma^{-1}\mathbb{S}_{\infty}(\mathbb{E}\mathbb{R}/\mathbb{Z})(W) \to \text{Diff}(E)(W) . \]

We have a similar diagram for \( B \). In order to prove Proposition 4.6 we must verify that the diagram (118) is obtained as a pull-back of the diagram (98) along these maps. In fact, these pull-backs induce the identity between the left squares of (118) and (98). For
the right squares we apply the functors $\mathbb{N}(\mathbf{Ch}) \to \mathbb{N}(\mathbf{Ch})[W^{-1}]$ and $H$ to the strictly commutative diagram of chain complexes

4.7 Functoriality of the transfer

We consider two composable proper submersions

$$W \xrightarrow{\pi} B \xrightarrow{\sigma} Z.$$ The composition $\kappa := \sigma \circ \pi$ is again a proper submersion. Let $g_{\pi}$ and $g_{\sigma}$ be Riemannian structures on $\pi$ and $\sigma$ (Definition 4.1). Then we get a Riemannian structure $g_{\kappa}$ on the composition as follows. The decompositions $TW \cong T^v\pi \oplus \pi^* TB$ and $TB \cong T^v\sigma \oplus \sigma^* T Z$ given by the horizontal distributions of $g_{\pi}$ and $g_{\sigma}$ induce a decomposition $T^v\kappa \cong T^v\pi \oplus \pi^* T^v\sigma$, where $\pi^* T^v\sigma$ is canonically identified with a subbundle of $T^h\pi$. We define the vertical metric of the Riemannian structure $g_{\kappa}$ by

$$g^{T^v\kappa} := g^{T^v\pi} \oplus \pi^* g^{T^v\sigma}.$$ Furthermore, we let

$$\pi^* T^h\sigma =: T^h\kappa \subset T^h\pi$$

be the horizontal subspace of $g_{\kappa}$.

We now have two connections on $T^v\kappa$, namely the connection $\nabla^{T^v\kappa}$ induced by $g_{\kappa}$ and the direct sum connection $\nabla^{\oplus} := \nabla^{T^v\pi} \oplus \pi^* \nabla^{T^v\sigma}$. In general they do not coincide. We define the transgression Euler form

$$\eta := \tilde{e}(\nabla^{T^v\kappa}, \nabla^{\oplus}) \in \Omega^{n-1}(W, \Lambda), \quad (119)$$

where $n := \dim(W) - \dim(Z)$. It satisfies

$$d\eta = e(g_{\kappa}) - e(g_{\pi}) \wedge \pi^* e(g_{\sigma}).$$

We now choose transfer data $b_{\sigma}$ and $b_{\pi}$ for $\sigma$ and $\pi$. 

76
Lemma 4.11 There is a canonical choice of transfer data $b_\kappa$ such that
\[ \text{tr}(b_\sigma) \circ \text{tr}(b_\pi) = \text{tr}(b_\kappa). \]

Proof. Let $W \hookrightarrow B \times \mathbb{R}^{k_\pi}$ and $B \hookrightarrow Z \times \mathbb{R}^{k_\sigma}$ be the fibrewise embeddings for $b_\pi$ and $b_\sigma$. They induced a fibrewise embedding $W \hookrightarrow Z \times \mathbb{R}^{k_\pi + k_\sigma}$ by composition. Let $\text{emb}_\pi : N_\pi \hookrightarrow B \times \mathbb{R}^{k_\pi}$ and $\text{emb}_\sigma : N_\sigma \hookrightarrow Z \times \mathbb{R}^{k_\sigma}$ be the extensions to open embeddings of normal bundles. Then we get an open embedding
\[ \text{emb}_\kappa : N_\kappa := N_\pi \oplus \pi^* N_\sigma \hookrightarrow Z \times \mathbb{R}^{k_\kappa} \]
with $k_\kappa := k_\pi + k_\sigma$. With these choices the diagram
\[ \Sigma^{k_\kappa} Z_+ \xrightarrow{\text{clps}_\kappa} W_{N_\kappa} \xrightarrow{z_\kappa} \Sigma^{k_\kappa} W_+ \]
\[ \Sigma^{k_\pi} \Sigma^{k_\sigma} Z_+ \xrightarrow{\text{clps}_\pi \text{clps}_\sigma} \Sigma^{k_\pi} B_{N_\sigma} \xrightarrow{z_\sigma} \Sigma^{k_\kappa} B_+ \]
commutes. This implies the assertion. \qed

We let $\hat{\text{tr}}_\pi$, $\hat{\text{tr}}_\sigma$, and $\hat{\text{tr}}_\kappa$ denote the differential Becker-Gottlieb transfers associated by Theorem 4.2 to the proper submersions $\pi$, $\sigma$, and $\kappa$ with the geometries $g_\pi$, $g_\sigma$, and $g_\kappa$ as above and transfer data $b_\pi$, $b_\sigma$, and $b_\kappa$ as described in Lemma 4.11. The following proposition clarifies the functoriality of the differential Becker-Gottlieb transfer with respect to iterated fibre bundles. Recall the definition of the transgression Euler form $\eta$ in (119).

Proposition 4.12 For $x \in \hat{E}^0(W)$ we have
\[ \hat{\text{tr}}_\kappa(x) = \hat{\text{tr}}_\sigma(\hat{\text{tr}}_\pi(x)) + a\left( \int_{W/Z} R(x) \wedge \eta \right). \]

Proof. The map $\hat{\text{tr}}_\sigma \circ \hat{\text{tr}}_\pi$ is induced by the composition of two diagrams of the form (98)
\[ \text{Sm}_\infty(E)(W) \xrightarrow{\text{rat}} H(\Omega A_\infty(W)) \xleftarrow{} H(\sigma \Omega A_\infty(W)) \]
\[ \text{Sm}_\infty(E)(B) \xrightarrow{\text{rat}} H(\Omega A_\infty(B)) \xleftarrow{} H(\sigma \Omega A_\infty(B)) \]
\[ \text{Sm}_\infty(E)(Z) \xrightarrow{\text{rat}} H(\Omega A_\infty(Z)) \xleftarrow{} H(\sigma \Omega A_\infty(Z)) \]
In general, given an element $\omega \in \Omega A^{-1}(M)$, the transformation
\[ \cdots + a(R(\ldots) \wedge \omega) : \hat{E}^0(M) \rightarrow \hat{E}^0(M) \]
is induced by the diagram

\[
\begin{array}{c}
\text{Sm}_\infty(E) \xrightarrow{\text{rat}} H(\Omega A_\infty(M)) \leftrightarrow H(\sigma \Omega A_\infty(M)) \\
\text{Sm}_\infty(E) \xrightarrow{\text{rat}} H(\Omega A_\infty(M)) \leftrightarrow H(\sigma \Omega A_\infty(M))
\end{array}
\]

where the homotopy in the right square is induced from the chain homotopy between \(\text{id}_{\Omega A(M)}\) and \(\text{id}_{\Omega A(M)} + \cdots \wedge d\omega\) given by \(\omega\). We conclude that the right-hand side of Equation (121) is induced by

\[
\begin{array}{c}
\text{Sm}_\infty(E)(W) \xrightarrow{\text{rat}} H(\Omega A_\infty(W)) \leftrightarrow H(\sigma \Omega A_\infty(W)) \\
\text{Sm}_\infty(E)(B) \xrightarrow{\text{rat}} H(\Omega A_\infty(B)) \leftrightarrow H(\sigma \Omega A_\infty(B)) \\
\text{Sm}_\infty(E)(Z) \xrightarrow{\text{rat}} H(\Omega A_\infty(Z)) \leftrightarrow H(\sigma \Omega A_\infty(Z))
\end{array}
\]

It can be written as

\[
\begin{array}{c}
\text{Sm}_\infty(E)(W) \xrightarrow{\text{rat}} H(\Omega A_\infty(W)) \leftrightarrow H(\sigma \Omega A_\infty(W)) \\
\text{Sm}_\infty(E)(B) \xrightarrow{\text{rat}} H(\Omega A_\infty(B)) \leftrightarrow H(\sigma \Omega A_\infty(B)) \\
\text{Sm}_\infty(E)(Z) \xrightarrow{\text{rat}} H(\Omega A_\infty(Z)) \leftrightarrow H(\sigma \Omega A_\infty(Z))
\end{array}
\]

where now the right square comes from a strictly commuting diagram of complexes. We therefore must show that the left part of the diagram (124) is equivalent to the square

\[
\begin{array}{c}
\text{Sm}_\infty(E)(W) \xrightarrow{\text{rat}} H(\Omega A_\infty(W)) \leftrightarrow H(\sigma \Omega A_\infty(W)) \\
\text{Sm}_\infty(E)(Z) \xrightarrow{\text{rat}} H(\Omega A_\infty(Z)) \leftrightarrow H(\sigma \Omega A_\infty(Z))
\end{array}
\]

used in the construction of \(\text{tr}_\kappa\). Recall that we can expand this to

\[
\begin{array}{c}
\text{Sm}_\infty(E)(W) \xrightarrow{\text{rat}} H(\Omega A_\infty(W)) \leftrightarrow H(\sigma \Omega A_\infty(W)) \\
\text{Sm}_\infty(E)(Z) \xrightarrow{\text{rat}} H(\Omega A_\infty(Z)) \leftrightarrow H(\sigma \Omega A_\infty(Z))
\end{array}
\]
Since $\text{tr}_\kappa = \text{tr}_\sigma \circ \text{tr}_\pi$ is functorial in the spectrum argument of $\text{Sm}_\infty$, the argument boils down to showing that the outer square of

\[
\begin{array}{ccc}
H(\Omega A_\infty(W)) & \longrightarrow & \text{Sm}_\infty(H(A))(W) \\
H(f_{W/Z} \cdot e(g_\kappa)) & \downarrow & \text{tr}_\pi \\
H(\Omega A_\infty(Z)) & \longrightarrow & \text{Sm}_\infty(H(A))(Z)
\end{array}
\]

obtained by the composition of the four small squares of (125) is equivalent to the square constructed in Proposition 4.8 evaluated at the submersion $\kappa : W \to Z$ with geometry $g_\kappa$. In detail the composition of the two inner squares of (125) expands to the composition of two diagrams of the form (114), one for $\pi$ and another one for $\sigma$. One checks that the assertion follows from the two Lemmas 4.13 and 4.14.

**Lemma 4.13** The composition of the three homotopies

\[
\begin{array}{ccc}
H(f_{W/Z} \cdot e(g_\kappa)) & \downarrow & H(f_{W/B} \cdot e(g_\pi)) \\
H(\Omega A_\infty(W)) & \longrightarrow & H(\Omega A_\infty(B)) \\
\eta & \downarrow & \eta \\
H(f_{B/Z} \cdot e(g_\sigma)) & \downarrow & H(f_{B/Z} \cdot e(g_\kappa)) \\
H(\Omega A_\infty(B)) & \longrightarrow & H(\Omega A_\infty(B))
\end{array}
\]

given by $\eta$ and two applications of Lemma 4.9 is homotopic to the homotopy

\[
\begin{array}{ccc}
H(f_{W/Z} \cdot e(g_\kappa)) & \downarrow & H(f_{W/B} \cdot e(g_\pi)) \\
H(\Omega A_\infty(W)) & \longrightarrow & H(\Omega A_\infty(B)) \\
\eta & \downarrow & \eta \\
H(f_{B/Z} \cdot e(g_\sigma)) & \downarrow & H(f_{B/Z} \cdot e(g_\kappa)) \\
H(\Omega A_\infty(B)) & \longrightarrow & H(\Omega A_\infty(B))
\end{array}
\]

for the composition $\kappa$ (again by Lemma 4.9), where $f_\sigma$, $f_\pi$, and $f_\kappa$ are the corresponding vertical compositions in the left column of (114).

**Proof.** This follows from the fact that all homotopies are obtained from transgressions and that the spaces of connections are contractible. \qed

**Lemma 4.14** The composition of the two middle squares in (125) is equivalent to the corresponding middle square in (114) for $\kappa$. 79
Proof. We expand the small squares in (125) as a composition of two diagrams of the form (114) (one on top of the other) and look at the sequence from up to down of small squares obtained from the middle parts. The upper middle square of the second and the lower middle square of the first (those squares are dealt with in Lemma 4.10) are inverse to each other and cancel out. The remaining ladder defines a square

\[
\begin{array}{ccc}
H(\Omega A_\infty(W)) & \longrightarrow & \text{Sm}_\infty(H(A))(W) \\
\downarrow & & \downarrow \\
H(\Omega A_\infty(Z)) & \longrightarrow & \text{Sm}_\infty(H(A))(Z)
\end{array}
\]

Since (120) commutes this square is equivalent to the square for \(\kappa\). This finishes the proof of Proposition 4.12.

4.8 Proof of Proposition 4.5

The idea is the same as in the proof of Theorem 3.35. We use the notation introduced there. We consider a maps \(\text{Diff}(E) \rightarrow \text{Diff}(E')\), \(\Omega A \rightarrow \Omega A'\), ... as maps of smooth \(I\)-spectra, \(I\)-complexes etc. On checks that the proof of Theorem 4.2 can be modified by replacing spaces, spectra etc. by corresponding \(I\)-objects to give a map of bundle-\(I\)-groups whose evaluation at \(W \rightarrow B\) is the diagram (99).

5 A transfer index conjecture

5.1 Introduction

An index theorem relates two different constructions of fibre integration, usually an analytic or geometric one and a homotopy theoretic version. In the present section we discuss an index theorem for the fibre integration of locally constant sheaves of finitely generated projective \(R\)-modules, where \(R\) the ring of integers in a number field. The index theorem which we are going to present here is not yet a proven theorem but the Conjecture 5.3 about a differentials refinement of the index theorems of Dwyer-Weiss-Williams [DWW03] and its characteristic class version proven by Bismut-Lott [BL95]. We will subsequently refer to it as the transfer index conjecture (TIC). The main innovation of the present paper is the construction of all elements necessary to state the transfer index conjecture 5.3. We are far from having a proof which will probably be quite complicated. In order to support the validity of the TIC 5.3 in Subsection 5.4 we discuss interesting consequences of the TIC which can be verified independently.
5.2 The statement of the transfer index conjecture

In this subsection give the statement of the transfer index conjecture, leaving the details of the construction of the analytic index to Subsection 5.3.

For a start we formulate the Dwyer-Weiss-Williams theorem 5.1 in the language of the present paper. We consider a proper submersion \( \pi : W \to B \). Let \( V \) be a locally constant sheaf of finitely generated \( R \)-modules on \( W \). It gives rise to a \( K \)-theory class \([V] \in KR^0(W)\), see Definition 3.29. We have a Becker-Gottlieb transfer \( (95) \)

\[
tr^* : KR^0(W) \to KR^0(B),
\]

and we define

\[
\text{index}^{top}(V) := tr^*[V].
\]

In order to recapture the naturality of this construction for pull-backs along cartesian squares

\[
\begin{array}{ccc}
W' & \xrightarrow{F} & W \\
\downarrow{\pi'} & & \downarrow{\pi} \\
B' & \xrightarrow{f} & B
\end{array}
\]

we will again use the language of \( \text{Bundle}_{\text{geom}}^\delta \)-sets (see Subsection 4.3 for more details about this notation). We have the bundle sets \( \text{Loc} d \) and \( KR^0 r \) which associate to a proper submersion \( \pi : W \to B \) the set \( \text{Loc}(W) \) of isomorphism classes of locally constant sheaves of finitely generated \( R \)-modules on \( W \), and the set \( KR^0(B) \), respectively. Then we can consider the topological index as a map of \( \text{Bundle}_{\text{geom}}^\delta \)-sets

\[
\text{index}^{top} : \text{Loc} d \to KR^0 r.
\]

We can form the higher derived images \( R^i\pi_*(V) \in \text{Sh}_{\text{Mod}(R)}(B) \) of the sheaf \( V \in \text{Sh}_{\text{Mod}(R)}(W) \). They are again locally constant sheaves of finitely generated \( R \)-modules (see Subsection 5.3 for details). The analytic index is defined as the \( K \)-theory class

\[
\text{index}^{an}(V) := \sum_{i \geq 0} (-1)^i[R^i\pi_*(V)] \in KR^0(B).
\]

In order to encode the naturality of the analytic index with respect to diagrams of the form (126) we again consider it as a map of \( \text{Bundle}_{\text{geom}}^\delta \)-sets

\[
\text{index}^{an} : \text{Loc} d \to KR^0 r.
\]

The index theorem of Dwyer-Weiss-Williams [DWW03] (in particular Cor. 8.12 in the revised 2001 version) now has as a consequence:

**Theorem 5.1** We have the equality of the topological and analytical index

\[
\text{index}^{top} = \text{index}^{an} : \text{Loc} d \to KR^0 r.
\]
The main purpose of the present section is to state a refinement of this theorem to differential algebraic $K$-theory. We fix an object

$$(\pi : W \to B, g) \in \text{Bundle}^\delta_{\text{geom}}$$

consisting of a proper submersion $\pi : W \to B$ with a Riemannian structure $g \in \text{Riem}^\delta(W \to B)$, see Definition 4.1. We furthermore consider an isomorphism class of locally constant sheaves of finitely generated $R$-modules with geometry

$$(\mathcal{V}, h^\mathcal{V}) \in \text{Loc}_{\text{geom}}(W),$$

see Definition 3.10. We have a cycle map

$$\text{cycl} : \text{Loc}_{\text{geom}}(W) \to \hat{K}R^0(W)$$

(Theorem 3.13) and a transfer

$$\hat{\text{tr}} : \hat{K}R^0(W) \to \hat{K}R^0(B)$$

(Theorem 4.2).

**Definition 5.2** The topological index of $(\mathcal{V}, h^\mathcal{V})$ is the element

$$\hat{\text{index}}_{\text{top}}(\mathcal{V}, h^\mathcal{V}) := \hat{\text{tr}}(\text{cycl}(\mathcal{V}, h^\mathcal{V})) \in \hat{K}R^0(B).$$

By construction, the topological index can be understood as a map of $\text{Bundle}^\delta_{\text{geom}}$-sets

$$\hat{\text{index}}_{\text{top}} : \text{Loc}_{\text{geom}} \to \hat{K}R^0.$$ 

In Subsection 5.3 we will construct an analytic index

$$\hat{\text{index}}_{\text{an}} : \text{Loc}_{\text{geom}} \to \hat{K}R^0.$$ 

The principal idea is that fibrewise Hodge theory provides geometries $h^{R^i\pi_*(\mathcal{V})}$ on the locally constant sheaves of finitely generated $R$-modules $R^i\pi_*(\mathcal{V})$ so that we can define (using the notation (85))

$$\hat{\text{index}}_{\text{an}}(\mathcal{V}, h^\mathcal{V}) := \sum_{i \geq 0} (-1)^i \text{cycl} \left( R^i\pi_*(\mathcal{V}), h^{R^i\pi_*(\mathcal{V})} \right) + a(T),$$

where the correction $a(T)$ involves the higher torsion form of Bismut-Lott. It is added in order to adjust the curvature of $\hat{\text{index}}_{\text{an}}(\mathcal{V}, h^\mathcal{V})$ so that it coincides with that of $\hat{\text{index}}_{\text{top}}(\mathcal{V}, h^\mathcal{V})$.

We have now all elements to formulate the TIC.
Conjecture 5.3 (Transfer index conjecture) The two maps
\[ \widetilde{\text{index}}_{\text{top}} = \widetilde{\text{index}}_{\text{an}} : \text{Loc}_{\text{geom}} \to \widetilde{KR}^0 \]
of Bundle_{\text{geom}}-sets are equal.

This conjecture appears to be very natural. The equality holds true if we specialize to the curvature or the underlying algebraic K-theory classes. Recall the structure maps \( R \) and \( I \) of differential cohomology, Definition 2.2. For the curvature we get essentially a reformulation of the theorem of Bismut-Lott [BL95]:

Theorem 5.4 We have the equality
\[ R \circ \widetilde{\text{index}}_{\text{top}} = R \circ \widetilde{\text{index}}_{\text{an}} : \text{Loc}_{\text{geom}} \to Z^0(\Omega A)r . \]

We will verify this theorem at the end of Subsection 5.3 after the construction of the analytic index. On the level of underlying algebraic K-theory classes we get the theorem of Dwyer-Weiss-Williams 5.1:

Theorem 5.5
\[ I \circ \widetilde{\text{index}}_{\text{top}} = I \circ \widetilde{\text{index}}_{\text{an}} : \text{Loc}_{\text{geom}} \to KR^0 r . \]

Proof. This immediately follows from
\[ I \circ \widetilde{\text{index}}_{\text{top}} = \text{index}_{\text{top}} \circ F , \quad I \circ \text{index}_{\text{an}} = \text{index}_{\text{an}} \circ F \]
and Theorem 5.1 where
\[ F : \text{Loc}_{\text{geom}} \to \text{Loc} \]
forgets the geometry. \( \square \)

The difference \( \hat{\delta} := \widetilde{\text{index}}_{\text{top}} - \widetilde{\text{index}}_{\text{an}} \) is therefore an additive natural transformation
\[ \text{Loc}_{\text{geom}}(W) \ni (\mathcal{V}, h^\mathcal{V}) \mapsto \hat{\delta}(\mathcal{V}, h^\mathcal{V}) \in \frac{HA^{-1}(B)}{\text{im}(KR^{-1}(B) \to HA^{-1}(B))} . \]

It actually factors over a transformation
\[ \delta : \text{Loc} \to \frac{HA^{-1}(B)}{\text{im}(KR^{-1}(B) \to HA^{-1}(B))} , \]
of Bundle-spaces. The Conjecture 5.3 can now be reformulated to say that
\[ \delta = 0 . \]

One possible approach to a proof of Conjecture 5.3 could be to show that such a transformation is necessarily zero. This approach works successfully in similar situations, e.g. for a differential refinement of the index theorem of Atiyah-Singer in [BS09]. The basic reason there was that the rational cohomology of the classifying space of complex vector bundles is concentrated in even degrees. But in the present case of algebraic K-theory of a number ring the situation is quite different since the rational cohomology of the corresponding classifying space is generated as a ring by odd-degree classes.
5.3 The analytic index

For a manifold $W$ we let $\text{Sh}_{\text{Mod}(R)}(W)$ denote the abelian category of sheaves of $R$-modules on the site of open subsets of $W$. The proper submersion $\pi : W \to B$ induces a push-forward $\pi_* : \text{Sh}_{\text{Mod}(R)}(W) \to \text{Sh}_{\text{Mod}(R)}(B)$. For a sheaf of $R$-modules $\mathcal{V} \in \text{Sh}_{\text{Mod}(R)}(W)$ we have the sequence of higher-derived images

$$R^i\pi_*(\mathcal{V}) \in \text{Sh}_{\text{Mod}(R)}(B), \quad i \geq 0.$$  

We consider a point $b \in B$ and let $W_b := \pi^{-1}(b)$ denote the fibre of $\pi$ over $b$. For a sheaf $\mathcal{F} \in \text{Sh}_{\text{Mod}(R)}(B)$ we let $\mathcal{F}_b$ denote the stalk of $\mathcal{F}$ at $b$. Then we have a canonical isomorphism of $R$-modules

$$R^i\pi_*(\mathcal{V})_b \cong H^i(W_b, \mathcal{V}|_{W_b}).$$

We now assume that $\mathcal{V}$ is locally constant. Then the sheaves $R^i\pi_*(\mathcal{V})$ are again locally constant sheaves of $R$-modules. If $\mathcal{V}$ is a locally constant sheaf of finitely generated $R$-modules, then so are the higher derived images $R^i\pi_*(\mathcal{V})$. Indeed, since $W_b$ is compact, the cohomology group can be calculated in the Čech picture using a finite good covering of $W_b$. It follows that the cohomology is a subquotient of a finitely generated $R$-module.

We define the analytic index

$$\text{index}^{an} : \text{Loc} \to KR^0$$

by the formula (127).

Following the ideas of [Lot00] we now refine this construction to an analytic index in differential algebraic $K$-theory

$$\text{index}^{an} : \text{Loc}_{\text{geom}} \to \hat{KR}^0.$$  

For an embedding $\sigma : R \to C$ we let $H^i(W/B, V_\sigma) \to B$ denote flat complex vector bundle associated to the locally constant sheaf $R^i\pi_*(\mathcal{V}) \otimes_\sigma C$ of finite-dimensional complex vector spaces. As the notation indicates, the fibre $H^i(W/B, V_\sigma)_b$ of this bundle at $b \in B$ can canonically be identified with the cohomology $H^i(W_b, V \otimes_\sigma \mathbb{C})$ and hence with the $i$'th cohomology of the Rham complex $\Omega(W_b, V_\sigma|_{W_b})$ of $W_b$ twisted by the flat bundle $V_\sigma|_{W_b}$ associated to the local system $V \otimes_\sigma \mathbb{C}$.

The following constructions are similar as in [BL95, Sec.3(d)]. The Riemannian structure $g$ on $\pi$ and the geometry $h^\nabla$ on $\mathcal{V}$ supply a Riemannian metric on $W_b$ and a metric on $V_b$. These structures induce an $L^2$-scalar product on the de Rham complex $\Omega(W_b, V_\sigma|_{W_b})$. We can now apply Hodge theory which provides an isomorphism of complex vector spaces

$$H^i(W/B, V_\sigma)_b \cong \mathcal{H}(W_b, V_\sigma|_{W_b}),$$

where the right-hand side denotes the subspace $\mathcal{H}^i(W_b, V_\sigma|_{W_b}) \subseteq \Omega^i(W_b, V_\sigma|_{W_b})$ of harmonic $i$-forms on $W_b$ with coefficients in $V_\sigma|_{W_b}$. The $L^2$-metric on $\mathcal{H}^i(W_b, V_\sigma|_{W_b})$ thus induces a
metric $h_{L^2(W/B,V)}^H(b)$ on $H^i(W/B,V)$. This metric depends smoothly on the base point and is therefore a hermitean metric on the bundle $H^i(W/B,V)$. The collection of metrics

$$h^{R^i\pi_*}(V) := \left(h_{L^2(W/B,V)}^H\right)_{\sigma \in \Sigma}$$

(128)

serves as a geometry (Definition 3.10) on $R^i\pi_*(V)$. Indeed, the condition (41) for $h^{R^i\pi_*}(V)$ is easily implied by the corresponding condition for $h^V$.

As a first approximation, we define

$$\text{index}^{an}_0(V,h^V) := \sum_i (-1)^i \text{cycl} \left(R^i\pi_*(V), h^{R^i\pi_*}(V)\right) \in \tilde{KR}^0(B),$$

(129)

where \text{cycl} is as in Definition 3.32. The subscript "0" indicates that this definition is not yet the final one. It has to be corrected by the Bismut-Lott analytic torsion form $T(\pi,g,V,h^V) \in \Omega A^{-1}(B)$ (see (131)) in order to match the curvature forms.

For each embedding $\sigma : R \to \mathbb{C}$ we have a Bismut-Lott analytic torsion form $[BL95, \text{Def. 3.22}]$

$$T(T^h\pi,g^{T^v\pi},h^V) \in \Omega(B).$$

By $[BL95, \text{Thm. 3.23}]$ it satisfies

$$dT(T^h\pi,g^{T^v\pi},h^V) = \int_{W/B} e(b) \wedge f(\nabla V, h^V) - \sum_i (-1)^i f(\nabla H^i(W/B,V), h_{L^2}^{H^i(W/B,V)}),$$

(130)

where the forms $f(\nabla V, h^V) \in \Omega(W)$ can be expressed in terms of Borel forms (35) in the form (compare $[BL95, (3.102)]$)

$$f(\nabla V, h^V) = \sum_{k \geq 0} \frac{1}{k!} \omega_{2k+1}(h^V).$$

We consider the decomposition

$$T(T^h\pi,g^{T^v\pi},h^V) = \sum_{k \geq 0} T_{2k}(T^h\pi,g^{T^v\pi},h^V), \quad T_{2k} \in \Omega^{2k}(B)$$

into homogeneous components and define

$$T(\pi,g,V,h^V) := \sum_{\sigma} \sum_{k \geq 0} \frac{1}{k!} T_{2k}(T^h\pi,g^{T^v\pi},h^V)b_{2k+1}(\sigma) \in \Omega A^{-1}(B),$$

(131)

where the basis elements $b_{2k+1}(\sigma)$ were defined after Proposition 3.9. Then we can rewrite (130) as follows:

$$dT(\pi,g,V,h^V) = \int_{W/B} e(b) \wedge \omega(h^V) - \sum_i (-1)^i \omega(h^{R^i\pi_*(V)}).$$

(132)

The final definition of the analytic index in differential algebraic $K$-theory will be the following.
Definition 5.6 If $W \to B$ is a proper submersion equipped with a geometry $g$ and $(\mathcal{V}, h^\mathcal{V})$ is a locally constant sheaf of finitely generated projective $R$-modules on $W$ with geometry $h^\mathcal{V}$, then we define
\[
\hat{\text{index}}^\text{an}(\mathcal{V}, h^\mathcal{V}) := \hat{\text{index}}^\text{an}_0(\mathcal{V}, h^\mathcal{V}) + a(\mathcal{T}(\pi, g, \mathcal{V}, h^\mathcal{V})) .
\]

We now verify Theorem 5.4 by a curvature calculation. By (43) we have
\[
R(\hat{\text{index}}^\text{an}_0(\mathcal{V}, h^\mathcal{V})) = \sum_i (-1)^i \omega(h^R \pi_*(\mathcal{V})) .
\]
From (132) we therefore get
\[
R(\hat{\text{index}}^\text{an}(\mathcal{V}, h^\mathcal{V})) = \int_{W/B} e(b) \wedge \omega(h^\mathcal{V}) .
\]
Equation (133) implies Theorem 5.4.

The construction of the metric $h^R \pi_*(\mathcal{V})$ and the torsion form $\mathcal{T}(\pi, g, \mathcal{V}, h^\mathcal{V})$ is compatible with pull-backs along morphisms in $\text{Bundle}^\delta_{\text{geom}}$. Hence our construction of the refined analytic index gives a map of $\text{Bundle}^\delta_{\text{geom}}$-sets
\[
\hat{\text{index}}^\text{an} : \text{Loc}_{\text{geom}}d \to \hat{KR}^0_r .
\]

5.4 Discussion of the transfer index conjecture

5.4.1 The relation with the work of Lott

In [Lot00] Lott constructed contravariant functors
\[
\text{Mf}^{\text{op}} \ni M \mapsto \hat{K}^0_{\text{Lott},\sigma}(M) \in \text{Ab} , \quad \text{Mf}^{\text{op}} \ni M \mapsto \hat{K}^0_{\text{Lott},\sigma}(M) \in \text{Ab}
\]
by cycles and relations. They depend on the choice of an embedding $\sigma : R \to \mathbb{C}$ which we added to the notation. A cycle ([Lot00 Def. 2]) for $\hat{K}^0_{\text{Lott},\sigma}(M)$ is a triple $(\mathcal{V}, h^\mathcal{V}_\sigma, \eta)$ of a locally constant sheaf of finitely generated $R$-modules on $M$, a metric $h^\mathcal{V}_\sigma$ on the complex vector bundle $V_\sigma \to M$ associated to $\mathcal{V}$ and $\sigma$, and a form $\eta \in \Omega^\text{ev}(M)/\text{im}(d)$. The set of isomorphism classes of cycles becomes a monoid with respect to direct sum. The group $\hat{K}^0_{\text{Lott},\sigma}(M)$ is then defined by group completion and factorization modulo the equivalence relation ([Lot00 Def. 3]) generated by the following. If $(\mathcal{V}_i, h^{V_i,\sigma}, \eta_i)$, $i = 0, 1, 2$, are three cycles and
\[
0 \to \mathcal{V}_0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to 0
\]
is an exact sequence of sheaves, then
\[
(\mathcal{V}_0, h^{V_0,\sigma}, \eta_0) + (\mathcal{V}_2, h^{V_2,\sigma}, \eta_2) \sim (\mathcal{V}_1, h^{V_1,\sigma}, \eta_1)
\]
if
\[
\eta_0 + \eta_2 - \eta_1 = \mathcal{T}_\sigma \in \Omega^\text{ev}(M)/\text{im}(d) ,
\]

86
were $T_\sigma$ is the analytic torsion form associated to the exact sequence \([134]\) equipped with metrics $h^{V,\sigma}$, see \([Lot00, A.2]\) for details. We let $[\mathcal{V}, h^{V,\sigma}, \eta] \in \hat{K}^0_{Lott,\sigma}(M)$ denote the class represented by the cycle $(\mathcal{V}, h^{V,\sigma}, \eta)$.

There are natural transformations
\[ b : \hat{K}_{Lott,\sigma}(M) \to KR^0(M), \quad c : \hat{K}_{Lott,\sigma}(M) \to Z^{odd}(\Omega(M)) \]
induced by
\[ (\mathcal{V}, h^{V,\sigma}, \eta) \mapsto \hat{I}(\mathcal{V}), \quad (\mathcal{V}, h^{V,\sigma}, \eta) \mapsto \sum_{k \geq 0} \omega_{2k+1}(h^{V,\sigma}) - d\eta \]
(see \((35)\) for the definition of the forms $\omega_{2k+1}(h^{V,\sigma})$), and a transformation
\[ a : \Omega^{ev}(M)/\text{im}(d) \to \hat{K}_{Lott,\sigma}(M), \quad \eta \mapsto (0, 0, -\eta). \]

Following \([Lot00, Def. 5]\) we define the functor $\tilde{K}^0_{Lott,\sigma}$ by
\[ \tilde{K}^0_{Lott,\sigma}(M) := \ker \left( c : \hat{K}_{Lott,\sigma}(M) \to Z^{odd}(\Omega(M)) \right), \]
which turns out to homotopy invariant.

These two functors $\tilde{K}^0_{Lott,\sigma}$ and $\hat{K}_{Lott,\sigma}$ bear some similarities with $\tilde{KR}^0_{flat}$ and $\tilde{KR}^0$. In order to set up a precise comparison we shall modify Lott’s definition in order to get rid of the dependence on the embedding $\sigma$. The idea is to consider all embeddings at once.

We define the modified group $\tilde{KR}^0_{Lott}(M)$ again by cycles and relations as above, where now a cycle is a triple $(\mathcal{V}, h^{V}, \eta)$ of a locally constant sheaf $\mathcal{V}$ of finitely generated $R$-modules with geometry $h^{V}$ on $M$ and a form $\eta \in \Omega^{A^{-1}}(M)$. This data is the same as a collection of cycles $(\mathcal{V}, h^{V,\sigma}, \eta_{\sigma})_{\sigma \in \Sigma^*}$ with the fixed underlying locally constant sheaf $\mathcal{V}$.

The equivalence relation for $\tilde{KR}^0_{Lott}(M)$ is generated as follows. If $(\mathcal{V}_0, h^{V_0}, \eta_0), (\mathcal{V}_1, h^{V_1}, \eta_1), (\mathcal{V}_2, h^{V_2}, \eta_2)$, are three cycles and
\[ 0 \to \mathcal{V}_0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to 0 \]
is an exact sequence of sheaves, then
\[ (\mathcal{V}_0, h^{V_0}, \eta_0) + (\mathcal{V}_2, h^{V_2}, \eta_2) \sim (\mathcal{V}_1, h^{V_1}, \eta_1) \]
if
\[ \eta_0 + \eta_2 - \eta_1 = T \in \Omega^{A^{odd}}(M)/\text{im}(d). \]

The torsion form $T$ on the right-hand side of \((135)\) is given by
\[ T = \sum_{\sigma \in \Sigma^*} b_{2k+1}(\sigma) T_{\sigma,2k}, \]
\[(136)\]
where the components $T_{\sigma,2k} \in \Omega^{2k}(M)/\text{im}(d)$ are determined by
\[ T_\sigma = \sum_{k \geq 0} T_{\sigma,2k}. \]
The torsion form satisfies
\[ d\mathbf{T} = \omega(h^{V_0}) + \omega(h^{V_2}) - \omega(h^{V_1}). \] (137)

As above there are natural transformations
\[ b : \hat{KR}^0_{\text{Lott}}(M) \to KR^0(M), \quad c : \hat{KR}^0_{\text{Lott}}(M) \to Z^0(\Omega A(M)) \]
induced by
\[ (\mathcal{V}, h^V, \eta) \mapsto \hat{I}(\mathcal{V}), \quad (\mathcal{V}, h^V, \eta) \mapsto \omega(h^V) - d\eta, \]
(see Definition 3.7 for \( \hat{I} \)) and a transformation
\[ a : \Omega A^{-1}(M)/\text{im}(d) \to \widehat{KR}^0_{\text{Lott}}(M), \quad \eta \mapsto (0, 0, -\eta). \]

The functor \( M \mapsto \overline{KR}^0_{\text{Lott}}(M) \) is defined as
\[ \overline{KR}^0_{\text{Lott}}(M) := \ker \left( c : \hat{KR}^0_{\text{Lott}}(M) \to Z^0(\Omega A(M)) \right). \]
It is again homotopy invariant.
A cycle \( (\mathcal{V}, h^V, \eta) \) gives rise to a class
\[ z(\mathcal{V}, h^V, \eta) := \text{cycl}(\mathcal{V}, h^V) + a(\eta) \in \hat{KR}^0(M) \]
such that
\[ R(z(\mathcal{V}, h^V, \eta)) = c([\mathcal{V}, h^V, \eta]), \quad b(z(\mathcal{V}, h^V, \eta)) = I([\mathcal{V}, h^V, \eta]). \]

The question is now whether this map \( z \) factorizes over \( \overline{KR}^0_{\text{Lott}}(M) \). To this end we must know that the following relation holds true in \( \overline{KR}^0(M) \).

**Theorem 5.7 (Lott’s relation)** If \( (\mathcal{V}_i, h^{V_i}), i = 0, 1, 2 \), are locally constant sheaves of finitely generated \( R \)-modules on a manifold \( M \) with geometries and
\[ 0 \to \mathcal{V}_0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to 0 \] (138)
is an exact sequence of sheaves, then we have the following relation in \( \overline{KR}^0(M) \):
\[ \text{cycl}(\mathcal{V}_0, h^{V_0}) + \text{cycl}(\mathcal{V}_2, h^{V_2}) - \text{cycl}(\mathcal{V}_1, h^{V_1}) = a(\mathbf{T}). \] (139)

**Proof.** A proof of this theorem for exact sequence of sheaves of finitely generated projective \( R \)-modules appears [BT]. In Lemma 5.20 we will show that one can drop the assumption of projectivity.

The proof given in [BT] depends upon a generalization of the theory presented in the present paper. Therefore, in Subsection 5.5 we in addition verify some special cases and weaker statements which can easily be obtained in the framework of the present paper.
From Lott’s relation Theorem 5.7 we get the dotted arrow in the following commutative diagram:

\[
\begin{array}{c}
\Omega A^{-1}(M)/\text{im}(d) \xrightarrow[\phi]{a} \hat{KR}^0_{Lott}(M) \\
\downarrow \quad \downarrow \quad \downarrow \\
\Omega A^{-1}(M)/\text{im}(d) \xrightarrow[\phi]{a} \hat{KR}^0_{Lott}(M)
\end{array}
\]

\[
\begin{array}{l}
\text{KR}^0_{Lott}(M) \xrightarrow{(R,I)} Z^0(\Omega A(M)) \times_{H^{Av}(M)} KR^0(M) \xrightarrow{} 0 \ .
\end{array}
\]

Note that we do not expect that the map

\[
\phi : \hat{KR}^0_{Lott}(M) \rightarrow \hat{KR}^0_M
\]

is an isomorphism since in general not all classes in \(KR^0(M)\) are represented by locally constant sheaves of finitely generated \(R\)-modules on \(M\) (consider e.g. the case where \(M\) is simply connected!). By restriction of \(\phi\) we obtain a transformation

\[
\phi : \hat{KR}^0_{Lott}(M) \rightarrow \hat{KR}^0_{flat}(M) \sim \text{Prop. 2.6} \rightarrow \text{KR}_{R/Z}^{-1}(M) .
\]

One of the main results of [Lot00] is the construction of the analytic index. In the following we present the adaptation of this construction to our groups \(\hat{KR}^0_{Lott}\). Let \(W \rightarrow B\) be a proper submersion with geometry \(g\). Given a cycle \((\mathcal{V}, h^\mathcal{V}, \eta)\) for \(\hat{KR}^0_{Lott}(W)\) we set

\[
\hat{\text{index}}^{an}_{Lott}(\mathcal{V}, h^\mathcal{V}, \eta) := \sum_i (-1)^i [R^i\pi^*(\mathcal{V}), h^{R^i\pi^*(\mathcal{V})}, 0]
\]

\[
+ [0, 0, \int_{W/B} e(b) \wedge \eta - T(\pi, g, \mathcal{V}, h^\mathcal{V})]
\]

The same argument as for [Lot00] Prop. 7 shows that

\[
\hat{\text{index}}^{an}_{Lott} : \hat{KR}^0_{Lott}(W) \rightarrow \hat{KR}^0_B
\]

is well-defined. The following diagram commutes

\[
\begin{array}{c}
\hat{KR}^0_{Lott}(W) \xrightarrow{\phi} \hat{KR}^0_B \\
\downarrow \hat{\text{index}}^{an}_{Lott} \quad \downarrow \hat{\text{index}}^{an}
\end{array}
\]

We now restrict to the flat part and extend the diagram to

\[
\begin{array}{l}
\hat{KR}^0_{Lott}(W) \xrightarrow{\phi} \hat{KR}^0_{flat}(W) \xrightarrow{\text{top}} \hat{KR}^0_{flat}(W) \xrightarrow{\text{tr}^*} \text{KR}_{R/Z}^{-1}(W) \ .
\end{array}
\]
The commutativity of the right square is the assertion of Proposition 4.4. The transfer index conjecture (5.3) implies that the middle square commutes. We now remark that the existence of the transformation $KR_{\text{Lott}}^0 \rightarrow KR_{\mathbb{R}/\mathbb{Z}}^{-1}$ (given by the compositions from the left to the right in the diagram) such that the outer square of (141) commutes was conjectured by Lott [Lot00, Conj. 1]. We see that a solution of the transfer index conjecture (5.3) together with Lott’s relation (5.7) would imply Lott’s conjecture [Lot00, Conj. 1].

5.4.2 The relation with the Cheeger-Muller theorem

The goal of the present subsection is to support the transfer index conjecture through its compatibility with the Cheeger-Muller theorem about the equality of Ray-Singer analytic torsion and Reidmeister torsion. Let $\pi : W \rightarrow B$ be a proper submersion with a Riemannian structure $g$ (Definition 4.1). We consider a local system $\mathcal{V}$ of finitely generated free (for simplicity) $\mathbb{R}$-modules on $W$. We further assume that we can choose the geometry $h^V$ (Definition 3.10) such that the metrics $h^V_\sigma$ are parallel for all places $\sigma \in \Sigma$ of $R$. Then we have $\omega(h^V) = b_0 \dim(\mathcal{V})$ (see (42)). We set $1_W := q^*1 \in \widehat{KR}^0(W)$, where $q : W \rightarrow *$ is the projection and $1$ is defined in (76). We have

$$\text{cycl}(\mathcal{V}, h^V) - \dim(\mathcal{V}) \ 1_W \in \widehat{KR}^0_{\text{flat}}(W) \cong KR_{\mathbb{R}/\mathbb{Z}}^{-1}(W).$$

By Proposition 4.4 the restriction of the differential cohomology transfer $\text{tr}$ of the bundle $\pi$ to the flat subfunctor $\widehat{KR}^0_{\text{flat}}$ coincides with the Becker-Gottlieb transfer $\text{tr}^*$ for the cohomology group $KR_{\mathbb{R}/\mathbb{Z}}^{-1}$. The latter is homotopy invariant. Hence the element

$$\text{tr}^*(\text{cycl}(\mathcal{V}, h^V) - \dim(\mathcal{V}) \ 1_W) \overset{\text{Prop.}4.4}{=} \text{tr}(\text{cycl}(\mathcal{V}, h^V) - \dim(\mathcal{V}) \ 1_W) \in \widehat{KR}^0_{\text{flat}}(B)$$

does neither depend on the Riemannian structure $g$ nor on the geometry $h^V$.

We now assume that $B$ is a point and that $W$ is connected. In this case the Becker-Gottlieb transfer can be calculated using a base point $s : * \rightarrow W$ as

$$\text{tr}^* = \chi(W) \ s^*.$$  

We get

$$\text{index}_{\text{top}}^{\text{top}}(\mathcal{V}, h^V) - \dim(\mathcal{V}) \ \hat{\text{tr}}(1_W) = \chi(W) \left( \text{cycl}(s^*(\mathcal{V}, h^V)) - \dim(\mathcal{V}) \ 1 \right) \overset{\text{Lemma}3.26}{=} \chi(W) \left( \text{cycl}(\det(s^*\mathcal{V}), h^{\det(s^*\mathcal{V})}) - 1 \right).$$  

We now consider the analytic index. For every $i \geq 0$ the cohomology group $H^i(W, \mathcal{V})$ of the compact manifold $W$ with coefficients in the sheaf $\mathcal{V}$ is a finitely generated $R$-module. Moreover, we have a collection of metrics $h^{H^i(W, \mathcal{V})} := (h^{H^i(W, \mathcal{V})}_{L^2})_{\sigma \in \Sigma}$ induced by Hodge
theory. We set \( h^{H(V)} = (h^{H(W,V)})_{i=0,\ldots,n} \). We have by Definition 5.6
\[
\text{index}^\text{an} (V, h^V) = \sum_{i \geq 0} (-1)^i \text{cyc} \left( H^i(W, V), h^{H^i(W,V)} \right) + a \left( T(\pi, g, V, h^V) \right).
\]
The zero form component of the analytic torsion form is related with the classical Ray-Singer torsion \([RS71]\) \( T^{\text{an}}(W, g^{TW}V_\sigma, h^{V_\sigma}) \in \mathbb{R} \) of the flat bundle \((V_\sigma, \nabla^{V_\sigma}, h^{V_\sigma})\) on the Riemannian manifold \((W, g^{TW})\) by
\[
T(\pi, g, V, h^V) = 2T^{\text{an}}(W, g^{TW}, V, h^V) := 2 \sum_{\sigma} b_1(\sigma) T^{\text{an}}(W, g^{TW}, V_\sigma, h^{V_\sigma})
\]
(see [BL95, Thm. 3.29]). We abbreviate
\[
D(V, h^V) := \sum_{i \geq 0} (-1)^i \text{cyc} \left( H^i(W, V), h^{H^i(W,V)} \right).
\]
The transfer index conjecture 5.3 now implies
\[
\chi(W) \left[ \text{cyc} \left( \det(s^*V), h^{\det(s^*V)} \right) - 1 \right] = D(V, h^V) - \dim(V) D(R_W, h^{R_W})
+ 2a \left( T^{\text{an}}(W, g^{TW}, V, h^V) \right) - 2 \dim(V) a \left( T^{\text{an}}(W, g^{TW}, R_W, h^{R_W}) \right).
\]
(143)
The flat bundle \((V_\sigma, \nabla^{V_\sigma})\) of complex vector spaces with the parallel metric \( h^{V_\sigma} \) gives rise to a Reidemeister torsion \( \tau_{R\varphi}(W, g^{TW}, V_\sigma, h^{V_\sigma}) \in \mathbb{R}^+ \) defined as follows. Let \( C(V) \) be the cochain complex of some finite smooth triangulation of \( W \) with coefficients in \( V \). It acquires a geometry \( h^{C(V)} \) from \( h^{V} \). The de Rham isomorphism induces isomorphisms \( H^i(C_\sigma) \cong H^i(W, V_\sigma) \) and therefore the isomorphisms \( \kappa_\sigma \) in the following chain of isomorphisms (see [87] for \( \phi_\sigma(C(V)) \))
\[
\psi_\sigma : \det(C(V)_\sigma) \overset{\phi_\sigma(C(V)_\sigma)}{\sim} \det(H(C(V)_\sigma)) \overset{\kappa_\sigma}{\sim} \det(H(W; V_\sigma)).
\]
We let \( h^{\text{det}(C(V)_\sigma)} \) and \( h^{L_2(H(W; V_\sigma))} \) denote the metrics induced on the determinants by the metrics \( h^{C(V)} \) and \( h^{H(W; V_\sigma)} \). The Reidemeister torsion \( \tau_{R\varphi}(W, g^{TW}, V_\sigma, h^{V_\sigma}) \in \mathbb{R}^+ \) is now uniquely defined by
\[
\psi_{\sigma, h^{\text{det}(C(V)_\sigma)}} = \tau_{R\varphi}(W, g^{TW}, V_\sigma, h^{V_\sigma}) h^{L_2(H(W; V_\sigma))}.
\]
It does not depend on the choice of the triangulation. We refer to [M"{u}93, Sec. 1] for more details.
Recall from [87] that
\[
\phi_\sigma(C(V)) h^{\text{det}(C(V)_\sigma)} = \tau_\sigma(C(V), h^{C(V)}, \kappa^*_\sigma h^{H(V)}) h^{L_2(\text{det}(H(W; V_\sigma)))}.
\]
and therefore
\[ \tau_{\text{RM}}(W, \mathcal{V}_\sigma, h_{\mathcal{V}_\sigma})^2 = \tau_{\sigma}(C(\mathcal{V}), h^{C(\mathcal{V})}, \kappa^* h^{H(\mathcal{V})})^2. \]

We define
\[ \ln \tau_{\text{RM}}(W, g^{TW}, \mathcal{V}, h^{\mathcal{V}}) := \sum_{\sigma \in \Sigma^*} \ln \tau_{\text{RM}}(W, g^{TW}, \mathcal{V}_\sigma, h_{\mathcal{V}_\sigma}) b_1(\sigma). \]

We conclude that
\[ \chi(W) [\text{cycl} \left( \det(s^* \mathcal{V}), h^{\det(s^* \mathcal{V})} \right) - 1] \]
\[ = \sum_{i} (-1)^i \text{cycl}(C^i(\mathcal{V}), h^{C^i(\mathcal{V})}) - \dim(\mathcal{V}) \sum_{i} (-1)^i \text{cycl}(C^i(R_W), h^{C^i(R_W)}) \]
\[ = D(\mathcal{V}, h^{\mathcal{V}}) - \dim(\mathcal{V}) D(R_W, h^{R_W}) + 2 a(\ln \tau(C(\mathcal{V}), h^{C(\mathcal{V})}, \kappa^* h^{H(\mathcal{V})}) - 2 \dim(\mathcal{V}) a(\ln \tau(R_W, h^{C(R_W)}, \kappa^* h^{R(W)}, \kappa^*(H))) \]
\[ + 2 a(\ln \tau_{\text{RM}}(W, g^{TW}, \mathcal{V}, h^{\mathcal{V}}) - 2 \dim(\mathcal{V}) a(\ln \tau_{\text{RM}}(W, g^{TW}, R_W, h^{R_W}))) \]

The Cheeger-Müller theorem \cite{Che79, Mül78} (see \cite{Mül93, BZ92} for generalizations) states

**Theorem 5.8**

\[ T^m(W, g^{TW}, \mathcal{V}_\sigma, h_{\mathcal{V}_\sigma}) = \tau_{\text{RM}}(W, g^{TW}, \mathcal{V}_\sigma, h_{\mathcal{V}_\sigma}). \]

Plugging this equality into (145) we get exactly (143). The Cheeger-Müller theorem is therefore compatible with the transfer index conjecture. Vice versa, observe that the transfer index conjecture implies an $\mathbb{R}/\mathbb{Z}$-version of the Cheeger-Müller theorem.

### 5.4.3 $S^1$-bundles

The analytic torsion form for $U(1)$-bundles equipped with a flat line bundle has been calculated in \cite[Prop. 4.13]{BL95}. In this paragraph we discuss the consequences of this calculation for the transfer index conjecture \[5.3\]

We consider an $U(1)$-principal bundle $p : P \to B$ over a simply connected base manifold $B$ with non-trivial first Chern class $c_1(p) \in H^2(B; \mathbb{Z})$. The second cohomology group $H^2(B; \mathbb{Z})$ of the base space $B$ is torsion-free, and we assume that the Chern class $c_1(p)$ is a primitive element. This implies that the total space $P$ of the bundle is simply connected, too. For a prime $r \geq 3$ we have an embedding of groups $\mathbb{Z}/r\mathbb{Z} \hookrightarrow U(1)$, $[n] \mapsto e^{2\pi i n}$. The group $\mathbb{Z}/r\mathbb{Z}$ therefore acts on $P$ via restriction of the $U(1)$-action. The quotient $\pi : W := P/(\mathbb{Z}/r\mathbb{Z}) \to B$ again has the structure of an $U(1)$-principal bundle. Its first Chern class satisfies $c_1(\pi) = r c_1(p)$.

We consider the ring
\[ R := \mathbb{Z}[\xi]/(1 + \xi + \cdots + \xi^{r-1}). \]
This is the ring of the integers in the cyclotomic number field $\mathbb{Q}(\xi)$ of rationals extended by an $r'$th root of unity. We get a representation $\mathbb{Z}/r\mathbb{Z} \to GL(1, R)$ by $[n] \mapsto \xi^n$. This representation determines a local system $\mathcal{V}$ of free $R$-modules of rank one on $W$.

Each choice of a nontrivial complex $r'$th root of unity $\sigma(\xi) \in \mathbb{C}$ defines a place $\sigma \in \Sigma$ of $\mathbb{Q}(\xi)$. Since we assume that $r$ is prime and $r \geq 3$, all these places are complex. Therefore $r_R = 0$ and $r_C = \frac{r-1}{2}$. For each place $\sigma$ the image of $\mathbb{Z}/r\mathbb{Z} \to GL(1, R)$ is contained in the subgroup $U(1) \subset GL(1, \mathbb{C})$ so that the bundle $V_{\sigma} \to W$ associated to $\sigma$ and $\mathcal{V}$ has a canonical flat hermitean metric $h^{\mathcal{V}_{\sigma}}$. The collection of these metrics forms a geometry $h^{\mathcal{V}}$ on $\mathcal{V}$.

We now consider the class

$$\text{cycl}(\mathcal{V}, h^{\mathcal{V}}) - 1_W \in \overset{\bigwedge}{KR}^0(W) .$$

(146)

This class is flat.

We equip the proper submersion $\pi : W \to B$ with the fibrewise Riemannian metric $g^{T^W\pi}$ which is $U(1)$-invariant and normalized such that the fibres have unit volume. We furthermore choose a principal bundle connection $\nabla^W$ on $W$. It defines and is defined by a $U(1)$-invariant horizontal distribution $T^h\pi$. The pair $g := (g^{T^W\pi}, T^h\pi)$ is a Riemannian structure on the proper submersion $\pi$. These choices together fix the transfer

$$\overline{\text{tr}} : \overset{\bigwedge}{KR}^0(W) \to \overset{\bigwedge}{KR}^0(B)$$

in differential algebraic $K$-theory by Theorem 4.2.

Since the class (146) is flat we can interpret it in $KR\mathbb{R}/\mathbb{Z}^{-1}(W)$, and by Proposition 4.4 we can express the differential cohomology transfer of flat classes in terms of the usual Becker-Gottlieb transfer:

$$\overline{\text{tr}} \left( \text{cycl}(\mathcal{V}, h^{\mathcal{V}}) - 1_W \right) = \text{tr}^\ast \left( \text{cycl}(\mathcal{V}, h^{\mathcal{V}}) - 1_W \right) .$$

The fundamental vector field generating the $U(1)$-action is a vertical vector field on $W$ without zeros. This implies (compare e.g. [Dou06]) that the Becker-Gottlieb transfer for $\pi$ vanishes. It follows that

$$0 = \overline{\text{tr}} \left( \text{cycl}(\mathcal{V}, h^{\mathcal{V}}) - 1_W \right) = \text{index}^{\text{top}}(\text{cycl}(\mathcal{V}, h^{\mathcal{V}})) - \text{index}^{\text{top}}(1) .$$

(147)

We now consider the analytic indices. The holonomy of $\mathcal{V}$ along the fibres $W_b \cong S^1$, $b \in B$, is the non-trivial element $\xi$. We get an exact sequence

$$0 \to H^0(W_b, \mathcal{V}|_{W_b}) \to R \overset{1-\xi}{\to} R \to H^1(W_b, \mathcal{V}|_{W_b}) \to 0 .$$

Since $R$ is an integral domain we see that $H^0(W_b, \mathcal{V}|_{W_b}) = 0$. It follows that $H^1(W_b, \mathcal{V}|_{W_b})$ is torsion. Recall the decomposition (81) of a sheaf of finitely generated $R$-modules into a torsion and a projective part. We conclude that

$$\text{Proj}(R^i\pi_\ast(\mathcal{V})) = 0 , \quad i \geq 0 .$$

93
We define the torsion $R$-module $L$ by

$$0 \to R \xrightarrow{\xi} R \to L \to 0.$$  

Since $B$ is simply connected we have

$$R^1\pi_* (V) = \text{Tors}(R^1\pi_* (V)) \cong L_B.$$  

Let $q : B \to *$ be the projection and $\hat{Z}(L) \in K^0 R(*)$, see (85). Combining [86] with Definition 5.6 we can write

$$\text{index}^an (V, h^V) = a(T(\pi, g, V, h^V)) + q^* \hat{Z}(L).$$

Next we calculate $\text{index}^an (1)$. Let $R_W$ denote the constant sheaf on $W$ generated by $R$ with its canonical geometry $h_R$. Then we have

$$1_W = \text{cyl}(R_W, h_R) \in K^0 R(W).$$

We have

$$R^i\pi_* (R_W) \cong \begin{cases} R & i = 0, 1 \\ 0 & i \geq 2 \end{cases}.$$  

Since we have normalized the volume of the fibres to one it is easy to check that $h_{L^2}(W/B, R)$ is the standard metric on $(R_B)_\sigma$ for $i = 0, 1$. This implies

$$\text{cyl}(R^i\pi_* (R_W), h_{R^i\pi_* (R_W)}) = 1_B, \quad i = 0, 1.$$  

The contributions of $i = 0$ and $i = 1$ to $\text{index}_0 (1_W)$ thus cancel out in (129), and we get

$$\text{index}^an (1_W) = a(T(\pi, g, R_W, h^R)) .$$

These torsion forms have explicitly been calculated by Bismut-Lott [BL95, Prop. 4.13] and Lott [Lot94]. In the case of non-trivial holonomy we get

$$T(\pi, g, V, h^V) = \sum_{\sigma \in \Sigma^*} \left( \sum_{j \text{even}} (-1)^{j/2} \frac{1}{(2\pi)^j} \frac{(2j + 1)!}{2^{2j}(j!)^2} \text{Re}(Li_{j+1}(\sigma(\xi))) c_1^j(\nabla^W) b_{2j+1}(\sigma) \\ + \sum_{j \text{odd}} (-1)^{(j-1)/2} \frac{1}{(2\pi)^j} \frac{(2j + 1)!}{2^{2j}(j!)^2} \text{Im}(Li_{j+1}(\sigma(\xi))) c_1^j(\nabla^W) b_{2j+1}(\sigma) \right).$$

Similarly, in the case of trivial holonomy we get

$$T(\pi, g, R_W, h^R) = \sum_{\sigma \in \Sigma^*} \sum_{j \text{even}} (-1)^{j/2} \frac{1}{(2\pi)^j} \frac{(2j + 1)!}{2^{2j}(j!)^2} Li_{j+1}(1) c_1^j(\nabla^W) b_{2j+1}(\sigma).$$  

94
The transfer index conjecture \[5.3\] together with the vanishing \((147)\) of the topological index now implies that

\[
a(T(\pi, g, \nu, h^V)) + q^* \hat{Z}(L) = a(T(\pi, g, R_W, h^{R_W})) .
\]

It has been checked in Subsection \[5.4.2\] that this, as a consequence of the Cheeger-Müller theorem, holds true after restriction to a point. Therefore it is now interesting to consider the contribution of higher-degree forms. Then the contribution of \(q^* \hat{Z}(L)\) drops out. The transfer index conjecture predicts that

\[
a \left( \sum_{\sigma \in \Sigma^*} \frac{1}{(2\pi)^j} \frac{(2j + 1)!}{2^{2j}(j!)^2} \Im(L_{j+1}(\sigma(\xi))) \ c_1^j(\nabla^W) \ b_{2j+1}(\sigma) \right) = 0
\]

for odd \(j\), and

\[
a \left( \sum_{\sigma \in \Sigma^*} \frac{1}{(2\pi)^j} \frac{(2j + 1)!}{2^{2j}(j!)^2} (\Re(L_{j+1}(\sigma(\xi))) - L_{j+1}(1)) \ c_1^j(\nabla^W) \ b_{2j+1}(\sigma) \right) = 0
\]

for even \(j \geq 1\). Recall that we have an exact sequence

\[
KR^{-1}(B) \xrightarrow{c} HA^{-1}(B) \xrightarrow{a} KR^0(B) \xrightarrow{R} Z^0(\Omega A(B)) .
\]

We now use the fact that \(\mathbb{C}P^\infty := \text{colim}_n \mathbb{C}P^n\) classifies \(U(1)\)-principal bundles. If we apply the above reasoning to the bundles \(S^{2n+1} \to \mathbb{C}P^n\) for all \(n\) in place of \(P \to B\), then we see that the transfer index conjecture \[5.3\] implies:

**Conjecture 5.9** There exists an element \(x \in KR^{-1}(\mathbb{C}P^\infty)\) such that

\[
c(x) = \sum_{j \geq 1} u_j c_1^j \in HA^{-1}(\mathbb{C}P^\infty)
\]

with

\[
u_j := \frac{1}{(2\pi)^j} \frac{(2j + 1)!}{2^{2j}(j!)^2} \begin{cases} \sum_{\sigma \in \Sigma^*} \Im(L_{j+1}(\sigma(\xi))) \ b_{2j+1}(\sigma) & j \text{ odd} \\ \sum_{\sigma \in \Sigma^*} (\Re(L_{j+1}(\sigma(\xi))) - L_{j+1}(1)) \ b_{2j+1}(\sigma) & j \geq 1 \text{ even} \end{cases}
\]

and the generator \(c_1 \in H^2(\mathbb{C}P^\infty; \mathbb{Z})\).

We can calculate

\[
KRQ^*(\mathbb{C}P^\infty) \cong KRQ^*[c_1] .
\]

Therefore the image \(x_Q \in KRQ^*(\mathbb{C}P^\infty)\) of the class \(x \in KR^{-1}(\mathbb{C}P^\infty)\) can be written in the form

\[
x_Q = \sum_{j \geq 1} \frac{(2j + 1)!}{2^{2j}(j!)^2} x_{2j+1} c_1^j
\]

95
with \( x_{2j+1} \in K_{2i+1}(R) \otimes \mathbb{Q} \) satisfying
\[
\frac{1}{(2\pi)^2} \frac{(2j+1)!}{2^{2j}(j!)^2} \sum_{\sigma \in \Sigma} \langle x_{2j+1}, \omega_{2j+1}(\sigma) \rangle \ b_{2j+1}(\sigma) = u_j .
\]

Finally, let us express the sum on the right-hand side in terms of the Borel regulator. We define the element \( \text{Li}^R_{j+1} \in X_{2j+1}(R) \) (see 232 for notation) such that
\[
\text{Li}^R_{j+1}(\sigma) := [\text{Li}_{j+1}(\sigma(\xi))] \in \mathbb{C}/\mathbb{R}(j+1) \cong \mathbb{R}(j) .
\]
We define \( \text{Li}^Z_{j+1} \in X_{2j+1}(\mathbb{Z}) \) by the same formula with \( \xi = 1 \). The inclusion \( i : \mathbb{Z} \to R \) induces a map \( i_* : X_{2j+1}(\mathbb{Z}) \to X_{2j+1}(R) \). By Proposition 6.29 we have
\[
\psi(\text{Li}^R_{j+1} - i_*\text{Li}^Q_{j+1}) = (-1)^j j! 2^{2j} u_j .
\]

Observe that the prefactors are rational. Recall the definition (233) of the normalized Borel regulator map \( r_{\text{Bor}} \). Then our Conjecture 5.3 via 5.9 implies:

**Fact 5.10** For every \( j \geq 1 \) there exists an element \( y_{2j+1} \in K_{2j+1}(R) \otimes \mathbb{Q} \) such that
\[
r_{\text{Bor}}(y_{2j+1}) = \text{Li}^R_{j+1} - i_*\text{Li}^Q_{j+1} .
\]

**Proof.** This fact has indeed been proven in arithmetic geometry. In fact, one can realize both terms, \( \text{Li}^R_{j+1} \) and \( i_*\text{Li}^Q_{j+1} \), separately as Borel regulators.

For the second, applying the main result of Borel [Bor77] in the case of the field \( \mathbb{Q} \), we conclude that there exists an element \( z_{2j+1} \in K_{2j+1}(\mathbb{Z}) \otimes \mathbb{Q} \) such that \( r_{\text{Bor}}(z_{2j+1}) = [\text{Li}^Q_{j+1}] \in X_{2j+1}(\mathbb{Z}) \). We then have
\[
i_*\text{Li}^Q_{j+1} = r_{\text{Bor}}(i_*z_{2j+1})
\]
by the naturality of the Borel regulator.

The existence of an element \( w_{2j+1} \in K_{2j+1}(R) \otimes \mathbb{Q} \) with \( r_{\text{Bor}}(w_{2j+1}) = \text{Li}^R_{j+1} \) is ensured by a Theorem of Beilinson, see [HK03, Thm. 5.2.1], [Bei80], [Neu88], [Esn89]. In these papers an element in motivic cohomology with prescribed Beilinson regulator is constructed. Now one can identify this particular motivic cohomology group with \( K_{2j+1}(R) \otimes \mathbb{Q} \) so that the Beilinson regulator is half of the Borel regulator [BG02]. Then the combination \( y_{2j+1} := w_{2j+1} - i_*z_{2j+1} \) does the job.

As an alternative to the arithmetic geometric construction of the elements \( w_{2k+1} \) and \( y_{2j+1} \) above one could also obtain the existence of these elements by an argument based on Fact 5.16 and Theorem 5.14. The details will be written up elsewhere.³

The validity of the consequence 5.10 further supports the transfer index conjecture 5.3.⁴

³We thank G. Kings for explaining this result.
⁴This is a result from the diploma thesis of Aron Strack.
5.4.4 Higher analytic and Igusa-Klein torsion

In this subsection we collect further consequences of the transfer index conjecture 5.3 which are provable by independent means. They involve the topological version of higher torsion defined by Igusa and Klein [Igu02] and its comparison with the analytic torsion due to Bismut and Goette [BG01], [Goe09].

These consequences come out of a comparison between the analytical and topological index for geometric locally constant sheaves of finitely generated projective $R$-modules whose underlying topological $K$-theory classes are in the image of the unit (148). This property allows an explicit calculation, as we shall see below.

The unit of the algebraic $K$-theory spectrum is a map of spectra

$$
\epsilon : S \to KR.
$$

(148)

We define the differential sphere spectrum $\text{Diff}(S)$ using the canonical differential data (Definition 2.10). Since the sphere spectrum is rationally even, by [BS10] the axioms (Definition 2.2) determine the functor $\hat{S}^0$ up to unique isomorphism. We can extend the unit $\epsilon$ to a map of canonical differential data (unique up the action of $K_1(R) \otimes \mathbb{Q}$) and thus obtain a map

$$
\hat{\epsilon} : \text{Diff}(S) \to \text{Diff}(KR)
$$

of differential function spectra. The $\mathbb{Z}$-graded group $\pi_*(S) \otimes \mathbb{R} \cong \mathbb{R}$ is trivial in non-zero degrees. Therefore the natural transformation $I$ (see Definition 2.2) induces an isomorphism of smooth groups:

$$
I_S : \hat{S}^0 \cong S^0.
$$

(149)

We consider a proper submersion $\pi : W \to B$ and assume, for simplicity, that $B$ is connected. It follows from the naturality of the differential transfer in the data that on $\hat{S}^0$ we have

$$
\hat{\text{tr}} \circ \hat{\epsilon} = \hat{\epsilon} \circ I_{S}^{-1} \circ \text{tr}^* \circ I_S,
$$

In particular, it follows from the relation $\text{tr}^*(1_W) = \chi(F)1_B$ that we have

$$
\hat{\text{tr}}(1_W) = \text{tr}(\hat{\epsilon}(I_{S}^{-1}(1_W))) = \hat{\epsilon}(I_{S}^{-1}(\text{tr}^*(1_W))) = \hat{\epsilon}(I_{S}^{-1}(\chi(F)1_B)) = \chi(F)1_B,
$$

where $\chi(F)$ is the Euler characteristic of the fibre of $\pi : W \to B$, and $1_M \in S^0(M)$ is the unit, and $1_M := \hat{\epsilon}(I_{S}^{-1}(1_M))$. We now consider a geometric locally constant sheaf $(\mathcal{V}, h^\mathcal{V})$ of free $R$-modules of rank $n$ such that $|\mathcal{V}| = \dim(\mathcal{V})1_W \in KR^0(W)$. Then there exists a form $\eta \in \Omega A^{-1}(W)$ such that

$$
\text{cycl}(\mathcal{V}, h^\mathcal{V}) + a(\eta) = \text{dim}(\mathcal{V})1_W
$$

holds true in $KR^0(W)$. In this case we can calculate the topological index explicitly:

$$
\text{index}^{\text{top}}(\mathcal{V}, h^\mathcal{V}) = \chi(F)\dim(\mathcal{V})1_B - a\left(\int_{W/B} \eta \wedge e(g)\right).
$$

(151)
In order to calculate the analytic index \( \text{index}^{an}(\mathcal{V}, h^\mathcal{V}) \) we need some preparations. Let \( \mathcal{V} \) be a locally constant sheaf of finitely generated projective \( R \)-modules on a manifold \( M \) which has a filtration

\[
0 = F^r \mathcal{V} \subseteq F^{r-1} \mathcal{V} \subseteq \cdots \subseteq F^1 \mathcal{V} \subseteq F^0 \mathcal{V} = \mathcal{V}
\]

by locally constant subsheaves such that the quotients \( F^{i-1} \mathcal{V}/F^i \mathcal{V} \) are trivialized, i.e. identified with \( R^n \) for all \( i \geq 1 \). Such a sheaf is called unipotent. The trivializations give the canonical flat geometries \( h^{F^{i-1} \mathcal{V}/F^i \mathcal{V}} \) on the subquotients. Let \( h^\mathcal{V} \) be any geometry on \( \mathcal{V} \). It induces for all \( i \) geometries \( h^{F^i \mathcal{V}} \) on the subsheaves \( F^i \mathcal{V} \) by restriction. We use the sum of Lott’s relations (Theorem 5.7)

\[
\text{cycl}(F^{i+1} \mathcal{V}, h^{F^{i+1} \mathcal{V}}) + \text{cycl}(F^i \mathcal{V}/F^{i+1} \mathcal{V}, h^{F^i \mathcal{V}/F^{i+1} \mathcal{V}}) - \text{cycl}(F^i \mathcal{V}, h^{F^i \mathcal{V}}) = -a(T_i) \quad (152)
\]

and equalities

\[
\text{cycl}(F^i \mathcal{V}/F^{i+1} \mathcal{V}, h^{F^i \mathcal{V}/F^{i+1} \mathcal{V}}) = \dim(F^i \mathcal{V}/F^{i+1} \mathcal{V}) 1_M
\]

in order to obtain the equality

\[
\text{cycl}(\mathcal{V}, h^\mathcal{V}) = \dim(\mathcal{V}) 1_M + a(T(F^*, h^\mathcal{V})) \quad , \quad T(F^*, h^\mathcal{V}) := \sum_i T_i .
\]

We now come back to our original situation and assume that the geometry \( h^\mathcal{V} \) on the sheaf \( \mathcal{V} \) on \( W \) is flat and that the sheaves \( R^i \pi_* (\mathcal{V}) \) are unipotent. Using the geometries (128) we define the form

\[
\tau(\pi, g, \mathcal{V}, h^\mathcal{V}) := T(\pi, g, \mathcal{V}, h^\mathcal{V}) + \sum_i (-1)^i a(T(F^*_i, h^{R^i \pi_* (\mathcal{V})})) \in \Omega A^{-1}(B) .
\]

One checks using (132) and (137) that \( \tau(\pi, g, \mathcal{V}, h^\mathcal{V}) \) is closed. By the usual homotopy argument its cohomology class

\[
\tau(\pi, \mathcal{V}) \in H(A)^{-1}(M)
\]

does not depend on the Riemannian geometry \( g \) of \( \pi \) and the flat geometry \( h^\mathcal{V} \).

\textbf{Definition 5.11} Assume that \( \mathcal{V} \) is a locally constant sheaf of finitely generated projective \( R \)-modules on \( W \) which admits a flat geometry \( h^\mathcal{V} \) and whose cohomology sheaves \( R^i \pi_* (\mathcal{V}) \) are unipotent. Then we have a higher analytic torsion class of the bundle \( \pi : W \to B \) and the locally constant sheaf \( \mathcal{V} \) given by

\[
\tau(\pi, \mathcal{V}) := [\tau(\pi, g, \mathcal{V}, h^\mathcal{V})] \in H(A)^{-1}(M) .
\]

A similar definition has been given in \cite[Def. 2.8]{Goe09}. Note that here we use the normalization fixed by Bismut-Lott. We have by the above construction

\[
\widehat{\text{index}}^{an}(\mathcal{V}, h^\mathcal{V}) = \lambda(F) \dim(\mathcal{V}) 1_B + a(\tau(\pi, \mathcal{V})) . \quad (153)
\]
Assuming the transfer index conjecture \[5.3\] by comparison with \((151)\) we get the following consequence:

\[
a \left( \int_{W/B} \eta \wedge e(g) + \tau(\pi, \mathcal{V}) \right) = 0 .
\]  
\((154)\)

In particular, if we take the canonical geometry \(h\mathbb{R}^W\) on \(\mathcal{V} = \mathbb{R}^W\), then in \((150)\) we can take \(\eta = 0\) and get

\[
a(\tau(\pi, R_W)) = 0 .
\]  
\((155)\)

**Definition 5.12** We call a proper submersion \(\pi : W \to B\) unipotent if the cohomology sheaves \(R^i\pi_*(R_W)\) are unipotent for all \(i \geq 0\).

From \((\cdot)\) and \((155)\) we conclude

**Corollary 5.13** We assume that \(\pi : W \to B\) is unipotent. If the transfer index conjecture \[5.3\] holds true, then there exists an element \(x \in KR^{-1}(B)\) such that \(c(x) = \tau(\pi, R_W)\) in \(H(A)^{-1}(B)\).

This is an integrality statement for the higher analytic torsion class.

We now return to the situation where \(\mathcal{V}\) is a locally constant sheaf of finitely generated projective \(R\)-modules on \(W\). In addition we now assume that the sheaves \(R^i\pi_*(\mathcal{V})\) are trivializable for all \(i \geq 0\). This is the case e.g. if the base manifold \(B\) is simply connected and the cohomology groups of the fibres of \(\pi\) with coefficients in \(\mathcal{V}\) are free. We further assume that \(h\mathbb{V}\) is flat so that we can take \(\eta = 0\).

In this case the Igusa-Klein torsion \(\tau_{IK}^{Igusa}(\pi, \mathcal{V})\) of a flat complex bundle \(\mathcal{V}\) is defined \[\text{[Goe09, Def. 4.4]}, \text{[Igu08]}, \text{[Igu02]}, \text{and (157)].}\] The superscript indicates the Igusa normalization. Using \((239)\) we define for all \(j \geq 0\) the renormalized version

\[
\tau_{IK}(\pi, \mathcal{V})_{2j} := N_{Igusa}(2j + 1) \tau_{IK}^{Igusa}(\pi, \mathcal{V})_{2j}
\]

and

\[
\tau_{IK}(\pi, \mathcal{V}) := \sum_{\sigma \in \Sigma_r} \sum_{j=1}^\infty \tau_{IK}(\pi, \mathcal{V})_{2j} b_{2j+1}(\sigma) \in H(A)^{-1}(M) .
\]

We further define the Igusa normalized characteristic class of a real vector bundle \(U \to M\)

\[
0^{Igusa}_J(U) := \sum_{j=1}^\infty (-1)^j \zeta(2j + 1) \left[ \text{ch}(U \otimes \mathbb{C}) \right]_{4j} .
\]

Its Bimut-Lott normalization is then

\[
0_J(U) := \sum_{j=1}^\infty N_{Igusa}(4j + 1) 0^{Igusa}_J(U)_{4j} .
\]
We furthermore define

\[ 0 J_R(U) := \sum_{\sigma \in \Sigma^*} 0 J(U)_{\sigma} b_{4j+1}(\sigma) \in H(A)^{-1}(M). \]

The following theorem has been announced by Goette [Goe09, Thm. 5.5] (note that he uses the Chern normalization).

**Theorem 5.14 (Goette)**

\[ \tau(\pi, V) = \tau_{IK}(\pi, V) + \text{tr}^* 0 J_R(T^v \pi) \in H(A)^{-1}(M). \]

If the fibre dimension is odd, then the transfer \( \text{tr}^* \) vanishes. Since \( \eta = 0 \) from (6) and (154) we now conclude:

**Corollary 5.15** Assume that the fibre dimension of \( \pi : W \to B \) is odd, that \( V \) satisfies \( [V] = \dim(V)1_W \), that the geometry \( h^V \) is flat, and that the sheaves \( R^i \pi_* (V) \) are trivializable. If the transfer index conjecture holds true, then there exists an element \( x \in KR^{-1}(B) \) such that \( c(x) = \tau_{IK}(\pi, V) \).

At the moment we can verify the existence of an element \( x \in KR^{-1}(B) \) such that \( c(x) = \tau_{IK}(\pi, V) \) in certain cases by independent means, see Proposition 5.19. But we can show in general that such an element exists rationally.

**Proposition 5.16** Under the assumption of Corollary 5.15 there exists an element \( x \in KR^{-1}(B) \otimes \mathbb{Q} \) such that \( c(x) = \tau_{IK}(\pi, V) \).

*Proof.* We first recall some details of the construction of the Igusa-Klein torsion. The unit of the algebraic K-theory spectrum \( \epsilon : S \to KR \) extends to a fibre sequence

\[ \ldots \Omega KR \to Wh^R(\ast) \to S \xrightarrow{\epsilon} KR \to \Sigma Wh^R(\ast) \ldots, \]

defining the Whitehead spectrum \( Wh^R(\ast) \). Since the homotopy groups \( \pi_i(S) \) for \( i \geq 1 \) are torsion the map of infinite loop spaces

\[ \Omega^{\infty+1} KR \to \Omega^{\infty} Wh^R(\ast) \]

is a rational equivalence. In particular we have an isomorphism of real cohomology groups

\[ H^*(\Omega^{\infty} Wh^R(\ast); \mathbb{R}) \xrightarrow{\sim} H^*(\Omega^{\infty+1} KR; \mathbb{R}). \]  

(156)

The spectrum cohomology classes (39)

\[ \omega_{2k+1}(\sigma) \in H^{2k+1}(KR; \mathbb{R}), \quad \sigma \in \Sigma \]

induce classes \( \tau_{2k}(\sigma) \in H^{2k}(\Omega^{\infty+1} KR; \mathbb{R}) \). Under the isomorphism (156) they correspond to the universal torsion classes

\[ \tau_{IK, 2k}(\sigma) \in H^{2k}(\Omega^{\infty} Wh^R(\ast); \mathbb{R}). \]
The main ingredient of the construction of the Igusa-Klein torsion of a bundle $\pi : W \to B$ equipped with a locally constant sheaf $\mathcal{V}$ of finitely generated projective $R$-modules is the construction of a well-defined homotopy class of maps

$$m_{IK}^\mathcal{V} : B \to \Omega^\infty Wh^R(*)$$

using the theory of framed Morse functions. Then by definition

$$\tau_{IK}(\pi, V_\sigma)_{2k} = (m_{IK}^\mathcal{V})^* \tau_{IK,2k}(\sigma). \quad (157)$$

We now consider the diagram of spaces

$$\begin{array}{c}
\Omega^{\infty+1} KR \\
\downarrow \phi \\
\Omega^\infty Wh^R(*) \\
\downarrow \chi \\
\Omega^\infty S \\
\downarrow \epsilon_{H\mathbb{Q}} \\
\Omega^\infty H\mathbb{Q}
\end{array}$$

The class $[\chi_\mathbb{Q}] \in H^0(B; \mathbb{Q}) \cong \mathbb{Q}$ is the Euler characteristic of the fibre which vanishes by assumption. Since $\epsilon_{H\mathbb{Q}}$ is a rational equivalence we conclude that the class $[\chi] \in S^0(B)$ vanishes rationally. We choose $0 \neq l \in \mathbb{N}$ such that $l[\chi] = 0$. Then there exists a lift $\hat{f} : B \to \Omega^{\infty+1} KR$ of the map $l m_{IK}^\mathcal{V}$. We consider this map as a class $[\hat{f}] \in KR^{-1}(B)$. Then we have $l \tau_{IK}(\pi, \mathcal{V}) = c([\hat{f}])$. Hence

$$x := [\hat{f}] \otimes l^{-1} \in KR^{-1}(B) \otimes \mathbb{Q}$$

does the job.

We now consider the special case of bundles over the base $B = S^{2j}$. Then we have $KR^{-1}(S^{2j}) \cong K_{2j+1}(R)$. In this case Corollary 5.15 is equivalent to:

**Corollary 5.17** Let $\pi : W \to S^{2j}$ be a proper submersion with odd-dimensional fibres and $(\mathcal{V}, h^\mathcal{V})$ be a locally constant sheaf of finitely generated projective $R$-modules with flat geometry $h^\mathcal{V}$. We assume that the sheaves $R^i \pi_*(\mathcal{V})$ are trivializable for all $i \geq 0$. If the transfer index conjecture holds true, then there exists an element $x \in K_{2j+1}(R)$ such that

$$\omega_{2j+1}(\sigma)(x) = \langle \tau_{IK}(\pi, V_\sigma), [S^{2j}] \rangle$$

for all $\sigma \in \Sigma$.

Hatcher’s construction provides examples of bundles $\pi : W \to S^{4k}$ with non-trivial Igusa-Klein torsion. Let

$$a_k := \text{denom}(\frac{B_k}{4k}) \in \mathbb{N},$$

101
where $B_k$ is the $k$’th Bernoulli number. By a Theorem of Adams [Ada63] this number is the order of the $J$-homomorphism

$$\pi_{4k-1}(O) \to \pi_{4k-1}(S).$$

Let

$$\kappa_k := \begin{cases} 1 & k \text{ is odd} \\ \frac{1}{2} & k \text{ is even} \end{cases}.$$

Then we have the following theorem shown in [GI10, Cor. 1.2.4]:

**Theorem 5.18** For sufficiently large $n$ we there exists a bundle $\pi : W \to S^{4k}$ with fibre $S^n$ such that

$$\langle \tau^{fgua}_I(\pi, C), [S^{4k}] \rangle = a_k \kappa_k \zeta(2k + 1).$$

For those examples we can verify the consequence of the transfer index conjecture formulated in Corollary 5.17 by independent means.

**Proposition 5.19** If the bundle $\pi : W \to S^{4k}$ obtained by Hatcher’s construction and $V = R_W$, then there exists an element $x \in K_{4k+1}(R)$ such that

$$\omega_{4k+1}(\sigma)(x) = \langle \tau_I(\pi, V_\sigma), [S^{4k}] \rangle$$

for all $\sigma \in \Sigma$.

**Proof.** Homotopy theoretically, Hatcher’s example can be understood by the following diagram which we take from [BDKW11, Sec. 8] with slight modifications.

Here $SG = SL_1(S) \subseteq GL_1(S)$ is the connected component of the identity of the group of units of the sphere spectrum, $A(*)$ is Waldhausen’s $K$-theory of spaces [Wal85] evaluated at the point, and $Wh^{Diff}(*)$ is the spectrum appearing in Waldhausen’s splitting [Wal85], [BW87].

$$A(*) \cong S \vee Wh^{Diff}(*) .$$
The map \( \lambda^R : A(*) \to KR \) is the linearization, and the map \( \Omega^{\infty+1}Wh^{Diff}(*) \to \Omega^{\infty}Wh^R(*) \) is induced from the lower right commutative square.

The map \( J : BSO \to \Omega^\infty S \) induces the \( J \)-homomorphism [Ada63]. We choose \( f \in \pi_{4k}(BSO) \) such that it represents an element the kernel of the \( J \)-homomorphism. Because of the splitting (159) the induced map \( S^{4k} \to BSG \) is zero homotopic and we can find a lift \( \tilde{f} : S^{4k} \to SG/SO \). Since \( J \circ f \) is zero homotopic we can further find a factorization of \( \hat{f} : S^{4k} \to \Omega^{\infty}Wh^R(*) \) through a map \( \check{f} : S^{4k} \to \Omega^{\infty+1}KR \). This map represents a class

\[ [\check{f}] \in K_{4k+1}(R). \]

If we build Hatcher’s example from the map \( f \), then we get a bundle \( W \to S^{4k} \) such that

\[ m_{IK}^{RW} \sim \hat{f}. \]

This implies that

\[ \tau_{IK}(\pi, R_W) = \sum_{\sigma \in \Sigma^*} \hat{f}^* \tau_{4k}(\sigma)b_{4k+1}(\sigma) \]

and therefore

\[ \langle \tau_{IK}(\pi, R_W), [S^{4k}] \rangle = \sum_{\sigma \in \Sigma^*} \omega_{4k+1}(\sigma)([\tilde{f}])b_{4k+1}(\sigma). \]

\[ \square \]

5.5 Discussion of Lott’s relation (Theorem 5.7)

We first remove the projectivity assumption in Lott’s relation and thus finish the proof of Theorem 5.7.

**Lemma 5.20** If Lott’s relation holds true for exact sequence of sheaves of finitely generated projective \( R \)-modules, then it also holds true for exact sequences of sheaves of finitely generated \( R \)-modules.

**Proof.** We consider an exact sequence

\[ V : 0 \to V_0 \to V_1 \to V_2 \to 0 \]
of finitely generated $R$-modules. It induces a diagram of sheaves (see (81) for notation)

\[
\begin{array}{c}
0 \\
\downarrow \\
\vdots \\
\downarrow \\
0 \rightarrow \text{Tors}(\mathcal{V}_0) \rightarrow \text{Tors}(\mathcal{V}_1) \rightarrow \text{Tors}(\mathcal{V}_2) \rightarrow 0 \\
\downarrow \\
\vdots \\
\downarrow \\
\text{Proj}(\mathcal{V}) : \\
0 \rightarrow \text{Proj}(\mathcal{V}_0) \rightarrow \text{Proj}(\mathcal{V}_1) \rightarrow \text{Proj}(\mathcal{V}_2) \rightarrow 0 \\
\end{array}
\]

with exact rows and columns. Indeed, the arrow named $!$ is surjective since the arrow named $!!$ locally splits.

We now assume that we are given geometries $h^{\mathcal{V}_i}$ for $i = 0, 1, 2$. By Definition 3.32 we have the relation

\[
\sum_{i=0}^{2} (-1)^i \text{cycl}(\mathcal{V}_i, h^{\mathcal{V}_i}) = \sum_{i=0}^{2} (-1)^i \text{cycl}(\text{Proj}(\mathcal{V}_i), h^{\mathcal{V}_i}) + \sum_{i=0}^{2} (-1)^i \hat{Z}(\text{Tors}(\mathcal{V}_i)) .
\]

Since the torsion forms for the sequences with geometries $(\mathcal{V}, h^{\mathcal{V}})$ and $(\text{Proj}(\mathcal{V}), h^{\mathcal{V}})$ coincide it remains to show that

\[
\sum_{i=0}^{2} (-1)^i \hat{Z}(\text{Tors}(\mathcal{V}_i)) = 0 . \tag{160}
\]

To this end we consider the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
B : \\
0 \rightarrow E(\text{Tors}(\mathcal{V}_0)) \rightarrow E(\text{Tors}(\mathcal{V}_1)) \rightarrow B_2 \rightarrow 0 \\
\downarrow \\
0 \\
\downarrow \\
\mathcal{A} : \\
0 \rightarrow F(\text{Tors}(\mathcal{V}_0)) \rightarrow F(\text{Tors}(\mathcal{V}_1)) \rightarrow \mathcal{A}_2 \rightarrow 0 \\
\downarrow \\
0 \\
\downarrow \\
0 \rightarrow \text{Tors}(\mathcal{V}_0) \rightarrow \text{Tors}(\mathcal{V}_1) \rightarrow \text{Tors}(\mathcal{V}_2) \rightarrow 0 \\
\end{array}
\]

where we use the notation introduced at the beginning of Section 3.10. Now we observe that $\mathcal{A}_2$ is a sheaf of projective $R$-modules. Indeed, let us consider stalks at a given point.
Then we can identify the stalk of \( A_2 \) canonically with the free \( R \)-module generated by those elements of the stalk of \( V_1 \) which do not belong to the stalk of \( V_0 \). In particular, \( A_2 \) is a sheaf of projective \( R \)-modules and the right vertical sequence is a resolution of \( \text{Tors}(V_2) \).

We now choose geometries \( h^{A_i}, i = 0, 1, 2 \) for the sheaves in the complex \( A \). They induce geometries \( h^{B_i} \) on the sheaves in the complex \( B \) by restriction. Then we have by Lemma \ref{lem:3.31}

\[
\sum_{i=0}^{2} (-1)^i \hat{Z} \left( \text{Tors}(V_i) \right) = \sum_{i=0}^{2} (-1)^i \text{cycl}(A_i, h^{A_i}) - \sum_{i=0}^{2} (-1)^i \text{cycl}(B_i, h^{B_i}).
\]

By [BT] we can assume Lott’s relation for the sequences of projective \( R \)-modules \( A \) and \( B \). Since their torsion forms coincide (since their complexifications for every place \( \sigma \) are isometric), we conclude the equality (160).

We now discuss some cases of Lott’s which can be verified independently of [BT]. We consider three locally constant sheaves of finitely generated projective \( R \)-modules on a manifold \( M \) with geometries \( (V_i, h^{V_i}), i = 0, 1, 2 \), which fit into an exact sequence

\[
V : 0 \to V_0 \to V_1 \to V_2 \to 0. \tag{161}
\]

Then we define

\[
\delta(V) := \text{cycl}(V_0, h^{V_0}) + \text{cycl}(V_2, h^{V_2}) - \text{cycl}(V_1, h^{V_1}) + a(T) \in \text{KR}^0(M), \tag{162}
\]

where \( T \) is the analytic torsion form defined in \((136)\). Lott’s relation asserts the equality \( \delta(V) = 0 \). We start with some general properties of \( \delta(V) \).

**Lemma 5.21** We have \( \delta(V) \in \text{im}(H(A)^{-1}(M) \to \text{KR}^0(M)) \). In particular, the class \( \delta(V) \) only depends on the sequence \( V \) and not on the choice of geometries.

**Proof.** We have the relation \( [V_0] + [V_2] = [V_1] \) in \( \text{KR}^0(M) \) and therefore \( I(\delta(V)) = 0 \). Furthermore, by [Lot00, Eq. (292)] we have the equality (137)

\[
\omega(h^{V_0}) + \omega(h^{V_2}) - \omega(h^{V_1}) = dT
\]

and therefore \( R(\delta(V)) = 0 \). This implies that \( \delta(V) \in \text{im}(H(A)^{-1}(M) \to \text{KR}^0(M)) \). Since this image is a homotopy invariant functor we conclude that \( \delta(V) \) does not depend on the geometries by a standard deformation argument. \( \square \)

**Lemma 5.22** We have \( \delta(V) = 0 \) if the sequence \( V \) splits.
Proof. We use the isomorphism \( V_1 \cong V_0 \oplus V_2 \), the additivity of the cycle map, and that the torsion form \( T \) vanishes for a metrically split sequence in order to conclude that
\[
\delta(V) = \text{cycl}(V_0, h^{V_0}) + \text{cycl}(V_2, h^{V_2}) - \text{cycl}(V_1, h^{V_0} \oplus h^{V_2}) = 0.
\]

Here are further cases in which we know that \( \delta(V) = 0 \).

Lemma 5.23 Assume that \( R \) is the ring of integers in a totally real number field (resp. arbitrary number field). Then we have \( \delta(V) = 0 \) if \( V \) is pulled back from a \( CW \)-complex of rational cohomological dimension \( \leq 3 \) (\( \leq 1 \)).

Proof. We discuss the totally real case. Let \( X \) be a \( CW \)-complex of rational cohomological dimension \( \leq 3 \) with a sequence \( V' \) and a map \( f : M \to X \) such that \( V \cong f^*V' \). By approximation of a finite skeleton of \( X \) we can assume that \( X \) is a manifold with \( H^l(X; \mathbb{Q}) = 0 \) for \( 4 \leq l \leq \dim(M) \), and that \( f \) is smooth. Then we have \( \delta(V) = \delta(f^*V') = f^*\delta(V') \). We claim that \( f^*\delta(V') = 0 \). Note that by Theorem 3.3 we have \( K_i(R) \otimes \mathbb{R} = 0 \) for \( i = 2, 3, 4 \). It follows that
\[
f^*(H(A)^{-1}(X)) \cong f^*(H^0(X; \mathbb{R}) \otimes K_1(R)).
\]

By (163) it suffices to show that \( \delta(V') \) is zero after restriction to a point. But after restriction to a point the sequence \( V' \) splits and we know that the conjecture holds true in this case by Lemma 5.22.

The assumption of Lemma 5.23 in the totally real case is e.g. satisfied if \( \dim(M) \leq 3 \) or \( \text{coht}_{\mathbb{Q}} \dim(\pi_1(M)) \leq 3 \).

Using some results in group cohomology one can verify Lott’s relation rationally.

Proposition 5.24 \( \delta(V) \in \text{im}(H(A)^{-1}(M) \overset{\alpha}{\to} \widehat{KR}_0(M)) \) is a torsion element.

Proof. After adding some split extension of trivial locally constant sheaves of finitely generated projective \( R \)-modules we can assume that the \( V_i \) are free. We fix a prime \( l \in \mathbb{Z} \) and a prime ideal \( \mathfrak{m} \subset R \) such that \( l \notin \mathfrak{m} \). We let \( R^\mathfrak{m} \) be the localization of \( R \) obtained by inverting the elements of \( \mathfrak{m} \). Then \( l \in (R^\mathfrak{m})^* \). As a consequence of the localization sequence (see Weibel’s \( K \)-book project V.6) and the finiteness of the higher \( K \)-theories of finite fields the inclusion \( R \to R^\mathfrak{m} \) induces an isomorphism
\[
K_i(R) \otimes \mathbb{Q} \simto K_i(R^\mathfrak{m}) \otimes \mathbb{Q}
\]
for all \( i \geq 0 \) with the exception of \( i = 1 \). The additional unit \( l \in (R^\mathfrak{m})^* \) increases the rank of \( K_1(R^\mathfrak{m}) \) (compared with \( K_1(R) \)) by one (see [Neu99, Kor. 11.7]). The construction of the cycle map given in Section 3 can be applied to the ring \( R^\mathfrak{m} \) with the only modification that the classes \( \omega_1(\sigma), \sigma \in \Sigma^* \) now become linearly independent. Of course, the statement of the theorem of Borel 3.3 and its subsequent refinements have to be modified in degree.
one, correspondingly. Furthermore, the elements \((b_1(\sigma))_{\sigma \in \Sigma^*}\) now form an honest basis of \(K_1(R^\ell) \otimes \mathbb{R}\) which is dual to the basis \((\omega_1(\sigma))_{\sigma \in \Sigma^*}\) of \(H^{\mathbb{R}}_1(KR^\ell)\).

The morphism of rings \(\phi : R \to R^\ell\) induces a map of smooth sets

\[ \phi_* : \text{Loc}_{\text{geom}}^\text{proj}(R) \to \text{Loc}_{\text{geom}}^\text{proj}(R^\ell) \]

by the same construction as in (89). We let \((KR^\ell, A', c')\) be the canonical data for \(\hat{KR}^0\).

The canonical map of data (Theorem 3.35) \((KR, A, c) \to (KR^\ell, A', c')\) induces a map of differential extensions

\[ \hat{\phi}_* : \hat{KR}^0 \to \hat{KR}^\ell_0. \]

The same argument as for the proof of Theorem 3.35 shows that

\[ \text{Loc}_{\text{geom}}^\text{free}(R) \xrightarrow[\phi_*]{} \text{Loc}_{\text{geom}}^\text{free}(R^\ell) \]

commutes. We furthermore have a square

\[
\begin{array}{ccc}
\text{im}(H(A)^{-1}(M) \to \hat{KR}^0(M)) & \xrightarrow{\hat{\phi}} & \text{im}(H(A')^{-1}(M) \to \hat{KR}^\ell_0(M)) \\
\cong & & \cong \\
H(A)^{-1}(M)/KR^{-1}(M) & \xrightarrow{\cong} & H(A')^{-1}(M)/KR^\ell^{-1}(M)
\end{array}
\]

Since \(A \to A'\) is injective the lower map rationally injective. Since by (164) we have the relation \(\bar{\phi}_*(\delta(V)) = \delta(\phi_* V)\) it suffices to show that \(\delta(\phi_* V)\) is a torsion element. It is at this point where the additional unit \(l \in R^\ell\) helps.

We assume that \(V^\ell\) is an extension of locally constant sheaves of finitely generated free \(R^\ell\)-modules on a connected manifold. We are going to show that \(\delta(V^\ell)\) is torsion. Let \(m := \dim(V^\ell_0)\) and \(n := \dim(V^\ell_2)\) be the ranks of the subsheaf and the quotient. Then \(V^\ell\) can be pulled back from the universal extension \(\hat{V}^\ell\) with these dimensions which lives on the classifying space \(BP\) of the group

\[ \hat{P} := \left( \begin{array}{cc} GL(m, R^\ell) & \text{Mat}(m, n, R^\ell) \\ 0 & GL(n, R^\ell) \end{array} \right) \subseteq GL(n + m, R^\ell). \]

It suffices to show that \(\delta(\hat{V}^\ell)\) is torsion. In order to define this element we approximate the classifying space by smooth manifolds. In the following we suppress these approximations. We consider the subgroups

\[ L := \left( \begin{array}{cc} GL(m, R^\ell) & 0 \\ 0 & GL(n, R^\ell) \end{array} \right), \quad N := \left( \begin{array}{cc} 1 & \text{Mat}(m, n, R^\ell) \\ 0 & 1 \end{array} \right) \]

107
of $GL(n + m, R^t)$. Then we have a split extension of groups

$$0 \to N \to P \to L \to 0.$$  

We claim that the projection $P \to L$ induces an isomorphism in rational group homology. Following [Knu01, Rem. 2.2.3] we consider the Leray-Serre spectral sequence

$$E^2_{p,q} := H_p(L, H_q(N; \mathbb{Q})) \Rightarrow H_{p+q}(P; \mathbb{Q}).$$

It suffices to show that $E^2_{p,q} = 0$ for $q \geq 1$. Since $N$ is abelian we have

$$H_q(N; \mathbb{Q}) \cong \Lambda^n_q(\text{Mat}(m, n, R^t) \otimes \mathbb{Q}).$$

The diagonal matrix $(l_1 m, l_2 n) \in \mathbb{Z}(L)$ acts on $N$ as multiplication by $l$ and therefore on $H_q(N; \mathbb{Q})$ by multiplication by $l^q$. On the other hand, $l^q - 1$ acts trivially on $E^2_{p,q} = H_p(L, H_q(N; \mathbb{Q}))$ for $q \geq 1$. This implies that $E^2_{p,q} = 0$. This finishes the verification of the claim.

It follows that the map $B i : BL \to BP$ induced by the embedding $i : L \to P$ is a rational homology equivalence. Consequently, the induced maps

$$B i^* : (KR^t)^{-1}(BP) \to (KR^t)^{-1}(BL), \quad B i^* : H(A')^{-1}(BP) \to H(A')^{-1}(BL)$$

are rational isomorphisms. Since $B i^* \tilde{\nabla}^t$ splits we have $B i^* \delta(\tilde{\nabla}^t) = \delta(B i^* \tilde{\nabla}^t) = 0$. This implies $\delta(\tilde{\nabla}^t)$ is torsion.

We conclude this Section with the following remark involving nil-smooth deformations. We again consider an exact sequence $V$ on $M$ as in (161). We shall see that one can connect $(V_0 \oplus V_2, h_{V_0} \oplus h_{V_2})$ with $(V_1, h_{V_1})$ by a nil-smooth family (Definition 3.38). Indeed, the composition of maps

$$M \stackrel{v_2}{\to} B GL(R) \to B GL(R)^+$$

can be deformed to

$$M \stackrel{v_0 \oplus v_2}{\to} B GL(R) \to B GL(R)^+ .$$

The deformation yields an element $x \in \text{Loc}^+(M)[1]$ (see [91]) which restricts to the classes of $V_1$ and $V_0 \oplus V_2$ at the ends of the interval considered here as points in $\text{Loc}^+(M)[0]$. We now choose a geometry $\hat{x} \in \text{Loc}_{geom}^+(M)[1]$ (i.e. a preimage of $x$ under the map (92)) which restricts to given geometries on the ends. The element $\hat{x}$ represents a class $\hat{x} \in \text{Loc}_{geom}^+(M \times [0, 1])$. This element connectes $(V_0 \oplus V_2, h_{V_0} \oplus h_{V_2})$ with $(V_1, h_{V_1})$ by a nil-smooth family. We define the form

$$S(\hat{x}) := \int_{M \times [0, 1]/M} R(\text{cycl}^+(\hat{x})) \in \Omega A^{-1}(M) .$$

It satisfies

$$dS(\hat{x}) = R(\text{cycl}(V_1, h_{V_1})) - R(\text{cycl}(V_0, h_{V_0})) - R(\text{cycl}(V_2, h_{V_2})) .$$

108
by Stoke’s theorem. By the homotopy formula we have
\[
\text{cycl}(V_0, h^\nu_0) + \text{cycl}(V_2, h^\nu_2) - \text{cycl}(V_1, h^\nu_1) + a(S) = 0 .
\]
(165)
Therefore Lott’s relation \[5.7\] boils down to the following fact.

**Proposition 5.25** There exists an element \( x \in KR^{-1}(M) \) such that
\[
\text{rat}(I(x)) = [S - \mathcal{T}] \in H(A)^{-1}(M) .
\]

We hope that this observation will be useful for the construction of interesting \( K \)-theory classes starting from a sequence \( V \).

### 6 Technicalities

#### 6.1 Categories with weak equivalences and \( \infty \)-categories

In the present paper we work with the notion of \( \infty \)-categories as developed in detail in [Lur09]. Thus an \( \infty \)-category \( C \) is a simplicial set \( C \in \text{sSet} \) which satisfies an inner horn filling condition. A functor \( C \to C' \) between \( \infty \)-categories \( C \) and \( C' \) is a map of simplicial sets, and the \( \infty \)-category \( \text{Fun}(C, C') \) of functors from \( C \) to \( C' \) is the simplicial set whose \( n \)-simplices are the set of maps \( \Delta^n \times C \to C' \).

If \( C \) is an ordinary category, then its nerve \( N(C) \in \text{sSet} \) is an example of an \( \infty \)-category. In the present paper, most \( \infty \)-categories arise from localizations. If \( C \) is an \( \infty \)-category and \( W \) is a collection of arrows of \( C \), then the localization \( l : C \to C[W^{-1}] \) is characterized by the following universal property: For all \( \infty \)-categories \( C' \) the induced map
\[
\text{Fun}(C[W^{-1}], C') \to \text{Fun}_{W^{-1}}(C, C')
\]
is an equivalence, where \( \text{Fun}_{W^{-1}}(C, C') \subseteq \text{Fun}(C, C') \) is the subcategory of those functors which map arrows in \( W \) to equivalences in \( C' \). Note that we can choose some inverse equivalence
\[
\text{Fun}_{W^{-1}}(C, C') \to \text{Fun}(C[W^{-1}], C') .
\]
(166)

In the present paper we mainly consider localizations which arise from a category with weak equivalences, i.e. a pair \( (C, W) \) consisting of a category \( C \) and a collection of arrows \( W \) of \( C \). We consider the set of arrows as a subset \( W \subseteq N(C)[1] \) of the one-simplices of the nerve of \( C \). The \( \infty \)-category associated to the category with weak equivalences \( (C, W) \) is the localization \( N(C)[W^{-1}] \) of \( N(C) \).

Often, in practice, categories with weak equivalences arise from Quillen model structures. We always assume that the model categories in question are equipped with functorial factorizations. If \( C \) is a category with a collection of weak equivalences \( W \), then the homotopy category of the localization \( N(C)[W^{-1}] \) is equivalent to the localization \( C[W^{-1}] \), although these may only be categories in a larger universe. Indeed, enlarging the universe
if necessary, if $D$ is an ordinary category then, by the universal property of $C[\ast^{-1}]$, we see that

$$N(\mathsf{Fun}(C[\ast^{-1}], D)) \simeq N(\mathsf{Fun}_{\ast^{-1}}(C, D)) \simeq \mathsf{Fun}_{\ast^{-1}}(N(C), N(D)) \simeq \mathsf{Fun}(N(C)[\ast^{-1}], D).$$

If $f : C \to C'$ is a functor between $\infty$-categories which carries the arrows of $W$ to the arrows of $W'$, then we obtain a functor

$$f : C[\ast^{-1}] \to C'[\ast'^{-1}]$$

between the localizations. If $f : C \to C'$ is a left Quillen functor between model categories, then it preserves weak equivalences between cofibrant objects. In this case we define the induced functor between the localizations by

$$f : N(C)[\ast^{-1}] \xrightarrow{\sim} N(C'_{\ast^{-1}}) \xrightarrow{f^c} N(C'_{\ast'^{-1}}) \xrightarrow{\sim} N(C')[\ast'^{-1}],$$

where $f^c$ denotes the restriction of $f$ to cofibrant objects. A similar construction works with right Quillen functors using the the subcategories of fibrant objects.

For two objects $x, y$ in an $\infty$-category we write

$$\mathsf{map}(x, y) \in N(\mathsf{sSet})[\ast^{-1}]$$

for the mapping space between $x$ and $y$. In the stable case we use a different convention, see \cite{188}.

### 6.2 $\infty$-categories of spaces, spectra, and chain complexes

The goal of the present subsection is to introduce the main examples of categories and $\infty$-categories used in the present paper.

We write $\mathsf{Top}$ for the model category of topological spaces (with the usual weak equivalences and Serre fibrations \cite[Ex. 9.1.15]{Hir03}) and $\mathsf{sSet}$ for the model category of simplicial sets (with the usual weak equivalences and Kan fibrations \cite[Ex. 9.1.13]{Hir03}). We also have the pointed versions $\mathsf{sSet}_*$ (cf. \cite[Ex. 9.1.14]{Hir03}) and $\mathsf{Top}_*$ (cf. \cite[Ex. 9.1.16]{Hir03}). Geometric realization is a left Quillen functor from $\mathsf{sSet}$ to $\mathsf{Top}$. Its right adjoint is the singular complex. This adjoint pair induces Quillen equivalences

$$|\ldots| : \mathsf{Top} \leftrightarrows \mathsf{sSet} : \mathsf{sing}, \quad |\ldots| : \mathsf{Top}_* \leftrightarrows \mathsf{sSet}_* : \mathsf{sing}$$

and therefore equivalences of $\infty$-categories

$$N(\mathsf{sSet})[\ast^{-1}] \simeq N(\mathsf{Top})[\ast^{-1}] , \quad N(\mathsf{sSet}_*)[\ast^{-1}] \simeq N(\mathsf{Top}_*)[\ast^{-1}].$$

We write $\mathsf{Sp}$ for the model category of spectra in simplicial sets. Specifically we take Bousfield-Friedlander spectra \cite{BF78}. We have a Quillen adjunction

$$\Sigma^\infty : \mathsf{sSet}_* \leftrightarrows \mathsf{Sp} : \Omega^\infty$$
which induces an adjunction of $\infty$-categories

$$\Sigma^\infty : N(sSet_\ast)[W^{-1}] \rightleftarrows N(Sp)[W^{-1}] : \Omega^\infty.$$  

We will also need the category $Sp_{\text{Top}}$ of spectra in topological spaces. Again, geometric realization gives a left Quillen functor $Sp \to Sp_{\text{Top}}$ which is a Quillen equivalence, and therefore induces an equivalence of $\infty$-categories

$$N(Sp)[W^{-1}] \simeq N(Sp_{\text{Top}})[W^{-1}]. \quad (168)$$

We let $\mathbf{Ch}$ denote the category of chain complexes of $\mathbb{Z}$-modules. We index our objects homologically

$$\ldots \to A_{i+1} \to A_i \to A_{i-1} \to \ldots$$

and adopt the convention that $A^i := A_{-i}$ in order to consider cochain complexes as chain complexes. It carries the injective model structure (compare [Hov01, Sec. 2]), in which the weak equivalences are the quasi-isomorphisms and all objects are cofibrant. For chain complexes $E$ and $F$ we write $\text{Hom}(E,F) \in \mathbf{Ch}$ for the chain complex of homomorphisms from $E$ to $F$. Furthermore, by $\text{Map}(E,F) \in N(\mathbf{Ch})[W^{-1}]$ we denote the derived mapping object. If we assume that $E$ is cofibrant and $F$ is fibrant, then it is represented by $\text{Hom}(E,F)$. Chain complexes and spectra are related by an Eilenberg-MacLane spectrum functor which will be discussed in Subsection 6.8.

### 6.3 Commutative and non-commutative group completion

The cartesian product turns $N(sSet)[W^{-1}]$ into a symmetric monoidal $\infty$-category. Hence we have well-defined categories of monoids, groups, commutative monoids and commutative groups. In greater detail (see [Lur, Def. 2.0.0.7]), a symmetric monoidal $\infty$-category is defined as a cocartesian fibration $C^\otimes \to \text{Comm}^\otimes$ over the commutative $\infty$-operad $\text{Comm}^\otimes$. Similarly, a monoidal $\infty$-category is a cocartesian fibration $D^\otimes \to \text{Ass}^\otimes$ over the associative $\infty$-operad (see [Lur, Def. 4.1.1.3]). A symmetric monoidal (respectively, monoidal) structure on an $\infty$-category $C$ (respectively, $D$) is then an embedding into a cocartesian fibration $C^\otimes \to \text{Comm}^\otimes$ (respectively, $D^\otimes \to \text{Ass}^\otimes$); that is,

$$\begin{array}{c}
C \to C^\otimes \\
\downarrow \\
* \to \text{Comm}^\otimes \\
\downarrow \\
\text{Alg}(C) \to C
\end{array} \quad \begin{array}{c}
D \to D^\otimes \\
\downarrow \\
* \to \text{Ass}^\otimes \\
\downarrow \\
\text{Alg}(C) \to C
\end{array}.$$

The $\infty$-category $C\text{Alg}(C)$ of commutative algebras in $C^\otimes$ is then defined as a full subcategory of the $\infty$-category $\text{Fun}_{\text{Comm}^\otimes}(\text{Comm}^\otimes, C^\otimes)$ of sections of $C^\otimes \to \text{Comm}^\otimes$ (see [Lur, Def. 2.4.2.1] for details). Similarly, the $\infty$-category $\text{Mon}(C)$ of algebras is a full subcategory of the $\infty$-category $\text{Fun}_{\text{Ass}^\otimes}(\text{Ass}^\otimes, D^\otimes)$ of sections $D^\otimes \to \text{Ass}^\otimes$. The forgetful functors

$$C\text{Alg}(C) \to \text{Alg}(C) \to C$$

111
which forget commutativity and the monoid structure are then given by pullback along

\[ \ast \to \text{Ass}^\otimes \to \text{Comm}^\otimes. \]

If \( C \) is an \( \infty \)-category with finite products, then \( C \) naturally determines a symmetric monoidal \( \infty \)-category \( C^x \to \text{Comm}^\otimes \) in which the multiplication is the cartesian product. In this case, we may write

\[ \text{CommMon}(C) = C\text{Alg}(C^x), \quad \text{Mon}(C) = C\text{Alg}(C^x \times_{\text{Comm}^\otimes} \text{Ass}^\otimes), \]

and refer to the (commutative) algebra objects as (commutative) monoids.

We now specialize to the category \( C = \mathbb{N}(\text{sSet})[W^{-1}] \). We have a symmetric monoidal functor \( \pi_0 : \mathbb{N}(\text{sSet})[W^{-1}] \to \mathbb{N}(\text{Set}) \). If \( X \in \text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]) \) is a monoid in the \( \infty \)-category of simplicial sets, then \( \pi_0(X) \) is a monoid in the ordinary sense. The \( \infty \)-category \( \text{Groups}(\mathbb{N}(\text{sSet})[W^{-1}]) \) of groups in \( \mathbb{N}(\text{sSet})[W^{-1}] \) is defined as the full subcategory of \( \text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]) \) of those monoids \( X \) which have the property that \( \pi_0(X) \) is a group.

Finally, the \( \infty \)-category of commutative groups in simplicial sets fits into pull-back square of \( \infty \)-categories

![Pull-back diagram](https://via.placeholder.com/150)

The inclusion functors \( I \) and \( I^\text{nc} \) are fully faithful, and all these functors are right-adjoint. We get a corresponding push-out square in \( \infty \)-categories of the left adjoints

\[ \text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]) \xrightarrow{\Omega B^\text{nc}} \text{Groups}(\mathbb{N}(\text{sSet})[W^{-1}]) \] \[ \downarrow ab \] \[ \text{CommMon}(\mathbb{N}(\text{sSet})[W^{-1}]) \xrightarrow{\Omega B} \text{CommGroups}(\mathbb{N}(\text{sSet})[W^{-1}]) \] \[ \downarrow Ab \]

where \( \Omega B \) and \( \Omega B^\text{nc} \) are the commutative and non-commutative versions of group completions, and \( ab, Ab \) are versions of abelization.

**Lemma 6.1** The non-commutative group completion \( \Omega B^\text{nc} \) is symmetric monoidal with respect to the cartesian symmetric monoidal structures.

**Proof.** According to [Lurie, Example 2.2.1.7], we must check that if \( X \to Y \) is a local equivalence (that is, \( \Omega B^\text{nc}(X) \to \Omega B^\text{nc}(Y) \) is an equivalence) then, for any monoid \( Z \), \( X \times Z \to Y \times Z \) is a local equivalence. This is clear because \( \Omega B^\text{nc} \cong \Omega \circ B \) and both \( \Omega \) and \( B \) commute with finite products. \( \square \)
Lemma 6.2 The square

\[
\begin{array}{ccc}
\text{CommMon}(N(sSet)[W^{-1}]) & \xrightarrow{\Omega B} & \text{CommGroups}(N(sSet)[W^{-1}]) \\
\downarrow V & & \downarrow U \\
\text{Mon}(N(sSet)[W^{-1}]) & \xrightarrow{\Omega B^{nc}} & \text{Groups}(N(sSet)[W^{-1}])
\end{array}
\]

commutes.

Proof. By Lemma 6.1 we get a commutative diagram

\[
\begin{array}{ccc}
\text{CommMon}(\text{Mon}(N(sSet)[W^{-1}])) & \xrightarrow{\Omega B^{nc}} & \text{CommMon}(\text{Groups}(N(sSet)[W^{-1}])) \\
\downarrow V & & \downarrow U \\
\text{Mon}(N(sSet)[W^{-1}]) & \xrightarrow{\Omega B^{nc}} & \text{Groups}(N(sSet)[W^{-1}])
\end{array}
\]

where the vertical arrows forget the commutative monoid structure and \(\Omega B^{nc}\) is the functor induced by \(\Omega B\) on commutative monoids. It is the left-adjoint to the inclusion. This diagram can now be extended to

\[
\begin{array}{ccc}
\text{CommMon}(N(sSet)[W^{-1}]) & \xrightarrow{\Omega B} & \text{CommGroups}(N(sSet)[W^{-1}]) \\
\downarrow \cong & & \downarrow \cong \\
\text{Mon}(\text{CommMon}(N(sSet)[W^{-1}])) & \xrightarrow{\Omega B^{nc}} & \text{Groups}(\text{CommMon}(N(sSet)[W^{-1}])) \\
\downarrow \cong & & \downarrow \cong \\
\text{CommMon}(\text{Mon}(N(sSet)[W^{-1}])) & \xrightarrow{\Omega B^{nc}} & \text{CommMon}(\text{Groups}(N(sSet)[W^{-1}])) \\
\downarrow \cong & & \downarrow \cong \\
\text{Mon}(N(sSet)[W^{-1}]) & \xrightarrow{\Omega B^{nc}} & \text{Groups}(N(sSet)[W^{-1}])
\end{array}
\]

The vertical equivalences are \(\infty\)-categorical versions of the Eckmann-Hilton argument, namely \(\text{Comm}^\otimes \simeq \text{colim}_n (\text{Ass}^\otimes)^{\otimes n}\) (see [Lur, Corollary 5.1.1.5]) and therefore

\[\text{Comm}^\otimes \simeq \text{Ass}^\otimes \otimes \text{Comm}^\otimes \simeq \text{Comm}^\otimes \otimes \text{Ass}^\otimes.\]

The upper three horizontal arrows are the left adjoints of the obvious inclusions which justifies identifying the top one with \(\Omega B\). \(\square\)

We will frequently use the fact that the \(\infty\)-category \(\text{CommGroups}(N(sSet)[W^{-1}])\) is a model for the \(\infty\)-category of connective spectra. More precisely, the \(\infty\)-loop space functor is part of an adjoint equivalence of \(\infty\)-categories

\[\text{sp} : \text{CommGroups}(N(sSet)[W^{-1}]) \xrightarrow{\sim} N(\text{Sp}^{>0})[W^{-1}] : \Omega^\infty.\]
6.4 Smooth objects

A contravariant functor $\text{Mf}^{\text{op}} \to \text{sSet}$ from the category of smooth manifolds to simplicial sets will be called a smooth object in the category of simplicial sets, or simply a smooth simplicial set. Homotopy theoretic constructions often produce homotopy coherent diagrams instead of functors. In order to capture these more general objects we consider the category of manifolds $\text{Mf}$ as an $\infty$-category via its nerve $N(\text{Mf})$. The category of contravariant functors from manifolds to simplicial sets can then be identified with

$$\text{Sm}(N(\text{sSet})) := \text{Fun}(N(\text{Mf})^{\text{op}}, N(\text{sSet})),$$

and the $\infty$-category of homotopy coherent diagrams of simplicial sets over the category of manifolds now acquires a precise meaning as

$$\text{Sm}(N(\text{sSet})[W^{-1}]) := \text{Fun}(N(\text{Mf})^{\text{op}}, N(\text{sSet})[W^{-1}]).$$

In general, if $\text{C}$ is an $\infty$-category, then we write

$$\text{Sm}(\text{C}) := \text{Fun}(N(\text{Mf})^{\text{op}}, \text{C})$$

for the $\infty$-category of smooth objects in $\text{C}$. Some of our basic smooth objects arise as follows. Let $(\text{C}, W)$ be a category with weak equivalences. Then a functor $f : \text{Mf}^{\text{op}} \to \text{C}$ induces a smooth object

$$(N(\text{Mf})^{\text{op}} \xrightarrow{N(f)} N(\text{C})) \in \text{Sm}(N(\text{C})) \quad (171)$$

which we will also denote by $f$. Its composition with the localization $N(\text{C}) \to N(\text{C})[W^{-1}]$ will usually be denoted by $f_\infty \in \text{Sm}(N(\text{C})[W^{-1}])$. More examples are constructed in Subsection 6.6.

We have an evaluation map

$$N(\text{Mf})^{\text{op}} \times \text{Sm}(\text{C}) \longrightarrow \text{C}.$$ 

Given a smooth object $E \in \text{Sm}(\text{C})$ we write $E(M) \in \text{C}$ for the evaluation of $E$ at the manifold $M \in \text{Mf}$. We need the following criterion for equivalences between smooth objects. Let $\text{D}$ be an $\infty$-category.

**Lemma 6.3** If the evaluation of $f \in \text{Fun}(\text{D}, \text{C})$ at every object of $\text{D}$ is an equivalence, then $f$ is an equivalence. In particular, if $f : E \to F$ is a morphism between smooth objects $E, F \in \text{Sm}(\text{C})$ such that the induced maps on evaluations $E(M) \to F(M)$ are equivalences for all manifolds $M \in \text{Mf}$, then $f$ is an equivalence in $\text{Sm}(\text{C})$.

**Proof.** Let $I = \Delta^1$ denote the nerve of the category

$$\xymatrix{ & w \\ e \ar@/^/[ur] & f \ar@/^/[ul] }$$

114
with two objects and a single nonidentity arrow $w$. A morphism $f : E \to F$ between two functors $E, F \in \text{Fun}(D, C)$ between $\infty$-categories is a functor $I \to \text{Fun}(D, C)$ which evaluates to $E, F$ at the objects of $I$. The morphism is an equivalence if and only if it has a factorization

$$
\begin{array}{c}
I \longrightarrow \text{Fun}(D, C) \\
\downarrow \\
I[w]^{-1}
\end{array}
$$

The space of such factorizations is the same as the space of factorizations

$$
\begin{array}{c}
\text{Fun}(I[w]^{-1}, C) \longrightarrow \text{Fun}(I[w]^{-1}, C) \\
\downarrow \\
\text{Fun}(I, C)
\end{array}
$$

We complete this diagram with by choosing the upper horizontal map as in (166)

$$
\begin{array}{c}
\text{Fun}_{(w)}^{-1}(I, C) \longrightarrow \text{Fun}(I[w]^{-1}, C) \\
\downarrow \\
\text{Fun}(I, C)
\end{array}
$$

If $f$ is an equivalence for each object of $D$, then the lower horizontal map has the factorization over the left vertical map. This yields the dotted arrow by composition.

The category $\text{Mf}$ has a Grothendieck pre-topology given by open coverings of manifolds. This induces a Grothendieck topology on the $\infty$-category $\mathcal{N}(\text{Mf})$, see [Lur09, Def. 6.2.2.1 and Rem. 6.2.2.3]. Given a covering $U \to M$ in $\text{Mf}$ we form the simplicial manifold $U^\bullet_M \in \text{Mf}^{\text{op}}$ of iterated fibre products of $U$ over $M$, i.e.

$$
U^k_M := \underbrace{U \times_M \cdots \times_M U}_{(k+1)\times}.
$$

The evaluation of a smooth object $E \in \text{Sm}(C)$ on $U^\bullet_M \to M$ gives rise to a cosimplicial object $E(U^\bullet_M) \in \text{Fun}(\mathcal{N}(\Delta), C)$ of $C$ equipped with a coaugmentation $E(M) \to E(U^\bullet_M)$.

**Definition 6.4** A smooth object $E \in \text{Sm}(C)$ of $C$ satisfies descent if the natural map

$$
E(M) \longrightarrow \lim_{\mathcal{N}(\Delta)} E(U^\bullet_M)
$$

is an equivalence in $C$ for all $M$ and coverings $U \to M$.

We let $\text{Sm}^{\text{desc}}(C) \to \text{Sm}(C)$ be the full subcategory of smooth objects which satisfy descent. This inclusion is the right adjoint of an adjunction

$$
L : \text{Sm}(C) \rightleftarrows \text{Sm}^{\text{desc}}(C) : \mathfrak{i},
$$

where $L$ is the sheafification functor [Lur09, Def. 6.2.2.6, Lem. 6.2.2.7, Not. 6.2.2.8].
6.5 Constant sheaves

In this subsection we characterize smooth simplicial sets (with the weak equivalences inverted) with descent as constant sheaves of simplicial sets. This useful property is a consequence of the fact that a manifold admits a covering whose iterated intersections are contractible. We will further observe that the function space construction induces a fully faithful embedding of ∞-categories.

Let \( p : Mf \to * \) be the projection. It induces a pull-back functor

\[
N(sSet)[W^{-1}] \cong \text{Fun}(N(*)^{op}, N(sSet)[W^{-1}]) \xrightarrow{p^*} \text{Sm}(N(sSet)[W^{-1}])
\]

which we will denote \( p^* \).

Definition 6.5 We say that a smooth simplicial set \( X \in \text{Sm}_{\text{desc}}(N(sSet)[W^{-1}]) \) is a constant sheaf if there exists a simplicial set \( T \in N(sSet)[W^{-1}] \) together with an equivalence \( L(p^*T) \cong X \), where \( L \) is the sheafification \((175)\). We let

\[
\text{Sm}^{\text{desc, const}}(N(sSet)[W^{-1}]) \subseteq \text{Sm}^{\text{desc}}(N(sSet)[W^{-1}])
\]

denote the full ∞-subcategory of constant sheaves.

We have a functor \( \text{space} : Mf \to \text{Top} \) which maps a manifold to its underlying topological space. We consider its composition with the singular complex \( \text{sing} : \text{Top} \to sSet \) at the level of nerves, and further compose with the localization \( l : N(sSet) \to N(sSet)[W^{-1}] \). In this way we get a functor

\[
\Pi_\infty \in \text{Fun}(N(Mf), N(sSet)[W^{-1}])
\]

which we call the fundamental ∞-groupoid functor. We define the functor

\[
\text{Sm}_\infty : N(sSet)[W^{-1}] \to \text{Sm}(N(sSet)[W^{-1}])
\]

as the adjoint of

\[
N(Mf)^{op} \times N(sSet)[W^{-1}] \xrightarrow{\text{map}(\Pi_\infty \times \text{id})} N(sSet)[W^{-1}].
\]

Anticipating Subsection \[6.6\] we state:

**Lemma 6.6** For \( T \in N(sSet)[W^{-1}] \) we have \( \text{Sm}_\infty(T) \in \text{Sm}^{\text{desc}}(N(sSet)[W^{-1}]) \).

**Proof.** This is Lemma \[6.11\] \( \Box \)

**Lemma 6.7** A smooth space \( X \in \text{Sm}^{\text{desc}}(N(sSet)[W^{-1}]) \) is a constant sheaf if and only if \( X \cong \text{Sm}_\infty(T) \) for some simplicial set \( T \in N(sSet)[W^{-1}] \).
Proof. It suffices to show that the natural map \( L(p^*T) \to \text{Sm}_\infty(T) \) is an equivalence. Let \( M \) be a manifold. We choose a covering \( U \to M \) such that the connected components of \( U^k_M \) (see (174)) are contractible for all \( k \geq 0 \). Then we consider the commutative diagram

\[
\begin{array}{ccc}
L(p^*T)(M) & \to & \text{Sm}_\infty(T)(M) \\
\downarrow & & \downarrow \\
\lim_{\Delta}\text{Sm}_\infty(T)(U^\bullet_M) & \to & \lim_{\Delta}\text{Sm}_\infty(T)(U^\bullet_M)
\end{array}
\]

The vertical arrows are equivalences since \( L(p^*T) \) and \( \text{Sm}_\infty(T) \) both satisfy descent. The lower horizontal arrow is an equivalence since \( L(p^*T)(V) \to \text{Sm}_\infty(T)(V) \) is an equivalence for every contractible manifold \( V \) as both sides are equivalent to \( T \). Thus the upper horizontal map is an equivalence. We now use that \( M \) was arbitrary and apply Lemma 6.3. 

The following Lemma seems to be related to [Dug01, Prop. 8.3].

**Lemma 6.8** For any \( \infty \)-category \( I \), the map

\[
\text{Sm}_\infty : \text{Fun}(I, \mathcal{N}(\text{sSet})[W^{-1}]) \to \text{Fun}(I, \text{Sm}^{\text{desc, const}}(\mathcal{N}(\text{sSet})[W^{-1}]))
\]

is an equivalence of \( \infty \)-categories. Similarly,

\[
\text{Sm}_\infty : \text{Fun}(I, \text{Mon}(\mathcal{N}(\text{sSet})[W^{-1}])) \to \text{Fun}(I, \text{Sm}^{\text{desc, const}}(\text{Mon}(\mathcal{N}(\text{sSet})[W^{-1}])))
\]

is an equivalence.

**Proof.** The case of a general \( I \) follows from the case of \( I = \Delta^0 \). In the proof of Lemma 6.7 we have seen that the natural map \( L(p^*T) \to \text{Sm}_\infty(T) \) is an equivalence for all \( T \). By Lemma 6.3 we get an equivalence \( L \circ p^* \to \text{Sm}_\infty \). It therefore suffices to show that

\[
L \circ p^* : \mathcal{N}(\text{sSet})[W^{-1}] \to \text{Sm}^{\text{desc, const}}(\mathcal{N}(\text{sSet})[W^{-1}])
\]

is an equivalence of \( \infty \)-categories. This functor is essentially surjective by definition of the subcategory of constant sheaves, so it suffices to show that the induced map

\[
\text{map}(X, Y) \to \text{map}(L \circ p^*(X), L \circ p^*(Y))
\]

is an equivalence.

We have an adjunction

\[
p^* : \mathcal{N}(\text{sSet})[W^{-1}] \rightleftarrows \text{Sm}(\mathcal{N}(\text{sSet})[W^{-1}]) : p_* ,
\]

where \( p_* \) is the evaluation at a point. Therefore we have an adjunction

\[
L \circ p^* : \mathcal{N}(\text{sSet})[W^{-1}] \rightleftarrows \text{Sm}^{\text{desc}}(\mathcal{N}(\text{sSet})[W^{-1}]) : p_* \circ i ,
\]

(176)
where $i$ is as in (175). The unit $\text{id} \to (p_* \circ i) \circ (L \circ p^*)$ is an equivalence because $\ast$ has no nontrivial covering sieves (this follows from the formula for the sheafification, which according to the proof of [Lur09, Proposition 6.2.2.7] is given by a possibly transfinite number of applications of the functor of [Lur09, Remark 6.2.2.12]). It follows that

$$\text{map}(L \circ p^*(X), L \circ p^*(Y)) \cong \text{map}(X, (p_* \circ i) \circ (L \circ p^*)(Y)) \cong \text{map}(X, Y).$$

This proves the first assertion.

For the second statement, we may again suppose $I = \Delta^0$. By definition, a smooth monoid object is constant if and only if it is of the form $Lp^*(M)$ for some $M \in \text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}])$ (note that this makes sense as both $L$ and $p^*$ preserve products). In other words, the map

$$\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]) \to \text{Sm}^{\text{desc, const}}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$$

is essentially surjective. Since limits of smooth objects with descent are again smooth objects with descent, we have an equivalence

$$\text{Sm}^{\text{desc}}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}])) \simeq \text{Mon}(\text{Sm}^{\text{desc}}(\mathbb{N}(\text{sSet})[W^{-1}])),$$

from which it follows, using the first assertion, that

$$\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]) \to \text{Sm}^{\text{desc, const}}(\text{Mon}(\mathbb{N}(\text{sSet})[W^{-1}]))$$

is also fully faithful.

\[ \square \]

### 6.6 Smooth function objects

Let $C^\otimes$ be a closed symmetric monoidal $\infty$-category equipped with a functor

$$W : \mathbb{N}(\text{Mf}) \to C,$$

where we write $C$ for the underlying $\infty$-category of $C^\otimes$ (see Subsection 6.3 for more details). The closed symmetric monoidal structure on $C$ gives the internal mapping object functor

$$\text{Map} : C^{\text{op}} \times C \longrightarrow C.$$

Precomposing it with $W^{\text{op}} : \mathbb{N}(\text{Mf})^{\text{op}} \to C^{\text{op}}$, we obtain a functor

$$\mathbb{N}(\text{Mf})^{\text{op}} \times C \xrightarrow{W^{\text{op}} \times \text{id}} C^{\text{op}} \times C \xrightarrow{\text{Map}} C.$$

It is adjoint to the functor

$$\text{Sm} : C \longrightarrow \text{Sm}(C), \quad \text{Sm}(M) := \text{Map}(W(M), X) \tag{177}$$

which associates to each object $c \in C$ the associated smooth function object $\text{Sm}(c)$.

118
**Definition 6.9** We say that $W$ preserves coverings if the natural map

$$\text{colim}_{(\Delta^\text{op})} W(U^\bullet_M) \to W(M)$$

is an equivalence in $C$ for every covering $U \to M$ of a manifold $M$.

**Lemma 6.10** If $W$ preserves coverings, then the functor

$$\text{Sm} : C \to \text{Sm}(C)$$

factors through the subcategory $\text{Sm}^\text{desc}(C)$ of smooth objects which satisfy descent.

**Proof.** Let $X \in C$. We will show that $\text{Sm}(X)$ satisfies descent (Def. 6.4). Let $U \to M$ be a covering of a manifold $M$. Since $\text{Map}(\ldots, X)$ turns colimits into limits the equivalence (178) induces an equivalence

$$\text{Sm}(X)(M) \xrightarrow{\sim} \text{lim}_{(\Delta)} \text{Sm}(X)(U^\bullet_M).$$

We now list some examples of the above construction.

1. Let $C =: N(sSet)$ and $W := \text{sing} \circ \text{space} : N(Mf) \to N(sSet)$. In this case we obtain

$$\text{Sm} : N(sSet) \to \text{Sm}(N(sSet)).$$

2. If instead we consider $C := N(sSet)[W^{-1}]$ and $W := \Pi_\infty : N(Mf) \to N(sSet)[W^{-1}]$, then

$$\text{Sm}_\infty : N(sSet) \to \text{Sm}(N(sSet)[W^{-1}]).$$

Note that $\text{Sm}_\infty \simeq l \circ \text{Sm}$, for $l : \text{Sm}(N(sSet)) \to \text{Sm}(N(sSet)[W^{-1}])$ the localization.

**Lemma 6.11** The fundamental $\infty$-groupoid functor $\Pi_\infty$ preserves coverings. Hence, for every $X \in N(sSet)[W^{-1}]$, we have $\text{Sm}_\infty(X) \in \text{Sm}_\text{desc}(N(sSet)[W^{-1}])$.

**Proof.** Let $M$ be a manifold and let $U \to M$ be an open cover, so that $M$ is the realization, in the category of smooth manifolds, of the simplicial manifold $U^\bullet_M$ (which in degree $n$ is the $(n+1)$-fold fibred product $U \times_M \cdots \times_M U$). Since $\Pi_\infty(M)$ is by [Seg68, Prop 4.1] also the realization, in the $\infty$-category of $\infty$-groupoids, of the simplicial object $\Pi_\infty(U^\bullet_M)$, we see that $\Pi_\infty$ preserves coverings and therefore that $\text{Sm}_\infty(X)$ satisfies descent. \qed
3. If we take $C := N(\text{Top})$ and $W := \text{space} : N(\text{Mf}) \to N(\text{Top})$, then we get a functor 

$$\text{Sm} : N(\text{Top}) \to \text{Sm}(N(\text{Top}))^{\text{sing}} \to \text{Sm}(N(\text{sSet})) .$$

We let 

$$\text{Sm}_\infty := l \circ \text{Sm} : N(\text{Top}) \to \text{Sm}(N(\text{sSet})[W^{-1}])$$

be the image of $\text{Sm}(X)$ under localization. For topological spaces $X, Y \in \text{Top}$ there is a natural equivalence of simplicial sets 

$$\text{sing}(\text{Map}(Y,X)) \simeq \text{map}(\text{sing}(Y),\text{sing}(X)) .$$

It induces an equivalence of smooth spaces 

$$\text{Sm}_\infty(X) \simeq \text{Sm}_\infty(\text{sing}(X))$$

in $\text{Sm}(N(\text{sSet})[W^{-1}])$ which is natural in $X$.

4. If we take $C := N(\text{Sp})[W^{-1}]$ and $W := \Sigma^\infty_+ \circ \Pi_\infty$, then we get the function spectrum construction 

$$\text{Sm}_\infty : N(\text{Sp})[W^{-1}] \to \text{Sm}(N(\text{Sp})[W^{-1}]) .$$

Since the left-adjoint $\Sigma^\infty_+$ preserves colimits we see from Lemma 6.11 that $\Sigma^\infty_+ \circ \Pi_\infty$ preserves coverings. Hence for every spectrum $E$ we have 

$$\text{Sm}_\infty(E) \in \text{Sm}^{\text{desc}}(N(\text{Sp})[W^{-1}]) .$$

The homotopy groups of the evaluation of the function spectrum at a manifold $M$ are given by 

$$\pi_n(\text{Sm}_\infty(E)(M)) \cong E^{-n}(M) .$$

The functor 

$$\Omega^\infty : N(\text{Sp})[W^{-1}] \to N(\text{sSet})[W^{-1}]$$

induces a functor 

$$\Omega^\infty : \text{Sm}(N(\text{Sp})[W^{-1}]) \to \text{Sm}(N(\text{sSet})[W^{-1}]) .$$

**Lemma 6.12** For a spectrum $E$ we have a natural equivalence of smooth spaces 

$$\text{Sm}_\infty(\Omega^\infty E) \simeq \Omega^\infty \text{Sm}_\infty(E) .$$

**Proof.** For every manifold $M$ we have the natural equivalence of evaluations 

$$\Omega^\infty \text{Sm}_\infty(E)(M) \simeq \Omega^\infty \text{Map}(\Sigma^\infty_+ \Pi_\infty M, E) \simeq \text{map}(\Pi_\infty M, \Omega^\infty E) \simeq \text{Sm}_\infty(\Omega^\infty E)(M) .$$

We now apply Lemma 6.3 \qed

120
5. An alternative route to a function spectrum is via the composition of functors

\[ W : N(Mf)^{\text{space}} \rightarrow N(\text{Top}) \xrightarrow{\Sigma^\infty} N(\text{Sp}_{\text{Top}})^{\text{sing}} \rightarrow N(\text{Sp}) \xrightarrow{l} N(\text{Sp})[W^{-1}] . \]

The resulting function spectrum will be denoted by

\[ \tilde{\text{Sm}}_{\infty}(E) \in \text{Sm}(N(\text{Sp})[W^{-1}]) . \] (184)

The equivalence in \( \text{Fun}(N(Mf), N(\text{Sp})[W^{-1}]) \)

\[ l \circ \Sigma^\infty \circ \text{sing} \circ \text{space} \simeq l \circ \text{sing} \circ \Sigma^\infty \circ \text{space} \]

(note that the functors denoted by the symbols \( \text{sing} \) and \( \Sigma^\infty \) on the two sides are similar, but act between different categories) induces an equivalence

\[ \tilde{\text{Sm}}_{\infty}(E) \simeq \text{Sm}_{\infty}(E) \] (185)

which is also natural in the spectrum \( E \).

6. The functor

\[ \text{map} : N(\text{sSet})[W^{-1}] \times N(\text{sSet})[W^{-1}] \rightarrow N(\text{sSet})[W^{-1}] \]

is symmetric monoidal in the second argument and hence induces a functor

\[ \text{map} : N(\text{sSet})[W^{-1}] \times \text{Mon}(N(\text{sSet})[W^{-1}]) \rightarrow \text{Mon}(N(\text{sSet})[W^{-1}]) . \]

We therefore get a smooth monoid functor

\[ \text{Sm}_{\infty} : \text{Mon}(N(\text{sSet})[W^{-1}]) \rightarrow \text{Sm}^{\text{desc}}(\text{Mon}(N(\text{sSet})[W^{-1}])) \]

such that

\[ \text{Sm}_{\infty}(G)(M) = \text{map}(\Pi_{\infty}(M), G) . \]

A similar construction applies to commutative monoids and groups.

Recall that the infinite loop space functor can be refined to a functor

\[ \Omega^\infty : N(\text{Sp})[W^{-1}] \rightarrow \text{CommGroups}(N(\text{sSet})[W^{-1}]) \]

(see Subsection 6.3). If we forget commutativity and inverses, then we get a functor

\[ \Omega^\infty : N(\text{Sp})[W^{-1}] \rightarrow \text{Mon}(N(\text{sSet})[W^{-1}]) . \]

This version of the infinite loop space functor appears in the following lemma.

**Lemma 6.13** The equivalence (182) refines to an equivalence

\[ \Omega^\infty(\text{Sm}_{\infty}(E)) \simeq \text{Sm}_{\infty}(\Omega^\infty E) \]

in \( \text{Sm}(\text{Mon}(N(\text{sSet})[W^{-1}])) \).

**Proof.** The calculation (183) can be interpreted in \( \text{Mon}(N(\text{sSet})[W^{-1}]) \). \( \Box \)

Of course, there are similar statements in which the \( \infty \)-category \( \text{Mon}(N(\text{sSet})[W^{-1}]) \) is replaced by one of the \( \infty \)-categories \( \text{Groups}(N(\text{sSet})[W^{-1}]) \), \( \text{CommMon}(N(\text{sSet})[W^{-1}]) \), or \( \text{CommGroups}(N(\text{sSet})[W^{-1}]) \), but Lemma 6.13 is what we use in Section 3.

121
6.7 The de Rham complex

We consider the category \( \text{Sh}_{\text{Ch}}(\text{Mf}) \) of sheaves of chain complexes on the site \( \text{Mf} \) of smooth manifolds with the open covering topology. A chain complex \( A \in \text{Ch} \) of real vector spaces gives rise to a constant sheaf of complexes \( A \in \text{Sh}_{\text{Ch}}(\text{Mf}) \). Another example of a sheaf of complexes on \( \text{Mf} \) is the de Rham complex \( \Omega \in \text{Sh}_{\text{Ch}}(\text{Mf}) \) which associates to a smooth manifold \( M \) the chain complex \( \Omega(M) \) of real differential forms. Note that we put the \( p \)-forms in degree \( -p \) so that the de Rham differential has degree \(-1\).

**Definition 6.14** Let \( A \in \text{Ch} \) be a chain complex of real vector spaces. The de Rham complex \( \Omega A \) with coefficients in \( A \) is defined as the tensor product in \( \text{Sh}_{\text{Ch}}(\text{Mf}) \)

\[
\Omega A := \Omega \otimes_{\mathbb{R}} A.
\]

The de Rham complex is a functor

\[
\Omega A : \text{Mf}^{op} \to \text{Ch}
\]

and therefore can also be considered as a smooth chain complex \( \Omega A \in \text{Sm}(\mathbb{N}(\text{Ch})) \) by [171]. We let

\[
\Omega A_\infty \in \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}])
\]

be its image under the localization \( \text{Sm}(\mathbb{N}(\text{Ch})) \to \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}]) \).

**Lemma 6.15** The smooth chain complex \( A_\infty \in \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})[W^{-1}]) \) associated to a complex of levelwise fine sheaves \( A \in \text{Sm}(\mathbb{N}(\text{Ch})) \) satisfies descent; in other words, \( A_\infty \in \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})[W^{-1}]) \).

**Proof.** Let \( U \to M \) be a covering and \( U_M^\bullet \) be the associated simplicial manifold. We must show that the natural map

\[
A_\infty(M) \to \lim_{\text{dir}(\Delta)} A_\infty(U_M^\bullet)
\]

is an equivalence. Now \( A(U_M^\bullet) \) is a cosimplicial chain complex. It gives rise to a double complex \( \check{C}^\bullet(A) \), the Čech complex associated to the covering \( U \). The limit in (187) is realized as the associated total complex \( \text{Tot} \check{C}^\bullet(A) \). A fine sheaf is Čech acyclic. Hence the natural inclusion \( A(M) \to \text{Tot} \check{C}^\bullet(A) \) is a quasi-isomorphism, i.e. an equivalence in \( \mathbb{N}(\text{Ch})[W^{-1}] \).

**Corollary 6.16** The de Rham complex \( \Omega A_\infty \) with coefficients in \( A \) satisfies descent.

**Proof.** The sheaf \( \Omega A^k \) of \( A \)-valued forms of total degree \( k \) is a sheaf of modules over \( C^\infty \) and therefore fine. \( \square \)
6.8 The Eilenberg-MacLane spectrum functor

The $\infty$-categories of chain complexes and spectra are related by an Eilenberg-MacLane spectrum functor. If $\mathbf{C}$ is a stable $\infty$-category (see [Lur, 1.1.1.9]), then for objects $x, y \in \mathbf{C}$ we adopt the convention that

$$\text{map}(x, y) \in \mathbb{N}^{\text{Sp}}[W^{-1}]$$

(188)

denotes the mapping spectrum. The $\infty$-category $\mathbb{N}(\text{Ch})[W^{-1}]$ is a stable $\infty$-category. It is compactly-generated as a stable $\infty$-category by the complex $\mathbb{Z}$ which is viewed as a complex concentrated in degree zero. Its endomorphism spectrum is a ring spectrum

$$HZ \simeq \text{map}(\mathbb{Z}, \mathbb{Z}) .$$

We can form the $\infty$-category of $HZ$-module spectra $\text{Mod}(HZ)$. By [Lur, Proposition 7.1.2.7] we have an equivalence of $\infty$-categories

$$H : \mathbb{N}(\text{Ch})[W^{-1}] \simeq \text{Mod}(HZ)$$

(189)

which we call the Eilenberg-MacLane spectrum functor.

We emphasize that for a chain complex $A$ we often use the abbreviation $H(A) := H(A_{\infty})$, where $A_{\infty}$ is $A$ considered as an object in $\text{Ch}[W^{-1}]$.

For an explicit construction of the Eilenberg-MacLane spectrum functor we refer to [Shi07]. There it is obtained from a zig-zag of Quillen adjunctions and therefore properly defined only on the level of $\infty$-categories.

It is also useful to consider the composite

$$H : \mathbb{N}(\text{Ch})[W^{-1}] \xrightarrow{H} \text{Mod}(HZ) \xrightarrow{F} \mathbb{N}(\text{Sp})[W^{-1}] ,$$

where $F : \text{Mod}(HZ) \to \mathbb{N}(\text{Sp})[W^{-1}]$ denotes the functor which forgets the $HZ$-module structure (it will always be clear from the context which version of $H$ is ment). For a complex $A$ we have a canonical isomorphism of abelian groups

$$\pi_n(H(A)) \cong H_n(A) .$$

(190)

**Lemma 6.17** The Eilenberg-MacLane spectrum functor $H : \mathbb{N}(\text{Ch})[W^{-1}] \to \text{Mod}(HZ)$ is an equivalence of closed symmetric monoidal $\infty$-categories. In particular, $H$ preserves the internal mapping objects in the sense that for $A, B \in \mathbb{N}(\text{Ch})[W^{-1}]$ there is a natural equivalence

$$H(\text{Map}(A, B)) \cong \text{Map}(H(A), H(B))$$

(191)

in $\text{Mod}(HZ)$. 

123
Proof. The functor $H$ is the equivalence of symmetric monoidal $\infty$-categories whose existence is asserted in [Lur Proposition 7.1.2.7] applied to the $E_\infty$-ring $HZ$. As an add-on, since the categories are compactly generated and the symmetric monoidal product commutes with colimits on both sides, this equivalence preserves the closed structure, hence the internal mapping objects.

Lemma 6.18 There is a unique (up to contractible choice) symmetric monoidal colimit-preserving functor

$$\mathbb{N}(sSet)[W^{-1}] \to \text{Mod}(HZ)$$

which sends the point to $HZ$.

Proof. Indeed, $\mathbb{N}(sSet)[W^{-1}]$ is freely generated under colimits by the terminal object, which must be mapped to $HZ$. See [Lur09 Theorem 5.1.5.6] for a precise statement.

We have two symmetric monoidal functors

$$\Phi, \Psi : \mathbb{N}(sSet)[W^{-1}] \to \text{Mod}(HZ).$$

The functor $\Phi$ is given by

$$\Phi(X) = \Sigma^\infty_+ X \wedge HZ.$$ (192)

It is symmetric monoidal since it is the composition of symmetric monoidal functors

$$\mathbb{N}(sSet)[W^{-1}] \xrightarrow{\Sigma^\infty_+} \mathbb{N}(Sp)[W^{-1}] \xrightarrow{\wedge^H \epsilon} \text{Mod}(HZ).$$

The functor $\Psi$ is defined as the composition

$$\Psi := H \circ C_* : \mathbb{N}(sSet)[W^{-1}] \to \mathbb{N}(Ch)[W^{-1}] \to \text{Mod}(HZ),$$ (193)

where $C_* : \mathbb{N}(sSet)[W^{-1}] \to \mathbb{N}(Ch)[W^{-1}]$ is the simplicial chain complex functor. It is symmetric monoidal, since it is the composition of symmetric monoidal functors

$$\mathbb{N}(sSet)[W^{-1}] \xrightarrow{C_*} \mathbb{N}(Ch)[W^{-1}] \xrightarrow{H} \text{Mod}(HZ).$$

The equivalence

$$C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$ (194)

is the shuffle map.

Corollary 6.19 There is a unique (up to contractible choice) equivalence $\alpha : \Phi \to \Psi$ of symmetric monoidal functors

$$\mathbb{N}(sSet)[W^{-1}] \to \text{Mod}(HZ).$$

Proof. Both functors preserve colimits and send the point to $HZ$. We can thus apply Lemma 6.18.

\hfill \Box

124
6.9 A spectrum level de Rham isomorphism

If we apply the smooth function object constructions of Subsection 6.6 to the closed symmetric monoidal ∞-category Mod(HZ) with W := (Σ∞ ⊕ Π∞) ∧ HZ, we get a functor

\[ \text{Sm}_\infty : \text{Mod}(HZ) \to \text{Sm}^{\text{desc}}(\text{Mod}(HZ)) . \]

For \( F : \text{Mod}(HZ) \to \mathbb{N}(\text{Sp})[W^{-1}] \) the forgetful functor we have a natural equivalence

\[ F \circ \text{Sm}_\infty(HZ) \cong \text{Sm}_\infty(F(HZ)) , \]

where \( \text{Sm}_\infty \) on the right-hand side is defined as in case 4 of Subsection 6.6. Indeed, this equivalence is induced by the natural equivalence in \( \mathbb{N}(\text{Sp})[W^{-1}] \)

\[ F(\text{Map}_{\text{Mod}(HZ)}(E \smash HZ, M)) \cong \text{Map}_{\mathbb{N}(\text{Sp})[W^{-1]}}(E, F(M)) , \]

where \( E \in \mathbb{N}(\text{Sp})[W^{-1}] \) and \( M \in \text{Mod}(HZ) \).

Let \( A \in \text{Ch} \) be a chain complex of real vector spaces. Associated to \( A \) are two smooth HZ-module spectra: we have \( H(A) \in \text{Mod}(HZ) \) and the associated smooth HZ-module

\[ \text{Sm}_\infty(H(A)) \in \text{Sm}(\text{Mod}(HZ)) , \]

and we also have the smooth HZ-module

\[ H(\Omega A_\infty) \in \text{Sm}(\text{Mod}(HZ)) \]

associated to the smooth chain complex \( \Omega A_\infty \), the de Rham complex \((186)\). The following proposition yields a spectrum level de Rham isomorphism.

**Proposition 6.20** There is a natural equivalence in \( \text{Sm}(\text{Mod}(HZ)) \)

\[ j : H(\Omega A_\infty) \sim \text{Sm}_\infty(H(A)) . \]  

(195)

**Proof.** Write \( C_* : \text{sSet} \to \text{Ch} \) for the functor (of ordinary categories) which associates to a simplicial set its chain complex. On smooth manifolds, the singular complex functor has a subfunctor

\[ \text{sing}_\infty \subseteq \text{sing} : \text{Mf} \to \text{sSet} \]

which associates to a manifold \( M \) its smooth singular complex. We define the composition

\[ S_* := C_* \circ \text{sing}_\infty : \text{Mf} \to \text{sSet} \to \text{Ch} . \]

We define the integration as the map of complexes

\[ I : \Omega A(M) \to \text{Hom}(S_*(M), A) \]

(196)

by

\[ I(\omega)(\sigma) = \int_{\Delta^n} \sigma^* \omega , \]
where \( \sigma : \Delta^n \to M \) is a smooth simplex, \( \omega \in (\Omega A)^m(M) \). Then we have \( I(\omega)(\sigma) \in A^{m-n} \), and we have by Stoke’s theorem \( I(d\omega)(\sigma) = (-1)^{m-n} dI(\omega)(\sigma) + I(\omega)(\partial \sigma) \). It follows that \( I \) is a chain map which, by de Rham theory, is a quasi-isomorphism. Thus the integration map \( I \) provides by Lemma 6.3 an equivalence

\[
\text{Rham} : \Omega A_{\infty} \to l(\text{Hom}(S_*, A))
\]

in \( \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}]) \), where \( l : \text{Sm}(\mathbb{N}(\text{Ch})) \to \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}]) \) is the localization. Moreover, we claim that the natural map

\[
l(\text{Hom}(S_*, A)) \to \text{Map}(S_*, A) \quad (197)
\]

is an equivalence. To see this, we’ll show that the complex \( \text{Hom}(S_*, A) \) represents the derived mapping complex. Note that \( \mathbb{N}(\text{Ch})[W^{-1}] \) is the \( \infty \)-category associated to \( \text{Ch} \), equipped with the projective model structure [Hov99, 2.3.11]. In this model structure, every complex is fibrant, and \( S_*(M) \) is cofibrant since it is bounded below and consists of free \( \mathbb{Z} \)-modules. Now apply \( H \) to get an equivalence in \( \text{Sm}(\text{Mod}(H\mathbb{Z})) \)

\[
H(\Omega A_{\infty}) \xrightarrow{H(\text{Rham})} H(l(\text{Hom}(S_*, A)))
\]

\[
\sim (198)
\]

\[
H(\text{Map}(S_*, A))
\]

\[
\text{Lemma 6.17}
\]

\[
\cong \text{Map}(H(S_*), H(A))
\]

\[
= \text{Map}(\Psi(\text{sing}_\infty), H(A))
\].

Formally we should write \( H(S_{*, \infty}) \) but we drop the subscript \( \infty \) here and in similar formulas below in order to ease notation.

The equivalence \( \alpha \) of Lemma 6.19 gives an equivalence

\[
\alpha(\text{sing}_\infty) : \Psi(\text{sing}_\infty) \xrightarrow{\sim} \Phi(\text{sing}_\infty)
\]

and therefore

\[
\text{Map}(\Psi(\text{sing}_\infty), H(A)) \xrightarrow{\sim} \text{Map}(\Phi(\text{sing}_\infty), H(A)) \quad (200)
\]

The inclusion of functors \( \text{sing}_\infty \subseteq \text{sing} \) is an objectwise equivalence and therefore induces by Lemma 6.3 an equivalence

\[
\text{Map}(\Phi(\text{sing}_\infty), H(A)) \xleftarrow{\sim} \text{Map}(\Phi(\text{sing}), H(A)) \xleftarrow{\text{def}} \text{Sm}_{\infty}(H(A)) \quad (201)
\]

We get the desired equivalence as a composition of (198), (200), and 201.

If we apply the forgetful map \( \text{Mod}(H\mathbb{Z}) \to \mathbb{N}(\text{Sp})[W^{-1}] \), then we get an equivalence

\[
j : H(\Omega A_{\infty}) \xrightarrow{\sim} \text{Sm}_{\infty}(H(A)) \quad (202)
\]

in \( \text{Sm}(\mathbb{N}(\text{Sp})[W^{-1}]) \). If we evaluate the spectrum level de Rham isomorphism given in Proposition 6.20 on a manifold \( M \) and take homotopy groups, we obtain, using the isomorphism (181), a functorial isomorphism of groups

\[
j : H^*(\Omega A(M)) \xrightarrow{\sim} H(A)^*(M) \quad (203)
\]

This is the classical de Rham isomorphism.
6.10 Function spectra with proper support

We consider a chain complex $A \in \text{Ch}$ of real vector spaces and a smooth manifold $W$. For a subset $K \subseteq W$ we define the subcomplex

$$\Omega A_K(W) := \{ \omega \in \Omega A(W) \mid \text{supp}(\omega) \subseteq K \} \subseteq \Omega A(W)$$

of $A$-valued differential forms on $W$ with support in $K$. For an ascending pair of subsets $K \subseteq K'$ we then have an inclusion of complexes $\Omega A_K(W) \subseteq \Omega A_{K'}(W)$.

Let now $\pi : W \to B$ be a smooth map between manifolds. We say that a subset $K \subseteq W$ is proper over $B$ if the restriction $\pi|_K : K \to B$ of $\pi$ to $K$ is a proper map. We define the complex of $A$-valued forms on $W$ with proper support over $B$ as the colimit in the category $\text{Ch}$

$$\Omega A_{c/\pi}(W) := \text{colim}_K \Omega A_K(W) ,$$

where $K \subseteq W$ runs over all subsets of $W$ which are proper over $B$. If

$$\begin{array}{ccc}
W' & \xrightarrow{F} & W \\
\downarrow{\pi'} & & \downarrow{\pi} \\
B' & \xrightarrow{f} & B
\end{array}$$

is a cartesian square of manifolds and the subset $K \subseteq W$ is proper over $B$, then the preimage $F^{-1}(K) \subseteq W'$ is proper over $B'$. The pull-back map

$$F^* : \Omega A(W) \to \Omega A(W')$$

is compatible with the support conditions and thus induces maps

$$F^* : \Omega A_K(W) \to \Omega A_{F^{-1}(K')}(W') , \quad F^* : \Omega A_{c/\pi}(W) \to \Omega A_{c/\pi'}(W') .$$

In general we do not introduce extra symbols for the underlying projection of a real vector bundle. Therefore, if $\pi : W \to B$ is a real vector bundle, then we will replace the subscript $c/\pi$ by $c$ (do not confuse this with the condition of having compact support).

Recall the Bundle-notation from Section [4.3] The association

$$\text{Bundle} \ni (\pi : W \to B) \mapsto \Omega A_{c/\pi}(W) \in \text{Ch}$$

defines a Bundle-chain complex

$$\Omega A_{c,...} \in \text{Bundle}(\mathbb{N}(\text{Ch})) .$$

We now introduce a corresponding notion of function spectra with proper support. We explain the idea by giving the construction evaluated at a bundle $\pi : W \to B$. Let $E \in \mathbb{N}(\text{Sp})[W^{-1}]$ be a spectrum. We define the function spectrum with support in $K$ by

$$\text{Sm}_{\infty,K}(E)(W) := \text{Map}(\Sigma^\infty(\text{sing}(W)/\text{sing}(W \setminus K)), E) .$$
We furthermore set
\[ \mathrm{Sm}_{c/\pi}(E)(W) := \colim_K \mathrm{Sm}_{\infty,K}(E)(W) , \]
where again \( K \) runs over all subsets \( K \subseteq W \) which are proper over \( B \), and the colimit is taken in the \( \infty \)-category \( \mathrm{Sm}(\mathbb{N}(\mathbf{Sp})[W^{-1}]) \). For a cartesian diagram (204) we have pull-backs
\[ F^* : \mathrm{Sm}_{\infty,K}(E)(W) \rightarrow \mathrm{Sm}_{\infty,K'}(W') , \quad F^* : \mathrm{Sm}_{c/\pi}(E)(W) \rightarrow \mathrm{Sm}_{c/\pi'}(E)(W') . \]
As in the examples given in Subsection 6.6 we can interpret this construction such that it gives a functor
\[ \mathrm{Sm}_{c/..} : \mathbb{N}(\mathbf{Sp})[W^{-1}] \rightarrow \mathbf{Bundle}(\mathbb{N}(\mathbf{Sp})[W^{-1}]) \]
which associates to a spectrum \( E \) the \( \mathbf{Bundle} \)-spectrum \( \mathrm{Sm}_{c/..}(E) \). The details of the precise construction are as follows. We let \( \mathbf{Bundle} \) denote the category of pairs \( (\pi : W \rightarrow B, K \subseteq W) \), where \( \pi : W \rightarrow B \) belongs to \( \mathbf{Bundle} \) and the subset \( K \subseteq W \) is proper over \( B \). A morphism \( (\pi : W \rightarrow B, K \subseteq W) \rightarrow (\pi : W' \rightarrow B', K' \subseteq W') \) in \( \mathbf{Bundle} \) is a cartesian diagram (204) such that \( F^{-1}(K) = K' \). We have two functors
\[ S, CS : \mathbf{Bundle} \rightarrow \mathbf{sSet} \]
given by
\[ S(W \rightarrow B, K) := \text{sing}(W) , \quad CS(W \rightarrow B, K) := \text{sing}(W \setminus K) . \]
They give rise to \( \mathbf{Bundle} \)-spaces
\[ S, CS \in \mathbf{Bundle}(\mathbb{N}(\mathbf{sSet})[W^{-1}]) . \]
We now define the functor
\[ X : \mathbb{N}(\mathbf{Sp})[W^{-1}] \times \mathbb{N}(\mathbf{Bundle})^\text{op} \rightarrow \mathbb{N}(\mathbf{Sp})[W^{-1}] , \quad \text{id} \times \Sigma^\infty(S/CS) \rightarrow \mathbb{N}(\mathbf{Sp})[W^{-1}] \times \mathbb{N}(\mathbf{Sp})^\text{op[W^{-1}}} \rightarrow \mathbb{N}(\mathbf{Sp})[W^{-1}] . \]
We finally define \( \mathrm{Sm}_{c/..} \) as the left Kan extension of \( X \) along the functor
\[ \mathbf{Bundle}^\text{op} \rightarrow \mathbf{Bundle}^\text{op} \]
which forgets the compact subset:
\[ \mathbb{N}(\mathbf{Sp})[W^{-1}] \times \mathbb{N}(\mathbf{Bundle})^\text{op} \xrightarrow{X} \mathbb{N}(\mathbf{Sp})[W^{-1}] \]
\[ \mathbb{N}(\mathbf{Sp})[W^{-1}] \times \mathbb{N}(\mathbf{Bundle}^\text{op}) \xrightarrow{\mathrm{Sm}_{c/..}} \mathbb{N}(\mathbf{Sp})[W^{-1}] \]
If \( \pi \) is the projection of a vector bundle, then we again write \( \text{Sm}_c(E)(W) := \text{Sm}_{c/\pi}(E)(W) \).

We now extend the spectrum level de Rham isomorphism to the properly supported case.

**Proposition 6.21** There exists an equivalence between bundle spectra

\[
  j : H(\Omega A_{c/\ldots}) \to \text{Sm}_{c/\ldots}(H(A)) \tag{205}
\]

which extends the spectrum level de Rham equivalence \((202)\).

**Proof.** In the following we describe the construction of \( j \) after evaluation at a bundle \( \pi : W \to B \). This makes it more readable, and it should become clear how to construct the natural transformation in general. Recall that for a manifold \( W \) we let \( \Omega A(W) = C_*(\text{sing}_\infty(W)) \) denote the smooth singular chain complex of \( W \). If \( \omega \in \Omega A_K(W) \), then the restriction of the integration map \((196)\) along the inclusion \( S_*(W \setminus K) \to S_*(W) \) vanishes so that we get an induced map of chain complexes

\[
I : \Omega A_K(W) \to \text{Hom}(S_*(W)/S_*(W \setminus K), A) . \tag{206}
\]

In the following we assume that \( K \subseteq W \) is a closed codimension zero submanifold with boundary which is proper over \( B \). We let \( \Omega A(W \setminus K)_\delta \subseteq \Omega A(W \setminus K) \) denote the subspace of forms which extend smoothly to \( W \). This inclusion is a quasi-isomorphism.

We consider the following diagram of chain complexes

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega A_K(W) & \longrightarrow & \Omega A(W) & \longrightarrow & \cdots & \longrightarrow & \Omega A(W \setminus K)_\delta & \longrightarrow & 0 \\
\downarrow I & & \downarrow I & & \downarrow I & & & & \downarrow I & & \\
0 & \longrightarrow & \text{Hom}(S_*(W)/S_*(W \setminus K), A) & \longrightarrow & \text{Hom}(S_*(W), A) & \longrightarrow & \text{Hom}(S_*(W \setminus K), A) & \longrightarrow & 0
\end{array}
\]

The upper horizontal sequence is exact by the definitions of \( \Omega A_K(W) \) and \( \Omega A(W \setminus K)_\delta \). The lower sequence is exact since \( A \) consists of real vector spaces which are injective as abelian groups. The middle and the right vertical maps are equivalences by versions of the de Rham isomorphism for \( W \) and \( W \setminus K \). It follows by the Five Lemma that \((206)\) is an equivalence for such subsets \( K \). Note that the closed codimension zero submanifolds with boundary \( K \subseteq W \) which are proper over \( B \) are cofinal among all subsets \( K \subseteq W \) which are proper over \( B \). We require the following lemma, where \( H \) denotes the Eilenberg-MacLane spectrum functor as in \((189)\).

**Lemma 6.22** There is an equivalence of \( H\mathbb{Z}\)-modules

\[
H((S_*(W)/S_*(W \setminus K))_\infty) \cong \Sigma^\infty(\text{sing}(W)/\text{sing}(W \setminus K)) \wedge H\mathbb{Z} . \tag{207}
\]

**Proof.** The equivalence \( \text{sing}_\infty \to \text{sing} \) in \( \text{Sm}(\mathbb{N}(\text{sSet})[W^{-1}]) \) applied to \( W \) and \( W \setminus K \) induces the first equivalence in \( \mathbb{N}(\text{Ch})[W^{-1}] \) in

\[
(S_*(W)/S_*(W \setminus K))_\infty \Rightarrow C_*(\text{sing}(W))/C_*(\text{sing}(W \setminus K)) \cong C_*(\text{sing}(W)/\text{sing}(W \setminus K)) .
\]
For the second equivalence we use the fact that \( C_* \) preserves quotients. We now apply the Eilenberg-MacLane spectrum functor \( H \) and compose the result with the equivalence given by Lemma 6.19. In this way we get the desired equivalence. \( \square \)

We can now define the de Rham map in \( \text{Mod}(HZ) \)

\[
j : H(\Omega A_{K,\infty})(W) \to \text{Sm}_{\infty,K}(H(A))(W)
\]
as the composition

\[
\begin{align*}
H(\Omega A_{K,\infty})(W) & \xrightarrow{H(I)} H(\text{Hom}(S_*(W)/S_*(W \setminus K), A)) \\
& \cong \text{Map}(H((S_*(W)/S_*(W \setminus K))_\infty), H(A)) \\
& \cong \text{Map}(\Sigma^\infty(\text{sing}(W)/\text{sing}(W \setminus K)) \wedge HZ, H(A)) \\
& \overset{\text{def}}{=} \text{Sm}_{\infty,K}(H(A))(W).
\end{align*}
\]

In (208) we use the equivalence (197) and the fact that \( S_*(W)/S_*(W \setminus K) \) is a chain complex of free \( \mathbb{Z} \)-modules in order to replace \( \text{Hom} \) by \( \text{Map} \). These maps are functorial in \( K \) and equivalences if \( K \) is a closed codimension zero submanifold with boundary.

The poset of these \( K \) is filtered. Recall that we indicate the localization \( \mathbb{N}(\text{Ch}) \mapsto \mathbb{N}(\text{Ch})[W^{-1}] \) by a subscript \( (\ldots)_\infty \). In general, if \( E : I \to \text{Ch} \) is a filtered system of complexes, then we have an equivalence

\[
\text{colim}_I(E_\infty) \cong (\text{colim}_I E)_\infty
\]
in \( \mathbb{N}(\text{Ch})[W^{-1}] \).

We thus get an equivalence in \( \text{Mod}(HZ) \) of colimits

\[
j : H(\Omega A_{c/\pi}(W)_\infty) \cong H(\text{colim}_K(\Omega A_K(W)_\infty)) \cong \text{Sm}_{c/\pi}(H(A))(W).
\]

Lastly, we observe that the whole construction above is natural in the bundle \( (\pi : W \to B) \in \text{Bundle} \). This completes the construction of the map (205). \( \square \)

### 6.11 The desuspension map

We now consider the special case of the projection \( B \times \mathbb{R}^k \to B \) of a trivial vector bundle. For a spectrum \( E \) the desuspension map is the equivalence defined by

\[
\begin{align*}
\text{Sm}_c(\Sigma^k E)(B \times \mathbb{R}^k) & \cong \text{colim}_R \text{Sm}_{c,B \times B(R)}(\Sigma^k E)(B \times \mathbb{R}^k) \\
& \cong \text{colim}_R \text{Map}(\Sigma^\infty\text{sing}(B_+ \setminus (\mathbb{R}^k / (\mathbb{R}^k \setminus B(R)))), \Sigma^k E) \\
& \cong \text{Map}(\Sigma^\infty\text{sing}(B)_+ \wedge \Sigma^\infty\text{sing}(S^k), \Sigma^k E) \\
& \cong \text{Map}(\Sigma^\infty\text{sing}(B)_+, \Sigma^k E) \\
& \cong \text{Sm}_{\infty}(E)(B).
\end{align*}
\]
The equivalence (212) involves an identification $\Sigma^\infty \text{sing}(S^k)_+ \cong \Sigma^k S$. Furthermore, the equivalence (211) employs the facts that $\text{sing}$ preserves the symmetric monoidal structure, and that the projections $\text{sing}(S^k) \to \text{sing}(\mathbb{R}^k/(\mathbb{R}^k \setminus B(R)))$ are all homotopy equivalence so that the whole system under the colimit is homotopy constant. Since $H : \text{Nil}(\text{Ch})[W^{-1}] \sim \text{Mod}(HZ)$ is an equivalence of stable $\infty$-categories we get the equivalence $\Sigma^k H(A) \cong H(A[k])$ and therefore the desuspension map

$$\text{desusp} : \text{Sm}_c(H(A[k]))(B \times \mathbb{R}^k) \to \text{Sm}_\infty(H(A))(B)$$

which appears as the right vertical arrow in the diagram (214) below.

Desuspension on the level of the de Rham complex can be realized by integration of differential forms, and the following proposition considers the compatibility of the de Rham isomorphism with these two versions of desuspension.

**Proposition 6.23** We have a natural commutative diagram in $\text{Sm}(\text{Mod}(HZ))$ (written evaluated at $B$)

$$
\begin{array}{ccc}
H(\Omega A_c(B \times \mathbb{R}^k)[k]) & \xrightarrow{j} & \text{Sm}_c(H(A[k]))(B \times \mathbb{R}^k) \\
\downarrow & & \downarrow \text{desusp} \\
H(\Omega A_\infty(B)) & \xrightarrow{j} & \text{Sm}_\infty(H(A))(B)
\end{array}
$$

(214)

**Proof.** Let $B^c(R) := \mathbb{R}^k \setminus B(R)$ denote the complement of the ball of radius $R$ centered at 0. We consider the following diagram:

$$
\begin{array}{ccc}
H(\Omega A_c(B \times \mathbb{R}^k)[k]) & \xrightarrow{j} & \text{Sm}_c(H(A[k]))(B \times \mathbb{R}^k) \\
\downarrow & & \downarrow \text{desusp} \\
H(\Omega A_\infty(B)) & \xrightarrow{j} & \text{Sm}_\infty(H(A))(B)
\end{array}
$$

(215)

In the following we explain the maps and verify that this diagram commutes. The upper and lower right vertical maps are part of the definition of the $j$-maps (e.g. the equivalence between the right-hand sides of (208) and (209)) and the upper and lower triangles commute. We now explain the version $\tilde{I}$ of the integration map which is defined for products:

$$\tilde{I} : \Omega A(M \times N) \to \text{Hom}(S_*(M) \otimes S_*(N), A)$$
If $\sigma \in \text{sing}_\infty(M)[p]$ and $\kappa \in \text{sing}_\infty(N)[q]$, then we have $\sigma \otimes \kappa \in S_*(M) \otimes S_*(N)$. For $\omega \in \Omega A(M \times N)$ we define
\[
I(\omega)(\sigma \otimes \kappa) := \int_{\Delta^p \times \Delta^q} (\sigma \times \kappa)^* \omega \in A.
\]
Finally we extend this map linearly. The map denoted by the symbol $\tilde{I}$ in the diagram is induced by this version of the integration in the obvious way.
The map denoted by $P^*$ in (215) is the pull-back along the shuffle map. Its explicit description is as follows. We can choose for all pairs $i,j \geq 0$ the fundamental chains $[\Delta^i \times \Delta^j] \in S_{i+j}(\Delta^i \times \Delta^j)$
defined as the sum of non-degenerated simplices in the canonical decomposition of the product. Then we have
\[
\partial [\Delta^i \times \Delta^j] = \sum_{\alpha=0}^i (-1)^\alpha (\partial_\alpha \times \text{id}_{\Delta^j})_* [\Delta^{i-1} \times \Delta^j] + \sum_{\beta=0}^j (-1)^{i+j+\beta} (\text{id}_{\Delta^i} \times \partial_{\beta})_* [\Delta^i \times \Delta^{j-1}].
\] (216)
We now consider two smooth manifold $M,N$ and singular simplices $(\kappa : \Delta^i \to M) \in \text{sing}_\infty(M)[i]$ and $(\sigma : \Delta^j \to N) \in \text{sing}_\infty(N)[j]$. Then we define
\[
P(\kappa \otimes \sigma) := (\kappa, \sigma)_* [\Delta^i \times \Delta^j] \in S_{i+j}(M \times N).
\] By linear extension we get a map of graded $\mathbb{Z}$-modules
\[
P : S_*(M) \otimes S_*(N) \to S_*(M \times N).
\] This is exactly the shuffle map (194). The relation (216) ensures that $P$ is a map of chain complexes. The map $P^*$ in (215) is then defined as the pull-back along $P$. If $\omega \in (\Omega A(M \times N)[-k])^{i+j}$, $(\kappa : \Delta^i \to M) \in \text{sing}_\infty(M)[i]$ and $(\sigma : \Delta^j \to N) \in \text{sing}_\infty(N)[j]$, then we have
\[
(P^* \circ I)(\omega)(\kappa \otimes \sigma) = I(\omega)(P(\kappa \otimes \sigma)) = \int_{\Delta^i \times \Delta^j} (\kappa \times \sigma)^* \omega = \tilde{I}(\omega)(\kappa \otimes \sigma).
\] This shows that the triangle containing $P^*$ commutes.
The map denoted by $f_c$ is induced by the fundamental class of $\mathbb{R}^k$. We let
\[
[\mathbb{R}^k] \in S^{\text{loc-fin}}_*(\mathbb{R}^k)
\] be a locally finite representative of the fundamental class of $\mathbb{R}^k$. It projects to cycles
\[
[\mathbb{R}^k]_R \in S_k(\mathbb{R}^k)/S_k(B^c(R))
\] for all $R > 0$. If $\phi \in \text{Map}(S_*(B) \otimes (S_*(\mathbb{R}^k)/S_*(B^c(R))), A[k])$, then we define
\[
f_c(\phi)(x) := \phi(x \otimes [\mathbb{R}^k]_R).
\]
This map factorizes over the colimit over $R$ and gives
\[ fc : \text{colim}_R \text{Map}(S_*(B) \otimes (S_*(\mathbb{R}^k)/S_*(B^c(R))), A[k]) \rightarrow \text{Map}(S_*(B), A) . \]

By construction, the square in the middle of the diagram (215) commutes strictly on the level of forms, i.e. before applying $H$. This finishes the verification that (215) commutes.

The right vertical composition in (215) is a map
\[ \text{desusp}' : \text{Sm}_c(H(A[k]))(B \times \mathbb{R}^k) \rightarrow \text{Sm}_\infty(H(A))(B) . \] (217)

The commutativity of the diagram (215) together with the following Lemma implies the assertion of Proposition 6.23.

**Lemma 6.24** There is a natural equivalence of desuspension maps
\[ \text{desusp}' \cong \text{desusp} \]
(with desusp as in (213)).

**Proof.** In order to shorten the notation we will denote these two maps (213) and (217) by
\[ \sigma, \sigma' : \text{Sm}_c(H(A))(B \times \mathbb{R}^k) \rightarrow \text{Sm}_\infty(H(A)[-k])(B) . \]

In order to ensure naturality of the equivalence with respect to $B$ we reformulate the construction of the two versions of the desuspension map in a functorial way. We define functors
\[ X, Y \in \text{Fun}(\mathbb{N}(\text{MF} \times \mathbb{N}^{op}), \mathbb{N}(\text{sSet}_*))[W^{-1}] \]
by
\[
X(M, n) := \text{sing}_\infty(M \times \mathbb{R}^k)/\text{sing}_\infty(M \times (\mathbb{R}^k \setminus B(0,n))), \\
Y(M, n) := \text{sing}_\infty(M \times \mathbb{R}^k)/\text{sing}_\infty(M \times (\mathbb{R}^k \setminus B(0,n))).
\]

The inclusion $\mathbb{R}^k \rightarrow \tilde{\mathbb{R}}^k$ induces an equivalence $Y(M, n) \rightarrow X(M, n)$ for all $n$ and $M$. By Lemma 6.3 the map $Y \rightarrow X$ in $\text{Fun}(\mathbb{N}(\text{MF} \times \mathbb{N}^{op}), \mathbb{N}(\text{sSet}_*))[W^{-1}]$ is an equivalence. We can choose an inverse
\[ \phi : X \rightarrow Y \]
in $\text{Fun}(\mathbb{N}(\text{MF} \times \mathbb{N}^{op}), \mathbb{N}(\text{sSet}_*))[W^{-1}]$. We fix an identification of the $k$-sphere $S^k$ with $\tilde{\mathbb{R}}^k$ and define the functor
\[ Z \in \text{Fun}(\mathbb{N}(\text{MF}), \mathbb{N}(\text{sSet}_*)) , \quad Z(M) := \text{sing}_\infty(M_+ \wedge S^k) . \]

We further define
\[ Z \in \text{Fun}(\mathbb{N}(\text{MF} \times \mathbb{N}^{op}), \mathbb{N}(\text{sSet}_*)) \]
as the constant extension of $\tilde{Z}$ along $\mathbb{N}^{op}$. We then have a transformation
\[ \psi : Z \rightarrow X \]

133
given by

\[ Z(M, n) := \text{sing}_\infty(M_+ \wedge S^k) \xrightarrow{\sim} \text{sing}_\infty(M_+ \wedge \mathbb{R}^k) \to X(M, n), \]

where the last map is induced by the projection \( \mathbb{R}^k \to \mathbb{R}^k/(\mathbb{R}^k \setminus \{0\}) \).

Let \( A \in \mathbb{N}(\text{Ch})[W^{-1}] \) be a chain complex. By definition we have

\[ \text{Sm}_c(H(A))(\cdots \times \mathbb{R}^k) = \text{colim}_{\mathbb{N}op} \text{Map}(\Sigma^\infty Y, H(A)). \]

In the following we will frequently use the canonical equivalence \( J \) between objects of \( \text{Fun}(\mathbb{N}(\text{sSet}_*)[W^{-1}], \text{Mod}(H\mathbb{Z})) \) (written evaluated at \( U \in \mathbb{N}(\text{sSet}_*) \))

\[ J : \text{Map}_{\mathbb{N}op}(\Sigma^\infty U, H(A)) \cong \text{Map}_{\text{Mod}(H\mathbb{Z})}(\Sigma^\infty U \wedge H\mathbb{Z}, H(A)) \]

\[ \cong \text{H}(\text{Map}(\text{C}_*(U), H(A))) \cong \text{H}(\text{Map}(\Sigma^\infty \mathcal{Z}, H(A))). \]

Here \( \alpha^{-1,*} \) is induced by the equivalence given in Corollary 6.19. The first equivalence uses the fact that \( E \wedge H\mathbb{Z} \) is the free \( H\mathbb{Z} \)-module generated by the spectrum \( E \), and the last equivalence is (191). In the first step of the construction we get rid of the colimit by

\[ \sigma_1 : \text{Sm}_c(H(A))(\cdots \times \mathbb{R}^k) \overset{\sigma_1}{=} \text{colim}_{\mathbb{N}op} \text{Map}(\Sigma^\infty Y, H(A)) \]

\[ \overset{\phi^*}{\longrightarrow} \text{colim}_{\mathbb{N}op} \text{Map}(\Sigma^\infty X, H(A)) \]

\[ \overset{\psi^*}{\longrightarrow} \text{colim}_{\mathbb{N}op} \text{Map}(\Sigma^\infty Z, H(A)) \]

\[ \cong \text{H}(\text{Map}(\Sigma^\infty \mathcal{Z}, H(A))). \]

Alternatively we could immediately go to chain complexes using the notation

\[ C_* : \mathbb{N}(\text{sSet}_*)[W^{-1}] \to \mathbb{N}(\text{Ch})[W^{-1}], \quad C_*(X) := N(\mathbb{Z}[X])/\mathbb{Z}. \]

\[ \sigma'_1 : \text{Sm}_c(H(A))(\cdots \times \mathbb{R}^k) \overset{\sigma'_1}{=} \text{colim}_{\mathbb{N}op} \text{Map}(\Sigma^\infty Y, H(A)) \]

\[ \overset{J}{\longrightarrow} \text{colim}_{\mathbb{N}op} H(\text{Map}(C_*(Y), A_\infty)) \]

\[ \overset{\phi^*}{\longrightarrow} \text{colim}_{\mathbb{N}op} H(\text{Map}(C_*(X), A_\infty)) \]

\[ \overset{\psi^*}{\longrightarrow} \text{colim}_{\mathbb{N}op} H(\text{Map}(C_*(Z), A_\infty)) \]

\[ \cong \text{H}(\text{Map}(C_*(\mathcal{Z}), A)). \]

Since \( J \) is natural, the following diagram commutes:

\[ \text{Sm}_c(H(A))(\cdots \times \mathbb{R}^k) \overset{\sigma_1}{\longrightarrow} \text{Map}(\Sigma^\infty \mathcal{Z}, H(A)) \]

\[ \downarrow \]

\[ H(\text{Map}(C_*(\mathcal{Z}), A)). \]
Since $\text{sing}_\infty$ preserves products we can write $Z(M) \cong \text{sing}_\infty(M_+) \wedge \text{sing}_\infty(S^k)$. We define the functor $U : \mathbb{N}(\text{Mf}) \rightarrow \mathbb{N}(\text{sSet}_*)[W^{-1}]$ such that $U(M) := \text{sing}_\infty(M_+)$. Then we have

$$Z = U \wedge \text{sing}_\infty(S^k).$$

Using Lemma 6.25 we can extend the diagram (218) to

$$\text{Sm}_c(H(A))(\cdots \times \mathbb{R}^k) \xrightarrow{\sigma_1} \text{Map}(\Sigma^\infty Z, H(A)) \xrightarrow{\sigma'_{J}} \text{Map}(U, \text{Map}(\Sigma^\infty \text{sing}_\infty(S^k), H(A))) \xrightarrow{J,J} \text{Map}(U, \text{Map}(\Sigma^\infty \text{sing}_\infty(S^k), H(A))).$$

Using Lemma 6.26 we can extend the diagram further to

$$\text{Sm}_c(HA)(\cdots \times \mathbb{R}^k) \xrightarrow{\sigma} \text{Map}(U, \text{Map}(\Sigma^\infty \text{sing}_\infty(S^k), H(A))) \xrightarrow{J,J} \text{Map}(U, H(A[k])) \xrightarrow{J} \text{Map}(U, H(A[k]))[s^k] = \text{Map}(\Sigma^\infty U, H(A[k])) = \text{Sm}_\infty(H(A[k]))(219)$$

We have

$$\text{Map}(\Sigma^\infty U, H(A[k])) \overset{def}{=} \text{Sm}_\infty(H(A[k]))$$

and the upper horizontal composition is the version $\sigma$ of the desuspension denoted by $\text{desusp}$ in (214). The other path defines the desuspension $\sigma'$ which is the map denoted by $\text{desusp}'$ in the diagram (215).

This finishes the proof of Proposition 6.23.

\textbf{Lemma 6.25} We have a natural commutative diagram of functors

$$\text{Fun}(\mathbb{N}(\text{sSet}_*)[W^{-1}]^{op} \times \mathbb{N}(\text{sSet}_*)[W^{-1}]^{op} \times \mathbb{N}(\text{Ch})[W^{-1}], \text{Mod}(HZ))$$

$$\begin{array}{ccc}
\text{Map}(\Sigma^\infty (X \wedge Y), H(A)) & \xrightarrow{J} & \text{Map}(\Sigma^\infty (X) \wedge \Sigma^\infty (Y), H(A)) \xrightarrow{J,J} \text{Map}(\Sigma^\infty (X), \text{Map}(\Sigma^\infty (Y), H(A)) \\
H(\text{Map}(C_*(X \wedge Y), A_\infty)) & \xrightarrow{P^*} & H(\text{Map}((C_*(X) \otimes C_*(Y), A_\infty)) \xrightarrow{J,J} H(\text{Map}(C_*(X), \text{Map}(C_*(Y), A_\infty))).
\end{array}$$

\textbf{Proof.} This is a consequence of the fact that the transformation $\alpha$ in Corollary 6.19 is a transformation of symmetric monoidal functors, and that $P^*$ is exactly the pull-back along the shuffle map which implements the monoidal structure on the chain complex functor $C_*$. Furthermore we use that $H$ is a symmetric monoidal functor.

Let $S^k = (S^1)^\wedge k$ be the simplicial $k$-sphere with $S^1 = \Delta^1/\partial \Delta^1$. We have an equivalence $\text{sing}_\infty(S^1) \cong S^1$ which induces

$$\text{sing}_\infty(S^k) \cong \text{sing}_\infty((S^1)^\wedge k) \cong \text{sing}_\infty(S^1)^\wedge k \cong (S^1)^\wedge k \cong S^k \quad (221)$$
We fix a representative \([S^k] : \mathbb{Z}[\mathbb{Z}] \xrightarrow{\sim} C_\ast(\text{sing}_\infty(S^k))\) of the fundamental class of the \(k\)-sphere \(S^k\).

**Lemma 6.26** We have the following commutative diagram in \(\text{Fun}(\mathbb{N}(\text{Ch})[W^{-1}], \text{Mod}(\mathbb{H}\mathbb{Z}))\):

\[
\begin{array}{ccc}
\text{Map}(\Sigma^\infty \text{sing}_\infty(S^k), H(A)) & \longrightarrow & \Omega^k H(A) \\
\downarrow{\sim} & & \downarrow{}
\end{array}
\]

where the right vertical arrow is given by the compatibility of \(H\) with the translations on \(\mathbb{N}(\text{Ch})[W^{-1}]\) and \(\text{Mod}(\mathbb{H}\mathbb{Z})\).

**Proof.** We rewrite the upper line (using the transformation \(\alpha\) from Corollary 6.19) as

\[
\begin{array}{ccc}
\text{Map}(\Sigma^\infty \text{sing}_\infty(S^k) \wedge \mathbb{H}\mathbb{Z}, H(A)) & \longrightarrow & \Omega^k H(A) \\
\downarrow{\sim} & & \\
H(\text{Map}(C_\ast(\text{sing}_\infty(S^k), A_\infty))^{[S^k]} & \longrightarrow & H(\text{Map}(\mathbb{Z}[\mathbb{Z}], A_\infty)) \longrightarrow H(A[-k])
\end{array}
\]

The commutativity of the right square expresses exactly the compatibility of the Eilenberg-MacLane spectrum functor with the shifts. The left square commutes since \(\alpha\) is natural and the diagram

\[
\begin{array}{ccc}
H(\mathbb{Z}[\mathbb{Z}]) & \longrightarrow & H(C_\ast(\text{sing}_\infty(S^k))) \\
\downarrow{\sim} & & \downarrow{\alpha^{-1}} \\
\Sigma^\infty S^k \wedge \mathbb{H}\mathbb{Z} & \longrightarrow & \Sigma^\infty \text{sing}_\infty(S^k) \wedge \mathbb{H}\mathbb{Z}
\end{array}
\]

commutes. \(\square\)

### 6.12 Representability of characteristic classes

Let \(B\) be a topological space of the homotopy type of a countable \(CW\)-complex. Then we define the smooth set

\[
\pi_0(\text{Sm}_\infty(B)) \in \text{Sm}(\mathbb{N}(\text{Set})).
\]

It associates to a smooth manifold \(M\) the set of homotopy classes of maps \(M \to B\). Let \(E\) be a spectrum with homotopy bounded from above, i.e. such there exists an \(N \in \mathbb{N}\) such that \(\pi_i(E) = 0\) for \(i \geq N\). Then we obtain another smooth set

\[
E^0 \in \text{Sm}(\mathbb{N}(\text{Set}))
\]

which associates to a smooth manifold \(M\) the underlying set of the cohomology group \(E^0(M)\). We have a canonical map

\[
\Phi : E^0(B) \to \text{hom}(\pi_0(\text{Sm}_\infty(B)), E^0)
\]
Proposition 6.27 The map $\Phi$ is an isomorphism.

Proof. There exists a smooth manifold $M$ and a map $f : M \to B$ which is an $N$-equivalence (this uses that a CW-model of $B$ has countable skeleta). It follows from the Atiyah-Hirzebruch spectral sequence that the restriction $f^* : E^0(B) \to E^0(M)$ is an isomorphism. This implies that $\Phi$ is injective.

Let now $\phi \in \text{hom}(\pi_0(\text{Sm}_\infty(B)), E^0)$ be given. The map $f : M \to B$ represents a class $[f] \in \pi_0(\text{Sm}_\infty(B))(M)$. Then we set $e := \phi([f]) \in E^0(M) \cong E^0(B)$. One easily checks that $\Phi(e) = \phi$. This shows that $\Phi$ is surjective. \qed

6.13 Thom and Euler forms

In this subsection, which is a review of [BGV92, Ch. 1.6], we recall the constructions of the Euler and Thom forms.

If $\pi : V \to M$ is an $n$-dimensional real vector bundle, then we can form the orientation bundle $\Lambda \to M$. It is a one-dimensional real vector bundle with structure group reduced to the subgroup $\{\pm 1\} \subset GL(1, \mathbb{R})$. If $\text{Fr}(V) \to M$ denotes the $GL(n, \mathbb{R})$-principal bundle of frames of $V$, then we can write the orientation bundle as an associated vector bundle

$$
\Lambda := \text{Fr}(V) \times_{GL(n, \mathbb{R}), \text{sign(det)}} \mathbb{R},
$$

where $\text{sign(det)} : GL(n, \mathbb{R}) \to \{\pm 1\}$ is the “sign of the determinant” representation. The main reason for introducing the orientation bundle is that we have an integration map

$$
\int_{V/M} : \Omega_c(V, \pi^*\Lambda)[n] \to \Omega(M)
$$

without any condition of orientability. Here $\Omega_c(V, \pi^*\Lambda)[n]$ denotes the de Rham complex of differential forms on $V$ with coefficients in $\Lambda$ which are properly supported over $M$, and which is shifted such that $n$-forms are in degree 0.

The bundle $\Lambda$ has a natural flat connection, and by $\underline{\Lambda}$ we denote the associated locally constant sheaf of parallel sections. The Thom class of $V$ is a cohomology class

$$
U(V) \in H^\dim(V)(\pi^*\Lambda),
$$

where the subscript $c$ indicates cohomology with proper support over $M$ as discussed in Subsection 6.10, and we have an isomorphism

$$
H^\dim(V)(\pi^*\Lambda) \cong H^\dim(V)(\Omega_c(V, \pi^*\Lambda)).
$$
The Thom class \( U(V) \) is uniquely determined by the condition that \( \int_{V/M} U(V) = 1 \), where the integration in cohomology is induced by the map \((223)\) on form level. The restriction of the Thom class to the zero section

\[
e(V) := 0^*_U U(V) \in H^{\dim(V)}(M, \Lambda)
\]

is the by definition the Euler class of \( V \).

We need a refinement of the Thom class on the level of differential forms for real vector bundles with metric and metric connection \((V, h^V, \nabla^V)\). We let \( MQ(\nabla^V) \in \Omega^{\dim(V)}(V, \pi^*\Lambda) \) be the Mathai-Quillen form (see [BGV92, Ch. 1.6, (1.37)] but note the different normalization). For the sake of completeness we briefly recall its construction.

We consider the bigraded algebra \( A \) with constituents \( A^{i,j} := \Omega^i(V, \pi^*\Lambda) \). We have the tautological section \( x \in A^{0,1} \). Using the metric on \( V \) we define the section \( |x|^2 \in A^{0,0} \). We further have \( \nabla^{\pi^*V} x \in A^{1,1} \). The curvature \( R^V \) can be considered as an element in \( \Omega^2(M, \mathfrak{so}(V)) \). Using the metric again we identify \( \mathfrak{so}(V) \cong \Lambda^2 V \). With this identification we get \( F := \pi^* R^V \in A^{2,2} \). The metric on \( V \) induces a canonical element \( \text{vol} \in C^\infty(M, \Lambda^{\dim(V)} V^* \otimes \Lambda) \). The Berezin integral \( T : A \to \Omega^*(V, \Lambda) \) is given by the composition of projection from \( A \) to \( A^{*, \dim(V)} \) and the pairing with \( \text{vol} \). In terms of these constructs the Mathai-Quillen form is given by

\[
MQ(\nabla^V) := \frac{1}{(2\pi)^{\dim(V)/2}} T \exp\left(-\frac{|x|^2}{2} + i \nabla^{\pi^*V} x + F\right)
\]

The Mathai-Quillen form does not have proper support over \( M \) but decays rapidly at infinity. We modify this form in order to obtain a form with proper support over \( M \). We define the diffeomorphism \( f : B_1(V) \to V \) of the unit ball bundle \( B_1(V) \subset V \) with \( V \) by

\[
f(v) := \frac{v}{(1 - |v|^2)^{1/2}} , \quad v \in B_1(V) .
\]

The form \( f^* MQ(\nabla^V) \) is first defined on \( B_1(V) \) but can be extended by zero smoothly to all of \( V \). We define

\[
U(\nabla^V) := f^* MQ(\nabla^V) \in \Omega^{\dim(V)}_c(V, \pi^*\Lambda) .
\]

The Euler form \( e(\nabla^V) \in \Omega^{\dim(V)}(M, \Lambda) \) of the real euclidean vector bundle \((V, h^V)\) with connection \( \nabla^V \) is defined in terms of the curvature \( R^V \in \Omega^2(W, \mathfrak{so}(V)) \) by

\[
e(\nabla^V) = \frac{1}{(2\pi)^{\dim(V)/2}} T \exp(F) ,
\]

see [BGV92, Sec. 1.6] for details. In the following we list some obvious properties of the Thom form.

1. We have

\[
[U(\nabla^V)] = U(V) \in H^{\dim(V)}_c(V, \pi^*\Lambda) .
\]
2. For a pull-back diagram

\[
\begin{array}{ccc}
V' & \xrightarrow{F} & V \\
\downarrow{\pi'} & & \downarrow{\pi} \\
M' & \xrightarrow{f} & M
\end{array}
\]

we have an isomorphism \( \Lambda' \cong f^*\Lambda \) and (under this isomorphism)

\[ F^*U(\nabla V) = U(\nabla V') \in \Omega^\dim(V)(V', \pi'^*\Lambda') . \]

Here we assume that \( V' \) has the induced metric and metric connection \( \nabla V' = f^*\nabla V \).

3. We have

\[ 0_V^*(U(\nabla V)) = e(\nabla V) \in \Omega^\dim(V)(M, \Lambda) , \]

where \( e(\nabla V) \) denotes the Euler form (224).

4. For a two bundles \((V, h^V, \nabla V)\) and \((V', h^V', \nabla V')\) we have

\[ 0_V^*U(\nabla V \oplus \nabla V') = \pi'^*e(\nabla V) \wedge U(\nabla V') . \]

### 6.14 The normalized Borel regulator map

The goal of the present subsection is to understand the relation between the Kamber-Tondeur forms introduced in (35) and the normalized Borel regulator map. This becomes important if one wants to connect to developments in arithmetic geometry. The results of this subsection are only used in Subsubsection 5.4.3.

We follow [BG02] for the description of the normalized Borel regulator map. For \( n \geq 1 \) we consider \( G_n := SL(n, \mathbb{C}) \) as a semisimple real Lie group. Its maximal compact subgroup is \( K_n = SU(n) \), and we let \( X_n := G_n/K_n \) denote the associated symmetric space. The complexification of \( G_n \) can be identified with \( G_n,\mathbb{C} := SL(n, \mathbb{C}) \times SL(n, \mathbb{C}) \) with the maximal compact subgroup \( G_{u,n} := SU(n) \times SU(n) \). The embedding \( G_n \to G_{n,\mathbb{C}} \) is given by

\[ SL(n, \mathbb{C}) \ni A \mapsto (A, \bar{A}) \in SL(n, \mathbb{C}) \times SL(n, \mathbb{C}) . \]

The space \( X_{u,n} := G_{u,n}/K_n \) is the compact dual of \( X_n \). We have \( X_{u,n} = SU(n) \times SU(n)/SU(n) \) with the action \( (A, B)U = (AU, B\bar{U}) \) and therefore a diffeomorphism \( X_{u,n} \to SU(n) \) given by \( (A, B) \mapsto AB^t \). The induced left action of \( SU(n) \times SU(n) \) on \( SU(n) \) is given by \( (A, B)X = AXB^t \).

The rational cohomology of the \( h \)-space \( SU(n) \) is the exterior algebra

\[ H^*(SU(n); \mathbb{Q}) \cong \Lambda(\beta(1)^Q, \ldots, \beta(n)^Q) \]

generated by primitive elements \( \beta(n)^Q_{2j-1} \in H^{2j-1}(SU(n); \mathbb{Q}) \), where \( x \in H^*(SU(n); \mathbb{Q}) \) is called primitive if \( \mu^*(x) = x \times 1 + 1 \times x \) for the multiplication map \( \mu : SU(n) \times SU(n) \to SU(n) \).
We consider $\mathbb{C}$ as a real vector space with an action of $\mathbb{Z}/2\mathbb{Z}$ by complex conjugation and form the invariant subspace $\mathbb{R}(j) := (2\pi i)^j \mathbb{R} \subset \mathbb{C}$. We define normalized elements

$$
\beta(n)_{2j-1} := c(n)_{2j-1} \beta(n)_{2j-1}^\sigma \in H^{2j-1}(SU(n); \mathbb{R}(j)), \quad c(n)_{2j-1} \in \mathbb{C}
$$

such that $\beta(n)_{2j-1}(\pi_{2j-1}(SU(n))) = \mathbb{Z}(j) := (2\pi i)^j \mathbb{Z}$. This fixes these elements up to sign. We choose the signs such that for $3 \leq j \leq m \leq n$ we have the compatibility $\beta(n)_{2j-1}(U(m)) = \beta(m)_{2j-1}$. An explicit choice of the $\beta(n)_{2j-1}$ will be described in (236).

In order to distinguish the cohomology of a Lie group as a space from group cohomology we will denote the group-cohomology by $H^*_{gr}$. We write $H^*_{cont}$ for the continuous group cohomology. There is a canonical morphism

$$
\phi_n : H^{2j-1}(SU(n); \mathbb{R}(j)) \to H^{2j-1}_{gr}(SL(n, \mathbb{C}); \mathbb{R}(j-1))
$$

defined as the composition of the following maps:

1. The de Rham isomorphism and Hodge theory identify the cohomology of a compact Lie group with its biinvariant differential forms. Hence we have

$$
H^*(SU(n); \mathbb{R}(j)) = (2\pi i)^j H^*(SU(n); \mathbb{R}) \cong (2\pi i)^j \Omega^{2j-1}(SU(n))^{SU(n) \times SU(n)}, \quad (225)
$$

viewed as subspaces in the corresponding complexifications.

2. We have an isomorphism

$$
(2\pi i)^j \Omega^{2j-1}(SU(n))^{SU(n) \times SU(n)} = (2\pi i)^j \Omega^{2j-1}(X_{u,n})^{G_{u,n}} \cong (2\pi i)^j H^{2j-1}(\mathfrak{g}_{u,n}, K_n; \mathbb{R}), \quad (226)
$$

where the last group is relative Lie algebra cohomology. We use the general convention that small letters denote the Lie algebras of the groups denoted by the corresponding capital symbols.

3. We have Cartan decompositions of the Lie algebras $\mathfrak{g}_{u,n}, \mathfrak{g}_n \subset \mathfrak{g}_{n,\mathbb{C}}$

$$
\mathfrak{g}_{u,n} = \mathfrak{k}_n \oplus \mathfrak{p}_{u,n}, \quad \mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n,
$$

and the relation $\mathfrak{p}_{u,n} = i\mathfrak{p}_n$. Note that

$$
H^{2j-1}(\mathfrak{g}_{u,n}, K_n; \mathbb{R}) \cong (\Lambda^{2j-1} \mathfrak{p}_{u,n})^{K_n}, \quad H^{2j-1}(\mathfrak{g}_n, K_n; \mathbb{R}) \cong (\Lambda^{2j-1} \mathfrak{p}_n)^{K_n}. \quad (227)
$$

By complex linear extension and restriction of forms we now have an isomorphism

$$
(2\pi i)^j H^{2j-1}(\mathfrak{g}_{u,n}, K_n; \mathbb{R}) \cong (2\pi i)^j H^{2j-1}(\mathfrak{g}_n, K_n; \mathbb{R}) \cong (2\pi i)^j H^{2j-1}(\mathfrak{g}_n, K_n; \mathbb{R}) \quad (228)
$$

4. We have the van Est isomorphism

$$
H^{2j-1}(\mathfrak{g}_n, K_n; \mathbb{R}) \cong H^{2j-1}_{cont}(G_n; \mathbb{R}) = H^{2j-1}_{cont}(SL(n, \mathbb{C}); \mathbb{R}). \quad (229)
$$
5. We have a morphism
\[(2\pi i)^{j-1} H^{2j-1}_{cont}(SL(n, \mathbb{C}); \mathbb{R}) \to (2\pi i)^{j-1} H^{2j-1}_{gr}(SL(n, \mathbb{C}); \mathbb{R}) = H^{2j-1}_{gr}(SL(n, \mathbb{C}); \mathbb{R}(j-1)).\]

**Definition 6.28** For \( j \geq 2 \) a class \( c^{\text{Bor}}_{2j-1} \in H^{2j-1}_{gr}(GL(\mathbb{C}); \mathbb{R}(j-1)) \) is called a Borel element if its restriction to \( SL(\mathbb{C}, n) \) coincides with \( \phi_n(\beta(n)_{2j-1}) \) for some (hence all) \( n \geq j \).

This definition fixes the Borel elements up to decomposables. For every complex embedding \( \sigma \in \Sigma \) we get a map \( GL(R) \to GL(\mathbb{C}) \) which induces a map
\[ H^{2j-1}(GL(\mathbb{C}); \mathbb{R}(j-1)) \to H^{2j-1}(GL(R); \mathbb{R}(j-1)) \to \text{Hom}(K_{2j-1}(R); \mathbb{R}(j-1)), \]
where the last map is induced by the Hurewicz map, the fact that Quillen’s +-construction produces a homology isomorphism
\[ K_*(R) \xrightarrow[Hurewicz]{\text{Hurewicz}} H_*(BGL(R)^+; \mathbb{Z}) \cong H_*(BGL(R); \mathbb{Z}), \]
and the evaluation of cohomology against homology. We let
\[ \tilde{c}^{\text{Bor}}_{2j-1}(\sigma) : K_{2j-1}(R) \to \mathbb{R}(j-1) \]
denote the image under (231) of a Borel element \( c^{\text{Bor}}_{2j-1} \). This homomorphism \( \tilde{c}^{\text{Bor}}_{2j-1}(\sigma) \) does not depend on the choice of the Borel element since decomposable classes evaluate trivially against classes in the image of the Hurewicz map.

We have a \( \mathbb{Z}/2\mathbb{Z} \)-invariant decomposition
\[ \mathbb{C} \cong \mathbb{R}(j) \oplus \mathbb{R}(j-1) \]
and therefore an isomorphism of real vector spaces \( \mathbb{C}/\mathbb{R}(j) \cong \mathbb{R}(j-1) \). We define the real vector space of \( \mathbb{Z}/2\mathbb{Z} \)-invariants
\[ X_{2j-1}(R) := ((\mathbb{C}/\mathbb{R}(j)^{\Sigma})^{\mathbb{Z}/2\mathbb{Z}} \]
where the group \( \mathbb{Z}/2\mathbb{Z} \) acts by complex conjugation on both, \( \mathbb{C}/\mathbb{R}(j) \) and \( \Sigma \). Then we define the Borel regulator map
\[ r_{\text{Bor}} : K_{2j-1}(R) \to X_{2j-1}(R), \quad r_{\text{Bor}}(x)(\sigma) := \tilde{c}^{\text{Bor}}_{2j-1}(\sigma)(x). \]

Theorem 3.3 of Borel [Bor74] and its refinement Proposition 3.9 can be reformulated to say that the Borel regulator map \( r_{\text{Bor}} \) induces an isomorphism
\[ K_{2j-1}(R) \otimes \mathbb{R} \xrightarrow{\sim} X_{2j-1}(R) \]
for \( j \geq 2 \).
We have the following commutative diagram

$$\begin{array}{c}
K_{2j-1}(R) \\
\downarrow c \\
A^{2j-1}
\end{array} \quad \begin{array}{c}
\downarrow \psi \\
\downarrow \psi
\end{array} \quad \begin{array}{c}
X_{2j-1}(R)
\end{array}$$

with the isomorphism of real vector spaces $\psi$. We will need the explicit form of the isomorphism $\psi$ in the discussion 5.4.3. We have

$$\psi(f) = \frac{1}{(2\pi i)^{j-1}} \sum_{\sigma, \sigma' \in \Sigma^*} \psi_{\sigma, \sigma'} f_{2j-1}(\sigma)$$

for some real matrix $(\psi_{\sigma, \sigma'})_{\sigma, \sigma' \in \Sigma^*}$.

**Proposition 6.29** We assume that $j \geq 2$. Then we have

$$\psi_{\sigma, \sigma'} = (-1)^{j-1} (2j-1)! (j-1)! \delta_{\sigma, \sigma'}.$$

**Proof.** We have

$$\psi(r_{\text{Bor}}(x)) = \sum_{\sigma \in \Sigma^*} \omega_{2j-1}(\sigma)(x)b_{2j-1}(\sigma) = \frac{1}{(2\pi i)^{j-1}} \sum_{\sigma, \sigma' \in \Sigma^*} \psi_{\sigma, \sigma'} c_{2j-1}^{\text{Bor}}(\sigma')(x)b_{2j-1}(\sigma'),$$

and hence the relation

$$\omega_{2j-1}(\sigma) = \frac{1}{(2\pi i)^{j-1}} \sum_{\sigma, \sigma' \in \Sigma^*} \psi_{\sigma, \sigma'} c_{2j-1}^{\text{Bor}}(\sigma'), \quad \forall \sigma \in \Sigma^*,$$

determines the matrix $(\psi_{\sigma, \sigma'})_{\sigma, \sigma' \in \Sigma^*}$.

In order to compare $\omega_{2j-1}(\sigma)$ and $c_{2j-1}^{\text{Bor}}(\sigma)$ we give a description of the latter which is similar to the former.

The standard hermitean metric $h_0$ on $\mathbb{C}^n$ induces a normalization of the volume form. We let $\text{Met}_1(n)$ be the space of hermitean metrics on $\mathbb{C}^n$ inducing the normalized volume form. The group $SL(n; \mathbb{C})$ acts on $\mathbb{C}^n$ and therefore on $\text{Met}_1(n)$. The stabilizer of $h_0$ is $SU(n)$ so that we get an identification $X_n \cong \text{Met}_1(n)$. The trivial vector bundle $V := \text{Met}_1(n) \times \mathbb{C}^n \to \text{Met}_1(n)$ has a trivial connection $\nabla^V$ and a tautological metric $h_{\text{taut}}^V$.

Both structures are $SL(n, \mathbb{C})$-invariant. Hence the associated Kamber-Tondeur form is an invariant form $\omega_{2j-1}(h_{\text{taut}}^V) \in \Omega^{2j-1}(\text{Met}_1(n))^{SL(n, \mathbb{C})}$. We have an isomorphism

$$\Omega^{2j-1}(\text{Met}_1(n))^{SL(n, \mathbb{C})} \cong \Omega^{2j-1}(X_n)^{G_n} \cong (\Lambda^{2j-1}p_n)^{K_n}.$$

We let $\tilde{\omega}_{2j-1} \in H^{2j-1}(SL(n, \mathbb{C}); \mathbb{R})$ be the image of the form $\omega_{2j-1}(h_{\text{taut}}^V)$ under the composition of (235) with (229) and (230). It is clear that the following Lemma implies Proposition 6.29.
Lemma 6.30 We have
\[ \tilde{\omega}_{2j-1} = \frac{(-1)^{j-1}(j-1)!}{(2j-1)!(2\pi i)^{j-1}} \phi_n(\beta(n)_{2j-1}) . \]

Proof. Let \( \hat{\beta}_{2j-1} \in (\Lambda^{2j-1}p_n)^{K_n} \) be the image of \( \beta(n)_{2j-1} \) under (225), (226), (228), and (227). It suffices to compare \( \hat{\beta}_{2j-1} \) with the image \( \hat{\omega}_{2j-1} \) of \( \omega_{2j-1}(h_{\text{taut}}) \) under (235). The Cartan decomposition \( g_n = k_n \oplus p_n \) is the decomposition of complex trace-free \( n \times n \)-matrices into the anti-hermitan and hermitean parts. We thus identify
\[ p_n \cong \{ A \in \text{Mat}(n, \mathbb{C}) \mid A = A^*, \text{tr} A = 0 \} . \]

We use the composition of the exponential map and the projection \( p_n \to G_n \to X_n \) as a chart. In these coordinates the metric \( h_{\text{taut}} \) is a symmetric matrix valued function
\[ h_{\text{taut}}(A) = \exp(A) \exp(A)^* = 1 + A + A^* + \cdots = 1 + 2A + \cdots . \]

We have \( \nabla^V = d \) and \( \nabla^{V,*} - \nabla^V = (h_{\text{taut}}^{-1}d h_{\text{taut}}) \). At the origin \( A = 0 \) we get
\[ (\nabla^{V,*} - \nabla^V)(0)(X) = 2X , \quad X \in p_n . \]

We insert this into (35) and get
\[ \tilde{\omega}_{2j-1}(X_1, \ldots, X_{2j-1}) = \frac{1}{(2\pi i)^{j-1}} \sum_{s \in \Sigma^{2j-1}} \text{sign}(s) \text{Tr}(X_{s(1)} \ldots X_{s(2j-1)}) . \]

We now calculate \( \hat{\beta}_{2j-1} \) explicitly. We first give a cohomological description of a primitive element in \( H^{2j-1}(SU(n); \mathbb{Q}) \). We identify \( S^1 \cong \mathbb{R}/\mathbb{Z} \). On \( S^1 \times SU(n) \) we consider the suspension bundle \( V_{\text{susp}} \to S^1 \times SU(n) \) given as a quotient of the \( \mathbb{Z} \)-equivariant trivial bundle
\[ \mathbb{R} \times SU(n) \times \mathbb{C}^n \to \mathbb{R} \times SU(n) , \quad (t, g, v) \mapsto (t, g) \]
where the actions of \( \mathbb{Z} \) are given by
\[ n(t, g, v) := (t + n, g, g^n v) , \quad n(t, g) := (t + n, g) . \]

Then
\[ \text{ch}_{2j-1} := \int_{S^1 \times SU(n)/SU(n)} \text{ch}_{2j}(V_{\text{susp}}) \in H^{2j-1}(SU(n); \mathbb{Q}) \]
is primitive. The normalization of the Chern character is such that
\[ \text{ch}_{2j-1}(\pi_{2j-1}(SU(n))) = \mathbb{Z} . \]

We conclude that we can take
\[ \beta(n)_{2j-1} := (2\pi i)^j \text{ch}_{2j-1} . \] (236)
We now calculate a differential form representative. We choose a function \( \chi \in C^\infty([0, 1]) \) which is constant near the end points of the interval and satisfies \( \chi(0) = 0 \) and \( \chi(1) = 1 \). We define a connection on the trivial bundle \([0, 1] \times SU(n) \times \mathbb{C}^n \to [0, 1] \times SU(n)\) by

\[
\nabla := d + \chi(t)g^{-1}dg .
\]

It can be extended to a \( \mathbb{Z} \)-invariant connection on \( \mathbb{R} \times SU(n) \times \mathbb{C}^n \to \mathbb{R} \times SU(n) \). This \( \mathbb{Z} \)-invariant connection induces a connection \( \nabla_{\text{V_{susp}}} \) on \( V_{\text{susp}} \), which we use to calculate a form representative

\[
\text{ch}_2(\nabla_{\text{V_{susp}}}) = \frac{(-1)^j}{j!(2\pi i)^j} \text{Tr}(R^{\nabla_{\text{V_{susp}}}})^j .
\]

We have on \([0, 1] \times SU(n)\) that

\[
R^{\nabla_{\text{V_{susp}}}} = \chi'(t) dt \wedge g^{-1} dg + (\chi(t)^2 - \chi(t)) g^{-1} dg \wedge g^{-1} dg .
\]

The form \( \text{ch}_2(\nabla_{\text{V_{susp}}}) \) is thus bi-invariant under the action of \( SU(n) \times SU(n) \) on \( S^1 \times SU(n) \). For further calculation we therefore restrict to the origin \( g = 1 \), where

\[
R^{\nabla_{\text{V_{susp}}}}(t, 1) = \chi' dt \wedge dg + (\chi(t)^2 - \chi(t)) dg \wedge dg .
\]

We have for \( t \in [0, 1] \) that

\[
\text{Tr} \ R^{\nabla_{\text{V_{susp}}}}(t, 1)^{2j} = j \chi'(t)(\chi(t)^2 - \chi(t))^{j-1} dt \wedge \text{Tr}(dg)^{2j-1} + \ldots,
\]

where the terms not written do not contain \( dt \). Using

\[
\int_0^1 (x^2 - x)^{j-1} dx = (-1)^{j-1} \int_0^1 x^{j-1}(1 - x)^{j-1} dx
\]

we get

\[
\int_{S^1 \times SU(n)} \text{ch}_2(\nabla_{\text{V_{susp}}}) = \frac{j}{j!(2\pi i)^j} \text{Tr}(dg)^{2j-1} \int_0^1 \chi'(t)(\chi(t)^2 - \chi(t))^{j-1}
\]

\[
= \frac{1}{(2\pi i)^j} \frac{(-1)^{j-1}(j-1)!}{(2j-1)!} \text{Tr}(dg)^{2j-1} .
\]

This gives

\[
\hat{\beta}_{2j-1}(X_1, \ldots, X_{2j-1}) = \frac{(-1)^{j-1}(j-1)!}{(2j-1)!} \sum_{s \in \Sigma^{2j-1}} \text{sign}(s) \text{Tr}(X_{s(1)} \ldots X_{s(2j-1)}) .
\]

This implies that

\[
\frac{(-1)^{j-1}(2j-1)!}{(j-1)!(2\pi i)^j} \hat{\beta}_{2j-1} = \hat{\omega}_{2j-1} .
\]
6.15 More normalizations

In order to compare calculations of higher torsion invariants one must be careful with normalizations of characteristic forms for flat bundles and the corresponding normalizations of torsion forms. In this subsection we compare various normalizations occuring in the literature. We give explicit formulas for the renormalizing factors needed to transfer between different normalizations.

The Bismut-Lott normalization is related to the choice\(^{(35)}\)

\[
\omega_{2j+1}(h^V) = \frac{1}{(2\pi i)^{j}2^{2j+1}} \text{Tr} \omega(h^V)^{2j+1}.
\]

This is the standard choice adopted in the present paper. Another normalization, the Chern normalization, is fixed by taking

\[
\omega_{2j+1}^{\text{Chern}}(h^V) := \text{Im} \left( \hat{\text{ch}}(\nabla^V, \nabla^V)_{2j+1} \right).
\]

Here we use the notation introduced in Subsection 3.5 and \(\hat{\text{ch}}(\nabla_0, \nabla_1)\) is the transgression Chern form such that \(d\hat{\text{ch}}(\nabla_0, \nabla_1) = \text{ch}(\nabla_0) - \text{ch}(\nabla_1)\). The proof of Lemma 6.30 gives

\[
\hat{\text{ch}}(\nabla^V, \nabla^V)_{2j+1} = (-1)^j \frac{j!}{(2\pi i)^{j+1}(2j+1)!} \text{Tr} \omega(h^V)^{2j+1}
\]

and therefore

\[
\omega_{2j+1}^{\text{Chern}}(h^V)_{2j+1} = (-1)^j \frac{2^{2j+1}j!}{2\pi(2j+1)!} \omega_{2j+1}(h^V).
\]

We define the factor

\[
N_{\text{Chern}}(2j + 1) := (-1)^j \frac{2\pi(2j + 1)!}{2^{2j+1}j!}.
\]

The Chern normalization is used e.g. in the work of Goette \cite{Goe09}. The normalization of Igusa \cite{Igu02} is

\[
\omega_{2j+1}^{\text{Igusa}}(h^V) = \frac{1}{(2j + 1)!} 2^j \text{Tr} \omega(h^V)^{2j+1}.
\]

It follows that

\[
\omega_{2j+1}^{\text{Igusa}}(h^V) = \frac{(2\pi)^{j}2^{2j}}{(2j + 1)!} \omega_{2j+1}(h^V).
\]

The Igusa normalization is used for the topological version of higher torsion, the higher Reidemeister-Franz torsion, which is also called Igusa-Klein torsion. We define the factor

\[
N_{\text{Igusa}}(2j + 1) := \frac{(2j + 1)!}{(2\pi)^{j}2^{2j}}.
\]

Finally we have the Borel normalization fixed by the normalization of the Borel regulator \cite{BG02}. We have by Proposition 6.29

\[
\omega_{2j+1}^{\text{Borel}}(h^V) := \frac{(-1)^j(2\pi)^{j}j!}{(2j + 1)!} \omega_{2j+1}(h^V).
\]
Hence we define

\[ N_{\text{Borel}}(2j + 1) := \frac{(-1)^j(2j + 1)!}{(2\pi i)^j j!} \]  

(240)

References

[Ada63] J.F. Adams. On the groups \( J(X) \). I. Topology, 2:181–195, 1963.

[BDKW11] B. Badzioch, W. Dorabiala, J.R. Klein, and B. Williams. Equivalence of higher torsion invariants. Adv. Math., 226(3):2192–2232, 2011.

[Bei86] A.A. Beilinson. Higher regulators of modular curves. In Applications of algebraic \( K \)-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), volume 55 of Contemp. Math., pages 1–34. Amer. Math. Soc., Providence, RI, 1986.

[BF78] A.K. Bousfield and E.M. Friedlander. Homotopy theory of \( \Gamma \)-spaces, spectra, and bisimplicial sets. In Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, volume 658 of Lecture Notes in Math., pages 80–130. Springer, Berlin, 1978.

[BG75] J.C. Becker and D.H. Gottlieb. The transfer map and fiber bundles. Topology, 14:1–12, 1975.

[BG01] J.-M. Bismut and S. Goette. Families torsion and Morse functions. Astérisque, (275):x+293, 2001.

[BG02] J.I. Burgos Gil. The regulators of Beilinson and Borel, volume 15 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2002.

[BGV92] N. Berline, E. Getzler, and M. Vergne. Heat kernels and Dirac operators, volume 298 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992.

[BKS10] U. Bunke, M. Kreck, and T. Schick. A geometric description of differential cohomology. Ann. Math. Blaise Pascal, 17(1):1–16, 2010.

[BL95] J.-M. Bismut and J. Lott. Flat vector bundles, direct images and higher real analytic torsion. J. Amer. Math. Soc., 8(2):291–363, 1995.

[BM06] M.-T. Benameur and M. Maghfoul. Differential characters in \( K \)-theory. Differential Geom. Appl., 24(4):417–432, 2006.

[Bor74] A. Borel. Stable real cohomology of arithmetic groups. Ann. Sci. École Norm. Sup. (4), 7:235–272 (1975), 1974.
[Bor77] A. Borel. Cohomologie de $\text{SL}_n$ et valeurs de fonctions zeta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(4):613–636, 1977.

[Bou79] A.K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[BS98] J.C. Becker and R.E. Schultz. Axioms for bundle transfers and traces. *Math. Z.*, 227(4):583–605, 1998.

[BS09] U. Bunke and T. Schick. Smooth K-theory. *Astérisque*, (328):45–135 (2010), 2009.

[BS10] U. Bunke and T. Schick. Uniqueness of smooth extensions of generalized cohomology theories. *J. Topol.*, 3(1):110–156, 2010.

[BSSW09] U. Bunke, T. Schick, I. Schröder, and M. Wiethaup. Landweber exact formal group laws and smooth cohomology theories. *Algebr. Geom. Topol.*, 9(3):1751–1790, 2009.

[BT] U. Bunke and G. Tamme. Regulators and cycle maps in higher-dimensional differential algebraic k-theory. In preparation.

[Bun] U. Bunke. Differential cohomology. Course notes, Universität Regensburg, 2012. [http://arxiv.org/abs/1208.3961](http://arxiv.org/abs/1208.3961).

[Bun10a] U. Bunke. Adams operations in smooth K-theory. *Geom. Topol.*, 14(4):2349–2381, 2010.

[Bun10b] U. Bunke. Chern classes on differential K-theory. *Pacific J. Math.*, 247(2):313–322, 2010.

[BW87] M. Bökstedt and F. Waldhausen. The map $B\text{SG} \to A(*) \to QS^0$. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 418–431. Princeton Univ. Press, Princeton, NJ, 1987.

[BZ92] J.-M. Bismut and W. Zhang. An extension of a theorem by Cheeger and Müller. *Astérisque*, (205):235, 1992. With an appendix by François Laudenbach.

[Che79] J. Cheeger. Analytic torsion and the heat equation. *Ann. of Math. (2)*, 109(2):259–322, 1979.

[CS1v] J. Cheeger and J. Simons. Differential characters and geometric invariants. In *Geometry and topology (College Park, Md., 1983/84)*, volume 1167 of *Lecture Notes in Math.*, pages 50–80. Springer, Berlin, 1v.
[Dou06] C.L. Douglas. On the fibrewise Poincare-Hopf theorem. In Recent developments in algebraic topology, volume 407 of Contemp. Math., pages 101–111. Amer. Math. Soc., Providence, RI, 2006.

[Dug01] D. Dugger. Universal homotopy theories. Adv. Math., 164(1):144–176, 2001.

[DWW03] W. Dwyer, M. Weiss, and B. Williams. A parametrized index theorem for the algebraic $K$-theory Euler class. Acta Math., 190(1):1–104, 2003.

[EKMM97] A.D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May. Rings, modules, and algebras in stable homotopy theory, volume 47 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[Esn89] H. Esnault. On the Loday symbol in the Deligne-Beilinson cohomology. $K$-Theory, 3(1):1–28, 1989.

[FH00] D.S. Freed and M. Hopkins. On Ramond-Ramond fields and $K$-theory. J. High Energy Phys., (5):Paper 44, 14, 2000.

[FL10] D.S. Freed and J. Lott. An index theorem in differential $K$-theory. Geom. Topol., 14(2):903–966, 2010.

[FMS07] D.S. Freed, G.W. Moore, and G. Segal. Heisenberg groups and noncommutative fluxes. Ann. Physics, 322(1):236–285, 2007.

[Fre00] D.S. Freed. Dirac charge quantization and generalized differential cohomology. In Surveys in differential geometry, Surv. Differ. Geom., VII, pages 129–194. Int. Press, Somerville, MA, 2000.

[GI10] S. Goette and K. Igusa. Exotic smooth structures on topological fibre bundles. ArXiv e-prints, November 2010, 1011.4653.

[Goe09] S. Goette. Torsion invariants for families. Astérisque, (328):161–206 (2010), 2009.

[Hir03] P.S. Hirschhorn. Model categories and their localizations, volume 99 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.

[HK03] A. Huber and G. Kings. Bloch-Kato conjecture and Main Conjecture of Iwasawa theory for Dirichlet characters. Duke Math. J., 119(3):393–464, 2003.

[Hov99] M. Hovey. Model categories, volume 63 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999.

[Hov01] M. Hovey. Model category structures on chain complexes of sheaves. Trans. Amer. Math. Soc., 353(6):2441–2457 (electronic), 2001.
[HS05] M.J. Hopkins and I.M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.

[Igu02] K. Igusa. *Higher Franz-Reidemeister torsion*, volume 31 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2002.

[Igu08] K. Igusa. Axioms for higher torsion invariants of smooth bundles. *J. Topol.*, 1:159–186, 2008.

[Kar87] M. Karoubi. Homologie cyclique et *K*-théorie. *Astérisque*, (149):147, 1987.

[Knu01] K.P. Knudson. *Homology of linear groups*, volume 193 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[KS10] M. Kreck and W. Singhof. Homology and cohomology theories on manifolds. *Münster J. Math.*, 3:1–9, 2010.

[Lot94] J. Lott. Equivariant analytic torsion for compact Lie group actions. *J. Funct. Anal.*, 125(2):438–451, 1994.

[Lot00] J. Lott. Secondary analytic indices. In *Regulators in analysis, geometry and number theory*, volume 171 of *Progr. Math.*, pages 231–293. Birkhäuser Boston, Boston, MA, 2000.

[Lur] J. Lurie. Higher algebra. [http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf](http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf).

[Lur09] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mil71] J. Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[Müll78] W. Müller. Analytic torsion and *R*-torsion of Riemannian manifolds. *Adv. in Math.*, 28(3):233–305, 1978.

[Müll93] W. Müller. Analytic torsion and *R*-torsion for unimodular representations. *J. Amer. Math. Soc.*, 6(3):721–753, 1993.

[Neu88] J. Neukirch. The Beilinson conjecture for algebraic number fields. In *Beilinson’s conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 193–247. Academic Press, Boston, MA, 1988.

[Neu99] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.
D. Quillen. Higher algebraic $K$-theory. I. In Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

D.B. Ray and I.M. Singer. $R$-torsion and the Laplacian on Riemannian manifolds. Advances in Math., 7:145–210, 1971.

J. Schlüter. Lokal filtrierte Vektorbündel als geometrische Interpretation der algebraischen $K$-Theorie. PhD thesis, Freiburg, http://www.freidok.uni-freiburg.de/volltexte/8162/, 2011.

G. Segal. Classifying spaces and spectral sequences. Inst. Hautes Études Sci. Publ. Math., (34):105–112, 1968.

B. Shipley. $HZ$-algebra spectra are differential graded algebras. Amer. J. Math., 129(2):351–379, 2007.

J. Simons and D. Sullivan. Axiomatic characterization of ordinary differential cohomology. J. Topol., 1(1):45–56, 2008.

J. Simons and D. Sullivan. Structured vector bundles define differential $K$-theory. In Quanta of maths, volume 11 of Clay Math. Proc., pages 579–599. Amer. Math. Soc., Providence, RI, 2010.

R.M. Switzer. Algebraic topology—homotopy and homology. Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)].

F. Waldhausen. Algebraic $K$-theory of spaces. In Algebraic and geometric topology (New Brunswick, N.J., 1983), volume 1126 of Lecture Notes in Math., pages 318–419. Springer, Berlin, 1985.

C. Weibel. The $K$-book: an introduction to algebraic $K$-theory. http://www.math.rutgers.edu/~weibel/Kbook.html.