Modeling for Non-Markovian Quantum Systems

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Abstract—This brief presents an augmented Markovian system framework that can be applied to model non-Markovian quantum systems. In this augmented system model, ancillary systems are designed to play the role of internal modes of the non-Markovian environment converting white noise to colored noise. To capture the non-classical mutual influence between a quantum system and non-Markovian environments, direct interactions between a principal system and the ancillary system are introduced. Consequently, we showed that the augmented Markovian formulation can be used to theoretically model the environment for observed non-Markovian behavior in a recent experiment on quantum dots.

Index Terms—Augmented system model, non-Markovian quantum systems, quantum colored noise, quantum dots, system realization.

I. INTRODUCTION

The control of open quantum systems has been a rapidly advancing area of quantum information technology in recent years [1]–[8], where an open quantum system refers to a quantum system interacting with an environment or other quantum systems.

Among open quantum systems, the most widely investigated class of quantum systems is that comprising of quantum systems coupled to a memoryless environment. The evolution of such systems can be described by master equations [1] in the Schrödinger picture and Langevin equations [1] or quantum stochastic differential equations [9] in the Heisenberg picture. Both mathematical models give rise to Markovian dynamics. Thus, these open quantum systems are commonly referred to as Markovian quantum systems. In addition, the Markovian quantum systems can be coupled to a field satisfying a singular commutation relation, e.g., quantum white noise [10]. These Markovian models underpin a number of successful control applications such as the design of real-time feedback control laws [11] for cooling a quantum particle [12] and stabilizing states [13] or entanglement [14], [15] of a quantum system.

However, many problems of interest involve more complicated environmental influences, which cannot be handled within the Markovian setting and require treating the environment as quantum colored noise. This necessitates the investigation of the non-Markovian quantum systems [17]–[23]. A non-Markovian quantum system is a quantum system interacting with an environment with memory effects. To describe the dynamics of the non-Markovian quantum systems involving quantum colored noise, several models have been developed. For example, such models include non-Markovian Langevin equations where the non-Markovian effects are embedded in a memory kernel function [24] and master equations where a time-varying damping function characterizes the non-Markovian damping processes [1]. However, these existing models for non-Markovian quantum systems are not compatible with quantum filtering theory [25], [26]. In addition, unlike quantum white noise, quantum colored noise does not satisfy the singular commutation relations. For that reason, non-Markovian models are difficult to use for processing quantum measurements. Once the quantum colored noise is measured, the states of the quantum system interacting with this noise will be demolished.

A standard approach in classical control systems analysis and design is whitening of the colored noise by introducing additional dynamics so as to model the system with the non-Markovian effects of colored noise as an augmented system driven by white noise, where the system does not affect the colored noise. Similar ideas have been explored for quantum systems as well. A pseudo-mode method was proposed for effectively simulating the non-Markovian effects by using a Monte Carlo wave function [28], [29], which was applied to model the energy transfer process in photosynthetic complexes [30]. Also, the dynamics of non-Markovian quantum systems can be described by using a hierarchy equation approach [31], which has been applied to the indirect measurement of a non-Markovian quantum system [32]. An augmented system approach has been applied to obtain a quantum filter for quantum systems interacting with non-classical fields using a field-mediated connection method in a situation where the non-Markovian system therein does not introduce backaction on the environment [33].

In this brief, we present a general and systematic augmented Markovian system framework. We show that
this framework can be applied to model the non-Markovian quantum systems. To capture the effects of the non-Markovian environment, we introduce ancillary systems to augment the principal system of interest, which is realized by linear open quantum systems. Compared to the principal system, the augmented system model is defined on an augmented Hilbert space. Also, we introduce a spectral factorization method to determine the structure of the linear ancillary systems to ensure that their fictitious output has a power spectral density that is identical to that of the non-Markovian environment under consideration. Nevertheless, while these elements of our model follow the classical system modeling approach, the proposed model reflects distinctively quantum features in that the quantum plant and its non-Markovian environment mutually influence each other. This feature distinguishes quantum system-environment interactions from the classical case where the classical colored noise disturbs a plant but not vice versa [34]. To account for this special feature of non-Markovian quantum systems, in the proposed approach, the proposed model reflects distinctively quantum elements of our model follow the classical system modeling method [33], we connect them by a direct interaction mediated method [33], we connect them by a direct interaction.

II. GENERAL AUGMENTED MARKOV SYSTEM FRAMEWORK

In this section, we will introduce the structure of the general augmented Markovian system framework by using an \((S, L, H)\) description, where the internal energy, the couplings to the environment, and the scattering process of the environmental field for a quantum system are captured by a Hamiltonian \(H\), a coupling operator \(L\), and a scattering matrix \(S\), respectively. With this description, the corresponding general dynamical equations, including the quantum stochastic differential equations and the master equation, can be obtained. Without loss of generality, we will not specify the operators in the description so as to obtain general dynamics of the augmented system. We refer the reader to [40] for the details about the \((S, L, H)\) description of the Markovian quantum systems.

A. \((S, L, H)\) Description of the Augmented Markovian System Model

In the augmented system, we first introduce a principal system. We describe the principal system as

\[
G_p = (I, L_p, H_p)
\]

where \(H_p\) is the Hamiltonian of the principal system. The symbol \(L_p\) is the coupling operator vector of the principal system with respect to a probing field defined on a Fock space \(\mathcal{F}_p\). The coupling operator \(L_p\) is to allow the principal system to be probed for measurement, by shining an input field through probing channels and observing the output of the probing field. This operator does not describe how the system is coupled with the environment. Both the operators \(H_p\) and \(L_p\) are defined on a system Hilbert space \(\mathcal{H}_p\).

Moreover, we introduce the ancillary system as the environment of the principal system, which is characterized by the compact \((S, L, H)\) notation

\[
G_a = (I, L_a, H_a)
\]

with a Hamiltonian operator \(H_a\) and a collection of coupling operators with respect to ancillary quantum white noises, combined into a coupling operator vector \(L_a\). As noted earlier, the scattering matrix is assumed to be identical. As the ancillary system is driven by the quantum white noise, we define an output operator \(c\) on the Hilbert space \(\mathcal{H}_a\) that carries the response of the ancillary system to the quantum white noise. Note that the ancillary system evolves on a Hilbert space \(\mathcal{H}_a \otimes \mathcal{F}_a\), where the ancillary system and the corresponding white noise are defined on the Hilbert space \(\mathcal{H}_a\) and the Fock space \(\mathcal{F}_a\), respectively.

We have described the principal system and the ancillary system, and thus, the next problem is how to connect these two systems to capture complicated dynamics in some quantum system, for example, the nonclassical mutual influence between the system and its environment in a non-Markovian quantum system. Different from the existing field-mediated method [33], we connect them by a direct interaction Hamiltonian

\[
H_{pa} = i(e^+\z - \z^+e)
\]

where the operator vector \(e\) of the ancillary system and \(\z\) is a coupling operator vector defined on the Hilbert space \(\mathcal{H}_p\) of the principal system.

Using the general quantum feedback network theory [40], we write the total system including the principal quantum system (1) and the ancillary system model (2) for the non-Markovian environment interacting via the Hamiltonian (3) in a compact \((S, L, H)\) form as

\[
G_T = \left(I, \left(\begin{array}{c} L_a \\ L_p \end{array}\right), H_p + H_a + H_{pa}\right).
\]

The augmented system \(G_T\) is defined on the tensor product Hilbert space \(\mathcal{H}_p \otimes \mathcal{H}_a \otimes \mathcal{F}_p \otimes \mathcal{F}_a\). Since this augmented system only interacts with quantum white noise fields, namely, with the ancillary white noise field and the probing field, the overall system is Markovian. However, the principal subsystem is not Markovian since it interacts with the ancillary system.

B. Quantum Stochastic Differential Equations for the Augmented System

With the \((S, L, H)\) description (4), the Heisenberg picture stochastic differential equation of the evolution operator \(U(t)\) for the augmented system (4) can be directly obtained as

\[
\begin{align*}
\text{d}U(t) &= \left(-i(H_p + H_a + H_{pa})dt - \frac{1}{2}L_p^\dagger L_p dt + \frac{1}{2}L_p^\dagger L_a dt + dB_p^\dagger(t)L_p - L_p^\dagger dB_p(t) + dB_a^\dagger(t)L_a - L_a^\dagger dB_a(t)\right) U(t)
\end{align*}
\]
where $dB_p$ and $dB_a$ are the quantum infinitesimal increments for the probing field of the principal system and the noise process of the ancillary system, respectively. Then, using (5), the evolution of an augmented system operator $X'$ can be defined as $X'(t) = U^*(t)X'U(t)$ that satisfies the quantum stochastic differential equation

$$dX'(t) = -i[X'(t), H(t)]dt + (L_{L_p}(t)[X'(t)] + L_{L_a}(t)[X'(t)])dt$$

$$+ ((X'(t), c^\dagger(t)z(t)) + [c^\dagger(t)c(t), X'(t)])dt$$

$$+ dB_{L_p}(t)[X'(t), L_p(t)] + [L_{L_p}^*(t), X'(t)]dB_p(t)$$

$$+ dB_{L_a}(t)[X'(t), L_a(t)] + [L_{L_a}^*(t), X'(t)]dB_a(t)$$

(6)

where $H(t) = U^*(t)(H_p + H_a)U(t)$ and other time-varying operators are obtained in the same way as $X'(t)$.

Note that any operator of the augmented system $X'$ can be written as $X' = X_p \otimes X_a$; i.e., a tensor product of a principal system operator $X_p$ and an ancillary system operator $X_a$. Thus, the generator for the augmented system can be expressed as

$$G_T = G_p(X_p) \otimes X_a + X_p \otimes G_a(X_a) - i[X', H_{pa}]$$

(7)

where

$$G_p(X_p) = -i[X_p, H_p] + L_{L_p}(X_p)$$

(8)

$$G_a(X_a) = -i[X_a, H_a] + L_{L_a}(X_a)$$

(9)

are the generators for the principal system and the ancillary system, respectively.

In particular, for $X' = X_p \otimes I$, i.e., a principal system operator, (6) reduces to

$$dX_p(t) = -i[X_p(t), H_p(t)]dt + L_{L_p}(t)(X_p(t))$$

$$+ (c^\dagger(t)[X_p(t), z(t)] + [c^\dagger(t), X_p(t)]c(t))dt$$

$$+ dB_{L_p}(t)[X_p(t), L_p(t)] + [L_{L_p}^*(t), X_p(t)]dB_p(t)$$

(10)

with $H_p(t) = U^*(t)H_pU(t)$. When $X' = I \otimes X_a$, i.e., an operator of the ancillary system, we have

$$dX_a(t) = -i[X_a(t), H_a(t)]dt + L_{L_a}(t)(X_a(t))dt$$

$$+ ([X_a(t), c^\dagger(t)]z(t) + z^\dagger c(t)][c(t), X_a(t)])dt$$

$$+ dB_{L_a}(t)[X_a(t), L_a(t)] + [L_{L_a}^*(t), X_a(t)]dB_a(t)$$

(11)

with $H_a(t) = U^*(t)H_aU(t)$. Here, we have used a conventional notation of quantum mechanics where $X_p \otimes I$ and $I \otimes X_a$ as $X_p$ and $X_a$, respectively.

One can see from (10) that due to the direct coupling term induced by the ancillary system in the second line of (10), the principal system will not behave as a Markovian quantum system when it is coupled to the ancillary system. Indeed, the ancillary system operator $c(t)$ of the ancillary field process can be treated as an input into the principal system generated by the ancillary system. Also, it can be seen from (11) that the evolution of the ancillary system operator $X_a$ depends on the principal system operator $z$, which shows that the principal system acts back on the ancillary system. This explains the non-Markovian nature of the environment model.

C. Master Equation for the Augmented System

Alternatively, the dynamics of the augmented system can be described using a master equation. In particular, it is convenient to use master equations for the principal system involving nonlinear dynamics, e.g., a qubit system. This will be shown in Section IV.

For the augmented system (4), the master equation for the density matrix of the augmented system $\rho(t)$ takes the form

$$\dot{\rho}(t) = -i[H_p + H_a, \rho(t)] + [c^\dagger z, \rho(t)] + [\rho(t), z^\dagger c]$$

$$+ L_{L_p}^*(\rho(t)) + L_{L_a}^*(\rho(t))$$

(12)

where the adjoint of the Lindblad superoperator $L_L^*$ is calculated as $L_L^*(\rho(t)) = (1/2)[L^\dagger(t), L(t)] + (1/2)[L(t), L^\dagger(t)]$. The first three terms on the right-hand side of (12) determine the internal dynamics of the augmented system, and the other terms are induced by the ancillary white noise and probing field processes. Once again, we observe that the state evolution of the augmented system is Markovian since future values of the density matrix $\rho(t)$ only depend on the present density matrix. One can also obtain the density matrix $\rho_p(t)$ of the principal system as a partial trace with respect to the ancillary system

$$\rho_p(t) = \text{tr}_{a}[\rho(t)]$$

(13)

The detailed calculation of the partial trace $\text{tr}_{a}[\cdot]$ can be found in [1].

III. AUGMENTED MODEL FOR NON-MARKOVIAN SYSTEMS

In Section II, we have presented a general augmented Markovian system framework that can represent a quantum system involving complicated dynamics. As mentioned, due to the direct interaction we introduced, the ancillary system in the augmented model can induce the non-Markovian dynamics to the principal system. In this section, we target to represent a non-Markovian quantum system by using the augmented Markovian system model.

A. Augmented System Representation for Non-Markovian Systems

A non-Markovian quantum system is a quantum system in an environment with memory and is typically regarded as a quantum system disturbed by quantum colored noise [see Fig. 1(a)]. The quantum system is disturbed by the quantum colored noise arising in the non-Markovian environment, which is characterized by a power spectral density $S(\omega)$. This power spectral density indicates that in the environment, there exist interacting internal modes. These modes and their interactions are symbolically represented as dashed circles and dashed straight lines in Fig. 1(a), respectively. In the existing studies [16], [24], [37], the non-Markovian quantum systems can be described by an integral–differential Langevin equation, where the power spectral density of the environment is embedded in a memory kernel function.
In this brief, we model the non-Markovian quantum system by using the augmented Markovian system model where we represent the system in an augmented Hilbert space. A schematic of the augmented system is shown in Fig. 1(b). This approach resembles the classical approach to modeling colored noise signals using shaping filters where the shape of the spectrum is related to internal modes of the filter. We represent the system in the non-Markovian environment as the principal system where we require their Hamiltonian to be identical. When the non-Markovian quantum system is probed by a field, we can characterize the probing process by using the coupling operator. The coupling mechanism of environment–system interactions, backaction on the environment can be explained using a direct interaction (3) that can capture the energy exchanges between the non-Markovian system and its environment.

As opposed to the classical case, we show that the non-Markovian system behavior and nonclassical system backaction on the environment can be explained using a direct coupling mechanism of environment–system interactions, within an augmented Markovian system picture. In addition, due to the ancillary system and the direct interactions, the principal component of the augmented Markovian system corresponding to the original system will retain its non-Markovian features.

B. Linear Quantum Systems Models for Non-Markovian Environments With Rational Power Spectral Densities

We now demonstrate that the proposed modeling of non-Markovian systems is sufficiently rich in the sense that for a broad class of environment power spectral densities $S(s)$, an ancillary system can be constructed whose characteristics match $S(s)$. Specifically, we show that when the environment has a rational power spectral density (expressed as a rational function of frequency), the corresponding ancillary system can be realized within the class of linear quantum systems. Clearly, when $S(s)$ is not rational but can be approximated by a rational power spectral density, an approximating ancillary linear quantum system model can be constructed. From a practical perspective, such an approximation is often sufficient and leads to meaningful results as will be demonstrated in Section IV.

First, let us consider a linear quantum system comprised of $n$ harmonic oscillators interacting with $m$ channels of quantum white noise fields and obtain an expression of the associated power spectral density. Since our aim is to represent a non-Markovian environment in a form of an ancillary system and such environments do not generate energy, we restrict attention to the class of linear quantum systems whose quantum stochastic differential equations only involve annihilation operators.

Now we recall the Heisenberg picture equations for a linear annihilation only system. Specifically, the Hamiltonian of such a linear system has the form

$$H_a = a \dagger \Omega a$$

where $a = [a_1, a_2, \ldots, a_n]^T$ is a column vector of annihilation operators with the annihilation operator $a_j$ as its $j$th component and $a^T = [a_1^T, a_2^T, \ldots, a_n^T]$ is the corresponding row vector of creation operators. These operators are defined on the common Hilbert space $h_a$ and satisfy the singular commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk}, \quad [a_j, a_k] = 0, \quad j, k = 1, \ldots, n.$$  

The diagonal and non-diagonal elements of the Hermitian matrix $Ω \in \mathbb{C}^{n \times n}$ represent the internal angular frequencies and the couplings between the harmonic oscillators, respectively. Also, the system is coupled with a white noise field, and the corresponding coupling operator $L_a$ of such a linear annihilation only system with respect to the quantum white noise fields $b_a(t)$ can be expressed as

$$L_a = N_a a$$

with a matrix $N_a \in \mathbb{C}^{m \times n}$. Also, according to our standing assumption, an identity scattering matrix is assumed.

With these Hamiltonian and coupling operators, the evolution operator $\tilde{U}(t)$ of the linear ancillary system satisfies

$$d\tilde{U}(t) = \left(-\frac{1}{2}L_a^\dagger L_a + ia \dagger \Omega a\right)dt$$

$$+ dB_a^t(t)L_a - L_a^t dB_a(t)\right)\tilde{U}(t)$$
where $dB_a$ is the quantum infinitesimal increment for the white noise field process. Then, the annihilation operators $a(t) = \hat{U}(t) a \hat{U}(t)$ of the system evolve according to the linear quantum stochastic differential equation

$$da(t) = F_a a(t) dt + G_a dB_a(t)$$

where $F_a = -i\Omega - (1/2)N_0^a N_a$ and $G_a = -N_a^\dagger$.

Our result concerning the modeling of a colored noise environment in terms of linear systems of the form (19) is summarized in the following theorem. Since the non-Markovian quantum systems normally involve only one kind of colored noise, for simplicity, we restrict attention to single input ancillary systems; that is, $m = 1$, and the corresponding spectral density $S(s)$ is scalar. The extension to the matrix case is quite straightforward, as will be seen from the proof.

**Theorem 1:** Suppose that the power spectral density $S(s)$ of an environment colored noise process is rational and satisfies $S(s) = S^\dagger(-s)$, $S(s)$ has no poles on the imaginary axis, $\lim_{t \to \infty} S(s) = 0$, and $S(0 \omega) \geq 0$ for all $\omega$. Then, there exists a linear quantum system in the form of (19) and a matrix $K_a \in \mathbb{C}^{1 \times n}$, such that the process

$$c(t) = K_a a(t)$$

has the desired power spectral density $S(s)$.

Note that although the operator (20) is expressed in terms of operators of the system (19), unlike the input defined in [10], it is not an output field of the system available for quantum measurement. However, it represents a physical quantity whereby the ancillary system can interact with other quantum systems.

The Proof of Theorem 1 will be given later in this section. Before proving the theorem, it is instructive to compute the power spectral density of the operator (20). Since the system (19) is to represent the internal modes of the environment, its dynamics can be assumed to start from a long time ago. Formally, this means that the initial time $t_0$ when the systems (19) and (20) were at rest is $-\infty$. Hence, $c(t)$ can be expressed as

$$c(t) = \int_{-\infty}^{t} \Xi_a(t-\theta) dB_a(\theta)$$

where

$$\Xi_a(t) = \begin{cases} K_a e^{-F_a t} G_a, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

is the inverse Laplace transform of the $r \times m$ transfer function matrix

$$\Gamma(s) = K_a (s I - F_a)^{-1} G_a.$$  (23)

That is, $c(t)$ is analogous to the stationary response of a linear system with the transfer function matrix $\Gamma(s)$ to a white noise input. Using definition (14), its power spectral density can be computed to be

$$\int_{-\infty}^{+\infty} \langle c^\dagger(t + \tau) c(t) \rangle e^{-\frac{s}{2} \tau} d\tau = \Gamma^\dagger(-s) \Gamma(s)$$

where $\Gamma^\dagger(-s) = \Gamma^\dagger(-s^*)$ is the adjoint of the transfer function $\Gamma(s)$. The calculation is analogous to the calculation of the power spectral density for an output of a classical linear system driven by a white noise input [27].

It follows from (24) that the spectral factorization method can be employed to obtain a linear quantum system representation of the environment with a positive rational spectral density $S(s)$.

**Proof of Theorem 1:** Given a power spectral density $S(s)$, we will determine the corresponding Hamiltonian (15) and the coupling operator (17) that define the linear quantum system (19), (20). First, we observe that according to [39, Ths. 5 and 7], $S(s)$ can be factorized as in (24) with a stable transfer function

$$\Gamma(s) = \frac{\beta_n s^{n-1} + \cdots + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$  (25)

Next, a stable transfer function $\Gamma(s)$ of the form (25) has a state-space realization with matrices

$$F_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad G_0 = \begin{bmatrix} \vdots \\ 0 \end{bmatrix}, \quad K_0 = \begin{bmatrix} -\beta_0, & \cdots, & -\beta_{n-1} \end{bmatrix}$$

i.e., $\Gamma(s) = K_0 (s I - F_0)^{-1} G_0$ (see [27]). Since such a realization is Hurwitz and controllable, the Lyapunov equation

$$F_0 P + P F_0^\dagger + G_0 G_0^\dagger = 0$$

for this realization has a unique and invertible solution $P > 0$ [27]. Hence, we can find a factorization for the inverse of $P$, i.e., $P^{-1} = T^\dagger T$, and thus, (27) can be reexpressed as

$$T F_0 T^{-1} + (T^{-1})^\dagger F_0^\dagger T^\dagger + T G_0 G_0^\dagger T^\dagger = 0.$$  (28)

Define

$$F_a = T F_0 T^{-1}, \quad G_a = T G_0, \quad K_a = K_0 T^{-1}.$$  (29)

Substituting this into (28), we obtain

$$F_a + F_a^\dagger + G_a G_a^\dagger = 0.$$  (30)

Equation (30) shows that the new realization (29) obtained from (26) by applying the coordinate transformation $T$ satisfies the physical realizability condition of [38]. Also, it follows from [38] that we can obtain expressions for the Hermitian matrix $\Omega$ and the coupling operator $L_a$ as:

$$\Omega = \frac{i}{2} (F_a - F_a^\dagger), \quad L_a = -G_a G_a^\dagger.$$  (31)

As an example, let us apply the result of Theorem 1 to a Lorentzian power spectral density of the form

$$S_0(\omega) = \frac{\frac{\gamma_0^2}{4} \frac{2\gamma_0}{\omega^2}}{\frac{\gamma_0^2}{4} + (\omega - \omega_0)^2}$$

which commonly arises in solid-state systems [41].
Corollary 1: The power spectral density (32) of the Lorentzian noise can be realized by a single-mode linear quantum system

\[
\begin{align*}
\dot{a}_0(t) &= -\left(\frac{\gamma_0}{2} + i\omega_0\right) a_0(t)dt - i\sqrt{2}\theta B_a(t) \\
\dot{c}_0(t) &= -\frac{\sqrt{\gamma_0}}{2} a_0(t)
\end{align*}
\]

with the annihilation operator \(a_0\).

Proof: The Lorentzian spectrum (32) can be factorized as in (24) with

\[
\Gamma_0(s) = -\frac{\gamma_0}{s + i\omega_0 + \frac{\delta^2}{4}}.
\]

The order of the denominator of this transfer function is one, i.e., \(n = 1\). This means that the ancillary system can be realized by a single-mode linear quantum system. Then, the matrices \(F_0, G_0, K_0,\) and \(T\) reduce to scalars. From (26), we obtain

\[
F_0 = -\left(\frac{\gamma_0}{2} + i\omega_0\right), \quad G_0 = 1, \quad K_0 = \frac{\gamma_0}{2}.
\]

Substituting (35) into the Lyapunov equation (28), we have \(T^T = \gamma_0\), which can be solved to give

\[
T = -\sqrt{\gamma_0}.
\]

According to (29), the realizability of the system (19) with

\[
F_a = -\left(\frac{\gamma_0}{2} + i\omega_0\right), \quad G_a = -\sqrt{\gamma_0}, \quad K_a = -\frac{\sqrt{\gamma_0}}{2}
\]

satisfies the physical realizability condition (30). Hence, (33) corresponds to a physically realizable linear quantum system. Indeed, from (31), the Hamiltonian and the coupling operator of this system can be obtained as

\[
H_a = \omega_0 a_0^\dagger a_0, \quad L_a = \sqrt{\gamma_0} a_0
\]

where \(a_0\) is the annihilation operator of the system.

Remark 1: The central frequency of the Lorentzian power spectral density (32) determines the angular frequency \(\omega_0\) of the ancillary system, and the bandwidth of the power spectral density determines the system damping rate \(\gamma_0\) with respect to the white noise field.

IV. APPLICATION TO AN EXPERIMENT ON A QUANTUM DOT SYSTEM

In some recent experiments involving solid-state systems, the Markovian system models cannot completely explain some of the observed phenomena. For example, in an experiment for effectively measuring a double quantum dot qubit in a superconducting waveguide resonator [35], [36], the experimental data on the broadened resonator linewidth under suitable parameters disagree with the calculation based on a Markovian system model. This discrepancy in [35] and [36] was predicted to be caused by colored noise. In this section, we assume that the discrepancies in the experiment are caused by colored noise and apply our augmented system model to explore the effect of the colored noise assumption.

A block diagram of the quantum dot system in [35] and [36] is shown in Fig. 2. In the Markovian system model considered in [35] and [36], a double quantum dot qubit is directly coupled to a superconducting waveguide resonator (i.e., a cavity), which can be described by a Jaynes–Cummings Hamiltonian as

\[
H_d = \left(v_0 - v_d\right)a_+^\dagger a_r + \frac{v_{qb} - v_d}{2}\sigma_z + g_c\sin\theta(a_+^\dagger \sigma_- + \sigma_+ a_r)
\]

\[
+ \frac{\epsilon}{2}(a_+^\dagger + a_r)
\]

where the angular frequency of the resonator is \(v_0 = 6.775\ \text{GHz}\) and the frequency of the qubit is \(v_{qb} = (4\mu^2 + \delta^2)^{1/2}\) with \(\mu = 4.5\ \text{GHz}\). Here, \(\delta\) is the detuning frequency between the qubit states, which can be varied. The coupling strength \(g_c\) is \(0.05 \times 2\pi\ \text{GHz}\) between the resonator and the qubit can be modulated via a sine function \(\sin\theta = (2\mu/(4\mu^2 + \delta^2)^{1/2})\). The annihilation and creation operators for the resonator are denoted as \(a_r\) and \(a_r^\dagger\), respectively. The symbols \(v_d\) and \(\epsilon\) represent the frequency and amplitude of the driving field, which is a weak and fixed strength field (see [36] for more details).

Furthermore, the qubit is coupled with one dissipative channel and one dephasing channel that are characterized by coupling operators \(\tilde{\gamma}_-\) and \(\tilde{\gamma}_+\), where \(\tilde{\gamma}_- = \sin^2\theta\tilde{\gamma}_- + \cos^2\theta\tilde{\gamma}_+\) and \(\tilde{\gamma}_+ = \cos^2\theta\tilde{\gamma}_- + \sin^2\theta\tilde{\gamma}_+\) with \(\tilde{\gamma}_- = 3.3 \times 2\pi\ \text{GHz}\) and \(\tilde{\gamma}_+ = 100 \times 2\pi\ \text{MHz}\). In addition, the cavity is probed by a field; the corresponding coupling operator is \((\kappa_+)^{1/2}a_r\) with \(\kappa_+ = 2.6 \times 2\pi\ \text{MHz}\). Hence, the dynamics of this system obey the master equation

\[
\dot{\rho}(t) = -i[H_d, \rho(t)] + \mathcal{L}_{\kappa_+}^{\star}\rho(t)
\]

where \(\rho(t)\) is the density matrix for the resonator and qubit system.

With this model, the transmission amplitude through the cavity is calculated as \(\langle a_r \rangle = \text{tr}[a_r \rho(t)]\), where \(\rho(t)\) represents the steady-state solution of (40). The shift of the resonance frequency \(\Delta\nu_{0j}\) and the broadened linewidth of the cavity \(\kappa^\star\) can be obtained from the square of the transmission amplitude, which are plotted as red lines in Fig. 3(a) and (b), respectively. Compared to the experimental data plotted as blue dots, the shift of the resonance frequency curve in Fig. 3(a) is in
good agreement, but the broadened linewidth curve in Fig. 3(b) shows a large discrepancy. This discrepancy is potentially caused by colored noise as conjectured in [35] and [36].

To explore the reason for the discrepancy in the broadened linewidth curves and obtain better curve matching, an augmented system model is utilized to describe the system dynamics. We assume that the dissipative channel of the quantum dot system is a colored noise channel. However, since the spectrum of the colored noise is unknown for this quantum dot system, it is difficult to realize the ancillary system in the augmented system by using the spectral factorization method. From Corollary 1, we know that the Lorentzian noise can be generated using a single-mode linear factorization method. From Corollary 1, we know that the Lorentzian noise can be generated using a single-mode linear factorization method. From Corollary 1, we know that the Lorentzian noise can be generated using a single-mode linear factorization method. From Corollary 1, we know that the Lorentzian noise can be generated using a single-mode linear factorization method. From Corollary 1, we know that the Lorentzian noise can be generated using a single-mode linear factorization method.

The block diagram for the model with two ancillary systems is shown in Fig. 2. This modified model with refined parameters can be described as follows. The Hamiltonian of this modified model can be written as

$$H_d = H_d + \sum_{j=1,2} \left( \frac{i\sqrt{\kappa_j}}{2} (\sigma_j a_j + a_j^\dagger \sigma_j) + (\nu_j - \nu_d) a_j^\dagger a_j \right)$$

where $a_1$ and $a_2$ and $a_1^\dagger$ and $a_2^\dagger$ denote the annihilation and creation operators for the ancillary systems, respectively, and the respective angular frequencies of the two ancillary systems are $\nu_1 = 6.775$ GHz and $\nu_2 = 1.2$ GHz. The coupling strengths between the ancillary systems and the quantum dot are $\kappa_1 = 0.5\tilde{\gamma}_-$ and $\kappa_2 = 1.125\tilde{\gamma}_-$, respectively. The coupling operators of the ancillary systems with respect to quantum white noise fields are $\sqrt{\gamma_1 a_1}$ and $\sqrt{\gamma_2 a_2}$ with $\gamma_1 = 35$ MHz and $\gamma_2 = 20$ GHz, respectively. Thus, the dynamics of the augmented system can also be described by a master equation

$$\dot{\rho}(t) = -i[H_d, \rho(t)] + \mathcal{L}_d^{\sigma_{\rho}}(\rho(t)) + \sum_{j=1,2} \mathcal{L}_d^{\sqrt{\gamma_j a_j}}(\rho(t)) + \mathcal{L}_d^{\sigma_{\rho}}(\tilde{\rho}(t))$$

where $\tilde{\rho}(t)$ is the density matrix of the augmented system.

The transmission amplitude through the cavity is now $(a_t)^\dagger = \text{tr}(a_t \tilde{\rho}(t))$, where $\tilde{\rho}(t)$ is the steady-state solution of (42); thus, with the modified model, the shift of the resonance frequency and the broadened linewidth of the cavity can be obtained as well.

After several trials using the two-ancillary model, we obtained a broadened linewidth that matches the experimental data significantly closer than the Markovian model (see the green lines in Fig. 3). At the same time, the curve for the shift of the resonance frequency deviates only slightly from the experimental data. The spectrum of the colored noise generated by the two ancillary systems is shown in Fig. 4; it has a double Lorentzian shape. One Lorentzian spectrum is sharp with an identical frequency to the cavity, and the other one is very broad with a center frequency far away from that of both the resonator and the quantum dot. These results indicate that the discrepancies in [35] and [36]
can indeed be attributed to colored noise. Possibly even better results could be achieved if more ancillary systems were utilized. However, the computation time increases considerably in this case due to the dimensions of the augmented system.

V. CONCLUSION

In this brief, we have presented an augmented Markovian system model that can represent non-Markovian quantum systems. Also, a spectral factorization approach has been used to systematically represent a non-Markovian environment by means of linear ancillary quantum systems. Importantly, we show that the non-Markovian dynamics of a system can be interpreted as a result of direct interactions between the system and an ancillary system. Next, the proposed augmented Markovian system model has been applied to explore the structure of unknown colored noises in an experiment involving a resonator and quantum dot system.

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