The refractive index of reciprocal electromagnetic media

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Abstract

We study the electromagnetics of media described by identical inhomogeneous relative dielectric and magnetic tensors, $\varepsilon = \mu$. Such media occur generically as spatial transformation media, i.e. electromagnetic media that are defined by a deformation of space. We show that such media are completely described by a refractive index $n(r, \hat{s})$ that depends on position $r$ and direction $\hat{s}$, but is independent of polarization. The phase surface is always ellipsoidal, and $n(r, \hat{s})$ is therefore represented by the radius vector to the surface of the ellipsoid. We apply our method to calculate the angular dependence of the refractive index in the well-studied cylindrical cloak and to a new kind of structurally chiral medium induced by a twist deformation. By way of a simple example we also show that media for which $\varepsilon = \mu$ do not in general preserve the impedance properties of vacuum. The implications of this somewhat surprising conclusion for the field of transformation optics are discussed.

Keywords: transformation optics, reciprocal media, cloaking

(Some figures may appear in colour only in the online journal)

1. Introduction

Spatial transformation optics relates deformations of Euclidean space to an electromagnetic medium, so that essentially any desired continuous mapping of the electromagnetic field can be achieved. The technique has been employed in a variety of contexts, most notably to propose [1–3], and then implement [4] an electromagnetic cloak, in which linear rays are distorted in such a way as to avoid a certain region of space, rather like water flowing around an obstruction in a river.

The electromagnetic medium arising from applying the spatial transformation optics algorithm is one in which the relative permittivity and permeability tensors are equal, i.e. $\varepsilon = \mu$. This can be understood by noting that transformation optics mimics the deformation of vacuum, so that in order for the transformed medium to respond to electric and magnetic fields in the same proportion as for vacuum, inevitably we have that

$$\varepsilon = \mu \equiv \kappa,$$

say. We refer to such a medium as being electromagnetically reciprocal or a ‘$\kappa$’ medium. The spatial deformation in general changes $\kappa$ from $\kappa_0$, its value in flat vacuum, to $\tilde{\kappa}$, a value that is both inhomogeneous and anisotropic. In the ideal case $\tilde{\kappa}$ will be real-valued, just like $\kappa_0$ is real for vacuum, though a perfectly lossless medium can only be approximated in practice. The required complexity can in principle be accessed through metamaterials’ technology, which precisely seeks to engineer the appropriate anisotropy and inhomogeneity via effective medium parameter values that are not found in nature. However, for most demonstrations of transformation optics to date some level of approximation has been invoked to bring the desired functionality within reach of current technology. A common approach is the so-called reduced parameter scheme [5], in which the precise medium values are replaced with ones which at least yield the desired refractive index distribution. Rays are then refracted appropriately, although light is also scattered due...
to impedance changes. In the first demonstration of the electromagnetic cloak, for example, the reduced parameter scheme was used to eliminate the spatial dependence of one of the permeability components, and the geometry restricted to TE polarization [4].

Despite these compromises, the transformation medium defined by $\kappa$ is of intrinsic interest since, from a technological viewpoint, the rapid advance of metamaterials technology brings ‘perfect’ transformation optics media closer to reality. More generally, a $\kappa$-medium can mimic gravitational curvature without the need for the enormous mass densities required to actually distort space [6]. A curious feature of electromagnetic media described by the single tensor $\kappa$ is that even if its principal values are all distinct, it is not birefringent. A detailed analysis of the geometric origin of such non-birefringence was made by Favaro et al [7] which embraced, for example, the well studied example of anisotropic unirefringence encountered in moving media [8, 9].

The purpose of this paper is to study the electromagnetic characteristics of media described by $\kappa$, and thereby to increase our understanding of generic electromagnetic media formulated through spatial transformation optics. After some mathematical preliminaries in section 2 we will show in section 3 that media described by $\kappa$ are completely characterized by a refractive index function $n(r, \hat{s})$ which depends on position $r$ and propagation direction $\hat{s}$, but is independent of polarization. The index is obtained from a simple ellipsoid construction, distinct from the usual constant energy index ellipsoid associated with birefringent media [10]. Thereby, we develop a geometrical optics of spatial transformation optics. To illustrate our approach, in section 4 we calculate the index along the rays of the traditional electromagnetic cloak, and then, in section 5, for a novel structurally chiral medium induced by a twist deformation. For this medium it is shown that it is integral lines of the Poynting vector that are modified by the deformation, the integral lines of the wave vector being unaffected.

Before concluding in section 7, we address the problem of impedance matching in spatial transformation optics in section 6. We show by a simple example that although spatial dilations of vacuum can be successfully mimicked by the polarization-independent index $n(r, \hat{s})$, in general it is not possible to preserve the polarization independence of the impedance. This has significant implications for the future of transformation optics, as it shows that perfect cloaking cannot be achieved, even in principle.

2. Preliminaries

It is common to treat the design of a cloak or other device as a coordinate transformation [1]. However, this conflicts with the notion of the coordinate invariance of any physical process. Consequently, it is more accurate mathematically to replace the notion of design by coordinate transformation with that of design by deformation, in which one physical solution on a manifold is taken into another, and coordinates are relegated to the local labelling of points.

![Figure 1](image_url)

**Figure 1.** (a) Morphism. Points in the manifold are mapped from $\mathcal{P}$ to $\varphi(\mathcal{P})$ under the mapping $\varphi$. The coordinate representation of the morphism is $\phi_{\varphi(\mathcal{P})} \circ \varphi \circ \phi_{\mathcal{P}}^{-1} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

(b) Coordinate transformation. Coordinates of point $P$ in $\mathcal{M}$ can be assigned coordinates either via $\phi : \mathcal{U} \rightarrow \mathbb{R}^3$, or via $\psi : \mathcal{V} \rightarrow \mathbb{R}^3$. Coordinate transformation is associated with the map $\psi \circ \phi^{-1} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

Consider figure 1, where the deformation $\varphi$ of a manifold $\mathcal{M}$ may be described as the mapping of $\mathcal{M}$ onto itself, i.e. as $\varphi : \mathcal{M} \rightarrow \mathcal{M}$. A point $P \in \mathcal{M}$ is thus mapped under $\varphi$ to $\varphi(P)$. A coordinate system at $P$, or chart, is a map $\phi_{\mathcal{P}}$ from a neighborhood $\mathcal{U}_{\mathcal{P}} \subset \mathcal{M}$ of $P \in \mathcal{P}$ to a neighborhood of Euclidean space $\mathcal{U}(\mathcal{P}) \subset \mathbb{R}^3$. Likewise, coordinates at $\varphi(P)$ may be described by the chart $\phi_{\varphi(\mathcal{P})} : \mathcal{U}_{\varphi(\mathcal{P})} \rightarrow \mathbb{R}^3$. The coordinate representation of $\varphi$ is therefore given by the composite map

$$\phi_{\varphi(\mathcal{P})} \circ \varphi \circ \phi_{\mathcal{P}}^{-1} : \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

as shown in figure 1(a). If $x'$ are the coordinates of $P$ under $\phi_{\mathcal{P}}$, and $\hat{x}^m$ the coordinates of $\varphi(P)$ under $\phi_{\varphi(\mathcal{P})}$, then the map of (2) is represented by the functions $\tilde{x}^m(x')$. As an example of a physical quantity that can be morphed under $\varphi$, the components $\epsilon_0^{ij}$ of the dielectric tensor $\epsilon_0$ will be taken under $\varphi$ to [11],

$$\epsilon_0^{ij} = \det \left(\frac{\partial \tilde{x}^m}{\partial x'}\right)^{-1} \frac{\partial \tilde{x}^m}{\partial x'} \frac{\partial \hat{x}^l}{\partial \tilde{x}^j} \epsilon_0^{lj},$$

where the summation convention is assumed and we denote morphed quantities with a tilde.

By contrast, a coordinate transformation relates distinct charts at $P$, and is described by the composition

$$\psi \circ \phi^{-1} : \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

as shown in figure 1(b). The transformation rule for the dielectric tensor $\epsilon_0$ between coordinate systems (i.e. $x' \rightarrow \hat{x}'$) is given by [12]

$$\epsilon_0^{ij} = \det \left(L_{ij}^l\right)^{-1} L_{ij}^l L_{lj}^l \epsilon_0^{lj},$$

with an analogous equation for transforming the permeability tensor components $\mu_0^{ij}$.
vectors used at \( \mathcal{P} \). If a coordinate basis is associated with both systems (i.e. \( \{ \mathbf{e}_i = \partial / \partial x^i \} \), \( \{ \mathbf{e}'_j = \partial / \partial x'^j \} \)) then \( L^i_j = \partial x^i / \partial x'^j \). In other bases, such as an orthonormal basis \( \{ \mathbf{e}_i \} \), wherein the basis vectors satisfy \( \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \), the \( L^i_j \) are prescribed by functions at \( \mathcal{P} \) that determine the transformation rule \( \mathbf{e}'_j = \mathbf{e}_j L^i_j \). Elements of the matrix inverse to \( L^i_j \) are denoted \( L^{i'}_{j'} \), i.e. \( L^i_j L^{i'}_{j'} = \delta_i^{i'} \). An orthonormal basis requires the additional structure of a metric on \( \mathcal{M} \). We will assume the usual Euclidean metric given in Cartesian coordinates by \( g_{ij} = \delta_{ij} \).

Transformation optics is concerned with maps of the form of (2), represented by the functions \( \tilde{x}^m(x') \). An electromagnetic medium that produces the same deformations as if flat vacuum were distorted by \( \varphi \) is one whose dielectric tensor is given by equation (3) and whose permeability tensor is analogously given by

\[
\tilde{\mu}^{lm} = \left| \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) \right|^{-1} \tilde{x}^i \frac{\partial \tilde{x}^m}{\partial x^j} \mu_0^{ij}.
\]  

(6)

In Cartesian coordinates the vacuum constitutive parameters are just \( \epsilon_0^0 = \epsilon_0 \delta^{ij} \) and \( \mu_0^0 = \mu_0 \delta^{ij} \). Since the permittivity and permeability induced by a spatial deformation are the same up to a constant, we may characterize the electromagnetic medium by a single tensor \( \kappa \) whose components are

\[
\tilde{\kappa}^{lm} = \left| \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) \right|^{-1} \tilde{x}^i \frac{\partial \tilde{x}^m}{\partial x^j} \kappa_0^{ij},
\]  

(7)

where in Cartesian coordinates \( \kappa_0^{ij} = \delta^{ij} \). We note that provided \( \varphi \) is a diffeomorphism, \( \tilde{\kappa} \) is invertible. We also note that a lossy transformed \( \kappa \)-medium results if \( \kappa_0 \) is complex-valued.

In the following, the distinctions between deformations and coordinate transformations, and between coordinate and orthonormal bases will all be important in calculating the generalized refractive index associated with spatial deformations.

It is known that the eigen-indices associated with the Fresnel equation for media formed from equation (7) are always degenerate [7], i.e. for light whose wave vector lies parallel to the unit vector \( \hat{s} \) there is just one refractive index \( n(\mathbf{r}, \hat{s}) \) which depends on position \( \mathbf{r} \) and direction \( \hat{s} \), but is independent of the field polarization. In the following section we calculate \( n(\mathbf{r}, \hat{s}) \) explicitly for a general morphism \( \varphi \).

3. The refractive index \( n(\mathbf{r}, \hat{s}) \)

When calculating the index \( n(\mathbf{r}, \hat{s}) \) it is most convenient to use Cartesian coordinates. Consider a plane wave propagating in a medium characterized by \( \kappa \). The idea of a plane wave propagating through an inhomogeneous medium is an approximation that is useful when the wavelength is much greater than the scale of the inhomogeneity of \( \kappa \). Since any

\[
\kappa = \tilde{\kappa} \text{ induced by deformation is inevitably inhomogeneous (see equation (7)) our results are only strictly valid in the geometric optics limit. If the local wave-vector of such a wave is \( \mathbf{k} \), then from the frequency domain Maxwell curl relations}
\]

\[
\mathbf{k} \times \mathbf{E} = \omega \mu_0 \kappa \cdot \mathbf{H},
\]  

(8)

\[
\mathbf{k} \times \mathbf{H} = -\omega \epsilon_0 \kappa \cdot \mathbf{E},
\]  

(9)

as illustrated in figure 2.

Momentarily, let us set the propagation direction to be the \( z \)-axis i.e. \( \mathbf{k} = kj \hat{z} \). Denoting \( \perp \) as the projection of vectors and operators onto the transverse \( x-y \) plane we find, since \( \kappa \) is invertible, that

\[
(\kappa \cdot \mathbf{E})_{\perp} = (\kappa^{-1})_{\perp} \cdot \mathbf{E}_{\perp},
\]  

(10)

\[
(\kappa \cdot \mathbf{H})_{\perp} = (\kappa^{-1})_{\perp} \cdot \mathbf{H}_{\perp}.
\]  

(11)

The Maxwell curl relations become

\[
k \times \mathbf{E}_{\perp} = \omega \mu_0 (\kappa^{-1})_{\perp} \cdot \mathbf{H}_{\perp},
\]  

(12)

\[
k \times \mathbf{H}_{\perp} = -\omega \epsilon_0 (\kappa^{-1})_{\perp} \cdot \mathbf{E}_{\perp},
\]  

(13)

where in \( x-y \) coordinates the operator \( \times : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is defined via

\[
\times = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]  

(14)

An eigen-equation can then be formed in \( \mathbf{E}_{\perp} \) as:

\[
\left[(\kappa^{-1})_{\perp}\right] \times \mathbf{E}_{\perp} = -n^2 \mathbf{E}_{\perp},
\]  

(15)
where \( n = k (\omega^2 \varepsilon_0 \mu_0)^{-1/2} \). In fact, it is easy to show that since \( \kappa \) is symmetric
\[
\left[ (\kappa^{-1})_\perp \right]^2 = -\det (\kappa^{-1})_\perp I,
\]
where \( I \) is the \( 2 \times 2 \) identity. Hence the refractive index \( n \) associated with the wave vector \( \mathbf{k} \) is given by
\[
n = \left[ \det (\kappa^{-1})_\perp \right]^{-1/2}.
\]
Although this key result has been calculated in Cartesian coordinates, it is valid in any orthonormal basis. It relates the refractive index for light propagating in a particular direction in a transformation medium to the deformation \( \varphi \) (see equations (2), (7) and (17)). Equation (17) is also valid for a complex \( \kappa \)-medium, in which case the imaginary part of \( n \) corresponds to absorption.

Aligning \( \mathbf{k} \) along the \( z \)-axis requires generally two rotations (say \( \phi \) followed by \( \theta \)—see figure 3), so that
\[
\kappa = R_z(\theta) R_x(\phi) \kappa_c R_x(-\phi) R_z(-\theta),
\]
where \( \kappa_c \) denotes the pre-rotated representation of \( \kappa \) and
\[
R_x(\phi) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
and
\[
R_z(\theta) = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}.
\]

Now since \( \kappa_c \) is symmetric it will be diagonal in an appropriately oriented Cartesian system. If we choose the local Cartesian axes to be along the principal axes of \( \kappa_c = \text{diag}(\kappa_1, \kappa_2, \kappa_3) \), then setting \( C_\theta = \cos \theta \) etc, \((\kappa^{-1})_\perp \) is straightforwardly calculated to be
\[
(\kappa^{-1})_\perp = \begin{pmatrix} \kappa_1^{-1} - C_\theta^2 C_\phi^2 + \kappa_2^{-1} S_\phi^2 + \kappa_3^{-1} S_\phi^2 & C_\theta S_\phi C_\phi (\kappa_2^{-1} - \kappa_1^{-1}) \\ C_\theta S_\phi C_\phi (\kappa_2^{-1} - \kappa_1^{-1}) & \kappa_1^{-1} S_\phi^2 + \kappa_2^{-1} C_\phi^2 \end{pmatrix}
\]
From equation (17) it is then found that
\[
n(\theta, \phi) = \left[ \kappa_1^{-1} - \kappa_2^{-1} C_\theta^2 + \kappa_3^{-1} S_\phi^2 \left( \kappa_1^{-1} S_\phi^2 + \kappa_2^{-1} C_\phi^2 \right) \right]^{-1/2}.
\]
Setting \( n_x = n S_\theta S_\phi, n_y = n S_\theta C_\phi, n_z = n C_\theta \) the above equation can be rewritten as:
\[
\frac{n_x^2}{n_1^2} + \frac{n_z^2}{n_3^2} + \frac{n_y^2}{n_2^2} = 1,
\]
where \( n_1^2 = \kappa_2 \kappa_3, n_2^2 = \kappa_3 \kappa_1, n_3^2 = \kappa_1 \kappa_2 \). For the lossless case where all the principal values are real, equation (22) becomes an ellipsoid as illustrated in figure 4. It represents the phase surface, and the refractive index, for arbitrary polarization, is equal to the radius vector to the surface of the ellipsoid. It represents a quite distinct index ellipsoid construction to the standard one associated with the constant energy surface \( W = \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \) [10, 13]. There, the two polarizations and their associated indices for a given wave vector are given by the orientation and length of the sectional ellipse orthogonal to \( \mathbf{k} \). The phase surface in standard birefringent media, where \( \mu \) is proportional to the identity and \( \varepsilon \) is tensorial, can self-intersect [10], supporting conical refraction along the direction of such ‘diabolical’ points [14]. By contrast, we have shown that in electromagnetically reciprocal media (\( \kappa \)-media) the phase surface is always
ellipsoidal and consequently conical refraction cannot be supported in transformation media. Note that electromagnetically \( n_0(r, \mathbf{s}) \) completely describes the medium. Given this function\(^1\) the electromagnetic medium can be reconstructed by finding its three maximal radii \((n_1, n_2, n_3)\), each of which occurs along a principal axis, and then calculating the principal constitutive parameters as

\[
\kappa_1 = \frac{n_2 n_3}{n_1}, \quad \kappa_2 = \frac{n_3 n_1}{n_2}, \quad \kappa_3 = \frac{n_1 n_2}{n_3},
\]

(23)

An interesting question is whether the map \( \psi \circ \varphi \circ \psi^{-1} \) can be reconstructed from a knowledge of the above local principal values \((\kappa_1(r), \kappa_2(r), \kappa_3(r))\), together with the three parameters that specify the orientation at \( r \) of the local principal axes \((\Phi(r), \Theta(r), \Psi(r), \text{say})\). Since \( \kappa \) is symmetric, the specification of equation (7) results in six differential equations in terms of the six known quantities \([\kappa_1(r), \kappa_2(r), \kappa_3(r), \Theta(r), \Phi(r), \Psi(r)]\). Solving these throughout \( \mathcal{M} \) reconstructs a coordinate representation of \( \varphi \) up to some constant function on \( \mathcal{M} \). Hence, up to a constant, knowledge of the refractive index function \( n(r, \mathbf{s}) \) is equivalent to knowledge of the morphism \( \varphi \).

4. Cylindrical cloak

It is instructive to apply the previous results to the cylindrical cloak [4], which expands the origin to a circle of radius \( a \), compressing the disc of radius \( b \) into an annulus of inner radius \( a \) and outer radius \( b \). In cylindrical polar coordinates the deformation is described via

\[
\tilde{r} = \left(1 - \frac{a}{b}\right)r + a, \quad \tilde{\theta} = \theta, \quad \tilde{z} = z.
\]

(24)

Transforming the Cartesian representation of \( \kappa \) (i.e. \( \kappa^\theta = \delta^\theta \)) to cylindrical polar coordinates (see equation (5)) we obtain \( \kappa^\theta = \text{diag}(r, r^{-1}, r) \). Then applying equation (7) to the deformation given by equation (24) yields

\[
\tilde{\kappa}^{rr} = \tilde{r}
\left(1 - \frac{a}{b}\right)
\]
\[\tilde{\kappa}^{\theta\theta} = \tilde{r}^{-1}
\left(1 - \frac{a}{b}\right)^{-1} \]
\[\tilde{\kappa}^{zz} = \tilde{r}
\left(1 - \frac{a}{b}\right)
\left(1 - \frac{a}{b}\right)^{-2}.
\]

(25)

The orthonormal basis in polar coordinates is \([\partial_r, r^{-1}\partial_\theta, \partial_z]\), so that \( L_r^r = 1, L_\theta^\theta = r, L_z^z = 1 \) and, according to the transformation rule analogous to equation (5),

\[
\tilde{\kappa}^{rr} = \left(1 - \frac{a}{b}\right) \equiv \kappa_r,
\]
\[
\tilde{\kappa}^{\theta\theta} = \left(1 - \frac{a}{r}\right)^{-1} \equiv \kappa_\theta,
\]
\[
\tilde{\kappa}^{zz} = \left(1 - \frac{a}{r}\right)^2 \equiv \kappa_z,
\]

(26)

where, for notational simplicity, \( \tilde{r} \) has been replaced by \( r \). These principal values yield the principal indices according to equation (17) as

\[
n_r = \left(\kappa_r\kappa_\theta\right)^{1/2} = \left(1 - \frac{a}{b}\right)^{-1},
\]
\[
n_\theta = \left(\kappa_\theta\kappa_z\right)^{1/2} = \left(1 - \frac{a}{r}\right)^{-1},
\]
\[
n_z = \left(\kappa_r\kappa_z\right)^{1/2} = 1.
\]

(27)

The only refractive index component that varies in space inside the cloak is \( n_\theta \), which depends only on \( r \). Choosing \( a = 1 \) and \( b = 2 \) and restricting to the \( x-y \) plane, the index ellipses at different points inside the cloak are illustrated in

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1 Note \( \kappa : S^2 \times \mathcal{M} \rightarrow \mathbb{R} \); i.e. it is not a function on \( \mathcal{M} \).
where $dx = (dx^2 + dy^2)^{1/2}$. The other point to note is that the inclusion of isotropic absorption simply results in an exponential decay in the field intensity that is the same along any given geodesic.

5. Twist deformation—a new kind of structurally chiral medium

The above formalism can also be used to design a new kind of structurally chiral medium where rays parallel to the $z$-axis starting at a point in the plane $z = 0$ are twisted so that they form helices about the $z$-axis for $z > 0$. Consider the following transformation, expressed in Cartesian coordinates as

\[
\begin{pmatrix}
  \tilde{x} \\
  \tilde{y} \\
  \tilde{z}
\end{pmatrix} = R_z(Kz) \begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  \cos Kz & \sin Kz & 0 \\
  -\sin Kz & \cos Kz & 0 \\
  0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix},
\]

where $K$ is a constant.

Still in Cartesians, we then readily calculate from equation (7) that the $\kappa$-medium is specified by

\[
\kappa = \begin{pmatrix}
  1 + a^2 & ab & a \\
  ab & 1 + b^2 & b \\
  a & b & 1
\end{pmatrix},
\]

where $a = -K(x \sin Kz - y \cos Kz)$ and $b = -K(x \cos Kz + y \sin Kz)$. The eigenvalues and corresponding eigen-indices are

\[
k_1 = 1,
\]
\[
k_2 = \beta + (\beta^2 - 1)^{1/2},
\]
\[
k_3 = \beta - (\beta^2 - 1)^{1/2},
\]
\[
m_1 = 1,
\]
\[
m_2 = \left[\beta - (\beta^2 - 1)^{1/2}\right]^{1/2},
\]
\[
m_3 = \left[\beta + (\beta^2 - 1)^{1/2}\right]^{1/2},
\]

where $\beta = 1 + K^2(x^2 + y^2)/2$. Note that all indices throughout the medium are independent of $z$. However, the principal axes of the ellipsoid vary, both within the $x$-$y$ plane, and with $z$. The fact that $n_1n_2n_3 = 1$ indicates that the map of equation (29) is volume preserving.

Figure 7 shows the evolution of the eigen-directions of $\tilde{\kappa}$ at $(x, y) = (1, 1)$ for $K = 0.3$. The principal directions associated with $\tilde{\kappa}$ are calculated, and to each principal direction an arrow of length equal to the corresponding refractive index is plotted. The direction associated with the index $n_1 = 1$ is always radial in the $x$-$y$ plane. The other two directions are associated with an index that is $< 1$ (respectively $> 1$), which points in the same (contra-) direction to the twist, though out of the $x$-$y$ plane. Together the three directions are mutually orthogonal. The triad at other points the same distance from the axis can be adduced by rotation.

Figure 8 shows the evolution of light propagating along an eigenvector, in this case that associated with the smallest refractive index. The trajectories shown result from the local propagation direction in the plane $z = 0$ lying along the eigen-direction associated with $n = 0.8$. The resulting trajectories are helical and the figure shows the tumbling evolution of the index ellipsoid as it travels along one of these trajectories.

As well as the radial direction in the $x$-$y$ plane, there is another direction in which $n = 1$. From equation (30) we have that

\[
\tilde{\kappa}^{-1} = \begin{pmatrix}
  1 & 0 & -a \\
  0 & 1 & -b \\
  -a & b & 1 + a^2 + b^2
\end{pmatrix}.
\]

For propagation along the axis of the twist (i.e. along the $z$-axis) we see that

\[
(\tilde{\kappa}^{-1})_z = \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix},
\]

from which $n = 1$ follows from equation (17). An axial monochromatic plane wave incident from vacuum to the medium occupying the half-space $z > 0$ has an electric and magnetic field for $z < 0$ given by

\[
\mathbf{E} = \begin{pmatrix}
  E_{x0} \\
  E_{y0} \\
  0
\end{pmatrix} e^{ikz}, \quad \mathbf{B} = c^{-1} \begin{pmatrix}
  -E_{y0} \\
  E_{x0} \\
  0
\end{pmatrix} e^{ikz},
\]

where $c = (\epsilon_0 \mu_0)^{-1/2}$ and the time dependence is implicit. Inside the medium the electric field is (for derivation see appendix)

\[
\mathbf{E}(z > 0) = \begin{pmatrix}
  \cos Kz & \sin Kz & 0 \\
  -\sin Kz & \cos Kz & 0 \\
  d & f & 1
\end{pmatrix} \begin{pmatrix}
  E_{x0} \\
  E_{y0} \\
  0
\end{pmatrix} e^{ikz},
\]

where $d = -K(x \sin Kz + y \cos Kz)$ and $f = K(x \cos Kz - y \sin Kz)$. The magnetic field inside the medium turns out to be

\[
\mathbf{H}(z > 0) = (\epsilon_0 \mu_0)^{-1} \begin{pmatrix}
  \cos Kz & \sin Kz & 0 \\
  -\sin Kz & \cos Kz & 0 \\
  d & f & 1
\end{pmatrix} \begin{pmatrix}
  -E_{x0} \\
  E_{y0} \\
  0
\end{pmatrix} e^{ikz}.
\]
The Poynting vector inside the medium, defined as \( \mathbf{S} = \Re (\mathbf{E} \times \mathbf{H}^*) \), is then calculated as

\[
\mathbf{S}(\mathbf{r}) = \frac{\mu_0 c}{2} \left( \begin{array}{c} \frac{\partial}{\partial x} \\
 \frac{\partial}{\partial y} \\
 1 
\end{array} \right)
\]

In other words the Poynting vector is tangent to the helical curve resulting from the morphing of a straight line parallel to the z-axis. These are the helical curves shown in figure 8.

We emphasise the distinction between the medium discussed here and previously studied structurally chiral media (SCM). The most commonly studied SCM consists of a medium whose dielectric tensor is expressed as \( \mathbf{\varepsilon}_{\text{ref}}(K) \mathbf{\varepsilon}_{\text{ref}}(K) \) where \( \mathbf{\varepsilon}_{\text{ref}} \) has distinct principal values. Such a medium has been widely studied both in the context of sculptured thin films [15–21], and in cholesteric liquid crystals [22]. The progressive rotation of the principal axes of a birefringent medium produces a Bragg grating that for axial propagation reflects one circular polarization while transmitting the other [23]. Plane waves, whether transmitted or reflected, follow linear paths.

However, for the case of the structurally chiral medium considered here, the pre-transformed medium is vacuum and there is no birefringence. Anisotropy is induced as a result of the helical transformation. As shown above, the energy flow associated with axial propagation follows a helical path as shown in figure 8, while the integral lines of the wave-vector are straight lines parallel to the z-axis. The medium described here is also distinct from the ‘field rotator’ implemented by Chen et al [24], and from an optically active medium.

Although difficult to manufacture, the medium discussed here would have interesting properties. Since the transformation transports the electromagnetic field, an image input to a slab of the proposed medium would emerge rotated, but with the polarization preserved. This is in contrast to standard methods of optically rotating an image (e.g. a dove prism [10]), which do not preserve polarization in general.
6. Impedance anisotropy

In this section we consider whether the constraint $\epsilon = \mu$ implies that all plane waves propagating in the $\kappa$ medium experience the same impedance as vacuum. We will show by way of a simple example that this is not the case. Consider the trivial morphism that dilates one spatial direction

$$\mathbf{r} \rightarrow \lambda \mathbf{r}$$

for any transformation optics design, and cannot be reduced parameter scaling. Since every transformation optics device is built out of similar spatial compressions/expansions that vary in degree at each point, no transformation device can preserve the impedance properties of vacuum.

$$\hat{\kappa} = \text{diag}(\lambda^{-1}, \lambda^{-1}, \lambda).$$

Consider an off-axis ray or plane wave propagating in the above $\kappa$-medium with $s = (\sin \theta, 0, \cos \theta)$. Rotating so that the $z$-direction becomes the temporary propagation direction we have that

$$\hat{\kappa} = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix}.$$

(41)

where $c = \cos \theta$ and $s = \sin \theta$. Hence the transverse medium properties are given by

$$\left(\hat{\kappa}^{-1}\right)_\perp = \begin{pmatrix} c^2 \lambda + s^2 \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

(42)

and the index seen by the ray along $\theta$ (irrespective of polarization) is given by

$$n(\theta) = \sqrt{\det \left(\hat{\kappa}^{-1}\right)_\perp} = \left(\lambda^2 \cos^2 \theta + \sin^2 \theta\right)^{1/2}.$$

(43)

This index behaviour is largely unremarkable; along the direction of dilation ($\theta = 0$) it is just $\lambda$, and orthogonal to this direction ($\theta = \pi/2$) the vacuum value $n = 1$ is maintained. However, despite the fact that $\epsilon = \mu$, the impedance behaviour of the medium is in general polarization dependent.

For a field propagating along the $z$-direction, the impedance is independent of polarization and equal to its vacuum value $\eta_0 = (\mu_0/\epsilon_0)^{1/2}$, but for a field propagating along the $x$-direction we have according to equation (12) that

$$\begin{pmatrix} E_x \\ E_z \end{pmatrix} = \frac{\omega \eta_0}{k_0} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} H_x \\ H_z \end{pmatrix}.$$

(44)

and

$$\begin{pmatrix} E_x \\ E_z \end{pmatrix} = \eta_0 \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} H_x \\ H_z \end{pmatrix}.$$

(45)

So if the electric field points along $y$ the impedance is $\lambda^{-1}$ times its vacuum value, while if the field points along $z$, the impedance is $\lambda$ times its vacuum value. It is worth noting that this polarization dependence of the impedance is intrinsic to any transformation optics design, and cannot be fixed by reduced parameter scaling. Since every transformation optics device is built out of similar spatial compressions/expansions that vary in degree at each point, no transformation device can preserve the impedance properties of vacuum.

7. Conclusion

In this paper we have analysed electromagnetically reciprocal media for which $\epsilon = \mu = \kappa$ that arise canonically as the electromagnetic media that replicate a morphing of flat space. By clearly distinguishing between a morphing of space and a coordinate transformation, we emphasise the intrinsic, coordinate-free nature of transformation optics, and avoid the possible error of associating a transformation medium with a coordinate transformation such as a switch between Cartesian and polar coordinates.

We showed that the electromagnetics of a $\kappa$-medium can be described in terms of a refractive index function $n(\mathbf{r}, \hat{s})$, which can be represented by an ellipsoidal phase surface. A plane wave propagating in the $\hat{s}$ direction sees just one refractive index, independent of polarization. If the $\kappa$-medium is inhomogeneous, then the index at $\hat{s}$ direction was calculated. A new kind of structurally chiral medium resulted when a $\kappa$-medium was designed by morphing space with a progressive twist along the $z$-axis. It was found that the wave vector of an axially incident plane wave to such a medium is undeviated, while the Poynting vector morphs exactly as prescribed by the deformation.

Finally, we showed that whatever the deformation, preserving the impedance properties of vacuum is not generally possible. Propagation transverse to the deformation will inevitably result in a dependence of the impedance on polarization. In other words, perfect cloaking is impossible.

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Appendix

Here we derive equations (36) and (37).

The components of the electromagnetic field tensor are, in Cartesian coordinates

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_z \\ E_x & 0 & B_z \\ E_z & -B_z & 0 \end{pmatrix},$$

(46)

where the Greek indices now run over the four dimensions of spacetime, i.e. $\alpha = 0, 1, 2, 3$. After deformation the new components of the field tensor are

$$\tilde{F}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu} F_{\alpha\beta}.$$

(47)
where, for the deformation of equation (29)

$$\frac{\partial x^\alpha}{\partial \xi^\mu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos K^2 & -\sin K^2 & d \\
0 & \sin K^2 & \cos K^2 & f \\
0 & 0 & 0 & 1
\end{pmatrix},$$

(48)

and

$$\bar{d} = -K(\xi \sin K^2 + \bar{\eta} \cos K^2)$$

and

$$\bar{f} = K(\xi \cos K^2 - \bar{\eta} \sin K^2).$$

After inserting equation (48) into equation (47) we find, for the plane wave of equation (35), that after dropping tildes on $(\xi, \bar{\eta}, \zeta)$ in the resulting expression, the morphed electric field is given by

$$E = \begin{pmatrix}
\cos K^2 & \sin K^2 & 0 \\
-\sin K^2 & \cos K^2 & 0 \\
d & f & 1
\end{pmatrix} \begin{pmatrix}
E_{0x} \\
E_{0y} \\
e^{ikz}\zeta
\end{pmatrix},$$

(49)

where

$$d = -K(x \sin K^2 + y \cos K^2)$$

and

$$f = K(x \cos K^2 - y \sin K^2).$$

This is just equation (36). For the morphed magnetic $B$-field we obtain

$$B = e^{-1} \begin{pmatrix}
\cos K^2 & \sin K^2 & 0 \\
-\sin K^2 & \cos K^2 & 0 \\
d & f & 1
\end{pmatrix} \begin{pmatrix}
E_{0x} \\
E_{0y} \\
e^{ikz}\zeta
\end{pmatrix},$$

(50)

The fields of equations (49) and (50) are readily shown to satisfy $\nabla \times \mathbf{E} = ick_0 \mathbf{B}$.

The components of the excitation tensor are, in Cartesian coordinates

$$G^{\alpha\beta} = \begin{pmatrix}
0 & D_x & D_y & D_z \\
-D_x & 0 & H_z & -H_y \\
-D_y & -H_z & 0 & H_x \\
D_z & H_y & -H_x & 0
\end{pmatrix}.$$  

(51)

The transformation law is given by [12]

$$G^{\mu\nu} = \begin{vmatrix}
\frac{\partial \xi^\alpha}{\partial x^\mu}
\end{vmatrix}^{-1} \frac{\partial x^\alpha}{\partial \xi^\nu} G^{\alpha\beta},$$

(52)

where, for the deformation of equation (29)

$$\frac{\partial \xi^\alpha}{\partial x^\mu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos K^2 & \sin K^2 & a \\
0 & -\sin K^2 & \cos K^2 & b \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{vmatrix}
\frac{\partial x^\alpha}{\partial \xi^\mu}
\end{vmatrix} = 1,$$  

(53)

and $a$ and $b$ were defined after equation (30). We then find that the morphed displacement vector is given by

$$\mathbf{D} = e_0 \begin{pmatrix}
\cos K^2 & \sin K^2 & 0 \\
-\sin K^2 & \cos K^2 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
E_{0x} \\
E_{0y} \\
e^{ikz}\zeta
\end{pmatrix},$$

(54)

and the magnetic field

$$\mathbf{H} = (c\mu_0)^{-1} \begin{pmatrix}
\cos K^2 & \sin K^2 & 0 \\
-\sin K^2 & \cos K^2 & 0 \\
d & f & 1
\end{pmatrix} \begin{pmatrix}
-E_{0x} \\
-E_{0y} \\
e^{ikz}\zeta
\end{pmatrix},$$

(55)

which is just equation (37). It is readily checked that $\mathbf{D} = e_0 k \cdot \mathbf{E}$ and $\mathbf{B} = \mu_0 k \cdot \mathbf{H}$, where $k$ is given by equation (30).

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