Discrete isoperimetric problems in spaces of constant curvature

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Abstract
The aim of this paper is to prove isoperimetric inequalities for simplices and polytopes with \( d + 2 \) vertices in Euclidean, spherical and hyperbolic \( d \)-space. In particular, we find the minimal volume \( d \)-dimensional hyperbolic simplices and spherical tetrahedra of a given inradius. Furthermore, we investigate the properties of maximal volume spherical and hyperbolic polytopes with \( d + 2 \) vertices with a given circumradius, and the hyperbolic polytopes with \( d + 2 \) vertices with a given inradius and having a minimal volume or minimal total edge length. Finally, for any \( 1 \leq k \leq d \), we investigate the properties of Euclidean simplices and polytopes with \( d + 2 \) vertices having a fixed inradius and a minimal volume of its \( k \)-skeleton. The main tool of our investigation is Euclidean, spherical and hyperbolic Steiner symmetrization.

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1 | INTRODUCTION

The classical discrete isoperimetric inequality, stating that among convex \( n \)-gons of unit area in the Euclidean plane \( \mathbb{E}^2 \), the ones with minimal perimeter are the regular ones, was already observed by Zenodorus [3]. Since that time, many similar problems, often called isoperimetric problems,
have appeared in the literature [6], asking about the extremal value of a geometric quantity in a certain family of convex polytopes. In particular, results about convex polyhedra in Euclidean 3-space with a given number of vertices or faces and having a fixed inradius or circumradius can be found, for example, in [8] or [1].

Much less is known about convex polytopes with a given number of vertices in the $d$-dimensional Euclidean space $\mathbb{E}^d$. Among the known results for polytopes with $d + 2$ vertices, which in most problems seems to be the first interesting case, we can mention the paper [5] of Böröczky and Böröczky Jr. finding the minimum surface area polytopes of unit volume in this family, [15] of Klein and Wessler determining the maximum volume polytopes with unit diameter and [12] of G. Horváth and the second named author about maximum volume polytopes with unit circumradius. We note that both [15] and [12] contain partial results for polytopes with $d + 3$ vertices (see also [16]).

In the authors’ knowledge, in hyperbolic and spherical spaces there are just a few examples of solutions of isoperimetric problems. In particular, Peyerimhoff [18] proved that among hyperbolic simplices inscribed in a given ball, the regular ones have maximal volume, and Böröczky [4] proved the analogous statement for spherical simplices.

Our goal in this paper is to examine isoperimetric problems in $d$-dimensional Euclidean, hyperbolic and spherical space for polytopes with $d + 1$ or $d + 2$ vertices. Our main tool is the Steiner symmetrization of convex sets, whose hyperbolic and spherical counterparts were introduced by Schneider [21] and Böröczky [4], respectively.

In Section 2, we introduce the necessary notation and prove some lemmas needed in our investigation, describing also the properties of Steiner symmetrization. In Section 3, we present our main results. In Section 4, we apply our methods to prove statements for measures generated by rotationally symmetric density functions. Finally, in Section 5, we present some additional remarks and pose some open problems.

2 | PRELIMINARIES

Throughout the paper, we denote the $d$-dimensional Euclidean, hyperbolic and spherical space by $\mathbb{E}^d$, $\mathbb{H}^d$ and $\mathbb{S}^d$, respectively. We regard $\mathbb{S}^d$ as the unit sphere centered at the origin $o$ of the space $\mathbb{E}^{d+1}$.

Let $M \in \{ \mathbb{E}^d, \mathbb{H}^d, \mathbb{S}^d \}$. For any two points $p, q \in M$, which are not antipodal if $M = \mathbb{S}^d$, we denote the shortest geodesic connecting $p$ and $q$ by $[p, q]$, and call it the closed segment with endpoints $p$ and $q$; the distance $d(p, q)$ of $p$ and $q$ is defined as the length of $[p, q]$. If $M = \mathbb{E}^d$, then for any point $x$ its distance from the origin $o$ is the Euclidean norm of $x$, which we denote by $||x||$.

We say that $K \subset M$ is convex if for any two points $p, q \in K$, we have $[p, q] \subseteq K$; here, if $M = \mathbb{S}^d$, we also assume that $K$ is contained in an open hemisphere of $\mathbb{S}^d$. Furthermore, for any $X \subseteq M$, where $X$ is contained in an open hemisphere if $M = \mathbb{S}^d$, the intersection of all convex sets containing $X$ is a convex set, called the convex hull of $X$, and denoted by $\text{conv}(X)$. A compact, convex set with non-empty interior is called a convex body. A convex polytope in $M$ is the convex hull of finitely many points. A face of a convex polytope $P$ is the intersection of the polytope with a supporting hyperplane of $P$, 0-dimensional faces of $P$ are called vertices of $P$, and we denote the vertex set of $P$ with $V(P)$.

Let $P \subset M$ be a $d$-dimensional convex polytope. The radius of a largest ball contained in $P$ is called the inradius of $P$, and is denoted by $\text{ir}(P)$, and the radius of a smallest ball containing $P$ is called the circumradius of $P$, denoted by $\text{cr}(P)$. For any $1 \leq k \leq d - 1$, the $k$-skeleton of $P$ is
the union of its $k$-dimensional faces; we denote it by $\text{skel}_k(P)$. For any $1 \leq k \leq d$, we denote $k$-dimensional volume by $\text{vol}_k(\cdot)$, and call $\text{vol}_k(\text{skel}_k(P))$ the total $k$-content of $P$, or if $k = 1$, then the total edge length of $P$.

Note that in the projective ball model of $\mathbb{H}^d$, hyperbolic line segments are represented by Euclidean line segments, implying that hyperbolic convex sets are exactly those represented by Euclidean convex sets. Furthermore, applying central projection onto a tangent hyperplane of $\mathbb{S}^d$, the same holds in spherical space as well. This shows that Remarks 1–4, which are well known in $\mathbb{E}^d$, hold in $\mathbb{H}^d$ and $\mathbb{S}^d$ as well. We note that for $\mathbb{M} = \mathbb{E}^d$, Remarks 1 and 2 can be found in [11, section 6.1] (see also [12]).

**Remark 1.** Every $d$-dimensional simplicial polytope $P \subset \mathbb{M}$ with $d + 2$ vertices is the convex hull of two simplices $S_1, S_2$ with dimensions $\dim S_1 = k$ and $\dim S_2 = d - k$ with some $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$, respectively, such that $S_1 \cap S_2 = \{p\}$ for some point $p$ in the relative interior of both $S_1$ and $S_2$. Furthermore, every $d$-dimensional polytope $Q \subset \mathbb{M}$ is the convex hull of $r$ points and a $(d - r)$-dimensional simplicial polytope with $d - r + 2$ vertices, for some $0 \leq r \leq d - 2$.

We note that in Remark 1, $S_1$ and $\text{conv}(S_2 \cup Q)$ are two simplices which intersect at a point which is a relative interior point of $S_1$ and a point of $\text{conv}(S_2 \cup Q)$.

**Remark 2.** The combinatorial structure of a convex polytope with a few vertices can be described, for example, by a Gale diagram of the polytope. If $P$ is a $d$-dimensional convex polytope with $d + 2$ vertices, and $P = \text{conv}(S_1 \cup S_2 \cup \{p_1, ..., p_r\})$, where $\dim S_1 = k_1$, $\dim S_2 = k_2$ with $k_1 + k_2 + r = d$ and $k_1, k_2 \geq 1$, then $P$ can be represented by a multiset consisting of the points $-1, 0, 1$ on the real line, with multiplicities $k_1 + 1, r, k_2 + 1$. Here, the elements equal to $-1$ represent the vertices of $S_1$, those equal to $1$ represent the vertices of $S_2$, and those equal to $0$ represent the $p_i$. In this representation, a subset of the vertex set of $P$ is the vertex set of a face of $P$ if and only if the complement of the vertex set of $P$ is associated to points in the diagram whose convex hull contains $0$ in its relative interior. In particular, if $k_1, k_2 \geq 2$, then every pair of vertices of $P$ is connected by an edge, and the same holds for any pair apart from the vertices of $S_1$ and/or $S_2$ if $k_1 = 1$ and/or $k_2 = 1$, respectively.

**Remark 3.** Let $P = \text{conv}(S_1 \cup S_2)$ be the convex hull of two simplices with $\dim S_1 + \dim S_2 = d$ such that the subspaces generated by them intersect at a single point $p$ which belongs to $S_1 \cap S_2$, and is different from any vertex of $S_1$ and $S_2$. Then, for any vertices $q_1, q_2$ of $S_1$, there is a hyperplane $H$ containing $V(P) \setminus \{q_1, q_2\}$ and intersecting $[q_1, q_2]$.

**Proof.** Observe that $V(P) \setminus \{q_1, q_2\}$ contains $d$ points, and thus, there is a hyperplane containing it. We show that this hyperplane can be chosen in such a way that it intersects $[q_1, q_2]$. Indeed, any hyperplane $H'$ containing $V(P) \setminus \{q_1, q_2\}$ satisfies $S_2 \subset H'$, implying also $p \in H'$. Furthermore, since $V(S_1) \setminus \{q_1, q_2\} \subset H' \cap L_1$, if $p \notin \text{conv}(V(S_1) \setminus \{q_1, q_2\})$, then there is a unique hyperplane $H'$ containing $V(P) \setminus \{q_1, q_2\}$, and this hyperplane intersects $[q_1, q_2]$. If $p \in \text{conv}(V(S_1) \setminus \{q_1, q_2\})$, then the subspace spanned by $V(P) \setminus \{q_1, q_2\}$ is $(d - 2)$-dimensional, and thus, we may choose $H'$ such that it contains a point of $[q_1, q_2]$. 

**Remark 4.** Let $H$ be a hyperplane of $\mathbb{M}$. Let $Q$ be a compact, convex set in $H$ and let $[p_1, p_2]$ be a segment such that $H \cap [p_1, p_2]$ is a singleton $\{p\}$. Let $P = \text{conv}(Q \cup [p_1, p_2])$. Then, for any
plane $F$ containing $[p_1, p_2]$, $F \cap P$ is either $[p_1, p_2]$, or a triangle $\text{conv}\{q, p_1, p_2\}$, or a quadrangle $\text{conv}\{q_1, q_2, p_1, p_2\}$, where $q, q_1, q_2$ are relative boundary points of $Q$.

2.1 | Euclidean Steiner symmetrization

**Definition 1.** Let $H$ be a hyperplane in $\mathbb{E}^d$. The Steiner symmetrization $\sigma$ with respect to $H$ is the geometric transformation which assigns to any convex body $K \subset \mathbb{E}^d$ the unique compact set $K'$ symmetric to $H$ with the property that for any line $L$ orthogonal to $H$, $K \cap L$ and $K' \cap L$ are both non-degenerate segments of equal length, or singletons, or the empty set (see Figure 1). In this case we call the set $\sigma(K)$ the Steiner symmetral of $K$.

**Lemma 1.** Let $P \subset \mathbb{E}^d$ be a convex polytope with vertices $p_1, p_2, \ldots, p_n$. Assume that for some hyperplane $H' \subset \mathbb{E}^d$ intersecting $[p_1, p_2]$, we have $p_3, \ldots, p_n \in H'$. Let $H$ be the hyperplane bisecting $[p_1, p_2]$. Let $\pi : \mathbb{E}^d \to H$ denote the orthogonal projection onto $H$ and let $\sigma$ denote the Steiner symmetrization with respect to $H$. Then,

$$\sigma(P) = \text{conv}([p_1, p_2] \cup \{\pi(p_3), \ldots, \pi(p_n)\}).$$

Furthermore,

$$\text{ir}(\sigma(P)) \geq \text{ir}(P)$$

with equality if and only if $P$ is symmetric to $H$. Finally, if for some $1 \leq k \leq d - 1$, $P$ satisfies the property that for any $3 \leq i_1 \leq i_2 \leq \ldots \leq i_m$, $\text{conv}\{p_1, p_{i_1}, \ldots, p_{i_m}\}$ is a $k$-face of $P$ if and only if $\text{conv}\{p_2, p_{i_1}, \ldots, p_{i_m}\}$ is a $k$-face of $P$, then

$$\text{vol}_k(\text{skel}_k(\sigma(P))) \leq \text{vol}_k(\text{skel}_k(P)).$$

For the proof of Lemma 1 we need Lemma 2. Lemma 2 explores the properties of the so-called linear parameter systems or shadow systems, introduced by Rogers and Shephard in [20]. Lemma 2 can be found in [13], and hence, we omit its proof.
Lemma 2. Let \( p_1, \ldots, p_k, v \in \mathbb{E}^d \) and \( \lambda_1, \ldots, \lambda_k \in \mathbb{R} \). For \( i = 1, 2, \ldots, k \) and all \( t \in \mathbb{R} \), set \( p_i(t) = p_i + \lambda_i tv \) and \( S(t) = \text{conv}(p_1(t), \ldots, p_k(t)) \). Then the function \( f : \mathbb{R} \to \mathbb{R}, f(t) = \text{vol}_{k-1}(S(t)) \) is a convex function of \( t \), and if the points \( p_1, \ldots, p_k, v \) are affinely independent, then \( f \) is strictly convex.

Proof of Lemma 1. Clearly, to prove (1), it is sufficient to prove the first equality, which is the immediate consequence of Remark 4. The inequality in (2) follows from the fact that the Steiner symmetral of a ball of radius \( r \) is a ball of radius \( r \). The equality case in (2) is as follows if \( P \) is a triangle in \( \mathbb{E}^d \), and applying this observation for the sections of \( P \) with planes through \( [p_1, p_2] \) yields the statement in the general case.

Now we prove (3). Assume that \( P \) is not symmetric to \( H \). Let \( v \) be a unit normal vector of \( H \), and for any \( p_i \), where \( i \geq 3 \), let \( d_i \) denote the signed distance of \( p_i \) from \( H \) in such a way that \( v \) points towards the positive side of \( H \). We define the linear parameter system \( L(t) = \{p_i(t) : i = 1, 2, \ldots, n\} \), where \( p_1(t) = p_1 \), \( p_2(t) = p_2 \) and for \( i = 3, 4, \ldots, n \), \( p_i(t) = p_i - (1 - t)d_iv \) and set \( P(t) = \text{conv}(L(t)) \). Observe that \( P(1) = P \), \( P(-1) \) is the reflected copy of \( P \) about \( H \), and \( P(0) = \text{conv}([p_1, p_2] \cup \{\pi(p_i), i = 3, 4, \ldots, n\} \) is the Steiner symmetral of \( P \) with respect to \( H \). Note that for any value of \( t \), the points \( p_i(t) \), where \( i = 3, 4, \ldots, n \), lie in a rotated copy of \( H' \) around \( H \cap H' \). Thus, by Lemma 2, the function \( g : [-1, 1] \to \mathbb{R}, g(t) = \text{vol}_k(\text{skel}_k(P(t))) \) is strictly convex. Since \( g(-t) = g(t) \) for any \( t \in [-1, 1] \), it implies that the unique minimum of \( g \) is attained at \( t = 0 \). This readily yields (3). □

2.2 | Hyperbolic Steiner symmetrization

In the following definition, for a hyperbolic line \( L \subset \mathbb{H}^d \), the line \( L \) and the hypercycles with axis \( L \) that are contained in a plane through \( L \) are called the \( g \)-lines of \( \mathbb{H}^d \) with axis \( L \) (see [18]).

Definition 2. Let \( H \) be a hyperplane in \( \mathbb{H}^d \), and \( L \) be a line orthogonal to \( H \). The Steiner symmetrization \( \sigma \) with respect to \( H \) and with axis \( L \) is the geometric transformation which assigns to any convex body \( K \subset \mathbb{H}^d \) the unique compact set \( K' \) symmetric to \( H \) with the property that for any \( g \)-line \( L' \) with axis \( L, K \cap L' \) and \( K' \cap L' \) are both non-degenerate segments of equal length, or singletons or the empty set (see Figure 2). In this case we call the set \( \sigma(K) \) the Steiner symmetral of \( K \).

We note that hyperbolic Steiner symmetrization preserves volume [18, Proposition 8].

In the next lemmas, if \( H \) is a hyperplane in \( \mathbb{H}^d \) orthogonal to a line \( L \), the \( g \)-orthogonal projection \( \pi : \mathbb{H}^d \to H \) onto \( H \) with axis \( L \) is defined by \( \pi(p) = q \) for any \( p \in \mathbb{H}^d \), where \( q \) is the intersection point of \( H \) and the unique \( g \)-line through \( p \) with axis \( L \). Our next lemma is proved by Peyerimhoff [18, Proposition 9], and hence, we omit its proof.

Lemma 3. Let \( T = \text{conv}(p_1, p_2, p_3) \) be a triangle in \( \mathbb{H}^2 \). Let \( L \) be the line through \( [p_1, p_2] \), and let \( H \) be the bisector of \( [p_1, p_2] \). Let \( \pi : \mathbb{H}^2 \to H \) denote the \( g \)-orthogonal projection onto \( H \) with axis \( L \), and let \( \sigma \) denote the Steiner symmetrization with respect to \( H \) and with axis \( L \). Then

\[
\sigma(T) \subseteq \text{conv}(p_1, p_2, \pi(p_3)), \quad \text{and} \quad d(p_1, p_3) + d(p_2, p_3) \geq d(p_1, \pi(p_3)) + d(p_2, \pi(p_3)). \tag{4} \tag{5}
\]

Furthermore, in any of (4) or (5), we have equality if and only if \( p_3 \in H \).
Lemma 4. Let $P \subset \mathbb{H}^d$ be a convex polytope with vertices $p_1, p_2, \ldots, p_n$. Assume that for some hyperplane $H' \subset \mathbb{H}^d$ intersecting $[p_1, p_2]$, we have $p_3, \ldots, p_n \in H'$. Let $L$ be the straight line through $[p_1, p_2]$ and let $H$ be the bisector of $[p_1, p_2]$. Let $\pi : \mathbb{H}^d \to H$ denote the $g$-orthogonal projection onto $H$ with axis $L$ and let $\sigma$ be the Steiner symmetrization with respect to $H$ with axis $L$. Then,

\[ \sigma(P) \subseteq \text{conv}(\{p_1, p_2\} \cup \{\pi(p_3), \ldots, \pi(p_n)\}), \]

(6)

\[ \text{vol}_d(P) = \text{vol}_d(\sigma(P)) \leq \text{vol}_d(\text{conv}(\{p_1, p_2\} \cup \{\pi(p_3), \ldots, \pi(p_n)\})) \quad \text{and} \]

(7)

\[ \text{ir}(P) \leq \text{ir}(\text{conv}(\{p_1, p_2\} \cup \{\pi(p_3), \ldots, \pi(p_n)\})), \]

(8)

with equality in any of (6), (7) or (8) if and only if $P$ is symmetric to $H$.

Proof. Note that $P = \bigcup F(F \cap P)$, where $F$ runs over the planes containing $L$, and these intersections are mutually disjoint apart from $[p_1, p_2]$. Let $F$ be a plane with $L \subset P$. By Remark 4, $F \cap P = \text{conv}(p_1, p_2, q)$ for some relative boundary point $q$ of $\text{conv}(p_3, \ldots, p_n)$. By Lemma 3, $\sigma(F \cap P) \subseteq \text{conv}(\sigma([p_1, p_2]) \cup \pi(q))$. Thus, we have $\sigma(P) \subseteq \text{conv}(\sigma([p_1, p_2]) \cup \pi(\text{conv}(p_3, \ldots, p_n)))$. On the other hand, the relation $\pi(\text{conv}(p_3, \ldots, p_n)) \subseteq \text{conv}(\pi(p_3), \ldots, \pi(p_n))$ follows from [18, Lemma 10], which states that if $X$ is a convex set in $H$ containing the intersection point of $H$ and $L$, then $\pi^{-1}(X)$ is convex. Thus, we have

\[ \sigma(P) \subseteq \text{conv}(\sigma([p_1, p_2]) \cup \{\pi(p_3), \ldots, \pi(p_n)\}). \]

Here, by Lemma 3, there is equality if and only if all $p_i$ with $i \geq 3$ are contained in $H$. The second inequality in Lemma 4 readily follows from the first one and the fact that hyperbolic volume is invariant under Steiner symmetrization [18]. The inequality in (8), together with the equality case, follows from the fact that the Steiner symmetral of a ball of radius $r$ is a ball of radius $r$ [18, Proposition 8]. \qed
2.3 | Spherical Steiner symmetrization

**Definition 3.** Let $H$ be a hyperplane in $\mathbb{S}^d$. The *Steiner symmetrization* $\sigma$ with respect to $H$ is the geometric transformation which assigns to any convex body $K \subset \mathbb{S}^d$, disjoint from the two poles of $H$, the unique compact set $K'$ symmetric to $H$ with the property that for any open half circle $L$ orthogonal to $H$ and ending at the two poles of $H$, $K \cap L$ and $K' \cap L$ are both non-degenerate segments of equal length, or singletons, or the empty set (see Figure 3). In this case we call the set $\sigma(K)$ the *Steiner symetral* of $K$.

We observe that a convex body $K$ disjoint from the poles of $H$ is contained in an open hemisphere whose center lies in $H$; by our definition $\sigma(K)$ is contained in the same open hemisphere. Our next remark is the spherical counterpart of [18, Lemma 10].

**Remark 5.** Let $H$ be a hyperplane in $\mathbb{S}^d$ with poles $\pm p$, and let $\pi : \mathbb{S}^d \setminus \{\pm p\} \to H$ denote the orthogonal projection onto $H$. Then for any convex set $K \subset H$, the set $\pi^{-1}(K)$ is convex.

The proof of our next lemma can be found in [4].

**Lemma 5.** Let $H$ be the bisector of the segment $[p_1, p_2] \in \mathbb{S}^2$, and let $p_3$ be different from the poles of $H$. Let $\pi$ and $\sigma$ denote the orthogonal projection onto $H$ and the Steiner symmetrization with respect to $H$, respectively. Then

$$\sigma(\text{conv}\{p_1, p_2, p_3\}) \subseteq \text{conv}\{p_1, p_2, \pi(p_3)\},$$

with equality if and only if $p_3 \in H$.

**Lemma 6.** Let $P \subset \mathbb{S}^d$ be a convex polytope with vertices $p_1, p_2, \ldots, p_n$. Assume that for some hyperplane $H' \subset \mathbb{S}^d$ intersecting $[p_1, p_2]$, we have $p_3, \ldots, p_n \in H'$. Let $H$ be the bisector of $[p_1, p_2]$, and
assume that $P$ is disjoint from the poles of $H$. Let $\pi : \mathbb{S}^d \to H$ denote the orthogonal projection onto $H$, and let $\sigma$ be the Steiner symmetrization with respect to $H$. Then,

$$\sigma(P) \subseteq \text{conv}([p_1, p_2] \cup \{\pi(p_3), \ldots, \pi(p_n)\}),$$

(9)

$$\text{vol}_d(P) \leq \text{vol}_d(\sigma(P)) \leq \text{vol}_d(\text{conv}([p_1, p_2] \cup \{\pi(p_3), \ldots, \pi(p_n)\}))$$

(10)

with equality in any of (9) and (10) if and only if $P$ is symmetric to $H$.

Proof. By Remark 4 and Lemma 5, $\sigma(P) \subseteq \text{conv}([p_1, p_2] \cup \pi(P))$. On the other hand, by Remark 5, $\pi(P) \subseteq \text{conv}\{q, \pi_H(p_3), \ldots, \pi_H(p_n)\}$, where $q$ is the midpoint of $[p_1, p_2]$. Thus, we have $\sigma(P) \subseteq \text{conv}([p_1, p_2] \cup \{\pi(p_3), \ldots, \pi(p_n)\})$, where, by Lemma 5, we have equality if and only if $p_3, \ldots, p_n \subset H$, that is, if $P$ is symmetric to $H$. This, and the fact that spherical volume does not decrease under Steiner symmetrization implies (10).

\[ \square \]

3 | MAIN RESULTS

We start with Theorem 1, which was proved for $\mathbb{M} = \mathbb{E}^d$ in [12].

**Theorem 1.** Let $\mathbb{M} \in \{\mathbb{E}^d, \mathbb{H}^d, \mathbb{S}^d\}$. Let $B \subseteq \mathbb{M}$ be a ball, and let $P \subset B$ be a convex polytope with $d + 2$ vertices. Then,

$$\text{vol}_d(P) \leq \text{vol}_d(Q),$$

where $Q = \text{conv}(S_1 \cup S_2)$ for some regular simplices $S_1, S_2$ inscribed in $B$, with $\dim S_1 = k$ and $\dim S_2 = d - k$ for some $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$, and contained in mutually orthogonal subspaces of $\mathbb{M}$ intersecting at the center of $B$. Furthermore, for any $d > 0$ there is a value $\varepsilon > 0$ such that if the radius of $B$ is at most $\varepsilon$, or if $\mathbb{M} = \mathbb{E}^d$, then $k = \lfloor \frac{d}{2} \rfloor$.

Proof. By compactness argument, in the family of convex polytopes $P \subset B$ with at most $d + 2$ vertices, $\text{vol}_d(\cdot)$ attains its maximum, and any polytope with maximal volume has exactly $d + 2$ vertices. Let $P$ be such a polytope. Then, by Remark 1, $P = \text{conv}(S_1 \cup S_2) \subset B$ for some simplices $S_1, S_2$ of dimensions $k$ and $d - k$, respectively, where $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$, and the subspaces spanned by them intersect in a single common point $p$ of $S_1$ and $S_2$. Let $L_k$ (resp. $L_{d-k}$) denote the $k$-dimensional (resp. $(d - k)$-dimensional) subspace spanned by $S_1$ (resp. $S_2$). Note that without loss of generality, we may assume that every vertex of $P$ lies on the boundary of $B$.

Consider an edge $[q_1, q_2]$ of $S_1$. We show that $P$ is symmetric to the bisector $H$ of $[q_1, q_2]$.

Suppose for contradiction that $P$ is not symmetric to $H$. Let us choose a hyperplane $H'$ containing $V(P) \setminus \{q_1, q_2\}$ such that $H'$ intersects $[q_1, q_2]$, and recall that by Remark 3, such a hyperplane exists. Let $L$ denote the line through $[q_1, q_2]$. Note that since $P$ is inscribed in $B$, the bisector $H$ of $[q_1, q_2]$ passes through the center of $B$. Let $\sigma$ denote the Steiner symmetrization with respect to $H$ (or if $\mathbb{M} = \mathbb{H}^d$, with respect to $H$ and with axis $L$), and let $\pi$ denote the orthogonal projection onto $H$.

Let $P' = \text{conv}([q_1, q_2] \cup \pi(V(P) \setminus \{q_1, q_2\}))$. Then, for $\mathbb{M} = \mathbb{S}^d$ and $\mathbb{M} = \mathbb{H}^d$, by Lemmas 6 and 4, respectively, we have that $\text{vol}_d(P) < \text{vol}_d(P')$, contradicting our assumption that $P$ has maximal volume. In the remaining case, if $\mathbb{M} = \mathbb{E}^d$, then $P'$ is a convex polytope in $B$ with $(d + 2)$
vertices satisfying $\text{vol}_d(P') = \text{vol}_d(P)$ such that at least one vertex of $P'$ lies in the interior of $B$. Thus, there is a convex polytope $P'' \subset B$ with $(d + 2)$ vertices and satisfying $\text{vol}_d(P'') > \text{vol}_d(P)$, a contradiction.

We have shown that $P$ is symmetric to the bisector of the edge $[q_1, q_2]$ of $S_1$. We obtain similarly that $P$ is symmetric to any edge of $S_1$ and $S_2$, implying that $S_1$ and $S_2$ are regular simplices inscribed in $B$ contained in orthogonal subspaces such that the centers of $S_1$, $S_2$ and $B$ coincide.

The fact that if $\mathcal{M} = \mathbb{E}^d$, the volume of $\text{conv}(S_1 \cup S_2)$ is maximal if $k = \lfloor \frac{d}{2} \rfloor$ follows by a simple computation (see, for example, [12]). The existence of some $\epsilon > 0$ depending on $d$ only, such that for any ball $B$ of radius at most $\epsilon$, the volume of $\text{conv}(S_1 \cup S_2)$ is maximal if $k = \lfloor \frac{d}{2} \rfloor$ follows from the well-known observation that in ‘small scale’, hyperbolic and spherical spaces are ‘almost’ Euclidean, and can be proved by a standard limit argument.

Our next two theorems deal with the volumes of polytopes in Euclidean and hyperbolic spaces, containing a given ball.

**Theorem 2.** Let $\mathcal{M} \in \{\mathbb{E}^d, \mathbb{H}^d\}$ and let $B \subset \mathcal{M}$ be a ball. Then, among simplices containing $B$, the ones with maximal volume are the regular simplices circumscribed about $B$.

**Theorem 3.** Let $\mathcal{M} \in \{\mathbb{E}^d, \mathbb{H}^d\}$, let $B \subset \mathcal{M}$ be a ball and let $P$ be a convex polytope containing $B$ with at most $d + 2$ vertices. Then,

$$\text{vol}_d(P) \geq \text{vol}_d(Q),$$

for a convex polytope $Q = \text{conv}(S_1 \cup S_2)$ containing $B$, where $S_1, S_2$ are regular simplices with $\text{dim} S_1 = k$ and $\text{dim} S_2 = d - k$ for some $1 \leq k \leq \frac{d}{2}$, and contained in mutually orthogonal subspaces of $\mathcal{M}$ intersecting at the center of $B$. Furthermore, for any $d > 0$ there is a value $\epsilon > 0$ such that if the radius of $B$ is at most $\epsilon$, or $\mathcal{M} = \mathbb{E}^d$, then $k = \lfloor \frac{d}{2} \rfloor$, or $d = 4$ and $k = 1$.

We note that while the natural choice to solve isoperimetric problems in $\mathbb{H}^d$ or $\mathbb{S}^d$ is to use hyperbolic or spherical Steiner symmetrization, sometimes Euclidean Steiner symmetrization can also be used in these spaces. We illustrate it in the proof of Theorems 2 and 3 in hyperbolic space. The proof presented in this paper is a variant of the proof of Lemma 9 in [2].

**Proof of Theorems 2 and 3.** We start with the proof of Theorem 2.

First, assume that $\mathcal{M} = \mathbb{H}^d$. We represent $\mathbb{H}^d$ in the projective ball model. As we already remarked in Section 2, in this model the points of $\mathbb{H}^d$ correspond to the points of the open unit ball $\text{int}(B^d)$ of $\mathbb{E}^d$ centered at the origin, and a set is convex in $\mathbb{H}^d$ if and only if it is represented by a convex set in $\text{int}(B^d)$. The volume element at point $x$ in this model is

$$\frac{1}{(1+||x||^2)^{d+2}}dx.$$ Hence, if $A$ is a Borel set in $\text{int}(B^d)$, then

$$\text{vol}(A) = \int_A \frac{1}{(1+||x||^2)^{d+2}}dx,$$

where $dx$ denotes integration with respect to $d$-dimensional Lebesgue measure [19].

Let $S$ be a simplex in $\mathbb{H}^d$ containing $B$. Without loss of generality, we may assume that the center of $B$ is $o$. Observe that by compactness, $\text{vol}_d(\cdot)$ attains its minimum on the family of hyperbolic simplices in $\mathbb{H}^d$. Let $S$ be a simplex in $B$ minimizing this volume, and let $[p_1, p_2]$ be an edge of
S. Let $H$ denote the hyperplane perpendicular to the line through $[p_1, p_2]$ that passes through $o$. Let $\sigma$ denote the (Euclidean) Steiner symmetrization with respect to $H$. Then, clearly, $\sigma(S)$ is a hyperbolic simplex in $H^d$ containing $B$. Furthermore, since the density function $\frac{1}{1 + ||x||^2}$ is a strictly increasing function of $||x||$, by Fubini’s theorem we have that

$$\text{vol}_d(\sigma(S)) = \int_{\sigma(S)} \frac{1}{1 + ||x||^2} \, dx \leq \int_S \frac{1}{1 + ||x||^2} \, dx = \text{vol}_d(S),$$

with equality if and only if $\sigma(S) = S$, that is, if $S$ is symmetric to $H$. Since $S$ has a minimal hyperbolic volume on the family of simplices containing $B$, it follows that the bisector of any edge of $S$ contains $o$, and thus, $S$ is a regular simplex with $o$ as its center. We obtain also that $S$ is circumscribed about $B$, as otherwise $\text{vol}_d(S)$ is clearly not minimal. This proves Theorem 2 for $\mathbb{M} = H^d$. For $\mathbb{M} = \mathbb{E}^d$, and to obtain the inequality in Theorem 3, we can apply an analogous argument.

Now we investigate the equality part of Theorem 3. By a standard limit argument, it is sufficient to prove the statement for $\mathbb{M} = \mathbb{E}^d$, where we may assume that $B$ is the unit ball centered at the origin.

The statement clearly holds if $d = 2$, and hence, we assume that $d \geq 3$. Let $P$ be a convex polytope in $\mathbb{E}^d$ with $B \subset P$ and minimal volume. We show that then

$$\text{vol}_d(P) \leq \frac{d^{d/2}}{d!} \left( \left\lfloor \frac{d}{2} \right\rfloor \right)^{\left\lfloor \frac{d}{2} \right\rfloor + 1} \left( \left\lceil \frac{d}{2} \right\rceil \right)^{\left\lceil \frac{d}{2} \right\rceil + 1}.$$  \hspace{1cm} (11)

First, an elementary computation shows that the volume of a regular simplex in $\mathbb{E}^d$ with unit inradius is $\frac{d^{d/2}(d+1)^{d/2}}{d!}$, which is strictly smaller than the quantity in (11). Thus, in the following we consider the case that $P$ has exactly $d + 2$ vertices.

Similarly like in the proof of Theorem 2, we have that if $P$ has minimal volume, then $P = \text{conv}(S_1 \cup S_2)$ for two regular simplices $S_1$ and $S_2$, where the affine hulls of $S_1$ and $S_2$ are perpendicular and meet at the common centerpoint of $S_1$ and $S_2$. Let $1 \leq \dim S_1 = k \leq \dim S_2 = d - k \leq d - 1$. Without loss of generality, we may assume that $o$ is the center of $S_1$ and $S_2$.

We determine the edge lengths and the dimensions of $S_1$ and $S_2$. For $i = 1, 2$, let the inradius of $S_i$ be denoted by $r_i$. Let the vertices of $S_1$ be $p_1, \ldots, p_{k+1}$, and the vertices of $S_2$ be $q_1, \ldots, q_{d-k+1}$. Note that for any value of $i$, $||p_i|| = kr_i$, and $||q_i|| = (d-k)r_2$. Furthermore, since a facet of $P$ is the convex hull of a facet of $S_1$ and a facet of $S_2$, the volume of $P$ can be obtained as $(k + 1)(d - k + 1)$ times the volume of $\text{conv}\{o, p_1, \ldots, p_k, q_1, \ldots, q_{d-k}\}$. In addition, the latter volume is $d!$ times the volume of the parallelootope $P'$ spanned by the vectors $p_1, \ldots, p_k, q_1, \ldots, q_{d-k}$. Let $G$ denote the Gram matrix of these vectors, and note that for any values of $i, j$, $\langle p_i, p_j \rangle = k^2r_1^2$ and $\langle q_i, q_j \rangle = (d-k)^2r_2^2$. It is well known that $\text{vol}_d(P') = \sqrt{\det(G)}$. Thus, an elementary computation shows that $\text{vol}_d(P') = \frac{k^{d-k}r_1^d r_2^k (d-k)^{d-k+1/2}}{d!}$, which yields that

$$\text{vol}_d(P) = \frac{1}{d!} \left( \frac{k^{d-k}r_1^d r_2^k (d-k)^{d-k+1/2}}{d!} \right) (d-k)^{d-k+1/2} (k+1)(d-k+1).$$  \hspace{1cm} (12)

Note that since the unit ball centered at $o$ touches at least one facet of $P$, it touches every facet of $P$. Let $c_1$ and $c_2$ be the centers of a facet of $S_1$ and a facet of $S_2$, respectively. Then the unit ball centered at $o$ touches the segment $[c_1, c_2]$. Since the triangle $\text{conv}\{o, c_1, c_2\}$ is a right triangle with legs of lengths $r_1$ and $r_2$, respectively, we obtain that $(r_1^2 - 1)(r_2^2 - 1) = 1$. Thus, there is a real
value $t \in \mathbb{R}$ such that $r_1 = \sqrt{1+e^t}$ and $r_2 = \sqrt{1+e^{-t}}$. Substituting these expressions into (12), we obtain that $\text{vol}_d(P) = f_k(t)$, where

$$f_k(t) = \frac{1}{d!} (1 + e^t)^{k/2} (1 + e^{-t})^{(d-k)/2} k^{k+1/2} (d - k + 1).$$

We find the minimum of $f_k(t)$ for all real $t \in \mathbb{R}$ and integer $1 \leq k \leq d - 1$. Observe that for any value of $k$, $f_k(t)$ attains its minimum for some value of $t$. On the other hand, an elementary computation shows that the only real solution of the equation $f'_k(t) = 0$ is $t = \frac{d-k}{k}$, which shows that for any fixed value of $k$, $f_k(t)$ is minimal if $t = \frac{d-k}{k}$. Let $g_d(k) = f_k(\frac{d-k}{k})$. Then

$$g_d(k) = \frac{d^{d/2}}{d!} k^{(d+1)/2} (d - k)^{(d-k+1)/2} (d - k + 1).$$

We remark that for $d = 3$ we have proved the statement, and hence, we assume that $d \geq 4$. Let us extend the domain of $g_d$ to the interval $k \in [1, d - 1]$ using the same formula. Then by differentiating twice and applying algebraic transformations, we obtain that the function $\ln g_d$ is strictly convex on $[2, d - 2]$, implying that for any integer $2 \leq k < \lceil \frac{d}{2} \rceil$, we have $g_d(k) > g_d(\lceil \frac{d}{2} \rceil)$. On the other hand, an elementary computation shows that $g_d(1) \geq g_d(2)$, with equality if and only if $d = 4$. This yields the assertion.

**Corollary 1.** Let $P$ be a convex polytope in $\mathbb{E}^d$ with at most $d + 2$ vertices and unit inradius. Then,

$$\text{vol}_d(P) \leq \frac{d^{d/2}}{d!} \left( \left\lfloor \frac{d}{2} \right\rfloor \right)^{(d/2)+1/2} \left( \left\lceil \frac{d}{2} \right\rceil + 1 \right)^{(d/2)+1/2} \left( \left\lfloor \frac{d}{2} \right\rfloor + 1 \right)^{(d/2)+1/2},$$

with equality if and only if there are regular simplices $S_1, S_2$ lying in orthogonal affine subspaces that meet at the common center of both $S_1$ and $S_2$, such that one of the following holds:

- $\dim S_1 = \lfloor \frac{d}{2} \rfloor$ and $\dim S_2 = \lceil \frac{d}{2} \rceil$;
- $d = 4$, $\dim S_1 = 1$ and $\dim S_2 = 3$.

We are unable to extend Theorem 2 for $\mathbb{M} = \mathbb{S}^d$. Nevertheless, we note that the spherical version of Dowker’s theorem implies that if $\mathbb{M} = \mathbb{S}^2$, then any spherical $m$-gon of minimal area containing the spherical cap $B$ is a regular $m$-gon circumscribed about $P$ [9]. We present a proof for $\mathbb{M} = \mathbb{S}^3$ using László Fejes Tóth’s famous ‘moment lemma’, which we quote as follows (see [7]).

**Lemma 7 (Moment lemma).** Let $\rho(\tau)$ be a strictly increasing function for $0 < \tau < \frac{\pi}{2}$. Furthermore, let $p_1, p_2, \ldots, p_k$ be $k$ points of $\mathbb{S}^2$ not all of them lying on a hemisphere, and let $\tau(p)$ denote the minimal spherical distance of the point $p \in \mathbb{S}^2$ from a point of the set $\{p_1, p_2, \ldots, p_k\}$. Let $\Delta$ be an equilateral spherical triangle with vertices $\bar{p}_1, \bar{p}_2, \bar{p}_3$ and having area $\frac{2\pi}{k-2}$, and let $\bar{\tau}(p)$ denote the minimum spherical distance of $p$ from one of the vertices of $\Delta$. Then we have

$$\int_{\mathbb{S}^2} \rho(\tau(p)) \, d\omega_s \leq (2k - 4) \int_{\Delta} \rho(\bar{\tau}(p)) \, d\omega_s,$$

where $d\omega_s$ denotes integration with respect to spherical volume. Equality in (13) holds if and only if $\{p_i : i = 1, 2, \ldots, k\}$ is the vertex set of a regular tetrahedron, octahedron or icosahedron.
**Theorem 4.** Let $B \subset S^3$ be a spherical ball. Let $T \subset S^3$ be a tetrahedron containing $B$, and let $T_{\text{reg}}$ be a regular tetrahedron circumscribed about $B$. Then,

\[ \text{vol}_3(T) \geq \text{vol}_3(T_{\text{reg}}), \]

with equality if and only if $T$ is congruent to $T_{\text{reg}}$.

**Proof.** Clearly, we may assume that $T$ is circumscribed about $B$. Let $p_i$, where $i = 1, 2, 3, 4$, denote the points where a face of $T$ touches $B$. Observe that $\text{bd}(B)$ is a 2-dimensional spherical space, and that the central projections of the faces of $T$ onto $\text{bd}(B)$ coincide with the Voronoi cells of the point set $\{p_1, p_2, p_3, p_4\}$; that is, any point of the cell containing $p_i$ is not farther from $p_i$ than from any other point of the set. Furthermore, if for any $p \in \text{bd}(B)$, $\tau(p)$ denotes the minimum spherical distance from the points $p_i$, then there is a strictly increasing function $\rho(\tau)$ such that the volume of $T$ coincides with $\int_{\text{bd}(B)} \rho(\tau(p)) \, d\omega_3$. Thus, the assertion follows from Lemma 7. \(\square\)

It is well known (see, for example, [14]) that among simplices $S$ in $E^d$, the ratio of the circumradius of $S$ to its inradius is minimal if and only if $S$ is regular. Our next result is a partial generalization of this for hyperbolic and spherical spaces.

**Theorem 5.** Let $d \geq 2$. For any simplex $S$ in the hyperbolic space $H^d$,

\[ \tanh \text{cr}(S) \geq d \tanh \text{ir}(S), \]

with equality if, and only if $S$ is regular. Furthermore, for any simplex $S$ in the spherical space $S^d$ with $d \leq 3$,

\[ \tan \text{cr}(S) \geq d \tan \text{ir}(S), \]

with equality if, and only if $S$ is regular.

**Proof.** Among simplices $S$ of a given volume in $H^d$ or $S^d$, the regular ones have minimal circumradius (see [18] and [4], respectively). Furthermore, among simplices of a given volume in $H^d$, the regular ones have maximal inradius (see Theorem 2), and the same holds for $d \leq 3$ for simplices in $S^d$ (see Theorem 4, and the remark preceding it). Theorem 5 readily follows from these observations. \(\square\)

We remark that the $d = 2$ case of Theorem 5 can be found, for example, in [22]. Before stating our next result, we recall that by [18], among simplices in $H^d$ containing a given ball $B$ the regular simplices circumscribed about $B$ have minimal total edge length. In our next statement we show that there is no similar statement for spherical simplices, and prove a variant of this statement for hyperbolic polytopes with $d + 2$ vertices.

**Proposition 1.** Let $d \geq 2$, let $B$ be a ball of radius $r$ in $S^d$, and let $S_{\text{reg}}$ be a regular simplex circumscribed about $B$. We have the following.

(i) For any $d \geq 3$, if $r$ is sufficiently close to $\frac{\pi}{2}$, then there is a simplex $S$ containing $B$ and satisfying $\text{vol}_1(\text{skel}_1(S)) < \text{vol}_1(\text{skel}_1(S_{\text{reg}}))$. 

\[ \text{vol}_3(T) \geq \text{vol}_3(T_{\text{reg}}), \]

with equality if and only if $T$ is congruent to $T_{\text{reg}}$. 

**Proof.** Clearly, we may assume that $T$ is circumscribed about $B$. Let $p_i$, where $i = 1, 2, 3, 4$, denote the points where a face of $T$ touches $B$. Observe that $\text{bd}(B)$ is a 2-dimensional spherical space, and that the central projections of the faces of $T$ onto $\text{bd}(B)$ coincide with the Voronoi cells of the point set $\{p_1, p_2, p_3, p_4\}$; that is, any point of the cell containing $p_i$ is not farther from $p_i$ than from any other point of the set. Furthermore, if for any $p \in \text{bd}(B)$, $\tau(p)$ denotes the minimum spherical distance from the points $p_i$, then there is a strictly increasing function $\rho(\tau)$ such that the volume of $T$ coincides with $\int_{\text{bd}(B)} \rho(\tau(p)) \, d\omega_3$. Thus, the assertion follows from Lemma 7. \(\square\)

It is well known (see, for example, [14]) that among simplices $S$ in $E^d$, the ratio of the circumradius of $S$ to its inradius is minimal if and only if $S$ is regular. Our next result is a partial generalization of this for hyperbolic and spherical spaces.

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\[ \tanh \text{cr}(S) \geq d \tanh \text{ir}(S), \]

with equality if, and only if $S$ is regular. Furthermore, for any simplex $S$ in the spherical space $S^d$ with $d \leq 3$,

\[ \tan \text{cr}(S) \geq d \tan \text{ir}(S), \]

with equality if, and only if $S$ is regular.

**Proof.** Among simplices $S$ of a given volume in $H^d$ or $S^d$, the regular ones have minimal circumradius (see [18] and [4], respectively). Furthermore, among simplices of a given volume in $H^d$, the regular ones have maximal inradius (see Theorem 2), and the same holds for $d \leq 3$ for simplices in $S^d$ (see Theorem 4, and the remark preceding it). Theorem 5 readily follows from these observations. \(\square\)

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**Proposition 1.** Let $d \geq 2$, let $B$ be a ball of radius $r$ in $S^d$, and let $S_{\text{reg}}$ be a regular simplex circumscribed about $B$. We have the following.

(i) For any $d \geq 3$, if $r$ is sufficiently close to $\frac{\pi}{2}$, then there is a simplex $S$ containing $B$ and satisfying $\text{vol}_1(\text{skel}_1(S)) < \text{vol}_1(\text{skel}_1(S_{\text{reg}}))$. 

(ii) For any $0 < r < \frac{\pi}{2}$, if $d$ is sufficiently large, then there is a simplex $S$ containing $B$ and satisfying $\text{vol}_1(\text{skel}_1(S)) < \text{vol}_1(\text{skel}_1(S_{\text{reg}}))$.

Proof. First, we compute $\text{vol}_1(\text{skel}_1(S_{\text{reg}}))$. Let $\pi$ denote the central projection onto the tangent hyperplane $H$ of $S^d$ in $E^{d+1}$ at the center of $S_{\text{reg}}$, and note that $S_{\text{reg}}$ is contained in the open hemisphere centered at this point. Thus, $\pi(S_{\text{reg}})$ is a regular Euclidean simplex with inradius $\tan r$, implying that its circumradius is $d \tan r$. Regarding the center $c$ of $S_{\text{reg}}$ as the origin of the Euclidean $d$-space $H$ and taking the inner product of two vertices of $\pi(S_{\text{reg}})$, we have that if $p_i$ and $p_j$ are two vertices of $S_{\text{reg}}$, then the angle of the Euclidean triangle with vertices $c, \pi(p_i)$ and $\pi(p_j)$ at the vertex $c$ is $\arccos(-\frac{1}{d})$. This quantity is equal to the angle of the spherical triangle with vertices $p_i, p_j$ and $c$ at $c$. Since the length of the edges $[c, p_i]$ and $[c, p_j]$ in this triangle is $\arctan(d \tan r)$, by the spherical Law of Cosines and using trigonometric identities we obtain that the length of the edges of $S_{\text{reg}}$ is $\arccos \frac{1 - d \tan^2 r}{1 + d^2 \tan^2 r}$. Thus, $\text{vol}_1(\text{skel}_1(S_{\text{reg}}))$ is equal to $f(r, d) = \left(\frac{d+1}{2}\right) \arccos \frac{1 - d \tan^2 r}{1 + d^2 \tan^2 r}$.

We show that for any $\varepsilon > 0$, there is a simplex $S$ containing $B$ with $\text{vol}_1(\text{skel}_1(S)) < d \pi + \varepsilon$. Let $S_0$ be a regular $(d - 1)$-dimensional simplex whose total edge length is strictly less than $\varepsilon$. We assume that the spherical segment connecting the center $c'$ of $S_0$ to $c$ is perpendicular to $S_0$, and denote the length of this segment by $x$. Assume that $x < \frac{\pi}{2}$, and let $p$ be a point on the line through this segment such that $c$ is the midpoint of $[c', p]$. Observe that if $x$ is sufficiently close to $\frac{\pi}{2}$, then $S = \text{conv}(S_0 \cup \{p\})$ contains $B$. Furthermore, $\text{vol}_1(\text{skel}_1(S)) < d \pi + \varepsilon$.

Thus, to show the statement it is sufficient to show that if $d \geq 3$ is fixed and $\tan r$ is sufficiently large, or if $0 < r < \frac{\pi}{2}$ is fixed and $d$ is sufficiently large, then $f(d, r) - d \pi$ is positive, which can be checked by an elementary computation. \hfill $\square$

Before Theorems 6–8, we remark that unlike volume and surface area, the quantity $\text{vol}_k(\text{skel}_k(P))$ is not continuous in the family of $d$-dimensional convex polytopes, if $1 \leq k \leq d - 2$.

**Theorem 6.** Let $B \subset H^d$ be a ball with center $c$, and let $P$ be a convex polytope containing $B$ and with $V(P) \leq d + 2$ such that $P$ has minimal total edge length among the polytopes in $H^d$ containing $B$ and having at most $d + 2$ vertices. Then we have one of the following.

(i) There are regular simplices $S_1$ and $S_2$ with $\dim S_1 + \dim S_2 = d$ such that the subspaces spanned by $S_1$ and $S_2$ are orthogonal complements of each other, $S_1 \cap S_2 = \{c\}$, and $P = \text{conv}(S_1 \cup S_2)$.

(ii) There are perpendicular segments $S_1, S_2$ meeting at the common midpoint $q_1$ of both segments, and a regular $(d - 2)$-dimensional simplex $Q$ with center $q_2$ such that $c$ lies on the segment $[q_1, q_2]$, $Q$ is perpendicular to $[q_1, q_2]$ and $P = \text{conv}(S_1 \cup S_2 \cup Q)$.

(iii) $P$ is a regular simplex circumscribed about $B$.

**Proof.** We note that if $V(P) = d + 1$, then the statement follows from [18, Theorem 3]. Thus, we assume that $V(P) = d + 2$. By Remark 1, there are simplices $S_1$ and $S_2$ with $\dim S_i = k_i$ for $i = 1, 2$ and intersecting at a singleton which is a relative interior point of both, and there are points $p_1, p_2, \ldots, p_r$ such that $P = \text{conv}(S_1 \cup S_2 \cup \{p_1, \ldots, p_r\})$, and $k_1 + k_2 + r = d$. If $r > 0$, let $Q = \text{conv}\{p_1, \ldots, p_r\}$.

Let $[q_1, q_2]$ be any edge of $S_1$. Then there is a unique hyperplane $H'$ containing all other vertices of $P$ such that $H'$ intersects $[q_1, q_2]$ at a point $q$. Let $L$ be the line containing $[q_1, q_2]$, and let $H$
denote the bisector of \([q_1, q_2]\). Let \(\sigma\) denote Steiner symmetrization with respect to \(H\) with axis \(L\). We show that if \(P\) is not symmetric to \(H\), then \(\text{conv}(\sigma(P))\) is a convex polytope with \(\text{ir}(P) \leq \text{ir}(\text{conv}(\sigma(P)))\) and strictly smaller total edge length. Here, the first property follows from (8). Furthermore, if \(k_2 > 1\), then any pair of vertices not in \(S_1\) are connected by an edge, which, by Lemma 3 and the fact the length of a segment does not increase under \(g\)-orthogonal projection, implies the second property. Finally, assume that \(k_2 = 1\), that is, \(S_2 = [w_1, w_2]\). By Lemma 3, it is sufficient to show that \([\pi(w_1), \pi(w_2)]\) is not an edge of \(\sigma(P)\), where \(\pi : \mathbb{H}^d \to H\) denotes \(g\)-orthogonal projection onto \(H\) with axis \(L\).

Observe that \(\text{conv}(V(P) \setminus \{w_1, w_2\})\) intersects \(S_2\) in a relative interior point of \(S_2\). Let this point be \(w\). The line \(L' \subset H'\) through \(q\) and \(w\) intersects \(\text{bd}(P)\) in \(q\) and another point, which we denote by \(w'\). Then, by [18, Lemma 10] (see also the proof of Lemma 4), \(\pi(\text{conv}\{w_1, w_2, q, w'\})\) is a non-degenerate quadrangle in \(H\) with \(\text{conv}\{\pi(w_1), \pi(w_2)\}\) as a diagonal. This, as \(\pi(w_1), \pi(w_2)\) are vertices of \(\sigma(P)\), yields also that the segment connecting them is not an edge of \(\sigma(P)\), and thus, if \(P\) is not symmetric to \(H\), \(\text{vol}_1(\text{skel}_1(\sigma(P))) < \text{vol}_1(\text{skel}_1(P))\).

We have shown that \(P\) is symmetric to the bisector of every edge of \(S_1\). Note that if \(r > 0\) and \(k_1 > 1\), then \(S' = \text{conv}(S_1 \cup Q')\) is a simplex intersecting \(S_2\) at a relative interior point of \(S_2\), and thus, by the same argument, \(P\) is symmetric to the bisector of every edge of \(S_2\). Thus, if \(k_1 > 1\) or \(k_2 > 1\), then \(r = 0\), and the properties in (i) are satisfied for \(P\). Furthermore, if \(k_1 = k_2 = 1\), then we obtain similarly that \(P\) is symmetric to the bisector of \(S_1, S_2\) and every edge of \(Q\), implying the properties in (ii).

We note that the statement of Theorem 7 for simplices can be found in [13].

**Theorem 7.** For any \(1 \leq k \leq d - 1\), among simplices in \(\mathbb{E}^d\) with unit volume, the regular ones have minimal total \(k\)-content. Furthermore, for any \(1 \leq k \leq d - 1\), among the convex polytopes in \(\mathbb{E}^d\) with at most \(d + 2\) vertices and unit volume, if \(P\) has minimal total \(k\)-content, then it satisfies one of the following.

(i) There are regular simplices \(S_1, S_2\) with \(\dim S_1 + \dim S_2 = d\) such that the subspaces spanned by \(S_1\) and \(S_2\) are orthogonal complements of each other, \(S_1 \cap S_2 = \{c\}\), and \(P = \text{conv}(S_1 \cup S_2)\).

(ii) There are regular simplices \(S_1, S_2\), of dimensions \(\dim S_1 = k_1 \leq k\) and \(\dim S_2 = k_2 \leq k\) and lying in orthogonal subspaces that meet at the common center \(q_1\) of both simplices, and a regular \((d - k_1 - k_2)\)-dimensional simplex \(Q\) with center \(q_2\) such that \(Q\) is perpendicular to the segment \([q_1, q_2]\), and \(P = \text{conv}(S_1 \cup S_2 \cup Q)\).

(iii) \(P\) is a regular simplex of unit volume.

**Proof.** Let \(P\) be a simplex of unit volume in \(\mathbb{E}^d\), and let \([p_1, p_2]\) be an edge of \(S\). Let \(H\) denote the bisector of \([p_1, p_2]\), and let \(\sigma\) denote Steiner symmetrization with respect to \(H\). If \(P\) is not symmetric to \(H\), then \(\sigma(P)\) is a unit volume simplex with strictly smaller total \(k\)-content. Thus, a simplex with minimal total \(k\)-content is symmetric to the bisector of any of its edges, implying that it is regular.

Now, let \(P\) be a \(d\)-dimensional convex polytope of unit volume and having exactly \(d + 2\) vertices. Then there are simplices \(S_1, S_2\) of dimensions \(k_1\) and \(k_2\), respectively, and possibly a simplex \(Q = \text{conv}\{p_1, p_2, \ldots, p_r\}\) such that the affine hulls of \(S_1\) and \(S_2\) intersect in a singleton \(q_1\) which lies in the relative interior of both \(S_1\) and \(S_2\), and \(k_1 + k_2 + r = d\).
Consider the case that one of \( k_1 \) and \( k_2 \), say \( k_1 \), is greater than \( k \). Let \([p_1, p_2]\) be an edge of \( \text{conv}(S_1 \cup Q) \). It is easy to see that any face of \( P \) not containing \( \text{conv}(S_1 \cup S_2) \) is a simplex. Furthermore, by the properties of the Gale diagram of \( P \) described in Remark 2, for any \( k \)-element subset \( S' \) of the vertices of \( P \) with \( p_1, p_2 \notin S' \), we have that \( \text{conv}(\{p_1\} \cup S') \) is a \( k \)-face of \( P \) if and only if \( \text{conv}(\{p_2\} \cup S') \) is a \( k \)-face of \( P \). Thus, we can apply Lemma 1, and conclude that if \( P \) has minimal \( k \)-content, then it is symmetric to the bisector of \([p_1, p_2]\). Applying the same argument for all edges of \( \text{conv}(S_1 \cup Q) \) we obtain that \( \text{conv}(S_1 \cup Q) \) is a regular simplex and the affine hull of \( S_2 \) meets it at its center, showing that in this case \( Q = \emptyset \).

Using the same argument, we obtain that for any values of \( k_1 \) and \( k_2 \), if \( P \) has minimal total \( k \)-content, then it is symmetric to the bisector of every edge of \( S_1, S_2 \), and if it exists, of \( Q \). This yields the assertion.  

By the idea in the proof of Theorem 7, we obtain Theorem 8.

**Theorem 8.** For any \( 1 \leq k \leq d - 1 \), among simplices in \( \mathbb{E}^d \) with unit inradius, the regular ones have minimal total \( k \)-content. Furthermore, for any \( 1 \leq k \leq d - 1 \), among the convex polytopes in \( \mathbb{E}^d \) with \( d + 2 \) vertices and unit inradius, if \( P \) has minimal total \( k \)-content, then it satisfies one of the following.

(i) There are regular simplices \( S_1 \) and \( S_2 \) with \( \dim S_1 + \dim S_2 = d \) such that the subspaces spanned by \( S_1 \) and \( S_2 \) are orthogonal complements of each other, \( S_1 \cap S_2 = \{c\} \), and \( P = \text{conv}(S_1 \cup S_2) \).

(ii) There are regular simplices \( S_1, S_2 \), of dimensions \( \dim S_1 = k_1 \leq k \) and \( \dim S_2 = k_2 \leq k \) and lying in orthogonal subspaces that meet at the common center \( q_1 \) of both simplices, and a regular \((d - k_1 - k_2)\)-dimensional simplex \( Q \) with center \( q_2 \) such that \( Q \) is perpendicular to the segment \([q_1, q_2]\), and \( P = \text{conv}(S_1 \cup S_2 \cup Q) \).

(iii) \( P \) is a regular simplex of unit volume.

**4 | APPLICATIONS FOR MEASURES WITH ROTATIONALLY SYMMETRIC DENSITY FUNCTIONS**

Theorems 2 and 3 are immediate consequences of the following, more general theorems, which can be proved by the same idea (for the proof of Theorem 9, see also the proof of [2, Lemma 9]).

**Theorem 9.** Let \( D \) be an open ball centered at \( o \), or let \( D = \mathbb{E}^d \). Let \( \rho : [0, R_0) \) be a non-negative function, where \( R_0 \in \mathbb{R} \cup \{\infty\} \) denotes the radius of \( D \). Let \( S_{\text{reg}} \subset D \) be a regular \( d \)-dimensional simplex centered at \( o \).

(i) If \( \rho \) is a decreasing function, then for any simplex \( S \subset D \) with \( \text{cr}(S) = \text{cr}(S_{\text{reg}}) \), we have

\[
\int_S \rho(||x||) \, dx \geq \int_{S_{\text{reg}}} \rho(||x||) \, dx.
\]

Furthermore, if \( \rho \) is strictly decreasing, then equality holds if and only if \( S \) is congruent to \( S_{\text{reg}} \), and its center is \( o \).
(ii) If \( \rho \) is increasing, then for any \( d \)-dimensional simplex \( S \subset D \) with \( \text{ir}(S) = \text{ir}(S_{\text{reg}}) \), we have

\[
\int_S \rho(||x||) \, dx \leq \int_{S_{\text{reg}}} \rho(||x||) \, dx.
\]

Furthermore, if \( \rho \) is strictly increasing, then equality holds if and only if \( S \) is congruent to \( S_{\text{reg}} \), and its center is \( o \).

**Theorem 10.** Let \( D \) be an open ball centered at \( o \), or let \( D = \mathbb{E}^d \). Let \( \rho : [0, R_0) \) be a non-negative function, where \( R_0 \in \mathbb{R} \cup \{\infty\} \) denotes the radius of \( D \). Let \( P \subset D \) be a convex polytope with \( d + 2 \) vertices.

(i) If \( \rho \) is a decreasing function, then there is a convex polytope \( Q = \text{conv}(S_1 \cup S_2) \) of circumradius equal to \( \text{cr}(P) \), where \( S_1 \) and \( S_2 \) are regular simplices centered at \( o \) such that the subspaces spanned by \( S_1 \) and \( S_2 \) are orthogonal complements of each other, and

\[
\int_P \rho(||x||) \, dx \geq \int_Q \rho(||x||) \, dx.
\]

(ii) If \( \rho \) is a decreasing function, then there is a convex polytope \( Q = \text{conv}(S_1 \cup S_2) \) of inradius equal to \( \text{ir}(P) \), where \( S_1 \) and \( S_2 \) are regular simplices centered at \( o \) such that the subspaces spanned by \( S_1 \) and \( S_2 \) are orthogonal complements of each other, and

\[
\int_P \rho(||x||) \, dx \leq \int_Q \rho(||x||) \, dx.
\]

We remark that since by central projection, an open hemisphere in \( S^d \) can be represented as the Euclidean space \( \mathbb{E}^d \) equipped with a strictly decreasing rotationally symmetric density function, Theorems 2 and 3 can be also applied for spherical polytopes. As an example for another application, we present Corollary 2.

**Corollary 2.** Recall that the density function of the standard Gaussian measure on \( \mathbb{E}^d \) is \( \rho(||x||) \) with \( \rho(\tau) = \frac{1}{(2\pi)^{d/2}} e^{-\tau^2/2} \). Since \( \rho(\tau) \) is a strictly decreasing function of \( \tau \), by Theorem 9 we have that among simplices in \( \mathbb{E}^d \) with a given circumradius, the regular simplex centered at the origin has maximal Gaussian measure. By Theorem 10, we obtain a similar statement for the Gaussian measures of polytopes in \( \mathbb{E}^d \) with \( d + 2 \) vertices and a given circumradius.

5 CONCLUDING REMARKS AND OPEN PROBLEMS

Motivated by Theorem 9, it is a natural question to ask if for any measure generated by a rotationally symmetric density function \( \rho \), if \( B \) is a ball centered at \( o \), then among simplices \( S \) contained in \( B \), \( \int_S \rho(||x||) \) is maximal if \( S \) is a regular simplex inscribed in \( B \). Our next example shows that this is not true in general.
Example 1. Let $B$ be the closed unit disk centered at $o$, let $0 < \varepsilon \leq 1$, and set

$$\rho(\tau) = \begin{cases} 1, & \text{if } \tau \in [1-\varepsilon,1], \\ 0, & \text{otherwise.} \end{cases}$$

Let $T_{\text{reg}}$ denote a regular triangle inscribed in $B$, and let $T$ denote an isosceles triangle inscribed in $B$ such that the length of its altitude belonging to its base is $\varepsilon$. An elementary computation shows that $\int_{T_{\text{reg}}} \rho(||x||)dx = \Theta(\varepsilon^2)$, and $\int_{T} \rho(||x||)dx = \Theta(\varepsilon^{3/2})$. Thus, if $\varepsilon$ is sufficiently small, $\int_{T_{\text{reg}}} \rho(||x||)dx < \int_{T} \rho(||x||)dx$.

We could not prove that among simplices in $\mathbb{S}^d$ containing a given ball $B$, the regular ones circumscribed about $B$ have minimal volume. Nevertheless, we show that an affirmative answer to this problem implies a spherical variant of the so-called Simplex Mean Width Conjecture (see, for example, [17]), stating that among simplices contained in a given ball $B$ in $\mathbb{E}^d$, the regular ones inscribed in $B$ have maximal mean width. Before introducing it, we remark that the mean width of a convex body is proportional to its first intrinsic volume, and note that, apart from the equality case, by the property described in Remark 6 an affirmative answer to Problem 1 implies the Simplex Mean Width Conjecture by a standard limit argument.

Remark 6. A detailed description of possible variants of intrinsic volumes of convex bodies in $\mathbb{S}^d$ can be found in [10]. One of these, based on the geometric observation that the first intrinsic volume of a convex body in $\mathbb{E}^d$ is proportional to the total rotation invariant measure of the hyperplanes intersecting it, is defined as follows. Let $S$ denote the space of spherical hyperplanes in $\mathbb{S}^d$, let $\mu$ denote the rotation invariant probability measure on $S$, and let $\chi(\cdot)$ define Euler characteristic. For any convex body in $\mathbb{S}^d$, set

$$U_1(K) = \frac{1}{2} \int_S \chi(K \cap S) d\nu(S).$$

By (20) of [10], for any spherically convex set $K \subset \mathbb{S}^d$, we have

$$U_1(K) = \frac{1}{2} - \frac{\text{vol}_d(K^*)}{\text{vol}_d(\mathbb{S}^d)}$$

where $K^*$ denotes the polar of $K$. Thus, in the family of simplices in $\mathbb{S}^d$ contained in a given ball $B \subset \mathbb{S}^d$, the functional $U_1$ attains its maximum at $S$ if and only if among the simplices containing $B^*$, the simplex $S^*$ has minimal volume.

Problem 1. Let $d \geq 4$. Prove or disprove that for any ball $B \subset \mathbb{S}^d$, among the simplices containing $B$ the regular ones circumscribed about $B$ have minimal volume.

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