Non-universality for first passage percolation on the
exponential of log-correlated Gaussian fields

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Abstract

We consider first passage percolation (FPP) where the vertex weight is given by the exponential of two-dimensional log-correlated Gaussian fields. Our work is motivated by understanding the discrete analog for the random metric associated with Liouville quantum gravity (LQG), which roughly corresponds to the exponential of a two-dimensional Gaussian free field (GFF).

The particular focus of the present paper is an aspect of universality for such FPP among the family of log-correlated Gaussian fields. More precisely, we construct a family of log-correlated Gaussian fields, and show that the FPP distance between two typically sampled vertices (according to the LQG measure) is \( N^{1+O(\varepsilon)} \), where \( N \) is the side length of the box and \( \varepsilon \) can be made arbitrarily small if we tune a certain parameter in our construction. That is, the exponents can be arbitrarily close to 1. Combined with physics predictions on related exponents when the underlying field is GFF, our result suggests that such exponent is not universal among the family of log-correlated Gaussian fields.

1 Introduction

For an \( N \times N \) box \( V_N \subseteq \mathbb{Z}^2 \) with left bottom corner at the origin, we consider a log-correlated Gaussian field \( \{ \varphi_{N,v} : v \in V_N \} \) (see below for precise definitions). The main object investigated in this article is the first passage percolation (FPP) on the exponential of \( \{ \varphi_{N,v} : v \in V_N \} \).

More precisely, for \( \gamma > 0 \) and \( u,v \in V_N \), we define the FPP distance by

\[
d_\gamma(u,v) = \min_P \sum_{w \in P} e^{\gamma \varphi_{N,w}},
\]

where the minimum is taken over all paths in \( V_N \) connecting \( u \) and \( v \).

Our motivation is from the two-dimensional Liouville quantum gravity (LQG) introduced in [30]. The mathematical model for LQG can be formally described as a Riemannian “manifold” with tensor of the form

\[
e^{\gamma X(x)}dx^2,
\]

where \( X \) is a Gaussian free field (GFF) on (say) a torus \( \mathbb{T} \) and \( \gamma \in [0,2) \) is a parameter. Note the realization of GFF is a generalized function rather than a function. A fundamental question yet to understand is on the LQG geometry. The volume of this “manifold” is well-defined, say set up by

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the theory of Gaussian multiplicative chaos for log-correlated Gaussian fields \[24\] (see also \[31\]), and is (in some context) refereed to as the LQG measure. In a celebrated work \[21\], it is proved that the famous KPZ formula \[25\] (for more background and references, see \[21\]) holds for the LQG measure. However, people know little about the metric of this “manifold”. Despite intensive research, it remains an important open problem to even rigorously make sense of \(\text{(2)}\). For seminal works in this direction, see \[29, 20\].

In light of \(\text{(2)}\), it is natural to consider the discrete analogue of the LQG metric on two-dimensional discrete Gaussian free field \(\{\eta_{N,v}\}\) (say, with Dirichlet boundary condition by convention), which is a special instance of log-correlated Gaussian fields with covariance \(\mathbb{E}_{\eta_{N,u}}\eta_{N,v}\) given by the Green function of simple random walk killed at exiting the boundary \(\partial V_N\). For \(u, v \in V_N\), we define the discrete analog of the LQG measure as a random probability measure \(\mu_{\gamma}\) on \(V_N\) by

\[
\mu_{\gamma}(v) \propto e^{\gamma \eta_{N,v}}
\]

(we remark that the LQG measure has also been studied from the spin glass point of view, see \[6, 7\]). Furthermore, by resemblance in formulation between \(\text{(1)}\) and \(\text{(2)}\), one way to define the discrete analog for LQG metric is to consider the FPP distance, i.e., replacing the field \(\{\varphi_{N,v}\}\) in \(\text{(1)}\) by \(\{\eta_{N,v}\}\) (such metric was explicitly mentioned in \[9\]). It is interesting to understand the FPP metric, part of it due to its natural and close connection to the continuous LQG metric. A major open problem is to prove that (a suitable) discrete LQG metric under appropriate normalization converges in some sense to the continuous LQG metric. While FPP metric might not eventually be “the suitable” one with all properties that are desirable for LQG metric, we believe it should capture a large part of fundamental mathematical structure of the suitable notion.

We choose to work with the FPP metric for its simple formulation, as well as its connection to classical first passage percolation with independent weights (see, e.g., \[23, 8\] for recent surveys). Another motivation of studying the FPP metric is its connection to the heat kernel estimate for Liouville Brownian motion (LBM), whose mathematical construction is provided in \[22, 10\]. The LBM is closely related to the geometry of LQG; in \[17, 11\] the KPZ formula is derived from Liouville heat kernel. In \[28\] some nontrivial bounds for LBM heat kernel are established. A very interesting direction is to compute the heat kernel of LBM with high precision. It is plausible that understanding the FPP metric as well as its geodesics is of crucial importance in computing the LBM heat kernel.

In this paper, we study an aspect of universality for FPP metric when the underlying random media is in the class of log-correlated Gaussian fields. Our motivation is two-fold. On one hand, universality is interesting on its own. There has been an intensive research recently on maxima of log-correlated Gaussian fields, and many universality aspects have been observed and rigorously established. It is shown that the limiting law of the centered maximum is in the same universality class for all log-correlated Gaussian fields under mild assumptions (see \[19, 27\] and references therein). Furthermore, it is highly expected that the limiting law of the extremal process should also be in the same universality class (see \[12, 13\] for the case of discrete Gaussian free field and see \[5, 2\] for the case of branching Brownian motion). From the spin glass point of view, the works \[6, 7\] strongly suggest certain universality behavior of the Gibbs measure (corresponding to the LQG measure in our context) among log-correlated Gaussian fields. Furthermore, the work of \[32\] suggests that the KPZ formula for the LQG measure is universal among log-correlated Gaussian fields (in the continuous set up). Since the exponent of FPP metric between a typical pair is largely determined by the geometry of the random set whose Gaussian values are in the same order of the
maximum, it seems plausible that this may also have a universal exponent among the family of log-correlated Gaussian fields. On the other hand, the universality, either holding or not, is a useful guide for computing the exponent of FPP metric for GFF since it will suggest what are the “correct” properties to focus on in order for such computations.

However, despite the fact that universality is of much interest, we are not aware of a precise formulation in this particular case. In what follows, we propose a natural candidate, for which we refer to as the strong universality of FPP metric.

Following [19], we say a suitably normalized version of Gaussian field \( \{ \varphi_{N,v} : v \in V_N \} \) is log-correlated if it satisfies the following two properties. Write \( w = (w_1, w_2) \) for all \( w \in \mathbb{Z}^2 \), and denote by \( |u - v| = \max\{|u_1 - v_1|, |u_2 - v_2|\} \) the \( \ell_\infty \)-distance.

(A.0) (Logarithmically bounded fields) There exists a constant \( \alpha_0 > 0 \) such that for all \( u, v \in V_N \),

\[
\text{Var} \varphi_{N,v} \leq \log_2 N + \alpha_0
\]

and

\[
\mathbb{E}(\varphi_{N,v} - \varphi_{N,u})^2 \leq 2 \log_2(|u - v| \vee 1) - |\text{Var} \varphi_{N,v} - \text{Var} \varphi_{N,u}| + 4 \alpha_0,
\]

(A.1) (Logarithmically correlated fields) For any \( \delta > 0 \) there exists a constant \( \alpha(\delta) > 0 \) such that for all \( u, v \in V_N^\delta, |\text{Cov}(\varphi_{N,v}, \varphi_{N,u}) - (\log_2 N - \log_2(|u - v| \vee 1))| \leq \alpha(\delta) \). Here we denote by \( V_N^\delta = \{ u \in V_N : d(u, V_N^c) > \delta \} \), where \( d(u, V_N^c) = \min\{|u - v| : v \in V_N^c\} \).

The seemingly odd assumption on (A.1) (with introduction of \( V_N^\delta \)) is to give enough flexibility to incorporate the influence from the boundary, seen in the GFF as well as four-dimensional Gaussian membrane model. At this point, it is natural to define the measure \( \mu_\gamma \) and the metric \( d_\gamma \) with respect to the field \( \{ \varphi_{N,v} : v \in V_N \} \), in the same manner as in [30] (where we simply replace the field \( \eta_{N,v} \) by \( \varphi_{N,v} \)) and [1].

**Definition 1.1** (Strong universality). For each \( \gamma > 0 \), there exists \( \beta = \beta(\gamma) \) such that the following holds for any sequence of log-correlated Gaussian fields (and in particular, \( \beta \) does not depend on \( \alpha_0 \) or \( \alpha(\delta) \) in (A.0) and (A.1)). For any fixed \( \varepsilon > 0 \), with probability tending to 1 as \( N \to \infty \),

\[
\mu_\gamma \times \mu_\gamma(\{(u,v) : N^{\beta - \varepsilon} \leq d_\gamma(u,v) \leq N^{\beta + \varepsilon}\}) > 1 - \varepsilon_N,
\]

for a sequence of numbers \( \varepsilon_N \to 0 \) (as \( N \to \infty \)).

In this paper, we construct a family of log-correlated Gaussian fields. The main result Theorem [12] is that for \( 0 < \gamma < 1/2 \), we can make \( \beta(\gamma) \) arbitrarily close to 1 if we tune a certain parameter in our construction. Previously, precise formulae on exponents for highly related random metrics associated with GFF were predicted [35], which suggested that \( \beta(\gamma) < 1 \) for the case of GFF. In addition, in a recent work [18] it was proved that \( \beta(\gamma) < 1 \) for small fixed \( \gamma > 0 \) when the underlying field is a two-dimensional branching random walk (which, approximately, is a log-correlated Gaussian field). Altogether, this suggests that the strong universality does not hold.

**K-coarse modified branching random walk.** Our construction is based on the modified branching random walk (MBRW) introduced by [14]. We first briefly review the definition of MBRW in \( \mathbb{Z}^2 \). Suppose \( N = 2^n \) for some \( n \in \mathbb{N} \). For \( j = 0, 1, \ldots, n \) we define \( B_j \) to be the set of boxes of side length \( 2^j \) with corners in \( \mathbb{Z}^d \). For \( v \in V_N \), we define \( B_j(v) \) to be those elements of \( B_j \) which contain
v. Denote by \( \{ b_{j,B} : j \geq 0, B \in B_j \} \) a family of independent centered Gaussian variables such that \( \text{Var}(b_{j,B}) = 2^{-2j} \) for all \( B \in B_j \). Then the MBRW \( \{ S_{N,z} \}_{z \in V_N} \) is defined by

\[
S_{N,z} = \sum_{j=0}^{n} \sum_{B \in B_j(z)} b_{j,B}.
\]

Note that our definition of MBRW is slightly different from that in [14] since we do not view the box as a torus, but this difference is only for our technical convenience and is immaterial. For an integer \( K = 2^k \), we define the \( K \)-coarse MBRW by

\[
\varphi_{N,z} = \varphi_{N,z}(K) = \sum_{j=0}^{[n/k] - 1} \sum_{B \in B_{jk}(z)} \sqrt{kb_{jk,B}},
\]

where \( [x] \) is the largest integer which is less than or equal to \( x \) (and analogously, we will use the notation \( \lceil x \rceil \) for the smallest integer which is greater than or equal to \( x \)). An easy adaption of [14, Lemma 2.2] yields that for \( u,v \in V_N \),

\[
|\text{Cov}(\varphi_{N,u}, \varphi_{N,v}) - (n - \log_2(|u-v| \lor 1))| \leq 6k + 1,
\]

(see Lemma 2.1 below). Therefore, a sequence of \( K \)-coarse MBRW (for fixed \( K \) with \( N \to \infty \)) is a sequence of log-correlated Gaussian fields (where all the \( \alpha_0 \) and \( \alpha(\delta) \) are bounded by \( 6k + 1 \)).

**Theorem 1.2.** Fix \( 0 < \gamma < \frac{1}{2} \). For any \( \varepsilon > 0 \) there exists \( K(\varepsilon) \), \( \rho = \rho(\varepsilon) > 0 \) such that for all \( K \geq K(\varepsilon) \) the FPP metric for \( K \)-coarse MBRW satisfies

\[
\lim_{N \to \infty} \mathbb{P} \left( \mu_\gamma \times \mu_\gamma \left( \{ (u,v) : N^{1-\varepsilon} \leq d_\gamma(u,v) \leq N^{1+\varepsilon} \} \right) \right) \geq 1 - N^{-\rho} = 1.
\]

**Remark 1.3.** The assumption for \( \gamma < 1/2 \) is in some sense necessary in order to control the influence to the field in the local neighborhood of \( u \) and \( v \) when conditioning on the values of \( \varphi_{N,v} \) and \( \varphi_{N,u} \). The larger the \( \gamma \) is, the larger the typical values of \( \varphi_{N,v} \) and \( \varphi_{N,u} \) are when \( v \) and \( u \) are sampled according to the measure \( \mu_\gamma \). It is clear that when \( \gamma \) exceeds a certain threshold, the statement in the preceding theorem no longer holds. However, we do not attempt to achieve the sharp threshold in this work.

**Remark 1.4.** It remains to see what the non-universality result in our work would imply in the continuous set up. An interesting future direction is to investigate the behavior of the geodesic for \( K \)-coarse MBRW, and to decide whether its dimension is strictly larger than 1 (which is predicted to be the case for GFF). From the current work, we know that there exists a path of cardinality at most \( N^{1+\varepsilon} \) which has sum of weights less than \( N^{1+\varepsilon} \) (see Theorem 1.7), where \( \varepsilon \) vanishes as \( K \to \infty \).

We next describe the main ingredient for the proof of Theorem 1.2. For possible future applications in computing the heat kernel for the LBM on the continuous-analog of our \( K \)-coarse MBRW, we introduce a Bernoulli process \( \{ \xi_{N,w}, w \in V_N \} \) (For the purpose of proving Theorem 1.2, we simply set \( \xi_{N,w} = 1 \) for all \( w \)). Let \( q \) be positive integer, \( 0 < p \leq 1 \), and we assume

- \( \{ \xi_{N,w}, w \in V_N \} \) is independent of \( \{ \varphi_{N,w}, w \in V_N \} \).


• \( \{\xi_{N,w}\} \) is \( q \)-dependent, i.e., \( \{\xi_{N,w}, w \in A\} \) independent of \( \{\xi_{N,w}, w \in B\} \) provided \( d(A, B) > q \). Here \( d(A, B) = \min\{|z - w| : z \in A, w \in B\} \).

• \( \mathbb{P}(\xi_{N,w} = 1) \geq p \) for all \( w \in V_N \).

Using the language of percolation, we say a site \( w \) is open if \( \xi_{N,w} = 1 \), and a path is open if all sites on it are open. For convenience, we define the diameter of a path to be the maximal \( \ell_{\infty} \)-distance of two vertices on the path.

**Theorem 1.5.** Let \( \delta \in (0, 1), \kappa \in (0, \frac{1}{2}\delta^2) \). Then, there exists \( K_0 = K_0(\delta, q) > 0 \) and \( p_0 = p_0(\delta, q) \in (0, 1) \) such that the following holds. Suppose \( K \geq K_0 \), and \( p \geq p_0 \). Then, for \( N \) is large enough, with probability at least \( 1 - 6e^{-n/10} \),

\[
|\{w \in P : \varphi_{N,w}(K) \leq 4\delta \log N, \xi_{N,w} = 1\}| \geq N^{1-\delta}
\]

simultaneously for all paths \( P \) with diameter at least \( N^{1-\kappa} \).

**Remark 1.6.** The geometric property established in Theorem 1.5 is formulated without referring to any notion of LQG metric. We believe this is the key property making \( K \)-coarse MBRW differ from GFF and leading to the non-universality of the FPP metric. Furthermore, we feel that due to such difference the non-universality should hold for any reasonable notion of discrete LQG metric.

By duality, Theorem 1.5 implies that for a rectangle with height at least \( N^{1-\kappa} \), there exist \( N^{1-\delta} \) horizontal cut sets with \( w \) being open and \( \varphi_{N,w}(K) \) being small. This implies the existence of a “short” open horizontal path with small Gaussian values on it. The following result can be deduced from Theorem 1.5 combined with Russo-Seymour-Welsh kind of constructions 33 34.

**Theorem 1.7.** Let \( 0 < \delta < 1/2, \kappa \in (0, \frac{1}{2}\delta^2), 0 < \zeta < \frac{1}{4}\delta^4 \log^2 2 \). Then, there exists \( K_1 = K_1(\delta, q) > 0 \) and \( p_1 = p_1(\delta, q) \in (0, 1) \) such that the following holds. Suppose \( K \geq K_1 \), and \( p \geq p_1 \). For \( N \) is large enough, if \( |u - v| > N^{1-\kappa} \), then with probability at least \( 1 - 128(1 - p)^{\frac{1}{(2q+1)^2}} - e^{-\zeta n} \), one can find an open path \( P \) connecting \( u, v \) such that \( |P| \leq N^{1+2\delta}, \varphi_{N,w}(K) \leq \delta \log N, \forall w \in P \).

**Organization.** Theorem 1.5, the core theorem of the article, is proved in Section 2. In Section 3 we show that (see Proposition 3.1) a typical pair of vertices \( u, v \) sampled from \( \mu, \times \mu, \) has \( \ell_{\infty} \)-distance at least \( N^{1-\kappa} \). Combined with Theorem 1.5, it yields the desired lower bound in Theorem 1.2. In Proposition 3.3, we show the existence of a certain good path connecting two vertices, which implies Theorem 1.7. This gives the desired upper bound in Theorem 1.2 modulo the difference between two randomly sampled vertices according to the LQG measure and two fixed vertices (which is addressed in Proposition 3.5).

**Notation convention.** Throughout the paper, \( \delta, \kappa, \zeta \) are small positive parameters, \( q \) is a fixed positive integer, \( p \in (0, 1] \) and \( k \) is a large parameter chosen depending on \( \delta \) and \( q \). Denote \( m = \lfloor n/k \rfloor \). We consider the limiting behavior when \( n \to \infty \), thus we can assume without loss of generality that the inequalities such as \( 2\delta n - \log n + 2^k > \delta n \) hold.

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2 Connectivity for level sets

This section is devoted to the proof of Theorem 1.5. We give the heuristic argument in the simpler case where \( p = 1 \). The proof is by contradiction. Suppose there exists a path \( P \) of diameter \( \geq N^{1-\kappa} \) such that all the Gaussian values along the path are \( \geq 4\delta \log N \) except for \( N^{1-\delta} \) vertices (recall that \( \kappa < \delta^2/2 \)). We consider the average of all the Gaussian variables along the path \( P \), i.e., \( \text{Av}_P = \frac{1}{|P|} \sum_{z \in P} \varphi_{N,z} \). Then \( \text{Av}_P \geq 3\delta \log N \). For notation convenience, we write

\[
\psi_{j,z} = \sum_{B \in B_{jk}(z)} \sqrt{k} b_{jk,B}.
\]

(5)

Recall \( m = \lfloor n/k \rfloor \) and \( \varphi_{N,z} = \sum_{j=0}^{m-1} \psi_{j,z} \). Then it follows that

\[
\text{Av}_{P,j^*} \geq 3\delta k
\]

for some \( j^* \), where \( \text{Av}_{P,j^*} \) is the average in the \( j^* \)-th level, i.e., \( \text{Av}_{P,j^*} = \frac{1}{|P|} \sum_{z \in P} \psi_{j^*,z} \). We consider the sequence of boxes \( B_1, \ldots, B_H \) (with \( H \geq N^{1-\kappa} 2^{-j^*k} \)) of side length \( 2^{j^*k} \) along \( P \), and let

\[
T_h = \max_{z \in B_h} \psi_{j^*,z}.
\]

It is then natural (but incorrect as we point out later) to expect that

\[
\text{Av}_{T,\{B_1, \ldots, B_H\},j^*} = \frac{1}{H} (T_1 + \ldots + T_H) \geq 3\delta k.
\]

But this can be ruled out by the following first moment computation. One can show that \( T_1, \ldots, T_H \) are essentially independent of each other and each \( T_h \) has mean \( O(\sqrt{k}) \) with sub-Gaussian tail of variance \( O(k) \). This implies that (when \( k \) is sufficiently large) for each such sequence of boxes \( B_1, \ldots, B_H \)

\[
\Pr(\text{Av}_{T,\{B_1, \ldots, B_H\},j^*} \geq 3\delta k) \leq 2e^{-4\delta^2 k H}.
\]

However, the number of such sequences of boxes is at most \( 4^H \). Therefore, a simple union bound implies that with overwhelming probability, there exists no sequence of boxes with \( \text{Av}_{T,\{B_1, \ldots, B_H\},j^*} \geq 3\delta k \), arriving at a contradiction.

A crucial gap in this heuristics is that the path \( P \) could intersect certain \( B_h \) for many times (up to \( 2^{2j^*k} \)) and intersect another \( B_{h'} \) for \( O(2^{j^*k}) \) times, thus \( \text{Av}_{P,j^*} \) is dominated by a weighted average of \( T_1, \ldots, T_H \) rather than \( \text{Av}_{T,\{B_1, \ldots, B_H\},j^*} \). In order to address this issue, in Subsection 2.2 we extract a subset \( Q \) of a path \( P \), which stretches uniformly in some sense. Thus, the average of \( \psi_{j^*,z} \) along \( Q \) is indeed dominated by the average \( \text{Av}_{T,\{B_1, \ldots, B_H\},j^*} \), where \( B_1, \ldots, B_H \) is a sequence of boxes associated with \( Q \). Afterwards, the proof of Theorem 1.5 is carried out in Section 2.3 following the heuristics on \( Q \).

2.1 Preliminary lemmas

This subsection records a number of preliminary lemmas.

**Lemma 2.1.** Denote \( \sigma_{z,w} = \mathbb{E} \varphi_{N,z} \varphi_{N,w} \). Then, \( 0 \leq (km - \log_2 (|z - w| \lor 1)) - \sigma_{z,w} \leq 5k + 1 \).
Proof. The proof follows that of Lemma 2.2 in [14]. The case for \( z = w \) is obvious and in what follows we focus on the case when \( z \neq w \). Note \( E\psi_{j,z}\psi_{j,w} = 0 \) if \( |z - w| \geq 2^j \). Suppose \( |z - w| < 2^j \), equivalently, \( j \geq \lceil \frac{1}{2} \log_2(|z - w| + 1) \rceil \). Then

\[
E\psi_{j,z}\psi_{j,w} = k(1 - \frac{|z_1 - w_1|}{2^j})(1 - \frac{|z_2 - w_2|}{2^j}),
\]

which implies that

\[
k - 2k \times \frac{|z - w|}{2^j} \leq E\psi_{j,z}\psi_{j,w} \leq k. \tag{6}
\]

Note \( \sigma_{z,w} = \sum_{j=\lceil \frac{1}{2} \log_2(|z-w|+1) \rceil}^{m-1} E\psi_{j,z}\psi_{j,w} \). On one hand,

\[
\sigma_{z,w} \leq k \left( m - \frac{1}{k} \log_2(|z - w| + 1) \right) \leq km - \log_2 |z - w|.
\]

On the other hand,

\[
\sigma_{z,w} \geq k \sum_{j=\lceil \frac{1}{2} \log_2(|z-w|+1) \rceil}^{m-1} (1 - 2 \times \frac{|z - w|}{2^j}) = k \left( m - \frac{1}{k} \log_2(|z - w| + 1) \right) - 4k|z - w| \times 2^{-k \lceil \frac{1}{2} \log_2(|z-w|+1) \rceil} \geq km - (\log_2 |z - w| + k + 1) - 4k = km - \log_2 |z - w| - 5k - 1,
\]

completing the verification of the lemma.

\[ \square \]

Lemma 2.2. ([27, Theorem 7.1, Equation (7.4)]) Let \( \{G_z : z \in B\} \) be a Gaussian field on a finite index set \( B \). Set \( \sigma^2 = \max_{z \in B} \text{Var}(G_z) \). Then

\[
P(\max_{z \in B} G_z - \mathbb{E} \max_{z \in B} G_z \geq x) \leq 2e^{-\frac{x^2}{2\sigma^2}}.
\]

Lemma 2.3. We have that

\[
P\left( \sum_{j=(1-\delta^2)m}^{m-1} \psi_{j,z} \leq \frac{5}{2} \delta \log N \text{, for all } z \in V_N \right) \geq 1 - 2e^{-n/10}.
\]

In addition, we have

\[
P(\varphi_{N,u} - \varphi_{N,v} \leq 100 \sqrt{K \log N} \text{, for all } u, v \in V_N, u \sim v) \geq 1 - e^{-n/10}. \tag{7}
\]

Proof. The proof of these two equalities are highly similar to each other, and thus we omit the proof of (7). Denote \( \theta_z = \sum_{j=(1-\delta^2)m}^{m-1} \psi_{j,z} \) for short. Note for each \( z \), \( \text{Var}(\theta_z) \leq k \delta^2 m \leq \delta^2 n \). Thus,

\[
P(\theta_z > \frac{5}{2} \delta \log N) \leq 2e^{-\text{Var}(\theta_z) \frac{\delta^2 \log^2 N}{8}} \leq 2e^{-\frac{25 \log^2 N}{8k}}.
\]

Therefore, a simple union bound yields that

\[
P(\exists z \in V_N \text{ such that } \theta_z > \frac{5}{2} \delta \log N) \leq 2e^{-\frac{25 \log^2 n}{8k} N^2} \leq 2e^{-n/10}. \square
\]
Lemma 2.4. Suppose $B \subset V_N$. Then for all $a > 1/\sqrt{2\pi k}$,

$$\mathbb{E} \exp \left( a(\max_{z \in B} \psi_{j,z} - \mathbb{E} \max_{z \in B} \psi_{j,z}) \right) \leq 3\sqrt{2\pi k} a^2 e^{\frac{1}{2}ka^2}.$$

Proof. Denote $T = \max_{z \in B} \psi_{j,z}$ for short. Then,

$$\mathbb{E} e^{a(T - ET)} \leq 1 + \int_{1}^{\infty} \mathbb{P} \left( e^{a(T - ET)} > x \right) \, dx \leq 1 + \int_{0}^{\infty} \mathbb{P} \left( T - ET > \frac{x}{a} \right) e^x \, dx.$$

By Lemma 2.2, $\mathbb{P} \left( T - ET > \frac{x}{a} \right) \leq 2 \exp\left( -\frac{(x/a)^2}{2} \right)$. It follows that

$$\mathbb{E} e^{a(T - ET)} \leq 1 + \int_{0}^{\infty} 2e^{-\frac{1}{2k}x^2} e^x \, dx = 1 + \int_{0}^{\infty} 2e^{-\frac{1}{2k}(x-ka)^2 + \frac{1}{2}ka^2} \, dx \leq 1 + 2e^{\frac{1}{2}ka^2} \sqrt{2\pi k}a^2 \leq 3e^{\frac{1}{2}ka^2} \sqrt{2\pi k}a^2,$$

where in the last inequality we use $1 \leq \sqrt{2\pi k}a^2$. \qed

Lemma 2.5. ([19, Theorem 1.2]) Let $m_N = 2\sqrt{\log 2n - \frac{3}{4\log 2} \log n}$. Then there exists a constant $C_K$, depending on $K$, such that

$$|\mathbb{E} \max_{z \in V_N} \psi_{N,z} - m_N| \leq C_K.$$

Lemma 2.6. ([10, Theorem 4.1]) There exists a universal constant $C_F$ with the following property. Let $B \subset \mathbb{Z}^2$ be a box of side length $b$ and $\{G_w : w \in B\}$ be a mean zero Gaussian field satisfying

$$\mathbb{E}(G_z - G_w)^2 \leq |z - w|/b \quad \text{for all } z, w \in B.$$

Then $\mathbb{E} \max_{w \in B} G_w \leq C_F$.

Lemma 2.7. For all $j$ and $B \in B_{jk}$ we have that $\mathbb{E} \max_{z \in B} \psi_{j,z} \leq 2C_F \sqrt{k}$.

Proof. For any $z, w \in B_{jk}$, $|z - w| < 2^k$. By (6),

$$\mathbb{E}(\psi_{j,z} - \psi_{j,w})^2 = 2k - 2\mathbb{E} \psi_{j,z} \psi_{j,w} \leq 4k \times \frac{|z - w|}{2^k}.$$

By Lemma 2.6, $\mathbb{E} \max_{z \in B} \psi_{j,z} \leq 2C_F \sqrt{k}$. \qed
2.2 The \( k \)-block-nest

This section is devoted to extract a subset \( Q \) of a path \( P \), which stretches uniformly. Our construction is based on an inductive procedure, and we first give a description on the mental picture behind. Let \( \mathcal{BD}_{jk} \) be those elements in \( B_{jk} \) which has lower left corner in \( 2^{jk}\mathbb{Z}^2 \). We see that \( (\mathcal{BD}_{jk})_{j=m,\ldots,0} \) forms an increasing sequence of nested uniform partitions of \( V_N \). A natural attempt for the initial round of the induction is to consider the intersections of \( P \) with all boxes in \( \mathcal{BD}_{(m-1)k} \), which lead to a few subpaths of \( P \). However, there are a couple of issues with this oversimplified attempt:

- Despite that the path \( P \) is connected and self-avoiding, the intersection of \( P \) with a box in \( \mathcal{BD}_{(m-1)k} \) could contain a number of disconnected subpaths (the disconnectedness can potentially result in difficulties in entropy estimates).

- Some of the boxes in \( \mathcal{BD}_{(m-1)k} \) may intersect with \( P \) substantially more heavily than other boxes, which would violate our desired “uniformly stretching” property.

Fix a box \( B \) in \( \mathcal{BD}_{(m-1)k} \). In order to address the issue of disconnectedness, we only take the “first” connected piece of \( P \) in \( B \), denoted by \( P^{(1)} \). Furthermore, in order to address the issue of non-uniformity, we only keep an initial segment \( P^{(2)} \) of \( P^{(1)} \) so that \( P^{(2)} \) intersects with precisely \( K_1 \) boxes in \( \mathcal{BD}_{(m-2)k} \) for a pre-fixed number \( K_1 \) (depending on \( K \)). However, in the fix for the second issue, we face the difficulty that \( P^{(1)} \) might be short so it is impossible to extract such \( P^{(2)} \). In order to address this issue and carry out the induction, we give a rigorous meaning on “traversing” a box, and replace the concept of intersection by traversing. Then, from a traversing \( P^{(1)} \), we are able to extract \( P^{(2)} \) to traverse exact \( K_1 \) boxes in \( \mathcal{BD}_{(m-2)k} \). This ensures that our induction would in the end give us a large subset \( Q \subseteq P \) which stretches uniformly. In what follows, we provide the formal construction and verification for the aforementioned inductive procedure, which we will refer to as \( k \)-block-nest program.

Suppose \( B \) is a box of side length \( \ell \). Let \( B^* \) and \( B^{**} \) be the boxes centered at \( B \) of side lengths \( 3\ell \) and \( 7\ell \), respectively. That is,

\[
B^* = \{ z : d(z, B) \leq \ell \}, \quad B^{**} = \{ z : d(z, B) \leq 3\ell \}.
\]

By traversing of a box \( B \), we mean a path in \( B^* \) from \( \partial B \) to \( \partial B^* \). By distance of a path, we mean the \( \ell_\infty \)-distance between the two end points of the path. Then each traversing of \( B \) has distance at least the side length of \( B \). First, let \( j \geq 1 \). We aim to extract traversings of \( K_1 \) (to be defined) boxes in \( \mathcal{BD}_{jk} \) from a traversing of a box in \( \mathcal{BD}_{(j+1)k} \).

**Definition 2.8.** Suppose \( B_1, B_2, \ldots, B_H \in \mathcal{BD}_{jk} \). They are called \( j \)-separated if \( d(B^*_h, B^*_h') > 2^{jk} \), \( \forall h \neq h' \), and coherent if in addition \( d(B_{h+1}, B^{**}_h) = d(B_{h+1}, \bigcup_{s=1}^h B^{**}_s) = 1 \), \( \forall h = 1, \ldots, H - 1 \).

**Remark 2.9.** The definition of \( j \)-separation is to ensure the independence of \( \max_{z \in B^*_h} \psi_{j,z} \), \( 1 \leq h \leq H \). The further definition of coherence makes us to calculate the number of all possible such sequences of boxes. These two aspects are both essential in the heuristic argument at the beginning of Section 2.

**Definition 2.10.** Let \( B_h \in \mathcal{BD}_{jk} \) and \( P^{(h)} \) be a traversing of \( B_h \), \( \forall h = 1, \ldots, H \). We call \( P^{(h)} \)'s \( j \)-sections with number \( H \) and \( B_h \)'s the associated \( j \)-blocks if \( B_1, \ldots, B_H \) are \( j \)-separated.
Assume that $P$ has distance at least $N^{1-\kappa}$. In order to define the $k$-block-nest of a path, we give the induction operation first. Suppose $B \in BD_{(j+1)k}$, and $P$ is a traversing of $B$. We denote by $P_0, P_1, P_2, \ldots$ the sequence of vertices along the path $P$. Let $s_1 = 0$ and $B_h = BD_{jk}(P_h)$, where $BD_{jk}(z)$ is the unique element in $BD_{jk}$ containing $z$. Let $s_{h+1}$ be the first time $P$ departing $\cup_{1 \leq i \leq h} B_i^{**}$ forever, i.e.,

$$s_{h+1} = \inf \{s \geq s_h + 1 : P_r \notin \cup_{1 \leq i \leq h} B_i^{**}, \text{ for all } r \geq s \},$$

where we use the convention that $\inf \emptyset = \infty$. For $h = 1, 2, \ldots$, define

$$t_h = \inf \{r \geq s_h + 1 : P_r \in \partial B_h^{*}\}.$$

We denote by $\tau$ the first $h$ so that $t_{h+1} = \infty$. Since $B_1, B_2, \ldots, B_r$ are coherent and $P$ has distance $\geq 2^{(j+1)k}$, we see that $|P_{s_{h+1}} - P_{s_h}| \leq 4 \times 2^k$ and furthermore $\tau$ is no less than

$$K_1 = \frac{2^{(j+1)k}}{4 \times 2^k} = 2^{k-2}.$$

Note $t_h < s_{h+1}$. Then, we can obtain $j$-sections $P_{[s_h, t_h]}$, $h = 1, \ldots, K_1$ from $P$, as well as the associated blocks $B_h$’s which are coherent (and disregard the boxes $B_{K_1+1, \ldots, B_r}$). Notice that $B_1$ is on the boundary of $B$, which means $\partial B_1 \cap \partial B \neq \emptyset$. In general, suppose that $P$ consists of $(j+1)$-sections with number $H$. We then deal with each section of $P$ as above, and obtain $H$ families of $j$-sections as well as their associated blocks. Since different sections of $P$ have distance larger than $2^{(j+1)k}$, $j$-sections and associated $j$-blocks in different families have distance larger than $2^k$. Finally, combining all families together, we obtain $j$-sections with number $HK_1$. In addition, the associated blocks $B_{(r-1)K_1+1, \ldots, B_rK_1}$ (in each family) are coherent for $r = 1, \ldots, H$.

Now we are ready to construct the $k$-block-nest of the path $P$. Denote

$$j_0 = \lfloor (1 - \frac{1}{2} \delta^2) m \rfloor.$$ 

Then $N^{1-\kappa} > 4 \times 2^{j_0}$ since $\kappa < \frac{1}{2} \delta^2$ and $n$ is large enough. Consequently, we can delete the parts of $P$ before it hits the boundary of $B_{j_0, 1} = BD_{j_0k}(P_0)$, and after it hits $\partial B_{j_0, 1}$. Then we obtain a traversing of $B_{j_0, 1}$, denoted by $P^{(j_0)}$. By the induction operation, for each $j = j_0 - 1, \ldots, 1$ we obtain a family of $j$-sections $P^{(j)}$ and the associated $j$-blocks $\{B_{j,h}, h = 1, \ldots, K_1^{j_0-j}\}$. We call $P^{(j)}$ the family of $j$-sections of $P$, and $\{B_{j,h}\}$ the family of $j$-blocks of $P$.

Now we bound the number of possible families of $j$-blocks. Note $B_{j_0, 1}$ has $2^{(m+1-j_0)k}$ possible choices. Note $B_{j,(r-1)K_1+1}$ is on the boundary of $B_{j+1,r}$ by induction operation. Given each $B_{j+1,r}$ there exist at most $4^{2k+1}$ possible choices of $B_{j,(r-1)K_1+1}$. By the property of coherent, there are at most $32$ possible choices of $B_{j,h+1}$ for each $B_{j,h}, h = (r-1)K_1+1, \ldots, rK_1-1$. Thus, there are at most

$$4 \times 2^k \times 32 K_1^{-1} \leq 2^{2k+5(K_1-1)} \leq 2^{5K_1+k} = : c$$

possible families of coherent blocks $B_{j,(r-1)K_1+1, \ldots, B_{j,r}K_1}$, given $B_{j+1,r}$. By induction, for $j = j_0 - 1, \ldots, 1$ there are at most

$$2^{(m+1-j_0)k}c^{K_1}c^{K_2} \cdots c^{K_i^{j_0-1-j}} \leq 2^{(m+1-j_0)k}\frac{K_1\cdots K_i^{j_0-1-j}}{c^{K_1+1}} = \frac{5K_1+k}{c^{K_1+1}} \leq 2^{(m+1-j_0)k+6K_1^{j_0-1-j}}.$$
possible families of $j$-blocks, where the last inequality holds for $k \geq 6$. The above upper bound is also valid for $j = j_0$.

At last, we set $j = 0$ and give $Q$. For each 1-block $B$, we wish to use the induction operation introduced above to extract the part $Q \cap B^*$ along the corresponding 1-section, which is a traversing of $B$. Note each $B_h$ in the induction operation is a single-point set since $j = 0$ now. In order to allow us to take advantage of the $q$-dependence of the Bernoulli process $\{\xi_{N,z}, z \in V_N\}$ later, we replace $B_i^{\ast}$ by $\{w : d(w, B) \leq q\}$ in (8). This is to ensure each pair of the $s_h$’s defined in (8) has distance at least $q + 1$, so that $\xi_{N,w}$, $w \in Q$ are independent. Then, we obtain

$$K_2 := \left\lfloor \frac{2^k}{q+1} \right\rfloor$$

points of $s_h$’s in $B^*$, and they compose $Q \cap B^*$. We call $Q$ the output of $P$ via the $k$-block-nest program. Similar reasoning as above implies that given a 1-block $B$, there are at most

$$4 \times 2^k \times (4(2q + 2))^{K_2-1} \leq 2^k (8q + 4)^{K_2} =: \tilde{c}$$

possible $Q \cap B^*$. Thus there are at most

$$2^{(m+1-j_0)k+6K_1^{j_0-1}} \times \tilde{c}K_1^{j_0-1} = 2^{(m+1-j_0)k+(k+6)K_1^{j_0-1}} (8q + 4)K_1^{j_0-1} K_2 \leq 2^{(m-j_0)k}(8q + 5)K_1^{j_0-1} K_2$$

possible outputs for $k \geq k'(q)$, where $k'(q)$ is taken to ensure $2^k K_2 \leq \frac{8q+5}{8q+1}$ and $K_1, K_2 \geq 1$.

Furthermore, the points in the output $Q$ have distances at least $q + 1$ to each other. Then, there exists $k'(q)$ such that following propositions hold.

**Proposition 2.11.** Let $k \geq k'(q)$. Suppose $P$ is a path of distance at least $N^{1-\kappa}$, $Q$ is the output of $P$, and $B_{j,h}, 1 \leq h \leq K_1^{j_0-j}$ are the $j$-blocks of $P$, $\forall j = 1, \ldots , j_0$. Then, the following properties hold.

a) For $z, w \in Q$, $z \neq w$, we have $d(z, w) > q$,

b) For $1 \leq j \leq j_0 - 1$ and $h \neq h'$, we have $d(B_{j,h}^*, B_{j,h'}^*) > 2^j k$,

c) $|Q| = K_1^{j_0-1} K_2$, $Q = \bigcup_{h=1}^{K_1^{j_0-j}} (Q \cap B_{j,h}^*)$, and $|Q \cap B_{j,h}^*| = K_1^{j_0-1} K_2$, $\forall h$.

**Proposition 2.12.** Let $k \geq k'(q)$. For all paths with distance at least $N^{1-\kappa}$, there are at most $2^{(m+1-j_0)k+6K_1^{j_0-j}}$ possible families of $j$-blocks, $\forall j = 1, \ldots , j_0$, and at most $2^{(m-j_0)k}(8q + 5)K_1^{j_0-1} K_2$ possible outputs.

### 2.3 Proof of Theorem 1.5

Let $k \geq k'(q)$ and $n$ large enough such that the $k$-block-nest program works and Proposition 2.11 as well as Proposition 2.12 holds. Let $Q$ be the $k$-output of $P$. Denote

$$\tilde{Q} := \{w \in Q : \varphi_{N,w} \leq 4 \delta \log N\}, \quad \hat{Q} := \{w \in Q : \xi_{N,w} = 1\}.$$

We will show the following lemmas for $k$ large than some $k(\delta, q)$ and $p$ large than some $p_0(\delta, q)$.

**Lemma 2.13.** $P(|\tilde{Q}| \geq \frac{\delta}{5+4 \delta}|Q| \text{ for all possible output } Q) \geq 1 - 5e^{-n/10}$ for $n$ large enough.

**Lemma 2.14.** $P(|\hat{Q}| \geq (1 - \frac{\delta}{6+8 \delta})|Q| \text{ for all possible output } Q) \geq 1 - e^{-n/10}$ for $n$ large enough.
Proof of Theorem 1.5. Since every path of diameter at least \( N^{1-\kappa} \) contains a subpath of distance at least \( N^{1-\kappa} \), we can assume without loss of generality that \( P \) has distance at least \( N^{1-\kappa} \). By Lemmas 2.13 and 2.14, we obtain that

Thus, in order to prove Theorem 1.5, it suffices to check that

\[
\mathbb{P}(|\hat{Q} \cap \hat{Q}| \geq \frac{\delta}{6 + 8\delta} |Q| \text{ for all possible output } Q) \geq 1 - 6e^{-n/10}.
\]

Thus, in order to prove Theorem 1.5, it suffices to check that \( \frac{\delta}{6 + 8\delta} |Q| \geq N^{1-\delta} \). By Proposition 2.11, \( |Q| = K_1^{j_0-1} K_2 \geq K_1^{j_0-1} \), where \( K_1 = 2^{k_2-2} \) and \( j_0 = [\left( 1 - \frac{1}{2} \delta^2 \right) m] \). This implies that

\[
\frac{N^{1-\delta}}{|Q|} \leq 2(1-\delta)k(m+1)-(k-2)\left( 1-\frac{\delta^2}{2} \right)^m = 2^{k(1-\delta)+2(k-2)} \left( 2-\left( \delta^2 \right) k+2-\delta^2 \right)^m.
\]

Take \( k_0 := k_0(\delta, q) \geq k'(q) \lor k(\delta, q) \) such that \( (\delta - \frac{\delta^2}{2})k_1 - 2 + \delta^2 > 0 \). Then for all \( k \geq k_0 \) and \( p \geq p_0 \), we have \( N^{1-\delta}/|Q| \leq \frac{\delta}{6 + 8\delta} \) provided that \( n \) is sufficiently large.

It remains to provide proofs for Lemmas 2.13 and 2.14.

Proof of Lemma 2.13. It is clear from a simple union bound that

\[
\mathbb{P}(\varphi_{N,z} \geq -3 \log N, \ \forall z \in V_N) \geq 1 - 6e^{-n/10}.
\]

Together with Lemma 2.3, we can suppose without loss of generality that \( \varphi_{N,w} \geq -3 \log N \) and \( \theta_w = \sum_{j=[(1-\delta^2)m]}^{m-1} \psi_{j,w} \leq \frac{\delta}{2} \log N \) for all \( w \in V_N \). Thus, we have

\[
\frac{1}{|Q|} \sum_{w \in Q} \varphi_{N,w} \geq \frac{|\hat{Q}|}{|Q|}(-3 \log N) + \frac{|Q| - |\hat{Q}|}{|Q|} \times 4\delta \log N = \left( 4\delta - (3 + 4\delta) \frac{|\hat{Q}|}{|Q|} \right) \log N.
\]

Suppose \( |\hat{Q}| \leq \frac{\delta}{3+4\delta} |Q| \), i.e., \( (3 + 4\delta) \frac{|\hat{Q}|}{|Q|} \leq \delta \). Then

\[
\frac{1}{|Q|} \sum_{w \in Q} \varphi_{N,w} \geq 3\delta \log N.
\]

Combined with \( \frac{1}{|Q|} \sum_{w \in Q} \theta_w \leq \max_{z \in V_N} \theta_z \leq \frac{\delta}{2} \log N \), it yields that (recall 3)

\[
\frac{1}{|Q|} \sum_{w \in Q} \left( \sum_{j=0}^{[(1-\delta^2)m]-1} \psi_{j,w} \right) \geq \frac{1}{2} \log N \geq \frac{1}{2} \delta k \log 2 \times m.
\]

Hence, there exists \( 0 \leq j \leq [(1-\delta^2)m] - 1 \) such that

\[
\frac{1}{|Q|} \sum_{w \in Q} \psi_{j,w} \geq \frac{1}{2} \delta k \log 2.
\]

For \( j \geq 1 \), let \( \{ B_{j,h} : h = 1, \ldots, K_1^{j_0-j} \} \) be the family of \( j \)-blocks of \( P \). By Proposition 2.11, \( Q \) can be decomposed into a disjoint union of \( Q \cap B_{j,h}^* \)'s, each of which has the same cardinality. It follows that

\[
\frac{1}{K_1^{j_0-j}} \sum_{h=1}^{K_1^{j_0-j}} T_{j,h} \geq \frac{1}{2} \delta k \log 2, \tag{10}
\]
where $T_{j,h} := \max_{z \in B^*_h} \psi_{j,z}$. It remains to check that this event happens with small probability.

We deal with a specific family of $T_{j,h}$’s first. They are independent and identically distributed, since $d(B^*_j, h, B^*_h) > 2^j$ by Proposition 2.11. It follows that for $a > 0$,

$$
P \left( \frac{1}{K_1^{j_0-j}} \sum_{h=1}^{K_1^{j_0-j}} T_{j,h} \geq \frac{1}{2} \delta k \log 2 \right) \leq \mathbb{E} \exp \left( a \sum_{h=1}^{K_1^{j_0-j}} (T_{j,h} - \frac{1}{2} \delta k \log 2) \right) = \left( \mathbb{E} e^{a(T - \frac{1}{2} \delta k \log 2)} \right)^{K_1^{j_0-j}},$$

where $T = T_{j,1}$. By Lemma 2.7, $\mathbb{E} T \leq 9 \mathbb{E} \max_{z \in B^*_h} \psi_{j,z} \leq 18C_F \sqrt{k}$. Combined with Lemma 2.4, it follows that for all $a > 1/\sqrt{2\pi k}$,

$$
\mathbb{E} e^{a(T - \frac{1}{2} \delta k \log 2)} = \mathbb{E} e^{a(\mathbb{E} T - \mathbb{E} T)} \times e^{a(\mathbb{E} T - \frac{1}{2} \delta k \log 2)} \\
\leq 3 \sqrt{2\pi ka^2} e^{\frac{1}{2} \delta k a^2} \times e^{a(18C_F \sqrt{k} - \frac{1}{2} \delta k \log 2)} \\
= 3 \sqrt{2\pi ka^2} e^{\frac{1}{2} k a(36C_F / \sqrt{k} + a - \delta \log 2)}.
$$

Especially, we aim to set $a = \frac{1}{2} \delta \log 2$. There exists $k_2 := k_2(\delta, q) \geq k'(q)$ such that $a := \frac{1}{2} \delta \log 2 > 1/\sqrt{2\pi k}$ and $36C_F / \sqrt{k} + a - \delta \log 2 \leq -\frac{1}{4} \delta \log 2$ for all $k \geq k_2$. It follows that

$$
\mathbb{E} e^{a(T - \frac{1}{2} \delta k \log 2)} \leq 3 \sqrt{2\pi k} \times \frac{1}{2} \delta \log 2 \times e^{-\frac{1}{4} k \times \frac{1}{2} \delta \log 2} \times e^{-\frac{1}{4} \delta \log 2} \\
= \frac{3}{2} \sqrt{2\pi k} \delta \log 2 \times e^{-\frac{1}{4} \delta \log 2}.
$$

Take $k_3 := k_3(\delta, q) \geq k_2$ such that the right hand side above is less than $e^{-\frac{3}{36} \delta k}$ for all $k \geq k_3$. Consequently,

$$
P \left( \frac{1}{K_1^{j_0-j}} \sum_{h=1}^{K_1^{j_0-j}} T_{j,h} \geq \frac{1}{2} \delta k \log 2 \right) \leq e^{-\frac{3}{36} \delta k} K_1^{j_0-j} \tag{11}
$$

for any specific family of $T_{j,h}$’s. By Proposition 2.12, there are at most $2^{(m+1-j_0)k+6K_1^{j_0-j}}$ possible families of $T_{j,h}$’s. Therefore for each fixed $j$

$$
P \left( \exists P \text{ such that } \frac{1}{K_1^{j_0-j}} \sum_{h=1}^{K_1^{j_0-j}} T_{j,h} \geq \frac{1}{2} \delta k \log 2 \right) \leq 2^{(m+1-j_0)k+6K_1^{j_0-j}} e^{-\frac{1}{36} \delta ^2 k K_1^{j_0-j}} \\
\leq 2^{(m+1-j_0)k} \left( 2^6 e^{-\frac{1}{36} \delta ^2 k} \right) K_1^{j_0-j}.
$$

It follows that

$$
P \left( \exists P \text{ such that } \frac{1}{K_1^{j_0-j}} \sum_{h=1}^{K_1^{j_0-j}} T_{j,h} \geq \frac{1}{2} \delta k \log 2 \text{ for some } j = 1, \ldots, [1 - \delta ^2] m - 1 \right) \\
\leq \sum_{j=1}^{[(1-\delta^2)m]-1} 2^{(m+1-j_0)k} \left( 2^6 e^{-\frac{1}{36} \delta ^2 k} \right) K_1^{j_0-j} \\
= 2^{(m+1-j_0)k} \sum_{j=j_0+1-[1,(1-\delta ^2)m]}^{j_0-1} \left( 2^6 e^{-\frac{1}{36} \delta ^2 k} \right) K_1^j \\
\leq 2^{(m+1-j_0)k} \left( 2^6 e^{-\frac{1}{36} \delta ^2 k} \right) K_1^{j_0-[(1-\delta ^2)m]+1} / (1 - 2^6 e^{-\frac{1}{36} \delta ^2 k}).$$
Recall \( j_0 = \lceil (1 - \frac{1}{2}\delta^2) m \rceil \), which implies \( j_0 - \lceil (1 - \delta^2) m \rceil + 1 \geq \frac{1}{2}\delta^2 m - 1 \). Thus \( K_1^{j_0} - \lceil (1 - \delta^2) m \rceil + 1 \geq 2^{\frac{1}{2}\delta^2 n} \). Therefore, the right hand side above is less than \( \frac{1}{2}e^{-n/10} \) provided that \( k \geq k_4 := k_4(\delta, q)(\geq k_3) \) and \( n \) is large enough.

For \( j = 0 \), we denote \( \{ \psi_{0,w}, w \in Q \} \) by \( \{ T_{0,h}, h = 1, \ldots , |Q| \} \). By a) of Proposition 2.11 they are independent and distributed as \( N(0, k) \). Then, the same reasoning above implies that

\[
\mathbb{P}\left( \exists P \text{ such that } 1 \leq \frac{1}{|Q|} \sum_{w \in Q} \psi_{0,w} \geq \frac{1}{2}\delta k \log 2 \right) \leq 2^{(m-j_0)k}(8q + 5)K_1^{j_0 - 1}K_2 e^{-\frac{1}{2}k(\delta \log 2)^2}K_1^{j_0 - 1}K_2
\]

Note \( K_2 = \lfloor \frac{2p}{q + 1} \rfloor \). The right hand side above is less than \( \frac{1}{2}e^{-n/10} \) provided \( k \geq \tilde{k}(\delta, q)(\geq k_4) \). \( \square \)

Proof of Lemma 2.14. Investigate a specific output \( Q \) first. By Proposition 2.11 \( \xi_{N,w}, w \in Q \) are independent. Denote \( \tilde{p} = 1 - \frac{\delta}{b + 8q} \). By Chebyshev inequality, \( \forall a > 0 \),

\[
\mathbb{P}(|\tilde{Q}| \leq \tilde{p}|Q|) \leq \left( \prod_{w \in Q} \mathbb{E}e^{a(\tilde{p} - \xi_{N,w})} \right) \leq \left( \mathbb{E}e^{a(\tilde{p} - \xi)} \right)^{|Q|},
\]

where \( \mathbb{P}(\xi = 1) = 1 - \mathbb{P}(\xi = 0) = p > \tilde{p} \). Take a such that \( e^a = \frac{p(1 - \tilde{p})}{\tilde{p}(1 - p)} \). Then \( \mathbb{E}e^{a(\tilde{p} - \xi)} = e^{-\rho} \),

where

\[
\rho = \rho(\delta) = f(\tilde{p}) - f(p) + (p - \tilde{p}) f'(p) > 0,
\]

and \( f(x) := x \log x + (1 - x) \log(1 - x) \). Note there are at most \( 2^{(m-j_0)k}(8q + 5)K_1^{j_0 - 1}K_2 \) possible outputs. Therefore,

\[
\mathbb{P}(|\tilde{Q}| \leq \tilde{p}|Q| \text{ for some output } Q) \leq e^{-\rho K_1^{j_0 - 1}K_2} \times 2^{(m-j_0)k}(8q + 5)K_1^{j_0 - 1}K_2
\]

\[
\leq 2^{(m-j_0)k} \left( e^{-\rho (8q + 5)} \right)K_1^{j_0 - 1}K_2.
\]

Note \( \rho \) increases to \( \infty \) as \( p \) increase to \( 1 \), i.e. \( e^\rho > 8q + 5 \) for \( p \) being larger than some \( p_0(\delta, q) \in (\tilde{p}, 1) \). Consequently, the right hand side above is less than \( e^{-n/10} \) for \( n \) large enough provided. \( \square \)

3 Proof of Theorem 1.2

In section 3.1, we show that with high probability a typical pair of vertices sampled from \( \mu \gamma \times \mu \gamma \) has \( \ell_\infty \)-distance at least \( N^{1-\kappa} \). Therefore, the lower bound on their FPP distance follows readily from Theorem 1.5. Much of the work goes to the proof of the upper bound, for which we need to convert boundary-to-boundary crossings to a vertex-to-vertex crossing. The proof technique is a Russo-Seymour-Welsh kind of construction \[33, 34].

3.1 Macroscopic \( \ell_\infty \)-distance for a typical pair

The goal of this subsection is to prove the following statement.
Proposition 3.1. For any fixed $\kappa > 0$, we have
\[ P\left( \mu_\gamma \times \mu_\gamma \left( \{ (u, v) : |u - v| < N^{1-\kappa} \} \right) > N^{-\rho} \right) \to 0, \]
for some $\rho > 0$.

By definition, we have
\[ \mu_\gamma \times \mu_\gamma \left( \{ (u, v) : |u - v| < N^{1-\kappa} \} \right) = \frac{\sum_{|u-v|<N^{1-\kappa}} e^{\gamma(\varphi_{N,u}+\varphi_{N,v})}}{(\sum w e^{\gamma \varphi_{N,w}})^2}. \]

(12)

Thus, in order to prove Proposition 3.1, we need to control both the numerator and the denominator in (12). We start with the denominator.

Lemma 3.2. For any $\tau > 0$,
\[ \lim_{N \to \infty} P \left( \sum_{w \in V_N} e^{\gamma \varphi_{N,w}} \geq N^2 e^{\frac{1}{2} \gamma^2 n e^{-\tau n}} \right) = 1. \]

Proof. The statement corresponds to an estimate on the cardinality for the level sets of the Gaussian field, which has been well-understood for log-correlated Gaussian fields. For instance, the dimension of level sets for Gaussian free field was computed in [16]. While the technique in [16] extends to our $K$-coarse MBRW, we choose to apply a more handy theorem from [15].

Take $R > 2\sqrt{\frac{1}{\gamma} \log 2}$. Since $\gamma < 2\sqrt{\log 2}$, one can take $1 \leq r \leq R-1$ such that $|\frac{r}{R} - \frac{\gamma}{2\sqrt{\log 2}}| < \frac{1}{R}$. By Lemma 2.5, we have $E \max_{z \in V_N} \varphi_{N,z} \geq 2\sqrt{\log 2} n - \frac{1}{\sqrt{\log 2}} \log n$, which implies that $\{ \varphi_{N,w} : w \in V_N \}$ is an extremal (using the notion of [15]) field. Thus, by [15, Theorem 1.6], with probability tending to 1 as $N \to \infty$ the level set

\[ U_{N,R} := \{ w \in V_N : \varphi_{N,w} \geq \frac{r}{R} \max_{z \in V_N} \varphi_{N,z} \} \]

has cardinality at least $N^{2\left(1-\frac{1}{R^2}-\frac{1}{R^2}\right)}$. On this event, it then follows that
\[ \sum_{w \in V_N} e^{\gamma \varphi_{N,w}} \geq \sum_{w \in U_{N,R}} e^{\gamma \varphi_{N,w}} \geq N^{2\left(1-\frac{1}{R^2}-\frac{1}{R^2}\right)} e^{\gamma^2 \left(2\sqrt{\log 2} n - \frac{1}{\sqrt{\log 2}} \log n \right)} \]
\[ \geq n - \frac{1}{\sqrt{\log 2}} N^{2\left(1-\frac{1}{R^2}\right)} e^{\gamma^2 \left(2\sqrt{\log 2} - \frac{1}{\sqrt{\log 2}} \log 2 \right)} \]
\[ \geq n - \frac{1}{\sqrt{\log 2}} N^{2\left(1-\frac{1}{R^2}\right)} e^{-\frac{2\log 2 n}{R^2} \gamma^2 n}, \]

where in the last inequality we used
\[ \frac{r}{R} 2\gamma \sqrt{\log 2} - \frac{r^2}{R^2} 2 \log 2 = -2 \log 2 \left( \frac{r}{R} - \frac{\gamma}{2\sqrt{\log 2}} \right)^2 + \frac{1}{2} \gamma^2 \geq -2 \log 2 \left( \frac{r}{R} - \frac{\gamma}{2\sqrt{\log 2}} \right)^2 + \frac{1}{2} \gamma^2. \]

Recalling that $\tau > \frac{4}{R^2} \log 2$, we obtain that $\sum_{w \in V_N} e^{\gamma \varphi_{N,w}} \geq N^2 e^{\frac{1}{2} \gamma^2 n e^{-\tau n}}$ as desired. \hfill \Box
Proof of Proposition 3.1. In light of (12) and Lemma 3.2, it remains to bound the numerator in (12) from above. For this purpose, we apply a straightforward first moment calculation. Note that
\[ E e^{2\gamma^2 \text{Var}(\varphi_{N,u}+\varphi_{N,v})} = e^{2\gamma^2 \text{Var}(\varphi_{N,u}+\varphi_{N,v})} = e^{2\gamma^2 (km + E \varphi_{N,u} \varphi_{N,v})}. \]

By Lemma 2.1 we have that \( E \varphi_{N,u} \varphi_{N,v} \leq n - \log_2(|u-v| \vee 1) \). Consequently,
\[ E e^{2\gamma^2 \text{Var}(\varphi_{N,u}+\varphi_{N,v})} \leq e^{2n\gamma^2} e^{-\gamma^2 (n-\log_2(|u-v| \vee 1))} \leq e^{2n\gamma^2} e^{-\gamma^2 (\log_2(|u-v| \vee 1))}. \]

Grouping pairs of vertices in terms of a Dyadic decomposition for their \( \ell_\infty \)-distance and summing over all possible groups, we get that
\[ E \sum_{u \neq v, |u-v| < N^{1-\kappa}} e^{\gamma (\varphi_{N,u}+\varphi_{N,v})} \leq e^{2n\gamma^2} 16N^2 \sum_{r=0}^{|n(1-\kappa)|} 2^{2r} e^{-\gamma^2 r} \leq 16e^{2n\gamma^2} N^2 (2^2 e^{-\gamma^2})^{|n(1-\kappa)|+1} \frac{2^2 e^{-\gamma^2} - 1}{2^2 e^{-\gamma^2} - 1} \]
\[ \leq \frac{64e^{-\gamma^2}}{4e^{-\gamma^2} - 1} e^{2n\gamma^2} N^2 (2^2 e^{-\gamma^2})^{|n(1-\kappa)|+1} \leq \frac{64e^{-\gamma^2}}{4e^{-\gamma^2} - 1} N^{-2\kappa} e^{\kappa \gamma^2} N^4 e^{\gamma^2 n}, \]
where \( 2^2 e^{-\gamma^2} > 1 \) since \( \gamma < \sqrt{\log 4} \). It follows that
\[ E \sum_{|u-v| < N^{1-\kappa}} e^{\gamma (\varphi_{N,u}+\varphi_{N,v})} \leq N^2 e^{2\gamma^2 n} + \frac{64e^{-\gamma^2}}{4e^{-\gamma^2} - 1} N^{-2\kappa} e^{\kappa \gamma^2} N^4 e^{\gamma^2 n}. \]

Combined with Markov inequality, it yields that for any fixed \( \rho > 0 \),
\[ P \left( \sum_{|u-v| < N^{1-\kappa}} e^{\gamma (\varphi_{u}+\varphi_{v})} > N^{-\rho} e^{-2\gamma n} N^4 e^{\gamma^2 n} \right) \leq \frac{N^4 e^{2\gamma n} e^{\gamma (\varphi_{u}+\varphi_{v})}}{N^4 e^{\gamma^2 n}} \leq N^4 e^{2\gamma n} \left( \frac{e^{\gamma^2 n}}{N^2} + \frac{64e^{-\gamma^2}}{4e^{-\gamma^2} - 1} \right)^{\kappa}, \]
which tends to 0 as \( n \to \infty \) provided that \( \gamma < \sqrt{2 \log 2} \) and \( \tau, \rho \) small enough. Combined with Lemma 3.2 this completes the proof of the proposition. \( \square \)

3.2 Lower bound for Theorem 1.2

Set \( p = 1 \) and \( \delta = \frac{\varepsilon}{1 + 4\gamma} \). By Theorem 1.5 and symmetry, with probability at least \( 1 - 6e^{-n/10} \), we have that \( \{|w \in P : \varphi_{N,w} \geq -4\delta \log N\} \geq N^{1-\delta} \) for any path \( P \) with diameter at least \( N^{1-\kappa} \). In what follows, we assume without loss that this event occurs. Thus, for any pair \( u, v \) with \( |u - v| \geq N^{1-\kappa} \) we have that
\[ d_\gamma(u, v) \geq N^{1-\delta} \times e^{\gamma (-4\delta \log N)} = N^{1-(1+4\gamma)\delta} = N^{1-\varepsilon}. \]
Therefore, we get that
\[ \mu_\gamma \times \mu_\gamma \left( \left\{(u, v) : d_\gamma(u, v) \geq N^{1-\varepsilon} \right\} \right) \geq \mu_\gamma \times \mu_\gamma \left( \left\{(u, v) : |u - v| \geq N^{1-\kappa} \right\} \right). \]

Combined with Proposition 3.1, this yields that for some \( \rho > 0 \)
\[ \lim_{N \to \infty} \mathbb{P}(\mu_\gamma \times \mu_\gamma \{(u, v) : d_\gamma(u, v) \geq N^{1-\varepsilon}\} \geq 1 - N^{-\rho}) = 1, \]
completing the proof of the lower bound of Theorem 1.2.

### 3.3 Upper bound for Theorem 1.2

This section is devoted to the upper bound of \( d_\gamma(u, v) \). Namely, we prove
\[ \lim_{N \to \infty} \mathbb{P} \left( \mu_\gamma \times \mu_\gamma \left( \left\{(u, v) : d_\gamma(u, v) > N^{1+\varepsilon} \right\} \right) > N^{-\rho} \right) = 0. \]

**Outline of the proof.** First, we show that for some fixed \( \rho > 0 \) and a certain \( \kappa > 0 \),
\[ \lim_{N \to \infty} \mathbb{P} \left( \mu_\gamma \times \mu_\gamma \left( \left\{(u, v) : |u - v| > N^{1-\kappa}, \varphi_{N,u} \varphi_{N,v} \leq 2\gamma \log N, d_\gamma(u, v) > N^{1+\varepsilon} \right\} \right) \right) > N^{-\rho} = 0. \]

By Lemma 3.2 and Markov inequality, we only need to show for a certain \( \tau > 0 \),
\[ \lim_{N \to \infty} \frac{\sum_{|u - v| > N^{1-\kappa}} \mathbb{E} e^{\gamma(\varphi_{N,u} + \varphi_{N,v})} 1_{\varphi_{N,u} \varphi_{N,v} \leq 2\gamma \log N, d_\gamma(u, v) > N^{1+\varepsilon}}}{N^{-\rho} e^{-\frac{2\gamma n}{N}} N^{4e^{\gamma^2n}}} = 0. \]

To bound the numerator, we consider a pair \( u, v \) with \( |u - v| > N^{1-\kappa} \) and write
\[ \mathbb{E} e^{\gamma(\varphi_{N,u} + \varphi_{N,v})} 1_{\varphi_{N,u} \varphi_{N,v} \leq 2\gamma \log N, d_\gamma(u, v) > N^{1+\varepsilon}} = \mathbb{E} e^{\gamma(\varphi_{N,u} + \varphi_{N,v})} 1_{\varphi_{N,u} \varphi_{N,v} \leq 2\gamma \log N} \mathbb{P}(d_\gamma(u, v) > N^{1+\varepsilon} | \varphi_{N,u}, \varphi_{N,v}). \]

Then, we show that if \( |u - v| > N^{1-\kappa} \), one can find a “short” path \( P \) connecting \( u, v \), on which the Gaussian variables \( \varphi_{N,z} \)'s are all small. Hence, \( d_\gamma(u, v) \) is less likely to be larger than \( N^{1+\varepsilon} \).

For convenience, an open path \( P \) is called **good** if \( \varphi_{N,z} \leq 7\delta \log N \) for all \( z \in P \). The following proposition on the existence of good path between a typical pair is a key ingredient.

**Proposition 3.3.** Suppose \( 0 < \delta < 1/2 \), \( 0 < \kappa < \frac{1}{4} \delta^2 \), \( 0 < \zeta < \frac{1}{4} \delta^4 \log^2 2 \). Denote
\[ \ell_0 = k|\delta^2 m|, \quad \ell_1 = \left| \log_2 ([N^{1-2\kappa}] + 2\ell_0 + 1) \right| - 1. \]

There exists \( p_1 = p_1(\delta, q) \in (0, 1) \) and \( K_1 = K_1(\delta, q) \) such that the following holds for \( K \geq K_1 \), \( p \geq p_1 \) and \( N \) large enough. If \( |u - v| > N^{1-\kappa} \), then with probability at least \( 1 - 128(1-p)^{(\frac{1}{2}+\delta)^2} - e^{-\zeta n} \), one can find a good path \( P \) connecting \( u, v \) such that \( |P| \leq N^{1+2\delta} \). Furthermore, there exist \( P^N \), \( P^F \) and \( Q^\ell \), \( \ell = \ell_0 + 1, \ldots, \ell_1 \) such that \( P \subseteq P^N \cup P^F \cup (\cup_{\ell = \ell_0+1}^{\ell_1} Q^\ell) \), and the followings hold.

(i) \( P^N \leq 40N^{(1+\delta)^2} \), \( |P^F| \leq N^{1+2\delta} \), \( |Q^\ell| \leq 40 \times 2^{(1+\delta)\ell} \), for all \( \ell = \ell_0 + 1, \ldots, \ell_1 \).

(ii) \( \sigma_{z,u} + \sigma_{z,v} \) is less than \( 4nk \) for all \( z \in P^F \), and it is less than \( n(1+\kappa) + 1 - \ell \) for all \( z \in Q^\ell \), \( \ell = \ell_0 + 1, \ldots, \ell_1 \). (Recall that \( \sigma_{z,u} = \mathbb{E} \varphi_{N,z} \varphi_{N,u} \))
Remark 3.4. (1) The assumption that \(|u - v| > N^{1-\kappa}\) is immaterial and merely for the convenience of reusing results obtained in previous sections. (2) The additional properties for the good path in Proposition 3.3 are for the purpose of controlling how conditioning on the value at the endpoints \(u\) and \(v\) would change the behavior of the path.

Finally, we set \(p = 1\) and show that for \(\varphi_{N,u}, \varphi_{N,v} \leq 2\gamma \log N\), the good path given in Proposition 3.3 leads to \(d_\gamma(u, v) \leq N^{1+\epsilon}\). This is incorporated in the following proposition.

**Proposition 3.5.** Set \(p = 1\). Suppose the assumptions in Proposition 3.3 hold, and in addition \(0 < \delta < \delta_0(\epsilon) > 0\). If \(|u - v| \geq N^{1-\kappa}\) and \(x, y \leq 2\gamma \log N\), we have

\[
P (d_\gamma(u, v) > N^{1+\epsilon} | \varphi_{N,u} = x, \varphi_{N,v} = y) \leq e^{-\zeta n}.
\]

This together with (17) and (13) implies that for \(|u - v| \geq N^{1-\kappa}\),

\[
\mathbb{E} e^{\gamma (\varphi_{N,u} + \varphi_{N,v})} 1_{\{\varphi_{N,u}, \varphi_{N,v} \leq 2\gamma \log N, d_\gamma(u, v) > N^{1+\epsilon}\}} \leq e^{-\zeta n} \mathbb{E} e^{\gamma (\varphi_{N,u} + \varphi_{N,v})} 1_{\{\varphi_{N,u}, \varphi_{N,v} \leq 4\gamma \log N, d_\gamma(u, v) > N^{1+\epsilon}\}} \leq e^{-\zeta n} e^{2(2\log 2 - \log_2 |u - v|)} \leq e^{-\zeta n} e^{\gamma^2 n(1+\kappa)}.
\]

It follows that

\[
\sum_{|u - v| > N^{1-\kappa}} \mathbb{E} e^{\gamma (\varphi_{N,u} + \varphi_{N,v})} 1_{\{\varphi_{N,u}, \varphi_{N,v} \leq 4\gamma \log N, d_\gamma(u, v) > N^{1+\epsilon}\}} \leq N e^{-\zeta n} e^{\gamma^2 n(1+\kappa)} \leq N e^{-\zeta n} e^{\gamma^2 n (1+\kappa)}.
\]

Therefore, (16) follows for \(\kappa < \zeta / \gamma^2\) and \(\tau, \rho\) small enough.

The rest of the paper is devoted to the proof of (15), Propositions 3.3 and 3.5

### 3.3.1 Proof of (15)

Suppose \(Z \sim N(0, 1), b > a > 0\). Then

\[
\mathbb{E} e^{aZ} 1_{Z > b} = \int_{b}^{\infty} e^{az} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz = \int_{b}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (z-a)^2} e^{\frac{1}{2} a^2} dz = P(Z > b-a) e^{\frac{1}{2} a^2} \leq 2 e^{-\frac{1}{2} (b-a)^2} e^{\frac{1}{2} a^2}.
\]

It follows that

\[
\mathbb{E} e^{\gamma \varphi_{N,u}} 1_{\varphi_{N,u} > 2\gamma \log N} \leq \mathbb{E} e^{\gamma \sqrt{km} Z} 1_{Z > \gamma \sqrt{km} 2 \log 2} \leq 2 e^{-\frac{1}{2} (2 \log 2 - 1)^2 \gamma^2 km} e^{\frac{1}{2} a^2} \leq 2 e^{-\frac{1}{2} (2 \log 2 - 1)^2 \gamma^2 (n-k)} e^{\frac{1}{2} a^2}.
\]

Therefore,

\[
\sum_{u \in V} \mathbb{E} e^{\gamma \varphi_{N,u}} 1_{\varphi_{N,u} > 2\gamma \log N} \leq 2 N \rho e^{-\frac{1}{2} (2 \log 2 - 1)^2 \gamma^2 (n-k)} e^{\gamma n}.
\]

By Lemma 5.2 and Markov inequality, (15) holds provided \(\tau, \rho\) are sufficiently (but fixed) small.
3.3.2 Proof of Proposition 3.3

In order to prove Proposition 3.3, we first show that with high probability there exists a good crossing from the left boundary to the right boundary of a large box. Namely, we prove this for boxes of side length \( L = 2^\ell \) for all \( \ell \geq \ell_0 \) (recall \( \ell_0 = k[\delta^2 m] \)).

For any rectangle \( B \), denote by \( LR(B) \) all the paths in \( B \) from the left boundary to the right one, and by \( UD(B) \) all the paths in \( B \) from the top boundary of \( B \) to the bottom one. If \( B \) is a box of side length \( L = 2^\ell \), we denote \( \cup_{a=0}^{3}(B + (aL, 0)) \) by \( \hat{B} \). That is, \( \hat{B} \) is a rectangle with width \( 4L \) and height \( L \), which shares the same left bottom corner with \( B \). We call \( Q \) a \( B \)-crossing if it consists of two good paths \( Q^V \in UD(B) \) and \( Q^H \in LR(B) \) with \( |Q^V| \leq 4L^{1+\delta} \) and \( |Q^H| \leq 16L^{1+\delta} \).

**Lemma 3.6.** Suppose \( 0 < \delta < 1/2 \) and \( B \) is a box of side length \( L = 2^\ell \) with \( \ell_0 \leq \ell \leq n - 2k \). Then

\[
\mathbb{P}(\text{there exists a } B\text{-crossing}) \geq 1 - 9e^{-\zeta_1 \ell}, \quad \text{where } \zeta_1 = \frac{\log^2 2}{2} \delta^2.
\]

To prove Lemma 3.6, the following lemma will be useful.

**Lemma 3.7.** Let \( \tilde{N} = 2^{k\tilde{m}} \), where \( 1 \leq \tilde{m} < m \). Then \( \mathbb{E} \max_{z \in V_{\tilde{N}}}(\varphi_{N,z} - \varphi_{\tilde{N},z}) \leq \sqrt{8kC_F} \).

Furthermore,

\[
\mathbb{P}\left( \max_{z \in V_{\tilde{N}}}(\varphi_{N,z} - \varphi_{\tilde{N},z}) \geq 2\delta \log N \right) \leq 2e^{-\frac{\delta^2 \log^2 2}{2k(m - \tilde{m})} n^2}. \quad (18)
\]

**Proof.** Denote

\[
\theta_z := \varphi_{N,z} - \varphi_{\tilde{N},z} = \sum_{j=m}^{m-1} \psi_{j,z}.
\]

Then \( \theta_z \sim N(0, k(m - \tilde{m})) \). For \( z, w \in V_{\tilde{N}} \), \( |z - w| < 2^{k\tilde{m}} \), thus by (13),

\[
\mathbb{E}\theta_z \theta_w = \sum_{j=m}^{m-1} \mathbb{E}\psi_{j,z} \psi_{j,w} \geq k \sum_{j=m}^{m-1} \left( 1 - 2\frac{|z - w|}{2j} \right)
\]

\[
= k(m - \tilde{m}) - 2k|z - w| \sum_{j=m}^{m-1} 2^{-jk} = k(m - \tilde{m}) - \frac{4k}{2^{km}} |z - w|.
\]

It follows that

\[
\mathbb{E}(\theta_z - \theta_w)^2 \leq \frac{8k}{2^{km}} |z - w| = 8k \frac{|z - w|}{N}.
\]

By Lemma 2.6, we have

\[
\mathbb{E} \max_{z \in V_{\tilde{N}}} \theta_z \leq \sqrt{8kC_F}.
\]

At this point, (18) follows from the preceding inequality and an application of Lemma 2.2. \( \quad \square \)

**Proof of Lemma 3.6.** Suppose \( kr \leq \ell + 1 < k(r + 1) \), where \( |\delta^2 m| \leq r \leq m - 2 \). Denote \( \tilde{N} = 2^{k(r+1)} \).

We would like to mention that as \( m \to \infty \), one has \( \ell_0 \to \infty \) hence \( r \to \infty \).

Denote by \( UD_c(B) \) the collection of all cut sets that separate the top and bottom boundaries of \( B \) (with respect to the induced graph on \( B \)), and by \( LR_c(B) \) the collection of all cut sets that separate the left and right boundaries of \( B \) (with respect to the induced graph on \( B \)). We first show

\[
\mathbb{P}(E) \geq 1 - 9e^{-\zeta_1 \ell}, \quad (19)
\]
\[ E := \left\{ \{w \in C : \varphi_{N,w} \leq 7\delta \log N, \, \xi_{N,w} = 1\} \mid \tilde{N}^{1-\delta}/4, \text{ for all } C \in UD_c(B) \cup LR_c(\tilde{B}) \right\}. \]

Let \( \tilde{V} \) be the box of side length \( \tilde{N} \) and sharing the same left bottom corner with \( B \). Then \( \tilde{B} \subseteq \tilde{V} \). Consider the Gaussian field \( \{\varphi_{N,z} : z \in \tilde{V}\} \). By [7], the event

\[ E_1 := \left\{ |\varphi_{N,z} - \varphi_{\tilde{N},z}| \leq \delta \log N \text{ for all } z, w \in \tilde{V}, z \sim w \right\} \]

happens with probability at least \( 1 - e^{-k(r+1)/10} \geq 1 - e^{-\zeta_1} \), where we would like to mention \( \zeta_1 < 1/10 \). By Lemma [5.1] the event

\[ E_2 := \left\{ \max_{z \in \tilde{V}} (\varphi_{N,z} - \varphi_{\tilde{N},z}) \leq 2\delta \log N \right\} \]

happens with probability \( 1 - 2e^{-\frac{\delta^2 \log^2 2 \cdot n^2}{2(k(m-r-1)/n)^2}} \geq 1 - 2e^{-\frac{\delta^2 \log^2 2}{2}} \geq 1 - 2e^{-\zeta_1} \). Denote

\[ \Psi(A) := \left\{ v \in A : \varphi_{N,v} \leq 4\delta \log N, \, \xi_{N,v}^* = 1 \right\} \]

for any set \( A \), where \( \xi_{N,v}^* = 1 \) if \( \xi_{N,v} = 1 \) for all \( |z - w| \leq 1 \); and \( \xi_{N,v}^* = 0 \) otherwise. Note \( \{\xi_{N,v}^*, w \in \tilde{V}\} \) is \( (q + 2) \)-dependent, and \( P(\xi_{N,v}^* = 1) \geq p_0(\delta, q + 2) \) if we set \( p_1(\delta, q) \) such that

\[ 1 - 9(1 - p_1(\delta, q)) \geq 9p_0(\delta, q + 2), \]

i.e.

\[ p_1(\delta, q) \geq 1 - (1 - p_0(\delta, q + 2))/9. \]

By Theorem [1.5] (applied to box \( \tilde{V} \)), since \( r \) is large enough, with probability at least \( 1 - 6e^{-k(r+1)/10} \geq 1 - 6e^{-\zeta_1} \), we have the event

\[ E_3 := \{|\Psi(P)| \geq \tilde{N}^{1-\delta} \text{ simultaneously for all paths } P \subseteq \tilde{V}\} \]

happens, for \( K \geq K_1(\delta, q) := K_0(\delta, q + 2) \) and \( p \geq p_1(\delta, q) \). Then, we have \( \mathbb{P}(E_1 \cap E_2 \cap E_3) \geq 1 - 9e^{-\zeta_1} \). Suppose \( E_1, E_2, E_3 \) all happen. We call a sequence of vertices \( P_0, P_1, \ldots, P_h \) a \( * \)-sequence if \( |P_i - P_{i+1}| = 1 \) for all \( 0 \leq i < h \), and a \( * \)-path if in addition \( |P_i - P_{i'}| > 1 \) for \( |i' - i| > 1 \). Suppose \( C \in LR_c(\tilde{B}) \). Then it contains a \( * \)-sequence \( \tilde{P} = \tilde{P}_0, \tilde{P}_1, \ldots, \tilde{P}_h \) from the top boundary of \( B \) to the bottom one. We extract a \( * \)-path \( P^* = P_0^*, P_1^*, \ldots, P_h^* = \tilde{P}_{s_0}, \tilde{P}_{s_1}, \ldots, \tilde{P}_{s_h} \) by defining \( s_0 := 0, \]

\[ s_{r+1} := \inf\{s \geq s_r + 1 : |\tilde{P}_t - P_r^*| > 1, \forall t > s\} \]

til \( P^* \) hits the bottom boundary. We then take a path \( P \) such that \( P \supset P^* \), by inserting one vertex between \( P_s^* \) and \( P_{s+1}^* \) if they are not neighbours step by step for \( s = 0, 1, \ldots, h - 1 \). Note if \( P_s^* \) and \( P_{s+1}^* \) are not neighbours, they have two common neighbours, at most one of which is used in the steps \( 0, 1, \ldots, s - 1 \). So we can succeed in finding the path \( P \). Let

\[ Q := \{v \in P^* : v \text{ is in } \Psi(P) \text{ or neighbouring } \Psi(P)\}. \]

Then we have \( |Q| \geq |\Psi(P)|/4 \). For any \( w \in Q \), by the definition of \( \Psi(P) \) and \( Q \), we have \( \varphi_{N,w} \leq 7\delta \log N \) on \( E_1 \). This together with \( E_2 \) implies that \( \varphi_{N,w} \leq 7\delta \log N \). By definition of \( \Psi(P) \), \( \xi_{N,w}^* \) and \( Q \), we have \( \xi_{N,w} = 1, \forall w \in Q \). It follows that

\[ |\{w \in C : \varphi_{N,w} \leq 7\delta \log N, \, \xi_{N,w} = 1\}| \geq |Q| \geq \tilde{N}^{1-\delta}/4, \]
where the last inequality holds on $E_3$. The same reasoning implies that the above event also holds for all $C \in \mathcal{UD}_c(B)$. Hence (19) holds.

By min-cut-max-flow theorem, on $E$, there are at least $\tilde{N}^{1-\delta}/4$ disjoint good paths in $\mathcal{UD}(B)$. The shortest one, denoted by $Q^c$, has cardinality at most $4L^2/\tilde{N}^{1-\delta} \leq 4L^{1+\delta}$, since $|B| = L^2$. By the same reasoning, there exists a good path $Q^B$ in $\mathcal{LR}(\tilde{B})$ with cardinality at most $16 \times L^{1+\delta}$. This completes the proof of the lemma.

**Proof of Proposition 3.3.** Recall $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Without loss of generality, we suppose $v_1 - u_1 = |u - v| \geq \tilde{N}^{1-\delta}$. We first show that one can find a good path in $V_N$ from $u$ to $\{z = (z_1, z_2) : z_1 = v_1 - \lfloor \tilde{N}^{1-2\delta} \rfloor\}$, using Russo-Seymour-Welsh type of technique. Without loss of generality, we suppose $u_2 < N/2$. Recall $u_{u,\ell}$ (for $\ell \geq \ell_0$) be the box of side length $2^\ell$ and with the left bottom corner $u + (2^\ell - 1, 0)$. Then, $u_{u,\ell+1}$ and $\hat{B}_{u,\ell}$ share the same right bottom corner. Let $\ell_2$ be the minimum $\ell$ such that $\hat{B}_{u,\ell}$ intersects $\{z : z_1 = v_1 - \lfloor \tilde{N}^{1-2\delta} \rfloor\}$, i.e.,

$$\ell_2 := \lfloor \log_2 \frac{v_1 - u_1 - \lfloor \tilde{N}^{1-2\delta} \rfloor + 2^{\ell_0+1} + 1}{6} \rfloor.$$  

By a simple union bound, we see that

$$\mathbb{P}(E_1) \leq 2^{2\ell_0} e^{-24\delta^2 \log^2 2} n \leq e^{-\delta^2 n},$$

where

$$E_1 = \{ \varphi_{N,x} \leq 7\delta \log N, \forall z \in B_{u,\ell_0} \}.$$  

On the other hand, denote

$$E_2 = \left\{ \text{there exists an open path in } B_{u,\ell_0} \text{ from } u \text{ to } u + (0, 2^\ell_0) \right\}.$$

Note $u + (0, 2^\ell_0)$ is the left top corner of $B_{u,\ell_0}$. Then on $E_2$, one can find a closed cut set (and thus a $\ast$-path) in $B_{u,\ell_0}$, separating $u$ and $u + (0, 2^\ell_0)$. On one hand, there are at most $C_h = 2^h \times 8^h$ possible $\ast$-paths in $B_{u,\ell_0}$ with length $h$. On the other hand, suppose $P^* \ast$ is a $\ast$-path with length $h$. Then we can find $Q \subset P^*$ such that $|Q| \geq \lceil |P^*|/(2q + 1) \rceil$ and each pair of $Q$ has distance at least $q + 1$. It follows that

$$\mathbb{P}(E_2) \leq \sum_{h=1}^{\infty} C_h \times (1-p)^{\frac{h}{(2q+1)^2}} \leq \sum_{h=1}^{\infty} 2h \times (8(1-p)^{\frac{1}{(2q+1)^2}})^h \leq \frac{16(1-p)^{\frac{1}{(2q+1)^2}}}{1 - 8(1-p)^{\frac{1}{(2q+1)^2}}} \leq 64(1-p)^{\frac{1}{(2q+1)^2}},$$

provided that $8(1-p)^{\frac{1}{(2q+1)^2}} \leq \frac{1}{2}$. Thus the above inequality holds for $p$ great than or equal to

$$p_1(\delta, p) := \max \left\{ 1 - \frac{1}{9} (1 - p_0(\delta, q + 2)), 1 - 16^{-(2q+1)^2} \right\}.$$

Denote

$$E_3 = \{ \text{there exists a } B_{u,\ell,\ell_2} \text{-crossing, } \forall \ell = \ell_0, \ldots, \ell_2 \}.$$  

Then by the above reasoning as well as Lemma 3.6, we conclude

$$P(E_1 \cap E_2 \cap E_3) \geq 1 - e^{-\delta^2 n} - 64(1-p)^{\frac{1}{(2q+1)^2}} - \frac{9}{1 - e^{-\zeta_1}} e^{-\zeta_1 k[\delta^2 m]} \geq 1 - 64(1-p)^{\frac{1}{(2q+1)^2}} - \frac{1}{3} e^{-\zeta n},$$

21
where the last inequality holds for $\zeta < \zeta_1\delta^2$.

Finally, it is sufficient to check that on the events as $E = E_1 \cap E_2 \cap E_3$ we can find a good path as required. Assume $E$ occurs. Denote by $Q^{u,\ell} = Q^{u,\ell,V} \cup Q^{u,\ell,H}$ the $B_{u,\ell}$-crossings. On $E_2$, we can find an open path in $B_{u,\ell_0}$ from $u$ to $u + (0, 2^\ell_0)$. It is furthermore a good path since $E_1$ occurs. We choose it as $Q^{u,\ell_0,V}$. Then,

$$\left(\bigcup_{\ell_0 = 0}^{\ell_2 - 1} Q^{u,\ell_0} \right) \cup Q^{u,\ell_2,V} \cup \left( Q^{u,\ell_2,H} \cap \{z : z_1 \leq v_1 - \lceil N^{1-2\kappa}\rceil\} \right)$$

contains a good path $P^u$ from $u$ to $\{z : z_1 = v_1 - \lceil N^{1-2\kappa}\rceil\}$. In addition,

$$P^u \subseteq V_N \cap \{z : z_1 \leq v_1 - \lceil N^{1-2\kappa}\rceil\}$$

since $B_{u,\ell_2}$ is located at the left of $\{z : z_1 = v_1 - \lceil N^{1-2\kappa}\rceil\}$ and has height $2^{\ell_2} < \frac{N}{2} < N - u_1$. The good path $P^u$ has cardinality

$$|P^u| \leq 20 \left( (2\ell_0)^{1+\delta} + \ldots + (2\ell_2)^{1+\delta} \right) = 20\sum_{\ell_0 = 0}^{\ell_2} (2^{1+\delta})^\ell \leq 20 \times 2^{(1+\delta)(\ell_2+1)} \leq 80N^{1+\delta}. \quad (21)$$

By symmetry, with probability at least $1 - 64(1 - p)^{(2\ell_0 + 1)^2} - \frac{1}{4}e^{-\zeta n}$, there exists a good path $P^v$ from $v$ to $\{z : z_1 = u_1 + \lceil N^{1-2\kappa}\rceil\}$, which has cardinality at most $80N^{1+\delta}$. Denote

$$B_{u,v} := \{w \in V_N : u_1 + \lceil N^{1-2\kappa}\rceil \leq w_1 \leq v_1 - \lceil N^{1-2\kappa}\rceil\}. \quad (22)$$

Then each path in $\mathcal{L}\mathcal{R}(B_{u,v})$ has distance at least $2^{(1+\kappa)n} - 2^{(1-2\kappa)n} > 2^{(1-2\kappa)n}$. By Theorem 1.5 and the same reasoning in the proof of Lemma 3.6, with probability at least $1 - 7e^{-n/10} \geq 1 - \frac{4}{3}e^{-\zeta n}$ there exists a good path $P^v$ in $\mathcal{U}\mathcal{D}(B_{u,v})$ with cardinality at most $4N^{1+\delta}$. Provided paths $P^u$, $P^v$ and $P^V$, there exists a good path $P$ connecting $u$ and $v$ such that

$$|P| \leq 160 \times N^{1+\delta} + 4N^{1+\delta} \leq N^{1+2\delta}. \quad (23)$$

This event happens with probability at least $1 - 128(1 - p)^{(2\ell_0 + 1)^2} - e^{-\zeta n}$ for $\zeta < \zeta_1\delta^2$.

Finally, we decompose $P$ into three parts $P^u$, $P^v$ and $Q$ (which may overlap) as stated in the proposition. Let $P^N$ consist of $Q^{u,\ell_0}$ as well as its analog $Q^{v,\ell_0}$, $Q$ consist of $Q^{u,\ell}$ as well as the analogs $Q^{v,\ell}$ for all $\ell = \ell_0 + 1, \ldots, \ell_1$, and $P^F = (P^u \cup P^v \cup P^V) \setminus (P^N \cup Q)$.

Observe that $|P^N| \leq |Q^{u,\ell_0}| + |Q^{v,\ell_0}| \leq 40 \times 2^{\ell_0(1+\delta)} \leq 40 \times 2^{(1+\delta)\delta n} = 40N^{(1+\delta)\delta^2}$. This together with the fact that $Q^{\ell} := Q^{u,\ell} \cup Q^{v,\ell}$ has cardinality at most $40 \times 2^{(1+\delta)\ell}$ implies (i) of Proposition 3.3. For (ii), suppose $z \in Q^{u,\ell}$, for $\ell = \ell_0 + 1, \ldots, \ell_1$. We have $|z - u| \geq 2^\ell$, and $|z - v| \geq \frac{1}{2}N^{1-\kappa}$. This together with Lemma 2.1 implies that

$$\sigma_z, u + \sigma_z, v \leq n(1 + \kappa) + 1 - \ell. \quad (23)$$

The same result holds for $z \in Q^{v,\ell}$ by symmetry. For any $z \in P^F$, note that $B_{u,\ell_1 + 1}$ is located at the right of $\{z : z_1 = u_1 + \lceil N^{1-2\kappa}\rceil\}$ as well as $P^u$ is located at the left of $\{z : z_1 = v_1 - \lceil N^{1-2\kappa}\rceil\}$. It follows that $P^u \setminus (P^N \cup Q) \subseteq B_{u,v}$. By symmetry, $P^v \setminus (P^N \cup Q) \subseteq B_{u,v}$. These together with $P^V \in \mathcal{U}\mathcal{D}(B_{u,v})$ imply that $P^F \subseteq B_{u,v}$. It follows that for any $z \in P^F$, $|z - u|, |z - v| \geq N^{1-2\kappa}$, which implies that $\sigma_z, u + \sigma_z, v \leq 4n\kappa$. Thus, we completed the proof of Proposition 3.3. \qed
3.3.3 Proof of Proposition 3.5

Set
\[ a_z = \frac{km \sigma_{z,u} - \sigma_{z,v} \sigma_{u,v}}{(km)^2 - \sigma_{u,v}^2}, \quad \text{and} \quad b_z = \frac{km \sigma_{z,v} - \sigma_{z,u} \sigma_{u,v}}{(km)^2 - \sigma_{u,v}^2}. \]

We see that \( G_z := \varphi_{N,z} - (a_z \varphi_{N,u} + b_z \varphi_{N,v}) \) is independent of \( (\varphi_{N,u}, \varphi_{N,v}) \). Denote
\[ \phi_{N,z} = \varphi_{N,z} - (a_z \varphi_{N,u} + b_z \varphi_{N,v}) + (a_z x + b_z y). \]

Then, the conditional distribution of \( \{\varphi_{N,z} : z \in V_N\} \) given \( \varphi_{N,u} = x, \varphi_{N,v} = y \) is the same as the distribution of \( \{\phi_{N,z} : z \in V_N\} \). That is,
\[ \mathcal{L}(\{\varphi_{N,z} : z \in V_N\} | \varphi_{N,u} = x, \varphi_{N,v} = y) = \mathcal{L}(\{\phi_{N,z} : z \in V_N\}). \]

Hence, we consider the FPP distance between \( u, v \) according to the Gaussian field \( \{\phi_{N,z} : z \in N\} \), which is denoted by \( d_{\gamma,\delta}(u, v) \).

Let \( P \) be the good path (with respect to the field \( \{\varphi_{N,w} : w \in V_N\} \) given in Proposition 3.3) which exists with probability at least \( 1 - e^{-n} \) (recall \( p = 1 \)). Clearly, \( d_{\gamma,\delta}(u, v) \leq \sum_{z \in P} e^{\gamma \phi_{N,z}} \). For all \( z \in P \),
\[ \phi_{N,z} \leq (1 + |a_z| + |b_z|)7\delta \log N + (|a_z| + |b_z|)2\gamma \log N. \]

By Lemma 2.1 and \( |u - v| > N^{1-\kappa} \), one has \( km - \sigma_{u,v} \geq n(1 - \kappa) \). Therefore,
\[ |a_z| + |b_z| \leq \left( \frac{km + \sigma_{u,v}}{km - \sigma_{u,v}} \right)(\sigma_{z,u} + \sigma_{z,v}) \leq \frac{\sigma_{z,u} + \sigma_{z,v}}{n(1 - \kappa)}, \]
which is less than \( \frac{2}{\kappa n} \) since \( \sigma_{z,u}, \sigma_{z,v} \leq n \). It follows that
\begin{equation}
    e^{\gamma \phi_{N,z}} \leq N^{\frac{3n}{2} - 7\gamma \delta \log N(|a_z| + |b_z|)^2}. \tag{24}
\end{equation}

Take \( \delta \) (consequently, \( \kappa \)) small enough, and we only need to check
\begin{equation}
    \sum_{z \in P} N(|a_z| + |b_z|)^2 \gamma^2 < N^{1 + \frac{1}{3} \varepsilon}. \tag{25}
\end{equation}

To this end, let \( P^N, P^F, Q^\ell \) (for \( \ell = \ell_0 + 1, \ldots, \ell_1 \)) be given in Proposition 3.3. Then
\[ \sum_{z \in P} N(|a_z| + |b_z|)^2 \gamma^2 \leq \sum_{z \in P^N} N(|a_z| + |b_z|)^2 \gamma^2 + \sum_{z \in P^F} N(|a_z| + |b_z|)^2 \gamma^2 + \sum_{\ell = \ell_0 + 1}^{\ell_1} \sum_{z \in Q^\ell} N(|a_z| + |b_z|)^2 \gamma^2. \]

We next show that each term in the right hand side above is less than \( \frac{1}{3} N^{1 + \frac{1}{3} \varepsilon} \), thus \( 25 \) hold. For \( z \in P^N \), we use \( |a_z| + |b_z| \leq \frac{2}{\kappa n} \). This together with (i) of Proposition 3.3 implies that
\[ \sum_{z \in P^N} N(|a_z| + |b_z|)^2 \gamma^2 \leq 40N^{(1 + \delta)^2 + \frac{1}{2} - 2\gamma^2} \leq \frac{1}{3} N^{1 + \frac{1}{3} \varepsilon}, \]
where we use \( \gamma < \frac{1}{2} \) as well as \( \delta \) and \( \kappa \) being small enough.
For \( z \in P^F \), Proposition 3.3 implies that
\[
\sum_{z \in P^F} N^\left(\|a_z\| + \|b_z\|\right)^{2\gamma^2} \leq N^{1 + 2\delta + \frac{1}{1 - \kappa} \times 2\gamma^2} \leq \frac{1}{3} N^{1 + \frac{1}{2}\varepsilon}.
\]

For \( z \in Q^\ell \), Proposition 3.3 implies that
\[
\sum_{z \in Q^\ell} N^\left(\|a_z\| + \|b_z\|\right)^{2\gamma^2} \leq 40 \times 2^{(1 + \delta)\ell} N^{\left(\frac{n(1 + \kappa) + 1 - \ell}{\kappa + 1}\right)} 2\gamma^2 = 40 N^{\frac{1 + \kappa}{1 - \kappa} 2\gamma^2} \left(2^{1 + \delta - \frac{1}{1 - \kappa} 2\gamma^2}\right)^\ell.
\]

It follows that for \( C = \frac{40 \times \frac{2^{1 + \delta - \frac{1}{1 - \kappa} 2\gamma^2}}{2^{1 + \delta - \frac{1}{1 - \kappa} 2\gamma^2} - 1}}{2^{1 + \delta - \frac{1}{1 - \kappa} 2\gamma^2} - 1} \),
\[
\sum_{\ell = \ell_0 + 1}^{\ell_1} \sum_{z \in Q^\ell} N^\left(\|a_z\| + \|b_z\|\right)^{2\gamma^2} \leq C N^{\frac{1 + \kappa}{1 - \kappa} 2\gamma^2} \left(2^{1 + \delta - \frac{1}{1 - \kappa} 2\gamma^2}\right)^{\ell_1 + 1}.
\]
Recall \( \ell_1 = \lfloor \log_2(\lceil N^{1 - 2\kappa} \rceil + 2^{\ell_0 + 1}) \rfloor - 1 \), where \( \ell_0 \leq \delta^2 n \). Hence, \( 2^{\ell_1 + 1} \leq \lceil N^{1 - 2\kappa} \rceil + 2N^{\delta^2} \leq N \).

Therefore,
\[
\sum_{\ell = \ell_0 + 1}^{\ell_1} \sum_{z \in Q^\ell} N^\left(\|a_z\| + \|b_z\|\right)^{2\gamma^2} \leq C N^{\frac{1 + \kappa}{1 - \kappa} 2\gamma^2} N^{1 + \delta - \frac{1}{1 - \kappa} 2\gamma^2} = C N^{1 + \delta + \frac{\kappa}{1 - \kappa} 2\gamma^2} \leq \frac{1}{3} N^{1 + \frac{1}{2}\varepsilon},
\]
provided that \( N \) is sufficiently large. Altogether, this completes the proof of the proposition.

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