On algebraic properties of topological full groups

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Abstract. We discuss the algebraic structure of the topological full group \([T]\) of a Cantor minimal system \((X, T)\). We show that \([T]\) has a structure similar to a union of permutational wreath products of the group \(\mathbb{Z}\). This allows us to prove that the topological full groups are locally embeddable into finite groups, give an elementary proof of the fact that the group \([T]'\) is infinitely presented, and provide explicit examples of maximal locally finite subgroups of \([T]\). We also show that the commutator subgroup \([T]'\), which is simple and finitely-generated for minimal subshifts, is decomposable into a product of two locally finite groups, and that \([T]\) and \([T]'\) possess continuous ergodic invariant random subgroups.

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§1. Introduction

In the paper we study algebraic properties of a completely new, from the geometric group theory point of view, class of groups—the full groups of dynamical systems. Our goals are to develop machinery for the study of full groups, to establish a number of new results, and to survey known facts. We also hope to attract the attention of specialists in group theory to full groups as they possess very unusual algebraic properties, which seem never to have appeared in the literature before. The algebraic properties of full groups make them appear in other fields of mathematics; see, for example, a recent preprint [1], where full groups were applied to the complexity problem of the isomorphism relation between finitely-generated simple groups. Since the theory of full groups lies on the intersection of the theory of dynamical systems and group theory, we take extra care when discussing dynamical results so that the paper is accessible to non-specialists in dynamics.

Full groups first appeared 50 years ago in the seminal paper of Dye [2] as an algebraic invariant of orbit equivalence for measure-preserving dynamical systems. Even though full groups contain complete information about the orbits of the underlying systems, their algebraic structure has remained largely unknown.

Let \(G\) be a group acting on a set \(X\). The set \(X\) may be considered along with some structure such as a measure, a Borel \(\sigma\)-algebra, or a topology. Consider the
group \([G]\), termed the **full group** of \(G\), consisting of all automorphisms \(S\) of \(X\) preserving the structure such that for every \(x \in X\), \(S(x) = g(x)\) for some \(g \in G\). It turns out that in many cases \([G]\) is so 'rich' that any isomorphism \(\alpha\) between \([G]\) and \([H]\), where \(H\) is a group which also acts on \(X\) by structure-preserving automorphisms, will always be spatially generated in the sense that there is an automorphism \(\Lambda: X \to X\) with \(\alpha(g) = \Lambda \circ g \circ \Lambda^{-1}\) for every \(g \in [G]\). Such results have been established for groups acting on a Cantor set [3] (Theorem 4.5(c))–[8], on noncompact zero-dimensional spaces [9], standard Borel spaces [10], and measure spaces ([2] and [11]). As an application to the theory of dynamical systems, these results show that algebraic (group) properties of full groups completely determine the class of orbit equivalence for the underlying dynamical systems. This raises the question of interaction between dynamical characteristics of the system and algebraic properties of full groups.

In the present paper, we focus on algebraic properties of full groups for systems acting on Cantor sets. Consider a pair \((X, T)\), where \(X\) is a Cantor set and \(T: X \to X\) is a homeomorphism of \(X\). Denote by \([T]\) the full group of \(\{T^n\}_{n \in \mathbb{Z}}\). Consider a subgroup \([[[T]]]\), termed the **topological full group**, consisting of all homeomorphisms \(S\) of \(X\) such that \(Sx = T^{f_S(x)}x\) for every point \(x\), where \(f_S: X \to \mathbb{Z}\) is a continuous function. We would like to emphasize that no group topology is involved and the word ‘topological’ refers to the setting of topological dynamics. The group \([[[T]]]\) is countable and its commutator subgroup is simple and finitely generated under some assumptions on \((X, T)\) (listed later in the text).

One of the main techniques in Cantor dynamics is the method of periodic approximation, which mimics the behaviour of \((X, T)\) by periodic transformations. This means roughly that there exists a refining sequence of clopen (Kakutani-Rokhlin) partitions \(\{\Xi_n\}_{n \geq 1}\) (§3) such that the action of \(T\) can be viewed as nearly a permutation of \(\Xi_n\)-atoms. We will use the Kakutani-Rokhlin tower analysis to show that every element of \([[[T]]]\) can be represented uniquely as a product of a permutation from Sym(\(\Xi_n\)-atoms) and induced transformations of \(T\) on some clopen sets (Theorem 4.7). The proof of this result indicates that \([[[T]]]\) has a structure similar to an increasing union of wreath products \(\mathbb{Z} \text{ Wr Sym}(\Xi_n\text{-atoms})\). We then use Theorem 4.7 to establish that the topological full group of a Cantor minimal system is locally embeddable into finite groups (LEF groups) (Theorem 2.6). We notice that Theorem 4.7 has recently been used to show that the closure of the full group in Homeo(\(X\)) is topologically simple [12].

Theorem 4.7 can also be used for creating an effective algorithm for finding the normal form of group elements when the underlying dynamical system is effectively defined (such as substitution systems).

As corollaries of Theorem 2.6, we find:

(1) that topological full groups are sofic;

(2) a new and elementary proof of the fact that topological full groups are not finitely presented (established originally by Matui [13]) (here we use arguments from [14] related to convergence in the space of marked groups); and

(3) that the universal theory of topological full groups coincides with that of finite groups.

We will also prove that the commutator subgroup \([[[T]]]'\), while being simple and finitely generated for minimal subshifts, can be represented as \([[[T]]]' = \Gamma_1 \Gamma_2\),
where $\Gamma_i, i = 1, 2$, is a (maximal) locally finite subgroup. To the best of our knowledge, this is the first example of a simple finitely generated group with such a factorization property. We will also construct explicit examples of maximal locally finite subgroups.

The structure of the paper is the following. In §2 we list known, and establish several new, algebraic properties of topological full groups.

Section 3 is devoted to the basics of Cantor dynamics. We introduce all necessary definitions from the theory of dynamical systems and explain the essence of Kakutani-Rokhlin tower analysis, which will be the main technical ingredient in our proofs. We would like to mention that the present paper is perhaps the first work that uses Kakutani-Rokhlin partitions for the study of algebraic properties of full groups.

In §4 we establish that the structure of a topological full group is similar to that of permutational wreath products (Theorem 4.7). This result was inspired by the paper of Bezuglyi and Kwiatkowski [6].

Section 5 is devoted to the proof of the LEF property and to the discussion of locally finite subgroups.

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§2. Algebraic properties of topological full groups

This section can be considered as a short survey of algebraic properties of full groups as we list results scattered over numerous papers. Throughout the present paper the symbol $X$ will stand for a Cantor set, that is, for any zero-dimensional compact metric space without isolated points, and $T: X \to X$ will denote a homeomorphism of $X$. Recall that all Cantor sets are homeomorphic to each other and, in particular, to the space of sequences $\{0, 1\}^\mathbb{N}$.

Consider a pair $(X, T)$, where $X$ is a Cantor set and $T: X \to X$ is a homeomorphism. The pair $(X, T)$ is called a Cantor dynamical system. If $X$ has no proper closed $T$-invariant subsets, then $T$ is called minimal. We will always assume that a dynamical system $(X, T)$ is minimal, that is, if $Y \subset X$ is a nonempty closed set with $T(Y) = Y$, then $Y = X$. The minimality of $T$ is equivalent to the property that every $T$-orbit $\{T^n(x) : n \in \mathbb{Z}\}$ is dense in $X$. Observe that if $T$ is a minimal homeomorphism, then $T^k, k \neq 0$, might be nonminimal, although it will still have no finite orbits. A homeomorphism that has no finite orbits is called aperiodic. The following is an equivalent definition of the topological full group.

Definition 2.1. A topological full group $[[T]]$ is defined as a group consisting of all homeomorphisms $S$ for which there is a finite clopen partition $\{C_1, \ldots, C_m\}$ of $X$ and a set of integers $\{n_1, \ldots, n_m\}$ such that $S|C_i = T^{n_i}|C_i$ for every $i = 1, \ldots, m$.

Consider the topological full group $[[T]]$. For a point $x \in X$, denote by $[[T]]_x$ the set of all elements $S \in [[T]]$ with

$S(\{T^n(x) : n \geq 0\}) = \{T^n(x) : n \geq 0\}$. 
It follows from the compactness and zero-dimensionality of the Cantor set that $[[T]]$ is countable. The groups $[[T]]$ and $[[T]]'$ (the commutator subgroup) are complete algebraic invariants for flip conjugacy of the system $(X, T)$: see [5] and [7]. We recall that two dynamical systems $(X, T)$ and $(Y, S)$ are flip conjugate if either $T$ and $S$ or $T$ and $S^{-1}$ are topologically conjugate. This implies that dynamical invariants of flip conjugacy such as topological entropy ([15], Theorem 7.3), spectral characteristics, the number of invariant ergodic measures, and others, can be used to distinguish full groups and their commutator subgroups up to isomorphism. Observe that for every positive real number $\alpha$, there exists a minimal subshift with topological entropy equal to $\alpha$: see [16] and [17]. This implies that there are uncountably many nonisomorphic topological full groups as well as their commutator subgroups. In particular, there are uncountably many nonisomorphic finitely generated simple groups satisfying the properties listed in the following theorem and other results of the present paper.

**Theorem 2.2.** (1) For every $S \in [[T]]$, let $\varphi(S) = \int_X f_S(x) \, d\mu(x)$, where $\mu$ is a $T$-invariant probability measure. Then $\varphi$ is a homomorphism from $[[T]]$ onto $\mathbb{Z}$ ([5], §5). Thus, $[[T]]$ is indicable.

(2) For every point $x \in X$, $[[T]]_x$ is locally finite [5].

(3) If $x, y \in X$ are points lying in different $T$-orbits, then $\text{Ker}(\varphi) = [[T]]_x \cdot [[T]]_y$ is a product of two locally finite groups ([13], Lemma 4.1). (4) Any finite group can be embedded into $[[T]]'$ (see Remark 4.3). The group $[[T]]'$ contains a subgroup isomorphic to an infinite direct sum of the group $\mathbb{Z}$ (again, see Remark 4.3).

(5) The commutator subgroup $[[T]]'$ is simple ([13], Theorem 4.9). Furthermore, if $H$ is a normal subgroup in $[[T]]$, then $[[T]]' \subset H$ ([7], Theorem 3.4).

(6) The commutator subgroup $[[T]]'$ is finitely generated if and only if $(X, T)$ is topologically isomorphic to a minimal subshift over a finite alphabet ([13], Theorem 5.4).

(7) If $(X, T)$ is a minimal subshift, then $[[T]]'$ is not finitely presented ([13], Theorem 5.7).

(8) The commutator subgroup $[[T]]'$ contains the lamplighter subgroup if and only if $(X, T)$ is not conjugate to an odometer (a rotation on $\{p_n\}$-adic integers).

Matui [13] was apparently the first author who obtained purely algebraic results in the theory of full groups of Cantor systems. We notice, though, that the existence of connections between properties of full groups and those of the underlying dynamical systems could already be seen from results of the type of Dye on orbit equivalence: see [5] and the aforementioned references.

In the first version of the present paper, we conjectured that the topological full group for a minimal Cantor system is amenable. Juschenko and Monod [18] have recently announced an affirmative solution to our conjecture.

**Theorem 2.3 ([18]).** The topological full group of a Cantor minimal system is amenable.

We notice that the minimality (or, at least, aperiodicity) of a dynamical system seems to be crucial for the amenability of topological full groups as, for example, the
topological full group of any shift of finite type contains a free non-abelian subgroup [H. Matui, private communication]. We also mention that there are examples of minimal Cantor $\mathbb{Z}^2$-actions, whose topological full groups contain free non-abelian subgroups [19].

Denote by $AG$ the class of amenable groups, and by $SAG$, termed the class of subexponentially amenable groups, the class of groups containing groups of sub-exponential growth and closed under (I) direct unions and (II) group extensions: see [20] for details. The family $SAG$ can be described by transfinite induction. Namely, denote by $SAG_0$ the class of groups of sub-exponential growth. If $\alpha$ is not a limit ordinal, set $SAG_\alpha$ to be the class of groups obtained from $SAG_{\alpha-1}$ either by operation (I) or by operation (II). If $\alpha$ is a limit ordinal, set $SAG_\alpha = \bigcup_{\beta < \alpha} SAG_\beta$. Then $SAG = \bigcup_\alpha SAG_\alpha$, where $\alpha$ runs over all ordinals; see [21] for a related construction of elementary amenable groups.

**Proposition 2.4.** Let $(X, T)$ be a minimal subshift over a finite alphabet. Then neither of the groups $[[T]]$ or $[[T]]'$ is subexponentially amenable.

**Proof.** It is enough to show that $[[T]]'$ is not subexponentially amenable as the class $SAG$ is closed under passing to subgroups. Assume the converse. Choose the least ordinal $\alpha > 0$ with $[[T]]' \in SAG_\alpha$. Since the lamplighter group has exponential growth, we get that $[[T]]' \not\in SA_0$ (Theorem 2.2, (8)). Thus, $\alpha > 0$.

As $[[T]]'$ is finitely generated (Theorem 2.2), $\alpha$ cannot be a limit ordinal. So, $[[T]]'$ has to be an extension of a group from $SAG_{\alpha-1}$ by a group from the same class. This is impossible in view of the simplicity of $[[T]]'$. This contradiction yields the result.

**Definition 2.5.** A group $G$ is called locally embeddable into finite groups (abbr. LEF) if for every finite set $F \subset G$ there is a finite group $H$ and a map $\varphi: G \to H$ such that

1. $\varphi$ is injective on $F$; and
2. $\varphi(gh) = \varphi(g)\varphi(h)$ for every $g, h \in F$.

The notion of an LEF group was introduced by Stepin in the 1980s as a group property equivalent to the existence of uniform free approximations for group actions. We refer the reader to [22] and [23], Ch. 7, for a detailed exposition of the theory of LEF groups; see also the references therein for complete historical information. The proof of the following result is presented in §5.

**Theorem 2.6.** The topological full group of any Cantor minimal system is an LEF group.

We mention that the LEF property and amenability do not imply each other ([20], [22] and [23], Ch. 7), yet both imply soficity ([23], Corollary 7.5.11). A group is called sofic if it embeds into an ultraproduct of a sequence of finite symmetric groups endowed with a normalized Hamming metric.

**Corollary 2.7.** The topological full group $[[T]]$ and its commutator subgroup $[[T]]'$ are sofic.

We would like to mention that in [24] Elek recently suggested a different approach, related to some ideas from [18], to obtain the LEF property for the topological full group of minimal $\mathbb{Z}$-subshifts.
Remark 2.8. Since there are uncountably many nonisomorphic finitely generated groups $[[T]]'$, we get that the class $AG \setminus SAG$ contains a continuum of simple finitely generated LEF groups.

Each finitely generated LEF group can also be defined as a limit of finite groups in the topological space of marked groups (or Cayley graphs) introduced in [14]. If a finitely presented group $G$ is a limit of a sequence of finite groups $\{G_n\}$, then starting from some index $n$, all groups $G_n$ are quotients of $G$. Thus, finitely presented infinite simple groups are isolated points in the space of marked groups ([25], §2) and, therefore, cannot have the LEF property. Hence, Theorem 2.6 and the simplicity of $[[T]]'$ imply that $[[T]]'$, when $T$ is a minimal subshift over a finite alphabet, is a finitely generated, but infinitely presented, group. The latter result was first obtained by Matui by different (dynamical) methods.

Corollary 2.9. For every minimal subshift $T$ over a finite alphabet, the commutator subgroup $[[T]]'$ is infinitely presented.

In §5, we will modify the proof of Lemma 4.1 from [13] to establish that the commutator subgroup can be represented as a product of two locally-finite subgroups. We notice that groups admitting factorizations $G = A \cdot B$ with $A$ and $B$ being locally finite subgroups have been studied in numerous papers; see, for example, the publications of Amberg, Chernikov, Kegel, Subbotin, Sushchanskyy, Sysak and many others. See, in particular, [26] and the references therein. To the best of our knowledge, commutators of topological full groups are the first examples of finitely-generated simple groups that can be factorized into a product of two locally-finite subgroups. For a point $x \in X$, set $\Gamma_x = [[T]]_x \cap [[T]]'$. Note that each group $\Gamma_x$ is locally finite [5].

Theorem 2.10. Let $x, y \in X$ be points with disjoint $T$-orbits. Then $[[T]]' = \Gamma_x \cdot \Gamma_y$.

2.1. Group laws and universal theories.

Definition 2.11. (1) A group $G$ is said to satisfy a group law if there exists $k \geq 1$ and a word $w \in F_k$, where $F_k$ is the free group of rank $k$, such that $w(g_1, \ldots, g_k) = 1$ for any $g_1, \ldots, g_k \in G$.

(2) If $G$ acts on a set $Y$, $G$ is said to separate $Y$ if for any finite set $C \subset Y$, the pointwise stabilizer $st_G(C)$ does not stabilize any other point outside $C$.

In [27] Abért shows that if a permutation group $G$ separates $Y$, then $G$ does not satisfy any group law. We notice that $[[T]]'$ separates $X$ since for any clopen set $O$ and a point $x \in O$ there is $\gamma \in [[T]]'$ with $\text{supp}(g) \subset O$ and $g(x) \neq x$: see, for example, [8], Proposition 2.4. Thus, we immediately get the following result.

Proposition 2.12. The topological full group $[[T]]$ satisfies no group law.

The following observation was suggested by Mark Sapir. Recall that a universal sentence in a first order language $\mathcal{L}$ is any sentence of the form $\forall x_1, \ldots, \forall x_k \Phi$, where $\Phi$ is a quantifier-free formula. The universal theory for a class of groups $\mathcal{K}$ is the family of all universal sentences that are valid in all groups from $\mathcal{K}$. The

\footnote{This type of argument was used by the first author to show that all groups of intermediate growth constructed in [14] are infinitely presented.}
universal theory of \( \mathcal{K} \) is denoted by \( \text{Th}_\forall(\mathcal{K}) \). A group \( G \) is called a model of a set of sentences \( W \) if every sentence from \( W \) holds in \( G \). We recall that a group \( G \) is LEF if and only if it is embeddable into an ultraproduct of finite groups [23], Theorem 7.2.5. The latter is a model for \( \text{Th}_\forall(\mathcal{F}) \), where \( \mathcal{F} \) is the class of finite groups. We refer the reader to [28] for a survey on local embeddability and universal models.

**Proposition 2.13.** For any minimal Cantor system \((X, T)\), the universal theory of \( [[T]]' \) coincides with the universal theory of the class of finite groups and hence is undecidable.

**Proof.** For any Cantor minimal system \((X, T)\), \( [[T]]' \) contains any finite group (Theorem 2.2). Thus, \( \text{Th}_\forall([[T]]') \subset \text{Th}_\forall(\mathcal{F}) \). On the other hand, since \( [[T]]' \) is LEF, it can be embedded into an ultraproduct of finite groups. Hence, \( \text{Th}_\forall(\mathcal{F}) \subset \text{Th}_\forall([[T]]') \).

The result of Slobodskoi [29] on the undecidability of the universal theory for finite groups completes the proof.

### 2.2. Word problem.

Consider a subshift \((X, T)\) over a finite alphabet \( \mathcal{A} \). Denote by \( L(X) \) the language of \( X \), that is, the set of all finite words appearing in sequences of \( X \).

**Definition 2.14.** The language \( L(X) \) is called recursive if there exists a Turing machine which for any input of a word \( w \) over the alphabet \( \mathcal{A} \), decides whether \( w \) lies in \( L(X) \).

For example, the languages of primitive substitution dynamical systems are recursive [30].

**Theorem 2.15.** Let \((X, T)\) be a minimal subshift with recursive language. Then the word problem in \( [[T]]' \) is decidable.

**Proof.** Fix generators \( \{g_1, \ldots, g_p\} \) for \( [[T]]' \) (Theorem 2.2). The explicit definition of generators can be found in the proof of Theorem 5.4 in [13]. This definition shows that if the language is recursive, then the generators can be effectively defined. We leave the checking of the details to the reader.

Find \( n > 1 \) such that the cocycles \( f_{g_i} \) are constant on each cylinder word (centred at zero) of length \( n \). Let \( M = \max_i \max_{x \in X} |f_{g_i}(x)| \). Fix a group element \( w = g_{i_1} \cdots g_{i_k} \), where \( \{g_{i_j}\} \) are generators. Find all words \( \mathcal{W} \) in \( L(X) \) of length \( Mnk \). We note that by assumption there is an effective algorithm for writing out all such words.

Fix a word \( v \in L(X) \) of length \( n \). For each generator, \( g_i[v] = T^{n_i}v \), for some (unique) \( n_i \), where \( [v] \) is the cylinder set centred at zero, defined by \( v \). Note that \( |n_i| \leq M \). Thus, knowing the action of each generator on every word of length \( Mnk \), we can check whether the action of \( w \) is trivial on each cylinder set \( [v] \), \( v \in \mathcal{W} \). If so, then \( w \equiv \text{id} \). Otherwise, \( w \not\equiv \text{id} \).

### 2.3. Invariant random subgroups and totally nonfree actions.

Consider the space \( \text{SUB}_T \) of all subgroups of \( [[T]] \) (identified with a closed subspace of \( \{0, 1\}[[T]] \) via characteristic functions). Recall that any group acts on the set of its subgroups by conjugation. This action will be referred to as adjoint.
Definition 2.16. A probability measure on the space of all subgroups invariant with respect to the adjoint action is called an invariant random subgroup.

An action of a group $G$ on the set $X$ is called completely nonfree if for any two distinct points $x, y \in X$, their pointwise stabilizers $\text{st}_G(x)$ and $\text{st}_G(y)$ are different subgroups of $G$.

Proposition 2.17. The actions of $[[T]]$ and $[[T]]'$ on $X$ are completely nonfree.

Proof. Fix two distinct points $x, y \in X$. Find a clopen set $U$ and integers $n < m < k$ such that $x \in U$, and the sets $U, T^n U, T^m U$ and $T^k U$ are disjoint and do not contain $y$. Define an element $Q$ as follows:

$$Q|U = T^n|U, \quad Q|T^n U = T^{-n}|T^n U, \quad Q|T^m U = T^{k-m}T^m(U), \quad Q|T^k U = T^{m-k}|T^k U$$

and $Q = \text{id}$ elsewhere. Note that $Q \in \text{St}_{[[T]]}(y) \setminus \text{St}_{[[T]]}(x)$. Since $Q$ is a product of two conjugate involutions, we get that $Q \in [[T]]'$.

In [31] Vershik introduced the notion of a totally nonfree action as an action of a group $G$ on a measure space $(X, \mu)$ such that the family of sets $\{x \in X : g(x) = x\}$, $g \in G$, generates the Borel $\sigma$-algebra. We note that every completely nonfree action is automatically totally nonfree. In [31] Vershik also posed the question of which countable groups admit continuous invariant random subgroups.

Proposition 2.18. Let $(X, T)$ be a Cantor minimal system. Then both $[[T]]$ and $[[T]]'$ have nonatomic (ergodic) invariant random subgroups.

Proof. Since distinct points have different stabilizers, the map $\alpha : X \to \text{SUB}_T$ given by $\alpha(x) = \text{st}_{[[T]]}(x)$ is injective. We notice that any element $g \in \text{st}_{[[T]]}(x)$ also stabilizes some neighbourhood of $x$, as the action of the group $\{T^n\}$ on $X$ is free. Hence the map $\alpha : X \to \text{SUB}_T$ is a homeomorphism on its image: see [32], Lemma 5.4. This implies that the dynamical systems $(\alpha(X), [[T]])$ and $(X, [[T]])$ are conjugate. So any $T$-invariant measure on $X$ gives rise to a random subgroup on $[[T]]$. The proof for $[[T]]'$ is identical. A $T$-ergodic measure will produce an ergodic random subgroup.

Since $[[T]]'$ is simple, it has no atomic invariant subgroups except for $\{\text{id}\}$. The group $[[T]]'$, on the contrary, has normal subgroups $[[T]]'$ and $\text{Ker}(\varphi)$: see Theorem 2.2. Thus, Dirac measures supported by these subgroups are nontrivial invariant random subgroups of $[[T]]$.

§ 3. Kakutani-Rokhlin partitions

In this section we fix our notation and introduce the necessary definitions from the theory of Cantor dynamical systems. The interested reader may also consult the papers [4]–[7], [13], [33] and [34].

We will sometimes need to use the diameter of a set. By this we mean that the diameter is defined by some metric on $X$ compatible with the topology. If $d$ is a metric on $X$ and $\mathcal{U}$ is an open cover of $X$, then, due to the compactness of $X$, there exists a number $\delta > 0$ such that any set $A \subset X$ with $d$-diameter less than $\delta$
is contained in at least one member of \( \mathcal{U} \). The number \( \delta \) is called the Lebesgue number of the cover \( \mathcal{U} \).

**Definition 3.1.** Consider a clopen set \( B \) such that the sets 
\[ \xi = \{ B, TB, \ldots, T^{n-1}B \} \]
are disjoint. The family \( \xi \) is called a \( T \)-tower with base \( B \) and height \( n \). A clopen partition of \( X \) of the form 
\[ \Xi = \{ T^iB_v : 0 \leq i \leq h_v - 1, \ v = 1, \ldots, q \} \]
is called a Kakutani-Rokhlin partition. The sets \( \{ T^iB_v \} \) are called atoms of \( \Xi \).

Fix an arbitrary clopen set \( A \subset X \). Define a function \( t_A : A \to \mathbb{N} \) by setting 
\[ t_A(x) = \min\{ k \geq 1 : T^kx \in A \} \]
Using the minimality of \( T \), one can check that half-orbits \( \{ T^nx : n \geq 0 \}, \ x \in X \), are dense in \( X \). This shows that the function \( t_A \) is well-defined and takes only finite values. Since \( T \) is a homeomorphism of a Cantor set, \( t_A \) is continuous and, hence, bounded. Sometimes, \( t_A \) is referred to as the function of the first return.

Denote by \( K \) the set of all integers \( k \in \mathbb{N} \) such that the set \( A_k = \{ x \in A : t_A(x) = k \} \) is nonempty. It follows from the continuity of \( t_A \) that \( K \) is finite and 
\[ A = \bigsqcup_{k \in K} A_k \]
is a clopen partition. The definition of \( t_A \) implies that, for every \( k \in K \), the sets \( \{ A_k, TA_k, \ldots, T^{k-1}A_k \} \) are disjoint. Indeed if \( T^iA_k \cap T^jA_k \neq \emptyset \) for \( 0 \leq i < j \leq k - 1 \), then \( A_k \cap T^{j-i}A_k \neq \emptyset \). It follows that there exists \( x \in A_k \) with \( T^{j-i}x \in A_k \). Hence \( k = t_A(x) \leq j - i \leq k - 1 \), which is a contradiction. Similarly, one can check that the family 
\[ \Xi = \{ T^iA_k : 0 \leq i \leq k - 1, \ k \in K \} \]
consists of disjoint sets and 
\[ X = \bigsqcup_{k \in K} \bigsqcup_{i=0}^{k-1} T^iA_k. \]
Thus, \( \Xi \) is a Kakutani-Rokhlin partition of \( X \).

**Definition 3.2.** The union of the sets \( B(\Xi) = \bigsqcup_{k \in K} A_k \) is called the base of the partition \( \Xi \) and the union of the top levels \( H(\Xi) = \bigsqcup_{k \in K} T^{k-1}A_k \) is called the top or roof of the partition.

**Remark 3.3.** (1) A convenient way to look at Kakutani-Rokhlin partitions is by tracing the trajectories of points. Namely, if \( x \in A_k \) for some \( k \in K \), then \( x \) moves to the level \( TA_k \) under the action of \( T \). Consequently, applying \( T \) to the point \( x \), we see that \( x \) is moving up until it gets to the top level, that is, \( T^{k-1}x \in H(\Xi) \). When we apply \( T \) once again, the point \( T(T^{k-1}x) \) returns to the set \( A = \bigsqcup_{i \in K} A_i \). However, we cannot predict the exact set \( A_i, i \in K \), to which the point \( T(T^{k-1}x) = T^kx \) returns.
(2) Note that $T(H(\Xi)) = B(\Xi)$.

(3) Let $B$ be an arbitrary clopen set. We can partition each $A_k$, $k \in K$, into a finite number of clopen subsets $\{C_{j,k}\}$, $j = 0, \ldots, p_k$, so that each set $T^iC_{j,k}$, $0 \leq i \leq k - 1$, is either disjoint from $B$ or is a subset of $B$. Thus, $\Xi' = \{T^iC_{j,k} : k \in K, j = 0, \ldots, p_k, \ i = 0, \ldots, k - 1\}$ is a clopen partition refining $\Xi$ and $B$.

Fix a point $x_0 \in X$. Choose a decreasing sequence of clopen sets $\{E_n\}_{n \geq 1}$ with $\bigcap_{n \geq 1} E_n = \{x_0\}$. Denote by $\Xi_n$ the clopen partition constructed by the function $t_{E_n}$, as above. Refining partitions $\Xi_n$ as in Remark 3.3(3) if necessary, we can assume that the sequence $\{\Xi_n\}_{n \geq 1}$ satisfies the following conditions.

1) The partitions $\{\Xi_n\}_{n \geq 1}$ generate the topology of $X$.

2) The partition $\Xi_{n+1}$ refines $\Xi_n$.

3) $\bigcap_{n \geq 1} B(\Xi_n) = \{x_0\}$ and $B(\Xi_{n+1}) \subset B(\Xi_n)$ for every $n \geq 1$.

Since $\Xi_n$ is a Kakutani-Rokhlin partition, we can represent it as

$$\Xi_n = \{T^iB_v^{(n)} : 0 \leq i \leq h_v^{(n)} - 1, \ v = 1, \ldots, v_n\}$$

for some clopen set $B_v^{(n)}$ and positive integers $v_n$ and $h_v^{(n)}$, $v \in V_n = \{1, \ldots, v_n\}$. Set $\xi_v^{(n)} = \{B_v^{(n)}, \ldots, T^{h_v^{(n)}-1}B_v^{(n)}\}$. Then $\Xi_n$ is a disjoint union of towers $\xi_v^{(n)}$, $v \in V_n$. Note that $h_v^{(n)}$ is the height of the tower $\xi_v^{(n)}$.

Fix a sequence of positive integers $\{m_n\}$ such that $m_n \to \infty$ as $n \to \infty$. Take a subsequence of $\{\Xi_n\}_{n \geq 1}$ (we will omit an extra subindex) so that each Kakutani-Rokhlin partition $\Xi_n$ additionally meets the following conditions.

4) $h_n \geq 2m_n + 2$, where $h_n = \min_{v \in V_n} h_v^{(n)}$.

5) The sets $T^iB(\Xi_n)$ have the property

$$\text{diam}(T^iB(\Xi_n)) < \frac{1}{n} \quad \text{for} \quad -m_n - 1 \leq i \leq m_n. \quad (\dagger)$$

Remark 3.4. Observe that we do not need the minimality of $T$ to get partitions satisfying properties (1)–(4). Such partitions exist for any aperiodic$^2$ homeomorphism [35]. However, condition (5) holds only for minimal (or, at least, essentially minimal$^3$) systems [33].

§ 4. Rotations and permutations

In this section, we introduce two kinds of group elements: permutations (Definition 4.1) and rotations (Definition 4.5). We then show that every element can be factored uniquely into a product of a permutation and a rotation (Theorem 4.7). The results of this section are inspired by the paper [6]: cf. [36], Theorem 3.3. Theorem 2.2 of [6] describes how elements of [[T]] move atoms of Kakutani-Rokhlin partitions. Fix a sequence of partitions $\{\Xi_n\}_{n \geq 1}$ that satisfies properties (1)–(5) from § 3. We use the same notation as in § 3.

**Definition 4.1.** Fix an integer $n \geq 1$. We say that a homeomorphism $P \in [[T]]$ is an $n$-permutation if (1) its orbit cocycle $f_P$ is compatible (constant on atoms) with $\Xi_n$, and (2) for any point $x \in T^iB_v^{(n)}$ ($0 \leq i \leq h_v^{(n)} - 1, \ v \in V_n$) we have that

2A homeomorphism is called aperiodic if every orbit is infinite.

3A system is called essentially minimal if it has only one minimal component.
0 \leq f_P(x) + i \leq h_v^{(n)} - 1. The latter condition means that \( P \) permutes atoms only within each tower without moving points over the top or the base of the tower. We will call \( P \) simply a permutation when the partition \( \Xi_n \) is clear from the context.

Recall that the set \( V_n = \{1, \ldots, v_n\} \) stands for the index set enumerating \( T \)-towers of \( \Xi_n \). Then each permutation \( P \) can be factored uniquely into a product of permutations \( P_1, \ldots, P_{v_n} \) such that \( P_i \) acts only within the tower \( \xi_i^{(n)} \), \( i = 1, \ldots, v_n \).

**Definition 4.2.** Fix a clopen set \( A \). Let \( t_A \) be the function of the first return to the set \( A \). Define a homeomorphism \( T_A \) by \( T_A(x) = T^{t_A(x)}x \) when \( x \in A \), and \( Tx = x \) otherwise. The homeomorphism \( T_A \) belongs to \( \|T\| \) and is called an induced transformation of \( T \).

**Remark 4.3.** (1) Note that the minimality of \( T \) implies the minimality of the induced transformation \( T_A \) on \( A \).

(2) Since minimal homeomorphisms have no periodic points, the group generated by \( T_A \) is isomorphic to \( \mathbb{Z} \). Choose a sequence of disjoint clopen sets \( \{A_n\}_{n \geq 1} \) such that \( A_n = B'_n \sqcup B''_n \) with \( B''_n = T^{q_n}(B_n) \) for some \( q_n \). Then the element \( Q_n = T_{B''_n} \cdot T_{B'_n}^{-1} \) belongs to \( \|T\|' \), as \( T_{B''_n} \) and \( T_{B'_n} \) are conjugate in \( \|T\| \). Hence, the subgroup of \( \|T\|' \) generated by \( \{Q_n\}_{n \geq 1} \) is isomorphic to the infinite direct sum \( \bigoplus_{i=1}^{\infty} \mathbb{Z} \).

(3) Given an integer \( n > 0 \), choose a clopen set \( U \) such that the sets \( \mathscr{F} = \{U, Tu, \ldots, T^n U\} \) are disjoint. Then using powers of \( T \), we can embed a symmetric group \( \text{Sym}(n) \) into \( \|T\| \) by acting on \( \mathscr{F} \). In particular, any finite group is embeddable into \( \|T\| \). One can modify these arguments to show that any finite group also embeds into \( \|T\|' \).

For each integer \( 0 \leq i \leq h_n - 1 \), set
\[
U(i) = \bigsqcup_{v \in V_n} T^{h_v^{(n)} - i - 1} B_v^{(n)}, \quad D(i) = \bigsqcup_{v \in V_n} T^i B_v^{(n)}.
\]

The set \( U(i) \) (\( D(i) \)) consists of atoms that are at a distance \( i \) from the top (base) of \( \Xi_n \). Notice that the induced transformation \( T_{U(i)} \) has the form
\[
T_{U(i)}(x) = \begin{cases} 
    x & \text{if } x \notin U(i); \\
    T^{h_v^{(n)} - i - 1} B_v^{(n)} & \text{if } x \in T^{h_v^{(n)} - i - 1} B_v^{(n)} \text{ and } T^{i+1} x \in B_w^{(n)}. 
\end{cases}
\]

Similarly, the transformation \( T_{D(i)}^{-1} \) has the form
\[
T_{D(i)}^{-1}(x) = \begin{cases} 
    x & \text{if } x \notin D(i); \\
    T^{-h_w^{(n)}} x & \text{if } x \in T^{-h_w^{(n)}} B_v^{(n)} \text{ and } T^{-i-1} x \in T^{h_w^{(n)} - 1} B_w^{(n)}.
\end{cases}
\]

**Remark 4.4.** (1) Informally, the action of \( T_{U(i)} \) can be expressed as follows: \( T_{U(i)} \) moves any point \( x \in T^{h_v^{(n)} - i - 1} B_v^{(n)} \) to the top of the tower \( \xi_v^{(n)} \), then to the base of some tower \( \xi_w^{(n)} \); eventually this point is moved up to the level \( T^{h_w^{(n)} - 1 - i} B_w^{(n)} \).

This can be seen from the decomposition
\[
h_w^{(n)} = i + 1 + (h_v^{(n)} - 1 - i).
\]
Here the first summand shows the number of levels to the top of the tower $\xi_v^{(n)}$; the number 1 tells us that the point is mapped onto the base of $\Xi_n$; the last summand shows the number of steps needed to move the point to the level of the tower $\xi_w^{(n)}$ that is at the same distance from the top as the level $T^iB_v^{(n)}$. Observe that a similar description can be also applied to $T_D^{-1}(i)$.

(2) Notice that $T_U(i)$ and $T_D(i)$ belong to $[[T]]$. Observe also that their orbit cocycles might not be compatible with $\Xi_n$, though the cocycles will be compatible with some other partition $\Xi_m$, $m > n$.

In the following definition, the sequence of integers $\left\{m_n\right\}_{n \geq 1}$ satisfies Equation (1) of §3.

**Definition 4.5.** A homeomorphism $R \in [[T]]$ is called an $n$-rotation with rotation number less than or equal to $r > 0$ if there are two sets $S_d, S_u \subset \{0, \ldots, m_n\}$ such that

$$R = \prod_{i \in S_u} (T_U(i))^{l_i} \times \prod_{j \in S_d} (T_D(j))^{k_j}$$

for some integers $\{l_i\}$ and $\{k_j\}$ with $|l_i| \leq r$ and $|k_j| \leq r$. The sets $S_u$ and $S_d$ will be called supportive sets for $R$.

Since $U(i) \cap D(j) = \emptyset$ for all $0 \leq i \neq j \leq m_n < h_n/2$, $R$ is well-defined and its definition does not depend on the order of the product.

The supportive set $S_d$ shows which levels within a distance $m_n$ from the base of $\Xi_n$ are ‘occupied’ by the support of $R$. The set $S_u$ shows the same, but for levels within a distance $m_n$ of the top of $\Xi_n$. We would like to emphasize that the numbers $\{l_i\}$ and $\{k_j\}$ in the definition of rotations can take on arbitrary (positive and negative) integer values with absolute values not exceeding $r$. One can construct rotations whose rotation number is any given integer. If $R_1$ and $R_2$ are rotations with rotation numbers not exceeding $r_1$ and $r_2$, respectively, then the rotation number of $R_1R_2$ does not exceed $r_1 + r_2$.

**Example 4.6.** We illustrate the definitions of rotations and permutations on the example of the topological full group of the 2-odometer. Furthermore, we will completely describe the structure of this group. The description was suggested by Matui in private communications.

Set $Y = \{0, 1\}^\mathbb{N}$. Recall that the 2-odometer $O: Y \to Y$ is defined as $O(0^n1w) = 1^n0w$ for any sequence $1^n0w \in Y$ and $O(1^\infty) = 0^\infty$. For every $n \geq 1$, set $B_0^{(n)} = \{x \in Y : x_0, \ldots, x_{n-1} = 0^n\}$ and $B_i^{(n)} = O^iB_0^{(n)}$, $0 < i < 2^n - 1$. Then $\Xi_n = \{B_0^{(n)}, \ldots, B_{2^n-1}^{(n)}\}$ is a sequence of Kakutani-Rokhlin partitions $\left\{\Xi_n\right\}_{n \geq 1}$ satisfying conditions (1)–(5) from §3. Observe that $\bigcap_{n \geq 1} B(\Xi_n) = \{0^\infty\}$.

Fix $n \geq 1$. Consider a subset $G_n \subset [[O]]$ of elements $S$ such that the orbit cocycle $f_S$ is compatible with $\Xi_n$. Since the odometer $O$ permutes the atoms $\{B_0^{(n)}, \ldots, B_{2^n-1}^{(n)}\}$ cyclically, $S$ will also act as a permutation of these atoms. Denote the induced permutation on $\{0, \ldots, 2^n - 1\}$ by $\mathcal{P}_n(S)$. We notice that $G_n$ is, in fact, a subgroup of $[[O]]$ and $\mathcal{P}_n: G_n \to \text{Sym}(2^n)$ is a homomorphism.

If $S \in \text{Ker}(\mathcal{P}_n)$, that is, $\mathcal{P}_n(S) = \text{id}$, then for every $i = 0, \ldots, 2^n - 1$, there exists $k_i \in \mathbb{Z}$ such that

$$S|B_i^{(n)} \equiv O_i^{k_i}|B_i^{(n)},$$

(1)
where $O_i$ is the induced transformation of $O$ onto $B_i^{(n)}$. Since $\Xi_n$ consists of only one tower, we get that $D(i) = U(i)$ and $O_i = O_{D(i)} = O_{U(i)}$ for every $i = 0, \ldots, 2^n - 1$.

It follows from Equation (1) that $\text{Ker}(\mathcal{P}_n)$ is isomorphic to $\mathbb{Z}^{2^n}$. Observe that $\text{Ker}(\mathcal{P}_n)$ consists of exactly $n$-rotations in our terminology. The rotation number of $S \in \text{Ker}(\mathcal{P}_n)$ is

$$\max_{0 \leq i \leq 2^n - 1} |k_i|.$$ 

Denote by $\text{Perm}_n$ the set of all $n$-permutations of $[[O]]$. Observe that $\text{Perm}_n$ is a subgroup of $G_n$. As $\text{Ker}(\mathcal{P}_n) \cap \text{Perm}_n = \{1\}$ and $\mathcal{P}_n(\text{Perm}_n) = \text{Sym}(2^n)$, $G_n$ is a semidirect product of $\text{Perm}_n$ and $\text{Ker}(\mathcal{P}_n)$, which, at the same time, can be represented as a permutational wreath product of $\mathbb{Z}$ and $\text{Sym}(2^n)$. Notice that any element $S \in G_n$ can be factored uniquely as $S = PR$, where $P \in \text{Perm}_n$ is an $n$-permutation and $R \in \text{Ker}(\mathcal{P}_n)$ is an $n$-rotation. We will show in Theorem 4.7 that a similar factorization exists in the topological full group of any Cantor minimal system.

We fix the following notation. Let $P_v$ be a permutation of levels within the tower $\xi_v^{(n)}$. Note also that $P_v$ naturally induces a permutation of the set $\{0, \ldots, h_v^{(n)} - 1\}$. For permutations $P_v$ and $P_w$ of distinct towers, the equality $k = P_v(i) = P_w(i)$ will mean that the images of the atoms $T^i B_v^{(n)}$ and $T^i B_w^{(n)}$ are the atoms $T^k B_v^{(n)}$ and $T^k B_w^{(n)}$.

1. We will treat the set $\{0, \ldots, h_v^{(n)} - 1\}$ as a cyclic group (mod) $h_v^{(n)}$ and we will denote by $[-a, b]$ the set

$$[-a, b] = \{0, \ldots, b\} \cup \{h_v^{(n)} - 1, h_v^{(n)} - 2, \ldots, h_v^{(n)} - a\}$$

for $a, b \geq 0$.

2. We can naturally identify the group $\{0, \ldots, h_v^{(n)} - 1\}$ with $\{z^{k} : 1 \leq z \leq h_v^{(n)} = 1\}$. Denote by $d$ the metric on the unit circle normalized in such a way that $h_v^{(n)}$ is the total length of the circle. Denote by $d_v^{(n)}(0, h_v^{(n)} - 1)$ the metric induced on the set $\{0, \ldots, h_v^{(n)} - 1\}$. Observe that $d_v^{(n)}(0, h_v^{(n)} - 1) = 1$.

The following proposition establishes that every element of $[[T]]$ can be represented as a product of permutations and rotations.

**Theorem 4.7.** Let $Q \in [[T]]$. (1) There exists $n_0 > 0$ such that for all $n \geq n_0$, the homeomorphism $Q$ can be represented as $Q = PR$, where $P$ is an $n$-permutation and $R$ is an $n$-rotation with rotation number not exceeding 1. Furthermore, $P$ can be factorized (in a unique way) as a product of permutations $P_1, \ldots, P_v$ meeting the following conditions:

(i) $P_v$ acts only within the $T$-tower $\xi_v^{(n)}$, $v \in V_n$;
(ii) $P_v(i) = P_w(i)$ for all $i \in [-m_n, m_n]$ and for all $v, w \in V_n$;
(iii) $P_v$ induces a permutation of $\{0, \ldots, h_v^{(n)} - 1\}$ (denoted by the same symbol) with the property that $d_v^{(n)}(P_v(i), i) \leq n_0$ for all $i \in \{0, \ldots, h_v^{(n)} - 1\}$;
(iv) $R$ acts only on levels which are within a distance $n_0$ of the top or the bottom of the partition. In other words, the supportive sets of $R$ are contained in $\{0, 1, \ldots, n_0 - 1\}$.
(2) If $Q = P_2R_2$ is another factorization with $P_2$ being an $n$-permutation and $R_2$ being an $n$-rotation, then $P = P_2$ and $R = R_2$.

(3) For any finite set $F \subseteq \{[T]\}$, there exists $n_0 > 0$ such that for all $n \geq n_0$, the factorizations $Z = P_2R_Z$ of elements $Z \in F$ into $n$-permutations and $n$-rotations satisfy $P_{Z_1} \neq P_{Z_2}$ for $Z_1, Z_2 \in F$ and $Z_1 \neq Z_2$.

Proof. (1) We will split the proof into several steps.

(1-a) Choose an integer $n_0$ such that the orbit cocycle $f_Q$ is compatible (constant on atoms) with $\Xi_{n_0}$, and $n_0 \geq \max_{x \in X} |f_Q(x)|$. Furthermore, in view of the compactness of $X$, we can assume that $n_0$ is chosen so large that $1/n_0$ is a Lebesgue number for the clopen partition $\{f_Q^{-1}(k) : k \in \mathbb{Z}\}$, that is, any set having diameter less than $1/n_0$ is contained in the set $f_Q^{-1}(k)$ for some $k \in \mathbb{Z}$. Notice that the choice of $n_0$ implies that $f_Q$ is constant on each set $T^i B(\Xi_{n_0})$ for $-m_n - 1 \leq i \leq m_n$: see Equation (1) in §3.

Thus, for all $n \geq n_0$, if for some $x \in T^i B^{(n)}$ we have that $f_Q(x) + i \geq h_v^{(n)}$, then $f_Q[U(h_v^{(n)} - i)] \equiv \text{const}$. Since $n_0 \geq \max_{x \in X} |f_Q(z)|$, we have that $h_v^{(n)} - i \leq f_Q(x) \leq n_0$. Similarly, if $x \in T^i B^{(n)}$ is such that $f_Q(x) + i < 0$, then $f_Q[D(i)] \equiv \text{const}$ and $i \leq n_0$. These properties say that if an atom $T^i B^{(n)}$ is moved over the top or the bottom of $\Xi_n$ by $Q$, then this atom is taken from the $n_0$ lowest or $n_0$ highest levels of $\Xi_n$, that is, $d_v^{(n)}(i, 0) \leq n_0$ or $d_v^{(n)}(i, h_v^{(n)} - 1) \leq n_0$.

(1-b) Define a rotation $R \in \{[T]\}$ as follows:

$$R(x) = \begin{cases} T_{U(h_v^{(n)} - 1 - i)}(x) & \text{if } f_Q(x) + i \geq h_v^{(n)}, \\ T_{D(i)}^{-1}(x) & \text{if } f_Q(x) + i < 0, \\ x & \text{otherwise.} \end{cases}$$

Observe that the rotation number of $R$ is equal to 1 and that $R$ may act non-trivially only on the $n_0$ lowest levels and the $n_0$ highest levels of $\Xi_n$.

Define a permutation $P$ as a product of $v_n$ permutations $P_1 \ldots P_{v_n}$, where each permutation $P_v$ acts only within the tower $\xi_v^{(n)}$ as follows:

$$P_v|_{T^i B^{(n)}_v} = T^p|_{T^i B^{(n)}_v}, \quad \text{where } p = -i + (i + f_Q|_{T^i B^{(n)}}) \mod h_v^{(n)},$$

for $i = 0, \ldots, h_v^{(n)} - 1$. In other words, we define $P_v$ as $Q$ if $Q$ does not move the set $T^i B^{(n)}_v$ over the top or the base of the tower, that is, $0 \leq f_Q(x) + i \leq h_v^{(n)} - 1$ for $x \in T^i B^{(n)}_v$. If $f_Q(x) + i \geq h_v^{(n)}$ for $x \in T^i B^{(n)}_v$, then the entire level $U(h_v^{(n)} - 1 - i)$ will go over the top of $\Xi_n$ under the action of $Q$. This implies that $Q U(h_v^{(n)} - 1 - i) = D(j)$ for some (unique) $j$. In this case, we require that $P_v$ send $T^i B^{(n)}_v$ to the atom $T^j B^{(n)}_v$ under the action of $T^j$. Now if $f_Q(x) + i < 0$ for $x \in T^i B^{(n)}_v$, then $Q D(i) = U(j)$ for some $j$. We set $P = T^{h_v^{(n)} - 1 - j - i}$ on such a set.

(1-c) We claim that $Q = PR$. Fix $x \in X$. Find unique $v \in V_n$ and $0 \leq i \leq h_v^{(n)} - 1$ with $x \in T^i B^{(n)}_v$. We will consider three cases depending on the value of $f_Q(x) + i$.

Assume that $0 \leq f_Q(x) + i \leq h_v^{(n)} - 1$. It follows from the definitions of $P$ and $R$ that $R(x) = x$ and $P(x) = Q(x)$. Thus, $Q(x) = PR(x)$. 

---
Assume that \( i + f_Q(x) \geq h_v^{(n)} \). Then \( y := R(x) = T^{h_w^{(n)}}(x) \), where \( w \) is uniquely defined by the relation \( T^{h_w^{(n)}-i}x \in B_w^{(n)} \). Note that \( h_v^{(n)} - i \leq n_0 \) and \( y, x \in U(h_v^{(n)}-1-i) \). Hence \( f_Q(y) = f_Q(x) \) by the choice of \( n_0 \). Note also that \( y \in T^jB_w^{(n)} \) with \( j = h_w^{(n)} - (h_v^{(n)} - i) \). Hence, by the definition of \( P \), we get that \( P(y) = T^p(y) \) with

\[
p = -j + (f_Q(y) + j \mod h_w^{(n)}) = -j + (f_Q(x) + (h_w^{(n)} - (h_v^{(n)} - i)) \mod h_w^{(n)}) = -j + (f_Q(x) - (h_v^{(n)} - i) - j) = f_Q(x) - h_w^{(n)}.
\]

It follows that \( PR(x) = T^{f_Q(x)-h_w^{(n)}}T^{h_w^{(n)}x} = T^{f_Q(x)}x = Qx \).

The case \( i + f_Q(x) < 0 \) can be checked similarly. We leave the details to the reader.

(2) Assume that \( PR = P_2R_2 \). Hence \( P_2^{-1}P = R_2R^{-1} \). Observe that \( P_2^{-1}P \) is an \( n \)-permutation and that \( R_2R^{-1} \) is an \( n \)-rotation. Notice also that if \( P \) is an \( n \)-permutation and \( R \) is an \( n \)-rotation, then \( P \neq R \) unless they are both trivial transformations. This immediately yields the uniqueness of the decomposition.

(3) Consider two distinct elements \( Z_1 \) and \( Z_2 \) of \( [[T]] \). Find a clopen set \( C \) such that \( Z_1(y) \neq Z_2(y) \) for every \( y \in C \). Consider factorizations \( Z_1 = P_n^{(1)}R_n^{(1)} \) and \( Z_2 = P_n^{(2)}R_n^{(2)} \) into \( n \)-permutations and \( n \)-rotations. We notice that for all sufficiently large \( n \) the \( n \)-rotations \( R_n^{(1)} \) and \( R_n^{(2)} \) are supported by the sets \( [-k, k] \), for some \( k > 0 \) independent of \( n \). Hence, we can find a clopen subset \( C' \subset C \) such that for large \( n \) the supports of \( R_n^{(1)} \) and \( R_n^{(2)} \) are disjoint from \( C' \). This implies that \( P_n^{(1)}|C' \neq P_n^{(2)}|C' \) for all sufficiently large \( n \). This completes the proof.

\[\textbf{§ 5. Applications}\]

In this section, we show that the topological full group of any Cantor minimal system is locally embeddable into finite groups and that its commutator is a product of two locally-finite subgroups.

\textbf{Theorem 5.1.} Let \((X,T)\) be an arbitrary Cantor minimal system. Then the topological full group \( [[T]] \) is an LEF group.

\textbf{Proof.} Fix a finite set \( F \subset [[T]] \). We can assume that the group identity is contained in \( F \). Set \( F^2 = F \cdot F \). Using Theorem 4.7, find \( n \in \mathbb{N} \) such that every \( Q \in F^2 \) can be factored as \( Q = S_QR_Q \), where \( R_Q \) is an \( n \)-rotation and \( S_Q \) is an \( n \)-permutation such that \( S_Q \neq S_Z \) for \( Q, Z \in F^2 \) with \( Q \neq Z \).

Since \( F \) is a finite set, there is an integer \( d \) such that all \( n \)-rotations \( \{R_Q : Q \in F\} \) are supported by levels \([-d, d]\) for all \( n \) large enough (Theorem 4.7, (iv)). Using Theorem 4.7, (ii), we can choose \( n \) so that for every \( Q \in F \), we have \( S_{Q,v}^{-1}(i) = S_{Q,w}^{-1}(i) \) for all \( i \in [-d,d] \) and \( v, w \in V_n \), where \( S_Q = \prod_{v \in V_n}S_{Q,v} \). This implies that \( S_Z^{-1}R_QS_Z \) is an \( n \)-rotation for all \( Z, Q \in F \).

Denote by \( H \) the group of all \( n \)-permutations. Then \( H \) is a finite group. Define a map \( \varphi : F^2 \rightarrow H \) by \( \varphi(Q) = \varphi(S_QR_Q) = S_Q \). Observe that in view of Theorem 4.7, the factorization \( Q = S_QR_Q \) is unique for every \( Q \in F^2 \). Thus, the map \( \varphi \) is well-defined. Note that \( \varphi \) is injective on \( F \).
If $Q, Z \in F$, then
\[ \varphi(QZ) = \varphi(S_Q R_Q S_Z R_Z) = \varphi((S_Q S_Z) S_Z^{-1} R_Q S_Z R_Z) = S_Q S_Z, \]
where the last equality follows from the uniqueness of the factorization and the fact that $S_Z^{-1} R_Q S_Z R_Z$ is an $n$-rotation and $S_Q S_Z$ is an $n$-permutation. Therefore, $\varphi(QZ) = S_Q S_Z = \varphi(Q) \varphi(Z)$. Set $\varphi(g) = \mathrm{id}$ for all $g \notin F^2$. This shows that $[[T]]$ is an LEF group.

Recall that the group $[[T]]_x$ is defined as the set of all elements $S \in [[T]]$ with $S(\{T^n(x) : n \geq 1\}) = (\{T^n(x) : n \geq 1\})$. It was observed in [5] that $[[T]]_x$ is locally finite. In the language of the present paper, this can also be derived from the fact that $[[T]]_x$ coincides with the set of all $n$-permutations associated with a sequence $\{\mathcal{P}_n\}_{n \geq 1}$ of Kakutani-Rokhlin partitions converging to $\{x\}$, that is, $\bigcap_{n \geq 1} B(\mathcal{P}_n) = \{x\}$ (see §3 for details). Indeed, if $P$ is an $n$-permutation consistent with $\mathcal{P}_n$, then $P \in [[T]]_x$ as $P$ does not move points over the top or bottom of the partition. Conversely, if $Q \in [[T]]_x$, then $Q = PR$ is a product of an $n$-permutation and an $n$-rotation associated with $\mathcal{P}_n$ (Theorem 4.7). Notice that if $R \neq \mathrm{id}$, then $Q$ maps points from the backward $T$-orbit of $x$ onto the forward $T$-orbit. So $Q$ must be an $n$-permutation. The following result is an immediate corollary of Theorem 4.7.

**Proposition 5.2.** For any point $x \in X$, $[[T]]_x$ is a maximal locally-finite subgroup of $[[T]]$.

*Proof.* Consider any element $Q \in [[T]] \setminus [[T]]_x$. Using Theorem 4.7, we can write down $Q = PR$, where $P \in [[T]]_x$ and $R$ is a rotation. Since $Q \notin [[T]]_x$, we get that $R \neq \mathrm{id}$. Notice that the group generated by $[[T]]_x$ and the element $Q$ contains $R$, which is an element of infinite order. This establishes the result.

Recall that $\Gamma_x = [[T]]_x \cap [[T]]'$, where $x \in X$. Each group $\Gamma_x$ is locally finite. The proof of Theorem 5.4 is a modification of that of Lemma 4.1 from [13]. We notice that Lemma 4.1 in [13] is based heavily on Remark 5.6 in [5], whose proof, in its turn, uses operator-algebra techniques. In the following lemma we give a purely combinatorial proof of Remark 5.6 in [5]. For a point $x \in X$, set $O^+(x) = \{T^n(x) : n \geq 0\}$ and $O^-(x) = \{T^n(x) : n < 0\}$. These are the forward and backward $T$-orbits of $x$, respectively. For each $Q \in [[T]]$, set
\[ a(Q) = \mathrm{card}(Q(O^-(x)) \cap O^+(x)), \quad b(Q) = \mathrm{card}(Q^{-1}(O^+(x)) \cap O^-(x)). \]

Fix a $T$-invariant measure $\mu$ and set $\varphi(Q) = \int_X f_Q \, d\mu$. Notice that $\varphi : [[T]] \to \mathbb{R}$ is a nontrivial group homomorphism: see [5].

**Lemma 5.3.** For any $Q \in [[T]]$, we have that $\varphi(Q) = a(Q) - b(Q)$. In particular, $\varphi([[T]]) = \mathbb{Z}$ and the definition of $\varphi$ is independent of the choice of $\mu$.

*Proof.* Let $P$ be an $n$-permutation. Then $\varphi(P) = 0$. Notice that $\varphi(T) = 1$ and $\varphi(T^{-1}) = -1$. If $T_C$ is the induced transformation on a clopen set $C$, then $T = T_C P$ for some periodic transformation $P$: see, for example, [7], Proposition 2.1. Hence $\varphi(T_C) = 1$ and $\varphi(T_C^{-1}) = -1$. 

Consider an arbitrary \( Q \in [[T]] \). Using Theorem 4.7, find an \( n \)-permutation \( P \) and an \( n \)-rotation \( R \) with \( Q = PR \) and \( R \) having rotation number at most 1. Then \( \varphi(Q) = \varphi(R) \). Writing down \( R \) as a product of induced transformations,

\[
R = \prod_{a \in A} T_{U(a)} \times \prod_{b \in B} T_{D(b)}^{-1}
\]

for some subsets \( A, B \subset \{0, 1, \ldots, m_n\} \), we get that

\[
\varphi(Q) = \text{card}(A) - \text{card}(B).
\]

Since the set \( A \) (the set \( B \)) represents the levels that are mapped by \( Q \) over the top (bottom) of a partition, we get that

\[
\text{card}(A) = a(Q) \quad \text{and} \quad \text{card}(B) = b(Q).
\]

**Theorem 5.4.** Let \((X, T)\) be a Cantor minimal system. Then \([[T]]' = \Gamma_x \cdot \Gamma_y \), where \( x \) and \( y \) are points from distinct \( T \)-orbits.

**Proof.** Let \( Q \in \Gamma \). Let \( \{\Xi_n\}_{n \geq 1} \) be a Kakutani-Rokhlin partition built over the point \( x \). Applying Theorem 4.7 to the element \( Q \) and partitions \( \{\Xi_n\} \), we can represent \( Q \) as \( Q = P_nR_n \), where \( P_n \) is an \( n \)-permutation and \( R_n \) is an \( n \)-rotation (with respect to these partitions). Clearly, \( P_n \in [[T]]_x \).

Set

\[
I^+ = \{n \geq 0 : Q(T^n(x)) \in O^-(x)\} \quad \text{and} \quad I^- = \{n < 0 : Q(T^n(x)) \in O^+(x)\}.
\]

It follows from the lemma above that \( \text{card}(I^+) = \text{card}(I^-) = m \) for some \( m \). Using the minimality of \( T \), we can find a clopen set \( Y \) of the form \( Y = C \cup T^p(C) \) (for some \( p \in \mathbb{Z} \)) such that \( x \in Y \) and \( T^i(Y) \cap T^j(Y) = \emptyset \) for any \(-q \leq i \neq j \leq q\), where \( q = \max_{x \in X} |f_Q(x)| \), and such that none of the sets \( \{T^j(Y) : -q \leq j \leq q\} \) contains the point \( y \).

Fix any bijection \( \pi \) that maps \( I^- \) onto \( I^+ \) and \( I^+ \) onto \( I^- \). Set \( I = I^- \cup I^+ \). Define a homeomorphism \( P_2 \) as follows: \( P_2[T^n(Y)] = T^{\pi(n)-n}[T^n(Y)] \) for any \( n \in I \) and \( P_2 = \text{id} \) elsewhere. Observe that \( P_2 \in [[T]]_y \) and that \( P_2 \) is an involution. We notice that \( P_2 \in [[T]]_y \), as \( Y \) was chosen so that \( \{T^j(Y) : -q \leq j \leq q\} \) did not contain the point \( \{y\} \). Since \( Y = C \cup T^p(C) \), \( P_2 \) is a product of two isomorphic involutions (built over \( C \) and \( T^p(C) \), respectively). This implies that \( P_2 \in [[T]]'_y \). Hence \( P_2 \in [[T]]'_y \).

Set \( P_1 = QP_2^{-1} \in [[T]]'_y \). Let \( x_n = T^n(x) \). If \( n \in I^+ \), then \( P_2^{-1}x_n \) is moved by \( Q \) to the forward orbit of \( x \). Thus, \( P_1x_n = QP_2^{-1}x_n \in O^+(x) \). Similarly, if \( n \notin I^+ \), then \( P_2(x_n) \in O^+(x) \). This implies that \( P_1 \in [[T]]_x \).

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