Preempting to Minimize Age of Incorrect Information under Random Delay

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Abstract

We consider the problem of optimizing the decision of a preemptive transmitter to minimize the Age of Incorrect Information (AoII) when the channel has a random delay. In the system, a transmitter observes a dynamic source and makes decisions based on the system status. When the channel is busy, the transmitter can choose whether to preempt to transmit a new update. When the channel is idle, the transmitter can choose whether or not to transmit a new update. We assume that the channel has a random delay and that this delay is independent and identically distributed for any update. At the other end of the channel is a receiver that estimates the state of the dynamic source based on the updates it receives. We adopt AoII to measure the performance of the system. Therefore, this paper aims to optimize the transmitter’s action in each time slot to minimize AoII. We first use the Markov decision process to formulate the optimization problem and give the corresponding value iterative algorithm to obtain the optimal policy. However, the value iteration algorithm is computationally demanding, and some approximations are made to realize the algorithm. Hence, we theoretically analyze some canonical delay distributions and obtain the corresponding optimal policies by leveraging the policy improvement theorem.

I. INTRODUCTION

With the development of 5G communications, our requirements for communication networks are changing. For example, we need communication networks to be more efficient and intelligent. At the same time, we question whether traditional performance metrics such as delay can still meet these higher requirements. Therefore, researchers have recently proposed semantic communication, a new design paradigm for networked systems. Semantic communication considers the semantics of the information, defined as the importance of the transmitted information for the purpose of transmission. Semantic measures are the core of semantic communication. In [1], the authors present several representative semantic measures. The first is freshness, which captures
how fresh information is. In other words, it measures the time elapsed between the generation of the latest information at the destination and arrival at the destination. Age of Information (AoI), introduced in [2], is a good and widely studied example. The second is relevance, which captures the amount of change in the process since the last sampling, and is very important in remote estimation. For example, when the process changes slowly, we can reduce the transmission of information to save valuable resources. When the process changes dramatically, we need to increase the transmission of information so that the distant receiver can have better knowledge of the process. Relevance is different from freshness since freshness ignores the specific content of the transmitted information. Hence, as shown in [3], optimizing AoI does not achieve optimal system performance for communication purposes. The third is value, which captures the value of information transmission for communication purposes. Value of Information (VoI) is a good example that quantifies the difference between the benefit of transmitting this information and its cost. One of the essential parts of semantic communication is designing semantic metrics that quantify semantic measures. We note that AoI and VoI capture only one part of the semantic measure. However, a more refined semantic metric needs to consider more than one semantic measure and integrate multiple semantic measures well into a single semantic metric. The Age of Incorrect Information (AoII), introduced in [4], is one example.

AoII combines freshness and relevance of information. As presented in [4], AoII is dominated by two penalty functions. The first one is the time penalty function, which is based on the idea that information only ages if it is wrong. If information provides accurate information, we consider it fresh no matter when it was generated. Through the time penalty function, AoII captures the time elapsed since the last time the receiver had the correct information. The second is the information penalty function, the idea behind which is that different information mismatches will cause different damage to the system. When the difference between the information on the receiver and the correct information is slight, we think it will not cause significant harm to the system. In contrast, a significant difference will cause great damage to the system. Therefore, the information penalty function captures the mismatch between the information on the receiver side and the correct information. By combining the two penalty functions, AoII not only captures the mismatch between the information at the receiver and the correct information but also reflects the aging process of the incorrect information.

With the introduction of AoII, researchers have devoted themselves to revealing its characteristics and performance in networked systems. In [4], the authors study the minimization of AoII
in the presence of average transmission rate limits. Then, in [5], the result is extended to the case when the time penalty function is general. In [6], the authors study a system setup similar to the one described above but involving a source process with multiple states and an AoII that incorporates the quantified information mismatches between the source and receiver. Different from the previous papers, [7] studies AoII in the context of scheduling. In this type of problem, a base station sends updates to multiple users and tries to ensure that each user’s information is as accurate as possible. In [7], the authors study the problem of minimizing AoII when channel state information is available and the time penalty function is general. The authors of [8] consider a similar system, but the base station has no way of knowing the true state of the event before deciding to transmit. In the above papers, the update transmission time is constant, usually one time slot. However, in practice, the communication channel will usually suffer a random delay due to the influence of various factors. In such a system setting, the authors of [9] compare three performance metrics: AoII, AoI, and real-time error by extensive numerical results. [10] and [11] consider the problem of minimizing AoII in the presence of channel delay by controlling when the transmitter transmits a new update. The transmitter in both papers can decide whether to initiate a new transmission only when the channel is idle. When the channel is busy, the transmitter must wait for the update being transmitted to complete transmission before it can initiate the subsequent transmission. In this paper, we also consider the problem of minimizing AoII by controlling the action of the transmitter when the channel has a random delay. However, the transmitter can preempt updates being transmitted and immediately transmit new updates. In this way, we have to consider the transmitter’s decision when the channel is idle and whether the transmitter needs to terminate the transmission to transmit new updates when the channel is busy.

The main contribution of this paper can be summarized as follows. 1) We consider optimizing the system that takes AoII as the performance measure, and the communication channel suffers a random delay. 2) We formulate the problem using the Markov decision process and propose the value iteration algorithm approximating the optimal solution. 3) We theoretically analyze four canonical channel delays and find the optimal solution. 4) We obtain the closed-form expression of the expected AoII achieved by the optimal solutions.

The paper is organized in the following way. Section [II] describes the system we consider and the specific choice of two penalty functions in AoII. At the same time, we also formulate the optimization problem considered in this paper. In Section [II], we model the problem considered
in this paper by the Markov decision process and propose an algorithm to approximate the answer to the problem considered. In Section [IV] we conduct theoretical analysis when the probability distribution of the delay follows four canonical distributions and obtain the exact answer to the considered problem.

II. SYSTEM OVERVIEW

A. System Model

We consider a slotted-time system with a transmitter-receiver pair. In the system, the transmitter observes the dynamic source by receiving updates from the dynamic source and controls the transmission of updates over a communication channel with a random delay so that the distant receiver has the best real-time knowledge of the dynamic source. We use a two-state symmetric Markov chain to model the dynamic source and denote the state transition probability by $p$. The Markov chain is illustrated in Fig. 1. The dynamic source sends updates to the transmitter in each time slot. Therefore, the transmitter can accurately know the state of the dynamic source in every time slot. In the remainder of this paper, we denote the update received at time slot $k$ by $X_k$. After receiving a new update, the transmitter will discard the old update and decide whether to transmit the newly received update based on the system status. Since the communication channel has a random delay, the action the transmitter can choose depends on the channel status. When the channel is idle, the transmitter chooses between transmitting the new update and remaining idle. While the channel is transmitting a previous update, the transmitter can preempt the channel to transmit the newly received update or skip the update. We assume that terminating the current transmission and starting a new transmission can be done in the same time slot. All transmitted updates will be accurately transmitted to the distant receiver over a communication channel with random delay. We denote the random delay of the communication channel as $T$, which is a random variable and can be fully characterized by a probability distribution. For simplicity, we assume $T$ is independent and identically distributed for each update. More precisely, the update

![Fig. 1: The two-state symmetric Markov chain with state transition probability $p$.](image-url)
will be received by the distant receiver after $t$ time slots with probability $p_t \triangleq Pr(T = t)$ where $t \in \mathbb{N}$.

When an update is accurately transmitted to the distant receiver, the receiver modifies its estimate of the dynamic source state based on the update. We denote the receiver’s estimate at time slot $k$ as $\hat{X}_k$. According to [12], when $p \leq \frac{1}{2}$, the receiver’s best estimate is simply the last update it received. When $p > \frac{1}{2}$, the best estimate will be related to the transmission time of the update. In the rest of this paper, we only consider the case of $p \leq \frac{1}{2}$. Therefore, the receiver will simply take the last update it received as its estimate of the current state of the dynamic source. The methodology presented in this paper can be extended to the case of $p > \frac{1}{2}$ by adopting the corresponding optimal estimator. At the same time, in order for the transmitter to make a better decision, the receiver will inform the transmitter of its reception of the new update through the transmission of the $\text{ACK/NACK}$ packet. Specifically, whenever the receiver receives a new update, it sends an $\text{ACK}$ packet to the transmitter. Otherwise, a $\text{NACK}$ packet is transmitted. For simplicity, we assume that $\text{ACK/NACK}$ packets can be received accurately and instantaneously by the transmitter. This assumption is widely used in relevant literature [4], [13]. This way, the transmitter will always know the receiver’s current estimate. An illustration of the system model is given by Fig. 2. In a typical time slot, the transmitter will receive an update from the dynamic source and discard the old one. The transmitter will then determine whether to transmit the new update based on the system status. When the transmitter chooses not to transmit, it will do nothing. When the transmission is chosen, the update will be transmitted directly if the channel is idle. Otherwise, the ongoing transmission will be terminated so that the new update can be transmitted. If not preempted, the update will be received by the receiver after a random amount of time. The receiver then modifies its estimate based on this update and sends an $\text{ACK}$ packet to inform the transmitter of its reception of the update.

![Fig. 2: An illustration of the system model.](image-url)
B. Age of Incorrect Information

The system adopts the Age of Incorrect Information (AoII) as the measure of performance. AoII is first introduced in [4] and can be fully characterized by the information penalty function \( g(X_k, \hat{X}_k) \) and the time penalty function \( f(k) \). In this paper, we choose \( g(X_k, \hat{X}_k) = |X_k - \hat{X}_k| \) and \( f(k) = k \). Then, in a slotted time system, AoII can be written as

\[
\Delta_{AoII}(X_k, \hat{X}_k, k) = \sum_{h=U_k+1}^{k} \left( g(X_h, \hat{X}_h) F(h - U_k) \right),
\]

where \( F(k) \triangleq f(k) - f(k - 1) = 1 \) and \( U_k \) is the last time slot where the receiver’s estimate is correct. Mathematically, \( U_k \) can be defined by

\[
U_k \triangleq \max\{h : h \leq k, X_h = \hat{X}_h\}.
\]

We notice that the dynamic source has only two states. Hence, \( g(X_k, \hat{X}_k) \in \{0, 1\} \). Consequently, AoII can be simplified as

\[
\Delta_{AoII}(k) = k - U_k \triangleq \Delta_k.
\]

Leveraging the definition of \( U_k \), the evolution of \( \Delta_k \) can be characterized by the following two cases.

- When the receiver’s estimate is correct at time slot \( k \), \( U_k = k \) by definition. Hence, \( \Delta_k = 0 \).
- When the receiver’s estimate is erroneous at time slot \( k \), \( U_k = U_{k-1} \) by definition. Hence, \( \Delta_k = k - U_k = \Delta_{k-1} + 1 \).

Combining together, the dynamics of \( \Delta_k \) can be summarized as follows

\[
\Delta_k = \mathbb{1}\{X_k \neq \hat{X}_k\}(\Delta_{k-1} + 1),
\]

where \( \mathbb{1}\{\cdot\} \) is an indicator function. A sample path of \( \Delta_k \) is given in Fig. [3].

C. Problem Formulation

In this paper, we investigate the problem of controlling the decision of the transmitter at each time slot to minimize the AoII of the system. To this end, we define a policy as one that specifies the transmitter’s decision at each time slot based on the current system status. Therefore, the goal of this paper is to find a policy that minimizes the AoII of the system. Mathematically, the problem can be formulated as the following minimization problem.

\[
\arg\min_{\psi \in \Psi} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_\psi \left( \sum_{t=0}^{T-1} \Delta_t \right),
\]
Fig. 3: A sample path of $\Delta_k$. In the figure, $T_i$ and $D_i$ are the transmission time and the delivery time of the $i$-th update, respectively. At $T_1$, the transmitted update is $X_3$. The estimate at time slot 6 (i.e., $\hat{X}_6$) changes due to the reception of the update transmitted at $T_2$. Note that the transmission decisions in the sample path are taken randomly.

where $\Psi$ is the set of all admissible policies.

**Definition 1** (Optimal policy). A policy is said to be optimal if it minimizes the AoII of the system. The action specified by the optimal policy is called the optimal action.

### III. MDP Characterization

The problem reported in (2) can be characterized by an infinite horizon with average cost Markov Decision Process (MDP) $\mathcal{M}$, which consists of the following components.

- **The state space $\mathcal{S}$.** The state can be represented by the triplet $s = (\Delta, t, i)$ where $\Delta \in \mathbb{N}_0$ is the current AoII, $t \in \mathbb{N}_0$ indicates the time the transmission has been in progress. We define $t = 0$ if there is no transmission in progress. The last element $i \in \{-1, 0, 1\}$ indicates the status of the channel. $i = -1$ if the channel is idle, $i = 0$ if the transmitting update is the same as receiver’s estimate, and $i = 1$ otherwise. In order to better distinguish between the state of the system, we will use $s$ and $(\Delta, t, i)$ to represent the state interchangeably throughout the rest of the paper.

- **The feasible action $\mathcal{A}$.** The feasible actions are $a \in \{0, 1\}$. When $i \neq -1$, $a = 1$ if the transmitter decides to terminate the current transmission and immediately start a new one. Otherwise, $a = 0$. When $i = -1$, $a = 1$ if the transmitter decides to transmit the new update and $a = 0$ otherwise.
- The state transition probabilities \( P \). The probability that action \( a \) at state \( s \) leads to state \( s' \) is denoted by \( P_{s,s'}(a) \). The dynamics of \( P_{s,s'}(a) \) will be detailed in the following section.

- The instant cost \( C \). The instant cost for being at state \( s \) is \( C(s) = \Delta \).

**Remark 1.** Note that the state space of \( \mathcal{M} \) defined above is infinite because \( \Delta \) can theoretically grow infinitely. Also, according to the definition of \( t \) and \( i \), \( i = -1 \) if and only if \( t = 0 \), in which case the communication channel is idle.

Let \( V(s) \) be the value function of state \( s \). Then, the optimal action at state \( s \), which is denoted by \( a^*(s) \), can be determined using the following equation.

\[
a^*(s) = \arg\min_{a \in A} \left\{ \sum_{s' \in S} P_{s,s'}(a)V(s') \right\}.
\]

Hence, it is sufficient to calculate the value function for each state in \( S \). It well know that, for each \( s \in S \), \( V(s) \) satisfies the following Bellman equation.

\[
V(s) + \theta = \min_{a \in A} \left\{ C(s) + \sum_{s' \in S} P_{s,s'}(a)V(s') \right\},
\]

\[ (3) \]

where \( \theta \) is the expected AoII achieved by the optimal policy. In the Bellman equation, the only thing that is not specified is the state transition probability \( P_{s,s'}(a) \). Therefore, in the next section, we will delve into the expression of \( P_{s,s'}(a) \).

### A. State Transition Probabilities

First, we define and compute an auxiliary quantity \( Pr(T = t + 1 \mid t) \), which is the probability that an update will be delivered in the next time slot after the transmission has been in progress for \( t \) time slots. Using \( p_t \)'s defined in Section II-A, it is easy to get that

\[
Pr(T = t + 1 \mid t) = \frac{p_{t+1}}{1 - \sum_{i=1}^{t} p_i}.
\]

For simplicity, we will abbreviate \( Pr(T = t + 1 \mid t) \) as \( q_{t+1} \) for the remainder of this paper. Leveraging \( q_{t+1} \), we can proceed with deriving the state transition probabilities. We first note that the evolution of \( \Delta \) follows (1). Hence, in the following, we will omit the discussion on the evolution of \( \Delta \). For the sake of reading, let’s write \( P_{s,s'}(a) \) as \( Pr[s' \mid s, a] \). Then, we distinguish between different states.

- \( s = (0, 0, -1) \). In this case, the channel is idle. Hence, the transmitter can choose to initiate a transmission or be idle. To start with, we consider the case where the transmitter decides
to initiate the transmission (i.e., \( a = 1 \)). In the next slot, the update will either be delivered or still in transit. When the update is delivered in the next time slot, which happens with probability \( q_1 \), the receiver’s estimate will not change.

\[
Pr[(0, 0, -1) \mid (0, 0, -1), a = 1] = q_1(1 - p).
\]

\[
Pr[(1, 0, -1) \mid (0, 0, -1), a = 1] = q_1 p.
\]

When the update is still in transit in the next time slot, which happens with probability \( 1 - q_1 \), \( t \) will increase by 1 as the transmission continues. \( i \) will be zero since the transmission started when \( \Delta = 0 \). Meanwhile, receiver’s estimate will not change as no update is delivered.

\[
Pr[(0, 1, 0) \mid (0, 0, -1), a = 1] = (1 - q_1)(1 - p).
\]

\[
Pr[(1, 1, 0) \mid (0, 0, -1), a = 1] = (1 - q_1)p.
\]

Then, we consider the case where the transmitter chooses to be idle (i.e., \( a = 0 \)). In this case, \( t \) and \( i \) will remain unchanged. Meanwhile, receiver’s estimate will not change as no update is delivered.

\[
Pr[(0, 0, -1) \mid (0, 0, -1), a = 0] = 1 - p.
\]

\[
Pr[(1, 0, -1) \mid (0, 0, -1), a = 0] = p.
\]

- \( s = (\Delta, 0, -1) \) where \( \Delta > 0 \). In this case, the analysis is similar to the previous case, except that the transmission takes place when \( \Delta > 0 \) and the receiver’s estimate will flip when the update arrives. Hence, we will omit the details and present the resulting dynamics directly.

\[
Pr[(0, 0, -1) \mid (\Delta, 0, -1), a = 1] = p_1(1 - p).
\]

\[
Pr[(\Delta + 1, 0, -1) \mid (\Delta, 0, -1), a = 1] = p_1 p.
\]

\[
Pr[(0, 1, 1) \mid (\Delta, 0, -1), a = 1] = (1 - p_1)p.
\]

\[
Pr[(\Delta + 1, 1, 1) \mid (\Delta, 0, -1), a = 1] = (1 - p_1)(1 - p).
\]

\[
Pr[(\Delta + 1, 0, -1) \mid (\Delta, 0, -1), a = 0] = 1 - p.
\]

\[
Pr[(0, 0, -1) \mid (\Delta, 0, -1), a = 0] = p.
\]

- \( s = (0, t, 0) \) where \( t \geq 1 \). In this case, the channel is busy transmitting another update. Therefore, the transmitter either terminates the ongoing transmission and immediately sends
a new transmission or stay idle. When the transmitter chooses to terminate and initiate a new transmission (i.e., \(a = 1\)), the update will be delivered to the receiver in the next time slot with probability \(q_1\). In this case, the receiver’s estimate will not change. Meanwhile, \(t\) and \(i\) will reset to 0 and \(-1\), respectively, as the channel becomes idle in the next time slot.

\[
Pr[(0, 0, -1) \mid (0, t, 0), a = 1] = q_1(1 - p).
\]

\[
Pr[(1, 0, -1) \mid (0, t, 0), a = 1] = q_1 p.
\]

When the update is still in transmission at the next time slot, \(t\) will become 1, because this is a new transmission. \(i\) will be 0 because the transmission is initiated when \(\Delta = 0\). The receiver’s estimate will not change as it does not receive any new updates.

\[
Pr[(0, 1, 0) \mid (0, t, 0), a = 1] = (1 - q_1)(1 - p).
\]

\[
Pr[(1, 1, 0) \mid (0, t, 0), a = 1] = (1 - q_1)p.
\]

When the transmitter chooses to do nothing, the update currently being transmitted will be delivered in the next time slot with probability \(q_{t+1}\). In this case, receiver’s estimate will not change as \(i = 0\) indicates that the newly arrived update brings no new information to the receiver. Meanwhile, \(t\) and \(i\) will reset to 0 and \(-1\), respectively.

\[
Pr[(0, 0, -1) \mid (0, t, 0), a = 0] = q_{t+1}(1 - p).
\]

\[
Pr[(1, 0, -1) \mid (0, t, 0), a = 0] = q_{t+1}p.
\]

The update currently being transmitted will still in transmission in the next time slot with probability \(1 - q_{t+1}\). In this case, \(t\) will increase by 1 as the transmission continues. Apparently, \(i\) and receiver’s estimate will not change.

\[
Pr[(0, t + 1, 0) \mid (0, t, 0), a = 0] = (1 - q_{t+1})(1 - p).
\]

\[
Pr[(1, t + 1, 0) \mid (0, t, 0), a = 0] = (1 - q_{t+1})p.
\]

\(s = (0, t, 1)\) where \(t \geq 1\). When the transmitter chooses to terminate the ongoing transmission and start a new one (i.e., \(a = 1\)), the dynamics of the system are exactly the same as in the previous case. Therefore, we omitted the details of the analysis.

\[
Pr[(0, 0, -1) \mid (0, t, 1), a = 1] = q_1(1 - p).
\]
\[ Pr[(1, 0, -1) \mid (0, t, 1), a = 1] = q_1 p. \]

\[ Pr[(0, 1, 0) \mid (0, t, 1), a = 1] = (1 - q_1)(1 - p). \]

\[ Pr[(1, 1, 0) \mid (0, t, 1), a = 1] = (1 - q_1)p. \]

When the transmitter chooses to do nothing (i.e., \( a = 0 \)), the dynamics of the system are very similar to the previous case except that the receiver’s estimate will flip when the update is delivered. Hence, we will skip the details and give the dynamics directly.

\[ Pr[(0, 0, -1) \mid (0, t, 1), a = 0] = q_{t+1}p. \]

\[ Pr[(1, 0, -1) \mid (0, t, 1), a = 0] = q_{t+1}(1 - p). \]

\[ Pr[(0, t + 1, 1) \mid (0, t, 1), a = 0] = (1 - q_{t+1})(1 - p). \]

\[ Pr[(1, t + 1, 1) \mid (0, t, 1), a = 0] = (1 - q_{t+1})p. \]

- \( s = (\Delta, t, 0) \) where \( \Delta > 0 \) and \( t \geq 1 \). When the transmitter chooses to terminate an ongoing transmission and initiate a new one (i.e., \( a = 1 \)), the dynamics of the system are similar to the corresponding scenario when \( s = (0, t, 0) \) except that the new update is different from receiver’s estimate. Hence, receiver’s estimate will flip when the new update is delivered. We have omitted the detailed analysis here.

\[ Pr[(0, 0, -1) \mid (\Delta, t, 0), a = 1] = q_1(1 - p). \]

\[ Pr[(\Delta + 1, 0, -1) \mid (\Delta, t, 0), a = 1] = q_1 p. \]

\[ Pr[(0, 1, 1) \mid (\Delta, t, 0), a = 1] = (1 - q_1)p. \]

\[ Pr[(\Delta + 1, 1, 1) \mid (\Delta, t, 0), a = 1] = (1 - q_1)(1 - p) \]

When the transmitter chooses to do nothing (i.e., \( a = 0 \)), the dynamics of the system are similar to the corresponding dynamics when \( s = (0, t, 0) \) except that the value of \( \Delta \) is not the same. Hence, the dynamics of \( \Delta \) will be different.

\[ Pr[(0, 0, -1) \mid (\Delta, t, 0), a = 0] = q_{t+1}p. \]

\[ Pr[(\Delta + 1, 0, -1) \mid (\Delta, t, 0), a = 0] = q_{t+1}(1 - p). \]

\[ Pr[(0, t + 1, 0) \mid (\Delta, t, 0), a = 0] = (1 - q_{t+1})p. \]
\[ Pr[(\Delta + 1, t + 1, 0) \mid (\Delta, t, 0), a = 0] = (1 - q_{t+1})(1 - p). \]

- \( s = (\Delta, t, 1) \) where \( \Delta > 0 \) and \( t \geq 1 \). Here, the analysis will be very similar to the previous case except that the update being transmitted is not the same as the receiver’s estimate.

\[ Pr[0, 0, -1] \mid (\Delta, t, 1), a = 1 = q_1(1 - p). \]
\[ Pr[(\Delta + 1, 0, -1) \mid (\Delta, t, 1), a = 1 = q_1 p. \]
\[ Pr[0, 1, 1] \mid (\Delta, t, 1), a = 1 = (1 - q_1)p. \]
\[ Pr[(\Delta + 1, 1, 1) \mid (\Delta, t, 1), a = 1 = (1 - q_1)(1 - p). \]
\[ Pr[0, 0, -1] \mid (\Delta, t, 1), a = 0 = q_{t+1}(1 - p). \]
\[ Pr[(\Delta + 1, 0, -1) \mid (\Delta, t, 1), a = 0 = q_{t+1}p. \]
\[ Pr[0, t + 1, 1] \mid (\Delta, t, 1), a = 0 = (1 - q_{t+1})p. \]
\[ Pr[(\Delta + 1, t + 1, 1) \mid (\Delta, t, 1), a = 0 = (1 - q_{t+1})(1 - p). \]

Transitions between states that are not discussed above default to 0. Then, combing the cases together, we fully characterized the dynamics of the state transition probability \( P_{s,s'}(a) \).

**B. Relative Value iteration Algorithm**

Relative Value Iterative (RVI) algorithm is a classical method to approximate the optimal policy for the Markov decision process. Before presenting the Relative Value Iteration (RVI), we first show that the optimal policy exists.

**Theorem 1** (Existence of optimal policy). There exists a stationary policy \( \psi \) that is optimal for \( M \). Moreover, the minimum expected AoII is independent of initial state.

**Proof.** The proof is very similar to proof of Theorem 1 of [11]. Hence, the details are omitted here. \( \Box \)
In the sequel, we present the RVI that approximates the optimal policy for $M$. Direct application of RVI becomes impractical as the state space $S$ of $M$ is infinite. Hence, we use Approximating Sequence Method (ASM) \[14\]. To this end, we construct another $M^{(m)} = (S^{(m)}, A, P^{(m)}, C)$ by truncating the value of $\Delta$ and $t$. More precisely, we impose

\[
S^{(m)} : \begin{cases} \\
\Delta \in \{0, 1, ..., m\}, \\
i \in \{-1, 0, 1\}, \\
t \in \{0, 1, ..., t_{\text{max}} - 1\}, 
\end{cases}
\]

where $m$ is the predetermined maximal value of $\Delta$ and $t_{\text{max}}$ is the predetermined maximum transmission time when the transmission time for an update can be infinite. The transition probabilities from $s \in S^{(m)}$ to $z \in S - S^{(m)}$ are redistributed to the states $s' \in S^{(m)}$ in the following way.

\[
P_{s,s'}^{(m)}(a) = \begin{cases} \\
P_{s,s'}(a) & s' = (\Delta', t', i') \text{ where } \Delta' < m, \ t' < t_{\text{max}} - 1, \\
P_{s,s'}(a) + \sum_{G(z,s')} P_{s,z}(a) & s' = (\Delta', t', i') \text{ where } \Delta' = m, \ t' = t_{\text{max}} - 1,
\end{cases}
\]

where $G(z, s') = \{ z : i = i', \Delta > m, t > t_{\text{max}} - 1 \}$, $z = (\Delta, t, i)$, and $s' = (\Delta', t', i')$. The action space $A$ and the instant cost $C$ are the same as defined in $M$.

**Remark 2.** We can rigorously show that the sequence of optimal policies for $M^{(m)}$ will converge to the optimal policy for $M$ as $m \to \infty$ and $t_{\text{max}} \to \infty$. The proof will be very similar to the proof of Theorem 1 of \[6\]. Hence, the details are omitted here.

Then, we can apply RVI algorithm to the truncated MDP $M^{(m)}$ and treat the resulting optimal policy as an approximation of the optimal policy for $M$. The pseudocode of RVI algorithm is given in Algorithm 1 of Appendix I.

**IV. Optimal Policy under Special Delay Distributions**

In the section, we will make a theoretical analysis of the system under some special delay distributions. Before deriving the optimal policies under the special delay distributions, we first introduce the policy iteration algorithm, which will be used to derive the corresponding optimal policy. Policy iteration algorithm is an iterative algorithm that iterates between the following two steps.
• The first step is policy evaluation step. In this step, we will calculate the resulting value function \( V_\psi(s) \) and the expected cost \( \theta_\psi \) for a given policy \( \psi \). To be more specific, \( V_\psi(s) \) and \( \theta_\psi \) satisfy the following system of linear equations.

\[
V_\psi + \theta_\psi = C(s) + \sum_{s' \in S} P_{s,s'}(\psi) V_\psi(s') \quad s \in S,
\]

where \( P_{s,s'}(\psi) \) is the probability that the system will transition from state \( s \) to state \( s' \) under policy \( \psi \). Note that (4) forms a underdetermined system. Hence, we can select any state \( s \) as a reference state and set the corresponding value function as 0. In this way, we can obtain a unique solution.

• The second step is policy improvement step. In this step, we will obtain a new policy \( \psi' \) based on the \( V_\psi(s) \) obtained in the previous policy evaluation step. Let \( \psi'(s) \) be the action suggested by the new policy \( \psi' \) at state \( s \). Then, \( \psi'(s) \) is given by the following equation.

\[
\psi'(s) = \arg\min_{a \in A} \left\{ \sum_{s' \in S} P_{s,s'}(a) V_\psi(s') \right\}.
\]

The pseudocode of policy iteration algorithm is given in Algorithm 2 of Appendix I.

Theorem 2 (Policy improvement theorem). Suppose that we have obtained the value function resulting from the operation of a policy \( A \) and that the policy improvement step has produced a policy \( B \). When policy \( A \) and policy \( B \) are identical, both \( A \) and \( B \) are optimal.

Proof. The proof is provided in Theorem 2 of [10].

With policy iteration algorithm and Theorem 2 in mind, we are able to proceed with deriving the optimal policies when the communication delay follows some special probability distributions.

A. Geometric Delay

In this section, we consider the case where the transmission time for an update is Geometrically distributed with success probability \( p_s \). More precisely, we have

\[
p_t = p_s(1 - p_s)^{t-1}, \quad t \geq 1.
\]

Hence, \( q_t = p_s \) for all feasible \( t \).

Definition 2 (Preemptive Policy). A preemptive policy always preempts updates that are being transmitted and always starts a new transmission when the channel is idle.
An illustration of the preemptive policy is given in Fig. 4.

**Theorem 3** (Optimal policy under Geometric delay). *The preemptive policy is optimal when the communication delay is Geometrically distributed.*

**Proof.** According to the Theorem 2, it is sufficient to prove that the policy iteration algorithm will eventually converge to the preemptive policy defined by Definition 2. The complete proof can be found in Appendix A.

**Remark 3.** *The optimality of preemptive policy is intuitive since \( q_t \) is independent of \( t \), which means that the probability of an update being delivered is independent of how long it has been transmitted. Hence, it is always optimal to send a new update.*

**Proposition 1** (Performance of Preemptive Policy). *The expected AoII achieved by the preemptive policy \( \bar{\Delta}_p \) is given by the following equation.*

\[
\bar{\Delta}_p = \frac{p}{(p + p_s - 2p_s p)(p_s + 2p - 2p_s p)}.
\]  

**Proof.** The expected AoII achieved by the preemptive policy can be calculated as

\[
\bar{\Delta}_p = \sum_{s \in S} C(s) \pi_s,
\]

where \( \pi_s \) is the steady-state probability of state \( s \) in the Markov chain induced by the preemptive policy. The complete proof can be found in Appendix B.

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Fig. 4: An illustration of the preemptive policy. In the plot, green dots means \( a = 1 \).
Remark 4. Note that the expected AoII achieved by the preemptive policy is given as a closed-form expression, which is very easy to calculate.

B. Zipf Delay

Here, we consider the case where the transmission time for an update is bounded by $t_{\text{max}}$ and is distributed following a Zipf distribution. More precisely, we assume that $p_t$ satisfies the following equation.

\[
p_t = \begin{cases} \frac{t^{-a}}{\sum_{i=1}^{t_{\text{max}}} i^{-a}} & 1 \leq t \leq t_{\text{max}}, \\ 0 & \text{otherwise}, \end{cases}
\]

where $a > 0$ is a predetermined constant. In this case, $q_t$ can be calculated as

\[
q_t = \frac{t^{-a}}{\sum_{i=t}^{t_{\text{max}}} i^{-a}}, \quad 1 \leq t \leq t_{\text{max}}.
\]

Note that when $t > t_{\text{max}}$, $q_t = 0$ by definition. Since the transmission time for an update is upper bounded by $t_{\text{max}}$, the state $s = (\Delta, t, i)$ in the corresponding $\mathcal{M}$ satisfies $1 \leq t \leq t_{\text{max}} - 1$.

Remark 5. In linguistics, the Zipf distribution is a frequency distribution of the occurrences of words found in a text. A transmission delay that follows Zipf distribution is also considered in [9].

Definition 3 (Weak preemptive policy). A weak preemptive policy will not preempt updates only at state $s = (\Delta, t_{\text{max}} - 1, 1)$ where $\Delta \geq 1$ and always starts a new transmission when the channel is idle.

An illustration of the weak preemptive policy is given in Fig. 5.

Proposition 2 (Performance of weak preemptive policy). The expected AoII achieved by the weak preemptive policy $\bar{\Delta}_{\rho'}$ is give by the following equation.

\[
\bar{\Delta}_{\rho'} = \frac{p}{(2p + q_1 - 2q_1 p)(p + q_1 - 2q_1 p)}.
\]

Proof. We follow a similar methodology to that adopted in the proof of Proposition [1]. The complete proof can be found in Appendix [C].

Remark 6. Although preemptive policy and weak preemptive policy are not exactly the same, they yield the same expected AoII. The reason behinds this is that the actions specified by the
two policies differ only in the transient states, which does not affect the long-term average performance.

**Condition 1.** We define the following quantity.

\[
Q = \frac{(1 - q_1)(2p - 1) - p(1 - q_{t_{\text{max}} - 1})}{(2p + q_1 - 2q_1p)(p + q_1 - 2q_1p)} - \frac{(1 - q_1)(1 - 2p) + p(2p - 1)(1 - q_{t_{\text{max}} - 1})}{q_1 + p - 2q_1p} - 2(1 - q_{t_{\text{max}} - 1})(1 - p).
\]

Then, the condition is \(Q \geq 0\).

**Theorem 4** (Optimal policy under Zipf distribution). Under Condition 7, the weak preemptive policy is optimal when the communication delay follows the Zipf distribution.

**Proof.** We follow the same methodology presented in the proof of Theorem 3. The complete proof can be found in Appendix D.

**Remark 7.** From the proof of Theorem 4 we can conclude that weak preemptive policy is optimal if the system satisfies the following conditions.

1) The communication delay is bounded by \(t_{\text{max}}\).
2) \(q_1 \geq q_t\) for \(2 \leq t < t_{\text{max}} - 1\).
3) The system verifies Condition 7.
C. Uniform Delay

In this section, we consider the case where the transmission time for an update is uniformly distributed with a maximum transmission time denoted by $t_{max}$. More precisely, we have

$$p_t = \begin{cases} \frac{1}{t_{max}} & 1 \leq t \leq t_{max}, \\ 0 & \text{otherwise.} \end{cases}$$

Since we have assumed an upper bound on the transmission time, the state $s = (\Delta, t, i)$ in the corresponding $\mathcal{M}$ satisfies $1 \leq t \leq t_{max} - 1$.

**Definition 4** (Threshold preemptive policy). A threshold preemptive policy always starts a new transmission when the channel is idle and will not preempt updates only when $\Delta > 0$ and $i = 1$.

An illustration of the threshold preemptive policy is given in Fig. 6.

**Proposition 3** (Performance of threshold preemptive policy). The expected AoII achieved by the threshold preemptive policy $\bar{\Delta}_{tp}$ is given by the following equation.

$$\bar{\Delta}_{tp} = \Sigma + \sum_{t=1}^{t_{max}-1} \Sigma(t),$$
where

\[
\Sigma = \frac{\Pi + \sum_{t=1}^{t_{\text{max}}-1} \left\{ p_{t+1} p_1 \left[ \sum_{i=1}^{t} \left( \prod_{j=i+1}^{t} p_j \right) \Pi(i) \right] \right\}}{1 - p_{t_{\text{max}}-1} \sum_{t=1}^{t_{\text{max}}-1} \left[ p_{t+1} \left( \prod_{l=1}^{t} p_l \right) \right]},
\]

\[
\Sigma(t) = \left( \prod_{l=1}^{t} p_l \right) \Sigma + \sum_{i=1}^{t} \left[ \left( \prod_{j=i+1}^{t} p_j \right) \Pi(i) \right], \quad 1 \leq t \leq t_{\text{max}} - 1,
\]

\[
\Pi = \frac{1}{1 - q_1 - \sum_{t=1}^{t_{\text{max}}-1} \left[ q_{t+1} \left( \prod_{l=1}^{t} p_l \right) \right] + 1 + \sum_{t=1}^{t_{\text{max}}-1} \left( \prod_{l=1}^{t} p_l \right)},
\]

\[
\Pi(t) = \prod_{i=1}^{t} p_i \Pi, \quad 1 \leq t \leq t_{\text{max}} - 1,
\]

\[P_t = (1 - q_t) (1 - p), \quad 1 \leq t \leq t_{\text{max}} - 1.\]

Proof. We recall that the dynamics of system under threshold preemptive policy can be fully characterized by a Markov chain with state space \( S \). Hence, the expected AoII achieved by the threshold preemptive policy can be calculated as

\[
\bar{\Delta}_{tp} = \sum_{s \in S} C(s) \pi_s,
\]

where \( \pi_s \) is the steady-state probability of state \( s \) in the induced Markov chain. The complete proof can be found in Appendix \( E \). \( \square \)

**Condition 2.** The conditions are the following.

\[
\begin{cases}
\gamma(1) - \frac{\bar{\Delta}_{tp}}{p} \leq 0, \\
(1 - q_{t+1}) \bar{\Delta}_{tp} - \sigma_t \leq 0, \quad 1 \leq t \leq t_{\text{max}} - 2, \\
(1 - q_{t+1}) p \gamma(t + 1) + \left( q_{t+1} \left( 1 - p \right) - p \right) \bar{\Delta}_{tp} \geq 0, \quad 1 \leq t \leq t_{\text{max}} - 2, \\
\sigma_t \leq \sigma, \quad 1 \leq t \leq t_{\text{max}} - 1,
\end{cases}
\]

where \( \bar{\Delta}_{tp} \) is given by Proposition \( 3 \) and

\[
\sigma = \frac{\sum_{i=1}^{t_{\text{max}}-2} \left( \frac{1}{\prod_{j=i+1}^{t_{\text{max}}-2} p_j} \right) + 1 + P_{t_{\text{max}}-1}}{\left[ \frac{1 - q_i p_1}{\prod_{l=1}^{t_{\text{max}}-2} p_l} - \sum_{i=2}^{t_{\text{max}}-2} \left( \frac{q_i p}{\prod_{j=i+1}^{t_{\text{max}}-2} p_j} \right) \right] - q_{t_{\text{max}}-1} p - P_{t_{\text{max}}-1} p},
\]
\[ \sigma_t = \left[ 1 - q_1 p \prod_{i=1}^{t} P_i \right] - \sum_{i=2}^{t} \left( \frac{q_i p}{\prod_{j=1}^{t} P_j} \right) \sigma - \sum_{i=1}^{t-1} \left( \frac{1}{\prod_{j=i}^{t} P_j} \right), \quad 1 \leq t \leq t_{\max} - 2, \]
\[ \sigma_{t_{\max} - 1} = 1 + p\sigma; \]
\[ \gamma(1) = \frac{\bar{\Delta}_{t_1}}{1-p} + \frac{\bar{\Delta}_{t_1}}{p(1-q_1)(1-p)} - \frac{\sigma}{(1-q_1)(1-p)}, \]
\[ \gamma(t) = \frac{\gamma(t-1) - \sigma_{t-1}}{(1-q_t)(1-p)} + \frac{\bar{\Delta}_{t_1}}{1-p}, \quad 2 \leq t \leq t_{\max} - 1. \]

**Theorem 5** (Optimal policy under uniform distribution). *Under Condition 2 the threshold preemptive policy is optimal when the communication delay is bounded by \( t_{\max} \) and is uniformly distributed.*

**Proof.** We follow the same methodology as presented in the proof of Theorem 3. The complete proof can be found in Appendix F. \qed

**Remark 8.** *From the proof the Theorem 5 we can conclude that threshold preemptive policy is optimal if the system satisfies the following conditions.*

1) *The delay is bounded by \( t_{\max} \).*

2) *The system verifies Condition 2.*

**D. Constant Delay**

In this section, we consider the case where the update transmission time is deterministic. To do this, we assume that each transmission will take \( t_{\max} \) time slots. Specifically, we have \( p_{t_{\max}} = 1 \) and \( p_t = 0 \) for \( t \neq t_{\max} \). Consequently, \( q_t \)'s are given by the following equation.

\[ q_t = \begin{cases} 
1 & t = t_{\max}, \\
0 & \text{otherwise}. 
\end{cases} \]

When the delay is constant, the transmission time for an update can not surpass \( t_{\max} \). Hence, the state \( s = (\Delta, t, i) \) in the corresponding \( M \) satisfies \( 1 \leq t \leq t_{\max} - 1. \)

**Definition 5** (Preempt-at-consistency policy). *A preempt-at-consistency policy initiates new transmission when the channel is idle, preempt the transmitting update when the update is the same as receiver’s estimate, and stay idle when the update is different from the receiver’s estimate except when \( t = t_{\max} - 1 \).*

An illustration of the preempt-at-consistency policy is given in Fig. 7. We first define two
Fig. 7: An illustration of the preempt-at-consistency policy. In the plot, green dots means \( a = 1 \) and red dots means \( a = 0 \).

auxiliary quantities \( f(t, p) \) and \( f_0(t, p) \), where \( f(1, p) = (1 - p) \) and \( f_0(1, p) = p \). Then, \( f(t, p) \) and \( f_0(t, p) \) for \( t \geq 2 \) can be obtained using the following iterative form.

\[
f(t, p) = (1 - p)f(t - 1, p) + p^2 \sum_{i=1}^{t-2} (1 - p)^{t-2-i} f(i, t) + p^2(1 - p)^{t-2}, \quad 2 \leq t \leq t_{\text{max}} - 1,
\]

\[
f_0(t, p) = (1 - p)^{t-1}p + p \sum_{i=1}^{t-1} (1 - p)^{t-i-1} f(i, p), \quad 2 \leq t \leq t_{\text{max}} - 2.
\]

For the sake for consistency, we define \( f_0(t_{\text{max}} - 1, p) = 0 \). Then, we have the following proposition.

**Proposition 4** (Performance of preempt-at-consistency policy). The expected AoII resulting from the adoption of preempt-at-consistency \( \bar{\Delta}_{pc} \) is given by the following equation.

\[
\bar{\Delta}_{pc} = \Sigma + \sum_{t=1}^{t_{\text{max}} - 1} \Sigma(t),
\]

where \( t_{\text{max}} \geq 3 \) and

\[
\Sigma = \sum_{i=0}^{t_{\text{max}} - 2} p(1 - p)^i \Pi(t_{\text{max}} - 1 - i) + \Pi \frac{1 - p(1 - p)^{t_{\text{max}} - 1}}{1 - p(1 - p)^{t_{\text{max}} - 1}},
\]

\[
\Sigma(t) = (1 - p)^t \Sigma + \sum_{i=0}^{t-1} (1 - p)^i \Pi(t - i), \quad 1 \leq t \leq t_{\text{max}} - 1,
\]

Figure 7: An illustration of the preempt-at-consistency policy. In the plot, green dots mean \( a = 1 \) and red dots mean \( a = 0 \).
\[
\Pi = \frac{1}{p - f(t_{max} - 1, p) + 1 + \sum_{t=1}^{t_{max} - 1} \left( f_0(t, p) + f(t, p) \right)}, \\
\Pi(t) = f(t, p)\Pi, \quad 1 \leq t \leq t_{max} - 1.
\]

**Proof.** We follow the same methodology as presented in the proof of Proposition 3. The complete proof can be found in Appendix G. \qed

**Remark 9.** When \( t_{max} = 2 \), by following a very similar procedure, we are able to obtain

\[
\bar{\Delta}_{pc} = \frac{3p - p^2 - p^3}{(1 + p)(1 - p + p^2)}.
\]

**Condition 3.** The conditions are the following.

\[
\begin{cases}
\bar{\Delta}_{pc} \leq \frac{(1 - p)\sigma + 1}{2}, \\
\bar{\Delta}_{pc} \geq p\sigma_1, \\
V(0, t, 1) \leq 0, \quad 1 \leq t \leq t_{max} - 2, \\
\bar{\Delta}_{pc} \geq pV(1, t, 1), \quad 1 \leq t \leq t_{max} - 2, \\
\sigma_t \leq \sigma, \quad 1 \leq t \leq t_{max} - 2,
\end{cases}
\]

where \( \bar{\Delta}_{pc} \) is given by Proposition 4 and

\[
\sigma = \frac{\sum_{i=0}^{t_{max} - 1} (1 - p)^i}{1 - p(1 - p)^{t_{max} - 1}}, \\
\sigma_t = \frac{\sigma - 1}{(1 - p)^t} - \sum_{i=1}^{t-1} \frac{1}{(1 - p)^i}, \quad 1 \leq t \leq t_{max} - 1,
\]

for each \( 1 \leq t \leq t_{max} - 2, \)

\[
V(0, t, 1) = -\bar{\Delta}_{pc} + (1 - p)V(0, t + 1, 1) + pV(1, t + 1, 1), \\
V(1, t, 1) = 1 - \bar{\Delta}_{pc} + (1 - p)\sigma_t + (1 - p)V(1, t + 1, 1) + pV(0, t + 1, 1).
\]

**Theorem 6 (Optimal policy under constant delay).** Under Condition 3 the preempt-at-consistency policy is optimal when the delay is \( t_{max} \) for all the updates.
Proof. The follow along the same methodology as presented in the proof of Theorem 5. The complete proof can be found in Appendix H.

E. Condition Check

In this section, we verify the conditions given in Condition 1, Condition 2, and Condition 3 under certain system parameters and show the corresponding results. The results are visualized in Fig. 8.

Remark 10. We also verified the condition when the communication channel delay follows Zipf distribution with \( a \in \{2, 3, ..., 10\} \). The results show that the system under consideration always satisfies the conditions in these cases. Therefore, the results of the condition verification when \( a \in \{2, 3, ..., 10\} \) are not shown here.

In Fig. 8a, we verify the conditions given by Condition 1 for the system with parameters specified in the plot. When \( p \) becomes larger, the conditions are satisfied only when \( t_{max} \) is sufficiently large. In other words, the optimal policy is weakly preemptive only when \( t_{max} \) is large enough. Also, a larger \( p \) requires a larger \( t_{max} \) to satisfy the conditions. The verification results for the conditions given in Condition 2 are visualized in Fig. 8b. Contrary to the case of Zipf distribution, when \( p \) becomes larger, the condition is satisfied only when \( t_{max} \) is small. In other words, the optimal policy is threshold preemptive only when \( t_{max} \) is small. Meanwhile, a larger \( p \) requires a smaller \( t_{max} \) to satisfy the conditions. In Fig. 8c, we show the results for the verification of Condition 3. From the figure, we can see that all of the system settings we considered meet the conditions. Hence, we can conclude that the corresponding optimal policy is the one defined by Definition 5 according to Theorem 6.

We emphasize here that the optimal policy depends not only on the type of probability distribution of the delay but also on the probability distribution parameters.

V. CONCLUSION

In this paper, we consider optimizing the performance of the transmitter-receiver system measured by AoII. In the system, the transmitter decides when to transmit updates to a distant receiver over a channel with a random delay to achieve the minimum AoII. The transmitter we consider can preempt the transmitting update to transmit a new update when the channel is busy, and the receiver will predict the state of the dynamic source based on the received
(a) The communication channel delay follows Zipf distribution with $\alpha = 1$ and $t_{max}$. The plot shows our results for verifying all the conditions given in Condition 1.

(b) The communication channel delay is uniformly distributed with maximum delay $t_{max}$. The plot shows our results for verifying all the conditions given in Condition 2.

(c) The communication channel delay is $t_{max}$. The plot shows our results for verifying all the conditions given in Condition 3.

Fig. 8: A visual representation of the condition validation. In the plots, a green dot indicates that all conditions are met while a red dot indicates that at least one condition is not met. Therefore, for the system with green dot, the optimal policy can be obtained easily using Theorem 4, Theorem 5, or Theorem 6.
updates. In this paper, the receiver will use the latest update received as his estimate. Therefore, this paper aims to optimize the transmitter decision in each time slot to minimize the AoII of the system. First, we cast the optimization problem into an infinite horizon with average cost Markov decision process and introduce the value iteration algorithm for finding the optimal policy. Then, to implement the value iteration algorithm, we simplify the Markov decision process so that its state space becomes finite. However, the optimal policies resulting from the value iteration algorithm are only approximations. We further conduct a theoretical analysis of the system when the delay distribution of the communication channel follows some typical distributions. Specifically, we consider the Geometric distribution, Zipf distribution, Uniform distribution, and the case where the delay is deterministic. To this end, we introduce the policy iteration algorithm. Then, leveraging the policy improvement theorem and under some easy-to-verify conditions, we theoretically find the optimal policy for each case described above. We notice that it is optimal to transmit new updates when the channel is idle. When the channel is transmitting a previous update, whether the transmitter preempts is closely related to whether the transmitted update can bring new information to the receiver and whether the transmitted update carries correct information about the dynamic source.

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According to the Theorem 2, it is sufficient to prove that the policy iteration algorithm will eventually converge to the preemptive policy defined by Definition 2. Therefore, in practice, we only need to prove that the policy derived from the expected AoII and the value function eventually converge to the preemptive policy defined by Definition 2. To this end, we first calculate the expected AoII \( \theta(\psi) \) and the value function \( V(\psi(s)) \) resulting from the adoption of the preemptive policy \( \psi \), which satisfy the following system of linear equations.

\[
V(\psi(0, 0, -1)) = -\theta(\psi) + (1 - p_s)(1 - p)V(\psi(0, 1, 0)) + (1 - p_s)pV(\psi(1, 1, 0)) + \\
p_s(1 - p)V(\psi(0, 0, -1)) + p_sV(\psi(1, 0, -1)).
\]

\[
V(\psi(0, t, 0)) = -\theta(\psi) + (1 - p_s)(1 - p)V(\psi(0, 1, 0)) + (1 - p_s)pV(\psi(1, 1, 0)) + \\
p_s(1 - p)V(\psi(0, 0, -1)) + p_sV(\psi(1, 0, -1)), \quad 1 \leq t \leq t_{max} - 1.
\]

\[
V(\psi(0, t, 1)) = -\theta(\psi) + (1 - p_s)(1 - p)V(\psi(0, 1, 0)) + (1 - p_s)pV(\psi(1, 1, 0)) + \\
p_s(1 - p)V(\psi(0, 0, -1)) + p_sV(\psi(1, 0, -1)), \quad 1 \leq t \leq t_{max} - 1.
\]

For each \( \Delta \geq 1 \),

\[
V(\psi(\Delta, 0, -1)) = \Delta - \theta(\psi) + (1 - p_s)(1 - p)V(\psi(\Delta + 1, 1, 1)) + (1 - p_s)pV(\psi(0, 1, 1)) + \\
p_s(1 - p)V(\psi(0, 0, -1)) + p_sV(\psi(\Delta + 1, 0, -1)).
\]

\[
V(\psi(\Delta, t, 0)) = \Delta - \theta(\psi) + (1 - p_s)(1 - p)V(\psi(\Delta + 1, 1, 1)) + (1 - p_s)pV(\psi(0, 1, 1)) + \\
p_s(1 - p)V(\psi(0, 0, -1)) + p_sV(\psi(\Delta + 1, 0, -1)), \quad 1 \leq t \leq t_{max} - 1.
\]
\[
V^\psi(\Delta, t, 1) = \Delta - \theta^\psi + (1 - p_s)(1 - p)V^\psi(\Delta + 1, 1, 1) + (1 - p_s)pV^\psi(0, 1, 1) + \\
p_s(1 - p)V^\psi(0, 0, -1) + p_s pV^\psi(\Delta + 1, 0, -1), \quad 1 \leq t \leq t_{\text{max}} - 1.
\]

We notice that this system of linear equations consists of an infinite number of equations, so it is difficult to directly obtain the expressions. However, some structural properties of \(V^\psi(s)\) are sufficient for us to complete the proof. These structural properties are presented in the following lemma.

**Lemma 1.** The value function \(V^\psi(s)\) possesses the following structural properties.

1) \(V^\psi(0, t, 0) = V^\psi(0, t, 1) = V^\psi(0, 0, -1) \triangleq V^\psi(0)\) for any \(t\).

2) \(V^\psi(\Delta, t, 0) = V^\psi(\Delta, t, 1) = V^\psi(\Delta, 0, -1) \triangleq V^\psi(\Delta)\) for any \(t\).

3) \(V^\psi(\Delta)\) is non-decreasing in \(\Delta\).

**Proof.** The first two properties are obvious. Hence, we will focus on proving the third property.

We note that the above system of linear equations can be solved using iterative method [15]. Since we only need to compare the relationship between different value functions without knowing the specific value, we can ignore \(\theta^\psi\) in the iterative equations without affecting the conclusion.

Then, the iterations are given by

\[
V^{\psi}_{\nu+1}(\Delta, t, i) = \Delta + \sum_{s' \in S} P_{s, s'}(\psi) V^{\psi}_{\nu}(s'),
\]

where \(V^{\psi}_{\nu}(s)\) is the value function at iteration \(\nu\) of the iteration. We know that \(\lim_{\nu \to \infty} V^{\psi}_{\nu}(s) = V^\psi(s)\). Then, leveraging the iterative nature, we can use mathematical induction to prove the desired results. We initialize \(V^{\psi}_{0}(\Delta) = 0\) for all \(\Delta\). Then, the base case \(\nu = 0\) is true by initialization. We assume the property is true at iteration \(\nu\). Then, we check whether it holds at iteration \(\nu + 1\). When \(\Delta \geq 1\), we have

\[
V^{\psi}_{\nu+1}(\Delta + 1) - V^{\psi}_{\nu+1}(\Delta) = 1 + (1 - p_s)(1 - p)[V^{\psi}_{\nu}(\Delta + 2) - V^{\psi}_{\nu}(\Delta + 1)] + p_s p[V^{\psi}_{\nu}(\Delta + 2) - V^{\psi}_{\nu}(\Delta + 1)].
\]

Applying the assumption for iteration \(\nu\), we can easily conclude that \(V^{\psi}_{\nu+1}(\Delta + 1) \geq V^{\psi}_{\nu+1}(\Delta)\).
Then, we only need to consider the case of $\Delta = 0$.

$$V^{\psi}_{\nu+1}(1) - V^{\psi}_{\nu+1}(0) = 1 + (1 - ps)(1 - p)[V^{\psi}_{\nu}(2) - V^{\psi}_{\nu}(0)] + (1 - ps)p[V^{\psi}_{\nu}(0) - V^{\psi}_{\nu}(1)] +$$

$$+ ps[V^{\psi}_{\nu}(2) - V^{\psi}_{\nu}(1)]$$

$$\geq (1 - ps)(1 - p)[V^{\psi}_{\nu}(1) - V^{\psi}_{\nu}(0)] + (1 - ps)p[V^{\psi}_{\nu}(0) - V^{\psi}_{\nu}(1)] +$$

$$+ ps[V^{\psi}_{\nu}(2) - V^{\psi}_{\nu}(1)]$$

$$= (1 - ps)(1 - 2p)[V^{\psi}_{\nu}(1) - V^{\psi}_{\nu}(0)] + ps[V^{\psi}_{\nu}(2) - V^{\psi}_{\nu}(1)]$$

$$\geq 0.$$  

Hence, the property also holds at iteration $\nu + 1$. Then, by mathematical induction, we can conclude that property 3 is true. □

Equipped with Lemma 1, we can proceed with deriving the optimal policy resulting from the value function $V^{\psi}(s)$ and the expected AoII $\theta^{\psi}$. To this end, we define $V^{\psi,a}(s)$ as the value function resulting from taking action $a$ at state $s$, which can be calculated using the following equations.

$$V^{\psi,a}(s) = \Delta - \theta^{\psi} + \sum_{s' \in S} P_{s,s'}(a)V^{\psi}(s)$$

$$= \Delta - \theta^{\psi} + \sum_{\Delta' \geq 0} P_{\Delta,\Delta'}(a)V^{\psi}(\Delta'),$$

where $P_{\Delta,\Delta'}(a)$ is the probability that executing action $a$ in case of $\Delta$ will leads to $\Delta'$. To determine the optimal action, we only need to determine the sign of $\delta V^{\psi}(s) \triangleq V^{\psi,0}(s) - V^{\psi,1}(s)$ for each state $s$. When $\delta V^{\psi}(s) < 0$, the optimal action is $a = 0$. Otherwise, $a = 1$ is optimal. Hence, we distinguish between the following states.

- $s = (0, 0, -1)$.

$$\delta V^{\psi}(0, 0, -1) = (1 - p)V^{\psi}(0) + pV^{\psi}(1) - (1 - ps)(1 - p)V^{\psi}(0) -$$

$$(1 - ps)pV^{\psi}(1) - ps(1 - p)V^{\psi}(0) - psV^{\psi}(1) = 0.$$

- $s = (\Delta, 0, -1)$ where $\Delta \geq 1$.

$$\delta V^{\psi}(\Delta, 0, -1) = pV^{\psi}(0) + (1 - p)V^{\psi}(\Delta + 1) - (1 - ps)(1 - p)V^{\psi}(\Delta + 1) -$$

$$(1 - ps)pV^{\psi}(0) - psV^{\psi}(\Delta + 1) - ps(1 - p)V^{\psi}(0)$$

$$= ps(1 - 2p)[V^{\psi}(\Delta + 1) - V^{\psi}(0)]$$

$$\geq 0.$$
• \( s = (0, t, 0) \) where \( t \geq 1 \).
\[
\delta V^\psi(0, t, 0) = (1 - p_s)(1 - p)V^\psi(0) + (1 - p_s)pV(1) + p_s(1 - p)V^\psi(0) + p_s p V^\psi(1) - (1 - p_s)(1 - p)V^\psi(0) - (1 - p_s)pV^\psi(1) - p_s (1 - p)V^\psi(0) - p_s p V^\psi(1)
\]
\[= 0\]

• \( s = (0, t, 1) \) where \( t \geq 1 \).
\[
\delta V^\psi(0, t, 1) = (1 - p_s)(1 - p)V^\psi(0) + (1 - p_s)pV^\psi(1) + p_s(1 - p)V^\psi(1) + p_s p V^\psi(0) - (1 - p_s)(1 - p)V^\psi(0) - (1 - p_s)pV^\psi(1) - p_s (1 - p)V^\psi(0) - p_s p V^\psi(1)
\]
\[= p_s(1 - 2p)[V^\psi(1) - V^\psi(0)]
\]
\[\geq 0.\]

• \( s = (\Delta, t, 0) \) where \( \Delta \geq 1 \) and \( t \geq 1 \).
\[
\delta V^\psi(\Delta, t, 0) = (1 - p_s)(1 - p)V^\psi(\Delta + 1) + (1 - p_s)pV^\psi(0) + p_s(1 - p)V^\psi(\Delta + 1) + p_s p V^\psi(0) - (1 - p_s)(1 - p)V^\psi(\Delta + 1) - (1 - p_s)pV^\psi(0) - p_s (1 - p)V^\psi(0) - p_s p V^\psi(\Delta + 1)
\]
\[= p_s(1 - 2p)[V^\psi(\Delta + 1) - V^\psi(0)]
\]
\[\geq 0.\]

• \( s = (\Delta, t, 1) \) where \( \Delta \geq 1 \) and \( t \geq 1 \).
\[
\delta V(\Delta, t, 1) = (1 - p_s)(1 - p)V^\psi(\Delta + 1) + (1 - p_s)pV^\psi(0) + p_s(1 - p)V^\psi(0) + p_s p V^\psi(\Delta + 1) - (1 - p_s)(1 - p)V^\psi(\Delta + 1) - (1 - p_s)pV^\psi(0) - p_s (1 - p)V^\psi(0) - p_s p V^\psi(\Delta + 1)
\]
\[= 0.\]

Hence, we can conclude that the policy iteration algorithm convergences to the preemptive policy. Then, by Theorem 2, we can claim that the optimal policy is preemptive policy.

**APPENDIX B**

**PROOF OF PROPOSITION 1**

The expected AoII achieved by the preemptive policy can be calculated as

\[
\bar{\Delta}_p = \sum_{s \in S} C(s)p_s,
\]
where $\pi_s$ is the steady-state probability of state $s$ in the Markov chain induced by the preemptive policy. According to Definition 2, the state space of the induced Markov chain can be classified in the following way.

- The system will be in states with $t = 0$ and $i = -1$ each time an update is delivered. Hence, state $s = (\Delta, 0, -1)$ where $\Delta \geq 0$ is recurrent. We denote the corresponding steady-state probability as $\pi_{-1}(\Delta)$.
- Under the preemptive policy, the transmitting updates are either delivered or preempted by new updates after one time slot. Hence, state with $t > 1$ is transient. Then, we focus on the state with $t = 1$.
  - We note that $i = 0$ will only occur if the transmitter initiates the transmission in the states with $\Delta = 0$. Hence, state $s = (\Delta, 1, 0)$ where $\Delta > 1$ is transient and state $s = (\Delta, 1, 0)$ where $\Delta \in \{0, 1\}$ is recurrent. We denote the steady-state probability of the recurrent state as $\pi_0(\Delta)$. For the sake of writing consistency, we define $\pi_0(\Delta) = 0$ when $\Delta > 1$.
  - We notice that $i = 1$ will occur if the transmitter initiates the transmission in the states with $\Delta \neq 0$. Hence, state $s = (\Delta, 1, 1)$ where $\Delta \geq 0$ is recurrent. We denote the corresponding steady-state probability as $\pi_1(\Delta)$.

In the following, we will calculate the steady-state probability of the recurrent states. Combining with the system dynamic, these steady-state probabilities satisfy the following balance equations.

\[
\pi_{-1}(0) = p_s(1 - p) \sum_{i=0}^{\infty} \pi_{-1}(i) + p_s(1 - p) \sum_{i=0}^{\infty} \pi_0(i) + p_s(1 - p) \sum_{i=0}^{\infty} \pi_1(i). \tag{6}
\]

\[
\pi_{-1}(\Delta) = p_s p \left( \pi_{-1}(\Delta - 1) + \pi_0(\Delta - 1) + \pi_1(\Delta - 1) \right) \quad \Delta \geq 1.
\]

\[
\pi_0(0) = (1 - p_s)(1 - p) \left( \pi_{-1}(0) + \pi_0(0) + \pi_1(0) \right). \tag{7}
\]

\[
\pi_0(1) = (1 - p_s) p \left( \pi_{-1}(0) + \pi_0(0) + \pi_1(0) \right).
\]

\[
\pi_0(\Delta) = 0 \quad \Delta \geq 2.
\]

\[
\pi_1(0) = (1 - p_s) p \sum_{i=1}^{\infty} \left( \pi_{-1}(i) + \pi_0(i) + \pi_1(i) \right). \tag{8}
\]

\[
\pi_1(1) = 0.
\]

\[
\pi_1(\Delta) = (1 - p_s)(1 - p) \left( \pi_{-1}(\Delta - 1) + \pi_0(\Delta - 1) + \pi_1(\Delta - 1) \right) \quad \Delta \geq 2.
\]
\[
\sum_{i=0}^{\infty} \pi_{-1}(i) + \sum_{i=0}^{\infty} \pi_0(i) + \sum_{i=0}^{\infty} \pi_1(i) = 1 \tag{9}
\]

In the following, we will solve this system of linear equation. First of all, combining (6) and (9) yields

\[
\pi_{-1}(0) = p_s(1 - p).
\]

We define \(\pi(\Delta) \triangleq \pi_{-1}(\Delta) + \pi_0(\Delta) + \pi_1(\Delta)\) for \(\Delta \geq 0\). Note that, using this definition, \(\bar{\Delta}_p = \sum_{i=0}^{\infty} i\pi(i)\) and \(\sum_{i=0}^{\infty} \pi(i) = 1\). Then, when \(\Delta = 0\), we have

\[
\pi(0) = p_s(1 - p) + \pi_0(0) + \pi_1(0).
\]

Combining with (7) and (8) yields

\[
\pi(0) = p_s(1 - p) + (1 - p_s)(1 - p)\pi(0) + (1 - p_s)p(1 - \pi(0)).
\]

Then, we obtain

\[
\pi(0) = 1 - \frac{p}{1 - (1 - p_s)(1 - 2p)}.
\]

Likewise, we can obtain

\[
\pi(1) = p\pi(0).
\]

\[
\pi(\Delta) = (p_s p + (1 - p_s)(1 - p))\pi(\Delta - 1), \quad \Delta \geq 2. \tag{10}
\]

We define \(\Pi = \sum_{i=2}^{\infty} \pi(i)\). Then, we sum (10) from 2 to \(\infty\), which gives us the following

\[
\Pi = (p_s p + (1 - p_s)(1 - p))(\Pi + \pi(1)).
\]

After some algebraic manipulations, we have

\[
\Pi = \frac{(p_s p + (1 - p_s)(1 - p))}{1 - (p_s p + (1 - p_s)(1 - p))}\pi(1).
\]

We also define \(\Sigma = \sum_{i=2}^{\infty} i\pi(i)\). Then, the expected AoII achieved by the preemptive policy can be calculated as

\[
\bar{\Delta}_p = \pi(1) + \Sigma. \tag{11}
\]

To obtain \(\Sigma\), we multiply both size of (10) by \(\Delta - 1\).

\[
(\Delta - 1)\pi(\Delta) = (p_s p + (1 - p_s)(1 - p))(\Delta - 1)\pi(\Delta - 1), \quad \Delta \geq 2.
\]

Then, we sum the above equation from 2 to \(\infty\), which gives

\[
\Sigma - \Pi = (p_s p + (1 - p_s)(1 - p))(\Sigma + \pi(1)).
\]
Then, we obtain
\[
\Sigma = \frac{(p_s p + (1 - p_s)(1 - p)) \pi(1) + \Pi}{1 - (p_s p + (1 - p_s)(1 - p))},
\]
Combining with (11), the expected AoII is given by
\[
\tilde{\Delta}_p = \pi(1) + \frac{(p_s p + (1 - p_s)(1 - p)) \pi(1) + \Pi}{1 - (p_s p + (1 - p_s)(1 - p))},
\]
where
\[
\Pi = \frac{p(p_s p + (1 - p_s)(1 - p))}{1 - (p_s p + (1 - p_s)(1 - p))} \pi(0),
\]
\[
\pi(1) = p\pi(0),
\]
\[
\pi(0) = 1 - \frac{p}{1 - (1 - p_s)(1 - 2p)}.
\]
After some algebraic manipulations, we have
\[
\tilde{\Delta}_p = \frac{p}{(p + p_s - 2p_s p)(p_s + 2p - 2p_s p)}.
\]

APPENDIX C

PROOF OF PROPOSITION 2

Same as we did in the proof of Proposition 1, the expected AoII can be calculated as
\[
\tilde{\Delta}_{p'} = \sum_{s \in \mathcal{S}} C(s) \pi_s,
\]
where \( \pi_s \) is the steady-state probability of state \( s \) in the Markov chain induced by the weak preemptive policy. Similar to what we did in the proof of Proposition 1, the state space of the induced Markov chain can be classified as follows.

- The system will be in states with \( t = 0 \) and \( i = -1 \) each time an update is delivered. Hence, state \( s = (\Delta, 0, -1) \) where \( \Delta \geq 0 \) is recurrent. We denote the steady-state probability for these states as \( \pi_{-1}(\Delta) \).
- We recall that when \( i \neq -1 \), the weak preemptive policy will not preempt the transmitting update only in state \( s = (\Delta, t_{max} - 1, 1) \) where \( \Delta \geq 1 \). Hence, similar to the discussion detailed in the proof of proposition 1, state with \( t > 1 \) is transient. When \( t = 1 \), states are transient except state \( s = (\Delta, 1, 0) \) where \( \Delta \in \{0, 1\} \) and state \( s = (\Delta, 1, 1) \) where \( \Delta \geq 0 \). We denote the steady-state probability for the recurrent state when \( t = 1 \) as \( \pi_i(\Delta) \). For the sake of writing consistency, we define \( \pi_0(\Delta) = 0 \) when \( \Delta > 1 \).
We notice that the state classification of Markov chain induced by weak preemption policy is the same as that of Markov chain induced by preemption policy. At the same time, the state transition probabilities are also the same. Hence, the steady-state probability of the recurrent state can be obtained directly by replacing $p_s$’s in the proof of Proposition 1 with $q_1$. Meanwhile, the expected AoII $\bar{\Delta}_p'$ can also be obtained directly by replacing $p_s$’s in (5) with $q_1$.

**APPENDIX D**

**PROOF OF THEOREM 4**

We follow the same methodology presented in the proof of Theorem 3. First of all, we calculate the expected AoII $\theta^\psi$ and the value function $V^\psi(s)$ resulting from the adoption of the weak preemptive policy $\psi$.

$$V^\psi(0,0,-1) = \theta^\psi + (1-q_1)(1-p)V^\psi(0,1,0) + (1-q_1)pV^\psi(1,1,0) + q_1(1-p)V^\psi(0,0,-1) + q_1pV^\psi(1,0,-1).$$

$$V^\psi(0,t,0) = \theta^\psi + (1-q_1)(1-p)V^\psi(0,1,0) + (1-q_1)pV^\psi(1,1,0) + q_1(1-p)V^\psi(0,0,-1) + q_1pV^\psi(1,0,-1), \quad 1 \leq t \leq t_{max} - 1.$$

$$V^\psi(0,t,1) = \theta^\psi + (1-q_1)(1-p)V^\psi(0,1,0) + (1-q_1)pV^\psi(1,1,0) + q_1(1-p)V^\psi(0,0,-1) + q_1pV^\psi(1,0,-1), \quad 1 \leq t \leq t_{max} - 1.$$

For each $\Delta \geq 1$,

$$V^\psi(\Delta,0,-1) = \Delta - \theta^\psi + (1-q_1)(1-p)V^\psi(\Delta + 1,1,1) + (1-q_1)pV^\psi(0,1,1) + q_1(1-p)V^\psi(0,0,-1) + q_1pV^\psi(\Delta + 1,0,-1).$$

$$V^\psi(\Delta,t,0) = \Delta - \theta^\psi + (1-q_1)(1-p)V^\psi(\Delta + 1,1,1) + (1-q_1)pV^\psi(0,1,1) + q_1(1-p)V^\psi(0,0,-1) + q_1pV^\psi(\Delta + 1,0,-1), \quad 1 \leq t \leq t_{max} - 1.$$

$$V^\psi(\Delta,t,1) = \Delta - \theta^\psi + (1-q_1)(1-p)V^\psi(\Delta + 1,1,1) + (1-q_1)pV^\psi(0,1,1) + q_1(1-p)V^\psi(0,0,-1) + q_1pV^\psi(\Delta + 1,0,-1), \quad 1 \leq t \leq t_{max} - 2.$$

$$V^\psi(\Delta,t_{max}-1,1) = \Delta - \theta^\psi + (1-p)V^\psi(0,0,-1) + pV^\psi(\Delta + 1,0,-1). \quad (12)$$

Instead of solving the above system of linear equations with infinitely many equations, some structural properties of the solution will be sufficient for the following analysis.
Lemma 2. The value function $V^\psi(s)$ possesses the following properties.

1) $V^\psi(\Delta, 0, -1) = V^\psi(\Delta, t, 0)$ for any $\Delta$ and $t$. We abbreviate them as $V^\psi(\Delta)$.

2) $V^\psi(\Delta, t, 1) = V^\psi(\Delta)$ for $t < t_{max} - 1$. $V^\psi(0, t, 1) = V^\psi(0)$ for any $t$.

3) $V^\psi(\Delta)$ is increasing in $\Delta \geq 0$.

4) $V^\psi(1) - V^\psi(0) = \frac{\theta^\psi}{p}$ and $V^\psi(\Delta + 1) - V^\psi(\Delta) \triangleq \sigma$ is independent of $\Delta$, where

$$\sigma = \frac{1}{q_1 + p - 2q_1p}.$$

Proof. The first two properties are obvious as we can verify directly by comparing the corresponding equations. For the third property, the proof is based on the mathematical induction as presented in the proof of Lemma 1. Hence, we omit the proof here for the sake of space. In the following, we will focus on proving the last property. To this end, applying the first two properties to the above system of linear equations yields the following.

$$V^\psi(0) = -\theta^\psi + (1 - p)V^\psi(0) + pV^\psi(1).$$  \(13\)

$$V^\psi(\Delta) = \Delta - \theta^\psi + (1 - q_1)(1 - p)V^\psi(\Delta + 1) + (1 - q_1)pV^\psi(0) + q_1(1 - p)V^\psi(0) + q_1pV^\psi(\Delta + 1).$$

From (13), we can easily conclude that

$$V^\psi(1) - V^\psi(0) = \frac{\theta^\psi}{p}.$$ 

In the following, we proof that $V^\psi(\Delta + 1) - V^\psi(\Delta)$ is independent of $\Delta$. We recall that $V^\psi(\Delta)$ can be obtained using iterative method. Let $V^\psi_\nu(\Delta)$ be the intermediate value function at iteration $\nu$. Then, the intermediate value function is updated in the following way.

$$V^\psi_{\nu+1}(0) = -\theta^\psi + (1 - p)V^\psi_\nu(0) + pV^\psi_{\nu}(1).$$

$$V^\psi_{\nu+1}(\Delta) = \Delta - \theta^\psi + (1 - q_1)(1 - p)V^\psi_{\nu}(\Delta + 1) + (1 - q_1)pV^\psi_{\nu}(0) + q_1(1 - p)V^\psi_{\nu}(0) + q_1pV^\psi_{\nu}(\Delta + 1).$$

Then, we know that $\lim_{\nu \to \infty} V^\psi_\nu(\Delta) = V^\psi(\Delta)$. Consequently, we can use mathematical induction to prove the desired results. To this end, we initialize the $V^\psi_0(\Delta) = 0$ for all $\Delta$. Then, the base case $\nu = 0$ is true by initialization. We assume the result holds at iteration $\nu$ and examine whether it still hold at iteration $\nu + 1$. To this end, we have

$$V^\psi_{\nu+1}(\Delta + 1) - V^\psi_{\nu+1}(\Delta) = 1 + (1 - q_1)(1 - p)[V^\psi_{\nu}(\Delta + 2) - V^\psi_{\nu}(\Delta + 1)] + q_1p[V^\psi_{\nu}(\Delta + 2) - V^\psi_{\nu}(\Delta + 1)].$$
According to our assumption, \( V_{\nu}^\psi(\Delta + 1) - V_{\nu}^\psi(\Delta) \) is independent of \( \Delta \). Hence, we can conclude that \( V_{\nu+1}^\psi(\Delta + 1) - V_{\nu+1}^\psi(\Delta) \) is also independent of \( \Delta \). Then, to calculate the constant \( \sigma \), we have

\[
V^\psi(\Delta + 1) - V^\psi(\Delta) = 1 + (1 - q_1)(1 - p)[V^\psi(\Delta + 2) - V^\psi(\Delta + 1)] + \\
q_1 p [V^\psi(\Delta + 2) - V^\psi(\Delta + 1)] \\
= 1 + (1 - q_1)(1 - p)\sigma + q_1 p \sigma.
\]

Finally, we obtain

\[
\sigma = \frac{1}{q_1 + p - 2q_1 p}.
\]

With Lemma 2 in mind, we can proceed with the policy improvement step in the policy iteration algorithm. In this step, we will verify that the optimal policy derived from \( \theta^\psi \) and \( V^\psi(s) \) is still weak preemptive policy. Same as we did in the proof of Theorem 3, we define \( V^\psi,a(s) \) as the value function resulting from taking action \( a \) at state \( s \). To determine the optimal policy, we only need to determine the sign of \( \delta V^\psi(s) \triangleq V^\psi,0(s) - V^\psi,1(s) \) for each state \( s \). When \( \delta V^\psi(s) < 0 \), the optimal action is \( a = 0 \). Otherwise, \( a = 1 \) is optimal. Then, we distinguish between the following states.

- \( s = (0, 0, -1) \).
  \[
  \delta V^\psi(0, 0, -1) = (1 - p)V^\psi(0) + pV^\psi(1) - (1 - q_1)(1 - p)V^\psi(0) - \\
  (1 - q_1)pV^\psi(1) - q_1(1 - p)V^\psi(0) - q_1 p V^\psi(1) \\
  = 0.
  \]

- \( s = (\Delta, 0, -1) \) where \( \Delta \geq 1 \).
  \[
  \delta V^\psi(\Delta, 0, -1) = pV^\psi(0) + (1 - p)V^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1) - \\
  (1 - q_1)pV^\psi(0) - q_1 p V^\psi(\Delta + 1) - q_1(1 - p)V^\psi(0) \\
  = q_1(1 - 2p)[V^\psi(\Delta + 1) - V^\psi(0)] \\
  \geq 0.
  \]
• $s = (0, t, 0)$ where $t \geq 1$.

$$\delta V^\psi(0, t, 0) =$$

$$(1 - q_{t+1})(1 - p)V^\psi(0) + (1 - q_{t+1})pV^\psi(1) + q_{t+1}(1 - p)V^\psi(0) + q_{t+1}pV^\psi(1) -$$

$$(1 - q_1)(1 - p)V^\psi(0) - (1 - q_1)pV^\psi(1) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(1))$$

$$= 0.$$  

• $s = (0, t, 1)$ where $1 \leq t \leq t_{\text{max}} - 3$.

$$\delta V^\psi(0, t, 1) =$$

$$(1 - q_{t+1})(1 - p)V^\psi(0) + (1 - q_{t+1})pV^\psi(1) + q_{t+1}(1 - p)V^\psi(0) -$$

$$(1 - q_1)(1 - p)V^\psi(0) - (1 - q_1)pV^\psi(1) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(1))$$

$$= q_{t+1}(1 - 2p)[V^\psi(1) - V^\psi(0)] \geq 0.$$

• $s = (0, t_{\text{max}} - 2, 1)$.

$$\delta V^\psi(0, t_{\text{max}} - 2, 1) = (1 - q_{t_{\text{max}} - 1})(1 - p)V^\psi(0) + (1 - q_{t_{\text{max}} - 1})pV^\psi(1, t_{\text{max}} - 1, 1) +$$

$$q_{t_{\text{max}} - 1}(1 - p)V^\psi(1) + q_{t_{\text{max}} - 1}pV^\psi(0) - (1 - p)V^\psi(0) - pV^\psi(1).$$

We notice that $V^\psi(1, t_{\text{max}} - 1, 1)$ satisfies (12). Hence, replacing $V^\psi(1, t_{\text{max}} - 1, 1)$ with the corresponding expression yields

$$\delta V^\psi(0, t_{\text{max}} - 2, 1) = (1 - q_{t_{\text{max}} - 1})p[1 - V^\psi(0) + pV^\psi(2)] +$$

$$q_{t_{\text{max}} - 1}(2p - 1)V^\psi(0) + q_{t_{\text{max}} - 1}(1 - p)V^\psi(1) - pV^\psi(1).$$

According to Lemma 2, $V^\psi(2) = V^\psi(1) + \sigma$. Hence, $\delta V^\psi(0, t_{\text{max}} - 2, 1)$ can be written as

$$\delta V^\psi(0, t_{\text{max}} - 2, 1) = q_{t_{\text{max}} - 1}(2p - 1)V^\psi(0) + (1 - q_{t_{\text{max}} - 1})p[1 - \theta^\psi + (1 - p)V^\psi(0) +$$

$$(1 - p)V^\psi(1) + \sigma] + q_{t_{\text{max}} - 1}(1 - p)V^\psi(1) - pV^\psi(1)$$

$$= [(1 - p)[p - q_{t_{\text{max}} - 1} + q_{t_{\text{max}} - 1}p^2][V^\psi(0) - V^\psi(1)] +$$

$$(1 - q_{t_{\text{max}} - 1})p(1 - \theta^\psi + p\sigma).$$

Plugging the expressions for $\sigma$, $\theta$, and using the fact that $V^\psi(1) - V^\psi(0) = \frac{\sigma^\psi}{p}$ yield

$$\delta V^\psi(0, t_{\text{max}} - 2, 1) = \frac{(q_{t_{\text{max}} - 1} - q_{t_{\text{max}} - 1}p - p) + (1 - q_{t_{\text{max}} - 1})p(q_1 + 2p - 2q_1p)^2}{(q_1 + 2p - 2q_1p)(q_1 + p - 2q_1p)}. \quad (14)$$

We notice that the denominator of (14) is positive. Hence, examining the sign of the numerator of (14) is sufficient to determine the sign of $\delta V^\psi(0, t_{\text{max}} - 2, 1)$. To this end, let
\[ \alpha(p) = (q_{t_{\text{max}} - 1} - q_{t_{\text{max}} - 1}p - p) + (1 - q_{t_{\text{max}} - 1})p(q_1 + 2p - 2q_1p)^2, \] which is the numerator of (14). Then, we take the derivative of \( \alpha(p) \) with respect to \( p \). More precisely, we have

\[ \frac{d\alpha(p)}{p} = (1 - q_{t_{\text{max}} - 1}) \left( 12p_1^2p^2 + 12p^2 - 24q_{t_{\text{max}} - 1}p^2 + 8p_1p - 8p_1^2p + p_1^2 \right) - q_{t_{\text{max}} - 1} - 1 \]

We notice that \( \frac{d\alpha(p)}{p} \) is quadratic with

\[ -\frac{8p_1 - 8p_1^2}{24p_1^2 + 24 - 48q_{t_{\text{max}} - 1}} = -\frac{p_1 - p_1^2}{3(p_1^2 - 2q_{t_{\text{max}} - 1} + 1)} \leq 0. \]

Hence, we can conclude that \( \frac{d\alpha(p)}{p} \) is decreasing in \( p \) when \( 0 \leq p \leq 0.5 \). Moreover, \( \frac{d\alpha(p)}{p} \bigg|_{p=0} \leq 0 \). Hence, the \( \alpha(p) \) is decreasing in \( p \). At the same time, when \( p = \frac{1}{2} \), \( \alpha(p) = 0 \). Hence, we can conclude that \( \alpha(p) \geq 0 \) when \( 0 \leq p \leq 0.5 \). Consequently, \( \delta V^\psi(0, t_{\text{max}} - 2, 1) \geq 0 \).

- \( s = (0, t_{\text{max}} - 1, 1) \).

\[ \delta V^\psi(0, t_{\text{max}} - 1, 1) = (1 - p)V^\psi(1) + pV^\psi(0) - (1 - q_1)(1 - p)V^\psi(0) - (1 - q_1)pV^\psi(1) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(1) \]

\[ = (1 - p)V^\psi(1) + pV^\psi(0) - (1 - p)V^\psi(0) - pV^\psi(1) \]

\[ \geq 0. \]

- \( s = (\Delta, t, 0) \) where \( \Delta \geq 1 \) and \( 1 \leq t \leq t_{\text{max}} - 1 \).

\[ \delta V^\psi(\Delta, t, 0) = (1 - q_{t+1})(1 - p)V^\psi(\Delta + 1) + (1 - q_{t+1})pV^\psi(0) + \]

\[ q_{t+1}(1 - p)V^\psi(\Delta + 1) + q_{t+1}pV^\psi(0) - (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) - (1 - q_1)pV^\psi(0, 1, 1) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(\Delta + 1) \]

\[ = (1 - p)V^\psi(\Delta + 1) + pV^\psi(0) - (1 - q_1)(1 - p)V^\psi(\Delta + 1) - (1 - q_1)pV^\psi(0) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(\Delta + 1) \]

\[ = q_1(1 - 2p)[V^\psi(\Delta + 1) - V^\psi(0)] \]

\[ \geq 0. \]
\( s = (\Delta, t, 1) \) where \( \Delta \geq 1 \) and \( 1 \leq t < t_{\text{max}} - 2 \).

\[
\delta V^\psi(\Delta, t, 1) = (1 - q_{t+1})(1 - p)V^\psi(\Delta + 1) + (1 - q_{t+1})pV^\psi(0) + q_{t+1}(1 - p)V^\psi(0) + \\
q_{t+1}pV^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1) - (1 - q_1)pV^\psi(0) - \\
q_1(1 - p)V^\psi(0) - q_1pV^\psi(\Delta + 1) \\
= (q_1 - q_{t+1})(1 - 2p)[V^\psi(\Delta + 1) - V^\psi(0)] \\
\geq 0.
\]

\( s = (\Delta, t_{\text{max}} - 2, 1) \) where \( \Delta \geq 1 \).

\[
\delta V^\psi(\Delta, t_{\text{max}} - 2, 1) = (1 - q_{t_{\text{max}} - 1})(1 - p)V^\psi(\Delta + 1, t_{\text{max}} - 1, 1) + \\
(1 - q_{t_{\text{max}} - 1})pV^\psi(0) + q_{t_{\text{max}} - 1}(1 - p)V^\psi(0) + \\
q_{t_{\text{max}} - 1}pV^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1) - \\
(1 - q_1)pV^\psi(0) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(\Delta + 1).
\]

We recall that \( V^\psi(\Delta + 1, t_{\text{max}} - 1, 1) \) satisfies the expression given by (12). Hence, replacing \( V^\psi(\Delta + 1, t_{\text{max}} - 1, 1) \) with corresponding expression yields

\[
\delta V^\psi(\Delta, t_{\text{max}} - 2, 1) = (1 - q_{t_{\text{max}} - 1})(1 - p)(\Delta + 1 - \theta^\psi + (1 - p)V^\psi(0) + pV^\psi(\Delta + 2)) + \\
(1 - q_{t_{\text{max}} - 1})pV^\psi(0) + q_{t_{\text{max}} - 1}(1 - p)V^\psi(0) + \\
q_{t_{\text{max}} - 1}pV^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1) - \\
(1 - q_1)pV^\psi(0) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(\Delta + 1).
\]

Meanwhile, \( V^\psi(\Delta + 1) = V^\psi(1) + \Delta \sigma \) and \( V^\psi(1) - V^\psi(0) = \frac{\theta^\psi}{p} \). Hence, we have

\[
\delta V^\psi(\Delta, t_{\text{max}} - 2, 1) = \left( \frac{(1 - q_1)(2p - 1) - p(1 - q_{t_{\text{max}} - 1})}{p} \right) \theta - \\
[((1 - q_1)(1 - 2p) + p^2(1 - q_{t_{\text{max}} - 1})))\Delta - \\
(1 - q_{t_{\text{max}} - 1})(1 - p)p]\sigma + (1 - q_{t_{\text{max}} - 1})(1 - p)(\Delta + 1).
\]

We notice that the coefficient before \( \Delta \) is

\[
[((1 - q_1)(1 - 2p) + p^2(1 - q_{t_{\text{max}} - 1})))\sigma + (1 - q_{t_{\text{max}} - 1})(1 - p) \geq 0.
\]
Hence, $\delta V^\psi(\Delta, t_{\text{max}} - 2, 1)$ is increasing in $\Delta$. When $\Delta = 1$, we have

$$\delta V(1, t_{\text{max}} - 2, 1) = \frac{(1 - q_1)(2p - 1) - p(1 - q_{t_{\text{max}} - 1})}{(2p + q_1 - 2q_1p)(p + q_1 - 2q_1p)} - \frac{(1 - q_1)(1 - 2p) + p(2p - 1)(1 - q_{t_{\text{max}} - 1})}{q_1 + p - 2q_1p} + 2(1 - q_{t_{\text{max}} - 1})(1 - p).$$

Since we assume that Condition [I] holds, we can conclude that $\delta V(1, t_{\text{max}} - 2, 1) \geq 0$. Consequently, $\delta V(\Delta, t_{\text{max}} - 2, 1) \geq \delta V(1, t_{\text{max}} - 2, 1) \geq 0$ for $\Delta \geq 1$.

- $s = (\Delta, t_{\text{max}} - 1, 1)$ where $\Delta \geq 1$. 

$$\delta V^\psi(\Delta, t_{\text{max}} - 1, 1) = (1 - p)V(0) + pV(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1) - (1 - q_1)pV^\psi(0) - q_1pV^\psi(\Delta + 1) = (1 - q_1)(1 - 2p)[V^\psi(0) - V^\psi(\Delta + 1)] \leq 0.$$ 

Combining the above cases, we can conclude that the weak preemptive policy is optimal.

**APPENDIX E**

**PROOF OF PROPOSITION 3**

We recall that the dynamics of system under threshold preemptive policy can be fully characterized by a Markov chain with state space $\mathcal{S}$. Hence, the expected AoII achieved by the threshold preemptive policy can be calculated as

$$\bar{\Delta}_{tp} = \sum_{s \in \mathcal{S}} C(s)\pi_s,$$

where $\pi_s$ is the steady-state probability of state $s$ in the induced Markov chain. We notice from the state transition probability detailed in Section [III-A] that when $a = 1$, the transition probabilities depends only on $\Delta$. Leveraging this observation, we can combine states with the same $\Delta$ and the same action specified by the policy into one large state to reduce the state space of the induced Markov chain. More precisely, we have the following.

- The action specified by threshold preemptive policy at state $s = (0, 0, -1)$ and state $s = (0, t, i)$ for any $t$ and $i$ is $a = 1$. Hence, these states share the same dynamics. Consequently, we denote the steady state probability for being in these states as $\pi_0$. 

September 29, 2022 DRAFT
• For $\Delta > 0$, the action suggested by threshold preemptive policy at state $s = (\Delta, 0, -1)$ and state $s = (\Delta, t, 0)$ for any $t$ is $a = 1$. Hence, similar to the above case, we denote the steady state probability for being in these states as $\pi_\Delta$.

• At state $s = (\Delta, t, 1)$, the suggested action is $a = 0$. In this case, the dynamic depends on all three quantities in the triplet. Hence, we denote the steady state probability for being in these states as $\pi_\Delta(t)$.

Then, combining with the system dynamics detailed in Section III-A, the steady-state probabilities satisfy the following balance equations.

$$
\pi_0 = (1 - p)\pi_0 + \left( (1 - q_1)p + q_1(1 - p) \right) \sum_{\Delta=1}^{\infty} \pi_\Delta + \sum_{t=1}^{t_{\text{max}}-1} \left[ (1 - q_{t+1})p + q_{t+1}(1 - p) \right] \sum_{\Delta=1}^{\infty} \pi_\Delta(t).
$$

$$
\pi_1 = p\pi_0.
$$

$$
\pi_\Delta = q_1 p \pi_{\Delta-1} + \sum_{t=1}^{t_{\text{max}}-1} q_{t+1} p \pi_{\Delta-1}(t), \quad \Delta \geq 2.
$$

(15)

$$
\pi_1(t) = 0, \quad 1 \leq t \leq t_{\text{max}} - 1.
$$

$$
\pi_\Delta(1) = (1 - q_1)(1 - p)\pi_{\Delta-1}, \quad \Delta \geq 2.
$$

(16)

$$
\pi_\Delta(t) = (1 - q_t)(1 - p)\pi_{\Delta-1}(t - 1), \quad 2 \leq t \leq t_{\text{max}} - 1 \text{ and } \Delta \geq 2.
$$

(17)

$$
\pi_0 + \sum_{\Delta=1}^{\infty} \pi_\Delta + \sum_{t=1}^{t_{\text{max}}-1} \sum_{\Delta=1}^{\infty} \pi_\Delta(t) = 1.
$$

We define $\Pi \triangleq \sum_{\Delta=1}^{\infty} \pi_\Delta$ and $\Pi(t) \triangleq \sum_{\Delta=1}^{\infty} \pi_\Delta(t)$. Then, the balance equations can be written as the following.

$$
\pi_0 = (1 - p)\pi_0 + \left( (1 - q_1)p + q_1(1 - p) \right) \Pi + \sum_{t=1}^{t_{\text{max}}-1} \left[ (1 - q_{t+1})p + q_{t+1}(1 - p) \right] \Pi(t).
$$

$$
\pi_1 = p\pi_0.
$$

$$
\Pi - \pi_1 = q_1 p \Pi + \sum_{t=1}^{t_{\text{max}}-1} q_{t+1} p \Pi(t).
$$

(18)

$$
\Pi(1) = (1 - q_1)(1 - p)\Pi.
$$

(19)

$$
\Pi(t) = (1 - q_t)(1 - p)\Pi(t - 1), \quad 2 \leq t \leq t_{\text{max}} - 1.
$$

(20)

$$
\pi_0 + \Pi + \sum_{t=1}^{t_{\text{max}}-1} \Pi(t) = 1.
$$

(21)
Note that (18), (19), and (20) is obtained by summing (15), (16), and (17) over $\Delta$ from 2 to $\infty$, respectively. To solve the above system of linear equations, we first combine (19) and (20), which yields the following.

$$\Pi(t) = \prod_{l=1}^{t}(1 - q_l)(1 - p) \Pi \triangleq \prod_{l=1}^{t} P_l \Pi, \quad 1 \leq t \leq t_{max} - 1.$$ 

Then, plugging in the result into (18) gives us the following.

$$\Pi - p\pi_0 = q_1 p \Pi + \left( \sum_{t=1}^{t_{max}-1} q_{t+1} p \prod_{l=1}^{t} P_l \right) \Pi.$$ 

Then, we can obtain

$$\pi_0 = \frac{1 - q_1 p - p \left( \sum_{t=1}^{t_{max}-1} q_{t+1} \prod_{l=1}^{t} P_l \right)}{p} \Pi.$$ 

Since we have expressed $\pi(0)$ and $\Pi(t)$ for $1 \leq t \leq t_{max} - 1$ in terms of $\Pi$, combining with (21) yields

$$\Pi = \frac{1}{1 - q_1 - \left( \sum_{t=1}^{t_{max}-1} q_{t+1} \prod_{l=1}^{t} P_l \right)} + 1 + \sum_{t=1}^{t_{max}-1} \prod_{l=1}^{t} P_l.$$ 

So far, we have obtained the closed-form expression for $\pi(0)$, $\Pi$, and $\Pi(t)$ for $1 \leq t \leq t_{max} - 1$. In the following, we calculate the expected AoII $\bar{\Delta}_{tp}$. We first notice that $\bar{\Delta}_{tp}$ can be calculated using the following equation.

$$\bar{\Delta}_{tp} = \sum_{\Delta=1}^{\infty} \left( \Delta \left[ \pi_{\Delta} + \sum_{t=1}^{t_{max}-1} \pi_{\Delta}(t) \right] \right) \triangleq \Sigma + \sum_{t=1}^{t_{max}-1} \Sigma(t). \quad (22)$$

Here, we define $\Sigma \triangleq \sum_{\Delta=1}^{\infty} \Delta \pi_{\Delta}$ and $\Sigma(t) \triangleq \sum_{\Delta=1}^{\infty} \Delta \pi_{\Delta}(t)$. Hence, it is sufficient to obtain the closed-form expression of $\Sigma$ and $\Sigma(t)$ for $1 \leq t \leq t_{max} - 1$. To this end, we first multiply both sides of (17) by $\Delta - 1$.

$$(\Delta - 1)\pi_{\Delta}(t) = (1 - q_t)(1 - p)(\Delta - 1)\pi_{\Delta-1}(t-1), \quad 2 \leq t \leq t_{max} - 1 \text{ and } \Delta \geq 2.$$ 

Then, we sum the above equation over $\Delta$ from 2 to $\infty$.

$$\sum_{\Delta=2}^{\infty} (\Delta - 1)\pi_{\Delta}(t) = (1 - q_t)(1 - p) \sum_{\Delta=1}^{\infty} \Delta \pi_{\Delta}(t-1), \quad 2 \leq t \leq t_{max} - 1.$$ 

After some algebra, we obtain

$$\Sigma(t) - \Pi(t) = (1 - q_t)(1 - p) \Sigma(t-1), \quad 2 \leq t \leq t_{max} - 1. \quad (23)$$
By applying the same steps to (16), we can obtain the following.

\[
\Sigma(1) - \Pi(1) = (1 - q_1)(1 - p) \Sigma = \mathcal{P}_1 \Sigma. 
\]

(24)

Combining (23) and (24) gives us the following.

\[
\Sigma(t) = \left( \prod_{l=1}^{t} \mathcal{P}_l \right) \Sigma + \sum_{i=1}^{t} \left( \left( \prod_{j=i+1}^{t} \mathcal{P}_j \right) \Pi(i) \right), \quad 1 \leq t \leq t_{\text{max}} - 1.
\]

(25)

Then, we apply again the same steps to (15).

\[
\Sigma - \Pi = p_1 \Sigma + \sum_{t=1}^{t_{\text{max}} - 1} p_{t+1} \Sigma(t).
\]

(26)

When combining (25) and (26), we can obtain

\[
\left[ 1 - p_1 p - \sum_{t=1}^{t_{\text{max}} - 1} p_{t+1} \left( \prod_{l=1}^{t} \mathcal{P}_l \right) \right] \Sigma = \Pi + \sum_{t=1}^{t_{\text{max}} - 1} \left[ p_{t+1} \left( \sum_{i=1}^{t} \prod_{j=i+1}^{t} \mathcal{P}_j \Pi(i) \right) \right].
\]

Rearranging the terms yields

\[
\Sigma = \frac{\Pi + \sum_{t=1}^{t_{\text{max}} - 1} \left( p_{t+1} \left( \sum_{i=1}^{t} \prod_{j=i+1}^{t} \mathcal{P}_j \Pi(i) \right) \right)}{1 - p_1 p - \sum_{t=1}^{t_{\text{max}} - 1} p_{t+1} \left( \prod_{l=1}^{t} \mathcal{P}_l \right)}.
\]

(27)

Finally, plugging (26) and (27) into (22) gives us the closed-form expression of the expected AoII $\bar{\Delta}_{t\cdot p}$.

APPENDIX F

PROOF OF THEOREM 5

We follow the same methodology as presented in the proof of Theorem 3. First of all, we calculate the expected AoII $\theta^\psi$ and the value function $V^\psi(s)$ resulting from the adoption of the threshold preemptive policy $\psi$.

\[
V^\psi(0, 0, -1) = -\theta^\psi + (1 - q_1)(1 - p)V^\psi(0, 1, 0) + (1 - q_1)pV^\psi(1, 1, 0) + q_1(1 - p)V^\psi(0, 0, -1) + q_1 p V^\psi(1, 0, -1).
\]

\[
V^\psi(0, t, 0) = -\theta^\psi + (1 - q_1)(1 - p)V^\psi(0, 1, 0) + (1 - q_1)pV^\psi(1, 1, 0) + q_1(1 - p)V^\psi(0, 0, -1) + q_1 p V^\psi(1, 0, -1), \quad 1 \leq t \leq t_{\text{max}} - 1.
\]

\[
V^\psi(0, t, 1) = -\theta^\psi + (1 - q_1)(1 - p)V^\psi(0, 1, 0) + (1 - q_1)pV^\psi(1, 1, 0) + q_1(1 - p)V^\psi(0, 0, -1) + q_1 p V^\psi(1, 0, -1), \quad 1 \leq t \leq t_{\text{max}} - 1.
\]
For each $\Delta \geq 1$,

\[
V^\psi(\Delta, 0, -1) = \Delta - \theta^\psi + (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) + (1 - q_1)pV^\psi(0, 1, 1) + q_1(1 - p)V^\psi(0, 0, -1) + q_1pV^\psi(\Delta + 1, 0, -1).
\]

\[
V^\psi(\Delta, t, 0) = \Delta - \theta^\psi + (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) + (1 - q_1)pV^\psi(0, 1, 1) + q_1(1 - p)V^\psi(0, 0, -1) + q_1pV^\psi(\Delta + 1, 0, -1), \quad 1 \leq t \leq t_{\text{max}} - 1.
\]

\[
V^\psi(\Delta, t, 1) = \Delta - \theta^\psi + (1 - q_{t+1})(1 - p)V^\psi(\Delta + 1, t + 1, 1) + (1 - q_{t+1})pV^\psi(0, t + 1, 1) + q_{t+1}(1 - p)V^\psi(0, 0, -1) + q_{t+1}pV^\psi(\Delta + 1, 0, -1), \quad 1 \leq t \leq t_{\text{max}} - 1.
\]

To solve the above system of linear equations, we first investigate some structural properties of the value function $V^\psi(s)$.

**Lemma 3.** The value function $V^\psi(s)$ possesses the following properties.

1) $V^\psi(0, 0, -1) = V^\psi(0, t, i) \triangleq V^\psi(0)$ for all $i$ and $t$.
2) $V^\psi(\Delta, 0, -1) = V^\psi(\Delta, t, 0) \triangleq V^\psi(\Delta)$ for $\Delta \geq 1$ and all $t$.

**Proof.** The properties can be verified easily by comparing the corresponding expressions. Hence, the details are omitted here.

Leveraging the structural properties detailed in Lemma 3, the above system of linear equations can be simplified as the following.

\[
V^\psi(0) = -\theta^\psi + (1 - p)V^\psi(0) + pV^\psi(1).
\]

\[
V^\psi(\Delta) = \Delta - \theta^\psi + (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) + (1 - q_1)pV^\psi(0) + q_1(1 - p)V^\psi(0) + q_1pV^\psi(\Delta + 1), \quad \Delta \geq 1.
\]

\[
V^\psi(\Delta, t, 1) = \Delta - \theta^\psi + (1 - q_{t+1})(1 - p)V^\psi(\Delta + 1, t + 1, 1) + (1 - q_{t+1})pV^\psi(0) + q_{t+1}(1 - p)V^\psi(0) + q_{t+1}pV^\psi(\Delta + 1), \quad \Delta \geq 1 \text{ and } 1 \leq t \leq t_{\text{max}} - 1.
\]

**Lemma 4.** The value function possesses the following properties.

- $V^\psi(\Delta + 1) - V^\psi(\Delta) \triangleq \sigma$ is independent of $\Delta \geq 1$.
- $V^\psi(\Delta + 1, t, 1) - V^\psi(\Delta, t, 1) \triangleq \sigma_t$ is independent of $\Delta \geq 1$ for $1 \leq t \leq t_{\text{max}} - 1$. 

Proof. We recall that the system of linear equations defined by (28), (29), and (30) can be solved using iterative method. Hence, we can use mathematical induction to prove the desired results. The proof will be very similar to those detailed in the proofs of Lemma 1 and Lemma 2. Hence, the details are omitted here.

In the following, we will obtain the closed-form expression of $\sigma$ and $\sigma_t$ for $1 \leq t \leq t_{\text{max}} - 1$. To this end, combining the definitions of $\sigma$ and $\sigma_t$ with (29) and (30), we can easily obtain the following.

$$
\sigma = 1 + (1 - q_1)(1 - p)\sigma_1 + q_1 p \sigma.
$$

(31)

$$
\sigma_t = 1 + (1 - q_{t+1})(1 - p)\sigma_{t+1} + q_{t+1} p \sigma, \quad 1 \leq t \leq t_{\text{max}} - 1.
$$

(32)

Solve the above system of linear equations yields

$$
\sigma_t = \frac{1 - q_1 p}{\prod_{i=1}^{t_{\text{max}} - 2} p_i} + \sum_{i=2}^{t_{\text{max}} - 2} \left( \frac{q_i p}{\prod_{j=i}^{t_{\text{max}} - 2} p_j} \right) - q_{t_{\text{max}} - 1} p - p_{t_{\text{max}} - 1} p
$$

For $1 \leq t \leq t_{\text{max}} - 2$.

With $\sigma$ and $\sigma_t$ in mind, we can proceed with obtaining the closed form expression of the value functions. We notice that $\theta^\psi = \Delta_{tp}$. Without loss of generality, we can set $V^\psi(0) = 0$. Then, we have $V^\psi(1) = \frac{\theta^\psi}{p}$ and

$$
V^\psi(\Delta) = \frac{\theta^\psi}{p} + (\Delta - 1) \sigma \quad \Delta \geq 1.
$$

Then, combining (30), (31), and (32), we have

$$
V^\psi(1, 1, 1) = \frac{\theta^\psi}{1 - p} + \frac{\theta^\psi}{p(1 - q_1)(1 - p)} - \frac{\sigma}{(1 - q_1)(1 - p)},
$$

$$
V^\psi(1, t, 1) = \frac{V^\psi(1, t - 1, 1) - \sigma_{t-1}}{(1 - q_t)(1 - p)} + \frac{\theta^\psi}{1 - p}, \quad 2 \leq t \leq t_{\text{max}} - 1.
$$

Hence, $V^\psi(\Delta, t, 1) = V^\psi(\Delta, t, 1) + (\Delta - 1) \sigma_t$ for $\Delta \geq 1$ and $1 \leq t \leq t_{\text{max}} - 1$.

Since we have obtained the closed for expressions of both $V^\psi(s)$ and $\theta^\psi$, we proceed with obtaining the resulting policy. To this end, we define $V^{\psi, a}(s)$ as the value function resulting from taking action $a$ at state $s$. To determine the optimal policy, we only need to determine the
sign of $\delta V^\psi(s) \triangleq V^\psi,0(s) - V^\psi,1(s)$ for each state $s$. When $\delta V^\psi(s) < 0$, the optimal action is $a = 0$. Otherwise, $a = 1$ is optimal. Hence, we distinguish between the following states.

- $s = (0, t, 0)$ where $1 \leq t \leq t_{\text{max}} - 1$.
  \[
  \delta V^\psi(0, t, 0) = (1 - q_{t+1})(1 - p)^\psi V(0) + (1 - q_{t+1})pV^\psi(1) + (1 - q_{t+1})(1 - p)V^\psi(0) + q_{t+1}pV^\psi(1) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(1) = 0.
  \]

- $s = (\Delta, t, 0)$ where $\Delta \geq 1$ and $1 \leq t \leq t_{\text{max}} - 1$.
  \[
  \delta V^\psi(\Delta, t, 0) = (1 - q_{t+1})(1 - p)^\psi V(\Delta + 1) + (1 - q_{t+1})pV^\psi(0) + q_{t+1}pV^\psi(0) + q_{t+1}(1 - p)V^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) -
  \]
  \[
  (1 - q_1)pV^\psi(0) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(\Delta + 1) = (1 - p)V^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) -
  \]
  \[
  q_1pV^\psi(\Delta + 1) \geq 0.
  \]

Then, we will show that $\beta(\Delta) \triangleq V^\psi(\Delta, 1, 1) - V^\psi(\Delta) \leq 0$ for $\Delta \geq 1$. To this end, we first notice that

$$\beta(\Delta + 1) - \beta(\Delta) = \sigma_1 - \sigma.$$  

According to Condition 2, we know that $\sigma_1 \leq \sigma$. Hence, $\beta(\Delta + 1) \leq \beta(\Delta)$. Then, it is sufficient to show that $\beta(1) \leq 0$.

$$\beta(1) = \frac{\theta^\psi}{1 - p} + \frac{\theta^\psi}{p(1 - q_1)(1 - p)} - \frac{\sigma}{(1 - q_1)(1 - p)} - \frac{\theta^\psi}{p}.$$  

Hence, according to Condition 2, $\beta(1) \leq 0$. Consequently, we know $V^\psi(\Delta, 1, 1) \leq V^\psi(\Delta)$. Then, we have

$$\delta V^\psi(\Delta, t, 0) \geq (1 - p)V^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1) -
  \]
  \[
  q_1pV^\psi(\Delta + 1) = (1 - 2p)q_1V^\psi(\Delta + 1)
  \]
  \[
  \geq 0.
  $$
• $s = (0, t, 1)$ where $1 \leq t \leq t_{\text{max}} - 1$.

\[
\delta V^\psi(0, t, 1) = (1 - q_{t+1})(1 - p)V^\psi(0) + (1 - q_{t+1})pV^\psi(1, t + 1, 1) + q_{t+1}pV^\psi(0) + q_{t+1}(1 - p)V^\psi(1) - (1 - q_1)(1 - p)V^\psi(0) - (1 - q_1)pV^\psi(1) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(1)
\]

\[
= (1 - q_{t+1})pV^\psi(1, t + 1, 1) + q_{t+1}(1 - p)V^\psi(1) - pV^\psi(1)
\]

\[
= (1 - q_{t+1})pV^\psi(1, t + 1, 1) + \frac{q_{t+1}(1 - p) - p}{p} \theta^\psi.
\]

Note that $\gamma(t)$ in Condition 2 is equal to $V^\psi(1, t, 1)$. Hence, according to Condition 2 we know that $\delta V^\psi(0, t, 1) \geq 0$ for $1 \leq t \leq t_{\text{max}} - 2$. When $t = t_{\text{max}} - 1$, $\delta V^\psi(0, t, 1) = (1 - 2p)V^\psi(1) \geq 0$. Then, we can conclude that $\delta V^\psi(0, t, 1) \geq 0$ for any feasible $t$.

• $s = (\Delta, t, 1)$ where $\Delta \geq 1$ and $1 \leq t \leq t_{\text{max}} - 1$.

\[
\delta V^\psi(\Delta, t, 1) = (1 - q_{t+1})(1 - p)V^\psi(\Delta + 1, t + 1, 1) + (1 - q_{t+1})pV^\psi(0) + q_{t+1}pV^\psi(\Delta + 1) + q_{t+1}(1 - p)V^\psi(0) - (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) - (1 - q_1)pV^\psi(0) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(\Delta + 1)
\]

\[
= V^\psi(\Delta, t, 1) - V^\psi(\Delta).
\]

To determine the sign of $\gamma(\Delta, t) \triangleq V^\psi(\Delta, t, 1) - V^\psi(\Delta)$, we first notice that

\[
\gamma(\Delta + 1, t) - \gamma(\Delta, t) = \sigma_t - \sigma.
\]

According to Condition 2, we know that $\gamma(\Delta + 1, t) \leq \gamma(\Delta, t)$. Meanwhile, we have

\[
V^\psi(1, t + 1, 1) - V^\psi(1, t, 1) = \frac{(1 - q_{t+1})\theta^\psi - \sigma_t + (q_{t+1} + p - q_{t+1})V^\psi(1, t, 1)}{(1 - q_{t+1})(1 - p)}
\]

\[
= \frac{(1 - q_{t+1})\theta^\psi - \sigma_t}{(1 - q_{t+1})(1 - p)} - \frac{[q_{t+1}(1 - p) + p]V^\psi(1, t, 1)}{(1 - q_{t+1})(1 - p)}
\]

\[
\leq \frac{\theta^\psi}{1 - p} - \frac{\sigma_t}{(1 - q_{t+1})(1 - p)}
\]

Then, by Condition 2 we know $V^\psi(1, t + 1, 1) \leq V^\psi(1, t, 1)$ for $1 \leq t \leq t_{\text{max}} - 2$. Then, for $\Delta \geq 1$ we have $\gamma(\Delta, t) \leq \gamma(1, t) \leq \gamma(1, 1) = \beta(1) \leq 0$. Consequently,

\[
\delta V(\Delta, t, 1) = V(\Delta, t, 1) - V(\Delta) \leq 0.
\]
\[ s = (0, 0, -1). \]
\[
\delta V^\psi(0, 0, -1) = (1 - p)V^\psi(0) + pV^\psi(1) - (1 - q_1)(1 - p)V^\psi(0) -
\]
\[ (1 - q_1)pV^\psi(1) - q_1(1 - p)V^\psi(0) - q_1pV^\psi(1) \]
\[ = 0. \]
\[ s = (\Delta, 0, -1) \text{ where } \Delta \geq 1. \]
\[
\delta V^\psi(\Delta, 0, -1) = pV^\psi(0) + (1 - p)V^\psi(\Delta + 1) - (1 - q_1)(1 - p)V^\psi(\Delta + 1, 1, 1) -
\]
\[ (1 - q_1)pV^\psi(0) - q_1pV^\psi(\Delta + 1) - q_1(1 - p)V^\psi(0). \]

We know that \( V^\psi(\Delta + 1, 1, 1) \leq V^\psi(1) \). Hence, we have
\[
\delta V^\psi(\Delta, 0, -1) \geq q_1(1 - 2p)[V^\psi(\Delta + 1) - V^\psi(0)]
\]
\[ \geq 0. \]

Combining together, we can conclude that the policy iteration algorithm converges to the threshold preemptive policy. Hence, by Theorem 2, the threshold preemptive policy is optimal.

**Appendix G**

**Proof of Proposition 4**

Similar to what we did in the proof of Proposition 3, we group the states with same dynamics under preempt-at-consistency policy to reduce the state space of the induced Markov chain. More precisely, the grouping can be divided into the following three cases.

- **We notice that when the transmitter decides to preempt, the dynamic of system at the next time slot depends only on \( \Delta \). Then, we group \( s = (0, 0, -1), s = (0, t, 0) \) where \( 1 \leq t \leq t_{\text{max}} - 1 \), and \( s = (0, t_{\text{max}} - 1, 1) \) and denote the steady state probability for being in these states as \( \pi_0 \). Meanwhile, we group \( s = (\Delta, 0, -1) \) and \( s = (\Delta, t, 0) \) where \( 1 \leq t \leq t_{\text{max}} - 1 \) and denote the steady state probability for being in these states as \( \pi_\Delta \).

- **When the transmitter decides not to preempt, the system dynamics in the next time slot depends on all three components of \( s \). Hence, we denote the steady state probability for being in \( s = (0, t, 1) \) where \( 1 \leq t \leq t_{\text{max}} - 2 \) and \( s = (\Delta, t, 1) \) where \( \Delta \geq 1 \) and \( 1 \leq t \leq t_{\text{max}} - 1 \) as \( \pi_\Delta(t) \).

Then, the steady-state probabilities satisfies the following balance equations.
\[
\pi_0 = (1 - p) \left( \pi_0 + \sum_{\Delta=1}^{\infty} \pi_\Delta(t_{\text{max}} - 1) + \pi_0(t_{\text{max}} - 2) \right) + p \sum_{\Delta=1}^{\infty} \pi_\Delta(t_{\text{max}} - 2). \]
$$\pi_1 = p\pi_0.$$  

$$\pi_\Delta = p\pi_{\Delta-1}(t_{\text{max}} - 1), \quad \Delta \geq 2.$$  

$$\pi_0(1) = p\sum_{\Delta=1}^{\infty} \pi_\Delta.$$  

$$\pi_0(t) = (1-p)\pi_0(t-1) + p\sum_{\Delta=1}^{\infty} \pi_\Delta(t-1), \quad 2 \leq t \leq t_{\text{max}} - 2.$$  

$$\pi_1(1) = 0$$  

$$\pi_1(t) = p\pi_0(t-1), \quad 2 \leq t \leq t_{\text{max}} - 1.$$  

$$\pi_\Delta(1) = (1-p)\pi_{\Delta-1}, \quad \Delta \geq 2.$$  

$$\pi_\Delta(t) = (1-p)\pi_{\Delta-1}(t-1), \quad \Delta \geq 2 \text{ and } 2 \leq t \leq t_{\text{max}} - 1.$$  

$$\pi_0 + \sum_{\Delta=1}^{\infty} \pi_\Delta + \sum_{t=1}^{t_{\text{max}}-2} \pi_0(t) + \sum_{t=1}^{t_{\text{max}}-1} \sum_{\Delta=1}^{\infty} \pi_\Delta(t) = 1.$$  

We define $$\Pi = \sum_{\Delta=1}^{\infty} \pi_\Delta$$ and $$\Pi(t) = \sum_{\Delta=1}^{\infty} \pi_\Delta(t).$$ Then, above system of linear equations can be rewritten as the following.

$$\pi_0 = (1-p)(\pi_0 + \Pi(t_{\text{max}} - 1) + \pi_0(t_{\text{max}} - 2)) + p\Pi(t_{\text{max}} - 2).$$  

$$\pi_1 = p\pi_0.$$  

$$\Pi - \pi_1 = p\Pi(t_{\text{max}} - 1).$$  

$$\pi_0(1) = p\Pi.$$  

$$\pi_0(t) = (1-p)\pi_0(t-1) + p\Pi(t-1), \quad 2 \leq t \leq t_{\text{max}} - 2.$$  

$$\pi_1(1) = 0.$$  

$$\pi_1(t) = p\pi_0(t-1), \quad 2 \leq t \leq t_{\text{max}} - 1.$$  

$$\Pi(1) = (1-p)\Pi.$$  

$$\Pi(t) - \pi_1(t) = (1-p)\Pi(t-1), \quad 2 \leq t \leq t_{\text{max}} - 1.$$  

$$\pi_0 + \Pi + \sum_{t=1}^{t_{\text{max}}-2} \pi_0(t) + \sum_{t=1}^{t_{\text{max}}-1} \Pi(t) = 1.$$
Combining (39) and (40) yields
\[ \pi_0(t) = (1 - p)^{t-1} p \Pi + p \sum_{i=1}^{t-1} (1 - p)^{t-1-i} \Pi(i), \quad 1 \leq t \leq t_{\text{max}} - 2. \] (46)

Then, we combine the above equation with (42) and (44).
\[ \Pi(t) = (1 - p)\Pi(t - 1) + p \left[(1 - p)^{t-2} p \Pi + p \sum_{i=1}^{t-2} (1 - p)^{t-2-i} \Pi(i)\right], \quad 2 \leq t \leq t_{\text{max}} - 1. \]

With the definition of \( f(t, p) \) in mind and after some algebraic manipulations, we obtain
\[ \Pi(t) = f(t, p) \Pi, \quad 1 \leq t \leq t_{\text{max}} - 1. \] (47)

Then, using (37) and (38), we have
\[ \pi_0 = \left[\frac{1}{p} - f(t_{\text{max}} - 1, p)\right] \Pi. \]

Leveraging (46), (47), and the definition of \( f_0(t, p) \), we obtain
\[ \pi_0(t) = f_0(t, p) \Pi, \quad 1 \leq t \leq t_{\text{max}} - 2. \]

Finally, using (45), we have
\[ \Pi = \frac{1}{1/p - f(t_{\text{max}} - 1, p) + 1 + \sum_{t=1}^{t_{\text{max}}-1} \left(f_0(t, p) + f(t, p)\right)}. \]

With the above results in mind, we derive the closed-form expression of \( \Delta_{pc} \). We first notice that
\[ \Delta_{pc} = \sum_{\Delta=1}^{\infty} \Delta \left(\pi_\Delta + \sum_{t=1}^{t_{\text{max}}-1} \pi_\Delta(t)\right) \triangleq \Sigma + \sum_{t=1}^{t_{\text{max}}-1} \Sigma(t), \]
where we define \( \Sigma \triangleq \sum_{\Delta=1}^{\infty} \Delta \pi_\Delta \) and \( \Sigma(t) \triangleq \sum_{\Delta=1}^{\infty} \Delta \pi_\Delta(t) \). We first multiple both side of (33) by \( \Delta - 1 \) and sum over \( \Delta \) from 2 to \( \infty \).
\[ \sum_{\Delta=2}^{\infty} (\Delta - 1) \pi_\Delta = \sum_{\Delta=2}^{\infty} (\Delta - 1)p\pi_{\Delta-1}(t_{\text{max}} - 1). \]

Plugging in the definition yields
\[ \Sigma - \Pi = p\Sigma(t_{\text{max}} - 1). \] (48)

Likewise, for each \( 2 \leq t \leq t_{\text{max}} - 1 \), we multiple both side of (35) by \( \Delta - 1 \) and sum over \( \Delta \) from 2 to \( \infty \).
\[ \sum_{\Delta=2}^{\infty} (\Delta - 1) \pi_\Delta(t) = \sum_{\Delta=2}^{\infty} (\Delta - 1)(1 - p)\pi_{\Delta-1}(t - 1), \quad 2 \leq t \leq t_{\text{max}} - 1. \]
Plugging in the definition yields
\[ \sum(t) - \Pi(t) = (1 - p)\sum(t - 1), \quad 2 \leq t \leq t_{max} - 1. \] (49)

Again, we multiply both sides of (34) by \( \Delta - 1 \) and sum over \( \Delta \) from 2 to \( \infty \).
\[ \sum_{\Delta=2}^{\infty} (\Delta - 1)\pi_{\Delta}(1) = \sum_{\Delta=2}^{\infty} (\Delta - 1)(1 - p)\pi_{\Delta-1}. \]

Plugging in the definition yields
\[ \sum(1) - \Pi(1) = (1 - p)\sum. \] (50)

Combining (49) and (50) yields
\[ \sum(t) = (1 - p)^t\sum + \sum_{i=0}^{t-1} (1 - p)^i\Pi(t - i), \quad 1 \leq t \leq t_{max} - 1. \] (51)

Then, we compare the above equation with (48).
\[ \sum(t_{max} - 1) = (1 - p)^{t_{max}-1}\sum + \sum_{i=0}^{t_{max}-2} (1 - p)^i\Pi(t_{max} - 1 - i) = \frac{\sum - \Pi}{p}. \]

Finally, we get
\[ \sum = \frac{\sum_{i=0}^{t_{max}-2} p(1 - p)^i\Pi(t_{max} - 1 - i) + \Pi}{1 - p(1 - p)^{t_{max}-1}}. \] (52)

Then, \( \bar{\Delta}_{pc} \) can be calculated using (51) and (52).

**APPENDIX H**

**PROOF OF THEOREM 6**

The follow along the same methodology as presented in the proof of Theorem 5. First of all, we calculate the value function \( V^\psi(s) \) resulting from preempt-at-consistency policy \( \psi \). Let \( \theta^\psi \) be the expected AoII achieved by \( \psi \). Leveraging the fact that the transmission delay is constant, the value functions satisfy the following system of linear equations.
\[ V^\psi(0, t, 0) = -\theta^\psi + (1 - p)V^\psi(0, 1, 0) + pV^\psi(1, 1, 0), \quad 1 \leq t \leq t_{max} - 1. \]
\[ V^\psi(0, t, 1) = -\theta^\psi + (1 - p)V^\psi(0, t + 1, 1) + pV^\psi(1, t + 1, 1), \quad 1 \leq t \leq t_{max} - 2. \]
\[ V^\psi(0, t_{max} - 1, 1) = -\theta^\psi + (1 - p)V^\psi(0, 1, 0) + pV^\psi(1, 1, 0). \]
\[ V^\psi(0, 0, -1) = -\theta^\psi + (1 - p)V^\psi(0, 1, 0) + pV^\psi(1, 1, 0). \]
For each $\Delta \geq 1$,

$$V_{(\Delta, t, 0)} = \Delta - \theta_\psi + (1 - p)V_{{(\Delta + 1, 1, 1)}} + pV_{{(0, 1, 1)}}, \quad 1 \leq t \leq t_{\text{max}} - 1.$$  

$$V_{{(\Delta, t, 1)}} = \Delta - \theta_\psi + (1 - p)V_{{(\Delta + 1, t + 1, 1)}} + pV_{{(0, t + 1, 1)}}, \quad 1 \leq t \leq t_{\text{max}} - 2.$$  

$$V_{{(\Delta, t_{\text{max}} - 1, 1)}} = \Delta - \theta_\psi + (1 - p)V_{{(0, 0, -1)}} + pV_{{(\Delta + 1, 0, -1)}}.$$  

$$V_{{(\Delta, 0, -1)}} = \Delta - \theta_\psi + (1 - p)V_{{(\Delta + 1, 1, 1)}} + pV_{{(0, 1, 1)}}.$$  

Before solving the above system of linear equations, we investigate some structural properties of the value functions.

**Lemma 5.** The value function $V_\psi(s)$ satisfies the following structural properties.

1) $V_\psi(0, 0, -1) = V_\psi(0, t_{\text{max}} - 1, 1) = V_\psi(0, t, 0)$ where $1 \leq t \leq t_{\text{max}}$. We abbreviate them as $V_\psi(0)$.

2) For $\Delta \geq 1$, $V_\psi(\Delta, 0, -1) = V_\psi(0, 0, 0)$ for $1 \leq t \leq t_{\text{max}}$. We abbreviate them as $V_\psi(\Delta)$.

3) $V_\psi(\Delta + 1) - V_\psi(\Delta) \triangleq \sigma$ is independent of $\Delta \geq 1$ and, for $1 \leq t \leq t_{\text{max}} - 1$, $V_\psi(\Delta + 1, t, 1) - V_\psi(\Delta, t, 1) \triangleq \sigma_t$ is also independent of $\Delta \geq 1$. $\sigma$ and $\sigma_t$ can be calculated by

$$\sigma = \sum_{i=0}^{t_{\text{max}} - 1} (1 - p)^i \frac{1}{1 - p(1 - p)^{t_{\text{max}} - 1}}.$$  

$$\sigma_t = \sigma - \frac{1}{(1 - p)^t} \sum_{i=1}^{t_{\text{max}} - 1} (1 - p)^i, \quad 1 \leq t \leq t_{\text{max}} - 1.$$  

**Proof.** We can verify the first two properties by comparing the expressions they satisfy. Hence, the details are omitted here. In the following, we will focus on proving the third property. First of all, using the first two properties, the system of linear equations the value satisfies can be simplified as the following.

$$V_\psi(0) = -\theta_\psi + (1 - p)V_\psi(0) + pV_\psi(1).$$  

$$V_\psi(0, t, 1) = -\theta_\psi + (1 - p)V_\psi(0, t + 1, 1) + pV_\psi(1, t + 1, 1), \quad 1 \leq t \leq t_{\text{max}} - 2.$$  

For each $\Delta \geq 1$,

$$V_\psi(\Delta) = \Delta - \theta_\psi + (1 - p)V_\psi(\Delta + 1, 1, 1) + pV_\psi(0, 1, 1).$$  

$$V_\psi(\Delta, t, 1) = \Delta - \theta_\psi + (1 - p)V_\psi(\Delta + 1, t + 1, 1) + pV_\psi(0, t + 1, 1), \quad 1 \leq t \leq t_{\text{max}} - 2.$$  

September 29, 2022  DRAFT
\[ V^\psi(\Delta, t_{\text{max}} - 1, 1) = \Delta - \theta^\psi + (1 - p) V^\psi(0) + p V^\psi(\Delta + 1). \]

We recall that the system of linear equations can be solved using iterative method. Hence, we can use mathematical induction to prove the desired results. A similar procedures are presented in the proof of Proposition \[\square\] Hence, the details are omitted here. Then, we proceed with obtaining the closed-form expression of \( \sigma \) and \( \sigma_t \) for \( 1 \leq t \leq t_{\text{max}} - 1 \). To this end, we have the following.

\[ V^\psi(\Delta + 1) - V^\psi(\Delta) = 1 + (1 - p)[V^\psi(\Delta + 2, 1, 1) - V^\psi(\Delta + 1, 1, 1)]. \]

Hence, we have

\[ \sigma = 1 + (1 - p)\sigma_1. \tag{53} \]

Likewise, we can contain

\[ \sigma_t = 1 + (1 - p)\sigma_{t+1}, \quad 1 \leq t \leq t_{\text{max}} - 2. \tag{54} \]

\[ \sigma_{t_{\text{max}} - 1} = 1 + p\sigma. \tag{55} \]

Then, solving the system of linear equations defined by (53), (54), and (55) yields

\[ \sigma = \frac{\sum_{i=0}^{t_{\text{max}} - 1} (1 - p)^i}{1 - p(1 - p)^{t_{\text{max}} - 1}}. \]

\[ \sigma_t = \frac{\sigma - 1}{(1 - p)^t} - \sum_{i=1}^{t-1} \frac{1}{(1 - p)^i}, \quad 1 \leq t \leq t_{\text{max}} - 1. \]

Equipped with the above lemma, we can calculate the value functions. Without loss of generality, we set \( V^\psi(0) = 0 \). Then, the system of linear equations the value functions satisfy becomes the following.

\[ \theta^\psi = p V^\psi(1). \]

\[ V^\psi(1) = 1 - \theta^\psi + (1 - p)[V^\psi(1, 1, 1) + \sigma_1] + p V^\psi(0, 1, 1). \]

\[ V^\psi(0, t, 1) = -\theta^\psi + (1 - p) V^\psi(0, t + 1, 1) + p V^\psi(1, t + 1, 1), \quad 1 \leq t \leq t_{\text{max}} - 2. \]

\[ V^\psi(0, t_{\text{max}} - 1, 1) = -\theta^\psi + p V^\psi(1). \]

\[ V^\psi(1, t, 1) = 1 - \theta^\psi + (1 - p)[V^\psi(1, t + 1, 1) + \sigma_{t+1}] + p V^\psi(0, t + 1, 1), \quad 1 \leq t \leq t_{\text{max}} - 2. \]

\[ V^\psi(1, t_{\text{max}} - 1, 1) = 1 - \theta^\psi + p[V^\psi(1) + \sigma]. \]
Solving the above equations yields.

\[ V^\psi(0, t_{max} - 1, 1) = 0, \]

\[ V^\psi(1, t_{max} - 1, 1) = \sigma_{t_{max} - 1}, \]

and for each \( 1 \leq t \leq t_{max} - 2, \)

\[ V^\psi(0, t, 1) = -\theta^\psi + (1 - p)V^\psi(0, t + 1, 1) + pV^\psi(1, t + 1, 1), \]

\[ V^\psi(1, t, 1) = 1 - \theta^\psi + (1 - p)\sigma_{t+1} + (1 - p)V^\psi(1, t + 1, 1) + pV^\psi(0, t + 1, 1). \]

Note that \( \theta^\psi = \bar{\Delta}_{pc} \), which can be calculated easily using Proposition 4.

Since the closed-form expressions for the value function are obtained, we can proceed with the second step in policy iteration algorithm. In this step, we derive the policy resulting from the value functions calculated above. To this end, we define \( V^{\psi,a}(s) \) as the value function resulting from taking action \( a \) at state \( s \). To determine the policy, it is sufficient to determine the sign of \( \delta V^\psi(s) \triangleq V^\psi,0(s) - V^\psi,1(s) \) for each state \( s \). When \( \delta V^\psi(s) < 0 \), the optimal action is \( a = 0 \). Otherwise, \( a = 1 \) is optimal. Hence, we distinguish between the following states.

- \( s = (0, t, 0) \) where \( 1 \leq t \leq t_{max} - 1. \)

\[ \delta V^\psi(0, t, 0) = (1 - p)V^\psi(0) + pV^\psi(1) - (1 - p)V^\psi(0) - pV^\psi(1) \]

\[ = 0. \]

- \( s = (\Delta, t, 0) \) where \( \Delta \geq 1 \) and \( 1 \leq t \leq t_{max} - 1. \)

\[ \delta V^\psi(\Delta, t, 0) = (1 - p)V^\psi(\Delta + 1) + pV^\psi(0) - (1 - p)V^\psi(\Delta + 1, 1) - pV^\psi(0, 1, 1) \]

\[ = (1 - p)V^\psi(\Delta + 1) - V^\psi(\Delta) + \Delta - \theta \]

\[ = (1 - p)\Delta \sigma - (\Delta - 1)\sigma + \Delta - 2\theta. \]

According to Condition 3, we know that \( \delta V^\psi(\Delta, t, 0) \geq 0. \)

- \( s = (0, t, 1) \) where \( 1 \leq t \leq t_{max} - 2. \)

\[ \delta V^\psi(0, t, 1) = (1 - p)V^\psi(0, t + 1, 1) + pV^\psi(1, t + 1, 1) - (1 - p)V^\psi(0) - pV^\psi(1) \]

\[ = V^\psi(0, t, 1). \]

According to Condition 3, we know that \( \delta V^\psi(0, t, 1) \leq 0. \)
\( s = (0, t_{\text{max}} - 1, 1) \)
\[
\delta V^\psi(0, t_{\text{max}} - 1, 1) = pV^\psi(0) + (1 - p)V^\psi(1) - (1 - p)V^\psi(0) - pV^\psi(1)
\]
\[
= (1 - 2p)[V^\psi(1) - V^\psi(0)]
\]
\[\geq 0.\]

\( s = (\Delta, t, 1) \) where \( \Delta \geq 1 \) and \( 1 \leq t \leq t_{\text{max}} - 2. \)
\[
\delta V^\psi(\Delta, t, 1) = (1 - p)V^\psi(\Delta + 1, t + 1, 1) + pV^\psi(0, t + 1, 1) -
\]
\[
(1 - p)V^\psi(\Delta + 1, 1, 1) - pV^\psi(0, 1, 1)
\]
\[
= V^\psi(\Delta, t, 1) - V^\psi(\Delta).
\]

We notice that
\[
\delta V^\psi(\Delta + 1, t, 1) - \delta V^\psi(\Delta, t, 1) = \sigma_t - \sigma.
\]

From Condition 3, we know that \( \sigma_t \leq \sigma \) for all feasible \( t \). Hence, \( \delta V^\psi(\Delta + 1, t, 1) \leq \delta V^\psi(\Delta, t, 1) \) for all feasible \( t \). Meanwhile, by Condition 3, we have \( \delta V^\psi(1, t, 1) \leq 0 \) for all feasible \( t \). Then, we can conclude that \( \delta V^\psi(\Delta, t, 1) \leq 0 \) for all feasible \( \Delta \) and \( t \).

\( s = (\Delta, t_{\text{max}} - 1, 1) \) where \( \Delta \geq 1. \)
\[
\delta V^\psi(\Delta, t_{\text{max}} - 1, 1) = pV^\psi(\Delta + 1) + (1 - p)V^\psi(0) -
\]
\[
(1 - p)V^\psi(\Delta + 1, 1, 1) - pV^\psi(0, 1, 1)
\]
\[
= V^\psi(\Delta, t_{\text{max}} - 1, 1) - V^\psi(\Delta)
\]
\[
= \sigma_1 + (\Delta - 1)\sigma_{t_{\text{max}} - 1} - \frac{\theta^\psi}{p} - (\Delta - 1)\sigma
\]
\[
\leq \sigma_1 - \frac{\theta^\psi}{p}.
\]

According to Condition 3, we know that \( \delta V^\psi(\Delta, t, 1) \leq 0. \)

\( s = (0, 0, -1). \)
\[
\delta V^\psi(0, 0, -1) = (1 - p)V^\psi(0) + pV^\psi(1) - (1 - p)V^\psi(0) - pV^\psi(1)
\]
\[
= 0.
\]

\( s = (\Delta, 0, -1) \) where \( \Delta \geq 1. \)
\[
\delta V^\psi(\Delta, 0, -1) = pV^\psi(0) + (1 - p)V^\psi(\Delta + 1) - pV^\psi(0, 1, 1) - (1 - p)V^\psi(\Delta + 1, 1, 1).
\]
According to Condition 3, we have

\[ \delta V^\psi(\Delta, 0, -1) \geq (1 - p)[V^\psi(\Delta + 1) - V^\psi(\Delta + 1, 1)]. \]

As discussed in the case of \( s = (\Delta, t, 1) \), \( \delta V^\psi(\Delta, 0, -1) \geq 0 \).

Combining all the cases together, we can conclude that the policy iteration algorithm will converge to the preempt-at-consistency policy. Then, by policy improvement theorem detailed in Theorem 2, the preempt-at-consistency policy is optimal when the transmission delay is \( t_{\text{max}} \).
Algorithm 1 Improved Relative Value Iteration

Require:

MDP $\mathcal{M} = (\mathcal{S}, \mathcal{P}, \mathcal{A}, \mathcal{C})$

Convergence Criteria $\epsilon$

1: procedure RELATIVEVALUEITERATION($\mathcal{M}$)

2: Initialize $V_0(s) = 0$ for $s \in \mathcal{S}$; $\nu = 0$

3: Choose $s_{\text{ref}} \in \mathcal{S}$ arbitrarily

4: while $V_\nu$ is not converged[1] do

5: for $s \in \mathcal{S}$ do

6: for $a \in \mathcal{A}$ do

7: $H_s = C(s) + \sum_{s'} P_{s,s'}(a) \cdot V_\nu(s')$

8: $a^*(s) = \text{argmin}_a \{H_s,a\}$

9: $Q_{\nu+1}(s) = H_s,a^*$

10: for $s \in \mathcal{S}$ do

11: $V_{\nu+1}(s) = Q_{\nu+1}(s) - Q_{\nu+1}(s_{\text{ref}})$

12: $\nu = \nu + 1$

return $\psi \leftarrow a^*(s)$

---

[1] RVI converges when the maximum difference between the results of two consecutive iterations is less than $\epsilon$. 

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Algorithm 2 Policy Iteration Algorithm

Require:

Markov Decision Process: $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{C})$

1: procedure POLICYITERATIONALGORITHM($\mathcal{M}$)
2: \hspace{1em} Initialize $\psi(s) \in \mathcal{A}$ and $V^\psi(s) \in \mathbb{R}$ for all $s \in \mathcal{S}$.
3: \hspace{1em} $(V^\psi(s), \theta^\psi) \leftarrow$ PolicyEvaluationStep($\mathcal{M}, \psi$).
4: \hspace{1em} $\psi \leftarrow$ PolicyImprovementStep($\mathcal{M}, V^\psi(s)$).
5: \hspace{1em} if $\psi$ does not converges then
6: \hspace{2em} go to line 3
7: \hspace{1em} return $(\psi, \theta^\psi)$.