A continuous-time asset market game with short-lived assets

Mikhail Zhitlukhin

Received: 30 August 2020 / Accepted: 26 January 2022 / Published online: 8 June 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We propose a continuous-time game-theoretic model of an investment market with short-lived assets. The first goal of the paper is to obtain a stochastic equation which determines the wealth processes of investors and to provide conditions for the existence of its solution. The second goal is to show that there exists a strategy such that the logarithm of the relative wealth of an investor who uses it is a submartingale regardless of the strategies of the other investors, and the relative wealth of any other essentially different strategy vanishes asymptotically. This strategy can be considered as an optimal growth portfolio in the model.

Keywords Asset market game · Relative growth optimal strategy · Martingale convergence · Evolutionary finance

Mathematics Subject Classification (2020) 91A25 · 91B55

JEL Classification C73 · G11

1 Introduction

This paper studies a dynamic game-theoretic model of an asset market and constructs a strategy which allows an investor to achieve a faster growth of wealth compared to other market participants. Our results belong to the strand of research on evolutionary finance, the field which studies financial markets from a point of view of evolutionary dynamics and investigates properties of investment strategies like survival, extinction, dominance and how they affect the structure of the market (see e.g. Evstigneev et al. [22] and Holtfort [24] for an introduction to this field and a review
of models). This theory differs from the well-known theory of growth-optimal portfolios in a single-investor setting, which originated with Kelly [29] and Breiman [15] (see also Algoet and Cover [1], Karatzas and Kardaras [27], Platen and Heath [33] for a modern exposition of the subject), by considering markets where the growth rate of wealth of one agent depends on the actions of other agents, e.g. through their influence on asset prices. While the majority of models in evolutionary finance are in discrete time, the novelty and one of the goals of this paper consists in developing a continuous-time model.

The model considered here describes a market with several assets and investors. The assets yield random payoffs which are divided between the investors proportionally to the amount of money invested in each asset. Consequently, the investors’ wealth depends not only on their own strategies and realised assets’ payoffs, but also on the strategies of the other investors in the market. One of the main results of the paper consists in proving the existence of a strategy, called relative growth optimal (and we will sometimes call it simply optimal for brevity), such that the logarithm of the relative wealth of an investor who uses it is a submartingale, regardless of the strategies used by the other investors (by the relative wealth, we mean the share of wealth of one investor in the total wealth of the market). Remarkably, the strategy that we find needs only to know the current total market wealth and the probability distribution of future payoffs, but does not require the knowledge of the other investors’ strategies or their individual wealth. Such a strategy can be attractive for possible applications, since quantitative information about individual market agents is always scarce. Besides this submartingale property, the strategy has other good characteristics (similar to those of growth-optimal portfolios in single-investor models) which are also investigated in the paper.

Not necessarily all investors would prefer to use the relative growth optimal strategy as they may have other economic goals. But the fact of its existence allows describing the asymptotic structure of the market, in particular the asymptotic distribution of wealth. We prove that if at least one investor uses the optimal strategy, then the relative wealth of other investors who use essentially different strategies will vanish asymptotically and those investors will have a vanishing impact on the market. Consequently, the market becomes determined by investors who use the relative growth optimal strategy or strategies which are asymptotically close to it. Results of this type are well known in the literature, beginning with the seminal paper of Blume and Easley [7] (see below for a review of other papers).

An important simplifying assumption made in the present paper consists in that we consider a market which includes only short-lived assets. Such assets can be thought of as financial contracts which can be bought at some moment of time, yield payoffs at the next time instant, and then expire. Short-lived assets in our model have no liquidation value; so investors can get a profit (or loss) only from receiving asset payoffs and paying for buying new assets, which is different from usual models in mathematical finance with long-lived assets (e.g. common shares). Obtaining analogous results in a model with long-lived assets seems to be a considerably more difficult problem.

Short-lived assets have a clear interpretation in discrete-time models, where they live for one period of time. In our continuous-time model, it will be assumed that
assets live for infinitesimally short periods of time. Roughly speaking, one can think that there are two types of flow of money. On the one hand, each asset yields a payoff flow which is distributed between the investors in this asset; on the other hand, there are flows of money which the investors spend for investing in the assets – in order to get a greater share of the payoffs, an investor needs to increase the investment intensity. This is a somewhat simplified description, and we consider a more general scheme. In particular, the model can include atom payoffs, which allow embedding in our model a discrete-time model with payoffs at fixed moments of time. It can also represent a market where atom payoffs are made at random moments of time (e.g., governed by a Poisson process). The proposed model can be useful as an approximation of discrete-time models with payoffs yielded frequently but being small in a certain sense. In such a situation, it is natural to approximate cumulative payoff sequences with some nondecreasing continuous-time processes. We do not formally prove convergence of discrete-time to continuous-time models, but provide some illustrative examples.

This paper is connected with the paper of Drokin and Zhitlukhin [18] which considers a related model in discrete time. Regarding mathematical methods, both papers are based on the approach proposed by Amir et al. [6], which directly shows (in discrete time) that the logarithm of the wealth of an investor who uses the optimal strategy is a submartingale regardless of the other investors’ strategies (see also the paper of Amir et al. [5] where similar but technically more involved ideas were used for a model with long-lived assets). Then, using martingale convergence theorems, we can obtain results about the asymptotic structure of the market. This martingale approach is more general compared to the methods used in earlier works, which were based on the assumptions that payoff sequences and/or strategies are stationary (as in e.g., Hens and Schenk-Hoppé [23], Evstigneev et al. [20]). An essential difference of our model and the model of Amir et al. [6] (in addition to that we consider a continuous-time model) is that they assume that market agents spend their whole wealth for purchasing assets in each time period; so the total market wealth is always equal to the most recent total payoff of the assets. On the other hand, our model includes a risk-free asset (cash or a bank account with zero interest rate) that can be used by investors to store capital. This leads to more complicated wealth dynamics, but is necessary for the consideration of a continuous-time model where asset payoffs can be infinitesimal but yielded in a continuous way. Moreover, adding the possibility to store capital in cash opens interesting questions about the asymptotics of the total market wealth, which do not arise in models where the whole wealth is spent for purchasing assets. For example, as was observed in [18], greater uncertainty in asset payoffs may result in faster growth of investors’ wealth. In the present paper, we consider similar questions for our continuous-time model.

Let us mention how this paper is related to the existing literature. Among various lines of research in evolutionary finance, our work is most closely related to results on stability and survival of investment strategies, which focus on evolutionary dynamics and properties in models where investors’ strategies are specified exogenously (i.e., they do not necessarily arise from solutions of individual optimisation problems). Central to this direction are strategies that perform well irrespectively of competitors’ actions. One of the main results is that a strategy which splits its investment budget
between assets proportionally to their expected dividends (often also called the Kelly strategy, like in the classical capital growth theory) survives in the market provided that the agent’s beliefs about the dividends are correct. Survival means that a strategy maintains a share of market wealth bounded away from zero over an infinite time horizon. Typically, the Kelly strategy turns out to be the only surviving strategy in a market, i.e., the share of wealth of all other asymptotically different strategies converges to zero. First results in this direction were obtained by Blume and Easley [7] for a model with Arrow securities. The results of that paper essentially depend on the assumption that the corresponding market is complete. Models of incomplete markets were considered later, among other papers, in Alós-Ferrer and Ania [3], Amir et al. [4], Amir et al. [5], Evstigneev et al. [19, 20], Hens and Schenk-Hoppé [23], which established the optimality of the Kelly strategy under different assumptions. If no agent uses the Kelly strategy, there might be several survivors, even if some agent uses a strategy which is strictly closer to the optimal strategy than the strategies of the other agents (this largely depends on whether the market is complete or incomplete; in a complete market, there is typically a single survivor, except in some uninteresting cases). A simple example of coexistence of surviving investors in an incomplete market is provided in Evstigneev et al. [21, Chap. 9.3.3]. Survival and coexistence in a related setting but with investors’ strategies depending on one-step equilibrium asset prices were studied by Bottazzi and Dindo [12], Bottazzi and Giachini [13, 14].

The above-mentioned papers deal with discrete-time models. In evolutionary finance with exogenous investor strategies, there are relatively few models with continuous time. One can mention the papers of Palczewski and Schenk-Hoppé [31, 32] in which a continuous-time model with long-lived assets is constructed. The paper [31] proves that the model can be obtained as a limit of discrete-time models, and [32] investigates questions of survival of investments strategies in it. However, their results are obtained only for time-independent strategies and under the assumption that cumulative dividend processes are pathwise absolutely continuous. In the present paper, we allow strategies to be time-dependent and asset payoffs to be represented by arbitrary processes (but assets are short-lived). A continuous-time model with short-lived assets was also constructed in Zhitlukhin [37]. An essential limitation of that model consists in the assumption that all investors spend the same exogenously specified proportion of their wealth for purchasing assets.

Another large body of literature consists of results on market selection of investment strategies by market forces in the framework of general equilibrium, when agents maximise utility from consumption. For discrete-time models, see e.g. Blume and Easley [8, 9], Sandroni [34], Coury and Sciubba [16] and the references therein. Among continuous-time models, one can mention Borovička [11], Yan [36].

The paper is organised as follows. In Sect. 2, we briefly describe a discrete-time model which helps to explain the main ideas of the paper. The general model is formulated in Sect. 3. In Sect. 4, we define the notion of relative growth optimality and construct a candidate optimal strategy. In Sect. 5, we formulate the main results, which show that this strategy is indeed optimal and study its properties. Section 6 contains the proofs of the results. In the Appendix, we provide auxiliary technical facts which are used in the paper.
2 Preliminary consideration: a discrete-time model

In this section, we describe the main ideas of the paper using a simple model with discrete time which avoids the technical details of continuous time. Based on the discrete-time model, a general continuous-time model is formulated in Sect. 3. The model presented here is a slightly simplified version of the model from Drokin and Zhitlukhin [18].

Let $(\Omega_1, F, P)$ be a probability space with a filtration $F = (F_t)_{t=0,1,2,...}$. The model includes $M \geq 2$ investors and $N \geq 1$ assets which yield nonnegative random payoffs at times $t = 1, 2, \ldots$ The assets live for one period: they are purchased by the investors at time $t$, yield payoffs at $t + 1$, and then the cycle repeats. The asset prices are determined endogenously by a short-run equilibrium of supply and demand. The supply of each asset is normalised to 1, and the demand depends on the actions of the investors. The payoffs are specified in an exogenous way, i.e., do not depend on the investor’s actions. Each investor receives a part of the payoff yielded by an asset which is proportional to the owned share of this asset.

The asset payoffs are specified by random sequences $A^n = (A^n_t)_{t=1,2,...}$ with $A^n_t \geq 0$, adapted to the filtration. The wealth of investor $m$ is described by an adapted random sequence $Y^m = (Y^m_t)_{t=0,1,2,...}$ with $Y^m_t \geq 0$. The initial wealth $Y^m_0$ of each investor is non-random and strictly positive. The wealth $Y^m_t$ at subsequent times $t \geq 1$ is determined by the investors’ strategies and the asset payoffs.

A strategy of investor $m$ is a plan according to which this investor allocates the available budget $Y^m_t$ towards the purchase of assets. Such an allocation is specified by a sequence of vectors $\ell^m = (\ell^m_t)_{t=1,2,...}$, $\ell^m_t = (\ell^m_1, \ldots, \ell^m_N)$, with $\ell^m_t$ being the budget allocated for purchasing asset $n$ at time $t - 1$. At each moment of time, the vectors $\ell^m_t$ are selected by the investors simultaneously and independently; so the model represents a simultaneous-move $N$-person dynamic game, and the $\ell^m_t$ represent the investors’ actions. These actions may depend on a random outcome $\omega$ and current and past wealth of the investors; so we define a strategy of investor $m$ as a sequence $\ell^m = (\ell^m_t)_{t=1,2,...}$ consisting of $(F_{t-1} \otimes B(R^{tM}))$-measurable functions $\ell^m_t : \Omega \times \mathbb{R}^{tM} \rightarrow \mathbb{R}^N$.

(We use boldface letters to distinguish between strategies and their realisations, see below.) The arguments $y_s = (y_s^1, \ldots, y_s^M) \in \mathbb{R}^M$, $s \leq t - 1$, correspond to the wealth of the investors at past moments of time. It is assumed that short sales are not allowed, so $\ell^m_t \geq 0$, and it is not possible to borrow money, so $\sum_n \ell^m_t \leq y_{t-1}^m$. The amount of wealth $y_{t-1}^m - \sum_n \ell^m_t$ is held in cash and carried forward to the next time period.

After the selection of the investment budgets $\ell^m_t$ by the investors, the equilibrium asset prices $p^n_{t-1}$ are determined from the market clearing condition that the aggregate demand for each asset is equal to the aggregate supply, which is assumed to be 1. At time $t - 1$, investor $m$ can buy $x^m_{t-1} = \ell^m_t / p^n_{t-1}$ units of asset $n$; so its price at time $t - 1$ should be equal to the total amount of capital invested in this asset, i.e.,

$$p^n_{t-1} = \sum_{m=1}^M \ell^m_t.$$

(2.1)
If $\sum_m \ell_t^{m,n} = 0$, i.e., no one buys asset $n$, we can put $p_t^n = 0$ and $x_t^{m,n} = 0$ for all $m$.

Thus, investor $m$’s portfolio between times $t - 1$ and $t$ consists of $x_t^{m,n}$ units of asset $n$ and $\ell_t^m := y_t^m - \sum_n \ell_t^{m,n}$ units of cash. At time $t$, the total payoff received by this investor from the assets in the portfolio is equal to $\sum_n x_t^{m,n} A_t^n$. In our model, the assets have no liquidation value; so the budgets used at time $t - 1$ for buying assets are not returned to the investors. Consequently, investor $m$’s wealth is described by the adapted sequence $Y_t^m = (Y_t^m)_{t=0,1,2,...}$ which is defined by the recursive relation

$$Y_t^m(\omega) = Y_{t-1}^m(\omega) - \sum_{n=1}^N \ell_t^{m,n}(\omega) n + \sum_{n=1}^N \sum_k \ell_t^{k,n}(\omega) A_t^n(\omega), \quad t = 1, 2, \ldots, (2.2)$$

where $\ell_t^{m,n}(\omega) = \ell_t^{m,n}(\omega, Y_0, Y_1(\omega), \ldots, Y_{t-1}(\omega))$ are the realisations of the investors’ strategies, with $0/0 = 0$ on the right-hand side of (2.2).

Note that the investors’ actions precede the asset prices; so they first “announce” the budgets they plan to allocate for buying the assets, and then the prices are adjusted to clear the market. This modelling approach is analogous to market games of Shapley–Shubik type (see Shapley and Shubik [35]).

**Remark 2.1** One can see that the asset prices do not enter Equation (2.2). In view of this, it is convenient to give another interpretation of (2.2) which will be useful for the continuous-time model developed below: the agents compete for the assets’ payoffs by investing their capital in each asset, and those who spend more money receive a greater share of the payoff, proportionally to their share in the total investment in this asset made at time $t - 1$.

The main results of the paper mainly concern the relative wealth of investors. For investor $m$, we define the relative wealth as the adapted sequence $r_t^m = (r_t^m)_{t=0,1,2,...}$ with

$$r_t^m = \frac{Y_t^m}{\sum_k Y_t^k}. $$

Our goal is to identify a strategy such that the relative wealth of an investor who uses it grows in the following sense: for any strategies of the other investors and any initial wealth, the sequence $\ln r_t^m$ is a submartingale (as a consequence, $r_t^m$ will be a submartingale as well). Such a strategy exhibits several asymptotic optimality properties, which we consider in Sects. 4 and 5.

### 3 The general model

In this section, we extend the above discrete-time model to continuous time. Based on the interpretation of Equation (2.2) given in Remark 2.1, we assume that the assets yield payoffs continuously, and in order to receive the payoffs, the agents need to continuously invest money in the assets. The payoffs are divided between the agents proportionally to their investment intensities.
Observe that (2.2) can be written in the form

$$\Delta Y^m_t(\omega) = -\sum_{n=1}^{N} \Delta L^{m,n}_t(\omega) + \sum_{n=1}^{N} \Delta L^{m,n}_t(\omega) \Delta X^n_t(\omega),$$

(3.1)

where

$$L^{m,n}_t(\omega) = \sum_{s=1}^{t} \ell^{m,n}_s(\omega), \quad X^n_t(\omega) = \sum_{s=1}^{t} A^n_s(\omega)$$

are, respectively, the process of the cumulative wealth invested by investor \( m \) in asset \( n \) and the cumulative payoff process of asset \( n \). The symbol \( \Delta \) denotes a one-step increment, e.g. \( \Delta Y^m_t = Y^m_t - Y^m_{t-1} \).

The form of (3.1) suggests that an analogous model in continuous time can be obtained by considering continuous-time processes \( X, Y, L \) and replacing one-step increments with infinitesimal increments, e.g. \( \Delta X_t \) with \( dX_t \). Our next goal is to define such a model properly. The model we are about to formulate includes the above discrete-time model as a particular case.

**Notation.**

We work on a filtered probability space \( (\Omega,\mathcal{F},\mathbb{P}) \) with a continuous-time filtration \( \mathcal{F}_t \) satisfying the usual conditions. By \( \mathcal{P} \), we denote the predictable \( \sigma \)-algebra on \( \Omega \times \mathbb{R}_+ \).

As usual, equalities and inequalities for random variables are assumed to hold with probability one. For random processes, an equality \( X = Y \) is understood to hold up to \( \mathbb{P} \)-indistinguishability, i.e., \( \mathbb{P}[\exists t : X_t \neq Y_t] = 0 \); in the same way, we treat inequalities. Pathwise properties (continuity, monotonicity, etc.) are assumed to hold for all \( \omega \).

For vectors \( x, y \in \mathbb{R}^N \), we denote by \( xy = \sum_n x^n y^n \) the scalar product, by \( |x|_1 \) or simply \( |x| \) the \( \ell^1 \)-norm \( |x|_1 = \sum_n |x^n| \), and by \( |x|_2 = \sqrt{x^Tx} \) the \( \ell^2 \)-norm. For a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) and a vector \( x \), the notation \( f(x) \) means the application of the function to each coordinate of the vector, so that \( f(x) = (f(x^1), \ldots, f(x^N)) \). If \( x \in \mathbb{R}_+^{MN} \), we denote by \( x^m \) the vector \( (x^{m,1}, \ldots, x^{m,N}) \in \mathbb{R}^N \) and by \( x^{i,n} \) the vector \( (x^{1,n}, \ldots, x^{M,n}) \in \mathbb{R}^M \). The maximum of two numbers \( a, b \) is written as \( a \lor b \), and the minimum as \( a \land b \).

The notation \( \xi \cdot G = (\xi \cdot G_t)_{t \geq 0} \) is used for the integral process of a process \( \xi \) with respect to a process \( G \). In what follows, all the integrators are nondecreasing càdlàg processes so that integrals are understood in the pathwise Lebesgue–Stieltjes sense as \( f \cdot G_t(\omega) = \int_0^t f_s(\omega) dG_s(\omega) \). If \( f, G \) are vector-valued, then \( f \cdot G_t = \sum_n f^n \cdot G^n_t \).

### 3.1 Payoff processes and investment strategies

There are \( N \geq 1 \) assets yielding random payoffs which are distributed between \( M \geq 2 \) investors. The cumulative payoffs are represented by exogenous adapted nondecreasing càdlàg processes \( X = (X_t)_{t \geq 0} \) with values in \( \mathbb{R}^N_+ \). Without loss of generality, \( X_0 = 0 \).

A strategy of investor \( m \) is identified with a function \( L \) which represents the cumulative wealth invested in each asset and takes values in \( \mathbb{R}^N_+ \). In order to specify how a
strategy may depend on the past history of the market, let \((D, \mathcal{D}, (\mathcal{D}_t)_{t \geq 0})\) denote the
filtered measurable space consisting of the space \(D\) of nonnegative càdlàg functions
\(y: \mathbb{R}_+ \to \mathbb{R}_+^M\), the filtration \(\mathcal{D}_t = \sigma (d_u, u \leq t)\), where \(d_u\) is the mapping \(d_u(y) = y_u\)
for \(y \in D\), and \(\mathcal{D} = \bigvee_{t \geq 0} \mathcal{D}_t\). Elements \(y\) of the space \(D\) represent possible paths of
the wealth processes of the investors (which are yet to be defined) on the whole time
axis \(\mathbb{R}_+\). The wealth of each investor cannot become negative (this assumption is im-
posed on a solution of the wealth equation in the next section); hence the \(y\) assume
values in \(\mathbb{R}_+^N\).

Let \((E, \mathcal{E}, (\mathcal{E}_t)_{t \geq 0})\) be the filtered measurable space with
\[E = \Omega \times D, \quad \mathcal{E}_t = \mathcal{F}_t \otimes \mathcal{D}_t, \quad \mathcal{E} = \bigvee_{t \geq 0} \mathcal{E}_t.\]

Let \(\mathcal{P}^E\) denote the predictable \(\sigma\)-algebra on \(E \times \mathbb{R}_+\), i.e., \(\mathcal{P}^E\) is generated by all
measurable functions \(\xi(\omega, y, t): E \times \mathbb{R}_+ \to \mathbb{R}\) which are left-continuous in \(t\) for any
fixed \((\omega, y)\) and \(\mathcal{E}_t\)-measurable for any fixed \(t\). In what follows, functions \(\xi(\omega, y, t)\)
are often written as \(\xi_t(\omega, y)\), or \(\xi_t(y)\) when omitting \(\omega\) does not lead to confusion.

**Definition 3.1** A strategy is a \(\mathcal{P}^E\)-measurable function \(L_t(\omega, y)\) with values in \(\mathbb{R}_+^N\)
and \(L_0(\omega, y) = 0\), which is nondecreasing and càdlàg in \(t\).

The following lemma will be used further in the construction of the model.

**Lemma 3.2** Let \(L_t(y)\) be a \(\mathcal{P}^E\)-measurable function and \(Y\) an adapted càdlàg
process with values in \(\mathbb{R}_+^M\). Then the process \(L_t(\omega) = L_t(\omega, Y(\omega))\) is predictable
(\(\mathcal{P}\)-measurable).

**Proof** The \(\sigma\)-algebra \(\mathcal{P}^E\) is generated by the sets \(C \times [s, \infty)\), where \(s \geq 0\) and
\(C \in \mathcal{E}_{s-}\) (as usual, \(\mathcal{E}_{s-} = \bigvee_{u < s} \mathcal{E}_u\) and \(\mathcal{E}_{0-} = \mathcal{E}_0\)); see Liptser and Shiryaev [30,
§ 1.2]. Hence, approximating \(L_t(y)\) by simple \(\mathcal{P}^E\)-measurable functions, it is enough
to prove the lemma for functions
\[L_t(\omega, y) = I_{((\omega, y) \in C)} I_{[t \geq s]}, \quad C \in \mathcal{E}_{s-}, s \geq 0. \quad (3.2)\]

Using that \(\mathcal{D}_s\) is generated by the sets
\[\{y \in D : y_{s_i} \in B_i, \ i = 1, \ldots, n\},\]
where \(s_1 < \cdots < s_n \leq s\) and \(B_i \in B(\mathbb{R}_+^M)\), one can see that in (3.2), it is enough to
consider only sets \(C\) of the form
\[C = A \times \{y \in D : y_{s_i} \in B_i, \ i = 1, \ldots, n\}, \quad A \in \mathcal{F}_{s_n}, s_1 < \cdots < s_n < s.\]

For such sets, \(I_{((\omega, Y(\omega)) \in C)} = I_{(\omega \in A)} I_{\{Y_{s_i}(\omega) \in B_i, \ i \leq n\}}\) is measurable with respect
to \(\mathcal{F}_{s_n}\), and hence is \(\mathcal{F}_{s--}\)-measurable. This implies the predictability of the process
\(L_t(\omega) = I_{((\omega, Y(\omega)) \in C)} I_{[t \geq s]}\). \(\square\)
3.2 The wealth equation

The wealth of the investors is described by an \( \mathbb{R}_+^M \)-valued càdlàg adapted process \( Y = (Y^1, \ldots, Y^M), Y^m = (Y^m_t), t \geq 0 \). In this section, we state the equation which defines \( Y \). We always assume that the initial wealth \( Y^m_0 \) of each investor is non-random and strictly positive. The set of vectors \( y \in \mathbb{R}_+^M \) with all coordinates strictly positive is denoted by \( \mathbb{R}_+^{M+} \).

Let \( X^c = (X^c_t)_{t \geq 0} \) denote the continuous part of the payoff process \( X \), i.e., the nondecreasing process with values in \( \mathbb{R}_+^N \) defined as

\[
X^c_t = X_t - \sum_{s \leq t} \Delta X_s,
\]

where \( \Delta X_s = X_s - X_{s-} \) and we put \( \Delta X_0 = 0 \) for \( s = 0 \). Denote by \( \mu \) the measure of jumps of \( X \) and by \( \nu \) its compensator. Define the predictable scalar process \( G = (G_t)_{t \geq 0} \) (the so-called operational time process) as

\[
G_t = |X_t^c| + (|x| \wedge 1) \ast v_t,
\]

where the star denotes integration with respect to \( \nu \), i.e., for a measurable function \( f(\omega, t, x) \) on \( \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^N \),

\[
f \ast \nu_t(\omega) = \int_0^t f(\omega, s, x) \nu(\omega, ds, dx).
\]

Note that \( X^c \) is not the continuous martingale part of \( X \), as usually denoted in stochastic calculus. Actually, all the martingales in our paper will have zero continuous part, and so the notation \( X^c \) should not lead to any confusion.

Let \( H = (H_t)_{t \geq 0} \) be an arbitrary scalar predictable càdlàg nondecreasing process such that \( G \ll H \) (i.e., for a.a. \( \omega \), the measure on \( \mathbb{R}_+ \) generated by the function \( G_t(\omega) \) is absolutely continuous with respect to the measure generated by \( H_t(\omega) \)).

**Definition 3.3** We call a strategy profile \( (L^1, \ldots, L^M) \) and a vector of initial wealths \( y^0 \in \mathbb{R}_+^{M+} \) feasible if there exists a unique (up to \( P \)-indistinguishability) nonnegative càdlàg adapted process \( Y \), called the wealth process, which assumes values in \( \mathbb{R}_+^M \) and satisfies the following conditions:

1) \( Y \) solves the wealth equation

\[
dY^m_t = -d|L^m_t| + \sum_{n=1}^N \frac{\ell^m_{t,n}}{\ell_{t,n}} dX^n_t, \quad Y^m_0 = y^m_0, \quad (3.3)
\]

for \( m = 1, \ldots, M \), where \( L^m_t(\omega) = L^m_t(\omega, Y(\omega)) \) (and \( |L^m_t| \) denotes the \( \ell^1 \)-norm of the vector \( L^m_t \)) and \( \ell \) is any \( (P \otimes H) \)-version of the \( \mathbb{R}_+^{MN} \)-valued process of predictable Lebesgue derivatives (see the Appendix for details on Lebesgue derivatives; the measure \( P \otimes H \) is defined as in (A.8) there)

\[
\ell_{t,n}^m = \frac{dL_{t,n}^m}{dH_t}. \quad (3.4)
\]
2) If \( Y^m_t(\omega) = 0 \) or \( Y^m_{t-}(\omega) = 0 \), then \( L^m_t(\omega) = L^m_{t-}(\omega) \) and \( Y^m_t(\omega) = 0 \) for all \( s \geq t \).

When \(|\ell^{m,n}_t(\omega)| = 0\) in (3.3) for some \( \omega, t, n \), we put \( \ell^{m,n}_t(\omega)/|\ell^{m,n}_t(\omega)| = 0 \). Observe that the derivatives \( \ell \) are well defined, since if \( Y \) is an adapted càdlàg process, then \( L^{m,n} \) is a predictable process according to Lemma 3.2.

As usual, (3.3) should be understood in the integral sense (a.s. for all \( t \)), i.e.,

\[
Y^m_t = Y^m_0 - |L^m_t| + \sum_{n=1}^{N} \int_0^t \ell^{m,n}_s / |\ell^{m,n}_s| \, dX^n_s,
\]

where the integral is understood as a pathwise Lebesgue–Stieltjes integral. It is well defined since the process \( X \) is càdlàg and nondecreasing, and the integrand is non-negative and bounded.

Let us clarify that we use Lebesgue derivatives in the wealth equation and not Radon–Nikodým derivatives (e.g. \( dL^{m,n}_t / d|\ell^{m,n}_t(\omega)| \)) for two reasons. First, this allows differentiating with respect to a process \( H \) not depending on the solution of the equation, which is yet to be found. Second, it is natural to require that the solution should not depend on what particular version of the derivative is used. This is so if \( G \ll H \) (see Proposition 3.4 below). Thus, if one would like to use Radon–Nikodým derivatives, the process \( H \) should dominate both the processes \(|L|\) and \( G \), which would make formulas rather cumbersome.

Sufficient conditions for the existence and uniqueness of a solution of (3.3) are provided in the next section. Now we prove a result which shows that the solution, if it exists, does not depend on the choice of the process \( H \) and the versions of the derivatives \( \ell \).

**Proposition 3.4** Suppose \( Y \) is a solution of (3.3), where the derivative process \( \ell \) is defined as in (3.4) with respect to some càdlàg nondecreasing predictable process \( H \) such that \( G \ll H \). Then for any càdlàg nondecreasing predictable process \( \tilde{H} \) such that \( G \ll \tilde{H} \) and any \((P \otimes \tilde{H})\)-version of the derivative \( \tilde{\ell} = dL/d\tilde{H} \), the process \( Y \) also solves (3.3) with \( \tilde{\ell} \) in place of \( \ell \).

**Proof** Let \( F : \mathbb{R}^{MN} \to \mathbb{R}^{MN} \) denote the function which specifies the distribution of payoffs in (3.3), i.e.,

\[
F(\ell)^{m,n} = \frac{\ell^{m,n}}{|\ell^{m,n}|},
\]

where \( F(\ell)^{m,n} = 0 \) if \(|\ell^{m,n}| = 0\). As follows from (3.5), we have to show for each \( m, n \) that

\[
F(\ell)^{m,n} \cdot X^n = F(\tilde{\ell})^{m,n} \cdot X^n,
\]

where \( F(\ell) \) denotes the process given by \( F(\ell_t(\omega)) \), and \( F(\tilde{\ell}) \) denotes \( F(\tilde{\ell}_t(\omega)) \).
One can see that if \( f, f' \geq 0 \) are predictable processes such that \( f = f' \) \((P \otimes G)\)-a.e., then \( f \cdot X^n = f' \cdot X^n \). We have

\[
\ell^{m,n} = \frac{dL^{m,n}}{dH} = \frac{dL^{m,n}}{d\tilde{H}} \frac{d\tilde{H}}{dH} = \frac{\tilde{\ell}^{m,n}}{d\tilde{H}} \frac{d\tilde{H}}{dH} \quad (P \otimes G)\)-a.e.,
\]

where the second equality holds in view of claim (b) of Proposition A.3 from the Appendix. Since \( \frac{d\tilde{H}}{dH} > 0 \) \((P \otimes G)\)-a.e. by claim (c) of Proposition A.3, we have \( F(\ell)^{m,n} = F(\tilde{\ell})^{m,n} \); so \((3.7)\) holds, which finishes the proof. \( \square \)

**Remark 3.5**

1) In the discrete-time model in Sect. 2, the wealth equation can be interpreted by introducing asset prices via \((2.1)\). A counterpart of \((2.1)\) in the general model under consideration is the family of processes \( p^n = (p^n_t)_{t \geq 0}, n = 1, \ldots, N \),

given by

\[
p^n_t = \sum_{m=1}^{M} \ell^{m,n}_t.
\]

The process \( p^n \) can be viewed as the combined investment intensity of all the investors in asset \( n \) with respect to the process \( H \), rather than a price which should be paid if one wants to buy this asset. Then \((3.3)\) means that the payoffs are distributed proportionally to the investment intensities (cf. Remark 2.1).

2) Short sales and borrowing of cash were not allowed in the discrete-time model. In the present general model, short sales are also not allowed because the cumulative investment functions \( L \) are nondecreasing in \( t \). On the other hand, borrowing of cash is not explicitly prohibited (by borrowing we mean that \(|\Delta L^m_t| > Y^m_t\)). However, all our results will be stated for feasible strategy profiles, in which wealth processes must be nonnegative. So if an investor borrows cash at time \( t \), she should be immediately compensated by the received payoffs at the same time \( t \) (this is possible if some asset makes a payoff \( \Delta X^n_t > 0 \)).

### 3.3 A sufficient condition for the existence and uniqueness of a solution of the wealth equation

The following theorem provides a sufficient condition for the existence and uniqueness of a solution of \((3.3)\). Note that the main results of our paper, formulated in Sect. 5, do not require this condition to hold (they only require a unique solution to exist), and may be valid under less strict assumptions.

**Theorem 3.6** Suppose that for each \( m \) the strategy \( L^m \) of investor \( m \) satisfies the following two conditions:

1. **(C1)** There exists a \((\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^M))\)-measurable function \( v^m(\omega, z) \) with values in \( \mathbb{R}_+^N \) such that for all \( \omega \in \Omega, y \in D, t \in \mathbb{R}_+, n = 1, \ldots, N \),

\[
L^{m,n}_t(\omega, y) = \int_0^t v^{m,n}_s(\omega, y) I_{\{\inf_{u<s} y^m_u > 0\}} dG_s(\omega) \quad (3.8)
\]

2. **(C2)** There exists a \((\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^M))\)-measurable function \( u^m(\omega, z) \) with values in \( \mathbb{R}_+^N \) such that for all \( \omega \in \Omega, y \in D, t \in \mathbb{R}_+, n = 1, \ldots, N \),

\[
F(L) = F(L^m) \quad (P \otimes G)\)-a.e.
\]

3. **(C3)** There exists a \((\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^M))\)-measurable function \( z^m(\omega, z) \) with values in \( \mathbb{R}_+^N \) such that for all \( \omega \in \Omega, y \in D, t \in \mathbb{R}_+, n = 1, \ldots, N \),

\[
H^m_t(\omega, y) = \int_0^t z^{m,n}_s(\omega, y) I_{\{\inf_{u<s} y^m_u > 0\}} dG_s(\omega) \quad (3.9)
\]

Then there exists a unique solution \( (X^n)_{n \in \{1, \ldots, N\}} \) of the wealth equation \((3.3)\).
(for $s = 0$, we put $y_0 = y_0$), and for all $\omega \in \Omega$, $z \in \mathbb{R}_+^M$, $t \in \mathbb{R}_+$,

$$|v^m_t(\omega, z)| \Delta G_t(\omega) \leq z^m.$$  \hspace{1cm} (3.9)

(C2) There exist sets $\Pi^{m,n} \in \mathcal{P}$ for $n = 1, \ldots, N$, a predictable càdlàg process $\delta^m > 0$ and a non-random function $C^m : \mathbb{R}_+^M \to (0, \infty)$ such that

$$v^{m,n}_t(\omega, z) = 0 \quad \text{for all } (\omega, t) \in \Pi^{m,n} \text{ and } z \in \mathbb{R}_+^N,$$  \hspace{1cm} (3.10)

and for all $(\omega, t) \notin \Pi^{m,n}$ and $z, \tilde{z}, a \in \mathbb{R}_+^M$ such that $z^k, \tilde{z}^k \in [a^k/2, 2a^k]$ for all $k$, it holds that

$$v^{m,n}_t(\omega, z) \geq \left(C^m(a)\delta^m_t(\omega)\right)^{-1} \text{ if } z^m > 0,$$  \hspace{1cm} (3.11)

$$v^{m,n}_t(\omega, z) \leq C^m(a)\delta^m_t(\omega),$$  \hspace{1cm} (3.12)

$$|v^{m,n}_t(\omega, z) - v^{m,n}_t(\omega, \tilde{z})| \leq C^m(a)\delta^m_t(\omega)|z - \tilde{z}|.$$  \hspace{1cm} (3.13)

Then for any vector $y_0 \in \mathbb{R}_+^M$ of initial wealths and any predictable nondecreasing càdlàg process $H$ such that $G \ll H$, Equation (3.3) has a unique solution (up to $P$-indistinguishability).

The proof is provided in Sect. 6. Let us comment on the conditions imposed in the theorem.

In condition (C1), (3.8) restricts the class of strategies under consideration to strategies that from the whole information of investors’ past wealth use only the knowledge of the current wealth $y_{t-}$, on which depend the “instantaneous” investment rates $v^{m,n}_t$. The indicator in the integrand appears for the purpose of ensuring that the process $Y^m$ is nonnegative: if $Y^m_u$ or $Y^m_{u-}$ become zero for some $u$, such a strategy stops investing after $u$. For the same reason, we require (3.9) to hold, which means that an investor cannot spend more money than available (i.e., borrowing of cash is not allowed). Note that (C1) implies that the realisation of the strategy is absolutely continuous with respect to $G$, i.e., $L^m \ll G$, which is a reasonable requirement since if a strategy does not have this property, then it “wastes” money (it invests in assets when the expected payoff is zero). As will become clear from the proof (and can be seen from (6.7)), the function $v^{m,n}_t$ specifies the Lebesgue derivative of $L^{m,n}$ with respect to $G$, namely $\ell^{m,n}_t(Y_{t-}) = v^{m,n}_t(Y_{t-})I_{\{Y^m_{t-} > 0\}}$; see (6.7).

Condition (C2) is needed because the proof is based on a contraction mapping argument. The inequalities (3.11) and (3.12) are analogous to similar upper and lower bounds on the equation coefficients in such proofs, while (3.13) is a Lipschitz-continuity condition. Note that it would be too restrictive to require $v^{m,n}_t$ to be bounded away from zero globally in (3.11). Indeed, if asset $n$ does not yield a payoff “predictably” at time $t$, it would be natural to take $v^{m,n}_t = 0$. Therefore, we relax the lower bound on $v$ by introducing the sets $\Pi^{m,n}$ where $v^{m,n}_t$ may vanish.

The conditions of the theorem may look cumbersome, but it is possible to verify that certain strategies satisfy them. In particular, we do that in Sect. 4.2 for a candidate optimal strategy under mild additional assumptions.
4 Optimal strategies

4.1 Definition

If a strategy profile and a vector of initial wealths are feasible, we define the relative wealth of investor $m$ as the process $r^m_t = (r^m_t)_{t \geq 0}$ with

$$r^m_t = \frac{Y^m_t}{|Y_t|},$$

where $r^m_t(\omega) = 0$ if $|Y_t(\omega)| = 0$.

We are interested in finding strategies for which the relative wealth of an investor does not decrease on average in the following sense.

**Definition 4.1** For a given payoff process $X$, we call a strategy $L$ relative growth optimal for investor $m$ if for any feasible initial wealth and any strategy profile where investor $m$ uses this strategy, it holds that $Y^m_t > 0$ for all $t \geq 0$ and $\ln r^m$ is a submartingale.

The following proposition states some properties of a relative growth optimal strategy which easily follow from the definition.

**Proposition 4.2** Suppose investor $m$ uses a relative growth optimal strategy. Then for any strategies used by the other investors in a feasible strategy profile, the following claims hold true:

1) The relative wealth $r^m$ is a submartingale.

2) Investor $m$ survives in the market in the sense that her relative wealth always stays bounded away from zero, i.e.,

$$\inf_{t \geq 0} r^m_t > 0 \quad \text{a.s.} \quad (4.1)$$

3) Investor $m$ achieves an asymptotic growth rate of wealth which is not slower than that of any other investor, i.e., for any $k$, we have

$$\limsup_{t \to \infty} \frac{1}{t} \ln Y^m_t \geq \limsup_{t \to \infty} \frac{1}{t} \ln Y^k_t \quad \text{a.s.} \quad (4.2)$$

(This is analogous to the notion of asymptotic growth optimality in single-investor market models; see e.g. Karatzas and Shreve [28, Sect. 3.10].)

**Proof** Claim 1) follows from Jensen’s inequality. Claim 2) follows from the fact that $\ln r^m$ is a nonpositive submartingale and hence has a finite limit $z = \lim_{t \to \infty} \ln r^m_t$. Therefore, $\lim_{t \to \infty} r^m_t = e^z > 0$.

To prove (4.2), observe that $\sup_{t \geq 0} |Y_t|/Y^m_t < \infty$ by (4.1). Consequently, we have $\sup_{t \geq 0} Y^k_t/Y^m_t < \infty$ for any $k$, so $\limsup_{t \to \infty} \frac{1}{t} \ln (Y^k_t/Y^m_t) \leq 0$, from which one can obtain (4.2).
4.2 A candidate relative growth optimal strategy

Denote by \( \nu_t \) the predictable random measure on \( B(\mathbb{R}_+^N) \) defined by

\[
\nu_t(\omega, A) := \nu(\omega, \{t\} \times A), \quad A \in B(\mathbb{R}_+^N),
\]

and introduce the predictable process

\[
\bar{\nu}_t = \nu_t(\mathbb{R}_+^N).
\]

One can see that \( \bar{\nu}_t \) is the conditional probability for a jump of the process \( X \) given the \( \sigma \)-algebra \( \mathcal{F}_t \) (see Jacod and Shiryaev [25, Proposition II.1.17]), i.e., that we have

\[
\bar{\nu}_t = \mathbb{P}[\Delta X_t \neq 0 | \mathcal{F}_t].
\]

We always assume that a “good” version of the compensator is chosen, i.e., such that \( \bar{\nu}_t(\omega) \in [0, 1] \) for all \( \omega, t \).

The candidate relative growth optimal strategy which we define below behaves differently at points \( t \) where \( \bar{\nu}_t = 0 \) and where \( \bar{\nu}_t > 0 \) and has three “regimes” of operation. In order to define it, we need some auxiliary objects which we now introduce.

Let us partition \( \Omega \times \mathbb{R}_+ \times (0, \infty) \) into three sets \( \Gamma_0, \Gamma_1, \Gamma_2 \) which belong to \( \mathcal{P} \otimes B(\mathbb{R}_+) \) and are defined by

\[
\Gamma_0 = \{(\omega, t, c) : \bar{\nu}_t(\omega) = 0\},
\]

\[
\Gamma_1 = \left\{(\omega, t, c) : 0 < \bar{\nu}_t(\omega) < 1, \text{ or } \bar{\nu}_t(\omega) = 1 \text{ and } \int_{\mathbb{R}_+^N} \frac{c}{|x|} \nu_t(\omega, dx) > 1 \right\},
\]

\[
\Gamma_2 = \left\{(\omega, t, c) : \bar{\nu}_t(\omega) = 1 \text{ and } \int_{\mathbb{R}_+^N} \frac{c}{|x|} \nu_t(\omega, dx) \leq 1 \right\}.
\]

In the definition of the optimal strategy, the argument \( c \) in the triple \( (\omega, t, c) \) corresponds to the value of the total wealth of all the investors right before time \( t \), i.e., \( |Y_{t-}| \) (points \( (\omega, t, c) \) with \( c = 0 \) are not included in any of the sets; it is easier to deal with them separately). Roughly speaking, the sets \( \Gamma_i \) differ in the conditional size of possible jumps of the payoff process \( X \) relative to the current total market wealth. These three sets correspond to the above-mentioned three “regimes” of the optimal strategy. We comment on their structure after the definition of the optimal strategy in (4.9) below.

The next lemma defines an auxiliary function \( \zeta \) which is needed to specify what proportion of wealth the optimal strategy keeps in cash.

**Lemma 4.3** For each \( (\omega, t, c) \in \Gamma_1 \), the equation

\[
\int_{\mathbb{R}_+^N} \frac{c}{z + |x|} \nu_t(\omega, dx) = 1 - \frac{c}{z} (1 - \bar{\nu}_t(\omega)) \quad (4.3)
\]

has a unique solution \( z^*(\omega, t, c) \in (0, c) \). The function \( \zeta(\omega, t, c) \) which is defined on \( \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \) by

\[
\zeta = cI_{\Gamma_0} + z^* I_{\Gamma_1} \quad (4.4)
\]

is \( (\mathcal{P} \otimes B(\mathbb{R}_+)) \)-measurable.
Proof For \((\omega, t, c) \in \Gamma_1\), the left-hand side of (4.3) is a strictly decreasing continuous function of \(z\), while the right-hand side is a nondecreasing continuous function of \(z\). The existence and uniqueness of the solution \(z^*\) then follows from comparison of their values at \(z = c\) and \(z \to 0\).

To prove the measurability of \(\zeta\), consider the function \(f : \Omega \times \mathbb{R}^3_+ \to \mathbb{R}\) defined by

\[
f(\omega, t, c, z) = \left(\int_{\mathbb{R}^N_+} \frac{c}{z + |x|} v_{|t|}(\omega, dx) - 1 + \frac{c}{z} (1 - \bar{v}_t(\omega))\right) I_{\{(\omega, t, c) \in \Gamma_1\}} \wedge 1.
\]

Observe that \(f\) is a Carathéodory function, i.e., \((\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+))\)-measurable in \((\omega, t, c)\) and continuous in \(z\). Then by Filippov’s implicit function theorem (see e.g. Aliprantis and Border [2, Theorem 18.17]), the set-valued function \(\phi(\omega, t, c) = \{z \in [0, c] : f(\omega, t, c, z) = 0\}\) admits a measurable selector. Since \(\phi\) is single-valued on the set \(\Gamma_1\) (because we have \(\phi(\omega, t, c) = \{z^*(\omega, t, c)\}\)), this implies the \((\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+))\)-measurability of \(\zeta\). \(\Box\)

It is known that there exist a predictable process \(b\) with values in \(\mathbb{R}^N_+\) and a transition kernel \(K_{\omega, t}(dx)\) from \((\Omega \times \mathbb{R}_+, \mathcal{P})\) to \((\mathbb{R}^N_+, \mathcal{B}(\mathbb{R}^N_+))\) such that up to \(P\)-indistinguishability,

\[
X^c_t(\omega) = b \cdot G_t(\omega), \quad v(\omega, dt, dx) = K_{\omega, t}(dx) dG_t(\omega).
\]

(4.5)

Since the filtration is complete, we can assume (4.5) holds for all \(\omega, t\). Also, it will be convenient to select “good” versions of \(b\) and \(K\) which for all \((\omega, t)\) satisfy the conditions

\[
|b_t(\omega)| = 0 \quad \text{if} \quad \Delta G_t(\omega) > 0, \quad K_{\omega, t}([0]) = 0, \quad (4.6)
\]

\[
|b_t(\omega)| + \int_{\mathbb{R}^N_+} (1 \wedge |x|) K_{\omega, t}(dx) = 1
\]

(4.7)

(it is always possible to select such versions; see e.g. Jacod and Shiryaev [25, Proposition II.2.9]). Define the \((\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+))\)-measurable function \(\hat{\lambda}(\omega, t, c)\) with values in \(\mathbb{R}^N_+\) by

\[
\hat{\lambda}_\ell(0) = 0, \quad \hat{\lambda}_\ell(c) = \frac{b_t}{c} + \int_{\mathbb{R}^N_+} \frac{x}{\xi_t(c) + |x|} K_t(dx) \quad \text{for} \ c > 0 \quad (4.8)
\]

(the argument \(\omega\) is omitted for brevity). Thanks to (4.6) and (4.7), the integral in the above formula is well defined.

Now we are in a position to introduce the strategy which will be shown to be relative growth optimal. When used by investor \(m\), its cumulative investment process is defined by

\[
\widehat{L}_t(y) = \int_0^t y^{m}_{s-} \hat{\lambda}_s(|y_s|) dG_s
\]

(4.9)
(for \( s = 0 \), put \( y_{0-} = y_0 \)). The function \( \hat{\lambda} \) can be interpreted as a “rate” of investment in each asset. When it is necessary to emphasise that the strategy \( \hat{L} \), as a function of \( y \), depends on which investor uses it, we use the notation \( \hat{L}_m(\hat{\lambda}) \).

In order to get a clearer picture of the structure of the strategy \( \hat{L} \), let us explain the role of the sets \( \Gamma_i \) in an informal way, and indicate connections with earlier results in the literature. On the set \( \Gamma_0 \), we have \( \Delta G_t = 0 \), so \( \Delta \hat{L}_t = 0 \). In this case, the conditional probability of receiving a strictly positive instantaneous payoff is zero, i.e., \( P[\Delta X_t = 0 \mid F_{t-}] = 1 \), and the assets yield only “infinitesimal” payoffs (from the continuous part \( X^c_t \)) or inaccessible “jump” payoffs, and the strategy invests only “infinitesimal” amounts of wealth in the assets.

On the sets \( \Gamma_1 \) and \( \Gamma_2 \), the conditional probability of receiving a strictly positive payoff is positive, and the strategy invests in the assets a positive amount of wealth. On \( \Gamma_1 \), either a jump of \( X \) occurs with conditional probability greater than 0 but strictly less than 1, or it occurs with conditional probability 1, but the magnitude of the jump is not large compared to the current total wealth of all the agents (in the sense that \( E[|\Delta X_t|^{-1} \mid F_{t-}] > c^{-1} \)). Consequently, the strategy \( \hat{L} \) invests not the entire wealth in the assets, but keeps a part of it in cash, which can be seen from the fact that \( |\Delta L^m_n(Y_{t-})| < Y^m_m \). On \( \Gamma_2 \), a jump occurs with probability 1 and its magnitude is sufficiently large, and the optimal strategy invests the entire wealth in the assets. It is easy to see that we have

\[
\Delta L^m_n(Y_{t-}) = Y^m_t E \left[ \Delta X^n_t \left\{ \frac{\Delta X^n_t}{\zeta_t(Y_{t-}) + |\Delta X_t|} \right\} \mid F_{t-} \right] \quad \text{on } \Gamma_1, \tag{4.10}
\]

\[
\Delta L^m_n(Y_{t-}) = Y^m_t E \left[ \Delta X^n_t \left\{ \frac{\Delta X^n_t}{|\Delta X_t|} \right\} \mid F_{t-} \right] \quad \text{on } \Gamma_2.
\]

In particular, on the set \( \Gamma_2 \), the optimal strategy splits its wealth between the assets proportionally to their expected relative payoffs at time \( t \) (where the relative payoff of asset \( n \) is \( \Delta X^n_t / |\Delta X_t| \)). The optimality of this strategy in evolutionary finance models with short-lived assets is well known when there is no risk-free asset; see Amir et al. [6]. An analogue of (4.10) for a discrete-time model with a risk-free asset was obtained in Drokin and Zhitlukhin [18]. The structure of the optimal strategy on the set \( \Gamma_0 \) is a new result.

It is natural to ask how to find the formula for \( \hat{L} \). Essentially, as will become clear from the proof of Theorem 5.1, we need to look at the drift coefficient of the process \( \ln r^m \), and then it turns out to be possible to find the investment proportions \( \hat{\lambda} \) which make this coefficient nonnegative for any strategies of the other investors by using some known inequalities. Unfortunately, our method of proof does not show directly how to obtain (4.8) (like for example in the capital growth theory for markets with exogenous prices, where an optimal strategy maximises the growth rate of a portfolio); so it looks like a “guess” of the correct answer. However, given previous results in the literature, including the papers mentioned above, it becomes intuitively clear in what from the optimal strategy should be looked for, which leads to (4.8).

To conclude this section, we state a proposition which provides sufficient conditions for the feasibility of a strategy profile where one or several investors use the strategy \( \hat{L} \). It is based on Theorem 3.6, but we show that the conditions of that theorem hold automatically for \( \hat{L} \) under some mild additional assumptions on the payoff.
process. In particular, if these assumptions hold, then a strategy profile where all the investors use the strategy \( \hat{L} \) is feasible (we consider such profiles in Theorem 5.3 in the next section).

Define the predictable process \( h = (h_t)_{t \geq 0} \) with values in \( \mathbb{R}^N_+ \) by

\[
    h_t = b_t + \int_{\mathbb{R}^N_+} \frac{x}{1 + |x|} K_t(dx),
\]

and define the scalar predictable process \( p = (p_t)_{t \geq 0} \) by

\[
    p_t = \int_{\mathbb{R}^N_+} \frac{\nu_t(dx)}{(1 + |x|)^2}.
\]

**Proposition 4.4** Suppose \( (p_t \Delta G_t)^{-1} I_{\{\Delta G_t > 0\}} \) is a locally bounded process and for each \( n \), the process \( (h^n_t)^{-1} I_{\{h^n_t > 0\}} \) is locally bounded (where \( 0/0 = 0 \) for these processes). Then any strategy profile in which every investor uses either the strategy \( \hat{L} \) or a strategy which satisfies the conditions of Theorem 3.6 is feasible for any initial wealth \( y_0 \in \mathbb{R}^M_+ \).

The proof is given in Sect. 6.

### 4.3 Examples

In this section, we provide two examples which clarify the structure of the relative growth optimal strategy \( \hat{L} \). The first example shows how a discrete-time model can be embedded in continuous time, and in the second example, we consider a model with payoffs given by Lévy processes.

**Example 4.5** Suppose that the assets yield atom payoffs at integer moments of time \( t = 1, 2, \ldots \), and the filtration updates also only at these moments, i.e.,

\[
    \Delta X_t = 0 \text{ for } t \notin \mathbb{N} \text{ and } X_t = \sum_{s=0}^{[t]} \Delta X_s, \quad \mathcal{F}_t = \mathcal{F}_{[t]} \text{ for } t \geq 0.
\]

Note that in order to define \( \hat{L} \) by (4.8), (4.9), we can use instead of the process \( G \) any other process \( H \) such that \( G \ll H \) and define \( \hat{L} \) by (4.8) and (4.9) with the process \( b \) and the kernel \( K \) now computed with respect to \( H \) as in (4.5).

It will be convenient to take \( H_t = \lfloor t \rfloor \). Then we have \( b = 0 \) and for integer \( t \), \( K_t(dx) = \nu_t(dx) \) is the conditional distribution of the jump \( \Delta X_t \) on \( \mathbb{R}^N_+ \setminus \{0\} \), i.e., for any set \( A \in \mathcal{B}(\mathbb{N}^N_+ \setminus \{0\}) \), we have \( P[\Delta X_t \in A \mid \mathcal{F}_{t-1}] = K_t(A) \), and \( P[\Delta X_t = 0 \mid \mathcal{F}_{t-1}] = 1 - K_t(\mathbb{R}^N_+) \). The process \( \hat{L}_t(y) \) is piecewise constant and may jump only at integer moments of time. The value of \( \hat{\lambda}_t^n(c) \) at integer \( t \) shows the proportion of current wealth the investor puts in asset \( n \).

Assume that \( \tilde{\nu}_t > 0 \) for all integers \( t \). Then the function \( \hat{\lambda}_t(c) \) can be defined as follows. For non-integer \( t \), we have \( \lambda_t(c) = 0 \). For integer \( t \), let

\[
    \tilde{\Gamma}_t = \left\{ (\omega, c) : c E \left[ \frac{1}{|\Delta X_t|} \mid \mathcal{F}_{t-1} \right] > 1 \right\}.
\]
where the conditional expectation is computed with respect to the distribution \( K_t(dx) \), i.e., for any function \( f \),

\[
E[f(\Delta_1 X_t) | \mathcal{F}_{t-1}] = \int_{\mathbb{R}^n_+} f(x) K_t(dx) + f(0)(1 - K_t(\mathbb{R}^n_+)).
\]

One can see that

\[
\tilde{\Gamma}_t = \{ (\omega, c) : (\omega, t, c) \in \Gamma_1 \}
\]

with the set \( \Gamma_1 \) defined in Sect. 4.2. Consequently, we have

\[
\hat{\lambda}_t(c) = E\left[ \frac{\Delta X_t}{\zeta_t(c) + |\Delta X_t|} \bigg| \mathcal{F}_{t-1} \right],
\]

where the function \( \zeta_t(c) \) on the set \( \tilde{\Gamma}_t \) solves the equation

\[
E\left[ \frac{1}{\zeta_t(c) + |\Delta X_t|} \bigg| \mathcal{F}_{t-1} \right] = \frac{1}{c},
\]

and \( \zeta_t(c) = 0 \) on the complement of \( \tilde{\Gamma}_t \) (which consists of all those \( (\omega, c) \) such that \( (\omega, t, c) \in \Gamma_2 \)). This follows from (4.3) and (4.4).

Observe that \( \zeta_t(c) \) is related to the proportion of wealth an investor keeps in cash (i.e., \( 1 - |\hat{\lambda}_t(c)| \)), namely \( \zeta_t(c) = c(1 - |\hat{\lambda}_t(c)|) \). In particular, if all the investors in the market use the strategy \( \hat{L} \), then \( \zeta_t \) is equal to the total amount of wealth they keep in cash. Now one can see the following connection with the Kelly strategy from evolutionary finance. An investor who uses \( \hat{L} \) acts like she assumes that the other investors also use the strategy \( \hat{L} \) and invests her wealth in the assets proportionally to their expected relative payoffs at the next time instant. This is so because the “payoff” of cash is \( \zeta_t(c) \) (invested money is just returned in the same amount), and the total payoff of the assets is \( |\Delta X_t| \), so that \( \zeta_t(W_{t-1}) + |\Delta X_t| \) is the total amount of market wealth at time \( t \). Therefore, the random vector under the expectation in (4.12) can be viewed as the vector of relative asset payoffs.

Among related results in the literature, let us mention the paper of Amir et al. [6], where the optimal strategy also invests proportionally to the expected relative payoffs. However, in that paper, all the wealth is reinvested (there is no cash account), and so the total market wealth is always equal to the last total payoff. A similar model with cash account was considered by Drokin and Zhitlukhin [18], and the strategy (4.12) was found there. So essentially the present paper includes the model of [18] as a particular case.

**Example 4.6** Suppose \( X \) is a nondecreasing Lévy process. Then we can take \( H_t = t \) and choose a deterministic and time-independent process \( b \) and kernel \( K \), i.e., \( X^c_t = bt \), where \( b \in \mathbb{R}^n_+ \), and \( \nu(dt, dx) = K(dx)dt \). Since \( \bar{v}_t = 0 \), we have \( \Gamma_1 = \Gamma_2 = \emptyset \). Consequently \( \zeta_t(c) = c \), and the optimal strategy is defined by the formulas

\[
\hat{\lambda}(c) = \frac{b}{c} + \int_{\mathbb{R}^n_+} \frac{x}{c + |x|} K(dx), \quad \hat{L}_m^m(y) = \int_0^t y_{s-} \hat{\lambda}(\lfloor y_{s-} \rfloor) ds.
\]
As a particular case, suppose the payoffs $X^n_t, n = 1, \ldots, N$, are independent Poisson processes with intensities $\rho_1, \ldots, \rho_N$ and jumps of size 1. Then

$$b = 0, \quad K(dx) = \sum_{n=1}^{N} \rho_i \delta_{e_i}(dx),$$

where $e_i = (0, \ldots, 1, \ldots, 0)$ is the $i$-th basis vector in $\mathbb{R}^N$ and $\delta_{e_i}$ is the Dirac measure with unit mass at $e_i$. The function $\hat{\lambda}(c)$ has the simple form

$$\hat{\lambda}(c) = \frac{\rho_n}{c+1}. \quad (4.13)$$

One can see that this strategy continuously spends money with intensity $|\hat{\lambda}|$, but at the moments of jumps of the Poisson process, it receives payoffs which compensate for the money spent. To illustrate this, let us further simplify the model. Assume that there is only one asset with Poisson payoffs with intensity $\rho_1 = 1$ and two investors. Investor 1 uses the strategy $\hat{L}$, while investor 2 keeps all the money in cash (i.e., uses the strategy $L = 0$). Suppose the initial wealths are $Y^1_0 = Y^2_0 = 1$. Then $W_t = Y^1_t + 1$ and

$$\hat{\lambda}(W_t) = \frac{1}{Y^1_t + 2}.$$

Consequently, between two consecutive moments $\tau_n$ and $\tau_{n+1}$ of jumps of the process $X$, the process $Y^1$ follows the equation

$$dY^1_t = -\frac{Y^1_t}{Y^1_t + 2} dt, \quad t \in (\tau_n, \tau_{n+1}),$$

and at $\tau_n$ the process $Y^1$ jumps up by 1. Then the relative wealth satisfies the equation

$$d \ln r^1_t = -\frac{1}{(1 + Y^1_t)(2 + Y^1_t)} dt + \ln \left(1 + \frac{1}{Y^1_t(2 + Y^1_t)}\right) dX^1_t.$$

Using that $X^1_t - t$ is a martingale, we see that $\ln r^1$ is a submartingale with drift

$$\ln \left(1 + \frac{1}{Y^1_{t-}(2 + Y^1_t)}\right) - \frac{1}{(1 + Y^1_t)(2 + Y^1_t)} \geq 0$$

(the inequality follows from the relation $\ln(1 + z^{-1}) \geq (1 + z)^{-1}$ for $z > 0$).

Note that if investor 1 knows the strategy of investor 2, then $\hat{L}$ may be not the best strategy to use (in the above example where investor 2 is inactive, it would be better to invest in the asset with some small positive intensity). However, $\hat{L}$ guarantees that $\ln r^1$ remains a submartingale if investor 2 switches to another strategy.

To conclude this example, observe that the market with Poisson payoffs and the optimal strategy in it can be obtained by passing to the limit in the discrete-time model. Let us briefly show this, without giving a rigorous proof.
Consider a series of models from the first example indexed by a parameter \( k \) such that the assets may yield independent payoffs only at times \( t = i/k, \ i = 1, 2, \ldots \), and only one asset yields a payoff at a time. The amount of payoff is 1, i.e.,

\[
\Delta X^{(k)}_{i/k} \in \{e_1, \ldots, e_N\}
\]

and the vectors \( \Delta X^{(k)}_{i/k}, \ i = 1, 2, \ldots \), are independent.

Suppose that in the \( k \)th model, the probabilities of the payoffs are

\[
P(\Delta X^{(k)}_{i/k} = e_n) = \rho_n / k,
\]

where all \( \rho_n > 0 \). Then the cumulative payoff processes \( X^{(k), n} \) converge in distribution to Poisson processes with intensities \( \rho_n \) as \( k \to \infty \). The sets \( \Gamma_i^{(k)} \) are equal to \( \Omega \times \mathbb{R}_+ \); hence the optimal investment proportion function \( \hat{\lambda}^{(k)}(c) \) in the \( k \)th model is defined by (4.12) with the function \( \zeta^{(k)}(c) \) solving the equation

\[
\frac{1}{\zeta^{(k)}(c)} \left( 1 - \frac{|\rho|}{k} \right) + \frac{|\rho|}{k(\zeta^{(k)}(c) + 1)} = \frac{1}{c},
\]

where \( |\rho| = \sum_n \rho_n \). Its solution lying in the interval \((0, c)\) is \( \zeta^{(k)}(c) = c - o(1) \), and we have

\[
\hat{\lambda}^{(k), n}(c) = \frac{\rho_n}{(c + 1)k} + o(k^{-1}),
\]

which agrees with (4.13) in the limit.

5 The main results

This section contains three theorems which are the main results of the paper on relative growth optimal strategies. For convenience, we divide this section into four parts, each of the first three containing a theorem and comments, and the last providing examples. The proofs are given in Sect. 6.

5.1 Existence and uniqueness of a relative growth optimal strategy

The first result establishes the existence of a relative growth optimal strategy (\( \hat{L} \) is such a strategy) and shows that it is in a certain sense unique.

**Theorem 5.1** 1) The strategy \( \hat{L} \) is relative growth optimal.

2) Suppose \( L \) is a strategy of investor \( M \) with the property that the strategy profile \( (\hat{L}^1, \ldots, \hat{L}^{M-1}, L) \) and a vector \( y_0 \in \mathbb{R}_+^M \) of initial wealths are feasible and \( r^M \) is a submartingale. Then \( L_t(Y) = \hat{L}^M_t(Y) \) for all \( t \geq 0 \), where \( Y \) is the solution of the wealth equation for this strategy profile and initial wealth \( y_0 \) (i.e., the process \( L_t(\omega) = L_t(\omega, Y(\omega)), t \geq 0 \), coincides up to \( P \)-indistinguishability with the process \( \hat{L}^M_t(\omega) = \hat{L}^M_t(\omega, Y(\omega)), t \geq 0 \) ).

\( \Box \)
Let us comment on the second part of Theorem 5.1. It can be regarded as a uniqueness result for a relative growth optimal strategy: If \( M - 1 \) investors use the strategy \( \hat{L} \), then the remaining investor, if she wants her relative wealth to be a submartingale, has to act as if using the strategy \( \hat{L} \) in the sense that the realisation of her cumulative investment process will be the same as if she used \( \hat{L} \). As a consequence, the relative wealth of each investor will stay constant.

However, note that the strategy \( L_t(\omega, y) \), as a function on \( \Omega \times D \times \mathbb{R}_+ \), may be different from \( \hat{L}_t(\omega, y) \). Let us provide an example. Suppose there is only one asset with the non-random payoff process \( X_t = t \) and two investors with initial wealth \( y_1^0 = y_2^0 = 1 \). In this case, \( G_t = t \) and the strategy \( \hat{L} \), if used by investor 2, has the form

\[
\hat{L}_t(y) = \int_0^t \frac{y_2^2 - y_1^2}{y_s^{-1} + y_s^{2-}} \, ds.
\]

On the other hand, consider the strategy \( L \) for investor 2 defined as

\[
L_t(y) = \int_0^t \left( \frac{1}{3} I\{y_u^1 = 1, \forall u < s\} + \frac{y_2^2 - y_1^2}{y_s^{-1} + y_s^{2-}} I\{\exists u < s; y_u^1 \neq 1\} \right) \, ds.
\]

It is not hard to see that \( L \) is also relative growth optimal. However, it leads to a different wealth process of investor 2 compared to \( \hat{L} \), if for example \( L_t^1 \equiv 0 \).

5.2 Asymptotics of the representative strategy

The second main result shows that the strategy \( \hat{L} \) asymptotically determines the structure of the market in the sense that if there is an investor who uses \( \hat{L} \), then the representative strategy of all the investors is asymptotically close to \( \hat{L} \). (By the representative strategy, we mean the weighted sum of the investors’ strategies with their relative wealth as the weights; see below.) Moreover, if the representative strategy of the other investors is asymptotically different from \( \hat{L} \), they will be driven out of the market – their relative wealth will vanish as \( t \to \infty \).

In order to state the theorem, let us introduce auxiliary processes. Suppose a unique solution of the wealth equation exists. Let \( L_t^m(\omega, Y(\omega)) = L_t^m(\omega) \) be the realisations of the investors’ strategies and as above, \( \ell_t^m = dL_t^m/dG_t \). For each \( m \), define the predictable process \( L_t^{(s), m} = L_t^m - \ell_t^m \cdot G_t \), which is the singular part of the Lebesgue decomposition of \( L_t^m \) with respect to \( G \) (hence the superscript “(s)”).

Define the proportion \( \lambda_t^m = (\lambda_t^m)_{t \geq 0} \) of wealth invested in the assets by investor \( m \) as the predictable process with values in \( \mathbb{R}_+^N \) and the components

\[
\lambda_t^{m,n} = \frac{\ell_t^{m,n}}{Y_t^m},
\]

with \( 0/0 = 0 \). Note that by condition 2) of Definition 3.3, we have \( \ell_t^{m,n} = 0 \) on the set \( \{ (\omega, t) : Y_t^m(\omega) = 0 \} \) \( (P \otimes G) \)-a.e. Introduce also the processes of cumulative proportions of invested wealth \( \Lambda_t^m = (\Lambda_t^m)_{t \geq 0} \) and their singular parts.
$\Lambda^{(s),m} = (\Lambda^{(s),m}_t)_{t \geq 0}$ by

$$\Lambda^m_t = \frac{1}{Y^-_m} \cdot L^m_t, \quad \Lambda^{(s),m}_t = \Lambda^m_t - \lambda^m_t \cdot G_t = \frac{1}{Y^-_m} \cdot L^{(s),m}_t.$$  

They are nondecreasing, predictable, càdlàg and with values in $[0, +\infty)^N$.

For a set $\mathbb{M} \subseteq \{1, \ldots, M\}$ of investors, let us denote their total wealth by $Y^\mathbb{M}_t = \sum_{m \in \mathbb{M}} Y^m_t$, their relative wealth by $r^\mathbb{M}_t = \sum_{m \in \mathbb{M}} r^m_t$, and the processes associated with the realisation of their representative strategy by $L^\mathbb{M}_t = \sum_{m \in \mathbb{M}} L^{(s),m}_t$ and

$$\ell^\mathbb{M}_t = \frac{dL^\mathbb{M}_t}{dG_t} = \sum_{m \in \mathbb{M}} \ell^m_t, \quad L^{(s),\mathbb{M}}_t = \sum_{m \in \mathbb{M}} L^{(s),m}_t.$$  

To shorten the notation, for the set $\mathbb{M}_1 = \{1, \ldots, M\}$ of all investors, we write $\bar{\lambda}^n_t = \lambda^{\mathbb{M}_1,n}_t$, and for the set $\mathbb{M}_2 = \{2, \ldots, M\}$, we write $\tilde{\lambda}^n_t = \lambda^{\mathbb{M}_2,n}_t$, and similarly for the other processes. (It is clear that $\bar{r}_t = 1$ and $\tilde{r}_t = 1 - r^1_t$, which simplifies the above formulas.)

**Theorem 5.2** Suppose investor 1 uses the strategy $\hat{L}$, the other investors use arbitrary strategies $L^{\mathbb{M}}_m$, and the strategy profile $(\hat{L}_1, L^2, \ldots, L^M)$ is feasible for some initial wealth $y_0 \in \mathbb{R}_+^M$. Then

$$|\lambda^1_t - \bar{\lambda}_t|^2 \cdot G_\infty + |\bar{\Lambda}^{(s)}_\infty| < \infty \quad a.s., \quad (5.1)$$

and as $t \to \infty$,

$$r^1_t \longrightarrow 1 \quad a.s. \text{ on } \{\omega : |\lambda^1_t - \bar{\lambda}_t|^2 \cdot G_\infty(\omega) = \infty \text{ or } |\bar{\Lambda}^{(s)}_\infty(\omega)| = \infty\}. \quad (5.2)$$

Equation (5.1) expresses the idea that the investment proportions $\bar{\lambda}$ of the representative strategy of all investors together are close to $\lambda^1 = \hat{\lambda}$ asymptotically in the sense that the integral $\int_0^t |\hat{\lambda}_s - \bar{\lambda}_s|^2 dG_s$ converges as $t \to \infty$ and the singular part $\bar{\Lambda}^{(s)}_\infty$ stays bounded. If $G_\infty = \infty$, this roughly speaking means that $|\hat{\lambda}_t - \bar{\lambda}_t|^2$ is small asymptotically. This convergence has a clear interpretation if the process $G$ is of some simple form. We illustrate this below in the last part of this section for some particular models.

Equation (5.2) shows that the strategy $\hat{L}$ drives other strategies out of the market if their realisations are asymptotically different from it. This result can also be regarded as asymptotic uniqueness of a survival strategy: If investors $m = 2, \ldots, M$ want to survive against investor 1 who uses the strategy $\hat{L}$, they should also act, at least collectively, in such a way that the realisation of their representative strategy is asymptotically close to $\hat{L}$.
5.3 Asymptotics of the market wealth when all investors use the optimal strategy

Now we consider the situation when all the investors use the strategy $\hat{L}$. Obviously, in this case, their relative wealth will remain the same. However, it is interesting to look at the asymptotic behaviour of the absolute wealth $W_t := |Y_t|$. A priori, it is even not obvious whether it will grow. Our third result partly answers this question: we prove that $1/W$ is a supermartingale and provide a condition for $W_t \to \infty$ as $t \to \infty$.

**Theorem 5.3** Suppose that all the investors use the strategy $\hat{L}$, and that the initial wealth $y_0 \in \mathbb{R}_+^M$ and the strategy profile $(\hat{L}, \ldots, \hat{L})$ are feasible. Then the process $V_t := \frac{1}{W_t}$, $t \geq 0$, is a supermartingale and the limit $W_\infty := \lim_{t \to \infty} W_t$ exists in $(0, \infty]$ a.s. Moreover, if $X$ is quasi-continuous (i.e., $\hat{v} \equiv 0$), then we have $\{W_\infty = \infty\} = \{(1 \wedge |x|^2) \ast \nu_\infty = \infty\}$ a.s.

Observe that under the conditions of the theorem, if $E[|X_t|] < \infty$ for all $t$, then also $E[W_t] < \infty$ (since $W_t \leq |y_0| + |X_t|$), and so the process $W$ is a submartingale by Jensen’s inequality.

It is interesting to note that if one investor uses the strategy $\hat{L}$ and the other investors use arbitrary strategies, then it does not necessarily hold that the wealth of such an investor will grow. In particular, it may happen that $W_t \to 0$ as $t \to \infty$, which is remarkable because an investor always has a trivial strategy which guarantees that the wealth will not vanish – just keep all the wealth in cash. An example can be found in Drokin and Zhitlukhin [18].

Observe also that, as will become clear from the proof of the theorem, the continuous part of the payoff process $X$ does not affect the process $W$ if all the investors use the strategy $\hat{L}$, i.e., $W$ is the same for any payoff processes $X$ and $X'$ such that $X - X'$ is a continuous process. For example, if $X$ is continuous, then $W_t = W_0$ for all $t \geq 0$ even if $X$ is a strictly increasing process. In particular, one can see that the continuous part of $X$ does not enter the condition for having $W_\infty = \infty$ in the second claim of Theorem 5.3.

5.4 Examples

Let us illustrate Theorems 5.2 and 5.3 with examples using the models from Sect. 4.3.

Assume that $\bar{\lambda}^{(x)} = 0$, which eliminates strategies that aimlessly waste money. Observe that in the discrete-time model, we have $G_t \leq t$. Consequently, the set $\{\lambda^1 - \bar{\lambda}_t^2 \cdot G_\infty = \infty\}$ is included in the set $\{\sum_{t \geq 1} |\lambda^1_t - \bar{\lambda}_t|^2 = \infty\}$, and so (5.2) simplifies to

$$r^1_t \to 1 \text{ a.s. on } \left\{ \omega : \sum_{t \geq 1} |\lambda^1_t(\omega) - \bar{\lambda}_t(\omega)|^2 = \infty \right\}.$$ 

This means that investor 1 becomes the single survivor in the market if the realisation of the representative strategy of the other investors asymptotically differs from $\hat{L}$ in the sense that the above series diverges. In particular, this is so if $|\lambda^1_t - \bar{\lambda}_t|^2$ does not converge to zero as $t \to \infty$. 

 Springer
Now consider the model with Lévy payoffs. We have \( G_t = c t \) where \( c \) is an appropriate constant, hence
\[
r^1_t \to 1 \quad \text{a.s. on } \left\{ \omega : \int_0^\infty |\lambda^1_t(\omega) - \tilde{\lambda}_t(\omega)|^2 dt = \infty \right\}.
\]

For this model, Theorem 5.3 also takes a simple form. Assume that the payoff process has a non-zero jump part (i.e., it consists not only of a deterministic drift). Then Theorem 5.3 states that if all the investors use \( \hat{\lambda} \), then \( \frac{1}{W} \) is a supermartingale and \( \lim_{t \to \infty} W_t = \infty \), since \( (1 \wedge |x|^2) \ast \nu_{\infty} = \infty \). Note that if the payoff process is deterministic \( (X_t = b t \text{ for some } b \in \mathbb{R}_+^N) \), then \( \hat{\lambda}_t(c) = b/c \), and one can check that the total wealth process stays constant.

6 Proofs

6.1 Proof of Theorem 3.6

Without loss of generality, we assume that the functions \( C^m \) and the processes \( \delta^m \) are the same for all the investors, since otherwise one can take \( C(a) = \max_m C^m(a) \) and \( \delta_t = \max_m \delta^m_t \). Moreover, we can assume that \( \delta \) is a nondecreasing process, or otherwise take \( \delta'_t = \sup_{s \leq t} \delta_s \) (\( \delta' \) is finite-valued since \( \delta \) is predictable and càdlàg and hence locally bounded; see e.g. Dellacherie and Meyer [17, VII.32]). Proposition 3.4 implies that it is enough to prove the existence and uniqueness of a solution for some particular choice of the process \( H \) such that \( G \ll H \). We do this for \( H = G \).

We are going to construct the process \( Y \) by induction on stochastic intervals \( [0, \tau_{i,j}] \) with appropriately chosen stopping times \( \tau_{i,j} (i \in \{0, 1, \ldots, M\} \text{ and } j \in \mathbb{Z}_+) \) such that \( \tau_{i,j} \leq \tau_{i',j'} \) if \( (i, j) \leq (i', j') \) lexicographically (i.e., \( i < i' \), or \( i = i' \) and \( j \leq j' \)), and \( \sup_{i,j} \tau_{i,j} = \infty \). Here, “by induction” means that we construct processes \( Y^{i,j} \) such that on the set \( \{(\omega, t) : t \leq \tau_{i,j}(\omega)\} \), they satisfy (3.5) and \( Y^{i,j} = Y^{i',j'} \) for any \( (i', j') \geq (i, j) \) on this set. From these processes, we can form a single process \( Y \) satisfying (3.5) on the whole set \( \Omega \times \mathbb{R}_+ \).

Before providing an explicit construction, let us briefly explain the role that \( \tau_{i,j} \) will play. The stopping times \( \tau_{i,0} \) for \( i \geq 1 \) will be the times when the wealth of one or several investors reaches zero “in a continuous way” (i.e., for some \( m \), we have \( Y^m_t > 0 \) for \( t < \tau_{i,0} \), but \( Y^m_{\tau_{i,0}-} = 0 \)). The index \( i \) will correspond to the \( i \)th such event. Not necessarily all the investors will eventually have zero wealth; in that case, we put \( \tau_{i,j}(\omega) = \infty \) for \( i \) starting from some \( i' \) and all \( j \).

Between \( \tau_{i,0} \) and \( \tau_{i+1,0} \), we construct a sequence of stopping times \( \tau_{i,j} \to \tau_{i+1,0} \) as \( j \to \infty \) such that on each interval \( [\tau_{i,j}, \tau_{i,j+1}] \), the wealth of all the investors who have non-zero wealth at \( \tau_{i,j} \) can be bounded away from zero by a \( \mathcal{F}_{\tau_{i,j}} \)-measurable variable. The wealth of some of those investors may become zero at \( \tau_{i,j+1} \), but only “by a jump” (i.e., when they invest their whole wealth, but do not receive any asset payoffs). In our construction, if \( \tau_{i,0}(\omega) < \infty \), it will also hold that \( \tau_{i,j}(\omega) < \infty \) for all \( j \) (we just bound \( \tau_{i,j+1} \leq \tau_{i,j} + 1 \), see (6.3) below). However, it may be the case that \( \tau_{i,j} \to \infty \) as \( j \to \infty \) if the wealth of investors who have positive wealth at \( \tau_{i,j} \) always stays positive.
The reason why we need to treat differently the times when the wealth reaches zero in a continuous way and by a jump is that we do not assume that the function $C(a)$ is bounded in a neighbourhood of zero (this is necessary, for example, to apply the theorem to the strategy $\hat{L}$; see the proof of Proposition 4.4).

Now we proceed to the construction of $\tau_{i,j}$ and $Y_{i,j}$. Let $\tau_{0,0} = 0$ and for all $t \geq 0$, put $Y_{t,0} = y_0$, where $y_0 \in \mathbb{R}_+^M$ is the given initial wealth. Suppose $\tau_{i,j}$ and $Y_{i,j}$ are constructed. We now show how to construct $\tau_{i,j+1}$ and $Y_{i,j+1}$. For brevity, $i$ is assumed fixed and omitted in the notation; so we simply write $\tau_j$, $Y_j$, while $Y_{j,m}$ denotes the $m$th coordinate of $Y_j$.

Let $A(\omega) = \{m : Y_{j,m}^j(\omega) > 0\}$ denote the set of investors who are still active, i.e., have positive wealth, at $\tau_j$; for $\omega$ such that $\tau_j(\omega) = \infty$, we put $A(\omega) = \emptyset$. Observe that $A$ is an $\mathcal{F}_{\tau_j}$-measurable random set (since it is finite, the measurability means that $I_{\{m \in A\}}$ are $\mathcal{F}_{\tau_j}$-measurable functions).

On the set $\{\omega : A(\omega) = \emptyset\}$, define $\tau_{j+1} = \tau_j + 1$ (with $\tau_{j+1}(\omega) = \infty$ if $\tau_j(\omega) = \infty$), and on the set $\Omega' = \{\omega : A(\omega) \neq \emptyset\}$, define

$$
\gamma = (\delta_{\tau_j} + 1)C(Y_{j}^j)
$$

and

$$
\tau_{j+1} = \inf\left\{ t > \tau_j : |X_t - X_{\tau_j}| \geq \frac{1}{4M} Y_{j}^j \wedge \min_{m \in A} Y_{j,m}^j \right\}
$$

or $G_t - G_{\tau_j} \geq \frac{1}{2\gamma} \left( \frac{1}{2M} \wedge \min_{m \in A} Y_{j,m}^j \right)$ (6.1)

or $\delta_t \geq \delta_{\tau_j} + 1$ or $t \geq \tau_j + 1$. (6.2)

Observe that we have the strict inequality $\tau_{j+1} > \tau_j$ on $\Omega'$ since the processes $X$, $G$, $\delta$ are càdlàg. Also, $\tau_{j+1} \leq \tau_j + 1$ by the condition in (6.3).

For each $\omega$, define the complete metric space $E(\omega)$ consisting of càdlàg functions $f : \mathbb{R}_+ \to \mathbb{R}_+^M$ satisfying the conditions

$$
f_t = Y_t^j(\omega) \quad \text{for} \quad t \leq \tau_j(\omega),
$$

$$
f_t^m \in \left[ \frac{1}{2} Y_{j,m}^j(\omega), \ 2 Y_{j,m}^j(\omega) \right] \quad \text{for} \quad t > \tau_j(\omega), \ m = 1, \ldots, M,
$$

and the metric

$$
d(f, \tilde{f}) = \sup_{t \geq 0} |f_t - \tilde{f}_t|
$$

(note that if $A(\omega) = \emptyset$, then $E(\omega)$ consists of one element).

From now on, we assume that $\omega$ is fixed and omit it in the notation. Consider the operator $U$ on $E$ which maps a function $f \in E$ to the càdlàg function.

 Springer
\( g := U(f) \) for \( U : \mathbb{R}_+ \rightarrow \mathbb{R}^M_+ \) defined by the formula

\[
g^m_t = Y^{j,m}_t - \int_0^t |v^m_s(f_{s-})| I_{(\tau_j < s < \tau_{j+1}, m \in A)} dG_s
+ \int_0^t F^m(\ell_s(f_{s-})) I_{(\tau_j < s < \tau_{j+1})} dX_s,
\]

where \( F : \mathbb{R}^{MN} \rightarrow \mathbb{R}^{MN} \) is the function defined in (3.6) and

\[
\ell^{m,n}_s(z) = v^{m,n}_s(z) I_{\{m \in A\}}.
\]

Let us show that \( U \) is a contraction mapping from \( \mathbb{E} \) to itself. If \( A = \emptyset \), this is obvious since then \( \mathbb{E} \) has only one element as noted above; so consider the case \( A \neq \emptyset \).

Suppose \( f \in \mathbb{E}, g = U(f) \). First we show that \( g \in \mathbb{E} \). It is clear that \( g \) satisfies (6.4), and if \( m \notin A \), then \( g^m \) satisfies (6.5). To show that the lower bound in (6.5) is satisfied for \( m \in A \), consider the first integral in (6.6). Since \( f \in \mathbb{E} \), we have \( v^{m,n}_s(f_{s-}) \leq C(Y^j_{\tau_j}) \delta_s \leq \gamma \) by condition (3.12), using that \( \delta_s < \delta_{\tau_j} + 1 \) for \( s < \tau_{j+1} \). Hence the integral can be bounded from above by \( \gamma (G_{\tau_j+1} - G_{\tau_j}) \), and this quantity does not exceed \( \frac{1}{2} Y^{j,m}_{\tau_j} \) by the choice of \( \tau_{j+1} \) (see (6.2)). Therefore \( g^m_t \geq \frac{1}{2} Y^{j,m}_{\tau_j} \) for \( t \geq \tau_j \). The upper bound from (6.5) for \( m \in A \) follows from observing that the second integral in (6.6) is bounded from above by \( |X_{\tau_{j+1}} - X_{\tau_j}| \) since \( F^{m,n}(\ell) \leq 1 \) and \( |X_{\tau_{j+1}} - X_{\tau_j}| \leq Y^{j,m}_{\tau_j} \) by the choice of \( \tau_{j+1} \) (see (6.1)). Thus \( g \) satisfies conditions (6.4) and (6.5); so \( g \in \mathbb{E} \).

Now we show that \( U \) is contracting. Fix \( \omega \) and consider \( f, \tilde{f} \in \mathbb{E} \) and \( m \in A \). Then

\[
|U(f)^m_t - U(\tilde{f})^m_t| \leq \int_{(\tau_j(\omega), \tau_{j+1}(\omega))} |v^m_s(f_{s-}) - v^m_s(\tilde{f}_{s-})| dG_s
+ \sum_{n=1}^N \int_{(\tau_j(\omega), \tau_{j+1}(\omega))} |F^{m,n}(\ell_s(f_{s-})) - F^{m,n}(\ell_s(\tilde{f}_{s-}))| dX^n_s
= : T^m_1 + T^m_2.
\]

By conditions (3.13) and (6.3), we have \( |v^{m,n}_s(f_{s-}) - v^{m,n}_s(\tilde{f}_{s-})| \leq \gamma d(f, \tilde{f}) \) for \( s \in (\tau_j(\omega), \tau_{j+1}(\omega)) \). Hence

\[
T^m_1 \leq \gamma d(f, \tilde{f})(G_{\tau_{j+1}} - G_{\tau_j}) \leq \frac{1}{4M} d(f, \tilde{f}),
\]

where the last inequality is due to (6.2). To bound \( T^m_2 \), observe that for each \( n \) and \( t \) such that \( (\omega, t) \in [\tau_j, \tau_{j+1}] \setminus \Pi^{m,n} \), we have by (3.11) that

\[
|\ell^{m,n}_s(f_{s-})| \geq \min_{m \in A} v^{m,n}_s(f_{s-}) \geq \frac{1}{\gamma},
\]

\( \Box \) Springer
and a similar inequality is true for \(|\ell_s^n(\tilde{f}_{s-})|\). It is straightforward to check that \(F\) satisfies the property

\[
\left| \frac{\partial F_{m,n}}{\partial p,q}(\ell) \right| \leq 1 \quad \text{for any } m, n, p, q.
\]

Hence for any \(\ell, \tilde{\ell} \in \mathbb{R}^{MN}_{+}\) such that \(|\ell^n| \geq \alpha\) and \(|\tilde{\ell}^n| \geq \alpha\) for all \(n\) with some \(\alpha > 0\), we have \(|F_{m,n}(\ell) - F_{m,n}(\tilde{\ell})| \leq \alpha^{-1}|\ell - \tilde{\ell}|\). From this and (6.8), we find that on the set \(\tau_j, \tau_{j+1}\) \(\setminus \Pi^{m,n}_{1}\),

\[
|F_{m,n}(\ell_s^n(f_{s-})) - F_{m,n}(\ell_s^n(\tilde{f}_{s-}))| \leq \gamma|\ell_s^n(f_{s-}) - \ell_s^n(\tilde{f}_{s-})| \leq \gamma^2 d(f, \tilde{f}).
\]

On \(\Pi^{m,n}_{1}\), we have

\[
F_{m,n}(\ell_s^n(f_{s-})) - F_{m,n}(\ell_s^n(\tilde{f}_{s-})) = 0.
\]

Consequently, we obtain the bound

\[
\mathcal{T}_{2} \leq \gamma^2 d(f, \tilde{f}) |X_{\tau_{j+1}} - X_{\tau_j}| \leq \frac{1}{4M} d(f, \tilde{f}),
\]

and we see that \(U\) is a contraction mapping because \(d(U(f), U(\tilde{f})) \leq \frac{1}{2} d(f, \tilde{f})\).

As a result, \(U(\omega)\) has a fixed point \(f^*(\omega)\) for any \(\omega\). Observe that the operator \(U\) preserves adaptedness, i.e., if \(f = (f_t(\omega))_{t \geq 0}\) is a càdlàg adapted process with values in \(\mathbb{R}^M_{+}\) and satisfies conditions (6.4) and (6.5), then \(U(\omega, f(\omega))\) is such a process as well. Hence \(f^*\) is a càdlàg adapted process since it can be obtained for example as the limit \(U^{(n)}(Y_t^j)\) as \(n \to \infty\), where the superscript \((n)\) stands for the \(n\)-fold application of \(U\).

Now we can define the process \(Y_t^{j+1}\) as follows. For each \(m\), put

\[
Y_t^{j+1,m} = f_{t}^{*,m} \quad \text{for } t < \tau_{j+1},
\]

\[
Y_t^{j+1,m} = f_{\tau_{j+1}}^{*,m} - \int_{(\tau_{j+1}, t]} \left| v_{\tau_{j+1}}^{m}\big( f_{\tau_{j+1}}^{*,m-}\big) \right| I_{[m \in A]} \Delta G_{\tau_{j+1}}
\]

\[
+ \sum_{n=1}^{N} F_{m,n}(v_{\tau_{j+1}}^{m}\big( f_{\tau_{j+1}}^{*,m-}\big)) I_{[m \in A]} \Delta X_{\tau_{j+1}}^n
\]

for \(t \geq \tau_{j+1}\)

(note that \(Y_t^{j+1} = Y_{\tau_{j+1}}^{j+1}\) for all \(t \geq \tau_{j+1}\)). Inserting \(Y_t^{j+1,m}\) in (6.6), we obtain on \([\tau_{j}(\omega), \tau_{j+1}(\omega)]\) the equation

\[
Y_t^{j+1,m}(\omega) = Y_{\tau_{j}}^{j+1}(\omega) - \int_{(\tau_{j}(\omega), t]} v_{\tau_{j}}^{m}(Y_s^{j+1}) I_{[Y_s^{j+1} > 0]} dG_{\tau_{j}}(\omega)
\]

\[
+ \int_{(\tau_{j}(\omega), t]} F_{m}(\ell_s^n(Y_s^{j+1})) dX_s(\omega).
\]

The indicator here can be equivalently replaced by \(I_{[\inf_s Y_s^{j+1,m} > 0]}\), so that the first integral becomes equal to \(|L_t^{m}(Y^{j+1})| - |L_{\tau_{j}}^{m}(Y^{j+1})|\) by (3.8). In the second integral,
on \((\tau_j(\omega), \tau_{j+1}(\omega))\), we have (as follows from (3.8))
\[
\ell_{t}^{m,n}(Y_{t}^{j+1})(\omega) = \frac{dL_{t}^{m,n}(Y_{t}^{j+1})}{dG_{t}}(\omega).
\]
Consequently, (6.9) implies that the process \(Y^{j+1}\) satisfies (3.5) for \(t \leq \tau_{j+1}(\omega)\).

Proceeding by induction, we obtain for fixed \(i\) a nondecreasing sequence of stopping times \(\tau_{i,j}\) and processes \(Y_{i,j}\). Let \(\tau_{i+1,0} = \lim_{j \to \infty} \tau_{i,j} \in [0, \infty]\). On \([0, \tau_{i+1,0}]\), define the process \(Y^{i+1,0}\) by joining up the \(Y^{i,j}\), i.e., for \((\omega, t)\) such that \(t < \tau_{i+1,0}(\omega)\), put
\[
Y_{t}^{i+1,0}(\omega) = Y_{t}^{i,0}(\omega)I_{[t < \tau_{i,0}(\omega)]} + \sum_{j=1}^{\infty} Y_{t}^{i,j}(\omega)I_{[\tau_{i,j-1}(\omega) \leq t < \tau_{i,j}(\omega)]}.
\]
Observe that on the set \(\{\tau_{i+1,0} < \infty\}\), the limit \(Y_{t}^{i+1,0}\) exists since for \(t < \tau_{i+1,0}(\omega)\), the process \(Y^{i+1,0}\) satisfies (3.5), in which the integral processes are nondecreasing and bounded by \(X^{n}_{\tau_{i+1,0}}\) and the term \(|L^{m}|\) is nondecreasing and bounded by \(Y_{0}^{m} + |X^{n}_{\tau_{i+1,0}}|\). For \(t \geq \tau_{i+1,0}(\omega)\), put
\[
Y_{t}^{i+1,0}(\omega) = Y_{\tau_{i+1,0}^{-}}^{i+1,0}(\omega) - |\ell_{m}|(\omega) + \sum_{n=1}^{N} F_{n}(\ell)\Delta X^{n}_{\tau_{i+1,0}}(\omega)
\]
with
\[
\ell_{m} = v_{\tau_{i+1,0}^{-}}^{i+1,0}(\omega)I_{[\inf_{s < \tau_{i+1,0}} Y_{s}^{i+1,0,m} > 0]}\Delta G_{\tau_{i+1,0}}
\]
(the process \(Y^{i+1,0}\) stays constant after \(\tau_{i+1,0}\)). One can see that now \(Y^{i+i,0}\) satisfies (3.5) for \(t \leq \tau_{i+1,0}(\omega)\). So the proof of the existence of a solution can be finished by induction. Uniqueness follows from the uniqueness of the fixed point of the operator \(U\) on each step of the induction. \(\square\)

6.2 Proof of Proposition 4.4

As follows from Theorem 5.1 (see Remark 6.1 after its proof), if a solution of the wealth equation exists and investor \(m\) uses the strategy \(\hat{L}\), then the wealth of this investor does not vanish (\(Y_{m} > 0\) and \(Y_{m}^{-} > 0\)). Therefore, it is enough to prove Proposition 4.4 for a strategy profile in which every investor uses either a strategy satisfying the conditions of Theorem 3.6, or a strategy \(\hat{L}'\) such that when used by investor \(m\), its cumulative investment process is
\[
\hat{L}'_{i}(m; y) = \int_{0}^{t} y_{s}^{m} \hat{\lambda}_{s}(|y_{s}^{-}|)I_{[\inf_{u < s} y_{u} = 0]}dG_{s}
\]
(it differs from the strategy \(\hat{L}\) only by the presence of the indicator). In order to show that such a profile is feasible, we verify conditions (C1), (C2) of Theorem 3.6 for \(\hat{L}'(m)\).
So let us fix $m$ and assume that investor $m$ uses the strategy $\hat{L}^t(m)$. Let

$$v_{t}^{m,n}(\omega, z) = z^m \hat{\lambda}^t_n(\omega, |z|),$$

so that $\hat{L}^t(m)$ can be represented in the form (3.8). Inequality (3.9) is satisfied because if $\Delta G_t(\omega) > 0$ (and therefore $\bar{\nu}_t(\omega) > 0$), then

$$|\hat{\lambda}_t(\omega, c)| = \int_{\mathbb{R}^N_+} |x| (\zeta_t(\omega, c) + |x|)^{-1} v_{t}^{m,n}(\omega, dx) \leq 1$$

as follows from the definition of $\hat{\lambda}$. Hence condition (C1) holds.

In order to verify condition (C2), consider the sets

$$\Pi^{m,n} = \{(\omega, t) : h^n_t(\omega) = 0\}$$

and define the function $C^m(a)$ by

$$C^m(a) = \max \left( \frac{2|a| \lor 1}{a^m / 2}, \frac{2|a| \lor 1}{|a|^3 \land 1} \right) \text{ if } a^m > 0,$$

$$C^m(a) = 1 \text{ if } a^m = 0,$$

and the process $\delta_t = (\delta_t^m)_{t \geq 0}$ by

$$\delta_t^m = \sup_{s \leq t} \left( \max_{1 \leq r \leq N} \frac{h^n_s}{h^r_n} \lor \frac{I_{\{h^n_s > 0\}}}{p_s \Delta G_s} \right) \lor 1.$$

The local boundedness assumptions imply that $\delta$ is finite-valued.

Equality (3.10) clearly holds. To prove inequalities (3.11) and (3.12), consider $z, a \in \mathbb{R}^N_+$ such that $z^k \in [a^k / 2, 2a^k]$ for all $k$. Suppose $z^m > 0$ (and hence $a^m > 0$). Then (3.11) follows from the fact that outside the set $\Pi^{m,n}$,

$$v_{t}^{m,n}(z) = z^m \hat{\lambda}_t^n (|z|) \geq \frac{z^m}{|z| \lor 1} h^n_t \geq \frac{a^m / 2}{(2|a| \lor 1) \delta_t^m} \geq \frac{1}{C^m(a) \delta_t^m},$$

where the first inequality uses the bound $\hat{\lambda}_t^n (c) \geq h_t^n / (c \lor 1)$ for any $c > 0$, which can be obtained from (4.8), (4.11) using that $\zeta_t(c) \in [0, c]$. To prove (3.12), we can use on the set $\Gamma_0$ the estimate

$$\hat{\lambda}_t^n (c) = \frac{b^n_t}{c} + \int_{\mathbb{R}^N_+} \frac{x^n}{c + |x|} K_t(dx) \leq \frac{h^n_t}{c \land 1} \leq \frac{1}{c \land 1}. \quad (6.10)$$

The last inequality here holds since $|h_t| \leq |b_t| + \int_{\mathbb{R}^N_+} (1 \land |x|) K_t(dx) = 1$; see (4.7).

On the set $\Gamma_1 \cup \Gamma_2$, we can use the estimate

$$\hat{\lambda}_t^n (c) = \int_{\mathbb{R}^N_+} \frac{x^n}{\zeta_t(c) + |x|} K_t(dx) \leq K_t(\mathbb{R}^N_+) = \frac{\bar{\nu}_t}{\Delta G_t} \leq \frac{1}{\Delta G_t} \quad (6.11)$$

(\text{\copyright} \ Springer)
(note that if \((\omega, t, c) \in \Gamma_1 \cup \Gamma_2\), then \(\Delta G_t > 0\) and \(b_t = 0\)). Therefore we obtain

\[
v_{t}^{m, n}(z) \leq z^m \max \left( \frac{1}{|z| \wedge 1}, \frac{I_{(\Delta G_t > 0)}}{\Delta G_t} \right) \leq (2|a| \vee 1)\delta_t^m \leq C^m(a)\delta_t^m, \tag{6.12}\]

and so (3.12) holds.

To prove (3.13), suppose \(z, \tilde{z}, a \in \mathbb{R}_+^N\) and \(z^k, \tilde{z}^k \in [a^k / 2, 2a^k]\) for all \(k\). If \(z^m = \tilde{z}^m = 0\), then \(v_{t}^{m, n}(z) = v_{t}^{m, n}(\tilde{z}) = 0\) so that (3.13) holds. If \(\tilde{z}^m = 0\) but \(z^m > 0\), then using (6.12), we obtain

\[
|v_{t}^{m, n}(z) - v_{t}^{m, n}(\tilde{z})| = v_{t}^{m, n}(z) \leq (2|a| \vee 1)\delta_t^m \leq C^m(a)\delta_t^m |z - \tilde{z}|,
\]

where we used the inequality \(|z - \tilde{z}| \geq z^m \geq a^m / 2\). In a similar way, (3.13) is satisfied if \(zm = 0\) but \(\tilde{z}^m > 0\).

Let us consider the case \(z^m > 0, \tilde{z}^m > 0\). Denote \(c = |z|, \tilde{c} = |\tilde{z}|\). Then

\[
|v_{t}^{m, n}(z) - v_{t}^{m, n}(\tilde{z})| \leq \lambda^n(t)(c)|z^m - \tilde{z}^m| + z^m |\lambda^n(t)(c) - \lambda^n(t)(\tilde{c})|. \tag{6.13}
\]

Using (6.10) and (6.11), the first term on the right-hand side can be bounded by

\[
\lambda^n(t)|z^m - \tilde{z}^m| \leq \frac{2\delta_t^m}{|a| \wedge 1} |z - \tilde{z}|.
\]

For the second term on the right-hand side of (6.13), we have

\[
|z^m \lambda^n(t)(c) - \lambda^n(t)(\tilde{c})| \leq 2a^m \left( \frac{|c - \tilde{c}|}{c\tilde{c}} b_t^n + |c - \tilde{c}| \int_{\mathbb{R}_+^N} \frac{x^n}{(c + |x|)(\tilde{c} + |x|)} K_t(dx) I_{\{\Delta G_t = 0\}} \right. \\
+ \left. |\xi_t(c) - \xi_t(\tilde{c})| \int_{\mathbb{R}_+^N} \frac{x^n}{(\xi_t(c) + |x|)(\xi_t(\tilde{c}) + |x|)} K_t(dx) I_{\{\Delta G_t > 0\}} \right) =: 2a^m(A_1 + A_2 + A_3).
\]

Using the relations \(|c - \tilde{c}| \leq |z - \tilde{z}|, |b_t| \leq 1, |h_t| \leq 1\), we obtain

\[
A_1 \leq \frac{4|c - \tilde{c}|}{|a|^2} b_t^n \leq \frac{4|z - \tilde{z}|}{|a|^2}
\]

and

\[
A_2 \leq \frac{|c - \tilde{c}|}{c(\tilde{c} \wedge 1)} \int_{\mathbb{R}_+^N} \frac{x^n}{1 + |x|} K_t(dx) \leq \frac{4|z - \tilde{z}|}{|a|(|a| \wedge 1)} h_t^n \leq \frac{4|z - \tilde{z}|}{|a|^2 \wedge 1}.
\]
To bound $A_3$, assume $c \geq \tilde{c}$ (hence also $\zeta_t(c) \geq \zeta_t(\tilde{c})$) and $\zeta_t(c) > 0$. Then we have

$$A_3 \leq \left(\zeta_t(c) - \zeta_t(\tilde{c})\right) \left(\int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(c) + |x|} K_t(dx)\right) I_{\{\Delta G_t > 0\}}$$

$$\leq \frac{\zeta_t(c) - \zeta_t(\tilde{c})}{c \Delta G_t} I_{\{\Delta G_t > 0\}} \leq \left(\zeta_t(c) - \zeta_t(\tilde{c})\right) \frac{2 I_{\{\Delta G_t > 0\}}}{|a| \Delta G_t},$$

(6.14)

where the second inequality uses the bound

$$\int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(c) + |x|} K_t(dx) = \frac{1}{c \Delta G_t} \int_{\mathbb{R}_+^N} c \frac{v_t(dx)}{\zeta_t(c) + |x|} = \frac{1}{c \Delta G_t} \left(1 - \frac{c}{\zeta_t(c)} (1 - \bar{v}_t)\right) \leq \frac{1}{c \Delta G_t}.$$  

Here the second equality follows from (4.3); notice that $(\omega, t, c) \in \Gamma_1$ because we assume $\zeta_t(c) > 0$.

Now we need to bound $\zeta_t(c) - \zeta_t(\tilde{c})$ in (6.14). Let $Q_t$ be the random measure on $\mathbb{R}_+^N$ defined by $Q_t(A) = v_t(A) + (1 - \bar{v}_t) I_{\{0 \in A\}}$. Note that $Q_t(\mathbb{R}_+^N) = 1$. Since $(\omega, t, c) \in \Gamma_1$ and $(\omega, t, \tilde{c}) \in \Gamma_1 \cup \Gamma_2$, we find from (4.3) and (4.4) that

$$\int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(c) + |x|} Q_t(dx) = \frac{1}{c}, \quad \int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(\tilde{c}) + |x|} Q_t(dx) \leq \frac{1}{\tilde{c}}.$$  

From this, we obtain

$$\frac{1}{\tilde{c}} - \frac{1}{c} \geq \left(\zeta_t(c) - \zeta_t(\tilde{c})\right) \int_{\mathbb{R}_+^N} \frac{Q_t(dx)}{(\zeta_t(c) + |x|)(\zeta_t(\tilde{c}) + |x|)}$$

$$\geq \left(\zeta_t(c) - \zeta_t(\tilde{c})\right) \int_{\mathbb{R}_+^N} \frac{Q_t(dx)}{(c + |x|)^2}$$

$$\geq \left(\zeta_t(c) - \zeta_t(\tilde{c})\right) \int_{\mathbb{R}_+^N} \frac{\bar{v}_t(dx)}{(c + |x|)^2} \geq \left(\zeta_t(c) - \zeta_t(\tilde{c})\right) \frac{c^2 \vee 1}{c^2 \vee 1} p_t.$$

Hence we conclude that

$$\zeta_t(c) - \zeta_t(\tilde{c}) \leq \frac{(c - \tilde{c})(c^2 \vee 1)}{c \tilde{c} p_t} \leq \frac{4|z - \tilde{z}|}{(|a|^2 \vee 1)p_t}. \quad (6.15)$$

From (6.14) and (6.15), we find

$$A_3 \leq \frac{8|z - \tilde{z}| I_{\{\Delta G_t > 0\}}}{(|a|^3 \vee 1)p_t \Delta G_t}.$$  

This implies that (3.13) is satisfied when $z^m > 0$ and $\tilde{z}^m > 0$, because

$$|v(z)_t^{m,n} - v(\tilde{z})_t^{m,n}| \leq \left(\frac{2}{|a| \vee 1} + 2|a| \left(\frac{4}{|a|^2 \vee 1} + \frac{4}{|a|^3 \vee 1} + \frac{8}{|a|^4 \vee 1}\right)\right) \delta_t^m |z - \tilde{z}|$$

$$\leq \frac{2 + 32|a|}{|a|^3 \vee 1} \delta_t^m |z - \tilde{z}| \leq C^m(a) \delta_t^m |z - \tilde{z}|.$$
Thus condition (C2) holds, and this finishes the proof. □

6.3 Proof of Theorem 5.1

The key idea of the proof of 1) is to show that $\ln r$ is a $\sigma$-submartingale by showing that its drift rate is nonnegative. Since $\ln r$ is a nonpositive process, it is then a usual submartingale (see Kallsen [26, Proposition 3.1]). For the reader's convenience, the necessary technical results are provided in the Appendix; for more details, see e.g. [26].

Let us state one auxiliary inequality which generalises the well-known Gibbs inequality and plays an important role in the proof. Suppose $\alpha, \beta \in \mathbb{R}^N_+$ are two vectors such that $|\alpha|, |\beta| \leq 1$ and for each $n$, it holds that if $\beta^n = 0$, then also $\alpha^n = 0$. Then

$$
\sum_{n=1}^{N} \alpha^n (\ln \alpha^n - \ln \beta^n) \geq \frac{|\alpha - \beta|^2}{4} + |\alpha| - |\beta|,
$$

(6.16)

where $\alpha^n (\ln \alpha^n - \ln \beta^n) = 0$ if $\alpha^n = 0$. A short direct proof can be found in Drokin and Zhitlukhin [18, Lemma 2].

Now we can proceed to the proof of 1). Assume that the strategy $\hat{L}$ is used by investor $m = 1$ and the wealth equation has a unique solution $Y$. We use the notation of Sect. 5 and introduce the predictable $\mathbb{R}^N_+$-valued processes $\lambda^1$, $\Lambda^1$ for investor 1 and $\tilde{\lambda}$, $\tilde{\Lambda}$, $\tilde{\Lambda}^{(s)}$ for the other investors. To keep the notation concise, the superscript “1” for investor 1 is omitted from now on; so we simply write $\lambda$, $\Lambda$. The total market wealth is denoted by $W = Y + \tilde{Y}$. It will be also convenient to assume that a particular version of $\lambda$ is selected, namely $\lambda_t(\omega) = \hat{\lambda}_t(\omega, W_t(\omega))$ for all $(\omega, t)$, with the function $\hat{\lambda}_t(c)$ defined in (4.8).

Let $r_t = Y_t / W_t$ denote the relative wealth of investor 1. Define the predictable process $F = (F_t)_{t \geq 0}$ with values in $\mathbb{R}^N_+$ by

$$
F^n_t = \frac{\lambda^n_t}{r_t - \lambda^n_t + (1 - r_t)\tilde{\lambda}^n_t},
$$

where $0/0 = 0$. Then $Y$ and $W$ can be written as stochastic exponentials,

$$
Y = Y_0 \mathcal{E} \left( -|\Lambda| + \frac{F}{W_-} \cdot X \right),
$$

$$
W = W_0 \mathcal{E} \left( -r_- \cdot |\Lambda| - (1 - r_-) \cdot |\tilde{\Lambda}| + \frac{1}{W_-} \cdot |X| \right).
$$

(6.17)

Recall that the stochastic exponential of a semimartingale $S$ is the process $\mathcal{E}(S)$ which solves the equation $d\mathcal{E}(S)_t = \mathcal{E}(S)_{t-} dS_t$ with $\mathcal{E}(S)_0 = 1$. It is known that $\mathcal{E}(S) > 0$ and $\mathcal{E}(S)_- > 0$ if $\Delta S > -1$. From the definition of $\hat{\lambda}$, one can check that

$$
\Delta \left( -|\Lambda| + \frac{F}{W_-} \cdot X \right) > -1
$$

up to an evanescent set; hence $Y > 0$ and $Y_- > 0$. 
Let $\zeta = (\zeta_t(\omega))_{t \geq 0}$ denote the predictable process given by $\zeta_t(\omega, W_t - (\omega))$. As follows from the definition of $\tilde{L}$ and $\zeta$, we have $\zeta_t = (1 - |\Delta \tilde{\Lambda}_t|)W_t\_\_$. Let

$$\tilde{\zeta}_t = (1 - |\Delta \tilde{\Lambda}_t|)W_t\_\_. $$

Define the predictable function $f(\omega, t, x)$ by

$$f_t(x) = \ln \frac{\zeta_t + F_t x}{r_t - \zeta_t + (1 - r_{t\_\_})\tilde{\zeta}_t + |x|}. $$

Using the Doléans-Dade formula, which for a process of bounded variation $S$ takes the form $E(S)_t = \exp(S^c_t + \sum_{u \leq t} \ln(1 + \Delta S_u))$, we obtain

$$\ln r_t = \ln r_0 + (1 - r\_{\_\_}) \cdot (|\tilde{\Lambda}_t^c| - |\Lambda_t^c|) + \frac{F - 1}{W^-} \cdot X_t^c + \sum_{s \leq t} f_s(\Delta X_s). $$

For the further analysis, it will be convenient to split the process $\ln r$ into several parts. Let $f_t(x) = f^1_t(x) + f^2_t(x) + f^3_t(x)$, where

$$f^1_t(x) = f_t(x)I_{\{\Delta G_t = 0, \Delta \tilde{\Lambda}_t = 0\}},
$$

$$f^2_t(x) = f_t(x)I_{\{\Delta G_t > 0\}},
$$

$$f^3_t(x) = f_t(x)I_{\{\Delta G_t = 0, \Delta \tilde{\Lambda}_t > 0\}}. $$

Then

$$\ln r_t = \ln r_0 + Z_t + \tilde{Z}_t $$

with the processes

$$Z_t = (1 - r\_{\_\_}) \cdot (|\tilde{\Lambda}_t^c| - |\Lambda_t^{(s)c}| - |\Lambda_t^c|) + \frac{F - 1}{W^-} \cdot X_t^c + \sum_{s \leq t} (f_s^1 + f_s^2) (\Delta X_s), $$

$$\tilde{Z}_t = (1 - r\_{\_\_}) \cdot |\tilde{\Lambda}_t^{(s)c}| + \sum_{s \leq t} f_s^3 (\Delta X_s), $$

where $\tilde{\Lambda}_t^{(s)c} = \tilde{\Lambda}_t^{(s)} - \sum_{u \leq t} \Delta \tilde{\Lambda}_u^{(s)}$ is the continuous part of the singular part of the Lebesgue decomposition of $\Lambda$ with respect to $G$.

Observe that we have $I_{\{\Delta X \neq 0, \Delta G = 0, \Delta \tilde{\Lambda} \neq 0\}} = 0$ since the set $\{\Delta X \neq 0, \Delta G = 0\}$ is totally inaccessible and the process $\tilde{\Lambda}$ is predictable. Therefore,

$$\sum_{s \leq t} f_s^3 (\Delta X_s) = \sum_{s \leq t} f_s^3 (0) = - \sum_{s \leq t} \ln(1 - (1 - r\_{s\_\_}) |\Delta \tilde{\Lambda}_s^c|). $$

\textcopyright Springer
From this formula and (6.20), it follows that $\tilde{Z}$ is a nondecreasing predictable càdlàg process. So in order to show that $\ln r$ is a $\sigma$-submartingale, it is enough to show that $Z$ is a $\sigma$-submartingale.

Next we use condition (A.1) (see Proposition A.1 in the Appendix). Since the process $Z$ is of bounded variation, it is not difficult to see (from e.g. the canonical representation of a semimartingale) that its continuous part can be represented as $Z^c = b^0 \cdot G$, where $b^0$ is the predictable process from (A.1). From (6.19), we find

$$b^0_t = (1 - r_t)(|\tilde{\lambda}_t| - |\lambda_t|)I_{\Delta G_t = 0} + \frac{(F_t - 1)b_t}{W_{t-}}.$$

The jump measure $\mu^Z$ of $Z$ is such that for a function $g(\omega, t, z)$ with $g(\omega, t, 0) = 0$, we have

$$g * \mu^Z_t = g(f^1_t + f^2_t) * \mu_t + \sum_{s \leq t} g(f^2_s(0))I_{\Delta X_s = 0},$$

where $g(f^1 + f^2)$ denotes the composition of $g$ and $f^1 + f^2$. Then the compensator of $\mu^Z$ can be represented in the form $\nu^Z = K^Z dG$ with the kernel $K^Z$ such that

$$\int_{\mathbb{R}} g_t(z)K^Z_t(dz) = \int_{\mathbb{R}^N_+} g_t(f^1_t(x) + f^2_t(x))K_t(dx) + \frac{1 - \tilde{v}_t}{\Delta G_t} g_t(f^2_t(0)).$$

(When $\Delta G_t(\omega) = 0$, we have $f^2_t(\omega, x) = 0$; so we treat the last term on the right-hand side as zero.) Consequently, the drift rate of $Z$ with respect to $G$ is

$$\varrho_t = b^0_t + \int_{\mathbb{R}} zK^Z_t(dz) = h^1_t + h^2_t$$

with the predictable processes

$$h^1_t = (1 - r_t)(|\tilde{\lambda}_t| - |\lambda_t|)I_{\Delta G_t = 0} + \frac{(F_t - 1)b_t}{W_{t-}} + \int_{\mathbb{R}^N_+} f^1_t(x)K_t(dx),$$

$$h^2_t = \int_{\mathbb{R}^N_+} f^2_t(x)K_t(dx) + \frac{1 - \tilde{v}_t}{\Delta G_t} f^2_t(0).$$

(6.22)

We want to show that $h^1, h^2 \geq 0$. For $h^1$, using the inequality $x - 1 \geq \ln x$ for $x > 0$, we find that

$$(F_t - 1)b_t \geq b_t \ln F_t,$$

(6.23)

where we put $b^n_t \ln F^n_t = 0$ if $F^n_t = 0$ (notice that if $F^n_t = 0$, then $\lambda^n_t = 0$, so also $b^n_t = 0$). Introduce the set $X_t(\omega) = \{x \in \mathbb{R}^N_+ : x^n = 0 \text{ if } F^n_t(\omega) = 0, \ n = 1, \ldots, N\}$. On the set \{$(\omega, t, x) : \Delta G_t(\omega) = 0, \ x \in X_t(\omega)$\}, using the concavity of the logarithm, the equality $\Delta \Lambda_t = 0$ if $\Delta G_t = 0$ and the inequality $\xi_t \leq W_{t-}$, we obtain

$$f^1_t(x) \geq \ln \frac{W_{t-} + F_t x}{W_{t-} + |x|} \geq \frac{x \ln F_t}{W_{t-} + |x|},$$

(6.24)
A continuous-time asset market game with short-lived assets

where we put \( x^n \ln F^n_t = 0 \) if \( F^n_t = x^n = 0 \). Denote

\[
a_t = \int_{\mathbb{R}^N_+} \frac{x W_{t-}}{W_{t-} + |x|} K_t(dx).
\]

(6.25)

As follows from (4.8), we have \( K_t(\omega, \mathbb{R}^N_+ \setminus \mathcal{X}_t(\omega)) = 0 \). Then from (6.24) and (6.25), we obtain

\[
\int_{\mathbb{R}^N_+} f_1^1(x) K_t(dx) = \int_{\mathcal{X}_t} f_1^1(x) K_t(dx) \geq \frac{a_t \ln F_t}{W_{t-}} I_{\{\Delta G_t = 0\}}.
\]

Together with (6.23), this implies

\[
h_1^1 \geq \left( (1 - r_{t-})(|\tilde{\lambda}_t| - |\lambda_t|) + \frac{(a_t + b_t) \ln F_t}{W_{t-}} \right) I_{\{\Delta G_t = 0\}}.
\]

From (4.8), it follows that we have \( \lambda_t = (a_t + b_t)/W_{t-} \) when \( \Delta G_t = 0 \); so on the set \( \{\Delta G = 0\} \),

\[
h_1^1 \geq (1 - r_{t-})(|\tilde{\lambda}_t| - |\lambda_t|) + \lambda_t \ln F_t
\]

\[
= (1 - r_{t-})(|\tilde{\lambda}_t| - |\lambda_t|) + \lambda_t \left( \ln \lambda_t - \ln \left( r_{t-} \lambda_t + (1 - r_{t-}) \tilde{\lambda}_t \right) \right).
\]

Applying (6.16), we obtain

\[
h_1^1 \geq \frac{1}{4} (1 - r_{t-})^2 |\lambda_t - \tilde{\lambda}_t|^2 I_{\{\Delta G_t = 0\}} \geq 0.
\]

(6.26)

Let us prove that \( h^2 \geq 0 \). Consider the set \( \{\Delta G > 0\} \), on which we have

\[
f_2^2(x) = \ln \frac{\zeta_t + F_t x}{r_{t-} \zeta_t + (1 - r_{t-}) \tilde{\zeta}_t + |x|}
\]

\[
= \ln \frac{\zeta_t + F_t x}{\zeta_t + |x|} + \ln \frac{\zeta_t + |x|}{r_{t-} \zeta_t + (1 - r_{t-}) \tilde{\zeta}_t + |x|}.
\]

Using the concavity of the logarithm, we find that for \( x \in \mathcal{X}_t(\omega) \),

\[
f_2^2(x) \geq \frac{x \ln F_t}{\zeta_t + |x|} + \ln \frac{\zeta_t + |x|}{r_{t-} \zeta_t + (1 - r_{t-}) \tilde{\zeta}_t + |x|} =: A_t(x) + B_t(x).
\]

(6.27)

For the term \( A_t(x) \), applying (6.16), we get

\[
\int_{\mathbb{R}^N_+} A_t(x) K_t(dx) = \lambda_t \ln F_t = \lambda_t \left( \ln \lambda_t - \ln \left( r_{t-} \lambda_t + (1 - r_{t-}) \tilde{\lambda}_t \right) \right)
\]

\[
\geq \frac{1}{4} (1 - r_{t-})^2 |\lambda_t - \tilde{\lambda}_t|^2 + (1 - r_{t-})(|\lambda_t| - |\tilde{\lambda}_t|).
\]

© Springer
For the term $B_t(x)$, using the inequality $\ln x \geq 1 - x^{-1}$, we obtain

$$B_t(x) \geq \frac{(1 - r_{t-})(\zeta_t - \tilde{\zeta}_t)}{\zeta_t + |x|}.$$

From the definition of $\zeta$ (see (4.3)), it follows that

$$\int_{\mathbb{R}_+^N} \frac{1}{\zeta_t + |x|} K_t(dx) \geq \frac{1}{W_t - \Delta G_t} - \frac{1 - \tilde{v}_t}{\zeta_t \Delta G_t}.$$

So we have

$$\int_{\mathbb{R}_+^N} B_t(x) K_t(dx) \geq (1 - r_{t-})(|\tilde{\lambda}_t| - |\lambda_t|) - \frac{(1 - r_{t-})(1 - \tilde{v}_t)(\zeta_t - \tilde{\zeta}_t)}{\zeta_t \Delta G_t}, \quad (6.28)$$

where for the first term on the right-hand side, we used that

$$\zeta_t - \tilde{\zeta}_t = (|\tilde{\lambda}_t| - |\lambda_t|)W_t - \Delta G_t.$$

Thus, using (6.27) and (6.28) and $K_t(\omega, \mathbb{R}_+^N \setminus X_t(\omega)) = 0$, we find

$$\int_{\mathbb{R}_+^N} f^2_t(x) K_t(dx) \geq \frac{1}{4} (1 - r_{t-}) (|\lambda_t - \tilde{\lambda}_t|)^2 - \frac{(1 - r_{t-})(1 - \tilde{v}_t)(\zeta_t - \tilde{\zeta}_t)}{\zeta_t \Delta G_t}, \quad (6.29)$$

Also, using again the inequality $\ln x \geq 1 - x^{-1}$, we obtain

$$f^2_t(0) = \ln \frac{\zeta_t}{r_{t-} - \zeta_t + (1 - r_{t-}) \zeta_t} \geq \frac{(1 - r_{t-})(\zeta_t - \tilde{\zeta}_t)}{\zeta_t}, \quad (6.30)$$

Hence from (6.22), (6.29) and (6.30), we find that

$$h^2_t \geq \frac{1}{4} (1 - r_{t-}) (|\lambda_t - \tilde{\lambda}_t|)^2 I_{\{\Delta G_t > 0\}} \geq 0. \quad (6.31)$$

Thus we have proved that $h^1, h^2 \geq 0$ so that $\ln r$ is a submartingale. This finishes the proof of 1).

To prove 2), suppose that the investors $m = 1, \ldots, M - 1$ use the strategy $\hat{L}$ and investor $M$ uses some strategy $L$. If $r^M$ is a submartingale, then $\ln r^1$ is a supermartingale by Jensen’s inequality, and hence a martingale by the first claim of the theorem. Consequently, we find from (6.18) (with the same notation as above) that

$$\tilde{Z}_t = 0 \quad \text{a.s. for all } t \geq 0, \quad h^1 + h^2 = 0 \quad (P \otimes G)\text{-a.e.}$$

The first equality implies that $L^{(s), M} = 0$, so $L^M \ll G$. The second equality together with (6.26) and (6.31) implies that $\tilde{\lambda}_t = \hat{\lambda}_t(W_{t-})$ $(P \otimes G)$-a.e., and therefore $\lambda^M_t = \hat{\lambda}_t(W_{t-})$ $(P \otimes G)$-a.e. Then (4.9) gives $L^M = \hat{L}^M(Y)$, which finishes the proof. \hfill $\square$

**Remark 6.1** As can be seen from the proof, the wealth of an investor who uses the strategy $\hat{L}$ does not vanish ($Y^m > 0$ and $Y^m_0 > 0$) on any solution of the wealth equation (if it exists). This fact is needed in the proof of Proposition 4.4.
6.4 Proof of Theorem 5.2

We use the same notation as in the proof of Theorem 5.1. Since \( \ln r \) is a non-positive submartingale, there exists the limit \( r_\infty = \lim_{t \to \infty} r_t \). As we have shown, \( \ln r_t = \ln r_0 + Z_t + \tilde{Z}_t \), where \( Z \) is a submartingale with drift rate

\[
\delta_t = h^1_t + h^2_t \geq \frac{1}{4} (1 - r_{t-})^2 |\lambda_t - \tilde{\lambda}_t|^2 = \frac{1}{4} |\lambda_t - \tilde{\lambda}_t|^2.
\]

Hence the compensator \( A_t = \delta \cdot G_t \) of \( Z_t \) (see (A.2)) satisfies the inequality

\[
A_t \geq \frac{1}{4} |\lambda - \tilde{\lambda}|^2 \cdot G_t.
\]

Since \( Z \) is bounded from above as \( Z_t \leq -\ln r_0 \), the process \( A \) converges to a finite limit \( A_\infty \), and so \( |\lambda - \tilde{\lambda}|^2 \cdot G_\infty < \infty \). Moreover, on the set \( \{ |\lambda - \tilde{\lambda}|^2 \cdot G_\infty = \infty \} \), we necessarily have \( r_\infty = 1 \), because otherwise we should have \( A_\infty = \infty \) on this set.

From the inequality \( \ln(1 - (1 - r_{\infty}) |\Delta \tilde{\lambda}_t|) \leq -(1 - r_{\infty}) |\Delta \tilde{\lambda}_t| \) combined with (6.20), (6.21), we obtain

\[
\tilde{Z}_t \geq (1 - r_{\infty}) |\tilde{\lambda}_t| = |\tilde{\lambda}_t|.
\]

Since \( \tilde{Z} \) converges, we have \( |\tilde{\lambda}_\infty| < \infty \), and on the set \( \{ |\tilde{\lambda}_\infty| = \infty \} \), we have \( r_\infty = 1 \). \( \square \)

6.5 Proof of Theorem 5.3

Suppose all the investors use the strategy \( \hat{L} \). By (6.17), \( W = W_0 \mathcal{E}(S) \) with the process

\[
S_t = -|\tilde{\lambda}(W_\infty)| \cdot G_t + \frac{1}{W_-} |X_t| = -\frac{|x|}{\zeta + |x|} \cdot \nu_t + \sum_{s \leq t} \frac{\Delta X_s}{W_s},
\]

where \( \zeta \) denotes the predictable process \( \zeta_t(W_t \cdot \nu_t) \). In particular, the continuous part \( S^c \) and the jumps \( \Delta S \) are given by

\[
S^c_t = -\frac{|x| I_{\{\nu_t = 0\}}}{W_- + |x|} \cdot \nu_t,
\]

\[
\Delta S_t = -\int_{\mathbb{R}^N_+} \frac{|x|}{\zeta_t + |x|} \nu_t(dx) + \frac{\Delta X_t}{W_{t-}} = \frac{\zeta_t + \Delta X_t}{W_{t-}} - 1.
\]

From the formula \( \mathcal{E}(S)_t = \exp(S^c_t + \sum_{s \leq t} \ln(1 + \Delta S_s)) \), we find \( V = V_0 \mathcal{E}(U) \) with the process

\[
U_t = -S^c_t - \sum_{s \leq t} \frac{\Delta S_s}{1 + \Delta S_s} = -S^c_t - \sum_{s \leq t} \left( \frac{W_{t-}}{\zeta_s + |X_s|} - 1 \right). \quad (6.32)
\]

The continuous part of \( U \) is \( U^c_t = -S^c_t = b^U \cdot G_t \) with the predictable process

\[
b^U_t = \int_{\mathbb{R}^N_+} \frac{|x|}{W_{t-} + |x|} K_t(dx) I_{\{G_t = 0\}},
\]
and the jump measure $\mu^U$ acts on functions $f(\omega, t, u)$ with $f(\omega, t, 0) = 0$ as

$$f \ast \mu^U_t = f\left(\frac{W_-}{\xi + |x|} - 1\right) \ast \mu_t + \sum_{s \leq t} f_s\left(\frac{W_s^-}{\xi_s} - 1\right) I_{\{\Delta X_s = 0, \vec{\nu}_s > 0\}}.$$ 

Therefore its compensator $\nu^U$ is such that

$$f \ast \nu^U_t = f\left(\frac{W_-}{\xi + |x|} - 1\right) \ast \nu_t + \sum_{s \leq t} f_s\left(\frac{W_s^-}{\xi_s} - 1\right) (1 - \vec{\nu}_s) I_{\{\vec{\nu}_s > 0\}}.$$ 

In particular, $\nu^U = K^U \otimes G$ with the transition kernel $K^U$ such that

$$\int_{\mathbb{R}} f_t(u) K^U_t(du) = \int_{\mathbb{R}} f_t\left(\frac{W_t^-}{\xi_t + |x|} - 1\right) K_t(dx) + f\left(\frac{W_t^-}{\xi_t} - 1\right) (1 - \vec{\nu}_t) \frac{I_{\{\vec{\nu}_t > 0\}}}{\Delta G_t}.$$ 

From the definition of $\zeta$ in Lemma 4.3, it follows that $\int_{\mathbb{R}} |u| K^U_t(du) < \infty$, and hence the drift rate of $U$ with respect to $G$ is given by

$$\partial_t^U = b_t^U + \int_{\mathbb{R}} u K^U_t(du) \leq 0,$$

where the inequality follows from the observation that $\zeta_t = W_{t-}$ on the set $\{\Delta G = 0\}$ and that on the set $\{\Delta G > 0\}$, we have

$$\int_{\mathbb{R}_+} \frac{W_t^-}{\xi_t + |x|} K_t(dx) \leq \left(1 - \frac{W_t^-}{\xi_t}\right) \frac{(1 - \vec{\nu}_t)}{\Delta G_t},$$

in view of the fact that $K_t(dx) = (\Delta G_t)^{-1} \nu_{\{t\}}(dx)$ and the definition of $\zeta$. Consequently, $U$ is a $\sigma$-supermartingale. This implies that $V$ is also a $\sigma$-supermartingale, because it is nonnegative. In particular, it has an a.s. limit $V_\infty = \lim_{t \to \infty} V_t \in [0, \infty)$, and therefore $W_\infty = 1/V_\infty \in (0, \infty]$, which proves the first claim of the theorem.

If $\vec{\nu} \equiv 0$, we have $\zeta_t = W_{t-}$ for all $t$, so (6.32) becomes

$$U_t = -\frac{|x|}{W_- + |x|} \ast (\mu_t - \nu_t),$$

and hence $U$ is a purely discontinuous local martingale with bounded jumps; in fact, $\Delta U_t \in (-1, 0]$. Consequently, according to Karatzas and Kardaras [27, Proposition 7.1], we have the equality $\{V_\infty = 0\} = \{|u|^2 \ast \nu^U_\infty = \infty\}$ a.s., or equivalently $\{W_\infty = \infty\} = \{(\frac{|x|}{W_- + |x|})^2 \ast \nu^U_\infty = \infty\}$ a.s. Together with the existence of the limit $W_\infty$, this yields the second claim of the theorem. \qed
Appendix

The first part of this appendix provides a known condition for a semimartingale to be a submartingale, in a form convenient for our applications in the proof of Theorem 5.1. The next two parts assemble several facts about the Lebesgue decomposition and Lebesgue derivatives of \( \sigma \)-finite measures, and prove auxiliary results for random measures generated by predictable nondecreasing càdlàg processes.

A.1 Submartingality conditions

A scalar semimartingale \( Z \) with \( Z_0 = 0 \) is called a \( \sigma \)-submartingale if there exists a nondecreasing sequence of predictable sets \( \Pi_n \in \mathcal{P} \) such that \( Z_{\Pi_n}^t := \int_0^t I_s(\Pi_n) dZ_s \) is a submartingale for each \( n \) and \( \bigcup_n \Pi_n = \Omega \times \mathbb{R}_+ \). Suppose the triplet \((B^h, C, \nu)\) of predictable characteristics of \( Z \) with respect to a truncation function \( h(z) \) admits the representation

\[
B^h = b^h \cdot G, \quad C = c \cdot G, \quad \nu = K \otimes G,
\]

where \( b^h = (b^h_t)_{t \geq 0}, c = (c_t)_{t \geq 0} \) are predictable processes, \( K = (K_t(dz))_{t \geq 0} \) is a transition kernel and \( G = (G_t)_{t \geq 0} \) is a nondecreasing predictable càdlàg process. Then \( Z \) is a \( \sigma \)-submartingale if and only if \((P \otimes G)\)-a.e. on \( \Omega \times \mathbb{R}_+ \),

\[
\int_{\{|z| > 1\}} |z| K_t(dz) < \infty \quad \text{and} \quad \partial_t := b^h_t + \int_{\mathbb{R}} (z - h(z)) K_t(dz) \geq 0
\]

(see Karatzas and Kardaras [27, Proposition 11.2], Kallsen [26, Lemma 3.1]). The predictable process \( \partial \) is called the drift rate of \( Z \) with respect to \( G \). One can see that it does not depend on the choice of the truncation function \( h \) (see Jacod and Shiryaev [25, Proposition II.2.24]).

Observe that if we have for all \( t \geq 0 \) that

\[
\int_{\mathbb{R}} |z| K_t(dz) < \infty,
\]

then \( \partial_t = b^0_t + \int_{\mathbb{R}} z K_t(dz) \), where \( b^0_t = b^h_t - \int_{\mathbb{R}} h(z) K_t(dz) \) is a well-defined predictable process which does not depend on the choice of \( h \). From this, we obtain the following result which is used in the proof of Theorem 5.1.

**Proposition A.1** Suppose \( Z \) is a nonpositive semimartingale. Then \( Z \) is a submartingale if \((P \otimes G)\)-a.e.,

\[
\int_{\{z < 0\}} z K_t(dz) > -\infty \quad \text{and} \quad \partial_t = b^0_t + \int_{\mathbb{R}} z K_t(dz) \geq 0. \tag{A.1}
\]

In particular, if (A.1) is satisfied, then the process \( \partial \) is \( G \)-integrable and the compensator of \( Z \) is

\[
A_t = \partial \cdot G_t. \tag{A.2}
\]

Formula (A.2) follows from observing that for a nonpositive semimartingale, we have \( \int_{\{z > 0\}} z K_t(dz) < \infty \) for \( t \geq 0 \) since \( K_t(\{z : z > -Z_{t-}\}) = 0 \).
A.2 Lebesgue decomposition of σ-finite measures

Let \((\Omega, \mathcal{F})\) be a measurable space. First recall the following known result, which can be found (in a slightly different form) e.g. in Bogachev [10, Chap. 3.2].

**Proposition A.2** Let \(P, \tilde{P}\) be σ-finite measures on \((\Omega, \mathcal{F})\). Then there exist a measurable function \(Z \geq 0\) (\(P\)-a.s. and \(\tilde{P}\)-a.s.) and a set \(\Gamma \in \mathcal{F}\) such that

\[
\tilde{P}[A] = \int_A Z dP + \tilde{P}[A \cap \Gamma] \quad \text{for any } A \in \mathcal{F}
\]

(A.3)

and

\[
P[\Gamma] = 0.
\]

(A.4)

Such a \(Z\) is \(P\)-a.s. unique and \(\Gamma\) is \(\tilde{P}\)-a.s. unique, i.e., if \(Z'\) and \(\Gamma'\) also satisfy the above properties, then \(Z = Z'\) \(P\)-a.s. and \(\tilde{P}[\Gamma \Delta \Gamma'] = 0\) (where we denote by \(\Gamma \Delta \Gamma' = (\Gamma \setminus \Gamma') \cup (\Gamma' \setminus \Gamma)\) the symmetric difference of the two sets).

The function \(Z\) – the Lebesgue derivative of \(\tilde{P}\) with respect to \(P\) – is denoted in this paper by \(d\tilde{P}/dP\). If \(\tilde{P} \ll P\), the Lebesgue derivative coincides with the Radon–Nikodým derivative and one can take \(\Gamma = \emptyset\). When it is necessary to emphasise that the set \(\Gamma\) is related to \(\tilde{P}\) and \(P\), we use the notation \(\Gamma_{P/\tilde{P}}\).

In an explicit form, \(Z\) and \(\Gamma\) can be constructed as follows. Let \(Q\) be any σ-finite measure on \((\Omega, \mathcal{F})\) such that \(P \ll Q\), \(\tilde{P} \ll Q\) (for example, \(Q = P + \tilde{P}\)). Then

\[
Z = \left(\frac{dP}{dQ}\right)^{-1}I_{\left\{\frac{dP}{dQ} > 0\right\}} , \quad \Gamma = \left\{\omega : \frac{dP}{dQ}(\omega) = 0\right\},
\]

where the derivatives are in the Radon–Nikodým sense.

By approximating a measurable function with simple functions, it follows from (A.3) that for any \(\mathcal{F}\)-measurable function \(f \geq 0\), we have

\[
\int_\Omega f d\tilde{P} = \int_\Omega f \frac{d\tilde{P}}{dP} dP + \int_\Omega f I_\Gamma d\tilde{P}
\]

(A.5)

(where the integrals may assume the value \(+\infty\)).

The following proposition contains facts about Lebesgue derivatives that are used in the paper.

**Proposition A.3** Let \(P, \tilde{P}, Q\) be σ-finite measures on \((\Omega, \mathcal{F})\). Then the following statements are true:

(a) Suppose \(Q\) is representable in the form \(Q[A] = \int_A f dP + \int_A \tilde{f} d\tilde{P}\), where \(f, \tilde{f} \geq 0\) are measurable functions and \(\tilde{f} = 0\) \(P\)-a.s. Then

\[
\frac{dP}{dQ} = \frac{1}{f} I_{\{f > 0, \tilde{f} = 0\}}, \quad \Gamma_{P/Q} = \{f = 0, \tilde{f} = 0\}.
\]
(b) If $R$ is a $\sigma$-finite measure such that $R \ll P$ and $R \ll Q$, then

\[
\frac{d\tilde{P}}{dP} = \frac{d\tilde{P}}{dQ} \frac{dQ}{dP} \quad R\text{-a.s.} \tag{A.6}
\]

(c) If $R$ is as in (b), then $dQ/dP > 0$ $R$-a.s. and $dP/dQ > 0$ $R$-a.s.

**Proof** (a) is obtained by straightforward verification of (A.3) and (A.4).

(b) Observe that for any $A \in \mathcal{F}$, we have

\[
\tilde{P}[A] = \int_A \frac{d\tilde{P}}{dQ} dQ + \tilde{P}[A \cap \Gamma_{\tilde{P}/Q}]
\]

\[
= \int_A \frac{d\tilde{P}}{dQ} dP + \int_{\Omega} I_{A \cap \Gamma_{\tilde{P}/P}} \frac{d\tilde{P}}{dQ} dQ + \tilde{P}[A \cap \Gamma_{\tilde{P}/Q}] \tag{A.7}
\]

where we applied (A.5) to obtain the second equality, and the third follows by expressing the second integral in the second line from the equality

\[
\tilde{P}[A \cap \Gamma_{\tilde{P}/P}] = \int_{\Omega} I_{A \cap \Gamma_{\tilde{P}/P}} \frac{d\tilde{P}}{dQ} dQ + \tilde{P}[A \cap \Gamma_{\tilde{P}/Q} \cap \Gamma_{\tilde{P}/P}].
\]

Suppose for $A = \{\frac{d\tilde{P}}{dP} > \frac{d\tilde{P}}{dQ} \frac{dQ}{dP}\}$ that $R[A] > 0$. Then we also have $R[A'] > 0$ for $A' = A \cap (\Gamma_{\tilde{P}/P} \cup \Gamma_{\tilde{P}/Q} \cup \Gamma_{\tilde{P}/P})^c$ because $R[\Gamma_{\tilde{P}/P}] = R[\Gamma_{\tilde{P}/Q}] = R[\Gamma_{\tilde{P}/P}] = 0$. Consequently, $P[A'] > 0$. But this leads to a contradiction between the decomposition (A.3) and the equality (A.7) for $\tilde{P}[A']$ since according to them, we should have

\[
\int_{A'} \frac{d\tilde{P}}{dP} dP = \int_{A'} \frac{d\tilde{P}}{dQ} dQ dP,
\]

which is impossible due to the choice of $A$. Hence $R[\frac{d\tilde{P}}{dP} > \frac{d\tilde{P}}{dQ} \frac{dQ}{dP}] = 0$. In the same way, we show that $R[\frac{d\tilde{P}}{dP} < \frac{d\tilde{P}}{dQ} \frac{dQ}{dP}] = 0$.

(c) follows from (A.6) if one takes $\tilde{P} = P$. \qed

### A.3 Lebesgue decomposition of nondecreasing predictable processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space satisfying the usual assumptions, and $\mathcal{P}$ the predictable $\sigma$-algebra on $\Omega \times \mathbb{R}_+$. For a scalar nondecreasing càdlàg predictable process $G$, denote by $P \otimes G$ the measure on $\mathcal{P}$ defined as

\[
(P \otimes G)[A] = E[I(A) \cdot G_\infty], \quad A \in \mathcal{P}. \tag{A.8}
\]

Observe that $P \otimes G$ is $\sigma$-finite on $\mathcal{P}$. Indeed, this can be shown by considering the predictable stopping times $\tau_n = \inf\{t \geq 0 : G_t \geq n\}$. The stochastic intervals $A_n = [0, \tau_n) := \{(w, t) : t < \tau_n(w)\}$ are predictable, i.e., $A_n \in \mathcal{P}$, and we have $(P \otimes G)[A_n] \leq n$ and $\bigcup_n A_n = \Omega \times \mathbb{R}_+$. \(\square\ Springer
Proposition A.4 (a) For any scalar nondecreasing càdlàg predictable processes $G, \tilde{G}$, there exist a predictable process $\xi \geq 0$ and a set $\Gamma \in \mathcal{P}$ such that up to $P$-indistinguishability,

$$\tilde{G} = \tilde{G}_0 + \xi \cdot G + I_\Gamma \cdot \tilde{G} \quad \text{and} \quad I_\Gamma \cdot G = 0. \quad (A.9)$$

(b) A predictable process $\xi \geq 0$ and a set $\Gamma \in \mathcal{P}$ satisfy (A.9) if and only if $\xi$ is a version of the Lebesgue derivative $d(P \otimes \tilde{G})/d(P \otimes G)$ and $\Gamma$ is the corresponding set from the Lebesgue decomposition.

We denote any $(P \otimes G)$-version of such a process $\xi$ by $d\tilde{G}/dG$ or $d\tilde{G}_t/dG_t$ and call it a predictable Lebesgue derivative of $\tilde{G}$ with respect to $G$. When it is necessary to emphasise that the set $\Gamma$ is related to $\tilde{G}$ and $G$, we use the notation $\Gamma \tilde{G}/G$.

**Proof** Without loss of generality, assume $\tilde{G}_0 = 0$.

(a) Let $\xi = d(P \otimes \tilde{G})/d(P \otimes G)$ and $\Gamma$ be the corresponding set from the Lebesgue decomposition. Define the process

$$\tilde{G}' = \xi \cdot G + I_\Gamma \cdot \tilde{G}.$$ 

We have to show that $\tilde{G}' = \tilde{G}$. Since $\tilde{G}'$ and $\tilde{G}$ are càdlàg, it is enough to show that $\tilde{G}'_t = \tilde{G}_t$ a.s. for any $t \geq 0$, and this is equivalent to having

$$E[\tilde{G}'_t I_B] = E[\tilde{G}_t I_B] \quad \text{for any } B \in \mathcal{F}_t. \quad (A.10)$$

Let $M$ be the bounded càdlàg martingale given by $M_u = E[I_B | \mathcal{F}_u]$. We have

$$E[\tilde{G}'_t I_B] = E[\tilde{G}'_t M_t] = E[M_- \cdot \tilde{G}'_t],$$

and similarly

$$E[\tilde{G}_t I_B] = E[M_- \cdot \tilde{G}_t], \quad (A.11)$$

where we used the following fact: If $A$ is a nondecreasing càdlàg predictable process and $M$ is a bounded càdlàg martingale, then for any stopping time $\tau$, we have $E[M_\tau A_\tau] = E[M_- \cdot A_\tau]$. The latter result can be found in Jacod and Shiryaev [25, Lemma I.3.12] for the case $E[A_\infty] < \infty$, from which our case follows by a localisation procedure.

Finally, from the definition of $\tilde{G}'$ and the Lebesgue decomposition of the measure $P \otimes \tilde{G}$, it follows that the measures $P \otimes \tilde{G}$ and $P \otimes \tilde{G}'$ coincide. Hence for any nonnegative $\mathcal{P}$-measurable function $f$, we have $E[f \cdot \tilde{G}_t] = E[f \cdot \tilde{G}'_t]$, which finishes the proof by (A.10) and (A.11).

(b) In view of the construction in (a), it only remains to show that if $\xi, \Gamma$ satisfy (A.9), then $\xi$ is the Lebesgue derivative and $\Gamma$ is the corresponding predictable set. This follows from straightforward verification of properties (A.3) and (A.4).

**Acknowledgements** I thank the referees for carefully reading the paper and providing valuable comments which have helped improve the quality of the paper.
References

1. Algoet, P.H., Cover, T.M.: Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. Ann. Probab. 16, 876–898 (1988)
2. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis: A Hitchhiker’s Guide, 3rd edn. Springer, Berlin (2006)
3. Alos-Ferrer, C., Ania, A.B.: The asset market game. J. Math. Econ. 41, 67–90 (2005)
4. Amir, R., Evstigneev, I.V., Hens, T., Schenk-Hoppé, K.R.: Market selection and survival of investment strategies. J. Math. Econ. 41, 105–122 (2005)
5. Amir, R., Evstigneev, I.V., Hens, T., Xu, L.: Evolutionary finance and dynamic games. Math. Financ. Econ. 5, 161–184 (2011)
6. Amir, R., Evstigneev, I.V., Schenk-Hoppé, K.R.: Asset market games of survival: a synthesis of evolutionary and dynamic games. Ann. Finance 9, 121–144 (2013)
7. Blume, L.E., Easley, D.: Evolution and market behavior. J. Econ. Theory 58, 9–40 (1992)
8. Blume, L.E., Easley, D.: Optimality and natural selection in markets. J. Econ. Theory 107, 95–135 (2002)
9. Blume, L.E., Easley, D.: If you’re so smart, why aren’t you rich? Belief selection in complete and incomplete markets. Econometrica 74, 929–966 (2006)
10. Borovička, J.: Survival and long-run dynamics with heterogeneous beliefs under recursive preferences. J. Polit. Econ. 128, 206–251 (2020)
11. Bottazzi, G., Dindo, P.: Evolution and market behavior with endogenous investment rules. J. Econ. Dyn. Control 48, 121–146 (2014)
12. Bottazzi, G., Giachini, D.: Wealth and price distribution by diffusive approximation in a repeated prediction market. Physica A 471, 473–479 (2017)
13. Coury, T., Sciubba, E.: Belief heterogeneity and survival in incomplete markets. Econ. Theory 49, 37–58 (2012)
14. Dellacherie, C., Meyer, P.A.: Probabilities and Potential B. Theory of Martingales. North-Holland, Amsterdam (1982)
15. Kallsen, J.: σ-localization and σ-martingales. Theory Probab. Appl. 48, 152–163 (2004)
16. Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. Finance Stoch. 11, 447–493 (2007)
17. Karatzas, I., Shreve, S.E.: Methods of Mathematical Finance. Springer, Berlin (1998)
18. Kelly, J.L., Jr.: A new interpretation of information rate. Bell Syst. Tech. J. 35, 917–926 (1956)
19. Liptser, R.S., Shiryaev, A.N.: Theory of Martingales. Kluwer Academic, Dordrecht (1989)
20. Palczewski, J., Schenk-Hoppé, K.R.: From discrete to continuous time evolutionary finance models. J. Econ. Dyn. Control 34, 913–931 (2010)
32. Palczewski, J., Schenk-Hoppé, K.R.: Market selection of constant proportions investment strategies in continuous time. J. Math. Econ. 46, 248–266 (2010)
33. Platen, E., Heath, D.: A Benchmark Approach to Quantitative Finance. Springer, Berlin (2006)
34. Sandroni, A.: Do markets favor agents able to make accurate predictions? Econometrica 68, 1303–1341 (2000)
35. Shapley, L., Shubik, M.: Trade using one commodity as a means of payment. J. Polit. Econ. 85, 937–968 (1977)
36. Yan, H.: Natural selection in financial markets: does it work? Manag. Sci. 54, 1935–1950 (2008)
37. Zhitlukhin, M.: Survival investment strategies in a continuous-time market model with competition. Int. J. Theor. Appl. Finance 24, 2150001 (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.