On the Closed Graph Theorem and the Open Mapping Theorem

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Abstract

Let $E, F$ be two topological spaces and $u : E \to F$ be a map. If $F$ is Hausdorff and $u$ is continuous, then its graph is closed. The Closed Graph Theorem establishes the converse when $E$ and $F$ are suitable objects of topological algebra, and more specifically topological groups, topological vector spaces (TVS’s) or locally vector spaces (LCS’s) of a special type. The Open Mapping Theorem, also called the Banach-Schauder theorem, states that under suitable conditions on $E$ and $F$, if $v : F \to E$ is a continuous linear surjective map, it is open. When the Open Mapping Theorem holds true for $v$, so does the Closed Graph Theorem for $u$. The converse is also valid in most cases, but there are exceptions. This point is clarified. Some of the most important versions of the Closed Graph Theorem and of the Open Mapping Theorem are stated without proof but with the detailed reference.

1 Introduction

Let $E, F$ be two topological spaces and $u : E \to F$ be a map. If $F$ is Hausdorff and $u$ is continuous, then its graph is closed (see Lemma 2 below). The Closed Graph Theorem establishes the converse when $E$ and $F$ are suitable objects of topological algebra, and more specifically topological groups, topological vector spaces (TVS’s) or locally vector spaces (LCS’s) of a special type. Banach’s theorem states that when $E$ and $F$ are Fréchet spaces and $u$ is linear, this map is continuous if, and only if its graph is closed ([3], p. 41, Thm. 7). The Open Mapping Theorem, also called the Banach-Schauder theorem, states that under

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suitable conditions on $E$ and $F$, if $v : F \to E$ is a continuous linear surjective map, it is open, i.e. for every open subset $\Omega$ of $F$, $v(\Omega)$ is an open subset of $E$ ([27], p. 6, Satz 2). These two results are easy consequences of Banach’s isomorphism theorem ([2], p. 238, Thm. 7).

All these results still hold true when $E, F$ are metrizable complete TVS’s over a non-discrete valued division ring $k$ [7].

Several generalizations of the Closed Graph Theorem and the Open Mapping Theorem have been established when $k = \mathbb{R}$ or $\mathbb{C}$ and when $E, F$ are special kinds of LCS’s over $k$, or when $E, F$ are suitable topological groups. When the Open Mapping Theorem holds true for $v$, so does the Closed Graph Theorem for $u$. The converse is also valid in most cases, but there are exceptions, e.g. Pták’s classical result [25] which can be stated as follows: (i) Let $E$ be barrelled and $F$ be infra-Pták (i.e. $B_r$-complete). A $k$-linear map $u : E \to F$ is continuous if its graph is closed. (ii) Let $F$ be Pták (i.e. $B$-complete) and $E$ be barrelled, then a $k$-linear continuous surjection $v : F \to E$ is open. A Pták space is infra-Pták ([25], (4.2)), but Valdivia has shown in [30] that there exist infra-Pták spaces which are not Pták. If ”Pták” is replaced by ”infra-Pták” in (ii), this statement is no longer correct.

We review in this paper some of the most important versions of the Closed Graph Theorem and of the Open Mapping Theorem. Their connection is expressed in Theorem 14 below (Section 3). Some preliminaries are given in Section 2. A typology of topological spaces, topological groups and topological LCS’s is given in Section 4. Versions of the Closed Graph Theorems and of the Open Mapping Theorems in topological groups and in LCS’s are stated in Sections 5 and 6 respectively.

2 Preliminaries

2.1 Continuity and closed graph

In this paper, we use nets [18] rather than filters [5] in topological spaces. These two notions can be used in an equivalent way for convergence studies, and they are connected as follows: denoting by $I$ a preordered set, with preorder relation $\succeq$, a net $(x_i)_{i \in I}$ in $E$ is a map $I \to E : i \mapsto x_i$. The family of sets $\mathcal{B} = (B_i)_{i \in I}$ where $B_i = \{x_i \in E ; j \succeq i\}$ is a base of the elementary filter associated with $(x_i)_{i \in I}$, and $(x_i)$ converges to $x \in E$ if, and only if $\lim_{B} = x$. Conversely, let $\mathcal{B}$ be a filter base and, using the choice axiom, for every $B \in \mathcal{B}$ let $x_B \in B$. Then $(x_B)_{B \in \mathcal{B}}$ is a net associated to $\mathcal{B}$ and $\lim_{\mathcal{B}} = x$ if, and only if every net associated to $\mathcal{B}$ converges to $x$. A sequence is a net with a countable set of indices which can be identified with $\mathbb{N}$.

Let $E, F$ be two topological spaces and $u : E \to F$ be a map. The
map $u$ is continuous (resp. sequentially continuous) if, and only if for any point $x \in E$, whenever $(x_i)$ is a net (resp. a sequence) in $E$ converging to $x$, the net (resp. the sequence) $(u(x_i))$ converges to $u(x)$. The graph $Gr(u)$ of $u$ is closed (resp. sequentially closed) if, and only if whenever $(x_i, u(x_i))$ is a net (resp. a sequence) in $E \times F$ converging to $(x, y)$, necessarily $(x, y) \in Gr(u)$, i.e. $y = u(x)$. If $E$ is a topological group, the point $x$ here above can be replaced by the neutral element $e$.

**Lemma 1** Let $E, F$ be two topological spaces, $u : E \to F$ be a map, and $Gr(u)$ be its graph. Then $u$ is continuous if, and only if the map $p : Gr(u) \to E : (x, f(x)) \mapsto x$ is open.

**Proof.** The map $p$ is bijective with inverse $p^{-1} : x \mapsto (x, f(x))$. Thus $p^{-1}$ is continuous (i.e., $p$ is open) if, and only if $f$ is continuous. ■

The following result is a slight generalization of ([5], p. I.53, Cor. 2 of Prop. 2):

**Lemma 2** Let $E, F$ be two topological spaces and $u : E \to F$ be a map. If $F$ is Hausdorff and $u$ is continuous (resp. sequentially continuous), then $Gr(u)$ is closed (resp. sequentially closed) in $E \times F$.

**Proof.** Let $((x_i, u(x_i)))_{i \in I}$ be a net (resp. a sequence) in $Gr(u)$ converging to $(x, y) \in E \times F$. Then, $(x_i) \to x$. If $u$ is continuous (resp. sequentially continuous), this implies $(u(x_i)) \to u(x)$. If $F$ is Hausdorff, $u(x) = y$, thus $(x, y) \in Gr(u)$ and $Gr(u)$ is closed (resp. sequentially closed). ■

The Closed Graph Theorem is the converse of Lemma 2 when $E, F$ and $u$ satisfy the suitable conditions.

Our first example will be:

**Lemma 3** Let $E, F$ be two topological spaces, $u : E \to F$ be a map, and assume that $F$ is (Hausdorff and) compact. Then $u$ is continuous if, and only if $Gr(u)$ is closed.

**Proof.** (If): Let $(x_i)$ be a net converging to $x$ in $E$. Since $F$ is compact, $(u(x_i))$ has a cluster point in $F$, thus $((x_i, u(x_i)))$ has a cluster point $(x, y)$ in $E \times F$. Since $Gr(u)$ is closed, $(x, y) \in Gr(u)$, i.e. $y = u(x)$. Therefore $u(x)$ is the unique cluster point of $(u(x_i))$ and $(u(x_i)) \to u(x)$, which implies that $u$ is continuous. ■

For the second example we will give, the following is necessary:

**Definition 4** A subset $A$ of a topological space $X$ is called nearly open if $A \subset \overline{A}$, i.e. $A$ belongs to the interior of its closure. Let $u : X \to Y$
where $X$ and $Y$ are topological spaces. The map $u$ is called nearly open (resp. nearly continuous) if for any open set $\Omega$ in $X$ (resp. $Y$), $u(\Omega)$ (resp. $u^{-1}(\Omega)$) is nearly open. (In the above, the term "almost" is sometimes used in place of "nearly").

The following result, the proof of which is less trivial than that of Lemma 3, is due to Weston and Pettis [31], [24]:

**Theorem 5** Let $X, Y$ be two metrizable complete topological spaces. Let $u : X \to Y$. The following conditions are equivalent:
(a) $u$ is continuous.
(b) $\text{Gr}(u)$ is closed and $u$ is nearly continuous.

### 2.2 Inductive limits and locally convex hulls

**Notation 6** $\text{TCat}$ is a category used in topological algebra and is a subcategory of $\text{TSet}$, the category of topological spaces ([5], Chap. II); there exists a forgetful functor $\text{Forget} : \text{TCat} \to \text{Cat}$ where $\text{Cat}$ is a subcategory of $\text{Set}$, the category of sets. The three cases considered below are:
(1) $\text{TGrp} \xrightarrow{\text{Forget}} \text{Grp}$ where $\text{TGrp}$ is the category of topological groups, and $\text{Grp}$ is the category of groups;
(2) $\text{TVsp}_k \xrightarrow{\text{Forget}} \text{Vsp}_k$ where $\text{TVsp}_k$ the category of right topological spaces over a topological division ring $k$ and $\text{Vsp}_k$ is the category of right vector spaces over the division ring $k$;
(3) $\text{LCS} \xrightarrow{\text{Forget}} \text{Vsp}_k$ where $\text{LCS}$ the category of locally convex spaces over $k = \mathbb{R}$ or $\mathbb{C}$;
For short, a morphism of $\text{Cat}$ (resp. $\text{TCat}$) is called a map (resp. a morphism).

Let us recall the general notions of inductive (or direct) limit and of filtrant inductive limit ([22], Chap 2 & 3).

Let $C$ be a category and $(F_\alpha)_{\alpha \in A}$ be a family of objects of $C$. An inductive limit $\lim \overrightarrow{F_\alpha}$ is (if it exists) a functor $C \to \mathcal{C}$, uniquely determined up to isomorphism, consisting of one object and a family of morphisms $\varphi_\alpha : F_\alpha \to \lim \overrightarrow{F_\alpha}$ such that for any object $X$ and morphisms $f_\alpha : F_\alpha \to X$, there exists a morphism $f : \lim \overrightarrow{F_\alpha} \to X$ such that $f_\alpha$ factors through $f$ according to $f_\alpha = f \circ \varphi_\alpha$.

Let $A$ be equipped with a binary relation $\preceq$. The latter is a preorder relation if (and only if) it is transitive and reflexive. The preorder set $A$ is called right filtrant if each pair of elements has an upper bound. Let $(F_\alpha)_{\alpha \in A}$ be a family of objects of $C$ indexed by the right-filtrant set
A and, for each pair \((\alpha, \beta) \in A \times A\) such that \(\alpha \preceq \beta\), let \(\varphi^\alpha_\beta : F_\alpha \to F_\beta\) be a morphism such that \(\varphi^\alpha_\beta = \text{id}_{F_\alpha}\) and \(\varphi^\alpha_\beta = \varphi^\alpha_\gamma \circ \varphi^\beta_\alpha\) for all \(\alpha, \beta, \gamma \in A\) such that \(\alpha \preceq \beta \preceq \gamma\). Then \((F_\alpha, \varphi^\alpha_\beta)\) is called a direct system with index set \(A\). The inductive limit \(G = \lim_{\to} G_\alpha\) (if it exists) is called filtrant if the compatibility relation \(\varphi_\alpha = \varphi_\beta \circ \varphi^\alpha_\beta\) holds whenever \(\alpha \preceq \beta\).

### 2.2.1 Inductive limits of topological groups

Let \(\text{TGr} = \text{TGr}\). A topology on a group \(G\) which turns \(G\) into a topological group will be called called a group-topology.

Let \(A\) be a set, \(G\) be a topological group, and \((G_\alpha, \varphi_\alpha)_{\alpha \in A}\) be a family of pairs of topological groups and morphisms \(G_\alpha \to G\) such that \(G = \bigcup_{\alpha \in A} \varphi_\alpha(G_\alpha)\). Then \(G\) is the inductive limit of the family \((G_\alpha)\) relative to the group-maps \(\varphi_\alpha\) in \(\text{Grp}\) (written \(G = \lim_{\to} \varphi_\alpha :[\varphi_\alpha, \text{Grp}]\)).

Let \(\Sigma_\alpha\) be the topology of \(G_\alpha\). The final group-topology \(\Sigma\) of the family \((\Sigma_\alpha)\) relative to the group-maps \(\varphi_\alpha\) is defined to be the finest group-topology which makes all maps \(\varphi_\alpha\) continuous. This group topology exists, i.e. inductive limits exist in \(\text{TGr}\). Nevertheless, this intricate group-topology is not Hausdorff in general, even if the topologies of the topological groups \(G_\alpha\) are Hausdorff ([1], p. 132) and Abelian [1].

**Lemma and Definition 7** The following conditions are equivalent:

(i) \(G\) is endowed with the above final group-topology \(\Sigma\) (the topological group obtained in this way is written \(G[\Sigma]\), or \(G\) when there is no confusion).

(ii) \(G[\Sigma]\) is the inductive limit of the family \((G_\alpha)\) relative to the group-maps \(\varphi_\alpha\) in \(\text{TGr}\) (written \(G[\Sigma] = \lim_{\to} \varphi_\alpha :[\varphi_\alpha, \text{TGr}]\)), i.e. a map \(\rho : G \to H\), where \(H\) is any topological group, is a morphism in \(\text{TGr}\) if, and only if for each \(\alpha \in A\), \(\rho \circ \varphi_\alpha : G_\alpha \to H\) is a morphism.

Since the groups \(\varphi_\alpha(G_\alpha)\) and \(G_\alpha/\ker \varphi_\alpha\) are canonically isomorphic, they are identified; then the group-map \(\varphi_\alpha\) and the canonical surjection \(\pi_\alpha : G_\alpha \twoheadrightarrow G_\alpha/\ker \varphi_\alpha\) are identified too. The final group-topology \(\Sigma^g_\alpha\) on \(G_\alpha/\ker \varphi_\alpha\) relative to \(\pi_\alpha\) (resp. the final group-topology \(\Sigma^f_\alpha\) on \(\varphi_\alpha(G_\alpha)\) relative to \(\varphi_\alpha\)) is the finest group-topology on \(G_\alpha/\ker \varphi_\alpha\) (resp. \(\varphi_\alpha(G_\alpha)\)) which makes \(\pi_\alpha\) (resp. \(\varphi_\alpha\)) continuous. Obviously, \(\Sigma^g_\alpha\) and \(\Sigma^f_\alpha\) coincide and \(G = \lim_{\to} \varphi_\alpha(G_\alpha) [j_\alpha, \text{TGr}]\) where \(j_\alpha : \varphi_\alpha(G_\alpha) \hookrightarrow G\) is the inclusion.

**Corollary and Definition 8** (i) Let \(A\) be right-filtrant and let \((G_\alpha, \varphi^\alpha_\beta)\) be a direct system in \(\text{TGr}\). Let \(G = \lim_{\to} [\varphi_\alpha, \text{TGr}]\) be the filtrant inductive limit of this system and \(\alpha \preceq \beta\). The compatibility relation \(\varphi_\alpha = \varphi_\beta \circ \varphi^\alpha_\beta\) implies \(\varphi_\alpha(G_\alpha) \subset \varphi_\beta(G_\beta)\). The group-map
\( j^\alpha_\beta : \varphi_\alpha(G_\alpha) \to \varphi_\beta(G_\beta) \) induced by \( \varphi_\beta^\alpha \) (for \( \alpha \preceq \beta \)) is injective and continuous and is called the canonical monomorphism. Therefore the topological group \( G \) is the filtrant inductive limit of the direct system \((\varphi_\alpha(G_\alpha), j^\alpha_\beta)\).

(ii) An inductive limit of topological groups is called countable if the set of indices \( A \) is countable.

(iii) An inductive limit of topological groups is called strict if it is filtrant, \( \varphi_\beta^\alpha \) is the inclusion \( G_\alpha \hookrightarrow G_\beta \), \( \varphi_\alpha \) is the inclusion \( G_\alpha \hookrightarrow G \), and the topology of \( T_\alpha \) and that induced in \( G_\alpha \) by \( T_\beta \) coincide whenever \( \alpha \preceq \beta \).

Lemma 9 Let \( N \) be a normal subgroup of \( G = \varprojlim G_\alpha [\varphi_\alpha, \text{TGr}] \), let \( N_\alpha = \varphi_\alpha^{-1}(N) \) and let \( \bar{\varphi}_\alpha : G_\alpha/N_\alpha \to G/N \) be the induced map. (i) Then
\[
\varprojlim G_\alpha/N_\alpha [\bar{\varphi}_\alpha, \text{TGr}] = G/N.
\]

(ii) Considering in \( \text{TGr} \) a direct system \((G_\alpha, \varphi_\beta^\alpha)_{\alpha \preceq \beta, \alpha, \beta \in A} \) and the filtrant inductive limit \( G = \varprojlim G_\alpha \), the map \( \bar{\varphi}_\beta^\alpha : F_\alpha/N_\alpha \to F_\beta/N_\beta \) induced by \( \varphi_\beta^\alpha \) exists and is a morphism. In addition, \((G_\alpha/N_\alpha, \bar{\varphi}_\beta^\alpha)_{\alpha \preceq \beta, \alpha, \beta \in A} \) is a direct system in \( \text{TGr} \) whose filtrant inductive limit is \( G/N \).

Proof. (i) Consider the commutative diagram
\[
\begin{array}{ccc}
G_\alpha & \xrightarrow{\varphi_\alpha} & G \\
\pi_\alpha \downarrow & & \downarrow \pi \\
G_\alpha/N_\alpha & \xrightarrow{\bar{\varphi}_\alpha} & G/N
\end{array}
\]
where \( \pi_\alpha : G_\alpha \to G_\alpha/N_\alpha \) and \( \pi : G \to G/N \) are the canonical epimorphisms. We have
\[
G/N = \varprojlim G [\pi, \text{TGr}] = \varprojlim \left( \varprojlim G_\alpha [\varphi_\alpha, \text{TGr}] \right) [\pi, \text{TGr}],
\]
therefore (\cite{I}, CST19, p. IV.20)
\[
G/N = \varprojlim G_\alpha [\pi \circ \varphi_\alpha, \text{TGr}] = \varprojlim G_\alpha [\bar{\varphi}_\alpha \circ \pi_\alpha, \text{TGr}]
= \varprojlim G_\alpha/N_\alpha [\bar{\varphi}_\alpha, \text{TGr}].
\]

(ii) We know that \( \varphi_\alpha = \varphi_\beta \circ \varphi_\beta^\alpha \) whenever \( \alpha \preceq \beta \). Let \( x \in N_\alpha \), i.e. \( \varphi_\alpha(x) \in N \); equivalently, \( \varphi_\beta(\varphi_\beta^\alpha(x)) \in N \), i.e. \( \varphi_\beta^\alpha(x) \in N_\beta \). Therefore, \( \varphi_\beta^\alpha(N_\alpha) \subset N_\beta \), and the map \( \bar{\varphi}_\beta^\alpha : F_\alpha/N_\alpha \to F_\beta/N_\beta \) induced by \( \varphi_\beta^\alpha \) exists. This map is easily seen to be continuous, and \((F_\alpha/N_\alpha, \bar{\varphi}_\beta^\alpha)_{\alpha \preceq \beta, \alpha, \beta \in A} \) is a direct system in \( \text{TGr} \). By (i), the filtrant inductive limit of that direct system is \( G/N \).
2.2.2 Inductive limits of locally convex spaces

The above definitions, rationales and results still hold, mutatis mutandis, in the category LCS. Final topologies hence inductive limits exist in that category. Moreover – contrary to what happens in TGrp – these topologies are easily described ([7], p. II.29). The following is classical:

**Proposition 10** Assuming that $F = \lim_{\rightarrow} F_n$ is a strict countable inductive limit, then (i) the topology induced in $F_n$ by that of $F$ coincides with that of $F$ (so that the topology of $F$ is Hausdorff if the topologies of the $F_n$ is Hausdorff), (ii) if $F_n$ is closed in $F_{n+1}$ for every $n$, then $F_n$ is closed in $F$, (iii) if $F_n$ is complete for every $n$, then $F$ is complete ([7], §II.4, Prop. 9). In addition ([7], §II.4, Exerc. 14) the topology of $F$ is the finest topology among all topologies compatible with the vector space structure of $F$ (locally convex or not) which induce in $F_n$ a coarser topology than the given topology $T_n$.

In what follows, we will say that the strict inductive limit $F = \lim_{\rightarrow} F_n$ is nontrivial if $F_n \subsetneq F_{n+1}$.

2.2.3 Locally convex hulls of locally convex spaces

Let $(F_\alpha)_{\alpha \in A}$ be a family of LCS’s, $F$ be an LCS, and $(\varphi_\alpha)_{\alpha \in A}$ be a family of linear maps $F_\alpha \to F$. Then, algebraically,

$$E := \sum_{\alpha \in A} \varphi_\alpha (F_\alpha) \cong \left( \bigoplus_{\alpha \in A} F_\alpha \right) / H$$

where $H = \ker (\varphi)$. $\varphi : (x_\alpha)_{\alpha \in A} \in \bigoplus_{\alpha \in A} F_\alpha \to \sum_\alpha \varphi_\alpha (F_\alpha)$. The hull topology of $E$ is defined to be the finest locally convex topology which makes all maps $\varphi_\alpha$ continuous; $E$, endowed with this topology, is called the locally convex hull of $(F_\alpha, \varphi_\alpha)$ ([19], p. 215). Then (1) is an isomorphism of LCS’s, $H$ is a subspace of $\bigoplus_{\alpha \in A} F_\alpha$, and $H$ is closed if, and only if $E$ is Hausdorff.

Conversely, let $(F_\alpha)_{\alpha \in A}$ be a family of LCS’s and $H$ be a subspace of $\bigoplus_{\alpha \in A} F_\alpha$. Then $E = \left( \bigoplus_{\alpha \in A} F_\alpha \right) / H$ is a locally convex hull of $(F_\alpha, \varphi_\alpha)$ where $\varphi_\alpha = \varphi | F_\alpha : \bigoplus_{\alpha \in A} F_\alpha \to E$ (canonical map).

Assuming that $(F_\alpha, \varphi_\alpha^\alpha)_{\alpha \in A}$ is a direct system and $\varphi_\alpha (F_\alpha) \subset \varphi_\beta (F_\beta)$ for $\alpha \ll \beta$, the locally convex hull $\sum_{\alpha \in A} \varphi_\alpha (F_\alpha) = \bigcup_{\alpha \in A} \varphi_\alpha (F_\alpha)$ coincides with $\lim_{\rightarrow} F_\alpha$. Since $\varphi_\beta^\alpha$ is continuous, the canonical injection $j_\beta^\alpha : \varphi_\alpha (F_\alpha) \to \varphi_\beta (F_\beta)$ is continuous. Therefore, a locally convex hull is a generalization of an inductive limit in the category LCS.
3 Properties $\mathcal{G}_C$, $\mathcal{O}_C$ and their relations

Let $E, F$ be topological spaces and $\mathcal{C} \subset \mathcal{P}(E \times F)$ be such that

$$\{\text{closed subsets of } E \times F\} \subset \mathcal{C}$$

For example, $\mathcal{C}$ can be the set of all closed, or of all sequentially closed, or of all Borel subsets of $E \times F$.

Let $\mathbf{T\text{Cat}}$ be any of the categories $\mathbf{T\text{Grp}}$, $\mathbf{T\text{Vsp}}_k$, and $\mathbf{LCS}$; then the notion of strict morphism is well-defined in $\mathbf{T\text{Cat}}$ [5, p. III.16, Def. 1]. A morphism $f : E \to F$, where $E, F \in \mathbf{T\text{Cat}}$, is strict if, and only if for any open subset $\Omega$ of $E$, $f(\Omega)$ is open in $f(E)$. Therefore, if $f$ is a surjective morphism, $f$ is strict if, and only if it is an open mapping.

**Definition 11** $\mathcal{G}_C$ and $\mathcal{O}_C$ are the following properties of pairs of objects of $\mathbf{T\text{Cat}}$:

(i) $(E, F)$ is $\mathcal{G}_C$ if whenever a map $u : E \to F$ is such that $\text{Gr}(u) \in \mathcal{C}$, then $u$ is a morphism.

(ii) $(E, F)$ is $\mathcal{O}_C$ if whenever $G \in \mathbf{T\text{Cat}} \cap \mathcal{C}$, $G \subset E \times F$, and $v : G \to E$ is a surjective morphism, then $v$ is strict.

**Example 12** Let $\mathbf{T\text{Cat}} = \mathbf{T\text{Vsp}}_k$, let $E, F$ be two metrizable complete TVS’s and let $\mathcal{C}$ be the set of all closed subsets of $E \times F$.

(1) Let $u : E \to F$ be a map. If $\text{Gr}(u) \in \mathcal{C}$, $u$ is a morphism by the usual Closed Graph Theorem, thus $(E, F)$ is $\mathcal{G}_C$.

(2) Every space $G \in \mathcal{C}$ is metrizable and complete, thus every surjective morphism $v : G \to E$ is strict by the usual Open Mapping Theorem, thus $(E, F)$ is $\mathcal{O}_C$.

**Lemma 13** If $E$ is Hausdorff, $(E, F)$ is $\mathcal{G}_C$, and $v : F \to E$ is a bijective morphism, then $v$ is an isomorphism.

**Proof.** The graph of $v$ is closed (Lemma 2), thus so is the graph of $v^{-1} : E \to F$. Therefore, this graph is $\mathcal{C}$ in $E \times F$, and since $(E, F)$ is $\mathcal{G}_C$, $v^{-1}$ is continuous. $\blacksquare$

The relations between $\mathcal{G}_C$ and $\mathcal{O}_C$ are expressed in the following theorem when $\mathbf{T\text{Cat}} = \mathbf{T\text{Grp}}$:

**Theorem 14** Let $E, F$ be topological groups.

(i) $\mathcal{O}_C \Rightarrow \mathcal{G}_C$.

(ii) Conversely, if $E$ is Hausdorff and $(E, F/N)$ is $\mathcal{G}_C$ for every closed normal subgroup $N$ of $F$, then every surjective morphism $v : F \to E$ is strict.

(iii) Therefore whenever a morphism $v : F \to E$ is such that $(E, \text{im}(v)/\text{im}(N \cap \text{im}(v)))$ is $\mathcal{G}_C$ for every normal subgroup $N$ of $F$ such that $N \cap \text{im}(v)$ is closed in $\text{im}(v)$, this morphism $v$ is strict.
Proof. (i): This proof is a variant of that of [28], Prop. 17.3. Let \( u : E \to F \) be a map (i.e. a morphism of groups). Its graph \( \text{Gr}(u) \) is a subgroup \( G \) of \( E \times F \). Let us endow \( E \times F \) with the product topology \( \pi \) and \( G \) with the topology induced by \( \pi \). Then the two projections

\[
p : G \to E : (x, u(x)) \mapsto x, \quad q : G \to F : (x, u(x)) \mapsto u(x)
\]

are continuous and

\[u = q \circ p^{-1}.
\]

Assume that \((E, F)\) is \( \mathcal{O}_C \). Then, if \( \text{Gr}(u) \in \mathcal{C} \), \( p \) is open, thus \( p^{-1} \) is continuous, and \( u \) is continuous too. Therefore \((E, F)\) is \( \mathcal{G}_C \).

(ii): Let \( v : F \twoheadrightarrow E \) be a surjective morphism and consider the following commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{v} & E \\
\downarrow{\pi} & & \uparrow{\bar{v}} \\
F/\ker(v) & \end{array}
\]

where \( \bar{v} : F/\ker(v) \to E \) is a bijective morphism.

Assume that \( E \) is Hausdorff. Then the normal subgroup \( N = \ker(v) \) of \( F \) is closed. If in addition \((E, F/N)\) is \( \mathcal{G}_C \), \( \bar{v} \) is an isomorphism by Lemma 13. As a result, the morphism \( v \) is strict.

(iii) The subgroup \( N' = N \cap \text{im}(v) \) is normal in \( F' = \text{im}(v) \), thus (iii) is a consequence of (ii).

4 Typology of topological spaces, groups and vector spaces

4.1 Typology of topological spaces

A subspace \( A \) of a topological space \( X \) is rare when \( X \setminus A \) is everywhere dense (in other words, \( A \) is empty). A subspace \( B \) of \( X \) is meagre (or ”of first category”) if it is a countable union of rare subspaces. A subset of \( X \) is called non-meagre (or ”of second category”) when it is not meagre (i.e. when it is not of first category). The following is classical ([6], p. IX.54, Def. 3).

Lemma and Definition 15 Let \( X \) be a topological space.

(1) The following conditions are equivalent:

(a) every open subset of \( X \) which is non-empty is non-meagre;
(b) if a subset \( B \) is meagre, then \( X \setminus B \) is everywhere dense, i.e. \( \hat{B} \) is empty;
(c) every countable union of closed sets with empty interior has an empty
interior;
(c') every countable intersection of open everywhere dense subsets in $X$ is everywhere dense in $X$.

(2) A topological space $X$ is Baire if the above equivalent conditions hold.

A complete metrizable topological space is Baire, and a locally compact topological space as well ("Baire’s theorem").

**Definition 16** A topological space $X$ is

- first countable (or satisfies the first axiom of countability) if every point has a countable base of neighborhoods;
- second countable (or satisfies the second axiom of countability) if its topology has a countable base;
- separable if there exists a countable dense subspace.

In general,

second countable $\Rightarrow$ first countable and separable

Let $X$ be a topological space and $\mathcal{P}(X)$ (resp. $\mathcal{K}(X)$) be the set of all subsets (resp. of all compact subsets) of $X$. The topological space $X$ is

- **completely regular** if it is uniformizable and Hausdorff ([9], p. IX.8, Def. 4),

- **inexhaustible** if for every countable family of closed subsets $A_n \subset X$, the relation $X = \bigcup_n A_n$ implies that there exists $n$ such that $A_n$ has nonempty interior ([15], Def. 3.3.a).

- **Polish** if it is metrizable, complete and separable ([9], p. IX.57, Def. 1),

- **Suslin** if it is Hausdorff and there exist a Polish space $P$ and a continuous surjection $P \to X$ ([9], p. IX.59, Def. 1),

- **Lindelöf** if, from any open covering, one can extract a countable covering ([9], p. IX.76, Def. 1),

- **$K_{\sigma\delta}$** if it is a countable intersection of countable unions of compact sets ([8], §3),

- **$K$-analytic** if there exists a $K_{\sigma\delta}$ space $Y$ and a continuous surjection $f : Y \to X$. 

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The definitions given below are due to Martineau [21] (K-Suslin spaces) and Valdivia (29, p. 52) (quasi-Suslin spaces).

**Definition 17** A topological space $X$ is called

(1) K-Suslin if there exist a Polish space $P$ and a map $T : P \rightarrow \mathcal{S}(X)$ such that

(i) $X = \bigcup_{p \in P} T(p)$, and

(ii) $T$ is semi-continuous, i.e. for every $p \in P$, and for every neighborhood $V$ of $T(p)$ in $X$, there exists a neighborhood $U$ of $p$ in $P$ such that $T(U) \subset V$;

(2) quasi-Suslin if there exist a Polish space $P$ and a map $T : P \rightarrow \mathcal{P}(X)$ such that

(i') $X = \bigcup_{p \in P} T(p)$, (ii') if $(p_n)$ is a sequence in $P$ converging to $p$ and if $x_n \in T(p_n)$ for every $n \geq 0$, then $(x_n)$ has an cluster point in $X$ belonging to $T(p)$.

A completely regular topological space is K-Suslin if, and only if it is K-analytic [26].

For a completely regular topological space,

| metrizable quasi-Suslin | Polish | Suslin | \(\Rightarrow\) | K-Suslin | \(\Rightarrow\) | quasi-Suslin |
|------------------------|--------|--------|-----------------|----------|-----------------|-----------------|
| Baire                  | separable |        | K-analytic      |          |                 |                 |
| inexhaustible \(\Rightarrow\) 2nd category \(\Leftarrow\) Baire  K-Suslin |

and

| K-Suslin | \(\Rightarrow\) | Lindelöf | \(\Rightarrow\) | paracompact |
|----------|-----------------|----------|-----------------|-------------|
| quasi-Suslin Lindelöf | 2nd countable \(\Rightarrow\) 1st countable separable |

**Proof.** Polish \(\Rightarrow\) Suslin \(\Rightarrow\) K-Suslin \(\Rightarrow\) quasi-Suslin is clear [21], (29, p. 60, (3)). Metrizable quasi-Suslin \(\Rightarrow\) K-Suslin by (29, p. 67, (26)). Polish \(\Rightarrow\) Baire (Baire’s theorem) \(\Rightarrow\) nonmeagre in itself \(\Leftrightarrow\) inexhaustible (6, p. IX.112, Exerc. 7) \(\Rightarrow\) of second category. K-Suslin \(\Rightarrow\) Lindelöf \(\Rightarrow\) paracompact by (21, Prop. 2) and (6, p. IX.76, Prop. 2). 2nd countable \(\Rightarrow\) Lindelöf and 2nd countable \(\Rightarrow\) 1st countable separable according to the definitions. A space is K-Suslin if, and only if it is quasi-Suslin and Lindelöf (23, p. 148).

In addition, every space which is metrizable, locally compact and countable at infinity is Polish.
A subset \( S \) of a topological space is called \( \text{Borel} \) (resp. \( \text{sq-Borel} \)) if it belongs to the smallest \( \sigma \)-algebra \( \mathfrak{B} \) of \( X \) generated by the closed (resp. sequentially closed) subsets of \( X \) ([9], p. 130, Def. 7.2.6). (A Borel set is sq-Borel.)

Let \( S \) be a subset of a topological space \( X \); \( S \) is called \( F\)-Borel if it belongs to the smallest subset \( F\mathfrak{B} \) of \( \mathcal{P}(X) \) which is stable by countable union, countable intersection and which contains the closed subsets of \( X \) ([21]). Note that \( F\mathfrak{B} \) is not a \( \sigma \)-algebra in general, for \( A \in F\mathfrak{B} \) does not imply \( X \setminus A \in F\mathfrak{B} \) (in particular, open subsets of \( X \) need not belong to \( F\mathfrak{B} \)). Obviously, \( F\mathfrak{B} \subset \mathfrak{B} \).

Considering Hausdorff spaces, the following hereditary properties hold ([6],§IX.6), ([29], §I.4):

- An open subset of a Polish (resp. Suslin, Baire) space is such.
- A closed subspace of a Suslin (resp. \( K \)-Suslin, quasi-Suslin) space is such. Moreover, a sequentially closed subspace of a Suslin space is Suslin.
- A sequentially closed subspace and the image by a linear continuous surjection of a semi-Suslin space are again semi-Suslin.
- A countable sum or product of Polish (resp. Suslin) spaces is again such.
- A countable union of Suslin spaces is again Suslin.
- A finite product of \( K \)-Suslin (resp. quasi-Suslin) spaces is again such.
- A countable product, a countable intersection, a countable projective limit of Suslin (resp. \( K \)-Suslin, quasi-Suslin, semi-Suslin) spaces is such.
- A countable intersection of Polish spaces is again Polish.
- A product of metrizable complete spaces is a Baire space (but a product of Baire spaces need not be Baire).
- The image by a continuous surjection of a Suslin (resp. \( K \)-Suslin, quasi-Suslin, Lindelöf) space is such. Moreover the image of a Suslin space by a sequentially continuous surjection is Suslin.
- If \( (X_n) \) is a sequence of \( K \)-Suslin (resp. quasi-Suslin) spaces covering \( X \), then \( X \) is \( K \)-Suslin (resp. quasi-Suslin).
• sq-Borel subspaces of Suslin spaces are Suslin ([9], p. 130, Prop. VII.2.7).

• A space which contains a dense Baire subspace is Baire [14].

• $F$-Borel subspaces of $K$-Suslin spaces are $K$-Suslin [21].

4.2 Typology of topological groups

A topological group is uniformizable, thus a Hausdorff topological group
is completely regular. A topological group $G$ is metrizable if, and only
if it is Hausdorff and first countable ([6], p. IX.23, Prop. 1).

**Lemma 18** A topological group is second countable if, and only if it is
both first countable and separable ([13], Exerc. E3.3).

Consider a Hausdorff topological group $G$ and a subgroup $H$. The
homogeneous space $G/H$ is Hausdorff if, and only if $H$ a closed subspace
of $G$. Since $G/H$ is the image of the continuous canonical map $G \to G/H$
we see that

• The quotient of a Suslin (resp. $K$-Suslin, quasi-Suslin, Lindelöf)
group by a closed subgroup is Suslin (resp. $K$-Suslin, quasi-Suslin,
Lindelöf).

Let $(G_\alpha)$ be a family of topological groups, and $G = \lim \rightarrow G_\alpha$ in $\text{TGr}$. The topological group $G$ is called

• of type $\beta$ if the groups $G_\alpha$ are Suslin and Baire.

• of type $K.\beta$ if the groups $G_\alpha$ are $K$-Suslin and Baire [21].

A metrizable separable group is second countable, thus Lindelöf. If
in addition it is locally compact and countable at infinity, it is Polish.

4.3 Typology of locally convex spaces

Let $E$ be a TVS and $A$ be a subset of $E$. This set is called $CS$-compact
if every series of the form

$$\sum_{n \geq 0} a_n x_n, \quad x_n \in A, \quad a_n \geq 0, \quad \sum_{n \geq 0} a_n = 1$$

converges to a point of $A$. If $E$ is an LCS, every CS-compact set is
bounded and convex. The set of all CS-compact subsets of $E$ is denoted
by $\mathcal{S}(E)$.

The following definition is due to Valdivia ([29], p. 78):
Definition 19 A locally convex space $E$ is called semi-Suslin if it is locally convex and there exist a Polish space $P$ and a map $T : P \to \mathcal{S}(E)$ such that

(i") $E = \bigcup_{p \in P} T(p)$,

(ii") if $(p_n)$ is a sequence in $P$ converging to $p$, there exists $A \in \mathcal{S}(E)$ such that $T(p_n) \in A$ for all $n$.

For an LCS ([29], p. 80, (3)) every quasi-Suslin and locally complete is semi-Suslin. A Fréchet space and the strong dual of a metrizable locally convex space are semi-Suslin ([29], p. 81). Conversely, a convex-Baire semi-Suslin space is Fréchet ([29], p. 92, (26)).

A Hausdorff LCS is called ultrabornological if it is an inductive limit of Banach spaces. A Fréchet space, the strong dual of a reflexive Fréchet space, a quasi-complete bornological space, an inductive limit of ultrabornological spaces, are all ultrabornological ([7], n°III.6.1 and §III.4, Exercise 20 & 21; [13], Chap. 4, Part 1, §5). A $\beta$ LCS is ultrabornological, but not conversely.

A $K.\beta$ LCS is called $K$-ultrabornological.

A barrel in a TVS is a set which is convex, balanced, absorbing and closed. A barrelled space is an LCS in which every barrel is a neighborhood of 0 ([7], §III.4, Def. 2). An inductive limit and a product of barrelled spaces, a quotient of a barrelled space, are all barrelled ([7], p. III.25). A Banach space is barrelled, therefore an ultrabornological space is barrelled. A nontrivial ($\mathcal{LF}$)-space (i.e. a nontrivial strict inductive limit of Fréchet spaces) is ultrabornological but not Baire (assuming that $E = \bigcup_n E_n$, $E_n \subsetneq E_{n+1}$ and $E_n$ is closed in $E$, then $E_n = \bar{E}_n$ and $\bar{E}_n = \emptyset$ because $E_n \subsetneq E$ is not a neighborhood of 0 in $E$; thus $E_n$ is rare and $E$ is not Baire). Thus an ultrabornological space need not be Baire.

Lemma and Definition 20 ([29], §I.2.2). Let $E$ be an LCS. (1) The following conditions are equivalent:

(a) Given any sequence $(A_n)$ of closed convex subsets with empty interior, $\bigcup_n A_n$ has empty interior.

(b) If $(A_n)$ is a covering of $E$ by closed convex sets, there exists an index $n$ such that $A_n$ is nonempty.

(2) The space $E$ is called convex-Baire if these equivalent conditions hold.

Every convex-Baire space is barrelled ([29], p. 94).

The notion of webbed space is due to De Wilde [9].

Definition 21 Let $E$ be an LCS, $W = \{C_{n_1,\ldots,n_k}\}$ be a collection of subsets $C_{n_1,\ldots,n_k}$ of $E$ where $k$ and the $n_1,\ldots,n_k$ run through the natural
integers. \( W \) is called a web if
\[
E = \bigcup_{n_1=1}^{\infty} C_{n_1}, \quad C_{n_1,\ldots,n_{k-1}} = \bigcup_{n_k=1}^{\infty} C_{n_1,\ldots,n_k}
\]
and a \( C \)-web if in addition for every sequence \( (n_k) \) there exists a sequence of positive numbers \( \rho_k \) such that for all \( \lambda_k \) such that \( 0 \leq \lambda_k \leq \rho_k \), and all \( x_k \in C_{n_1,\ldots,n_k} \), the series \( \sum_{k=1}^{\infty} \lambda_k x_k \) converges in \( E \).

\( E \) is called a webbed space if there exists a \( C \)-web in \( E \).

A Fréchet space, a sequentially complete (\( DF \)) space (in the sense of Grothendieck \([13]\)), the strong or weak dual of a countable inductive limit of metrizable locally convex spaces, are webbed (\([20]\), §35.4), (\([9]\), p. 57, Prop. IV.3.3). In addition, an LCS space is Fréchet if, and only if it is both webbed and Baire (\([17]\), 5.4.4).

Barrelled spaces, convex-Baire spaces, \( \beta \) spaces, ultrabornological spaces and webbed spaces enjoy the following hereditary properties:

- A quotient of a barrelled (resp. semi-Suslin, webbed) space is barrelled (resp. semi-Suslin, webbed), and a Hausdorff quotient of a convex-Baire space is convex-Baire.

- The image of a semi-Suslin (resp. webbed) space by a continuous map is semi-Suslin (resp. webbed) (\([29]\), p. 79), (\([20]\), p. 61, (2)). Since a Suslin locally convex space is the continuous image of a separable Fréchet space, and since a Fréchet space is webbed, a Suslin space is webbed.

- A product of barrelled (resp. convex-Baire) spaces is barrelled (resp. convex-Baire). A countable product of ultrabornological (resp. webbed) spaces is ultrabornological (resp. webbed). A finite product of semi-Suslin spaces is semi-Suslin (\([29]\), p. 80).

- An inductive limit of barrelled (resp. \( \beta \), ultrabornological) spaces is barrelled (resp. \( \beta \), ultrabornological). A locally convex hull of barrelled spaces is barrelled (\([19]\), p. 368, (3)). An inductive limit and a locally convex hull of countably many webbed spaces is webbed. A countable inductive limit of Suslin spaces is Suslin (since it is the countable union of continuous images of Suslin spaces).

- A dense subspace of a convex-Baire space is convex-Baire (therefore a convex-Baire space need not be complete).

- A sequentially closed subspace of a semi-Suslin space is semi-Suslin (\([29]\), p. 79).
Let $E$ be an LCS and $\mathcal{T}$ its topology. Then $\mathcal{T}^f$ is defined to be the finest topology on $E'$ which coincides with the topology induced by the weak topology $\sigma(E',E)$ on the $\mathcal{T}$-equicontinuous subsets of $E'$. Therefore, a linear subspace $Q \subset E'$ is $\mathcal{T}^f$-closed if, and only if for each equicontinuous subset $A \subset E'$, $Q \cap A$ is $\sigma(E',E)$-closed in $A$ (equivalently, $Q$ is $\mathcal{T}^f$-closed if, and only if for every neighborhood $U$ of $0$ in $E$, $Q \cap U^0$ is $\sigma(E',E)$-closed in $U^0$, where $U^0$ is the polar of $U$).

**Definition 22** An LCS $E$ is called a Pták space (or a $B$-complete space) if every $\mathcal{T}^f$-closed linear subspace of $E'$ is $\sigma(E',E)$-closed. An LCS $E$ is called an infra-Pták space (or a $B_r$-complete space) if every weakly dense $\mathcal{T}^f$-closed linear subspace of $E'$ coincides with $E'$.

A Pták space is infra-Pták space ([25], (4.2)) but not conversely [30], and an infra-Pták space is complete ([20], §34.2).

A closed subspace of an infra-Pták (resp. a Pták) space is again infra-Pták (resp. Pták) ([25], (3.9); [16], §3.17).

A Fréchet space, the strong dual of a reflexive Fréchet space, and the Hausdorff quotient of a Pták space, are all Pták spaces ([16], §3.17). A product of two Pták spaces need not be Pták.

A separable Fréchet space is Polish, thus $K$-Suslin, and Baire, hence $\beta$. Conversely, a locally convex space which is Baire and $K$-Suslin is separable Fréchet ([29], p. 64, (21)). If $E$ is a Fréchet space, its weak dual is $K$-Suslin (but not separable in general, thus not Suslin), and $E''[\sigma(E'',E')]$ is quasi-Suslin; if $E$ is reflexive, $(E,\sigma(E,E'))$ is $K$-Suslin ([28], p. 557), ([29], p. 66, (24)).

The above is summarized below, where $(\mathcal{F})$ means Fréchet, $(s\mathcal{F})$ means separable Fréchet, $(c\mathcal{LSF})$ means a countable inductive limit (non necessarily strict) of Fréchet spaces, $(c\mathcal{LSF})$ means countable inductive limit of separable Fréchet spaces, and l.c.h. means locally convex hull:

| quasi-Suslin | locally complete | Baire | $K$-Suslin $\Leftrightarrow$ $(s\mathcal{F})$ |
|-------------|-----------------|-------|---------------------------------|
| $\Downarrow$ | $\Downarrow$ | $\Downarrow$ | $\Downarrow$ |
| semi-Suslin | $\Leftarrow$ $(\mathcal{F})$ | $(c\mathcal{LSF})$ $\Rightarrow K\beta$ $\Rightarrow \beta$ |
| $\Downarrow$ | $\Downarrow$ | $\Downarrow$ | $\Downarrow$ |
| Pták | $\Leftarrow$ $(\mathcal{F})$ | $\Rightarrow$ ultrabornological |
| $\Downarrow$ | $\Downarrow$ | $\Downarrow$ | $\Downarrow$ |
| infra-Pták | Baire $\Rightarrow$ l.c.h. Baire |
| $\Downarrow$ | $\Downarrow$ | $\Downarrow$ | $\Downarrow$ |
| complete convex-Baire | $\Rightarrow$ barrelled |
and

\[(sF) \Rightarrow (cLsF) \Rightarrow \text{Suslin}\]
\[\downarrow \]
\[(F) \leftrightarrow \text{webbed Baire} \leftrightarrow \text{webbed}\]
\[\downarrow \]
\[\text{ultrabornological} \Rightarrow \text{l.c.h. metrizable Baire} \Leftarrow \text{Baire}\]
\[\downarrow \]
\[\text{l.c.h. Baire} \Rightarrow \text{barrelled}\]

5 Theorems in topological groups

Let us review three important versions of the (generalized) Closed Graph Theorem and of the Open Mapping Theorem for topological groups. Let \(G, H\) be Hausdorff topological groups and \(u : G \rightarrow H\) be a map. The table below summarizes the results. The first column is the statement number, the last column contains the reference. In the other columns, the conditions on \(G, H\) and \(\text{Gr}(u)\) under which \(u\) is a morphism are indicated.

5.1 Closed Graph Theorems

Let \(u : G \rightarrow H\) be a map. Then \(u\) is continuous under any of the conditions on \(G, H\) and \(\text{Gr}(u)\) in the table below:

|     | \(G\)   | \(H\)   | \(\text{Gr}(u)\) | Ref. |
|-----|---------|---------|------------------|------|
| (1) | Baire   |         | Suslin           | [6]  |
| (2) | \(\beta\) |         | Suslin           | Borel [21] |
| (3) | \(K,\beta\) | \(K\)-Suslin | \(F\)-Borel | [21] |
| (4) | 2nd category | metrizable, complete, Lindelöf | closed | [32] |
| (5) | Suslin, Baire | Suslin | Borel |      |
| (6) | Polish | Polish | closed | [15] |

Remark 23 (A). The main statements are (1)-(4). Indeed, (1)\(\Rightarrow\)(5): Let \(G\) and \(H\) be Suslin. Then \(G \times H\) is Suslin. If \(\text{Gr}(u)\) is Borel in \(G \times H\), then \(\text{Gr}(u)\) is Suslin. If in addition \(G\) is Baire, by (1) \(u\) is continuous.

(2)\(\Rightarrow\)(5)\(\Rightarrow\)(6). If \(G\) is Suslin Baire, it is \(\beta\), thus (2)\(\Rightarrow\)(5). If \(G\) and \(H\) are Polish, they are Suslin Baire, and if \(\text{Gr}(u)\) is closed, it is Borel, thus (5)\(\Rightarrow\)(6).

(3)\(\Rightarrow\)(6): If \(G\) and \(F\) are Polish, they are respectively \(K,\beta\) and \(K\)-Suslin, and if \(\text{Gr}(u)\) is closed, it is \(F\)-Borel. Therefore by (3) \(u\) is continuous.

(4)\(\Rightarrow\)(6): Let \(G\) be Polish, hence Baire, hence 2nd category. Let \(G\) be
Polish, hence metrizable, complete and Lindelöf. By (5), if \( \text{Gr} (u) \) is closed, \( u \) is continuous.

(B). Let \( E, F \) be two metrizable, complete and separable TVS's over a non-discrete valued division ring and \( u : E \to F \) be linear with closed graph. By (6), \( u \) is continuous. In Banach’s Closed Graph Theorem, \( E \) and \( F \) are not assumed to be separable thus cannot be deduced of any of the above statements which, apart from (3), all imply a separability condition. In addition, metrizable TVS is not \( K \)-Suslin in general.

5.2 Open Mapping Theorems

Let \( u : H \to G \) be a surjective map where \( G, H \) are Hausdorff topological groups. Then \( u \) is open under any of the following conditions on \( H \) and \( G \):

|       |       |       |
|-------|-------|-------|
| \((2')\) | \(\beta\) | Suslin |
| \((3')\) | \(K, \beta\) | \(K\)-Suslin |
| \((4')\) | 2nd category | metrizable, complete, Lindelöf |
| \((6')\) | inexhaustible | Polish |

Remark 24 \((2'), (3'), (4')\) are consequences of \((2), (3), (4)\) above and of Theorem 14. \((4')\) is not stated in [32].

\((4') \Rightarrow (6')\): If \( G \) is inexhaustible, it is of 2nd category, and if \( H \) is Polish, it is metrizable, complete and Lindelöf.

6 Theorems in locally convex spaces

6.1 Closed Graph Theorems

The Closed Graph Theorem was generalized by Dieudonné-Schwartz [10], Köthe, Grothendieck ([12], Introduction, IV, Théorème B) and then De Wilde ([20], p. 57):

Theorem 25 Let \( E \) be ultrabornological, \( F \) be webbed, and the graph of \( u : E \to F \) be sequentially closed. Then \( u \) is continuous.

Remark 26 The above result is completely symmetric when both \( E \) and \( F \) are countable inductive limits of Fréchet spaces (the case considered by Köthe) and a fortiori when they are strict countable inductive limits of Fréchet spaces (the case considered by Dieudonné-Schwartz).
Pták’s Closed Graph Theorem ([25], Theorem 4.8), proved in 1953, and its generalization obtained in 1956 by A. and W. Robertson ([20], p. 41) can be stated as follows:

**Theorem 27** Let $E$ be barrelled (resp. a locally convex hull of Baire spaces) and $F$ be an infra-Pták space (resp. a locally convex hull of a sequence of Pták spaces). A linear map $u : E \rightarrow F$ is continuous if its graph is closed.

Schwartz’s Borel graph theorem can be stated as follows ([9], p. 136, Prop. VII.4.1):

**Theorem 28** Let $E$ be ultrabornological, $F$ be Suslin, and the graph of $u : E \rightarrow F$ is a sq-Borel subset of $E \times F$. Then $u$ is continuous.

In Schwartz’s Borel graph theorem, $F$ is necessarily separable. Martineau’s and Valdivia’s generalizations do not require this restriction. Martineau’s result ([21], Thm. 6) involves $K$-ultrabornological spaces which are locally convex hulls of Baire spaces of a particular type, so this theorem is a consequence of Valdivia’s result (3) in the table below.

**Theorem 29** Let $E, F$ be two LCS’s and $u : E \rightarrow F$. Then $u$ is continuous if one of the following conditions hold:

- $E$ is the locally convex hull of Baire spaces (this happens if $E$ is $K$-ultrabornological), $F$ is $K$-Suslin, $u$ has closed graph ([29], p. 58, (11)).
- $E$ is the locally convex hull of metrizable Baire spaces, $F$ is quasi-Suslin, $u$ has closed graph ([29], p. 63, (17)).
- $E$ is the locally convex hull of metrizable Baire (resp. metrizable convex-Baire) spaces, $E$ is Suslin (resp. semi-Suslin), $u$ has sequentially closed graph ([29], p. 71, (15) (resp. p. 84, (13))).

Let us summarize these results in a table where the larger the row number, the stronger the condition on $E$. This condition could hardly be weaker than barrelledness since, as proved by Mahowald in 1961 (see [20], p. 38), if every linear mapping of the LCS $E$ into an arbitrary
Banach space is continuous, then $E$ is barrelled.

| $E$                  | $F$                  | $\text{Gr}(u)$ |
|----------------------|----------------------|----------------|
| (1) barrelled        | infra-Pták           | closed        |
| (2) l.c.h. Baire     | $K$-Suslin           | closed        |
| (3) l.c.h. Baire     | l.c.h. Pták          | closed        |
| (4) l.c.h. metrizable Baire | quasi-Suslin   | closed        |
| (5) l.c.h. metrizable Baire | Suslin       | sq-closed    |
| (6) l.c.h. metrizable convex-Baire | semi-Suslin | sq-closed    |
| (7) ultrabornological | webbed              | sq-closed    |
| (8) ultrabornological | Suslin              | sq-Borel     |

The sq-closedness of $\text{Gr}(u)$ is hardly a restriction in practice. If only maps with sq-closed graph are considered, (4)$\Rightarrow$(5), (7)$\Rightarrow$(8).

### 6.2 Open Mapping Theorems

Let $u : F \to E$ be a surjective morphism. In the 3rd column of the above table $F$ and $F/H$ ($H \subset F$) have the same property except for row (1) since a quotient of an infra-Pták space need not be infra-Pták. Therefore, by Theorem[14] if $E$ and $F$ are as in this table, except in row (1), $u$ is open. Row (1) must be changed to:

| $E$                  | $F$                  |
|----------------------|----------------------|
| (1') barrelled       | Pták                 |

which is Pták’s result ([25], p. 59, Thm. (4.10)).

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