Finite Percolation at a Multiple of the Threshold

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Bond percolation on infinite heavy-tailed power-law random networks lacks a proper phase transition; or one may say, there is a phase transition at zero percolation probability. Nevertheless, a finite size percolation threshold \( q_c(N) \), where \( N \) is the network size, can be defined. For such heavy-tailed networks, one can choose a percolation probability \( q(N) = \rho q_c(N) \) such that \( \lim_{N \to \infty} (q - q_c(N)) = 0 \), and yet \( \rho \) is arbitrarily large (such a scenario does not exist for networks with non-zero percolation threshold). We find that the critical behavior of random power-law networks is best described in terms of \( \rho \) as the order parameter, rather than \( q \). This paper makes the notion of the phase transition of the size of the largest connected component at \( \rho = 1 \) precise. In particular, using a generating function based approach, we show that for \( \rho > 1 \), and the power-law exponent, \( 2 \leq \tau < 3 \), the largest connected component scales as \( \sim N^{1-1/\tau} \), while for \( 0 < \rho < 1 \) the scaling is \( \sim N^{2\tau-1} \); here, the maximum degree of any node, \( k_{max} \), has been assumed to scale as \( N^{1/\tau} \). In general, our approach yields that for large \( N, \rho \gg 1, 2 \leq \tau < 3 \), and \( k_{max} \sim N^{1/\tau} \), the largest connected component scales as \( \sim \rho^{1/(3-\tau)} N^{1-1/\tau} \). Thus, for any fixed but large \( N \), we recover, and make it precise, a recent result that computed a scaling behavior of \( q^{1/(3-\tau)} \) for “small \( q \)”. We also provide large-scale simulation results validating some of these scaling predictions, and discuss applications of these scaling results to supporting efficient unstructured queries in peer-to-peer networks.

I. INTRODUCTION

Percolation on random graphs with heavy tailed power-law degree distributions is known to possess certain unique properties. Unlike most cases of finite and infinite dimensional percolation, the percolation on an infinitely large, heavy tailed power-law (PL) random graph lacks a percolation threshold, i.e., the phase transition happens at zero percolation probability. This is the source of many interesting critical behavior in heavy-tailed random PL networks that are remarkably different from those observed in networks with non-zero percolation thresholds. In this paper, we are particularly interested in the scaling of the size of the largest connected component close to the criticality, for networks of finite but large size \( N \). To that end, we first make the notion of closeness to criticality precise, and provide a summary of our results.

A. Summary of the Results

Consider a heavy-tailed PL network with exponent \( 2 \leq \tau < 3 \). A finite size percolation threshold \( q_c(N) \) can be defined for these networks that goes to zero as \( N \) goes to infinity.

Consider percolation at a probability \( q(N) = \rho q_c(N) \). After bond percolation with probability \( q(N) \), a total of around \( \mathcal{H}(N, \rho) \sim q(N) N \langle k \rangle / 2 \) links remain in the network, where \( \langle k \rangle \) is the average degree. Of these \( \mathcal{H}(N, \rho) \) links, a number \( \mathcal{G}(N, \rho) \) of them belong to the same largest connected component. This paper answers the following questions: How does \( \mathcal{G}(N, \rho) \) scale with \( N \) and \( \rho \)? How does the fraction \( \mathcal{G}(N, \rho) / \mathcal{H}(N, \rho) \) scale with \( N \) and \( \rho \)? In other words, what is the chance that a link left after the percolation belongs to the giant connected component? Does this probability converge to some constant?

The answers to these questions are known for networks with finite percolation threshold \( q_c = q_c(\infty) > 0 \) (note that \( \rho \) is meaningful only when \( \rho \leq 1 / q_c \), otherwise one has \( q > 1 \)). In particular, \( \mathcal{G}(N, \rho) \sim N \) for \( 1 < \rho \leq 1 / q_c \), and \( \mathcal{G}(N, \rho) = O(k_{max} \log N) \) for \( 0 < \rho < 1 \), where \( k_{max} \) is the largest degree of the nodes in the network. When \( \rho = 1 \), \( \mathcal{G}(N, \rho) \) is known to scale as \( N^{2/3} \) [4] for most random networks under certain constraints. The probability that a random link after percolation belongs to the largest connected component, \( \mathcal{G}(N, \rho) / \mathcal{H}(N, \rho) \), converges to a constant \( \omega(\rho) > 0 \) when \( 1 / q_c \geq \rho > 1 \), and goes to zero at least as fast as \( \frac{k_{max} \log N}{N} \) when \( 0 \leq \rho < 1 \).

The answers to the above questions for heavy-tailed PL networks turn out to be distinctly different from those for random networks with finite percolation thresholds. In order to capture these differences succinctly, we define a scaling exponent, \( \lambda_\tau(\rho) \equiv \lim_{N \to \infty} \log \mathcal{G}(N, \rho) / \log(N) \). Fig. [4] shows a schematic of some of the results that we derive, based on our application of the generating function approach.
For $1 < \rho \ll N^{1/\tau}$, we show that $G(N, \rho) \sim (k)N^{1-1/\tau}$ (see Eqs. [18, 19]). In other words, the scaling exponent is $\lambda_c(\rho) = 1 - 1/\tau$ for any $\rho > 1$, and thus, the size of the giant connected component does not scale linearly with $N$ for constant $\rho > 1$. For $0 < \rho < 1$, the scaling exponent $\lambda_c(\rho) = 1 - 2/\tau$. This last statement follows from the observation that the percolation threshold $q_c$ is fixed large component scales linearly as $|q - q_c|$, and using the fact that $q_c \sim N^{1-3/\tau}$ (see Eqn. [11]) gives the desired result.

Therefore, if we consider $\rho$ as the order parameter, then there is a phase transition at $\rho = 1$ for all random networks (irrespective of whether it is heavy-tailed with unbounded variance or not) except that the phase transition levels are different (Fig. 1). This provides a unified view of the phase transition phenomenon for largest connected components in random networks.

The above result enables us to calculate the following interesting quantity: $G(N, \rho)/H(N, \rho)$ (see Eqn. [20]). We find that for $\rho > 1$, the ratio $G(N, \rho)/H(N, \rho) \sim N^{(\tau-2)/\tau}$ for $2 \leq \tau < 3$. In other words, the probability that a randomly chosen link after the percolation belongs to the giant connected component goes to zero as $N^{(\tau-2)/\tau}$ for $2 < \tau < 3$. Only for $\tau = 2$, a finite fraction of all the links that remain after the percolation belong to the giant connected component.

### B. Relation to Previous Work

The finite-size scaling of the cardinality of the largest percolation components close to criticality has always been a source of interest. While for finite dimensional percolation problems, the issue is still subject to a great deal of controversy [2], it has been successfully resolved for percolation on most random graphs (that are examples of infinite dimensional percolation). In particular, for an Erdős-Reyni random graph with percolation threshold $q_c$, the size of the largest connected component is known to scale as $\Theta(N^{2/3})$ when $q = q_c + O(N^{-2/3})$. Thus, right at the percolation threshold $q_c$, the size of the largest connected component scales with the network size as $\sim N^{2/3}$ [3]. Furthermore, it can be shown that for any fixed large $N$, the size of the largest connected component scales linearly as $|q - q_c|$ when $q - q_c$ is small. This indicates, among other things, the continuity of the size of the largest connected component as a function of $q$ at $q = q_c$. Note that in such cases, the percolation threshold $q_c$ is finite, and independent of $N$. In [1], this result has been extended to random networks on any given degree distribution, subject to certain constraints.

Recent results in [4] have shown that critical scaling properties of random power-law networks with exponents $\tau$ in the range $[2, 4]$, are significantly different from those of most other random networks. For heavy-tailed PL networks (i.e., $2 \leq \tau < 3$), which is the case of interest in this paper, results in [4] suggest that the size of the largest connected component scales as $|q - q_c|^{1/(3 - \tau)}$ for any fixed but large $N$ (as opposed to a linear scaling of $|q - q_c|$ for most other random networks). The approach used in this work is also based on the generating function approach adopted in this paper. Our goal, however, is to study the scaling laws as a function of $N$ for a fixed $q/q_c$.

We first argue that the relevant order parameter in heavy-tailed power-law random networks is $\rho = q/q_c$, and critical percolation properties are best explained in terms of $\rho$. We then derive, in a unified way, the scaling of the size of the largest connected component as a function of $\rho$ and $N$. We show that for any fixed $\rho > 1$, the size of the largest connected component scales as $N^{1-1/\tau}$, in contrast to the universal scaling of $O(N^{2/3})$ for most random networks with finite size percolation threshold. Interestingly,
for \( \tau \) approaching 3, where the network assumes a finite size percolation threshold, the universal scaling exponent 2/3 is recovered. On the other hand, for \( \rho < 1 \), the size of the largest connected component scales at most as fast as \( N^{(\tau-2)/\tau} \). For fixed but large \( N \), we find that the size of the largest component as a function of \( \rho \) for all \( \rho \), and we recover the results in [4] for \( 1 \ll \rho \ll N^{1/\tau} \). In general, for large \( \rho \gg 1 \) and \( N \), the size of the largest connected component is shown to scale as \( \sim \rho^{1/(3-\tau)} N^{1-1/\tau} \).

C. Applications

The results in this paper also have applications to peer-to-peer (P2P) data mining or search algorithms on unstructured complex networks. Percolation at a multiple of the threshold is at the heart of a scalable P2P search algorithm, called percolation search algorithm, introduced by the authors in [5]. Percolation search uses a probabilistic broadcast algorithm to reduce the number of communications necessary for finding contents in a large scale complex network that has a heavy-tailed power-law degree distribution. The percolation probability \( q \) will correspond to the probability with which nodes of the network communicate a message to any of their neighbors. For the search algorithm to work efficiently, one needs to choose a value of \( q \) that is as small as possible and yet results in a large enough connected component; that is, choose \( q \) to be a constant multiple of the threshold \( q_c \). The scaling of this connected component directly specifies the scaling of the P2P search traffic with network size. The results in this paper form the theoretical basis for showing that this traffic scales sub-linearly with the network size for any heavy-tailed random power-law network.

The rest of this paper is organized as follows: In Section II we briefly review the generating function formalism [1] for calculating the percolation properties of random networks on given degree distributions. We then specialize the approach to the case of percolation at a multiple of the threshold for heavy-tailed random power-law networks and derive the scaling relation of the size of the connected component as a function of both \( q/q_c \) and the network size. A number of simulations are reported in Section III which verify our results. Concluding remarks are made in Section IV.

II. CALCULATION OF THE CLUSTER SIZES

In this section, we introduce the analytic approach of the paper.

A. Modeling Finite Size Effects

Consider a random graph with degree distribution \( p_k \), \( k = 1, 2, \ldots, k_{\text{max}} \). Thus, the probability that a randomly chosen node of this graph has degree \( k \) is \( p_k \), where \( 1 \leq k \leq k_{\text{max}} \). We assume that the maximum degree is such that every network is expected to have at least one node of that degree: \( np_{k_{\text{max}}} \geq 1 \). For a power-law random graph \( p_k = A \tau k^{-\tau} \) for \( k \in (1, k_{\text{max}}) \), we have:

\[
A_\tau \approx \zeta(\tau) \\
(A_{\tau})_{\text{max}} \approx \frac{N^{1/\tau} \zeta(\tau)}{A_\tau^{1/\tau}}
\]

This choice of \( k_{\text{max}} \) reflects the maximum scaling of \( k_{\text{max}} \) that results in the expected number of high degree nodes to be one.

Thus, as \( N \) increases, \( k_{\text{max}} \) will increase and the variance of the distribution will become unbounded for \( \tau < 3 \). We focus on the parameter regime, \( 2 \leq \tau < 3 \), where the networks have degree distributions with unbounded variance but bounded mean (except at \( \tau = 2 \), where it increases only logarithmically). Though we are interested in finite size effects, we focus on the case of large values of \( N \) where \( \langle k^2 \rangle \gg \langle k \rangle \).

An accompanying assumption we make is that the generating function approach, which is meant to deal with infinite size networks, is still applicable to random power-law networks of finite, but large size, \( N \). The same assumption is implicitly made in [4], which also uses the same generating function formalism. A formal treatment of the validity of this assumption is beyond the scope of this paper. It is not, however, difficult to see that the scaling of the variance (and, hence, of \( k_{\text{max}} \)) with \( N \) will play an important role in any such formal approach, and for a discussion on choosing \( k_{\text{max}} \) see [6, 7]. We have verified the predictions made by the generating function approach via large-scale simulation results, some of which are presented in Section III and in Appendix [B]. The simulation results in Appendix [B] show that the predicted infinite-size percolation thresholds are matched closely by numerical estimates obtained from synthetic finite-size networks; the match is very good even for relatively small size networks.

B. Percolation as a Branching Process

Just above the criticality, the largest component of the network is a tree. Percolation on a tree can be viewed as a branching process. Using this fact, we will treat percolation as branching process using the generating functions method introduced in [1].
generating function for the degree of a randomly chosen node can be defined as \( G_0(x) = \sum_{k=1}^{k_{\text{max}}} p_k x^k \). The generating function for the degree of a node arrived at by following one end of a randomly chosen link is \( G_1(x) = G'_0(x)/G'_0(1) \).

The bond percolation with probability \( q \) on a given graph is as follows: For each link of the network, delete the link with probability \( 1-q \) and retain it with probability \( q \), independently. The distribution of the size of the connected components of a random graph after bond percolation with probability \( q \) can in principle be found. Throughout the rest of the paper, we use results in \([8],[1]\).

Let \( u \) be the probability that a random link does not lead to an infinite set of nodes, one then has:

\[
\begin{align*}
  u &= 1 - q + q G_1(u) \\
  \text{This equation is understood as follows: Take any random link. The probability that the link is deleted} & \text{ (and thus does not lead to a giant component) is } 1-q \text{ (hence the first term on the right hand side). Now follow the link to one of its random ends to arrive at a node } V. \ G_1(x) \text{ is the generating function of the distribution of the number of links that go out of } V. \text{Being a random graph, the probability of each of those links to lead into a giant connected component is again } u. \text{ The probability that none of the remaining links of } V \text{ go to an infinite component is } G_1(u). \text{ Hence we have (Eqn. } 3) \text{.}
\end{align*}
\]

As far as the nodes are concerned, the probability that a random node does not lead to an infinite number of nodes is thus \( G_0(u) \), and hence the fraction of nodes in the infinitely large component is:

\[
S = 1 - G_0(u) \tag{4}
\]

Below the percolation threshold, the only solution to (Eqn. 3) is \( u = 1 \). In fact, this can be used to show that the percolation threshold of any random graph on a given degree distribution is given by:

\[
q_c = G'_1(1)^{-1} = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle} \tag{5}
\]

When working just above the percolation threshold (as is the subject of this paper), one expects that \( u \lesssim 1 \), and hence we look for solutions of the form \( u = 1 - \delta \) with \( \delta \approx 0 \):

\[
\begin{align*}
  u &= 1 - q + q G_1(u) \\
  1 - \delta &= 1 - q + q G_1(1 - \delta) \\
  \frac{\delta}{q} &= 1 - G_1(1 - \delta) \\
  \text{Follow a particular edge, define } \delta \text{ the probability} & \text{ that it leads to an infinite component. In order to solve for } \delta, \text{ one can write a Taylor series expansion of } G_1(1 - \delta):
\end{align*}
\]

\[
G_1(1 - \delta) = \sum_{n=0}^{\infty} G_1^{(n)}(1) \frac{(-\delta)^n}{n!} \tag{7}
\]

For heavy tailed random graphs, all these moments (except for the mean \( G_1^0(1) \)) are large (in fact infinite as \( N \) goes to infinity). Nevertheless, for any finite \( N \), these moments are still finite and the expansions can be carried out. In this work, we deal with heavy-tail power-law random graphs. For these graphs we have \( \langle k^{n+1} \rangle \approx \left( \frac{n-\tau}{n+1-\tau} \right) k_{\text{max}}^n \). We use this property to make the following approximation:

\[
G_1^{(n)}(1) = \frac{\prod_{i=0}^{n} (k - n)}{\langle k \rangle} \approx \frac{\langle k^{n+1} \rangle}{\langle k \rangle} \tag{8}
\]

Finally, by approximating the sum as an integral we can get that for \( n \geq 1 \):

\[
\langle k^{n+1} \rangle \approx A_c k^{n+2-\tau} \frac{n!}{n+2-\tau} \tag{9}
\]

While these approximations may seem crude, we show in Section III that the predictions we make match the simulation results.

### C. Solving for the Scaling Exponent

We can put (Eqn. 5) into (Eqn. 7):

\[
G_1(1 - \delta) = \sum_{n=0}^{\infty} G_1^{(n)}(1) \frac{(-\delta)^n}{n!} \approx \sum_{n=0}^{\infty} \langle k^{n+1} \rangle \frac{(-\delta)^n}{n!} \approx 1 + \sum_{n=1}^{\infty} \frac{A_c}{\langle k \rangle} k_{\text{max}}^{n+2-\tau} \frac{(-\delta)^n}{n!} \approx 1 + A_c k_{\text{max}}^{2-\tau} \sum_{n=1}^{\infty} \frac{(-\delta k_{\text{max}})^n}{n! (n+2-\tau)}
\]

Defining a finite size percolation threshold like any other random network as \( q_c = 1/G'_1(1) \), one gets:

\[
\frac{A_c k_{\text{max}}^{2-\tau}}{\langle k \rangle} \approx \frac{(3-\tau)(k^2)}{k_{\text{max}} q_c} \approx \frac{3-\tau}{k_{\text{max}} q_c}
\]

\[
G_1(1 - \delta) = \sum_{n=0}^{\infty} G_1^{(n)}(1) \frac{(-\delta)^n}{n!} \tag{7}
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For heavy tailed random graphs, all these moments (except for the mean \( G_1^0(1) \)) are large (in fact infinite as \( N \) goes to infinity). Nevertheless, for any finite \( N \), these moments are still finite and the expansions can be carried out. In this work, we deal with heavy-tail power-law random graphs. For these graphs we have \( \langle k^{n+1} \rangle \approx \left( \frac{n-\tau}{n+1-\tau} \right) k_{\text{max}}^n \). We use this property to make the following approximation:

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\]

While these approximations may seem crude, we show in Section III that the predictions we make match the simulation results.
which gives:

\[ G_1(1-\delta) \approx 1 + \frac{3-\tau}{k_{\text{max}}q_c} \sum_{n=1}^{\infty} \frac{(-\delta k_{\text{max}})^n}{n!(n+2-\tau)} \]

Inserting back into (Eqn. 9):

\[ \frac{\delta}{q} = 1 - G_1(1-\delta) \approx -\frac{3-\tau}{k_{\text{max}}q_c} \sum_{n=1}^{\infty} \frac{(-\delta k_{\text{max}})^n}{n!(n+2-\tau)} - \frac{q_c}{q(3-\tau)}(\delta k_{\text{max}}) \approx \sum_{n=1}^{\infty} \frac{(-\delta k_{\text{max}})^n}{n!(n+2-\tau)} \]

For each constant \( \tau \), define \( \alpha = \frac{\delta}{q(3-\tau)} \) and \( z = \delta k_{\text{max}} \) to get an equation that is independent of \( k_{\text{max}} \) in the limit of large \( k_{\text{max}} \):

\[ -\alpha z = \sum_{n=1}^{\infty} \frac{(-z)^n}{n!(n+2-\tau)} \]  

(Eqn. 9) is our first result of this paper. It clearly states that as far as \( k_{\text{max}} \) is large, \( z \) would be independent of \( k_{\text{max}} \) and hence \( N \). Thus, the fraction of links in the giant connected component will scale as \( \delta = zk_{\text{max}}^{-1} \) for some constant \( z \) depending only on \( \tau \) and \( q/q_c \).

The number of links in the largest connected component after percolation, will therefore scale as:

\[ G(\tau, \rho) \triangleq \langle k \rangle \delta N = \langle k \rangle k_{\text{max}}^{-1} z N \]

where \( z \) depends only on \( \tau, \rho \) and \( k_{\text{max}} \) depends only on \( N, \tau \). The above relation is valid as far as the ratio \( \rho = q/q_c \) for

\[ q_c \approx \langle k \rangle/\langle k^2 \rangle \approx \langle k \rangle k_{\text{max}}^{-3} = \langle k \rangle N^{1-3/\tau} \]

When \( 2 < \tau < 3 \), (Eqn. 9) can be simplified in terms of incomplete gamma functions. The incomplete gamma function is defined for \( \Re(a) > 0 \) as:

\[ \Gamma(a, z) = \int_{z}^{\infty} x^{a-1} e^{-x} dx \]

At \( z = 0 \) it takes on the familiar value: \( \Gamma(a, 0) = \Gamma(a) \). For all \( z \), the following holds true:

\[ \Gamma(a, z) = \frac{1}{a} (\Gamma(a+1, z) - e^{-z} z^a) \]

The series expansion for \( \Gamma(a, z) \) is as below:

\[ \Gamma(a, z) = \Gamma(a) - z^a \sum_{n=0}^{\infty} \frac{(-z)^n}{n!(n+a)} \]

Or equivalently using (Eqn. 14):

\[ \sum_{n=1}^{\infty} \frac{(-z)^n}{n!(n+a)} = \frac{\Gamma(a) - \Gamma(a, z)}{z^a} - \frac{1}{a} \]

Applying this identity to (Eqn. 9) with \( a = 2-\tau \):

\[ -\alpha z = \frac{\Gamma(2-\tau) - \Gamma(2-\tau, z)}{z^{2-\tau}} - \frac{1}{2-\tau} \]

Applying the above to (Eqn. 9), with \( a = 2-\tau \), and setting \( \beta = \frac{q}{q_c} \frac{\tau^2}{3-\tau} \)

\[ \beta z = \frac{\Gamma(3-\tau) - \Gamma(3-\tau, z)}{z^{2-\tau}} + e^{-z} - 1 \]

when \( 2 < \tau < 3 \).

To see how the scaling results of \( \beta \) follow from \( \tau \), lets consider the case \( z \gg 1 \). In this case, \( \Gamma(a, z) \to 0 \), hence:

\[ -\alpha z \approx z^{\tau-2} \Gamma(2-\tau) - \frac{1}{2-\tau} \]

\[ \beta z \approx (2-\tau)z^{\tau-2} \Gamma(2-\tau) - 1 \]

\[ = z^{\tau-2} \Gamma(3-\tau) - 1 \]

If \( z^{\tau-2} \Gamma(3-\tau) \gg 1 \), we can neglect 1 to obtain:

\[ \beta z \approx z^{\tau-2} \Gamma(3-\tau) \]

\[ z \approx \left( \frac{\Gamma(3-\tau)}{\beta} \right)^{1/(3-\tau)} \]

\[ = \left( \frac{\Gamma(3-\tau)}{q_c(\tau-2)} \right)^{1/(3-\tau)} \]

\[ = \rho^{1/(3-\tau)} \left( \frac{\Gamma(4-\tau)}{(\tau-2)} \right)^{1/(3-\tau)} \]

In other words, the fraction of the links in the largest connected component \( \delta \) scales as:

\[ \delta = k_{\text{max}}^{-1} z \approx \left( \frac{\Gamma(4-\tau)}{(\tau-2)} \right)^{1/(3-\tau)} \rho^{1/(3-\tau)} N^{-1/\tau} \]

When \( 1 < \rho \ll k_{\text{max}} \) and \( 2 < \tau < 3 \) and when the scaling of \( k_{\text{max}} \) is chosen according to Eqn. 9 (i.e., \( N^{1/\tau} \)). The number of links in the largest connected component will therefore scale as 

\[ G(\rho, \tau) \sim \rho^{1/(3-\tau)} N^{1-1/\tau} \]

Similar result follows for \( \tau = 2 \) (see Appendix A). In particular, the fraction of links in the largest connected component can be shown to scale as \( N^{-1/2} \rho \ln \rho \) and therefore, the total number of links in the largest connected component scales as 

\[ G(\rho, 2) \sim \rho \ln \rho \sqrt{N} \ln N \]
Another interesting regime is where $\rho \to 1^+$ or $z \ll 1$. This will correspond to percolation very close to the threshold $q_c$. If $z \ll 1$ we can look at the first few terms of (Eqn. 9):

$$-\alpha z = \sum_{n=1}^{\infty} \frac{(-z)^n}{n!(n+2-\tau)}$$

$$\approx -\frac{z}{3-\tau} + \frac{z^2}{2(4-\tau)}$$

$$\delta = zk_{\max}^{-1} = 2(1-\alpha(3-\tau))\frac{4-\tau}{3-\tau}k_{\max}^{-1}$$

$$= 2\left(1 - \frac{q_c}{q}\right)\frac{4-\tau}{3-\tau}N^{-1/\tau}$$

$$= 2\frac{4-\tau}{3-\tau}((\rho - 1)/\rho)N^{-1/\tau}$$

In other words, the scaling with the order parameter is again linear close to the finite size percolation threshold.

D. Links in the Largest Component After Percolation

The parameter $\delta$ is the fraction of total links before percolation that are in the largest connected component after percolation. With percolation threshold being small, most of the links are deleted during the percolation process. We are interested in the probability that a link not deleted during the percolation process is part of the largest connected component. Equivalently, what fraction of the links that remain after percolation are part of the largest connected component?

To answer this, note that the number of links that remain after the percolation is closely approximated by

$$H(N, \rho) = \rho q_c(k)$$

Using (Eqn. 11) to calculate $q_c$ and the scaling results for $\delta$, the probability that a link left after the percolation is part of the largest connected component can be calculated as follows.

$$\mathcal{G}(N, \rho) \sim \left\{ \begin{array}{ll}
\rho \ln \rho & \tau = 2 \\
\rho^{1/(3-\tau)}N^{2/\tau-2} & 2 < \tau < 3
\end{array} \right. \quad (20)$$

Interestingly, for $\tau > 2$, the fraction of links that are part of the giant connected component constitute to an infinitesimally small fraction of all links after the percolation. Only for $\tau = 2$ a finite fraction of all the links that remain after the percolation belong to a single connected component.

III. SIMULATIONS

Figure 2 shows the scaling of the largest component after bond percolation as a function of the size of the original network $N$ when $q/q_c$ is a constant [4] using network sizes ranging from 1,000 – 1,000,000 nodes for different values of $\tau$. We compare the simulation result to the scaling we would predict (i.e., $N^{1-1/\tau}$). Simulation results suggest that our scaling law is correct as long as $k_{\max}$ is very large compared to $z$ and $q/q_c$ is in the order of one.

IV. CONCLUDING REMARKS

We investigated the properties of the size of the giant connected component just above the percolation threshold in heavy-tailed power-law random graphs, for which the percolation probability is known to be vanishingly small. By normalizing the percolation probability by the percolation threshold, we were able to trace the scaling behavior of the size of the giant connected component at very small percolation probabilities. In particular, we showed that $\delta$, the fraction of links in the giant connected component close to the percolation threshold, is proportional to the factor $1/k_{\max}$, for $k_{\max} \propto N^{1/\tau}$.

A. High Degree Nodes and the Giant Connected Component

Let us address the question of which nodes are most likely to be in the giant connected component when

$$\beta = \rho^{-1} - \frac{2}{3-\tau}$$

is in the order of one. Note that the percolation probability is $q_c = \langle k \rangle / ((\langle k^2 \rangle - \langle k \rangle)) \propto k_{\max}^{-\tau-3}$ for large $k_{\max}$, when $2 \leq \tau < 3$. As such, the probability that any node with $k$ links will have any edges left after percolation (with $\beta$ in the order of one) is around $kk_{\max}^{\tau-3}$. For this probability not to be negligible, we must have $k \sim k_{\max}^{3-\tau}$ or greater. Put in other words, take any node with degree $k_{\max}^{3-\tau}$ for any finite $\epsilon > 0$, then with probability one this node will lose all its edges after the bond percolation and will not be in the largest connected component after percolation at a multiple of the threshold.

B. Percolation Search

The percolation search algorithm, developed by the authors, is based on a probabilistic broadcasting scheme, with as small a probability of broadcast as possible. The success of this algorithm depends on finding a percolation probability which would result
FIG. 2: Examples of the scaling of the size of the largest connected component with the network size $N$ for different values of $\tau$ and $q/q_c$: From top down, $\tau = 2.0, q/q_c = 1.1$, $\tau = 2.2, q/q_c = 20.0$ and $\tau = 2.5, q/q_c = 10.0$. The scaling $N^{1-1/\tau}$ is depicted on all the plots. Even in the case of the top plot, where the largest component is as small as 100 our scaling predictions provide good matches to the simulations.
in most of the high-degree nodes to fall into one giant connected component, while as few low-connectivity nodes would be present in the same component. This will ensure that most of the search traffic will be carried out by high capability nodes that have assumed large degrees, while the low connectivity nodes will only occasionally participate in a search. Moreover, since a query message will be passed only along a few edges, the protocol will result in low overall traffic. The results in this paper (see the preceding discussions) show that random power-law networks are ideally suited for percolation search, and broadcasting with probability just above the percolation threshold leads to high query hit rates. For more details see [3].

APPENDIX A: CASE OF $\tau = 2$

In the case of $\tau = 2$, (Eqn. 9) reads as:

$$- \alpha z = \sum_{n=1}^{\infty} \frac{(-z)^n}{n!}$$

(A1)

The above series is related to the $Ei(z)$, or exponential integral function:

$$Ei(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} \, dt.$$  

The series representation of $Ei(z)$ is:

$$Ei(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} + \gamma + \ln|z|$$

Where $\gamma$ is the Euler-Mascheroni constant and is approximately 0.5772. This gives us:

$$\sum_{n=1}^{\infty} \frac{(-z)^n}{n!} = Ei(-z) - \gamma - \ln|z|$$

(A2)

We can then write (Eqn. A1) as:

$$- \alpha z = Ei(-z) - \gamma - \ln z$$

(A3)

when $\tau = 2$. For $z \gg 1$, $Ei(-z) \to 0$, or $z/\ln(z) \approx \alpha^{-1} = \rho$ or equivalently, $z \approx \rho \ln(\rho)$.

APPENDIX B: VERIFYING THE PERCOLATION THRESHOLD

The discussions in this paper highlighted the significance of $q_c(N) = (k)/(k^2 - k)$ as the true finite size percolation threshold. Strictly speaking, however, $q_c(N) > 0$ (for any fixed $N$) is the percolation threshold of an infinite size random graph whose degree distribution is given by the truncated PL: $p_k = A r k^{-\tau}$ for $k \in (1, k_{max})$, and $p_k = 0$ for $k > k_{max}$ (we let $k_{max}$ scale as $N^{1/\tau}$, and hence is fixed for a given $N$). We know from percolation theory [4, 5] that for a random graph with a fixed $k_{max}$, and the degree distribution satisfying certain

FIG. 3: For a finite network we define $S(q)$ as the size of the largest component as function of percolation probability $q$ and find the value of $q$ where $dS/dq$ is maximized (marked by the Simulation points). We compare that value with the percolation threshold of an infinite network with the same degree distribution: $q_{c_c} = (k)/(k^2 - k)$. We see that the ratio of these two values is approximately constant and tends towards unity as the network grows in size.
additional technical conditions, the infinite size percolation threshold can be shown to be the limit of a uniformly convergent series, and that the phase transition for any network of finite but large size, $M$, will also happen close to the infinite size percolation threshold. The exact dependence of the network size $M_0$ (above which the infinite size percolation threshold will be a good approximation for finite size percolation) on the nature of the degree distribution and $k_{\text{max}}$ (especially when it is large) is currently unknown. In this paper, we are assuming that for any fixed but large $N$ (which determines $k_{\text{max}}$, and hence $q_c(N)$, for the family of random networks), the infinite size percolation threshold is also a good approximation for a finite network of size $N$.

Figure 3 directly compares what we might call the empirical threshold to the infinite network model. If $S(q)$ is the size of the largest component for a fixed network size with a bond percolation probability $q$, we define the empirical threshold as the point where $dS(q)/dq$ is maximized. For all networks, the empirical threshold is very close to the infinite network model, and as $N$ grows, these two agree more closely.