Real eigenvalues in symmetrised multi-well potentials with gain and loss

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Abstract. Balanced gain and loss leads to stationary dynamics in open systems. This occurs naturally in \( PT \)-symmetric systems, where the imaginary part of the potential describing gain and loss is perfectly antisymmetric. While this case seems intuitive, stationary dynamics is also possible in asymmetric open systems. We will show that open multi-well quantum systems can possess completely (or partly) real spectra if their Hamiltonian is (semi-)symmetrised. A simple matrix model for the description of two and three-mode systems is used as an example, for which analytical symmetrised solutions are derived. We will further explicitly show how symmetrisation can be used to systematically find two-mode systems with a stable, stationary ground state. It will also become clear why only \( PT \)-symmetric two-mode systems can also have stationary excited states.

1. Introduction

Non-Hermitian Hamiltonians are used in physics since the emergence of quantum mechanics (QM) to solve certain problems which are either not solvable within the framework of ordinary QM or only with great difficulty [1]. They are, for example, particularly suited to effectively describe open quantum systems (see e.g. [2]), which may be arbitrarily hard to treat by explicitly taking the environment into account. However, the interest in non-Hermitian Hamiltonians rapidly increased after Bender and Boettcher introduced the concept of \( PT \) symmetry in 1998 [3]. A \( PT \)-symmetric open quantum system only requires a symmetric real potential and an antisymmetric imaginary potential and possesses either a completely or partly real spectrum. Since its first observation in optical wave guides [4], \( PT \) symmetry was also observed, among other fields, in simple mechanical [5] and electrical systems [6, 7]. In recent years advances were made towards technical applications in superconducting wires [8, 9], \( PT \) lasers [10, 11], synthetic materials [12, 13], NMR [14], and even in wireless power
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Of particular interest is also the study of exceptional points in $\mathcal{PT}$-symmetric open quantum systems. Even a $\mathcal{PT}$-symmetric Hamiltonian was proposed quite recently to approach the Riemann hypothesis.

The concept of $\mathcal{PT}$ symmetry is intriguingly powerful, yet simple and versatile in application. It must be noted, however, that already several years before its introduction, Scholtz, Geyer and Hahne had published a paper on quasi-Hermitian Hamiltonians in non-Hermitian QM. Quasi-Hermiticity does not rely on the strict symmetry conditions on the potentials which $\mathcal{PT}$ symmetry requires, but allows for the occurrence of real eigenvalues in asymmetric potentials and even in cases with either pure gain or loss. The latter was experimentally observed a while ago in anti-$\mathcal{PT}$-symmetric systems. In a similar fashion as $\mathcal{PT}$ symmetry, the applicability of quasi-Hermitian QM ranges from scattering problems to constant-intensity waves, which were recently experimentally realised with pressure waves. Quasi-Hermiticity can also be used to define a generalised entropy functional for non-Hermitian quantum systems.

There are other types of non-$\mathcal{PT}$-symmetric systems with similar properties. Anti-$\mathcal{PT}$ symmetry was, for example, observed for coupled atomic spin waves, in electrical circuits, and in diffusive systems. Another type of non-$\mathcal{PT}$-symmetric potentials is based on the connection between the Zakharov–Shabat spectral problem and the Schrödinger eigenvalue problem which, for example, allows for the construction of unidirectionally invisible asymmetric potentials.

The characteristic feature of such non-$\mathcal{PT}$-symmetric potentials is clearly their lack of any obvious symmetry. Hence they can be applied to situations where gain or loss are either uncontrolled or even uncontrollable. This could be useful for quantum transport in chain potentials with gain and loss. Such systems were recently discussed in the context of $\mathcal{PT}$-symmetric quantum dot chains. However, the realisation of stable $\mathcal{PT}$-symmetric potentials is demanding and small perturbations will immediately break the symmetry. A first step towards such applications with asymmetric potentials has been made by Lunt et al., who reported on the formation of a steady ground state in a non-$\mathcal{PT}$-symmetric, two-mode Bose-Einstein condensate. Due to the non-linear properties of the condensate the system is stable with respect to small asymmetries in gain and loss. The aim of the present paper is to continue this line of thought and discuss asymmetric systems in the broader context of quasi-Hermitian QM by introducing the concept of symmetrisation, which allows for a generalisation to few-mode systems.

The paper is organised as follows. In section 2 we will give a short and rather general overview of the concept of symmetrisation, which leads to a complex-conjugated eigenvalue structure. However, in contrast to other works (see e.g. and references therein) we will consistently use right-handed and left-handed representations which allows for a generalisation we call semi-symmetrisation. Section 2.1 will introduce the basic formalism in non-Hermitian QM and establish its connection to the presence of an antiunitary symmetry. In section 2.2 the properties of the corresponding symmetrisation operators will be discussed for cases with real and complex spectra.
We will briefly investigate the application of symmetrisation to non-linear quantum systems. Section 2.3 deals with the application of this concept to open quantum systems with complex symmetric potentials. In section 3 the concept of symmetrisation will be applied to linear two-mode and three-mode quantum systems. It is shown that a two-mode open quantum system can either only be $\mathcal{PT}$-symmetric or semi-symmetrisable. In the latter case there occur isolated complex resonances in the spectrum of the system. Systems with more than three modes can be symmetrised completely. This allows for the occurrence of second and third-order exceptional points. Last but not least an anti-$\mathcal{PT}$-symmetric potential will serve as an example for a completely symmetrised three-mode system.

2. Theory

In the following we will discuss under which circumstances a non-Hermitian Hamiltonian has a spectrum which consists either entirely or partly of real or complex-conjugated eigenvalues. We will start with an intuitive motivation for the concept of symmetrisation and make connections to the fundamental symmetries in QM. Later we will discuss properties of this concept in different scenarios.

2.1. Symmetrisation

Basically, there are two possible approaches to ensure that the spectrum of an operator is complex conjugate to itself, which is the statement of Wigner’s theorem [46,47]. Since these concepts are essential for the present paper, we will give a short review of this topic. For the sake of simplicity and clarity we will assume that the spectra considered are discrete and non-degenerate. The case with degeneracy is, for example, described in [48].

Antiunitary symmetry: Consider a non-Hermitian operator $\mathcal{H}$ and the corresponding right-hand Schrödinger equation

$$\mathcal{H}|\psi_n\rangle_R = E_n |\psi_n\rangle_R,$$

where $E_n$ are the complex eigenvalues of $\mathcal{H}$ with right eigenstates $|\psi_n\rangle_R$. By introducing an antilinear operator $\mathcal{A}$, where antilinear means

$$\mathcal{A}(\alpha |\alpha\rangle + \beta |\beta\rangle) = \alpha^* \mathcal{A} |\alpha\rangle + \beta^* \mathcal{A} |\beta\rangle,$$

one finds a right-hand eigenvalue equation $\mathcal{H} |\tilde{\psi}_n\rangle_R = E_n^* |\tilde{\psi}_n\rangle_R$ with $|\tilde{\psi}_n\rangle_R = \mathcal{A} |\psi_n\rangle_R$ if

$$[\mathcal{A}, \mathcal{H}] = 0.$$

Therefore, if $E_n \in \mathbb{C}$ is in the spectrum of $\mathcal{H}$ so is $E_n^*$; the eigenvalues arise in complex-conjugate pairs. Note that this is in particular true for real eigenvalues [49], which can be considered as their own complex conjugates.
In physical terms this antiunitary symmetry can be related to time-reversal invariance [27], as the time-reversal operator is antilinear [46]. This does, however, not exclude more general antiunitary symmetries, of which \( PT \) symmetry [3] is a prominent recent example. It is well known that the spectra of \( PT \)-symmetric Hamiltonians, i.e. which satisfy \([\mathcal{H}, PT] = 0\), are in general complex conjugate and can thus also be partially or entirely real [3, 50].

**Symmetrisation:** Instead of (1) we will now consider its adjoint equation. The complex conjugate of \( E_n \) then appears naturally on the right-hand side and we can introduce a linear operator \( \hat{\rho}_L \) in such a way that
\[
\langle \psi_n | R \mathcal{H}^\dagger \hat{\rho}_L^\dagger = \langle \psi_n | R \hat{\rho}_L^\dagger E_n^* . \tag{4}
\]
Requiring that the operator \( \hat{\rho}_L \) satisfies the relation
\[
\hat{\rho}_L \mathcal{H} = \mathcal{H}^\dagger \hat{\rho}_L \tag{5}
\]
we find a left-hand eigenvalue equation
\[
\langle \tilde{\psi}_n | L \mathcal{H} = \langle \tilde{\psi}_n | L E_n^* \tag{6}
\]
with a left eigenstate \( |\tilde{\psi}_n\rangle_L = \hat{\rho}_L |\psi_n\rangle_R \neq 0 \), which must not necessarily be normalised. Since the Hamiltonian considered is not Hermitian, its eigenbasis is bi-orthogonal (cf. [1, 27, 51]). This means that the left and right eigenstates of the same eigenvalue do not coincide, as it would be the case for Hermitian operators.

Equation (5) shows that the combination of \( \mathcal{H} \) and \( \hat{\rho}_L \) is Hermitian if \( \hat{\rho}_L \) is Hermitian. A Hamiltonian satisfying (5) is thus called *symmetrisable* (see [33, 52]) with respect to the left symmetrisation operator \( \hat{\rho}_L \). This corresponds to the method proposed by Darboux [53, 54], which relates the spectra of the Hamiltonians \( \mathcal{H} \) and \( \mathcal{H}^\dagger \). The left symmetrisation operator can also be understood as a symmetry of \( \mathcal{H} \) within a superoperator framework [47]. It transforms a right eigenstate of \( \mathcal{H} \) into a left eigenstate of \( \mathcal{H} \) with the complex conjugate eigenvalue. This seems to stand in sharp contrast to the presence of an antiunitary symmetry with respect to \( A \), where \( |\tilde{\psi}_n\rangle_R = A |\psi_n\rangle_R \).

However, let us introduce the antilinear operator
\[
\mathcal{T}_L = \sum_m L|\psi_m\rangle K \langle \psi_m|_L , \tag{7}
\]
where \( K \) is the operator associated with complex conjugation. It is a simple exercise to check that \( \mathcal{T}_L \) also satisfies the relation (5), that is \( \mathcal{T}_L \mathcal{H} = \mathcal{H}^\dagger \mathcal{T}_L \). Then
\[
\hat{\rho}_L = \mathcal{T}_L A \tag{8}
\]
is linear and solves (5) if \( A \) satisfies (3). This shows that symmetrisation of \( \mathcal{H} \) is equivalent to generalised \( PT \) symmetry with some linear operator \( \mathcal{P} \) and some antilinear
operator $\mathcal{T}$ \cite{55}. However, this requires $\hat{\rho}_L$ to be unitary; only then $\mathcal{A}$ is antiunitary and thus an actual symmetry.

Analogously to (5) we define the right symmetrisation operator $\hat{\rho}_R$ satisfying

\[ \mathcal{H} \hat{\rho}_R = \hat{\rho}_R \mathcal{H}^\dagger \] (9)

via the left-hand eigenvalue equation

\[ \mathcal{H}^\dagger |\psi_n\rangle_L = E^*_n |\psi_n\rangle_L . \] (10)

The operator $\hat{\rho}_R$ transforms left eigenstates of $\mathcal{H}$ into right eigenstates of $\mathcal{H}$, $\hat{\rho}_R |\psi_n\rangle_L = |\tilde{\psi}_n\rangle_R$, thus complementing $\hat{\rho}_L$. As before, we can define an antilinear operator

\[ \mathcal{T}_R = \sum_m |\psi_m\rangle_K \langle \psi_m|_R \] (11)

satisfying $\mathcal{H} \mathcal{T}_R = \mathcal{T}_R \mathcal{H}^\dagger$, so that

\[ \hat{\rho}_R = \mathcal{A} \mathcal{T}_R \] (12)

satisfies (9).

Further, the combination $\hat{\rho}_R \hat{\rho}_L$ commutes with $\mathcal{H}$ by definition,

\[ \hat{\rho}_R \hat{\rho}_L \mathcal{H} = \hat{\rho}_R \mathcal{H}^\dagger \hat{\rho}_L = \mathcal{H} \hat{\rho}_R \hat{\rho}_L . \] (13)

Therefore, $\hat{\rho}_R \hat{\rho}_L$ and $\mathcal{H}$ share the same eigenbasis. Thus, $\hat{\rho}_R \hat{\rho}_L$ is diagonal with respect to the bi-orthogonal product

\[ \langle \psi_n | \psi_m \rangle = \langle \psi_n | \psi_m \rangle_R = \delta_{nm} \] (14)

defined in [1]. With a suitable choice of the symmetrisation operators the eigenvalues of $\hat{\rho}_R \hat{\rho}_L$ are unity. If $\hat{\rho}_L$ is invertible, then a Hamiltonian is, if at all, always symmetrisable from both sides due to

\[ \hat{\rho}_R \propto \hat{\rho}_L^{-1} \]. (15)

For finite-dimensional Hilbert spaces there exists another important property, which will be used in section 3. With the cyclic property of the determinant, $\det(AB) = \det A \det B$ for square matrices $A$ and $B$, one can easily show that \[56\]

\[ \det(\mathcal{H} - E_n \mathbb{1}) \det(\hat{\rho}_L \hat{\rho}_R) = \det(\hat{\rho}_L) \det(\mathcal{H} - E_n \mathbb{1}) \det(\hat{\rho}_R) = \det(\hat{\rho}_L \mathcal{H} \hat{\rho}_R - \hat{\rho}_L E_n \hat{\rho}_R) = \det(\mathcal{H}^\dagger - E_n \mathbb{1}) \det(\hat{\rho}_L \hat{\rho}_R) \] (16)

for any Hamiltonian $\mathcal{H}$ satisfying (5). Thus, if the determinants of $\hat{\rho}_L$, $\hat{\rho}_R$, and $\hat{\rho}_R \hat{\rho}_L$ are nonzero, i.e. if the operators are invertible, then $\mathcal{H}$ and $\mathcal{H}^\dagger$ have the same characteristic equations and therefore share the same set of eigenvalues. Further, because

\[ \det(\mathcal{H} - E_n \mathbb{1})^* = \det(\mathcal{H}^* - E_n^* \mathbb{1}) = \det(\mathcal{H}^\dagger - E_n^* \mathbb{1}) = \det(\mathcal{H} - E_n^* \mathbb{1}) \], \label{17}
where we used \( \det A^\top = \det A \) and \( (\det A)^* = \det A^* \), the coefficients of the characteristic polynomial must be real. Because of the fundamental theorem of algebra the spectrum consists only of complex-conjugated pairs. In contrast to [5], which is only a necessary condition [24], the reality of the coefficients of the characteristic polynomial is a sufficient condition to obtain a complex-conjugated spectrum [57], as it ensures that the kernel of the symmetrisation operators are empty. We will return to this discussion in section 2.2.3.

An operator satisfying (13) is called either quasi-Hermitian [23, 58], \( \hat{\rho}_L \)-Hermitian [59–61], pseudo-Hermitian [62–67], crypto-Hermitian [68–70], or generalised Hermitian [71], depending on the properties of \( \hat{\rho}_L \). However, in contrast to these works the concept of symmetrisation considered here does not require the symmetrisation operators to be necessarily invertible [24, 72] or their kernel to be empty.

Further, we want to emphasise that we do not understand symmetrisation as a generalisation or extension of quantum mechanics as in [45, 73], where the inner product is redefined by (14). By doing so, one defines a new ‘physical’ Hilbert space in which Hamiltonians satisfying (5) are effectively Hermitian. In this context the operators \( \hat{\rho}_L \) and \( \hat{\rho}_R \) are often referred to as metric operators. Instead, the systems we consider are non-Hermitian because of boundary conditions, i.e. they effectively describe open quantum systems with non-unitary evolution (see e.g. the concluding remarks in [74]). Such Hamiltonians can, for example, be derived by using projection-operator methods [75, 76]. They are described by a complex potential [27], so that the resulting Hamiltonian is non-Hermitian but can be written in a complex-symmetric form [77]. We then consider (5) and (9) as properties of a given open quantum system where \( \hat{\rho}_L \) and \( \hat{\rho}_R \) are constructed specifically for the Hamiltonian [23].

2.2. Symmetrisation operators

2.2.1. Real spectrum Due to the property of transforming between left and right eigenstates, the symmetrisation operators can simply be written in terms of projection operators (as shown in [48, 65]). If the spectrum of \( \mathcal{H} \) is real, then a suitable choice of symmetrisation operators is given by

\[
\hat{\rho}_L = \sum_n p_n^L |\psi_n\rangle\langle\psi_n|_L, \tag{18a}
\]

\[
\hat{\rho}_R = \sum_n p_n^R |\psi_n\rangle\langle\psi_n|_R, \tag{18b}
\]

where the coefficients \( p_n^L \) and \( p_n^R \) are assumed to be constant. Equations (18a) and (18b) can fulfil both the trace condition (i.e. trace equal to one) and the positivity condition (i.e. the operator is positive, cf. [78]) if we choose real coefficients \( p_n^L, p_n^R \geq 0 \) satisfying

\[
\sum_n p_n^L = 1 = \sum_n p_n^R. \tag{19}
\]

Thus, (18a) and (18b) are the density operators associated with the ensembles \( \{p_n^L, |\varphi_n\rangle_L\} \) and \( \{p_n^R, |\varphi_n\rangle_R\} \), where \( p_n^L \) and \( p_n^R \) are ‘probabilities’ to be in the left or right
eigenstate with energy \( E_n \), respectively. The corresponding time evolution is governed by the generalised von Neumann equations

\[
\begin{align*}
\frac{i}{\hbar} \frac{d\hat{\rho}_L}{dt} &= \left[ \hat{H}^\dagger \hat{\rho}_L - \hat{\rho}_L \hat{H} \right], \quad (20a) \\
\frac{i}{\hbar} \frac{d\hat{\rho}_R}{dt} &= \left[ \hat{H} \hat{\rho}_R - \hat{\rho}_R \hat{H}^\dagger \right]. \quad (20b)
\end{align*}
\]

If the Hamiltonian is left and right symmetrisable by the symmetrisation operators (18a) and (18b) then (20a) and (20b) vanish due to (5) and (9), respectively, so that \( |\hat{\rho}_L|^2 \) and \( |\hat{\rho}_R|^2 \) are conserved. This is in agreement with the reality of the spectrum if we consider (18a) and (18b) as density matrices with respect to ordinary QM.

Another possible choice of the coefficients in (18a) and (18b) is given by

\[
p_n^L = p_n^R = \sqrt{p_n}, \quad \sum_n p_n = 1.
\]

(21)

Then

\[
\hat{\rho}_R \hat{\rho}_L = \sum_n p_n^R |\psi_n\rangle_L \langle \psi_n|_R
\]

(22)

is a density operator with respect to the bi-orthogonal product (14), which is governed by the von Neumann equation

\[
\frac{i}{\hbar} \frac{d}{dt} \hat{\rho}_R \hat{\rho}_L = [\hat{H}, \hat{\rho}_R \hat{\rho}_L].
\]

(23)

Note that \( |\hat{\rho}_R \hat{\rho}_L|^2 \) is also conserved if the Hamiltonian is symmetrisable because of (13).

Both these choices are generalisations of the Hermitian case, where left and right eigenstates coincide, depending only on the inner product one relies on. For a Hermitian system either the operators (18a) and (18b) or the operator (22) correspond to the usual definition of the density operator, while the respective other choice must be considered unphysical due to the coefficients being either square roots or squares of the probabilities. However, whether or not any of these operators is, in the end, physically meaningful depends solely on the probabilities and their interpretation.

Since we use the ordinary inner product, we view \( \hat{\rho}_L \) defined by (18a) as the usual density operator which is diagonal with respect to the right eigenstates. It corresponds to a mixed state reflecting our incomplete knowledge about the state of the non-Hermitian, open quantum system. By considering the time derivative of the expectation value of \( \hat{\rho}_L \) with respect to a right state,

\[
\frac{i}{\hbar} \frac{d}{dt} \langle \psi | \hat{\rho}_L | \psi \rangle_R = \langle \psi | \hat{\rho}_L \hat{H} - \hat{H}^\dagger \hat{\rho}_L | \psi \rangle_R + \langle \psi | i \frac{d}{dt} \hat{\rho}_L | \psi \rangle_R,
\]

(24)

one finds that \( \langle \hat{\rho}_L \rangle_R \) is a conserved quantity if (5) holds. The statement (24) is true in general [64] and thus is also applicable if the spectrum is not entirely real.

We want to point out again that symmetrisation operators are not unique [79,80]. In fact, one can construct an infinite number of symmetrisation operators for one and the
same Hamiltonian \[81\]. The specific choice of the symmetrisation operators is, however, not important as long as the Hamiltonian and the boundary conditions are the same. Since any operators of the form \(18a\) and \(18b\) satisfy \(5\) and \(9\), which are necessary for the properties of \(H\), the choice of the coefficients can be considered as a gauging.

2.2.2. Complex spectrum If the spectrum is not entirely real, we know from the discussions above that the complex eigenvalues must arise in conjugate pairs if the Hamiltonian is symmetrisable. The corresponding symmetrisation operators \[48, 79\] are generalisations of the operators \(18a\) and \(18b\),

\[
\hat{\rho}_L = \sum_n p_{n}^{L} |\psi^0_n\rangle\langle\psi^0_n|_L + \sum_m \left[ p_{m}^{L} |\psi^{-}_n\rangle\langle\psi^{+}_n|_L + p_{m}^{-L} |\psi^{+}_n\rangle\langle\psi^{-}_n|_L \right], \tag{25a}
\]

\[
\hat{\rho}_R = \sum_n p_{n}^{R} |\psi^0_n\rangle\langle\psi^0_n|_R + \sum_m \left[ p_{m}^{R} |\psi^{-}_n\rangle\langle\psi^{+}_n|_R + p_{m}^{-R} |\psi^{+}_n\rangle\langle\psi^{-}_n|_R \right], \tag{25b}
\]

where \(n\) runs over all real energies \((0)\) and \(m\) runs over all complex-conjugate pairs of energies with positive (+) and negative (−) imaginary parts. Their evolution is still governed by \(20a\) and \(20b\).

The operators \((25a)\) and \((25b)\) can no longer be considered as density operators. While the first terms are diagonal with respect to the energy eigenstates, the second terms are non-diagonal. If we calculate the trace of either \(\hat{\rho}_L\) or \(\hat{\rho}_R\) with respect to the right eigenstates, we find \((19)\). Thus, the additional terms corresponding to the complex part of the spectrum are traceless; \(\hat{\rho}_L\) and \(\hat{\rho}_R\) are indefinite. For this reason there occur states of the Hamiltonian with negative norm, which must to be considered unphysical \[23\] and may be excluded by superselection rules \[82\]. Nevertheless, the occurrence of symmetric pairs of complex eigenvalues can be considered physical in general, as these can be understood as emission and absorption phenomena \[34\].

The concept of indefinite metrics is not new both in physics \[83, 84\] and mathematics \[85, 86\]. Such an indefinite metric gives rise to a Krein space \[59, 87\], in which, for example, non-Hermitian operators with real spectrum are self-adjoint.

2.2.3. Isolated complex resonances In this section we consider the case in which the symmetrisation operators are not invertible. However, as it was not directly required for them to be invertible in the first place, symmetrisation of \(\mathcal{H}\) is still possible to a certain degree.

Consider a left symmetrisation operator \(\hat{\rho}_L\) with \(\text{det} \hat{\rho}_L = 0\). Clearly there is no inverse of \(\hat{\rho}_L\) and its kernel is non-empty. Nevertheless it is still possible to satisfy \(5\) if the elements in the kernel of \(\hat{\rho}_L\) are exclusively right eigenstates \(|\varphi_n\rangle_R\) of \(\mathcal{H}\). For these states \((4)\) is fulfilled trivially as \(\hat{\rho}_L |\varphi_n\rangle_R = 0\). However, the corresponding eigenvalues are neither real nor part of a conjugate pair; these energies form isolated complex resonances in the spectrum of \(\mathcal{H}\).

Since \((13)\) still remains valid, \(\hat{\rho}_R\hat{\rho}_L\) can be considered as an identity with respect to the right eigenstates of \(\mathcal{H}\) which are not in the kernel of \(\hat{\rho}_L\) and acts as annihilation.
operator for states from the kernel of the left symmetrisation operator. Though $\hat{\rho}_R$ is not the inverse of $\hat{\rho}_L$, they are semi-inverse. Following the definition $ABA = A$ \cite{88}, where $B$ is the semi-inverse of $A$, we find
\begin{equation}
\hat{\rho}_L \hat{\rho}_R \hat{\rho}_L |\psi_n\rangle_R = \hat{\rho}_L \hat{\rho}_R |\tilde{\psi}_n\rangle_L = \hat{\rho}_L |\psi_n\rangle_R.
\end{equation}
Hence we call such a system semi-symmetrisable.

Semi-symmetrisable operators possess the properties of symmetrisable operators on a subspace spanned by eigenstates of the Hamiltonian which are not in the kernels of the symmetrisation operators. The dimension of this subspace is given by the rank of $\hat{\rho}_L$. With respect to the definitions \ref{18a} and \ref{25a}, it is obvious that, because of bi-orthogonality, for any left eigenstate we exclude from the sums, the corresponding right eigenstate is part of the kernel of the symmetrisation operator. The same applies to the definitions \ref{18a} and \ref{25a} of the right symmetrisation operator. Since the rank of the symmetrisation operators can take any value between 1 and the full rank, we conclude that any operator with a discrete number of real eigenvalues is semi-symmetrisable, even if the spectrum is not entirely discrete. For this reason one can apply symmetrisation to a specific subspace, for example obtained by means of some approximation, without the need to care for the entirety of the spectrum. A specific example for a physical system which is solely semi-symmetrisable is the two-mode system with arbitrary gain and loss discussed in section 3.1.

\subsection{2.2.4. Non-linear operators}
In analogy with \cite{89}, where a Bose-Einstein condensate in a $\mathcal{PT}$-symmetric double-well potential is considered, we will now briefly investigate the existence of metric operators in non-linear systems of the form
\begin{equation}
\mathcal{H} = \mathcal{H}_{\text{lin}} + f(\psi),
\end{equation}
where $f(\psi)$ is a non-linear functional of the wave function $\psi$ and $\mathcal{H}_{\text{lin}}$ is a linear Hamiltonian. By symmetrising \ref{27} via \ref{5} and assuming that \ref{5} holds for the linear part we find
\begin{equation}
\hat{\rho}_L f(\psi) = f(\hat{\rho}_L \psi) \hat{\rho}_L.
\end{equation}
If the non-linear functional $f$ fulfils this relation then the spectrum of the non-linear operator $\mathcal{H}$ is also conjugate.

We will now consider a specific type of non-linearity,
\begin{equation}
f(\psi) = |\psi|^2,
\end{equation}
which, for example, describes the contact interaction in the Gross-Pitaevskii equation \cite{90, 91} for Bose-Einstein condensates in optical potentials \cite{2, 89, 92, 94}. However, beyond ultra-cold atoms there is a broad range of other systems described by such a discrete non-linear Schrödinger equation \cite{95, 96}, such as light in paraxial approximation with a Kerr non-linearity \cite{92, 97, 98}, polarons \cite{99, 101}, excitons \cite{102}, or even the
contraction of proteins \[103\]. This has two immediate consequences: first, since \(|\psi|^2\)
is scalar, we only have to consider complex conjugation in (28), and second, since symmetrisation operators are linear, \(\hat{\rho}_L\) commutes with \(|\psi|^2\). Because the absolute square is invariant under complex conjugation, we conclude that it must be invariant with respect to the insertion of \(\hat{\rho}_L\). Therefore \((\hat{\rho}_L)^\dagger \hat{\rho}_L = 1\), i.e. the symmetrisation operator must be unitary, which is the case if it is connected to an antiunitary symmetry via (8).

In fact, because of their Hermiticity, unitary symmetrisation operators are involutory. Since \(\hat{\rho}_R\) is the inverse of \(\hat{\rho}_L\) (if the inverse exists at all), this implies the equality of the left and right symmetrisation operators, \(\hat{\rho}_R = (\hat{\rho}_L)^\dagger = \hat{\rho}_L\). This leaves us only with a small class of symmetrisation operators which allow for the treatment of non-linear, non-Hermitian open quantum systems. For the contact interaction considered above this class is given by the cyclic group \(\mathbb{Z}_2\), which corresponds to generalisations of the parity operator \[34\]. The symmetry connected with it are \(\mathcal{PT}\) symmetry and its generalisations.

2.3. Complex potentials

We now want to focus on open quantum systems described by Hamiltonians with a complex potential, which describes gain and loss in the system. In the simplest case such Hamiltonians are complex symmetric, i.e. \(\mathcal{H}^\dagger = \mathcal{H}\), because only the diagonal is complex. We want to emphasise that this involves no loss of generality, since any matrix can be transformed into a complex-symmetric form \[77\].

For complex-symmetric Hamiltonians the set of bi-orthogonal states is complete \[27,104\]. For any right eigenstate of \(\mathcal{H}\) the corresponding left eigenstate is then easy to find: conjugating (10) yields

\[
\mathcal{H}^\dagger \tilde{\psi}_n^* \rangle_L = \mathcal{H} \tilde{\psi}_n^* \rangle_L = E_n^* \tilde{\psi}_n^* \rangle_L .
\]

(30)

It is immediately clear that left and right eigenstates of a complex-symmetric Hamiltonian are complex conjugate with respect to their bra or ket forms, respectively. Note that the left bra and right ket eigenstates for complex-symmetric Hamiltonians are transpose of each other, because they share the symmetry of the Hamiltonian.

An interesting property arises in the context of complex potentials. If one considers \[5\] as a property of a given system with an imaginary potential, then \(9\) is a property of the same system with the negative imaginary potential. In other words, if the system is symmetrisable from both sides simultaneously, then there exist two suitable gain and loss distributions, each corresponding to one symmetrisation operator. The inversion of gain and loss corresponds to time reversal, so that

\[
\mathcal{H} = \mathcal{T} \mathcal{H}^\dagger \mathcal{T}
\]

(31)

with the involutory time-reversal operator \(\mathcal{T}\). By plugging this into \[5\] and comparing the result to \(9\) we find

\[
\hat{\rho}_R = \mathcal{T} \hat{\rho}_L \mathcal{T}.
\]

(32)
This is a consequence of the dual character of $H$ and $H^\dagger$, which is in case of complex-symmetric Hamiltonians defined with respect to the complex conjugate rather than the Hermitian adjoint. An interesting discussion on this duality can be found in [47], where it is shown that left eigenstates evolve backwards in time compared to right eigenstates and vice versa.

**Symmetric potentials** There is a distinguished choice for the symmetrisation operators that appears as a natural extension of the Hermitian case. Since there is no bi-orthogonal basis in Hermitian quantum systems, (18a) and (18b) coincide in this case. We may assume that there exists an operator $P$ with the property

$$P^2 |\psi_n\rangle = \exp(i\varphi_n) |\psi_n\rangle,$$

(33)

where $|\psi_n\rangle$ can be any eigenstate of the bi-orthogonal basis. Equation (33) defines a parity operator, which can be chosen such that $P^2 = 1$. This immediately implies that

$$\hat{\rho}_R \hat{\rho}_L = P^2$$

and thus

$$P = \hat{\rho}_L = \hat{\rho}_R = P^{-1},$$

(34)

i.e. $P$ is involutory.

Although in general in non-Hermitian systems there does not exist a pair of operators of this form obeying $\hat{\rho}_L = \hat{\rho}_R$, we may find a system with states obeying $P |\psi_n\rangle_R = |\tilde{\psi}_n\rangle_L$, where $|\tilde{\psi}_n\rangle_L$ equals the left eigenstate up to a phase factor. In this case we can expect $P$ to satisfy both (5) and (9). Combined with an antilinear operator $T = T_R = T_L$ this corresponds to $PT$ symmetry, where $T$ can be identified with the time-reversal operator and $P$ defines the indefinite inner product [105,106].

It is well known that $PT$-symmetric Hamiltonians must possess a real symmetric and antisymmetric imaginary potential (see e.g. [107]), causing the symmetry in the states used above. Due to this and in particular because $P$ defined by (34) is independent of the states of the Hamiltonian, which is not the case for $\hat{\rho}_L$ defined in (18a) and (25a), generalised $PT$ symmetry appears to be a distinguished choice.

Another example for symmetrisation operators which are independent of the states are the differential operators discussed in [24]. Similar to the $PT$-symmetric case this allows for all real spectra but with arbitrary gain and loss distributions. However, this requires gain and loss to depend locally on the changes of the potential, which makes such potentials hard to interpret physically and the corresponding symmetrisation operators are unbound.

### 3. Application to matrix models

In this section we apply the concept of symmetrisation discussed in section 2 to complex few-well potentials, where the imaginary part describes in and out-coupling of particles in each well. By assuming low temperatures and deep potential wells such systems can be described in the mean-field approximation by a matrix model with a complex
potential (see e. g. [93]). This specifically means that we deal with finite-dimensional Hilbert spaces.

3.1. Two-mode systems with arbitrary gain and loss

We will start by investigating the formation of steady states in an open linear two-mode system described by a non-linear non-Hermitian matrix model. Two-mode systems are well studied both for $\mathcal{PT}$-symmetric potentials [2,108–111] and recently also for arbitrary gain and loss potentials [112,113]. The most general symmetrisation operator for two-dimensional systems is also already known [114,115]. In the following we will discuss the two-mode system once more with respect to symmetrisation.

The Hamiltonian $H$ of the system reads in atomic units

$$H = \begin{pmatrix} \epsilon_1 + i\gamma_1 & -J \\ -J & \epsilon_2 + i\gamma_2 \end{pmatrix},$$

(35)

where $J > 0$ is the transition rate between the two sites and the real potential is given by the on-site energies $\epsilon_1$ and $\epsilon_2$. The imaginary part of the potential makes the Hamiltonian non-Hermitian and allows for the description of an effectively open system via the in- and out-coupling parameters $\gamma_1$ and $\gamma_2$. Note that the tunnelling rate could without loss of generality be set to $J = 1$, which corresponds to a specific choice of the energy scale. However, we will keep the parameter in the following, so that the reader may keep track of where the off-diagonal elements enter into calculations.

The reality of the characteristic polynomial of the Hamiltonian (35) allows for checking efficiently whether the Hamiltonian is symmetrisable or not. This evaluation yields the two conditions

$$\gamma_1 + \gamma_2 = 0,$$

(36a)

$$\epsilon_1 \gamma_2 + \epsilon_2 \gamma_1 = 0.$$  

(36b)

Of these (36a) is only satisfied for an antisymmetric imaginary potential and thus by a $\mathcal{PT}$-symmetric Hamiltonian. Since the reality of the characteristic polynomial is a sufficient condition for a real or complex-conjugate spectrum, we can conclude that there are no asymmetric potentials with entirely real spectrum. The symmetrisability condition (5) is, however, not sufficient. We will therefore continue investigating the symmetrisability of the Hamiltonian (35).

In two-dimensional spaces we may conveniently write the symmetrisation operator in terms of Pauli matrices,

$$\hat{\rho}_L = \sum_{n=0}^{3} \hat{\rho}_{L,n} \sigma_n,$$

(37)

where $\sigma_0 = 1$ and $\hat{\rho}_{L,n} \in \mathbb{R}$, so that $\hat{\rho}_L$ is Hermitian.
With (35) and (37) the symmetrisation condition (5) yields
\[
\begin{pmatrix}
\gamma_1 + \gamma_2 & 0 & \gamma_1 - \gamma_2 \\
0 & \gamma_1 + \gamma_2 & \epsilon_1 - \epsilon_2 \\
\gamma_1 - \gamma_2 & -(\epsilon_1 - \epsilon_2) & 2J \\
\end{pmatrix}
\begin{pmatrix}
\hat{\rho}_{L,0} \\
\hat{\rho}_{L,1} \\
\hat{\rho}_{L,2} \\
\end{pmatrix}
= 0.
\] (38)

A solution for this equation only exists if the determinant of the coefficient matrix vanishes, i.e. if
\[
(\gamma_1 + \gamma_2)^2 \left[ (\gamma_1 + \gamma_2)^2 - (\gamma_1 - \gamma_2)^2 + 4J^2 \right] + (\epsilon_1 - \epsilon_2)^2 \left[ (\gamma_1 + \gamma_2)^2 - (\gamma_1 - \gamma_2)^2 \right] = 0.
\] (39)

For a symmetric potential with \(\epsilon_1 = \epsilon_2\) and \(\gamma_1 = -\gamma_2 = \gamma\) both terms in (39) vanish identically. The solution of (38) then is
\[
\hat{\rho}_L = \begin{pmatrix}
\hat{\rho}_{L,0} & \hat{\rho}_{L,1} - i\frac{\gamma}{J}\hat{\rho}_{L,0} \\
\hat{\rho}_{L,1} + i\frac{\gamma}{J}\hat{\rho}_{L,0} & \hat{\rho}_{L,0}
\end{pmatrix},
\] (40)
which has the two degrees of freedom \(\hat{\rho}_{L,0}\) and \(\hat{\rho}_{L,1}\). We can obtain the corresponding right symmetrisation operator by simply changing the sign of \(\gamma\), which corresponds to the adjoint Hamiltonian. Hence, \(\hat{\rho}_R = (\hat{\rho}_L)^\dagger\). For \(\hat{\rho}_{L,0} = 0\) and \(\hat{\rho}_{L,1} = 1\) we retrieve the parity operator \(\mathcal{P}\).

For \(J \neq 0\) there is no other choice of parameters for which the first term in (39) vanishes. However, by choosing
\[
\epsilon_1 - \epsilon_2 = \pm i(\gamma_1 + \gamma_2)\sqrt{1 + \frac{J^2}{\gamma_1 \gamma_2}}
\] (41)
with \(-J^2 < \gamma_1 \gamma_2 < 0\), the condition (39) is satisfied. The lower bound of the product \(\gamma_1 \gamma_2\) stems from the fact that if we assume that only either \(\gamma_1\) or \(\gamma_2\) is nonzero, then (39) can only be satisfied if the modes are decoupled, i.e. \(J = 0\), or if there is no gain or loss at all, which contradicts the assumption. The upper bound is required for the expression under the square root to be negative, so that the difference of the on-site energies is a real quantity. Hence, a symmetrisable operator of the form (35) always requires the presence of both gain and loss. This appears to be intuitive, in particular if we recall that here real energies describe stationary states in open quantum systems. However, as shown in [24], there exist potentials in unbound systems which do not require gain at all, which is hard to grasp from a physical point of view.

From a mathematical point of view, there is an infinite number of systems which can fulfil the relation (41), as we only require a specific difference in the on-site energies. Since we are free, however, to gauge the energy scale, we only have to consider two physically different situations which correspond to a system \(\mathcal{H}\) with parameters \((\epsilon_1, \epsilon_2)\) according to (41) and its time-reversed counterpart \(\mathcal{H}^\dagger\). We suppose a system to evolve forward in time if particles in the system move from left to right. Consequently, if the particles are moving from right to left the system is evolving backwards in time.
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Figure 1. Real and imaginary parts of the eigenvalues (43) for (41) with $\epsilon_1 \geq \epsilon_2$ and the different parameterisations (a) $\gamma_1 = -\gamma_2 = \gamma$, (b) $\gamma_1 = 2\gamma$, $\gamma_2 = -\gamma/2$, and (c) $\gamma_1 = \gamma + 1/2$, $\gamma_2 = -\gamma + 1/2$. The solutions $\mu^+$ and $\mu^-$ are drawn as solid and dashed lines, respectively. For antisymmetric gain and loss there are two real solutions for $|\gamma| < J$, which form a conjugate pair for $|\gamma| > J$. For asymmetric gain and loss there exists one real and one complex solution, respectively, if the Hamiltonian is symmetrisable. The shaded areas indicate the region where no symmetrisation is possible.

By plugging (41) into (38) we can calculate the symmetrisation operator, which in matrix form reads

$$\hat{\rho}_L = \begin{pmatrix}
-\frac{J}{\gamma_1} & -i \left(1 \mp \sqrt{1 + \frac{J^2}{\gamma_1 \gamma_2}}\right) \\
-i \left(1 \mp \sqrt{1 + \frac{J^2}{\gamma_1 \gamma_2}}\right) & \frac{J}{\gamma_2}
\end{pmatrix} \hat{\rho}_{L,2} \quad (42)$$

and has only the one degree of freedom $\hat{\rho}_{L,2}$ in contrast to (40). This matrix is non-invertible as $\det \hat{\rho}_L = 0$ independently of the choice of $\hat{\rho}_{L,2}$. Hence, rank $\hat{\rho}_L = 1$, which immediately implies that there exists one real eigenvalue $\mu$. By suitably shifting the energy scale this real eigenvalue can take the form $\mu = 0$ and its existence is determined by $\det \mathcal{H} = 0$. The solutions of this condition are given by (41). We can thus state the situation as follows: there are two physically different systems with $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$, respectively, which satisfy (41). For both these systems there exist two imaginary potentials $\gamma_1 > \gamma_2$ (forward in time) and $\gamma_1 < \gamma_2$ (backwards in time), respectively, so that each system has one real eigenvalue. Semi-symmetrisation thus connects four Hamiltonians $\mathcal{H}_+, \mathcal{H}_-^+, \mathcal{H}_-, \mathcal{H}_-^+$, where the index refers to the sign in (41).

The general solutions for the eigenvalue of the two-mode system are given by

$$\mu = \frac{1}{2} \left[(\epsilon_1 + \epsilon_2) + i(\gamma_1 + \gamma_2) \pm \sqrt{[(\epsilon_1 - \epsilon_2) + i(\gamma_1 - \gamma_2)]^2 + 4J^2} \right]. \quad (43)$$

These solutions $\mu^+$ and $\mu^-$ are plotted in figure 1 for different parameterisations of gain and loss. Figure 1(a) shows the case with a symmetric potential, i.e. $\gamma_1 = -\gamma_2 = \gamma,
Figure 2. Real and imaginary parts of the eigenvalues (43) as functions of $\gamma_1$ and $\gamma_2$, where $\epsilon_1 > \epsilon_2$ are determined by (41). The two solid straight lines show the $\gamma$ axes for the parameterisations in figure 1 (b) and figure 1 (c). The black dashed lines indicate the borders between the regions where the system is symmetrisable and where it is not, and the white dashed lines are contour lines. The exceptional points are marked by dots. The case $\epsilon_1 < \epsilon_2$ looks identical but with (c) and (d) exchanged.

which is well known from $\mathcal{PT}$ symmetry and will not be discussed in the following; instead we refer to [110] for a detailed discussion on the $\mathcal{PT}$-symmetric two-mode system.

Figure 1 (b) shows a system with asymmetric gain and loss

$$\gamma_1 = 2\gamma, \quad \gamma_2 = -\frac{\gamma}{2}.$$  \hspace{1cm} (44)

The on-site energies are chosen symmetrically as

$$\epsilon_1 = -\epsilon_2 = \frac{\Delta\epsilon}{2},$$  \hspace{1cm} (45)

where $\Delta\epsilon$ corresponds to (41). Thus, $\epsilon_1 + \epsilon_2 = 0$ in (43). As discussed before there is an ambiguity in the sign of $\Delta\epsilon$. In the following we will use this freedom to choose the sign
such that $\epsilon_1 > \epsilon_2$. For $\gamma < J$ the spectrum resembles the symmetric case in figure 1(a) but with the imaginary part of $\mu^+$ rotated about the origin $\gamma = 0$, where the system is Hermitian, while $\mu^-$ remains entirely real. The real parts of $\mu^+$ and $\mu^-$ are still symmetric but not circular anymore. Instead they appear elliptic and are stretched by a factor of 1.25 at $\gamma = 0$. In the domain $|\gamma| > J$ there exist no real solutions at all, since (41) becomes imaginary and cannot be used to determine the on-site energies anymore. At $\gamma = \pm J$ no bifurcation occurs in comparison to the symmetric case, as the imaginary parts of $\mu^+$ are nonzero.

In figure 1(c) another parameterisation of asymmetric gain and loss is shown with

$$
\gamma_1 = \gamma + \frac{1}{2}, \quad \gamma_2 = -\gamma + \frac{1}{2},
$$

and (45), so that $\gamma_1 + \gamma_2 = 1$. We again find a deformation of the solutions of the symmetric case in figure 1(a). However, there are no solutions on the interval $[-1/2, 1/2]$, where the gain and loss parameter (46) have the same sign; i.e. for $-1/2 < \gamma < 0$ there is only gain and for $0 < \gamma < 1/2$ there is only loss. For $|\gamma| > J$ there are no solutions either, for the same reasons discussed above. In the remaining parameter regions there are solutions with either $\text{Im} \mu^+ = 0$ ($\gamma < 0$) or $\text{Im} \mu^- = 0$ ($\gamma > 0$). The offset of the respective other imaginary parts are given by $i(\gamma_1 + \gamma_2)$, which is zero in the symmetric case. We further find that the real parts of the energy eigenvalues diverge at $\gamma = \pm 1/2$. By approaching the regions where only either gain or loss occurs the pairs $(\gamma_1, \gamma_2)$ become increasingly unsuitable for sustaining the properties we want to impose onto the system. This is reflected by the divergence of the term (41) at $\gamma = \pm 1/2$ due to

$$
\gamma_1 \gamma_2 = \frac{1}{4} - \gamma^2,
$$

and the subsequent divergence of $\text{Re} \mu^+$ and $\text{Re} \mu^-$. 

Figure 2 shows the real and imaginary parts of the energy eigenvalues (43) as functions of $(\gamma_1, \gamma_2)$. The area where real solutions of (41) exist is enclosed by hyperbolas and the axes in the second and forth quadrants. In the first and third quadrants, there exists no solution at all, as they correspond to pure gain or loss. For $\gamma_1 = \gamma_2$, which corresponds to the off-diagonal, both $\mu^+$ and $\mu^-$ are real and $\Delta \epsilon = 0$, which corresponds to the case of a symmetric potential.

The slices in the $(\gamma_1, \gamma_2)$ space corresponding to the parameterisations (44) and (46) are shown by the two solid, straight lines. Figures 2 (c) and (d) clearly show that the imaginary parts of $\mu^+$ and $\mu^-$ are growing linearly along directions parallel to the diagonal. Hence we find a constantly shifted imaginary part by slicing parallel to the off-diagonal and a, as it were, rotated imaginary part by slicing along any other direction through the origin. Since both real parts diverge towards the axes, the energy eigenvalues may be arbitrarily large by rotating the slice around the origin.

It is worth mentioning that results similar to the results discussed above were previously found for Bose-Einstein condensates in a two-well potential with gain and loss [44]. By balancing asymmetries in the gain and loss with an asymmetric trapping
potential the formation of a real ground state is possible. This is in agreement with our results and corresponds to semi-symmetrisation. However, the discussions in [44] provide some insight into the physical meaning of this mathematically rather abstract concept. Lunt et al also discuss a two-mode system with asymmetric gain and loss but with a symmetric trapping potential, which corresponds to \( \epsilon_1 = \epsilon_2 = 0 \) and

\[
\gamma_1 = \pm J \sqrt{\frac{1+a}{1-a}}, \quad \gamma_2 = \mp J \sqrt{\frac{1-a}{1+a}},
\]

where \( a \in \mathbb{R} \) is a free parameter. This Hamiltonian also yields a single real eigenvalue, though it is not semi-symmetrisable because of (41). Hence, semi-symmetrisation is neither necessary nor sufficient for the occurrence of real eigenvalues.

### 3.2. Few-mode systems

Although few-mode systems with up to four dimensions can, in principle, be treated analytically as described in section 3.1, the number of equations grows quadratically, which is not feasible. However, by demanding the reality of the characteristic polynomial, the number of equations grows only linearly and enables us to investigate a three-dimensional model analytically. Since this condition is sufficient, we will deal with fully symmetrisable systems at first.

The Hamiltonian of the three-mode system possesses the same structure as the Hamiltonian (35),

\[
\mathcal{H} = \begin{pmatrix}
\epsilon_1 + i \gamma_1 & -J & 0 \\
-J & \epsilon_2 + i \gamma_2 & -J \\
0 & -J & \epsilon_3 + i \gamma_3
\end{pmatrix}.
\]

Evaluation of the imaginary parts of the coefficients of the characteristic polynomial yields

\[
\begin{align}
\gamma_1 + \gamma_2 + \gamma_3 &= 0, \\
\epsilon_1 \gamma_2 + \epsilon_1 \gamma_2 + \epsilon_2 \gamma_3 + \gamma_2 \epsilon_3 + \epsilon_1 \gamma_3 + \gamma_1 \epsilon_3 &= 0, \\
\gamma_1 \epsilon_2 \epsilon_3 + \epsilon_1 \gamma_2 \epsilon_3 + \epsilon_1 \epsilon_2 \gamma_3 + \gamma_1 \gamma_2 \gamma_3 - J^2 (\gamma_1 + \gamma_3) &= 0.
\end{align}
\]

Equation (50a) shows, as already stated in [116], that the sum of all gain and loss terms must vanish. This arises from the condition that the trace of the Hamiltonian must be real, which corresponds to highest-order non-trivial coefficients of the characteristic polynomial (cf. (36a)), and appears to be a necessary consequence of balanced gain and loss (see e.g. [24]). However, as we showed in the two-dimensional case, there are stationary solutions with real energy even if (36a) is not satisfied. Hence, we emphasise that balanced gain and loss has to be considered per state, meaning that one also has to consider the occupation numbers \( n_k \) in each well. Thus, balanced gain and loss means

\[
\sum_k n_k \gamma_k = 0.
\]
It is easy to spot the trivial solutions of (50a)–(50c): the Hermitian potential, \( \gamma_1 = \gamma_2 = \gamma_3 = 0 \) and the symmetric potential, \( \gamma_1 = -\gamma_3, \gamma_2 = 0, \epsilon_1 = \epsilon_3, \) where \( \epsilon_2 \) can be chosen arbitrarily in the latter case. However, if either only \( \gamma_1 \) or \( \gamma_3 \) is zero, (50c) cannot be satisfied for \( J \neq 0 \). The other solutions of (50a)–(50c) are given by

\[
\begin{align*}
\gamma_1 & = -(\epsilon_2 - \epsilon_3)\gamma_0, \\
\gamma_2 & = (\epsilon_1 - \epsilon_3)\gamma_0, \\
\gamma_3 & = -(\epsilon_1 - \epsilon_2)\gamma_0,
\end{align*}
\]

where

\[
\gamma_0 = \pm \sqrt{\frac{\Delta\epsilon_{12}^3 + \Delta\epsilon_{23}^3 - \Delta\epsilon_{13}^3 + 3J^2\Delta\epsilon_{13}}{3\Delta\epsilon_{12}\Delta\epsilon_{23}\Delta\epsilon_{13}}}
\]

with \( \Delta\epsilon_{kl} = \epsilon_k - \epsilon_l \neq 0 \).

Of course the solutions (52a)–(52c) again only depend on the difference of the on-site energies; otherwise we would not be able to freely gauge the energy. Further, those solutions only exist if the term under the square root in (52d) is positive. By assuming that \( \Delta\epsilon_{12}\Delta\epsilon_{23} > 0 \), we find

\[
\Delta\epsilon_{12}\Delta\epsilon_{23} \leq J^2.
\]

However, for \( \Delta\epsilon_{12}\Delta\epsilon_{23} < 0 \), there are no solutions at all. Hence, \( \Delta\epsilon_{12} \) and \( \Delta\epsilon_{23} \) must have the same sign, which yields

\[
\epsilon_1 \leq \epsilon_2 \leq \epsilon_3,
\]

with two possible gain and loss distributions

\[
\gamma_1 \geq 0, \quad \gamma_2 \leq 0, \quad \gamma_3 \geq 0
\]

in each case.

The regions in the \((\epsilon_1, \epsilon_2, \epsilon_3)\) parameter space where solutions occur are shown in figure 3 for slices along the coordinate planes, which are symmetric in the sense that \( \epsilon_1 \) and \( \epsilon_3 \) are fully interchangeable. As in the two-dimensional case, an antisymmetric imaginary potential, i.e. \( \gamma_1 = -\gamma_2 \) and \( \gamma_2 = 0 \), is required by a symmetric real potential with \( \epsilon_1 = \epsilon_3 \) and arbitrary \( \epsilon_2 \). The spectrum for a symmetric real potential looks similar to the spectrum of the symmetric two-mode system in figure 1 (a), but with an additional state, which has zero energy everywhere. In contrast to the two-dimensional case, where the \( \mathcal{PT} \)-symmetric potentials are special cases of symmetrisable systems as shown in figure 2, \( \mathcal{PT} \)-symmetric and symmetrisable systems are completely exclusive for non-trivial choices of the parameters as shown in figure 3. That is, they only coincide for \( \epsilon_1 = \epsilon_2 = \epsilon_3 = 0 \).

Since all three states can now be potentially real, exceptional points occur in the spectrum where at least two states coalesce. The two independent second-order exceptional points between ground and first excited states and the first and second excited states are shown in figure 3 respectively. They correspond to tangent
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Figure 3. Slices along the coordinate planes in the \((\epsilon_1, \epsilon_2, \epsilon_3)\) space. The grey shaded areas again indicate the parameter regions where no real solutions of (52d) exist. In contrast to figure 2 symmetric potentials \((\epsilon_1 = \epsilon_3)\) and non-symmetric potentials are completely exclusive. Exceptional points occur between the ground and the first excited states (EP\(_{12}\)) and between the first and second excited states (EP\(_{23}\)), respectively. Their trajectories meet (marked by dots), which leads to a cusp catastrophe. Inside the cusps in the blue shaded area all three states are real while in the other areas only one real state is found. The solid straight line in (b) indicates the parametrisation used in figure 4.

Bifurcations, where two stationary states coalesce and give birth to a pair of states with complex-conjugated energies. The trajectories of these exceptional points in the coordinate planes meet in the vicinity of the origin and create the characteristic form of a cusp. Hence, such a bifurcation scenario is called cusp catastrophe \([117]\). The cusp points are again exceptional points but of third order, i.e. the coalescence of three states in a pitchfork bifurcation \([118, 121]\). However, we will not discuss this phenomenon in detail here and instead refer the interested reader to \([121]\), where we already reported on the occurrence of a cusp bifurcation in \(\mathcal{PT}\)-symmetric Bose-Einstein condensates.

Note that there is a strong resemblance between figure 3(b) and the area in figure 2, despite the fact, that completely different parameter spaces are shown. This is because the on-site energies and the gain and loss parameters differ only by an imaginary unit on entering the Schrödinger equation. Hence, real and imaginary parts of the potential are connected in a certain way. This is most obvious in \(\mathcal{PT}\)-symmetric systems, where the real potential is symmetric and the imaginary potential is antisymmetric, corresponding to the first bisector in figure 3(b). However, this relation also holds vice versa, that is if the imaginary potential is symmetric and the real potential is antisymmetric. Hence, we will now choose an antisymmetric real potential with \(\epsilon_1 = -\epsilon_3\) and \(\epsilon_2 = 0\), which corresponds to the second bisector in figure 3(b). Note that this type of potentials is partially embedded into the region of symmetrisable Hamiltonians in figure 4(b) in
the same way as the $\mathcal{PT}$-symmetric potentials are in figure 2. The spectrum along the second bisector is shown in figure 4 (a) as a function of the on-site energy $\epsilon_1$, and the corresponding imaginary potential is chosen according to (52a)–(52d), which is shown in figure 4 (b). Remarkably, the imaginary potential is always symmetric with $\gamma_1 = \gamma_3$. Such potentials are called anti-$\mathcal{PT}$-symmetric \cite{25,26}, since the corresponding Hamiltonians satisfy
\begin{equation}
\{\mathcal{PT}, \mathcal{H}\} = 0,
\end{equation}
where the curly braces indicate the anti-commutator. $\mathcal{PT}$ and anti-$\mathcal{PT}$ symmetry are two special cases which show that symmetrised systems indeed bear certain symmetries, although these symmetries may not always be obvious.

Apart from $\mathcal{PT}$ and anti-$\mathcal{PT}$-symmetric configurations there again exist also completely asymmetric potentials, which still lead to symmetrised Hamiltonians. This holds, in particular, also for semi-symmetrised Hamiltonians. In principle one can find potentials with any number of real or complex-conjugated energies starting from one up to the number of wells. However, there is no straightforward way of constructing such semi-symmetrised Hamiltonians in multi-well potentials. Even in a three-mode system it is already feasible, though for three dimensions not yet being essential, to resort to numerical methods for calculating the potential parameters with a variational approach.

4. Conclusions

In this paper we investigated the symmetrisation of Hamiltonians for few-mode systems. We introduced symmetrisation in a similar manner as quasi-Hermiticity. However,
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we focused on the bi-orthogonal nature of non-Hermitian quantum systems and their spectral properties. If a Hamiltonian is symmetrisable, i.e. if (5) and (9) hold, a pair of left and right symmetrisation operators exist. The corresponding spectrum then consists only of real or pairwise complex-conjugated eigenvalues. This is equivalent to quasi-Hermitian Hamiltonians if the kernels of the symmetrisation operators are empty, so that they are inverse to each other. We also showed, that a system can be semi-symmetrisable if at least one eigenstate is in the kernel of the symmetrisation operators, so that the left and right symmetrisation operators are only semi-inverse to each other. The energies corresponding to such states are isolated spectral resonances.

We then applied the concept of symmetrisation to few-mode systems with complex potentials described by a discrete, linear Schrödinger equation. The symmetrisation of a two-mode system yields only $\mathcal{PT}$-symmetric systems. However, the two-mode system is semi-symmetrisable, so that the ground state is always real. All few-mode systems with at least three dimensions are fully symmetrisable, so that there exist non-$\mathcal{PT}$-symmetric Hamiltonians with all real spectra. A special class of such Hamiltonians shown here correspond to anti-$\mathcal{PT}$ symmetry, where the imaginary potential is symmetric and the real potential is antisymmetric.

The concept of symmetrisation discussed in this paper can be applied, in principle, to most situations already well studied and understood in the presence of $\mathcal{PT}$ symmetry. However, there is always a trade-off between the gain in generality and the loss in simplicity. That is, while symmetrisation allows for more general and especially non-obvious ‘symmetries’, it is mathematically more abstract and physically not necessarily meaningful. This probably accounts for the success of the concept of $\mathcal{PT}$ symmetry, which is simple and intuitive. Nevertheless, symmetrisation allows for new applications as, for example, shown in [31], which are impossible with $\mathcal{PT}$ symmetry.

Another example for an application is discussed in [44], where the authors argue that asymmetric gain and loss is inevitable in real experiments. The authors also find further limitations by a stability analysis, which was not yet considered in this paper. Moreover, they use a Bose-Einstein condensate with contact interaction, viz. a non-linear quantum system. Though the first system discussed has a symmetric real potential and is thus not symmetrised, Lunt et al had to introduce an asymmetric real potential to stabilise their system. This seems to be a good starting point to investigate the symmetrisation of non-linear quantum systems.

Since the focus of this paper lies on the concept of symmetrisation itself, it would be interesting to investigate the phenomena occurring in symmetrised systems and their parameter dependencies in more detail in the future. This applies in particular to the exceptional points found in non-$\mathcal{PT}$-symmetric parameter regions. In the spectrum shown in figure 4 (a) two bifurcation scenarios are shown. They occur, however, not at the edge of the symmetrisable parameter region as in figure 2, which seems intuitive. Instead the exceptional points are always in the middle of the regions shown in figure 3 (b).

It should further be noted that the occurrence of the borders in figures 2 and 3
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is clearly indicated by the parameters becoming complex numbers. Nevertheless, the spectra in figures 1 and 4 show no sign of this whatsoever. It might thus be interesting to continue the parameters of the symmetrisable matrix models analytically into the bi-complex plane to investigate the borders of the symmetrisable parameter regions.

Further it seems crucial to extend the concept of symmetrisation to extended multi-well potentials, that is, going from a finite-dimensional to an infinite-dimensional Hilbert space. Since the matrix model discussed here is an approximation of a real, extended multi-well potential, which can, for example, be realised with ultra-cold Bose-Einstein condensates in a deep, one-dimensional optical lattice potential (see e.g. [94] and references therein), it is to be expected that at least a finite number of stationary states can be realised in asymmetric extended potentials.

Another possible application for symmetrisation might be in many-body quantum systems described by quantum master equations [122, 123]. It was shown that the characteristic and dynamical properties of such systems correspond to the $\mathcal{PT}$-symmetric models in mean-field approximation [122]. However, the concept of $\mathcal{PT}$ symmetry is not applicable directly to many-body quantum systems. Moreover, for a small number of particles a natural imbalance between gain and loss terms occurs. For these reasons it might be worthwhile to investigate many-body systems with respect to symmetrisability.

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References

[1] Moiseyev N 2011 *Non-Hermitian Quantum Mechanics* (Cambridge University Press) URL https://doi.org/10.1017/CBO9780511976186
[2] Graefe E M, Korsch H J and Niederle A E 2008 *Phys. Rev. Lett.* **101**(15) 150408 URL https://doi.org/10.1103/PhysRevLett.101.150408
[3] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80**(24) 5243–5246 URL https://doi.org/10.1103/PhysRevLett.80.5243
[4] Guo A, Salamo G J, Duchesne D, Morandotti R, Volatier-Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 *Phys. Rev. Lett.* **103**(9) 093902 URL https://doi.org/10.1103/PhysRevLett.103.093902
[5] Bender C M, Berntson B K, Parker D and Samuel E 2013 *Am. J. Phys.* **81** 173–179 URL https://doi.org/10.1119/1.4789549
[6] Schindler J, Li A, Zheng M C, Ellis F M and Kottos T 2011 *Phys. Rev. A* **84**(4) 040101 URL https://doi.org/10.1103/PhysRevA.84.040101
[7] Schindler J, Lin Z, Lee J M, Ramezani H, Ellis F M and Kottos T 2012 *J. Phys. A* **45** 444029 URL https://doi.org/10.1088/1751-8113/45/44/444029
[8] Rubinstein J, Sternberg P and Ma Q 2007 *Phys. Rev. Lett.* **99**(16) 167003 URL https://doi.org/10.1103/PhysRevLett.99.167003
[9] Chtchelkatchev N M, Golubov A A, Baturina T I and Vinokur V M 2012 *Phys. Rev. Lett.* **109**(15) 150405 URL https://doi.org/10.1103/PhysRevLett.109.150405
Real eigenvalues in symmetrised multi-well potentials with gain and loss

[10] Chong Y D, Ge L and Stone A D 2011 *Phys. Rev. Lett.* **106**(9) 093902 URL [https://doi.org/10.1103/PhysRevLett.106.093902](https://doi.org/10.1103/PhysRevLett.106.093902)

[11] Liertzer M, Ge L, Cerjan A, Stone A D, Türeci H E and Rotter S 2012 *Phys. Rev. Lett.* **108**(17) 173901 URL [https://doi.org/10.1103/PhysRevLett.108.173901](https://doi.org/10.1103/PhysRevLett.108.173901)

[12] Regensburger A, Bersch C, Míri M A, Onishchukov G, Christodoulides D N and Peschel U 2012 *Nature* **488** 167–171 URL [https://doi.org/10.1038/nature11298](https://doi.org/10.1038/nature11298)

[13] Castaldi G, Savoia S, Galdi V, Alù A and Engheta N 2013 *Philos. Trans. R. Soc. London, Ser. A* **371** 20120053 URL [https://doi.org/10.1098/rsta.2012.0053](https://doi.org/10.1098/rsta.2012.0053)

[14] Zheng C, Hao L and Long G L 2013 *Philos. Trans. R. Soc. London, Ser. A* **371** 20120053 URL [https://doi.org/10.1098/rsta.2012.0053](https://doi.org/10.1098/rsta.2012.0053)

[15] Assawaworrarit S, Yu X and Fan S 2017 *Nature* **546** 387–390 URL [https://doi.org/10.1038/nature22404](https://doi.org/10.1038/nature22404)

[16] Heiss W D 2012 *J. Phys. A* **45** 444016 URL [https://doi.org/10.1088/1751-8113/45/44/444016](https://doi.org/10.1088/1751-8113/45/44/444016)

[17] Gutöhrlein R, Cartarius H, Main J and Wunner G 2016 *J. Phys. A* **49** 485301 URL [https://doi.org/10.1088/1751-8121/49/48/485301](https://doi.org/10.1088/1751-8121/49/48/485301)

[18] Schnabel J, Cartarius H, Main J, Wunner G and Heiss W D 2017 *Acta Polytech.* **57** 454 URL [https://doi.org/10.14311/ap.2017.57.0454](https://doi.org/10.14311/ap.2017.57.0454)

[19] Pan L, Chen S and Cui X 2019 *Phys. Rev. A* **99**(1) 011601(R) URL [https://doi.org/10.1103/PhysRevA.99.011601](https://doi.org/10.1103/PhysRevA.99.011601)

[20] Sweeney W R, Hsu C W, Rotter S and Stone A D 2019 *Phys. Rev. Lett.* **122**(9) 093901 URL [https://doi.org/10.1103/PhysRevLett.122.093901](https://doi.org/10.1103/PhysRevLett.122.093901)

[21] Bender C M, Brody D C and Müller M P 2017 *Phys. Rev. Lett.* **118**(13) 130201 URL [https://doi.org/10.1103/PhysRevLett.118.130201](https://doi.org/10.1103/PhysRevLett.118.130201)

[22] Bender C M and Brody D C 2018 *J. Phys. A* **51** 135203 URL [https://doi.org/10.1088/1751-8121/aaab68](https://doi.org/10.1088/1751-8121/aaab68)

[23] Scholtz F, Geyer H and Hahne F 1992 *Ann. Phys.* **213** 74–101 URL [https://doi.org/10.1016/0003-4916(92)90284-s](https://doi.org/10.1016/0003-4916(92)90284-s)

[24] Nixon S and Yang J 2016 *Phys. Rev. A* **93**(3) 031802 URL [https://doi.org/10.1103/PhysRevA.93.031802](https://doi.org/10.1103/PhysRevA.93.031802)

[25] Ge L and Türeci H E 2013 *Phys. Rev. A* **88**(5) 053810 URL [https://doi.org/10.1103/PhysRevA.88.053810](https://doi.org/10.1103/PhysRevA.88.053810)

[26] Wu J H, Artoni M and La Rocca G C 2014 *Phys. Rev. Lett.* **113**(12) 123004 URL [https://doi.org/10.1103/PhysRevLett.113.123004](https://doi.org/10.1103/PhysRevLett.113.123004)

[27] Sternheim M M and Walker J F 1972 *Phys. Rev. C* **6**(1) 114–121 URL [https://doi.org/10.1103/PhysRevC.6.114](https://doi.org/10.1103/PhysRevC.6.114)

[28] Ruschhaupt A, Dowdall T, Simón M A and Muga J G 2018 *Europhys. Lett.* **120** 20001 URL [https://doi.org/10.1209/0295-5075/120/20001](https://doi.org/10.1209/0295-5075/120/20001)

[29] Simón M A, Buendía A, Kieldy A, Mostafazadeh A and Muga J G 2019 *Phys. Rev. A* **99**(5) 052110 URL [https://doi.org/10.1103/PhysRevA.99.052110](https://doi.org/10.1103/PhysRevA.99.052110)

[30] Makris K G, Musslimani Z H, Christodoulides D N and Rotter S 2015 *Nat. Commun.* **6** URL [https://doi.org/10.1038/ncomms8257](https://doi.org/10.1038/ncomms8257)

[31] Brandstöttter A, Makris K G and Rotter S 2019 *Phys. Rev. B* **99**(11) 115402 URL [https://doi.org/10.1103/PhysRevB.99.115402](https://doi.org/10.1103/PhysRevB.99.115402)

[32] Rivet E, Brandstöttter A, Makris K G, Lissek H, Rotter S and Fleury R 2018 *Nat. Phys.* **14** 942–947 URL [https://doi.org/10.1038/s41567-018-0188-7](https://doi.org/10.1038/s41567-018-0188-7)

[33] Stenholm S 2002 *Ann. Phys.* **302** 142–157 URL [https://doi.org/10.1006/aphy.2002.6309](https://doi.org/10.1006/aphy.2002.6309)

[34] Jakob M and Stenholm S 2003 *Phys. Rev. A* **67**(3) 032111 URL [https://doi.org/10.1103/PhysRevA.67.032111](https://doi.org/10.1103/PhysRevA.67.032111)

[35] Peng P, Cao W, Shen C, Qu W, Wen J, Jiang L and Xiao Y 2016 *Nat. Phys.* **12** 1139–1145 URL [https://doi.org/10.1038/nphys3594](https://doi.org/10.1038/nphys3594)
https://doi.org/10.1038/nphys3842

[36] Choi Y, Hahn C, Yoon J W and Song S H 2018 Nat. Commun. 9 URL https://doi.org/10.1038/s41467-018-04690-y

[37] Li Y, Peng Y G, Han L, Miri M A, Li W, Xiao M, Zhu X F, Zhao J, Ali A, Fan S and Qiu C W 2019 Science 364 170–173 URL https://doi.org/10.1126/science.aaw6259

[38] Wadati M 2008 J. Phys. Soc. Jpn. 77 074005 URL https://doi.org/10.1143/JPSJ.77.074005

[39] Tsoy E N, Allayarov I M and Abdullaev F K 2014 Opt. Lett. 39 4215–4218 URL https://doi.org/10.1364/OL.39.004215

[40] Konotop V V and Zezyulin D A 2014 Opt. Lett. 39 5535–5538 URL https://doi.org/10.1364/OL.39.005535

[41] Nixon S and Yang J 2016 Opt. Lett. 41 2747–2750 URL https://doi.org/10.1364/OL.41.002747

[42] Yang J 2017 Opt. Lett. 42 4067–4070 URL https://doi.org/10.1364/OL.42.004067

[43] Zhang L L, Li Z Z, Zhan G H, Yi G Y and Gong W J 2019 Phys. Rev. A 99 URL https://doi.org/10.1103/physreva.99.032119

[44] Lunt P, Haag D, Dast D, Cartarius H and Wunner G 2017 Phys. Rev. A 96(2) 023614 URL https://doi.org/10.1103/PhysRevA.96.023614

[45] Mostafazadeh A 2010 Int. J. Geom. Methods Mod. Phys. 07 1191–1306 URL https://doi.org/10.1142/S0219887810004816

[46] Wigner E 1931 Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren (Die Wissenschaft; Sammlung von Einzeldarstellungen aus den Gebieten der Naturwissenschaft und Technik vol 85) (Braunschweig, Germany: Friedrich Vieweg und Sohn)

[47] Simón M, Álvaro Buendía and Muga J 2018 Mathematics 6 111 URL https://doi.org/10.3390/math6070111

[48] Solombrino L 2002 J. Math. Phys. 43 5439–5445 URL https://doi.org/10.1063/1.1504485

[49] Wigner E P 1960 J. Math. Phys. 1 409–413 URL https://doi.org/10.1063/1.1703672

[50] Bender C M 2007 Rep. Prog. Phys. 70 947–1018 URL https://doi.org/10.1088/0034-4885/70/6/035

[51] Brody D C 2013 J. Phys. A 47 035305 URL https://doi.org/10.1088/1751-8113/47/3/035305

[52] Reid W T 1951 Duke Math. J. 18 41–56 URL https://doi.org/10.1215/S0012-7094-51-01805-4

[53] Darboux G 1882 C. R. Acad. Sci. Paris 94 1456–1459

[54] Cannata F, Junker G and Trost J 1998 Phys. Lett. A 246 219–226 URL https://doi.org/10.1016/s0375-9601(98)00517-9

[55] Bender C M, Berry M V and Mandilara A 2002 J. Phys. A 35 L467–L471 URL https://doi.org/10.1088/0305-4470/35/31/101

[56] Bender C, Tateo R, Dorey P, Dunning T, Lévai G, Kuzhel S, Fring A, Jones H and Hook D 2019 PT Symmetry in Quantum and Classical Physics (WORLD SCIENTIFIC Publishing Company Incorporated) ISBN 9781786345950

[57] Bender C M and Mannheim P D 2010 Phys. Lett. A 374 1616–1620 URL https://doi.org/10.1016/j.physleta.2010.02.032

[58] Dieudonné J 1961 Quasi-Hermitian operators Proceedings of the International Symposium on Linear Spaces (Pergamon Press) pp 115–122

[59] Langer H 1962 Math. Ann. 146 60–85 URL https://doi.org/10.1007/BF01396668

[60] Kühne R 1964 Math. Ann. 154 56–69 URL https://doi.org/10.1007/BF01360725

[61] Pease M 1965 Methods of Matrix Algebra Mathematics in science and engineering (Academic Press) ISBN 9780080955223

[62] Froissart M 1959 Il Nuovo Cimento 14 197–204 URL https://doi.org/10.1007/BF02724848

[63] Lee T and Wick G 1969 Nucl. Phys. B 9 209–243 URL https://doi.org/10.1016/0550-3213(69)90098-4
[95] Eilbeck J, Lomdahl P and Scott A 1985 *Physica D* **16** 318–338 URL https://doi.org/https://doi.org/10.1016/0167-2789(85)90012-0

[96] Kenkre V M and Campbell D K 1986 *Phys. Rev. B* **34**(7) 4959–4961 URL https://doi.org/10.1103/PhysRevB.34.4959

[97] Agrawal G 2001 *Applications of Nonlinear Fiber Optics* (Amsterdam: Elsevier) ISBN 978-0-080-49922-2

[98] Ramezani H, Kottos T, El-Ganainy R and Christodoulides D N 2010 *Phys. Rev. A* **82**(4) 043803 URL https://doi.org/10.1103/PhysRevA.82.043803

[99] Holstein T 1959 *Ann. Phys.* **8** 325–342 URL https://doi.org/10.1016/0003-4916(59)90002-8

[100] Holstein T 1959 *Ann. Phys.* **8** 343–389 URL https://doi.org/10.1016/0003-4916(59)90003-x
[123] Dast D, Haag D, Cartarius H and Wunner G 2016 Phys. Rev. A 93(3) 033617 URL https://doi.org/10.1103/PhysRevA.93.033617