Abstract. An assortment of ‘species’ of families of subsets of a set and some of their properties are investigated, with an eye on the logic and limit role they may play as ‘relaxed parallels’ to ultrafilters.

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1. Introduction, Points vs. Families

Let $X$ be a set. For points $x \in X$ and subsets $S \subseteq X$, there is the truth-value whether $X \in S$ or not.

But with families of subsets $F \subseteq \mathcal{P}(X)$, where $\mathcal{P}(x)$ is the set of subsets of $X$, there is also a truth-value – whether $S \in F$ or not.

Indeed, for every $x \in X$, the family

$$U_x := \{ S \subseteq X \mid x \in S \}$$

‘does the same as $x$’ for that matter: by definition, the truth-value whether $x \in S$ is the same as the truth-value whether $S \in U_x$.

Key words and phrases. families of subsets, ultrafilters and ‘relaxing them’, logical connectives, filters, eventual families, limits with respect to them, inner and outer (finitely-additive) ‘measures’, multi-sets and multi-families, common fixed-points, Hausdorff topological spaces.
In this sense, judging by these truth-values, the map (obviously an injection) \( x \mapsto U_x \) embeds \( X \) into \( \mathcal{P}(X) \), making \( \mathcal{P}(X) \) a kind of extension of \( X \) – ‘pseudo-points’ defined by to which subsets they belong, in a way other than all \( x \in X \).

Indeed, \( U_x \) in an instance of an ultrafilter in \( X \), and using the Axiom of Choice one proves that in an infinite set there are many other ultrafilters.

To see what we aim at, note that any subset of \( X \) can be viewed as a property of elements of \( X \), namely the property for \( x \) to be in \( S \), thus for \( S \) to be the set of elements satisfying the property.

Logical connectives among properties will make operations on the sets:

**I. NOT** – giving the complement \( S^c \), defined as \( S^c := \{ x \in X \mid x \notin S \} \). Indeed, to say that \( x \in S^c \) is to say that \( \text{NOT} \ x \in S \).

**II. AND** – giving the intersection \( S \cap T \) of two subsets. Indeed, to say that \( x \in S \) AND \( x \in T \) is to say that \( x \in S \cap T \).

**III. OR** – giving the union \( S \cup T \). To say that \( x \in S \) OR \( x \in T \) is to say that \( x \in S \cup T \).

**IV. IMPPLICATION** – giving inclusion of subsets. Indeed, to say that \( x \in S \) always implies \( x \in T \), is to say that \( S \subset T \).

**V. TRUE** – giving \( S \) as the whole \( X \) for which \( x \in X \) is always true.

**VI. FALSE** – giving \( S \) as the empty set \( \emptyset \) for which \( x \in \emptyset \) is always false.

But given a family of subsets \( \mathcal{F} \subset \mathcal{P}(S) \). Can we plug \( \mathcal{F} \) instead of \( x \) in I - VI, i.e.,

**I** Does \( S^c \in \mathcal{F} \) hold if and only if \( \text{NOT} \ S \in \mathcal{F} \)?

**II** Does \( S \cap T \in \mathcal{F} \) hold if and only if \( S \in \mathcal{F} \) AND \( S \in \mathcal{F} \)?

**III** Does \( S \cup T \in \mathcal{F} \) hold if and only if \( S \in \mathcal{F} \) OR \( S \in \mathcal{F} \)? \( S \in \mathcal{F} \)?

**IV** Does \( S \subset T \) make \( S \in \mathcal{F} \) always imply \( T \in \mathcal{F} \)?

**V** Are we guarantied that \( X \) belongs to \( \mathcal{F} \)?

**VI** Are we guarantied that \( \emptyset \) does not belong to \( \mathcal{F} \)?

Certainly these may not hold. Clearly they hold for the \( U_x \).

An ultrafilter is characterized by all \( I - IV \) holding.

This means that for ultrafilters first-order logical operations carry over.

And since in an infinite set we will have them besides the \( U_x \), one applies that to make, say, from the natural numbers \( \mathbb{N} \) non-standard models of its first-order theory, i.e. genuine extensions of \( \mathbb{N} \) where first-order statements hold if and only if they hold in \( \mathbb{N} \) (alas, the induction axiom, inasmuch as it refers to ‘any subset of \( \mathbb{N} \)’ is not first-order – that’s much of the point).

But suppose we relax our requirements, assume that for \( \mathcal{F} \) only some of \( I - IV \) hold. In this note we try to remark on such families.

For instance, suppose only IV holds: if \( S \subset T \) and \( S \in \mathcal{F} \) then \( T \in \mathcal{F} \). Then \( \mathcal{F} \) is called an eventual family (see [CenLe] and [1.2]).

Note that for eventual \( \mathcal{F} \), the only if in II above and the if in III are automatic. II just says that \( \mathcal{F} \) respects intersections – if \( S \) and \( T \) belong to \( \mathcal{F} \) also \( S \cap T \) belongs – an eventual family satisfying that is called a filter.

Suppose only I holds: \( S^c \in \mathcal{F} \) if and only if \( S \notin \mathcal{F} \). These will be referred to as self-Aso in §3. That, of course, holds for ultrafilters, but not only for them. To take an example: let \( X \) be finite with \( 2n + 1 \) elements and let \( \mathcal{F} \) be the family of sets with \( > n \) elements. This example is also eventual.

Recall that one easily proves that an ultrafilter in finite \( X \) must be ‘fixed’ – an \( U_x \).

1.1. **Limits.** Families of subsets \( \mathcal{F} \) just give to any \( S \subset X \) a truth value, whether \( S \in \mathcal{F} \) or not.

To speak ‘fancily’: they extract from any \( \{0, 1\} \)-valued function on \( X \) a value in \( \{0, 1\} \).
The merit of that way of speaking is that, with respect to some $F$, one often can do similar things with some space $Y$ instead of $\{0, 1\}$: extract a limit in $Y$ from $Y$-valued functions on $X$.

Let's see how that goes with $F$ an ultrafilter.

Let $Y$ be a bounded interval in $\mathbb{R}$: we want $F$ to extract a limit from a bounded real-value function $f$ on $X$.

Now, partition $Y$ into a finite number of disjoint small intervals $Y_1, Y_2, \ldots, Y_k$.

Then $f^{-1}(Y_1), f^{-1}(Y_2), \ldots, f^{-1}(Y_k)$ will be a disjoint partition of $X$. Now, $F$ cannot contain two of these, because then it would also contain their intersection – empty. It cannot contain none of them, then it will not contain their union – the whole $X$. Therefore it contains exactly one of them. So, modulo the ultrafilter $F$, i.e. outside some subset not in the ultrafilter, $f$ takes values in some numerical set of as small diameter as we wish.

Things that hold modulo $F$ cannot be in contradiction, indeed any finite conjunction of them also hold modulo $F$ since as an ultrafilter it contains the intersection of any two of its members.

So one concludes that for any such bounded real valued function $f$ on $X$, and an ultrafilter $F$ there is a unique ‘limit’ $a \in \mathbb{R}$ characterized by: for any neighborhood $U$ of $x$ in the reals, modulo the ultrafilter $f$ takes values in $U$.

And one easily sees that this limit depends linearly on $f$, nay, for any continuous real function $\phi$ of $n$ variables, the limit of $\phi$ applied to $n$ functions equals $\phi$ applied to their limits!

But these notions can be applied to much more general families $F$. Suppose we retain the definition of a limit point of a $Y$-valued function on $X$, $Y$ a Haudorff topological space, as any point $y \in Y$ such that

For any neighborhood $U$ of $y$ in $Y$, any set containing $\{x \in X \mid f(x) \in U\}$ belongs to $F$.

Thus that limit notion will be the same for $F$ and for the eventual core of $F$, defined as the maximal eventual family contained in $F$, given by

$$F' := \{ S \subset X \mid \text{any } S' \supset S \text{ is in } F \}$$

So we here assume from the start $F$ eventual.

Then we have

- For $F$ an ultrafilter and $Y$ a bounded real interval, we saw that a unique limit exists always, whatever the function $f : X \to Y$. That holds for any compact $Y$.
- Clearly, with $F$ a filter a limit is always unique (yet need not exist).

1.2. Focusing on Eventual Families. This section includes some points mentioned in [CenLe], recapitulated here for the sake of completeness.

Above the notion of eventual families of subsets was introduced, namely,

**Definition 1.** Let $X$ be a set and let $F$ be a family of subsets of $X$. The family $F$ is called an eventual family if it is upper hereditary with respect to inclusion, i.e., if

$$S \in F, S' \supseteq S \Rightarrow S' \in F.$$  \hfill (1)

The family $F$ is called a co-eventual family if it is lower hereditary with respect to inclusion, i.e., if

$$S \in F, S' \subseteq S \Rightarrow S' \in F.$$  \hfill (2)

We mention in passing that Borg [Bo] uses the term ‘hereditary family’, in his work in the area of combinatorics, for exactly what we call here ‘co-eventual family’.

Several simple observations regarding such families can be made.
Proposition 2. (i) A family \( F \) of subsets of \( X \) is co-eventual iff its complement, i.e., the family of subsets of \( X \) which are not in \( F \), is eventual.

(ii) The empty family and the family of all subsets of \( X \) are each both eventual and co-eventual, and they are the only families with this property.

Proof. (i) This follows from the definitions. (ii) That the empty family and the family of all subsets of \( X \) are each both eventual and co-eventual is trivially true. We show that if \( F \) is eventual and co-eventual and is nonempty then it must contain all subsets of \( X \). Let \( S \in F \) and distinguish between two cases. If \( S = \emptyset \) then \( F \) must contain all subsets of \( X \) because \( F \) is eventual. If \( S \neq \emptyset \) let \( x \in S \), then, since \( F \) is co-eventual it must contain the singleton \( \{x\} \). Consequently, the set \( \{x, y\} \), for any \( y \), is also in \( F \) and so \( \{y\} \in F \), thus, all subsets of \( X \) are contained in \( F \). Alternatively, if we look at \( S \in F \), then for any subset \( S' \) of \( X \), \( F \) contains \( S \cup S' \) since \( F \) is eventual. Then since \( F \) is co-eventual, it must contain \( S' \), leading to the conclusion that it contains all subsets.

Remark 3. An eventual family \( F \) need not contain the intersection of two of its members. If it does so for every two of its members then it is a filter.

Similarly to the notion used in \([1]\) and \([LCen]\) in the finite-dimensional space setting, we make here the next definition.

Definition 4. Given a family \( F \) of subsets of a set \( X \), the star set associated with \( F \), denoted by \( \text{Star}(F) \), is the subset of \( X \) that consists of all \( x \in X \) such that the singletons \( \{x\} \in F \), namely,

\[
\text{Star}(F) := \{ x \in X \mid \{x\} \in F \}.
\]  

Suppose now that \( X \) is a Hausdorff Topological space.

Definition 5. Let \( F \) be an eventual family of subsets of \( X \). A point \( x \in X \) is called an accumulation (or limit) point of \( F \) if every (open) neighborhood of \( x \) belongs to \( F \). The set of all accumulation points of \( F \) is called the limit set of \( F \).

Proposition 6. The limit set of an eventual family \( F \) is always closed.

Proof. We show that the complement of the limit set, i.e., the set of all non-accumulation points, is open. The point \( y \) is a non-accumulation point iff it has an open neighborhood which does not belong to \( F \), i.e., when it is a member of some open set not in \( F \). Hence the complement of the limit set is the union of all open sets not in \( F \), and by definition, in a topological space, the union of any family of open sets is open.

We turn our attention now to sequences in \( X \), i.e., maps \( \mathbb{N} \to X \), where \( \mathbb{N} \) denotes the integers.

Definition 7. Given are a family \( F \) of subsets of \( X \) and a mapping between sets \( f : X \to Y \). The family of subsets of \( Y \) whose inverse image sets \( f^{-1}(S) \) belong to \( F \) will be denoted by \( \text{Push}(f, F) \) and called the push of \( F \) by \( f \), namely,

\[
\text{Push}(f, F) := \{ S \subseteq Y \mid f^{-1}(S) \in F \}.
\]  

Combining Definitions 5 and 7 the following remark is obtained.

Remark 8. Let \( \mathcal{E} \) be an eventual family of subsets of \( \mathbb{N} \) and let \( f : \mathbb{N} \to X \) be defined by some given sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \). The accumulation points and the limit set of \( (x_n)_{n \in \mathbb{N}} \) with respect to \( \mathcal{E} \) are those defined with respect to the push of \( \mathcal{E} \) by \( f \).

\footnote{Since, by definition, a neighborhood always contains an open neighborhood, considering all neighborhoods or just the open ones does not make a difference here.}
The next examples emerge by using two different eventual families in \( \mathbb{N} \). The same ‘machinery’ yields both ‘cases’ via changing the eventual family \( E \) in \( \mathbb{N} \).

**Examples 9.**

1. Take as \( E \) the family \( H \) of all subsets of \( \mathbb{N} \) with finite complement. Then accumulation points/limits with respect to \( H \) are the usual limits, and if there is a limit point then it is unique. This is the case, as one clearly sees, in a Hausdorff space \( X \) whenever \( E \) is a filter, as here \( E \) clearly is.

2. Now take as \( E \) the family \( G \) of all infinite subsets of \( \mathbb{N} \). Then being an accumulation point means being some accumulation point of the sequence in the usual sense, which in general, need not be unique. Indeed, here \( E \) is not a filter.

When considering eventual families in \( \mathbb{N} \) it is often desirable to assume that they are finitely-insensitive, as we define next. All our examples have this property.

**Definition 10.** A family \( E \) of subsets of \( \mathbb{N} \) is called a finitely-insensitive family if for any \( S \in E \), finitely changing \( S \), which means here adding and/or deleting a finite number of its members, will result in a set \( S' \in E \).

**Definition 11.** Let \( X \) be a Hausdorff topological space and let \( F \) be an eventual family in \( X \). The closure of an eventual family \( F \) in \( X \), denoted by \( \text{cl} F \), consists of all subsets \( S \subseteq X \) such that all the open subsets \( U \subseteq X \) which contain \( S \) belong to \( F \).

Clearly, \( F \) is always a subfamily of \( \text{cl} F \), and the set of limit points of an eventual family \( F \), in a Hausdorff topological space \( X \), is just \( \text{Star} (\text{cl} F) \), given in Definition 4.

**2. Multi-sets and Multi-families**

Here too some points mentioned in \[CenLe\] are recapitulated.

A **multi-set** (sometimes termed bag, or mset) is a modification of the concept of a set that allows for multiple instances for each of its elements. The number of instances given for each element is called the multiplicity of that element in the multi-set. The multiplicities of elements are any number in \( \{0, 1, \ldots, \infty\} \), see the corner-stone review of Blizard \[Bl\].

Yet in contrast to \[Bl\], we shall not pursue the distinction, from our point of view of purely philosophical motivation, between a multi-set and its ‘representing function’ – the \( \{0, 1, \ldots, \infty\} \)-valued function giving the multiplicities. Thus,

**Definition 12.** (i) A **multi-set** \( M \) in a set \( X \) is a \( \{0, 1, \ldots, \infty\} \)-valued function \( M \) on \( X \), its value \( M(x) \) at some \( x \in X \) considered as the multiplicity of \( x \) in \( M \). In particular \( M(x) = 0 \) – the multiplicity being 0, means ‘\( x \) not belonging’.

A subset \( S \subseteq X \) is identified with a multi-set which is the characteristic function or indicator function of \( S \), i.e.,

\[
\iota_S(x) := \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S \end{cases}
\]  

(ii) A **multi-family** \( M \) on a set \( X \) is a multi-set in the powerset \( 2^X \) of \( X \) (i.e., all the subsets of \( X \)). For such subset \( S \) \( M(S) \) is the multiplicity of \( S \) in \( M \). A family \( F \) of subsets of \( X \) is thus identified with the, here \( \{0, 1\} \)-valued, multi-family on \( X \) \( \iota_F \), the characteristic function or indicator function of \( F \).

\[
\iota_F(S) := \begin{cases} 1, & \text{if } S \in F, \\ 0, & \text{if } S \notin F. \end{cases}
\]

(iii) A multi-family \( M \) on a set \( X \) is called **increasing** if

\[
S, S' \subseteq X, S \subseteq S' \Rightarrow M(S) \leq M(S'),
\]
and called **decreasing** if

\[ S, S' \subseteq X, S \subseteq S' \Rightarrow \mathcal{M}(S) \geq \mathcal{M}(S'). \] (8)

Clearly, a family of subsets of \( X \) is an eventual (resp. co-eventual) family if and only if the multi-family that defines it is increasing (resp. decreasing).

The next example shows why these notions may be useful.

**Example 13.** Considering the set \( \mathbb{N} \), for a, finite or infinite, subset \( S \subseteq \mathbb{N} \) write \( S \) as

\[ S = \{ n^S_1, n^S_2, \ldots \}, \] (9)

where \( n^S_\ell \in \mathbb{N} \) for all \( \ell \), and the sequence \( (n^S_\ell)_{\ell=1}^{L} \) (where \( L \) is either finite or \( \infty \)) is strictly increasing, i.e., \( n^S_1 < n^S_2 < \ldots \). The **gaps** between consecutive elements in \( S \) will be the sequence of differences

\[ n^S_2 - n^S_1 - 1, n^S_3 - n^S_2 - 1, \ldots, \] (10)

where, if \( S \) is finite add \( \infty \) at the end. Defining

\[ \text{Gap} (S) := \limsup_{k} (n^S_{k+1} - n^S_{k} - 1), \] (11)

makes \( \text{Gap} \) a multi-family on \( \mathbb{N} \), thus taking values in \( \{0, 1, \ldots, \infty\} \), in particular, taking the value \( \infty \) for (among others) any finite \( S \).

Note that if \( \text{Gap} (S) \) is finite then there must be an infinite number of differences \( (n^S_{k+1} - n^S_{k} - 1) \) equal to \( \text{Gap} (S) \), but this is not true for any larger integer - because by the definition of \( \limsup \) and because we are dealing with integer-valued items, a finite \( \limsup \) must actually be attained an infinite number of times.

Observe further that the larger the set \( S \) is – the smaller (or equal) is \( \text{Gap} (S) \). Thus, \( \text{Gap} \) is a decreasing multi-family.

Define the complement-multi-family for some multi-family \( G \) on the subsets of a set \( X \) by

\[ (\text{co}G)(S) := G(S^c), \quad \forall S \subseteq X \] (12)

where \( S^c \) is the complement of \( S \) in \( X \).

We will focus on \( \text{coGap} := \text{coGap} \). For any \( S \subseteq \mathbb{N} \), let us denote by \( c_S \) the maximal number of integers between consecutive elements of \( S \), namely, between \( n^S_\ell \in S \) and \( n^S_{\ell+1} \in S \). If \( S \) has arbitrarily big such ‘intervals’ between consecutive elements then we write \( c_S = \infty \). With this in mind, \( \text{coGap} = \text{Gap}^c \) is an increasing multi-family equal to \( (c_S) \forall S \subseteq \mathbb{N} \).

### 2.1. Extensions of Notions Pertaining to Families to Multi-families

We now extend some of the notions of Subsection 1.2 to multi-families.

**Definition 14.** Given a multi-family \( \mathcal{M} \) on the subsets of a set \( X \). The **star set associated with** \( \mathcal{M} \), denoted by \( \text{Star} (\mathcal{M}) \), is the multi-set \( M \) on \( X \) whose value on some \( x \) – the multiplicity of \( x \) with respect to it, is defined to be the multiplicity \( \mathcal{M}(\{x\}) \) of the singleton \( \{x\} \) with respect to \( \mathcal{M} \).

\[ \text{Star} (\mathcal{M})(x) := M(\{x\}). \] (13)

**Definition 15.** Given are a multi-family \( \mathcal{M} \) on the subsets of \( X \) and a mapping between sets \( f : X \rightarrow Y \). The **push** of \( \mathcal{M} \) by \( f \) is defined as the multi-family on the subsets of \( Y \) given by

\[ \text{Push} (f, \mathcal{M})(S) := \mathcal{M}(f^{-1}(S)). \] (14)

**Definition 16.** A multi-family \( \mathcal{M} \) of subsets of \( \mathbb{N} \) a **finitely-insensitive multi-family** if for any \( S \in \mathcal{M} \), finitely changing \( S \), i.e., adding and/or deleting a finite number of its members, will not change its multiplicity, i.e., will result in a set \( S' \in \mathcal{M} \) such that \( \mathcal{M}(S) = \mathcal{M}(S') \).
Definition 17. Let $X$ be a Hausdorff topological space. The **closure of an increasing multi-family** $M$ in $X$, denoted by $\text{cl } M$, is defined to be the (increasing) multi-family given by

$$\text{cl } M(S) := \min\{M(U) \mid \text{all open subsets } U \subseteq X \text{ such that } S \subseteq U\}. \quad (15)$$

Definition 18. Let $X$ be a Hausdorff topological space and let $M$ be an increasing multi-family. The multi-set $M := \text{Star}(\text{cl } M)$ will be called the **multi-set-limit** of $M$ and denoted by $\lim M$. It is thus given by, for any $x \in X$,

$$M(x) = \min\{M(U) \mid \text{all open subsets } U \subseteq X \text{ such that } x \in U\}. \quad (16)$$

Given a multi-family $M$ on the subsets of $\mathbb{N}$, the limiting notions with respect to $M$ for a sequence $(x_n)_{n \in \mathbb{N}}$, are defined as those with respect to $\text{Push} (f, M)$ of $M$ to $X$ by the function $f : \mathbb{N} \to X$ which represents the sequence $(x_n)$. In particular, for an increasing multi-family $M$ on the subsets of $\mathbb{N}$, the multi-set limit of $\text{Push} (f, M)$ will be called the **multi-set-limit** of $(x_n)$, denoted by $\lim_M x_n$.

This multi-set $G$ on $X$ can be described as follows. Given a point $x \in X$, consider the following subsets of $\mathbb{N}$

$$S(U) := \{n \in \mathbb{N} \mid x_n \in U\}, \text{ for open neighborhoods } U \text{ of } x. \quad (17)$$

Then,

$$G(x) = \min\{M(S(U)) \mid \text{all open subsets } U \subseteq X \text{ such that } x \in U\}. \quad (18)$$

Remark 19. Let us focus on the increasing multi-family $\text{coGap}$ in $\mathbb{N}$ of example $\text{I.3}$.

Note, that for a set $S$ not to belong to $\text{coGap}$, i.e., to have $\text{coGap}(S) = 0$, just means that $S$ is finite - as a ‘family, ignoring multiplicities’ $\text{coGap}$ is just the family of infinite sets of natural numbers.

Thus, when we turn to the limit of a sequence $(x_n)_{n \in \mathbb{N}}$ in a Hausdorff Space $X$ (a notion which is obviously dependent on the topology. In a Banach or Hilbert space we will have strong and weak limits etc.); and we take the $\text{coGap}$-limit (it will be a multi-set on $X$, to which for some $x$ in $X$ to belong (at least) $n$ times, one must have, for every neighborhood $U$ of $x$, that the $x_n$ stay in $U$ for some $n$ consecutive places as far as we go; then the $\text{coGap}$-limit of $(x_n)_{n \in \mathbb{N}}$, ‘forgetting the multiplicities’ is just the set of accumulation points of $(x_n)_{n \in \mathbb{N}}$ (which is, recalling the examples $\text{I.3}$ in Section $\text{I.2}$ just its $G$-limit for $G$ the eventual family of the infinite subsets of $\mathbb{N}$).

Note that, in general, if the sequence has a limit $x^*$ (in the good old sense) then its $\text{coGap}$-limit ‘includes $x^*$ infinitely many times and does not include any other point’. This sort of indicates to what extent the $\text{coGap}$-limit may be viewed as ‘more relaxed’ than the usual limit.

The inverse implication does not always hold (it holds however in a compact space) as the following counterexample shows. In $\mathbb{R}$ (the reals), define a sequence by

$$x_{2n} := n \text{ and } x_{2n-1} := -1 \quad (19)$$

then its $\text{coGap}$-limit contains $-1$ infinitely often and does not contain others, but $-1$ is not a limit.

2.2. Outer and Inner Increasing Multi-Families, The Outer Core and Inner Hull.

Let us try to relate a multi-family, a $((0, 1, \ldots, \infty)$-valued) function on sets, with the likes of measures,

**Definition 20.** An increasing multi-family $M$ on (the subsets) of a set $X$ will be called **outer** if it is a ‘finitely outer measure’ on the subsets, i.e. if it satisfies, for any finite number of
subsets \( S_1,\ldots,S_k \)

\[
\mathcal{M} \left( \bigcup_{i=1}^{k} S_i \right) \leq \sum_{i=1}^{k} \mathcal{M}(S_i). \quad (20)
\]

(for an infinite number of sets – no way – cf. the example of \( \mathbb{N} \) as union of singletons.)

Similarly,

**Definition 21.** An increasing multi-family \( \mathcal{M} \) on (the subsets) of a set \( X \) will be called **inner** if it is a ‘finitely inner measure’ on the subsets, i.e. if it satisfies, for any finite number of **disjoint** subsets \( S_1,\ldots,S_k \),

\[
\mathcal{M} \left( \bigcup_{i=1}^{k} S_i \right) \geq \sum_{i=1}^{k} \mathcal{M}(S_i). \quad (21)
\]

(which here easily implies the same for any infinite number of disjoint sets!)

For a general increasing multi-family \( \mathcal{M} \), its **outer core** \( \text{Out} \mathcal{M} \) is defined as the biggest outer multi-family less than \( \mathcal{M} \), namely, (as easily seen)

\[
\text{Out} \mathcal{M}(S) := \min \left\{ \sum_{i=1}^{k} \mathcal{M}(S_i) \mid \bigcup_{i=1}^{k} S_i = S \right\}.
\]

And its **inner hull** \( \text{Inn} \mathcal{M} \) is defined as the smallest inner multi-family greater than \( \mathcal{M} \), namely, (as easily seen)

\[
\text{Inn} \mathcal{M}(S) := \max \left\{ \sum_{i=1}^{k} \mathcal{M}(S_i) \mid \bigcup_{i=1}^{k} S_i = S, \text{the } S_i \text{ disjoint} \right\}.
\]

Clearly \( \text{Out} \mathcal{M} \) and \( \text{Inn} \mathcal{M} \) thus defined will also be increasing.

**Remark 22.** For a mapping \( X \rightarrow Y \), the definition of a push of a multi-family, and the fact that the inverse image of a complement, union, intersection,... is the complement, union, intersection,... of the inverse image(s), imply that the **push of an outer (resp. inner)** multi-family is always outer (resp. inner).

**Remark 23.** As a little exercise, what about the example of the \( (0,1) \)-valued multi-family \( \text{coGap} \) in \( \mathbb{N} \) (example 13)?

As a brief reflection will show, it is not outer, rather its outer core is given, somewhat surprisingly, by:

\[
\text{Out} \text{coGap}(S) := \min(2, \text{coGap}(S)).
\]

Indeed, any set \( S \subset \mathbb{N} \) can be decomposed into two sets with \( \text{coGap} \leq 1 \), that do not contain any intervals of 2 or more consecutive numbers – take the subsets of the even (resp. odd) members of \( S \). Also, since \( \text{coGap} \) is both finitely insensitive and vanishing only for finite sets, only decompositions into sets with non-zero value of \( \text{coGap} \) matter. This makes an (infinite) set \( S \) with \( \text{coGap} = 1 \) have the same value also by \( \text{OutcoGap} \), and if \( \text{coGap}(S) \geq 2 \) to have \( \text{OutcoGap}(S) = 2 \).

By contrast, the inner hull \( \text{InncoGap}(S) \) turns out ‘trivially’, to be the multi-family equal infinity for any infinite subset of \( \mathbb{N} \) while 0 for a finite \( S \). That follows from the following ‘trick’: every set \( S \) with ‘intervals’ of \( n \) consecutive numbers as far as we go (\( n > 0 \)) we may partition into any number \( K \) of **disjoint** sets with the same property (the \( i \)'th part as if picks the \( mK + i \)'th such intervals for \( m = 0,1,\ldots \)),
2.3. Image of a Multi-set by a Mapping. We wish to consider what image a mapping can give to a multi-set on $X$ (such as a star of a multi-family, or a limit of a sequence with respect to a multi-family in $\mathbb{N}$, etc.)

**Definition 24.** Let $f : X \rightarrow Y$ be a mapping between sets, and let $L$ be a multi-set on $X$. Its multi-image \textbf{multi-}$f(L)$ is defined as the multi-set in $Y$:

$$\text{multi-}f(L)(y) := \sum_{x \in f^{-1}(y)} L(x).$$

(Note that there is no problem in the definition of the sum, since we are summing integers – if, say, the sum comprises an infinite number of non-zero terms the sum is infinity.)

Note that if $L$ takes only $\{0, 1\}$-values, i.e. is (the characteristic function of) a set $S$, still $f(S)$ may be different from \textbf{multi-}$f(S)$, the latter may take values bigger than 1. Hence the special notation \textbf{multi-}$f(L)$.

2.4. Some properties.

**Proposition 25.** (i)a. For a mapping between sets $f : X \rightarrow Y$, and an increasing multi-family $\mathcal{M}$ on (the subsets of) $X$,

$$\text{Out} (f \ast \mathcal{M}) \geq f \ast (\text{Out} \mathcal{M}).$$

(i)b. For a mapping between sets $f : X \rightarrow Y$, and an increasing multi-family $\mathcal{M}$ on (the subsets of) $X$,

$$\text{Inn} (f \ast \mathcal{M}) \leq f \ast (\text{Inn} \mathcal{M}).$$

(ii). For an increasing multi-family $\mathcal{M}$ on (the subsets of) a Hausdorff topological space $X$,

$$\text{Out} (\text{cl} \mathcal{M}) \geq \text{cl}(\text{Out} \mathcal{M}).$$

(iii). For a continuous mapping between Hausdorff topological spaces $f : X \rightarrow Y$, and an increasing multi-family $\mathcal{M}$ on (the subsets of) $X$,

$$\text{cl} (f \ast \mathcal{M}) \geq f \ast (\text{cl} \mathcal{M}).$$

**Proof.** (i)a: In computing $(\text{Out} (f \ast \mathcal{M}))(S)$, we minimize, for $S = \bigcup_{i=1}^{k} S_i$, on $\sum_{i=1}^{k} (f \ast \mathcal{M})(S_i) = \sum_{i=1}^{k} \mathcal{M}(f^{-1}(S_i))$. Now $\sum_{i=1}^{k} (f^{-1}(S_i)) = f^{-1} \left( \bigcup_{i=1}^{k} S_i \right) = f^{-1}(S)$. So here we are minimizing the sum of $\mathcal{M}$ on some ways of writing $f^{-1}(S)$ as a finite union, namely the union of $(f^{-1}(S_i))_{i=1}^{k}$. So we will get a $\leq$ value if we minimize on all ways of writing $f^{-1}(S)$ as a finite union. But the latter gives $(\text{Out} \mathcal{M})(f^{-1}(S)) = (f \ast (\text{Out} \mathcal{M}))(S)$.

(i)b: In computing $(\text{Inn} (f \ast \mathcal{M}))(S)$, we maximize, for $S = \bigcup_{i=1}^{k} S_i$, $S_i$ disjoint, on $\sum_{i=1}^{k} (f \ast \mathcal{M})(S_i) = \sum_{i=1}^{k} \mathcal{M}(f^{-1}(S_i))$. Now $\sum_{i=1}^{k} (f^{-1}(S_i)) = f^{-1} \left( \bigcup_{i=1}^{k} S_i \right) = f^{-1}(S)$. So here we are maximizing the sum of $\mathcal{M}$ on some ways of writing $f^{-1}(S)$ as a disjoint finite union, namely the union of $(f^{-1}(S_i))_{i=1}^{k}$. So we will get a $\geq$ value if we maximize on all ways of writing $f^{-1}(S)$ as a finite union. But the latter gives $(\text{Inn} \mathcal{M})(f^{-1}(S)) = (f \ast (\text{Inn} \mathcal{M}))(S)$.

(ii): In computing $(\text{Out} (\text{cl} \mathcal{M}))(S)$ we minimize, for $S = \bigcup_{i=1}^{k} S_i$, on $\sum_{i=1}^{k} (\text{cl} \mathcal{M})(S_i)$. Each $(\text{cl} \mathcal{M})(S_i)$ is the minimum of $\mathcal{M}(U_i)$ for all open $U_i \supset S_i$. So we are minimizing on $\sum_{i=1}^{k} \mathcal{M}(U_i)$ for all $S_i$ such that $S = \bigcup_{i=1}^{k} S_i$ and open $U_i \supset S_i$. But each $\sum_{i=1}^{k} \mathcal{M}(U_i)$ $\geq$ than $(\text{Out} \mathcal{M})(\bigcup_{i=1}^{k} U_i)$ and as $\bigcup_{i=1}^{k} U_i$ is open and contains $\bigcup_{i=1}^{k} S_i = S$, that is $\geq$ than $(\text{cl}(\text{Out} \mathcal{M}))(S)$. Hence the assertion.

(iii): In computing $(\text{cl} (f \ast \mathcal{M}))(S)$ for sets $S \subset Y$, we minimize on $(f \ast \mathcal{M})(U)$ for open sets $U$ in $Y$ containing $S$. But $(f \ast \mathcal{M})(U) = \mathcal{M}(f^{-1}U)$ and, $f$ being continuous, $f^{-1}U$ is open in
X, and of course contains \( f^{-1}S \). Therefore the latter is \( \geq (\text{cl } \mathcal{M})(f^{-1}S) = (f \ast (\text{cl } \mathcal{M}))(S) \). And we are done. \( \Box \)

**Remark 26.** Here we stated facts holding generally and with ‘straightforward’ proofs. Maybe for special cases (such as compact topological spaces) these may be strengthened (say, equality in the inequalities) with more sophisticated arguments.

Yet we mention one such case

**Proposition 27.** (ii) *. For an increasing multi-family \( \mathcal{M} \) on (the subsets of) a Hausdorff topological space \( X \), and a finite set \( F \),

\[
\text{Inn}(\text{cl } \mathcal{M})(F) \leq (\text{cl } \text{Inn } \mathcal{M})(F).
\]

**Proof.** In computing \( (\text{Inn } \text{cl } \mathcal{M}))(F) \) we maximize, for \( F = \bigcup_{i=1}^{k} F_i \), \( F_i \) disjoint (and necessarily finite), on \( \sum_{i=1}^{k} (\text{cl } \mathcal{M})(F_i) \).

Pick a decomposition \( F = \bigcup_{i=1}^{k} F_i \) which gives the maximum, that being the value of \( (\text{Inn } \text{cl } \mathcal{M})(F) \).

Each \( (\text{cl } \mathcal{M})(F_i) \) is the minimum of \( \mathcal{M}(U_i) \) for all open \( U_i \supset F_i \), and, in a Hausdorff space, we may assume also the \( U_i \) disjoint. Then we have \( \sum_{i=1}^{k} \mathcal{M}(U_i) \leq (\text{Inn } \mathcal{M})(\bigcup_{i=1}^{k} U_i) \), making (for that maximizing decomposition)

\[
(\text{Inn } \text{cl } \mathcal{M}))(F) \leq (\text{Inn } \mathcal{M})(\bigcup_{i=1}^{k} U_i).
\]

On the other hand, in computing \( (\text{cl } \text{Inn } \mathcal{M}))(F) \) we minimize on \( (\text{Inn } \mathcal{M}))(U) \) for \( U \) an open set containing (the finite) \( F \), and, again since the space is Hausdorff and we speak about finite sets, for every decomposition \( F = \bigcup_{i=1}^{k} F_i \), in particular for the above maximizing one, and every such \( U \) there are disjoint open \( U_i, U_i \) containing \( F_i \), such that \( \bigcup_{i=1}^{k} U_i \subset U \), implying, by equation (22), that for each of our \( U \)’s \( (\text{Inn } \mathcal{M})(U) \geq (\text{Inn } \text{cl } \mathcal{M}))(F) \). \( \Box \)

2.5. **Action of a Continuous Mapping on Limits with respect to Inner Multi-Families.** Thus, Propositions 25 and 27 yield, but (the arguments I have work ...) only for inner multi-families,

**Theorem 28.** Let \( X \) and \( Y \) be Hausdorff topological spaces, \( f : X \to Y \) be continuous, and \( \mathcal{M} \) be an inner increasing multi-family on (the subsets of ) \( X \). Then

\[
\lim(f \ast \mathcal{M}) \geq \text{multi- } f(\lim \mathcal{M}).
\]

**Proof.** In Proposition 25 (iii), take the star of both sides to get

\[
\text{Star } (\text{cl } (f \ast \mathcal{M})) \geq \text{Star } (f \ast (\text{cl } \mathcal{M})).
\]

The LHS here is the (multi-set) \( \lim(f \ast \mathcal{M}) \). The RHS is the multi-set whose value at some \( y \in Y \) is

\[
(f \ast (\text{cl } \mathcal{M}))(\{y\}) = (\text{cl } \mathcal{M})(f^{-1}(y)) = (\text{cl } (\text{Inn } \mathcal{M}))(f^{-1}(y))
\]
(\mathcal{M} \text{ being inner}). Then, by, Proposition 27 for every finite \( F \subset f^{-1}(y) \)

\[
\begin{align*}
\geq & \quad (\text{Inn}(\text{cl} \mathcal{M}))(F) \\
= & \quad (\text{Inn}(\text{cl} \mathcal{M}))(\bigcup_{x \in F}\{x\}) \\
\geq & \quad \sum_{x \in f^{-1}(y)} (\text{cl} \mathcal{M})(\{x\}) \\
= & \quad \sum_{x \in F} (\text{Star}(\text{cl} \mathcal{M}))(x) \\
= & \quad \sum_{x \in F} (\text{lim} \mathcal{M})(x).
\end{align*}
\]

That holding for any finite \( F \subset f^{-1}(y) \) means

\[
\geq \sum_{x \in f^{-1}(y)} (\text{lim} \mathcal{M})(x) = (\text{multi-f}(\text{lim} \mathcal{M}))(y).
\]

\[\square\]

For sequences, noting remark 22 we get

**Corollary 29.** Let \( X \) and \( Y \) be Hausdorff topological spaces, \( f : X \to Y \) be continuous, and suppose \( \mathcal{M} \) is an inner increasing multi-family in \( \mathbb{N} \),

Then for a sequence \( (x_n) \) in \( X \), (recall these limits are multi-sets on the spaces)

\[
\mathcal{M}-\text{lim}(f(x_n)) \geq \text{multi-f}(\mathcal{M}-\text{lim} x_n).
\]

And compare what will be said in §2.6

2.6. **Special Properties of Inner Eventual Families.** An inner eventual family, i.e. to which two disjoint sets cannot both belong ((1) in (1)), shares with a filter (by the same reasoning) the fact that in a Hausdorff topological space it can have at most one limit point, consequent for an inner eventual family \( \mathcal{E} \) the \( \mathcal{E} \)-limit of a sequence in a Hausdorff space, if it exists, is unique.

Thus, the statements about an inner \( \mathcal{M} \), of Theorem 28

\[
\lim(f \ast \mathcal{M}) \geq \text{multi-f}(\text{lim} \mathcal{M}),
\]

and of Corollary 29

\[
\mathcal{M}-\text{lim}(f(x_n)) \geq \text{multi-f}(\mathcal{M}-\text{lim} x_n),
\]

when restricted to an inner eventual family (not general multi-families), will really speak about a single limit point (if exists), and say that, as with limits in the (much more restrictive) good old sense,

**Theorem 30.** Let \( X \) and \( Y \) be Hausdorff topological spaces, \( f : X \to Y \) be continuous.

(i) Let \( \mathcal{F} \) is an inner eventual family on (the subsets of) \( X \).

Then, if \( \lim \mathcal{F} \) exists (an element of \( X \), necessarily unique),

then \( \lim(f \ast \mathcal{M}) \) also exists (an element of \( Y \) and is equal to \( f(\lim \mathcal{M}) \)).

(ii) Let \( \mathcal{E} \) is an inner eventual family in \( \mathbb{N} \).

Given a sequence \( (x_n) \) in \( X \), if its \( \mathcal{E}-\lim x_n \) in \( X \) exists (an element of \( X \), necessarily unique),

then \( \mathcal{E}-\lim(f(x_n)) \) also exists and is equal to \( f(\mathcal{E}-\lim x_n) \).
3. More about Eventual Families, Associated and Self-Associated, Inner, Outer – Generalization of Ultrafilters?

3.1. Dually Associating an Eventual Family $F$ with the Family of Sets with Complements not in $F$. For an eventual family $F$ in some set $X$, Proposition 2 told us that its complement – the family of sets not in $F$, is a co-eventual family and vice-versa. But clearly also the set of complements of the members of $F$ is a co-eventual family and vice-versa.

These are different. For instance, for the family $G$ of all infinite subsets of $\mathbb{N}$ and the family $H$ of all subsets of $\mathbb{N}$ with finite complement (cf. examples 9), the complement of $G$ is the family of finite sets – indeed just the set of complements of the members of $H$, while the family of complements of the members of $G$ – all subsets of $\mathbb{N}$ with infinite complements – is just the complement of $H$!

This may be generalized. For any family $F$ in some set $X$, as if ‘go to the co-eventual one way and return the other way’ to get what we call here its associate eventual family $F$ in $X$, defined as

$$\text{Aso} (F) := \text{the family of all subsets whose complement is not in } F.$$

Clearly $\text{Aso} \circ \text{Aso} = \text{id}$ – it is an involution, a kind of duality. And as we saw it pairs, in particular, the above $G$ and $H$ on (the subsets of) $\mathbb{N}$.

Note that The ‘larger’ $F$ is the ‘smaller’ $\text{Aso} F$ will be.

For a mapping $X \to Y$, the definition of a push of an eventual family, and the fact that the inverse image of a complement is the complement of the inverse image, imply that the associate of the push is the push of the associate.

3.2. Partial Generalization to Multi-Families. For a multi-family on (the subsets) of a set $X$, one readily generalizes taking the family of complements (cf. example 13) to $M \mapsto \text{co } M$ defined by $\text{co } M(S) := M(S^c)$ ($S^c$ denotes the complement of $S$).

The other way – ‘the complement of $F$, i.e. the family of sets not in $(E)$’ does not seem to generalize readily, so we do not have a duality such as $\text{Aso}$.

Yet we can easily see that

**Proposition 31.** If $M$ is an increasing multi-family on (the subsets of) some set $X$, then the associate of the (eventual family) the (upper) level set $\{S \mid M(S) \geq n + 1\}$ is the (eventual family) the (lower) level set (of, here, a decreasing multi-family) $\{S \mid \text{(co } M)(S) \leq n\}$. (recall, these are integers, $\leq n$ is the negation of $\geq n + 1$ – one need not bother with $< \text{ etc.}$)

Proof. □

3.3. Inner and Outer. Let us focus on an eventual family $F$ on (the subsets of) a set $X$, identified with its characteristic functions, to be thought of as just a $(0,1)$-valued increasing multi-family.

And one may ask: when are these outer, inner, ‘finitely additive’? Also, recalling §3.1 when such an eventual families is the associate of itself?.

These are related to viewing the above characteristic function of $F$, as in §2.2 as $((0,1)$-valued) likes of measures.

(1) Let us clarify in what way the condition for being a (finitely-additive) ‘measure’; ‘outer (resp. inner) measure’ applies here. That concerns two disjoint subsets $S$ and $T$, comparing the values the characteristic function of $F$ gives to $S, T$ and $S \cup T$, i.e. do these belong to $F$ or not.

When one of $S$ and $T$ has ‘measure’ 0 (i.e. is not in $F$) while the other has ‘measure’ 1 (i.e. does belong to $F$), there is only one possibility: since $F$ is eventual, the union
$S \cup T$ also belongs to $\mathcal{F}$, has thus ‘measure’ 1, and there is always ‘full’ finite additivity: $0 + 1 = 1$.

So we need to focus on the cases where both $S$ and $T$ have measure 0 or both have measure 1.

(2) When both $S$ and $T$ have ‘measure’ 0, i.e. are not in $\mathcal{F}$, the union, $S \cup T$, of course, has ‘measure’ 0 or 1. So the condition (21) for ‘inner’ always holds here, while for equality, i.e. finite additivity, equivalently the condition (20) for ‘outer’ here, we need that also $S \cup T$ has ‘measure’ 0, i.e. is not in $\mathcal{F}$. Thus for finite additivity, equivalently ‘outer’, to always hold here we need that

\[(O) \quad \text{the union of two sets (hence of a finite number of sets) not in } \mathcal{F} \text{ is also not in } \mathcal{F} \]

(3) Analogously, when both $S$ and $T$ (hence, of course, also $S \cup T$) have ‘measure’ 1, i.e. belong to $\mathcal{F}$, the condition (20) for ‘outer’ will always hold here, we will never have equality, i.e. finite additivity, and also the condition (21) for ‘inner’ never holds here.

(4) Therefore when $\mathcal{F}$ is inner, in particular ‘finitely additive’, the last case cannot occur, that is

\[(I) \quad \text{two disjoint subsets cannot both belong to } \mathcal{F} \]

(5) So a filter is inner (unless it is the ‘trivial’ family of all subsets) – if it has disjoint members, it will contain their intersection $\emptyset$.

(6) And we may conclude: $\mathcal{F}$ is outer if and only if (O) always holds, is inner if and only if (I) always holds, and gives a ‘finitely additive measure’ if and only if both always hold.

(7) The condition for outer here, Condition (O), may be phrased thus: A set does not belong to $\mathcal{F}$ iff its complement belongs to $\text{Aso } \mathcal{F}$ (see §3.1). But (de Morgan laws) the union of complements is the complement of the intersection. So (O) may be translated into: in $\text{Aso } \mathcal{F}$ the intersection of members is also a member, i.e. $\text{Aso } \mathcal{F}$ is a filter. Hence the condition (O) for outer is equivalent to

\[(O') \quad \text{the associate } \text{Aso } \mathcal{F} \text{ is a filter} \]

(8) Now, which eventual families are the associate of themselves? For that $\mathcal{F}$ must have the property that a set $S$ belongs to $\mathcal{F}$ iff its complement $S^c$ does not.

That implies, yet is stronger than, the ‘inner’ condition (I).

If $\mathcal{F}$ is both inner and outer (and is not the trivial empty family), thus furnishes a ‘finitely additive measure’, it clearly will be self-Aso.

Indeed, the ‘measures’ of $S$ and $S^c$, each 0 or 1, must add up to the measure of the whole set $X$ which is 1 since $X \in \mathcal{F}$. Hence each is 0 iff the other is 1.

(9) But a self-Aso eventual family may not be ‘finitely additive’, and we shall see (Proposition 34) that it is ‘finitely additive’ iff it is moreover a filter.

**Example 32.** A simple example (which will have some general significance) of a self-Aso eventual family $\mathcal{F}$ which is not finitely additive and not a filter.

In a finite set $X$ with odd number $2N + 1 > 1$ of elements, let $\mathcal{F}$ be the family of all sets with $\geq N + 1$ elements, i.e. which include more than half the elements of $X$.

It (i.e. its characteristic function) is not a ‘finitely additive probability measure’ – the singletons have measure 0 and their union – the whole $X$ has measure 1.

And it clearly is not a filter – the intersection of sets with $> N$ elements can very well have $\leq N$ elements.

As we shall see, one may use this example to construct similar examples in an infinite $X$.

In particular, we may transform that to an example in an infinite set $X$: pick a finite subset $F \subset X$ with $2N + 1 > 1$ elements and define $\mathcal{F}$ on $X$ by: $\mathcal{F}$ is the family of all subsets of $X$ whose intersection with $F$ has $\geq N + 1$ elements.
Also, in the example of the pair \( G \) and \( F \), the latter is a filter, so \( G \) is outer. About \( F \) – the family of sets with finite complement – the same trick as in Remark 23 of decomposing a set to its even (resp. odd) elements which both will not belong to \( F \), shows that Out \( F \) is the zero multi-family, identified with the empty family.

This shows that one should not ‘dismiss’ the two ‘trivial’ families: the empty and the one comprising all subsets – they may have significant roles!

3.4. A ‘Litmus Paper’: Measured and non-Measured Partitions.

Definition 33. Let \( F \) be a self-Aso eventual family on (the subsets) of a set \( X \).

Let \( X = X_1 \cup X_2 \cup \ldots X_k \) (the \( X_i \) disjoint) be a finite partition of \( X \).

If some \( X_i \) is in \( F \) the partition is called measured. As \( F \) is a self-Aso eventual family, the complement of \( X_i \) is then not in \( F \). As the latter contains all the unions of the \( X_j, j \neq i \), also these are not in \( F \). While the other finite unions – those including \( X_j \) must be in \( F \) since \( X_i \) is.

If no \( X_i \) is in \( F \) call the partition non-measured. Then \( F \) cannot be a ‘finitely additive measure’ since the \( X_i \) have ‘measure’ 0 and their union \( X \) has ‘measure’ 1.

Note that when the partition is into three parts, the two-element unions are the complements of the parts, so the value the ‘measure’ gives to the parts determines everything concerning the unions of parts.

Proposition 34. Let \( F \) be a self-Aso eventual family on (the subsets) of a set \( X \). TFAE

1. (The characteristic function) of \( F \) is finitely additive
2. There are no non-measured partitions
3. There are no non-measured partitions into three sets.
4. \( F \) is a filter (thus an ultrafilter)

Proof. That \( 1 \Rightarrow 2 \) we have noted above: The existence of a non-measured partition, with parts of ‘measure’ 0 and their (finite) union \( X \) of ‘measure’ 1, clearly violates finite additivity.

3 \( \Rightarrow \) 1 Let \( S \) and \( T \) be disjoint. Then we have the three-part partition \( S, T, (S \cup T)^c = S^c \cap T^c \). The partition is measured, therefore one of the three parts have ‘measure’ 1 and the others ‘measure’ 0. As in Definition 33 this determines the values for all partial unions of the parts and means that ‘finite additivity’ holds whenever these are concerned, in particular for \( S, T \) and their union.

2 \( \Rightarrow \) 3 is obvious.

3 \( \Rightarrow \) 2 Firstly, to a non-measured partition with 2 parts, just add \( \emptyset \) as a part.

We show that if there is a non-measured partition into \( K > 3 \) parts, one can find one with \( < K \) parts. Indeed, unite two parts \( S \) and \( S' \) to get a partitions into \( K - 1 \) parts. If that union of two parts is not in \( F \), the new partition is non-measured with \( K - 1 \) parts. If it is in \( F \), its complement, i.e. the union of all parts other than \( S \) and \( S' \), is not in \( F \), so the latter complement, together with \( S \) and \( S' \), make a non-measured with 3 parts.

3 \( \Rightarrow \) 1 For \( F \) to be a filter, it must contain \( S \cap T \) if \( S, T \in F \). Now, we have a partition of \( X \) into three sets, \( S \cap T, S^c \cap T^c \). of these, the union of \( S \cap T \) and \( S^c \) includes \( T \), so is in \( F \). And the union of \( S \cap T \) and \( S \cap T^c \) includes \( S \), so is in \( F \) too. Their respective complements \( S \cap T^c \) and \( S^c \) thus cannot be in \( F \). So, for the partition to be measured \( S \cap T \) must be in \( F \).

4 \( \Rightarrow \) 1 Assume \( F \) is a filter. For finite additivity we need that if \( S \) and \( T \) are disjoint, the ‘measure’ of \( S \cup T \) is the sum of the ‘measures’ of \( S \) and \( T \).

As we saw in (1) above, if one ‘measure’ is 0 and the other is 1 thing are always OK.

Also, a filter being inner, the measures cannot be both 1 – see (4) and (5) above.

If these both ‘measures’ are 0, i.e. both \( S \) and \( T \) are not in \( F \), then \( F \) includes their complements \( S^c \) and \( T^c \), hence, being a filter, includes \( S^c \cap T^c = (S \cup T)^c \) (De Morgan’s laws), so \( S \cup T \) is not in \( F \) – has ‘measure’ 0.

\( \square \)
3.5. Ultrafilters vs. Inner; Outer; Self-Aso Eventual Families. So, we have inner eventual families, including the filters and self-Aso, which include the self-Aso filters, equivalently those furnishing a ‘finitely additive measure’

The latter may be characterized as ‘self-Aso filters’ – filters to which a set belongs iff its complement does not. Alternatively as maximal filters – the lack of a set \( S \) with both \( S \) and \( S^c \) not in \( \mathcal{U} \) is exactly what hampers extending to a larger filter. These are the ultrafilters.

As mentioned in the end of §3.5 self-Aso eventual families share part of the properties of ultrafilters:

They both ‘extract’ a \((0, 1)\) value – yes/no if you wish – from such value at each member of a set, and consequently (see item 4 below) may ‘extract’ a numerical value – a limit, from numerical values at each member (i.e. a numerical function).

Self-Aso eventual families, like ultrafilters, would respect inclusion/implication and complement/negation, but not the binary Boolean ‘and’ and ‘or’. Thus they would not do for most of the roles ultrafilters have in Logic, etc., but maybe would still provide a generalization for some.

Note also that

1. In a finite set every ultrafilter is ‘fixed’ at some element \( x \) – is the family of subsets containing \( x \). Thus to a finite set they will not add any ‘limit points’ in the sense of §3.3 As our ‘majority’ example 32 shows, that is certainly not the case with self-Aso.

2. With a self-Aso family \( \mathcal{F} \), in particular an ultrafilter, by the way things (limits, integrals) are defined, if \( Q \) is a subset which belongs to \( \mathcal{F} \), equivalently \( Q^c \) does belong there, then any change in \( Q^c \) does not matter at all – indeed it is a set of measure 0 here – and one may view \( \mathcal{F} \) as a self-Aso family in \( Q \), ‘and forget about \( Q^c \).’ Conversely, given a self-Aso family \( \mathcal{F}' \) on a \( Q \subset X \) one ‘equates’ it with the family on \( X \) \( \mathcal{F} := \{ S \subset X \mid S \cap Q \in \mathcal{F}' \} \).

3. Yet inner eventual families (including the self-Aso) do not share with filters the fact that In a (Hausdorff) compact space, they must have some limit points, and a unique limit point must be a limit - for that one need that every finite partition be measured.

4. In any (Hausdorff) compact space an ultrafilter \( \mathcal{U} \) has a unique limit. This limit may be viewed also as an integral. The limit is an \( x_0 \in X \) all whose neighborhoods belong to \( \mathcal{U} \). Thus their complements are not in \( \mathcal{U} \), so ‘can be completely forgotten’ according to item 2. It is as if ‘everything is’ in any neighborhood of \( x_0 \), which, of course, can be taken as small as we wish).

This hallmark ultrafilters, seems to be largely shared by self-Aso families. (Just, contrary to ultrafilters, one may not have a limit).

3.6. Constructing new Self-Aso from Old. Similarly to ultrafilters, one may construct new self-Aso from old (by sort of ‘concatenating’ the, if you wish, ‘extracting a \((0, 1)\) value – yes/no, from a family of such’). When one of the ingredients is self-Aso but not an ultrafilter (as in example 32) one would get many examples of such.

Of course, any function \( X \to Y \) would push ultrafilters (resp. self-Aso) in \( X \) to such in \( Y \).

Another example is a product: Let \( X \) and \( Y \) be sets, \( \mathcal{E} \) and \( \mathcal{F} \) self-Aso eventual families on \( (\text{the subsets of}) \) \( X \) and \( Y \) respectively.

One may ‘straightforwardly’ define such on the Cartesian product \( X \times Y \) as follows:

To check whether a subset \( S \) of \( X \times Y \) (i.e. – its characteristic function – a \((0, 1)\)-valued function on \( X \times Y \)) belongs, one just ‘integrates’ on the \( X \)-slices using \( \mathcal{E} \) – ‘extracts’ a \((0, 1)\)-value for each slice, and then ‘integrates’ these on \( Y \) using \( \mathcal{F} \).

The problem is, that as with ultrafilters, the order \( Y \) after \( X \) or \( X \) after \( Y \) matters very much – no analog of Fubini’s theorem.
While, as said, one does get many examples, That ‘asymmetry’ with product hampers ‘fully’ using ultrafilters, let alone self-Aso to ‘extend sets by limits’. One needs to extend ‘non-canonically’ to get what is called Non-standard Analysis (see references in the bibliography, say my [Le]).

One tries to modify that product scenario:

Let $X$ be a set, let us be given a self-Aso eventual family $F$ on $X^{\{1,2,\ldots,2N+1\}}$ – the $2N + 1$-tuples in $X$ (or, see item 2 above, – on some subset $Q$ therein), then one ‘projects’ to a self-Aso eventual family in $X$ by ‘deciding whether a subset $S \subset X$ belongs’ by considering the set of $2N + 1$-tuples with ‘majority’ lying in $S$, and seeing whether that belongs to $F$.

(At the end of example 32 we, in fact, did that with $F$ a fixed ultrafilter)

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