Exact controllability in minimal time of the Navier-Stokes periodic flow in a 2D-channel
Gabriela Marinoschi

“Gheorghe Mihoc-Caius Iacob” Institute of Mathematical Statistics and
Applied Mathematics of the Romanian Academy,
Calea 13 Septembrie 13, Bucharest, Romania
gabriela.marinoschi@acad.ro

Abstract. This work is concerned with the necessary conditions of optimality for a minimal time control problem \((P)\) for the linearized Navier-Stokes periodic flow in a 2D-channel, subject to a boundary input which acts on the transversal component of the velocity. The objective in this problem is the reaching of the laminar regime in a minimum time, as well as its preservation after this time. The determination of the necessary conditions of optimality relies on the analysis of intermediate minimal time control problems \((P_k)\) for the Fourier modes \(\hat{k}\) associated to the Navier-Stokes equations and on the proof of the maximum principle for them. Also it is found that one can construct, on the basis of the optimal controllers of problems \((P_k)\), a small time called here quasi minimal and a boundary controller which realizes the required objective in \((P)\).

Key words: minimal time controllability, boundary control, necessary conditions of optimality, Navier-Stokes equations.

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1 Introduction

In this paper, we focus on the determination of the necessary conditions of optimality for the linearized Navier-Stokes periodic flow in a channel, driven in minimal time towards a stationary laminar regime by a boundary control acting upon the transversal flow velocity.

In the following, we briefly review some titles in the literature devoted to this subject. A maximum principle for the time optimal control of the 2D Navier-Stokes equations is presented in [1]. For some aspects concerning the Navier-Stokes controllability and stabilization we refer the reader to the monographs [5], [6] and to the papers [2] and [3], investigating the stabilization of the Navier-Stokes flow in a channel by controllers with a vertical velocity observation which acts on the normal component of velocity, and by noise wall normal controllers, respectively. For techniques referring to minimal time controllability results we also mention the papers [18], [19], [14]. Moreover, we cite the more recent monograph [20] where a detailed investigation of time optimal control problems is developed.

On the other hand, controllability of Navier-Stokes in finite small-time flow gave rise in the literature to a set of reference works. The small-time global exact null controllability problem for the Navier-Stokes equation was suggested by J.-L. Lions in [13]. The control was a source term supported within a small subset of the domain, which is similar to controlling only a part of the boundary, with the Dirichlet boundary condition on the uncontrolled part of the boundary. An exhaustive presentation of various controllability problems, including also that of Navier-Stokes equations, is found in [9]. Here, we shall indicate only a few titles related especially to the controllability by using a boundary control. In [12], a small-time global exact null controllability is proved for a control supported on the whole boundary, while the paper [11] is devoted to the proof of the local exact controllability of the 2D Navier-Stokes system in a bounded domain in the case when the control function is concentrated on the whole boundary or on some part of it. The small-time global exact null controllability for Navier-Stokes under an irrotational flow boundary condition on the uncontrolled boundaries in a 2D rectangular domain it is proved in [8]. The exact boundary controllability of the Navier-Stokes system where the controls are supported in a given open subset of the boundary is provided in [16]. More recently, the paper [10] focuses on the small-time controllability presenting a new method, which takes into account the boundary layer for getting the control determination.
Let us consider the fluid flow in a 2-D infinitely long channel, governed by the incompressible Navier-Stokes equations:

\[
\begin{align*}
    u_t - \nu \Delta u + uu_x + v u_y &= \theta_x, & v_t - \nu \Delta v + uv_x + v v_y &= \theta_y, \\
    u_x + v_y &= 0, \\
    u(t, x + 2\pi, y) &= u(t, x, y), & v(t, x + 2\pi, y) &= v(t, x, y), \\
    u(t, x, 0) &= u(t, x, L) = 0, & v(t, x, 0) &= 0, & v(t, x, L) &= 0, \\
    u(0, x, y) = u_0, & v(0, x, y) = v_0, & \text{for } t \in \mathbb{R}_+ = (0, \infty), x \in \mathbb{R}, y \in (0, L).
\end{align*}
\]  

(1.1)

Here, \((u, v)\) is the fluid velocity, \(\theta\) is the pressure, the subscripts \(t, x, y\) represent the partial derivatives with respect to these variables.

We also consider the steady-state flow with zero vertical velocity, governed by (1.1). This flow velocity turns out to be of the form \((U(y), 0)\), where \(U(y) = -\frac{a}{2\nu} \left(\frac{y^2}{L} - y\right)\), \(a \in \mathbb{R}_+\) (see e.g., [17]).

The problem we are concerned with is to steer the flow (1.1) to the stationary regime \((U(y), 0)\), within a minimal finite time, by means of a boundary control \(w\) acting at \(y = L\) upon the transversal velocity component \(v\), namely by \(v(t, x, L) = w(t, x)\). More precisely, the objective is to characterize the boundary control \(w\) which could force the flow \((u, v)\) starting from \((u_0, v_0) \neq (U(y), 0)\) to reach the laminar regime \((U(y), 0)\) at a minimal time and, moreover, to preserve it at this value after that time. We stress that we are concerned with the determination of the necessary conditions of optimality and with the proof of the controllability result. However, we succeed to prove that this action can be done within a quasi-minimal time, provided by optimal minimal times for the problems in modes of the Fourier transform of the Navier-Stokes linearized system. Further, we shall describe in detail the arguments.

We shall study this problem for the linearized flow around the laminar steady-state \((U(y), 0)\). Also, since the flow is periodic along the longitudinal axis, we shall consider it on a period \((0, 2\pi)\). Thus, we linearize (1.1) around \((U(y), 0)\) relying on the change of function \(u \to u - U\) and continue to keep the same notation \(u\) for the linearized longitudinal velocity. Then, the linearized controlled system reads

\[
\begin{align*}
    u_t - \nu \Delta u + U u_x + U_y v &= \theta_x, & v_t - \nu \Delta v + U v_x &= \theta_y, \\
    u_x + v_y &= 0, \\
    u(t, 2\pi, y) &= u(t, 0, y), & v(t, 2\pi, y) &= v(t, 0, y), \\
    u(t, x, 0) &= u(t, x, L) = 0, & v(t, x, 0) &= 0, & v(t, x, L) &= w(t, x), \\
    u(0, x, y) &= u_0, & v(0, x, y) &= v_0, & \text{for } t \in \mathbb{R}_+, x \in (0, 2\pi), y \in (0, L).
\end{align*}
\]  

(1.2)

We express the flow controllability in minimum time by the problem

\((P)\) Minimize \(J(T, w) = T; T > 0, w \in H^1((0, T; L^2(0, 2\pi))) \cap L^\infty(0, \infty; L^2(0, 2\pi))\),

\[
\begin{align*}
    w(0, y) &= 0, & \int_0^T \int_0^{2\pi} |w(t, x)|^2 dx dt \leq \rho^2, \\
    u(T, x, y) &= 0, & v(T, x, y) &= 0, & \text{a.e. } (x, y) \in (0, 2\pi) \times (0, L),
\end{align*}
\]

subject to system (1.2), with \(u_0 \neq 0, v_0 \neq 0\). In addition, by resetting \(w\) after \(t = T\) one ensures that the null regime is preserved, as we shall see.

It is obvious that the minimal time should be positive. Indeed, by absurd, if \(T = 0\), we would have \(0 = v(T) = v(0) = v_0\), and similarly for \(u\), which contradicts the hypothesis. The requirement \(w(0, x) = 0\) is done especially for technical purposes, but it is also in agreement with the fact that at the initial time the boundary condition at \(y = L\) is no-slip. Finally, it is clear that in problem \((P)\) it is important to find information about the controller \(w\) only on the interval \((0, T)\) within which the objective is reached. On the interval \((T, \infty)\) the function \(w\) can take a whatever value; in particular it is of no effect the preserving the zero value for \(v\) after the time \(T\). That is why the property of \(w\) of belonging to \(H^1\) is required only for \(t \in (0, T)\).

Our purpose is to find the necessary conditions of optimality for \((P)\). To this end, the following controllability assumption, which will allow the derivation of an observability result for the adjoint system, will be in effect:

\((H)\) For each \((t_0, T)\), \(0 \leq t_0 < T < \infty\), and each \((u_0^0, v_0^0)\), \(\|(u_0^0, v_0^0)\|_{L^2(0, 2\pi; L^2(0, L))} \leq 1\), there exists \(w \in H^1(0, T; L^2(0, 2\pi))\) and \(\gamma_{(t_0, T)} \in L^1(0, T)\) satisfying \(w(t) = 0\) for \(0 \leq t \leq t_0 < T\) and
Similarly, conjugate. The notation work. (by the objective is expected to be reached in a smaller time. but \( \gamma \) (which ensures that \( \theta \)) to minimization problems for the Fourier coefficients of the velocity. To this end, we write

\[
\int_0^T \int_0^{2\pi} |w_1(t, x)|^2 \, dx \, dt \leq \int_0^T |\gamma(t_0, \tau)(t_0)| \, dt_0,
\]
such that \( u^{t_0, w}(T, x, y) = 0, v^{t_0, w}(T, x, y) = 0 \) a.e. \( (x, y) \in (0, 2\pi) \times (0, L) \).

Here, \( (u^{t_0, w}, v^{t_0, w}) \) is the solution to \( (1.2) \) starting from \( (u^0, v^0) \neq (0, 0) \) at time \( t = t_0 \), and controlled by \( w \). We note that \( \gamma(t_0, \tau) \) depends on the interval \( (t_0, \tau) \), it is bounded on \( (0, \tau) \) for all \( \delta > 0 \), but \( \gamma(t_0, \tau) \to \infty \) as \( t_0 \to \tau \). In fact \( \gamma(t_0, \tau) \) represents the controllability cost, which should be larger if the objective is expected to be reached in a smaller time.

In what concerns the controllability hypothesis \( (H) \), its verification is beyond the purpose of this work.

Next, we are going to describe the organization of the paper. It is convenient to reduce problem \( (P) \) to minimization problems for the Fourier coefficients of the velocity. To this end, we write

\[
f(t, x, y) = \sum_{k \in \mathbb{Z}, k \neq 0} f_k(t, y)e^{ikx}, \quad f_k = \bar{f}_{-k}, \quad f_0 = 0,
\]

(1.3)

(which ensures that \( f \) is real), where \( i = \sqrt{-1} \in \mathbb{C} \), the set of complex numbers and \( \bar{f} \) is the complex conjugate. The notation \( f \) stands for \( u, v, \theta \), and \( f_k \) stands for \( u_k, v_k, \theta_k \). Obviously, \( f_k(t, y) \in \mathbb{C} \). Similarly,

\[
w(t, x) = \sum_{k \in \mathbb{Z}, k \neq 0} w_k(t)e^{ikx}, \quad w_k = \bar{w}_{-k}
\]

\[
u_0(t, y) = \sum_{k \in \mathbb{Z}, k \neq 0} \nu_{k0}(y)e^{ikx}, \quad \nu_0(x, y) = \sum_{k \in \mathbb{Z}, k \neq 0} \nu_{k0}(y)e^{ikx}, \quad \nu_{k0} = \bar{\nu}_{-k0}, \quad \nu_{k0} = \bar{\nu}_{-k0}.
\]

Replacing in \( (1.2) \) the functions by their Fourier series and identifying the coefficients we obtain the system

\[
\begin{align*}
(u_k)_t - \nu u_k'' + (\nu k^2 + ikU)u_k + U'v_k &= ik\theta_k, \\
(v_k)_t - \nu v_k' + (\nu k^2 + ikU)v_k &= \theta_k', \\
iv u_k + v_k' &= 0, \\
u_k(t, 0) &= u_k(t, L) = v_k(t, 0) = 0, \quad v_k(t, L) = w_k(t), \\
u_k(0, y) &= u_{k0}, \quad v_k(0, y) = v_{k0}.
\end{align*}
\]

(1.5)

For simplicity, we denote by the superscripts \( ', '' \), \( iv \) the first four partial derivatives with respect to \( y \) of the functions \( u_k \) and \( v_k \).

By the Parseval identity, in particular for

\[
\sum_{k \in \mathbb{Z}, k \neq 0} \int_0^T |(w_k)_t(t)|^2 \, dt = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^T |w(t, x)|^2 \, dt \right) \, dx,
\]

(1.6)

we can consider that

\[
\int_0^T |(w_k)_t(t)|^2 \, dt \leq \rho_k^2, \quad \text{where} \quad \sum_{k \in \mathbb{Z}, k \neq 0} \rho_k^2 \leq \rho^2.
\]

Eliminating \( \theta_k \) between the first equations in \( (1.5) \) by using

\[
v_k' = -iku_k
\]

(1.7)

we obtain the following equation in \( v_k \):

\[
(k^2 v_k - v_k')_t + \nu v_k'' - (2\nu k^2 + ikU)v_k' + (\nu k^4 + ik^3 U + ikU'')v_k = 0,
\]

(1.8)

\[
v_k(t, 0) = 0, \quad v_k(t, L) = w_k(t),
\]

(1.9)

\[
v_k'(t, 0) = v_k'(t, L) = 0,
\]

(1.10)

\[
v_k(0, y) = v_{k0}(y),
\]

(1.11)
for $t \in \mathbb{R}_+, y \in (0, L)$.

Consequently, for each $k \in \mathbb{Z}, k \neq 0$ we can consider the minimization problem for the mode "$k$" :

$$(P_k) \quad \text{Minimize } \left\{ J_k(T, w) = T; \ T > 0, \ w \in H^1(0, T) \cap L^\infty(0, \infty), \right.$$ 

$$\left. \int_0^T |(w_k)_x(t)|^2 \, dt \leq \rho_k^2, \ v_k(T, y) = 0 \ a.e. \ y \in (0, L) \right\},$$

subject to (1.8)-(1.11), with $v_{k0} \neq 0$.

The controllability hypothesis $(H)$ and the Parseval identity provide for each mode "$k$" the following consequence:

$$(H_k) \quad \text{For each } (t_0, T), \ 0 \leq t_0 < T, \text{ and each } v^0 \neq 0, \ ||v^0||_{L^2(0, T)} \leq 1, \text{ there exists } w \in H^1(0, T) \text{ and }$$

$$\gamma(t_0, T) \in L^1(0, T), \text{satisfying } w(t) = 0 \text{ for } 0 \leq t \leq t_0, \left(\int_0^T |w_t(t)|^2 \, dt\right)^{1/2} \leq \|\gamma(t_0, T)\|_{L^1(0, T)}, \text{ such that }$$

$$v^{0, w}(T, y) = 0, \text{ where } v^{0, w} \text{ is the solution to (1.8)-(1.11)} \text{ starting from } v^0 \text{ at time } t = t_0.$$

Here, we included the constant $\frac{1}{\pi}$ in (1.6) in $\gamma(t_0, T)$.

As announced previously, the main part of the paper is directed to the determination of the necessary conditions of optimality for $(P)$, which will be deduced from those found for $(P_k)$.

Here there is the structure of the paper. For technical reasons, by using an appropriate variable transformation, we shall study instead of $(P_k)$, another problem $(\tilde{P}_k)$ set on a fixed time interval. In Section 2, we shall prove, in Theorem 2.2, the well-posedness of the transformed state system, and conclude with the existence of a solution $(T^*_k, w^*_k)$ to $(\tilde{P}_k)$. A characterization of the optimality conditions cannot be done directly for problem $(\tilde{P}_k)$, so that we have to resort to an approximating problem $(P_{k, \varepsilon})$, indexed along a small positive parameter $\varepsilon$, and prove the existence of a solution $(T^*_{k, \varepsilon}, w^*_{k, \varepsilon})$, in Theorem 3.1. A convergence result, formally expressed by $(P_{k, \varepsilon}) \to (\tilde{P}_k)$, as $\varepsilon \to 0$, will be proved in Theorem 3.2. The latter actually shows that if we fix an optimal pair $(T^*_k, w^*_k)$ in $(\tilde{P}_k)$ we can recover it as a limit of a sequence of solutions $(T^*_{k, \varepsilon}, w^*_{k, \varepsilon})$ to $(P_{k, \varepsilon})$. Based on these results, we proceed to the calculation of the necessary conditions of optimality for the approximating minimization problem $(P_{k, \varepsilon})$ in Proposition 3.4. They can be established if $T^*_k$ is small enough and $\rho_k$ is chosen sufficiently large. Appropriate fine estimates following by an observability result allow to pass to the limit in the approximating optimality conditions to get those corresponding to problem $(\tilde{P}_k)$ in Theorem 4.1. Finally, by relying on the Fourier characterization of $u, v$ and $w$ we prove in Theorem 5.2 that, if $(P)$ has an admissible pair $(T_*, w_*)$, there exists $(T^*, w^*)$ which steers $(u_0, v_0)$ into $u(T^*) = v(T^*) = 0$ and $T^* \leq T_*$. This pair is constructed on the basis of $(T^*_k, w^*_k)$ with $T^*_k$ minimal in problems $(P_k)$. For this reason we call it a quasi minimal time for $(P)$. However, it is not clear if it is precisely the minimal one.

**Notation.** Let $X_\mathbb{R}$ be a real Banach space and let $T > 0$. We denote by $X$ the complexified space $X_\mathbb{R} + iX_\mathbb{R}$ and by $L^p(0, T; X), W^{1,p}(0, T; X), C^i([0, T]; X)$ the complexified spaces containing functions of the form $f_1 + if_2$, with $f_1, f_2 \in L^2(0, T; X_\mathbb{R}), W^{k,p}(0, T; X_\mathbb{R})$ and $C^i([0, T]; X_\mathbb{R})$, respectively, for $p \in [1, \infty], i \in \mathbb{N}$.

The space $W^{k,p}(0, T; X_\mathbb{R}) = \{ f \in L^p(0, T; X_\mathbb{R}); \frac{\partial^m f}{\partial t^m} \in L^p(0, T; X_\mathbb{R}), m = \{1, \ldots, k\} \}.$

We shall use the standard Sobolev spaces $(H^1(0, 1))_\mathbb{R}, (H^2(0, 1))_\mathbb{R}$ and denote

$$H_\mathbb{R} := (L^2(0, L))_\mathbb{R}, (H_0^2(0, L))_\mathbb{R} = \{ f \in (H^2(0, L))_\mathbb{R}; f(0) = f(L) = f'(0) = f'(L) = 0 \}. $$

We have $(H_0^2(0, L))_\mathbb{R} \subset (H^1_0(0, L))_\mathbb{R} \subset H_\mathbb{R} \subset (H^1_0(0, L))_\mathbb{R} \subset ((H_0^2(0, L))_\mathbb{R})^*$ with compact injections, where $(H^1_0(0, L))_\mathbb{R}$ and $((H_0^2(0, L))_\mathbb{R})^*$ are the duals of $(H^1_0(0, L))_\mathbb{R}$ and $(H_0^2(0, L))_\mathbb{R}$, respectively. Their corresponding complexified spaces $H_0^2(0, L), H_0^1(0, L), H, (H^1_0(0, L))^*, (H_0^2(0, L))^*$ are defined as before, and satisfy

$$H^2_0(0, L) \subset H^1_0(0, L) \subset H \subset (H^1_0(0, L))^* \subset (H^2_0(0, L))^*$$

with compact injections. Also, we define the spaces

$$V_T := \{ f \in H^1(0, T); f(0) = 0 \}, \text{ with the norm } ||w||^2_{V_T} = \int_0^T |w(t)|^2 \, dt, \quad (1.12)$$

$$V_1 := \{ f \in H^1(0, T); f(0) = 0 \}, \text{ with the norm } ||w||^2_{V_1} = \int_0^1 |w(t)|^2 \, dt.$$
We denote by \(|\zeta|\) the norm of \(\zeta = \zeta_1 + i\zeta_2 \in \mathbb{C}\), the space of complex numbers. The scalar product on \(\mathbb{C}\) is defined as
\[
(a, b)_{\mathbb{C}} = \overline{a}b, \quad \text{for } a, b \in \mathbb{C}, \text{ with } \overline{b} \text{ the complex conjugate,}
\] (1.13)
The scalar product and norm in \(H\) are defined by
\[
(\zeta, z)_{H} = \int_{0}^{L} \zeta(y)\overline{z}(y)dy, \quad \|z\|_{H} = \left( \int_{0}^{L} \left| \zeta(y) \right|^2 dy \right)^{1/2}, \quad \zeta, z \in H.
\] (1.14)
The first and second derivatives of a function \(w\) depending only on \(t\) will be denoted by \(\dot{w}\) and \(\ddot{w}\).

2 Problems \((P_k)\) and \((\widehat{P_k})\)

In order to handle in a more convenient way the arguments in the proofs of the next results, and especially for calculating the approximating necessary conditions of optimality, we shall use a state system and a new minimization problem for the modes \(k\), on a fixed time interval, by making an appropriate transformation in order to bring the interval \((0, T)\) into \((0, 1)\). To this end, we set in the state system
\[
t = \widehat{t}T, \quad v_k(t, y) = \widehat{v}_k(\widehat{t}T, y) := \hat{v}_k(\widehat{t}, y), \quad \widehat{w}_k(t) = w_k(\widehat{t}),
\] (2.1)
such that \(\widehat{t} \in [0, 1]\) when \(t \in [0, T]\).

Then, the restriction \(|w|_{V_1} \leq \rho_k\) becomes \(|\widehat{w}|_{V_1} \leq \rho_k\sqrt{T}\), with \(V_1\) defined in (1.12).

The state system (1.8)-(1.11) is transformed into the appropriate system for \(\hat{v}_k\)
\[
(k^2\hat{v}_k'' - \hat{v}_k''')\widehat{t} + T \left( \nu \hat{v}_k'' + (2\nu k^2 + ikU)\hat{v}_k'' + (\nu k^4 + ik^3U + U'')\hat{v}_k \right) = 0,
\] (2.2)
\[
\hat{v}_k(\widehat{t}, 0) = 0, \quad \hat{v}_k(\widehat{t}, L) = \hat{w}_k(\widehat{t}),
\] (2.3)
\[
\hat{v}_k'(\widehat{t}, 0) = 0, \quad \hat{v}_k'(\widehat{t}, L) = 0,
\] (2.4)
\[
\hat{v}_k(0, y) = v_{k0}(y),
\] (2.5)
for \(\widehat{t} \in (0, 1), y \in (0, L)\).

In this way, problem \((P_k)\) becomes \((\widehat{P_k})\) below:

\((\widehat{P_k})\) Minimize \(\{J_k(T, w) = T; \ T > 0, \ w \in V_1 \cap L^\infty(0, \infty),
\]
\[
\int_{0}^{L} |\widehat{w}_k(t)|^2 dt \leq \rho_k^2 T, \quad \hat{v}_k(1, y) = 0 \ a.e. \ y \in (0, L)\},
\]
subject to (2.2)-(2.5), with \(v_{k0} \neq 0\), where \(V_1\) was defined in (1.12).

The controllability hypothesis \((H_k)\) will be correspondingly written on the interval \((\hat{t}_0, 1)\).

**Note.** However, for not overloading the notation, we shall skip in sections 2-4 the notation with the decoration “\(\widehat{\cdot}\)” and will resume it in Theorem 5.2. Thus, in system (2.2)-(2.5) we shall write \(t, v_k, w_k, \) instead of \(\widehat{t}, \hat{v}_k, \hat{w}_k\).

In this section we prove the well-posedness for the state system derived from (2.2)-(2.5) and the existence of a solution to \((\widehat{P_k})\), which obviously imply the same results for (1.8)-(1.11) and \((P_k)\).

We begin with some definitions. For each \(k \in \mathbb{Z} \setminus \{0\}\) let us define the operators
\[
E_{0k} : D(E_{0k}) \subset H \rightarrow H, \ D(E_{0k}) = H^2(0, L) \cap H^1_0(0, L), \ E_{0k}z := k^2z - z''
\] (2.6)
and
\[
F_{0k} : D(F_{0k}) \subset H \rightarrow H, \ D(F_{0k}) = H^4(0, L) \cap H^2_0(0, L),
\]
\[
F_{0k}z = \nu z'' + (2\nu k^2 + ikU)z'' + (\nu k^4 + ik^3U + ikU'')z.
\] (2.7)

Since \(E_{0k}\) is \(m\)-accretive, coercive, hence invertible, with the inverse continuous on \(H\), we can define the operator
\[
A_k := F_{0k}E_{0k}^{-1}, \ A_k : D(A_k) \subset H \rightarrow H, \ D(A_k) = \{v \in H; \ E_{0k}^{-1}v \in D(F_{0k})\}.
\] (2.8)
We also observe that,
\[ v \in D(A_k) \text{ iff } v = E_{0k} \varphi, \text{ for } \varphi \in H^4(0, L) \cap H_0^2(0, L). \]  

(2.9)

By Lemma 1 in [3] we know that \( A_k \) is closed and densely defined in \( H \), and \(-A_k \) generates a \( C_0 \)-analytic semigroup on \( H \), that is, its resolvent has the property
\[ \| (\sigma I + A_k)^{-1} f \| \leq \frac{\| f \|}{|\sigma| - \sigma_0}, \text{ for all } f \in H \text{ and } |\sigma| > \sigma_0. \]  

(2.10)

**Definition 2.1.** We call a solution to (2.2)-(2.5) a function
\[ v_k \in C([0, 1]; H^2(0, L)) \cap W^{1,2}(0, 1; H^2(0, L)) \cap L^2(0, 1; H^4(0, L)), \]

which satisfies (2.2)-(2.5) for a.a. \( t > 0 \).

**Theorem 2.2.** Let \( v_{k0} \in H^4(0, L) \cap H_0^2(0, L) \), \( w_k \in V_1 \), \( \| w_k \|_{V_1} \leq \rho k \sqrt{T} \), \( T > 0 \). Then, problem (2.2)-(2.5) has a unique solution
\[ v_k \in C([0, 1]; H^2(0, L)) \cap W^{1,2}(0, 1; H^2(0, L)) \cap L^2(0, 1; H^4(0, L)), \]

(2.11)

which satisfies the estimate
\[ \| v_k(t) \|_{H^2(0,L)}^2 + \int_0^1 \left\| \frac{dv_k}{dt}(t) \right\|_{H^2(0,L)}^2 dt + T \int_0^1 \| v_k(t) \|_{H^4(0,L)}^2 dt \leq C \left( \| v_{k0} \|_{H^2(0,L)}^2 + T \int_0^1 |w_k(t)|^2 dt + \int_0^1 \| w_k(t) \|_{H^4(0,L)}^2 dt \right), \text{ for all } t \geq 0. \]  

(2.12)

The solution is continuous with respect to the data, that is, two solutions \((v_k^1, v_k^2)\) corresponding to the data \((v_{k0}^1, w_k^1)\) and \((v_{k0}^2, w_k^2)\) satisfy the estimate
\[ \| (v_k^1 - v_k^2)(t) \|_{H^2(0,L)}^2 + \int_0^1 \left\| \frac{dv_k^1 - dv_k^2}{dt}(t) \right\|_{H^2(0,L)}^2 dt + T \int_0^1 \| (v_k^1 - v_k^2)(t) \|_{H^4(0,L)}^2 dt \leq C \left( \| (v_{k0}^1 - v_{k0}^2) \|_{H^2(0,L)}^2 + \int_0^1 \left( \| (w_k^1 - w_k^2)(t) \|_{H^4(0,L)}^2 + T \| (w_k^1 - w_k^2)(t) \|_{H^4(0,L)}^2 \right) dt \right), \]

for all \( t \geq 0 \).

**Proof.** We recall that \( V_1 := \{ f \in H^1(0, 1); f(0) = 0 \} \). Let us introduce a function transformation in order to homogenize the boundary conditions, namely
\[ \tilde{v}_k(t,y) = v_k(t,y) - \beta(y)w_k(t), \]

(2.14)

where
\[ \beta(y) = -\frac{2}{T} y^3 + \frac{3}{T^2} y^2. \]

(2.15)

This transformation is chosen such that \( \tilde{v}_{k,c}(t,0) = \tilde{v}_{k,c}(t,L) = \tilde{v}_{k,c}'(t,0) = \tilde{v}_{k,c}'(t,L) = 0 \). Equation (2.2) is transformed into
\[ (k^2 \tilde{v}_k - \tilde{v}_k')_t + T \left[ \nu \tilde{v}_k'' - (2

(2.16)

where
\[ a_k = -\left( \nu \beta'' - (2 \nu k^2 + ikU) \right) \beta + (\nu k^4 + ik^3 U + ikU') \beta, \]

(2.17)
and $a_k, b_k \in C^\infty(0, L)$. We denote $\tilde{v}_k(t) = k^2 \tilde{v}_k(t) - \tilde{w}_k(t)$, for $t \in (0, 1)$, and note that since $\tilde{v}_k$ vanishes at the boundaries, we have in fact
\[
\tilde{v}_k(t) = E_{0k} \tilde{v}_k(t), \quad t \in (0, 1).
\] (2.18)

Since $\tilde{v}_k(0) = (k^2 v_{k0} - v_{k0}''(0)) - (k^2 \beta - \beta') w_k(0)$ and $w_k(0) = 0$, equation (2.10) can be written as the equivalent Cauchy problem
\[
\frac{d\tilde{v}_k}{dt}(t) + T A_k \tilde{v}_k(t) = T a_k w_k(t) + b_k \dot{w}_k(t), \quad \text{a.e. } t \in (0, 1),
\] (2.19)
where $A_k = F_{0k} E_{0k}^{-1}$, by (2.8). We recall that $-A_k$ generates a $C_0$-analytic semigroup and so the solution to (2.19) is given by
\[
\tilde{v}_k(t) = v_1(t) + v_2(t) + v_3(t),
\]
where
\[
v_1(t) = e^{-t TA_k} \tilde{v}_k(0), \quad v_2(t) = T \int_0^t (e^{-\tau TA_k} a_k) w_k(s) ds, \quad v_3(t) = \int_0^t (e^{-\tau TA_k} b_k) \dot{w}_k(s) ds.
\]

Since $v_{k0} \in H^1(0, L) \cap H^2_0(0, L)$, it follows that $\tilde{v}_k(0) = E_{0k} v_{k0} \in D(A_k)$. Then, using the existence theorems for the solutions to equations with a $C_0$-analytic semigroup (see e.g. [15], Theorem 3.5, p. 114 and [7], Proposition 1.148, p. 60), it follows for the first term that
\[
v_1 \in C([0, 1]; D(A_k)) \cap C^1([0, 1]; H),
\]
\[
\|v_1(t)\|_{D(A_k)} + \left\| \frac{dv_1}{dt}(t) \right\|_H \leq C \|E_{0k} v_{k0}\|_{D(A_k)} \leq C \|v_{k0}\|_{H^4(0, 1) \cap H^2_0(0, 1)}, \text{ for all } t \in [0, 1].
\]
The second term can be viewed as the solution to the Cauchy problem
\[
\frac{dv_2}{dt}(t) + T A_k v_2(t) = T a_k w_k(t), \quad \text{a.e. } t \in (0, 1),
\]
\[
v_2(0) = 0,
\]
where $w ka_k \in L^2(0, 1; H)$, whence $v_2 \in C([0, 1]; H) \cap W^{1, 2}(0, 1; H) \cap L^2(0, 1; D(A_k))$ and
\[
\|v_2(t)\|^2_H + \int_0^1 \left\| \frac{dv_2}{dt}(t) \right\|^2_H dt + T \int_0^1 \|A_k v_2(t)\|^2_H dt \leq CT \int_0^1 |w_k(t)|^2 dt.
\]
Similarly, $v_3 \in C([0, 1]; H) \cap W^{1, 2}(0, 1; H) \cap L^2(0, 1; D(A_k))$ and
\[
\|v_3(t)\|^2_H + \int_0^1 \left\| \frac{dv_3}{dt}(t) \right\|^2_H dt + T \int_0^1 \|A_k v_3(t)\|^2_H dt \leq C \int_0^1 |\dot{w}_k(t)|^2 dt.
\]
Gathering the results for $v_1, v_2, v_3$, we get
\[
\tilde{v}_k \in C([0, 1]; H) \cap W^{1, 2}(0, 1; H) \cap L^2(0, 1; D(A_k)),
\]
\[
\left\| \tilde{v}_k(t) \right\|^2_H + \int_0^1 \left\| \tilde{v}_k(t) \right\|^2_H dt + T \int_0^1 \|A_k \tilde{v}_k(t)\|^2_H dt \leq C \left( \|v_{k0}\|^2_{H^4(0, 1) \cap H^2_0(0, 1)} + T \int_0^1 |w_k(t)|^2 dt + \int_0^1 |\dot{w}_k(t)|^2 dt \right).
\]
This implies (2.11), whence by (2.18) we obtain
\[
\tilde{v}_k \in C([0, 1]; H^2(0, L) \cap H^1_0(0, L)) \cap W^{1, 2}(0, 1; H^2(0, L) \cap H^1_0(0, L)) \cap L^2(0, 1; H^4(0, L) \cap H^1_0(0, L)),
\]
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the last space being derived by $E_{\text{ok}}^{-1}(D(A_k)) \subset H^4(0, L) \cap H^3_0(0, L)$. Finally, by (2.11) we obtain (2.12). Estimate (2.12) follows by the estimate for $\tilde{v}_k$. Also, 
\[ |w_k(t)|^2 = \left| \int_0^t \tilde{w}_k(\tau) d\tau \right|^2 \leq \int_0^1 |\tilde{w}_k(\tau)|^2 d\tau = \rho_k^2 T. \]

Since the equation is linear we also get (2.13). This implies still the uniqueness. The proof is ended. \[ \square \]

Even if we are only interested in obtaining the conditions of optimality, we also provide later, for the reader convenience, the proof of the existence of a solution to $(P_k)$.

To this end, we define an admissible pair for $(P)$ a pair $(P_{\epsilon}, w_{\epsilon})$ with $w_{\epsilon} \in H^1(0, T; L^2(0, 2\pi)) \cap L^\infty(0, \infty; L^2(0, 2\pi))$, $w(0, y) = 0$, $\int_0^T \int_0^{2\pi} |w_k(t, x)|^2 dt dx \leq \rho^2$ and $u(T, x, y) = 0$, $v(T, x, y) = 0$. By the Parseval identity this implies that $(P_{\epsilon}, w_{\epsilon})$ is an admissible pair for $(\hat{P}_k)$, with $w_{\epsilon}$ being the mode $k$ of $w_{\epsilon}$. We are not concerned here with the proof of the existence of an admissible pair. Related results can be found in the literature already cited referring to the controlability in small-time.

**Theorem 2.3.** Let $v_{k0} \in H^1(0, 1) \cap H^2_0(0, 1)$, $v_{k0} \neq 0$. If $(P)$ has an admissible pair, then, problem $(\hat{P}_k)$ has at least a solution $(T_k^n, w_k^n)$ with the corresponding optimal state $v_k^n$. Moreover, let us set $\hat{w}_k(t) := w_k(t)$ for $t \in [0, 1]$, and $\hat{w}_k(t) = 0$ for $t \in (1, \infty)$. Then, $v_k^n(t) = 0$ for $t > 1$.

**Proof.** Following the assumption before, $(\hat{P}_k)$ has an infimum denoted $T_k^\ast$ which is positive. We consider a minimizing sequence $(T_k^n, w_k^n)$ such that $T_k^n > 0$, $\|w_k^n\|_{V_1} \leq \rho_k \sqrt{T_k^n}$, with $v_k^n(1, y) = 0$ and
\[ T_k^\ast \leq J(T_k^n, w_k^n) = T_k^n + \frac{1}{n}, \quad n \geq 1. \]

This yields $T_k^n \rightarrow T_k^\ast$ as $n \rightarrow \infty$. Also, there exists $w_k^\ast \in H^1(0, 1)$ such that, on a subsequence, $w_k^n \rightarrow w_k^\ast$ weakly in $H^1(0, 1)$, strongly in $C([0, 1])$ by Arzelà theorem, and $\|w_k^n\|_{V_1} \leq \rho_k \sqrt{T_k^n}$. Thus, $w_k^n(0) \rightarrow w_k^\ast(0) = 0$. The solution to (2.2)-(2.5) corresponding to $(T_k^n, w_k^n)$ is denoted $v_k^n$, and has the properties (2.11), (2.13) with $T = T_k^n$. Thus, by a simple calculation, handling the property (2.13) we get

$\{v_k^n \rightarrow v_k^\ast \text{ strongly in } C([0, 1]; H^2(0, 1)) \cap W^{1, 2}(0, 1; H^2(0, 1)) \cap L^2(0, 1; H^4(0, 1)) \}$

We can pass to the limit in (2.2)-(2.5) written for $(T_k^n, w_k^n)$ to obtain that $v_k^\ast$ is the solution to (2.2)-(2.5) corresponding to $(T_k^\ast, w_k^\ast)$. Moreover, since $v_k^n \rightarrow v_k^\ast$ strongly in $C([0, 1]; H)$ we also have $v_k^\ast(1, y) = 0$, so that $(T_k^\ast, w_k^\ast)$ is optimal in $(\hat{P}_k)$.

Next, we prove the last assertion in the statement of the theorem. If $w_k^\ast$ is extended by 0 on $(1, \infty)$ the system for the variable $\chi_k$ starting from the initial datum at $t = 1$ reads
\[
(k^2 \chi_k - \chi_k^iv - (2\nu k^2 + ikU)\chi_k^o + (\nu k^4 + ik^3Uik + U''\chi_k = 0, \chi_k(t, 0) = 0, \chi_k(t, L) = 0, \chi_k(t, 0) = \chi_k(t, L) = 0, \chi_k(0, y) = v_k^\ast(T_k^\ast) = 0,
\]
for $(t, y) \in (0, 1) \times (0, L)$. Obviously, it has the unique solution 0, which extends the solution $v_k^\ast$. \[ \square \]

**3 The approximating problem $(P_k, \epsilon)$**

In this section, we introduce an approximating minimization problem $(P_k, \epsilon)$, prove the existence of a solution, its convergence to $(\hat{P}_k)$ and determine the approximating necessary conditions of optimality.

Let $(T_k^\ast, w_k^\ast)$ be a solution to $(\hat{P}_k)$ and let $\epsilon > 0$. We introduce the following approximating problem $(P_k, \epsilon)$:
\[
(P_k, \epsilon) \quad \text{Minimize} \left\{ J_{k, \epsilon}(T, w) = T + \frac{1}{2\epsilon} \left\| (\sigma I + A_k)^{-1} v_k(1) \right\|_H^2 \right\}
\]
subject to the approximating system (2.2)-(2.5). We underline that \( v_k(1) \) is \( v_k(1, y) \).

**Theorem 3.1.** Let \( v_{k0} \in H^2(0, L) \cap H^2_0(0, L) \), \( v_{k0} \neq 0 \). Then, problem \((P_{k, \varepsilon})\) has at least a solution \((T_{k, \varepsilon}^*, w_{k, \varepsilon}^*)\) with the corresponding optimal state \( v_{k, \varepsilon}^* \).

**Proof.** For \((P_{k, \varepsilon})\) we see that there exists at least an admissible pair, which is \((T_{k, \varepsilon}^*, w_{k, \varepsilon}^*)\), the optimal pair in \((\tilde{P}_k)\). Then, \( J_{k, \varepsilon}(T, w) \) is positive and so there exists \( d_{\varepsilon} = \inf J_{k, \varepsilon}(T, w) \) and it is positive. Indeed, by absurd if \( J_{k, \varepsilon}(T, w) = 0 \), then each term, including \( T \), should be equal to 0. This implies that in the second term of \( J_{k, \varepsilon}, v_k(T = 0, y) = 0 \), which contradicts \( v_{k0} \neq 0 \).

We consider a minimizing sequence \((T_{k, \varepsilon}^n, w_{k, \varepsilon}^n)\) with \( T_{k, \varepsilon}^n > 0 \), \( \| w_{k, \varepsilon}^n(t) \|_{V_1} \leq \rho_k \sqrt{T_{k, \varepsilon}^n} \), satisfying

\[
d_{\varepsilon} \leq J_{k, \varepsilon}(T_{k, \varepsilon}^n, w_{k, \varepsilon}^n) \leq d_{\varepsilon} + \frac{1}{n}, \quad n \geq 1.
\] (3.1)

Hence, there exists \( T_{k, \varepsilon}^* > 0 \) such that \( T_{k, \varepsilon}^* \to T_{k, \varepsilon}^* \) as \( n \to \infty \). On a subsequence, denoted still by \( n \), we have \( w_{k, \varepsilon}^n \to w_{k, \varepsilon}^* \) weakly in \( V_1 \) and strongly in \( C([0, 1]) \), so that \( w_{k, \varepsilon}^n(0) \to w_{k, \varepsilon}^*(0) = 0 \) and

\[
\| w_{k, \varepsilon}^* \|_{V_1} \leq \rho_k \sqrt{T_{k, \varepsilon}^*}.
\]

By (3.1)

\[
\int_0^t (w_{k, \varepsilon}^n(\tau) - w_{k, \varepsilon}^n) d\tau \to \int_0^t (w_{k, \varepsilon}^*(\tau) - w_{k, \varepsilon}^*) d\tau, \quad \text{uniformly for all } t \in [0, 1],
\] (3.2)

according to Arzelà theorem, because the sequence \((\int_0^t (w_{k, \varepsilon}^n - w_{k, \varepsilon}^*) d\tau)_n\) is bounded and its derivative converge weakly.

Then, the state system (2.2)-(2.5) corresponding to \( T_{k, \varepsilon}^n \) and \( w_{k, \varepsilon}^n \) has, by Theorem 2.2, a unique solution continuous in time on \([0, 1]\). This solution \( v_{k, \varepsilon}^n \) has the properties (2.11)-(2.14) with \( T = T_{k, \varepsilon}^n \), \( w_{k, \varepsilon}^n \). By (2.13) we deduce that \( v_{k, \varepsilon}^n \to v_{k, \varepsilon}^* \) strongly in \( C([0, 1]; H^2(0, L)) \cap W^{1,2}(0, 1; H^3(0, L)) \cap H^2(0, L) \) as \( n \to \infty \). Passing to the limit in (2.2)-(2.5) written for \((T_{k, \varepsilon}^n, w_{k, \varepsilon}^n)\) we get that \( v_{k, \varepsilon}^* \) is the solution to (2.2)-(2.5) corresponding to \( T_{k, \varepsilon}^* \) and \( w_{k, \varepsilon}^* \). Moreover, \( (\sigma I + A_k)^{-1}v_{k, \varepsilon}^n(1) \to (\sigma I + A_k)^{-1}v_{k, \varepsilon}^*(1) \) strongly in \( H \). Passing to the limit in (3.1), as \( n \to \infty \), we get on the basis of the previous convergences and of the weakly lower semicontinuity of the norms, that \( J_{k, \varepsilon}(T_{k, \varepsilon}^*, w_{k, \varepsilon}^*) = d_{\varepsilon} \), that is \((T_{k, \varepsilon}^*, w_{k, \varepsilon}^*)\) is an optimal controller in \((P_{k, \varepsilon})\).

We note that in problem \((P_{k, \varepsilon})\), the optimal state satisfies (2.2)-(2.5) with \( T = T_{k, \varepsilon}^* \) and \( w_{k, \varepsilon}^* \).

**Theorem 3.2.** Let \((T_{k, \varepsilon}^*, w_{k, \varepsilon}^*, v_{k, \varepsilon}^*)\) be optimal in \((P_{k, \varepsilon})\) and \((T_k^*, w_k^*, v_k^*)\) be optimal in \((\tilde{P}_k)\). Then,

\[
T_{k, \varepsilon}^* \to T_k^*, \quad w_{k, \varepsilon}^* \to w_k^* \text{ weakly in } H^1(0, 1) \text{ and strongly in } C([0, 1]),
\]

\[
v_{k, \varepsilon}^* \to v_k^* \text{ strongly in } C([0, 1]; H^2(0, L)) \cap W^{1,2}(0, 1; H^3(0, L)) \cap L^2(0, 1; H^4(0, L)).
\]

**Proof.** Let \((T_{k, \varepsilon}^*, w_{k, \varepsilon}^*, v_{k, \varepsilon}^*)\) be optimal in \((P_{k, \varepsilon})\) and denote

\[
h_{k, \varepsilon}^*(t) = \int_0^t (w_{k, \varepsilon}^* - w_k^*) d\tau.
\]

The fact that \( (T_{k, \varepsilon}^*, w_{k, \varepsilon}^*) \) is optimal in \((P_{k, \varepsilon})\) implies that

\[
J_{k, \varepsilon}(T_{k, \varepsilon}^*, w_{k, \varepsilon}^*) = J_{k, \varepsilon}^* + \frac{1}{2\varepsilon} \| (\sigma I + A_k)^{-1}v_{k, \varepsilon}^* - 1 \|_H^2 + \frac{1}{2} \int_0^1 |h_{k, \varepsilon}^*(t)|^2 dt \leq J_{k, \varepsilon}(T, w) = T + \frac{1}{2\varepsilon} \| (\sigma I + A_k)^{-1}v(1) \|_H^2 + \frac{1}{2} \int_0^1 |w - w_k^*(\tau)|^2 dt
\]

(3.5)
for any $T > 0$ and $w \in V_1$, $\|w\|_{V_1} \leq \rho_k \sqrt{T}$, where $v$ is the solution to the state system corresponding to $(T, w)$. Let us set in (3.3), $T = T_k^*$ and $w = w_k^*$, the chosen optimal controller in $(\hat{P}_k)$. Thus, the second and the last terms on the right-hand side of (3.3) vanish and

$$J_{k, \varepsilon}(T_{k, \varepsilon}^*, w_{k, \varepsilon}^*) = T_{k, \varepsilon}^* + \frac{1}{2\varepsilon} \|\sigma A_k^{-1}v_{k, \varepsilon}^*(1)\|_H^2 + \frac{1}{2} \int_0^1 |h_{k, \varepsilon}^*(t)|^2 dt \leq T_k^*.$$

(3.6)

Then, $T_{k, \varepsilon}^* \rightarrow T_k^*$, and on a subsequence denoted still by $\varepsilon$, we have $w_{k, \varepsilon}^* \rightarrow w_k^*$ weakly in $V_1$, strongly in $C([0,1])$, and $w_{k, \varepsilon}(0) \rightarrow w_k^*(0) = 0$. Also, $\|w_k^*\|_{V_1} \leq \rho_k \sqrt{T_k^*}$.

The solution $v_{k, \varepsilon}^*$ corresponding to $(T_k^*, w_k^*)$ exists, it is unique, according to Theorem 2.2 and has the properties (2.11-2.13) with $T = T_{k, \varepsilon}^*$ and $w_{k, \varepsilon}^*$. Therefore, by handling some calculations based on (2.13) we get

$$v_{k, \varepsilon}^* \rightarrow v_k^{T_k^*, w_k^*} := v_k^* \text{ strongly in } W^{1,2}(0,1; H^2(0,L)) \cap L^2(0,1; H^4(0,L))$$

(3.7)

where $v_k^*$ is the solution to (2.2)-2.5 corresponding to $(T_k^*, w_k^*)$. By (3.6) we have

$$\| (\sigma A_k^{-1}v_{k, \varepsilon}^*(1)) \|_H^2 \leq 2\varepsilon T_k^*,$$

(3.8)

so that

$$\lim_{\varepsilon \rightarrow 0} \| (\sigma A_k^{-1}v_{k, \varepsilon}^*(1)) \|_H^2 = 0$$

(3.9)

which implies that $v_{k, \varepsilon}^*(1, \cdot) \rightarrow 0$ strongly in $H$. On the other hand, by (3.7), $v_{k, \varepsilon}^*(1) \rightarrow v_k^*(1)$, so that $v_k^*(1) = 0$. Again by (3.6), $T_{k, \varepsilon} \leq J_{k, \varepsilon}(T_k^*, w_k^*) \leq T_k^*$, implying at limit that $T_k^* \leq T_k^*$. Since $T_k^*$ and $w_k^*$ satisfy the restrictions required in problem $(\hat{P}_k)$, that is $T_k^* > 0$, $\|w_k^*\|_{V_1} \leq \rho_k \sqrt{T_k^*}$, and $v_k^*(1) = 0$, recalling that $T_k^*$ is the infimum in $(\hat{P}_k)$ it follows that $T_k^* = T_k^*$.

Again by (3.6) we see that $T_k^* + \limsup_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \| (\sigma A_k^{-1}v_{k, \varepsilon}^*(1)) \|_H^2 \leq T_k^*$, which implies

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \| (\sigma A_k^{-1}v_{k, \varepsilon}^*(1)) \|_H^2 = 0.$$  

(3.10)

Also, $T_k^* + \limsup_{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^1 |h_{k, \varepsilon}^*(t)|^2 dt \leq T_k^*$ inferring that

$$\limsup_{\varepsilon \rightarrow 0} \int_0^1 |h_{k, \varepsilon}^*(t)|^2 dt = 0.$$

(3.11)

Therefore,

$$h_{k, \varepsilon}^* \rightarrow 0 \text{ strongly in } L^2(0,1), \text{ as } \varepsilon \rightarrow 0$$

(3.12)

and so it follows that

$$h_{k, \varepsilon}^*(t) = \int_0^t (w_{k, \varepsilon}^* - w_k^*)(\tau)d\tau \rightarrow \int_0^t (w_k^* - w_k^*)(\tau)d\tau = 0, \text{ for all } t \in [0,1], \text{ as } \varepsilon \rightarrow 0.$$

Thus, we get $w_k^*$ for all $[0,1]$ and so $v_k^*(t) = v_k^*(t)$ for all $t \in [0,1]$.

Based on the previous convergences, we pass to the limit in (3.6) and conclude that

$$\lim_{\varepsilon \rightarrow 0} J_{k, \varepsilon}(T_{k, \varepsilon}^*, w_{k, \varepsilon}^*) = T_k^* = J_k(T_k^*, w_k^*) = T_k^*.$$

□
3.1 Systems in variations and the adjoint system for \((P_{k,\varepsilon})\)

Since the minimization problem depends on two controllers, \(T_{k,\varepsilon}^*\) and \(w_{k,\varepsilon}^*\), we shall study separate variations with respect to them. First, let us keep \(T_{k,\varepsilon}^*\) fixed and give variations to \(w_{k,\varepsilon}^*\). We shall obtain a first system in variations.

Let \((T_{k,\varepsilon}^*, w_{k,\varepsilon}^*)\) be an optimal controller in \((P_{k,\varepsilon})\). For \(\lambda > 0\), we set

\[
w_{k,\varepsilon}^\lambda = w_{k,\varepsilon}^* + \lambda \omega, \quad \text{where} \quad \omega = \tilde{w} - w_{k,\varepsilon}^*, \quad \tilde{w} \in V_1, \quad \|\tilde{w}\|_{V_1} \leq \rho_k \sqrt{T_{k,\varepsilon}^*}.
\]  

(3.13)

We note that the state system satisfies (2.2)-(2.5) with \(T = T_{k,\varepsilon}^*\) and \(w_{k,\varepsilon}^*\). We define \(Y_{\lambda} = \frac{T_{k,\varepsilon}^* - w_{k,\varepsilon}^*}{\lambda}\), where \(v_{k,\varepsilon}^\lambda w_{k,\varepsilon}^*\) is the solution to (2.2)-(2.5) corresponding to \(T_{k,\varepsilon}^*\) and \(w_{k,\varepsilon}^\lambda\). Taking into account the estimates of Theorem 2.2, we deduce by a straightforward calculation that

\[
Y_{\lambda} \to Y \quad \text{as} \quad \lambda \to 0, \quad \text{strongly in} \quad C([0, 1]; H^2(0, L)) \cap W^{1, 2}(0, 1; H^2(0, L)) \cap L^2(0, 1; H^4(0, L)),
\]

and that \(Y\) satisfies the equations

\[
(k^2 Y - Y'')_t + T_{k,\varepsilon}^* \left(\nu Y^{IV} - (2\nu k^2 + ikU)Y'' + (\nu k^4 + ik^3 U + ikU'')Y\right) = 0, \quad (3.14)
\]

\[
Y(t, 0) = 0, \quad Y(t, L) = \omega, \quad Y'(t, 0) = Y'(t, L) = 0, \quad (3.15)
\]

\[
Y(0, y) = 0, \quad (3.16)
\]

for \((t, y) \in (0, 1) \times (0, L)\). Moreover, following Theorem 2.2, we state that (3.14)-(3.16) has a unique solution

\[
Y \in C([0, 1]; H^2(0, L)) \cap W^{1, 2}(0, 1; H^2(0, L)) \cap L^2(0, 1; H^4(0, L)).
\]  

(3.17)

We introduce the dual system

\[
-(k^2 p_{k,\varepsilon} - p''_{k,\varepsilon})_t + T_{k,\varepsilon}^* \left(\nu p^{IV}_{k,\varepsilon} - (2\nu k^2 + ikU)p''_{k,\varepsilon} - 2ikU'p'_{k,\varepsilon} + (\nu k^4 + ik^3 U)p_{k,\varepsilon}\right) = 0, \quad (3.18)
\]

\[
p_{k,\varepsilon}(t, 0) = p_{k,\varepsilon}(t, L) = p'_{k,\varepsilon}(t, 0) = p'_{k,\varepsilon}(t, L) = 0, \quad (3.19)
\]

\[
k^2 p_{k,\varepsilon}(1, y) - p''_{k,\varepsilon}(1, y) = (\sigma I + A_k)^{-2} \left(\frac{1}{\varepsilon} v_{k,\varepsilon}(1)\right),
\]  

(3.20)

for \((t, y) \in (0, 1) \times (0, L)\).

Now, in order to simplify the writing in the next calculations, we use a formal notation

\[
Ev := k^2 v - v'', \quad Fz := \nu v^{IV} - (2\nu k^2 + ikU) v'' + (\nu k^4 + ik^3 U + ikU'') v
\]  

(3.21)

for \(v \in H^4(0, L)\) and rewrite the state equation (2.2)-(2.5) for the solution \(v_{k,\varepsilon}\) corresponding to \(w_{k,\varepsilon}\) as

\[
E(v_{k,\varepsilon}^t)_t + T_{k,\varepsilon}^* Fv_{k,\varepsilon}^t(t) = 0, \quad \text{a.e.} \quad t \in (0, 1),
\]  

(3.22)

\[
v_{k,\varepsilon}^t(0) = v_{k0}.
\]  

(3.23)

Proposition 3.3. The adjoint system (3.18)-(3.20) has a unique solution

\[
p_{k,\varepsilon} \in C([0, 1]; H^6(0, L) \cap H_0^2(0, L)) \cap C^1([0, 1]; H^4(0, L) \cap H_0^2(0, L)).
\]  

(3.24)

Proof. We recall the definition (2.6) and introduce, similarly to (2.7), the following operator:

\[
F_{0k}^* : D(F_{0k}^*) = D(F_{0k}) \subset H \to H,
\]

(3.25)

\[
F_{0k}^* z = \nu z^{IV} - (2\nu k^2 + ikU) z'' - 2ikU' z' + (\nu k^4 + ik^3 U) z.
\]
We can interpret $F_{0k}$ as the dual of $F$ in the sense of distributions, that is $(Fv, \varphi) = (\varphi, F_{0k}^*v)$, for $\varphi \in C_0^\infty(0,L)$ and $v \in H^4(0,L)$. System (3.13)-(3.20) can be written
\begin{equation}
-E_{0k}(p_{k,\varepsilon}) (t) + T_{k,\varepsilon}^* F_{0k}^* p_{k,\varepsilon} (t) = 0, \text{ a.e. } t \in (0,1),
\end{equation}
\begin{equation}
E_{0k} p_{k,\varepsilon} (1) = (\sigma I + A_k)^{-2} \left( \frac{1}{\varepsilon} v_{k,\varepsilon}^* (1) \right).
\end{equation}

Also, we introduce
\begin{equation}
B_k = F_{0k} E_{0k}^{-1}, \quad B_k : D(B_k) \subset H \to H, \quad D(B_k) = \{ v \in H^2(0,L); \ E_{0k}^{-1} v \in H^2_0(0,L) \} = D(A_k).
\end{equation}

By the same argument as for $A_k$ we infer that $-B_k$ generates a $C_0$-analytic semigroup.

We write the equivalent equation for $\tilde{p}_{k,\varepsilon} (t) := E_{0k} p_{k,\varepsilon} (t)$,
\begin{equation}
-\frac{d \tilde{p}_{k,\varepsilon}}{dt} (t) + T_{k,\varepsilon}^* B_k \tilde{p}_{k,\varepsilon} (t) = 0, \text{ a.e. } t \in (0,1),
\end{equation}
\begin{equation}
\tilde{p}_{k,\varepsilon} (1) = (\sigma I + A_k)^{-2} \left( \frac{1}{\varepsilon} v_{k,\varepsilon}^* (1) \right).
\end{equation}

The solution is
\begin{equation}
\tilde{p}_{k,\varepsilon} (t) = e^{T_{k,\varepsilon} B_k (1-t)} \tilde{p}_{k,\varepsilon} (1), \text{ for all } t \in [0,1]
\end{equation}
and since $\bar{v}_{k,\varepsilon} (1) \in H$, by (2.11), we have $\tilde{p}_{k,\varepsilon} (1) \in D(A_k^2) = D(B_k^2)$. Therefore,
\begin{equation}
\tilde{p}_{k,\varepsilon} \in C([0,1]; D(B_k^2)) \cap C^1([0,1]; D(B_k))
\end{equation}
and
\begin{equation}
p_{k,\varepsilon} (t) = E_{0k}^{-1} (\tilde{p}_{k,\varepsilon} (t)), \text{ for all } t \in [0,1]
\end{equation}
turns out to be in the spaces (3.24), by the observation (2.9) made before Theorem 2.2. The proof is ended.

Now, let us keep $w_{k,\varepsilon}^*$ fixed and give variations to $T_{k,\varepsilon}^*$. For $\lambda > 0$, we define $Z^\lambda = \frac{T_{k,\varepsilon}^* \lambda + \gamma_{k,\varepsilon}^*}{\gamma_{k,\varepsilon}^* - \gamma_{k,\varepsilon}^*}$ and by a straightforward calculation we have $Z^\lambda \to Z$ as $\lambda \to 0$, where $Z$ satisfies the system in variations
\begin{equation}
EZ(t) + T_{k,\varepsilon}^* FZ(t) = -T_{k,\varepsilon}^* Fv_{k,\varepsilon}^*(t), \text{ a.e. } t \in (0,1),
\end{equation}
\begin{equation}
Z(t,0) = Z(t,L) = Z'(t,0) = Z'(t,L) = 0,
\end{equation}
\begin{equation}
Z(0,y) = 0.
\end{equation}

Since the right hand side in the first equation in (3.30) is in $L^2(0, T_{k,\varepsilon}^*; H)$ it follows that (3.30) has the unique solution
\begin{equation}
Z \in C([0,1]; H^2(0,L)) \cap W^{1,2}(0,1; H^2(0,L)) \cap L^2(0,1; H^4(0,L)).
\end{equation}

### 3.2 Necessary conditions of optimality for $(P_{k,\varepsilon})$

We recall that $V_1 = \{ f \in H^1(0,1); f(0) = 0 \}$. Let us introduce the set
\begin{equation}
K_T = \{ w \in V_1; \|w\|_{V_1} \leq \rho_k \sqrt{T} \}
\end{equation}
and denote the normal cone to $K_T$ at $w$ by
\begin{equation}
N_{K_T}(w) = \left\{ \chi \in V_1^*; \text{ Re}\langle \chi(t), (w-w_1)(t)\rangle_{V_1^*, V_1} \geq 0, \text{ for all } w_1 \in K \right\},
\end{equation}
where $V_1^*$ is the dual of $V_1$. 

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We make a parenthesis for a discussion about this cone. It is known that

\[ N_K^\tau(w) = \begin{cases} \cup_{\alpha > 0} \alpha \Lambda (w), & \text{if } \|w\|_{V_1} = \rho_k \sqrt{T} \\ \{0\}, & \text{if } \|w\|_{V_1} < \rho_k \sqrt{T} \\ \emptyset, & \text{if } \|w\|_{V_1} > \rho_k \sqrt{T}, \end{cases} \]

where \( \alpha \in \mathbb{R}_+ \), and \( \Lambda : V_1 \to V_1^* \) is the duality mapping from \( V_1 \) to \( V_1^* \). Here, \( \Lambda w = -\dot{w} \) (see e.g., [1], pp. 2-4).

By abuse of notation, we denote still by \( N_K^\tau(w) \) the restriction of \( N_K^\tau(w) \) on \( L^2(0,1) \). In this case

\[ \Lambda w = -\dot{w}, \quad \Lambda : D(\Lambda) := \{ w \in H^2(0,1); \ w(0) = \dot{w}(1) = 0 \} \subset L^2(0,1) \to L^2(0,1). \]

Let \( (T_{k,\varepsilon}^*, w_{k,\varepsilon}^*) \) be an optimal controller in \( (P_{k,\varepsilon}) \), with \( w_{k,\varepsilon}^* \in K_{T_{k,\varepsilon}}^* \). For \( \lambda > 0 \), we have set in

\[ w_{k,\varepsilon}^\lambda = w_{k,\varepsilon} + \lambda \omega, \text{ where } \omega = \bar{w} - w_{k,\varepsilon}, \quad \bar{w} \in K_{T_{k,\varepsilon}}^*, \]

that is, \( \bar{w} \in V_1, \|\bar{w}\|_{V_2} \leq \rho_k \sqrt{T_{k,\varepsilon}} \). We recall the notation

\[ h_{k,\varepsilon}^*(t) = \int_0^t (w_{k,\varepsilon}^* - w_k^*) (\tau) d\tau. \tag{3.34} \]

**Proposition 3.4.** Let \( (T_{k,\varepsilon}^*, w_{k,\varepsilon}^*) \) be an optimal control in \( (P_{k,\varepsilon}) \) with the optimal state \( v_{k,\varepsilon}^* \). Then,

\[ \alpha_{k,\varepsilon} \bar{w}_{k,\varepsilon}^*(t) = T_{k,\varepsilon}^* \nu p_{k,\varepsilon}^m(t,1) + \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau, \quad \text{for all } t \in [0,1], \tag{3.35} \]

where \( w_{k,\varepsilon}^*(0) = \bar{w}_{k,\varepsilon}(1) = 0, \bar{w}_{k,\varepsilon} \in H^2(0,1), \|w_{k,\varepsilon}^*\|_{V_1} = \rho_k \sqrt{T_{k,\varepsilon}} \), \( \alpha_{k,\varepsilon} \in \mathbb{R}_+ \) and

\[ \alpha_{k,\varepsilon} \rho_k \sqrt{T_{k,\varepsilon}} \|\bar{w}_{k,\varepsilon}^*\|_{V_2} + T_{k,\varepsilon}^* \text{Re} \int_0^1 \left( \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau \right) - T_{k,\varepsilon}^* \text{Re} \int_0^1 \left( \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau \right) \text{dt}, \tag{3.36} \]

with \( p_{k,\varepsilon} \) the solution to the dual backward equation \( 3.18 - 3.20 \).

**Proof.** The proof will be done in two steps.

**Step 1. The first condition of optimality.** Let \( (T_{k,\varepsilon}^*, w_{k,\varepsilon}^*) \) be an optimal controller in \( (P_{k,\varepsilon}) \). As already said, we keep \( T_{k,\varepsilon}^* \) fixed and give variations to \( w_{k,\varepsilon}^* \). Using the fact that \( w_{k,\varepsilon}^* \) is optimal we can write

\[ J_{k,\varepsilon}(T_{k,\varepsilon}^*, w_{k,\varepsilon}^*) \leq J_{k,\varepsilon}(T, w), \quad \text{for all } w \in K_T. \]

This holds true if we replace \( T \) by \( T_{k,\varepsilon}^* \) and \( w \) by \( w_{k,\varepsilon}^* \). By calculating

\[
\begin{align*}
\lim_{\lambda \to 0} \frac{J_{k,\varepsilon}(T_{k,\varepsilon}^*, w_{k,\varepsilon}^\lambda) - J_{k,\varepsilon}(T_{k,\varepsilon}^*, w_{k,\varepsilon}^*)}{\lambda} \\
= \lim_{\lambda \to 0} \frac{1}{\lambda} \left( \frac{1}{2} \left( \int_0^1 \left( \|\sigma I + A_k\| \left\| \bar{w}_{k,\varepsilon}^\lambda(1) \right\|^2_H - \|\sigma I + A_k\| \left\| \bar{w}_{k,\varepsilon}^\lambda(1) \right\|^2_H \right) \right) + \frac{1}{2} \left( \int_0^1 \left( \int_0^1 (w_{k,\varepsilon}^\lambda - w_k^*)(\tau) d\tau \right)^2 dt - \int_0^1 \left( \int_0^1 (w_{k,\varepsilon}^\lambda - w_k^*)(\tau) d\tau \right)^2 dt \right) \right)
\end{align*}
\]

we obtain

\[
\begin{align*}
\lim_{\lambda \to 0} \frac{J_{k,\varepsilon}(T_{k,\varepsilon}^*, w_{k,\varepsilon}^\lambda) - J_{k,\varepsilon}(T_{k,\varepsilon}^*, w_{k,\varepsilon}^*)}{\lambda} \\
= \text{Re} \left\{ \int_0^1 \frac{1}{\varepsilon} (\sigma I + A_k)^{-2} w_{k,\varepsilon}^\lambda(1) \cdot Y(1) dy + \int_0^1 h_{k,\varepsilon}^*(t) \left( \int_0^t \nu(s) ds \right) dt \right\} \geq 0.
\end{align*}
\]
Here we used the definition of the scalar product in $\mathbb{C}$ and $\text{Re}(a \cdot \overline{b}) = \text{Re}(\overline{\alpha} \cdot b)$, for $a, b \in \mathbb{C}$. Then, we calculate the last term in (3.37)

$$\text{Re} \int_0^1 h_{k,\varepsilon}^*(t) \left( \int_0^t \overline{\omega}(s) ds \right) dt = \text{Re} \int_0^1 \int_0^1 h_{k,\varepsilon}^*(t) \overline{\omega}(s) dt ds$$

(3.38)

and replacing it in (3.37) we obtain

$$\text{Re} \left\{ \int_0^1 \langle \sigma I + A_k \rangle^{-2} \left( \frac{1}{\varepsilon^2} \overline{v_{k,\varepsilon}}(1) \right) Y(1, y) dy + \int_0^1 \left( \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau \right) \overline{\omega}(t) dt \right\} \geq 0.$$ (3.39)

Using the notation (3.21) we can write eq. (3.11) in the equivalent form

$$EY(t) + T_{k,\varepsilon}^*FY(t) = 0$$ (3.40)

with the boundary and initial conditions. Now, we multiply this equation scalarly in $H$ by $p_{k,\varepsilon}(t)$ and integrate with respect to $t$ over $(0, 1)$. While performing all the integrals by parts, we put first into evidence the relation

$$\int_0^L FY(t, y) \cdot p_{k,\varepsilon}(t, y) dy = \int_0^L F_{0k}^* p_{k,\varepsilon}(t, y) \cdot Y(t, y) dy - \nu p_{k,\varepsilon}'''(t, 1) \omega(t)$$

(3.41)

and obtain

$$\int_0^1 \int_0^L \left\{ - (k^2 p_{k,\varepsilon} - p_{k,\varepsilon}''') \right\} Y dy dt$$

+ $T_{k,\varepsilon}^* \int_0^1 \int_0^L \left\{ - (2\nu k^2 + ikU)p_{k,\varepsilon}''' - 2ikU'p_{k,\varepsilon}'' + (\nu k^4 + ik^3 U)p_{k,\varepsilon} \right\} Y dy dt$

+ $\int_0^L \left\{ k^2 p_{k,\varepsilon}(1) - p_{k,\varepsilon}''(1) \right\} Y(1, y) dy - T_{k,\varepsilon}^* \int_0^1 \omega(t) \nu p_{k,\varepsilon}'''(t, 1) dt = 0.$

By (3.24), we see that $p_{k,\varepsilon}''' \in C([0, 1]; H^1(0, L))$ and so the trace $p_{k,\varepsilon}'''(t, 1)$ is well defined and belongs to $C([0, 1]; \mathbb{C})$. Recalling now the equations in the adjoint system (3.18)-(3.20) and comparing with the integrands in (3.42) we get

$$\int_0^L \langle \sigma I + A_k \rangle^{-2} \left( \frac{1}{k} \overline{v_{k,\varepsilon}}(1) \right) \cdot Y(1) dy = T_{k,\varepsilon}^* \int_0^1 \omega(t) \nu p_{k,\varepsilon}'''(t, 1) dt.$$ (3.43)

Then plugging the latter into (3.39) and using again $\text{Re}(a \cdot b) = \text{Re}(\overline{a} \cdot b)$ we deduce the relation

$$\text{Re} \left\{ \int_0^1 T_{k,\varepsilon}^* \nu p_{k,\varepsilon}'''(t, 1) \overline{\omega}(t) dt + \int_0^1 \left( \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau \right) \overline{\omega}(t) dt \right\}$$

= $\text{Re} \int_0^1 \left( - T_{k,\varepsilon}^* \nu p_{k,\varepsilon}'''(t, 1) - \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau \right) (w_{k,\varepsilon} - \overline{\omega})(t) dt \geq 0,$

for all $\overline{\omega} \in K_{T_{k,\varepsilon}^*}$. The immediate result is that

$$\eta_{k,\varepsilon} := - T_{k,\varepsilon}^* \nu p_{k,\varepsilon}'''(\cdot, 1) - \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau \in N_{K_{T_{k,\varepsilon}^*}^*} (w_{k,\varepsilon}^*)$$

(3.45)

where $N_{K_{T_{k,\varepsilon}^*}^*} (w_{k,\varepsilon}^*)$ is viewed as the cone from $V_1$ to $V_1^*$. However, since $\eta_{k,\varepsilon} \in L^2(0, 1)$, we can consider it in the restriction of $N_{K_{T_{k,\varepsilon}^*}^*} (w_{k,\varepsilon}^*)$ to $L^2(0, 1)$, still denoted by $N_{K_{T_{k,\varepsilon}^*}^*} (w_{k,\varepsilon}^*)$. Because an element of this cone is of the form $\eta_{k,\varepsilon} = -\alpha_{k,\varepsilon} \omega_{k,\varepsilon}^*$ with $w_{k,\varepsilon}(0) = \omega_{k,\varepsilon}(1) = 0$ and $\alpha_{k,\varepsilon} > 0$, we have

$$- \alpha_{k,\varepsilon} \omega_{k,\varepsilon}^*(t) = - T_{k,\varepsilon}^* \nu p_{k,\varepsilon}'''(t, 1) - \int_0^1 h_{k,\varepsilon}^*(\tau) d\tau, \text{ for all } t \in [0, 1],$$

(3.46)
which is just (3.35). We note that \( \tilde{w}_{k,\varepsilon} \) turns out to be in \( H^2(0,1) \).

The situation in which \( \|w^{*,\varepsilon}_{k,\varepsilon}\|_{V_i} < \rho_k \sqrt{T^{*,\varepsilon}_{k,\varepsilon}} \) provides \( \tilde{w}_{k,\varepsilon}(t) = 0 \), which gives the solution \( w^{*,\varepsilon}_{k,\varepsilon}(t) = 0 \) which is not relevant for our problem. In particular, it means that \( w^*_k = 0 \), that is the flow would not be controlled.

Thus, (3.35) follows for the case when \( \|w^{*,\varepsilon}_{k,\varepsilon}\|_{V_i} = \rho_k \sqrt{T^{*,\varepsilon}_{k,\varepsilon}} \).

**Step 2. The second condition of optimality.** Here we keep \( w^{*,\varepsilon}_{k,\varepsilon} \) fixed and give variations for \( T^{*,\varepsilon}_{k,\varepsilon} \). Since \( T^{*,\varepsilon}_{k,\varepsilon} \) realizes the minimum in \( (P_{k,\varepsilon}) \) we write

\[
J_{k,\varepsilon}(T^{*,\varepsilon}_{k,\varepsilon}, w^{*,\varepsilon}_{k,\varepsilon}) \leq J_{k,\varepsilon}(T^{*,\varepsilon}_{k,\varepsilon} + \lambda, w^{*,\varepsilon}_{k,\varepsilon}), \quad \lambda > 0,
\]

that is,

\[
J_{k,\varepsilon}(T^{*,\varepsilon}_{k,\varepsilon}, w^{*,\varepsilon}_{k,\varepsilon}) = T^{*,\varepsilon}_{k,\varepsilon} + \frac{1}{2\varepsilon} \left\| (\sigma I + A_k)^{-2} v^{*,\varepsilon}_{k,\varepsilon} w^{*,\varepsilon}_{k,\varepsilon}(1) \right\|^2_H + \frac{1}{2} \int_0^1 |h^{*,\varepsilon}_{k,\varepsilon}(t)|^2 \, dt
\]

\[
\leq J_{k,\varepsilon}(T^{*,\varepsilon}_{k,\varepsilon} + \lambda, w^{*,\varepsilon}_{k,\varepsilon}) = T^{*,\varepsilon}_{k,\varepsilon} + \lambda + \frac{1}{2\varepsilon} \left\| (\sigma I + A_k)^{-2} v^{*,\varepsilon}_{k,\varepsilon} + \lambda w^{*,\varepsilon}_{k,\varepsilon}(T^{*,\varepsilon}_{k,\varepsilon} + \lambda) \right\|^2_H + \frac{1}{2} \int_0^1 |h^{*,\varepsilon}_{k,\varepsilon}(t)|^2 \, dt.
\]

By performing the computations as before we obtain

\[
1 + \text{Re} \int_0^1 \frac{1}{\varepsilon} (\sigma I + A_k)^{-2} v^{*,\varepsilon}_{k,\varepsilon}(1,y)Z(1,y)dy = 0,
\]

(3.47)

where \( Z \) is the solution to the system in variations (3.30). We consider the same adjoint system (3.26-3.27). By multiplying the first equation in (3.30) by \( p_k,\varepsilon(t) \) scalarly in \( H \), integrating along \( t \in (0,1) \) and taking the real part, we obtain

\[
\text{Re} \int_0^1 \int_0^L T^{*,\varepsilon}_{k,\varepsilon} F v^{*,\varepsilon}_{k,\varepsilon} p_k,\varepsilon \, dydt = 1.
\]

(3.49)

By using the final condition in the adjoint system and comparing (3.47) and (3.48) we get

\[
\text{Re} \int_0^1 \int_0^L T^{*,\varepsilon}_{k,\varepsilon} F v^{*,\varepsilon}_{k,\varepsilon} p_k,\varepsilon \, dydt = 1.
\]

Finally, applying a relation similar to (3.41) in the left-hand side of (3.49) we obtain

\[
\text{Re} \int_0^1 \int_0^L T^{*,\varepsilon}_{k,\varepsilon} F v^{*,\varepsilon}_{k,\varepsilon} \cdot v^{*,\varepsilon}_{k,\varepsilon} \, dydt - \text{Re} \int_0^1 T^{*,\varepsilon}_{k,\varepsilon} \nu p^{*,\varepsilon}_{k,\varepsilon}(t,1) w^{*,\varepsilon}_{k,\varepsilon} \, dt = 1
\]

and recalling again that \( \text{Re}(a \cdot b) = \text{Re}(\overline{a} \cdot b) \) we get

\[
- \text{Re} \int_0^1 T^{*,\varepsilon}_{k,\varepsilon} \nu p^{*,\varepsilon}_{k,\varepsilon}(t,1) w^{*,\varepsilon}_{k,\varepsilon} \, dt + \text{Re} \int_0^1 T^{*,\varepsilon}_{k,\varepsilon} \nu p^{*,\varepsilon}_{k,\varepsilon}(t,1) w^{*,\varepsilon}_{k,\varepsilon} \, dt = 1.
\]

(3.50)

Using (3.35) we still can write

\[
\text{Re} \int_0^1 \left( -T^{*,\varepsilon}_{k,\varepsilon} \nu p^{*,\varepsilon}_{k,\varepsilon}(t,1) - \int_t^1 h^{*,\varepsilon}_{\tau}(\tau) \, d\tau \right) w^{*,\varepsilon}_{k,\varepsilon}(t) \, dt + \text{Re} \int_0^1 T^{*,\varepsilon}_{k,\varepsilon} \nu p^{*,\varepsilon}_{k,\varepsilon}(t,1) w^{*,\varepsilon}_{k,\varepsilon} \, dt = 1 - \text{Re} \int_0^1 \int_t^1 h^{*,\varepsilon}_{\tau}(\tau) \, d\tau \, w^{*,\varepsilon}_{k,\varepsilon} \, dt.
\]

(3.51)

But \( \eta_{k,\varepsilon}(t) \in N_{K_{k,\varepsilon}^*}(w^{*,\varepsilon}_{k,\varepsilon}) \) and so we have

\[
\text{Re} \int_0^1 \eta_{k,\varepsilon}(t) w^{*,\varepsilon}_{k,\varepsilon}(t) \, dt = \text{Re} \left( \eta_{k,\varepsilon}(t), w^{*,\varepsilon}_{k,\varepsilon}(t) \right)_{V_i^*, V_i} = \rho_k \sqrt{T^{*,\varepsilon}_{k,\varepsilon}} \| \eta_{k,\varepsilon} \|_{V_i}.
\]

(3.52)
Let \( \| w^*_k \|_{V_1} = \rho_k \sqrt{T^*_k} \). Taking into account that in this case there is \( \alpha_{k,\varepsilon} > 0 \) such that \( \eta_{k,\varepsilon}(t) = -\alpha_{k,\varepsilon} \bar{w}_{k,\varepsilon} \), we get by (3.51) and (3.52), relation (3.36), as claimed.

Moreover, on the one hand, \( \langle -\alpha_{k,\varepsilon} \bar{w}_{k,\varepsilon}, \bar{w}_{k,\varepsilon} \rangle_{V_1^*, V_1} = \alpha_{k,\varepsilon} \int_0^1 \| \bar{w}_{k,\varepsilon}(t) \|^2 \, dt = \alpha_{k,\varepsilon} \rho_k^2 T^*_k \). On the other hand by (3.52) we get

\[
\int_0^1 |\bar{w}_{k,\varepsilon}(t)|^2 \, dt = \alpha_{k,\varepsilon} \rho_k^2 T^*_k,
\]

hence \( \| \bar{w}_{k,\varepsilon} \|_{V_1^*} = \rho_k \sqrt{T^*_k} \) is verified and (3.51) implies then (3.36).

The situation \( \| \bar{w}_{k,\varepsilon} \|_{V_1^*} < \rho_k \sqrt{T^*_k} \) was excluded, so that (3.36) and (3.35) follow for the case when \( \| \bar{w}_{k,\varepsilon} \|_{V_1^*} = \rho_k \sqrt{T^*_k} \). This ends the proof.

\[\Box\]

4 The maximum principle for \((P_k)\)

Theorem 4.1. Let \((T^*_k, w^*_k, v^*_k)\) be optimal in \((\hat{P}_k)\). If

\[2 \max \left\{ T^*_k, \sqrt{T^*_k} \right\} T^*_k \| \gamma \|_{L^1(0,1)} < 1,\]

\[\rho_k \left( 1 - T^*_k \| \gamma \|_{L^1(0,1)} (T^*_k + \sqrt{T^*_k}) \right) \geq T^*_k \| \gamma \|_{L^1(0,1)} \| v_{k,0} \|_{H^s(0,L)\cap H^2(0,L)},\]

then, there exist \( \alpha^*_k > 0 \) such that

\[\alpha^*_k \bar{w}^*_k(t) = T^*_k v^*_k \bar{w}_{k,\varepsilon}(t, 1), \text{ a.e. } t \in (0,1), w^*_k(0) = w^*_k(1) = 0, \| w^*_k \|_{V_1} = \rho_k \sqrt{T^*_k},\]

\[\alpha^*_k \rho_k^2 T^*_k + T^*_k \Re \int_0^1 \langle \bar{v}_k(t), F_{0k} p_k(t) \rangle_H dt = 1,\]

where \( p_k \) is the solution to the adjoint equation

\[- E_{0k}(p_k)(t) + T^*_k F_{0k} p_k(t) = 0, \text{ a.e. } t \in (0,1).\]

\[\text{Proof.}\] Using the notation (3.21) let us consider the system

\[Ev(t) + T^*_k Fv(t) = 0,\]

\[v(t, 0) = 0, v(t, L) = w(t), v'(t, 0) = v'(t, L) = 0,\]

\[v(0) = v_{0k} \in L^2(0,L)\]

and repeat a similar calculus providing (4.42). Thus, we multiply the first equation in (4.5) scalarly by \( p_{k,\varepsilon}(t) \), integrate along \( (t, 1) \) and get

\[\int_0^L v(1, y) E_{0k} p_{k,\varepsilon}(1, y) dy - \int_0^L v(t, y) E_{0k} p_{k,\varepsilon}(t, y) dy = 0, \text{ for } t \in (0,1).\]

We recall that by \((H_k)\), written in the new variable \((2.1)\), the system \((2.2) - (2.5)\) is controllable and also \((4.5)\) is. Namely, for each initial datum \( v^{t_0} \in L^2(0,1), \| v^{t_0} \|_{H} \leq 1 \), there exists \( w \in H^2(0,1) \), and \( \gamma(t_{0},1) \in L^1(0,1) \), satisfying \( w(\tau) = 0 \) for \( 0 \leq \tau \leq t_0 \), \( \left( \int_0^1 \| \dot{w}(t) \|^2 \, dt \right)^{1/2} \leq \| \gamma \|_{L^1(0,1)} \), such that \( v^{t_0,w}(1, y) = 0, \text{ a.e. } y \in (0,1) \). Here, \( v^{t_0,w} \) is the solution starting at time \( t_0 \).
We apply \((H_k)\), for \(t_0 = t, v^{t,w} = v\) the solution to \((3.15), v^{t_0} = v^t := v(t)\), and \(\gamma_{1,1} = \gamma(t)\) with \(\gamma \in L^2(0,1)\). Thus, we have \(v(1,y) = 0\), and by taking the real part in \((1.10)\), we get
\[
\text{Re}(\mathbf{v}(t), E_{0,k}p_{k,e}(t))_H = -\nu T_{k,e}^* \text{Re} \int_0^1 \overline{p_{k,e}''(\tau,1)} \mathbf{v}(\tau) d\tau.
\]
We choose now \(\mathbf{v}(t) = \frac{E_{0,k}p_{k,e}(t)}{\|E_{0,k}p_{k,e}(t)\|_{B^2}}\) which ensures that \(\|\mathbf{v}(t)\|_H \leq 1\), and use the fact the \(\|E_{0,k}p_{k,e}(t)\|^2 = k^4 \|p_{k,e}(t)\|^2_H + 2k^2 \|p_{k,e}(t)\|^2_H \geq \|p_{k,e}(t)\|_{H^2(0,L)}^2\), for \(k \geq 1\). Since \(p_{k,e}''(\cdot,1) \in L^2(0,1)\)

\[
\|p_{k,e}(t)\|_{H^2(0,1)} \leq \|E_{0,k}p_{k,e}(t)\|_H \leq \nu T_{k,e}^* \text{Re} \int_0^1 \overline{p_{k,e}''(\tau,1)} \mathbf{v}(\tau) d\tau
\]

Finally, we obtain the observability relation
\[
\|p_{k,e}(t)\|_{H^2(0,1)} \leq \|\gamma\|_{L^1(0,1)} T_{k,e}^* \|\overline{p_{k,e}''(\cdot,1)}\|_{V_1^*}, \quad \text{a.e. } t \in (0,1).
\]
In the second term on the left-hand side of \((3.36)\) we observe that by integration by parts we can write
\[
\overline{(v_{k,e}^*(t), E_{0,k}p_{k,e}(t))} \leq \|v_{k,e}^*(t)\|_{H^2(0,1)} \|p_{k,e}(t)\|_{H^2(0,1)}.
\]
Now, we replace \(-T_{k,e}^* \overline{p_{k,e}''(\tau,1)}\) in \((3.36)\) by \((3.35)\). Thus, we find via \((1.17)\) that
\[
\rho_k \|T_{k,e}^* v_{k,e}^*(\tau,1)\|_{V_1^*} \leq 1 + C(\rho_k, T_{k,e}^*) + T_{k,e}^* \|\gamma\|_{L^1(0,1)} \left(1 + \frac{\|v_{k,0}\|_{H^4(0,L) \cap H^2_0(0,L)} + \rho_k (T_{k,e}^* + \sqrt{T_{k,e}^*}) T_{k,e}^* \|\overline{p_{k,e}''(\cdot,1)}\|_{V_1^*}\right).
\]
with \(C(\rho_k, T_{k,e}^*) \leq C(\rho_k, T_k)\), by \((3.6)\). Therefore, by denoting
\[
\text{the, we have }
\]
we get
\[
T_{k,e}^* \|\overline{p_{k,e}''(\cdot,1)}\|_{V_1^*} (\rho_k D_k(T_{k,e}^*) - G_k(T_{k,e}^*)) \leq 1 + C(\rho_k, T_k^*).
\]
On the other hand, on the basis of \((1.1)\) we have that
\[
1 > 2 \max \left\{ T_{k,e}^* \sqrt{T_k^*} T_{k}^* \|\gamma\|_{L^1(0,1)} \right\} \geq \left( T_{k,e}^* + \sqrt{T_{k,e}^*} \right) T_{k,e}^* \|\gamma\|_{L^1(0,1)}
\]

since \(T_{k,e}^* \leq T_k^*\) by Theorem 3.2, so \(D_k(T_{k,e}^*) > 0\). On the other hand, \(G_k(T_{k,e}^*) \geq G_k(T_{k,e}^*)\) and
\[
D(T_k^*) \leq D(T_{k,e}^*),
\]
since \(\rho_k D_k(T_{k,e}^*) - G_k(T_{k,e}^*) \geq \rho_k D_k(T_k^*) - G_k(T_k^*)\) and so it follows that
\[
\|\overline{p_{k,e}''(\cdot,1)}\|_{V_1^*} \leq \frac{1 + C(\rho_k, T_k^*)}{\rho_k D_k(T_k^*) - G_k(T_k^*)} < \frac{1 + C(\rho_k, T_k)}{\rho_k D_k(T_k^*) - G_k(T_k^*)}.
\]

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independent of $\varepsilon$. Going back to (4.7), we deduce that
\[
\int_0^1 \| p_{k,\varepsilon}(t) \|_{H^2(0,L)}^2 \, dt \leq C. 
\] (4.11)

Next, we establish an estimate for $\| p_{k,\varepsilon}(t) \|_{H^2(0,L)}^2$. Since $p_{k,\varepsilon}(t) \in H^2(0,L)$ a.e. $t \in (0,1)$, let us choose a $t_0$ in the neighborhood of 1, such that $p_{k,\varepsilon}(t_0) \in H^2(0,L)$. By (4.4), since $T_{k,\varepsilon}^* \leq T_k^*$ we have
\[
\| p_{k,\varepsilon}(t_0) \|_{H^2(0,L)} \leq C\gamma(t_0).
\]

Let us consider the adjoint system with $p_{k,\varepsilon}(t_0)$ as the final value,
\[
-E_{0k}(p_{k,\varepsilon})_t(t) + F_{0k}^* p_{k,\varepsilon}(t) = 0, 
\]
and denote $E_{0k} p_{k,\varepsilon}(t) = q_{k,\varepsilon}(t)$. We have
\[
-(q_{k,\varepsilon})_t(t) + B_k q_{k,\varepsilon}(t) = 0, \text{ a.e. } t \in (0,t_0), \ t_0 < 1,
\]
\[
\| q_{k,\varepsilon}(t_0) \|_{H} \leq C\gamma(t_0).
\]
Since $-B_k$ generates a $C_0$-analytic semigroup, we have (see [13])
\[
\| B_k q_{k,\varepsilon}(t) \| \leq \frac{C\gamma(t_0)}{t_0 - t}, \quad 0 \leq t < t_0 < 1
\]
hence
\[
\| q_{k,\varepsilon}(t) \|_{H^2(0,L)} \leq \frac{C\gamma(t_0)}{t_0 - t}, \quad \text{for } 0 \leq t < t_0 < 1
\]
and so
\[
\| p_{k,\varepsilon}(t) \|_{H^2(0,L)} \leq \frac{C\gamma(t_0)}{t_0 - t}, \quad \text{for } 0 \leq t < t_0 < 1, \quad (4.12)
\]
where $C$ denotes several constants. Therefore, on a subsequence we have
\[
p_{k,\varepsilon} \rightarrow p_k \text{ weakly in } L^2(0,1; H^2(0,L)) \text{ and weak-star in } L^\infty(0,1-\delta; H^4(0,L)),
\]
for all $\delta > 0$. By the adjoint equation
\[
E_{0k}(p_{k,\varepsilon}) = T_{k,\varepsilon}^* F_{0k}^* p_{k,\varepsilon} \rightarrow T_k^* F_{0k}^* p_k \text{ weak-star in } L^\infty(0,1-\delta; H),
\]

hence
\[
(p_{k,\varepsilon})_t \rightarrow E_{0k}^{-1} T_{k,\varepsilon}^* F_{0k}^* p_k := (p_k)_t \text{ weak-star in } L^\infty(0,1-\delta; H^2(0,L) \cap H^1_0(0,L)). \quad (4.14)
\]

At limit we get the adjoint equation
\[
-E_{0k}(p_k)_t(t) + T_{k,\varepsilon}^* F_{0k}^* p_k(t) = 0 \text{ a.e. } t \in (0,1), \quad (4.15)
\]
which is (4.4).

Since $H^4(0,1)$ is compact in $H^{4-\varepsilon'}(0,1)$ for any $\varepsilon' > 0$, it follows by (4.13), (4.14) and the Aubin-Lions lemma that
\[
p_{k,\varepsilon} \rightarrow p_k \text{ strongly in } L^\infty(0,1-\delta; H^4(0,L)). \quad (4.16)
\]
By Ascoli-Arzelà theorem we have
\[
p_{k,\varepsilon} \rightarrow p_k \text{ strongly in } C([0,1-\delta]; H^2(0,L)).
\]
Therefore, (4.16) implies
\[
p_{k,\varepsilon}'' \rightarrow p_k'' \text{ strongly in } L^2(0,1-\delta; H^{4-\varepsilon'}(0,L)) \text{ for all } \varepsilon' > 0, \ \delta > 0
\]

and also, by the trace convergence
\[ p_{k,ε}''''(\cdot, 1) \rightarrow p''''_k(\cdot, 1) \text{ strongly in } L^2(0, 1 - δ), \text{ for all } δ > 0. \]

On the other hand, by (4.10)
\[ p_{k,ε}''''(\cdot, 1) \rightarrow χ \text{ weakly in } V_1^*, \]
which combined with the previous convergence yields
\[ χ(t) = p''''_k(t, 1) \text{ a.e. } t \in (0, 1). \]  

The convergence of \( (v_{k,ε}^*) \) is ensured by Theorem 2.2, namely
\[ v_{k,ε}^* \rightarrow v_k^* \text{ strongly in } C([0, 1]; H^2(0, L)) \cap W^{1,2}(0, 1; H^2(0, L)) \cap L^2(0, 1; H^4(0, L)). \]

We recall that \( w_{k,ε}^* \rightarrow w_k^* \text{ weakly in } H^1(0, 1) \text{ and uniformly in } C([0, 1]). \)

We have already viewed that the case \( w_{k,ε}^* \rightarrow 0 \) is not acceptable.

Thus, we have to discuss only the case \( w_{k,ε}^* \rightarrow 0^+ \), that is \( w_k^* = 0 \), which we have already seen that it is not compatible with our problem.

Since \( \| \hat{w}_{k,ε}^* \|_{V_1^*} = ρ_k \sqrt{\Gamma_{k,ε}} \), on the one hand, \( \hat{w}_{k,ε}^* \rightarrow \hat{w}_k^* \) weakly in \( V_1^* \).

On the other hand, by (5.35) we have that
\[ \alpha_{k,ε} \hat{w}_{k,ε}^* \rightarrow T_k^* p''''_k(\cdot, 1) \text{ weakly in } V_1^* \]
because \( h_{k,ε}^* \rightarrow 0 \text{ strongly in } L^2(0, 1) \text{ and so in } V_1^* \), by (5.12). We immediately infer that (4.22) takes place.

Then, by passing to the limit in (5.35), where \( \| \hat{w}_{k,ε}^* \|_{V_1^*} = ρ_k \sqrt{\Gamma_{k,ε}} \), we obtain (4.30). This ends the proof.

5 Problem (P)

Definition 5.1. We call a quasi minimal solution to problem (P) a pair \( (T^*, w^*) \) given by

\[ T^* := \sup_{k ∈ Z, k \neq 0} \{ T_k^*; v_k^*(T_k^*) = 0 \}, \quad w^*(t, x) = \sum_{k ∈ Z, k \neq 0} w_k^*(t)e^{ikx} \]  

where \( (T_k^*, w_k^*) \) is optimal in \( (P_k) \), for each \( k ∈ Z, k \neq 0 \).

We recall, as specified before Theorem 2.3, that an admissible pair for (P) is a pair \( (T^*, w^*) \) with \( w^* ∈ H^1(0, T; L^2(0, 2π)) \cap L^∞(0, ∞; L^2(0, 2π)), w(0, y) = 0, ∫_T^∞ ∫_0^{2π} |w_i(t, x)|^2 \, dx \, dt = ρ^2 \) and \( u(T, x, y) = 0, v(T, x, y) = 0. \)

Further, we shall resume the notation "̃"" corresponding to the functions in \( (P_k) \) by the transformation (2.1).

Theorem 5.2. Let \( (u_0, v_0) \in L^2(0, 2π; H^3(0, L) \cap H^4_0(0, L)) \) \( × L^2(0, 2π; H^4(0, L) \cap H^3_0(0, L)) \) and \( (u_0, v_0) \neq (0, 0) \). If (P) has an admissible pair \( (T^*, w^*) \), there exists a quasi minimal solution \( (T^*, w^*) \) to (P) given by (5.1), with the corresponding state

\[ u^* ∈ W^{1,2}(0, T^*; L^2(0, 2π; H^1(0, L))) \cap L^2(0, T^*; L^2(0, 2π; H^3(0, L))) \]
\[ v^* ∈ W^{1,2}(0, T^*; L^2(0, 2π; H^2(0, L))) \cap L^2(0, T^*; L^2(0, 2π; H^4(0, L))). \]
Moreover, if
\[
2 \max \left\{ T^*, \sqrt{T^*} \right\} \| \gamma \|_{L^1(0,T^*)} < 1, \tag{5.3}
\]
\[
\rho_k \left( 1 - (T^* + \sqrt{T^*}) \right) \geq \| \gamma \|_{L^1(0,T^*)} \| v_k \|_{H^4(0,L) \cap H^2(0,L)}
\]
for all \( k \in Z \setminus \{0\} \), then the optimal pairs \( (T_k^*, w_k^*) \) in \((P_k)\) satisfy
\[
\alpha_k w_k^* = \nu_{k}^* (t, 1), \quad \text{a.e. } t \in (0, 1), \quad w_k^*(0) = w_k^*(T_k^*) = 0, \quad \| w_k^* \|_{V_T} = \rho_k, \tag{5.4}
\]
\[
\alpha_k \rho_k T_k^* + \Re \int_0^{T_k^*} (v_k^*(t), F_{ok} p_k(t))_{H^1} dt = 1, \tag{5.5}
\]
where \( \alpha_k > 0, \) \( v_k^* \) is the solution to \((1.8)-(1.11)\) and \( p_k \) is the solution to the adjoint equation
\[
- E_{ok}(p_k) t + F_{ok}^* p_k(t) = 0, \quad \text{a.e. } t \in (0, T_k^*). \tag{5.6}
\]

**Proof.** By the hypothesis, there exists \( T_\ast \) such that \( u(T_\ast, x, y) = v(T_\ast, x, y) = 0 \). Then, by Parseval identity, \( u_k(T_\ast, y) = v_k(T_\ast, y) = 0 \), hence \((T_\ast, w_{k\ast})\) is an admissible pair for \((P_k)\), where \( w_{k\ast} \) is the mode "\( k \)" of \( w_k \). Correspondingly, \((T_\ast, w_{k\ast})\) is an admissible pair in \((\hat{P}_k)\). Then, by Theorem 2.3, it follows that \((\hat{P}_k)\) has a solution \((T_k^*, w_k^*)\), and since \( T_k^* \) is minimal it follows that \( T_k^* < T_\ast \). The state \( \hat{v}_k \) has the properties \((2.11)\) and \( \hat{v}_k(1) = 0 \). Moreover, by extending \( \hat{w}_k \) by 0 after \( \hat{t} = 1 \) we get \( \hat{v}_k^*(t) = 0 \) for \( \hat{t} > 1 \).

Now, we resume the transformation \((2.1)\) and set \( \hat{t} = \frac{t}{T_k^*} \), where \( t \in (0, T_k^*) \) if \( \hat{t} \in (0, 1) \). All the results obtained for \((\hat{P}_k)\) will be correspondingly transported to \((P_k)\) on \((0, T_k^*)\). Thus, \((P_k)\) has an optimal pair \((T_k^*, w_k^*), \hat{v}_k^*\) on \((0, T_k^*)\).

By the third equation in \((1.5)\) we have \( u_k^*(t, y) = \frac{1}{H_k^2} (v_k^*)'(t, y) \) and so \( u_k^*(T_k^*) = 0 \) and \( u_k^*(t) = 0 \) for \( t > T_k^* \).

We set \( T^* \) and \( w^* \) as in \((5.1)\). It follows that \((T^*, w^*)\) is a quasi minimal solution to \((P)\) and since \( T_k^* < T_\ast \), it follows that \( T^* < T_\ast \). By the Parseval identity, it follows that \( u^* \) and \( v^* \) constructed by \((1.3)\) satisfy \( u^*(T^*) = v(T^*) = 0 \), as required in \((P)\), and \( u^*(t) = v^*(t) = 0 \) for \( t > T^* \).

Now, let \((5.3)\) hold. We have
\[
1 > 2 \max \left\{ T^*, \sqrt{T^*} \right\} \| \gamma \|_{L^1(0,T^*)} \geq 2 \max \left\{ T_k^*, \sqrt{T_k^*} \right\} \| \gamma \|_{L^1(0,T_k^*)}.
\]

We note that \((1.1), (1.2), (1.3)\) should be read now for the variable \( \hat{t} \) and functions \( \hat{v}_k^*, \hat{w}_k^*, \hat{p}_k \). It means that the previous relation \( 1 > 2 \max \{ T_k^*, \sqrt{T_k^*} \} \| \gamma \|_{L^1(0, T_k^*)} \) implies, by the transformation \( t = T_k^* \hat{t} \) that \( 1 > 2 \max \{ T_k^*, \sqrt{T_k^*} \} \| \gamma \|_{L^1(1)} \), that is the first condition in \((1.1)\). Similarly, one can prove that the second condition in \((5.3)\) implies the second condition in \((1.1)\), so that the results in Theorem 4.1 will follow. Next, by the transformation \( \hat{t} = \frac{t}{T_k^*} \) in \((1.2)-(1.3)\) we get \((5.3)-(5.6)\).

Also, the solution \( v_k^* \) to \((1.8)-(1.11)\) satisfies
\[
\| v_k(t) \|_{H^4(0,L)} + \int_0^{T_k^*} \| \frac{dv_k(t)}{dt} \|_{H^2(0,L)} dt + \int_0^{T_k^*} \| v_k(t) \|_{H^4(0,L)} dt \leq C \left( \| v_{k0} \|_{H^4(0,L) \cap H^2(0,L)} + \int_0^{T_k^*} |w_k(t)|^2 dt + \int_0^{T_k^*} |\hat{w}_k(t)|^2 dt \right), \quad \text{for all } t \geq 0.
\]

Then,
\[
\frac{1}{2\pi} \int_0^{2\pi} \| v^*(t, x) \|_{H^2(0,1)}^2 dx = \sum_{k \in Z, k \neq 0} \| v_k^*(t) \|_{H^2(0,L)}^2 \leq C \sum_{k \in Z, k \neq 0} \left( \| v_{k0} \|_{H^4(0,L) \cap H^2(0,L)} + \rho_k^2 (T_k^*)^2 + \rho_k^2 T_k^* \right) \leq C \left( \frac{1}{2\pi} \int_0^{2\pi} \| v_0(x) \|_{H^4(0,L) \cap H^2(0,L)} dx + \rho^2 \max \{ (T_k^*)^2, T_k^* \} \right) := I_1, \quad \text{for all } t \in [0, T^*],
\]

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since \( \sum_{k \in \mathbb{Z}, k \neq 0} \rho_k^2 \leq \rho^2 \). Similarly, we proceed for showing that \( v^* \) belongs to the other two spaces.

For \( u^* \) we calculate e.g.,

\[
\frac{1}{2\pi} \int_0^{2\pi} \| u^*(t,x) \|_{H^1(0,L)}^2 \, dx = \sum_{k \in \mathbb{Z}, k \neq 0} \| u^*_k(t) \|_{H^1(0,L)}^2 = \sum_{k \in \mathbb{Z}, k \neq 0} \left\| \frac{i}{k} (v^*_k)'(t) \right\|_{H^1(0,L)}^2
\]

\[
= \sum_{k \in \mathbb{Z}, k \neq 0} \frac{1}{k^2} \| (v^*_k)'(t) \|_{H^1(0,L)}^2 \leq \sum_{k \in \mathbb{Z}, k \neq 0} \| v^*_k(t) \|_{H^2(0,L)}^2 \leq I_1,
\]

and proceed similarly for the other norms, so that \( u^* \) and \( v^* \) belong to the spaces \([5.2]\). This ends the proof.

\[\square\]

### 6 Conclusions

We conclude that the study of the controllability problem \((P)\) returns the fact that, if \((P)\) has an admissible pair \((T^*, w^*)\) one can prove that there exists \((T^*, w^*)\) which ensures the flow stabilization towards the stationary laminar regime, with \( T^* \leq T_* \). The pair \((T^*, w^*)\) has an important property, namely it is constructed via the solutions of minimal time controllability problems for the modes of the Fourier transforms of the Navier-Stokes linearized system. The first condition \([5.3]\) agrees a known fact, that is, when the time necessary to reach the objective is smaller, the effort \( \gamma \) is greater.

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