FORWARD–BACKWARD SDES WITH JUMPS AND CLASSICAL SOLUTIONS TO NONLOCAL QUASILINEAR PARABOLIC PDES

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**Abstract.** We obtain an existence and uniqueness theorem for fully coupled forward–backward SDEs (FBSDEs) with jumps via the classical solution to the associated quasilinear parabolic partial integro-differential equation (PIDE), and provide the explicit form of the FBSDE solution. Moreover, we embed the associated PIDE into a suitable class of non-local quasilinear parabolic PDEs which allows us to extend the methodology of Ladyzhenskaya et al (1968) to non-local PDEs of this class. Namely, we obtain the existence and uniqueness of a classical solution to both the Cauchy problem and the initial–boundary value problem for non-local quasilinear parabolic second-order PDEs.

1. Introduction

One of the well-known tools to solve FBSDEs driven by a Brownian motion is their link to quasilinear parabolic PDEs which, by means of Itô’s formula, allows to obtain the explicit form of the FBSDE solution via the classical solution to the associated PDE [3, 12, 14, 6]. However, if we are concerned with FBSDEs with jumps, the associated PDE becomes a PIDE whose coefficients contain non-local dependencies on the solution. To the best of our knowledge, there are no results on the solvability, in the classical sense, of PIDEs appearing in connection to FBSDEs with jumps.

In this work, we obtain the existence and uniqueness of a classical solution for a class of non-local quasilinear parabolic PDEs, which includes PIDEs of interest, and apply this result to obtain the existence and uniqueness of solution to fully coupled FBSDEs driven by an $n$-dimensional Brownian motion and a compensated Poisson random measure on an arbitrary time interval $[0, T]$:

\[
\begin{align*}
X_t &= x + \int_0^t f(s, X_s, Y_s, Z_s, \tilde{Z}_s) \, ds + \int_0^t \sigma(s, X_s, Y_s) \, dB_s + \int_0^t \int_{\mathbb{R}_*^l} \varphi(s, X_{s-}, Y_{s-}, y) \, \tilde{N}(ds, dy), \\
Y_t &= h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s, \tilde{Z}_s) \, ds - \int_t^T Z_s \, dB_s - \int_t^T \int_{\mathbb{R}_*^l} \tilde{Z}_s(y) \, \tilde{N}(ds, dy).
\end{align*}
\]

Above, the forward SDE is $n$-dimensional and the backward SDE (BSDE) is $m$-dimensional; the coefficients $f(t, x, u, w)$, $g(t, x, v, p, w)$, $\sigma(t, x, u)$, and $\varphi(t, x, u, y)$ are functions of appropriate dimensions whose argument $(t, x, u, p, w)$ belongs to the space $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times L_2(\nu, \mathbb{R}_*^l) \rightarrow \mathbb{R}^m$, where $\nu$ is the intensity of the Poisson random measure involved in (1) and $\mathbb{R}_*^l = \mathbb{R}^l - \{0\}$. 

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The coefficients of \((2)\) are expressed via the coefficients of \((1)\) as follows
\[
\begin{aligned}
&-\sum_{i,j=1}^{n} a_{ij}(t, x, u)\partial_{x_i,x_j} u + \sum_{i=1}^{n} a_{i}(t, x, u, \partial_x u, \vartheta_u)\partial_{x_i} u + a(t, x, u, \partial_x u, \vartheta_u) + \partial_t u = 0.
\end{aligned}
\]

Our second object of interest is the following \(\mathbb{R}^m\)-valued non-local quasilinear parabolic PDE on \([0, T] \times \mathbb{R}^n\) associated to FBSDE \((1)\)
\[
(2)
\]

where the support \(Z\) of the function \(y \mapsto \varphi(t, x, u, y)\) is assumed to have a finite \(\nu\)-measure for each \((t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m\). In \((2)\), \(\partial^2_{x_i,x_j} u, \partial_{x_i} u, \vartheta_u, u, \) and \(\vartheta_u\) are evaluated at \((t, x)\). Non-local PDE \((2)\) is assumed to be uniformly parabolic, i.e., for all \(\xi \in \mathbb{R}^n\), it holds that \(\mu(|u|)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t, x, u)\xi_i \xi_j \leq \mu(|u|)|\xi|^2\), where \(\mu\) and \(\tilde{\mu}\) are non-decreasing, and, respectively, non-increasing functions.

BSDEs and FBSDEs with jumps have been studied by many authors, e.g., [2, 9, 10, 11, 13, 17, 18, 19]. Existence and uniqueness results for fully coupled FBSDEs with jumps were previously obtained in [18], [19], and, on a short time interval, in [11]. The main assumption in [18] and [19] is the so-called monotonicity assumption (see, e.g., [18], p. 436, assumption (H3.2)). This is a rather technical condition that fails to hold for most of naturally appearing functions.

Importantly, we obtain a link between the solution to FBSDE \((1)\) and the solution to the associated PIDE. A similar link in the case of FBSDEs driven by a Brownian motion was established by Ma, Pottor, and Yong [12], and the related result on the solvability of FBSDEs is known as the four step scheme. The main tool to establish this link (and, consequently, to solve Brownian FBSDEs) was the existence of a classical solution to traditional quasilinear parabolic second-order PDEs, which, however, is well known; and as the main reference used in [12], stands the monograph of Ladyzhenskaya et al [8]. Since the consideration of FBSDEs with jumps leads to PDEs of type \((2)\), i.e., containing the non-local dependence \(\vartheta_u\), the theory developed in [8] is not applicable anymore.

It is rather doubtless that a Lévy analog of the four step scheme is an interesting problem on its own; yet due to the technical complexity of the proofs, a direct generalization of the results of [8] to non-local PDEs is not obvious. Indeed, it has been more than twenty years since the original four step scheme was published.

Thus, this article has the following two main contributions. First of all, we define a class of non-local quasilinear parabolic PDEs containing the PIDE associated to FBSDE \((1)\) and establish the existence and uniqueness of a classical solution to the Cauchy problem and the initial–boundary value problem for PDEs of this class; and, secondly, we prove the existence and uniqueness theorem for fully coupled FBSDEs with jumps \((1)\) and provide the formulas that express the solution to FBSDE \((1)\) via the solution to associated non-local PDE \((2)\) with the coefficients and the function...
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\( \vartheta u \) given by (3). The major difficulty of this work appears in obtaining the first of the aforementioned results.

The following scheme is used to obtain the existence and uniqueness result for nonlocal PDEs. We start with the initial–boundary value problem on an open bounded domain. The maximum principle, the gradient estimate, and the Hölder norm estimate are obtained in order to show the existence of solution by means of the Leray–Schauder theorem. Further, the diagonalization argument is employed to prove the existence of solution to the Cauchy problem. Remark that obtaining the gradient estimate is straightforward and can be obtained from the similar result in [8] by freezing the non-local dependence \( \vartheta u \). However, the estimate of Hölder norms cannot be obtained in the similar manner because bounds for the time derivatives of the coefficients of (2) have to be known in advance. It is well known, however, that the Hölder norm estimates are crucial for application of the Leray–Schauder theorem and the diagonalization argument.

The organization of the article is as follows. Section 2 is dedicated to the existence and uniqueness of solution to abstract multidimensional non-local quasilinear parabolic PDEs of form (2). We consider both the Cauchy problem and the initial–boundary value problem. In Section 3, we show that by means of formulas (3), the PIDE associated to FBSDE (1) is included into the class of non-local PDEs considered in Section 2. Finally, in the same section we obtain the existence and uniqueness theorem for FBSDEs with jumps and provide the formulas connecting the solution to an FBSDE with the solution to the associated PIDE.

2. Multidimensional non-local quasilinear parabolic PDEs

In this section, we obtain the existence and uniqueness of solution for the initial–boundary value problem and the Cauchy problem for an abstract \( \mathbb{R}^n \)-valued non-local quasilinear parabolic PDE of form (2), where \( \vartheta u(t, x) \) is a function built via \( u \), taking values in a normed space \( E \), and satisfying additional assumptions to be discussed later. Remark that the function \( \vartheta u \) considered in this section is not necessary of the form specified in (3).

Let \( F \subset \mathbb{R}^n \) be an open bounded domain with a piecewise-smooth boundary and non-zero interior angles. For a more detailed description of the forementioned class of domains we refer the reader to [8] (p. 9). Further, in the case of the initial–boundary value problem we consider the boundary condition

\[
(4) \quad u(t, x) = \psi(t, x), \quad (t, x) \in \{(0, T) \times \partial F\} \cup \{t = 0\} \times F,
\]

where \( \psi \) is the boundary function defined as follows

\[
(5) \quad \psi(t, x) = \begin{cases} 
\varphi_0(x), & x \in \{t = 0\} \times F, \\
0, & (t, x) \in [0, T] \times \partial F.
\end{cases}
\]

In the case of the Cauchy problem, we consider the initial condition

\[
(6) \quad u(0, x) = \varphi_0(x), \quad x \in \mathbb{R}^n.
\]

Further, in the case of the initial–boundary value problem, the coefficients of PDE (2) are defined as follows: \( a_{ij} : [0, T] \times F \times \mathbb{R}^m \to \mathbb{R}, a_i : [0, T] \times F \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times E \to \mathbb{R}, \) \( i, j = 1, \ldots, n, a : [0, T] \times F \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times E \to \mathbb{R}^m \). In the case of the Cauchy problem, everywhere in the above definitions, \( F \) should be replaced with the entire space \( \mathbb{R}^n \).

We remark that due the presence of the function \( \vartheta u \), the existence and uniqueness results of Ladyzenskaya et al [8] for initial–boundary value problem (2)-(4) and Cauchy problem (2)-(6) are not applicable to the present case.
Remark 1. Without loss of generality we assume that \( \{a_{ij}\} \) is a symmetric matrix. Indeed, since we are interested in \( C^{1,2} \)-solutions of (2), then for all \( i,j, \partial^2_{x_i x_j} u = \partial^2_{x_j x_i} u \). Therefore, \( \{a_{ij}\} \) can be replaced with \( \frac{1}{2} (a_{ij} + a_{ji}) \) for non-symmetric matrices.

2.1 Steps involved in solving the problem

In this subsection, we explain the scheme for obtaining the existence and uniqueness theorem for non-local PDE (2). In each step, we mention whether it is an adaptation of the similar result in [8], or the differences are essential.

1. Maximum principle. We start with the maximum principle for an initial–boundary value problem for PDE (2) on a bounded domain. This step can be viewed as an adaptation of the similar result in [8].

2. Gradient estimate. To obtain an a priori estimate for the gradient of a classical solution, we freeze the function \( \vartheta_u \) in (2). After this, we are able to apply the result from [8] on the gradient estimate.

3. Hölder norm estimate. This estimate cannot be obtained by simply freezing \( \vartheta_u \), as in the previous step, since to do so, we need bounds for the time derivatives of the coefficients of (2), where the letter will involve the time derivative \( \partial_t \vartheta_u \). To overcome this issue, we have to find an a priori bound for \( \partial_t u \). At this step, the difference with [8] becomes essential since it requires to estimate an \( L^2 \)-type norm of \( \vartheta_u \), given by (3), via the similar norm of \( u \).

4. Existence and uniqueness theorem for an initial–boundary value problem. The main tools in our proof are a priori estimates of Hölder norms obtained in the previous steps and the version of the Leray-Schauder theorem from [5]. The proof essentially relies on the scheme outlined in [8].

5. Existence theorem for a Cauchy problem. The main technique here is the diagonalization argument, described in [8] for the case of one equation, together with the existence theorem obtained in the previous step.

6. Uniqueness theorem for a Cauchy problem. To prove the uniqueness, we use the results on fundamental solutions from the book [4] along with Gronwall’s inequality and estimates for \( \vartheta_u \).

2.2 Notation and terminology

In this subsection we introduce the necessary notation that will be used throughout this article.

\( T > 0 \) is a fixed real number, not necessarily small.

\( \mathbb{F} \subset \mathbb{R}^n \) is an open bounded domain with a piecewise-smooth boundary \( \partial \mathbb{F} \) and non-zero interior angles.

\( \mathbb{F}_T = (0, T) \times \mathbb{F} \) and \( \mathbb{F}_t = (0, t) \times \mathbb{F} \), \( t \in (0, T) \).

\( \partial^2 \mathbb{F} = [0, T] \times \partial \mathbb{F} \) and \( \partial \mathbb{F}_t = [0, t] \times \partial \mathbb{F} \), \( t \in (0, T) \).

\( \overline{\mathbb{F}} = \mathbb{F} \cup \partial^2 \mathbb{F} \), \( \overline{\mathbb{F}_t} = \{ t = 0 \} \times \overline{\mathbb{F}} \), \( \overline{\mathbb{F}_t} \) is the closure of \( \mathbb{F} \).

\( \Gamma_i = \{ t = 0 \} \times \mathbb{F} \cup \partial \mathbb{F}_t, t \in [0, T] \).

\( (E, \| \cdot \|) \) is a normed space.

For a function \( \phi(t, x, u, p, w) : [0, T] \times \mathbb{F} \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times E \rightarrow \mathbb{R}^k \), where \( k = 1, 2, \ldots \),

\( \partial_x \phi \) or \( \phi_x \) denotes the partial gradient with respect to \( x \in \mathbb{R}^n \);

\( \partial_{x_i} \phi \) or \( \phi_{x_i} \) denotes the partial derivative \( \frac{\partial}{\partial x_i} \phi \);

\( \partial^2_{x_i x_j} \phi \) or \( \phi_{x_i x_j} \) denotes the second partial derivative \( \frac{\partial^2}{\partial x_i \partial x_j} \phi \);

\( \partial_t \phi \) or \( \phi_t \) denotes the partial derivative \( \frac{\partial}{\partial t} \phi \);

\( \partial_u \phi \) denotes the partial gradient with respect to \( u \in \mathbb{R}^m \);

\( \partial_{u_i} \phi \) or \( \phi_{u_i} \) denotes the partial derivative \( \frac{\partial}{\partial u_i} \phi \);

\( \partial_p \phi \) denotes the partial gradient with respect to \( p \in \mathbb{R}^{m \times n} ; \)
\[ \partial_p \phi \] or \( \phi_{p} \) denotes the partial gradient with respect to the \( i \)th column \( p_i \) of the matrix \( p \in \mathbb{R}^{m \times n} \).
\[ \partial_w \phi \] denotes the partial Gâteaux derivative of \( \phi \) with respect to \( w \in E \).
\[ \bar{\mu}(s) \], \( s \geq 0 \), is a positive non-increasing continuous function.
\[ \mu(s) \] and \( \bar{\mu}(s) \), \( s \geq 0 \), are positive non-decreasing continuous functions.
\[ P(s, r, t) \] and \( \varepsilon(s, r) \), \( s, r, t \geq 0 \), are positive and non-decreasing with respect to each argument, whenever the other arguments are fixed.
\( \varphi_0(x) \) is the initial condition.
\( M \) is the a priori bound on \( \mathbb{P}_T \) for the solution \( u \) to problem (2)-(4) (as defined in Remark 3).
\( M_1 \) is the a priori bound for \( \partial_x u \) on \( \mathbb{P}_T \).
\( M \) is the a priori bound for \( \| \partial_u \|_E \) on \( \mathbb{P}_T \).
\( K \) is the common bound for the partial derivatives and the Hölder constants, mentioned in Assumption (A8), over the region \( \mathbb{P}_T \times \{ |u| \leq M \} \times \{ \| w \|_E \leq M \} \times \{ |p| \leq M_1 \} \), as defined in Remark 5.
\( K_{\xi} \) is the constant defined in Assumption (A10).

The Hölder space \( C^{2+\beta}(\mathbb{F}) \), \( \beta \in (0, 1) \), is understood as the (Banach) space with the norm

\[ \| \phi \|_{C^{2+\beta}(\mathbb{F})} = \| \phi \|_{C^{2}(\mathbb{F})} + [\phi'']_\beta, \quad \text{where} \quad [\phi'']_\beta = \sup_{x, y \in \mathbb{F}, 0 < |x - y| < 1} \frac{|\phi(x) - \phi(y)|}{|x - y|^\beta}. \]

(7)

For a function \( \varphi(x, \xi) \) of more than one variable, the Hölder constant with respect to \( x \) is defined as

\[ [\varphi]_\beta^x = \sup_{x, x' \in \mathbb{F}, 0 < |x - x'| < 1} \frac{|\varphi(x, \xi) - \varphi(x', \xi)|}{|x - x'|^\beta}, \]

i.e., it is understood as a function of \( \xi \).

The parabolic Hölder space \( C^{1+\frac{\beta}{2}, 2+\beta}(\mathbb{P}_T) \), \( \beta \in (0, 1) \), is defined as the Banach space of functions \( u(t, x) \) possessing the finite norm

\[ \| u \|_{C^{1+\frac{\beta}{2}, 2+\beta}(\mathbb{P}_T)} = \| u \|_{C^1(\mathbb{P}_T)} + \sup_{t \in [0, T]} [\partial_t u]^x_\beta + \sup_{t \in [0, T]} [\partial_{xx} u]^x_\beta + \sup_{x \in \mathbb{P}} [\partial_t u]^x_\beta + \sup_{x \in \mathbb{P}} [\partial_{xx} u]^x_\beta. \]

(9)

\( C^{1+\frac{\beta}{2}, 2+\beta}(\mathbb{P}_T) \), \( \beta \in (0, 1) \), denotes the space of functions \( u \in C(\mathbb{P}_T) \) possessing the finite norm

\[ \| u \|_{C^{1+\frac{\beta}{2}, 2+\beta}(\mathbb{P}_T)} = \| u \|_{C(\mathbb{P}_T)} + \sup_{t \in [0, T]} [u]^x_\beta + \sup_{x \in \mathbb{P}} [u]^x_\beta. \]

(10)

\( C^{0,1}(\mathbb{P}_T) \) and \( C^{1,2}(\mathbb{P}_T) \) denotes the space of functions from \( C^{0,1}(\mathbb{P}_T) \) and \( C^{1,2}(\mathbb{P}_T) \), respectively, vanishing on \( \partial \mathbb{P} \).

\( \mathcal{H}(E, \mathbb{R}^m) \) is the Banach space of bounded positively homogeneous maps \( E \to \mathbb{R}^m \) with the norm \( \| \phi \|_\mathcal{H} = \sup_{\| w \|_E \leq 1} |\phi(w)| \).

The Hölder space \( C^{2+\beta}(\mathbb{R}^n) \), \( \beta \in (0, 1) \), is understood as the (Banach) space with the norm

\[ \| \phi \|_{C^{2+\beta}(\mathbb{R}^n)} = \| \phi \|_{C^2(\mathbb{R}^n)} + [\phi'']_\beta, \]

where \( C^2(\mathbb{R}^n) \) denotes the space of twice continuously differentiable functions on \( \mathbb{R}^n \) with bounded derivatives up to the second order. The second term in (10) is the
Hölder constant which is defined as in (7) but the domain $F$ has to be replaced with the entire space $\mathbb{R}^n$.

Similarly, for a function $\varphi(x, \xi)$, $x \in \mathbb{R}^n$, of more than one variable, the Hölder constant with respect to $x$ is defined as in (8) but $F$ should be replaced with $\mathbb{R}^n$.

Further, the parabolic Hölder space $C^{1+\frac{\beta}{2},2+\beta}_0([0,T] \times \mathbb{R}^n)$ is defined as the Banach space of functions $u(t,x)$ possessing the finite norm

$$
\|u\|_{C^{1+\frac{\beta}{2},2+\beta}_0([0,T] \times \mathbb{R}^n)} = \|u\|_{C^{2,\beta}(\{t=0\})} + \sup_{t \in [0,T]} \|\partial_t u\|_{L^\infty} + \sup_{t \in [0,T]} \|\partial_{xx} u\|_{L^\infty}
$$

where $C^{1+\frac{\beta}{2},2+\beta}_0([0,T] \times \mathbb{R}^n)$ denotes the space of bounded continuous functions whose mixed derivatives up to the second order in $x \in \mathbb{R}^n$ and first order in $t \in [0,T]$ are bounded and continuous on $[0,T] \times \mathbb{R}^n$.

We say that a smooth surface $S \subset \mathbb{R}^n$ (or $S \subset [0,T] \times \mathbb{R}^n$) is of class $C^{\gamma}$ (resp. $C^{\gamma_1,\gamma_2}$), where $\gamma, \gamma_1, \gamma_2 > 1$ are not necessarily integers, if at some local Cartesian coordinate system of each point $x \in S$, the surface $S$ is represented as a graph of function of class $C^{\gamma}$ (resp. $C^{\gamma_1,\gamma_2}$). For more details on surfaces of the classes $C^{\gamma}$ and $C^{\gamma_1,\gamma_2}$, we refer the reader to [8] (pp. 9–10).

Furthermore, we say that a piecewise smooth surface $S \subset \mathbb{R}^n$ is of class $C^{\gamma}$, $\gamma > 1$, if each of its smooth components is of this class.

The Hölder norm of a function $u$ on $\Gamma_T$ is defined as follows

$$
\|u\|_{C^{1+\frac{\beta}{2},2+\beta}(\Gamma_T)} = \max \left\{ \|u\|_{C^{2,\beta}(\mathbb{F})}, \|u\|_{C^{1+\frac{\beta}{2},2+\beta}(\partial \mathbb{F})} \right\},
$$

where the norm $\|u\|_{C^{1+\frac{\beta}{2},2+\beta}(\partial \mathbb{F})}$ is defined in [8] (p. 10). However, since we restrict our consideration only to functions vanishing on the boundary $\partial \mathbb{F}$, we do not need the details on the definition of Hölder norms on $(\partial \mathbb{F})$, i.e., in our case it always holds that

$$
\|u\|_{C^{1+\frac{\beta}{2},2+\beta}(\Gamma_T)} = \|u\|_{C^{2,\beta}(\mathbb{F})}.
$$

**Remark 2.** Some notation of this article is different than in the book of Ladyzhenskaya et al. [8]. For reader’s convenience, we provide the correspondence of the most important notation: $\Omega = \mathbb{F}$, $S = \partial \mathbb{F}$, $S_T = (\partial \mathbb{F})_T$, $Q_T = \mathbb{F}_T$, $\Gamma_T = \Gamma_T$, $N = m$.

### 2.3 Maximum principle

In this subsection, we obtain the maximum principle for problem (2)-(4) under assumptions (A1)–(A4) below. Obtaining an a priori bound for the solution to problem (2)-(4) is an essential step for obtaining other a priori bounds, as well as proving the existence of solution.

We agree that the functions $\mu(s)$ and $\hat{\mu}(s)$ in the assumptions below are non-decreasing and, respectively, non-increasing, continuous, defined for positive arguments, and taking positive values. Further, $L_E, c_1, c_2, c_3$ are non-negative constants.

**Assume the following.**

(A1) For all $(t,x,u) \in \mathbb{F}_T \times \mathbb{R}^m$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$
\hat{\mu}(\|u\|)\|\xi\|^2 \leq \sum_{i,j=1}^n a_{ij}(t,x,u)\xi_i\xi_j \leq \mu(\|u\|)\|\xi\|^2.
$$

(A2) The function $\vartheta_u : \mathbb{F}_T \rightarrow E$ is defined for each $u \in C^{1,1}_0(\mathbb{F}_T)$, and such that

$$
\sup_{\mathbb{F}_T} \|e^{-\lambda t}\vartheta_u(t,x)\|_E \leq L_E \sup_{\mathbb{F}_T} \|e^{-\lambda t}u(t,x)\|_E
$$

for all $\lambda \geq 0$. 

There exists a function \( \zeta : \mathcal{R}_0 \times \mathbb{R}^n \rightarrow [0, \infty) \), where \( \mathcal{R}_0 = \mathbb{F}_T \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times E \), such that for all \( (t, x, u, p, w) \in \mathcal{R}_0 \), \( \zeta(t, x, u, p, w, 0) = 0 \) and \( (a(t, x, u, p, w), u) \geq -c_1 - c_2 ||u||^2 - c_3 ||w||^2_E - \zeta(t, x, u, p, w, p^T u) \).

(A4) The function \( \varphi_0 : \mathbb{F} \rightarrow \mathbb{R}^m \) is of class \( C^{2+\beta}(\mathbb{F}) \) with \( \beta \in (0, 1) \).

**Lemma 1.** Assume (A1). If a twice continuously differentiable function \( \varphi(x) \) achieves a local maximum at \( x_0 \in \mathbb{F} \), then for any \( (t, u) \in [0, T] \times \mathbb{R}^m \),

\[
\sum_{i,j} a_{ij}(t, x_0, u)\varphi_{x_i x_j}(x_0) \leq 0.
\]

**Proof.** For each \( (t, u) \in [0, T] \times \mathbb{R}^m \), we have

\[
\sum_{i,j=1}^n a_{ij}(t, x_0, u)\varphi_{x_i x_j}(x_0) = \sum_{i,j=k,l=1}^n \varphi_{y_k y_l}(x_0) a_{ij}(t, x_0, u)v_{ik}v_{jl} = \sum_{k=1}^n \lambda_k \varphi_{y_k y_k}(x_0),
\]

where \( \{v_{ij}\} \) is the matrix whose columns are the vectors of the orthonormal eigenbasis of \( \{a_{ij}(t, x_0, u)\} \), \( \{y_1, \ldots, y_n\} \) are the coordinates with respect to this eigenbasis, and \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( \{\varphi_{y_k y_k}(x_0)\} \).

Note that by (A1), \( \lambda_k = \sum_{i,j=1}^n a_{ij} v_{ik}v_{jk} \geq \hat{\mu}(||u||) > 0 \). Let us show that \( \varphi_{y_k y_k}(x_0) \leq 0 \). Since \( \varphi(y_1, \ldots, y_n) \) has a local maximum at \( x_0 \), then \( \varphi_{y_k}(x_0) = 0 \) for all \( k \). Suppose for an arbitrary fixed \( k \), \( \varphi_{y_k y_k}(x_0) > 0 \). Then, by the second derivative test, the function \( \varphi(y_1, \ldots, y_n) \), considered as a function of \( y_k \) while the rest of the variables is fixed, would have a local minimum at \( x_0 \). The latter is not the case. Therefore, \( \varphi_{y_k y_k}(x_0) \leq 0 \). The lemma is proved.

**Lemma 2.** For a function \( \varphi : \mathbb{F}_T \rightarrow \mathbb{R} \), one of the mutually exclusive conditions 1)–3) necessarily holds:

1) \( \sup_{\mathbb{F}_T} \varphi(t, x) \leq 0 \);
2) \( 0 < \sup_{\mathbb{F}_T} \varphi(t, x) = \sup_{\Gamma_{\mathbb{F}_T}} \varphi(t, x) \);
3) \( \exists (t_0, x_0) \in (0, T] \times \mathbb{F} \) such that \( \varphi(t_0, x_0) = \sup_{\mathbb{F}_T} \varphi(t, x) > 0 \).

**Proof.** The proof is straightforward. \( \square \)

**Theorem 1 (Maximum principle for initial–boundary value problem (2)–(4)).** Assume (A1)-(A4). If \( u(t, x) \) is a \( C^{1,2}(\mathbb{F}_T) \)-solution to problem (2)–(4), then

\[
\sup_{\mathbb{F}_T} |u(t, x)| \leq e^{\lambda T} \max \{ \sup_{\mathbb{F}_T} |\varphi_0(x)|, \sqrt{\lambda} \}, \quad \text{where} \quad \lambda = c_2 + c_3 L_E^2 + 1.
\]

**Proof.** Let \( v(t, x) = u(t, x)e^{-\lambda t} \). Then, \( v \) satisfies the equation

\[
-\sum_{i,j=1}^n a_{ij}(t, x, u)v_{x_i x_j} + e^{-\lambda t} a(t, x, u, u_x, \vartheta_u) + \sum_{i=1}^n a_i(t, x, u, u_x, \vartheta_u) v_{x_i} + \lambda v + v_t = 0.
\]

Multiplying the above identity scalarly by \( v \), and noting that \( (v_{x_i x_j}, v) = \frac{1}{2} \partial^2_{x_i x_j} |v|^2 - (v_{x_i}, v_{x_j}) \), we obtain

\[
-\sum_{i,j=1}^n a_{ij}(t, x, u)\partial^2_{x_i x_j} |v|^2 + e^{-\lambda t} (a(t, x, u, u_x, \vartheta_u), v)
\]

\[+ \sum_{i,j=1}^n a_{ij}(t, x, u)(v_{x_i}, v_{x_j}) + \frac{1}{2} \sum_{i=1}^n a_i(t, x, u, u_x, \vartheta_u) \partial_{x_i} |v|^2 + \lambda |v|^2 + \frac{1}{2} \partial_t |v|^2 = 0,
\]
where $u$ and $v$ are evaluated at $(t, x)$. If $t = 0$, then (11) follows trivially. Otherwise, for the function $w = |v|^2$, one of the conditions 1)–3) of Lemma 2 necessarily holds. Note that condition 1) is excluded. Furthermore, if 2) holds, then

\[ \sup_{\mathbb{F}_T} |u(t, x)| \leq e^{\lambda T} \sup_{\mathbb{F}_T} |v(t, x)| \leq e^{\lambda T} \sup_{\mathbb{F}_T} |\varphi_0(x)|. \]

Suppose now that 3) holds, i.e., the maximum of $|v|^2$ is achieved at some point $(t_0, x_0) \in (0, T) \times \mathbb{F}$. Then, we have

\[ \partial_x w(t_0, x_0) = 0 \quad \text{and} \quad \partial_t w(t_0, x_0) \geq 0. \]

By Lemma 1, the first term in (12) is non-negative at $(t_0, x_0)$. Further, assumption (A1) and identities (14) imply that the third, fourth, and the last term on the left-hand side of (12), evaluated at $(t_0, x_0)$, are non-negative. Consequently, substituting $v(t_0, x_0) = u(t_0, x_0)e^{-\lambda t_0}$, we obtain

\[ u \text{ on } \mathcal{M}, \quad \text{throughout the text, by (A2),} \]

\[ 0 \geq e^{-2\lambda t_0} a(t_0, x_0, u_x(t_0, x_0), \partial_u(t_0, x_0), u(t_0, x_0)) + \lambda |v(t_0, x_0)|^2 \]
\[ \geq -c_1 e^{-2\lambda t_0} - c_2 |v(t_0, x_0)|^2 - c_3 e^{-\lambda t_0} \partial_u(t_0, x_0) \parallel \nabla \xi ||^2_E - c_7 (\ldots, 0) + \lambda |v(t_0, x_0)|^2 \]
\[ \geq -c_1 - c_2 |v(t_0, x_0)|^2 - c_3 L_E^2 |v(t_0, x_0)|^2 + \lambda |v(t_0, x_0)|^2. \]

Picking $\lambda = c_2 + c_3 L_E^2 + 1$, we obtain that $|v(t_0, x_0)|^2 \leq c_1$. Therefore,

\[ \sup_{\mathbb{F}_T} |u(t, x)| \leq \sqrt{c_1} e^{\lambda T}. \]

The above inequality together with (13) implies (11).

\[ \square \]

**Corollary 1.** Let assumptions of Theorem 1 hold. If, in (A3), $c_1 = 0$, then

\[ \sup_{\mathbb{F}_T} |u(t, x)| \leq e^{\lambda T} \sup_{\mathbb{F}_T} |\varphi_0(x)| \quad \text{with} \quad \lambda = c_2 + c_3 L_E^2 + 1. \]

If $c_1 > 0$, then

\[ \sup_{\mathbb{F}_T} |u(t, x)| \leq e^{(\lambda + c_1)T} \max\{\sup_{\mathbb{F}_T} |\varphi_0(x)|, 1\}. \]

**Remark 3.** Let $M$ denote the biggest of right-hand sides of (11) and (18). By Theorem 1 and Corollary 1, $M$ is an a priori bound for $|u(t, x)|$ on $\mathbb{F}_T$. Everywhere below throughout the text, by $M$ we understand the quantity defined above. Furthermore, by (A2), $L_E M$ is a bound for $\|\partial_u(t, x)\|_E$, which we denote by $\bar{M}$.

**Remark 4.** The function $\zeta$ was added to the right-hand side of the inequality in (A3) just for the sake of generality (it is not present in the similar assumption in [8]). Indeed, the presence of this function does not give any extra work in the proof.

### 2.4 Gradient estimate

We now formulate assumptions (A5)–(A7) which, together with previously introduced assumptions (A1)–(A4), will be necessary for obtaining an a priori bound for the gradient $\partial_x u$ of the solution $u$ to problem (2)–(4). Obtaining the gradient estimate is crucial for obtaining estimates of Hölder norms, as well as for the proof of existence. Everywhere below, $\mathcal{R}$ and $\mathcal{R}_1$ are regions defined as follows

\[ \mathcal{R} = \mathbb{F}_T \times \{ |u| \leq M \} \times \mathbb{R}^{m \times n} \times \{ \|u\|_E \leq \bar{M} \}; \quad \mathcal{R}_1 = \mathbb{F}_T \times \{ |u| \leq M \}. \]
Further, the functions $\hat{\mu}(s), \eta(s, r), P(s, r, t), \varepsilon(s, r)$ in the assumptions below are continuous, defined for positive arguments, taking positive values, and non-decreasing with respect to each argument, whenever the other arguments are fixed.

Assumptions (A5)–(A7) read:

(A5) For all $(t, x, u, p, w) \in \mathcal{R}$ it holds that
\[
|a_i(t, x, u, p, w)| \leq \eta(|u|, \|w\|_E)(1 + |p|), \quad i \in \{1, \ldots, n\},
\]
and
\[
|a(t, x, u, p, w)| \leq (\varepsilon(|u|, \|w\|_E) + P(|u|, |p|, \|w\|_E)) (1 + |p|)^2,
\]
where $\lim_{r \to \infty} P(s, r, q) = 0$ and $2(M + 1)\varepsilon(M, \tilde{M}) \leq \hat{\mu}(M)$.

(A6) $a_{ij}$ and $a_i$ are continuous on $\mathcal{R}$; $\partial_{x} a_{ij}$ and $\partial_{u} a_{ij}$ exist and are continuous on $\mathcal{R}_1$; moreover, $\max \{|\partial_{x} a_{ij}(t, x, u)|, |\partial_{u} a_{ij}(t, x, u)|\} \leq \hat{\mu}(|u|)$.

(A7) The boundary $\partial \mathcal{F}$ is of class $C^{2+\beta}$.

In Theorem 2 below, we obtain the gradient estimate for a $C^{1,2}(\overline{\mathcal{F}})$-solution $u(t, x)$ of problem (2)–(4). The main idea is to freeze $\vartheta_u$ in the coefficients $a_i$ and $a$ and apply the result of [8] on the gradient estimate of a classical solution to a system of quasilinear parabolic PDEs.

**Theorem 2.** (Gradient estimate) Let (A1)–(A7) hold, and let $u(t, x)$ be a $C^{1,2}(\overline{\mathcal{F}})$-solution to problem (2)–(4). Further let $\tilde{M}$ be the a priori bound for $|u(t, x)|$ on $\overline{\mathcal{F}}$ whose existence was established by Theorem 1. Then, there exists a constant $M_1 > 0$, depending only on $M, \tilde{M}, \sup_{\mathcal{R}} |\vartheta_0|, \mu(M), \hat{\mu}(M), \eta(M, M), \sup_{q \geq 0} P(M, q, \tilde{M})$, and $\varepsilon(M, \tilde{M})$ such that
\[
\sup_{\overline{\mathcal{F}}} |\partial_x u| \leq M_1.
\]

**Proof.** In (2), we freeze $\vartheta_u$ in the coefficients $a_i$ and $a$. Non-local PDE (2) is, therefore, reduced to the following quasilinear parabolic PDE with respect to $v$
\[
- \sum_{i,j=1}^{n} a_{ij}(t, x, v) \partial_{x_i x_j}^2 v + \sum_{i=1}^{n} a_i(t, x, v, \partial_x v, \partial_u(t, x)) \partial_x v + a(t, x, v, \partial_x v, \partial_u(t, x)) + \partial_t v = 0
\]
with initial–boundary condition (4). Since $\tilde{M}$ is an a priori bound for $\|\vartheta_u(t, x)\|_E$ (see Remark 3), we are in the assumptions of Theorem 6.1 from [8] (p. 592) on the gradient estimate for solutions of PDEs of form (21). Indeed, assumptions (A1) and (A5) are the same as in Theorem 6.1, and (A6) immediately implies the continuity of functions $(t, x, v, p) \to a(t, x, v, p, \vartheta_u(t, x))$ and $(t, x, v, p) \to a_i(t, x, v, p, \vartheta_u(t, x))$ in the region $\overline{\mathcal{F}} \times \{|v| \leq \tilde{M}\} \times \mathbb{R}^{m \times n}$. Further, (A5) implies conditions (6.3) on p. 588 and inequality (6.7) on p. 590 of [8]. It remains to note that by (A3),
\[
(a(t, x, v, p, \vartheta_u(t, x)), v) \geq -c_1' - c_2 |v|^2 + \zeta(t, x, v, p, \vartheta_u(t, x), p^T v),
\]
where $c_1' = c_1 + c_3 \tilde{M}^2$. Therefore, by Corollary 1, any solution $v(t, x)$ of (21) satisfies the estimate $\sup_{\overline{\mathcal{F}}} |v(t, x)| \leq e^{(c_2 + 1 + c_1')T} \max \{\sup_{\overline{\mathcal{F}}} |\vartheta_0(x)|, 1\} \leq \tilde{M}$.

Since $v(t, x) = u(t, x)$ is a $C^{1,2}(\overline{\mathcal{F}})$-solution to (21), then by Theorem 6.1 of [8], estimate (20) holds true. By the same theorem, the constant $M_1$ only depends on $\tilde{M}$, $\sup_{\mathcal{R}} |\vartheta_0|$, $\mu(M)$, $\hat{\mu}(M)$, $\eta(M, \tilde{M})$, $\sup_{q \geq 0} P(M, q, \tilde{M})$, and $\varepsilon(M, \tilde{M})$. □

**2.5 Estimate of $\partial_t u$**

Now we complete the set of assumptions (A1)–(A7) with assumptions (A8)–(A10) below. All together, these assumptions are necessary to obtain an a priori bound for the time derivative $\partial_t u$ which is crucial for proving that any $C^{1,2}(\overline{\mathcal{F}})$-solution
to problem (2)–(4) belongs to class $C^{1+\frac{\beta}{2},1+\beta}(\mathbb{F}_T)$ and obtaining a bound for the $C^{1+\frac{\beta}{2},1+\beta}(\mathbb{F}_T)$-norm of this solution. The region $\mathcal{R}_1$ is defined, as before, by (19), and the region $\mathcal{R}_2$ is defined as follows

$$\mathcal{R}_2 = \mathbb{F}_T \times \{ |u| \leq M \} \times \{ |p| \leq M \} \times \{ \|w\|_E \leq M \}.$$  

Assumptions (A8)–(A10) read:

(A8) $\partial_i a_{ij}, \partial^2_{i\alpha}a_{ij}, \partial^2_{\alpha\beta}a_{ij}, \partial^2_{\alpha i}a_{ij}, \partial^2_{\alpha\alpha}a_{i\alpha}, \partial^2_{\beta\alpha}a_{i\alpha}, \partial^2_{\beta\beta}a_{i\alpha}$ exist and are continuous on $\mathcal{R}_1$; $\partial_\alpha a, \partial_{\alpha\alpha} a, \partial_{\alpha\beta} a, \partial_{\beta\beta} a, \partial_{\alpha i} a, \partial_{\alpha\alpha} a_i, \partial_{\alpha\beta} a_i, \partial_{\beta\beta} a_i$ exist and are continuous and bounded on $\mathcal{R}_2$; $a$ and $a_i$ are $\beta$-Hölder continuous in $x$, $\beta \in (0, 1)$, and locally Lipschitz in $w$ with the Hölder and Lipschitz constants bounded over $\mathcal{R}_2$.

(A9) For each $u \in C^1_0(\mathbb{F}_T)$, $\partial_\alpha u$ and $\partial_x u$ exist and are continuous and bounded; moreover, the bounds for $\|\partial_\alpha u\|$ and $\|\partial_x u\|$ only depend on the bounds for $|\partial_\alpha u(t,x)|$ and $|\partial_x u(t,x)|$ in $\mathbb{F}_T$.

(A10) For all $u \in C^{1,2}_0(\mathbb{F}_T)$, $(t,x) \in \mathbb{F}_T$, it holds that

$$\frac{\partial u(t + \Delta t, x) - \partial u(t, x)}{\Delta t} = \hat{\partial} u(t,x) + \zeta_{u,u}(t,x) v(t,x) + \zeta_{u,x}(t,x),$$

where $v(t,x) = (\Delta t)^{-1}(u(t + \Delta t, x) - u(t, x))$, $\zeta_{u,u}, \zeta_{u,x}$ are bounded functions with values in $L^2(\mathbb{R}^m, E)$ and $E$, respectively, depending non-locally on $u$ and $u_x$ (their common bound will be denoted by $K_{\zeta}$), and $\hat{\partial} : \mathbb{F}_T \to E$, defined for each $v \in C^{1,2}_0(\mathbb{F}_T)$, is such that for all $a > 0$ and $\tau \in (0, T)$,

$$\int_{\mathbb{S}^2(|v|^2)} \|\hat{\partial} v(t,x)\|_E^2 dt dx \leq \hat{L}_E \left( \int_{\mathbb{S}^2(|v|^2)} |v(t,x)|^4 dt dx + a^2 \xi \right),$$

where $\hat{L}_E > 0$ is a constant depending on $\|u\|_{C^{0,1}(\mathbb{F}_T)}$, $E^v(|v|^2) = \{ (t,x) \in \mathbb{F}_T : |v(t,x)|^2 > a \}$, and $\lambda$ is the Lebesgue measure on $\mathbb{R}^{n+1}$.

Remark 5. The common bound over $\mathcal{R}_2$ for the partial derivatives and the Hölder constants mentioned in assumption (A8) and related to the functions $a$ and $a_i$ will be denoted by $K$.

Remark 6. According to the results of [16] (p. 484), for locally Lipschitz mappings in normed spaces, the Gâteaux and Hadamard directional differentiability are equivalent. Moreover, the local Lipschitz constant of a function is the same as the global Lipschitz constant of its Gâteaux derivative. Thus, under (A8), the chain rule holds for the Gâteaux derivatives $\partial_\alpha a$ and $\partial_x a_i$ which, moreover, are globally Lipschitz and positively homogeneous.

The following maximum principle for non-local linear-like parabolic PDEs written in the divergence form, is crucial for obtaining the a priori bound for $\partial_t u$.

Consider the following system of non-local PDEs in the divergence form

$$\partial_t u - \sum_{i=1}^n \partial_{x_i} \left[ \sum_{j=1}^n \hat{a}_{ij}(t,x) \partial_{x_j} u + A_i(t,x)u + f_i(t,x) \right] + \sum_{i=1}^n B_i(t,x) \partial_{x_i} u$$

$$+ A(t,x)u + C(t,x)(\hat{\partial} u(t,x)) + f(t,x) = 0,$$

where $\hat{a}_{ij} : \mathbb{F}_T \to \mathbb{R}$, $A_i : \mathbb{F}_T \to \mathbb{R}^{m \times m}$, $B_i : \mathbb{F}_T \to \mathbb{R}^{m \times m}$, $f_i : \mathbb{F}_T \to \mathbb{R}^m$, $i, j = 1, \ldots, n$, $A : \mathbb{F}_T \to \mathbb{R}^{m \times m}$, $f : \mathbb{F}_T \to \mathbb{R}^m$, and $C : \mathbb{F}_T \to \mathcal{H}(E, \mathbb{R}^m)$, where $\mathcal{H}(E, \mathbb{R}^m)$ is the Banach space of bounded positively homogeneous maps $E \to \mathbb{R}^m$ with the norm $\|\hat{\phi}\|_\mathcal{H} = \sup_{\|w\|_E \leq 1} |\hat{\phi}(w)|$. In (25), the function $u$ together with its partial derivatives, as usual, is evaluated at $(t,x)$ and $\hat{\partial} u(t,x)$ is an $E$-valued function.
built via $u$ and satisfying inequality (24). Remark that all terms in (25), except the term containing $\hat{\sigma}_u(t,x)$, are linear in $u$.

The lemma below, which is a version of the integration-by-parts formula, can be found in [8] (p. 60).

**Lemma 3.** Let $f$ and $g$ be real-valued functions from the Sobolev spaces $W^{1,p}(G)$ and $W^{1,q}(G)$ ($\frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{n}$), respectively, where $G \subset \mathbb{R}^n$ is a bounded domain. Assume that the boundary $\partial G$ is piecewise smooth and that $fg = 0$ on $\partial G$. Then,

$$\int_G f \partial_x g \, dx = -\int_G g \partial_x f \, dx.$$

Further, for each $\tau, \tau' \in [0,T]$, $\tau < \tau'$, we define the squared norm

$$\|v\|_{\tau,\tau'}^2 = \sup_{t \in [\tau,\tau']} \|v(t, \cdot)\|^2_{L^2(F)} + \|\partial_x v\|^2_{L^2(F,\tau,\tau')}$$

(26)

where $F_{\tau,\tau'} = F \times [\tau, \tau']$. Furthermore, for an arbitrary real-valued function $\phi$ on $F_T$ and a number $\alpha > 0$, we define $\phi^\alpha = (\phi - \alpha)^+$ and $F_{\alpha}^\phi = \{(t,x) \in F_{\tau} : \phi > \alpha\}$, where $\tau \in (0,T]$. The following result was obtained in [8] (Theorem 6.1, p. 102). It will be used in Lemma 4.

**Proposition 1.** Let $\phi(t,x)$ be a real-valued function of class $C(F_{\tau})$ such that $\sup_{F_{\tau}} \phi \leq \hat{\alpha}$, where $\hat{\alpha} \geq 0$. Assume for all $\alpha > \hat{\alpha}$ and for a positive constant $\gamma$, it holds that $\|\phi^\alpha\|_{0,T} \leq \gamma \alpha \sqrt{\lambda_{n+1}(F_{\tau}^\phi)}$, where $\lambda_{n+1}$ is the Lebesgue measure on $\mathbb{R}^{n+1}$. Then, there exists a constant $\delta > 0$, depending only on $n$, such that

$$\sup_{F_{\tau}} \phi(t,x) \leq 2 \hat{\alpha} (1 + \delta \gamma^2 \tau \lambda_n(F)).$$

**Remark 7.** We attributed the values $1 + \kappa = r = q = 4$ for the space dimensions $n = 1, 2$ and $1 + \kappa = r = q = \frac{2n}{n+1}$ for $n \geq 3$ to the constants $r$, $q$, and $\kappa$ appearing in the original version of Theorem 6.1 in [8] (p. 102), since for our application we do not need Theorem 6.1 in the most general form. Also, we remark that by our choice of the parameters, $1 + \frac{1}{n} < 2$ for all space dimensions $n$.

**Lemma 4.** Assume the coefficients $\hat{a}_{ij}$, $A_i$, $B_i$, $f_i$, $A$, and $C$ are of class $C(F_T)$ and that $\sum_{i,j=1}^n \hat{a}_{ij}(t,x)\xi_i \xi_j \geq g|\xi|^2$ for all $(t,x) \in F_T$, $\xi \in \mathbb{R}^m$, and for some constant $q > 0$. Let $u(t,x)$ be a generalized solution to problem (25) which is of class $C^{1,1}(F_T)$ and such that $\partial_u$ satisfies (24). Further let $v = |u|^2$. Then, there exist a number $\tau \in (0,T]$ and a constant $\gamma > 0$, where $\gamma$ depends on the common bound $A$ over $F_T$ for the coefficients $A_i$, $B_i$, $f_i$, $A$, $C$, and also on $\hat{L}_E$, $\varphi$, $n$, and $\lambda_n(F)$, and $\gamma$ depends on the same quantities as $\tau$ and on $\sup_{F_T} |u_0|$, such that

$$\|u^\alpha\|_{0,T} \leq \gamma \alpha \sqrt{\lambda_{n+1}(F_{\tau}^\phi)}$$

for all $\alpha \geq \sup_{F_T} |u_0|^2 + 1$.

**Proof.** Let $\tau \in (0,T]$. Multiplying PDE (25) scalarly by a $W^{1,p}(F_T)$-function $\eta(t,x)$ ($p > 1$) vanishing on $\partial F_T$ and applying the integration-by-parts formula (Lemma 3), we obtain

$$\int_{F_{\tau}} \left[ u(t,x), \eta(t,x) \right] + \sum_{i=1}^n \left( \sum_{j=1}^n \hat{a}_{ij}(t,x)u_{x_j} + A_i(t,x)u + f_i(t,x), \eta_{x_i}(t,x) \right)$$

$$+ \left( \sum_{i=1}^n B_i(t,x)u_{x_i} + A(t,x)u + f(t,x) + C(t,x)(\hat{\sigma}_u(t,x), \eta(t,x)) \right) \, dt \, dx = 0.$$

(28)
For simplicity of notation, we write \( F^n \) for \( F^n(v) \). Define \( \eta(t, x) = 2u(t, x)q(t, x) \) and note that \( v^\alpha \) and its derivatives vanish outside of \( F^n \). Since \( D_u \eta = 2(D_u u, u)q^\alpha = (\partial_t u)q^\alpha = \frac{1}{2} \partial_t (q^\alpha) \), we rewrite (28) as follows

\[
\frac{1}{2} \int_\Omega (\alpha^\alpha)^2 \, dx + 2 \int_{\mathbb{R}^2} \left( \sum_{i=1}^n \sum_{j=1}^n a_{ij}(t, x)u_{x_j} + A(t, x)v + f(t, x) \right) \, dx = 0.
\]

Note that the following inequalities hold on \( F^n \):

\[
2 \sum_{i,j=1}^n a_{ij}(t, x)u_{x_j} + \left( \sum_{i=1}^n n \rho u_{x_i} + A(t, x) + f(t, x) \right) \eta_x = 0.
\]

(29)

Furthermore, since \( \eta(t, x) = 2u(t, x)q(t, x) \) is a small constant and the last inequality holds by (24) with \( \tilde{A} \) being a constant that depends only on \( A \) and the constant \( L_E \) from (24). By virtue of these inequalities, from (29) we obtain

\[
(30) \quad \frac{1}{2} \int_\Omega (\alpha^\alpha)^2 \, dx + 2 \int_{\mathbb{R}^2} |\eta u_x|^2 \alpha + (\alpha^\alpha)^2 \, dx + 2 \int_{\mathbb{R}^2} \left( 1 + \eta^2 \right) \, dx + 2 \int_{\mathbb{R}^2} \left( 1 + \eta^2 \right) \, dx \leq \tilde{A}_\eta \int_{\mathbb{R}^2} (1 + \eta^2) \, dx + \tilde{A} \alpha^2 \lambda(F^n),
\]

where \( \tilde{A}_\eta = \epsilon^{-1} \sup_{\mathbb{R}^2} \left( 2 \sum_{i=1}^n |A_i|^2 + 2 \sum_{i=1}^n |f_i|^2 + 2 \sum_{i=1}^n |B_i|^2 + 2 \sum_{i=1}^n |A|^2 + 2 \sum_{i=1}^n f_i \right) \). Picking \( \epsilon = \frac{\delta}{4} \) and defining \( \tilde{\eta} = \min \left( \frac{1}{2}, \frac{\delta}{2} \right) \), for \( \alpha \geq \sup_{\mathbb{R}^2} |u_0|^2 + 1 \), (26) and (30) imply

\[
(31) \quad \tilde{\eta} \int_{\mathbb{R}^2} \alpha^\alpha(t, x)^2 \, dx = \tilde{\eta} \left( \int_{\mathbb{R}^2} \eta \alpha^\alpha(t, x)^2 \, dx + \int_{\mathbb{R}^2} (\alpha^\alpha)^2 \, dx \right) \leq \tilde{A}_\eta \int_{\mathbb{R}^2} (1 + \eta^2) \, dx + \tilde{A} \alpha^2 \lambda_{n+1}(F^n) \leq \tilde{A} \left( \int_{\mathbb{R}^2} (\alpha^\alpha)^2 \, dx + \lambda_{n+1}(F^n) \right),
\]

where \( \tilde{A} = 3\tilde{A}_\eta + \tilde{A}_\eta \) and \( \tilde{\eta} = \min \left( \frac{1}{2}, \frac{\delta}{2} \right) \). Further, from inequality (3.7) (p. 76) in [8] it follows that \( \int_{\mathbb{R}^2} |\alpha^\alpha|^2 \, dx \leq \gamma \lambda_{n+1}(F^n) \tau^\alpha \), where \( \gamma > 0 \) is a constant depending on the space dimension \( n \). Furthermore, since \( \lambda_{n+1}(F^n) \leq \tau \lambda_n(\mathbb{R}^2) \), we can pick \( \tau \) sufficiently small such that \( \tilde{A} \tau^\alpha \leq \tilde{\eta} / 2 \). This implies (27) with \( \gamma = (2 \tilde{A} \tilde{\eta}^{-1})^{1/2} \).

\( \square \)

Lemma 5. If, under assumptions of Lemma 4, \( f_i = f = 0 \), then (27) holds for all \( \alpha \geq \sup_{\mathbb{R}^2} |u_0|^2 \).
Proof. If \( f_i = f = 0 \), then the arguments preceding inequality (31) imply that

\[
\tilde{u}^i\|v^\alpha\|_{0,T}^2 \leq \frac{1}{2} \int_T (v^\alpha(0, x))^2 \, dx + \tilde{A}_{\sigma} \int_{\mathcal{F}_T} v^2 \, dt \, dx + \tilde{A}^2 \lambda_{n+1}(\mathcal{F}_T)
\]

with \( \alpha \geq \sup_\tau |u_0|^2 \). The rest of the proof is the same. \( \square \)

**Theorem 3** (Maximum principle for systems of non-local PDEs of form \((25)\)). Let assumptions of Lemma 4 be fulfilled. Further let a solution \( u \) to problem \((25)\) vanishes on \( \partial \mathcal{F} \). Then \( \sup_{\mathcal{F}_T} |u| \) is bounded by a constant depending only on \( A, \nu, n, T, \lambda_n(F), \tilde{L}_E \), and linearly depending on \( \sup_{\mathcal{F}_T} |u_0| \).

**Proof.** It follows from Proposition 1 and Lemma 4 that there exists a bound for \( \sup_{\mathcal{F}_T} |u| \) depending only on \( A, \nu, n, \lambda_n(F), \tilde{L}_E \), and \( \sup_{\mathcal{F}_T} |u_0| \), where \( \tau \in (0, T] \) is sufficiently small and depends on \( A, \nu, n, \lambda_n(F) \), and \( \tilde{L}_E \). Remark that by Proposition 1, the above bound is a multiple of \( \tilde{\alpha} = \sup_{\mathcal{F}_T} |u_0| + 1 \). It is important to emphasize that \( \tau \) does not depend on \( \sup_{\mathcal{F}_T} |u_0| \). By making the time change \( \tau = \tau - \tau \) in problem \((25)\), we obtain a bound for \( \sup_{\mathcal{F}_T, \tau} |u| \) depending on \( A, \nu, n, \lambda_n(F) \), and \( \sup_{\mathcal{F}_T} |u(\tau, x)| \), where the latter quantity was proved to have a bound which is a multiple of \( \sup_{\mathcal{F}_T} |u_0| + 1 \). On the other hand, by Proposition 1, the bound for \( \sup_{\mathcal{F}_T, \tau} |u| \) is a multiple of \( \sup_{\mathcal{F}_T} |u(\tau, x)| + 1 \). In a finite number of steps, depending on \( T \), we obtain a bound for \( |u| \) in the entire domain \( \mathcal{F}_T \). This bound will depend linearly on \( \sup_{\mathcal{F}_T} |u_0| \) by Proposition 1. The theorem is proved. \( \square \)

**Corollary 2.** Let assumptions of Theorem 3 be fulfilled, and let \( f = f_i = u_0 = 0 \) on \( \mathcal{F}_T \). Then, \( u = 0 \).

**Proof.** Under assumptions of the corollary, we can set \( \alpha = 0 \) in Lemma 5. This implies that \( \|u\|_{0,T}^2 = 0 \) for sufficiently small \( \tau \in (0, T] \). Since \( u \) is continuous on \( \mathcal{F}_T \), then it is zero on \( \mathcal{F}_T \). The same argument as in Theorem 3 implies that \( u = 0 \) on \( \mathcal{F}_T \). \( \square \)

Since the maximum principle for systems of non-local PDEs of form \((25)\) is obtained, we can prove the theorem on existence of an a priori bound for \( \partial_t u \).

**Theorem 4.** Let \((A1)-(A10)\) hold, and let \( u(t, x) \) be a \( C^{1,2} \)-solution to problem \((2)-(4)\). Then, there exists a constant \( M_2 \), depending only on \( M, M_1, K, K_{\xi, \zeta}, T, \lambda_n(F), \tilde{L}_E, \|\tilde{\varphi}_0\|_{C^{2+, \beta}(\mathcal{F})} \), such that

\[
\sup_{\mathcal{F}_T} |\partial_t u| \leq M_2.
\]

**Proof.** Rewrite \((2)\) in the divergence form, i.e.,

\[
\partial_t u - \sum_{i=1}^n \partial_{x_i} \left[ \sum_{j=1}^n a_{ij}(t, x, u, u_x, \partial_u) u_{x_j} \right] + \hat{a}(t, x, u, u_x, \partial_u) = 0 \quad \text{with}
\]

\[
\hat{a}(t, x, u, p, w) = \sum_{i=1}^n a_i(t, x, u, p, w) + a(t, x, u, p, w) + \sum_{i, j=1}^n \partial_{x_i} a_{ij}(t, x, u) p_j + \sum_{i, j=1}^n (\partial_u a_{ij}(t, x, u, p_i) p_j), \quad \text{where} \quad p_i \text{ is the } i\text{th column of the matrix } p, \text{ and } u, u_x \text{ and } \partial_u \text{ are evaluated at } (t, x). \quad \text{Further, we define } v(t, x) = (\Delta t)^{-1} (u(t + \Delta t, x) - u(t, x)) \text{ and } t' = t + \Delta t, \text{ where } \Delta t \text{ is fixed. If } t = 0, \text{ we assume that } \Delta t > 0, \text{ and if}
\]
\( t = T \), then \( \Delta t < 0 \). The PDE for the function \( v \) takes form (25) with

\[
\begin{aligned}
\hat{a}_{ij}(t, x) &= a_{ij}(t', x, u(t', x)); \\
A_i(t, x) &= \sum_{j=1}^{n} u_{x_j}(t, x) \int_0^1 \lambda \partial_{\theta} a_{ij}(t, x, \lambda u(t', x) + (1 - \lambda) u(t, x)) \beta(t, x) x_i \lambda \cdot x_j \chi(t, x) \, \lambda \, d\lambda; \\
f_i(t, x) &= \sum_{j=1}^{n} a_{ij}(t, x, \lambda u(t', x) + (1 - \lambda) u(t, x)) x_j(t, x); \\
f(t, x) &= \phi(t, x, u(t', x), x_i(t', x), \partial u(t', x)) + \int_0^1 \lambda \partial_{\theta} \hat{a}(t, x, u(t', x), x_i(t', x), \partial u(t, x)) x_i(t, x) \, d\lambda; \\
\beta(t, x) &= \sum_{i} a_{ij}(t, x, u(t', x), x_i(t', x), \partial u(t', x)); \\
C(t, x) &= \int_0^1 \lambda \partial_{\theta} \hat{a}(t, x, u(t', x), x_i(t', x), \partial u(t, x)) x_i(t, x) \, d\lambda.
\end{aligned}
\]

Above, \( \xi_{u, u} \) and \( \zeta_{u, u} \) are bounded functions from representation (23). Remark that the above coefficients are bounded by a constant, say \( A \), depending on \( M, M_1, K \), and \( K_{\xi, \zeta} \) (where the latter is the bound for \( \xi_{u, u} \) and \( \zeta_{u, u} \), defined in (A10)). By Theorem 3, \( \sup_{\mathcal{F}} |v| \) is bounded by a constant depending only on \( A, T, \lambda_\nu(F), \hat{L}_E \), and \( \sup_{\mathcal{F}} |v(0, x)| \). Moreover, the dependence on \( \sup_{\mathcal{F}} |v(0, x)| \) is linear. Letting \( \Delta t \) go to zero, we obtain that the bound for \( |\partial u| \) on \( \mathcal{F}_T \) depends only on \( A, T, \lambda_\nu(F), \hat{L}_E \), and \( \sup_{\mathcal{F}} |\partial u(0, x)| \). Finally, equation (2) implies that \( |\partial u(0, x)| \) can be estimated via \( \|v_0\|_{C^{1, \beta}(\mathcal{F})} \), and the bounds for the coefficients \( a_{ij}, a_i, \) and \( a \) over \( \mathcal{R}_2 \), defined by (22). Further, by virtue of (A1) and (A5), the latter bounds can be estimated by a constant depending only on \( M, M \), and \( M_1 \). The theorem is proved.

2.6 Hölder norm estimates

In this subsection, we prove that any \( C^{1,2} \)-solution to problem (2)–(4) is, in fact, of class \( C^{1+\frac{\beta}{2}+\beta} \). Moreover, we obtain a bound for its \( C^{1+\frac{\beta}{2}+\beta} \)-norm. Unlike the bound for the gradient, this bound cannot be obtained directly from the results of [8] by freezing \( \partial_{\theta} u \). Our proof essentially relies on the estimate of the time derivative \( \partial_t u \) obtained in the previous subsection.

**Theorem 5.** (Hölder norm estimate) Let (A1)–(A10) hold, and let \( u(t, x) \) be a \( C^{1,2}(\mathcal{F}_T) \)-solution to problem (2)–(4). Further let \( M \) and \( M_1 \) be the a priori bounds for \( u \) and, respectively, \( \partial_{\theta} u \) on \( \mathcal{F}_T \) (whose existence was established by Theorems 1 and 2). Then, \( u(t, x) \) is of class \( C^{1+\frac{\beta}{2}+\beta}(\mathcal{F}_T) \). Moreover, there exists a constant \( M_3 > 0 \) depending only on \( M, M_1, K, K_{\xi, \zeta}, T, \lambda_\nu(F), \hat{L}_E, \|v_0\|_{C^{2+\beta}(\mathcal{F})} \), and on the \( C^{2+\beta} \)-norms of the functions defining the boundary \( \partial \mathcal{F} \), such that

\[
\|u\|_{C^{1+\frac{\beta}{2}+\beta}(\mathcal{F}_T)} \leq M_3.
\]

**Proof.** Freeze the function \( \partial_{\theta} u \) in the coefficients \( a_{ij}, a_i, \) and consider the following PDE with respect to \( v \)

\[
\begin{aligned}
- \sum_{i,j=1}^{n} a_{ij}(t, x, v) \partial_{x_{ij}}^2 v + \hat{a}(t, x, v, \partial_x v) + \partial_t v &= 0,
\end{aligned}
\]

where \( \hat{a}(t, x, v, p) = a(t, x, v, p, \partial_u(t, x)) + \sum_{i=1}^{n} a_i(t, x, v, p, \partial_{\theta} u(t, x)) p_i \). Let us prove that the coefficients of (33) satisfy the assumptions of Theorem 5.2 from [8] (p. 587) on the Hölder norm estimate. First we show that the assumptions on the continuity of the partial derivatives \( \partial_{\theta} \hat{a}, \partial_u \hat{a}, \partial_{\theta} \partial_{\theta} \hat{a} \) and on the \( \beta \)-Hölder continuity of \( \hat{a} \) in \( x \), mentioned in the formulation of Theorem 5.2 in [8], are fulfilled. Indeed, they follow from (A8) and (A9). To see this, we first note that \( a \) and \( a_i \) depend on \( t \) and \( x \) not just via their first two arguments but also via the function \( \partial_u(t, x) \) (assumed known
a priori) whose differentiability in $t$ and $x$ follows from $(A9)$. Therefore, by $(A8)$ and $(A9)$, $a$ and $a_i$ are $\beta$-Hölder continuous in $x$ and differentiable in $t$.

Further, Theorem 5.2 of [8] introduces a common bound (denote it by $C$) for the partial derivatives $\partial_t a, \partial_u a, \partial_p a$ and the Hölder constant $[\tilde{a}]^\beta$ which, in case of [8], exists due to the continuity of the above functions on $\mathbb{F}_T \times \{ |u| \leq M \} \times \{ |p| \leq M_1 \}$. However, in our case, the expression for $\partial_t a$ will contain $\partial_t \vartheta_u$, and the expression for $[\tilde{a}]^\beta$ will contain $\partial_x \vartheta_u$. Therefore, by $(A9)$, the bound $C$, required for the application of Theorem 5.2, will depend on $M_1$ and $M_2$, i.e., the bounds for $\partial_x \vartheta_u$ and $\partial_t \vartheta_u$. That is why the existence of a bound for $\partial_t \vartheta_u$ is indispensable and must be obtained in advance.

The verification of the rest of the assumptions of Theorem 5.2 in [8] is straightforward and follows from assumptions $(A1)$, $(A4)$, $(A7)$, and $(A8)$. Since $v = u$ is a $C^{1,2,2,2,2,2}(\mathbb{F}_T)$-solution to problem (33)-(4), by aforementioned Theorem 5.2, $u$ belongs to class $C^{1+\frac{1}{2},2+\beta,2+\beta}(\mathbb{F}_T)$, and its Hölder norm $\|u\|_{C^{1+\frac{1}{2},2+\beta,2+\beta}(\mathbb{F}_T)}$ is bounded by a constant $M_3$, depending on the constants specified in the formulation of this theorem. \hfill $\square$

The rest of this subsection deals with estimates of other Hölder norms of the solution $u$ under assumptions that do not require the a priori bound $M_2$ for $\partial_t \vartheta_u$. These estimates will be useful in the proof of existence of solution to Cauchy problem (2)-(6). The need of these bounds comes from the fact that $M_2$ depends on $\lambda_n(\mathbb{F})$, the Lebesgue measure of the domain $\mathbb{F}$.

**Theorem 6.** Assume $(A1)$–$(A7)$. Let $u(t, x)$ be a generalized $C^{0,1}(\mathbb{F}_T)$-solution to equation (2) such that $|u| \leq M$ and $|\partial_x u| \leq M_1$ on $\mathbb{F}_T$. Then, there exist a number $\alpha \in (0, \beta)$ and a constant $M_4$, both depending only on $M$, $M_1$, $M$, $\beta$, $n$, $m$, and $\sup_\mathbb{F} \| \varphi_0 \|_{C^{2+\beta,2+\beta}(\mathbb{F}_T)}$ such that

$$\|u\|_{C^{\frac{1}{2},\alpha}(\mathbb{F}_T)} \leq M_4.$$ 

**Proof.** Freeze the functions $u$, $\partial_x u$, and $\partial_x \vartheta_u$ inside the coefficients $a_{ij}$, $a_i$, and $a$, and consider the linear PDE with respect to $v$

$$\partial_t v - \sum_{i,j=1}^{n} \tilde{a}_{ij}(t,x) \partial^2_{x_i,x_j} v + \sum_{i=1}^{n} \tilde{a}_i(t,x) \partial_{x_i} v + \tilde{a}(t,x) = 0$$

with

$$\tilde{a}_{ij}(t,x) = a_{ij}(t,x,u,\partial_x u, \partial_x \vartheta_u), \quad \tilde{a}_i(t,x) = a_i(t,x,u,\partial_x u, \partial_x \vartheta_u), \quad \tilde{a}(t,x) = a_{ij}(t,x,u),$$

where $v, u, \partial_x u, \text{ and } \partial_x \vartheta_u$ are evaluated at $(t, x)$. Note that by $(A1)$, $(A5)$, and $(A6)$, $a_{ij}, \partial_x a_{ij}, \partial_\vartheta a_{ij}, a_i$, and $a$ are bounded in the region $\mathbb{R}_2$, defined by (22), and the common bound depends on $M$, $M_1$, and $M$. The existence of the bound $M_4$ follows now from Theorem 3.1 of [8] (p. 582). \hfill $\square$

**Theorem 7.** Assume $(A1)$–$(A7)$. Further, assume the following conditions are satisfied in the region $\mathbb{R}_2$, defined by (22):

(i) $a_{ij}, a_i, a$ are Hölder continuous in $t, x, u, p$, with exponents $\frac{\beta}{2}, \beta, \beta, \beta$, respectively, and, moreover, locally Lipschitz and Gâteaux differentiable in $v$; all Hölder and Lipschitz constants are bounded (say, by a constant $M_4$);

(ii) For any $C^{1,2}(\mathbb{F}_T)$-solution $u(t,x)$ to problem (2)-(4) and for some $\beta' \in (0, \beta)$, the bound for $[\tilde{a}_u]^\beta_\mathbb{F}$ is determined only by the bound for $[u]^\beta_\mathbb{F}$ and $M_1$; and the bound for $[\tilde{a}_{\vartheta u}]^{\beta'}_\mathbb{F}$ is determined only by $M_1$.

Let $u(t,x)$ be a $C^{1,2}(\mathbb{F}_T)$-solution to equation (2) such that $|u| \leq M$ and $|\partial_x u| \leq M_1$ on $\mathbb{F}_T$, and let $\mathbb{G} \subset \mathbb{F}$ be a strictly interior open domain. Then, there exist a number $\alpha \in (0, \beta \land \beta')$ and a constant $M_5$, both depending only on $M$, $M_1$, 


\[ M, M, \| \varphi_0 \|_{C^{2+\beta}(\mathbb{T})}, \text{and the distance between } \mathbb{T} \text{ and } \partial \mathbb{E}, \text{such that } u \text{ is of class } C^{1+\frac{\varepsilon}{2},2+\alpha}(\mathbb{T}), \text{ and} \]

\[
\| u \|_{C^{1+\frac{\varepsilon}{2},2+\alpha}(\mathbb{T})} \leq M.
\]

**Proof.** Freeze the function \( \vartheta_u \) in the coefficients \( a_i \), and \( a \), and consider PDE (33) with respect to \( v \). Let \( \alpha \) be the smallest of \( \beta \) and the exponent whose existence was established by Theorem 6. Assumptions (i) and (ii) imply that the coefficient \( \tilde{a} \) in PDE (33) is Hölder continuous in \( t, x, u \), and \( p \) with exponents \( \frac{\varepsilon}{2}, \alpha, \alpha, \text{ and } \alpha \), respectively. Moreover, the Hölder constants are bounded and their common bound depends on \( M, M_1, \text{ and } M_4 \). The constant \( M_4 \), in turn, depends on \( M, M_1, M, \beta, \text{ and } \sup_{\mathbb{T}} \| \varphi_0 \|_{C^{2+\beta}(\mathbb{T})} \). Thus, by Theorem 5.1 of [8] (p. 586), the solution \( u \) is of class \( C^{1+\frac{\varepsilon}{2},2+\alpha}(\mathbb{T}) \) and the bound for the norm \( \| u \|_{C^{1+\frac{\varepsilon}{2},2+\alpha}(\mathbb{T})} \) depends only on \( M, M_1, M, \sup_{\mathbb{T}} \| \varphi_0 \|_{C^{2+\beta}(\mathbb{T})}, \text{ and the distance between } \mathbb{T} \text{ and } (\partial \mathbb{E})_T \). The theorem is proved. \( \square \)

### 2.7 Existence and uniqueness for the initial–boundary value problem

To obtain the existence and uniqueness result for problem (2)–(4), we need the two additional assumptions below:

(A11) The following compatibility condition holds for \( x \in \partial \mathbb{E} \):

\[
- \sum_{i,j=1}^{n} a_{ij}(0, x, 0) \partial_{x, x, x} \varphi_0(x) + \sum_{i=1}^{n} a_i(0, x, 0, \partial_x \varphi_0(x), \partial_{\varphi_0}(0, x)) \partial_{x_i} \varphi_0(x) \\
+ a(0, x, 0, \partial_x \varphi_0(x), \partial_{\varphi_0}(0, x)) = 0.
\]

(A12) For any \( u, u' \in C^{1,2}_0(\mathbb{T}) \), it holds that

\[
\partial_u(t, x) - \partial_u(t, x) = \hat{\partial}_{u-u'}(t, x) + \zeta_{u, u', u, u'}(t, x)(u(t, x) - u'(t, x)),
\]

where \( \zeta_{u, u', u, u'} : \mathbb{T} \rightarrow L(\mathbb{R}^m, E) \) is bounded and may depend non-locally on \( u, u', u, \text{ and } u' \); \( \hat{\partial}_v : \mathbb{T} \rightarrow E \) is defined for each \( v \in C^{1,2}_0(\mathbb{T}) \) and satisfies (A2) (in the place of \( \partial_u \)).

The main tool in the proof of existence for initial–boundary value problem (2)-(4) is the following version of the Leray–Schauder theorem proved in [5] (Theorem 11.6, p. 286). First, we recall that a map is called **completely continuous** if it takes bounded sets into relatively compact sets.

**Theorem 8. (Leray-Schauder theorem)** Let \( X \) be a Banach space, and let \( \Phi \) be a completely continuous map \( [0, 1] \times X \rightarrow X \) such that for all \( x \in X \), \( \Phi(0, x) = c \in X \).

Assume there exists a constant \( K > 0 \) such that for all \( (\tau, x) \in [0, 1] \times X \) solving the equation \( \Phi(\tau, x) = x \), it holds that \( \| x \|_X < K \). Then, the map \( \Phi_1(x) = \Phi(1, x) \) has a fixed point.

**Remark.** Theorem 11.6 in [5] is, in fact, proved for the case \( c = 0 \). However, let us observe that the assumptions of Theorem 11.6 are fulfilled for the map \( \Phi(\tau, x) = \Phi(\tau, x + c) - c \), whenever \( \Phi \) satisfies the assumptions of Theorem 8. To see this, we first check that \( \Phi \) is completely continuous. Let \( B \subset [0, 1] \times X \) be a bounded set, then \( B' = \{ (\tau, x + c) \text{ s.t. } (\tau, x) \in B \} \) is also a bounded set with the property \( \Phi(B) = \Phi(B') - c \). Therefore, \( \Phi \) is completely continuous if and only if \( \Phi \) is completely continuous. Next, it holds that \( \Phi(0, x) = 0 \) for all \( x \in X \). It remains to note that \( x \) is a fixed point of the map \( \Phi(\tau, \cdot) \) if and only if \( x + c \) is a fixed point of the map \( \Phi(\tau, \cdot) \).
Now we are ready to prove the main result of Section 2 which is the existence and uniqueness theorem for non-local initial–boundary value problem (2)-(4).

**Theorem 9** (Existence and uniqueness for initial–boundary value problem). Let (A1)–(A11) hold. Then, there exists a $C^{1+\frac{2+\beta}{2}}(\overline{F_T})$-solution to non-local initial–boundary value problem (2)-(4). If, in addition, (A12) holds, then this solution is unique.

**Proof.** Existence. For each $\tau \in [0, 1]$, consider the initial–boundary value problem

$$
\begin{aligned}
\partial_t u - \sum_{i,j=1}^n (\tau a_{ij}(t, x, u) + (1-\tau)\delta_{ij}) \partial_{x_i x_j}^2 u + (1-\tau)\Delta \varphi &= 0, \\
u(t, x, u, \partial_x u, \vartheta, \partial_u) &= 0.
\end{aligned}
$$

(36)

where $u$, $u_x$, and $\vartheta$ are evaluated at $(t, x)$. In the above equation, we freeze $u \in C^{1,2}(\overline{F_T})$ whenever it is in the arguments of the coefficients $a_{ij}(t, x, u)$, $\nu(t, x, u, \partial_x u, \vartheta, \partial_u)$, and consider the following linear initial–boundary value problem with respect to $v$:

$$
\begin{aligned}
\partial_t v^k - \sum_{i,j=1}^n (\tau a_{ij}(t, x, u) + (1-\tau)\delta_{ij}) \partial_{x_i x_j}^2 v^k + (1-\tau)\Delta \varphi^k &= 0, \\
\nu^k(t, x, u, \partial_x u, \vartheta, \partial_u) &= 0, \\
v^k(0, x) = \varphi^k_0(x), &\quad v^k(t, x)|_{(\partial F)_T} = 0,
\end{aligned}
$$

(37)

where $v^k$, $\varphi^k_0$, and $a^k$ are the $k$th components of $v$, $\varphi_0$, and $a$, respectively. Remark that the assumptions of Theorem 5.2, Chapter IV in [8] (p. 320) on the existence and uniqueness of solution for linear parabolic PDEs are fulfilled for problem (37).

Indeed, the assumptions of Theorem 5.2 in [8] require that the coefficients of (37) are of class $C^{1+\frac{2+\beta}{2}}(\overline{F_T})$ for some $\beta \in (0, 1)$. This holds by (A8), (A9), and (A4). The assumption about the boundary $\partial F$ and the boundary function $\psi$ is fulfilled by (A4) and (A7). Finally, the compatibility condition on the boundary $\partial F$, required by Theorem 5.2, follows from (A11). Therefore, by Theorem 5.2 (p. 320) in [8], we conclude that there exists a unique solution $v^k(t, x)$ to problem (37) which belongs to class $C^{1+\frac{2+\beta}{2}}(\overline{F_T})$. Clearly, $v^k$ is also of class $C^{1,2}(\overline{F_T})$, and, therefore, for each $\tau \in [0, 1]$, we have the map $\Phi : C^{1,2}(\overline{F_T}) \to C^{1,2}(\overline{F_T})$, $\Phi(\tau, u) = v$. Note that, fixed points of the map $\Phi(\tau, \cdot)$, if any, are solutions of (36). In particular, fixed points of $\Phi(1, \cdot)$ are solutions to original problem (2)-(4).

To prove the existence of fixed points of the map $\Phi(1, \cdot)$, we apply the Leray–Schauder theorem (Theorem 8). Let us verify its conditions. First we note that if $\tau = 0$, then the PDE in (37) takes the form $\partial_t v^k - \Delta v^k + \Delta \varphi^k_0 = 0$. Therefore, it holds that $\Phi(0, u) = \varphi_0$ for all $u \in C^{1,2}(\overline{F_T})$. Let us prove that $\Phi$ is completely continuous. Suppose $B \subset [0, 1] \times C^{1,2}(\overline{F_T})$ is a bounded set, i.e., for all $(\tau, u) \in B$, it holds that $\|u\|_{C^{1,2}(\overline{F_T})} \leq \gamma_B$ for some constant $\gamma_B$ depending on $B$. By aforementioned Theorem 5.2 from [8] (p. 320), the solution $v_{\tau, u}(t, x) = \{v_{\tau, u}^k(t, x)\}_{k=1}^m$ to problem (37), corresponding to the pair $(\tau, u) \in B$, satisfies the estimate

$$
\|v_{\tau, u}\|_{C^{1+\frac{2+\beta}{2}}(\overline{F_T})} \leq \gamma_1(\|\varphi(0, t, x, u, \partial_x u, \vartheta, \partial_u(t, x))\|_{C^{1,2}(\overline{F_T})} + \|\varphi_0\|_{C^{2+\beta}(\overline{F_T})}),
$$

where the first term on the right-hand side is bounded by (A8), (A9), and by the boundedness of $\|u\|_{C^{1,2}(\overline{F_T})}$ for all $(\tau, u) \in B$. Moreover, the bound for this term depends only on $\gamma_B$ and $K$ (where $K$ is the constant defined in Remark 5). This implies that $\|v_{\tau, u}\|_{C^{1+\frac{2+\beta}{2}}(\overline{F_T})}$ is bounded by a constant that depends only on $K$, $\gamma_B$, $\gamma_1$, and $\gamma_B$. By the definition of the norm in $C^{1+\frac{2+\beta}{2}}(\overline{F_T})$ (see (9)), the family $v_{\tau, u}$, $(\tau, u) \in B$, is uniformly bounded and uniformly continuous in $C^{1,2}(\overline{F_T})$.\]
By the Arzelá–Ascoli theorem, \( \Phi(B) \) is relatively compact, and, therefore, the map \( \Phi \) is completely continuous.

It remains to prove that there exists a constant \( K > 0 \) such that for each \( \tau \in [0,1] \) and for each \( C^{1,2}(\mathbb{F}_T) \)-solution \( u_\tau \) to problem (36), it holds that \( \|u_\tau\|_{C^{1,2}(\mathbb{F}_T)} \leq K \).

Remark that the coefficients of problem (36) satisfy (A1)–(A10). Hence, by Theorem 5, the Hölder norm \( \|u_\tau\|_{C^{1,2+\beta}(\mathbb{F}_T)} \), and, therefore, the \( C^{1,2}(\mathbb{F}_T) \)-norm of \( u_\tau \), is bounded by a constant depending only on \( M, \hat{M}, \hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{K}_\xi, \xi, T, \lambda_n(\mathbb{F}), \hat{L}_E, \|\varphi_0\|_{C^{2+\beta}(\mathbb{F})}, \) and on the \( C^{2+\beta} \)-norms of the functions defining the boundary \( \partial \mathbb{F} \).

Thus, the conditions of Theorem 8 are fulfilled. This implies the existence of a fixed point of the map \( \Phi(1, \cdot, \cdot) \), and, hence, the existence of a \( C^{1,2}(\mathbb{F}_T) \)-solution to problem (2)-4. Further, by Theorem 5, any \( C^{1,2}(\mathbb{F}_T) \)-solution to problem (2)-4 is of class \( C^{1,2+\beta}(\mathbb{F}_T) \).

**Existence and uniqueness for the Cauchy problem**

In this subsection, we consider Cauchy problem (2)–(6). The results of the previous subsection will be used to prove the existence theorem for this problem.

Below, we formulate assumptions (A1')–(A12') needed for the existence and uniqueness theorem. Assumptions (A1')–(A3') are the same as (A1)–(A3) but \( \mathbb{F} \) should be replaced with \( \mathbb{R}^n \), and \( C^{1,2}(\mathbb{F}_T) \) with \( C^{1,2}_b([0,T] \times \mathbb{R}^n) \). Also, \( \vartheta_u \) is defined for all \( u \in C^{1,2}_b([0,T] \times \mathbb{R}^n) \).

As before, the functions \( \mu(s), \hat{\mu}(s), \eta(s,r), P(s,r,t), \varepsilon(s,r) \) are continuous, defined for positive arguments, taking positive values, and non-decreasing (except \( \hat{\mu}(s) \)) with respect to each argument, whenever the other arguments are fixed; the function \( \hat{\mu}(s) \) is non-increasing. Further, \( \hat{\mathcal{R}}, \hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2, \) and \( \hat{\mathcal{R}}_3 \) are defined as follows

\[
\hat{\mathcal{R}} = [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathbb{E}; \quad \hat{\mathcal{R}}_1 = [0,T] \times \mathbb{R}^n \times \mathbb{R}^m; \\
\hat{\mathcal{R}}_2 = [0,T] \times \mathbb{R}^n \times \{u \leq C_1\} \times \{|p| \leq C_2\} \times \{|w|_E \leq C_3\}; \\
\hat{\mathcal{R}}_3 = [0,T] \times \{|x| \leq C_1\} \times \{|u| \leq C_2\} \times \{|p| \leq C_3\} \times \{|w|_E \leq C_4\},
\]

where \( C_1, C_2, C_3, C_4 > 0 \) are arbitrary constants. Assumptions (A4')–(A12') read:

**A4'** The initial condition \( \varphi_0 : \mathbb{R}^n \to \mathbb{R}^m \) is of class \( C^{2+\beta}_b(\mathbb{R}^n) \), \( \beta \in (0,1) \).

**A5'** For all \( (s, x, u, p, w) \in \hat{\mathcal{R}}, \)

\[
|a_1(t, x, u, p, w)| \leq \eta(|u|, \|w\|_E)(1 + |p|), \\
|a(s, x, u, p, w)| \leq (\varepsilon(|u|, \|w\|_E) + P(|u|, \|w\|_E, |p|))(1 + |p|)^2,
\]
where $\lim_{r \to \infty} P(s, r, q) = 0$ and $2(s + 1)\varepsilon(s, r) \leq \hat{\mu}(s)$.

(A6) $\partial_{x}a_{ij}, \partial_{u}a_{ij}, \partial_{tu}a_{ij}, \partial_{xx}a_{ij}, \partial_{ux}a_{ij},$ and $\partial_{uu}a_{ij}$ exist and are continuous on $\overline{\mathcal{R}}_{1}$; moreover, max $\{ |\partial_{x}a_{ij}(t, x, u)|, |\partial_{u}a_{ij}(t, x, u)| \} \leq \hat{\mu}(|u|)$.

(A7) $\partial_{x}a, \partial_{u}a, \partial_{tu}a, \partial_{uu}a, \partial_{x}u, \partial_{uu}u$, and $\partial_{u}u$ exist and are bounded and continuous on regions of form $\overline{\mathcal{R}}_{2}$; $a$ and $a_{i}$ are $\beta$-Hölder continuous in $x$ and locally Lipschitz in $w$ with the Hölder and Lipschitz constants bounded in regions of form $\overline{\mathcal{R}}_{2}$. 

(A8) The same as (A9) but valid for any bounded domain $\mathbb{F}$. 

(A9) The same as (A10) but valid for any bounded domain $\mathbb{F}$.

(A10) For any bounded domain $\mathbb{F} \subset \mathbb{R}^{n}$, for some $\alpha \in (0, \beta)$, the bound for $|\partial_{u}u|^{\alpha}_{\mathbb{F}}$ on $\mathbb{F}_{T}$ is determined only by the bounds for $|u|_{x}$ and $\partial_{x}u$, and the bound for $|\partial_{u}u|^{\alpha}_{\mathbb{F}}$ is determined only by the bound for $\partial_{x}u$.

(A11) For all $u, u' \in C^{1,2}_{b}([0, T] \times \mathbb{R}^{n})$, representation (35) holds with $\tilde{\partial}_{x}: [0, T] \times \mathbb{R}^{n} \rightarrow E$, $\tilde{\partial}_{u}: [0, T] \times \mathbb{R}^{n} \rightarrow E$, and $\tilde{\partial}_{tu}: [0, T] \times \mathbb{R}^{n} \rightarrow E$, defined for each $v \in C^{1,2}_{b}(\mathbb{R}^{n} \times [0, T])$ and satisfying the inequality $\sup_{[0, T] \times \mathbb{R}^{n}} \| \tilde{\partial}_{x}v \| E \leq L_{E} \sup_{[0, T] \times \mathbb{R}^{n}} |v|$ for all $t \in (0, T]$; $s_{u, u', u_{x}, u_{x}}^{\alpha}(t, x)$ in (35) are bounded, continuous, and $\alpha$-Hölder continuous in $x$.

(A12) $\partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}, \partial_{x}a_{ij}$, $\partial_{x}a_{ij}$ exist and are bounded and continuous on regions of form $\overline{\mathcal{R}}_{2}$; and, moreover, $\alpha$-Hölder continuous in $x, u, p, w$, for some $\alpha \in (0, 1); \partial_{x}a$ and $\partial_{x}a_{i}$ are locally Lipschitz in $w$. Furthermore, all the Lipschitz constants are bounded over regions of form $\overline{\mathcal{R}}_{2}$, and all the Hölder constants are bounded over regions of form $\overline{\mathcal{R}}_{3}$.

Assumptions (A11)–(A12) are required only for the proof of uniqueness. Unlike initial–boundary value problems, we do not prove a maximum principle for Cauchy problems. The uniqueness result for problem (2)-(6) follows from the possibility to solve linear parabolic systems via fundamental solutions.

Theorem 10 below is one of our main results.

**Theorem 10 (Existence and uniqueness for the Cauchy problem).** Let (A1'–(A10')) hold. Then, there exists a $C^{1,2}_{b}([0, T] \times \mathbb{R}^{n})$-solution to non-local Cauchy problem (2)-(6) which, moreover, belongs to class $C^{1, \frac{\beta}{2} + \alpha}_{b}([0, T] \times \mathbb{R}^{n})$ for some $\alpha \in (0, \beta)$. If, additionally, (A11') and (A12') hold, then this solution is unique.

**Proof.** Existence. We employ the diagonalization argument similar to the one presented in [8] (p. 493) for the case of one equation. Consider PDE (2) on the ball $B_{r}$ of radius $r > 1$ with the boundary function

$$
\psi(t, x) = \begin{cases} 
\varphi_{0}(x)\xi(x), & x \in \{ t = 0 \} \times \partial B_{r}, \\
0, & (t, x) \in [0, T] \times \partial B_{r},
\end{cases}
$$

where $\xi(x)$ is a smooth function such that $\xi(x) = 1$ if $x \in B_{r-1}$, $\xi(x) = 0$ if $x \notin B_{r}$; further, $\xi(x)$ decays from 1 to 0 along the radius on $B_{r} \setminus B_{r-1}$ in a way that $\xi^{(l)}(x)$, $l = 1, 2, 3$, does not depend on $r$ and are zero on $\partial B_{r}$. Let $u_{r}(t, x)$ be the $C^{1, \frac{\beta}{2} + 2 + \alpha}_{b}(\overline{B}_{r+1})$-solution to problem (2)-(38) in the ball $B_{r+1}$ whose existence was established by Theorem 9. Remark, that since $u_{r}$ is zero on $\partial B_{r+1}$, it can be extended by zero to the entire space $\mathbb{R}^{n}$, and, therefore, $\partial_{x}u_{r}$ is well-defined. Moreover, by Theorem 1, on $B_{r+1}$ the solution $u_{r}$ is bounded by a constant $M$ that only depends on $T, L_{E}, \sup_{B_{r+1}} |\varphi_{0}|$, and the constants $c_{1}, c_{2}, c_{3}$ from (A3'). Next, by Theorem 2, the gradient $\partial_{x}u_{r}$ possesses a bound $M_{1}$ on $B_{r+1}$ which only depends on $M, M, \sup_{B_{r+1}} |\varphi_{0}|, \mu(M), \hat{\mu}(M), \hat{\mu}(M), \eta(M, M), \sup_{q \geq 0} P(M, q, M)$, and $\varepsilon(M, M)$. Thus, both bounds $M$ and $M_{1}$ do not depend on $r$. 


Remark that the partial derivatives and Hölder constants mentioned in assumption (A7') are bounded in the region $[0, T] \times \mathbb{R}^n \times \{|u| \leq M \} \times \{|p| \leq M_1 \} \times \{\|w\|_E \leq M \}$. Let $K$ be their common bound.

Fix a ball $B_R$ for some $R$. By Theorem 7, there exist $\alpha \in (0, \beta)$ and a constant $C > 0$, both depend only on $M$, $M_1$, $M$, $K$, and $\|\varphi_0\|_{C^{2+\beta}(\mathbb{R}^n)}$, such that $\|u_r\|_{C^{1+\frac{\alpha}{2}+\alpha}(0, T) \times \overline{B_R}} \leq C$ (remark that the distance between $\overline{B_R}$ and $\partial B_{R+1}$ equals one). Therefore, $\|u_r\|_{C^{1+\frac{\alpha}{2}+\alpha}(0, T) \times \overline{B_R}} \leq C$ for all $R > R$. It is important to mention that the constant $C$ does not depend on $r$. By the Arzelà-Ascoli theorem, the family of functions $u_r(t, x)$, parametrized by $r$, is relatively compact in $C^{1,2}([0, T] \times \overline{B_R})$. Hence, the family $\{u_r\}$ contains a sequence $\{u^{(i)}_r\}_{k=1}^\infty$ which converges in $C^{1,2}([0, T] \times \overline{B_R})$.

Further, we can choose a subsequence $\{u^{(i)}_r\}_{k=1}^\infty$ of $\{u^{(0)}_r\}_{k=1}^\infty$ with $r_k > R+1$ that converges in $C^{1,2}([0, T] \times \overline{B_{R+1}})$. Proceeding this way we find a subsequence $\{u^{(i)}_r\}_{k=1}^\infty$ with $r_k > R+l$ that converges in $C^{1,2}([0, T] \times \overline{B_{R+l}})$. The diagonal sequence $\{u^{(k)}_r\}_{k=1}^\infty$ converges pointwise on $[0, T] \times \mathbb{R}^n$ to a function $u(t, x)$, while its derivatives $\partial_t u^{(k)}_r$, $\partial_x u^{(k)}_r$, and $\partial^2 u^{(k)}_r$ converge pointwise on $[0, T] \times \mathbb{R}^n$ to the corresponding derivatives of $u(t, x)$. Therefore, $u(t, x)$ is a $C^{1,2}$-solution to problem (2)-(6).

Let us prove that $u \in C^{1,2}_b([0, T] \times \mathbb{R}^n)$. Note that $|u| \leq M$ and $|\partial_t u| \leq M_1$, where $M$ and $M_1$ are bounds for $|u_r|$ and, respectively $|\partial_t u_r|$, that are independent of $r$. By Theorem 7, $\|u\|_{C^{1+\frac{\alpha}{2}+\alpha}(0, T) \times \overline{B_R}} \leq C$, where the constants $\alpha$ and $C$ are the same as for $u_r$. Moreover, the above estimate holds for any ball $B_R$. Therefore, $\|u\|_{C^{1+\frac{\alpha}{2}+\alpha}(0, T) \times \mathbb{R}^n} \leq C$.

Uniqueness. Rewrite (2) in the form

$$\sum_{i,j=1}^n a_{ij}(t, x, u) \partial^2_{x,x_j} u + \tilde{a}(t, x, u, \partial_x u, \partial_t u) + \partial_t u = 0,$$

where $\tilde{a}(t, x, u, p, w) = a(t, x, u, p, w) + \sum_{i=1}^n a_i(t, x, u, p, w)p_i$ with $p_i$ being the $i$th column of the matrix $p$. As before, $u$, $\partial_t u$, $\partial_x u$, and $\partial^2 u$ are evaluated at $(t, x)$.

Suppose we have two $C^{1,2}$-solutions $u$ and $u'$ to Cauchy problem (39)-(6). Then $v = u - u'$ is a solution to

$$\begin{cases}
\partial_t u - \sum_{i,j=1}^n \tilde{a}_{ij}(t, x) \partial^2_{x,x_j} u + \tilde{a}(t, x, u, \partial_x u, \partial_t u) + \partial_t u = f(t, x),
\end{cases}$$

with the coefficients

$$\begin{align*}
\tilde{a}_{ij}(t, x) &= a_{ij}(t, x, u(t, x)), \\
A(t, x) &= -\sum_{i,j=1}^n \partial^2_{x,x_j} u(t, x) \int_0^1 d\lambda \partial_x a_{ij}(t, x, \lambda u(t, x)) + (1 - \lambda)u(t, x) \right)^T \\
+ \int_0^1 d\lambda \partial_x a(t, x, u(t, x), \lambda u(t, x)) + (1 - \lambda)u(t, x) \\
+ \sum_{i=1}^n \lambda \partial_{x_i} a(t, x, u(t, x), \lambda u(t, x)) + (1 - \lambda)\partial_{u(t, x)} a(t, x, u(t, x), \lambda u(t, x)) + (1 - \lambda)\partial_{\lambda u(t, x)} a(t, x, u(t, x), \lambda u(t, x)) + (1 - \lambda)\partial_{\lambda^2 u(t, x)} a(t, x, u(t, x), \lambda u(t, x)) + (1 - \lambda)\partial_{\lambda^3 u(t, x)} a(t, x, u(t, x), \lambda u(t, x)), \\
B(t, x) &= \int_0^1 d\lambda \partial_p a(t, x, u(t, x)) + (1 - \lambda)\partial_{u(t, x)} a(t, x, u(t, x)), \\
C(t, x) &= \int_0^1 d\lambda \partial_p a(t, x, u(t, x), \lambda u(t, x)) + (1 - \lambda)\partial_{\lambda u(t, x)} a(t, x, u(t, x)), \\
f(t, x) &= 0,
\end{align*}$$

where $\zeta_{u, u', u, u', \lambda}$ is defined by decomposition (35). Assumptions (A1'), (A6'), (A7'), and (A10')–(A12') imply the conditions of Theorems 3 and 6 in [4] (Chapter 9, pp. 256 and 260) on the existence and uniqueness of solution to a system of linear parabolic PDEs via the fundamental solution $G(t, x; \tau, z)$. Namely, the aforementioned
Theorems 3 and 6 imply that the function $v$ satisfies the equation
\[
v(t, x) = \int_0^t \int_{\mathbb{R}^n} G(t, x; \tau, z) C(\tau, z) \left( \hat{\theta}_v(\tau, z) \right) d\tau dz.
\]

Further, (A11') and (A12') imply the boundedness of $C(t, z)$ and $\hat{\theta}_v(\tau, z)$. Finally, taking into account the estimate $sup_{[0, t] \times \mathbb{R}^n} ||\hat{\theta}_v|| \leq L_E sup_{[0, t] \times \mathbb{R}^n} sup \{v\}$, as well as Theorem 2 in [4] (Chapter 9, p. 251) which provides an estimate for the fundamental solution via a Gaussian-density-type function, by Gronwall’s inequality, we obtain that $v(t, x) = 0$. Therefore, a $C_b^{1,2}$-solution to (2)-(6) is unique. \hfill $\square$

3. Fully-coupled FBSDEs with jumps

In this section, we obtain an existence and uniqueness theorem for FBSDEs with jumps by means of the results of Section 2.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space with the augmented filtration $\mathcal{F}_t$ satisfying the usual conditions. Assume that the filtration $\mathcal{F}_t$ is generated by the following two mutually independent processes: an $n$-dimensional standard Brownian motion $B_t$ and a Poisson random measure $N(t, \cdot)$ on $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_i)$, where $\mathbb{R}_i = \mathbb{R}^l - \{0\}$ and $\mathcal{B}(\mathbb{R}_i)$ is the $\sigma$-algebra of Borel sets. Further let $\hat{N}(t, A) = N(t, A) - \nu(A)$ be the associated compensated Poisson random measure on $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_i)$, and $\nu(A)$ be its intensity which is assumed to be a Lévy measure.

Fix an arbitrary $T > 0$ and consider FBSDE (1). By a solution to (1) we understand an $\mathcal{F}_t$-adapted quadruple $(X_t, Y_t, \tilde{Z}_t)$ with values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times L_2(\nu, \mathbb{R}_i^l \rightarrow \mathbb{R}^m)$ such that (1) is fulfilled a.s. The latter includes the existence of all stochastic integrals involved in (1). In particular, $\tilde{Z}_t$ is $\mathcal{F}_t$-predictable and such that $\mathbb{E} \int_0^T \int_{\mathbb{R}_i^l} |\tilde{Z}(t, y)|^2 \nu(dy) dt < +\infty$ and, furthermore, $\mathbb{E} \int_0^T |\tilde{Z}_t|^2 dt < +\infty$. Also remark that we implicitly assume the existence of the left limits for the pair $(X_t, Y_t)$, and, in fact, we will be interested in càdlàg versions of $(X_t, Y_t)$, so the aforementioned requirement would be automatically fulfilled.

Together with FBSDE (1), we consider the associated final value problem for the following partial integro-differential equation:

\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} \theta(t, x, \theta, \partial_x \theta \sigma(t, x, \theta), \partial_x \theta(t, x)) - \int_{\mathbb{R}_i^l} \varphi(t, x, \theta, y) \nu(dy) \\
+ \frac{1}{2} \text{tr}(\partial^2_{xx} \theta \sigma(t, x, \theta) \sigma(t, x, \theta)^\top) + g(t, x, \theta, \partial_x \theta \sigma(t, x, \theta), \partial_x \theta(t, x)) \\
+ \int_{\mathbb{R}_i^l} \theta(t, x)(y) \nu(dy) + \partial_t \theta = 0; \quad \theta(T, x) = h(x).
\end{cases}
\end{align*}
\]

In (40), $x \in \mathbb{R}^n$, and the equation is $\mathbb{R}^m$-valued. Further, $\theta$, $\partial_x \theta$, $\partial_t \theta$, and $\partial^2_{xx} \theta$ are everywhere evaluated at $(t, x)$ (we omit the arguments $(t, x)$ to simplify the equation). As before, $\partial_x \theta$ is understood as a matrix whose $(ij)$th component is $\partial_{x_i} \theta_{x_j}$, and the first term in (40) is understood as the multiplication of the matrix $\partial_x \theta$ by the vector-valued function following after it. Furthermore, $\text{tr}(\partial^2_{xx} \theta \sigma(t, x, \theta) \sigma(t, x, \theta)^\top)$ is the vector whose $i$th component is the trace of the matrix $\partial^2_{xx} \theta^i \sigma(t, x, \theta)^\top$. Finally, for any $\nu \in C_b([0, T] \times \mathbb{R}^n)$, we define the function

\[
\partial_x \theta(t, x) = \nu(t, x + \varphi(t, x, v(t, x), \cdot)) - \nu(t, x).
\]

By introducing the time-changed function $u(t, x) = \theta(T-t, x)$, we transform problem (40) to the following Cauchy problem:

\[
\begin{align*}
\begin{cases}
\begin{aligned}
\partial_t u \left\{ &\int_{\mathbb{R}_i^l} \varphi(t, x, u, y) \nu(dy) - \int_t^T g(t, x, u, \partial_x u \sigma(t, x, u), \partial_x u(t, x)) \\
&- \frac{1}{2} \text{tr}(\partial^2_{xx} u \sigma(t, x, u) \sigma(t, x, u)^\top) - \int_{\mathbb{R}_i^l} \theta_u(t, x)(y) \nu(dy) + \partial_t u = 0; \quad u(0, x) = h(x).
\end{aligned}
\end{cases}
\end{align*}
\]
In (42), \( \hat{f}(t, x, u, p, w) = f(T - t, x, u, p, w) \), and the functions \( \sigma, \varphi \), and \( \hat{g} \) are defined via \( \sigma, \varphi \), and, respectively, \( g \) in the similar manner. Furthermore, the function \( \vartheta_u \) is defined by (41) via the function \( \hat{\varphi} \) (however, we use the same character \( \vartheta \)).

Let us observe that problem (42) is, in fact, non-local Cauchy problem (2)-(6) if we define the coefficients \( a_{ij}, a_i, a \), and the function \( \vartheta_u \) by formulas (3), and assume that the normed space \( E = L^2(\nu, \mathbb{R}^m) \). In other words, formulas (3) embed the PIDE in (42) into the class of non-local PDEs considered in the previous section. Further, it will be shown that assumptions (B1)–(B8) below imply (A1')–(A12').

As before, \( \mu(s), \hat{\mu}(s), \hat{\mu}(s), P(s, r, t), \zeta(r), \) and \( \varepsilon(s, r) \) are continuous functions, defined for positive arguments, taking positive values, and non-decreasing (except \( \hat{\mu}(s) \)) with respect to each argument, whenever the other arguments are fixed; the function \( \hat{\mu}(s) \) is non-increasing. Further, \( \tilde{R}, \tilde{R}_1, \tilde{R}_2, \) and \( \tilde{R}_3 \) are regions defined as in the previous section with \( E = L^2(\nu, \mathbb{R}^m) \):

\[
\tilde{R} = [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times L^2(\nu, \mathbb{R}^m) \rightarrow \mathbb{R}^m; \\
\tilde{R}_1 = [0, T] \times \mathbb{R}^n \times \mathbb{R}^m; \\
\tilde{R}_2 = [0, T] \times \mathbb{R}^n \times \{ |u| \leq C_1 \} \times \{ |p| \leq C_2 \} \times \{ ||w||_\nu \leq C_3 \}; \\
\tilde{R}_3 = \mathbb{R}_T \times \{ |u| \leq C_1 \} \times \{ |p| \leq C_2 \} \times \{ ||w||_\nu \leq C_3 \},
\]

where, \( C_1, C_2, C_3 \) are constants, and \( || \cdot ||_\nu \) is the norm in \( L^2(\nu, \mathbb{R}^m) \).

We assume:

(B1) \( \mu(|u|)I \leq \sigma(t, x, u, \sigma(t, x, u, u)\nu(t, x, u, y) \leq \mu(|u|)I \) for all \( (t, x, u) \in \tilde{R}_1 \).

(B2) \( (t, x, u) \mapsto \varphi(t, x, u, \cdot) \) is a map \( \tilde{R}_1 \rightarrow L^2(\nu, Z \rightarrow \mathbb{R}^n) \), where \( Z \subset \mathbb{R}^l \) is a common support of the \( L^2 \)-functions \( y \mapsto \varphi(t, x, u, y) \), which is assumed to be of finite \( \nu \)-measure. Further, \( \partial_t \varphi, \partial_u \varphi \) exist for \( \nu \)-almost each \( y \); \( \partial_x \varphi, \partial_{ux} \varphi, \) and \( \partial_{uxu} \varphi \) exist w.r.t. the \( L^2(\nu, Z \rightarrow \mathbb{R}^n) \)-norm. Moreover, all the mentioned derivatives are bounded as maps \( \tilde{R}_1 \rightarrow L^2(\nu, Z \rightarrow \mathbb{R}^n) \).

(B3) There exist constants \( c_1, c_2, c_3 > 0 \) and a function \( \zeta : \tilde{R} \times \mathbb{R}^n \rightarrow (0, +\infty) \) such that for all \( (t, x, u, p, w) \in \tilde{R}, (t, x, u, p, w, 0) = 0 \) and

\[
(g(t, x, u, p, w), u) \leq c_1 + c_2 |u|^2 + c_3 ||w||^2 + \zeta(t, x, u, p, w, p^uw).
\]

(B4) The final condition \( h : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is of class \( C^{2+\beta}_b(\mathbb{R}^n), \beta \in (0, 1) \).

(B5) For all \( (t, x, u, p, w) \in \tilde{R}, \)

\[
\int_Z \varphi(t, x, u, y)\nu(dy) \leq \zeta(|u|); \\
|f(t, x, u, p, w)| \leq \eta(|u|, ||w||_\nu)(1 + |p|);
\]

\[
|g(t, x, u, p, w)| \leq (\varepsilon(|u|, ||w||_\nu) + P(|u|, ||p||_\nu)(1 + |p|)^2),
\]

where \( \lim_{r \rightarrow \infty} P(s, r, q) = 0 \) and \( 4(1 + s)(1 + \mu(s))\varepsilon(s, r) < \hat{\mu}(s) \).

(B6) There exist continuous derivatives \( \partial_x \sigma, \partial_u \sigma \) such that

\[
\max \{ ||\partial_x \sigma(t, x, u)||, ||\partial_u \sigma(t, x, u)|| \} \leq \mu(|u|).
\]

(B7) For any bounded domain \( \mathbb{F} \subset \mathbb{R}^n \) and for any \( u \in C^{0,1}_b(\mathbb{F}_T) \), it holds that

(1) \( D(t, x, y) > 0 \) for \( \lambda_{n+1} \) almost all \( (t, x, y) \in \mathbb{F}_T \times Z \) and (2) \( D \) is \( \mathbb{F}_T \). Further, \( \hat{\mathbb{F}} \) and \( D(t, x, y) = |\det [I + \partial_x \varphi(t, x, u, t, y)] + \partial_u \varphi(t, x, u, t, y)]|\partial_x u(t, x) \).

(B8) The functions \( \partial f, \partial g, \partial f, \partial g, \partial^2 f, \partial^2 g, \partial^2 f, \partial^2 g, \partial^2 f, \partial^2 g, \partial^2 f, \partial^2 g, \partial^2 f, \partial^2 g, \partial^2 f, \partial^2 g, \partial^2 f, \partial^2 g \) are bounded and continuous in regions of form \( \tilde{R}_2 \): the derivatives of group \( b \) are \( \alpha \)-Hölder continuous in \( x, u, p, w \) for some \( \alpha \in (0, 1) \), and all the Hölder constants are bounded over regions of form \( \tilde{R}_3 \). Further, \( f, g, \partial f, \) and \( \partial g \) are locally Lipschitz in \( w \), and all the Lipschitz constants are bounded over regions of form \( \tilde{R}_2 \).
Theorem 11 below is the existence and uniqueness result for final value problem (40) which involves a PIDE. It can be regarded as a particular case of Theorem 10 and is the main tool to show the existence and uniqueness for FBSDEs with jumps. In particular, it is shown that assumptions (A8')–(A12'), including decompositions (23), (35), and inequality (24), are fulfilled when \( \vartheta \) is given by (41).

**Theorem 11.** Let (B1)–(B8) hold. Then, final value problem (40) has a unique \( C_b^{1,2}([0,T] \times \mathbb{R}^n) \)-solution.

**Proof.** Since problem (40) is equivalent to problem (42), it suffices to prove the existence and uniqueness for the latter. As we already mentioned, introducing functions (3), letting the normed space \( E \) be \( L_2(\nu, \mathbb{R}_+^n \rightarrow \mathbb{R}^m) \), and defining \( \vartheta_u \) by (41), we rewrite Cauchy problem (42) in form (2)-(6).

Let us prove that (A1')–(A12') are implied by (B1)–(B8). Indeed, (B1) implies (A1'). Next, we note that by (B2), the function \( \varphi(t,x,u,\cdot) \) is supported in \( Z \) and \( \nu(Z) < \infty \). This implies (A2') since for any \( \lambda \geq 0 \) and for any \( u \in C_b([0,T] \times \mathbb{R}^n) \),

\[
\|e^{-\lambda t}\vartheta_u(t,x)\|_{\nu} \leq 2 \nu(Z) \sup_{[0,T] \times \mathbb{R}^n} |e^{-\lambda t}u(t,x)|.
\]

Further, (A3') follows from (B3) and (3), since for any \( u \in \mathbb{R}^m \), \( \int_2(w(y),u)\nu(dy) \leq \frac{1}{2} \|w\|^2 + \frac{\nu(Z)}{2} \|u\|^2 \). Next, by (B5) and (B1),

\[
|\hat{f}(t,x,u,p \vartheta(t,x,u),w)| \leq (\|p\|_{\nu}) (1 + |p|) \|

which, together with the inequality for \( \varphi \) in (B5), implies the first inequality in (A5'). The second inequality in (A5') follows, again, from (B5) and (B1) by virtue of the following estimates

\[
|\hat{g}(t,x,u,p \vartheta(t,x,u),w)| \leq (\|w\|_{\nu}) + \frac{P(|u|,\|w\|_{\nu},|p|\sqrt{\mu(|u|)})}{\mu(|u|)} (1 + |p|)^2,
\]

and

\[
\int_2(w(y)\nu(dy)) \leq \tilde{P}(\|w\|_{\nu},|p|)(1 + |p|)^2,
\]

where \( \tilde{\xi}(s,r) = 2\xi(s,r)(1 + \mu(s)), \tilde{P}(s,r,q) = 2P(s,s,p\sqrt{\mu(s)}) (1 + \mu(s)) \), and \( \tilde{P}(s,r) = \nu(Z)^{\frac{1}{2}} s(1 + r)^{-2} \). Further, (B6) and (B8) imply (A6'), (A7'), and (A12'). Remark, that (A7') is implied, in particular, by the fact that the function \( \nu(Z) \rightarrow \mathbb{R}^m \rightarrow \mathbb{R}^n \), \( w \rightarrow \int_2(w(y)\nu(dy) \) is Gâteaux-differentiable and Lipschitz.

It remains to verify assumptions (A8')–(A11'). Let us start with (A8'). First remark that if \( u \in C_b^{1,1}(\mathcal{F}_T) \), where \( \mathcal{F} \) is a bounded domain, then it can be extended by 0 outside of \( \mathcal{F} \) defining a bounded continuous function on \([0,T] \times \mathbb{R}^n \). Therefore, \( \vartheta_u(t,x) \) is well-defined on \([0,T] \times \mathbb{R}^n \) for any function \( u \) which is zero on \( \partial \mathcal{F} \). Further, note that by (B2), \( \vartheta_u(t,x) \) takes values in \( L_2(\nu, Z \rightarrow \mathbb{R}^m) \) for any \( u \in C_b^{1,2}([0,T] \times \mathbb{R}^n) \). Furthermore, (B2) implies that \( \partial_t \vartheta_u(t,x) \) and \( \partial_x \vartheta_u(t,x) \) exist in \( L_2(\nu, Z \rightarrow \mathbb{R}^m) \) and are expressed via \( \partial_t u, \partial_x u, \partial_t \varphi, \partial_x \varphi, \) and \( \partial_u \varphi \). Hence, (A8') is fulfilled.

Let us verify (A9'). Recall that (A9') is assumption (A10) from subsection 2.5 valid for any bounded domain \( \mathcal{F} \). Let \( u \in C_b^{0,1}(\mathcal{F}_T) \) and \( v(t,x) = (\Delta t)^{-1} (u(t',x) - u(t,x)) \) with \( t' = t + \Delta t \). The immediate computation implies decomposition (23) with

\[
\begin{aligned}
\hat{\vartheta}_v &= v(t,x + \varphi(t,x,u(t,x),\cdot)) - v(t,x), \\
\zeta_{u,u} &= \int_0^1 d\lambda \partial_x u(t',x + \lambda \Delta \varphi) \int_0^0 d\Lambda \partial_u \varphi(t,x + \Lambda \Delta \varphi) \int_0^1 d\lambda \partial_t \varphi(t + \lambda \Delta t,x + u(t',x),\cdot), \\
\xi_{u,u} &= \int_0^0 d\Lambda \partial_x u(t',x + \lambda \Delta \varphi) \int_0^1 d\lambda \partial_t \varphi(t + \lambda \Delta t,x + u(t',x),\cdot).
\end{aligned}
\]
where \( \Delta \hat{\varphi} = \hat{\varphi}(t', x, u(t', x), \cdot) - \hat{\varphi}(t, x, u(t, x), \cdot) \). Further, inequality (24) follows from (B8). Indeed, define the functions \( \Phi_{t,y}(x) = x + \hat{\varphi}(t, x, u(t, x), y) \) and \( \tilde{v}(t, x, y) = v(t, \Phi_{t,y}(x)) \) on \([0, T] \times \mathbb{R}^n \times Z\). By the definition (see (B8)), \( D(t, x, y) = |\det \partial_x \Phi_{t,y}(x)| \). We have

\[
(43) \int_{\mathbb{F}_Z^2} \left( \int_Z |\tilde{v}|^2(t, x, y)\nu(dy) \right)^2 dt dx \leq \int_{\mathbb{F}_Z^2} \left( \int_{\{y:|\tilde{y}|<|v|\}} |\tilde{v}|^2(t, x, y)\nu(dy) \right)^2 dt dx \\
+ \int_{\mathbb{F}_Z^2} \left( \int_{\{y:|\tilde{y}|>|v|\}} |\tilde{v}|^2(t, x, y)\sqrt{D(t, x, y)}\sqrt{D^{-1}(t, x, y)}\nu(dy) \right)^2 dt dx \\
\leq \nu(Z)^2 \int_{\mathbb{F}_Z^2} |v|^4 dx dt + \Lambda \nu(Z)^2 \int_Z \nu(dy) \int_0^\tau dt \int_{\{x:|\tilde{x}|<\alpha;\beta>0\}} |\tilde{v}|^4 dt dx \\
(43) \leq (\nu(Z)^2 + \Lambda \nu(Z)) \int_{\mathbb{F}_Z^2} |v|^4 dt dx,
\]

where \( \Lambda \) is the constant from (B8) depending on \( F \) and \( u \). The second integral in the third line is estimated as follows. First, we remark that since \( D(t, x, y) > 0 \), by Theorem 1.2 in [7] (p. 190), the map \( \Phi_{t,y} : \mathbb{R}^n \to \mathbb{R}^n \) is invertible. Therefore, we can transform this integral by the change of variable \( x = \Phi_{t,y}(x) \). Thus, inequality (43) implies (24), and (A9') is verified.

Further, (A10') is verified immediately by (41). To verify (A11'), we note that decomposition (35) holds with

\[
\begin{align*}
\tilde{\sigma}_v &= v(t, x, \hat{\varphi}(t, x, u(t, x), \cdot)) - v(t, x), \\
\tilde{\sigma}_{u,u_u} &= \int_0^\tau d\lambda \partial_x u(t, x + \lambda \hat{\varphi}) \int_0^\tau d\lambda \partial_y \hat{\varphi}(t, x, \lambda u(t, x) + (1 - \lambda) u(t, x), \cdot),
\end{align*}
\]

where \( v = u - u' \) and \( \delta \hat{\varphi} = \hat{\varphi}(t, x, u(t, x), \cdot) - \hat{\varphi}(t, x, u'(t, x), \cdot) \). By (B2), \( \tilde{\sigma}_{u,u_u} \) is bounded, continuous, and has a bounded derivative in \( x \). This verifies (A11').

Thus, we conclude, by Theorem 10, that there exists a unique \( C^1_{b,1/2}([0, T] \times \mathbb{R}^n) \)-solution to problem (42). \( \square \)

**Remark 9.** As in Theorem 10, assumptions (B1)–(B7) imply the existence of a \( C^1_{b,1/2} \)-solution to problem (40), and (B8) is required only for the proof of uniqueness.

**Remark 10.** We formulate Theorem 11 just as a result sufficient for the application to FBSDEs. However, we note that, by Theorem 10, the \( C^1_{b,1/2} \)-solution to problem (40) also belongs to class \( C^{1+\frac{2}{b},2+\alpha}_{b,1/2}([0, T] \times \mathbb{R}^n) \) for some \( \alpha \in (0, \beta) \).

Before we prove our main result (Theorem 12 below), which is the existence and uniqueness theorem for FBSDE (1), we state a version of Itô’s formula (Lemma 6) used in the proof of Theorem 12. We give the proof of the lemma since we do not know a reference for the time-dependent case.

**Lemma 6.** Let \( X_t \) be an \( \mathbb{R}^n \)-valued semimartingale with càdlàg paths of the form

\[
X_t = x + \int_0^t F_s ds + \int_0^t G_s dB_s + \int_0^t \int_Z \Phi_s(y)\tilde{N}(ds dy),
\]

where the \( d \)-dimensional Brownian motion \( B_t \) and the compensated Poisson random measure \( \tilde{N} \) are defined as above. Further, let \( Z \subseteq \mathbb{R}^d \) be such that \( \nu(Z) < \infty \), and \( F_t, G_t, \) and \( \Phi_t \) be stochastic processes with values in \( \mathbb{R}^n \), \( \mathbb{R}^{n \times d} \), and \( L_2(\nu, Z \to \mathbb{R}^n) \), respectively. Then, for a real-valued function \( \phi(t, x) \) of class \( C^1_{b,1/2}([0, T] \times \mathbb{R}^n) \), a.s., it holds that

\[
\phi(t, X_t) = \phi(0, x) + \int_0^t \partial_x \phi(s, X_s) ds + \int_0^t (\partial_x \phi(s, X_s), F_s) ds + \int_0^t (\partial_x \phi(s, X_s), G_s dB_s)
\]
\[ + \frac{1}{2} \int_0^t \text{tr} (\partial_{xx}^2 \phi(s, X_s) G_s G_s^T) \, ds + \int_0^t \int_Z \left[ \phi(s, X_s - + \Phi_s(y)) - \phi(s, X_s - ) \right] \tilde{N}(ds \, dy) \]

(44)

\[ + \int_0^t \int_Z \left[ \phi(s, X_s - + \Phi_s(y)) - \phi(s, X_s - ) - (\partial_x \phi(s, X_s - ), \Phi_s(y)) \right] \nu(dy) \, ds. \]

Remark 11. In the above lemma, we agree that \( X_{0-} = X_0 = x \).

Proof of Lemma 6. Let us first assume that the function \( \phi \) does not depend on \( t \). Applying Itô’s formula (see Theorem 33 in [15], p. 74), we obtain

\[ \phi(X_t) - \phi(x) = \int_0^t (\partial_x \phi(X_s), F_s) \, ds + \int_0^t (\partial_x \phi(X_s - ), dX_s) \]

(45)

\[ + \frac{1}{2} \int_0^t \text{tr} (\partial_{xx}^2 \phi(X_s) G_s G_s^T) \, ds + \sum_{0 < s \leq t} \left( \phi(X_s) - \phi(X_s - ) - (\partial_x \phi(X_s - ), X_s - X_s - ) \right). \]

Note that the last summand in (45) equals \( \int_0^t \int_Z \left[ \phi(X_s - + \Phi_s(y)) - \phi(X_s - ) - (\partial_x \phi(X_s - ), \Phi_s(y)) \right] N(ds \, dy) \). By the standard argument (see, e.g., [1], p. 256), we obtain formula (44) without the term containing \( \partial_x \phi(s, X_s) \).

Now take a partition of the interval \([0, t] \). Then, for each pair of successive points,\[ \phi(t_{n+1}, X_{t_{n+1}}) - \phi(t_n, X_{t_n}) = \left[ \phi(t_{n+1}, X_{t_n}) - \phi(t_n, X_{t_n}) \right] \]

(46)

\[ + \left[ \phi(t_{n+1}, X_{t_{n+1}}) - \phi(t_{n+1}, X_{t_n}) \right]. \]

The first difference on the right-hand side equals \( \int_{t_n}^{t_{n+1}} \partial_x \phi(s, X_s) \, ds \), while the second difference is computed by formula (45). Assume, the mesh of the partition goes to zero as \( n \to \infty \). Then, summing identities (46) and letting \( n \to \infty \), we arrive at formula (44). Indeed, the convergence of the stochastic integrals holds in \( L_2(\Omega) \) by Lebesgue’s dominated convergence theorem, implying the convergence almost surely for a subsequence. Further, in the term containing the time derivative \( \partial_x \phi \), we have to take into account that \( X_t \) has càdlàg paths. \( \square \)

Let \( S \) denote the class of processes \((x_t, y_t, z_t, \tilde{z}_t)\) with values in \( \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m \times n} \), and \( L_2(\nu, \mathbb{R}^d \to \mathbb{R}^m) \), respectively, such that \( x_t, y_t \), and \( z_t \) are \( \mathcal{F}_t \)-adapted, \( \tilde{z}_t \) is \( \mathcal{F}_t \)-predictable, and

\[ \sup_{t \in [0, T]} \left\{ \mathbb{E}|x_t|^2 + \mathbb{E}|y_t|^2 \right\} + \int_0^T \left( \mathbb{E}|z_t|^2 + \mathbb{E}||\tilde{z}_t||^2 \right) dt < \infty. \]

(47)

The main result of this work is the following.

Theorem 12. Assume (B1)–(B8). Then, there exists a solution \((X_t, Y_t, Z_t, \tilde{Z}_t)\) to FBSDE (1), such that \( X_t \) is a càdlàg solution to

\[ X_t = x + \int_0^t f(s, x_s, \theta(s, x_s), \partial_x \theta(s, x_s) \sigma(s, x_s, \theta(s, x_s)), \partial_x \theta(s, x_s) ) \, ds \]

(48)

\[ + \int_0^t \sigma(s, x_s, \theta(s, x_s)) \, dB_s + \int_0^t \int_{\mathbb{R}^d} \varphi(s, x_{s-}, \theta(s, x_{s-}), y) \tilde{N}(ds \, dy), \]

where \( \theta(t, x) \) is the unique \( C_{b, 1}^{2, 2}([0, T], \mathbb{R}^n) \)-solution to problem (40), whose existence was established by Theorem 11, and \( \theta_0 \) is given by (41). Furthermore, \( Y_t, Z_t, \text{ and } \tilde{Z}_t \) are explicitly expressed via \( \theta \) by the formulas

\[ Y_t = \theta(t, X_t), \quad Z_t = \partial_x \theta(t, X_t) \sigma(t, X_t, \theta(t, X_t)), \text{ and } \tilde{Z}_t = \partial_x \theta(t, X_{t-}). \]

Moreover, the solution \((X_t, Y_t, Z_t, \tilde{Z}_t)\) is unique in the class \( S \).
Proof. Existence. First we prove that SDE (48) has a unique càdlàg solution. Define $\hat{f}(t, x) = f(t, x, \theta(t, x), \partial_{\theta}(t, x)\sigma(t, x, \theta(t, x)), (t, x) = \sigma(t, x, \theta(t, x)), \text{and} \hat{\varphi}(t, x, y) = \varphi(t, x, \theta(t, x), y).$ With this notation, SDE (48) becomes

$$X_t = x + \int_0^t \hat{f}(t, X_s)ds + \int_0^t \hat{\sigma}(s, X_s)dB_s + \int_0^t \int_{\mathbb{R}^2} \hat{\varphi}(s, X_s, y)\tilde{N}(ds dy).$$

Note that since $\theta$ is of class $C_1^{1,2}([0, T] \times \mathbb{R}^n)$, (B1) and (B5) imply that $\hat{f}(t, x), \hat{\sigma}(t, x),$ and $\int \hat{\varphi}(t, x, y)\nu(dy)$ are bounded. Further, (B6) implies the boundedness of $\partial_x \hat{\sigma}(t, x)$, while (B1), (B2), (B6), and (B8) imply the boundedness of $\partial_x \hat{f}(t, x).$ Furthermore, (B2) implies the boundedness of $\partial_x \int \hat{\varphi}(t, x, y)\nu(dy)$, therefore, the Lipschitz condition and the linear growth conditions, required for the existence and uniqueness of a càdlàg adapted solution to (50) (see [1]), Theorem 2.6.9, p. 374, are fulfilled. By Theorem 2.6.9 in [1] (more precisely, by its time-dependent version considered in Exercise 2.6.10, p. 375), there exists a unique càdlàg solution $X_t$ to SDE (50).

Further, define $Y_t, Z_t,$ and $\tilde{Z}_t$ by formulas (49). Applying Itô’s formula (Lemma 6) to $\theta(t, X_t)$, we obtain

$$\theta(t, X_t) = \theta(T, X_T) - \int_t^T \partial_x \theta(s, X_{s-})\sigma(s, X_{s-}, \theta(s, X_{s-})) dB_s$$

$$- \int_t^T \left\{ \partial_x \theta(s, X_s)f(s, X_s, \theta(s, X_s), \partial_{\theta}(s, X_s)\sigma(s, X_s, \theta(s, X_s)), \theta(s, X_s)) \right. \right.$$

$$+ \partial_x \theta(s, X_s) \int_{\mathbb{R}^2} \varphi(s, X_s, \theta(s, X_s), y)\nu(dy) + \partial_{\theta}(s, X_s) \left. \right. \right.$$

$$+ \frac{1}{2} \text{tr} [\partial_{xx} \theta(s, X_s)\sigma(s, X_s, \theta(s, X_s))\sigma(s, X_s, \theta(s, X_s))^{T}] + \int_{\mathbb{R}^2} \partial_{\theta}(s, X_s)(y)\nu(dy)] ds$$

$$- \int_t^T \int_{\mathbb{R}^2} \partial_{\theta}(s, X_{s-})(y)\tilde{N}(ds dy).$$

Since $\theta(t, x)$ is a solution to PIDE (40), then $Y_t, Z_t,$ and $\tilde{Z}_t$, defined by (49), are solution processes for the BSDE in (1). Furthermore, $Y_t$ and $Z_t$ are càdlàg, and $\tilde{Z}_t$ is left-continuous with right limits.

Uniqueness. Suppose $(X'_t, Y'_t, Z'_t, \tilde{Z}'_t)$ is another solution to FBSDE (1) satisfying (47). As before, $\theta(t, x)$ is the unique $C_1^{1,2}([0, T], \mathbb{R}^n)$-solution to PIDE (40). Define $(Y''_t, Z''_t, \tilde{Z}''_t)$ by formulas (49) via $\theta(t, x)$ and $X'_t$. Therefore, $(Y'_t, Z'_t, \tilde{Z}_t)$ and $(Y''_t, Z''_t, \tilde{Z}''_t)$ are two solutions to the BSDE in (1) with the process $X'_t$ being fixed. By the results of [17] (Lemma 2.4, p.1455), the solution to the BSDE in (1) is unique in the class of processes $(Y_t, Z_t, \tilde{Z}_t)$ whose square norm $\text{sup}_{t \in [0, T]} \mathbb{E}|Y_t|^2 + \int_0^T (\mathbb{E}|Z_t|^2 + \mathbb{E}\|\tilde{Z}_t\|^2)dt$ is finite. Without loss of generality, we can assume that $Y'_t$ is càdlàg by considering, if necessary, its càdlàg modification. Since both $Y'_t$ and $Y''_t$ are càdlàg, then $Y'_t = \theta(t, X'_t)$ for all $t \in [0, T]$ a.s. Further, $Z'_t = Z''_t$ and $\tilde{Z}_t = \tilde{Z}''_t$ as elements of $L_2([0, T] \times \Omega \rightarrow \mathbb{R}^m)$ and $L_2([0, T] \times \Omega \rightarrow E)$, respectively, where $E = L_2(\nu, \mathbb{F}_t \rightarrow \mathbb{R}^m)$. Therefore, $\lambda \otimes \mathbb{P}$-a.e. (the Lebesgue measure on $\mathbb{R}$), $f(t, X'_t, Y'_t, Z'_t, \tilde{Z}'_t) = f(t, X'_t, Y''_t, Z''_t, \tilde{Z}''_t)$ and consequently, $\int_0^T f(s, X'_s, Y'_s, Z'_s, \tilde{Z}'_s)ds = \int_0^T f(s, X'_s, Y''_s, Z''_s, \tilde{Z}''_s)ds$ for each $t \in [0, T]$ a.s. Hence, $X'_t$ is a solution to SDE (48). By uniqueness, $X'_t = X_t.$ Thus, the quadruple $(X_t, Y_t, Z_t, \tilde{Z}_t)$ is unique in the class $S.$

Remark 12. Finally, we would like to remark that the monotonicity assumption in [18] (Assumption H3.2, p. 436) is not fulfilled for FBSDE (1) if $m \neq n$. Indeed,
rewrite H3.2 in our notation and separate the terms on its left-hand side. We obtain
\[ (52) \quad -(\Delta g, \Delta x) + (\Delta f, \Delta y) + (\Delta \sigma, \Delta z) + \int_{\mathbb{R}_+^l} (\Delta \varphi(\xi), \Delta \tilde{z}(\xi)) \nu(d\xi) \leq -\beta_1|\Delta x|^2 - \beta_2(|\Delta y|^2 + |\Delta z|^2 + \|\Delta \tilde{z}\|^2), \]

where \( \beta_1, \beta_2 \geq 0 \) are constants, \( \Delta x = G(x-x'), \Delta y = G'(y-y'), \Delta z = G'(z-z'), \Delta \tilde{z} = G'(\tilde{z} - \tilde{z}'), \Delta \sigma = \sigma(t, x, y - \sigma(t, x', y'), \Delta \varphi = \varphi(t, x, y, \cdot) - \varphi(t, x', y', \cdot), \Delta g = g(t, x, y, z, \tilde{z}) - g(t, x', y', z', \tilde{z}'), \Delta f = f(t, x, y, z, \tilde{z}) - f(t, x', y', z', \tilde{z}'), \) and \( G \) is an \( m \times n \) matrix of full rank. The above inequality is assumed to be fulfilled for all \( (t, x', y', z', \tilde{z}') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times L_2(\nu, \mathbb{R}_+^l \to \mathbb{R}^m). \)

Let \( m < n \). Then, according to [18], \( \beta_2 > 0 \). Consider (52) in the case \( x = x', y = y', z \neq z' \). Then, the left-hand side of (52) equals zero while the right-hand side is negative.

If \( m > n \), we have to additionally assume that \( \sigma(t, x, y) \) explicitly depends on \( y \). Consider the case \( x = x', \tilde{z} = \tilde{z}', y - y' \in \ker G_T, z \neq z' \). Then, (52) implies
\[ (G(\sigma(t, x, y) - \sigma(t, x, y')), z - z') \leq 0. \]

Since \( \ker G = \{0\} \), this inequality cannot be fulfilled for all \( y, y', z, z' \) as above.

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