On the relation between nuclear and nucleon Structure Functions and their moments

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Calculations of nuclear Structure Functions (SF) $F_A^k(x, Q^2)$ routinely exploit a generalized convolution, involving the SF for nucleons $F^N_k$ and the linking SF $f^{P,N,A}$ of a fictitious nucleus, composed of point-particles, with the latter usually expressed in terms of hadronic degrees of freedom. For finite $Q^2$ the approach seemed to be lacking a solid justification and the same is the case for recently proposed, effective nuclear parton distribution functions (pdf), which exactly reproduce the above-mentioned hadronically computed $F_A^k$. Many years ago Jaffe and West proved the above convolution in the Plane Wave Impulse Approximation (PWIA) for the nuclear components in the convolution. In the present note we extend the above proof to include classes of nuclear Final State Interactions (FSI). One and the same function appears to relate parton distribution functions (pdf) in nuclei and nucleons, and SF for nuclear targets and for nucleons. That relation is the previously conjectured one, with an entirely different interpretation of $f^{P,N,A}$. We conclude with an extensive analysis of moments of nuclear SF based on the generalized convolution. Characteristics of those moments are shown to be quite similar to the same for a nucleon. We conclude that the above evidences asymptotic freedom of a nucleon in a medium and not of a composite nucleus.

I. INTRODUCTION.

This note concerns two related topics. The first is a generalized convolution, involving Structure Functions (SF) $F_A^k$ and $F^N_k$, which compose cross sections for inclusive scattering of unpolarized leptons from composite targets $A$ and for a nucleon. The second one deals with implications of the above for moments of $F_A^k$.

Standard approaches employing hadronic degrees of freedom have used generalized convolutions of the form

$$F^A = f^A * F^N$$

(1.1)

$$F_k^A(x, Q^2) = \sum_a \int_x^A d_z \frac{dz}{z^2-k} f^{a,A}(z, Q^2) F_k^a \left(\frac{x}{z}, Q^2\right)$$

(1.2)

$$F_k^{(N)}(x, Q^2) = \frac{Z F_p^p + N F_n^p}{A} = \frac{1}{2} \left[1 - \frac{\delta N}{A}\right] F_k^p + \frac{1}{2} \left[1 + \frac{\delta N}{A}\right] F_k^n$$

(1.3)

The involved SF depend on the squared 4-momentum transfer $q^2 = -Q^2 = -(|q|^2 - \nu^2)$ and on the Bjorken variable $x$ in terms of the nucleon mass $M$ with support $0 \leq x = Q^2/2M\nu \leq M_A/M \approx A$.

Eq. (1.2) decomposes $F_k^A$ into contributions from various constituents $'a'$, such as nucleons, virtual bosons, etc. For the kinematic region of our main interest, $x \gtrsim 0.2$, it suffices to retain only nucleons, or more precisely, the averaged nucleon with SF $F_k^{(N)}$, Eq. (1.3), obtained by weighting $F_{p,n}^a$ with $Z:N: \delta N/A$ is the relative neutron excess.

Within the framework of hadron dynamics, the convolution (1.3) can be proven in the PWIA [1]. In that approximation the linking function $f$ in Eq. (1.3), which in general is the SF of a fictitious nucleus composed of point-nucleons, is approximated by $f^{P,N,A} \rightarrow f^{PWIA}$, with the latter related to the spectral function of the knocked-out nucleon in the target [2]. For finite $Q^2$, Eq. (1.3) stood as a conjecture.

The same is the case for an alternative, non-perturbative Gersch-Rodriguez-Smith (GRS) approach [3], which has originally been formulated for a non-relativistic system of point-particles [4]. It has subsequently been extended to systems of composite constituents, such as quantum gases and liquids H$_2$, D$_2$, He etc. Since the energy scales for electronic, rotation-vibration modes, etc. differ appreciably, the Born-Oppenheimer approximation applies. As a consequence the SF (or 'linear response') of the composite system, is accurately given as a repeated regular convolution [14], involving SF of the translation of the centers of mass of inert molecules and of internal modes of each molecule

$$F^{\text{qu,gas}}(|q|, \nu) = \int d\nu_1 F^{\text{trans}}(|q|, \nu - \nu_1) \int d\nu_2 F^{\text{rot}}(|q|, \nu_1 - \nu_2) \ast \int d\nu_3 F^{\text{vib}}(|q|, \nu_2 - \nu_3) \ast ...$$

(1.5)
The next step in the development has been a covariant generalization of the above GRS theory, first for the SF of a system of point-particles, i.e. for \( f \) in \( \text{GRS} \). For increasing \( Q^2 \), internal degrees of freedom need ultimately to be included through \( F_k^{(N)} \), as described by the generalized convolution \( \text{GRS} \).

It stands to reason, that in general the convolution Eq. \( \text{GRS} \) for a composite nucleus rests on different energy scales for the participating modes. In fact, Eq. \( \text{GRS} \) was proven for a model with quarks clustered in nucleons, where the energy scale for internal excitations is much in excess of the same for \( NN \) forces \( \text{GRS} \). For higher, but not asymptotic \( Q^2 \), it seemed difficult to derive a covariant version and Eq. \( \text{GRS} \) has been considered a conjecture.

Calculations were based on data for \( F^p_2 \) and on some adopted \( F_k^N \), such that a calculation of \( F_k^A \) amounts to the same of \( f^{\text{GRS}} \). The latter can be evaluated using purely hadronic notions, such as single-nucleon spectral functions, nuclear density matrices of various orders, \( NN \) forward scattering amplitudes, etc. Support for the validity of Eq. \( \text{GRS} \) came mainly from the satisfactory description of a large body of inclusive scattering cross sections data for \( Q^2 \gtrsim 2.5 \text{GeV}^2 \). \( \text{GRS} \) also has its deficiencies. For example, \( F_k^{(N)} \) is taken to be the SF of a free averaged nucleon, which generally is off its mass-shell. In addition, Eq. \( \text{GRS} \) lacks explicit spin, iso-spin structure and in particular \( f \) is usually computed from spin, iso-spin averaged input.

Next we recall an alternative representation of nuclear SF, which uses nuclear parton distribution functions \( q_i^A(x, Q^2) \) for finite \( Q^2 \), which has to be computed from their nucleonic analogs \( q_i(x, Q^2) \). Those nuclear pdf are effective ones: we do not aim for an underlying theory, and in particular not for accounting of \( Q^2 \) dependence, compatible with evolution from a scale \( Q_0^2 \). The only requirement is the exact reproduction of \( F^A_2(x, Q^2) \), as computed in the hadronic representation \( \text{GRS} \).

The above requirement is nowhere sufficient to determine those pdf, and the apparent freedom is exploited by making two deliberate choices \( \text{GRS} \). An inessential one assumes \( F^A_2 \) to be the same combination of nuclear parton dfs, as \( F^{(N)}_2 \) is of nucleon ones, thus (for clarity we drop the \( x, Q^2 \) dependence in arguments)

\[
F^N_2 = \sum_i a_i x q_i = \frac{5x}{18} \left[ u_v + d_v + 2u + 2d - \frac{3s}{5A} (u_v - d_v + 2u - 2d) \right],
\]

\[
F^A_2 \equiv \sum_i a_i x q_i^A = \frac{5x}{18} \left[ u_v^A + d_v^A + 2u^A + 2d^A + \frac{3s^A}{5A} (u_v^A - d_v^A + 2u^A - 2d^A) \right]
\]

Next we chose to relate nuclear pdf of given species \( i \) to its analog for the averaged nucleon \( \langle N \rangle \), in precisely the same way as the hadronic representation \( \text{GRS} \) links nuclear and nucleon SF, thus

\[
q_{i/A}(x, Q^2) = \sum_a \int_0^A dz f_{a/A}(z, Q^2) q_{i/a}(\frac{x}{z}, Q^2)
\]

\[
\approx \int_0^A dz f_{N/A}(x, Q^2) q_{i/N}(\frac{x}{z}, Q^2)
\]

Eq. \( \text{GRS} \) does not mix flavors and uses a single linking function \( f_{N/A} = f^{P_N,A} \), independent of the species, whether valence, sea quarks or gluons. By construction the computed nuclear SF \( F^A_2 \) in the parton representation \( \text{GRS} \) are identical to their hadronic analog \( \text{GRS} \), provided the same input is used. In practice the input \( F^p_2 \) for the two differs (see Ref. \( \text{GRS} \) for a discussion). Between parenthood we add, that being the same SF as in Eq. \( \text{GRS} \), \( f \) carries along the above-mentioned deficiencies.

For both the hadron and pdf representations of \( F^A_2 \), there seemed to be missing a proof of Eqs. \( \text{GRS} \), as well an estimate of the lower limit \( Q_0^2 \), beyond which Eq. \( \text{GRS} \) is approximately valid. However, we recently stumbled upon 20 year old papers by Jaffe and West, which contain the basics of the desired proof \( \text{GRS} \). Judging from the lack of citations, even cognoscenti apparently overlooked or forgot the above papers, possibly because those were published in the proceedings of a summer school and of an AIP meeting. In the above publications the generalized convolution is derived, using a parton model as well as pQCD, both in the special case of the PWIA. In the following we generalize their results to include nuclear FSI.

Since the article of Jaffe is fairly self-contained, it will suffice to only cite some essentials, and in particular the central relation between forward \( \gamma \)-target scattering amplitudes and pdf. We then show that, although the inclusion of general FSI usually spoils their accommodation in a convolution for \( F^A_2 \), this is not the case for some nuclear FSI not involving partons. The above holds for instance for the Distorted Wave Impulse Approximation (DWIA) in the form given in Ref. \( \text{GRS} \). The same is the case for the GRS version for finite, relatively large \( Q^2 \) and those a \( f^{P_N,A} \) are precisely the ones contained in Eqs. \( \text{GRS} \) and \( \text{GRS} \), the above indeed completes the proof for, what previously was called a conjecture.
We conclude this note with the analysis of data on moments of high-$Q^2$ nuclear SF and present pQCD results for the above as done in the past for a proton.

II. DERIVATION OF NUCLEAR PARTON DISTRIBUTION FUNCTIONS.

We start with a proton and consider the forward scattering amplitude (fsa) $a(\gamma p)$ as a two-step process, where the proton emits a quark, which in turn absorbs the virtual photon (Fig. 1). That amplitude can be evaluated, given an expression for the current in terms of parton fields. For instance in a model with free parton fields, the result is \[ [12] \]

\[
F_2^p(x, Q^2) = x \sum_i e_i^2 q_i(x, Q^2) \tag{2.1}
\]

\[
q_i(x, Q^2) = \int \frac{d^4k}{(2\pi)^4} \delta \left( x - \frac{k.q}{p.q} \right) \chi_i(k, p) \tag{2.2}
\]

\[
q_i^{\text{scal}}(x) = \lim_{Q^2 \to \infty} q_i(x, Q^2) = \int \frac{d^4k}{(2\pi)^4} \delta \left( x - \frac{k^+}{p^+} \right) \chi_i(k, p), \tag{2.3}
\]

with the sum in Eq. (2.1) over quarks with charge $e_i$, $\chi_i = \chi_{i/p}$ in Eq. (2.2). Above one neglects spin and color: their inclusion is straight-forwarded, and is immaterial for the reasoning. To lowest order, i.e. in the PWIA, Eq. (2.2) is proportional to, what in nuclear physics parlance is called, the spectral function of a parton $i$ in the $p$. The $\delta$-function in the integrand of Eq. (2.2) selects the momentum fraction $x$ of the quark in the proton as determined by the 4-momenta $k, p, q$ of the quark, proton and virtual photon and the integrand in Eq. (2.2) holds for finite $Q^2$. Eq. (2.3) is the Bjorken limit of the above, in which case the argument of the $\delta$-function can be expressed in terms of the dominant light-cone components $k^+, p^+$: the resulting $q_i, F_2^p$ are pdf and SF in the scaling limit and depend only on $x$.

Of an entirely different nature is the $Q^2$-dependence generated by 'FSI' beyond the PWIA, coming from quarks which emit gluons, from gluon pair-production, triple gluon coupling, etc. Those add $\ln(Q^2)$ and $[1/Q^2]_n$ corrections to the above scaling limits for pdf and SF. For the present purpose it is irrelevant whether those ultimately derive from the Operator Product Expansion (OPE), or are calculated in pQCD by evolution.

Much of the above for a $p$ target, a neutron, or averaged $N$, holds also for a general target $A$: One can copy Eqs. (2.1), (2.2) and (2.3) replacing $p(N)$ by a composite target. However, it is awkward to deal with the spectral function of a parton in a nucleus, as is the fsa $\chi_{i/A}$ in the PWIA.

A more natural way is the evaluation of that amplitude upon insertion of an intermediate set of states for free nucleons and a fully interacting daughter nucleus. The product of the fsa $a(\gamma q_i)$ and $a(q_iN)$ is subsequently integrated over the intermediate momentum in order to form $a(\gamma N) \propto F^N$. The result, illustrated in Fig. 2, amounts to the following relation between the three involved fsa

\[
\chi_{i/A}(k, P) = \sum_a \int \frac{d^4p}{(2\pi)^4} \chi_{i/a}(k, p) \chi_{a/A}(p, P), \tag{2.4}
\]

where the two sub-amplitudes for $\gamma q$ and $N$-Sp are in the PWIA. The fsa $a(N$-Sp) in the PWIA is now related to the familiar spectral function of a nucleon in the target.

As in Eq. (2.2) for a $p$, one now projects out of each fsa the appropriate pdf, Eq. (2.4) is converted to

\[
q_{i/A}(x) = \sum_a \int_x^1 dz \int dp_0^2 f_{a/A}(z; p_0^2) q_{i/a} \left( \frac{x}{z}; p_0^2 \right) \tag{2.5}
\]

\[
\approx \int_x^1 dz f_{N/A}(z) q_{i/N} \left( \frac{x}{z} \right) \tag{2.6}
\]

Again Eq. (2.6) relates to several constituents/clusters, all of which may be off their mass-shell ($p_0^2 \neq M_i^2$), while in Eq. (2.5) one only retains nucleons, and in addition disregards those off-shell effects. Eq. (2.6) is clearly Eq. (1.3) in the Bjorken limit.

Next, upon inclusion of gluon emissions from quarks, nuclear pdf acquire $Q^2$-dependence, changing Eqs. (2.5) and (2.6) into

\[
q_{i/A}(x, Q^2) = \sum_a \int_x^1 dz \int dp_0^2 f_{a/A}(z, Q^2; p_0^2) q_{i/a} \left( \frac{x}{z}; Q^2; p_0^2 \right) \tag{2.7}
\]
\[ f_{N/A} \approx \int_x^A dz f_{N/A}(z, Q^2) q_{i/N} \left( \frac{x}{z}, Q^2 \right) \]  

(2.8)

Above \( f_{N/A} \) is the df of nucleons in the nucleus in the PWIA, while \( q_{i/N} \) are pdf beyond their scaling limit. Now just as gluon effects may be viewed as FSI on the fsa \( a(\gamma q_i) \) in the in the scaling limit, one should consider FSI pertinent to the nuclear part.

As emphasized by Jaffe, most classes of those FSI cannot be accommodated in a generalized convolution. However, the above does not hold for selected, nuclear FSI, generated by the interaction between the above-assumed free \( N \) and the spectator nucleus. An illustrative example is a ladder of Eqs. (3.2) are expressed in terms of the NS anomalous dimension \( \gamma \).

\[ Eqs. (2.8), (2.9) \] are manifestly the same as Eqs. (1.9), (1.3), but Eq. (2.3) is a choice, whereas Eq. (2.8) is the result of a derivation. Just as for the descriptions outlined in Sections I, II, one deals with one, species-independent \( f_{N/A} \), which relates df of partons in nuclei and nucleons without flavor mixing. We recall, that the above correspondence holds for those FSI, generated by the interaction between the above-assumed free nucleons, respectively nuclear pdf, and obtains

\[ F_2^A(x, Q^2) = \int_x^A dz f_{P_{N,A}}(z, Q^2) F_2^{(N)} \left( \frac{x}{z}, Q^2 \right) \]  

(2.9)

In a last step one takes the proper combinations (1.6), (1.7) of nucleon, respectively nuclear pdf, and obtains

\[ M^p(n, Q^2) = (M^p_2(n, Q^2) = ) \int_0^A dx x^{n-2} f_2^p(x, Q^2) \]  

(3.1)

For lowest twist (LO), non-singlets (NS) and large enough \( Q^2 \), asymptotic freedom of QCD predicts that moments of various rank raised to known powers are linear in \( \ln(Q^2) \). In terms of the strong coupling constant \( \alpha_c \)

\[ \frac{M^p(n, Q^2_0)}{M^p(n, Q^2)} \approx \left[ \frac{\alpha_c(Q_0^2)}{\alpha_c(Q^2)} \right]^{-d(n)} \]  

(3.2)

\[ \frac{\alpha_c(Q_0^2)}{\alpha_c(Q^2)} \approx 1 + \frac{\beta_0}{4\pi} \alpha_c(Q_0^2) \ln \left( \frac{Q^2}{Q_0^2} \right) + O \left( \alpha_c(Q_0^2)^2 \right) \]  

(3.3)

with \( Q_0 \) some scale and \( \beta_0(N_f) = 11 - 2N_f/3 \) in terms of the number of flavors \( N_f \). The exponents \( d^{NS}(n, N_f) \) in Eq. (3.2) are expressed in terms of the NS anomalous dimension \( \gamma_0^{NS}(n) \)

\[ d^{NS}(n, N_f) = \frac{\gamma_0^{NS}(n)}{2\beta_0(N_f)} \]  

(3.4)

\[ \gamma_0^{NS}(n) = \frac{8}{3} \left[ 1 - \frac{2}{n(n+1)} + 4 \sum_{2 \leq j \leq n} \frac{1}{j} \right] \]  

(3.5)

This concludes our generalization of the proofs of Jaffe and West on the 'factorization' of nuclear pdf and SF. The next Section deals with moments or Mellin transforms of nuclear SF in their obvious relation to \( F_2^{(N)}(x, Q^2) \).

### III. MOMENTS OF NUCLEAR STRUCTURE FUNCTIONS.

We recall the role played by moments \( M^p \) for a \( p \), for instance the Cornwall-Norton moments \( M^p \)

\[ M^p(n, Q^2) = (M^p_2(n, Q^2) = ) \int_0^A dx x^{n-2} f_2^p(x, Q^2) \]  

For lowest twist (LO), non-singlets (NS) and large enough \( Q^2 \), asymptotic freedom of QCD predicts that moments of various rank raised to known powers are linear in \( \ln(Q^2) \). In terms of the strong coupling constant \( \alpha_c \)
For conciseness we define

\[ S^A = [M^A]^{-1/d(n)} \]
\[ \mathcal{L}^A = \ln(S^A) \]  

and find in view of \(|\frac{\beta_n}{4\pi} \alpha_c(Q_0^2)| \ll 1\)

\[
S^p(n, Q^2) \approx S^p(n, Q_0^2) \left[ 1 + \frac{\beta_n}{4\pi} \alpha(Q_0^2) \ln \left( \frac{Q^2}{Q_0^2} \right) \right] \\
= c^p(n) \ln(Q^2) + b^p(n) \approx c^p(n) \ln(Q^2) + b^p 
\]

\[
\mathcal{L}^p(n, Q^2) \approx \mathcal{L}^p(n, Q_0^2) + \frac{\beta_n}{4\pi} \alpha(Q_0^2) \ln \left( \frac{Q^2}{Q_0^2} \right) \\
\approx \zeta^p(n) \ln(Q^2) + \eta^p(n) 
\]  

(3.7) (3.8)

Slopes \(c^p(n)\) for order \(n\) and the common intercept \(b^p\), are in principle determined by the scale or coupling constant, Eqs. 3.7, 3.8. The predictions 3.7, 3.8 have decades ago been checked against available proton data [17]. A recent JLab experiment, covering \(Q^2 \lesssim 4.5\text{GeV}^2\) and \(x \lesssim x_M(Q^2) \approx 0.8\) for \(Q^2 = 4.5\text{ GeV}^2\) allowed a more detailed analysis of the effects of higher twist components in the moments \(M^p(n, Q^2)\) [18].

There has been hardly any interest in moments of nuclear SF [9] for moderate [19] and large \(Q^2\). In a straightforward way one can generalize the above for any target \(A\), including for the averaged \(N\), which requires in addition to \(F_2^n\) knowledge of SF \(F_2^A\) for which there is no direct experimental information. We refer to Ref. 7 for the description of an indirect extraction of \(F_2^n\) or \(C(x, Q^2) = F_2^n(x, Q^2) / F_2^p(x, Q^2)\) from inclusive scattering data on various targets. Once obtained,

\[
F^{(N)}(x, Q^2) = \frac{Z + N C(x, Q^2)}{Z + N} F_2^p(x, Q^2) 
\]  

(3.9)

Since \(F_2^n \neq F_2^p\), the parameter functions \(c, b\) in 3.7 for a neutron will differ from those for a proton and the same is the case for the averaged \(N\), or for any target \(A\). In order to relate the latter two, one naturally exploits Eq. 1.9 and its Mellin transform \((m^{(N)} \equiv 1)\)

\[
M^A(n, Q^2) = m^A(n + 1, Q^2) M^{(N)}(n, Q^2) 
\]  

(3.10)

with

\[
M^A(n, Q^2) = \int_0^A dxx^{n-2} F_2^A(x, Q^2) \\
m^A(n, Q^2) = \int_0^A dxx^{n-2} f^{PN,A}(x, Q^2) \\
\sigma^A(n, Q^2) = [m^A(n, Q^2)]^{-1/d(n)} 
\]  

(3.11)

A remark on \(m^A\) is in order here. First, for \(Q^2 \gtrsim 20\text{ GeV}^2\) one may neglect FSI parts in the calculated SF \(f^{PN,A}\) from which \(m^A\) is computed. Next, as moments of a peaked, normalized \(f^{PN,A}\), \(m^A(n = 2, Q^2)\) has a minimum value 1, independent of \(A\) and \(Q^2\). For increasing \(n\), \(m^A(n)\) slowly increases, least for \(D\), He and about to the same measure for all \(A \gtrsim 12\). Those moments moreover carry the weak \(Q^2\)-dependence of \(f\) [20] and reach for \(n = 7\) the asymptotic limits \(\approx 1.027\) for \(D\) and \(\approx 1.082\) for medium and heavy A.

For use below we also briefly discuss the behavior of \(\sigma^A\), Eq. 3.11. For \(n\) between 2 and 7, the exponent \(d(n, N_f = 6)\) increases from 0.507 to 1.397 and causes \(\sigma^A\) for D to barely decrease from 1.000 to 0.997, and for \(A \gtrsim 12\), from 1.000 to \(\approx 0.931\). It suffices to illustrate (Fig. 4) the \(n\)-dependence of \(m^A(n, Q^2 = 20\text{ GeV}^2)\) for D and Fe, representative for a target with \(A \gtrsim 12\): The choice made for \(Q^2\) is irrelevant, since the \(Q^2\)-dependence of \(m^A\) is negligible for all practical purposes.

For target-independent anomalous dimensions, Eqs. 3.7, 3.11, 3.11 enable the generalizations of Eqs. 3.7 and 3.8

\[
S^{(N)}(n, Q^2) \approx c^{(N)}(n) \ln(Q^2) + b^{(N)}(n) 
\]
\[
\mathcal{L}^{(N)}(n, Q^2) \approx \zeta^{(N)}(n) \ln(Q^2) + \eta^{(N)}(n) 
\]  

(3.12) (3.13)
as well as
\[ S^A(n, Q^2) \approx c^A(n) \ln(Q^2) + b^A(n) \] (3.14)
\[ \mathcal{L}^A(n, Q^2) \approx \zeta^A(n) \ln(Q^2) + \eta^A(n) \] (3.15)

For given \( Q^2 \) we compared the separate expansions Eqs. (3.14), (3.15) and found that the logarithm of the first is closely the second.

From the above one infers, that slopes and intercepts \( c, b \) will differ for \( p, n \) and thus for \( \left\langle N \right\rangle \), while for general \( A \) one checks from Eq. (3.10) the following approximation
\[ c^A(2, Q^2) \approx \sigma^A(3, Q^2) \zeta^A(n) \approx c^A(n) \] (3.16)
\[ b^A(n, Q^2) \approx \sigma^A(n + 1, Q^2) \eta^A(n) \approx b^A(n) \] (3.17)
\[ \zeta^A(n) \approx \zeta^A(n) \] (3.18)
\[ \eta^A(n, Q^2) \approx \eta^A(n) + \ln\left(\sigma^A(n + 1, Q^2)\right) \approx \eta^A(n) \] (3.19)

Medium changes are governed by \( \sigma^A \), Eq. (3.10), target-to-target differences between slopes and intercepts for general targets and \( \left\langle N \right\rangle \) never exceed a few \% (see Fig. 4, the text after Eq. (3.11) and also point 5) below).

In what follows we distinguish between computed and experimental \( S^F \) and their moments, as well as for ratios \( \rho^A \), which derive from the Mellin transform (3.10) of the convolution (1.3). For \( M^{A,th} \) one has
\[ \rho^{A,th}(n, Q^2) = \frac{M^{A,th}(n, Q^2)}{m^A(n + 1, Q^2)} = M^A(n, Q^2) \] (3.20)
\[ [\rho^{A,th}(n, Q^2)]^{-1/d(n)} \approx c^A(n, Q^2) \ln(Q^2) + b^A(n, Q^2) \]
\[ \approx c^A(n, Q^2) \ln(Q^2) + b^A(n, Q^2) \] (3.21)
where we used Eq. (3.10). Clearly \( \rho^{A,th}(n, Q^2) = M^A(n, Q^2) \). For iso-singlet targets \( M^A(n, Q^2) \) does not depend on \( A \), whereas for \( I \neq 0 \), there is a weak \( A \)-dependence due to the small neutron excess \( \delta N/A \), Eq. (1.3).

Using the measured \( F^{A,dat} \), we consider the corresponding moments \( M^{A,dat} \), Eq. (3.11), and the ratios \( \rho^{A,dat} \), Eq. (3.18). In contrast to \( \rho^{A,th} \), the ratios \( \rho^{A,dat} \) do depend on \( f^{P,NA} \). A reliable computation of the latter and thus indirectly of \( m^A \), is presently only possible for \( A \leq 4 \).

Understanding the \( n \)-dependence of \( M^A(n, Q^2) \) relies on the knowledge, that all \( S^F \) \( x, Q^2 \) reach maxima for the smallest \( x \), then decrease with increasing \( x \) and become negligibly small beyond \( x \approx 0.8 \). The derived moments \( M^A(n, Q^2) \) of lowest order thus critically depend on the values of \( F^A \) \( x, Q^2 \) for very small \( x \) and on their accuracy. For growing \( n \), \( M^A(N) \) draws more and more from increasing \( x \). Since for medium \( x \), \( F^A \) have fallen by at least an order of magnitude from their maxima, it becomes increasingly difficult to reliably compute \( M^{A,dat}(n, Q^2) \) for large \( n \). We now mention results for \( N_f = 6 \).

1) \( M^{A,th}(n, Q^2) \) is barely \( A \)-dependent, and for various \( n \) slowly approaches its asymptotic \( Q^2 \) limit. In particular
\[ M^{A,th}(n = 2, Q^2 \to \infty) \to 5 \frac{N_f}{6(3N_f + 16)} = 0.1471 \] (3.22)

2) There is only meager experimental information available on \( F^A_2 \) for large \( Q^2 \). In spite of the fact, that second generation EMC ratios \( \mu^A = F^A_2/F^D_2 \) have been measured for large \( Q^2 \), the individual \( F^A_2 \) are only rarely available. We know of CERN NA-4 data on \( F^A_2, A = D, C \) [21, 22] and NA-2 for Fe [21, 23], which are not dense and do not extend over the entire required critical \( x \) range. To those we added a few data points from a JLab experiment [22], although the relevant \( Q^2 \) is low for a LO analysis. In detail: The NA-4 \( F^D_2 \) data show substantial scatter [21], which reflects in their moments and in \( S^D = [M^D]^{-1/d(n)} \). In spite of the above remarks, \( S^D \) for low \( n \) accurately follows the theoretical curves, but for increasing \( n \), data overshoots predictions up to \( \approx 15\% \) (Fig. 5).

It is instructive to make a similar comparison for a large body of D data, which have been parametrized by Arneodo et al. [22]. Very good agreement now obtains for \( n \leq 4 \). Discrepancies grow again with \( n \), but are definitely smaller than for the above mentioned data (Fig. 6). The cause is clearly few percent differences between the two data sets. The comparison also illustrates the effect of experimental scatter.

ii) The above data for \( F^D_2 \) lack values for small \( x \) [21, 22] without which one cannot compute low-order moments. We therefore took recourse to a previously proposed method, which is based on the observation that all \( F^A_2( x \approx 0.18, Q^2) \)
have a common value $\approx 0.30$, approximately independent of $A$ and $Q^2$ (see for instance Ref. [7]). If nuclear SF are well-known for $x_m > x_0$, one may extrapolate $F^A$ down to $x_0$ (Fig. 7).

iii) Before discussing Fe, we mention the result of a comparison of the high-$Q^2$ data of Ref. [21, 22, 23] for $F_2^A(x, Q^2)$ a) $F_2^D \approx F_2^C$ b) For small $x$ both D and C are a few % lower than $F_2^F$, but for $x \gtrsim 0.18$ the situation appears reversed and $F_2^{D,C} \approx (15-20)\%$ larger than for Fe. No similar behavior has been observed for lower $Q^2$. It is conceivable that the above Fe data have a normalization error of the order of (15-20)\% for $x \gtrsim 0.22$. In Fig. 8 we entered adjusted $S_{Fe,dat}$.

3) All $S^{A,th}(n, Q^2)$ intersect around $Q^2 \approx (0.6 - 1.0)\text{ GeV}^2$, which is reflected in the approximate equality of all $b^A(n)$. The exception is $n = 2$ for which $S^A$ has a very small slope, which may reflect the sensitivity of $M^A(n = 2, 3)$ to the small $x$-behavior of the nuclear SF. The fact that there is little $A$-dependence seems to exclude screening effects in $F_2^A$ for $x \lesssim 0.15$ as a cause, but quarks emitted by virtual bosons in the same small-$x$ range may contribute [20].

4) Results for $\rho^{A, dat}$ and for $\rho^{A,(N);th}$ are assembled in Table I. There is overall agreement for D and C and a deficiency for the non-adjusted Fe data.

5) We tested, whether the expansions (3.10) and (3.11) for $S^{A,th}$ and $\mathcal{L}^{A,th}$ and varying $n$, are approximately linear functions of $\ln(Q^2)$ with only weakly $A$-dependent coefficients. Table II confirms the above for our targets.

The small, but marked influence of $m^A$ is manifest in a comparison between $S^{I(n)}(n, Q^2)$ (for which $m^{I(n)} = 1$) and $S^A$. As Eq. (3.10) predicts, intercepts $b^{A}(n) \approx b^A(n)$ are quite similar, while for the slopes one has approximately $c^{(N)}(n) \approx c^A(n = 2)$.

6) Finally we exploited the fact, that $\eta^A(n) \gg \eta(n) \ln(Q^2)$ in Eq. (3.16). Consequently

$$\frac{\mathcal{L}^A(n, Q^2)}{\mathcal{L}^A(k, Q^2)} \approx \frac{\eta^A(n)}{\eta^A(k)} \left[ 1 + \left( \frac{\xi^A(n)}{\eta^A(n)} - \frac{\zeta^A(k)}{\eta^A(k)} \right) \ln(Q^2) \right]$$

$$\approx \frac{\eta^A(n)}{\eta^A(k)} \left( 1 + \frac{\xi^A(n)}{\eta^A(n)} \ln(Q^2) \right)$$

(3.20)

The bracketed form exceeds 1 by less than 10% and predicts only a weak $A$ and $Q^2$-dependence of the above ratios for pairs $n, k$, which gently grows with $n-k$. It thus suffices to illustrate the above for one species. We chose Fe and the pairs $n, k = (4, 2), (5, 3), (6, 4), (7, 5), (7, 3)$ (Fig. 9). The data are seen to follow the predictions (3.20) remarkably well, including the weak $\ln(Q^2)$ dependence in (3.20).

For a proton the linear dependence of $S^P$ on $\ln(Q^2)$ has been regarded as support for asymptotic freedom. With quite similar results for nuclei, we do not tend to conclude the same for composite systems. It is more likely that the de facto separation of nuclear and nucleon components ascribes the above to the propagation of asymptotic freedom of (on-shell) nucleons and allocates to the medium, controled modifications of slopes and intercepts (see Ref. [27] for a differently argued separation).

It is instructive to compare the above with an extention of the bag model of nucleons to nuclei with comparable inter-nucleon spacings and sizes of bags, which may overlap and cause conceptual complications. No such problems occur in the above interpretation of the convolution (1.8).

Finally we remark that the above analysis is complicated by the presence of color singlet contributions, which are coupled to those for gluons. Only for sufficiently high $n \gtrsim 4 - 5$ are those approximately decoupled [17], allowing an analysis of actual moments and not of the assumed non-singlets. This is also the reason, why we do not study medium effects on slopes and intercepts in greater detail.

IV. CONCLUSION.

The present note generalizes old work by Jaffe and West, who by means of a parton model and pQCD in the PWIA for large $Q^2$ proved that parton distribution functions and Structure Functions of composite targets and of nucleons are related by a generalized convolution. Their publications did not appear in the standard literature and have apparently been forgotten or disregarded. The present note is therefore in part an amende honorable to their work.

We first reviewed facets of the conjectured convolution for finite $Q^2$, working in both a hadronic and an effective nuclear pdf representation. In those we did not aim to check, whether the $Q^2$ dependence is actually reproduced by, or in agreement with evolution from a scale $Q_0^2$.

Next, we mentioned crucial points in the publications of Jaffe and West. Those are foremost the general relation between forward scattering amplitudes and parton distribution functions. Next we cited the decomposition of fsa $a(\gamma A)$ into the fsa $a(\gamma N)$ and $a(N, \text{spectator-nucleus})$. Jaffe and West studied those first in the PWIA and in the Bjorken limit, leading to the scaling results. Those have subsequently been supplemented by contributions, due to gluon emission by quarks, etc. which, as regards photon-parton scattering, extend results beyond the above limit.
We generalized the above and included classes of FSI between a nucleon, immediately emitted by a target and the remaining spectator nucleus. The LO expressions, relating nuclear and nucleon pdf, and consequently the same for Structure Functions, continue to be of the convolution type. Moreover, those are identical to the same, previously conjectured ones in the above hadron and effective pdf representations. That correspondence is a formal one: the interpretation of the two results is entirely different.

The existence of an 'ultimate' description does not imply a preference over an 'effective' one under all circumstances. It is not only relatively easy to compute the SF $f^{PN,A}$ from nuclear physics concepts than from pQCD, or to use data on nucleon SF, as opposed to a calculation of $F^{N}$: results from effective theories are frequently quite accurate.

The above is not at all specific for descriptions of nuclear SF, but holds for many 'effective' theories. A classical example is the inter-atomic interaction of the centers of the atoms in di-atomic molecules. The 'true' potential ought in principle to be derived from Schrödinger QM, which is extremely laborious, but in practice one uses Lennard-Jones or Morse potentials. Those do contain the essentials of the physics, including a short-range repulsion, which mimics the effect of the Pauli principle for overlapping electron configurations. The spectroscopy of di-atomic molecules, and the physics of gases and liquids of di-atomic molecules is accurately accounted for by effective dynamics.

The last part of this note concerns moments of nuclear SF. The behavior of moments $M^p$ of the SF $F^p_2$, specifically the linear dependence of $\langle S^p \rangle$ on $\ln (Q^2)$, has in the past been shown to be related to asymptotic freedom of QCD. Quite similar properties are shared by nuclear moments. However, rather than concluding that inclusive scattering data on nuclei supports asymptotic freedom for composite systems, we prefer a sober point of view. The formal factorization of $F^A_2$ (or the actual one of moments $M^A(n, Q^2)$) separates nucleonic and nuclear dependencies, without changing the required separation of parts with hard and soft $Q^2$-dependence as is the case of a proton. The observed $\ln(Q^2)$ behavior simply reflects the propagation of asymptotic freedom of isolated nucleons, with characteristic medium modifications of nucleon parameters.

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TABLE I: Ratios $\rho^{A,\text{dat}}(n, Q^2)$, Eq. 3.18, $A= D, C, Fe$ for $n = 2 - 7$ and a number of roughly common $Q^2$ values. Also shown are $\rho^{A,\text{th}}(n, Q^2)$ for the averaged $N$, pertinent to an iso-scalar nucleus and for Fe.

| target | $n$ | $\rho^{A,\text{dat}}(n, Q^2)$ | $Q^2$ | 3.5 | 17 | 35 | 50 | 72 |
|--------|----|-------------------------------|------|-----|----|----|----|----|
|        | 2  | 0.1548 | 0.1400 | 0.1372 | 0.1354 | 0.1315 |
| D      | 3  | 0.0401 | 0.0309 | 0.0298 | 0.0290 | 0.0276 |
|        | 4  | 0.0156 | 0.0104 | 0.0099 | 0.0096 | 0.0090 |
|        | 5  | 0.0073 | 0.0043 | 0.0040 | 0.0039 | 0.0036 |
|        | 6  | 0.0021 | 0.0010 | 0.0010 | 0.0009 | 0.0008 |
| C      | 2  | 0.1555 | 0.1400 | 0.1373 | 0.1351 | 0.1347 |
|        | 3  | 0.0398 | 0.0311 | 0.0291 | 0.0284 | 0.0281 |
|        | 4  | 0.0152 | 0.0111 | 0.0094 | 0.0091 | 0.0089 |
|        | 5  | 0.0069 | 0.0044 | 0.0037 | 0.0036 | 0.0035 |
|        | 6  | 0.0034 | 0.0023 | 0.0017 | 0.0016 | 0.0016 |
|        | 7  | 0.0017 | 0.0010 | 0.0008 | 0.0008 | 0.0008 |
| Fe     | 2  | 0.1499 | 0.1325 | 0.1290 | 0.1270 | 0.1267 |
|        | 3  | 0.0354 | 0.0276 | 0.0260 | 0.0253 | 0.0251 |
|        | 4  | 0.0121 | 0.0087 | 0.0080 | 0.0077 | 0.0076 |
|        | 5  | 0.0049 | 0.0034 | 0.0030 | 0.0029 | 0.0029 |
|        | 6  | 0.0021 | 0.0014 | 0.0013 | 0.0012 | 0.0012 |
|        | 7  | 0.0009 | 0.0007 | 0.0006 | 0.0006 | 0.0006 |

|        | $\langle N \rangle_{I=0}$ | $\rho^{A,\text{th}}(n, Q^2)$ | $\langle N \rangle_{Fe}$ | $\rho^{A,\text{th}}(n, Q^2)$ |
|--------|-----------------------------|-------------------------------|-----------------------------|-------------------------------|
|        | 2  | 0.1469 | 0.1414 | 0.1393 | 0.1380 | 0.1369 | 0.1468 | 0.1396 | 0.1380 | 0.1369 |
|        | 3  | 0.0376 | 0.0315 | 0.0296 | 0.0285 | 0.0275 | 0.0368 | 0.0308 | 0.0295 | 0.0288 | 0.0272 |
|        | 4  | 0.0149 | 0.0111 | 0.0100 | 0.0096 | 0.0092 | 0.0145 | 0.0109 | 0.0098 | 0.0094 | 0.0090 |
|        | 5  | 0.0073 | 0.0050 | 0.0044 | 0.0041 | 0.0038 | 0.0071 | 0.0048 | 0.0044 | 0.0041 | 0.0037 |
|        | 6  | 0.0041 | 0.0026 | 0.0022 | 0.0020 | 0.0019 | 0.0039 | 0.0025 | 0.0022 | 0.0020 | 0.0018 |
|        | 7  | 0.0025 | 0.0015 | 0.0012 | 0.0011 | 0.0010 | 0.0024 | 0.0014 | 0.0012 | 0.0011 | 0.0010 |

FIG. 1: The decomposition of the forward $\gamma p$ amplitude in the PWIA and its link to the quark-$p$ scattering amplitude.
FIG. 2: Same as Fig. 1 for a composite target. Inclusion of an intermediate set of free nucleon and spectator states, and recombination of terms (marked by dashed horizontal), leads to a generalized convolution of forward amplitudes $a(\gamma N)$ and $a(N-\text{Sp})$.

FIG. 3: Ladder of $N-\text{Sp}$ nucleus collisions, which are accommodated in a convolution, and an example of nuclear FSI which cannot.
FIG. 4: Moments $m^A(n, Q^2)$ and its characteristic power $\sigma^A(n, Q^2)$ Eq. $A=\text{D, Fe}; n = 2 - 7, Q^2 = 20\,\text{GeV}^2$. 

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- $m^D$
- $m^{\text{Fe}}$
- $\sigma^D$
- $\sigma^{\text{Fe}}$
FIG. 5: Characteristic powers of moments $S^D(n,Q^2)$, Eq. (3.6), for $D$ as function of $\ln Q^2$. Data points for underlying SF are from Ref. [21] $n$ increases for lines with increasing slopes.
FIG. 6: Same as Fig. 5 for parametrizations for the average of a vast body of D data.
FIG. 7: Same as Fig. 5 for C. Data are from Ref. [21, 24].
FIG. 8: Same as Fig. 5 for Fe for partly renormalized $F_2^{Fe}$ data.
FIG. 9: $\mathcal{L}^{F.e.th}(n, Q^2)$ versus $\mathcal{L}^{F.e.th}(k, Q^2)$, Eq. (3.20) for $(n, k) = (4, 2), (5, 3), (6, 4), (7, 5), (7, 3)$. Data points as in Fig. 7.
TABLE II: Expansion coefficients of $S^{(N)}_{A;th}(n, Q^2)$, Eqs. 3.12, 3.13, 3.14, 3.15 for $n = 2 - 7$, for $A = D, C, Fe$ compared with the same for the average nucleon $\langle N \rangle_{I=0}$.

| target | $n$ | $c^A(n)$ | $b^A(n)$ | $\zeta^A(n)$ | $\eta^A(n)$ |
|--------|-----|----------|----------|-------------|-------------|
| D      | 2   | 2.085    | 46.503   | 0.0426      | 3.840       |
|        | 3   | 9.937    | 73.035   | 0.1175      | 4.297       |
|        | 4   | 14.273   | 82.415   | 0.1444      | 4.421       |
|        | 5   | 17.717   | 87.177   | 0.1607      | 4.480       |
|        | 6   | 19.379   | 89.868   | 0.1731      | 4.512       |
|        | 7   | 21.111   | 91.332   | 0.1831      | 4.530       |
| C      | 2   | 2.284    | 46.017   | 0.0469      | 3.830       |
|        | 3   | 10.286   | 71.608   | 0.1232      | 4.278       |
|        | 4   | 14.690   | 79.540   | 0.1526      | 4.386       |
|        | 5   | 17.641   | 82.447   | 0.1723      | 4.425       |
|        | 6   | 19.875   | 82.875   | 0.1891      | 4.433       |
|        | 7   | 21.655   | 81.655   | 0.2049      | 4.420       |
| Fe     | 2   | 2.241    | 47.245   | 0.0449      | 3.856       |
|        | 3   | 10.404   | 73.705   | 0.1214      | 4.307       |
|        | 4   | 14.917   | 82.088   | 0.1505      | 4.418       |
|        | 5   | 17.937   | 85.366   | 0.1697      | 4.460       |
|        | 6   | 20.210   | 86.173   | 0.1856      | 4.471       |
|        | 7   | 22.002   | 85.414   | 0.2002      | 4.465       |
| $\langle N \rangle_{I=0}$ | 2   | 2.137    | 40.989   | 0.0443      | 3.725       |
|        | 3   | 9.918    | 49.888   | 0.1182      | 4.029       |
|        | 4   | 14.246   | 50.006   | 0.1443      | 4.093       |
|        | 5   | 17.205   | 48.929   | 0.1598      | 4.126       |
|        | 6   | 19.504   | 47.498   | 0.1711      | 4.144       |
|        | 7   | 21.379   | 46.007   | 0.1800      | 4.154       |