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AN ELLIPTIC PROBLEM WITH DEGENERATE COERCIVITY
AND A SINGULAR QUADRATIC GRADIENT LOWER ORDER TERM

Gisella Croce
Laboratoire de Mathématiques Appliquées du Havre
Université du Havre
25, rue Philippe Lebon
76063 Le Havre (FRANCE)

Abstract. In this paper we study a Dirichlet problem for an elliptic equation
with degenerate coercivity and a singular lower order term with natural growth
with respect to the gradient. The model problem is
\begin{equation}
\begin{cases}
-\text{div} \left( \frac{\nabla u}{1 + |u|^p} \right) + \frac{[\nabla u]^2}{|u|^p} = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
where $\Omega$ is an open bounded set of $\mathbb{R}^N$, $N \geq 3$ and $p, \theta > 0$. The source $f$
is a positive function belonging to some Lebesgue space. We will show that,
even if the lower order term is singular, it has some regularizing effects on the
solutions, when $p > \theta - 1$ and $\theta < 2$.

1. Introduction

In this paper we study the following problem:
\begin{equation}
\begin{cases}
-\text{div} \left( \frac{b(x)\nabla u}{1 + |u|^p} \right) + B\frac{[\nabla u]^2}{|u|^p} = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
where $\Omega$ is an open bounded set of $\mathbb{R}^N$, $N \geq 3$, $B, p > 0$ and $\theta > 0$. We assume
that $b : \Omega \to \mathbb{R}$ is a measurable function such that for some positive constants $\alpha$ and $\beta$
\begin{equation}
\alpha \leq b(x) \leq \beta \quad \text{for a.e. } x \in \Omega.
\end{equation}
Moreover $f$ is a positive function belonging to some Lebesgue space $L^m(\Omega)$, with
$m \geq 1$. We point out three characteristics of this problem: the operator $A(v) =
-\text{div} \left( \frac{b(x)v}{(1 + |v|^p)} \right)$ is defined on $H^1_0(\Omega)$ but is not coercive on this space when
$v$ is large, as proved in [10]. The lower order term has a quadratic growth with
respect to the gradient and is singular in the variable $u$. As we will see, existence
and summability of solutions to problem (1) depend on these features.

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distributional solution, singular lower order term, quadratic growth.
It is known that the degenerate coercivity has in some sense a bad effect on the summability of the solutions to problem

\[
\begin{aligned}
-\text{div} (a(x,u) \nabla u) = f & \quad \text{in } \Omega, \\
u = 0 & \quad \text{on } \partial \Omega,
\end{aligned}
\]  

as proved in [3]. There \( f \in L^m(\Omega) \) was not assumed to be positive, \( a : \Omega \times \mathbb{R} \to \mathbb{R} \) was a Carathéodory function such that \( \frac{\alpha}{(1 + |s|)^p} \leq a(x,s) \leq \beta \), for \( p \in (0,1) \) and \( \alpha, \beta > 0 \). Apart from the case where \( m > \frac{N}{2} \), the summability of the solutions is lower than the summability of the solutions to elliptic coercive problems. Indeed, in [3] it is shown that if \( \frac{N}{2} < m < \frac{N}{2} \) there exists a \( H^1_0(\Omega) \cap L^r(\Omega) \) distributional solution, with \( r = \frac{Nm(1-p)}{N-2m} \); if \( \frac{N}{2} < m < \frac{N}{2} \), there exists a \( W^{1,s}_0(\Omega) \) distributional solution, with \( s = \frac{Nm(1-p)}{N-m(1+p)} \). For \( p > 1 \) the authors prove a non-existence result for constant sources \( f \). Note that a bad effect on the regularity of the solutions appears even when the right hand side of (3) is an element of \( H^{-1}(\Omega) \), as \( -\text{div}(F) \), with \( F \in L^2(\Omega) \). As a matter of fact, in this case the solutions are in general not in \( H^1_0(\Omega) \) (see [13]).

The presence of lower order terms can have a regularizing effect on the solutions. In [3] and [4] three kinds of lower order terms are considered for elliptic problems with degenerate coercivity, with no restriction on \( p \). In the first paper the author analyses a lower order term defined by a Carathéodory function \( g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N \) with the following properties. There exists \( d \in L^1(\Omega) \), two positive constants \( \mu_1, \mu_2 > 0 \) and a continuous increasing real function \( h \) such that \( g(x,s,\xi)s \geq 0 \), \( \mu_1 |\xi|^2 \leq |g(x,s,\xi)| \) when \( |s| \geq \mu_2 \) and \( |g(x,s,\xi)| \leq d(x)h(|s|)|\xi|^2 \). It is proved that for a \( L^1(\Omega) \) source there exists a \( H^1_0(\Omega) \) distributional solution to

\[
\begin{aligned}
-\text{div} (a(x,u) \nabla u) + g(x,u,\nabla u) = f & \quad \text{in } \Omega, \\
u = 0 & \quad \text{on } \partial \Omega.
\end{aligned}
\]

This proves that the summability of the gradient of the solutions is much larger than that of the solutions of problem (3). It is even larger than the summability of the gradient of the solutions to elliptic coercive problems with \( L^1(\Omega) \) sources, which is \( L^s(\Omega) \) for every \( s < \frac{N}{N-1} \) (see [13] for example). We remark moreover that the lower order term gives the existence of a solution for \( p \geq 1; \) for these values of \( p, \) (3) has no solution.

In a previous article [13] we consider two kinds of lower order terms \( h(u) \). For \( h(u) = |u|^q-1 u \), with \( q > p+1 \), we establish the existence of a distributional solution \( u \in W^{1,t}_0(\Omega) \cap L^q(\Omega) \), \( t < \frac{2q}{p+1+q} \), for any \( L^1(\Omega) \) source \( f \). If \( f \in L^m(\Omega), m > 1 \) and \( q \geq \frac{p+1}{m-1} \), then there exists a distributional solution \( u \) in \( H^1_0(\Omega) \cap L^m(\Omega) \). If \( \frac{p+1}{2m-1} < q < \frac{p+1}{m-1} \), there exists a distributional solution \( u \) in \( W^{1,\frac{2qm}{q+p+1}}_0(\Omega) \) such that \( |u|^m \in L^1(\Omega) \). These results show that if \( q \) is sufficiently large, there
exists a distributional solution for any source; this is not the case f or problem (3).

The second lower order term analysed in [14] is
\[ h(u), \]
where \( h : [0, s_0) \to \mathbb{R} \) is a continuous, increasing function such that \( h(0) = 0 \) and \( \lim_{s \to s_0} h(s) = +\infty \) for some \( s_0 > 0 \). The regularizing effects of this lower order term are even better than the previous one. Indeed for a positive \( L^1(\Omega) \) source, there exists a bounded \( H^1_0(\Omega) \) solution.

In the literature we find several papers about elliptic coercive problems with lower order terms having a quadratic growth with respect to the gradient (see [6, 10, 11, 12, 8] for example and the references therein), that is, for problem
\[
\begin{cases}
-\text{div}(M(x)\nabla u) + g(u)|\nabla u|^2 = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

In these works it is assumed that \( M : \Omega \to \mathbb{R}^{N \times N} \) is a bounded elliptic Carathéodory map, so that there exists \( \alpha > 0 \) such that
\[ \alpha |\xi|^2 \leq M(x)\xi \cdot \xi \]
for every \( \xi \in \mathbb{R}^N \).

Various assumptions are made on \( g \). With no attempt of being exhaustive, we will describe some recent results where a singular \( g \) has been considered, namely
\[ g(u) = \frac{1}{|u|^\theta}. \]
The case where \( 0 < \theta \leq 1 \), introduced in [2, 3, 4], has been studied in [6, 7, 8, 13, 15]. From this body of literature one can deduce that for a positive source \( f \in L^m(\Omega) \), if \( \frac{2N}{2N - \theta(N - 2)} \leq m < \frac{N}{2} \) then there exists a strictly positive solution \( u \in H^1_0(\Omega) \cap L^{(2-\theta)m^*}(\Omega) \); if \( 1 < m < \frac{2N}{2N - \theta(N - 2)} \) then the solution \( u \) belongs to \( W^{1,q}_0(\Omega) \), with \( q = \frac{Nm(2-\theta)}{N - m\theta} \). The authors of [8] consider the general case \( \theta < 2 \), assuming that \( f \) is a strictly positive function on every compactly contained subset of \( \Omega \). They prove that if \( f \in L^{\frac{2N}{N-2\theta}}(\Omega) \) there exists a positive \( H^1_0(\Omega) \) solution. Finally, in [13] the lower order term is taken to be
\[ \lambda u + \mu \frac{|\nabla u|^2}{|u|^\theta} \chi_{\{u > 0\}} \]
where \( \chi_{\{u > 0\}} \) denotes the characteristic function of the set \( \{u > 0\} \), \( \lambda > 0 \) and \( \mu \in \mathbb{R} \).

In this paper we consider the same lower order term as above in an elliptic problem defined by an operator with degenerate coercivity. We will see that if \( 0 < \theta < 2 \), then \( \frac{|\nabla u|^2}{|u|^\theta} \) has a regularizing effect, even if it is singular in \( u \). We are going to state our results. We will distinguish the cases \( 0 < \theta < 1 \) and \( 1 \leq \theta < 2 \).

**Theorem 1.1.** Let \( 0 < \theta < 1 \). Assume that \( f \) is a positive function belonging to \( L^m(\Omega) \), with \( m \geq \frac{2N}{2N - \theta(N - 2)} \). Then there exists a function \( u \in H^1_0(\Omega) \), strictly positive on \( \Omega \), such that
\[ \frac{|\nabla u|^2}{|u|^\theta} \in L^1(\Omega) \]
and
\[
\int_{\Omega} \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{|u|^\theta} \varphi = \int f \varphi,
\]
for every \( \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \).
In the case where \( m < \frac{2N}{2N - \theta(N - 2)} = \left(\frac{2}{\theta}\right)^\ast \), we are able to prove the existence of an infinite energy solution, belonging to \( W^{1,\sigma}_0(\Omega) \), with \( \sigma = \frac{mN(2 - \theta)}{N - \theta m} \) (smaller than 2).

**Theorem 1.2.** Let \( 0 < \theta < 1 \). Assume that \( f \) is a positive function belonging to \( L^m(\Omega) \), with \( \frac{N}{2N - \theta(N - 1)} < m < \frac{2N}{2N - \theta(N - 2)} \). Then there exists a function \( u \in W^{1,\sigma}_0(\Omega) \), strictly positive on \( \Omega \), such that

\[
\int_{\Omega} \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{\nabla u^2}{u^\theta} \varphi = \int_{\Omega} f \varphi,
\]

for every \( \varphi \in C^1_0(\Omega) \).

In the case where \( 1 \leq \theta < 2 \), we are able to prove the same results as in the case \( 0 < \theta < 1 \), under a stronger hypothesis on \( f \).

**Theorem 1.3.** Let \( 1 \leq \theta < 2 \) and \( p > \theta - 1 \). Assume that \( f \in L^m(\Omega) \), with \( m \geq \frac{N}{2N - \theta(N - 2)} \), and satisfies

\[
\text{ess inf}\{f(x) : x \in \omega\} > 0
\]

for every \( \omega \subset \subset \Omega \). Then there exists a function \( u \in H^1_0(\Omega) \), strictly positive on \( \Omega \), such that \( \frac{\nabla u^2}{u^\theta} \in L^1(\Omega) \) and

\[
\int_{\Omega} \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{\nabla u^2}{u^\theta} \varphi = \int_{\Omega} f \varphi
\]

for every \( \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \).

**Theorem 1.4.** Let \( 1 \leq \theta < 2 \) and \( p > \theta - 1 \). Assume that \( f \in L^m(\Omega) \), with \( \frac{N}{2N - \theta(N - 1)} < m < \frac{2N}{2N - \theta(N - 2)} \), and satisfies

\[
\text{ess inf}\{f(x) : x \in \omega\} > 0
\]

for every \( \omega \subset \subset \Omega \). Then there exists a function \( u \in W^{1,\sigma}_0(\Omega) \), strictly positive on \( \Omega \), such that \( \frac{\nabla u^2}{u^\theta} \in L^1(\Omega) \) and

\[
\int_{\Omega} \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{\nabla u^2}{u^\theta} \varphi = \int_{\Omega} f \varphi
\]

for every \( \varphi \in C^1_0(\Omega) \).

We remark that if \( \theta < \frac{N}{N - 1} \) we are able to prove the existence of solutions when the source \( f \) belongs to \( L^1(\Omega) \).

We would like to point out the regularizing effects of the lower order term, in the case where \( p > \theta - 1 \) and \( 0 < \theta < 2 \). Our results furnish \( H^1_0(\Omega) \) solutions for less summable sources than for problem (3), since

\[
\frac{2N}{2N - \theta N + 2\theta} < \frac{2N}{N(1 - p) + 2(p + 1)}.
\]
Even in the case where the source $f$ is less summable, we get a better regularity of solutions than for problem (3): indeed
$$\sigma = \frac{mN(2-\theta)}{N-\theta m} \geq \frac{Nm(1-p)}{N-m(1+p)}$$ as $m \leq \frac{N}{2}$ and $p \leq \theta - 1$.

In the case where $0 < p \leq \theta - 1$, we are able to prove the existence of a solution to problem (1) with the same regularity as the solutions of problem (3).

**Theorem 1.5.** Let $1 \leq \theta < 2$ and $0 < p \leq \theta - 1$. Assume that $f \in L^m(\Omega)$ and satisfies
$$\text{ess inf} \{ f(x) : x \in \omega \} > 0$$ for every $\omega \subset \subset \Omega$.

1. If $m > \frac{N}{2}$, then there exists a strictly positive $H^1_0(\Omega) \cap L^\infty(\Omega)$ solution to problem (4).
2. If $\frac{N}{N+2-p(N-2)} \leq m < \frac{N}{2}$, then there exists a strictly positive $H^1_0(\Omega) \cap L^r(\Omega)$ solution to problem (4), where $r = \frac{Nm(1-p)}{N-2m}$.
3. If $\frac{N}{N+1-p(N-1)} < m < \frac{2N}{N+2-p(N-2)}$, then there exists a strictly positive $W^{1,s}_0(\Omega)$ solution to problem (4), where $s = \frac{Nm(1-p)}{N-m(1+p)}$.

Moreover $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$.

In the case where $\theta \geq 2$, the situations changes. Indeed we will prove a nonexistence result of finite energy solutions. Let $\lambda_1(f)$ denote the first positive eigenvalue of
$$\begin{cases} -\Delta u = \lambda fu & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$
where $f \in L^q(\Omega)$, with $q > \frac{N}{2}$. Using a result of [3], it is quite easy to prove the following

**Theorem 1.6.** Let $f \geq 0$, $f \not\equiv 0$, be a $L^q(\Omega)$ function, with $q > \frac{N}{2}$. If either $\theta > 2$, or $\theta = 2$ and $\lambda_1(f) > \frac{\beta}{\alpha^2}$, then there is no $H^1_0(\Omega)$ solution to problem (4).

## 2. A priori estimates

To prove the existence of solutions to problem (4) we use the following approximating problems:
$$\begin{cases} -\text{div} \left( \frac{b(x)}{1+|T_n(u_n)|^p} \nabla u_n \right) + B \frac{|\nabla u_n|^2}{(|u_n| + \frac{1}{n})^{\sigma+1}} = T_n(f) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial \Omega, \end{cases}$$
where, for $n \in \mathbb{N}$ and $s \in \mathbb{R}$
$$T_n(s) = \max\{-n, \min\{n, s\}\}.$$ These problems are well-posed due to the following result proved in [3, 11, 12].
Lemma 2.3. Let \( \theta \) be the Lebesgue measure of a set \( E \subset \mathbb{R}^N \) that
\[
M(x, s) \xi \cdot \xi \geq \alpha_0 |s|^2, \quad |M(x, s)| \leq \beta_0
\]
for a.e. \( x \in \Omega \), for every \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \). Let \( g(s) \) be a Carathéodory function such that \( g(s) \geq 0 \), \( |g(s)| \leq \gamma(s) \), where \( \gamma \) is a continuous, non-negative and increasing function. Then there exists a \( H^1_0(\Omega) \) bounded solution to
\[
\begin{cases}
-\text{div}(M(x, u)\nabla u) + g(u)|\nabla u|^2 = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Proof. Let us consider
\[
\begin{align*}
0 & \leq \gamma(\theta) - \theta = \gamma(\theta) - \theta \leq |\theta| + \gamma(\theta - \theta)
\end{align*}
\]
We prove now two a priori estimates on \( u_n \), which are true for every \( p > 1 \) and \( \theta \in (0, 2) \). In the sequel \( C \) will denote a positive constant independent of \( n \); \( \mu(E) \) will be the Lebesgue measure of a set \( E \subset \mathbb{R}^N \).

Lemma 2.2. Let \( u_n \) be the solutions to problems (6). Then it results
\[
B \int_{\Omega} \frac{u_n|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} \leq \int_{\Omega} f.
\]

Proof. Let us consider \( \frac{T_h(u_n)}{h} \), \( h > 0 \), as a test function in (6). We have, dropping the non-negative operator term,
\[
B \int_{\Omega} \frac{|\nabla u_n|^2 u_n}{h} \frac{T_h(u_n)}{h} \leq \int_{\Omega} f \frac{T_h(u_n)}{h}.
\]
It is now sufficient to pass to the limit as \( h \to 0 \), using Fatou’s lemma and the fact that \( \frac{T_h(u_n)}{h} \to 1 \) as \( h \to 0 \).

Lemma 2.3. Let \( 0 < \theta < 2 \). Let \( f \) be a positive function belonging to \( L^m(\Omega) \), with \( m \geq \frac{2N}{2N - \theta(N - 2)} \). Then the solutions \( u_n \) to problems (6) are uniformly bounded in \( H^1_0(\Omega) \). Thus there exists a function \( u \in H^1_0(\Omega) \) such that, up to a subsequence, \( u_n \rightharpoonup u \) weakly in \( H^1_0(\Omega) \) and a.e. in \( \Omega \).

Proof. The assertion follows by proving that the solutions \( u_n \) to problems (6) are uniformly bounded in \( H^1_0(\Omega) \). If we take \( (u_n + 1)^\theta - 1 \) as a test function in problem (6) we obtain
\[
B \int_{\Omega} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} u_n (u_n + 1)^\theta \leq B \int_{\Omega} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} u_n + \int_{\Omega} f u_n^\theta + C,
\]
dropping the positive operator term. We can estimate the right hand side using (7) in order to get
\[ B \int_{\Omega} |\nabla u_n|^2 \, u_n (u_n + 1)^\theta \leq \int_{\Omega} f u_n^\theta + C. \]

By working in \( \{ u_n \geq 1 \} \), the previous inequality gives
\[ B \int_{\{ u_n \geq 1 \}} |\nabla u_n|^2 \leq \int_{\Omega} f u_n^\theta + C \leq \int_{\{ u_n \geq 1 \}} f (u_n - 1)^\theta + C. \]

We use the Sobolev inequality in the left hand side and the Hölder inequality with exponent \( \frac{2^*}{\theta} \) in the last term, recalling that \( f \) belongs to \( L^m(\Omega) \) with \( m \geq \frac{2N}{2N - \theta (N - 2)} = \left( \frac{2^*}{\theta} \right)^{'} \).

Thus
\[ \frac{2N}{2N - \theta (N - 2)} = \left( \frac{2^*}{\theta} \right)^{'} \]

Thus
\[ S \frac{B}{2} \left[ \int_{\{ u_n \geq 1 \}} (u_n - 1)^{2^*} \right] \leq B \int_{\{ u_n \geq 1 \}} |\nabla u_n|^2 \leq C \left[ \int_{\{ u_n \geq 1 \}} (u_n - 1)^{2^*} \right] + C. \]

Since we are assuming \( \theta < 2 \), we deduce that
\[ \int_{\{ u_n \geq 1 \}} (u_n - 1)^{2^*} \leq C. \]

It follows from (8) that
\[ \int_{\{ u_n \geq 1 \}} |\nabla u_n|^2 \leq C. \]

Let us search for the same kind of estimate in \( \{ u_n < 1 \} \). Taking \( T_1(u_n) \) as a test function in problem (6), we get
\[ \frac{\alpha}{2^p} \int_{\{ u_n < 1 \}} |\nabla T_1(u_n)|^2 \leq \alpha \int_{\{ u_n < 1 \}} \frac{|\nabla T_1(u_n)|^2}{(1 + u_n)^p} \leq \int_{\Omega} f T_1(u_n) \leq \int_{\Omega} f \]

using hypothesis (2) and dropping the non-negative lower order term. As a consequence of estimates (9) and (10), \( u_n \) is uniformly bounded in \( H^1_0(\Omega) \). By compactness, there exists a function \( u \in H^1_0(\Omega) \) such that, up to a subsequence, \( u_n \rightarrow u \) weakly in \( H^1_0(\Omega) \) and a.e. in \( \Omega \). □

Lemma 2.4. Let \( 0 < \theta < 2 \). Let \( f \) be a positive function belonging to \( L^m(\Omega) \), with \( \frac{N}{2N - \theta (N - 1)} < m \leq \frac{2N}{2N - \theta (N - 2)} \). Then the solutions \( u_n \) to problems (3) are uniformly bounded in \( W^{1,\sigma}_0(\Omega) \), \( \sigma = \frac{m N (2 - \theta)}{N - \theta m} \). Thus there exists a function \( u \in W^{1,\sigma}_0(\Omega) \) such that, up to a subsequence, \( u_n \rightarrow u \) weakly in \( W^{1,\sigma}_0(\Omega) \) and a.e. in \( \Omega \).

Proof. The assertion follows by proving that the solutions \( u_n \) to problems (3) are uniformly bounded in \( W^{1,\sigma}_0(\Omega) \). Take \((u_n + 1)^{\theta + 1} - 1\), with \( \gamma = \frac{2^* - \theta m}{2m' - 2} \), as a test function in problem (6), we get
\[ \frac{\alpha}{2^p} \int_{\{ u_n < 1 \}} |\nabla T_1(u_n)|^2 \leq \alpha \int_{\{ u_n < 1 \}} \frac{|\nabla T_1(u_n)|^2}{(1 + u_n)^p} \leq \int_{\Omega} f T_1(u_n) \leq \int_{\Omega} f \]

using hypothesis (2) and dropping the non-negative lower order term. As a consequence of estimates (9) and (10), \( u_n \) is uniformly bounded in \( H^1_0(\Omega) \). By compactness, there exists a function \( u \in H^1_0(\Omega) \) such that, up to a subsequence, \( u_n \rightarrow u \) weakly in \( H^1_0(\Omega) \) and a.e. in \( \Omega \). □
function in problems (8). Note that $\gamma < 0$; indeed $2^* - \theta m' < 0$ and $2m' - 2^* > 0$, since $m < \frac{N}{2}$. Moreover, $\theta + 2\gamma = \frac{2^*(2 - \theta)}{2m' - 2^*} > 0$, as $\theta < 2$. Dropping the non-negative operator term and using estimate (12), we get

$$B \int_{\Omega} \frac{|
abla u_n|^2}{(u_n + \frac{1}{n})^\theta +1} u_n (u_n + 1)^{2\gamma + \theta} \leq \int_{\Omega} f(u_n + 1)^{2\gamma + \theta} + C.$$ 

By working in \{ $u_n \geq 1$ \} the previous inequality gives

$$\frac{B}{2(\gamma + 1)^2} \int_{\{ u_n \geq 1 \}} |\nabla [(u_n + 1)^{\gamma+1} - 2\gamma^1]|^2 \leq \frac{B}{2} \int_{\{ u_n \geq 1 \}} |\nabla u_n|^2 (u_n + 1)^{2\gamma}$$

$$\leq \int_{\{ u_n \geq 1 \}} f(u_n + 1)^{2\gamma+\theta} + \int_{\{ u_n \leq 1 \}} f(u_n + 1)^{2\gamma+\theta} + C \leq \int_{\{ u_n \geq 1 \}} f(u_n + 1)^{2\gamma+\theta} + C.$$ 

The Hölder inequality on the right hand side and the Sobolev inequality on the left one imply

$$S \left[ \int_{\{ u_n \geq 1 \}} [(u_n + 1)^{\gamma+1} - 2\gamma^1] \right] \leq C \int_{\{ u_n \geq 1 \}} |\nabla u_n|^2 (u_n + 1)^{2\gamma}$$

$$\leq C + C \left[ \int_{\{ u_n \geq 1 \}} (u_n + 1)^{(2\gamma+\theta)m'} \right]^{\frac{1}{\theta}}.$$

We remark that the choice of $\gamma$ is equivalent to require $(\gamma + 1)2^* = (2\gamma + \theta)m'$; moreover $\frac{2}{2^*} \geq \frac{1}{m'}$, due to the hypotheses on $m$ and $\theta$. Hence

$$\int_{\{ u_n \geq 1 \}} (u_n + 1)^{(\gamma+1)2^*} = \int_{\{ u_n \geq 1 \}} (u_n + 1)^{(2\gamma+\theta)m'} \leq C \quad \forall n \in \mathbb{N}.$$ 

Now, with $\sigma = \frac{mN(2 - \theta)}{N - \theta m}$ as in the statement, and recalling that $\gamma < 0$, let us write

$$\int_{\{ u_n \geq 1 \}} |\nabla u_n|^\sigma = \int_{\{ u_n \geq 1 \}} \frac{|\nabla u_n|^\sigma}{(u_n + 1)^{-\gamma\sigma}} (u_n + 1)^{-\gamma\sigma}.$$ 

The Hölder inequality with exponent $\frac{2}{\sigma}$ and estimates (12) and (13) give

$$\int_{\{ u_n \geq 1 \}} |\nabla u_n|^\sigma \leq \left[ \int_{\{ u_n \geq 1 \}} \frac{|\nabla u_n|^2}{(u_n + 1)^{2\gamma}} \right]^{\frac{1}{2}} \left[ \int_{\{ u_n \geq 1 \}} (u_n + 1)^{-\gamma\sigma} \right]^{\frac{1}{2\sigma}} \leq C$$

since $-\gamma \frac{2\sigma}{2 - \sigma} = (\gamma + 1)2^*$. It remains to analyse the behaviour of $\nabla u_n$ on \{ $u_n \leq 1$ \}. Taking $T_1(u_n)$ as a test function in (8) and dropping the non-negative the lower
order term, we get
\[ \frac{\alpha}{2^p} \int_{\{u_n \leq 1\}} |\nabla T_1(u_n)|^2 \leq \frac{\alpha}{(1 + u_n)^p} \int_{\Omega} |\nabla T_1(u_n)|^2 \leq \int_{\Omega} f T_1(u_n) \leq \int_{\Omega} f \]
by hypothesis (4). This last estimate and (14) imply that \( u_n \) is uniformly bounded in \( W_0^{1,\sigma}(\Omega) \). Since \( \sigma > 1 \), there exists a function \( u \in W_0^{1,\sigma}(\Omega) \) such that, up to a subsequence, \( u_n \to u \) weakly in \( W_0^{1,\sigma}(\Omega) \) and a.e. in \( \Omega \).

In the following lemma, we will assume some hypotheses on \( p \). This will give, in some cases, better estimates than Lemmata 2.3 and 2.4.

**Lemma 2.5.** Let \( 0 < p < 1 \). Let \( f \in L^m(\Omega), r = \frac{Nm(1 - p)}{N - 2m} \), and \( s = \frac{Nm(1 - p)}{N - m(1 + p)} \).

1. If \( m > \frac{N}{2} \), the solutions of (4) are uniformly bounded in \( H^1_0(\Omega) \cap L^\infty(\Omega) \).
   Thus there exists a function \( u \in H^1_0(\Omega) \cap L^\infty(\Omega) \) such that, up to a subsequence, \( u_n \to u \) weakly in \( H^1_0(\Omega) \) and a.e. in \( \Omega \).
2. If \( \frac{2N}{N + 2 - p(N - 2)} \leq m < \frac{N}{2} \), the solutions of (4) are uniformly bounded in \( H^1_0(\Omega) \cap L^r(\Omega) \).
   Thus there exists a function \( u \in H^1_0(\Omega) \cap L^r(\Omega) \) such that, up to a subsequence, \( u_n \to u \) weakly in \( H^1_0(\Omega) \) and a.e. in \( \Omega \).
3. If \( \frac{N}{N + 1 - p(N - 1)} < m < \frac{2N}{N + 2 - p(N - 2)} \), the solutions of (4) are uniformly bounded in \( W^{1,s}_0(\Omega) \).
   Thus there exists a function \( u \in W^{1,s}_0(\Omega) \) such that, up to a subsequence, \( u_n \to u \) weakly in \( W^{1,s}_0(\Omega) \) and a.e. in \( \Omega \).

**Proof.** In problems (4) consider as a test function the same test functions as in (3). With this choice, the lower order term is non-negative and we can take into account only the term given by the operator. Therefore one can follow the same proofs as in (3) to get the above estimates.

**Remark 1.** Let \( p > \theta - 1 \). Lemmata 2.3 and 2.4 give a further uniform estimate on \( u_n \) than Lemma 2.5. Indeed, if one chooses \( u_n \) as a test function in (3), then, by hypothesis (4),
\[ \int_{\Omega} |\nabla u_n|^2 \left[ \frac{\alpha}{(1 + |u_n|)^p} + \frac{Bu_n}{(u_n + \frac{1}{n})^{\theta + 1}} \right] \leq \int_{\Omega} fu_n. \]
If \( p > \theta - 1 \), the lower order term has a leading role in the left hand side of the previous inequality.

We are going to prove the a.e. convergence of the gradients of \( u_n \). We will follow the same technique as in (3). Remark that a similar technique was used for elliptic degenerate problems in (3).

**Lemma 2.6.** Let \( u_n \) be the solutions to problems (4) and \( u \) be the function found in Lemmata 2.3, or 2.4 or 2.5, according to the summability of \( f \). Up to a subsequence, \( \nabla u_n \) converges to \( \nabla u \) a.e. in \( \Omega \).
Proof. Let \( h, k > 0 \). In the sequel \( C \) will denote a constant independent of \( n, h, k \). Let us consider \( T_h(u_n - T_k(u)) \) as a test function in problems (3). Then

\[
\int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla T_h(u_n - T_k(u)) \leq h \int_{\Omega} f + B \int_{\Omega} |\nabla u_n|^2 u_n (u_n + \frac{1}{n})^{p+1} h.
\]

By estimate (6) on the right hand side and by hypothesis (2) on the left one, we get

\[
\int_{\Omega} \frac{\nabla u_n \cdot \nabla T_h(u_n - T_k(u))}{(1 + T_n(u_n))^p} \leq C h.
\]

Then we can write

\[
\int_{\{u_n - T_k(u) \leq h\}} \frac{|\nabla(u_n - T_k(u))|^2}{(1 + u_n)^p} \leq \int_{\Omega} \frac{\nabla(u_n - T_k(u)) \cdot \nabla T_h(u_n - T_k(u))}{(1 + T_n(u_n))^p} \leq C h - \int_{\Omega} \frac{\nabla T_h(u_n - T_k(u))^2}{(1 + u_n)^p}.
\]

At the limit as \( n \to \infty \) one has

\[
\limsup_{n \to \infty} \int_{\{u_n - T_k(u) \leq h\}} \frac{|\nabla T_h(u_n - T_k(u))^2}{(1 + u_n)^p} \leq C h.
\]

Since \( u_n \leq h + k \) in \( \{u_n - T_k(u) \leq h\} \), we get

(15) \[
\limsup_{n \to \infty} \int_{\{u_n - T_k(u) \leq h\}} |\nabla T_h(u_n - T_k(u))^2 \leq C h(1 + h + k)^p.
\]

We recall that \( u_n \) is uniformly bounded in \( W^{1,\eta}(\Omega) \), where \( \eta \) equals 2 or \( \sigma \) or \( s \), according to the statements of Lemmata 2.3, 2.4 and 2.5. Let \( q \in (1, \eta) \). We can write

\[
\int_{\Omega} |\nabla(u_n - u)|^q = \int_{\{u_n - u \leq h, |u| \leq k\}} |\nabla(u_n - u)|^q + \int_{\{u_n - u \leq h, |u| > k\}} |\nabla(u_n - u)|^q + \int_{\{|u_n - u| > h\}} |\nabla(u_n - u)|^q.
\]

Using the Hölder inequality with exponent \( \frac{2}{q} \) on the first term of the right hand side and exponent \( \frac{2}{q} \) on the other ones, we have

\[
\int_{\Omega} |\nabla(u_n - u)|^q \leq C \left[ \int_{\{u_n - u \leq h, |u| \leq k\}} |\nabla(u_n - u)|^2 \right]^{\frac{q}{2}} + C \mu(\{|u| > k\})^{1 - \frac{q}{2}} + \mu(\{|u_n - u| > h\})^{1 - \frac{q}{2}},
\]

where we have used that \( u_n \) is uniformly bounded in \( W^{1,\eta}_0(\Omega) \) to estimate the last two terms. By (15) the limit as \( n \to \infty \) gives

\[
\limsup_{n \to \infty} \int_{\Omega} |\nabla(u_n - u)|^q \leq [Ch(1 + h + k)^p]^{\frac{q}{2}} + C \mu(\{|u| > k\})^{1 - \frac{q}{2}}.
\]
The limit as \( h \to 0 \) implies
\[
\lim_{n \to \infty} \sup_{\Omega} \int |\nabla (u_n - u)|^q \leq C \mu(\{ |u| > k \})^{1 - \frac{q}{p}}.
\]

At the limit as \( k \to +\infty \), \( \mu(\{ |u| > k \}) \) converges to 0. Therefore \( \nabla u_n \to \nabla u \) in \( L^q(\Omega) \). Up to a subsequence, \( \nabla u_n \to \nabla u \) a.e. in \( \Omega \).

### 3. Existence results in the case \( 0 < \theta < 1 \)

To prove the existence of solutions to problem (4), the key point is to prove that the function \( u \) found by compactness in the lemmata of Section 2 is strictly positive. In the case \( 0 < \theta < 1 \), we use a technique similar to that in [3].

**Proposition 1.** Let \( 0 < \theta < 1 \). Let \( u_n \) and \( u \) be as in Lemma 2.4. Then \( u > 0 \).

**Proof.** We define, for \( s \geq 0 \),
\[
H_n(s) = \int_0^s \frac{t(1 + T_n(t))^p}{\alpha(t + \frac{1}{s})^{p+1}} dt, \quad H(s) = \int_0^s \frac{(1 + t)^p}{\alpha t^{p+1}} dt.
\]

Observe that \( H \) is well-defined, since \( \theta < 1 \). We choose \( e^{-BH_n(u_n)} \phi \), where \( \phi \) is a positive \( C_0^\infty(\Omega) \) function, as a test function in (4). This gives
\[
\int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \phi e^{-BH_n(u_n)} - \int T_n(f)e^{-BH_n(u_n)} \phi =
\]
\[
= B \int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} e^{-BH_n(u_n)} |\nabla u_n|^2 \phi H_n(u_n) - B \int_\Omega \frac{e^{-BH_n(u_n)} |\nabla u_n|^2 u_n}{(\frac{1}{n} + u_n)^{p+1}} \phi
\]
\[
\geq B \int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} e^{-BH_n(u_n)} |\nabla u_n|^2 \phi H_n(u_n) - B \int_\Omega \frac{e^{-BH_n(u_n)} |\nabla u_n|^2 u_n}{(\frac{1}{n} + u_n)^{p+1}} \phi
\]
by hypothesis (3). The last quantity is positive, due to the choice of \( H_n \) and \( \phi \). As a consequence
\[
\int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \phi e^{-BH_n(u_n)} \geq \int T_n(f)e^{-BH_n(u_n)} \phi \geq \int T_1(f)e^{-BH_n(u_n)} \phi.
\]

Now, we set
\[
P_n(s) = \int_0^s \frac{e^{-BH_n(t)}}{(1 + T_n(t))^p} dt, \quad P(s) = \int_0^s \frac{e^{-BH(t)}}{(1 + t)^p} dt.
\]

With these definitions, we remark that we have just proved that the inequality
\[-\text{div}(b(x)\nabla (P_n(u_n))) \geq T_1(f)e^{-BH_n(u_n)} \]
holds distributionally. Observe that for every \( n \in \mathbb{N} \), \( P_n(u_n) \in H^1_0(\Omega) \), since \( P_n \) is bounded and \( u_n \in H^1_0(\Omega) \). Let \( z_n \) be the \( H^1_0(\Omega) \) solution to
\[-\text{div}(b(x)\nabla z_n) = T_1(f)e^{-BH_n(u_n)} ;
\]
let \( z \) be the \( H^1_0(\Omega) \) solution to
\[-\text{div}(b(x)\nabla z) = T_1(f)e^{-BH(u)} .
\]

Then
\[-\text{div}(b(x)\nabla (P_n(u_n))) \geq -\text{div}(b(x)\nabla z_n) .
\]
The comparison principle in $H_0^1(\Omega)$ implies that $P_n(u_n(x)) \geq z_n(x)$ for a.e. $x \in \Omega$. Up to a subsequence, $z_n \to z$ weakly in $H_0^1(\Omega)$ and a.e. in $\Omega$. At the limit a.e. in $\Omega$, as $n \to +\infty$, we have $P(u) \geq z$. By the strong maximum principle $z > 0$ and so $P(u) > 0$. Since $P$ is strictly increasing, $u > 0$ in $\Omega$.

**Corollary 1.** Let $0 < \theta < 1$. Let $u_n$ and $u$ be as in Lemma 2.6. Then $\frac{\nabla u^2}{u^\theta} \in L^1(\Omega)$.

**Proof.** We pass to the limit in (7). The a.e. convergence of $u_n$ to $u$ (see Lemmata 2.3, 2.4 and 2.5), the a.e. convergence of $\nabla u_n$ to $\nabla u$ (see Lemma 2.6) and Proposition 1 imply

$$B \int_\Omega \frac{\nabla u^2}{u^\theta} \leq \int_\Omega f$$

by Fatou’s lemma.

We are going to prove Theorem 1.1.

**Proof.** We are going to prove that the function $u$ found in Lemma 2.3, and studied in Lemma 2.6, Proposition 1 and Corollary 1, is a weak solution to problem (1). We use the same technique as in [8].

We will prove that (4) holds true for every positive and bounded $\varphi \in H_0^1(\Omega)$. The general case follows from the fact that every such function $\varphi$ can be written as $\varphi_+ - \varphi_-$ with $\varphi_\pm$ bounded, positive and belonging to $H_0^1(\Omega)$.

We pass to the limit as $n \to \infty$ in

$$\int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \varphi + B \int_\Omega \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} \varphi = \int_\Omega T_n(f) \varphi,$$

where $\varphi$ is a positive bounded $H_0^1(\Omega)$ function. Regarding the first term we observe that $\frac{b(x)}{(1 + T_n(u_n))^p} \nabla \varphi$ strongly converges to $\frac{b(x)}{(1 + u)^p} \nabla \varphi$ in $L^2(\Omega)$ and $\nabla u_n$ weakly converges to $\nabla u$ in $L^2(\Omega)$. For the second one we use the a.e. convergence of $\nabla u_n$, proved in Lemma 2.6. Fatou’s lemma implies

$$\int_\Omega \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_\Omega \frac{\nabla u^2}{u^\theta} \varphi \leq \int_\Omega f \varphi. \quad (16)$$

The proof of the opposite inequality is more delicate. To this aim, we define, for $n \in \mathbb{N}$ and $s \geq 0$,

$$H_{\frac{\alpha}{\beta}}(t) = \int_0^t \frac{B(1 + s)^p}{\alpha(s + \frac{1}{n})^\theta} ds, \quad H_0(t) = \int_0^t \frac{B(1 + s)^p}{\alpha s^\theta} ds.$$

$H_0$ is well-posed, since $\theta < 1$. Let us consider

$$v = e^{-H_{\frac{\alpha}{\beta}}(u_n)} e^{H_0(T_j(u))} \varphi,$$

where $j \in \mathbb{N}$ and $\varphi$ is a positive bounded $H_0^1(\Omega)$ function, as a test function in (8). Then

$$\int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \varphi e^{-H_{\frac{\alpha}{\beta}}(u_n)} e^{H_0(T_j(u))}$$
\[ + \frac{B}{\alpha} \int_\Omega \frac{b(x)}{(1 + T_n(u_n))} \varphi e^{-\frac{H_\alpha}{2} u_n} e^{\frac{H_\alpha}{2} T_j(u)} \nabla u_n \cdot \nabla T_j(u) \frac{(T_j(u) + \frac{1}{2})^p}{T_j(u) + \frac{1}{2} + 1} \varphi \, \frac{(1 + u_n)^p}{(1 + T_n(u_n))^p} \frac{(\frac{1}{n} + u_n)^p}{(\frac{1}{n} + u_n)^p} \varphi e^{-\frac{H_\alpha}{2} (u_n)} e^{\frac{H_\alpha}{2} (T_j(u))} \]

\[ = \int_\Omega T_n(f) e^{-\frac{H_\alpha}{2} (u_n)} e^{\frac{H_\alpha}{2} (T_j(u))} \varphi + \frac{B}{\alpha} \int_\Omega \frac{b(x) |\nabla u_n|^2}{(1 + T_n(u_n))^p} (\frac{1}{n} + u_n)^p \varphi e^{-\frac{H_\alpha}{2} u_n} e^{\frac{H_\alpha}{2} T_j(u)} \]

\[ - B \int_\Omega \frac{|\nabla u_n|^2 u_n}{(\frac{1}{n} + u_n)^p} e^{-\frac{H_\alpha}{2} (u_n)} e^{\frac{H_\alpha}{2} (T_j(u))} \varphi. \]

Note that by hypothesis (2) and inequality

\[ \left( \frac{u_n + 1}{1 + T_n(u_n)} \right) \geq 1 \frac{u_n}{u_n + \frac{1}{n}}, \]

the sum of the last two terms is non-negative. At the limit as \( n \to \infty \) we have

\[ \int_\Omega \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi e^{-H_\alpha(u)} e^{\frac{H_\alpha}{2} T_j(u)} \]

\[ + \frac{B}{\alpha} \int_\Omega \frac{b(x)}{(1 + u)^p} \varphi e^{-H_\alpha(u)} e^{\frac{H_\alpha}{2} T_j(u)} \nabla u \cdot \nabla T_j(u) \frac{1}{(T_j(u) + \frac{1}{2})^p} (T_j(u) + 1)^p \]

\[ \geq \int_\Omega f e^{-H_\alpha(u)} e^{\frac{H_\alpha}{2} T_j(u)} \varphi + \frac{B}{\alpha} \int_\Omega \frac{b(x) |\nabla u|^2}{u^p} \varphi e^{-H_\alpha(u)} e^{\frac{H_\alpha}{2} T_j(u)} \]

\[ - B \int_\Omega \frac{|\nabla u|^2}{u^p} e^{-H_\alpha(u)} e^{\frac{H_\alpha}{2} T_j(u)} \varphi, \]

using the weak convergence of \( u_n \) to \( u \) in \( H^1_\alpha(\Omega) \) in the left hand side and Fatou’s lemma in the right one. Now we pass to the limit as \( j \to \infty \), using that \( e^{-H_\alpha(u)} e^{\frac{H_\alpha}{2} T_j(u)} \leq 1 \) and Corollary 4. We obtain

\[ (17) \int_\Omega \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi \geq \int_\Omega f \varphi - \frac{B}{\alpha} \int_\Omega \varphi \frac{|\nabla u|^2}{u^p}. \]

Inequalities (16) and (17) imply that

\[ \int_\Omega \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_\Omega \varphi \frac{|\nabla u|^2}{u^p} = \int_\Omega f \varphi \]

for every positive and bounded \( \varphi \in H^1_\alpha(\Omega). \)

We are going to prove Theorem 1.2.

**Proof.** We are going to prove that the function \( u \) found in Lemma 2.3 and studied in Lemma 2.4, Proposition 3 and Corollary 4, is a weak solution to problem (1). We use the same technique as in [13, 23].

We first prove (3) for every positive \( C^1_0(\Omega) \) function \( \varphi \). With the same argument as in the previous theorem (i.e., using Fatou’s lemma) one can prove that

\[ (18) \int_\Omega \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_\Omega \varphi \frac{|\nabla u|^2}{u^p} \leq \int_\Omega f \varphi. \]
To prove the opposite inequality, we slightly modify the previous proof, since we no longer have uniform estimates of \( u_n \) in \( H^1_0(\Omega) \). Observe that, however, \( T_k(u_n) \) is uniformly bounded in \( H^1_0(\Omega) \). Indeed, it is sufficient to consider \( T_k(u_n) \) as a test function in (3): we obtain

\[
\int_{\{u_n \leq k\}} |\nabla T_k(u_n)|^2 \leq Ck(1+k)^p \quad \forall n \in \mathbb{N}
\]

by hypothesis (4). We will use, for \( k \in \mathbb{N} \) and \( s \in \mathbb{R} \)

\[
R_k(s) = \begin{cases} 
1, & s \leq k \\
k + 1 - s, & k \leq s \leq k + 1 \\
0, & s > k + 1,
\end{cases}
\]

to define a test function. We set, for \( t \geq 0 \),

\[
H^+_{\frac{t}{p}}(t) = \int_0^t \frac{B(1+s)^p}{\alpha(s+\frac{1}{p})^\theta} ds, \quad H^-_{\frac{t}{p}}(t) = \int_0^t \frac{B(1+s)^p}{\alpha s^\theta} ds.
\]

This is possible, since \( \theta < 1 \). We consider

\[
v = e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u_n) \varphi,
\]

where \( \varphi \) is a positive \( C^2_0(\Omega) \) function and \( j \in \mathbb{N} \), as a test function in (3). Then

\[
\int_{\Omega} \frac{b(x)}{(1+T_n(u_n))^p} \nabla u_n \cdot \nabla \varphi e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u_n)
\]

\[
+ \frac{B}{\alpha} \int_{\Omega} \frac{b(x)}{(1+T_n(u_n))^p} \varphi e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} \nabla u_n \cdot \nabla T_j(u) \left( \frac{1}{T_j(u)} + \frac{1}{\alpha} \right) \varphi R_k(u_n) + \int_{\{k \leq u_n \leq k+1\}} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \varphi e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u_n)
\]

\[
= \int_{\Omega} T_n(f) e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} \varphi R_k(u_n) + \int_{\{k \leq u_n \leq k+1\}} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \varphi e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u_n)
\]

\[
+ \frac{B}{\alpha} \int_{\Omega} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \left( \frac{1}{T_n(u_n)} + \varphi \right) e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u_n)
\]

\[-B \int_{\Omega} \frac{\nabla u_n \cdot \nabla \varphi}{(1+T_n(u_n))^p} e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u_n) \varphi.
\]

The sum of the last two terms is positive, since \( b(x) \geq \alpha \) by hypothesis (3) and by inequality

\[
\left( \frac{u_n + 1}{1+T_n(u_n)} \right)^p \geq \frac{u_n}{u_n + \frac{1}{\alpha}}.
\]

Dropping the non-negative term

\[
\int_{\{k \leq u_n \leq k+1\}} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \varphi e^{-H^+_{\frac{t}{p}}(u_n)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u_n),
\]

at the limit as \( n \to \infty \) we have, by Fatou’s lemma, the weak convergence of \( u_n \) in \( W^{1,\infty}_0(\Omega) \) and the weak convergence of \( T_k(u_n) \) in \( H^1_0(\Omega) \),

\[
\int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi e^{-H_+_{\frac{t}{p}}(u)} e^{H^-_{\frac{t}{p}}(T_j(u))} R_k(u) +
\]
+ \frac{B}{\alpha} \int_{\Omega} \frac{b(x)}{(1 + u)^p} \varphi e^{-H_0(u)} e^{\frac{H_p(T_j(u))}{2}} \frac{\nabla u \cdot \nabla T_j(u)}{(T_j(u) + \frac{1}{j})^p} (T_j(u) + 1)^p R_k(u) \\
 \geq \int_{\Omega} \frac{b(x)}{(1 + u)^p} \varphi e^{-H_0(u)} e^{\frac{H_p(T_j(u))}{2}} \varphi R_k(u) + \frac{B}{\alpha} \int_{\Omega} \frac{b(x)|\nabla u|^2}{u^\theta} \varphi e^{-H_0(u)} e^{\frac{H_p(T_j(u))}{2}} R_k(u) \\
 - B \int_{\Omega} \frac{\nabla u^2}{u^\theta} e^{-H_0(u)} e^{\frac{H_p(T_j(u))}{2}} R_k(u) \varphi.
}

As in the previous proof, it is now sufficient to pass to the limit as \( j \to \infty \) first, using that \( e^{-H_0(u)} e^{\frac{H_p(T_j(u))}{2}} \leq 1 \) and Corollary 1, and then to the limit as \( k \to \infty \), using that \( R_k(u) \) tends to 1. We thus obtain

\[
\int_{\Omega} \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi \geq \int_{\Omega} f \varphi - B \int_{\Omega} \frac{\nabla u^2}{u^\theta} \varphi.
\]

Inequalities (18) and (19) imply that

\[
\int \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int \frac{\nabla u^2}{u^\theta} \varphi = \int f \varphi
\]

for every positive \( \varphi \in C^1_0(\Omega) \). Now, let \( \varphi \) any \( C^1_0(\Omega) \) function. We define \( \varphi_\pm = \rho^\varepsilon * \varphi_\pm \) as the convolution of a mollifier \( \rho^\varepsilon \), for \( \varepsilon > 0 \), with \( \varphi_\pm \). Then \( \varphi_\varepsilon \) is a positive \( C^1_0(\Omega) \) function, for \( \varepsilon \) sufficiently small. By (20) we have

\[
\int \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla (\varphi_\varepsilon - \varphi_\pm) + B \int \frac{\nabla u^2}{u^\theta} (\varphi_\varepsilon - \varphi_\pm) = \int f (\varphi_\varepsilon - \varphi_\pm).
\]

Since \( \varphi_\varepsilon \to \varphi \) uniformly in \( \Omega \) and in \( W^{1,q}_0(\Omega) \) for every \( q \geq 1 \), as \( \varepsilon \to 0 \), the result follows.

\[\square\]

4. Existence results in the case \( 1 \leq \theta < 2 \)

As in the above case, we need to prove that the function \( u \) found in Section 3 is not 0 in \( \Omega \). To this aim, we are going to prove that for every \( \omega \subset \subset \Omega \) there exists a positive constant \( c_\omega \) such that the solutions \( u_n \) to problems (3) satisfy \( u_n \geq c_\omega \) in \( \omega \) for every \( n \in \mathbb{N} \). We will follow a similar technique to that one in [3]. The following theorem, proved in [3] (and in [1]), will be useful to us.

**Theorem 4.1.** Let \( B : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function such that for every \( \omega \subset \subset \Omega \) there exists \( m_\omega > 0 \) such that \( B(x,s) \geq m_\omega l(s) \) for a.e. \( x \in \Omega \) and for every \( s \geq 0 \). Assume that \( l : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous increasing function such that \( l(s)/s \) is increasing for \( s \) sufficiently large and for some \( t_0 > 0 \)

\[
\int_{t_0}^{+\infty} \frac{dt}{\sqrt{\int_0^t l(s)ds}} < +\infty.
\]

Then for every \( \omega \subset \subset \Omega \) there exists a constant \( C_\omega > 0 \) such that every subsolution \( v \in H^{1,loc}_\omega(\Omega) \) of \( -\text{div}(b(x)\nabla v) + B(x,v) = 0 \) such that \( v^+ \in L^\infty(\omega) \) and \( B(x,v^+) \in L^1_\text{loc}(\Omega) \) satisfies \( v \leq C_\omega \) in \( \omega \).
Remark 2. We recall that a sub-solution of \(-\text{div}(b(x)\nabla v) + l(v)g(x) = 0\) is a $W^{1,1}_{\text{loc}}(\Omega)$ function such that
\[
\int_{\Omega} b(x)\nabla v \cdot \nabla \phi + \int_{\Omega} l(v)g(x)\phi \leq 0
\]
for every $C^\infty(\Omega)$ positive function $\phi$.

Remark 3. In the literature condition (21) is called the Keller-Osserman condition, due to the papers [17, 19] on semilinear equations.

Proposition 2. Let $1 \leq \theta < 2$. Let $u_n$ be the solutions of (4). Then for every $\omega \subset \subset \Omega$ there exists a strictly positive constant $c_\omega$ such that $u_n \geq c_\omega$ in $\omega$ for every $n \in \mathbb{N}$.

Proof. Step 1. Let $u_n$ be a $H^1_0(\Omega) \cap L^\infty(\Omega)$ solution to (3). We perform a change of variable in order to get a sub-solution of an elliptic semi-linear problem, as in Theorem 4.1.

We set $a_n(s) = \frac{1}{(1 + T_n(s))^\theta}$. Then $u_n$ satisfies, distributionally,
\[
-\text{div}(b(x)a_n(u_n)\nabla u_n) + \frac{B}{u_n^\theta}|\nabla u_n|^2 \geq T_1(f),
\]
that is,
\[
(b(x)a_n(u_n) - a_n'(u_n)b(x)\nabla u_n)^2 + \frac{B}{u_n^\theta}|\nabla u_n|^2 \geq T_1(f). \tag{22}
\]

Let $k_n(t) = \int_1^t \frac{B}{\alpha \theta a_n(r)}dr$ and $\psi_n(s) = \int_s^1 e^{-k_n(t)}a_n^2(t)dt$. We remark that
\[
\psi_n'(s) = -\alpha^2(s)e^{-k_n(s)}, \quad \frac{\psi_n'(s)}{\psi_n(s)} = \frac{\beta a_n'(s)}{a_n(s)}.
\]

We define $v_n = \psi_n(u_n)$. Then
\[
\text{div}(b(x)\nabla v_n) = \text{div}(b(x)\psi_n'(u_n)\nabla u_n) = \psi_n'(u_n)\text{div}(b(x)\nabla u_n) + b(x)\psi_n'(u_n)|\nabla u_n|^2
\]
and therefore
\[
-a_n(u_n)\text{div}(b(x)\nabla u_n) = -a_n(u_n)\frac{\text{div}(b(x)\nabla v_n)}{\psi_n'(u_n)} + a_n(u_n)b(x)\frac{\psi_n'(u_n)}{\psi_n(u_n)}|\nabla u_n|^2.
\]

By inequality (22) we have
\[
T_1(f) \leq -a_n(u_n)\frac{\text{div}(b(x)\nabla v_n)}{\psi_n'(u_n)} + a_n(u_n)b(x)\frac{\psi_n'(u_n)}{\psi_n(u_n)}|\nabla u_n|^2 - a_n'(u_n)b(x)|\nabla u_n|^2 + \frac{B}{u_n^\theta}|\nabla u_n|^2.
\]

Using that $a_n'(s) \leq 0$, $\frac{\psi_n'(s)}{\psi_n(s)} \leq 0$ and hypothesis (3) we obtain
\[
T_1(f) \leq -a_n(u_n)\frac{\text{div}(b(x)\nabla v_n)}{\psi_n'(u_n)} + a_n(u_n)\frac{\psi_n'(u_n)}{\psi_n(u_n)}|\nabla u_n|^2 - a_n'(u_n)\beta|\nabla u_n|^2 + \frac{B}{u_n^\theta}|\nabla u_n|^2.
\]

Due to (23)
\[
T_1(f) \leq -a_n(u_n)\frac{\text{div}(b(x)\nabla v_n)}{\psi_n'(u_n)}.
\]
Observing that $\psi'_n(s) = -\frac{\alpha}{a_n^2}(s)e^{-k_n(s)} \leq 0$, $v_n$ satisfies

$$0 \geq -\text{div}(b(x)\nabla v_n) + T_1(f)e^{-k_n(\psi^{-1}(v_n))} a_n^{-1}(\psi^{-1}(v_n)).$$

**Step 2.** We now study, for $s \geq 0$

$$s \rightarrow e^{-k_n(\psi^{-1}(s))} a_n^{-1}(\psi^{-1}(s)).$$

We remark that $\psi^{-1}(s) \leq 1$, since $s \geq 0 = \psi(1)$ and $\psi$ is decreasing. Therefore

$$\text{(24)} \quad a_n^{-1}(\psi^{-1}(s)) \geq a_n^{-1}(1) = a_0,$$

as $a_n$ is decreasing.

Recalling that

$$\psi_n(s) = \int_s^1 e^{-k_n(t)} a_n^{-1}(t) dt, \quad k_n(t) = \int_t^1 \frac{B}{\alpha s^\theta a_n(r)} dr$$

and

$$\left\{ \begin{array}{ll}
  a_n(s) = a_1(s), & s \leq 1 \\
  a_n(s) \leq a_1(s), & s > 1
\end{array} \right.$$

it is not difficult to prove that

$$\text{(25)} \quad \psi_n(s) \geq \psi_1(s),$$

distinguishing the cases $s \leq 1$ and $s > 1$. Now, inequality (25) and the fact that $\psi_n$ is decreasing imply that $\psi_n^{-1}(s) \leq \psi_1^{-1}(s)$ for every $s \geq 0$. Recalling that $\psi_n^{-1}(s) \leq 1$ and $a_n(s) = a_1(s) \geq 0$ for $s \geq 1$, we deduce easily that

$$\text{(26)} \quad e^{-k_n(\psi_n^{-1}(s))} \geq e^{-k_1(\psi_1^{-1}(s))}.$$

Due to (24) and (26), $v_n$ satisfies

$$0 \geq -\text{div}(b(x)\nabla v_n) + B(x, v_n)$$

with

$$B(x, s) = \left\{ \begin{array}{ll}
  T_1(f)a_0 l(s), & s \geq 0 \\
  0, & s \leq 0
\end{array} \right.,$$

where $l(s) = e^{-k_1(\psi_1^{-1}(s))} - 1$, $s \geq 0$.

**Step 3.** We are going to prove that $l$ satisfies the hypotheses of Theorem 4.1.

We observe that $l$ is continuous and increasing, since $\psi_1^{-1}$ is decreasing and $k_1$ is increasing. We claim that $l(s)/s$ is increasing for $s$ sufficiently large. This is equivalent to prove that $Y(t) = \frac{l(\psi_1(t))}{\psi_1(t)}$ is decreasing for small positive $t$. Now, $Y'(t) < 0$ if and only if

$$\text{(27)} \quad l'(\psi_1(t))\psi_1(t) - \int_t^1 l'(\psi_1(s))\psi_1'(s) ds > 0.$$

We remark that $l'(\psi_1(t)) = \frac{B}{\alpha s^\theta a_1^{\frac{\theta}{1 + \theta}}(t)}$ is decreasing in $(0, w_0]$. Therefore

$$l'(\psi_1(t))\psi_1(t) - \int_t^1 l'(\psi_1(s))\psi_1'(s) ds = \int_0^1 e^{-k_1(s)} a_1^{-1}(s) [h(t) - h(s)] ds$$
due to the choice of \( w_0 \). Let
\[
M_1 = \int_{w_0}^{1} e^{-k_2(s)} a_1^2(s) \{ h(t) - h(s) \} ds , \quad M_2 = \int_{w_0}^{1} e^{-k_1(s)} a_1^2(s) h(s) ds .
\]
We have proved that
\[
l'(\psi_1(t))\psi_1(t) - \int_{1}^{t} l'(\psi_1(s))\psi_1'(s)ds \geq M_1 h(t) - M_2 .
\]
If \( t \) is sufficiently small, the last quantity is positive, since \( h \) is decreasing for small positive \( t \). Therefore (27) holds.

We are going to study the last condition on \( l \), that is, the existence of a positive \( t_0 \) such that
\[
(28) \quad \int_{t_0}^{+\infty} \frac{dt}{\sqrt{\int_{t_0}^{1} l(s)ds}} < \infty .
\]
Using the change of variable \( \tau = \psi_1^{-1}(s) \) we get
\[
\int_{0}^{1} l(s)ds = \int_{0}^{1} [e^{-k_1(\psi_1^{-1}(s))} - 1]ds = \int_{\psi_1^{-1}(t)}^{1} [e^{-k_1(\tau)} - 1]a_1^2(\tau)e^{-k_1(\tau)}d\tau .
\]
It is easy to see that 
\[
e^{-k_1(\tau)} - 1 \geq \frac{1}{2} e^{-k_1(\tau)} \quad \text{for} \quad \tau \leq \tau_0 \quad \text{sufficiently small. Moreover} \quad a_1(\tau) \geq \frac{1}{2}, \quad \text{for} \quad \tau \leq 1. \quad \text{Therefore it suffices to find} \quad t_0 \quad \text{sufficiently large} \quad (t_0 > \psi_1(\tau_0)) \quad \text{such that}
\]
\[
\int_{t_0}^{+\infty} \frac{dt}{\sqrt{\int_{t_0}^{1} e^{-2k_1(\tau)}d\tau}} < \infty .
\]
The last integral can be estimated, using the change \( w = \psi_1^{-1}(t) \) and the fact that \( a_1(s) \leq 1 \), in the following way:
\[
\int_{\psi_1^{-1}(t_0)}^{1} \frac{\psi_1'(w)dw}{\sqrt{\int_{t_0}^{1} e^{-2k_1(\tau)}d\tau}} = \int_{0}^{\psi_1^{-1}(t_0)} e^{-k_2(w)} a_1^2(w) dw \leq \int_{0}^{\psi_1^{-1}(t_0)} \frac{dw}{\sqrt{\int_{t_0}^{1} e^{-2k_1(\tau)}d\tau}} \leq \int_{w_0}^{1} k_1'(w) e^{2[k_1(w) - k_1(\tau)]} d\tau .
\]
where \( w_0 \) is chosen in such a way that \( k_1'(w) \) is decreasing in \( (0, w_0] \). We observe that
\[
\int_{0}^{1} \sqrt{k_1'(w)}dw < \infty , \quad \text{as} \quad \theta < 2 . \quad \text{Hence it suffices to prove that there exists a strictly positive constant} \quad c \quad \text{such that}
\]
\[
k_1'(w) \int_{w}^{w_0} e^{2[k_1(w) - k_1(\tau)]} d\tau \geq c .
\]
Now, since \( k_1'(w) \) is decreasing in \( (0, w_0] \),
\[
k_1'(w) \int_{w}^{w_0} e^{2[k_1(w) - k_1(\tau)]} d\tau \geq \int_{w}^{w_0} k_1'(w) e^{2[k_1(w) - k_1(\tau)]} d\tau = \frac{1}{2} \int_{w}^{w_0} e^{2[k_1(w) - k_1(\tau)]} d\tau .
\]
Observe that \( e^{2k_1(w)} \to 0 \) as \( w \to 0 \), since \( k_1(w) = \int_{1}^{w} \frac{B}{\alpha t^\alpha a_1(t)}dt \to -\infty \) as \( w \to 0 \), by hypothesis \( \theta \geq 1 \). Therefore (28) is proved.

**Step 4.** Theorem 4.1 applies and gives, for every \( \omega \subset \subset \Omega \), the existence of a constant \( C_\omega > 0 \) such that \( v_\omega \leq C_\omega . \) Recalling that \( \psi_n(s) \geq \psi_1(s) \) by (25), we have
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As in Lemmata 2.3, 2.4, 2.5. Then, up to a subsequence, $u_n \to u$ in $L^1(\Omega)$.

Proof. As in the proof of Corollary [1], we pass to the limit in (1) using the a.e. convergence of $u_n$ to $u$ (see Lemmata 2.3, 2.4 and 2.5), the a.e. convergence of $\nabla u_n$ to $\nabla u$ (see Lemma 2.6) and Proposition 2. 

Corollary 3. For every $\omega \subset\subset \Omega$ there exists a positive constant $\tilde{c}_\omega$ such that

$$\frac{u_n}{u_n + \frac{1}{n}} \leq \tilde{c}_\omega \ \forall x \in \omega.$$ 

Proof. It is sufficient to observe that in every subset $\omega \subset\subset \Omega$

$$\frac{u_n}{u_n + \frac{1}{n}} \leq \frac{1}{u_n - \frac{1}{n}} = \tilde{c}_\omega,$$

since $u_n \geq c_\omega > 0$ in $\omega$ by Proposition 3.

As in [3] we prove the strong convergence of $T_k(u_n)$ in $H^1_{loc}(\Omega)$. This will be used to compute the limit of the lower order term in problems (1).

Lemma 4.2. Let $u_n$ be the solutions to problems (17) and $u$ be the function found in Lemmata 2.3, 2.4, 2.5. Then, up to a subsequence, $T_k(u_n) \to T_k(u)$ in $H^1_{loc}(\Omega)$.

Proof. We are going to prove that

$$\lim_{n \to \infty} \int_{\Omega} |\nabla (T_k(u_n) - T_k(u))|^2 \phi = 0$$

for all positive $\phi \in C^\infty_c(\Omega)$. Let $\varphi_\lambda(s) = se^{\lambda s^2}$, $\lambda > 0$. As in [1], we will consider as a test function $\varphi_\lambda(T_k(u_n) - T_k(u))\phi$, where $\lambda$ will be chosen later. In the sequel $\varepsilon(n)$ will denote any quantity converging to 0, as $n \to \infty$. From (19) we get

$$\int_{\Omega} \frac{b(x)}{1 + T_k(u_n))^p} \nabla u_n \cdot \nabla (T_k(u_n) - T_k(u)) \varphi_\lambda'(T_k(u_n) - T_k(u)) \phi$$

$$+ B \int_{\Omega} \frac{u_n |\nabla u_n|^2}{(u_n + \frac{1}{n})^{p+1}} \varphi_\lambda(T_k(u_n) - T_k(u)) \phi = - \int_{\Omega} \frac{b(x)}{(1 + T_k(u_n))^p} \nabla u_n \cdot \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u)) + \int_{\Omega} T_n(f) \varphi_\lambda(T_k(u_n) - T_k(u)) \phi.$$ 

It is not difficult to prove that

$$\int_{\Omega} T_n(f) \varphi_\lambda(T_k(u_n) - T_k(u)) \phi \to 0, \quad \int_{\Omega} \frac{b(x)}{(1 + T_k(u_n))^p} \nabla u_n \cdot \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u)) \to 0,$$

as $n \to \infty$. Indeed for the first limit one can use the Lebesgue Theorem. For the second one it is sufficient to observe that $\nabla u_n$ converges weakly in some Sobolev space given by the statements of Lemmata 2.3, 2.4 and 2.5 and

$$\frac{b(x)}{(1 + T_k(u_n))^p} \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u))$$

is uniformly bounded with respect to $n$. 

\[\Box\]
We are going to treat the left hand side of (29). We choose \( \omega_\phi \subset \Omega \), with \( \text{supp}\phi \subset \omega_\phi \). Then

\[
B \int_\Omega \frac{u_n \nabla u_n}{(u_n + \frac{1}{n})^p} \cdot \nabla \phi \leq -B\tilde{c}_\omega \int_\Omega |\nabla \nabla \phi(T_k(u_n) - T_k(u))| \phi
\]

by Corollary \( 3 \). We deduce from (29) that

\[
\int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla (T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi
- B\tilde{c}_\omega \int_\Omega |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).
\]

We remark that

\[
\int_\{u_n \geq k\} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla (T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi = \varepsilon(n).
\]

Hence inequality (30) is equivalent to

\[
\int_\{u_n \leq k\} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla (T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi
- B\tilde{c}_\omega \int_\Omega |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).
\]

Remark that

\[
\int_\{u_n \leq k\} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla T_k(u) \cdot \nabla (T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi \to 0, \ n \to \infty.
\]

Adding the above quantity in both sides of (31) we get

\[
\int_\{u_n \leq k\} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla (u_n - T_k(u)) \cdot \nabla (T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi
- B\tilde{c}_\omega \int_\Omega |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).
\]

By hypothesis \( 2 \) on \( b \), we obtain

\[
\int_\{u_n \leq k\} \frac{\alpha}{(1 + k)^p} |\nabla (T_k(u_n) - T_k(u))|^2 \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi
- B\tilde{c}_\omega \int_\Omega |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).
\]

It is easy to prove that

\[
\int_\Omega |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \int_\Omega 2 |\nabla (T_k(u_n) - T_k(u))|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi + \varepsilon(n).
\]
We deduce from (32) that the quantity
\[
\int_{\{u_n \leq k\}} \left[ \frac{\alpha}{(1 + k)^p} \varphi'_{\lambda}(T_k(u_n)) - \frac{2B\tilde{c}_\omega}{\lambda} |\varphi(T_k(u_n)) - T_k(u)| \right] |\nabla(T_k(u_n) - T_k(u))|^2 \phi
\]
tends to 0. Now, \( \varphi_\lambda \) has the following property: for every \( a, b > 0 \), \( a\varphi'_\lambda(s) - b|\varphi_\lambda(s)| \geq \frac{a}{2} \) if \( \lambda > \frac{b^2}{4a^2} \). Therefore there exists \( \lambda > 0 \) such that
\[
\frac{\alpha}{(1 + k)^p} \varphi_\lambda(s) - \frac{2B\tilde{c}_\omega}{\lambda} |\varphi_\lambda(s)| \geq \frac{\alpha}{2(1 + k)^p} \quad \forall s \in \mathbb{R}.
\]
Applying this inequality to the quantity (33), the statement of the theorem is proved.

We are now going to prove Theorems 1.3 and 1.4 in a unique proof. As we will see the only difference is the choice of the test functions \( \varphi \). Theorem 1.3 can be proved with the same technique.

Proof. By Lemmata 2.3 and 2.4 the solutions \( u_n \) to (3) are uniformly bounded in \( H^1_0(\Omega) \) and \( W_0^{1,\sigma}(\Omega) \) respectively; moreover \( \nabla u_n \) converges to \( \nabla u \) a.e. in \( \Omega \) up to a subsequence, by Lemma 2.6. The solutions \( u_n \) satisfy
\[
\int_\Omega \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n : \nabla \varphi + B \int_\Omega \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}} \varphi = \int_\Omega T_n(f) \varphi.
\]
For the proof of Theorem 1.3 we consider for \( \varphi \) a bounded \( H^1_0(\Omega) \) function. For the proof of Theorem 1.4 \( \varphi \) is a \( C^1_0(\Omega) \) function. To compute the limit of the first term in the case where \( u_n \) weakly converges to \( u \) in \( H^1_0(\Omega) \) (Theorem 1.3) it is sufficient to use that \( \frac{b(x)}{(1 + T_n(u_n))^p} \nabla \varphi \) strongly converges to \( \frac{b(x)}{(1 + u)^p} \nabla \varphi \) in \( (L^2(\Omega))^N \) for every \( \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \). In the case where \( u_n \) weakly converges to \( u \) in \( W_0^{1,\sigma}(\Omega) \), with \( \sigma < 2 \) (Theorem 1.4), one uses that \( \frac{b(x)}{(1 + T_n(u_n))^p} \nabla \varphi \) strongly converges to \( \frac{b(x)}{(1 + u)^p} \nabla \varphi \) in \( (L^r(\Omega))^N \) for every \( r \geq 1 \) and for every \( \varphi \in C^1_0(\Omega) \).

To compute the limit of \( \int_\Omega \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}} \varphi \) we will use the same technique as in [3]. We are going to prove that \( \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}} \) is equi-integrable. Let \( E \subset \subset \omega \subset \subset \Omega \). Then
\[
\int_E \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}} \leq \int_{E \cap \{u_n \leq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}} + \int_{E \cap \{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}}
\]
\[
\leq \tilde{c}_\omega \int_{E \cap \{u_n \leq k\}} |\nabla T_k(u_n)|^2 + \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}} + \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\sigma}},
\]
where we have used Corollary 3 to estimate the first term. Now, if we choose \( T_1(u_n - T_{k-1}(u_n)) \) in problems (1) we have, dropping the non-negative operator
term,

\[ B \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{q+1}} \leq \int_{\{u_n \geq k-1\}} f. \]

Observe that there exists a constant \( C > 0 \) such that \( \mu(\{u_n \geq k-1\}) \leq \frac{C}{k-1} \), as \( u_n \) are uniformly bounded in \( L^1(\Omega) \). This implies that the right hand side of (34) converges to 0 as \( k \to \infty \), uniformly with respect to \( n \). We deduce that there exists \( k_0 > 1 \) such that

\[ \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{q+1}} \leq \frac{\varepsilon}{2} \quad \forall k \geq k_0, \forall n \in \mathbb{N}. \]

Moreover, since \( T_k(u_n) \to T_k(u) \) in \( H^1_{loc}(\Omega) \) by Lemma \ref{lem:b} there exist \( n_\varepsilon, \delta_\varepsilon \) such that for every \( E \subset \subset \Omega \) with \( \mu(E) < \delta_\varepsilon \) we have

\[ \int_{E \cap \{u_n \leq k\}} |\nabla T_k(u_n)|^2 = \int_{E} |\nabla T_k(u_n)|^2 \leq \frac{\varepsilon}{2c_\omega} \quad \forall n \geq n_\varepsilon. \]

This and (35) imply that \( \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} \) is equi-integrable. Now, recall that \( \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} \) converges a.e. to \( \frac{|\nabla u|^2}{u^\theta} \), belonging to \( L^1(\Omega) \) by Corollary \ref{cor:b}. By Vitali’s theorem we have the result.

\[ \square \]

5. A NON-EXISTENCE RESULT IN THE CASE \( \theta \geq 2 \)

We are going to prove Theorem \ref{thm:1} about the non-existence of finite energy solutions to problem (1) when \( \theta \geq 2 \). We will use the following result of \( \ref{cor:b} \):

**Theorem 5.1.** Let \( M(x,s) \) be a \( N \times N \) matrix whose entries are Carathéodory functions \( m_{ij} : \Omega \times \mathbb{R} \to \mathbb{R} \), for every \( i, j = 1, \ldots, N \). Assume that there exist two positive constants \( \alpha, \beta \) such that \( M(x,s) \xi \cdot \xi \geq \alpha |\xi|^2 \) and \( |M(x,s)| \leq \beta \) for a.e. \( x \in \Omega \), and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \). Let \( g : \Omega \times (0, +\infty) \to \mathbb{R}^+ \) be a Carathéodory function such that for some constants \( s_0, \Lambda > 0 \) and \( \theta \geq 2 \) it holds

\[ g(x,s) \geq \frac{\Lambda}{s^\theta} \quad \forall s \in (0, s_0]. \]

Let \( f \geq 0, f \neq 0 \), be a \( L^q(\Omega) \) function, with \( q > \frac{N}{\theta} \). If one of the following conditions holds:

\begin{enumerate}
  \item \( \theta > 2 \)
  \item \( \theta > 2 \) and \( \lambda_1(f) > \frac{\beta}{\Lambda s_0^\theta} \),
\end{enumerate}

then there is no \( H^1_0(\Omega) \) solution to problem

\[ \begin{cases}
  -\text{div}(M(x,u) \nabla u) + g(x,u) |\nabla u|^2 = f & \text{in } \Omega, \\
  u = 0 & \text{on } \partial\Omega.
\end{cases} \]

**Proof.** (of Theorem 1.6) By the change of variables

\[ v = \begin{cases}
  \frac{1 - (1 + u)^{1-p}}{p - 1} & p \neq 1 \\
  \ln(1 + u) & p = 1,
\end{cases} \]
problem (1) is equivalent to
\[
\begin{aligned}
-\text{div} \left( b(x) \nabla v \right) + Bg(v) |\nabla v|^2 &= f \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
with
\[
g(s) = \begin{cases} 
\frac{[1 - (p - 1)s] \frac{2p}{2-p}}{([1 - (p - 1)s]^{\frac{2p}{2-p}} - 1)^p}, & p \neq 1 \\
\frac{e^{2s} - 1}{(e^{2s} - 1)^p}, & p = 1.
\end{cases}
\]
It is easy to prove that \( g(s)s^\theta \to 1, \) as \( s \to 0^+ \). Hence for every fixed \( 0 < \varepsilon < B \) there exists \( s_\varepsilon > 0 \) such that \( Bg(s) \geq \frac{\varepsilon^\theta}{B} \) for every \( s \in (0, s_\varepsilon] \). Theorem 5.3 therefore applies to problem (36). We deduce that there is no \( H^1_0(\Omega) \) solution to problem \( (1) \) if either \( \theta > 2 \), or \( \theta = 2 \) and \( \lambda_1(f) > \frac{\beta}{\alpha (B - \varepsilon)} \), for every \( 0 < \varepsilon < B \). As a consequence there is no \( H^1_0(\Omega) \) solution to problem (1) if either \( \theta > 2 \), or \( \theta = 2 \) and \( \lambda_1(f) > \frac{\beta}{\alpha B} \). □

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References

[1] (MR1970464) A. Alvino, L. Boccardo, V. Ferone, L. Orsina and G. Trombetti, Existence results for nonlinear elliptic equations with degenerate coercivity, *Annali di Matematica* 182 (2003), pp. 53-79.

[2] (MR2476925) D. Arcoya, S. Barile, P.J. Martínez-Aparicio, Singular quasilinear equations with quadratic growth in the gradient without sing condition, *J. Math. Anal. Appl.* 350 (2009), pp. 401-408.

[3] (MR2308041) D. Arcoya, J. Carmona, P.J. Martínez-Aparicio, Elliptic obstacle problems with natural growth on the gradient and singular nonlinear terms, *Adv. Nonlinear Stud.* 7 (2007), pp. 299-317.

[4] (MR2459205) D. Arcoya, P.J. Martínez-Aparicio, Quasilinear equations with natural growth, *Rev. Mat. Iberoam.* 24 (2008), pp. 597-616.

[5] (MR2514734) D. Arcoya, J. Carmona, T. Leonori, P.J. Martínez-Aparicio, L. Orsina and F. Petitta, Existence and non-existence of solutions for singular quadratic quasilinear equations, *J. Differential Equations* 246 (2009), pp. 4006-4042.

[6] (MR0963104) A. Bensoussan, L. Boccardo and F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solutions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 5 (1988), pp. 347-364.

[7] (MR2287532) L. Boccardo, Quasilinear elliptic equations with natural growth terms: the regularizing effects of lower order terms, *J. Nonlin. Conv. Anal.* 7 no.1 (2006), pp. 355-365.

[8] (MR2434059) L. Boccardo, Dirichlet problems with singular and gradient quadratic lower order terms, *ESAIM Control Optim. Calc. Var.* 14 (2008), pp. 411-426.

[9] (MR1645710) L. Boccardo, A. Dall’Aglio and L. Orsina, Existence and regularity results for some elliptic equations with degenerate coercivity, Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996), *Atti Sem. Mat. Fis. Univ. Modena* 46 suppl. no. 5 (1998), pp. 1-81.

[10] (MR1183664) L. Boccardo and T. Gallouët, Strongly nonlinear elliptic equations having natural growth terms and \( L^1 \) data, *Nonlinear Anal.* 19 (1992), pp. 573-579.

[11] (MR0766873) L. Boccardo, F. Murat and J.-P. Puel, Existence de solutions non bornées pour certaines équations quasi-linéaires, *Port. Math.* 41 (1982), pp. 507-534.
L. Boccardo, F. Murat and J.-P. Puel, $L^\infty$ estimate for some nonlinear elliptic partial differential equations and application to an existence result, *SIAM J. Math. Anal.* 23 (1992), pp. 326-333.

L. Boccardo, L. Orsina and M.M. Porzio, *Existence results for quasilinear elliptic and parabolic problems with quadratic gradient terms and sources*, preprint.

G. Croce, The regularizing effects of some lower order terms on the solutions in an elliptic equation with degenerate coercivity, *Rendiconti di Matematica*, Serie VII, 27 (2007), pp. 299-314.

D. Giachetti and F. Murat, An elliptic problem with a lower order term having singular behaviour, *Boll. Unione Mat. Ital. Sci. B*, in press.

D. Giachetti and M.M. Porzio, Existence results for some nonuniformly elliptic equations with irregular data, *J. Math. Anal. Appl.*, 257 (2001), pp. 100-130.

J.B. Keller, On the solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.* 10 (1957), pp. 503-510.

F. Leoni and B. Pellacci, Local estimates and global existence for strongly nonlinear parabolic equations with locally integrable data, *J. Evol. Equ.* 6 (2006), pp. 113-144.

R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* 7 (1957), pp. 1641-1647.

A. Porretta, Uniqueness and homogenization for a class of noncoercive operators in divergence form, *Atti Sem. Mat. Fis. Univ. Modena* 46 suppl. (1998), pp. 915-936.

A. Porretta, Existence for elliptic equations in $L^1$ having lower order terms with natural growth, *Port. Math.* 57 (2000), pp. 179-190.

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E-mail address: gisella.croce@univ-lehavre.fr