Reissner–Mindlin shell theory based on Tangential Differential Calculus

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We propose a reformulation of the linear Reissner–Mindlin shell theory in terms of tangential differential calculus. An advantage of our approach is that shell analysis on implicitly defined surfaces is enabled and a parametrization of the surface is not required. In addition, the implementation is more compact and intuitive compared to the classical approach. The numerical results confirm, that this approach is equivalent to the classical theory based on local coordinates.

1 Introduction

The standard approach of modelling shells is based on curvilinear coordinates with co- and contra-variant base vectors implied by a parametrization, see, e.g., [1]. Nevertheless, surfaces may also be defined implicitly, e.g., following the level-set method. In this case, the classical approach is not applicable. Therefore, a parametrization-free formulation based on tangential differential calculus is proposed, which can handle both explicitly and implicitly defined surfaces [2]. The obtained boundary value problem (BVP) may also be discretized by new finite element techniques such as TraceFEM or CutFEM, where a parametrization dose not exist.

2 Tangential differential calculus (TDC)

The TDC provides a framework to define surface operators in terms of the global Cartesian coordinates system. These operators are well defined independently of the surface definition, i.e., explicit or implicit. The tangential gradient of a scalar-valued function \( u(x) \) on the surface \( \Gamma \) is defined as \( \nabla_{\Gamma}u(x) = P \cdot \nabla u \), where \( P = I - Q \cdot n_\Gamma \otimes n_\Gamma \) are projection operators, \( n_\Gamma \) is the unit normal vector and \( u \) is a smooth extension of \( u \) in a neighbourhood of \( \Gamma \). In the case of explicitly defined (parametrized) surfaces, the equivalent expression for the tangential gradient is \( \nabla_{\Gamma}u(x) = J \cdot (J^\top \cdot J)^{-1} \cdot \nabla_{\Gamma}u(r) \), where \( J(r) \) is the Jacobi matrix. For vector-valued functions \( \mathbf{v}(x) \) one may distinguish between the directional gradient \( \nabla_{\Gamma}^{\text{dir}} \mathbf{v}(x) = \nabla_{\Gamma} \mathbf{v} \cdot P \) and the covariant gradient \( \nabla_{\Gamma}^{\text{cov}} \mathbf{v}(x) = P \cdot \nabla_{\Gamma}^{\text{cov}} \mathbf{v}(x) \). For a more detailed introduction to the TDC, we refer to, e.g., [3,4].

3 Reissner–Mindlin shell equations

The Reissner–Mindlin shell equations are derived in the frame of the TDC using a global Cartesian coordinates system. In the following, a linear elastic material governed by Hooke’s law is assumed and, herein, we restrict ourselves to infinitesimal displacements and rotations. The shell of thickness \( t \) is defined with the median surface \( \Gamma \) and a thickness parameter in normal direction \( \zeta \leq \lfloor t/2 \rfloor \).

In contrast to the Kirchhoff-Love shell theory [3], transverse shear deformations are considered and the displacement field \( \mathbf{u}_{\Omega} \) is defined as \( \mathbf{u}_{\Omega} = \mathbf{u} + \zeta \mathbf{w} \), where \( \mathbf{u} \) is the deflection of the middle surface and \( \mathbf{w} \) is the difference vector, modelling the rotation of the shell director. As usual in the Reissner–Mindlin theory, the out of plane drilling moment is neglected and therefore, the difference vector is tangential.

The linear strain tensor \( \varepsilon_{\Gamma} = \text{sym} (\nabla_{\Gamma}^{\text{cov}} \mathbf{u}_{\Omega}) \) is split into in-plane strains \( \varepsilon_{\Gamma}^P = P \cdot \varepsilon_{\Gamma} \cdot P \) and transverse shear strains \( \varepsilon_{\Gamma}^S = Q \cdot \varepsilon_{\Gamma} + \varepsilon_{\Gamma} \cdot Q \). With the Lamé constants \( \mu \) and \( \lambda \), the in-plane and transverse shear stresses follow. Analytical pre-integration with respect to the thickness leads to the stress resultants: effective membrane forces \( \mathbf{n}_\Gamma \), bending moments \( \mathbf{m}_\Gamma \), transverse shear forces \( \mathbf{q}_\Gamma \). For curved shells, the physical membrane force is given as \( \mathbf{n}_{\text{real}} = \mathbf{n}_\Gamma + \mathbf{H} \cdot \mathbf{m}_\Gamma \), where \( \mathbf{H} = \nabla_{\Gamma}^{\text{cov}} n_\Gamma \) is the Weingarten map. The force and moment equilibrium in strong form based on the stress resultants is

\[
\text{div}_{\Gamma} \mathbf{n}_{\text{real}}^{\text{dir}} + Q \cdot \text{div}_{\Gamma} \mathbf{q}_{\Gamma} + \mathbf{H} \cdot (\mathbf{q}_{\Gamma} \cdot n_\Gamma) = -f, \tag{1}
\]

\[
P \cdot \text{div}_{\Gamma} \mathbf{m}_{\Gamma} - \mathbf{q}_{\Gamma} \cdot n_\Gamma = -c, \tag{2}
\]

with \( f \) being the load vector per area and \( c \) being a distributed moment vector on the middle surface \( \Gamma \). The obtained equilibrium is valid on explicitly and implicitly defined surfaces, which might be seen as a generalization of the classical shell equations. In combination with boundary conditions as shown in detail in [2], the complete second-order BVP is defined. The equilibrium in strong from is converted to a weak form discretized with isogeometric analysis (IGA) using NURBS as trial...
and test functions [5]. The discrete displacement of the middle surface \( u^h \) is defined by \( u^h = (N^h \cdot \tilde{u}^h)E_i \), where \( E_i \) are the global Cartesian base vectors. The discretization of the difference vector \( w^h \) is, in general, not trivial due to the additional constraint that the difference vector needs to be tangential. In [2] different approaches of discretizing tangential vector fields are outlined. Herein, the discrete difference vector \( w^h \) is defined in the global Cartesian coordinate system and the constraint is enforced weakly with a Lagrange multiplier. In the context of Lagrange multiplier methods, the resulting weak form has a saddle point structure and the discrete Babuška-Brezzi condition needs to be satisfied in order to obtain useful solutions in the involved fields. Here, based on numerical experiments, the same order of the trial and test functions are employed for all fields. Another important issue in the context of Reissner-Mindlin shells is locking. The discrete system tends to transverse shear locking in the case of thin shells. Herein, only order elevation is chosen as a measure against locking.

### 4 Numerical results

As an example the partly clamped hyperbolic paraboloid from [6] is chosen. The middle surface is defined by \( z = x^2 - y^2 \) with \((x, y) \in [-0.5, 0.5]^2\). The thickness of the shell is set to \( t = 0.01 \). The material parameters are Young’s modulus \( E = 2 \times 10^{11} \), Poisson’s ratio \( \nu = 0.3 \). The edge at \( x = -0.5 \) is clamped and the other edges are free. The shell is loaded by gravity forces \( f = [0, 0, -8000 \cdot t]^T \). For the convergence analyses, the problem is computed with uniform meshes \( n = [2, 4, 8, 16, 32, 64] \) elements per side and different orders up to \( p = 6 \).

Fig. 1: Partly clamped hyperbolic paraboloid: (a) Displacement field \( u \) (scaled by a factor of 2000), (b) Normalized convergence of reference displacement \( u_{z,i,\text{Ref}} = -9.3355 \times 10^{-5} \) at point \( i = [0.5, 0, 0.25]^T \).

In Fig. 1(a), the grey surface represents the undeformed configuration and the colours of the deformed shell are the Euclidean norm of the displacement field \( u \). In Fig. 1(b), the normalized convergence of the reference displacement \( u_{z,i} \) is plotted as a function of the element size \( 1/n \). The convergence is as expected and in agreement with the results shown, e.g., in [6, 7]. In [2], additional test cases are investigated, also enabling optimal higher-order convergence rates.

### 5 Conclusion and Outlook

We have presented a parametrization-free reformulation of the linear Reissner–Mindlin shell theory based on the TDC. The shell equations are applicable to both explicitly and implicitly defined surfaces. The numerical results are obtained with surface FEM using NURBS as trial and test functions. The results of the convergence analyses confirm that the reformulation is equivalent compared to the classical approach in the case of parametrized surfaces. In future works, the obtained shell equations are discretized on implicitly defined surfaces using recent finite element techniques such as TraceFEM, where the classical formulation of shells fails.

### References

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