MULTIPRESSURE POLYTROPES AS MODELS FOR THE STRUCTURE AND STABILITY OF MOLECULAR CLOUDS. I. THEORY

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ABSTRACT

We present a theoretical formalism for determining the structure of molecular clouds and the precollapse conditions in star-forming regions. The model consists of a pressure-bounded, self-gravitating sphere of an ideal gas that is supported by several distinct pressures. Since each pressure component is assumed to obey a polytropic law \( P_i(r) \propto \rho^{\gamma_i} \), we refer to these models as “multipressure polytropes.” We treat the case without rotation. The time evolution of one of these polytropes depends additionally on the adiabatic index \( \gamma_i \) of each component, which is modified to account for the effects of any thermal coupling to the environment of the cloud. We derive structure equations as well as perturbation equations for performing a linear stability analysis. Special attention is given to representing properly the significant pressure components in molecular clouds: thermal motions, static magnetic fields, and turbulence. The fundamental approximation in our treatment is that the effects of turbulent motions in supporting a cloud against gravity can be approximated by a polytropic pressure component. In particular, we approximate the turbulent motions as a superposition of Alfvén waves. We generalize the standard treatment of the stability of polytropes to allow for the flow of entropy in response to a perturbation, as expected for the entropy associated with wave pressure. In contrast to the pressure components within stars, the pressure components within interstellar clouds are “soft,” with polytropic indices \( \gamma_{pl} \leq 4/3 \) and (except for Alfvén waves) adiabatic indices \( \gamma_i \leq 4/3 \). This paper focuses on the characteristics of adiabatic polytropes with a single pressure component that are near the brink of gravitational instability as a function of \( \gamma_{pl} \) and \( \gamma_i \) for \( \gamma_{pl} \leq 4/3 \). The properties of such polytropes are generally governed by the conditions at the surface. We obtain upper limits for the mass and size of polytropes in terms of the density and sound speed at the surface. The mean-to-surface density and pressure drops are limited to less than a factor 4 for \( \gamma_{pl} \leq 1 \), regardless of the value of \( \gamma_i \). The central-to-surface density and pressure drops in isentropic clouds \( (\gamma_i = \gamma_{pl}) \) are also limited, but they can become quite large (as observed) in non-isentropic clouds, which have \( \gamma_i > \gamma_{pl} \). We find that the motions associated with Alfvén waves are somewhat less effective in supporting clouds than are the kinetic motions in an isothermal gas.

Subject headings: ISM: clouds — ISM: molecules — ISM: structure — stars: formation

1. INTRODUCTION

Star formation plays a crucial role in galactic evolution and in determining the structure of the interstellar medium. In general, star-forming regions appear to be in rough hydrostatic equilibrium since the observed supersonic line widths in giant molecular clouds (GMCs) are due to disordered macroscopic motions that contribute to dynamical support rather than to large-scale infall (Zuckerman & Evans 1974). While the large-scale turbulent motions within GMCs lead to evolution of the shape of the cloud (Ballesteros-Paredes, Vazquez-Semadeni, & Scalo 1999), they do not lead to an overall expansion or contraction on a dynamical timescale. Correspondingly, the lifetime of a GMC typically exceeds its free-fall timescale \((\sim 10^6-5 \text{ yr})\) by about an order of magnitude (Blitz & Shu 1980). To a good approximation, then, most observed GMCs are in a steady state. Structures within GMCs also appear to be in approximate equilibrium: the density in and around a typical ammonia core within a GMC varies roughly as \( r^{-2} \) (e.g., Snell 1981; Fulkerson & Clark 1984; Zhou et al. 1990; Ladd et al. 1991), a relationship that is consistent with a simple, stably stratified model. Turner (1993) finds that the majority of cores he observes in high-latitude cirrus clouds have characteristics that are consistent with hydrostatic equilibrium.

We find that spherical models accurately represent many of the properties of these equilibrium structures, aside, of course, from details about the multidimensional cloud geometry. In particular, the time-honored problem of the structure and gravitational stability of gas spheres bears upon several unresolved issues in star formation. What is the dependence of density and line width on size scale for a cloud in stable hydrostatic equilibrium? What are the initial conditions appropriate for gravitational collapse, particularly on large scales where thermal pressure is relatively unimportant? How can GMCs be stable at the high mean pressures often observed?

1.1. Molecular Cloud Structure

Because we must specify an equilibrium structure before performing a stability analysis, our first step is to determine the density and line-width profiles of a model molecular cloud. For a cloud in hydrostatic equilibrium, these profiles can be determined theoretically once the nature of the pressure supporting the cloud is specified. We consider three
sources of pressure: thermal motions, static magnetic fields, and turbulence.

The thermal sound speed in a given molecular cloud is roughly constant and corresponds to a temperature of 10–30 K. Thermal motions typically have significant dynamical influence only in cloud cores (Myers & Benson 1983). This pressure component may dominate support in these limited regions if the static magnetic field is nearly straight and uniform or if the field strength is low.

Typically, though, static magnetic fields play a substantial role in stabilizing molecular clouds (Mouschovias 1976b; Heiles et al. 1993). Magnetic support is particularly significant in regions that are forming clusters of stars. Since the mass of a star-forming core exceeds its thermal Jeans mass, a clump containing multiple cores must contain many thermal Jeans masses; the material in the clump thus owes much of its dynamical support to the static magnetic field.

Turbulent motions provide another source of nonthermal support of molecular clouds. On large scales, turbulent motions have an energy density comparable to that of the static magnetic field (Myers & Goodman 1988). These motions are supersonic and are observed to be an increasing function of map size (Larson 1981; Cernicharo, Bachiller, & Duvert 1985; Myers & Fuller 1992). It was suggested some time ago that these motions are large-amplitude MHD waves (Arons & Max 1975); recent observations (Myers & Goodman 1988) and theoretical results (Bonazzola et al. 1987; Pudritz 1990; McKee & Zweibel 1995) are consistent with this idea. Alfven waves, being noncompressive, are expected to be the dominant form of MHD waves present (McKee & Zweibel 1995). The pressure due to the waves includes both the dynamic pressure associated with the motion of the gas and the magnetic pressure associated with a time-dependent magnetic field. The importance of the wave pressure increases with size scale for two reasons: first, there is an increase in the range of wavenumbers for the MHD waves (Bonazzola et al. 1987), and, second, the density decreases with size (Larson 1981), which leads to an increase in the amplitude of the waves (Fatuzzo & Adams 1993; McKee & Zweibel 1995). Numerical simulations confirm that MHD waves are an important contributor to the support of molecular clouds, but they also indicate that the waves damp out very rapidly (Gammie & Ostriker 1996; Vazquez-Semadeni et al. 1999; but see McKee 1999).

The primary pressure components that support molecular clouds against gravitational collapse have been identified, and there is substantial agreement as to when each plays an important role: static fields are important throughout molecular clouds, whereas thermal pressure is important primarily in small, dense regions and turbulent, or wave, pressure is important on large scales. However, there is no quantitative understanding as to how these three sources of pressure interact to produce the density and velocity structure that is observed in molecular clouds. To date there has been only one study of the case in which all three components are present (Lizano & Shu 1989). In this work, as well as in subsequent work (Gehman et al. 1996; McLaughlin & Pudritz 1996), a phenomenological model is adopted for the wave pressure, in which it varies as the logarithm of the density (a “logatrophe”). The disadvantage of such phenomenological models is that it is impossible to study the stability of the clouds in a physically motivated way. Our intention is to develop a theoretical framework in which the distinct pressure components can be modeled in such a manner that both the structure and the stability of the clouds can be determined.

1.2. Precollapse Conditions in Star-forming Regions

The state of a cloud at the onset of collapse determines the development of the density and velocity profiles once collapse begins, the fragmentation of the cloud, the luminosity of the protostars that form, and the final stellar configuration. We can illustrate the effects of the precollapse conditions by comparing self-similar solutions for the collapse of an isothermal sphere, which is often used to model low-mass star formation. There is an infinite number of such solutions (Hunter 1977), with the solution discovered by Larson (1969) and Penston (1969) and the solution found by Shu (1977) demonstrating the ranges of velocity profiles and densities that are possible. The Larson-Penston (LP) solution starts from a static, uniform gas that fills a large spherical volume. As the gas collapses under the influence of gravity, a rarefaction front propagates inward from the outer boundary at the sound speed. When the rarefaction reaches the origin, a condensed object forms. At this instant, the inflow is spatially constant at 3.3 times the sound speed; the density has the same scaling as a singular isothermal sphere (SIS), \( \rho \propto r^{-2} \), but its magnitude is 4.4 times greater. Shu (1977) considered the other extreme, in which collapse begins from the equilibrium state described by the SIS. This solution is “inside-out” in the sense that, after a perturbation at the origin, information that internal regions are evolving dynamically propagates outward in an expansion wave. Since the initial state is static, with \( \rho \propto r^{-2} \), core formation begins immediately, before a flow field develops. Following the formation of a condensed object in both the LP solution and the Shu solution, the region around this object acquires power-law profiles of the form \( \rho \propto r^{-3/2} \) and \( v \propto r^{-1/2} \); however, the density and velocity at comparable points remain larger in the LP solution. As a result, the mass accretion rate and the accretion luminosity are larger in the LP solution than in the Shu solution.

Results from studies of the dynamical evolution of collapsing isothermal clouds often resemble the analytic solutions described above. An isothermal sphere in an unstable, but nonsingular, equilibrium has a density profile that is approximately flat (like the initial state of the LP solution) in a central region but that resembles the \( r^{-2} \) form of the SIS in an outer region. As a result, simulations of collapse from such initial conditions (Hunter 1977; Fiedler & Mouschovias 1993; Foster & Chevalier 1993) display densities and flows near their incipient condensed objects that closely resemble those in the LP solution. By contrast, the material in the outer part of the sphere maintains acoustic contact while adjusting to imbalances between pressure gradients and gravity; thus, in this region, the simulation more closely reproduces the zero flow beyond the expansion wave in the SIS collapse solution. If the part of the sphere that approximates \( \rho \propto r^{-2} \) prior to collapse is sufficiently large, the accretion rate eventually approaches the constant mass accretion rate of the SIS (Foster & Chevalier 1993). Similar results are found for the gravitational collapse of a cloud supported by a magnetic field undergoing ambipolar diffusion (Safier, McKee, & Stahler 1997). The initial state of the cloud also affects the stability of the collapse with respect to fragmentation, an essential process in the formation of star systems and clusters (Boss 1988). The collapse of a uniform-
density sphere is unstable to fragmentation, while that of an isothermal sphere is much less so (Shu 1977; Silk & Suto 1988). Simulations (e.g., Boss 1987) confirm that a more centrally concentrated initial density profile inhibits fragmentation during dynamical collapse.

This discussion illustrates the impact of the precollapse conditions on star formation in the idealized case of an isothermal sphere. In reality, the gas may not be isothermal, and magnetic fields and turbulent pressure both contribute to the support of the cloud. One simplification remains possible for low-mass star formation, however: it is believed to proceed quiescently, as ambipolar diffusion gradually reduces the pressure support due to static magnetic fields (Mestel & Spitzer 1956; Shu et al. 1987; Mouschovias 1991). In this case, the collapse proceeds from a critical point—an equilibrium state on the verge of gravitational instability. By this criterion, neither a homogeneous sphere nor an SIS are accurate representations of the precollapse state, since the first is not an equilibrium and the second is an unstable equilibrium. Nevertheless, highly centrally concentrated initial states are possible; calculations indicate that static magnetic fields provide a substantial stabilizing influence both in terms of the critical mass they support and the maximum center-to-surface density ratio that they allow (Mouschovias 1976b; Tomisaka, Ikeuchi, & Nakamura 1988b, hereafter TIN; Lizano & Shu 1989). One of the objectives of this paper is to develop a framework for determining how these equilibria, particularly those at a critical point, are affected by Alfvén waves.

1.3. Pressures in Molecular Clouds

GMCs appear to maintain a mean pressure that is substantially above the pressure in the local interstellar medium. Observations indicate that the mean kinetic pressures of nearby GMCs range from $4 \times 10^4$ to $2.4 \times 10^5$ K cm$^{-3}$ (Bertoldi & McKee 1992), and the typical GMC in the molecular ring of the Galaxy has a mean pressure of $3 \times 10^3$ K cm$^{-3}$ based on the column densities inferred by Solomon et al. (1987). By comparison, the pressure in the local interstellar medium is only about $2 \times 10^4$ K cm$^{-3}$ (Boulares & Cox 1990; we have subtracted out the cosmic-ray pressure since it is approximately uniform and cannot contribute to the support of GMCs).

The origin of the relatively high mean pressures in GMCs is not understood. Consider the simple, well-studied example of the isothermal sphere: the maximum ratio of mean to surface pressure for a stable isothermal sphere is only about 2.5 (e.g., Spitzer 1968), well below what is deduced from observations. The polytropes suggested by Maloney (1988) have even smaller pressure ratios. In the magnetized models of TIN, the maximum stable pressure ratio is also about 3. Center-to-surface pressure ratios can be larger but are also often limited—for example, a stable isothermal sphere has a maximum pressure drop of 14. As we shall see in § 5, however, it is possible to achieve larger center-to-surface pressure drops in polytropic models (see also Curry & McKee 1999); large center-to-surface pressure drops were also found in the FUV-heated clouds studied by Falgarone of Puget (1985, 1986).

The resolution of this issue is crucial. Because the mean pressure of GMCs is representative of the pressure of the star-forming regions within them, it is intimately coupled to the star formation process. The mean pressure determines, in part, the mean density in the star-forming region, which in turn affects the density of the star cluster that forms from the cloud. The line width–size relation (Larson 1981) suggests that the mean pressure of self-gravitating GMCs is constant: in GMCs, the line width–size relation has the form $\sigma \propto R^{1/2}$, where $\sigma$ is the one-dimensional velocity dispersion and $R$ is the cloud radius (e.g., Solomon et al. 1987); for self-gravitating clouds, we also have $\sigma^2 \propto M/R$, so that $M \propto R^2$, and hence the mean pressure, $\bar{P} \approx \rho \sigma^2 \propto (M/R^3)R \propto R^4$, should be the same for all GMCs. Chièze (1987) and Elmegreen (1989) have suggested that this constancy can be understood if the ambient pressure of GMCs is constant since, in their models, $\bar{P}$ is proportional to the ambient pressure.

For both problems—the origin of the line width–size relation and the stellar number density ultimately produced by the cloud—an understanding of the high mean pressure of GMCs is essential. Part of the explanation lies in the fact that the pressure in the molecular gas is increased by the weight of the overlying atomic gas. Elmegreen (1989) pointed out the importance of this effect for the atomic hydrogen; Holliman (1995) showed that the effect is even larger if one is attempting to explain CO data on molecular clouds, since the layer in which the carbon is atomic is considerably thicker (and therefore heavier) than the atomic hydrogen layer. However, it is not clear that this effect is sufficient. Two other effects that could contribute to the large pressures observed in GMCs are (1) that the ambient pressures of GMCs could be several times greater than the local interstellar value, perhaps because of the effects of massive star formation; and (2) that the structure of the clouds could permit a mean pressure in the molecular gas significantly above the values allowed in existing models. The formalism developed herein can be used to evaluate the effect of the overlying atomic gas and to determine the mean pressure in the molecular gas.

1.4. Previous Models of Self-gravitating Gas Clouds

In this paper we appeal to the relative simplicity of polytropic spherical models, in which $P(r) \propto \rho^{\gamma_p}(r)$, to determine the structure and stability of self-gravitating gas clouds. Some of the earliest work on this problem was carried out by Ebert (1955) and Bonnor (1956), who investigated the stability of pressure-bounded, gravitating spheres supported exclusively by isothermal gas pressure. These models become susceptible to dynamic instability when self-gravity induces a center-to-surface density ratio exceeding 14. Non-isothermal polytropes ($\gamma_p \neq 1$) have been considered by Shu et al. (1972), Viala & Horedt (1974), and Chièze (1987) under the assumption that there is no internal heat flow (a locally adiabatic system; see § 2.2). These results show that the maximum stable center-to-surface pressure ratio increases with $\gamma_p$. Maloney (1988) considered polytropes with $\gamma_p < 1$ in order to account for the observed line width–size relation, since such polytropes have line widths that increase outward. He assumed that some unspecified heating mechanism was able to maintain a constant central temperature when the cloud was compressed; with this assumption, clouds are stable to gravitational collapse. Lizano & Shu (1989), Gehman et al. (1996), and McLaughlin & Pudritz (1996) considered polytropes in the limit $\gamma_p \to 0$ (logatropes) to model turbulent pressure in clouds. Isothermal polytropes ($\gamma_p = 1$) with nonisothermal specific heats were analyzed by Yabushita (1968).

Lynden-Bell & Wood (1968, hereafter LBW), used poly-
tropes with $\gamma_p = 1$ to study self-gravitating star clouds (i.e., globular clusters) in order to understand the gravothermal catastrophe discovered by Antonov (1962). In contrast to the polytropic models of gas clouds cited above, internal heat flow is allowed, although there is no heat transfer from the cloud to its environment (a globally adiabatic system; see §2.2). Indeed, it is efficient thermal conduction inside the star cluster that establishes the isothermal temperature profile. They modeled the cluster as a gas with no internal degrees of freedom. If such a gas is confined by an ambient pressure, the onset of dynamical instability occurs at a center-to-surface density ratio of 389 (more precisely, we find 389.6); the fact that the gas heats up when it contracts permits the cloud to evolve to much greater densities before it becomes unstable. However, for a cluster of stars it is appropriate to consider a gas confined to a fixed volume rather than by a fixed pressure since the environment of the cluster performs no work on the stellar fluid. In this case, they showed that the maximum center surface density for a stable configuration is 709.

The structure of rotating, self-gravitating clouds has been discussed by Stahler (1983). Most clouds in equilibrium are thought to derive significant support from static magnetic fields, and magnetic braking should diminish the dynamical influence of rotation (Mestel & Paris 1984; Mouschovias 1987). The direct impact of static magnetic fields on the structure and stability of gas clouds is explored in the axisymmetric models of Mouschovias (1976a, 1976b), Tomisaka, Ikeuchi, & Nakamura (1988a), TIN, and Lizano & Shu (1989).

In §3 we present an approach for determining the structure of a spherical cloud supported by multiple pressure components. The stability analysis in §4 then allows us to determine the range of stable equilibria that can represent cloud structure. In many cases, these models are close to the critical equilibrium that defines the limit of gravitational stability, since it has been argued that ammonia cores (Myers & Benson 1983; Foster & Chevalier 1993), massive star-forming clumps (Bertoldi & McKee 1992), and GMCs as a whole (McKee 1989) all verge on gravitational collapse.

2. FORMULATION OF THE MODELS

2.1. Multipressure Polytropes

Our intention in this work is to determine the structure and stability of gas clouds in hydrostatic equilibrium that are supported by several pressure components. The basic assumption underlying our work is that such a quasi-static model can be used to model turbulent molecular clouds. We assume that in equilibrium, each pressure component satisfies a polytropic relation of the form $P_i(r) = K_{pi} \rho_i^{\gamma_i}(r)$, where $K_{pi}$ is independent of position. In general, one can generalize the classical analysis of polytropes to multiple components in several ways.

1. Multilayered, or composite, polytropes have spatially distinct pressure components (Chandrasekhar 1939, p. 170), as in the core-envelope stellar models of Schönberg & Chandrasekhar (1942). Applications to gas clouds are considered by Curry & McKee (1999).

2. Multifluid polytropes have several different components that interact only gravitationally. Examples include multimass models for star clusters (Taff et al. 1975), models for gas in dark matter halos (Umemura & Ikeuchi 1985; Gerhard & Silk 1996), and models for molecular clouds including embedded stars.

3. Multipressure polytropes consist of a single self-gravitating fluid with several pressure components, so that the total pressure $P(r)$ satisfies the relation

$$P(r) = \sum P_i(r) = \sum K_{pi} \rho_i^{\gamma_i}(r).$$

It is this last type of polytrope that is of interest here. As discussed above, the pressure components relevant to molecular clouds are thermal pressure, static magnetic pressure, and the pressure due to MHD waves.

We shall restrict our attention to spherical polytropes. Treating molecular clouds as spherical is clearly a substantial idealization, but it enables us to explore some of the essential physics underlying their structure. There are several effects that could lead to deviations from spherical symmetry.

1. Rotation.—As discussed in §1.4 above, rotation is expected to have a relatively small effect on the structure of molecular clouds because of the effects of magnetic braking.

2. Tidal gravitational fields.—These also appear to have a minor effect on Galactic molecular clouds (Scoville & Sanders 1987).

3. Massive star formation.—Massive stars can significantly disrupt molecular clouds, leading to violations of both hydrostatic equilibrium and spherical symmetry. Our models apply only to clouds that are not being disrupted by this process; fortunately, in the Galaxy there are many more GMCs than large OB associations (Williams & McKee 1997), so this will be a reasonable approximation for many clouds.

4. Anisotropic pressure.—The stress due to an ordered magnetic field is intrinsically anisotropic and necessarily leads to a violation of our assumption of spherical symmetry. Nonetheless, many of the qualitative features of magnetic stresses can be captured by treating them as spherically symmetric (Saftor et al. 1997), and it is possible to obtain reasonably accurate quantitative results for the structure of magnetized clouds using the approximation of a spherically symmetric magnetic stress (Holliman 1995).

5. Large-scale turbulent motions.—Such motions can make clouds quite nonspherical (Ballesteros-Paredes et al. 1999); however, the time average of the cloud shape is much closer to being spherical.

Polytropes with a single pressure component are often described in terms of a parameter $n$ such that $\rho(r) \propto T^n(r)$; the polytropic index $\gamma_p$ is related to $n$ by

$$\gamma_p = 1 + \frac{1}{n}.$$  

"Negative-index polytropes" with $n < 0$ have been discussed by Shu et al. (1972), Viala & Horedt (1974), and Maloney (1988). Such polytropes are hotter on the outside than at the center, just the opposite of the case in a star. An example of such a polytrope that is relevant for our models is the pressure due to Alfvén waves, which satisfy $\gamma_p = 1/2$ (McKee & Zweibel 1995), corresponding to $n = -2$. We shall assume that $\gamma_p > 0$, since polytropes with $\gamma_p < 0$ are unstable for all values of the external pressure (Viala & Horedt 1974).
2.2. Locally Adiabatic and Globally Adiabatic Pressure Components

In order to determine the stability of multipressure polytropes, we assume that the response of each pressure component to an adiabatic perturbation is the same as that of an ideal gas that is either "locally adiabatic" or "globally adiabatic." Physically, a locally adiabatic pressure component has a timescale for internal heat transfer that is long compared to the dynamical timescale, whereas a globally adiabatic component has a short timescale for internal heat transfer. The distinction between local and global adiabaticity is necessitated by the fact that we are treating self-gravitating systems that are inhomogeneous.

The response of a locally adiabatic pressure component to an adiabatic density perturbation satisfies the usual adiabatic relation,

$$\delta \ln P_i = \gamma_i \delta \ln \rho .$$

(3)

The specific entropy associated with a pressure component in our ideal gas model is

$$s_i = \frac{k}{\gamma_i - 1} \ln \left( \frac{P_i}{\rho_i^\gamma} \right) + \text{const}$$

(4)

(e.g., Landau & Lifschitz 1958). The entropy is thus determined by the entropy parameter

$$K_i \equiv \frac{P_i}{\rho_i^\gamma} ;$$

(5)

this does not change with time for a locally adiabatic pressure component. Just as in the analysis of the stability of stars (Leduc 1965), the adiabatic indices $\gamma_i$ that describe the temporal variations of the pressure components need not be the same as the polytropic indices $\gamma_{pi}$ that describe the spatial variations. The polytropic indices $\gamma_{pi}$ help to determine the equilibrium structure of a cloud (§3); stability depends additionally on the adiabatic indices $\gamma_i$ (§4). A pressure component with $\gamma_i = \gamma_{pi}$ is isentropic, since it has a spatially constant specific entropy that remains constant during an adiabatic perturbation: $K_i = P_i(r)\rho_i(r)^{-\gamma_i} \propto \rho_i(r)^{1-\gamma_i}$ is spatially and temporally constant for $\gamma_i = \gamma_{pi}$. Chandrasekhar (1939) termed such components as being in "convective adiabatic equilibrium" since efficient convection produces such a structure in a thermally insulated system. On the other hand, if a locally adiabatic component has $\gamma_i \neq \gamma_{pi}$, then it cannot remain polytropic after a pressure perturbation; Yabushita (1968) has considered this case for spatially isothermal spheres ($\gamma = 1$).

If internal heat transfer is significant for a pressure component, then we assume that the component is globally adiabatic. Such a pressure component retains its polytropic form during an adiabatic perturbation because of efficient internal heat flow in the cloud. The distinction between locally adiabatic and globally adiabatic components is important only for $\gamma_i \neq \gamma_{pi}$; if the component is isentropic, then no heat transfer is required to maintain the polytropic structure and the component is locally adiabatic as well. The model of a globular cluster considered by LBW, in which the stars interact dynamically so as to maintain an isothermal distribution, provides an example of a globally adiabatic system. In our work, Alfvén waves, which have $\gamma_{pi} = 1/2$ (Walén 1944; Weinberg 1962) and $\gamma_i = 3/2$ (McKee & Zweibel 1995), are modeled as a globally adiabatic pressure component. (Note that since $\gamma_{pi} \neq 1$, the "heat flow" associated with the Alfvén waves does not make the gas isothermal.) The polytropes considered by Maloney (1988), like our model of a polytrope supported by Alfvén waves, have $\gamma_{pi} < 1$ and are not locally adiabatic; however, Maloney's models require an injection of heat to maintain a constant central temperature during a compression, whereas our model redistributes heat that is already present.

It is important to distinguish the physical pressure components from the model pressure components. We model each pressure component as a thermally insulated, ideal gas with a local ratio of specific heats equal to $\gamma_i$. If the pressure component is actually thermally insulated, then $\gamma_i$ is the ratio of specific heats, as usual. On the other hand, if the gas is subject to heating and cooling, then it can be modeled as a locally adiabatic component with an adiabatic index $\gamma_i$ that is generally not equal to the physical ratio of specific heats; the effects of heating and cooling are treated by invoking hypothetical internal degrees of freedom. For example, if the rate for heating is proportional to $\propto n T^\alpha$ and the rate for cooling varies as $\propto n T^\beta$, then it can be shown that, in equilibrium, the model has $\gamma_i = 1 + 1/(a - b)$, whereas the gas (if it is monatomic) actually has a ratio of specific heats of 5/3. In such a case, $P(r, t)$ depends only on density, and as a result the pressure component is isentropic ($\gamma_i = \gamma_{pi}$).

3. STRUCTURE EQUATIONS

The elegance of polytropic models of self-gravitating clouds is due to the small number of physical constants and parameters that are needed to specify the structure of the cloud. The structure is determined by the mass equation,

$$\frac{dM}{dr} = 4\pi r^2 \rho ,$$

(6)

and the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = - \frac{GM}{r^2} \rho .$$

(7)

Note that these equations remain valid even if the sphere evolves quasi-statically.

It is customary to combine these equations into a single second-order differential equation, the Lane-Emden equation (e.g., Chandrasekhar 1939; see Appendix A). However, this equation admits a homology transformation, so that one of the two boundary conditions serves merely to set the density scale (Chandrasekhar 1939). As a result, it is possible to write this equation as a first-order equation in terms of the scale-free variables

$$u \equiv \frac{4\pi \rho r^3}{M(r)} , \quad (n + 1) v \equiv \frac{GM(r)\rho}{Pr} ,$$

(8)

with a separate first-order equation for the dimensionless radius (Chandrasekhar 1939).

We have chosen a variation of this approach, in which we place the two first-order equations on an equal footing. We adopt

$$\chi \equiv \ln \left( \rho_c / \rho \right) ,$$

(9)

as an independent variable, where $\rho_c$ is the central density of the equilibrium cloud (i.e., it does not vary when the cloud is perturbed). $\chi$ is generally a monotonically increasing function of $r$: since $P(r)$ is monotonically decreasing in a
self-gravitating cloud, \( \rho(r) \) will also be monotonically decreasing provided only that the pressure varies as a positive power of the density (\( \gamma_p > 0 \)), which is necessary for stability in any case (Viala & Horedt 1974).

As dependent variables, we adopt one that is proportional to the mass measured in Jeans masses (cf. Stahler 1983),

\[
\mu \equiv \frac{M(r)}{c^3(r)/[G^{\mu} \rho(r)]^{1/2}}, \quad (10)
\]

and one proportional to the radius measured in Jeans lengths,

\[
\lambda \equiv \frac{r}{c(r)/[G^{\mu} \rho(r)]^{1/2}}, \quad (11)
\]

where \( c \equiv (P/\rho)^{1/2} \) is the isothermal sound speed. More specifically, \( \mu \sim M(r)/M_J(r) \), where \( M_J(r) \) is the local generalized Jeans mass—i.e., the maximum mass that the concerted action of all pressure components can maintain in equilibrium, given the values of specified parameters, such as the entropy and ambient pressure. Similarly, \( \lambda \sim r/R_J(r) \), where \( R_J(r) \) is the radius of a uniform-density sphere containing \( M_J(r) \). These variables are related to the standard homology variables by

\[
u = \frac{4\pi \lambda^3}{\mu}, \quad (n + 1)\nu = \frac{\mu}{\lambda}. \quad (12)
\]

An advantage of our dependent variables (and of the homology variables) is that they depend upon the properties at the surface of the cloud, not at the center. For interstellar clouds, the surface pressure is generally known, whereas the conditions at the center of the cloud are not. Further discussion of the relationship between our variables and the standard nondimensional variables can be found in Appendix A.

Since individual polytropic pressure components can be expressed as \( P_i = K_{pi} \rho_i \exp (-\chi_i) \gamma_p^i \), the total pressure is

\[
P = \sum_i K_{pi} \rho_i \gamma_p^i \exp (-\chi_i) \gamma_p^i. \quad (13)
\]

Since \( d\ln \rho = -d\chi_i \), the overall polytropic index is

\[
\gamma_p = \frac{d\ln P}{d\ln \rho} = -\frac{d\ln P}{d\chi_i} = 1 - \frac{d\ln c^2}{d\chi_i}. \quad (14)
\]

For a single pressure component, \( \gamma_p \) is spatially constant for polytropes. However, for a multicomponent system, \( \gamma_p \) can be a function of position, since equations (13) and (14) imply that

\[
\gamma_p = \frac{\sum_i K_{pi} \rho_i \gamma_p^i \exp (-\chi_i) \gamma_p^i}{\sum_i K_{pi} \rho_i \gamma_p^i \exp (-\chi_i) \gamma_p^i} = \sum_i \left( \frac{P_i}{P} \right) \gamma_p^i \quad \text{(polytrope)}, \quad (15)
\]

where we have indicated that this equation is valid only for polytropes, which have \( K_{pi} \) constant. In terms of our dimensionless variables, the mass equation (6) yields

\[
\frac{d\ln M}{d\chi} = 4\pi \gamma_p \frac{\lambda^4}{\mu^2}, \quad (16)
\]

whereas equation (7) becomes

\[
\frac{d\ln r}{d\chi} = \frac{\gamma_p \lambda}{\mu}. \quad (17)
\]

Appearances of \( \ln M \) and \( \ln r \) in equations (16) and (17) can now be eliminated in favor of expressions involving \( \mu \) and \( \lambda \) with the aid of equations (10) and (11),

\[
d\ln M = d\ln \mu + 3d\ln c - \frac{1}{2} d\ln \rho \quad (18)
\]

and

\[
d\ln r = d\ln \lambda + d\ln c - \frac{1}{2} d\ln \rho. \quad (19)
\]

We then use equations (14), (18), and (19) in equations (16) and (17) to obtain the structure equations for multipressure polytropes:

\[
\frac{d\ln \mu}{d\chi} = 4\pi \gamma_p \frac{\lambda^4}{\mu^2} - \frac{1}{2} (4 - 3\gamma_p) \quad (20)
\]

and

\[
\frac{d\ln \lambda}{d\chi} = \frac{\gamma_p \lambda}{\mu} - \frac{1}{2} (2 - 3\gamma_p). \quad (21)
\]

If desired, one can obtain a single first-order equation by taking the ratio of these two equations; once \( \mu(\lambda) \), say, is found from integrating this equation, the density can be evaluated by integrating equation (21).

The structure equations are integrated outward from the center of the cloud, which has \( \mu = \lambda = 0 \) and \( \chi = 0 \). In order to interpret \( \lambda/\mu \) near the center, we also need a relation between these two variables there. Since the cloud is not singular, the density approaches a constant at the center, and as a result

\[
\frac{\mu}{\lambda^3} = \frac{M(r)}{\rho \rho^3} \Rightarrow 4\pi \frac{\lambda}{\mu^3}, \quad (\lambda \to 0). \quad (22)
\]

Equations (20) and (21), together with equation (15) for \( \gamma_p \), describe the general structure of multipressure polytropes. By inspecting these equations, we see that for a given \( \gamma_p \) equilibria are distinguished only by how far integration is carried in \( \chi \) and, thus, by how centrally concentrated they are. Figure 1 shows the structure of a spatially isothermal sphere; the origin corresponds to the center of the sphere,

![Figure 1](image-url)
Observe that the sound speed $c$ increased outward for $\gamma_p < 1$, as discussed in §2. In order for the mass to be finite, we require $\rho$ to fall off more slowly than $1/r^3$, which implies $\gamma_p < 4/3$. One can then readily show that an SPS is described by single values of $\mu$ and $\lambda$:

$$\mu = \left(\frac{2}{\pi}\right)^{1/2} \frac{4 - 3\gamma_p^{1/2} \gamma_p^{3/2}}{(2 - \gamma_p^2)^{3/2}},$$

$$\lambda = \left(\frac{1}{2\pi}\right)^{1/2} \frac{4 - 3\gamma_p^{1/2} \gamma_p^{3/2}}{2 - \gamma_p}.$$  

An important parameter describing the thermal structure of a cloud is the ratio of the mean square value of $c$:

$$\left<c^2\right> \equiv \frac{1}{M} \int c^2 \, dM = \frac{1}{M} \int \rho \, dV = \frac{P}{\bar{\rho}}$$

(26)

to the surface value, where $P$ and $\bar{\rho}$ are the volume-averaged pressure and density, respectively. For an SPS, we find

$$\psi \equiv \frac{\left<c^2\right>}{c_s^2} = \frac{4 - 3\gamma_p}{6 - 5\gamma_p}, \quad \left(\gamma_p < \frac{6}{5}\right),$$

(27)

where we have added the subscript “$s$” on $c^2$ to emphasize that it is measured at the surface of the cloud.

Note that $\psi$, which is equivalent to the mean energy per gram divided by the surface value, diverges as $\gamma_p \to 6/5$: for $\gamma_p \geq 6/5$, the energy of an SPS is concentrated at the center. Physical polytropes must satisfy the boundary condition $dP/dr = 0$ at the origin, since the gravitational force vanishes there. The fact that SPSs do not satisfy this condition is not important for $\gamma_p < 6/5$, since the mass and energy of such polytropes are dominated by the outer layers. As a result, the solutions for physical polytropes approach the SPS solutions for large values of $\gamma$. On the other hand, for $\gamma_p \geq 6/5$, the fact that SPSs do not satisfy the proper central boundary condition means that they cannot serve as approximations for physical polytropes. For example, equation (23) shows that an SPS with $\gamma_p = 6/5$ has $\rho \propto r^{-5/2}$, whereas the analytic solution for this case gives $\rho \propto r^{-3}$ at large radii (Chandrasekhar 1939).

The mean density in an SPS is

$$\frac{\bar{\rho}}{\rho_s} = \frac{3\mu}{4\pi \lambda^3} \frac{3(2 - \gamma_p)}{(4 - 3\gamma_p)}, \quad \left(\gamma_p < \frac{4}{3}\right),$$

(28)

whereas the mean pressure is

$$\frac{P}{P_s} = \frac{3\mu}{4\pi \lambda^3} \frac{3(2 - \gamma_p)}{(4 - 3\gamma_p)}, \quad \left(\gamma_p < \frac{6}{5}\right).$$

(29)

(In each case, the first equation is general and the expression after the arrow is for the SPS.) As emphasized in the Introduction, polytropes do not have large mean density or pressure contrasts with the ambient medium: for $\gamma_p < 1$, both are less than a factor of 3 for an SPS.

The gravitational energy $W$ can be described by the parameter $a$ defined by

$$W \equiv -\frac{3}{5} a \frac{GM^2}{r}.$$  

(30)

For an SPS, we have

$$a = \frac{5}{3} \left(\frac{4 - 3\gamma_p}{6 - 5\gamma_p}\right) = \frac{5}{3} \psi, \quad \left(\gamma_p < \frac{6}{5}\right).$$

(31)

3.2. The Virial Theorem

Before passing on to a consideration of the stability of multipressure polytropes, it is worthwhile to determine the implications of the virial theorem for our problem. The virial theorem for an unmagnetized gas sphere is

$$3 \int P \, dV - 3PV + W = 0.$$  

(32)

Note that this equation applies at any point within the sphere as well as at its surface. In our notation, this becomes

$$3\psi \lambda \mu - 4\pi \lambda^4 - \frac{3}{2} a a^2 = 0.$$  

(33)

This can be solved to give an explicit relation for $\mu(\lambda)$ in terms of two parameters, $a$ and $\psi$, that are usually of order unity:

$$\mu = \frac{5\psi}{2a} \left[1 \pm \left(1 - \frac{16\pi a^2 \lambda^3}{15\psi^2}\right)^{1/2}\right].$$

(34)

Note that $\mu$ has the correct limit (22) as $\lambda \to 0$ if the minus sign is chosen (since $a$ and $\psi \to 1$ as $\lambda \to 0$). The solution with the minus sign is the correct solution for all values of $\lambda$ for $\gamma_p < 0.82$; it is correct for all stable isentropic spheres for $\gamma_p < 0.90$. For $\gamma_p > 0.82$, the solution can switch sign as $\lambda$ increases: for $0.87 > \gamma_p > 0.82$ there can be multiple sign changes, whereas for $\gamma_p > 0.87$ there is only one change in sign, from minus to plus. Equation (34) is exact, and it is not restricted to polytropes.

4. THE STABILITY OF MULTIPRESSURE POLYTROPES

4.1. General Analysis

LBW have given a general discussion of the gravitational stability of spatially isothermal spheres ($\gamma_p = 1$). They considered both the case of a sphere subject to a given pressure, which is appropriate for a cloud of gas, and that of a sphere confined to a fixed volume, which is a model for a cluster of stars. They focused on the cases $\gamma_p = 1$ (an isothermal sphere, which is the Bonnor-Ebert problem) and $\gamma_p = 1$, $\gamma = 5/3$. Here, we shall generalize their discussion of pressure-bounded spheres to the case of arbitrary $\gamma$ and $\gamma_p$. Our objective is to find the criterion for determining the critical point, which is the most centrally concentrated
structure that is gravitationally stable. Equivalently, the critical point gives the maximum mass for which an equilibrium exists at the specified conditions.

The first law of thermodynamics for the cloud reads \( \delta E = \delta Q - P_s \delta V \), where \( E \) is the total energy of the cloud, \( \delta Q \) is the net heat flow into the cloud, and \( P_s \) is the pressure at the surface of the cloud. We shall focus on systems that are in equilibrium. Such systems are characterized by having a stationary value of the entropy subject to the given constraints. The equilibrium is stable if the entropy is a maximum.

We wish to determine the stability criterion for a pressure-bounded gas cloud. The stability of a system is determined by its thermodynamic free energy. For an adiabatic, pressure-bounded system, this free energy is the enthalpy determined by its thermodynamic free energy. For an adiabatic, pressure-bounded system, this free energy is the enthalpy determined by its thermodynamic free energy.

The equilibrium is stable if the entropy is a maximum.

For simplicity, however, we shall focus on adiabatic pressure perturbations. Even though we assume that the perturbations in our model are adiabatic, we can treat heat transfer from the environment by allowing \( \gamma \) to differ from the physical ratio of specific heats (see §2.2).

A complementary approach is to consider a sequence of spheres with a monotonically varying mass, all with the same entropy (or entropy per unit mass) and all embedded in a medium of constant pressure. In this case, the critical point corresponds to the sphere with an extremum in the mass \( \delta M = 0 \). For self-gravitating clouds, this extremum (if it exists) is the maximum stable mass under the specified conditions, and we label this \( M_{cr} \). Since \( \delta M = 0 \) at the critical point, it does not matter whether it is the entropy or entropy per unit mass that is held constant along the sequence.\(^1\)

The existence of a critical point depends on the value of \( \gamma \) and on the structure of the cloud, which is parameterized by \( \gamma_p \) for a polytrope. For example, isothermal spheres \( (\gamma = \gamma_p = 1) \); see Fig. 2), in which adiabatic compression increases the magnitude of the gravitational energy faster than the thermal energy, have critical points, whereas spheres with \( \gamma = \gamma_p > 4/3 \), in which the opposite is true, do not. Bonnor (1956) and Ebert (1955) used the condition \( \delta P_s = 0 \) to determine the stability of isothermal spheres.

Our approach in determining the stability of self-gravitating clouds is quite different from that of Maloney

\(^1\) The approach of considering a sequence of equilibria is also used in assessing the stability of degenerate stars (Shapiro & Teukolsky 1983). Stars have zero pressure at their surfaces, so the criterion \( \delta P_s = 0 \) that we are using for clouds is irrelevant. Although degenerate stars have negligible entropy, they have a well-defined entropy parameter \( K \) and adiabatic index \( \gamma \), which in general depend on the density but approach constants in the nonrelativistic and extreme relativistic limits. The mass sequence can be arranged with either the central density or the radius as the independent parameter. The dependence of the equation of state on density leads to the possibility of multiple critical masses along the sequence (the Chandrasekhar mass and the maximum neutron star mass). In principle, such behavior can occur for multipressure polytropes as well.
adiabatic spheres with collapse and global collapse occurs at globally adiabatic clouds, the dividing line between core required to effect core collapse, and as a result it cannot heat from the core (LBW). Such a heat transfer is generally the envelope actually expands because of the transfer of spatially constant temperature and its temperature(considered in the interstellar literature. Such a sphere has a remains constant if it is perturbed (LBW). When an isothermal sphere collapses, the collapse is due to rarefaction of the cloud, not compression.

4.1. Isothermal Spheres

The special case of an isothermal sphere is frequently considered in the interstellar literature. Such a sphere has a spatially constant temperature ($T_0$), and its temperature remains constant if it is perturbed ($\gamma = 1$). By setting the ratio of specific heats for the gas to unity, we can treat isothermal pressure perturbations as being adiabatic; the work done on the gas goes into the hypothetical internal degrees of freedom. As a result, the critical point for this problem can be determined by extremizing $H$ in the usual manner. It is customary, however, to use the Gibbs free energy $G = H - TS$ and an adiabatic index equal to the actual ratio of specific heats. For an isothermal sphere, $\delta Q = T \delta S$, so that equation (35) implies $\delta G = -S \delta T + V \delta P_s$. When $\delta T = 0$, the critical point, which is characterized by $\delta P_s = 0$, also corresponds to the condition $\delta G = 0$. Either way, one finds that the maximum possible density contrast for a stable isothermal sphere is 14.0 (Bonnor 1956; Ebert 1955).

4.1.2. Global Collapse versus Core Collapse

When an isothermal sphere collapses, the collapse is global: all flow is inward. On the other hand, when a globular cluster collapses, the collapse is restricted to the core; the envelope actually expands because of the transfer of heat from the core (LBW). Such a heat transfer is generally required to effect core collapse, and as a result it cannot occur in clouds with only locally adiabatic pressure. For globally adiabatic clouds, the dividing line between core collapse and global collapse occurs at $\gamma = 4/3$. Globally adiabatic spheres with $\gamma = 4/3$ evolve homologously (Shapiro & Teukolsky 1983): if one were to compress a $\gamma = 4/3$ sphere, the rate at which the work done on the sphere is transformed into kinetic energy would exactly balance the rate of increase of the magnitude of the gravitational energy. Because the fraction of $P \bar{d}V$ work that is converted to kinetic energy increases with $\gamma$, compression of a sphere with $\gamma > 4/3$ everywhere increases the ratio of kinetic to gravitational energy, reducing the center-to-surface density ratio $\rho_i/\rho_s$. Conversely, if the surface of such a globally adiabatic sphere were to expand, the ratio of kinetic to gravitational energy would decrease. If this evolution were carried sufficiently far, the cloud would become unstable, resulting in collapse of the core and dynamical expansion of the envelope. Thus, for $\gamma > 4/3$, gravitational collapse is due to rarefaction of the cloud, not compression.

Globally adiabatic clouds with $\gamma < 4/3$ everywhere exhibit the opposite behavior. For such clouds, compression increases the magnitude of the gravitational energy faster than it does the kinetic energy, so that $\rho_i/\rho_s$ increases. If the compression is sufficiently great, the cloud becomes unstable, and the entire cloud collapses.

4.2. Equations for the Perturbed Structure

In order to assess the stability of multipressure polytropes quantitatively, it is necessary to perturb the structure equations. This is done in Appendix B; we summarize the results here. For simplicity, we assume that there is only one globally adiabatic component.

The equations governing the variations in $\mu$ and $\lambda$ are

$$\frac{d \delta \ln \mu}{d \chi} = 2 \delta \ln \gamma_p + (4 - 3\gamma_p)(2 \delta \ln \lambda - \delta \ln \mu)$$

and

$$\frac{d \delta \ln \lambda}{d \chi} = \delta \ln \gamma_p + \left[\gamma_p \left(\frac{\lambda}{\mu} + 1\right) - 2\right] \delta \ln \mu$$

$$+ \left[4 - 3\gamma_p \left(\frac{\lambda}{\mu} + 1\right) - 2\right] \delta \ln \lambda.$$ (37)

These equations involve the perturbation in the polytropic index, $\delta \gamma_p$, which is also evaluated in Appendix B. There we find it convenient to introduce a new adiabatic index

$$\Gamma \equiv \sum_{i \in i} \left(\frac{P_i}{P} \gamma_i + \frac{P_s}{P} \gamma_{pg}\right),$$

which is the weighted mean of the adiabatic indices for the locally adiabatic components $\gamma$ and the polytropic index for the globally adiabatic component $g$. In terms of $\Gamma$, one finds

$$\frac{\delta \ln P}{\delta \ln \rho} = -\frac{\delta \ln P}{\delta \chi} = \Gamma = \sum_{i \in i} \left(\frac{P_i}{P} \gamma_i\right) \left(\frac{P_s}{P} \gamma_{pg}\right) = 0,$$ (40)

where $\gamma_{pg}$ is the perturbation in the polytropic coefficient for the globally adiabatic component and is evaluated in Appendix C. If there are no globally adiabatic com-
ponents, then this expression simplifies to

$$
\delta \ln \gamma_p = \frac{1}{\gamma} \left[ 4(\gamma_p - \gamma) \delta \ln \lambda - 2 \left( \gamma_p - \gamma + \frac{\Gamma}{4 - 3\gamma} \right) \delta \ln \mu \right].
$$

(43)

The solution for the perturbed structure is then given by inserting equation (C17) into equation (42), then inserting that into equations (36) and (37) for $\delta \ln \mu$ and $\delta \ln \lambda$, and then solving them simultaneously with equations (20) and (21) for $\mu$ and $\lambda$. This procedure is more complicated than the conventional treatment of the stability of hydrostatic clouds, which permits the stability to be determined from the solution of a single second-order equation involving the spatial gradient of the adiabatic index $\gamma$ (Ledoux 1965). The additional complexity arises both because we have dropped the assumption that the perturbations are locally adiabatic and because we have in effect evaluated the gradient of $\gamma$.

As we found in § 4.1, the critical point is given by the condition that $\delta \ln P_\rho = 0$. Equation (39) then implies that the critical point is determined by

$$
\delta \ln \mu = \frac{-2}{\Gamma} \left( \frac{P_\rho}{P} \right) \delta \ln K_{\rho}. 
$$

(44)

This condition can be used to determine the critical mass for clouds supported by locally adiabatic pressure components, for those supported by globally adiabatic pressure components and for multipressure polytropes that may be supported by both types of pressure.

5. Locally Adiabatic Polytropes

In multipressure polytrope models of GMCs, the thermal gas pressure and the magnetic stresses are modeled as locally adiabatic pressure components: the entropy associated with the component remains constant in each mass element during a perturbation. Here we first determine how to characterize the average entropy associated with a locally adiabatic pressure component, and then use this to describe the critical mass. As we shall see, nonisentropic clouds are particularly interesting since they can have large central-to-surface pressure ratios.

5.1. Entropy of Locally Adiabatic Pressure Components

For locally adiabatic pressure components (labeled "l"), the entropy parameter $K_{l} = P_{l}/\rho^{\gamma_l}$ remains constant in each mass element during the evolution of the cloud (see eq. [4] for the relation between $K_{l}$ and $s_{l}$, the specific entropy associated with pressure component $l$). In general, $K_{l}$ depends on position unless the system is isentropic ($\gamma_l = \gamma_{pe}$).

Particularly for the case in which magnetic fields are important, it is convenient to introduce a spherically symmetric reference state of constant pressure $P_{\rho, ref}$ (Mouschovias 1976a). We shall generalize Mouschovias's concept of a reference state by allowing the density in the reference state $\rho_{ref}$ to depend on position in the cloud; we do not assume that the reference state is a polytrope. For a locally adiabatic component, the entropy parameter is the same for each mass element in the cloud as it is for that mass element in the reference state,

$$
K_{l}(M) = \frac{P_{l}(M)}{[\rho(M)]^{\gamma_l}} = \frac{P_{l, ref}}{[\rho_{ref}(M)]^{\gamma_l}}.
$$

(45)

In the isentropic case ($\gamma_l = \gamma_{pe}$), the reference density is constant, as Mouschovias assumed: since the entropy parameter $K_{l}$ is identical to the factor $K_{pl}$ that governs the polytropic structure, it is constant; with both $K_{l}$ and $P_{l, ref}$ constant, it follows that $\rho_{l, ref}$ is as well.

For nonisentropic systems, we define the average entropy parameter by

$$
\langle K_{l} \rangle^{1/\gamma_l} \equiv \frac{1}{M} \int K_{l}^{1/\gamma_l} dM.
$$

(46)

Introducing the power $1/\gamma_l$ into the definition enables us to evaluate the integral for the reference state,

$$
\langle K_{l, ref} \rangle = \frac{P_{l, ref} V_{ref}^{4/3}}{M^{4/3}} = \frac{P_{l, ref}}{\rho_{ref}^{4/3}},
$$

(47)

where $dM = \rho_{ref} dV_{ref}$ and $V_{ref}$ is the volume in the reference state. Thus $K_{l, ref}^{1/\gamma_l}$ is a measure of the local entropy per unit mass, and $\langle K_{l} \rangle^{1/\gamma_l}$ is a measure of the mean entropy per unit mass of an entire cloud.

For an isothermal gas, we have $P_{l, ref} = \rho_{l, ref}^2$ with $c_{th} = const$, corresponding to $\gamma_l = 1$. As a result, the entropy parameter is simply

$$
K_{th} = \langle K_{th} \rangle = c_{th}^2.
$$

(48)

Next, consider a cloud supported by a poloidal magnetic field. We treat the magnetic field as a gas with an adiabatic index $\gamma_B = 4/3$. This relation is exact for a tangled field. For a poloidal field, it is approximate, but it ensures that the magnetic flux $\Phi = \int 2n r B dr$ is conserved during a homologous, spherical compression, since then $B \propto r^{2/3}$ and $r B dr \propto r^{5/3}(dr/r) = const$. The parameter that measures the mean specific entropy for the field is

$$
\langle K_B \rangle = \frac{P_{B, ref} V_{ref}^{4/3}}{M^{4/3}} \propto \frac{B_{ref}^2}{M^{4/3}}.
$$

(49)

in terms of quantities measured in the reference state. Since the flux $\Phi$ is the same in the cloud and in the reference state, we can rewrite this as

$$
\langle K_B \rangle \propto \frac{\Phi^2}{M^{4/3}} \propto \frac{B^2}{\rho^{4/3}},
$$

(50)

where $\Phi \equiv \Phi/r R^2$ is the mean magnetic field and $\rho$ is the mean density. It follows that the parameter that measures the total entropy of the cloud depends only on the magnetic flux,

$$
M \langle K_B \rangle^{3/4} \propto \Phi^{3/2}.
$$

(51)

Adiabatic evolution of the cloud thus corresponds to evolution with constant magnetic flux.

5.2. The Critical Mass

As discussed in § 4.1, the critical mass is the mass of a cloud at the critical point: for given values of the ambient pressure and component entropies, there is no nearby equilibrium state with a mass greater than $M_{cr}$. In order to determine the value of the critical mass, we use the approach adopted in § 4, based on a sequence of equilibria with constant mass and entropy but variable pressure. (It should be kept in mind that this sequence of equilibria is not a sequence of polytropes for nonisentropic clouds.) For simplicity, we consider a cloud supported by a single pressure component. We can express the mass of the cloud (eq.
The critical point occurs at an extremum of the surface that the quantities are evaluated at the surface of the cloud. The critical point occurs at an extremum of the surface pressure, $\delta P = 0$. If we adiabatically perturb a cloud of fixed mass, equation (52) shows that the critical point occurs at

$$\delta \ln \mu = 0.$$  

(53)

This is just the condition implied by equation (44) in the absence of globally adiabatic components. The critical mass is then

$$M_{cr} = \frac{\mu_{cr} c_{s}^{3/2}}{G^{3/2} \rho_{s}^{1/2}} = \frac{\mu_{cr} c_{s}^{2}}{G^{3/2} \rho_{s}^{1/2}},$$  

(54)

where $\mu_{cr}$ is a number that depends on $\gamma$ and $\gamma_{cr}$.

For a nonisothermal gas, it is convenient to have an expression for the critical mass in terms of $\langle c_{s}^{2} \rangle \equiv M^{-1} \int c_{s}^{2} dM$, which is often more readily observable. To accomplish this, we reexpress equation (10) as

$$M = \frac{\mu}{\psi^{3/2}},$$  

(56)

recall that $\psi \equiv \langle c_{s}^{2} \rangle / c_{s}^{2}$ (see eq. [27]). At the critical point, we have $\mu_{cr} = \mu_{cr}/\psi_{cr}^{3/2}$. For an isothermal cloud, $\psi = 1$ and $\mu_{cr} = \mu_{cr}$.

It is important to keep in mind that $M_{cr}$ is the maximum stable mass for given values of $\langle K_{s} \rangle$ (or, equivalently, $K_{s}$) and $P_{s}$; it is the maximum stable mass for given values of $c_{s}$ and $P_{s}$ (or, equivalently, $\rho_{s}$) only for isentropic spheres. In § 5.3 below we show that $\mu$ is indeed a maximum at the critical point for isentropic spheres. For nonspherical polytropes, which are discussed in greater detail below (§ 5.4), $\mu_{s}$ is less than the value for isentropic spheres. This can be seen graphically in Figure 1, which portrays all the equilibria for isothermal spheres: as $\gamma_{r}$ rises above unity, our numerical results show that the critical point moves down and to the left, following this curve as the maximum stable value of $\chi$ increases. (This result for isothermal spheres was first found by Yabushita 1968.) How is it then possible to satisfy the critical point condition $\delta \ln \mu = 0$ at a point other than the maximum in Figure 1? The answer is that the perturbed sphere is not a polytrope for $\gamma_{r} \neq \gamma$ and is therefore not represented in this figure.

The expressions for the critical mass above depend on the specific entropy. It is also possible to express the critical mass in terms of the total entropy in the cloud, which is related to $M \langle K_{s} \rangle^{1/r}$. We define this alternative form for the critical mass, which we label “$M_{cr,t}$,” by

$$M_{cr,t} = \frac{M_{cr}^{3}}{M}.$$  

(57)

One can show that $M_{cr,t}$ indeed depends on $M$ and $\langle K_{s} \rangle$ only through the combination $M \langle K_{s} \rangle^{1/r}$ with the aid of equation (52); it also depends on the numerical parameter $(K_{s} / \langle K_{s} \rangle)_{cr}$. At the critical mass ($M = M_{cr}$), the two forms for the critical mass are identical ($M_{cr} = M_{cr,t}$).

For an isothermal gas, the critical mass is the Bonnor-Ebert mass,

$$M_{cr} = M_{BE} = 1.1822 \left(\frac{c_{s}^{2}}{G^{3/2} \rho_{s}^{1/2}}\right),$$  

(58)

which follows from equation (54) with $\mu_{cr} = 1.1822$. The alternate form for the critical mass, $M_{cr,t}$, depends on the total thermal energy in the cloud, $M_{th}$.

Magnetically supported clouds have $\gamma = 4/3$ as discussed above. The critical mass can thus be expressed as

$$M_{B} \equiv M_{cr,t} \propto M_{th} \propto \left(\frac{M^{2} \langle K_{s} \rangle^{3/2}}{G^{3/2}}\right)^{1/3} \propto \frac{\Phi}{G^{1/2}},$$  

(60)

The numerical value for $M_{B}$ depends on the distribution of the flux in the cloud; for the standard case in which a uniform field threads a spherical cloud, it is given as $0.126 \rho_{s}^{1/2}$ by Mouschovias & Spitzer (1976). The relation between the two forms of the magnetic critical mass appears to have been first given by Mouschovias & Spitzer (1976); equation (57) shows that this relation is a general one that applies to all forms of support by an adiabatic pressure.

5.3. Results for Single-Component, Isentropic Polytropes ($\gamma = \gamma_{p}$)

The pressure components in interstellar clouds are “soft,” with polytropic indices $\gamma_{p} \leq 4/3$, and this constrains the mass, density, and pressure of self-gravitating interstellar clouds; for example, a stable isothermal sphere must be less than the Jeans mass and has a maximum density ratio of 14.04. Here we consider the limitations on the mass of single component, isentropic polytropic polytropes with $\gamma_{p} < 4/3$.

The stability analysis of isentropic polytropes, previously considered by Shu et al. (1987), Viala & Horedt (1974), and Chièze (1987), is a subset of our work, and we have plotted the results in Figures 3 and 4. Note that the critical density and pressure ratios increase with $\gamma$. The points marked on these figures denote the values for an isothermal sphere, the case originally considered by Bonnor (1956) and Ebert (1955). When $\gamma > 4/3$, an isentropic polytrope is unconditionally stable. Such a polytrope can be stable even if the density and pressure vanish at its surface, and as such they can provide a simple model for stars. Systems with $\gamma = 4/3$ evolve homologously in response to either a pressure perturbation or an entropy perturbation (Shapiro & Teukolsky 1983). The critical configuration for such a polytrope
clouds have constant $\gamma_p$, so integration of equations (20) and (21) yields a unique solution. The perturbed cloud is also isentropic, so the perturbed cloud must lie on this solution as well. As a result, isentropic clouds have identical spatial and Lagrangian variations, so that $\delta \ln \mu = d \ln \mu = 0$ at the critical point. (Recall from the discussion in § 4.1 that this extremum is a maximum.) Equation (20) then implies

$$\frac{\mu^2}{\mu_c^2} = \frac{8\pi\gamma_p}{4 - 3\gamma_p} \quad (\gamma_p = \gamma).$$  \hspace{1cm} (61)

Chièze (1987) previously obtained this relation (his eq. [A18]) when he considered the stability of single component polytropes, under the implicit assumption that $\gamma = \gamma_p$. The equivalence of the equations follows from the variational transformations in Appendix A. With the aid of the expression for $\mu$ obtained from the virial theorem (eq. [34]) it is possible to evaluate $\mu_c$ explicitly:

$$\mu_c = \frac{9}{8(2\pi)^{1/2}} \frac{\psi_c^2 (4 - 3\gamma_p)^{1/2} \gamma_p^{3/2}}{[1 - (3/4)(1 - 2\alpha_c)^{5}/\gamma_p]^{12}} \quad (\gamma_p = \gamma).$$ \hspace{1cm} (62)

The parameters $\alpha_c$ and $\psi_c$ are of order unity for $\gamma_p < 1.3$; as $\gamma_p \rightarrow 4/3$, we find $\alpha_c \rightarrow 2.5$ and $\mu_c \rightarrow 4.555$, whereas $\psi_c \propto (4 - 3\gamma_p)^{-1/4}$ grows without bound. One can readily show that equation (62) agrees with equation (24) for an SPS if the values of $\alpha$ and $\psi$ for an SPS are inserted into this expression. Note that $\mu_c \rightarrow 0$ as $\gamma_p \rightarrow 0$: the critical mass shrinks with $\gamma_p$ since it becomes impossible to maintain a substantial pressure gradient as $\gamma_p$ approaches zero.

5.4. Results for Nonisentropic, Locally Adiabatic Polytropes ($\gamma \neq \gamma_p$)

5.4.1. Stellar versus Pressure-confined Polytropes

The behavior of locally adiabatic polytropes is summarized in Figure 5. Since such polytropes adhere to the Schwarzschild criterion for stability against convection, those with $\gamma < \gamma_p$ are convectively unstable; we do not consider them further. Locally adiabatic polytropes can be divided into two classes.

**Stellar polytropes ($\gamma > 4/3, \gamma_p > 6/5$).**—These can serve as models for stable stars. Polytropes with $\gamma > 4/3$ are unconditionally stable (Ledoux 1965), since a locally adiabatic gas with $\gamma > 4/3$ becomes hotter at every point under compression and therefore cannot collapse under its own gravity. (Polytropes with $\gamma = 4/3$ and zero pressure at their boundaries are neutrally stable.) Furthermore, polytropes with $\gamma_p > 6/5$ have zero-pressure boundaries at finite mass and radius, like stars (Chandrasekhar 1939). (Recall that polytropes with $\gamma_p > 6/5$ can have all their energy concentrated in the core—§ 3.1.) As the surface pressure approaches zero ($\chi \rightarrow \infty$) the dimensionless mass $\mu \rightarrow 0$ for $6/5 < \gamma_p < 4/3$, 4.555 for $\gamma_p = 4/3$, and $\infty$ for $\gamma_p > 4/3$, as can be inferred from Chandrasekhar (1939). For fixed $\gamma$, there is no limit to the dimensional mass of a stellar polytrope; stars have limited masses because $\gamma$ depends on the mass, approaching 4/3 for massive, radiation-dominated stars or for degenerate stars that approach the Chandrasekhar limit.

**Pressure-confined polytropes ($\gamma \leq 4/3$ and/or $0 < \gamma_p \leq 6/5$).**—First consider polytropes with $\gamma_p < 6/5$. All such polytropes must be confined by the pressure of an ambient medium (Chandrasekhar 1939). In the limit of a large
center-to-surface density ratio ($\chi \geq 1$), the pressure at the surface of such a polytrope approaches that of an SPS,

$$P_s = \frac{\gamma_p (4 - 3\gamma_p) c_s^4}{2\pi G (2 - \gamma_p^2)^2 r_s^2}.$$  \hspace{1cm} (63)

(Polytropes with $\gamma_p = 6/5$ do not satisfy this relation because their mass and energy are concentrated at the center, but they too must be pressure confined if they have a finite radius.) Next, consider polytropes with $4/3 \geq \gamma_p \geq 6/5$ and $\gamma < 4/3$. Such polytropes can have arbitrarily large values of $\chi$, but they become unstable for $\chi > \chi_{cr}$; as a result, stable polytropes in this region of parameter space must also be pressure confined. The mass and energy of these polytropes are concentrated near the center when $\chi$ becomes large. Finally, polytropes with $\gamma = 4/3$ must be pressure confined if they are to be stable (and not just neutrally stable) against small perturbations. Polytropic models for thermal pressure in interstellar gas and for magnetic pressure are in the pressure-confined regime, and, as we shall see below, polytropic models for Alfvén waves are equivalent to locally adiabatic polytropes in this regime. Insofar as the pressure in interstellar clouds can be represented as the sum of these pressure components, it follows that polytropes used to model interstellar clouds must be pressure confined.

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Fig. 5.—Characteristics of locally adiabatic polytropes in the $\gamma\gamma_p$ plane. We do not consider polytropes in the region $\gamma < \gamma_p$ since they are convectively unstable. Above $\gamma = 4/3$, the polytropes are always stable, even for arbitrarily large values of $\chi = \ln(p_c/p_s)$. Stellar polytropes have $\gamma > 4/3$ so that they are stable and in addition have $\gamma_p > 6/5$ so that they can have $P_s = 0$ at finite radius; as a result, they can be used as models for stable stars. Stable, finite-size polytropes in the rest of the parameter space with $\gamma \geq \gamma_p$ must be confined by the pressure of an ambient medium. The locally adiabatic pressure components in interstellar clouds have $\gamma_s \leq 4/3$. For polytropes in the region labeled "Finite $\chi_{cs}$" the polytropes become unstable for a central-to-surface density contrast greater than $\exp(\chi_{cs})$. In the region labeled "SPS" (singular polytropic sphere), polytropes are formally stable even for an infinite density contrast ($\chi = \infty$). These regions are separated by the curve labeled "$\gamma_{cs}$" which is given in eq. (69). It is possible that clouds in part of the SPS region, particularly near the $\gamma_{cs}$ curve at small $\gamma_p$, are unstable to finite perturbations (§5.4.3).

---

Fig. 6.—Value of $\mu_{cs}$, which determines the critical mass, is plotted for isentropic polytropes as a function of $\gamma_p$. Also shown is the value of $\mu$ for singular polytropic spheres ("SPS"). For polytropes with a critical point, we have $\mu_{cs}(\text{isentropic}) \geq \mu_{cs}(\text{SPS})$.

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5.4.2. Limitations on the Mass and Radius of Pressure-confined Polytropes

The parameters $\mu_{cs}$ and $\mu_{cs}'$, which describe the critical mass of a cloud, are plotted in Figures 6 and 7. Recall that for the isentropic case, $\mu_{cs}$ is a local maximum in $\mu(\chi)$ for $\gamma_p < 4/3$ (§5.3); numerical solutions show that this is a global maximum as well, so that $\mu_{cs}(\text{nonisentropic}) < \mu_{cs}(\text{isentropic})$. Furthermore, as indicated in Figure 6, $\mu_{cs}(\text{isentropic})$ is a monotonically increasing function of $\gamma_p$, reaching $\mu_{cs} = 4.555$ at $\gamma_p = 4/3$. The maximum value of $\mu_{cs}$ generally occurs for nonisentropic clouds and so cannot be read off Figure 7. We find that it too is a monotonically increasing function of $\gamma_p$, reaching $\mu_{cs} = 1.686$ at
\( \gamma_p = 4/3 \). We conclude that all polytropes with \( \gamma_p \leq 4/3 \), whether isentropic or not, satisfy \( \mu \leq 4.555 \) and \( \mu' \leq 1.686 \), so that

\[
M \leq 4.555 \left[ \frac{c_s^4}{(G^3 \rho^3)^{1/2}} \right], \quad M \leq 1.686 \left[ \frac{\langle c_s^2 \rangle^{3/2}}{(G^3 \rho^3)^{1/2}} \right]. \tag{64}
\]

In terms of the surface pressure instead of surface density, we find

\[
M \leq 4.555 \left[ \frac{c_s^4}{(G^3 P^3)^{1/2}} \right], \quad M \leq 1.409 \left[ \frac{\langle c_s^2 \rangle^{2}}{(G^3 P^3)^{1/2}} \right]. \tag{65}
\]

If attention is restricted to negative-index polytropes, the same line of reasoning that led to an upper limit of 4.555 on \( \mu \) for \( \gamma_p < 4/3 \) leads to an upper limit of \( \mu' \leq \mu_{c\alpha}(\gamma = \gamma_p = 1) = 1.1822 \) for \( \gamma_p < 1 \). Our results show that for such polytropes \( \mu' \leq \mu_{c\alpha}(\gamma = 1) = 1.1822 \) as well. Note that since \( \mu_{c\alpha} \) decreases below \( \mu_{c\alpha}(\text{isentropic}) \) as \( \gamma \) increases above \( \gamma_p \), it is possible for \( \mu \) to exceed \( \mu_{c\alpha} \) for stable clouds (similarly, \( \mu' \) can exceed \( \mu_{c\alpha} \)), but this is by less than a factor of 1.5 for \( \gamma_p \leq 1 \).

Our results also set limits on the size of polytropes. As indicated in Figure 1, the maximum value of \( \lambda \) occurs prior to the critical point. This maximum value increases monotonically with \( \gamma_p \) for \( \gamma_p < 4/3 \), reaching 0.6603 at \( \gamma_p = 4/3 \). Therefore, all polytropes with \( \gamma_p \leq 4/3 \) satisfy

\[
r = 0.6603 \left[ \frac{c_s^4}{(G^3 \rho^3)^{1/2}} \right] = 0.6603 \left[ \frac{\langle c_s^2 \rangle^{2}}{(G^3 P^3)^{1/2}} \right]. \tag{66}
\]

For negative index polytropes, the upper limit on the radius has the same form, but the coefficient is reduced to 0.5142.

5.4.3. Pressure and Density Drops in Nonisentropic Polytropes

As discussed in the Introduction, an important question to be addressed by the polytropic models is whether they can account for the observed densities and pressures in molecular clouds. First consider the mean densities and pressures. These quantities have been plotted for isentropic polytropes in Figure 4, and they diverge as \( \gamma_p \to 4/3 \). For \( 6/5 \leq \gamma_p < 4/3 \), the mean-to-surface density and pressure ratios can become arbitrarily large as \( \gamma \to 4/3 \), since such polytropes have their mass and energy concentrated near the center, as discussed above (§ 5.4.1). For \( \gamma_p < 6/5 \), the mean-to-surface density and pressure ratios can no longer diverge. For negative index polytropes \( (\gamma_p < 1) \), it is convenient to express our results for the mean densities and pressures in terms of the values for SPSs (eqs. \([28]\) and \([29]\)),

\[
\frac{\bar{\rho}}{\rho_s} \leq 1.26 \frac{(2 - \gamma_p)}{(4 - 3\gamma_p)}, \tag{67}
\]

\[
\frac{\bar{P}}{P_s} \leq 1.26 \frac{(2 - \gamma_p)}{(6 - 5\gamma_p)}. \tag{68}
\]

We conclude that negative-index polytropes cannot have large mean-to-surface density or pressure ratios, regardless of the value of the adiabatic index \( \gamma \).

Next, consider the central-to-surface density and pressure ratios, which can become quite large for nonisentropic clouds. For fixed \( \gamma_p \), the maximum value of \( \chi \) for a stable equilibrium (i.e., \( \chi_{c\alpha} \)) increases as \( \gamma \) increases above \( \gamma_p \), and it is possible for \( \chi_{c\alpha} \) to reach infinity if \( \gamma \) exceeds a value we label \( \gamma_{c\alpha} \). We can determine the value of \( \gamma_{c\alpha} \) analytically by noting that the cloud becomes a singular polytropic sphere with constant values of \( \mu \) and \( \lambda \) as \( \gamma \to \gamma_{c\alpha} \). After a little algebra, we find that the requirement that the critical point \( (\delta \ln \mu \to 0) \) occur in the limit \( \chi \to \infty \) implies

\[
\gamma_{c\alpha} = \frac{32\gamma_p(2 - \gamma_p)}{(6 - 5\gamma_p)^2}, \tag{69}
\]

For \( \gamma_p = 1 \), this yields \( \gamma_{c\alpha} = 32/25 = 1.28 \). Yabushita (1968) previously pointed out that spatially isothermal spheres with \( \chi = \infty \) are unstable for \( \gamma < 32/25 \). For \( \gamma_p = 6/5 \), \( \gamma_{c\alpha} = 4/3 \); for \( 4/3 \geq \gamma_p > 6/5 \), \( \gamma_{c\alpha} \) remains at \( 4/3 \). Note that polytropes with \( \gamma > \gamma_{c\alpha} \) and \( \gamma_p < 6/5 \) satisfy equation (63) for the surface pressure, so the infinite pressure drop is attained only at infinite radius. Since actual clouds are finite in extent, they are pressure confined as discussed above.

There is a subtlety in the behavior of nonsentropic clouds with large density drops, however. Recall that as \( \chi \to \chi_{c\alpha} \), the pressure perturbation \( \delta \ln P_s \to 0 \) according to the condition for the critical point (§ 4.1). For \( \gamma_p = 1 \), for example, the pressure perturbation within the sphere remains comparable to the central value \( \delta \ln P_c \) throughout most of the volume and approaches 0 only near the edge. For negative-index polytropes, though, the pressure perturbation can drop to very small values well inside the sphere. Consider the example of a sphere with \( \gamma_p = 0.5 \) and \( \gamma = 0.79 \) (slightly smaller than \( \gamma_{c\alpha} = 0.793 \)). We find that \( \delta \ln P \) drops to 0.1 \( \delta \ln P_c \) at \( \chi \sim 4 \), far smaller than \( \chi_{c\alpha} \approx 32 \). Thus, for any such polytrope with \( \gamma > 4 \), the surface perturbations must be very small in order to prevent the perturbations at the center from becoming nonlinear. Although our linear analysis cannot determine what would occur if the central pressure perturbation became nonlinear, it is quite possible that the cloud would become unstable. If so, the stability of the cloud in this case could be likened to that of a pencil balanced on a flattened point—as \( \gamma \) increases, the size of the flattened region shrinks so that it becomes susceptible to smaller and smaller perturbations, and as a result, such a cloud would be unlikely to survive in a medium with large pressure fluctuations. As \( \gamma \) increases, the value of \( \chi \) at which the pressure fluctuation becomes substantially smaller than the central value grows. We find that a sufficient condition for density drops of at least \( 10^3 \) to occur in clouds with \( \delta \ln P_s \delta \ln P_c > 0.1 \) is \( \gamma > 1.3 \gamma_{c\alpha}^{1/2} \). For \( \gamma_p < 0.8 \), clouds with \( \gamma \) just above \( \gamma_{c\alpha} \) do not satisfy this condition even though they are formally stable for arbitrarily large density drops. It is interesting to note that insofar as logatropes can be represented by polytropes with \( \gamma \to 0 \), they do become stable for very large density drops provided \( \gamma \) is sufficiently large; the central temperature is not constant unless \( \gamma = 1 \), however.

Finally, we consider the response of locally adiabatic polytropes to pressure perturbations. For isothermal polytropes, \( M_{c\alpha} \) is reduced somewhat by an increase in the external pressure, \( M_{c\alpha} \propto P_s^{-1/2} \) (eq. [54]). For negative-index polytropes, however, \( M_{c\alpha} \) can decrease much more sharply with \( P_s \) because of the decrease in \( c_s \) (Shu et al. 1972; Tohline, Bodenheimer, & Christodoulou 1987). Since the decrease in the temperature is bounded (it is difficult to cool below 10 K in a typical molecular cloud, for example), it is more convenient to express the critical mass in terms of quantities after the compression,

\[
M_{c\alpha} = \mu_{c\alpha} \left( \frac{c_s^4}{G^{3/2} P_s^{1/2}} \right), \tag{70}
\]
where \( c_{s,t} \) is the final value of \( c_s \) etc. This result shows that the reduction in the critical mass due to cooling, which reduces \( c_s \), can be large, but the reduction due to the compression is limited by the weak \( P_{\text{cr}}^{-1/2} \) dependence. For example, in a radiative shock the final pressure is related to the initial value \( P_{\text{cr}} \) by \( P_{\text{cr}} = (v_{\text{shock}}/c_{s,t})^2 P_{\text{cr}} \). In spherical implosions even higher compressions, and correspondingly greater reductions in \( M_{\text{cr}} \), are possible (Tohline et al. 1987), although in practice it may be difficult to maintain the high degree of spherical symmetry required to achieve very large compressions.

6. GLOBALLY ADIABATIC POLYTROPES

When the flow of energy is significant on a dynamical timescale, the assumption that the perturbation is locally adiabatic breaks down. The perturbation can be approximated as being globally adiabatic provided that no heat is supplied to or removed from the cloud during the perturbation. This is the approximation that we adopt for the Alfvén waves we are using as a model for the turbulence in molecular clouds. Precisely because heat can flow during a perturbation, determining the condition that the perturbation be adiabatic is nontrivial, and it is this problem we now address.

6.1. Entropy of a Globally Adiabatic Pressure Component

For a globally adiabatic pressure component \( g \), heat is assumed to flow within the system so as to maintain the polytropic condition \( P_g = K_{pg} \rho^{s(g)} \), but there is no exchange of heat with the surroundings, nor with the other pressure components. As a result of the heat flow, \( K_{pg} \) may change with time, although it is independent of position. In the actual globally adiabatic system (as opposed to our model of the system), this heat flow is reversible. For example, if a nonuniform cloud with Alfvén waves is adiabatically compressed and then decompressed back to its original size, the Alfvén waves will be unchanged by the process, since the wave action is conserved (Dewar 1970). This leads to a problem: in our model for the Alfvén waves, heat flows across a temperature gradient— an irreversible process. As a result, the entropy as conventionally defined may not be conserved for globally adiabatic processes.

To define an entropy for a globally adiabatic pressure component, we again introduce a spherically symmetric reference state that is related to the cloud by a reversible adiabatic transition (cf. Mouschovias 1976a). In this reference state, the globally adiabatic pressure component has a uniform pressure \( P_{g,\text{ref}} \); since globally adiabatic components are assumed to always be polytropic, it follows that the density \( \rho_{g,\text{ref}} \) is uniform as well. The entropy parameter for this component in the reference state is

\[
K_{g,\text{ref}} \equiv \frac{P_{g,\text{ref}}}{\rho_{g,\text{ref}}^s}.
\]

If the cloud then returns to its initial state by a reversible adiabatic process, the entropy will be unchanged. Thus, the entropy of a globally adiabatic pressure component is characterized by \( K_{g,\text{ref}} \). In our case, we are interested in self-gravitating clouds, so the transition to the reference state can be visualized as being due to slowly decreasing the value of the gravitational constant \( G \) to zero.

In Appendix C, we evaluate \( K_{g,\text{ref}} \) by extending an argument due to McKee & Zweibel (1995). There we demon-
6.3. Results for Globally Adiabatic Clouds

There are two key differences between the stability of globally adiabatic clouds and that of locally adiabatic clouds. First, there is no convective instability, since it is assumed that the heat flow is rapid compared to fluid motions; as a result, it is possible to have stable systems with \( \gamma_g < \gamma_p \), although we have not identified one in practice. Second, as discussed in § 4.1.2, clouds with \( \gamma_g > 4/3 \), which are stable if they are locally adiabatic, can be unstable to core collapse. In core collapse, instability is triggered when a perturbation causes a sufficiently large heat flow from the center of the cloud to the envelope. This is the case that is relevant to clouds supported by Alfvén waves, which have \( \gamma_w = 3/2 \) (see § 6.4). Globally adiabatic polytropes with \( \gamma_p > 6/5 \) are not subject to core collapse, however: as shown in § 5.4, polytropes with \( \gamma_p > 6/5 \) can have arbitrarily large density contrasts, and our numerical results show that they are stable for \( \gamma_g > 4/3 \).

Results for the center-to-surface pressure and density ratios, \( P_c/P_s \) and \( \rho_c/\rho_s \), are plotted in Figures 8 and 9, respectively. Note that for the cases of greatest interest, in which \( \gamma_g > \gamma_{p\text{tr}} \), these ratios are determined primarily by the value of \( \gamma_{p\text{tr}} \). Comparing these results with those for locally adiabatic polytropes discussed in § 5, we find that globally adiabatic polytropes have smaller critical density and pressure contrasts (provided \( \gamma > \gamma_{p\text{tr}} \), the only case of relevance for locally adiabatic polytropes).

It is important to note that the structure of a cloud supported by a globally adiabatic pressure component is determined by the value of \( \gamma_p \) and is therefore identical to that of a cloud supported by a locally adiabatic pressure component with the same value of \( \gamma_p \). For a locally adiabatic cloud, the density drop at the critical point \( (\rho_c/\rho_s)_{\text{cr}} \) rises smoothly from the isentropic value (Fig. 3) to \( \infty \) as \( \gamma_p \) rises from \( \gamma_{p\text{tr}} \) to \( \gamma_w \). Figure 9 shows that \( (\rho_c/\rho_s)_{\text{cr}} \) increases above the isentropic value as \( \gamma \) increases above \( \gamma_{p\text{tr}} \) for a globally adiabatic component as well. Hence, for any globally adiabatic cloud with \( \gamma_g > \gamma_p \), it is always possible to find a value of \( \gamma_p \), which we label \( \gamma_p(\text{equiv}) \), such that the critical point of the locally adiabatic cloud is identical to that of the globally adiabatic cloud. The value of \( \gamma_p(\text{equiv}) \) for the equivalent locally adiabatic cloud is in the range \( \gamma_w > \gamma_p(\text{equiv}) > \gamma_p \). We have evaluated \( \gamma_p(\text{equiv}) \) for several cases: for \( \gamma = 5/3 \) and \( \gamma_p = 0.5 \) (the Alfvén wave case—see below), \( \gamma_p(\text{equiv}) = 0.60564 \); for \( \gamma = 3/2 \) and \( \gamma_p = 0.1 \), \( \gamma_p(\text{equiv}) = 0.11985 \); and for \( \gamma = 5/3 \) and \( \gamma_p = 1.0 \) (the LBW case), \( \gamma_p(\text{equiv}) = 1.22042 \). It must be kept in mind that although it is possible to locate the critical point in a globally adiabatic polytrope with a locally adiabatic equivalent, the nature of the instability in the two cases is quite different: for \( \gamma_p > 4/3 \), the globally adiabatic polytrope is subject to core collapse, which is impossible for a locally adiabatic polytrope.

How does this result carry over to the case of a cloud with multiple pressure components? Just as in the single-component case, the structure of the cloud is determined by the values of \( \gamma_p \) for each of the components, with the values of \( \gamma_g \) and \( \gamma_p \) determining which of the structures corresponds to the critical point. The critical point for a cloud supported by one or more locally adiabatic pressure components plus a globally adiabatic pressure component is therefore the same as that of a cloud supported by the same locally adiabatic components plus a locally adiabatic component with a suitable \( \gamma_p(\text{equiv}) \). Numerical calculations for two-component clouds show that the value of \( \gamma_p(\text{equiv}) \) is not constant, but instead depends on the fraction of the pressure in the globally adiabatic component. This is reason-
able, since although the structure equations do not depend on the values of $\gamma_p$ and $\gamma_c$, the equations for the perturbed structure do (§ 42). In contrast to the single-component case, the value of $\gamma_{\alpha}$ (equiv) can exceed $\gamma_{\alpha}^\text{c}$, although for Alfvén waves it is generally $\lesssim 1$.

### 6.4. Clouds Supported by Alfvén Waves

We now focus on Alfvén waves, which we are using to model the turbulence in molecular clouds. As shown by Dewar (1970), Alfvén waves exert an isotropic pressure and can thus provide support parallel to the background magnetic field. For small-amplitude waves, McKee & Zweibel (1995) showed that $\gamma^\text{p,w} = 3/2$ and $\gamma^\text{p,w} = 1/2$; this latter result was originally derived by Walén (1944). While Alfvén waves are undoubtedly an important component of the MHD waves that contribute to the turbulent pressure in molecular clouds, there are several limitations to our model that should be borne in mind. First, we assume that the waves are adiabatic, whereas simulations suggest that the waves damp rapidly (Vazquez-Semadeni et al. 1999). Second, following Dewar, we assume that the waves are in the WKB limit (short wavelength), whereas the observations suggest that the largest amplitudes are in waves with wavelengths comparable to the size of the cloud; as a result, we expect our models to be more accurate in reproducing the mean value of the wave pressure than the value of the wave pressure at the edge of the cloud, for example. Finally, whereas simulations of one-dimensional MHD turbulence by Gammie & Ostriker (1996) confirmed that $\gamma^\text{p,w} = 0.5$ for small amplitudes, they indicate that $\gamma^\text{p,w}$ becomes significantly smaller for large amplitudes. It is not clear that this result can be applied to GMCs, however: much of the reduction in $\gamma^\text{p,w}$ they found was due to the fact that the gas became highly clumped at large amplitude, and the wave pressure was not much larger inside the clumps than outside. However, the question for GMCs is whether Alfvén waves can maintain a pressure gradient in a clumpy medium, and this would require a much larger scale simulation.

It is possible to infer $\gamma^\text{p,w}$ from observation, but the existing results are not definitive. If observations show that the density scales as $\rho \propto r^{-\rho}$ and the one-dimensional nonthermal velocity dispersion scales as $\sigma^\text{n} \propto r^\rho$, then $P^\text{w} \propto \rho \sigma^2$ implies

$$\gamma^\text{p,w} = 1 - \frac{2q}{p}. \quad (80)$$

The Myers (1985) reanalysis of the Larson (1981) heterogeneous data gave $p = 1.2$ and $q = 0.3$, corresponding to $\gamma^\text{p,w} = 0.5$. When Caselli & Myers (1995) focused on low-mass cores, they found $p = 1.1$ and $q = 0.53$, corresponding to $\gamma^\text{p,w} \approx 0$; such cores are believed to be largely supported by static magnetic fields and thermal pressure, however. (It should also be noted that their value of $p$ applies well outside the inner, isothermal core.) For massive cores, where the nonthermal motions are more significant, they found $p = 1.6$ and $q = 0.21$, corresponding to $\gamma^\text{p,w} \approx 0.75$. With the exception of low-mass cores, then, it appears that $\gamma^\text{p,w} \sim 0.5$ is not inconsistent with the data.

Adopting $\gamma^\text{p,w} = 0.5$ and $\gamma_w = 1.5$, we find that the maximum possible pressure ratio for a cloud supported by Alfvén waves is only 4.15; the corresponding density ratio of 17.26 is somewhat greater than that for a critical isothermal cloud. The polytropic constant varies as $K^\text{p} = \langle c^2 \rangle / \langle K^\text{p} \rangle = \langle \rho^{5/2} \rangle / \langle \rho \rangle - 2$ (eqs. [76] and [77]). The critical mass for a cloud supported by Alfvén waves can be expressed in terms of the nonthermal velocity dispersion $\sigma^\text{n}$ since the Alfvén wave pressure, including the pressure of the fluctuating fields, is

$$P^\text{w} = \rho \sigma^2 \quad (81)$$

(McKee & Zweibel 1995). For $\gamma^\text{p,w} = 3/2$ and $\gamma^\text{p,w} = 1/2$, we find $\mu^\text{p} = 0.21334$, $\psi^\text{cr} = \langle \rho \sigma^2 \rangle / \langle \rho \rangle = 0.71324$, and $\mu^\text{c} = 0.35405$. The critical mass in this case is then

$$M^\text{c} = 0.3919 \frac{\sigma^2}{G^{3/2} \rho_0^{1/2}} \quad (82)$$

This result for the critical mass is less than the estimate of McKee & Zweibel (1992) based on the virial theorem; they found a coefficient of 2.17 in the second expression. This discrepancy is due to the fact that the maximum possible value of $\mu$ in equilibrium (independent of stability) is governed by $\gamma^\text{p,w}$, which does not enter the virial theorem argument. In inferring the maximum mass allowed by the virial theorem, one implicitly assumes that the three terms in equation (32) can be of similar magnitude; however, for values of $\gamma^\text{p,w}$ significantly less than 1, the pressure becomes approximately constant in the cloud and this is not allowed.

Although the results of this paper are primarily for single-component polytropes, we shall present two results for multipressure polytropes. We find that the critical mass for a cloud consisting of a cloud supported by isothermal gas pressure and Alfvén waves is approximately

$$M^\text{c} = (M^\text{BE} + M^\text{w}^{2/3})^{3/2} \quad (83)$$

$$= 1.182 \left( \frac{\sigma^3}{G^{3/2} \rho_0^{1/2}} \right), \quad (84)$$

where the effective velocity dispersion $\sigma^\text{eff}$ is given by

$$\sigma^2 \equiv \sigma^2 + \langle \mu^\text{p,cr}/\mu^\text{BE} \rangle^{2/3} \langle c^2 \rangle \quad (85)$$

For Alfvén waves with $\gamma^\text{p} = 1/2$, the effective velocity dispersion becomes

$$\sigma^2 \equiv \sigma^2 + 0.67 \langle \sigma^2 \rangle \quad (86)$$

equation (84) is accurate to within 2% in this case. Thus Alfvén waves are less effective at supporting clouds despite the fact that their pressure is 1.5 times larger than $\rho \sigma^2$ because of the fluctuating magnetic fields.

Second, in Figure 10 we show the density distribution in a multipressure polytrope that reproduces some of the features observed in molecular clouds. We have chosen the polytropic index of the magnetic pressure to be $\gamma^\text{p} = 1$ in order to illustrate the dramatic effect of a nonisentropic pressure component (recall that $\gamma^\text{p} = 4/3$). The magnetic pressure is comparable with the Alfvén wave pressure, and the thermal pressure is 10% of the magnetic pressure. The case shown is a subcritical cloud, with a center-to-surface density contrast of 545 compared to the critical value of 1737; this density drop is far larger than that of the Bonnor-Ebert sphere shown for comparison, which has a density drop of 14. On the other hand, the mean-to-surface density contrast of 2.59 for the multipressure polytrope is only slightly greater than the value of 2.46 for the Bonnor-Ebert...
spherical symmetry, both of which become better approximations if a time average is taken. Rotation is neglected, which appears to be a reasonably good approximation (Goodman et al. 1993).

Several features of our analysis bear mention. First, since interstellar clouds are confined by the pressure of the ambient medium, we describe the cloud in terms of variables \( \mu \equiv M(G^{3/2}p^{1/2}/c) \) and \( \lambda \equiv r(G^{1/2}p^{1/2}/c) \), where \( c \equiv (P/\rho)^{1/2} \) that depend only on local properties of the cloud, not on the properties at the center of the cloud. The variable \( \mu \) is proportional to the ratio of the mass to the local value of the Jeans mass, whereas \( \lambda \) is proportional to the ratio of the radius to the local value of the Jeans radius. The resulting description is simpler than the standard one (see Appendix A and Stahler 1983). Next, we assume that the perturbations are adiabatic. As is often done in the study of stellar structure, we allow the adiabatic index \( \gamma_i \), which measures how the pressure of the \( i \)th component responds to a perturbation in density, to differ from the polytropic index \( \gamma_p \). The structure of the cloud depends on the values of \( \gamma_{pi} \), whereas the stability depends on the values of \( \gamma_i \) as well. Third, for pressure components that are not isentropic (i.e., those for which \( \gamma_i \neq \gamma_p \)), we consider two cases: for a locally adiabatic component, such as the thermal pressure or the magnetic pressure, the entropy of each mass element remains constant during a perturbation, but as a result the perturbed cloud is no longer a polytrope; on the other hand, for a globally adiabatic component, the entropy flows so as to maintain the polytropic relation between the pressure and density. This case is appropriate for Alfvén waves under the assumptions that the perturbation is applied quasi-statically and that there are no sources or sinks for the waves. We describe both locally and globally adiabatic pressure components in terms of the entropy parameter \( K_i = \frac{P}{\rho^{\gamma_i}} \). For a thermal gas, \( K_i \) is just \( c_{th}^2 \); for the magnetic field, the appropriate average of \( K_i \) is related to the magnetic flux; and for Alfvén waves, it is related to the wave action. In Appendix C, we show how to define the mean entropy parameter for a globally adiabatic pressure component so that it remains constant during an adiabatic change.

The essential feature of the three pressure components that support molecular clouds is that their equations of state are soft, with a limited ability to resist gravity: The thermal gas is approximately isothermal (\( \gamma_p \approx 1 \approx 1 \)) in regions of high extinction. The magnetic field must have \( \gamma = 4/3 \) in order to ensure flux freezing in spherical symmetry (§ 5.1); stability against the interchange instability requires \( \gamma_p \leq \gamma = 4/3 \). The turbulent pressure is modeled with Alfvén waves, which contribute a significant fraction of the pressure in MHD turbulence, and these waves can be rigorously shown to obey \( \gamma = 3/2 \) and \( \gamma_p = 1/2 \) in the limit of small amplitude, small wavelength, and negligible damping (McKee & Zweibel 1995). In contrast to the gas and the field, Alfvén waves are globally adiabatic, and clouds supported by Alfvén waves are subject to core collapse if the pressure drop inside the clouds is too large. As discussed in § 6.3, Alfvén waves are equivalent to a locally adiabatic pressure component with \( \gamma_p \text{(equiv)} \leq 1 \) insofar as determining the conditions for gravitational collapse.

The results in this paper are primarily for polytropes with only one pressure component; results for multipressure polytropes are deferred to another paper. We extend previous work on polytropes with a locally adiabatic pressure...
to the nonisentropic case \( \gamma \neq \gamma_p \). The parameter space for the stability of locally adiabatic polytropes is portrayed in Figure 5. Clouds with \( \gamma < \gamma_p \) are unstable to convection and are not considered here. Locally adiabatic polytropes are classified either as stellar polytropes (\( \gamma > 4/3 \) and \( \gamma_p > 6/5 \)) or as pressure-confined polytropes. Stellar polytropes can serve as models for stars since they are not subject to gravitational collapse and they can have a zero pressure boundary at a finite radius. On the other hand, pressure-confined polytropes must have a finite pressure at their surfaces, either because \( \gamma_p \leq 6/5 \) or in order to be stable against gravitational collapse with \( \gamma \leq 4/3 \). A polytrope supported by any of the pressure components in a molecular cloud must be pressure confined: In contrast to the case of a star, the stability of a molecular cloud is determined by conditions at its surface.

We introduced the concept of a globally adiabatic pressure component as an idealized model for treating the turbulent motions in molecular clouds. Molecular clouds are magnetized, and as a result the turbulent motions can be considered to be a superposition of MHD waves. The pressure associated with wave motions cannot be locally adiabatic since the waves can move from one part of a cloud to another under the influence of a perturbation. However, in the absence of sources and sinks, the total entropy of the waves remains constant—the system is globally adiabatic. A gravitationally bound cluster of stars is another example of a globally adiabatic system. We have determined the appropriate entropy parameter for a nonuniform, globally adiabatic pressure component; for the spatially isothermal case (\( \gamma_p = 1 \)), it is just the exponential of the usual entropy evaluated by LBW. We then generalized the standard treatment of the stability of polytropes to include a globally adiabatic component with an arbitrary value of \( \gamma_p \). In contrast to locally adiabatic systems, globally adiabatic systems can be unstable to gravitational collapse even when \( \gamma > 4/3 \); provided \( \gamma_p < 6/5 \), they undergo core collapse beyond the critical point (the "gravo-thermal catastrophe" of LBW). This is in contrast to the phenomenological model used by Maloney (1988) in which gravitational instability is impossible. Our results for the critical pressure and density ratios are shown in Figures 8 and 9; these are smaller than for the corresponding locally adiabatic cases. Alfvén waves, which we have used to model for the turbulent pressure, are particularly simple because their pressure is isotropic; as shown in equations (84) and (86) and as discussed at the end of § 6.4, the nonthermal motions associated with such waves are somewhat less effective in supporting a cloud than are the thermal motions of an isothermal gas.

The critical mass—i.e., the maximum mass that is stable against gravitational collapse for a given entropy distribution and ambient pressure—is evaluated in terms of the parameters \( \mu_{ct} \) and \( \mu_{c}' \),

\[
M_{ct} = \frac{\mu_{ct} c_s^3}{G^{3/2} \rho_\odot^{1/2}} = \frac{\mu_{ct} (\varepsilon^2 / \gamma)^{3/2}}{G^{3/2} \rho_\odot^{1/2}}. \tag{87}
\]

The values of \( \mu_{ct} \) and \( \mu_{c}' \) for isentropic polytropes are plotted in Figures 6 and 7; the values of \( \mu \) and \( \mu_{c}' \) for singular polytropic spheres are also shown. The values of \( \mu_{ct} \) and \( \mu_{c}' \) depend primarily on the value of \( \gamma_p \); both are of order unity for \( \gamma_p \sim 1 \) and approach zero as \( \gamma_p \rightarrow 0 \). The maximum value of \( \mu \) for any polytrope with \( \gamma_p \leq 4/3 \) is 4.555, the value of \( \mu_{ct} \) at \( \gamma_p = 4/3 \). Negative-index polytropes have \( \mu < 1.1822 \), the value of \( \mu_{c}' \) for an isothermal sphere. The ratio of the mean density or pressure to the surface value is less than 4 for any negative-index polytrope. On the other hand, the density and pressure contrasts between the center and the edge of a critically stable polytrope rise from the values shown in Figure 3 as \( \gamma \) becomes greater than \( \gamma_p \). In the region denoted "SPS" in Figure 5, locally adiabatic, singular polytropic spheres are formally stable against collapse; the region in which spheres with large density contrasts are stable against finite perturbations may be more limited, as discussed in § 5.4. Non-isentropic polytropes may thus provide an explanation for the very large density contrasts observed between cores within molecular clouds and the edges of the clouds (see Fig. 10 and Curry & McKee 1999).

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APPENDIX A

STANDARD NONDIMENSIONAL VARIABLES AND EQUATIONS

A set of "standard" dimensionless variables and associated equations is generally used to describe the structure of polytropic gas spheres (e.g., Chandrasekhar 1939). Polytropes satisfy

\[
P(r) = K_p \rho(r)^\gamma \tag{A1}
\]

where \( K_p \) is independent of position. The standard equations for isothermal spheres (with \( \gamma_p = 1 \)) are different from those for other polytropic spheres. An advantage of the variables we use (or of the homology variables \( u \) and \( (n + 1)u \)) is that a single set of equations applies to both cases. Furthermore, our formulation applies to multiple pressure components as well, whereas the standard formulation is restricted to single pressure components. Here we compare the standard isothermal and polytropic variables and equations to those introduced in this work. Some of these relations have been given previously by Stahler (1983), who focused on the case \( n > 0 \); his \( M_p \) and \( R_p \) are proportional to our \( \mu \) and \( \lambda \), respectively.

In the standard equations, the polytropic constant \( n \) describes the structure. The relationship between \( n \) and the polytropic index \( \gamma_p \) is \( \gamma_p = (n + 1)/n \). Isothermal spheres have \(|n| \to \infty \); the case \( n = -1 \) corresponds to an isobaric cloud.
A1. DIMENSIONLESS VARIABLES AND STRUCTURE EQUATIONS

The standard dependent variables are

\[ \Psi \equiv \ln \frac{\rho_c}{\rho} \quad \text{(isothermal spheres)} \]  \hspace{1cm} (A2)

and

\[ \theta \equiv \left( \frac{\rho}{\rho_c} \right)^{1/n} \quad \text{(other polytropes)}, \]  \hspace{1cm} (A3)

where \( \rho_c \) is the central density. In this work we introduce a nondimensional mass \( \mu \) in equation (10) and a nondimensional radius \( \lambda \) in equation (11) as dependent variables.

The standard independent variable is a dimensionless radius:

\[ \xi \equiv R \left( \frac{c^2}{4\pi G \rho_c} \right)^{-1/2} \quad \text{(isothermal spheres)}, \]  \hspace{1cm} (A4)

\[ \xi \equiv R \left( \frac{|n+1| c^2}{4\pi G \rho_c} \right)^{-1/2} \quad \text{(other polytropes)}, \]  \hspace{1cm} (A5)

the independent variable in our analysis is \( \chi = \ln (\rho_c/\rho) \).

The differential structure equations are derived using the equation of hydrostatic equilibrium (eq. [7]). In the standard nondimensional formulation, the equilibrium condition is reflected in the Lane-Emden equation,

\[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Psi}{d\xi} \right) = e^{-\Psi} \quad \text{(isothermal spheres)}, \]  \hspace{1cm} (A6)

\[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\frac{|n+1|}{n+1} \theta^n \quad \text{(other polytropes)}, \]  \hspace{1cm} (A7)

The structure equations used in this work are equations (20), (21), and, for multipressure polytropes, equation (14).

The boundary conditions for integration are \( \Psi = 0, d\Psi/d\xi = 0 \) at \( \xi = 0 \) for isothermal spheres, and \( \theta = 1, d\theta/d\xi = 0 \) at \( \xi = 0 \) for other polytropes. In our formulation, \( \mu = 0 \) and \( \lambda = 0 \) at \( \chi = 0 \). Note that both \( \xi \) and \( \chi \) vanish at the origin.

A2. DIMENSIONAL QUANTITIES

Equation (A3) states \( \rho \propto \theta^n \); together with the ideal gas law, this proportionality implies that the temperature \( T \propto \theta \), so \( c^2 = c_s^2 \theta \). Chandrasekhar (1939) and Viala & Horedt (1974) identify the following additional proportionals for polytropic structures: \( \xi \propto \text{distance}, \theta^\prime \propto \text{temperature gradient}, \theta^{n+1} \propto \text{pressure}, \) and \([-\left(n+1\right)/(n+1)]\xi^2 \theta^\prime \propto M(\xi)\).

In the standard problem, the dimensional radius can be determined using equations (A4) and (A5) together with the equation of hydrostatic equilibrium; alternate forms are

\[ R = \frac{GM}{c^2} \left( \xi \frac{d\Psi}{d\xi} \right)^{-1} \quad \text{(isothermal spheres)}, \]  \hspace{1cm} (A8)

\[ R = -\frac{1}{n+1} \frac{GM}{c^2} \left( \xi \frac{d\theta}{d\xi} \right)^{-1} \quad \text{(other polytropes)}, \]  \hspace{1cm} (A9)

and

\[ R = \frac{GM}{c^2} \frac{\lambda}{\mu} \quad \text{(this work)}. \]  \hspace{1cm} (A10)

We may write the pressure as

\[ P = \frac{1}{4\pi} \frac{c^8}{M^2 G^3} \xi^4 \left( \frac{d\Psi}{d\xi} \right)^2 e^{-\Psi} \quad \text{(isothermal spheres)}, \]  \hspace{1cm} (A11)

\[ P = \frac{|n+1|^3}{4\pi} \frac{c^8}{M^2 G^3} \xi^4 \theta^n \left( \frac{d\theta}{d\xi} \right)^2 \quad \text{(other polytropes)}, \]  \hspace{1cm} (A12)

and

\[ P = \frac{c^8}{M^2 G^3} \mu^2 \quad \text{(this work)}. \]  \hspace{1cm} (A13)

We have chosen to express \( R \) and \( P \) in terms of \( M(R) \) and \( c(R) \) for our variables, since in our approach the dependent variables depend only on properties at the surface. Alternatively, one can express the results in our variables in terms of the central sound speed; for a single-component polytrope, the required relation is \( c^2 = c_s^2 \exp \left[ -(\gamma - 1)\lambda \right] \).
Finally, the mass can be expressed as

$$M = \frac{1}{(4\pi)^{1/2}} \frac{c^3}{G^{1/2} \rho_c^{1/2}} \frac{\zeta^2 d\Psi}{d\zeta} \quad \text{(isothermal spheres),}$$  \hspace{1cm} (A14)

$$M = - \frac{|n + 1|^{5/2}}{(4\pi)^{1/2}(n + 1)} \frac{c^3}{G^{1/2} \rho_c^{1/2}} \frac{\zeta^2 d\theta}{d\zeta} \quad \text{(other polytropes),}$$  \hspace{1cm} (A15)

and

$$M = \frac{c^3}{G^{3/2} \rho_c} \quad \text{(this work).}$$  \hspace{1cm} (A16)

If desired, the last result can be expressed in terms of the central density by using the relation $\rho = \rho_c \exp(\chi)$.

### A3. CONVERSION AMONG DEPENDENT VARIABLES

**Isothermal case.**—Since $\Psi = \chi$, we find

$$\mu = \frac{1}{(4\pi)^{1/2}} \frac{c^3}{G^{1/2} \rho_c^{1/2}} \frac{\zeta^2 d\Psi}{d\zeta} e^{-\Psi/2},$$  \hspace{1cm} (A17)

$$\lambda = \frac{1}{(4\pi)^{1/2}} \frac{c^3}{G^{1/2} \rho_c^{1/2}} \frac{\zeta e^{-\Psi/2}}{\xi},$$  \hspace{1cm} (A18)

and

$$\xi = (4\pi)^{1/2} \lambda e^{\chi/2},$$  \hspace{1cm} (A19)

$$\frac{d\Psi}{d\xi} = \frac{1}{(4\pi)^{1/2}} \frac{c^3}{G^{1/2} \rho_c^{1/2}} \frac{\mu}{\lambda^2} e^{-\chi/2}.$$  \hspace{1cm} (A20)

**Polytropic case.**—Since $\theta = \exp(-\chi/n)$, we find

$$\mu = - \frac{|n + 1|^{1/2}}{(4\pi)^{1/2}} \frac{c^3}{G^{1/2} \rho_c^{1/2}} \frac{\xi^2 \theta^{n-3/2}}{\xi^2 \theta^{n-1/2}} \frac{d\theta}{d\xi},$$  \hspace{1cm} (A21)

$$\lambda = \frac{|n + 1|^{1/2}}{(4\pi)^{1/2}} \frac{c^3}{G^{1/2} \rho_c^{1/2}} \frac{\xi \theta^{n-1/2}}{\xi^2 \theta^{n-1/2}},$$  \hspace{1cm} (A22)

and

$$\xi = (4\pi)^{1/2} \frac{\lambda e^{\xi/2}}{|n + 1|^{1/2}} \frac{\theta^{n-1/2}}{\theta^{n-1/2}},$$  \hspace{1cm} (A23)

$$\frac{d\theta}{d\xi} = \frac{|n + 1|^{1/2}}{(4\pi)^{1/2}(n + 1)} \frac{\mu}{\lambda^2} e^{-\gamma_p \chi/2}. $$  \hspace{1cm} (A24)

### APPENDIX B

**DERIVATION OF EQUATIONS FOR THE PERTURBED STRUCTURE**

#### B1. PERTURBATIONS IN $\mu$ AND $\lambda$

To assess the stability of multipressure polytropes, we must determine the structure they assume after a perturbation. We consider Lagrangian perturbations so that $\delta M = 0$. For each pressure component we have $P_i = K_{pi} \rho_i^{\gamma_i}$. Prior to the perturbation, $K_{pi}$ is constant in space; after the perturbation, $K_{pi}$ will generally have a different value, and it may depend on position. In general, we find

$$\delta \ln P_i = \delta \ln K_{pi} - \gamma_p \delta \chi,$$  \hspace{1cm} (B1)

where we have used the fact that $\rho_c$ is defined to be the central density of the unperturbed sphere, so that $\delta \ln \rho = -\delta \chi$. The perturbation in the total pressure is then

$$\delta \ln P = \sum_i \left(\frac{P_i}{P}\right) \delta \ln P_i = \sum_i \left(\frac{P_i}{P}\right) \delta \ln K_{pi} - \gamma_p \delta \chi.$$  \hspace{1cm} (B2)
We can evaluate \( \delta \chi \) from the fact that, at constant mass, the definition of \( \mu \) implies that \( \mu \propto \rho^2 P^{-3/2} \). Varying this relation and combining it with equation (B2), we find

\[
\delta \chi = -\frac{2}{4 - 3\gamma_p} \left[ \delta \ln \mu + \frac{3}{2} \sum_i \left( \frac{P_i}{P} \right) \delta \ln K_{pi} \right],
\]

(B3)

and

\[
\delta \ln P = \frac{2}{4 - 3\gamma_p} \left[ \gamma_p \delta \ln \mu + 2 \sum_i \left( \frac{P_i}{P} \right) \delta \ln K_{pi} \right].
\]

(B4)

We now apply Lagrangian variations to the structure equations. Starting with equation (16), we have

\[
\frac{d\delta \chi}{d\chi} = -\frac{\delta \gamma_p}{\gamma_p} - \delta \ln \left( \frac{\lambda^2}{\mu^2} \right).
\]

(B5)

It is convenient to insert equation (16) into equation (20) before carrying out its variation:

\[
d\ln \mu = d\ln M - \frac{1}{2}(4 - 3\gamma_p) d\chi.
\]

(B6)

Varying equations (B6) and (21), and using equation (B5), we obtain the equations for \( d \delta \mu /d\chi \) (eq. [36]) and \( d \delta \lambda /d\chi \) (eq. [37]) in the text.

**B2. EVALUATION OF \( \delta \gamma_p \) FOR ADIABATIC PERTURBATIONS**

To complete the set of equations for the perturbed structure, we must determine the perturbation in the polytropic index, \( \delta \gamma_p \). Recall that a perturbed locally adiabatic polytrope is not itself a polytrope unless it is isentropic (§ 2.2). As a result, we must allow for a variation in \( K_{pi} \) when we evaluate the polytropic index:

\[
\gamma_p \equiv \frac{d \ln P}{d \ln \rho} = \sum_i \left( \frac{P_i}{P} \right) \left( \gamma_{pi} - \frac{d \ln K_{pi}}{d \chi} \right),
\]

(B7)

from equation (B1). (In equilibrium, the cloud is a polytrope, \( K_{pi} \) is independent of position, and \( \gamma_p \) reduces to \( \Sigma \gamma_{pi}(P_i/P) \) as it must—see eq. [15].) Varying equation (B7), we obtain

\[
\delta \gamma_p = \sum_i \left( \frac{P_i}{P} \right) \left[ (\gamma_{pi} - \gamma_p) \delta \ln K_{pi} - \gamma_{pi}(\gamma_{pi} - \gamma_p) \delta \chi + \frac{d \ln K_{pi}}{d \chi} \right],
\]

(B8)

with the aid of equation (B1). We have used the fact that \( K_{pi} \) is constant \( (d \ln K_{pi}/d\chi = 0) \) in the initial equilibrium state in order to simplify this result.

For a locally adiabatic pressure component \( \ell \), we have \( P_{\ell} = K_{\ell} \rho^{\gamma_{p\ell}} \), where the entropy parameter \( K_{\ell} \) is constant during a perturbation. As a result, we have \( \delta \ln P_{\ell} = \gamma_{p\ell} \delta \ln \rho = -\gamma_{p\ell} \delta \chi \), so that

\[
\delta \ln K_{p\ell} = (\gamma_{p\ell} - \gamma_p) \delta \chi,
\]

(B9)

from equation (B1). As a result, such a component contributes a term

\[
(\delta \gamma_{p\ell})_{\ell} = \left( \frac{P_{\ell}}{P} \right) \left[ (\gamma_{p\ell} - \gamma_p) \frac{d \delta \chi}{d \chi} - \gamma_{p\ell}(\gamma_{p\ell} - \gamma_p) \delta \chi \right]
\]

(B10)

to \( \delta \gamma_p \).

For a globally adiabatic pressure component, it is the total entropy of the cloud that remains constant during a perturbation. In order to be distinct from the locally adiabatic case, the component must be nonisentropic \( (\gamma_{p\ell} \neq \gamma_p) \). Redistribution of energy in the cloud during the perturbation maintains the polytropic form, \( P_{\ell} \propto \rho^{\gamma_{p\ell}} \), but changes the value of \( K_{p\ell} \) (the entropy parameter \( K_{p\ell} \) changes as well). The change in \( K_{p\ell} \) needed to maintain constant entropy will be discussed in § 6 below. All that we need to note here is that since \( K_{p\ell} \) is independent of position for a globally adiabatic pressure component, such a component contributes a term

\[
(\delta \gamma_{p\ell})_g = \frac{P_{\ell}}{P} (\gamma_{p\ell} - \gamma_p)(\delta \ln K_{p\ell} - \gamma_{p\ell} \delta \chi)
\]

(B11)

to \( \delta \gamma_p \) from equation (B8).

In a multipressure polytrope, both types of pressure components may be present. For simplicity, we shall assume that only one globally adiabatic component is present; the generalization to more than one such component is straightforward. We find it convenient to introduce a new adiabatic index

\[
\Gamma = \sum_i \left( \frac{P_i}{P} \right) \gamma_i + \left( \frac{P_g}{P} \right) \gamma_{p\ell},
\]

(B12)
which is the weighted mean of the adiabatic indices for the locally adiabatic components \( \ell \) and the polytropic index for the globally adiabatic component \( g \). This is equivalent to

\[
\Gamma = \gamma_p + \sum_{i=l} \left( \frac{P_i}{P} \right) (\gamma_i - \gamma_p) .
\]

A related quantity that we shall need is

\[
\Gamma' = \sum_{i=l} \left( \frac{P_i}{P} \right) \gamma_i (\gamma_p - \gamma_i) + \left( \frac{P_g}{P} \right) \gamma_p (\gamma_p - \gamma_p) .
\]

In terms of these quantities, we find

\[
\delta \chi = -\frac{2}{4 - 3\Gamma} \left[ \delta \ln \mu + \frac{3}{2} \left( \frac{P_g}{P} \right) \delta \ln K_{pg} \right] ,
\]

from equations (B3) and (B4).

Evaluating \( \delta \ln \gamma_p \) from equation (B8), we then obtain equation (42) in the text.

**APPENDIX C**

**ENTROPY FOR GLOBALLY ADIABATIC PRESSURE COMPONENTS**

**C1. DETERMINATION OF THE ENTROPY**

Our objective is to evaluate the entropy parameter \( K_{pg,ref} \) (eq. [71]) for a globally adiabatic pressure component \( g \) for an arbitrary nonuniform cloud. To do this, we generalize the McKee & Zweibel (1995) treatment of Alfvén waves to arbitrary \( Alfvén \) values of and The equation for the internal energy of an ideal gas is

\[
\frac{\partial u_g}{\partial t} + \nabla \cdot (u_g v + q_g) + P_g \nabla \cdot v = \mathcal{S}_g ,
\]

where \( q_g \) is the heat flux and \( \mathcal{S}_g \) is the source term for the globally adiabatic pressure component (e.g., McKee et al. 1987). (Note that it is possible for \( \mathcal{S}_g \) to vanish in our model even in the presence of heating and cooling, provided the effects of the heating and cooling are represented by internal degrees of freedom.) The total internal energy in the cloud associated with globally adiabatic pressure component is \( U_g = \int u_g dV \), where the integral extends over the entire cloud. The rate of change of this energy is

\[
\frac{dU_g}{dt} = \int \frac{\partial u_g}{\partial t} dV + \int u_g v \cdot dS .
\]

Eliminating \( \partial u_g/\partial t \) with equation (C1) and using the divergence theorem, this becomes

\[
\frac{dU_g}{dt} = \int \left[ -P_g \nabla \cdot v + \mathcal{S}_g \right] dV - \int q_g \cdot dS ,
\]

where \( dS \) is an element of the surface bounding the system. For an adiabatic process, the last two terms vanish. Note that we do not assume that the heat flux is zero inside the cloud, only at the surface. We do assume that there is no exchange of heat with the other pressure components; for example, we do not allow for the possibility that energy would be transferred from the waves to the thermal motions of the gas by wave damping. The equation of continuity then allows us to rewrite equation (C3) as

\[
\frac{dU_g}{dt} = \int P_g \frac{d\rho}{dt} dV = K_{pg}(t) \int \rho^{\gamma_g - 2} \frac{d\rho}{dt} dM ,
\]

where we have included the argument in \( K_{pg}(t) \) to emphasize that it may change with time. Since the internal energy is given by

\[
U_g = \frac{1}{\gamma_g - 1} \int P_g dV = \frac{K_{pg}}{\gamma_g - 1} \int \rho^{\gamma_g - 1} dM ,
\]

we can eliminate the unknown \( K_{pg} \) from equation (C4):

\[
\frac{d\ln U_g}{dt} = \left( \frac{\gamma_g - 1}{\gamma_g - 1} \right) \frac{d\ln \left( \int \rho^{\gamma_g - 1} dM \right)}{dt} .
\]
The internal energy can also be expressed in terms of the mean square sound speed $\langle c_s^2 \rangle$:

$$U_g = \frac{1}{\gamma_g - 1} \int \left( \frac{P_g}{\rho} \right) dM \equiv \frac{1}{\gamma_g - 1} M \langle c_s^2 \rangle .$$

As a result, equation (C6) can be reexpressed as

$$\frac{d \ln \langle c_s^2 \rangle}{dt} = \left( \frac{\gamma_g - 1}{\gamma_{ps} - 1} \right) \frac{d \ln \left( \langle c_s^2 \rangle^2 / K_{ps} \right)}{dt} .$$

Integrating this and evaluating the integration constant in the reference state, we find

$$K_{\gamma, ref}^{-1} = K_{ps}^{-1} \langle c_s^2 \rangle \left( \gamma_g - \gamma_{ps} \right) ,$$

which is the desired result.

Next, we express the entropy parameter for a globally adiabatic component in a form similar to that for a locally adiabatic component, $\langle K_g \rangle$. We define the mean entropy parameter as

$$\langle K_g \rangle \left( \gamma_{ps} - \gamma_g \right) \equiv \int \frac{1}{M} \int K_{\gamma, ref}^{-1} \langle c_s^2 \rangle dM .$$

Equation (C9) relates the entropy parameter $K_{\gamma, ref}$ to the polytropic parameter $K_{ps}$ in terms of $\langle c_s^2 \rangle$, which is proportional to the mean temperature of the pressure component. In the particular case of an isentropic component, we have simply $\langle K_g \rangle = K_{ps}$, as was clear from the definitions. For the globally adiabatic case of greatest interest to us, Alfven waves, we have $\gamma_g = 3/2$ and $\gamma_{ps} = 1/2$, so that $\langle K_g \rangle = \langle c_s^2 \rangle / K_{ps}$. With the aid of equation (C9), we can express the entropy parameter in terms of directly observable quantities if we choose the pressure in the reference state to be the surface pressure $P_{gs}$:

$$\langle K_g \rangle = \frac{c_s^{2/3}}{\psi \left( \gamma_g - \gamma_{ps} \right) / (\gamma_{ps} - 1) \psi_{gs}^{-1}} ,$$

where $\psi_{gs} \equiv \langle c_s^2 \rangle / c_s^2$ is usually of order unity, and we have included the subscript "s" to emphasize that the quantity is to be evaluated at the surface of the cloud.

**C2. EVALUATION OF $\delta \ln K_{ps}$ AT CONSTANT ENTROPY**

To evaluate $\langle \delta c_g \rangle$ in equation (B11), we need to know how $K_{ps}$ changes as the result of a globally adiabatic perturbation. Variation of equation (C9) at constant entropy gives

$$\left( \gamma_g - 1 \right) \delta \ln K_{ps} = \left( \gamma_g - \gamma_{ps} \right) \delta \ln \langle c_s^2 \rangle .$$

Now, with the aid of equation (B1) we find

$$\delta c_g^2 = \frac{\rho}{P} \left( \delta \ln P_g - \delta \ln \rho \right) = c_g^2 \left[ \delta \ln K_{ps} - \left( \gamma_{ps} - 1 \right) \delta \xi \right] .$$

Noting that $\delta \ln K_{ps}$ is constant inside the cloud, we obtain

$$\delta \ln \langle c_g^2 \rangle = \delta \ln K_{ps} - \frac{\gamma_{ps} - 1}{M \langle c_g^2 \rangle} \int c_g^2 \delta \xi dM .$$

Inserting this result into equation (C14), we then find

$$\delta \ln K_{ps} = \frac{2 \left( \gamma_g - \gamma_{ps} \right)}{M \langle c_g^2 \rangle} \left[ \delta \ln \mu \left[ (4 - 3 \Gamma) c_g^2 \right] dM \int \left\{ 4 - 3 \sum \gamma \left( \rho / \rho \right) \left[ (4 - 3 \Gamma) c_g^2 \right] dM \right\} .$$

with the aid of equation (B15) for $\delta \xi$.

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